Quantization of borderline Levi conjugacy classes of orthogonal groups.

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Abstract

We construct equivariant quantization of a special family of Levi conjugacy classes of the complex orthogonal group $SO(N)$, whose stabilizer contains a Cartesian factor $SO(2) \times SO(P)$, $1 \leq P < N$, $P \equiv N \mod 2$.

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1 Introduction

This continuation of [1] is devoted to equivariant quantization of a special family of conjugacy classes in the complex algebraic group $G = SO(N)$. This work completes construction of quantum semisimple conjugacy classes of $SO(N)$ and, generally, of all simple groups of the infinite series. Classes of our present concern have isotropy subgroups with a Cartesian factor $SO(2) \times SO(P)$, where $P$ is of the same parity as $N$. Due to the isomorphism $GL(1) \simeq SO(2)$, they form a borderline between the Levi and non-Levi families, whose bulk cases have been processed in [1, 2, 3].

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A solution of the classical Yang-Baxter equation makes \( G \) a Poisson group with the Drinfeld-Sklyanin Poisson structure on it. It also gives rise to another Poisson bracket on \( G \) making it a Poisson manifold over \( G \) with respect to the conjugacy transformation. This Poisson structure restricts to any conjugacy class of \( G \). We construct a quantization of its polynomial algebra along that structure, which is equivariant under the action of the quantized universal enveloping algebra \( U_q(\mathfrak{g}) \). In the present paper we deal with the standard or Drinfeld-Jimbo classical \( r \)-matrix and the standard quantum group \( U_q(\mathfrak{g}) \). The constructed quantization can be automatically generalized for all other factorizable \( r \)-matrices on \( G \). For details, the reader is referred to [1].

Observe that semisimple conjugacy classes in \( SO(N) \) can be categorized by their sets of eigenvalues: whether they include both \( \pm 1 \) or not. The stabilizer subgroup of the second type is Levi, and such a class is isomorphic to an adjoint orbit in \( \mathfrak{so}(N) \) as an affine variety. Their quantization has been constructed in [2]. The stabilizer of the first type contains a Cartesian factor \( SO(2m) \times SO(P) \), where \( 2m \) and \( P \) are the multiplicities of the eigenvalues \(-1 \) and \(+1 \), respectively. If \( m \geq 2 \) (one should also assume \( P \geq 4 \) for even \( N \)), the subgroup \( L \) is not Levi. Such classes have been quantized in [1]. The remaining classes corresponding to \( m = 1 \) form a special family, which was not covered before.

The quantization method of the borderline Levi classes is similar to that used in [1] and [2]: a realization of its quantized polynomial algebra in a \( U_q(\mathfrak{g}) \)-module of highest weight. In the case of interest, it is a parabolic Verma module of special weight. Due to this constrain, it is not a deformation of a Verma module over \( U(\mathfrak{g}) \). The boundary classes were not covered in [2] because the analysis was based on the properties of the Shapovalov form derived by deformation arguments from its classical counterpart. The specialization of the highest weight in our present approach requires a special study of the module \( \mathbb{C}^N \otimes M_\lambda \) carried out in this paper.

Consider the borderline class \( O \) passing through the diagonal matrix \( o \) with entries

\[
\mu_1, \ldots, \mu_1, \ldots, \mu_\ell, \ldots, \mu_\ell, -1, 1, \ldots, 1, -1, 1, \ldots, 1, -1, 1, \ldots, 1, -1, 1, \ldots, 1, -1, 1, \ldots, 1, -1, 1, \ldots, 1,
\]

where \( P = 2m \) if \( N = 2n \) and \( P = 2m + 1 \) if \( N = 2n + 1 \). The complex numbers \( \{\mu_i\}_{i=1}^\ell \) and \( \mu_{\ell+1} = -1, \mu_{\ell+2} = 1 \) satisfy the conditions \( \mu_i \neq \mu_j \pm 1 \) for \( i < j \leq \ell \) and \( \mu_i^2 \neq 1 \) for \( 1 \leq i \leq \ell \). The centralizer of the point \( o \in G \) is the subgroup

\[
L = GL(n_1) \times \ldots \times GL(n_\ell) \times SO(2) \times SO(P),
\]

(1.1)
whose Lie algebra $l$ is a Levi subalgebra in $g$,

\[ l = \mathfrak{gl}(n_1) \oplus \cdots \oplus \mathfrak{gl}(n_\ell) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(P). \]

The subgroup $L$ is determined by an integer valued vector $n = (n_i)_{i=1}^{\ell+2}$ subject to $\sum_{i=1}^{\ell+2} n_i = n$. We reserve the integer $l$ for $\sum_{i=1}^{\ell} n_i$, so that $l + 1 + p = n$. Here $n_{\ell+1} = 1$ and $n_{\ell+2} = p$.

Let $\mathcal{M}_L$ denote the moduli space of conjugacy classes with the fixed isotropy subgroup (1.1), regarded as Poisson spaces as fixed in [1]. We introduce the subspace $\mathcal{M}'_L$ of classes with $\mu_{\ell+1} = -1$. The sets of all $\ell + 2$-tuples $\mu$ as specified above parameterize $\mathcal{M}_L$ and $\mathcal{M}'_L$ although not uniquely. We denote these sets by $\hat{\mathcal{M}}_L$ and, respectively, $\hat{\mathcal{M}}'_L$.

As a variety, the class $O$ associated with $\mu$ and $n$ is determined by the set of equations

\[
(A - \mu_1) \cdots (A - \mu_\ell)(A + 1)(A - 1)(A - \mu_1^{-1}) \cdots (A - \mu_\ell^{-1}) = 0, \tag{1.2}
\]

\[
\text{Tr}(A^k) = \sum_{i=1}^{\ell} n_i (\mu_i^k + \mu_i^{-k}) + 2(-1)^k + P, \quad k = 1, \ldots, N, \tag{1.3}
\]

where the matrix multiplication in the first line is understood. This system is polynomial in the matrix entries $A_{ij}$ and defines an ideal of $\mathbb{C}[\text{End}(\mathbb{C}^N)]$ vanishing on $O$.

**Theorem 1.1.** The system of polynomial relations (1.2) and (1.3) generates the defining ideal of the class $O$ in $\mathbb{C}[\text{SO}(N)]$.

**Proof.** The proof is similar to [3], Theorem 2.3. \qed

Our goal is a generalization of this statement for the quantized polynomial algebra of $O$.

### 2 Parabolic Verma module $M_\lambda$

We adopt certain conventions concerning representations of quantum groups, which are similar to [1]. Unless otherwise stated, the quantum group $U_q(g)$ and its modules are considered over the complex field, upon specialization of $q$ to not a root of unit. Extension of the ring of scalars via $q = e^{\hbar}$ determines the embedding $U_q(g) \subset U_h(g)$, where the latter is considered over the ring $\mathbb{C}[\hbar]$ of formal power series in $\hbar$. We assume that $U_h(g)$-modules are free over $\mathbb{C}[\hbar]$ and their rank will be referred to as dimension. Finite dimensional $U_h(g)$-modules are deformations of their classical counterparts, and we drop the reference to $\hbar$ to simplify notation. For instance, the natural $N$-dimensional representation of $U_h(g)$ will be denoted simply by $\mathbb{C}^N$. 

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Let $U_h(h)$ be the Cartan subalgebra in $U_h(g)$. We shall deal with $U_h(h)$-diagonalizable, i.e. weight modules. If $V$ is an $h$-invariant subspace, we mean by $[V]_\alpha$ the subspace of weight $\alpha \in h^*$. We stick to the additive parametrization of weights facilitated by the embedding $U_q(h) \subset U_h(h)$. Under this convention, weights belong to $\frac{1}{i} h^*[h]$ and are well defined on $t_i^{\pm 1} \in q^h$. It is sufficient for our needs to confine them to the subspace $h^{-1} h^* \oplus h^* \subset h^{-1} h^*[h]$.

We denote by $\mathfrak{c}_l \subset h$ the center of $l$ and realize its dual $\mathfrak{c}_l^*$ as a subspace in $h^*$ thanks to the canonical inner product. Let $p^+ = l + g_+ \subset g$ denote the parabolic subalgebra, where $g_+$ are the subalgebras generated by the positive and negative Chevalley generators. An element $\lambda \in \mathfrak{c}_l^* = h^{-1} \mathfrak{c}_l^* \oplus \mathfrak{c}_l^*$ defines a one-dimensional representation of $U_q(l)$ denoted by $C_\lambda$. Its restriction to $U_q(h)$ acts by the assignment $g^{\pm h_{\alpha}} \mapsto g^{\pm (\alpha, \lambda)}$, $\alpha \in \Pi^+$. Since $q = e^h$, the pole in $\lambda$ is compensated, and the representation is correctly defined. It extends to $U_q(p^+)$ by setting it zero on $g_+ \subset p^+$. Denote by $M_\lambda$ the parabolic Verma module $U_q(g) \otimes_{U_q(p^+)} C_\lambda$, \[2\]. Regarded as a $U_q(g_-)$-module by restriction from $U_q(g)$, $M_\lambda$ is isomorphic to the quotient $U_q(g_-)/U_q(g_-)l_-$, which we denote by $U^-_l$.

The vector space $C^N$ is regarded as a $U_q(g)$-module supporting its natural representation. Of key importance for us is the structure of the tensor product $C^N \otimes M_\lambda$. The element $R_{12}R$ expressed through the universal R-matrix $R \in U_h(g) \otimes U_h(g)$ operates on $C^N \otimes M_\lambda$ as an invariant matrix $Q \in \text{End}(C^N) \otimes U_q(g)$, which commutes with $\Delta(x)$ for all $x \in U_q(g)$. The normal form of $Q$ is determined by the submodule structure of $C^N \otimes M_\lambda$, the study of which takes the majority of this paper. The eigenvalues of $Q$ are found in [2]. It is also known that $Q$ is semisimple for generic $\lambda \in \mathfrak{c}_l^*$. Then we are going to check that $Q$ remains semisimple for a certain set of $\lambda$ of our interest.

Let $\{\varepsilon_i\}_{i=1}^N$ be the weights of the natural $U_q(g)$-module $C^N$. Then $\{\varepsilon_i\}_{i=1}^n$, $n = \lfloor \frac{N}{2} \rfloor$ (the integer part of $\frac{N}{2}$), form an orthogonal basis in $h^*$, and $\varepsilon_i = -\varepsilon_{N+1-i}$. The positive roots are expressed through $\{\varepsilon_i\}_{i=1}^n$ in the standard way as fixed in [1]. Denote by $w_i \in C^N$ the standard basis elements of weight $\varepsilon_i$, $i = 1, \ldots, N$. The natural $U_q(g)$-module splits into irreducible $U_q(l)$-modules,

$$C^N = (C^{n_1} \oplus \cdots \oplus C^{n_\ell}) \oplus C \oplus C^P \oplus C \oplus (C^{n_{\ell+1}} \oplus \cdots \oplus C^{n_1}),$$

which decomposition is compatible with the basis $\{w_i\}_{i=1}^N = \cup_{i=1}^{2\ell+3} \{w_k\}_{k=m_i}^{m_i-1}$ counting from the left. Here $m_i = n_1 + \cdots + n_{i-1} + 1$ for $i = 1, \ldots, \ell + 2$, and $m_{2\ell+4-i} = N + 1 - \sum_{k=1}^{i} n_k$, $i = 1, \ldots, \ell$. Note that $w_{m_i}$ is the highest weight vector of the corresponding irreducible $l$-submodule in $C^N$.

For $\lambda \in \mathfrak{c}_l^*$, the operator $Q \in \text{End}(C^N \otimes M_\lambda)$ satisfies the equation $\prod_{i=1}^{2\ell+3}(Q - x_i) = 0$.
with the roots
\[ x_i = q^{2(\lambda, \varepsilon_k) - 2(m_i - 1)}, \quad i = 1, \ldots, \ell + 2, \]
\[ x_{2\ell+4-i} = q^{-2(\lambda, \varepsilon_k) - 2N + 2(m_i + m_k)}, \quad i = 1, \ldots, \ell + 1, \] (2.5)
see [2], Theorem 4.2. The root \( x_i \) corresponds to a submodule \( M_i \subset \mathbb{C}^{N} \otimes M_\lambda \), where \( Q \) acts as multiplication by \( x_i \). For generic \( \lambda \in \mathfrak{c}_l^* \) and \( q \), the roots \( x_i \) are pairwise distinct, and \( \mathbb{C}^{N} \otimes M_\lambda = \oplus_{i=1}^{2\ell+3} M_i \).

In this paper, we are interested in special \( \lambda \) making \( x_{\ell+1} = q^{2(\lambda, \varepsilon_{\ell+1}) - 2l} \) equal to \( x_{\ell+3} = q^{-2(\lambda, \varepsilon_{\ell+1}) - 2l - 2P} \). In particular, this condition is satisfied if
\[ q^{2(\lambda, \varepsilon_{\ell+1})} = -q^{-P}. \] (2.6)

Let \( \mathfrak{c}_{\ell,r}^* \) be the subset of all weights \( \lambda \in \mathfrak{c}_l^* \) subject to (2.6). We prove that, for generic \( \lambda \in \mathfrak{c}_{\ell,r}^* \) and generic \( q \) including \( q \to 1 \), the direct sum decomposition of \( \mathbb{C}^{N} \otimes M_\lambda \) still holds, and the operator \( Q \) is semisimple. To this end, we study the submodules \( M_{\ell+1} \) and \( M_{\ell+3} \) and show that their sum is direct for all \( \lambda \) satisfying (2.6). Our analysis is based on calculation of singular vectors generating \( M_{\ell+1} \) and \( M_{\ell+3} \).

As in [2], we introduce a subspace of weights that we use for the parametrization of \( \mathcal{M}_{\ell,r} \), the moduli space of borderline conjugacy classes with fixed \( L \). Put \( \mu_k^0 = e^{2(\lambda, \varepsilon_k)} \), for \( k = 1, \ldots, \ell + 2 \). The subset \( \mathfrak{c}_{\ell,r}^* \subset \mathfrak{c}_l^* \) is specified by the condition \( \mu_0^{\ell+1} = -1 \). Let \( \mathfrak{c}_{\ell,r}^{*,reg} \) denote the set of all weights \( \lambda \in \mathfrak{c}_{\ell,r}^* \) such that \( \mu^0 \in \mathcal{M}_L \) and similarly define \( \mathfrak{c}_{\ell,r}^{*,reg'} \subset \mathfrak{c}_{\ell'}^* \), by the requirement \( \mu^0 \in \mathcal{M}_{L'} \). Finally, we introduce \( \mathfrak{c}_{\ell,r}^{*,reg'} = \mathfrak{c}_{\ell,r}^{*,reg} \cap (h^{-1}\mathfrak{c}_{\ell,r}^{*,reg'} \oplus \mathfrak{c}_l^*) \). The subset \( \mathfrak{c}_{\ell,r}^{*,reg'} \) is dense in \( \mathfrak{c}_{\ell,r}^{*} \).

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3 On singular vectors in \( \mathbb{C}^{N} \otimes M_\lambda \)

In this section, \( l \) is the Levi subalgebra \( \mathfrak{h} + \mathfrak{so}(P) \), which can be otherwise put as \( \ell = l \). The parabolic Verma module \( M_\lambda \) is relative to this subalgebra. In other words, \( \lambda \in \mathfrak{c}_l^* \) if and only if \( (\lambda, \varepsilon_i) = 0 \), \( i = l + 2, \ldots, n \).

Given weight \( \lambda \in \frac{1}{\ell} \mathfrak{h}^* \oplus \mathfrak{h}^* \) we denote \( \lambda_i = (\lambda, \varepsilon_i) \), for all \( i = 1, \ldots, N \). The natural representation of \( U_q(\mathfrak{g}) \) on \( \mathbb{C}^{N} \) is determined by the action \( f_{\varepsilon_j - \varepsilon_k} w_i = (-1)^{s_i} \delta_{ij} w_k, e_{\varepsilon_j - \varepsilon_k} w_i = (-1)^{s_i} \delta_{ij} w_j \), for \( \varepsilon_j - \varepsilon_k \in \Pi^+ \), where \( \varepsilon_i = 0 \) if \( i \leq \frac{N}{2} \) and \( \varepsilon_i = 1 \) otherwise. Note that Chevalley generators are normalized so that their representation matrices are independent of \( q \).

For \( \mathfrak{g} = \mathfrak{so}(2n + 1) \), the natural representation is determined, up to scalar multipliers, by the graph
Proof. Choose a weight basis \( \{ w_i \} \). Since Chevalley generators, so that \( \psi \) scalar multiplier. Such \( \psi \) is unique, which is obvious for odd \( N \) and still true for even \( N \), since \( f_{\alpha_n} f_{\alpha_{n-1}} = f_{\alpha_{n-1}} f_{\alpha_n} \). We denote this monomial \( \psi_{ji} \) and write \( i \prec j \). This makes the integer interval \([1, N]\) a poset with the Hasse diagram above.

In what follows, we also use the monomial \( \psi^{ij} \) obtained from \( \psi_{ji} \) by reversing the order of Chevalley generators, so that \( v^i = \psi^{ij} v^j \). We also put \( \psi^{ii} = 1 \) for all \( i \). It is clear that \( \psi^{ij} = \psi^{im} \psi^{mj} \) for any \( m \) such that \( i \preceq m \preceq j \).

**Definition 3.1.** We call \( \psi^{ij} \), \( i \preceq j \), the principal monomial of weight \( \varepsilon_j - \varepsilon_i \).

Remark that all Chevalley monomials of weight \( \varepsilon_j - \varepsilon_i \) are obtained from \( \psi^{ij} \) by permutation of factors.

Recall that a non-zero weight vector \( v \) in a \( U_q(\mathfrak{g}) \)-module is called singular if it generates the trivial \( U_q(\mathfrak{g}_+) \)-submodule, i.e. \( e_\alpha v = 0 \), for all \( \alpha \in \Pi^+ \). Since the weights of \( e_\alpha v \) are pairwise distinct, this is equivalent to the equation \( Ev = 0 \), where \( E = \sum_{m=1}^n e_{\alpha_m} \). We will also work with the operator \( E' = \sum_{m=2}^n e_{\alpha_m} \), in view of Corollary 3.3 below.

**Lemma 3.2.** Let \( W \) be a finite dimensional \( U_q(\mathfrak{g}) \)-module and \( W^* \) its right dual module. Let \( Y \) be a \( U_q(\mathfrak{g}) \)-module. Singular vectors in \( W \otimes Y \) are parameterized by homomorphisms \( W^* \rightarrow Y \) of \( U_q(\mathfrak{g}_+) \)-modules.

Proof. Choose a weight basis \( \{ w_i \} \). Suppose that \( u \in W \otimes Y \) is a singular vector, \( u = \sum_{i=1}^d w_i \otimes y_i \), for some \( y_i \in Y \). Let \( \pi : U_q(\mathfrak{g}) \rightarrow \text{End}(W) \) denote the
representation homomorphism, \( \pi(u)w_i = \sum_{j=1}^{N} \pi(u)_{ij}w_j \). We have, for \( \alpha \in \Pi^+ \),

\[
e_{\alpha}u = \sum_{i=1}^{d} \sum_{j=1}^{d} \pi(e_{\alpha})_{ij}w_j \otimes y_i + \sum_{i=1}^{d} q^{(\alpha, \varepsilon_i)}w_i \otimes e_{\alpha}y_i.
\]

(3.7)

So \( e_{\alpha}u = 0 \) is equivalent to \( e_{\alpha}y_i = -q^{(\alpha, \varepsilon_i)} \sum_{j=1}^{d} \pi(e_{\alpha})_{ji}y_j \). The vector space \( \text{Span}\{y_i\}_{i=1}^{d} \) supports the right dual representation of \( U_q(g_+) \), provided \( y_i \) are linear independent. In general, it is a quotient of the right dual representation.

Formula (3.7) can be more explicitly rewritten as

\[
y_j = (-1)^{\varepsilon_i+1}q^{(\varepsilon_i-\varepsilon_j, \varepsilon_i)}e_{\varepsilon_i-\varepsilon_j}y_i
\]

for all \( i, j \in [1, N] \) such that \( \varepsilon_i - \varepsilon_j \in \Pi^+ \). In the following corollary, \( M_\lambda \) is a parabolic Verma module relative to arbitrary \( l \).

**Corollary 3.3.** Singular vectors \( \{u_i\} \in \mathbb{C}^N \otimes M_\lambda \) are parameterized by weight elements \( y \in M_\lambda \) satisfying \( e_{\alpha_1}^3y = 0 \) if \( N = 3 \), \( e_{\alpha_1}^2y = e_{\alpha_2}^2y = 0 \) if \( N = 4 \) and \( e_{\alpha_1}^2y = E'y = 0 \) for \( N > 4 \).

**Proof.** The weight \( \varepsilon_1 \) is integral dominant. The dual natural representation of \( U_q(g) \) is generated by the vector of lowest weight \(-\varepsilon_1\). When restricted to \( U_q(g_+) \), it is isomorphic to a quotient of the left regular \( U_q(g_+) \)-module. It is the quotient by the left ideal in \( U_q(g_+) \) generated by \( e_{\alpha_1}^3 \) if \( N = 3 \), by \( e_{\alpha_1}^2, e_{\alpha_2}^2 \) if \( N = 4 \), and by \( e_{\alpha_1}^2, e_{\alpha_i}, i = 2, \ldots, n \) if \( N > 4 \). Therefore, all homomorphisms from the co-natural module to \( M_\lambda \) are generated by the assignment \( U_q(g_+) \ni 1 \rightarrow y \in M_\lambda \), where \( y \) satisfies the hypothesis.

Singular vectors generate \( U_q(l) \)-submodules of highest weight. It is known that, for generic \( \lambda \), singular vectors in \( \mathbb{C}^N \otimes M_\lambda \) are parameterized by the highest weights \( \nu \) of the irreducible \( U_q(l) \)-submodules in \( \mathbb{C}^N \) and carry the weights \( \lambda + \nu \). We denote by \( u_j \) the singular vector of weight \( \lambda + \varepsilon_j, j = 1, \ldots, N \), which is defined up to a non-zero scalar factor. We can write

\[
u_j = \sum_{j=1}^{N} w_j \otimes y_{ji}, \quad j = 1, \ldots, N,
\]

where \( y_j \in M_\lambda \) is an element of weight \( \lambda + \varepsilon_j - \varepsilon_i, i \leq j \). For each \( j \) the linear span \( \{y_{j,i}\}_{i=1}^{j} \) supports a quotient of the co-natural representation of \( U_q(g_+) \), which is cyclicly generated by \( \{y_{j,1}\} \).
Singular vectors \( u_i, i = 1, \ldots, n - 1 \), are related to the subalgebra \( \mathfrak{gl}(n) \subset \mathfrak{g} \) and can be found in [4]. Singular vector \( u_{n+1} \) in the case of \( \mathfrak{g} = \mathfrak{so}(2n) \) is related to another copy of \( \mathfrak{gl}(n) \) with \( \alpha_{n-1} \) replaced by \( \alpha_n \). Singular vector \( u_{n+1} \) for \( \mathfrak{g} = \mathfrak{so}(2n + 1) \) can be constructed as follows. Define the ”dynamical root vectors” \( f_{\varepsilon_k} \) by setting \( f_{\varepsilon_n} = f_{\alpha_n} \) and

\[
f_{\varepsilon_{k+1}} = f_{\alpha_{k+1}} f_{\varepsilon_k} [h_{\varepsilon_k} + n - k + 1] - f_{\varepsilon_k} f_{\alpha_{k+1}} [h_{\varepsilon_k} + n - k] q
\]

for all \( k = n - 1, \ldots, 1 \). It is also convenient to put \( f_0 = 1 \) taking into account \( \varepsilon_{n+1} = 0 \). Let \( M_\lambda \) be a Verma module and \( v_\lambda \) its canonical generator. One can check the identity

\[
e_{\alpha_k} f_{\varepsilon_i} v_\lambda = \delta_{ki} [\lambda_i + n - i] f_{\varepsilon_{i+1}} v_\lambda,
\]

by induction on \( i \). Setting \( y_{n+1, i} = f_{\varepsilon_i} v_\lambda \), one obtains \( y_{n+1, i} = (-q)^{i-1} \prod_{k=1}^{i-1} [\lambda_i + n - k] f_{\varepsilon_i} v_\lambda, \)

\( i = 1, \ldots, n + 1 \).

We need not know all singular vectors for the purpose of this study. We are especially interested in \( u_{N-l} \) carrying the weight \( \lambda - \varepsilon_{l+1} \). It is expanded over the basis \( \{ w_i \}_{i=1}^{N} \subset \mathbb{C}^N \) as \( u_{N-l} = \sum_{i=1}^{N-l} w_i \otimes y_i \) with coefficients \( y_i = y_{N-l, i} \) of weight \( \lambda - \varepsilon_i - \varepsilon_{l+1}, i = 1, \ldots, l + 1 \). They are generated by \( y_1 \) via the co-natural action of \( U_q(\mathfrak{g}_+) \). We call \( y_1 \) the generating coefficient. Our next goal is to evaluate \( y_1 \).

Consider the graph corresponding to the co-natural representation of \( U_q(\mathfrak{g}_+) \) for \( N > 3 \).

One can readily write down \( y_i \) for \( l + 2 \leq i \leq N - l \), up to a scalar factor. Indeed, the corresponding weight spaces in \( M_\lambda \) have dimension 1. Suppose that \( \psi^{i,N-l} = f_{\alpha_i} \psi^{j,N-l} \) for \( \alpha = \varepsilon_i - \varepsilon_j \in \Pi^+ \) (for odd \( N \), \( j \) is always \( i - 1 \), while for even \( N \) \( j \) may be also \( i - 2 \) for \( i = n + 1, n + 2 \)). Then \( e_{\alpha_i} \psi^{i,N-l} v_\lambda \sim \psi^{i+1,N-l} v_\lambda \) and \( y_i \sim \psi^{i,N-l} v_\lambda \):

\[
y_{N-l} \sim v_\lambda, \quad y_{N-l-1} \sim f_{l+1} v_\lambda, \quad y_{N-l-2} \sim f_{l+2} f_{l+1} v_\lambda, \quad \ldots
\]

In particular, \( y_{l+1} \sim f_{l+1} \cdots f_{n-1} f_n f_{n-1} \cdots f_{l+1} v_\lambda \) for odd \( N \) and a similar expression with \( f_{n-1} f_n \) in place of \( f_n \) for even \( N \).

The problem essentially boils down to finding \( y_i \) with \( i \leq l + 1 \). These coefficients feature the following chain property. Let \( \mathfrak{g}_i' \subset \mathfrak{g} \) denote the subalgebra with simple roots \( \{ \alpha_j \}_{j=i}^n \) and let \( M_{l,\lambda}' \subset M_{\lambda} \) be the \( U_q(\mathfrak{g}_i') \)-submodule generated by \( v_\lambda \). If \( y_i \in M_{l,\lambda}' \), then \( y_i \) is the generating coefficient for a \( U_q(\mathfrak{g}_i') \)-singular vector in \( \mathbb{C}^{N-2i+2} \otimes M_{l,\lambda}' \), as follows from the representation graph. This observation enables construction of \( y_i \) by descending induction starting from \( y_{l+1} \in M_{l+1,\lambda}' \), which is done in the next section.
3.1 Symmetric classes

In this section, we fix \( l = 0 \) or equivalently \( n = 1 + p \). This assumption corresponds to the symmetric conjugacy class of matrices with eigenvalues \(-1\) and \(+1\) of multiplicities \(2\) and \(P\), respectively. The singular vector of interest has weight \( \lambda + \varepsilon_{l+1} = \lambda - \varepsilon_1 \).

We introduce the following basis in \([U^-_1]_{-2\varepsilon_1}\). Observe that \(d_P^0 = \dim[U^-_1]_{-2\varepsilon_1}\) is \(p + 1\) for odd \(P\) and \(p\) for even \(P\) (recall that \(P \equiv N \mod 2\) is the multiplicity of \(+1\) in the spectrum of the conjugacy class). Define monomials \(\phi_m, m = 1, \ldots, d_P^0\), by

\[
\phi_m = \begin{cases}
    f_{\alpha_m} \cdot f_{\alpha_1} f_{\alpha_{m+1}} \cdot f_{\alpha_{p+1}} f_{\alpha_{p+1}} \cdot f_{\alpha_1}, & 1 \leq m \leq p + 1 \\
    f_{\alpha_{m-1}} \cdot f_{\alpha_1} f_{\alpha_{m+1}} \cdot f_{\alpha_{p+1}} f_{\alpha_{p+1}} \cdot f_{\alpha_1}, & 1 \leq m \leq p - 1 \\
    f_{\alpha_{p-1}} \cdot f_{\alpha_1} f_{\alpha_{p+1}} f_{\alpha_{p+1}} \cdot f_{\alpha_1}, & m = p \\
    f_{\alpha_{p-1}} \cdot f_{\alpha_1} f_{\alpha_{p+1}} \cdot f_{\alpha_1}, & m = p + 1
\end{cases}
\]

for odd \(N\) and

\[
\phi_m = \begin{cases}
    f_{\alpha_m} \cdot f_{\alpha_1} f_{\alpha_{m+1}} \cdot f_{\alpha_{p+1}} f_{\alpha_{p+1}} \cdot f_{\alpha_1}, & 1 \leq m \leq p - 1 \\
    f_{\alpha_{p-1}} \cdot f_{\alpha_1} f_{\alpha_{p+1}} f_{\alpha_{p+1}} \cdot f_{\alpha_1}, & m = p \\
    f_{\alpha_{p-1}} \cdot f_{\alpha_1} f_{\alpha_{p+1}} \cdot f_{\alpha_1}, & m = p + 1
\end{cases}
\]

for even \(N\).

All \(\phi_m\) have weight \(-2\varepsilon_1\). Using the Serre relations, one can check for even \(N\) that \(\phi_{p+1} = f_{\alpha_p} f_{\alpha_{p-1}} \cdot f_{\alpha_1} f_{\alpha_{p+1}} f_{\alpha_{p+1}} \cdot f_{\alpha_1} = \phi_p\), so the number of independent \(\phi_m\) is equal to \(d_P^0 = \dim[U^-_1]_{-2\varepsilon_1}\). Still it is convenient to consider both \(\phi_p\) and \(\phi_{p+1}\).

The leftmost position in all \(\phi_m\) is occupied by \(f_{\alpha_m}\). We define vectors \(\phi'_m\) of weight \(-2\varepsilon_1 + \alpha_m\) obtained from \(\phi_m\) by deleting \(f_{\alpha_m}\):

\[
\phi'_m = f_{\alpha_{m-1}} \cdot f_{\alpha_1} f_{\alpha_{m+1}} \cdot f_{\alpha_{p+1}} f_{\alpha_{p+1}} \cdot f_{\alpha_1}
\]

for odd \(N\), and

\[
\phi'_m = f_{\alpha_{p-1}} \cdot f_{\alpha_1} f_{\alpha_{p+1}} f_{\alpha_{p+1}} \cdot f_{\alpha_1}, \quad m = p
\]

for even \(N\).

Abusing notation, we will also identify \(\phi_m\) and \(\phi'_m\) with their images in the quotient \(U^-_1\).

Lemma 3.4. The monomial \(\phi'_m\) spans \([U^-_1]_{-2\varepsilon_1 + \alpha_m}\) for each \(m = 1, \ldots, p + 1\).

*Proof.* One can check that \(\dim[U^-_1]_{-2\varepsilon_1 + \alpha_m} = 1\), so to prove the statement, we must prove that \(\phi'_m \neq 0\). The squared norm \(\langle \phi'_m v_\lambda, \phi'_m v_\lambda \rangle\) with respect to the Shapovalov form on \(M_\lambda\) is equal to \([\lambda_1]_q\) for \(m = 1\) and to \([\lambda_1]_q[\lambda_1 - 1]_q\) otherwise. It is not zero if \([\lambda_1]_q[\lambda_1 - 1]_q \neq 0\). Due to the isomorphism \(M_\lambda \simeq U^-_1\), \(\phi'_m \neq 0\) as well as its projection in \(U^-_1\) for generic \(\lambda\). But \(\phi'_m\) is independent of \(\lambda\), which completes the proof. \(\Box\)

Let \(\Phi^0\) denotes the linear span of \(\{\phi_m v_\lambda\}_{m=1}^{d_P^0} \subset M_\lambda\). Denote by \(\hat{E}\) the composition \(\mathbb{C}^{d_P^0} \to \Phi^0 \to M_\lambda\) of linear maps, \((A_m) \mapsto \sum_{m=1}^{d_P^0} A_m \phi_m = y \mapsto Ey\). For \(N \geq 5\), the operator \(\hat{E}\) acts on \(\mathbb{C}^{d_P^0}\) by \((A_m)_{m=1}^{d_P^0} \mapsto \sum_{m=1}^{p+1} B_m \phi'_m v_\lambda\), where the scalar coefficients \(B_m\) are given in Appendix A.
Lemma 3.5. Suppose that \( N \geq 5 \). Then the map \( \hat{E} \) is injective for generic \( \lambda \).

Proof. Define

\[
A_m = (-1)^{m-1} \left[ \frac{P}{2} - m + 1 \right] q, \quad m = 1, \ldots, d^0_P. \tag{3.8}
\]

For \( N \geq 5 \), one can check that (3.8) is a unique solution of the system of equations \( B_i = 0 \), \( i = 2, \ldots, p + 1 \), up to a common scalar factor. This makes \( B_1 = A_1[\lambda_1]_q + A_2[\lambda_1 - 1]_q \) into \( [\lambda_1 + \frac{P}{2} - 1]_q \), which does not vanish for generic \( \lambda \).

\[ \square \]

Corollary 3.6. a) The system \( \{ \phi_m v_\lambda \}_{m=1}^{d^0_P} \) forms a basis in \( [M_\lambda]_{-2 \xi_1} \). b) The vector \( f_{2\xi_1}^{(P)} v_\lambda = \sum_{m=1}^{d^0_P} A_m \phi_m v_\lambda \), where \( A_m \) are given by (3.8), is a generating coefficient. c) It is a unique generating coefficient of weight \( \lambda - 2 \xi_1 \), up to a scalar factor.

Proof. The statement is obvious for \( N = 3, 4 \) with \( p = 0 \) and, respectively, \( p = 1 \). Then \( d^0_P = 1 \) and the vectors \( f_{2\xi_1}^{(1)} v_\lambda = [\frac{1}{2}] q f_{\alpha_1}^2 v_\lambda, f_{2\xi_1}^{(2)} v_\lambda = f_{\alpha_1} f_{\alpha_2} v_\lambda \) satisfy the conditions \( e_{\alpha_1}^2 f_{2\xi_1}^{(1)} v_\lambda = 0 \) and \( e_{\alpha_1}^2 f_{2\xi_1}^{(2)} v_\lambda = e_{\alpha_2}^2 f_{2\xi_1}^{(2)} v_\lambda = 0 \), as required.

Now suppose that \( N \geq 5 \). Since the operator \( \hat{E} \) is injective, the map \( \mathbb{C}^{d^0_P} \to \Phi^0 \) is injective too. It is surjective by construction, hence it is a bijection. For generic \( \lambda \), the vectors \( \{ \phi_m v_\lambda \}_{m=1}^{d^0_P} \) form a basis in \( \Phi^0 \) and hence in \( [M_\lambda]_{-2 \xi_1} \), as the latter has dimension \( d^0_P \). The vectors \( \{ \phi_m \}_{m=1}^{d^0_P} \) form a basis in \( [U^-_{\xi_1}]_{-2 \xi_1} \), due to the linear isomorphism \( [M_\lambda]_{\mu} \simeq [U^-_{\xi_1}]_{\mu - \lambda} \). These vectors are independent of \( \lambda \), hence they form a basis at all \( \lambda \), as well as \( \{ \phi_m v_\lambda \}_{m=1}^{d^0_P} \). This implies that \( f_{2\xi_1}^{(P)} v_\lambda \neq 0 \), and it is a unique generating coefficient, up to a scalar factor.

\[ \square \]

3.2 The case \( l = 1 \)

To keep reference to the symmetric case considered in the previous section, we enumerate the simple roots \( \Pi_0 = \{ \alpha_i \}_{i=0}^{p+1} \). Then the roots \( \{ \alpha_i \}_{i=1}^{p+1} \) correspond to the subalgebra \( U_q(\mathfrak{g}') \subset U_q(\mathfrak{g}) \). Under this embedding, we regard \( \phi_m \) and \( f_{2\xi_1}^{(P)} \) constructed in the previous section as elements of \( U_q(\mathfrak{g}) \).

Observe that \( d^0_P = \dim[M_\lambda]_{\lambda - \xi_0 - \xi_1} \) is equal to \( 3p + 3 \) for odd \( N \) and \( 3p + 1 \) for even \( N \). The only generator which does not commute with \( f_{\alpha_0} \) is \( f_{\alpha_1} \), and it enters \( \phi_m \) twice. There are three possible ways to allocate \( f_{\alpha_0} \) relative to these \( f_{\alpha_1} \). We use this observation to construct the basis in \( [U^-_{\xi_1}]_{-\xi_0 - \xi_1} \) from the basis in \( [U^-_{\xi_1}]_{-2 \xi_1} \). For all \( m = 1, \ldots, p + 1 \), define \( \phi^1_m = f_{\alpha_0} \phi_m \) and \( \phi^3_m = \phi_m f_{\alpha_0} \). Define \( \phi^2_m \) to be the monomial obtained from \( \phi_m \) by replacing the rightmost copy of \( f_{\alpha_1} \) with \( f_{\alpha_0} f_{\alpha_1} \). For even \( N \), the equality \( \phi_{p+1} = \phi_p \) implies
$\phi_{p+1}^1 = \phi_p^1$ and $\phi_{p+1}^3 = \phi_p^3$, so we have effectively $3p + 1$ monomials for even $N$ and $3p + 3$ monomials for odd $N$.

As in the symmetric case, for all $m \in [1, p + 1]$ we define $\phi_m^i \in U_q(\mathfrak{g}_-)$ of weight $-\varepsilon_0 - \varepsilon_1 + \alpha_m$ by deleting the leftmost copy of $f_{\alpha_m}$ from $\phi_m^i$. Note that $\phi_1^1 = \phi_2^2$ and, for even $N$, $\phi_{p+1}^1 = \phi_p^1$, $\phi_{p+1}^3 = \phi_p^3$. Put $r_m = 2$ for $m = 1$ and $r_m = 1$ for $m > 1$.

**Lemma 3.7.** For all $m = 1, \ldots, p + 1$, the vectors $\{\phi_m^i v_\lambda\}_{i=r_m} \subset [U^-]_{-\varepsilon_0 - \varepsilon_1 + \alpha_m}$ are linearly independent.

**Proof.** One can check that the Gram matrix of the system $\{\phi_m^i v_\lambda\}_{i=r_m}$ with respect to the Shapovalov form on $M_\lambda$ is

$$
\begin{pmatrix}
[\lambda_1 q \lambda_0 - \lambda_1 + 1]_q & [\lambda_1 q \lambda_0 - \lambda_1 + 1]_q \\
[\lambda_1 q \lambda_0 - \lambda_1]_q & [\lambda_1 q \lambda_0 - \lambda_1]_q \\
[\lambda_1 q \lambda_0 - \lambda_1 + 2]_q & [\lambda_1 q \lambda_0 - \lambda_1 + 1]_q & [\lambda_1 q \lambda_0 - \lambda_1]_q \\
[\lambda_1 q \lambda_0 - \lambda_1]_q & [\lambda_1 q \lambda_0 - \lambda_1]_q & [\lambda_1 q \lambda_0 - \lambda_1]_q \\
[\lambda_1 q \lambda_0 - \lambda_1 + 1]_q & [\lambda_1 q \lambda_0 - \lambda_1 + 1]_q & [\lambda_1 q \lambda_0 - \lambda_1]_q \\
[\lambda_1 q \lambda_0 - \lambda_1]_q & [\lambda_1 q \lambda_0 - \lambda_1]_q & [\lambda_1 q \lambda_0 - \lambda_1]_q \\
\end{pmatrix},
$$

$m = 1, 2, \ldots, n$, for either parity of $N$. Its determinant is equal to

$$
\begin{align*}
[\lambda_0 - \lambda_1]_q [\lambda_1]_q [\lambda_0 + 1]_q, & \quad m = 1, \\
[\lambda_0 - \lambda_1]_q [\lambda_1]_q [\lambda_1]_q [\lambda_0 + 1]_q, & \quad m = 2, \ldots, p + 1.
\end{align*}
$$

It does not vanish for generic $\lambda$, hence $\{\phi_m^i v_\lambda\}_{i=r_m}$ are linearly independent. This is also true for all $\lambda$, since $\phi_m^i$ are independent of $\lambda$. \hfill $\Box$

All $\phi_m^i v_\lambda$ are annihilated by $e_{\alpha_0}^2$, as $f_{\alpha_0}$ enters only once. Therefore their linear combination annihilated by $e_{\alpha_i}, i > 1$, is a generating coefficient.

Present $\mathbb{C}^{d_p} = \mathbb{C}^{p+1} \oplus \mathbb{C}^{p+1} \oplus \mathbb{C}^{p+1}$ for odd $N$ and $\mathbb{C}^{d_p} = \mathbb{C}^{p} \oplus \mathbb{C}^{p+1} \oplus \mathbb{C}^{p}$ for even $N$. Let the upper index of $(A_m^i) \in \mathbb{C}^{d_p}$ label the summand in this decomposition while the lower index mark the coordinate within this summand.

Denote by $\hat{E}$ the composition $\mathbb{C}^{d_p} \to \Phi^1 \to M_\lambda$ of linear maps acting by $(A_m^i) \mapsto \sum_{m,i} A_i^i \phi_m^i = y \mapsto Ey$. It acts by $\hat{E}$: $(A_m^i) \mapsto \sum_{i=r_m} B^i_m \phi_m^i v_\lambda$, where the scalar factors $B^i_m$ are given in Appendix. Define $\bar{f}_{\alpha_0 + \varepsilon_1}^{(p)} = \sum_{m,i} A^{i}_m \bar{\phi}^{i}_m$, where $A^i_m$ are as follows:

$$
A^k_m = \begin{cases}
(-1)^{m+1} \lambda_1 + P - m]_q [\lambda_1 + \frac{P}{2}]_q, & k = 1, \\
(-1)^{m} q^{-m \frac{P}{2} + q^{-m - \frac{P}{2}}} [\lambda_1 + \frac{P}{2} - 1]_q [\lambda_1 + \frac{P}{2}]_q, & k = 2, \\
(-1)^{m+1} \lambda_1 + m - 1]_q [\lambda_1 + \frac{P}{2} - 1]_q, & k = 3,
\end{cases}
$$

for $m = 1, \ldots, d_p^1$ apart from $A^2_{p+1}, A^3_{p+1}$ for even $N$, which are set to $(-1)^{p}[\lambda_1 + \frac{P}{2} - 1]_q [\lambda_1 + \frac{P}{2}]_q$.  

Lemma 3.8. Up to a scalar factor, the vector \( f_{\varepsilon_0+\varepsilon_1}^{(P)} v_\lambda \) is a unique solution of the system \( e_{\alpha_i} f_{\varepsilon_0+\varepsilon_1}^{(P)} v_\lambda = 0 \) for all \( i = 1, \ldots, p+1 \). Furthermore, \( e_{\alpha_0} f_{\varepsilon_0+\varepsilon_1}^{(P)} v_\lambda = [\lambda_0 + \lambda_1 + P]_q f_{2\varepsilon_1}^{(P)} v_\lambda \).

Proof. The first part of the statement is proved by a lengthy straightforward calculation, which is omitted here. Let us prove the second statement. Observe the identities

\[
\sum_{i=1}^{3} A_m^i [\lambda_0 - \lambda_1 + 3 - i]_q = [\lambda_0 + \lambda_1 + P]_q A_m,
\]

which hold for \( m = 1, \ldots, p + 1 \), odd \( N \), and for \( m = 1, \ldots, p - 1 \), even \( N \). This readily implies the statement for odd \( N \):

\[
e_{\alpha_0} f_{\varepsilon_0+\varepsilon_1}^{(P)} = \sum_{m=1}^{p+1} \sum_{i=1}^{3} A_m^i e_{\alpha_0} \phi_m^i v_\lambda = \sum_{m=1}^{p+1} \left( \sum_{i=1}^{3} A_m^i [\lambda_0 - \lambda_1 + 3 - i]_q \phi_m v_\lambda \right) = [\lambda_0 + \lambda_1 + P]_q f_{2\varepsilon_1}^{(P)}.
\]

If \( N \) is even, we have also

\[
\sum_{i=1}^{3} A_p^i [\lambda_0 - \lambda_1 + 3 - i]_q + A_{p+1}^2 [\lambda_0 - \lambda_1 + 1] = [\lambda_0 + \lambda_1 + P]_q A_p.
\]

Then, for even \( N \),

\[
e_{\alpha_0} f_{\varepsilon_0+\varepsilon_1}^{(P)} v_\lambda = \sum_{m=1}^{p} \sum_{i=1}^{3} A_m^i e_{\alpha_0} \phi_m^i v_\lambda + A_{p+1}^2 e_{\alpha_0} \phi_{p+1}^2 v_\lambda = \sum_{m=1}^{p-1} \left( \sum_{i=1}^{3} A_m^i [\lambda_0 - \lambda_1 + 3 - i]_q \phi_m v_\lambda \right) + \left( \sum_{i=1}^{3} A_p^i [\lambda_0 - \lambda_1 + 3 - i]_q + A_{p+1}^2 [\lambda_0 - \lambda_1 + 1] \right) \phi_p v_\lambda = [\lambda_0 + \lambda_1 + P]_q f_{2\varepsilon_1}^{(P)},
\]

as required. \( \square \)

Proposition 3.9. The vectors \( \phi_m^i \) form a basis in \([U_1^-]_{-\varepsilon_0-\varepsilon_1} \). Up to a scalar factor, \( f_{\varepsilon_0+\varepsilon_1}^{(P)} v_\lambda \) is a unique generating coefficient of the weight \( \lambda - \varepsilon_0 - \varepsilon_1 \).

Proof. Observe that \( d_p^i \) is equal to the dimension of \([U_1^-]_{-\varepsilon_0-\varepsilon_1} \), so we need to prove only linear independence. Fix a constant \( c \) and restrict \( \lambda \) to the hyperplane \( \lambda_1 = c \). By Lemma 3.8, the map \( \tilde{E} : \mathbb{C}^{d_p^i} \to \Phi^1 \to M_\lambda \) is injective for all \( \lambda \) such that \([\lambda_0 + c + 2n - 1]_q \neq 0 \). Since the map \( \mathbb{C}^{d_p^i} \to \Phi^1 \) is surjective, the map \( E : \Phi^1 \to M_\lambda \) is injective too. This implies that \( \phi_m^i v_\lambda \) are linearly independent for such \( \lambda \). Since \( \phi_m^i \) are independent of \( \lambda_0 \), they are linearly independent at all \( \lambda \) subject to \( \lambda_1 = c \), and so are \( \phi_m^i v_\lambda \). As \( c \) is arbitrary, the statement holds true for all \( \lambda \). \( \square \)
3.3 The case \( l = 2 \)

In order to relate our calculation to already considered cases \( l = 0, 1 \), we enumerate the roots as \( \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_{p+1} \). We are looking for the generating coefficient of weight \( \lambda - \varepsilon_{-1} - \varepsilon_1 \).

It is an element of \( M_\lambda \) satisfying the equations \( e_{\alpha_{-1}}y = e_{\alpha_j}y = 0, \ j \geq 0 \).

Define the element

\[
 f^{(P)}_{\varepsilon_{-1} + \varepsilon_1} = f_{\alpha_{-1}}f^{(P)}_{\varepsilon_0 + \varepsilon_1}[h_{\varepsilon_0} + h_{\varepsilon_1} + P + 1]_q - f^{(P)}_{\varepsilon_0 + \varepsilon_1}f_{\alpha_{-1}}[h_{\varepsilon_0} + h_{\varepsilon_1} + P]_q \in U_q(\mathfrak{b}_-), \quad (3.9)
\]

of weight \( -\varepsilon_{-1} - \varepsilon_0 - \varepsilon_1 \).

**Proposition 3.10.** The element \( f^{(P)}_{\varepsilon_{-1} + \varepsilon_1}v_\lambda \in M_\lambda \) is a unique generating coefficient of weight \( -\varepsilon_{-1} - \varepsilon_0 - \varepsilon_1 \). Furthermore,

\[
 e_{\alpha_{-1}}f^{(P)}_{\varepsilon_{-1} + \varepsilon_1}v_\lambda = [\lambda_{-1} + \lambda_1 + P + 1]_q f^{(P)}_{\varepsilon_0 + \varepsilon_1}v_\lambda.
\]

**Proof.** We are looking for the generating coefficient in the form

\[
 y = \sum_{m,k} (A_m^{(1)}f_{\alpha_{-1}}\varphi^k_m - A_m^{(2)}\varphi^k_m f_{\alpha_{-1}}),
\]

where \( (A_m^{(1)}), (A_m^{(2)}) \in \mathbb{C}^{d_P} \). Since \( f_{\alpha_{-1}}\varphi^k_m \) and \( \varphi^k_m f_{\alpha_{-1}} \) are independent, the conditions \( e_{m}f^{(P)}_{\varepsilon_{-1} + \varepsilon_1}v_\lambda = 0 \) for positive \( m \) give \( A_m^{(j)} = A_m^{(j)}C^j \) for some scalars \( C^j, j = 1, 2 \). That is, \( y = C^1 f_{\alpha_{-1}}f^{(P)}_{\varepsilon_0 + \varepsilon_1}v_\lambda - C^2 f^{(P)}_{\varepsilon_0 + \varepsilon_1}f_{\alpha_{-1}}v_\lambda \).

The coefficients \( C^1, C^2 \) are found from the condition \( e_{\alpha_0}y = \sum_{m=1}^n E_m f_{\alpha_{-1}}\varphi_m = 0 \), where \( E_m \) are equal to

\[
 \left( A_m^{(1)}[\lambda_0 - \lambda_1 + 2]_q + A_m^{(2)}[\lambda_0 - \lambda_1 + 1]_q + A_m^{(3)}[\lambda_0 - \lambda_1]_q \right) C^1 - \left( A_m^{(1)}[\lambda_0 - \lambda_1 + 3]_q + A_m^{(2)}[\lambda_0 - \lambda_1 + 2]_q + A_m^{(3)}[\lambda_0 - \lambda_1 + 1]_q \right) C^2.
\]

This boils down to \( m \) equations \( E_m = 0 \) on \( C^i \). One can check that system is consistent and \( C^1 = [\lambda_0 + \lambda_1 + P + 1]_q, \ C^2 = [\lambda_0 + \lambda_1 + P]_q \), up to a common scalar factor. Thus, \( y = f^{(P)}_{\varepsilon_{-1} + \varepsilon_1}v_\lambda \) is a generating coefficient. \(\square\)

3.4 Generating coefficients for arbitrary \( l \geq 0 \)

Now we return to the usual enumeration of simple roots, \( \alpha_1, \ldots, \alpha_n \). The algebra \( \mathfrak{g} = \mathfrak{so}(2l + 2 + P) \) includes the subalgebra \( \mathfrak{so}(6 + P) \) via the assignment \( \alpha_i \mapsto \alpha_{l+i} \), i.e.

\[
\alpha_{-1} \mapsto \alpha_{l-1}, \quad \alpha_0 \mapsto \alpha_l, \quad \ldots, \quad \alpha_{p+1} \mapsto \alpha_{l+p+1} = \alpha_n.
\]
Under this embedding, \( f_{\xi_{i+1}+\xi_{l+1}}^{(P)}(\lambda), \ i = 1, 2, 3, \) become elements of \( U_q(\mathfrak{g}_-) \) of weights \(-\varepsilon_{l+2-i} - \varepsilon_{l+1}\). The subalgebra \( \mathfrak{so}(6 + P) \) corresponds to already considered case \( l = 2 \).

Define an element \( f_{\xi_{l-1} + \xi_{l+1}}^{(P)} \in U_q(\mathfrak{b}_-) \) by setting
\[
f_{\xi_{l-1} + \xi_{l+1}}^{(P)} = f_{\alpha_l-1}f_{\xi_{l-1}+\xi_{l+1}}^{(P)}[h_{\xi_l} + h_{\xi_{l+1}} + P + 1]q - f_{\xi_{l+1} + \xi_{l+1}}^{(P)}f_{\alpha_l-1}[h_{\xi_l} + h_{\xi_{l+1}} + P]q, \tag{3.11}
\]
so that \( f_{\xi_{l-1} + \xi_{l+1}}^{(P)}(\lambda) \) is indeed the evaluation of \( f_{\xi_{l-1} + \xi_{l+1}}^{(P)} \) at the point \( \lambda \in \mathfrak{h}^* \). Observe that
\[
e_{\alpha_k}f_{\xi_{k+1} + \xi_{l+1}}^{(P)}v_\lambda = [\lambda_k + \lambda_{l+1} + P + l - k]qf_{\xi_{k+1} + \xi_{l+1}}^{(P)}v_\lambda,
\]

once \( k = l - 1, l \). Suppose we have defined \( f_{\xi_{k+1} + \xi_{l+1}}^{(P)} \) for some \( k \in [1, l - 1] \). Then put
\[
f_{\xi_{k+1} + \xi_{l+1}}^{(P)} = f_{\alpha_k}f_{\xi_{k+1} + \xi_{l+1}}^{(P)}[h_{\xi_{k+1}} + h_{\xi_{l+1}} + P + l - k]q - f_{\xi_{l+1} + \xi_{l+1}}^{(P)}f_{\alpha_k}[h_{\xi_{k+1}} + h_{\xi_{l+1}} + P + l - k - 1]q.
\]

**Proposition 3.11.** The vectors \( f_{\xi_{k+1} + \xi_{l+1}}^{(P)}v_\lambda \in M_\lambda \) satisfy the equations
\[
e_{\alpha_j}f_{\xi_{k+1} + \xi_{l+1}}^{(P)}v_\lambda = \delta_{jk}[\lambda_k + \lambda_{l+1} + P + l - k]qf_{\xi_{k+1} + \xi_{l+1}}^{(P)}v_\lambda, \quad k = 1, \ldots, l,
\]
\[
e_{\alpha_j}f_{2\xi_{l+1}}^{(P)}v_\lambda = \delta_{j,l+1}[\lambda_{l+1} + P/2 - 1]qf_{\xi_{l+2} + \xi_{l+1}}^{(P)}v_\lambda,
\]
where \( f_{\xi_{l+2} + \xi_{l+1}} = \phi'_1 \). Then \( f_{\xi_{1} + \xi_{l+1}}^{(P)}v_\lambda \) is a unique generating coefficient of the singular vector in \( \mathbb{C}^N \otimes M_\lambda \) of weight \( \lambda - \varepsilon_1 - \varepsilon_{l+1} \).

**Proof.** The case of \( k = l - 1, l \) has been worked out in Sections 3.1 - 3.3. We suppose that the statement is proved for some \( k + 1 \leq l + 1 \) and prove it for \( k \). Clearly \( e_{\alpha_j}f_{\xi_{k+1} + \xi_{l+1}}^{(P)}v_\lambda = 0 \) for \( j > k + 1 \) by the induction assumption and \( j < k \) by construction. The element \( f_{\xi_{k+1} + \xi_{l+1}}^{(P)} \) of weight \(-\varepsilon_{k+1} - \varepsilon_{l+1}\) commutes with \( e_{\alpha_k} \) modulo \( U_q(\mathfrak{b}_-)e_{\alpha_{k-1}} \), which readily implies the formula for \( j = k \). Then the remaining equality \( e_{\alpha_{k+1}}f_{\xi_{k+1} + \xi_{l+1}}^{(P)}v_\lambda = 0 \) easily follows from the induction assumption
\[
e_{\alpha_{k+1}}f_{\xi_{k+1} + \xi_{l+1}}^{(P)} = [\lambda_{k+1} + \lambda_{l+1} + P + l - k - 1]qf_{\xi_{k+2} + \xi_{l+1}}^{(P)}v_\lambda.
\]

Finally, we argue that \( f_{\xi_{1} + \xi_{l+1}}^{(P)}v_\lambda \) does not turn zero for all \( \lambda \). We showed in Sections 3.1 - 3.3 that \( f_{\xi_{k+1} + \xi_{l+1}}^{(P)}v_\lambda \neq 0 \) for \( k = l, l + 1, l + 2 \). Assuming it is true for all \( k \leq l \), observe that \( f_{\xi_{k+1} + \xi_{l+1}}^{(P)} \) is a “modified commutator” of \( f_{\xi_{k+1} + \xi_{l+1}}^{(P)} \) with \( f_{\alpha_k} \) and that \( (\alpha_k, \varepsilon_{k+1} + \varepsilon_{l+1}) \neq 0 \). Further arguments are based on [9], Lemma 9.1, and are similar to the proof of Corollary 9.2 therein. \( \square \)
Next we determine the principal terms of the generating coefficients. This will be of importance for our further analysis. Observe that
\[
\begin{align*}
&f^{(P)}_{2\varepsilon_i+1} v_\lambda = \left[ \frac{P}{2} \right] q^{l+1,N-l} v_\lambda + \ldots, \\
&f^{(P)}_{\varepsilon_i+\varepsilon_i} v_\lambda = [\lambda_{l+1} + P] q^{l,N-l} v_\lambda + \ldots, \\
&f^{(P)}_{\varepsilon_i+\varepsilon_i+1} v_\lambda = [\lambda_{l+1} + P - 1] q^{l,N-l} v_\lambda + \ldots,
\end{align*}
\]
where \( m < l \). The omitted terms contain only non-principal monomials.

Now we can express the principal terms of the coefficients \( y_i = y_{N-l,i} \) of the singular vector \( u_{N-l} \). Introduce scalar coefficients \( c'_i \) via the equality \( y_i = c'_i \psi^{i,N-l} v_\lambda + \ldots \), where the omitted terms do not contain \( \psi^{i,N-l} v_\lambda \). Note that we have exact equality \( y_i = c'_i \psi^{i,N-l} v_\lambda \) for \( i = l+2, \ldots, N-l \). Formula (3.7) can be rewritten as
\[
y_j = (-1)^{\varepsilon_i+1} q^{(\varepsilon_i-\varepsilon_j,\varepsilon_i)} e_{\varepsilon_j-\varepsilon_i} y_i = (-1)^{\varepsilon_i+1} q^{(\varepsilon_i,\varepsilon_i)} e_{\varepsilon_j-\varepsilon_i} y_i
\]
for all \( i, j \in [1, N] \) such that \( \varepsilon_j - \varepsilon_i \in \Pi^+ \). Then
\[
\begin{align*}
c'_m &= (-q)^{m-1} [\lambda_{l+1} + P - 1] q^{l,\lambda_{l+1} + P - 1} \frac{P}{2} q^{l,\lambda_{l+1} + P - 1} \prod_{i=1}^{l,\lambda_{l+1} + P - 1} \prod_{i=m+1}^{l,\lambda_{l+1} + P - 1} [\lambda_i + \lambda_{l+1} + P + l - i + 1],
\end{align*}
\]
where \( m = 1, \ldots, l \). Assuming \( g = \mathfrak{so}(2n+1) \), we continue as
\[
\begin{align*}
c'_{l+2+k} &= (-q)^k c'_{l+2}, & k &= 1, \ldots, p, \\
c'_{n+1+k} &= (-q)^{n-l-1} q^{k-1} c'_{l+2}, & k &= 1, \ldots, p, \\
c'_{n+2+p} &= (-q)^{n-l-1} q^p c'_{l+2}, & k &= n+1, \ldots, p.
\end{align*}
\]
For \( g = \mathfrak{so}(2n) \), we have
\[
\begin{align*}
c'_{l+2+k} &= (-q)^k c'_{l+2}, & k &= 1, \ldots, p - 1, \\
c'_{n+1+k} &= (-q)^{n-l-2} q^{k} c'_{l+2}, & k &= 0, \ldots, p, \\
c'_{n+2+p} &= (-q)^{n-l-2} q^{p+1} c'_{l+2}.
\end{align*}
\]
We use these formulas in the next section.
In this section we deal with two Levi subalgebras, \( \mathfrak{l} \) and \( \hat{\mathfrak{l}} = \mathfrak{h} + \mathfrak{so}(P) \subset \mathfrak{l} \). All objects related to \( \mathfrak{l} \) will be marked with hat. In particular, \( M_\lambda \) is a parabolic Verma module induced from \( U_q(\hat{\mathfrak{l}} + \mathfrak{g}_+) \), while \( M_\lambda \) stands for the one induced from \( U_q(\mathfrak{l} + \mathfrak{g}_+) \).

Given a weight \( \lambda \in \mathfrak{c}_i^* \) define \( \hat{V}_i \subset \mathbb{C}^N \otimes \hat{M}_\lambda \) to be the submodule generated by \( \{ w_k \otimes v_\lambda \}_{k=1}^i \). The sequence \( \{0\} = \hat{V}_0 \subset \hat{V}_1 \subset \ldots \subset \hat{V}_N \) forms a filtration, \( \hat{V}_\bullet \), of \( \mathbb{C}^N \otimes \hat{M}_\lambda \). Its graded component \( \text{gr} \hat{V}_j = \hat{V}_j/\hat{V}_{j-1} \) is generated by \( (\text{the image of}) \ w_j \otimes v_\lambda \).

Now assume that \( \lambda \in \mathfrak{c}_i^* \subset \mathfrak{c}_i^* \). Recall that \( \{ w_m \}_{k=1}^{2\ell+3} \) are the highest weight vectors of the irreducible \( \mathfrak{t} \)-blocks in (2.4). Since \( \text{Span}(w_m)_{k=m_i}^{m_i-1} = U_q(\mathfrak{t}) w_m \), the image of \( \hat{V}_k \) under projection \( \mathbb{C}^n \otimes \hat{M}_\lambda \to \mathbb{C}^n \otimes M_\lambda \) coincides with the image \( V_m \) of \( \hat{V}_m \) for all \( k = m_i, \ldots, m_i - 1 \).

The sequence \( \{0\} = V_0 \subset V_1 \subset \ldots \subset V_N \) forms a filtration \( V_\bullet \) of \( \mathbb{C}^N \otimes M_\lambda \) with the graded module

\[
\text{gr} V_\bullet = (\oplus_{i=1}^l \text{gr} V_i) \oplus \text{gr} V_{l+1} \oplus \text{gr} V_{l+2} \oplus \text{gr} V_{N-l} \oplus (\oplus_{i=1}^{\ell} \text{gr} V_{l+3+i}).
\]

(4.15)

The graded components \( \text{gr} V_i = V_i/\text{gr} V_{i-1} \) are labeled with irreducible \( \mathfrak{t} \)-submodules of (2.4), and generated by the images of \( w_m \otimes v_\lambda \) carrying the highest weight \( \lambda + \varepsilon_m \).

**Proposition 4.1.** As a filtration of \( U_q(\mathfrak{g}_-) \)-modules, \( V_\bullet \) is independent of \( \lambda \in \mathfrak{c}_i^* \).

For a proof, see e.g. [4].

For generic \( \lambda \in \mathfrak{c}_i^* \), the graded component \( \text{gr} V_{m_i} \) is a parabolic Verma module induced from \( U_q(\mathfrak{t}) w_{m_i} \subset \mathbb{C}^N \), hence that is true for all \( \lambda \). The operator \( \mathcal{Q} \) is scalar on each \( \text{gr} V_{m_i} \), which is a cyclic module of highest weight \( \lambda + \varepsilon_{m_i} \). Therefore (4.15) determines the spectrum of \( \mathcal{Q} \) and a polynomial equation on \( \mathcal{Q} \). For generic \( \lambda \) this polynomial is minimal, but may not be so for special values of \( \lambda \). In particular, we are interested in \( \lambda \in \mathfrak{c}_{i,\text{reg}}^* \).

Suppose that \( i \leq j \) and fix a path from \( i \) to \( j \) on the Hasse diagram. We define \( \sum_{m=i}^j \) as summation over all nodes of that path. We shall use it only when it is path-independent.

**Proposition 4.2 ([5]).** Suppose that \( i, j \in [1, N] \) are such that \( i \prec j \). Then

\[
w_i \otimes \psi^{ij} v_\lambda = (-1)^{j-i+\sum_{k=i}^{j-1} \epsilon_k q^{(e_j-e_i,e_j)}} - \sum_{k=i+1}^{j} \epsilon_k q^{(e_k,e_k)} q^{\lambda_i - \lambda_j} w_j \otimes v_\lambda \mod V_{j-1}.
\]

(4.16)

If \( \psi \) is a Chevalley monomial of weight \( \varepsilon_j - \varepsilon_i \) and \( \psi \neq \psi^{ij} \), then \( w_i \otimes \psi v_\lambda \in V_{j-1} \).

It is also convenient to use an equivalent local version of formula (4.16):

\[
w_i \otimes \psi^{ij} v_\lambda = (-1)^{\epsilon_i+1} q^{\lambda_i - \lambda_i + (\epsilon_k - \varepsilon_i, e_j - e_k)} w_k \otimes \psi^{kj} v_\lambda \mod V_{j-1}
\]

(4.17)
where \( \varepsilon_i - \varepsilon_k = \alpha \in \Pi^+ \) is a positive simple root for some \( i, k \in [1, N] \), and \( j \geq k \). Note that (4.16) holds true for \( \mathfrak{g} = \mathfrak{gl}(n) \) and \( I = \bigoplus_{i=1}^{l+1} \mathfrak{gl}(n_i) \) via the embeddings \( U_q(\mathfrak{gl}(n)) \subset U_q(\mathfrak{so}(N)) \), \( \mathbb{C}^n \subset \mathbb{C}^N \), of algebras and their natural representations.

We consider yet another system of \( U_q(\mathfrak{g}) \)-submodules and compare it with \( \{ V_i \}_{i=1}^{l+3} \). As we mentioned, for generic \( \lambda \) the tensor product \( \mathbb{C}^N \otimes M_\lambda \) decomposes into the direct sum

\[
\mathbb{C}^N \otimes \hat{M}_\lambda = \bigoplus_{i=1}^{2l+3} \hat{M}_i, \quad \lambda \in \mathfrak{c}_1^*, \\
\mathbb{C}^N \otimes M_\lambda = \bigoplus_{i=1}^{2l+3} M_i, \quad \lambda \in \mathfrak{c}_1^*,
\]

where \( \hat{M}_i \) and \( M_i \) are generated by singular vector \( \hat{u}_i \) and, respectively, by the projection \( u_{m_i} \) of \( \hat{u}_{m_i} \) (which otherwise might turn zero). The left decomposition holds if the Shapovalov forms of \( \hat{M}_\lambda \) and all \( \hat{M}_i \) are not degenerate; the same is true for the right decomposition. The operator \( Q \) is scalar multiple on \( \hat{M}_i \) and \( M_i \) with the eigenvalues \( \hat{x}_i \) and, respectively, \( x_i = \hat{x}_{m_i} \). Denote \( W_i = \sum_{k=1}^I M_k \). For generic \( \lambda \), \( M_i \) is the parabolic Verma modules induced from the corresponding irreducible \( \mathfrak{l} \)-submodule of \( \mathbb{C}^N \). Therefore, it is independent of \( \lambda \) regarded as an \( U_q(\mathfrak{g}_-) \)-module.

**Proposition 4.3.** There is an inclusion \( W_i \subset V_i \). Further, \( W_i = V_i \) if and only if \( W_i = \bigoplus_{k=1}^I M_k \). Consequently, \( W_i = V_i \) if and only if \( W_k = V_k \) for all \( k \leq i \).

**Proof.** The last statement readily follows from the second. The inclusion \( W_i \subset V_i \) follows from Proposition 4.2. Since \( M_k \) and \( \text{gr} V_k \) are cyclic modules of the same highest weight, either the projection \( \pi_k : M_k \to \text{gr} V_k \) is zero or coincides with \( \text{gr} V_k \), which is the case for generic \( \lambda \in \mathfrak{c}_1^* \). In particular, \( M_k \) is isomorphic to \( \text{gr} V_k \) for all \( \lambda \). Denote \( M'_k = W_{k-1} \cap M_k \). For each \( k \) the projection \( \pi_k \) factorizes to the composition

\[
M_k \twoheadrightarrow M_k/M'_k \simeq W_k/W_{k-1} \twoheadrightarrow W_k/(W_k \cap V_{k-1}) \hookrightarrow \text{gr} V_k,
\]

where the left and middle arrows are surjective and the right one is injective. As argued, \( \pi_k \) is either an isomorphism or \( \pi_k = 0 \). If \( M'_k = \{0\} \) for all \( k \leq i \), then, by ascending induction on \( k \), all these maps are isomorphisms, and \( V_k = W_k \) including \( k = i \). Conversely, assuming \( V_i = W_i \), we get \( M'_i = \{0\} \) and \( V_{i-1} = W_{i-1} \). Descending induction on \( i \) completes the proof. \( \square \)

**Corollary 4.4.** For all \( j \in [1, N] \), decomposition \( W_j = \bigoplus_{i=1}^I M_i \) holds if and only if \( \pi_i(u_i) \neq 0 \) for all \( i = 1, \ldots, j \).

In particular, if the eigenvalues \( \{x_k\}_{k=1}^N \) are pairwise distinct, the sum \( W_{2\ell+3} = \bigoplus_{k=1}^{2\ell+3} M_k \) is direct, and \( W_{2\ell+3} = V_{2\ell+3} = \mathbb{C}^N \otimes M_\lambda \). However, we are interested in the situation when \( x_{\ell+1} = x_{\ell+3} \). To address this case, we need to calculate the \( \pi_{\ell+3}(u_{\ell+3}) \in \text{gr} V_{\ell+3} \).
Let $C_i$, $i = 1, \ldots, 2\ell + 3$, be the scalar coefficient in the presentation $u_i = C_i w_i \otimes v_\lambda \mod V_{i-1}$ and $\hat{C}_i$ be similarly defined for $i = 1, \ldots, 2\ell + 3$. Note that the image of $\hat{u}_i$ may turn zero in $C^N \otimes M_\lambda$, so $u_i$ is obtained from $\hat{u}_m$, after an appropriated rescaling. This implies that $C_i$ is proportional to $\hat{C}_m$, up to a factor turning zero at $\lambda \in C^*_i$.

Our next goal is to calculate $\hat{C}_i$ for some $i$ of importance. We do it first for $i = n + 1$ in the case of odd $N$. Retaining the principal term, we write

$$y_{n+1,i} = (-q)^{i-1} \prod_{k=1}^{i-1} [\lambda_k + n - k] \prod_{k=i+1}^{n} [\lambda_k + n - k + 1] q^{i,n+1} v_\lambda + \ldots$$

**Proposition 4.5.** $\hat{C}_{n+1} = \prod_{j=1}^{n} [\lambda_j + 1 + n - j] q$.

**Proof.** One can check that

$$\hat{C}_{n+1} = \sum_{i=1}^{n+1} q^{i-1} q^{-\lambda_i + i-n-\delta_{i+1}} \prod_{j=1}^{i-1} [\lambda_j + n - j] q \prod_{j=i+1}^{n} [\lambda_j + n + 1 - j] q$$

Replacing $\lambda_i$ with $\lambda_i - \lambda_{n+1}$ one gets the expression, which is shown in [4], Lemma 6.1, to be equal to $\prod_{j=1}^{n} [\lambda_j - \lambda_{n+1} + 1 + n - j] q$, for any $\lambda_i$, $i = 1, \ldots, n+1$. This proves the lemma. \[ \square \]

Next we calculate $\hat{C}_{N-1}$. First we assume $l = 0$. The coefficient $\hat{C}_N$ is $\sum_{i=1}^{N} c_i''$, where $c_1' = [\frac{P}{2}]_q$, $c_2' = -q[\lambda_1 + \frac{P}{2} - 1]_q$, and $c_i'$ for $i > 2$ are given by formulas (3.13) and (3.14) (one should put $l = 0$ there). The coefficients $c_i''$ are obtained by specialization of (4.16). For $N = 2n + 1$ they are $c_N'' = 1$ and

$$c_1'' = (-1)^{n-1} q^{-2\lambda_1 - 2n+1}, \quad c_{1+k}'' = (-1)^{n-k-1} q^{-\lambda_1 - 2n+1}, \quad c_{n+m}'' = q^{-\lambda_1 - n+m}, \quad c_{n-1}' = q^{-\lambda_1},$$

where $k = 1, \ldots, p$, $m = 1, \ldots, m + 1$. For $N = 2n$, they are $c_N'' = 1$ and

$$c_1'' = (-1)^{n-1} q^{-2\lambda_1 - 2n+2}, \quad c_{1+k}'' = (-1)^{n-k} q^{-\lambda_1 - 2n+2}, \quad c_{n+k}'' = q^{-\lambda_1 - n+k}, \quad c_{n-1}' = q^{-\lambda_1},$$

where $k = 1, \ldots, p$.

**Lemma 4.6.** In the symmetric case $l = 0$, the singular vector $\hat{u}_N$ is equal to $\hat{C}_N w_N \otimes v_\lambda \mod V_{N-1}$, where $\hat{C}_N = (-1)^{[\frac{P+1}{2}]}[\lambda_1 + \frac{P}{2}]_q[\lambda_1 + P - 1]_q$.

**Proof.** The coefficient $(-1)^{[\frac{P+1}{2}]} \hat{C}_N$ is equal to

$$q^{-2\lambda_1 - 2n+1} [\frac{P}{2}]_q + [\lambda_1 + \frac{P}{2} - 1]_q q^{-\lambda_1} \left( -\frac{q^{2n+1}}{q - q^{-1}} + q + q^{2n+1} \right) + q^{2n+1} [\lambda_1 + \frac{P}{2} - 1]_q [\lambda_1]_q$$

if $P = 2p + 1$. For $P = 2p$, it is equal to

$$q^{-2\lambda_1 - 2n+2} [\frac{P}{2}]_q + [\lambda_1 + \frac{P}{2} - 1]_q q^{-\lambda_1} q^{-\lambda_1} \left( -\frac{q^{2n+1}}{q - q^{-1}} + q + q^{2n+1} \right) + q^{2n} [\lambda_1 + \frac{P}{2} - 1]_q [\lambda_1]_q.$$

Counting the coefficients before $q^{\pm 2\lambda_1}$ and $\lambda$-independent terms proves the statement. \[ \square \]
Now consider the general case \( l > 0 \).

**Proposition 4.7.** The singular vector \( \hat{u}_{N-l} \) is equal to \( \hat{C}_{N-l} w_{N-l} \otimes v_\lambda \) modulo \( \hat{V}_{N-1} \), where

\[
\hat{C}_{N-l} = (-1)^{\lfloor \frac{l+1}{2} \rfloor + l} [\lambda_{l+1} + \frac{P}{2} q] \prod_{j=1}^{l+2} [\lambda_j + \lambda_{l+1} + P + 1 + l - j]_q.
\]

**Proof.** The second sum in the expansion \( \hat{u}_{N-l} = \sum_i w_i \otimes y_i + \sum_{i=l+1}^l w_i \otimes y_i \) can be replaced with \( (-q)^l \prod_{i=1}^l [\lambda_i + \lambda_{l+1} + P + l - i]_q \hat{C}_{N-l} w_{N-l} \otimes y_{N-l} \) mod \( \hat{V}_{N-l-1} \), where the factor before \( \hat{C}_{N-l} \) comes from a different normalization of \( c'_l \) and \( c'_1 \) in Lemma 4.6. We have

\[
c'_i = (-1)^{\lfloor \frac{l+1}{2} \rfloor} (-q)^{l-i} [\lambda_l + \lambda_{l+1} + P + l - j]_q \prod_{j=1}^{l+1} [\lambda_j + \lambda_{l+1} + P + l - j + 1]_q
\]

and \( c''_i = (-1)^{\lfloor \frac{l+1}{2} \rfloor - i + 1} q^{-l+1-i} q^{-\lambda_i - \lambda_{l+1}} \) for \( i = 1, \ldots, l \). Note with care that \( c''_i = -q^{-2} q^{-\lambda_i + \lambda_{l+1}} c''_{l+1} \). Summing up the products

\[
c'_i c''_i = (-1)^l q^{-l+1} q^{-\lambda_i - \lambda_{l+1}} [\lambda_l + \lambda_{l+1} + P + l - j]_q \prod_{j=1}^{l+1} [\lambda_j + \lambda_{l+1} + P + l - j + 1]_q
\]

from \( i = 1 \) to \( i = l \) and adding \( (-q)^l \prod_{i=1}^l [\lambda_i + \lambda_{l+1} + P + l - i]_q \hat{C}_{N-l} \) one gets \( \hat{C}_{N-l} (-1)^l \) times the right-hand side of (4.18), where one should replace \( n \) with \( l \) and \( \lambda_i \) with \( \lambda_i + \lambda_{l+1} + P \) for \( i = 1, \ldots, l \). Finally, since \( \lambda_{l+2} = 0 \), the factor \( [\lambda_l + P - 1]_q \) is included in the product as \( [\lambda_j + \lambda_{l+1} + P + 1 + l - j]_q, j = l+2 \).

The operator \( Q \) satisfies on \( \mathbb{C}^N \otimes \hat{M}_\lambda \) the polynomial equation \( \prod_{i=1}^{2l+3} (Q - \hat{x}_i) = 0 \). When projected to \( \text{End}(\mathbb{C}^N \otimes M_\lambda) \), it satisfies the equation \( \prod_{i=1}^{2l+3} (Q - x_i) = 0 \), where \( x_i = \hat{x}_{m_i} \). Denote by \( \hat{C}_{l+3} \) the product of \( \hat{x}_{l+1} - \hat{x}_k \) over all \( k \leq l \) such that \( k \neq m_i \), \( i = 1, \ldots, \ell \). Put \( C_{l+3} = \frac{\hat{C}_{l+3}}{\hat{C}_{l+3}} \). Using arguments similar to [4], Lemma 6.6, one can prove that the image of \( u_{l+3} = \frac{1}{C_{l+3}} \hat{u}_{l+3} \) in \( \mathbb{C}^N \otimes M_\lambda \) is regular in \( q \) and \( \lambda \in \mathfrak{c}_l^* \). Then \( u_{l+3} = C_{l+3} w_{l+3} \otimes v_\lambda \mod V_{l+2} \) is a singular vector. Similarly we define \( u_{n+1} \) for the case \( N = 2n+1, P = 1 \).

**Proposition 4.8.** Suppose that \( \lambda \in \mathfrak{c}_l^* \) and \( q \in \mathbb{C} \) are such that \( \{x_i\}_{i=1}^{2l+3} \setminus \{x_{l+3}\} \) are pairwise distinct. Then \( \mathbb{C}^N \otimes M_\lambda = \sum_{i=1}^{2l+3} M_i \).

**Proof.** All we need to check is that the sum \( M_{l+1} + M_{l+3} \) is direct. We have \( W_{l+2} = \sum_{i=1}^{l+2} M_i \) hence \( W_{l+2} = V_{l+2} \), by Proposition 4.3. Further, \( C_{l+3} \neq 0 \) implies \( W_{l+3} = V_{l+3} \), hence \( M_{l+1} \cap M_{l+3} \subset W_{l+2} \cap M_{l+3} = \{0\} \), again by Proposition 4.3.

**Corollary 4.9.** For \( \lambda \in \mathfrak{c}_l^* \), the operator \( Q \in \text{End}(\mathbb{C}^N \otimes M_\lambda) \) satisfies a polynomial equation of degree \( 2l + 2 \) with roots \( \{x_i\}_{i=1}^{2l+3} \setminus \{x_{l+3}\} \).
5  Quantization of borderline Levi classes.

Fix $\lambda \in \mathfrak{c}_{t,\text{reg}}$ and define $\mu \in \mathbb{C}^{\ell+2}[\hbar]$ by

$$\mu_i = x_i, \quad i = 1, \ldots, \ell + 2. \quad (5.19)$$

The eigenvalues of $\mathcal{Q}$ on $\text{End}(\mathbb{C}^N \otimes M_\lambda)$ are expressed through $\mu$ by

$$\mu_i, \quad \mu_i^{-1} q^{-2N+2(n_i+1)}, \quad i = 1, \ldots, \ell, \quad \mu_{\ell+1} = -q^{-N+2}, \quad \mu_{\ell+2} = q^{-N+P}, \quad (5.20)$$

cf. (2.5). By construction, $\lim_{\hbar \rightarrow 0} \mu \in \tilde{\mathcal{M}}_{K}$.

Define central elements $\tau_k \in U_q(\mathfrak{g})$ by

$$\tau_k = \text{Tr}(q^{2\hbar_k}Q^k) \in A, \quad (5.21)$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \mathbb{R}_{+}} \alpha = \sum_{i=1}^{n} (N/2 - i) \varepsilon_i$ is the half-sum of positive roots. A module $M$ of highest weight $\lambda$ determines a one dimensional representation $\chi^\lambda$ of the center of $U_q(\mathfrak{g})$, which assigns a scalar to each $\tau_k$.

$$\chi^\lambda(\tau_k) = \sum_{\nu} q^{2k(\lambda + \nu) - 2k(\rho, \varepsilon_1) + k(\nu, \nu) - k} \prod_{\alpha \in \mathbb{R}_{+}} \frac{q^{(\lambda + \nu + \rho, \alpha)} - q^{-(\lambda + \nu + \rho, \alpha)}}{q^{(\lambda + \rho, \alpha)} - q^{-(\lambda + \rho, \alpha)}}, \quad (5.22)$$

cf. [2], formula (24). The summation is taken over weights $\nu$ of the module $\mathbb{C}^N$. Restriction of $\lambda$ to $\mathfrak{c}_{t,\text{reg}}$ makes the right hand side a function of the vector $\mu$ defined in (5.19). We denote this function by $\vartheta^k_{n,q}(\mu)$, where $n = (n_1, \ldots, n_\ell, 1, p)$ is the integer valued vector of multiplicities. In the limit $\hbar \rightarrow 0$, $\vartheta^k_{n,q}(\mu)$ goes over into the right hand side of (1.3), where $\mu_i = \lim_{\hbar \rightarrow 0} q^{2(\lambda, \varepsilon_{m_i})}, i = 1, \ldots, \ell$.

In general, $\tau^k \mod \hbar$ do not separate classical conjugacy classes of $SO(2n)$. That is done by an additional invariant which nevertheless turns zero on a class with eigenvalues $\pm 1$. Its quantum counterpart $\tau^-$ yields $\chi^\lambda(\tau^-) = \prod_{i=1}^{n} (q^{2(\lambda + \rho, \varepsilon_i)} - q^{-2(\lambda + \rho, \varepsilon_i)})$, cf. [2], Proposition 7.4. It vanishes for $\lambda \in \mathfrak{c}_{t,\text{reg}}$ and can be ignored.

Denote by $S \in \text{End}(\mathbb{C}^N) \otimes \text{End}(\mathbb{C}^N)$ the product of the ordinary flip on $\mathbb{C}^N \otimes \mathbb{C}^N$ and the R-matrix of the form of [8]. It is $U_{h}(\mathfrak{g})$-invariant, i.e. commutes with $\Delta(x)$ for all $x \in U_{h}(\mathfrak{g})$. Let $\kappa \in \text{End}(\mathbb{C}^N) \otimes \text{End}(\mathbb{C}^N)$ be the one-dimensional projector to the trivial $U_{h}(\mathfrak{g})$-submodule. Denote by $\mathbb{C}_{h}[O(N)]$ the associative algebra generated by the matrix entries $A = (A_{ij})_{i,j=1}^{N} \in \text{End}(\mathbb{C}^N) \otimes \mathbb{C}_{h}[O(N)]$ modulo the relations

$$S_{12}A_{2}S_{12}A_{2} = A_{2}S_{12}A_{2}S_{12}, \quad A_{2}S_{12}A_{2}\kappa = q^{-N+1}\kappa = \kappa A_{2}S_{12}A_{2}, \quad (5.23)$$

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These relations are understood in \( \text{End}(\mathbb{C}^N) \otimes \text{End}(\mathbb{C}^N) \otimes \mathbb{C}_h[O(N)] \), and the indices distinguish the two copies of \( \text{End}(\mathbb{C}^N) \), in the usual way.

The algebra \( \mathbb{C}_h[O(N)] \) is an equivariant quantization of \( \mathbb{C}[O(N)] \). The algebra \( C_\hbar[G], \quad G = SO(N) \), is a quotient of \( \mathbb{C}_h[O(N)] \) setting a quantized determinant to 1. Its explicit form is immaterial, because it is automatically fixed by the equations of conjugacy class. The algebra \( \mathbb{C}_\hbar[G] \) can be realized as a \( U_\hbar(g) \)-invariant subalgebra in \( U_q(g) \), with respect to the adjoint action. The embedding is implemented via the assignment

\[
\text{End}(\mathbb{C}^N) \otimes \mathbb{C}_h[G] \ni A \mapsto Q \in \text{End}(\mathbb{C}^N) \otimes U_q(g).
\]

**Theorem 5.1.** Suppose that \( \lambda = C_{t,\text{reg}}^* \) and let \( \mu \) be as in (5.19). The quotient of \( \mathbb{C}_h[G] \) by the ideal of relations

\[
\prod_{i=1}^{\ell} (Q - \mu_i) \times (Q - \mu_{i+1})(Q - \mu_{i+2}) \times \prod_{i=1}^{\ell} (Q - \mu_i^{-1} q^{-2N+2(n_i+1)}) = 0,
\]

\[\text{Tr}_q(Q^k) = \varphi_{n,q}^k(\mu) \]

is an equivariant quantization of the class \( \lim_{\hbar \to 0} \mu \in \hat{M}_L \). It is the image of \( \mathbb{C}_h[G] \) in the algebra of endomorphisms of the \( U_q(g) \)-module \( M_\lambda \).

**Proof.** The proof is similar to [1], Theorem 10.1. and [2], Theorem 8.2. \( \square \)

Theorem 5.1 describes the ideal in \( \mathbb{C}_h[G] \). To describe the ideal in \( \mathbb{C}_h[O(N)] \), one should replace \( Q \) with \( A \) in (5.24) and (5.25) and add the relations (5.23).

### Appendix

**Lemma A.1.** Suppose that \( N \geq 5 \) and \( y = \sum_{m=1}^{d_p} A_m \phi_m v_\lambda \in \Phi^0 \). Then, for all \( m = 1, \ldots, p+1 \), one has \( e_m y = B_m \phi_m v_\lambda \), where the scalar factors \( B_m \) are

\[
B_1 = A_1[\lambda]_q + A_2[\lambda_1 - 1]_q,
B_i = A_{i-1} + [2]_q A_i + A_{i+1}, \quad i = 2, \ldots, d_p - 1,
B_p = B_{p+1} = A_{p-1} + [2]_q A_p, \quad \text{for even } N,
B_{p+1} = A_p + (1 + [2]_q) A_{p+1}, \quad \text{for odd } N.
\]

**Proof.** A straightforward calculation. \( \square \)
Lemma A.2. Suppose that \( y = \sum_{m,i} A^i_m x^m v_{\lambda} \in \Phi^1 \), where \( (A^i_m) \in \mathbb{C}^d \). Then, for all \( m = 1, \ldots, p + 1 \), one has \( e_{\alpha_m} y = \sum_{i=\alpha_m}^3 B^i_m x^m v_{\lambda} \), where the scalar factors \( B^i_m \) are as follows.

a.1) \( P = 3 \)

\[
B^1_2 = A^1_1([\lambda_1]_q + [\lambda_1 - 1]_q) + A^1_1[\lambda_1]_q, \quad B^1_3 = A^3_1([\lambda_1]_q + [\lambda_1 + 1]_q) + A^2_1[\lambda_1]_q,
\]

a.2) \( P = 2p + 1 \geq 5 \)

\[
\begin{align*}
B^1_1 &= A^1_1[\lambda_1]_q + A^2_1[\lambda_1 - 1]_q + A^3_1[\lambda_1 + 1]_q, \\
B^1_2 &= A^2_1[\lambda_1 + 1]_q + A^2_2[\lambda_1]_q, \\
B^k_m &= A^k_{m-1} + A^k_m [2]_q + A^k_{m+1}, \quad 2 \leq m \leq p, \\
B^i_{p+1} &= A^i_p + (1 + [2]_q) A^i_{p+1} + A^2_{p+1}, \quad i = 1, 3, \\
B^2_{p+1} &= A^2_p + A^2_{p+1}.
\end{align*}
\]

b.1) \( P = 4 \)

\[
B^1_1 = A^1_1[\lambda_1]_q + A^3_1[\lambda_1 + 1]_q, \quad B^3_1 = A^3_1([\lambda_1]_q + [\lambda_1 + 1]_q) + A^2_1[\lambda_1]_q,
\]

\[
B^1_2 = A^1_1[\lambda_1]_q + A^1_2[\lambda_1 + 1]_q, \quad B^3_2 = A^2_1[\lambda_1]_q + A^2_2[\lambda_1 + 1]_q.
\]

b.2) \( P = 2p \geq 6 \)

\[
B^k_i = A^k_{i-1} + [2]_q A^k_i + A^k_{i+1}, \quad i = 1, \ldots, p - 1, \quad \text{(A.26)}
\]

whenever the pair \((i, k)\) is distinct from specified below, in which case \( B^k_i \) are

\[
\begin{align*}
B^2_{p-1} &= A^2_{m-3} + A^2_{p-1} [2]_q + A^2_p + A^2_{p+1}, \\
B^2_p &= A^2_{p-1} + A^2_p [2]_q, \\
B^2_{p+1} &= A^2_{p-1} + A^2_{p+1} [2]_q, \\
B^i_p &= A^i_{p-1} + A^i_p [2]_q + A^2_{p+1}, \quad i = 1, 3, \\
B^i_{p+1} &= A^i_{p-1} + A^i_p [2]_q + A^2_p, \quad i = 1, 3.
\end{align*}
\]

This is verified by a straightforward brute force calculation, which is omitted here.

References

[1] Mudrov, A.: Non-Levi closed conjugacy classes of \( SO_q(N) \), J.Math.Phys, 54, 081701 (2013), http://dx.doi.org/10.1063/1.4816625

[2] Mudrov, A.: Quantum conjugacy classes of simple matrix groups, Commun. Math. Phys. 272, 635 – 660 (2007).
[3] Mudrov, A.: Non-Levi closed conjugacy classes of $SP_q(N)$, Commun.Math.Phys., 317 (2013) 317 – 345

[4] Ashton, T. and Mudrov, A.: On representations of quantum conjugacy classes of GL(n). Lett. Math. Phys., 103, 1029-1045 (2013).

[5] Ashton, T. and Mudrov, A.: Tensor product of natural and generalized Verma modules. in preparation.

[6] Drinfeld, V.: Quantum Groups. In Proc. Int. Congress of Mathematicians, Berkeley 1986, Gleason, A. V. (eds) pp. 798–820, AMS, Providence (1987).

[7] Jantzen, J. C.: Lectures on quantum groups. Grad. Stud. in Math., 6, AMS, Providence, RI (1996).

[8] Faddeev, L., Reshetikhin, N., and Takhtajan, L.: Quantization of Lie groups and Lie algebras. Leningrad Math. J., 1, 193–226 (1990).

[9] Mudrov, A.: Orthogonal basis for the Shapovalov form on $A_n$, arXiv:1206.3647.