DEFORMATION OF DIRAC STRUCTURES ALONG ISOTROPIC SUBBUNDLES

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Abstract. Given a Dirac subbundle and an isotropic subbundle of a Courant algebroid, we provide a canonical method to obtain a new Dirac subbundle. When the original Dirac subbundle is involutive (i.e., a Dirac structure) this construction has interesting applications, for instance to Dirac’s theory of constraints and to the Marsden-Ratiu reduction in Poisson geometry.

1. Introduction

The concept of Dirac structure generalizes Poisson and presymplectic structures by embedding them in the framework of the geometry of $TM \oplus T^*M$ or, more generally, the geometry of a Courant algebroid. Dirac structures were introduced in a remarkable paper by T. Courant [7]. Therein, they are related to the Marsden-Weinstein reduction [14] and to the Dirac bracket [9] on a submanifold of a Poisson manifold. More recently, Dirac structures have been considered in connection to the reduction of implicit Hamiltonian systems (see [2], [1]). This simple but powerful structure allows to deal with mechanical situations in which we have both gauge symmetries and Casimir functions.

We present a construction which takes an isotropic subbundle $S$ and a Dirac subbundle $D$ of an exact Courant algebroid, and produces a new Dirac subbundle $D^S$ (Def. 3.1). This construction, which we refer to as stretching, was introduced by the first two authors in [6]. When both $S$ and $D$ are involutive, we find conditions ensuring that $D^S$ is also involutive, i.e. a Dirac structure (Thm. 4.1).

We further show that three prominent classes of Dirac structures are indeed stretched Dirac structures: the Dirac brackets that appeared in Dirac’s theory of constraints, the Dirac structures underlying the Marsden-Ratiu quotients in Poisson geometry [13], and coupling Dirac structures on Poisson fibrations [3].

The paper is organized as follows. In Section 2 we review basic definitions, in Section 3 we describe our stretching construction, in Section 4 we discuss when the stretched structure is involutive, and in Section 5 we present examples and applications.

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2. COURANT ALGEBROIDS AND DIRAC STRUCTURES

Definition 2.1. A Courant algebroid \cite{12} over a manifold \( M \) is a vector bundle \( E \to M \) equipped with an \( \mathbb{R} \)-bilinear bracket \([\cdot, \cdot]\) on \( \Gamma(E) \), a non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on the fibers and a bundle map \( \pi : E \to TM \) (the anchor) satisfying, for any \( e_1, e_2, e_3 \in \Gamma(E) \):

(i) \([e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]\)
(ii) \(\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]\)
(iii) \([e_1, fe_2] = f[e_1, e_2] + (\pi(e_1)f)e_2\)
(iv) \(\pi(e_1)e_2, e_3] = ([e_1, e_2], e_3] + [e_2, [e_1, e_3]]\)
(v) \([e, e] = D(e, e)\)

where \( D : C^\infty(M) \to \Gamma(E) \) is defined by \( D = \frac{1}{2}\pi^* \circ d \), using the bilinear form to identify \( E \) and its dual.

We see from axiom (v) that the bracket is not skew-symmetric, but rather satisfies \([e_1, e_2] = -[e_2, e_1] + 2D(e_1, e_2)\).

A Courant algebroid is called \textbf{exact} if

\[ 0 \to T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \to 0 \]

is an exact sequence. Choosing a splitting \( TM \to E \) of the above sequence with isotropic image allows one to identify the exact Courant algebroid with \( TM \oplus T^*M \) endowed with the natural symmetric pairing

\[ \langle (X, \xi), (X', \xi') \rangle = \frac{1}{2}(i_{X'}\xi + i_X\xi') \]

and the Courant bracket

\[ [(X, \xi), (X', \xi')] = ([X, X'], \mathcal{L}_X\xi' - i_Xd\xi + i_{X'}i_XH) \]

for some closed 3-form \( H \). In fact, the Courant algebroid uniquely determines the cohomology class of \( H \), called \( \check{\text{Severa}} \) class. The anchor \( \pi \) is given by the projection onto the first component. When it is important to stress the value of the 3-form \( H \) we shall use the notation \( E_H \) for \( TM \oplus T^*M \) equipped with this Courant algebroid structure.

Definition 2.2. A \textbf{Dirac subbundle} or \textbf{almost Dirac structure} in an exact Courant algebroid \( E \) is a subbundle \( D \subset E \) which is maximal isotropic with respect to \( \langle \cdot, \cdot \rangle \). The maximal isotropicity condition implies that \( D^\perp = D \), where \( D^\perp \) stands for the orthogonal subspace of \( D \). In particular, \( \text{rank}(D) = \dim(M) \).

A \textbf{Dirac structure} is an involutive Dirac subbundle, i.e. a Dirac subbundle \( D \) whose sections closed under the Courant bracket. In this case the restriction to \( D \) of the Courant bracket is skew-symmetric and \( D \) with anchor \( \pi \) is a Lie algebroid.

The two basic examples of Dirac structures are:

\textbf{Example 2.1.} For any 2-form \( \omega \), the graph \( L_\omega \) of \( \omega^\flat : TM \to T^*M \) is a Dirac subbundle such that \( \pi(L_\omega) = TM \). \( L_\omega \) is a Dirac structure in \( E_H \) if and only if \( d\omega = -H \). In particular, \( L_\omega \) is a Dirac structure in \( E_0 \) if and only if \( \omega \) is closed.
Example 2.2. Let $\Pi$ be a bivector field on $M$. The graph $L_\Pi$ of the map $\Pi^2 : T^* M \to TM$ is always a Dirac subbundle. In this case the natural projection from $L_\Pi$ to $T^* M$ is one-to-one. $L_\Pi$ is a Dirac structure in $E_H$ if and only if $\Pi$ is a twisted Poisson structure. In particular, $L_\Pi$ is a Dirac structure in $E_0$ if and only $\Pi$ is a Poisson structure.

3. Stretched Dirac subbundles

Until the end of this note we will assume the following setup:

- $E$ is an exact Courant algebroid
- $S \subset E$ is an isotropic subbundle (i.e. $S \subset S^\perp$)
- $D \subset E$ is a Dirac subbundle.

We further assume that $D \cap S$ (or equivalently $D \cap S^\perp$) has constant rank along $M$.

Definition 3.1. The stretching of $D$ along $S$ is the Dirac subbundle

$$D^S := (D \cap S^\perp) + S.$$ 

To justify the fact that $D^S$ is maximal isotropic we use

$$(D^S)^\perp = (D^\perp + S) \cap S^\perp = (D \cap S^\perp) + S = D^S,$$

where in the last line we have used that $D$ is maximal isotropic and $S$ is a subset of $S^\perp$. It is also clear that $D^S$, as the sum of two subbundles whose intersection has constant rank, is a (smooth) subbundle.

$D^S$ is the Dirac subbundle closest to $D$ among those containing $S$, as stated in the following

Proposition 3.1. Let $D, S$ and $D^S$ be as above and let $D'$ be a Dirac subbundle such that $S \subset D'$. Then, $D' \cap D \subset D^S \cap D$. In addition, $D' \cap D = D^S \cap D$ if and only if $D' = D^S$.

Proof. From the isotropicity of $D'$ and $S \subset D'$ we deduce that $D' \subset S^\perp$. Hence,

$$D' \cap D \subset S^\perp \cap D = D^S \cap D.$$ 

If the equality $D' \cap D = D^S \cap D$ holds, then $D' \supset D' \cap D = S^\perp \cap D$. Since $S \subset D'$, we find that $D^S = (D \cap S^\perp) + S \subset D'$. But $D^S$ and $D'$ have the same dimension, so that they are equal. \qed

4. Integrability

In this section we determine various properties of $D^S$, in particular conditions under which $D^S$ is a Dirac structure.

Lemma 4.1. Assume that $S$ and $D$ are closed under the Courant bracket. Then the set of $S$-invariant sections of $D^S$

$$\{ e \in D^S : [\Gamma(S), e] \subset \Gamma(S) \}$$

is closed under the Courant bracket.
Prop. 4.1 are all equivalent.

Theorem 4.1. Given a Dirac subbundle $D$ and an involutive isotropic subbundle $S \subset E$, we say that $S$ is canonical for $D$ if there exists a local $S$-invariant section $S$ of $D$ passing through any of its points.

Proposition 4.1. Assume that $S$ and $D$ are closed under the Courant bracket. We have the following chain of implications:

a) $S$ is canonical for $D$ ⇒

b) $D^S$ is a Dirac structure ⇒

c) $[\Gamma(S), \Gamma(D^S)] \subset \Gamma(D^S)$ (i.e. $S$ preserves $D^S$)

Proof. a) ⇒ b): We have to show that the Courant bracket of two sections $v, v' \in \Gamma(D^S)$ is again a section of $D^S$. We write $v$ and $v'$ in terms of a local basis, $\{e_i\}$, of $S$-invariant sections (such a basis always exists due to the canonicity of $S$). From Def. 2.1 iii) and Lemma 4.1 one immediately obtains that $[v, v']$ is a linear combination of the $e_i$'s and hence belongs to $\Gamma(D^S)$.

b) ⇒ c): holds because $S \subset D^S$.

The following theorem gives sufficient conditions to ensure that $D^S$ is a Dirac structure.

Theorem 4.1. Assume that $S$ and $D$ are closed under the Courant bracket and additionally that $\pi(S^\perp)$ is an integrable regular distribution. Then items a), b), c) of Prop. 4.1 are all equivalent.

\footnote{Recall that a section $e$ is $S$-invariant iff $[\Gamma(S), e] \subset \Gamma(S)$.}
Proof. We just need to show c) ⇒ a) in Prop. 4.1. Notice first that $\pi(S)$ is a regular integrable distribution. Indeed

$$\text{Ker}(\pi) \cap S = \pi^*(\pi(S^\perp)^\circ),$$

and given that $\pi(S^\perp)$ is regular and $\pi^*$ is injective for exact Courant algebroids, it follows that $\text{Ker}(\pi) \cap S$ is a subbundle. Now, the fact that $S$ is also a subbundle implies the regularity of $\pi(S)$. Integrability follows from the assumption that $\Gamma(S)$ is closed under the Courant bracket.

Take a commuting basis of local sections of $\pi(S)$ denoted by $\{\partial_i\}$. Fix lifts $s_i$ of $\partial_i$ to $S$, i.e. $s_i \in \Gamma(S)$ and $\pi(s_i) = \partial_i$. Since we are assuming c) of Prop. 4.1 we can define a partial $\pi(S)$-connection on $D^S$ by imposing

$$\nabla_i e := [s_i, e].$$

The involutivity of $S$ and Def. 2.1 ii) imply that $[\Gamma(S), \Gamma(\text{Ker}(\pi) \cap S)] \subset \Gamma(\text{Ker}(\pi) \cap S)$, so we can use $\nabla$ to define a partial $\pi(S)$-connection on $D^S/(\text{Ker}(\pi) \cap S)$. We now argue that $\tilde{\nabla}$ is flat.

The curvature of the connection $\nabla$, with components $F_{ij}$, is given by

$$F_{ij} e = \nabla_i \nabla_j e - \nabla_j \nabla_i e = [s_i[s_j, e]] - [s_j[s_i, e]] = [[s_i, s_j], e],$$

and given that $\partial_i$ and $\partial_j$ commute and $S$ is involutive we have $[s_i, s_j] \in \text{Ker}(\pi) \cap S$.

Next we want to show that

$$[\Gamma(\text{Ker}(\pi) \cap S), \Gamma(D^S)] \subset \Gamma(\text{Ker}(\pi) \cap S).$$

For that, take a section $s \in \Gamma(\text{Ker}(\pi) \cap S)$ and write $s = \pi^*(\eta)$ with $\eta \in \Gamma(\pi(S^\perp)^\circ)$. Also take arbitrary sections $e \in \Gamma(D^S)$ and $s^\perp \in \Gamma(S^\perp)$. Now

$$\langle [s, e], s^\perp \rangle = \langle \pi^*(\eta), [e, s^\perp] \rangle - \pi(e) \langle s, s^\perp \rangle = i_{[\pi(e), \pi(s^\perp)]}\eta = 0,$$

where in the first equality we used Def. 2.1 iv) and in last equality we used that $D^S \subset S^\perp$ and $\pi(S^\perp)$ is integrable.

Eq. (4.2) and eq. (4.3) together imply that $\tilde{\nabla}$ is a flat connection. Hence through any point of $D^S/(\text{Ker}(\pi) \cap S)$ passes a local horizontal sections for $\tilde{\nabla}$, and any lift of it to a section $e_h$ of $D^S$ satisfies $\nabla_i e_h \in \Gamma(\text{Ker}(\pi) \cap S)$ for all $i$. But the sections $\{s_i\}$ used to build the connection $\nabla$, together with $\Gamma(\text{Ker}(\pi) \cap S)$, span $\Gamma(S)$. Hence using Def. 2.1 iii) and eq. (4.3) we get

$$[\Gamma(S), e_h] \subset \Gamma(S),$$

completing the proof.

Remark 4.1. With the hypotheses of Theorem 4.1, if $[\Gamma(S), \Gamma(D)] \subset \Gamma(D)$ then $S$ is canonical for $D$ (because condition c) in Prop. 4.1 is satisfied). The converse is not true, see e.g. [15] for a counterexample in the context of Poisson manifolds.

\footnote{The connection $\tilde{\nabla}$ depends on the choice of lifts $s_i \in \Gamma(S)$.}
5. Examples and applications

In this section we work only with the exact Courant algebroid $E_0$. The first two examples show that two well-known constructions in Poisson geometry, once phrased in tensorial terms, correspond to the stretching of Poisson structures.

5.1. Dirac brackets. We give a description of the classical Dirac bracket in tensorial terms, i.e. in terms of Dirac structures, thereby giving a clear geometric interpretation to the Dirac bracket. Further we present a natural generalization.

We recall first the construction of the Dirac bracket on a Poisson manifold $(M, \Pi)$.

**Definition 5.1.** Given a regular foliation $R$ on an open set $U \subset M$ whose leaves $N$ are the level sets of second class constraints $\varphi^1, \ldots, \varphi^m$ (i.e. independent functions for which the matrix $C^{ab} := \{\varphi^a, \varphi^b\}_\Pi$ is invertible, with inverse $C_{ab}$), the **Dirac bracket** is defined as

\[
\{f, g\}_{\text{Dirac}} := \{f, g\}_\Pi - \{f, \varphi^a\}_\Pi C^{ab}\{\varphi^b, g\}_\Pi.
\]

We denote by $\Pi_{\text{Dirac}}$ the bivector field corresponding to the bracket $\{\cdot, \cdot\}_{\text{Dirac}}$.

**Lemma 5.1.**

i) The level sets $N$ of $\varphi$ are cosymplectic submanifolds of $(M, \Pi)$ and therefore have a Poisson structure induced by $\Pi$.

ii) $(M, \Pi_{\text{Dirac}})$ is obtained putting together the level sets $N$ of $\varphi$, endowed with the Poisson structure induced by $\Pi$. In particular the Dirac bracket (5.1) depends only on the level sets of the constraints (and not on the constraints themselves).

**Remark 5.1.** 1) Here and in the following we use repeatedly the following fact: a Poisson (Dirac) manifold is determined by its foliation into symplectic (presymplectic) leaves.

2) Lemma 5.1 ii) recovers the fact that (5.1) is a Poisson bracket.

**Proof.**

i) Since the $\varphi^i$ are second class constraints, the leaves $N$ of $R$ satisfy $\Pi^*TN = TM|_N$, which by definition means that they are cosymplectic submanifolds. There is an induced Poisson structure on $N$ [8, Sect. 8], obtained pulling back to $N$ the Dirac structure given by the graph of $\Pi$. The corresponding Poisson bracket of functions $f, g$ on $N$ is $\{\tilde{f}, \tilde{g}\}_\Pi|_N$, where $\tilde{f}, \tilde{g}$ are extensions of $f, g$ to $M$ and $d\tilde{f}$ is required to annihilate $\Pi^*TN$ at points of $N$.

ii) One checks easily that $\{\varphi^i, g\}_{\text{Dirac}} = 0$ for all $g \in C^\infty(U)$, i.e. that the $\varphi^i$ are Casimir functions for $\Pi_{\text{Dirac}}$, hence the level sets $N$ of $\varphi$ are Poisson submanifolds (i.e. unions of symplectic leaves) w.r.t. $\Pi_{\text{Dirac}}$. The Poisson structure on $N$ as a Poisson submanifold of $(M, \Pi_{\text{Dirac}})$ agrees with the one induced by $\Pi$ in the way described in i). Indeed for all functions $f, g$ on $N$ and extensions $\tilde{f}, \tilde{g}$ as above we have $\{\tilde{f}, \tilde{g}\}_{\text{Dirac}}|_N = \{\tilde{f}, \tilde{g}\}_\Pi|_N$ since $\{\tilde{f}, \varphi^a\}_\Pi|_N = 0$ for all constraints $\varphi^a$.

Now consider an integrable distribution $R \subset TM$ and let $D$ be a Dirac structure on $E_0 \to M$ so that $D \cap R^0$ has constant rank\(^3\). We consider the stretched Dirac subbundle $DR^0$.

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\(^3\)Here $R^0 \subset T^*M$ denotes the annihilator of $R$, i.e., the sections of $R^0$ are the 1-forms that kill all sections of $R$. 
The following proposition shows that in the special case that \( D \) is the graph of a Poisson structure \( \Pi \) and the leaves of \( R \) are cosymplectic in \((M, \Pi)\), the Dirac subbundle \( DR \) gives exactly the classical Dirac bracket (Def. 5.1). Hence \( DR \) can be considered as a generalization of the classical Dirac bracket.

**Proposition 5.1.** 1) \( DR \) is a Dirac structure. It is constructed putting together the integral submanifolds \( N \) of \( R \), with the (smooth) Dirac structure induced pulling back \( D \).

Now assume that \( D \) is the graph of a Poisson structure \( \Pi \) and the leaves of \( R \) are cosymplectic in \((M, \Pi)\), so that the Dirac bracket (5.1) can be defined (see Lemma 5.1).

Then \( DR \) is the graph of the Poisson structure \( \Pi_{\text{Dirac}} \).

**Proof.** 1) Notice that since \( R \) is integrable we can choose a frame for \( R^o \) consisting of closed 1-forms, which act trivially under the Courant bracket. Hence \( R^o \) is canonical for \( D \) (see Def. 4.1). So from Prop. 4.1 we conclude that \( DR \) is a Dirac structure.

Since \( \pi(DR) \) is everywhere tangent to the foliation \( R \), the integral submanifolds \( N \) of \( R \) are unions of presymplectic leaves of \( DR \). The Dirac structure \( D \) can be restricted to any leaf \( N \) of the foliation induced by \( R \), delivering a smooth subbundle since \( D \cap R^o \) has constant rank [7] [5]. Further, a simple computation shows that the pullback to \( N \) of \( D \) is equal to the pullback to \( N \) of \( DR \).

2a) The Dirac structure \( DR \) is the graph of a bivector field if and only if \( DR + TM = TM \oplus T^*M \). Taking orthogonals we obtain \((D + R^o) \cap R = \{0\}\). This can be rewritten as \( \Pi^f(R^o) \cap R = \{0\} \), which by definition [8, Sect. 8] means that the leaves of \( R \) are Poisson-Dirac submanifolds of \((M, \Pi)\).

2b) This follows from Lemma 5.1 ii) and part i) of this Proposition. \( \square \)

**Remark 5.2.** Prop. 5.1 2a) shows that even within the framework of Poisson geometry, i.e. in the case that both \( D \) and \( DR \) correspond to Poisson structures, our construction of Dirac structure \( DR \) is more general than the classical Dirac bracket.

5.2. **The Marsden-Ratiu reduction.** We show that the reduced Poisson structure induced via Marsden-Ratiu reduction [13] from a Poisson manifold \((M, \Pi)\) is obtained pushing forward not \( \Pi \) itself but rather a suitable stretching of \( \Pi \).

We start by recalling the Poisson reduction by distributions as it was stated by Marsden and Ratiu in [13], see also [15]. The set-up is the following:

\[(M, \{\cdot, \cdot\}) \text{ is a Poisson manifold},
\]

\(N\) is a submanifold with embedding \( \iota : N \hookrightarrow M \),

\(B \subset TN \) is a smooth subbundle of \( TM \) restricted to \( N \).

We shall also assume that \( F := B \cap TN \) is an integrable regular distribution on \( N \) and \( N := N/F \) is a smooth manifold.
Definition 5.2. \[13\] \((M, \{\cdot, \cdot\}, N, B)\) is Poisson reducible if there is a Poisson bracket \(\{\cdot, \cdot\}_N\) on \(N\) such that for any \(f_1, f_2 \in C^\infty(N) \cong C^\infty(N)_F\) we have:

\[\{f_1, f_2\}_N = \iota^* \{f_1^B, f_2^B\}\]

for all extensions \(f_i^B \in C^\infty(M)_B\) of \(f_i\).

Here \(C^\infty(N)_F := \{f \in C^\infty(N) \mid df|_F = 0\}\) and \(C^\infty(M)_B := \{f \in C^\infty(M) \mid df|_B = 0\}\).

Given \((M, \{\cdot, \cdot\}, N, B)\) clearly there is at most one Poisson bracket \(\{\cdot, \cdot\}_N\) on \(N\) satisfying the requirement of Def. 5.2. The following proposition (which is essentially \[10, Prop. A.2\]) describes the reduced Poisson structure on \(N\) in terms of bivector fields rather than in terms of brackets: it is obtained from \(\Pi\) by stretching along \(B\), pulling back to \(N\) and then pushing forward to \(N\).

Proposition 5.2. Assume that the prescription of Def. 5.2 gives a well-defined bivector field on \(N\), denote by \(L_N\) its graph, and denote \(L_\Pi = \text{graph}(\Pi)\). Then the pullback of the almost Dirac structure \(L_N\) under \(p : N \rightarrow N\) is \(\iota^*(L_B^\Pi)\).

Consequently \(L_N\) is given by the push-forward under \(p\) of \(\iota^*(L_B^\Pi)\).

5.3. Couplings on Poisson fibrations. Given a Dirac subbundle \(D\) and an isotropic subbundle \(S\), there are situations in which one wants to “deform” \(D\) to a new Dirac subbundle which contains \(S\). A natural candidate for the new Dirac structure is the stretching \(D_S\). An instance is provided by our next example, inspired by the results of Brahic and Fernandes \[3\] which (in the case of a flat connection) can be rephrased in our formalism.

Our data are a manifold \(M\) and:

- A splitting of the tangent bundle into two regular, integrable distributions: \(TM = \text{Hor} \oplus \text{Vert}\).
- A two-form in \(\text{Hor}\): \(\omega \in \Omega^2(\text{Hor})\).
- A bivector field in \(\text{Vert}\): \(\pi_\text{V} \in \wedge^2(\text{Vert})\).

The question is how two combine these data and which are the conditions that produce a Dirac structure. In principle there are two dual ways of doing this by using the deformation by stretching. The two different procedures give the same result.

1) Consider \(\pi_\text{V}^+ : T^* M \rightarrow \text{Vert}\) and take \(D = \text{graph}(\pi_\text{V}^+ )\).

\(D\) is a Dirac structure if and only if

\[i) \ [\pi_\text{V}, \pi_\text{V}] = 0.\]

To define \(S\), the stretching direction, consider the bundle map \(\hat{\omega}^\flat : \text{Hor} \rightarrow \text{Vert}^\circ\) induced by \(\omega\) and take \(S = \text{graph}(\hat{\omega}^\flat)\).

\(S\) is involutive if and only if:

\[ii) \ \omega\ is \ horizontally \ closed,\]

\[iii) \ \mathcal{L}_v(\omega(u_1, u_2)) = 0 \ for \ v \in \Gamma(\text{Vert}) \ and \ u_i \in \Gamma(\text{Hor}) \ such \ that [v, u_i] \in \Gamma(\text{Vert}).\]

4References \[13\], and subsequently \[10\], formulate conditions which ensure that \((M, \{\cdot, \cdot\}, N, B)\) is Poisson reducible.
Now, assuming that the other conditions hold, we can show that $S$ preserves $D^S$ if and only if

$$iv) \mathcal{L}_u \pi_V = 0 \text{ for any } u \in \Gamma(Hor) \text{ s. t. } [v, u] \in \Gamma(Vert), \forall v \in \Gamma(Vert).$$

If conditions $i)-iv$ are satisfied then, using Thm. 4.4 ($S^1 = S + Vert + Hor^\circ$ and therefore $\pi(S^1) = TM$), we have that $D^S$ defines a Dirac structure. In the next paragraph we shall show an alternative way of obtaining the same result with the roles of $\omega$ and $\pi_V$ exchanged.

b) We introduce first the bundle map $\omega^b : Hor \to Hor^*$. Consider the Dirac subbundle $D' \subset TM \oplus T^*M$ induced by graph($\omega^b$), i.e.

$$D' = \{(v, \xi)|v \in Hor, \xi|_{Hor} = \omega^b v\}.$$  

One can show that $D'$ is a Dirac structure if and only if condition $ii)$ above holds. Now we proceed to define the new stretching subbundle $S'$. Take

$$S'_x = \{((\pi_V^* \xi)_x, \xi_x)|\xi_x \in (Hor^\circ)_x\}.$$  

$S'$ is involutive if and only if conditions $i)$ and $iv)$ hold. Finally one can show that, assuming all previous conditions, the stretching $D'^{S'}$ is a Dirac structure if and only if

$$iii)' \mathcal{L}_v (\omega(u_1, u_2)) = 0 \text{ for } v \in \Gamma(\pi_V^*(T^*M)) \text{ and } u_i \in \Gamma(Hor) \text{ s. t. } [v, u_i] \in \Gamma(Vert).$$

It is interesting to compare the two construction. First it is clear that $D^S = D'^{S'}$. Further, condition $iii)$ in construction a) implies condition $iii)'$ in b). Therefore, even if both give the same final result, the second construction has a broader range of application.

Remark 5.3. We establish the connection between the above and the coupling of Poisson fibrations of ref. [3]. Suppose that $M$ is the total space of a fibration so that $Vert$ is the distribution tangent to the fibers. One computes easily that $D^S$ agrees with the fiber non-degenerate almost Dirac structure associated to the triple ($\pi_V$, $Hor$, $\omega$) in Cor. 2.6 of [3]. Brahic and Fernandes compute the necessary and sufficient conditions for this to be a Dirac structure in Cor. 2.8 of [3]. If the horizontal connection is flat their conditions are equivalent to our $i)$, $ii)$, $iii)'$ and $iv)$ above, i.e. the conditions for having a Dirac structure following the stretching procedure introduced in the paper.

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