CHARGE RELAXATION IN THE PRESENCE OF SHOT NOISE IN
COULOMB COUPLED MESOSCOPIC SYSTEMS

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In the presence of shot noise the charge on a mesoscopic conductor fluctuates. We are interested in the charge fluctuations which arise if the conductor is in the proximity of a gate to which it is coupled by long range Coulomb forces only. Specifically we consider a gate coupled to the edge of a Hall bar subject to a quantizing magnetic field which contains a quantum point contact. The gate is located away from the quantum point contact. We evaluate the charge relaxation resistance for this geometry. The charge relaxation resistance determines the current fluctuations and potential fluctuations induced into the gate. If there is only one edge channel the charge relaxation resistance is determined by transmission and reflection probabilities alone, but in the presence of many channels the density of states of all edge states determines this resistance. This work is to appear in "Quantum Physics at Mesoscopic Scale" edited by D.C. Glattli, M. Sanquer and J. Tran Thanh Van Editions "Frontieres", 1999.

1 Introduction

Coulomb coupled mesoscopic systems are of interest from a conceptual point of view but increasingly also because one of the components can serve as a measurement probe testing the other components. The simplest Coulomb coupled system consists of two conductors, which form the plates of a mesoscopic capacitor. The author, in collaboration with Thomas and Prêtre, investigated a capacitor which is formed by two conductors each coupled via a single lead to an electron reservoir. In this work it is shown that instead of a simple geometrical capacitance the admittance is determined by an electrochemical capacitance \( C \) and a charge relaxation resistance \( R_q \) which together determine the RC-time of the capacitor. The charge relaxation resistance is not simply a series resistance but instead of transmission probabilities depends on the energy derivatives of the scattering amplitudes.

If one or both of the conductors are connected to more than one reservoir several novel aspects arise. Even if we consider only the equilibrium admittance the distribution of the displacement currents on the different contacts is now an interesting problem which requires a self-consistent treatment of the long range Coulomb interactions. A dramatic aspect of such a system is the fact that capacitance coefficients need no longer to be even functions of the magnetic field. An experiment highlighting such asymmetric capacitance coefficients was realized by Chen et al. who investigated the low frequency admittance of a Hall bar with a gate overlapping an edge of the conductor. Similar experiments in the regime of the fractional quantized Hall effect have subsequently also been carried out. We mention here only for completeness that at large frequencies an arrangement of three or more purely capacitive coupled conductors exhibits a dynamic capacitance matrix with elements which are not even functions of magnetic field. Still a different type of experiment, pioneered by Field et al. uses a single electron transistor to measure capacitively the electron charge of a mesoscopic conductor in proximity to the transistor.

The conductor can be brought into a transport state, if it is connected to several leads. Instead of equilibrium noise, now the shot noise due to the granularity of the charge, is the source of fluctuations. Now the fluctuations of the charge and their relaxation are governed by a non-equilibrium charge relaxation resistance \( R_v \). Non-equilibrium charge relaxation resistances for fluctuations generated by shot noise have been discussed by Pedersen, van Langen and the author for quantum point contacts (QPC) and chaotic cavities. An experiment in which the fluctuations of the charge in the presence of shot noise is of paramount importance has been carried out by Buks et al. In this work an Aharonov-Bohm ring with a quantum dot which is
in turn capacitively coupled to a QPC is investigated. With increasing current through the QPC, 
Buks et al. have observed that the visibility of the Aharonov-Bohm effect is diminished. The 
charge fluctuations associated with the shot noise of the system provide an additional dephasing 
mechanism. In addition to the discussion of the dephasing time given by Buks et al. theoretical 
work by Levinson and Aleiner has addressed this problem. The author and A. M. Martin, 
have related the dephasing time in Coulomb coupled conductors to the nonequilibrium charge 
relaxation resistance. Here we analyze a simpler geometry: We consider a Hall bar with a QPC which acts as a source of shot noise and consider a capacitively coupled gate which overlaps 
the edge of the Hall bar at some distance away from the QPC (see Fig. 1). This geometry is 
thus similar to the one used in the experiment of Chen et al. It is also a geometry, which is 
of interest in a new experiment by Buks et al. where instead of a gate it is planned to 
couple to a single or double quantum dot. The purpose of this work is to present a discussion 
of the non-equilibrium charge relaxation resistance which determines the potential fluctuations 
between the conductor and the gate and determines the current induced into the gate. The 
dephasing rate which is of interest in the experiment of Buks et al. is the subject of Ref. and 
will not be discussed here.

2 The scattering matrix

To be specific we consider the conductor shown in Fig. 1. Of interest is the current $dI_\alpha(\omega)$ at contact $\alpha$ of this conductor if an oscillating voltage $dV_\beta(\omega)$ is applied at contact $\beta$. Here $\alpha$ and $\beta$ label the contacts of the conductor and the gate and take the values 1, 2, 3. Furthermore, 
we are interested in the current noise spectrum $S_{I_\alpha I_\beta}(\omega)$ defined as $2\pi S_{I_\alpha I_\beta}(\omega)\delta(\omega + \omega') = 1/2\langle \hat{I}_\alpha(\omega)\hat{I}_\beta(\omega') + \hat{I}_\beta(\omega')\hat{I}_\alpha(\omega) \rangle$ and the fluctuation spectrum of the electrostatic potential. We assume that the charge dynamics is relevant only in the region underneath the gate. Everywhere else we assume the charge to be screened completely. This is a strong assumption: The QPC is made with the help of gates (capacitors) and also exhibits its own capacitance. Edge states might generate long range fields, etc. Thus the results presented below can only be expected to capture the main effects but can certainly be refined. We assume that the gate is a macroscopic conductor and screens perfectly. To be brief, we consider the case of one edge channel only.

The scattering matrix of the QPC alone can be described by $r \equiv s_{11} = s_{22} = -i R^{1/2}$ and $t \equiv s_{21} = s_{12} = T^{1/2}$ where $T = 1 - R$ is the transmission probability through the QPC. Here the indices 1 and 2 label the reservoirs of Fig. 1. A carrier traversing the region underneath the gate acquires a phase $\phi(U)$ which depends on the electrostatic potential $U$ in this region. Since we consider only the charge pile up in this region all additional phases in the scattering
problem are here without relevance. The total scattering matrix of the QPC and the traversal of the region Ω is then simply

$$s = \begin{pmatrix} r & t \ e^{i\phi} \\ t^* e^{-i\phi} & r^* \end{pmatrix}. \quad (1)$$

If the polarity of the magnetic field is reversed the scattering matrix is given by $s_{\alpha\beta}(B) = s_{\beta\alpha}(-B)$, i.e. in the reversed magnetic field it is only the second column of the scattering matrix which contains the phase $\phi(U)$. In what follows the dependence of the scattering matrix on the phase $\phi$ is crucial. We emphasize that the approach presented here can be generalized by considering all the phases of the problem and by considering these phases and the amplitudes to depend on the entire electrostatic potential landscape.

3 Density of States Matrix Elements

To describe the charge distribution due to carriers in an energy interval $dE$ in our conductor, we consider a current of unit amplitude incident from contact $\beta$ and simultaneously a current of unit amplitude incident from contact $\gamma$ which leave the conductor through contact $\alpha$. This gives rise to a charge in the region of interest given by $N_{\beta\gamma}(\alpha) dE$ where $dE$ is a small energy interval, and $N_{\beta\gamma}(\alpha)$ is a density of states matrix element given by

$$N_{\beta\gamma}(\alpha) = -(1/4\pi i) [s_{\alpha\beta}^*(\partial s_{\alpha\gamma}/\partial e U) - (\partial s_{\alpha\beta}^*/\partial e U)s_{\alpha\gamma}]. \quad (2)$$

The low frequency charge dynamics can be found if these density of states matrix elements are known. For the specific example given by Eq. (1) we find that all elements with $\alpha = 1$ vanish: $N_{11}(1) = N_{21}(1) = N_{12}(1) = N_{22}(1) = 0$. There are no carriers incident from contact 1 or 2 which pass through region Ω and leave the conductor through contact 1. The situation is different if we demand that the current leaves the sample through contact 2. Now we find

$$N_{\beta\gamma}(2) = \begin{pmatrix} TN & t^* r N \\ r^* t N & RN \end{pmatrix}, \quad (3)$$

where $N = -(1/2e\pi d\phi/dU$ is the density of states of carriers in the edge state underneath the gate. For the reverse magnetic field polarity all components of the matrix vanish except the elements $N_{22}(1) = TN$ and $N_{22}(2) = RN$.

For the charge and its fluctuations underneath the gate it is not relevant through which contact carriers leave. The charge pile up and its fluctuations are thus governed by a matrix

$$N_{\beta\gamma} = \sum_\alpha N_{\beta\gamma}(\alpha) \quad (4)$$

which is obtained by summing over the contact index $\alpha$ from the elements given by Eq. (2). For our example the density matrix elements for the charge are thus evidently given by $N_{\beta\gamma} = N_{\beta\gamma}(2)$ whereas for the reversed magnetic field polarity we have $N_{11} = TN$, $N_{22} = RN$ and $N_{21} = N_{21} = 0$.

Furthermore, we will make use of the injectivity of a contact into the region Ω and will make use of the emissivity of the region Ω into a contact. The injectivity of contact is the charge injected into a region in response to a voltage variation at the contact $\alpha$, independently through which contact the carriers leave the sample. The injectivities of contact 1 and 2 are

$$N_1 = N_{11}(1) + N_{11}(2) = TN \quad (5)$$

$$N_2 = N_{22}(1) + N_{22}(2) = RN \quad (6)$$

Note that the sum of the injectivities of both contacts is just the density of states $N$ underneath the gate. The emissivity of region Ω is the portion of the density of states of carriers which leave
the conductor through contact $\alpha$ irrespectively from which contact they entered the conductor. We find emissivities
\begin{align}
N^{(1)} &= N_{11}(1) + N_{22}(1) = 0, \\
N^{(2)} &= N_{11}(2) + N_{22}(2) = N.
\end{align}
(7) (8)

Any charge accumulation or depletion is only felt in contact 2. The injectivities and emissivities in the magnetic field $B$ are related by reciprocity to the emissivities and injectivities in the reversed magnetic field, $N_{\alpha}(B) = N^{(\alpha)}(-B)$ and $N^{(\alpha)}(B) = N_{\alpha}(-B)$. In contrast, the density of states $N$ is an even function of magnetic field.

4 The Poisson Equation: The effective interaction

Thus far we have only considered bare charges. The true charge, however, is determined by the long range Coulomb interaction. First we consider the screening of the average charges and in a second step we consider the screening of charge fluctuations. We describe the long range Coulomb interaction between the charge on the edge state and on the gate with the help of a geometrical capacitance $C$. The charge on the edge state beneath the gate is determined by the voltage difference between the edge state and the gate $dQ = C(dU - dV_g)$, where $dU$ and $dV_g$ are deviations from an equilibrium reference state. On the other hand the charge beneath the gate can also be expressed in terms of the injected charges $e^2 N_1 dV_1$ in response to a voltage variation at contact 1 and $e^2 N_2 dV_2$ in response to a voltage variation at contact 2. The injected charge leads to a response in the internal potential $dU$ which in turn generates a screening charge $-e^2 N dU$ proportional to the density of states. Thus the Poisson equation for the charge underneath the gate is
\begin{equation}
dQ = C(dU - dV_g) = e^2 N_1 dV_1 + e^2 N_2 dV_2 - e^2 N dU
\end{equation}
(9)
and the charge on the gate is given by $-dQ = C(dV_g - dU)$. Solving Eq. (9) for $dU$ gives
\begin{equation}
dU = G_{eff}(C dV_g + e^2 N_1 dV_1 + e^2 N_2 dV_2),
\end{equation}
(10)
where $G_{eff} = (C + e^2 N)^{-1}$ is an effective interaction which gives the potential underneath the gate in response to an increment in the charge.

5 Admittance

Consider now the low-frequency conductance: To leading order in the frequency $\omega$ we write
\begin{equation}
G_{\alpha\beta}(\omega) = G_{\alpha\beta}(0) - i\omega E_{\alpha\beta} + \omega^2 K_{\alpha\beta} + O(\omega^3).
\end{equation}
(11)
Here $G_{\alpha\beta}(0)$ is the dc-conductance matrix, $E_{\alpha\beta}$ is the emittance matrix, and $K_{\alpha\beta}$ is a second order dissipative contribution to the frequency dependent admittance. The zero-frequency dc-conductance matrix has only four non-vanishing elements which are given by $G = G_{11} = G_{22} = -G_{12} = -G_{21} = (e^2/h)T$. Ref. showed that the emittance matrix $E$ is given by
\begin{equation}
E_{\alpha\beta} = e^2 N_{\beta\beta}(\alpha) - e^2 N^{(\alpha)} G_{eff} N_{\beta}
\end{equation}
(12)
As it is written, Eq. (12) applies only to the elements where $\alpha$ and $\beta$ take the values 1 or 2. The remaining elements can be obtained from current conservation (which demands that the elements of each row and column of this matrix add up to zero) or can be obtained directly by using a more general formula. For our example we find an emittance matrix,
\begin{equation}
E = C\mu \begin{pmatrix}
0 & 0 & 0 \\
T & R & -1 \\
-\bar{T} & -\bar{R} & 1
\end{pmatrix},
\end{equation}
(13)
with an electrochemical capacitance of the conductor vis-a-vis the gate given by \( C_\mu = C e^2 N/(C+e^2 N) \). Eq. (13) determines the displacement currents in response to an oscillating voltage at one of the contacts. There is no displacement current at contact 1 (the elements of the first row vanish) which is consequence of our assumption that charge pile up occurs only underneath the gate. The emittance matrix in the magnetic field \( B \) and in the magnetic field \(-B\) are related by reciprocity, \( E_{\alpha\beta}(B) = E_{\beta\alpha}(-B) \). For the reverse polarity, a voltage oscillation at contact 1 generates no displacement currents (the elements of the first column vanish).

The emittance matrix element \( E_{21} \) is positive and thus has the sign not of a capacitive but of an inductive response. The elements of row 3 and column 3 are a consequence of purely capacitive coupling and have the sign associated with the elements of a capacitance matrix. Thus these elements represent the capacitance matrix elements which can be measured in an ac-experiment. Note that the capacitances \( E_{31} \equiv C_{31} \) and \( E_{32} \equiv C_{32} \) depend not only on the density of states and geometrical capacitances but also on transmission and reflection probabilities. Measurement of these capacitances provides thus a direct confirmation of the concept of partial density of states. Furthermore, we see that for instance \( C_{31}(B) \equiv E_{31} = TC_\mu \) but \( C_{31}(-B) = 0 \). A similarly striking variation of the capacitance coefficients was observed in the experiment of Chen et al. in the integer quantum Hall effect and in Refs. in the fractional quantum Hall effect.

6 Bare charge fluctuations

Let us now turn to the charge fluctuations. With the help of the charge density matrix the low frequency limit of the bare charge fluctuations can be obtained. It is given by

\[
S_{NN}(\omega) = h \sum_{\delta\gamma} \int dE F_{\gamma\delta}(E,\omega) N_{\gamma\delta}(E,E+\hbar\omega) N_{\gamma\delta}^\dagger(E,E+\hbar\omega) \tag{14}
\]

where the elements of \( N_{\gamma\delta} \) are in the zero-frequency limit of interest here given by Eq. (4) and and \( F_{\gamma\delta} = f_\gamma(E)(1-f_\delta(E+\hbar\omega)) + f_\delta(E+\hbar\omega)(1-f_\gamma(E)) \) is a combination of Fermi functions. Using only the zero-frequency limit of the elements of the charge operator determined above gives,

\[
S_{NN}(\omega) = hN^2 \left[ T^2 \int dE \int F_{11}(E,\omega) + T\mathcal{R} \int dE F_{12}(E,\omega) + \mathcal{R}^2 \int dE F_{22}(E,\omega) \right]. \tag{15}
\]

At equilibrium all the Fermi functions are identical and we obtain \( S_{NN}(\omega) = hN^2 \int dE F(E,\omega) \) which in the zero-frequency limit is

\[
S_{NN}(\omega) = hN^2 kT \tag{16}
\]

and at zero-temperature to leading order in frequency is,

\[
S_{NN}(\omega) = hN^2 \hbar \omega. \tag{17}
\]

In the zero-temperature, zero-frequency limit, in the presence of a current through the sample, we find for the charge fluctuations associated with shot noise

\[
S_{NN}(\omega) = hN^2 T \Re e[V]. \tag{18}
\]

However, the bare charge fluctuations are not by themselves physically relevant.
7 Fluctuations of the true charge

To find the fluctuations of the true charge we now write the Poisson equation for the fluctuating charges. All contact potentials are at their equilibrium value, $dV_1 = dV_2 = dV_g = 0$. The fluctuations of the bare charge now generate fluctuations in the electrostatic potential. Thus the electrostatic potential has also to be represented by an operator $\hat{U}$. Furthermore, the potential fluctuations are also screened. As in the case of the average charges we take the screening to be proportional to the density of states $N$ but replace the c-number $U$ by its operator expression $\hat{U}$. The equation for the fluctuations of the true charge is thus

$$d\hat{Q} = C\hat{U} = e\hat{N} - e^2\hat{N}\hat{U}$$  \hspace{1cm} (19)

whereas the fluctuation of the charge on the gate is simply $-d\hat{Q} = -C\hat{U}$. Thus $d\hat{Q}$ is the charge operator of the dipole which forms between the charge on the edge state and the charge on the gate. Solving Eq. (19) for the potential operator $\hat{U}$ and using this result to find the fluctuations of the charge $d\hat{Q}$ gives

$$S_{QQ}(\omega) = e^2C^2G_{e,f}^2S_{NN}(\omega) = 2C^2(1/2e^2)(S_{NN}(\omega)/N^2).$$  \hspace{1cm} (20)

We now discuss three limits of this result.

8 Equilibrium and non-equilibrium charge relaxation resistance

At equilibrium, in the zero-frequency limit, the charge fluctuation spectrum can be written with the help of the equilibrium charge relaxation resistance $R_q$,

$$S_{QQ}(\omega) = 2C^2R_qkT.$$  \hspace{1cm} (21)

For our specific example we find using Eqs. (16) and (20),

$$R_q = h/2e^2.$$  \hspace{1cm} (22)

The charge relaxation resistance is universal and equal to half a resistance quantum as expected for a single edge state. At equilibrium the fluctuation spectrum is via the fluctuation dissipation theorem directly related to the dissipative part of the admittance. We could also have directly evaluated the element $K_{33}$ of Eq. (11) to find $K_{33} = C^2R_q$. Second at equilibrium, but for frequencies which are large compared to the thermal energy, but small compared to any intrinsic excitation frequencies, we find that zero-point fluctuations give rise to a noise power spectral density

$$S_{QQ}(\omega) = 2C^2R_qh\omega$$  \hspace{1cm} (23)

which is determined by the charge relaxation resistance Eq. (22). Third, in the presence of transport, we find in the zero-frequency, zero-temperature limit, a charge fluctuation spectrum determined by the non-equilibrium charge relaxation resistance $R_v$,

$$S_{QQ}(\omega) = 2C^2R_v|V|,$$  \hspace{1cm} (24)

where $|V|$ is the voltage applied between the two contacts of the sample and a non-equilibrium charge relaxation resistance

$$R_v = (h/e^2)TR,$$  \hspace{1cm} (25)

which is maximal for a semi-transparent QPC. We emphasize that the universality of the equilibrium charge relaxation resistance and the property that the non-equilibrium charge relaxation
resistance follows directly the low frequency shot noise spectrum are due to the special geometry investigated here and are due to the fact that we considered the case of one edge state only. If two or more edge states are present both $R_q$ and $R_v$ depend on the density of states of all the edge channels underneath the gate.

The current at the gate due to the charge fluctuations is $dI_g = -iωdQ(ω)$ and thus its fluctuation spectrum is given by $S_{1g1g}(ω) = ω^2S_{QQ}$. The potential fluctuations are related to the charge fluctuations by $d\hat{U} = d\hat{Q}/C$ and thus the spectral density of the potential fluctuations is $S_{UU}(ω) = C^{-2}S_{QQ}$. Thus the charge relaxation resistance determines, together with the electrochemical and geometrical capacitance, the fluctuations of the charge, the potential and the current induced into the gate. Since dephasing rates can be linked to the low frequency limit of the potential fluctuations the charge relaxation resistance also determines the dephasing rate in Coulomb coupled mesoscopic conductors.

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