A Note on the Closed-Form Solution for the Longest Head Run Problem of Abraham de Moivre

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In a series of $n$ independent trials, an event $E$ has a probability $p$ of occurring for each trial. If event $E$ occurs at least $r$ times in succession in these trials, then we say that we have a run of size $r$. What is the probability $y_{n,r}$ of having a run of size $r$ in $n$ trials? This problem was formulated and solved by Abraham de Moivre in the second edition of his book *The Doctrine of Chances: Or, a Method of Calculating the Probabilities of Events in Play* [1, Problem LXXXVIII, p. 243]. Although more than 280 years have passed since then, de Moivre’s problem and its variations remain of great interest in probability and statistics; see, for example, [7] and references therein.

Formally, let $X_1, \ldots, X_n$ be a sequence of independent and identically distributed (iid) Bernoulli random variables with the probability of success $p = P(X_i = 1) = P(E)$ and the probability of failure $1 - p = P(X_i = 0)$. We denote by

$$L_n = \max\{k : X_{i+1} = \cdots = X_{i+k} = 1, \ i = 0, 1, \ldots, n - k\}$$

the length of the longest consecutive series of trials in which event $E$ occurs. Then, recalling that having a run of size $r$ is defined as the occurrence of at least $r$ consecutive successes in $n$ iid Bernoulli trials with corresponding probability $y_{r,n}$, we obtain

$$y_{n,r} = P(L_n \geq r).$$

De Moivre did not provide a proof, but he demonstrated a method for finding $y_{n,r}$. Reviewing that method, one can see that he used the method of generating functions. He demonstrated the method with an example of ten trials having $p = 1/2$, in which he obtained a probability of a run of size 3 equals 65/128.

A closed-form solution of $y_{n,r}$ was given by J. V. Uspensky in [10] as a polynomial with binomial coefficients, arrived at by first obtaining a difference equation and then using the method of generating functions to solve it. Surprisingly, a simple closed form of $y_{n,r}$ can be obtained as a corollary to the difference equation given by Uspensky. This closed-form solution seems to have never been reported in the literature. In this note, we present it along with Uspensky’s original derivations.

**Uspensky’s Solution**

We present Uspensky’s solution [10, pp. 77–79], keeping his original notation. This solution demonstrates the power of the use of ordinary linear difference equations along with generating functions.

Let us consider $n + 1$ trials with the corresponding probability $y_{n+1,r}$. A run of size $r$ in $n + 1$ trials can happen in two mutually exclusive ways:

- **(W1)** the run is obtained in the first $n$ trials;
- **(W2)** the run is obtained on trial $n + 1$.

The second way, (W2), means that among the first $n - r$ trials, there is no run of size $r$, event $E^c$ occurred at trial $n - r + 1$; and event $E$ occurred in the trials $n - r + 2, \ldots, n + 1$. Combining (W1) and (W2), we obtain a linear difference equation of order $r + 1$,

$$y_{n+1,r} = y_{n,r} + (1 - y_{n-r,r})pq^r,$$  \hspace{1cm} (1)

with initial conditions

$$y_{0,r} = y_{1,r} = \cdots = y_{r-1,r} = 0, \quad y_{r,r} = p^r,$$

where $q = 1 - p$.

Using the method of generating functions, Uspensky obtained the following closed-form solution of (1):

$$y_{n,r} = 1 - \beta_{n,r} + p^r \beta_{n-r,r},$$

$$\beta_{n,r} = \sum_{l=0}^{\left[\frac{n}{r}\right]} (-1)^l \binom{n - lr}{l} (qpr)^l,$$  \hspace{1cm} (2)

where $[x]$ is defined as the greatest integer less than or equal to $x$.

Returning to de Moivre’s original example, where $n = 10$, $r = 3$, and $p = 1/2$, and using (2), we obtain $y_{10,3} =$...
and from (3), we obtain the positions 2
with initial conditions Sze´kely and Tusna´dy [9]. That problem was then extended to solution of the problem in the case of a fair coin was given by Moivre’s longest head run problem. An interesting recursive extended in [3]. A large collection of classical problems in another problem of de Moivre’s that was later discussed and probability with historical comments and original citations is motivations [5]. The problem has also found applications in the Markov chain setting, where Uspensky’s generating Doctrine of Chances

Surprisingly, a simple closed-form solution follows from Uspensky’s equation (1) as follows: Replacing

\[ y_{n,r} = y_{n-1,r} + (1 - y_{n-1,r-r}) q p^r, \]  

with initial conditions \( y_{0,r} = \cdots = y_{r-1,r} = 0 \), \( y_{r,r} = p^r \).

If \( n - 1 - r \leq r - 1 \) (i.e., \( n/2 \leq r \)), then \( y_{n-1,r-r} = 0 \), and from (3), we obtain

\[ y_{n,r} = y_{n-1,r} + q p^r, \quad \text{for } n \leq 2, \]

with initial conditions \( y_{0,r} = \cdots = y_{r-1,r} = 0 \), \( y_{r,r} = p^r \).

On iterating (4), we obtain the following corollary.

**COROLLARY 1** If \( n/2 \leq r \leq n \), where \( r \) is an integer, then

\[ y_{n,r} = p^r + (n - r) q p^r. \]

An alternative explanation for (5) is that when \( n/2 \leq r \leq n \), we cannot have two separate runs of size \( r \). Therefore, a run of size \( r \) in \( n \) trials begins either at position 1 (this happens with probability \( p^r \)) or at one of the positions \( 2, \ldots, n -(r-1) \) (at each such position, this happens with probability \( q p^r \)). Then denote by \( R_i \) an event in which a run of size \( r \) begins at position \( i \), \( i = 1, 2, \ldots, n -(r-1) \). Since the events \( R_1, R_2, \ldots, R_{n-(r-1)} \) are disjoint, we obtain

\[ y_{n,r} = P(R_1) + P(R_2) + \cdots + P(R_{n-(r-1)}) = p^r + (n - r) q p^r. \]

**Comments**

There are a number of interesting problems discussed in Uspensky’s book [10], many of which have roots in the classics of probability, their origins tracing back to founders of modern probability such as Blaise Pascal, Pierre de Fermat, Christiaan Huygens, Jacob Bernoulli, Abraham de Moivre, Pierre-Simon Laplace, Andrey Andreyevich Markov, Sergei Natanovich Bernstein, and others. For example, Uspensky [10] considered another problem of de Moivre’s that was later discussed and extended in [3]. A large collection of classical problems in probability with historical comments and original citations is nicely presented in the book [4].

There are many follow-ups on and extensions of de Moivre’s longest head run problem. An interesting recursive solution of the problem in the case of a fair coin was given by Székely and Tusnády [9]. That problem was then extended to the Markov chain setting, where Uspensky’s generating function was generalized to the case of dependent observations [5]. The problem has also found applications in numerous fields, among which are reliability [2], computational biology [8], and finance, where time-dependent sequences naturally occur (see [6] and references therein).

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