Abstract. The classical theory of the cross-ratio is a beautiful case study of the moduli of ordered points of the projective line and of invariants of the action of PGL$_2$. We generalize the theory of the cross-ratio to the setting of $S$-valued points for an arbitrary scheme $S$. To accomplish this goal, we provide a comprehensive and computationally focused treatment of automorphisms of projective space over $S$, of equalizers in the category of schemes, and of vanishing loci of sections of line bundles. Most of these ideas exist in the literature, though not with the level of detail or generality that we require. After introducing the notion of a “strongly distinct” pair of morphisms, we define the cross-ratio of 4-tuples of pairwise strongly distinct $S$-valued points of the projective line — which is valued in the units of the ring of global functions on the scheme $S$ — and show that it enjoys all of the familiar properties of the cross-ratio.

1. Introduction

On the projective line over a field $k$, the action of PGL$_2(k)$ is triply transitive: Any triple of distinct points can be mapped to any other triple. However, the action is not 4-transitive. Indeed, given a 4-tuple of distinct points, $a, b, c, d$, its orbit under the PGL$_2(k)$ action is determined by the cross-ratio, $\chi(a, b, c, d) := \frac{(a-c)(b-d)}{(a-d)(b-c)}$. That is, if $a', b', c', d'$ is another 4-tuple of distinct points, there is an element $M \in$ PGL$_2(k)$ such that $a' = M.a$, $b' = M.b$, $c' = M.c$, and $d' = M.d$, if and only if $\chi(a, b, c, d) = \chi(a', b', c', d')$.

The purpose of this note is to show how the above scenario can be generalized to the case of the projective line $\mathbb{P}^1_S$ over an arbitrary scheme $S$. There are several obstacles that must be dealt with in this quest. First, we must understand what takes the place of PGL$_2(k)$. That is, we must identify the automorphism group of $\mathbb{P}^1_S$. This is well known, but as we will need to understand the action in detail, we give an account of this story in §2. Second, we need to be clear what we mean by “point”. For us, this will mean an $S$-valued point; i.e., a section of the structure map $\mathbb{P}^1_S \to S$.

Finally, and most crucially, we run into the problem that, in general, the action of Aut$_S(\mathbb{P}^1_S)$ on $S$-points is not even singly transitive! However, we can view $\mathbb{P}^1_S$ as a family of $\mathbb{P}^1$’s, varying over the base $S$, and through that lens we see that the right generalization of “4 distinct points of $\mathbb{P}^1(k)$” should be “4 sections that are distinct in each fiber of $\mathbb{P}^1_S$”. In §5 we introduce the notion of strong distinctness for a pair of $S$-points of $\mathbb{P}^1_S$, or more generally for a pair of scheme morphisms $f, g: S \to X$. Morally, $f$ and $g$ are strongly distinct if they take distinct values at all points of $S$. This slogan turns out to be equivalent to the definition in some cases, such as when $S$ is a variety over an algebraically closed field or when $X$ is an $S$-scheme and $f, g$ are $S$-morphisms (Corollary 5.4).

Having overcome all of these obstacles, our main result (Theorem 6.4) is that the automorphism group of $\mathbb{P}^1_S$ acts simply transitively on triples of pairwise strongly distinct $S$-points. Moreover, there is a natural generalization of the cross-ratio to 4-tuples of pairwise strongly distinct points (Definition 6.2), such that two of these 4-tuples are in the same Aut$_S(\mathbb{P}^1_S)$-orbit if and only if their cross-ratios are equal (Theorem 6.5).

The definition of strong distinctness amounts to a condition on the equalizer of the two morphisms. To deal with this, we discuss equalizers in §3. Finally, to get a useful criterion for strong distinctness (the
Determinant Criterion, Corollary 5.10), we need some information about the zero locus of a section of a line bundle, which we discuss in §4.

Some notation and terminology. As usual, we write $\mathcal{O}_X$ for the structure sheaf of a scheme $X$, and $\Gamma(X, \mathcal{O}_X)$ for its ring of global sections. We also write $\kappa(X)$ for the function field of $X$, when $X$ is integral. In particular, if $x \in X$ is a point, then $\kappa(x)$ is its residue field. The term vector bundle will refer to a locally free sheaf of finite rank. A line bundle is a vector bundle of rank 1.

2. Automorphisms of Projective Space

Fix an integer $n \geq 2$ for the duration of this section. Our present goal is to give a complete description of the automorphisms of $\mathbb{P}^{n-1}_S$ for a general scheme $S$. This is discussed briefly in [10, pp.19-21] for noetherian schemes (possibly requiring that $S$ be a variety over an algebraically closed field) and in [6, §6.2] for connected affine schemes. But we want to avoid placing any restrictions on the scheme $S$. While many of the results stated here are well known, we do not believe there is a complete account in the literature without some restrictions placed on $S$. For that reason, we give an exposition of the theory for the benefit of the reader.

Over a field $k$, it is a staple in every algebraic geometry course to prove that the automorphism group of $\mathbb{P}^{n-1}_k$ is isomorphic to $\text{GL}_n(k)/k^\times = \text{PGL}_n(k)$. If we wish to replace $k$ with a more general base scheme $S$, we encounter a problem. The functor $\text{Spec} A \rightsquigarrow \text{GL}_n(A)/A^\times$ from commutative rings to groups is not a sheaf in the Zariski topology. Put another way, the functor $S \rightsquigarrow \text{GL}_n(S)/\mathbb{G}_m(S)$ is not representable, whereas the functor
denoting the group of $S$-automorphisms of $\mathbb{P}^{n-1}_S \times S$, is representable by a certain group scheme over $\mathbb{Z}$. We will reserve the name $\text{PGL}_n$ to refer to that group scheme. Then for any scheme $S$, $\text{PGL}_n(S)$ denotes the group of morphisms from $S$ to $\text{PGL}_n$. Thus we have three functors from $\text{Sch}$ to $\text{Grp}$:

\[
S \rightsquigarrow \text{GL}_n(R)/R^\times \\
\text{with } R = \Gamma(S, \mathcal{O}_S) \text{ the ring of global functions on } S,
\]

\[
S \rightsquigarrow \text{PGL}_n(S),
\]

\[
S \rightsquigarrow \text{Aut}_S(\mathbb{P}^{n-1}_S).
\]

When $S = \text{Spec } R$ is the spectrum of a local ring, all three of these functors give the same group. More generally, we will see that the first functor is a subgroup of the second. We will also show that the last two are naturally isomorphic, which is what it means for $\text{Aut}_S(\mathbb{P}^{n-1}_S)$ to be represented by $\text{PGL}_n$. Along the way, we will introduce a fourth functor $S \rightsquigarrow \text{G}_n(S)$ that is useful for explicitly writing down elements of $\text{PGL}_n(S)$. Finally, we will close this section with an example that illustrates the difference between $\text{GL}_2(R)/R^\times$ and $\text{PGL}_2(R)$ for a general ring $R$.

Remark 2.1. A fifth functor ought to be considered in the above story. Let $M_n(R)$ be the algebra of $n \times n$ matrices with coefficients in $R = \Gamma(S, \mathcal{O}_S)$. The Skolem-Noether theorem [8, IV.1.4] shows that when $R$ is local, every automorphism of $M_n(R)$ is realized as conjugation by an element of $\text{GL}_n(R)$. Consequently, the functor

denotes the “group quotient” $\text{GL}_n/\text{GL}_1$ exists, and represents the functor $S \rightsquigarrow \text{Aut}_S(\mathbb{P}^{n-1}_S)$, “du moins pour $S$ noethérien.”

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1If $S \rightarrow T$ is a morphism of schemes, then base-extending an automorphism of $\mathbb{P}^{n-1}_T$ to $S$ yields an automorphism of $\mathbb{P}^{n-1}_S$.

2Grothendieck raises this issue in [3, §8], where he asserts that the “group quotient” $\text{GL}_n/\text{GL}_1$ exists, and represents the functor $S \rightsquigarrow \text{Aut}_S(\mathbb{P}^{n-1}_S)$, “du moins pour $S$ noethérien.”
2.1. Properties of PGL.

Definition 2.2. Set $N = n^2 - 1$. Define the group scheme $\text{PGL}_n$ as the open subscheme of $\mathbb{P}^N = \text{Proj} \mathbb{Z}[\{x_{i,j} : 1 \leq i, j \leq n\}]$ obtained by removing the zero locus of the determinant form $\Delta = \det((x_{i,j})) \in \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n))$. For a scheme $S$, we define $\text{PGL}_n(S)$ to be the set of $S$-points of the scheme $\text{PGL}_n$. For a ring $R$, we write $\text{PGL}_n(R)$ instead of $\text{PGL}_n(\text{Spec } R)$.

The usual matrix multiplication formulas define a rational map $\mathbb{P}^N \times \mathbb{P}^N \dasharrow \mathbb{P}^N$, and the fact that the determinant is multiplicative shows that away from $\Delta = 0$, this is a morphism $m: \text{PGL}_n \times \text{PGL}_n \to \text{PGL}_n$.

Proposition 2.3. $\text{PGL}_n$ is a group scheme with multiplication map $m$.

Proof. Clearly $m$ is associative, since the same is true of matrix multiplication. $\mathbb{P}^N$ has the affine open subset $D(x_{1,1}) = \text{Spec} \mathbb{Z}[\{x_{i,j}/x_{1,1} : (i, j) \neq (1, 1)\}]$, and the map $\text{Spec} \mathbb{Z} \to D(x_{1,1})$ given by $x_{i,j}/x_{1,1} \mapsto \delta_{i,j}$ (Kronecker delta) defines an element of $\text{PGL}_n(\mathbb{Z})$ that serves as the group identity. Finally, to define the inversion morphism, let $M = (x_{i,j})$ and consider the usual adjugate formula for the inverse of a matrix, $M^{-1} = \Delta^{-1}A$, where $A$ is the adjugate matrix to $M$. The entries of $A$ are homogeneous polynomials of degree $n - 1$ in $\{x_{i,j}\}$, so $A$ defines a rational map $i: \mathbb{P}^N \dasharrow \mathbb{P}^N$. From the formula, and properties of det, we see that $\det(A) = \det(M)^{n-1} = \Delta^{n-1}$, so that $i$ induces a morphism from $\text{PGL}_n$ to itself. Since multiplication by $\Delta^{-1}$ induces the identity morphism on $\text{PGL}_n$, we find that $i$ serves as the inverse morphism for the group law $m$. That is, $\text{PGL}_n$ is a group scheme. \hfill $\square$

Definition 2.4. For any scheme $S$, let $G_n(S)$ be the set of equivalence classes $^3$ of pairs $(L; M)$, where:

- $L$ is a line bundle on $S$ such that $L^\otimes n \cong \mathcal{O}_S$;
- $M = (m_{i,j})$ is an $n \times n$ matrix of global sections of $L$ such that the determinant
  $$\det(M) := \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} m_{1,\sigma(1)} \otimes \cdots \otimes m_{n,\sigma(n)}$$
  is a nowhere-vanishing section of $L^\otimes n$;
- $(L; M) \sim (L'; M')$ if there exists an isomorphism $\varphi: L \to L'$ such that $\varphi(M) = M'$, with $\varphi(M)$ computed by applying $\varphi$ to each entry of $M$.

We endow $G_n(S)$ with a group structure via $(L; M) \cdot (L'; M') = (L \otimes L'; M \star M')$, where $\star$ denotes “tensor multiplication”. That is, $(M \star M')_{i,j} = \sum_k m_{i,k} \otimes m'_{k,j} \in \Gamma(S; L \otimes L')$.

Remark 2.5. The identity element of $G_n(S)$ is $(\mathcal{O}_S; I)$, where $I_{i,j} = \delta_{i,j}$. Given $(L; M) \in G_n(S)$, write $\text{ad}(M)$ for the “tensor adjugate” of $M$. That is, $\text{ad}(M)_{i,j} = (-1)^{i+j} \det(M^{j,i})$, where $M^{j,i}$ is the matrix of global sections of $L$ obtained by removing the $j$-th row and $i$-th column from $M$. Then one verifies that $(L^\otimes (n-1); \text{ad}(M))$ is the inverse of $(L; M)$ in the group $G_n(S)$.

Proposition 2.6. The functors $S \mapsto \text{PGL}_n(S)$ and $S \mapsto G_n(S)$ are naturally isomorphic.

Proof. Fix a scheme $S$ and set $N = n^2 - 1$. By definition, $\text{PGL}_n(S) = \text{Hom}(S, \text{PGL}_n)$ is the subset of $\mathbb{P}^N(S)$ comprising morphisms $S \to \mathbb{P}^N$ with image disjoint from the zero locus of $\Delta$. Using [5, II.7.1], we see that up to a natural notion of equivalence, such maps are in bijective correspondence with data $(L; (s_{i,j}))_{1 \leq i,j \leq n}$ where $L$ is a line bundle on $S$, and $s_{i,j}$ are global sections of $L$ that generate $L$ and that satisfy $\Delta(s_{1,1}, \ldots, s_{n,n}) \notin m_x L_x^\otimes n$ locally at every point $x \in S$. This last condition is the same as saying $\det(\{s_{i,j}\})$ generates $L^\otimes n$, that is, $(L; M)$ represents an element of $G_n(S)$, where $M$ is the matrix with

$^3$More precisely, we choose a set of representatives, one from each equivalence class. Such a set of representatives exists because line bundles are locally finitely presented. But, the equivalence classes themselves are proper classes, not eligible for set membership without expanding the universe. We do not elaborate the difference any further in this paper.
entries $M_{i,j} = s_{i,j}$. Moreover, the criterion for data $(L; M)$ and $(L'; M')$ to define the same point of $\mathbb{P}^N(S)$, hence of $\text{PGL}_n(S)$, is exactly the equivalence relation defining $G_n(S)$. Thus, we get a bijection of sets $\text{PGL}_n(S) \cong G_n(S)$. It is an isomorphism of groups because the group operation on each side is defined by matrix multiplication.

For any morphism of schemes $T \to S$, line bundles and sections pull back, so naturality of the isomorphisms $\text{PGL}_n(\bullet) \cong G_n(\bullet)$ follows immediately. \hfill \Box

The functor $G_n$ is equivalent to $\text{PGL}_n$, but the former is much easier to work with. Henceforth, we will identify them without comment. In particular, we will treat elements of $\text{PGL}_n(S)$ as being represented by pairs $(L; M) \in G_n(S)$.

**Corollary 2.7.** For any scheme $S$ with ring of global functions $R = \Gamma(S, \mathcal{O}_S)$, we have a natural exact sequence of groups

$$1 \to R^\times \to \text{GL}_n(R) \to \text{PGL}_n(S) \to \text{Pic}(S)[n].$$

(2.1)

In particular, if $\text{Pic}(S)[n] = 0$, then $\text{PGL}_n(S) \cong \text{GL}_n(R)/R^\times$.

**Proof.** Proposition 2.6 shows that $\text{PGL}_n \cong G_n$ as functors, so we identify them without further comment.

The map $R^\times \to \text{GL}_n(R)$ takes $r$ to the scalar matrix $rI$, where $I$ is the $n \times n$ identity matrix. The map $\text{GL}_n(R) \to \text{PGL}_n(S)$ is given by $M \mapsto (\mathcal{O}_S; M)$. The map $\text{PGL}_n(S) \to \text{Pic}(S)[n]$ is given by $(L; M) \mapsto L$. Since $L \cong \mathcal{O}_S$ for $(L; M) \in G_n(S)$, we find that $L \in \text{Pic}(S)[n]$. Naturality of the sequence (2.1) in $S$ is immediate from the definition of the maps.

It is obvious that $R^\times \to \text{GL}_n(R)$ is injective. Let us show that the sequence (2.1) is exact at $\text{GL}_n(R)$. First, the composition $R^\times \to \text{GL}_n(R) \to \text{PGL}_n(S)$ is trivial because it carries $r$ to $(\mathcal{O}_S; rI) \sim (\mathcal{O}_S; I)$. Next, if the matrix $M$ lies in the kernel of $\text{GL}_n(R) \to \text{PGL}_n(S)$, then there is an $\mathcal{O}_S$-module automorphism $\varphi: \mathcal{O}_S \to \mathcal{O}_S$ such that $\varphi(I) = M$. Since $\varphi$ must be multiplication by some $r \in R^\times$, we see that $M = rI$ is a scalar matrix. So $M$ is in the image of $R^\times \to \text{GL}_n(R)$, as desired.

Now we show that the sequence (2.1) is exact at $\text{PGL}_n(S)$. The composition $\text{GL}_n(R) \to \text{PGL}_n(S) \to \text{Pic}(S)[n]$ is trivial because it carries a matrix $M$ to the trivial bundle $\mathcal{O}_S$. Now suppose that $(L; M) \in \text{PGL}_n(S)$ has trivial image in the Picard group. Then $L \cong \mathcal{O}_S$ via some isomorphism $\varphi: L \to \mathcal{O}_S$. It follows that $(L; M) \sim (\mathcal{O}_S; \varphi(M))$ is the image of $\varphi(M) \in \text{GL}_n(R)$.

**Remark 2.8.** An alternate proof of Corollary 2.7 can be given using sheaf cohomology. By the Skolem-Noether theorem, the sequence of sheaves

$$1 \to \mathbb{G}_m \to \text{GL}_n \to \text{PGL}_n \to 1$$

is exact for the Zariski topology. Passing to the long exact sequence on sheaf cohomology gives

$$1 \to R^\times \to \text{GL}_n(R) \to \text{PGL}_n(R) \to H^1(S, \mathbb{G}_m) \to H^1(S, \text{GL}_n) \to \cdots$$

It is well known that $H^1(S, \mathbb{G}_m) \cong \text{Pic}(S)$, and that $H^1(S, \text{GL}_n)$ is isomorphic to the set of vector bundles on $S$ of rank $n$. The final map is given by $L \mapsto L \otimes \mathcal{O}_S$. One can use the determinant bundle to show that the kernel of this map lies inside the $n$-torsion of $\text{Pic}(S)$.

**Corollary 2.9.** If $R$ is a commutative ring, then $\text{GL}_n(R)/R^\times$ is a normal subgroup of $\text{PGL}_n(R)$. If $R$ is a local ring, then $\text{PGL}_n(R) = \text{GL}_n(R)/R^\times$.

**Proof.** If $S = \text{Spec} R$, then $R = \Gamma(S, \mathcal{O}_S)$, and the first result follows from the preceding corollary. If $R$ is local, then $\text{Pic}(S) = \text{Pic}(R) = 0$, so $\text{GL}_n(R) \to \text{PGL}_n(R)$ is surjective with kernel $R^\times$. \hfill \Box
At the end of this section, we will exhibit an example of a ring $R$ and an element of $\text{PGL}_n(R)$ with nontrivial image in $\text{Pic}(R)[n]$. However, the map $\text{PGL}_n(S) \to \text{Pic}(S)[n]$ is not always surjective. In fact we have

**Corollary 2.10.** Let $S$ be a complete integral variety over an algebraically closed field $k$. Then

1. $\text{PGL}_n(S) = \text{PGL}_n(k)$; and
2. the image of $\text{PGL}_n(S)$ in $\text{Pic}(S)[n]$ is zero.

**Proof.** Let $(L; M)$ be an element of $G_n(S) = \text{PGL}_n(S)$. Since $\det(M)$ generates $L^\otimes n$, some entry of $M$ is a nonzero global section of $L$. Suppose $s$ is such a section. Then $s^\otimes n$ is a nonzero global section of $L^\otimes n \cong \mathcal{O}_S$. As $S$ is complete and integral, we have $\Gamma(S, \mathcal{O}_S) = k$, and every nonzero section of $\mathcal{O}_S$ is nowhere vanishing. But because $S$ is reduced, $s$ has a nonzero germ wherever $s^\otimes n$ does, so this means that $s$ is also nowhere vanishing. Hence $L \cong \mathcal{O}_S$, and $(L; M)$ maps to zero in $\text{Pic}(S)[n]$. This proves (2). Now (1) follows from (2) together with the preceding two corollaries. \hfill $\square$

2.2. **Automorphisms and PGL.** Now we turn to the proof that the functors

$$S \rightsquigarrow \text{PGL}_n(S) \quad \text{and} \quad S \rightsquigarrow \text{Aut}(\mathbb{P}_S^{n-1})$$

are naturally isomorphic. Since maps to projective space are associated with line bundles, it stands to reason that computing $\text{Aut}_S(\mathbb{P}_S^{n-1})$ involves understanding line bundles on $\mathbb{P}_S^{n-1}$. The following is the key lemma.

**Lemma 2.11.** Let $S$ be a connected scheme, $\pi_S : \mathbb{P}_S^n \to S$ the projection, and $\mathcal{L}$ any line bundle on $\mathbb{P}_S^n$. Then there is a unique integer $d \in \mathbb{Z}$, and a line bundle $L$ on $S$, unique up to isomorphism, such that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}_S^n}(d) \otimes \pi_S^* L$.

**Sketch of proof.** If $Z$ is any locally noetherian scheme and $X$ is a flat and projective $Z$-scheme with geometrically integral fibers, a result of Grothendieck [4, no.232,Thm. 3.1] says that a certain functor from $Z$-schemes to groups, called the relative Picard functor, $\text{Pic}_{X/Z}$, is representable by a group scheme. In general, $\text{Pic}_{X/Z}$ is defined by a sheafification process applied to the functor $S \rightsquigarrow \text{Pic}(X \times_Z S)$. But under certain hypotheses, $\text{Pic}_{X/Z}$ is isomorphic to the functor

$$S \rightsquigarrow \text{Pic}(X \times_Z S)/\pi_S^* \text{Pic}(S),$$

where $\pi_S : X \times_Z S \to S$ is the projection. By [1, 8.1:Prop.4], it suffices that $X$ satisfy the following three conditions:

1. $X$ is quasi-compact and quasi-separated;
2. the structure map $f : X \to Z$ admits a section; and
3. $\tilde{f}_*(\mathcal{O}_X) = \mathcal{O}_Z$ where $\tilde{f} : \tilde{X} \to \tilde{Z}$ is the base change of $f$ via any morphism $\tilde{Z} \to Z$.

These conditions, as well as the hypotheses for Grothendieck’s result, are all satisfied when $Z = \text{Spec}Z$ and $X = \mathbb{P}_Z^n$. Therefore $S \rightsquigarrow \text{Pic}(\mathbb{P}_Z^n)/\pi_S^* \text{Pic}(S)$ is representable by a group scheme $\mathcal{P}$. Moreover, by [1, 8.2:Thm.5], $\mathcal{P}$ is a disjoint union of quasi-projective schemes (each corresponding to line bundles with a fixed Hilbert polynomial). And by [1, 8.4:Thm.3] (a restatement of [4, no.236:Thm.2.1]), each of these components is in fact projective. Moreover, since $H^2(\mathbb{P}_Z^n, \mathcal{O}_{\mathbb{P}_Z^n}) = 0$ for any field $k$, by [1, 8.4:Prop.2] we have that $\mathcal{P}$ is

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4For example, the result is stated in [10, pp.19-21], but at a crucial point they refer to Mumford’s “Lectures on Curves on an Algebraic Surface” [9]. Unfortunately, in that work one of Mumford’s standing assumptions is that all schemes are of finite type over an algebraically closed field. See also Footnote 2.
formally smooth over Spec \( \mathbb{Z} \). Since \( P \) is also locally of finite presentation (being locally quasi-projective), it is in fact smooth ([11, 00TN]).

Now if \( k \) is any field, we have \( P(k) = \text{Pic}(P^d_S) \cong \mathbb{Z} \), with \( d \in \mathbb{Z} \) corresponding to the line bundle \( \mathcal{O}_{P^d_S}(d) \). This implies that each irreducible component of \( P \) has relative dimension zero over Spec \( \mathbb{Z} \), since otherwise, when \( k \) is uncountable and algebraically closed, \( P(k) \) would also be uncountable. Thus each component of \( P \) is smooth of relative dimension zero, hence étale over Spec \( \mathbb{Z} \). But Spec \( \mathbb{Z} \) has no nontrivial étale cover, so we conclude that \( P \) is a disjoint union of copies of Spec \( \mathbb{Z} \), indexed by \( \mathbb{Z} \). That is, \( P \) is the group scheme \( \mathbb{Z} \) associated to the group \( \mathbb{Z} \). This means that \( P(S) = \mathbb{Z} \) for any connected scheme \( S \), with \( d \in \mathbb{Z} \) representing the class of \( \mathcal{O}_{P^d_S}(d) \) in \( \text{Pic}(P^d_S)/\pi^*_S \text{Pic}(S) \).

Finally, for any scheme \( S \), since \( P(S) = \text{Pic}(P^d_S)/\pi^*_S(\text{Pic}(S)) \), we have a short exact sequence of abelian groups:

\[
0 \rightarrow \pi^*_S(\text{Pic}(S)) \rightarrow \text{Pic}(P^d_S) \rightarrow P(S) \rightarrow 0.
\]

When \( S \) is connected, since \( P(S) = \mathbb{Z} \) this sequence splits, and we conclude \( \text{Pic}(P^d_S) \cong \pi^*_S\text{Pic}(S) \oplus \mathbb{Z} \), with the \( \mathbb{Z} \) component generated by the class of \( \mathcal{O}_{P^d_S}((1)) \).

We can now prove the desired isomorphism of functors. This is also well known, but with Lemma 2.11 in hand the proof is straightforward, so we will present it.

**Theorem 2.12.** There is a natural isomorphism between the following two functors from the category of schemes to the category of groups:

\[
S \mapsto PGL_n(S) \quad \text{and} \quad S \mapsto \text{Aut}_S(P^n_{S^{-1}}).
\]

**Proof.** For a scheme \( S \), we will construct an isomorphism of groups \( PGL_n(S) \cong \text{Aut}_S(P^n_{S^{-1}}) \). The reader will easily verify that the entire construction is compatible with base change along a morphism \( T \rightarrow S \), so that our isomorphisms are natural in the scheme \( S \).

We first reduce to the case of \( S \) connected. If \( S \) is a disjoint union \( S = \coprod_i S_i \) with each \( S_i \) connected, then it is clear that \( PGL_n(S) = \coprod_i PGL_n(S_i) \), since \( PGL_n(S) \) is the set of morphisms from \( S \) to the group scheme \( PGL_n \). It is also clear that \( P^n_{S^{-1}} = \prod_i P^n_{S_i^{-1}} \), and that any \( S \)-automorphism must stabilize each \( P^n_{S_i^{-1}} \).

It follows that also \( \text{Aut}_S(P^n_{S^{-1}}) = \prod_i \text{Aut}(P^n_{S_i^{-1}}) \). Thus if we prove that \( PGL_n(S_i) \cong \text{Aut}(P^n_{S_i^{-1}}) \) for each \( i \), we get that \( PGL_n(S) \cong \text{Aut}(P^n_{S^{-1}}) \).

Henceforth we assume \( S \) is connected. To define a map \( PGL_n(S) \rightarrow \text{Aut}_S(P^n_{S^{-1}}) \), let \( (L; M) \) represent an element of \( PGL_n(S) \), and choose an affine open cover \( \{U_i\} \) of \( S \) such that \( L|_{U_i} \) is trivial, with isomorphisms \( \phi_i : L|_{U_i} \cong \mathcal{O}_S|_{U_i} \). Let \( R_i = \mathcal{O}_S(U_i) \). Then \( P^n_{U_i^{-1}} = \text{Proj} R_i[x_1, \ldots, x_n] \), and the matrix \( \phi(M) \), with entries in \( R_i \), gives an automorphism of \( P^n_{U_i^{-1}} \). On \( U_i \cap U_j \), with \( R_{i,j} = \mathcal{O}_S(U_i \cap U_j) \), the \( R_{i,j} \)-module automorphism \( \phi_j\phi_i^{-1} \) is multiplication by some element of \( R_{i,j}^* \), so we see that \( \phi_i(M) \) and \( \phi_j(M) \) induce the same automorphism on \( \text{Proj} R_{i,j}(x_1, \ldots, x_n) \), and we get a well defined automorphism of \( P^n_{U_i^{-1}} \).

To go the other way, an automorphism \( \alpha \in \text{Aut}_S(P^n_{S^{-1}}) \) is specified by the line bundle \( \mathcal{L} = \alpha^*(\mathcal{O}_{P^n_{S^{-1}}}(1)) \) and a basis of global sections \( \{s_i = \alpha^*(x_i) : 1 \leq i \leq n\} \). By Lemma 2.11, there is a line bundle \( L \) on \( S \) and an integer \( d \) such that \( \mathcal{L} \cong \pi^*_S(L) \otimes \mathcal{O}_{P^n_{S^{-1}}}(d) \). Note that \( \dim_{\kappa(s)} H^0(s, \mathcal{O}_{P^n_{S^{-1}}}(1)) = n \) for any point \( s \in S \). Therefore also

\[
n = \dim_{\kappa(s)} H^0((P^n_{S^{-1}}), \mathcal{O}_{P^n_{S^{-1}}}(1)) = \dim_{\kappa(s)} H^0((P^n_{S^{-1}}, \alpha^*(\mathcal{O}_{P^n_{S^{-1}}}(1))) = \dim_{\kappa(s)} H^0(P^n_{S^{-1}}, \mathcal{O}_{P^n_{S^{-1}}}(d))
\]

since \( \pi^*_S(L) \) is trivial on \( P^n_{S^{-1}} \). This clearly implies \( d = 1 \).
Now we have $L \cong \pi_S^*(L) \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, and so the global sections $s_i$ defining $\alpha$ are linear combinations of $x_1, \ldots, x_n$ with coefficients in $\Gamma(S, L)$. Thus there are elements $M_{i,j} \in \Gamma(S, L)$ such that $s_i = \sum_j M_{i,j} \otimes x_j$. Let $M$ be the matrix with entries $M_{i,j}$. Then we map $\alpha$ to the class of $(L; M)$ in $\text{PGL}_n(S)$. It is straightforward to show that this is inverse to the map given above.

Our primary interest is the action of $\text{Aut}_S(\mathbb{P}^{n-1})$ on the $S$-points of $\mathbb{P}^{n-1}$. Identifying the automorphism group with $G_n(S)$ as in the Theorem, we can make the action explicit as follows. An $S$-point is a morphism $\alpha : S \to \mathbb{P}^{n-1}$. By [5, II.7.1], such morphisms are in bijective correspondence with equivalence classes of data $(A; a_1, \ldots, a_n)$, where $A = a^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ is a line bundle on $S$, and $a_i = a^*(x_i)$ are global sections of $A$ that generate $A$. A second set of data $(A; a'_1, \ldots, a'_n)$ determines the same morphism if and only if there is an $\mathcal{O}_S$-isomorphism $\varphi : A \to A'$ such that $\varphi(a_i) = a'_i$ for $i = 1, \ldots, n$. Then tracing through the map $G_n(S) \to \text{Aut}_S(\mathbb{P}^{n-1})$ defined in the proof of Theorem 2.12 gives the following description.

**Corollary 2.13.** Let $\sigma \in G_n(S) \cong \text{PGL}_n(S)$ be given by the data $(L; M)$. Then $\sigma$ acts on a point $\alpha = (A; a_1, \ldots, a_n) \in \mathbb{P}^{n-1}(S)$ via

$$\sigma(\alpha) = \left( L \otimes A; M \ast \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right),$$

where the column vector $M \ast \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ can be viewed as $n$ global sections of $L \otimes A$ that generate $L \otimes A$.

**Remark 2.14.** This shows how to find an example of an $S$ such that $\text{Aut}_S(\mathbb{P}^{n-1})$ does not act transitively on $\mathbb{P}^{n-1}(S)$. Indeed, a consequence of Corollary 2.13 is that if $a = (A; a_1, \ldots, a_n)$ and $b = (B; b_1, \ldots, b_n)$ are in the same orbit under $\text{Aut}_S(\mathbb{P}^{n-1})$, then $A \otimes B^{-1}$ must represent an $n$-torsion class in $\text{Pic}(S)$. For example, let $K$ be a number field with class number divisible by 3, and let $A \subset R := \mathcal{O}_K$ be an ideal representing an element of order 3 in the class group. Let $a_1, a_2$ be generators of $A$, so that $(A; a_1, a_2)$ represents a point $a \in \mathbb{P}^1(R)$. We also have the point $b = (B; b_1, b_2)$ with $B = R$, $b_1 = 0$, and $b_2 = 1$. Then the class of $A \otimes B^{-1} = A$ is not 2-torsion, so $A$ and $B$ are in different orbits.

### 2.3. The Case of a Domain.

If $S$ is an integral scheme, then every line bundle $L$ on $S$ embeds into the constant sheaf of rational functions $\mathcal{K}_S$. Global sections of $L$ map to elements of the rational function field $K = \mathcal{K}(S)$. Via these identifications, it is possible to identify $\text{PGL}_n(S)$ with a subgroup of $\text{GL}_n(K) / K^\times$. We prove this after a few preliminary results.

**Lemma 2.15.** Let $X$ be a reduced scheme, and let $\alpha, \beta : X \to Y$ be a pair of morphisms. Suppose $\alpha(x) = \beta(x)$ for every $x \in X$, and the induced maps on local rings agree at every generic point of $X$. Then $\alpha = \beta$.

**Proof.** Since $\alpha$ and $\beta$ agree as continuous maps, it suffices to work locally and compare the induced morphisms of ringed structures. To that end, we may assume that $X = \text{Spec} B$ and $Y = \text{Spec} A$. Then $\alpha$ and $\beta$ are induced by ring homomorphisms $f, g : A \to B$, respectively. For any prime ideal $\mathfrak{p}$ of $B$, we know that $f^{-1}(\mathfrak{p}) = g^{-1}(\mathfrak{p}) =: \mathfrak{q}$. We have the following commutative diagram (for $f$ or for $g$):

$$
\begin{array}{ccc}
A & \xrightarrow{f,g} & B \\
\downarrow & & \downarrow \\
A_{\mathfrak{q}} & \xrightarrow{f_{\mathfrak{q}}, g_{\mathfrak{q}}} & B_{\mathfrak{p}}.
\end{array}
$$

The hypothesis that $\alpha$ and $\beta$ agree on generic points shows that $f_{\mathfrak{p}} = g_{\mathfrak{p}}$ when $\mathfrak{p}$ is a minimal prime ideal. Suppose that $a \in A$, and consider the element $b = f(a) - g(a)$. The above diagram shows that $b$ vanishes in $B_{\mathfrak{p}}$ for every minimal prime ideal $\mathfrak{p}$ of $B$. In particular this means that $b$ is contained in $\mathfrak{q}$. Since every prime ideal contains a minimal prime (using Zorn’s Lemma), it follows that $b$ lies in every prime ideal of $B$, so that $b$ is nilpotent. As $X$ is reduced, we conclude that $b = 0$. Hence $\alpha$ and $\beta$ agree. \qed
Remark 2.16. The hypothesis that $X$ is reduced is essential in Lemma 2.15. For example, take

$$X = \text{Spec } \mathbb{Z}_p[t]/(t^2, pt).$$

Then the automorphism induced by $t \mapsto -t$ is the identity map on points, and induces the identity map on the local ring at the generic point $(t)$, but is clearly not the identity map on $X$.

Lemma 2.17. Let $S$ be an irreducible scheme with generic point $\eta \in S$, and let $X$ be an irreducible $S$-scheme with separated structure map $\pi : X \to S$. Given an automorphism $\alpha \in \text{Aut}_S(X)$, let $\alpha_\eta$ be the restriction of $\alpha$ to $X_\eta = \pi^{-1}(\eta)$. If $\alpha_\eta$ is the identity morphism, then $\alpha(x) = x$ for all $x \in X$.

Proof. Let $\Gamma \subset X \times_S X$ be the graph of $\alpha$ — i.e., $\Gamma$ is the image of the morphism $1 \times \alpha : X \to X \times_S X$. Also, let $\Delta$ be the diagonal morphism. Suppose we knew that $\Gamma = \Delta(X)$. Then for any $x \in X$ we would have $1 \times \alpha(x) = \Delta(x')$ for some $x' \in X$. Then applying the first projection gives $x = x'$, and applying the second projection gives $\alpha(x) = x'$, so we would get $\alpha(x) = x$. Thus it suffices to prove that $\Gamma = \Delta(X)$.

Note that $1 \times \alpha = (1, \alpha) \circ \Delta$, where $(1, \alpha)$ is the map from $X \times_S X$ to itself that is the identity on the first factor and $\alpha$ on the second. Evidently $(1, \alpha)$ is an automorphism, as it has the inverse $(1, \alpha^{-1})$. Since $X$ is separated over $S$, $\Delta$ is a closed immersion, and we find that $1 \times \alpha$ is also a closed immersion. Thus $\Gamma$ is the closure of $(1 \times \alpha)(\xi)$ where $\xi$ is the generic point of $X$. Since $\Delta(X)$ is also the closure of $\Delta(\xi)$, it suffices to prove that $(1 \times \alpha)(\xi) = \Delta(\xi)$. But on $X_\eta$ we have $1 \times \alpha_\eta = \Delta_{X_\eta}$, so $(1 \times \alpha)(\xi) = (1 \times \alpha_\eta)(\xi) = \Delta_\eta(\xi) = \Delta(\xi)$. \square

Proposition 2.18. Let $S$ be an integral scheme with field of fractions $K = \kappa(S)$. Let $X$ be a reduced and separated $S$-scheme such that every irreducible component of $X$ dominates $S$. Then there is a natural inclusion $\text{Aut}_S(X) \hookrightarrow \text{Aut}_K(X_K)$, where $X_K$ is the generic fiber of $X$.

Proof. The restriction of an automorphism of $X \to S$ to the generic fiber is an automorphism, so we obtain a natural homomorphism $\text{Aut}_S(X) \to \text{Aut}_K(X_K)$. We must show that this homomorphism is injective.

Every generic point of $X$ lives in the fiber over $\text{Spec } K$. If $\alpha \in \text{Aut}_S(X)$ is trivial when restricted to the generic fiber, then $\alpha$ agrees with the identity at all generic points of $X$. Let $\zeta \in X$ be one of those generic points, and let $Z \subset X$ be its closure, with the reduced induced scheme structure. Since the inclusion $Z \hookrightarrow X$ is a closed immersion, in particular it is separated, and therefore $Z$ is separated over $S$. Then by Lemma 2.17, we have $\alpha(z) = z$ for every point $z \in Z$. Since every point of $X$ is in the closure of some generic point, we find that $\alpha(x) = x$ for all $x \in X$. Then Lemma 2.15 shows that $\alpha$ is the identity morphism. \square

Remark 2.19. The hypothesis that $X$ be separated is necessary in both Lemma 2.17 and Proposition 2.18. A counterexample to both results is provided by the affine line with doubled origin: Let $S = \mathbb{A}^1$, and let $X = X_1 \cup X_2$ where $X_1 = X_2 = \mathbb{A}^1$ and $X_1 \setminus \{0\}$ is identified with $X_2 \setminus \{0\}$, and $X \to S$ is the obvious map. The automorphism defined by interchanging points of $X_1$ and $X_2$ acts as the identity on the dense open subset $X_1 \cap X_2$, but swaps the two copies of the point 0.

Corollary 2.20. Let $R$ be a domain with field of fractions $K$. There is a natural inclusion

$$\text{PGL}_n(R) \hookrightarrow \text{PGL}_n(K) \cong \text{GL}_n(K)/K^\times.$$ 

Remark 2.21. To make the inclusion more explicit, let $(L; M) \in G_n(\text{Spec } R)$ represent an element of $\text{PGL}_n(R)$, where $L$ is a line bundle and $M$ a matrix of elements of $L$ with $\text{det}(M)$ generating $L^{\otimes n}$. As each line bundle is isomorphic to an $R$-submodule of $K$ [5, Prop. II.6.15], we may assume that $L \subset K$. Then $M \in \text{GL}_n(K)$.

When $R$ is a Dedekind ring, we can give a numerical characterization of the image of $\text{PGL}_n(R)$ in $\text{GL}_n(K)/K^\times$. (Thanks to Aaron Gray for this result in the case $n = 2$.)
Proposition 2.22. Let $R$ be a Dedekind ring with field of fractions $K$. Let $A = (A_{i,j}) \in \text{GL}_n(K)$ represent an element of $\text{PGL}_n(K)$. Then $A$ lies in the image of the homomorphism $\text{PGL}_n(R) \hookrightarrow \text{PGL}_n(K)$ if and only if for each nonzero prime ideal $\mathfrak{p}$ of $R$, the following holds:

$$n \mid \text{ord}_\mathfrak{p} (\det(A)) \quad \text{and} \quad \text{ord}_\mathfrak{p}(A_{i,j}) \geq \frac{1}{n} \text{ord}_\mathfrak{p}(\det(A)) \quad \text{for all } i, j.$$

Proof. Identify $\text{PGL}_n(R)$ with $G_n(\text{Spec } R)$. Suppose first that $A$ is in the image and fix a nonzero prime ideal $\mathfrak{p}$ of $R$. As every line bundle on $\text{Spec } R$ is isomorphic to a nontrivial ideal, we see there is an ideal $I \subset R$ and a matrix $M = (M_{i,j})$ with entries in $I$ such that $A = xM$ for some $x \in K^\times$. Then $\det(A) = x^n \det(M)$. Since $I^{\otimes n} \cong I^n$ is generated by $\det(M)$, we see that

$$\text{ord}_\mathfrak{p}(\det(A)) = n(\text{ord}_\mathfrak{p}(x) + \text{ord}_\mathfrak{p}(I)). \tag{2.2}$$

Hence, $n \mid \text{ord}_\mathfrak{p}(\det(A))$. As each entry $M_{i,j}$ lies in $I$, we have $\text{ord}_\mathfrak{p}(M_{i,j}) \geq \text{ord}_\mathfrak{p}(I)$. Thus, by (2.2), we have

$$\text{ord}_\mathfrak{p}(A_{i,j}) = \text{ord}_\mathfrak{p}(x) + \text{ord}_\mathfrak{p}(M_{i,j}) \geq \frac{1}{n} \text{ord}_\mathfrak{p}(\det(A)).$$

For the converse, suppose that for every nonzero prime ideal $\mathfrak{p}$ of $R$ we know that

$$n \mid \text{ord}_\mathfrak{p}(\det(A)) \quad \text{and} \quad \text{ord}_\mathfrak{p}(A_{i,j}) \geq \frac{1}{n} \text{ord}_\mathfrak{p}(\det(A)) \quad \text{for all } i, j.$$

Set $m_\mathfrak{p} = \frac{n}{\text{ord}_\mathfrak{p}(\det(A))}$, and define

$$I = \prod_{\mathfrak{p} \in \text{Spec } R} \mathfrak{p}^{m_\mathfrak{p}} \subset K^\times.$$  

Then $I$ is a fractional ideal, and $A_{i,j} \in I$ because $\text{ord}_\mathfrak{p}(A_{i,j}) \geq m_\mathfrak{p}$ by hypothesis. Moreover, since $\text{ord}_\mathfrak{p}(\det(A)) = m_\mathfrak{p} n = \text{ord}_\mathfrak{p}(I^n)$, it follows that $\det(A)$ generates $I^n$. We conclude that $(I; A) \in G_n(\text{Spec } R)$. \qed

2.4. Example: Non-matrix Element of $\text{PGL}$. We close this section with the promised example of an element of $\text{PGL}_2(R)$ that is not an element of $\text{GL}_2(R)/R^\times$. Of course, by the exact sequence (2.1), we know that it must arise from a nontrivial element of $\text{Pic}(R)[2]$. The 2-torsion points of elliptic curves naturally give rise to such line bundles.

Let $k$ be a field of characteristic different from 2, and consider the projective elliptic curve $E/k$ with affine equation $y^2 = x^3 - x$. Its affine coordinate ring

$$R = k[x, y]/(y^2 - x^3 + x)$$

is a Dedekind domain.

Let $L = Rx + Ry$ be an ideal of $R$, and set

$$M = \begin{pmatrix} x & y \\ y & x^2 \end{pmatrix}.$$  

Since $L^{\otimes 2} \cong L^2 = Rx$ and $\det(M) = x$, we have $(L; M) \in G_2(\text{Spec } R) = \text{PGL}_2(R)$. If $(L; M)$ were induced by a matrix in $\text{GL}_2(R)$, then $L$ would have trivial image in $\text{Pic}(R)$ by (2.1). But the divisor of $L$ is the point $(0, 0)$, which has order 2 in $\text{Pic}(R)$.
3. Equalizers in the Category of Schemes

In this section we present a number of results about equalizers in the category of schemes. It is likely that all of these results are available in the literature in one form or another. For convenience, and for want of a suitable reference, we state and prove them in the form that we will need.

Throughout this section, \( S \) denotes a fixed scheme. All schemes should be viewed as \( \mathbb{Z} \)-schemes unless specified otherwise.

In a category \( C \), the equalizer of two morphisms \( f, g: X \to Y \) is a morphism \( i: E_C(f, g) \to X \) satisfying the following two properties:

- \( f \circ i = g \circ i \), and
- If \( j: W \to X \) is any morphism such that \( f \circ j = g \circ j \), then \( j \) factors uniquely through \( i \). That is, there exists a unique morphism \( k: W \to E_C(f, g) \) such that \( j = i \circ k \).

If an equalizer exists, then it is unique up to canonical isomorphism over \( X \). We will sometimes abuse terminology and refer to the source object \( E_C(f, g) \) as the equalizer of \( f \) and \( g \).

If \( C \) is the category of schemes over \( S \), we will write \( E_S(f, g) \) for the equalizer of two morphisms \( f, g: X \to Y \). If \( C \) is the category of all schemes, we write \( E_Z(f, g) \) for the equalizer. (This notational distinction will turn out to be irrelevant; see Remark 3.3.)

**Proposition 3.1.** Equalizers exist in the category of \( S \)-schemes. Given \( S \)-morphisms \( f, g: X \to Y \), the equalizer of \( f \) and \( g \) is given by the fiber product \( E(f, g) = X \times_Y S \) relative to the map \( f \times g: X \to Y \times S \) and the diagonal \( \Delta: Y \to Y \times S \), with \( E(f, g) \to X \) projection onto the first factor.

**Proof.** Write \( \pi_X, \pi_Y \) for the projection morphisms onto the two factors of \( X \times_Y S \). To argue that \( \pi_X: X \times_Y S \to X \) is the equalizer of \( f \) and \( g \) in the category of \( S \)-schemes, we must show three things:

(a) \( f \circ \pi_X = g \circ \pi_X \);
(b) If \( j: W \to X \) is an \( S \)-morphism such that \( f \circ j = g \circ j \), then there exists an \( S \)-morphism \( k: W \to X \times_Y S \) such that \( j = \pi_X \circ k \); and
(c) The morphism \( k \) in (b) is unique.

For (a), write \( p, q \) for the first and second projections of \( Y \times S \) to \( Y \). Combining the fiber product square for \( X \times_Y S \) with the diagram defining the map \( f \times g: X \to Y \times S \), we obtain the following commutative diagram:

\[
\begin{array}{ccc}
X \times_Y S & \xrightarrow{\pi_X} & X \\
\downarrow{f \times g} & & \downarrow{\pi} \\
Y \times S & \xrightarrow{\Delta} & Y \\
\downarrow{g} & & \downarrow{\Delta} \\
Y & \xrightarrow{p} & S
\end{array}
\]  

(3.1)

Using the fact that \( p \circ \Delta = q \circ \Delta = 1_Y \), we can chase around the diagram to see that

\[ f \circ \pi_X = p \circ \Delta \circ \pi_Y = 1_Y \circ \pi_Y = q \circ \Delta \circ \pi_Y = g \circ \pi_X. \]

This proves (a).
For (b), suppose that \( j: W \to X \) is such that \( f \circ j = g \circ j \). Note that \( j: W \to X \) and \( f \circ j: W \to Y \) can be used to define a morphism \( W \to X \times_{(Y \times_{S} Y)} Y \) provided we can show that
\[
(f \times g) \circ j = \Delta \circ f \circ j.
\] (3.2)
As the target of these morphisms is \( Y \times_{S} Y \), it suffices to show equality in (3.2) after applying the projections \( p \) and \( q \). Noting that \( f = p \circ (f \times g) \) and \( g = q \circ (f \times g) \), we have
\[
p \circ \Delta \circ f \circ j = 1_Y \circ f \circ j = [p \circ (f \times g)] \circ j
\]
\[
q \circ \Delta \circ f \circ j = 1_Y \circ f \circ j = g \circ j = [q \circ (f \times g)] \circ j.
\]
We conclude that (3.2) holds. By the universal property for the fiber product \( X \times_{(Y \times_{S} Y)} Y \), there is a unique morphism \( k: W \to X \times_{(Y \times_{S} Y)} Y \) such that \( \pi_X \circ k = j \) and \( \pi_Y \circ k = f \circ j \). This completes the proof of (b).

For (c), suppose that \( k': W \to X \times_{(Y \times_{S} Y)} Y \) is another \( S \)-morphism such that \( \pi_X \circ k' = j \). Then (3.1) and (3.2) show that
\[
\Delta \circ \pi_Y \circ k' = (f \times g) \circ \pi_X \circ k' = (f \times g) \circ j = \Delta \circ f \circ j.
\]
Applying \( p \) to the far ends of this equality gives \( \pi_Y \circ k' = f \circ j \). That is, \( k' \) satisfies both equalities \( \pi_X \circ k' = j \) and \( \pi_Y \circ k' = f \circ j \). In (b), we argued that there is a unique such morphism, so \( k' = k \).

The next result shows that formation of the equalizer is both agnostic to the choice of base scheme and also compatible with base extension.

**Proposition 3.2.** Let \( \rho: T \to S \) be a morphism of schemes.

1. If \( f, g: X \to Y \) are morphisms of \( T \)-schemes, then they are also morphisms of \( S \)-schemes, and we have a canonical isomorphism \( E_S(f, g) \cong E_T(f, g) \) over \( X \).
2. If \( f, g: X \to Y \) are morphisms of \( S \)-schemes, then we have a canonical isomorphism \( E_T(f, g) \cong E_S(f, g)_T \) over \( X_T \).

**Proof.** We begin with assertion 1. Let \( i: E_T(f, g) \to X \) be the equalizer of \( f \) and \( g \) in the category of \( T \)-schemes. We claim it is also the equalizer in the category of \( S \)-schemes. To that end, we must show three things:

- (a) \( f \circ i = g \circ i \);
- (b) If \( j: W \to X \) is an \( S \)-morphism such that \( f \circ j = g \circ j \), then there exists \( k: W \to E_T(f, g) \) over \( S \) such that \( j = i \circ k \); and
- (c) The morphism \( k \) in (b) is unique.

That \( i \) is the equalizer of \( f \) and \( g \) (in any category) immediately implies (a).

For (b), suppose that \( j: W \to X \) is an \( S \)-morphism such that \( f \circ j = g \circ j \). Write \( \rho_W: W \to S \) and \( \rho_X: X \to T \) for the structure morphisms of \( W \) and \( X \), respectively. To say \( j \) is an \( S \)-morphism is to say that the following diagram commutes:

\[
\begin{array}{ccc}
W & \xrightarrow{j} & X \\
\rho_W \downarrow & & \downarrow \rho_X \\
S & \xleftarrow{\rho_X \circ j} & T
\end{array}
\]

We endow \( W \) with a \( T \)-scheme structure via the map \( \rho_X \circ j \); in this way, \( j \) becomes a morphism of \( T \) schemes. By the universal property for \( E_T(f, g) \), there is a \( T \)-morphism \( k: W \to E_T(f, g) \) such that \( j = i \circ k \). Any \( T \)-morphism is also an \( S \)-morphism, so (b) is complete.

For (c), we cannot immediately lean on the universal property for \( E_T(f, g) \) to assert that \( k \) is unique because the \( T \)-scheme structure on \( W \) is not unique. Suppose that \( k': W \to E_T(f, g) \) is a second \( S \)-morphism
such that \( j = i \circ k' \). Let \( \rho_E : E_T(f,g) \to T \) be the structure morphism for \( E_T(f,g) \). Then \( k' \) is a \( T \)-morphism if we give \( W \) the structure map \( \rho_E \circ k' \). As above, this gives a commutative diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{k'} & E_T(f,g) \\
\rho_W & \searrow & \nearrow \rho_E \\
S & \xleftarrow{\rho} & T
\end{array}
\]

Since \( j = i \circ k' \), composition across the top of this diagram shows that the \( T \)-scheme structures \( \rho_E \circ k' \) and \( \rho_X \circ j \) agree on \( W \). The latter is the structure we used in part (b), so the uniqueness part of the universal property for \( E_T(f,g) \) applies to show that \( k' = k \). This completes the proof of the first part of the proposition.

For the second part of the proposition, let \( i : E_S(f,g) \to X \) be the equalizer of \( f \) and \( g \) in the category of \( S \)-schemes. After base extension to \( T \), we obtain the \( T \)-morphism \( i_T : E_S(f,g)_T \to X_T \), which immediately satisfies \( f_T \circ i_T = g_T \circ i_T \). It suffices to show that \( i_T : E_S(f,g)_T \to X_T \) satisfies the definition of the equalizer in the category of \( T \)-schemes. The proof strategy is similar to the above argument, so we omit it. \( \square \)

Remark 3.3. As a consequence of the first part of Proposition 3.2, we no longer need to make it explicit which category of schemes we use to compute the equalizer. If \( f, g : X \to Y \) are morphisms of \( S \)-schemes, we will now write \( E(f,g) \to X \) for the equalizer.

The formation of the equalizer of two morphisms \( f, g : X \to Y \) behaves well with respect to precomposition.

**Proposition 3.4.** Let \( i : E(f,g) \to X \) be the equalizer of two morphisms \( f, g : X \to Y \). If \( h : Z \to X \) is any morphism, then the equalizer of \( f \circ h \) and \( g \circ h \) is given by \( \pi_Z : E(f,g) \times_X Z \to Z \).

**Proof.** One proves that the map \( \pi_Z : E(f,g) \times_X Z \to Z \) satisfies the universal property for the equalizer of \( f \circ h \) and \( g \circ h \). The strategy is virtually the same as in the proof of Proposition 3.2(1), so we omit it. \( \square \)

Let \( X \) be a scheme over \( S \). The universal property for the fiber product shows that for any scheme \( Y \), we have a canonical bijection

\[
\text{Hom}_Z(X,Y) \cong \text{Hom}_S(X,Y_S).
\]

It turns out that this bijection preserves equalizers.

**Proposition 3.5.** Let \( X \) be a scheme over \( S \), and \( Y \) an arbitrary scheme. Let \( f, g : X \to Y \) be morphisms, and let \( f', g' : X \to Y_S \) be the induced morphisms of \( S \)-schemes. Then \( E(f,g) \) is canonically isomorphic to \( E(f',g') \) as schemes over \( X \).

**Proof.** Let \( \rho_X : X \to S \) be the structure morphism for \( X \). Write \( i : E(f,g) \to X \) for the equalizer of \( f \) and \( g \). Note that \( i \) becomes an \( S \)-morphism if we endow \( E(f,g) \) with the \( S \)-scheme structure given by \( \rho_X \circ i \). We will show that \( i \) also satisfies the universal property to be the equalizer of \( f' \) and \( g' \).

First we show that

\[
f' \circ i = g' \circ i. \tag{3.3}
\]

Since the target is the product \( Y_S = Y \times_{\text{Spec} \mathbb{Z}} S \), it suffices to show that (3.3) holds after applying the two projections \( \pi_Y \) and \( \pi_S \). Using the fact that \( \pi_Y \circ f' = f \) and \( \pi_Y \circ g' = g \), we have

\[
\pi_Y \circ f' \circ i = f \circ i = g \circ i = \pi_Y \circ g' \circ i
\]

\[
\pi_S \circ f' \circ i = \rho_X \circ i = \pi_S \circ g' \circ i.
\]

Hence (3.3) holds.
Next, let \( j : W \to X \) be an \( S \)-morphism such that \( f' \circ j = g' \circ j \). Applying \( \pi_Y \) to both sides of this equality shows that \( f \circ j = g \circ j \), from which we conclude that there is a unique morphism \( k : W \to E(f,g) \) such that \( i \circ k = j \). To conclude the proof, we must show that \( k \) is an \( S \)-morphism with the given structure on \( E(f,g) \). Write \( \rho_W : W \to S \) for the structure morphism for \( W \). Then
\[
\rho_W = \rho_X \circ j = \rho_X \circ i \circ k,
\]
where the first equality follows from the fact that \( j \) is an \( S \)-morphism. Since \( \rho_X \circ i \) is the structure map for \( E(f,g) \), we conclude that \( k \) is an \( S \)-morphism, and we are finished. \( \square \)

The next few results give various characterizations of the points of a fiber product, and specifically of the equalizer of two morphisms.

**Lemma 3.6.** Let \( X,Y \) be \( S \)-schemes with structure morphisms \( \rho_X : X \to S \) and \( \rho_Y : Y \to S \). Then there is a bijection between the points of \( X \times_S Y \) and triples \((x,y,p)\) where \( x \in X \) and \( y \in Y \) satisfy \( \rho_X(x) = \rho_Y(y) =: s \), and \( p \) is a prime ideal of \( \kappa(x) \otimes_{\kappa(s)} \kappa(y) \).

**Proof.** For \( z \in X \times_S Y \), set \( x = \pi_X(z) \), \( y = \pi_Y(z) \), and \( s = \rho(z) \) where \( \rho = \rho_X \pi_X = \rho_Y \pi_Y \). Then we associate to \( z \) the triple \((x,y,\ker(\kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(z)))\). Conversely, given \((x,y,p)\) satisfying the stated conditions, we get maps
\[
\begin{align*}
\kappa(x) &\to \kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(p) \\
\kappa(s) &\to \kappa(y) \to \kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(p)
\end{align*}
\]
which agree on the image of \( \kappa(s) \). These give rise to morphisms:
\[
\begin{align*}
\xi : \text{Spec} \kappa(p) &\to \text{Spec} \kappa(x) \to X \\
\eta : \text{Spec} \kappa(p) &\to \text{Spec} \kappa(y) \to Y.
\end{align*}
\]
The composition of either of these with the morphism to \( S \) factors through \( \text{Spec} \kappa(s) \), and hence the two compositions agree. Thus we get a morphism \( \text{Spec} \kappa(p) \to X \times_S Y \). Its image is a single point \( z \in X \times_S Y \), which we associate to \((x,y,p)\). It is straightforward to show that these these maps are inverse to each other. \( \square \)

**Remark 3.7.** This result is similar to the more familiar statement that for any morphism of schemes \( f : X \to S \), the set of points \( f^{-1}(s) \subset X \) mapping to a given point \( s \in S \) is in bijection with the points of \( \text{Spec} \kappa(s) \times_S X \). In fact, this latter result can be used to give another proof of Lemma 3.6. See also [7, Ex. 3.1.7].

Since equalizers are fiber products with diagonal morphisms, we will want to use a special property of diagonals.

**Definition 3.8.** A morphism of schemes \( f : X \to Y \) is a **locally closed immersion** if for each point \( x \in X \), there exist neighborhoods \( U \) and \( V \) of \( x \) and \( f(x) \), respectively, such that \( f(U) \subset V \) and \( f|_U : U \to V \) is a closed immersion.

**Lemma 3.9.** If \( S \) is a scheme and \( Y \) is an \( S \)-scheme, then the diagonal \( \Delta : Y \to Y \times_S Y \) is a locally closed immersion.

**Proof.** By the previous lemma, for \( y \in Y \), the point \( \Delta(y) \) corresponds to the triple \((y,y,p)\) for some prime ideal \( p \) of \( \kappa(y) \otimes_{\kappa(s)} \kappa(y) \), where \( s = \rho_Y(y) \). In particular, the image of \( \Delta \) is covered by open affine sets \( V \times_U V = \text{Spec} B \otimes_A B \) where \( U = \text{Spec} A \) and \( V = \text{Spec} B \) are affine open subsets of \( S \) and \( Y \), respectively. Then \( \Delta|_V \) is given by the ring homomorphism \( B \otimes_A B \to B : b \otimes b' \mapsto bb' \). This homomorphism is clearly surjective, hence \( \Delta|_V \) is a closed immersion. \( \square \)
Locally closed immersions have the following useful properties.

**Lemma 3.10.** Let $f: X \to Y$ be a locally closed immersion. Then $f$ preserves residue fields: For each point $x \in X$, $\kappa(x) = \kappa(f(x))$.

**Proof.** Let $y = f(x)$. The statement is local, so we may assume $X = \text{Spec } A$, $Y = \text{Spec } B$, and the corresponding homomorphism $B \to A$ is surjective with kernel $I$. Now $y$ is a prime ideal of $B$ containing $I$, and $x = y/I$, so that $A/x = B/y$. But $\kappa(x)$ is the fraction field of $A/x$ and $\kappa(y)$ is the fraction field of $B/y$, which therefore coincide. □

**Convention 3.11.** For a scheme $X$, let us write $|X|$ for the underlying topological space. If $f: X \to Y$ is morphism of schemes, we write $|f|: |X| \to |Y|$ for the underlying continuous map.

**Lemma 3.12.** Let $f: X \to Z$ and $g: Y \to Z$ be morphisms of schemes, and assume $g$ is a locally closed immersion. Then the canonical continuous map $|X \times_Z Y| \to |X| \times_{|Z|} |Y|$ given by $q \mapsto (\pi_X(q), \pi_Y(q))$ is a bijection.

**Proof.** By Lemma 3.6, the points of $X \times_Z Y$ are given by triples $(x, y, p)$ with $f(x) = g(y)$ and $p$ a prime ideal in $\kappa(x) \otimes_{\kappa(y)} \kappa(y)$, where $z = g(y)$. Since $g$ is a locally closed immersion, by Lemma 3.10, we have $\kappa(y) = \kappa(z)$, so that $\kappa(x) \otimes_{\kappa(y)} \kappa(y) = \kappa(x)$, and we must have $p = 0$. It follows that the correspondence $p \mapsto (\pi_X(p), \pi_Y(p))$ is a bijection. The universal property for the topological fiber products shows that this map is continuous. □

**Remark 3.13.** Though we will not need this, the map in Lemma 3.12 is actually a homeomorphism. In fact, $|X \times_Z Y|$ has a basis of open sets of the form $U \times_W V$ where $U = \text{Spec } A$, $V = \text{Spec } B$ and $W = \text{Spec } C$ are affine open subsets of $X, Y, Z$, respectively, and where $A$ and $B$ are $C$-algebras. Moreover, the fact that $Y \to Z$ is a locally closed immersion means that we can take $B = C/I$ for some ideal $I$ of $C$. Then $U \times_W V = \text{Spec } A/AI$. Using this, one can show that the image of $|U \times_W V|$ in $|X| \times_{|Z|} |Y|$ is equal to $|U| \times_{|W|} |V|$, which is open in $|X| \times_{|Z|} |Y|$. Thus the map is open as well as being a continuous bijection, and hence it is a homeomorphism.

Now we can describe the points of the equalizer.

**Theorem 3.14.** Let $i: E(f, g) \to X$ be the equalizer of two morphisms $f, g: X \to Y$. Then the set map $|i|$ is injective with image equal to 

\[ \{ x \in X : f(x) = g(x) \Rightarrow y \text{ and the maps } \kappa(y) \to \kappa(x) \text{ induced by } f \text{ and } g \text{ agree} \} \]

**Proof.** By Proposition 3.1, we may identify the equalizer with $\pi_X: X \times_{(Y \times_S Y)} Y \to X$. Since $\Delta: Y \to Y \times_S Y$ is a locally closed immersion, by Lemma 3.12 we have a continuous bijection

\[ |X \times_{(Y \times_S Y)} Y| \xrightarrow{i} |X| \times_{|Y \times_S Y|} |Y| \]

induced by the diagram

\[
\begin{array}{ccc}
|X \times_{(Y \times_S Y)} Y| & \xrightarrow{\pi_Y} & |Y| \\
\downarrow j & & \downarrow \pi_Y \\
|X| \times_{|Y \times_S Y|} |Y| & \xrightarrow{\pi_Y \circ g} & |Y| \\
\downarrow \pi_X & & \downarrow \Delta \\
|X| & \xrightarrow{f \times g} & |Y \times_S Y| \\
\end{array}
\]

(3.4)
First we show that \( \pi_X \) is injective on points. Since \( j \) is a bijection, it suffices to show that \( \pi_{|X|} \) is injective. Suppose that \( (x_i, y_i) \in |X| \times |Y| \) for \( i = 1, 2 \) have the same image under \( \pi_{|X|} \). Then \( x_1 = x_2 =: x \). Going around the other way in diagram (3.4) shows that
\[
|\Delta| \circ \pi_{|Y|}(x, y_1) = |\Delta| \circ \pi_{|Y|}(x, y_2).
\]
That is, \( \Delta(y_1) = \Delta(y_2) \). Since \( \Delta \) followed by either projection to \( Y \) is the identity, we conclude that \( y_1 = y_2 \). Hence \( \pi_{|X|} \) is injective.

Now we describe the image of \( i = \pi_X \). Suppose that \( x \in \text{im}(i) \). Let \( h \colon \text{Spec} \kappa(x) \to X \) be the canonical morphism. Since \( i \) is a locally closed immersion, it is an isomorphism on residue fields, and we find that \( E(f, g) \times_X \text{Spec} \kappa(x) \cong \text{Spec} \kappa(x) \). It follows from Proposition 3.4 that \( E(f \circ h, g \circ h) \cong \text{Spec} \kappa(x) \). Hence, \( E(f \circ h, g \circ h) \to \text{Spec} \kappa(x) \) is the identity, and we conclude that \( f \circ h = g \circ h \). That is, \( f(x) = g(x) \) and the maps \( \kappa(y) \to \kappa(x) \) induced by \( f \) and \( g \) agree.

Conversely, suppose that \( x \in X \) is such that \( f(x) = g(x) \), and let us write \( y \) for this common value. We suppose further that the maps \( \varphi, \psi \colon \kappa(y) \to \kappa(x) \) induced by \( f \) and \( g \) agree. As above, write \( h \colon \text{Spec} \kappa(x) \to X \). Since the fiber \( i^{-1}(x) \) is homeomorphic to \( E(f, g) \times_X \text{Spec} \kappa(x) \), and since the latter is isomorphic to \( E(f \circ h, g \circ h) \) by Proposition 3.4, it suffices to show that \( E(f \circ h, g \circ h) \) is nonempty. To that end, we may replace \( X \) with \( \text{Spec} \kappa(x) \) and \( h \) with the identity. As we are assuming that \( f(x) = g(x) = y \), the question is now local on \( S \) and \( Y \), so we may assume \( S = \text{Spec} R \) and \( Y = \text{Spec} A \). Then \( f \) and \( g \) correspond to \( R \)-algebra homomorphisms \( \varphi, \psi \colon A \to \kappa(x) \), respectively. Since \( f(x) = g(x) \), these two homomorphisms have the same kernel \( m \). Moreover, the corresponding homomorphisms \( A/m \to \kappa(x) \) agree, so we conclude that \( \varphi = \psi \). That is, \( f = g \), and hence \( E(f, g) \) is nonempty, as desired.

**Example 3.15.** Let \( k \) be a field and set \( X = \text{Spec} k[[e]]/(e^2) \). Choose two elements \( a, b \in k \) with \( a \neq b \) and define morphisms \( f, g \colon X \to A_k^1 = \text{Spec} k[t] \) by \( t \to at \) and \( t \to bt \), respectively. Then we claim that the equalizer is \( E(f, g) = \text{Spec} k \), given as a closed subscheme of \( X \). To see this, first observe that the unique point \( x \in X \) maps to the origin \( 0 \in A_k^1 \), and the induced maps of residue fields \( k = \kappa(0) \to \kappa(x) = k \) are the identity. Hence, the theorem shows that \( E(f, g) \) consists of a single point. We also know that \( E(f, g) \) is a locally closed subscheme of \( X \), so it is either \( \text{Spec} k \) or \( \text{Spec} k[[e]]/(e^2) \). It is evidently not the latter because \( f \) and \( g \) disagree as morphisms of schemes.

**Corollary 3.16.** Let \( X \) be an \( S \)-scheme and \( f, g \in X(S) = \text{Hom}_S(S, X) \). Then the equalizer \( i \colon E(f, g) \to S \) is injective as a map of sets, and identifies \( |E(f, g)| \) with the set \( \{ s \in S : f(s) = g(s) \} \).

**Proof.** Write \( \rho \colon X \to S \) for the structure morphism of \( X \). Since \( f \) is an \( S \)-morphism, the composition \( \rho \circ f \) is the identity. For \( s \in S \), this yields a canonical isomorphism \( \kappa(f(s)) \iso \kappa(s) \). In particular, if \( g(s) = f(s) = x \), then the induced isomorphisms of residue fields \( \kappa(x) \to \kappa(s) \) agree. Now apply the theorem.

Finally, we close with an intuitive way to characterize the equalizer, at least in the case where the target is separated.

**Proposition 3.17.** Let \( f, g \colon X \to Y \) be morphisms of \( S \)-schemes. If \( Y \) is separated over \( S \), then the equalizer \( E(f, g) \to X \) is a closed immersion. In particular, the equalizer is the largest closed subscheme of \( X \) on which \( f \) and \( g \) agree.

**Proof.** By Proposition 3.1 the equalizer \( E(f, g) \to X \) is a base extension of the diagonal \( \Delta \colon Y \to Y \times_S Y \) via the morphism \( f \times g \colon X \to Y \times_S Y \). Closed immersions are stable under base extension.

**Example 3.18.** If \( Y \) is not separated, then the equalizer of two morphisms \( f, g \colon X \to Y \) can fail to be a closed subscheme of \( X \). For example, let \( Y \) be the affine line with doubled origin. Write \( Y = Y_1 \cup Y_2 \) with \( Y_1 = Y_2 = k^1 \), where \( Y_1 \setminus \{ 0 \} \) is identified with \( Y_2 \setminus \{ 0 \} \). Let \( f, g \colon k^1 \to Y \) be the open immersions of \( Y_1 \) and \( Y_2 \), respectively. Then the set of points on which \( f \) and \( g \) agree is \( k^1 \setminus \{ 0 \} \), which is not closed.
4. Vanishing Loci for Sections of Vector Bundles

Let $X$ be a scheme, and let $E$ be a vector bundle on $X$. For any global section $s$ of $E$, we know intuitively what the zero locus of $s$ should be:

“zeroes of $s$” $= \{ x \in X \mid s_x \in \mathfrak{m}_x E_x \}$.

However, this is only a set, and we would like to endow it with a scheme structure. To that end, we should look for functions on $X$ that characterize the vanishing of $s$. If we view $s$ geometrically as a map from $X$ to (the total space of) $E$, then for all functions $f$ on $E$, the composition $f \circ s$ is a function on $X$ that vanishes precisely when $s$ does. This motivates the following definition.

**Definition 4.1.** Given a scheme $X$, a vector bundle $E$ on $X$, and a global section $s$ of $E$, we define the zero scheme of $s$ to be the closed subscheme of $X$ cut out by the ideal sheaf $\text{im}(s^\vee)$. Here $s$ determines a homomorphism of $\mathcal{O}_X$-modules $s: \mathcal{O}_X \to E$, and we take $s^\vee: E^\vee \to \mathcal{O}_X$ to be the dual homomorphism.

**Remark 4.2.** Global sections of $E$ can be identified with $X$-morphisms from $X$ to the geometric vector bundle $\text{Spec}(\text{Sym}_X E^\vee)$ associated to $E$ (see [5, II:ex.5.18]). If $z$ is the zero section, then we could also have defined the zero scheme of $s$ to be the equalizer of $s$ and $z$, which is a closed immersion by Proposition 3.17. In fact, Proposition 4.7 below will show that this agrees with our definition.

The next proposition explains why this definition captures our intuition of the zero locus of $s$.

**Proposition 4.3.** Let $X$ be a scheme, $E$ a vector bundle on $X$, and $s$ a global section of $E$. As sets, we have

$$Z(\text{im}(s^\vee)) = \{ x \in X \mid s_x \in \mathfrak{m}_x E_x \}.$$

**Proof.** It suffices to work locally, so we may assume that $X = \text{Spec} R$, $E = R^m$, and $s = (s_1, \ldots, s_m) \in R^m$. Then $\text{im}(s^\vee) = (\sum r_i s_i : r_i \in R)$. So

$$p \in Z(\text{im}(s^\vee)) \iff p \supseteq \text{im}(s^\vee) \iff s_i \in p \text{ for all } i \iff s_p \in pE_p. \Box$$

**Remark 4.4.** Definition 4.1 is meaningful—but not correct—for sections of general sheaves. Indeed, the intuitive notion of zero locus of a section may not even be closed. For example, let $X = \text{Spec} \mathbb{Z}_p$, $E = \mathbb{F}_p$, and $s \in E$ any nonzero element. Then at the generic point $x = (0)$, we have $E_x = 0$, so $s_x \in \mathfrak{m}_x E_x$; but at the closed point $x = (p)$, we have $\mathfrak{m}_x = 0$ and $s_x = s \neq 0$, so $s_x \notin \mathfrak{m}_x E_x$. Thus the zero locus of $s$ is $X - \{(p)\}$, which is open and not closed. The zero scheme, as given by Definition 4.1, is $Z(\text{im}(s^\vee)) = X$ since $E^\vee = 0$.

In the case of line bundles, we have an alternate characterization in terms of annihilators. Though this is interesting in its own right, we do not use it in what follows.

**Proposition 4.5.** Let $X$ be a scheme, $L$ a line bundle on $X$, and $s$ a global section of $L$. We have an equality of ideal sheaves

$$\text{im}(s^\vee) = \text{Ann}(L/(s)),$$

where $s^\vee: L^\vee \to \mathcal{O}_X$ is the dual to the morphism $s: \mathcal{O}_X \to L$.

**Proof.** It suffices to work locally, in which case we take $X = \text{Spec} R$ and $L = R$. Then $\text{Ann}(L/(s)) = (s)$. An element of $L^\vee$ is given by multiplication-by-$r$ for some $r \in R$. So

$$\text{im}(s^\vee) = \{ rs : r \in R \} = (s). \Box$$
Remark 4.6. The same “zero locus” description does not apply to $\text{Ann}(E/(s))$ for a vector bundle $E$ of rank larger than 1. For example, if $X = \text{Spec } R$, $E = R \oplus R$, and $s = (1, 0)$, we see that the map $s^\vee: E^\vee \to R$ is given by dot product with $s$. Hence,

$$\text{im}(s^\vee) = R \implies Z(\text{im}(s^\vee)) = \emptyset$$

$\text{Ann}(E/(s)) = 0 \implies Z(\text{Ann}(E/(s))) = X.$

So the annihilator is not capturing the fact that $s$ is nowhere vanishing.

To conclude this section, we show that the zero scheme of a section of a vector bundle is characterized by a universal property.

Proposition 4.7. Let $X$ be a scheme, $E$ a vector bundle on $X$, and $s$ a global section of $E$. Write $Z$ for the zero scheme of the section $s$, and $i: Z \to X$ for the canonical closed immersion. Then

1. $i^*(s) = 0$;
2. If $j: W \to X$ is a morphism such that $j^*(s) = 0 \in j^*(E)$, then $j$ factors uniquely through $i$.

Proof. The first statement is local so it suffices to take $X$ affine, $X = \text{Spec } R$, $E = R^m$, and $s = (s_1, \ldots, s_m)$. By the proof of Proposition 4.3, the zero scheme of $s$ is cut out by the ideal

$I = (s_1, \ldots, s_m) \subset R$,

so $Z = \text{Spec } R/I$. It follows that $i^*E = (R/I)^m$ and $i^*(s) = (0, \ldots, 0)$, proving (1). For (2), by taking suitable open covers, we can reduce to the case of $W$ and $X$ affine and $E$ free. Say $X = \text{Spec } R$ and $W = \text{Spec } T$. Then $s = (s_1, \ldots, s_m)$ is an element of $R^m$, and $Z = \text{Spec } R/I$ as above. Now $j$ gives a ring homomorphism $\varphi: R \to T$, and $j^*(s) = 0$ becomes $\varphi(s_k) = 0$ for $k = 1, \ldots, m$, so that $\varphi$ factors uniquely through $R/I$. That is, $j$ factors through $i$, at least locally. But by the uniqueness of factoring through a quotient ring, these locally defined maps $W \to Z$ must agree on intersections, hence define a global morphism. \(\square\)

As a final remark, we note that everything in this section can be generalized to the case of several sections of a vector bundle $E$. If $s_0, \ldots, s_n$ are global sections of $E$, then the common zero scheme of these sections is the closed subscheme associated with the ideal sheaf $\text{im}(s_0^\vee \oplus \cdots \oplus s_n^\vee)$, where $s_0^\vee \oplus \cdots \oplus s_n^\vee$ is the dual of the morphism of sheaves $O_X \to \bigoplus_{i=0}^n E$ given by $1 \mapsto (s_0, \ldots, s_n)$. Analogues of Propositions 4.3 and 4.7 hold.

5. Strongly Distinct Morphisms

Definition 5.1. Let $X$ and $Y$ be schemes. We say that two morphisms $f, g: X \to Y$ are strongly distinct if the equalizer of $f$ and $g$ is $\emptyset \to X$. (Here $\emptyset$ is the empty scheme, which is the initial object in the category of schemes.) We write $f \neq g$ to denote that $f$ and $g$ are strongly distinct.

A consequence of Theorem 3.14 is that two morphisms $f, g: X \to Y$ are strongly distinct if they “disagree at every point of $X$”:

Corollary 5.2. Let $f, g: X \to Y$ be morphisms of schemes. If $f(x) \neq g(x)$ for all $x \in X$, then $f$ and $g$ are strongly distinct.

Remark 5.3. The converse of this corollary does not hold in general. For example, consider the two morphisms $f, g: \text{Spec } \mathbb{Q}(i) \to \text{Spec } \mathbb{Z}[t] = \mathbb{A}^1_{\mathbb{Z}}$ given by $t \mapsto \pm i$. Then $f$ and $g$ have the same image, namely the prime ideal $(t^2 + 1)$. But in fact they are strongly distinct. To see this, consider the induced $\mathbb{Q}(i)$ morphisms $f', g': \text{Spec } \mathbb{Q}(i) \to \text{Spec } \mathbb{Q}(i)[t] = \mathbb{A}^1_{\mathbb{Q}(i)}$. By Proposition 3.5, these have the same equalizer as $f$ and $g$. But now $f'$ and $g'$ have distinct images in $\mathbb{A}^1_{\mathbb{Q}(i)}$, so $E(f, g) = E(f', g') = \emptyset.$

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By Corollary 3.16, the converse of the previous corollary does hold if we restrict our attention to $S$-points of an $S$-scheme:

**Corollary 5.4.** Let $X$ be a scheme over $S$. Two points $f,g \in X(S) = \text{Hom}_S(S,X)$ are strongly distinct if and only if $f(s) \neq g(s)$ for all $s \in S$.

**Example 5.5.** Let $X$ be a scheme of finite type over a field $k$ — i.e., a variety. Two points $f,g \in X(k)$ are strongly distinct if and only if they are distinct in the usual sense.

The property of being strongly distinct is preserved under precomposition by an arbitrary morphism.

**Proposition 5.6.** Let $X,Y,Z$ be schemes, and let $h : Z \to X$ be a morphism. The induced map

$$h^* : \text{Hom}(X,Y) \to \text{Hom}(Z,Y)$$

preserves strong distinctness.

**Proof.** Proposition 3.4 shows that the equalizer of $f \circ h$ and $g \circ h$ satisfies $E(f \circ h, g \circ h) \cong E(f,g) \times_X Z$. Since $f$ and $g$ are strongly distinct, the first projection gives a morphism $E(f \circ h, g \circ h) \to E(f,g) = \emptyset$. It follows that $E(f \circ h, g \circ h) = \emptyset$ since that is the only scheme that admits a morphism to the empty scheme. (A map $0 \to R$ is a ring homomorphism if and only if $R = 0$.) Hence $f \circ h$ and $g \circ h$ are strongly distinct, as desired. 

**Remark 5.7.** Definition 5.1 can be generalized in an obvious way to talk about strongly distinct morphisms in a category with an initial object $\emptyset$. If we insist that the initial object is strictly initial — i.e., any morphism $X \to \emptyset$ is an isomorphism — then Proposition 5.6 and its proof are valid in this more general setting.

We now give an alternate description of the equalizer in the case of morphisms to $\mathbb{P}^1$.

**Theorem 5.8.** Let $a,b : S \to \mathbb{P}^1$ be morphisms determined by line bundles $A$ and $B$ with global sections $(a_0,a_1)$ and $(b_0,b_1)$, respectively. Let $Z \subset S$ be the zero scheme of the section $a_0 \otimes b_1 - a_1 \otimes b_0$ of $A \otimes B$. Then $Z \to S$ is the equalizer of $a$ and $b$.

We give the proof after first addressing a preliminary statement.

**Lemma 5.9.** Let $Z$ be a scheme and $A,B$ be line bundles on $Z$ generated by global sections $(a_0,a_1)$ and $(b_0,b_1)$, respectively, and suppose $a_0 \otimes b_1 = a_1 \otimes b_0 \in A \otimes B$. Then there exists a unique isomorphism $A \to B$ mapping $a_0 \mapsto b_0$ and $a_1 \mapsto b_1$.

**Proof.** We can find an affine open cover $\{U_i\}$ of $Z$ such that $A$ and $B$ are both trivial on each $U_i$. We can further assume that on each $U_i$, either $a_0$ or $a_1$ generates $A$. Suppose $a_0$ generates $A$ on $U_i$, and let $a_1 = r_1a_0$. Then

$$0 = a_0 \otimes b_1 - a_1 \otimes b_0$$

$$= a_0 \otimes b_1 - r_1a_0 \otimes b_0$$

$$= a_0 \otimes (b_1 - r_1b_0).$$

Now $A|_{U_i}$ is a rank-one free module with generator $a_0$, so $a_0 \otimes u = 0$ implies $u = 0$, and we get $b_1 - r_1b_0$, from which it also follows that $b_0$ generates $B$ on $U_i$. Then we can define an isomorphism $\varphi_i : A|_{U_i} \to B|_{U_i}$ by sending $a_0$ to $b_0$, which then also maps $a_1$ to $b_1$. If instead $a_1$ generates $A$ on $U_i$, a symmetrical argument shows that also $b_1$ generates and there is again an isomorphism $\varphi_i$ sending $a_0$ to $b_0$ and $a_1$ to $b_1$. Clearly these isomorphisms glue to form a global isomorphism $A \to B$. 

**Proof of Theorem 5.8.** Let $i : Z \to S$ be the canonical closed immersion, where $ZS$ is the zero scheme of $u := a_0 \otimes b_1 - a_1 \otimes b_0$. By Proposition 4.7 and the definition of $Z$, we have $i^*(u) = 0$. Then by Lemma 5.9, there is an isomorphism $i^*(A) \to i^*(B)$ taking $i^*(a_0)$ to $i^*(b_0)$ and $i^*(a_1)$ to $i^*(b_1)$. It follows that the data
is the following important criterion for strong distinctness, shows that we may be pairwise strongly distinct. Then the Determinant Criterion shows that \( \Delta(a, b) \) is a nowhere vanishing section of \( A \otimes B \). In particular, if \( a \neq b \), then \( B \cong A^{-1} \).

6. Cross-ratios

Let \( S \) be a fixed scheme throughout this section.

**Lemma 6.1.** Let \( a, b, c, d : S \to \mathbb{P}_S^1 \) be pairwise strongly distinct. Then \( a^* \mathcal{O}(1) \cong b^* \mathcal{O}(1) \cong c^* \mathcal{O}(1) \), and each of these line bundles is 2-torsion in \( \text{Pic}(S) \).

**Proof.** Set \( A = a^* \mathcal{O}(1) \), \( B = b^* \mathcal{O}(1) \), and \( C = c^* \mathcal{O}(1) \). Since \( a \neq b \neq c \neq a \), by Corollary 5.10, we have \( A \cong B^{-1} \cong C \cong A^{-1} \), and consequently also \( A \cong B \). \( \square \)

If \( a, b \in \mathbb{P}_S^1 \) are represented by the data \( (A; a_0, a_1) \) and \( (B; b_0, b_1) \), respectively, define the section \( \Delta(a, b) \in A \otimes B \) by \( a_0 \otimes b_1 - a_1 \otimes b_0 \). If \( a, b \) are strongly distinct, then the Determinant Criterion shows that \( \Delta(a, b) \) is a nowhere vanishing section of \( A \otimes B \).

**Definition 6.2** (Cross-Ratio). Let \( a, b, c, d \in \mathbb{P}_S^1 \) be four pairwise strongly distinct points, represented by the data \( (L; a_0, a_1) \), \( (L; b_0, b_1) \), \( (L; c_0, c_1) \), and \( (L; d_0, d_1) \), respectively. (Lemma 6.1 shows that we may choose all of the line bundles to be equal.) Choose an isomorphism \( \varphi : L^{\otimes 2} \to \mathcal{O}_S \). The cross-ratio of these four points, denoted \( (a, b; c, d) \), is the element of \( \Gamma(S, \mathcal{O}_S) \) defined by 

\[
(a, b; c, d) = \frac{\varphi(\Delta(a, c)) \cdot \varphi(\Delta(b, d))}{\varphi(\Delta(a, d)) \cdot \varphi(\Delta(b, c))}.
\]

The following are simple but useful consequences of the definition:

**Proposition 6.3.** With notation as in Definition 6.2,

1. \( (a, b; c, d) \) is independent of the choice of isomorphism \( \varphi \)
2. \( (a, b; c, d) \) is a unit in \( \Gamma(S, \mathcal{O}_S) \)
Proof. As every automorphism of \( \mathcal{O}_S \) is given by multiplication by a unit of \( \Gamma(S, \mathcal{O}_S) \), the definition makes it clear that \( (a, b; c, d) \) is independent of the choice \( \varphi \). The Determinant Criterion shows that the numerator and denominator of \( (a, b; c, d) \) are units of \( \Gamma(S, \mathcal{O}_S) \), so that \( (a, b; c, d) \) is itself a unit. \( \square \)

In order to see the connection with the usual cross-ratio, suppose that \( S = \text{Spec } k \) is the spectrum of a field. Then we may take \( L = \mathcal{O}_{\text{Spec } k} \) and \( \varphi : L \otimes L \to \mathcal{O}_{\text{Spec } k} \) is multiplication in \( k \). In that case, the cross-ratio of the four points is given by the quotient

\[
(a, b; c, d) = \frac{(a_0c_1 - a_1c_0)(b_0d_1 - b_1d_0)}{(a_0d_1 - a_1d_0)(b_0c_1 - b_1c_0)}
\]

If we dehomogenize by setting \( a_1 = b_1 = c_1 = d_1 = 1 \), this becomes the familiar formula for the cross-ratio:

\[
(a, b; c, d) = \frac{(a - c)(b - d)}{(a - d)(b - c)}.
\]

We set the following notation for standard points of \( \mathbb{P}^1(S) \):

\[
\infty = (\mathcal{O}_S; 1, 0) \quad \quad 0 = (\mathcal{O}_S; 0, 1) \quad \quad 1 = (\mathcal{O}_S; 1, 1).
\]

Note that \( \infty, 0, \) and \( 1 \) are pairwise strongly distinct by the Determinant Criterion.

**Theorem 6.4.** Let \( a, b, c \in \mathbb{P}^1(S) \) be pairwise strongly distinct. Then there is a unique \( \sigma \in \text{PGL}_2(S) \) such that \( \sigma(a) = \infty, \sigma(b) = 0, \) and \( \sigma(c) = 1 \).

Proof. By Lemma 6.1, we may assume that \( L := a^* \mathcal{O}(1) = b^* \mathcal{O}(1) = c^* \mathcal{O}(1) \), and that \( L \) is 2-torsion in \( \text{Pic}(S) \). Let \( x_0, x_1 \) be the standard generators of \( \mathcal{O}(1) \), and let \( a_i = a^*(x_i), b_i = b^*(x_i), \) and \( c_i = c^*(x_i) \). Fix an isomorphism \( \varphi : L^\otimes 2 \to \mathcal{O}_S \). Define \( \sigma \) by the pair \( (L; M) \), where \( M \in M_2(\mathcal{O}(L, S)) \) is given by:

\[
M = \begin{pmatrix}
\varphi(\Delta(a, c)) & 0 \\
0 & \varphi(\Delta(c, b))
\end{pmatrix} \star \begin{pmatrix}
b_1 & -b_0 \\
-a_1 & a_0
\end{pmatrix}.
\]

Applying this to \( a, b, c \) we find:

\[
\sigma(a) = (L^\otimes 2; \varphi(\Delta(a, c))\Delta(a, b), 0) = (\mathcal{O}_X; \varphi(\Delta(a, c))\varphi(\Delta(a, b)), 0) = \infty
\]

\[
\sigma(b) = (L^\otimes 2; 0, \varphi(\Delta(c, b))\Delta(a, b)) = (\mathcal{O}_X; 0, \varphi(\Delta(c, b))\varphi(\Delta(a, b))) = 0
\]

\[
\sigma(c) = (L^\otimes 2; \varphi(\Delta(a, c))\Delta(c, b), \varphi(\Delta(c, b))\Delta(a, c)) = (\mathcal{O}_X; \varphi(\Delta(a, c))\varphi(\Delta(c, b)), \varphi(\Delta(c, b))\varphi(\Delta(a, c))) = 1.
\]

If \( \sigma' \) is any other automorphism satisfying the desired property, then \( \tau := \sigma' \circ \sigma^{-1} \) is an automorphism fixing \( \infty, 0, 1 \in \mathbb{P}^1(S) \). We show that \( \tau \) is the identity. Write \( \tau = (T; A) \), where \( T \) is a line bundle on \( S \) and \( A = (t_{00}, t_{11}) \) is a matrix of sections of \( T \). Then

\[
\tau(\infty) = (T; A * (\frac{1}{0})) = (T; t_{00}, t_{10})
\]

\[
\tau(0) = (T; A * (\frac{0}{1})) = (T; t_{01}, t_{11})
\]

\[
\tau(1) = (T; A * (\frac{1}{1})) = (T; t_{00} + t_{01}, t_{10} + t_{11})
\]

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Since $\tau(\infty) = \infty$, we conclude that there is an isomorphism $\theta: T \cong O_S$ such that $\theta(t_{00}) = 1$ and $\theta(t_{10}) = 0$. As $\tau(0) = 0$, we may use this same isomorphism to find $\theta(t_{01}) = 0$ and $\theta(t_{11}) = r \in \Gamma(S, O_S)^\times$. Finally, since $\tau(1) = 1$, we conclude that

$$1 = \theta(t_{00}) + \theta(t_{01}) = \theta(t_{10}) + \theta(t_{11}) = r.$$ 

Thus, $\theta$ witnesses the equivalence between $(T; A)$ and $(O_S; 1)$; that is, $\tau$ is the identity. \hfill $\square$

**Theorem 6.5.** Let $a, b, c, d \in \mathbb{P}^1(S)$ be pairwise strongly distinct, and let $\sigma \in \text{PGL}_2(S)$ be the unique automorphism such that $\sigma(a) = \infty$, $\sigma(b) = 0$, and $\sigma(c) = 1$. Then $\sigma(d) = (O_S; z, 1)$, where $z = (a, b, c, d)$.

*Proof.* Apply the formula for $\sigma$ given by (6.1) to see that $\sigma(d) = (O_S; z_0, z_1)$, where

$$z_0 = \varphi(\Delta(a, c)) \varphi(\Delta(b, d)), \quad z_1 = \varphi(\Delta(a, d)) \varphi(\Delta(b, c)).$$

Since $a, b, c, d$ are pairwise strongly distinct, it follows from the Determinant Criterion that $z_1$ is a unit in $\Gamma(S, O_S)$. Applying the automorphism of $O_S$ given by multiplication by $z_1^{-1}$, we see that $\sigma(d) = (O_S; z_0/z_1, 1)$. Looking at the formula for the cross-ratio, we see that $z_0/z_1 = (a, b, c, d)$. \hfill $\square$

**Corollary 6.6.** Given two 4-tuples, $(x_1, x_2, x_3, x_4)$ and $(y_1, y_2, y_3, y_4)$, of pairwise strongly distinct points in $\mathbb{P}^1(S)$, the following are equivalent:

1. There exists $\gamma \in \text{Aut}_S(\mathbb{P}^1_S) \cong \text{PGL}_2(S)$ such that $y_i = \gamma(x_i)$ for $i = 1, 2, 3, 4$;
2. $(x_1, x_2; x_3, x_4) = (y_1, y_2; y_3, y_4)$.

*Proof.* There is a unique automorphism $\alpha$ taking $(x_1, x_2, x_3, x_4)$ to $(\infty, 0, 1, a)$ for some $a \in \mathbb{P}^1(S)$. Similarly there is a unique automorphism $\beta$ taking $(y_1, y_2, y_3, y_4)$ to $(\infty, 0, 1, b)$ for some $b \in \mathbb{P}^1(S)$. Thus if (1) holds, we must have $a = b$. But we also know that $a = (O_S; a, 1)$ and $b = (O_S; b, 1)$ with $a = (x_1, x_2; x_3, x_4)$ and $b = (y_1, y_2; y_3, y_4)$, so $a = b$ implies (2). Conversely, if (2) holds, then we have $a = b$ and we can set $\gamma = \beta^{-1} \alpha$. \hfill $\square$

**Proposition 6.7.** Let $a, b, c, d \in \mathbb{P}^1(S)$ be pairwise strongly distinct. The cross-ratio enjoys the following symmetries:

1. $(c, d; a, b) = (a, b; c, d)$;
2. $(b, a; c, d) = (a, b; c, d)^{-1}$; and
3. $(a, c; b, d) = 1 - (a, b; c, d)$.

*Remark 6.8.* Note that the permutation group $S_4$ is generated by the elements (12), (23), and (13)(24). The three formulas in the proposition give a complete description of how the $S_4$ action on a 4-tuple of pairwise strongly distinct points impacts the value of its cross-ratio. In particular, it shows that the Klein 4-group \{1, (12)(34), (13)(24), (14)(23)\} stabilizes the cross-ratio. This is well known in the classical setting.

*Proof of Proposition 6.7.* The first symmetry is immediate upon writing down the formula for $(c, d; a, b)$ and flipping the signs of the four factors.

For the second symmetry, write down the formula for $(b, a; c, d)$; it is visibly the inverse of the formula for $(a, b; c, d)$.

For the third symmetry, let $\sigma$ be the unique automorphism of $\mathbb{P}^1_S$ that satisfies $\sigma(a) = \infty$, $\sigma(b) = 0$, $\sigma(c) = 1$. By Theorem 6.5, we have $(a, b; c, d) = z$, where $\sigma(d) = (O_S; z, 1)$. Define

$$\tau = (O_S; \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \in \text{PGL}_2(S).$$

Then

$$\tau \circ \sigma(a) = \infty, \quad \tau \circ \sigma(c) = 0, \quad \tau \circ \sigma(b) = 1,$$ 

and

$$\tau \circ \sigma(d) = z.$$ 


\[ \tau \circ \sigma(d) = (\mathcal{O}_S; \left( \begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \ast (\mathcal{I})) = (\mathcal{O}_S; 1 - z, 1). \]

A second application of Theorem 6.5 shows that

\[(a, c; b, d) = 1 - z = 1 - (a, b; c, d). \]

We close with the observation that the cross-ratio behaves well under change of coefficients:

**Proposition 6.9.** Let \( f: T \to S \) be a morphism of schemes, and let \( a, b, c, d \in \mathbb{P}^1(S) \) be pairwise strongly distinct. If we write \( f^* \) for the pullback map on sections, then we have

\[ (f^*(a), f^*(b); f^*(c), f^*(d)) = f^*((a, b; c, d)). \]

**Proof.** As the automorphism in Theorem 6.4 is unique, we can build it locally and then glue. By Theorem 6.5, we can compute the cross-ratio locally. So it suffices to assume that \( S = \text{Spec} \, A, \, T = \text{Spec} \, B \), and that \( f \) is given by \( \psi: A \to B \). Moreover, we may assume that

\[ a = (\mathcal{O}_S; a_0, a_1), \quad b = (\mathcal{O}_S; b_0, b_1), \quad c = (\mathcal{O}_S; c_0, c_1), \quad d = (\mathcal{O}_S; d_0, d_1). \]

(Here it is important to note that strong distinctness implies that all four line bundles are isomorphic, so we may simultaneously trivialize them.) Now the result is clear from a computation:

\[ f^*(a, b; c, d) = \psi \left( \frac{\Delta(a, c) \Delta(b, d)}{\Delta(a, d) \Delta(b, c)} \right) = \frac{\Delta(\psi(a), \psi(c)) \Delta(\psi(b), \psi(d))}{\Delta(\psi(a), \psi(d)) \Delta(\psi(b), \psi(c))} = (f^*(a), f^*(b); f^*(c), f^*(d)). \]

\[ \square \]

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