Higher-order Coulomb Corrections to the Threshold $e^+e^- \rightarrow W^+W^-$ Cross Section

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Abstract

The QED Coulomb correction is one of the most important corrections to the $e^+e^- \rightarrow W^+W^-$ total cross section near threshold. We calculate these corrections through second order, and discuss the implications for extracting $M_W$ from a threshold cross section measurement at LEP2. Analytic expressions are derived in various kinematic limits.
1 Introduction

The magnitude of the $e^+e^- \rightarrow W^+W^-$ cross section near threshold at LEP2 provides a powerful method for measuring $M_W$ at the LEP2 collider, see for example Ref. [1]. It is sufficient to make a single measurement of the cross section at the ‘collision energy of maximal sensitivity’, which is approximately 0.5 GeV above the nominal $\sqrt{s} = 2M_W$ threshold. The method relies on a high-precision calculation of the cross section as a function of $M_W$, and it is therefore important that higher-order corrections are under control. The QED Coulomb correction is one of the most important of these, especially at threshold. Note that since Coulomb physics — which is associated with large space-time intervals — is so different from the other radiative effects, it is possible to study its impact separately.

As was emphasized in Refs. [2, 3] (see also [4]), the Coulomb corrections to the total $e^+e^- \rightarrow W^+W^-$ cross sections are relatively small because the instability effects mask the Coulomb singularity. In fact the corrections are numerically more important in the case of differential distributions, for example of the $W$ invariant mass or momentum [5], but even so their exact (i.e. all-orders) calculation appears to be unnecessary for the level of precision required at LEP2.

The first-order Coulomb correction for the (off-shell) $W^+W^-$ total cross section was first presented in Ref. [2, 3], and its quantitative effects were discussed in Refs. [1, 4]. It was shown that the correction is positive in the threshold region, attaining a maximum value of approximately $+6\%$ at threshold. In terms of using the size of the measured threshold cross section to measure $M_W$, the inclusion of the first-order Coulomb correction is equivalent to a shift in $M_W$ of order 100 MeV. In view of the target LEP2 precision on $M_W$ of better than $\pm 50$ MeV, it is clearly important to investigate the size of the higher-order contributions.

In this paper we present a quantitative analysis of these Coulomb contributions, and in particular their impact on the threshold $W^+W^-$ cross section. We do not include initial-state radiation or ‘hard’ radiative corrections, but their effects would be straightforward to include, at least in principle (see for example Ref. [2]). In the following section we recall the results of Ref. [3] for the higher-order Coulomb contributions to the $W^+W^-$ cross section, and derive an explicit expression for the $O(\alpha^2)$ contribution. In Section 3 we obtain a closed formula for the $O(\alpha^2)$ correction which is valid in the non-relativistic limit, and in Section 4 we derive numerical results for the first- and second-order Coulomb corrections over the range of LEP2 energies. Some brief conclusions are presented in Section 5.
2 Higher-order Coulomb effects

The appropriate formalism for analysing the all-orders Coulomb effects in the production of a pair of heavy unstable particles, based on the technique of non-relativistic Green’s functions, has been known for some time, see Refs. 7, 8. It was developed substantially because of the requirements of $t\bar{t}$ threshold production physics 9, 10, 11, 12. The application to unstable $W$ boson production was first considered in Ref. 3. A simplified prescription for incorporating high-order Coulomb corrections in this case has been proposed in Ref. 4, based on the direct substitution of the one-loop off-shell correction into the exact all-orders on-shell result 13. Although this procedure may provide some qualitative understanding of the size of the higher-order contributions, it cannot be justified on theoretical grounds. In particular, it certainly cannot be applied at or just below threshold energies, $E \approx \sqrt{s} - 2M_W \leq 0$. More generally, the first-order correction is related only to the real part of the Coulomb factor $f(\vec{p}, E)$ (see 8, 11, 3), whereas at higher-orders both the real and imaginary parts contribute.

In what follows, we will quantitatively compare results for the Coulomb corrections to the total $W^+W^-$ cross section obtained from the ‘exact’ non-relativistic all-orders prescription of Ref. 3 with the approximate procedure of Ref. 4.

Note that for energies $E \gg \Gamma_W$ the modification of the Coulomb contribution to the total cross section arising from the instability of the $W$ bosons is, at most, of relative order $O(\alpha \Gamma_W / E)$. The modifications due to all other final-state QED interaction effects are cancelled in the total cross section, up to terms of relative order $\alpha \Gamma_W / M_W$ 10, 15, 2 (see also 16).

The all-orders Coulomb contribution $\delta_C$ to the radiative correction to the $e^+e^- \rightarrow W^+W^-$ off-shell Born cross section can be written as

$$1 + \delta_C = |f(\vec{p}, E)|^2,$$

where the Coulomb enhancement factor $f(\vec{p}, E)$ is given by 3, 11, 3

$$f(\vec{p}, E) = 1 + \alpha \sqrt{s} \kappa \int_0^1 dx \frac{x^{-\alpha \sqrt{s}/4\kappa}}{\kappa^2(1 + x)^2 + p^2(1 - x)^2},$$

with

$$p^2 = \frac{1}{4s} \left[ s^2 - 2s(s_1 + s_2) + (s_1 - s_2)^2 \right]$$

$$\kappa = \sqrt{-M_W(E + i\Gamma_W)} \equiv p_1 - ip_2$$

3Recall that for stable particles the imaginary part of the one-loop correction contains an infrared divergence which is related to the unobservable Coulomb phase. This singularity leads to some specific consequences for the case of unstable $W$ bosons 10, 3. In some sense, the remnant of the Coulomb phase could be important in differential distributions, see for example the discussion in Ref. 14.
\[ p_{1,2} = \left[ \frac{1}{2} M_W \left( \sqrt{E^2 + \Gamma_W^2} + E \right) \right]^{\frac{1}{2}} \]  

(5)

\[ E = \frac{s - 4 M_W^2}{4 M_W} \]  

(6)

As discussed in Ref. [3], the representation (2) is convergent for all values of \( E \), both above \( (E > 0) \) and below \( (E < 0) \) the nominal \( W^+W^- \) threshold. Expanding \( f(p, E) \) as a power series in \( \alpha \) up to \( O(\alpha^2) \) terms one finds

\[ f(p, E) \approx 1 + \frac{\alpha \sqrt{s}}{4 ip} \ln D + \frac{\alpha^2 s}{16 ip \kappa} \int_0^1 dx \ln \left( \frac{1 + x D}{1 + x/D} \right), \]  

(7)

with

\[ D = \frac{\kappa + ip}{\kappa - ip}. \]  

(8)

It is straightforward to verify that the real part of the second term on the right-hand side in Eq. (7) coincides with the \( O(\alpha) \) correction calculated in Refs. [2, 3]:

\[ |f|^2 \approx 1 + 2 \text{Re} \left( \frac{\alpha \sqrt{s}}{4 ip} \ln D \right) + O(\alpha^2) \]

\[ = 1 + \frac{\alpha \sqrt{s}}{4p} \left[ \pi - 2 \text{arctan} \left( \frac{|\kappa|^2 - p^2}{2p \text{Re}(\kappa)} \right) \right] + O(\alpha^2). \]  

(9)

Eqs. (2, 7) demonstrate explicitly how the Coulomb singularity is screened by instability effects. In fact the expansion parameter of the Coulomb series is effectively \( \alpha \pi \sqrt{s}/(2|\kappa|) \), which attains a maximum value of \( \alpha \pi \sqrt{M_W/\Gamma_W} \approx \frac{1}{7} \) at the exact threshold \( (E = 0) \) in contrast to the expansion parameter \( \alpha \pi \sqrt{s}/(2p) \) of the stable \( W \) case. In particular, in the limit \( p \ll |\kappa| \), which corresponds to the lower \( W \) momentum tail, we have explicitly

\[ f(p, E) = 1 - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n (n-1)!} \left( \frac{\alpha \sqrt{s}}{\kappa} \right)^n \int_0^1 dx \frac{\ln^{n-1} x}{(1 + x)^2} \]

\[ = 1 + \frac{\alpha \sqrt{s}}{2\kappa} + \frac{\alpha^2 s}{4\kappa^2} \ln 2 + O(\alpha^3). \]  

(10)

### 3 Closed formula in the non-relativistic limit

In Refs. [7, 8] a general approach has been proposed which allows one to obtain a closed formula for the dominant contribution to the all-orders Coulomb correction
to the total unpolarized cross section for the production of a pair of heavy unstable particles. The application of this to the case of \( e^+e^- \rightarrow W^+W^- \) has been discussed in Ref. [3]. The Coulomb effects are incorporated through the imaginary part of the Green’s function \( G_{E+i\Gamma_W}(\vec{r}=0,\vec{r}'=0) \) of the Schrödinger equation for the interacting \( W^+W^- \) system. The result for the total \( W^+W^- \) cross section is

\[
\sigma(s) = C \left[ \frac{2p^2}{\sqrt{s}} + \frac{\alpha}{\beta^2} \arctan \left( \frac{p}{\beta} \right) \sqrt{E^2 + \Gamma_W^2} + O(\alpha^3) \right],
\]

where \( \zeta(2) = \pi^2/6 \). The factor \( C \) is related to the expansion of the off-shell Born cross section \( \sigma_0(s,s_1,s_2) \) \([17]\) in the non-relativistic region \( \beta = 2p/\sqrt{s} \ll 1 \) (where \( p^2 \approx (\sqrt{s} - \sqrt{s_1} - \sqrt{s_2})M_W \)),

\[
\sigma_0 = C\beta + O(\beta^3),
\]

which, when folded with the usual Breit-Wigner factors to give the total Born cross section,

\[
\sigma_B(s) = \int_0^s ds_1 \int_0^{(\sqrt{s}-\sqrt{s_1})^2} ds_2 \rho(s_1)\rho(s_2) \sigma_0(s,s_1,s_2).
\]

leads to the first term on the right-hand side of (11),

\[
\sigma_B(s) \approx C \frac{2p^2}{\sqrt{s}}.
\]

An explicit numerical calculation (see below) gives \( C \approx 54 \text{ pb} \) and \( a \approx 80 \text{ pb} \).

Finally, it is straightforward to show that for \( E \gg \Gamma_W \) the right-hand side of Eq. (11) reproduces the expansion of the Coulomb factor for stable particles \([13]\),

\[
|\psi(0)|^2 = \frac{Z}{1 - \exp(-Z)} = 1 + \frac{Z^2}{2} + \frac{Z^4}{12} + \ldots,
\]

Note that in practice the role of these \( \Gamma_W/M_W \) effects — which are related to the large ‘intrinsic’ momenta of the \( W \) bosons — is likely to be suppressed by experimental cuts on the \( \sqrt{s_i} \), which are necessary in order to separate the \( W^+W^- \) events from the non-\( W^+W^- \) background.
with $Z = \alpha \pi / \beta$. The modification caused by the $W$ instability is important only for $E \lesssim \Gamma_W$. Far below threshold, $E < 0$ with $|E| \gg \Gamma_W$, Eq. (11) becomes

\begin{equation}
\sigma(s) \simeq \frac{C \Gamma_W}{2 \sqrt{|E| M_W}} \left[ 1 + \alpha \sqrt{M_W/|E|} + \frac{\alpha^2}{2} \zeta(2) \frac{M_W}{|E|} + \ldots \right].
\end{equation}

### 4 Numerical results

In this section we present the numerical results of a calculation of the $O(\alpha)$ and $O(\alpha^2)$ Coulomb corrections to the total $e^+e^- \to W^+W^-$ cross section near threshold, i.e.

\begin{equation}
\sigma(s) = \int_0^s ds_1 \int_0^{(\sqrt{s} - \sqrt{s_1})^2} ds_2 \rho(s_1)\rho(s_2) \sigma_0(s, s_1, s_2) \left[ 1 + \delta_C(s, s_1, s_2) \right],
\end{equation}

with $1 + \delta_C = |f|^2$ where $f$ is given by Eq. (7). Our calculation is an extension of that presented in Ref. [1], which incorporated the $O(\alpha)$ Coulomb correction, and all parameters are identical except that here we use an updated value for the $W$ mass, $M_W = 80.41$ GeV/$c^2$. 

Fig. 1 shows the $O(\alpha)$ and $O(\alpha^2)$ Coulomb corrections normalized to the Born cross section, as a function of the $e^+e^-$ collider energy $\sqrt{s}$. The position of the nominal threshold $\sqrt{s} = 2M_W$ is marked. We see that the energy dependence of the first and second order corrections is similar — both attain a maximum just below threshold of approximately $+6\%$ and $+0.2\%$ respectively. In Ref. [1] it was shown that at threshold a change in the cross section of $\Delta \sigma$ was equivalent to a shift in the $W$ mass of

\begin{equation}
\Delta M_W = 17 \text{ MeV} \cdot \left[ \frac{\Delta \sigma}{\sigma} \times 100\% \right].
\end{equation}

Evidently, the inclusion of the second-order Coulomb correction is equivalent to $\Delta M_W = 3.4$ MeV, which is negligible in comparison to the anticipated precision on $M_W$ using the threshold cross section measurement method.

How do these results compare with the analytic approximations derived in the previous section? The expansion of Eq. (11) at $E = 0$ is

\begin{equation}
\sigma(s = 4M_W^2) = C \sqrt{\frac{\Gamma_W}{2M_W}} \left[ 1 + \frac{X}{2} + \frac{X^2}{6} + \ldots \right],
\end{equation}

where

\begin{equation}
X = \alpha \pi \sqrt{\frac{M_W}{2\Gamma_W}} \approx 0.1005.
\end{equation}

This gives first- and second-order corrections of $+5\%$ and $+0.17\%$ respectively, in good agreement with the exact result. In fact the difference between these values and the
‘exact values’ is due simply to the negative $O(\Gamma_W/M_W)$ corrections to the Born cross section (16) which are not taken into account in (21). Note also from Fig. 1 that the first and second order corrections decrease rapidly below threshold, consistent with the analytic result for $E \ll -\Gamma_W$ of Eq. (18).

Returning to the expression for $f$ in Eq. (7), we see that the second-order correction contains contributions from both the real and imaginary parts of the first-order contribution to $f$, and from the real part of the second-order contribution. Schematically, from (1) and (7) we have

$$
\delta_C^{(1)} \leftrightarrow 2f_1^R
\delta_C^{(2)} \leftrightarrow 2f_2^R + (f_1^R)^2 + (f_1^I)^2.
$$

The following table gives the corresponding breakdown of the second-order correction (in %) at several collider energies.

| $\sqrt{s}$ (GeV) | $2f_2^R$ | $(f_1^R)^2$ | $(f_1^I)^2$ |
|-----------------|--------|-------------|-------------|
| 155             | 0.056  | 0.039       | 0.001       |
| 160             | 0.109  | 0.092       | 0.016       |
| 165             | -0.056 | 0.055       | 0.083       |
| 170             | -0.072 | 0.031       | 0.080       |

We see that the real parts are numerically dominant at threshold, whereas for $E \gg \Gamma_W$ there is a strong cancellation between $(f_1^I)^2$ and $2f_2^R$.

In Ref. [4] it was suggested that a reasonable approximation to the higher-order Coulomb corrections could be obtained by using the stable particle expansion (17) with the ‘exact’ first-order off-shell correction as the expansion parameter. This corresponds to

$$
|f|^2 \approx 1 + \frac{X}{2} + \frac{X^2}{12} + \ldots
$$

with (see Eq. (3))

$$
X = \frac{\alpha \sqrt{s}}{2p} \left[ \pi - 2 \arctan \left( \frac{|\kappa|^2 - p^2}{2p \text{Re}(\kappa)} \right) \right].
$$

Fig. 2 shows the exact and approximate second-order corrections as a function of $\sqrt{s}$. Formally, the two results must become equal far above threshold, where both must coincide with the stable-W result. This behaviour is evident for $\sqrt{s} \gtrsim 170$ GeV. However, the approximation clearly breaks down around and below threshold as expected (for example, by comparing Eqs. (24) and (25)).
5 Conclusions

In this note we have presented analytic and numerical results for the first- and second-order Coulomb corrections to the $e^+e^- \rightarrow W^+W^-$ cross section in the threshold region. In fact the corrections are known to all orders, see Eq. (2), although it is clear from Fig. 1 that in practice the first- and second-corrections are sufficient for phenomenology at the LEP2 collider. In terms of determining $M_W$ from a precision measurement of the threshold cross section, the inclusion of the first- and second-order Coulomb corrections is equivalent to a shift in $M_W$ of 100 MeV and 3.4 MeV respectively.

We have also derived analytic expressions for the corrections far above, far below, and close to threshold. For the former, the well-known stable-W results are reproduced. For the latter, we have shown that the effective expansion parameter right at threshold is $\alpha \pi \sqrt{M_W/\Gamma_W} \approx \frac{1}{17}$ rather than $\alpha \approx \frac{1}{137}$, and this explains the overall size of the first- and second-order corrections in this region.

Finally, we have studied the validity of the approximation [4] in which the stable-W all-orders result is combined with the unstable-W first-order result. As expected the approximation works well away from threshold, but is seen to break down for $\sqrt{s} \lesssim 165$ GeV.

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Figure Captions

[1] The $O(\alpha)$ and $O(\alpha^2)$ Coulomb corrections of Eq. (19) normalized to the Born cross section, as a function of the $e^+e^-$ collider energy $\sqrt{s}$. The position of the nominal threshold $\sqrt{s} = 2M_W$ is marked.

[2] The $O(\alpha^2)$ Coulomb correction of Fig. 1 (solid line) compared to the approximate form of Eqs. (24,25) [4] (dashed line).
Fig. 1

\[
\delta_C (\%) \quad \sqrt{s} \quad \text{(GeV)}
\]

- \(O(\alpha)\)
- \(O(\alpha^2) \times 10\)
Fig. 2

\[ O(\alpha^2) \approx O(\alpha^2) \]

\[ \delta_C \text{ (\%)} \]

\[ \sqrt{s} \text{ (GeV)} \]