More M-branes on product of Ricci-flat manifolds

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Abstract

Partially supersymmetric intersecting (non-marginal) composite M-brane solutions defined on the product of Ricci-flat manifolds $M_0 \times M_1 \times \ldots \times M_n$ in $D = 11$ supergravity are considered and formulae for fractional numbers of unbroken supersymmetries are derived for the following configurations of branes: $M_2 \cap M_2$, $M_2 \cap M_5$, $M_5 \cap M_5$ and $M_2 \cap M_2 \cap M_2$. Certain examples of partially supersymmetric configurations are presented.

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1 Introduction

In this paper we consider a class of partially supersymmetric solutions in 11-dimensional supergravity [1]. The renewed interest in $D = 11$ supergravity has been appeared due to unification of string theories in terms of the conjectured $M$-theory [2, 3]. Supergravity solutions with less than maximal supersymmetry may be of interest also in a context of investigations of dual strongly coupled field theories [4] and possible modeling of multiple $M2$ and $M5$ configurations [5, 6, 7, 8, 9].

Here we deal with non-marginal intersecting $M$-brane solutions defined on product of Ricci-flat manifolds $[10] [11]

\[ M_0 \times M_1 \times \ldots \times M_n. \] (1.1)

These solutions are governed by several harmonic functions defined on the manifold $M_0$ which is called as transverse space (see also [12] [13] [14] for more general solutions with composite branes on product of Ricci-flat manifolds). Here we extend an approach suggested earlier in [15] where supersymmetric configurations with one brane ($M2$ and $M5$) on product of two Ricci-flat spaces were considered.

We note that for flat factor spaces $M_i = \mathbb{R}^{k_i}$, $i = 0, \ldots, n$, the supersymmetric solutions with intersecting $M$-branes were considered intensively in numerous publications, see [16]– [26] and references therein. The basic $M2$-brane [16] and $M5$-brane [17] solutions defined on $\mathbb{R}^{k_0} \times \mathbb{R}^{k_1}$ (with $(k_0, k_1) = (8, 3), (5, 6)$, respectively) preserve $1/2$ of supersymmetries (SUSY). The classification of supersymmetric $M$-brane configurations on product of flat factor spaces $M_i = \mathbb{R}^{k_i}$ was done in the work of E. Bergshoeff et al [22]. The fractional numbers of preserved SUSY are given by the relation [22]

\[ \mathcal{N} = 2^{-k} \] (1.2)

with $k = 1, 2, 3, 4, 5$. (We note that in proving (1.2) the so-called “$2^{-k}$-splitting” theorem from [15] may be used.)

The relation (1.2) is no more valid if composite $M$-brane configurations on product of Ricci-flat manifolds (1.1) are taken into consideration. In this case $\mathcal{N}$ depends upon certain numbers of chiral parallel (i.e. covariantly constant) spinors on $M_i$ and brane sign factors $c_s$. Here a composite Ansatz for the 4-form corresponding to $m$ intersecting $M$-branes

\[ F = \sum_{s=1}^{m} c_s \mathcal{F}_s, \] (1.3)

is assumed, where $\mathcal{F}_s$ is an elementary 4-form corresponding to $s$-th $M$-brane and $c_s = \pm 1$ is the sign (charge density) factor of $s$-th brane, $s = 1, \ldots, m$.

For $M2$-brane solution on product of two Ricci-flat factor spaces $M_0 \times M_1$ ($n = 1, m = 1$) the number of unbroken SUSY is (at least) [15]

\[ \mathcal{N} = n_0(c)n_1/32, \] (1.4)
where $n_0(c)$ is the number of chiral parallel spinors on 8-dimensional transverse manifold $M_0$ with the chirality $c = \pm 1$, $n_1$ is the number of parallel spinors on 3-dimensional world-volume $M_1$ and the chirality number $c = \pm 1$ coincides with the $M/2$-brane sign. \(^2\)

For $M_1 = \mathbb{R}^3$ relation (1.4) was used implicitly in [27] (for $M_0$ being the cones over certain 7-dimensional Einstein manifolds and for $M_0 = \mathbb{R}_+ \times M_{00}$, with certain Ricci-flat $M_{00}$) and in [28] (for hyper-Kähler 8-dimensional manifold with holonomy group $Sp(2)$).

For $M/5$-brane solution on product of two Ricci-flat factor spaces $M_0 \times M_1$ the number of unbroken SUSY is (at least) the following one [15]

$$\mathcal{N} = n_0 n_1(c)/32,$$

where $n_0$ is the number of parallel spinors on transverse 5-dimensional manifold $M_0$ and $n_1(c)$ is the number of chiral parallel spinors on 6-dimensional world-volume manifold $M_1$. Here the chirality $c = \pm 1$ is coinciding with the $M/5$-brane sign. Relation (1.5) was used implicitly in [29] (for $M_0 = \mathbb{R}^5$), [30] (for $M_1 = \mathbb{R}^2 \times K^3$) and [31] (for $M_1$ with holonomy group $\mathbb{R}^4$ or $Sp(1) \ltimes \mathbb{R}^4$).

Here we start the classification of partially supersymmetric $M$-brane solutions defined on the products of Ricci-flat manifolds, i.e. we are aimed at the generalization of the classification from [22].

The paper is organized as follows. In Section 2 we describe the basic notations and definitions in arbitrary dimension $D$, e.g. splitting relations for spin connection and spinorial covariant derivative on (warped) product manifolds. Section 3 is devoted to SUSY “Killing-like” equations in 11-dimensional supergravity and splitting procedure for the flux term in these equations. In Section 4 the general class of non-marginal composite $M$-brane solutions defined on product of Ricci-flat manifolds [10, 11] is considered. Here a key proposition concerning the solutions to SUSY equations is presented. This proposition (Proposition 1) is proved in Appendix. It gives a background for the calculation of the fractional numbers of preserved SUSY for (non-marginal) composite $M$-brane solutions. In Section 5 the relations for these fractional numbers are derived for the following sets of brane: $M_2, M_5, M_2 \cap M_2, M_2 \cap M_5, M_5 \cap M_5$ and $M_2 \cap M_2 \cap M_2$ and numerous examples for various factor-spaces $M_i$ are considered.

## 2 Basic notations

Here we describe the basic notations in arbitrary dimension $D$.

### 2.1 Product of manifolds

Let us consider the manifold

$$M = M_0 \times M_1 \times \ldots \times M_n,$$

\(^2\)Here and in what follows the phrase “... is the number of ... spinors” means “... is the number of linear independent... spinors” (or, more rigorously, “... is the dimension of space of ... spinors”).
with the metric
\[ g = e^{2\gamma(x)} \hat{g}^0 + \sum_{i=1}^{n} e^{2\phi^i(x)} \hat{g}^i, \]  
(2.2)

where \( g^0 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu \) is a metric on the manifold \( M_0 \) and \( g^i = g^i_{m_i n_i} (y_i) dy^{m_i}_i \otimes dy^{n_i}_i \) is a metric on the manifold \( M_i, i = 1, \ldots, n \). Here and in what follows \( \hat{g}^i = p^*_i g^i \) is the pullback of the metric \( g^i \) to the manifold \( M \) by the canonical projection: \( p_i : M \rightarrow M_i, i = 0, \ldots, n \).

The functions \( \gamma, \phi^i : M_0 \rightarrow \mathbb{R} \) are smooth. We denote \( d_\nu = \dim M_\nu; \nu = 0, \ldots, n; D = \sum_{\nu=0}^{n} d_\nu \). We put any \( M_\nu, \nu = 0, \ldots, n \), to be oriented and connected spin manifold. Then the volume \( d_i \)-form
\[ \tau_i \equiv \sqrt{|g^i(y_i)|} dy^{1}_i \wedge \ldots \wedge dy^{d_i}_i, \]  
(2.3)
is correctly defined for any \( i = 1, \ldots, n \).

Let \( \Omega = \Omega(n) \) be a set of all non-empty subsets of \( \{1, \ldots, n\} \) \( (|\Omega| = 2^n - 1) \). For any \( I = \{i_1, \ldots, i_k\} \in \Omega, i_1 < \ldots < i_k \), we denote
\[ \tau(I) \equiv \hat{\tau}_{i_1} \wedge \ldots \wedge \hat{\tau}_{i_k}, \]  
(2.4)
\[ d(I) \equiv \sum_{i \in I} d_i. \]  
(2.5)

Here and in what follows \( \hat{\tau}_{i} = p^*_i \tau_i \) is the pullback of the form \( \tau_i \) to the manifold \( M \) by the canonical projection: \( p_i : M \rightarrow M_i, i = 1, \ldots, n \).

For \( I \in \Omega \) we define an indicator of \( i \) belonging to \( I \)
\[ \delta^i_I \equiv \sum_{j \in I} \delta^i_j = 1, \quad i \in I, \]
0, \quad i \notin I. 
(2.6)

### 2.2 Diagonalization of the metric

For the metric \( g = g_{MN}(x)dx^M \otimes dx^N \) from (2.2), \( M, N = 0, \ldots, D - 1 \), defined on the manifold \( \{2.1\} \), we define the diagonalizing \( D \)-bein \( e^A = e^A_M dx^M \)
\[ g_{MN} = \eta_{AB} e^A_M e^B_N, \quad \eta_{AB} = \eta^{AB} = \eta_A \delta_{AB}, \]  
(2.7)
\( \eta_A = \pm 1; A, B = 0, \ldots, D - 1 \).

We choose the following frame vectors
\[ (e^A_M) = \text{diag}(e^\gamma e^{(0)\alpha}_\mu, e^{\phi^i e^{(1)a_i}_{m_i}}, \ldots, e^{\phi^n e^{(n)a_n}_{m_n}}), \]  
(2.8)
where
\[ g_{\mu\nu}^0 = \eta_{ab} e^{(0)\alpha}_\mu e^{(0)\beta}_\nu, \quad g^i_{m_i n_i} = \eta^{(i)}_{ab} e^{(i)\alpha}_{m_i} e^{(i)\beta}_{n_i}, \]  
(2.9)
i = 1, \ldots, n, and
\[ (\eta_{AB}) = \text{diag}(\eta^{(0)}_{ab}, \eta^{(1)}_{a_1 b_1}, \ldots, \eta^{(n)}_{a_n b_n}). \]  
(2.10)

For \( (e^A_M) = (e^A_M)^{-1} \) we get
\[ (e^A_M) = \text{diag}(e^{-\gamma} e^{(0)\alpha}_a, e^{-\phi^i} e^{(1)m_i}_{a_i}, \ldots, e^{-\phi^n} e^{(n)m_n}_{a_n}), \]  
(2.11)
where \((e^{(0)}_{\mu}) = (e^{(0)}_{\mu})^{-1}, (e^{(i)}_{m,a_i}) = (e^{(i)}_{a_i,m})^{-1}, i = 1, \ldots, n.\)

Indices. For indices we also use an alternative numbering: \(A = (a, a_1, \ldots, a_n), B = (b, b_1, \ldots, b_n),\) where \(a, b = 1, 2, \ldots, (d_0)_0; a_1, b_1 = 1, 2, \ldots, (d_1)_1; \ldots; a_n, b_n = 1, 2, \ldots, (d_n)_n;\) and \(M = (\mu, m_1, \ldots, m_n), N = (\nu, n_1, \ldots, n_n),\) where \(\mu, \nu = 1, 2, \ldots, (d_0)_0; m_1, n_1 = 1, 2, \ldots, (d_1)_1; \ldots; m_n, n_n = 1, 2, \ldots, (d_n)_n.\)

### 2.3 Gamma-matrices

In what follows \(\hat{\Gamma}_A\) are “frame” \(k \times k\) gamma-matrices satisfying

\[
\hat{\Gamma}_A \hat{\Gamma}_B + \hat{\Gamma}_B \hat{\Gamma}_A = 2 \eta_{AB} I_k,
\]

\(A, B = 0, \ldots, D - 1.\) Here \(I_k\) is unit \(k \times k\) matrix and \(k = 2^{[D/2]} .\)

We also use “world” \(\Gamma\)-matrices

\[
\Gamma_M = e^A_M \hat{\Gamma}_A, \quad \Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2 g_{MN} I_k,
\]

\(M, N = 0, \ldots, D - 1,\) and the matrices with upper indices: \(\hat{\Gamma}^A = \eta^{AB} \hat{\Gamma}_B\) and \(\Gamma^M = g^{MN} \Gamma_N.\) In what follows we will use the relation

\[
\hat{\Gamma}^A \hat{\Gamma}^B + \hat{\Gamma}^B \hat{\Gamma}^A = 2 \eta^{AB} I_k,
\]

where \(\eta^{AB} = \eta_{AB}.\)

For any manifold \(M_l\) with the metric \(g^l\) we will also consider \(k_l \times k_l\) \(\Gamma\)-matrices with \(k_l = 2^{[d_l/2]}\) obeying

\[
\hat{\Gamma}^a_{(l)} \hat{\Gamma}^b_{(l)} + \hat{\Gamma}^b_{(l)} \hat{\Gamma}^a_{(l)} = 2 \eta^{a b} I_k_{l},
\]

and

\[
\hat{\Gamma}_{(l)} = \hat{\Gamma}^1_{(l)} \ldots \hat{\Gamma}^{(d_l)}_{(l)},
\]

\(l = 0, \ldots, n.\)

### 2.4 Spin connection

Here we use the standard definition for the spin connection

\[
\omega^A_{BM} = \omega^A_{BM} (e, \eta) = e^A_N \nabla_M [g(e, \eta)] e^N_B,
\]

where the covariant derivative \(\nabla_M [g]\) corresponds to the metric \(g = g(e, \eta)\) from \((2.7)\). The spinorial covariant derivative reads

\[
D_M = \partial_M + \frac{1}{4} \omega_{ABM} \hat{\Gamma}^A \hat{\Gamma}^B,
\]

where \(\omega_{ABM} = \eta_{AA'} \omega^{A'}_{BM}, \omega_{ABM} = -\omega_{BAM}.\) For \(D = 4\) it was introduced by V.A. Fock and D.D. Ivanenko in [32].
The non-zero components of the spin connection (2.17) in the frame (2.8) read

\[ \omega^a_{\ b \mu} = \omega^a_{\ b \mu} (e^{(0)}, \eta^{(0)}) - e^{(0)} \nu a \gamma_{\ \nu} e^{(0)} b \gamma_{\ \mu} e^{(0)} a \mu, \]  
(2.19)

\[ \omega^a_{\ a m j} = -\delta_{ij} e^{(0)} a \gamma (e^{(0)} a \nu \nabla^{(0)} \phi^{(0)} e^{(i)} a m i), \]  
(2.20)

\[ \omega^a_{\ b i m} = \delta_{ij} e^{(0)} b \gamma (e^{(0)} b \nu \partial^{(0)} \phi^{(0)} e^{(i)} a m i), \]  
(2.21)

\[ \omega^a_{\ b j m i} = \delta_{ij} e^{(0)} b \gamma (e^{(0)} a \nu \nabla^{(0)} \partial^{(0)} \phi^{(0)} e^{(i)} a m i), \]  
(2.22)

\[ i, j, k = 1, \ldots, n, \] where \( \omega^a_{\ b \mu} (e^{(0)}, \eta^{(0)}) \) and \( \omega^a_{\ b i m} (e^{(i)}, \eta^{(i)}) \) are components of the spin connections corresponding to the metrics from (2.9).

Let

\[ A_M \equiv \omega_{ABM} \hat{\Gamma}^A \hat{\Gamma}^B. \]  
(2.23)

For \( A_M = A_M (e, \eta, \hat{\Gamma}^C) \) in the frame (2.8) we get

\[ A_{\mu} = \omega^{(0)}_{\ ab \mu} + (\Gamma_{\mu} \Gamma^\nu - \Gamma^\nu \Gamma_{\mu}) \gamma_{\ \nu}, \]  
(2.24)

\[ A_{m i} = \omega^{(i)}_{\ a b m i} \hat{\Gamma}^a \hat{\Gamma}^b + 2 \Gamma_{m i} \gamma \hat{\phi}^i, \]  
(2.25)

where \( \omega^{(0)}_{\ ab \mu} = \omega_{ab \mu} (e^{(0)}, \eta^{(0)}) \) and \( \omega^{(i)}_{\ a b m i} = \omega_{a b m i} (e^{(i)}, \eta^{(i)}), \) \( i = 1, \ldots, n. \)

Relations (2.24) and (2.25) imply the following decomposition of the covariant derivative (2.18)

\[ D_{\mu} = \bar{D}^{(0)}_{\ mu} + \frac{1}{4} (\Gamma_{\mu} \Gamma^\nu - \Gamma^\nu \Gamma_{\mu}) \gamma_{\ \nu}, \]  
(2.26)

\[ D_{m i} = \bar{D}^{(i)}_{\ mi} + \frac{1}{2} \Gamma_{m i} \gamma \hat{\phi}^i, \quad i > 0, \]  
(2.27)

where

\[ \bar{D}^{(l)}_{\ mi} = \partial_{m i} + \frac{1}{4} \omega^{(l)}_{\ ab m i} \hat{\Gamma}^a \hat{\Gamma}^b, \]  
(2.28)

\[ l = 0, \ldots, n \] \( (\mu = m_0). \)

In what follows operators (2.28) will generate the covariant spinorial derivatives corresponding the manifolds \( M_l \)

\[ D^{(l)}_{\ mi} = \partial_{m i} + \frac{1}{4} \omega^{(l)}_{\ ab m i} \hat{\Gamma}^a \hat{\Gamma}^b, \]  
(2.29)

\[ l = 0, \ldots, n. \]

3 SUSY equations

We consider the \( D = 11 \) supergravity with the action in the bosonic sector \[ S = \int d^{11}z \sqrt{|g|} \left\{ R[g] - \frac{1}{2(4!)} F^2 \right\} - \frac{1}{6} \int A_{\mu} F_{\mu} \right\} \]  
(3.1)
where $F = dA$ is 4-form. Here we consider pure bosonic configurations in $D = 11$ supergravity (with zero fermionic fields) that are solutions to the equations of motion corresponding to the action (3.1).

The number of supersymmetries (SUSY) corresponding to the bosonic background $(e_A^M, A_{M_1M_2M_3})$ is defined by the dimension of the space of solutions to (a set of) linear first-order differential equations (SUSY eqs.)

\[(D_M + B_M)\varepsilon = 0, \tag{3.2}\]

where $D_M$ is covariant spinorial derivative from (2.18), $\varepsilon = \varepsilon(z)$ is 32-component spinor field (see Remarks 1 and 2 below) and

\[B_M = \frac{1}{288}(\Gamma_M\Gamma^N\Gamma^P\Gamma^Q\Gamma^R - 12\delta_M^N\Gamma^P\Gamma^Q\Gamma^R)F_{NPQR}. \tag{3.3}\]

Here $F = dA = \frac{1}{4!}F_{NPQR}dz^N \wedge dz^P \wedge dz^Q \wedge dz^R$, and $\Gamma_M$ are world $\Gamma$-matrices.

The number of unbroken SUSY is

\[N = N/32, \tag{3.4}\]

where $N$ is the dimension of linear space of solutions to differential equations (3.2).

**Remark 1.** In this paper we put for simplicity $\varepsilon(z) \in \mathbb{C}^{32}$. The imposing of Majorana condition $\bar{\varepsilon} = B\varepsilon$, where $\hat{B}$ denotes the complex conjugation and $B$ is non-degenerate matrix obeying $\hat{\Gamma}_A = \mp B\hat{\Gamma}_A B^{-1}$, will give the same number $N$ for the dimension of real linear space of parallel Majorana spinors obeying (3.2).

**Remark 2.** A possible consistent approach to superanalysis implies the use of infinite-dimensional super-commutative Banach algebras, e.g. Grassmann-Banach ones [33, 34, 35, 36] (see also [37]). One may put $\varepsilon(z) \in (G_1)^{32}$, where $G_1$ is an odd part of an infinite-dimensional Grassmann-Banach algebra $G = G_0 \oplus G_1$ [33, 36, 38]. In this case the complex-valued solution $\varepsilon(z)$ should be replaced by $\varepsilon(z)g_1$, where $g_1$ is arbitrary element of $G_1$.

Here we consider the decomposition of matrix-valued field $B_M$ on the product manifold (2.1) in the frame (2.8) for electric and magnetic branes.

**M2-brane.** Let the 4-form be

\[F = d\Phi \wedge \tau(I) \tag{3.5}\]

where $\Phi = \Phi(x), I = \{i_1, \ldots, i_k\}, i_1 < \ldots < i_k, d(I) = 3$. The calculations give [15]

\[B_{mi} = \frac{1}{12}S(I) \exp(-\sum_{i \in I} d_i \phi^i)[(1 - 3\delta^l_0)\Gamma_m^l\Gamma^\nu\Phi_{,\nu} - 3\delta^l_0\Phi_{,m}]\hat{\Gamma}(I), \tag{3.6}\]

\[l = 0, \ldots, n, \text{ where } m_0 = \mu, \]

\[S(I) = \text{sign}\left(\prod_{i \in I} \text{det}(e^{(i)m_i}_{a_i})\right) \tag{3.7}\]

and
\[ \hat{\Gamma}(I) = \hat{\Gamma}^{A_1} \hat{\Gamma}^{A_2} \hat{\Gamma}^{A_3} \] (3.8)

with \((A_1, A_2, A_3) = (1_i, \ldots, (d_i)_i, \ldots, 1_k, \ldots, (d_k)_k)\).

**M5-brane.** Let

\[ F = (s_0 d\Phi) \wedge \tau(\bar{I}), \] (3.9)

where \(s_0\) is the Hodge operator on \((M_0, g^0)\) and \(\bar{I} = \{1, \ldots, n\} \setminus I = \{j_1, \ldots, j_l\}, j_1 < \ldots < j_l\). It follows from (3.9) that \(d_0 + d(\bar{I}) = 5\) and \(d(I) = 6\). We get

\[ B_{m_i} = \frac{1}{24} S(\{0\}) S(\bar{I}) \exp[-(d_0 - 2)\gamma - \sum_{i \in \bar{I}} d_i \phi^i] \times \] (3.10)
\[ \times [2\Gamma_m \Gamma^{\nu} \Phi, \nu - 3 \delta^0_0 (\Gamma_m \Gamma^{\nu} - \Gamma^{\nu} \Gamma_m) \Phi, \nu + 6 \delta^0_0 \Gamma^{\nu} \Gamma_m] \hat{\Gamma}(\{0\}) \hat{\Gamma}(\bar{I}), \]

\[ l = 0, \ldots, n, \] where

\[ S(\{0\}) = \text{sign}(\det(e^{(0)\nu}_{\ a})), \] (3.11)

and

\[ \hat{\Gamma}(\{0\}) = \hat{\Gamma}^{(d_0)}_0, \]
\[ \hat{\Gamma}(\bar{I}) = \hat{\Gamma}^{B_1} \ldots \hat{\Gamma}^{B_k} \] (3.13)

with \((B_1, \ldots, B_k) = (1_{j_1}, \ldots, (d_{j_1})_{j_1}, \ldots, 1_{j_l}, \ldots, (d_{j_l})_{j_l})\) and \(d_0 + k = 5\).

## 4 Composite M-brane solutions

We consider a classical solution corresponding to the action (3.1) \[10, 11, 12\]. The metric is defined on the product manifold (2.1) and has the following form

\[ g = e^{2\gamma(x)} g^0 + \sum_{i=1}^n e^{2\phi^i(x)} g^i, \] (4.1)

\[ e^{2\gamma} = \left( \prod_{s \in S_c} H_s \right)^{1/3} \left( \prod_{s \in S_m} H_s \right)^{2/3}, \] (4.2)

\[ e^{2\phi^i} = e^{2\gamma} \prod_{s \in S} H_s^{-\delta_{s s_i}}, \] (4.3)

\[ i = 1, \ldots, n. \]

Here \(g^0 = g^0_{\mu\nu}(x) dx^\mu \otimes dx^\nu\) is a Ricci-flat metric on the manifold \(M_0\) and \(g^i = g^i_{m_n}(y_i) dy_i^m \otimes dy_i^n\) is a Ricci-flat metric on \(M_i, i = 1, \ldots, n\).

The 4-form reads

\[ F = \sum_{s \in S_c} c_s dH_s^{-1} \wedge \tau(I_s) + \sum_{s \in S_m} c_s (s_0 dH_s) \wedge \tau(\bar{I}_s), \] (4.4)
where $c_s^2 = 1$; $*_0$ is the Hodge operator on $(M_0, g^0)$, $H_s$ are harmonic functions on $(M_0, g^0)$ and

$$\bar{I}_s = \{1, \ldots, n\} \setminus I_s$$

is dual set.

Here the set of indices $S_e \subset \Omega(n)$ describes electric branes with worldvolume dimensions $d(I_s) = 3$, $I_s \in S_e$ and $S_m \subset \Omega(n)$ describes magnetic branes with worldvolume dimensions $d(I_s) = 6$, $I_s \in S_m$. The intersections rules are standard ones

$$d(I_s \cap I_{s'}) = \frac{1}{9} d(I_s) d(I_{s'}),$$

$s \neq s', s, s' \in S$, where $S = S_e \cup S_m$, or, explicitly,

$$d(I_s \cap I_{s'}) = 1, 2, 4,$$

for $M_2 \cap M_2$, $M_2 \cap M_5$ and $M_5 \cap M_5$ cases, respectively.

Only one manifold $(M_{i_0}, g^{i_0})$ ($i_0 > 0$) has a pseudo-Euclidean signature while all others $(M_i, g^i)$ ($i \neq i_0$) should be of Euclidean signature. The index $i_0$ is contained by all brane sets: $i_0 \in I_s$, $s \in S$.

It should be noted that here the Chern-Simons term in the action (3.1) does not give the contribution into equations of motion due to

$$F \wedge F = 0.$$  

(4.8)

Using the decomposition of matrix-valued field $B_M$ on the product manifold (2.1) in the frame (2.8) for electric and magnetic branes from (3.6) and (3.10) we obtain

$$B_M = \sum_{s \in S} B^s_M,$$

(4.9)

where

$$B^s_{ml} = \frac{c_s}{12} H_s[(1 - 3\delta^l_{I_s})\Gamma_{ml} \Gamma^{\nu} \partial_{\nu} H_s^{-1} - 3\delta_0^l \partial_{ml} H_s^{-1}]\hat{\Gamma}^s_{[s]},$$

(4.10)

for $s \in S_e$, and

$$B^s_{ml} = \frac{c_s}{24} H_s^{-1}[2\Gamma_{ml} \Gamma^{\nu} \partial_{\nu} H_s - 3\delta_0^l (\Gamma_{ml} \Gamma^{\nu} - \Gamma^{\nu} \Gamma_{ml}) \partial_{\nu} H_s + 6\delta^l_{I_s} \Gamma^{\nu} \Gamma_{ml} \partial_{\nu} H_s]\hat{\Gamma}^s_{[s]},$$

(4.11)

for $s \in S_m$; $l = 0, \ldots, n$.

Here we denote

$$c_s = c_s S(I_s), \quad s \in S_e;$$

$$c_s = c_s S(\{0\}) S(I_s), \quad s \in S_m,$$

and

$$\hat{\Gamma}^s_{[s]} = \hat{\Gamma}(I_s), \quad s \in S_e;$$

$$\hat{\Gamma}^s_{[s]} = \hat{\Gamma}(\{0\}) \hat{\Gamma}(I_s), \quad s \in S_m.$$
In derivation of (4.9)-(4.11) the following formulae are used
\[- \sum_{i \in I_s} d_i \phi^i = \ln H_s, \quad s \in S_e; \quad (4.15)\]
\[- \gamma(d_0 - 2) - \sum_{i \in \bar{I}_s} d_i \phi^i = -\ln H_s, \quad s \in S_m. \quad (4.16)\]

The proof of these relations is given in Appendix A. It is based on intersection rules (4.6). The definitions (4.13) and (4.14) imply
\[\hat{\Gamma}[s] = \hat{\Gamma}^A_1 \hat{\Gamma}^A_2 \hat{\Gamma}^A_3, \quad \text{for } s \in S_e, \quad (4.17)\]
where \((\hat{\Gamma}^A_{i_0})^2 = -1\) for some \(i_0 \in \{1, 2, 3\}\), \((\hat{\Gamma}^A_i)^2 = 1\) for \(i \neq i_0\), and
\[\hat{\Gamma}[s] = \hat{\Gamma}^B_1 \hat{\Gamma}^B_2 \hat{\Gamma}^B_3 \hat{\Gamma}^B_5 \hat{\Gamma}^B_5, \quad \text{for } s \in S_m, \quad (4.18)\]
where \((\hat{\Gamma}^B_i)^2 = 1\) for all \(i\).

It follows from the relations (4.13) and (4.14), that
\[(\hat{\Gamma}[s])^2 = 1, \quad (4.19)\]
\[1 = 1_{32}. \quad \text{The matrices } \hat{\Gamma}[s] \text{ commute with each other due to intersection rules (4.7).} \]
In what follows the following proposition plays a key role.

**Proposition 1.** Let
\[\varepsilon = \left( \prod_{s \in S_e} H_s \right)^{-1/6} \left( \prod_{s \in S_m} H_s \right)^{-1/12} \eta, \quad (4.20)\]
where
\[\bar{D}^{(l)}_{m_l} \eta = 0, \quad l = 0, \ldots, n, \quad (4.21)\]
and
\[\hat{\Gamma}[s] \eta = c[s] \eta \quad (4.22)\]
for all \(s \in S\). Then the relation (3.2)
\[(D_M + B_M) \varepsilon = 0 \quad (4.22)\]
is satisfied identically.

The proof of this proposition is given in Appendix B. For flat factor-spaces \(M_i = \mathbb{R}^{k_i}\) (and suitably chosen beins \((e^{(i)m_a}_a)\)), \(i = 0, \ldots, n\), relations (4.21) are satisfied identically for constant \(\eta\) and hence we should deal only with the set of equations (4.22) [22].

When all \(H_s = 1\) and \(F = 0\) (i.e. all branes are removed) we get a more simple proposition.

**Proposition 2.** \(\eta\) is parallel spinor (i.e. \(D_M \eta = 0\)) on the product manifold (L.1) if and only if
\[\bar{D}^{(l)}_{m_l} \eta = 0, \quad l = 0, \ldots, n. \quad (4.23)\]

Proposition 2 may be used for constructing chiral parallel spinors on product manifolds. An example describing chiral parallel spinors on product of two 4-dimensional manifolds (of Euclidean signatures) is given Appendix C.
5 Supersymmetric composite $M$-brane solutions

Here we consider certain examples of supersymmetric solutions. We put for simplicity that
\[ \det(e^{(l)\mu \eta \over a}) > 0, \]  
(5.1)
\[ l = 0, \ldots, n, \]  
and hence
\[ c_s = c \]  
(5.2)
for all $s \in S$ (see (3.7), (3.11) and (4.12)).

5.1 M2-brane

Let us we consider (in detail) the electric 2-brane solution defined on the manifold
\[ M_0 \times M_1. \]  
(5.3)
The solution reads
\[ g = H^{1/3} \{ \hat{g}^0 + H^{-1} \hat{g}^1 \}, \]  
(5.4)
\[ F = cdH^{-1} \land \hat{\tau}_1, \]  
(5.5)
where $c^2 = 1$, $H = H(x)$ is a harmonic function on $(M_0, g^0)$, $d_1 = 3$, $d_0 = 8$ and the metrics $g^i, i = 0, 1$, are Ricci-flat; $g^0$ has the Euclidean signature and $g^1$ has the signature $(-, +, +)$.

We consider $\Gamma$-matrices
\[ (\hat{\Gamma}^A) = (\hat{\Gamma}^{a_0} \otimes 1_2, \hat{\Gamma}^{a_1} \otimes \hat{\Gamma}^{a_i}(1)), \]  
(5.6)
where $16 \times 16$ $\Gamma$-matrices $\hat{\Gamma}^{a_0}(0), a_0 = 1_0, \ldots, 8_0$, correspond to $M_0$:
\[ \hat{\Gamma}^{a_0}(0) \hat{\Gamma}^{b_0}(0) + \hat{\Gamma}^{a_0}(0) \hat{\Gamma}^{b_0}(0) = 2 \delta_{a_0 b_0} 1_{16}, \]  
(5.7)
2 $\times$ 2 $\Gamma$-matrices $\hat{\Gamma}^{a_1}(1), a_1 = 1_1, 2_1, 3_1$, correspond to $M_1$:
\[ \hat{\Gamma}^{a_1}(1) \hat{\Gamma}^{b_1}(1) + \hat{\Gamma}^{a_1}(1) \hat{\Gamma}^{b_1}(1) = 2 \eta_{a_1 b_1} 1_2 \]  
(5.8)
with $\eta_{a_1 b_1} = \text{diag}(-1, +1, +1)$ and $\hat{\Gamma}(0) = \hat{\Gamma}^{1_0}(0) \ldots \hat{\Gamma}^{8_0}(0)$.

The $\Gamma$-matrices (5.6) obey the relations (2.14) due to (5.7), (5.8) and the identity
\[ (\hat{\Gamma}(0))^2 = 1_{16}. \]  
(5.9)
It follows from (2.28), (5.0) and (5.9) that
\[ \bar{D}^{(0)}_{m_0} = \partial_{m_0} + \frac{1}{4} \omega^{(0)}_{\alpha_0 \beta_0 \mu_0} \hat{\Gamma}^{\alpha_0} \hat{\Gamma}^{\beta_0} \otimes 1_2, \]  
(5.10)
\[ \bar{D}^{(1)}_{m_1} = \partial_{m_1} + \frac{1}{4} \omega^{(1)}_{\alpha_1 \beta_1 \mu_1} 1_{16} \otimes \hat{\Gamma}^{\alpha_1} \hat{\Gamma}^{\beta_1}. \]  
(5.11)
Let
\[ \eta = \eta_0(x) \otimes \eta_1(y), \]  
where \( \eta_0 = \eta_0(x) \) is 16-component spinor on \( M_0 \), and \( \eta_1 = \eta_1(y) \) is 2-component spinor on \( M_1 \). Then we get from (5.10) and (5.11)
\[ \hat{D}^{(0)}_{m_0} \eta = (D^{(0)}_{m_0} \eta_0) \otimes \eta_1, \quad \hat{D}^{(1)}_{m_1} \eta = \eta_0 \otimes (D^{(1)}_{m_1} \eta_1), \]  
where \( D^{(0)}_{m_0} = \partial_{m_0} + \frac{1}{4} \hat{\kappa}^{a_0} \hat{\kappa}^{b_0} \hat{\Gamma}^a_{(0)} \hat{\Gamma}^b_{(0)} \) and \( D^{(1)}_{m_1} = \partial_{m_1} + \frac{1}{4} \hat{\kappa}^{a_1} \hat{\kappa}^{b_1} \hat{\Gamma}^a_{(1)} \hat{\Gamma}^b_{(1)} \) are covariant (spinorial) derivatives (2.29) corresponding to the manifolds \( M_0 \) and \( M_1 \), respectively.

The operator (4.17) corresponding to \( M^2 \)-brane reads
\[ \hat{\Gamma}^s = \hat{\Gamma}^{a_1} \hat{\Gamma}^{a_2} \hat{\Gamma}^{a_3} = \hat{\Gamma}^{(0)} \otimes \hat{\Gamma}^{(1)}, \]  
where \( \hat{\Gamma}^{(1)} = \hat{\Gamma}^{a_1} \hat{\Gamma}^{a_2} \hat{\Gamma}^{a_3} \).

Choosing (real) matrices
\[ \hat{\Gamma}^{a_1} = i\sigma_2, \quad \hat{\Gamma}^{a_2} = \sigma_1, \quad \hat{\Gamma}^{a_3} = \sigma_3 = \hat{\Gamma}^{a_1} \hat{\Gamma}^{a_2}, \]  
where \( \sigma_i \) are the standard Pauli matrices, we get
\[ \hat{\Gamma}^{(1)} = 1_2. \]  

Due to (5.12), (5.14) and (5.16) the chirality restriction (4.22) in Proposition 1 is satisfied if
\[ \hat{\Gamma}^{(0)} \eta_0 = c \eta_0, \]  
see (5.2).

Using the Proposition 1 we are led to the following solution to SUSY equations (3.2) corresponding to the field configuration from (5.14), (5.5)
\[ \varepsilon = H^{-1/6} \eta_0(x) \otimes \eta_1(y). \]  

Here \( \eta_0(x) \) is a 16-component parallel (Killing) chiral spinor (field) on \( M_0 (D^{(0)}_{m_0} \eta_0 = 0) \) obeying (5.17) and \( \eta_1(y) \) is a 2-component parallel spinor on \( M_1 (D^{(1)}_{m_1} \eta_1 = 0) \).

Hence the number of unbroken SUSY is at least \[ [15] \]
\[ \mathcal{N} = n_0(c)n_1/32, \]  
where \( n_0(c) \) is the number of chiral parallel (Killing) spinors on \( M_0 \) satisfying (5.17), and \( n_1 \) is the number of parallel spinors on \( M_1 \).

Since the 3-dimensional space \( (M_1, g^1) \) is considered to be Ricci-flat, it is flat, i.e. the Riemann tensor corresponding to \( g^1 \) is zero.

Case \( n_1 = 2 \). Let us consider flat pseudo-Euclidean space \( M_1 = \mathbb{R}^3, g^1 = -dy_1^2 \otimes dy_1^2 + dy_2^2 \otimes dy_2^2 + dy_3^2 \otimes dy_3^2 \). In this case \( n_1 = 2 \) and
\[ \mathcal{N} = n_0(c)/16. \]  

For flat \( M_0 = \mathbb{R}^8 \) we get \( n_0(c) = 8 \) and hence \( \mathcal{N} = 1/2 \) in agreement with [16].
Example: $M_0$ with holonomy groups $Spin(7)$, $SU(4)$, $Sp(2)$. According to M. Wang’s classification [39] an irreducible, simply-connected, Riemannian 8-dimensional manifold $M_0$ admitting parallel spinors must have precisely one of the following holonomy groups: $Spin(7)$, $SU(4)$ or $Sp(2)$. For suitably chosen orientation on $M_0$ the numbers of chiral parallel spinors are the following ones $(n_0(+1), n_0(-1)) = (k, 0)$, where $k = 1, 2, 3$ for $H = Spin(7), SU(4), Sp(2)$, respectively. Hence $\mathcal{N} = k/16$ for $c = +1$ and $\mathcal{N} = 0$ for $c = -1$. The hyper-Kähler case with $H = Sp(2)$ was studied in [28].

Example: $M_0$ with holonomy groups $SU(2)$ and $SU(2) \times SU(2)$. Let us consider $K3 = CY_2$ which is a 4-dimensional Ricci-flat Kähler manifold with a holonomy group $SU(2) = Sp(1)$ and self-dual (or anti-self-dual) curvature tensor. $K3$ has two Killing spinors of the same chirality, say, $+1$. Let $M_0 = \mathbb{R}^3 \times K3$, then we get $n_0(+1) = n_0(-1) = 4$ and hence $\mathcal{N} = 1/4$ for any $c = \pm 1$. (See Appendix C.) For $M_0 = K3 \times K3$, we obtain $n_0(+1) = 4, n_0(-1) = 0$ and hence $\mathcal{N} = 1/4$ for $c = 1$ and $\mathcal{N} = 0$ for $c = -1$.

Example: $M_0$ with holonomy group $SU(3)$. Let $M_0 = \mathbb{R}^2 \times CY_3$ be 6-dimensional Calabi-Yau manifold (3-fold) of holonomy $SU(3)$. Since $CY_3$ has two parallel spinors of opposite chiralities (and the same for $\mathbb{R}^2$) we get $n_0(+1) = n_0(-1) = 2$ and hence $\mathcal{N} = 1/8$ for any $c = \pm 1$.

5.2 M5-brane

Now we consider the magnetic 5-brane solution defined on the manifold (5.3) with $d_0 = 5$ and $d_1 = 6$:

$$g = H^{2/3} \{ \hat{g}^0 + H^{-1} \hat{g}^1 \},$$

$$F = c(\ast_0 dH),$$

where $c^2 = 1, \ast_0$ is the Hodge operator on $(M_0, \hat{g}^0)$, $H = H(x)$ is a harmonic function on $(M_0, \hat{g}^0)$, and metrics $\hat{g}^i, i = 0, 1$, are Ricci-flat, $\hat{g}^0$ has a Euclidean signature and $\hat{g}^1$ has the signature $(-, +, +, +, +, +)$.

Let us consider $\Gamma$-matrices

$$(\hat{\Gamma}^A) = (\hat{\Gamma}_{(0)}^{a_0} \otimes \hat{\Gamma}_{(1)}^{a_1}, \mathbf{1}_4 \otimes \hat{\Gamma}_{(1)}^{a_1}),$$

where $\hat{\Gamma}_{(0)}^{a_0}, a_0 = 1_0, \ldots, 5_0$, are $4 \times 4$ $\Gamma$-matrices corresponding to $M_0$ and $\hat{\Gamma}_{(1)}^{a_1}, a_1 = 1_1, \ldots, 6_1$, are $8 \times 8$ $\Gamma$-matrices corresponding to $M_1$. Here $\hat{\Gamma}_{(1)}^{a_1} = \hat{\Gamma}_{(1)}^{a_1} \cdots \hat{\Gamma}_{(1)}^{a_1}$ and $(\hat{\Gamma}_{(1)}^{a_1})^2 = \mathbf{1}_8$.

We put $\eta = \eta_0(x) \otimes \eta_1(y_1)$, where $\eta_0 = \eta_0(x)$ is 4-component spinor on $M_0$, and $\eta_1 = \eta_1(y_1)$ is 8-component spinor on $M_1$. Then relations (5.13) are valid in magnetic case too.

The operator (4.18) corresponding to M5-brane reads

$$\hat{\Gamma}_{[a]} = \hat{\Gamma}_{10}^{a_0} \hat{\Gamma}_{20} \hat{\Gamma}_{30}^{a_0} \hat{\Gamma}_{40}^{a_0} \hat{\Gamma}_{50} = \hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)},$$

where $\hat{\Gamma}_{(0)} = \hat{\Gamma}_{10}^{a_0} \cdots \hat{\Gamma}_{50}^{a_0} = \mathbf{1}_4$, if we put $\hat{\Gamma}_{50}^{a_0} = \hat{\Gamma}_{10}^{a_0} \cdots \hat{\Gamma}_{0}^{a_0}$. Then the chirality restriction (4.22) is satisfied if

$$\hat{\Gamma}_{(1)} \eta_1 = c \eta_1.$$
Using the Proposition 1 we get the following solution to equations (3.2) corresponding to the field configuration from (5.21) and (5.22)

\[ \varepsilon = H^{-1/12}\eta_0(x) \otimes \eta_1(y). \]  

(5.26)

Here \( \eta_0(x) \) is a 4-component parallel (Killing) spinor (field) on \( M_0 (\mathcal{D}_{m_0} \eta_0 = 0) \) and \( \eta_1(y) \) is a 8-component chiral parallel spinor on \( M_1 (\mathcal{D}_{m_1} \eta_1 = 0) \) obeying (5.25).

Hence the number of unbroken SUSY is at least \( [15] \)

\[ N = n_0 n_1(c)/32, \]  

(5.27)

where \( n_0 \) is the number of parallel spinors on \( M_0 \), and \( n_1(c) \) is the number of chiral parallel spinors on \( M_1 \) satisfying (5.25).

For flat factor-spaces with \( M_0 = \mathbb{R}^5 \) (and Euclidean metric) and \( M_1 = \mathbb{R}^6 \) (and Minkowskian metric of \( \mathbb{R}^{1,5} \)) we get \( n_0 = 4, n_1(c) = 4 \) and hence \( N = 1/2 \) in agreement with [17].

Example: generalized Kaya solution. Let \( M_1 = \mathbb{R}^2 \times K3 \). In this case we obtain from (5.27)

\[ N = n_0/16 \]

since \( n_1(c) = 2 \). For flat \( M_0 = \mathbb{R}^5 \) (\( n_0 = 4 \)) we get \( N = 1/4 \) [30]. For \( M_0 = \mathbb{R} \times K3 \) we have \( n_0 = 2 \) and consequently \( N = 1/8 \).

Example: generalized Figueroa-O’Farrill solutions. Let \((M_1, g^1)\) be a 6-dimensional Ricci-flat pp-wave solution from [31] (page. 5) of holonomy \( Sp(1) \ltimes \mathbb{R}^4 \) or \( \mathbb{R}^4 \) for \( k = 1, 2 \), respectively. Then the numbers of chiral parallel spinors are \( (n_1(+1), n_1(-1)) = (k, k) \), \( k = 1, 2 \). Due to (5.27) we have

\[ N = n_0 k/32. \]

For flat \( M_0 = \mathbb{R}^5 \) we have \( N = 1/8, 1/4 \) for \( k = 1, 2 \), respectively, in agreement with [31]. For \( M_0 = \mathbb{R} \times K3 \) we reduce these numbers to \( N = 1/16, 1/8 \), for \( k = 1, 2 \), respectively.

5.3 \( M2 \cap M5\)-branes

Let us consider a solution with intersecting \( M2\)- and \( M5\)-branes defined on the manifold

\[ M_0 \times M_1 \times M_2 \times M_3, \]  

(5.28)

where \( d_0 = 4, d_1 = 1, d_2 = 4 \) and \( d_3 = 2 \).

The solution reads

\[ g = H_{1}^{1/3}H_{2}^{2/3}\{g^0 + H_{1}^{-1}g^1 + H_{2}^{-1}g^2 + H_{1}^{-1}H_{2}^{-1}g^3\}, \]  

(5.29)

\[ F = c_1 dH_{1}^{-1} \wedge \tilde{\tau}_1 \wedge \tilde{\tau}_3 + c_2 (\ast_0 dH_2) \wedge \tilde{\tau}_1, \]  

(5.30)

where \( c_1^2 = c_2^2 = 1; H_1, H_2 \) are harmonic functions on \((M_0, g^0)\), metrics \( g^0, g^2 \) are Ricci-flat and \( g^1, g^3 \) are flat. (Since \((M_3, g^3)\) is 2-dimensional Ricci-flat space, it is flat.) The
metrics $g^i$, $i = 0, 1, 2$ have Euclidean signatures and the metric $g^3$ has the signature $(-, +)$. The brane sets are $I_1 = \{1, 3\}$ and $I_2 = \{2, 3\}$ for $M2$ and $M5$ branes, respectively. We put here $M_1 = \mathbb{R}$.

Let us introduce the following set of $\Gamma$-matrices

$$\hat{\Gamma}^A = (\hat{\Gamma}^{a_0}_{(0)} \otimes 1 \otimes \mathbf{1}_4 \otimes \mathbf{1}_2),$$

$$\hat{\Gamma}^{a_0}_{(0)} \otimes 1 \otimes \hat{\Gamma}^{a_2}_{(2)} \otimes \hat{\Gamma}^{a_3}_{(3)},$$

$$\hat{\Gamma}^{a_0}_{(0)} \otimes 1 \otimes \hat{\Gamma}^{a_2}_{(2)} \otimes \mathbf{1}_2,$$

$$\hat{\Gamma}^{a_0}_{(0)} \otimes 1 \otimes \hat{\Gamma}^{a_3}_{(3)}).$$

Here

$$\hat{\Gamma}^{a_0}_{(0)} = \hat{\Gamma}^{a_1}_{(0)} \hat{\Gamma}^{a_2}_{(0)} \hat{\Gamma}^{a_3}_{(0)} \hat{\Gamma}^{a_4}_{(0)}, \quad \hat{\Gamma}^{a_0}_{(0)} = \hat{\Gamma}^{a_1}_{(0)} \hat{\Gamma}^{a_2}_{(2)} \hat{\Gamma}^{a_3}_{(2)} \hat{\Gamma}^{a_4}_{(2)}, \quad \hat{\Gamma}^{a_0}_{(0)} = \hat{\Gamma}^{a_1}_{(3)} \hat{\Gamma}^{a_2}_{(3)}$$

obey

$$(\hat{\Gamma}^{a_0}_{(0)})^2 = (\hat{\Gamma}^{a_0}_{(2)})^2 = \mathbf{1}_4, \quad (\hat{\Gamma}^{a_0}_{(3)})^2 = \mathbf{1}_2.$$ (5.33)

Let

$$\eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3),$$ (5.34)

where $\eta_0 = \eta_0(x)$ is 4-component spinor on $M_0$, $\eta_1 = \eta_1(y_1)$ is 1-component spinor on $M_1$, $\eta_2 = \eta_2(y_2)$ is 4-component spinor on $M_2$, and $\eta_3 = \eta_3(y_3)$ is 2-component spinor on $M_3$.

It follows from (2.28), (5.31) and (5.33) that

$$\hat{D}^{(a_0)}_{m_0} \eta = (D^{(0)}_{m_0} \eta_0) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3, \quad \hat{D}^{(a_1)}_{m_1} \eta = \eta_0 \otimes (D^{(1)}_{m_1} \eta_1) \otimes \eta_2 \otimes \eta_3,$$ (5.35)

$$\hat{D}^{(a_2)}_{m_2} \eta = \eta_0 \otimes \eta_1 \otimes (D^{(2)}_{m_2} \eta_2) \otimes \eta_3, \quad \hat{D}^{(a_3)}_{m_3} \eta = \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes (D^{(3)}_{m_3} \eta_3),$$

where $D^{(i)}_{m_i}$ are covariant (spinorial) derivative (2.29) corresponding the manifold $M_i$, $i = 0, 1, 2, 3$. Here $D^{(a_1)}_{m_1} = \partial_{m_1}$.

The operator (4.17) corresponding to $M2$-brane reads

$$\hat{\Gamma}_{[s]} = \hat{\Gamma}^{a_1}_{(0)} \hat{\Gamma}^{a_2}_{(2)} \hat{\Gamma}^{a_3}_{(3)} \hat{\Gamma}^{a_3}_{(0)} \hat{\Gamma}^{a_0}_{(0)} \otimes 1 \otimes \hat{\Gamma}^{a_0}_{(2)} \otimes \mathbf{1}_2,$$ (5.36)

$s = I_1$ and the operator (4.18) corresponding to $M5$-brane has the following form

$$\hat{\Gamma}_{[s]} = \hat{\Gamma}^{a_1}_{(0)} \hat{\Gamma}^{a_2}_{(2)} \hat{\Gamma}^{a_3}_{(3)} \hat{\Gamma}^{a_4}_{(0)} \hat{\Gamma}^{a_0}_{(0)} \otimes 1 \otimes \hat{\Gamma}^{a_0}_{(2)} \otimes \hat{\Gamma}^{a_3}_{(3)},$$ (5.37)

$s = I_2$. Then the restrictions (4.22) are satisfied if

$$\hat{\Gamma}_{(j)} \eta_j = c_{(j)} \eta_j, \quad c_{(j)}^2 = 1,$$ (5.38)

$$j = 0, 2, 3,$$ and

$$c_{(0)} c_{(2)} = c_1, \quad c_{(2)} c_{(3)} = c_2.$$ (5.39)

Using the Proposition 1 we obtain the following solution to equations (3.2) corresponding to the field configuration from (5.29), (5.30)

$$\varepsilon = H_1^{-1/6} H_2^{-1/12} \eta_0(x) \otimes \eta_1 \otimes \eta_2(y_2) \otimes \eta_3(y_3).$$ (5.40)
Here $\eta_i$, $i = 0, 2, 3$, are parallel chiral spinors defined on $M_i$, respectively $(D_{m_i}^i \eta_i = 0)$, obeying \[(5.38)\) and \[(5.39)\); $\eta$ is constant (1-dimensional spinor).

It follows from \[(5.39)\] that either i) $c(2) = 1$, $c(0) = c_1$, $c(3) = c_2$, or ii) $c(2) = -1$, $c(0) = -c_1$, $c(3) = -c_2$.

Thus, the number of linear independent solutions given by \[(5.40)\] is

$$N = 32\mathcal{N} = n_0(c_1)n_2(1)n_3(c_2) + n_0(-c_1)n_2(-1)n_3(-c_2).$$

\[(5.41)\]

Here $n_j(c_j)$ is the number of chiral parallel spinors on $M_j$ obeying \[(5.38), j = 0, 2, 3.$$  

**\(\mathbb{R}^{1,1}\)-intersection.** Let $M_3 = \mathbb{R}^2$, $g^3 = -dy_3^1 \otimes dy_3^1 + dy_3^2 \otimes dy_3^2$. Thus we deal with 2-dimensional pseudo-Euclidean space $\mathbb{R}^{1,1}$. In this case we get $n_3(c) = 1$ and the number of unbroken SUSY is at least

$$\mathcal{N} = \frac{1}{32} \sum_{c = \pm 1} n_0(cc_1)n_2(c).$$

\[(5.42)\]

For flat $M_0 = M_2 = \mathbb{R}^4$ we have $n_0(c) = n_2(c) = 2$ and hence $\mathcal{N} = 1/4$.

**Example: one K3 factor-space.** Let $M_0 = \mathbb{R}^4$ and $M_2 = K3$. We get from \[(5.42)\]: $\mathcal{N} = 1/8$. The same number of fractional SUSY will be obtained for $M_0 = K3$ and $M_2 = \mathbb{R}^4$.

**Example: two K3 factor-spaces.** Let $M_0 = M_2 = K3$. We put for chiral numbers $(n_i(+1), n_i(-1)) = (2, 0)$, $i = 0, 2$. Then we get $\mathcal{N} = 1/8$ for $c_1 = +1$ and $\mathcal{N} = 0$ for $c_1 = -1$. It should be noted that here the number of fractional SUSY depends only upon the sign-factor of electric $M2$-brane $c_1$. It is independent upon the sign-factor of electric $M5$-brane $c_2$. The change of the orientation of one of the manifolds, say $M_2$, i.e. when $M_2 = K3$, with $(n_2(+1), n_2(-1)) = (0, 2)$ will lead to $\mathcal{N} = 1/8$ for $c_1 = -1$ and $\mathcal{N} = 0$ for $c_1 = +1$.

### 5.4 $M2 \cap M2$-branes

Consider a solution with two intersecting $M2$-branes defined on the manifold \[(5.28)\] 

$$M_0 \times M_1 \times M_2 \times M_3,$$

with $d_0 = 6$, $d_1 = d_2 = 2$ and $d_3 = 1$.

The solution reads

$$g = H_1^{1/3}H_2^{1/3}\{g^0 + H_1^{-1}g^1 + H_2^{-1}g^2 + H_1^{-1}H_2^{-1}g^3\},$$

\[(5.43)\]

$$F = c_1dH_1^{-1} \wedge \hat{\tau}_2 \wedge \hat{\tau}_3 + c_2dH_2^{-1} \wedge \hat{\tau}_2 \wedge \hat{\tau}_3,$$

\[(5.44)\]

where $c_1^2 = c_2^2 = 1$; $H_1$, $H_2$ are harmonic functions on 6-dimensional Ricci-flat Riemann manifold $(M_0, g^0)$, metrics $g^i$, $i = 1, 2$, are 2-dimensional flat metrics of Euclidean signature $(+, +)$. The brane sets are $I_1 = \{1, 3\}$ and $I_2 = \{2, 3\}$. We put here $M_3 = \mathbb{R}$ and $g^3 = -dt \otimes dt$ ($\tau_3 = dt$).

Let us consider the following set of $\Gamma$-matrices

$$(\hat{\Gamma}^A) = (\hat{\Gamma}^{\tau_0}_{(0)} \otimes 1_2 \otimes 1_2 \otimes 1),$$

\[(5.45)\]
\[ i \hat{\Gamma}_0 \otimes \hat{\Gamma}_1 \otimes 1_2 \otimes 1, \]
\[ \hat{\Gamma}_0 \otimes \hat{\Gamma}_1 \otimes \hat{\Gamma}_2 \otimes 1, \]
\[ \hat{\Gamma}_0 \otimes \hat{\Gamma}_1 \otimes \hat{\Gamma}_2 \otimes 1. \]

Here
\[ \hat{\Gamma}_0 = \hat{\Gamma}_1 \otimes \ldots \otimes \hat{\Gamma}_n, \quad \hat{\Gamma}_1 = \hat{\Gamma}_1 \otimes \hat{\Gamma}_2, \quad \hat{\Gamma}_2 = \hat{\Gamma}_2 \otimes \hat{\Gamma}_2 \]

obey
\[ (\hat{\Gamma}_0)^2 = -1_8, \quad (\hat{\Gamma}_1)^2 = (\hat{\Gamma}_2)^2 = -1_2. \]

As in the previous case we consider the ansatz (5.34) \( \eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \),
where \( \eta_0 = \eta_0(x) \) is 8-component spinor on \( M_0 \), \( \eta_1 = \eta_1(y_1) \) is 2-component spinor on \( M_1 \),
\( \eta_2 = \eta_2(y_2) \) is 2-component spinor on \( M_2 \), and \( \eta_3 = \eta_3(y_3) \) is 1-component spinor on \( M_3 \).

Due to (2.28), (5.45), and (5.47) the relations (5.35) are satisfied identically. Here \( D_{m_3}^{(3)} = \partial_{m_3} \).

The operators (4.17) corresponding to \( M_2 \)-branes read
\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}_1 \hat{\Gamma}_2 \hat{\Gamma}_3 = -\hat{\Gamma}_1 \otimes \hat{\Gamma}_2 \otimes 1, \]
for \( s = I_1 \) and
\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}_1 \hat{\Gamma}_2 \hat{\Gamma}_3 = -\hat{\Gamma}_1 \otimes \hat{\Gamma}_1 \otimes 1_2 \otimes 1, \]
for \( s = I_2 \). Then the chirality restrictions (4.22) are satisfied if
\[ \hat{\Gamma}_j \eta_j = c_{(j)} \eta_j, \quad c_{(j)}^2 = -1, \]
\[ j = 0, 1, 2, \]
and
\[ -c_{(0)} c_{(2)} = c_1, \quad -c_{(0)} c_{(1)} = c_2. \]

Using the Proposition 1 we get the following solution to SUSY equations (3.2) corresponding to the field configuration from (5.43), (5.44)
\[ \varepsilon = H_1^{-1/6} H_2^{-1/6} \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3. \]

Here \( \eta_i, i = 0, 1, 2, \) are chiral parallel spinors defined on \( M_i \), respectively: \( D_{m_i}^{(i)} \eta_i = 0 \),
obeying (5.50) and (5.51); \( \eta_3 \) is constant (1-component spinor).

It follows from (5.51) that either i) \( c_{(0)} = i, c_{(2)} = \imath c_1, c_{(1)} = \imath c_2 \), or ii) \( c_{(2)} = -i, c_{(1)} = -\imath c_1, c_{(0)} = 1 \).

Thus, the number of linear independent solutions given by (5.52) is
\[ N = 32 N = n_0(+i) n_1(\imath c_2) n_2(\imath c_1) + n_0(-i) n_1(-\imath c_2) n_2(-\imath c_1), \]
where \( n_j(c_{(j)}) \) is the number of chiral parallel spinors on \( M_j \) obeying (5.50), \( j = 0, 1, 2. \)

We remind that here \( M_1 \) and \( M_2 \) are 2-dimensional flat spaces of Euclidean signature.

Case \( M_1 = M_2 = \mathbb{R}^2 \). Let \( M_1 = M_2 = \mathbb{R}^2 \). Then due to \( n_1(c) = n_2(c) = 1 \) the number of unbroken SUSY is at least
\[ N = \frac{1}{32} n_0. \] (5.54)

Here \( N \) does not depend upon brane signs \( c_s \). For flat \( M_0 = \mathbb{R}^6 \) we have \( n_0 = 8 \) and hence \( N = 1/4 \).

**Example:** \( M_0 = CY_3 \). Let \( M_0 = CY_3 \) be 6-dimensional Calabi-Yau manifold (3-fold) of holonomy \( SU(3) \). Then \( n_0(i) = n_0(-i) = 1, n_0 = 2 \) and hence \( N = 1/16 \).

**Example:** \( M_0 = \mathbb{R}^2 \times K3 \). For \( M_0 = \mathbb{R}^2 \times K3 \) we get \( n_0(i) = n_0(-i) = 2, n_0 = 4 \) and consequently \( N = 1/8 \).

### 5.5 \( M5 \cap M5 \)-branes

Now we deal with \( M5 \cap M5 \)-solution defined on the manifold (5.28)

\[ M_0 \times M_1 \times M_2 \times M_3, \]

with \( d_0 = 3, d_1 = d_2 = 2 \) and \( d_3 = 4 \).

The solution reads

\[ g = H_1^{2/3} H_2^{2/3} \{ g^0 + H_1^{-1} g^1 + H_2^{-1} g^2 + H_1^{-1} H_2^{-1} g^3 \}, \]

\[ F = c_1 (*_0 dH_1) \land \hat{\tau}_2 + c_2 (*_0 dH_2) \land \hat{\tau}_1, \] (5.56)

where \( c_1^2 = c_2^2 = 1 \); \( H_1, H_2 \) are harmonic functions on \( (M_0, g^0) \), metrics \( g^i, i = 0, 1, 2, 3 \), are Ricci-flat (the first three metrics are flat). The metrics \( g^i, i = 0, 1, 2 \), have Euclidean signatures and the metric \( g^3 \) has the signature \((-++,+,-+)\). The brane sets are \( I_1 = \{1, 3\} \) and \( I_2 = \{2, 3\} \).

We choose the following \( \Gamma \)-matrices

\[ (\hat{\Gamma}^A) = (i \hat{\Gamma}^{a_0}_{(0)} \otimes \hat{\Gamma}^{(1)} \otimes \hat{\Gamma}^{(2)} \otimes \hat{\Gamma}^{(3)}), \]

\[ 1_2 \otimes \hat{\Gamma}^{a_1}_{(1)} \otimes \hat{\Gamma}^{(2)} \otimes \hat{\Gamma}^{(3)}, \]

\[ i1_2 \otimes 1_2 \otimes \hat{\Gamma}^{a_2}_{(2)} \otimes \hat{\Gamma}^{(3)}, \]

\[ 1_2 \otimes 1_2 \otimes 1_2 \otimes \hat{\Gamma}^{a_3}_{(3)}, \] (5.57)

where

\[ \hat{\Gamma}^{(0)} = \hat{\Gamma}^{(1)} \hat{\Gamma}^{(2)} \hat{\Gamma}^{(3)}, \quad \hat{\Gamma}^{(1)} = \hat{\Gamma}^{(1)} \hat{\Gamma}^{(2)}, \quad \hat{\Gamma}^{(2)} = \hat{\Gamma}^{(1)} \hat{\Gamma}^{(2)} \hat{\Gamma}^{(3)}, \quad \hat{\Gamma}^{(3)} = \hat{\Gamma}^{(1)} \hat{\Gamma}^{(2)} \hat{\Gamma}^{(3)} \] (5.58)

obey

\[ (\hat{\Gamma}^{(0)})^2 = (\hat{\Gamma}^{(1)})^2 = (\hat{\Gamma}^{(2)})^2 = -1_2, \quad (\hat{\Gamma}^{(3)})^2 = -1_4. \] (5.59)

We put \( \hat{\Gamma}^{a_0}_{(0)} = (\sigma_1, \sigma_2, \sigma_3) \) and hence

\[ \hat{\Gamma}^{(0)} = i1_2. \] (5.60)
Here we consider the decomposition \([5.31]\) \(\eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3)\), where \(\eta_0 = \eta_0(x)\) is 2-component spinor on \(M_0\), \(\eta_1 = \eta_1(y_1)\) is 2-component spinor on \(M_1\), \(\eta_2 = \eta_2(y_2)\) is 2-component spinor on \(M_2\), and \(\eta_3 = \eta_3(y_1)\) is 4-component spinor on \(M_3\).

Due to \([2.28]\), \([5.31]\) and \([5.39]\) relations \([5.35]\) are also satisfied in this case.

The operators \([4.18]\) corresponding to \(M5\)-branes read

\[
\hat{\Gamma}_c = \hat{\Gamma}_1^0 \hat{\Gamma}_2^0 \hat{\Gamma}_3 \hat{\Gamma}_1^2 \hat{\Gamma}_2^2 = 1_2 \otimes \hat{\Gamma}_1 \otimes 1_2 \otimes \hat{\Gamma}_3,
\]

for \(s = I_1\) and

\[
\hat{\Gamma}_c = \hat{\Gamma}_1^0 \hat{\Gamma}_2^0 \hat{\Gamma}_3 \hat{\Gamma}_1^2 \hat{\Gamma}_2^2 = 1_2 \otimes 1_2 \otimes \hat{\Gamma}_2 \otimes \hat{\Gamma}_3,
\]

for \(s = I_2\). Then the restrictions \([4.22]\) are satisfied if

\[
\hat{\Gamma}_j \eta_j = c_{(j)} \eta_j, \quad c_{(j)}^2 = -1,
\]

\(j = 1, 2, 3\), and

\[
c_{(1)} c_{(3)} = c_1, \quad c_{(2)} c_{(3)} = c_2.
\]

According to Proposition 1 we get the following solution to equations \([3.2]\) corresponding to the field configuration from \([5.55]\), \([5.56]\)

\[
\varepsilon = H_1^{-1/12} H_2^{-1/12} \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3).
\]

Here \(\eta_0\) is parallel spinor on \(M_0\) and \(\eta_i, i = 1, 2, 3\), are chiral parallel spinors defined on \(M_i\), respectively \((D_{_{\eta_i}}^0 \eta_i = 0)\), obeying \([5.63]\) and \([5.64]\).

It follows from \([5.64]\) that either i) \(c_{(3)} = i, c_{(1)} = -ic_1, c_{(2)} = -ic_2\), or ii) \(c_{(3)} = -i, c_{(1)} = ic_1, c_{(2)} = ic_2\).

Thus, the number of linear independent solutions given by \([5.65]\) is

\[
N = 32 \mathcal{N} = n_0 n_1 (-ic_1) n_2 (-ic_2) n_3 (+i) + n_0 n_1 (ic_1) n_2 (ic_2) n_3 (-i).
\]

Here \(n_j(c_{(j)})\) is the number of chiral parallel spinors on \(M_j\) (see \([5.63]\)), \(j = 1, 2, 3\), and \(n_0\) is the number of parallel spinors on \(M_0\).

**Case** \(M_1 = M_2 = \mathbb{R}^2\).

Let \(M_1 = M_2 = \mathbb{R}^2\). We get \(n_1(c) = n_2(c) = 1\) and hence

\[
\mathcal{N} = \frac{1}{32} n_0 n_3.
\]

For flat factor-spaces with \(M_0 = \mathbb{R}^3\) (and Euclidean metric) and \(M_3 = \mathbb{R}^4\) (and Minkowski metric of \(\mathbb{R}^{1,3}\)) we get \(\mathcal{N} = 1/4\).

**Example: 4-dimensional pp-wave metric.** Let \((M_3, g^3)\) be a 4-dimensional Ricci-flat pp-wave solution from \([31]\) with the holonomy group \(H = \mathbb{R}^2\). Then we get \(n_3(1) = n_3(-1) = 1, n_3 = 2\) (see \([40]\), p. 53) and hence \(\mathcal{N} = n_0/16\). For \(M_0 = \mathbb{R}^3\) we are led to \(\mathcal{N} = 1/8\).
5.6 $M2 \cap M2 \cap M2$-branes

The last example is related to the solution with three intersecting $M2$-branes defined on the manifold

$$M_0 \times M_1 \times M_2 \times M_3 \times M_4,$$

(5.68)

with $d_0 = 4$, $d_1 = d_2 = d_3 = 2$ and $d_4 = 1$.

The solution reads

$$g = H_1^{1/3} H_2^{1/3} H_3^{1/3} \{g'^0 + H_1^{-1} g^1 + H_2^{-1} g^2 + H_3^{-1} g^3 + H_1^{-1} H_2^{-1} H_3^{-1} g^4\},$$

(5.69)

$$F = c_1 d H_1^{-1} \wedge \hat{\tau}_1 \wedge \hat{\tau}_3 + c_2 d H_2^{-1} \wedge \hat{\tau}_2 \wedge \hat{\tau}_4 + c_3 d H_3^{-1} \wedge \hat{\tau}_3 \wedge \hat{\tau}_4,$$

(5.70)

where $c_1^2 = c_2^2 = c_3^2 = 1$; $H_1$, $H_2$, $H_3$ are harmonic functions on $(M_0, g'^0)$, metrics $g'^i$, $i = 0, 1, 2, 3, 4$, are Ricci-flat (the last four metrics are flat). The metrics $g'^i$, $i = 0, 1, 2, 3$, have Euclidean signatures and the metric $g^4$ has the signature $(-)$. Here we put $M_4 = \mathbb{R}$, $g^4 = -dt \otimes dt$ ($\tau_4 = dt$). The brane sets are $I_1 = \{1, 4\}$, $I_2 = \{2, 4\}$ and $I_3 = \{3, 4\}$.

Let us consider the following set of $\Gamma$-matrices

$$(\hat{\Gamma}^A) = (\hat{\Gamma}_{\alpha_0}^i) \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes 1,$$

(5.71)

$$\hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)}^i \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes 1,$$

$$i \hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)} \otimes \hat{\Gamma}_{(2)} \otimes \mathbf{1}_2 \otimes \mathbf{1},$$

$$\hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)} \otimes \hat{\Gamma}_{(2)} \otimes \hat{\Gamma}_{(3)} \otimes 1,$$

$$\hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)} \otimes \hat{\Gamma}_{(2)} \otimes \hat{\Gamma}_{(3)} \otimes 1,$$

where

$$\hat{\Gamma}_{(0)} = \hat{\Gamma}_{(0)}^0 \ldots \hat{\Gamma}_{(0)}^4, \quad \hat{\Gamma}_{(i)} = \hat{\Gamma}_{(i)}^1 \hat{\Gamma}_{(i)}^2,$$

(5.72)

obey

$$(\hat{\Gamma}_{(0)})^2 = \mathbf{1}_4, \quad (\hat{\Gamma}_{(i)})^2 = -\mathbf{1}_2,$$

(5.73)

$i = 1, 2, 3$.

Here we put

$$\eta = \eta_0 (x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4),$$

(5.74)

where $\eta_0 = \eta_0(x)$ is 4-component spinor on $M_0$, $\eta_i = \eta_i(y_i)$ is 2-component spinor on $M_i$, $i = 1, 2, 3$, and $\eta_4 = \eta_4(y_4)$ is 1-component spinor on $M_4$.

Due to (2.28), (5.71) and (5.73) the following relations take place

$$\hat{D}_{m_0}^{(0)} \eta = (D_{m_0}^{(0)} \eta_0) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4, \quad \hat{D}_{m_1}^{(1)} \eta = \eta_0 \otimes (D_{m_1}^{(1)} \eta_1) \otimes \eta_2 \otimes \eta_3 \otimes \eta_4,$$

(5.75)

$$\hat{D}_{m_2}^{(2)} \eta = \eta_0 \otimes \eta_1 \otimes (D_{m_2}^{(2)} \eta_2) \otimes \eta_3 \otimes \eta_4, \quad \hat{D}_{m_3}^{(3)} \eta = \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes (D_{m_3}^{(3)} \eta_3) \otimes \eta_4,$$

$$\hat{D}_{m_4}^{(4)} \eta = \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes (D_{m_4}^{(4)} \eta_4),$$

where $D_{m_i}^{(i)}$ are covariant derivative corresponding to $M_i$, $i = 0, 1, 2, 3$. Here $D_{m_4}^{(4)} = \partial_{m_4}$.

The operators (1.17) corresponding to $M2$-branes read

$$\hat{\Gamma}_{[s]} = \hat{\Gamma}_{11}^1 \hat{\Gamma}_{22}^2 \hat{\Gamma}_{14} = -\hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(2)} \otimes \hat{\Gamma}_{(3)} \otimes 1$$

(5.76)
for $s = I_1$,  
\[ \hat{\Gamma}_s = \hat{\Gamma}^{12} \hat{\Gamma}^{22} \hat{\Gamma}^{14} = -\hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)} \otimes 1_2 \otimes \hat{\Gamma}_{(3)} \otimes 1 \]  
(5.77)

for $s = I_2$ and  
\[ \hat{\Gamma}_s = \hat{\Gamma}^{13} \hat{\Gamma}^{23} \hat{\Gamma}^{14} = -\hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)} \otimes 1_2 \otimes 1 \]  
(5.78)

for $s = I_3$.

We put  
\[ \hat{\Gamma}_{(0)} \eta_0 = c_{(0)} \eta_0, \quad c_{(0)}^2 = 1, \]  
(5.79)
\[ \hat{\Gamma}_{(j)} \eta_j = c_{(j)} \eta_j, \quad c_{(j)}^2 = -1, \]  
(5.80)

$j = 1, 2, 3$. Then the chirality restrictions (1.22) are satisfied if  
\[ -c_{(0)} c_{(2)} c_{(3)} = c_1, \quad -c_{(0)} c_{(1)} c_{(3)} = c_2, \quad -c_{(0)} c_{(1)} c_{(2)} = c_3. \]  
(5.81)

Due to Proposition 1 we obtain the following solution to SUSY equations (3.2) corresponding to the field configuration from (5.69), (5.70)

\[ \varepsilon = H_1^{-1/6} H_2^{-1/6} H_3^{-1/6} \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4. \]  
(5.82)

Here $\eta_i, i = 0, 1, 2, 3$, are chiral parallel spinors defined on $M_i$, respectively $(D_{m_i} \eta_i = 0)$, obeying (5.79), (5.80) and (5.81); $\eta_4$ is constant.

Equations (5.81) have the following solutions

\[ c_{(0)} = c_1 c_2 c_3, \quad c_{(j)} = \pm ic_j, \]  
(5.83)

$j = 1, 2, 3$.

Thus, the number of linear independent solutions given by (5.82) and (5.83) is

\[ N = 32 N = n_0(c_1 c_2 c_3) \sum_{c = \pm 1} n_1(i c c_1) n_2(i c c_2) n_3(i c c_3), \]  
(5.84)

where $n_j(c_{(j)})$ is the number of chiral parallel spinors on $M_j, j = 0, 1, 2, 3$; see (5.79) and (5.80).

Here 2-dimensional manifolds $M_1, M_2$ and $M_3$ are flat.

Case $M_1 = M_2 = M_3 = \mathbb{R}^2$. Let $M_1 = M_2 = M_3 = \mathbb{R}^2$. Then all $n_j(i c) = 1, j = 1, 2, 3, c = \pm 1$, and hence we get from (5.84) we get

\[ N = \frac{1}{16} n_0(c_1 c_2 c_3). \]  
(5.85)

For $M_0 = \mathbb{R}^4$ we have $n_0(c) = 2$ and hence $N = 1/8$ for all values of $c_i, i = 1, 2, 3$.

Example: $M_0 = K3$. Let $M_0 = K3$ with $n_0(1) = 2$ and $n_0(-1) = 0$. Then we get $N = 1/8$ if $c_1 c_2 c_3 = 1$ and $N = 0$ if $c_1 c_2 c_3 = -1$.  

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6 Conclusions and discussions

In this paper we have considered the “Killing-like” SUSY equations in $D = 11$ supergravity for non-marginal $M$-brane solutions defined on the product of Ricci-flat manifolds $M_0 \times M_1 \times \ldots \times M_n$ [10, 11].

By proving the Proposition 1 we have found the solutions to these “Killing-like” equations which are defined up to the solutions to first-order differential equations: $\bar{D}^{(l)}_{m_l} \eta = 0$, $l = 0, \ldots , n$, with brane “chirality” conditions $\tilde{\Gamma}_{[l]} \eta = c_s \eta$ imposed. The operators $\bar{D}^{(l)}_{m_l}$ after a proper choice of $\Gamma$-matrices written in the tensor product form acts on 32-component spinor $\eta = \eta_0 \otimes \ldots \otimes \eta_n$ as following $\bar{D}^{(l)}_{m_l} \eta = \ldots \otimes \eta_{l-1} \otimes D^{(l)}_{m_l} \eta_l \otimes \eta_{l+1} \otimes \ldots$, where $D^{(l)}_{m_l}$ is the spinorial covariant derivative corresponding to the manifold $M_l$. Thus, the problem of finding the solutions to SUSY equations is reduced here to the search of chiral parallel spinors on factor-spaces $M_l$ and to the (technical) task of finding suitable sets of $\Gamma$-matrices written in the tensor product form corresponding to the product manifold $M_0 \times M_1 \times \ldots \times M_n$.

This program was successfully fulfilled here for the following brane configurations: $M_2, M_5, M_2 \cap M_2, M_2 \cap M_5, M_5 \cap M_5$ and $M_2 \cap M_2 \cap M_2$ and formulae for fractional numbers of unbroken supersymmetries $N$ were obtained. (The formulae for $M_2$-, $M_5$-brane solutions were obtained earlier in [15].) Here we have considered certain examples of partially supersymmetric configurations for various factor-spaces $M_i$.

In the next publications we plan to complete the list of all partially supersymmetric non-marginal $M$-brane configurations on product of Ricci-flat factor-spaces along a line as it was done by E. Bergshoeff et al [22] for $M_i = \mathbb{R}^{k_i}, i = 0, \ldots , n$. The results of this paper may be used in studies of partially supersymmetric solutions defined on product of Ricci-flat manifolds which take place in $IIA$, $IIB$ and other low-dimensional supergravities. One can extend this formalism to the so-called “pseudo-supersymmetric” p-brane solutions, suggested recently in [41].

Another topic of interest may be in analyzing of special solutions with moduli functions $H_s = C_s + Q_s/r^{d_0-2}$ ($C_s \geq 0, Q_s > 0$) which are defined on $(M_0, g^0)$ being a cone over certain Einstein space $(X, h)$, i.e. $g^0 = dr \otimes dr + r^2 h$ $(X = S^{d_0-1}$ for $M_0 = \mathbb{R}^{d_0})$. In the “near-horizon” case $C_s = 0$ the fractional numbers of unbroken SUSY might actually be larger (e.g. twice larger) then “at least” numbers $N$ obtained here for generic $H_s$-functions. Here we will get Freund-Rubin-type solutions with composite $M$-branes (see [42] and references therein), e.g. partially supersymmetric ones, that may of interest in a context of AdS/CFT approach and its modifications.
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Appendix

A The proof of relations (4.15) and (4.16)

Here we prove relation (4.15)

$$E_s = - \sum_{i \in I_s} d_i \phi^i = \ln H_s, \quad s \in S_e,$$

and relation (4.16)

$$M_s = - \gamma (d_0 - 2) - \sum_{i \in I_s} d_i \phi^i = - \ln H_s, \quad s \in S_m.$$

For the solution under consideration we have

$$\gamma = \sum_{s \in S} \gamma_s, \quad \gamma_s = a_s \ln H_s \quad (A.1)$$

and

$$\phi^i = \sum_{s \in S} \phi^i_s, \quad \phi^i_s = (a_s - \frac{1}{2} \delta^i_{I_s}) \ln H_s, \quad (A.2)$$

where \(a_s = 1/6\) for \(s \in S_e\) and \(a_s = 1/3\) for \(s \in S_m\).

Relation (4.15) and (4.16) just follow from the identities

$$E^s_{s'} = - \sum_{i \in I_s} d_i \phi^i_{s'} = \delta^s_{s'} \ln H_s, \quad s \in S_e,$$

and

$$M^s_{s'} = - \gamma_s (d_0 - 2) - \sum_{i \in I_s} d_i \phi^i_{s'} = - \delta^s_{s'} \ln H_s, \quad s \in S_m,$$

respectively.

Let us prove (A.3). We get

$$E^s_{s'} = E^s_{s'} \ln H_s,$$

where

$$E^s_{s'} = - \sum_{i \in I_s} d_i (a_{s'} - \frac{1}{2} \delta^i_{I_{s'}}) = - a_{s'} d(I_s) + \frac{1}{2} d(I_s \cap I_{s'}). \quad (A.6)$$
Here the following identity was used
\[ \sum_{i \in I} d_i \delta_j^i = d(I \cap J). \]  
(\text{A.7})

(See (2.5), (2.6)). In what follows the relation \( d(I_s) = 3 \) for \( s \in S_e \) is used.

In order to prove (A.3) one should verify the equality
\[ \mathcal{E}_s^{s'} = \delta_s^{s'}, \]  
(\text{A.8})

for \( s \in S_e \) and all \( s' \).

Let \( s' \in S_m \), then \( a_{s'} = 1/3 \) and \( d(I_s \cap I_{s'}) = 2 \) (due to intersection rules (4.6)). From (A.6) we get \( \mathcal{E}_s^{s'} = 0 \) in agreement with (A.8).

Let \( s' \in S_e \), then \( a_{s'} = 1/6 \) and \( d(I_s \cap I_{s'}) = 3 \) if \( s = s' \) and \( d(I_s \cap I_{s'}) = 1 \) if \( s \neq s' \) (due to intersection rules (4.6)). Hence we obtain from (A.6): \( \mathcal{E}_s^{s'} = 1 \) for \( s = s' \) and \( \mathcal{E}_s^{s'} = 0 \) for \( s \neq s' \) in agreement with (A.8). Thus, relation (A.6) is proved and hence relations (A.3) and (4.15) are also proved.

Now we prove (A.4). We get
\[ M_s^{s'} = M_s^{s'} \ln H_s, \]  
(\text{A.9})

where
\[ M_s^{s'} = -(d_0 - 2)a_{s'} - \sum_{i \in I_s} d_i(a_{s'} - \frac{1}{2} \delta_i^s), \]  
(\text{A.10})

\[ = -(d_0 - 2 + d(I_s)a_{s'}) + \frac{1}{2} d(I_s \cap I_{s'}). \]

Using the definition of the dual set (4.5) we have
\[ d(\bar{I}_s) = 11 - d_0 - d(I_s) \]  
(\text{A.11})

and
\[ d(\bar{I}_s \cap I_{s'}) = d(I_{s'}) - d(I_s \cap I_{s'}). \]  
(\text{A.12})

In what follows the relation \( d(I_s) = 6 \) for \( s \in S_m \) is used.

In order to prove (A.4) one should verify the equality
\[ M_s^{s'} = -\delta_s^{s'}, \]  
(\text{A.13})

for \( s \in S_m \) and all \( s' \).

Let \( s' \in S_e \), then \( a_{s'} = 1/6 \) and \( d(I_s \cap I_{s'}) = 2 \) (due to intersection rules (4.6)). From (A.10), (A.11) and (A.12) we obtain \( M_s^{s'} = 0 \) in agreement with (A.13).

Let \( s' \in S_m \), then \( a_{s'} = 1/3 \), \( d(I_{s'}) = 6 \) and \( d(I_s \cap I_{s'}) = 6 \) if \( s = s' \) and \( d(I_s \cap I_{s'}) = 4 \) if \( s \neq s' \) (due to intersection rules (4.6)). Hence we get from (A.10), (A.11) and (A.12): \( M_s^{s'} = -1 \) for \( s = s' \) and \( M_s^{s'} = 0 \) for \( s \neq s' \) in agreement with (A.13). Thus, relation (A.13) is proved and hence relations (A.4) and (4.16) are also proved.
B  The proof of the Proposition 1

Relations (2.26) and (2.27) may be written in a condensed form as follows

\[ D_{nl} = \tilde{D}_{nl}^{(l)} + \sum_{s \in S} A_{nl}^s, \]  

(B.14)

\[ l = 0, 1, \ldots, n, \ (m_0 = \mu) \] where

\[ A_{nl}^s = \frac{1}{4} \delta_0^i (\Gamma_{m_0} \Gamma^\nu - \Gamma^\nu \Gamma_{m_0}) \gamma_{s,\nu} + \frac{1}{2} (1 - \delta_0^i) \Gamma_{m_0} \phi_{s,\nu}, \]  

(B.15)

with \( \gamma_s \) and \( \phi_s^i \) defined in (A.1) and (A.2).

Due to ansatz (4.20) and (B.14) the SUSY equations (3.2) read

\[ [\tilde{D}_{nl}^{(l)} + \sum_{s \in S} (A_{nl}^s + B_{nl}^s + b_s \partial_{nl} \ln H_s)] \eta = 0, \]  

(B.16)

\[ l = 0, 1, \ldots, n, \] where \( b_s = -1/6 \) if \( s \in S_e \) and \( b_s = -1/12 \) if \( s \in S_m \).

Relation (B.16) is valid due to condition (4.21) (\( \tilde{D}_{nl}^{(l)} \eta = 0 \)) and the following identities

\[ (A_{nl}^s + B_{nl}^s + b_s \partial_{nl} \ln H_s) \eta = 0, \]  

(B.17)

\[ l = 0, 1, \ldots, n, \ s \in S, \] which are valid when the chirality restrictions (4.22) are imposed.

The verification of identities (B.17) is a straightforward one for electric and magnetic branes by using formulae (4.10) and (4.11).

C  Example: chiral parallel spinors on product of two 4-dimensional manifolds

Let us consider chiral parallel spinors on \( M_0 \times M_1 \), where \( M_0 \) and \( M_1 \) are 4-dimensional Ricci-flat manifolds of Euclidean signatures.

We consider \( \Gamma \)-matrices

\[ (\hat{\Gamma}^A) = (\hat{\Gamma}_{(0)}^{a_0} \otimes 1_4, \hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)}^{a_1}), \]  

(C.18)

where \( \hat{\Gamma}_{(0)}^{a_0} \), \( a_0 = 1_0, \ldots, 4_0 \), correspond to \( M_0 \), and \( \hat{\Gamma}_{(1)}^{a_1} \), \( a_1 = 1_1, \ldots, 4_1 \), correspond to \( M_1 \).

Here \( \hat{\Gamma}_{(0)} = \hat{\Gamma}_{(0)}^{1_0} \hat{\Gamma}_{(0)}^{2_0} \hat{\Gamma}_{(0)}^{3_0} \hat{\Gamma}_{(0)}^{4_0} \) obeys \( (\hat{\Gamma}_{(0)})^2 = 1_4 \).

We obtain

\[ \tilde{D}_{m_0}^{(0)} = \partial_{m_0} + \frac{1}{4} \omega_{a_0b_0m_0} \hat{\Gamma}_{(0)}^{a_0} \hat{\Gamma}_{(0)}^{b_0} \otimes 1_4, \]  

(C.19)

\[ \tilde{D}_{m_1}^{(1)} = \partial_{m_1} + \frac{1}{4} \omega_{a_1b_1m_1} \mathbf{1}_4 \otimes \hat{\Gamma}_{(1)}^{a_1} \hat{\Gamma}_{(1)}^{b_1}, \]  

(C.20)

Let

\[ \eta = \eta_0(x) \otimes \eta_1(y_1), \]  

(C.21)

where \( \eta_0 = \eta_0(x) \) is 4-component spinor on \( M_0 \) and \( \eta_1 = \eta_1(y_1) \) is 4-component spinor on \( M_1 \). Then

\[ \tilde{D}_{m_0}^{(0)} \eta = (D_{m_0}^{(0)} \eta_0) \otimes \eta_1; \]  

\[ \tilde{D}_{m_1}^{(1)} \eta = \eta_0 \otimes (D_{m_1}^{(1)} \eta_1), \]  

(C.22)
where $D^{(0)}_{m_0} = \partial_{m_0} + \frac{1}{4} \omega^{(0)}_{a_0 b_0 m_0} \hat{\Gamma}^{a_0} \hat{\Gamma}^{b_0}$ and $D^{(1)}_{m_1} = \partial_{m_1} + \frac{1}{4} \omega^{(1)}_{a_1 b_1 m_1} \hat{\Gamma}^{a_1} \hat{\Gamma}^{b_1}$ are covariant (spinorial) derivatives corresponding to the manifolds $M_0$ and $M_1$, respectively.

It follows from the Proposition 2 and (C.22) that $\eta = \eta_0(x) \otimes \eta_1(y_1)$ is parallel spinor if and only if $\eta_0$ and $\eta_1$ are parallel spinors on $M_0$ and $M_1$, respectively.

The chirality operator $\hat{\Gamma}$ (which is a product of all $\hat{\Gamma}^A$) reads

$$\hat{\Gamma} = \hat{\Gamma}^{(0)} \otimes \hat{\Gamma}^{(1)},$$

(C.23)

where $\hat{\Gamma}^{(1)} = \hat{\Gamma}^{a_1} \hat{\Gamma}^{b_1} \hat{\Gamma}^{c_1} \hat{\Gamma}^{d_1}$ obeys $(\hat{\Gamma}^{(1)})^2 = 1_4$.

If $\eta$ is chiral parallel spinor on $M_i$ with chirality $c_{(i)} = \pm 1$: $\hat{\Gamma}^{(i)} \eta_i = c_{(i)} \eta_i$, $i = 0, 1$, then $\eta = \eta_0 \otimes \eta_1$ is chiral parallel spinor on $M_0 \times M_1$, with the chirality $c = c_{(0)} c_{(1)}$: $\hat{\Gamma} \eta = c \eta$.

The generalization of this example to arbitrary even dimensions $d_0$ and $d_1$ is a straightforward one.

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