Super Fivebranes near the boundary of $AdS_7 \times S^4$

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Abstract

We determine, to the first order in the radius of Anti-de-Sitter, the realisation of the $OSp(6,2|2)$ superconformal algebra on vector fields. We then calculate, to this order, the superspace metric describing the background of $AdS_7 \times S^4$. The coordinates we work with are adapted to a 6+5 splitting of the eleven dimensional superspace. Finally, we deduce in a manifestly supersymmetric form the equations governing the dynamics of the fivebrane near the boundary of $AdS_7$. 

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1 Introduction

The relation between Anti-de-Sitter space (AdS) and superconformal field theories defined on its boundary has been a subject of interest for a relatively long time [1–3] (see [4] for a review). The AdS/CFT conjecture [5–7] is the latest proposed such relation and has, if correct, far reaching consequences: it allows, in principle, a non-perturbative definition of string or M-theory on AdS backgrounds. The conjecture states [5], for example, that M-theory on $AdS_7 \times S^4$ is equivalent to a six-dimensional superconformal $(2,0)$ field theory [8] defined on the boundary of $AdS$. Little is known about M-theory besides that its low energy approximation is given by eleven-dimensional supergravity, its compactification on a circle leads to type IIA superstring [9] and that it contains membranes [10] and fivebranes [11–15]. The worldvolume action of super p-branes and/or their equations of motion are formulated in a superspace background. So the study of p-branes in AdS space necessitates the determination of the superbackground. The bosonic $SO(6,2) \times SO(5)$ symmetry of $AdS_7 \times S^4$ is promoted, when one considers the fermionic coordinates in addition to the bosonic ones, to a $Osp(6,2|2)$ superconformal symmetry. The expression of the moving basis and the rest of the background depend on the eleven-dimensional supercoordinates one chooses to work with or equivalently on the realisation of the isometries as super vector fields in superspace. Such a realisation was found in [16–19] using the coset approach [20]. With a particular choice of super coordinates, this approach leads to closed expressions for the isometries of $AdS_7 \times S^4$ as well as the moving basis. However these coordinates are not particularly adapted to a 6+5 splitting of eleven dimensional spacetime, the six dimensions representing the fivebrane’s worldvolume and the five remaining dimensions the transverse dimensions. The aim of this paper is propose such coordinates and use the resulting isometries and background to deduce in a manifestly supersymmetric form the equations of motion of a fivebrane near the boundary of $AdS$.

In order to study the super fivebrane in the $AdS_7 \times S^4$ background there exists a privileged system of coordinates which can be seen as follows: the fivebrane super worldvolume is spanned by the $(2,0)$ six-dimensional supercoordinates $(x^\mu, \theta^\alpha)$ where $\mu = 0, \ldots, 5$ and $\alpha$ is a spin index of the $(4_+, 4)$ spinorial representation of $SO(5,1) \times SO(5)$. The transverse fluctuations of the fivebrane are described by a $(2,0)$ multiplet which comprises five scalars $\phi^i$ in the vectorial representation of $SO(5)$ and a symplectic-Weyl-Majorana fermion $\psi^{a\alpha'}$ in the $(4_-, 4)$ spinorial representation of $SO(5,1) \times SO(5)$. A convenient set of supercoordinates to parametrise the eleven-dimensional superspace is thus given by $(x^k, \phi^i, \theta^\alpha, \psi^{a\alpha'})$. The $(2,0)$ super Poincaré six-
dimensional group is a subgroup of $Osp(6, 2|2)$ and so it has natural realisation on the six-dimensional super worldvolume of the fivebrane. In this paper, we shall use these coordinates to parametrise the eleven-dimensional superspace. This will allow us to find, in a manifestly supersymmetric way, the interacting $(2,0)$ theory describing the fivebrane near the boundary of $AdS_7$. In order to do so, we determine first the realisation of the superconformal algebra on vector fields and the invariant metric.

In section 2, we briefly review the $AdS_7$ metric and how it arises in eleven-dimensional supergravity. In section 3, we determine from [21, 22] a realisation of the superconformal algebra on vector fields. This realisation does not provide us with the Killing vector fields, as we show in section 4, but will be the zero order approximation to it. In section 5, we look for a modification of the realisation found in section 3. We demand that the action of the super-Poincaré group be not modified and we show, in section 6, that the vector fields have an expansion in the radius, $R$, of $AdS$. In this section we also determine explicitly the first order expansion which we use in section 7 to construct the invariant metric. In section 8, relying on the doubly supersymmetric formalism of the superembedding approach [11], we determine from the invariant metric the equations governing the dynamics of the fivebrane to the first order in $R$; this constitutes a good approximation near the boundary of $AdS$. Due to our choice of coordinates, we get manifestly worldvolume superconformal equations of motions. At order zero in $R$ the equations, that is at the boundary of $AdS$, the equations reduce to the doubleton equations [2] and at the next order we get interaction terms in the $(2,0)$ theory. The equations we get are the supersymmetric version of the bosonic ones found for radial coordinates in [3, 23] and all bosonic degrees of freedom in [24]. Our conventions and some of the technical tools are collected in Appendix A; the superconformal $Osp(6, 2|2)$ algebra is presented in Appendix B.

## 2 AdS and near horizon geometry

The $d + 1$ dimensional Anti-de-Sitter manifold is conveniently described as the submanifold of flat $d + 2$ dimensional flat manifold with signature $(-, -, +, \cdots +)$ embedded by the equation

$$\phi \chi + \eta_{\mu \nu} X^\mu X^\nu = - R^2,$$

where $\eta_{\mu \nu}$ is a $d$-dimensional metric with signature $(-, +, \ldots, +)$ and $R$ is the radius of $AdS_{d+1}$ spacetime. The boundary of $AdS$ is obtained by considering points at infinity subject to (2.1). More precisely define the primed coordinates by $\phi = \lambda \phi', \chi = \lambda \chi$, and $X^\mu = \lambda X'^\mu$ and take the limit $\lambda \to \infty$. 

then the boundary is described by the surface

$$\phi' \chi' + \eta_{\mu\nu} X'^\mu X'^\nu = 0, \quad (2.2)$$

subject to the equivalence relation $$(\phi', \chi', X'^\mu) \equiv s(\phi', \chi', X'^\mu)$$ where $s$ is any real non-zero number. The surface (2.2) is a compactification of a d-dimensional Minkowski spacetime: when $\phi' \neq 0$ the coordinates $X'^\mu$ span $\mathbb{R}^d$ which when added to the $\phi' = 0$ part provides the compactification, $S^1 \times S^{d-2}$, whose universal cover is $\mathbb{R} \times S^{d-1}$. Convenient coordinates of $AdS$ are given by $\phi$ and $x^\mu = RX^\mu/\phi$. These are well defined if $\phi \neq 0$ and $x^\mu$ remain finite in the scaling limit $\lambda \to \infty$. So the boundary at $\phi = \infty$ can be parametrised by $x^\mu$. In these coordinates the $AdS$ metric takes the form

$$g = \frac{\phi^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{\phi^2} d\phi d\phi. \quad (2.3)$$

This metric has an apparent singularity at $\phi = 0$, however as noted above, this is merely a coordinate singularity. Note that the $\phi = 0$ part of $AdS_{d+1}$ is given by $\mathbb{R} \times AdS_{d-1}^E$ where $AdS^E$ is Euclidean Anti-de-Sitter which is topologically a ball and its boundary is the sphere $S^{d-2}$. In the following, we shall see that $\phi$ has a natural interpretation as the radial distance from a fivebrane source and so the fivebrane is at $\phi = 0$. Note also that by a change of coordinates $\phi \to \beta \phi^\alpha$ the metric becomes $\beta^2 \phi^{2\alpha} / R^2 dx^dx + \alpha^2 R^2 d\phi d\phi / \phi^2$ which when $\alpha = 1/2$ and $\beta^2 / R^2 = 2/R$ will be the form we shall use in the following.

The eleven-dimensional fivebrane is a solitonic solution to the eleven-dimensional supergravity low energy equations of motion which preserves sixteen supersymmetries. The solution is given by

$$g_{p=5} = H^{-1/3} \eta_\mu dx^\mu dx^\nu + H^{2/3} \eta_{ij} d\phi^i d\phi^j, \quad G_{i_1...i_4} = \epsilon_{i_1...i_5} \partial^5 H, \quad (2.4)$$

where $x^\mu, \mu = 0, \ldots, 5$ are the coordinates along the fivebrane and $\phi^i$ are the transverse coordinates. The function $H$ entering the solution is radial and harmonic, $\eta^{ij} \partial_i \partial_j H = 0$, and so it reads

$$H = c + \frac{R^3}{8\phi^3}, \quad (2.5)$$

where $\phi^2 = \eta_{ij} \partial^i \partial^j, c$ is a constant and

$$R^3 = N l_p^3, \quad (2.6)$$

where $l_p$ is the eleven-dimensional Planck length and $N$ is the charge of the field strength $G$ or the number of fivebranes.
If one insists on having an asymptotically flat eleven dimensional spacetime then one has to impose that $c$ is not zero and one can set it equal to one. However, another solution is to set $c = 0$. This does not change the charge of the $G$ field strength and preserves one-half of the eleven-dimensional supersymmetries. In the latter case, $c = 0$, the metric takes the form of $AdS_7 \times S^4$. The radial coordinate $\phi$ combines with the worldvolume coordinates $x^\mu$ to form $AdS_7$. In fact in this case, the number of supersymmetries is greater than sixteen since the six-dimensional Poincaré invariance of the general ($c \neq 0$) case is transformed to a conformal invariance under the group $SO(6, 2)$ which when combined with the sixteen supersymmetries gives sixteen other special conformal supersymmetries. Another way to get the $AdS$ spacetime is to consider the decoupling limit $l_p \to 0$ and $\phi/l_p^3$ finite so that $\phi \ll R$ and one can neglect the constant $c$ in $H$. It is this limit that suggested the relation between the $AdS_{d+1}$ bulk theory and the worldvolume $d$-dimensional effective theory [5].

More generally, the $AdS/CFT$ conjecture states that M-theory in $AdS_7 \times S^4$ is holographically equivalent to the six-dimensional $(2,0)$ theory at the boundary of $AdS$ [4]. In the following, we shall show that it is possible to determine as an expansion in $R^3/\phi$ the metric and isometries of the superbackground whose bosonic limit is $AdS_7 \times S^4$. We shall calculate explicitly the first order terms which give the correction to the free dynamics of the fivebrane near the (part of the) boundary at $\phi = \infty$.

3 A realisation of $Osp(6, 2|2)$ on vector fields

In this section we determine, from the free six-dimensional $(2,0)$ multiplet, a realisation of $Osp(6, 2|2)$ on eleven-dimensional spacetime.

The six-dimensional $(2,0)$ superspace is spanned by the coordinates $(x^\mu, \theta^\alpha)$ where $\mu = 0, \ldots, 5$ and $\alpha$ is a spinorial index in the $(4_+, 4)$ of $SO(5, 1) \times SO(5)$, $SO(5)$ being the R-symmetry group. The “flat” superconformal transformations are defined as in ref. [21] by the requirement that the flat six-dimensional supersymmetric metric

$$g = \eta_{\mu\nu}E^\mu \otimes E^\nu,$$

where $E^\mu = dx^\mu - \bar{\theta}\Gamma^\mu d\theta$, transforms by a scale factor, that is

$$\delta g = L_\xi g = -\alpha g,$$

this guarantees that the tension of the string obtained from the open membrane stretched between the origin and $\phi$ is finite
where $\alpha$ is the scale factor. The realisation of the transformations in super-space was determined and is given by $\xi = \xi^\mu E_\mu + \xi^\alpha E_\alpha$. Here, $E_\mu = \partial_\mu$ and $E_\alpha = D_\alpha = \partial_\alpha - (\bar{\theta} \Gamma^\alpha)_{\alpha} \partial_\mu$ are the supersymmetric basis of vectors. The components of $\xi$ are determined in terms of $\zeta^\mu$, $\zeta^\alpha$ and $\zeta_{ij}$ which are defined by

$$\zeta^\mu = a_\mu + a_{\mu \nu} x^\nu + \lambda x_\mu + (x^2 \eta_{\mu \nu} - 2 x_\mu x_\nu) k^\nu,$$

and

$$\zeta^\beta = e^\beta + x_\mu (\Gamma^\mu)^{\beta \alpha'} \eta_{\alpha'}^\mu, \quad \zeta_{ij} = \frac{e_{ij}}{4}.$$ (3.3)

The parameters $a_\mu$, $a_{\mu \nu}$, $\lambda$, $k_\mu$, $\epsilon_{ij}$, $e^\alpha$ and $\eta^\alpha$ are those of infinitesimal translations, Lorentz transformations, dilatation, special conformal transformations, rotations, supersymmetries and fermionic special conformal transformations. The six-dimensional super Killing vector fields are given by [21]

$$\xi^\mu_0 = \zeta^\mu - 2 \bar{\theta} \Gamma^\mu \zeta + \bar{\theta} \left( \Gamma^{\mu \nu} \zeta^\nu + \frac{1}{4} \Gamma^{\mu \nu \rho} \partial_{\mu_1} \zeta_{\mu_2} \right) \theta
+ \frac{1}{2} \bar{\theta} \Gamma^{\mu_1 \mu_2} \theta \bar{\theta} \Gamma_\mu \partial_\mu \zeta - \frac{1}{64} \bar{\theta} \Gamma^{\mu_1 \mu_2} \theta \bar{\theta} \Gamma_{\mu_1 \mu_2 \mu_3} \theta \partial_\rho \partial^\sigma \zeta^{\mu_3},$$ (3.5)

and

$$\xi^\alpha_0 = - \left[ \zeta - \Gamma_{ij} \theta \zeta^i \right] + \frac{1}{12} \theta \partial_\mu \zeta^\mu - \frac{1}{4} \Gamma^{\mu \nu} \theta \partial_\mu \zeta_{\nu}
- \frac{1}{2} \theta \bar{\theta} \partial_\mu \zeta - \frac{1}{2} \Gamma^{\mu \nu} \theta \bar{\theta} \Gamma_\mu \partial_\nu \zeta - \frac{1}{24} \Gamma_{\mu_1 \mu_2} \partial_\mu \zeta_{\mu_3} \theta \Gamma^{\mu_1 \mu_2 \mu_3} \theta
+ \frac{1}{32} \Gamma^{\mu_1 \mu_2} \theta \bar{\theta} \Gamma_{\mu_1 \mu_2 \mu_3} \theta \partial_\rho \partial^\sigma \zeta_{\mu_3}.\] (3.6)

The scale factor, $\alpha$, is given by

$$\alpha = \frac{1}{3} \partial_\mu \zeta^\mu - \frac{2}{3} \bar{\theta} \Gamma^\alpha \partial_\alpha \zeta,$$ (3.7)

which reads explicitly

$$\alpha = 2 \lambda - 4 x^\mu k_\mu - 4 \bar{\theta} \eta.$$ (3.8)

The $(2,0)$ on-shell multiplet is described by a superfield $\Phi^i$ in the vectorial representation of $SO(5)$ subject to the constraint [23]

$$D_\alpha \Phi^i = \frac{1}{4} (\Gamma^i_{\ j})_\alpha^\beta D_\beta \Phi^j.$$ (3.9)
It has been shown in [21] that (3.9) is invariant under the superconformal group provided the scalar superfield $\Phi^i$ transforms as

$$\delta_0 \Phi^i = \xi_0(\Phi^i) + \Lambda^{ij}_0(\xi) \Phi^j,$$  \hspace{1cm} (3.10)

where

$$\Lambda_{ij}(\xi) = 4\zeta_{ij} - \frac{2}{3} \bar{\theta} i \bar{\Gamma}_j \Gamma^\sigma \partial_\sigma \zeta + \frac{1}{4} \bar{\theta} \Gamma^\nu \Gamma_{ij} \theta \partial_\sigma \partial^\rho \zeta^\nu + \frac{1}{3} \eta_{ij} \partial_\mu \zeta^\mu.$$  \hspace{1cm} (3.11)

From (3.10) and (3.11) we get

$$\delta_0 \Phi = \xi_0(\Phi) + \alpha \Phi,$$  \hspace{1cm} (3.12)

Where $\Phi^2 = \Phi^i \Phi_i$. The fermionic superfield whose $\theta = 0$ component, $\psi^{\prime \alpha}$ is the superpartner of $\phi^i$, the $\theta = 0$ component of $\Phi^i$, is $\Psi = 1/5 \Gamma_i D \Phi^i$. Under a superconformal transformation, it transforms as

$$\delta \Psi = \xi(\Psi) + \alpha \Psi + \Phi^j \Gamma_j \partial_\alpha + D \xi \Psi,$$  \hspace{1cm} (3.13)

where $D \xi$ denotes the matrix $D_\alpha \xi^\beta$. We can construct a realisation of the superconformal group in eleven-dimensional superspace as follows. We consider

$$x^\mu = (x^\mu, \phi^i)$$

and

$$\theta^\alpha = (\theta^\alpha, \psi^{\prime \alpha}),$$

to be the even coordinates and the odd coordinates. The two spinors of $SO(5,1) \times SO(5)$ of opposite six-dimensional chirality combine to form one Majorana spinor of $SO(10, 1)$. The vector fields

$$\xi_0 = \xi_0^\alpha \partial_\alpha + \xi_0^i \partial_i - \Lambda^j_0(\xi) \partial_j - \left((\alpha + \phi^j \Gamma_j \partial_\alpha + D \xi) \psi^{\prime \alpha}\right) \partial_{\alpha'} ,$$  \hspace{1cm} (3.14)

where $\partial_\alpha = \partial / \partial \phi^i$ and $\partial_{\alpha'} = \partial / \partial \psi^{\prime \alpha}$, provide a realisation of the superconformal algebra on eleven dimensional superspace. Note that $\phi^i$ and $\psi^{\prime \alpha}$ are no longer fields but become coordinates in the eleven-dimensional superspace.

4 Modified bosonic realisation

In this section, we show that the realisation found in the previous section does not provide us with the super Killing vectors of $AdS^7 \times S^4$. In order to see that, it suffices to examine the bosonic part of the metric and the transformations.

The bosonic metric of $AdS_7 \times S^4$ is given by

$$g = \frac{2\phi}{R^2} \eta_{\mu \nu} dx^\mu dx^\nu + \frac{R^2}{4\phi^2} \eta_{ij} d\phi^i d\phi^j ,$$  \hspace{1cm} (4.1)
where $\phi^2 = \phi^i \phi_i$. This metric is not invariant under the transformations generated by the bosonic part of $\xi_0$ constructed above. In fact we have

$$\delta_0(\eta_{\mu\nu} dx^\mu dx^\nu) = -\alpha \eta_{\mu\nu} dx^\mu dx^\nu$$

(4.2)

and

$$\delta_0 \phi = \alpha \phi,$$

(4.3)

so the first piece of the metric is invariant. However the second part is not invariant and its variation is given by

$$\delta_0 \left( \frac{R^2}{4\phi^2} \eta_{ij} d\phi^i d\phi^j \right) = \frac{R^2}{4\phi^2} \phi^j \left( d\alpha d\phi^j + d\phi^j d\alpha \right) \eta_{ij}$$

(4.4)

This variation can be cancelled by modifying the transformation of $x^\mu$ by

$$\delta x^\mu = \delta_0 x^\mu + \frac{R^3}{8\phi} \partial^\mu \alpha.$$ (4.5)

The modified transformations, as will be seen in the next section, still close on the superconformal algebra. In order to extend the bosonic metric to the superconformal case we have to modify the realisation found in the previous section of the superconformal algebra on eleven dimensional superspace. This will be the goal of the next section.

5 Modified superconformal realisations

Let $g$ be an element of the superconformal algebra which is presented in Appendix B, the vector field which gives a realisation of $Osp(6,2|2)$ must verify

$$[\xi(g), \xi(g')] = \xi([g, g']),$$

(5.1)

In terms of the generators, $\xi(g)$ is of the form

$$\xi = a^\mu P_\mu + \epsilon^a Q_a + \frac{1}{2} a^{\mu\nu} M_{\mu\nu} + \frac{1}{2} \epsilon^{ij} J_{ij} + k^\mu K_\mu + \eta^{\alpha'} S_{\alpha'} + \lambda D,$$

(5.2)

We write $\xi$ as $\xi_0 + \Delta \xi$, where $\xi_0$ is the previously determined solution to the closure relations. So $\Delta \xi$ must be a solution of

$$[\Delta \xi(g), \xi_0(g')] + [\xi_0(g), \Delta \xi(g')] + [\Delta \xi(g), \Delta \xi(g')] = \Delta \xi([g, g']).$$

(5.3)
We require that \((x^\mu, \theta^\alpha)\) span the six-dimensional superspace so that translations and supersymmetric transformations are unchanged

\[
\Delta P_\mu = \Delta Q_\alpha = 0, \tag{5.4}
\]

and we also require that the Lorentz and \(R\)-group \(SO(5)\) realisations be unchanged

\[
\Delta M_{\mu\nu} = \Delta J_{ij} = 0. \tag{5.5}
\]

The weights of the different coordinates remain the same so \(\Delta D = 0\).

\[
\Delta = 0. \tag{5.6}
\]

So the modification concerns only \(\Delta K_\mu \equiv X_\mu\) and \(\Delta S_\alpha \equiv X_\alpha\). These have to obey the following constraints which are the closure relations (5.1) expressed in terms of the generators

\[
\begin{align*}
[P_\mu, X_\alpha] &= 0, & [P_\mu, X_\nu] &= 0, \\
\{X_\alpha, Q_\beta\} &= 0, & [X_\mu, Q_\alpha] &= -(\Gamma_\mu)_\alpha^\beta X_\beta, \\
[X_\mu, K_\nu] + [K_\mu, X_\nu] + [X_\mu, X_\nu] &= 0, \\
[X_\alpha, S_\beta] + [S_\alpha, X_\beta] + \{X_\alpha, X_\beta\} &= 2(\Gamma^\mu)_\alpha^\beta X_\mu.
\end{align*} \tag{5.7}
\]

The other commutation relations give the Lorentz, \(SO(5)\) and dilatation transformations of \(X_\mu\) and \(X_\alpha\). The first two equations imply that the components of \(X_\mu\) and \(X_\alpha\) are independent of \(x\) in the \((\partial_\mu, D_\alpha, \partial_i, \partial_\alpha')\) basis. Similarly the next two equations imply that the components of \(X_\alpha\) are independent of \(\theta\) and that those of \(X_\mu\) are of the form

\[
X_\mu = Y_\mu + \theta^\alpha (\Gamma_\mu)_\alpha^\beta X_\beta, \tag{5.8}
\]

where the components of \(Y_\mu\) are independent of \(\theta\). The last equation turns out, after the use of Jacobi identities, to be a consequence of the fifth relation. Finally we end with the equation

\[
[X_\mu, K_\nu] + [K_\mu, X_\nu] + [X_\mu, X_\nu] = 0, \tag{5.9}
\]

where the components of \(X_\mu\) are independent of \(x\) and depend linearly on \(\theta\) as in equation (5.8).

Let us examine first the bosonic case, where \(\theta\) and \(\psi\) are absent. The most general form of \(X_\mu\) whose components are independent of \(x\) and have the correct transformations under \(SO(5,1)\), \(SO(5)\) and dilatations is given by

\[
X_\mu^{(b)} = a \frac{1}{\theta} \partial_\mu, \tag{5.10}
\]
where \( a \) is a constant. Note that there are no components on \( \partial_i \) because there are no Lorentz vectors independent of \( x \). This \( X_\mu \) satisfies [\( X_\mu, X_\nu \)] = 0 and a non-trivial calculation gives, for any value of \( a \),

\[
[X_\mu^{(b)}, K_\nu] + [K_\mu, X_\nu^{(b)}] = 0. \tag{5.11}
\]

So in the bosonic case we have a one parameter deformation of the realisation of the conformal Lie algebra on space-time.

In the supersymmetric case, the most general expression for \( X_\mu \) is much more complicated, it has an expansion in powers of \( \psi \) which a priori stops at sixteen \( \psi \)'s. However, this power series in \( \psi \) is correlated, as will be shown in the next section, with the dependence on \( R \) and \( \phi \). This will help simplifying the resulting expressions.

6 Expansion in \( R \)

Define \( \phi'^i \) by \( \phi'^i = R^3 \phi^i \) then the bosonic metric, expressed in the new coordinates, when divided by \( R^2 \) does not depend on \( R \). From the expression of \( \xi_0 \), we deduce that if we define \( \psi' = R^{-3} \psi \) then \( \xi_0 \), when expressed in terms of the primed quantities is independent of \( R \). Since the bosonic metric has only an overall scale dependence on \( R \) the full \( \xi \), which are the Killing vectors of the full superconformal metric, are independent of \( R \) when expressed in terms of the primed quantities.

Let us first consider \( Y_\mu \), it has an expansion in \( \psi' \) of the form

\[
Y_\mu = \sum_n \left( \frac{y_n^{(n)}(\psi')}{\phi^{1+5n/4}} \partial_\nu + \frac{y_n^\alpha(\psi')}{\phi^{3/4+5n/4}} D_\alpha \\
+ \frac{y_n^i(\psi')}{\phi^{1/2+5n/4}} \partial_i + \frac{y_n^\alpha(\psi')}{\phi^{3/4+5n/4}} D_\alpha' \right), \tag{6.1}
\]

where the \( y_n^{(n)}(\psi') \) are homogeneous polynomials in \( \psi' \) of degree \( n \). The powers of \( \phi \) in the denominators are determined from the behaviour under dilatations. When expressed in terms of unprimed quantities we get

\[
Y_\mu = \sum_n \left( \mathcal{R}^{4+n} \frac{y_n(\psi)}{\phi^{1+5n/4}} \partial_\nu + \mathcal{R}^{3+n} \frac{y_n^\alpha(\psi)}{\phi^{3/4+5n/4}} D_\alpha \\
+ \mathcal{R}^{2+n} \frac{y_n^i(\psi)}{\phi^{1/2+5n/4}} \partial_i + \mathcal{R}^{1+n} \frac{y_n^\alpha(\psi)}{\phi^{3/4+5n/4}} D_\alpha' \right), \tag{6.2}
\]
where $R = R^{3/4}$. Lorentz invariance constrains the degrees of polynomials appearing in (6.2). For example, all terms quadratic in $\psi$ are of the form $\bar{\psi} \Gamma^{\mu_1 \nu_{2\mu_3} \psi} \psi \Gamma^{\mu_{ij} \psi}$, $\bar{\psi} \Gamma^{\mu_1 \nu_{2\mu_3} \psi} \Gamma^{\mu_{ij} \psi}$, so they all have an odd number of six-dimensional vector indices and thus cannot contribute to $Y_{\mu}$. Using this kind of arguments one can show that if a polynomial of degree $n$ contributes to one component then the other polynomials appearing in the component have degrees $n + 4p$. The lowest degree polynomials appearing in each component are given by

\begin{align}
 y_{(0)\mu} &= a_1 \frac{\eta_{\mu} \psi}{\phi}, \\
y_{(1)\mu} &= a_2 (\Gamma_{\mu} \psi)^{\alpha} \frac{\phi^i}{\phi^3} + a_3 (\Gamma_{\mu} \psi)^{\alpha} \frac{1}{\phi^2}, \\
y_{(2)\mu} &= a_4 \bar{\psi} \Gamma_{\mu} \psi^{\phi_j} \phi^3, \\
y_{(3)\mu} &= a_5 (\Gamma^{\mu_1 \nu_{2\mu} \psi})^{\alpha} (\bar{\psi} \Gamma_{\mu_1 \nu_2} \psi) \frac{1}{\phi^3} + a_6 (\Gamma^{\mu_1 \nu_{2\mu} \psi})^{\alpha} (\bar{\psi} \Gamma_{\mu_1 \nu_2} \psi) \frac{\phi_i}{\phi^4} \\
 &+ a_7 (\Gamma^{\mu_1 \nu_{2\mu} \psi})^{\alpha} (\bar{\psi} \Gamma_{\mu_1 \nu_2} \psi) \frac{\phi^i}{\phi^4} + a_8 (\Gamma^{\mu_1 \nu_{2\mu} \psi})^{\alpha} (\bar{\psi} \Gamma_{\mu_1 \nu_2} \psi) \frac{\phi_i \phi^j}{\phi^5},
\end{align}

(6.3)

where $a_i$, $i = 1, \ldots, 8$ are arbitrary constants. We have used Fierz rearrangements to eliminate some of the terms with three powers of $\psi$ (see Appendix A). Note that the leading power of $r$ in (6.2) for all the components is $R^{3/4}$; from (6.2) and (6.3) we see that the general form of $Y_{\mu}$ is given by

\begin{align}
 Y_{\mu} &= \sum_{p=0}^{4} R^{4(p+1)} Y_{\mu}^{(p)},
\end{align}

(6.4)

with

\begin{align}
 Y_{\mu}^{(p)} &= y_{(4p)\mu} \frac{1}{(\phi)^{1+5p}} \partial_{\nu} + y_{(1+4p)\mu} \frac{1}{(\phi)^{2+5p}} D_{\alpha} \\
 &+ y_{(2+4p)\mu} \frac{1}{(\phi)^{2+5p}} \partial_{i} + y_{(3+4p)\mu} \frac{1}{(\phi)^{3+5p}} \partial_{\alpha}. \quad (6.5)
\end{align}

Similarly, one can make an analogous analysis for $X_{\alpha}$. Here, the lowest degree
polynomial contributing to $X_\alpha$ are given by

\begin{align}
x^{\mu}_{(1)\alpha} &= b_1 (\Gamma^\mu \psi)_\alpha \frac{1}{\phi^2} + b_2 (\Gamma^{\mu i} \psi)_\alpha \frac{\partial i}{\phi^3} \\
x^{\beta}_{(2)\alpha} &= b_3 (\Gamma_{\nu ij})_\alpha \beta (\bar{\psi} \Gamma^{\nu ij} \psi) \frac{1}{\phi^4} + b_4 (\Gamma_{\nu i j} \psi)_\alpha \beta (\bar{\psi} \Gamma^{\nu i j} \psi) \frac{\partial i}{\phi^5} + b_5 (\Gamma_{\nu i j} \psi)_\alpha \beta (\bar{\psi} \Gamma^{\nu i j} \psi) \frac{\partial^i}{\phi^4} + b_6 (\Gamma_{\nu i j} \psi)_\alpha \beta (\bar{\psi} \Gamma^{\nu i j} \psi) \frac{\partial^j}{\phi^5} \\
x^{\alpha}_{(3)\alpha} &= b_7 (\Gamma_{\nu} \psi)_\alpha (\bar{\psi} \Gamma^{\nu ij} \psi) \frac{\partial i}{\phi^4} + b_8 (\Gamma_{\nu i j} \psi)_\alpha (\bar{\psi} \Gamma^{\nu ij} \psi) \frac{\partial^i}{\phi^5} + b_9 (\Gamma_{\nu i j} \psi)_\alpha (\bar{\psi} \Gamma^{\nu i j} \psi) \frac{\partial^j}{\phi^6} \\
x^{\alpha'}_{(0)\alpha} &= b_0 \delta_{\alpha} \alpha' \phi + b_{10} (\Gamma_{i})_\alpha \alpha' \phi^i.
\end{align}

(6.6)

The corresponding expansion in $r$ follows with the difference with respect to $Y_\mu$ that the last term in (6.6) contributes at order zero and the contribution to the first order comes from terms with four $\psi$’s. We deduce that $X_\alpha$ and $X_\mu$ have an expansion of the form

\begin{align}
X_\alpha &= \sum_{p=1}^{4} R^{4p} X^{(p)}_\alpha, \\
X_\mu &= \sum_{p=1}^{4} R^{4p} X^{(p)}_\mu,
\end{align}

(6.7)

and the closure relation gives

\begin{align}
[K_{\mu}, X^{(p)}_\nu] + [X^{(p)}_\mu, K_{\nu}] &= \sum_{q+q'=p} [X^{(q)}_\nu, X^{(q')}_{\mu}].
\end{align}

(6.8)

Since $K_{\mu}$ has an $x$ and $\theta$ dependence, each component of the equation above gives upon identification of terms dependent on $x$ and $\theta$ a series of equations which consistency is not guaranteed a priori. To the lowest order, equation (6.8) yields the linear equation

\begin{align}
[K_{\mu}, X^{(1)}_\nu] + [X^{(1)}_\mu, K_{\nu}] &= 0,
\end{align}

(6.9)

where $X^{(1)}_\mu = Y^{(1)}_\mu + \bar{\theta} \Gamma_{\mu} X^{(1)}$, with the components of $Y^{(1)}_\mu$ given in (5.3) and those of $X^{(1)}_\alpha$ given in (5.6). The detailed analysis of the equations is long but straightforward. The number of equations is much greater than the number of coefficients. We have checked the compatibility of all the equations. Some of the resulting equations are homogeneous and are identically satisfied, if we set $a_1 \neq 0$, then the inhomogeneous equations allow the determination of the other coefficients in terms of $a_1$. We obtain the following results for the coefficients in $Y_\mu$:

\begin{align}
a_3 &= a_5 = a_6 = a_7 = 0 \\
a_1 &= -4a_2 = 8a_4 = 32a_8
\end{align}

(6.10)
and for $X_\alpha$

$$
\begin{align*}
  b_1 &= b_5 = b_6 = b_7 = b_9 = b_{10} = 0, \\
  b_2 &= -\frac{a_1}{4}, \quad b_3 = \frac{a_1}{128}, \quad b_4 = -\frac{a_1}{32}, \quad b_8 = -\frac{a_1}{32}.
\end{align*}
$$

(6.11)

7 Invariant metric

In this section we shall use the Killing vector fields determined in the preceding section to deduce the invariant metric. We write the metric in the form

$$
\begin{align*}
g &= \frac{2\phi}{R}\eta_{\mu\nu}\bar{E}^\mu \otimes \bar{E}^\nu + \frac{R^2}{4\phi^2}\eta_{ij}\bar{E}^i \otimes \bar{E}^j.
\end{align*}
$$

(7.1)

taking into account the behaviour under translation, supersymmetry, dilations, rotation and Lorentz transformations as well as a $\mathbb{Z}_2$ symmetry acting as $\phi^i \to -\phi^i$ and $\psi \to -\psi$, we get, to order $R^4$ the following expression for $\bar{E}^\mu$

$$
\begin{align*}
\bar{E}^\mu &= E^\mu + R^4 \left( c_1 \frac{1}{\phi^5} (\bar{\psi} \Gamma^{\mu\nu_1\nu_2} \psi)(\bar{\psi} \Gamma_{\nu_1\nu_2} \psi)E^\nu + c_2 (\bar{\psi} \Gamma^\mu d\psi) \frac{1}{\phi^3} \right. \\
&\quad + c_3 (\bar{\psi} \Gamma^{\mu ij} \psi)(\bar{\psi} \Gamma_{ij} d\theta) \frac{1}{\phi^4} + c_4 (\bar{\psi} \Gamma^{\mu ij} \psi)(\bar{\psi} \Gamma_{kj} d\theta) \frac{\phi_k}{\phi^6} \\
&\quad + c_5 (\bar{\psi} \Gamma^{\mu ij} \psi)(\bar{\psi} \Gamma_{ij} d\theta) \frac{\phi_j}{\phi^4} + c_6 (\bar{\psi} \Gamma^{\mu ij} \psi)(\bar{\psi} \Gamma_{ij} d\theta) \frac{\phi_j}{\phi^4} \\
&\quad + c_7 (\bar{\psi} \Gamma^{\mu i} \psi)(\bar{\psi} \Gamma_{i} d\theta) \frac{\phi_j}{\phi^5} + c_8 (\bar{\psi} \Gamma^{\mu i} \psi)(\bar{\psi} \Gamma_{i} d\theta) \frac{\phi_j}{\phi^5} \\
&\left. \quad + c_9 (\bar{\psi} \Gamma^{\mu i} \psi)(\bar{\psi} \Gamma_{i} d\theta) \frac{\phi_j}{\phi^5} + c_{10} (\bar{\psi} \Gamma^{\mu ij} \psi)(\bar{\psi} \Gamma_{ij} d\theta) \frac{\phi_j}{\phi^5} \right). 
\end{align*}
$$

(7.2)

The terms in $c_3$, $c_4$, $c_5$ and $c_6$ are redundant since, by Fierz rearrangements they can be cast into the form of already present terms (see Appendix A); so we can set these coefficients to zero. Similarly, we get for $\bar{E}^i$, to the order zero

$$
\begin{align*}
\bar{E}^i &= d\phi^i + d_1 (\bar{\psi} \Gamma^i d\theta) + d_2 (\bar{\psi} \Gamma^{ij} d\theta) \frac{\phi_j}{\phi} \\
&\quad + d_3 (\bar{\psi} \Gamma^{ij} d\theta) \frac{\phi_j}{\phi^2} + d_4 (\bar{\psi} d\theta) \frac{\phi^i}{\phi} + d_5 (\bar{\psi} \Gamma^{ij} \psi) \frac{\phi_j}{\phi^2} E^\mu.
\end{align*}
$$

(7.3)

The coefficients $c_i$ and $d_i$ are determined by the requirement of invariance of the metric under conformal and special conformal transformations which
were determined in the preceding section. The metric can be decomposed as a term of order zero and a term of order one

\[ g = g_0 + g_1, \]  

where

\[ g_0 = \frac{2\phi}{R} \eta_{\mu\nu} E^\mu \otimes E^\nu \]  

and

\[ g_1 = \frac{2\phi}{R} \eta_{\mu\nu} \left( (\bar{E}^\mu - E^\mu) \otimes E^\nu + E^\mu \otimes (\bar{E}^\nu - E^\nu) \right) + \frac{R^2}{4\phi^2} \eta_{ij} \bar{E}^i \otimes \bar{E}^j. \]  

The equations governing the invariance of the metric give at order zero

\[ L_{\xi_0} g_0 = 0, \]  

which is satisfied by the construction of \( \xi_0 \) presented in section 3. At the next order we get

\[ L_{\Delta \xi} g_0 + L_{\xi_0} g_1 = 0. \]  

This equation is satisfied by construction for all generators except for conformal and superconformal transformations (the invariance under the other transformations are implemented in the ansatz (7.2)–(7.3)). These give the values of the constants appearing in \( \bar{E}^\mu \):

\[ c_3 = c_4 = c_5 = c_6 = c_9 = 0 \]
\[ c_1 = -\frac{1}{512}, \quad c_2 = -\frac{1}{32} \]
\[ c_7 = -\frac{1}{16} + \frac{\sqrt{3}}{128}, \quad c_8 = -\frac{1}{32} + \frac{\sqrt{3}}{128} \]  

\[ c_{10} = -\frac{3}{64} + \frac{\sqrt{3}}{32}, \]  

and those appearing in \( \bar{E}^i \):

\[ d_2 = d_3 = d_4 = 0, \]
\[ d_1 = 1, \quad d_5 = \frac{\sqrt{3}}{4}. \]  

In addition the value of \( a_1 \) is fixed as

\[ a_1 = \frac{1}{2}. \]
8 Interaction in the six-dimensional (2,0) theory

In this section we derive the equations governing a fivebrane in the \(AdS_7 \times S^4\) background to the first order in \(R^4\). This will give the first order interaction terms in the \((2,0)\) theory. We shall use the superembedding approach which is particularly convenient for our purposes. This approach is manifestly worldvolume and target space supersymmetric. It has been shown that, by fixing the worldvolume coordinates, it leads to the manifestly target space supersymmetric Green–Schwarz equations for p-branes. Here, we shall choose to work in the “physical” gauge which leads to manifestly worldvolume supersymmetric equations in terms of the superfields \(\Phi^i\) and \(\Psi\) which appeared in the free \((2,0)\) theory reviewed in section 3.

In the physical gauge, the super worldvolume of the fivebrane is spanned by the \((x^\mu, \theta^\alpha)\) coordinates. The dynamics is described by the superfields \(\Phi^i(x, \theta)\) and \(\Psi(x, \theta)\). Let \(e_\alpha\) be a basis of odd vector fields on the worldvolume. And let \(e^\mu\) be the pull-back of the eleven-dimensional even moving basis:

\[
e^\mu(\alpha) = \delta^\mu_\alpha D_\alpha + \Delta^\mu_\alpha \partial_\mu.
\] (8.4)

Note that \(\Delta^\mu_\alpha\) is determined from (8.2) as well as the equation relating \(\Phi^i\) to \(\Psi\).

The \(\mu\) components of the equation (8.2) give

\[
0 = \Delta^\mu_\alpha + R^4 \left( c_1 \frac{1}{\Phi_5} (\bar{\Psi} \Gamma^{\mu \nu_1 \nu_2} \Psi)(\bar{\Psi} \Gamma_{\nu_1 \nu_2} \Psi) \Delta^\nu_\alpha - c_2 (e_\alpha(\bar{\Psi}) \Gamma^{\mu} \Psi) \frac{1}{\Phi_5} \\
+ [c_7 (\bar{\Psi} \Gamma^{\mu \nu_1 \nu_2} \Psi)(\Gamma_{\nu_1 \nu_2} \Psi) \frac{\Phi^j}{\Phi_5} - c_8 (\bar{\Psi} \Gamma^{\mu \nu_1 \nu_2} \Psi)(\Gamma_{\nu_1 \nu_2} \Psi) \frac{\Phi_i}{\Phi_5}]_\alpha \\
+ c_{10} (\bar{\Psi} \Gamma^{\mu \nu} \Psi) \frac{\Phi_i e_\alpha(\Phi_j)}{\Phi_5} \right),
\] (8.5)
where we have kept non-zero constants. The $i$ components of equation (8.2) give

$$0 = e^\alpha (\Phi^i) - d_1(\Gamma^i\Psi\alpha) + d_5(\bar{\Psi}\Gamma_\mu^{ij}\Psi)\frac{\Phi_j}{\Phi^5}\Delta^\mu_\alpha.$$  (8.6)

Equation (8.5) shows that, to the first order in $R^4$, $\Delta^\mu_\alpha$ is given by

$$\Delta^\mu_\alpha = -R^4 \left[ -c_2(D_\alpha(\bar{\Psi})\Gamma^\mu\Psi)\frac{1}{\Phi^3} + \left[c_7(\bar{\Psi}\Gamma^{\mu\nu_2}\Psi)(\Gamma_{\nu_1\nu_2})\frac{\Phi_i}{\Phi^5}\Phi^j - c_8(\bar{\Psi}\Gamma^{\mu\nu_2}\Psi)(\Gamma_{\nu_1\nu_2})\Phi_j + c_{10}(\bar{\Psi}\Gamma^{\mu\nu_2}\Psi)\frac{\Phi_i}{\Phi^5}\right] e^\alpha \right].$$  (8.7)

Substituting the expression of $\Delta^\mu_\alpha$ in equation (8.6), we get

$$D_\alpha \Phi^i = d_1(\Gamma^i\Psi\alpha) + R^4 \left[ -c_2(D_\alpha(\bar{\Psi})\Gamma^\mu\Psi)\frac{1}{\Phi^3} + \left[c_7(\bar{\Psi}\Gamma^{\mu\nu_2}\Psi)(\Gamma_{\nu_1\nu_2})\frac{\Phi_i}{\Phi^5}\Phi^j - c_8(\bar{\Psi}\Gamma^{\mu\nu_2}\Psi)(\Gamma_{\nu_1\nu_2})\Phi_j + c_{10}(\bar{\Psi}\Gamma^{\mu\nu_2}\Psi)\frac{\Phi_i}{\Phi^5}\right] e^\alpha \right].$$  (8.8)

To order zero we have [21]

$$D_\beta \Psi = -(\Gamma^\mu)_{\beta\alpha} \partial_\mu \Phi^i + \frac{1}{5} H_{\beta\alpha},$$

$$D_\gamma H_{\alpha\beta} = 5(\Gamma^\mu)_{\gamma\alpha} \partial_\mu \Psi + 5(\Gamma^\mu)_{\gamma\alpha} \partial_\mu \Psi$$  (8.9)

where

$$H_{\alpha\beta} = H_{\mu_{1}\mu_{2}\mu_{3}}(\Gamma_{\alpha\beta\gamma})_{\mu_{1}\mu_{2}\mu_{3}},$$  (8.10)

$H_{\mu_{1}\mu_{2}\mu_{3}}$ being a self-dual 3-form. So to order $R^4$, we can replace in the right hand side of equation (8.8) $D_\Phi$ by $\Gamma^i\Psi$ and $D_\Psi$ by its zero order expression (8.9). The equation of motion to the first order becomes

$$D_\alpha \Phi^i = (\Gamma^i\Psi) + R^4 \left[ -c_2(D_\alpha(\bar{\Psi})\Gamma^\mu\Psi)\frac{1}{\Phi^3} + \left[c_7(\bar{\Psi}\Gamma^{\mu\nu_2}\Psi)(\Gamma_{\nu_1\nu_2})\Phi_j - c_8(\bar{\Psi}\Gamma^{\mu\nu_2}\Psi)(\Gamma_{\nu_1\nu_2})\Phi_j + c_{10}(\bar{\Psi}\Gamma^{\mu\nu_2}\Psi)\Phi_j\right] \frac{\Phi_i}{\Phi^3} \right],$$  (8.11)

where $c'_2 = c_7 - c_{10}/4$ and $c'_8 = c_8 - c_{10}/4$. These equations are invariant under the superconformal transformations which read

$$\delta \Phi^i = \delta_0 \Phi^i + a_1 R^4 \left[ \Phi^\mu - \frac{1}{4\Phi^3}(\bar{\eta}\Gamma^{\mu\nu_2}\Psi)\Phi_j \right] \left[ \partial_\mu \Phi^i - \frac{1}{8}(\bar{\Psi}\Gamma^{\mu\nu_2}\Psi)\Phi_j \right]$$  (8.12)
where $\eta' = \eta - \kappa \Gamma^\mu \theta$ and $\delta_0 \Phi^i$ is given in (3.10).

On the boundary of $AdS$, the resulting equations are the doubleton equation of the free multiplet, the first correction in $R$ yields an interacting (2,0) theory with a non-linear realisation of the superconformal algebra.

A Conventions and Fierz rearrangements

The eleven dimensional superalgebra reads

$$\{Q_{\hat{\alpha}}, Q_{\hat{\beta}}\} = 2(\Gamma^{\hat{\mu}}C)_{\hat{\alpha}\hat{\beta}}\hat{P}_{\hat{\mu}},$$

(A.1)

where $\hat{\alpha} = 1, \ldots, 32$, $\hat{\mu} = 0, \ldots 10$ and $C_{\hat{\alpha}\hat{\beta}}$ is an antisymmetric matrix verifying

$$C^{-1}\Gamma^{\hat{\mu}}C = -\Gamma^{\hat{\mu}T}.$$  

(A.2)

The reality condition on 11D fermions reads

$$\Psi = C\bar{\Psi}^T,$$

(A.3)

or equivalently

$$\bar{\Psi}^{\hat{\alpha}} = C^{\hat{\alpha}\hat{\beta}}\Psi_{\hat{\beta}} \equiv \Psi^{\hat{\alpha}},$$

(A.4)

where $C^{\hat{\alpha}\hat{\beta}}$ is the inverse of $C_{\hat{\alpha}\hat{\beta}}$. We shall use $C$ to raise and lower indices and the notation $(\Gamma^{\hat{\mu}})_{\hat{\alpha}\hat{\beta}}$ for $(\Gamma^{\hat{\mu}}C)_{\hat{\alpha}\hat{\beta}}$.

We are interested in the $6 + 5$ splitting the eleven dimensional spacetime. A representation of the Gamma matrices is conveniently given by

$$\Gamma^\mu = \gamma^\mu \otimes 1, \quad \Gamma^{5+i} = \tilde{\gamma} \otimes \gamma^i,$$

(A.5)

where $\mu$ and $i$ are respectively six and five dimensional vector indices, and $\tilde{\gamma}$ is the chirality matrix in six dimensions:

$$\tilde{\gamma} = \gamma^0 \ldots \gamma^5.$$  

(A.6)

This matrix allows to decompose 11D Majorana fermions into two 6D $Sp(2)$ symplectic Majorana Weyl fermions: $\psi^{\hat{\alpha}} = (\psi^\alpha, \psi^{\prime \alpha})$. We shall also denote $\Gamma^{5+i}$ by $\Gamma^i$. In this representation the charge conjugaison matrix $C$ may be written as

$$C = \mathcal{C} \otimes \Omega,$$

(A.7)
where $C$ is symmetric and verifies
\begin{equation}
C^{-1} \gamma^\mu C = -\gamma^\mu T, \tag{A.8}
\end{equation}
whereas $\Omega$ is antisymmetric and verifies
\begin{equation}
\Omega^{-1} \gamma^i \Omega = \gamma^i T. \tag{A.9}
\end{equation}

The antisymmetrised product of $n$ gamma matrices is denoted by $\hat{\Gamma}^\mu_{\mu_1...\mu_n}$. We have
\begin{equation}
(\hat{\Gamma}^\mu_{\mu_1...\mu_n} C)^T = -(-1)^{n(n+1)/2} \Gamma^\mu_{\mu_1...\mu_n} C. \tag{A.10}
\end{equation}

A useful relation is the Fierz rearrangement formula which reads for four Weyl-Majorana fermions $\bar{\psi}$
\begin{equation}
(\bar{\psi}_1 \Pi^+ \gamma_{\nu_1} \psi_2) (\bar{\psi}_3 \Pi^+ \gamma_{\nu_2} \psi_4) = \sum_{n_1=0,2} \sum_{n_2=0,1,2} \frac{2}{c_{n_1} c_{n_2}} (\bar{\psi}_1 \Gamma_{\mu_1...\mu_n_1} \Gamma_{\nu_1...\nu_n_2} \Pi^+ \epsilon_4) (\bar{\psi}_3 \Gamma_{\mu_1...\mu_n_1} \Gamma_{\nu_1...\nu_n_2} \Pi^+ \epsilon_2),
\end{equation}
with coefficients $c_n$ and $\tilde{c}_n$ given by
\begin{equation}
c_n = 8 (-1)^{(n-1)/2} \text{ and } \tilde{c}_n = 4 (-1)^{n(n-1)/2}. \tag{A.12}
\end{equation}

$\Pi^\pm$ are the chirality projection operators in six dimensions. The formula (A.11) has been used in our derivations to restrict the number of independent terms appearing in the Killing vectors and in the metric, particularly those involving three fermions $\psi$. First, in the expression (6.3) of $y^0_{(3)}$ and in the expression (6.2) of $\tilde{E}^\mu$, Fierz rearrangements eliminate terms with combinations like $(\Gamma^i \psi)(\bar{\psi} \Gamma_{\nu_1} \psi)$ or $(\Gamma^{j_1 j_2} \psi)(\bar{\psi} \Gamma_{\nu_1 j_1 j_2} \psi)$, indeed
\begin{equation}
(\Gamma^i \psi)(\bar{\psi} \Gamma_{\nu_1} \psi) = \frac{1}{4} (\Gamma_{i\sigma_1 \sigma_2} \psi)(\bar{\psi} \Gamma_{\nu} \sigma_1 \sigma_2 \psi) + \frac{1}{4} (\Gamma_{\sigma_1 \sigma_2} \psi)(\bar{\psi} \Gamma_{i\nu \sigma_1 \sigma_2} \psi),
\end{equation}
\begin{equation}
(\Gamma^{j_1 j_2} \psi)(\bar{\psi} \Gamma_{\nu_1 j_1 j_2} \psi) = -(\Gamma_{\sigma_1 \sigma_2} \psi)(\bar{\psi} \Gamma_{i\nu \sigma_1 \sigma_2} \psi). \tag{A.13}
\end{equation}

\footnote{The summation over the indices $\mu$ and $i$ is ordered in the formula (A.11): $\mu_1 < \ldots < \mu_{n_1}$ and $i_1 < \ldots < i_{n_2}$. Otherwise some factorials appear in the normalization.}
The expression (6.6) of $x_{(3)\alpha}^i$ involves terms of the form
\[
(\Gamma_{\sigma} N_{[i_1 i_2]}^i \psi)(\bar{\psi} \Gamma^{\sigma i_1 i_2} \psi).
\] (A.14)

There are \textit{a priori} twelve terms that can be constructed using $\phi^i$ and gamma matrices:
\[
N_{[i_1 i_2]}^i = a(\delta_i^i \phi_{i_2} - \delta_i^j \phi_{i_1}) + a'(\delta_i^i \phi_{i_2} - \delta_i^j \phi_{i_1}) \frac{\phi_m \Gamma^m}{\phi}
+ b\phi(\delta_i^i \Gamma_{i_2} - \delta_i^j \Gamma_{i_1}) + b'(\delta_i^i \Gamma_{i_2 m} - \delta_i^j \Gamma_{i_1 m}) \phi^m
+ c\frac{\phi^j}{\phi}(\Gamma_{i_1} \phi_{i_2} - \Gamma_{i_2} \phi_{i_1}) + c'\frac{\phi^j \phi^m}{\phi^2}(\Gamma_{mi_1} \phi_{i_2} - \Gamma_{mi_2} \phi_{i_1})
+ d\phi^j \Gamma_{i_1 i_2} + d'\frac{\phi^j \phi^m}{\phi^2} \Gamma_{mi_1 i_2}
+ e(\Gamma_{i_1} \phi_{i_2} - \Gamma_{i_2} \phi_{i_1}) + e'(\Gamma_{mi_1} \phi_{i_2} - \Gamma_{mi_2} \phi_{i_1}) \frac{\phi^m}{\phi}
+ f\phi \Gamma_{i_1 i_2} + f'\Gamma_{mi_1 i_2 m} \phi^m.
\] (A.15)

However a Fierz rearrangement insures that $N_{[i_1 i_2]}^i$ and $(QN)_{[i_1 i_2]}^i$ where $Q$ is defined as
\[
(QN)_{[i_1 i_2]}^i = -\frac{1}{6} N_{[i_1 j] [i_2 k]}^i \Gamma_{j k} - \frac{1}{3} \left( N_{[i_1 j]}^i \Gamma_{j i_2} - N_{[i_2 j]}^i \Gamma_{j i_1} \right),
\] (A.16)
are equivalent in the sense that the expression (A.14) computed with $N_{[i_1 i_2]}^i$ and $(QN)_{[i_1 i_2]}^i$ are equal. Examining the eigenvalues of the operator $Q$ allows to conclude that
\[
b \sim c \sim c' \sim d \sim d' \sim f \sim 0
a \sim b' \sim -e \sim -f'
\] (A.17)
a' \sim e'.
Finally, we end up with only the two terms written in (3.6).
B  The \( Osp(6,2|2) \) algebra

\[
\begin{align*}
[p_{\mu}, p_{\nu}] &= 0 \quad (B.1) \\
[p_{\mu}, m_{\rho\sigma}] &= -\eta_{\mu\rho} p_{\sigma} + \eta_{\mu\sigma} p_{\rho} \quad (B.2) \\
[p_{\mu}, j_{ij}] &= 0 \quad (B.3) \\
[p_{\mu}, d] &= p_{\mu} \quad (B.4) \\
[p_{\mu}, k_{\nu}] &= -2\eta_{\mu\nu} d - 2m_{\mu\nu} \quad (B.5) \\
[p_{\mu}, q_{\alpha'}] &= 0 \quad (B.6) \\
[m_{\mu\nu}, m_{\rho\sigma}] &= -\eta_{\mu\sigma} m_{\nu\rho} + \eta_{\mu\rho} m_{\nu\sigma} - \eta_{\nu\rho} m_{\mu\sigma} + \eta_{\nu\sigma} m_{\mu\rho} \quad (B.8) \\
[m_{\mu\nu}, j_{ij}] &= 0 \quad (B.9) \\
[m_{\mu\nu}, d] &= 0 \quad (B.10) \\
[m_{\mu\nu}, k_{\sigma}] &= \eta_{\mu\sigma} k_{\nu} - \eta_{\nu\sigma} k_{\mu} \quad (B.11) \\
[m_{\mu\nu}, q_{\alpha'}] &= \frac{1}{2} (\Gamma_{\mu\nu})_{\alpha'\beta'} q_{\beta'} \quad (B.12) \\
[m_{\mu\nu}, s_{\alpha}] &= \frac{1}{2} (\Gamma_{\mu\nu})_{\alpha\beta} s_{\beta} \quad (B.13) \\
[j_{ij}, j_{kl}] &= \eta_{il} j_{jk} - \eta_{ik} j_{jl} + \eta_{jk} j_{il} - \eta_{jl} j_{ik} \quad (B.14) \\
[j_{ij}, d] &= 0 \quad (B.15) \\
[j_{ij}, k_{\mu}] &= 0 \quad (B.16) \\
[j_{ij}, q_{\alpha'}] &= -\frac{1}{2} (\Gamma_{ij})_{\alpha'\beta'} q_{\beta'} \quad (B.17) \\
[j_{ij}, s_{\alpha}] &= -\frac{1}{2} (\Gamma_{ij})_{\alpha\beta} s_{\beta} \quad (B.18) \\
[d, d] &= 0 \quad (B.19) \\
[d, k_{\mu}] &= k_{\mu} \quad (B.20) \\
[d, q_{\alpha'}] &= -\frac{1}{2} q_{\alpha'} \quad (B.21) \\
[d, s_{\alpha}] &= \frac{1}{2} s_{\alpha} \quad (B.22) \\
[k_{\mu}, k_{\nu}] &= 0 \quad (B.23) \\
[k_{\mu}, q_{\alpha'}] &= -(\Gamma_{\mu})_{\alpha'\beta} s_{\beta} \quad (B.24) \\
[k_{\mu}, s_{\alpha}] &= 0 \quad (B.25) \\
\{q_{\alpha'}, q_{\beta'}\} &= 2 (\Gamma_{\mu})_{\alpha'\beta'} p_{\mu} \quad (B.26) \\
\{q_{\alpha'}, s_{\beta}\} &= 2 c_{\alpha'\beta} + 2 (\Gamma_{ij})_{\alpha'\beta} j_{ij} + (\Gamma_{\mu\nu})_{\alpha'\beta} m_{\mu\nu} \quad (B.27) \\
\{s_{\alpha}, s_{\beta}\} &= 2 (\Gamma_{\mu})_{\alpha\beta} k_{\mu} \quad (B.28)
\end{align*}
\]
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