THE 8-UNIVERSALITY CRITERION IS UNIQUE

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Abstract. Using the methods developed for the proof that the 2-universality criterion is unique, we partially characterize criteria for the $n$-universality of positive-definite integer-matrix quadratic forms. We then obtain the uniqueness of Oh’s 8-universality criterion as an application of our characterization results.

1. Introduction

A degree-two homogenous polynomial in $n$ independent variables is called a quadratic form (or just form) of rank $n$. For a rank-$n$ quadratic form $Q(x_1, \ldots, x_n) = \sum_{i,j} a_{ij} x_i x_j$ (where $a_{ij} = a_{ji}$), the matrix given by $L = (a_{ij})$ is the Gram Matrix of a $\mathbb{Z}$-lattice $L$ equipped with a symmetric bilinear form $B(\cdot, \cdot)$ such that $B(L, L) \subseteq \mathbb{Z}$. Then, $Q(x) = x^T L x = B(L x, x)$ for $x \in \mathbb{R}^n$.

A rank-$n$ quadratic form $Q$ is said to represent an integer $k$ if there exists an $x \in \mathbb{Z}^n$ such that $Q(x) = k$. More generally, a $\mathbb{Z}$-lattice $L$ represents another $\mathbb{Z}$-lattice $\ell$ if there exists a $\mathbb{Z}$-linear, bilinear form-preserving injection $\ell \to L$. A quadratic form is called universal if it represents all positive integers. Analogously, a lattice is called $n$-universal if it represents all positive-definite integer-matrix rank-$n$ quadratic forms. Connecting these two notions of universality, we observe that a rank-$n$ quadratic form $Q$ is universal if and only if it is 1-universal, as for an integer $k$,

$$k = Q(x_1, \ldots, x_n) \iff Q(x_1 x, \ldots, x_n x) = kx^2.$$ 

In 1993, Conway and Schneeberger announced their celebrated Fifteen Theorem, giving a criterion characterizing the universal positive-definite integer-matrix quadratic forms. Specifically, they showed that any positive-definite integer-matrix form which represents the set of nine critical numbers

$$S_1 = \{1, 2, 3, 5, 6, 7, 10, 14, 15\}$$

is universal (see [C2, Bh]). Kim, Kim, and Oh [KKO2] presented an analogous criterion for 2-universality, showing that a positive-definite integer-matrix lattice is 2-universal if and only if it represents the set of forms

$$S_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \right\}.$$

Oh [Oh] gave a similar criterion for 8-universality, which we state in Theorem 3 of Section 4.
A set $\mathcal{S}$ of rank-$n$ lattices having the property that a lattice $L$ is $n$-universal if and only if $L$ represents every lattice in $\mathcal{S}$ is called an $n$-criterion set. Thus, for example, the set $\mathcal{S}_2$ obtained by Kim, Kim, and Oh [KKO2] is a 2-criterion set and the set $\mathcal{S}_1$ found by Conway [C2] naturally gives the 1-criterion set

$$\{x^2, 2x^2, 3x^2, 5x^2, 7x^2, 10x^2, 14x^2, 15x^2\}.$$ 

The set $\mathcal{S}_1$ of the Fifteen Theorem is known to be unique (see [KKO1]), in the sense that if $\mathcal{S}_1'$ is a set of integers such that a quadratic form is universal if and only if it represents the full set $\mathcal{S}_1'$, then $\mathcal{S}_1 \subseteq \mathcal{S}_1'$. The author [Ko] recently obtained an analogous uniqueness result for the 2-criterion set $\mathcal{S}_2$.

Kim, Kim, and Oh [KKO1] have proven that $n$-criterion sets exist for all positive integers $n$. However, the problems of finding and determining the uniquenesses of criterion sets have both proven to be difficult (see [KKO1]). Here, we advance both problems: We obtain the first characterization results for arbitrary $n$-criterion sets, from which we obtain the uniqueness of Oh’s 8-universality criterion as a corollary.

2. Notations and Terminology

We use the lattice-theoretic language of quadratic form theory. A complete introduction to this approach may be found in [O’M].

For a $\mathbb{Z}$-lattice (or hereafter, just lattice) $L$ with basis $\{x_1, \ldots, x_n\}$, we write $L \cong \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$. If $L$ is of the form $L = L_1 \oplus L_2$ for sublattices $L_1, L_2$ of $L$ and $B(L_1, L_2) = 0$ then we write $L \cong L_1 \perp L_2$ and say that $L_1$ and $L_2$ are orthogonal.

For a sublattice $\ell$ of $L_1 \perp L_2$ which can be expressed in the form

$$\ell \cong \mathbb{Z}(x_{1,1} + x_{2,1}) + \cdots + \mathbb{Z}(x_{1,n} + x_{2,n})$$

with $x_{i,j} \in L_i$, we denote $\ell(L_i) := \mathbb{Z}x_{i,1} + \cdots + \mathbb{Z}x_{i,n}$. We naturally extend this notation to lattices $\ell$ represented by $L_1 \perp L_2$. We then say that a lattice is additively indecomposable if either $\ell(L_1) \cong 0$ or $\ell(L_2) \cong 0$ whenever $L_1 \perp L_2$ represents $\ell$. Otherwise, we say that $\ell$ is additively decomposable.

Finally, we use the lattice notation of Conway [C1]. In particular, $I_n$ is the rank-$n$ lattice of the form $\langle 1, \ldots, 1 \rangle$ and $E_8$ is the unique even unimodular lattice of rank 8.

3. Characterization Results for $n$-criterion Sets

In this section, we prove two results which partially characterize the contents of arbitrary $n$-criterion sets.

**Proposition 1.** Any $n$-criterion set must include the lattice $I_n$.

**Proof.** If $T$ is a finite, nonempty set of rank-$n$ lattices not containing $I_n$, then every lattice $T \in T$ may be written in the form $T \cong I_k \perp T'$, where $0 \leq k < n$, the sublattice $T'$ is of rank $n-k$, and the first minimum of $T'$ is larger than 1. Indeed, any $I_k$-sublattice of $T$ is unimodular and therefore splits $T$; the condition on $T'$ follows from Minkowski reduction.

We may therefore write $T$ in the form

$$T = \bigcup_{k=0}^{n-1} \{I_k \perp T_{k,1}\}_{i=0}^{i_k},$$
where \(0 < |T| = \sum_{k=0}^{n-1} t_k\) and each \(T_{k,i}\) is a rank-\((n - k)\) lattice with first minimum greater than 1. Then, the lattice

\[
I_{n-1} \perp \left( \perp_{i=0}^{i=n_1} T_{0,i} \right) \perp \cdots \perp \left( \perp_{i=0}^{i=n-n-1} T_{n-1,i} \right)
\]

represents all of \(T\) but has does not represent \(I_n\). It follows that \(T\) is not an \(n\)-criterion set, so any \(n\)-criterion set must contain \(I_n\).

**Proposition 2.** Let \(E\) be the set of additively indecomposable lattices of rank \(n\). If \(|E| > 0\), then any \(n\)-criterion set must include at least one lattice \(E \in E\).

**Proof.** If \(T = \{T_i\}_{i=1}^k\) is a finite, nonempty set of rank-\(n\) lattices with \(T \cap E = \emptyset\), then every lattice \(T_i \in T\) is additively decomposable. It follows that the lattice

\[
T_1 \perp \cdots \perp T_k
\]

represents all of \(T\) but does not represent any lattice in \(E\), since \(T_1 \perp \cdots \perp T_k\) has no rank-\(n\) additively indecomposable sublattices. Thus, \(T\) is not an \(n\)-criterion set. It then follows that any \(n\)-criterion set must contain some lattice \(E \in E\). \(\square\)

**Remark.** It is clear that direct analogues of these two propositions hold in the more general setting of \(S\)-universal lattices discussed in \(\{KKO1\}\). In particular, suppose that \(S\) is an infinite set of lattices. Then, if \(n = \max \{k : I_k \in S\} > 0\), any finite set \(S_8 \subset S\) with the property that a lattice \(L\) represents every lattice \(\ell \in S_8\) if and only if \(L\) represents every \(\ell \in S\) must contain \(I_n\). Similarly, such a set \(S_8\) must contain an additively indecomposable lattice if \(S\) does.

### 4. Uniqueness of the 8-criterion Set

Oh obtained the following 8-criterion set in \([Oh]\) remark on Theorem 3.1:

**Theorem 3** (Oh). The set \(S_8 = \{I_8, E_8\}\) is an 8-criterion set.

The set \(S_8\) is clearly a minimal 8-criterion set, as for each \(\ell \in S_8\) there is a lattice which represents \(S_8 \setminus \ell\) but does not represent \(\ell\). (The single lattice in \(S_8 \setminus \ell\) suffices.) Meanwhile, our characterization results imply the following corollary which strengthens Theorem \(3\).

**Corollary 4.** Every 8-criterion set must contain \(S_8\) as a subset.

**Proof.** Since \(E_8\) is the unique additively indecomposable lattice of rank 8, the result follows directly from Propositions \(1\) and \(2\). \(\square\)

Corollary \(4\) when combined with Theorem \(3\) shows that \(S_8\) is the unique minimal 8-criterion set.

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