Quantum Field Theory of Open Spin Networks and New Spin Foam Models

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Abstract

We describe how a spin-foam state sum model can be reformulated as a quantum field theory of spin networks, such that the Feynman diagrams of that field theory are the spin-foam amplitudes. In the case of open spin networks, we obtain a new type of state-sum models, which we call the matter spin foam models. In this type of state-sum models, one labels both the faces and the edges of the dual two-complex for a manifold triangulation with the simple objects from a tensor category. In the case of Lie groups, such a model corresponds to a quantization of a theory whose fields are the principal bundle connection and the sections of the associated vector bundles. We briefly discuss the relevance of the matter spin foam models for quantum gravity and for topological quantum field theories.

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1 Introduction

The prototype of the state sum models was the Ponzano-Regge (PR) model of three-dimensional Euclidian quantum general relativity [23]. The idea was to construct a theory of quantum gravity by using the simplicial decomposition of the space-time manifold and group theory. The PR state sum was a sum of amplitudes for all possible colorings of the edges of a given triangulation with the irreducible representations (irreps) of \( SU(2) \). This sum was divergent, so that a simple, but topologically non-invariant cut-off procedure was used to regularize it. Twenty years later, Turaev and Viro discovered a topologically invariant way to regularize the PR state sum, by using the quantum \( SU(2) \) group [25]. Their motivation was not quantum gravity, but they wanted to find new three-manifold invariants.

Since the Turaev-Viro (TV) state sum gave new three-manifold invariants, it was natural to generalize the TV construction to four-manifolds, which was done by Crane and Yetter [9, 10]. Unfortunately, the Crane-Yetter (CY) state sum does not yield new manifold invariants. However, by modifying the CY state sum model, Barrett and Crane have obtained a non-topological model which describes a theory of four-dimensional quantum gravity [4, 5]. This is an interesting model since there are indications that it could have the general relativity as a part of its classical limit [6, 12]. Moreover, it has been proven that a slightly modified BC model [20, 21], has finite amplitudes for regular triangulations [22, 11].

On the other hand, the work on the canonical quantization of the connection formulation of general relativity, for a review and references see [24], has revealed the importance of the spin networks for quantum gravity. It was also realized that the spin networks are closely related to the spin foams [1], while the spin foams are the basic objects of the quantum gravity state-sum models [3].

The third important development was a realization that the spin-foam state sum models for Lie groups are the Feynman diagrams for a partition function of a field theory defined over the group [6, 18, 13]. Since the usual perturbative quantum field theories (QFT) have an operator formulation [26], it was not difficult to construct the states and the operators which give the transition amplitudes corresponding to the spin-foam amplitudes for the manifolds with boundaries [14]. These boundary states correspond to the boundary spin networks induced by the space-time spin foam, and it was shown in [10] that including the open-ended spin nets was equivalent to
introducing matter field operators. In this way one obtains a generalization of the spin-foam, which we call a matter spin-foam.

In this paper we give a more general description of the results obtained in [13, 16]. We show that these results can be understood in a simple way from a category theory perspective, and therefore they can be generalized to the case of tensor categories associated to quantum groups, or to other tensor categories.

2 Spin network amplitudes

Let $\Gamma$ be an oriented graph with a finite set of vertices $V$ and a finite set of edges $E$. We can label each edge $e \in E$ with an irreducible representation (irrep) $\Lambda$ of a Lie algebra $g$, and each vertex $v \in V$ with an intertwiner operator $i_v$ corresponding to the irreps of the edges which terminate or start from $v$. Such a graph is called a spin network, or a spin net for a short.

In order to be able to make generalizations, for example to the case of a Hopf algebra $U_q(g)$, one can give a more general definition of a spin net. It is a diagram representing a composition of the morphisms from a tensor category, where the lines represent simple objects from the category. In the case of the Lie algebra $g$, the relevant category is the category of representations of $g$, $\text{Cat}(g)$, or an appropriate tensor sub-category. This interpretation then allows us to associate a set of complex numbers to a spin net, which we will call a spin net amplitude, simply by evaluating the components of this morphism in the bases $\{e_\alpha\}$ for the vector spaces $\Lambda$ associated to the edges of $\Gamma$. In this way we obtain

$$A_{\beta_1... \beta_V}(\Gamma, \Lambda, i) = \sum_{\ldots} C_{\ldots(\alpha_1)} \cdots C_{\ldots(\alpha_k)} \cdots C_{\ldots(\beta_1)} \cdots C_{\ldots(\beta_V)} , \quad (1)$$

where $\Lambda = \{\Lambda_1, ..., \Lambda_E\}$ is the set of irreps of $g$ labeling the edges of $\Gamma$ and $i = \{i_1, ..., i_V\}$ is the set of intertwiners associated to the vertices of $\Gamma$. The sum in (1) is over the representation indices of the internal edges, while $\alpha, ...$ and $\beta, ...$ are the representation indices of the external edges. The numbers $C_{\ldots(\alpha_i)}$ are the components of the intertwiner at the vertex $v$. The up indices represent the domain of a morphism, and the down indices represent the image of the morphism. One can lower the up indices with the morphism $C_{\alpha\beta}$, which maps $\Lambda^*$ (dual of $\Lambda$) to $\Lambda$ ($e_\alpha = C_{\alpha\beta} e_\beta$), or rise the lower indices with the morphism $C^{\alpha\beta}$, which maps $\Lambda$ to $\Lambda^*$ ($e^\alpha = C^{\alpha\beta} e_\beta$). This means
that all the internal indices are contracted (for each down index there is an identical up index).

In order to evaluate (1), it is sufficient to now the three-vertex amplitude, or the Clebsh-Gordan coefficients $C_{\alpha}^{\beta\gamma(i3)}$. The higher-vertex amplitudes $C_{\ldots(i_n)}$, where $n > 3$, are determined by the minimal three-valent spin net with $n$ external edges.

One can also give a slightly more general definition of the spin net, by inserting in the edges non-trivial endomorphisms of the corresponding simple objects. This type of spin nets is obtained as a basis of square-integrable invariant functions on $G^E$, where $G$ is a Lie group whose Lie algebra is $g$ \(g\). In this case the edge endomorphism is given by the representation matrix $D_{\alpha}^\beta(g)$ of a group element $g$. Then the spin-net amplitude (1) is simply the value of the spin-net function at the point $g_1 = \ldots = g_E = e$ where $e$ is the group identity element, or a normalized integral of the spin-net function over $G^E$. This can be generalized to the case of quantum groups \([19]\). However, the simpler Lie algebra approach will be sufficient for our purposes. Still, it will be useful to be aware of the functions on the group approach in order to understand some of our constructions.

In the case of the quantum group, we use the category of representations of the Hopf algebra $U_q(g)$, so that the formula for the spin-net amplitude will be again (1), but now the intertwiner coefficients will be determined by the Clebsh-Gordan coefficients for the quantum group.

Hence the amplitude for an open spin net is not an invariant, but it is a tensorial quantity. The case of a closed spin net graph can be understood as a composition of morphisms between the trivial one-dimensional irreps. In this case one obtains an invariant.

The expression (1) will be always finite in the case of the category of finite-dimensional irreps. In the case of infinite-dimensional irreps, one may have to do some modifications to make it finite. The simplest modification is to use an edge endomorphism $\epsilon_{\alpha}^{\beta\delta}$. For example, the Feynman diagrams of relativistic particle quantum field theories \([26]\), are the spin net amplitudes for the category of unitary representations of the Poincare group $ISO(3,1)$. These irreps are infinte-dimensional, and the edge endomorphism is given by the Feynman propagator.

In the case of the BC state-sum models, a different type of modification occurs \([14]\). Since one can think of a spin net as an invariant function on $G$, one can also consider invariant functions on the coset space $G/H$, where $H$ is a subgroup of $G$ \([14]\). Consequently one obtains spin nets where the
edges are labelled only by the simple irreps $N$, i.e. the irreps which have an invariant vector under $H$. Hence the corresponding spin-net amplitudes can be expressed as multiple integrals of a propagator

$$K_N(x,y) = P_0 D^{(N)}(g_x^{-1}g_y)P_0,$$  \hspace{1cm} (2)$$

where $x$ and $y$ are the points in the coset space $G/H$, which label the vertices of the spin net graph, and $P_0$ is a projector from a simple irrep $N$ onto a trivial irrep of $H$.

For the BC model, one has $G = SO(4)$ in the Euclidian gravity case, or $G = SO(3,1)$ in the Lorentzian gravity case, and $H = SO(3)$. In the Lorentzian case, the unitary irreps of $SL(2,C)$ are used, which are infinite-dimensional. The Lorentzian spin net amplitudes require a trivial regulator which is a delta-function of one of the coset-space coordinates \cite{5}.

The coset space construction can be generalized to the case of open spin nets \cite{16}. Let $\Lambda$ be an irrep of $G$ which contains a finite-dimensional irrep $\tau$ of $H$. Then one can define a matrix function $K_{\Lambda,\tau}(x,y) \in \text{End}(\tau, \tau)$ on the coset space $G/H$ \cite{8}. Let $\Gamma$ be a spin net graph labelled with the irreps $\tau$ and the corresponding intertwiners $i$. One can label each vertex of $\Gamma$ with points from $G/H$, and associate to each internal edge a propagator $K_{\Lambda,\tau}(x_j, x_k)$.

Then a spin-net amplitude is given by

$$A(\Gamma, \Lambda, \tau, i) = \int \prod_{v \in V} dx_v \prod_{v \in V} C_{\ldots(i_v)} \prod_{e \in E_{int}} K_{\Lambda_e,\tau_e}(x_{v1(e)}, x_{v2(e)}) ,$$  \hspace{1cm} (3)$$

where $E_{int}$ is the set of internal edges of $\Gamma$.

### 3 Spin foam amplitudes

The state sum models which had been used for constructing $d$-dimensional quantum gravity and topological models were based on a two-complex $J = (V,E,F)$, consisting of a finite number of vertices $v \in V$, edges $e \in E$ and faces $f \in F$ \cite{11}. The two complex $J$ can be based on an arbitrary graph $(E,V)$, but it is usually taken to be the dual two-skeleton of a triangulation of the spacetime manifold $M$. One then labels the faces of $J$ with the irreps of $G$, or with the irreps of the $q$-deformed $G_q$, while the edges of $J$ will be labelled with the interwiners for the irreps of the faces that meet at a given edge. One can regard this object as a generalization of the spin net, where
the one-complex \( \Gamma = (V, E) \) is replaced by a set \((E, F)\) from the two complex 
J, so that \( J \) is called a spin foam \[1\].

An amplitude for a spin foam can be written as

\[
A(J) = \sum_{A, i} \prod_{f \in F} A_2(\Lambda_f) \prod_{e \in E} A_1(\Lambda_{f_1(e)} \cdots \Lambda_{f_{d}(e)}) \prod_{v \in V} A_0(\Lambda_{f_1(v)} \cdots \Lambda_{f_{d}(v)}) \quad , (4)
\]

where the sum is over the labelling set of the irreps and the intertwiners. \( A_2 \) is the amplitude for a face, and \( A_2 \) depends on the irrep of that face. \( A_1 \) is the amplitude for an edge, and \( A_1 \) depends on the irreps of the faces which share that edge. \( A_0 \) is the amplitude for a vertex of the two complex \( J \), and \( A_0 \) depends on the irreps of the faces which intersect at that vertex.

One can choose the amplitudes \( A_{0,1,2} \) to be the amplitudes for the spin nets one can associate to the faces, edges and the vertices of \( J \). A simple way to do this is to replace each edge in the graph \((V, E)\) by a bundle of \( d \) parallel lines, and then to connect at each vertex of \( J \) the \( d + 1 \) bundles of lines into a one-skeleton graph of a \( d \)-simplex. In this way, each face of \( J \) will correspond to a closed smooth loop of the "bundle" graph. Hence we can associate to each line in the "bundle" graph an irrep \( \Lambda \). Therefore \( A_2 \) will be the amplitude for the single loop spin net, which is \( \dim \Lambda \) for the finite irreps of \( G \), or \( \dim_q \Lambda \) for the irreps of \( G_q \). \( A_0 \) will be the amplitude for the \( d \)-simplex spin net, which will also depend on \( d \) intertwiners. For the edge amplitude there is some freedom, i.e. it could be one, or a given function of \( \Lambda_1, ..., \Lambda_d \).

In analogy to the spin net case, one can think of a \( d \)-dimensional spin foam as a composition of morphisms in a category where the intertwiner morphisms \( i_d \) from \( \Cat(g) \) belong to the set of objects. Hence a spin foam will be described by the \((d + 1)\)-valent graph, where the edges are labelled by the morphisms \( i_d \), and each vertex will represent the elementary spin-foam morphism among \( d + 1 \) objects \( i_d \). More generally, a spin foam is a spin net in a 2-category whose objects are the elements of a tensor category.

The expression \( (4) \) is generically not finite for the \( \Cat(g) \) case. When it is finite, then it defines a topological theory, which happens in \( d = 2 \) compact \( G \) case \[3\]. Compact \( G \) spin-foam amplitudes are not finite in \( d = 3 \) and \( d = 4 \) dimensions. In the quantum group case, when \( q \) is a root of unity then the corresponding tensor category has only a finite number of simple objects. Hence the corresponding spin-foam amplitude is finite. This gives a topological theory, i.e. the amplitude is invariant under the Pachner moves. TV and CY state-sum models are examples of such topological theories.
In analogy to the coset-space spin networks, one can also construct the coset-space spin foams, where the faces are labelled by the simple irreps, like in the case of the BC models \([4, 5]\). BC spin-foam amplitudes can be made finite by using the edge amplitude \(A_1 = A(\Theta_4)\), where \(\Theta_4\) is the theta graph with four edges \([22, 11]\). These models are relevant for quantum gravity, and they are not topological, because the simple irreps do not form a tensor category.

### 4 QFT of closed spin nets

In order to evaluate the spin-foam amplitude (4), we used the ”bundle graph” technique. However, this is the same as the Feynman diagram formalism from QFT which was developed in the field theory over the group approach \([7, 18, 13, 15]\).

Let us describe this in the \(d = 4\) case. Consider the action

\[
S[\phi] = \frac{1}{2} \sum_{\Lambda, J} Q_{\Lambda_1 \Lambda_2}(x_1, x_2) \phi_{\Lambda_1 \Lambda_2}^{\delta_1 \delta_2} + \frac{1}{5!} \sum_{\Lambda, J} V_{\Lambda_1 \cdots \Lambda_5}(x_1, \ldots, x_5) \phi_{x_1}^{\Lambda_1 \cdots \Lambda_5}
\]

\[
= S_0 + S_I ,
\]

(5)

where the complex numbers \(\phi_{x}^{\Lambda_1 \cdots \Lambda_4(i)}\) are tensors transforming like \(C^{\alpha_1 \cdots \alpha_4(i)}\). In the group theory approach, the modes \(\phi\) are the Fourier modes of the field \(f(g_1, ..., g_4)\), but in the general case these are tensors transforming like intertwiners. The vertex functions \(Q\) and \(V\) are given by

\[
Q^{-1}(x_1, x_2) = \delta(i_1, i_2) A_1(\Lambda_1, \ldots, \Lambda_4)
\]

\[
V(x_1, \ldots, x_5) = \delta^{10} A_0(\Lambda_1, \ldots, \Lambda_{10}; i_1, ..., i_5)
\]

(6)

(7)

where \(\delta\) are the appropriate delta functions of the indices.

The propagator \(Q^{-1}\) defines the edge endomorphism \(i_1 \rightarrow i_1\) and \(V\) defines the morphism \(i_1 \otimes ... \otimes i_5 \rightarrow C\). The amplitude for a composition of these morphisms is the same as a Feynman diagram for the perturbative Green’s functions for the path-integral

\[
\int \prod_i d\phi_i \exp(-S[\phi]) .
\]

(8)

In this way one obtains exactly (4).
Alternatively, the Fourier modes $\phi^{\vec{\alpha}}$ can be promoted into creation and annihilation operators $[\phi^{\vec{\alpha}}, \phi^{\dagger \vec{\beta}}] = \delta^{\vec{\alpha} \vec{\beta}}$, such that
\[
[\phi^{\vec{\alpha}}, \phi^{\dagger \vec{\beta}}] = \delta^{\vec{\alpha} \vec{\beta}},
\]
where $(\phi^{\dagger \vec{\alpha}})^{\dagger} = \phi^{\vec{\alpha}}$. These operators act in the Hilbert space with a basis
\[
\{|0\rangle, \phi^{\dagger \vec{\alpha}}|0\rangle, \phi^{\dagger \vec{\alpha}} \phi^{\dagger \vec{\beta}}|0\rangle, \cdots\},
\]
where the vacuum is defined by $\phi^{\vec{\alpha}}|0\rangle = 0$. The spin foam amplitude can be represented as an matrix element of an evolution operator $(S_I)^n$, where $n$ is the number of the spin-foam vertices, in analogy with the particle field theory case. As shown in [15], one can construct the “in” and the “out” states which describe the spatial spin nets corresponding to the boundary triangulations of the spacetime manifold. Such spin net states can be represented by
\[
|\gamma\rangle = \prod_{v \in V(\gamma)} \phi^{\dagger \vec{\alpha}_v}|0\rangle,
\]
where $\gamma$ is a four-valent spin net graph dual to the boundary triangulation, and all the representation indices $\alpha_v$ are appropriately contracted.

In the quantum group at root of unity case, one has a well-defined topological QFT . The formula (10) then gives a map from a boundary manifold to a Hilbert space. This TQFT should satisfy a generalization of Atiyah’s axioms for manifolds equipped with a principal $G$-bundle.

5 QFT of open spin nets

The construction in the previous section describes only the states associated to closed spin nets. In the Lie group case it is known that the closed spin nets are related to the connections on the principal bundle $(M,G)$. The matter fields can be considered as sections of the associated vector bundles $(M,\Lambda_s)$, where $\Lambda_s$ is a finite-dimensional representation of $G$. Then it is natural to consider an open spin net, where a free edge at a vertex $v$ carrying the representation $\Lambda_s$ can be interpreted as a matter field of "spin" $\Lambda_s$ at $v$. Hence the transition amplitude between two open $(d-1)$-dimensional spin nets will be described by a $d$-dimensional spin foam interpolating between them. This spin foam can be thought of as an morphism in a category where the objects are the intertwiners $i_d$ and the matter irreps $\Lambda_s$. Therefore this
new morphism, which we will call a matter spin foam \footnote{16}, will be described by a graph where the edges are labelled by $i_d$ or by $\Lambda_s$.

We will only consider the graphs where the sub-graphs formed by the edges labelled by $i_d$ correspond to triangulations of the space-time manifold and the edges labelled by the matter irreps connect only the vertices of that subgraph. This corresponds to the physical requirement that the matter fields always propagate in a space-time background. Hence the basic spin-foam vertices will have $d+1$ edges labeled with the intertwiner morphisms $i_d$ and $n$ edges labelled by the matter irreps of $G$. The corresponding amplitude will be determined by the spin net with $n$ external edges labelled by the matter irreps, while the internal edges labelled by the background irreps will form a $(d+1)$-simplex spin net.

Let us consider the case of a single matter irrep $\Lambda_s$ and $d = 4$. In the first quantization formalism \footnote{17, 2}, the matter can be introduced by replacing the spin net function $\gamma$ with an open spin net function $\gamma_s$, which is obtained from $\gamma$ by putting an external edge carrying a matter irrep $\Lambda_s$ at each site of $\gamma$ where a matter quantum is located.

In the QFT formalism \footnote{16}, one can construct the state $|\gamma_s\rangle$ by introducing the matter creation and annihilation operators $\psi_\sigma(\times_v)$ and $\psi_\sigma^\dagger(\times_v)$, where $\sigma$ is the representation index of $\Lambda_s$, and the label $\times_v$ denotes the intertwiner of the spin net site where the matter quantum is located. Hence the corresponding state would be given as

$$|\gamma_s\rangle = \prod_{v' \in V'(\gamma)} \prod_{\sigma,\sigma'} \psi_{\sigma,\sigma'}(\times_{v'}) \prod_{v \in V(\gamma)} \phi_{\times_v} |0\rangle , \quad (11)$$

where $V'$ is the set of vertices where the matter quanta are located.

The corresponding matter spin foam amplitude can be expressed as a matrix element in a QFT determined by the action $S[\phi] + S_s[\psi, \phi]$, where

$$S_s = \sum_\Lambda \psi_\sigma(x)\psi_{\sigma'}(x') \Psi^{\sigma\sigma'}(x, x', x_1, \ldots, x_5) \phi_{x_1} \cdots \phi_{x_5} + (h.c.) , \quad (12)$$

and (h.c.) stands for the hermitian conjugate term. Hence

$$\Psi^{\sigma\sigma'}(x, x', x_1, \ldots, x_5) = \delta^{10} \delta_{x,x_1} \delta_{x',x_k} A^{\sigma\sigma'}(A_1, \ldots, A_{10}) , \quad (13)$$

where $A^{\sigma\sigma'}$ is a spin net amplitude for the pentagram with two external edges attached at the vertices 1 and $k$, where $1 \leq k \leq 5$. In the quantum gravity models \footnote{16}, this graph is zero, and one needs to consider a modified graph with an extra internal matter edge between the vertices 1 and $k$ ($k \neq 1$).
We will also add to $S_s$ a purely quadratic matter term
\[ \sum_{\Lambda} \psi_{\sigma}(x) Q^{\sigma\sigma'}(x, x') \psi_{\sigma'}(x') + (h.c.) , \] (14)
in order to have a well-defined perturbative expansion. The propagator $Q^{-1}$ will be determined by the endomorphism $i_x \otimes \Lambda_s \rightarrow i_x \otimes \Lambda_s$. We denote the propagator amplitude as $G_{\sigma\sigma'}$.

Note that $(\psi_{\sigma})^* = \psi^\sigma = C^{\sigma\sigma'} \psi_{\sigma'}$, and the vector space duality $*$ is in general different from the complex conjugation (reality) properties. The reality properties determine relation between the creation and annihilation operators $\psi$ and $\psi^\dagger$, see [15].

Hence a relevant Feynman diagram $\Gamma_s$, which is generated by the action $S + S_s$, will be determined by a spacetime skeleton diagram $\Gamma$ (a 5-valent graph dual to a spacetime triangulation) plus a line connecting the vertices of $\Gamma$. Therefore $\Gamma_s$ will be a graph consisting of 5-valent and 7-valent vertices. Out of these diagrams we will consider only those for which the matter irreps form a line which connects the centers of a string of the adjacent 4-simplices, extending from the initial to the final boundary. In this case, the matter spin foam $J_s$ will be given by the usual spin foam $J$ and a line $L$ of matter edges with two external edges. The corresponding amplitude will be given by
\[ A_{\sigma_i}^{\sigma_i'}(J_s) = \sum_{\Lambda} \prod_{f \in F} A_f(\Lambda) A_{\sigma_i}^{\sigma_i'}(L) \prod_{e \in E'} A_e(\Lambda) \prod_{v \in V'} A_v(\Lambda) , \] (15)
where
\[ A_{\sigma_i}^{\sigma_i'}(L) = A_{\sigma_i}^{\sigma_i'}(v_1) G_{\sigma_1}^{\sigma_2}(e_{12}) A_{\sigma_2}^{\sigma_3}(v_2) \cdots G_{\sigma_{k-1}}^{\sigma_k}(e_{k-1,k}) A_{\sigma_k}^{\sigma_i}(v_k) . \] (16)
The number of the initial and the final matter quanta has to be the same for a free theory, and the above formula can be then easily generalized to the case when there are several lines $L$, or a loop of matter edges.

One can also take a set of different irreps $\Lambda_{s_1} = S_1, \ldots, \Lambda_{s_k} = S_k$ to represent fields of different spins. The corresponding QFT action can be written as
\begin{align*}
S_m &= \sum_S \sum_{\Lambda} \psi_S(x) \psi_S(x') Q_{SS}(x, x') \\
&\quad + \sum_S \sum_{\Lambda} \psi_S(x) \psi_S(x') V_{SS}(x, x', x_1, \ldots, x_5) \phi_{x_1} \cdots \phi_{x_5} \\
&\quad + \sum_S \sum_{\Lambda} \psi_S(x) \cdots \psi_S(x^{(k)}) V_{S_1\ldots S_k}(x, \ldots, x^{(k)}; x_1, \ldots, x_5) \phi_{x_1} \cdots \phi_{x_5} \\
&\quad + (h.c.) , \tag{17}
\end{align*}

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where the vertex $V$ will be determined by the spin net amplitudes for the open graphs based on the pentagram. A new feature is that the set of possible graphs for the vertex $V_{S_1...S_k}$ will contain graphs with more than five vertices, i.e. there will be vertices formed by the matter irreps describing the interactions among the matter fields $[16]$.

The corresponding Feynman diagrams, or the spin-foam amplitudes, will be generically divergent in the Lie group case. By going to tensor categories with finitely many simple objects, like those for quantum groups at root of unity, one will obtain finite numbers. Since such amplitudes are topologically invariant when there are no matter irreps, then it will be interesting to explore the topological invariance of the corresponding amplitudes with the matter irreps, since one then expects to obtain topological models.

Another way of obtaining finite expressions is by using the coset space spin networks. The corresponding models will be relevant for quantum gravity with matter fields $[16]$, and they will not be topological. This requires a specification of the irreps $s$ of $H$ which are contained in the irreps $S$ of $G$. In that case the matter fields will have the indices from the $H$ irreps, and the action can be written as

$$S_m = \sum_S \sum_N \psi_s(x)\psi_s(x')Q_{ss}^S(x, x') + \sum_S \sum_N \phi_{s_1}(x')\phi_{s_k}(x^{(k)})V_{s_1...s_k}^S(x,...,x^{(k)};x,...,5)\phi_{x_1}...\phi_{x_5} \quad ,$$

where $N$ are the simple background irreps and the vertices $Q$ and $V$ will be determined by the spin net amplitudes for open graphs based on the $\Theta_4$, $\Theta_5$ and the pentagram graphs. However, these amplitudes will be now given as multiple integrals of the propagators $K_N(x, y)$ and $K_{S,s}(x, y)$, as in the equation (3).

In the case of the Lorentzian BC model, $G = SO(3, 1)$, $H = SO(3)$, and the background irreps are the simple irreps $N = (0, \rho)$, $\rho \geq 0$, which are unitary and infinite dimensional. The matter can be introduced by using the finite-dimensional irreps of $SO(3, 1)$ $[16]$, which can be labelled as $(j, k)$, $j, k \in \frac{1}{2}\mathbb{Z}_+$, and these irreps are non-unitary. To each $S = (j, k)$ irrep, we will associate a $SU(2)$ irrep $s$ contained in $S$. Hence one will have spin foams...
where the faces will be labelled with the $N$ irreps, while the edges will be labelled by the $S$ and $s$ irreps.

6 Concluding remarks

Note that by generalizing the notion of the spin network as a composition of morphisms among simple objects from a tensor category, one is able to generalize the notion of a spin foam as a spin net in a 2-category. In this way one obtains matter spin foams, where the edges of the graph are labelled by the simple objects or by the intertwiners. If we think of this graph as a dual one-skeleton of a triangulation, then in the Lie group case, we label the edges with the irreps of $G$, or with the intertwiners, while the faces (loops of the graph) are labelled by the irreps corresponding to the edge intertwiners. It would be interesting to explore the topological properties of these amplitudes, especially in the quantum group case, where the amplitudes are finite. This may give topological invariants for manifolds equipped with principal and associated vector bundles.

In the case of the BC model with matter, we expect that the amplitudes are finite, because the amplitudes without matter are finite, and adding matter introduces the matter propagators $K_{s,s}$, which are similar functions of the coset space points as the $K_N$ propagators \cite{16}. In addition, one can consider the amplitude \cite{16} as an amplitude for matter propagation in a fixed set of the background irreps, which could be interpreted as an amplitude of a quantum field theory in a curved background.

References

[1] Baez J, Spin Foam Models, Class. Quant. Grav. 15 (1998) 1827-1858.

[2] Baez J and Krasnov K, Quantization of Diffeomorphism-Invariant Theories with Fermions, J. Math. Phys. 39 (1998) 1251-1271.

[3] Baez J, An Introduction to Spin Foam Models of Quantum Gravity and BF Theory, Lect. Notes Phys. 543 (2000) 25-94.

[4] Barrett J W and Crane L, Relativistic Spin Networks and Quantum Gravity, J. Math. Phys. 39 (1998) 3296-3302.
[5] Barrett J W and Crane L, *A Lorentzian Signature Model for Quantum General Relativity*, Class. Quant. Grav. **17** (2000) 3101-3118.

[6] Barrett J W and Williams R M, *The Asymptotics of an Amplitude for the 4-simplex*, Adv. Theor Math. Phys. **3** (1999) 209-215.

[7] Boulatov D V, *A Model of Three-Dimensional Lattice Gravity*, Mod. Phys. Lett. A **7** (1992) 1629-1646.

[8] Camporesi R and Higuchi A, *On the Eigenfunctions of the Dirac Operator on Spheres and Real Hyperbolic Spaces*, J. Geom. Phys. **15** (1996) 1

[9] Crane L and Yetter D N, *A Categorical Construction of 4d Topological Quantum Field Theory*. In: Quantum Topology, L H Kauffman and R Baadhio (Eds), World Scientific, Singapore, 1993.

[10] Crane L, Kauffman L H and Yetter D N, *State-Sum Invariants of 4-Manifolds I*, J. Knot Th. Ramif. **6** (1997) 177-234.

[11] Crane L, Perez A and Rovelli C, *A Finiteness Proof for the Lorentzian State Sum Spinfoam Model for Quantum General Relativity*, Phys. Rev. Lett. **87** (2001) 181301

[12] De Pietri R and Freidel L, *so(4) Plebanski Action and Relativistic Spin Foam Model*, Class. Quant. Grav. **16** (1999) 2187-2196.

[13] De Pietri R, Freidel L, Krasnov K and Rovelli C, *Barrett-Crane Model from a Boulatov-Ooguri Field Theory over a Homogeneous Space*, Nucl. Phys. B **574** (2000) 785

[14] Freidel L and Krasnov K, *Simple Spin Networks as Feynman Graphs*, J.Math.Phys. **41** (2000) 1681-1690.

[15] Miković A, *Quantum Field Theory of Spin Networks*, Class. Quant. Grav. **18** (2001) 2827-2851.

[16] Miković A, *Spin Foam Models of Matter Coupled to Gravity*, hep-th/0108099

[17] Morales-Tecotl H A and Rovelli C, *Fermions in Quantum Gravity*, Phys. Rev. Lett. **72** (1994) 3642-3645.
[18] Ooguri H, Topological Lattice Models in Four Dimensions, Mod. Phys. Lett. A 7 (1992) 2799-2810.

[19] Oeckl R, Generalized Lattice Gauge Theory, Spin Foams and State Sum Invariants, hep-th/01100259

[20] Perez A and Rovelli C, A Spin Foam Model without Bubble Divergences, Nucl. Phys. B 599 (2001) 255

[21] Perez A and Rovelli C, Spin Foam Model for Lorentzian General Relativity, Phys. Rev. D 63 (2001) 041501

[22] Perez A, Finiteness of a Spinfoam Model for Euclidean Quantum General Relativity, Nucl. Phys. B 599 (2001) 427

[23] Ponzano G and Regge T, Semiclassical Limit of Racah Coefficients. In: Spectroscopy and Group Theoretical Methods in Physics, F. Block et al (Eds), North-Holland, Amsterdam, 1968.

[24] Rovelli C, Loop Quantum Gravity, Living Rev. Relativity, 1 (1998) 1-34.

[25] Turaeev V G and Viro O Y, State Sum Invariants of 3-manifolds and Quantum 6j-symbols, Topology 31 (1992) 865-902.

[26] Weinberg S, The Quantum Theory of Fields, vol. 1, Cambridge University Press, Cambridge, 1995.