Construction of fuzzy $S^4$

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Abstract

We construct a fuzzy $S^4$, utilizing the fact that $\mathbb{CP}^3$ is an $S^2$ bundle over $S^4$. We find that the fuzzy $S^4$ can be described by a block-diagonal form whose embedding square matrix represents a fuzzy $\mathbb{CP}^3$. We discuss some pending issues on fuzzy $S^4$, i.e., precise matrix-function correspondence, associativity of the algebra, and, etc. Similarly, we also obtain a fuzzy $S^8$, using the fact that $\mathbb{CP}^7$ is a $\mathbb{CP}^3$ bundle over $S^8$. 
1 Introduction

As we have witnessed for more than a decade, the idea of fuzzy $S^2$ [1] has been one of the guiding forces for us to investigate fuzzy spaces. For example, the fuzzy complex projective spaces $\mathbb{CP}^k$ ($k = 1, 2, \ldots$) [2, 3] are successfully constructed in the same spirit as the fuzzy $S^2$. From physicists’ point of view, it is of great interest to obtain a four-dimensional fuzzy space. The well-defined fuzzy $\mathbb{CP}^2$ is not suitable for this purpose, since $\mathbb{CP}^2$ does not have a spin structure [2]. Construction of fuzzy $S^4$ is then physically well motivated. (Notice that fuzzy spaces are generally obtained for compact spaces and that $S^4$ is the simplest four-dimensional compact space that allows a spin structure.) Since $S^4$ naturally leads to $\mathbb{R}^4$ at a certain limit, the construction of fuzzy $S^4$ would also shed light on the studies of noncommutative Euclidean field theory.

There have been several attempts to construct the fuzzy $S^4$ from a field theoretic point of view [4, 5, 6] as well as from a rather mathematical interest [7, 8, 9], however, it would be fair to say that the construction of fuzzy $S^4$ is not yet satisfactory. In [7, 8], the construction is carried out with a projection from some matrix algebra (which in fact coincides with the algebra of fuzzy $\mathbb{CP}^3$) and, owing to this forcible projection, it is advocated that fuzzy $S^4$ obeys a non-associative algebra. Although, in the commutative limit, the associativity is recovered, the non-associativity limits the use of the fuzzy $S^4$ for physical models. (Non-associativity is not compatible with unitarity of the algebra for symmetry operations in any physical models.) In [5, 6], the fuzzy $S^4$ is alternatively considered in a way of constructing a scalar field theory on it, based on the fact that $\mathbb{CP}^3$ is a $\mathbb{CP}^1$ (or $S^2$) bundle over $S^4$. While the resulting action leads to a correct commutative limit, it is, as a matter of fact, made of a scalar field on fuzzy $\mathbb{CP}^3$. Its non-$S^4$ contributions are suppressed by an additional term. (Such a term can be obtained group theoretically.) The action is interesting but the algebra of fuzzy $S^4$ is still unclear. In this sense, the approach in [5, 6] is related to that in [7, 8]. Either approach uses a sort of brute force method which eliminates unwanted degrees of freedom from fuzzy $\mathbb{CP}^3$. Such a method gives a correct counting for the degrees of freedom of fuzzy $S^4$, but it does not clarify the construction of fuzzy $S^4$ per se, as a matrix approximation to $S^4$. This is precisely what we attempt to do in this paper. (Notice that the term “fuzzy $S^4$” is also used, mainly in the context of M(atrix) theory, e.g., in [10, 11], for the space developed in [12]. This space actually obeys the constraints for fuzzy $\mathbb{CP}^3$.)

In [9], the construction of fuzzy $S^4$ is considered through fuzzy $S^2 \times S^2$. This allows one to describe the fuzzy $S^4$ with some concrete matrix configurations. However, the algebra is still non-associative and one has to deal with non-polynomial functions on the fuzzy $S^4$. Since those functions do not naturally become polynomials on $S^4$ in the commutative limits, there is not a proper matrix-function correspondence. The matrix-function correspondence is a correspondence between functions on a fuzzy space (which are represented by some matrices) and truncated functions on the corresponding commutative space. In the case of fuzzy $\mathbb{CP}^k$, the fuzzy functions are represented by full $(N \times N)$-matrices, so the product of them is given by matrix multiplication which leads to the associativity for the algebra of fuzzy $\mathbb{CP}^k$. Defining the symbols of functions on it, one can show that their star products reduce to the
ordinary commutative products of functions on $\mathbb{CP}^k$ in the large $N$ limit [3, 15]. In this case, the matrix-function correspondence may be checked by the matching between the number of matrix elements and that of truncated functions. This matching, however, is not enough to warrant the matrix-function correspondence of fuzzy $S^4$; further we need to confirm the correspondence between the product of fuzzy functions and that of truncated functions. In order to do so, it is important to construct a fuzzy $S^4$ with a clear matrix configuration (which should be different from the proposal in [9]).

The plan of this paper is as follows. In section 2, following Medina and O’Connor in [5], we propose a construction of fuzzy $S^4$ by use of the fact that $\mathbb{CP}^3$ is an $S^2$ bundle over $S^4$. We will obtain a fuzzy $S^4$, imposing a further constraint on the fuzzy $\mathbb{CP}^3$. This extra constraint is expressed by a matrix language and essentially plays the role of a projection in a less forcible fashion. The advantage of this constraint is that it enable us to describe the algebra of fuzzy $S^4$ in terms of the generators of $SU(4)$. (This eventually leads to a closed associative algebra for fuzzy $S^4$.) The emerging algebra is not a subalgebra of fuzzy $\mathbb{CP}^3$. This is because we construct the fuzzy $\mathbb{CP}^3$ as embedded in $\mathbb{R}^{15}$. The structure of algebra becomes clearer in the commutative limit which is considered in terms of homogeneous coordinates of $\mathbb{CP}^3$. With these coordinates we also explicitly show that the extra constraint for fuzzy $S^4$ has a correct commutative limit. The idea of constructing a fuzzy space from another by means of an additional constraint has been considered by Nair and Randjbar-Daemi in obtaining a fuzzy $S^3/\mathbb{Z}_2$ out of fuzzy $S^2 \times S^2$ [13]. Our construction of fuzzy $S^4$ is inspired by their work.

In section 3, we show the matrix-function correspondence of fuzzy $S^4$. After a brief review of the case in fuzzy $S^2$, we start with different calculations of the number of truncated functions on $S^4$. We then show that this number agrees with the number of degrees of freedom for fuzzy $S^4$. This number turns out to be a sum of absolute squares, and hence we can choose a block-diagonal matrix configuration for the function of fuzzy $S^4$. This form is also induced from the structure of the fuzzy functions. The star product is based on the product of such matrices and naturally reduces to the commutative product, similarly to what happens in fuzzy $\mathbb{CP}^3$. This leads to the precise matrix-function correspondence of fuzzy $S^4$. Of course, this matrix realization of fuzzy $S^4$ is not the only one that leads to this correspondence; there are a number of ways related to the ways of allocating the absolute squares to form any block-diagonal matrices. Our construction is, however, useful in comparison with the fuzzy $\mathbb{CP}^3$.

The fact that $\mathbb{CP}^3$ is an $S^2$ bundle over $S^4$ can be seen by a Hopf map, $S^7 \to S^4$ with the fiber being $S^3$. One can derive the map, noticing that the $S^4$ is the quaternion projective space. In the same reasoning, octonions define a Hopf map, $S^{15} \to S^8$ with its fiber being $S^7$, giving us another fact that $\mathbb{CP}^7$ is a $\mathbb{CP}^3$ bundle over $S^8$. Following these mathematical facts, in section 4, we apply our construction to fuzzy $S^8$ and outline its construction. We conclude with some brief comments.
2 Construction of fuzzy $S^4$

We begin with the construction of fuzzy $\mathbb{CP}^3$. The constructions of fuzzy $\mathbb{CP}^k$ ($k = 1, 2, \cdots$) are generically given in an appendix; here we briefly rephrase it in the case of $k = 3$. The coordinates $Q_A$ of fuzzy $\mathbb{CP}^3$ can be defined by

$$Q_A = \frac{L_A}{\sqrt{C_2^{(3)}}}$$

(1)

where $L_A$ are $N^{(3)} \times N^{(3)}$-matrix representations of $SU(4)$ generators in the $(n, 0)$-representation (the totally symmetric representation of order $n$). The coordinates satisfy the following constraints of fuzzy $\mathbb{CP}^3$

$$Q_A Q_A = 1$$
$$d_{ABC} Q_A Q_B = c_{3,n} Q_C$$

(2)

(3)

As is shown explicitly in the appendix, in the large $n$ limit these constraints become (algebraic) equations which represent $\mathbb{CP}^3$ embedded in $\mathbb{R}^{15}$. (Notice the number of $SU(4)$ generators is 15.) In equations (1)-(3), $C_2^{(3)}$, 1, $d_{ABC}$ and $c_{3,n}$ are all defined in the appendix, including the relation

$$N^{(3)} = \frac{1}{6}(n + 1)(n + 2)(n + 3)$$

(4)

Now let us consider the decomposition, $SU(4) \to SU(2) \times SU(2) \times U(1)$, where the two $SU(2)$’s and one $U(1)$ are defined by

$$\begin{pmatrix} SU(2) & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & SU(2) \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(5)

in terms of the $(4 \times 4)$-matrix generators of $SU(4)$ in the fundamental representation. (Each $SU(2)$ denotes the algebra of $SU(2)$ group in the $(2 \times 2)$-matrix representation.) As we will see in this section, functions on $S^4$ are functions on $\mathbb{CP}^3 = SU(4)/U(3)$ which are invariant under transformations of $H \equiv SU(2) \times U(1)$, $H$ being relevant to the above decomposition of $SU(4)$. In order to obtain functions on fuzzy $S^4$, we thus need to require

$$[\mathcal{F}, L_\alpha] = 0$$

(6)

where $\mathcal{F}$ denote matrix-functions of $Q_A$’s and $L_\alpha$ are generators of $H$ represented by $N^{(3)} \times N^{(3)}$-matrices. (A construction of fuzzy $S^4$ can be carried out by imposing the additional constraint (6) onto the fuzzy functions of $\mathbb{CP}^3$.) What we claim is that the further condition (6) makes the functions $\mathcal{F}$ (and $Q_A$) become functions on fuzzy $S^4$. This does not mean that the fuzzy $S^4$ is a subset of fuzzy $\mathbb{CP}^3$. Notice that $Q_A$’s are defined in $\mathbb{R}^{15}$ ($A = 1, \cdots, 15$) with the algebraic constraints (2) and (3). While locally, say around the pole of $A = 15$ in eqn (3), one can specify the six coordinates of fuzzy $\mathbb{CP}^3$, globally they are embedded in $\mathbb{R}^{15}$. Equation (6) is a global constraint in this sense. An emerging algebraic structure of fuzzy $S^4$ will be clearer when we consider the commutative limit of our construction.
As $n$ becomes large, we can approximate $Q_A$ by the commutative coordinates on $\mathbb{CP}^3$, 

$$Q_A \approx \phi_A = -2 \text{Tr}(g^tl_Agt_15)$$

(7)

which indeed obey the following constraints for $\mathbb{CP}^3$

$$\phi_A \phi_A = 1, \quad d_{ABC} \phi_A \phi_B = \sqrt{2/3} \phi_C$$

(8)

(Algebraic constraints for $\mathbb{CP}^k$ embedded in $\mathbb{R}^{k^2+2k}$ are generically given in the appendix; see equations (55)-(57).) In (7), $t_A$ are the $SU(4)$ generators in the fundamental representation and $g$ is a group element of $SU(4)$ given as a $4 \times 4$-matrix. Functions on $\mathbb{CP}^3$ are then written as

$$f_{\mathbb{CP}^3}(u, \bar{u}) \sim f_{i_1i_2\cdots i_l}^1 \bar{u}_{i_1} \bar{u}_{i_2} \cdots \bar{u}_{i_l} u_{j_1} u_{j_2} \cdots u_{j_l}$$

(9)

where $l = 0, 1, 2, \ldots, n$, $u_j = g_{ij4}$, $\bar{u}_i = (\bar{g})^i_4$ and $\bar{u}_i u_i = 1$ ($i, j = 1, 2, 3, 4$). $\mathbb{CP}^3$ can be described by four complex coordinates $Z_i$ with the identification $Z_i \sim \lambda Z_i$ where $\lambda$ is any complex number except zero ($\lambda \in \mathbb{C} - \{0\}$). Following Penrose and MacCallum [14], we now write $Z_i$ in terms of two spinors $\omega, \pi$ as

$$Z_i = (\omega_a, \pi_{\bar{a}}) = (x_{a\bar{a}}, \pi_{\bar{a}})$$

(10)

where $a = 1, 2, \bar{a} = 1, 2$ and $x_{a\bar{a}}$ can be defined with the coordinates $x_{ij}$ on $S^4$ via $x_{a\bar{a}} = (\mathbf{1} x_4 - i\sigma \cdot \bar{x})$, $\sigma$ being $2 \times 2$ Pauli matrices. The scale invariance $Z_i \sim \lambda Z_i$ can be realized by the scale invariance $\pi_{\bar{a}} \sim \lambda \pi_{\bar{a}}$. The $\pi_{\bar{a}}$’s then describe a $\mathbb{CP}^1 = S^2$. This shows that $\mathbb{CP}^3$ is an $S^2$ bundle over $S^4$ (or Penrose’s projective twistor space). In (9), we can parametrize $u_i$ by the homogeneous coordinates $Z_i$, i.e., $u_i = \frac{Z_i}{\sqrt{Z \cdot Z}}$.

Functions on $S^4$ can be considered as functions on $\mathbb{CP}^3$ which satisfy

$$\frac{\partial}{\partial \pi_{\bar{a}}} f_{\mathbb{CP}^3}(Z, \bar{Z}) = \frac{\partial}{\partial \bar{\pi}_{\bar{a}}} f_{\mathbb{CP}^3}(Z, \bar{Z}) = 0$$

(11)

This implies $f_{\mathbb{CP}^3}$ are further invariant under transformations of $\pi_{\bar{a}}, \bar{\pi}_{\bar{a}}$. In terms of the 4-spinor $Z$, such transformations are expressed by

$$Z \rightarrow e^{it_\alpha} \theta_\alpha Z$$

(12)

where $t_\alpha \in H$, $H$ being generators (or algebra) of $H = SU(2) \times U(1)$ defined by the last two matrices in (5). (Interchanging the roles of indices $a$ and $\bar{a}$, we may choose the first matrix in (5) for the $SU(2)$ of $H$.) The coordinates $\phi_A$ in (7) can be written by $\phi_A(Z, \bar{Z}) \sim \bar{Z}_i (t_A)_{ij} Z_j$. Under an infinitesimal ($\theta_\alpha \ll 1$) transformation as in (12), the coordinates transform as

$$\phi_A \rightarrow \phi_A + \theta_\alpha f_{\alpha AB} \phi_B$$

(13)

where $f_{ABC}$ is the structure constant of $SU(4)$. The constraint (11) is then rewritten as

$$f_{\alpha AB} \phi_B \frac{\partial}{\partial \phi_A} f_{\mathbb{CP}^3} = 0$$

(14)
where $f_{\mathbb{C}P^3}$ are functions of $\phi_A$’s. Note that $\phi_A$’s in (14) are defined solely by (7), i.e., they are defined on $\mathbb{R}^{15}$.

From (7) we find $f_{\alpha AB} \phi_B \sim \tilde{Z}_i([t_A, t_\alpha])_{ij} Z_j$, where $t_\alpha$ are the generators of $H = SU(2) \times U(1) \subset SU(4)$ as before. Since any $(4 \times 4)$-matrix function is linear in $t_A$, the constraint (11) or (14) is then realized by $[t_A, t_\alpha] = 0$ which can be considered as a commutative implementation of the fuzzy constraint (6). Specifically, we may choose $t_\alpha = \{t_1, t_2, t_3, \sqrt{2} t_8 + \sqrt{2} t_{15}\}$ in the conventional choices of the $SU(4)$ generators in the fundamental representation. The constraint $[t_A, t_\alpha] = 0$ then restricts $A$ to be $A = 8, 13, 14, \text{and } 15$. Of course, this is a local analysis. The constraint $[t_A, t_\alpha] = 0$ globally defines $S^4$ as embedded in $\mathbb{R}^{15}$ similarly to how we have defined $\mathbb{C}P^3$. The number of $\mathbb{C}P^3$ coordinates $\phi_A$ is locally restricted to be six because of the algebraic constraints in (8). Similarly, the constraint $[t_A, t_\alpha] = 0$ further restricts the number of coordinates to be four, which is correct for the coordinates on $S^4$.

Functions on $S^4$ are polynomials of $\phi_A = -2Tr(g^4 t_A g t_{15})$ which obey $[t_A, t_\alpha] = 0$. A product of functions is based on the products of such $t_A$’s. Extension to the fuzzy $S^4$ is essentially done by replacing the fundamental representation, $t_A$, by any symmetric representation $(n, 0)$ of $SU(4)$, $L_A$. Then, the algebra of fuzzy $S^4$ naturally becomes associative in the commutative limit, while the associativity of fuzzy $S^4$, itself, will be discussed in the next section. There, we present a concrete matrix configuration of fuzzy $S^4$ so that the associativity is obviously seen. Even without any such matrix realizations, we can extract another property of the algebra from the condition (6). Since functions on fuzzy $S^4$ are represented by matrices which obey this condition, it is easily seen that the product of such functions also obeys the same condition. This leads to the closure of the algebra. One of the main results of this paper is that we can construct a fuzzy $S^4$ such that its algebra is closed and associative. The condition (6) plays an essential part in our construction. Imposing such an additional condition to obtain a fuzzy space from another was first considered by Nair and Randjbar-Daemi in the construction of fuzzy $S^3/\mathbb{Z}_2$ from fuzzy $S^2 \times S^2$ [13]. Our construction is very similar to theirs.

3 Matrix-function correspondence

In this section we examine our construction of fuzzy $S^4$ by confirming its matrix-function correspondence. To show a one-to-one correspondence, one needs to show two things: (a) a matching between the number of matrix elements for the fuzzy $S^4$ and the number of truncated functions on $S^4$; (b) a correspondence between the product of functions on fuzzy $S^4$ and that on $S^4$. Now, it would be suggestive to take a moment to review how (a) and (b) are fulfilled in the case of fuzzy $S^2 = SU(2)/U(1)$. Let $D^{(j)}_{mn}(g)$ be the Wigner $D$-functions for $SU(2)$. These are the spin-$j$ matrix representations of an $SU(2)$ group element $g$, $D^{(j)}_{mn}(g) = \langle jm | g | jn \rangle$ ($m, n = -j, \cdots, j$). Functions on $S^2$ can be expanded in terms of particular Wigner $D$-functions, $D^{(j)}_{m0}$, which are invariant under the right action of $U(1)$. (Since the state $|j0\rangle$ has no $U(1)$ charge, the right action of the $U(1)$ operator, $R_3$, on $g$ makes $D^{(j)}_{m0}(g)$ vanish, $R_3 D^{(j)}_{m0}(g) = 0$; in fact one can choose any fixed value ($m = -j, \cdots, j$) for this $U(1)$
charge.) These $D$-functions are essentially the spherical harmonics, $D^{(l)}_{m0} = \sqrt{\frac{4\pi}{2l+1}}(-1)^mY^l_{-m}$, and so a truncated expansion can be written as $f_{S^2} = \sum_{l=0}^{n} \sum_{m=-l}^{l} f^l_m D^{(l)}_{ml}$. The number of coefficients $f^l_m$ are counted by $\sum_{l=0}^{n}(2l+1) = (n+1)^2$. This relation implements the condition (a) by defining the functions on fuzzy $S^2$ of coefficients $f$ and so a truncated expansion can be written as $D$-functions. These $f$-functions are essentially the spherical harmonics, $D^2_{lj}$. These $f$-functions are dominated and this leads to an ordinary commutative product of $D$-functions on some space is given by an absolute square. Then, following the above procedure, we may establish the matrix-function correspondence. This is true for fuzzy $\mathbb{C}P^k$. In the case of fuzzy $\mathbb{C}P^3$, the absolute square appears from

$$N^{(3)} \times N^{(3)} = \sum_{l=0}^{n} \text{dim}(l, l)$$  \hspace{1cm} (18)

$$\text{dim}(l, l) = \frac{1}{12}(2l+3)(l+1)^2(l+2)^2$$  \hspace{1cm} (19)
where \( \text{dim}(l, l) \) is the dimension of \( SU(4) \) in the real \((l, l)\)-representation. This arises from the fact that a general function on \( \mathbb{CP}^3 = SU(4)/U(3) \) can be expanded by \( D_{M0}(g) \), the Wigner \( D \)-functions of \( SU(4) \) belonging to the \((l, l)\)-representation \((l = 0, 1, 2, \cdots) \). Here, \( g \) is an element of \( SU(4) \). The lower index \( M \) \((M = 1, \cdots, \text{dim}(l, l)) \) labels the state in this representation, while the index \( 0 \) represents any suitably fixed state in this representation. Like in (15), the symbol of fuzzy \( \mathbb{CP}^3 \) is defined by \( D_{IN(3)}^{(n,0)}(g) \) and its complex conjugate, where \( D_{IN(3)}^{(n,0)}(g) = \langle (n, 0), I | g | (n, 0), N^{(3)} \rangle \) are the \( D \)-functions belonging to the symmetric \((n, 0)\)-representation. While the index \( I \) \((I = 1, 2, \cdots, \text{dim}(n, 0) = N^{(3)}) \) labels the state in this representation, the index \( N^{(3)} \) indicates some highest weight state, which is a singlet under \( SU(3) \) and is \( U(1) \) invariant. The states of fuzzy \( \mathbb{CP}^3 \) are then expressed by \(|(n, 0), I \rangle \). Notice that one can alternatively express the states by \( \phi_{i_1 \cdots i_n} \), where the sequence of \( i_m = \{1, 2, 3, 4\} \) \((m = 1, \cdots, n) \) is in a totally symmetric order. The matrix-function correspondence for fuzzy \( \mathbb{CP}^k \), in the form of (18) and (19), is generically given in the appendix.

Let us now return to the conditions (a) and (b) of fuzzy \( S^4 \). In the following subsections, we present (i) different ways of counting the number of truncated functions on \( S^4 \), (ii) the one-to-one matrix-function correspondence, and (iii) a concrete matrix configuration for a function on fuzzy \( S^4 \). In (ii), the condition (a) is shown; we find the number of matrix elements for fuzzy \( S^4 \) agrees with the number calculated in (i). The condition (b) is also shown in (ii) by considering the commutative limits of the symbols and star products on fuzzy \( S^4 \). In (iii), we confirm the one-to-one correspondence by choosing a block-diagonal matrix realization of fuzzy \( S^4 \). With this construction, it becomes obvious that the algebra of fuzzy \( S^4 \) is closed and associative.

\( (i) \) Ways of Counting

A direct counting of the number of truncated functions on \( S^4 \) can be made in terms of the spherical harmonics \( Y_{l_1 l_2 l_3 m} \) on \( S^4 \) with a truncation at \( l_1 = n \) \cite{9}

\[
N^{S^4}(n) = \sum_{l_1=0}^{n} \sum_{l_2=0}^{l_1} \sum_{l_3=0}^{l_2} (2l_3 + 1) = \frac{1}{12} (n + 1)(n + 2)^2(n + 3)
\]

(20)

Alternatively, one can count \( N^{S^4}(n) \) by use of a tensor analysis. The number of truncated functions on \( \mathbb{CP}^3 \) is given by the totally symmetric and traceless tensors \( f_{j_1 \cdots j_l}^{i_1 \cdots i_l} \) \((i, j = 1, \cdots, 4) \) in (9) with the summation over \( l = 0, 1, \cdots, n \). Now we split the indices by \( i = a, \hat{a} \) \((a = 1, 2, \hat{a} = 3, 4) \) and similarly for \( j \). The additional constraint (11) for the extraction of \( S^4 \) from \( \mathbb{CP}^3 \) means that the tensors are independent of any combinations of \( \hat{a} \)'s in the sequence of \( i \)'s. (We will not lose generality by assuming that the number of \( \hat{a} \)'s in the sequence \( i \)'s is not less than that in \( j \)'s.) In other words, in the transformation (12), \( Z \rightarrow e^{i\alpha_ \theta_ \alpha} Z \), functions on \( S^4 \) are invariant under the transformations involving \( (t_\alpha)_{\hat{a}b} \) where \( t_\alpha \) are the \((4 \times 4)\)-generators of \( H = SU(2) \times U(1) \). There are \( N^{(2)}(l) = \frac{1}{2} (l + 1)(l + 2) \) ways of having a symmetric order \( i_1, i_2, \cdots, i_l \) for \( i = \{1, 2, \hat{a}\} \) \((\hat{a} = 3, 4) \). This can be regarded as an \( N^{(2)}(l) \)-degeneracy due to an \( S^2 \) internal symmetry for the extraction of \( S^4 \) out of \( \mathbb{CP}^3 \sim S^4 \times S^2 \). This \( S^2 \) symmetry is relevant to the above \((t_\alpha)_{\hat{a}b}\)-transformations.
the number of truncated functions on CP^3 is given by (19), the number of those on S^4 may be calculated by
\[ N_{S^4}(n) = \sum_{l=0}^{n} \frac{\text{dim}(l, l)}{N^{(2)}(l)} = \sum_{l=0}^{n} \frac{1}{6} (l+1)(l+2)(2l+3) = \frac{1}{12} (n+1)(n+2)^2(n+3) \] (21)
which reproduces (20). This is also in accordance with a corresponding calculation in the context of \( S^4 = SO(5)/SO(4) \) [5, 6].

(ii) One-to-one matrix-function correspondence

As discussed earlier, the states of fuzzy CP^3 can be denoted by \( \phi_{i_1 i_2 \cdots i_n} \) where the sequence of \( i_m = \{1, 2, 3, 4\} \) \( (m = 1, \cdots , n) \) is in a totally symmetric order. Let us denote a function on fuzzy CP^3 by an \( N^{(3)} \times N^{(3)} \) matrix, \((\hat{F})_{IJ}(I, J = 1, 2, \cdots , N^{(3)}) \). Likewise in quantum mechanics, the matrix element of the function operator \( \hat{F} \) on fuzzy CP^3 can be defined by \( \langle I | \hat{F} | J \rangle \), where we denote \( \phi_{i_1 \cdots i_n} = |i_1 \cdots i_n\rangle \equiv |I\rangle \). What we like to find is an analogous matrix expression \((\hat{F}_{S^4})_{IJ} \) for a function on fuzzy S^4. Let us now consider the states on fuzzy S^4 in terms of \( \phi_{i_1 i_2 \cdots i_n} \). Splitting each \( i \) into \( a \) and \( \dot{a} \), we may express \( \phi_{i_1 i_2 \cdots i_n} \) as
\[ \phi_{i_1 i_2 \cdots i_n} = \{ \phi_{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_n} , \phi_{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_{n-1} a_1} , \cdots , \phi_{a_1 a_2 \cdots a_n} \} \] (22)
From the analysis in the previous section, one can obtain the states corresponding to fuzzy S^4 by imposing an additional condition on (22), i.e., the invariance under the transformations involving any \( \dot{a}_m \). Transformations of the states on fuzzy S^4, under this particular condition, can be considered as follows. On the set of states \( \phi_{a_1 a_2 \cdots a_n} \), which are \( (n+1) \) in number, the transformations must be diagonal because of (11), but we can have an independent transformation for each state. (The number of the states are \( (n+1) \), since the sequence of \( \dot{a}_m = \{3, 4\} \) is in a totally symmetric order.) Thus we get \( (n+1) \) different functions proportional to identity. On the set of states \( \phi_{a_1 a_2 \cdots a_{n-1}} \), we can transform the \( a_1 \) index (to \( b_1 = \{1, 2\} \) for instance), corresponding to a matrix function \( f_{a_1 b_1} \) which have \( 2^2 \) independent components. But we can also choose the matrix \( f_{a_1 b_1} \) to be different for each choice of \( (\dot{a}_1 \cdots \dot{a}_{n-1}) \) giving \( 2^2 \times n \) functions in all, at this level. We can represent these as \( f_{a_1 b_1}^{(\dot{a}_1 \cdots \dot{a}_{n-1})} \), the extra composite index \( (\dot{a}_1 \cdots \dot{a}_{n-1}) \) counting the multiplicity. Continuing in this way, we find that the set of all functions on fuzzy S^4 is given by
\[ (\hat{F}_{S^4})_{IJ} = \{ f_{a_1 a_2 \cdots a_n}^{(\dot{a}_1 \cdots \dot{a}_{n-1})} \hat{\delta}_{\dot{a}_1 \cdots \dot{a}_{n-1} , \dot{a}_n , b_1 \cdots b_n}^{(\dot{a}_1 \cdots \dot{a}_{n-1})} , f_{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_{n-2}}^{(\dot{a}_1 \cdots \dot{a}_{n-2})} , \cdots , f_{a_1 a_2 \cdots a_n}^{(a_1 \cdots a_n)} \hat{\delta}_{\dot{a}_1 \cdots \dot{a}_{n-1} , \dot{a}_n , b_1 \cdots b_n}^{(a_1 \cdots a_n)} \} \] (23)
where we split \( i_m \) into \( a_m , \dot{a}_m \) and \( j_m \) into \( b_m , \dot{b}_m \). Each operator \( \hat{\delta}_{\dot{a}_1 \cdots \dot{a}_{m-1} , \dot{a}_m , b_1 \cdots b_m} \) \( (m = 1, 2, \cdots , n) \) indicates an identity operator such that the corresponding matrix is invariant under transformations from \( \{\dot{a}_1 \cdots \dot{a}_m\} \) to \( \{\dot{b}_1 \cdots \dot{b}_m\} \). The structure in (23) shows that \( \hat{F}_{S^4} \) is effectively composed of \( (l+1) \times (l+1) \)-matrices \( (l = 0, 1, \cdots , n) \) with the number of those matrices for fixed \( l \) being \( (n+1-l) \). Thus the number of matrix elements for fuzzy S^4 is also counted by
\[ N_{S^4}(n) = \sum_{l=0}^{n} (l+1)^2(n+1-l) = \frac{1}{12} (n+1)(n+2)^2(n+3) \] (24)
The relation (24) implements the condition (a). In order to show the precise matrix-function correspondence, we further need to show the condition (b), the correspondence of products. We carry out this in analogy with the case of fuzzy $S^2$ in (15)-(17). The symbol of the function $\hat{F}$ on fuzzy $\mathbb{CP}^3$ can be defined as

$$\langle \hat{F} \rangle = \sum_{IJ} \langle N^{(3)} | g | I \rangle \langle \hat{F} \rangle_{IJ} \langle J | g | N^{(3)} \rangle$$

(25)

where $\langle J | g | N^{(3)} \rangle$ is the previous $D$-function, $D_{IJ}^{(n,0)}(g)$. $|N^{(3)}\rangle$ is a $U(3)$ invariant state in the $(n,0)$-representation of $SU(4)$. (We have defined it as $|(n,0),N^{(3)}\rangle$ before.) The symbol of a function on fuzzy $S^4$ is defined in the same way except that $\langle \hat{F} \rangle_{IJ}$ is replaced with $(\hat{F}s^4)_{IJ}$ in (25). Now let us consider a product of two functions on fuzzy $S^4$. As we have seen, a function on fuzzy $S^4$ can be described by $(l+1) \times (l+1)$-matrices. From the structure of $\hat{F}s^4$ in (23), we are allowed to treat these matrices independently. The product is then considered as a set of matrix multiplications. This leads to a natural definition of the product, since the product of functions also becomes a function, retaining the same structure as in (23). The symbol of a product of two functions on fuzzy $S^4$ is written as

$$\langle \hat{F}s^4 \hat{G}s^4 \rangle = \sum_{IJK} (\hat{F}s^4)_{IJ} (\hat{G}s^4)_{JK} \langle N^{(3)} | g | I \rangle \langle K | g | N^{(3)} \rangle \equiv \langle \hat{F}s^4 \rangle \ast \langle \hat{G}s^4 \rangle$$

(26)

where the product $(\hat{F}s^4)_{IJ} (\hat{G}s^4)_{JK}$ is defined by the set of matrix multiplications. With the orthogonality of the $D$-functions, the associativity of the star products is easily seen.

Similarly to what happens in (17), the star product on fuzzy $\mathbb{CP}^3$ becomes the corresponding commutative product on $\mathbb{CP}^3$ in the large $n$ limit [15]. (In the case of fuzzy $\mathbb{CP}^4$, it is known that a symbol of any matrix function in a polynomial form becomes a corresponding commutative function in the large $n$ limit; this is rigorously shown in [16].) The symbols and star products of fuzzy $S^4$ can be obtained from those of fuzzy $\mathbb{CP}^3$ by simply replacing the function operator $\hat{F}$ with $\hat{F}s^4$. We can therefore find the correspondence between fuzzy and commutative products for $S^4$. We can in fact directly check this correspondence even at the level of finite $n$ from the following discussion.

Let us consider a parametrization of functions on $S^4$ in terms of the homogeneous coordinates on $\mathbb{CP}^3$, $Z_i = (\omega_a, \pi_a) = (x_{a\bar{a}}, \pi_{a\bar{a}})$, as in (10). The functions on $S^4$ can be constructed from $x_{a\bar{a}}$ under the constraint in (11) which implies that the functions are independent of $\pi_{a\bar{a}}$ and $\bar{\pi}_{a\bar{a}}$. Expanding in powers of $x_{a\bar{a}}$, we can express the functions in terms of $\{1, x_{a\bar{a}}, x_{a1\bar{a}1}, x_{a2\bar{a}2}, x_{a1\bar{a},a2\bar{a}2}, x_{a3\bar{a}3}, \ldots\}$, where the indices $a$’s (and $\bar{a}$’s) are symmetric in order (as in the case of functions on $\mathbb{CP}^3$; see (9)). Owing to the extra constraint (11), one can consider that all the factors involving $\pi_{a\bar{a}}$ and $\bar{\pi}_{a\bar{a}}$ can be absorbed into the coefficients of these terms. By iterative use of the relations, $x_{a\bar{a}} \pi_{a\bar{a}} = \omega_a$ and its complex conjugation, the above set of powers in $x_{a\bar{a}}$ can be expressed in terms of $\omega$’s and $\bar{\omega}$’s as

$$\begin{pmatrix}
\omega_{a1} \\
\bar{\omega}_{b1}
\end{pmatrix}_{2\times2}, \begin{pmatrix}
\omega_{a1} \omega_{a2} \\
\bar{\omega}_{a1} \omega_{b1}
\end{pmatrix}_{3\times3}, \begin{pmatrix}
\omega_{a1} \omega_{a2} \omega_{a3} \\
\bar{\omega}_{a1} \omega_{a2} \omega_{b1} \\
\omega_{b1} \omega_{b2} \omega_{b1}
\end{pmatrix}_{4\times4}, \ldots$$

(27)
where the indices $a$ and $b$ are simply used to distinguish $\bar{\omega}$ and $\omega$, respectively. Because the indices need to be symmetric, the number of independent terms in each column should be counted as indicated in (27).

Notice that even though the functions on $S^4$ can be parametrized by $\omega$'s (and $\bar{\omega}$'s), the overall variables of the functions should be the coordinates on $S^4$, $x_\mu$, instead of $\omega_a = \pi_\alpha x_{\alpha \bar{a}}$. The coefficients of the terms in (27) need to be accordingly chosen. For instance, the term $\omega_a$ with a coefficient $c_a$ will be expressed as $c_a \omega_a = c_a \pi_\alpha x_{\alpha \bar{a}} \equiv h_{\alpha \bar{a}} x_{\alpha \bar{a}}$, where $h_{\alpha \bar{a}}$ is considered as some arbitrary set of constants. Now we like to define truncated functions on $S^4$ in the present context. The functions on $S^4$ are generically expanded in powers of $\bar{\omega}_a$ and $\omega_b$ ($a = 1, 2$ and $b = 1, 2$)

$$ f_{S^4}(\omega, \bar{\omega}) \sim f^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_\beta} \bar{\omega}_{a_1} \bar{\omega}_{a_2} \cdots \bar{\omega}_{a_n} \omega_{b_1} \omega_{b_2} \cdots \omega_{b_\beta} $$

(28)

where $\alpha, \beta = 0, 1, 2, 3, \cdots$ and the coefficients $f^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_\beta}$ should be understood as generalizations of the above-mentioned $c_a$. The truncated functions on $S^4$ may be obtained by putting an upper bound for the value ($\alpha + \beta$). We choose this by setting $\alpha + \beta \leq n$. In (27), this choice corresponds to a truncation at the column which is to be labelled by $(n+1) \times (n+1)$. In order to count the number of truncated functions in (28), we have to notice the following relation between $\omega_a$ and $\bar{\omega}_a$

$$ \bar{\omega}_{a\omega_a} \sim x_\mu x_\mu = x^2 $$

(29)

Using this relation, we can contract $\bar{\omega}_a$'s in (27). For example, we begin with the contractions involving $\bar{\omega}_{a_1}$ with all terms in (27), which yield the following new set of terms

$$ 1, \begin{pmatrix} \bar{\omega}_{a_2} \\ \omega_{b_1} \end{pmatrix}, \begin{pmatrix} \bar{\omega}_{a_2} \bar{\omega}_{a_3} \\ \bar{\omega}_{a_2} \omega_{b_1} \\ \omega_{b_1} \omega_{b_2} \end{pmatrix}, \cdots $$

(30)

The coefficients for the terms in (30) are independent of those for (27), due to the scale invariance $\pi_\alpha \pi_\bar{a} \sim |\lambda|^2$ $(\lambda \in \mathbb{C} - \{0\})$ in the contracting relation (29). Consequently, we can make a similar contraction at most $n$-times. The total number of truncated functions on $S^4$ is then counted by

$$ N^{S^4}(n) \equiv \sum_{l=0}^{n} \left[ 1^2 + 2^2 + \cdots + (l+1)^2 \right] = \frac{1}{12} (n+1)(n+2)^2(n+3) $$

(31)

which indeed equals to the previously found results in (20) and (21).

From (27)-(31), we find that all the coefficients in $f_{S^4}(\omega, \bar{\omega})$ correspond to the number of the matrix elements for $F^{S^4}$ given in (24). Further, since any products of fuzzy functions do not alter their structure in (23), such products correspond to the commutative products of $f_{S^4}(\omega, \bar{\omega})$'s. This leads to the precise correspondence between the functions on fuzzy $S^4$ and the truncated functions on $S^4$ at any level of truncation.

(iii) A block-diagonal matrix realization of fuzzy $S^4$

Although we have analyzed the structure of functions on fuzzy $S^4$ and their products in some detail, we haven’t presented an explicit matrix configuration for those fuzzy functions.
But, by now, it is obvious that we can use a block-diagonal matrix to represent them and this choice makes the associativity of the algebra automatic. Let us write down the equation (24) in a explicit form as

\[
N^{S^4}(n) = 1 + 1 + 2^2 + 1 + 2^2 + 3^2 + 1 + 2^2 + 3^2 + 4^2 + \cdots + (n + 1)^2
\]

(32)

If we locate all the squared elements block-diagonally, then the dimension of the embedding square matrix is given by

\[
\sum_{l=0}^{n} [1 + 2 + \cdots + (l + 1)] = \frac{1}{6} (n + 1)(n + 2)(n + 3) = N^{(3)}
\]

(33)

The coordinates of fuzzy $S^4$ are then represented by these $N^{(3)} \times N^{(3)}$ block-diagonal matrices, $X_A$, which satisfy

\[
X_A X_A \sim 1
\]

(34)

where $I$ is the $N^{(3)} \times N^{(3)}$ identity matrix and $A = 1, 2, 3, 4$ and 5, four of which are relevant to the coordinates of fuzzy $S^4$. The fact that $N^{S^4}$ is a sum of absolute squares does not necessarily warrant the associative algebra. (Every integer is a sum of squares, $1 + 1 + \cdots + 1$, but this does not mean any linear space of any dimension is an algebra.) It is the structure of $\hat{F}^{S^4}$ as well as the matching between (31) and (24) that lead to these matrices $X_A$.

Of course, $X_A$ are not the only matrices that describe fuzzy $S^4$. Instead of diagonally locating every block one by one, we can also put the same-size blocks into a single block, using matrix multiplication (or matrix addition). Then, the final form has a dimension of $\sum_{l=0}^{n} (l + 1) = \frac{1}{2} (n + 1)(n + 2) = N^{(2)}$. This implies an alternative description of fuzzy $S^4$ in terms of $N^{(2)} \times N^{(2)}$ block-diagonal matrices, $\overline{X}_A$, which are embedded in $N^{(3)}$-dimensional square matrices and satisfy $\overline{X}_A \overline{X}_A \sim \overline{1}$, where $\overline{1} = diag(1, 1, \cdots, 1, 0, 0, \cdots, 0)$ is an $N^{(3)} \times N^{(3)}$ diagonal matrix, with the number of 1’s being $N^{(2)}$. Our choice of $X_A$ is, however, convenient in the context where we extract the fuzzy $S^4$ from fuzzy CP$^3$. The number of 1’s in $X_A$ is $(n + 1)$. This corresponds to the dimension of $SU(2) \subset SU(4)$ in our $N^{(3)}(n)$-dimensional matrix representation. (Notice that a fuzzy $S^2 = CP^1$ is conventionally described by $(n + 1) \times (n + 1)$ matrices.) Using the coordinates $X_A$, we can then confirm the constraint in (6), i.e.,

\[
[\mathcal{F}(X), L_\alpha] = 0
\]

(35)

where $\mathcal{F}(X)$ are matrix-functions of $X_A$’s and $L_\alpha$ are the generators of $H = SU(2) \times U(1) \subset SU(4)$, represented by $N^{(3)} \times N^{(3)}$ matrices. If both $\mathcal{F}(X)$ and $G(X)$ commute with $L_\alpha$, so does $\mathcal{F}(X)G(X)$. Thus, there is the closure of such “functions” under multiplication. This indicates that the fuzzy $S^4$ follows a closed associative algebra.
4 Construction of fuzzy $S^8$

We outline a construction of fuzzy $S^8$ in a way of reviewing our construction of fuzzy $S^4$. As mentioned in the introduction, $\text{CP}^7$ is a $\text{CP}^3$ bundle over $S^8$. We expect that we can similarly construct the fuzzy $S^8$ by factoring out a fuzzy $\text{CP}^3$ out of fuzzy $\text{CP}^7$.

The structure of fuzzy $S^4$ as a block-diagonal matrix has been derived, based on the following two equations

\begin{align}
N^{S^4}(n) &= \sum_{l=0}^{n} \left( N^{(1)}(l) \right)^2 N^{(1)}(n-l) \quad (36) \\
N^{(3)}(n) &= \sum_{l=0}^{n} N^{(1)}(l) N^{(1)}(n-l) \quad (37)
\end{align}

where $N^{(k)}(l) = \frac{(l+k)!}{k!}$ as in the appendix. The fuzzy-$S^8$ analogs of these equations are

\begin{align}
N^{S^8}(n) &= \sum_{l=0}^{n} N^{S^4}(l) N^{(3)}(n-l) \quad (38) \\
N^{(7)}(n) &= \sum_{l=0}^{n} N^{(3)}(l) N^{(3)}(n-l) \quad (39)
\end{align}

where $N^{S^8}(n)$ is the number of truncated functions on $S^8$, which can be calculated in terms of the spherical harmonics as in the case of $S^4$ in (20)

\begin{align}
N^{S^8}(n) &= \sum_{a=0}^{n} \sum_{b=0}^{a} \sum_{c=0}^{b} \sum_{d=0}^{c} \sum_{e=0}^{d} \sum_{f=0}^{e} \sum_{g=0}^{f} (2g + 1) \\
&= \frac{1}{4 \cdot 7!} (n+1)(n+2)(n+3)(n+4)^2(n+5)(n+6)(n+7) \quad (40)
\end{align}

This number is also calculated by a tensor analysis as in (21)

\begin{align}
N^{S^8}(n) &= \sum_{l=0}^{n} \frac{\text{dim}(l, l)}{N^{(6)}(l)} = \sum_{l=0}^{n} \frac{1}{7!} (2l + 7)(l+1)(l+2)(l+3)(l+4)(l+5)(l+6) \\
&= \frac{n+4}{4} \frac{(n+7)!}{7! n!} \quad (41)
\end{align}

where $\text{dim}(l, l)$ is the dimension of $\text{SU}(8)$ in the $(l, l)$-representation, i.e., $\text{dim}(l, l) = \frac{1}{7!} (2l + 7)(l+1)(l+2)(l+3)(l+4)(l+5)(l+6)$. All these calculations from (36) to (41) are carried out by use of Mathematica.

The equations (38) and (39) indicate that the fuzzy $S^8$ is composed of $N^{(3)}(l)$-dimensional block-diagonal matrices of fuzzy $S^4$ ($l = 0, 1, \cdots, n$) with the number of those matrices for fixed $l$ being $N^{(3)}(n-l)$. Thus the fuzzy $S^8$ is also described by a block-diagonal matrix whose embedding square matrix of dimension $N^{(7)}(n)$ represents the fuzzy $\text{CP}^7$. Notice that we have a nice matryoshka-like structure for fuzzy $S^8$, namely, a fuzzy-$S^8$ box is composed
of a number of fuzzy-$S^4$ blocks and each of those blocks is further composed of a number of fuzzy-$S^2$ blocks. The fuzzy $S^8$ is then represented by $N(7) \times N(7)$ block-diagonal matrices $X_A$ which satisfy $X_A X_A \sim 1$ ($A = 1, 2, \ldots, 9$), where $1$ is the $N(7) \times N(7)$ identity matrix. Similarly to the case of fuzzy $S^4$, the fuzzy $S^8$ should also obey a closed associative algebra.

Let us now consider the decomposition

$$SU(8) \rightarrow SU(4) \times SU(4) \times U(1) = H^{(4)}$$

where the two $SU(4)$’s and one $U(1)$ are defined similarly to (5) in terms of the generators of $SU(8)$ in the fundamental representation. Noticing the fact that the number of 1-dimensional blocks in the coordinate $X_A$ of fuzzy $S^8$ is $N(3)$($n$), we find $[X_A, L_\alpha] = 0$ where $L_\alpha$ are now the generators of $H^{(4)}$ which are represented by $N(7) \times N(7)$ matrices. This is in accordance with the statement that functions on $S^8$ are functions on $CP^7 = SU(8)/U(7)$ which are invariant under transformations of $H^{(4)} = SU(4) \times U(1)$. Coming back to the original idea, we can then construct the fuzzy $S^8$ out of fuzzy $CP^7$ by imposing the particular constraint $[F, L_\alpha] = 0$, where $F$ are matrix-functions of coordinates $Q_A$ on fuzzy $CP^7$, $Q_A$ being defined as in the appendix. (This constraint further restricts the function $F$ to be a function on fuzzy $S^8$, that is, a polynomial of $X_A$’s.)

Following the same method, we may construct higher dimensional fuzzy spheres [8, 17, 18]. But we are incapable of doing so as far as we utilize bundle structures analogous to $CP^3$ or $CP^7$. This is because, as far as complex number coefficients are used, there are no division algebra allowed beyond octonions. The fact that $CP^7$ is a $CP^3$ bundle over $S^8$ is based on the fact that octonions provide the Hopf map, $S^{15} \rightarrow S^8$ with its fiber being $S^7$. Since this map is the final Hopf map, there are no more bundle structures available to construct fuzzy spheres in a direct analogy with the constructions of fuzzy $S^8$, $S^4$ and $S^2$.

5 Conclusions

We have presented a construction of fuzzy $S^4$, utilizing the fact that $CP^3$ is an $S^2$ bundle over $S^4$. A fuzzy $S^4$ is obtained by an imposition of an additional constraint on a fuzzy $CP^3$. We find the constraint is appropriate by considering commutative limits of functions on fuzzy $S^4$ in terms of homogeneous coordinates of $CP^3$.

We propose that coordinates on fuzzy $S^4$ be described by block-diagonal matrices whose embedding square matrix represents the fuzzy $CP^3$. Along the way, we have shown a precise matrix-function correspondence for fuzzy $S^4$, providing different ways of counting the number of truncated functions on $S^4$. Because of its structure, the fuzzy $S^4$ should follow a closed and associative algebra.

Finally, we have also seen that an analogous construction can be made for fuzzy $S^8$.  

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Appendix: Construction of fuzzy $\text{CP}^k$

Here we present the construction of fuzzy $\text{CP}^k$ ($k = 1, 2, \cdots$) in the framework of the creation-annihilation operators [19, 2]. The coordinates $Q_A$ of fuzzy $\text{CP}^k$ can be defined in terms of $L_A$ which are $N(k) \times N(k)$-matrix representations of $SU(k+1)$ generators in the $(n, 0)$-representation (the totally symmetric representation of order $n$)

$$Q_A = \frac{L_A}{\sqrt{C_2^{(k)}}}$$

with two constraints

$$Q_A Q_A = 1$$
$$d_{ABC} Q_A Q_B = c_{k,n} Q_C$$

where $1$ is the $N(k) \times N(k)$ identity matrix, $d_{ABC}$ is the totally symmetric invariant tensor for $SU(k+1)$, $C_2^{(k)}$ is the quadratic Casimir for $SU(k+1)$ in the $(n, 0)$-representation

$$C_2^{(k)} = n k (n + k + 1)$$

and $N(k)$ is the dimension of $SU(k+1)$ in the $(n, 0)$-representation

$$N(k) = \text{dim}(n, 0) = \frac{(n + k)!}{k! n!}.$$  

In order to determine the coefficient $c_{k,n}$ in (45), we now notice that the $SU(k+1)$ generators in the $(n, 0)$-representation can be written by

$$\Lambda_A = a_i^\dagger (t_A)_{ij} a_j$$

where $t_A$ ($A = 1, 2, \cdots, k^2 + 2k$) are the $SU(k+1)$ generators in the fundamental representation with normalization $Tr(t_A t_B) = \frac{1}{2} \delta_{AB}$ and $a_i^\dagger, a_i$ ($i = 1, \cdots, k+1$) are the creation and annihilation operators acting on the $SU(k+1)$ states in the $(n, 0)$-representation which are spanned by

$$\left| n_1, n_2, \cdots, n_{k+1} \right> = (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \cdots (a_{k+1}^\dagger)^{n_{k+1}} \left| 0 \right>$$

with the following relations

$$a_i^\dagger a_i \left| n_1, n_2, \cdots, n_{k+1} \right> = (n_1 + n_2 + \cdots + n_{k+1}) \left| n_1, n_2, \cdots, n_{k+1} \right>$$
$$= n \left| n_1, n_2, \cdots, n_{k+1} \right>$$

$$a_i \left| 0 \right> = 0.$$
Using the completeness relation for $t_A$'s
\[ (t_A)_{ij} (t_A)_{kl} = \frac{1}{2} \left( \delta_{il} \delta_{jk} - \frac{1}{k+1} \delta_{ij} \delta_{kl} \right) \] (52)
and the commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$, we can check $\Lambda_A \Lambda_A = C^{(k)}_2$, where the creation and annihilation operators act on the states of the form (49) from the left. We also find
\[ d_{ABC} \Lambda_B \Lambda_C = (k - 1) \left( \frac{n}{k+1} + \frac{1}{2} \right) a_i^\dagger (t_A)_{ij} a_j \]
\[ = (k - 1) \left( \frac{n}{k+1} + \frac{1}{2} \right) \Lambda_A . \] (53)
(This result is also obtained in [20].) Representing $\Lambda_A$ by $L_A$, we can determine the coefficient $c_{k,n}$ in (45) by
\[ c_{k,n} = \frac{(k - 1)}{\sqrt{C^{(k)}_2}} \left( \frac{n}{k+1} + \frac{1}{2} \right) \] (54)
For $k \ll n$, we have
\[ c_{k,n} \longrightarrow c_k = \sqrt{\frac{2}{k(k+1)}} (k - 1) \] (55)
and this leads to the constraints for the coordinates $q_A$ of $\mathbb{CP}^k$
\[ q_A q_A = 1 \] (56)
\[ d_{ABC} q_A q_B = c_k q_C \] (57)
The second constraint (57) restricts the number of coordinates to be $2k$ out of $k^2 + 2k$. For example, in the case of $\mathbb{CP}^2 = \text{SU}(3)/U(2)$ this constraint around the pole of $A = 8$ becomes $d_{8BC} q_8 q_B = \frac{1}{\sqrt{3}} q_C$. Normalizing the 8-coordinate to be $q_8 = -2$, we find the indices of the coordinates are restricted to 4, 5, 6, and 7 with the conventional choice of the generators of $SU(3)$ as well as with the definition $d_{ABC} = 2Tr(t_A t_B t_C + t_A t_C t_B)$.

**Matrix-Function Correspondence**

The matrix-function correspondence for fuzzy $\mathbb{CP}^k$ can be expressed by
\[ N^{(k)} \times N^{(k)} = \sum_{l=0}^{n} \text{dim}(l,l) \] (58)
where $\text{dim}(l,l)$ is the dimension of $SU(k+1)$ in the $(l,l)$-representation. This real $(l,l)$-representations are required so that we have scalar functions on $\mathbb{CP}^k = \frac{SU(k+1)}{U(k)}$ [3]. Symbolically the correspondence is written as
\[ (n, 0) \otimes (0, n) = \bigoplus_{l=0}^{n} (l,l) \] (59)
in terms of the dimensionality of $SU(k+1)$. The l.h.s. of (59) can be interpreted from the fact that $\Lambda_A = a_i^\dagger (t_A)_{ij} a_j \sim a_i^\dagger a_j$ transforms like $(n,0) \otimes (0,n)$. The r.h.s. of (59), on the other hand, can be interpreted by a usual tensor analysis, i.e., $\text{dim}(l,l)$ is the number of ways to construct tensors of the form $T_{i_1 i_2 \cdots i_l}$ such that the tensor is traceless and totally symmetric with $i$ and $j$ being $1, 2, \cdots, k+1$.
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