A DUALITY THEORY FOR UNBOUNDED HERMITIAN OPERATORS IN HILBERT SPACE

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Abstract. We develop a duality theory for unbounded Hermitian operators with dense domain in Hilbert space. As is known, the obstruction for a Hermitian operator to be selfadjoint or to have selfadjoint extensions is measured by a pair of deficiency indices, and associated deficiency spaces; but in practical problems, the direct computation of these indices can be difficult. Instead, in this paper we identify additional structures that throw light on the problem. While duality considerations are a tested tool in mathematics, we will attack the problem of computing deficiency spaces for a single Hermitian operator with dense domain in a Hilbert space which occurs in a duality relation with a second Hermitian operator, often in the same Hilbert space.

I. Introduction

The theory of unbounded Hermitian operators with dense domain in Hilbert space was developed by H. M. Stone and John von Neumann with view to use in quantum theory; more precisely to put the spectral theory of the Schrödinger equation on a sound mathematical foundation. Early in the theory, it was realized that a Hermitian operator may not be selfadjoint. It was given a quantitative formulation in the form of deficiency indices and deficiency spaces, and we refer the reader to the books [20] and [21], and more recently [11] and [19]. In physical problems, see e.g., [6], these mathematical notions of defect take the form of “boundary conditions;” for example waves that are diffracted on the boundary of a region in Euclidean space; the scattering of classical waves on a bounded obstacle [16]; a quantum mechanical “particle” in a repulsive potential that shoots to infinity in finite time; or in more
recent applications (see e.g., [13], [7], [8], [18]) random walk on infinite weighted graphs $G$ that “wonder off” to points on an idealized boundary of $G$. In all of the instances, one is faced with a dynamical problem: For example, the solution to a Schrödinger equation, represents the time evolution of quantum states in a particular problem in atomic physics. The operators in these applications will be Hermitian, but in order to solve the dynamical problems, one must first identify a selfadjoint extension of the initially given operator. Once that is done, von Neumann’s spectral theorem can then be applied to the selfadjoint operator. A choice of selfadjoint extension will have a spectral resolution, i.e., it is an integral of an orthogonal projection valued measure; with the different extensions representing different “physical” boundary conditions. Since non-zero deficiency indices measure that degree of non-selfadjointness, the question of finding selfadjoint extensions takes on some urgency.

Now the variety of applied problems that lend themselves to computation of deficiency indices and the study of selfadjoint extensions are vast and diverse. As a result, it helps if one can identify additional structures that throw light on the problem. Here duality considerations within the framework of Hilbert space are tested tools in applied mathematics. In this paper we will device such a geometric duality theory: We will attack the problem of computing deficiency spaces for a single Hermitian operator with dense domain in a Hilbert space which occurs in a duality relation with a second Hermitian operator, often in the same Hilbert space. We will further use our duality to prove essential selfadjointness of families of Hermitian operators that arise naturally in reproducing kernel Hilbert spaces. The latter include graph Laplacians for infinite weighted graphs $(G, w)$ with the Laplacian in this context presented as a Hermitian operator in an associated Hilbert space of finite energy functions on the vertex set in $G$. Other examples include Hilbert spaces of band-limited signals. Further applications enter into the techniques used in discrete simulations of stochastic integrals, see [12]. We encountered the present operator theoretic duality in our study of discrete Laplacians, which in turn have part of its motivation in numerical analysis. A key tool in applying numerical analysis to solving partial differential equations is discretization, and use of repeated differences; see e.g., [5].

Specifically, one picks a grid size $h$, and then proceeds in steps: (1) Starting with a partial differential operator, then study an associated discretized operator with the use of repeated differences on the $h$-lattice in $\mathbb{R}^d$. (2) Solve the discretized problem for $h$ fixed. (3) As $h$ tends to
zero, numerical analysts evaluate the resulting approximation limits, and they bound the error terms.

Our present approach, based on reproducing kernels and unbounded operators, fits into a larger framework in applied operator theory, for example the use of reproducing kernel Hilbert spaces in the determination of optimal spectral estimation: Here the problem is to estimate some sampled signal represented as the sum of a deterministic (time-)
function and a term representing noise, for example white noise; see e.g., [10, 12]. For the multivariable case, the process under study is indexed by some prescribed discrete set $X$ (representing sample points; it could be the vertex set in an infinite graph). The choice of statistical distribution, modeling the noise term, then amounts to a selection of a reproducing kernel (representing function differences) with vectors $v_x$ (dipoles in the present context), and linear combinations of these vectors $v_x$ in this approach then represents a spectral estimator. The problem becomes that of selecting samples which minimize error terms in a prediction of a signal.

II. Reproducing kernel-Hilbert spaces

For this purpose, one must use a metric, and the norm in Hilbert space has proved an effective tool, hence the Hilbert spaces and the operator theory. This procedure connects to our present graph-Laplacians: When discretization is applied to the Laplace operator in $d$ continuous variables, the result is the graph of integer points $\mathbb{Z}^d$ with constant weights. But if numerical analysis is applied instead to a continuous Laplace operator on a Riemannian manifold, the discretized Laplace operator will instead involve infinite graph with variable weights, so with vertices in other configurations than $\mathbb{Z}^d$. Inside the technical sections we will use standard tools from analysis and probability. References to the fundamentals include [10], [15], [17] and [22]. There is a large literature covering the general theory of reproducing kernel Hilbert spaces and its applications, see e.g., [3], [1], [2], [4], and [23]. Such applications include potential theory, stochastic integration, and boundary value problems from PDEs among others.

In brief summary, a reproducing kernel Hilbert space consists of two things: a Hilbert space of functions $f$ on a set $X$, and a reproducing kernel $k$, i.e., a complex valued function $k$ on $X \times X$ such that for every $x$ in $X$, the function $k(\cdot, x)$ is in and reproduces the value $f(x)$ from the inner product $< f, k(\cdot, x) >$ in $\mathcal{H}$ so the formula

$$f(x) = < f, k(\cdot, x) >$$
holds for all $x$ in $X$. Moreover, there is a set of axioms for a function $k$ in two variables that characterizes precisely when it determines a reproducing kernel Hilbert space. And conversely there are necessary and sufficient conditions that apply to Hilbert spaces $\mathcal{H}$ and decide when $\mathcal{H}$ is a reproducing kernel Hilbert space. Here we shall restrict these “reproducing” axioms and obtain instead a smaller class of reproducing kernel Hilbert spaces. We add two additional axioms: Firstly, we will be reproducing not the values themselves of the functions $f$ in $\mathcal{H}$, but rather the differences $f(x) - f(y)$ for all pairs of points in $X$; and secondly we will impose one additional axiom to the effect that the Dirac mass at $x$ is contained in $\mathcal{H}$ for all $x$ in $\mathcal{H}$. In more precise form, the axioms are as follows:

(i) For all $x, y \in X$, $\exists w_{x,y} \in \mathcal{H}$ such that $f(x) - f(y) = \langle f, w_{x,y} \rangle$; and

(ii) For all $x \in X$, we have $\delta_x \in \mathcal{H}$.

Quantum states in physics are represented by norm-one vectors $v$ in some Hilbert space $\mathcal{H}$, i.e., $\|v\|_\mathcal{H} = 1$. Hence the significance of assumption (ii) is to allow us to “place” quantum states on the points in some prescribed set $X$ which allows a reproducing kernel-Hilbert space $\mathcal{H}$, subject to condition (ii): If $x \in X$, then the corresponding quantum state is $\|\delta_x\|_\mathcal{H}^{-1} \delta_x$; and the transition probability $x \mapsto y$ is

$$p_{x,y} := \|\delta_x\|_\mathcal{H}^{-1} \|\delta_y\|_\mathcal{H}^{-1} |\langle \delta_x, \delta_y \rangle_\mathcal{H}|.$$  

When these two additional conditions (i)–(ii) are satisfied, we say that $\mathcal{H}$ is a relative reproducing kernel Hilbert space. It is known that every weighted graph (the infinite case is of main concern here) induces a relative reproducing kernel Hilbert space, and an associated graph Laplacian. A main result in section VII below is that the converse holds: Given a relative reproducing kernel Hilbert space $\mathcal{H}$ on a set $X$, it is then possible in a canonical way to construct a weighted graph $G$ such that $X$ is the set of vertices in $G$, and such that its energy Hilbert space coincides with $\mathcal{H}$ itself. In our construction, the surprise is that the edges in $G$ as well as the weights on the edges may be built directly from only the Hilbert space axioms defining the initially given relative reproducing kernel Hilbert space. Since this includes all infinite graphs of electrical resistors and their potential theory (boundaries, harmonic functions, and graph Laplacians) the result has applications to these fields, and it serves to unify diverse branches in a vast research area.
III. Other Applications

One additional application of our relative reproducing kernel-Hilbert spaces to infinite graphs \( G \) entails the concept of "graph-boundary." This is part of the study of discrete dynamical systems and their harmonic analysis, i.e., following infinite paths in the vertex set of \( G \), and computing probabilities of sets of infinite paths.

While there is already a substantial literature on "boundaries" in the case of bounded harmonic functions on infinite weighted graphs \((G, w)\), our present setting has a quite different flavor. We are concerned with harmonic functions \( h \) of finite energy, and our reproducing kernel Hilbert spaces are chosen such as to make this precise, as well as serving as a computational device. An important technical point is that these "finite-energy Hilbert spaces" do not come equipped with an \textit{a priori} realization as \( L^2 \)-spaces.

This fact further explains why the resulting boundary theory is somewhat more subtle than is the better known and better understood theory for the case of bounded harmonic functions. Moreover there does not appear to be a direct way of comparing the two "boundary theories."

There are some good intuitive reasons why stochastic integrals should "have something to" do with boundaries and finite energy for infinite weighted graphs \((G, w)\); indeed be a crucial part of this theory. Indeed, a fixed choice of weights on edges in \( G \) (for example conductance numbers) yields probabilities for a random walk. Going to the "boundary" for \((G, w)\) involves a subtle notion of limit, and it is a well known principle that suitable limits of random walk yield Brownian motion realized in \( L^2 \)-spaces of global measures (e.g., Wiener measure), and so corresponding to the stochastic nature of Brownian motion.

The discreteness of vertex sets in infinite graphs, has a quantum aspect as well [9], [14]. It enters when inner products from a chosen reproducing kernel-Hilbert space is used in encoding transition probabilities, i.e., computing a transition between two vertices in \( G \) as the absolute value of the inner product of the corresponding Dirac-delta functions. Hence, vertices in \( G \) play the role of quantum states.

Let \( \mathcal{H} \) be a reproducing kernel Hilbert space of functions on some fixed set \( X \); we assume properties (i)–(ii) above. There is then a dense linear subspace \( \mathcal{D} \subseteq \mathcal{H} \), and a hermitian operator \( \Delta: \mathcal{D} \to \mathcal{H} \) determined by

\[
(\Delta u)(x) := \langle \delta_x, u \rangle, \quad \forall u \in \mathcal{D},
\]

where \( \langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathcal{H}} \) refers to the inner product in \( \mathcal{H} \).
Definition 3.1. Let \( \mathcal{H}, X, \Delta \) be as described above; and let \( x_0 \in X \) be given. A vector \( w (= w_{x_0}) \) is said to be a monopole if
\[
\langle w, \Delta u \rangle = \langle \delta_{x_0}, u \rangle \quad \text{for all } u \in \mathcal{D}.
\] (Contrast (2) with condition (i) above. A function \( w_{x,y} \in \mathcal{H} \) satisfying (i) is called a bipole. We will see that bipoles always exist, while monopoles do not.)

Example 3.2. Consider functions \( u \) on the integers \( \mathbb{Z} \) subject to the condition
\[
\| u \|_2^2 = \sum_{x \in \mathbb{Z}} | u(x) - u(x+1) |^2 < \infty.
\] Moding out with the constant functions on \( \mathbb{Z} \), note that \( \| \cdot \| \) in (3) is then a Hilbert norm. The corresponding Hilbert space will be denoted \( \mathcal{H} \).

It is convenient to realize \( \mathcal{H} \) via the following Fourier series representation
\[
\tilde{u}(\theta) = \sum_{x \in \mathbb{Z}} u(x) e^{ix\theta},
\] i.e., a \( 2\pi \)-periodic function. Note that the same construction works \textit{mutatis mutandis} in \( d \) variables for \( d > 1 \).

Lemma 3.3. A \( 2\pi \)-periodic function \( \tilde{u} \) as in (4) represents an \( u \in \mathcal{H} \) if and only if
\[
\sin \left( \frac{\theta}{2} \right) \tilde{u}(\theta) \in L^2(-\pi, \pi);
\] and in that case
\[
\| u \|_2^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \sin^2 \left( \frac{\theta}{2} \right) | \tilde{u}(\theta) |^2 \, d\theta.
\] (5)

Remark 3.4. Note that the constant function \( u_1 \equiv 1 \) on \( \mathbb{Z} \) does not contribute to (3); and as a result \( \tilde{u}_1(\theta) = \delta(\theta - 0) \) does not contribute to (4). The last fact can be verified directly.

Proof of Lemma 3.3. For the RHS in (3) we have
\[
\left| \left( \tilde{u}(\cdot) - \tilde{u}(\cdot+1) \right)(\theta) \right| = \left| (1-e^{-i\theta}) \tilde{u}(\theta) \right| = 2 \left| \sin \left( \frac{\theta}{2} \right) \tilde{u}(\theta) \right|,
\] and the conclusion in (3) now follows from Parseval’s formula for Fourier series.

Lemma 3.5. The Hilbert space \( (\mathcal{H},\mathbb{Z}) \) in Example 3.2 has dipoles, but not monopoles.
Proof. Let \( x, y \in \mathbb{Z} \). We may assume without loss that \( 0 \leq y < x \). Now set
\[
w_{x,y} (n) := \begin{cases} 
0, & \text{if } |n| \leq y \\
|n| - y, & \text{if } y < |n| \leq x \\
x - y, & \text{if } x < |n|.
\end{cases}
\]
(6)

Then the reproducing formula holds, i.e., we have (i):
\[
\langle w_{x,y}, u \rangle_{\mathcal{H}} = u(x) - u(y), \quad \forall u \in \mathcal{H}.
\]
(7)

Setting \( v_x := w_{x,0} \), we get
\[
\langle v_x, v_y \rangle_{\mathcal{H}} = |x| \land |y|.
\]
(8)

The fact that there are no monopoles follows from the observation that
\[
\tilde{w} (\theta) = \frac{e^{ix \cdot \theta}}{4 \sin^2 \left( \frac{\theta}{2} \right)}
\]
does not satisfy the finiteness condition in (5).

Remark 3.6. Set \( X = \mathbb{Z} \), and let \( \mathcal{H} \) be the Hilbert space in Example 3.2 of functions \( f : \mathbb{Z} \to \mathbb{C} \), modulo the constant functions, such that
\[
\|f\|_{\mathcal{H}}^2 = \sum_{x \in \mathbb{Z}} |f(x) - f(x+1)|^2 < \infty.
\]

A computation reveals the following three facts (details in section VII below):

(a) For all \( x \in \mathbb{Z} \setminus (0) \), there is a \( v_x \in \mathcal{H} \) such that
\[
\langle v_x, f \rangle_{\mathcal{H}} = f(x) - f(0) \text{ holds for all } f \in \mathcal{H}.
\]

(b) There is no \( w \in \mathcal{H} \) such that
\[
\langle w, f \rangle_{\mathcal{H}} = f(x) \text{ holds for all } f \in \mathcal{H}.
\]

(c) Functions in \( \mathcal{H} \) may be unbounded: Take for example \( f(x) := \log (1 + |x|) \), defined for all \( x \in \mathbb{Z} \).

A glance at the defining conditions (i) – (ii) for “relative reproducing kernel Hilbert spaces” suggests applications to “boundaries” of infinite discrete configurations, such as infinite weighted graphs.

One of the aims of our paper is to study precisely this: The introduction of a suitable reproducing kernel Hilbert space into the analysis of an infinite configuration \( X \) leads to an associated “boundary”, i.e., to a compactification of \( X \), so the boundary consisting of the points in the compactification not already in \( X \); hence notions not present in the finite case; see especially our operator theoretic formulation of “boundary” in section IV below.
The study of “boundary terms” is central to our approach. In contrast to other related but different notions in the literature of “boundary” for random walks, we employ here tools intrinsic to unbounded operators with dense domain in Hilbert space. To start this, we must first, for a given infinite configuration $X$, identify the “right” Hilbert space; see sections IV and V below. Our boundary “$bd X$” (section VIII) is comparable to, but different from, other boundaries in the literature.

IV. Extensions of Unbounded Operators

**Definition 4.1.**

- $\mathcal{H}$: some given complex Hilbert space with fixed inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$.
- $\mathcal{D} \subset \mathcal{H}$: some given dense linear subspace in $\mathcal{H}$.
- $\Delta : \mathcal{D} \to \mathcal{H}$: a given linear operator; typically unbounded.

We say that $\Delta$ is **Hermitian** iff

$$\langle u , \Delta v \rangle = \langle \Delta u , v \rangle , \forall u , v \in \mathcal{D};$$

and we say that $\mathcal{D}$ is the **domain** of $\Delta$; written

- $\text{dom } (\Delta) := \mathcal{D}$.

The **adjoint** operator $\Delta^*$ is defined as follows: Let

$\text{dom } (\Delta^*) := \{ \psi \in \mathcal{H} | \text{ such that } \exists C < \infty \text{ with } |\langle \psi , \Delta v \rangle| \subseteq C \| v \|, \forall v \in \mathcal{D} \}$.

If $\psi \in \text{dom } (\Delta^*)$, then by Riesz’ lemma, there is a unique $w \in \mathcal{H}$ such that

$$\langle \psi , \Delta v \rangle = \langle w , v \rangle , \forall v \in \mathcal{D};$$

and we set $\Delta^* \psi := w$.

The graph $G$ of an operator $\Delta$ is defined by

$$G (\Delta) := \left\{ \left( \begin{array}{c} v \\ \Delta v \end{array} \right) | v \in \text{dom } (\Delta) \right\} \subseteq \mathcal{H} \times \mathcal{H}.$$

If $\Delta$ is hermitian, there is a closed hermitian operator $\Delta^{\text{clo}}$ such that

$$G (\Delta^{\text{clo}}) := G (\Delta) \cap \| \cdot \| \times \| \cdot \| - \text{closure}. $$

One checks that

$$\left( \Delta^{\text{clo}} \right)^* = \Delta^*.$$

For a pair of operators $\Delta_1$ and $\Delta_2$ we say that

$$\Delta_1 \subseteq \Delta_2 \iff G (\Delta_1) \subseteq G (\Delta_2).$$
Lemma 4.2. Let $T$ be a hermitian extension of $\Delta$. Then the following containments hold:
\[ \Delta \subseteq T \subseteq T^\text{clo} \subseteq T^* \subseteq \Delta^*. \]  
(16)

Corollary 4.3. Let $T$ be a hermitian extension of $\Delta$; then
\[ \text{dom (} T^* \text{)} \subseteq \text{dom (} \Delta^* \text{)}. \]  
(17)

Proof. Immediate from (15) and (16). \qed

We now turn to a specific family of hermitian extensions of a fixed densely defined operator $\Delta$.

Definition 4.4. Let $\Delta$ be a hermitian operator with dense domain $D$ in a Hilbert space $H$. Let $C$ be a closed subspace in $H$, and assume that $C \subset \text{dom (} \Delta^* \text{)}$. \( \text{(18)} \)

On the space
\[ D + C = \{ v + h | v \in D, \ h \in C \} \]  
(19)
set
\[ \Delta_C (v + h) := \Delta v, \text{ for } v \in D \text{ and } h \in C. \]  
(20)

Lemma 4.5. Let $\Delta$, $H$, and $C$ be as in the definition. Then the following two conditions are equivalent:

(i) $\Delta_C$ in (20) is a well defined hermitian extension operator; and
(ii) $C \subseteq \ker (\Delta^*)$.

Proof. (i)$\Rightarrow$(ii). From (i) we conclude that the following implication holds:
\[ (v \in D, \ h \in C, \ v + h = 0) \Rightarrow \Delta v = 0. \]  
(21)
Now use (18), and apply $\Delta^*$ to $v + h$ in (21): We get
\[ 0 = \Delta^* (v + h) = \Delta^* v + \Delta^* h = \Delta v + \Delta^* h = \Delta^* h; \]
so $h \in \ker (\Delta^*)$. This applies to all $h \in C$ so (ii) holds.

(ii)$\Rightarrow$(i). Assume (ii). We must then prove the implication (21). Then it follows that $\Delta_C$ is a well defined extension operator. If $v \in D$, $h \in C$, and $v + h = 0$, then
\[ \Delta^* (v + h) = 0 = \Delta^* v + \Delta^* h = \Delta v \]
since $h \in \ker (\Delta^*)$. Hence $\Delta v = 0$ which proves (21).

To prove that $\Delta_C$ is hermitian consider vectors $\psi_i := v_i + h_i$, $v_i \in D$, $h_i \in C$, $i = 1, 2$. 

Then

\[ \langle \Delta_C \psi_1, \psi_2 \rangle = \langle \Delta v_1, v_2 + h_2 \rangle \]
\[ = \langle \Delta v_1, v_2 \rangle + \langle \Delta v_1, h_2 \rangle \]
\[ = \langle v_1, \Delta v_2 \rangle + \langle v_1, \Delta^* h_2 \rangle \]
\[ = \langle v_1, \Delta v_2 \rangle \]
\[ = \langle v_1 + h_1, \Delta v_2 \rangle \]
\[ = \langle \psi_1, \Delta_C \psi_2 \rangle , \]

which is the desired conclusion; in other words, \( \Delta_C \) is a hermitian extension operator.

\( \square \)

**Theorem 4.6.** Let \( \Delta \) be a hermitian operator with dense domain \( \mathcal{D} \) in a Hilbert space, and let \( \mathcal{C} \) be a closed subspace such that \( \mathcal{C} \subset \ker(\Delta^*) \). Let \( \Delta_C \) be the corresponding hermitian extension operator.

Then

\[ \text{dom}(\Delta_C^*) = \{ \psi \in \text{dom}(\Delta^*) | \Delta^* \psi \in \mathcal{H} \cap \mathcal{C} \} \] (22)

where

\[ \mathcal{H} \cap \mathcal{C} := \{ \varphi \in \mathcal{H} | \langle \varphi, h \rangle = 0, \forall h \in \mathcal{C} \} . \] (23)

Proof. Whenever \( \mathcal{C} \) is a closed subspace, the corresponding orthogonal projection will be denoted \( P_C \). Recall \( P_C \) satisfies

\[ P_C = P_C^* = P_C^2, \quad \text{and} \]
\[ P_C \mathcal{H} = \mathcal{C} . \] (24) (25)

Set \( P_C^\perp = I - P_C \); then \( P_C^\perp \) is the projection onto \( \mathcal{H} \cap \mathcal{C} \).

For a one-dimensional subspace spanned by a single vector \( h \neq 0 \), we have

\[ P_h v = \|h\|^{-2} \langle h, v \rangle h, \quad \forall v \in \mathcal{H} . \] (26)

We now turn to the proof of (22). First this inclusion: (\( \subseteq \)) (Easy direction!) So let \( \psi \in \text{dom}(\Delta^*) \), and assume that \( \Delta^* \psi \in \mathcal{C}^\perp \). Then we get the following estimate:

\[ |\langle \psi, \Delta_C (v + h) \rangle| = |\langle \psi, \Delta v \rangle| \]
\[ = |\langle \Delta^* \psi, v \rangle| \]
\[ = |\langle \Delta^* \psi, v + h \rangle| \]
\[ \leq \|\Delta^* \psi\| \cdot \|v + h\| \]

valid for all \( v \in \mathcal{D} \), and all \( h \in \mathcal{C} \).

We conclude from Definition 4.1 that \( \psi \in \text{dom}(\Delta_C^*) \).

(\( \supseteq \)) Conversely, if some fixed vector \( \psi \) is in the \( \text{dom}(\Delta_C^*) \), then there is a constant \( C < \infty \) such that

\[ |\langle \psi, \Delta_C (v + h) \rangle| \leq C \|v + h\| \]

for all \( v \in \mathcal{D} \) and \( h \in \mathcal{C} \).
Since $\Delta_c (v + h) = \Delta v$, we get
\[ |\langle \psi, \Delta v \rangle|^2 \leq C^2 \|v + h\|^2 = C^2 \left( \|v\|^2 + 2 \text{Re} \langle v, h \rangle + \|h\|^2 \right). \tag{27} \]

Now replacing $h$ with $\lambda h$ for $\lambda \in \mathbb{C}$, we arrive at the following estimate; essentially an application of Schwarz' inequality:

As a result we get the following estimate:
\[ \|h\|^2 |\langle \psi, \Delta v \rangle|^2 \leq C^2 \cdot \left( \|h\|^2 \cdot \|v\|^2 - |\langle h, v \rangle|^2 \right), \]
valid for all $v \in \mathcal{D}$ and $h \in \mathcal{C}$. Or equivalently:
\[ |\langle \psi, \Delta v \rangle|^2 \leq C^2 \cdot \left( \|v\|^2 - \frac{|\langle h, v \rangle|^2}{\|h\|^2} \right). \]

Introducing the rank-one projection $P_h$ this then reads as follows:
\[ |\langle \psi, \Delta v \rangle|^2 \leq C^2 \cdot \left( \|v\|^2 - \|P_h v\|^2 \right); \tag{28} \]

or equivalently:
\[ |\langle \psi, \Delta v \rangle|^2 \leq C^2 \cdot \|P_h^\perp v\|^2; \tag{28} \]
See equation [26].

An application of Riesz' theorem then yields a vector $\varphi^* \in \{h\}^\perp$ such that
\[ \langle \psi, \Delta v \rangle = \langle \varphi^*, P_h^\perp v \rangle \tag{29} \]
valid for all $v \in \mathcal{D}$.

But $\langle \varphi^*, P_h^\perp v \rangle = \langle \varphi^*, v \rangle$, and we conclude that
\[ \langle \psi, \Delta v \rangle = \langle \varphi^*, v \rangle \tag{30} \]
for all $v \in \mathcal{D}$. From [30], and Definition 4.1, we conclude that $\psi \in \text{dom}(\Delta^*)$, and that
\[ \langle \Delta^* \psi - \varphi^*, v \rangle = 0 \text{ for all } v \in \mathcal{D}. \]

Since $\mathcal{D}$ is dense in $\mathcal{H}$, we get
\[ \Delta^* \psi = \varphi^* \in \{h\}^\perp, \]
and therefore
\[ \Delta^* \psi \in \bigcap_{h \in \mathcal{C}} \{h\}^\perp = \mathcal{C}^\perp = \mathcal{H} \ominus \mathcal{C}. \]

\[ \square \]
V. Pairs of Hermitian operators in duality

In Lemma 4.5 we introduced the following fundamental properties for a pair \((\Delta, C)\) where \(\Delta\) is a given Hermitian operator with dense domain \(D\) in a fixed Hilbert space \(\mathcal{H}\); and where \(C\) is a closed subspace in \(\mathcal{H}\).

**Definition 5.1.** Let \((\Delta, C)\) be a pair as described above, and let \(\mathcal{H}\) be the ambient Hilbert space. We say the \((\Delta, C)\) is a duality pair iff the inclusion
\[
C \subseteq \ker (\Delta^*)
\]
holds.

Let \(R(\Delta) = \{\Delta v | v \in D\}\) be the range of \(\Delta\), and
\[
R(\Delta)^\text{clo} = R(\Delta)^{\perp \perp}
\]
the norm closure in \(\mathcal{H}\).

**Lemma 5.2.** For a pair \((\Delta, C)\) in \(\mathcal{H}\), the following conditions are equivalent:

(i) \((\Delta, C)\) is a duality pair; and

(ii) \(R(\Delta)^\text{clo} \subseteq \mathcal{H} \ominus C\).

**Proof.** (i) \(\Rightarrow\) (ii). Given \((31)\), we may take ortho-complements, and
\[
(\ker (\Delta^*))^\perp \subseteq C^\perp
\]
The desired \((ii)\) now follows from
\[
R(\Delta)^\text{clo} = R(\Delta)^{\perp \perp} = (\ker (\Delta^*))^\perp \text{ and } C^\perp = \mathcal{H} \ominus C.
\]

(ii) \(\Rightarrow\) (i). The above argument works in reverse: Take perpendicular on both sides in \((33)\), and note that the containment reverses, so \((ii)\) implies \((i)\). \(\square\)

The following family of reproducing kernel Hilbert spaces includes duality pairs. This in turn includes all graph-Laplacians on infinite weighted graphs, as we will show.

**Definition 5.3.** Let \(X\) be a set. Pick some \(o \in X\), and set \(X^* := X \setminus \{0\}\). A Hilbert space \(\mathcal{H}\) is said to be a reproducing kernel Hilbert space with base-point if there is a function
\[
k : X \times X^* \to \mathbb{C}
\]
such that
\[
v_x(\cdot) := k(\cdot, x) \in \mathcal{H}, \forall x \in X^*; \tag{35}\n\]
\[
\langle v_x, f \rangle_\mathcal{H} = f(x) - f(o), \forall f \in \mathcal{H}, \forall x \in X^*. \tag{36}\n\]
In particular, $\mathcal{H}$ is a space of functions on $X$. The inner product in $\mathcal{H}$ is denoted $\langle \cdot, \cdot \rangle_\mathcal{H}$ or simply $\langle \cdot, \cdot \rangle$.

In addition, we require

\[
\text{closed span } \{ v_x | x \in X^* \} = \mathcal{H}; \quad \text{and} \quad \{ \delta_x | x \in X \} \subset \mathcal{H}. \tag{37}
\]

Hence

\[
\delta_x (y) = \begin{cases} 
1 & \text{if } y = x \\
0 & \text{if } y \neq x \text{ in } X;
\end{cases} \tag{39}
\]

i.e., the Dirac-functions on $X$.

**Remark 5.4.** (a) Because of (36), $\mathcal{H}$ is really a space of functions modulo the constant functions.

(b) Not all reproducing kernel Hilbert spaces have property (38): Take for example $X := [0,1]$, $o = 0$,

\[
k (x, y) := x \wedge y, \tag{40}
\]

i.e., the smallest two numbers.

Let $\mathcal{H}$ be the space of measurable functions $f$ on $X$ such that the distribution derivative $f' = \frac{df}{dx}$ is in $L^2 (0, 1)$. Set

\[
\| f \|_\mathcal{H}^2 := \int_0^1 |f'(x)|^2 \, dx. \tag{41}
\]

It is easy to check then that conditions (35)–(37) will be satisfied; but that (38) will not hold.

On the other hand, energy Hilbert spaces for weighted graphs will satisfy (38). Specifically, let $(G, c) = (G^0, G^1, c)$ be an (infinite) weighted graph, i.e.,

- $G^0$ is the vertex set (discrete);
- $G^1 \subset G^0 \times G^0$ is the set of edges in $G$;
- $c : G^1 \to \mathbb{R}$ a fixed weight function such that $c(xy) = c(yx)$ for all $(xy) \in G^1$.

For functions $u$ and $v$ on $G^0$ set

\[
\langle u, v \rangle_\mathcal{H} := \frac{1}{2} \sum_{\text{s.t. } (xy) \in G^1} c(xy) \left( u(x) - u(y) \right) \left( v(x) - v(y) \right); \tag{42}
\]

and $\| u \|_\mathcal{H}^2 = \langle u, u \rangle_\mathcal{H}$. (We choose our inner product to be linear in the second variable.)
The Hilbert space $\mathcal{H}$ consists of all functions $u$ such that
\[
\|u\|_{\mathcal{H}}^2 = \sum_{x,y} c(xy) |u(x) - u(y)|^2 < \infty.
\] (43)

We proved in [13] that $\mathcal{H}$ is a reproducing kernel Hilbert space with base-point; in particular, if $o \in G^0$ is chosen, then conditions (35)–(38) are satisfied.

Here we shall include (38) as part of our definition.

More precisely:

**Proposition 5.5.** Let $(G, c)$ be a weighted graph with energy Hilbert space $\mathcal{H} = \mathcal{H}_E$.

Pick a base-point $o \in G^0$, and let $(v_x)_{x \in G^0 \setminus \{0\}}$ be the family (dipoles) from (36).

Suppose, for all $x \in G^0$,
\[
c(x) = \sum_{y, \text{such that } (xy) \in G^1} c(xy) < \infty;
\] (44)

then (38) holds, and
\[
\delta_x = c(x) v_x - \sum_{(xy) \in G^1} c(xy) v_y.
\] (45)

**Proof.** By a direct computation, using (43) and (44), we get
\[
\|\delta_x\|_{\mathcal{H}}^2 = \sum_{y, \text{such that } (xy) \in G^1} c(xy) = c(x) < \infty.
\] (46)

\[\square\]

**Lemma 5.6.** Let $(\mathcal{H}, X, \Delta)$ be a reproducing kernel system as outlined in section II. Let $x_0 \in X$; then some $w_0 \in \mathcal{H}$ is a monopole at $x_0$ if and only if $w_0 \in \text{dom}(\Delta^*)$ and
\[
\Delta^* w_0 = \delta_{x_0}.
\] (47)

**Proof.** If $w_0 \in \mathcal{H}$ is a monopole at $x_0$, then (2) holds, i.e., $\langle w_0, \Delta u \rangle = \langle \delta_{x_0}, u \rangle$ is satisfied for all $u \in D$. Since $|\langle \delta_{x_0}, u \rangle| \leq \|\delta_{x_0}\| \cdot \|u\|$ holds by Schwarz, it follow that (47) is satisfied.

The argument for the converse implication is an application of Riesz’ lemma to the Hilbert space $\mathcal{H}$. \[\square\]

**Remark 5.7.** It is not true in general that the truncated summations on the R.H.S. in (45) converge in the norm (43) of $\mathcal{H}_E$. But it is if
each \( x \in G^0 \) has at most a finite number of neighbors. For pairs of points in \( G^0 \), set

\[
x \sim y \text{ iff there is an edge } e \in G^1 \text{ with } e = (xy).
\]

Set

\[
\text{Nbh}_{G}(x) := \{ y \in G^0 | y \sim x \}.
\]

We say that \( G \) has finite degrees if

\[
\# \text{Nbh}_{G}(x) < \infty, \forall x \in G^0.
\]

**Theorem 5.8.** Let \((\mathcal{H}, X)\) be a reproducing kernel Hilbert space with base point \( o \), and assume \((38)\) is satisfied. For \( x \in X \), and \( f \in \mathcal{H} \), set

\[
(\Delta f)(x) := \langle \delta_x, f \rangle;
\]

then \( \Delta \) is a hermitian operator with dense domain

\[
\mathcal{D}_V := \text{span} \{ v_x | x \in X^* \}.
\]

It satisfies:

\[
\Delta v_x = \delta_x - \delta_0; \text{ and}
\]

\[
\langle u, \Delta u \rangle \geq 0, \forall u \in \mathcal{D}_V.
\]

Moreover, in the case of weighted graphs \((G, c)\), the identity

\[
(\Delta u)(x) = \sum_{y \sim x} c(xy)(u(x) - u(y))
\]

holds.

**Proof of \((52)\).**

\[
(\Delta v_x)(y) = (\text{by } (50)) \langle \delta_y, v_x \rangle
\]

\[
= (\text{by } (36)) \delta_y(x) - \delta_y(o)
\]

\[
= (\delta_x - \delta_0)(y),
\]

which is \((52)\).
Proof of (53). Consider \( u = \sum_x \xi_x v_x \), a finite linear combination, \( \xi_x \in \mathbb{C} \); then

\[
\langle u, \Delta u \rangle = \sum_{x,y} \xi_x \xi_y \langle v_x, \delta_y - \delta_0 \rangle
\]

\[
= (\text{by } (52)) \sum_{x,y} \xi_x \xi_y \langle v_x, \delta_y - \delta_0 \rangle
\]

\[
= (\text{by } (56)) \sum_{x,y} \xi_x \xi_y (\delta_{x,y} + 1)
\]

\[
= \sum_x |\xi_x|^2 + \sum_x |\xi_x|^2 \geq 0.
\]

□

Proof of (54). In the case of weighted graphs \((G,c)\)

\[
(\Delta u) (x) = (\text{by } (50)) \langle \delta_x, u \rangle_{\mathcal{H}_E}
\]

\[
= (\text{by } (42)) \frac{1}{2} \sum_{s,t} \sum_{x \sim x} c(s,t) (\delta_x(s) - \delta_x(t)) (u(s) - u(t))
\]

\[
= \sum_{s \sim x} c(sx) (u(x) - u(s))
\]

which is the desired formula (54). □

Corollary 5.9. Let \((\mathcal{H}, X, o)\) be as in the theorem, and let \( \Delta \) be the operator in (50). Let \( \Delta^{\text{clo}} \) be the graph-closure of \( \Delta \).

Then the domain of \( \Delta^{\text{clo}} \) is contained in \( \ell^2(X) \cap \ell^1(X) \) where \( \ell^2 \cap \ell^1 \) is understood with regard to counting measure on \( X \).

Note that

\[
\{ \delta_x \} \subseteq \mathcal{H}
\]

(55)

is part of the assumption in the corollary.

Proof. Step 1. We saw that if

\[
u = \sum_{x \in G^0 \setminus \{0\}} \xi_x v_x
\]

is a finite summation with \( \xi_x \in \mathbb{C} \), then \( \xi_x = (\Delta u) (x) \). Hence by the theorem,

\[
\langle u, \Delta u \rangle_E = \sum_{x \in X^*} |(\Delta u) (x)|^2 + \left| \sum_{x \in X^*} \Delta u (x) \right|^2.
\]
Step 2. A simple approximation argument shows that (56) extends to be valid also for all $u \in \text{dom} \,(\Delta^{\text{clo}})$.

To see this, note that by (36), the $\|\cdot\|_E$-norm convergence implies pointwise convergence. If a sequence $(u_n) \subset D_V$ is chosen such that $\|u - u_n\|_E \to 0; \|\Delta u_n - \Delta^{\text{clo}} u\|_E \to 0,$ then we have pointwise of the corresponding functions on $X$, and we may apply Fatou to the summations $\sum_x |(\Delta u_n)(x)|^2$, and $\sum_x (\Delta u_n)(x)$. □

Theorem 5.10. Let $(\mathcal{H},X)$ be a reproducing kernel Hilbert space with base point $o$, and assume (38) holds. Set

$$C: = \mathcal{H} \ominus \{ \delta_x | x \in X \}$$

$$= \{ u \in \mathcal{H} | \langle u, \delta_x \rangle = 0, \forall x \};$$

then $(\Delta, C)$ is a duality pair.

Proof. The claim is that $C \subseteq \text{ker} \,(\Delta^*)$; see (31). But by (52), $\Delta$ maps its domain $D_V$ into $C^\perp$, so if $h \in C$, then

$$\langle \Delta u, h \rangle = 0 \text{ for } \forall u \in D_V.$$ 

Hence $h \in \text{dom} \,(\Delta^*)$, and

$$\langle u, \Delta^* h \rangle = 0, \forall u \in D_V.$$ 

But $D_V$ is dense in $\mathcal{H}$ by (37), and we conclude that $\Delta^* h = 0$, i.e., that $h \in \text{ker} \,(\Delta^*)$. □

Remark 3.4(a) revisited. Even though in Example (41), $\delta_x$ is not in $\mathcal{H}$, the operator $\Delta := - \left( \frac{d}{dx} \right)^2$ on the domain $\mathcal{D} := C^\infty_c (0,1)$ is still hermitian and (52) holds. However, this candidate for domain $\mathcal{D}$ is not dense in $\mathcal{H}$; in fact the function $f(x) = x$ is in $\mathcal{H} \ominus C^\infty_c (0,1)$.

VI. THE ESSENTIAL SELFADJOINTNESS PROBLEM FOR A PAIR OF HERMITIAN OPERATORS IN DUALITY

Let $(\mathcal{H},X)$ be a reproducing kernel Hilbert space with base point $o$, and assume that

$$\delta_x \in \mathcal{H} \text{ for all } x \in X.$$ 

Let $\Delta$ be the associated hermitian operator.

$$(\Delta f)(x) := \langle \delta_x, f \rangle, \ x \in X.$$ 

(59)
Let \( v_x := k(\cdot, x) \) be the functions in \( \mathcal{H} \) derived from the reproducing kernel \( k(\cdot, \cdot) \) for \( \mathcal{H} \). Set

\[
\mathcal{F} := \text{closed span} \{ \delta_x | x \in X \}; \\
\mathcal{C} := \mathcal{H} \ominus \mathcal{F}; \\
\mathcal{D}_F := \text{span} \{ \delta_x | x \in X \}; \quad \text{and} \\
\mathcal{D}_V := \text{span} \{ v_x | x \in X^* \};
\]

where \( X^* := X \setminus \{0\} \), and where “span” means “finite linear combinations.”

It follows from Theorem 4.6 and Theorem 5.8 that the prescriptions

\[
\Delta_F := \Delta|_{\mathcal{D}_F}, \quad (60)
\]

meaning restriction; and

\[
\Delta_V := \Delta|_{\mathcal{D}_V} \quad (61)
\]

yield hermitian operators with dense domain; the domain \( \mathcal{D}_F \) of \( \Delta_F \) is dense in \( \mathcal{H} \).

Let \( \Delta^{\text{clo}}_F \) be the closure of \( \Delta_F \) with domain \( \text{dom} (\Delta^{\text{clo}}_F) \); and similarly \( \Delta^{\text{clo}}_V \) for the closure of \( \Delta_V \) as a densely defined hermitian operator in \( \mathcal{H} \).

Finally, set

\[
\mathcal{D}_H := \mathcal{D}_V + \mathcal{C}, \quad (62)
\]

and

\[
\Delta_H (u + h) := \Delta_V u \quad \text{for all} \ u \in \mathcal{D}_V \ \text{and} \ h \in \mathcal{C}. \quad (63)
\]

We proved in Theorem 4.6 that

\[
\text{dom} (\Delta^*_H) = \{ \psi \in \text{dom} (\Delta^*_V) | \Delta^*_V \psi \in \mathcal{F} \}. \quad (64)
\]

**Definition 6.1.** A hermitian operator \( \Delta \) with dense domain \( \mathcal{D} \) is a Hilbert space \( \mathcal{H} \) is said to be selfadjoint iff \( \Delta = \Delta^* \); and it is said to be essentially selfadjoint iff \( \Delta^{\text{clo}} \) is selfadjoint, where \( \Delta^{\text{clo}} \) means the (graph) closure of \( \Delta \).

By a theorem of von Neumann \((21, 19)\), \( \Delta \) is essentially selfadjoint iff there are values \( \lambda_{\pm} \in \mathbb{C}, \ \text{Im} \lambda_+ > 0, \ \text{Im} \lambda_- < 0 \) such that the two equations

\[
\Delta^* \psi_\pm = \lambda_\pm \psi_\pm \quad (65)
\]

in \( \mathcal{H} \) only have the zero-solutions \( \psi_\pm = 0 \). The solutions \( \psi_\pm \) to (65) form the deficiency spaces, and their respective dimensions are called the deficiency indices.
If $\Delta$ is further semibounded, i.e., $\langle u, \Delta u \rangle \geq 0$ for all $u \in \mathcal{D}$, then for essential selfadjointness it is enough to verify that the equation

$$\Delta^* \psi = -\psi$$

has only the zero-solution $\psi = 0$ in $\mathcal{H}$. (It is understood in (65) and (66) that the vectors $\psi_\pm$ and $\psi$ are assumed to be in dom $(\Delta^*)$.

**Theorem 6.2.** Consider the two operators $\Delta_F$ in $\mathcal{F}$, and $\Delta_H$ in $\mathcal{H}$ above, equation, (60) and (63), respectively.

Fix $\lambda \in \mathbb{C}$, with $\operatorname{Im} \lambda \neq 0$, or if $\Delta_V$ is semibounded, $\operatorname{Re} \lambda < 0$; then a function $\psi \in \mathcal{H}$ satisfies

$$\Delta_H^* \psi = \lambda \psi$$

if and only if $\psi \in \mathcal{F}$, and

$$\Delta_F^* \psi = \lambda \psi.$$ 

**Proof.** The reasoning is on (64) and the previous considerations. Indeed we have the following two-way implications:

- $\psi$ satisfies (67).
- $\Delta_V^* \psi \in \mathcal{F}$, and $\Delta_V^* \psi = \lambda \psi$. (69)
- $\psi \in \mathcal{F}$, and $\Delta_F^* \psi = \lambda \psi$.

In the last step we used that $\lambda \neq 0$, so a solution $\psi$ to (69) with $\Delta_V^* \psi \in \mathcal{F}$ must be in $\mathcal{F}$. □

**Corollary 6.3.** Let the two operators $\Delta_F$ in $\mathcal{F}$, and $\Delta_V$ in $\mathcal{H}$ be as above; and let $\Delta_H$ be the extension of $\Delta_V$ defined in (63).

Then the following two properties are equivalent:

1. $\Delta_H$ is essentially selfadjoint; and
2. $\Delta_F$ is essentially selfadjoint.

**Proof.** The equivalence (i)$\Leftrightarrow$(ii) is immediate from the theorem; given von Neumann’s theory of deficiency spaces; see (67) and (68) above. □

**Remark 6.4.** In many applications (see [13]) it’s easier to verify essential selfadjointness for $\Delta_F$ than it is for $\Delta_H$.

This is an instance of our duality theory: A comparison of restrictions and extensions.

**Corollary 6.5.** Consider the two operators $\Delta_F$ in $\mathcal{F}$ and $\Delta_V$ in $\mathcal{H}$. Let $\Delta_H$ be the extension of $\Delta_V$ defined in (63). We assume that $\mathcal{C} \subseteq \ker (\Delta_V^*)$. Then the following four affirmations are equivalent:
(i) $\Delta^*_V$ maps its domain into $\mathcal{F} = \mathcal{H} \ominus \mathcal{C}$.
(ii) $\Delta^*_V = \Delta^*_H$.
(iii) $\Delta^{clo}_V = \Delta^{clo}_H$.
(iv) $\mathcal{C} \subseteq \text{dom} \left( \Delta^{clo}_V \right)$.

Proof. (i)$\Rightarrow$(ii). In general $\Delta_V \subseteq \Delta_H$ so $\Delta^*_V \subseteq \Delta^*_H$, and
$$\text{dom} \left( \Delta^*_H \right) = \{ \psi \in \text{dom} \left( \Delta^*_V \right) | \Delta^*_V \psi \in \mathcal{F} \} , \quad (70)$$
so if (i) holds, then $\text{dom} \left( \Delta^*_H \right) = \text{dom} \left( \Delta^*_V \right)$, and (ii) follows.
(ii)$\Rightarrow$(iii). Take adjoints in (ii), and we get
$$\Delta^{clo}_V = \Delta^{**}_V = \Delta^{**}_H = \Delta^{clo}_H$$
which is condition (iii).
(iii)$\Rightarrow$(iv). A simple limit consideration applied to (63) yields the following:
$$\text{dom} \left( \Delta^{clo}_H \right) = \text{dom} \left( \Delta^{clo}_V \right) + \mathcal{C} \quad (71)$$
which proves (iii)$\Leftrightarrow$(iv). Since $\Delta^{*}_H$ maps into $\mathcal{F}$ by (70), it follows that (iv)$\Rightarrow$(i). $\square$

There are many applications of selfadjoint extension operators; a major reason being that the Spectral Theorem applies to each selfadjoint extension, while it does not apply to a hermitian non-selfadjoint operator.

The operator $\Delta$ we consider here is semibounded on its dense domain, so it has semibounded selfadjoint extensions with the same bound, for example the Friedrichs extension $\Delta_{Fr}$; see [11].

The following applies to any one of the semibounded selfadjoint extension $\Delta_S$ of $\Delta$. Given $\Delta_S$, there is a projection valued measure $E_S(\cdot)$ defined on the Borel-sets $\mathcal{B}$ in $[0, \infty)$ and mapping into projections in $\mathcal{H}$; i.e., each $P := E_S(B)$, $B \in \mathcal{B}$ satisfies $P = P^* = P^2$; and we have
$$\Delta_S = \int_0^\infty \lambda E_S(d\lambda) , \quad I_\mathcal{H} = \int_0^\infty E_S(d\lambda) , \quad \text{and} \quad (72)$$
$$\int_0^\infty \| E_S(d\lambda) u \|^2_\mathcal{H} = \| u \|_\mathcal{H}^2 \quad \text{for all } u \in \mathcal{H} . \quad (73)$$

**Definition 6.6.** Let $(\mathcal{H}, X, o)$ be a relative reproducing kernel Hilbert space satisfying the conditions above, and let $\Delta$, $\Delta_S$ be associated operators with the listed properties. Let $\mathcal{H}_R$ be a real form of $\mathcal{H}$, and
set
\[ S := \left\{ u \in \mathcal{H}_R \mid \int_0^\infty \lambda^{2p} \left\| E_S (d\lambda) u \right\|^2 < \infty, \text{ for all } p \in \mathbb{N} \right\}. \quad (74) \]

Recall
\[ u \in \text{dom} (\Delta^p_S) \Leftrightarrow \left\| \Delta^p_S u \right\|^2 = \int_0^\infty \lambda^{2p} \left\| E_S (d\lambda) u \right\|^2 < \infty. \quad (75) \]

We turn \( S \) into a Fréchet space with the seminorms
\[ \left\| u \right\|_p := \left\| \Delta^p_S u \right\|_{\mathcal{H}}, \text{ for } u \in S, \text{ and } p \in \mathbb{N}. \quad (76) \]
and we denote the dual of \( (S, \left\| \cdot \right\|_p)_{p \in \mathbb{N}} \) by \( S' \) for tempered distributions.

As a result we get the following Gelfand triple \[ S \subseteq \mathcal{H}_R \subseteq S' \]
with the two inclusions in (77) representing continuous embeddings.

The cylinder sets \((\subseteq S')\) generate a sigma-algebra \( \mathcal{B} := \mathcal{B} (S') \), and there is an associated (Wiener-) measure \( W \) defined on \( \mathcal{B} \) determined uniquely by the following identity:
\[ \int_{S'} e^{i \langle u, \xi \rangle} dW (\xi) = e^{-\frac{i}{2} \left\| u \right\|^2_{\mathcal{H}}}, \text{ for all } u \in \mathcal{H}_R. \quad (78) \]

In the exponent on the LHS in (78), the expression \( \langle u, \cdot \rangle \) will be denoted as a function \( \tilde{u} \) on \( S' \). We have
\[ \langle u_1, u_2 \rangle_{\mathcal{H}_R} = \int_{S'} \tilde{u}_1 \tilde{u}_2 dW, \quad (79) \]
and
\[ \int_{S'} \tilde{u} dW = 0, \quad (80) \]
for all \( u_1, u_2, u \in \mathcal{H}_R \).

If \( \mu \) is a signed measure on \( S' \), we denote its Fourier transform
\[ \hat{\mu} (u) := \int_{S'} e^{i \langle u, \xi \rangle} d\mu (\xi); \quad (81) \]
or simply \( \int_{S'} e^{i \tilde{u} (\cdot)} d\mu (\cdot). \)
Definition 6.7. We shall need the Hermite polynomials \((H_n)_{n \in \mathbb{N}_0}\) given by \(H_0 \equiv 1, \ H_1 (x) = 1 - x, \) and
\[
\frac{d}{dx} e^{-\frac{x^2}{2}} = H_n (x) e^{-\frac{x^2}{2}}; \tag{82}
\]
so
\[
H_{n+1} (x) = \frac{d}{dx} H_n (x) - x H_n (x).
\]

Lemma 6.8. Let \(\mathcal{H}, \mathcal{S}, \mathcal{S}'\) be as in \([77]\), let \(f \in \mathcal{H}_\mathbb{R}\) be given; and set
\[
d\mu (\cdot) := \tilde{f} (\cdot) dW (\cdot). \tag{83}
\]
Then for the Fourier transform, we have
\[
\hat{d\mu} (u) = \frac{i}{\pi} \langle f, u \rangle_{\mathcal{H}} e^{-\frac{|u|^2}{2}} \quad \text{for all } u \in \mathcal{S}. \tag{84}
\]

Proof. Let \(f \in \mathcal{H}_\mathbb{R}, \ u \in \mathcal{S}, \) and \(\varepsilon \in \mathbb{R}_+.\) By \([78]\), we then have
\[
\int_{\mathcal{S}'} e^{i(u+\varepsilon f, \cdot)} dW (\cdot) = e^{-\frac{1}{2}\varepsilon^2 |u|^2}. \tag{85}
\]
An application of \(\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0}\) to both sides in \([85]\) then yields
\[
\int_{\mathcal{S}'} i \tilde{f} (\cdot) e^{i(u, \cdot)} dW (\cdot) = -\langle u, f \rangle e^{-\frac{|u|^2}{2}}. \tag{86}
\]
By virtue of \([81]\) and \([82]\), this formula is equivalent to \([84]\); i.e., the conclusion in the lemma. \(\square\)

Proposition 6.9. Let \((\mathcal{H}, X, o)\) be a relative reproducing kernel Hilbert space. For \(x, y \in X, \ x \neq y,\) let \(w_{x,y} \in \mathcal{H}\) be the solution to
\[
\langle w_{x,y}, u \rangle = u (x) - u (y), \quad \forall u \in \mathcal{H}. \tag{86}
\]
Then
\[
\hat{(w_{x,y} dW)} (u) = i \langle u (x) - u (y) \rangle e^{-\frac{|u|^2}{2}}, \quad \forall u \in \mathcal{H}_\mathbb{R}. \tag{87}
\]

Proof. We have
\[
(w_{x,y} dW) (u) = \langle \tilde{f} (\cdot), u \rangle e^{-\frac{|u|^2}{2}} = \langle \tilde{f} (\cdot), i (u (x) - u (y)) e^{-\frac{|u|^2}{2}} \rangle
\]
which is the desired formula \([87]\). \(\square\)

Theorem 6.10. Let \(\mathcal{H}, \ f, \ u, \) and \(W\) be as described above. Then
\[
\hat{f} (\cdot)^n dW (\cdot) (u) = H_n (f, u) e^{-\frac{1}{2}|u|^2} \tag{88}
\]
where $H_n, n \in \mathbb{N}_0$, are the Hermite polynomials in (82):

$$H_1(f, u) = i \langle f, u \rangle,$$
$$H_2(f, u) = \|f\|^2 - \langle u, f \rangle^2,$$

etc.

**Proof.** The reader may check that the theorem follows from Lemma 6.8 combined with the recursive Hermite formulas (82) in Definition 6.7. Indeed we must apply \( \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \) recursively to the RHS in (85):

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} e^{-\frac{1}{2} \|u+\varepsilon f\|^2} = -\langle u, f \rangle e^{-\frac{1}{2} \|u\|^2},$$

and

$$\left. \left( \frac{d}{d\varepsilon} \right)^2 \right|_{\varepsilon=0} e^{-\frac{1}{2} \|u+\varepsilon f\|^2} = (\langle u, f \rangle^2 - \|f\|^2) e^{-\frac{1}{2} \|u\|^2}.$$ 

□

**Definition 6.11.** Let \((\mathcal{H}, X, o)\) be a relative reproducing kernel Hilbert space satisfying condition (58). (It follows then that \(X\) is discrete!) Let \(S'\) be the real space in the Gelfand triple (77), and let \(W\) be the corresponding Wiener measure determined by (78). By a boundary point for \((\mathcal{H}, X, o)\) we mean a measure \(\beta\) on \(S'\) such that there is a sequence \(x_1, x_2, \cdots\) in \(X\) satisfying

$$\lim_{n \to \infty} (u(x_n) - u(o)) = \int_{S'} \hat{u} \, d\beta \text{ for all } u \in \mathcal{H}. \quad (89)$$

The following result is different from the classical Riemann-Lebesgue theorem, but it is inspired by it. First, for \(x \in X\) set \(v_x := w_{x, o}\) the dipole for the pair of points \(x, o\) in \(X\).

**Corollary 6.12.** Let \((\mathcal{H}, X, o)\) and \(\{v_x\}_{x \in X^*}\) be a reproducing kernel system subject to the conditions listed above; and let \(\beta\) be a boundary point.

Then there is a sequence \(x_1, x_2, \cdots\) in \(X\) such that

$$\lim_{n \to \infty} (v_{x_n} dW)(u) = i \int_{S'} \hat{u} \, d\beta \cdot e^{-\frac{1}{2} \|u\|^2}. \quad (90)$$

**Proof.** Let \(\beta\) be a boundary point as indicated. Pick \((x_n)_{n \in \mathbb{N}} \subset X\) such that (89) holds. When this is substituted into (87) from Proposition 6.9, the desired conclusion (90) follows. □
VII. Computing deficiency-spaces for a pair of Hermitian operators in duality

The setting will be as in the two previous sections. We are given a reproducing kernel Hilbert space \((\mathcal{H}, X, o)\) with base-point, and we introduce the three associated hermitian operators \(\Delta_F\) in \(\mathcal{F}\); and \(\Delta_V\) (with \(\Delta_V \subseteq \Delta_H\)) and \(\Delta_H\) densely defined operators in \(\mathcal{H}\).

We saw in Theorem 6.2 that it is frequently easier to compute deficiency spaces for \(\Delta_F\) than it is for the other two operators in the larger ambient Hilbert space \(\mathcal{H}\). But in all cases \(\mathcal{H}\) may be somewhat intractable because it is determined by a fixed reproducing kernel \(k(\cdot, \cdot)\), and the spanning functions \(v_x(\cdot) := k(\cdot, x)\) are far from forming an orthogonal system in \(\mathcal{H}\); in fact in my examples, turning \(\{v_x | x \in X^*\}\) into a frame still leaves with poor frame-bound estimates.

As before, here we set \(X^* := X \setminus \{o\}\).

We will now examine the fundamental property,

\[
\delta_x \in \mathcal{H}, \forall x \in X.
\]  

(91)

Our aim is to represent \(\delta_x\) as an expansion in \(\{v_y | y \in X^*\}\).

To avoid difficulties with "bad" frame-bounds, we add the following restricting assumption,

For all \(x\), we have

\[
\# \{y \in X | \langle \delta_x, \delta_y \rangle \neq 0\} < \infty.
\]  

(92)

We will see in the next section that (92) corresponds to a finite-degree restriction on a graph build from the system \((\mathcal{H}, X, o)\).

**Proposition 7.1.** Let \((\mathcal{H}, X, o)\) and \(\{v_x | x \in X^*\}\) be as specified above; see also section VI for additional details.

Then

\[
\delta_x = \|\delta_x\|^2_{\mathcal{H}} v_x + \sum_{y \in X \setminus \{o, x\}} \langle \delta_x, \delta_y \rangle v_y.
\]  

(93)

**Proof.** Note that with assumption (92), we have ruled out infinite summations occurring on the R.H.S. in (93). However, it is possible to relax condition (92), and this will be taken up in a subsequent paper. \(\square\)

We will need the following:

**Lemma 7.2.** If some \(u \in \mathcal{H}\) has a finite representation

\[
u = \sum_{x \in X^*} \xi_x v_x,
\]  

finite summation, and with \(\xi_x \in \mathbb{C}\); then

\[
\xi_x = (\Delta u)(x) := \langle \delta_x, u \rangle_{\mathcal{H}}.
\]  

(95)
Proof. Let $y \in X^*$, and compute
\[
\langle \delta_y, u \rangle_H = (by \ 94) \sum_{x \in X^*} \xi_x (\delta_y (x) - \delta_y (0))
\]
and therefore
\[
\xi_y = \langle \delta_y, u \rangle_H = (\Delta u) (y)
\]
which is the desired conclusion. \qed

Proof of Proposition 6.1 resumed. With the lemma, we now compute the L.H.S. in (93):
\[
\delta_x = (by \ the \ lemma) \sum_y (\Delta \delta_x) (y) v_y
\]
\[
= (\Delta \delta_x) (x) v_x + \sum_{y \neq x} (\Delta \delta_x) (y) v_y
\]
\[
= (by (95)) \langle \delta_x, \delta_x \rangle_H v_x + \sum_{y \neq x} \langle \delta_y, \delta_x \rangle_H v_y
\]
which is the desired formula (93). \qed

Definition 7.3. Let $(H, X, o)$ be as above, and set
\[
c(x) = \max \left( \|\delta_x\|_H^2, \sum_{y \neq x} |\langle \delta_x, \delta_y \rangle_H|^2 \right). \quad (96)
\]
Let $\ell^2(X, c)$ be the $\ell^2$-space with $c(\cdot)$ as weight, i.e., all $\xi: X \to \mathbb{C}$ such that
\[
\sum_x c(x)|\xi_x|^2 = :\|\xi\|_{\ell^2(c)}^2 < \infty. \quad (97)
\]

Theorem 7.4. Let $(H, X, o)$ be as above, and let the function $c(\cdot)$ be defined by (96). Then $\ell^2(c)$ is contractively embedded in $H$.

Proof. Since $H$ is a Hilbert space, we shall state the embedding of $\ell^2(c)$ into $H$ instead as a mapping into $H^* = (the \ dual \ of \ H) \simeq H$.

For $\xi \in \ell^2(c)$, set
\[
L(\xi): u \mapsto \sum_x \xi_x \cdot (\Delta u) (x). \quad (98)
\]
We will show that the summation on the R.H.S. in (98) is absolutely convergent and that
\[
\sum_{x} |\xi_{x} \cdot (\Delta u)(x)|^{2} \leq \|\xi\|_{\ell^{2}(c)}^{2} \cdot \|u\|_{\mathcal{H}}^{2}.
\] (99)

The conclusion in the theorem follows from this, and an application of Riesz’ lemma to \(\mathcal{H}\).

By the theorem, we have
\[
\delta_{x} = \sum_{y} \langle \delta_{x}, \delta_{y} \rangle_{\mathcal{H}} w_{x,y}
\] (100)

where \(w_{x,y} := v_{x} - v_{y}\); and
\[
\sum_{x} |\xi_{x} (\Delta u)(x)|
= \sum_{x} |\xi_{x} \langle \delta_{x}, u \rangle_{\mathcal{H}}|
= (by (100)) \sum_{x} \sum_{y} |\xi_{x} \langle \delta_{x}, \delta_{y} \rangle_{\mathcal{H}} \langle w_{x,y}, u \rangle_{\mathcal{H}}|
= (\text{Fubini}) \sum_{y} \sum_{x} |\xi_{x} \langle \delta_{x}, \delta_{y} \rangle_{\mathcal{H}} \langle w_{x,y}, u \rangle_{\mathcal{H}}|
= \sum_{y} \sum_{x} |\xi_{x}| \sqrt{\langle \delta_{x}, \delta_{y} \rangle_{\mathcal{H}}} \sqrt{|\langle \delta_{x}, \delta_{y} \rangle_{\mathcal{H}}|} |u(x) - u(y)|
\leq (\text{Schwarz}) \sum_{y} \left( \sum_{x} |\xi_{x}|^{2} \cdot |\langle \delta_{x}, \delta_{y} \rangle| \right)^{\frac{1}{2}} \cdot \left( \sum_{x} |\langle \delta_{x}, \delta_{y} \rangle_{\mathcal{H}}| \cdot |u(x) - v(y)|^{2} \right)^{\frac{1}{2}}
\leq (\text{Schwarz}) \left( \sum_{x} |\xi_{x}|^{2} c(x) \cdot \sum_{y} |\langle \delta_{x}, \delta_{y} \rangle_{\mathcal{H}}| \cdot |u(x) - v(y)|^{2} \right)^{\frac{1}{2}}
\leq \|\xi\|_{\ell^{2}(c)}^{2} \cdot \|u\|_{\mathcal{H}},
\]
which is the desired estimate (99). \(\square\)

Let \((\mathcal{H}, X, o)\) be a relative reproducing kernel Hilbert space such that (91) is satisfied; and let \(\Delta\) be the associated operator from (59). Since vectors in \(\mathcal{H}\) are determined from differences (via dipoles, see equation (86)) intuitively one would expect the constant function 1 to be represented by zero in \(\mathcal{H}\).

The next result offers an operator theoretic answer to this question.

Definition 7.5. A family of finite subsets \((F_k)_{k \in \mathbb{N}}\) is said to be an exhaustion or a filtration in \(X\) if

\[
F_1 \subset F_2 \subset \cdots F_k \subset F_{k+1} \subset \cdots
\]  

and

\[
\bigcup_{k=1}^{\infty} F_k = X.
\]

Let

\[
f_k = \chi_{F_k} = \sum_{x \in F_k} \delta_x.
\]

Now define the boundaries \(\text{bd} F_k\) for each \(k\) as follows:

\[
\text{bd} F_k = \{x \in F_k | \exists y \in F_k^c \text{ with } \langle \delta_x, \delta_y \rangle \neq 0\}.
\]

For simplicity we will assume finite degrees, i.e., assume that \(\mathcal{H}_F\) is satisfied for all \(x \in X\). By \(F_k^c = X \setminus F_k\).

For functions \(\psi\) on \(X\), define a normal derivative \(\frac{\partial \psi}{\partial n}\) referring to the filtration:

\[
\frac{\partial \psi}{\partial n}(x) = \sum_{y \sim x} \langle \delta_x, \delta_y \rangle (\psi(x) - \psi(y))
\]

where \(y \sim x\) means \(y \neq x\) and \(\langle \delta_x, \delta_y \rangle \neq 0\). Moreover for \(x \in F_k\), set

\[
\left(\frac{\partial \psi}{\partial n}\right)_k(x) = \sum_{y \in F_k^c} \langle \delta_x, \delta_y \rangle (\psi(x) - \psi(y)).
\]

Lemma 7.6. Let \(\langle \cdot, \cdot \rangle\) be the inner product in \(\mathcal{H}\), and let \(F \subset X\) be a finite subset. Then for \(\psi \in \mathcal{H}\), we have the identity

\[
\langle \chi_F, \psi \rangle = \sum_{x \in F} \left(\frac{\partial \psi}{\partial n}\right)_F(x).
\]

Proof.

\[
\langle \chi_F, \psi \rangle = \sum_{x \neq y} \langle \delta_x, \delta_y \rangle (\chi_F(x) - \chi_F(y)) (\psi(x) - \psi(y))
\]

\[
= \sum_{x \in F} \sum_{y \in F^c} \langle \delta_x, \delta_y \rangle (\psi(x) - \psi(y))
\]

\[
= \sum_{x \in F} \left(\frac{\partial \psi}{\partial n}\right)_F(x).
\]

\(\square\)
Theorem 7.7. Let $\mathcal{H}$, $X$, $\sigma$ be as above, and let $(F_k)_{k \in \mathbb{N}}$ be a filtration. Then

$$f_k := \chi_{F_k} \in \mathcal{H}$$

converges to zero weakly if and only if

$$\lim_{k \to \infty} \sum_{x \in F_k} \frac{\partial \psi}{\partial n}(x) = 0 \text{ for all } \psi \in \mathcal{H}. \quad (108)$$

Proof. The conclusion (108) is immediate from the lemma. \hfill \square

Corollary 7.8. Let $\mathcal{H} = \mathcal{H}_E$ where $\mathcal{H}_E$ is the energy Hilbert space coming from a weighted graph $(G, c)$ with $G^0 = \text{the set of vertices}$, and $G^1 = \text{the set of edges}$, i.e.,

$$\langle u, v \rangle_{\mathcal{H}_E} = \sum_{x \sim y} c(xy) \left( \overline{u(x)} - u(y) \right) \left( v(x) - v(y) \right)$$

and

$$(\Delta u)(x) = \sum_{y \sim x} c(xy) \left( u(x) - u(y) \right).$$

Then for every filtration $(F_k)$ in $G^0$, $\chi_{F_k} \to 0$ as $k \to \infty$, with weak convergence in $\mathcal{H}_E$.

Proof. By Theorem 7.7, we only need to prove that the limit property (108) is satisfied; but this follows in turn from the proof of Theorem 4.7; specifically the proof of (52) in this theorem. \hfill \square

Example 7.9. Let $G$ be the graph $\mathbb{Z}^d$ with nearest neighbors; i.e., $x \sim y$ for pairs of points $x$ and $y$ in $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$ and the two only differ on one coordinate place, i.e., $\exists i$ such that $|x_i - y_i| = 1$. For $x \sim y$ set $c(xy) = 1$.

For filtration, let

$$F_k := [-k, k]^d \cap \mathbb{Z}^d,$$

and $f_k := \chi_{F_k}$. Then

$$\|f_k\|_{\mathcal{H}_E}^2 = (2d) \cdot (2k)^{d-1}.$$

In particular, it follows that $(f_k)_{k \in \mathbb{N}}$ is not a Cauchy sequence in $\mathcal{H}_E$.

VIII. Concluding remarks and applications

We saw that every weighted graph $(G, c)$ with finite degrees gives rise to a reproducing kernel Hilbert space $(\mathcal{H}, X)$ with $X = G^0 \setminus \{o\}$. 
Here $G^c$ denotes the set of vertices in $G$, and $o$ is a chosen (and fixed) base-point for $G^0$. To see this we introduce the graph Laplacian

$$\Delta (u) (x) := \sum_{y \sim x} c(xy)(u(x) - u(y)) \quad (109)$$

with

$$c(x) := \sum_{y \sim x} c(xy). \quad (110)$$

Equation (109) then takes the form

$$\Delta (u) (x) = c(x)u(x) - \sum_{y \sim x} c(xy)u(y) \quad (111)$$

for all functions $u$ on $G^0$.

In section IV, and in [13], we introduced the associated energy Hilbert space $H_E$ with its inner product $\langle \cdot, \cdot \rangle_E$ and norm $\| \cdot \|_E$. We showed that for every $x$, there is a unique $v_x \in H_E$ such that

$$\langle v_x, f \rangle = f(x) - f(0), \quad \forall f \in H_E. \quad (112)$$

Setting $w_{x,y} := v_x - v_y$, we get

$$\langle w_{x,y}, f \rangle = f(x) - f(y). \quad (113)$$

Furthermore, the Dirac functions $\delta_x$ satisfy

$$\delta_x \in H_E, \text{ and } \|\delta_x\|_E^2 = c(x) \text{ for all } x \in G^0. \quad (114)$$

**Theorem 8.1.** (a) Let $(H, X)$ be a reproducing kernel Hilbert space of functions on a set $X$. Let $o \in X$ be a base-point. Let $k(\cdot, \cdot)$ be the reproducing kernel for $(H, X, o)$, and set

$$v_x(y) := k(y, x) \text{ for } x \in X^*.$$  

Then $(v_x)_{x \in X^*}$ satisfies

$$\langle v_x, f \rangle = f(x) - f(0), \quad \forall f \in H, \forall x \in X^*. \quad (116)$$

(b) The following two affirmations are equivalent:

(i) $(H, X, o)$ satisfies:

- $\delta_x \in H, \forall x \in X$.
- For every $x \in X$, we have
  \[ \# \{ y \in X \mid \langle \delta_x, \delta_y \rangle \neq 0 \} < \infty. \quad (117) \]

- The following identity holds:
  \[ \|\delta_x\|^2 = -\sum_{y \in X} \langle \delta_x, \delta_y \rangle. \quad (118) \]
(ii) There is a weighted graph \((G, c)\) with finite degrees such that \(X = G^0\);

\[
G^1 = \{(x, y) \mid \langle \delta_x, \delta_y \rangle \neq 0\};
\]

(119)

and

\[
c(xy) = -\langle \delta_x, \delta_y \rangle, \quad \forall (xy) \in G^1.
\]

(120)

(c) If the conditions in (i) or (ii) are satisfied, the Laplace operator

\[
(\Delta u)(x) = \langle \delta_x, u \rangle
\]

satisfies (109).

Proof. (a) This is already proved in section IV.

(b) (i)⇒(ii). Assume that some reproducing kernel Hilbert space \((H, X, o)\) with base-point \(o\) satisfies the three conditions (bullet points) listed in (i). We will construct a weighted graph \((G, c)\) with \(G^0 := X\); \(G^1\) we take to be the set in (119); and we set

\[
c(xy) = -\langle \delta_x, \delta_y \rangle
\]

as in (120); and \(c(x) := \|\delta_x\|^2\). We then have the implication: (10)⇒(12).

As a result, the axioms for weighted graphs are satisfied for this particular \((G, c)\), and the degrees are finite by assumption (117).

We set

\[
(\Delta u)(x) = \langle \delta_x, u \rangle, \quad u \in H
\]

(121)

which is possible by the first assumption in (i).

It remains to prove that then (109) is satisfied. □

Lemma 8.2. Let \(x, y \in X\), and suppose \(x \neq y\). Let

\[
u \in \text{span}\{\delta_z \mid z \in X\}, \quad u := \sum_z \xi_z \delta_z
\]

(a finite linear combination \(\xi \in \mathbb{C}\)).

Then

\[
\xi_x - \xi_y = \langle w_{x,y}, u \rangle
\]

(122)

with \(w_{x,y}\) from (113).

Proof. We have

\[
\langle w_{x,y}, u \rangle = \sum_z \xi_z \langle w_{x,y}, \delta_z \rangle
= (by (113)) \sum_z \xi_z (\delta_z (x) - \delta_z (y))
= \xi_x - \xi_y.
\]

□
Proof of Theorem 8.1 resumed.

Case 1. If \( u \in \mathcal{H} \), and
\[
u \perp \{ \delta_z | z \in X \}, \tag{123}
\]
then set \( \Delta u = 0 \).

Case 2. If \( u = \sum_z \xi_z \delta_z \) is a finite linear combination as in Lemma 8.2, we may compute \( (\Delta u)(x) \) from the assumptions as follows:
\[
(\Delta u)(x) = \sum_z \xi_z (\Delta \delta_z)(x)
= \xi_x (\Delta \delta_x)(x) + \sum_{z \neq x} \xi_z (\Delta \delta_z)(x)
= (\text{by } (121)) \xi_x \langle \delta_x, \delta_x \rangle + \sum_{z \neq x} \xi_z \langle \delta_x, \delta_z \rangle
= (\text{by } (118)) -\xi_x \sum_{z \neq x} \langle \delta_x, \delta_z \rangle + \sum_{z \neq x} \xi_z \langle \delta_x, \delta_z \rangle
= \sum_{z \neq x} (\xi_z - \xi_x) \langle \delta_x, \delta_z \rangle
= (\text{by } (122)) - \sum_{z \neq x} \langle w_{x,z}, u \rangle \langle \delta_x, \delta_z \rangle
= (\text{by } (113)) - \sum_{z \neq x} (u(x) - u(z)) \langle \delta_x, \delta_z \rangle
= (\text{by } (120)) \sum_{z \neq x} c(xz)(u(x) - u(z)),
\]
which is the desired formula \( (109) \).

The proof of the converse formula \( (ii) \Rightarrow (i) \) amounts to showing that every weighted graph \((G, c)\) with finite degrees yields a reproducing kernel Hilbert space representation as stated. But with \((G, c)\) given, we may take \( \mathcal{H} := \mathcal{H}_E \), as in section 11V; \( X := G^0 \) = the set of vertices. A direct computation then yields the formulas
\[
\langle \delta_x, \delta_y \rangle_E = \begin{cases} 
c(x) & \text{if } y = x 
-c(xy) & \text{if } y \sim x 
0 & \text{for other cases}, \text{i.e., } y \neq x \text{ and } y \not\sim x.
\end{cases}
\]
and, as a result, we get \((\mathcal{H}_E, G^0, o)\) as a reproducing kernel Hilbert space with base point 0, and reproducing kernel
\[
k(x, y) = \langle v_x, v_y \rangle_E.
\]
Finally, it follows that equation \( (118) \) will then be satisfied. \qed
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