Trimmed Harrell-Davis quantile estimator based on the highest density interval of the given width

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ABSTRACT

Traditional quantile estimators that are based on one or two order statistics are a common way to estimate distribution quantiles based on the given samples. These estimators are robust, but their statistical efficiency is not always good enough. A more efficient alternative is the Harrell-Davis quantile estimator which uses a weighted sum of all order statistics. Whereas this approach provides more accurate estimations for the light-tailed distributions, it’s not robust. To be able to customize the tradeoff between statistical efficiency and robustness, we could consider a trimmed modification of the Harrell-Davis quantile estimator. In this approach, we discard order statistics with low weights according to the highest density interval of the beta distribution.

1. Introduction

We consider a problem of quantile estimation for the given sample. Let $x$ be a sample with $n$ elements: $x = \{x_1, x_2, \ldots, x_n\}$. We assume that all sample elements are sorted ($x_1 \leq x_2 \leq \ldots \leq x_n$) so that we could treat the $i$th element $x_i$ as the $i$th order statistic $x_{(i)}$. Based on the given sample, we want to build an estimation of the $p$th quantile $Q(p)$.

The traditional way to do this is to use a single order statistic or a linear combination of two subsequent order statistics. This approach could be implemented in various ways. A classification of the most popular implementations could be found in Hyndman and Fan (1996). In this paper, Rob J. Hyndman and Yanan Fan describe nine types of traditional quantile estimators which are used in statistical computer packages. The most popular approach in this taxonomy is Type 7 which is used by default in R, Julia, NumPy, and Excel:

$$Q_{\text{HF7}}(p) = x_{[h]} + (h - [h])(x_{[h]+1} - x_{[h]}), \quad h = (n - 1)p + 1.$$  

Traditional quantile estimators have simple implementations and a good robustness level. However, their statistical efficiency is not always good enough: the obtained estimations could noticeably differ from the true distribution quantile values. The gap between the estimated and true values could be decreased by increasing the number of used order statistics. Harrell and Davis (1982) suggest estimating quantiles using a weighted sum of all order statistics:

$$Q_{\text{HD}}(p) = \sum_{i=1}^{n} W_{\text{HD},i} \cdot x_i, \quad W_{\text{HD},i} = I_{i/n}(\alpha, \beta) - I_{(i-1)/n}(\alpha, \beta),$$

where $I_{x}(\alpha, \beta)$ is the regularized incomplete beta function, $\alpha = (n + 1)p$, $\beta = (n + 1)(1 - p)$. To get a better understanding of this approach, we could look at the probability density function of...
the beta distribution Beta($\alpha, \beta$) (see Figure 1). If we split the $[0; 1]$ interval into $n$ segments of equal width, we can define $W_{\text{HD},i}$ as the area under curve in the $i$th segment. Since $I_x(\alpha, \beta)$ is the cumulative distribution function of Beta($\alpha, \beta$), we can express $W_{\text{HD},i}$ as $I_{i/n}(\alpha, \beta) - I_{(i-1)/n}(\alpha, \beta)$.

The Harrell-Davis quantile estimator is suggested in David and Nagaraja (2003), Grissom and Kim (2012), Wilcox (2016), and Dickinson Gibbons and Chakraborti (2020) as an efficient alternative to the traditional estimators. Yoshizawa, Sen, and Davis (1985), asymptotic equivalence for the traditional sample median and the Harrell–Davis median estimator is shown.

The Harrell-Davis quantile estimator shows decent statistical efficiency in the case of light-tailed distributions: its estimations are much more accurate than estimations of the traditional quantile estimators. However, the improved efficiency has a price: $Q_{\text{HD}}$ is not robust. Since the estimation is a weighted sum of all order statistics with positive weights, a single corrupted element may spoil all the quantile estimations, including the median. It may become a severe drawback in the case of heavy-tailed distributions in which it’s a typical situation when we have a few extremely large outliers. In such cases, we use the median instead of the mean as a measure of central tendency because of its robustness. Indeed, if we estimate the median using the traditional quantile estimators like $Q_{\text{HF7}}$, its asymptotic breakdown point is 0.5. Unfortunately, if we switch to $Q_{\text{HD}}$, the breakdown point becomes zero so that we completely lose the median robustness.

Another severe drawback of $Q_{\text{HD}}$ is its computational complexity. If we have a sorted array of numbers, a traditional quantile estimation could be computed using $O(1)$ simple operations. For an unsorted sample, there are approaches that allow getting an order statistic using $O(n)$ operations (e.g., see Alexandrescu 2017). If we estimate the quantiles using $Q_{\text{HD}}$, we need $O(n)$ operations for a sorted array. Moreover, these operations involve computation of $I_x(\alpha, \beta)$ values which are pretty expensive from the computational point of view.

Alternatively, we could consider the Sfakianakis-Verginis quantile estimator (see Sfakianakis and Verginis 2008) or the Navruz-Özdemir quantile estimator (see Navruz and Özdemir 2020) which are also based on a weighted sum of all order statistics. However, they also have the disadvantages listed above.

Neither $Q_{\text{HF7}}$ nor $Q_{\text{HD}}$ fit all kinds of problem. $Q_{\text{HF7}}$ is simple, robust, and computationally fast, but its statistical efficiency doesn’t always satisfy the business requirements. $Q_{\text{HD}}$ could provide better statistical efficiency, but it’s computationally slow and not robust.

To get a reasonable tradeoff between $Q_{\text{HF7}}$ and $Q_{\text{HD}}$, we consider a trimmed modification of the Harrell-Davis quantile estimator. The core idea is simple: we take the classic Harrell-Davis quantile estimator, find the highest density interval of the underlying beta distribution, discard all the order statistics outside the interval, and calculate a weighted sum of the order statistics within the interval. The obtained quantile estimation is more robust than $Q_{\text{HD}}$ (because it doesn’t use...
extreme values) and typically more statistically efficient than $Q_{HF7}$ (because it uses more than only two order statistics). Let’s discuss this approach in detail.

2. The trimmed Harrell-Davis quantile estimator

The estimators based on one or two order statistics are not efficient enough because they use too few sample elements. The estimators based on all order statistics are not robust enough because they use too many sample elements. It looks reasonable to consider a quantile estimator based on a variable number of order statistics. This number should be large enough to ensure decent statistical efficiency but not too large to exclude possible extreme outliers.

A robust alternative to the mean is the trimmed mean. The idea behind it is simple: we should discard some sample elements at both ends and use only the middle order statistics. With this approach, we can customize the tradeoff between robustness and statistical efficiency by controlling the number of the discarded elements. If we apply the same idea to $Q_{HD}$, we can build a trimmed modification of the Harrell-Davis quantile estimator. Let’s denote it as $Q_{THD}$.

In the case of the trimmed mean, we typically discard the same number of elements on each side. We can’t do the same for $Q_{THD}$ because the array of order statistic weights $\{W_{HD, i}\}$ is asymmetric. It looks reasonable to drop the elements with the lowest weights and keep the elements with the highest weights. Since the weights are assigned according to the beta distribution, the range of order statistics with the highest weight concentration could be found using the beta distribution highest density interval. Thus, once we fix the proportion of dropped/kept elements, we should find the highest density interval of the given width. Let’s denote the interval as $[L; R]$ where $R - L = D$. The order statistics weights for $Q_{THD}$ should be defined using a part of the beta distribution within this interval. It gives us the truncated beta distribution $T\text{Beta}(\alpha, \beta, L, R)$ (see Figure 2).

We know the CDF for Beta($\alpha, \beta$) which is used in $Q_{HD}$: $F_{HD}(x) = I_\alpha(x, \beta)$. For $Q_{THD}$, we need the CDF for $T\text{Beta}(\alpha, \beta, L, R)$ which could be easily found:

$$F_{THD}(x) = \begin{cases} 
0 & \text{for } x < L, \\
(F_{HD}(x) - F_{HD}(L))/(F_{HD}(R) - F_{HD}(L)) & \text{for } L \leq x \leq R, \\
1 & \text{for } R < x.
\end{cases}$$

Figure 2. Truncated beta distribution $B(5.5, 5.5)$ probability density function.
The final $Q_{THD}$ equation has the same form as $Q_{HD}$:

$$Q_{THD} = \sum_{i=1}^{n} W_{THD,i} \cdot x_i, \quad W_{THD,i} = F_{THD}(i/n) - F_{THD}((i-1)/n).$$

There is only one thing left to do: we should choose an appropriate width $D$ of the beta distribution highest density interval. In practical application, this value should be chosen based on the given problem: researchers should carefully analyze business requirements, describe desired robustness level via setting the breakdown point, and come up with a $D$ value that satisfies the initial requirements.

However, if we have absolutely no information about the problem, the underlying distribution, and the robustness requirements, we can use the following rule of thumb which gives the starting point: $D = 1/\sqrt{n}$. We denote $Q_{THD}$ with such a $D$ value as $Q_{THD-SQR}$. In most cases, it gives an acceptable tradeoff between the statistical efficiency and the robustness level. Also, $Q_{THD-SQR}$ has a practically reasonable computational complexity: $O(\sqrt{n})$ instead of $O(n)$ for $Q_{HD}$. For example, if $n = 10\,000$, we have to process only 100 sample elements and calculate 101 values of $I_x(x, \beta)$.

### An example

Let’s say we have the following sample:

$$x \approx \{-0.565, -0.106, -0.095, 0.363, 0.404, 0.633, 1.371, 1.512, 2.018, 100\,000\}.$$

Nine elements were randomly taken from the standard normal distribution $N(0,1)$. The last element $x_{10}$ is an outlier. The weight coefficient for $Q_{HD}$ an $Q_{THD-SQR}$ are presented in Table 1.

| $i$ | $x_i$   | $W_{HD,i}$ | $W_{THD-SQR,i}$ |
|-----|---------|------------|-----------------|
| 1   | -0.565  | 0.0005     | 0               |
| 2   | -0.106  | 0.0146     | 0               |
| 3   | -0.095  | 0.0727     | 0               |
| 4   | 0.363   | 0.1684     | 0.1554          |
| 5   | 0.404   | 0.2438     | 0.3446          |
| 6   | 0.633   | 0.2438     | 0.3446          |
| 7   | 1.371   | 0.1684     | 0.1554          |
| 8   | 1.512   | 0.0727     | 0               |
| 9   | 2.018   | 0.0146     | 0               |
| 10  | 100\,000\,000 | 0.0005     | 0               |

Table 1. Weight coefficients of $Q_{HD}$ and $Q_{THD-SQR}$ for $n = 10$.

As we can see, $Q_{HD}$ is heavily affected by the outlier $x_{10}$. Meanwhile, $Q_{THD}$ gives a reasonable median estimation because it uses a weighted sum of four middle order statistics.

### 3. Beta distribution highest density interval of the given width

In order to build the truncated beta distribution for $Q_{THD}$, we have to find the Beta($\alpha, \beta$) highest density interval of the required width $D$. Thus, for the given $\alpha, \beta, D$, we should provide an interval $[L; R]$:

$$\text{BetaHDI}(\alpha, \beta, D) = [L; R].$$

Let’s briefly discuss how to do this. First of all, we should calculate the mode $M$ of Beta($\alpha, \beta$) (non-degenerate cases are presented in Figure 3):
The actual value of BetaHDI($\alpha$, $\beta$, $D$) depends on the specific case from the above list which defines the mode location. Three of these cases are easy to handle:

- **Degenerate case** $\alpha \leq 1$, $\beta \leq 1$: There is only one way to get such a situation: $n = 1, p = 0.5$. Since such a sample contains a single element, it doesn’t matter how we choose the interval.
- **Left border case** $\alpha \leq 1$, $\beta > 1$: The mode equals zero, so the interval should be “attached to the left border”: BetaHDI($\alpha$, $\beta$, $D$) = $[0; D]$.
- **Right border case** $\alpha > 1$, $\beta \leq 1$: The mode equals one, so the interval should be “attached to the right border”: BetaHDI($\alpha$, $\beta$, $D$) = $[1 - D; 1]$

The fourth case is the middle case ($\alpha > 1$, $\beta > 1$), the HDI should be inside $(0; 1)$. Since the density function of the beta distribution is a unimodal function, it consists of two segments: a monotonically increasing segment $[0, M]$ and a monotonically decreasing segment $[M, 1]$. The HDI $[L; R]$ should contain the mode, so

$$L \in [0; M], \quad R \in [M; 1].$$

Since $R - L = D$, we could also conclude that

$$L = R - D \in [M - D; 1 - D], \quad R = L + D \in [D; M + D].$$

Thus,

$$L \in [\max(0, M - D); \min(M, 1 - D)], \quad R \in [\max(M, D); \min(1, M + D)].$$
The density function of the beta distribution is also known (see Croarkin and Tobias 2021):

\[
f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a, b)}, \quad B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.
\]

It’s easy to see that for the highest density interval \([L; R]\), the following condition is true:

\[f(L) = f(R).\]

The left border \(L\) of this interval could be found as a solution of the following equation:

\[f(t) = f(t + D), \quad \text{where} \quad t \in [\max(0, M - D); \min(M, 1 - D)].\]

The left side of the equation is monotonically increasing, the right side is monotonically decreasing. The equation has exactly one solution which could be easily found numerically using the binary search algorithm.

### 4. Simulation study

Let’s perform a few numerical simulations and see how \(Q_{\text{THD}}\) works in action.\(^1\)

#### 4.1. Simulation 1

Let’s explore the distribution of estimations for \(Q_{\text{HF7}}, Q_{\text{HD}},\) and \(Q_{\text{THD-SQRT}}\). We consider a contaminated normal distribution which is a mixture of two normal distributions: \((1 - \varepsilon)\mathcal{N}(0, \sigma^2) + \varepsilon\mathcal{N}(0, c\sigma^2)\). For our simulation, we use \(\varepsilon = 0.01, \sigma = 1, c = 1\ 000\ 000\). We generate 10 000 samples of size 7 randomly taken from the considered distribution. For each sample, we estimate the

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\(^1\)The source code of all simulations is available on GitHub: https://github.com/AndreyAkinshin/paper-thdqe
median using $Q_{HF7}$, $Q_{HD}$, and $Q_{THD-SQRT}$. Thus, we have 10 000 median estimations for each estimator. Next, we evaluate lower and higher percentiles for each group of estimations. The results are presented in Table 2.

We also perform the same experiment using the Fréchet distribution (shape = 1) as an example of a right-skewed heavy-tailed distribution. The results are presented in Table 3.

In Table 2, approximately 2% of all $Q_{HD}$ results exceed 10 by their absolute values (while the true median value is zero). Meanwhile, the maximum absolute value of the $Q_{THD-SQRT}$ median estimations is approximately 1.7. In Table 3, the higher percentiles of $Q_{HD}$ estimations are also noticeably higher than the corresponding values of $Q_{THD-SQRT}$.

Thus, $Q_{THD-SQRT}$ is much more resistant to outliers than $Q_{HD}$.

Table 4. Distributions from Simulation 2.

| Distribution                      | Support             | Skewness     | Tailness     |
|-----------------------------------|---------------------|--------------|--------------|
| Uniform ($a = 0, b = 1$)          | [0; 1]              | Symmetric    | Light-tailed |
| Triangular ($a = 0, b = 2, c = 1$) | [0; 2]              | Symmetric    | Light-tailed |
| Triangular ($a = 0, b = 2, c = 0.2$) | [0; 2]             | Right-skewed | Light-tailed |
| Beta ($a = 2, b = 4$)             | [0; 1]              | Right-skewed | Light-tailed |
| Beta ($a = 2, b = 10$)            | [0; 1]              | Right-skewed | Light-tailed |
| Normal ($m = 0, sd = 1$)          | $(-\infty; +\infty)$ | Symmetric    | Light-tailed |
| Weibull (scale = 1, shape = 2)    | [0; +\infty)        | Right-skewed | Light-tailed |
| Student (df = 3)                  | $(-\infty; +\infty)$ | Symmetric    | Light-tailed |
| Gumbel (loc = 0, scale = 1)       | $(-\infty; +\infty)$ | Right-skewed | Light-tailed |
| Exp (rate = 1)                    | (0; +\infty)        | Right-skewed | Light-tailed |
| Cauchy (x0 = 0, gamma = 1)        | $(-\infty; +\infty)$ | Symmetric    | Heavy-tailed |
| Pareto (loc = 1, shape = 0.5)     | [1; +\infty)        | Right-skewed | Heavy-tailed |
| Pareto (loc = 1, shape = 3)       | [1; +\infty)        | Right-skewed | Heavy-tailed |
| LogNormal (mlog = 0, sdlog = 1)   | (0; +\infty)        | Right-skewed | Heavy-tailed |
| LogNormal (mlog = 0, sdlog = 2)   | (0; +\infty)        | Right-skewed | Heavy-tailed |
| LogNormal (mlog = 0, sdlog = 3)   | (0; +\infty)        | Right-skewed | Heavy-tailed |
| Weibull (shape = 0.3)             | (0; +\infty)        | Right-skewed | Heavy-tailed |
| Weibull (shape = 0.5)             | (0; +\infty)        | Right-skewed | Heavy-tailed |
| Frechet (shape = 1)               | (0; +\infty)        | Right-skewed | Heavy-tailed |
| Frechet (shape = 3)               | (0; +\infty)        | Right-skewed | Heavy-tailed |

Figure 4. Relative statistical efficiency of quantile estimators for $n = 5$. 

In Table 2, approximately 2% of all $Q_{HD}$ results exceed 10 by their absolute values (while the true median value is zero). Meanwhile, the maximum absolute value of the $Q_{THD-SQRT}$ median estimations is approximately 1.7. In Table 3, the higher percentiles of $Q_{HD}$ estimations are also noticeably higher than the corresponding values of $Q_{THD-SQRT}$.

Thus, $Q_{THD-SQRT}$ is much more resistant to outliers than $Q_{HD}$.
4.2. Simulation 2

Let’s compare the statistical efficiency of $Q_{\text{HD}}$ and $Q_{\text{THD}}$. We evaluate the relative efficiency of these estimators against $Q_{\text{HF7}}$ which is a conventional baseline in such experiments. For the $p$th quantile, the classic relative efficiency can be calculated as the ratio of the estimator mean squared errors (MSE) (see Dekking et al. 2005):

$$\text{Efficiency}(p) = \frac{\text{MSE}(Q_{\text{HF7}}, p)}{\text{MSE}(Q_{\text{Target}}, p)} = \frac{\mathbb{E}[(Q_{\text{HF7}}(p) - \theta(p))^2]}{\mathbb{E}[(Q_{\text{Target}}(p) - \theta(p))^2]}.$$
where $\theta(p)$ is the true quantile value. We conduct this simulation according to the following scheme:

- We consider a bunch of different symmetric and asymmetric, light-tailed and heavy-tailed distributions listed in Table 4.
- We enumerate all the percentile values $p$ from 0.01 to 0.99.
- For each distribution, we generate 200 random samples of the given size. For each sample, we estimate the $p$th percentile using $Q_{HF7}$, $Q_{HD}$, and $Q_{THD}$—$\text{SQRT}$. For each estimator, we calculate the arithmetic average of $(Q(p) - \theta(p))^2$.
- MSE is not a robust metric, so we won’t get reproducible output in such an experiment. To achieve more stable results, we repeat the previous step 101 times and take the median across $E[(Q(p) - \theta(p))^2]$ values for each estimator. This median is our estimation of $\text{MSE}(Q, p)$.
- We evaluate the relative efficiency of $Q_{HD}$ and $Q_{THD}$—$\text{SQRT}$ against $Q_{HF7}$.

The results of this simulation for $n = \{5, 10, 20\}$ are presented in Figures 4–6. As we can see, $Q_{THD}$—$\text{SQRT}$ is not so efficient as $Q_{HD}$ in the case of light-tailed distributions. However, in the case of heavy-tailed distributions, $Q_{THD}$—$\text{SQRT}$ has better efficiency than $Q_{HD}$ because the estimations of $Q_{HD}$ are corrupted by outliers.

5. Conclusion

There is no perfect quantile estimator that fits all kinds of problems. The choice of a specific estimator has to be made based on the knowledge of the domain area and the properties of the target distributions. $Q_{HD}$ is a good alternative to $Q_{HF7}$ in the light-tailed distributions because it has higher statistical efficiency. However, if extreme outliers may appear, estimations of $Q_{HD}$ could be heavily corrupted. $Q_{THD}$ could be used as a reasonable tradeoff between $Q_{HF7}$ and $Q_{HD}$. In most cases, $Q_{THD}$ has better efficiency than $Q_{HF7}$ and it’s also more resistant to outliers than $Q_{HD}$. By customizing the width $D$ of the highest density interval, we could set the desired breakdown point according to the research goals. Also, $Q_{THD}$ has better computational efficiency than $Q_{HD}$ which makes it a faster option in practical applications.

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Disclosure statement

The author reports there are no competing interests to declare.

Reference implementation

Here is an R implementation of the suggested estimator:

```r
getBetaHdi <- function(a, b, width) {
  eps <- 1e-9
  if (a < 1 + eps & b < 1 + eps) # Degenerate case
    return(c(NA, NA))
  if (a < 1 + eps & b > 1) # Left border case
    return(c(0, width))
  if (a > 1 & b < 1 + eps) # Right border case
    return(c(1 - width, 1))
  if (width > 1 - eps)
```
return(c(0, 1))

# Middle case
mode <- (a - 1)/(a + b - 2)
pdf <- function(x) dbeta(x, a, b)
l <- uniroot{
    f = function(x) pdf(x) - pdf(x + width),
    lower = max(0, mode - width),
    upper = min(mode, 1 - width),
    tol = 1e-9
}$$root
r <- l + width
return(c(l, r))
}

thdquantile <- function(x, probs, width = 1/sqrt(length(x)))
    sapply(probs, function(p) {
        n <- length(x)
        if (n == 0) return(NA)
        if (n == 1) return(x)
        x <- sort(x)
        a <- (n + 1) * p
        b <- (n + 1) * (1 - p)
        hdi <- getBetaHdi(a, b, width)
        hdiCdf <- pbeta(hdi, a, b)
        cdf <- function(xs) {
            xs[xs <= hdi[1]] <- hdi[1]
            xs[xs >= hdi[2]] <- hdi[2]
            (pbeta(xs, a, b) - hdiCdf[1])/(hdiCdf[2] - hdiCdf[1])
        }
        iL <- floor(hdi[1] * n)
        iR <- ceiling(hdi[2] * n)
        cdfs <- cdf(iL:iR/n)
        W <- tail(cdfs, -1) - head(cdfs, -1)
        sum(x[(iL + 1):iR] * W)
    })

An implementation of the original Harrell-Davis quantile estimator could be found in the Hmisc package (see Harrell 2021).

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