One-Dimensional Flows in the Quantum Hall System

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Abstract

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We construct the $c$ function whose gradient determines the RG flow of the conductivities ($\sigma_{xy}$ and $\sigma_{xx}$) for a quantum Hall system, subject to two assumptions. (1) We take the flow to be invariant with respect to the infinite discrete symmetry group, $\Gamma_H$, recently proposed by several workers to explain the ‘superuniversality’ of the delocalization exponents in these systems. (2) We also suppose the flow to be ‘quasi-holomorphic’ (which we make precise) in the sense that it is as close as possible to a one-dimensional flow in the complex parameter $\sigma_{xy} + i\sigma_{xx}$. These assumptions together with the known asymptotic behaviour for large $\sigma_{xx}$, completely determine the $c$ function, and so the phase diagram, for these systems. A complete description of the RG flow also requires a metric in addition to the $c$ function, and we identify the features which are required for this by the RG. A similar construction produces the $c$ function for other systems enjoying an infinite discrete symmetry, such as for supersymmetric QED.
In this article we present a conjecture concerning the exact beta function describing the renormalization group (RG) flow of the conductivities for quantum Hall systems. We do so by deriving the most general possible c function which is consistent with three properties which we assume for this flow. (The RG beta function may be obtained from the c function by taking its gradient.) The properties which we use are: (i) the flow commutes with an infinite discrete symmetry group, $\Gamma_H$; (ii) it has a beta function with no singularities and which is consistent with the large-$\sigma_{xx}$ form as predicted by the weak-coupling expansion of the effective sigma-model description of the RG \cite{1}; and (iii) the flow is ‘quasi-holomorphic’ in the complex conductivity, $z$, defined by: $\sigma_{xy} + i\sigma_{xx} = \left(\frac{\epsilon}{\pi}\right) z$. These assumptions are sufficiently strong to completely determine the c function, and so permit the identification of the phase diagram in both the strong-coupling (small $\sigma_{xx}$) and weak-coupling (large $\sigma_{xx}$) regimes.

1. The Assumptions

Before describing our results, some justification for the three requirements are in order. We deal with each in turn.

- *(1) The Symmetry:* Convincing experimental evidence and theoretical arguments support the existence of an infinite discrete symmetry, $\Gamma_H$, of the parameter space in the quantum Hall system. There are three main lines of evidence.

  First, the delocalization exponent which describes the transition between two quantum Hall phases, as found both in real scaling experiments \cite{2} and numerical simulations \cite{3}, exhibits a “superuniversality” in the sense that it does not depend on the which phases are involved. This otherwise mysterious equivalence is easily understood as a consequence of a discrete symmetry, $\Gamma_H$, mapping the critical points into one other.\footnote{See Appendix A for this argument in detail.}

  Furthermore, the positions of these critical points are consistent with what would be expected if $\Gamma_H$ were a large subgroup of the modular group, $\Gamma_H \subseteq SL(2,\mathbb{Z})$, where the modular group is defined to act on the complex conductivity, $z$, by holomorphic fractional
linear transformations [4]:

\[ z \to \gamma(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}. \]  

(1)

In fact, the very existence of the quantum Hall plateaux at fractional values of \( \sigma_{xy} \) is an inevitable and striking consequence of such a symmetry if \( \Gamma_H \) is a sufficiently large subgroup of the modular group.

Secondly, such a symmetry is also very plausible on theoretical grounds. On the Hall plateaux the long wavelength electromagnetic response is described by a Chern-Simons action, \( \mathcal{L}_{CS} = -\frac{1}{2} \sigma_{xy} \epsilon^{\mu\nu\lambda} A_{\mu} \partial_\nu A_\lambda + ej^{\mu} A_{\mu} \). But general arguments [5] imply that such an action is invariant under the replacement \( \sigma_{xy}^{-1} \to \sigma_{xy}^{-1} + 2n \frac{e}{h} \), where \( n \) is any integer. Similar arguments as applied to the dual action [6] imply a symmetry with respect to shifts of \( \sigma_{xy} \) rather than its inverse. Furthermore, a convincing picture of the full symmetry group also emerges from more microscopic considerations, such as expressed by the “Law of Corresponding States” as formulated within the context of the Chern-Simons Landau-Ginzburg (CSLG) theory of the quantum Hall effect [7], [8].

Finally, a remarkable recent experiment [9] has shown that quantum Hall systems in different phases, even far from the transition point, have nonlinear \( I - V \) response curves which are dual to one another in a well-defined way that is consistent with the particle-vortex duality that one expects from the CSLG picture. Remarkably, the persistence of duality far from the transition point seems to indicate that its action is not destroyed by renormalization effects.

All of these considerations point to the symmetry group being the subgroup \( \Gamma_H = \Gamma_T(2) \subset SL(2, \mathbb{Z}) \), which is generated by the two transformations [4]:

\[ T(z) = z + 1 \quad \text{and} \quad R(z) = \frac{z}{1 - 2z}. \]  

(2)

• (2) The Large-\( \sigma_{xx} \) Limit: The transport properties for quantum Hall systems have been argued to be described in the far infrared by a two-dimensional sigma model. The beta

\[ ^2 \text{Duality and discrete symmetries in condensed matter systems have recently been treated from a different point of view in ref. [10].} \]
function for the complex conductivity, \( z \), may be evaluated in this model for large \( y = \text{Im} \ z \) using semiclassical methods. In perturbation theory the result is independent of \( x = \text{Re} \ z \) and is given by [1]:

\[
\left( \frac{dz}{dt} \right)_{\text{p.t.}} \equiv \beta^z(x, y)_{\text{p.t.}} = -\frac{i}{2\pi^2 y} + O(\frac{1}{y^2}).
\] (3)

The real part, \( x \), first enters the evolution equations in the large-\( y \) limit through non-perturbative instanton contributions, which contribute to \( \beta^z \) a series in the quantity \( \overline{q} = e^{-2\pi(y+ix)} \), with each term of this series premultiplied by its own series in \( 1/y \). We therefore demand a similar large-\( y \) limit for \( \beta^z \) — i.e. a Taylor series in \( q \) and \( \overline{q} \) whose coefficients might themselves be a series in \( 1/y \). For later purposes notice that once the perturbative series in \( 1/y \) is put aside, the instanton contributions to \( \frac{dz}{dt} \) depend on the anti-holomorphic quantity \( \overline{z} = x - iy \).

- (3) Quasi-Holomorphy: If \( \beta^z \) were a holomorphic function — that is, independent of \( \overline{z} \) — then powerful results of complex analysis could be used to rule out the existence of a beta function satisfying the previous two assumptions. Even if this were not so, the asymptotic expansion in powers of \( 1/y \) explicitly breaks holomorphy and so excludes a holomorphic \( \beta^z \). The assumption of ‘quasi-holomorphy’ is meant to capture the predictive nature of holomorphy in a way which is consistent with the asymptotic form of the sigma model at large \( y \). We postpone its detailed definition until further tools are introduced, and simply motivate it here as a very predictive ansatz which is motivated by the asymptotic expression, eq. (3). It is also suggested by the experimental observation [9] of duality far from the critical points, since these seem to indicate that the holomorphic group action of eq. (1) is not corrupted by renormalization effects. It is hard to imagine how this would be possible if the complex conductivities \( z \) and \( \overline{z} \) were to mix under rescalings of the system.

In any case, quasi-holomorphy incorporates the physical requirements of the system in the simplest possible way. If its predictions are borne out, then we will acquire an important constraint on microscopic descriptions of these systems.

We next turn to the consequences of these assumptions, including a precise definition of quasiholomorphy. As shall become clear, the resulting conditions on the RG flow are
very restrictive, and at present we have only been able to solve them for the \( c \) function, and not also for the delocalization exponents, \( \nu \), at the critical points.

2. The Consequences

We are therefore led to investigate the properties of nonsingular beta functions whose flows are compatible with a sub-modular group \( \Gamma \).\(^3\) The basic requirement is that the action of the symmetry, \( z \rightarrow \gamma(z) \), should commute with the RG flow, \( z \rightarrow r_t(z, \overline{z}) \). That is (see Appendix A for details): \( \gamma \circ r_t = r_t \circ \gamma \). For infinitesimal flows this implies the beta function, \( \beta^z = \frac{dz}{dt} \), must satisfy:

\[
\beta^z(\gamma(z), \overline{\gamma(z)}) = \frac{d\gamma}{dz} \beta^z(z, \overline{z}) = (cz + d)^w \beta^z(z, \overline{z}),
\]

with \( w = -2 \). For \( \beta^z \) holomorphic, the second equality in eq. (4) defines an automorphic function of weight \( w \). It is a theorem \([12]\) that no such function having negative weight exists which is analytic throughout the upper half plane,\(^4\) and which admits an expansion in integer powers of \( q \) as \( z \rightarrow i\infty \). No such candidate beta function can therefore exist.

We must therefore relax the condition that \( \beta^z \) be holomorphic. To this end imagine using a metric\(^5\) \( G_{ij} \) to convert the contravariant quantity, \( \beta^i \), into a covariant vector:

\[
\beta_z = G_{zz} \beta^z + G_{z\overline{z}} \beta^{\overline{z}}.
\]

Since \( G_{ij} \) is typically not holomorphic it is clear that we cannot demand that both \( \beta^z \) and \( \beta_z \) be holomorphic, or almost so.

We may now more precisely state the condition of quasi-holomorphy. We assume the \textit{covariant} quantity, \( \beta_z \), to be a holomorphic function of \( z \) (and not \( \overline{z} \)), apart from the addition of a possible series in \( 1/y \). We also take it to be nonsingular throughout the upper

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\(^3\) Ref. [11] contains similar applications of finite discrete groups to constraining beta functions.

\(^4\) Recall \( \text{Im } z \propto \sigma_{xx} \geq 0 \).

\(^5\) Although we do not use any particular metric in what follows, a natural metric for these purposes is the Zamolodchikov metric, which is defined in terms of the correlations of the relevant operators of the sigma model \([13]\). Since the sigma model becomes weakly coupled in the large-\( y \) regime, this metric might be expected to approach the flat Cartesian metric \( ds^2 = dz d\overline{z} \) in this limit.
half complex plane. There are three motivations for making this ansatz for $\beta_z$ rather than for $\beta^\zbar$. First, recall that sigma-model calculations produce instanton corrections to $\beta^\zbar$ which depend on $\zbar$ instead of $z$. This might be expected if $\beta^\zbar$ were obtained as $G^{z\zbar}\beta^\zbar$ with $\beta^\zbar$ antiholomorphic. Second, it is believed [14], [13], [15] that all RG flows in two (and possibly higher) dimensions are gradient flows, in that $\beta_i = \partial_i \Phi$ for a potential function, $\Phi$. In this sense it is $\beta_i$ which is more fundamental, and so which may be expected to have simpler properties. Third, the function $\beta_z$ transforms as an automorphic function of weight $w = +2$ with respect to $\Gamma_T(2)$ transformations, and it is a theorem [12] that this is the lowest possible weight for which such a holomorphic function exists.

2.1) The Conditions Governing $\beta_i$

The three conditions are now sufficiently precise to be used to constrain the possible form for the beta function. For brevity’s sake, we simply quote our main results here. $\beta_z$ may be conveniently constructed in terms of the generalized Eisenstein series:

$$E_k^{(p,q)}(z) \equiv N_k \sum_{mn} (mz + n)^{-k},$$

where the sum over $m$ and $n$ is over all pairs of integers $(m, n)$ — the prime indicating the omission of $(0, 0)$ — for which $(m, n) = (p, q) \pmod{2}$. The normalization constant, $N_k$, is chosen to ensure $E_k^{(p,q)} \to 1$ as $y \to \infty$, and is given in terms of the Riemann zeta function, $\zeta_\calr(k)$, by $N_k^{-1} = 2\zeta_\calr(k) (1 - 2^{-k})$. The series converges for $k \geq 2$, and is analytic throughout the upper half plane. For $k > 2$ the sum defines an automorphic function of weight $w = k$. The unique (up to normalization) $w = 2$ holomorphic function for the group $\Gamma_T(2)$ is given by [12]:

$$E_2^\calr(z) \equiv -\frac{3}{8} E_2^{(0,0)}(z) + \frac{3}{2} E_2^{(0,1)}(z) = 1 + 12 \sum_{n=1}^\infty \left[\sigma_+^+(n) - \sigma_-^-(n)\right] q^n,$$

where $q = e^{2\pi iz}$ and $\sigma_k^\pm(n) = \sum_{d|n} (\pm)^d d^k$ is a sum over all positive integers $d$ that divide $n$. A second weight-two function, which is quasiholomorphic, may be chosen to be:

$$H_2^\calr(z, \zbar) \equiv \frac{1}{\pi y} - \frac{3}{8} E_2^{(0,0)}(z) + \frac{1}{2} E_2^{(0,1)}(z) = \frac{1}{\pi y} + 4 \sum_{n=1}^\infty \left[3\sigma_+^+(n) - \sigma_-^-(n)\right] q^n.$$
We use $H$ to denote such quasiholomorphic quantities in recognition of Hecke, who first introduced them [16]. These are the only independent linear combinations of the $E_2^{(p,q)}(z)$ which transform with weight two after the addition of a function of $y$, and we assume these to form a basis for all quasiholomorphic functions having weight two.

If we use eq. (3) for the asymptotic form for large $y$, and if we suppose $G_{z\bar{z}} = 1 + O(1/y)$ and $G_{zz} = O(1/y)$ in this limit, then we must choose:

$$\beta_z = \frac{i}{2\pi} H_2^T(z, \bar{z}).$$

(8)

If, on the other hand, we permit $\beta_z = O(1)$ for large $y$, then it may be taken to be an arbitrary linear combination of $H_2^T(z, \bar{z})$ and $E_2^T(z)$.

Both $E_2^T$ and $H_2^T$ are nonsingular in the upper half plane, and have simple zeroes at the fixed point $\xi \equiv \frac{1}{2}(1+i)$ of $\Gamma_7(2)$, as well as at all its images under the group. Since $\det G_{ij}$ should never vanish, these are therefore the candidates for the delocalization critical point in the quantum Hall system. We obtain in this way the phase diagram of refs. [4] and [8]. Since these critical points are all related to one another by $\Gamma_7(2)$ transformations, they must all share the same critical exponents. The values for these exponents are found by expanding $\beta^j$ about $z = \xi$, and computing the inverses of the eigenvalues of the following matrix of derivatives:

$$\left. \left( \frac{\partial_i \beta_j}{z=\xi} \right) \right| = G^{jk}(\xi) \left. \left( \frac{\partial_i \beta_k}{z=\xi} \right) \right|.$$

(9)

Clearly a determination of these indices requires knowledge of the inverse metric evaluated at the critical point.

It is noteworthy that choosing $\beta_z$ to be any linear combination of $H_2^T$ and $E_2^T$ gives a gradient flow, $\beta_z = \partial \Phi$.\textsuperscript{6} This is because both $H_2^T$ and $E_2^T$ are the derivatives of appropriate functions: $iH_2^T = \partial \Phi_H$ and $iE_2^T = \partial \Phi_E$, with:

$$\Phi_H = \frac{1}{2\pi} \ln \left| \frac{\Delta_T}{y^4 \Delta} \right|, \quad \text{and} \quad \Phi_E = \frac{1}{2\pi} \ln \left| \frac{\Delta_3}{\Delta} \right|.$$

(10)

\textsuperscript{6} Here, and in what follows, $\partial$ without a subscript denotes $\partial/\partial z$, and $\bar{\partial}$ denotes $\partial/\partial \bar{z}$.
Here $\Delta$ and $\Delta_T$ are the so-called ‘cusp’ forms of lowest weight for $SL(2, \mathbb{Z})$ and $\Gamma_T(2)$ which are analytic everywhere, and vanish nowhere, in the upper half plane (excluding $i\infty$). They are given in terms of the Eisenstein series by:\footnote{We adopt here an unconventional normalization for $\Delta$.}

\[
\Delta(z) = (E_4)^3 - (E_6)^2, \quad \Delta_T(z) = E_4 - E_4^{(0,1)}.
\] (11)

These expressions use the $SL(2, \mathbb{Z})$ Eisenstein series, $E_k(z)$, which are defined in terms of eq. (5) by $E_k(z) = (1 - 2^{-k}) E_k^{(0,0)}(z)$.

2.2) The Conditions Governing $G_{ij}$

Obtaining the RG flow given $\beta_z$ requires raising the index using a metric. We now consider what requirements govern its possible form. We formulate two such conditions, which are required to ensure consistency with the assumed form for $\beta_i$. We require:

• (1) $\Gamma_T(2)$ Covariance: The symmetry, $\Gamma_T(2)$, acts on the coupling constants as indicated in eq. (1). $G_{ij}(z, \bar{z})$ must transform under this transformation as a covariant rank-two tensor. We therefore require the following transformation property under $\Gamma_T(2)$:

\[
G_{zz} \to (cz + d)^4 G_{zz}, \quad \text{and} \quad G_{z\bar{z}} \to |cz + d|^4 G_{z\bar{z}}.
\] (12)

• (2) RG Consistency: There is a consistency requirement that the metric must satisfy which arises because the quasiholomorphy assumption is imposed on the components of $\beta_i$ rather than on $\beta^i$. To understand the issue, imagine examining $\beta_i$ at two points, $z$ and $z'$, which are related by an RG transformation: $z' = r_t(z, \bar{z})$. There are then two ways to compute the components of $\beta_i$. First, one can simply evaluate the ansatz for $\beta_i$ at both $z$ and $z'$. Second, one can compute $\beta_i'(z', \bar{z}')$ by evolving $\beta_i(z, \bar{z})$ from $z$ to $z'$ using the RG evolution. These two alternatives need not agree with one another given an arbitrary ansatz for the metric, $G_{ij}(z, \bar{z})$. We must require the metric to be consistent, in this sense, with our ansatz for $\beta_z$. 
To formulate this condition more precisely, suppose we adopt an ansatz for the metric:

\[ G_{ij}(z, \bar{z}) = \sum_a c_a G^{(a)}_{ij}(z, \bar{z}), \quad (13) \]

where \( c_a \) are constants and each of the \( G^{(a)}_{ij} \)'s is a tensor having a specific functional form. For example, we might choose one of these to be the hyperbolic metric: \( G^{(1)}_{ij} dz^i dz^j = (2/y^2) dz d\bar{z}, \) or we could choose \( G^{(2)}_{ij} dz^i dz^j = V^2 dz^2 + \bar{V}^2 d\bar{z}^2 + 2V \bar{V} dz d\bar{z}, \) etc., with \( V \equiv H_T^2, \) say. (Notice that both of these examples transform properly with respect to \( \Gamma(2). \))

The consistency condition states that the change of \( G_{ij} \) due to the RG flow must also have the same form as eq. (13). That is:

\[
(\mathcal{L}_\beta G)_{ij} \equiv \beta^k \partial_k G_{ij} + G_{ik} \partial_j \beta^k + G_{kj} \partial_i \beta^k \\
\equiv \nabla_i \beta_j + \nabla_j \beta_i \\
= \sum_a \tilde{c}_a G^{(a)}_{ij}(z, \bar{z}), \quad (14)
\]

where \( \mathcal{L}_\beta G \) is defined by the first line of eqs. (14), and denotes the Lie derivative of \( G_{ij} \) along the RG flow. The second equality in eq. (14) is an identity, with the covariant derivative, \( \nabla_i, \) constructed using the Levi-Civita connection for the metric \( G_{ij}. \) The content of the consistency condition lies in the third of eqs. (14). When it is satisfied, the RG evolution simply makes the parameters of ansatz (13) into functions of scale: \( c_a = c_a(t). \)

To date we have been unsuccessful in finding a simple ansatz for \( G_{ij} \) which satisfies condition (14). It would be encouraging to think that this condition may be sufficiently restrictive as to ensure a unique solution.

3. The SL(2,\( \mathbb{Z} \)) Case

We pause here to briefly record the results which may be obtained using identical arguments for the case where the symmetry group is \( SL(2, \mathbb{Z}) \) itself, rather than just its subgroup, \( \Gamma(2). \) These results may have applications within supersymmetric gauge
theories, such as for supersymmetric QED with both light elementary charges and magnetic monopoles.

Only one linear combination of the two forms, $H^T_2(z, \bar{z})$ and $E^T_2(z)$, transforms with weight $w = 2$ under the full group, $SL(2, \mathbb{Z})$. This unique combination is given by:

$$H_2(z, \bar{z}) = -\frac{3}{\pi y} + E_2(z) = 1 - \frac{3}{\pi y} - 24 \sum_{n=1}^{\infty} \sigma_1^+(n) q^n,$$  \hspace{1cm} (15)

where, $E_2 = \frac{3}{4} E_2^{(0,0)}$. Such a covariant beta function may be written as the derivative of a real RG potential in the following way: $iH_2(z, \bar{z}) = \partial \Psi(z, \bar{z})$, with:

$$\Psi = \frac{1}{2 \pi} \ln \left( \frac{y^{12} \Delta \bar{\Delta}}{y_1 y_2} \right).$$  \hspace{1cm} (16)

The beta function defined by eq. (15) has two types of fixed points, corresponding to the points $z = i$ and $z = e^{i\pi/3}$, as well as the images of these under $SL(2, \mathbb{Z})$. The resulting phase diagram is described in more detail in ref. [4].

4. Conclusions

In this paper we have argued that plausible features of quantum Hall systems, together with the ansatz of quasiholomorphy, combine to strongly constrain the RG flow which is possible for these systems. We have used these constraints to determine the $c$ function, from which the beta function is obtained by taking the gradient. We found a two-parameter space of solutions, which can be further restricted by using the weak-coupling form which has been computed for quantum Hall systems for large $\sigma_{xx}$. For asymptotically flat metrics, the result is unique, and is given by eq. (10):

$$\beta_z = \frac{i}{2 \pi} H^T_2(z, \bar{z}) = \partial \Phi$$ \hspace{1cm} with \hspace{1cm} $\Phi = \frac{1}{2 \pi} \Phi_H = \left( \frac{1}{2 \pi} \right)^2 \ln \left| \frac{\Delta_T}{y_1 y_2} \right|. \hspace{1cm} (17)$

This function fixes the phase diagram for these systems even in the strong coupling (small $\sigma_{xx}$) regime. The extraction of more information requires knowledge of a Zamolodchikov-type metric, which relates $\beta^i$ to $\beta_j = \partial_j \Phi$. 

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We close by listing some of the many directions which remain to be pursued. First, an explicit solution of the consistency condition, eq. (14), for $G_{ij}$ would be useful for quantum Hall systems, since this would permit the calculation of the delocalization critical exponents, $\nu$. Should such a prediction succeed, it would immediately suggest searching for the underlying origin of the symmetry and quasiholomorphy properties. It is perhaps tantalizing in this regard that disordered systems such as these are often describable using supersymmetric models, since supersymmetric models often exhibit special holomorphy properties.

Appendix A. Beta functions and Discrete Symmetries

We start by reviewing properties shared by all beta functions.

4.1) Definitions and Notation

Consider a system which for some range of energies is adequately described by an effective Lagrangian parametrized by $n$ independent couplings \{${g^1, g^2, \ldots, g^n}$\}. Scale transformations on the system generates a flow in this parameter space:

$$g^i \to r^i_t(g) = g^i(t)$$

(18)

labelled by the scale parameter $t = \ln(\mu/m)$, where $\mu$ is the mass scale of interest and $m$ is a fiducial mass scale relative to which we choose to measure $\mu$. The beta function is a vector field tangent to this flow:

$$\beta^i(g) = \frac{dg^i}{dt}.$$  

(19)

Notice that $\beta^i$ transforms like a contravariant vector field under general coordinate transformations on the parameter space; i.e. if $g^i \to g'^i$ then

$$\beta^i \to \beta'^i = \frac{\partial g'^i}{\partial g^j} \beta^j = g'^i, j \beta^j.$$  

(20)
The covariant vector field $\beta_i$ is obtained from $\beta^i$ by lowering the index with a metric $G_{ij}$ on the parameter space, $\beta_i = G_{ij} \beta^j$. In a specific quantum field theory, such a metric can be obtained [13], [17] using the two-point correlation functions of the operators $O_i$ to which the $g^i$ couple. That is, if the interaction lagrangian is $\mathcal{L} = g^i O_i$, then $G_{ij} \simeq \int dx \langle O_i(x) O_j(0) \rangle$.

Since the RG flows physically express how the system responds as successive degrees of freedom are ‘integrated out’, they are believed to be always monotonic, i.e. flowlines flow from sources to sinks, and never close upon themselves. This behaviour is automatic if the flow is a gradient flow:

$$\beta^i = G^{ij} \partial_j \Phi,$$

since in this case $\beta^i$ is curl-free. The function $\Phi$ is known as the RG potential. For two-dimensional, unitary field theories, all RG flows are known to be gradient in the immediate vicinity of a fixed point [13]. Moreover, they have been found to be gradient in all other unitary field theories that have been checked.

4.2) Behaviour Near Fixed Points

$g = g_*$ is a fixed point of the RG flow if the beta function vanishes there. Expanding around such a point,

$$\beta^i(g) = B^i_j (g - g_*)^j + O((g - g_*)^2)$$

we obtain a differential equation

$$\frac{dg}{dt} = B^i_j (g - g_*)^j$$

whose solution is

$$(g - g_*)^i = (e^{B(t-t_0)})^i_j (g - g_*)^j,$$

where $g_0 = g(t_0)$ is the starting point (initial value) of the flow. Rotating to the basis $\tilde{g}^i$

where $B$ is diagonal,

$$(\tilde{g} - \tilde{g}_*)^i = e^{b_i(t-t_0)} (\tilde{g} - \tilde{g}_*)^i,$$
we see that the RG flow decouples into one-dimensional flows near the fixed points, and that the eigenvalues of \( B \) give the flow rates in the principal directions around the fixed point.

In order to see how the eigenvalues \( b_i \) are measured, we consider any physical quantity \( \xi \) defined such that it depends on the scale transformation

\[
\mu = \lambda \mu, \quad t = t_0 + \ln \lambda, \quad \lambda(t) = e^{t-t_0}
\]  

only through \( g \):

\[
\xi(g_0) = \xi(g(t_0)) \rightarrow \xi(g(t)) = \lambda^{-1}(t)\xi(g_0).
\]  

Differentiating with respect to \( t \) and letting \( t \rightarrow t_0 \) we obtain a differential equation for \( \xi \)

\[
(\xi, i \beta^i + \xi = 0) \text{ which to linear order in } g \text{ is}
\]

\[
\xi_i B^i_j (g - g^*)^j + \xi = 0.
\]  

In the basis which diagonalizes \( B \) the solution separates

\[
\frac{\xi}{\xi_0} = \prod_{i=1}^n \frac{\xi^i}{\xi^i_0}
\]  

where the \( \xi^i = (\tilde{g}^i - \tilde{g}^*_i)^{-\nu_i} \) can be regarded as “decoupled” correlations lengths which define the observable critical scaling exponents \( \nu_i = b_i^{-1} \). Any physical length scale \( l \) can be expressed in terms of the \( \xi^i \) through a homogeneous function \( F \) of weight one:

\[
l = F(\xi^1, \xi^2, \ldots, \xi^n) = \lambda^{-1}F(\lambda\xi^1, \lambda\xi^2, \ldots, \lambda\xi^n).
\]  

Equivalently, with quasi-homogeneous parameters \( x^i = (\tilde{g} - \tilde{g}^*_i)^i \) of quasi-weight \(-\nu_i^{-1}\), \( l \) is a quasi-homogeneous function \( f \) of total weight one:

\[
l = f(x^1, x^2, \ldots, x^n) = \lambda^{-1}f(\lambda^{-\nu_1} x^1, \lambda^{-\nu_2} x^2, \ldots, \lambda^{-\nu_n} x^n).
\]
Universality is the idea that all lagrangians with a given symmetry flow to the same infrared (IR: $t \rightarrow +\infty$, UV: $t \rightarrow -\infty$) fixed point. In physical systems exhibiting second order phase transitions, the critical points are RG fixed points with unstable ($\lambda_i > 0$) directions. It is an experimental fact that the critical exponents $\nu_i$ measured in such systems take only a few specific values, which therefore label the corresponding universality classes. In short, different theories can and do have the same critical behaviour.

4.3) RG and Discrete Symmetries

In the body of this paper we are concerned with a different type of universality, sometimes called "superuniversality", where different fixed points have the same critical behaviour. In particular, distinct critical points have identical scaling exponents. We next show in detail why such behaviour is to be expected when the system possesses a set of parameter space symmetries $g^i \rightarrow \gamma^i(g)$ which commute with the RG flow:

$$\gamma(r_t(g)) = r_t(\gamma(g)).$$

(32)

In order to see this, differentiate the equation with respect to $t$ and evaluate at $t = 0$, giving:

$$\gamma^i \cdot j(g) \beta^j(g) = \beta^i(\gamma(g))$$

(33)

for all $g$. This is just eq. (20) specialized to the symmetry transformations, $g^i'(g) = \gamma^i(g)$. Differentiating again gives

$$\gamma^i \cdot j(k)(g)\beta^j(g) + \gamma^i \cdot j(g)\beta^j \cdot k(g) = \beta^i \cdot j(\gamma(g))\gamma^j \cdot k(g).$$

(34)

If two distinct RG fixed points $g_*$ and $g_{**}$ are related by $\gamma$ then

$$g^i_{**} = \gamma^i(g_*) \quad \text{and} \quad \beta^i(g_*) = \beta^i(g_{**}) = 0,$$

(35)

and eq. (34) reduces to

$$\gamma^i \cdot j(g_*)\beta^j \cdot k(g_*) = \beta^i \cdot j(g_{**})\gamma^j \cdot k(g_*)$$

(36)
Finally, expanding $\beta(g)$ around $g_*$ and $g_{**}$ with linear matrix coefficients $B_*$ and $B_{**}$, respectively, and expanding $\gamma(g)$ around $g_*$,

$$\gamma^i(g) = g_{**}^i + D_{ij}^i (g - g_*)^j + \ldots \quad (37)$$

we find from eq. (36) that $B_{**} = DB_* D^{-1}$. Since $B_{**}$ is simply a similarity transformation of $B_*$ they must have the same eigenvalues, i.e. the same critical exponents.

Notice also that if $g_*$ is fixed by some transformation $\gamma_*$ in the symmetry group $\Gamma$, then eq. (33) shows that either $\gamma^i_{*,j}$ has 1 as an eigenvalue, or $\beta(g_*) = 0$ and $g_*$ must also be an RG fixed point. The former case arises below when we turn to modular transformations of the upper half of the complex plane. The point $z_* = i\infty$, which will play a central role in our discussion, is fixed by translations $T(z) = z + 1$, but $z_*$ need not be an RG fixed point and the beta function need not vanish at this point. Other conditions on the beta function, like holomorphicity, may force it to vanish, also at $i\infty$, but it is important to realize that it is not sufficient that $z_*$ is a fixed point of $\Gamma$ for it to be a fixed point of the RG flow.

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