The Bianchi Type $I$ minisuperspace model

Sigbjørn Hervik
Department of Physics, University of Oslo
P.O.Box 1048 Blindern
N-0316 Oslo, Norway

Abstract

The minisuperspace model of a Bianchi Type $I$ universe with compact spatial sections is investigated. The classical solutions are brought onto a form were the difference between compact and infinite spatial sections are manifest. One of the features of the compact case is that it has a non-trivial moduli space. The solution space of the compact Bianchi Type $I$ universe is 10 dimensional whereas the Kasner solutions only have a 1 dimensional solution space. We also include the classical solutions with dust and a cosmological constant. Solutions to the Wheeler-DeWitt equation are obtained in light of the tunneling boundary proposal by Vilenkin. Back reaction effects from a simple scalar field are also investigated.

I. INTRODUCTION

Quantum cosmology was initiated by DeWitt [1] in the late sixties. The canonical quantization procedure which in the twenties was the key to the theory of Quantum mechanics, was again used to derive a “Schrödinger equation” for gravity. The idea was that many of the cosmological mysteries which had their cause in the initial epoch of the universe, could be explained by Quantum cosmology (QC). Through the work of DeWitt, Misner [2], Ryan [3, 4] etc., QC had prosperous days in the next decade. Suddenly the interest came to a hold, and the number of articles was drastically reduced. But the interest for QC was again renewed in the eighties pioneered through the work of Hartle and Hawking [5]. They based their work on the boundary condition for a quantum state of the universe. Soon after, Vilenkin [7, 8] and Linde proposed their version of a suitable boundary condition.

Most of the previous work on QC is based on FRW and deSitter universe models although some authors have studied anisotropic models (for instance [14]). The spatial sections of the FRW and deSitter models correspond to globally homogeneous, isotropic, simply connected spaces. In recent years a paper by the mathematician Thurston [15] has drawn many a physicist’s attention. His main conjecture, which still remains to be proven, claims that there are essentially 8 types of 3-manifolds. Concerning cosmology and especially QC the most interesting are their compact quotients which show interesting properties [16].
the nineties several authors investigated locally homogeneous spacetimes with non-trivial topologies. Hawking and Turok \[17\] studied quantum creation of hyperbolic universes in the context of the no boundary proposal. Coule and Martin \[18\] in addition to Costa and Fagundes \[19,20\] studied cosmologies with compact hyperbolic spatial sections. We also want to mention the work of Fagundes \[21\] already in the early eighties, in which he studies compact cosmologies which in the Thurston classification are quotients of $\mathbb{R} \times \mathbb{H}^2$.

In this paper we shall use a locally homogeneous anisotropic model that in the Thurston classification has the covering space $E^3$. We will not consider in any detail the covering space $E^3$ itself (which is simply connected) but rather one of its compact quotients, the three torus, $T^3 \cong \mathbb{R}^3/\mathbb{Z}^3$ (which is a multiply connected space with a non-trivial fundamental group). The possibility of multiply connected spatial sections of the Bianchi metrics \[22,23\] has again renewed the interest for QC. The space $T^3$ allows a Bianchi Type I metric according to the Bianchi classification.

Using Misner’s notation the Bianchi metrics can be written as:

$$ds^2 = -N(t)^2 dt^2 + e^{2\alpha(t)} [e^{2\beta(t)}]_{ij} \chi^i \chi^j$$

where $\beta = \text{diag}(\beta_+, \sqrt{3}\beta_-, \beta_+, -\sqrt{3}\beta_-, -2\beta_+)$ and $\chi^i = R_{ik}\omega^k$ where $R_{ik}$ is an $SO(3)$ matrix which can be parameterized by its Euler angles $(\theta, \phi, \psi)$ i.e. $R = e^{\theta \kappa_3}e^{\phi \kappa_1}e^{\psi \kappa_3}$. The matrices $(\kappa_1, \kappa_2, \kappa_3)$ are the generators of the Lie group $SO(3)$. The invariant 1-forms $\omega^i$ obey the corresponding Bianchi Type Lie algebra: $d\omega^i = -\frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k$.

The Einstein-Hilbert action is given by $\int S = \int_M d^4x \sqrt{-g} (R - 2\Lambda) + 2 \int_{\partial M} d^3x \sqrt{h} K$. In the case of a spatial homogeneous spacetime like the Bianchi Types, an invariant basis of forms on the spatial hypersurfaces can be found. Thus, three of the dimensions in the action may be integrated. We assume that we have a topology of a 3-torus, $T^3$, and for simplicity’s sake the volume will be set equal to 1. Choosing compact spatial sections are done for two reasons: Firstly because the spatial integration of the action is then finite, but also because the solution space of non-isomorphic metrics has a dimension equal to the dimension of the (true) phase space \[24\].

II. THE MODULI SPACE OF THE TOROIDAL BIANCHI TYPE I UNIVERSE AND THE PRINCIPLE OF SYMMETRIC CRITICALITY

Let us briefly review how we construct the torus $T^3$ as a quotient space of $E^3$. It is important that we differentiate between the symmetry group that each hypersurface possesses and the symmetry group the hypersurfaces possess as spatial sections in the four dimensional manifold. For a more thorough investigation on this construction consult \[23\].

The full symmetry group of $E^3$ is $IO(3)$, translations and $O(3)$ rotations. The Bianchi metrics admit a 3 dimensional transitive group of isometries. The 3 dimensional group corresponding to the Bianchi Type I Lie algebra is translations in three dimensions, thus isomorphic to $\mathbb{R}^3$. $\mathbb{R}^3$ is a normal Lie subgroup of $IO(3)$ acting simply transitive on the spaces $\mathbb{R}^3$. We will therefore consider the manifold $\mathbb{R}^3$ with a simply transitivity symmetry

\[1\]In this paper we use the following conventions: $c = 16\pi G = \hbar = 1$. 

2
group $\mathbb{R}^3$ defined in the obvious way. We will call this construction $\hat{\mathbb{R}}^3$. These spaces have exactly the symmetry allowed by the Bianchi Type I Lie algebra. To construct a three torus $T^3$ as a quotient of $\hat{\mathbb{R}}^3$ we can do the following: We find a freely and properly discontinuous subgroup $\Gamma$ of $\text{Sym}(\hat{\mathbb{R}}^3)$, and identify points in $\mathbb{R}^3$ under the action of $\Gamma$. This will ensure that the resulting quotient space is a smooth manifold $\mathbb{25}$. We are interested in subgroups $\Gamma$ so that the resulting quotient has the topology of a three torus. These subgroups $\Gamma$ can be characterized by three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ so that $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \in GL^+(3)$. These vectors will be the generators of the fundamental group of the torus $T^3$ and points in $\hat{\mathbb{R}}^3$ are identified under the action (which in this case is addition) of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. $\Gamma$ will then be $\Gamma = \{(n\mathbf{a} + m\mathbf{b} + k\mathbf{c}) \in \mathbb{R}^3 | n, m, k \in \mathbb{Z}\}$. The moduli space can therefore be described by a matrix $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \in GL^+(3)$ apart from the freedom of discrete modular transformations represented by conjugation with respect to matrices in $GL^+(3, \mathbb{Z})$. We will not fix the gauge with respect to these transformations in this paper. We will simply ignore them. The actual parameter space is therefore Teichmüller space. Through the homeomorphism $GL^+(3) \approx \mathbb{R}^6 \times SO(3)$ we will parametrize the SO(3) sector by the Euler angles $(\theta, \phi, \psi)$.

In this paper we will follow a slightly different but equivalent point of view. We consider a cube $C$ in $\mathbb{R}^3$. We parametrize the cube by $C = \{(x, y, z) | 0 \leq x, y, z \leq 1\}$ and identify points on the boundary of the cube so that we have a toroidal topology. The moduli space will be determined by the linear mapping $L_x(x) = Lx$. In the above description the matrix $L$ is simply $L = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$, $L \in GL^+(3)$. For the Bianchi Type I universe the invariant 1-forms can be written locally as $\omega^i = K^i_j dx^j$ where $K^i_j$ is a constant matrix. After identification the coordinates $x^i$ become angular-like variables with periodicity 1.

For a minisuperspace model one always has to ensure that the equations of motion obtained from a locally variational principle are equivalent to the Einstein field equations. If this is true we say that the Principle of symmetric criticality holds. It has been known for a while that the Bianchi Class B models fail to satisfy this principle $\mathbb{26}$. To prove consistency of the minisuperspace model used in this paper we will use a theorem by Torre and collaborators $\mathbb{27,28}$. This theorem states

The principle of symmetric criticality (PSC) is valid for any metric field theory derivable from a local Lagrangian density if and only if the following 2 conditions are satisfied at each point $x$ in the region of spacetime under consideration.

1. $H^q(G, I_x) \neq 0$
2. $V^I_x \cap (V^I_x)_0 = 0$

We will explain the symbols as we verify the principle of symmetric criticality in our case.

$G$ is the symmetry group which is used for reduction. In our case we have $G = \mathbb{R}^3 \times D_2 \mathbb{23}$. The orbits of $G$ in the spacetime have dimension $q$, i.e. $q = 3$. $I_x$ is the isotropy group for a point $x$. For a Bianchi Type I universe with toroidal spatial sections each point $x$ has

\footnote{The symmetry group of $\hat{\mathbb{R}}^3$ will actually be $\mathbb{R}^3 \rtimes D_2$ where $D_2$ is the dihedral group and $\rtimes$ is the semi-direct product. $D_2$ has 4 elements: The unit element and rotations around the 3 axis by an angle of $\pi$.}
$I_x \cong \mathbb{Z}^3 \times D_2 \subset G$. $H^q(G, I_x)$ is the Lie algebra cohomology of $G$ relative to $I$. In our case it is easy to see that since $H^3(T^3) \neq 0$ where $H^3(T^3)$ is the deRham cohomology class of the $T^3$, we will have $H^q(G, I_x) \neq 0$. The first condition is therefore fulfilled. $V_x$ is the vector space of symmetric rank 2 tensors at a point $x$ in the spacetime, and $V_x^I$ is the vector space of $I_x$-invariant symmetric rank 2 tensors at $x$. $(V_x^I)_0$ is the annihilator of $V_x^I$, i.e. linear functions on $V$ which vanish on $V^I$. By for instance direct calculation it is not difficult by ordinary linear algebra to show that also the second condition holds. Alternatively we can use the fact that the group action acts transversally on the bundle of metrics. Thus the principle of symmetric criticality holds for the Bianchi Type $I$ universe with toroidal spatial topology.

III. THE KASNER UNIVERSE

Kasner [32] solved the Einstein field equations for a Bianchi Type $I$ universe with a vanishing energy-momentum tensor, $T^{\mu\nu} = 0$ and spatial sections homeomorphic to the Euclidean three-space $\mathbb{R}^3$. The solution which now bears his name can be written as:

$$ds^2 = -dt^2 + t^{\frac{2}{3}} \left[ \frac{1}{3} \cos(\gamma) dX^2 + t^{\frac{1}{3}} \cos(\gamma - \frac{2}{3} \pi) dY^2 + t^{\frac{1}{3}} \cos(\gamma + \frac{2}{3} \pi) dZ^2 \right]$$

where $\gamma$ is an angular variable. If the three (complex) roots of the cubic

$$z^3 - \frac{64}{27} e^{3i\gamma} = 0$$

are denoted by $p_1$, $p_2$ and $p_3$, the real parts of $p_1$, $p_2$ and $p_3$ are equal to the three exponents inside the square brackets in eq. 1. Due to eq. 2 the following will also be true:

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 0$$

The Kasner solutions are therefore a 1-parameter family of solutions parameterized by the angular variable $\gamma$. This solution space is therefore usually called the Kasner circle.

Let us denote the 2-parameter family parameterization of the flat Euclidean 3-space with metric

$$d\sigma^2_\gamma(t) = t^{\frac{1}{3}} \cos(\gamma) dX^2 + t^{\frac{1}{3}} \cos(\gamma - \frac{2}{3} \pi) dY^2 + t^{\frac{1}{3}} \cos(\gamma + \frac{2}{3} \pi) dZ^2$$

by $(\mathbb{R}^3, d\sigma^2_\gamma(t))$. For a fixed $X$, $Y$ and $Z$ this family will be volume preserving and homogeneous but it has an anisotropic expansion (in $t$).

Despite that Kasner solved the field equations for a vanishing source tensor, the solutions for a mixture of dust and a vacuum energy can be brought onto a similar form to that of Kasner eq. 1.

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3Due to the fact that a cosmological constant may be viewed as a vacuum energy, we will treat a cosmological constant as a part of the energy-momentum tensor [33].
IV. THE GENERAL SOLUTION OF THE BIANCHI TYPE I UNIVERSE WITH $T^3$ SPATIAL SECTIONS AND WITH DUST AND A COSMOLOGICAL CONSTANT

From [23] we get the Hamiltonian

$$\mathcal{H} = \frac{N}{24} \left[ e^{-3\alpha} \left( -p_\alpha^2 + p_+^2 + p_-^2 \right) + 48M + 48\Lambda e^{3\alpha} \right]$$  \hspace{1cm} (5)$$

where $M$ is a constant defined by $\int d^3x \sqrt{g} \rho_{dust} = 2M$. The canonical 1-form is given by:

$$\Theta = p_\alpha d\alpha + p_+ d\beta_+ + p_- d\beta_- + p_\theta d\theta + p_\phi d\phi + p_\psi d\psi$$

We have here treated the dust field non-dynamically, i.e. solved the classical equations for co-moving dust and thereafter inserted the solutions into the action. The conjugate momentum for the dust field is therefore absent in eq. 5 (compare with [30,31]).

The Einstein field equations are now equivalent to the set of Hamiltonian equations

$$-\dot{p}_i = \frac{\partial \mathcal{H}}{\partial q_i}, \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$  \hspace{1cm} (6)$$

together with the constraint equation

$$\mathcal{H} = 0$$

First we will choose the gauge $N = 1$. From the Hamiltonian equations of motion the following variables will be constants: The Euler angles $(\theta, \phi, \psi)$ and their conjugated momenta $(p_\theta, p_\phi, p_\psi)$ and the conjugated momenta $p_\pm$. Let us set

$$p_+ = 4a, \quad p_- = 4b.$$  \hspace{1cm} (7)$$

and introduce the anisotropy parameter $A^2 = a^2 + b^2$. In addition two more constants $C_\pm$ appear when the equations for $\dot{\beta}_\pm$ are integrated. The constants $(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi, A, C_+, C_-)$ have a clear geometrical meaning. These constants (for $A \neq 0$) specify a matrix $L \in GL^+(3)$ such that the regular solid cube is mapped into the spaces $(\mathbb{R}^3, d\sigma_i^2(t))$ by the linear mapping $l_L(x) \equiv Lx$. The cube is mapped onto a parallelepiped with volume $A = \det(L)$. Introducing the volume element $v = e^{3\alpha}$, the constraint equation yields the equation for $v$:

$$\dot{v}^2 = 3\Lambda v^2 + 3Mv + A^2$$  \hspace{1cm} (8)$$

The constant $M$ will scale as $A$, so it is more useful to introduce a new constant $m$ by $M = Am$. The volume element $v$ will also scale as $A$ so we introduce the function $V(t)$ by $v(t) = AV(t)$.

\footnote{For an explicit expression of this matrix consult [23].
A. The case $\Lambda > 0$

A singularity seems unavoidable since for $v = 0$, $\dot{v} = \pm A$, the volume element starts off at the initial singularity or collapses at the final singularity at a speed $A$. Our interest will be mainly on the expanding solutions. Defining $\kappa = \frac{m}{2|\Lambda|}$ and $\omega^2 = 3\Lambda$ we can write the solutions as:

$$V(t) = \frac{1}{\omega} [\sinh \omega t + \kappa \omega (\cosh \omega t - 1)]$$

where we have chosen $V(0) = 0$.

The remaining equations can be integrated in a straightforward manner, and introducing

$$\Sigma_+(t) = \frac{2}{\omega} \frac{(\cosh \omega t - 1)}{\sinh \omega t + \kappa \omega (\cosh \omega t - 1)}$$

the equations for $\beta_{\pm}$ can be solved to yield

$$\beta_+ = \frac{1}{3} \sin(\gamma - \frac{\pi}{6}) \ln \Sigma_+(t) + C_+$$

$$\beta_- = \frac{1}{3} \cos(\gamma - \frac{\pi}{6}) \ln \Sigma_+(t) + C_-$$

The line-element for a dust-filled Bianchi Type I with a positive cosmological constant is therefore

$$ds^2 = -dt^2 + V(t)^\frac{2}{3} \left[ \Sigma_+(t)^\frac{4}{3} \cos(\gamma) dX^2 + \Sigma_+(t)^\frac{4}{3} \cos(\gamma - \frac{2}{3}\pi) dY^2 + \Sigma_+(t)^\frac{4}{3} \cos(\gamma + \frac{2}{3}\pi) dZ^2 \right]$$

where $dX = Ldx$ and $0 \leq x,y,z \leq 1$ are “angular” variables.

B. The case $\Lambda < 0$

Let us now consider the case where $\Lambda$ is negative. Perhaps the simplest way to obtain the solution for $\Lambda < 0$ is to redefine $\omega^2 = 3|\Lambda|$, and $\kappa = \frac{m}{2|\Lambda|}$, and perform the following mapping:

$$\omega \mapsto i\omega$$

$$\kappa \mapsto -\kappa$$

to obtain for the volume function

$$V(t) = \frac{1}{\omega} [\sin \omega t + \kappa \omega (1 - \cos \omega t)].$$

Here the notation is a bit misleading (which it often is concerning angular variables). The “exact” differentials $dX$ and $dx$ are only exact in the covering space $\mathbb{R}^3$. On the torus $dx$ can be a globally defined 1-form which will be closed but not exact i.e. it corresponds to a non-trivial element in the deRahm cohomology class of the torus.
We can then define
\[
\Sigma^- (t) = \frac{2\omega (1 - \cos \omega t)}{\sin \omega t + \kappa \omega (1 - \cos \omega t)}
\]
The resulting line-element is then
\[
ds^2 = -dt^2 + V(t)^{\frac{2}{3}} \left[ \Sigma^- (t)^{\frac{4}{3}} \cos (\gamma) dX^2 + \Sigma^- (t)^{\frac{4}{3}} \cos (\gamma - \frac{2\pi}{3}) dY^2 + \Sigma^- (t)^{\frac{4}{3}} \cos (\gamma + \frac{2\pi}{3}) dZ^2 \right]
\]
(11)

This line element describes a universe that expands at first. After a while the cosmological term becomes dominant and turn the expanding phase into a contracting one, and it ends in a final singularity after an elapsed time \( t_F = \frac{2}{\omega} \left[ \pi - \arctan \left( \frac{2}{m} \sqrt{\frac{|\Lambda|}{3}} \right) \right] \).

C. The case \( \Lambda = 0 \), and a general form of the line element

The equations for \( \Lambda = 0 \) can be solved in a similar manner, and we will therefore just write down the result. Defining
\[
\Sigma_0 (t) = \frac{t}{1 + \frac{3}{4} mt}
\]
the line element for \( \Lambda = 0 \) takes the form
\[
ds^2 = -dt^2 + \left( t + \frac{3}{4} mt^2 \right)^{\frac{4}{3}} \left[ \Sigma_0 (t)^{\frac{4}{3}} \cos (\gamma) dX^2 + \Sigma_0 (t)^{\frac{4}{3}} \cos (\gamma - \frac{2\pi}{3}) dY^2 + \Sigma_0 (t)^{\frac{4}{3}} \cos (\gamma + \frac{2\pi}{3}) dZ^2 \right]
\]
(12)

Up to a simple rescaling these three line elements are those derived by Saunders [34]. The \( \Lambda = 0 \) case was also derived by Stephani [35] and the \( m = 0 \) case by Grøn [36]. Already now we can see the resemblance of these three cases and the Kasner line element. However, the line element can be brought into a more Kasner-like form by choosing another function \( N \).

We define a new time variable, \( \tau \), by:
\[
t \equiv \int_0^\tau dz \left( 1 - \frac{3}{4} m z^2 - \frac{3}{4} \Lambda z^2 \right)^{\frac{1}{2}} = \begin{cases} 
\Sigma_-^{-1} (\tau) , & \Lambda < 0 \\
\Sigma_0^{-1} (\tau) , & \Lambda = 0 \\
\Sigma_+^{-1} (\tau) , & \Lambda > 0 
\end{cases}
\]
(13)

where \( ^{-1} \) means the inverse function. Inserting this new time variable into the previous line elements, we see that all the three cases can be brought into a similar form. Thus we can write all the three line elements as:
\[
ds^2 = -\frac{d\tau^2}{(1 - \frac{3}{4} m \tau^2 - \frac{3}{4} \Lambda \tau^2)^{\frac{1}{2}}} + \frac{1}{(1 - \frac{3}{4} m \tau^2 - \frac{3}{4} \Lambda \tau^2)^{\frac{1}{3}}} \left[ \tau^{\frac{2}{3}} \cos (\gamma) dX^2 + \tau^{\frac{2}{3}} \cos (\gamma - \frac{2\pi}{3}) dY^2 + \tau^{\frac{2}{3}} \cos (\gamma + \frac{2\pi}{3}) dZ^2 \right]
\]
(14)
As we see, the line element is now manifestly cast onto a Kasner form. The solution space is now explicitly decomposed into the following sequence:

\[
SO(3) \times \mathbb{R}^6 \xrightarrow{l_\mathbf{L}} \mathbb{R}^3 \xrightarrow{s_\gamma(t)} (\mathbb{R}^3, ds_\gamma^2(t)) \xrightarrow{K} (\mathbb{R} \times \Sigma, ds^2)
\] (15)

These mappings can be given the following interpretations:

- **\(l_\mathbf{L} \):** Deformation of the cube. If we represent the unit cube \(C \) by the embedding \( C \cong \{(x^i), i = 1, 2, 3 | 0 \leq (x^i - x_0^i) \leq 1 \} \) for any \((x_0^i)\), then \( l_\mathbf{L}(\mathbf{x}) = \mathbf{Lx} \) for \( \mathbf{L} \in GL^+(3) \cong SO(3) \times \mathbb{R}^6 \). (See figure \[\text{II}\])

- **\(s_\gamma(t) \):** An explicit 2-parameter parameterization of the Euclidean 3-space with an anisotropic metric. Given \((X, Y, Z) \in \mathbb{R}^3 \) the metric is given by eq. \[\text{III}\].

- **\(K \):** The Kasner foliation, given explicitly by: \( K(\mathbb{R}^3, ds_\gamma^2(t)) = (\mathbb{R} \times \Sigma, ds^2) \) where \( \Sigma = \mathbb{R}^3 \) and \( ds^2 = -N(t)^2 dt^2 + V(t)^2 d\sigma_\gamma^2(t) \). The \( N \) and \( V \) are determined by the Einstein field equations and as shown they are given by \( N(t) = \frac{1}{(1 + mt^2)^2 - \Lambda^2} \) and \( V(t) = t \cdot N(t) \).

The solution space is now parameterized by these mappings, and the effect of choosing compact spatial sections is now explicitly shown. We have also factorized the solution space into a topological (or modular) sector (the \( SO(3) \times \mathbb{R}^6 \) sector) and a Kasner sector (the Kasner circle). Contributions from the dust and the cosmological terms are also factored out. Different matter configurations only affect the last mapping \( K \), all the other degrees of freedom affect only the mappings \( l_\mathbf{L} \) and \( s_\gamma \).

Interestingly, the Kasner limit, where \( A \longrightarrow \infty \) in such a way that \( \lim_{A \longrightarrow \infty} l_\mathbf{L}(C) = \mathbb{R}^3 \), is well defined. The topological sector becomes degenerate and effectively we have only a \( S^1 \) degree of freedom. On the other hand the FRW limit \( A \longrightarrow 0 \) is apparently badly defined. Again the topological sector becomes degenerate in addition to the \( S^1 \)-sector, but we do not seem to uniquely reproduce the FRW solutions.

There is also one more interesting consequence of a non-trivial topology. For the Kasner universe \( (T^{\mu\nu} = 0) \) the special case \( \gamma = 0 \):

\[
ds^2 = -dt^2 + t^2 dX^2 + dY^2 + dZ^3
\]
is through the transformation \( \tilde{T} = t \cosh X, \ \tilde{X} = t \sinh X \) seen to be equal to the inside of the future lightcone in the \((\tilde{T}, \tilde{X})\) Minkowski space. Thus it can be naturally expanded to a flat manifold containing the apparent singularity \( t = 0 \). The compact case on the contrary can not. Assuming \( \mathbf{L} = A^{\frac{1}{3}} \cdot (\text{id}) \) the differential structure of the \((\tilde{T}, \tilde{X})\) space is that of a cone. The singularity \( t = 0 \) is now the locus of the cone. It therefore represents a true singularity in the \( C^\infty \) structure of the maximal extended space (i.e. the cone including the locus). Interestingly we will actually know the true nature of the singularity. Taking the limit \( A \longrightarrow \infty \) to obtain the Kasner universe, topological considerations suggest that the lightcone in the “flat” \((\tilde{T}, \tilde{X})\) ought to be identified as one point. The geometry of the space \( \gamma = 0 \) is no longer globally flat, it is more like that of a cone. This explains perhaps why the “flat space” case \( \gamma = 0 \) does not evolve as a flat space through the factorization \[\text{[15]}\].
V. THE WHEELER-DEWITT EQUATION

In the previous sections we have solved the classical equations for a Bianchi Type I universe. We will now quantize the dynamical degrees of freedom according to the Wheeler-DeWitt procedure. The constraint equation will then turn into an operator equation on the wave function. The resulting equation is the so-called Wheeler-DeWitt (WD) equation. Let us also include a real scalar field \( \Phi \), which later will yield interesting back-reaction effects.

The WD equation for a Bianchi Type I with a scalar field reads:

\[
\frac{1}{2} (-\nabla^2 + V_E) \Psi = 0
\]

where

\[
\nabla^2 = \frac{e^{-3\alpha}}{12} \left( -e^{\xi\alpha} \frac{\partial}{\partial\alpha} e^{-\xi\alpha} \frac{\partial}{\partial\alpha} + \nabla^2_\beta + \frac{12}{\partial^2} \right)
\]

\[V_E = V(\Phi) e^{3\alpha} + 4M\]

Here \( \xi \) represents some of the factor ordering ambiguity and \( \nabla^2_\beta = \frac{\partial^2}{\partial\beta^2} + \frac{\partial^2}{\partial\beta^2} \) is the usual Laplace operator in \( \beta \)-space. To simplify the expressions we introduce the volume element \( v = e^{3\alpha} \) and do the rescaling: \( \tilde{\beta}_\pm = 3\beta_\pm, \tilde{\Phi} = \frac{\sqrt{2}}{3} \Phi, \mu = \frac{1}{3} M \) and \( \tilde{V}(\tilde{\Phi}) = \frac{2}{3} V(\Phi) \). We now drop the tildes to avoid unnecessary writing. The WD equation then turns into (with \( \zeta = 1 - \frac{\xi}{3} \))

\[
\left[ v^2 \frac{\partial^2}{\partial v^2} + \zeta v \frac{\partial}{\partial v} - \nabla^2_\beta - \frac{\partial^2}{\partial \Phi^2} + \mu v + V(\Phi) v^2 \right] \Psi(v, \beta_\pm, \Phi, \phi, \theta, \psi) = 0 \quad (16)
\]

Since the WD equation does not involve any of the Euler angles, the \( SO(3) \) sector can be treated separately.

A. The \( SO(3) \)-sector

Since \( SO(3) \cong \mathbb{P}^3 \), this section of superspace is naturally equipped with an elliptic Riemannian structure. Due to the compactness of \( SO(3) \) as a topological space there exists a complete countable set of orthogonal functions on this space. Thus the conjugated momenta are quantized. We may choose the eigenfunctions in such a way that they are characterized by three quantum numbers: \( (l, m, m') \). These numbers are all integers and obeys: \( -l \leq m', m \leq l \) and \( l = 0, 1, 2... \) The eigenfunctions are not surprisingly the irreducible representation matrices of the group \( SO(3) \), often written as \( D^{(l)}_{mm'}(\phi, \theta, \psi) \). These functions are given by:

\[
D^{(l)}_{mm'}(\phi, \theta, \psi) = e^{-im\phi} e^{-im'\psi} d^{(l)}_{mm'}(\theta)
\]

where

\[
d^{(l)}_{mm'}(\theta) = \sum_\lambda \frac{(-1)^\lambda [(l+m)!(l-m)!][(l+m')!(l-m')!]^\frac{1}{2}}{\lambda! [(l+m-\lambda)!(l-m'-\lambda)!][(\lambda+m'-m)!]^\frac{1}{2}}
\]

\[
\times \left( \cos \frac{\theta}{2} \right)^{2l-2\lambda-m'+m} \left( -\sin \frac{\theta}{2} \right)^{2\lambda+m'-m}
\]

(19)
where the sum is only over integer $\lambda$ which makes sense. The algebraic properties of these functions now follow from the group theoretical properties of the Lie group $SO(3)$. We could for instance create a “Trace wave function”. Every point in $P^3$ (or element in $SO(3)$) may be represented by a unit rotation vector $\mathbf{\hat{n}}$ in $\mathbb{R}^3$ and a rotation angle $\alpha$. Then the “Trace wave function” may be defined by:

$$
\Psi_{Tr}^{(l)}(\phi, \theta, \psi) = \sum_{m=-l}^{l} D_{mm}^{(l)}(\phi, \theta, \psi) = \frac{\sin(l + \frac{1}{2})\alpha}{\sin \frac{\alpha}{2}}
$$

These functions have their maximum at the unit element, will be zero where $\alpha = \frac{2\pi n}{2l+1}$ where $n$ is an integer $1 \leq n \leq 2l$. From a classical quantum description if the Bianchi Type I universe was in such a state, the fundamental cube would look as if it were “wobbling” around its identity mapping. It might suddenly do a quantum leap to a different orientation. Investigating the “Probability” amplitude $|\Psi_{Tr}^{(l)}|^2$ we see that the classical description describes a shell model (fig. 3). There are $2l$ regions where the Bianchi universe could be. These regions are separated by forbidden regions. The most probable shell is the one containing the identity element. Let us therefore call this the ground state. This situation becomes even more evident in the limit $l \to \infty$. It might appear that these wave functions allow a geometry change. If a Bianchi Type I universe was born in an “excited” state which is meta-stable, it could suddenly cross the barrier to the ground state. The state in the neighborhood of the identity element would be the most probable. Due to the representation decomposition

$$D^{(l_1)} \otimes D^{(l_2)} \cong \bigoplus_{l=|l_1-l_2|} D^{(l)}$$

the following will also be true:

$$\Psi_{Tr}^{(l_1 \otimes l_2)} = \Psi_{Tr}^{(l_1)} \cdot \Psi_{Tr}^{(l_2)} = \sum_{l=|l_1-l_2|} \Psi_{Tr}^{(l)}$$

Thus the solution space spanned by these Trace wave functions could be generated by the two generators: $\Psi_{Tr}^{(0)}$ and $\Psi_{Tr}^{(1)}$. These Trace wave functions are of course only a particular set of functions which is by no means complete (although orthogonal) but drawn to attention only because of the algebraic beauty they possess. The famous orthogonality theorem for the matrix elements of irreducible representations will ensure that our functions $D_{mm}^{(l)}(\phi, \theta, \psi)$ are orthogonal over the Riemannian space $P^3$. All of these identities follow from the Lie group properties of $SO(3)$ and there are of course a lot more than the ones mentioned here. Thus a lot of the properties of the Bianchi Type I are directly related to the structure of the superspace, at least in this description. The fact that the superspace has this Lie group structure provides a lot of information.

**B. Exact solutions for a zero-mass scalar field**

For a Klein-Gordon field the scalar potential can be written as $V(\Phi) = \lambda + m^2 \Phi^2$, where $\lambda$ correspond to a cosmological constant. For this type of potential exact solutions for the
WD equation is very difficult if not impossible to obtain. However, if we are dealing with a zero-mass scalar field $m = 0$, exact solutions are possible to obtain. Firstly we can separate out the $\beta_\pm$ and $\Phi$ dependent parts by introducing $\Psi(v, \beta_\pm, \Phi) = F(v)e^{i(k_+\beta_+ + k_-\beta_- + \Phi)}$. As we see $k^2 \equiv k_+^2 + k_-^2$ corresponds to the classical anisotropy parameter $A^2$ i.e. the volume of the fundamental domain. Thus, these particular solutions are planar wave solutions where $F(v)$ satisfy the equation:

$$v^2 \frac{d^2 F}{dv^2} + \zeta v \frac{dF}{dv} + (k^2 + n^2 + \mu v + \lambda v^2)F = 0 \quad (20)$$

This is Whittaker’s equation and its solutions can be written as

$$F(v) = v^{a-\frac{1}{2}} W_{\mp L, p}(\pm 2i\lambda^{\frac{1}{2}} v) \quad (21)$$

where $a = \frac{\xi}{6}$, $L = \frac{im}{2\lambda^{\frac{1}{2}}}$, $p^2 = a^2 - (k^2 + n^2)$ and $W_{L, p}(z)$ is the Whittaker function. These function have an essential singularity at infinity and have a branch point at the origin. This is due to the fact that the classical solutions have a physical singularity at vanishing 3-volume. In the absence of dust $\mu = 0$, the equation 20 turn into a Bessel equation. Thus the solution for $\mu = 0$ can be written as

$$F(v) = v^a Z_p(\lambda^{\frac{1}{2}} v)$$

where $Z_p(z)$ is one of the Bessel functions or a linear combination of them. This can also be seen from the identity

$$W_{0, p}(2z) = \sqrt{\frac{2z}{\pi}} K_p(z)$$

The behavior of these solutions are not drastically altered with the inclusion of a non-zero dust term. For simplicity’s sake we will set $\mu = 0$ in the rest of this paper. In the further investigations the dust term is not essential and because the properties of the Bessel functions is better known we will simply drop the dust term. The solutions are therefore (a linear combination of) the Bessel functions.

In the case of a classical state that is not allowed, for instance if we have a negative cosmological constant and a large enough $v$, we would expect an exponential decrease of the wave function. For negative $\lambda$ we can write the solutions as a linear combination of the modified Bessel functions $K_p(|\lambda|^{\frac{1}{2}} v)$ and $I_p(|\lambda|^{\frac{1}{2}} v)$. However, $I_p(|\lambda|^{\frac{1}{2}} v) \approx e^{\lambda^{\frac{1}{2}} v}$ for $|\lambda|^{\frac{1}{2}} v \gg |p|$, so we have to abandon these solutions if we demand an exponential decrease for $|\lambda|^{\frac{1}{2}} v \gg |p|$. The actual linear combination is a matter of boundary condition, which we will leave for another section.

C. A non-zero mass scalar field: Harmonic oscillator expansion

For a non-zero mass scalar field the WD equation is difficult if not impossible to solve exactly.
Let us try to find solutions of the form:

\[ \Psi(v, \Phi) = \sum_{n=0}^{\infty} c_n(v) \psi_n(v, \Phi) \quad (22) \]

where \( \psi_n(v, \Phi) \) are the eigenfunctions of the harmonic oscillator equation:

\[
\left[ -\frac{1}{2} \frac{\partial^2}{\partial \Phi^2} + \frac{1}{2} m^2 v^2 \Phi^2 \right] \psi_n = mv(n + \frac{1}{2}) \psi_n \quad (23)
\]

Thus, we can consider the wave equation \( \Psi \) as a path in the Hilbert space which is spanned by the basis \( \{|n\rangle\} \) where \( |n\rangle \) is the usual normalized eigenstate of the one-dimensional harmonic oscillator equation. In a coordinate representation these are given by:

\[
\psi_n(v, \Phi) = \left( \frac{mv}{\pi} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} e^{-\frac{1}{2}mv\Phi^2} H_n(\Phi \sqrt{mv}) \quad (24)
\]

where \( H_n \) is the \( n \)'th Hermite polynomial. Calculating the derivatives with respect to \( v \):

\[
v \frac{\partial}{\partial v} |n\rangle = \left[ \frac{1}{4} - \frac{1}{2} mv\Phi^2 \right] |n\rangle + \sqrt{\frac{n}{2}} \sqrt{mv\Phi} |n-1\rangle \quad (25)
\]

\[
v^2 \frac{\partial^2}{\partial v^2} |n\rangle = \left[ -\frac{3}{16} - \frac{1}{4} mv\Phi^2 + \frac{1}{4} m^2 v^2 \Phi^4 \right] |n\rangle
- \sqrt{\frac{n}{2}} (mv)^{\frac{3}{2}} \Phi^3 |n-1\rangle
+ \frac{1}{2} \sqrt{n(n-1)mv\Phi^2} |n-2\rangle \quad (26)
\]

where we have used the identity:

\[
H'_n = 2nH_{n-1}
\]

Using the annihilation and creation operators \( a \) and \( a^\dagger \) defined by:

\[
a = \sqrt{\frac{mv}{2}} \Phi - \frac{1}{\sqrt{2mv}} \frac{\partial}{\partial \Phi}
\]

\[
a^\dagger = \sqrt{\frac{mv}{2}} \Phi + \frac{1}{\sqrt{2mv}} \frac{\partial}{\partial \Phi} \quad (27)
\]

we can write any power of \( \sqrt{mv\Phi} \) as

\[
(mv)^{\frac{3}{2}} \Phi^k = \frac{1}{\sqrt{2^k}} (a + a^\dagger)^k
\]

Inserting eq. 22 into the original WD equation for a non-zero mass scalar field, and using eq. 23, 24, 25 and 26 we can get a differential equation which the coefficients \( c_n(v) \) shall
satisfy. If we instead introduce a new family of functions defined by $d_n(v) = \nu \zeta c_n(v)$ the resulting equation may be written as:

$$0 = v^2 d_n'' + \left( \lambda v^2 + m v (2n + 1) + k^2 + \frac{1}{8} - \frac{\xi^2}{36} - \frac{1}{8} n(n + 1) \right) d_n$$

$$- \frac{1}{2} v \left( \sqrt{n(n - 1)} d_{n-2}' - \sqrt{(n + 1)(n + 2)} d_{n+2}' \right)$$

$$+ \frac{1}{4} \left( \sqrt{n(n - 1)} d_{n-2}' - \sqrt{(n + 1)(n + 2)} d_{n+2}' \right)$$

$$+ \frac{1}{16} \left( \sqrt{n(n - 1)} (n - 2) d_{n-4}' + \sqrt{(n + 1)(n + 2)(n + 3)(n + 4)} d_{n+4}' \right)$$

The infinite set of equations divides into two, one odd set of equations and one even set of equations, which involve respectively odd or even coefficients only. The Hilbert space is therefore divided into two: odd and even states. Now it is time for some approximations. Let us investigate the properties of the solutions in the limit $v \to 0$. In the further calculations we will assume that the wave function $\Psi(v, \Phi)$ is independent of $\Phi$ in the limit $v \to 0$ in the following sense:

Given a compact interval $I \subset \mathbb{R}$ and a $\delta > 0$. Then there exists a $\epsilon > 0$ so that for $0 < v < \epsilon$, $|\frac{\partial \Psi}{\partial \Phi}(v, \Phi_0)| < \delta$ $\forall \Phi_0 \in I$.

This is exactly the right boundary condition for demanding that $c_n(v) \to a_n(v)$ as $v \to 0$ where $a_n(v)$ is the coefficients of a function independent of $\Phi$ (constant in $\Phi$) in the harmonic oscillator expansion. Using the test function $f = \left(\frac{mv}{4\pi}\right)^{\frac{1}{4}}$ we see that:

$$|f\rangle = \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} |2n\rangle$$

Based on these results we can split the total wave function into an even and an odd part $\Psi = \Psi_{\text{odd}} + \Psi_{\text{even}}$. We extract the solution for the constant function $f$ by defining a new family of coefficients by $d_{2n} = \sqrt{\frac{(2n)!}{2^n n!}} \mathcal{E}_n$. The equation for $\mathcal{E}_n$ will then be

$$0 = v^2 \mathcal{E}_n'' + \left( \lambda v^2 + m v (4n + 1) + k^2 + \frac{1}{8} - \frac{\xi^2}{36} - \frac{1}{2} n(n + 1) \right) \mathcal{E}_n$$

$$- v \left( n \mathcal{E}_{n-1}' - (n + \frac{1}{2}) \mathcal{E}_{n+1}' \right)$$

$$+ \frac{1}{2} \left( n \mathcal{E}_{n-1}' - (n + \frac{1}{2}) \mathcal{E}_{n+1}' \right)$$

$$+ \frac{1}{4} \left( n(n - 1) \mathcal{E}_{n-2}' + (n + \frac{1}{2})(n + \frac{3}{2}) \mathcal{E}_{n+2}' \right)$$

This equation can without difficulty be solved to first order in $v$. The result is:

$$\mathcal{E}_n = v^f \left( 1 - \frac{1}{4} \pm \frac{1}{\sqrt{\frac{\xi^2}{36} - k^2}} (4n + 1) m v + \mathcal{O}(v^2) \right)$$
where \( l = \frac{1}{4} \pm \sqrt{\frac{\xi^2}{36} - k^2} \). Assuming \( \xi < 6 \) the real value of the first order term will be negative, i.e. to first order the higher exited modes in the expansion will decay compared to the ground state.

**D. Backreaction Effects: The massless case**

The preceding results describe an even wave function in \( \Phi \). The expectation value of the scalar field vanishes, but the expectation value of \( \Phi^2 \) diverges as \( v \to 0 \). This will have consequences for the effective Hamiltonian in the theory. Let us try to replace the energy momentum tensor, \( T^{\mu \nu} \), by its expectation value \( \langle T^{\mu \nu} \rangle \) using the obtained results. This effectively means that the scalar field operator is replaced with its expectation value:

\[
-\frac{\partial^2}{\partial \Phi^2} + m^2 v^2 \Phi^2 \rightarrow \frac{\langle \Psi \left[ -\frac{\partial^2}{\partial \Phi^2} + m^2 v^2 \Phi^2 \right] \Psi \rangle}{\langle \Psi | \Psi \rangle}
\]

In the massless case the expectation value of the energy momentum tensor only contributes with a positive constant \( n^2 \). Keeping the volume of the universe fixed (i.e. keeping \( A^2 \) fixed), the term \( n^2 \) will reduce the radius of the Kasner circle (the “radius” in eq. 2). Note also that the equations [3] will still be true. Since the center of the Kasner circle represents an isotropic universe, we see that the inclusion of a massless scalar field effectively reduces the anisotropy of the universe.

**E. Backreaction Effects: The Harmonic oscillator expansion**

Using the harmonic oscillator expansion we can try to understand the effect of a non-zero mass scalar field on the classical equations. In the harmonic oscillator expansion we can write

\[
\langle \Psi | \left[ -\frac{\partial^2}{\partial \Phi^2} + m^2 v^2 \Phi^2 \right] | \Psi \rangle = mv \sum_{n=0}^{\infty} |c_n|^2 (2n + 1)
\]  

In our case we shall use only the even part, but unfortunately the results obtained can not be used as they stand; every sum will diverge. We have to regularize the result so that the sums may be performed. Let us write:

\[
|c_n|^2 \propto v^{2Re(l)} \left( 1 - \frac{2}{\eta} \kappa n \right)^{\eta(n + \frac{1}{2})}
\]

where \( \kappa = \text{Re} \left( \frac{1}{1 \pm \sqrt{\frac{\xi^2}{36} - k^2}} \right) \) and \( \eta \) is some unknown parameter which represents the uncertainty in the higher order contribution in the expansion (31). To first order, however, we see that this is exactly the previously obtained results. With this replacement the sums may be performed and the result is to first order in \( v \):

\[
\frac{\langle \Psi_{\text{even}} | \left[ -\frac{\partial^2}{\partial \Phi^2} + m^2 v^2 \Phi^2 \right] | \Psi_{\text{even}} \rangle}{\langle \Psi_{\text{even}} | \Psi_{\text{even}} \rangle} \approx mv \left( \frac{2\eta - 1}{\eta} \right) + \frac{1}{\kappa}
\]  

14
Luckily the unknown parameter $\eta$ does not contribute to lowest order, so the validity of the regularization procedure which we used is strengthened. If we assume $\xi = 0$ and we associate $k^2$ with the anisotropy parameter $A^2$, we see that we get an effective anisotropy parameter $A^2_{\text{eff}} = 2A^2 + \frac{9}{16}$. Comparing this to the classical solutions, an increase of the effective anisotropy parameter in the Hamiltonian constraint correspond to an increase of the speed of the volume element by $\frac{A_{\text{eff}}}{A}$ and a decrease of the anisotropy speed by the same amount.

Even though these considerations were very simple and only approximative, this indicates that the effect of a simple scalar field with mass is that it reduces the effective anisotropy. Comparing with the zero mass solutions, we see that a zero mass scalar field also has the same effect, it increases the anisotropy parameter but reduces the anisotropy. The result in the limit $v \to 0$ for a non-zero mass is thus reasonable.

VI. BOUNDARY CONDITIONS

Concerning boundary conditions, quantum cosmology has mostly invoked two different boundary conditions. One is that of Hartle-Hawking [3], which is stated in terms of the path integral formalism. The Hartle-Hawking wave function $\Psi_{HH}$ for a Bianchi Type I universe with a positive cosmological constant has been derived by Duncan and Jensen [4]. Duncan and Jensen also use three-torus as spatial sections, but they do not give any comments concerning the $SO(3)$ sector. Apparently the only reason they use compact spatial sections is to make the spatial integration in the action finite. In this paper we will however be more interested in the tunneling boundary proposal by Vilenkin [5].

A. The Bianchi Type I minisuperspace

Essential in this discussion is the description of the Bianchi Type I from a minisuperspace point of view. If we exclude a scalar field degree of freedom, the minisuperspace will be 6-dimensional. Topologically we can write the minisuperspace as $M^3 \times \mathbb{P}^3$ where $\mathbb{P}^3 \cong SO(3)$ and $M^3$ is the 3-dimensional Minkowski space. As the $SO(3)$-sector has already been discussed, we will only look at the evolution in the Minkowski sector. We will also choose the lapse to be unity $N = 1$. The DeWitt metric on the Minkowski sector will then be:

$$ds^2 = e^{3\alpha}(-d\alpha^2 + d\beta_+^2 + d\beta_-^2)$$  \hspace{1cm} (33)

Doing a conformal transformation we can map this metric onto a Penrose-Carter diagram. Mapping the classical solutions (with no dust) into this conformal diagram we get a picture like figure 3. Notice that all the three different cases start off at past light-like infinity $I^-$. They all start off as a Kasner universe (straight line). They end up however, at different places. For a positive-valued cosmological constant the universe ends as a deSitter universe: moving along the $t$-axis to the time-like future $i^+$. The Kasner universe ($\Lambda = 0$) ends at the future light-like infinity $I^+$, while the case $\Lambda < 0$ the universe collapses again in past light-like infinity $I^-$. The whole boundary except past time-like infinity $i^-$ represents singular space-time geometries, and thus according to the tunneling boundary condition, is not allowed. However, from the calculations done in the previous chapters, there may
be indications that the starting point $I^-$ is somewhat unstable concerning inclusion of a scalar field. Even a rather modestly behaved scalar field effectively reduces the speed of the universe near the initial singularity to below “the speed of light”. The reduction of the speed of a Kasner universe is shown in figure 4. This effectively changes the path’s end-points, so that the universe starts off at $i^-$. It ought to be emphasized that this is not enough to ensure regularity of the universe at the starting point, but this may indicate that there exists a homotopy of time-like paths connecting the Kasner universe with the deSitter solution. These different solutions will be universes with different speeds in superspace. One might say that we have given the Kasner universe a mass, and the homotopy can be defined by the action of the 3-dimensional Lorentz group.

In the light of this discussion we change the formulation of the tunneling proposal to include all universes which may be connected to universes which satisfy the regularity condition, by the action of the Lorentz group in superspace.

B. The tunneling wave function

The tunneling wave function will have a current in superspace which points outwards at the singular boundaries of superspace. The conserved current

$$J = -\frac{i}{2}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*)$$

which is analogous to the conserved Klein-Gordon current, states that its zero-component $J^0$ is a conserved entity. The zeroth component of the Klein-Gordon current can be interpreted as a charge, which locally can obtain both negative and positive values. Also the zeroth component superspace current can be both negative and positive. In light of the tunneling proposal, these correspond to expanding and contracting solutions.

Using the solutions to the WD-equation for a zero-mass scalar field, we get the “space” components of the superspace current to be:

$$J^\pm = -\frac{i}{2}v^{-\frac{2\xi}{\beta}}(\Psi^* \partial_\pm \Psi - \Psi \partial_\pm \Psi^*) = v^{-\frac{2\xi}{\beta}}k_\pm |\Psi|^2$$

(34)

At the singular boundary $v \to \infty$, we use the solutions for a positive cosmological constant in the big geometry limit. The two possible solutions are the Hankel functions $H^{(1)/(2)}_p(v) \approx \frac{1}{\sqrt{v}}e^{\pm iv}$. The zero-component then becomes:

$$J^0 \approx \mp |\lambda|^{\frac{1}{2}}v^2(1 - \frac{\xi}{\beta}) |\Psi|^2$$

(35)

The outgoing modes are therefore the second Hankel function $H^{(2)}_p(v)$ and the tunneling wave functions are are a linear combinations of the functions

$$\Psi(v, \beta_\pm, \Phi) = v^{\frac{\xi}{2}}H^{(2)}_p(\lambda^{\frac{1}{2}}v)e^{i(k_+\beta_+ + k_-\beta_-)}e^{i\Phi}$$

(36)

The solution space of the WD equation will therefore be spanned by the solutions (35) multiplied by the solutions from the $SO(3)$-sector $D^{(l)}_{mn'}$. 
C. Does the Tunneling boundary proposal predict an isotropic universe?

As we have constructed the Tunneling wave function we might wonder if the wave function predict an isotropic or an anisotropic universe. In the classical case we had trouble with reconstructing the FRW universe in the limit \(A \to 0\). The FRW universe appeared as if it was disconnected from the Bianchi Type I solutions. This picture is drastically improved in the Quantum case. If we hold \(v\) constant, that is keeping the volume of the universe constant, and let \(A\) approach zero from a classical point of view we have to let the function \(V(t)\) approach infinity. For a positive cosmological constant the limit \(V \to \infty\) is indeed a isotropic FRW limit with finite spatial volume. As \(v\) and \(A\) are independent variables we could interpret the \(A \to 0\) a FRW limit. Inserting a scalar field we saw that it also reduced the anisotropy. The anisotropy was reduced as the ratio \(\frac{k^2}{k^2+n^2}\). These two mechanisms for reducing the anisotropy affect the classical solutions in two different ways. Whereas the reduction of \(A\) isotropize the universe through the modular and evolutionary degrees of freedom, the inclusion of a scalar field reduces the Kasner circle. Thus a scalar field could also work for isotropy reduction for the Kasner universe (with infinite spatial sections). \(A \to 0\) for the Kasner universe is however more subtle because apparently we do not have any clear geometric meaning of the entity \(A\) in that case.

The \(k^2\) enters the wave functions through the order of the Hankel function. The order \(p\) is given by \(p^2 = \frac{\xi^2}{36} - k^2 - n^2\). The factor ordering parameter is assumed to be small, so we assume that \(p\) is purely imaginary: \(p = ir\) where \(r \geq 0\). A small \(r\) means an isotropic universe. From the integral representation of the Hankel functions \[38\] we can write

\[
H^{(2)}_{ir}(z) = \frac{ie^{-\frac{r\pi}{2}}}{\pi} \int_{-\infty}^{\infty} e^{-iz\cosh t - i rt} dt
\]

Apparently the wave functions are suppressed by an exponential factor as \(r\) grows. Thus the probability amplitude is exponentially damped in \(r\): \(|\Psi|^2 \sim e^{-r\pi}\). We do have to be a bit careful however because any linear combination of these wave functions is a solution. We might imagine that it had an exponentially growing prefactor in \(r\). Let us try to estimate a reasonable prefactor by “normalizing” the wave functions over superspace. Since the variable \(v\) will behave as a time variable in superspace we could also say that we divide by the “time average” of the particular solution. The “normalizing” integrals \(\int dv|\Psi(v)|^2\) diverge for the Hankel functions. We will therefore do the following: we Wick-rotate the volume element, \(v \mapsto iv\), so that the Hankel functions \(H^{(2)}_{ip}\) are “rotated” to the Bessel functions \(K_p\). This function will under reasonable conditions be square integrable. We estimate the prefactor by demanding that the wave function is normalized after a Wick-rotation of the volume element. The Bessel functions are related through the following equations

\[
K_p(z) = \frac{\pi}{2} ie^{\frac{r\pi}{2}} H^{(1)}_{ir}(iz)
\]

\[
H^{(2)}_{ip}(z) = H^{(1)}_{ip}(\overline{z})
\]

We also introduce the integration measure \(\sqrt{|G|}\) which reflects the geometry of minisuper-
space. The DeWitt metric (eq. [33]) will yield \( \sqrt{|G|} = v^{1/2} \) but if we assume \( \sqrt{|G|} = v^{1-2x} \) the resulting integral become surprisingly simple [38]:

\[
I = \int_0^{\infty} dv \sqrt{|G||\Psi|^2} = \frac{4}{\pi^2 \pi r} e^{\pi r} \int_0^{\infty} dv v K_{1r}(\lambda^{1/2} v)^2
\]

Using these wave functions after “normalization” the probability amplitude decays as

\[
|\Psi|^2 \approx e^{-\pi r / 2 \pi r / 1 - e^{-2\pi r}}
\]

These wave functions will peak for \( r = 0 \) so they predict an isotropic universe. They will exponentially decay for larger values of \( r \). Whether this “normalizing” function is the right weight function is so far only speculation, but if these functions are reasonable the wave functions predict an isotropic universe.

**VII. CONCLUSION AND SUMMARY**

In this paper we have discussed the Bianchi Type I minisuperspace model with compact spatial sections. In the classical case this removed some pathologies. The classical solutions were all written down in a way so that the solution space was manifestly factorized into three parts: a topological sector, a Kasner sector and a matter sector. The Kasner solutions were obtained in the limit \( \Lambda^{-1} \rightarrow \infty \) where one easily could see that the modular sector becomes degenerate. Choosing compact spatial sections also gave a better understanding of the topology of the 4-space. For instance the apparently flat case \( \gamma = 0 = \mathcal{T}^{\mu\nu} \) of the Kasner universe was shown to have a conical singularity at \( t = 0 \).

In the quantum mechanical case since the (true) phase space has a dimension equal to the dimension of the solution space, it is easier to give a precise meaning to the WD equation. This paper has also shown the necessity of introducing compact spatial sections when one wants to quantize the model, and provides a solution to the WD equation for \( T^3 \) spatial sections. Also a set of solutions in the \( SO(3) \) sector have been given which had particularly nice algebraic properties. These solutions have a structure which describes a shell structure in the \( SO(3) \) sector. It was also indicated how these solutions allow geometry changing solutions.

Inserting various matter configurations in general showed isotropization of the universe. The resulting tunneling wave functions appeared as if they peaked at small anisotropy. Under reasonable considerations we constructed a wave function which had an exponentially decaying probability amplitude for increasing anisotropy. This seems to agree with other authors [33, 40].

\[6\] Thus these two integration measures coincide iff \( \xi = \frac{3}{2} \). We would also mention that the integral is manageable for any power of \( v \) (provided that the integral converges).
It is quite clear that universe models with compact spatial sections will yield many interesting and surprising results. Since these possibilities only recently have come to physicist’s and cosmologist’s attention there is a lot of work still to be done. What we have shown in this paper is that even at the classical level, this may reveal interesting properties of cosmological models.

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FIG. 1. Deformation of the cube: The mapping $l_L : C \mapsto (\mathbb{R}^3, d\sigma^2(t))$

FIG. 2. The square of the Trace wavefunctions $\Psi^{(1)}_{Tr}$, $\Psi^{(3)}_{Tr}$ and $\Psi^{(10)}_{Tr}$ normalized so that $\Psi^{(l)}_{Tr} = 1$ at the unit element. Note also that the $x$-axis covers the range of $\alpha$ twice.
FIG. 3. The evolution of the Bianchi Type I in conformal superspace

FIG. 4. The evolution of a Kasner universe versus the evolution of a Kasner universe with a reduced “speed”