Integrable geodesic motion on 3D curved spaces from non-standard quantum deformations

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Abstract

The link between 3D spaces with (in general, non-constant) curvature and quantum deformations is presented. It is shown how the non-standard deformation of a \textit{sl}(2) Poisson coalgebra generates a family of integrable Hamiltonians that represent geodesic motions on 3D manifolds with a non-constant curvature that turns out to be a function of the deformation parameter \(z\). A different Hamiltonian defined on the same deformed coalgebra is also shown to generate a maximally superintegrable geodesic motion on 3D Riemannian and \((2 + 1)\)D relativistic spaces whose sectional curvatures are all constant and equal to \(z\). This approach can be generalized to arbitrary dimension.

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1 Introduction

Recently, a non-standard deformation of the Poisson $sl(2)$ coalgebra has been used to obtain a very large family of (super)integrable geodesic motions on certain two-dimensional (2D) spaces with (constant or variable) curvature \cite{1}. The spaces with constant curvature come from the superintegrable geodesic motion; these are the sphere, Euclidean, hyperbolic, (anti-)de Sitter and Minkowskian spaces. For all of them the deformation parameter $z$ coincides exactly with the Gaussian curvature. In turn, the spaces with variable curvature arise from the integrable geodesic motion and they can be considered as deformations of the abovementioned spaces for which the curvature is a function of both the deformation parameter $z$ and some intrinsic coordinates of the space. Moreover, some known and new (super)integrable potentials (such as oscillator and Kepler-type ones) defined on these curved spaces have also been obtained \cite{2} by making use of the dynamical coalgebra symmetry \cite{3}.

Although the quantum coproduct ensures the existence of the generalization of this “dynamical generation” of curvature to arbitrary dimension, the explicit geometric characterization of the spaces so obtained is far from being straightforward. The aim of this contribution is to present a first step in this direction by making fully explicit the class of 3D curved spaces together with their associated free Hamiltonians that arise from this $q$-deformed geodesic dynamics.

2 Integrable geodesic motion on 2D curved spaces

Let us briefly recall the 2D results by considering the non-standard quantum deformation of $sl(2)$ written as a Poisson coalgebra $(sl_z(2), \Delta_z)$ with real deformation parameter $z$; the Poisson brackets, coproduct and Casimir are given by \cite{4}

\[
\{J_3, J_+\} = 2J_+ \cosh z J_-, \quad \{J_3, J_-\} = -2 \frac{\sinh z J_-}{z}, \quad \{J_-, J_+\} = 4J_3, \quad (1)
\]

\[
\Delta_z(J_-) = J_- \otimes 1 + 1 \otimes J_-, \quad \Delta_z(J_l) = J_l \otimes e^{z J_-} + e^{-z J_-} \otimes J_l, \quad l = +, 3, \quad (2)
\]

\[
C_z = \frac{\sinh z J_-}{z} J_+ - J^2_3. \quad (3)
\]

Starting from the one-particle symplectic realization of (1) given by

\[
J^{(1)}_- = q_1^2, \quad J^{(1)}_+ \equiv \frac{\sinh z q_1^2}{z q_1^2} p_1^2, \quad J^{(1)}_3 = \frac{\sinh z q_1^2}{z q_1^2} q_1 p_1, \quad (4)
\]

(under the which $C^{(1)}_z = 0$), the coproduct (2) provides the following two-particle symplectic realization of (1):

\[
J^{(2)}_- = q_1^2 + q_2^2, \quad J^{(2)}_+ \equiv \frac{\sinh z q_1^2}{z q_1^2} p_1^2 e^{z q_2^2} + \frac{\sinh z q_2^2}{z q_2^2} p_2^2 e^{-z q_1^2}, \quad (5)
\]

\[
J^{(2)}_3 = \frac{\sinh z q_1^2}{z q_1^2} q_1 p_1 e^{z q_2^2} + \frac{\sinh z q_2^2}{z q_2^2} q_2 p_2 e^{-z q_1^2}.
\]
By substituting (5) in (3) we obtain the two-particle Casimir

\[ C_z^{(2)} = \frac{\sinh zq_1^2}{zq_1^2} \frac{\sinh zq_2^2}{zq_2^2} (q_1p_2 - q_2p_1)^2 e^{-zq_1^2} e^{zq_2^2}, \]

\[ \text{(6)} \]

which Poisson-commutes with the generators (5).

The coalgebra approach [3] ensures that any smooth Hamiltonian function \( H_z = H_z(J_+^{(2)}, J_-^{(2)}, J_3^{(2)}) \) gives rise to an integrable system, for which \( C_z^{(2)} \) is the constant of the motion. In particular, we find a large family of integrable deformations of the free motion of a particle on the 2D Euclidean space defined by

\[ \mathcal{H}_z = \frac{1}{2} J_+^{(2)} f(z J_-^{(2)}), \]

\[ \text{(7)} \]

where \( f \) is an arbitrary smooth function such that \( \lim_{z \to 0} f(z J_-^{(2)}) = 1 \), i.e., \( \lim_{z \to 0} \mathcal{H}_z = \frac{1}{2}(q_1^2 + p_2^2) \). Two relevant possibilities for \( \mathcal{H}_z \) have been studied in [1]:

- The integrable Hamiltonian \( \mathcal{H}_z^I = \frac{1}{2} J_+^{(2)} \), which defines the geodesic motion on a 2D Riemannian space with metric

\[ ds^2 = \frac{2zq_1^2}{\sinh zq_1^2} e^{-zq_2^2} dq_1^2 + \frac{2zq_2^2}{\sinh zq_2^2} e^{zq_1^2} dq_2^2, \]

\[ \text{(8)} \]

and whose non-constant Gaussian curvature reads

\[ K = -z \sinh \left(z(q_1^2 + q_2^2)\right). \]

- The superintegrable Hamiltonian \( \mathcal{H}_z^S = \frac{1}{2} J_+^{(2)} e^{z J_-^{(2)}} \), which is a Stäckel system [5] so that this is endowed with an additional constant of the motion [4]

\[ I_z^{(2)} = \frac{\sinh zq_1^2}{2zq_1^2} e^{zq_1^2} p_1^2. \]

\[ \text{(9)} \]

This Hamiltonian leads to a Riemannian metric of constant curvature which coincides with the deformation parameter, \( K = z \), namely

\[ ds_S^2 = \frac{2zq_1^2}{\sinh zq_1^2} e^{-zq_1^2} e^{-2zq_2^2} dq_1^2 + \frac{2zq_2^2}{\sinh zq_2^2} e^{-zq_2^2} dq_2^2. \]

\[ \text{(10)} \]

A suitable change of coordinates [1] (depending on an additional “contraction” parameter) have allowed us to derived the 2D sphere, Euclidean, hyperbolic, Minkowskian and (anti-)de Sitter spaces from the metric (10). Likewise, their “deformed” counterpart (understood as spaces with non-constant curvature) have been deduced from the integrable metric (8). The explicit solution of the geodesic flows for all these spaces has been studied in [2], as well as a method to introduce (super)integrable potentials on them by adding a potential term of the form \( U(z J^{(2)}) \); in this way, some known potentials are recovered (appearing in the classifications [6, 7]) and also new ones are obtained. We recall that another approach to superintegrability on 2D spaces of variable curvature can be found in [8, 9].
3 Integrable geodesic motion on 3D curved spaces

In order to perform the generalization of this construction to 3D spaces, a three particle symplectic realization of the deformed Poisson algebra \( \{ \} \) has to be obtained from the 3-sites coproduct map \( \delta \), which is defined as:

\[
\Delta_z^{(3)} = (\Delta_z \otimes \text{id}) \circ \Delta_z = (\text{id} \otimes \Delta_z) \circ \Delta_z. \tag{11}
\]

Hence from (2) and (4), we find that (11) reads

\[
\begin{align*}
J_z^{(3)} &= q_1^2 + q_2^2 + q_3^2 \equiv q^2, \\
J_z^{(3)} &= \frac{\sinh z q_1^2}{z q_1^2} p_1^2 e^{z q_1^2} e^{z q_3^2} + \frac{\sinh z q_2^2}{z q_2^2} p_2^2 e^{z q_2^2} e^{z q_3^2} + \frac{\sinh z q_3^2}{z q_3^2} p_3^2 e^{-z q_1^2} e^{-z q_2^2}, \\
J_z^{(3)} &= \frac{\sinh z q_1^2}{z q_1^2} q_1 p_1 e^{z q_2^2} e^{z q_3^2} + \frac{\sinh z q_2^2}{z q_2^2} q_2 p_2 e^{z q_1^2} e^{z q_3^2} + \frac{\sinh z q_3^2}{z q_3^2} q_3 p_3 e^{-z q_1^2} e^{-z q_2^2}.
\end{align*} \tag{12}
\]

By substituting these expressions in (5) we get the three-particle Casimir

\[
C_z^{(3)} = \frac{\sinh z q_1^2}{z q_1^2} \frac{\sinh z q_2^2}{z q_2^2} (q_1 p_2 - q_2 p_1)^2 e^{-2 z q_1^2} e^{2 z q_2^2} + \frac{\sinh z q_1^2}{z q_1^2} \frac{\sinh z q_3^2}{z q_3^2} (q_1 p_3 - q_3 p_1)^2 e^{-2 z q_1^2} e^{2 z q_3^2} + \frac{\sinh z q_2^2}{z q_2^2} \frac{\sinh z q_3^2}{z q_3^2} (q_2 p_3 - q_3 p_2)^2 e^{-2 z q_2^2} e^{2 z q_3^2}, \tag{13}
\]

which Poisson-commutes, by construction (3), with the three-particle generators (12) and also with the two-particle Casimir \( C_z^{(2)} \) (6). Hence the generic Hamiltonian \( H_z = H_z(J_z^{(3)}, J_z^{(3)}, J_z^{(3)}) \) determines a family of integrable systems as the three functionally independent functions \( \{ H_z, C_z^{(2)}, C_z^{(3)} \} \) are mutually in involution.

3.1 Integrable geodesic motion on spaces of non-constant curvature

As in the 2D case, we consider the kinetic energy \( T_z^{(1)}(q_i, \dot{q}_i) \) coming from the integrable Hamiltonian \( H_z^{(1)}(q_i, p_i) \) that can be rewritten as the free Lagrangian

\[
2T_z^{(1)} = \frac{z q_1^2}{\sinh z q_1^2} e^{-2 z q_2^2} e^{-2 z q_3^2} \dot{q}_1^2 + \frac{z q_2^2}{\sinh z q_2^2} e^{2 z q_1^2} e^{-2 z q_3^2} \dot{q}_2^2 + \frac{z q_3^2}{\sinh z q_3^2} e^{2 z q_1^2} e^{2 z q_2^2} \dot{q}_3^2, \tag{14}
\]

which defines a geodesic flow on a 3D Riemannian space with a definite positive metric given, up to a constant factor, by

\[
ds^2 = \frac{2 z q_1^2}{\sinh z q_1^2} e^{-2 z q_2^2} e^{-2 z q_3^2} d\dot{q}_1^2 + \frac{2 z q_2^2}{\sinh z q_2^2} e^{2 z q_1^2} e^{-2 z q_3^2} d\dot{q}_2^2 + \frac{2 z q_3^2}{\sinh z q_3^2} e^{2 z q_1^2} e^{2 z q_2^2} d\dot{q}_3^2. \tag{15}
\]

The sectional curvatures \( K_{ij} \) in the planes 12, 13 and 23 turn out to be

\[
\begin{align*}
K_{12} &= \frac{\hat{z}}{4} e^{-2 z q^2} \left( 1 + e^{2 z q^2} - 2 e^{2 z q^2} \right), \\
K_{13} &= \frac{\hat{z}}{4} e^{-2 z q^2} \left( 2 - e^{2 z q^2} + e^{2 z q^2} e^{2 z q^2} - 2 e^{2 z q^2} \right), \\
K_{23} &= \frac{\hat{z}}{4} e^{-2 z q^2} \left( 2 - e^{2 z q^2} e^{2 z q^2} - 2 e^{2 z q^2} \right),
\end{align*} \tag{16}
\]

4
while the scalar curvature $K$ fulfills

$$K = 2(K_{12} + K_{13} + K_{23}) = -5z \sinh(zq^2).$$

(17)

The geometric characterization of these spaces becomes much more clear if we introduce the new canonical coordinates $(\rho, \theta, \phi)$ and conjugated momenta $(p_\rho, p_\theta, p_\phi)$ defined by

$$\cosh^2(\lambda_1 \rho) = e^{2zq^2},$$

$$\sinh^2(\lambda_1 \rho) \cos^2(\lambda_2 \theta) = e^{2zq^2} e^{2zq^2} (e^{2zq^2} - 1),$$

$$\sinh^2(\lambda_1 \rho) \sin^2(\lambda_2 \theta) \cos^2(\phi) = e^{2zq^2} (e^{2zq^2} - 1),$$

$$\sinh^2(\lambda_1 \rho) \sin^2(\lambda_2 \theta) \sin^2(\phi) = e^{2zq^2} - 1,$$

(18)

where $z = \lambda_1^2$ and $\lambda_2 \neq 0$ is an additional parameter which can be either a real or a pure imaginary number. Thus the metric (15) is transformed into

$$ds^2 = \frac{1}{\cosh(\lambda_1 \rho)} \left( d\rho^2 + \lambda_2^2 \frac{\sinh^2(\lambda_1 \rho)}{\lambda_1^2} \left( d\theta^2 + \frac{\sin^2(\lambda_2 \theta)}{\lambda_2^2} d\phi^2 \right) \right),$$

(19)

which is just the metric of the 3D Riemannian and relativistic spacetimes written in geodesic polar coordinates multiplied by a global factor $1/\cosh(\lambda_1 \rho) \equiv e^{-zJ^3}$ that encodes the information concerning the variable curvature of the space. In the new coordinates the sectional and scalar curvatures read

$$K_{12} = K_{13} = -\frac{1}{2} \lambda_1^2 \frac{\sinh^2(\lambda_1 \rho)}{\cosh(\lambda_1 \rho)}, \quad K_{23} = \frac{1}{2} K_{12}, \quad K = -\frac{5}{2} \lambda_2^2 \frac{\sinh^2(\lambda_1 \rho)}{\cosh(\lambda_1 \rho)}.$$

(20)

Therefore, by considering the possible pairs $(\lambda_1, \lambda_2)$ we have obtained a quantum group deformation of the 3D sphere $(i, 1)$, hyperbolic $(1, 1)$, de Sitter $(1, i)$ and anti-de Sitter $(i, i)$ spaces. The “classical” limit $z \to 0$ corresponds to a zero-curvature limit which leads to the proper Euclidean $(0, 1)$ and Minkowskian $(0, i)$ spaces. The resulting integrable Hamiltonian on these six curved spaces with its two constants of motion in the latter phase space turn out to be

$$H^I_z = \frac{1}{2} \cosh(\lambda_1 \rho) \left( p_\rho^2 + \lambda_2^2 \frac{\sinh^2(\lambda_1 \rho)}{\lambda_1^2} \left( p_\theta^2 + \frac{\lambda_2^2}{\sin^2(\lambda_2 \theta)} p_\phi^2 \right) \right),$$

(21)

$$C^{(2)}_z = p_\phi^2, \quad C^{(3)}_z = p_\theta^2 + \frac{\lambda_2^2}{\sin^2(\lambda_2 \theta)} p_\phi^2,$$

(22)

provided that $H^I_z = 2\mathcal{H}^I_z$, $C^{(2)}_z = 4C^{(2)}_z$ and $C^{(3)}_z = 4\lambda_2^2 C^{(3)}_z$.

### 3.2 Superintegrable geodesic motion on spaces of constant curvature

We now consider the Hamiltonian $\mathcal{H}^S_z = \frac{1}{2} J^{(3)}_+$ which has four (functionally independent) constants of motion, namely $C^{(2)}_z$ [10], $C^{(3)}_z$ [12], $T^{(2)}_z$ [12] and $L^{(3)}_z$ [11]

$$T^{(3)}_z = \frac{\sinh zq_1^2}{2zq_1^2} e^{zq_1^2} e^{2zq_1^2} p_1^2 + \frac{\sinh zq_2^2}{2zq_2^2} e^{zq_2^2} p_2^2.$$

(23)
Consequently, $H_z^S$ determines a maximally superintegrable system with free Lagrangian and associated metric given, in terms of [13] and [15], by $T_z^S = T_z^R e^{-2\alpha^2}$ and $ds_z^2 = ds_R^2 e^{-2\alpha^2}$.

The 2D pattern that links maximal superintegrability and constant curvature is reproduced in the 3D case since the space defined by $ds_z^2$ is of Riemannian type with constant sectional and scalar curvatures $K_{ij} = z$, $\kappa = 6z$. Next, through the change of coordinates [18] and by introducing a new radial coordinate [11]

$$r = \int_0^\rho \frac{dx}{\cosh(\lambda_1 x)},$$

(i.e. $\cosh(\lambda_1 \rho) = 1/\cos(\lambda_1 r)$), we find that $ds_z^2$ is just transformed into a metric written in terms of geodesic polar coordinates [10]

$$ds_z^2 = dr^2 + \lambda_2^2 \frac{\sin^2(\lambda_1 r)}{\lambda_1^2} \left( d\theta^2 + \frac{\sin^2(\lambda_2 \theta)}{\lambda_2^2} d\phi^2 \right).$$

According to the pair $(\lambda_1, \lambda_2)$, this metric provides the usual (non-deformed) 3D sphere $(1, 1)$, Euclidean $(0, 1)$, hyperbolic $(i, 1)$, anti-de Sitter $(1, i)$, Minkowskian $(0, i)$, and de Sitter $(i, i)$ spaces. In the latter phase space, the maximal superintegrable Hamiltonian, $H_z^S = 2H_z^I$, is found to be

$$H_z^S = \frac{1}{2} \left( \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)} \left( p_r^2 + \frac{\lambda_2}{\lambda_1} \sin^2(\lambda_2 \theta) p_\theta^2 \right) \right),$$

while its four constants of motion are the same $C_z^{(2)}$ and $C_z^{(3)}$ [22], together with

$$I_z^{(2)} = \left( \frac{\lambda_2 \sin(\lambda_2 \theta)}{\tan(\lambda_1 r)} \sin \phi p_r + \frac{\lambda_1 \cos(\lambda_2 \theta)}{\tan(\lambda_1 r)} \sin \phi p_\theta + \frac{\lambda_1 \lambda_2 \cos \phi}{\tan(\lambda_1 r) \sin(\lambda_2 \theta)} p_\phi \right)^2,$$

$$I_z^{(3)} = \left( \lambda_2 \sin(\lambda_2 \theta) p_r + \frac{\lambda_1 \cos(\lambda_2 \theta)}{\tan(\lambda_1 r)} p_\theta \right)^2 + \lambda_1^2 \lambda_2^2 \left( \frac{\tan^2(\lambda_1 r) \sin^2(\lambda_2 \theta) + 1}{\tan^2(\lambda_1 r) \sin^2(\lambda_2 \theta)} \right) p_\phi^2,$$

where $I_z^{(2)} = 4\lambda_2^2 I_z^{(2)}$ and $I_z^{(3)} = 4\lambda_2^2 I_z^{(3)}$. Notice that the two sets \{ $H_z^S, C_z^{(2)}, C_z^{(3)}$ \} and \{ $H_z^I, I_z^{(2)}, I_z^{(3)}$ \} are formed by three functions which are mutually in involution.

To conclude, the ND generalization of all of these results should be obtained by following the same construction based on the $N$-th coproduct map. In particular, the ND variable curvature spaces will be defined through the $N$-particle symplectic realization of the free integrable Hamiltonian [11]

$$H_z^{1(N)} = \frac{1}{2} J_+^{(N)} = \frac{1}{2} \sum_{i=1}^N \sinh q_i^2 \frac{p_i^2}{z q_i^2} \exp \left( -z \sum_{k=1}^{i-1} q_k^2 + z \sum_{l=i+1}^N q_l^2 \right),$$

and the ND analogue of the change of coordinates [18] will be the cornerstone for the geometric interpretation of these spaces and their free integrable systems. In the superintegrable case, $H_z^{S(N)} = \frac{1}{2} J_+^{(N)} e^{z J_-^{(N)}} = H_z^{(N)} e^{z J_-^{(N)}}$, we conjecture that the ND analogues of the Riemannian and relativistic spaces here presented will be recovered as the corresponding constant curvature spaces defined by the maximal superintegrable geodesic motion. Work on this line is in progress.
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