On a Generalized Lamé-Navier system in $\mathbb{R}^3$

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Abstract

This paper is devoted to a fundamental system of equations in Linear Elasticity Theory: the famous Lamé-Navier system. The Clifford algebra language allows us to rewrite this system in terms of the euclidean Dirac operator, which at the same time suggests a very natural generalization involving the so-called structural sets. We are interested in finding some structures in the solutions of these generalized Lamé-Navier systems. Using MATLAB we also implement algorithms to compute with such partial differential operators as well as to verify some theoretical results obtained in the paper.

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1 Introduction

In the state of equilibrium the three-dimensional displacement vector $\vec{u}$ should satisfy the Lamé-Navier system

$$\mu \triangle \vec{u} + (\mu + \lambda) \text{grad}(\text{div}\vec{u}) = 0,$$

at any point within a homogeneous isotropic linear elastic body without volume forces. The quantities $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$ are called the Lamé constants.
This system was originally introduced by G. Lamé in 1837 [18] while studying the method of separation of variables for solving the wave equation in elliptic coordinates. Moreover, its applications cover many branches in the fields such as linear elastostatics, chaotic Hamiltonian systems, and the theory of Bose-Einstein condensates [3, 6, 15, 24, 25, 27, 23].

From [22] it is known that the Lamé equation (1) can be rewritten in the form

$$\left(\frac{\mu + \lambda}{2}\right) \partial u + \left(\frac{3\mu + \lambda}{2}\right) \partial^2 u = 0,$$

where

$$\partial := e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}$$

stands for the Dirac operator in \(\mathbb{R}^3\) constructed with the generators \(\{e_1, e_2, e_3\}\) of the real Clifford algebra \(\mathbb{R}_{0,3}\). The null-solutions of \(\partial u\) are referred in the literature as monogenic functions [4, 9].

The search for all linear partial differential operators of the form

$$\psi\partial := \psi^1 \frac{\partial}{\partial x_1} + \psi^2 \frac{\partial}{\partial x_2} + \psi^3 \frac{\partial}{\partial x_3},$$

such that solutions of the differential equation \(\psi\partial u = 0\) are always solutions of the Laplace equation \(\Delta u = 0\), goes back to Nono [17].

Let \(\psi\partial\) be a linear differential operator of the form (3) with coefficients \(\psi^i \in \mathbb{R}^3 \subset \mathbb{R}_{0,3}\). To fulfill the Laplacian factorization \(\psi\partial \psi\partial = -\Delta\) in \(\mathbb{R}^3\), the following relations hold

$$\psi^i \psi^j + \psi^j \psi^i = -2\delta_{ij} \ (i, j = 1, 2, 3).$$

The system \(\{\psi^1, \psi^2, \psi^3\}\) can be thought of as an orthonormal (in the usual Euclidean sense) basis in \(\mathbb{R}^3\). In this way, we obtain what will be referred to as structural set [26].

The \(\mathbb{R}_{0,3}\)-valued solutions of \(\psi\partial u = 0\) are the so-called \(\psi\)-hyperholomorphic functions. As pointed out in [11], the class of \(\psi\)-hyperholomorphic functions is wider than the one we get by rotations from the class of monogenic functions. The flexibility introduced by the structural sets allows us to look for new perspectives in several lines of research concerning the mapping properties of a related \(\Pi\)-operator, geometric conformal mappings and additive decompositions of harmonic functions [1, 2, 5, 7, 8, 10, 12, 16].

It is precisely in this scenario that a generalization of the Lamé equation naturally emerges. Indeed, the idea is to consider in (2) the generalized Dirac operator \(\psi\partial\), rather than the standard one.
In line with that way of thinking, we arrive at two possible generalizations of the Lamé system (1):

\[\alpha \left[\psi \partial \vec{u} \psi \partial \vec{u}\right] + \beta \left[\psi \partial \psi \partial \vec{u}\right] = 0 \tag{4}\]

and

\[\alpha \left[\phi \partial \vec{u} \phi \partial \vec{u}\right] + \beta \left[\phi \partial \phi \partial \vec{u}\right] = 0, \tag{5}\]

where \(\phi, \psi\) are two structural sets and for brevity we used the notation \(\alpha = \frac{\mu + \lambda}{2}, \beta = \frac{3\mu + \lambda}{2}\).

This paper aims to investigate the structure of the solutions of these generalized systems, as well as to determine the similarities and differences between them and the solutions of the classical Lamé equation.

Before going to the next section, we want to point out that even though the previous systems actually generalize the Lamé equation, the solutions of any of them remain biharmonic functions, as in classical linear elasticity theory.

## 2 Preliminaries

First we recall some definitions and basic properties of a Clifford algebra.

Let \(e_1, e_2, e_3\) be an orthonormal basis of \(\mathbb{R}^3\). Let \(\mathbb{R}_{0,3}\) be real Clifford algebra constructed over \(\mathbb{R}^3\). The basic multiplication rules are governed by

\[e_i^2 = -1, \quad e_i e_j = -e_j e_i, \quad i, j = 1, 2, 3, \quad i < j.\]

Any element \(a \in \mathbb{R}_{0,m}\) may thus be written as \(a = \sum_A a_A e_A, \quad a_A \in \mathbb{R}\), where \(e_A := e_{i_1} \cdots e_{i_k}\) with \(A = \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}\) is such that \(i_1 < \cdots < i_k\). Additionally, one puts \(e_\emptyset = 1\).

An element \(a \in \mathbb{R}_{0,3}\) can be alternatively written as

\[a = [a]_0 + [a]_1 + [a]_2 + [a]_3, \tag{6}\]

where \([\_]_k\) denotes the projection of \(\mathbb{R}_{0,3}\) onto the subspace \(\mathbb{R}_{0,3}^{(k)}\) of \(k\)-vectors defined by

\[\mathbb{R}_{0,3}^{(k)} = \text{span}_\mathbb{R}(e_A : |A| = k).\]

The conjugation in \(\mathbb{R}_{0,3}\) is defined as the anti-involution \(a \mapsto \overline{a}\) for which \(\overline{e_i} = -e_i\). A norm \(\|\_\|\) on \(\mathbb{R}_{0,3}\) is defined by \(\|a\|^2 = Sc[a\overline{a}]\) for \(a \in \mathbb{R}_{0,3}\). We remark that for \(\vec{a} \in \mathbb{R}^3\) we have \(\|\vec{a}\| = |\vec{a}|\), the usual Euclidean norm.

We will consider functions defined on subsets of \(\mathbb{R}^3\) and taking values in \(\mathbb{R}_{0,3}\). Those functions might be written as \(f = \sum_A f_A e_A\), where \(f_A\) are \(\mathbb{R}\)-valued functions. The notions of continuity, differentiability and integrability of a \(\mathbb{R}_{0,3}\)-valued function \(f\) have the usual component-wise meaning. In particular, the spaces of all
$k$-time continuous differentiable and $p$-integrable functions are denoted by $C^k(E)$ and $L^p(E)$ respectively, where $E$ can be any suitable subset of $\mathbb{R}^3$.

The so-called Dirac operator $\mathbf{\partial}$ is defined by

$$\mathbf{\partial} := e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}.$$ 

An $\mathbb{R}_{0,3}$-valued function $f$, defined and differentiable in an open region $\Omega$ of $\mathbb{R}^3$, is called left monogenic (right monogenic) in $\Omega$ if $\partial f = 0$ ($\mathbf{\partial} f = 0$) in $\Omega$. Functions that are both left and right monogenic are called two-sided monogenic.

More generally, for fixed orthonormal base $\psi := \{\psi^1, \psi^2, \psi^3\}$ in $\mathbb{R}^3$ (structural set) we introduce the so-called $\psi$-hyperholomorphic functions (left or right respectively), which belong to $\text{ker}[\psi \partial (\cdot)]$ or $\text{ker}[(\cdot) \psi \partial]$, where

$$\psi \partial := \psi^1 \frac{\partial}{\partial x_1} + \psi^2 \frac{\partial}{\partial x_2} + \psi^3 \frac{\partial}{\partial x_3}. \tag{7}$$

For further use we introduce for an open set $\Omega \subset \mathbb{R}^3$, the following subclasses of $\mathbb{R}_{0,3}$-valued functions

$$I_{\phi,\psi}(\Omega) = \{ u \in C^2(\Omega) : \phi \partial u = 0 \},$$

$$H_{\phi,\psi}(\Omega) = \{ u \in C^2(\Omega) : \phi \partial \psi \partial u = 0 \}.$$ 

It is wise to note here that for $\phi = \psi$, the class $H_{\phi,\psi}(\Omega)$ coincides with the space $H(\Omega)$ of harmonic functions in $\Omega$. On the other hand and no less important, we note that in case of being $\phi = \psi = \{e_1, e_2, e_3\}$, the class $I_{\phi,\psi}(\Omega)$ becomes the space $I(\Omega)$ of inframonogenic functions introduced in [13, 14] and studied more extensively in [20, 21, 22]. The above is reason enough to name the elements of $I_{\phi,\psi}$ as $(\phi, \psi)$-inframonomeric functions while the elements of $H_{\phi,\psi}$ as $(\phi, \psi)$-harmonic functions. For vector-valued functions we will use the alternative notations $I_{\phi,\psi}(\Omega)$ and $H_{\phi,\psi}(\Omega)$.

### 3 Auxiliary results

In order to prove the main results, we will establish some auxiliary results which are provided in this section.

**Proposition 1** An $\mathbb{R}_{0,3}$-valued function $f$ is $(\psi, \psi)$-inframonomeric in $\Omega \subset \mathbb{R}^3$ if and only if each $k$-vector valued function $[f]_k, 0 \leq k \leq 3$, is $(\psi, \psi)$-inframonomeric there.
Proof.
The proof is adapted from [20]. The only new ingredient to use is the representation of \( \psi^i \) \((i = 1, 2, 3)\) by
\[
\psi^i = \sum_{k=1}^{3} \psi_k^i e_k,
\]
where \( \psi_k^i \in \mathbb{R} \).

After this, the proof follows very similar lines of the proof of Proposition 1 in [20] and will be omitted. \(\square\)

In contrast to the particular case \( \phi = \psi \), the above nice property is no longer valid in general. As a simple counterexample consider the function given by
\[
g(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \sqrt{2}x_1x_2 + (e_3e_1 - 1)x_3^2.
\]

Let be \( \phi = \{e_1, e_3, e_2\} \) and \( \psi = \{\sqrt{\frac{2}{2}}(e_1 + e_3), \sqrt{\frac{2}{2}}(e_1 - e_3), e_2\} \) two structural sets. On the one hand, we have
\[
\phi \partial g \psi \partial = e_1 \left[ \frac{\sqrt{2}}{2}(e_1 + e_3) \right] + e_3 \left[ \frac{\sqrt{2}}{2}(e_1 - e_3) \right] + e_1 \sqrt{2} \left[ \frac{\sqrt{2}}{2}(e_1 - e_3) \right] + e_3 \sqrt{2} \left[ \frac{\sqrt{2}}{2}(e_1 + e_3) \right] + e_2 [2e_3e_1 - 2]e_2
\]
\[
= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}e_1e_3 + \frac{\sqrt{2}}{2}e_3e_1 + \frac{\sqrt{2}}{2} - 1 - e_1e_3 + e_3e_1 - 1 - 2e_3e_1 + 2
\]
\[
= 0,
\]
but, on the other
\[
\phi \partial g \psi \partial = \phi \partial \left[ \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_3^2 + \sqrt{2}x_1x_2 \right] \psi \partial
\]
\[
= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}e_1e_3 + \frac{\sqrt{2}}{2}e_3e_1 + \frac{\sqrt{2}}{2} - 1 - e_1e_3 + e_3e_1 - 1 + 2
\]
\[
= 2e_3e_1 \neq 0,
\]
\[
\phi \partial g \psi \partial = 0,
\]
\[
\phi \partial g \psi \partial = \phi \partial [e_3e_1] \psi \partial
\]
\[
= -2e_3e_1 \neq 0,
\]
\[
\phi \partial g \psi \partial = 0.
\]
Let as before \( \psi = \{ \psi^1, \psi^2, \psi^3 \} \) and \( \phi = \{ \phi^1, \phi^2, \phi^3 \} \) be two structural sets in \( \mathbb{R}^3 \).
With the notations

\[
\nu(f) = \sum_{i=1}^{3} \psi^i \cdot f \cdot \psi^i, \quad \omega(f) = \sum_{i=1}^{3} \phi^i \cdot f \cdot \psi^i, \quad \tilde{\omega}(f) = \sum_{i=1}^{3} \psi^i \cdot f \cdot \phi^i,
\]

we have

**Lemma 1** Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R}_{0,3} \) and \( x_{\psi} = \sum_{i=1}^{3} \psi^i \cdot x_i \). Then,

1. \( \psi \partial (f \cdot x_{\psi}) = (\psi \partial f) \cdot x_{\psi} + \nu(f), \quad (x_{\psi} f) \cdot \psi \partial = x_{\psi} (f \cdot \psi \partial) + \nu(f) \)
2. \( \psi \partial [\nu(f)] = -2 f \cdot \psi \partial - \nu(f \cdot \psi \partial), \quad [\nu(f)] \cdot \psi \partial = -2 f \cdot \psi \partial - \nu(f \cdot \psi \partial) \)
3. \( \psi \partial [\psi(f) \cdot \psi \partial] = \nu(f \cdot \psi \partial), \quad [\nu(f)] \cdot \psi \partial = -\Delta \nu(f) = \nu(f \cdot \psi \partial f) = -\nu(\Delta f) \)
4. \( \psi \partial [\omega(f)] \cdot \psi \partial = \omega(f \cdot \psi \partial), \quad [\omega(f)] \cdot \psi \partial = -\Delta \omega(f) = \omega(f \cdot \psi \partial f) = -\omega(\Delta f) \)
5. \( \nu(\bar{u}) = \bar{u} \)
6. \( \psi \partial \omega(f) = -2 f \cdot \psi \partial - \omega(f \cdot \psi \partial), \quad \omega(f) \cdot \psi \partial = -2 f \cdot \psi \partial - \omega(f \cdot \psi \partial) \)
7. \( \psi \partial (f \cdot x_{\psi}) = (\psi \partial f) \cdot x_{\psi} + \omega(f), \quad (x_{\psi} f) \cdot \psi \partial = x_{\psi} (f \cdot \psi \partial) + \tilde{\omega}(f) \)
8. \( \psi \partial \omega(f) = \omega(f) \cdot \psi \partial \omega(f) = \omega(f \cdot \psi \partial f) - (\psi \partial [\omega(f)]) \)

**Proof.**

For the sake of brevity we only include the proofs of (6) and (7). The remaining statements follow by similar arguments.

**Proof of (6):**

\[
\phi \partial \omega(f) = \sum_{1 \leq i,j \leq 3} \phi^i \phi^j (\partial x_i, f) \psi^j \\
= \sum_{1 \leq i,j \leq 3} \phi^i \phi^j (\partial x_i, f) \psi^j + \sum_{1 \leq i,j \leq 3} \phi^i \phi^j (\partial x_i, f) \psi^j \\
= - \sum_{i=1}^{3} (\partial x_i, f) \psi^j - \sum_{1 \leq i,j \leq 3, i \neq j} \phi^i \phi^j (\partial x_i, f) \psi^j \\
= - \sum_{i=1}^{3} (\partial x_i, f) \psi^j - \left( \sum_{1 \leq i,j \leq 3} \phi^i \phi^j (\partial x_i, f) \psi^j + \sum_{i=1}^{3} (\partial x_i, f) \psi^j \right) \\
= -2 \sum_{i=1}^{3} (\partial x_i, f) \psi^j - \sum_{1 \leq i,j \leq 3} \phi^i \phi^j (\partial x_i, f) \psi^j \\
= -2 f \cdot \psi \partial - \omega(f \cdot \psi \partial).
\]
Proof of (7):

\[
\phi_\psi(f_\psi) = \sum_{i=1}^{3} \phi_i \frac{\partial (f_\psi)}{\partial x_i} = \sum_{i=1}^{3} \left( \frac{\partial f}{\partial x_i} \phi_i + f \phi_i \right) = \sum_{i=1}^{3} \left( \psi f + x_\psi \frac{\partial f}{\partial x_i} \right) \phi_i = (\phi_\psi)_{\psi} + o(f).
\]

As was explicitly mentioned at the end of the introduction, the solutions of (5) are biharmonic functions. The following stronger result is in fact true.

**Proposition 2** If \( \vec{u} \in C^3(\Omega) \) satisfies in \( \Omega \) the generalized Lamé-Navier system (5), then \( \psi^3 \vec{u} = 0 \) in \( \Omega \).

**Proof.**

On applying \( \phi_\psi \) to both sides of (5) yields

\[
\alpha \phi_\psi \phi_\psi \phi_\psi \vec{u} + \beta \psi \phi_\psi \phi_\psi \vec{u} = 0,
\]

and hence

\[
\alpha \psi \phi_\psi \phi_\psi \vec{u} + \beta \psi \phi_\psi \phi_\psi \vec{u} = 0.
\]

Since \( \vec{u}^\psi \phi_\psi = -\frac{\alpha}{\beta} \phi_\psi \vec{u} \), it follows that

\[
-\frac{\alpha^2}{\beta} \psi \phi_\psi \vec{u} + \beta \psi \phi_\psi \vec{u} = 0
\]

or equivalently

\[
\left( \beta - \frac{\alpha^2}{\beta} \right) \psi \phi_\psi \vec{u} = 0.
\]

From this, we conclude that \( \psi^3 \vec{u} = 0 \), as otherwise would be \( \alpha = \beta \) or \( \alpha = -\beta \) and one is led to a contradiction with the original assumptions on the Lamé constants \( \mu, \lambda \).

\( \square \)
4 Additive decomposition of the generalized Lamé-Navier solutions

In this section our main results are stated and proved. We start by a rather simple generalization of [22, Theorem 3.1].

**Theorem 1** If a vector field \( \vec{u} \) satisfies in \( \Omega \subset \mathbb{R}^3 \) the generalized Lamé-Navier system (4), then it admits in \( \Omega \) the decomposition

\[
\vec{u} = \vec{h} + \vec{i},
\]

where \( \vec{h} \in H(\Omega) \) and \( \vec{i} \in I_{\psi,\psi}(\Omega) \). Moreover, this representation is unique up to a vector field in \( H(\Omega) \cap I_{\psi,\psi}(\Omega) \).

**Proof.** Let \( g = \alpha \vec{u} \psi + \beta \vec{u} \partial \psi, \vec{u} \) satisfying (4). Clearly \( g \) is a \( \mathbb{R}_{0,3} \)-valued (left) \( \psi \)-hyperholomorphic function in \( \Omega \). Moreover, in virtue of Lemma 1 (1) we have \( \psi \partial (gx \psi) = \nu (g) \), which by Lemma 1 (2)-(5) yields

\[
\psi \partial (gx \psi) = \psi \partial (g \vec{u} \psi) = -g \psi \partial = \left( \frac{\alpha^2}{\beta} - \beta \right) \psi \partial \vec{u} \psi \partial
\]

and

\[
\psi \partial \partial (g \vec{u}) = -2g \psi \partial = 2 \left( \frac{\beta^2}{\alpha} - \alpha \right) \psi \partial \psi \partial \vec{u}.
\]

Equivalently:

\[
\psi \partial \left[ g \vec{u} \psi - \left( \frac{\alpha^2}{\beta} - \beta \right) \vec{u} \right] \psi \partial = 0
\]

(9)

and

\[
\psi \partial \psi \partial \left[ g \vec{u} \psi - 2 \left( \frac{\beta^2}{\alpha} - \alpha \right) \vec{u} \right] = 0.
\]

(10)

Let be \( I := g \vec{u} \psi - (\frac{\alpha^2}{\beta} - \beta) \vec{u} \) and \( H := g \vec{u} \psi - 2(\frac{\beta^2}{\alpha} - \alpha) \vec{u} \). Of course, since (9)-(10) \( I \in I_{\psi,\psi}(\Omega), H \in H(\Omega) \) and

\[
\left( \frac{\alpha^2}{\beta} - \beta - \frac{2\beta^2}{\alpha} + 2\alpha \right) \vec{u} = H - I.
\]

Our next task is to prove that \( \frac{\alpha^2}{\beta} - \beta - \frac{2\beta^2}{\alpha} + 2\alpha \neq 0 \), or equivalently, that

\[
(\alpha + 2\beta)(\alpha^2 - \beta^2) \neq 0.
\]
Indeed, if $\alpha + 2\beta = 0$ we would have $\frac{3\alpha}{\mu} = -7$ and then $\frac{1}{\mu} < -\frac{2}{3}$, which contradicts the initial assumption on the Lamé coefficients $\lambda, \mu$. In a similar way the supposition $\alpha^2 - \beta^2 = 0$ leads to a contradiction.

Therefore, we have

$$\bar{u} = h + i,$$

(11)

where

$$h = \left( \frac{\alpha^2}{\beta} - \beta - \frac{2\beta^2}{\alpha} + 2\alpha \right)^{-1} H, \quad i = -\left( \frac{\alpha^2}{\beta} - \beta - \frac{2\beta^2}{\alpha} + 2\alpha \right)^{-1} I.$$

Since $h \in \mathcal{H}(\Omega)$ and $i \in \mathcal{I}_{\psi,\psi}(\Omega)$, the desired representation easily follows from Proposition 1 and taking the 1-vector part in both sides of (11).

The proof of the second part is obvious. Indeed, assume that $\bar{u}$, being a solution of (1) admits two different representations, say,

$$\bar{u} = \bar{h}_1 + \bar{i}_1, \quad \bar{u} = \bar{h}_2 + \bar{i}_2,$$

where $\bar{h}_1, \bar{h}_2 \in \mathcal{H}(\Omega)$ and $\bar{i}_1, \bar{i}_2 \in \mathcal{I}_{\psi,\psi}(\Omega)$.

Then by subtracting both representations we obtain that $\bar{h}_1 - \bar{h}_2 = \bar{i}_2 - \bar{i}_1$ are simultaneously harmonic and $(\psi, \psi)$-inframonogenic.

From now on we will be concerned with the much more general Lamé-Navier system (5). Since Proposition 1 is no longer available in this general situation, the next decomposition theorems involve $\mathbb{R}_{0,3}$-valued functions rather than simply vector-valued ones. We start with a technical lemma, whose proof is a matter of direct calculations.

**Lemma 2** Let $\bar{u}$ satisfy (5) in $\Omega \subset \mathbb{R}^3$ and put $g = \alpha \bar{u} \bar{\Omega}_\psi + \beta \bar{\Omega}_\psi \bar{u}$. Then

$$\phi_{\bar{\Omega}_\psi}^\psi \left( g \bar{\epsilon}_\psi \right) = (\phi_{\bar{\Omega}_\psi}^\psi \bar{g}) \bar{\epsilon}_\psi - \alpha \phi_{\bar{\Omega}_\psi}^\psi \bar{\epsilon}_\psi + \left( \frac{\alpha^2}{\beta} - \beta \right) \omega(\Delta \bar{u}) + 2 \left( \frac{\beta^2}{\alpha} - \alpha \right) \phi_{\bar{\Omega}_\psi}^\psi \bar{u},$$

and

$$\phi_{\bar{\Omega}_\psi}^\psi \left[ g \bar{\epsilon}_\psi - \frac{\alpha^2}{\beta} \bar{\epsilon}_\psi - \left( \frac{\alpha^2}{\beta} - \beta + \frac{2\beta^2}{\alpha} - 2\alpha \right) \bar{u} \right] = (\phi_{\bar{\Omega}_\psi}^\psi \bar{g}) \bar{\epsilon}_\psi + \left( \frac{\alpha^2}{\beta} - \beta \right) \omega(\Delta \bar{u}).$$

(12)

Here is a generalization of Theorem 1.

**Theorem 2** Let $\bar{u}$ satisfy (5) in $\Omega \subset \mathbb{R}^3$. If $\bar{u}$ is harmonic and $(\psi, \psi)$-inframonogenic in $\Omega$, then it admits there the splitting

$$\bar{u} = h + i,$$

where $h \in \mathcal{H}_{\phi,\psi}(\Omega)$ and $i \in \mathcal{I}_{\phi,\psi}(\Omega)$. Moreover, this representation is unique up to an element in $\mathcal{H}_{\phi,\psi}(\Omega) \cap \mathcal{I}_{\phi,\psi}(\Omega)$.  

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Proof.
Once again, let \( g = \alpha \bar{u} \vec{\phi} + \beta \vec{\psi} \partial \bar{u} \). Under the assumptions stated above, we have \( \frac{\phi}{\partial g} = 0, \frac{\psi}{\partial g} = 0 \) and \( g^\psi = 0 \).

Lemma 1 (6)-(7) now yields
\[
\frac{\phi}{\partial (g \bar{u})} = \omega (g) \frac{\psi}{\partial \bar{u}} = -2 \frac{\phi}{\partial g} - \omega (g) \frac{\psi}{\partial \bar{u}} = 0.
\]
Consequently, by (12) we have
\[
\frac{\phi}{\partial \bar{u}} (g \bar{u}) = 2 \left( \frac{\beta^2}{\alpha} - \alpha \right) \frac{\phi}{\partial \bar{u}} g \bar{u},
\]
or equivalently
\[
\frac{\phi}{\partial \bar{u}} \left[ g \bar{u} - 2 \left( \frac{\beta^2}{\alpha} - \alpha \right) \bar{u} \right] = 0.
\]
Since \( \alpha - \frac{\beta^2}{\alpha} \neq 0 \), the proof is completed after taking
\[
h := \left( 2 \alpha - \frac{2 \beta^2}{\alpha} \right)^{-1} \left[ g \bar{u} - 2 \left( \frac{\beta^2}{\alpha} - \alpha \right) \bar{u} \right],
\]
where \( h \in H_{\phi,\psi}(\Omega) \) and \( i^* \in I_{\psi,\phi}(\Omega) \). Moreover, this representation is unique up to an element in \( H_{\phi,\psi}(\Omega) \cap I_{\psi,\phi}(\Omega) \).

Now, we will show how to deal without imposing any assumption of inframonomonicity. As we will see, a subtle change is needed in replacing the space \( I_{\phi,\psi}(\Omega) \) of the previous theorem by \( I_{\psi,\phi}(\Omega) \).

**Theorem 3** Let \( \bar{u} \) satisfy (5) in \( \Omega \subset \mathbb{R}^3 \). If \( \bar{u} \) is harmonic in \( \Omega \), then it admits the decomposition
\[
\bar{u} = h + i^*,
\]
where \( h \in H_{\phi,\psi}(\Omega) \) and \( i^* \in I_{\psi,\phi}(\Omega) \). Moreover, this representation is unique up to an element in \( H_{\phi,\psi}(\Omega) \cap I_{\psi,\phi}(\Omega) \).

Proof.
Let \( g = \alpha \bar{u} \vec{\phi} + \beta \vec{\psi} \partial \bar{u} \) and \( \bar{g} = \alpha \vec{\phi} \partial \bar{u} + \beta \bar{u} \vec{\phi} \). Then
\[
\frac{\phi}{\partial \bar{u}} g = \frac{\phi}{\partial \bar{u}} \bar{g} = \left( \beta - \frac{\alpha^2}{\beta} \right) \frac{\phi}{\partial \bar{u}} \bar{u} \vec{\phi} = 0,
\]
since \( \bar{u} \) is harmonic.

By applying (13) we obtain
\[
\frac{\phi}{\partial \bar{u}} \left[ \left( g - \frac{\alpha}{\beta} \bar{g} \right) \bar{u} \vec{\phi} - \left( \frac{\alpha^2}{\beta} - \beta + 2 \frac{\beta^2}{\alpha} - 2 \alpha \right) \bar{u} \right] = 0.
\]
On the other hand, the relation \((g - \frac{\alpha}{\beta} \bar{g}) \bar{\psi} = (\beta - \frac{\alpha^2}{\beta}) (\bar{\psi} \partial \bar{u}) \bar{\psi}\) implies

\[
\psi \bar{\partial} \left[ \left( g - \frac{\alpha}{\beta} \bar{g} \right) \bar{x}_\psi \right] = \psi \bar{\partial} \left[ \left( \beta - \frac{\alpha^2}{\beta} \right) (\bar{\psi} \partial \bar{u}) \bar{\psi} \right]
\]

\[
= \left( \beta - \frac{\alpha^2}{\beta} \right) \left[ (\bar{\psi} \partial \bar{u}) \bar{x}_\psi + \nu (\bar{\psi} \partial \bar{u}) \right]
\]

\[
= \left( \beta - \frac{\alpha^2}{\beta} \right) (-2 \bar{\psi} \partial \bar{u} - \psi \partial \bar{u}).
\]

Consequently

\[
\psi \bar{\partial} \left[ \left( g - \frac{\alpha}{\beta} \bar{g} \right) \bar{x}_\psi \right] \phi \partial = \left( \beta - \frac{\alpha^2}{\beta} \right) \left( \frac{2\alpha}{\beta} - 1 \right) \psi \partial \bar{u} \bar{\partial}
\]

or equivalently

\[
\psi \bar{\partial} \left[ \left( g - \frac{\alpha}{\beta} \bar{g} \right) \bar{x}_\psi - \left( 2\alpha - \beta - \frac{2\alpha^3}{\beta^2} + \frac{\alpha^2}{\beta} \right) \bar{u} \right] \phi \partial = 0. \tag{15}
\]

Next let be

\[
I^* = \left( g - \frac{\alpha}{\beta} \bar{g} \right) \bar{x}_\psi - \left( 2\alpha - \beta - \frac{2\alpha^3}{\beta^2} + \frac{\alpha^2}{\beta} \right) \bar{u}
\]

and

\[
H = \left( g - \frac{\alpha}{\beta} \bar{g} \right) \bar{x}_\psi - \left( 2\alpha - \beta - \frac{2\alpha^3}{\beta^2} + \frac{\alpha^2}{\beta} \right) \bar{u} + \left( 4\alpha - \frac{2\alpha^3}{\beta^2} - \frac{2\beta^2}{\alpha} \right) \bar{u}.
\]

The proof is now easily completed from (14)-(15) by choosing \(h = (4\alpha - \frac{2\alpha^3}{\beta^2} - \frac{2\beta^2}{\alpha})^{-1} H\) and \(i^* = -(4\alpha - \frac{2\alpha^3}{\beta^2} - \frac{2\beta^2}{\alpha})^{-1} I^*\). The factor \(4\alpha - \frac{2\alpha^3}{\beta^2} - \frac{2\beta^2}{\alpha}\) is not 0, since one would otherwise obtain a contradiction with the assumptions on the Lamé parameters.

Once again, the proof of uniqueness is straightforward and will be omitted. \(\square\)

Similarly we have a corresponding theorem without recourse to the assumption of harmonicity.

**Theorem 4** If a \((\psi, \bar{\psi})\)-infra-monogenic vector field \(\bar{u}\) satisfies in \(\Omega \subset \mathbb{R}^3\) the generalized Lamé-Navier system \(\text{(5)}\), then it admits the representation

\[
\bar{u} = h + i^*,
\]

where \(h \in \mathcal{H}_{\psi, \bar{\psi}}(\Omega)\) and \(i^* \in \mathcal{I}_{\psi, \bar{\psi}}(\Omega)\). Moreover, this representation is unique up to an element in \(\mathcal{H}_{\psi, \bar{\psi}}(\Omega) \cap \mathcal{I}_{\psi, \bar{\psi}}(\Omega)\).
5 Construction of solutions

In this section we give a direct method for constructing solutions of (5) from harmonic and/or inframonogenic functions.

Theorem 5 If \( u \) is harmonic or \((\phi, \psi)\)-inframonogenic in \( \Omega \), then

\[
    w = u \frac{\psi}{\alpha} - \frac{\beta}{\alpha} \frac{\psi}{\alpha} u
\]

satisfies (5).

Proof. Indeed, we have

\[
    \alpha [\phi \frac{\psi}{\alpha} u] + \beta [\psi \frac{\psi}{\alpha} u] =
    \alpha \phi \frac{\psi}{\alpha} u - \beta \phi \frac{\psi}{\alpha} u + \beta \phi \frac{\psi}{\alpha} u
    - \frac{\beta^2}{\alpha} \phi \frac{\psi}{\alpha} u
    = \left( \alpha - \frac{\beta^2}{\alpha} \right) \phi \frac{\psi}{\alpha} u = 0,
\]

which is due to the fact that \( u \) is harmonic or \((\phi, \psi)\)-inframonogenic in \( \Omega \).

Notice that the above solution is in general \( \mathbb{R}_{0,3} \)-valued but it becomes vector-valued if \( u \) is a scalar function, as is easy to check. The same fact is valid in the following theorem.

Theorem 6 If \( u \) is \((\phi, \psi)\)-harmonic or \((\psi, \psi)\)-inframonogenic in \( \Omega \) then

\[
    \tilde{w} = u \frac{\psi}{\alpha} - \frac{\alpha}{\beta} \frac{\psi}{\alpha} u
\]

satisfies (5).

The following result shows how to a given harmonic and \((\phi, \psi)\)-harmonic vector field \( \tilde{h} \), corresponds a sort of \((\phi, \psi)\)-inframonogenic conjugate function \( i \) such that \( \tilde{h} + i \) represents a solution of (5).

Theorem 7 Let \( \tilde{h} \in \mathcal{H}(\Omega) \cap \mathcal{H}_{\phi, \psi}(\Omega) \) and suppose \( \omega(\psi \frac{\psi}{\alpha} \tilde{h}) = -\phi \frac{\psi}{\alpha} \tilde{h} \). Then, there exists a function \( i \in \mathcal{I}_{\phi, \psi}(\Omega) \) such that \( \tilde{h} + i \) solves (5). Moreover, \( i \) may be represented as

\[
    i = \frac{\alpha}{\beta^2} [\tilde{h} + (\psi \frac{\psi}{\alpha} \tilde{h}) \frac{\psi}{\alpha}].
\]
Proof. A direct calculation gives
\[
\phi \partial \psi = \frac{\alpha}{2\beta} [\phi \partial \tilde{\nu} + (\psi \partial \tilde{\nu}) \mathcal{E}_\psi + \omega (\psi \partial \tilde{\nu})] \psi = \frac{\alpha}{2\beta} [\phi \partial \tilde{\nu} + \omega (\psi \partial \tilde{\nu})] = 0.
\]

On the other hand,
\[
\alpha [\phi (\tilde{h} + i) \psi] + \beta [\phi \partial \psi (\tilde{h} + i)] = \alpha \phi \partial \tilde{\nu} + \beta \phi \partial \psi i
\]
\[
= \alpha \phi \partial \tilde{\nu} + \frac{\alpha}{2} \phi \partial \psi (\psi \partial \tilde{\nu})
\]
\[
= \alpha \phi \partial \tilde{\nu} + \frac{\alpha}{2} \phi \nu (\psi \partial \tilde{\nu})
\]
\[
= \alpha \phi \partial \tilde{\nu} + \frac{\alpha}{2} \phi [2\tilde{\nu} - \psi \nu (\tilde{h})]
\]
\[
= \frac{\alpha}{2} \phi \partial \nu (\tilde{h})
\]
\[
= \frac{\alpha}{2} \phi \psi \partial \tilde{h} + \frac{\alpha}{2} \phi \psi \partial \tilde{h}
\]
\[
= 0.
\]

And we are done. $\square$

It is worth noting that the above proof strongly depended on the assumption that $\tilde{u}$ is a vector-valued function.

The following result is also obtained in a similar way:

**Theorem 8** Let $\tilde{i} \in \mathbb{L}_{\psi, \psi}(\Omega) \cap \mathbb{L}_{\phi, \psi}(\Omega)$ and suppose $\phi \partial \psi (\tilde{i}) = \psi \partial \tilde{i}$. Then, there exists a function $\tilde{h} \in \mathcal{H}_{\phi, \psi}(\Omega)$ such that $\tilde{h} + \tilde{i}$ solves (5). Moreover, $\tilde{h}$ may be represented as $\tilde{h} = \frac{\beta}{\alpha} [2\tilde{i} + (\tilde{i} \psi \partial \tilde{\nu})] \psi$.

At the end of the paper (see Appendix) a function in MATLAB is provided for performing computations using the Clifford algebra reformulation of both classical and generalized Lamé-Navier systems. Moreover, the implemented MATLAB function is used to verify some algebraic results obtained in the paper.

As an example, consider the generalized Lamé-Navier system
\[
\alpha [\phi \partial \tilde{u}] + \beta [\phi \partial \tilde{u}] = 0,
\]
where $\alpha = 0.1$, $\beta = 0.2$ and $\phi = \{-e_1, e_2, e_3\}$.

Applying the function **Lame** to the harmonic vector field
\[
\tilde{u} = x_1 x_2 e_1 + (-2x_1^2 - 3x_2^2 + 5x_3^2)e_2 + x_3 e_3,
\]
we verify that it is a solution of (16).
On the other hand, after applying the procedure carried out in Theorem 3, we arrive to the decomposition
\[ \vec{u}(x) = h + i^*, \]
where
\[ h(x) = \frac{-20}{9} [(1.05x_2 - 0.15)x_1e_1 + (0.15x_1^2 + 2.10x_2^2 - 0.75x_3^2 - 0.15x_2)e_2 \]
\[ \quad - (0.60 + 0.75x_2)x_3e_3 - 2.25x_1x_3e_1e_2e_3], \]
\[ i^*(x) = \frac{20}{9} [(1.50x_2 - 0.15)x_1e_1 + (0.75x_2^2 + 1.50x_3^2 - 0.75x_1^2 - 0.15x_2)e_2 \]
\[ \quad - (0.15 + 0.75x_2)x_3e_3 - 2.25x_1x_3e_1e_2e_3], \]
satisfy \( \phi_2 \phi \partial h = 0 \) and \( \partial i^* \phi_2 \partial h = 0 \), respectively.

Finally, we remark that such a vector field (17) is also a particular solution of the inhomogeneous classical Lamé system
\[ \alpha [\partial \vec{u} \partial] + \beta [\partial^2 \vec{u}] = e_2. \] (18)

This fact suggests the idea that, if structural sets are conveniently chosen, some kinds of classical inhomogeneous Lamé systems (in presence of a constant volume force) may be rewritten as a homogeneous one. But we will not develop this point here.

6 Appendix: MATLAB implementation

function [DphiDpsif,DpsiDphif,DphifDpsi,DpsifDphi,DpsifDpsi,DphifDphi,DfD,D2f,Lcf,Lcphif,Lcpsif,Lgf,Lgif]= Lame(Phi1,Phi2,Phi3,Psi1,Psi2,Psi3,F,Cl)

syms 'x1' 'x2' 'x3';
phi11=Phi1(1);phi12=Phi1(2);phi13=Phi1(3);phi21=Phi2(1);phi22=Phi2(2);
phi23=Phi2(3);phi31=Phi3(1);phi32=Phi3(2);phi33=Phi3(3);
psi11=Psi1(1);psi12=Psi1(2);psi13=Psi1(3);psi21=Psi2(1);psi22=Psi2(2);
psi23=Psi2(3);psi31=Psi3(1);psi32=Psi3(2);psi33=Psi3(3);

alpha=Cl(1);beta=Cl(2);
alphaC=Clifford([0 3 0],[alpha 0 0 0 0 0 0 0]);
betaC=Clifford([0 3 0],[beta 0 0 0 0 0 0 0]);
if beta-alpha>0 && 7*alpha>beta;
    disp('Coefficients meet Lame’s restrictions')
else
    disp('Coefficients do not meet Lame’s restrictions')
end
k0=F(1);k1=F(2);k2=F(3);k3=F(4);k4=F(5);k5=F(6);k6=F(7);k7=F(8);
\[\text{dfx1} = \text{Clifford}([0 \ 3 \ 0], [dk0x1, dk1x1, dk2x1, dk3x1, dk4x1, dk5x1, dk6x1, dk7x1]);\]
\[\text{dfx2} = \text{Clifford}([0 \ 3 \ 0], [dk0x2, dk1x2, dk2x2, dk3x2, dk4x2, dk5x2, dk6x2, dk7x2]);\]
\[\text{dfx3} = \text{Clifford}([0 \ 3 \ 0], [dk0x3, dk1x3, dk2x3, dk3x3, dk4x3, dk5x3, dk6x3, dk7x3]);\]
\[\phi_{i1} = \text{Clifford}([0 \ 3 \ 0], [0 \ \phi_{11} \ \phi_{12} \ \phi_{13} \ 0 \ 0 \ 0 \ 0]);\]
\[\phi_{i2} = \text{Clifford}([0 \ 3 \ 0], [0 \ \phi_{21} \ \phi_{22} \ \phi_{23} \ 0 \ 0 \ 0 \ 0]);\]
\[\phi_{i3} = \text{Clifford}([0 \ 3 \ 0], [0 \ \phi_{31} \ \phi_{32} \ \phi_{33} \ 0 \ 0 \ 0 \ 0]);\]
\[\psi_{i1} = \text{Clifford}([0 \ 3 \ 0], [0 \ \psi_{11} \ \psi_{12} \ \psi_{13} \ 0 \ 0 \ 0 \ 0]);\]
\[\psi_{i2} = \text{Clifford}([0 \ 3 \ 0], [0 \ \psi_{21} \ \psi_{22} \ \psi_{23} \ 0 \ 0 \ 0 \ 0]);\]
\[\psi_{i3} = \text{Clifford}([0 \ 3 \ 0], [0 \ \psi_{31} \ \psi_{32} \ \psi_{33} \ 0 \ 0 \ 0 \ 0]);\]
\[
\text{Df} = \text{Clifford}([0 3 0],[0 0 1])
\]
\[
\text{D2f} = \text{Clifford}([0 3 0],[0 1 0 0 0 0 0 0]) \ast (\text{dfx11} + \text{dfx22} + \text{dfx33})
\]
\[
\text{DphiDpsi} = \phi_1 * \text{dfx11} + \phi_2 * \text{dfx22} + \phi_3 * \text{dfx33}
\]
\[
\text{Lcf} = \alpha_C * \text{D2f} + \beta_C * \text{DphiDpsi}
\]
\[
\text{Lcphif} = \alpha_C * \text{DphiDpsi} + \beta_C * \text{D2f}
\]
\[
\text{Lcpsif} = \alpha_C * \text{DpsiDphi} + \beta_C * \text{D2f}
\]
\[
\text{Lgf} = \alpha_C * \text{DphiDpsi} + \beta_C * \text{DphiDphi}
\]
\[
\text{Lgif} = \alpha_C * \text{DpsiDphi} + \beta_C * \text{DpsiDpsi}
\]

Example (16)-(17)

\[
\text{Lame}([-1 0 0],[0 1 0],[0 0 1],[1 0 0],[0 0 1],[0 1 0],[-2*x1^2-3*x2^2+5*x3^2 0 0 0 0],[0.1 0.2])
\]

Coefficients meet Lame's restrictions

\[
\text{D2f} = 0e0 \quad \text{DphiDpsi} = 0e0 + -10e2 \quad \text{DpsiDphi} = 0e0 + -6e2
\]
\[
\text{DphiDpsi} = 0e0 + 20e2 \quad \text{DpsiDphi} = 0e0 + 20e2 \quad \text{DpsiDpsi} = 0e0 + 14e2
\]
\[
\text{Lcf} = 0e0 + 1e2 \quad \text{Lcphif} = 0e0 + 7/5e2 \quad \text{Lcpsif} = 0e0 + 1e2
\]
\[ \text{Lgf} = 0e0 \quad \text{Lgif} = 0e0 + 4/5e2 \]

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