Entanglement has been recently studied very extensively as it impacts both foundational as well as applied aspects of quantum theory (for a review see [1]). Along with studies aimed at understanding the nature of entanglement, its presence in condensed matter systems, especially spin chains, has been used as a good platform to study various aspects of many-body systems including quantum phase transitions [2-7]. One such widely studied system is the Ising model which exhibits a quantum phase transition when a magnetic field is applied in a direction transverse to the interaction [8]. A variant of the transverse-field Ising model, when the field is applied impulsively kicked via a Dirac delta comb, has been discussed in the literature [9-11]. Even in the presence of time dependence the transverse Ising model remains integrable via the Jordan-Wigner transformation [12] as the resulting fermions are free. These models can lead to very highly entangled states with multipartite entanglement.

Multipartite entanglement, whose understanding is still incomplete, has been investigated in many recent works [13, 14]. The generalized cluster states, or graph states, with highly persistent distributed entanglement, has been proposed as a model of quantum computation, distinct from the circuit model, the so-called one-way quantum computer [15-17]. Cluster states have also been experimentally obtained, for example see [18] and have been applied for constructing quantum error correcting codes, which for example see [19] and have been applied for constructing quantum error correcting codes, which for example see [20].

We present a solvable model of iterating cluster state protocols that lead to entanglement production, between contiguous blocks, of 1 ebit per iteration. This continues till the blocks are maximally entangled at which stage an unravelling begins at the same rate till the blocks are unentangled. The model is a variant of the transverse field Ising model and can be implemented with CNOT and single qubit gates. The inter qubit entanglement as measured by the concurrence is shown to be zero for periodic chain realizations while for open boundaries there are very specific instances at which these can develop. Thus we introduce a class of simply produced states with very large multipartite entanglement content of potential use in measurement based quantum computing.

\[ \mathcal{H}(t) = \sum_{j=1}^{L-1} \sigma_j^x \sigma_{j+1}^x + \sum_{k=-\infty}^{\infty} \delta (k - \frac{t}{\tau}) \sum_{j=1}^{L} \sigma_j^z \]  
\[ U = \exp \left( -\frac{i \pi}{4} \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x \right) \exp \left( -\frac{i \pi}{4} \sum_{j=1}^{L} \sigma_j^z \right) \]  

The operator \( \exp (i \frac{\pi}{4} \sigma_j^x \sigma_{j+1}^x) \) is the Cartan form of a CNOT gate [23, 24]. Thus we can represent and implement quite simply the evolution operator as a quantum circuit comprising of CNOT gates and one-qubit gates. At time \( t = n\tau \), where \( \tau = \frac{\pi}{2} \) and \( n \) is total number of kicks, the time evolved state is \( U^n |0\cdots0\rangle \), where \( |0\rangle \) is the eigenstate of local \( \sigma_z \) operator. The non-triviality of the time evolution stems from the non-commutativity of the operators involving different Pauli matrices in Eq. [4]. If the \( \sigma_z \) single qubit operators were not present, the resulting state for \( n = 1 \) was first studied as a “cluster state” [21].

We divide the spin chain into two blocks \( A \) with \( M \) spins and \( B \) with \( N = L - M \) spins. For simplicity, we consider the case of \( M = L/2 \) first. The blocking of the lattice, along with relabeling of the sites is as shown in Fig. [4] for the open boundary conditions. Let us introduce the following notation...
for spin operators of different partitions, for \( j \leq M = L/2 \),
\[
\hat{A}_j \equiv \hat{\sigma}_{M+1-j}, \quad \text{and} \quad \hat{B}_j \equiv \hat{\sigma}_{M+j}.
\]
(3)

Thus \( \hat{A}_1 \) and \( \hat{B}_1 \) represent the Pauli matrices for spins at locations \( M \) and \( M + 1 \) respectively.

The entanglement between the two blocks is calculated from the entropy of block \( A \) (by tracing out the block \( B \) spins).

In Fig. 1 the evolution of this entanglement in \( |\psi_n\rangle \) is shown for open and periodic boundary conditions with \( L = 20 \) spins and equal blocks with \( M = 10 \). In a very remarkable way the entanglement increases with every iteration exactly by 1 ebit (entanglement between a maximally-entangled pair of qubits) in the open chain case and by 2 ebits for the periodic chain. Thus after \( L/4 \) \((L/2)\) iterations, blocks \( A \) and \( B \) are maximally entangled in the periodic (open) chains, with an entanglement of \( L/2 \) ebits. In the periodic case, the translational symmetry ensures that this is the case for \( M \) blocks + 2 ebits (entanglement between a maximally-entangled pair of qubits) in the open chain case and by 2 ebits for the periodic chain. Thus after \( L/4 \) \((L/2)\) iterations, blocks \( A \) and \( B \) are maximally entangled in the periodic (open) chains, with an entanglement of \( L/2 \) ebits. In the periodic case, the translational symmetry ensures that this is the case for \( M \) blocks + 2 ebits (entanglement between a maximally-entangled pair of qubits) in the open chain case and by 2 ebits for the periodic chain. Thus after \( L/4 \) \((L/2)\) iterations, blocks \( A \) and \( B \) are maximally entangled in the periodic (open) chains, with an entanglement of \( L/2 \) ebits. In the periodic case, the translational symmetry ensures that this is the case for \( M \) blocks + 2 ebits (entanglement between a maximally-entangled pair of qubits) in the open chain case and by 2 ebits for the periodic chain.

The unitary operator in Eq. (2) can be written using this notation and labeling as:
\[
U = X_{AB}X_{AA}X_{BB}Z_AZ_B \equiv U_AU_BV_1,
\]
(4)
where \( X_{AB} = e^{-i\frac{\pi}{4}A_1B_1^*}, \quad X_{AA} = e^{-i\frac{\pi}{4}}\sum_{j=0}^{L/2-1} A_j^*A_{j+1} \) and \( Z_A = e^{-i\frac{\pi}{4}}\sum_{j=1}^{L/2} A_j \), with similar definitions for \( X_{BB} \) and \( Z_B \). The operator \( U_A \equiv X_{AA}Z_A \) is a multi-spin unitary, acting on spins of partition \( A \), while the interaction bond between the two partitions is represented by
\[
V_1 \equiv U_A^\dagger U_B^\dagger X_{AB}U_AU_B = e^{-i\frac{\pi}{4}A_1^*B_1^*}.
\]
(5)

In this format the unitary operator’s nonlocal parts, as far as the partition \( A : B \), are concerned are laid threadbare. Now, the powers of the unitary operator can be written as
\[
U^n = U_A^nU_B^nV_nV_{n-1}V_{n-2} \cdots V_1
\]
(6)
where \( V_n = U_A^\dagger U_B^\dagger V_{n-1}U_AU_B = U_A^nU_B^nX_{AB}U_A^nU_B^n \).

This can serve as a protocol for generating multipartite nonlocal entanglement using only one nonlocal two-qubit gate

\[ X_{AB} \] acting always on the same two qubits. Given that \( |\psi_0\rangle = \otimes^L|0\rangle \), it helps to find the states during the first few steps of the iteration. The quantum circuit implementing this unitary operation in Eq. 2 for \( n \) steps on the initial unentangled state \( |\psi_0\rangle \) is shown in Fig. 2 along with the resultant multipartite entangled state with \( n \) nonlocal (inter-partition) Bell pairs. It is helpful to also define the state \( |\psi_n\rangle = \prod_{i=1}^n V_i |\psi_0\rangle \), which is well defined as the \( V_i \) are commuting operators as exhibited below.

We will calculate the von Neumann entropy \( S_M(n) \) of the block with \( M \) spins as function of the time steps \( n \). The state after the first step is straightforward to evaluate, only \( V_1 \) involves interaction between the partitions. The state after one time step is, \( |\psi_1\rangle = U|\psi_0\rangle = U_AU_B|\psi_1\rangle \), has exactly one ebit of entanglement distributed over all spins. Defining a Bell state between a pair of spins (located at \( l \) and \( m \)) as, \( |\Phi_{lm} \rangle \equiv (|00\rangle - i|11\rangle)/\sqrt{2} \), we have,
\[
|\psi_1\rangle = |00\rangle_{A_2...A_M}|\Phi_{A_1B_1}|00\rangle_{B_2...B_M}.
\]
(7)
In the state \( |\psi_1\rangle \), the spins \( A_1 \) and \( B_1 \) at the interface between the blocks are in a Bell state, which has one ebit of entanglement. The state \( |\psi_1\rangle \) has additional local unitary transformations \( U_A \) and \( U_B \) that act on individual blocks, which do not change the entanglement between the blocks. However, the two-party entanglement (calculated by the concurrence measure of entanglement) between spins \( A_1 \) and \( B_1 \) is zero. This implies that the entanglement in the state \( |\psi_1\rangle \) is of multipartite nature. Thus, we can use this protocol as a reverse-engineering tool to transform a local Bell pair entanglement into a nonlocal multipartite entanglement. Further, a local Bell pair entanglement can also be transferred to another local Bell pair, by changing the blocking scheme after generating the state \( |\psi_1\rangle \), so that a different pair of spins is at the interface of blocks \( A' \) and \( B' \), through \( |\psi'_2\rangle = U_A^\dagger U_B^\dagger |\psi_1\rangle \).

Turning to the calculation of \( |\psi_2\rangle \), the state at the second step, the operator \( V_2 \) is required. The special value \( \tau = \pi/4 \) renders a remarkably simple and transparent form for this operator (As shown in the Supplementary Material [26]):
\[
V_2 = \exp \left(-i\frac{\pi}{4}A_0^*B_0^*A_0^*B_0^*\right).
\]
(8)
In the state \( |\psi_2\rangle = U_A^2U_B^2V_2|\psi_0\rangle \equiv U_A^2U_B^2|\psi_2\rangle \), while \( V_1 \) flips the spins labelled \( A_1 \) and \( B_1 \), \( V_2 \) retains these while flip-
the entanglement is \( V_L \) and the entanglement entropy for this state is

\[
S_{\text{ent}} = \sum_i S_{\text{ent}}^i = N \log_2 \lambda.
\]

which may be proved for example by induction. Thus with every iteration the operator \( V \) acquires an additional two-spins in the interaction. This implies that the state \( |\psi_n\rangle \equiv U^n_A U^n_B |\tilde{\psi}_n\rangle \) is given by,

\[
|\psi_n\rangle = |0..0\rangle_{A_1..A_M} |\Phi_{A_1, B_1}..\Phi_{A_M, B_M}\rangle |0..0\rangle_{B_{n+1}..B_M},
\]

and the entanglement entropy for this state is \( n \) ebits. Thus in the open chain case when \( n = M = L/2 \), the maximum entanglement of \( L/2 \) ebits is achieved. In the closed chain, this time is reduced to \( L/4 \).

Beyond the time when the maximum entropy is reached, the expression for \( V_n \) is to be modified: In the open chain case with \( M = L/2 \), the operator \( V_{M+1} \) is given by (Details in the Supplementary Material [26]),

\[
V_{M+1} = \exp \left( -i \frac{\pi}{4} A_M^y B_M^x \prod_{j=1}^{M-1} A_j^x B_j^z \right),
\]

and subsequently till \( n = L \), the operators with the highest index get decimated such that at \( V_{M+k} (1 \leq k \leq M) \) the operator string involved is

\[
A_{M-k+1}^y B_{M-k+1}^x A_{M-k+2}^x B_{M-k+2}^y .. A_1^x B_1^z.
\]

It follows that \( V_L = \exp(-i\pi A_1^y B_1^x) \) and \( V_{L+1} = V_1 \). Thus the interaction picture operators \( \tilde{V}_n \) are periodic, and it is also easy to check that they all commute with each other.

These can be used along with the observation that \( \exp(-i\pi A_1^y B_1^x) \) arrives at the state after \( M + 1 \) kicks as \( |\psi_{M+1}\rangle = U_A^{M+1} U_B^{M+1} \tilde{\psi}_{M+1} \),

\[
|\tilde{\psi}_{M+1}\rangle = |0\rangle_{A_M} |\Phi_{A_1, B_1}..\Phi_{A_{M-1}, B_{M-1}}\rangle |0\rangle_{B_M}
\]

which is block-local unitarily equivalent to \( |\psi_{M-1}\rangle \) and thus the entanglement is \( M - 1 \) ebits which is consistent with the open boundary case in Fig. [3]. Thus further time evolution unravels the entanglement at the rate of 1 ebit per iteration, and in the periodic boundary case at the rate of 2 ebits per iteration till there is no entanglement at all. The block entanglement is a periodic function of \( n \), and can be summarized within a period \( L \) \( \text{(L/2)} \) for open (closed) chain, with \( M = L/2 \), as

\[
S_{M, \text{Open}}(n) = n + (M - n)\Theta(n - M)\Theta(2M - n), \quad S_{M, \text{Closed}}(n) = 2n + (M - 2n)\Theta(n - \frac{M}{2})\Theta(M - n),
\]

where \( \Theta(x) \) is the Heaviside step function. The entanglement reaches its maximum possible value \( L/2 \) at \( n = L/2 \) in the open chain case, while in the closed this happens at \( n = L/4 \) as the Bell pairs are formed from both the interfaces of the blocks, increasing the entanglement by 2 ebits every step. Fig. [3] shows the von Neumann entropy as a function of \( n \) in both cases for \( L = 20 \); the numerical calculation agrees exactly with the above calculation.

Now, let us consider the case of unequal block sizes \( M \) (for block \( A \) and \( N = L - M \), let \( M < L/2 \). The entanglement increases 1 ebit (2 ebits) for open (closed) chain till \( n = M \) (\( M/2 \)) as in the case of equal size blocks, and it remains constant till \( n = N \) (\( N/2 \)), and then decreases by 1 ebit (2 ebits) per step for further kicks. The reason for non-decreasing entanglement at step \( n = M + 1 \) is due the structure of the operator \( V_{M+1} \), given by,

\[
V_{M+1} = \exp \left( -i \frac{\pi}{4} A_M^y B_M^x \prod_{j=1}^{M-1} A_j^x B_j^z \right),
\]

which is different from the operator shown in Eq. [12] (operator when the block sizes are equal). This operator will act on the state \( |\tilde{\psi}_{M+1}\rangle \), with all sites from block \( A \) locked in Bell pairs with \( A_M \) number of sites from block \( B \), to give the state at the next step

\[
|\tilde{\psi}_{M+1}\rangle = U_A^{M+1} U_B^{M+1} |\psi_{M+1}\rangle
\]

where we have used another Bell state \( |\psi\rangle = (|01\rangle - i|10\rangle)/\sqrt{2} \) (See detailed derivation in Supplementary Material [26]). There are \( M-1 \) Bell pairs between the two blocks, and the spin \( A_M \) is entangled with the Bell pair state of the spins \( B_M \) and \( B_{M+1} \). It is straightforward to see that the reduced density matrix \( \rho_A \) is proportional to the identity matrix, thus the von Neumann entropy is \( M \) at this time step. Similarly, the entropy will stay at this value till \( n = N = L - M \),

\[
S_{M, \text{Closed}}(n) = 2n + (M - 2n)\Theta(n - \frac{M}{2})\Theta(M - n),
\]
and after that it decreases by 1 ebit (2 ebits) per step for open (closed) chain, showing a clipped sawtooth structure as seen in Fig.3 for the case of $M = 8, L = 20$.

The remarkable algebraic properties of the model considered here lead to the high entanglement content and the creation of Bell pairs up to block-local operations. However this must not be construed as the creation of maximum entanglement between qubits, indeed it reflects very well the general observation that high multiparticle entanglement, such as between the blocks $A$ and $B$ coexist with low entanglement amongst individual qubits. Let us examine how the two-qubit entanglement, viz. concurrence, varies with $n$ by focusing on a pair of qubits $A_i$ and $B_i$, one each from the two blocks. After $n > i$ kicks, but before the unraveling reaches these qubits, the state $|\psi_n\rangle$ has a Bell pair between the two marked qubits, but the state $|\psi_n\rangle$ has also the nontrivial action of block-local unitaries $U_A$ and $U_B$. The reduced density matrix for the marked pair can be written in the form of a particular Krauss representation,

$$\rho_{A_i,n} = \sum_k p_k Q^{A_i}_k(n)Q^{B_i}_k(n)\rho_{\text{Bell}}Q^{A_i\dagger}_k(n)Q^{B_i\dagger}_k(n),$$

where $Q^{A_i}_k(n)$ are local operators for the qubit $A_i$ and $p_k$ are probabilities. These local operators generate the action of a quantum channel which is in general decohering and entanglement breaking. We have checked numerically that all pair concurrences are zero for all time steps for both open and periodic boundary conditions for sizes up to $L = 20$. The only exception is the central pair of qubits $\frac{L}{2}$ and $\frac{L}{2} + 1$ in the open chain; this pair has a nonzero concurrence (of unity) at exactly $n = \frac{L}{2}$. To illustrate the nonzero concurrence for the central pair of qubits, we calculate the concurrence of the central pair (23) for the simple case of $L = 4$ explicitly. Using Eq. (7) and applying the local unitaries, the state after the first kick is,

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (-|0000\rangle + |1010\rangle + |1010\rangle +|1111\rangle + i |0011\rangle + |0110\rangle - |0011\rangle + |1100\rangle).$$

The reduced density matrix $\rho_{23}$ is proportional to the identity, which is naturally a separable state. The operators $Q_\lambda$ with $p_k = 1/4$ can be chosen as $Q^{A}_1 = e^{-i\pi/4}\sigma_x^{A}, Q^{B}_1 = e^{-i\pi/4}\sigma_x^{B}, Q^{A}_2 = \sigma_x^{A}, Q^{B}_2 = \sigma_y^{B}, Q^{A}_3 = Q^{B}_1, Q^{B}_3 = \sigma_y^{B} Q^{A}_1, Q^{A}_4 = \sigma_y^{B} Q^{B}_1$. Apart from a local rotation of the Bell state implemented by $Q^{A}_1 Q^{B}_1$, the channel is therefore a two-qubit Pauli channel. Similarly, the state after the second kick can be written as $|\psi_2\rangle = \frac{1}{2} (|0000\rangle - i |0110\rangle - i |1001\rangle - |1111\rangle)$, and therefore $\rho_{23} = \frac{1}{4} (|00\rangle \langle 00| - i |11\rangle \langle 11| + |11\rangle \langle 11| + i |00\rangle \langle 00|)$. For this reduced density matrix, the calculation of concurrence (25) gives the maximum value of 1, as quite simply the state corresponds to the pure maximally entangled state $(|00\rangle - i |11\rangle) / \sqrt{2}$. Similarly, the concurrence is zero when $n$ is even multiple of 2, and revives for odd multiples of 2. For a periodic chain, the concurrence remains zero for any number of kicks.

In conclusion, we have investigated a solvable model that may be interpreted as iteration of graph states and shows an entanglement growth of 1 ebit (2 ebits) per iteration for the open (closed) chain. The entanglement generated in the unitary evolution is multiparticle in nature. On reaching the maximum possible block entanglement, for a contiguous block of $L/2$, it unravels and reaches zero after $L$ iterations. These states can be used in protocols of quantum computation, considering their proximity to well investigated graph states. The state after each iteration reveals a very interesting algebraic structure through the commuting interaction picture operators $V_i$, that may be hidden in approaches using the Jordan-Wigner fermions. This study reveals a further interesting consequence of integrability of the transverse field Ising model, but we emphasize that the features presented here are unique to the value of the parameter $\tau = \pi/4$, and hence the impulsive field plays an important role, and the features are easily revealed while remaining with spin operators rather than fermions.

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Supplementary Materials: Protocol using kicked Ising dynamics for generating states with maximal multipartite entanglement

I. CALCULATION OF $V_n$

From Eq. (4) in the main text, we see $V_1 = U_B^1 U_A^1 X_{AB} U_A U_B$. We have,

$$U_A^1 X_{AB} U_A = Z_A X_{AA} \left( 1 - i A^1_x B^1_x \right) \frac{1}{\sqrt{2}} X_{AA} Z_A.$$

(S1)

As $X_{AA}$ commutes with the central quantity, we have

$$U_A^1 X_{AB} U_A = \frac{1}{\sqrt{2}} (1 + i A^1_y B^1_y),$$

(S2)

where the identity $e^{i\pi/4} \sigma^x e^{-i\pi/4} = -\sigma^y$ is used. Note the importance of the $\pi/4$ factor. Similarly the action of $U_B$ changes $B^1_x$ in $X_{AB}$ to $-B^1_y$, we get

$$V_1 = \left( 1 - \frac{i}{\sqrt{2}} A^1_y B^1_y \right) = \exp \left( -\frac{i\pi}{4} A^1_y B^1_y \right).$$

(S3)

The second interaction operator $V_2 = U_B^1 U_A^1 V_1 U_A U_B$ can be written as

$$\frac{1}{\sqrt{2}} \left( 1 - i \tilde{A}_1^y \tilde{B}_1^y \right),$$

(S4)

where $A^1_y = Z_A X_{AA} A^1_y X_{AA} Z_A$. Now $X_{AA} A^1_y X_{AA} = e^{i\pi/4} A^1_x A^1_x e^{-i\pi/4} A^1_x A^1_x = -A^1_x A^1_x$. which follows from the identity

$$e^{i\pi/4} \sigma^y \sigma^x e^{-i\pi/4} \sigma^y \sigma^x = -\sigma^y \sigma^x.$$

(S5)

Therefore finally $A^1_y = A^2_x A^1_x$ as

$$-e^{i\pi/4} (\sigma^y + \sigma^x) e^{-i\pi/4} (\sigma^y + \sigma^x) = \sigma^x \sigma^y.$$  

(S6)

Similarly $B^1_y$ transforms to $B^2_y B^1_y$ and

$$V_2 = \frac{1}{\sqrt{2}} (1 - i A^2_y A^1_x B^2_y B^1_y) = \exp \left( -\frac{i\pi}{4} A^2_y B^2_y A^1_x B^1_y \right).$$

(S7)

For $2 \leq n \leq L/2 - 1$ let

$$V_n = \frac{1}{\sqrt{2}} (1 - i A^y_n A^z_{n-1} \cdots A^x_1 B^y_n B^z_{n-1} \cdots B^1_x).$$

(S8)

From this we show that $U_A^1 V_n U_A$ has the form of $V_{n+1}$, and hence prove the statement by induction. Observe that

$$U_A^1 V_n U_A = (U_A^1 A^y_n A^z_{n-1} \cdots A^x_1 U_A) \cdots (U_A^1 A^y_2 U_A) (U_A^1 A^y_1 U_A) (U_A^1 A^z_{n-1} U_A) \cdots (A^y_2 A^z_{n-2} A^y_{n-1} \cdots A^y_1).$$

(S9)

The operators in the “interior” are mapped to a string of three operators, while the “edges” contribute two. Using properties of Pauli matrices, this simplifies to $(-1)^{n-2} A^y_{n+1} A^z_n A^z_{n-1} \cdots A^z_1$. A similar relation holds for the $B$ operators, and hence finally

$$V_{n+1} = \frac{1}{\sqrt{2}} (1 - i A^y_{n+1} A^x_n \cdots A^y_1 B^y_{n+1} B^z_n \cdots B^1_x),$$

as required.

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The operator $V_{\frac{L}{2}+1}$ for open chain case can also be calculated as above. Observe however that there are now two “edges” the one with $A_z^y$ and one with $A_z^{y'}$. One encounters $U_A^y A_{L/2}^y A_{L/2-1}^z \cdots A_1^y U_A$ which simplifies to $(-1)^{L/2-2} A_{L/2}^z A_{L/2-1}^z \cdots A_1^z$, with a similar expression for the $B$ string. Thus at this stage the two decimations turn $x$ and $y$, and $V_{L/2+1}$ is as given in Eq. (12). Further iteration requires $U_A^y A_{L/2}^y U_A = -A_y^y$, which along with $U_A^z A_{L/2-1}^z U_A = A_z^z A_{L/2-1}^z$ results in the decimation of operators at position $L/2$ with the consequence that

$$V_{L/2+2} = \exp \left( -\frac{i\pi}{4} A_{L/2-1}^z B_{L/2-1}^z \prod_{j=1}^{L/2-2} A_j^z B_j^z \right).$$

(S10)

Further decimations lead to $V_L = e^{-\frac{i\pi}{4} A_1^y B_1^y}$ and finally $V_{L+1} = V_1 = e^{-\frac{i\pi}{4} A_1^y B_1^y}$.

II. UNEQUAL SIZED BLOCKS

Let us generalise the formalism in the manuscript by considering unequal size blocks. We divide the chain into two blocks $A$ and $B$ with $M$ and $N = L - M$ number of spins, respectively, where $M < L/2$. Following the discussion in the preceding section for equal sized blocks, the spins in block $A$ ($1, 2, \ldots, M$) are relabelled as $M, M-1, \ldots, 1$ and spins in block $B$ ($M+1, M+2, \ldots, M+N$) are labelled as $1, 2, \ldots, N$.

$$\bar{A}_j = \tilde{\sigma}_{M+1-j} \text{ and } \bar{B}_j = \tilde{\sigma}_{M+j}. \quad (S11)$$

where the counting of spins is such that $j = 1, 2, \ldots, M$ in $A$ side and $j = 1, 2, \ldots, N$ in $B$ side ($N > M$). The equal block size can be retraced by $N = M = L/2$. The time evolution of the initial state $|\psi_0\rangle = \otimes |0\rangle$ can be calculated by applying powers of the unitary operator given by Eq. (6) in the main text. Till $n = M$, the time evolved state can be easily calculated by Eq. (12) of the main text and the block entanglement is given by $S_{\text{open}}^M(n) = n$ for open chain case.

Afterwards for $n = M+1$, the state $|\tilde{\psi}_{M+1}\rangle$ is calculated by applying operator $V_{M+1} = \exp(-\frac{i\pi}{4} A_M^z B_M^y B_{M+1}^z) \prod_{j=1}^{M-1} A_j^z B_j^z)$ on $|\psi_M\rangle$ as

$$\tilde{\psi}_{M+1} = \frac{1}{\sqrt{2^{M+1}}} \sum_{\{a_i\}} M \eta(a_i) |a_M \cdots a_1\rangle_A |a_1 \cdots a_M 00\rangle_B \otimes |0\cdots 0\rangle_{B_M+2 \ldots B_N}$$

$$+ (1 - 2a_M) |\bar{a}_M a_{M-1} \cdots a_1\rangle_A |a_1 \cdots a_M 10\rangle_B \otimes |0\cdots 0\rangle_{B_M+2 \ldots B_N},$$

where the operations $A_M^z |a_M\rangle = |\bar{a}_M\rangle$ and $B_M^y |0\rangle = |i1\rangle$ are performed. Since we are calculating states after $(M+1)^{th}$ and $(M+2)^{th}$ kicks, therefore, the last two qbits in the block $B$ $M+1$ and $M+2$ are retained and rest of the qbits are put together as $|0\cdots 0\rangle_{B_M+2 \ldots B_N}$. Expanding the $M^{th}$ bit in the summation of above equation leads to

$$|\tilde{\psi}_{M+1}\rangle = \frac{1}{\sqrt{2^{M+1}}} \sum_{\{a_i\}} M \eta(a_i) (|0a_{M-1} \cdots a_1\rangle_A |a_1 \cdots 00\rangle_B + i|a_{M-1} \cdots a_1\rangle_A |a_1 \cdots 10\rangle_B$$

$$+ |1a_{M-1} \cdots a_1\rangle_A |a_1 \cdots 01\rangle_B - i|0a_{M-1} \cdots a_1\rangle_A |a_1 \cdots 110\rangle_B \otimes |0\cdots 0\rangle_{B_M+2 \ldots B_N},$$

which can be rearranged in a nicer form as

$$|\tilde{\psi}_{M+1}\rangle = |\Phi_{A_1, B_1 \cdots \Phi_{A_{M-1}, B_{M-1}}\rangle (0\cdots 0\rangle_{B_M+2 \ldots B_N} (|0\rangle_{A_M} |\Phi_{B_M, B_{M+1}}\rangle + |1\rangle_{A_M} |\Phi'_{B_M, B_{M+1}}\rangle) / \sqrt{2},$$

(S12)

where $|\Phi_{l, m}\rangle = (|01\rangle - i|10\rangle)/\sqrt{2}$ and $|\Phi'_{l, m}\rangle = (|01\rangle + i|10\rangle)/\sqrt{2}$ are Bell states between a pair of spins located at $l$ and $m$.

The reduced density matrix $\rho_A$ (or $\rho_B$) has $2^M$ equal eigenvalues $1/2^M$ and the block entanglement is $S_{\text{open}}^M(M+1) = M$. This shows that $(M+1)^{th}$ kick does not generate any new ebit between blocks $A$ and $B$ and the entanglement is already exhausted to a saturation value $M$. In order to confirm the saturation of entanglement let us calculate one more iteration for $(M+2)^{th}$ kick. The $(M+2)^{th}$ power of unitary operator requires $V_{M+2} = U_B^y U_A^z V_M + U_A^y U_B^z$. The calculation of $V_{M+2}$ requires the same method as discussed in Section I and more specifically needs following operator relations $U_A^y A_M^y U_A = -A_M^y$, $U_A^z A_{M-1}^y U_A = A_M^y A_{M-1}^y$ and $U_B^y (B_M^y B_{M+1}^z \cdots B_{M+2}^z) U_B = B_M^y B_{M+2}^y B_{M+1}^y B_{M+2}^z \cdots B_{M+2}^z$. Using the properties of Pauli matrices, we can write the expression for $V_{M+2}$ as

$$V_{M+2} = \exp(-\frac{i\pi}{4} A_M^y A_{M+1}^y B_{M+2}^y B_{M+1}^z B_M^z \cdots B_{M+2}^z) \prod_{j=1}^{M-2} A_j^z B_j^z).$$

(S13)
Using the relation $|\tilde{\psi}_{M+2}\rangle = V_{M+2}V_{M+1}|\tilde{\psi}_M\rangle$ we calculate the state after $(M + 2)^{th}$ kick.

\[
|\tilde{\psi}_{M+2}\rangle = \frac{1}{\sqrt{2^M}} \sum_{\{a_i\}} \eta(\{a_i\}) |a_M a_{M-1} \cdots a_1\rangle_A |a_1 \cdots a_M 00\rangle + |\bar{a}_M \bar{a}_{M-1} a_{M-2} \cdots a_1\rangle_A |a_1 \cdots a_M 11\rangle_B \\
+ |a_M \bar{a}_{M-1} a_{M-2} \cdots a_1\rangle_A |a_1 \cdots a_M -2 a_{M-1} 01\rangle_B + (1 - 2a_{M-1}) |\bar{a}_M a_{M-1} a_{M-2} \cdots a_1\rangle_A |a_1 \cdots a_M 10\rangle_B.
\]

By expanding the $m^{th}$ and $(m - 1)^{th}$ bits in the summation, we can express the state as

\[
|\tilde{\psi}_{M+2}\rangle = \frac{1}{\sqrt{2^M}} \sum_{\{a_i\}} \eta(\{a_i\}) |a_M a_{M-1} \cdots a_1\rangle_A |a_1 \cdots a_M -2 \rangle_B |\chi(a_{M-1}, a_M)\rangle,
\]

or in a more compact manner as

\[
|\tilde{\psi}_{M+2}\rangle = |\Phi_{A_1, B_1} \cdots \Phi_{A_{M-2}, B_{M-2}}\rangle |\chi(a_{M-1}, a_M)\rangle |0 \cdots 0\rangle_{B_{M+2} \cdots B_N},
\]

where state $|\chi(a_{M-1}, a_M)\rangle$ is a composite state of $(M - 1)^{th}$ and $M^{th}$ qubits of block $A$ as well as $(M - 1)^{th}$, $M^{th}$, $(M + 1)^{th}$ and $(M + 2)^{th}$ qubits of block $B$ and can be easily calculated by expanding Eq. (S14). The explicit form of these states are given as $|\chi(0, 0)\rangle = \frac{1}{2}|0000\rangle + |1001\rangle + |0110\rangle + |1111\rangle$, $|\chi(0, 1)\rangle = \frac{1}{2}|0001\rangle + |1000\rangle + |0111\rangle - |1110\rangle$, $|\chi(1, 0)\rangle = \frac{1}{2}|0010\rangle + |1011\rangle + |0100\rangle - |1101\rangle$ and $|\chi(1, 1)\rangle = \frac{1}{2}|0011\rangle - |1010\rangle + |0101\rangle + |1100\rangle$. Again, the reduced density matrix has $2^M$ equal eigenvalues $1/2^M$ and block entanglement $S^{\text{open}}_M(M + 2) = M$. For further kicks the entanglement does not change from the value at kick $n = M$ and remains constant till $n = N$. Afterwards the entropy starts decreasing in a unit step and reaches to zero after $L$ kicks.