Perturbative breaking of the pseudospin symmetry in the relativistic harmonic oscillator

R. LISBOA and M. MALHEIRO
Instituto de Física, Universidade Federal Fluminense, 24210-340 Niterói, Brazil
ronas@if.uff.br

A. S. de CASTRO
Departamento de Física e Química, Universidade Estadual Paulista, 12516-410 Guaratinguetá, São Paulo, Brazil

P. ALBERTO and M. FIOLEAIS
Departamento de Física and Centro de Física Computacional, Universidade de Coimbra, P-3004-516 Coimbra, Portugal

We show that relativistic mean fields theories with scalar, $S$, and vector, $V$, quadratic radial potentials can generate a harmonic oscillator with exact pseudospin symmetry and positive energy bound states when $S = -V$. The eigenenergies are quite different from those of the non-relativistic harmonic oscillator. We also discuss a mechanism for perturbatively breaking this symmetry by introducing a tensor potential. Our results shed light into the intrinsic relativistic nature of the pseudospin symmetry, which might be important in high density systems such as neutron stars.

Keywords: Pseudospin symmetry; Harmonic oscillator; Tensor potential.

Recently, there has been much interest in understanding the nuclear pseudospin symmetry in terms of relativistic dynamics \(^1\). However, in the presence of nuclear scalar and vector mean-field potentials this symmetry is broken non-perturbatively as it has been discussed in recent articles \(^2,3\). In this work we will show that for quadratic radial scalar, $S$, and vector, $V$, potentials with $S = -V$ is possible to break the pseudospin symmetry by introducing a tensor potential that preserves the form of the harmonic oscillator central potential but generates a pseudospin orbit term.

1. Harmonic oscillator and exact pseudospin symmetry

In relativistic mean field theories with scalar and vector potentials the Dirac Hamiltonian for a fermion (nucleon) of mass $m$ is

$$
H_D = \alpha \cdot p + \beta m + \frac{1}{2}(1 + \beta)\Sigma + \frac{1}{2}(1 - \beta)\Delta ,
$$

(1)
where $\alpha$ and $\beta$ are the usual Dirac matrices. We have introduced the “sum” and the “difference” potentials defined by $\Sigma = V + S$ and $\Delta = V - S$. When $\Sigma = 0$ or $\Delta = 0$, due to the matrix structure of $\frac{1}{2}(1 \pm \beta)$, the Dirac Hamiltonian is invariant under a SU(2) symmetry. This is a general feature which does not depend on the particular form of $S$ and $V$. In this paper we will consider only the case $\Sigma = 0$ ($S = -V$) and a harmonic oscillator potential with angular frequency $\omega$ for the $\Delta$ potential, i.e., $\Delta = \frac{1}{2}m\omega^2 r^2$.

A more generalized form of the relativistic harmonic oscillator where the case $\Delta = 0$ is also discussed has been presented recently. Following the details of that paper it is easy to prove that, for the case $\Sigma = 0$, the lower component of the Dirac spinor satisfies the differential equation

$$\frac{d^2}{dr^2}f_\kappa(r) - \frac{\tilde{l}(\tilde{l} + 1)}{r^2} - \frac{m(E - m)}{2} \omega_1^2 r^2 - \frac{m^2 - E^2}{r^2} f_\kappa(r) = 0. \quad (2)$$

As shown in, $\tilde{l}$ is not the non-relativistic angular momentum $l$ but rather the pseudo-orbital angular momentum of the lower component of the Dirac spinor. They are related by $\tilde{l} = l - \kappa/|\kappa|$, where the quantum number $\kappa$ determines whether spins are parallel or antiparallel. Pseudospin symmetry is exact when doublets with $j = \tilde{l} \pm \tilde{s}$ are degenerate, where $\tilde{s} = s$ is the pseudospin quantum number.

The eigenenergies are given by

$$\langle E + m \rangle \sqrt{\frac{E - m}{2m}} = \omega_1 \left(2\tilde{n} + \frac{3}{2}\right), \quad (\tilde{n} = 0, 1, 2, \ldots). \quad (3)$$

where $\tilde{n}$ is the radial quantum number related to the nodes of the lower component $f_\kappa(r)$. From Eq. (3) one concludes that the real solutions must have positive binding energy $E = \mathcal{E} - m$. The second order equation (2), which only depends on $\tilde{l}$, and also the eigenenergy equation (3), show that there is no pseudospin–orbit coupling. Therefore the states with same $(\tilde{n}, \tilde{l})$, but with $j = \tilde{l} + 1/2$ and $j = \tilde{l} - 1/2$ are degenerate; they can be seen as pseudospin doublets as discussed in. Thus, when $\Delta$ is a harmonic oscillator potential and $\Sigma = 0$, pseudospin symmetry is exact and there are only positive-energy bound states, i.e., $\Delta$ acts as a binding potential. This is an interesting result in view of the fact that the pseudospin symmetry obtained in the limit $\Sigma(r) \to 0$ cannot be realized for nuclear vector and scalar mean-fields which go to zero as $r \to \infty$. In fact, in that case, $\Sigma$ acts as a binding negative central potential well and therefore no bound states may exist when $\Sigma = 0$. The spectrum of single particle states for the case $\Sigma = 0$, $\omega_1 = 2$ and $m = 10$ is shown in Fig. 1 (a) using the quantum numbers of the upper component, that can be related to the non-relativistic quantum numbers. In Fig. 1 (b) we classify the same energy levels by the quantum numbers of the lower components $f_\kappa$. The comparison between these two figures exhibits the pseudospin symmetry and its quantum numbers. For example, the doublets $[1s_{1/2} - 0d_{3/2}]$ and $[1p_{3/2} - 0f_{5/2}]$, which have the same pseudo angular momentum $\tilde{l}$ and the same $\tilde{n}$, are, in the new notation, $[0\tilde{p}_{1/2} - \tilde{p}_{3/2}]$ and $[0\tilde{d}_{3/2} - \tilde{d}_{5/2}]$, respectively. Therefore, the harmonic
oscillator with $\Sigma = 0$ and $\Delta = \frac{1}{2} m \omega^2 r^2$ provides an example of exact pseudospin symmetry. Fig. 1 (a) also shows the $(2n + \tilde{l})$ degeneracy, which means that not only the states with the same $\tilde{n}$, $\tilde{l}$ are degenerate (pseudospin partners) but also, for example, $(\tilde{n} - 1, \tilde{l} + 2)$ or $(\tilde{n} + 1, \tilde{l} - 2)$ have the same energy.

The non-relativistic limit of the eigenvalue equation in Eq.(3) is reached when $\omega_1 \ll m$, which, in turn, means that $E = E - m \ll 1$, giving

$$E = \frac{1}{2m} \left( 2\tilde{n} + \tilde{l} + \frac{3}{2} \right)^2 \omega_1^2.$$  

This equation, valid for $\Sigma = 0$, shows that the energy is of second order in the ratio $\omega_1 / m$, meaning that the energy is zero up to first order in $\omega_1 / m$. We can interpret this fact by saying that, up to this order, there is no non-relativistic limit for $\Sigma = 0$ and therefore the theory is intrinsically relativistic and so is the pseudospin symmetry.

2. Perturbative breaking of the pseudospin by a tensor potential

When $\Sigma \neq 0$ ($S \neq -V$) a pseudospin orbit term shows up in the equation for the lower component given by

$$- \frac{\Sigma'}{E - \Sigma(r)} \kappa,$$

where the prime denote the derivative with respect to $r$ and $\kappa$ is the quantum number referred to before. So when $\Sigma' \neq 0$ the denominator, $E - \Sigma(r)$, can become very small near the pole and the contribution of this term can be quite significant, as shown in 5. This manifests the non-perturbative breaking of this symmetry when $\Sigma \neq 0$, as it is the case for the nucleus $^2$. However, we may overcome this problem if we keep $\Sigma' = \Sigma = 0$ but a tensor potential $i\beta \alpha \cdot \hat{r} U$ is introduced in the Dirac equation. In this case a new term appears in the spin-orbit term, $-2\kappa U / r$, as shown in 4. Because of the product $\alpha \cdot \hat{r}$, the pole structure in the denominator discussed before is absent.
If we choose a linear form for $U(r) = m\omega_2 r$, we have an additional quadratic central potential with a frequency $\omega_2$ in Eq. (2), so that we still have a global harmonic oscillator central potential. Moreover, we generate a pseudospin orbit term $-2\kappa m \omega_2$. Thus, with this new term, the $\tilde{l}$ degeneracy (pseudospin symmetry) can be broken perturbatively in the sense that can be made so small as the $\omega_2$ frequency. We show in Fig. 2 how this symmetry is broken for the $\tilde{p}$ states, for positive $\omega_2$, and present the eigenenergies in the spectroscopic notation $\tilde{n}\tilde{l}$, where states with $\tilde{l} = 0, 1, 2, \ldots$ are denoted by $\tilde{s}, \tilde{p}, \tilde{d}, \ldots$ respectively. Interestingly, from Fig. 3 we see that with the breaking tensor potential there is a value of $\omega_2$ for which $\tilde{0}_{1/2}$ becomes the ground state.

Finally, the relativistic nature of the pseudospin symmetry with harmonic oscillator potentials shown here suggests that this symmetry may be more important in ultra-relativistic systems. Thus, this symmetry may play a role in high density matter, such as the matter inside compact stars.
Acknowledgements

R. L. and M. M. thanks the nice atmosphere during the IWARA (Olinda/PE) where this work has been presented. R. L. and M. M. acknowledge support from CNPq. A.S. de C. also thanks support CNPq and FAPESP. P.A. and M.F. acknowledges the financial support from FCT (POCTI), Portugal.

Bibliography

1. J. N. Ginocchio, Phys. Rev. Lett. 78, 436 (1997).
2. P. Alberto, M. Fiolhais, M. Malheiro, A. Delfino and M. Chiapparini, Phys. Rev. Lett. 86, 5015 (2001); P. Alberto, M. Fiolhais, M. Malheiro, A. Delfino and M. Chiapparini, Phys. Rev. C65, 034307 (2002).
3. R. Lisboa, M. Malheiro and P. Alberto, Phys. Rev. C67, 054305 (2003).
4. R. Lisboa, M. Malheiro, A. S. de Castro, P. Alberto and M. Fiolhais, Phys. Rev. C69, 024319 (2004).
5. S. Marcos, L. N. Savushkin, M. López-Quelle and P. Ring, Phys. Rev. C62, 054309 (2000).