On Liouville type theorems for the steady Navier-Stokes equations in $\mathbb{R}^3$

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Abstract

In this paper we prove three different Liouville type theorems for the steady Navier-Stokes equations in $\mathbb{R}^3$. In the first theorem we improve logarithmically the well-known $L^{32}_2(\mathbb{R}^3)$ result. In the second theorem we present a sufficient condition for the triviality of the solution ($v = 0$) in terms of the head pressure, $Q = \frac{1}{2}|v|^2 + p$. The imposed integrability condition here has the same scaling property as the Dirichlet integral. In the last theorem we present Fubini type condition, which guarantee $v = 0$.

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1 Introduction

We consider the following stationary Navier-Stokes equations (NS) on $\mathbb{R}^3$.

\[
\begin{aligned}
(NS) \quad \left\{ 
\begin{array}{l}
(v \cdot \nabla)v = -\nabla p + \Delta v, \\
\text{div } v = 0,
\end{array}
\right.
\end{aligned}
\]

where $v(x) = (v_1(x), v_2(x), v_3(x))$ and $p = p(x)$ for all $x \in \mathbb{R}^3$. The system is equipped with the boundary condition:

\[
|v(x)| \to 0 \quad \text{as} \quad |x| \to +\infty. \quad (1.1)
\]
In addition to (1.1) one usually also assume following finiteness of the Dirichlet integral.
\[ \int_{\mathbb{R}^3} |\nabla v|^2 dx < +\infty. \] (1.2)

A long standing open question is if any weak solution of (NS) satisfying the conditions (1.1) and (1.2) is trivial (namely, \( v = 0 \) on \( \mathbb{R}^3 \)). We refer the book by Galdi [2] for the details on the motivations and historical backgrounds on the problem and the related results. As a partial progress to the problem we mention that the condition \( v \in L^2_9(\mathbb{R}^3) \) implies that \( v = 0 \) (see Theorem X.9.5, pp.729 [2]). As shown in [1], a different condition \( \Delta v \in L^{6,5}(\mathbb{R}^3) \) also imply \( v = 0 \). Another interesting progress, which shows that a solution \( v \in BMO^{-1}(\mathbb{R}^3) \) to (NS), satisfying (1.2) is trivial is obtained very recently by Seregin in [6]. For the case of plane flows the problem is solved by Gilbarg and Weinberger in [3], while the special case of the axially symmetric 3D flows without swirl is studied recently by Korobkov, M. Pileckas and R. Russo in [5] (see also [4]). In this paper we present three theorems, which present sufficient conditions to guarantee the triviality of the solution to (NS).

In the first theorem below we improve the above mentioned \( L^2_9 \)-result logarithmically.

**Theorem 1.1.** Let \( v \in L^1_{\text{loc}}(\mathbb{R}^3) \) be a distributional solution to (NS) such that
\[ \int_{\mathbb{R}^3} |v|^{\frac{9}{2}} \left( \log \left( 2 + \frac{1}{|v|} \right) \right)^{-1} dx < +\infty. \] (1.3)
Then \( v \equiv 0 \).

For discussion of the next theorem we introduce the head pressure,
\[ Q = \frac{1}{2} |v|^2 + p, \]
which has an important role in the study of the stationary Euler equations via the Bernoulli theorem. It is known (see e.g. Theorem X.5.1, pp. 688 [2]) that under the condition (1.1)-(1.2) we have \( p(x) \to p_0 \) as \( |x| \to +\infty \), where \( p_0 \) is a constant, which implies that
\[ Q(x) \to 0 \quad \text{as} \quad |x| \to +\infty \] (1.4)
after re-defining \( Q - p_0 \) as the new head pressure. Our second theorem below assumes integrability of \( Q \) to conclude the triviality of \( v \).

**Theorem 1.2.** Let \( (v, p) \) be a smooth solution to (NS) satisfying (1.4). Let us set \( M := \sup_{x \in \mathbb{R}^3} |Q(x)| \). Then, we have the following inequality.
\[ \int_{\mathbb{R}^3} \frac{|
abla Q|^2}{|Q|} \left( \log \frac{eM}{|Q|} \right)^{-\alpha - 1} dx \leq \frac{1}{\alpha} \int_{\mathbb{R}^3} |\omega|^2 dx \quad \forall \alpha > 0. \] (1.5)
Moreover, suppose there holds the boundary conditions (1.1), (1.4) and
\[ \int_{\mathbb{R}^3} \frac{|
abla Q|^2}{|Q|} \left( \log \frac{eM}{|Q|} \right)^{-\alpha - 1} dx = o \left( \frac{1}{\alpha} \right) \quad \text{as} \quad \alpha \to 0, \] (1.6)
then \( v = 0 \) on \( \mathbb{R}^3 \).
Remark 1.1. Since $|\nabla \sqrt{|Q|}|^2 = \frac{1}{4} \frac{\nabla Q}{|Q|}$, and $\sqrt{|Q|}$ has the same scaling as the velocity the integral $\int_{\mathbb{R}^3} \frac{\nabla Q}{|Q|}^2 \, dx$ has the same scaling property as the Dirichlet integral in (1.2).

Our third result concerns on the Fubini type condition for suitable function $\Phi(x, y)$ for $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ to guarantee the triviality of the solution to (NS).

**Theorem 1.3.** Let $v$ be a smooth solution to (NS) on $\mathbb{R}^3$ satisfying (1.1) and set $\omega = \text{curl } v$. Suppose there exists $q \in \left(\frac{3}{2}, 3\right)$ such that $x \in L^q(\mathbb{R}^3)$. We set

$$
\Phi(x, y) := \frac{1}{4\pi} \frac{\omega(x) \cdot (x - y) \times (v(y) \times \omega(y))}{|x - y|^3}
$$

for all $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $x \neq y$. Then, it holds

$$
\int_{\mathbb{R}^3} |\Phi(x, y)| \, dy + \int_{\mathbb{R}^3} |\Phi(x, y)| \, dx < \infty \quad \forall (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3.
$$

(1.8)

Furthermore, if there holds

$$
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(x, y) \, dxdy = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(x, y) \, dydx,
$$

(1.9)

then, $v = 0$ on $\mathbb{R}^3$.

**Remark 1.2.** One can show that if $\omega \in L^{\frac{9}{5}}(\mathbb{R}^3)$ is satisfied together with (1.1), then (1.9) holds, and therefore $v$ is trivial. Although this result follows immediately by applying the $L^{\frac{9}{5}}$-result together with Sobolev inequality and the Calderon-Zygmund inequality, $\|v\|_{L^{\frac{9}{5}}} \leq C\|\nabla v\|_{L^{\frac{9}{5}}} \leq C\|\omega\|_{L^{\frac{9}{5}}}$. The above theorem provides us with different proof of this. In order to check this result we first recall the estimate of the Riesz potential on $\mathbb{R}^3$ (8),

$$
\|I_\alpha(f)\|_{L^q} \leq C\|f\|_{L^p}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{3}, \quad 1 \leq p < q < +\infty,
$$

(1.10)

where

$$
I_\alpha(f) := C \int_{\mathbb{R}^3} \frac{f(y)}{|x - y|^{3-\alpha}} \, dy, \quad 0 < \alpha < 3
$$

for a positive constant $C = C(\alpha)$. Applying (1.10) with $\alpha = 1$, we obtain by the
Hölder inequality,
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\Phi(x, y)| dy dx \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\omega(x)||\omega(y)||v(y)|}{|x-y|^2} dy dx
\]
\[
\leq \left( \int_{\mathbb{R}^3} |\omega(x)|^\frac{2}{5} dx \right)^\frac{5}{4} \left\{ \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|\omega(y)||v(y)|}{|x-y|^2} dy \right)^{\frac{2}{5}} dx \right\}^{\frac{5}{4}}
\]
\[
\leq C\|\omega\|_{L^\frac{2}{5}} \left( \int_{\mathbb{R}^3} |\omega|^{\frac{9}{5}} |v|^{\frac{9}{5}} dx \right)^\frac{5}{9}
\]
\[
\leq C\|\omega\|_{L^\frac{2}{5}} ^2 \|\nabla v\|_{L^\frac{2}{5}} \leq C\|\omega\|_{L^\frac{2}{5}} ^3 < +\infty,
\]
Thus, by the Fubini-Tonelli theorem, (1.9) holds.

2 Proof of the main theorems

Below we use the notation \( A \lesssim B \) if there exists an absolute constant \( \kappa \) such that \( A \leq \kappa B \).

2.1 Proof of Theorem 1.1

**Definition 2.1.** Let \( \phi \in C^2(\mathbb{R}) \) be an N-function, i.e. \( \phi \) is an even function such that
\[
\lim_{\tau \to 0} \phi'(\tau) = 0, \quad \text{and} \quad \lim_{\tau \to \infty} \phi'(\tau) = +\infty.
\]
We say \( \phi \) belongs to the class \( N(p_0, p_1) \) \( (1 < p_0 \leq p_1 < +\infty) \) if for all \( \tau \geq 0 \)
\[
(p_0 - 1)\phi'(\tau) \leq \tau\phi''(\tau) \leq (p_1 - 1)\phi'(\tau).
\]

**Remark 2.2.** It is well known that \( \phi \in N(p_0, p_1) \) implies for all \( \tau \geq 0 \)
\[
\phi(\tau) \leq \tau \phi'(\tau) \leq p_1 \phi(\tau).
\]

We now define for \( q > 1 \)
\[
\phi_q(\tau) = \int_0^\tau \frac{\xi^{q-1}}{\log \frac{1+2\xi}{\xi}} d\xi, \quad \tau \geq 0.
\]

We easily calculate,
\[
\phi_q'(\tau) = \frac{\tau^{q-1}}{\log \frac{1+2\tau}{\tau}},
\]
\[
\phi_q''(\tau) = (q-1) \frac{\tau^{q-2}}{\log \frac{1+2\tau}{\tau}} + \frac{\tau^{q-2}}{\log^2 \frac{1+2\tau}{\tau}} \frac{1}{1+2\tau}
\]
\[
= \frac{\phi_q'(\tau)}{\tau} \left( (q-1) + \frac{1}{(1+2\tau) \log \frac{1+2\tau}{\tau}} \right).
\]
Observing that \( \frac{1}{(1+2\tau)\log \frac{1+2\tau}{\tau}} \leq \frac{1}{\log 2} \), we get for all \( \tau \geq 0 \)
\[
(q - 1)\phi'_q(\tau) \leq \tau \phi''_q(\tau) \leq (q + (\log 2)^{-1} - 1)\phi'_q(\tau).
\]
This shows that \( \phi \in N(q,q + (\log 2)^{-1}) \), and according to (2.2) it holds
\[
\phi_q(\tau) \sim \tau \phi'_q(\tau) = \frac{\tau^q}{\log \frac{2+\tau}{\tau}}.
\]
Thus, (2.4) is equivalent to
\[
\int_{\mathbb{R}^3} \phi^2_q(|v|) dx < +\infty.
\]

**Lemma 2.1.** For any constant \( a > \frac{1}{2} \) we have
\[
\log a \frac{1+2\tau}{\tau} \sim \log \frac{1+2\tau}{\tau}.
\]

**Proof** In case \( a \geq 1 \) we immediately get \( \log a \frac{1+2\tau}{\tau} \geq \log \frac{1+2\tau}{\tau} \). For the reverse we get for all \( 0 < \tau \leq 1 \),
\[
\log a \frac{1+2\tau}{\tau} \leq \left( \frac{\log a}{\log 3} + 1 \right) \log \frac{1+2\tau}{\tau},
\]
and for all \( \tau > 1 \)
\[
\log a \frac{1+2\tau}{\tau} \leq \log a + \log 3 \leq \frac{\log a + \log 3}{\log 2} \frac{1+2\tau}{\tau},
\]
which proves the claim.

In case \( a < 1 \) we see that \( \log a \frac{1+2\tau}{\tau} \leq \log \frac{1+2\tau}{\tau} \). On the other hand, we may choose \( \tau_0 > 0 \), such that
\[
\log \frac{1+2\tau_0}{\tau_0} = \frac{1}{2} \left( 1 + \frac{\log 2}{\log a^{-1}} \right) \log a^{-1}.
\]
Then for \( \tau \leq \tau_0 \) we obtain
\[
\log a \frac{1+2\tau}{\tau} = - \log a^{-1} + \log \frac{1+2\tau}{\tau} = -2 \left( 1 + \frac{\log 2}{\log a^{-1}} \right)^{-1} \log 2 + \frac{\tau_0 + 1+2\tau}{\tau}.
\]
\[
\geq \left[ 1 - 2 \left( 1 + \frac{\log 2}{\log a^{-1}} \right)^{-1} \right] \log \frac{1+2\tau}{\tau}
\]
\[
= \frac{\log 2 - \log a^{-1}}{\log 2 + \log a^{-1}} \log \frac{1+2\tau}{\tau}.
\]
For \( \tau > \tau_0 \) we easily see that
\[
\log a \frac{1+2\tau}{\tau} \geq \log 2 - \log a^{-1} = \frac{\log 2 - \log a^{-1}}{\log \frac{1+2\tau_0}{\tau_0}} \log \frac{1+2\tau_0}{\tau_0} \]
\[
= \frac{\log 2 - \log a^{-1}}{\log 2 + \log a^{-1}} \log \frac{1+2\tau}{\tau}.
\]
Whence, the claim.
Lemma 2.2. For all $k \in \mathbb{N}$

$$\frac{1 + 2\tau}{\tau} \sim \frac{1 + 2\tau^k}{\tau^k},$$

where the hidden constants depend on $q$ and $k$ only.

Proof In fact having $1 + 2\tau^k \leq (1 + 2\tau)^k \leq 2^{k-1}(1 + 2\tau^k) \leq 2^{2k-2}(1 + 2\tau^k)$ along with Lemma 2.1 we obtain

$$\log \frac{1 + 2\tau^k}{\tau^k} \leq k \log \frac{1 + 2\tau}{\tau} \leq \log 2^{2k-2} \frac{1 + 2\tau^k}{\tau^k} \lesssim \log \frac{1 + 2\tau}{\tau^k}.$$  

This proves the claim. $lacksquare$

Lemma 2.3. Let $f \in L^1(\mathbb{R}^3)$. Then for every $\varepsilon > 0$, there exists $R > \varepsilon^{-1}$, such that

$$\int_{B_R \setminus B_{R/2}} |f| dx \leq \frac{\varepsilon}{\log R}.$$ (2.8)

Proof Assume the assertion of the lemma is not true. Then there exists $\varepsilon > 0$ such that for all $R \geq \varepsilon^{-1}$ (2.8) does not hold. This implies for all $k \geq N$ with $2^k \geq \varepsilon^{-1}$

$$\int_{B_{2^k} \setminus B_{2^k-1}} |f| dx \geq \frac{\varepsilon}{k \log 2}.$$  

However the sum of right-hand side from $k = N$ to $\infty$ is infinite which clearly contradicts to $f \in L^1(\mathbb{R}^3)$. Thus, the assumption is not true and therefore the assertion of the lemma holds. $lacksquare$

In view of (1.3) we easily see that $v \in L^9_{\text{loc}}(\mathbb{R}^3)$. By using a standard mollifying argument we verify that $v \in W^{1,2}_{\text{loc}}(\mathbb{R}^3)$, and therefore $v \in C^\infty(\mathbb{R}^3)$ and $p \in C^\infty(\mathbb{R}^3)$. In particular, we have for all $\zeta \in C^\infty_c(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} |\nabla v|^2 \zeta dx = \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 \Delta \zeta dx + \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 v \cdot \nabla \zeta dx + \int_{\mathbb{R}^3} pv \cdot \nabla \zeta dx.$$ (2.9)

On the basis of (2.9) we have the following Caccioppoli-type inequality.

$$\int_{B_R} |\nabla v|^2 dx \lesssim R^{-1} \left\{ 1 + \int_{B_{2R}} |v|^3 dx \right\}.$$ (2.10)

Proof of (2.10): Let $R \leq r < \rho \leq 2R$. Into (2.9) we insert a off function $\zeta \in C^\infty_c(B_\rho)$ such that $\zeta \equiv 1$ on $B_r$, $0 \leq \zeta \leq 1$ in $\mathbb{R}^3$ and $|\nabla \zeta|^2 + |\nabla^2 \zeta| \lesssim (\rho - r)^{-2}$. This together with Hölder’s inequality and Young’s inequality immediately gives

$$\int_{B_r} |\nabla v|^2 dx \lesssim (\rho - r)^{-2} \int_{B_{2\rho}} |v|^2 dx + (\rho - r)^{-1} \int_{B_\rho} |v|^3 dx + (\rho - r)^{-1} \int_{B_{2\rho}} |p - p_{B_\rho}| |v| dx$$

$$\lesssim (\rho - r)^{-1} \left\{ 1 + \int_{B_{2R}} |v|^3 dx \right\} + (\rho - r)^{-1} \int_{B_\rho} |p - p_{B_\rho}| |v| dx.$$ (2.11)
Using Hölder’s inequality, Young’s inequality and consulting Theorem III.3.1, Theorem III.5.2 of [2], we estimate the last integral involving the pressure as follows

\[
\begin{align*}
\int_{B_{\rho}} (\rho - r)^{-1} |p - p_{B_{\rho}}| |v| dx \\
\lesssim (\rho - r)^{-1} \left( \int_{B_{\rho}} |\nabla v|^2 dx + \int_{B_{\rho}} |v|^3 dx \right)^{\frac{2}{3}} \left( \int_{B_{\rho}} |v|^3 dx \right)^{\frac{1}{3}} \\
\lesssim \rho^{1/2}(\rho - r)^{-1} \left( \int_{B_{\rho}} |\nabla v|^2 dx \right)^{\frac{2}{3}} \left( \int_{B_{2\rho}} |v|^3 dx \right)^{\frac{1}{3}} + (\rho - r)^{-1} \int_{B_{2\rho}} |v|^3 dx \\
\lesssim \delta \int_{B_{\rho}} |\nabla v|^2 dx + \rho(\rho - r)^{-2} \left( \int_{B_{2\rho}} |v|^3 dx \right)^{\frac{2}{3}} + (\rho - r)^{-1} \int_{B_{2\rho}} |v|^3 dx \\
\lesssim \delta \int_{B_{\rho}} |\nabla v|^2 dx + (\rho - r)^{-1} \left\{ 1 + \int_{B_{2\rho}} |v|^3 dx \right\}.
\end{align*}
\]

Inserting this inequality into the right-hand side of (2.11), we arrive at

\[
\int_{B_{\rho}} |\nabla v|^2 dx \lesssim (\rho - r)^{-1} \left\{ 1 + \int_{B_{2\rho}} |v|^3 dx \right\} + \delta \int_{B_{\rho}} |\nabla v|^2 dx. \tag{2.12}
\]

In (2.12) taking \( \delta > 0 \) sufficiently small, and applying a well known iteration argument, we obtain (2.10). This completes the proof of (2.10).

**Proof of Theorem 1.1** Let \( \varepsilon > 0 \) be arbitrarily chosen, but fixed. Thanks to Lemma 2.3 in view of (2.8) we may choose \( R \geq \varepsilon^{-1} \) such that

\[
\int_{B_{R \setminus B_{R/2}}} \phi_2(|v|) dx \leq \frac{\varepsilon}{\log R}. \tag{2.13}
\]

Let \( \zeta \in C^\infty_c(B_R) \) be a cut off function such that \( 0 \leq \zeta \leq 1 \) in \( B_R \), \( \zeta \equiv 1 \) on \( B_{R/2} \), and \( |\nabla \zeta| \lesssim R^{-1} \), \( |\nabla^2 \zeta| \lesssim R^{-2} \). Then from (2.9) we deduce

\[
\int_{B_{R/2}} |\nabla v|^2 dx \lesssim R^{-2} \int_{B_{R \setminus B_{R/2}}} |v|^2 dx + R^{-1} \int_{B_{R \setminus B_{R/2}}} |v|^3 dx + R^{-1} \int_{B_{R \setminus B_{R/2}}} |p - p_{B_{R \setminus B_{R/2}}}| |v| dx. \tag{2.14}
\]

Using Hölder’s inequality, Young’s inequality and consulting [2], we estimate the last
integral involving the pressure as follows

\[ R^{-1} \int_{B_R \setminus B_{R/2}} |p - p_{B_R \setminus B_{R/2}}||v|dx \]

\[ \lesssim R^{-1} \left( \int_{B_R \setminus B_{R/2}} |\nabla v|^\frac{2}{3}dx + \int_{B_R \setminus B_{R/2}} |v|^3dx \right)^{\frac{2}{3}} \left( \int_{B_R \setminus B_{R/2}} |v|^3dx \right)^{\frac{1}{3}} \]

\[ \lesssim R^{-\frac{1}{2}} \left( \int_{B_R} |\nabla v|^2dx \right)^{\frac{1}{2}} \left( \int_{B_R \setminus B_{R/2}} |v|^3dx \right)^{\frac{1}{3}} + R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3dx \]

\[ \lesssim R^{-\frac{1}{3}} + R^{-\frac{1}{6}} \int_{B_R} |\nabla v|^2dx + R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3dx. \]

Once more using Hölder’s inequality along with Young’s inequality we easily find

\[ R^{-2} \int_{B_R \setminus B_{R/2}} |v|^2dx \leq R^{-1} + R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3dx. \]

Inserting the last two inequalities into the right-hand side of (2.14), we arrive at

\[ \int_{B_{R/2}} |\nabla v|^2dx \lesssim R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3dx + R^{-\frac{1}{8}} + R^{-\frac{7}{6}} \int_{B_{R}} |\nabla v|^2dx. \] (2.15)

We now estimate the last integral on the right-hand side of (2.15) by means of (2.10). This implies

\[ \int_{B_{R/2}} |\nabla v|^2dx \lesssim R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3dx + R^{-\frac{1}{8}} + R^{-\frac{7}{6}} \int_{B_{2R}} |v|^3dx. \] (2.16)

By our assumption (1.3) we know that \( v \in L^q(\mathbb{R}^3) \) for all \( q > \frac{9}{2} \). This follows from standard regularity theory of the steady Navier-Stokes equations (e.g. see [7]). For \( \frac{9}{2} < q < \frac{54}{11} \) we find with the help of Jensen’s inequality

\[ R^{-\frac{1}{8}} + R^{-\frac{7}{6}} \int_{B_{2R}} |v|^3dx \lesssim R^{-\frac{1}{8}} + R^{-\frac{3q-9}{4q-7}} \|v\|_q \rightarrow 0 \quad \text{as} \quad R \rightarrow +\infty. \] (2.17)
Noting that $\phi_{3/2}$ is convex, applying Jensen’s inequality, we get

$$
\phi_{3/2} \left( \frac{8}{7 R \operatorname{meas}(B)} \int_{B_R \setminus B_{R/2}} |v|^3 \, dx \right) = \phi_{3/2} \left( R^2 \int_{B_R \setminus B_{R/2}} |v|^3 \, dx \right)
$$

$$
\leq \int_{B_R \setminus B_{R/2}} \phi_{3/2} \left( R^2 |v|^3 \right) \, dx
$$

$$
\lesssim \int_{B_R \setminus B_{R/2}} \frac{|v|^\frac{3}{2}}{ \log \frac{1+2R^2}{R^2 |v|^3} } \, dx.
$$

We split the integral on the right-hand side into two parts by setting

$$
A_1 = \{ x \in B_R \setminus B_{R/2} \mid |v|^3 \leq \varepsilon R^{-2} \},
$$

$$
A_2 = \{ x \in B_R \setminus B_{R/2} \mid |v|^3 > \varepsilon R^{-2} \}.
$$

Firstly, we easily see that

$$
\int_{A_1} \frac{|v|^{\frac{3}{2}}}{ \log \frac{1+2R^2}{R^2 |v|^3} } \, dx \lesssim \varepsilon^{\frac{3}{2}}.
$$

Secondly, with help of Lemma 2.2 and recalling that $R \geq \frac{1}{\varepsilon}$ we have in $A_2$

$$
4 \log R \geq \log R^2 + \log \frac{1}{\varepsilon} + \log 2 = \log 2 \frac{R^2}{\varepsilon} \geq \log \frac{1 + 2\varepsilon R^{-2}}{\varepsilon R^{-2}}
$$

$$
\geq \log \frac{1 + 2 |v|^3}{|v|^3} \gtrsim \log \frac{1 + 2 |v|}{|v|}.
$$

With this estimate along with (2.13) we get

$$
\int_{A_2} \frac{|v|^{\frac{3}{2}}}{ \log \frac{1+2R^2}{R^2 |v|^3} } \, dx \lesssim \frac{1}{\log 2} \int_{A_2} |v|^{\frac{3}{2}} \, dx \lesssim \frac{\log R}{\log 2} \int_{B_R \setminus B_{R/2}} \frac{|v|^{\frac{3}{2}}}{1+2|v|} \, dx
$$

$$
\lesssim \log R \int_{B_R \setminus B_{R/2}} \phi_{3/2} (|v|) \, dx \lesssim \varepsilon.
$$

Accordingly,

$$
\phi_{3/2} \left( \frac{8}{7 R \operatorname{meas}(B)} \int_{B_R \setminus B_{R/2}} |v|^3 \, dx \right) \lesssim \varepsilon.
$$

Thus, in view (2.16) together with the estimates we have just obtained we are able to chose a sequence $R_k \to +\infty$ as $k \to +\infty$, such that

$$
\int_{B_{R_{k/2}}} |\nabla v|^2 \, dx \to 0 \quad \text{as} \quad k \to +\infty,
$$

which yields $\nabla v = 0$ and therefore $v \equiv \text{const} = 0$. 

\[\square\]
2.2 Proof of Theorem 1.2

Proof of Theorem 1.2 Let us denote the vorticity \( \omega = \text{curl} \, v \). Then, it is well-known that from (NS) that the following equation holds true.

\[
\Delta Q - v \cdot \nabla Q = |\omega|^2. \tag{2.18}
\]

Under the condition (1.4) we have \( Q(x) \leq 0 \) for all \( x \in \mathbb{R}^3 \) by the maximum principle applied to (2.18). Moreover, by the maximum principle again, either \( Q(x) \equiv 0 \) on \( \mathbb{R}^3 \), or \( Q(x) < 0 \) for all \( x \in \mathbb{R}^3 \). Indeed, any point \( x_0 \in \mathbb{R}^3 \) such that \( Q(x_0) = 0 \) is a point of local maximum, which is not allowed unless \( Q \equiv 0 \) by the maximum principle. Let \( Q(x) \not\equiv 0 \) on \( \mathbb{R}^3 \), then without the loss of generality we may assume \( |Q(x)| > 0 \) for all \( x \in \mathbb{R}^3 \).

Let \( \lambda \in [0, M] \) we set \( D_\lambda = \{ x \in \mathbb{R}^3 \mid |Q(x)| > \lambda \} \). Then, we compute

\[
\int_{D_\lambda} f(Q(x))v \cdot \nabla Q \, dx = \int_{D_\lambda} v \cdot \nabla \left( \int_0^{Q(x)} f(q) \, dq \right) \, dx
= \int_{D_\lambda} \text{div} \left( v \int_0^{Q(x)} f(q) \, dq \right) \, dx = \int_{\partial D_\lambda} \left( \int_0^{Q(x)} f(q) \, dq \right) v \cdot \nu \, dS
= \int_0^\lambda f(q) \, dq \int_{\partial D_\lambda} v \cdot \nu \, dS = \int_0^\lambda f(q) \, dq \int_{D_\lambda} \text{div} \, v \, dx = 0. \tag{2.19}
\]

where \( \nu = \nabla Q / |\nabla Q| \) is the outward unit normal vector on \( \partial D_\lambda \). For \( \lambda \in (0, M) \) we Integrate (2.18) over \( D_\lambda \). Then, using the fact (2.19), we have

\[
\int_{D_\lambda} |\omega|^2 \, dx = \int_{D_\lambda} \Delta Q \, dx = \int_{\partial D_\lambda} \frac{\partial Q}{\partial \nu} \, dS
= \int_{\partial D_\lambda} |\nabla Q| \, dS. \tag{2.20}
\]

Using the co-area formula, we obtain

\[
\int_{D_\lambda} \frac{|\nabla Q|^2}{|Q|} \left( \log \frac{eM}{|Q|} \right)^{-\alpha-1} \, dx = \int_{\lambda}^{M} \int_{\partial D_q} \frac{|\nabla Q|}{|Q|} \left( \log \frac{eM}{|Q|} \right)^{-\alpha-1} \, dSdq
= \int_{\lambda}^{M} \frac{1}{q} \left( \log \frac{eM}{q} \right)^{-\alpha} \int_{\partial D_q} |\nabla Q| \, dSdq
\leq \int_{\lambda}^{M} \frac{1}{q} \left( \log \frac{eM}{q} \right)^{-\alpha} \, dq \int_{\partial D_\lambda} |\nabla Q| \, dS
= \frac{1}{\alpha} \left\{ 1 - \left( \log \frac{eM}{\lambda} \right)^{-\alpha} \right\} \int_{D_\lambda} |\omega|^2 \, dx
\leq \frac{1}{\alpha} \int_{\mathbb{R}^3} |\omega|^2 \, dx,
\]

where we used (2.20) in the fourth line. Passing \( \lambda \to 0 \), and applying the monotone convergence theorem, we obtain (1.5). Next, we assume (1.6) holds. We consider a
standard cut-off function $\sigma \in C_0^\infty([0, \infty))$ such that $\sigma(s) = 1$ if $s < 1$, and $\sigma(s) = 0$ if $s > 2$, and $0 \leq \sigma(s) \leq 1$ for $1 < s < 2$. For each $\alpha \in (0, 1)$ we define $\sigma_\alpha(x) := \sigma_\alpha(Q(x)) \in C_0^\infty(\mathbb{R}^3)$ by

$$
\sigma_\alpha(x) = 1 - \sigma \left\{ 3 \left( \log \frac{eM}{|Q(x)|} \right)^{-\alpha} \right\}.
$$

We note that

$$
\begin{cases}
\sigma_\alpha(x) = 1, & \text{if } |Q(x)| \geq Me^{1-(\frac{2}{3})^\frac{1}{\alpha}}, \\
0 < \sigma_\alpha(x) < 1, & \text{if } Me^{1-3^\frac{1}{\alpha}} < |Q(x)| < Me^{1-(\frac{2}{3})^\frac{1}{\alpha}}, \\
\sigma_\alpha(x) = 0, & \text{if } |Q(x)| \leq Me^{1-3^\frac{1}{\alpha}}.
\end{cases}
$$

We multiply (2.18) by $\sigma_\alpha$, and integrate it over $\mathbb{R}^3$. Then, the convection term vanishes by (2.19). Let $\alpha_1 > 0$ be fixed. For all $\alpha > \alpha_1$ we have

$$
\int_{\mathbb{R}^3} |\omega|^2 \sigma_\alpha(x) dx \leq \int_{\mathbb{R}^3} |\omega|^2 \sigma_\alpha(x) dx = \int_{\mathbb{R}^3} \Delta Q \sigma_\alpha(x) dx
$$

$$
= -3\alpha \int_{\{Me^{1-3^\frac{1}{\alpha}} < |Q(x)| < Me^{1-(\frac{2}{3})^\frac{1}{\alpha}}\}} \frac{|\nabla Q|^2}{|Q|} \left( \log \frac{eM}{|Q|} \right)^{-\alpha-1} \sigma' \left\{ 3 \left( \log \frac{eM}{|Q(x)|} \right)^{-\alpha} \right\} dx
$$

$$
\leq 3\alpha \sup_{1 \leq s \leq 2} |\sigma'(s)| \int_{\{Me^{1-3^\frac{1}{\alpha}} < |Q(x)| < Me^{1-(\frac{2}{3})^\frac{1}{\alpha}}\}} \frac{|\nabla Q|^2}{|Q|} \left( \log \frac{eM}{|Q|} \right)^{-\alpha-1} dx
$$

$$
\to 0 \text{ as } \alpha \to 0.
$$

Hence, we have shown $\int_{\mathbb{R}^3} |\omega|^2 \sigma_\alpha(x) dx = 0$ for all $\alpha_1 > 0$, which implies that $\omega = 0$ on $\mathbb{R}^3$. This, combined with the fact $\text{div } v = 0$ implies that $v$ is a harmonic function on $\mathbb{R}^3$. The boundary condition, together with the Liouville theorem for harmonic function, leads us to conclude $v = 0$ on $\mathbb{R}^3$.

### 2.3 Proof of Theorem 1.3

We first establish integrability conditions on the vector fields for the Biot-Savart’s formula in $\mathbb{R}^3$.

**Proposition 2.1.** Let $\xi = (\xi_1(x), \xi_2(x), \xi_3(x))$ and $\eta = (\eta_1(x), \eta_2(x), \eta_3(x))$ be smooth vector fields on $\mathbb{R}^3$. Suppose there exists $q \in [1, 3)$ such that $\eta \in L^q(\mathbb{R}^3)$. Let $\xi$ solve

$$
\Delta \xi = -\nabla \times \eta,
$$

under the boundary condition; either

$$
|\xi(x)| \to 0 \quad \text{as } |x| \to +\infty,
$$

$$
(2.21)
$$

$$
(2.22)
$$
or

$$\xi \in L^s(\mathbb{R}^3) \quad \text{for some} \quad s \in [1, \infty).$$  

(2.23)

Then, the solution of (2.21) is given by

$$\xi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \eta(y)}{|x-y|^3} \, dy \quad \forall x \in \mathbb{R}^3. \quad (2.24)$$

**Proof** Let $\sigma \in C_0^\infty(\mathbb{R}^3)$ be the cut-off function defined in the proof of Theorem 1.1. For each $R > 0$ we define $\sigma_R(x) := \sigma\left(\frac{|x|}{R}\right)$. Given $\epsilon > 0$ we denote $B_\epsilon(y) = \{x \in \mathbb{R}^3| |x-y| < \epsilon\}$. Let us fix $y \in \mathbb{R}^3$ and $\epsilon \in (0, \frac{\epsilon}{2})$. We multiply (2.21) by $\frac{\sigma_R(|x-y|)}{|x-y|}$, and integrate it with respect to the variable $x$ over $\mathbb{R}^3 \setminus B_\epsilon(y)$. Then,

$$\int_{\{|x-y| > \epsilon\}} \frac{\Delta \xi \sigma_R}{|x-y|} \, dx = - \int_{\{|x-y| > \epsilon\}} \frac{\sigma_R \nabla \times \eta(y)}{|x-y|} \, dx. \quad (2.25)$$

Since $\Delta \frac{1}{|x-y|} = 0$ on $\mathbb{R}^3 \setminus B_\epsilon(y)$, one has

$$\frac{\Delta \xi \sigma_R}{|x-y|} = \sum_{i=0}^3 \partial_{x_i} \left( \frac{\partial_{x_i} \xi \sigma_R}{|x-y|} \right) - \sum_{i=0}^3 \partial_{x_i} \left( \frac{\xi \partial_{x_i} \sigma_R}{|x-y|} \right)$$

$$- \sum_{i=0}^3 \partial_{x_i} \left( \xi \sigma_R \partial_{x_i} \left( \frac{1}{|x-y|} \right) \right) + \frac{\xi \Delta \sigma_R}{|x-y|} + 2 \sum_{i=0}^3 \xi \partial_{x_i} \left( \frac{1}{|x-y|} \right) \partial_{x_i} \sigma_R.$$  

Therefore, applying the divergence theorem, and observing $\partial_{\nu} \sigma_R = 0$ on $\partial B_\epsilon(y)$, we have

$$\int_{\{|x-y| > \epsilon\}} \frac{\Delta \xi \sigma_R}{|x-y|} \, dx = \int_{\{|x-y| = \epsilon\}} \frac{\partial_{\nu} \xi}{|x-y|} \, dS$$

$$- \int_{\{|x-y| = \epsilon\}} \frac{\xi}{|x-y|^2} \, dS + \int_{\{|x-y| > \epsilon\}} \frac{\xi \Delta \sigma_R}{|x-y|} \, dS$$

$$- 2 \int_{\{|x-y| > \epsilon\}} \frac{(x-y) \cdot \nabla \sigma_R \xi}{|x-y|^3} \, dS \quad (2.26)$$

where $\partial_{\nu}(\cdot)$ denotes the outward normal derivative on $\partial B_\epsilon(y)$. Passing $\epsilon \to 0$, one can easily compute that

$$\text{RHS of (2.26)} \quad \rightarrow \quad -4\pi \xi(y) + \int_{\mathbb{R}^3} \frac{\xi \Delta \sigma_R}{|x-y|} \, dx - 2 \int_{\mathbb{R}^3} \frac{(x-y) \cdot \nabla \sigma_R \xi}{|x-y|^3} \, dx$$

$$:= \quad I_1 + I_2 + I_3 \quad (2.27)$$

Next, using the formula

$$\frac{\sigma_R \nabla \times \eta}{|x-y|} = \nabla \times \left( \frac{\sigma_R \eta}{|x-y|} \right) - \frac{\nabla \sigma_R \times \eta}{|x-y|} + \frac{(x-y) \times \eta \sigma_R}{|x-y|^3},$$

12
and using the divergence theorem, we obtain the following representation for the right hand side of (2.25).

\[
\int_{\{\|x-y\|>\epsilon\}} \frac{\sigma_R \nabla \times \eta}{|x-y|} dx = \int_{\{\|x-y\|=\epsilon\}} \nu \times \left( \frac{\eta}{|x-y|} \right) dS - \int_{\{\|x-y\|>\epsilon\}} \frac{\nabla \sigma_R \times \eta}{|x-y|} dx + \int_{\{\|x-y\|>\epsilon\}} \frac{(x-y) \times \eta \sigma_R}{|x-y|^3} dx, \tag{2.28}
\]

where we denoted \( \nu = \frac{y-x}{|y-x|} \), the outward unit normal vector on \( \partial B_{\epsilon}(y) \). Passing \( \epsilon \to 0 \), we easily deduce

\[
\text{RHS of (2.28)} \quad \rightarrow \quad - \int_{\mathbb{R}^3} \frac{\nabla \sigma_R \times \eta}{|x-y|} dx - 2 \int_{\mathbb{R}^3} \frac{(x-y) \times \eta \sigma_R}{|x-y|^3} dx
\]

\[
\quad := J_1 + J_2 \quad \text{as} \quad \epsilon \to 0. \tag{2.29}
\]

We now pass \( R \to \infty \) for each term of (2.27) and (2.29) respectively below. Under the boundary condition (2.22) we estimate:

\[
|I_2| \leq \int_{\{R \leq \|x-y\| \leq 2R\}} \frac{\|\Delta \sigma\|_{L^\infty}}{R^2} \sup_{R \leq \|x-y\| \leq 2R} |\xi(x)| \left( \int_{\{R \leq \|x-y\| \leq 2R\}} dx \right)^{\frac{1}{3}} \left( \int_{\{R \leq \|x-y\| \leq 2R\}} \frac{dx}{|x-y|^3} \right)^{\frac{1}{3}} \leq \|\Delta \sigma\|_{L^\infty} \left( \int_{R}^{2R} \frac{dr}{r} \right)^{\frac{1}{3}} \sup_{R \leq \|x-y\| \leq 2R} |\xi(x)| \to 0
\]

as \( R \to \infty \) by the assumption (2.22), while under the condition (2.23) we have

\[
|I_2| \leq \int_{\{R \leq \|x-y\| \leq 2R\}} \frac{\|\Delta \sigma\|_{L^\infty}}{R^2} \|\xi\|_{L^s} \left( \int_{\{R \leq \|x-y\| \leq 2R\}} \frac{dx}{|x-y|^3} \right)^{\frac{s}{3}} \leq R^{-\frac{s}{3}} \|\Delta \sigma\|_{L^\infty} \|\xi\|_{L^s} \to 0
\]

as \( R \to \infty \). Similarly, under (2.22)

\[
|I_3| \leq 2 \int_{\{R \leq \|x-y\| \leq 2R\}} \frac{\|\nabla \sigma\|_{L^\infty}}{R} \frac{|\nabla \sigma_R(x-y)|}{|x-y|^2} dx \leq \frac{\|\nabla \sigma\|_{L^\infty}}{R} \sup_{R \leq \|x-y\| \leq 2R} |\xi(x)| \left( \int_{\{R \leq \|x-y\| \leq 2R\}} dx \right)^{\frac{1}{3}} \left( \int_{\{R \leq \|x-y\| \leq 2R\}} \frac{dx}{|x-y|^3} \right)^{\frac{1}{3}} \leq \frac{\|\nabla \sigma\|_{L^\infty}}{R} \left( \int_{R}^{2R} \frac{dr}{r} \right)^{\frac{1}{3}} \sup_{R \leq \|x-y\| \leq 2R} |\xi(x)| \to 0
\]
as $R \to \infty$, while under the condition (2.23) we estimate

$$|I_3| \leq 2 \int_{\{R \leq |x-y| \leq 2R\}} \frac{\xi(x)||\nabla \sigma_R(x-y)||}{|x-y|^2} dx$$

$$\lesssim \frac{||\nabla \sigma||_{L^\infty}}{R^2} ||\xi||_{L^s} \left( \int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^\frac{2}{q}} \right)^\frac{q-1}{q}$$

$$\lesssim R^{-\frac{2}{q}} ||\nabla \sigma||_{L^\infty} ||\xi||_{L^s} \to 0$$

as $R \to \infty$. Therefore, the right hand side of (2.26) converges to $-4\pi \xi(y)$ as $R \to \infty$. For $J_1, J_2$ we estimate

$$|J_1| \leq \int_{\{R \leq |x-y| \leq 2R\}} \frac{||\nabla \sigma_R||_1}{|x-y|} dx$$

$$\leq \frac{||\nabla \sigma||_{L^\infty}}{R} ||\eta||_{L^q(\{R \leq |x-y| \leq 2R\})} \left( \int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^\frac{2}{q}} \right)^\frac{q-1}{q}$$

$$\lesssim ||\nabla \sigma||_{L^\infty} ||\eta||_{L^q(\{R \leq |x-y| \leq 2R\})} R^{-\frac{2}{q}} \to 0$$

as $R \to \infty$. In passing $R \to \infty$ in $J_2$ of (2.29), in order to use the dominated convergence theorem, we estimate

$$\int_{\mathbb{R}^3} \frac{(x-y) \times \eta(y)}{|x-y|^3} dx \leq \int_{\{|x-y| < 1\}} \frac{\eta}{|x-y|^2} dx + \int_{\{|x-y| \geq 1\}} \frac{\eta}{|x-y|^2} dx$$

$$:= J_{21} + J_{22}. \quad (2.30)$$

$J_{21}$ is easy to handle as follows.

$$J_{21} \leq ||\eta||_{L^\infty(B_1(y))} \int_{\{|x-y| < 1\}} \frac{dx}{|x-y|^2} = 4\pi ||\eta||_{L^\infty(B_1(y))} < +\infty \quad (2.31)$$

For $J_{22}$ we estimate

$$J_{22} \leq \left( \int_{\mathbb{R}^3} |\eta|^q \right)^\frac{1}{q} \left( \int_{\{|x-y| > 1\}} \frac{dx}{|x-y|^\frac{2}{q-1}} \right)^\frac{q-1}{q}$$

$$\lesssim ||\eta||_{L^q} \left( \int_{1}^{\infty} r^{\frac{2}{q-1}} dr \right)^\frac{q-1}{q} < +\infty, \quad (2.32)$$

if $1 < q < 3$. In the case of $q = 1$ we estimate simply

$$J_{22} \leq \int_{\{|x-y| > 1\}} |\eta| dx \leq ||\eta||_{L^1}. \quad (2.33)$$

Estimates of (2.30)-(2.33) imply

$$\int_{\mathbb{R}^3} \frac{(x-y) \times \eta(y)}{|x-y|^3} dx < +\infty.$$
Summarising the above computations, one can pass first \( \epsilon \to 0 \), and then \( R \to +\infty \) in (2.25), applying the dominated convergence theorem, to obtain finally (2.24).

**Corollary 2.1.** Let \( v \) be a smooth solution to (NS) satisfying (1.1) such that \( \omega \in L^q(\mathbb{R}^3) \) for some \( q \in \left[ \frac{3}{2}, 3 \right) \). Then, we have

\[
v(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y) \times \omega(y)}{|x - y|^3} dy,
\]

and

\[
\omega(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y) \times (v(y) \times \omega(y))}{|x - y|^3} dy.
\]

**Proof** Taking curl of the defining equation of the vorticity, \( \nabla \times v = \omega \), using \( \text{div} \ v = 0 \), we have

\[\Delta v = -\nabla \times \omega,\]

which provides us with (2.34) immediately by application of Proposition 2.1. In order to show (2.35) we recall that, using the vector identity \( \frac{1}{2} \nabla |v|^2 = (v \cdot \nabla)v + v \times (\nabla \times v) \), one can rewrite (NS) as

\[-v \times \omega = -\nabla \left( p + \frac{1}{2} |v|^2 \right) + \Delta v.\]

Taking curl on this, we obtain

\[\Delta \omega = -\nabla \times (v \times \omega).\]

The formula (2.35) is deduced immediately from this equations by applying the proposition 2.1. For the allowed rage of \( q \) we recall the Sobolev and the Calderon-Zygmund inequalities(\[8\]),

\[\|v\|_{L^q} \lesssim \|\nabla v\|_{L^q} \lesssim \|\omega\|_{L^q}, \quad 1 < q < 3,
\]

which imply \( v \times \omega \in L^{\frac{3q}{6-q}}(\mathbb{R}^3) \) if \( \omega \in L^q(\mathbb{R}^3) \). We also note that \( \frac{3}{2} \leq q < 3 \) if and only if \( 1 \leq \frac{3q}{6-q} < 3 \).

**Proof of Theorem 1.3** Under the hypothesis (1.1) and \( \omega \in L^q(\mathbb{R}^3) \) with \( q \in \left[ \frac{3}{2}, 3 \right) \) both of the relations (2.34) and (2.35) are valid. We first prove the following.

**Claim:** For each \( x, y \in \mathbb{R}^3 \)

\[0 \leq |\omega(x)|^2 = \int_{\mathbb{R}^3} \Phi(x, y) dy \leq \int_{\mathbb{R}^3} |\Phi(x, y)| dy < +\infty,\]

and

\[0 = \int_{\mathbb{R}^3} \Phi(x, y) dx \leq \int_{\mathbb{R}^3} |\Phi(x, y)| dx < +\infty.\]
Proof of (1.8): Decomposing the integral and using the H"older inequality, we estimate

\[
\int_{\mathbb{R}^3} |\Phi(x,y)| \, dy \leq |\omega(x)| \left( \int_{\{|x-y| \leq 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} \, dy + \int_{\{|x-y| > 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} \, dy \right)
\]

\[
\leq |\omega(x)||v||_{L^\infty(B_1(x))} ||\omega||_{L^\infty(B_1(x))} \int_{\{|x-y| \leq 1\}} \frac{dy}{|x-y|^2}
\]

\[
+ |\omega(x)||v||_{L^{\frac{6q}{3q-6}}(B_1(x))} ||\omega||_{L^\infty(B_1(x))} \left( \int_{\{|x-y| > 1\}} \frac{dy}{|x-y|^{\frac{6q}{3q-6}}} \right)^{\frac{3q-6}{3q}}
\]

\[
\lesssim |\omega(x)||v||_{L^\infty(B_1(x))} ||\omega||_{L^\infty(B_1(x))}
\]

\[
+ |\omega(x)||\omega||_{L^q}^2 \left( \int_1^{\infty} r^{\frac{6q-6}{3q}} \, dr \right)^{\frac{3q-6}{3q}} < +\infty,
\]

(2.39)

where we used (2.36) and the fact that \( \frac{q-6}{3q-6} < -1 \) if \( \frac{3}{2} < q < 3 \). In the case \( q = \frac{3}{2} \) we estimate, instead,

\[
\int_{\mathbb{R}^3} |\Phi(x,y)| \, dy \leq |\omega(x)| \left( \int_{\{|x-y| \leq 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} \, dy + \int_{\{|x-y| > 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} \, dy \right)
\]

\[
\leq |\omega(x)||v||_{L^\infty(B_1(x))} ||\omega||_{L^\infty(B_1(x))} + |\omega(x)||v||_{L^3} ||\omega||_{L^\frac{6}{5}} < +\infty.
\]

(2.40)

We also have

\[
\int_{\mathbb{R}^3} |\Phi(x,y)| \, dx \leq |v(y)||\omega(y)| \left( \int_{\{|x-y| \leq 1\}} \frac{|\omega(x)|}{|x-y|^2} \, dx + \int_{\{|x-y| > 1\}} \frac{|\omega(x)|}{|x-y|^2} \, dx \right)
\]

\[
\lesssim |v(y)||\omega(y)||\omega||_{L^\infty(B_1(y))} + |v(y)||\omega(y)||\omega||_{L^q} \left( \int_{\{|x-y| > 1\}} \frac{dx}{|x-y|^{\frac{2q}{q-1}}} \right)^{\frac{q-1}{q}}
\]

\[
\lesssim |v(y)||\omega(y)||\omega||_{L^\infty(B_1(y))} + |v(y)||\omega(y)||\omega||_{L^q} \left( r^{-\frac{2q}{q-1}} \, dr \right)^{\frac{q-1}{q}} < +\infty
\]

(2.41)

where we used the fact that \( -\frac{2}{q-1} < -1 \) if \( \frac{3}{2} \leq q < 3 \). From (2.35) we immediately obtain

\[
\int_{\mathbb{R}^3} \Phi(x,y) \, dy = |\omega(x)| \cdot \left( \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times (v(y) \times \omega(y))}{|x-y|^3} \, dy \right)
\]

\[
= |\omega(x)|^2 \geq 0, \quad \forall x \in \mathbb{R}^3
\]

(2.42)

and combining this with (2.39), we deduce (2.37). On the other hand, using (2.34), we find

\[
\int_{\mathbb{R}^3} \Phi(x,y) \, dx = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(x) \cdot (x-y) \times (v(y) \times \omega(y))}{|x-y|^3} \, dx
\]

\[
= \left( \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(x) \times (x-y)}{|x-y|^3} \, dx \right) \cdot v(y) \times \omega(y)
\]

\[
= v(y) \cdot v(y) \times \omega(y) = 0
\]

(2.43)
for all $y \in \mathbb{R}^3$, and combining this with (2.41), we have proved (2.38). This completes the proof of the claim.

If (1.9) holds, then from (2.42) and (2.43) provide us with

$$\int_{\mathbb{R}^3} |\omega(x)|^2 dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(x,y) dy dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(x,y) dx dy = 0.$$

Hence,

$$\omega = 0 \quad \text{on} \quad \mathbb{R}^3. \quad (2.44)$$

We remark parenthetically that in deriving (2.44) it is not necessary to assume that $\int_{\mathbb{R}^3} |\omega(x)|^2 dx < +\infty$, and therefore we do not need to restrict ourselves to $\omega \in L^2(\mathbb{R}^3)$. Hence, from (2.34) and (2.44), we conclude $v = 0$ on $\mathbb{R}^3$. ■

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