SOME FUNCTIONAL PROPERTIES ON CARTAN-HADAMARD MANIFOLDS OF VERY NEGATIVE CURVATURE

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Abstract. In this paper we consider Cartan-Hadamard manifolds (i.e. simply connected of non-positive sectional curvature) whose negative Ricci curvature grows polynomially at infinity. We show that a number of functional properties, which typically hold when the curvature is bounded, remain true in this setting. These include the characterization of Sobolev spaces on manifolds, the so-called Calderón-Zygmund inequalities and the $L^p$-positivity preserving property, i.e. $u \in L^p \& (-\Delta + 1)u \geq 0 \Rightarrow u \geq 0$. The main tool is a new class of first and second order Hardy-type inequalities on Cartan-Hadamard manifolds with a polynomial upper bound on the curvature.

In the last part of the manuscript we prove the $L^p$-positivity preserving property, $p \in [1, +\infty)$, on manifolds with subquadratic negative part of the Ricci curvature. This generalizes an idea of B. Güneysu and gives a new proof of a well-known condition for the stochastic completeness due to P. Hsu.

1. Introduction

A major task for geometric analysts consists in determining under which assumptions, and to what extent, certain properties typical of the Euclidean space have their counterparts on a given complete, non-compact Riemannian manifold. The properties one is interested in include for instance certain functional inequalities, the behavior of solutions of PDEs, the characterization of some functional spaces, spectral properties, and so on. A common set of assumptions which ensure that the manifold $M$ at hand is in a sense “similar” to the the Euclidean space (locally, but uniformly) is a constant lower bound on the Ricci curvature, or $|\text{Ric}| \in L^\infty$ together with a positive lower bound on the injectivity radius. In this spirit, we consider the following problems.

A. On a Riemannian manifold $(M, g)$ one disposes of several, a priori different, definitions for the Sobolev space of order $k \in \mathbb{N}$ and integrability class $p \in [1, +\infty]$. For instance, one can define $W^{k,p}(M)$ as the space of $L^p$-functions whose covariant (distributional) derivatives are in $L^p$ up to the order $k$:

$$W^{k,p}(M) := \{ f \in L^p(M) : \nabla^j f \in L^p(M), \quad j = 0, \ldots, k \}.$$ (1.1)

This turns out to be a Banach space once endowed with the usual norm

$$\|u\|_{W^{k,p}} := \sum_{j=0}^{k} \|\nabla^j u\|_{L^p}.$$ (1.2)

Thanks to a generalized Meyers-Serrin-type theorem, [17], if $p \in [1, +\infty)$ this space can be characterized as the closure of $W^{k,p}(M) \cap C^\infty(M)$ with respect to $\|\cdot\|_{W^{k,p}}$, which is quite useful in applications. Alternatively, one can define the space $W^{k,p}_0(M)$ as the closure of compactly supported smooth functions $C^\infty_0(M)$ with respect to the Sobolev norm $\|\cdot\|_{W^{k,p}}$:

$$W^{k,p}_0 := \overline{C^\infty_0(M)}_{W^{k,p}}.$$ (1.3)

Finally, for even orders one can consider $H^{2m,p}(M)$ as the space of $L^p$ functions whose iterations of the (distributional) Laplace-Beltrami operator are in $L^p$ up to order $m$, i.e.,

$$H^{2m,p}(M) := \{ f \in L^p(M) : \Delta^j f \in L^p(M), \quad j = 0, \ldots, m \},$$ (1.4)

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endowed with the norm:

\[ \|u\|_{H^{2m,p}} := \sum_{j=0}^{m} \|\Delta^j u\|_{L^p}. \]

In the Euclidean setting, \( M = \mathbb{R}^n \), and on closed manifolds, the three spaces coincide. On an arbitrary Riemannian manifold one always has \( W^{1,p}(M) = W^{1}_0(M) \), whereas \( k = 2 \) is the first non-trivial order where in general one can only conclude that

\[ W^{2,p}_0(M) \subseteq W^{2,p}(M) \subseteq H^{2,p}(M). \]

Nonetheless, if \( |\text{Ric}| \leq L^\infty \) and the injectivity radius does not vanishes, it is actually possible to prove that \( W^{2,p}_0(M) = W^{2,p}(M) = H^{2,p}(M) \), see [21, 24]. See also [12] for a detailed introduction to the problem. Both proofs rely on a computation in a harmonic coordinate system which, together with a covering argument, allows to reduce the Riemannian problem to the Euclidean setting. It is worth noticing that the result is also true for higher order \( k \) if we require also that \( |\nabla^j \text{Ric}| \leq L^\infty \) for \( j = 0, \ldots, k-2 \). In the Hilbert case \( (p = 2) \), where a Bochner formula is available, a lower bound on Ricci curvature is actually enough, [2].

**B.** The second problem we consider is the existence of \( W^{2,p} \) regularity estimates for the solutions of the Poisson equation on a Riemannian manifold \((M, g)\); see [33] for a nice recent survey on the topic. More specifically, we are interested in a-priori \( L^p \)-Hessian estimates of the form

\[ \|\nabla^2 \varphi\|_{L^p} \leq C \|\Delta \varphi\|_{L^p} + \|\varphi\|_{L^p} \quad \forall \varphi \in C_0^\infty(M) \]

where \( C > 0 \) is a positive constant. Here \( p \in (1, +\infty) \) and \( \nabla^2 \varphi \) denotes the Hessian of \( \varphi \), i.e., the second order covariant derivative. Such inequalities, known in literature as \( L^p \)-Calderón-Zygmund \((CZ(p))\) inequalities, were first established in a work by A. Calderón and A. Zygmund, [7], in the Euclidean setting, where in fact one has the stronger

\[ \|\nabla^2 \varphi\|_{L^p} \leq C \|\Delta \varphi\|_{L^p} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n). \]

Note that the limit cases \( CZ(1) \) and \( CZ(+\infty) \) have been left out as they fail even in the Euclidean space, [36, 10]. It turn out that the validity of a \( CZ(p) \) inequality on a Riemannian manifold implies the equality of the three Sobolev spaces defined in (A), for details we refer to [22, Remark 2.1] or Remark 5.3 below. As a matter of fact, one can ensure the validity of (1.1) under the same assumptions of \( |\text{Ric}| \leq L^\infty \) and non-vanishing injectivity radius, [21, Theorem C]. Furthermore, if \( p = 2 \) a lower bound on Ricci curvature is enough, [21, Theorem B].

**C.** Finally, we consider a positivity property for the solutions of \(-\Delta u + u \geq 0\) on a complete Riemannian manifold. Note that in this paper \(-\Delta\) has non-negative spectrum.

**Definition 1.1.** A complete Riemannian manifold \((M, g)\) is said to be \(L^p\)-positivity preserving, \( p \in [1, +\infty) \), if the following implication holds true for every \( u \in L^p(M) \)

\[ (-\Delta + 1)u \geq 0 \text{ as a distribution } \Rightarrow u \geq 0. \]

Recall that \((-\Delta + 1)u \geq 0\) in the sense of distributions if the following inequality holds

\[ \int_M u(-\Delta + 1)\phi dV_g \geq 0 \quad \forall \phi \in C_0^\infty(M), \phi \geq 0. \]

This definition was introduced by B. Güneysu in [20]. When \( p = +\infty \), the \(L^\infty\)-positivity preserving property implies stochastic completeness while the \(L^2\) case, yields the essential self-adjointness of the Schrödinger operator \(-\Delta + V : C_0^\infty(M) \to L^2(M)\) for any non-negative \(L^2_{\text{loc}}\) potential \(V\). This latter implication and the fact that \(-\Delta + V\) is known to be essentially self-adjoint in \(L^2(M)\) for \(L^2_{\text{loc}}\) non-negative potentials [6, 22], lead M. Braverman, O. Milatovic and M. Shubin to propose the following conjecture, [6, Conjecture P].

**Conjecture (BMS-conjecture).** If \((M, g)\) is geodesically complete, then \(M\) is \(L^2\)-positivity preserving.
In the Euclidean case, the \( L^2 \)-positivity preserving was proved by T. Kato using the fact that 
\(-\Delta + 1 : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\) induces an isomorphism on the space of tempered distributions 
whose inverse is positivity preserving, see [30]. In the Riemannian setting, even though the
BMS conjecture is still open in its full generality, one can prove that if \( \text{Ric} \) is bounded form 
below, then \( M \) is \( L^p \)-positivity preserving on the whole scale \( p \in [1, +\infty] \), [20] Theorem
XIV.31. For a complete introduction to the topic we refer to the survey [19] as well as [20] Section
XIV.5. [6] Appendix B or [18].

Some of the aforementioned results can be slightly improved by allowing a small explosion 
on the non-negative part of \( \text{Ric} \). For instance, the equivalence of the Sobolev spaces 
\( W^{2,p}(M) = W^{2,p}_0(M), \ p \in [1, \infty) \), still holds if we allow \( |\text{Ric}| \leq br^2 \) with a small decay of 
the injectivity radius, while to prove that \( W^{2,2}(M) = W^{2,2}_0(M) \) and the \( L^p \)-positivity pre-
serving property \( \text{Ric} \geq -br^2 \) is enough. For reference on the first problem see [28] p.
95, Section XIV.C] for \( p \in [2, \infty) \) and Theorem 6.3 below for the whole range \( p \in [1, +\infty) \).
Nevertheless, the above results fail in general if we drop the curvature (and injectivity) assumptions and allow the bound on \( |\text{Ric}| \) to grow very fast at \( \infty \). Counterexamples with very unbounded curvature have been found in [42] [20] for 
(A) and in [21] [33] [39] [41] for (B) while, to the best of our knowledge, the BMS conjecture 
remains open in its full generality. Note that the above counterexamples are characterized 
by an oscillatory behavior of the Ricci curvature which diverges in the negative and positive 
part in [21] [33] [12] and in the positive part only in [31] [26] [37].

In this paper we show that several of the above properties still hold if one allows the 
curvature to become increasingly negative at infinity, possibly very fast, but in a controlled 
way. In particular, we consider a Cartan-Hadamard manifold \((M, g)\) (i.e. a simply-connected 
complete Riemannian manifold of non-positive sectional curvature) and assume that the 
Ricci curvature of \( M, \text{Ric} \), is controlled both from above and below polynomially at infinity.
Namely,
\[
-\frac{b}{r} \beta(x) \leq \text{Ric}(x) \leq -a \frac{r^\alpha(x)}{r},
\]
outside a compact set, where \( r(x) \) is the Riemannian distance of \( x \) from a fixed reference 
point \( o \in M \) and \( a \) and \( b \) are positive constants. Then, for suitable choices of the exponents 
\( 0 \leq \alpha \leq \beta \) we are able to prove the following results.

**Theorem I.** Let \((M, g)\) be a Cartan-Hadamard manifold satisfying (1.6) for some \( a, b > 0 \).

(a) If \( \alpha \geq 0 \) and \( \beta = 2\alpha + 2 \), then \( W^{2,p}_0(M) = W^{2,p}(M) \) for every \( p \in (1, +\infty) \).

(b) If \( \alpha = \beta \geq 0 \), then the \( L^2 \)-Calderón-Zygmund inequality (i.e., (1.4) with \( p = 2 \))
holds on \( M \).

(c) If \( \alpha \geq 0 \) and \( \beta = \alpha + 2 \), then \( M \) is \( L^p \)-positivity preserving for all \( p \in [2, +\infty) \).
In particular, the BMS conjecture is satisfied for this class of manifolds.

**Remark 1.2.** Note that in (a) and (c) we only require the radial Ric curvature to satisfy
(1.6) while in (b) we need (1.6) to hold in the sense of quadratic forms. Naturally, if
one substitutes \( \text{Ric} \) in (1.6) with the sectional curvature, the results still hold and are
actually somewhat easier to prove. This is due to the fact that in the proof we use a
Laplace comparison theorem for \( \text{Ric} \) bounded from above which holds on Cartan-Hadamard
manifolds; see Subsection 2.2 and Remark 2.2.

**Remark 1.3.** An additional property that extends from the Euclidean setting to the case of
Riemannian manifolds with \( \text{Ric} \) curvature bounded form below and non-vanishing injec-
tivity radius is the validity of an \( L^p \)-Sobolev inequality of the form
\[
\|\varphi\|_{L^p(M)} \leq C(\|\nabla \varphi\|_{L^p(M)} + \|\varphi\|_{L^p(M)}), \quad \forall \varphi \in C_0^\infty(M),
\]
where \( 1 \leq p \leq n, \ q = np/(n-p), \) and \( C > 0 \). For reference see [24] Theorem 3.2. Also
in this case there are known counterexamples if one drops the curvature assumptions [24]
Proposition 3.4. On Cartan-Hadamard manifolds, however, the Sobolev inequality (1.7)
is satisfied without further curvature assumptions as a consequence of the isoperimetric inequality, \([25]\).

Unlike the case of manifolds with lower bounded Ricci curvature, it is impossible to obtain Theorem 1(c) for \(p = \infty\). Indeed, as observed by B. G"uneysu in [18], the \(L^\infty\)-positivity preserving property implies the stochastic completeness of the manifold at hand. See also Remark 5.4 below. It turns out that Cartan-Hadamard manifolds satisfying \(\text{Ric} \leq -ar^\alpha\) for \(\alpha > 2\) are not stochastically complete; see Theorem 5.5 below. Conversely, one can prove that

**Theorem II.** Let \((M, g)\) be a complete Riemannian manifold satisfying

\[-\lambda^2(r(x)) \leq \text{Ric}(x) \quad \forall x \in M \setminus B_R,\]

with \(\lambda\) given by

\[\lambda(t) = at^k \prod_{j=0}^k \log^{[j]}(t)\]

where \(a > 0\), \(k \in \mathbb{N}\) and \(\log^{[j]}(t)\) stands for the \(j\)-th iterated logarithm. Then \(M\) is \(L^p\)-positivity preserving for any \(p \in [1, \infty]\).

As a corollary of the \(p = \infty\) case, we get in particular that a manifold at hand is stochastically complete. This gives a new proof of a celebrated condition for the stochastic completeness due to P. Hsu, [27]. See Remark 6.3.

Beyond their obvious topological triviality, the Cartan-Hadamard manifolds we consider in Theorem 1 have also quite strong metrical properties. On the one hand, the lower bound \(-br^\beta(x)\) for the Ricci curvature implies a Laplacian comparison, i.e., an upper control on \(\Delta r\). This, in turn, permits to construct suitable Hessian and Laplacian cut-off functions. Namely, one gets the existence of a family of smooth cutoffs \(\{\chi_R\} \in C_0^\infty(M)\) with \(R >> 1\) such that

1. \(\chi_R \equiv 1\) on \(B_R\) and \(\chi_R \equiv 0\) on \(M \setminus B_{2R}\);
2. \(|\nabla \chi_R| \leq C_1\);
3. \(|\nabla^2 \chi_R| \leq C_2 R^{\frac{2}{\beta} - 1}\),

with \(C_1, C_2 > 0\) (see Lemma 4.2). Most of the strategies proposed in previous literature to approach the density problem or the \(L^p\)-positivity preservation are precisely based on the existence of suitable cut-off functions which have bounded covariant derivatives up to the second order, for instance in the subquadratic case. Conversely, the control that we get on \(|\nabla^2 \chi_R|\) under our assumptions is not strong enough to allow us to obtain Theorem 1(a) and (c) by this strategy alone. The reason is essentially that, when \(\beta > 2\), the sole lower bound \(\text{Ric} \geq -br^\beta\) cannot guarantee that for any function \(f\)

\[f \in W^{2,p} \implies |\nabla^2 \chi_R|f \in L^p.\]

Instead, assuming also that \(\text{Ric} \leq -ar^\alpha\), one gets

\[f \in W^{2,p} \implies (r^\alpha f) \in L^p,\]

see Theorem 3.6. This latter relation, combined with the properties of the Hessian cut-off functions, yields (1.8).

To obtain (1.9), we exploit the validity on \(\Omega \subset M\) of certain Hardy-type inequalities (obtained elaborating on ideas by L. D’Ambrosio and S. Dipierro, [9]) of the form

\[\int_\Omega \frac{|\nabla G|^p}{G^p} (-\log G)^\beta|f|^p dV_g \leq \left( \frac{p}{p-1} \right)^p \int_\Omega (-\log G)^\beta |\nabla f|^p dV_g \quad \forall f \in C_0^\infty(\Omega),\]

where \(G \in C^\infty(\Omega)\) satisfies

(i) \(-\Delta_g G \geq 0\) on \(\Omega\);
(ii) \(0 \leq G \leq c < 1\);

and \(p \in (1, +\infty)\), see Theorem 3.1 and Theorem 3.6. Using a Laplacian comparison for Cartan-Hadamard manifolds, it turns out that an appropriate choice for \(G\) is the Green function for the \(p\)-Laplacian of the model manifold \(\bar{M}\) whose (radial) Ricci curvature is precisely \(-ar^\alpha\).
In order to prove Theorem 1(b) a further ingredient is needed. Using a special conformal deformation of $M$ based on the distance function, see [28], we prove first the validity of the disturbed infinitesimal Calderón-Zygmund inequality

$$\|\nabla^2 \varphi\|_{L^2} \leq A_1(\varepsilon) \|\Delta \varphi\|_{L^2} + \|\varphi\|_{L^2} \leq A_2 \|\nabla^2 \varphi\|_{L^2} \quad \forall \varphi \in C_0^\infty(M),$$

when $\text{Ric} \geq -br^s$; see Theorem 5.1. Then, one can conclude using again the Hardy-type inequalities.

We conclude this introduction with some words on the novelty of our proof in Theorem 1. Following the strategy adopted in [6, Appendix B] and [20, Theorem XIV.31], a key step in the proof of the $L^p$-positivity preserving is to show that for a given $\psi \in C^\infty$, there exists a positive solution $v \in C^\infty \cap W^{1,q}(M)$ of $-\Delta v + v = \psi$, with $1/q = 1 - 1/p$. While standard elliptic regularity theory ensures that $v \in L^q(M)$ (and hence $\Delta v \in L^q(M)$), the fact that $\nabla v \in L^q(M)$ is non-trivial. When $q \in (1,2]$ (i.e. $p \in [2,\infty]$) it is a consequence of the $L^q$-gradient estimates $\|\nabla v\|_{L^q} \leq C(\|v\|_{L^q} + \|\Delta v\|_{L^q})$, [8]. However, these estimates are not known a priori for $q > 2$ when the negative part of $\text{Ric}$ is unbounded. Instead, we use a version of Li-Yau gradient estimates, [4], to prove that $|\nabla v|(x) \leq \lambda(r(x))v(x)$ outside a compact set. Hence, $\nabla v$ is “almost” in $L^q$, which is enough to our purpose.

The paper is organized as follows. In Section 2 we construct and estimate the Green function for the $p$-Laplacian, $G_p$, on a model manifold with radial Ricci curvature $-ar^\alpha$. Then, using a Laplacian comparison for Cartan-Hadamard manifolds, we show that this function is $p$-superharmonic on a Cartan-Hadamard manifold whose Ricci curvature is bounded from above by $-ar^\alpha$. Section 3 is devoted the proof of the Hardy-type inequalities whose weight is given in terms of $G_p$. In Section 3 we prove respectively part (a), (b) and (c) of Theorem 1. In Section 6 we also prove Theorem 1.

**Notational warning.** Throughout the paper, $C$ will denote a real positive constant whose value can change from line to line. Whenever appropriate, we will explicit its dependency on other constant or parameters.

2. Estimates on Cartan-Hadamard manifolds

The goal of this section is to obtain asymptotic estimates for several geometric objects on Cartan-Hadamard manifolds whose (radial) Ricci curvature is bounded from above by $-ar^\alpha$.

### 2.1. Model manifold case.

We begin by studying the geometry of model manifolds with prescribed Ricci curvature. By direct computation we obtain asymptotic estimates for the $p$-Green function and the Laplacian of the Riemannian distance.

Let $(\tilde{M}, \tilde{g}) = (0, +\infty) \times_j S^{n-1}$ be a model manifold in the sense of E. R. Greene and H. Wu [25], that is, $[0, +\infty) \times S^{n-1}$ endowed with the metric

$$\tilde{g} = dt^2 + j^2(t)d\theta^2,$$

where $d\theta^2$ is the standard metric on $S^{n-1}$ and $j \in C^\infty((0, +\infty))$ such that $j > 0$ on $(0, +\infty)$, $j(0) = 0$, $j'(0) = 1$ and $j^{(2k)}(0) = 0$ for $k \in \mathbb{N}$. Denote with $\nabla$, $\Delta$ and $\text{Ric}$ the covariant derivative, Laplacian and Ricci tensor of $(\tilde{M}, \tilde{g})$ respectively, similarly $\tilde{r}(x)$ is the Riemannian distance from the pole $o$ (so that $\tilde{r}(t, \theta) = t$).

Suppose $(\tilde{M}, \tilde{g})$ satisfies

$$\tilde{\text{Ric}}_\alpha(x) = -(n - 1)A^2e^{\alpha}(x),$$

where $A > 0$, $\alpha \geq 0$ and $\tilde{\text{Ric}}_\alpha$ denotes Ricci curvature in the radial direction $\tilde{\nabla}\tilde{r}$. Since on model manifolds

$$\tilde{\text{Ric}}_\alpha(x) = -(n - 1)\frac{j''(\tilde{r}(x))}{j(\tilde{r}(x))},$$

$j$ needs to solve

$$\begin{cases}
  j''(t) - A^2e^{\alpha}j(t) = 0 \\
  j(0) = 0, \quad j'(0) = 1
\end{cases}$$

(2.1)
for \( t \in [0, +\infty) \). By classical ODE theory we have

\[
(2.2) \quad j(t) = D\sqrt{I_\nu} \left( 2Avt^{1/2\nu} \right), \quad \nu = \frac{1}{\alpha + 2},
\]

where \( D \) is a positive constant and \( I_\nu(t) \) is the modified Bessel function of the first kind and order \( \nu \), i.e., a positive solution of the Bessel equation

\[
t^2I''_\nu(t) + tI'_\nu(t) - (t^2 + \nu^2)I_\nu(t) = 0,
\]

see for reference \([5]\) and \([32]\). Note that, since by \([41, \text{Corollary 5.2}]\) we deduce that

\[
(\ref{eq:order}) \quad G_\nu \quad (2.2)
\]

\[
\text{therefore, } D \equiv \frac{\Gamma(\nu+1)}{(2\nu A)^\nu}.
\]

Using the following asymptotic

\[
(2.3) \quad I_\nu(t) \sim \frac{t^\nu}{\Gamma(\nu+1)} \quad t \to 0,
\]

we have

\[
(2.4) \quad j(t) \sim D_0 t^{-\frac{\alpha}{2}} \exp \left( \frac{2A}{\alpha + 2} t^{1+\frac{\alpha}{2}} \right) \quad t \to +\infty, \quad D_0 = \frac{D}{4\nu A}.
\]

Since \( \text{vol}(\partial B_t) = \omega_n j(t)^{n-1} \) where \( \omega_n \) is the volume of the Euclidean \( n \)-dimensional unit ball, \((\ref{eq:order})\) implies that

\[
\left( \frac{1}{\text{vol}(\partial B_t)} \right)^{\frac{p-1}{p}} \in L^1(+\infty), \quad \forall p > 1.
\]

By \([41]\) Corollary 5.2 we deduce that \((\hat{M}, \hat{g})\) is \( p \)-hyperbolic, that is, there exists a symmetric positive Green function for the \( p \)-Laplacian. Specifically, the positive \( p \)-Green function with pole \( o \in \hat{M} \) is a radial function given by

\[
(2.5) \quad G_p(x) = G_p(t) := \int_t^{+\infty} \left( \frac{1}{j(s)} \right)^{\frac{p-1}{p}} ds, \quad x = (t, \theta) \in (0, +\infty) \times S^{n-1}.
\]

Using \((\ref{eq:order})\) we obtain

\[
(2.6) \quad \partial_t G_p(t) \sim -D_1 t^{\frac{n-1}{p-2}} \exp \left( -A \frac{2}{\alpha + 2} \frac{n - 1}{p - 1} t^{1+\frac{\alpha}{2}} \right) \quad t \to +\infty,
\]

and

\[
(2.7) \quad G_p(t) \sim D_2 t^{\frac{n-1}{p-2}} \exp \left( -A \frac{2}{\alpha + 2} \frac{n - 1}{p - 1} t^{1+\frac{\alpha}{2}} \right) \quad t \to +\infty,
\]

where \( D_1, D_2 \) are positive constants depending on \( D, \alpha, n \) and \( p \). Note that \( \partial_t G_p(t) < 0 \) for all \( t > 0 \).

Next, we compute the Laplacian of the Riemannian distance given by

\[
\Delta \tilde{r} = (n-1) \frac{j'(\tilde{r})}{j(\tilde{r})}.
\]

By a simple computation we have

\[
\frac{j'(t)}{j(t)} = \frac{1}{2t} + A t^\nu \frac{I'_\nu \left( 2Avt^{1/2\nu} \right)}{I_\nu \left( 2Avt^{1/2\nu} \right)}.
\]

Using the recurrence relation \( 2I'_\nu(t) = I_{\nu+1}(t) + I_{\nu-1}(t) \) and \((\ref{eq:order})\), we conclude that \( \frac{j'}{j} \sim \frac{\nu}{\nu+1} \) therefore

\[
(2.8) \quad \frac{j'(t)}{j(t)} \sim At^\nu \quad t \to +\infty.
\]
Finally, using (2.4) once again we deduce
\[
\int_0^t j^{n-1}(s)ds \sim D_4 t^{-\frac{2}{\alpha(n+1)}} \exp \left( \frac{2A}{\alpha + 2} (n-1)t^{1+\frac{2}{\alpha}} \right)
\]
for some positive constant $D_4$, so that
\[
\frac{\int_0^t j^{n-1}(s)ds}{j^{(n-1)}(t)} \sim D_4 t^{-\frac{2}{\alpha}}.
\]

2.2. Comparison results for Cartan-Hadamard manifolds. Next, we relate via the Laplacian comparison the above estimates to a Cartan-Hadamard manifold with a suitable bound on the Ricci curvature.

Let $(M, g)$ be a Cartan-Hadamard manifold of dimension $n \geq 2$ with a fixed pole $o \in M$ and suppose that
\[
\text{Ric}_o(x) \leq -2(n-1)^2 A^2 r^\alpha(x) \quad \forall x \in M \setminus B_{R_0}
\]
for some $A, R_0 > 0$ and $\alpha \geq 0$, here $r(x)$ denotes the Riemannian distance from the pole.

Let $(\hat{M}, \hat{g})$ be the model manifold of radial Ricci curvature
\[
\hat{\text{Ric}}_o(\hat{x}) = -2(n-1)^2 A^2 \hat{r}^{\alpha}(\hat{x}),
\]
that is, $(\hat{M}, \hat{g}) = [0, +\infty) \times_j \mathbb{S}^{n-1}$ where
\[
\hat{g}(t) = \hat{D} \sqrt{\hat{I}_{\nu}} \left( 2\hat{A} t^{1/2\nu} \right), \quad \nu = \frac{1}{\alpha + 2}, \quad \hat{A} = \sqrt{2(n-1)A}.
\]
Since
\[
\text{Ric}_o(x) \leq \frac{1}{n-1} \hat{\text{Ric}}_o(\hat{x}),
\]
for all $x \in M \setminus B_{R_0}$ and $\hat{x} \in \hat{M}$ with $r(x) = \hat{r}(\hat{x})$, by [43, Theorem 2.15] and estimate (2.8) we have
\[
\Delta r \geq \frac{\hat{j}'(r)}{\hat{j}(r)} \sim \hat{A} r^{\frac{2}{\alpha}} = \sqrt{2(n-1)A} r^{\frac{2}{\alpha}} \quad r \to +\infty.
\]

It follows that
\[
\frac{\hat{j}'(r)}{\hat{j}(r)} \sim \sqrt{2(n-1)} \frac{j'(r)}{j(r)}
\]
where $j$ is as in (2.3). In particular, if $r >> 1$ is large enough we can assume that
\[
\Delta r \geq \frac{\hat{j}'(r)}{\hat{j}(r)} \geq (n-1) \frac{j'(r)}{j(r)}.
\]

Note here that
\[
\hat{\Delta} \hat{r} = (n-1) \frac{j'(\hat{r})}{\hat{j}(\hat{r})}
\]
is the Laplacian of the Riemannian distance on the model $(\hat{M}, \hat{g}) = [0, +\infty) \times_j \mathbb{S}^{n-1}$ considered in Section 2.1. In summary, we have the following comparison result.

Proposition 2.1. Let $(M, g)$ be a Cartan-Hadamard manifold of dimension $n \geq 2$ with
\[
\text{Ric}_o(x) \leq -2(n-1)^2 A^2 r^\alpha(x) \quad \forall x \in M \setminus B_{R_0}
\]
for some $A, R_0 > 0, \alpha \geq 0$. Let $j$ be as in (2.2), i.e., $j$ is the warping function of $(\hat{M}, \hat{g}) = [0, +\infty) \times_j \mathbb{S}^{n-1}$, model manifold of radial Ricci curvature
\[
\hat{\text{Ric}}_o(\hat{x}) = -(n-1)A^2 \hat{r}^{\alpha}(\hat{x}).
\]
Then, if $r(x) >> 1$ we have
\[
\Delta r \geq (n-1) \frac{j'(r)}{j(r)}.
\]
Remark 2.2. The slightly uncommon bound we require in (2.11) is due to the fact that we make use of a Laplacian comparison result for Cartan-Hadamard manifolds which is different from the classical one and holds with an upper bound for the Ricci curvature instead of an upper bound for the sectional curvature. Note also that the constant $2$ in (2.11) is quite arbitrary: one could replace it with any constant strictly greater than $1$.

We begin with the following lemma.

Lemma 2.3. Let $(M,g)$ be a Cartan-Hadamard manifold and suppose that
\begin{equation}
\Delta r \geq \phi(r) \quad \text{on } \Omega \subseteq M,
\end{equation}
for some $\phi \in C^0((0, +\infty))$ and $\Omega$ open. Let $v \in C^2(\mathbb{R})$ nonnegative and define $u(x) = v(r(x))$ for $x \in \Omega$. If $v' < 0$, then for all $p > 1$ we have
\begin{equation}
\Delta_p u \leq |v'|^{p-2}(v' \phi(r) + (p-1)v''),
\end{equation}
on $\Omega \setminus \{o\}$.

Proof. Since $(M,g)$ is Cartan-Hadamard, then $r \in C^\infty(M \setminus \{o\})$ so that $u \in C^2(M \setminus \{o\})$. Suppose $v' < 0$, then
\[
\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) = \text{div}(|v'|^{p-2} v' \nabla v) = |v'|^{p-2}(v' \Delta r + (p-1)v''),
\]
on $\Omega \setminus \{o\}$. \qed

Remark 2.4. Although it is not relevant to our work, we observe that if $v' > 0$, then (2.14) holds with the opposite sign.

Combining Lemma 2.3 with Proposition 2.1 we obtain a comparison result for radial $p$-harmonic functions.

Proposition 2.5. Let $(M,g)$ be a Cartan-Hadamard manifold satisfying (2.11) and let $(\tilde{M}, \tilde{g})$ be the model manifold as in Section 2.1. Let $v \in C^2(\mathbb{R})$ non-negative with $v' < 0$ and define $u(x) = v(r(x))$ and $\tilde{u}(x) = v(\tilde{r}(x))$. Then $\Delta_p u(x) \leq \Delta_p \tilde{u}(\tilde{x})$ for all $x \in M$ and $\tilde{x} \in \tilde{M}$ such that $r(x) = \tilde{r}(\tilde{x}) >> 1$.

Proof. By Proposition 2.1 if $r(x) >> 1$, then $\Delta r \geq (n-1)j'(r)/j(r)$, hence
\[
\Delta_p u(x) \leq |v'(r(x))|^{p-2} \left[ v'(r(x))(m-1)j'(r(x))j''(r(x)) + (p-1)v''(r(x)) \right] = \Delta_p \tilde{u}(\tilde{x}).
\]
\qed

In particular if we take $r(t) = G_p(t)$ as in (2.5), since $G_p$ defines the $p$-Green function on $(\tilde{M}, \tilde{g})$ we conclude that $G_p(x)$ is $p$-superharmonic on $(M,g)$ provided that $r(x) >> 1$.

3. HARDY INEQUALITIES VIA GREEN FUNCTION ESTIMATES

We now turn to the study of a class of functional inequalities on Riemannian manifolds which go under the name of Hardy or Rellich-type inequalities. These inequalities have an interest of their own and are extensively studied in literature, especially in the case of Cartan-Hadamard manifolds. See [3, 9, 11, 12, 31, 55, 44] among others. With the help of a result by L. D’Ambrosio and S. Dipierro, [9], we establish a new Hardy-type inequality on complete Riemannian manifolds possessing a non negative $p$-superharmonic function $G$.

Theorem 3.1. Let $(M,g)$ be a complete Riemannian manifold and $\Omega \subseteq M$ open. Fix $p > 1$ and let $G \in C^\infty(\Omega)$ such that
\begin{enumerate}[(i)]
  \item $-\Delta_p G \geq 0$ on $\Omega$;
  \item $0 \leq G \leq c < 1$.
\end{enumerate}
Then, for any $\beta \geq 0$,
\begin{equation}
\int_{\Omega} \frac{|\nabla G|^p}{G^p} (- \log G)^\beta |f|^p dV_g \leq \left( \frac{p}{p-1} \right)^p \int_{\Omega} (- \log G)^\beta |\nabla f|^p dV_g, \quad \forall f \in C_0^\infty(\Omega).
\end{equation}
Proof. Let $\delta > 0$ such that $G_\delta := G + \delta < 1$ and define
\[
\begin{align*}
  h &= -\frac{\lvert \nabla G_\delta \rvert ^{p - 1} \nabla G_\delta}{G_\delta ^{p - 1}} (-\log G_\delta) ^{\beta p}, \\
  A_h &= (p - 1) \frac{\lvert \nabla G_\delta \rvert ^{p}}{G_\delta ^{p}} (-\log G_\delta) ^{\beta p}.
\end{align*}
\]
Since $G \in C^\infty(\Omega)$ and $G_\delta \geq \delta$ we have $|h|, A_h \in L^1_{\text{loc}}(\Omega)$, furthermore,
\[
\frac{|h|^p}{A_h ^{p - 1}} = (p - 1) ^{1 - p} (-\log G_\delta) ^{\beta p} \in L^1_{\text{loc}}(\Omega).
\]
Next, we estimate
\[
\begin{align*}
  \text{div}(h) &= -\frac{(-\log G_\delta) ^{\beta p}}{G_\delta ^{p - 1}} \Delta_p G_\delta + (p - 1) \frac{\lvert \nabla G_\delta \rvert ^{p}}{G_\delta ^{p}} (-\log G_\delta) ^{\beta p} + \beta p \frac{\lvert \nabla G_\delta \rvert ^{p}}{G_\delta ^{p}} (-\log G_\delta) ^{\beta p - 1} \\
  &\geq (p - 1) \frac{\lvert \nabla G_\delta \rvert ^{p}}{G_\delta ^{p}} (-\log G_\delta) ^{\beta p} = A_h.
\end{align*}
\]
Thanks to [9, Lemma 2.10] we have
\[
\int _{\Omega} \frac{\lvert \nabla G_\delta \rvert ^{p}}{G_\delta ^{p}} (-\log G_\delta) ^{\beta p} |f|^p dV_g \leq \left( \frac{p}{p - 1} \right) ^p \int _{\Omega} (-\log G_\delta) ^{\beta p} |\nabla f|^p dV_g \quad \forall f \in C^\infty _0(\Omega).
\]
Since $-\log G_\delta \leq -\log G$ and $\nabla G_\delta = \nabla G$, letting $\delta \to 0$ and using Fatou’s lemma yields (3.1). □

Remark 3.2. It is worth noticing that the results of Theorem (3.1) still hold even under more relaxed regularity assumptions. Notably, it suffices to have $G \in W^{p,\infty} _{\text{loc}}(\Omega)$ and $-\Delta_p G \geq 0$ weakly on $\Omega$ to have the validity of (3.1). Assumption (ii) still needs to hold although it is always satisfied in applications.

Once we have the quite general (3.1), we return to our setting, that is, $(M, g)$ is a Cartan-Hadamard manifold satisfying the Ricci upper bound (2.11). Under such curvature assumptions one easily gets that $(M, g)$ is a $p$-hyperbolic manifold, i.e., there exists a symmetric positive Green kernel for the $p$-Laplacian. Namely, if $G_p(x)$ is the $p$-Green function with pole $o \in M$, it satisfies $\Delta_p G_p(x) = 0$ for all $x \neq o$ and, thus, can be used as weight in Theorem (3.1). Our interest is then to look for asymptotic estimates for the $p$-Green function of $(M, g)$ and its gradient so to better control growth at infinity of the weights in (3.1). One possibility is to use Li-Yau type estimates which are ensured under several lower bounds on Ricci. These, however, are not sufficient because it provides only an upper bound on $\nabla \log G_p$.

Thus, instead of using the $p$-Green function of $(M, g)$ directly, we use the $p$-Green function of the model manifold $(\tilde{M}, \tilde{g})$ constructed in Section 2.1 which is $p$-superharmonic outside a large enough compact set and whose estimates are already available.

Notice also that $G_p(x) \to 0$ as $r(x) \to +\infty$, hence, $G_p(x)$ distant form 1 provided that $r(x) >> 1$. In other words, $G_p(x)$ is a suitable weight in Theorem (3.1) as long as $r(x) >> 1$.

Proposition 3.3. Let $(M, g)$ be a Cartan-Hadamard manifold satisfying (2.11).

For $p > 1$ and $\beta \geq 0$ there exists a compact $K$ containing the pole such that
\[
\int _{\Omega} \frac{\lvert \nabla G_p \rvert ^{p}}{|G_p| ^{p}} (-\log G_p) ^{\beta p} |f|^p dV_g \leq \left( \frac{p}{p - 1} \right) ^p \int _{\Omega} (-\log G_p) ^{\beta p} |\nabla f|^p dV_g,
\]
for all $f \in C^\infty _0(\Omega)$ where $\Omega = M \setminus K$.

Using estimates (2.6) and (2.7) we deduce
\[
\begin{align*}
  \frac{\lvert \nabla G_p \rvert }{|G_p|} (r(x)) &\sim D_5 r(x) ^{\beta \frac{p}{p - 1}}, \\
  (-\log G_p(r(x))) &\sim D_6 r(x) ^{1 + \frac{\beta}{p - 1}}
\end{align*}
\]
so that
\[
\log |G_p| ^{\beta} = O(|\nabla \log G_p|)
\]
provided that $\beta \leq \frac{p}{p - 1}$.
Let pole

We begin by considering Step 1. We proceed by steps, gradually weakening the assumptions on proof.

\[ \int_M \frac{|\nabla G_p|^p}{|G_p|^p} (- \log G_p)^\beta |f|^p dV_g \leq \left( \frac{p}{p-1} \right)^p \int_M (- \log G_p)^\beta |\nabla f|^p dV_g, \]

for all \( f \in W^{1,p}(M) \) with \( \text{supp}(f) \cap K = \emptyset \).

**Proof.** We proceed by steps, gradually weakening the assumptions on \( f \).

**Step 1** We begin by considering \( f \in W^{1,p}(M) \) compactly supported in \( \Omega = M \setminus K \) so that \( f \in W_0^{1,p}(\Omega) \), i.e., there exists \( \eta_n \in C_0^\infty(\Omega) \) such that \( \eta_n \to f \) in \( W^{1,p} \) norm. Note that \( \text{supp}(\eta_n) \) and \( \text{supp}(f) \) are all contained in a compact \( \Omega' \subset \Omega \). Then, by \eqref{3.2} we have

\[ \int_M \frac{|\nabla G_p|^p}{|G_p|^p} (- \log G_p)^\beta |\eta_n|^p dV_g \leq \left( \frac{p}{p-1} \right)^p \int_M (- \log G_p)^\beta |\nabla \eta_n|^p dV_g. \]

Note that

\[ \left| \int_M (- \log G_p)^\beta (|\nabla u|^p - |\nabla f|^p) dV_g \right| \leq \sup_{\Omega'} (- \log G_p)^\beta \int_M ||\nabla u|^p - |\nabla f|^p| dV_g, \]

so that

\[ \int_M (- \log G_p)^\beta |\nabla u|^p dV_g \to \int_M (- \log G_p)^\beta |\nabla f|^p dV_g. \]

Similarly

\[ \int_M \frac{|\nabla G_p|^p}{|G_p|^p} (- \log G_p)^\beta |\eta_n|^p dV_g \to \int_M \frac{|\nabla G_p|^p}{|G_p|^p} (- \log G_p)^\beta |f|^p dV_g. \]

Hence, passing to the limit in \eqref{3.4} we obtain the validity of \eqref{3.6} for all \( f \in W^{1,p}(M) \) compactly supported in \( \Omega \).

**Step 2** Next, let \( f \in W^{1,p}(M) \) such that \( \text{supp}(f) \cap K = \emptyset \) and consider a family of cutoffs \( \chi_R \in C_0^\infty(M) \) such that \( \chi_R \equiv 1 \) on \( B_R \), \( \chi_R \equiv 0 \) outside \( 2B_R \) and \( |\nabla \chi_R| \leq C \) uniformly on \( R \). Such a family exists on any complete Riemannian manifold, see \[ \text{Ludovico Marini* and Giona Veronelli} \]

\[ \text{Note that Proposition 3.3 requires} \ x_R \in C_0^\infty(M) \text{ such that } x_R \equiv 1 \text{ on } B_R, \ x_R \equiv 0 \text{ outside } 2B_R \text{ and } |\nabla x_R| \leq C \text{ uniformly on } R. \]

Consider \( f \chi_R \in W^{1,p}(M) \), clearly \( \text{supp}(f \chi_R) \subseteq M \setminus K \) is compact. Then, by Step 1 (with \( \beta = 0 \)) we have

\[ \int_M \frac{|\nabla G_p|^p}{|G_p|^p} |f|^p \chi_R dV_g \leq \left( \frac{p}{p-1} \right)^p \left( \int_M |f|^p |\nabla \chi_R|^p dV_g + \int_M |\nabla f|^p |\chi_R|^p dV_g \right) \]

\[ \leq \left( \frac{p}{p-1} \right)^p \left( \int_M |\nabla f|^p dV_g + \int_{B_2R \setminus B_R} |f|^p dV_g \right). \]

Note that the LHS converges to \( \int_M |\nabla \log G_p|^p |f|^p dV_g \) by monotone convergence, on the other hand \( \int_{B_2R \setminus B_R} |f|^p dV_g \rightarrow 0 \) since \( f \in L^p(M) \). We conclude that

\[ \int_M |\nabla \log G_p|^p |f|^p dV_g \leq \left( \frac{p}{p-1} \right)^p \left( \int_M |\nabla f|^p dV_g \right) \]

for all \( f \in W^{1,p}(M) \) with \( \text{supp}(f) \cap K = \emptyset \).

**Step 3** Using Step 2, we now prove the more general \eqref{3.8} under the assumptions that \( f \in W^{1,p}(M) \) and \( \text{supp}(f) \cap K = \emptyset \). Indeed, let \( \chi_R \in C_0^\infty(M) \) be as in Step 2 so that \( f \chi_R \) is compactly supported in \( M \setminus K \), by Step 1 we have

\[ \int_M |\nabla \log G_p|^p (- \log G_p)^\beta |f|^p \chi_R dV_g \leq \left( \frac{p}{p-1} \right)^p \left( \int_M (- \log G_p)^\beta |f|^p |\nabla \chi_R|^p dV_g \right. \]

\[ + \left. \int_M (- \log G_p)^\beta |\nabla f|^p |\chi_R|^p dV_g \right). \]
Here, we reason as in Step 2. The only difference is the following estimate which is a consequence of (3.5) and (3.8):

\[
\int_M (\log G_p)^\beta |f|^p dV_g \leq C \int_M |\nabla \log G_p|^\beta |f|^p dV_g \leq C \left( \frac{p}{p-1} \right)^2 \int_M |\nabla f|^p dV_g
\]

where \( C > 0 \). Since \( |\nabla f| \in L^p(M) \) we are still able to conclude that \( \int_{B_{2n} \cap R_n} (\log G_p)^\beta |f|^p dV_g \to 0 \) as \( R \to +\infty \).

If we require \( f \in W^{2,p}(M) \) and apply (3.6) twice, we obtain the following second order Hardy-type inequality.

**Theorem 3.5.** Let \((M,g)\) be a Cartan-Hadamard manifold satisfying (2.11). For \( p > 1 \) and \( 0 \leq \beta \leq \frac{\alpha}{2+\alpha} \) there exists a compact \( K \) containing the pole such that

\[
\int_M \frac{|\nabla G_p|^p}{|G_p|^p} (\log G_p)^\beta |f|^p dV_g \leq C \int_M |\nabla^2 f|^p dV_g,
\]

for all \( f \in W^{2,p}(M) \) such that \( \text{supp}(f) \cap K = \emptyset \), where and \( C = C(p,K) > 0 \).

**Proof.** Using Theorems 3.4 and 3.5 we have

\[
\int_M \frac{|\nabla G_p|^p}{|G_p|^p} (\log G_p)^\beta |f|^p dV_g \leq C \int_M |\nabla \log G_p|^\beta |f|^p dV_g.
\]

Since \( |\nabla f| \in W^{1,p}(M) \) with \( \text{supp}(|\nabla f|) \cap K = \emptyset \) we apply (3.6) with \( \beta = 0 \) to \( |\nabla f| \) and conclude using Kato’s inequality \( |\nabla^2 f| \leq |\nabla f| \) .

Note that inequality (3.9) is more of a second-order Hardy-type inequality rather than a proper Rellich inequality. The reason being that the RHS is estimated with the \( L^p \)-norm of the Hessian rather than the Laplacian of \( f \). The optimal value for \( \beta \) in (3.9) is \( \beta = \frac{\alpha}{2+\alpha} \), in this case we have:

\[
|\nabla G_p|^\beta (\log G_p)^{\frac{\alpha}{2+\alpha}} \sim D_{\tau(x)}^\alpha
\]

which is the fastest growth we are able to control via (3.9). Finally we observe that no assumption on the support of \( f \) is needed as long as the weight has support distant from the pole. This is the kind of control needed for applications.

**Theorem 3.6.** Let \((M,g)\) be a Cartan-Hadamard manifold satisfying (2.11). For \( p > 1 \) and \( K \) as in Theorem 3.5, let \( \omega \geq 0 \) be a measurable function such that \( \text{supp}(\omega) \cap K = \emptyset \) and \( \omega(x) = O(r^\sigma(x)) \) on \( M \), then \( W^{2,p}(M) \hookrightarrow L^p(M,\omega^p dV_g) \).

**Proof.** In order to extend the support of \( f \), we need to remove the possible problems around the pole. To do so, let \( K' \) a compact set such that \( K \subseteq K' \subseteq M \setminus \text{supp}(\omega) \), let \( \varphi \in C^\infty(M) \) be a cutoff function such that \( \varphi \equiv 0 \) on \( K \) and \( \varphi \equiv 1 \) outside of \( K' \). Note that \( |\nabla \varphi| \) and \( |\nabla^2 \varphi| \) are uniformly bounded and that \( f \varphi \in W^{2,p}(M) \) with \( \text{supp}(f \varphi) \cap K = \emptyset \), then by Theorem 3.5 we have

\[
\int_M \omega^p |f|^p dV_g = \int_\Omega \omega^p |f \varphi|^p dV_g \leq C' \int_M \frac{|\nabla G_p|^p}{|G_p|^p} (\log G_p)^{\frac{\alpha}{2+\alpha}} |f \varphi|^p dV_g
\]

\[
\leq C \int_\Omega |\nabla^2 (f \varphi)|^p dV_g
\]

\[
\leq C \int_\Omega |\nabla^2 f|^p dV_g + C \int_\Omega |\nabla \varphi|^p |\nabla f|^p dV_g + C \int_\Omega |\nabla^2 \varphi|^p |f|^p dV_g
\]

\[
\leq C \|f\|_{W^{2,p}(M)}^p.
\]

As a direct consequence, if we have a family of weights \( \{\omega_R\} \) whose growth is suitably controlled and whose supports vanish at \(+\infty\) then \( \|\omega_R f\|_{L^p} \to 0 \).
Corollary 3.7. Let $p > 1$ and $(M,g)$ as in Theorem \ref{thm:density}. Let $f ∈ W^{2,p}(M)$ and $\{ω_R\} ⊆ C^∞(M)$ non-negative such that $\text{supp}(ω_R) ⊆ M \setminus B_R$ with $R ≫ 1$ and $ω_R(x) ≤ Cr^α(x)$, then
\[ \int_M ω_R^p |f|^p dV_g \to 0 \]
as $R \to +∞$.

Remark 3.8. Note that if we assume lower regularity in $f$, namely, $f ∈ W^{1,p}(M)$ we are still able to control $\|ωf\|_{L^p(M)}$ as long as $ω(x) ≤ Cr^{\frac{α}{2}}(x)$. The strategy here is the same of Theorem \ref{thm:density} but instead of the second order Hardy \eqref{eq:hardy} we use the Hardy-type inequality \eqref{eq:hardy} with $β = 0$. Similarly, if we take a family of weights $\{ω_R\}$ such that $ω_R(x) ≤ Cr^{\frac{α}{2}}(x)$ and $\text{supp}(ω_R) ⊆ M \setminus B_R$, we are still able to conclude that $\|ω_Rf\|_{L^p} \to 0$.

4. Density in $W^{2,p}$

In the following section, we apply the estimates developed in Section \ref{sec:estimates} to the density problem of smooth and compactly supported functions in the Sobolev space $W^{2,p}(M)$. To aim this, we construct via the Riemannian distance a family of smooth cutoff functions $\{χ_R\}$ which control up to the second covariant derivative. On arbitrary Riemannian manifolds there are two obstacles to this construction: the Riemannian distance might fail to be smooth on $M \setminus \{∂\}$ and, while $|∇r|$ is always bounded, $|∇^2r|$ might grow uncontrollably. In the case of Cartan-Hadamard manifolds, however, both difficulties can be overcome. Indeed, the cut locus of $M$ is empty which implies smoothness of the Riemannian distance. Furthermore, a lower bound on the radial Ricci curvature allows to control the Hilbert-Schmidt norm of $∇r$.

Lemma 4.1. Let $(M,g)$ be a Cartan-Hadamard manifold satisfying
\[ \text{Ric}_g(x) ≥ -(n-1)B^2r^β(x) \quad ∀ x ∈ M \setminus B_{R_0}, \]
for some $B,R_0 > 0$ and $β ≥ 0$. Then, there exist $R_1 > R_0$ and $C > 0$ such that
\[ |∇^2r|(x) ≤ Cr^{\frac{α}{2}}(x) \quad ∀ x ∈ M \setminus B_{R_1}. \]

Proof. By the Hessian comparison theorem (\cite[Theorem 2.3]{[10]}), the Hessian of the Riemannian distance, $∇^2r$, has non negative eigenvalues at every point in $M \setminus B_{R_0}$ and in particular
\[ |∇^2r| ≤ Δr \]
on $M \setminus B_{R_0}$. Then, by Laplacian comparison we conclude that
\[ |∇^2r| ≤ (n-1)\frac{j''(r)}{j'(r)}, \]
where $j$ is smooth solution of
\[ \begin{cases} j''(t) - B^2t^β j(t) = 0 \\ j(0) = 0, \quad j'(0) = 1 \end{cases} \]
on $[0, +∞)$. With similar estimates as in Section \ref{sec:distance} we get $\frac{j'(t)}{j(t)} \sim Bt^{\frac{α}{2}}$ for $t \to +∞$, in particular, there exist some $R_1 > R_0$ and some positive constant $\tilde{C}$, depending on $B, β, R_1$ such that
\[ \frac{j'(t)}{j(t)} ≤ \tilde{C}t^{\frac{α}{2}} \]
for $t ≥ R_1$. It follows that
\[ |∇^2r|(x) ≤ (n-1)\frac{j''(r(x))}{j'(r(x))} ≤ Cr^{\frac{α}{2}}(x), \]
for all $x ∈ M \setminus B_{R_1}$, where $C = C(B, β, R_1, n)$. \hfill \rlap{□}

Once we have second order estimates on the Riemannian distance, we obtain $\{χ_R\}$ by composing with a sequence of real cutoffs.
Lemma 4.2. Let \((M, g)\) be a Cartan–Hadamard manifold satisfying (4.1) then, there exists a family of smooth cutoffs \(\{\chi_R\} \in C_0^\infty(M)\) with \(R > 1\) such that

1. \(\chi_R \equiv 1\) on \(B_R\) and \(\chi_R \equiv 0\) on \(M \setminus \overline{B_{2R}}\);
2. \(\nabla \chi_R \leq \frac{a}{R}\);
3. \(\nabla^2 \chi_R \leq C_2 R^{2 - \delta}\),

with \(C_1, C_2 > 0\).

**Proof.** Fix \(\phi: \mathbb{R} \to [0, 1]\) a smooth function such that \(\phi \equiv 1\) on \((-\infty, 1]\) and \(\phi \equiv 0\) on \([2, +\infty)\), and let \(a > 0\) such that \(|\phi'| + |\phi''| \leq a\) uniformly on \(\mathbb{R}\). For \(R > 1\) (it suffices \(R \geq R_1\), with \(R_1\) as in Lemma 4.1, let

\[
\phi_R(t) := \phi \left( \frac{t}{R} \right)
\]

so that

\[
|\phi_R'| \leq \frac{a}{R^2}, \quad |\phi_R''| \leq \frac{a}{R^3}.
\]

Then, define \(\chi_R(x) := \phi_R \circ r(x)\), we have \(\chi_R \equiv 1\) on \(B_R\) and \(\chi_R \equiv 0\) on \(M \setminus B_{2R}\). Furthermore,

\[
\nabla \chi_R \leq |\phi_R'(r(x))| \nabla r(x)| \leq \frac{C_1}{R} \quad \nabla^2 \chi_R \leq |\phi_R''(r(x))| \nabla^2 r(x)| + |\phi_R'(r(x))| |\nabla r(x)|^2 \leq C_2 R^{2 - \delta} - 1,
\]

where \(C_1, C_2\) depend on \(a\) and the constant \(C\) of Lemma 4.1. \(\square\)

**Remark 4.3.** The above construction of the Hessian cutoffs is not the only possible one. It is worth noticing that the family \(\{\chi_R\}\) can be constructed on Riemannian manifolds without any topological restrictions as long as one of the following assumptions holds:

(a) \(\text{Ric}(x) \leq B^2 r^p(x)\) and \(\text{inj}(x) \geq i_0 r^{-\beta}(x) > 0\); (b) \(\text{Sect}(x) \leq B^2 r^3(x)\),

for some \(B, i_0 > 0\) and \(\beta \geq 0\). In this setting, although the Riemannian distance might loose smoothness, it is possible to construct a distance-like function \(H \in C^\infty(M)\) such that

(i) \(C^{-2} r(x) \leq H(x) \leq \max\{r(x), 1\}\); (ii) \(\nabla H(x) \leq 1\); (iii) \(\nabla^2 H(x) \leq C \max\{r^\beta(x), 1\}\),

for some \(C > 1\), see [29] Theorem 1.2. Then, one defines \(\chi_R = \phi_R \circ H(x)\) where \(\phi_R\) is a family of real cutoffs in a similar fashion of Lemma 4.1.

We can now prove the density of smooth compactly supported functions in the Sobolev space \(W^{2, p}\), namely (a) of Theorem 1. To obtain this we assume a double bound on the radial Ricci curvature. The bound from below allows the construction of the smooth cutoff functions while the bound from above ensures the validity of the functional estimates in Section 4.

**Theorem 4.4.** Let \((M, g)\) be a Cartan–Hadamard manifold with a fixed pole \(o \in M\). Suppose that

\[-(n - 1) B^2 r^{2 + \delta}(x) \leq \text{Ric}_o(x) \leq -2(n - 1)^2 A^2 r^\alpha(x), \quad \forall x \in M \setminus B_{R_0}\]

for some \(A, B, R_0 > 0\) and \(\alpha \geq 0\). Then \(W^{2, p}(M) = W^{2, p}(M)\) for all \(p > 1\).

**Proof.** Since \(C^\infty(M) \cap W^{2, p}(M)\) is dense in \(W^{2, p}(M)\) (see [17], it suffices to show that \(C^\infty(M)\) is dense in \(C^\infty(M) \cap W^{2, p}(M)\) with respect to the \(W^{2, p}\). To this goal, take \(f \in C^\infty(M) \cap W^{2, p}(M)\) and consider a family of cutoffs \(\{\chi_R\} \subset C^\infty(M)\) as in Lemma 4.1. Define \(f_R := \chi_R f \in C_0^\infty(M)\) and observe that

\[
(4.3) \quad \|f - f\|_{L^p} = \|\chi_R - 1\|f\|_{L^p}
\]

\[
(4.4) \quad \|\nabla (f - f)\|_{L^p} \leq \|
abla \chi_R\|_{L^p} + \|\chi_R - 1\|\nabla f\|_{L^p}
\]

\[
(4.5) \quad \|\nabla^2 (f - f)\|_{L^p} \leq 2\|
abla f\|\nabla \chi_R\|_{L^p} + \|\chi_R - 1\|\nabla^2 f\|_{L^p}.
\]

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Since $\nabla \chi_R$ and $(\chi_R - 1)$ are uniformly bounded and supported in $M \setminus \overline{B}_R$, $f \in W^{2,p}(M)$ implies that the RHS of (1.3), (1.4), and (1.5) except the last term, vanish as $R \to +\infty$. We only need to show that $\|\nabla^2 \chi_R\|_{L^p} \to 0$ as $R \to 0$. To see this, it is sufficient to observe that $\|\nabla^2 \chi_R\| \leq Cr^n$ and $\text{supp}(\chi_R) \subseteq M \setminus \overline{B}_R$, then by Corollary 5.7 we conclude the proof. □

Remark 4.5. When $p = 1$ our strategy to construct Hardy-type inequalities fails. Note for instance that the constant in (5.4) and subsequent derived inequalities explodes as $p \to 1$. Nevertheless, we expect the result to hold even when $p = 1$.

5. AN $L^2$-CALDERÓN-ZYGMUND INEQUALITY

As a further application of the tools developed in Section 3, we prove the validity of a $L^2$-Calderón-Zygmund inequality on Cartan-Hadamard manifolds with bounds on Ricci curvature. In the spirit of [25], we first prove a weighted $CZ(2)$ inequality which holds under lower bound on Ricci curvature.

Theorem 5.1. Let $(M, g)$ be a Cartan-Hadamard manifold with a fixed pole $o \in M$. Suppose that

$$\text{Ric}(x) \geq -(n-1)B^n \beta(x) \quad \forall x \in M \setminus B_{R_0}$$

in the sense of quadratic forms for some $B, R_0$ and $\beta \geq 0$. Then, for every $\varepsilon > 0$ there exists a constant $A_1 = A_1(\varepsilon) > 0$ such that

$$\|\nabla^2 \varphi\|_{L^2} \leq A_1 \|\Delta \varphi\|_{L^2} + A_2 \varepsilon^2 \|r^2 \varphi\|_{L^2} \quad \forall \varphi \in C_0^\infty(M).$$

Here $A_2$ is a fixed positive constant independent of $\varepsilon$.

Proof. Let $R_1$ be the constant of Lemma 4.1 and let $h : \mathbb{R} \to [1, +\infty)$ be a smooth function such that $h(t) \equiv 1$ for $t \leq R_1$, and $h(t) = t-a$ for some $a > 0$ and $t \geq 2R_1$ and $|h'(t)| \leq 1$ for all $t$. Define $H(x) := h(r(x))$ so that

(i) $\max\{C^{-1}r(x), 1\} \leq H(x) \leq \max\{Cr(x), 1\}$;

(ii) $|\nabla H(x)| \leq 1$;

(iii) $|\nabla^2 H(x)| \leq C r^2 \frac{\varepsilon}{n}$

for some $C > 1$. Consider on $(M, g)$ the following conformal deformation:

$$\tilde{g} := e^{2\phi} g, \quad \text{where } \phi := \frac{\beta}{2} \log H.$$

Note that $M$ remains complete also with respect to $\tilde{g}$. In this proof we denote with $\tilde{\nabla}^2, \tilde{\nabla}, \tilde{\Delta}$, $\text{Ric}$ the Hessian, gradient, Laplacian and Ricci tensor with respect to $\tilde{g}$. Moreover, let $\tilde{L}^2 = L^2(M, dV_{\tilde{g}})$. Since

$$|\nabla \phi| \leq \frac{\beta}{2}, \quad |\nabla^2 \phi| \leq \beta \max\{Cr^2, 1\},$$

we immediately get that $\text{Ric}$ is bounded from below by some constant $\tilde{C}$ depending on $\beta, B, C$ and $n$, see (25) in [25]. Thanks to [41] Proposition 4.5, this implies the validity on $(M, \tilde{g})$ of the following infinitesimal $CZ(2)$ inequality: for every $\varepsilon > 0$

$$\|\tilde{\nabla}^2 u\|_{L^2} \leq \tilde{C} \varepsilon^2 \|u\|_{L^2} + \left(1 + \tilde{C} \varepsilon^2\right) \|\tilde{\Delta} u\|_{L^2} \quad \forall u \in C_0^\infty(M).$$

Throughout the rest of the proof, we denote with $A_i$ real positive constants depending on $\beta, B, n, C$ and, possibly, $\varepsilon$. By standard estimates (see [25] Section 8.3) we get

$$|\tilde{\nabla}^2 u|^2 dV_{\tilde{g}} \geq \varepsilon^{(n-4)\phi} \left\{ \frac{1}{2} |\tilde{\nabla}^2 u|^2 - A_3 |\tilde{\nabla} u|^2 - |\tilde{\Delta} u|^2 \right\} dV_{\tilde{g}},$$

(5.3)

$$|\tilde{\Delta} u|^2 dV_{\tilde{g}} \leq \varepsilon^{(n-4)\phi} \left\{ |\Delta u|^2 + A_4 |\nabla u|^2 \right\} dV_{\tilde{g}}.$$  (5.4)

Inserting (5.3) and (5.4) in (5.2) yields

$$\|H^{\frac{\phi}{2}} |\nabla^2 u||_{L^2} \leq \tilde{C} \varepsilon^2 \|H^{\frac{\phi}{2}} u\|_{L^2} + A_6 \|H^{\frac{\phi}{2}} \Delta u\|_{L^2} + A_6 \|H^{\frac{\phi}{2}} (n) |\nabla u||_{L^2}.$$  (5.5)
forms, that is curvature and is necessary to ensure the validity of (5.2). For any HJM, we can estimate the last term on (5.1) thus proving (b) of Theorem I.

\[ \|\nabla\| \leq 1 \quad |\nabla^2\| \leq C^2 r^{-1}, \]

we have

\[ H^{(n-4)/4} |\nabla u| \leq A_7 |\varphi| + |\nabla \varphi|; \]
\[ H^{(n-4)/4} |\nabla^2 u| \geq |\nabla^2 \varphi| - A_7 |\nabla \varphi| - A_8 H^{(n-4)/4} |\varphi|. \]

Using these estimates in (5.5) yields

\[ \|\nabla^2 \varphi\|_2^2 \leq C^2 |\nabla^2 \|H^\beta \varphi\|_2^2 + A_6 \|\nabla \varphi\|_2^2 + A_5 \|\Delta u\|_L^2 + A_{10} \|H^{\frac{n-4}{4}} \varphi\|_2^2. \]

By the divergence theorem and Cauchy-Schwarz inequality we also have

\[ \|\nabla \varphi\|_2^2 = \int_M |\nabla \varphi|^2 dV_g = - \int_M \nabla \varphi \Delta \varphi dV_g \leq 2 \|\varphi\|_2^2 + 2 \|\Delta \varphi\|_2^2. \]

Moreover, since \( H^{\frac{n-4}{4}} = o(\beta) \) as \( r(x) \to +\infty \), for all \( \varepsilon > 0 \) there exists some constant \( C_\varepsilon > 0 \) such that

\[ H^{\frac{n-4}{4}} \leq \varepsilon \sqrt{C \frac{\beta}{A_{10}}}, \]

hence, \( A_{10} \|H^{\frac{n-4}{4}} \varphi\|_2^2 \leq \varepsilon^2 C |\nabla \|H^\beta \varphi\|_2^2 + C_\varepsilon^2 |\varphi|_2^2. \) Using these latter estimates, (5.6) becomes

\[ \|\nabla^2 \varphi\|_2^2 \leq A_6^2 \left[ |\Delta \varphi|_2^2 + |\varphi|_2^2 \right] + C_\varepsilon^2 |\nabla \|H^\beta \varphi\|_2^2 \]

Finally, since \( H(x) \leq \max\{\text{Cr}(x), 1\} \) we have

\[ \int_M H^{2\beta} \varphi^2 dV_g \leq \int_{r \leq 1} \varphi^2 dV_g + C^{2\beta} \int_{r \geq 1} r^{2\beta} \varphi^2 dV_g = \|\varphi\|_L^2 + C^{2\beta} \|r^\beta \varphi\|_L^2. \]

which gives (5.6).

Remark 5.2. Note that in Theorem 5.1 we require a bound on Ricci in the sense of quadratic forms, that is

\[ \text{Ric}(X, X)(x) \geq - (n - 1) B^2 r^\beta(x) g(X, X) \]

for any \( X \in T_x M \). This is a stronger assumption than the previous bounds on radial Ricci curvature and is necessary to ensure the validity of (5.2).

If we also assume also an upper bound on the Ricci curvature, using the second order Hardy-type inequality (3.9) we can estimate the last term on (5.1) thus proving (b) of Theorem I.
Theorem 5.3. Let \((M,g)\) be a Cartan-Hadamard manifold with a fixed pole \(o \in M\). Suppose that
\[ - (n-1)B^2r^2(x) \leq \text{Ric}(x) \leq -2(n-1)^2A^2r^\alpha(x) \quad \forall x \in M \setminus B_{r_0}, \]
for some constants \(B > \sqrt{2(n-1)}A > 0\) and some \(\alpha \geq 0\). Then, the following \(L^2\)-Calderón-Zygmund inequality holds on \(M\):
\[ \|\nabla^2 \phi\|_{L^2} \leq C(\|\Delta \phi\|_{L^2} + \|\phi\|_{L^2}) \]
for all \(\phi \in C^0_0(M)\).

**Proof.** By Theorem 5.1 we have the validity of (5.1), thus, we only need to estimate the weighted term \(\|r^\beta \phi\|^2_{L^2}\). Let \(K\) be a compact large enough (see Theorem 5.2), then
\[ \|r^\beta \phi\|^2_{L^2} = \int_M r^{2\beta} \phi^2 dV_g \leq \max_K r^{2\beta} \int_K \phi^2 dV_g + \int_{M\setminus K} r^{2\beta} \phi^2 dV_g. \]
Thanks to Theorem 5.2 we have
\[ \int_{M\setminus K} r^{2\beta} \phi^2 dV_g \leq C' \int_M |\nabla^2 \phi|^2 dV_g, \]
so that
\[ \|\nabla^2 \phi\|_{L^2} \leq A' (\|\Delta \phi\|_{L^2} + \|\phi\|_{L^2}) + A'' \varepsilon^2 (\|\phi\|^2_{L^2} + \|\nabla^2 \phi\|^2_{L^2}). \]
Since \(\varepsilon\) can be made arbitrarily small and \(A''\) is a fixed constant this last estimate yields (5.9). \(\square\)

**Remark 5.4.** It would be interesting to obtain a \(CZ(p)\) estimate also in the general case \(p \in (1, +\infty)\). To do this, however, one would need an *infinitesimal* \(CZ(p)\) estimate similar to (5.2) which, to the best of our knowledge, is not known when \(p \neq 2\).

**Remark 5.5.** The validity of an \(L^2\)-Calderón-Zygmund inequality directly implies the density of \(C^0_0(M)\) in \(W^{2,2}(M)\). The observation is due to S. Pigola and goes as follows. Let \(\phi \in W^{2,2}(M) \subseteq H^{2,2}(M)\), thanks to a result by O. Milatovic [22, Appendix A], there exists a sequence of functions \(\{\phi_k\} \subseteq C^\infty_0(M)\) such that \(\phi_k \rightarrow \phi\) in \(H^{2,2}(M)\). It follows that \(\{\phi_k\}\) is Cauchy in \(H^{2,2}(M)\), using (5.9) and the validity on \((M,g)\) of an \(L^2\)-gradient estimate (see [21, Proposition 3.10b]) we deduce that \(\{\phi_k\}\) is Cauchy also in \(W^{2,2}(M)\). By completeness we have \(\phi_k \rightarrow \phi\) in \(W^{2,2}(M)\), however, \(\phi = \phi\) thanks to the continuous embedding \(W^{2,2}(M) \subseteq H^{2,2}(M)\). See also [22, Remark 2.1]. As a result, Theorem 5.3 provides an alternative proof of Theorem 4.4 although under heavier assumptions.

The above observation also implies the following corollary.

**Corollary 5.6.** Let \((M,g)\) be a Cartan-Hadamard manifold as in Theorem 5.3, then
\[ W^{2,2}_0(M) = W^{2,2}(M) = H^{2,2}(M). \]

6. \(L^p\)-positivity preserving and the BMS conjecture

This last section is devoted to the study of the \(L^p\)-positivity preserving property, and BMS conjecture, on a certain class of manifolds. Following 6 Theorem B.1, we first prove.

**Lemma 6.1.** Let \(\phi \in C^\infty_0(M)\), \(\phi \geq 0\), then there exists a unique \(v \in C^\infty(M) \cap L^p(M) \forall p \in [1, +\infty), v > 0\), such that
\[ (-\Delta + 1)v = \phi. \]

**Proof.** Let \(\{\Omega_k\}\) be an exhaustion of \(M\) by relatively compact, open sets of smooth boundary satisfying
\[ \Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega_k \subseteq \Omega_{k+1} \subseteq \cdots, \]
that is, \(\Omega_k\) is relatively compact in \(\Omega_{k+1}\) for all \(k \in \mathbb{N}\). Furthermore, assume \(\Omega_1\) is large enough so that \(\text{supp}(\phi) \subseteq \Omega_1\). Then let \(v_k\) be a smooth solution of the following Dirichlet problem:
\[ \begin{cases} (-\Delta + 1)v_k = \phi & \text{on } \Omega_k, \\ v_k = 0 & \text{on } \partial \Omega_k. \end{cases} \]
By the strong maximum principle (Theorem 3.5), we immediately get $v_k > 0$ in the interior of $\Omega_k$ and $v_{k+1} \geq v_k$ for all $k$, hence, $\{v_k\}$ is a monotone increasing sequence of functions and thus admits a (possibly infinite) pointwise limit

$$0 < v(x) = \lim_{k \to +\infty} v_k(x).$$

Next we prove that $v$ is actually everywhere finite, smooth and belongs to $L^p(M)$ for any $p \in [1, +\infty)$. To this end, we multiply (6.3) by $v_k^{-1}$ and integrate over $\Omega_k$

$$\int_{\Omega_k} v_k^{p-1}(-\Delta + 1)v_k dV_g = \int_{\Omega_k} v_k^p dV_g - \int_{\Omega_k} v_k^{p-1} \nabla v_k \cdot \nabla v_k dV_g$$

$$= \int_{\Omega_k} v_k^p dV_g + \int_{\Omega_k} \langle \nabla v_k^{-1}, \nabla v_k \rangle dV_g$$

$$= \int_{\Omega_k} v_k^p dV_g + (p - 1) \int_{\Omega_k} v_k^{p-2} |\nabla v_k|^2 dV_g \geq \int_{\Omega_k} v_k^p dV_g.$$  

On the other hand, by Hölder’s inequality

$$\int_{\Omega_k} v_k^{p-1}(-\Delta + 1)v_k dV_g = \int_{\Omega_k} v_k^{p-1} \phi dV_g \leq \left\{ \int_{\Omega_k} \phi^p dV_g \right\}^{1/p} \left\{ \int_{\Omega_k} v_k^p dV_g \right\}^{1-1/p}$$

hence $\|v_k\|_{L^p(\Omega_k)} \leq \|\phi\|_{L^p(M)}$. Since $\{v_k\}$ is uniformly bounded in $L^p$ on any compact set, by standard interior regularity we deduce that $\{v_k\}$ is uniformly bounded in $W^{h, p}_0(\Omega)$ for any order $h$ and $p \in [1, +\infty)$. As a consequence of Sobolev spaces compact embedding, all the covariant derivatives of $\{v_k\}$ converge up to a subsequence uniformly on compact sets, i.e., $v_k$ converges in $C^\infty(M)$ topology. In particular, $v$ is positive, smooth and satisfies (6.1). Moreover, by Fatou’s lemma we also have that $v \in L^p(M)$ for any $p \in [1, +\infty)$. For $p = +\infty$, let $x^*$ be such $v_k(x^*) = \max_{\Omega_k} v_k$, by the maximum principle we get $v_k(x^*) \leq \phi(x^*) \leq \|\phi\|_{L^\infty(M)}$, that is, $\|v_k\|_{L^\infty(\Omega_k)} \leq \|\phi\|_{L^\infty(M)}$. Letting $k \to +\infty$ we get $v \in L^\infty(M)$. We would like now to extend the above result to the case where $(-\Delta + 1)v$ is positive Radon measure. To do so, we first need the following $L^p$-gradient estimate which is a simple extension of a result by T. Coulhon and X. T. Duong [8].

**Lemma 6.2.** Let $(M, g)$ be a complete Riemannian manifold, then for all $1 < q \leq 2$ there exists a constant $C > 0$ such that

$$\|\nabla u\|_{L^q} \leq C(\|u\|_{L^p} + \|\Delta u\|_{L^p})$$

for all $u \in C^\infty(M) \cap H^{2,q}(M)$.

**Proof.** The validity of (6.3) on $C^\infty_0(M)$ is known thanks to a result by T. Coulhon and X. T. Duong [8]. Thanks to a result by O. Milatovic [22] Appendix A], for all $u \in C^\infty(M) \cap H^{2,q}(M)$ there exists a sequence of functions $\{u_k\} \subseteq C^\infty_0(M)$ such that $u_k \to u$ with respect to the $H^{2,q}(M)$ norm. Applying (6.3) to the Cauchy differences we deduce that $\nabla u_k \to \nabla u$ in $L^q(M)$. Then we obtain the desired result applying (6.3) to $u_k$ and taking the limit.

**Theorem 6.3.** Let $(M, g)$ be a Cartan-Hadamard manifold satisfying

$$- (n - 1)B^2 r^{n+2}(x) \leq \text{Ric}_0(x) \leq -2(n - 1)^2 A^2 r^\alpha(x) \quad \forall x \in M \setminus B_{R_0}$$

for some constants $B > \sqrt{2}(n - 1)A > 0$ and some $\alpha, R_0 > 0$. Then, $M$ is $L^p$-positivity preserving for all $2 \leq p < +\infty$.

**Proof.** Let $u \in L^p(M)$ such that $(-\Delta + 1)u \geq 0$ in the sense of distributions. In order to prove $L^p$-positivity preserving, we need to show that

$$\int_M \phi u dV_g \geq 0 \quad \forall \phi \in C^\infty_0(M), \phi \geq 0.$$
By Lemma 6.1 let \( v \in C^\infty(M), v > 0 \) such that \((-\Delta + 1)v = \phi\) and let \( \{\chi_R\} \in C^\infty(M) \) be a family of cutoffs as in Lemma 6.2. Since \( v\chi_R \in C_0^\infty(M), v\chi_R \geq 0 \) we have

\[
0 \leq \int_M u(-\Delta + 1)(v\chi_R)dV_g = \int_M [-u\Delta(v\chi_R) + v\chi_Ru]dV_g
\]

\[
= -\int_M u\chi_R\Delta vdV_g - \int_M uv\chi_RdV_g
\]

\[
- \int_M u(\nabla\chi_R, \nabla v)dV_g + \int_M u\chi_RvdV_g.
\]

Recall that \( v, \Delta v \in L^q(M) \) for all \( q \in [1, +\infty] \), in particular, this holds for \( q = \frac{p}{p-1} \) so that \( uv \in L^1(M) \) and \( u\Delta v \in L^1(M) \). By dominated convergence we conclude that

\[
\int_M u\chi_R\Delta vdV_g \to \int_M u\Delta vdV_g, \quad \int_M u\chi_RvdV_g \to \int_M uvdV_g
\]

for \( R \to +\infty \). On the other hand, by Lemma 6.2 we have \( \|\nabla v\| \notin L^q(M) \) so that \( u\nabla v \in L^1(M) \), by dominated convergence we conclude that

\[
\int_M u(\nabla\chi_R, \nabla v)dV_g \leq \int_M |u||\nabla v||\nabla\chi_R|dV_g \to 0
\]

for \( R \to +\infty \). Finally, by Hölder’s inequality we have

\[
\int_M uv\Delta \chi_RDvdV_g \leq \left\{ \int_M |u|^pdV_g \right\}^{\frac{1}{p}} \left\{ \int_M |v\Delta \chi_R|^qdV_g \right\}^{\frac{1}{q}}.
\]

The lower bound on Ricci implies that \( |\Delta \chi_R| \leq C\bar{v}^\frac{n}{2}(x) \) and \( v \in W^{1,q}(M) \), hence, by Remark 6.3 we have

\[
\int_M |v\Delta \chi_R|^qdV_g \to 0
\]

as \( R \to +\infty \). In conclusion, we have proved that

\[
\int_M \phi udV_g = \lim_{R \to +\infty} \int_M u(-\Delta + 1)(v\chi_R)dV_g \geq 0,
\]

hence, \( u \geq 0 \) in the sense of distributions. \( \square \)

Note that, although Lemma 6.1 holds on the whole \( L^p \) scale, the case \( p = +\infty \) and \( 1 \leq p < 2 \) have been left out in the previous theorem. The difficulty in these situations is that we generally lack the \( L^q \)-gradient estimates where \( q > 2 \) is the conjugate exponent of \( p \). On the other hand, an \( L^1 \) gradient estimate which corresponds to the case \( p = +\infty \) is false even in the Euclidean setting.

Remark 6.4. Recall that the \( L^\infty \)-positivity preserving property implies stochastic completeness of the manifold at hand. Indeed, \( (M, g) \) is stochastically complete if the only non-negative, bounded \( C^2 \) solution of

\[
\Delta u = u
\]

is \( u \equiv 0 \). We refer to [10] Section 6 for a survey of the equivalent definitions of stochastic completeness. Indeed, if \( u \in C^2(M) \cap L^\infty(M), u \geq 0 \) solves \( \Delta u = u \) then \(-u\) solves \(-\Delta(-u) = u \geq 0 \). By \( L^\infty \)-positivity preserving we deduce that \( u \leq 0 \) hence \( u \equiv 0 \). As a matter of fact, when the Ricci curvature is below a certain critical growth we can prove that \( (M, g) \) looses stochastic completeness, hence, the \( L^\infty \)-positivity preserving property needs to fail.

Theorem 6.5. Let \( (M, g) \) be a Cartan-Hadamard manifold satisfying

\[
\text{Ric}_{c_0}(x) \leq -2(n-1)^2A^2\bar{r}^\alpha(x) \quad \forall x \in M \setminus B_{R_0},
\]

with \( A, \alpha, R_0 > 0 \). If \( \alpha > 2 \), then \( (M, g) \) is not stochastically complete.

Proof. Let \( j \) be as in 2.2 and define

\[
v(t) = \int_0^t j^{(1-n)}(s) \left( \int_0^s j^{(n-1)}(\tau)d\tau \right) ds
\]
then $u(x) = v(r(x))$ is a $C^2$ function on $M$. By \eqref{2.10} we have

$$\int_0^r j^{n-1}(r) dr = L^1(+\infty),$$

hence, $u$ is bounded. Since $v' \geq 0$, by Proposition \[2.1\] we have

$$\Delta u(x) = v''(r(x)) + \Delta r(x)v'(r(x)) \geq v''(r(x)) + (n-1)\frac{j''(r(x))}{j(r(x))}v'(r(x))$$

for $r \gg 1$. By direct computation this implies that $\Delta u \geq 1$ outside a compact set and in particular, there cannot be a sequence of points $\{x_k\} \subset M$ such that $u(x_k)$ converges to $\sup_M u$ and $\Delta u(x_k) < 1/k$ which is an equivalent formulation of stochastic completeness, see \[39\].

It remains to investigate the subquadratic case. In this setting the cut-off functions constructed in Lemma \[4.2\] have a much better behavior, namely, the Hessian is uniformly bounded. As a consequence, one can easily avoid the use of the Hardy-type inequality to control the term containing $\Delta \chi_R$. It turns out that such Laplacian cut-off functions exist on arbitrary complete Riemannian manifolds as long as the Ricci curvature satisfies

$$\text{Ric}(x) \geq -\lambda^2(r(x)) \quad \forall x \in M \setminus B_{R_0}.$$  \hfill (6.4)

Here $\lambda$ is a $C^\infty$ function given by

$$\lambda(t) = a t \prod_{j=0}^k \log^{\lfloor j \rfloor}(t)$$

for $t$ large enough, where $a > 0$, $k \in \mathbb{N}$ and $\log^{\lfloor j \rfloor}(t)$ stands for the $j$-th iterated logarithm. The following is a joint result of the second author with D. Impera and M. Rimoldi \[29\], Corollary 4.1, which slightly generalizes \[1\] Corollary 2.3.

**Theorem 6.6.** Let $(M, g)$ be a complete Riemannian manifold satisfying \eqref{6.1} in the sense of quadratic forms. Then, there exists a family of smooth cut-off functions $\{\chi_R\} \subset C^\infty_0(M)$, $R > R_0$, such that

1. $\chi_R \equiv 1$ on $B_R$ and $\chi_R \equiv 0$ on $M \setminus \overline{B_{R_0}}$;
2. $|\nabla \chi_R| \leq \frac{C_1}{R}$;
3. $|\Delta \chi_R| \leq C_2;
$

where $C_1, C_2 > 0$, $\gamma > 1$ and $\lambda$ is the function defined in \eqref{6.5}.

Using these cut-off functions instead of the ones constructed in Lemma \[4.2\] allows to drop any topological assumptions on $M$. This was already observed by B. Güneysu in the setting where $L^p$-gradient estimates are available, i.e. for $p \in [2, \infty)$. However, there is no need to use $L^p$-gradient estimates, since we can use instead a uniform Li-Yau estimate which is a special case of a result by D. Bianchi and A. Setti \[1\] Theorem 2.8.

**Theorem 6.7.** Let $(M, g)$ be a complete Riemannian manifold satisfying \eqref{6.1} in the sense of quadratic forms. Let $R > r > 0$ and let $\gamma > 1$ and let $v : M \setminus \overline{B_r} \to \mathbb{R}$ be a $C^2$ function satisfying

$$\begin{cases} v > 0 & \text{on } M \setminus \overline{B_r} \\
\Delta v = v. \end{cases}$$

Then, there exists a positive constant $C = C(n, \gamma, B) > 0$ such that

$$\frac{|\nabla v(x)|}{\lambda(R)} \leq Cv(x) \quad \forall x \in B_{\gamma R} \setminus \overline{B_R}.$$ \hfill (6.7)

Using these two results we can prove the following.

**Theorem 6.8.** Let $(M, g)$ be a complete Riemannian manifold satisfying \eqref{6.1} in the sense of quadratic forms. Then $M$ is $L^p$-positivity preserving for all $p \in [1, +\infty]$. [99]
Proof. Let \( u \in L^p(M) \), \( p \in [1, +\infty] \) such that \(-\Delta u + u \geq 0\) in the sense of distributions. Take \( \phi \in C_0^\infty(M) \), \( \phi \geq 0 \) we need to show that

\[
\int_M u \phi dV_g \geq 0.
\]

To this end we take \( v \in C^\infty(M) \), \( v > 0 \) as in Lemma 6.1, such that \(-\Delta v + v = \phi \) and \( v, \Delta v \in L^q(M) \) \( \forall q \in [0, +\infty] \). Then we proceed as in Theorem 6.3, using the cut-off functions of Theorem 6.3 instead of the one of Lemma 1.2. The proof differs from the one of Theorem 6.3 only in the estimate of the terms containing \( \Delta \) \( \chi \) \( L \)

Remark

To this end we take

\[
\lambda \rightarrow +\infty.
\]

As a consequence of the case \( p = +\infty \), we immediately get that the manifold at hand is stochastically complete. P. Hsu in [27] proved the stochastic completeness assuming the \( \text{Ric}(x) \geq -\kappa(r(x)) \), where \( \kappa \) is non decreasing and \( \int_{-\infty}^{\kappa} \kappa^{-1} = \infty \). Keeping also in account that the choice of \( \lambda \) in our result can be slightly generalized, [29] Proposition 1.1], our function \( \lambda \) is essentially the maximal one admissible in order to fulfill \( \int_{-\infty}^{\kappa} \lambda^{-1} = \infty \).

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