A Modified Generalization of Fractional Calculus Operators in A Complex Domain

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Abstract: This investigation deals with a new generalization for fractional calculus operators in a complex domain based on the well-known hypergeometric function. Conditions are forced for these generalized operators such as the upper bounds. Other properties for the above operators are also presented. Besides, the employment of these operators is proposed in the geometric function theory.

Keywords: Fractional Integral operator, Fractional differential operators, Univalent function, Convex function, Hypergeometric function, Bessel function, Wright functions.

1. Introduction

Fractional Calculus is a powerful tool that has been recently applied to complex mathematical with linear operators. Despite its complicated mathematical background, fractional calculus came to open a new window of opportunity to mathematical and real-world, which has appeared many new problems and acceptable results. For instance, the concepts of fractional calculus operators and their generalizations of analytic and univalent functions have been successfully obtained to determine the basic geometric properties such as the coefficient estimates and distortion inequalities for numerous subclasses of analytic functions, adding to that studied some their topological properties in a complex plane (see [1-3]).

In [4] introduced an approach of the fractional integral operator defined for $|z| < 1$ and real numbers $\rho, \mu \in R, \Re(\omega) > 0$ by

$$\mathcal{I}^{\omega, \mu, \rho}_{0, z} \psi(z) := \frac{z^{-\omega+\mu}}{\Gamma(\omega)} \int_{0}^{z} (z - \zeta)^{\omega-1} \psi(\zeta) \, {}_{2}F_{1}(\omega + \mu, -\rho, \mu; 1 - \frac{\zeta}{z}) d\zeta,$$

(1)

where the function $\psi(z)$ is analytic in a simply-connected region of the $z-$ plane containing the origin, with the order $\psi(z) = O(|z|^\varepsilon)$, $(z \to 0)$, for $\varepsilon > \max\{0, \mu - \rho\} - 1$, and the multiplicity of $(z - \zeta)^{\omega-1}$ is removed by requiring $\text{log}(z - \zeta)$ to be real when $z - \zeta > 0$. Here, $\Gamma(\cdot)$ is the Gamma function and ${}_2F_1(a, b; c; z)$ is the absolutely convergent Gauss hypergeometric function given for $a, b, c \in C, c > 0$ by the power series [5]:

$${}_2F_1(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, \quad |z| < 1$$
where
\[(y)_m = \frac{\Gamma(m+y)}{\Gamma(y)} = \begin{cases} 1, & \text{if } m = 0 \\ y(y + 1) \cdots (y + m - 1), & \forall \ m \in \mathbb{N} \end{cases} \]
is the Pochhammer symbol defined in terms of Gamma function.

Recently, [6] defined a modification of the fractional integral $\Phi_z^{\alpha,\beta}$ and differential $\tau_z^{\alpha,\beta}$ operators of order two parameters $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ such that $0 \leq \alpha - \beta < 1$, respectively, are presented as follows:

$$\Phi_z^{\alpha,\beta} \psi(z) := \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha - \beta)} z^{1-\alpha} \int_0^z (z - \zeta)^{\alpha - 1} \zeta^{\beta - 1} \psi(\zeta) d\zeta$$

and

$$\tau_z^{\alpha,\beta} \psi(z) := \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(1 - \alpha + \beta)} z^{1-\beta} \int_0^z (z - \zeta)^{\beta - 1} \zeta^{\alpha - 1} \psi(\zeta) d\zeta$$

where the function $\psi(z)$ is analytic in a simply-connected region of the $z-$ plane containing the origin, both of the multiplicity of $(z - \zeta)^{\alpha - \beta - 1}$ and $(z - \zeta)^{\beta - \alpha}$ are respectively removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

In this study, we shall restrict our attention to define new fractional calculus operators in the complex plane. The upper bounds for these operators given in terms of the univalent and convex functions. Some geometric applications associated with the Bessel function of the first kind are presented by the generalized Wright functions in the sense of generalization.

2. New classes of generalized fractional calculus operators

In this section, we proposed to define generalized fractional integral and differential operators in the classical definitions, where the order of the fractional integral and fractional differential operators must be positive real numbers. Our definition has been based on important remarks concerning in equations (2) and (3).

Now, we employ equation (1) in (2) to introduce a new generalized fractional integral operator $\mathcal{M}_z^{\alpha,\beta,\mu,\rho}$ as follows:

**Definition 1.** Let $\mu > 0$ and $\rho > 0$ be real numbers and $0 < \alpha \leq 1$, $0 < \beta \leq 1$ such that

$0 < \alpha - \beta \leq 1$. Then the fractional integral operator $\mathcal{M}_z^{\alpha,\beta,\mu,\rho}$ is defined by

$$\mathcal{M}_z^{\alpha,\beta,\mu,\rho} \psi(z) := \frac{\Gamma(\alpha)z^{1-2\alpha+\mu+\beta}}{\Gamma(\beta)\Gamma(\alpha - \beta)} \int_0^z (z - \zeta)^{\alpha - 1} \zeta^{\beta - 1} \psi(\zeta) \frac{\zeta \Gamma(\mu + \rho, \alpha - \beta; 1 - \frac{\zeta}{z})}{\Gamma(\mu + \rho)} d\zeta$$

where the function $\psi(z)$ is analytic in a simply-connected region of the $z-$ plane containing the origin with the order $\psi(z) = O(|z|^{\varepsilon})$, $(z \to 0)$, for $\varepsilon > \max\{0, \mu - \rho\} - 1$ and the multiplicity of $(z - \zeta)^{\alpha - \beta - 1}$ is removed as in equations (2).
Remark 1. By setting \( \mu = \beta - \alpha \) in (4), we have
\[
\mathcal{M}_z^{a,b,\alpha-\beta,\rho} \psi(z) = Q_z^{a,\beta}\psi(z).
\]

Next, we applying equation (1) in (3) to define a new generalized fractional differential operator \( \mathcal{N}_z^{a,\beta,\mu,\rho} \) by the following formula.

Definition 2. Let \( \mu > 0 \) and \( \rho > 0 \) be real numbers and \( 0 < \alpha \leq 1, \ 0 < \beta \leq 1 \) such that \( 0 \leq \alpha - \beta < 1 \). The generalized fractional differential operator \( \mathcal{N}_z^{a,\beta,\mu,\rho} \) is defined by:
\[
\mathcal{N}_z^{a,\beta,\mu,\rho} \psi(z) := \frac{\Gamma(\beta)z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz} \left\{ z^{\alpha-\beta-\mu} \int_0^z \zeta^{\alpha-1}(z - \zeta)^{\beta-\alpha} \psi(\zeta) \right\}_2F_1 \left( \beta - \alpha + \mu, 1 - \rho, 1 - \alpha + \beta; \frac{z}{\zeta} \right) d\zeta
\]
where the function \( \psi(z) \) is analytic in a simply-connected region of the \( z \)-plane containing the origin with order as given by (3).

Remark 2. By setting \( \mu = \alpha - \beta \) in (5), then we obtain the following closed results:
\[
\mathcal{N}_z^{a,\beta,\alpha-\beta,\rho} \psi(z) = T_z^{a,\beta}\psi(z).
\]

We shall need the following Definition to present the next outcomes in our investigation.

Definition 3. [5] For the real numbers \( c > 0 \) and \( \sigma > 0 \), the hypergeometric function \( _2F_1 \) in the integral terms is shown as follows:
\[
_2F_1(a, b, c; z) := \int_0^1 \gamma(s)_2F_1(a, b, c; zs) ds
\]
where
\[
\gamma(s) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(c-s)}s^{\sigma-1}(1-s)^{c-\sigma-1}.
\]

Also, we use the familiar Gauss equation
\[
_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b) > 0.
\]

The next results are based on two formulas of generalized fractional integral (4) and generalized fractional differential (5) with a power function.

Lemma 1. If \( 0 < \alpha \leq 1, \ 0 < \beta \leq 1 \) such that \( 0 < \alpha - \beta \leq 1 \) and \( \nu > \mu - \rho - 1 \), then
\[
\mathcal{M}_z^{a,\beta,\mu,\rho} z^{\nu} = \frac{\Gamma(\alpha)\Gamma(\nu+\beta)\Gamma(\nu+\beta+\rho-\mu)}{\Gamma(\beta)\Gamma(\nu+\beta-\mu)\Gamma(\nu+\alpha+\rho)} z^{\beta-(\alpha+\mu)+\nu}, \quad |z| < 1
\]
Proof. By using equation (4) and applying Definition 3, we get
\[
\mathcal{M}_z^{\alpha, \beta, \mu, \rho, \nu} z^\nu = \Phi_z^{\alpha, \beta} z^\nu.
\]

Similarly to the proof of Lemma 1, it is proved the association of the generalized fractional differential operator (5) with a power function.

Lemma 2. If \(0 < \alpha \leq 1, \quad 0 < \beta \leq 1\) such that \(0 \leq \alpha - \beta < 1\) and \(v > \mu - \rho - 1\), then
\[
\mathcal{K}_z^{\alpha, \beta, \mu, \rho, \nu} z^\nu = \frac{\Gamma(\beta) \Gamma(v + \alpha) \Gamma(v + \alpha - \mu + \rho)}{\Gamma(\alpha) \Gamma(v + \alpha - \mu) \Gamma(v + \beta + \rho)} z^{v + \alpha - \mu - \rho}, \quad |z| < 1
\]
in particular,
\[
\mathcal{K}_z^{\alpha, \beta, \mu, \rho, \nu} z^\nu = T_z^{\alpha, \beta} z^\nu.
\]

Proof. By using equation (5) to the function \(z^\nu\), we have
\[
\mathcal{K}_z^{\alpha, \beta, \mu, \rho} z^\nu = \frac{\Gamma(\beta) z^{1-\beta}}{\Gamma(\alpha) \Gamma(1 - \alpha + \beta)} \frac{d}{dz} \left\{ z^{\alpha - \mu - 1} \tilde{\zeta}^1 \right\}
\]
\[
\mathcal{K}_z^{\alpha, \beta, \mu, \rho} z^\nu = \frac{\Gamma(\beta) z^{1-\beta}}{\Gamma(\alpha) \Gamma(1 - \alpha + \beta)} \frac{d}{dz} \left\{ z^{\alpha - \mu - 1} \tilde{\zeta}^1 \right\}
\]
by employing Definition 1 in the above expression, we get
\[
\mathcal{K}_z^{\alpha, \beta, \mu, \rho} z^\nu = \frac{\Gamma(\beta)}{\Gamma(\alpha) \Gamma(1 - \alpha + \beta)} \left\{ \frac{d}{dz} z^{\alpha - \mu - v + a} \right\} \int_0^1 (1 - t)^{v + \alpha - 1} t^{\beta - a} \tilde{\zeta}^1 F_1(\beta - \alpha + \mu, 1 - \rho, 1 - \alpha + \beta; t) dt.
\]
Hence, we arrive at the desired results.

3. Upper Bounds

In this section, we deal with some applications of the new generalizations of fractional operators (4) and (5) in view of the univalent and convex functions in the open unit disk

\[ U = \{ z : |z| < 1 \}. \]

Let \( A \) denote the class of all normalized functions \( f \) of the form

\[ f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad m \in \mathbb{N} \setminus \{0\}, \tag{8} \]

which are analytic in \( U \) of the complex plane \( \mathbb{C} \). A function \( f \) is called univalent and denoted by \( f \in S = \{ f \in A \mid f \text{ one-to-one in } U \} \). A function \( f \in A \) that maps \( U \) onto a convex domain is called a convex function. Let denote \( K \) the class of all functions \( f \in A \) that are convex. Further, the convolution product for two analytic functions is given by

\[ (f * g)(z) = z + \sum_{m=2}^{\infty} a_m \omega_m z^m, \]

where \( g(z) = z + \sum_{m=2}^{\infty} \omega_m z^m \) and \( z \in U \).

**Lemma 3.** [7] Let \( S \) and \( K \) be subclasses of \( A \). If \( f \) defined by (8) is in the class \( S \), then \( |a_m| \leq m \) for all \( m \in \mathbb{N} \setminus \{1\} \) and for \( z \in U \) the equality holds for the Koebe function defined by

\[ f(z) = \frac{z}{(1 - z)^2}. \]

Adding to that, if the function \( f \) presented by (8) is in the class \( K \), then \( |a_m| \leq 1 \) and for \( z \in U \) the equality holds for

\[ f(z) = \frac{z}{(1 - z)}. \]

**Theorem 1.** For \( |z| < r, \ r < 1 \), let \( f \in S \) then

\[ \left| \mathcal{M}_z^{\alpha, \beta, \mu, \rho} f(z) \right| \leq r^{1+\beta - (\alpha + \mu)} \sum_{m=0}^{\infty} \frac{(2)_m B(m + 1 + \rho, \alpha) B(m + 1, \rho)}{B(m + 1 + \beta - \mu, \rho) B(m + 1, \beta)} \frac{r^m}{m!} \]

the equality holds for the Koebe function.
Proof. Let the function $f(z) \in S$. Then, by utilizing Lemma 1, we have

$$M_z^{\alpha,\beta,\mu,\rho} f(z) = \sum_{m=1}^{\infty} a_m \frac{\Gamma(\alpha) \Gamma(m+\beta) \Gamma(m+\beta+\rho-\mu)}{\Gamma(\beta) \Gamma(m+\beta-\mu) \Gamma(m+\alpha+\rho)} z^{\beta-(\alpha+\mu)+m}.$$ 

Thus by using the fact that $|a_m| \leq m$ in Lemma 3, we obtain

$$|M_z^{\alpha,\beta,\mu,\rho} f(z)| \leq \sum_{m=1}^{\infty} |a_m| \frac{\Gamma(\alpha) \Gamma(m+\beta) \Gamma(m+\beta+\rho-\mu)}{\Gamma(\beta) \Gamma(m+\beta-\mu) \Gamma(m+\alpha+\rho)} |z|^{\beta-(\alpha+\mu)+m}$$

$$\leq r^{\beta-(\alpha+\mu)} \sum_{m=0}^{\infty} (m+1) \frac{\Gamma(\alpha) \Gamma(m+\beta+1) \Gamma(m+\beta+\rho-\mu+1)}{\Gamma(\beta) \Gamma(m+\beta-\mu+1) \Gamma(m+\alpha+\rho+1)} r^m$$

$$= r^{1+\beta-(\alpha+\mu)} \sum_{m=0}^{\infty} \frac{(2m)B(m+1+\rho,\alpha)B(m+1+\rho)}{B(m+1+\beta-\mu,\rho)B(m+1+\beta) m!} r^m$$

where $B(t_1,t_2)$ represents the Beta function in terms of Gamma function given by [5]

$$B(t_1,t_2) = \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(t_1+t_2)}.$$ 

This completes the proof.

Theorem 2. For $|z| < r, \ r < 1$, let $f \in K$ then

$$|M_z^{\alpha,\beta,\mu,\rho} f(z)| \leq r^{1+\beta-(\alpha+\mu)} \sum_{m=0}^{\infty} \frac{(1)m B(m+1+\rho,\alpha)B(m+1+\rho)}{B(m+1+\beta-\mu,\rho)B(m+1+\beta) m!} r^m$$

the equality holds for the Koebe function.

Proof. Let the function $f(z) \in K$. Then, by applying Lemma 1 and Lemma 3, we have

$$|M_z^{\alpha,\beta,\mu,\rho} f(z)| \leq \sum_{m=1}^{\infty} |a_m| \frac{\Gamma(\alpha) \Gamma(m+\beta) \Gamma(m+\beta+\rho-\mu)}{\Gamma(\beta) \Gamma(m+\beta-\mu) \Gamma(m+\alpha+\rho)} |z|^{\beta-(\alpha+\mu)+m}, \ |a_m| \leq 1$$

$$\leq r^{\beta-(\alpha+\mu)+1} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha) \Gamma(m+\beta+1) \Gamma(m+\beta+\rho-\mu+1)}{\Gamma(\beta) \Gamma(m+\beta-\mu+1) \Gamma(m+\alpha+\rho+1)} r^m$$

$$= r^{1+\beta-(\alpha+\mu)} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha) \Gamma(m+\beta+1) \Gamma(m+\beta+\rho-\mu+1)}{\Gamma(\beta) \Gamma(m+\beta-\mu+1) \Gamma(m+\alpha+\rho+1)} r^m.$$
This completes the proof.

Finally, we introduced some observations concerning the operator \( N_z^{\alpha,\beta, \mu, \rho} \) of (5) and by considering a similar manner of Theorem 1 and Theorem 2, respectively, we obtain the upper bounds of the above operator in classes of the univalent and convex functions.

**Theorem 3.** For \(|z| < r, \ r < 1\), let \( f \in S \) then

\[
\left| N_z^{\alpha,\beta, \mu, \rho} f(z) \right| \leq r^{1+\alpha-(\beta+\mu)} \sum_{m=0}^{\infty} \frac{(2)_m B(m+1+\rho, \beta)B(m+1, \rho)}{B(m+1+\alpha-\mu, \rho)B(m+1, \alpha)} \frac{r^m}{m!}
\]

the equality holds for the Koebe function.

**Proof.** Let the function \( f(z) \in S \) and \(|a_m| \leq m\). Then, by applying Lemma 2, we obtain

\[
\left| N_z^{\alpha,\beta, \mu, \rho} f(z) \right| \leq \sum_{m=1}^{\infty} |a_m| \frac{\Gamma(\beta)\Gamma(m+\alpha)\Gamma(m+\alpha-\mu+\rho)}{\Gamma(\alpha)\Gamma(m+\alpha-\mu)\Gamma(m+\beta+\rho)} |z|^{m+\alpha-(\beta+\mu)}, \ |a_1| \leq 1
\]

\[
\leq r^{\alpha-(\beta+\mu)+1} \sum_{m=0}^{\infty} \frac{(m+1) \Gamma(\beta)\Gamma(m+\alpha+1)\Gamma(m+\alpha-\mu+\rho+1)}{\Gamma(\alpha)\Gamma(m+\alpha-\mu+1)\Gamma(m+\beta+\rho+1)} r^m
\]

\[
= r^{1+\alpha-(\beta+\mu)} \sum_{m=0}^{\infty} \frac{(2)_m B(m+1+\rho, \beta)B(m+1, \rho)}{B(m+\alpha+1-\mu, \rho)B(m+1, \alpha)} \frac{r^m}{m!}.
\]

**Theorem 4.** For \(|z| < r, \ r < 1\), let \( f \in K \) then

\[
\left| N_z^{\alpha,\beta, \mu, \rho} f(z) \right| \leq r^{1+\alpha-(\beta+\mu)} \sum_{m=0}^{\infty} \frac{(1)_m B(m+1+\rho, \beta)B(m+1, \rho)}{B(m+1+\alpha-\mu, \rho)B(m+1, \alpha)} \frac{r^m}{m!}
\]

the equality holds for the Koebe function.

**Proof.** By supposing \( f(z) \in K \), such that \(|a_m| \leq 1\) and applying Lemma 2. Then, we conclude the proof.

**4. Applications in terms of generalized Wright functions**

In view of definitions of the fractional integral operator (4) and fractional differential operator (5), we investigate to present some generalized properties associated with the Bessel function of the first kind \( J_\nu(z) \) formulated for \( z, \nu \in \mathbb{C} \) such that \( z \neq 0 \) and \( \Re(\nu) > -1 \) by [8]:
\[
J_v(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(\frac{m}{2} + 1\right)}{m! \Gamma(v + m + 1)}, \quad v \neq -1, -2, \ldots.
\]

We demonstrate that such associated are expanded in terms of the generalized Wright function \(q^p(z)\) which is given by the following formula:

\[
q^p(z) = \prod_{i=1}^{p} \left[ \frac{-i \omega_i}{\beta_i} \right] \Gamma\left(\frac{\alpha_i}{\beta_i}, \gamma_i\right) \Gamma\left(\frac{1}{2}, \gamma_i\right) z^m \prod_{j=1}^{q} \left[ \prod_{i=1}^{p} \Gamma(b_j + \omega_j, m) m! \right],
\]

where \(z, a_i, b_j \in \mathbb{C}\) and \(\sigma_i, \omega_j\) real numbers in \(R (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)\), under the condition \(\sum_{j=1}^{q} \omega_j - \sum_{i=1}^{p} \sigma_i > 1\).

In the following, we provide the generalized fractional integral operator (4) associated with the Bessel functions (9).

**Theorem 5.** Let \(\mu, \rho\) be positive non-zero numbers, \(v > -1\) and \(0 < \alpha \leq 1, \ 0 < \beta \leq 1\) such that \(0 < \alpha - \beta \leq 1\). Then

\[
M_{z}^{\alpha, \beta, \mu, \rho} (J_v)(z) = \frac{2^{\beta - (\alpha + \mu) + v} \Gamma(\alpha)}{2^v \Gamma(\beta)} \Psi^p(z) \left[ \frac{(v + \beta, 2), (v + \beta + \rho - \mu, 2)}{(v + \beta - \mu, 2), (v + \alpha + \rho, 2), (v + 1, 1)} \left[ \frac{-z^2}{4} \right] \right].
\]

**Proof.** Utilizing equation (4) and equation (9), we obtain

\[
M_{z}^{\alpha, \beta, \mu, \rho} (J_v)(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(\frac{m}{2} + 1\right)}{m! \Gamma(v + m + 1)} \left[ M_{z}^{\alpha, \beta, \mu, \rho} z^v \right].
\]

Using Lemma 1, we obtain

\[
M_{z}^{\alpha, \beta, \mu, \rho} (J_v)(z) = \frac{2^{\beta - (\alpha + \mu) + v} \Gamma(\alpha)}{2^v \Gamma(\beta)} \sum_{m=0}^{\infty} \frac{\Gamma(v + \beta + 2m) \Gamma(v + \beta + \rho - \mu + 2m)}{\Gamma(v + \beta - \mu + 2m) + \Gamma(v + \alpha + \rho + 2m) \Gamma(v + m + 1)} \frac{(-z^2)^m}{4^m m!}.
\]

By applying Equation (10), we have

\[
M_{z}^{\alpha, \beta, \mu, \rho} (J_v)(z) = \frac{2^{\beta - (\alpha + \mu) + v} \Gamma(\alpha)}{2^v \Gamma(\beta)} \Psi^p(z) \left[ \frac{(v + \beta, 2), (v + \beta + \rho - \mu, 2)}{(v + \beta - \mu, 2), (v + \alpha + \rho, 2), (v + 1, 1)} \left[ \frac{-z^2}{4} \right] \right].
\]

**Corollary 1.** Let \(\mu, \rho, v \in \mathbb{C}\) be such that \(v > -1\), and \(0 < \alpha \leq 1, \ 0 < \beta \leq 1\) with \(0 < \alpha - \beta \leq 1\). Then

\[
\Phi_{z}^{\alpha, \beta} (J_v)(z) = \frac{2^v \Gamma(\alpha)}{2^v \Gamma(\beta)} \Psi^p(z) \left[ \frac{(v + \beta, 2), (v + \alpha + \rho, 2)}{(v + \alpha, 2), (v + \alpha + \rho, 2), (v + 1, 1)} \left[ \frac{-z^2}{4} \right] \right].
\]

Corollary 1 achieves from Theorem 5 in respective cases \(\mu = \beta - \alpha\).
The following Theorem 6 introduces the generalized fractional differential operator (5) of the Bessel function (9).

**Theorem 6.** Let $\mu$, $\rho$ be positive non-zero numbers, $\nu > -1$ and $0 < \alpha \leq 1, \ 0 < \beta \leq 1$ be such that $0 < \alpha - \beta \leq 1$. Then

$$\kappa_{z}^{\alpha,\beta,\mu,\rho} (J_{\nu})(z) = \frac{z^{\nu + \alpha - (\beta + \mu)} \Gamma(\beta)}{2^{\nu} \Gamma(\alpha)} x_{\Psi_{3}}(z) \left[ \frac{(v + \alpha, 2), (v + \alpha - \mu + \rho, 2)}{(v + \beta + \rho, 2), (v + 1, 1)} - \frac{z^{2}}{4} \right].$$

**Proof.** Applying Equation (5) and Equation (9), we have

$$\kappa_{z}^{\alpha,\beta,\mu,\rho} (J_{\nu})(z) = \sum_{m=0}^{\infty} \frac{(-1)^{m} \frac{1}{2}^{v + 2m}}{m! \Gamma(v + m + 1)} \left( \kappa_{z}^{\alpha,\beta,\mu,\rho} x_{\nu + 2m} \right).$$

By using Lemma 2

$$\kappa_{z}^{\alpha,\beta,\mu,\rho} (J_{\nu})(z) = \frac{z^{\nu + \alpha - \mu} \Gamma(\beta)}{2^{\nu} \Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(v + \alpha + 2m) \Gamma(v + \alpha - \mu + 2m)}{\Gamma(v + \alpha - \mu + 2m) \Gamma(v + \beta + \rho + 2m) \Gamma(v + m + 1)} (-z^{2})^{m}.\frac{1}{4^{m} m!}.$$

By Equation (10)

$$\kappa_{z}^{\alpha,\beta,\mu,\rho} (J_{\nu})(z) = \frac{z^{\nu + \alpha - (\beta + \mu)} \Gamma(\beta)}{2^{\nu} \Gamma(\alpha)} x_{\Psi_{3}}(z) \left[ \frac{(v + \alpha, 2), (v + \alpha - \mu + \rho, 2)}{(v + \beta + \rho, 2), (v + 1, 1)} - \frac{z^{2}}{4} \right].$$

**Corollary 2.** Let $\mu$, $\rho$, $\nu \in \mathbb{C}$ be such that $\nu > -1$, and $0 < \alpha \leq 1, \ 0 < \beta \leq 1$ with $0 < \alpha - \beta \leq 1$. Then

$$T_{z}^{\alpha,\beta} (J_{\nu})(z) = \frac{z^{\nu} \Gamma(\beta)}{2^{\nu} \Gamma(\alpha)} x_{\Psi_{3}}(z) \left[ \frac{(v + \alpha, 2), (v + \beta + \rho, 2)}{(v + \beta, 2), (v + \beta + \rho, 2), (v + 1, 1)} - \frac{z^{2}}{4} \right].$$

Corollary 2 achieves from Theorem 6 in particular cases $\mu = \alpha - \beta$.

**5. Conclusion**

Conditions for the new fractional calculus operators are obtained. Also, some characteristics for these operators are delivered. Some geometric applications are studied in the sense of generalization.

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