p-ADIC AND ADELIC GENERALIZATION OF QUANTUM COSMOLOGY

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1. Abstract

p-Adic and adelic generalization of ordinary quantum cosmology is considered. In [1], we have calculated p-adic wave functions for some minisuperspace cosmological models according to the "no-boundary" Hartle-Hawking proposal. In this article, applying p-adic and adelic quantum mechanics, we show existence of the corresponding vacuum eigenstates. Adelic wave function contains some information on discrete structure of space-time at the Planck scale.

1. Introduction

While measurement data always belong to the field of rational numbers $Q$, by mathematical reasons, standard physical models use the field of real numbers $R \equiv Q_\infty$, and the field of complex numbers $C$. Completion of $Q$ with respect to the absolute value $|\cdot|_\infty$ gives $R$, and its algebraic extension yields $C$. According to the Ostrowski theorem [2] any non-trivial norm on the field of rational numbers $Q$ is equivalent to the usual absolute value $|\cdot|_\infty$ or to the $p$-adic norm $|\cdot|_p$, $p$ = a prime number. $p$-Adic norm is the non-Archimedean (ultrametric) one and for a rational number, $0 \neq x \in Q$, $x = p^n \frac{m}{n}$, $0 \neq n, \gamma, m \in Z$, has the value $|x|_p = p^{-\gamma}$. Completing $Q$ with respect to the $p$-adic norm one gets the field of $p$-adic numbers $Q_p$.

According to quantum gravity there is an uncertainty of measuring distances [2,3]: $\Delta l \geq l_{pl} = \sqrt{\hbar G/c^3}$, where $l_{pl} \sim 10^{-33} \text{cm}$ is the Planck length, $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant, $G$ is Newton's gravitational constant and $c$ is the speed of light. Thus the Planck length is the smallest possible distance which can be in principle measured. For these very small distances the Archimedean axiom of the Euclidean geometry is no more valid. Impossibility to measure the Archimedean distances shorter than the Planck length and possible existence of non-Archimedean spaces at the Planck scale is one of the main motivations for the investigation of $p$-adic quantum models.

After successful application of $p$-adic models to string amplitudes in 1987 [3-5], and formulation of $p$-adic quantum mechanics [6-8], one-dimensional systems with quadratic action were considered: a free particle and a harmonic oscillator [2], a particle in a constant field [9], a harmonic oscillator with time-dependent frequency [10].

In $p$-adic quantum mechanics with complex-valued wave functions, the Schrödinger
equation cannot be written down, because \( x \in \mathbb{Q}_p \), \( \psi(x) \in C \) and derivative \( \frac{d\psi}{dx} \) as well as product \( x\psi(x) \) have no sense. However, finite transformations are meaningful and the corresponding Weyl and evolution operators are \( p \)-adically well defined. Ordinary and \( p \)-adic quantum mechanics are unified within adelic quantum mechanics [11].

The application of \( p \)-adic numbers and adeles in quantum cosmology is of special interest. It should give a new information about structure of space and time, as well as more profound approach to the very early universe.

2. \( p \)-Adic numbers and adeles

\( p \)-Adic number \( x \in \mathbb{Q}_p \), in the canonical form, is an infinite expansion

\[
x = p^\gamma \sum_{i=0}^{\infty} x_ip^i, \quad x_0 \neq 0, \quad 0 \leq x_i \leq p - 1.
\]  

(2.1)

The norm of \( p \)-adic number \( x \) in (2.1) is \( |x|_p = p^{-\gamma} \) and satisfies not only the triangle inequality but also the stronger inequality

\[
|x + y|_p \leq \max(|x|_p, |y|_p).
\]

(2.2)

Metric on \( \mathbb{Q}_p \) is defined by \( d_p(x, y) = |x - y|_p \). This metric is a non-Archimedean one and pair \( (\mathbb{Q}_p, d_p) \) is locally compact, complete, separable and totally disconnected metric space.

Real and \( p \)-adic numbers may be unified in the form of the adeles [12]. An adele is an infinite sequence

\[
a = (a_\infty, a_2, ..., a_p, ...),
\]

(2.3)

where \( a_\infty \in \mathbb{Q}_\infty \), and \( a_p \in \mathbb{Q}_p \), with restriction that \( a_p \in \mathbb{Z}_p \) (\( \mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \} \)) for all but a finite number of \( p \).

Let \( S \) be a finite set of prime numbers and \( \mathcal{A}(S) = \mathbb{Q}_\infty \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \not\in S} \mathbb{Z}_p \). Space of all adeles is then \( \mathcal{A} = \bigcup_{S} \mathcal{A}(S) \) and it is a topological ring. (\( \mathcal{A} \) is a ring with respect to the componentwise addition and multiplication.) A principal adele is a sequence \( (r, r, ..., r, ...) \in \mathcal{A} \), where \( r \in \mathbb{Q} \).

An important function on \( \mathcal{A} \) is the additive character \( \chi(x), x \in \mathcal{A} \), which is continuous and complex valued function with properties

\[
|\chi_v(x_v)|_\infty = 1, \quad \chi_v(x_v + y_v) = \chi_v(x_v)\chi_v(y_v),
\]

(2.4)

\( v \in V = \{ v : v = \infty, 2, ..., p, ... \} \). The additive character over the group \( \mathcal{A} \) is given by

\[
\chi(x) = \prod_v \chi_v(x_v) = \exp(-2\pi i x_\infty) \prod_p \exp(2\pi i \{x_p\}_p),
\]

(2.5)

where \( \{x\}_p \) is the fractional part of a \( p \)-adic number \( x \).
An integral of the Gaussian type over $Q_v$ is

$$
\int_{Q_v} \chi_v(ax^2 + bx)dx = \lambda_v(a)|2a|^{-1/2} \chi_v \left( -\frac{b^2}{4a} \right), \quad a \neq 0,
$$

(2.6)

where $\lambda_v(a) : Q_v \mapsto C$, is the number-theoretical function [2] which has the following properties:

$$
|\lambda_v(a)|_\infty = 1, \quad \lambda_v(0) = 1, \quad \lambda_v(ab^2) = \lambda_v(a),
$$

(2.7)

$$
\lambda_v(a)\lambda_v(b) = \lambda_v(a+b)\lambda_v(a^{-1} + b^{-1}),
$$

(2.8)

for any $v = \infty, 2, ..., p, ...$, and $a \neq 0, b \neq 0$.

$p$-Adic Gauss integral over the region of integration $|x|_p \leq p^{-\nu}$ is

$$
\int_{|x|_p \leq p^{-\nu}} \chi_p(\alpha x^2 + \beta x)dx = \begin{cases} 
\frac{p^{-\nu}\Omega(p^{-\nu}|\beta|_p), |\alpha|_p p^{-2\nu} \leq 1,}{} \\
\lambda_p(a)|2\alpha|^{-1/2} \chi_p \left( -\frac{\alpha^2}{4\alpha} \right) \Omega \left( p^{\nu}\left|\frac{\beta}{2\alpha}\right|_p, |\alpha|_p p^{-2\nu} > 1, \right)
\end{cases}
$$

(2.9)

where $\Omega(u)$ is defined as follows:

$$
\Omega(u) = \begin{cases} 
1, & u \leq 1, \\
0, & u > 1.
\end{cases}
$$

(2.10)

3. Adelic quantum mechanics

Standard quantum mechanics with complex functions of a real variable, $\psi \in L_2(Q_\infty)$, can be generalized to $p$-adic quantum mechanics with complex functions of a $p$-adic argument, $\psi \in L_2(Q_p)$. Both quantum mechanics (real and $p$-adic) can be unified in the form of adelic quantum mechanics with complex functions of adelic variables, $\psi \in L_2(\mathcal{A})$ [11].

For a system which classical, real and $p$-adic, dynamics can be represented in the form

$$
z(t) = \mathcal{T}(t, t_0)z(t_0), \quad z = \left( \begin{array}{c} q \\ k \end{array} \right)
$$

(3.1)

($z = (z_\infty, z_2, ..., z_p, ...)$), adelic quantum mechanics is a triple

$$
(L_2(\mathcal{A}), W(z), U(t)),
$$

(3.2)

where $z$ is a point in classical adelic phase space, $t$ is an adelic time, $L_2(\mathcal{A})$ is the Hilbert space of complex square-integrable functions with respect to the Haar measure on $\mathcal{A}$, $W(z)$ is a unitary representation of Heisenberg-Weyl group on $L_2(\mathcal{A})$ and $U(t)$ is a unitary representation of the evolution operator on $L_2(\mathcal{A})$.

In equation (3.1), $q$ and $k$ are position and momentum, respectively. $\mathcal{T}$ is a matrix evolution, which satisfies $\mathcal{T}(t_2, t_1)\mathcal{T}(t_1, t_0) = \mathcal{T}(t_2, t_0)$ and $B(\mathcal{T}(t, t_0)z, \mathcal{T}(t, t_0)z') = B(z, z')$, where $B(z, z') = -kq' + qk'$ is symplectic bilinear form on the adelic phase space.
The operator $W_\nu(z)$ acts on the wave function $\psi_\nu(x) \in L_2(Q_\nu)$ in the following way:

$$W_\nu(z)\psi_\nu(x) = \chi_\nu\left(\frac{kq}{2} + kx\right)\psi_\nu(x + q).$$

(3.3)

An evolution operator $U(t)$ is defined by

$$U(t)\psi(x) = \int_A K_t(x, y)\psi(y)dy = \prod_{\nu} \int_{Q_\nu} K_{t_\nu}^{(\nu)}(x_\nu, y_\nu)\psi^{(\nu)}(y_\nu)dy_\nu$$

(3.4)

and it describes dynamics in adelic quantum mechanics. The operator $U(t)$ and its kernel $K_{t_\nu}(x_\nu, y_\nu)$ satisfy product relations:

$$U(t + t') = U(t)U(t'),$$

$$K_{t+t'}^{(\nu)}(x, y) = \int_{Q_\nu} K_t^{(\nu)}(x, z)K_{t'}^{(\nu)}(z, y)dz.$$  

(3.5)

By analogy with standard quantum mechanics, kernel of adelic evolution operator is given by product of Feynman’s path integrals

$$K_t^{(\nu)}(x, y) = \int \chi_\nu\left(-\frac{t}{0} L_\nu(q, \dot{q}, t)dt\right)Dq(t)$$

(3.6)

(we use $h = 1$ for the Planck constant), where integration is performed over classical real and $p$-adic trajectories with the boundary conditions $q(0) = y$, $q(t) = x$.

The eigenvalue problem for $U(t)$ reads

$$U(t)\psi_{\alpha\beta}(x) = \chi(E_\alpha t)\psi_{\alpha\beta}(x),$$

(3.7)

where $\psi_{\alpha\beta}(x)$ are adelic wave eigenfunctions, $E = (E_\infty, E_2, \ldots, E_p, \ldots)$ is the corresponding adelic energy, $\alpha = (\alpha_\infty, \alpha_2, \ldots, \alpha_p, \ldots)$ and $\beta = (\beta_\infty, \beta_2, \ldots, \beta_p, \ldots)$ are indices for energy levels and their degeneration, respectively.

The eigenstates for adelic evolution operator (3.7) are infinite products of eigenfunctions from the corresponding real and $p$-adic counterparts of a quantum model. One has to point out that any adelic eigenfunction contains only finitely many $p$-adic eigenfunctions which are different from the vacuum state $\Omega(|x|_p)$ defined by (2.10).

4. Adelic quantum cosmology

Adelic quantum cosmology is an application of adelic quantum theory to the universe as a whole. In the path integral approach to standard quantum cosmology starting point is Feynman’s idea that the amplitude to go from one state with intrinsic metric $h_{ij}$, and matter configuration $\phi$ on an initial hypersurface $\Sigma$, to another state with metric
and matter configuration \( \phi' \) on a final hypersurface \( \Sigma' \), is given by a functional integral of \( \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi]) \) over all four-geometries \( g_{\mu\nu} \), and matter configurations \( \Phi \), which interpolate between the initial and final configurations \([13]\), i.e.

\[
\langle h_{ij}', \phi', \Sigma'|h_{ij}, \phi, \Sigma \rangle_\infty = \int \mathcal{D}(g_{\mu\nu})_\infty \mathcal{D}(\Phi)_\infty \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi]). \tag{4.1}
\]

The \( S_\infty[g_{\mu\nu}, \Phi] \) is the usual Einstein-Hilbert action

\[
S[\mu\nu, \Phi] = \frac{1}{16\pi G} \left( \int_M d^4x \sqrt{-g} (R - 2\Lambda) + 2 \int_{\partial M} d^3x \sqrt{h} K \right) - \frac{1}{2} \int_M d^4x \sqrt{-g} [g^\mu\nu \partial_\mu \Phi \partial_\nu \Phi + V(\Phi)] \tag{4.2}
\]

for the gravitational field and matter fields \( \Phi \). In (4.2), \( R \) is scalar curvature of four-manifold \( M \), \( \Lambda \) is cosmological constant, \( K \) is trace of the extrinsic curvature \( K_{ij} \) at the boundary \( \partial M \) of the manifold \( M \).

To perform \( p \)-adic and adelic generalization we first make \( p \)-adic counterpart of the action (4.2) using form-invariance under change of real to the \( p \)-adic number fields \([2,14]\).

Then we generalize (4.1) and introduce \( p \)-adic complex-valued cosmological amplitude

\[
\langle h_{ij}', \phi', \Sigma'|h_{ij}, \phi, \Sigma \rangle_p = \int \mathcal{D}(g_{\mu\nu})_p \mathcal{D}(\Phi)_p \chi_p(-S_p[g_{\mu\nu}, \Phi]). \tag{4.3}
\]

The space of all 3-metrics and matter field configurations \((h_{ij}(\vec{x}), \phi(\vec{x}))\) on a 3-surface is called superspace (this is the configuration space in quantum cosmology). Superspace is the infinite dimensional one with a finite number of coordinates \((h_{ij}(\vec{x}), \phi(\vec{x}))\) at each point \( \vec{x} \) of the 3-surface. In practice the work with the infinite dimensions is not possible. One useful approximation therefore is to truncate the infinite degrees of freedom to a finite number, thereby obtaining some particular minisuperspace model. Usually, one restricts the four-metric to be of the form \( ds^2 = -N^2(t)dt^2 + h_{ij}dx^idx^j \), where \( N(t) \) is the laps function. Three-metric \( h_{ij} \) and matter fields are restricted in such a way that they depend on a finite number of functions of \( t, q^\alpha(t) \), where \( \alpha = 1, 2, ..., n \). For such minisuperspaces, functional integrals (4.1) and (4.3) are reduced to functional integration over three-metrics, matter configurations and to one usual integral over the laps function. If one takes boundary condition \( q^\alpha(t_2) = q_2^\alpha, q^\alpha(t_1) = q_1^\alpha \) then integral in (4.1) and (4.3), in the gauge \( \dot{N} = 0 \), is a minisuperspace propagator. In this case it holds

\[
\langle q_2^\alpha|q_1^\alpha \rangle_\nu = \int dN \mathcal{K}_\nu(q_2^\alpha, N|q_1^\alpha, 0), \tag{4.4}
\]

where

\[
\mathcal{K}_\nu(q_2^\alpha, N|q_1^\alpha, 0) = \int \mathcal{D}q^\alpha \chi_\nu(-S_\nu[q^\alpha]) \tag{4.5}
\]
is an ordinary quantum-mechanical propagator between fixed $q^\alpha$ in fixed time $N$.

For one dimensional quantum systems $p$-adic path integral is investigated in [15], where it is shown that for quadratic classical action $S^{cl}_p(q_2, N|q_1, 0)$, (4.5) becomes

$$\mathcal{K}_p(q_2, N|q_1, 0) = \lambda_p \left( -\frac{\partial^2 S^{cl}_p}{2\partial q_2 \partial q_1} \right) \left| \frac{\partial^2 S^{cl}_p}{\partial q_2 \partial q_1} \right|^{1/2}_p \chi_p(-S^{cl}_p(q_2, N|q_1, 0)). \quad (4.6)$$

If system has $n$ decoupled degrees of freedom, its $p$-adic kernel is a product

$$\mathcal{K}_p(q_2, N|q_1, 0) = \prod_{\alpha=1}^n \lambda_p \left( -\frac{\partial^2 S^{cl}_p}{2\partial q_2^\alpha \partial q_1^\alpha} \right) \left| \frac{\partial^2 S^{cl}_p}{\partial q_2^\alpha \partial q_1^\alpha} \right|^{1/2}_p \chi_p(-S^{cl}_p(q_2^\alpha, N|q_1^\alpha, 0)). \quad (4.7)$$

Expressions (4.6) and (4.7) have the same form as in ordinary quantum mechanics [15].

$p$-Adic and adelic wave functions of the universe may be found by means of the equation (3.7). The corresponding adelic eigenstates have the form

$$\Psi(q^\alpha) = \psi_\infty(q_\infty^\alpha) \prod_{p \in S} \psi_p(q_p^\alpha) \prod_{p \notin S} \Omega(|q_p^\alpha|_p). \quad (4.8)$$

A necessary condition to construct an adelic model is existence of the $p$-adic (vacuum) state $\Omega(|q^\alpha|_p)$, which satisfies

$$\int \mathcal{K}_p(q_2^\alpha, N|q_1^\alpha, 0) dq_1^\alpha = \Omega(|q_0^\alpha|_p) \quad (4.9)$$

for all but a finite number of $p$.

5. Some $p$-adic and adelic minisuperspace models

5.1. $p$-Adic and adelic model with cosmological constant in $D = 3$ dimensions

This model in the real case is considered in the paper [16]. Its metric is

$$ds^2 = \sigma^2 \left( -N^2(t)dt^2 + a^2(t)(d\theta^2 + \sin^2 \theta d\varphi^2) \right), \quad (5.1)$$

where $\sigma = G$. The corresponding $\nu$-adic action is

$$S_\nu[a] = \frac{1}{2} \int_0^1 dt N a^2(t) \left( -\frac{\dot{a}^2}{N^2 a^2} + \frac{1}{a^2} - \lambda \right), \quad (5.2)$$

where $\lambda = \Lambda \sigma^2$. The Euler-Lagrange equation of motion

$$\ddot{a} - N^2 a \lambda = 0$$
has the solution
\[ a(t) = \frac{1}{2 \sinh(N\sqrt{\lambda})} \left( (a_2 - a_1 e^{-N\sqrt{\lambda} t}) e^{N\sqrt{\lambda} t} + (a_1 e^{N\sqrt{\lambda} t} - a_2) e^{-N\sqrt{\lambda} t} \right), \]
(5.3)
where the boundary conditions are: \( a(0) = a_1, a(1) = a_2 \). For the classical action it gives
\[ S_{cl}^\nu(a_2, N|a_1, 0) = \frac{1}{2\sqrt{\lambda}} \left[ N\sqrt{\lambda} + \lambda \left( \frac{2a_1a_2}{\sinh(N\sqrt{\lambda})} - \frac{a_1^2 + a_2^2}{\tanh(N\sqrt{\lambda})} \right) \right]. \]
(5.4)
Quantum-mechanical propagator has the form
\[ \mathcal{K}_\nu(a_2, N|a_1, 0) = \lambda_\nu \left( -\frac{\sqrt{\lambda}}{2\sinh(N\sqrt{\lambda})} \right) \left| \frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} \right|^{1/2} \chi_\nu \left( -S_{cl}^\nu(a_2, N|a_1, 0) \right). \]
(5.5)
The equation (4.9), in a more explicit form, reads
\[ \lambda_p \left( -\frac{\sqrt{\lambda}}{2\sinh(N\sqrt{\lambda})} \right) \left| \frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} \right|^{1/2} \chi_p \left( -\frac{N}{2} + \frac{\sqrt{\lambda}}{2\tanh(N\sqrt{\lambda})} a_2^2 \right) \]
\[ \times \int_{|a_1|p \leq 1} \chi_p \left( -\frac{\sqrt{\lambda}}{2\tanh(N\sqrt{\lambda})} a_1^2 - \frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} a_2 a_1 \right) da_1 = \Omega(|a_2|p). \]
(5.6)
Using lower part of the Gauss integral (2.9) for \( \nu = 0 \), we obtain
\[ \Omega(|a_2|p) = \chi_p \left( -\frac{N}{2} + \frac{\sqrt{\lambda}}{2\tanh(N\sqrt{\lambda})} a_2^2 \right) \Omega(|a_2|p) \]
(5.7)
with condition \( |\frac{\sqrt{\lambda}}{2\tanh(N\sqrt{\lambda})}|_p > 1 \), i.e. \( |N|_p < 1 \). For \( p \neq 2 \), l.h.s. is equal to \( \Omega(|a_2|p) \) if \( |\lambda|_p \leq 1 \) holds. Applying also the upper part of (2.9) to (5.6), we have
\[ \lambda_p \left( -\frac{\sqrt{\lambda}}{2\sinh(N\sqrt{\lambda})} \right) \left| \frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} \right|^{1/2} \]
\[ \times \chi_p \left( -\frac{N}{2} + \frac{\sqrt{\lambda}a_2^2}{2\coth(N\sqrt{\lambda})} \right) \Omega \left( \left| \frac{\sqrt{\lambda}a_2}{\sinh(N\sqrt{\lambda})} \right|_p \right) = \Omega(|a_2|p). \]
(5.8)
It becomes an equality if condition \( |N|_p \leq 1 \) takes place. Finally, if we take into account convergence domain of hyperbolic functions, i.e. \( |x|_p \leq \frac{1}{p}, p \neq 2, |x|_p \leq \frac{1}{2^p} \), we obtain
\( p \)-adic vacuum eigenstates
\[ \psi_p(a, N) = \begin{cases} \Omega(|a|_p), & |N|_p \leq 1, \quad |\lambda|_p^{1/2} \leq 1/p, \\ \Omega(|a|_2), & |N|_2 \leq \frac{1}{4}, \quad |\lambda|_2^{1/2} \leq 1. \end{cases} \]
(5.9)
In the form containing $\Omega$-function we also have eigenstates

$$\psi_p(a, N) = \begin{cases} 
\Omega(p'|a|p), & |N|_p \leq p^{-2\nu}, |\lambda|_p^{1/2} \leq p^{2\nu-1}, \\
\Omega(2\nu'|a|2), & |N|_2 \leq 2^{-2-2\nu}, |\lambda|_2^{1/2} \leq 2^{2\nu},
\end{cases} \quad (5.10)$$

where $\nu = 1, 2, \ldots$.

### 5.2. $p$-Adic and adelic de Sitter model

The de Sitter minisuperspace model in quantum cosmology is the simplest, nontrivial and exactly soluble model. This model is given by the Einstein-Hilbert action with cosmological term (4.2) without matter fields, and by the Robertson-Walker metric

$$ds^2 = \sigma^2(-N^2(t)dt^2 + a^2(t)d\Omega_3^2), \quad (5.11)$$

where $\sigma^2 = \frac{2G}{3\pi}$ and $a(t)$ is the scale factor. Instead of (5.11) we shall use

$$ds^2 = \sigma^2 \left( -\frac{N^2(t)}{q(t)}dt^2 + q(t)d\Omega_3^2 \right), \quad (5.12)$$

which was considered in the real case in [17] and leads to quadratic action. The corresponding $\nu$-adic action for this one-dimensional minisuperspace model is

$$S_{\nu}[q] = \frac{1}{2} \int_{t_1}^{t_2} dt N \left( -\frac{\dot{q}^2}{4N^2} - \lambda q + 1 \right), \quad (5.13)$$

where $\lambda = \frac{\Lambda\sigma^2}{3}$. The definite $p$-adic integrals of the form (5.13) are considered in [18] by means of antiderivatives of analytic functions, and they formally have the same structure like those in the real case. The classical equation of motion ($N = 1$)

$$\ddot{q} = 2\lambda$$

with the boundary conditions $q(0) = q_1$ and $q(T) = q_2$, ($T = t_2 - t_1$) gives

$$q(t) = \lambda t^2 + \left( \frac{q_2 - q_1}{T} - \lambda T \right)t + q_1. \quad (5.14)$$

After substitution (5.14) into (5.13) and integration, one obtains that the classical action is

$$S_{\nu}^{\text{cl}}(q_2, T|q_1, 0) = \frac{\lambda^2 T^3}{24} - \left[ \lambda(q_1 + q_2) - 2 \right] \frac{T}{4} - \frac{(q_2 - q_1)^2}{8T}. \quad (5.15)$$

Since (5.15) is quadratic in $q_2$ and $q_1$, quantum-mechanical propagator has the form

$$\mathcal{K}_\nu(q_2, T|q_1, 0) = \frac{\lambda\nu(-8T)^{1/2}}{|4T|^{1/2}_\nu} \chi^\nu(-S_{\nu}^{\text{cl}}(q_2, T|q_1, 0)). \quad (5.16)$$

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The equation (4.9) reads
\[
\lambda_p(-8T) |4T|^{1/2}_p \chi_p \left( -\frac{\lambda^2 T^3}{24} - \frac{T}{2} + \lambda q_2 T + \frac{q_2^2}{8T} \right) \times \int_{|q_1|_p \leq 1} \chi_p \left( \frac{q_1^2}{8T} + \left( \frac{\lambda T}{4} - \frac{q_2}{4T} \right) q_1 \right) dq_1 = \Omega(|q_2|_p). \tag{5.17}
\]

For the eigenstates we have
\[
\psi_p(q, T) = \begin{cases} 
\Omega(|q|_p), & |T|_p \leq 1, |\lambda|_p \leq 1, p \neq 2, \\
\Omega(|q|_2), & |T|_2 \leq \frac{1}{2}, |\lambda|_2 \leq 1.
\end{cases} \tag{5.18}
\]

and
\[
\psi_p(q, T) = \begin{cases} 
\Omega(p^\nu|q|_p), & |T|_p \leq p^{-2\nu}, |\lambda|_p \leq p^{3\nu}, p \neq 2, \\
\Omega(2^\nu|q|_2), & |T|_2 \leq 2^{-2\nu}, |\lambda|_2 \leq 2^{1+3\nu},
\end{cases} \tag{5.19}
\]

where \( \nu = 1, 2, \ldots \).

5.3 p-Adic and adelic model with a homogeneous scalar field

This model over the field of real numbers was considered in the paper [19]. Metric is
\[
ds^2 = \sigma^2 \left( -N^2(t) \frac{dt^2}{a^2(t)} + a^2(t)d\Omega^2_3 \right), \tag{5.20}
\]

and the corresponding real and \( p \)-adic action for the gravitational and homogeneous scalar field is
\[
S_v[a, \phi] = \frac{1}{2} \int_0^1 dt N \left[ -\frac{a^2 \dot{x}^2}{N^2} + \frac{a^4 \dot{\phi}^2}{N^2} - a^2 V(\phi) + 1 \right], \tag{5.21}
\]

\( \phi = \left( \frac{4\pi G}{3} \right)^{1/2} \Phi \). The change of variables \( x = a^2 \cosh(2\phi) \) and \( y = a^2 \sinh(2\phi) \) in class of models with the scalar field potential \( V(\phi) = \alpha \cosh(2\phi) + \beta \sinh(2\phi) \), where \( \alpha \) and \( \beta \) are arbitrary parameters, leads to the action
\[
S_v[x, y] = \frac{1}{2} \int_0^1 dt N \left[ \frac{1}{4N^2}(-\dot{x}^2 + \dot{y}^2) - \alpha x - \beta y + 1 \right]. \tag{5.22}
\]

Varying this action with respect to \( x \) and \( y \) for the boundary conditions \( x(0) = x_1, y(0) = y_1, x(1) = x_2, y(1) = y_2 \), gives equations of motion with the solutions
\[
x(t) = \alpha N^2 t^2 + (x_2 - x_1 - \alpha N^2) t + x_1,
\]
\[
y(t) = -\beta N^2 t^2 + (y_2 - y_1 + \beta N^2) t + y_1. \tag{5.23}
\]
The classical action for the solutions (5.23) is given by

\[ S_{cl}^{\nu}(x_2, y_2, N|x_1, y_1, 0) = \frac{\alpha^2 - \beta^2}{24} N^3 + \frac{1}{4} (2 - \alpha (x_1 + x_2) - \beta (y_1 + y_2)) N + \frac{-(x_2 - x_1)^2 + (y_2 - y_1)^2}{8N}, \]  

and applying the formula (4.7) we get the corresponding propagator

\[ \mathcal{K}_\nu(x_2, y_2, N|x_1, y_1, 0) = \frac{1}{|4N\nu|} \chi_\nu(-S_{cl}^{\nu}(x_2, y_2, N|x_1, y_1, 0)). \]  

Vacuum state for this two-dimensional minisuperspace model has the form \( \Omega(|x|_p)\Omega(|y|_p) \), and equation (4.9) reads

\[ \int_{|x|_p \leq 1} \int_{|y|_p \leq 1} \mathcal{K}_{p}(x_2, y_2, N|x_1, y_1, 0) dx_1 dy_1 = \Omega(|x|_p)\Omega(|y|_p). \]  

We obtain eigenstates

\[ \psi_p(x, y, N) = \begin{cases} 
\Omega(|x|_p)\Omega(|y|_p), |N|_p \leq 1, |\alpha|_p \leq 1, |\beta|_p \leq 1, p \neq 2, \\
\Omega(|x|_2)\Omega(|y|_2), |N|_2 \leq \frac{1}{2}, |\alpha|_2 \leq 2, |\beta|_2 \leq 2. 
\end{cases} \]  

For the states of the form \( \Omega(p^\nu|x|_p)\Omega(p^\mu|y|_p) \) relevant integral is

\[ \int_{|x|_p \leq p^-\nu} \int_{|y|_p \leq p^-\mu} \mathcal{K}_{p}(x_2, y_2, N|x_1, y_1, 0) dx_1 dy_1 = \Omega(p^\nu|x|_p)\Omega(p^\mu|y|_p). \]  

The corresponding \( p \)-adic eigenfunctions are:

\[ \psi_p(x, y, N) = \begin{cases} 
\Omega(p^\nu|x|_p)\Omega(p^\mu|y|_p), |N|_p \leq p^{-2\nu}, |N|_p \leq p^{-2\mu}, |\alpha|_p \leq p^{3\nu}, |\beta|_p \leq p^{3\mu} \\
\Omega(2^\nu|x|_2)\Omega(2^\mu|y|_2), |N|_2 \leq 2^{-2\nu}, |N|_2 \leq 2^{-2\mu}, |\alpha|_2 \leq 2^{3\nu}, |\beta|_2 \leq 2^{3\mu} 
\end{cases} \]  

where \( \nu, \mu = 1, 2, 3, \ldots \)  

5.4 \( p \)-Adic and adelic Bianchi I (k=0) model (with three scaling factors)

We also apply the above formalism to the three-dimensional anisotropic minisuperspace model, investigated in the real case in [20]. The metric

\[ ds^2 = a^2 \left[ \frac{N^2(t)}{a^2(t)} dt^2 + a^2(t) dx^2 + b^2(t) dy^2 + c^2(t) dz^2 \right] \]  

leads to the action

\[ S_{v}[a, b, c] = \frac{1}{2} \int_0^1 dt \left[ -\frac{a}{N} (\dot{a}bc + ab\dot{c} + \dot{a}bc) - Nbc\lambda \right] \]
which by the substitution

\[ x = \frac{bc + a^2}{2}, \quad y = \frac{bc - a^2}{2}, \quad z^2 = a^2b^c \]

(5.33)
gives classical action and propagator in the form

\[ S_{v}^{cl}(x_2, y_2, z_2, N|x_1, y_1, z_1, 0) = \]

\[-\frac{1}{4N} [(x_2 - x_1)^2 - (y_2 - y_1)^2 + 2(z_2 - z_1)^2] - \frac{\lambda N}{4} [(x_1 + x_2) + (y_1 + y_2)]. \]

(5.34)

\[ K_{v}(x_2, y_2, z_2, N|x_1, y_1, z_1, 0) = \frac{\lambda_v(-2N)}{|4^{1/3}N|^{3/2}} \times \chi_v \left(-S_{v}^{cl}(x_2, y_2, z_2, N|x_1, y_1, z_1, 0)\right). \]

(5.35)

By an analogous way to the previous models we get \( p \)-adic eigenstates

\[ \psi_p(x, y, z, N) = \begin{cases} \Omega(|x|_p)\Omega(|y|_p)\Omega(|z|_p), & |N|_p \leq 1, \quad |\lambda|_p \leq 1, \\ \Omega(|x|_2)\Omega(|y|_2)\Omega(|z|_2), & |N|_2 \leq \frac{1}{2}, \quad |\lambda|_2 \leq 2. \end{cases} \]

(5.36)

and

\[ \psi_p(x, y, z, N) = \begin{cases} \Omega(p^{\nu_1}|x|_p)\Omega(p^{\nu_2}|y|_p)\Omega(p^{\nu_3}|z|_p), & |N|_p \leq p^{-2\nu_1}, \quad |\lambda|_p \leq p^{3\nu_1}, p^{3\nu_2}, \\ \Omega(2^{\nu_1}|x|_2)\Omega(2^{\nu_2}|y|_2)\Omega(2^{\nu_3}|z|_2), & |N|_2 \leq 2^{-2\nu_1,2^{-1}}, 2^{-2\nu_2,2^{-2}}, |\lambda|_2 \leq 2^{3\nu_1,2+1} \end{cases} \]

(5.37)

for all \( \nu_i \in \mathbb{Z} \).

6. Conclusion

It is shown that there exist \( p \)-adic and adelic counterparts of all the above four minisuperspace cosmological models. Their adelic eigenfunctions have the form (4.8). In particular, there is a place eigenstate of the form

\[ \Psi(q^1, \ldots, q^n) = \prod_{a=1}^{n} \psi_\infty(q^a_\infty) \prod_{p} \prod_{a=1}^{n} \Omega(|q^a_p|_p), \]

(6.1)

where \( \psi_\infty(q^a_\infty) \) is the wave function of the universe in the standard quantum cosmology. To interpret \( \Psi(q^1, \ldots, q^n) \) consider \( |\Psi(q^1, \ldots, q^n)|^2_\infty \) in rational points \( q^1, \ldots, q^n \), that is

\[ |\Psi(q^1, \ldots, q^n)|^2_\infty = \prod_{a=1}^{n} |\psi_\infty(q^a_\infty)|^2_\infty \prod_{p} \prod_{a=1}^{n} \Omega(|q^a_p|_p). \]

(6.2)

Due to the properties of \( \Omega(|q^a_p|_p) \), we have

\[ |\Psi(q^1, \ldots, q^n)|^2_\infty = \begin{cases} \prod_{a=1}^{n} |\psi_\infty(q^a_\infty)|^2_\infty, & q^1, \ldots, q^n \in \mathbb{Z}, \\ 0, & q^a \in \mathbb{Q}\setminus\mathbb{Z}. \end{cases} \]

(6.3)
According to the usual interpretation of a wave function, from (6.3) follows discreteness of quantities $q^\alpha$ ($\alpha = 1, ..., n$) at the natural scale $h = c = G = 1$. This kind of discreteness is a $p$-adic effect and depends on adelic quantum state.

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