A new class of non-aligned Einstein-Maxwell solutions with a geodesic, shearfree and non-expanding multiple Debever-Penrose vector

Norbert Van den Bergh
Ghent University, Department of Mathematical Analysis FEA16,
Galglaan 2, 9000 Ghent, Belgium
E-mail: norbert.vandenbergh@ugent.be

Abstract. In a recent study of algebraically special Einstein-Maxwell fields[1] it was shown that, for non-zero cosmological constant, non-aligned solutions cannot have a geodesic and shearfree multiple Debever-Penrose vector $k$. When $\Lambda = 0$ such solutions do exist and can be classified, after fixing the null-tetrad such that $\Psi_0 = \Psi_1 = \Phi_0 = 0$ and $\Phi_1 = 1$, according to whether the Newman-Penrose coefficient $\pi$ is 0 or not. The family $\pi = 0$ contains the Griffiths solutions[2], with as sub-families the Cahen-Spelkens, Cahen-Leroy and Szekeres metrics. It was claimed in [2] and repeated in [1] that for $\pi = 0$ both null-rays $k$ and $\ell$ are non-twisting ($\bar{\rho} - \rho = \bar{\mu} - \mu = 0$): while it is certainly true that $\mu (\bar{\rho} - \rho) = 0$, the case $\mu = 0$ appears to have been overlooked. A family of solutions is presented in which $k$ is twisting but non-expanding.

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1. Introduction

In the quest for exact solutions of the Einstein-Maxwell equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = F_{ac} F_b^c - \frac{1}{4} g_{ab} F_{cd} F^{cd},$$

(1)

a large amount of research has been devoted to the study of aligned Einstein-Maxwell fields, in which at least one of the principal null directions (PND’s) of the electromagnetic field tensor $F$ is parallel to a PND of the Weyl tensor, a so called Debever-Penrose direction, see [6] and citations therein. A systematic attempt at classifying the algebraically special non-aligned solutions was initiated in [1]. One of the topics considered, dealing with the reverse of the Goldberg-Sachs theorem, enquired after the existence of algebraically special (non-conformally flat and non-null) Einstein-Maxwell fields with a possible non-zero cosmological constant for which the multiple Weyl-PND $k$ is geodesic and shear-free‡ ($\Psi_0 = \Psi_1 = \kappa = \sigma = 0$) and for which $k$ is not parallel to a PND of $F$ ($\Phi_0 \neq 0$). In order to avoid frequent referring to the equations of [1] I present the commutator relations, GHP,

‡ Throughout I use the sign conventions and notations of [6] §7.4, with the tetrad basis vectors taken as $k, \ell, m, \overline{m}$ with $-k^a \ell_a = 1 = m^a \overline{m}_a$. When using the Geroch-Held-Penrose formalism, I will write primed variables, such as $\kappa', \sigma', \rho'$ and $\bar{\tau}'$, as their Newman-Penrose equivalents $-\nu, -\lambda, -\mu$ and $-\pi$. 
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Bianchi and Maxwell equations in the Appendix and I repeat part of the reasoning of [1]: choosing a null-rotation about \( k \) such that \( \Phi_1 = 0 \), it follows that \( \Phi_2 \neq 0 \). Using the GHP formalism the Maxwell equations (64,65) and Bianchi equations (66-69) become then

\[
\begin{align*}
\partial\Phi_0 &= 0, \\
\partial'\Phi_0 &= -\pi\Phi_0, \\
\not\partial\Phi_0 &= 0, \\
\not\partial'\Phi_0 &= -\mu\Phi_0, \\
\partial\Phi_2 &= -\nu\Phi_0 + \tau\Phi_2, \\
\not\partial\Phi_2 &= -\lambda\Phi_0 + \rho\Phi_2, \\
\partial\Psi_2 &= -\pi\Phi_0\Phi_2 + 3\tau\Psi_2, \\
\not\partial\Psi_2 &= \mu|\Phi_0|^2 + 3\rho\Psi_2,
\end{align*}
\]

after which the commutators \([\partial', \not\partial], [\partial', \not\partial], [\not\partial, \not\partial']\) and \([\not\partial', \not\partial]\) applied to \( \Phi_0 \) give

\[
\begin{align*}
\partial\pi &= (3\rho - \overline{\rho})\mu - 2\Psi_2 + \frac{R}{12}, \\
\not\partial\pi &= 3\rho\pi, \\
\not\partial\mu &= \overline{\lambda}\pi + 3\mu\tau, \\
\not\partial\mu &= \pi(\overline{\pi} + 3\tau) + 2\Psi_2 - \frac{R}{12}.
\end{align*}
\]

Herewith GHP equation (63’) becomes a simple algebraic equation for \( \Psi_2 \),

\[
\Psi_2 = \rho\mu - \tau\pi + \frac{R}{12},
\]

the \( \not\partial \) derivative of which, using (11,13,57,59), results in \( \rho R = 0 \).

As \( \rho = 0 \) would imply \( \Phi_0 = 0 \), it follows that an algebraically special Einstein-Maxwell solution possessing a shear-free and geodesic multiple Debever-Penrose vector, which is not a PND of \( F \), necessarily has a vanishing cosmological constant. The corresponding class of solutions is non-empty, as it contains the Griffiths metrics[2], encompassing as special cases the metrics of [7, 8, 9, 10].

In [2] Griffiths claimed that for \( \pi = 0 \) both null-rays \( k \) and \( \ell \) are necessarily non-twisting \((\overline{\rho} - \rho = \overline{\mu} - \mu = 0)\). As a consequence it was also claimed in [1] that the Griffiths metrics are uniquely characterised by the condition \( \pi = 0 \). However, when \( \pi = 0 \) the only conclusion to be drawn from (10, 14) is that \( \mu(\overline{\rho} - \rho) = 0 \). When \( \rho \) is real this indeed leads to the metrics of [2], but the case \( \mu = 0 \) appears to have been overlooked and leads, as shown in the section below, to new classes of solutions\|. 

\( \Phi_2 = 0 \ell \) becomes geodesic and shear-free and the Goldberg-Sachs theorem implies \( \Psi_3 = \Psi_4 = 0 \). The Petrov type would then be D, in which case [4, 5] the only null Einstein-Maxwell solutions are given by the (doubly aligned) Robinson-Trautman metrics.

\( \| \) The case \( \mu = 0 \) is not to be regarded as a Kundt family, as the null ray generated by \( \ell \) is neither geodesic nor shear-free.

§ with \( \Phi_2 = 0 \ell \) becomes geodesic and shear-free and the Goldberg-Sachs theorem implies \( \Psi_3 = \Psi_4 = 0 \). The Petrov type would then be D, in which case [4, 5] the only null Einstein-Maxwell solutions are given by the (doubly aligned) Robinson-Trautman metrics.
2. The twisting and non-expanding family

When \( \pi = 0 \) and \( \mu = 0 \) the equations of the previous paragraph immediately imply \( \Psi_0 = \Psi_1 = \Psi_2 = 0 \) and \( \Psi_3 = \rho v - \lambda \tau \). As little progress appears to be possible in the general case, I restrict to solutions for which \( k \) is non-expanding (\( \rho + \rho = 0 \)). Acting on this condition with the \( \partial \) and \( \Phi \) operators, the GHP equations yield \( \tau = 0 \) and
\[
\rho^2 + |\Phi_0|^2 = 0, \tag{15}
\]
the \( \partial \) derivative of which implies \( \lambda = \Phi_2 \Phi_0 \rho^{-1} \). Translating these results into Newman-Penrose language and fixing a boost and spatial rotation in the \( k, \ell \) and \( m, \bar{m} \) planes such that \( \Phi_0 = 1 \) and \( \rho = i \), it follows that the only non-0 spin coefficients are \( \rho, v \) and \( \lambda = -i \Phi_2 \), with the only non-vanishing components of the Weyl spinor being \( \Psi_3 = iv \) and \( \Psi_4 \). As \( [D, \Delta] = 0 \) coordinates \( u, v \) and \( \xi, \bar{\zeta} \) exist such that \( D = \partial_u, \Delta = \partial_v \) and
\[
\delta = e^{-iu}(\xi \partial_\xi + \eta \partial_\eta + P \partial_u + Q \partial_v), \tag{16}
\]
\( \xi, \eta, P, Q \) being arbitrary functions. The \( e^{-iu} \) factor is included for convenience: applying the \( [\delta, D] \) commutator to \( u, v \) and \( \xi \) shows that \( \delta, \eta, P, Q \) are functions of \( v, \xi, \bar{\zeta} \) only. Introducing new variables \( n = e^{-iu}v \) and \( \phi = e^{-2iu} \Phi_2 \) it follows that also \( n \) and \( \phi \) depend on \( v, \xi, \bar{\zeta} \) only, with the full set of Jacobi and field equations reducing to the following system of pde’s:
\[
P_v + i\bar{P} \phi - n = 0, \tag{17}
\]
\[
Q_v + iQ \phi = 0, \tag{18}
\]
\[
\xi_v + i\bar{\eta} \phi = 0, \tag{19}
\]
\[
\eta_v + i\bar{\xi} \phi = 0, \tag{20}
\]
\[
e^{-iu} \delta P - e^{iu} \bar{\delta} \bar{P} - 2i|P|^2 = 0, \tag{21}
\]
\[
e^{-iu} \delta Q - e^{iu} \bar{\delta} \bar{Q} - 2i\Re(Q \bar{P} - 1) = 0, \tag{22}
\]
\[
e^{-iu} \delta \xi - e^{iu} \delta \bar{\eta} - i(\xi \bar{P} + \bar{\eta} P) = 0, \tag{23}
\]
\[
e^{iu} \delta n = -iPn + 2|\phi|^2, \tag{24}
\]
\[
e^{iu} \delta \phi = -2iP \phi - n, \tag{25}
\]
with the \( \Psi_4 \) component of the Weyl spinor given by \( \Psi_4 = ie^{2iu}(\bar{P} + \Delta \phi) + e^{iu} \delta n \).

3. The case \( \phi = \phi(\xi, \bar{\zeta}) \)

Assuming \( \phi = \phi(\xi, \bar{\zeta}) \), writing \( \phi = H^2 h^2 \) with \( H > 0 \) and \( |h| = 1 \), \((18,19,20)\) integrate to \( Q = q_1 e^{H^2} + q_2 e^{-H^2}, \xi = c_1 e^{H^2} + c_2 e^{-H^2}, \eta = ih^{2}(-c_1 e^{H^2} + c_2 e^{-H^2}) \) with \( q_A, c_A \) depending on \( \zeta, \bar{\zeta} \) only and \( q_J + ih^{-2}q_J = 0 \) \((J = 1, 2)\). A coordinate transformation \( \zeta \to \bar{\xi}(\xi, \bar{\zeta}) \) allows one then to put (writing \( \bar{\xi} = x + iy \) and re-defining \( q_J \)),
\[
\delta = e^{-iu}[\partial_u + e^{H^2} - e^{-H^2}c_1 - i\xi h^{-1}(\partial_x + q_1 \partial_y) + e^{H^2} - e^{-H^2}c_2 + i\xi h^{-1}(\partial_y + q_2 \partial_x)], \tag{26}
\]
with \( C_J \) and \( q_J \) real functions of \( x \) and \( y \).
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An expression for \( P \) is obtained from (23),

\[
P = e^{i \frac{\pi}{4} H^{-1} \left[ e^{H^2 \nu - C_1 \left( \frac{h_x}{h} - C_{2,x} - \nu (H^2)_x - q_1 H^2 \right)} + ie^{-H^2 \nu - C_2 \left( \frac{h_y}{h} - C_{1,y} + \nu (H^2)_y + q_2 H^2 \right)} \right]},
\]

(27)

after which \( n \) follows from (17),

\[
n = -2hH^2 \left\{ e^{H^2 \nu - C_1^{-i \frac{\pi}{4}}} \left[ (1 + 2\nu H^2) \frac{H_x}{H} + C_{2,x} + q_1 H^2 \right] + e^{-H^2 \nu - C_2^{-i \frac{\pi}{4}}} \left[ (1 - 2\nu H^2) \frac{H_y}{H} + C_{1,y} - q_2 H^2 \right] \right\}.
\]

(28)

Herewith (21) becomes a polynomial identity in powers of \( \nu \) and \( e^{H^2 \nu} \),

\[
v^2 e^{2H^2 \nu} H_x^2 - v^2 e^{-2H^2 \nu} H_y^2 + \ldots = 0,
\]

(29)

showing that \( H \) is necessarily constant.

Introducing new variables \( r_1 = -H^2 q_1 - C_{2,x}, \ r_2 = H^2 q_2 - C_{1,y} \) the remaining coefficients of (29) lead to the equations,

\[
\begin{align*}
    r_{1,x} - 2r_1^2 - r_1 (C_1 + C_2)_x &= 0, \\
    r_{2,y} - 2r_2^2 - r_2 (C_1 + C_2)_y &= 0.
\end{align*}
\]

(30, 31)

Substituting this in (28), equation (24) becomes a Liouville equation determining \( C_1 + C_2 \),

\[
(C_1 + C_2)_{xy} + 2H^2 e^{C_1 + C_2} = 0,
\]

(32)

while (25) reduces to an identity. A final equation is (22), which now becomes

\[
r_{2,x} + r_{1,y} - H^2 e^{C_1 + C_2} = 0.
\]

(33)

The general solution of the Liouville equation being given by

\[
e^{C_1 + C_2} = -\frac{a_x b_y}{H^2 (a + b)^2},
\]

(34)

\((a = a(x) \text{ and } b = b(y) \text{ arbitrary functions}), \ r_1 \text{ and } r_2 \text{ are given by (30,31) as,}

\[
\begin{align*}
    r_1 &= \frac{b_y}{2(a + b)(1 + A(a + b))}, \\
    r_2 &= \frac{a_x}{2(a + b)(1 + B(a + b))}.
\end{align*}
\]

(35)

with arbitrary functions \( A = A(x) \), \( B = B(y) \). Herewith (33) reduces to the condition

\[
\left( \frac{dA}{da} - A^2 \right)(1 + B(a + b))^2 + \left( \frac{dB}{db} - B^2 \right)(1 + A(a + b))^2 = 0,
\]

(36)

implying either \( A_{,a} - A^2 = B_{,b} - B^2 = 0 \), or \( \log \frac{1 + B(a + b)}{1 + A(a + b)} \) being separable in \( x \) and \( y \). As the latter condition again can be shown to imply \( A_{,a} - A^2 = B_{,b} - B^2 = 0 \), we conclude that the general solution is given by

\[
\begin{align*}
    r_1 &= \frac{a_x}{2(a + b)} \frac{k - k_0 b}{k + k_0 a}, \\
    r_2 &= \frac{b_y}{2(a + b)} \frac{k - k_0 a}{k + k_0 b},
\end{align*}
\]

(37)

with either \( k_0 = 1 \) and \( k \) an arbitrary (real) constant\( \| \), or \( k_0 = 0, k = 1 \) (corresponding to the special case \( A = B = 0 \)).

\( \| \) which can be taken to be 0 or 1
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The resulting metric appears to contain two arbitrary functions, being the phase factor $h(x, y)$ of $\Phi$ and the function $F(x, y)$ defined by $e^{C_1 - C_2} = -\frac{a_2}{b_3} e^{2F}$. These however can be eliminated by the coordinate transformations $u \rightarrow u - i \log h$ and $v \rightarrow H^2 v - F$, after which the dual basis takes the form,

\[
\omega^1 = \frac{e^{iu}}{2H(a + b)}(e^{i\frac{x}{2} - v} da - e^{-i\frac{x}{2} + v} db),
\]

\[
\omega^3 = \frac{1}{H^2}dv - \frac{1}{H^2(a + b)}[(2a + b)k_0 + k]da - \frac{(a + 2b)k_0 + k}{2(k_0 b + k)}db,
\]

\[
\omega^4 = du + \frac{1}{2(a + b)}[k_0 a - k]e^{-2v}da - \frac{k_0 b - k}{k_0 a + k}e^{2v}db,
\]

$(k_0, k) = (1, 0), (1, 1)$ or $(0, 1)$.

### 4. Discussion

The null tetrad (38-40) determines a (presumably) new family of Einstein-Maxwell solutions with zero cosmological constant and with Maxwell field and energy-momentum tensor given by

\[
F = iH^2(\omega^1 - \omega^2) \wedge \omega^3 + i(e^{-2iu}\omega^1 - e^{2iu}\omega^2) \wedge \omega^4,
\]

\[
T = 2H^2(e^{-2iu}\omega^1 \otimes \omega^1 + e^{2iu}\omega^2 \otimes \omega^2 + H^2\omega^3 \otimes \omega^3) + 2\omega^4 \otimes \omega^4.
\]

The Petrov type is III, with the multiple Debever-Penrose vector $k = \partial_u$ being geodesic, shear-free and twisting but non-expanding. The real null vector $\ell$, fixed by a null-rotation such that $\Phi_1 = 0$, is non-diverging, but is non-geodesic and has non-vanishing shear. It follows that figure (2) in [1] has to be amended as in Fig. 1 below.

For all solutions $\partial_u$ is clearly a null Killing vector. While in general (i.e. with $k$ and $k_0 \neq 0$) the isometry group is 2-dimensional, with the second Killing vector given by

\[
K_2 = k_0(a + b)\partial_v + (a^2k_0^2 - k^2)\partial_u - (b^2k_0^2 - k^2)\partial_b,
\]

the special cases $k_0 = 0, k = 1$ and $k_0 = 1, k = 0$ admit a 3-dimensional group of isometries, with third Killing vector

\[
K_3 = a\partial_u + b\partial_b.
\]

In the latter cases the isometry group has Bianchi type III, with the orbits being time-like hypersurfaces parametrized by the null coordinate $v$. For the case $k_0 = 0, k = 1$ the tetrad simplifies to

\[
\omega^1 = \frac{e^{iu}}{2H(a + b)}[e^{i\frac{x}{2} - v}da - e^{-i\frac{x}{2} + v}db],
\]

\[
\omega^3 = \frac{1}{H^2}[dv - \frac{1}{2(a + b)}d(a - b)],
\]

\[
\omega^4 = du - \frac{1}{2(a + b)}[e^{-2v}da - e^{2v}db]
\]
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Figure 1. Algebraically special non-null Einstein-Maxwell solutions for which the multiple Weyl-PND $k$ is not a PND of $F$.

and the line-element becomes

$$ds^2 = \frac{1}{H^2} \left[ -2dv + \frac{1}{a+b} d(a-b) ight] du + \frac{1}{H^2(a+b)} (e^{-2v} da - e^{2v} db) dv + \frac{\cosh 2v}{H^2(a+b)^2} dadb.$$  \hspace{1cm} (48)

The non-0 components of the Weyl spinor are then given by

$$\Psi_3 = -H^3 e^{i(u-v)} (e^v + ie^{-v})$$  \hspace{1cm} (49)

$$\Psi_4 = 2H^4 e^{2iu} \cosh 2v.$$  \hspace{1cm} (50)

All the Carminati-McLenaghan invariants are regular functions of the essential coordinate $v$ over the interval $]-\infty, +\infty[$. The same holds for the case $k_0 = 1, k = 0$ (in which the essential coordinate is $v + \log(b/a)$), but although all the (CM-) invariants are transformed into each other under the coordinate transformation $v \rightarrow v + \log(b/a)$ the two special cases are inequivalent.

$\partial_u$ being Killing vector, it might look peculiar that the Weyl spinor components $\Psi_3$ and $\Psi_4$ still depend on $u$, even though the frame was “invariantly” fixed. This is due to the fixation having been done by means of a null rotation putting $\Phi_0 = 1$: the resulting frame scalars are
then not genuine Cartan invariants and, as the Maxwell field itself does not inherit the space-time symmetries ($F$ is not Lie-propagated along the integral curves of the null Killing vector $\partial_u$), the frame scalars depend on $u$ as well. This remark also shows that the present solutions are distinct from the Lucács et al. solutions admitting null-Killing vectors [11], as there the Maxwell field does not inherit all the space-time symmetries.

**Appendix A**

Weights of the spin-coefficients, the Maxwell and Weyl spinor components and the GHP operators:

\[
\begin{align*}
\kappa : [3, 1], \nu : [-3, -1], \sigma : [3, -1], \lambda : [-3, 1], \\
\rho : [1, 1], \mu : [-1, -1], \tau : [1, -1], \pi : [-1, 1], \\
\Phi_0 : [2, 0], \Phi_1 : [0, 0], \Phi_2 : [-2, 0], \\
\Psi_0 : [4, 0], \Psi_1 : [2, 0], \Psi_2 : [0, 0], \Psi_3 : [-2, 0], \Psi_4 : [-4, 0], \\
\delta : [1, -1], \delta' : [-1, 1], \rho' : [-1, -1], \Phi : [1, 1].
\end{align*}
\]

The prime operation is an involution with

\[
\begin{align*}
\kappa' &= -\nu, \sigma' = -\lambda, \rho' = -\mu, \tau' = -\pi, \\
\Psi_0' &= \Psi_4, \Psi_1' = \Psi_3, \Psi_2' = \Psi_2, \\
\Phi_0' &= -\Phi_2, \Phi_1' = -\Phi_1.
\end{align*}
\]

The GHP commutators acting on $(p, q)$ weighted quantities are given by:

\[
\begin{align*}
[p, p'] &= (\pi + \tau)\delta + (\pi + \tau)\delta' + (\kappa \nu - \pi \tau + \frac{R}{24} - \Phi_{11} - \Psi_2)p \\
&\quad + (\kappa \nu - \pi \tau + \frac{R}{24} - \Phi_{11} - \Psi_2)q, \\
[\delta, \delta'] &= (\mu - \bar{\mu})\rho + (\rho - \bar{\rho})\rho' + (\lambda \sigma - \mu \rho - \frac{R}{24} - \Phi_{11} + \Psi_2)p \\
&\quad - (\lambda \sigma - \mu \rho - \frac{R}{24} - \Phi_{11} + \Psi_2)q, \\
[p, \delta] &= \bar{\pi}p - \kappa\rho' + \bar{\rho} \delta + \sigma \delta' + (\kappa \mu - \sigma \pi - \Psi_1)p \\
&\quad + (\kappa \lambda - \pi \rho - \Phi_{01})q.
\end{align*}
\]

GHP equations:

\[
\begin{align*}
\delta \rho - \delta' \kappa &= \rho^2 + \sigma \bar{\sigma} - \kappa \tau + \kappa \pi + \Phi_{00}, \\
\delta \sigma - \delta' \kappa &= (\rho + \bar{\rho}) \sigma + (\pi - \tau) \kappa + \Psi_0, \\
\delta \tau - \delta' \kappa &= (\tau + \bar{\tau}) \rho + (\bar{\tau} + \pi) \sigma + \Phi_{01} + \Psi_1, \\
\delta \nu - \delta' \pi &= (\pi + \tau) \mu + (\pi + \tau) \lambda + \Psi_3 + \Phi_{01} \Phi_2, \\
\delta \rho - \delta' \sigma &= (\rho - \bar{\rho}) \tau + (\mu - \bar{\mu}) \kappa + \Phi_{01} - \Psi_1.
\end{align*}
\]

+ Objects $x$ transforming under boosts and rotations as $x \rightarrow A_{\frac{p+q}{2}} e^{\frac{p}{2} \theta} x$ are called well-weighted of type $(p, q)$. 
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\[ b' \sigma - \bar{\sigma} \tau = - \sigma \mu - \bar{\lambda} \rho - \tau^2 + \kappa \nu - \Phi_{02}, \]  

(62)

\[ b' \rho - \bar{\rho} \tau = - \bar{\mu} \rho - \lambda \sigma - \tau \tau + \kappa \nu - \frac{R}{12} - \Psi_2. \]  

(63)

Maxwell equations:

\[ b \Phi_1 - \bar{\sigma} \Phi_0 = \pi \Phi_0 + 2 \rho \Phi_1 - \kappa \Phi_2, \]  

(64)

\[ b \Phi_2 - \bar{\sigma} \Phi_1 = - \lambda \Phi_0 + 2 \pi \Phi_1 + \rho \Phi_2. \]  

(65)

Bianchi equations (with \( \Phi_{IJ} = \Phi_I \Phi_J \) and \( \Lambda = R/4 = \text{constant} \)):

\[ \bar{b}' \Psi_0 - \bar{b} \Psi_1 + b \Phi_{01} - \bar{\sigma} \Phi_{00} = - \pi \Psi_0 - 4 \rho \Psi_1 + 3 \kappa \Psi_2 + \pi \Phi_{00} + 2 \bar{\rho} \Phi_{01} + 2 \sigma \Phi_{10} \]  

\[ - 2 \kappa \Phi_{11} - \bar{\kappa} \Phi_{02}, \]  

(66)

\[ b' \Psi_0 - \bar{\sigma} \Psi_1 + b \Phi_{01} - b' \Phi_{00} = - \mu \Psi_0 - 4 \tau \Psi_1 + 3 \sigma \Psi_2 - \bar{\lambda} \Phi_{00} + 2 \pi \Phi_{01} + 2 \sigma \Phi_{11} \]  

\[ + \bar{\rho} \Phi_{02} - 2 \kappa \Phi_{12}, \]  

(67)

\[ 3 \bar{b}' \Psi_1 - 3 \bar{b} \Psi_2 + 2 \bar{b} \Phi_{11} - 2 \bar{\sigma} \Phi_{10} + \bar{b}' \Phi_{01} - b' \Phi_{00} = 3 \lambda \Psi_0 - 9 \rho \Psi_2 + 6 \pi \Psi_1 + 6 \kappa \Psi_3 \]  

\[ + (\bar{\mu} - 2 \mu) \Phi_{00} + 2 (\tau + \bar{\pi}) \Phi_{01} + 2 (\tau - \pi) \Phi_{10} + 2 (\bar{\rho} - \rho) \Phi_{11} \]  

\[ + 2 \sigma \Phi_{20} - \sigma \Phi_{02} - 2 \bar{\kappa} \Phi_{12} - 2 \kappa \Phi_{21}, \]  

(68)

\[ 3 b' \Psi_1 - 3 b' \Psi_2 + 2 b' \Phi_{11} - 2 b' \Phi_{10} + \bar{b}' \Phi_{01} - b' \Phi_{00} = 3 \nu \Psi_0 - 6 \mu \Psi_1 - 9 \tau \Psi_2 + 6 \sigma \Psi_3 \]  

\[ - \nu \Phi_{00} + 2 (\bar{\mu} - \mu) \Phi_{01} - 2 \bar{\lambda} \Phi_{10} + 2 (\tau + 2 \pi) \Phi_{11} + (2 \pi + \tau) \Phi_{02} \]  

\[ + 2 (\bar{\rho} - \rho) \Phi_{12} + 2 \sigma \Phi_{21} - 2 \kappa \Phi_{22}. \]  

(69)

Acknowledgment

All calculations were done using the Maple symbolic algebra system, while the properties of the metric (48) were checked with the aid of Maple’s DifferentialGeometry package[12].

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