A Strong threshold for the size of random caps to cover a sphere.

by

Bhupendra Gupta
Faculty of Engineering and Sciences,
Indian Institute of Information Technology (DM)-Jabalpur, India.

Abstract

In this article, we consider ‘N’ spherical caps of area $4\pi p$ were uniformly distributed over the surface of a unit sphere. We are giving the strong threshold function for the size of random caps to cover the surface of a unit sphere. We have shown that for large $N$, if $\frac{Np}{\log N} > 1/2$ the surface of sphere is completely covered by the $N$ caps almost surely, and if $\frac{Np}{\log N} \leq 1/2$ a partition of the surface of sphere is remains uncovered by the $N$ caps almost surely.

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1Corresponding Author. email: gupta.bhupendra@gmail.com, bhupen@iiitdm.in
1 Introduction.

Let $V_1, V_2, \ldots, V_N$ be the spherical caps on the surface of a unit sphere with their centers $v_1, v_2, \ldots, v_N$ respectively, on the surface of a unit sphere. Also let $v_1, v_2, \ldots, v_N$ are independently and uniformly distributed on the surface of a unit sphere. H. Maehara, [2] gives the threshold function $p_0(N) = \frac{\log N}{N}$ for the coverage of the surface of a unit sphere. H. Maehara, proves that for $\frac{p(N)\cdot N}{\log N} < 1$, probability that $N$ spherical caps cover the entire surface of the unit sphere is converges to 0, and for $\frac{p(N)\cdot N}{\log N} > 1$, probability that each point of the sphere is covered by $n$ caps is converges to 1. Since both of these are convergence in probability sense, the threshold $p_0(N)$ is a weak threshold. Also in article [2], instead of exact bounds author use lose approximations. Due to these approximations threshold suggested in [2] is different from threshold suggested in this article.

Now using the same model and notations as in H. Maehara, [2], we are giving the strong threshold function for the coverage of the surface of a unit sphere.

2 Basic Model and Definitions.

Here, we recall the same model as it is given by H. Maehara, [2]. We made some modification in the language for making the things more clear.

Let ‘$S$’ be the surface area of a unit sphere in 3-dimensional space. Let $V_1, V_2, \ldots, \text{be the spherical caps on the surface of a unit sphere with their centers } v_1, v_2, \ldots, \text{respectively, and uniformly distributed on the surface of a unit sphere. The area of a spherical cap of angular distance (angular radius) ‘$a$’ is } 2\pi(1 - \cos(a)) = 4\pi \sin^2(a/2)$.

Let ‘$p$’ be the probability that any point on the surface of unit sphere covered by a
specified spherical cap of angular distance (angular radius) ‘a’. Then

\[ p := \frac{\text{Area of a spherical cap of angular distance ‘a’}}{\text{Surface area of unit sphere}} = \frac{4\pi \sin^2(a/2)}{4\pi} = \sin^2(a/2) \quad (2.1) \]

Let there are ‘N’ random caps of angular distance ‘a’ on the surface ‘S’ of unit sphere. Let \( U_0(N, p) \), be the set of those points which remains uncovered by ‘N’ spherical caps and \( u_0(N, p) \) be the proportion of the area covered by \( U_0(N, p) \), i.e., \( u_0(N, p) \) be the proportion of the area which remains uncovered by ‘N’ spherical caps:

\[ u_0(N, p) := \frac{\{\text{the area of } U_0(N, p)\}}{4\pi}. \quad (2.2) \]

Then

\[ E(u_0(N, p)) = \frac{1}{4\pi} \int_S P[x \in U_0(N, p)]dx. \quad (2.3) \]

Now consider,

\[ P[x \in U_0(N, p)] = P[x \text{ is remains uncovered }] = \prod_{i=1}^{N} (1 - P[x \in V_i]) = (1 - p)^N. \quad (2.4) \]

Hence, from (2.3), we have

\[ E(u_0(N, p)) = (1 - p)^N. \quad (2.5) \]

Similarly, as in H. Maehara, [2], we have

\[ E(u_0^2(N, p)) = \frac{1}{16\pi^2} \int_S \int_S P[x, y \in U_0(N, p)]dx = \frac{1}{4\pi} \int_S P[x_0, y \in U_0(N, p)]dy, \quad (2.6) \]

where \( x_0 \) is a fixed point on \( S \). Let \( x_0 \) and \( y \) subtend an angle ‘\( \theta \)’ at the center of sphere. Then

\[ P[x_0, y \in U_0(N, p)] = (1 - (2p - q(\theta)))^N, \quad (2.7) \]

where, \( q(\theta) \) be the area of intersection between two spherical caps of angular distance ‘a’. Substituting the above probability in (2.6), we get

\[ E(u_0^2(N, p)) = \frac{1}{4\pi} \int_S (1 - (2p - q(\theta)))^N dy. \]
Since points \( x_0 \) and \( y \) subtend an angle between \( \theta \) and \( \theta + d\theta \) at the center of the sphere. Then

\[
E(u_0^2(N, p)) = \int_0^\pi (1 - (2p - q(\theta)))^N (1/2) \sin(\theta) d\theta. \tag{2.8}
\]

Since \( q(\theta) = 0 \) for \( \theta > 2a \),

\[
E(u_0^2(N, p)) < \int_0^{2a} (1 - p)^N (1/2) \sin(\theta) d\theta + \int_{2a}^\pi (1 - 2p)^N (1/2) \sin(\theta) d\theta \\
< (1 - p)^N \left[ - (1/2) \cos(\theta) \right]_0^{2a} + (1 - 2p)^N \\
= (1 - p)^N \frac{1 - \cos(2a)}{2} + (1 - 2p)^N. \tag{2.9}
\]

Using (2.5), in (2.9), we have

\[
E(u_0^2(N, p)) < (1 - p)^N \frac{1 - \cos(2a)}{2} + (1 - 2p)^N \\
= 4p(1 - p)^{N+1} + (1 - 2p)^N, \tag{2.10}
\]

since \( \frac{1 - \cos(2a)}{2} = 4p(1 - p) \).

Now for the lower bound of \( E(u_0^2(N, p)) \), from (2.8) we have

\[
E(u_0^2(N, p)) > \int_{2a}^\pi (1 - (2p - q(\theta)))^N (1/2) \sin(\theta) d\theta, \tag{2.11}
\]

since \( q(\theta) = 0 \) for \( \theta > 2a \).

\[
E(u_0^2(N, p)) > \frac{(1 - 2p)^N}{2} \int_{2a}^\pi \sin(\theta) d\theta \\
= \frac{(1 - 2p)^N}{2} \left[ - \cos(\theta) \right]_0^{2a} \\
= (1 - 2p)^N \frac{\cos(2a) + 1}{2} \\
= (1 - 2p)^N (1 - 4p(1 - p)), \tag{2.12}
\]

since \( \frac{\cos(2a) + 1}{2} = 1 - 4p(1 - p) \).

Let \( \Theta \) be a fixed monotone property.
Definition 1 A function $δ_Θ(c) : Z^+ \rightarrow R^+$ is a strong threshold function for $Θ$ if the following is true for every fixed $\epsilon > 0$,

- if $P[δ_Θ(c - \epsilon) \in Θ] = 1 - o(1)$, and
- if $P[δ_Θ(c + \epsilon) \in Θ] = o(1)$,

where ‘c’ is some constant.

3 Main Result.

**Theorem 3.1** Let $p = \frac{c \log N}{N}$, then for $c > \frac{1}{2}$, we have the surface of unit sphere is completely covered by ‘N’ spherical caps, i.e.,

$$U_0(N, p) = \phi, \quad \text{almost surely},$$

and for $c \leq \frac{1}{2}$, surface of unit sphere is not completely covered by ‘N’ spherical caps, i.e.,

$$U_0(N, p) \neq \phi, \quad \text{almost surely}.$$  

**Proof.** For arbitrary small $\epsilon$. We have,

$$P[U_0(N, p) \neq \phi] \simeq P[\|u_0(N, p)\| \geq \epsilon]. \quad (3.13)$$

From the Markov’s inequality, we have

$$P[\|u_0(N, p)\| \geq \epsilon] \leq \frac{E[u_0^2(N, p)]}{\epsilon^2} < \frac{1}{\epsilon^2} \left(4p(1 - p)^{N+1} + (1 - 2p)^N \right), \quad (3.14)$$

using the upper bound of $E[u_0^2(N, p)]$ from (2.10). Now taking $p = \frac{c \log N}{N}$, where $c$ is some constant. Then

$$P[\|u_0(N, p)\| \geq \epsilon] < \frac{1}{\epsilon^2} \left(4c \log N \left(1 - \frac{c \log N}{N}\right)^{N+1} + \left(1 - \frac{2c \log N}{N}\right)^N \right)$$
\begin{align*}
&< \frac{1}{\epsilon^2} \left( \frac{4c \log N}{N} \left( 1 - \frac{c \log N}{N} \right) e^{-c \log N} + e^{-2c \log N} \right) \\
&< \frac{1}{\epsilon^2} \left( \frac{4c \log N}{N^{1+c}} + \frac{1}{N^{2c}} \right). 
\end{align*}

(3.15)

If \( c > 1/2 \), the above probability is summable, i.e.,
\[
\sum_{N=0}^{\infty} P[|u_0(N, p)| \geq \epsilon] < \infty,
\]
and hence from (3.13), we have
\[
\sum_{N=0}^{\infty} P[U_0(N, p) \neq \phi] < \infty.
\]

Then by the Borel-Cantelli’s Lemma, we have
\[
P[U_0(N, p) \neq \phi, \text{ i.o.}] = 0.
\]

Thus, the set \( U_0(N, p) \neq \phi \) for only finitely many time, i.e., eventually \( U_0(N, p) = \phi \) happens infinitely times with probability 1. Hence for \( c > \frac{1}{2} \), we have
\[
U_0(N, p) = \phi, \quad \text{almost surely.}
\]

Now, by the lower bound of Chebyshev’s inequality (Page 55, Shirayev [3]), we have
\[
P[u_0(N, p) \geq \epsilon] \geq \frac{E[u_0(N, p)^2] - \epsilon^2}{16\pi^2},
\]
since \( u_0(N, p) \geq 0 \) and \( |u_0(N, p)| \leq 4\pi \). Now using the lower bound of \( E[u_0(N, p)^2] \) from (2.12) and taking \( \epsilon = o\left(\frac{1}{N}\right) \), we have
\[
P[u_0(N, p) \geq \epsilon] \geq \frac{(1 - 2p)^N(1 - 4p(1 - p)) - \epsilon^2}{16\pi^2}
\geq C_1(1 - 2p)^N(1 - 4p(1 - p)) - C_2,
\]
(3.17)

where \( C_1 = \frac{1}{16\pi^2} \) and \( C_2 = \frac{\epsilon^2}{16\pi^2} \).
Substituting $p = \frac{c \log N}{N}$, in the above expression we get

$$P[u_0(N, p) \geq \epsilon] \geq C_1 \left(1 - \frac{2c \log N}{N}\right)^N \left(1 - \frac{4c \log N}{N} \left(1 - \frac{c \log N}{N}\right)\right) - C_2$$

$$\geq C_3 e^{-2c \log N} = \frac{C_3}{N^{2c}},$$

(3.18)

where $C_3$ is some constant. If we take $c \leq 1/2$, then the probability (3.18), is not summable with respect to $N$, i.e.,

$$\sum_{N=1}^{\infty} P[u_0(N, p) \geq \epsilon] = \infty.$$  

Then by the Borel-Cantelli’s Lemma, we have

$$P[u_0(N, p) \geq \epsilon, \ i.o.] = 1,$$

since $u_0(N, p)$ are independent. Thus $u_0(N, p) \geq \epsilon$ happens infinitely many time with probability 1. Hence for $c \leq \frac{1}{2}$, we have

$$u_0(N, p) \geq \epsilon, \quad \text{almost surely.}$$

This implies for $c \leq 1/2$, we have $U_0(N, p) \neq \phi$ almost surely. \hfill \Box

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