Mahler’s question for intrinsic Diophantine approximation on triadic Cantor set: the divergence theory

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Abstract
In this paper, we consider Mahler’s question for intrinsic Diophantine approximation on the triadic Cantor set $\mathcal{K}$, i.e., approximating the points in $\mathcal{K}$ by rational numbers inside $\mathcal{K}$:

$$W_\mathcal{K}(\psi) := \{ x \in \mathcal{K} : |x - p/q| < \psi(q), \text{ for infinitely many } p/q \in \mathcal{K} \}.$$ 

By using the intrinsic denominator $q_{\text{int}}$ instead of the regular denominator $q$ of a rational $p/q \in \mathcal{K}$ in $\psi(\cdot)$, we present a complete metric theory for this variant of the set $W_\mathcal{K}(\psi)$, which yields a divergence theory of Mahler’s question.

Keywords Intrinsic Diophantine approximation · Cantor set · Mahler’s question · Khintchine-type result

Mathematics Subject Classification 11J83 · 11K55

1 Introduction

Metric Diophantine approximation studies the distribution of rational numbers in the quantitative sense. To some extent, the distribution of rational numbers in $\mathbb{R}^d$ has been well understood, since the pioneering work of Khintchine (Lebesgue measure theory) [18], Jarník (Hausdorff measure theory) [17] until the recent outstanding contributions by Beresnevich and Velani [5], Beresnevich, Dickinson and Velani [4] and Koukoulopoulos and Maynard [23].
As a further study, instead of studying the distribution of rational numbers in the whole space $\mathbb{R}^d$, one can also study it restricted to some subsets of $\mathbb{R}^d$, for example in or near fractal sets. This is called Diophantine approximation on fractals.

The study of Diophantine approximation on fractals was raised by Mahler in 1984. Throughout, let $K$ be the triadic Cantor set, $\mu$ the standard Cantor measure supported on $K$, and $\gamma = \log_3 2$ the Hausdorff dimension of $K$. In Mahler’s paper entitled ‘some suggestions for further research’ [27], Mahler wrote the following moving statement: “At the age of 80, I cannot expect to do much more mathematics. I may however state a number of questions where perhaps further research might lead to interesting results”. One of these questions was regarding intrinsic and extrinsic approximation on the Cantor set $K$. In Mahler’s words, how close can irrational elements of $K$ be approximated by rational numbers

- inside the Cantor set $K$ (intrinsic), or
- outside the Cantor set $K$ (extrinsic)?

Mahler’s question was first studied by Weiss [33], where a convergence part of the Khintchine-type theorem is given which implies almost no points in $K$ are very well approximable with respect to the Cantor measure $\mu$. This opens up the extensive attention on Mahler’s question in the last decades.

Naturally, the study goes along two lines: intrinsic case and extrinsic case. In this paper, we focus on Mahler’s first question, i.e., intrinsic Diophantine approximation on $K$. The main purpose is to characterize the size of the set

$$W_K(\psi) = \{x \in K : |x - p/q| < \psi(q), \text{ i.m. } p/q \in K\}$$

where i.m. stands for infinitely many and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive function.

The work towards how well the points in $K$ can be approximated by rational numbers in $K$ was initiated by Levesley et al. [26]. Let

$$A = \{3^n : n \geq 1\}.$$

Define

$$W_{A,K}(\psi) = \{x \in K : |x - p/q| < \psi(q), \text{ i.m. } (p, q) \in \mathbb{Z} \times A \text{ with } p/q \in K\}.$$  

The size of $W_{A,K}(\psi)$ was characterized by the following Khintchine/Jarník-type result. Recall $\gamma = \log_3 2$ is the Hausdorff dimension of $K$.

**Theorem 1.1** [26, Theorem 1] Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$. Let $f$ be a dimension function such that $f(r)/r^\gamma$ is non-decreasing. Then

$$\mathcal{H}^f(W_{A,K}(\psi)) = \begin{cases} \mathcal{H}^f(K), & \text{if } \sum_{n \geq 1} 3^{n\gamma} f(\psi(3^n)) = \infty; \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathcal{H}^f$ denotes $f$-Hausdorff measure.

Soon after, Bugeaud [9] presented explicit examples of points in $K$ with any given approximation order.

**Theorem 1.2** [9, Theorem 2] For any $\nu \geq 2$, there exist uncountable many points in $K$ with exact approximation (irrationality) order $\nu$.

It should be mentioned that to simulate the possible distributions of rational numbers near $K$, Bugeaud and Durand [10] presented a probabilistic model by allowing random translations of $K$ and established the dimension theory. Their result may give a clue to Mahler’s question.
The result in Theorem 1.1 can give a divergence result for Mahler’s first question. However, as will be seen later, their applications to Mahler’s first question can be improved. The main reason is that the rational numbers \( p/q \in \mathcal{K} \) with \( q \in \mathcal{A} \) are just those with terminating 3-adic expansions, only a subfamily of rational points in \( \mathcal{K} \) (and they are well spaced). Therefore to attack Mahler’s first question, one has to be clear of the distributions of all (at least most) rational numbers in \( \mathcal{K} \), which is the core ingredient for intrinsic Diophantine approximation on \( \mathcal{K} \).

A simple observation is that every rational number \( p/q \in \mathcal{K} \) can be expressed in 3-adic expansion as an ultimately periodic series, denoted by

\[
p/q = [\epsilon_1, \ldots, \epsilon_\ell, (\epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m})^\infty], \quad \text{with } \epsilon_i \in \{0, 2\}.
\]  

By calculation, the number in the right-hand side of (1.1) can be written as a fraction

\[
\frac{p}{q} = \frac{p^*}{3^\ell(3^m-1)}.
\]  

At the current stage, the biggest obstacle in achieving a complete answer to Mahler’s first question is that the rational in the right-hand side of (1.2) is not reduced. Its reduced form depends on the length of the period of the 3-adic expansion of \( p/q \) and the divisors of \( 3^m-1 \) (see last section) which is far out of reach currently.

So instead of the denominator in the reduced form, Fishman and Simmons [15] use \( 3^\ell(3^m-1) \) given in (1.2), denoted by \( q_{\text{int}} \), as the height of \( p/q \) and call it the intrinsic denominator of \( p/q \). This leads to a variant form of Mahler’s first question: consider the size of the set

\[
\mathcal{W}_{\text{int}, \mathcal{K}}(\psi) := \left\{ x \in \mathcal{K} : |x - p/q| < \psi(q_{\text{int}}), \text{ i.m. } p/q \in \mathcal{K} \right\}.
\]  

\textbf{Theorem 1.3} [15, Theorem 4.12] Let \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a non-increasing positive function. Then

\[
\mu\left(\mathcal{W}_{\text{int}, \mathcal{K}}(\psi)\right) = \begin{cases} 
0, & \text{if } \sum_{n \geq 1} n(3^n \psi(3^n))^{\gamma} < \infty; \\
1, & \text{if } \sum_{n \geq 1} \frac{1}{n} \left( \frac{\psi(3^n)}{\log(3^n \psi(3^n))} \right)^{\gamma} = \infty.
\end{cases}
\]

The above two series deciding the size of \( \mathcal{W}_{\text{int}, \mathcal{K}}(\psi) \) are inconsistent with each other. In this paper, we give the following dichotomy law.

\textbf{Theorem 1.4} Let \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a non-increasing positive function. Then

\[
\mu\left(\mathcal{W}_{\text{int}, \mathcal{K}}(\psi)\right) = \begin{cases} 
0, & \text{if } \sum_{n \geq 1} n(3^n \psi(3^n))^{\gamma} < \infty; \\
1, & \text{if } \sum_{n \geq 1} n(3^n \psi(3^n))^{\gamma} = \infty.
\end{cases}
\]

Consequently, letting \( f \) be a dimension function such that \( f(r)/r^{\gamma} \) is non-decreasing, we have

\[
\mathcal{H}^f\left(\mathcal{W}_{\text{int}, \mathcal{K}}(\psi)\right) = \begin{cases} 
\mathcal{H}^f(\mathcal{K}), & \text{if } \sum_{n \geq 1} n^{3^n \gamma} f(\psi(3^n)) = \infty; \\
0, & \text{otherwise}.
\end{cases}
\]

As said before, the core ingredient of Mahler’s first question is about how well the rational numbers in \( \mathcal{K} \) are distributed. We use a completely different way from the method used in proving Theorem 1.3: we apply a natural mechanism to generate all rational numbers in \( \mathcal{K} \) and then consider the distribution of these rationals by proving a counting lemma about “well
"spaced" rational numbers in $\mathcal{K}$. The latter is the essential part of this paper which enables us to enhance the known results.

Since $\psi$ is non-increasing and $q \leq q_{\text{int}}$ for a rational number $p/q$, one has that

$$\mathcal{W}_{\text{int}, \mathcal{K}}(\psi) \subset \mathcal{W}_{\mathcal{K}}(\psi).$$

Thus we have the following divergence theory for Mahler’s first question.

**Corollary 1.5** Let $\psi$ be a non-increasing positive function. When $\sum_{n \geq 1} n(3^n \psi(3^n))^{1/\gamma} = \infty$, we have

$$\mu(\mathcal{W}_{\mathcal{K}}(\psi)) = 1,$$

i.e., almost all points in $\mathcal{K}$ are intrinsically $\psi$-well approximable.

The series for the divergence part in Theorem 1.3 is more restrictive than our series. For example, they fail to work for the function

$$\psi(n) = \frac{1}{n[(\log n)^2 \log \log n]^{1/\gamma}}, \quad n \in \mathbb{N}. \quad (1.3)$$

We would also like to compare the function in (1.3) with the following Dirichlet-type result presented by Broderick, Fishman and Reich [8].

**Theorem 1.6** [8, Theorem 2 and Corollary 2] For any $x \in \mathcal{K}$ and any $Q$ of the form $3^n$, there exists $p/q \in \mathcal{K}$ with $1 \leq q \leq 3Q^{1/\gamma}$ such that

$$|x - p/q| < \frac{1}{qQ}.$$  \hspace{1cm} (1.4)

Consequently, there are infinitely many integer pairs $(p, q)$ with $p/q \in \mathcal{K}$ such that

$$|x - p/q| < \frac{1}{q(\log_3 q)^{1/\gamma}}. \quad (1.4)$$

The function $\psi$ in (1.3) has one more log-factor than the function in (1.4) which coincides with the case in the classic Diophantine approximation in $\mathbb{R}$ when passing from Dirichlet’s theorem to Khintchine’s theorem. It seems reasonable to conjecture that the series given in Theorem 1.4 should be the right series governing the measure of intrinsic Diophantine approximation in $\mathcal{K}$.

To conclude this section, we give a brief account on other achievements related to Mahler’s question beyond the intrinsic case, however Mahler’s question is still open.

The result of Weiss [33] was extended to fractals supporting ‘friendly’ measures, see [20, 21, 28]. There are also rich results on the size of badly approximable points on fractals, for example, the measure theory in [13, 31], the dimension theory in [22, 25], the winning properties in [6, 7, 14]. The setting of Levesley et al. [26] was also generalized to conformal iterated function systems by Baker [3] and Allen and Bárány [2]. In [16], Fishman and Simmons considered the extrinsic Diophantine approximation, i.e., Mahler’s second question. The approximation of points in $\mathcal{K}$ by algebraic numbers or by dyadic rational numbers was studied by Kristensen [24] and Allen, Chow and Yu [1] respectively. In [19] Khalil and Luethi established the Khintchine’s theorem on self-similar set with large Hausdorff dimension; in [34] Yu established measure and dimension results for fractals with large $l_1$-dimension. However, the latter two results are not applicable to the case of triadic Cantor set.

The paper is organized as follows. We fix some notation in the next section, and prove the convergence theory in Sect. 3. After proving a counting lemma in Sect. 4, we show the divergence theory in Sect. 5. In the last section, we give some words on the potential convergence theory for Mahler’s first question.
2 Preliminaries

2.1 Notation

We begin with some notation. Throughout the paper,

- all rationals \( p/q \) are in the reduced form.
- the expansion of a real number is in the base 3. We admit that a point has multiple expansions, however, as far as the expansions of rationals in \( \mathbb{K} \) are concerned, we always choose the expansions with digits in \( \mathcal{E} = \{0, 2\} \). Also, all the digits appearing below, e.g., \( \epsilon_i, w_j \), lie in \( \mathcal{E} \).
- Given a finite word \( (\epsilon_1, \ldots, \epsilon_n) \) and an integer \( t \), we write \( (\epsilon_1, \ldots, \epsilon_n)_t \) for the word \( (\epsilon_1, \ldots, \epsilon_n, \ldots, \epsilon_1, \ldots, \epsilon_n) \) generated by concatenation of \( t \) copies of \( (\epsilon_1, \ldots, \epsilon_n) \); \( (\epsilon_1, \ldots, \epsilon_n)^\infty \) for the infinite periodic sequence \( (\epsilon_1, \ldots, \epsilon_n, \epsilon_1, \ldots, \epsilon_n, \ldots) \). Moreover, for \( M = tn + r \), we write \( (\epsilon_1, \ldots, \epsilon_n)_{M/n} \) for the prefix with length \( M \) of the sequence \( (\epsilon_1, \ldots, \epsilon_n)^\infty \).
- \( a \ll b \) if \( a \leq cb \) for some unspecified constant \( c > 0 \).
- \( \# \): a cardinality of a finite set.
- \( \operatorname{dist}(A, B) \): the distance of two sets in a metric space.
- \( \lfloor \xi \rfloor \): integer part of \( \xi \).
- \( \lceil \xi \rceil \): the smallest integer no less than \( \xi \).
- For a point \( x \in \mathbb{K} \), write its 3-adic expansion as
  \[ x = \frac{\epsilon_1}{3} + \frac{\epsilon_2}{3^2} + \cdots + \frac{\epsilon_n}{3^n} + \cdots. \]
  Call \( (\epsilon_i)_{i \geq 1} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots) \) the digit sequence of \( x \) and write the above series simply as
  \[ x = [\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots], \]
  and write
  \[ [\epsilon_1, \ldots, \epsilon_\ell] = \frac{\epsilon_1}{3} + \frac{\epsilon_2}{3^2} + \cdots + \frac{\epsilon_\ell}{3^\ell}. \]
  For a rational number \( p/q \) with ultimately periodic expansion, say
  \[ \frac{p}{q} = \frac{\epsilon_1}{3} + \cdots + \frac{\epsilon_\ell}{3^\ell} + \frac{\epsilon_{\ell+1}}{3^{\ell+1}} + \cdots + \frac{\epsilon_{\ell+m}}{3^{\ell+m}} + \frac{\epsilon_{\ell+1}}{3^{\ell+m+1}} + \cdots + \frac{\epsilon_{\ell+2m}}{3^{\ell+2m}} + \cdots, \quad (2.1) \]
  we write it as
  \[ \frac{p}{q} = [\epsilon_1, \ldots, \epsilon_\ell, (\epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m})^\infty \]. \]

There are multiple choices of \( \ell, m \) in the expansion of \( p/q \); to make the intrinsic denominator well defined, we give the following definition.

**Definition 2.1** Let \( p/q \) be a rational number in \( \mathbb{K} \) with the digit sequence \( (\epsilon_i)_{i \geq 1} \). Let \( \ell \geq 0 \) be the smallest integer such that \( \{\epsilon_i\}_{i > \ell} \) is purely periodic, and let \( m \) be the smallest period of \( \{\epsilon_i\}_{i > \ell} \). Define
\[ \mathcal{L}(p/q) = \ell, \quad \mathcal{P}(p/q) = m. \]
Call them the prelength and the period of \( p/q \) respectively.
Remark Let $p/q \in \mathcal{K}$ with digit sequence $(\epsilon_i)_{i \geq 1}$. Amongst all the pairs satisfying

$$\left(\epsilon_1, \ldots, \epsilon_{\ell'}, (\epsilon_{\ell'+1}, \ldots, \epsilon_{\ell'+m'})^\infty \right) = (\epsilon_i)_{i \geq 1}, \quad (2.2)$$

$(\mathcal{L}(p/q), \mathcal{P}(p/q))$ is the smallest, that is, the equality (2.2) implies that $\ell' \geq \mathcal{L}(p/q)$ and $m' \geq \mathcal{P}(p/q)$.

A direct calculation of the number in the right-hand side of (2.1) yields

$$\frac{p}{q} = \frac{3^{\ell+m}[\epsilon_1, \ldots, \epsilon_{\ell}, \epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m}] - 3^\ell[\epsilon_1, \ldots, \epsilon_{\ell}]}{3^\ell(3^m - 1)}.$$ 

Definition 2.2 Let $p/q$ be a rational in $\mathcal{K}$ with prelength $\mathcal{L}(p/q) = \ell$ and period $\mathcal{P}(p/q) = m$. Write

$$p_{\text{int}} = 3^{\ell+m}[\epsilon_1, \ldots, \epsilon_{\ell}, \epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m}] - 3^\ell[\epsilon_1, \ldots, \epsilon_{\ell}], \quad q_{\text{int}} = 3^\ell(3^m - 1).$$

Call them the intrinsic numerator/denominator of $p/q$ respectively.

So for a rational number $p/q \in \mathcal{K}$ with $\mathcal{L}(p/q) = \ell$ and $\mathcal{P}(p/q) = m$, one has that

$$3^{\ell+m-1} < q_{\text{int}} = 3^\ell(3^m - 1) < 3^{\ell+m}.$$ 

For each word $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 2\}^n$, call

$$I_n(\epsilon_1, \ldots, \epsilon_n) = \left\{ x \in [0, 1] : 3\text{-adic expansion of } x \text{ begins with } (\epsilon_1, \ldots, \epsilon_n) \right\}$$

a cylinder of order $n$. A cylinder is a closed interval (since multiple expansions are allowed for its endpoints). For $x \in \mathcal{K}$, denote by $I_n(x)$ the $n$th order cylinder containing $x$. We collect some well known facts for later use. Recall that all cylinders here are those with digits in $\{0, 2\}$.

Lemma 2.3

- Different cylinders of order $n$ are separated by a gap at least $3^{-n}$. So, for any

$$x = [\epsilon_1, \ldots, \epsilon_n, \ldots] \in \mathcal{K}, \quad y = [\omega_1, \ldots, \omega_n, \ldots] \in \mathcal{K} \quad \text{if } \epsilon_j \neq \omega_j \text{ for some } 1 \leq j \leq n,$$

$$|x - y| \geq 3^{-n}. \quad \text{if } \epsilon_j \neq \omega_j \text{ for some } 1 \leq j \leq n,$$

- Let $B$ be a ball in $[0, 1]$ with $B \cap \mathcal{K} \neq \emptyset$. Then there exists an integer $n$ such that $B$ contains a cylinder of order $n$ but $B$ intersects at most $4$ distinct cylinders of order $n$.

- For any cylinder $I_n(\epsilon_1, \ldots, \epsilon_n)$ of order $n$,

$$\mu(I_n(\epsilon_1, \ldots, \epsilon_n)) = 3^{-n} = 2^{-n}.$$ 

- The measure $\mu$ is $\gamma$-Ahlfors regular and thus doubling in the sense that there exist constants $c > 0$ and $r_0 > 0$ such that for all $x \in \mathcal{K}$ and $r < r_0$,

$$c \cdot r^\gamma \leq \mu(B(x, r)) \leq c^{-1} \cdot r^\gamma, \quad \mu(B(x, 2r)) \leq c^{-1} \mu(B(x, r)).$$

Proof The first item is immediate. Now we prove the second one. Let $x \in B \cap \mathcal{K}$. Since $B$ is open, one has $I_n(x) \subset B$ for $n$ large. Let $n$ be the smallest integer such that $B$ contains a cylinder of order $n$. We claim that $B$ can intersect at most $4$ cylinders of order $n$. Otherwise, $B$ will contain completely three consecutive cylinders of order $n$, say

$$I_n\left(\epsilon_1^{(1)}, \ldots, \epsilon_n^{(1)}\right), I_n\left(\epsilon_1^{(2)}, \ldots, \epsilon_n^{(2)}\right), I_n\left(\epsilon_1^{(3)}, \ldots, \epsilon_n^{(3)}\right)$$
arranged in increasing order. Let \( \ell \) be the smallest integer such that
\[
\epsilon_{\ell}^{(1)} \neq \epsilon_{\ell}^{(2)}.
\]
- If \( \ell = n \). Then the first two cylinders will be contained in \( I_{n-1} \left( \epsilon_1^{(1)}, \ldots, \epsilon_{n-1}^{(1)} \right) \) and so
\[
I_{n-1} \left( \epsilon_1^{(1)}, \ldots, \epsilon_{n-1}^{(1)} \right) \subset B,
\]
by comparing their endpoints. It contradicts the minimality of \( n \).
- If \( \ell < n \). Since the first two cylinders are consecutive cylinders of the same order \( n \) but contained in two different cylinders \( I_{\ell}^{(1)} \) and \( I_{\ell}^{(2)} \) of order \( \ell \), one has that \( I_n(\epsilon_1^{(1)}, \ldots, \epsilon_n^{(1)}) \) lies in the rightmost of \( I_{\ell}^{(1)} \) and \( I_n(\epsilon_1^{(2)}, \ldots, \epsilon_n^{(2)}) \) lies in the leftmost of \( I_{\ell}^{(2)} \). In other words,
\[
\left( \epsilon_1^{(2)}, \ldots, \epsilon_n^{(2)} \right) = \left( \epsilon_1^{(2)}, \ldots, \epsilon_{\ell}^{(2)}, 0, \ldots, 0 \right).
\]
And thus
\[
\left( \epsilon_1^{(3)}, \ldots, \epsilon_n^{(3)} \right) = \left( \epsilon_1^{(2)}, \ldots, \epsilon_{\ell}^{(2)}, 0, \ldots, 0, 2 \right).
\]
This shows the last two are contained in the same cylinder \( I_{n-1}(\epsilon_1^{(2)}, \ldots, \epsilon_{n-1}^{(2)}) \) of order \( n - 1 \). By comparing the endpoints, we deduce that \( B \) contains the cylinder \( I_{n-1}(\epsilon_1^{(2)}, \ldots, \epsilon_{n-1}^{(2)}) \), which contradicts the minimality of \( n \) again.

The third item follows directly from the definition of \( \mu \). For the last item, one notices that
\[
\left[ I_n(x) \cap \mathcal{K} \right] \subset \left[ B(x, 3^{-n}) \cap \mathcal{K} \right] \subset I_{n-1}(x).
\]
(2.3)

\( \square \)

2.2 Chung–Erdos inequality

To estimate the measure of a limsup set in a probability space, the following results are widely used. Let \((\Omega, B, \nu)\) be a probability space and \(\{E_n\}_{n \geq 1}\) a sequence of measurable sets. Define
\[
E = \limsup_{n \to \infty} E_n.
\]
The first is the Borel–Cantelli lemma. See [12, Theorems 4.2.1 and 4.2.4] for a proof.

**Lemma 2.4** (Borel–Cantelli lemma)

\[
\nu(E) = \begin{cases} 
0, & \text{if } \sum_{n \geq 1} \nu(E_n) < \infty; \\
1, & \text{if } \sum_{n \geq 1} \nu(E_n) = \infty \text{ and the events } \{E_n\}_{n \geq 1} \text{ are independent.}
\end{cases}
\]

In applications, the measurable sets \(\{E_n\}_{n \geq 1}\) are not always independent, so for the divergence part, one uses the Chung–Erdős inequality [11] instead. See [32, Lemma 5] for a proof.

**Lemma 2.5** (Chung–Erdős inequality) If \(\sum_{n \geq 1} \nu(E_n) = \infty\), then
\[
\nu(E) \geq \limsup_{n \to \infty} \frac{\left( \sum_{1 \leq n \leq N} \nu(E_n) \right)^2}{\sum_{1 \leq i \neq j \leq N} \nu(E_i \cap E_j)}.
\]
In many cases, the Chung–Erdős inequality enables one to conclude the positivity of \( \nu(E) \). To get a full measure result, one applies the Chung–Erdős inequality locally, i.e., apply it to the set \( E \cap B \) for any ball \( B \subset \Omega \). Then one arrives at the full measure of \( E \) in the light of the following result.

**Lemma 2.6** [4, Proposition 1] Let \((\Omega, |\cdot|)\) be a metric space and let \( \nu \) be a doubling probability measure on \( \Omega \) such that any open set is measurable. Let \( E \) be a Borel subset of \( \Omega \). Assume that there are constants \( r_0, c > 0 \) such that for any ball \( B \) of radius \( r(B) < r_0 \) and centered in \( \Omega \) we have

\[
\nu(E \cap B) \geq c \cdot \nu(B).
\]

Then \( \nu(E) = 1 \).

### 2.3 Mass transference principle

The outstanding mass transference principle established by Beresnevich and Velani [5] enables one to transfer a full (Lebesgue) measure statement for a limsup set to a full Hausdorff measure statement.

Let \( \Omega \) be a locally compact metric space, and \( g \) a doubling dimension function (\( \exists \lambda \geq 1 \) such that \( g(2r) \leq \lambda g(r) \) for all \( r > 0 \) small enough). Suppose that there exist \( 0 < c < \infty \) and \( r_o > 0 \) such that for all \( x \in \Omega \) and \( 0 < r < r_o \),

\[
c \cdot g(r) \leq \mathcal{H}^g(B(x, r)) \leq c^{-1} \cdot g(r).
\]

Let \( f \) be a dimension function and write \( B^f(x, r) \) for the ball \( B(x, g^{-1}(f(r))) \).

**Theorem 2.7** (Beresnevich and Velani [5]) Assume that \( \{B_i\}_{i \in \mathbb{N}} \) is a sequence of balls in \( \Omega \) with radii tending to 0, and that \( \frac{f(r)}{g(r)} \) increases as \( r \to 0_+ \). If for any ball \( B \) in \( \Omega \),

\[
\mathcal{H}^g(B \cap \limsup_{i \to \infty} B_i^f) = \mathcal{H}^g(B);
\]

then for any ball \( B \) in \( \Omega \),

\[
\mathcal{H}^f(B \cap \limsup_{i \to \infty} B_i^g) = \mathcal{H}^f(B).
\]

So, the Hausdorff measure result for the set \( \mathcal{W}_{int, \mathcal{K}}(\psi) \) in Theorem 1.4 is just a consequence of its \( \mu \)-measure result by applying Theorem 2.7.

### 3 Convergence part

Recall the set we are considering:

\[
\mathcal{W}_{int, \mathcal{K}}(\psi) := \left\{ x \in \mathcal{K} : |x - p/q| < \psi(q_{int}), \ \text{i.m.} \ p/q \in \mathcal{K} \right\}.
\]

In this section, we show the convergence part of Theorem 1.4, i.e.,

\[
\sum_{n=1}^{\infty} n \left( 3^n \psi(3^n) \right)^\gamma < \infty \implies \mu(\mathcal{W}_{int, \mathcal{K}}(\psi)) = 0. \quad (3.1)
\]
The convergence of the series in (3.1) implies that $\psi(3^n) \to 0$ as $n \to \infty$. To use the Ahlfors regularity of $\mu$, we let $n_0$ be an integer such that for all $n \geq n_0$, $\psi(3^{n_0-1}) \leq r_0$, where $r_0$ is the positive number appearing in the Ahlfors regular property of $\mu$ (Lemma 2.3).

We classify the rational numbers $p/q$ according to their prelength and period: for each $n \geq 1$, define

$$T_n = \left\{ p/q \in \mathbb{K} : \mathcal{L}(p/q) = \ell, \mathcal{P}(p/q) = m, \quad \ell + m = n \right\}.$$

Note that by Definition 2.2, for each $p/q \in T_n$,

$$3^{n-1} \leq q_{\text{int}} = 3^\ell(3^m - 1) \leq 3^n,$$

and moreover

$$\#T_n = \sum_{\ell + m = n} \left\{ p/q = [\epsilon_1, \ldots, \epsilon_\ell, (\epsilon_\ell + 1, \ldots, \epsilon_{\ell + m})^\infty] : \mathcal{L}(p/q) = \ell, \quad \mathcal{P}(p/q) = m, \quad \epsilon_i \in \mathcal{E}, \quad \forall i \geq 1 \right\} \leq \sum_{\ell + m = n} 2^{\ell + m} = n2^n.$$

Since $\psi$ is non-increasing and $q_{\text{int}} \geq 3^{n-1}$ for $p/q \in T_n$, one has

$$\mathcal{W}_{\text{int}, \mathbb{K}}(\psi) \subset \left\{ x \in \mathbb{K} : |x - p/q| < \psi(3^{n-1}), \quad p/q \in T_n, \quad \text{i.m. } n \in \mathbb{N} \right\}.$$

By $\gamma$-Ahlfors regularity of the measure $\mu$, one has

$$\sum_{n=1}^{\infty} \sum_{p/q \in T_n} \mu\left( B(p/q, \psi(3^{n-1})) \right) \leq \sum_{n<n_0} \#T_n + \sum_{n=n_0}^{\infty} n \cdot 2^n \cdot c^{-1} \left( \psi(3^{n-1}) \right)^\gamma. \quad (3.2)$$

By changing the variable $n - 1$ to $n$, one readily checks that

$$\sum_{n=1}^{\infty} n \cdot 2^n \left( \psi(3^{n-1}) \right)^\gamma < \infty \iff \sum_{n=1}^{\infty} n \left( 3^n \psi(3^n) \right)^\gamma < \infty.$$

Then by the condition in (3.1), the first series in (3.2) converges. Thus the convergence part of the Borel–Cantelli lemma is applied to conclude that $\mu(\mathcal{W}_{\text{int}, \mathbb{K}}(\psi)) = 0$.

### 4 A counting lemma

In this section, we give a mechanism to generate all rational numbers in $\mathbb{K}$, then group them according to their prelength and period. After that, we present a counting lemma to show that inside each group there is a positive proportion of well separated rationals.

At last, we make a selection among rational numbers in $\mathbb{K}$ based on the counting lemma. We separate the main task of this section into 4 steps.

(I) **MECHANISM.** Note first that every rational $p/q \in \mathbb{K}$ has the following 3-adic expansion:

$$p/q = [\epsilon_1, \ldots, \epsilon_\ell, (\epsilon_{\ell + 1}, \ldots, \epsilon_{\ell + m})^\infty].$$

So $p/q$ is uniquely determined by $(\epsilon_1, \ldots, \epsilon_\ell)$ and $(\epsilon_{\ell + 1}, \ldots, \epsilon_{\ell + m})$ with $\epsilon_i \in \{0, 2\}$. This simple observation inspires a mechanism to generate all rational numbers in $\mathbb{K}$ in the following way.
For each word \((\epsilon_1, \ldots, \epsilon_n) \in \{0, 2\}^n\), it can generate \((n + 2)\) rational numbers (may be not distinct) in \(K\):

- the endpoints of the 3-adic cylinder:
  \[ [\epsilon_1, \ldots, \epsilon_n, 0^\infty], [\epsilon_1, \ldots, \epsilon_n, 2^\infty]; \]
- ultimately periodic ones:
  \[ [\epsilon_1, \ldots, \epsilon_\ell, (\epsilon_{\ell+1}, \ldots, \epsilon_n)^\infty], 0 \leq \ell < n. \]

This gives all rational numbers in \(K\) by taking union over \((\epsilon_1, \ldots, \epsilon_n)\) and \(n \geq 1\).

(II) **GROUPING.** We group the rational numbers in \(K\) in the following way: for each \(n \geq 1\), define \(K_n\) to be the set

\[
\bigcup_{(\epsilon_1, \ldots, \epsilon_n) \in \{0, 2\}^n} \left[ [\epsilon_1, \ldots, \epsilon_n, 0^\infty], [\epsilon_1, \ldots, \epsilon_n, 2^\infty] \right] \\
\bigcup \left[ [\epsilon_1, \ldots, \epsilon_\ell, (\epsilon_{\ell+1}, \ldots, \epsilon_n)^\infty] : 0 \leq \ell < n \right]
\]

Then we have

\[
\mathcal{W}_{\text{int}, K}(\psi) = \limsup_{n \to \infty} E_n, \quad \text{where } E_n := \bigcup_{p/q \in K_n} B(p/q, \psi(q_{\text{int}})).
\]

However, for some \((\epsilon_1, \ldots, \epsilon_n)\), the \((n + 2)\) number of so-generated rationals may not be distinct, and some of them may be very close to each other (for example \((\epsilon_1, \ldots, \epsilon_n) = (2, \ldots, 2)\), i.e., \(\epsilon_i = 2\) for all \(1 \leq i \leq n\). So the number of the balls \(B(p/q, \psi(q_{\text{int}}))\) for \(p/q \in K_n\) may be smaller than \(2^n(n + 2)\), and some of these balls may overlap. This brings difficulty in estimating the measure of \(E_n\), let alone \(\mu(E_n \cap E_m)\).

To overcome this, we are led to seeking a large subfamily \(F_n\) of words \((\epsilon_1, \ldots, \epsilon_n)\) hoping that

- most of \((\epsilon_1, \ldots, \epsilon_n)\) are chosen: at least a positive proportion among the collection of all words of length \(n\);
- each word \((\epsilon_1, \ldots, \epsilon_n)\) in \(F_n\) can generate as many as possible well spaced rationals: at least \(cn\) for some absolute constant \(c > 0\).

The next counting lemma serves this purpose. Once this is achieved, one should believe that the subset defined later will not differ from \(\mathcal{W}_{\text{int}, K}(\psi)\) so much.

(III) **COUNTING.** The complexity function \(C_t(\epsilon_1, \ldots, \epsilon_n)\) is defined to be the number of different subwords of length \(t\) appearing in \((\epsilon_1, \ldots, \epsilon_n)\). Let \(c_1 = 1/16, c_2 = 1/4\). Let \(k_n = \lfloor \log_2 n \rfloor\). Define

\[
F_n := \left\{ (\epsilon_1, \ldots, \epsilon_n) \in \{0, 2\}^n : C_{k_n}(\epsilon_1, \ldots, \epsilon_n) \geq \lfloor n/8 \rfloor \right\}.
\]

**Lemma 4.1** (A counting lemma)

\[
\#F_n \geq c_2 \cdot 2^n.
\]

**Proof** We prove this by using a probabilistic model. In the proof, we put \(k = k_n\), and write \(x_1 x_2 \cdots x_n\) (rather than \((x_1, x_2, \ldots, x_n)\)) for a finite word in \(\{0, 2\}^n\). For \(\xi \in \{0, 2\}^k\), let

\[
|x_1 x_2 \cdots x_n|_\xi = \# \{i : 0 \leq i \leq n - k, x_{i+1} x_{i+2} \cdots x_{i+k} = \xi \}
\]
and

\[ \Sigma_k^n(x_1x_2 \cdots x_n) = \sum_{\xi \in \{0,2\}^k} |x_1x_2 \cdots x_n|^2_\xi. \]

Let \( X_1, X_2, \ldots \) be i.i.d. random variables on the probability space \((\{0,2\}, P)\) with the distribution \( P(X_i = 0) = P(X_i = 2) = 1/2 \). Write

\[ \Sigma_k^n = \sum_{\xi \in \{0,2\}^k} |X_1X_2 \cdots X_n|^2_\xi. \]

At first, we estimate the expectation of the random variable \( \Sigma_k^n \):

\[
\mathbb{E}(\Sigma_k^n) = \sum_{\xi \in \{0,2\}^k} \mathbb{E}\left[ |X_1X_2 \cdots X_n|^2_\xi \right]
= \sum_{\xi \in \{0,2\}^k} \mathbb{E}\left[ \left( \sum_{i=0}^{n-k} \mathbb{I}_\xi(X_{i+1}X_{i+2} \cdots X_{i+k}) \right)^2 \right]
= \sum_{\xi \in \{0,2\}^k} \mathbb{E}\left[ \sum_{i,j=0}^{n-k} \mathbb{I}_\xi(X_{i+1}X_{i+2} \cdots X_{i+k}) \cdot \mathbb{I}_\xi(X_{j+1}X_{j+2} \cdots X_{j+k}) \right]
= \sum_{i,j=0}^{n-k} \mathbb{E}\left[ \sum_{\xi \in \{0,2\}^k} \mathbb{I}_\xi(X_{i+1}X_{i+2} \cdots X_{i+k}) \cdot \mathbb{I}_\xi(X_{j+1}X_{j+2} \cdots X_{j+k}) \right]
= \sum_{i,j=0}^{n-k} \mathbb{E}\left[ \mathbb{I}_{X_{i+1}X_{i+2} \cdots X_{i+k}=X_{j+1}X_{j+2} \cdots X_{j+k}} \right].
\] (4.1)

Now we show that

\[ P\left( X_{i+1}X_{i+2} \cdots X_{i+k} = X_{j+1}X_{j+2} \cdots X_{j+k} \right) = \begin{cases} 1, & \text{when } i = j; \\ 2^{-k}, & \text{when } i \neq j. \end{cases} \]

It is trivial when \( i = j \), so we assume \( i < j \).

- When \( j \geq i + k \), by independence

\[ P\left( X_{i+1}X_{i+2} \cdots X_{i+k} = X_{j+1}X_{j+2} \cdots X_{j+k} \right) = \prod_{t=1}^{k} P(X_{i+t} = X_{j+t}) = 2^{-k}. \]

- When \( i < j \leq i + k \). The requirement

\[ X_{i+1}X_{i+2} \cdots X_{i+k} = X_{j+1}X_{j+2} \cdots X_{j+k} \]

implies that

\[ X_{i+t} = X_{j+t} = \frac{j-i+(j-i)}{j-i} X_{i+t+(j-i)}, \text{ for all } 1 \leq t \leq k. \] (4.2)

This shows the periodicity

\[ X_{i+1} \cdots X_j X_{j+1} \cdots X_{j+k} = \left( X_{i+1} \cdots X_j \right)^{\frac{j-i+k}{j-i}}. \]
As a consequence
\[
P\left(X_{i+1}X_{i+2} \cdots X_{i+k} = X_{j+1}X_{j+2} \cdots X_{j+k}\right)
= \sum_{(\epsilon_1, \ldots, \epsilon_{j-i}) \in \{0,2\}^{j-i}} P\left(X_{i+1} \cdots X_jX_{j+1} \cdots X_{j+k} = (\epsilon_1, \ldots, \epsilon_{j-i}) \right)
= 2^{j-i} \cdot 2^{-(j-k)} = 2^{-k}.
\]

Turning back to (4.1), we have that
\[
\mathbb{E}(\Sigma^n_k) = \sum_{i=0}^{n-k} 1 + \sum_{i,j=0}^{n-k} \mathbb{E}(\Sigma^n_k) = (n - k + 1) + (n - k)(n - k + 1)2^{-k}
\leq n + n^2 \cdot 2n^{-1} \leq 3n.
\]

(4.3)

On the other hand, let \(A\) be the event that
\[
\#\left\{ \xi \in \{0,2\}^k : |X_1X_2 \cdots X_n|_\xi \geq 1 \right\} < \lfloor n/8 \rfloor =: \ell,
\]
i.e., the event that the number of different subwords of length \(k\) in \(X_1X_2 \cdots X_n\) is less than \([n/8]\). We will show that the probability of the event \(A\) is bounded away from 1 uniformly in \(n\). This will imply that, among all words of length \(n\), there is a positive proportion of the collection of words with complexity at least \([n/8]\). More precisely,
\[
1 - P(A) = P\left\{ w \in \Omega : \#\{ \xi \in \{0,2\}^k : |X_1(w) \cdots X_n(w)|_\xi \geq 1 \} \geq \lfloor n/8 \rfloor \right\}
= \sum_{\epsilon_1, \ldots, \epsilon_n \in \{0,2\}^n} P\left\{ w \in \Omega : X_1(w) \cdots X_n(w) = \epsilon_1 \cdots \epsilon_n, \right.
\#\{ \xi \in \{0,2\}^k : |X_1(w) \cdots X_n(w)|_\xi \geq 1 \} \geq \lfloor n/8 \rfloor \}
= \sum_{\epsilon_1, \ldots, \epsilon_n \in \{0,2\}^n : \sum_{l=1}^{n} \ell \geq \lfloor n/8 \rfloor} \left\{ w \in \Omega : X_1(w) \cdots X_n(w) = \epsilon_1 \cdots \epsilon_n \right\}
= \#\mathcal{F}_n \cdot 2^{-n}.
\]

(4.4)

For any \(\omega \in A\), let \(\xi_1, \ldots, \xi_r\) be the all different subwords of length \(k\) in \(X_1(\omega)X_2(\omega) \cdots X_n(\omega)\) and denote by \(z_1, \ldots, z_r\) for their respective numbers of occurrences in \(X_1(\omega)X_2(\omega) \cdots X_n(\omega)\). Then
\[
r < \ell; \ z_i \in \mathbb{Z}_{\geq 1}, \ \text{for all} \ 1 \leq i \leq r; \ \text{and} \ z_1 + \cdots + z_r = n - k + 1.
\]

Thus
\[
\sum_{\xi \in \{0,2\}^k} |X_1(\omega)X_2(\omega) \cdots X_n(\omega)|_\xi^2 = z_1^2 + \cdots + z_r^2
\geq \min \left\{ x_1^2 + \cdots + y_\ell^2 : y_i \geq 0, \ 1 \leq i \leq \ell, \ y_1 + \cdots + y_\ell = n - k + 1 \right\}.
\]

As a consequence,
\[ \mathbb{E}(\Sigma_k^c) \geq \int_A \left[ \sum_{\xi \in \{0,2\}^k} |X_1 X_2 \cdots X_n|^2 \right] dP \]

\[ \geq P(A) \cdot \min \left\{ y_1^2 + \cdots + y_{\ell}^2 : y_i \geq 0, 1 \leq i \leq \ell, \ y_1 + \cdots + y_{\ell} = n - k + 1 \right\} \]

\[ = P(A) \cdot \frac{(n - k + 1)^2}{\ell} \geq P(A) \cdot \frac{8(n - k + 1)^2}{n} \]

\[ \geq P(A) \cdot \frac{8 \cdot 1/2 \cdot n^2}{n} = P(A) \cdot 4n. \]

Thus, by (4.3) it follows that

\[ P(A) \leq \frac{3}{4}. \]

Hence by (4.4), one has

\[ \#F_n = 2^n \cdot (1 - P(A)) \geq \frac{1}{4} \cdot 2^n. \]

**Remark** We give a remark about the choice of \( k_n = \lfloor \log_2 n \rfloor \). Indeed, the choice of \( k_n \) is designed for estimating the correlation \( \mu(E_m \cap E_n) \) below, not only for Lemma 4.1. One will see that we need such a choice in estimating \( \mu(E_m \cap E_n) \). Then turning back, we need a proposition like Lemma 4.1.Philosophically, Lemma 4.1 must be true in view of the theory for normal numbers. Since almost all real numbers are normal numbers, we can say roughly that almost all words are ‘normal words’. Thus for ‘almost all’ words \( (\epsilon_1, \ldots, \epsilon_n) \), every word of length \( k \) would appear in it (when \( k \) is small compared with \( n \)). Thus

\[ C_k(\epsilon_1, \ldots, \epsilon_n) \gg 2^k \sim n, \text{ if we choose } k \sim \log_2 n. \]

(IV) **Selection.** Now we carry out the selection of rational numbers in \( \mathcal{K}_n \). Taking a word \( (\epsilon_1, \ldots, \epsilon_n) \) in \( \mathcal{F}_n \), we define

\[ \mathcal{G}(\epsilon_1, \ldots, \epsilon_n) = \left\{ 0 \leq \ell \leq n - k_n : (\epsilon_{i+1}, \ldots, \epsilon_{i+k_n}) \neq (\epsilon_{i+1}, \ldots, \epsilon_{i+k_n}) \text{ for any } 0 \leq i < \ell \right\}, \]

which is the set of positions where a new subword of length \( k_n \) appears.

By Lemma 4.1, there are at least \( c_2 \cdot 2^n \) words in \( \mathcal{F}_n \), and for every \( (\epsilon_1, \ldots, \epsilon_n) \) in \( \mathcal{F}_n \), there are at least \( c_1 n \) positions in \( \mathcal{G}(\epsilon_1, \ldots, \epsilon_n) \). To simplify our later calculation, we may assume without loss of generality that \( \#F_n = [c_2 2^n] \) and \( \#G(\epsilon_1, \ldots, \epsilon_n) = [c_1 n] \) by discarding the superfluous elements if necessary.

In the later use, we will restrict our attention to those rationals lying in a fixed cylinder, say \( I = I_l(\eta_1, \ldots, \eta_\ell) \). So for each \( (\epsilon_1, \ldots, \epsilon_n) \in \mathcal{F}_n \), denote by \( \mathcal{P}_I(\epsilon_1, \ldots, \epsilon_n) \) the collection of rational points in \( \mathcal{K} \) given by

\[ \mathcal{P}_I(\epsilon_1, \ldots, \epsilon_n) = \left\{ [\eta_1, \ldots, \eta_\ell, \epsilon_1, \ldots, \epsilon_{\ell+i}, (\epsilon_{\ell+i+1}, \ldots, \epsilon_n)^\infty] : \ell_i \in \mathcal{G}(\epsilon_1, \ldots, \epsilon_n), \ 1 \leq i \leq [c_1 n] \right\}. \]

It contains \([c_1 n]\) different rational points and all of them are in \( \mathcal{K} \cap I_{l+i+n}(\eta_1, \ldots, \eta_\ell, \epsilon_1, \ldots, \epsilon_n) \). Finally the selected rationals from \( \mathcal{K}_{l+i+n} \) are those in

\[ \bigcup_{(\epsilon_1, \ldots, \epsilon_n) \in \mathcal{F}_n} \mathcal{P}_I(\epsilon_1, \ldots, \epsilon_n). \]
We conclude this section by listing the following properties of the sets $P_{I}(\epsilon_{1}, \ldots, \epsilon_{n})$, which will be crucial to the proof of the divergence case.

**Lemma 4.2** We have the following basic facts:

1. for any two different words $(\epsilon_{1}, \ldots, \epsilon_{n})$ and $(\epsilon'_{1}, \ldots, \epsilon'_{n})$ in $F_{n}$,
   \[
   \text{dist}(P_{I}(\epsilon_{1}, \ldots, \epsilon_{n}), P_{I}(\epsilon'_{1}, \ldots, \epsilon'_{n})) \geq 3^{-(t+n)};
   \]
   \[
   2. \text{for any two different points } p_{i}/q_{i} \text{ and } p'_{i}/q'_{i} \text{ in } P_{I}(\epsilon_{1}, \ldots, \epsilon_{n}),
   \]
   \[
   |p_{i}/q_{i} - p'_{i}/q'_{i}| \geq 3^{-(t+n+k_n)}.
   \]
   \[
   3. \text{for any } p/q \in P_{I}(\epsilon_{1}, \ldots, \epsilon_{n}),
   \]
   \[
   q_{\text{int}} \leq 3^t+n.
   \]

**Proof** 1. All the rationals in $P_{I}(\epsilon_{1}, \ldots, \epsilon_{n})$ are contained in the $(t+n)$th order cylinder $I_{t+n}(\eta_{1}, \ldots, \eta_{t}, \epsilon_{1}, \ldots, \epsilon_{n})$.

2. Recall the expansion of the rationals in $P_{I}(\epsilon_{1}, \ldots, \epsilon_{n})$:
   \[
   p/q = [\eta_{1}, \ldots, \eta_{t}, \epsilon_{1}, \ldots, \epsilon_{\ell}, \epsilon_{\ell+1}, \ldots, \epsilon_{n}, (\epsilon_{\ell+1}, \ldots, \epsilon_{n})^{\infty}].
   \]
   So all of them have different $(t+n+k_n)$ prefix.

3. This follows from the definition of prelength and period (see also the remark after Definition 2.1). $\square$

## 5 Divergence part

### 5.1 A subset of $\mathcal{W}_{\text{int}, K}(\psi)$

For a rational number $p/q \in K$ with prelength $L(p/q) = \ell$ and period $P(p/q) = m$, one has that

\[
q_{\text{int}} = 3^\ell (3^{m} - 1).
\]

Moreover, once $p/q$ can be written as

\[
p/q = [\epsilon_{1}, \ldots, \epsilon_{\ell}, (\epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m})^{\infty}],
\]
we have (see the remark after Definition 2.1)

\[
q_{\text{int}} \leq 3^{\ell+m}'.
\]

This will give rise to the following subset of $\mathcal{W}_{\text{int}, K}(\psi)$:

\[
\tilde{\mathcal{W}} := \{ x \in K : |x - [\epsilon_{1}, \ldots, \epsilon_{\ell}, (\epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m})^{\infty}] | < \psi(3^{\ell+m}),
\]
\[
\epsilon_{i} \in \{0, 2\}, 1 \leq i \leq \ell + m, \text{ i.m. } (\ell, m) \in \mathbb{Z}_{\geq 0}^2 \}.
\]

**Lemma 5.1** Let $\psi$ be a non-increasing positive function defined on $\mathbb{N}$ with $\psi(q) \rightarrow 0$ as $q \rightarrow \infty$. Then

\[
(\tilde{\mathcal{W}} \setminus \mathbb{Q}) \subset \mathcal{W}_{\text{int}, K}(\psi).
\]

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Proof Let $x \in \tilde{W} \setminus \mathbb{Q}$. Then there exists a sequence $\{(\ell, m_n) : n \geq 1\}$ and the words $\epsilon^{(n)} \in \{0, 2\}^{\ell_n}$, $w^{(n)} \in \{0, 2\}^{m_n}$, such that for all $n \geq 1$,

$$|x - [\epsilon^{(n)}, (w^{(n)})^\infty]| < \psi(3^{\ell_n+m_n}).$$

For each $n \geq 1$, let

$$\frac{p_n}{q_n} = [\epsilon^{(n)}, (w^{(n)})^\infty],$$

and $q_{n, \text{int}}$ = the intrinsic denominator of $\frac{p_n}{q_n}$.

Since $q_{n, \text{int}} \leq 3^{\ell_n+m_n}$ and then by the monotonicity of $\psi$, it follows that

$$|x - \frac{p_n}{q_n}| < \psi(q_{n, \text{int}}), \text{ for all } n \geq 1.$$  

We claim that the set

$$\{\frac{p_n}{q_n} : n \geq 1\}$$

is infinite and thus $x \in W_{\text{int}, K}(\psi)$. If this were not the case, there would exist an infinite subsequence $\{(\ell_{n_k}, m_{n_k}) : k \geq 1\}$ such that

$$\frac{p_{n_k}}{q_{n_k}} = \frac{p}{q}, \quad |x - \frac{p}{q}| < \psi(3^{\ell_{n_k}+m_{n_k}}), \text{ for all } k \geq 1.$$  

Letting $k \to \infty$, one would have $x = p/q \in \mathbb{Q}$, a contradiction. \qed

In other words, without the restrictions that $\ell$ and $m$ must be the prelength and period of a rational number respectively, we get a subset $\tilde{W} \setminus \mathbb{Q}$ of $W_{\text{int}, K}(\psi)$. So to estimate the measure of $W_{\text{int}, K}(\psi)$ from below, it suffices to estimate that of $\tilde{W}$ from below. In the following, we will focus on the $\mu$-measure of $\tilde{W}$.

5.2 Chung–Erdös inequality

In this part, we will use the Chung–Erdös inequality to show that for a certain absolute constant $C > 0$ and any cylinder $I$,

$$\mu(\tilde{W} \cap I) \geq C \cdot \mu(I) \quad (5.1)$$

under the condition that

$$\sum_{n=1}^{\infty} n \left(3^n \psi(3^n)\right)^{1/y} = \infty. \quad (5.2)$$

This is sufficient to conclude the full measure of $\tilde{W}$ by applying Lemma 2.6, because of the following observation.

Lemma 5.2 If (5.1) is true for all cylinders, then for any ball $B$,

$$\mu(\tilde{W} \cap B) \geq C/4 \cdot \mu(B).$$

Proof We assume $B \cap K \neq \emptyset$ otherwise there is nothing to prove. By Lemma 2.3, there are 4 cylinders of the same order, say

$I_t^{(0)}$, $I_t^{(1)}$, $I_t^{(2)}$, $I_t^{(3)}$.
such that

\[ I_t^{(0)} \subset B, \quad \text{and} \quad \left[ B \cap K \right] \subset \bigcup_{0 \leq i \leq 3} I_t^{(i)}. \]

Applying (5.1) to \( I_t^{(0)} \), one has

\[
\mu\left( \tilde{W} \cap B \right) \geq \mu\left( \tilde{W} \cap I_t^{(0)} \right) \geq C \mu\left( I_t^{(0)} \right) \geq C/4 \cdot \mu\left( \bigcup_{0 \leq i \leq 3} I_t^{(i)} \right) \geq C/4 \cdot \mu(B). 
\]

From now on we fix a cylinder \( I = I_t(\eta_1, \ldots, \eta_t) \) to show the validity of (5.1). Moreover, we can further assume that

\[
\psi(3^t+n) \leq 1/4 \cdot 3^{-t+n+k_n}, \quad \text{and} \quad \psi(3^t+n) \leq \frac{r_0}{4}, \quad \text{for all} \ n \geq 1,
\]

where \( r_0 \) appears in the Ahlfors regularity of \( \mu \). Otherwise, let

\[
\psi'(3^n) = \min\left\{ \psi(3^n), \frac{1}{4} \cdot 3^{-t+n+k_n}, \frac{r_0}{4} \right\}.
\]

Note that for those \( n \) with \( \psi'(3^n) = 1/4 \cdot 3^{-t+n+k_n} \), one has

\[
n\left(3^n \psi'(3^n)\right)^\gamma = n\left(3^n \cdot \frac{1}{4} \cdot 3^{-t+n+k_n}\right)^\gamma = n\left(\frac{1}{4} \cdot 3^{-k_n}\right)^\gamma \geq 4^{-\gamma},
\]

where for the last equality, we used \( 3^{-k_n} \gamma \geq n^{-1} \). Thus it follows that

\[
\sum_{n \geq 1} n\left(3^n \psi(3^n)\right)^\gamma = \infty \implies \sum_{n \geq 1} n\left(3^n \psi'(3^n)\right)^\gamma = \infty.
\]

To apply the Chung–Erdős inequality to \( \tilde{W} \cap I \), we proceed as follows:

1. construct a nice subset \( \limsup_{n \to \infty} E_n \) of \( \tilde{W} \cap I \) and check that the condition in Chung–Erdős lemma is fulfilled, i.e.,

\[
\sum_{n=1}^{\infty} \mu(E_n) = \infty;
\]

2. estimate the correlation

\[
\mu(E_m \cap E_n), \quad \text{for} \ m \neq n.
\]

**5.2.1 A subset of \( \tilde{W} \cap I \)**

For each \( n \geq 1 \), let

\[
E_n = \left\{ B\left( p/q, \psi(3^{t+n}) \right) : p/q \in \mathcal{P}_I(\epsilon_1, \ldots, \epsilon_n), \ (\epsilon_1, \ldots, \epsilon_n) \in \mathcal{F}_n \right\}
\]

and define

\[
E_n = \bigcup_{A_n \in \mathcal{E}_n} A_n = \bigcup_{(\epsilon_1, \ldots, \epsilon_n) \in \mathcal{F}_n} \bigcup_{p/q \in \mathcal{P}_I(\epsilon_1, \ldots, \epsilon_n)} B\left( p/q, \psi(3^{t+n}) \right).
\]
Lemma 5.3

$$\limsup_{n \to \infty} E_n \subset \tilde{\mathcal{W}} \cap I.$$  

Proof For any $$x \in \limsup_{n \to \infty} E_n$$, there exist infinitely many integers $$n$$ such that $$x \in E_n$$. For each of these integers $$n$$, there exists $$(\epsilon_1, \ldots, \epsilon_n) \in \mathcal{F}_n$$, and then $$p_n/q_n = [\eta_1, \ldots, \eta_t, \epsilon_1, \ldots, \epsilon_i, (\epsilon_{i+1}, \ldots, \epsilon_n)\infty] \in \mathcal{K} \cap I$$ such that

$$|x - [\eta_1, \ldots, \eta_t, \epsilon_1, \ldots, \epsilon_i, (\epsilon_{i+1}, \ldots, \epsilon_n)\infty]| < \psi(3^{t+n}).$$

Since $$p_n/q_n \in \mathcal{K}$$ and $$\psi(3^{t+n}) \to 0$$ as $$n \to \infty$$, this shows $$x \in \mathcal{K}$$ and then $$x \in \tilde{\mathcal{W}}$$. Moreover, since $$I$$ is a closed interval, $$p_n/q_n \in I$$ and $$\psi(3^{t+n}) \to 0$$ as $$n \to \infty$$, one has $$x \in I$$.

The assumption (5.3) implies the balls in $$E_n$$ are disjoint by Lemma 4.2, so

$$\mu(E_n) = \sum_{(\epsilon_1, \ldots, \epsilon_n) \in \mathcal{F}_n} \sum_{p/q \in \mathcal{P}_t(\epsilon_1, \ldots, \epsilon_n)} \mu(B(p/q, \psi(3^{t+n})))$$

$$\geq [c_2 2^n] \cdot [c_1 n] \cdot c(\psi(3^{t+n}))^\gamma \geq c_2 2^n \cdot c_1 n \cdot c(\psi(3^{t+n}))^\gamma.$$ 

Thus,

$$\sum_{n=1}^{\infty} \mu(E_n) \geq \sum_{n=1}^{\infty} c_2 2^n \cdot c_1 n \cdot c(\psi(3^{t+n}))^\gamma = \infty.$$ 

This shows the condition in the Chung–Erdös lemma is fulfilled.

5.2.2 Correlation

Now we estimate the measure of the intersection $$E_m \cap E_n$$ with $$m < n$$. It is trivial that

$$\mu(E_m \cap E_n) = \sum_{A_m \in \mathcal{E}_m} \mu(A_m \cap E_n).$$

So, we pay attention to the number of balls in $$E_n$$ which can have non-empty intersection with a general ball $$A_m$$ in $$E_m$$.

Recall that by (5.3), for each ball $$A_n \in \mathcal{E}_n$$, the intersection $$A_n \cap \mathcal{K}$$ is contained in a cylinder of order $$t + n$$. On the other hand, from the definition of $$\mathcal{P}_t(\epsilon_1, \ldots, \epsilon_n)$$, each cylinder of order $$t + n$$ can intersect at most $$[c_1 n] \leq n$$ balls in $$\mathcal{E}_n$$. Moreover, the number of balls in $$\mathcal{E}_n$$ is

$$c_1 c_2 \cdot n \cdot 2^n \leq \# \{A_n : A_n \in \mathcal{E}_n\} = [c_1 n] \cdot \# \mathcal{F}_n \leq n \cdot 2^n.$$ 

Now let $$A_m = B(z, \psi(3^{t+m}))$$ be a general ball in $$\mathcal{E}_m$$. Three cases will be distinguished according to the comparison of the radius of $$A_m$$ and the gaps between the balls $$A_n$$ in $$\mathcal{E}_n$$.

CASE (i). $$\psi(3^{t+m}) \geq \frac{1}{3} \cdot 3^{-(t+n)}$$. In this case, the ball $$A_m$$ may intersect many cylinders of order $$t + n$$. We denote by $$\mathcal{D}$$ the collection of cylinders of order $$t + n$$ which meet $$A_m$$. 

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All these cylinders, except possibly for the leftmost and rightmost ones, are contained in \( A_m \) completely, and thus

\[(\#D - 2) \cdot \mu(I,_{t+n}) \leq \mu(A_m).\]

By the Ahlfors regularity of \( \mu \) and the condition \( 3^{t+n} \cdot \psi(3^{t+m}) \geq 4^{-1} \) in this case, one has

\[
\#D \leq \frac{\mu(A_m)}{\mu(I,_{t+n})} + 2 \leq c^{-1}(3^{t+n} \cdot \psi(3^{t+m})) + 2 \leq 9c^{-1}(3^{t+n} \cdot \psi(3^{t+m}))^\gamma.
\]

Therefore \( A_m \) can intersect at most

\[
[c_1 n] \cdot \#D \leq n \cdot 9c^{-1}(3^{t+n} \cdot \psi(3^{t+m}))^\gamma = 9c^{-1} n 2^{t+n} \cdot (\psi(3^{t+m}))^\gamma
\]

balls \( A_n \) in \( E_n \), and thus

\[
\mu(E_m \cap E_n) \leq \sum_{A_m \in \mathcal{E}_m} \sum_{A_n \in \mathcal{E}_n, A_n \cap A_m \neq \emptyset} \mu(A_n)
\leq \sum_{A_m \in \mathcal{E}_m} 9c^{-1} n 2^{t+n} (\psi(3^{t+m}))^\gamma \cdot c^{-1}(\psi(3^{t+n}))^\gamma
\leq m 2^m \cdot 9c^{-2} n 2^{t+n} (\psi(3^{t+m}))^\gamma \cdot (\psi(3^{t+n}))^\gamma
\leq 9c^{-2} m 2^m (\psi(3^{t+m}))^\gamma \cdot n 2^n (\psi(3^{t+n}))^\gamma \cdot 2^t
\leq c_3 \cdot \frac{\mu(E_m) \cdot \mu(E_n)}{\mu(I)},
\]

for some absolute constant \( c_3 > 0 \). Note that \( \mu(I) = 2^{-t} \).

**CASE (ii).** \( \frac{1}{2} \cdot 3^{-(t+n+k_n)} \leq \psi(3^{t+m}) < \frac{1}{4} \cdot 3^{-(t+n)} \). In this case, the ball \( A_m \) can intersect at most one cylinder of order \( (t+n) \) by Lemma 4.2 (1). Thus the centers of balls in \( E_n \) which can intersect \( A_m \) are all contained in one single cylinder of order \( t+n \). Denote by \( C \) the collection of balls in \( E_n \) which intersect \( A_m \). Then there exists an element \((\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{F}_n \) such that

\[
C \subset \left\{ A_n = B \left( \frac{p}{q}, \psi(3^{t+n}) \right) : \frac{p}{q} \in \mathcal{P}_I(\varepsilon_1, \ldots, \varepsilon_n) \right\}.
\]

By Lemma 4.2 (2), the centers in \( \mathcal{P}_I(\varepsilon_1, \ldots, \varepsilon_n) \) are at least \( 3^{-(t+n+k_n)} \)-separated which is larger than twice the radius \( \psi(3^{t+n}) \) by (5.3). Thus the thickened balls

\[
\left\{ \tilde{A}_n = B \left( \frac{p}{q}, \frac{1}{2} \cdot 3^{-(t+n+k_n)} \right) : \frac{p}{q} \in \mathcal{P}_I(\varepsilon_1, \ldots, \varepsilon_n) \right\}
\]

are still disjoint.

Take a ball \( A_n = B \left( \frac{p}{q}, \psi(3^{t+n}) \right) \) from \( C \). Let \( y \in A_n \cap A_m \). Then for any \( x \) in the thickened ball \( \tilde{A}_n \) of \( A_n \),

\[
|x - z| \leq |x - p/q| + |p/q - y| + |y - z| < \frac{1}{2} \cdot 3^{-(t+n+k_n)} + \psi(3^{t+n}) + \psi(3^{t+m})
\leq 4\psi(3^{t+m}), \quad \text{by the assumption in Case (ii)}.
\]
In other words, for each $A_n \in C$, its thickening $\tilde{A}_n$ is contained in $4A_m = B(z, 4\psi(3^{t+m}))$. As a consequence

$$\sum_{A_n \in C} \mu(\tilde{A}_n) = \mu\left( \bigcup_{A_n \in C} \tilde{A}_n \right) \leq \mu(4A_m).$$

Applying the Ahlfors regularity of $\mu$, it follows that

$$\#C \cdot c\left( \frac{1}{2} \cdot 3^{-(t+n+k_n)} \right)^\nu \leq c^{-1}(4\psi(3^{t+m}))^\nu.$$

Thus the ball $A_m$ can intersect at most

$$c^{-28} \cdot \left( \psi(3^{t+m}) \cdot 3^{t+n+k_n} \right)^\nu$$

balls in $E_n$. By noting that $3^{k_n} \leq n$, one has

$$\mu(E_m \cap E_n) \leq m2^m \cdot c^{-28} \cdot \left( \psi(3^{t+m}) \cdot 3^{t+n+k_n} \right)^\nu \cdot c^{-1}(\psi(3^{t+n}))^\nu$$

$$= c^{-3} \cdot m2^m \left( \psi(3^{t+m}) \right)^\nu \cdot n2^n \left( \psi(3^{t+n}) \right)^\nu \cdot 2^t$$

$$\leq c_3 \cdot \frac{\mu(E_m) \cdot \mu(E_n)}{\mu(I)}.$$

CASE (iii). $\psi(3^{t+m}) < \frac{1}{4} \cdot 3^{-(t+n+k_n)}$. Notice that by Lemma 4.2 (1) (2) the gaps between the balls in $E_n$ are at least

$$3^{-(t+n+k_n)} - 2\psi(3^{t+n}) \geq \frac{1}{2} \cdot 3^{-(t+n+k_n)}.$$

So in this case, the ball $A_m$ can intersect at most one ball in $E_n$. Thus

$$\mu(E_m \cap E_n) \leq m2^m \cdot c^{-1} \cdot \left( \psi(3^{t+n}) \right)^\nu.$$

In summary, we have shown that for $m < n$,

$$\mu(E_m \cap E_n) \leq c_3 \cdot \frac{\mu(E_m) \cdot \mu(E_n)}{\mu(I)} + c^{-1} \cdot m2^m \cdot \left( \psi(3^{t+n}) \right)^\nu.$$

So,

$$\sum_{1 \leq m \neq n \leq N} \mu(E_m \cap E_n) \leq c_3 \cdot \frac{1}{\mu(I)} \left( \sum_{1 \leq m \leq N} \mu(E_m) \right)^2 + 2c^{-1} \sum_{n=1}^{N} \sum_{m=1}^{n-1} m2^m \left( \psi(3^{t+n}) \right)^\nu$$

$$\leq c_3 \cdot \frac{1}{\mu(I)} \left( \sum_{1 \leq m \leq N} \mu(E_m) \right)^2 + 4c^{-1} \sum_{n=1}^{N} n2^n \left( \psi(3^{t+n}) \right)^\nu$$

$$\leq c_3 \cdot \frac{1}{\mu(I)} \left( \sum_{1 \leq m \leq N} \mu(E_m) \right)^2 + c_3 \sum_{n=1}^{N} \mu(E_n).$$

Therefore, by the Chung–Erdős inequality, we conclude that

$$\mu(\tilde{\mathcal{W}} \cap I) \geq \mu(\limsup_{n \to \infty} E_n) \geq c_3^{-1} \mu(I),$$

where the constant $c_3^{-1}$ is independent of $I$. Thus the full measure property of $\tilde{\mathcal{W}}$ follows by Lemma 2.6.
6 More words on Mahler’s first question

Corollary 1.5 gives the divergence theory of Mahler’s first question. However, it is not clear for the convergence theory. The main difficulty is that we have no information on the reduced form of \( p_{\text{int}}/q_{\text{int}} \).

For each \( n \geq 1 \), define

\[
N_n = \left\{ \frac{p}{q} \in \mathbb{K} : 3^{n-1} < q \leq 3^n \right\}.
\]

Recall the set \( \mathcal{W}_{\mathbb{K}}(\psi) \) related to Mahler’s intrinsic Diophantine approximation:

\[
\mathcal{W}_{\mathbb{K}}(\psi) = \left\{ x \in \mathbb{K} : |x - p/q| < \psi(q), \text{ i.m. } p/q \in \mathbb{K} \right\}
= \limsup_{n \to \infty} \bigcup_{p/q \in \mathbb{K}, \, 3^{n-1} < q \leq 3^n} B(p/q, \psi(q)) \cap \mathbb{K}.
\]

Thus by the convergence part of Borel–Cantelli lemma,

\[
\mu(\mathcal{W}_{\mathbb{K}}(\psi)) = 0, \quad \text{if } \sum_{n=1}^{\infty} \#N_n \cdot \left( \psi(3^{n-1}) \right)^Y < \infty. \quad (6.1)
\]

In other words, the convergence theory is highly related to the number of reduced rational numbers in \( \mathbb{K} \) with a prescribed range of the denominators, e.g. \( \#N_n \).

Now we give some observations on \( N_n \). Let \( p/q \in \mathbb{K} \) with \( \mathcal{L}(p/q) = \ell \) and \( \mathcal{P}(p/q) = m \).

So we write it as

\[
p/q = [\epsilon_1, \ldots, \epsilon_{\ell}, (\epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m})^\infty].
\]

Then by Definition 2.2, \( q_{\text{int}} = 3^\ell (3^m - 1) \) and

\[
p_{\text{int}} = 3^{\ell + m}[\epsilon_1, \ldots, \epsilon_\ell, \epsilon_{\ell + 1}, \ldots, \epsilon_{\ell + m}] - 3^\ell [\epsilon_1, \ldots, \epsilon_\ell] \quad (6.2)
= (3^m - 1)3^\ell [\epsilon_1, \ldots, \epsilon_\ell] + 3^m [\epsilon_{\ell + 1}, \ldots, \epsilon_{\ell + m}] \quad (6.3)
\]

**Lemma 6.1** Use gcd to denote the greatest common divisor. Then

\[
gcd(p_{\text{int}}, 3^\ell) = 1, \quad \gcd(p_{\text{int}}, q_{\text{int}}) = \gcd(3^m[\epsilon_{\ell + 1}, \ldots, \epsilon_{\ell + m}], 3^m - 1).
\]

**Proof** If \( p_{\text{int}} \) is a multiple of 3, by (6.2), one has that

\[
\epsilon_{\ell + m} = \epsilon_\ell.
\]

As a result,

\[
[\epsilon_1, \ldots, \epsilon_\ell, (\epsilon_{\ell + 1}, \ldots, \epsilon_{\ell + m})^\infty] = [\epsilon_1, \ldots, \epsilon_{\ell - 1}, (\epsilon_\ell, \ldots, \epsilon_{\ell + m - 1})^\infty].
\]

This contradicts the definition of \( \mathcal{L}(p/q) \). So the first assertion follows.

By the definition of \( p_{\text{int}}, \ q_{\text{int}} \) and the first assertion, one has

\[
gcd(p_{\text{int}}, q_{\text{int}}) = \gcd(p_{\text{int}}, 3^m - 1).
\]

Then by (6.3) and a simple fact that \( \gcd(a + tb, b) = \gcd(a, b) \), one has

\[
gcd(p_{\text{int}}, q_{\text{int}}) = \gcd(3^m[\epsilon_{\ell + 1}, \ldots, \epsilon_{\ell + m}], 3^m - 1).
\]

\[\square\]

**Lemma 6.2** For any \( p/q \in N_n \), one has \( \mathcal{L}(p/q) \leq n \) and \( \mathcal{L}(p/q) + \mathcal{P}(p/q) \geq n. \)
Proof Let $L(p/q) = \ell$ and $P(p/q) = m$. Then by the first assertion in Lemma 6.1, one has
\[ q \geq 3^\ell. \]
Since $p/q \in N_n$, it follows that
\[ 3^\ell \leq q \leq 3^n, \quad 3^{n-1} < q_{\text{int}} \leq 3^\ell + m. \]

\[ \square \]

As a corollary of Lemma 6.2, by writing
\[ A = \left\{ p/q \in K : \, 3^{n-1} < q \leq 3^n, \, L(p/q) \leq M + \log_2 n \right\}, \]
for any fixed $M > 0$, we have the following.

Corollary 6.3 \# $A \leq 2^{M+2} \cdot n \cdot 2^n$.

Proof By Lemma 6.2,
\[ \# A = \# \left\{ p/q \in K : \, 3^{n-1} < q \leq 3^n, \, L(p/q) \leq n, \, P(p/q) \leq M + \log_2 n \right\} \]
\[ \leq \# \left\{ [\epsilon_1, \ldots, \epsilon_{\ell}, (\epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m})^\infty] : \, \ell \leq n, \, m \leq M + \log_2 n, \, \epsilon_i \in \{0, 2\}, \, 1 \leq i \leq \ell + m \right\} \]
\[ = \sum_{\ell \leq n} \sum_{m \leq M + \log_2 n} 2^{\ell + m} \leq 2^{M+2} \cdot n \cdot 2^n. \]

\[ \square \]

As a consequence, by writing
\[ W^{(1)}(\psi) = \left\{ x \in K : |x - p/q| < \psi(q), \, \text{i.m.} \, p/q \in K, \, P(p/q) \leq \log_2 \log q \right\} \]
we have the following corollary.

Corollary 6.4
\[ \sum_{n \geq 1} n \cdot 2^n \left( \psi(3^n) \right)^\gamma < \infty \implies \mu \left( W^{(1)}(\psi) \right) = 0. \]

Proof Because of the limsup nature of $W^{(1)}(\psi)$, it can be written as
\[ W^{(1)}(\psi) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{p/q \in K: 3^n < q \leq 3^{n+1}, \, P(p/q) \leq \log_2 \log q} B \left( \frac{p}{q}, \psi(q) \right). \]
Then by Corollary 6.3 on the number of rationals in the union over $p/q$ and $\gamma$-regularity of the measure $\mu$, one has
\[ \mu \left( \bigcup_{p/q \in K: 3^n < q \leq 3^{n+1}, \, P(p/q) \leq \log_2 \log q} B \left( \frac{p}{q}, \psi(q) \right) \right) \leq 8 \cdot (n + 1) \cdot 2^{n+1} \cdot c^{-1} \left( \psi(3^n) \right)^\gamma \leq c^{-1} 2^5 \cdot n \cdot 2^n \cdot \left( \psi(3^n) \right)^\gamma. \]
Then the claimed result follows by the convergence part of Borel–Cantelli lemma. \[ \square \]
Thus, the only problem left is to consider the set
\[ \mathcal{W}^{(2)}(\psi) = \left\{ x \in \mathbb{K} : |x - p/q| < \psi(q), \text{ i.m. } p/q \in \mathbb{K}, \text{ } \mathcal{P}(p/q) > \log_2 \log q \right\} \]

Lemma 6.1 indicates that to understand the reduced form of \( p_{\text{int}} / q_{\text{int}} \), one need to be clear about the reduced form of a purely periodic 3-adic expansion. Define
\[ \mathcal{P}_m := \left\{ p'/q' \in \mathbb{K} : \text{purely periodic, } 3^{m-1} < q' \leq 3^m \right\}. \]

**Lemma 6.5** One has
\[ \#N_n \leq \sum_{m \leq n} 2^{n-m} \#\mathcal{P}_m. \]

**Proof** By Lemma 6.2, one has
\[ \#N_n = \# \left\{ p/q \in \mathbb{K} : 3^{n-1} < q \leq 3^n \right\} = \sum_{\ell \leq n} \# \left\{ p/q \in \mathbb{K} : L(p/q) = \ell, 3^{n-1} < q \leq 3^n \right\}. \]

Let \( p/q \in N_n \) with \( L(p/q) = \ell \). Then we write it as
\[ p/q = [\epsilon_1, \ldots, \epsilon_{\ell}, (\epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m})^\infty] \quad (6.4) \]
for some \( m \) with \( \mathcal{P}(p/q) = m \). By cutting off the first \( \ell \) digits in the expansion of \( p/q \), we define
\[ p'/q' = [(\epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m})^\infty] = \frac{3^m [\epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m}]}{3^m - 1}. \]

It is trivial that
\[ q' = \frac{3^m - 1}{\gcd(3^m [\epsilon_{\ell+1}, \ldots, \epsilon_{\ell+m}], 3^m - 1)}, \quad q = \frac{(3^m - 1)3^\ell}{\gcd(p_{\text{int}}, q_{\text{int}})} \]

By Lemma 6.1, we know that the two denominators are the same, so it follows that
\[ q' = q/3^\ell. \]

This implies that every \( p/q \in N_n \) with \( L(p/q) = \ell \) corresponds to a reduced rational \( p'/q' \in \mathbb{K} \) purely periodic with
\[ 3^{n-1-\ell} < q' \leq 3^{n-\ell}. \]

On the other hand, by the form (6.4) there are at most \( 2^\ell \) rationals \( p/q \) with \( L(p/q) = \ell \) corresponds to a same rational \( p'/q' \). Thus one has
\[ \#N_n' \leq \sum_{\ell \leq n} 2^\ell \# \left\{ p'/q' \in \mathbb{K} : \text{purely periodic, } 3^{n-1-\ell} < q' \leq 3^{n-\ell} \right\} \]
\[ = \sum_{\ell \leq n} 2^\ell \# \mathcal{P}_{n-\ell} = \sum_{m \leq n} 2^{n-m} \#\mathcal{P}_m. \]
If it is true that 
\[ \#P_m \ll 2^m, \text{ for all } m \gg 1, \]
one would have that 
\[ \#N_n \leq 2^n \sum_{m \leq n} \frac{\#P_m}{2^m} \ll n 2^n. \]
Thus by (6.1), if one can show \( \#N_n \ll n 2^n \) or \( \#P_n \ll 2^n \) for all \( n \geq 1 \), one will have 
\[ \mu(W_K(\psi)) = 0, \text{ if } \sum_{n=1}^{\infty} n \cdot \left(3^n \psi(3^n)\right)^\gamma < \infty. \]
Thus a complete metric theory for Mahler’s first question could be established: *almost all or almost no points in \( K \) are intrinsically \( \psi \)-well approximable according to* 
\[ \sum_{n \geq 1} n \left(3^n \psi(3^n)\right)^\gamma = \infty \text{ or } < \infty. \]

For the cardinality of \( N_n \), there are some progress recently in [29, 30]. But it is still far from being clear. The following are some conjectures:

- **in [8]** by Broderick, Fishman and Reich
  \[ \#N_n \ll 2^{\tau n}, \text{ for some } \tau < 2. \]

- **in [15]** by Fishman and Simmons
  \[ \#N'_n \ll 2^{(1+\epsilon)n}, \text{ for any } \epsilon > 0. \]

- We also pose an ambitious conjecture
  \[ \#P_n \ll 2^n, \text{ or } \#N'_n \ll n \cdot 2^n. \]

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