Time transfer functions as a way to validate light propagation solutions for space astrometry

Stefano Bertone$^{1,2}$, Olivier Minazzoli$^3$, Mariateresa Crosta$^2$, Christophe Le Poncin-Lafitte$^1$, Alberto Vecchiato$^2$ and Marie-Christine Angonin$^1$

$^1$ Observatoire de Paris, SYRTE, CNRS/UMR 8630, UPMC, 61 avenue de l’Observatoire, F-75014 Paris, France
$^2$ INAF, Astrophysical Observatory of Torino, University of Torino, Via Osservatorio 20, 1-10025 Pino Torinese (Torino), Italy
$^3$ UMR ARTEMIS, CNRS, University of Nice Sophia-Antipolis, Observatoire de la Côte d’Azur, BP4229, F-06304, Nice Cedex 4, France

E-mail: stefano.bertone@obspm.fr

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Abstract

Given the extreme accuracy of modern space astrometry, a precise relativistic modeling of observations is required. Concerning light propagation, the standard procedure is the solution of the null-geodesic equations. However, another approach based on the time transfer functions (TTF) has demonstrated its capability to give access to key quantities such as the time of flight of a light signal between two point-events and the tangent vector to its null-geodesic in a weak gravitational field using an integral-based method. The availability of several models, formulated in different and independent ways, must not be considered like an oversized relativistic toolbox. Quite the contrary, they are needed as validation to put future experimental results on solid ground. The objective of this work is then twofold. First, we build the time of flight and tangent vectors in a closed form within the TTF formalism giving the case of a time-dependent metric. Second, we show how to use this new approach to obtain a comparison of the TTF with two existing modelings, namely the Gaia RElativistic Model (GREM) and the Relativistic Astrometric MODel (RAMOD). In this way, we highlight the mutual consistency of the three models, opening the basis for further links between all the approaches, which is mandatory for the interpretation of future space missions data. This will be illustrated through two recognized cases: a static gravitational field and a system of gravitational mass monopoles in uniform motion.

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1. Introduction

Modern astrometry relies on high-precision observations whose data need to be reduced and interpreted in the framework of general relativity (GR) [1–6]. To reach the demanded precision, several key points need to be considered: the definition of the observation in a proper reference frame, global reference systems allowing the comparison of observations made in each proper reference frame and a precise modeling for the propagation of the observed signal. Each of these issues has been deeply studied in the literature: the definition of global reference systems has been given by the IAU 2000 Resolution B1.3 in the post-Newtonian (PN) approximation of GR [4] while several relativistic definitions of physically adequate local reference frames of a test observer have been proposed in [7, 8]. As mentioned above, a precise modeling for the relativistic propagation of electromagnetic waves (EW) is also required. In fact, the behavior of the EW in the Solar System is intrinsically related to space-time curvature and therefore one has to take it into account for modern astrometry. For instance, the astrometric mission Gaia [9] is expected to reach an accuracy of several microarcseconds (μas) for the positions, parallaxes and proper motion of remote celestial sources while PN corrections to light direction due to the gravitational field of Solar System’s bodies can reach 16 milliarcseconds (mas) for a light ray grazing Jupiter [5].

In this paper, we will focus on modeling the propagation of EW. In the paradigm of Maxwell electromagnetism minimally coupled to gravitation through the space-time metric, EW in their geometric optics limit are known to follow null-geodesics [10]. Assuming that the metric is known, solving the null-geodesic equations is the standard method allowing to get all the information about light propagation between two point-events. Many solutions have been proposed in the PN and the post-Minkowskian (PM) approximations when dealing with a metric tensor taking into account the dynamical behavior of the Solar System [3, 11–14]. However, it has been demonstrated that solving the null-geodesic equations is not mandatory and can be replaced by another approach based on the time transfer functions (TTF) [15]. If the TTF approach does not provide the full trajectory of light, two of us presented in [16] that it gives the essential tools to model an astrometric observable, namely the time of flight of an EW between two point-events and what we call the direction triple, i.e. the ratio of the spatial and temporal covariant components of the tangent vector to the null-geodesic at its emission and reception point. The TTF have been formulated as a general PM series of ascending powers of the Newtonian gravitational constant $G$ [17], which has not yet been done using null-geodesic approaches; explicit solutions have been obtained and tested in two cases: the PN stationary axisymmetric gravitating body [18] and a static point body up to 3PM order [19–23]. Nevertheless, this formalism has not been applied yet to light propagation in a time-dependent space-time, which is the problem treated in [3, 24, 25] using different approaches.

At present time, two robust modelings have been developed for Gaia: the Gaia RElativistic Model (GREM) [5] and the Relativistic Astrometric MODel (RAMOD) [6]. Both are based on the solution of the null-geodesic equations even if starting from a different definition of the involved quantities. Briefly, GREM is formulated according to a parametrized PN (PPN) scheme accurate at the μas level, while RAMOD is a family of models based on a general weak field metric and on a relativistic measurement protocol [26].

Since they will operate on the same set of real data, it is fundamental to be able to compare them. From the experimental point of view, in fact, modern space astrometry is going to bring our knowledge into a widely unknown territory. Such a huge push-forward will not only come from high-precision measurements, which call for a suitable relativistic modeling, but also in form of absolute results which can hardly be validated by independent, ground-based
observations. In this sense, it is of capital importance to have different, and cross-checked models to interpret these experimental data.

The goal of this paper is then twofold. In its first part we will present the application of the TTF formalism to an EW propagating in a system of point bodies in uniform motion. Second, we show how to use these results to obtain a consistency check with GREM and RAMOD on two well recognized quantities, namely the time of flight and the light direction triple.

The paper is organized as follows. Section 2 gives the notations used in this paper. In section 3 we give a short review of the TTF formalism in the PN approximation while in section 4 we present a new method to obtain the light direction triple in a closed form within the TTF formalism. The equations describing EW propagation in a time-dependent system are then explicitly given in section 5. Section 6 shows the procedure to interface the geodesic approaches to the TTF and finally, in section 7 we give our concluding remarks.

2. Notation and conventions

In this paper $c$ is the speed of light in a vacuum and $G$ is the Newtonian gravitational constant. The Lorentzian metric of space-time $\mathbb{R}^4$ is denoted by $g$. The signature adopted for $g$ is $((-++)$. We suppose that space-time is covered by a global quasi-Galilean coordinate system 
\[ (x^\mu) = (x^0, x) \]
where $x^0 = ct$, $t$ being a time coordinate, and $x = (x^i)$. We assume that $g_{00} < 0$ anywhere. We employ the vector notation $a$ in order to denote $(a^1, a^2, a^3) = (a^i)$. Considering two such quantities $a$ and $b$ we use $a \cdot b$ to denote $a^i b_i$ (Einstein convention on repeated indices is used). The quantity $|a|$ stands for the ordinary Euclidean norm of $a$. For any quantity $f(x^\lambda)$, $f_\alpha$ and $\partial_\alpha f$ denote the partial derivative of $f$ with respect to $x^\alpha$.

The indices in parentheses characterize the order of perturbation. They are set up or down, depending on the convenience.

3. Time transfer functions formalism

In this section, we recall the basics and the properties of the TTF formalism. This method stands as a development of Synge World Function [28], an integral approach based on the principle of minimal action (see [17] and references herein) and containing all the informations about an EW. While the World Function is an implicit equation of the photon trajectory nearly impossible to solve, the TTF formalism gives up some generality to provide important information about the propagation of an EW between two points at finite distance: the coordinate time of flight $T_e$ which is important in various fields of astronomy and space science, such as the positioning of space probes or the lunar laser ranging; the knowledge of the direction triple to the light ray, required for astrometry. The reader can refer to [16, 29, 30] and references herein for more details.

Let us define $x_A = (ct_A, x_A)$ the event of emission $\mathcal{A}$ and $x_B = (ct_B, x_B)$ the event of reception $\mathcal{B}$ of a light signal. We denote $T_e$ and $T_r$ as two distinct (coordinate) TTF defined as

\[ t_B - t_A = T_e(t_A, x_A, x_B) = T_r(t_B, x_A, x_B), \]

where $T_e$ and $T_r$ are evaluated at the event of emission $\mathcal{A}$ and at the event of reception $\mathcal{B}$, respectively.

We shall consider a weak gravitational field so that we can write

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \]

with $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ the Minkowskian background and $h_{\mu\nu}$ a small perturbation. The general PM expansion of this formalism has been given in [17] but in this work we shall
consider only the slow-motion, PN approximation [31]—a case well adapted to our Solar System in which the two approximations coincide. So, we assume that the potentials \( h_{\mu \nu} \) may be expanded as [12]

\[
\begin{align*}
    h_{00} &= \frac{1}{c^2} h_{00}^{(2)} + \mathcal{O} \left( \frac{1}{c^3} \right), \\
    h_{0i} &= \frac{1}{c^2} h_{0i}^{(3)} + \mathcal{O} \left( \frac{1}{c^3} \right), \\
    h_{ij} &= \frac{1}{c^2} h_{ij}^{(2)} + \mathcal{O} \left( \frac{1}{c^3} \right).
\end{align*}
\]

Under these hypothesis, the coordinate time of flight \( T_{c/r} \) of a photon between \( x_A \) and \( x_B \) is given by the expressions [19, 32]

\[
\begin{align*}
    T_c(x_A, t_B, x_B) &= \frac{R_{AB}}{c} + \frac{1}{c} \Delta_r(x_A, t_B, x_B) + \mathcal{O}(c^{-5}), \\
    T_r(t_A, x_A, x_B) &= \frac{R_{AB}}{c} + \frac{1}{c} \Delta_c(x_A, t_A, x_B) + \mathcal{O}(c^{-5}),
\end{align*}
\]

where \( R_{AB} \equiv |R_{AB}| \) with \( R_{AB} = x_B - x_A \); \( \Delta_{c/r}/c \) are the so-called ‘emission/reception delay functions’ [17] and represent the gravitational delay in the coordinate time of flight of the photon with respect to the Newtonian time of flight, defined as

\[
\begin{align*}
    \Delta_r &= \frac{R_{AB}}{2c^2} \int_0^1 \left[ h_{00}^{(2)} + \frac{2}{c} N_{AB} h_{0i}^{(3)} + N_{AB} N_{AB} h_{ij}^{(2)} \right] \zeta_r^{(\lambda)} \, d\lambda, \\
    \Delta_c &= \frac{R_{AB}}{2c^2} \int_0^1 \left[ h_{00}^{(2)} + \frac{2}{c} N_{AB} h_{0i}^{(3)} + N_{AB} N_{AB} h_{ij}^{(2)} \right] \zeta_c^{(\mu)} \, d\mu,
\end{align*}
\]

with \( N_{AB} \equiv R_{AB}/R_{AB} \). The two integrals are taken along the Minkowskian paths

\[
\begin{align*}
    \zeta_r^{(\lambda)} &= (x_B^0 - \lambda R_{AB}, x_B^i - \lambda R_{AB}^i), \\
    \zeta_c^{(\mu)} &= (x_A^0 + \mu R_{AB}, x_A^i + \mu R_{AB}^i),
\end{align*}
\]

which represent the unperturbed ‘straight lines’ between \( x_A \) and \( x_B \).

### 4. Light direction triple in closed form

The TTF formalism also provides a direct way of defining the ratio of the spatial and temporal covariant components of the tangent vectors to a photon trajectory \( k^{\mu} \equiv dx^\mu/d\lambda \) at its reception event, as follows

\[
\left( \frac{\mathbf{k}}{\mathbf{k}_0} \right)_B = -c \frac{\partial T_r}{\partial x_B^0} = -c \frac{\partial T_r}{\partial x_B^0} \left[ 1 - \frac{\partial T_r}{\partial t_B} \right]^{-1}.
\]

We call \( \left( \frac{\mathbf{k}}{\mathbf{k}_0} \right)_B \) the light direction triple at reception event. Similarly, one can define the light direction triple \( \left( \frac{\mathbf{k}}{\mathbf{k}_0} \right)_A \) at emission event as shown in [15].

We can outline the following procedure. Let us expand equation (7) as function of the gradient of \( \Delta_r \) and \( \Delta_c \) using equation (4)

\[
\left( \frac{\mathbf{k}}{\mathbf{k}_0} \right)_B = N_{AB}^{\mu} + \frac{\partial \Delta_r}{\partial x_B^{\mu}} + \mathcal{O}(c^{-5}) = N_{AB}^{\mu} + \frac{\partial \Delta_r}{\partial x_B^{\mu}} + N_{AB}^{\mu} \frac{\partial \Delta_r}{\partial x_B^{\mu}} + \mathcal{O}(c^{-5}).
\]

\(^4\) with the signature recommended by the IAU [4] (− + + +) and whose contravariant form \( h^{\mu \nu} \) with signature (+ − − −) is given in [15] as a PM expansion.

\[\frac{\partial \Delta_r}{\partial x_B^{\mu}}\]
Then, equation (5) allows us to express the gradients of $\Delta_x$ and $\Delta_y$ as integrals of the metric and its derivatives taken along the Minkowskian straight line (6) in order to build up the definition of the tangent vectors in a closed form. By defining $(N_{\alpha\beta})_i = \delta^i_j(N_{\alpha\beta})_j$ and

$$m_{i,a} \equiv \delta_{i,a} + 2N_{\alpha i}^k h_{0k,a} + N_{\alpha\beta}^k h_{jk,a},$$

$$\tilde{h}_i \equiv -N_{\alpha i}^j h_{00} - N_{\alpha\beta}^k N_{\beta i}^j h_{jk} + 2h_0 + 2N_{\alpha i}^j h_j,$$

(9a, 9b)

the gradients appearing in equation (8) can be computed as

$$\frac{\partial \Delta_c}{\partial x^i}_c = -\frac{1}{2} \int_0^1 [R_{AB}^i \delta m_0 - R_{AB}(1 - \lambda)m_i - \tilde{h}_i]_{z=1(\lambda)} d\lambda,$$

$$\frac{\partial \Delta_x}{\partial x^i}_c = \frac{1}{2} \int_0^1 [m_0]_{z=1(\lambda)} d\lambda,$$

$$\frac{\partial \Delta_y}{\partial x^i}_c = -\frac{1}{2} \int_0^1 [-R_{AB}^i \delta m_0 - R_{AB}^i m_i - \tilde{h}_i]_{z=1(\lambda)} d\lambda. $$

(10a, 10b, 10c)

5. Time transfer and light propagation in the Solar System

We provide now explicit equations for the quantities presented in sections 3 and 4 in the case of point-like, slowly moving and non-rotating bodies. This system can be represented by a PPN metric tensor admitting the following perturbation $h_{\mu\nu}$

$$h_{00} = \frac{2G}{c^2} \sum_p \mathcal{M}_p R_p(t, x),$$

$$h_{0i} = -(1 + \gamma) \frac{2G}{c^2} \sum_p \mathcal{M}_p \beta_p(t) R_p(t, x),$$

$$h_{ij} = \delta_{ij} \gamma h_{00},$$

(11)

with $\mathcal{M}_p$ the mass of the perturbing body $P$,

$$R_p(t, x) = x - x_P(t)$$

(12)

and $R_p(t, x) = |R_p(t, x)|$, where $x_P(t)$ is the trajectory of the perturbing body $P$; $\beta_p(t) = v_p(t)/c$ is the ratio of the perturbing body barycentric velocity at coordinate time $t$ to the speed of light $c$ and $\gamma$ is a PPN parameter [31].

Following the usual assumption [3] regarding the trajectory of the perturbing bodies $x_P$, we consider that they are rectilinear and uniform so that

$$x_P(t) = x_P(t_c) + c(t - t_c) \beta_p(t_c) + O(\Delta_x),$$

(13)

where $\Delta_x$ is some typical error made on the position of the perturbing body due to the linear approximation chosen for its trajectory and which can be minimized by an accurate choice of the free parameter $t_c$. As suggested in [5], we shall then choose $t_c$ as the time of maximum approach of the photon to the perturbing body $P$.

5.1. Delay functions in the case of point bodies in motion

Taking into account equation (11), then equation (5a) writes at first order

$$\Delta_x^{(1)}(x_A, t_B, x_B) = (\gamma + 1)R_{AB} \frac{G}{c^2} \sum_p \mathcal{M}_p g_p \int_0^1 \left[ \frac{1}{R_p(t, x)} \right]_{z=1(\lambda)} d\lambda,$$

(14)

where the integration path $z^\mu(\lambda)$ is given by equation (6a) and where we define

$$g_p \equiv N_{\alpha\beta} - \beta_p(t_c).$$

(15)

Remembering equation (12) along with equations (6a) and (13), we get

$$R_p(z^{(1)}_{\alpha}(\lambda), z_{\alpha}(\lambda)) = R_p - \lambda R_{pB} g_p + O(R_p \Delta_x),$$

(16)
where for practical reasons we set the notation
\[ R_{PB} = x_B - x_P(t_C) - c(t_B - t_C)\beta_P(t_C) \]  
(17a)
and
\[ R_{AB} = x_B - x_A, \]  
(17b)
with \( R_{XY} = |R_{XY}| \) and \( N_{XY} = R_{XY}/R_{XY} \). Noting the boundary conditions
\[ R_p(0) = R_{FB}, \]  
(18a)
\[ R_p(1) = R_{PB} - R_{AB}\rho_p \equiv R_{PA}, \]  
(18b)
\[ R_{PB} - R_{PA} = g_p R_{AB} \]  
(18c)
and substituting for \( R_p \) from equation (16) into equation (14), after some algebra one gets
\[ \Delta^{(1)}_e(x_A, t_B, x_B; v_P) = (\gamma + 1) \frac{G}{c^2} \sum_p \mathcal{M}_p(g_p \cdot N_{AB}) \ln \left[ \frac{g_p R_{PA} - R_{PB} \cdot g_p}{g_p R_{FB} - R_{PB} \cdot g_p} \right]. \]  
(19)
Expanding \( g_p \) in equation (19) it is then possible to show explicitly the terms depending on \( \beta_p \) as
\[ \Delta^{(1)}_b(x_A, t_B, x_B; v_P) = (\gamma + 1) \frac{G}{c^2} \sum_p \mathcal{M}_p \left\{ \ln \left( \frac{R_{PA} - R_{PB} \cdot N_{AB}}{R_{PB} - R_{PB} \cdot N_{AB}} \right) \right\} \]
\[ + \beta_p(t_C) \left\{ N_{AB} \ln \left( \frac{R_{PA} - R_{PB} \cdot N_{AB}}{R_{PB} - R_{PB} \cdot N_{AB}} \right) \right\} \]
\[ + \frac{R_{PB} - N_{AB} R_{PB}}{R_{PB} - R_{PA} \cdot N_{AB}} - \frac{R_{PB} - R_{PA} \cdot N_{AB}}{R_{PA} - R_{PA} \cdot N_{AB}} \right\}. \]  
(20)
A similar reasoning allows to compute the emission delay function as
\[ \Delta^{(1)}_e(t_A, x_A, x_B; v_P) = (\gamma + 1) \frac{G}{c^2} \sum_p \mathcal{M}_p(g_p \cdot N_{AB}) \ln \left[ \frac{g_p R_{PA} + R_{PB} \cdot g_p}{g_p R_{PA} + R_{PA} \cdot g_p} \right]. \]  
(21)
By setting \( \beta_p = 0 \) and \( g_p = N_{AB} \) in equations (19)–(21), we retrieve the static case given in [19] in the field of a single gravitational source.

The equivalence \( \Delta^{(1)}_b(x_A, t_B, x_B; v_p) = \Delta^{(1)}_e(t_A, x_A, x_B; v_P) \) stated by equation (1) when we consider equation (4) can be checked by inserting into equation (19) the relation
\[ R_{AB}^2 \left[ R_{P}^2 - (g_p \cdot R_X)^2 \right] = (g_p R_{AB} R_X)^2 - (R_{AB} g_P \cdot R_X)^2 \]
\[ = (R_{PA} R_{PB})^2 - (R_{PA} \cdot R_{PB})^2, \]  
(22)
with \( X \) taking the values ‘PB’ or ‘PA’ and the boundary conditions (18). After some algebra, one retrieves equation (21).

Finally, we apply equations (19)–(21) to the simple configuration of a signal propagating from the outer Solar System to the Earth and grazing Jupiter. Our evaluation of the influence by the orbital motion of Jupiter on the coordinate time of flight of the photon is of the order of 10\( ps \), in accordance with previous results [19].

5.2. Light direction triple in the case of point bodies in motion

We provide here the steps to compute the light direction triple at reception event \( (\hat{k})_P(x_A, t_B, x_B, x_P, \beta_P, \gamma) \) in the case of a time-dependent metric. First, we need to compute the partial derivatives of \( h_{00}(x, t) \) as
\[ h_{00, i} = -\frac{2G}{c^2} \sum_p \mathcal{M}_p \frac{R_P}{R_p} R_{P}^i, \quad h_{00, 0} = \frac{2G}{c^2} \sum_p \mathcal{M}_p \frac{R_P \beta_p(t_C)}{R_p}. \]  
(23)
Then, using equations (8) and (10) with the metric (11) and equation (23), it yields the following equation for the light direction triple

$$\vec{k}_b = -N_{AB}^i + (\gamma + 1) \frac{G}{c^2} \sum_p M_p \int_0^1 \{ R_{AB}^2 \left[ \left( R_{AB} \beta_p \cdot (t_C) - R_{AB}^i g_p \right) \frac{\lambda - \lambda^2}{R_p^i(\lambda)} \right] + \left( R_{PB}^i - N_{AB}^i R_{PB} \cdot \beta_p(\lambda) \right) \frac{1 - \lambda}{R_p^i(\lambda)} + \frac{2 \beta_p^i(\lambda) - N_{AB}^i}{R_p(\lambda)} \} \, d\lambda, \quad (24)$$

where the terms $\beta$ and $g_p$ represent the impact of the velocity of the perturbing bodies on light propagation.

The explicit computation of the integrals appearing in the right-hand side (rhs) of equation (24) may be obtained by taking into account the boundary conditions set in equation (18). After some algebra, we get an explicit expression for the light direction triple in the case of multiple deflecting bodies in uniform motion as

$$\vec{k}_b = -N_{AB}^i + (\gamma + 1) \frac{G}{c^2} \sum_p M_p \left[ R_{AB}^2 R_{PB}^i \left[ N_{AB}^i - (R_{PB} \cdot g_p)^2 \right] \right. \times \left[ N_{AB}^i \left[ (R_{PB} \cdot N_{AB})(R_{PB}^2 - R_{PA} R_{PB} - R_{AB} R_{PB} \cdot \beta_p(t_C)) - R_{PB}^2 R_{AB}^2 g_p^2 \right] + R_{PB}^2 \left[ R_{PB} R_{PA} - R_{PB}^2 + R_{AB} R_{PB} \cdot g_p \right] \right. \right. \left. \beta_p^i(t_C) R_{PB} \left[ (R_{PB} - R_{PB})(R_{PB} \cdot N_{AB}) + R_{PB} R_{AB} \right] \right. \left. + (\gamma + 1) \frac{G}{c^2} \sum_p M_p \beta_p^i(t_C) - N_{AB}^i \beta_p^i(t_C) \cdot N_{AB} \right] \ln \frac{R_{PB} + R_{PB} \cdot N_{AB}}{R_{PA} + R_{PA} \cdot N_{AB}} + O(c^{-4}). \quad (25)$$

As far as we know, this result is new within the TTF approach. In section 6, we check our formulation with some previous results obtained in [3] through the analytical solution of the null-geodesic equations. We shall note that, from the point of view of the astrometric data analysis, the last equation is obtained as a function of all known quantities (i.e., the coordinates of the observing satellite and the mass distribution in the Solar System) and of the astrometric unknown (i.e., the source coordinates). By setting $\beta_p = 0$ and $g_p = N_{AB}$, the perturbing bodies are fixed at their position at time $t_C$ and we easily retrieve the static case presented in [17]. To evaluate the magnitude of the velocity terms contributing to the direction of light, we use the definition given by [22] in the context of a spherically symmetric space-time and for a signal coming from infinity as

$$\Delta \chi \approx |N_{AB} \times \vec{k}_b|. \quad (26)$$

The expression of $\vec{k}_b$ for a light ray coming from infinity is then deduced from equation (25) by setting $N_{AB} = N$ and $R_{PB} \approx R_{AB}$ in this case. Introducing the impact parameter $b_p$ and the angle $\alpha$ between $R_{PB}$ and $N$ we get $b_p = R_{PB} \sin \alpha$, so that

$$\Delta \chi = (\gamma + 1) \frac{G}{c^2} \sum_p M_p \beta_p^i(t_C) - N_{AB}^i \beta_p^i(t_C) \cdot N_{AB} \ln \frac{R_{PB} + R_{PB} \cdot N_{AB}}{R_{PA} + R_{PA} \cdot N_{AB}} + O(c^{-4}, R_{AB}^{-1}). \quad (27)$$

The logarithmic term in equation (25) is proportional to $R_{AB}^{-1}$, so that it can be neglected for sources at quasi-infinity. Moreover, numerical estimates of equation (27) for a light ray grazing the major Solar System bodies are in agreement with [5].
6. Relativistic models for astrometry at the cross-checking point

The astrometric core solution of the forthcoming Gaia mission [9] will be performed by the astrometric global iterative solution (AGIS) software [33]. At the same time, an independent verification unit for AGIS called global sphere reconstruction (GSR) [34] has been set within the Gaia data processing and analysis consortium (DPAC). Since both pipelines are intended to operate on the same real data, the comparison of their results is mandatory in order to validate the final astrometric catalog. In order to keep the two software as separate as possible, two different relativistic modelings of light propagation have been implemented: AGIS relies on GREM [5], while GSR implements RAMOD [14]. Both models are internally consistent at the $\mu$as level required by Gaia. However, due to their different conception, a good reciprocal understanding is a complex task and a comparison effort is already in place, focusing on the modeling of the astrometric observable and the aberration [35]. We contribute to this study by establishing here an original procedure to cross-check our results on the gravitational deflection of light with those of these Gaia models. In particular, concerning GREM, we compare our model to its seminal study [3] which we will name KK92 in the following. This choice allows us to check both the results of section 5 and the GREM model, which is a version of KK92 accurate at the $\mu$as level.

6.1. KK92 modeling

KK92 describes light propagation in a gravitational system close to the one described by the metric (11), where the trajectory of the perturbing bodies is given by equation (13). Considering only the terms relevant for our purpose and using our notation, the trajectory of the photon can be written as

$$x^i(t) = x^i(t_B) + c\sigma^i(t - t_B) + \Delta x^i(t, x^i, t_B, x^i_B),$$

(28)

where $(t_B, x^i(t_B))$ are the reception coordinates, $\sigma$ is a normalized vector giving the unperturbed direction of light at past null infinity and the gravitational perturbation is given by

$$\Delta x^i = -\frac{2G}{c^2} \sum_P M_P \left\{ \frac{g_P \cdot R_P(t)}{g_P \cdot R_{PB} + g_P R_{PB}} \ln \left( \frac{g_P \cdot R_P(t) + g_P R_{PB}}{g_P R_P(t) - g_P \cdot R_{PB}} \right) \right\} + \frac{(\sigma \times R_P(t) \times g_P)^i}{g_P R_P(t) - g_P \cdot R_{PB}} - \frac{(\sigma \times R_{PB} \times g_P)^i}{g_P R_{PB} - g_P \cdot R_{PB}}.$$

(29)

The TTF formalism being designed for light propagation between two points located at finite distance, one has first to set the boundary condition

$$x(x_B, \sigma, \Delta t) = x_A$$

(30)

in equation (28) to provide the ‘crossing trajectory equation’

$$x^i(t_A) = x^i(t_B) - c\Delta t \sigma^i + \Delta x^i(\Delta t, x^i_B, \sigma^i),$$

(31)

where $\Delta t \equiv t_B - t_A$ represents the lapse of coordinate time between the emission and reception of the signal. In the following, equations (28–31) will be used to find the equivalence of KK92 and TTF for the coordinate time of flight and the direction triple at reception coordinates. If the velocity terms are neglected in equation (13), we then retrieve the equivalence in the GREM case and at the $\mu$as accuracy.
6.1.1. Coordinate time of flight. Let us state the formal development

$$\Delta t = \sum_i \Delta t_i,$$  \hspace{1cm} (32)

where $\Delta t_i$ is of order $O(c^{-n})$. Substituting for $\Delta t$ from equation (32) into equation (31) and identifying terms of the same order, we find

$$\Delta t_i(1) = \frac{R_{AB}}{c},$$  \hspace{1cm} (33a)

and

$$\Delta t_i(2) = N_{AB} \cdot \Delta x(\Delta t, x_i^b, \sigma^b),$$  \hspace{1cm} (33b)

where we used the property $\sigma \cdot \sigma = 1$ and noted that $N_{AB} \cdot (N_{AB} \times R \times g) = 0$. Using equation (22) shows that equation (33) is strictly equivalent to equation (4) when the gravitational delay is given by equation (19) with $\gamma = 1$.

6.1.2. Light direction triple. The relation between the tangent vectors $k^a = dx^a/d\lambda$ and the photon velocity $\dot{x}^i/c$ used in KK92 is obtained by $\dot{x}^i/c = (dx^i/d\lambda)/(dx^j/d\lambda) = k^i/k^0$. It follows that

$$\dot{k}_i = \frac{g_{ij}k^j + g_{0i}k^0}{g_{00}k^0 + g_{0i}k^i} = -\frac{\dot{x}_i}{c} - 2\hbar \theta_0 \sigma^i - (\delta_{ij} + \sigma^i \sigma^j)\lambda_0 + O(c^{-4}).$$  \hspace{1cm} (34)

An explicit equation for $\dot{x}_i/c$ is obtained by deriving the photon trajectory in equation (28) with respect to coordinate time. Its application at reception coordinates $(t_B, x_B)$ gives then

$$\frac{\dot{x}_B^j}{c} = \sigma^j + \frac{\Delta x^j}{c},$$  \hspace{1cm} (35)

where $\Delta x^j$ represents the gravitational perturbation to the photon direction

$$\frac{\Delta x^j}{c} = \frac{2G}{c^2} \sum_P M_P \frac{g_P}{R_{PB}} \left\{ g_P + \frac{(N_{AB} \times R_{PA} \times g_P)^j}{g_P R_{PB} - g_P \cdot R_{PB}} \right\}.$$  \hspace{1cm} (36)

Let us state the formal development

$$\sigma = \sum_i \sigma_i,$$  \hspace{1cm} (37)

where $\sigma_i$ is of order $O(c^{-n})$. Substituting for $\sigma$ from equation (37) into equation (31) and identifying all terms of the same order, we find

$$\sigma_i(1) = \frac{x_i^b - x_A^b}{R_{AB}},$$  \hspace{1cm} (38a)

and

$$\sigma_i(2) = -\frac{1}{R_{AB}} \left[ \delta_{ij} - N_{ABi} N_{ABj} \right] \Delta x^l(\Delta t, x_i^b, \sigma^b),$$  \hspace{1cm} (38b)

where we used the property $\sigma \cdot \sigma = 1$. Using equation (22) and after some algebra, we find the following relation

$$\frac{(\sigma \times R_{PA} \times g_P)^j}{g_P R_{PA} - (g_P \cdot R_{PA})} = \frac{(\sigma \times R_{PB} \times g_P)^j}{g_P R_{PB} - (g_P \cdot R_{PB})} = \frac{g_P (\sigma \times R_{PB} \times g_P)^j}{R_{PA}^2 R_{PB}^2 - (R_{PA} \cdot R_{PB})^2} [R_{PA} - R_{PB} - g_P R_{AB}].$$  \hspace{1cm} (39)
Substituting for $\Delta x^i$ from equation (29) into equation (38) with the relation given in equation (39), we obtain
\[
\sigma_i = N_{iAB} - \frac{2G}{c^2} \sum_p \mathcal{M}_p \left[ \frac{(N_{iAB} \times R_{PB} \times g_P^i)}{s_P R_{PB}^2} - (g_P R_{PA} - g_P R_{PB} - g_P^2 R_{AB}) \right]
+ (g_P - N_{iAB}^l N_{AB}^l g_P) \ln \left( g_P R_{PB} - g_P \cdot R_{PB} \right) + O(c^{-4}).
\]
(40)

It is then straightforward to check that equation (25) is equivalent to equation (34) when using equations (35), (36), and (40) and the metric tensor (11) at reception event.

6.2. RAMOD modeling

Based on a complete PM background [36], RAMOD has been solved explicitly in the 1PM approximation needed for GSR [37]. RAMOD always relies on measurable quantities with respect to a local barycentric observer along the light ray [6]. The unknown is then the local line-of-sight, representing the direction of the light ray in the rest space of the observer $u$, and defined as
\[
\vec{e}^\mu = -\frac{k^\mu}{u_\mu k^\theta} - u^\mu,
\]
(41)
where $k^\mu$ represent the tangent vectors to the null-geodesic. In this formalism, the null-geodesic equation transforms, according to the measurement protocol procedure, into a set of coupled nonlinear differential equations, called ‘master equations’
\[
\frac{d \vec{e}^0}{d\xi} - \vec{e} \vec{e}^j h_{0,j,i} - \frac{1}{2} h_{00,0} = 0,
\]
(42a)
\[
\frac{d \vec{e}^i}{d\xi} - \frac{1}{2} \vec{e} \vec{e}^j (\vec{e}^j h_{0,j,0} - h_{0,i}) + \vec{e} \vec{e}^j \left( h_{i,j,i} - \frac{1}{2} h_{i,j,k} \right) + \vec{e} \left( h_{0,0,i} + k_{i,0} - h_{0,i,k} \right) - \frac{1}{2} h_{00,k} - \vec{e} \vec{e}^j h_{0,j,0} + h_{00,0} = 0,
\]
(42b)
where $\xi$ is a parameter along the null-geodesic. Comparisons between RAMOD and other PM/PN astrometric models can be found in [36], where the author shows how RAMOD master equations recover the analytical linearized case used in [24] once converted in a coordinate form while in [35] the authors present a study of the aberration in RAMOD and GREM. An analytical cross-check has not been done yet with the TTF. We perform it in the case of a fully analytical solution accurate at the GREM level [37], described by the gravitational perturbation (11), where we set $\beta_p = 0$, $\gamma = 1$ and the position of the perturbing bodies is fixed at the time of maximum approach with the photon $tC$.

Indeed, the inclusion of the velocities in RAMOD requires a careful application of the measurement protocol so that the case of a field of bodies in uniform motion needs to be treated in a separate and exhaustive paper.

6.2.1. Coordinate time of flight. The computation of the coordinate time of flight $\Delta t$ can be obtained within RAMOD by considering the time component of the local observer $u$ [14] as
\[
u^0 = c \frac{dr}{d\xi} = 1 + \frac{h_{00}}{2} + O(c^{-4}).
\]
(43)
Inserting equation (11) taken at the chosen approximation into equation (43) and integrating between the emission $\zeta$ and the reception $\zeta_B$, we get

$$c \Delta t = \int_{\zeta}^{\zeta_B} \left( 1 + \frac{G}{c^2} \sum_p M_p \frac{1}{R_p(\zeta)} \right) d\zeta + O(c^{-4})$$

$$= \Delta \zeta + \frac{G}{c^2} \sum_p M_p \ln \left( \frac{R_{PB} + N_{AB} \cdot R_{PB}}{R_{PA} + N_{AB} \cdot R_{PA}} \right) + O(c^{-4}),$$

(44)

where $\Delta \zeta \equiv \zeta_B - \zeta$ and we used definitions (17) and (18) with $\beta_P = 0$. We need now an explicit expression for $\Delta \zeta$. First, we rewrite following our notation in equation (18) of [37] as

$$\bar{\varepsilon}_B^i = \frac{x_0^i - x_B^i}{\Delta \zeta} + 2 \frac{G}{c^2} \sum_p M_p \left\{ \frac{N^k}{2} \frac{1}{\Delta \zeta} \ln \left[ \frac{N_{AB} \cdot R_{PB} + R_{PB}}{N_{AB} \cdot R_{PA} + R_{PA}} \right] - \frac{1}{R_{PB}} \right\}$$

$$+ \frac{d^k_i}{d \bar{\varepsilon}_B^k} \left\{ - \frac{N_{AB} \cdot R_{PB}}{R_{PB}} + \frac{R_{PB} - R_{PA}}{\Delta \zeta} \right\} + O(c^{-4})$$

(45)

with $d_B = R_{PB} - N_{AB}(R_{PB} \cdot N_{AB})$. Then, using the relation $d_B \cdot N_{AB} = 0$ and the normalization condition $e^a \ell_a = g_{ab} e^b \ell^b = 1$ on equation (45), we obtain

$$\bar{\varepsilon}_B^i \bar{\varepsilon}_B^i = 1 - \hbar_{00}^B + O(c^{-4}) = \frac{R_{AB}^2}{\Delta \zeta^2} + \frac{2G}{c^2} \sum_p M_p$$

$$\times \left\{ \frac{R_{PB}}{\Delta \zeta^2} \ln \left[ \frac{(N_{AB} \cdot R_{PB} + R_{PB})}{(N_{AB} \cdot R_{PA} + R_{PA})} - \frac{R_{AB}}{R_{PB} \Delta \zeta} \right] \right\} + O(c^{-4}).$$

(46)

Following equation (44), we assume that $\Delta \zeta$ admits a PN expansion

$$\Delta \zeta = R_{AB} + \Delta \zeta(2) + O(c^{-3}),$$

(47)

where $\Delta \zeta(2)$ is of order $O(c^{-3})$. Substituting for $\Delta \zeta$ from equation (47) into equation (46) and identifying the terms of the same order, we get straightforwardly

$$\Delta \zeta(2) = \frac{G}{c^2} \sum_p M_p \ln \left[ \frac{R_{PB} + N_{AB} \cdot R_{PB}}{R_{PA} + N_{AB} \cdot R_{PA}} \right].$$

(48)

Finally, substituting for $\Delta \zeta$ from equations (47) and (48) into equation (44) we retrieve the Shapiro-like term of equation (19) with $\beta_P = 0$.

6.2.2. **Light direction triple.** The relation between the light direction triple $\hat{k}_i$ and the local line-of-sight $\bar{\varepsilon}_B^i$ at reception point $x_B$ is obtained by expanding equation (41) with the metric (11) taken at the chosen approximation. Considering equation (43) for $u^i$, we get

$$\bar{\varepsilon}_B^i = - \frac{k^i}{u^i k_0} \Big|_{\beta_B} = - (\hat{k}_i)_B \left[ 1 - \frac{3}{2} \hbar_{00}^B \right] + O(c^{-3}) \approx - (\hat{k}_i)_B \left[ 1 - \frac{3G}{c^2} \sum_p M_p \frac{R_{PB}}{R_{PB}} \right].$$

(49)

Substituting for $(\hat{k}_i)_B$ from equation (25) into equation (49) with $\beta_P = 0$, the reader can easily retrieve equation (45) where $\Delta \zeta$ is given by equations (47)–(48).

7. **Conclusions**

This paper provides the coordinate time of flight of a photon between two points at finite distance and the direction triple at its reception coordinates as integrals taken along the Minkowskian straight line between its emission and reception points. It is remarkable that equation (5) and equations (8)–(10) give these quantities in closed form as function of just
few parameters and for any metric tensor describing a weak gravitational field in the PPN framework. We use our formulas to provide a first application of the TTF formalism in the case of the time-dependent metric tensor (11), describing a system of point bodies in uniform motion. Equations (19)–(21) and equation (25) extend the results previously obtained in [18] and are used here to find a procedure to relate and compare three independent approaches to relativistic light propagation. Section 6.1 shows that the results of TTF and KK92 are equivalent at 1PN in a time-dependent gravitational field. The comparison with GREM at the $\mu$as level required for the Gaia mission can then be retrieved by fixing the positions of the perturbing bodies at their closest approach with the photon while the equivalence of the results of TTF and RAMOD at the same accuracy level is demonstrated in section 6.2. Such a cross-checking procedure enters the same thread of model comparison started in [35], which is essential to fully understand the observational data coming from Gaia in a common experimental context.

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