Nonlinear dynamics of a system of particle-like wavepackets

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March 30, 2022

Abstract

This work continues our studies of nonlinear evolution of a system of wavepackets. We study a wave propagation governed by a nonlinear system of hyperbolic PDE’s with constant coefficients with the initial data being a multi-wavepacket. By definition a general wavepacket has a well defined principal wave vector, and, as we proved in previous works, the nonlinear dynamics preserves systems of wavepackets and their principal wave vectors. Here we study the nonlinear evolution of a special class of wavepackets, namely particle-like wavepackets. A particle-like wavepacket is of a dual nature: on one hand, it is a wave with a well defined principal wave vector, on the other hand, it is a particle in the sense that it can be assigned a well defined position in the space. We prove that under the nonlinear evolution a generic multi-particle wavepacket remains to be a multi-particle wavepacket with a high accuracy, and every constituting single particle-like wavepacket not only preserves its principal wave number but also it has a well-defined space position evolving with a constant velocity which is its group velocity. Remarkably the described properties hold though the involved single particle-like wavepackets undergo nonlinear interactions and multiple collisions in the space. We also prove that if principal wavevectors of multi-particle wavepacket are generic, the result of nonlinear interactions between different wavepackets is small and the approximate linear superposition principle holds uniformly with respect to the initial spatial positions of wavepackets.

1 Introduction

The principal object of our studies here is a general nonlinear evolutionary system which describes wave propagation in homogeneous media governed by a hyperbolic PDE’s in \( \mathbb{R}^d \), \( d = 1, 2, 3, \ldots \) is the space dimension, of the form

\[
\frac{\partial}{\partial \tau} U = -\frac{i}{\ell} L (-i \nabla) U + F(U), \quad U(r, \tau)|_{\tau=0} = h(r), \quad r \in \mathbb{R}^d, \tag{1.1}
\]
where (i) $\mathbf{U} = \mathbf{U}(\mathbf{r}, \tau)$, $\mathbf{r} \in \mathbb{R}^d$, $\mathbf{U} \in \mathbb{C}^{2J}$ is a 2J dimensional vector; (ii) $\mathbf{L}(-i\nabla)$ is a linear self-adjoint differential (pseudodifferential) operator with constant coefficients with the symbol $\mathbf{L}(\mathbf{k})$, which is a Hermitian $2J \times 2J$ matrix; (iii) $\mathbf{F}$ is a general polynomial nonlinearity; (iv) $\sigma > 0$ is a small parameter. The properties of the linear part are described in terms of dispersion relations $\omega_n(\mathbf{k})$ (eigenvalues of the matrix $\mathbf{L}(\mathbf{k})$). The form of the equation suggests that the processes described by it involve two time scales. Since the nonlinearity $\mathbf{F}(\mathbf{U})$ is of order one, nonlinear effects occur at times $\tau$ of order one, whereas the natural time scale of linear effects, governed by the operator $\mathbf{L}$ with the coefficient $1/\sigma$, is of order $\sigma$. Consequently, the small parameter $\sigma$ measures the ratio of the slow (nonlinear effects) time scale and the fast (linear effects) time scale. A typical example of an equation of the form (1.1) is the nonlinear Schrödinger equation (NLS) or a system of NLS’s. Many more examples including a general nonlinear wave equation and the Maxwell equations in periodic media truncated to a finite number of bands are considered in [8], [9].

As in our previous works [8], [9] we consider here the nonlinear evolutionary system (1.1) with the initial data $\mathbf{h}(\mathbf{r})$ being a sum of wavepackets. The special focus of this paper is particle-like localized wavepackets which can be viewed as quasi-particles. Recall that a general wavepacket is defined as such a function $\mathbf{h}(\mathbf{r})$ that its Fourier transform $\hat{\mathbf{h}}(\mathbf{k})$ is localized in $\beta$-neighborhood of a single wavevector $\mathbf{k}_s$, called principal wavevector, where $\beta$ is a small parameter. The simplest example of a wavepacket is a function of the form

$$\hat{\mathbf{h}}(\beta; \mathbf{k}) = \beta^{-d} e^{-i\mathbf{kr}_s} \hat{h} \left( \mathbf{k} - \mathbf{k}_s \right) \mathbf{g}_n(\mathbf{k}_s), \quad \mathbf{k} \in \mathbb{R}^d,$$

(1.2)

where $\mathbf{g}_n(\mathbf{k}_s)$ is an eigenvector of the matrix $\mathbf{L}(\mathbf{k}_s)$ and $\hat{h}(\mathbf{k})$ is a scalar Schwartz function (i.e. it is infinitely smooth and rapidly decaying one). Note that for $\hat{\mathbf{h}}(\beta, \mathbf{k})$ of the form (1.2) we have its inverse Fourier transform

$$\mathbf{h}(\beta; \mathbf{r}) = h(\beta(\mathbf{r} - \mathbf{r}_s)) e^{i\mathbf{k}_s(\mathbf{r}-\mathbf{r}_s)} \mathbf{g}_n(\mathbf{k}_s), \quad \mathbf{r} \in \mathbb{R}^d.$$

(1.3)

Evidently, $\mathbf{h}(\beta, \mathbf{r})$ described by the above formula is a plane wave $e^{i\mathbf{k}_s \mathbf{r}} \mathbf{g}_n(\mathbf{k}_s)$ modulated by a slowly varying amplitude $h(\beta(\mathbf{r} - \mathbf{r}_s))$ obtained from $h(\mathbf{z})$ by a spatial shift along the vector $\mathbf{r}_s$ with a subsequent dilation with a large factor $\frac{1}{\beta}$. Clearly, the resulting amplitude has a typical spatial extension proportional to $\beta^{-1}$ and the spatial shift produces a noticeable effect if $|\mathbf{r}_s| \gg \beta^{-1}$. The spatial form of the wavepacket (1.3) naturally allows to interpret $\mathbf{r}_s \in \mathbb{R}^d$ as its position and, consequently, to consider the wavepacket as a particle-like one with the position $\mathbf{r}_s \in \mathbb{R}^d$. But how one can define a position for a general wavepacket? Note that not every wavepacket is a particle-like one. For example, let, as before, the function $h(\mathbf{r})$ be a scalar Schwartz function and let us consider a slightly more general than (1.3) function

$$\mathbf{h}(\beta; \mathbf{r}) = \left[ h(\beta(\mathbf{r} - \mathbf{r}_{s1})) + h(\beta(\mathbf{r} - \mathbf{r}_{s2})) \right] e^{i\mathbf{k}_s \mathbf{r}} \mathbf{g}_n(\mathbf{k}_s), \quad \mathbf{r} \in \mathbb{R}^d,$$

(1.4)

where $\mathbf{r}_{s1}$ and $\mathbf{r}_{s2}$ are two arbitrary, independent vector variables. The wave $\mathbf{h}(\beta, \mathbf{r})$ defined by (1.4) is a wavepacket with the wave number $\mathbf{k}_s$ for any choice of vectors $\mathbf{r}_{s1}$ and $\mathbf{r}_{s2}$, but
it is not a particle-like wavepacket, since it does not have a single position \(r_0\), but rather it is a sum of two particle-like wavepackets with two positions \(r_{*1}\) and \(r_{*2}\).

Our way to introduce a general particle-like wavepacket \(h(\beta, k_*, r_0; r)\) with a position \(r_0\) is by treating it as a single element of a family of wavepackets \(h(\beta, k_*, r_*; r)\) with \(r_* \in \mathbb{R}^d\) being another independent parameter. In fact, we define the entire family of wavepackets \(h(\beta, k_*, r_*; r)\), \(r_* \in \mathbb{R}^d\) subject to certain conditions allowing to interpret any fixed \(r_* \in \mathbb{R}^d\) as the position of \(h(\beta, k_*, r_*; r)\). Since we would like, of course, a wavepacket to maintain its particle-like property under the nonlinear evolution, it is clear that its definition must be sufficiently flexible to accommodate the wavepacket evolutionary variations. In light of the above discussion the definition of the particle-like wavepacket with a transparent interpretation of its particle properties turns into the key element of the entire construction. It turns out that there is a precise description of a particle-like wavepacket, which is rather simple and physically transparent and such a description is provided in Definition 2.2 below, see also Remarks 2.4, 2.5. The concept of the position is applicable to very general functions, it does not require a parametrization of the whole family of solutions, which was used, for example in [32], [25], [26].

As in our previous works we are interested in nonlinear evolution not only a single particle-like wavepacket \(h(\beta, k_*, r_*; r)\) but a system \(\{h(\beta, k_L, r_L; r)\}\) of particle-like wavepackets which we call multi-particle wavepacket. Under certain natural conditions of genericity on \(k_L\) we prove here that under the nonlinear evolution: (i) the multi-particle wavepacket remains to be a multi-particle wavepacket; (ii) the principal wavevectors \(k_L\) remain constant; (iii) the spatial position \(r_L\) of the corresponding wavepacket evolves with the constant velocity which is exactly its group velocity \(\frac{1}{\varrho} \nabla \omega_n(k_L)\). The evolution of positions of wavepackets becomes the most simple in the case when at \(\tau = 0\) we have \(r_L = \frac{1}{\varrho} r_{*L}\), that is the case when spatial positions are bounded in the same spatial scale in which their group velocities are bounded. In this case the evolution of the positions is described by the formula

\[
\mathbf{r}_l(\tau) = \frac{1}{\varrho} \left[ r_{*L} + \tau \nabla \omega_n(k_L) \right], \quad \tau \geq 0.
\]

The rectilinear motion of positions of particle-like wavepackets is a direct consequence of the spatial homogeneity of the master system (1.1). If the system were not spatially homogeneous, the motion of the positions of particle-like wavepackets would not be uniform, but we don’t study that problem in this paper. In the rescaled coordinates \(y = \varrho r\) the trajectory of every particle is a fixed, uniquely defined straight line defined uniquely if \(\varrho, \beta \to 0\). Notice that under above mentioned genericity condition the uniform and independent motion (1.5) of the positions of all involved particle-like wavepackets \(\{h(\beta, k_L, r_L; r)\}\) persists though they can collide in the space. In the latter case they simply pass through each other without significant nonlinear interactions, and the nonlinear evolution with a high accuracy is reduced just to a nonlinear evolution of shapes of the particle-like wavepackets. In the case when the set of the principal wavevectors \(\{k_L\}\) satisfy certain resonance conditions some components of the original multi-particle wavepacket can evolve into a more complex structure which can be only partly localized in the space and, for instance, can be needle or pancake like. We do
not study in detail those more complex structures here.

Now let us discuss in more detail the superposition principle introduced and studied for general multi-wavepackets in [9] in the particular case when initially all \( r_{\ast l} = 0 \). Here we consider multi-particle wavepackets with arbitrary \( r_{\ast l} \) and develop a new argument based on the analysis of an averaged wavepacket interaction system introduced in [8]. Assume that the initial data \( h \) for the evolution equation (1.1) to be the sum of a finite number of wavepackets (particle-like wavepackets) \( h_l, l = 1, \ldots, N \), i.e.

\[
h = h_1 + \ldots + h_N \tag{1.6}
\]

where the monochromaticity of every wavepacket \( h_l \) is characterized by another small parameter \( \beta \). The well known superposition principle is a fundamental property of every linear evolutionary system, stating that the solution \( U \) corresponding to the initial data \( h \) as in (1.6) equals

\[
U = U_1 + \ldots + U_N, \text{ for } h = h_1 + \ldots + h_N, \tag{1.7}
\]

where \( U_l \) is the solution to the same linear problem with the initial data \( h_l \).

Evidently the standard superposition principle can not hold exactly as a general principle for a nonlinear system, and, at the first glance, there is no expectation for it to hold even approximately. We show though that, in fact, the superposition principle does hold with a high accuracy for general dispersive nonlinear wave systems such as (1.1) provided that the initial data are a sum of generic particle-like wavepackets, and this constitutes one of the subjects of this paper. Namely, the superposition principle for nonlinear wave systems states that the solution \( U \) corresponding to the multi-particle wavepacket initial data \( h \) as in (1.6) satisfies

\[
U = U_1 + \ldots + U_N + D, \text{ for } h = h_1 + \ldots + h_N, \text{ where } D \text{ is small.}
\]

More detailed statement of the superposition principle for nonlinear evolution of wavepackets is as follows. We study the nonlinear evolution equation (1.1) on a finite time interval

\[
0 \leq \tau \leq \tau^\ast, \text{ where } \tau^\ast > 0 \text{ is a fixed number} \tag{1.8}
\]

which may depend on the \( L^\infty \) norm of the initial data \( h \) but, importantly, \( \tau^\ast \) does not depend on \( \varrho \). We consider classes of initial data such that wave evolution governed by (1.1) is significantly nonlinear on time interval \([0, \tau^\ast]\) and the effect of the nonlinearity \( F(U) \) does not vanish as \( \varrho \to 0 \). We assume that \( \beta, \varrho \) satisfy

\[
0 < \beta \leq 1, \ 0 < \varrho \leq 1, \ \frac{\beta^2}{\varrho} \leq C_1 \text{ with some } C_1 > 0. \tag{1.9}
\]

The above condition of boundedness on the dispersion parameter \( \frac{\beta^2}{\varrho} \) ensures that the dispersive effects are not dominant and they do not suppress nonlinear effects, see [8], [9] for a discussion.
Let us introduce the solution operator $S(h)(\tau) : h \rightarrow U(\tau)$ relating the initial data $h$ of the nonlinear evolution equation (1.1) to its solution $U(t)$. Suppose that the initial state is a system of particle-like wavepackets or multi-particle wavepacket, namely $h = \sum h_l$, with $h_l, l = 1, \ldots, N$ being ”generic” wavepackets. Then for all times $0 \leq \tau \leq \tau_*$ the following superposition principle holds

$$S \left( \sum_{l=1}^{N} h_l \right)(\tau) = \sum_{l=1}^{N} S(\ h_l)(\tau) + D(\tau), \quad (1.10)$$

$$\|D(\tau)\|_{E} = \sup_{0 \leq \tau \leq \tau_*} \|D(\tau)\|_{L^\infty} \leq C_\delta \frac{\theta}{\beta^{1+\delta}}$$

for any small $\delta > 0$. (1.11)

Obviously, the right-hand side of (1.11) may be small only if $\rho \leq C_1 \beta$. There are examples (see [8]) in which $D(\tau)$ is not small for $\rho = C_1 \beta$. In what follows we refer to a linear combination of particle-like wavepackets as a multi-particle wavepacket, and to single particle-like wavepackets which constitutes the multi-particle wavepacket as component particle wavepackets.

Very often in theoretical studies of equations of the form (1.1) or ones reducible to it a functional dependence between $\rho$ and $\beta$ is imposed, resulting in a single small parameter. The most common scaling is $\rho = \beta^2$. The nonlinear evolution of wavepackets for a variety of equations which can be reduced to the form (1.1) was studied in numerous physical and mathematical papers, mostly by asymptotic expansions of solutions with respect to a single small parameter similar to $\beta$, see [13], [16], [20], [22], [27], [33], [34], [41], [45], [47], [48] and references therein. Often the asymptotic expansions are based on a specific ansatz prescribing a certain form to the solution. In our studies here we do not use asymptotic expansions with respect to a small parameter and do not prescribe a specific form to the solution, but we impose conditions on the initial data requiring it to be a wavepacket or a linear combination of wavepackets. Since we want to establish a general property of a wide class of systems, we apply a general enough dynamical approach. There is a number of general approaches developed for the studies of high-dimensional and infinite-dimensional nonlinear evolutionary systems of hyperbolic type, [12], [15], [21], [24], [31], [37], [42], [47], [49], [51], [53]) and references therein. The approach we develop here is based on the introduction of a wavepacket interaction system. We show in [9] and here that solutions to this system are in a close relation to solutions of the original system.

The superposition principle implies, in particular, that in the process of nonlinear evolution every single wavepacket propagates almost independently of other wavepackets (even though they may ”collide” in physical space for a certain period of time) and the exact solution equals the sum of particular single wavepacket solutions with a high precision. In particular, the dynamics of a solution with multi-wavepacket initial data is reduced to dynamics of separate solutions with single wavepacket data. Note that the nonlinear evolution of a single wavepacket solution for many problems is studied in detail, namely it is well approximated by its own nonlinear Schrodinger equation (NLS), see [20], [27], [34], [35], [47], [48], [49], [8] and references therein.

Let us give now an elementary physical argument justifying the superposition principle which goes as follows. If there would be no nonlinearity, the system would be linear and,
consequently, the superposition principle would hold exactly. Hence, any deviation from it is due to the nonlinear interactions between wavepackets, and one has to estimate their impact. Suppose that initially at time $\tau = 0$ the spatial extension $s$ of every involved wavepacket is characterized by the parameter $\beta^{-1}$ as in (1.3). Assume also (and it is quite an assumption) that the involved wavepackets evolving nonlinearly maintain somehow their wavepacket identities, including the group velocities and the spatial extensions. Then, consequently, the spatial extension of every involved wavepacket is propositional to $\beta^{-1}$ and its group velocity $v_1$ is proportional to $\varrho^{-1}$. The difference $\Delta v$ between any two different group velocities is also proportional to $\varrho^{-1}$. Then the time when two different wavepackets overlap in the space is proportional to $s/|\Delta v|$ and, hence, to $\varrho/\beta$. Since the nonlinear term is of order one, the magnitude of the impact of the nonlinearity during this time interval should be roughly proportional to $\varrho/\beta$, which results in the same order of the magnitude of $D$ in (1.10)-(1.11). Observe, that this estimate is in agreement with our rigorous estimate of the magnitude of $D$ in (1.11) if we set there $\delta = 0$.

The rigorous proof of the superposition principle presented here is not directly based on the above argument since it already implicitly relies on the principle. Though some components of the physical argument can be found in our rigorous proof. For example, we prove that the involved wavepackets maintain under the nonlinear evolution constant values of their wavevectors with well defined group velocities (the wavepacket preservation). The Theorem 6.12 allows to estimate spatial extensions of particle-like wavepackets under the nonlinear evolution. The proof of the superposition principle for general wavepackets provided in [9] is based on general algebraic-functional considerations and on the theory of analytic operator expansions in Banach spaces. Here we develop an alternative approach with a proof based on properties of the wavepacket interaction systems introduced in [8].

To provide a flexibility in formulating more specific statements related to the spatial localization of wavepackets we introduce a few types of wavepackets:

- a single particle-like wavepacket $w$ which is characterized by the following properties: (a) its modal decomposition involves only wavevectors from $\beta$-vicinity of a single wavevector $k_*$, where $\beta > 0$ is a small parameter; (b) it is spatially localized in all directions and can be assigned its position $r_*$;

- a multi-particle wavepacket which is a system $\{w_l\}$ of particle-like wavepackets with the corresponding sets of wavevectors $\{k_{il}\}$ and positions $\{r_{il}\}$;

- a spatially localized multi-wavepacket which is a system $\{w_l\}$ with $w_l$ being either a particle-like wavepacket or a general wavepacket.

We would like to note that a more detailed analysis, which is left for another paper, indicates that under certain resonance conditions nonlinear interactions of particle-like wavepackets may produce a spatially localized wavepacket $w$ characterized by the following properties: (i) its modal decomposition involves only wavevectors from $\beta$-vicinity of a single wavevector $k_*$, where $\beta > 0$ is a small parameter; (ii) it is only partly spatially localized in some, not necessarily all directions, and, for instance, it can be needle-like or pancake-like.
We also would like to point out that the particular form (1.1) of dependence on the small parameter \( \rho \) is chosen so that appreciable nonlinear effects occur at times of order one. In fact, many important classes of problems involving small or parameters can be readily reduced to the framework of (1.1) by a simple rescaling. It can be seen from the following examples. The first example is a system with a small nonlinearity

\[ \partial_t v = -iLv + \alpha f(v), \quad v|_{t=0} = h, \quad 0 < \alpha \ll 1, \quad (1.12) \]

where the initial data is bounded uniformly in \( \alpha \). Such problems are reduced to (1.1) by the time rescaling \( \tau = ta \). Note that here \( \rho = \alpha \) and the finite time interval \( 0 \leq \tau \leq \tau_\ast \) corresponds to the long time interval \( 0 \leq t \leq \tau_\ast / \alpha \).

The second example is a system with small initial data considered on long time intervals. The system itself has no small parameters but the initial data are small, namely

\[ \partial_t v = -iLv + f_0(v), \quad v|_{t=0} = \alpha_0 h, \quad 0 < \alpha_0 \ll 1, \quad (1.13) \]

where \( \alpha_0 \) is a small parameter and \( f^{(m)}(v) \) is a homogeneous polynomial of degree \( m \geq 2 \). After rescaling \( v = \alpha_0 V \) we obtain the following equation with a small nonlinearity

\[ \partial_t V = -iLV + \alpha_0^{m-1} \left[ f^{(m)}(V) + \alpha_0 f^{(m+1)}(V) + \ldots \right], \quad V|_{t=0} = h, \quad (1.14) \]

which is of the form of (1.12) with \( \rho = \alpha_0^{m-1} \). Introducing the slow time variable \( \tau = t\alpha_0^{m-1} \) we get from the above an equation of the form (1.1), namely

\[ \partial_{\tau} V = -\frac{i}{\alpha_0^{m-1}}LV + \left[ f^{(m)}(V) + \alpha_0 f^{(m+1)}(V) + \ldots \right], \quad V|_{t=0} = h, \quad (1.15) \]

where the nonlinearity does not vanish as \( \alpha_0 \to 0 \). In this case \( \rho = \alpha_0^{m-1} \) and the finite time interval \( 0 \leq \tau \leq \tau_\ast \) corresponds to the long time interval \( 0 \leq t \leq \frac{\tau_\ast}{\alpha_0^{m-1}} \) with small \( \alpha_0 \ll 1 \).

The third example is related to a high-frequency carrier wave in the initial data. To be concrete, we consider the Nonlinear Schrodinger equation

\[ \partial_{\tau} U = -i\partial_x^2 U + i\alpha |U|^2 U, \quad U|_{\tau=0} = h_1(M\beta x) e^{iM_kx_1} + h_2(M\beta x) e^{iM_kx_2} + c.c., \quad (1.16) \]

where \( c.c. \) stands for complex conjugate of the prior term, and \( M \gg 1 \) is a large parameter. The equation (1.16) can be readily recast into the form (1.1) by change of variables \( y = Mr \) yielding

\[ \partial_{\tau} U = -\frac{1}{\rho} \partial_x^2 U + i\alpha |U|^2 U, \quad U|_{\tau=0} = h_1(\beta r) e^{ikx_1} + h_2(\beta r) e^{ikx_2} + c.c., \quad (1.17) \]

where \( \rho = \frac{1}{M^2} \ll 1 \).

Summarizing the above analysis we list below important ingredients of our approach.
- The wave nonlinear evolution is analyzed based on the modal decomposition with respect to the linear part of the system. The significance of the modal decomposition to the nonlinear analysis is based on the following properties: (i) the wave modal amplitudes do not evolve under the linear evolution; (ii) the same amplitudes evolve slowly under the nonlinear evolution; (iii) modal decomposition is instrumental to the wavepacket definition including its spatial extension and the group velocity.

- Components of multi-particle wavepacket are characterized by their wavevectors \( k_{\ast l} \), band numbers \( n_l \) and spatial positions \( r_{\ast l} \). The nonlinear evolution preserves \( k_{\ast l} \) and \( n_l \) whereas the spatial positions evolve uniformly with the velocities \( \frac{1}{\varrho} \nabla \omega_{n_l}(k_{\ast l}) \).

- The problem involves two small parameters \( \beta \) and \( \varrho \) respectively in the initial data and coefficients of the master equation (1.1). These parameters scale respectively (i) the range of wavevectors involved in its modal composition, with \( \beta^{-1} \) scaling its spatial extension, and (ii) \( \varrho \) scaling the ratio of the slow and the fast time scales. We make no assumption on the functional dependence between \( \beta \) and \( \varrho \), which are essentially independent and are subject only to inequalities.

- The nonlinear evolution is studied for a finite time \( \tau_{\ast} \) which may depend on, say, the amplitude of the initial excitation, and, importantly, \( \tau_{\ast} \) is long enough to observe appreciable nonlinear phenomena which are not vanishingly small. The superposition principle can be extended to longer time intervals up to blow-up time or even infinity if relevant uniform in \( \beta \) and \( \varrho \) estimates of solutions in appropriate norms are available.

- In the chosen slow time scale there are two fast wave processes with typical time scale of order \( \varrho \) which can be attributed to the linear operator \( L \): (i) fast time oscillations resulting in time averaging and consequent suppression of many nonlinear interactions; (ii) fast wavepacket propagation with large group velocities resulting in effective weakening of nonlinear interactions which are not time-averaged because of resonances. It is these two processes provide mechanisms leading to the superposition principle.

The rest of the paper is organized as follows. In the following Subsection 2.1 we introduce definitions of wavepackets, multiwavepackets and particle wavepackets. In subsection 2.1 we also formulate and briefly discuss some important results of [8] which are used in this paper, and in Subsection 2.2 we formulate new results. In Section 3 we formulate conditions imposed on the linear and the nonlinear parts of the evolution equation (1.1), and also introduce relevant concepts describing resonance interactions inside of wavepackets. In Section 4 we introduce an integral form of the basic evolution equation and study basic properties of involved operators. In Section 5 we introduce wavepacket interaction system describing the dynamics of wavepackets. In Section 6 we, first, define averaged wavepacket interaction system which plays a fundamental role in the analysis of the dynamics of multiwavepackets and then prove that solutions to this system approximate solutions to the original equation with high accuracy. We also discuss there properties of averaged nonlinearities, in particular, for universally and conditionally universal invariant wavepackets, and prove the fundamental
theorems on preservation of multi-particle wavepackets, namely Theorems 6.13 and Theorem 2.10. In Section 7 we prove the superposition principle using an approximate decoupling of the averaged wavepacket interaction system. In the last subsection of this section we prove some generalizations to the cases involving non-generic resonance interactions such as the second-harmonic and the third-harmonic generations.

2 Statement of results

This section consists of two subsections. In the first one we introduce basic concepts and terminology and formulate relevant results from [8] which are used latter on, and in the second one we formulate new results of this paper.

2.1 Wavepackets and their basic properties

Since both linear operator \( L(-i\nabla) \) and the nonlinearity \( F(U) \) are translation invariant, it is natural and convenient to recast the evolution equation (1.1) by applying to it the Fourier transform with respect to the space variables \( r \), namely

\[
\partial_\tau \hat{U}(k) = -\frac{i}{\ell} L(k) \hat{U}(k) + \hat{F}(\hat{U}(k)), \quad \hat{U}(k) \big|_{\tau=0} = \hat{h}(k),
\]

where \( \hat{U}(k) \) is the Fourier transform of \( U(r) \), i.e.

\[
\hat{U}(k) = \int_{\mathbb{R}^d} U(r) e^{-ir\cdot k} dr, \quad U(r) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{U}(k) e^{ir\cdot k} dr,
\]

and \( \hat{F} \) is the Fourier form of the nonlinear operator \( F(U) \) involving convolutions, see (3.9) for details. The equation (2.1) is written in terms of Fourier modes, and we call it the modal form of the original equation (1.1). The most of our studies are conducted first for the modal form (2.1) of the evolution equation and carried over then to the original equation (1.1).

The nonlinear evolution equations (1.1), (2.1) are commonly interpreted as describing wave propagation in a nonlinear medium. We assume that the linear part \( L(k) \) is a \( 2J \times 2J \) Hermitian matrix with eigenvalues \( \omega_{n,\zeta}(k) \) and eigenvectors \( g_{n,\zeta}(k) \) satisfying

\[
L(k) g_{n,\zeta}(k) = \omega_{n,\zeta}(k) g_{n,\zeta}(k), \quad \zeta = \pm, \quad \omega_{n,+}(k) \geq 0, \quad \omega_{n,-}(k) \leq 0, \quad n = 1, \ldots, J,
\]

where \( \omega_{n,\zeta}(k) \) are real-valued, continuous for all non-singular \( k \) functions, and vectors \( g_{n,\zeta}(k) \in \mathbb{C}^{2J} \) have unit length in the standard Euclidean norm. The functions \( \omega_{n,\zeta}(k) \), \( n = 1, \ldots, J \), are called dispersion relations between the frequency \( \omega \) and the wavevector \( k \) with \( n \) being the band number. We assume that the eigenvalues are naturally ordered by

\[
\omega_{J,+}(k) \geq \ldots \geq \omega_{1,+}(k) \geq 0 \geq \omega_{1,-}(k) \geq \ldots \geq \omega_{J,-}(k),
\]
and for almost every \( k \) (with respect to the standard Lebesgue measure) the eigenvalues are distinct and, consequently, the above inequalities become strict. Importantly, we also assume the following diagonal symmetry condition

\[
\omega_{n,-\zeta}(-k) = -\omega_{n,\zeta}(k), \quad \zeta = \pm, \quad n = 1, \ldots, J,
\]

which is naturally present in many physical problems (see also Remark 3.3 below), and is a fundamental condition imposed on the matrix \( L(k) \). Very often we use the abbreviation

\[
\omega_{n,+(k)} = \omega_{n}(k).
\]

In particular we obtain from (2.5)

\[
\omega_{n,-}(k) = -\omega_{n}(-k), \quad \omega_{n,\zeta}(k) = \zeta \omega_{n}(\zeta k), \quad \zeta = \pm.
\]

In addition to that in many examples we also have

\[
g_{n,\zeta}(k) = g_{n,-\zeta}^{*}(-k), \quad \text{where } z^{*} \text{ is complex conjugate to } z.
\]

We also use rather often the orthogonal projection \( \Pi_{n,\zeta}(k) \) in \( \mathbb{C}^{2J} \) onto the complex line defined by the eigenvector \( g_{n,\zeta}(k) \), namely

\[
\Pi_{n,\zeta}(k) \hat{u}(k) = \hat{u}_{n,\zeta}(k) g_{n,\zeta}(k), \quad n = 1, \ldots, J, \quad \zeta = \pm.
\]

As it is indicated by the title of this paper we study the nonlinear problem (1.1) for initial data \( \hat{h} \) in the form of a properly defined particle-like wavepackets or, more generally, a sum of such wavepackets to which we refer as multi-particle wavepacket. The simplest example of a wavepacket \( w \) is provided by the following formula

\[
w(\beta; r) = \Phi_{+}(\beta (r - r_{*})) e^{ik_{*} \cdot (r - r_{*})} g_{n,+(k_{*})}, \quad r \in \mathbb{R}^{d}, \quad \beta > 0
\]

where \( k_{*} \in \mathbb{R}^{d} \) is wavepacket principal wavevector, \( n \) is band number, and \( \beta > 0 \) is a small parameter. We refer to the pair \((n, k_{*})\) in (2.10) as wavepacket nk-pair and \( r_{*} \) as wavepacket position. Observe that the space extension of the wavepacket \( w(\beta; r) \) is proportional to \( \beta^{-1} \) and it is large for small \( \beta \). Notice also that if \( \beta \to 0 \) the wavepacket \( w(\beta; r) \) as in (2.10) tends, up to a constant factor, to the elementary eigenmode \( e^{ik_{*} \cdot r} g_{n,\zeta}(k_{*}) \) of the operator \( L(-i\nabla) \) with the corresponding eigenvalue \( \omega_{n,\zeta}(k_{*}) \). We refer to wavepackets of the simple form (2.10) as simple wavepackets to underline the very special way the parameter \( \beta \) enters its representation. The function \( \Phi_{\zeta}(r) \), which we call wavepacket envelope, describes its shape and it can be any scalar complex-valued regular enough function, for example a function from Schwartz space. Importantly, as \( \beta \to 0 \) the \( L^{\infty} \) norm of a wavepacket (2.10) remains constant, hence nonlinear effects in (1.1) remain strong.

Evolution of wavepackets in problems which can be reduced to the form (1.1) were studied for a variety of equations in numerous physical and mathematical papers, mostly by asymptotic expansions with respect to a single small parameter similar to \( \beta \), see [13], [16], [20],...
We are interested in general properties of evolutionary systems of the form (1.1) with wavepacket initial data which hold for a wide class of nonlinearities and all values of the space dimensions $d$ and the number $2J$ of the system components. Our approach is not based on asymptotic expansions but involves the two small parameters $\beta$ and $\varrho$ with mild constraints (1.9) on their relative smallness. The constraints can be expressed either in the form of certain inequalities or equalities, and a possible simple form of such a constraint can be a power law
\[ \beta = C \varrho^\alpha \] where $C > 0$ and $\alpha > 0$ are arbitrary constants. (2.11)

Of course, general features of wavepacket evolution are independent of particular values of the constant $C$. In addition to that, some fundamental properties such as wavepacket preservation are also totally independent on particular choice of the values of $\alpha$ in (2.11), whereas other properties are independent of $\alpha$ as it varies in certain intervals. For for instance, dispersion effects are dominant for $\alpha < 1/2$, whereas the wavepacket superposition principle of [8] holds for $\alpha < 1$.

To eliminate unbounded (as $\varrho \to 0$) linear term in (2.1) by replacing it with a highly oscillatory factor we introduce the slow variable $\hat{u}(k, \tau)$ by the formula
\[ \hat{U}(k, \tau) = e^{-i \frac{\varrho}{\beta} L(k)} \hat{u}(k, \tau), \] (2.12)
and get the following equation for $\hat{u}(k, \tau)$
\[ \partial_\tau \hat{u} = e^{i \frac{\varrho}{\beta} L} \hat{F} \left( e^{-i \frac{\varrho}{\beta} L} \hat{u} \right), \quad \hat{u}|_{\tau=0} = \hat{h}, \] (2.13)
which, in turn, can be transformed by time integration into the integral form
\[ \hat{u} = F(\hat{u}) + \hat{h}, \quad F(\hat{u}) = \int_0^\tau e^{i \frac{\varrho}{\beta} L} \hat{F} \left( e^{-i \frac{\varrho}{\beta} L} \hat{u}(\tau') \right) d\tau' \] (2.14)
with explicitly defined nonlinear polynomial integral operator $F = F(\varrho)$. This operator is bounded uniformly with respect to $\varrho$ in the Banach space $E = C([0, \tau_*], L^1)$. This space has functions $\hat{v}(k, \tau), 0 \leq \tau \leq \tau_*$ as elements and has the norm
\[ \| \hat{v}(k, \tau) \|_E = \| \hat{v}(k, \tau) \|_{C([0, \tau_*], L^1)} = \sup_{0 \leq \tau \leq \tau_*} \int_{\mathbb{R}^d} |\hat{v}(k, \tau)| \, dk, \] (2.15)
where $L^1$ is the Lebesgue space of functions $\hat{v}(k)$ with the standard norm
\[ \| \hat{v}(\cdot) \|_{L^1} = \int_{\mathbb{R}^d} |\hat{v}(k)| \, dk. \] (2.16)
Sometimes we use more general weighted spaces $L^{1,a}$ with the norm
\[ \| \hat{v} \|_{L^{1,a}} = \int_{\mathbb{R}^d} (1 + |k|^a) |\hat{v}(k)| \, dk, \quad a \geq 0. \] (2.17)
The space \( C([0, \tau_*], L^{1,a}) \) with the norm
\[
\| \hat{V}(k, \tau) \|_{E_a} = \sup_{0 \leq \tau \leq \tau_*} \int_{\mathbb{R}^d} (1 + |k|)^a |\hat{V}(k, \tau)| \, dk.
\] (2.18)

is denoted by \( E_a \), and, obviously, \( E_0 = E \).

A rather elementary existence and uniqueness theorem (Theorem 4.8) implies that if \( \hat{h} \in L^{1,a} \) then for a small and, importantly, independent of \( \varrho \) constant \( \tau_* > 0 \) this equation has a unique solution
\[
\hat{u}(\tau) = \mathcal{G}(\mathcal{F}(\varrho), \hat{h})(\tau), \ \tau \in [0, \tau_*], \ \hat{u} \in C^1 \left([0, \tau_*], L^{1,a}\right),
\] (2.19)
where \( \mathcal{G} \) denotes the solution operator for the equation (2.14). If \( \hat{u}(k, \tau) \) is a solution to the equation (2.14) we call the function \( \mathcal{U}(\varrho) \) defined by (2.12), (2.2) an \( F \)-solution to the equation (1.1). We denote by \( \hat{L}^1 \) the space of functions \( \hat{V}(r) \) such that their Fourier transform \( \hat{V}(k) \) belongs to \( L^1 \), and define \( \| \mathcal{V} \|_{L^1} = \left\| \mathcal{V} \right\|_{L^1} \). Since
\[
\| \mathcal{V} \|_{L^\infty} \leq (2\pi)^{-d} \left\| \hat{\mathcal{V}} \right\|_{L^1}, \text{ and } \hat{L}^1 \subset L^\infty,
\] (2.20)

\( F \)-solutions to (1.1) belong to \( C^1 \left([0, \tau_*], \hat{L}^1\right) \subset C^1 \left([0, \tau_*], L^\infty\right) \).

We would like to define wavepackets in a form which explicitly allows them to be real valued. This is accomplished based on the symmetry (2.5) of the dispersion relations, which allows to introduce a doublet wavepacket
\[
\mathcal{W}(\beta; r) = \Phi_+(\beta (r - r_*)) e^{ik_\beta (r - r_*)} g_{n,+} (k_*), \ \Phi_-(\beta (r - r_*)) e^{-ik_\beta (r - r_*)} g_{n,-} (-k_*).
\] (2.21)

Such a wavepacket is real if \( \Phi_-(r), g_{n,-} (-k_*) \) are complex conjugate respectively to \( \Phi_+(r), g_{n,+} (k_*) \), i.e. if
\[
\Phi_-(r) = \Phi_+^*(r), \ g_{n,+} (k_*) = g_{n,-} (-k_*)^*.
\] (2.22)

Usually when considering wavepackets with \( nk \)-pair \( (n, k_*) \) we mean doublet ones as in (2.21), but sometimes we use the term wavepacket also for an elementary one as defined by (2.10). Note that the latter use is consistent with the former one since it is possible to take one of two terms in (2.21) to be zero.

Below we give a precise definition of a wavepacket. To identify characteristic properties of a wavepacket suitable for our needs, let us look at the Fourier transform \( \hat{\mathcal{W}}(\beta; k) \) of an elementary wavepacket \( \mathcal{W}(\beta; r) \) defined by (2.10), that is
\[
\hat{\mathcal{W}}(\beta; k) = \beta^{-d} e^{-ik_\beta r} \hat{\Phi}(\beta^{-1} (k - k_*)) g_{\infty} (k_*).
\] (2.23)

We call such \( \hat{\mathcal{W}}(\beta; k) \) wavepacket too and it possesses the following properties: (i) its \( L^1 \) norm is bounded (in fact, constant), uniformly in \( \beta \to 0 \); (ii) for every \( \epsilon > 0 \) the value \( \hat{\mathcal{W}}(\beta; k) \to 0 \) for every \( k \) outside a \( \beta^{1-\epsilon} \)-neighborhood of \( k_* \), and the convergence is faster.
than any power of $\beta$ if $\Phi$ is a Schwartz function. To explicitly interpret the last property we introduce a cutoff function $\Psi(\eta)$ which is infinitely smooth and such that

$$\Psi(\eta) \geq 0, \quad \Psi(\eta) = 1 \text{ for } |\eta| \leq 1/2, \quad \Psi(\eta) = 0 \text{ for } |\eta| \geq 1,$$

and its shifted/rescaled modification

$$\Psi(\beta^{1-\epsilon} k_s; k) = \Psi(\beta^{-1-\epsilon} (k - k_s)).$$

(2.25)

If an elementary wavepacket $w(\beta; r)$ is defined by (2.23) with $\Phi(r)$ being a Schwartz function then

$$\| (1 - \Psi(\beta^{1-\epsilon}, k_s; \cdot)) \hat{w}(\beta; \cdot) \| \leq C_{\epsilon,s} \beta^s, \quad 0 < \beta \leq 1,$$

and the inequality holds for arbitrarily small $\epsilon > 0$ and arbitrarily large $s > 0$. Based on the above discussion we give the following definition of a wavepacket which is a minor variation of [9, Definition 8].

**Definition 2.1 (single-band wavepacket)** Let $\epsilon$ be a fixed number, $0 < \epsilon < 1$. For a given band number $n \in \{1, \ldots, J\}$ and a principal wavevector $k_s \in \mathbb{R}^d$ a function $\hat{h}(\beta; k)$ is called a wavepacket with $nk$-pair $(n, k_s)$ and the degree of regularity $s > 0$ if for small $\beta < \beta_0$ with some $\beta_0 > 0$ it satisfies the following conditions: (i) $\hat{h}(\beta; k)$ is $L^1$-bounded uniformly in $\beta$, i.e.,

$$\| \hat{h}(\beta; \cdot) \|_{L^1} \leq C, \quad 0 < \beta < \beta_0 \text{ for some } C > 0;$$

(2.27)

(ii) $\hat{h}(\beta; k)$ is composed essentially of two functions $\hat{h}_\zeta(\beta; k)$, $\zeta = \pm$, which take values in $n$-th band eigenspace of $L(k)$ and are localized near $\zeta k_s$, namely

$$\hat{h}(\beta; k) = \hat{h}_-(\beta; k) + \hat{h}_+(\beta; k) + D_h, \quad 0 < \beta < \beta_0,$$

(2.28)

where the components $\hat{h}_\pm(\beta; k)$ satisfy the condition

$$\hat{h}_\zeta(\beta; k) = \Psi(\beta^{-1-\epsilon}/2, \zeta k_s; k) \Pi_{n,z}(k) \hat{h}_\zeta(\beta; k), \quad \zeta = \pm,$$

(2.29)

where $\Psi(\cdot, \zeta k_s, \beta^{-1})$ is defined by (2.25) and $D_h$ is small, namely it satisfies the inequality

$$\| D_h \|_{L^1} \leq C' \beta^s, \quad 0 < \beta < \beta_0, \text{ for some } C' > 0.$$

(2.30)

The inverse Fourier transform $h(\beta; r)$ of a wavepacket $\hat{h}(\beta; k)$ is also called a wavepacket.

Evidently, if a wavepacket has the degree of regularity $s$, it also has a smaller degree of regularity $s' \leq s$ with the same $\epsilon$. Observe that the degree of regularity $s$ is related to the smoothness of $\Phi_\zeta(r)$ as in (2.10) so that the higher is the smoothness the higher $s/\epsilon$ can be
taken. Namely, if $\hat{\Phi}_{\zeta} \in L^{1,\alpha}$ then one can take in \eqref{2.30} any $s < a\epsilon$ according to the following inequality:

$$\int \left| (1 - \Psi (\beta^s \eta)) \hat{\Phi}_{\zeta} (\eta) \right| d\eta \leq \beta^{as} \left\| \hat{\Phi}_{\zeta} \right\|_{L^{1,\alpha}} \leq C\beta^s. \quad \text{(2.31)}$$

For example, if we define $\hat{h}_{\zeta}$ similarly to \eqref{2.29} and \eqref{2.23} by the formula

$$\hat{h}_{\zeta} (\beta; k) = \Psi (\beta^{-1-\epsilon} (k - k_s)) \beta^{-d} \hat{\Phi}_{\zeta} (\beta^{-1} (k - k_s)) \Pi_{n, \zeta} (k) g \quad \text{(2.32)}$$

where $\hat{\Phi}_{\zeta} (k)$ is a scalar Schwartz function and $g$ is a vector, then according to \eqref{2.31} estimate \eqref{2.30} holds and $\hat{h}_{\zeta} (\beta; k)$ is a wavepacket with arbitrarily large degree of regularity $s$ for any given $\epsilon$ such that $0 < \epsilon < 1$.

Now let us define a particle-like wavepacket following to the ideas indicated in the Introduction.

**Definition 2.2 (single-band particle-like wavepacket)** We call a function $\hat{h} (\beta; k) = \hat{h} (\beta, r_\ast; k), r_\ast \in \mathbb{R}^d$, a particle-like wavepacket with the position $r_\ast$, nk-pair $(n, k_\ast)$ and the degree of regularity $s > 0$ if (i) for every $r_\ast$ it is a wavepacket with the degree of regularity $s$ in the sense of the above Definition 2.1 with constants $C, C'$ independent of $r_\ast \in \mathbb{R}^d$; (ii) $\hat{h}_{\zeta}$ in \eqref{2.28} satisfy the inequalities

$$\int_{\mathbb{R}^d} \left| \nabla_k \left( e^{i r \cdot k} \hat{h}_{\zeta} (\beta, r_\ast; k) \right) \right| dk \leq C_1 \beta^{-1-\epsilon}, \quad \zeta = \pm, \ r_\ast \in \mathbb{R}^d, \quad \text{(2.33)}$$

where $C_1 > 0$ is an independent of $\beta$ and $r_\ast$ constant, $\epsilon$ is the same as in Definition 2.1. The inverse Fourier transform $h (\beta; r)$ of a wavepacket $\hat{h} (\beta; k)$ is also called a particle-like wavepacket with the position $r_\ast$. We also introduce quantity

$$a \left( r_\ast, \hat{h}_{\zeta} (r_\ast) \right) = \left\| \nabla_k \left( e^{i r \cdot k} \hat{h}_{\zeta} (\beta, r_\ast; k) \right) \right\|_{L^1} \quad \text{(2.34)}$$

which we refer to as the position detection function for the wavepacket $\hat{h} (\beta, r_\ast; k)$.

Note that the left-hand side of \eqref{2.33} coincides with $a \left( r_\ast, \hat{h}_{\zeta} (r_\ast) \right)$.

**Remark 2.3** If $\hat{h} (\beta; k) = \hat{h} (\beta, r_\ast; k)$ is a particle-like wavepacket with a position $r_\ast$ then, applying inverse Fourier transform to $\hat{h}_{\zeta} (\beta, r_\ast; k)$ and $\nabla_k \hat{h}_{\zeta} (\beta, r_\ast; k)$ as in \eqref{2.2} we obtain a function $h (\beta, r_\ast; r)$ which satisfies

$$|r - r_\ast| |h_{\zeta} (\beta, r_\ast; r)| \leq (2\pi)^{-d} a \left( r_\ast, \hat{h}_{\zeta} \right) \quad \text{(2.35)}$$

implying that $|h_{\zeta} (\beta; r)| \leq a \left( r_\ast, \hat{h}_{\zeta} \right) |r - r_\ast|^{-1}$. This inequality is useful for large $|r - r_\ast|$, whereas for bounded $|r - r_\ast|$ \eqref{2.27} implies a simpler inequality

$$|h_{\zeta} (\beta, r_\ast; r)| \leq (2\pi)^{-d} \left\| \hat{h}_{\zeta} \right\|_{L^1} \leq C. \quad \text{(2.36)}$$
Inequalities (2.33) and (2.34) suggest that the quantity $a \left( r_s, \hat{h}_\zeta (r_s) \right)$ can be interpreted as a size of the particle-like wavepacket $\hat{h}_\zeta (\beta, r_s; k)$.

Evidently a particle-like wavepacket is a wave and not a point. Hence the above definition of its position has a degree of uncertainty, allowing, for example, to replace $r_s$ by $r_s + a$ with a fixed vector $a$ (but not allowing unbounded values of $a$). The above definition of particle-like wavepacket position was crafted to meet the following requirements: (i) a system of particle-like wavepackets remains to be such a system under the nonlinear evolution; (ii) it is possible (in an appropriate scale) to describe the trajectories traced out by the positions of a system of particle-wavepackets.

**Remark 2.4** Typical dependence of the inverse Fourier transform $h(\beta, r_s; r)$ of a wavepacket $\hat{h}(\beta, r_s; k)$ on $r_s$ is provided by spatial shifts by $r_s$ as in (2.27), namely

$$h(\beta, r_s; r) = \Phi(\beta, r - r_s) e^{ik\cdot(r-r_s)} g$$

with a constant $g$. For such a function $h$ and for any $r'_s \in \mathbb{R}^d$

$$a (r'_s, \hat{h} (r_s)) = \left\| \nabla_k \left( \beta^{-d} e^{ik\cdot r_s} \hat{h}(\beta, r_s; k) \right) \right\|_{L^1} = \left\| \nabla_k \left( \beta^{-d} e^{ik\cdot r_s} e^{-ik\cdot r_s} \hat{\Phi} (k) \right) \right\|_{L^1} \left\| g \right\|$$

$$= \left\| g \right\| \int \left| i (r'_s - r_s) \hat{\Phi} (k') + \frac{1}{\beta} \nabla_k \hat{\Phi} (k') \right| dk'.$$

Hence, taking for simplicity $\left\| g \right\| = 1$, we obtain

$$\left| r'_s - r_s \right| \left\| \hat{\Phi} \right\|_{L^1} + \frac{1}{\beta} \left\| \nabla \hat{\Phi} \right\|_{L^1} \geq a (r'_s, \hat{h} (r_s)) \geq \left| r'_s - r_s \right| \left\| \hat{\Phi} \right\|_{L^1} - \frac{1}{\beta} \left\| \nabla \hat{\Phi} \right\|_{L^1}. \quad (2.37)$$

For small $\left| r'_s - r_s \right| \ll \frac{1}{\beta}$ we see that the position detection function $a (r'_s, \hat{h})$ is of order $O \left( \beta^{-1} \right)$, which is in the agreement with (2.33). For large $\left| r'_s - r_s \right| \gg \frac{1}{\beta}$ the $a (r'_s, \hat{h})$ is approximately proportional to $\left| r'_s - r_s \right|$. Therefore if we know $a (r'_s, \hat{h}(r_s))$ as a function of $r'_s$ we can recover the value of $r_s$ with the accuracy of order $O \left( \beta^{-1-\epsilon} \right)$ with arbitrary small $\epsilon$. Namely, let us take arbitrary small $\epsilon > 0$ and some $C > 0$ and consider the set

$$B(\beta) = \left\{ r'_s : a (r'_s, \hat{h}(r_s)) \leq C \beta^{-1-\epsilon} \right\} \subset \mathbb{R}^d, \quad (2.38)$$

which should provide an approximate location of $r_s$. According to (2.37), $r_s$ lies in this set for small $\beta$. If $r'_s$ lies in this set then

$$C \beta^{-1-\epsilon} \geq a (r'_s, \hat{h}(r_s)) \geq \left| r'_s - r_s \right| \left\| \hat{\Phi} \right\|_{L^1} - \frac{1}{\beta} \left\| \nabla \hat{\Phi} \right\|_{L^1}$$

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and \(|r'_s - r_s| \leq C_1\beta^{-1-\epsilon} + C_2\beta^{-1}\). Hence the diameter of the \(B(\beta)\) is of order \(O(\beta^{-1-\epsilon})\).

Observe, taking into account Remark 2.3, that the accuracy of the wavepacket location obviously cannot be better than its size \(a\left(r_s, \hat{h}_c(r_s)\right) \sim \beta^{-1}\). The above analysis suggest that the function \(h(\beta, r_s; r)\) can be viewed as pseudoshifts of the function \(h(\beta, 0; r)\) by vectors \(r_s \in \mathbb{R}^d\) in the sense that the regular spatial shift by \(r_s\) is combined with a variation of the shape of \(h(\beta, 0; r)\) which is limited by the fundamental condition (2.33). In other words, according Definition 2.2 as wavepackets move from \(0\) to \(r_s\) by a corresponding spatial shift it is allowed to change its shape subject to the fundamental condition (2.33). The later is instrumental for capturing nonlinear evolution of particle-like wavepackets governed by an equation of the form \((1.1)\).

**Remark 2.5** The set \(B(\beta)\) defined by (2.38) gives an approximate location of the support of the function \(\hat{h}(\beta, r_s; k)\) not only in the special case considered in Remark 2.4 but also when \(h(\beta, r_s; r)\) is a general particle-like wavepacket. One can apply with obvious modifications the above argument for \(e^{ikr}\hat{h}(\beta, r_s; k)\) in place of \(\hat{\Phi}(k)\) using (2.33). Here we give an alternative argument based on (2.33). Notice that condition \(a\left(r_{s0}, \hat{h}(r_s)\right) \leq C\beta^{-1-\epsilon}\) obviously can be satisfied not only by \(r_{s0} = r_s\). But one can show that the diameter of the set of such \(r_{s0}\) is estimated by \(O(\beta^{-1-\epsilon})\). Indeed, assume that a given function \(h(\beta, r)\) does not vanish at a given point \(r_0\), that is \(|h(\beta, r_0)| \geq c_0 > 0\) for all \(\beta \leq \beta_0\). The fulfillment of (2.33) for the function \(h(\beta, r)\) with two different values of \(r_s\), namely \(r_s = r'_s\) and \(r_s = r''_s\) implies that

\[
a\left(r'_s, \hat{h}\right) \leq C_1\beta^{-1-\epsilon}, \quad a\left(r''_s, \hat{h}\right) \leq C_2\beta^{-1-\epsilon},
\]

and according to (2.33) for all \(r\)

\[
|r - r'_s| |h(\beta, r)| \leq (2\pi)^{-d} C_1\beta^{-1-\epsilon}, \quad |r - r''_s| |h(\beta, r)| \leq (2\pi)^{-d} C_2\beta^{-1-\epsilon},
\]

Hence,

\[
|r_0 - r'_s| \leq \frac{(2\pi)^{-d} C_1\beta^{-1-\epsilon}}{c_0}, \quad |r_0 - r''_s| \leq \frac{(2\pi)^{-d} C_1\beta^{-1-\epsilon}}{c_0},
\]

and

\[
|r'_s - r''_s| \leq C_3\beta^{-1-\epsilon}.
\]

Note that if we rescale variables \(r\) and \(r_s\) as in Example 2.13 namely \(\varrho r = y\) and \(\varrho r_s = y_s\) with \(\varrho = \beta^2\), the diameter of the set \(B(\beta)\) in \(y\)-coordinates is of order \(\beta^{1-\epsilon} \ll 1\), and, hence, this set gives a good approximation for the location of the particle-like wavepacket as \(\beta \to 0\). It is important to notice, that our method to locate the support of wavepackets is applicable to very general wavepackets and does not use their specific form. This flexibility allows us to prove that particle-like wavepackets and their positions are well defined during nonlinear dynamics of generic equations with rather general initial data which form infinite-dimensional function spaces. Another approaches to describe dynamics of waves are applied to situations where solutions under considerations can be parametrized by a finite number of parameters and the dynamics of the parameters describes dynamics of the solutions. See for example [25, 26] where dynamics of centers of solitons is described.
Remark 2.6 Note that for a single wavepacket initial data \( h(\beta, r - r'_* ) \) one can make a change of variables to a moving frame \( (x, \tau) \), namely \( (r, \tau) = (x + v \tau, \tau) \), where \( v = \frac{1}{c} \nabla \omega(k_\nu) \) is the group velocity; this change of variables makes the group velocity zero. Often it is possible to prove that dynamics preserves functions which decay at infinity, namely if the initial data \( h(\beta, x) \) decays at the spatial infinity then the solution \( U(\beta, x, \tau) \) also decays at infinity (though corresponding proofs can be rather technical). This property can be reformulated in rescaled \( y \) variables as follows: if initial data are localized about zero, then the solution is localized about zero as well. Then, using the fact that the equation has constant coefficients, we observe that the solution \( U(\beta, y - y'_* , \tau) \), corresponding to \( h(\beta, y - y'_* ) \), is localized about \( y'_* \) provided that \( h(\beta, y) \) was localized about the origin. Note that in this paper we consider much more complicated case of multiple wavepackets. Even in the simplest case of the initial multiwavepacket which involves only two components, namely the wavepacket \( h(\beta, r) = h_1(\beta, r - r'_*) + h_2(\beta, r - r''_*) \) with two principal wave vectors \( k_{1*} \neq k_{2*} \), one evidently cannot use the above considerations based on the change of variables and the translational invariance. Using other arguments developed in this paper we prove that systems of particle-like wavepackets remain localized in the process of the nonlinear evolution.

Note that similarly to (1.2) and (1.4) a function of the form

\[
\beta^{-d} \left( e^{-i kr_{1*}} + e^{-i kr_{2*}} \right) \left[ \hat{h} \left( \frac{k - k_\nu}{\beta} \right) \right] g_\nu(k_\nu),
\]

defined for any pair of \( r_{1*} \) and \( r_{2*} \) where \( \hat{h} \) is a Schwartz function and all constants in Definition 2.1 are independent of \( r_{1*}, r_{2*} \in \mathbb{R}^d \), is not a single particle-like wavepacket since it does not have a single wavepacket position \( r_* \), but rather it is a sum of two particle-like wavepackets with two positions \( r_{1*} \) and \( r_{2*} \).

We want to emphasize once more that a particle-like wavepacket is defined as the family \( \hat{h}(\beta, r_*; k) \) with \( r_* \) being an independent variable running the entire space \( \mathbb{R}^d \), see, for example, (1.2), (1.3) and (2.11). In particular, we can choose a dependence of \( r_* \) on \( \beta \) and \( g \). An interesting type of such a dependence is \( r_* = r_{0*}/\varrho \) where \( \varrho \) satisfies (2.11) as we discuss below in the Example 2.13.

Our special interest is in the waves that are finite sums of wavepackets which we refer to as multi-wavepackets.

**Definition 2.7 (multi-wavepacket)** Let \( S \) be a set of \( nk \)-pairs:

\[
S = \{ (n_l, k_{st}), \ l = 1, \ldots, N \} \subset \Sigma = \{ 1, \ldots, J \} \times \mathbb{R}^d, \ (n_l, k_{st}) \neq (n_{l'}, k_{s't'}) \text{ for } l \neq l', \quad (2.39)
\]

and \( N = |S| \) be their number. Let \( K_S \) be a set consisting of all different wavevectors \( k_{st} \) involved in \( S \) with \( |K_S| \leq N \) being the number of its elements. \( K_S \) is called wavepacket \( k \)-spectrum and without loss of genericity we assume the indexing of elements \( (n_{l}, k_{st}) \) in \( S \) to be such that

\[
K_S = \{ k_{si}, i = 1, \ldots, |K_S| \}, \ i.e. \ l = i \text{ for } 1 \leq i \leq |K_S|.
\]
A function \( \hat{h}(\beta) = \hat{h}(\beta; k) \) is called a multi-wavepacket with \( nk \)-spectrum \( S \) if it is a finite sum of wavepackets, namely

\[
\hat{h}(\beta; k) = \sum_{l=1}^{N} \hat{h}_l(\beta; k), \quad 0 < \beta < \beta_0 \text{ for some } \beta_0 > 0,
\]

where \( \hat{h}_l, l = 1, \ldots, N, \) is a wavepacket with \( nk \)-pair \( (k_d, n_l) \in S \) as in Definition 2.1. If all the wavepackets \( \hat{h}_l(\beta; k) = \hat{h}_l(\beta, r^*_l; k) \) are particle-like ones with respective positions \( r^*_l \), then the multi-wavepacket is called multi-particle wavepacket and we refer to \( (r^*_1, \ldots, r^*_N) \) as its position vector.

Note that if \( \hat{h}(\beta; k) \) is a wavepacket then \( \hat{h}(\beta; k) + O(\beta^s) \) is also a wavepacket with the same \( nk \)-spectrum, and the same is true for multi-wavepackets. Hence, we can introduce multi-wavepackets equivalence relation "\( \simeq \)" of the degree \( s \) by

\[
\hat{h}_1(\beta; k) \simeq \hat{h}_2(\beta; k) \text{ if } \left\| \hat{h}_1(\beta; k) - \hat{h}_2(\beta; k) \right\|_{L^1} \leq C \beta^s \text{ for some constant } C > 0.
\]

Note that condition (2.33) does not impose restrictions on the term \( D_h \) in (2.28), therefore this equivalence can be applied to particle wavepackets.

Let us turn now to the abstract nonlinear problem (2.14) where (i) \( F = F(\varrho) \) depends on \( \varrho \) and (ii) the initial data \( \hat{h} = \hat{h}(\beta) \) is a multi-wavepacket depending on \( \beta \). We would like to state our first theorem on multi-wavepacket preservation under the evolution (2.14) for \( \beta, \varrho \to 0 \), which holds provided its \( nk \)-spectrum \( S \) satisfies a natural condition called resonance invariance. This condition is intimately related to the so-called phase and frequency matching conditions for stronger nonlinear interactions, and its concise formulation is as follows. We define for given dispersion relations \( \{\omega_n(k)\} \) and any finite set \( S \subset \{1, \ldots, J\} \times \mathbb{R}^d \) another finite set \( R(S) \subset \{1, \ldots, J\} \times \mathbb{R}^d \) where \( R \) is a certain algebraic operation described in Definition 3.8 below. It turns out that for any \( S \) always \( S \subseteq R(S) \), but if \( R(S) = S \) we call \( S \) resonance invariant. The condition of resonance invariance is instrumental for the multi-wavepacket preservation, and there are examples showing that if it fails, i.e. \( R(S) \neq S \), the wavepacket preservation does not hold. Importantly, the resonance invariance \( R(S) = S \) allows resonances inside the multi-wavepacket, that includes, in particular, resonances associated with the second and the third harmonic generations, resonant four-wave interaction etc. In this paper we will use basic results on wavepacket preservation obtained in [8], and we formulate theorems from [8] we need here. Since we use constructions from [8], for completeness we provide also their proofs in the following subsections. The following two theorems are proved in [8].

**Theorem 2.8 (multi-wavepacket preservation)** Suppose that the nonlinear evolution is governed by (2.14) and the initial data \( \hat{h} = \hat{h}(\beta; k) \) is a multi-wavepacket with \( nk \)-spectrum \( S \) and the regularity degree \( s \), and assume \( S \) to be resonance invariant (see Definition 3.8 below). Let \( \rho(\beta) \) be any function satisfying

\[
0 < \rho(\beta) \leq C \beta^s, \quad \text{for some constant } C > 0,
\]
and let us set \( \varrho = \rho(\beta) \). Then the solution \( \hat{u}(\tau, \beta) = \mathcal{G} \left( \mathcal{F} (\rho(\beta)), \hat{u}(\beta) \right)(\tau) \) to \( (2.14) \) for any \( \tau \in [0, \tau_*] \) is a multi-wavepacket with \( nk \)-spectrum \( S \) and the regularity degree \( s \), i.e.

\[
\hat{u}(\tau, \beta; k) = \sum_{l=1}^{N} \hat{u}_l(\tau, \beta; k), \text{ where } \hat{u}_l \text{ is wavepacket with } nk \text{-pair } (n_l, k_l) \in S.
\]

(2.44)

The time interval length \( \tau_* > 0 \) depends only on \( L^1 \)-norms of \( \hat{h}_l(\beta; k) \) and \( N \). The presentation \( (2.44) \) is unique up to the equivalence \( (2.42) \) of degree \( s \).

The above statement can be interpreted as follows. Modes in \( nk \)-spectrum \( S \) are always resonance coupled with modes in \( \mathcal{R}(S) \) through the nonlinear interactions, but if \( \mathcal{R}(S) = S \) then (i) all resonance interactions occur inside \( S \) and (ii) only small vicinity of \( S \) is involved in nonlinear interactions leading to the multi-wavepacket preservation.

The statement of Theorems 2.8 directly follows from the following general theorem proved in [8].

**Theorem 2.9 (multi-wavepacket approximation)** Let the initial data \( \hat{h} \) in the integral equation \( (2.14) \) be a multi-wavepacket \( \hat{h}(\beta; k) \) with \( nk \)-spectrum \( S \) as in \( (2.39) \), the regularity degree \( s \) and with the parameter \( \epsilon > 0 \) as in Definition \( 2.7 \). Assume that \( S \) is resonance invariant in the sense of Definition 3.8 below. Let the cutoff function \( \Psi(\beta^{1-\epsilon}, k_l; k) \) and the eigenvector projectors \( \Pi_{n, \pm}(k) \) be defined by \( (2.23) \) and \( (2.9) \) respectively. For a solution \( \hat{u} \) of \( (2.14) \) we set

\[
\hat{u}_l(\beta; \tau, k) = \left[ \sum_{\zeta_{\pm}} \Psi(C\beta^{1-\epsilon}, \zeta_k; k) \Pi_{n, \zeta}(k) \right] \hat{u}(\beta; \tau, k), \text{ } l = 1, \ldots, N.
\]

(2.45)

Then every such \( \hat{u}_l(\beta; \tau, k) \) is a wavepacket and

\[
\sup_{0 \leq \tau \leq \tau_*} \left\| \hat{u}(\beta; \tau, k) - \sum_{l=1}^{N} \hat{u}_l(\beta; \tau, k) \right\|_{L^1} \leq C_1 \varrho + C_2 \beta^s
\]

(2.46)

where the constants \( C, C_1 \) do not depend on \( \epsilon, s \) and \( \beta \), and the constant \( C_2 \) does not depend on \( \beta \) and \( \varrho \).

We would like to point out also that Theorem 2.8 allows to take values \( \hat{u}(\tau_*) \) as new wavepacket initial data for \( (1.1) \) and extend the wavepacket invariance of a solution to the next time interval \( \tau_* \leq \tau \leq \tau_{*1} \). This observation allows to extend the wavepacket invariance to larger values of \( \tau \) (up to blow-up time or infinity) if some additional information about solutions with wavepacket initial data is available, see [8].

Note that the wavepacket form of solutions can be used to obtain long-time estimates of solutions. Namely, very often behavior of every single wavepacket is well approximated by its own nonlinear Schrodinger equation (NLS), see [19], [38], [20], [27], [34], [35], [45], [47], [48], [49] and references therein, see also Section 6. Many features of the dynamics governed by NLS-type equations are well-understood, see [17], [18], [39], [40], [50], [51] and references therein. These results can be used to obtain long-time estimates for every single wavepacket (as, for example, in [35]) and, with the help of the superposition principle, for the multiwavepacket solution.
2.2 Formulation of new results on particle wavepackets

In this paper we prove the following refinement of Theorem 2.8 for the case of multi-particle wavepackets.

**Theorem 2.10 (multi-particle wavepacket preservation)** Assume that conditions of Theorem 2.9 hold and, in addition to that, the initial data \( \hat{h} = \hat{h}(\beta; k) \) is a multi-particle wavepacket of degree \( s \) with positions \( r_{s1}, \ldots, r_{sN} \) and the multi-particle wavepacket is universally resonance invariant in the sense of Definition 3.8. Assume also that

\[
\rho(\beta) \leq C^0, \quad s_0 > 0.
\]

Then the solution \( \hat{u}(\beta; \tau) = G\left( F(\rho(\beta)), \hat{h}(\beta) \right)(\tau) \) to (2.14) for any \( \tau \in [0, \tau_\ast] \) is a multi-particle wavepacket with the same \( nk \)-spectrum \( S \) and the same positions \( r_{s1}, \ldots, r_{sN} \).

**Remark 2.11** Note that in the statement of the above theorem the positions \( r_{s1}, \ldots, r_{sN} \) of wavepackets which compose the solution \( \hat{u}(\beta; \tau, k) \) of (2.13) and (2.14) do not depend on \( \tau \) and, hence, do not move. Note also that the solution \( \hat{U}(\beta; \tau, k) \) of the original equation (2.1), related to \( \hat{u}(\beta; \tau, k) \) by the change of variables (2.12), is composed of wavepackets \( U_l(\beta; \tau, r) \), corresponding to \( u_l(\beta; \tau, r) \), have their positions moving with respective constant velocities \( \nabla k \omega_n(k_u) \) (see for details Remark 4.1, see also the following corollary).

Using Proposition 4.2 we obtain from Theorem 2.10 the following corollary.

**Corollary 2.12** Let conditions of Theorem 2.10 hold and \( \hat{U}(\beta; \tau, k) \) be defined by (2.12) in terms of \( \hat{u}(\beta; \tau, k) \). Let

\[
\frac{\beta^2}{\rho} \leq C, \quad 0 < \beta \leq \frac{1}{2}, \quad 0 < \rho \leq \frac{1}{2}.
\]

Then \( \hat{U}(\beta; \tau, k) \) is for every \( \tau \in [0, \tau_\ast] \) a particle multi-wavepacket in the sense of Definition 2.2 with the same \( nk \)-spectrum \( S \), regularity \( s_1 \) and with \( \tau \)-dependent positions \( r_{sl} + \frac{\rho}{\rho} \partial k \omega_n(k_u) \).

In the following example we consider the case where spatial positions of wavepackets have a specific dependence on parameter \( \rho \), namely \( r_s = r^0_s/\rho \).

**Example 2.13 (Wavepacket trajectories and collisions)** Let us rescale the coordinates in the physical space as follows

\[
\rho r = y.
\]
with the consequent rescaling of the wavevector variable (dual with respect to Fourier transform) \( k = \varrho \eta \). It follows then that under the evolution (1.7) the group velocity of a wavepacket with a wavevector \( k_* \) in the new coordinates \( y \) becomes \( \nabla_{k\omega}(k_*) \) and evidently is of order one. If we set the positions \( r_{sl} = r_{sl}^0 / \varrho \) with fixed \( r_{sl}^0 \), then according to (2.35) wavepackets \( | h(\beta, r)| \) in \( y \)-variables have characteristic spatial scale \( y - r_{sl}^0 \sim \varrho a(r_{sl}, \hat{h}) \sim \varrho \beta^{-1} \) which is small if \( \varrho / \beta \) is small. The positions of particle-like wavepackets (quasiparticles) \( \hat{U}(y/\varrho, \tau) \) are initially located at \( y_1 = r_{sl}^0 \) and propagate with the group velocities \( \nabla_{k\omega}(k_*) \). Their trajectories are straight lines in the space \( \mathbb{R}^d \) described by

\[
y = \tau \nabla_{k\omega}(k_*) + r_{sl}^0, \quad 0 \leq \tau \leq \tau_*
\]

(compare with (1.9)). The trajectories may intersect, indicating "collisions" of quasiparticles. Our results (Theorem 2.10) show that if a multi-particle wavepacket initially was universally resonance invariant, then the involved particle-like wavepackets preserve their identity in spite of collisions and the fact that the nonlinear interactions with other wavepackets (quasiparticles) are not small, if fact, they are of order one. Note that \( r_{sl}^0 \) can be chosen arbitrarily implying that up to \( N(N-1) \) collisions can occur on the time interval \([0, \tau_*]\) on which we study the system evolution.

To formulate the approximate superposition principle for multi-particle wavepackets, we introduce now the solution operator \( G \) mapping the initial data \( \hat{h} \) into the solution \( \hat{u} = G(\hat{h}) \) of the modal evolution equation (2.14). This operator is defined for \( \| \hat{h} \| \leq R \) according to the existence and uniqueness Theorem 4.7. The main result of this paper is the following statement.

**Theorem 2.14 (superposition principle)** Suppose that the initial data \( \hat{h} \) of (2.14) is a multi-particle wavepacket of the form

\[
\hat{h} = \sum_{l=1}^{N} \hat{h}_l, \quad N \max_l \| \hat{h}_l \|_{L^1} \leq R,
\]

satisfying Definition 2.7 and its nk-spectrum is universally resonance invariant in the sense of Definition 3.8. Suppose also that the group velocities of wavepackets are different, namely

\[
\nabla_{k\omega_{nl_1}}(k_{sl_1}) \neq \nabla_{k\omega_{nl_2}}(k_{sl_2}) \text{ if } l_1 \neq l_2
\]

and that (2.48) holds. Then the solution \( \hat{u} = G(\hat{h}) \) to the evolution equation (2.14) satisfies the following approximate superposition principle

\[
G \left( \sum_{l=1}^{N} \hat{h}_l \right) = \sum_{l=1}^{N} G(\hat{h}_l) + \tilde{D},
\]
with a small remainder $\tilde{D}(\tau)$ such that

$$\sup_{0 \leq \tau \leq \tau_*} \| \tilde{D}(\tau) \|_{L^1} \leq C_\varepsilon \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta|,$$  \hspace{1cm} (2.53)

where (i) $\varepsilon$ is the same as in Definition 2.1 and can be arbitrary small; (ii) $\tau_*$ does not depend on $\beta$, $\varrho$, $r_{st}$ and $\varepsilon$; (iii) $C_\varepsilon$ does not depend on $\beta$, $\varrho$ and positions $r_{st}$.

A particular case of the above Theorem in which there was no dependence on $r_{st}$ was proved in [9] by a different method based on the theory of analytic operators in Banach spaces. The condition (2.51) can be relaxed if the initial positions of involved particle-like wavepackets are far apart, and the corresponding results are formulated in the theorem below and in Example 2.13.

**Theorem 2.15 (superposition principle)** Suppose that the initial data $\hat{h}$ of (2.14) is a multi-particle wavepacket of the form (2.50) with a universally resonance invariant $nk$-spectrum in the sense of Definition 3.8 and (2.48) holds. Suppose also that either the group velocities of wavepackets are different, namely (2.51) holds, or the positions $r_{st}$ satisfy the inequality

$$\tau_* |r_{st_1} - r_{st_2}|^{-1} \leq \frac{\varrho}{2C_{\omega,2} \beta^{1-\varepsilon}} \hbar \nabla_{k\omega n_{l_1}} (k_{st_1}) = \nabla_{k\omega n_{l_2}} (k_{st_2}), \ l_1 \neq l_2,$$  \hspace{1cm} (2.54)

where the constant $C_{\omega,2}$ is the same as in (3.2). Then the solution $\hat{u} = G(\hat{h})$ to the evolution equation (2.14) satisfies the approximate superposition principle (2.52), (2.53).

We prove in this paper further generalizations of the particle-like wavepacket preservation and the superposition principle to the cases where the $nk$-spectrum of a multi-wavepacket is not universal resonance invariant such as the cases of multi-wavepackets involving the second and the third harmonic generation. In particular, we prove Theorem 7.5 showing that that many (but, may be, not all) components of involved wavepackets remain spatially localized. Another Theorem 7.7 extends the superposition principle to the case when resonance interactions between components of a multi-wavepackets can occur.

### 3 Conditions and definitions

In this section we formulate and discuss all definitions and conditions under which we study the nonlinear evolutionary system (1.1) through its modal, Fourier form (2.1). Most of the conditions and definitions are naturally formulated for the modal form (2.1), and this is one of the reasons we use it as the basic one.
3.1 Linear part

The basic properties of the linear part $L(k)$ of the system (2.1), which is a $2J \times 2J$ Hermitian matrix with eigenvalues $\omega_{n,\zeta}(k)$, has been already discussed in the Introduction. To account for all needed properties of $L(k)$ we define the singular set of points $k$.

**Definition 3.1 (band-crossing points)** We call $k_0$ a band-crossing point for $L(k)$ if $\omega_{n+1,\zeta}(k_0) = \omega_{n,\zeta}(k_0)$ for some $n, \zeta$ or $L(k)$ is not continuous at $k_0$ or if $\omega_{1,\pm}(k_0) = 0$. The set of such points is denoted by $\sigma_{bc}$.

In the next Condition we collect all constraints imposed on the linear operator $L(k)$.

**Condition 3.2 (linear part)** The linear part $L(k)$ of the system (2.1) is a $2J \times 2J$ Hermitian matrix with eigenvalues $\omega_{n,\zeta}(k)$ and corresponding eigenvectors $g_{n,\zeta}(k)$ satisfying for $k \not\in \sigma_{bc}$ the basic relations (2.3)-(2.5). In addition to that we assume:

(i) the set of band-crossing points $\sigma_{bc}$ is a closed, nowhere dense set in $\mathbb{R}^d$ and has zero Lebesgue measure;

(ii) the entries of the Hermitian matrix $L(k)$ are infinitely differentiable in $k$ for all $k \not\in \sigma_{bc}$ that readily implies via the spectral theory, [36], infinite differentiability of all eigenvalues $\omega_n(k)$ in $k$ for all $k \not\in \sigma_{bc}$;

(iii) $L(k)$ satisfies a polynomial bound

$$\|L(k)\| \leq C(1 + |k|^p), \; k \in \mathbb{R}^d, \text{ for some } C > 0 \text{ and } p > 0. \quad (3.1)$$

Note that since $\omega_{n,\zeta}(k)$ are smooth if $k \not\in \sigma_{bc}$ the following relations hold:

$$\max_{|k| \leq \pi_0, \; l = 1, \ldots, N} |\nabla_k \omega_{n,\zeta}| \leq C_{\omega,1}, \quad \max_{|k| \leq \pi_0, \; l = 1, \ldots, N} |\nabla^2_k \omega_{n,\zeta}| \leq C_{\omega,2}, \quad (3.2)$$

where $C_{\omega,1}$ and $C_{\omega,2}$ are positive constants and

$$\pi_0 = \frac{1}{2} \min_{l = 1, \ldots, N} \min (\text{dist}\{\pm k_{s,l}, \sigma_{bc}\}, 1). \quad (3.3)$$

**Remark 3.3 (dispersion relations symmetry)** The symmetry condition (2.5) on the dispersion relations naturally arise in many physical problems, for example Maxwell equations in periodic media, see [2]-[4], [6], or when $L(k)$ originates from a Hamiltonian. We would like to stress that this symmetry conditions are not imposed to simplify studies but rather to take into account fundamental symmetries of physical media. The symmetry causes resonant nonlinear interactions, which create non-trivial effects. Interestingly, many problems without symmetries can be put into the framework with the symmetry by a certain extension, [8].
Remark 3.4 (band-crossing points) Band-crossing points are discussed in more details in [2, Section 5.4], [3, Sections 4.1, 4.2]. In particular, generically the set $\sigma_{bc}$ of band-crossing point is a manifold of the dimension $d - 2$. Notice also that there is an natural ambiguity in the definition of a normalized eigenvector $g_{n,\xi}(k)$ of $L(k)$ which is defined up to a complex number $\xi$ with $|\xi| = 1$. This ambiguity may not allow an eigenvector $g_{n,\xi}(k)$ which can be a locally smooth function in $k$ to be a uniquely defined continuous function in $k$ globally for all $k \not\in \sigma_{bc}$ because of a possibility of branching. But, importantly, the orthogonal projector $\Pi_{n,\xi}(k)$ on $g_{n,\xi}(k)$ as defined by (2.9) is uniquely defined and, consequently, infinitely differentiable in $k$ via the spectral theory, [36], for all $k \not\in \sigma_{bc}$. Since we consider $\hat{U}(k)$ as an element of the space $L^1$ and $\sigma_{bc}$ is of zero Lebesgue measure considering $k \not\in \sigma_{bc}$ is sufficient for us.

We introduce for vectors $\hat{u} \in \mathbb{C}^{2J}$ their expansion with respect to the orthonormal basis $\{g_{n,\xi}(k)\}$:

$$
\hat{u}(k) = \sum_{n=1}^{J} \sum_{\xi=\pm} \hat{u}_{n,\xi}(k) g_{n,\xi}(k) = \sum_{n=1}^{J} \sum_{\xi=\pm} \hat{u}_{n,\xi}(k), \quad \hat{u}_{n,\xi}(k) = \Pi_{n,\xi}(k) \hat{u}(k)
$$

(3.4)

and we refer to it as the modal decomposition of $\hat{u}(k)$ and to $\hat{u}_{n,\xi}(k)$ as the modal coefficients of $\hat{u}(k)$. Evidently

$$
\sum_{n=1}^{J} \sum_{\xi=\pm} \Pi_{n,\xi}(k) = I_{2J}, \text{ where } I_{2J} \text{ is the } 2J \times 2J \text{ identity matrix.}
$$

(3.5)

Notice that we can define the action of the operator $L(-i\nabla_r)$ on any Schwartz function $Y(r)$ by the formula

$$
L(-i\nabla_r) Y(k) = L(k) \hat{Y}(k),
$$

(3.6)

where in view of the polynomial bound (3.1) the order of $L$ does not exceed $p$. In a special case when all the entries of $L(k)$ are polynomials (3.6) turns into the action of the differential operator with constant coefficients.

3.2 Nonlinear part

The nonlinear term $\hat{F}$ in (2.1) is assumed to be a general functional polynomial of the form

$$
\hat{F} \left( \hat{U} \right) = \sum_{m \in M_F} \hat{F}^{(m)} \left( \hat{U}^m \right), \text{ where } \hat{F}^{(m)} \text{ is } m\text{-homogeneous polylinear operator,}
$$

(3.7)

$$
M_F = \{m_1, \ldots, m_p \} \subset \{2, 3, \ldots \} \text{ is a finite set, and } m_F = \max \{m : m \in M_F \}.
$$

(3.8)

The integer $m_F$ in (3.8) is called the degree of the functional polynomial $\hat{F}$. For instance, if $M_F = \{2\}$ or $M_F = \{3\}$ the polynomial $\hat{F}$ is respectively homogeneous quadratic or cubic.
Every \( m \)-linear operator \( \hat{F}^{(m)} \) in (3.7) is assumed to be of the form of a convolution
\[
\hat{F}^{(m)} \left( \hat{U}_{1}, \ldots, \hat{U}_{m} \right) (\mathbf{k}, \tau) = \int_{\mathbb{D}_{m}} \chi^{(m)} \left( \mathbf{k}, \overrightarrow{\mathbf{k}} \right) \hat{U}_{1} (\mathbf{k}') \cdots \hat{U}_{m} \left( \mathbf{k}^{(m)} \right) \tilde{d}^{(m-1)d} \overrightarrow{\mathbf{k}}, \quad (3.9)
\]
where \( \mathbb{D}_{m} = \mathbb{R}^{(m-1)d} \), \( \tilde{d}^{(m-1)d} \overrightarrow{\mathbf{k}} = \frac{dk' \cdots dk^{(m-1)}}{(2\pi)^{(m-1)d}} \),
\[
\mathbf{k}^{(m)} \left( \mathbf{k}, \overrightarrow{\mathbf{k}} \right) = \mathbf{k} - \mathbf{k}' - \cdots - \mathbf{k}^{(m-1)}, \quad \overrightarrow{\mathbf{k}} = (\mathbf{k}', \ldots, \mathbf{k}^{(m)}).
\]
indicating that the nonlinear operator \( F^{(m)} (\hat{U}_{1}, \ldots, \hat{U}_{m}) \) is translation invariant (it may be local or non-local). The quantities \( \chi^{(m)} \) in (3.9) are called susceptibilities. For numerous examples of nonlinearities of the form similar to (3.7), (3.9) see [2]-[8] and references therein. In what follows the nonlinear term \( \hat{F} \) in (2.1) will satisfy the following conditions.

**Condition 3.5 (nonlinearity)** The nonlinearity \( \hat{F} \left( \hat{U} \right) \) is assumed to be of the form (3.7)-(3.8). The susceptibility \( \chi^{(m)} (\mathbf{k}, \mathbf{k}', \ldots, \mathbf{k}^{(m)}) \) is infinitely differentiable for all \( \mathbf{k} \) and \( \mathbf{k}^{(j)} \) which are not band-crossing points, and is bounded, namely for some constant \( C_{\chi} \)
\[
\| \chi^{(m)} \| = (2\pi)^{-(m-1)d} \sup_{\mathbf{k}, \mathbf{k}', \ldots, \mathbf{k}^{(m)} \in \mathbb{R}^{d} \setminus \sigma_{bc}} |\chi^{(m)} (\mathbf{k}, \mathbf{k}', \ldots, \mathbf{k}^{(m)})| \leq C_{\chi}, \quad m \in \mathcal{M}_{F}, \quad (3.11)
\]
where the norm \( |\chi^{(m)} (\mathbf{k}, \overrightarrow{\mathbf{k}})| \) of \( m \)-linear tensor \( \chi^{(m)} : (\mathbb{C}^{2J})^{m} \rightarrow (\mathbb{C}^{2J})^{m} \) for fixed \( \mathbf{k}, \overrightarrow{\mathbf{k}} \) is defined by
\[
|\chi^{(m)} (\mathbf{k}, \overrightarrow{\mathbf{k}})| = \sup_{|\mathbf{x}| \leq 1} |\chi^{(m)} (\mathbf{k}, \overrightarrow{\mathbf{k}}) (\mathbf{x}_{1}, \ldots, \mathbf{x}_{m})|, \quad \text{where } |\mathbf{x}| \text{ is the Euclidean norm.} \quad (3.12)
\]
Since \( \chi_{\mathbf{k}, \overrightarrow{\mathbf{k}}}^{(m)} (\mathbf{k}, \mathbf{k}', \ldots, \mathbf{k}^{(m)}) \) are smooth if \( \mathbf{k} \notin \sigma_{bc} \) the following relation holds:
\[
\max_{|\mathbf{k} \pm \mathbf{k}_{l}| \leq \pi_{0}, \quad l = 1, \ldots, N} \left| \nabla \chi_{\mathbf{k}, \overrightarrow{\mathbf{k}}}^{(m)} (\mathbf{k}, \mathbf{k}', \ldots, \mathbf{k}^{(m)}) \right| \leq C_{\chi}', \quad (3.13)
\]
if \( \mathbf{k}_{l} \notin \sigma_{bc}, \quad \pi_{0} \) is defined by (3.3). The case when \( \chi^{(m)} (\mathbf{k}, \overrightarrow{\mathbf{k}}) \) depend on small \( \vartheta \) or, more generally, on \( \vartheta^{q}, \quad q > 0 \), can be treated similarly, see [8].

### 3.3 Resonance invariant nk-spectrum

In this section, being given the dispersion relations \( \omega_{n} (\mathbf{k}) \geq 0, \quad n \in \{1, \ldots, J\} \), we consider resonance properties of \( nk \)-spectra \( S \) and the corresponding \( k \)-spectra \( K_{S} \) as defined in Definition (2.7), i.e.
\[
S = \{(n_{l}, \mathbf{k}_{l}) \mid l = 1, \ldots, N\} \subset \Sigma = \{1, \ldots, J\} \times \mathbb{R}^{d}, \quad K_{S} = \{\mathbf{k}_{l} \mid l = 1, \ldots, |K_{S}|\}. \quad (3.14)
\]
We precede the formal description of the resonance invariance (see Definition 3.8) with the following guiding physical picture. Initially at $\tau = 0$ the wave is a multi-wavepacket composed of modes from a small vicinity of the $nk$-spectrum $S$. As the wave evolves according to (2.1) the polynomial nonlinearity inevitably involves a larger set of modes $[S]_{out} \supseteq S$, but not all modes in $[S]_{out}$ are ”equal” in developing significant amplitudes. The qualitative picture is that whenever certain interaction phase function (see (4.23) below) is not zero, the fast time oscillations weaken effective nonlinear mode interaction and the energy transfer from the original modes in $S$ to relevant modes from $[S]_{out}$, keeping their magnitudes vanishingly small as $\beta, \varrho \to 0$. There is a smaller set of modes $[S]_{out}^{res}$ which can interact with modes from $S$ rather effectively and develop significant amplitudes. Now,

$$\text{if } [S]_{out}^{res} \subseteq S \text{ then } S \text{ is called resonance invariant.}$$

(3.15)

In simpler situations the resonance invariance conditions turns into the well-known in nonlinear optics phase and frequency matching conditions. For instance, if $S$ contains $(n_0, k_{s_0})$ and the dispersion relations allow for the second harmonic generation in another band $n_1$ so that $2\omega_{n_0}(k_{s_0}) = \omega_{n_1}(2k_{s_0})$, then for $S$ to be resonance invariant it must contain $(n_1, 2k_{s_0})$ too.

Let us turn now to the rigorous constructions. First we introduce necessary notations. Let $m \geq 2$ be an integer, $\tilde{l} = (l_1, \ldots, l_m), l_j \in \{1, \ldots, N\}$ be an integer vector from $\{1, \ldots, N\}^m$ and $\tilde{\zeta} = (\zeta^{(1)}, \ldots, \zeta^{(m)})$, $\zeta^{(j)} \in \{+1, -1\}$ be a binary vector from $\{+1, -1\}^m$. Note that a pair $((\zeta^{(j)}, l_j), (\zeta^{(m)}, l_m))$ naturally labels a sample string of the length $m$ composed of elements $(\zeta^{(j)}, n_j, k_{s_j})$ from the set $\{+1, -1\} \times S$. Let us introduce the sets

$$\Lambda = \{(\zeta, l) : l \in \{1, \ldots, N\}, \zeta \in \{+1, -1\}\},$$

(3.16)

$$\Lambda^m = \{\tilde{\lambda} = (\lambda_1, \ldots, \lambda_m), \lambda_j \in \Lambda, j = 1, \ldots, m\}.$$

There is a natural one-to-one correspondence between $\Lambda^m$ and $\{-1, 1\}^m \times \{1, \ldots, N\}^m$ and we will write, exploiting this correspondence

$$\tilde{\lambda} = ((\zeta^{(j)}, l_j), \ldots, (\zeta^{(m)}, l_m)) = (\tilde{\zeta}, \tilde{l}), \tilde{\nu} \in \{-1, 1\}^m, \tilde{l} \in \{1, \ldots, N\}^m \text{ for } \tilde{\lambda} \in \Lambda^m.$$  

(3.17)

Let us introduce the following linear combination

$$\kappa_m (\tilde{\lambda}) = \kappa_m (\tilde{\zeta}, \tilde{l}) = \sum_{j=1}^{m} \zeta^{(j)} k_{s_j} \text{ with } \zeta^{(j)} \in \{+1, -1\},$$

(3.18)

and let $[S]_{K, out}$ be the set of all its values as $k_{s_j} \in K_S, \tilde{\lambda} \in \Lambda^m$, namely

$$[S]_{K, out} = \bigcup_{m \in \omega_{np}} \bigcup_{\tilde{\lambda} \in \Lambda^m} \{\kappa_m (\tilde{\lambda})\}.$$  

(3.19)

We call $[S]_{K, out}$ output $k$-spectrum of $K_S$ and assume that

$$[S]_{K, out} \cap \sigma_{bc} = \emptyset.$$  

(3.20)

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We also define the output nk-spectrum of $S$ by

$$[S]_{\text{out}} = \left\{ (n, k) \in \{1, \ldots, J\} \times \mathbb{R}^d : n \in \{1, \ldots, J\}, \ k \in [S]_{K,\text{out}} \right\}. \quad (3.21)$$

We introduce the following functions

$$\Omega_{1,m} \left( \vec{\lambda}, \vec{k}_s \right) = \sum_{j=1}^{m} \zeta^{(j)} \omega^{(j)}(k_{sl_j}) \right), \quad \vec{k}_s = (k_{s1}, \ldots, k_{s|K_S|}), \quad \text{where} \ k_{sl_j} \in K_S, \quad (3.22)$$

$$\Omega \left( \zeta, n, \vec{\lambda} \right) \left( k_{ss}, \vec{k}_s \right) = -\zeta \omega_n(k_{ss}) + \Omega_{1,m} \left( \vec{\lambda}, \vec{k}_s \right), \quad (3.23)$$

where $\zeta = \pm 1, \ m \in M_F$ as in (3.7). We introduce these functions to apply later to phase functions (4.23).

Now we introduce the resonance equation

$$\Omega \left( \zeta, n, \vec{\lambda} \right) \left( \zeta \kappa_m(\vec{\lambda}), \vec{k}_s \right) = 0, \quad \vec{l} \in \{1, \ldots, N\}^m, \quad \zeta \in \{-1, 1\}^m, \quad (3.24)$$

denoting by $P(S)$ the set of its solutions $\left( m, \zeta, n, \vec{\lambda} \right)$. Such a solution is called $S$-internal if

$$\left( n, \zeta \kappa_m(\vec{\lambda}) \right) \in S, \quad \text{that is} \quad n = n_{l_0}, \quad \zeta \kappa_m(\vec{\lambda}) = k_{s\lambda_0}, \quad l_0 \in \{1, \ldots, N\}, \quad (3.25)$$

and we denote the corresponding $l_0 = I(\vec{\lambda})$. We also denote by $P_{\text{int}}(S) \subset P(S)$ the set of all $S$-internal solutions to (3.24).

Now we consider the simplest solutions to (3.24) which play an important role. Keeping in mind that the string $\vec{l}$ can contain several copies of a single value $l$, we can recast the sum in (3.22) as follows:

$$\Omega_{1,m} \left( \vec{\lambda} \right) = \Omega_{1,m} \left( \vec{\zeta}, \vec{l} \right) = \sum_{l=1}^{N} \delta_l \omega_l(k_{sl}), \quad \text{where} \ \delta_l = \left\{ \begin{array}{ll} \sum_{j \in \vec{l}^{-1}(l)} \zeta^{(j)} & \text{if} \ \vec{l}^{-1}(l) \neq \emptyset \\ 0 & \text{if} \ \vec{l}^{-1}(l) = \emptyset \end{array} \right., \quad (3.26)$$

$$\vec{l}^{-1}(l) = \{ j \in \{1, \ldots, m\} : l_j = l \}, \quad \vec{l} = (l_1, \ldots, l_m), \quad 1 \leq l \leq N.$$

**Definition 3.6 (universal solutions)** We call a solution $\left( m, \zeta, n, \vec{\lambda} \right) \in P(S)$ of (3.24) universal if it has the following properties: (i) only a single coefficient out of all $\delta_l$ in (3.26) is nonzero, namely for some $I_0$ we have $\delta_{I_0} = \pm 1$ and $\delta_l = 0$ for $l \neq I_0$; (ii) $n = n_{I_0}$ and $\zeta = \delta_{I_0}$.

We denote the set of universal solutions to (3.24) by $P_{\text{univ}}(S)$. A justification for calling such solution universal comes from the fact that if a solution is a universal solution for one $\vec{k}_s$, it is a solution for any other $\vec{k}_s \in \mathbb{R}^d$. Note that a universal solution is a $S$-internal solution with $I(\vec{\lambda}) = I_0$ implying

$$P_{\text{univ}}(S) \subseteq P_{\text{int}}(S). \quad (3.27)$$

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Indeed, observe that for \( \delta \) as in (3.26)

\[
\zeta_m \left( \lambda \right) = \zeta_{m} \left( \zeta, \delta \right) = \delta_{\lambda} \zeta_{m} \left( \lambda \right) = \sum_{j=1}^{m} \zeta_{m} \left( \lambda \right) = \sum_{l=1}^{N} \delta_{l} \zeta_{m} \left( \lambda \right)
\]

implying \( \zeta_m \left( \lambda \right) = \delta_{l_{0}} \zeta_{m} \left( \lambda \right) \) and \( \zeta_{m} \left( \lambda \right) = \delta_{l_{0}} \zeta_{m} \left( \lambda \right) = \delta_{l_{0}} \zeta_{m} \left( \lambda \right) \). Then equation (3.24) is obviously satisfied and \( \left( n, \zeta \zeta_{m} \left( \lambda \right) \right) = \left( n_{l_{0}}, k_{s_{l_{0}}} \right) \in S \).

**Example 3.7 (universal solutions)** Suppose there is just a single band, i.e. \( J = 1 \), a symmetric dispersion relation \( \omega_1 (-k) = \omega_1 (k) \), a cubic nonlinearity \( F \) with \( \mathfrak{M}_{F} = \{ 3 \} \). We take the \( nk \)-spectrum \( S = \{(1, k_{s}), (1, -k_{s})\} \), that is \( N = 2 \) and \( k_{s_{1}} = k_{s_{2}} = -k_{s} \). This example is typical for two counterpropagating waves. Then \( \Omega_{1, 3} \left( \lambda \right) \left( k_{s} \right) = \sum_{j=1}^{3} \zeta_{m} \left( \lambda \right) = \left( \delta_{1} + \delta_{2} \right) \omega_{1} \left( k_{s} \right) \) and \( \zeta_{m} \left( \lambda \right) = \sum_{j=1}^{m} \zeta_{m} \left( \lambda \right) = \delta_{1} k_{s_{1}} + \delta_{2} k_{s_{2}} = \delta_{1} - \delta_{2} \zeta_{m} \left( \lambda \right) \) where we use notation (3.24). The universal solution set has the form \( P_{univ} \left( S \right) = \left\{ \left( 3, \zeta, 1, \lambda \right) : \lambda \in \Lambda_{\zeta} \right\} \) where \( \Lambda_{\zeta} \) consists of vectors \( (\lambda_{1}, \lambda_{2}, \lambda_{3}) \) of the form \((+, -), (+, -), (+, 1), (+, -), (-, 1), (+, 2), (-, 2), (+, 1), (+, 2), (-, 2), (+, 2)\), and vectors obtained from the listed ones by permutations of coordinates \( \lambda_{1}, \lambda_{2}, \lambda_{3} \). The solutions from \( P_{int} \left( S \right) \) have to satisfy \( |\delta_{1} - \delta_{2}| = 1 \) and \( |\delta_{1} + \delta_{2}| = 1 \) which is possible only if \( \delta_{1} \delta_{2} = 0 \). Since \( \zeta = \delta_{1} + \delta_{2} \) we have \( \zeta_{\lambda_{m}} \left( \lambda \right) = \left( \delta_{1} - \delta_{2} \right) \zeta_{m} \left( \lambda \right) = k_{s_{1}} \) if \( |\delta_{1}| = 1 \) or \( \zeta_{\lambda_{m}} \left( \lambda \right) = k_{s_{2}} \) if \( |\delta_{2}| = 1 \). Hence \( P_{int} \left( S \right) = P_{univ} \left( S \right) \) in this case. Note that if we set \( S_{1} = \{(1, k_{s})\}, S_{2} = \{(1, -k_{s})\} \) then \( S = S_{1} \cup S_{2} \) but \( P_{int} \left( S \right) \) is larger than \( P_{int} \left( S_{1} \right) \cup P_{int} \left( S_{2} \right) \). This can be interpreted as follows. When only modes from \( S_{1} \) are excited, the modes from \( S_{2} \) remain non-excited. But when the both \( S_{1} \) and \( S_{2} \) are excited, there is a resonance effect of \( S_{1} \) onto \( S_{2} \), represented, for example, by \( \lambda = ((+, 1), (-, 1), (+, 2)) \), which involves the mode \( \zeta_{\lambda_{m}} \left( \lambda \right) = k_{s_{2}} \).

Now we are ready to define resonance invariant spectra. First, we introduce a subset \( [S]_{out}^{res} \) of \( [S]_{out} \) by the formula

\[
[S]_{out}^{res} = \left\{ \left( n, k_{s_{s}} \right) \in [S]_{out} : k_{s_{s}} = \zeta_{m} \left( \lambda \right), m \in \mathfrak{M}_{F}, \right\}
\]

where \( \left( m, \zeta, n, \lambda \right) \) is a solution of (3.24), calling it resonant output spectrum of \( S \), and then we define

\[
\text{resonance selection operation } \mathcal{R} \left( S \right) = S \cup [S]_{out}^{res}.
\]

**Definition 3.8 (resonance invariant nk-spectrum)** The \( nk \)-spectrum \( S \) is called resonance invariant if \( \mathcal{R} \left( S \right) = S \) or, equivalently, \( [S]_{out}^{res} \subseteq S \). The \( nk \)-spectrum \( S \) is called universally resonance invariant if \( \mathcal{R} \left( S \right) = S \) and \( P_{univ} \left( S \right) = P_{int} \left( S \right) \).
Obviously, \( nk \)-spectrum \( S \) is resonance invariant if and only if all solutions of (3.24) are internal, that is, \( P_{\text{int}}(S) = P(S) \).

It is worth noticing that even when a \( nk \)-spectrum is not resonance invariant often it can be easily extended to a resonance invariant one. Namely, if \( \mathcal{R}^j(S) \cap \sigma_{bc} = \emptyset \) for all \( j \) then the set

\[
\mathcal{R}^\infty(S) = \bigcup_{j=1}^{\infty} \mathcal{R}^j(S) \subset \Sigma = \{1, \ldots, J\} \times \mathbb{R}^d
\]

is resonance invariant. In addition to that, \( \mathcal{R}^\infty(S) \) is always at most countable. Usually it is finite i.e. \( \mathcal{R}^\infty(S) = \mathcal{R}^p(S) \) for a finite \( p \), see examples below; also \( \mathcal{R}^\infty(S) = S \) for generic \( K_S \).

**Example 3.9 (resonance invariant \( nk \)-spectra for quadratic nonlinearity)** Suppose there is a single band, i.e. \( J = 1 \), with a symmetric dispersion relation, and a quadratic nonlinearity \( F \), that is \( \mathfrak{M}_F = \{2\} \). Let us assume that \( k_s \neq 0, k_s, 2k_s, 0 \) are not band-crossing points and look at two examples. First, suppose that \( 2\omega_1(k_s) \neq \omega_1(2k_s) \) (no second harmonic generation) and \( \omega_1(0) \neq 0 \). Let us set the \( nk \)-spectrum to be the set \( S_1 = \{(1, k_s)\} \), then \( S_1 \) is resonance invariant. Indeed, \( K_{S_1} = \{k_s\} \), \( S_{1\text{out}} = \{0, 2k_s, -2k_s\} \), \( S_{1\text{out}} = \{(1, 0), (1, 2k_s), (1, -2k_s)\} \) and an elementary examination shows that \( \{S_{1\text{out}}\} = \emptyset \subset S_1 \) implying \( \mathcal{R}(S_1) = S_1 \). For the second example let us assume \( \omega_1(0) \neq 0 \) and \( 2\omega_1(k_s) = \omega_1(2k_s) \), that is the second harmonic generation is present. Here \( \{S_{1\text{out}}\} = \{(1, 2k_s)\} \) and \( \mathcal{R}(S_1) = \{(1, k_s), (1, 2k_s)\} \) implying \( \mathcal{R}(S_1) \neq S_1 \) and, hence, \( S_1 \) is not resonance invariant. Suppose now that \( 4k_s, 3k_s \notin \sigma_{bc} \) and \( \omega_1(0) \neq 0, \omega_1(4k_s) \neq 2\omega_1(2k_s), \omega_1(3k_s) \neq \omega_1(k_s) + \omega_1(2k_s) \) and let us set \( S_2 = \{(1, k_s), (1, 2k_s)\} \). An elementary examination shows that \( S_2 \) is resonance invariant. Note that \( S_2 \) can be obtained by iterating the resonance selection operator, namely \( S_2 = \mathcal{R}(\mathcal{R}(S_1)) \). Note also that \( P_{\text{univ}}(S_2) \neq P_{\text{int}}(S_2) \). Notice that \( \omega_1(0) = 0 \) is a special case since \( k = 0 \) is a band-crossing point, and it requires a special treatment.

**Example 3.10 (resonance invariant \( nk \)-spectra for cubic nonlinearity)** Let us consider one-band case with symmetric dispersion relation and a cubic nonlinearity that is \( \mathfrak{M}_F = \{3\} \). First we take \( S_1 = \{(1, k_s)\} \), we assume that \( k_s, 3k_s \) are not band-crossing points, implying \( \{S_{1\text{out}}\} = \{k_s, -k_s, 3k_s, -3k_s\} \). We have \( \Omega_{1,3}(\lambda) = \sum_{j=1}^{3} \zeta^{(j)} \omega_1(k_s) = \delta_1 \omega_1(k_s) \) and \( \zeta \lambda \) where we use notation (3.20), \( \delta_1 \) takes values \( 1, -1, 3, -3 \). If \( \omega_1(k_s) \neq \omega_1(3k_s) \) then (3.24) has a solution only if \( |\delta_1| = 1 \) and \( \delta_1 = \zeta \), hence \( \zeta = k_s \) and every solution is internal. Hence, \( \{S_{1\text{out}}\} = \emptyset \) and \( \mathcal{R}(S_1) = S_1 \). Now consider the case associated with the third harmonic generation, namely \( 3\omega_1(k_s) = \omega_1(3k_s) \) and assume that \( \omega_1(3k_s) + 2\omega_1(k_s) \neq \omega_1(5k_s), 3\omega_1(3k_s) \neq \omega_1(9k_s), 2\omega_1(3k_s) + \omega_1(k_s) \neq \omega_1(7k_s), 2\omega_1(3k_s) - \omega_1(k_s) \neq \omega_1(5k_s) \). An elementary examination shows that the set \( S_4 = \{(1, 3k_s), (1, k_s), (1, -k_s)(1, -3k_s)\} \) satisfies \( \mathcal{R}(S_1) = S_4 \). Consequently, a multi-wavepacket having \( S_4 \) as its resonance invariant \( nk \)-spectrum involves the third harmonic generation and, according to Theorem 2.8, it is preserved under nonlinear evolution. The above examples indicate that in simple cases the conditions on \( k_s \) which make \( S \) non-invariant with respect to \( \mathcal{R} \) have a form of several algebraic equations, Hence, for almost
all $k_s$ such spectra $S$ are resonance invariant. The examples also show that if we fix $S$ and dispersion relations then we can include $S$ in larger spectrum $S' = \mathcal{R}^p(S)$ using repeated application of the operation $\mathcal{R}$ to $S$, and often the resulting extended $nk$-spectrum $S'$ is resonance invariant. We show in the following section that $nk$-spectrum $S$ with generic $K_S$ is universally resonance invariant.

Note that the concept of resonance invariant $nk$-spectrum gives a mathematical description of such fundamental concepts of nonlinear optics as phase matching, frequency matching, four wave interaction in cubic media and three wave interaction in quadratic media. If a multi-wavepacket has a resonance invariant spectrum, all these phenomena may take place in the internal dynamics of the multi-wavepacket, but do not lead to resonant interactions with continuum of all remaining modes.

### 3.4 Genericity of the $nk$-spectrum invariance condition

In simpler situations, when the number of bands $J$ and wavepackets $N$ are not too large, the resonance invariance of $nk$-spectrum can be easily verified as above in Examples 3.9, 3.10 but what one can say if $J$ or $N$ are large, or if the dispersion relations are not explicitly given? We show below that in properly defined non-degenerate cases a small variation of $K_S$ makes $S$ universally resonance invariant, i.e. the resonance invariance is a generic phenomenon.

Assume that the dispersion relations $\omega_n(k) \geq 0, n \in \{1, \ldots, J\}$ are given. Observe then that $\Omega_m(\zeta, n, \tilde{\lambda}) = \Omega_m(\zeta, n, \tilde{\lambda})(k_{s1}, \ldots, k_{s|K_S|})$ defined by (3.23) is a continuous function of $k_{s} \notin \sigma_{bc}$ for every $m, \zeta, n, \tilde{\lambda}$.

**Definition 3.11 ($\omega$-degenerate dispersion relations)** We call dispersion relations $\omega_n(k)$, $n = 1, \ldots, J$, $\omega$-degenerate if there exists such a point $k_s \in \mathbb{R}^d \setminus \sigma_{bc}$ that for all $k$ in a neighborhood of $k_s$ at least one of the following four conditions holds: (i) the relations are linearly dependent, namely $\sum_{n=0}^{J} C_n \omega_n(k) = c_0$, where all $C_n$ are integers, one of which is nonzero, and the $c_0$ is a constant; (ii) at least one of $\omega_n(k)$ is a linear function; (iii) at least one of $\omega_n(k)$ satisfies equation $C \omega_n(k) = \omega_n(Ck)$ with some $n$ and integer $C \neq \pm 1$; (iv) at least one of $\omega_n(k)$ satisfies equation $\omega_n(k) = \omega_{n'}(-k)$ where $n' \neq n$.

Note that fulfillment of any of the four conditions in Definition 3.11 makes impossible turning some non resonance invariant sets into resonance invariant ones by a variation of $k_{s}$. For instance, if $\mathcal{M}_F = \{2\}$ as in Example 3.9 and $2\omega_1(k) = \omega_1(2k)$ for all $k$ in an open set $G$ then the set $\{1, k_s\}$ with $k_s \in G$ cannot be made resonance invariant by a small variation of $k_s$. Below we formulate two theorems which show that if dispersion relations are not $\omega$-degenerate, then a small variation of $k_{s}$ turns non resonance invariant sets into resonance invariant; the proofs of the theorems are given in [8].

**Theorem 3.12** If $\Omega_m(\zeta, n_0, \tilde{\lambda})(k_{s1}, \ldots, k_{s|K_S|}) = 0$ on a cylinder $G$ in $(\mathbb{R}^d \setminus \sigma_{bc})^{[K_S]}$ which is a product of small balls $G_i \subset (\mathbb{R}^d \setminus \sigma_{bc})$ then either $(m, \zeta, n_0, \tilde{\lambda}) \in \mathcal{P}_{uni}(S)$ or dispersive relations $\omega_n(k)$ are $\omega$-degenerate as in Definition 3.11.
Theorem 3.13 (genericity of resonance invariance) Assume that dispersive relations \( \omega_n(k) \) are continuous and not \( \omega \)-degenerate as in Definition 3.11. Let \( K_{rinv} \) be a set of points \( \{k_1, \ldots, k_{|K_S|}\} \) such that there exists a universally resonance invariant \( nk \)-spectrum \( S \) for which its \( k \)-spectrum \( K_S = \{k_1, \ldots, k_{|K_S|}\} \). Then \( K_{rinv} \) is open and everywhere dense set in \( \mathbb{R}^d \setminus \sigma_{bc} \) for some \( |K_S| \).

4 Integrated evolution equation

Using the variation of constants formula we recast the modal evolution equation (2.1) into the following equivalent integral form

\[
\hat{U}(k, \tau) = \int_0^\tau e^{-i\frac{(\tau-\tau')}{\varrho}L(k)} \hat{F}\left(\hat{U}\right)(k, \tau) \, d\tau' + e^{-i\frac{\varpi}{\varrho}L(k)}\hat{\nu}(k), \quad \tau \geq 0.
\]  

(4.1)

Then we factor \( \hat{U}(k, \tau) \) into the slow variable \( \hat{u}(k, \tau) \) and the fast oscillatory term as in (2.12), namely

\[
\hat{U}(k, \tau) = e^{-i\frac{\varpi}{\varrho}L(k)}\hat{u}(k, \tau), \quad \hat{U}_{n, \zeta}(k, \tau) = \hat{u}_{n, \zeta}(k, \tau) e^{-i\frac{\varpi}{\varrho}\omega_n(k)},
\]

(4.2)

where \( \hat{u}_{n, \zeta}(k, \tau) \) are the modal components of \( \hat{u}(k, \tau) \) as in (3.4). Notice that \( \hat{u}_{n, \zeta}(k, \tau) \) in (4.2) may depend on \( \varrho \) and (4.2) is just a change of variables and not an assumption.

Remark 4.1 Note that if \( \hat{u}_{n, \zeta}(k, \tau) \) is a wavepacket, it is localized near its principal wavevector \( k_* \). The expansion of \( \zeta \omega_n(k) \) near the principal wavevector \( \zeta k_* \) (we take \( \zeta = 1 \) for brevity) takes the form

\[
\omega_n(k) = \omega(k_*) + \nabla_k \omega_n(k_*) (k - k_*) + \frac{1}{2} \nabla_k^2 \omega_n(k_*) (k - k_*)^2 + \ldots
\]

To discuss the impact of the change of variables (4.2) we make the change of variables \( k - k_* = \xi \). The change of variables (4.2)

\[
\hat{U}_{n, +}(k, \tau) = \hat{u}_{n, +}(k, \tau) e^{-i\frac{\varpi}{\varrho}\omega_n(k)}
\]

(4.3)

\[
= \hat{u}_{n, \zeta}(k, \tau) e^{-i\frac{\varpi}{\varrho}\omega_n(k_*)} e^{-\frac{i}{\varrho}\nabla_k \omega_n(k_*) (k - k_*)} e^{-\frac{i}{\varrho} \frac{1}{2} \nabla_k^2 \omega_n(k_*) (k - k_*)^2 + \ldots}
\]

\[
= \hat{u}_{n, +}(k_* + \xi, \tau) e^{-i\frac{\varrho}{\varpi}\zeta \omega_n(k_*)} e^{-\frac{i}{\varrho}\nabla_k \omega_n(k_*) \xi} e^{-i\frac{\varpi}{\varrho} R(\xi)}
\]

\[
R(\xi) = \omega_n(k) - \omega_n(k_*) - \nabla_k \omega_n(k_*) (k - k_*) = \frac{1}{2} \nabla_k^2 \omega_n(k_*) (\xi)^2 + \ldots
\]

(4.4)

has the first factor \( e^{-i\frac{\varpi}{\varrho}\omega_n(k_*)} \) responsible for fast time oscillations of \( \hat{U}_{n, \zeta}(k, \tau) \) and \( U_{n, \zeta}(r, \tau) \)

The second factor \( e^{-\frac{i}{\varrho}\nabla_k \omega_n(k_*) \xi} \) is responsible for the spatial shifts of the inverse Fourier transform by \( \frac{i}{\varrho} \nabla_k \omega_n(k_*) \), since the shifts are time dependent they cause the rectilinear movement of the wavepacket \( U_{n, \zeta}(r, \tau) \) with the group velocity \( \frac{1}{\varrho} \nabla_k \omega_n(k_*) \), the third factor is responsible
for dispersive effects. Hence the change of variables (4.2) effectively introduces the moving coordinate frame for \( \hat{U}_{n,\zeta}(k, \tau) \) for every \( k \) and in this coordinate frame \( \hat{u}_{n,\zeta}(k, \tau) \) has zero group velocity and does not have high-frequency time oscillations. The following proposition shows that if \( \hat{u}_{n,\zeta}(k, \tau) \) is a wavepacket with a constant position, \( \hat{U}_{n,+}(k, \tau) \) is a particle wavepacket in the sense of Definition with position which moves with a constant velocity.

**Proposition 4.2** Let \( \hat{u}_l(k, \tau) \) be for every \( \tau \in [0, \tau_*] \) a particle wavepacket in the sense of Definition 2.2 with nk-pair \((n, k_*)\) regularity \( s \) and position \( r_* \in \mathbb{R}^d \) which does not depend on \( \tau \), assume also that and constants \( C \) in (2.33) and \( C'C \) in (2.27) and (2.30) do not depend on \( \tau \). Let \( \hat{U}_l(k, \tau) \) be defined in terms of \( \hat{u}_l(k, \tau) \) by (4.2). Assume that (2.48) holds. Then \( \hat{U}_l(k, \tau) \) for every \( \tau \in [0, \tau_*] \) a particle wavepacket in the sense of Definition 2.2 with nk-pair \((n, k_*)\) regularity \( s \) and with \( \tau \)-dependent position \( r_* + \frac{\tau}{\theta} \nabla \omega_n(k_*) \in \mathbb{R}^d \).

**Proof.** The wavepacket \( \hat{u}_l(k, \tau) \) involves two components \( \hat{u}_{n,\zeta}(k, \tau) \), \( \zeta = \pm 1 \) for which (2.29) holds

\[
\hat{u}_{n,\zeta}(k, \tau) = \Psi \left( \beta \frac{1-\epsilon}{2}, \zeta k_*; k \right) \Pi_{n,\zeta}(k) \hat{u}_{n,\zeta}(k, \tau),
\]

By (4.2)

\[
\hat{U}_{n,\zeta}(k, \tau) = \hat{u}_{n,\zeta}(k, \tau) e^{-\frac{i}{\theta} \zeta \omega_n(k)}.
\]

According to Definition (2.1) multiplication by a scalar bounded continuous function \( e^{-\frac{i}{\theta} \zeta \omega_n(k)} \) may only change a constant \( C' \) in (2.30), therefore it transforms wavepackets into wavepackets. To check that \( \hat{U}_l(k, \tau) \) is a particle-like wavepacket we consider (2.33) with \( \hat{u}_l(\beta, r_*; k) \) replaced by \( \hat{u}_{n,\zeta}(k, \tau) e^{-\frac{i}{\theta} \zeta \omega_n(k)} \) and \( r_* \) replaced by \( r_* + \frac{\tau}{\theta} \nabla \omega_n(k_*) \). We consider for brevity \( \hat{u}_n(k, \tau) = \hat{u}_{n,\zeta}(k, \tau) \) with \( \zeta = 1 \); the case \( \zeta = -1 \) is similar.

\[
\begin{align*}
\int_{\mathbb{R}^d} \left| \nabla_k \left( e^{i(r_* + \frac{\tau}{\theta} \nabla \omega_n(k_*))} k \hat{u}_n(k, \tau) e^{-\frac{i}{\theta} \zeta \omega_n(k)} \right) \right| dk &= \\
\int_{\mathbb{R}^d} \left| \nabla_k \left( e^{i(r_* + \frac{\tau}{\theta} \nabla \omega_n(k_*))} k \hat{u}_n(k, \tau) e^{-\frac{i}{\theta} \zeta \omega_n(k)} e^{\frac{i}{\theta} \zeta \omega_n(k_*)} \right) \right| dk \\
&= \int_{\mathbb{R}^d} \left| k \nabla \left( e^{ir_* k \hat{u}_n(k, \tau)} e^{-\frac{i}{\theta} \tau R(k-k_*)} \right) \right| dk \leq I_1 + I_2
\end{align*}
\]

where where \( R(\xi) \) is defined by (4.4),

\[
I_1 = \int_{\mathbb{R}^d} \left| e^{-\frac{i}{\theta} \tau R(k-k_*)} \nabla_k \left( e^{ir_* k \hat{u}_n(k, \tau)} \right) \right| dk,
\]

\[
I_2 = \int_{\mathbb{R}^d} \left| \left( e^{ir_* k \hat{u}_n(k, \tau)} \right) \nabla_k e^{-\frac{i}{\theta} \tau R(k-k_*)} \right| dk.
\]

The integral \( I_1 \) is bounded uniformly in \( r_* \) by \( C'_1 \beta^{-1-\epsilon} \) since \( \hat{u}_{n,\zeta}(k, \tau) \) satisfies (2.33). Note that

\[
I_2 = \int_{\mathbb{R}^d} \left| \left( e^{ir_* k \hat{u}_n(k, \tau)} \right) \nabla_k e^{-\frac{i}{\theta} \tau R(k-k_*)} \right| dk \leq \\
\int_{\mathbb{R}^d} \left| \hat{u}_n(k, \tau) \right| \frac{\tau}{\theta} \left| \nabla_k R(k-k_*) \right| dk
\]

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Note that according to (4.3) and (2.25) \( \hat{u}_{n,\xi}(k,\tau) \neq 0 \) only if \( |k - k_\ast| \leq 2\beta^{1-\epsilon} \), and for such \( k - k_\ast \) we have Taylor remainder estimate

\[
|\nabla_k R(k - k_\ast)| \leq C\beta^{1-\epsilon}.
\]

Therefore \( I_2 \leq C'\beta^{1-\epsilon}/\varrho \) and

\[
I_1 + I_2 \leq C'\beta^{1-\epsilon}/\varrho
\]

Using (2.48) we conclude that this inequality implies (2.33) for \( \hat{U}_l(k,\tau) \), therefore it is a particle-like wavepacket.  

From (4.1) and (4.2) we obtain the following integrated evolution equation for \( \hat{u} = \hat{u}(k,\tau) \), \( \tau \geq 0 \),

\[
\hat{u}(k,\tau) = \mathcal{F}_N(\hat{u})(k,\tau) + \hat{h}(k) \,, \quad \mathcal{F}_N(\hat{u}) = \sum_{m \in \mathbb{N}_F} \mathcal{F}^{(m)}(\hat{u}^m(k,\tau))
\]

\[
\mathcal{F}^{(m)}(\hat{u}^m)(k,\tau) = \int_0^\tau \left( e^{-i\omega L(k)} \hat{F}_m \left( \left( e^{i\left(L(k)\hat{u}\right)} \right)^m \right) \right) (k,\tau') \, d\tau',
\]

where \( \hat{F}_m \) are defined by (3.7) and (3.9) in terms of the susceptibilities \( \chi^{(m)} \), and \( \mathcal{F}^{(m)} \) are bounded as in the following lemma.

Recall that spaces \( L^{1,a} \) are defined by the formula (2.17). Below we formulate basic properties of the spaces. Recall Young’s inequality

\[
\|\hat{u} \ast \hat{v}\|_{L^1} \leq \|\hat{u}\|_{L^1} \|\hat{v}\|_{L^1}.
\]

This inequality implies boundedness of convolution in \( L^{1,a} \), namely the following Lemma holds.

**Lemma 4.3** Let \( \hat{H}_1, \hat{H}_2 \in L^{1,a} \) be two scalar functions, \( a \geq 0 \). Let

\[
\hat{H}_3(k) = \int_{\mathbb{R}^d} \hat{H}_1(k - k') \hat{H}_2(k') \, dk'.
\]

Then

\[
\left\| \hat{H}_3(k) \right\|_{L^{1,a}} \leq \left\| \hat{H}_1(k) \right\|_{L^{1,a}} \left\| \hat{H}_2(k) \right\|_{L^{1,a}}.
\]

**Proof.** We have

\[
(1 + |k|)^a \left| \hat{H}_3(k) \right| \leq \sup_{k',k''} \frac{(1 + |k' + k''|)^a}{(1 + |k'|)^a (1 + |k''|)^a} \int_{\mathbb{R}^d} \left( 1 + |k - k'| \right)^a \left| \hat{H}_1(k - k') \right| (1 + |k'|)^a \left| \hat{H}_2(k') \right| \, dk'.
\]

Obviously,

\[
\frac{1 + |k' + k''|}{(1 + |k'|) (1 + |k''|)} \leq \frac{(1 + |k'| + |k''|)}{(1 + |k'|) (1 + |k''|)} \leq 1.
\]
Applying Young’s inequality (4.8) we obtain
\[
\int_{\mathbb{R}^d} (1 + |k|^a) |\hat{H}_3(k)| \, dk \leq \int_{\mathbb{R}^d} (1 + |k'|)^a |\hat{H}_1(k')| \, dk' \int_{\mathbb{R}^d} (1 + |k''|) |\hat{H}_2(k'')| \, dk''
\]

Using (2.18) we obtain (4.9). □

Using Lemma 4.3 we derive boundedness of integral operators \( \mathcal{F}^{(m)} \).

**Lemma 4.4 (boundness of multilinear operators)** Operator \( \mathcal{F}^{(m)} \) defined by (3.9), (4.7) is bounded from \( E_a = C([0, \tau_s], L^{1,a}) \) into \( C^1([0, \tau_s], L^{1,a}) \), \( a \geq 0 \) and
\[
\left\| \mathcal{F}^{(m)}(\hat{u}_1 \ldots \hat{u}_m) \right\|_{E_a} \leq \tau_s \left\| \chi^{(m)} \right\| \prod_{j=1}^m \left\| \hat{u}_j \right\|_{E_a},
\]
(4.10)
\[
\left\| \partial_\tau \mathcal{F}^{(m)}(\hat{u}_1 \ldots \hat{u}_m) \right\|_{E_a} \leq \left\| \chi^{(m)} \right\| \prod_{j=1}^m \left\| \hat{u}_j \right\|_{E_a}.
\]
(4.11)

**Proof.** Notice that since \( L(k) \) is Hermitian, \( \left\| \exp \left\{ -iL(k) \frac{\tau_s}{\varphi} \right\} \right\| = 1 \). Using the inequality (4.9) together with (3.9), (4.7) we obtain
\[
\left\| \mathcal{F}^{(m)}(\hat{u}_1 \ldots \hat{u}_m) \right\|_{L^{1,a}} \leq \sup_{k,k'} \left\| \chi^{(m)}(k,k') \right\|
\int_{\mathbb{R}^d} \int_0^\tau \int_{E_m} \left( (1 + |k'|)^a \hat{u}_1(k') \right) \ldots \left( (1 + |k'^{m-1}|)^a \hat{u}_m(k^{(m)}(k,k')) \right) \, dk' \ldots dk'^{m-1} \, d\tau_1 \, dk \leq
\]
\[
\left\| \chi^{(m)} \right\| \int_0^\tau \left\| \hat{u}_1(\tau_1) \right\|_{L^{1,a}} \ldots \left\| \hat{u}_m(\tau_1) \right\|_{L^{1,a}} \, d\tau_1 \leq \tau_s \left\| \chi^{(m)} \right\| \left\| \hat{u}_1 \right\|_{E_a} \ldots \left\| \hat{u}_m \right\|_{E_a},
\]
proving (4.10). A similar estimate produces we prove (4.11). □

The equation (4.6) can be recast as the following abstract equation in a Banach space
\[
\hat{u} = \mathcal{F}(\hat{u}) + \hat{h}, \ \hat{u}, \hat{h} \in E_a,
\]
(4.12)
and it readily follows from Lemma 4.4 that \( \mathcal{F}(\hat{u}) \) has the following properties.

**Lemma 4.5** The operator \( \mathcal{F}(\hat{u}) \) defined by (4.6) satisfies the Lipschitz condition
\[
\left\| \mathcal{F}(\hat{u}_1) - \mathcal{F}(\hat{u}_2) \right\|_{E_a} \leq \tau_s C_F \left\| \hat{u}_1 - \hat{u}_2 \right\|_{E_a}
\]
(4.13)
where \( C_F \leq C_\chi \frac{m_F^2}{4R} (4R)^{m_F-1} \) if \( \left\| \hat{u}_1 \right\|_{E_a}, \left\| \hat{u}_2 \right\|_{E_a} \leq 2R \), with \( C_\chi \) as in (3.11).

We also will use the following form of the contraction principle.

**Lemma 4.6 (contraction principle)** Consider equation
\[
x = \mathcal{F}(x) + h, \ x, h \in B,
\]
(4.14)
where $B$ is a Banach space, $\mathcal{F}$ is an operator in $B$. Suppose that for some constants $R_0 > 0$ and $0 < q < 1$ we have
\begin{align}
\|h\| &\leq R_0, \quad \|\mathcal{F}(x)\| \leq R_0 \text{ if } \|x\| \leq 2R_0, \quad (4.15) \\
\|\mathcal{F}(x_1) - \mathcal{F}(x_2)\| &\leq q \|x_1 - x_2\| \text{ if } \|x_1\|, \|x_2\| \leq 2R_0. \quad (4.16)
\end{align}

Then there exists a unique solution $x$ to the equation (4.14) such that $\|x\| \leq 2R_0$. Let $\|h_1\|, \|h_2\| \leq R_0$ then the two corresponding solutions $x_1, x_2$ satisfy
\begin{align}
\|x_1\|, \|x_2\| \leq 2R_0, \quad \|x_1 - x_2\| \leq (1 - q)^{-1} \|h_1 - h_2\|. \quad (4.17)
\end{align}

Let $x_1, x_2$ be the two solutions of correspondingly two equations of the form (4.14) with $\mathcal{F}_1$, $h_1$ and $\mathcal{F}_2$, $h_2$. Assume that that $\mathcal{F}_1(u)$ satisfies (4.13), (4.10) with a Lipschitz constant $q < 1$ and that $\|\mathcal{F}_1(x) - \mathcal{F}_2(x)\| \leq \delta$ for $\|x\| \leq 2R_0$. Then
\begin{align}
\|x_1 - x_2\| \leq (1 - q)^{-1} (\delta + \|h_1 - h_2\|). \quad (4.18)
\end{align}

Lemma 4.3 and the contraction principle as in Lemma 4.6 imply the following existence and uniqueness theorem.

**Theorem 4.7** Let $\|\hat{h}\|_{E_\alpha} \leq R$, let $\tau_* < 1/C_F$ where $C_F$ is a constant from Lemma 4.3. Then equation (4.6) has a solution $\hat{u} \in E_\alpha = C([0, \tau_*], L^{1,\alpha})$ which satisfies $\|\hat{u}\|_{E_\alpha} \leq 2R$, and such a solution is unique. Hence the solution operator $\hat{u} = \mathcal{G}(\hat{h})$ is defined on the ball $\|\hat{h}\|_{E_\alpha} \leq R$.

The following existence and uniqueness theorem follows from Theorem 4.7.

**Theorem 4.8** Let $a \geq 0$, (2.7) satisfy (3.17) and $\hat{h} \in L^{1,\alpha}(\mathbb{R}^d) \quad \|\hat{h}\|_{L^{1,\alpha}} \leq R$. Then there exists a unique solution $\hat{u}$ to the modal evolution equation (2.7) in the functional space $C^1([0, \tau_*], L^{1,\alpha})$, $\|\hat{u}\|_{E_\alpha} + \|\partial \tau \hat{u}\|_{E_\alpha} \leq R_1(R)$. The number $\tau_*$ depends on $R$ and $C_\chi$.

Using the inequality (2.20) and applying the inverse Fourier transform we readily obtain the existence of an $F$–solution of (1.1) in $C^1([0, \tau_*], L^\infty(\mathbb{R}^d))$ from the existence of the solution of equation (2.1) in $C^1([0, \tau_*], L^1)$. The existence of $F$-solutions with $[a]$ bounded spatial derivatives ([a] being an integer part of $a$) follows from solvability in $C^1([0, \tau_*], L^{1,\alpha})$.

Let us recast now the system (4.6)-(4.7) into modal components using the projections $\Pi_{n,\xi}(k)$ as in (2.9). The first step to introduce modal susceptibilities $\chi_{n,\xi}^{(m)}$ having one-dimensional range in $\mathbb{C}^{2J}$ and vanishing if one of its arguments $\hat{u}_j$ belongs to a $(2J - 1)$-dimensional linear subspace in $\mathbb{C}^{2J}$ ($j$-th null-space of $\chi_{n,\xi}^{(m)}$) as follows.
Definition 4.9 (elementary susceptibilities) Let
\[ \tilde{\zeta} = (\tilde{n}, \tilde{\zeta}) \in \{1, \ldots, J\}^m \times \{-1, 1\}^m = \Xi^m, \ (n, \zeta) \in \Xi \] (4.19)
and \( \chi^{(m)}(k, \tilde{k}) [\hat{u}_1(k'), \ldots, \hat{u}_m(k^{(m)})] \) be \( m \)-linear symmetric tensor (susceptibility) as in (3.6). We introduce elementary susceptibilities \( \chi^{(m)}_{n,\zeta,\xi}(k, \tilde{k}) : (\mathbb{C}^2) \mathbb{C}^2 \mathbb{J}^m \rightarrow \mathbb{C}^2 \mathbb{J}^m \) as \( m \)-linear tensors defined for almost all \( k \) and \( \tilde{k} = (k', \ldots, k^{(m)}) \) by the following formula
\[ \chi^{(m)}_{n,\zeta,\xi}(k, \tilde{k}) [\hat{u}_1(k'), \ldots, \hat{u}_m(k^{(m)})] = \chi^{(m)}_{n,\zeta,\xi}(k, \tilde{k}) [\hat{u}_1(k'), \ldots, \hat{u}_m(k^{(m)})] = \Pi_{n,\zeta}(k) \chi^{(m)}(k, \tilde{k}) \left[ (\Pi_{n,\zeta}(k') \hat{u}_1(k'), \ldots, \Pi_{n,\zeta}(k^{(m)}) \hat{u}_m(k^{(m)})) \right]. \] (4.20)

Then using (3.5) and the elementary susceptibilities (4.20) we get
\[ \chi^{(m)}(k, \tilde{k}) [\hat{u}_1(k'), \ldots, \hat{u}_m(k^{(m)})] = \sum_{n,\zeta} \sum_{\xi} \chi^{(m)}_{n,\zeta,\xi}(k, \tilde{k}) [\hat{u}_1(k'), \ldots, \hat{u}_m(k^{(m)})]. \] (4.21)

Consequently the modal components \( \mathcal{F}^{(m)}_{n,\zeta,\xi}(\hat{u}_1 \ldots \hat{u}_m)(k, \tau) \) are \( m \)-linear oscillatory integral operators defined in terms of the elementary susceptibilities (4.21) as follows.

Definition 4.10 (interaction phase) Using notations from (3.3) we introduce for \( \tilde{\zeta} = (\tilde{n}, \tilde{\zeta}) \in \Xi^m \) operator
\[ \mathcal{F}^{(m)}_{n,\zeta,\xi}(\hat{u}_1 \ldots \hat{u}_m)(k, \tau) = \int_0^\tau \int_{\mathbb{C}^m} \exp \left\{ i \phi_{n,\zeta,\xi}(k, \tilde{k}) \frac{\tau_1}{\varrho} \right\} \] (4.22)
\[ \chi^{(m)}_{n,\zeta,\xi}(k, \tilde{k}) [\hat{u}_1(k', \tau_1), \ldots, \hat{u}_m(k^{(m)}) (k, \tilde{k}), \tau_1) \right] d^{(m-1)}k d\tau_1, \]
with the interaction phase function \( \phi \) defined by
\[ \phi_{n,\zeta,\xi}(k, \tilde{k}) = \phi_{n,\zeta,\xi}(k, \tilde{k}) = \zeta_\omega (\zeta k) - \zeta_\omega (\zeta k') - \ldots - \zeta^{(m)}(\zeta k^{(m)}), \ (m) = k^{(m)}(k, \tilde{k}) \] (4.23)
where \( k^{(m)}(k, \tilde{k}) \) is defined by (3.10).

Using \( \mathcal{F}^{(m)}_{n,\zeta,\xi} \) in (4.22) we recast \( \mathcal{F}^{(m)}(\hat{u}^m) \) in the system (4.6)-(4.7) as
\[ \mathcal{F}^{(m)}(\hat{u}_1 \ldots \hat{u}_m)(k, \tau) = \sum_{n,\zeta,\xi} \mathcal{F}^{(m)}_{n,\zeta,\xi}(\hat{u}_1 \ldots \hat{u}_m)(k, \tau), \] (4.24)
yielding the following system for the modal components \( \hat{u}_{n,\zeta}(k, \tau) \) as in (2.9)
\[ \hat{u}_{n,\zeta}(k, \tau) = \sum_{m \in \mathbb{M}_F} \sum_{\xi \in \Xi^m} \mathcal{F}^{(m)}_{n,\zeta,\xi}(\hat{u}^m)(k, \tau) + \hat{h}_{n,\zeta}(k), \ (n, \zeta) \in \Xi. \] (4.25)
5 Wavepacket interaction system

The wavepacket preservation property of the nonlinear evolutionary system in any of its forms (1.1), (2.1), (4.6), (4.12), (4.25) is not easy to see directly. It turns out though that dynamics of wavepackets is well described by a system in a larger space $E^{2N}$ based on the original equation (4.6) in the space $E$. We call it wavepacket interaction system, which is useful in three ways: (i) the wavepacket preservation is quite easy to see and verify; (ii) it can be used to prove the wavepacket preservation for the original nonlinear problem; (iii) it can be used to study more subtle properties of the original problem, such as NLS approximation.

We start with the system (4.6) where $\hat{h}(k)$ is a multiwavepacket with a given $nk$-spectrum $S = \{(k_{si}, n_{i}), l = 1, \ldots, N\}$ as in (2.39) and $k$-spectrum $K_{S} = \{k_{si}, i = 1, \ldots, |K_{S}|\}$ as in (2.40). Obviously, for any $l$ $(k_{si}, n_{i})$ with $i_{l} \leq |K_{S}|$ and indexing $i_{l} = l$ for $l \leq |K_{S}|$ according to (2.40).

When constructing the wavepacket interaction system it is convenient to have relevant functions to be explicitly localized about the $k$-spectrum $K_{S}$ of the initial data. We implement that by making up the following cutoff functions based on (2.24), (2.25)

$$\Psi_{i,\theta}(k) = \Psi(k, \partial k_{si}, \beta^{1-\epsilon}) = \Psi(\beta^{-(1-\epsilon)}(k - \partial k_{si})), \ k_{si} \in K_{S}, i = 1, \ldots, |K_{S}|$$

with $\epsilon$ as in Definition 2.1 and $\beta > 0$ small enough to satisfy

$$\beta^{1/2} \leq \pi_{0}, \ \text{where} \ \pi_{0} = \pi_{0}(S) < \frac{1}{2} \min_{k_{si} \in K_{S}} \text{dist}(k_{si}, \sigma_{bc}).$$

In what follows we use notations from (3.16) and

$$\bar{l} = (l_{1}, \ldots, l_{m}) \in \{1, \ldots, N\}^{m}, \ \bar{\theta} = (\theta', \ldots, \theta^{(m)}) \in \{-1, 1\}^{m}, \ \bar{\lambda} = (\bar{l}, \bar{\theta}) \in \Lambda^{m};$$

$$\bar{n} = (n_{1}, \ldots, n_{m}) \in \{1, \ldots, J\}^{m}, \ \bar{\zeta} = \bar{n} \cdot \bar{\lambda} \in \Xi^{m}, \ k_{i} = (k', k^{(m)}) \in \mathbb{R}^{m}, \ \widehat{w}_{i, \theta} \in E, \ F$$

Based on the above we introduce now the wavepacket interaction system

$$\tilde{\Phi}_{i,\theta}(\cdot) = \Psi(\cdot, \partial k_{si}) \Pi_{i,\theta}(\cdot) F\left(\sum_{i' \in \lambda} \tilde{w}_{i', \theta'} \right) + \Psi(\cdot, \partial k_{si}) \Pi_{i,\theta}(\cdot) \tilde{h}(l, \theta) \in \Lambda, \ F$$

$$\tilde{w} = (\tilde{w}_{1,+}, \tilde{w}_{1,-}, \ldots, \tilde{w}_{N,+}, \tilde{w}_{N,-}) \in E^{2N}, \ \tilde{w}_{i,\theta} \in E, \ \text{with} \ \Psi(\cdot, \partial k_{si}), \Pi_{n,\theta} \text{being as in (5.1), (2.9), F defined by (4.6), and the norm in E^{2N} defined based on (2.15) by the formula}$$

$$\|\tilde{w}\|_{E^{2N}} = \sum_{i,\theta} \|\tilde{w}_{i,\theta}\|_{E}, \ E = C([0, \tau_{*}], L^{1}).$$
We also use the following concise form of the wave interaction system (5.5)

$$\dot{\tilde{w}} = \mathcal{F}_\Psi (\tilde{w}) + \tilde{h}_\Psi, \text{ where}$$

$$\tilde{h}_\Psi = \left( \psi_{i_1,+} \Pi_{n_1,+} \dot{h}, \psi_{i_1,-} \Pi_{n_1,-} \dot{h}, \ldots, \psi_{i_N,+} \Pi_{n_N,+} \dot{h}, \psi_{i_N,-} \Pi_{n_N,-} \dot{h} \right) \in E^{2N}.$$  

The following lemma is analogous to Lemmas 4.4, 4.5.

**Lemma 5.1** Polynomial operator $\mathcal{F}_\Psi (\tilde{w})$ is bounded in $E^{2N}$, $\mathcal{F}_\Psi (0) = 0$, and it satisfies Lipschitz condition

$$\| \mathcal{F}_\Psi (\tilde{w}_1) - \mathcal{F}_\Psi (\tilde{w}_2) \|_{E^{2N}} \leq C \tau_* \| \tilde{w}_1 - \tilde{w}_2 \|_{E^{2N}},$$

where $C$ depends only on $C_\chi$ as in (3.11), on the degree of $\mathcal{F}$ and on $\| \tilde{w}_1 \|_{E^{2N}} + \| \tilde{w}_2 \|_{E^{2N}}$, and it does not depend on $\beta$ and $\varphi$.

**Proof.** We consider every operator $\mathcal{F}^{(n)}_{\alpha,\zeta,\xi}(\tilde{w})$ defined by (4.22) and prove its boundedness and the Lipschitz property as in Lemma 4.1 using the inequality $\| \exp \left\{ \i \phi_{n,\zeta,\xi} \right\} \| \leq 1$ and inequalities (2.24), (3.11). Note that the integration in $\tau_1$ yields the factor $\tau_*$ and consequent summation with respect to $n, \zeta, \xi$ yields (5.8). 

Lemma 5.1 the contraction principle as in Lemma 4.6 and estimate (4.11) for the time derivative yield the following statement.

**Theorem 5.2** Let $\| \tilde{h}_\Psi \|_{E^{2N}} \leq R$. Then there exists $\tau_* > 0$ and $R_1 (R)$ such that equation (5.2) has a solution $\tilde{w} \in E^{2N}$ which satisfies

$$\| \tilde{w} \|_{E^{2N}} + \| \partial_\tau \tilde{w} \|_{E^{2N}} \leq R_1 (R)$$

and such a solution is unique.

**Lemma 5.3** Every function $\tilde{w}_{l,\zeta}(k, \tau)$ corresponding to the solution of (5.7) from $E^{2N}$ is a wavepacket with $nk$-pair $(k_{si}, n_i)$ with the degree of regularity which can be any $s > 0$.

**Proof.** Note that according to (5.1) and (5.7) the function

$$\tilde{w}_{l,\vartheta}(k, \tau) = \Psi (k, \partial k_{si}, \beta^{1-\varrho}) \Pi_{n_1,\vartheta} \mathcal{F}(k, \tau), \| \mathcal{F}(\tau) \|_{L^1} \leq C, 0 \leq \tau \leq \tau_*$$

involves the factor $\Psi_{l,\vartheta}(k) = \Psi (\beta^{-(1-\epsilon)} (k - \partial k_{si}))$ where $\epsilon$ is as in Definition 2.1. Hence,

$$\Pi_{n,\vartheta} \tilde{w}_{l,\vartheta}(k, \tau) = 0 \text{ if } n \neq n_i \text{ or } \vartheta' \neq \vartheta,$$

$$\tilde{w}_{l,\vartheta}(k, \tau) = \Psi (k, \partial k_{si}, \beta^{1-\epsilon}) \tilde{w}_{l,\vartheta}(k, \tau), \tilde{w}_{l,\vartheta}(k, \tau) = 0 \text{ if } |k - \partial k_{si}| \geq \beta^{1-\epsilon},$$

Since

$$\Psi (k, \partial k_{si}, \beta^{1-\epsilon}) \Psi (k, \partial k_{si}, \beta^{1-\epsilon}/2) = \Psi (k, \partial k_{si}, \beta^{1-\epsilon}),$$

(5.12)
Definition 2.1 for $\hat{w}_{l,\vartheta}$ is satisfied with $D_h = 0$ for any $s > 0$ and $C' = 0$ in (2.30). 

Now we would like to show that if $\hat{h}$ is a multiwavepacket, then the function

$$\hat{w}(k, \tau) = \sum_{(l, \vartheta) \in \Lambda} \hat{w}_{l, \vartheta}(k, \tau) = \sum_{\lambda \in \Lambda} \hat{w}_\lambda(k, \tau)$$  \tag{5.13}$$

constructed based on a solution of (5.7) is an approximate solution of equation (4.12) (see notation (3.16)). We will follow the lines of [8]. We introduce

$$\Psi_\infty(k) = 1 - \sum_{\vartheta = \pm} \sum_{i=1}^{\left|K_S\right|} \Psi(k, \vartheta k_{*i}) = 1 - \sum_{\vartheta = \pm} \sum_{k_{*i} \in K_S} \Psi \left(\frac{k - \vartheta k_{*i}}{\beta^{1-\epsilon}}\right).$$  \tag{5.14}$$

Expanding $m$-linear operator $F^{(m)} \left( \left( \sum_{l, \vartheta} \hat{w}_{l, \vartheta} \right)^m \right)$ and using notations (3.16), (3.17) we get

$$F^{(m)} \left( \left( \sum_{l, \vartheta} \hat{w}_{l, \vartheta} \right)^m \right) = \sum_{\bar{\lambda} \in \Lambda^m} F^{(m)} (\hat{w}_{\bar{\lambda}}), \text{ where}$$

$$\hat{w}_{\bar{\lambda}} = \hat{w}_{\lambda_1} \ldots \hat{w}_{\lambda_m}, \quad \bar{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \Lambda^m. \tag{5.15}$$

The next statement shows that (5.13) defines an approximate solution to integrated evolution equation (4.6).

**Theorem 5.4** Let $\hat{h}$ be a multi-wavepacket with resonance invariant $nk$-spectrum $S$ with regularity degree $s$, $\hat{w}$ be a solution of (5.7) and $\hat{w}(k, \tau)$ be defined by (5.13). Let

$$D(\hat{w}) = \hat{w} - F(\hat{w}) - \hat{h}. \tag{5.17}$$

Then there exists $\beta_0 > 0$ such that we have the estimate

$$\|D(\hat{w})\|_E \leq C_\varrho + C\beta^s, \text{ if } 0 < \varrho \leq 1, \beta \leq \beta_0. \tag{5.18}$$

**Proof.** Let

$$F^- (\hat{w}) = \left(1 - \sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta}\right) F(\hat{w}), \quad \hat{h}^- = \hat{h} - \sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta} \hat{h}. \tag{5.19}$$

Summation of (5.3) with respect to $l, \vartheta$ yields

$$\hat{w} = \sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta} F(\hat{w}) + \sum_{l, \vartheta} \Psi_{i_l, \vartheta} \Pi_{n_l, \vartheta} \hat{h}. \tag{5.3}$$

Hence, from (5.3) and (5.17) we obtain

$$D(\hat{w}) = \hat{h}^- - F^-(\hat{w}). \tag{5.20}$$

Using (2.28) and (2.30) we consequently obtain

$$\left\|\Pi_{n_l, \vartheta} \hat{h}\right\|_{L^1} \leq C\beta^s \text{ if } n_l \neq n_i; \quad \left\|\Psi_{i_l, \vartheta} \hat{h}\right\|_{L^1} \leq C\beta^s \text{ if } k_{*i} \neq k_{*i},$$
Now, to show (5.18) it is sufficient to prove that
\[ \left\| \mathcal{F}^{-} (\hat{w}) \right\|_E \leq C_2 \vartheta. \] 
(5.22)

Obviously,
\[ \mathcal{F}^{-} (\hat{w}) = \left( 1 - \sum_{l,\vartheta} \Psi_{i_l,\vartheta} \Pi_{n_l,\vartheta} \right) \sum_m \mathcal{F}^{(m)} (\hat{w}^m). \] 
(5.23)

Note that
\[ \sum_{l,\vartheta} \Psi_{i_l,\vartheta} \Pi_{n_l,\vartheta} = \sum_{\vartheta = \pm} \sum_{(n,k_*)} \Psi (\cdot, \vartheta k_*) \Pi_{n,\vartheta}. \] 
(5.24)

Using (3.5) and (5.14) we consequently obtain
\[ \sum_{\vartheta = \pm} \sum_{(n,k_*)} \Psi (\cdot, \vartheta k_*) \Pi_{n,\vartheta} = 1, \] 
(5.25)

with \( \Sigma \) defined in (3.14). Let us expand now \( \mathcal{F}^{(m)} (\hat{w}^m) \) using (5.15). According to (5.23) and (5.26) to prove (5.22) it is sufficient to prove that for every string \( \vec{\lambda} \in \Lambda^m \) the following inequalities hold
\[ \left\| \Psi_{\infty} \Pi_{n',\vartheta} \mathcal{F}^{(m)} (\hat{w}_{\vec{\lambda}}) \right\| \leq C_3 \vartheta \] 
for \( (n, \vartheta) \in \Lambda \), and
\[ \left\| \Psi (\cdot, \vartheta k_*) \Pi_{n,\vartheta} \mathcal{F}^{(m)} (\hat{w}_{\vec{\lambda}}) \right\| \leq C_3 \vartheta, \] 
if \( (n, k_*) \in \Sigma \setminus S. \) (5.26)

We will use (5.10) and (5.11) to obtain the above estimates. According to (4.24)
\[ \mathcal{F}^{(m)} [\hat{w}_{\vec{\lambda}}] (k, \tau) = \sum_{n,\zeta} \sum_{\bar{\xi}} \mathcal{F}^{(m)} [\hat{w}_{\lambda_1} \ldots \hat{w}_{\lambda_m}] (k, \tau). \] 
(5.29)

Note that according to (5.10) if \( \lambda_i = (l, \vartheta') \)
\[ \hat{w}_{\lambda_i} = \Pi_{n,\vartheta} \hat{w}_{\lambda_i}, \text{ if } n = n_l \text{ and } \vartheta' = \vartheta. \] 
(5.30)

Let us introduce notation
\[ \vec{n} \left( \vec{l} \right) = (n_1, \ldots, n_m), \quad \vec{\xi} \left( \vec{\lambda} \right) = \left( \vec{n} \left( \vec{l} \right), \vec{\vartheta} \right), \text{ for } \vec{\lambda} = \left( \vec{l}, \vec{\vartheta} \right) \in \Lambda^m. \] 
(5.31)

Since
\[ \Pi_{n',\vartheta} \Pi_{n,\vartheta'} = 0, \text{ if } n \neq n' \text{ or } \vartheta' \neq \vartheta \] 
(5.32)

then (5.30) implies
\[ \mathcal{F}^{(m)} [\hat{w}_{\lambda_1} \ldots \hat{w}_{\lambda_m}] = 0 \text{ if } \vec{\xi} = \left( \vec{n}, \vec{\zeta} \right) \neq \vec{\xi} \left( \vec{\lambda} \right), \text{ and, hence,} \]
\[ \mathcal{F}^{(m)} [\hat{w}_{\vec{\lambda}}] (k, \tau) = \sum_{n,\zeta} \mathcal{F}^{(m)} [\hat{w}_{\lambda_1} \ldots \hat{w}_{\lambda_m}] (k, \tau), \] 
(5.33)
where we use notation (3.17), (5.31). Note also that

\[ \Pi_{n', \partial} F_{n, \xi}^{(m)} = 0 \text{ if } n' \neq n \text{ or } \partial \neq \zeta, \]  

and, hence, we have nonzero \( \Pi_{n', \partial} F_{n, \xi}^{(m)}(\tilde{w}) \) only if

\[ n' = n, \partial = \zeta. \]  

By (4.22)

\[ F_{n, \xi}^{(m)}(\tilde{w}) = \int_0^\infty \int_{D_m} \exp \left\{ i \phi_{n, \xi}^{(m)}(\tilde{x}) (k, \tilde{k}) \frac{\tau_1}{\hat{\varrho}} \right\} \hat{\zeta}(k, \tilde{k}) \left[ \hat{w}_{\lambda_1}(k', \tau_1), \ldots, \hat{w}_{\lambda_m}(m(k, \tilde{k}), \tau_1) \right] \overline{d^{(m-1)} \tilde{k} d \tau_1}, \]  

Now we use (5.11) and notice that according to the convolution identity in (3.9)

\[ |\hat{w}_{\lambda_1}(k', \tau_1)| \cdot \ldots \cdot |\hat{w}_{\lambda_m}(m(k, \tilde{k}), \tau_1)| = 0 \text{ if } |k - \sum_i \varrho_i k_{sl_i}| \geq m^{1-\epsilon}. \]  

Hence the integral (5.36) is nonzero only if \( (k, \tilde{k}) \) belongs to the set

\[ B_\beta = \left\{ (k, \tilde{k}) : |k^{(i)} - \varrho_i k_{sl_i}| \leq \beta^{1-\epsilon}, \ i = 1, \ldots, m, \ |k - \sum_i \varrho_i k_{sl_i}| \leq m^{1-\epsilon} \right\}. \]  

We will prove now that if \( (n, k_{ei}) \notin S \) then for small \( \beta \) one of the following alternatives holds:

either \( \Psi (\cdot, \vartheta k_{ei}) \Pi_{n', \partial} F_{n, \xi}^{(m)}(\tilde{w}) = 0 \) (5.39)

or (5.35) holds and \( \phi_{n, \xi}^{(m)}(k, \tilde{k}) \geq c > 0 \text{ for } (k, \tilde{k}) \in B_\beta. \) (5.40)

Note then since \( \phi_{n, \xi}^{(m)}(k, \tilde{k}) \) is smooth then using notation (3.18) we get

\[ \left| \phi_{n, \xi}^{(m)}(k, \tilde{k}) - \phi_{n', \xi}^{(m)}(k_{ee}, \tilde{k}_{ee}) \right| \leq C \beta^{1-\epsilon} \text{ for } (k, \tilde{k}) \in B_\beta, \]  

\[ \tilde{\vartheta} = (\vartheta_1, \ldots, \vartheta_m), \ k_{ee} = \zeta \sum_i \varrho_i k_{sl_i} = \zeta \varrho \left( \vartheta, \tilde{l} \right). \]  

Hence the alternative (5.40) holds if

\[ \phi_{n, \xi}^{(m)}(k_{ee}, \tilde{k}_{ee}) \neq 0, \]  

and, consequently, it suffices to prove that either (5.39) or (5.42) holds. Combining (5.38) with \( \Psi (k, \vartheta k_{ei}) = 0 \) for \( |k - \vartheta k_{ei}| \geq \beta^{1-\epsilon} \) we find that \( \Psi_{i, \vartheta} F_{\tilde{w}}^{(m)}(\tilde{w}) \) can be non-zero for
small $\beta$ only in a small neighborhood of a point $\zeta \kappa_m \left( \vec{\vartheta}, \vec{l} \right) \in [S]_{K_{out}}$, and that is possible only if
\[ k_{**} = \zeta \kappa_m \left( \vec{\vartheta}, \vec{l} \right) = \vartheta k_{*i}, \ k_{*i} \in K_S. \] (5.43)

Let us show that the equality
\[ \phi_{n,\zeta,\vec{\xi}} \left( k_{**}, \vec{k} \right) = 0 \] (5.44)
is impossible for $k_{**}$ as in (5.43) and $n' = n$ as in (5.34), keeping in mind that $(n, k_{*i}) \notin S$. It follows from (3.23) and (4.23) that the equation (5.44) has the form of the resonance equation (3.24). Since nk-spectrum $S$ is resonance invariant, in view of Definition 3.8 the resonance equation (5.44) may have a solution only if $k_{**} = k_{*i}$, $i = i_l$, $n = n_l$, with $(n_l, k_{*i}) \in S$. Since $(n, k_{*i}) \notin S$ that implies (5.44) does not have a solution and, hence, (5.42) holds when $(n, k_{*i}) \notin S$. Notice that (5.9) yields the following bounds
\[ \| \hat{w}_\lambda \|_E \leq R_1, \ |\partial_\tau \hat{w}_\lambda|_E \leq C. \] (5.45)

These bounds combined with Lemma 5.5, proven below, imply that if (5.42) holds then (5.42) holds. Now let us turn to (5.27). According to (5.14) and (5.37) the term $\Psi_{n',\vartheta} \mathcal{F}^{(m)} \left( \hat{w}_\lambda \right)$ can be non-zero only if $\zeta \kappa_m \left( \vec{\lambda} \right) = k_{**} \notin K_S$. Since nk-spectrum $S$ is resonance invariant we conclude as above that inequality (5.42) holds in this case as well. The fact that the set of all $\zeta \kappa_m \left( \vec{\lambda} \right)$ is finite, combined with inequality (5.42), imply (5.40) for sufficiently small $\beta$.

Using Lemma 5.5 as above we derive (5.27). Hence, all terms in the expansion (5.23) are either zero or satisfy (5.27) or (5.28) implying consequently (5.22) and (5.18).

Here is the lemma used in the above proof.

**Lemma 5.5** Let assume that
\[ \left| \Psi_{i,\vartheta} \Pi_{n',\vartheta} \phi^{(m)}_{n,\zeta,\vec{\xi}} \left( k, \vec{k} \right) \left[ \hat{w}_{\lambda_1} \left( k, \tau_1 \right), \ldots, \hat{w}_{\lambda_m} \left( k^{(m)} \left( k, \vec{k} \right), \tau_1 \right) \right] \right| = 0 \text{ for } (k, \vec{k}) \in B_\beta, \]
and
\[ \left| \phi_{n,\zeta,\vec{\xi}} \left( k, \vec{k} \right) \right| \geq \omega_* > 0 \text{ for } (k, \vec{k}) \notin B_\beta, \] where $B_\beta$ as in (5.38). (5.46)

Then
\[ \left\| \Psi \left( \cdot, \vartheta \right) k_{*i} \Pi_{n',\vartheta} \mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}} \left( \hat{w}_\lambda \right) \right\|_E \leq \frac{4 \varrho}{\omega_*} \left\| \phi \left( \cdot, \vartheta \right) \right\|_E \left\| \prod_j \left\| \hat{w}_\lambda_j \right\|_E + \frac{2 \varrho \tau_*}{\omega_*} \left\| \phi \left( \cdot, \vartheta \right) \right\|_E \sum_i \left\| \partial_\tau \hat{w}_\lambda_i \right\|_E \prod_{j \neq i} \left\| \hat{w}_\lambda_j \right\|_E. \] (5.47)

**Proof.** Notice that the oscillatory factor in (4.22) equals to
\[ \exp \left\{ i \phi \left( k, \vec{k} \right) \frac{\tau_1}{\vartheta} \right\} = \frac{\vartheta}{i \phi \left( k, \vec{k} \right)} \partial_\tau \exp \left\{ i \phi \left( k, \vec{k} \right) \frac{\tau_1}{\vartheta} \right\}. \]
Denoting \( \phi_{n,\xi} = \phi \), \( \psi_{i,0} \Pi_{n,\xi}^{(m)} \chi_{n,\xi}^{(m)} = \chi^{(m)} \) and integrating (4.22) by parts with respect to \( \tau_1 \) we obtain

\[
\Psi (k, \vartheta') \Pi_{n,\xi}^{(m)} \mathcal{F}_{(m)}^{(n,\xi)} \left( \hat{w}_\chi \right) (k, \tau) =
\]

\[
\int_B \Psi (k, \vartheta') \frac{\partial}{\partial \vartheta} \chi^{(m)}_{\hat{w}} (k, \tilde{k}) \hat{w}_{\lambda_1} (k', \tau) \cdots \hat{w}_{\lambda_m} (k^{(m)} (k, \tilde{k}), \tau) \tilde{d}^{(m-1)} \sigma
\]

\[
- \int_B \Psi (k, \vartheta') \frac{\partial}{\partial \vartheta} \chi^{(m)}_{\hat{w}} (k, \tilde{k}) \hat{w}_{\lambda_1} (k', 0) \cdots \hat{w}_{\lambda_m} (k^{(m)} (k, \tilde{k}), 0) \tilde{d}^{(m-1)} \sigma
\]

\[
- \int_0^\tau \int_B \Psi (k, \vartheta') \frac{\partial}{\partial \vartheta} \chi^{(m)}_{\hat{w}} (k, \tilde{k}) \partial_{\tau_1} \left[ \hat{w}_{\lambda_1} (k') \cdots \hat{w}_{\lambda_m} (k^{(m)} (k, \tilde{k})) \right] \tilde{d}^{(m-1)} \sigma \tilde{d} \tau_1,
\]

where \( B \) is the set of \( k^{(i)} \) for which (5.38) holds. The relations (3.11) and (2.24) imply

\[
\left| \chi^{(m)}_{\hat{w}} (k, \tilde{k}) \right| \leq \| \chi^{(m)} \|. \]

Using then (5.40), the Leibnitz formula, (5.9) and (4.8) we obtain (5.47). –

The main result of this subsection is the next theorem which, when combined with Lemma 5.3 implies the wavepacket preservation, namely that the solution \( \hat{u}_{n,\vartheta} (k, \tau) \) of (4.25) is a multi-wavepacket for all \( \tau \in [0, \tau_a] \).

**Theorem 5.6** Assume that conditions of Theorem 5.4 are fulfilled. Let \( \hat{u}_{n,\vartheta} (k, \tau) \) for \( n = n_l \) and \( \hat{w}_{l,\vartheta} (k, \tau) \) be the solutions to respective systems (4.25) and (5.3), \( \hat{w} \) be defined by (5.13). Then for sufficiently small \( \beta_0 > 0 \) we have

\[
\| \hat{u}_{n,\vartheta} - \Pi_{n,\vartheta} \hat{w} \|_E \leq C_\vartheta + C' \beta^s, \quad 0 < \beta \leq \beta_0, \quad l = 1, \ldots, N.
\]

**Proof.** Note that \( \hat{u}_{n,\vartheta} = \Pi_{n,\vartheta} \hat{u} \) where \( \hat{u} \) is a solution of (4.6) and, according to Theorem 4.7 \( \| \hat{u} \|_E \leq 2R \). Comparing the equations (4.6) and (5.17), which are \( \hat{u} = \mathcal{F} (\hat{u}) + \hat{h} \) and \( \hat{w} = \mathcal{F} (\hat{w}) + \hat{h} + \hat{D} (\hat{w}) \), we find that Lemma 4.6 can be applied. Then we notice that by Lemma 4.5 \( \mathcal{F} \) has the Lipschitz constant \( C_{\mathcal{F} \tau_s} \) for such \( \hat{u} \). Taking \( C_{\mathcal{F} \tau_s} < 1 \) as in Theorem 4.7 we obtain (5.46) from (4.17). –

Notice that Theorem 2.9 is a direct corollary of Theorem 5.6 and Lemma 5.3.

Analogous statement is proven in [8] for parameter-dependent equations (2.1) with

\[
\hat{F} (\hat{U}) = \hat{F} (\hat{U}, \vartheta).
\]

The following theorem shows that any multi-wavepacket solution to (4.6) yields a solution to the wavepacket interaction system (5.3).

**Theorem 5.7** Let \( \hat{u} (k, \tau) \) be a solution of (4.6) and assume that \( \hat{u} (k, \tau) \) and \( \hat{h} (k) \) are multiwavepackets with \( nk \)-spectrum \( S = \{(n_l, k_{ni}), l = 1, \ldots, N\} \) and the regularity \( s \).
Let also \( \Psi_{i,\vartheta} = \Psi_{i,\vartheta} \) be defined by (5.1). Then the functions \( \hat{w}_{l,\vartheta}(k, \tau) = \Psi_{i,\vartheta} \Pi_{n,\vartheta} \hat{u}(k, \tau) \) are a solution to the system (5.5) with \( \hat{h}(k) \) replaced by \( \hat{h}'(k, \tau) \) satisfying

\[
\left\| \hat{h}(k) - \hat{h}'(k, \tau) \right\|_{L^1} \leq C \beta^s, \quad 0 \leq \tau \leq \tau^*.
\]  

(5.50)

and if \( \hat{w}_{l,\vartheta} \) are solutions of (5.5) with original \( \hat{h}(k) \) we have the inequality

\[
\left\| \hat{w}'_{l,\vartheta}(k, \tau) - \hat{w}_{l,\vartheta} \right\|_{L^1} \leq C \beta^s, \quad 0 \leq \tau \leq \tau^*.
\]  

(5.51)

Proof. Multiplying (4.6) by \( \Psi_{i,\vartheta} \Pi_{n,\vartheta} \) we get

\[
\hat{w}'_{l,\vartheta} = \Psi(\cdot, \vartheta k_{s_i}) \Pi_{n,\vartheta} \mathcal{F}(\hat{u})(k, \tau) + \Psi(\cdot, \vartheta k_{s_i}) \Pi_{n,\vartheta} \hat{h}(k), \quad \hat{w}_{l,\vartheta} = \Psi(\cdot, \vartheta k_{s_i}) \Pi_{n,\vartheta} \hat{u}.  
\]  

(5.52)

Since \( \hat{u}(k, \tau) \) is a multiwavepacket with regularity \( s \) we have

\[
\left\| \hat{u}(\cdot, \tau) - \hat{w}'(\cdot, \tau) \right\|_{L^1} \leq C \beta^s \quad \text{where} \quad \hat{w}'(\cdot, \tau) = \sum_{l,\vartheta} \Psi(\cdot, \vartheta k_{s_i}) \hat{u}(\cdot, \tau).  
\]  

(5.53)

Let us recast (5.52) in the form

\[
\hat{w}'_{l,\vartheta} = \Psi(\cdot, \vartheta k_{s_i}) \Pi_{n,\vartheta} \mathcal{F}(\hat{w}') \mathcal{F}(\hat{u})(k, \tau) + \Psi(\cdot, \vartheta k_{s_i}) \Pi_{n,\vartheta} \left[ \hat{h}(k) + \hat{h}''(k, \tau) \right],
\]  

(5.54)

\[
\hat{h}''(k, \tau) = [\mathcal{F}(\hat{u}) - \mathcal{F}(\hat{w}')](k, \tau).
\]

Denoting \( \hat{h}(k) + \hat{h}''(k, \tau) = \hat{h}'(k, \tau) \) we observe that (5.54) has the form of (5.5) with \( \hat{h}(k) \) replaced by \( \hat{h}'(k, \tau) \). Inequality (5.50) follows then from (5.53) and (4.13). Using Lemma 4.6 we obtain (5.51).

6 Reduction of wavepacket interaction system to an averaged interaction system

Our goal in this section is to substitute the wavepacket interaction system (5.5) with a simpler averaged interaction system which describes the evolution of wavepackets with the same accuracy but has a simpler nonlinearity, and we follow here the approach developed in [8]. The reduction is a generalization of the classical averaging principle to the case of continuous spectrum, see [8] for a discussion and further simplification of the averaged interaction system. In the present paper we do not need the further simplification to a minimal interaction system leading to a system of NLS-type equations which is done in [8].

6.1 Time averaged wavepacket interaction system

Here we modify the wavepacket interaction system (5.5), substituting its nonlinearity with another one obtained by the time averaging, and prove that this substitution produces a
small error of order \( \rho \). As the first step we recast (5.3) in a slightly different form by using expansions (5.15), (5.29) together with (5.33) and (5.34) and writing the nonlinearity in the equation (5.5) in the form

\[
\Psi (\cdot, \vartheta k_{si}) \Pi_{ni, \vartheta} F (\cdot, \tau) = \sum_{m \in \mathbb{M}_F} \sum_{\lambda \in \Lambda^m} \Psi (\cdot, \vartheta k_{si}) F^{(m)}_{ni, \vartheta, \hat{\xi}} (\bar{\omega}, \lambda) \left( \bar{\omega}_{\lambda}, \lambda \right) \left( \bar{w}_\lambda, \lambda \right)
\]  

(6.1)

with \( F^{(m)}_{ni, \vartheta, \hat{\xi}} \) as in (4.22) and \( \hat{n} (\hat{\lambda}) \) as in (5.31), and we call \( F^{(m)}_{ni, \vartheta, \hat{\xi}} (\bar{\omega}, \lambda) \) a decorated monomial \( F^{(m)}_{ni, \vartheta, \hat{\xi}} (\bar{\omega}, \lambda) \) evaluated at \( \bar{\omega}_\lambda \). Consequently, the wavepacket interaction system (5.5) can be written in an equivalent form

\[
\bar{w}_{l, \vartheta} = \sum_{m \in \mathbb{M}_F} \sum_{\lambda \in \Lambda^m} \Psi (\cdot, \vartheta k_{si}) F^{(m)}_{ni, \vartheta, \hat{\xi}} (\bar{\omega}, \lambda) \left( \bar{w}_\lambda, \lambda \right) + \Psi (\cdot, \vartheta k_{si}) \Pi_{ni, \vartheta} \hat{u}, \quad l = 1, \ldots, N, \quad \vartheta = \pm.
\]  

(6.3)

The construction of the above mentioned time averaged equation reduces to discarding certain terms in the original system (6.3). First we introduce the following sets of indices related to the resonance equation (3.24) and \( \Omega_m \) defined by (3.23):

\[
\Lambda^m_{ni, \vartheta} = \left\{ \lambda = (\hat{\lambda}, \hat{\xi}) : \lambda \in \Lambda^m, \left( \vartheta, n_l, \lambda \right) = 0 \right\},
\]  

(6.4)

and then the time-averaged nonlinearity \( F_{av} \) by

\[
F_{av, ni, \vartheta} (\hat{w}) = \sum_{m \in \mathbb{M}_F} F^{(m)}_{ni, \vartheta, \hat{\xi}} (\bar{w}_\lambda, \lambda) + \sum_{\lambda \in \Lambda^m_{ni, \vartheta}} F^{(m)}_{ni, \vartheta, \hat{\xi}} (\bar{w}_\lambda, \lambda).
\]  

(6.5)

where \( F^{(m)}_{ni, \vartheta, \hat{\xi}} (\bar{w}_\lambda, \lambda) \) are defined in (6.2).

Remark 6.1 Note that the nonlinearity \( F^{(m)}_{av, ni, \vartheta} (\hat{w}) \) can be obtained from \( F^{(m)}_{ni, \vartheta} \) by an averaging formula using averaging operator \( A_T \) acting on polynomial functions \( F : (\mathbb{C}^2)^N \to (\mathbb{C}^2)^N \) as follows

\[
(A_T F)_{j, \xi} = \frac{1}{T} \int_0^T e^{-i\xi t} F_{j, \xi} (e^{i\phi t} u_{1, +}, e^{-i\phi t} u_{1, -}, \ldots, e^{i\phi N t} u_{N, +}, e^{-i\phi N t} u_{N, -}) dt.
\]  

(6.6)

Using this averaging we define for any polynomial nonlinearity \( G : (\mathbb{C}^2)^N \to (\mathbb{C}^2)^N \) averaged polynomial

\[
G_{av, j, \xi} (\bar{u}) = \lim_{T \to \infty} (A_T G)_{j, \xi} (\bar{u}).
\]  

(6.7)

If frequencies \( \phi_j \) in (6.6) are generic, \( G_{av, j, \xi} (\bar{u}) \) is always a universal nonlinearity. Note that \( F_{av, ni, \vartheta} (\hat{w}) \) defined by (6.3) can be obtained by the formula (6.7) where \( A_T \) is defined by formula (6.6) with frequencies \( \phi_j = \omega_{nj} (k_{si}) \) (it may be conditionally universal if the frequencies \( \phi_j \) are subjected to a condition of the form (6.22), see the following subsection for details, in particular for definitions of universal and conditionally universal nonlinearities).
Finally, we introduce the wave interaction system with time-average nonlinearity as follows:

$$\hat{v}_{l,\vartheta} = \Psi (\cdot, \vartheta k_{\vartheta l}) F_{av,n_l,\vartheta} (\tilde{v}) + \Psi (\cdot, \vartheta k_{\vartheta l}) \Pi_{n_l,\vartheta} \hat{h}, \ l = 1, \ldots, N, \vartheta = \pm. \quad (6.8)$$

Similarly to (5.7) we recast this system concisely as

$$\tilde{v} = F_{av,\Psi} (\tilde{v}) + \tilde{h}. \quad (6.9)$$

The following lemma is analogous to Lemmas 5.1, 4.5.

**Lemma 6.2** Operator $F_{av,\Psi} (\tilde{v})$ is bounded for bounded $\tilde{v} \in E^{2N}$, $F_{av,\Psi} (0) = 0$. Polynomial operator $F_{av,\Psi} (\tilde{v})$ satisfies the Lipschitz condition

$$\|F_{av,\Psi} (\tilde{v}_1) - F_{av,\Psi} (\tilde{v}_2)\|_{E^{2N}} \leq C \tau_* \|\tilde{v}_1 - \tilde{v}_2\|_{E^{2N}} \quad (6.10)$$

where $C$ depends only on $C_\chi a$ in (3.11), on the power of $F$ and on $\|\tilde{v}_1\|_{E^{2N}} + \|\tilde{v}_2\|_{E^{2N}}$, and, in particular, it does not depend on $\beta, \vartheta$.

From Lemma 6.2 and the contraction principle we obtain the following Theorem similarly to Theorem 5.2.

**Theorem 6.3** Let $\|\tilde{h}_\Psi\|_{E^{2N}} \leq R$. Then there exists $R_1 > 0$ and $\tau_* > 0$ such that equation (6.9) has a solution $\tilde{v} \in E^{2N}$ satisfying $\|\tilde{v}\|_{E^{2N}} \leq R_1$, and such a solution is unique.

The following theorem shows that the averaged interaction system introduced above provides a good approximation for the wave interaction system.

**Theorem 6.4** Let $\hat{v}_{l,\vartheta} (k, \tau)$ be solution of (6.8) and $\hat{w}_{l,\vartheta} (k, \tau)$ be the solution of (5.5). Then for sufficiently small $\beta$ the $\hat{v}_{l,\vartheta} (k, \tau)$ is a wavepacket satisfying (5.10), (5.11) with $\hat{w}$ replaced by $\hat{v}$. In addition to that, there exists $\beta_0 > 0$ such that

$$\|\hat{v}_{l,\vartheta} - \hat{w}_{l,\vartheta}\|_E \leq C \vartheta, \ l = 1, \ldots, N; \vartheta = \pm, \ \text{for} \ 0 < \vartheta \leq 1, \ 0 < \beta \leq \beta_0. \quad (6.11)$$

**Proof.** Formula (5.10), (5.11) for $\hat{v}_{l,\vartheta} (k, \tau)$ follow from (6.8). We note that $\hat{w}$ is an approximate solution of (6.8), namely we have an estimate for $D_{av} (\hat{w}) = \hat{w} - F_{av,\Psi} - \hat{h}_\Psi$ which is similar to (5.17), (5.18):

$$\|D_{av} (\hat{w})\| = \|\hat{w} - F_{av,\Psi} - \hat{h}\|_{E^{2N}} \leq C \vartheta, \ \text{if} \ 0 < \vartheta \leq 1, \beta \leq \beta_0. \quad (6.12)$$

The proof of (6.12) is similar to the proof of (5.22) with minor simplifications thanks to the absence of terms with $\Psi_\infty$. Using (6.12) we apply Lemma 4.6 and obtain (6.11).
6.1.1 Properties of averaged nonlinearities

In this section we discuss elementary properties of nonlinearities obtained by formula (6.5). A key property of such nonlinearities $F_{j,\zeta}$ is the following homogeneity-like property:

$$F_{j,\zeta}(e^{i\phi_1 t} u_{1,+}, e^{-i\phi_1 t} u_{1,-}, \ldots, e^{i\phi_N t} u_{N,+}, e^{-i\phi_N t} u_{N,-}) = e^{i\zeta \phi_1 t} F_{j,\zeta}(u_{1,+}, u_{1,-}, \ldots, u_{N,+}, u_{N,-}).$$  \hspace{1cm} (6.13)

The values of $\phi_i, i = 1, \ldots, N$ for which this formula holds depend on the resonance properties of the set $S$ which enters (6.5) through the index set $\Lambda_{n_l,\zeta}^m$. First, let us consider the simplest case when $\phi_i$ are arbitrary. An example of such a nonlinearity is the function

$$F_{2,\zeta}(u_{1,+}, u_{1,-}, u_{2,+}, u_{2,-}) = u_{1,+} u_{1,-} u_{2,+}.$$

We call a nonlinearity which is obtained by the formula (6.5) with a universal resonance invariant set $S$ a universal nonlinearity.

**Proposition 6.5** If $F_{j,\zeta}$ is a universal nonlinearity, then (6.13) holds for arbitrary set of values $\phi_i, i = 1, \ldots, N$.

**Proof.** Note that the definition (6.5) of the averaged nonlinearity essentially is based on the selection of vectors $\vec{\lambda} = ((\zeta', l'), \ldots, (\zeta^{(m)}, l_m)) \in \Lambda_{n_l,\zeta}^m$ as in (6.4), which is equivalent to the resonance equation (3.24) with $n = n_l, \zeta = \vartheta$. This equation has the form

$$-\zeta \omega_n (k_{**}) + \sum_{l=1}^N \delta_l \omega_l (k_{st}) = 0,$$  \hspace{1cm} (6.14)

with

$$k_{**} = -\zeta \sum_{l=1}^N \delta_l k_{st},$$  \hspace{1cm} (6.15)

where $\delta_l$ are the same as in (3.26). If $\vec{\lambda} \in \Lambda_{n_l,\zeta}^m$ and

$$\vec{w}_{\vec{\lambda}} = (\hat{w}_{\lambda_1} \ldots \hat{w}_{\lambda_m}) = (\hat{w}_{\zeta', l'} \ldots \hat{w}_{\zeta^{(m)}, l_m}) = (e^{-i\zeta' \phi_1 \hat{v}_{\zeta', l'} \ldots e^{-i\zeta^{(m)} \phi_m \hat{v}_{\zeta^{(m)}, l_m}}),$$

then, using (6.2) and the multilinearity of $F^{(m)}$ we get

$$F^{(m)}_{n_l,\zeta}(\vec{w}_{\vec{\lambda}}) = e^{-i\sum \zeta^{(j)} \phi_j} F^{(m)}_{n_l,\vartheta}(\hat{v}_{\vec{\lambda}}),$$

and

$$\sum_{j=1}^m \zeta^{(j)} \phi_{l_j} = \sum_{l=1}^N \delta_l \phi_l,$$  \hspace{1cm} (6.16)

where $\delta_l$ are the same as in (3.26). If we have a universal solution of (6.14), all coefficients at every $\omega_l (k_{st})$ cancel out ($\omega_n (k_{**})$ also equals one of $\omega_l (k_{st})$, namely $\omega_n (k_{**}) = \omega_{n_0} (k_{s_0})$).
Using notation (3.26) we see that a universal solution is determined by the system of equations on binary indices

\[ \delta_l = \sum_{j \in \vec{l}^{-1}(l)} \zeta^{(j)} = 0, \ l \neq I_0, \ \delta_{I_0} = \sum_{j \in \vec{l}^{-1}(I_0)} \zeta^{(j)} = \zeta. \] (6.17)

Obviously, the above condition does not involve values of \( \omega_l \) and hence if \( \delta_l, \zeta \) correspond to a universal solution of (6.14) then we have an identity

\[ -\zeta \phi_{I_0} + \sum_{i=1}^N \delta_l \phi_l = 0, \] (6.18)

which holds for any \( (\phi_1, \ldots, \phi_N) \in \mathbb{C}^N \).

Consider now the case where the \( nk \)-spectrum \( S \) is resonance invariant but may be not universal resonance invariant. Definition 3.8 of resonance invariant \( nk \)-spectrum implies that the set \( P(S) \) of all solutions of (3.24) coincides with the set \( P_{\text{int}}(S) \) of internal solutions. Hence, all solutions of (6.14), (6.15) are internal, in particular \( k_{\ast 

If we have a non-universal internal solution of (6.14), \( \omega_l (k_{\ast}) \) satisfy the following linear equation

\[ \zeta \omega_{n_{I_0}} (k_{\ast I_0}) + \sum_{l=1}^N \delta_l \omega_l (k_{\ast l}) = 0, \zeta k_{\ast I_0} + \sum_{l=1}^N \delta_l k_{\ast l} = 0 \] (6.19)

where at least one of \( b_j \) is non-zero. Note that if (6.19) is satisfied, we have additional (non-universal) solutions of (3.24) defined by

\[ \sum_{j \in \vec{l}^{-1}(l)} \zeta^{(j)} = \delta_l, \ l \neq I_0, \ \sum_{j \in \vec{l}^{-1}(I_0)} \zeta^{(j)} = \zeta + \delta_{I_0}. \] (6.20)

Now let us briefly discuss properties of equations (6.20). The right-hand sides of the above system form a vector \( \vec{b} = (b_1, \ldots, b_N) \) with \( b_l = \delta_l, \ l \neq I_0, \) and \( b_{I_0} = \zeta + \delta_{I_0} \). Note that \( \vec{l} = (l_1, \ldots, l_m) \) is uniquely defined by its level sets \( \vec{l}^{-1}(l) \). For every \( l \) the number \( \delta_{+l} \) of positive \( \zeta^{(j)} \) and the number \( \delta_{-l} \) of negative \( \zeta^{(j)} \) with \( j \in \vec{l}^{-1}(l) \) in (6.20) satisfy equations

\[ \delta_{+l} - \delta_{-l} = \delta_l, \ \delta_{+l} + \delta_{-l} = \left| \vec{l}^{-1}(l) \right| \] (6.21)

where \( \left| \vec{l}^{-1}(l) \right| = c_l \) is the cardinality (number of elements) of \( \vec{l}^{-1}(l) \). Hence, \( \delta_{+l}, \delta_{-l} \) are uniquely defined by \( \delta_l, \left| \vec{l}^{-1}(l) \right| \). Hence, the set of binary solutions \( \vec{\zeta} \) of (6.20) with a given \( \vec{b} \) and a given \( \vec{l} = (l_1, \ldots, l_m) \) is determined by subsets of \( \vec{l}^{-1}(l) \) with the cardinality \( \delta_{+l} \) elements. Hence, every solution with a given \( \vec{b} \) and a given \( \vec{l} \) can be obtained from one solution by permutations of indices \( j \) inside every level set \( \vec{l}^{-1}(l) \). If \( \vec{b} \) is given and the cardinalities \( \left| \vec{l}^{-1}(l) \right| = c_l \) are given, we can obtain different \( \vec{l} \) which satisfy (6.20) by choosing different decomposition of \( \{1, \ldots, m\} \) into subsets with given cardinalities \( c_l \). For given \( \vec{b} \)
and $\vec{c} = (c_1, \ldots, c_m)$ we obtain this way the set (may be empty for some $\vec{b}, \vec{c}$) of all solutions of (6.20). Solutions with the same $\vec{b}$ and $\vec{c}$ we call equivalent.

When for a given wavepacket there are several non-equivalent non-universal solutions, the number of which is denoted by $N_c$, we obtain from (6.19) a system of equations with integer coefficients
\[
\sum_{l=1}^{N} b_{l,i} \omega_l (k_{pl}) = 0, \quad i = 1, \ldots, N_c
\]
and solutions to (3.24) can be found from
\[
\sum_{j \in \vec{l} - 1(l)} \zeta^{(j)} = b_{l,i}, \quad \text{for some } i, \quad 0 \leq i \leq N_c
\]
where to include universal solutions we set $b_{l,0} = 0$.

Hence, when a wavepacket is universally resonance invariant, we conclude that all terms in (6.5) satisfy (6.17). Since (6.18) holds, we get (6.13) for arbitrary $(\phi_1, \ldots, \phi_N) \in \mathbb{C}^N$. If the wavepacket is conditionally universal with conditions (6.23), then using (6.16) and (6.23) we conclude that (6.18) and (6.13) hold if $(\phi_1, \ldots, \phi_N)$ satisfy the system the equations
\[
\sum_{l=1}^{N} b_{l,i} \phi_l = 0, \quad i = 1, \ldots, N_c.
\]

Now we would like to describe a special class of solutions of averaged equations. The evolution equation with an averaged nonlinearity has the form
\[
\partial_\tau U_{j,+} = -\frac{i}{\rho} L_j (-i \nabla) U_{j,+} + F_{j,+} (U_{1,+}, U_{1,-}, \ldots, U_{N,+}, U_{N,-}),
\]
\[
\partial_\tau U_{j,-} = \frac{i}{\rho} L_j (i \nabla) U_{j,+} + F_{j,-} (U_{1,+}, U_{1,-}, \ldots, U_{N,+}, U_{N,-}), \quad j = 1, \ldots, N,
\]
where $L (-i \nabla)$ is a linear scalar differential operator with constant coefficients. The characteristic property (6.13) implies that such a system admits special solutions of the form
\[
U_{j,\zeta} (r, \tau) = e^{-i\phi_j \tau/\rho} V_{j,\zeta} (r)
\]
where $V_{1,\zeta} (r)$ solve the time-independent nonlinear eigenvalue problem
\[
-i\phi_j V_{j,+} = -iL_j (-i \nabla) V_{j,+} + gF_{j,+} (V_{1,+}, V_{1,-}, \ldots, V_{N,+}, V_{N,-}),
\]
\[
i\phi_j V_{j,-} = iL_j (i \nabla) V_{j,+} + gF_{j,-} (V_{1,+}, V_{1,-}, \ldots, V_{N,+}, V_{N,-}), \quad j = 1, \ldots, N.
\]

### 6.1.2 Examples of universal and conditionally universal nonlinearities

Here we give a few examples of equations with averaged nonlinearities. When the multi-wavepacket is universal resonance invariant, the averaged wave interaction system involves NLS-type equations.
Example 6.6 The simplest example of (6.25) for one wavepacket \((N = 1)\) and one spatial dimension \((d = 1)\) is Nonlinear Schrodinger equation
\[
\begin{align*}
\partial_\tau U_{1,+} &= -\frac{i}{\varrho} a_2 \partial_x^2 U_{1,+} - \frac{i}{\varrho} a_0 U_{1,+} + a_1 \partial_x U_{1,+} - iq U_{1,-} U_{1,+}, \\
\partial_\tau U_{1,-} &= \frac{i}{\varrho} a_2 \partial_x^2 U_{1,-} + \frac{i}{\varrho} a_0 U_{1,-} + a_1 \partial_x U_{1,-} + iq U_{1,+} U_{1,-}.
\end{align*}
\] (6.28)

Note that by setting \(y = x + a_1 \tau / \varrho\) we can make \(a_1 = 0\). Obviously, the nonlinearity
\[
F_\zeta (U) = -i \zeta q U_{1,-} \zeta U_{1,\zeta}^2
\]
satisfies (6.13):
\[
i \zeta q e^{-i \zeta \phi_1} U_{1,-} \zeta \left( e^{i \zeta \phi_1} U_{1,\zeta} \right)^2 = e^{i \zeta \phi_1} i \zeta q U_{1,-} \zeta \left( U_{1,\zeta} \right)^2.
\]

The eigenvalue problem in this case takes the form
\[
\begin{align*}
i \phi_1 V_{1,+} &= -ia_2 \partial_x^2 V_{1,+} - ia_0 V_{1,+} + a_1 \partial_x V_{1,+} - iq V_{1,-} V_{1,+}, \\
-i \phi_1 V_{1,-} &= ia_2 \partial_x^2 V_{1,-} + ia_0 V_{1,-} + a_1 \partial_x V_{1,-} + iq V_{1,+} V_{1,-}.
\end{align*}
\] (6.29)

If \(a_1 = 0\) and we consider real-valued \(V_{1,+} = V_{1,-}\) we obtain the equation
\[
(\phi_1 + a_0) V_{1,+} = -a_2 \partial_x^2 V_{1,+} - \varrho q V_{1,+}^3
\]
or, equivalently,
\[
\frac{(\phi_1 + a_0)}{\varrho q} V_{1,+} + \frac{a_2}{\varrho q} \partial_x^2 V_{1,+} + V_{1,+}^3 = 0.
\]

If
\[
c^2 = \frac{a_2}{\varrho q} > 0, \quad \frac{(\phi_1 + a_0)}{\varrho q} = -b^2 < 0,
\] (6.30)

the last equation takes the form
\[
-b^2 V_{1,+} + c^2 \partial_x^2 V_{1,+} + V_{1,+}^3 = 0.
\]

with a family of classical soliton solutions
\[
V_{1,+} = 2^{1/2} \frac{b}{\cosh \left( b \left( x - x_0 \right) / c \right)}.
\]

Note that the norm of Fourier transform \(\hat{V}_{1,+} \|_{L^1} = C b\) where \(C\) is an absolute constant, Hence to have \(V_{1,+}\) bounded in \(L^1\) uniformly in small \(\varrho\) according to (6.30) we should take \(\phi_1 = -a_0 - b^2 \varrho q\) with a bounded \(b\).
If the universal resonance invariant multi-wavepacket involves two wavepackets \(N = 2\) and the nonlinearity \(F\) is cubic, that is \(\mathcal{M}_F = \{3\}\), a semilinear system PDE with averaged nonlinearity has the form

\[
\begin{align*}
\partial_t U_{2,+} &= -iL_2 (i\nabla) U_{2,+} + U_{2,+} \left( Q_{2,1,+} U_{1,+} + Q_{2,2,+} U_{2,-} \right), \\
\partial_t U_{2,-} &= iL_2 (-i\nabla) U_{2,-} + U_{2,-} \left( Q_{2,1,-} U_{1,+} + Q_{2,2,-} U_{2,-} \right), \\
\partial_t U_{1,+} &= -iL_1 (i\nabla) U_{1,+} + U_{1,+} \left( Q_{1,1,+} U_{1,+} + Q_{1,1,-} U_{2,-} \right), \\
\partial_t U_{1,-} &= iL_1 (-i\nabla) U_{1,-} + U_{1,-} \left( Q_{1,1,-} U_{1,+} + Q_{1,1,-} U_{2,-} \right).
\end{align*}
\]

Obviously, (6.13) holds with arbitrary \(\phi_1, \phi_2\).

Now let us consider quadratic nonlinearities. In particular, let us consider the one-band symmetric case \(\omega_n (k) = \omega_1 (k) = \omega_1 (-k)\), i.e. \(J = 1, \mathcal{M}_F = \{2\}\) and \(m = 2\). Suppose that there is a multi-wavepacket involving two wavepackets with wavevectors \(k_{s1}, k_{s2}\) i.e. \(N = 2\). The resonance equation (3.24) takes now the form

\[
-\zeta \omega_1 (\zeta' k_{s1} + \zeta'' k_{s2}) + \zeta' \omega_1 (k_{s1}) + \zeta'' \omega_1 (k_{s2}) = 0,
\]

(6.31)

where \(l_1, l_2 \in \{1, 2\}\), \(\zeta, \zeta', \zeta'' \in \{-1, 1\}\). All possible cases, and there are exactly four of them, correspond to the four well known effects in the nonlinear optics: (i) \(l_1 = l_2, \zeta' = \zeta''\) and \(\zeta' = -\zeta''\) correspond respectively to second harmonic generation and nonlinear optical rectification; (ii) \(l_1 \neq l_2, \zeta' = \zeta''\) and \(\zeta' = -\zeta''\) correspond respectively to sum-frequency and difference-frequency interactions.

Let us suppose now that \(k_{s1}, k_{s2} \neq 0\) and \(\omega_1 (k_{s1}) \neq 0, \omega_1 (k_{s2}) \neq 0\), where the last conditions exclude the optical rectification, and that \(k_{s1} \neq 0, k_{s2} \neq 0, k_{s1} \pm k_{s2}\) are not band-crossing points. Consider first the case when the wavepacket is universally resonance invariant.

**Example 6.7** Suppose there is a single band, i.e. \(J = 1\), with a symmetric dispersion relation, and a quadratic nonlinearity \(F\), that is \(\mathcal{M}_F = \{2\}\). Let us pick two points \(k_{s1}\) and \(k_{s2} \neq \pm k_{s1}\), and assume that \(k_{s1} \neq 0\) and \(k_{s2} \neq 0\). Assume also that (i) \(2 \omega_1 (k_{s1}) \neq \omega_1 (2k_{s1})\), \(i, j, l = 1, 2\), so there is no second harmonic generation; (ii) \(\omega_1 (k_{s1}) \pm \omega_1 (k_{s2}) \neq \omega_1 (k_{s1} \pm k_{s2})\), (no sum/difference-frequency interactions); (iii) \(\omega_1 (0) \neq 0, \omega_j (k_{s1}) \neq \omega_l (k_{s2}) \neq 0\). Let us set the \(nk\)-spectrum to be the set \(S_1 = \{(1, k_{s1}), (1, k_{s2})\}\). Then \(S_1\) is resonance invariant.

In this case (6.31) does not have solutions. Hence \(\Lambda^{m}_{\mathcal{N}, \varnothing} = \varnothing\) and the averaged nonlinearity equals zero.

Now let us consider the case where the wavepacket is not universal resonance invariant, but conditionally universal resonance invariant. In the following example a conditionally resonance invariant spectrum allows for second harmonic generation in the averaged system.

**Example 6.8** Suppose there is a single band, i.e. \(J = 1\), with a symmetric dispersion relation, and a quadratic nonlinearity \(F\), that is \(\mathcal{M}_F = \{2\}\). Let us pick two points \(k_{s1}\) and
such that \( k_2 = 2k_1 \), and assume that \( k_i \neq 0 \) and \( k_2, 2k_i, 0, \pm k_1 + k_2 \) are not band-crossing points. Assume also that (i) \( 2\omega_1 (k_1) = \omega_1 (2k_1) \) (second harmonic generation); (ii) \( \omega_i (k_1) \pm \omega_j (k_2) \neq \omega_i (k_1 \pm k_2) \), \( i, j, l = 1, 2 \) (no sum-/difference-frequencies interaction); (iii) \( \omega_1 (0) \neq 0 \), \( \omega_j (k_1) \pm \omega_i (k_2) \neq 0 \). Let us set the \( nk \)-spectrum to be the set \( S = \{(1, k_1), (1, k_2)\} \). Then \( S \) is resonance invariant. The condition (6.19) is takes here the form

\[
2\omega_1 (k_1) - \omega_1 (k_2) = 0, \quad 2k_1 - k_2 = 0,
\]

and the condition (6.24) turns into

\[
2\omega_1 (k_1) - \omega_1 (k_2) = 0.
\]

The wavepacket interaction system for such a multiwavepacket has the form

\[
\begin{align*}
\partial_t U_{2,+} &= -iL_2 (i\nabla) U_{2,+} + Q_{2,2,+} U_{1,+} U_{1,+}, \\
\partial_t U_{2,-} &= iL_2 (-i\nabla) U_{2,-} + Q_{2,2,-} U_{1,-} U_{1,-}, \\
\partial_t U_{1,+} &= -iL_1 (i\nabla) U_{1,+} + Q_{1,2,+} U_{2,+} U_{1,-}, \\
\partial_t U_{1,-} &= iL_1 (-i\nabla) U_{1,-} + Q_{1,2,-} U_{2,-} U_{1,+}.
\end{align*}
\]

### 6.2 Invariance of multi-particle wavepackets

The following Lemma shows that particle wavepackets are preserved under action of certain types of nonlinearities with elementary susceptibilities as in (4.20). In the following section we show in particular that universal nonlinearities are composed of such terms.

**Lemma 6.9** Let components \( \tilde{w}_{l,\xi} = \tilde{w}_{l_i} \) of \( \tilde{w}_{\lambda} = \tilde{w}_{l_1} \ldots \tilde{w}_{l_m} \) be particle-like wavepackets in the sense of Definition 2.2 and \( F_{n_l,\theta,\xi}(\tilde{w}_{\lambda}) \) be as in (6.3). Assume that

\[
\tilde{w}_{l,\xi} (k, \beta) = 0 \text{ if } |k - \xi l_i| \geq \beta^{1-\epsilon}, \quad \xi = \pm, \quad i = 1, \ldots, m. \tag{6.32}
\]

Assume that vector index \( \lambda \in \Lambda^m_{n_l,\theta} \) be such a vector which has at least one component \( \lambda_j = (\xi_j, l_j) \) such that

\[
\nabla \omega_{n_l} (k_{l_i}) = \nabla \omega_{n_j} (k_{l_j}). \tag{6.33}
\]

Then for any \( r_\ast \in \mathbb{R}^d \)

\[
\left\| \nabla_k \left( e^{-ir \cdot k} \psi (-, k_{l_i}, \beta^{1-\epsilon}) F_{n_l,\theta,\xi}(\tilde{w}_{\lambda}) \right) \right\|_E \leq C r_\ast \left\| \nabla_k e^{-ir \cdot k^{(i)}} w_{l_i} \right\|_E \prod_{i \neq j} \left\| w_{l_j,\xi_j} \right\|_E + C r_\ast \left( \beta^{-1+\epsilon} + \frac{\beta^{1-\epsilon}}{\theta} \right) \prod_{j=1}^m \left\| w_{l_j,\xi_j} \right\|_E,
\]

where \( C \) does not depend on \( r_\ast \) and small \( \beta, \theta \).
Proof. Note that
\[ r_* k = r_* (k' + \ldots + k^{(m)}). \]
We have by (4.22)
\[ \nabla_k e^{-i r_* k} f_n^{(m)} \left( \frac{\phi(\theta, \vec{\zeta})}{\bar{\lambda}} \right) (k, \tau) = \nabla_k \int_0^\tau \int_{[-\pi, \pi]^d} \exp \left\{ i \phi_{\theta, \vec{\zeta}} \left( k, \bar{k} \right) \frac{\tau_1}{\bar{\eta}} \right\} \] (6.35)
\[ \Psi e^{-i r_* k} \chi_{\theta, \vec{\zeta}}^{(m)} (k, \bar{k}) w_{l_1, \zeta'} (k') \ldots w_{l_m, \zeta'}^{(m)} (k^{(m)} (k, \bar{k})) \tilde{d}^{(m-1)d \bar{k} \bar{d} \tau_1}. \]
Without loss of generality we assume that in (6.33) \( l_j = l_m \) (the general case is reduced to this one by a reenumeration of variables of integration). By the Leibnitz formula
\[ \nabla_k \left[ \Psi e^{-i r_* k} f_n^{(m)} \left( \frac{\phi(\theta, \vec{\zeta})}{\bar{\lambda}} \right) \right] (k, \tau) = I_1 + I_2 + I_3, \] (6.36)
where
\[ I_1 = \int_0^\tau \int_{[-\pi, \pi]^d} \nabla_k \exp \left\{ i \phi_{\theta, \vec{\zeta}} \left( k, \bar{k} \right) \frac{\tau_1}{\bar{\eta}} - i r_* k \right\} \times \Psi \chi_{\theta, \vec{\zeta}}^{(m)} (k, \bar{k}) e^{-i r_* k'} w_{l_1, \zeta'} (k') \ldots e^{-i r_* k^{(m)} (k, \bar{k})} w_{l_m, \zeta'}^{(m)} (k^{(m)} (k, \bar{k})) \tilde{d}^{(m-1)d \bar{k} \bar{d} \tau_1}, \]
\[ I_2 = \int_0^\tau \int_{[-\pi, \pi]^d} \Psi \exp \left\{ i \phi_{\theta, \vec{\zeta}} \left( k, \bar{k} \right) \frac{\tau_1}{\bar{\eta}} - i r_* k \right\} \times \left[ \nabla_k \left( \Psi (k, k_m, \beta^{1-\epsilon}) \chi_{\theta, \vec{\zeta}}^{(m)} \left( k, \bar{k} \right) \right) \right] e^{-i r_* k'} w_{l_1, \zeta'} (k') \ldots e^{-i r_* k^{(m)} (k, \bar{k})} w_{l_m, \zeta'}^{(m)} (k^{(m)} (k, \bar{k})) \tilde{d}^{(m-1)d \bar{k} \bar{d} \tau_1}, \]
\[ I_3 = \int_0^\tau \int_{[-\pi, \pi]^d} \exp \left\{ i \phi_{\theta, \vec{\zeta}} \left( k, \bar{k} \right) \frac{\tau_1}{\bar{\eta}} - i r_* k \right\} \times \Psi \chi_{\theta, \vec{\zeta}}^{(m)} (k, \bar{k}) e^{-i r_* k'} w_{l_1, \zeta'} (k') \ldots \nabla_k \left( e^{-i r_* k^{(m)} (k, \bar{k})} w_{l_m, \zeta'}^{(m)} (k^{(m)} (k, \bar{k})) \right) \tilde{d}^{(m-1)d \bar{k} \bar{d} \tau_1}. \]
Since \( w_{j, \zeta} \) are bounded, we have
\[ \left\| e^{-i r_* k^{(j)}} w_{l_j, \zeta^{(j)}} (k^{(j)}) \right\|_{L^1} \leq \left\| w_{l_j, \zeta^{(j)}} (k^{(j)}) \right\|_{L^1} \leq C, \quad j = 1, \ldots, m. \] (6.37)
Using (4.3) and (6.37) we get
\[ |I_3| \leq \left\| \chi^{(m)} \right\| \prod_{j=1}^{m-1} \left\| w_{l_j, \zeta^{(j)}} \right\|_{L^1} \int_0^\tau \left\| \nabla_k e^{-i r_* k^{(m)} (k, \bar{k})} w_{l_m, \zeta'}^{(m)} (k^{(m)} (k, \bar{k})) \right\|_{L^1} d \tau_1. \] (6.38)
From (6.37), (2.26), (3.13) and the smoothness of \( \Psi (k, k_m, \beta^{1-\epsilon}) \) we get
\[ |I_2| \leq C_2 \beta^{-1+\epsilon} \prod_{j=1}^{m} \left\| w_{l_j, \zeta_j} \right\|_{L^1}. \] (6.39)
Now let us estimate $I_1$. Using (4.23) we obtain

$$I_1 = \int_0^\tau \int_{[-\pi,\pi]^{(m-1)d}} \left[ \exp \left\{ i\phi_{\theta,\xi} (\mathbf{k}, \mathbf{k}) \frac{T_1}{\theta} \right\} \right] \frac{T_1}{\theta} \left[ -\theta \nabla_k \omega_{n_l} (\mathbf{k}) + \zeta^{(m)} \nabla_k \omega_{n_{lm}} \left( \mathbf{k}^{(m)} \left( \mathbf{k}, \mathbf{k} \right) \right) \right] \chi_{\theta,\xi}^{(m)} \left( \mathbf{k}, \mathbf{k} \right) \mathbf{w}_{l_1,\xi'} (\mathbf{k}') \ldots \mathbf{w}_{l_m,\xi'}^{(m)} \left( \mathbf{k}_m^{(m)} \left( \mathbf{k}, \mathbf{k} \right) \right) \tilde{d}^{(m-1)d} d_k d\tau_1. \tag{6.40}$$

The difficulty in the estimation of the integral $I_1$ comes from the factor $\frac{T_1}{\theta}$ since $\theta$ is small. Since (6.32) holds, it is sufficient to estimate $I_1$ if

$$\left| k^{(j)} - \zeta^{(j)} k_{s n_j} \right| \leq \beta^{1-\epsilon} \text{ for all } j. \tag{6.41}$$

According to (3.18), since $\vec{\lambda} \in \Lambda^{m}_{n_1, \theta}$, we have

$$k^{(m)} \left( k_{s n_l}, \vec{k}_s \right) = k_{s n_{lm}}. \tag{6.33}$$

Hence, using (6.33) and (4.23) we obtain

$$\nabla_k \phi_{\theta,\xi} \left( k_{s n_l}, \vec{k}_s \right) = \left[ -\theta \nabla_k \omega_{n_l} (\mathbf{k}) + \zeta^{(m)} \nabla_k \omega_{n_{lm}} \left( \mathbf{k}^{(m)} \left( k_{s n_l}, \vec{k}_s \right) \right) \right] = 0. \tag{6.42}$$

Using (3.2) we conclude that in a vicinity of $\vec{k}_s$ defined by (6.41) we have

$$\left| -\theta \nabla_k \omega (\mathbf{k}) + \zeta^{(m)} \nabla_k \omega \left( \mathbf{k}^{(m)} \left( \mathbf{k}, \mathbf{k} \right) \right) \right| \leq 2 (m + 1) C_{\omega,2} \beta^{1-\epsilon}. \tag{6.42}$$

This yields the estimate

$$|I_1| \leq C_3 \beta^{1-\epsilon} / \theta. \tag{6.43}$$

Combining (6.43), (6.39) and (6.38) we obtain (6.48).

We introduce a $\beta$-dependent Banach space $E^1$ of differentiable functions of variable $\mathbf{k}$ by the formula

$$\| \mathbf{w} \|_{E^1(\mathbf{r}_s)} = \beta^{1+\epsilon} \| \nabla_k \left( e^{-i\mathbf{r}_s \cdot \mathbf{k}} \mathbf{w} \right) \|_E + \| \mathbf{w} \|_E. \tag{6.44}$$

We use for $2N$- component vectors with elements $\mathbf{w}_i (\mathbf{k}) \in E^2$ the following notation

$$\vec{\mathbf{w}} (\mathbf{k}) = (\mathbf{w}_1 (\mathbf{k}), \ldots, \mathbf{w}_N (\mathbf{k})), \quad \vec{\mathbf{r}}_s = (\mathbf{r}_{s_1}, \ldots, \mathbf{r}_{s_N}), \quad \mathbf{w}_i (\mathbf{k}) = (\mathbf{w}_{i,+} (\mathbf{k}), \mathbf{w}_{i,-} (\mathbf{k})), \tag{6.45}$$

$$e^{-i\vec{\mathbf{r}} \cdot \mathbf{k}} \vec{\mathbf{w}} (\mathbf{k}) = \left( e^{-i\mathbf{r}_{s_1} \cdot \mathbf{k}} \mathbf{w}_1 (\mathbf{k}), \ldots, e^{-i\mathbf{r}_{s_N} \cdot \mathbf{k}} \mathbf{w}_N (\mathbf{k}) \right).$$

Similarly to (5.6) we introduce the space $(E^1)^{2N} \left( \vec{\mathbf{r}}_s \right)$ with the norm

$$\| \vec{\mathbf{w}} \|_{(E^1)^{2N} \left( \vec{\mathbf{r}}_s \right)} = \sum_{l,\theta} \| \hat{\mathbf{w}}_{l,\theta} \|_{E^1(\mathbf{r}_{s_l})}. \tag{6.46}$$

The following proposition is obtained by comparing (6.44) and (2.33).
Proposition 6.10 A multi-wavepacket $\tilde{\Psi}$ is a multi-particle one with positions $r_{s1}, \ldots, r_{sN}$ if and only if
\[
\|\tilde{\Psi}\|_{(E^1)^{2N}(\tilde{r}_s)} \leq C
\]
where the constant $C$ does not depend on $\beta$, $0 < \beta \leq 1/2$, and $\tilde{r}_s$.

In view of the above we will call $E^1(\tilde{r}_s)$ and $(E^1)^{2N}(\tilde{r}_s)$ particle spaces. We also use notations
\[
\tilde{\Psi}_2\tilde{\Psi}_2^{-\lambda} = (\Psi(\cdot, k_{s1}, \beta^{1/2}) w_{\lambda_1}, \ldots, \Psi(\cdot, k_{sl}, \beta^{1/2}) w_{\lambda_m}) ,
\]
\[
\mathcal{F}^{(m)}_{n_1, \theta, \lambda, \psi_2} (\tilde{w}^m) = \Psi(\cdot, k_{s1}, \beta^{1/2}) \mathcal{F}^{(m)}_{n_1, \theta, \lambda, \psi_2} (\tilde{w}^m). 
\]

Lemma 6.11 Let $\tilde{v}, \tilde{v} \in (E^1)^{2N}(\tilde{r}_s)$ and $\mathcal{F}^{(m)}_{n_1, \theta, \lambda, \psi_2} (\tilde{w}_2)$ be as in (6.3a). Assume that vector index $\tilde{\lambda} \in \Lambda^m_{n_1, \theta}$ be such a vector which has at least one component $\lambda_j = (\zeta_j, l_j)$ with $l_j = l$. Assume that (1.9) holds and $\Psi(\cdot, k_s, \beta^{1/2})$ is defined in (2.25). Let $\|\tilde{v}\|_{(E^1)^{2N}(\tilde{r}_s)} \leq 2R$. Then
\[
\begin{equation}
\|\mathcal{F}^{(m)}_{n_1, \theta, \lambda, \psi_2} (\tilde{w}_2)\|_{E^1(\tilde{r}_s)} \leq C \tau_\ast \|\tilde{v}\|_{(E^1)^{2N}(\tilde{r}_s)}^{m-1} \|\tilde{w}_2\|_{(E^1)^{2N}(\tilde{r}_s)} (6.47)
\end{equation}
\]
where $C$ does not depend on $\beta$, $0 < \beta \leq 1/2$, and on $\tilde{r}_s$, $r_{sl}$, and $\tilde{r}_s$ is defined by (6.45). If $\|\tilde{v}\|_{(E^1)^{2N}(\tilde{r}_s)} \leq 2R$ the following Lipschitz inequality holds
\[
\begin{equation}
\|\mathcal{F}^{(m)}_{n_1, \theta, \lambda, \psi_2} (\tilde{w}_2) - \mathcal{F}^{(m)}_{n_1, \theta, \lambda, \psi_2} (\tilde{v})\|_{E^1(\tilde{r}_s)} \leq C \tau_\ast \|\tilde{w}_2 - \tilde{v}\|_{(E^1)^{2N}(\tilde{r}_s)} (6.48)
\end{equation}
\]
where $C$ does not depend on $\beta$, $0 < \beta \leq 1/2$, and on $\tilde{r}_s$, $r_{sl}$.

Proof. Note that $\tilde{\Psi}_2\tilde{\Psi}_2^{-\lambda}$ and $\tilde{\Psi}_2\tilde{\Psi}_2^{-\lambda}$ are wavepackets in the sense of Definition 2.2. To obtain (6.47) we apply inequality (6.34) and use (1.9); for the part of $E^1$-norm without $k$-derivatives we use (1.10). Using multilinearity of $\mathcal{F}^{(m)}_{n_1, \theta, \lambda, \psi_2}$ we observe that
\[
\mathcal{F}^{(m)}_{n_1, \theta, \lambda, \psi_2} (\tilde{w}_2) - \mathcal{F}^{(m)}_{n_1, \theta, \lambda, \psi_2} (\tilde{v}) = \sum_{j=1}^{m} \mathcal{F}^{(m)}_{n_1, \theta, \lambda, \psi_2} (w_{\lambda_1}, \ldots, w_{\lambda_j} - v_{\lambda_j}, v_{\lambda_{j+1}}, \ldots, v_{\lambda_m}) .
\]

We can apply to every term inequality (6.34). Multiplying (6.34) by $\beta^{1/2}$ and using (1.9) we deduce (6.48). □

Now we consider a system similar to (6.3),
\[
\tilde{v}_l, \theta = \mathcal{F}_{av, \theta, n_{1l}, \theta} (\tilde{v}) + \Psi(\cdot, k_{s_l}) \Pi_{n_l, \theta} \tilde{h}, \ l = 1, \ldots, N, \theta = \mp, (6.50)
\]
where $\mathcal{F}_{av, \theta, n_{1l}, \theta}$ is defined by a formula similar to (6.5):
\[
\mathcal{F}_{av, \theta, n_{1l}, \theta} (\tilde{v}) = \sum_{m \in \text{multi-} \mathcal{F}_F} \mathcal{F}^{(m)}_{n_{1l}, \theta, \lambda, \psi_2} (\tilde{v}).
\]

The system (6.50) can be written in the form similar to (6.3)
\[
\tilde{v} = \mathcal{F}_{av, \theta, n_{1l}, \theta} (\tilde{v}) + \tilde{h}. (6.52)
\]
Theorem 6.12 (solvability in particle spaces) Let the initial data \( \hat{h} \) in the averaged wavepacket interaction system (6.52) be a multi-particle wavepacket \( \hat{h}(\beta, k) \) with \( nk \)-spectrum \( S \) as in (2.39), the regularity degree \( s \) and with positions \( r_{st} \), \( l = 1, \ldots, N \). Let \( \| \hat{h} \|_{\left( E^1 \right)^{2N}(\bar{r}_s)} \leq R \). Assume that \( S \) is universally resonance invariant in the sense of Definition 3.8. Then there exists \( \tau_{ss} > 0 \) which does not depend on \( \bar{r}_s \), \( \beta \) and \( \rho \) such that if \( \tau_* \leq \tau_{ss} \) equation (6.52) has a unique solution \( \hat{v} \) in \( (E^1)^{2N}(\bar{r}_s) \), such that

$$
\| \hat{v} \|_{\left( E^1 \right)^{2N}(\bar{r}_s)} \leq 2R
$$

where \( R \) does not depend on \( \rho, \beta \) and on \( \bar{r}_s \). This solution is is a multi-particle wavepacket with positions \( r_{st} \).

Proof. Since \( S \) is universally resonance invariant every vector index \( \lambda \in \Lambda^m_{\rho, 0} \) has at least one component \( \lambda_j = (\zeta_j, l_j) \) with \( l_j = l \). Hence Lemma 6.11 is applicable and according to (6.48) the operator \( F_{av, \Psi} \) defined by (6.51) is Lipschitz in the ball \( \| \hat{v} \|_{\left( E^1 \right)^{2N}(\bar{r}_s)} \leq 2R \) with a Lipschitz constant \( C' \tau_* \) where \( C' \) which does not depend on \( \rho, \beta \), and on \( \bar{r}_s \). We choose \( \tau_{ss} \) so that \( C' \tau_{ss} \leq 1/2 \) and use Lemma 4.6. According to this Lemma equation (6.52) has a solution \( \hat{v} \) which satisfies (6.53). This solution is is a multi-particle wavepacket according to Proposition 6.10. \( \blacksquare \)

Theorem 6.13 (particle wavepacket approximation) Let the initial data \( \hat{h} \) in the integral equation (2.14) with solution \( \hat{u}(\tau, \beta; k) \) be an multi-particle wavepacket \( \hat{h}(\beta, k) \) with \( nk \)-spectrum \( S \) as in (2.39), the regularity degree \( s \) and with positions \( r_{st} \), \( l = 1, \ldots, N \), and components of \( \hat{h}(\beta, k) \) satisfy the inequality \( \| \hat{h} \|_{\left( E^1 \right)^{2N}(\bar{r}_s)} \leq R \). Let \( \tau_* \leq \tau_{ss} \). Assume that \( S \) is universally resonance invariant in the sense of Definition 3.8. We define \( \hat{v}(\tau, \beta; k) \) by the formula

$$
\hat{v}(\tau, \beta; k) = \sum_{l=1}^{N} \sum_{\xi = \pm} \hat{v}_{l, \xi}(\tau, \beta; k), \quad l = 1, \ldots, N,
$$

(6.54)

where \( \hat{v}_{l, \xi}(\tau, \beta; k) \) is a solution of (6.8). Then every such \( \hat{v}(k; \tau, \beta) \) is a particle-like wavepacket with the position \( r_{st} \) and

$$
\sup_{0 \leq \tau \leq \tau_*} \| \hat{u}(\tau, \beta; k) - \hat{v}(\tau, \beta; k) \|_{L_1} \leq C_1 \rho + C_2 \beta^s
$$

(6.55)

where the constant \( C_1 \) does not depend on \( \rho, s \) and \( \beta \), and the constant \( C_2 \) does not depend on \( \rho, \beta \).

Proof. Let \( \hat{v} \in (E^1)^{2N}(\bar{r}_s) \) be the solution of equation (6.52) which exists by Theorem 6.12 it is a particle-like wavepacket. Note that

$$
\Psi(\cdot, k_{st_1}, \beta^1 - \epsilon/2) \Psi(\cdot, k_{st_1}, \beta^1 - \epsilon) = \Psi(\cdot, k_{st_1}, \beta^1 - \epsilon)
$$

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and solution of (6.52) has the form $\hat{v}(\tau, k^1\ldots)$ and, consequently, for such solutions $\hat{v}_2 = \hat{v}$, the nonlinearity $F_{m_1,\vartheta,\tilde{x}}(\hat{v})$ coincides with and equation $\hat{v}(\cdot, k_{st}, \beta^{1-\epsilon}) F_{m_1,\vartheta,\tilde{x}}(\hat{v})$ and the equation (6.51) coincides with (6.8). Hence, $\hat{v}$ is a solution of (6.8). Estimate (6.55) follows from estimates (6.11) and (5.49).

**Corollary 6.14** If conditions of Theorem 2.10 are satisfied, the statement of Theorem 2.10 holds.

**Proof.** Note that functions $\hat{w}(\cdot, k, \tau) = \hat{u}_i(\cdot, k, \tau)$ in Theorem 5.7 are the two components of $\hat{u}_l(\tau, \beta; k)$ in (2.45). Hence, (5.51) implies that
\[
\|\hat{u}_l - \hat{w}_{l+} - \hat{w}_{l-}\|_E \leq C\beta^s, \ 0 < \beta \leq \beta_0,
\]
where $\hat{w}_{l,\vartheta}$ are solutions to (5.5). According to (6.11), if $\hat{v}_l(\cdot, \tau)$ is the solution of (6.8) we have
\[
\|\hat{v}_{l,\vartheta} - \hat{w}_{l,\vartheta}\|_E \leq C\vartheta, \ l = 1, \ldots, N; \ \vartheta = \pm.
\]
Hence,
\[
\|\hat{u}_l - \hat{v}_{l,+} - \hat{v}_{l,-}\|_E \leq C\vartheta + C\beta^s, \ 0 < \beta \leq \beta_0.
\]
This inequality implies (2.46). We have proved that $\hat{v}_{l,\vartheta}$ is a particle-like wavepacket as in Theorem 6.13. Estimate (6.58) implies that $\hat{u}_l$ is equivalent to $\hat{v}_l = \hat{v}_{l,+} + \hat{v}_{l,-}$ in the sense of (2.42) of the degree $s_1 = \min(s, s_0)$.

**7 Superposition principle and decoupling of the wavepacket interaction system**

In this section we give the proof of the superposition principle of [9] which is based on a study of the wavepacket interaction system (6.8). We will show that when we omit cross-terms in the averaged system wavepacket interaction system, the resulting error is estimated by $\frac{1}{\beta^{s_1}} \ln \beta$, that is component wavepackets evolve essentially independently and the time averaged wavepacket interaction system almost decouples.

Let $F_{av, n_l, \vartheta}$ be defined by (6.5) and let a decoupled nonlinearity $F_{av, n_l, \vartheta, diag}$ be defined by
\[
F_{av, n_l, \vartheta, diag}(\tilde{w}) = \sum_{m \in M_F} F_{n_l, \vartheta}^{(m)} F_{n_l, \vartheta, diag}(\tilde{w}) = \sum_{\tilde{x} \in \Lambda_{n_l, \vartheta, diag}} F_{n_l, \vartheta, diag}(\tilde{x}) (\tilde{w}_{\tilde{x}}).
\]
where the set of indices $\Lambda_{n_l, \vartheta, diag}$ consists of
\[
\Lambda_{n_l, \vartheta, diag} = \{ \tilde{x} = (\tilde{l}, \tilde{\xi}) \in \Lambda_{n_l, \vartheta} : l_j = l, \ j = 1, \ldots, m \}.
\]
Note that $\mathcal{F}_{n_i,\Theta,\text{diag}}^{(m)}$ in (7.1) depends only on $w_{l,+}$ and $w_{l,-}$:

$$\mathcal{F}_{n_i,\Theta,\text{diag}}^{(m)} (\tilde{w}) = \mathcal{F}_{\Theta,\text{diag},l}^{(m)} (w_l), \quad w_l = (w_{l,+}, w_{l,-}).$$  \hfill (7.3)

The coupling between different variables $v_l$ in (6.8) is caused by non-diagonal terms

$$\mathcal{F}_{\text{av},n_i,\Theta,\text{coup}}^{(m)} (\tilde{v}) = \mathcal{F}_{\text{av},n_i,\Theta}^{(m)} (\tilde{v}) - \mathcal{F}_{\text{av},n_i,\Theta,\text{diag}}^{(m)} (\tilde{v}).$$  \hfill (7.4)

Obviously, equation (6.9) can be written in the form

$$\tilde{v} = \mathcal{F}_{\text{av},\Psi,\text{diag}}^{(m)} (\tilde{v}) + \mathcal{F}_{\text{av},\Psi,\text{coup}}^{(m)} (\tilde{v}) + \tilde{h}_\Psi.$$  \hfill (7.5)

The system of decoupled equations has the form

$$\tilde{v}_{\text{diag}} = \mathcal{F}_{\text{av},\Psi,\text{diag}}^{(m)} (\tilde{v}_{\text{diag}}) + \tilde{h}_\Psi.$$  \hfill (7.6)

or, when written in components,

$$v_{\text{diag},l} = \mathcal{F}_{\text{av},\Psi,\text{diag},l}^{(m)} (v_{\text{diag},l}) + h_{\Psi,l}, \quad l = 1, \ldots, N.$$  \hfill (7.7)

We will prove that contribution of $\mathcal{F}_{\text{av},\Psi,\text{coup}}^{(m)}$ in (7.5) is small, the proof is based on the following lemma.

**Lemma 7.1 (small coupling terms)** Let $\mathcal{F}_{n_i,\Theta,\xi}(\tilde{w}_\lambda)$ be as in (6.5), let all components $w_\lambda$ of $\tilde{w}_\lambda$ satisfy (6.32) and be wavepackets in the sense of Definition 2.1, let also (7.2) hold. Assume also that: (i) the vector index $\lambda$ has at least two components $\lambda_i = (\zeta_i, l_i)$ and $\lambda_j = (\zeta_j, l_j)$ with $l_i \neq l_j$; (ii) both $w_{\lambda_i}$ and $w_{\lambda_j}$ are particle wavepackets in the sense of Definition 2.2; (iii) either (2.51) or (2.54) holds. Then for small $\beta$ and $\rho$

$$\left\| \mathcal{F}_{n_i,\Theta,\xi}(\tilde{w}_\lambda) \right\|_{E^N} \leq C \frac{\rho}{\beta^{1+\epsilon}} |\ln \beta|.$$  \hfill (7.8)

**Proof.** Since $k_{sl}$ are not band-crossing points, according to Definition 3.1 and Condition 3.2 the inequalities (3.2) and (3.13) hold. According to the assumption of the theorem at least two $\tilde{w}_{l_j}$ are different for different $j$. Let us assume that $l_{j_1} = l_1$, $l_{j_2} = l_m$, $l_1 \neq l_m$ (the general case can be easily reduced to this one by a relabeling of variables). Since $\tilde{w}_{l_1}$ and $\tilde{w}_{l_m}$ are particle wavepackets, they satisfy (2.33) with $r$ replaced by $r_{l_1}$ and $r_{l_m}$ respectively. Let us rewrite the integral with respect to $\tau_1$ in (3.22) as

$$\mathcal{F}_{n_i,\Theta,\xi}(\tilde{w}_\lambda)(k, \tau) =$$

$$\int_0^\tau \int_{\mathbb{D}_m} \exp \left\{ i \phi_{\zeta, \xi} \left( k, \bar{k} \right) \frac{\tau_1}{\theta} \right\} A_{\zeta, \xi}^{(m)} \left( k, \bar{k} \right) d^{(m-1)d-1} \bar{k} d\tau_1$$

where

$$A_{\zeta, \xi}^{(m)} \left( k, \bar{k} \right) = \chi_{\zeta, \xi}^{(m)} \left( k, \bar{k} \right) w_{l_1}(k') \ldots w_{l_m}(k^{(m)}),$$  \hfill (7.10)
and then rewrite (7.9) in the form

\[ I_{m}^{(m)} \left( \xi^{(m)} \right) (k, \tau) = I_{\xi}^{(m)} \left( \mathbf{w}_{1} \ldots \mathbf{w}_{l_{m}} \right) (k, \tau) = \int_{0}^{\tau} \int_{\mathbb{D}_{m}} \exp_{\phi} \left( \mathbf{k}, \mathbf{k}, \tau_{1}, \varrho, \mathbf{r}_{l_{1}}, \mathbf{r}_{l_{m}} \right) A \left( \mathbf{k}, \mathbf{k}, \mathbf{r}_{l_{1}}, \mathbf{r}_{l_{m}} \right) \tilde{d}^{(m-1)} d\mathbf{k} d\tau_{1}. \]

where

\[ \exp_{\phi} \left( \mathbf{k}, \mathbf{k}, \tau_{1}, \varrho, \mathbf{r}_{l_{1}}, \mathbf{r}_{l_{m}} \right) = \exp \left\{ i\phi_{\xi} \left( \mathbf{k}, \mathbf{k} \right) \frac{\tau_{1}}{\varrho} - i\mathbf{r}_{l_{1}} \mathbf{k}' - i\mathbf{r}_{l_{m}} \mathbf{k}^{(m)} \right\}, \]

\[ A \left( \mathbf{k}, \mathbf{k}, \mathbf{r}_{l_{1}}, \mathbf{r}_{l_{m}} \right) = e^{i\mathbf{r}_{l_{1}} \mathbf{k}' e^{i\mathbf{r}_{l_{m}} \mathbf{k}^{(m)}} A^{(m)}_{\xi} \left( \mathbf{k}, \mathbf{k} \right)}. \]

According to (3.10) \( \mathbf{k}^{(m)} \left( \mathbf{k}, \mathbf{k} \right) = \mathbf{k} - \mathbf{k}' - \ldots - \mathbf{k}^{(m-1)}. \) Hence, picking a vector \( \mathbf{p} \) with a unit length we obtain the formula

\[ \exp_{\phi} \left( \mathbf{k}, \mathbf{k}, \tau_{1}, \varrho, \mathbf{r}_{l_{1}}, \mathbf{r}_{l_{m}} \right) = \frac{\mathbf{p} \cdot \nabla_{\mathbf{k}'} \exp_{\phi} \left( \mathbf{k}, \mathbf{k}, \tau_{1}, \varrho, \mathbf{r}_{l_{1}}, \mathbf{r}_{l_{m}} \right)}{i \left[ \mathbf{p} \cdot \nabla_{\mathbf{k}'} \phi_{\xi, \xi} \left( \mathbf{k}, \mathbf{k} \right) \tau_{1} - \mathbf{p} \cdot (\mathbf{r}_{l_{1}} - \mathbf{r}_{l_{m}}) \right]}. \]

If we set

\[ \phi' = \nabla_{\mathbf{k}'} \phi_{\xi, \xi} \left( \mathbf{k}_{s}, \mathbf{k}_{s} \right) = \nabla_{\mathbf{k}'} \omega \left( \mathbf{c}^{(m)} \mathbf{k}_{s} \right) - \nabla_{\mathbf{k}^{(m)}} \omega \left( \mathbf{c}^{(m)} \mathbf{k}_{s}^{(m)} \right), \]

\[ c_{p} = \mathbf{p} \cdot \phi', q_{p} = \mathbf{p} \cdot (\mathbf{r}_{l_{1}} - \mathbf{r}_{l_{m}}), \]

\[ \theta_{0} \left( \mathbf{k}, \mathbf{k}, \varrho, \tau_{1} \right) = \frac{c_{p} \tau_{1} - q_{p}}{\mathbf{p} \cdot \nabla_{\mathbf{k}'} \phi_{\xi, \xi} \left( \mathbf{k}, \mathbf{k} \right) \tau_{1} - \mathbf{p} \cdot (\mathbf{r}_{l_{1}} - \mathbf{r}_{l_{m}})}, \]

then (7.13) can be recast as

\[ \exp_{\phi} \left( \mathbf{k}, \mathbf{k}, \tau_{1}, \varrho, \mathbf{r}_{l_{1}}, \mathbf{r}_{l_{m}} \right) = \frac{\mathbf{p} \cdot \nabla_{\mathbf{k}'} \exp_{\phi} \left( \mathbf{k}, \mathbf{k}, \tau_{1}, \varrho, \mathbf{r}_{l_{1}}, \mathbf{r}_{l_{m}} \right)}{i \left( c_{p} \tau_{1} - q_{p} \right)} \theta_{0} \left( \mathbf{k}, \mathbf{k}, \varrho, \tau_{1} \right). \]

If (2.51) holds the vector \( \phi' \neq 0 \) and to get \( |c_{p}| \neq 0 \) we can take

\[ \mathbf{p} = |\phi'|^{-1} \cdot \phi', |c_{p}| = p_{0} > 0. \]

If (2.54) holds we have \( \phi' = 0 \) and we set

\[ \mathbf{p} = (\mathbf{r}_{l_{1}} - \mathbf{r}_{l_{m}})^{-1} \cdot (\mathbf{r}_{l_{1}} - \mathbf{r}_{l_{m}}). \]

Let consider first the case when (2.51) holds. Notice that the denominator in (7.16) vanishes for

\[ \tau_{10} = \frac{q_{p}}{c_{p}}. \]

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We split the integral with respect to $\tau_1$ in (7.11) into a sum of two integrals, namely

$$F^{(m)}_{\zeta, \zeta} (w_1 \ldots w_m) (k, \tau) = F_1 + F_2,$$

$$F_1 = \int_{|\tau - \tau_1| \geq c_0 \beta^{1-\epsilon} |\ln \beta|} \int_{\mathbb{R}_m} \exp_\phi \left( k, \tilde{k}, \tau, \varphi, r_{l_1}, r_{l_m} \right) A \left( k, \tilde{k}, r_{l_1}, r_{l_m} \right) \tilde{d}^{(m-1)} \tilde{k} d\tau_1,$$

$$F_2 = \int_{|\tau - \tau_1| \leq c_0 \beta^{1-\epsilon} |\ln \beta|} \int_{\mathbb{R}_m} \exp_\phi \left( k, \tilde{k}, \tau, \varphi, r_{l_1}, r_{l_m} \right) A \left( k, \tilde{k}, r_{l_1}, r_{l_m} \right) \tilde{d}^{(m-1)} \tilde{k} d\tau_1,$$

where $c_0$ is a large enough constant which we estimate below in (7.22). Since $w_j$ are bounded in $E$ and (2.48) holds, we obtain similarly to (4.10) the estimate

$$\|F_2\|_L^1 \leq C \beta^{1-\epsilon} |\ln \beta| \prod_{j=1}^m \|w_j\|_E \leq C_1 \beta |\ln \beta| \beta^{1+\epsilon}. \quad (7.21)$$

To estimate the norm of $F_1$ we use (7.13) and integrate by parts the integral in (7.20) with respect to $k$. We obtain

$$F_1 = \int_{|\tau - \tau_1| \geq c_0 \beta^{1-\epsilon} |\ln \beta|} I (k, \tau_1) d\tau_1,$$

$$I (k, \tau_1) = - \int_{\mathbb{R}_m} \frac{\varphi \exp_\phi \left( k, \tilde{k}, \tau, \varphi, r_{l_1}, r_{l_m} \right)}{i (c_p \tau_1 - \varphi q_p)} \mathbf{p} \cdot \nabla k' \left[ \theta_0 A \left( k, \tilde{k}, r_{l_1}, r_{l_m} \right) \tilde{d}^{(m-1)} \tilde{k} \right].$$

According to (7.10) and (3.10) the expansion of the gradient $\nabla k'$ in the above formula involves derivatives of $\chi, \theta_0, e^{ir_1} k w_{l_1}$ and $e^{ir_m} k^{(m)} w_{l_m}$. To estimate $\theta_0$ and $\nabla \theta_0$ we note that

$$\theta_0 \left( k, \tilde{k}, \varphi, \tau \right) = \frac{(\mathbf{p} \cdot \phi' \tau - \varphi q_p)}{(\mathbf{p} \cdot \phi' \tau - \varphi q_p + \tau_1 \mathbf{p} \cdot \nabla k' \phi_{\zeta, \zeta} \left( k, \tilde{k} \right) - \phi')}$$

$$= \frac{1}{1 + \tau_1 \mathbf{p} \cdot \nabla k' \phi_{\zeta, \zeta} \left( k, \tilde{k} \right) / (\mathbf{p} \cdot \phi' \tau - \varphi q_p)}.$$

Since $|\tau_1 - \tau_1| \geq c_0 \beta^{1-\epsilon} |\ln \beta|$, from (7.19) we infer

$$|c_p \tau_1 - q_p| \geq c_p c_0 \beta^{1-\epsilon} |\ln \beta|. \quad (7.24)$$

From (6.32), we see that in the integral (7.22) integrands are nonzero only if

$$\left| k^{(j)} - \zeta^{(j)} k_{\star} \right| \leq \pi_0 \beta^{1-\epsilon}, |k - \zeta k_{\star}| \leq m \pi_0 \beta^{1-\epsilon},$$

where $\pi_0 \leq 1$. Using the Taylor remainder estimate for $\nabla k' \phi_{\zeta, \zeta}$ at $\tilde{k}_{\star}$ we obtain the inequality

$$\left| \nabla k' \phi_{\zeta, \zeta} \left( k, \tilde{k} \right) - \phi' \right| \leq 2 m C_{\omega, 2} \beta^{1-\epsilon}. \quad (7.26)$$
Hence in (7.23)
\[ |\tau_1 p \cdot [\nabla k\phi_0 (k, \vec{k}) - \phi'] / (c_p \tau_1 - q_p) | \leq 2m\tau_s C_{\omega,2} / (c_p c_0 |\ln \beta|). \] (7.27)

Suppose that \( \beta \leq 1/2 \) is small and \( c_0 \) satisfies
\[ \frac{m\tau_s C_{\omega,2}}{|\ln \beta|} \leq \frac{m\tau_s C_{\omega,2}}{\ln 2} \leq \frac{1}{4} |c_p| c_0. \] (7.28)

Then it follows from (7.23) with help of (7.28), (3.2), (7.24) and (7.27) that
\[ |\theta_0 (k, \vec{k}, \varrho, \tau_1)| \leq 2. \] (7.29)

Obviously,
\[ \nabla_k\theta_0 (k, \vec{k}, \varrho, \tau_1) = \frac{m\tau_s C_{\omega,2}}{1 + m\tau_s C_{\omega,2}} \left( \nabla_k \left[ p \cdot \left( \nabla k\phi_0 (k, \vec{k}) - \phi' \right) / (c_p \tau_1 - q_p) \right] \right)^2 \] (7.30)

Using (7.27), (7.28) and (3.2) we obtain that
\[ |\nabla_k\theta_0 (k, \vec{k}, \varrho, \tau_1)| \leq \frac{4\tau_s}{|c_p \tau_1 - q_p|} \left| \nabla_k \left[ p \cdot \left( \nabla k\phi_0 (k, \vec{k}) - \phi' \right) \right] \right| \leq \frac{4\tau_s C_{\omega,2}}{|c_p \tau_1 - q_p|}. \] (7.31)

To estimate \( \nabla_k \chi \) we use (3.13). We conclude that the absolute value of the integral (7.22) is not greater than
\[ |I (k, \tau_1)| \leq \frac{4\sigma \tau_s C_{\omega,2}}{|\tau_1 c_p - q_p|^2} \int_{\mathbb{R}^m} \left| A \left( k, \vec{k}, \vec{r}_l, r_{l_m} \right) \right| \tilde{d}^{(m-1)d} \vec{k} \tag{7.32} \]
\[ \leq \frac{4C_{\omega,2} \sigma \tau_s}{|\tau_1 c_p - q_p|^2} \left\| \chi^{(m)} (k, \cdot) \right\| + \frac{2\sigma \tau_s}{|\tau_1 c_p - q_p|^2} \left\| (\nabla_k' - \nabla_k^{(m)}) \chi^{(m)} (k, \cdot) \right\| \bigg| \prod_{j=1}^{m} ||w_j||_{L^1} \bigg| + \prod_{j=1}^{m-1} ||w_j||_{L^1} \bigg| \nabla_k^{(m)} e^{i\vec{r}_l m} \vec{k} \bigg| ||w_m||_{L^1}. \] (7.33)

Note that \( ||w_j||_{L^1} \) are bounded according to (2.27) and \( \nabla_k^{(m)} e^{i\vec{r}_l m} \vec{k} \), \( \nabla_k e^{i\vec{r}_l k} \), \( \omega \) by (2.33). Hence we obtain
\[ |I (k, \tau_1)| \leq \frac{C_2 \sigma \beta^{1-\epsilon}}{|\tau_1 c_p - q_p|^2} + \frac{\sigma C_2}{|\tau_1 c_p - q_p|^2}. \] (7.34)
Obviously,

\[
\int_{|\tau_1 - q_p/c_p| \geq c_0 \beta^{1-\epsilon} |\ln \beta|} |\tau_1 c_p - q_p|^{-1} d\tau_1 = \frac{1}{c_p} \int_{c_0 \beta^{1-\epsilon} |\ln \beta|}^{\tau_1 - q_p/c_p} d\tau_1 = \frac{1}{c_p} \ln \frac{\tau_1 - q_p/c_p}{c_0 \beta^{1-\epsilon} |\ln \beta|} \\
\leq \frac{1}{c_p} \left( C + |\ln \beta^{1-\epsilon} |\ln \beta|\right) \leq \frac{1}{c_p} \left[ C + |\ln \beta| + |\ln |\ln \beta||\right] \leq \frac{1}{c_p} \left[ C + 2 |\ln \beta| \right].
\]

Similarly, using (2.48) we get

\[
\int_{|\tau_1 - q_p/c_p| \geq c_0 \beta^{1-\epsilon} |\ln \beta|} |\tau_1 c_p - q_p|^{-2} d\tau_1 = \frac{1}{c_p} \int_{c_0 \beta^{1-\epsilon} |\ln \beta|}^{\tau_1 - q_p/c_p} d\tau_1 = \frac{1}{c_p} \left[ \frac{1}{c_0 \beta^{1-\epsilon} |\ln \beta|} - \frac{1}{\tau_1 - q_p/c_p} \right] \\
\leq \frac{1}{c_p c_0 \beta^{1-\epsilon} |\ln \beta|} \leq \frac{C_3 \beta}{\beta^{1-\epsilon} |\ln \beta|}
\]

Hence, we obtain for small $\beta$

\[
\left\| \mathcal{F}^{(m)}_{\zeta, \zeta'} (w_1 \ldots w_m) (k, \tau) \right\|_E \leq C_4 \frac{\beta}{\beta^{1+\epsilon} |\ln \beta|}.
\]

Now let us consider the case when (2.54) holds, $\phi' = 0$ and $p$ is defined by (7.18). Turning to expression (7.23) we notice that

\[
c_p \tau_1 - q_p = -\varrho |r_{l_1} - r_{l_m}| , \quad \tau_1 |c_p \tau_1 - q_p|^{-1} \leq \frac{1}{\beta^{1+\epsilon}}.
\]

and, according to (7.28),

\[
\left| \nabla_{k'} \phi_{\zeta, \zeta'} (k, \tilde{k}) - \phi' \right| \leq C_{\omega, 2} \beta^{1-\epsilon}.
\]

Then we estimate the denominator in (7.23) and (7.30) using (2.54):

\[
\left| \tau_1 p \cdot \left[ \nabla_{k'} \phi_{\zeta, \zeta'} (k, \tilde{k}) - \phi' \right] / (c_p \tau_1 - q_p) \right| \leq \tau_1 |c_p \tau_1 - q_p| / (\varrho |r_{l_1} - r_{l_m}|) \leq \frac{1}{2}.
\]

If $\beta$ is so small that (7.28) holds we again get (7.29) and (7.31). Hence, we obtain (7.33) in this case as well (in fact, in this case the logarithmic factor can be omitted). Finally, we obtain (7.36) after summing up over all $\tilde{\lambda}, \tilde{\zeta}$. $\blacksquare$

**Lemma 7.2** Let the $nk$-spectrum $S$ be universally resonance invariant. Let the operators $\mathcal{F}_{av, n_1, \varrho} (\tilde{w}), \mathcal{F}_{av, n_1, \varrho, \text{diag}} (\tilde{v})$ and $\mathcal{F}_{av, n_1, \varrho, \text{coup}}$ be defined respectively by (6.3), (7.7) and (7.4). Let $\tilde{v}, \|\tilde{v}\|_E \leq 2R$, be a multi-wavepacket solution of (6.9) with the $nk$-spectrum $S$. Then for small $\beta$ and $\varrho$

\[
\|\mathcal{F}_{av, n_1, \varrho, \text{coup}} (\tilde{v})\|_E \leq C \frac{\varrho}{\beta^{1+\epsilon} |\ln \beta|}.
\]

(7.36)
Proof. According to (6.5), (7.1) and (7.4) $\mathcal{F}_{av,n,\theta,\text{coup}}$ involves only terms with $\vec{\lambda} \in \Lambda_{n,\theta}^m \setminus \Lambda_{n,\theta}^{m,\text{diag}}$ and it is sufficient to prove the estimate (7.8) for indices $\vec{\lambda} \in \Lambda_{n,\theta}^m \setminus \Lambda_{n,\theta}^{m,\text{diag}}$. Such indices involve at least two components $\lambda_i = (\zeta_i, l_i)$ and $\lambda_j = (\zeta_j, l_j)$ with $l_i \neq l_j$ since the $nk$-spectrum is universally invariant, see (3.26). According to Theorem 6.13 the solution $\tilde{v}$ is a particle-like wavepacket, therefore all components of $\tilde{v}_{\vec{\lambda}}$ are particle-like; (6.32) holds according to (5.11). Hence, all conditions of Lemma 7.1 are fulfilled and (7.36) follows from (7.8). 

Note now that every equation (7.7) is an approximation of the equation (4.6) with single-wavepacket initial data $\hat{h}$, namely

$$\hat{u}_l(k, \tau) = F(\hat{u}_l)(k, \tau) + \hat{h}_l(k). \quad (7.37)$$

One can apply to this equation Theorems 5.6 and 6.4 formally restricted to the case $N = 1$ of a single wavepacket. Based on this observation and on the above Lemma we prove the following theorem which implies previously formulated Theorems 2.14 and 2.15.

**Theorem 7.3** Assume that the multiwavepacket $\tilde{h} = \sum \hat{h}_l$ is particle-like and its $nk$-spectrum is universally resonance invariant. Assume also that either (2.53) or (2.54) holds. Let $\hat{u}$ be solution of equation (4.6). Let $\tilde{v}_l$ be solutions of (7.37). Then the superposition principle holds, namely

$$\left\| \hat{u} - \sum_{l=1}^N \tilde{v}_l \right\| \leq C \frac{\theta}{\beta^{1+\epsilon}} |\ln \beta| + C \beta^s. \quad (7.38)$$

**Proof.** Let $v_{\text{diag},l}$ be a solution of the decoupled system (7.7). We compare systems (7.5) and (7.6). The difference between the systems is the term $\mathcal{F}_{av,n,\theta,\text{coup}}(\tilde{v})$. According to Theorem 6.12 the solution $\tilde{v}$ is a particle-like wavepacket and we can apply Lemma 7.2. According to this Lemma (7.36) holds. Applying Lemma 4.6 to the equations (7.5) and (7.6) and using (7.36) we conclude that the difference of their solutions satisfies the inequality

$$\left\| v_l - v_{\text{diag},l} \right\|_E \leq C' \frac{\theta}{\beta^{1+\epsilon}} |\ln \beta| + C' \beta^s. \quad (7.39)$$

According to Theorem 6.13 inequality (6.53) holds where $\tilde{v}$ is a solution of (6.9) which can be rewritten in the form of (7.3). From (6.53) and (7.39) we infer

$$\left\| \tilde{u} - \sum_{l=1}^N v_{\text{diag},l} \right\| \leq C_1 \frac{\theta}{\beta^{1+\epsilon}} |\ln \beta| + C_1 \beta^s. \quad (7.40)$$

Note that equation (7.7) for $v_{\text{diag},l}$ coincides with the averaged equation (6.9) obtained for the wave interaction system derived for (7.37). Therefore, applying Theorems 5.6 and 6.4 to the case $N = 1$ and $\tilde{h} = \tilde{h}_l$ we deduce from (6.49) and (6.11) the estimate

$$\left\| \tilde{u}_l - v_{\text{diag},l} \right\|_E \leq C_2 \theta + C_2' \beta^s. \quad (7.41)$$

Finally, from (7.40) and (7.41) we infer (7.38).
7.1 Generalizations

In this section we show that the particle-like wavepacket invariance can be extended to the case when \( nk \)-spectra \( S \) are not universally resonance invariant. So suppose that \( nk \)-spectrum \( S \) is resonance invariant and consider nonlinearities of the form similar to (6.5)

\[
\mathcal{F}_{\text{res},n_l,\theta} (\hat{w}) = \sum_{m \in \mathfrak{m}_F} F_{n_l,\theta}^{(m)} = \sum_{\tilde{x} \in \Lambda'_{n_l,\theta}} F_{n_l,\theta}^{(m)} (\hat{w}_{\tilde{x}}),
\]

where \( \Lambda'_{n_l,\theta} \subseteq \Lambda_{n_l,\theta}^m \) is a given subset of \( \Lambda_{n_l,\theta}^m \). Obviously, \( \mathcal{F}_{\text{av}} \) defined by (6.5) has the form of (7.42) with \( \Lambda_{n_l,\theta} = \Lambda_{n_l,\theta}^m \). Let us introduce a multi-wavepacket

\[
\hat{w} = (w_{n_1,+, \ldots, w_{n_N,+}, w_{n_N,-})
\]

with the \( nk \)-spectrum \( S = \{(n_l, \theta), l = 1, \ldots, N; \theta = \pm\} \).

We call a subset \( S' \subseteq S \) sign-invariant if with it has \( (n_l, \theta) \) as an element then \( (n_l, -\theta) \) is also its element. Suppose that \( S' \subseteq S \) is sign-invariant. It is easy to see that if a set \( S' \subseteq S \) is sign-invariant then it is uniquely defined by a subset of indices \( I' = I' (S') \subseteq I = \{1, \ldots, N\} \), namely

\[
S' = \{(n_l, \theta) : l \in I' (S'), \theta = \pm\}.
\]

Definition 7.4 We call index pair \((n_l, k_{sd})\) Group Velocity Matched (GVM) with \( \mathcal{F}_{\text{res},n_l,\theta} \) if every nonzero term \( F_{n_l,\theta}^{(m)} \) in the sum (7.42) has the index \( \tilde{x} \) such that for at least one component \( \lambda_j = \left( \xi^{(j)}, l_j \right) \) of this index

\[
\nabla \omega_{n_l} (k_{sd}) = \nabla \omega_{n_l} \left( k_{sd} l_j \right).
\]

We call \( S' \) a GVM set with respect to the nonlinearity \( \mathcal{F}_{\text{res}} \) defined by (7.42) if \( S' \subseteq S \) is sign-invariant and every \( (n_l, k_{sd}) \in S' \) is GVM.

Obviously, if \( S \) is universally resonance invariant and \( \Lambda'_{n_l,\theta} = \Lambda_{n_l,\theta}^m \) as in (6.5) then \( S \) is a GVM set, and in this case \( l_j = I_0 \) as in Definition 3.6. If \( S' \subseteq S \) is sign-invariant we call a multi-wavepacket \( \hat{w} \) as in (7.43) with the \( nk \)-spectrum \( S = \{(n_l, \theta), l = 1, \ldots, N; \theta = \pm\} \) partially \( S' \)-localized multi-wavepacket if for every \( (n_l, \theta) \in S' \) the wavepacket \( w_{n_l, \theta} \) is a spatially localized with a position \( r_{sd} \). Note that according to Definition 2.7 if \( S' = S \) is a partially \( S' \)-localized multi-wavepacket then it is a multi-particle wavepacket.

Theorem 2.10 on the particle-like wavepacket preservation can be generalized as follows.

Theorem 7.5 (preservation of spatially localized wavepackets) Assume that conditions of Theorem 2.9 hold, in particular the initial datum \( \hat{h} = \hat{h} (\beta, k) \) is a multi-wavepacket with \( nk \)-spectrum \( S \). Assume also that \( S' \subseteq S \) is a GVM set, \( \hat{h} = \hat{h} (\beta, k) \) is partially \( S' \)-localized wavepacket with positions \( r_{sd}, l \in I' (S') \), and that (2.47) holds. Then the solution \( \hat{u} (\tau, \beta) = G \left( F (\rho (\beta)), \hat{h} (\beta) \right) (\tau) \) to (2.14) for any \( \tau \in [0, \tau_\ast] \) is a multi-wavepacket with
nk-spectrum \(S\), and it is an \(S'\)-localized wavepacket with positions \(r_{st}, l \in I' (S')\). Namely, (2.40) holds where \(\hat{u}_t\) is wavepacket with nk-pair \((n_t, k_{st}) \in S'\) defined by (2.45), the constants \(C, C_1, C_2\) do not depend on \(r_{st}\), and every \(\hat{u}_t, l \in I' (S')\), is equivalent in the sense of the equivalence (2.42) of degree \(s_1 = \min (s, s_0)\) to a spatially localized wavepacket with position \(r_{st}\).

**Proof.** The proof of the Theorem is the same as the proof of Theorem 2.10 since it used only the fact that a universally resonance invariant set is a GVM one, that allows to apply Lemma 6.9. One also have to use the space \((E^1)^{2N} (\bar{r}_s, S')\) with the norm defined by the formula similar to (6.46):

\[
\| \tilde{w} \|_{(E^1)^{2N} (\bar{r}_s, S')} = \sum_{l, \theta} \| \tilde{w}_{l, \theta} \|_E + \beta^{1+\epsilon} \sum_{\vartheta = \pm} \sum_{l \in I' (S')} \| \nabla_k (e^{-ir_{st}k} \tilde{w}_{l, \theta}) \|_E.
\] (7.45)

After replacing \((E^1)^{2N} (\bar{r}_s)\) with \((E^1)^{2N} (\bar{r}_s, S')\) we can literally repeat all the steps of the proof of Theorem 2.10 and obtain the statement of Theorem 7.3.

Below we prove that the superposition principle can hold not only for universal resonance invariant multiwavepackets, but for other cases allowing resonant processes such as second and third harmonic generations, three-wave interaction etc. Here we prove a theorem applicable to such situations, which is more general than Theorem 2.14.

Let us consider a multi-wavepacket with resonance invariant nk-spectrum

\[ S = \{(n_t, k_{st}), l = 1, \ldots, N\} \]

as in (3.14), and assume that is a union of spectra \(S_p\):

\[ S = S_1 \cup \ldots \cup S_K, S_p \cap S_q = \emptyset \quad \text{if} \quad p \neq q. \] (7.46)

Recall that resonance interactions are defined in terms of vectors \(\bar{\lambda} \in \Lambda^m\) (see (3.10), (3.17)). We call a vector \(\bar{\lambda} = \left(\left(\zeta', l_1\right), \ldots, \left(\zeta^{(m)}, l_m\right)\right) \in \Lambda^m\) cross-interacting (CI) if there exist at least two indices \(\left(\zeta^{(i)}, l_i\right)\) and \(\left(\zeta^{(j)}, l_j\right)\) such that \(\left(\zeta^{(i)}, l_i\right) \in S_{p_i}\), \(\left(\zeta^{(j)}, l_j\right) \in S_{p_j}\) with \(p_i \neq p_j\).

**Definition 7.6 (partially GVM decomposition)** We call decomposition (7.40) partially GVM with respect to \(F_{\text{res}}\) defined by (7.42) if the following two conditions are satisfied: (i) every spectrum \(S_j, j = 1, \ldots, K\), is resonance invariant; (ii) a solution \((m, \zeta, n, \bar{\lambda}) \in P (S)\) of the resonance equation (3.24) with CI vector \(\bar{\lambda} = \left(\left(\zeta', l_1\right), \ldots, \left(\zeta^{(m)}, l_m\right)\right)\) has at least two indices \(\left(\zeta^{(i)}, l_i\right) \in S_{p_i}\) and \(\left(\zeta^{(j)}, l_j\right) \in S_{p_j}\) with \(p_i \neq p_j\) such that both \(l_i\) and \(l_j\) are GVM with respect to \(F_{\text{res}}\) and

\[
|\nabla_k \omega_{n_{st}} (k_{st}) - \nabla_k \omega_{n_{st}} (k_{st})| \neq 0.
\] (7.47)
Now we use Lemma 7.1 for small coupling. Being given a partially GVM decomposition (7.46) we introduce the set of coupling terms between \( S_p_i \) and \( S_p_j \) as follows:

\[
\Lambda_{m,\text{coup}}^{m,\theta} = \left\{ \tilde{\lambda} = \left( \tilde{l}, \tilde{\zeta} \right) \in \Lambda_{m,\theta}^m : \exists \ i \neq j \text{ such that } l_i \in S_{p_i}, l_j \in S_{p_j} \right\},
\]

(7.48)

We also introduce a set of interactions reducible to every \( S_p \) (block-diagonal) which is similar to (7.2):

\[
\Lambda_{m,\text{red}}^{m,\theta} = \Lambda_{m,\theta}^m \setminus \Lambda_{m,\text{coup}}^{m,\theta},
\]

(7.49)

and the reduced operator

\[
F_{av,n_l,\theta,\text{red}}(\tilde{\tilde{w}}) = \sum_{m \in \mathbb{M}_p} F_{av,n_l,\theta,\text{red}}(\tilde{w}^m), \quad F_{av,n_l,\theta}^{(m)}(\tilde{w}) = \sum_{\tilde{\lambda} \in \Lambda_{m,\theta}^m} F_{av,n_l,\theta}^{(m)}\left(\tilde{\lambda}\right)(\tilde{w}_\tilde{\lambda}),
\]

(7.50)

where \( \Lambda_{m,\theta}^m \) is defined by (7.49). Note that if the set \( S \) is universal resonance invariant and every \( S_p \) is a two-point set \( \{(+, l_i), (+, l_j)\} \) then \( \Lambda_{m,\theta}^m = \Lambda_{m,\text{diag}}^m \). We introduce also a partially decoupled, reduced system similar to (7.6)

\[
\tilde{v}_{\text{red}} = F_{av,\psi,\text{red}}(\tilde{v}_{\text{red}}) + \tilde{h}_\psi,
\]

(7.51)

which can be rewritten in the decoupled form, similar to (7.7):

\[
v_{\text{red},p} = F_{av,\psi,\text{red},p}(v_{\text{red},p}) + h_{\text{red},\psi,p}, \quad p = 1, \ldots K.
\]

(7.52)

Now \( v_{\text{red},p} \) may include more than one wavepacket, namely

\[
v_{\text{red},p} = \sum_{(n_l,\theta) \in S_p} (\tilde{v}_{\text{red}})_{n_l,\theta}, \quad h_{\text{red},\psi,p} = \sum_{(n_l,\theta) \in S_p} (\tilde{h}_\psi)_{n_l,\theta}, \quad p = 1, \ldots K.
\]

(7.53)

The following theorem is a generalization of the Theorem 2.14 on the superposition.

**Theorem 7.7 (general superposition principle)** Suppose that the initial data \( \tilde{h} \) of (2.14) is a multi-wavepacket of the form

\[
\tilde{h} = \sum_{p=1}^{K} \tilde{h}_{\text{red},p},
\]

(7.54)

where \( \tilde{h} \) is a multi-wavepacket in the sense of Definition 3.3 with resonance invariant nk-spectrum \( S \), \( \tilde{h}_{\text{red},p} \) is a multi-wavepacket with a resonance invariant nk-spectrum \( S_p \) and the decomposition (7.46) is partially GVM in the sense of Definition 7.6 with respect to the nonlinearity \( F_{av} \) defined by (6.5). Suppose also that (2.48) holds. Then the solution \( \tilde{u} = G(\tilde{h}) \) to the evolution equation (2.14) satisfies the approximate superposition principle:

\[
G\left(\sum_{p=1}^{K} \tilde{h}_{\text{red},p}\right) = \sum_{p=1}^{K} G(\tilde{h}_{\text{red},p}) + \tilde{\Delta},
\]

(7.55)
with a small remainder $\tilde{D}(\tau)$ satisfying the following estimate

$$
\sup_{0 \leq \tau \leq \tau_\ast} \left\| \tilde{D}(\tau) \right\|_{L^1} \leq C_\epsilon \frac{\varrho}{\beta^{1+\epsilon}} |\ln \beta| ,
$$

(7.56)

where $\epsilon$ is the same as in Definition 2.1 and can be arbitrary small, $\tau_\ast$ does not depend on $\beta, \varrho$ and $\epsilon$.

**Proof.** The proof of Theorem 7.7 is similar the proof of Theorem 7.3. Averaged system (6.9) can be written similarly to (7.5) in the form

$$
\tilde{v} = F_{av,\Psi,\text{red}}(\tilde{v}) + F_{av,\Psi,\text{coup}}(\tilde{v}) + \tilde{h}_\Psi.
$$

(7.57)

Comparing now systems (7.57) and (7.51) we find that the difference between them is the term $F_{av,n_l,\varrho,\text{coup}}(\tilde{v})$. According to Theorem 7.5 the solution $\tilde{v}$ is a spatially localized wavepacket and, hence, we can apply Lemma 7.2 getting the inequality (7.36). Applying Lemma 4.6 to the equations (7.57) and (7.51) and using (7.36) we conclude that the difference of their solutions satisfies the inequality

$$
\| v_p - v_{\text{red},p} \|_E \leq C' \frac{\varrho}{\beta^{1+\epsilon}} |\ln \beta| + C' \beta^s, \, p = 1, ..., K
$$

(7.58)

According to Theorem 6.13 inequality (6.55) holds where $\tilde{v}$ is a solution of (7.57) and we infer from (7.58)

$$
\left\| \tilde{u} - \sum_{p=1}^{K} v_{\text{red},p} \right\| \leq C_1 \frac{\varrho}{\beta^{1+\epsilon}} |\ln \beta| + C_1 \beta^s.
$$

(7.59)

Similarly to (7.37) we introduce equation for $\hat{u}_{\text{red},p} = G \left( \hat{h}_{\text{red},p} \right)$

$$
\hat{u}_{\text{red},p}(k, \tau) = F(\hat{u}_{\text{red},p})(k, \tau) + \hat{h}_{\text{red},p}(k).
$$

(7.60)

Applying Theorems 5.6 and 6.3 we infer similarly to (7.41) the inequality

$$
\| \hat{u}_{\text{red},p} - v_{\text{red},p} \|_E \leq C_2 \varrho + C_2 \beta^s.
$$

(7.61)

Finally, from (7.59) and (7.61) we infer (7.56).

**Acknowledgment:** Effort of A. Babin and A. Figotin is sponsored by the Air Force Office of Scientific Research, Air Force Materials Command, USAF, under grant number FA9550-04-1-0359.

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