Cayley-Type Conditions for Billiards within $k$ Quadrics in $\mathbb{R}^d$

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Abstract

The notions of reflection from outside, reflection from inside and signature of a billiard trajectory within a quadric are introduced. Cayley-type conditions for periodical trajectories for the billiard in the region bounded by $k$ quadrics in $\mathbb{R}^d$ and for the billiard ordered game within $k$ ellipsoids in $\mathbb{R}^d$ are derived. In a limit, the condition describing periodical trajectories of billiard systems on a quadric in $\mathbb{R}^d$ is obtained.

1 Introduction

We study periodic trajectories of the following well-known integrable mechanical system: motion of a free particle within an ellipsoid in the Euclidean space of any dimension $d$. On the boundary, the particle respects the billiard law. To be more precise, let us mention some basic notions.

Let $(Q, g)$ be a $d$-dimensional Riemannian manifold and let $D \subset Q$ be a domain with a piecewise smooth boundary $B$. Let $\pi : T^*Q \to Q$ be a natural projection and let $g^{-1}$ be the contravariant metric on the cotangent bundle.

Consider the reflection mapping $r : \pi^{-1}B \to \pi^{-1}B$, $p_- \mapsto p_+$, which associates the covector $p_+ \in T^*_xQ$, $x \in B$ to a covector $p_- \in T^*_xQ$ such that the following conditions hold: $|p_+| = |p_-|$, $p_+ - p_- \perp B$.

A billiard in $D$ is a dynamical system with the phase space $M = T^*D$ whose trajectories are geodesics given by the Hamiltonian $H(p, x) = \frac{1}{2}g_x^{-1}(p, p)$, reflected at points $x \in B$ according to the billiard law: $r(p_-) = p_+$. Here $p_-$ and $p_+$ denote the momenta before and after the reflection.

Integrability of the billiard system within quadrics is related to classical geometrical properties: the Chasles, Poncelet and Cayley theorems. From the Chasles theorem [1] every line in $\mathbb{R}^d$ is tangent to $d-1$ quadrics confocal to the
boundary; all segments of one billiard trajectory are tangent to the same \( d - 1 \) quadrics \([2]\). We refer to these \( d - 1 \) quadrics as caustics of the given trajectory.

From now on, we consider a billiard with a boundary which consists of the union of \( k \) confocal quadrics in \( \mathbb{R}^d \). According to the generalized Poncelet theorem \([3]\), with \( k, d \) arbitrary, \emph{there exists a closed trajectory with \( d - 1 \) given confocal caustics if and only if infinitely many such trajectories exist, and all of them have the same period.} The periodicity of a billiard trajectory depends on its caustic surfaces. An important question is to find an analytical connection between them and the corresponding period.

In \([4]\), Cayley found the analytical condition for caustic conics in the Euclidean plane case (\( d = 2 \)) with \( k = 1 \) conic as a boundary. The classical and algebro-geometric proofs of Cayley’s theorem can be found in Lebesgue’s book \([5]\) and the paper \([6]\), respectively. Moreover, in \([5]\) the complete Poncelet theorem for billiard systems in a plane (\( d = 2 \)) within \( k \) conics, with \( k \) arbitrary, was proved (see also \([7]\)).

The generalisation of Cayley’s condition for \( k = 1 \) is established by the authors for any dimension \( d \) \([8]\), by use of the Veselov-Moser discrete \( L - A \) pair \([9]\).

The main goal of this paper is to give Cayley-type conditions describing periodic trajectories of the billiard in the region bounded by \( k \) confocal quadrics in \( \mathbb{R}^d \) and of the billiard ordered game within \( k \) ellipsoids in \( \mathbb{R}^d \), for \( k, d \) arbitrary. The importance of these questions was underlined several times by experts; let us mention Arnol’d (see \([1, 10]\)), for example, in connection with applications in laser technology. In a limit case, we derive analytic conditions for periodic billiard trajectories on a quadric in \( \mathbb{R}^d \) bounded by any finite number of quadrics, solving in this way a problem explicitly posed by Abenda and Fedorov \([11]\).

## 2 Planar case: \( d = 2, k \) arbitrary

The derivation of Cayley-type conditions for the billiard in a plane within \( k \) conics can be done following Lebesgue. In \([5]\), he considered polygons inscribed in a conic \( \Gamma \), whose sides are tangent to \( \Gamma_1, \ldots, \Gamma_k \), where \( \Gamma, \Gamma_1, \ldots, \Gamma_k \) all belong to a pencil of conics. In the dual plane, such polygons correspond to billiard trajectories having caustic \( \Gamma^* \) with bounces on \( \Gamma_1^*, \ldots, \Gamma_k^* \). The main object of Lebesgue’s analysis in \([5]\) was the cubic Cayley curve, which parametrizes contact points of tangents drawn from a given point to all conics of the pencil.

We summarize Lebesgue’s results as follows. Let \( C \) and \( \Gamma \) be conics of a pencil \( \mathcal{F} \) and \( \Delta(x) \) be the discriminant of the conic \( C + x\Gamma = 0 \). If \( \lambda_1, \ldots, \lambda_k \) denote parameters corresponding to \( \Gamma_1, \ldots, \Gamma_k \), respectively, then the existence of the Poncelet polygon is equivalent to

\[
\det \begin{pmatrix}
1 & \lambda_1 & \lambda_2 & \ldots & \lambda_1^p \sqrt{\Delta(\lambda_1)} & \lambda_1 \sqrt{\Delta(\lambda_1)} & \ldots & \lambda_1^{p-2} \sqrt{\Delta(\lambda_1)} \\
\vdots & \vdots & \vdots & \ldots & \lambda_k & \lambda_k \sqrt{\Delta(\lambda_k)} & \ldots & \lambda_k^{p-2} \sqrt{\Delta(\lambda_k)} \\
1 & \lambda_k & \lambda_k^2 & \ldots & \lambda_k^p \sqrt{\Delta(\lambda_k)} & \lambda_k \sqrt{\Delta(\lambda_k)} & \ldots & \lambda_k^{p-2} \sqrt{\Delta(\lambda_k)}
\end{pmatrix} = 0
\]
for \( k = 2p \)
\[
\det \begin{pmatrix}
1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^p & \sqrt{\Delta(\lambda_1)} & \lambda_1 \sqrt{\Delta(\lambda_1)} & \ldots & \lambda_1^{p-1} \sqrt{\Delta(\lambda_1)} \\
\vdots & & & & & \vdots & & & \\
1 & \lambda_k & \lambda_k^2 & \ldots & \lambda_k^p & \sqrt{\Delta(\lambda_k)} & \lambda_k \sqrt{\Delta(\lambda_k)} & \ldots & \lambda_k^{p-1} \sqrt{\Delta(\lambda_k)}
\end{pmatrix} = 0
\]
for \( k = 2p + 1 \).

The case with two ellipses, when the billiard trajectory is placed between them and the particle bounces from one to the other of them alternately, is of special interest.

**Corollary 1** The condition for existence of a \( 2m \)-periodic billiard trajectory which bounces exactly \( m \) times to the ellipse \( \Gamma_1^* = C^* \) and \( m \) times to \( \Gamma_2^* = (C + \gamma \Gamma)^* \), having \( \Gamma^* \) for the caustic, is

\[
\det \begin{pmatrix}
f_0(0) & f_1(0) & \ldots & f_{2m-1}(0) \\
f'_0(0) & f'_1(0) & \ldots & f'_{2m-1}(0) \\
\vdots & \vdots & \ddots & \vdots \\
f_0^{(m-1)}(\gamma) & f_1^{(m-1)}(\gamma) & \ldots & f_{2m-1}^{(m-1)}(\gamma)
\end{pmatrix} = 0
\]

where \( f_j = x^j \) (0 \( \leq j \leq m \)), \( f_{m+i} = x^{i-1} \sqrt{\Delta(x)} \) (1 \( \leq i \leq m - 1 \)).

We consider a simple example with four bounces on each of the two conics.

**Example 1** The condition on a billiard trajectory placed between ellipses \( \Gamma_1^* \) and \( \Gamma_2^* \), to be closed after four alternating bounces to each of them is

\[
\det X = 0
\]

where the elements of the \( 3 \times 3 \) matrix \( X \) are

\[
X_{11} = -4B_0 + B_1 \gamma + 4C_0 + 3C_1 \gamma + 2C_2 \gamma^2 + C_3 \gamma^3 \\
X_{12} = -3B_0 + B_1 \gamma + 3C_0 + 2C_1 \gamma + C_2 \gamma^2 \\
X_{13} = -2B_0 + B_1 \gamma + 2C_0 + C_1 \gamma \\
X_{21} = -6B_0 + B_2 \gamma^2 + 6C_0 + 6C_1 \gamma + 4C_2 \gamma^2 + 3C_3 \gamma^3 \\
X_{22} = -6B_0 + B_1 \gamma + B_2 \gamma^2 + 6C_0 + 4C_1 \gamma + 3C_2 \gamma^2 \\
X_{23} = -5B_0 + 2B_1 \gamma + B_2 \gamma^2 + 5C_0 + 3C_1 \gamma \\
X_{31} = -4B_0 + B_3 \gamma^3 + 4C_0 + 4C_1 \gamma + 4C_2 \gamma^2 + 3C_3 \gamma^3 \\
X_{32} = -4B_0 + B_2 \gamma^2 + B_3 \gamma^3 + 4C_0 + 4C_1 \gamma + 3C_2 \gamma^2 \\
X_{33} = -4B_0 + B_1 \gamma + B_2 \gamma^2 + B_3 \gamma^3 + 4C_0 + 3C_1 \gamma
\]
with $C_i, B_i$ being coefficients in the Taylor expansions around $x = 0$ and $x = \gamma$, respectively

\[
\sqrt{\Delta(x)} = C_0 + C_1 x + C_2 x^2 + \ldots
\]
\[
\sqrt{\Delta(x)} = B_0 + B_1 (x - \gamma) + B_2 (x - \gamma)^2 + \ldots.
\]

3 Periodic billiard trajectories inside $k$ confocal quadrics in $\mathbb{R}^d$

The complete Poncelet theorem (CPT) was generalized to the case $d = 3$ by Darboux in [12] in 1870. Higher-dimensional generalizations of CPT were obtained quite recently in [3]. The main result of the present paper is the Cayley-type condition for generalized CPT for $d \geq 3$, although obtained results can be applied immediately in the case $d = 2$.

Consider an ellipsoid in $\mathbb{R}^d$

\[
\frac{x_1^2}{a_1} + \cdots + \frac{x_d^2}{a_d} = 1 \quad a_1 > \cdots > a_d > 0
\]

and a related system of Jacobian elliptic coordinates $(\lambda_1, \ldots, \lambda_d)$ ordered by the condition

$\lambda_1 > \lambda_2 > \cdots > \lambda_d$.

Any quadric from the corresponding confocal family is given by

\[
Q_{\lambda} : \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_d^2}{a_d - \lambda} = 1.
\]

Lemma 1 Suppose a line $\ell$ is tangent to quadrics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-1}}$ from the family (1). Then Jacobian coordinates $(\lambda_1, \ldots, \lambda_d)$ of any point on $\ell$ satisfy the inequalities $P(\lambda_s) \geq 0$, $s = 1, \ldots, d$, where

$P(x) = (a_1 - x) \cdots (a_d - x)(a_1 - x) \cdots (a_{d-1} - x)$.

Proof. Follows from [13].

Consider a billiard system within $\Omega$ and let $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-1}}$ be caustics of one of its trajectories. For any $s = 1, \ldots, d$, the set $\Lambda_s$ of all values taken by the coordinate $\lambda_s$ on the trajectory is, according to lemma 1, included in $\Lambda'_s = \{ \lambda \in [\beta'_s, \beta''_s] : P(\lambda) \geq 0 \}$. By [14], the set $\Lambda'_s$ is a closed interval and coincides with $\Lambda_s$. Denote $[\gamma'_s, \gamma''_s] := \Lambda_s = \Lambda'_s$.

Note that the trajectory touches quadrics of any pair $Q_{\gamma'_s}, Q_{\gamma''_s}$ alternately. Thus, any periodic trajectory has the same number of intersection points with each of them.
Theorem 1 A trajectory of the billiard system within $\Omega$ with caustics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d+1}}$ is periodic with exactly $n_s$ points at $Q_{\gamma'_s}$ and $n_s$ points at $Q_{\gamma''_s}$ ($1 \leq s \leq d$) if and only if
\[ \sum_{s=1}^{d} n_s \left( A(P_{\gamma'_s}) - A(P_{\gamma''_s}) \right) = 0 \] on the Jacobian of the curve
\[ \Gamma : y^2 = P(x) := (a_1 - x) \cdots (a_d - x)(a_1 - x) \cdots (a_{d+1} - x). \]
Here $A$ denotes the Abel-Jacobi map, where $P_{\gamma'_s}, P_{\gamma''_s}$ are points on $\Gamma$ with coordinates $P_{\gamma'_s} = \left( \gamma'_s, (\gamma'_s)^s \sqrt{P(\gamma'_s)} \right)$, $P_{\gamma''_s} = \left( \gamma''_s, (\gamma''_s)^s \sqrt{P(\gamma''_s)} \right)$.

Proof. Following Jacobi [15], let us consider the integrals
\[ \sum_{s=1}^{d} \int \frac{d\lambda_s}{\sqrt{P(\lambda_s)}}, \sum_{s=1}^{d} \int \frac{\lambda_s d\lambda_s}{\sqrt{P(\lambda_s)}}, \ldots, \sum_{s=1}^{d} \int \frac{\lambda_s^{d-1} d\lambda_s}{\sqrt{P(\lambda_s)}} \]
over the polygonal line $A_1 A_2 \ldots A_{k+1}$, which represents a billiard trajectory, where $k = 2(n_1 + \cdots + n_d)$. The last integral is equal to the total length of the polygonal line, while the others are equal to zero. Considering the behaviour of elliptic coordinates along each segment of the trajectory, we calculate values of the integrals and obtain that the condition $A_{k+1} = A_1$ is equivalent to (2). \hfill $\Box$

Our next step is to introduce a notion of bounces ‘from outside’ and ‘from inside’. More precisely, let us consider an ellipsoid $Q_{\lambda}$ from the confocal family (1) such that $\lambda \in (a_{s+1}, a_s)$ for some $s \in \{1, \ldots, d\}$, where $a_{d+1} = -\infty$.

Observe that along a billiard ray which reflects at $Q_{\lambda}$, the elliptic coordinate $\lambda_i$ has a local extremum at the point of reflection.

Definition 1 A ray reflects from outside at the quadric $Q_{\lambda}$ if the reflection point is a local maximum of the Jacobian coordinate $\lambda_s$, and it reflects from inside if the reflection point is a local minimum of the coordinate $\lambda_s$.

Let us remark that in the case when $Q_{\lambda}$ is an ellipsoid, the notions introduced in definition 1 coincide with the usual ones.

Assume now a $k$-tuple of confocal quadrics $Q_{\beta_1}, \ldots, Q_{\beta_k}$ from the confocal pencil (1) is given. We consider a billiard system with trajectories having bounces at $Q_{\beta_1}, \ldots, Q_{\beta_k}$ respectively. Such a trajectory has $d - 1$ caustics from the same family (1). We additionally assign to each trajectory the signature $\sigma = (i_1, \ldots, i_k)$ by the following rule:
\[ i_s = +1 \quad \text{if the reflection at } Q_{\beta_s} \text{ is from inside} \]
\[ i_s = -1 \quad \text{if the reflection at } Q_{\beta_s} \text{ is from outside}. \]

Suppose $Q_{\beta_1}, \ldots, Q_{\beta_k}$ are ellipsoids and consider a billiard ordered game with signature $\sigma = (i_1, \ldots, i_k)$. In order that trajectories of such a game stay bounded, the following condition has to be satisfied:
\[ i_s = -1 \quad \Rightarrow \quad i_{s+1} = i_{s-1} = 1 \quad \text{and} \quad \beta_{s+1} < \beta_s, \beta_{s-1} < \beta_s. \]
(Here, we identify indices 0 and $k + 1$ with $k$ and 1, respectively.)

**Theorem 2** Given a billiard ordered game within $k$ ellipsoids $Q_{\beta_1}, \ldots, Q_{\beta_k}$ with signature $\sigma = (i_1, \ldots, i_k)$. Its trajectory with caustics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-1}}$ is $k$-periodic if and only if

$$
\sum_{s=1}^{k} i_s (A(P_{\beta_s}) - A(P_{\alpha}))
$$

is equal to a sum of several expressions of the form $A(P_{\alpha_p}) - A(P_{\alpha_{p'}})$ on the Jacobian of the curve $\Gamma : y^2 = \mathcal{P}(x)$, where $P_{\beta_s} = \left(\beta_s, +\sqrt{\mathcal{P}(\beta_s)}\right)$, $\alpha = \min\{a_d, \alpha_1, \ldots, \alpha_{d-1}\}$ and $Q_{\alpha_p}, Q_{\alpha_{p'}}$ are pairs of caustics of the same type.

When $Q_{\beta_1} = \cdots = Q_{\beta_k}$ and $i_1 = \cdots = i_k = 1$ we obtain the Cayley-type condition for billiard motion inside an ellipsoid in $\mathbb{R}^d$. Such periodic trajectories were described in [8] using a different technique, based on a Veselov-Moser discrete Lax representation.

We are going to treat in more detail the case of billiard motion between two ellipsoids.

**Proposition 1** The condition that there exists a closed billiard trajectory between two ellipsoids $Q_{\beta_1}$ and $Q_{\beta_2}$, which bounces exactly $m$ times to each of them, with caustics $Q_{\alpha_1}, \ldots, Q_{\alpha_{d-1}}$, is

$$
\text{rank} \begin{pmatrix}
  f'_1(P_{\beta_2}) & f'_2(P_{\beta_2}) & \cdots & f'_{m-d+1}(P_{\beta_2}) \\
  f''_1(P_{\beta_2}) & f''_2(P_{\beta_2}) & \cdots & f''_{m-d+1}(P_{\beta_2}) \\
  \vdots & \vdots & \ddots & \vdots \\
  f^{(m-1)}_1(P_{\beta_2}) & f^{(m-1)}_2(P_{\beta_2}) & \cdots & f^{(m-1)}_{m-d+1}(P_{\beta_2})
\end{pmatrix} < m - d + 1.
$$

Here

$$
f_j = \frac{y - B_0 - B_1(x - \beta_1) - \cdots - B_{d+j-2}(x - \beta_1)^{d+j-2}}{x^{d+j-1}} \quad 1 \leq j \leq m - d + 1
$$

and $y = B_0 + B_1(x - \beta_1) + \cdots$ is the Taylor expansion around the point symmetric to $P_{\beta_1}$ with respect to the hyperelliptic involution of the curve $\Gamma$. (All notation is as in theorem 2.)

4 Periodic trajectories of billiards on quadrics in $\mathbb{R}^d$

In [16] the billiard systems on a quadric $\mathcal{E}$ in $\mathbb{R}^d$

$$
\frac{x_1^2}{a_1} + \cdots + \frac{x_d^2}{a_d} = 1, \quad a_1 > \cdots > a_d,
$$
Theorem 4

A trajectory of the billiard system constrained to the ellipsoid \( E \) within \( \Omega : \beta_1' \leq \lambda_1 \leq \beta_1'', \ldots, \beta_{d-1}' \leq \lambda_{d-1} \leq \beta_{d-1}'' \), with caustics \( Q_{\beta_1}, \ldots, Q_{\beta_k} \), is periodic with exactly \( n_s \) bounces at each of quadrics \( Q_{\gamma_1'}, Q_{\gamma_1''} \) (1 \( \leq s \leq d-2 \)) if and only if

\[
\sum_{s=1}^{d-1} n_s \left( \tilde{A}(P_{\gamma_1'}) - \tilde{A}(P_{\gamma_1''}) \right) = 0
\]

on the Jacobian of the curve

\( \Gamma_1 : y^2 = P_1(x) := -x(a_1 - x)(a_2 - x)(a_3 - x) \cdots (a_{d-2} - x) \).

Here \( P_{\gamma_1'}, P_{\gamma_1''} \) are the points on \( \Gamma_1 \) with coordinates \( P_{\gamma_1'} = \left( \gamma'_s, (-1)^s \sqrt{P_1(\gamma'_s)} \right) \),

\( P_{\gamma_1''} = \left( \gamma''_s, (-1)^s \sqrt{P_1(\gamma''_s)} \right) \), with \( [\gamma'_s, \gamma''_s] = \{ \lambda \in [\beta'_s, \beta''_s] : P_1(\lambda) \geq 0 \} \),

1 \( \leq s \leq d-2 \), and \( \tilde{A}(P) = (0, \int_0^P \frac{x \, dx}{y}, \int_0^P \frac{x^2 \, dx}{y}, \ldots, \int_0^P \frac{x^{d-2} \, dx}{y}) \).

In the same way as in the previous section, a billiard ordered game constrained to the ellipsoid \( E \) within given quadrics \( Q_{\beta_1}, \ldots, Q_{\beta_k} \) of the same type can be defined. The only difference is that now the signature \( \sigma = (i_1, \ldots, i_k) \) can be given arbitrarily, since trajectories are bounded, lying on the compact hypersurface \( E \). Denote by \( Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}} \) the caustics of a given trajectory of the game. Since quadrics \( Q_{\beta_1}, \ldots, Q_{\beta_k} \) are all of the same type, there exist \( \mu', \mu'' \) in the set \( S = \{ a_1, \ldots, a_d, a_1, \ldots, a_{d-2} \} \) such that \( \beta_1, \ldots, \beta_k \in [\mu', \mu''] \) and \( (\mu', \mu'') \cap S \) is empty.

Associate with the game the following divisors on the curve \( \Gamma_1 \):

\[
D_s = \begin{cases} 
P_{\mu''} & \text{if } i_s = i_{s+1} = 1 \\
0 & \text{if } i_s = -i_{s+1} = 1, \beta_s < \beta_{s+1} \text{ or } i_s = -i_{s+1} = -1, \beta_s > \beta_{s+1} \\
P_{\mu''} - P_{\mu'} & \text{if } i_s = -i_{s+1} = 1, \beta_s > \beta_{s+1} \\
P_{\mu'} - P_{\mu''} & \text{if } i_s = -i_{s+1} = -1, \beta_s < \beta_{s+1} \\
P_{\mu'} & \text{if } i_s = i_{s+1} = -1,
\end{cases}
\]

where \( P_{\mu'} \) and \( P_{\mu''} \) are its branching points with coordinates \( (\mu', 0) \) and \( (\mu'', 0) \), respectively.

Theorem 4

Given a billiard ordered game constrained to \( E \) within quadrics \( Q_{\beta_1}, \ldots, Q_{\beta_k} \) with signature \( \sigma = (i_1, \ldots, i_k) \). Its trajectory with caustics \( Q_{\alpha_1}, \ldots, Q_{\alpha_{d-2}} \).
..., $Q_{\alpha_d-2}$ is $k$-periodic if and only if

$$\sum_{s=1}^{k} t_s (\tilde{A}(P_{\beta_s}) - \tilde{A}(D_s))$$

is equal to a sum of several expressions of the form $\tilde{A}(P_{\alpha_p}) - \tilde{A}(P_{\alpha_{p'}})$ on the Jacobian of the curve $\Gamma_1: y^2 = P_1(x)$, where $P_{\beta_s} = (\tilde{\beta}_s, + \sqrt{P_1(\beta_s)})$ and $Q_{\alpha_p}$, $Q_{\alpha_{p'}}$ are pairs of caustics of the same type.

**Proposition 2** Consider the case $d = 3$ and a billiard system constrained to the ellipsoid $E$ with the boundary $Q_{\gamma}$ and caustic $Q_{\alpha}$, $a_3 < \gamma < \alpha < a_2$. A trajectory is $k$-periodic if:

$$\text{rank} \begin{pmatrix} C_{p+1} & C_{p+2} & \ldots & C_{2p-2} \\ C_{p+2} & C_{p+3} & \ldots & C_{2p-1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{2p} & C_{2p+1} & \ldots & C_{3p-3} \end{pmatrix} < \frac{p - 2}{k = 2p}$$

and

$$\text{rank} \begin{pmatrix} C_{p+1} & C_{p+2} & \ldots & C_{2p-1} \\ C_{p+2} & C_{p+3} & \ldots & C_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ C_{2p} & C_{2p+1} & \ldots & C_{3p-2} \end{pmatrix} < \frac{p - 1}{k = 2p + 1}$$

where

$$y = C_0 + C_1 \left( \tilde{x} - \frac{1}{\alpha - \gamma} \right) + C_2 \left( \tilde{x} - \frac{1}{\alpha - \gamma} \right)^2 + \ldots$$

is the Taylor expansion with respect to $\tilde{x} = \frac{1}{\alpha - x}$ around the point $P_{\gamma}$.

## 5 Conclusion

As an important historical remark, we would like to emphasize the significance of Darboux’s contribution to the study of problems related to generalized Poncelet theorem. The impression is that his work in the field (see [12]) is completely unknown nowadays. We shall present in another publication a more detailed overview of Darboux’s ideas with comparison to the Lebesgue geometric approach and applications to separable perturbed problems [17, 18].

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