UMBRAL CALCULUS ASSOCIATED WITH BERNOULLI POLYNOMIALS

DAE SAN KIM$^1$ AND TAEKYUN KIM$^2$

Abstract. Recently, D. S. Kim and T. Kim have studied applications of umbral calculus associated with $p$-adic invariant integrals on $\mathbb{Z}_p$ (see [6]). In this paper, we investigate some interesting properties arising from umbral calculus. These properties are useful in deriving some identities of Bernoulli polynomials.

1. Introduction

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\frac{te^{xt}}{e^t - 1} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$. In the special case, $x = 0$, $B_n(0) = B_n$ are called the $n$-th Bernoulli numbers. From (1), we note that

$$B_0 = 1, \quad (B + 1)^n - B^n = B_n(1) - B_n = \delta_{1,n}, \quad \text{(see [2, 3, 4])},$$

where $\delta_{m,k}$ is the Kronecker symbol.

In particular, by (1), we set

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l, \quad \text{(see [1, 2, 3, 4, 7])}. \quad (2)$$

By (2), we see that $B_n(x)$ is a monic polynomial of degree $n$. We recall the Euler polynomials are defined by the generating function to be

$$\frac{2e^{xt}}{e^t + 1} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [1, 7])}, \quad (3)$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the $n$-th Euler numbers. From (3), we can derive the following equation:

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} x^l, \quad \text{(see [7, 9])}. \quad (4)$$

Thus (4), we see that $E_n(x)$ is also a monic polynomial of degree $n$. By (4), we get

$$E_0 = 1, \quad (E + 1)^n + E^n = E_n(1) + E_n = 2 \delta_{0,n}. \quad (5)$$

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable $t$ over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k : a_k \in \mathbb{C} \right\}.$$

We use the notation $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ denotes the vector space of all linear functional on $\mathbb{P}$. 


Let $(L \mid p(x))$ be the action of a linear functional $L$ on a polynomial $p(x)$, and we remark that the vector space operation on $P^*$ are defined by

$$\langle L + M \mid p(x) \rangle = \langle L \mid p(x) \rangle + \langle M \mid p(x) \rangle,$$

$$\langle cL \mid p(x) \rangle = c \langle L \mid p(x) \rangle,$$  \hspace{1cm} (see [6, 15]),

where $c$ is any constant in $\mathbb{C}$.

Thus, by (6) and (7), we have

$$P_t \text{ is called the Sheffer sequence for } (S_n(x)).$$

Let us assume that

$$\delta_{n,k} = \begin{cases} 0, & n \neq k, \\ 1, & n = k, \end{cases}$$

and so an element $\delta_{n,k}$ of $F$ will be thought of as both a formal power series and a linear functional. We shall call $F$ the umbral algebra. The umbral calculus is the study of umbral algebra and modern classical umbral calculus can be described as a systematic study of the class of Sheffer sequences (see [15]).

The order $\text{ord}(f(t))$ of a nonzero power series $f(t)$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish. If a series $f(t)$ with $\text{ord}(f(t)) = 1$, then $f(t)$ is called a delta series. If a series $f(t)$ with $\text{ord}(f(t)) = 0$, then $f(t)$ is called an invertible series (see [6, 15]). For $f(t), g(t) \in F$, we have

$$\langle f(t)g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)p(x) \rangle,$$  \hspace{1cm} (see [6, 15]).

Let us assume that $S_n(x)$ denotes a polynomial of degree $n$. If $f(t)$ is a delta series and $g(t)$ is an invertible series, then there exists a unique sequence $\{S_n(x)\}$ such that

$$\langle g(t)f(t)k \mid S_n(x) \rangle = n! \delta_{n,k}, \hspace{1cm} n, k \geq 0$$

(see [6]). The sequence $\{S_n(x)\}$ is called the Sheffer sequence for $(g(t), f(t))$, denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the associated sequence for $f(t)$ or $S_n(x)$ is associated to $f(t)$. If $S_n(x) \sim (g(t), f(t))$, then $S_n(x)$ is called the Appell sequence for $g(t)$ or $S_n(x)$ is Appell for $g(t)$ (see [6, 15]). For $p(x) \in P$, it is known (see [6, 15]) that

$$\langle e^{yt} - 1 \mid p(x) \rangle = \int_0^y p(u) du,$$  \hspace{1cm} (10)

$$\langle f(t) \mid xp(x) \rangle = (\partial_t f(t) \mid p(x)) = \langle f'(t) \mid p(x) \rangle,$$  \hspace{1cm} (11)

and

$$\langle e^{yt} - 1 \mid p(x) \rangle = p(y) - p(0),$$  \hspace{1cm} (see [6, 15]).

Let us assume that $S_n(x) \sim (g(t), f(t))$. Then we have the following equations [13, 19]:

$$h(t) = \sum_{k=0}^{\infty} \frac{h(t) \mid S_k(x)}{k!} g(t)f(t)^k, \hspace{1cm} h(t) \in F,$$  \hspace{1cm} (13)
\[ p(t) = \sum_{k=0}^{\infty} \frac{\langle g(t)f(t)^k \mid p(x) \rangle}{k!} S_k(x), \quad p(t) \in \mathbb{P}, \quad (14) \]

\[ f(t)S_n(x) = nS_{n-1}(x), \quad (n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}), \quad (15) \]

and

\[ \frac{1}{g(f(t))} = \sum_{k=0}^{\infty} \frac{S_k(g(t))}{k!} t^k, \quad \text{for all } y \in \mathbb{C}. \quad (16) \]

where \( \bar{f}(t) \) is the compositional inverse of \( f(t) \) (see \([15]\)).

Let \( f_1(t), \ldots, f_m(t) \in \mathcal{F} \). Then as is well known, we have

\[ \langle f_1(t) f_2(t) \cdots f_m(t) \mid x^n \rangle = \sum_{i_1, \ldots, i_m} \binom{n}{i_1, \ldots, i_m} \langle f_1(t) \mid x^{i_1} \rangle \cdots \langle f_m(t) \mid x^{i_m} \rangle, \quad (17) \]

where the sum is over all nonnegative integers \( i_1, \ldots, i_m \) such that \( i_1 + \cdots + i_m = n \) (see \([6, 15]\)).

In \([6]\), D. S. Kim and T. Kim have studied applications of umbral calculus associated with \( p \)-adic invariant integrals on \( \mathbb{Z}_p \). In this paper, we derive some interesting properties of Bernoulli polynomials arising from umbral calculus. These properties will be used in studying identities on the Bernoulli polynomials.

2. Umbral Calculus and Bernoulli Polynomials

Let \( \mathbb{P}_n = \{ p(x) \in \mathbb{C}[x] : \deg p(x) \leq n \} \) and let \( S_n(x) \sim (g(t), t) \). From \([16]\), we have

\[ \frac{1}{g(t)} x^n = S_n(x) \Leftrightarrow x^n = g(t)S_n(x), \quad (n \geq 0). \quad (18) \]

Let us take \( g(t) = \frac{1}{t} (e^t - 1) \in \mathcal{F} \). Then \( g(t) \) is invertible series. By \([1]\), we get

\[ \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}. \quad (19) \]

Thus by \([19]\), we have

\[ \frac{1}{g(t)} x^n = B_n(x), \quad (n \geq 0), \quad (20) \]

and

\[ tB_n(x) = B'_n(x) = nB_{n-1}(x). \quad (21) \]

From \([20]\) and \([21]\), we note that \( B_n(x) \) is an Appell sequence for \( \frac{1}{t}(e^t - 1) \). By \([2]\), we get

\[ \int_x^{x+y} B_n(u)du = \frac{1}{n+1} \{ B_{n+1}(x+y) - B_{n+1}(x) \} \]

\[ = \sum_{k=1}^{\infty} \frac{y^k}{k!} k! k^{-1} B_n(x) = \frac{e^{yt} - 1}{t} B_n(x). \quad (22) \]

In particular, for \( y = 1 \), we have

\[ B_n(x) = \frac{t}{e^t - 1} \int_x^{x+1} B_n(u)du = \frac{t}{e^t - 1} x^n. \quad (23) \]

By \([15]\), we easily get

\[ B_n(x) = t \left\{ \frac{1}{n+1} B_{n+1}(x) \right\}. \quad (24) \]

From \([24]\), we can derive the following equation:

\[ \langle e^{yt} - 1 \mid B_n(x) \rangle = \langle e^{yt} - 1 \mid \frac{1}{n+1} B_{n+1}(x) \rangle = \int_0^y B_n(u)du. \quad (25) \]
For \( r \in \mathbb{N} \), the \( n \)-th Bernoulli polynomials of order \( r \) are defined by the generating function to be
\[
\left( \frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(26)}
\]

In the special case, \( x = 0 \), \( B_n^{(r)}(0) = B_n^{(r)} \) are called the \( n \)-th Bernoulli numbers of order \( r \). By (26), we get
\[
B_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}^{(r)} x^r, \quad \text{(27)}
\]

Note that
\[
B_n^{(r)} = \sum_{i_1 + \cdots + i_r = n} \binom{n}{i_1, \ldots, i_r} B_{i_1} \cdots B_{i_r}. \quad \text{(28)}
\]

From (27) and (28), we note that \( B_n^{(r)}(x) \) is a monic polynomials with coefficients in \( \mathbb{Q} \). By (24), we get
\[
\int_x^{x+y} B_n^{(r)}(u) du = \frac{1}{n+1} \left( B_{n+1}^{(r)}(x+y) - B_{n+1}^{(r)}(x) \right)
= \sum_{k=1}^{\infty} \frac{y^k}{k!} B_n(x) = \frac{e^{yt} - 1}{t} B_n^{(r)}(x),
\quad \text{(29)}
\]

and
\[
B_n^{(r)}(x+1) - B_n^{(r)}(x) = n B_{n-1}^{(r-1)}(x). \quad \text{(30)}
\]

From (29) and (30), we note that
\[
\frac{e^t - 1}{t} B_n^{(r)}(x) = \int_x^{x+1} B_n^{(r)}(u) du = B_n^{(r-1)}(x). \quad \text{(31)}
\]

By (31), we get
\[
B_n^{(r)}(x) = \left( \frac{t}{e^t - 1} \right)^r B_n^{(r-1)}(x) = \left( \frac{t}{e^t - 1} \right)^{r-1} B_n(x) = \left( \frac{t}{e^t - 1} \right)^r x^n, \quad \text{(32)}
\]

and
\[
t B_n^{(r)}(x) = n \left( \frac{t}{e^t - 1} \right)^r x^{n-1} = n B_{n-1}^{(r)}(x). \quad \text{(33)}
\]

It is easy to show that \( \left( \frac{e^t - 1}{t} \right)^r \) is an invertible series in \( \mathcal{F} \). Therefore, by (32) and (33), we obtain the following lemma.

**Lemma 1.** \( B_n^{(r)}(x) \) is the Appell sequence for \( \left( \frac{e^t - 1}{t} \right)^r \).

By (33), we get
\[
B_n^{(r)}(x) = t \left\{ \frac{1}{n+1} B_{n+1}^{(r)}(x) \right\}, \quad (n \geq 0). \quad \text{(34)}
\]

Thus, from (34), we have
\[
\left\langle \frac{e^{yt} - 1}{t} \mid B_n^{(r)}(x) \right\rangle = \left\langle e^{yt} - 1 \mid \frac{1}{n+1} B_{n+1}^{(r)}(x) \right\rangle = \int_0^y B_n^{(r)}(u) du. \quad \text{(35)}
\]

In the special case, \( y = 1 \), we have
\[
\left\langle \frac{e^t - 1}{t} \mid B_n^{(r)}(x) \right\rangle = \int_0^1 B_n^{(r)}(u) du = B_n^{(r-1)}. \]
By (17), we get
\[ \langle \left( \frac{t}{e^t - 1} \right)^{r} | x^n \rangle = \sum_{n=1+i_1, \ldots, i_r} \binom{n}{i_1, \ldots, i_r} \langle \left( \frac{t}{e^t - 1} \right)^{r} \rangle \cdots \langle \left( \frac{t}{e^t - 1} \right)^{r} \rangle \] (36)
and
\[ \langle \left( \frac{t}{e^t - 1} \right)^{r} | x^n \rangle = B_n, \quad \langle \left( \frac{t}{e^t - 1} \right)^{r} \rangle = B_n^{(r)}. \] (37)
Thus, from (36) and (37), we have
\[ \sum_{n=1+i_1, \ldots, i_r} \binom{n}{i_1, \ldots, i_r} B_{i_1} \cdots B_{i_r} = B_n^{(r)}. \]

Let us take \( p(x) \in \mathbb{P}_n \) with
\[ p(x) = \sum_{k=0}^{n} b_k B_k(x). \] (38)

From (20) and (21), we note that \( B_n(x) \sim \left( \frac{x^r}{r!}, t \right) \). By the definition of Appell sequences, we get
\[ \langle \frac{e^t - 1}{t} t^k | B_n(x) \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \] (39)
and, from (38), we have
\[ \langle \frac{e^t - 1}{t} t^k | p(x) \rangle = \sum_{l=0}^{n} b_l \langle \frac{e^t - 1}{t} t^k | B_l(x) \rangle = \sum_{l=0}^{n} b_l l! \delta_{l,k} = k! b_k. \] (40)
Thus, by (25) and (40), we get
\[ b_k = \frac{1}{k!} \langle \frac{e^t - 1}{t} t^k | p(x) \rangle = \frac{1}{k!} \langle \frac{e^t - 1}{t} t^k | p^{(k)}(x) \rangle = \frac{1}{k!} \int_0^1 p^{(k)}(u) du, \] (41)
where \( p^{(k)}(u) = \frac{d^k}{da^k} p(u) \). Therefore, by (38) and (41), we obtain the following theorem.

**Theorem 2.** Let \( p(x) \in \mathbb{P}_n \) with \( p(x) = \sum_{k=0}^{n} b_k B_k(x) \). Then we have
\[ b_k = \frac{1}{k!} \langle \frac{e^t - 1}{t} t^k | p^{(k)}(x) \rangle = \frac{1}{k!} \int_0^1 p^{(k)}(u) du, \]
where \( p^{(k)}(u) = \frac{d^k}{da^k} p(u) \).

Let \( p(x) = B_n^{(r)}(x) \in \mathbb{P}_n \) with \( p(x) = \sum_{k=0}^{n} b_k B_k(x) \). Then we have
\[ p^{(k)}(x) = k! \binom{n}{k} B_{n-k}^{(r)}(x), \]
and
\[ b_k = \frac{1}{k!} \langle \frac{e^t - 1}{t} t^k | p^{(k)}(x) \rangle = \binom{n}{k} \langle \frac{e^t - 1}{t} t^k | B_{n-k}^{(r)}(x) \rangle. \] (43)
Therefore, by Theorem 2 and (43), we obtain the following theorem.

**Corollary 3.** For \( n \geq 0 \), we have
\[ B_n^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} \langle \frac{e^t - 1}{t} t^k | B_{n-k}^{(r)}(x) \rangle B_k(x). \]
Thus, by (45), we have

**Theorem 4.**

Therefore, by (46), we obtain the following theorem.

In other words,

\[ B_n^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{(r-1)} B_k(x). \]

From the definition of Appell sequences, we note that

\[ \left\langle \left( \frac{e^t - 1}{t} \right)^r t^k \big| B_n^{(r)}(x) \right\rangle = n! \delta_{n,k}, \quad (n, k \geq 0). \] (44)

Let \( p(x) \in \mathbb{P}_n \) with \( p(x) = \sum_{k=0}^{n} b_k^{(r)} B_k^{(r)}(x) \). By (45), we get

\[
\left\langle \left( \frac{e^t - 1}{t} \right)^r t^k \big| p(x) \right\rangle = \sum_{l=0}^{n} b_l^{(r)} \left\langle \left( \frac{e^t - 1}{t} \right)^r t^k \big| B_l^{(r)}(x) \right\rangle = \sum_{l=0}^{n} b_l^{(r)} l! \delta_{l,k} = k! b_k^{(r)}.
\] (45)

Thus, by (45), we have

\[ b_k^{(r)} = \frac{1}{k!} \left\langle \left( \frac{e^t - 1}{t} \right)^r t^k \big| p(x) \right\rangle. \] (46)

Therefore, by (46), we obtain the following theorem.

**Theorem 4.** Let \( p(x) \in \mathbb{P}_n \) with \( p(x) = \sum_{k=0}^{n} b_k^{(r)} B_k^{(r)}(x) \). Then we have

\[ b_k^{(r)} = \frac{1}{k!} \left\langle \left( \frac{e^t - 1}{t} \right)^r t^k \big| p(x) \right\rangle. \]

Let us consider \( p(x) = B_n(x) \) with

\[ B_n(x) = p(x) = \sum_{k=0}^{n} b_k^{(r)} B_k^{(r)}(x). \] (47)

By Theorem 4 and (47), we get

\[ b_k^{(r)} = \frac{1}{k!} \left\langle \left( \frac{e^t - 1}{t} \right)^r t^k \big| p(x) \right\rangle = \frac{1}{k!} \left\langle \left( \frac{e^t - 1}{t} \right)^r t^k \big| B_n(x) \right\rangle. \] (48)

For \( k < r \), we have

\[
b_k^{(r)} = \frac{1}{k!} \left\langle \left( \frac{e^t - 1}{t} \right)^r t^k \big| B_n(x) \right\rangle = \frac{1}{k!} \left( \frac{e^t - 1}{t} \right)^r \left\langle \frac{B_{n+r-k}(x)}{(n+1) \cdots (n+r-k)} \big| B_n(x) \right\rangle = \frac{1}{k!(r-k)!} \binom{r}{r-k} \sum_{j=0}^{r-k} (-1)^{r-j} \left\langle e^{jt} \big| B_{n+r-k}(x) \right\rangle
\] (49)

\[ = \frac{1}{r!} \binom{r}{r-k} \sum_{j=0}^{r-k} (-1)^{r-j} e^{jt} B_{n+r-k}(j). \]
Let $k \geq r$. Then by (45), we get

$$b_k^{(r)} = \frac{1}{k!} \left< (e^t - 1)^r t^{k-r} \mid B_n(x) \right> = \frac{1}{k!} \left< (e^t - 1)^r \mid t^{k-r}B_n(x) \right>$$

$$= \frac{1}{k!} \binom{n}{k-r} (k-r)! \left< (e^t - 1)^r \mid B_{n+r-k}(x) \right>$$

$$= \frac{1}{k} \sum_{j=0}^{r} \binom{n}{j} (j) (-1)^{r-j} \left< e^{jt} \mid B_{n+r-k}(x) \right>$$

$$= \frac{1}{k} \sum_{j=0}^{r} \binom{n}{j} (j) (-1)^{r-j} B_{n+r-k}(j). \quad (50)$$

Therefore, by (47), (49) and (50), we obtain the following theorem.

**Theorem 5.** For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$B_n(x) = \sum_{k=0}^{r-1} \frac{1}{r!} \binom{n}{k} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} B_{n+r-k}(j) B_k^{(r)}(x)$$

$$+ \sum_{k=r}^{n} \frac{1}{r!} \binom{n}{k} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} B_{n+r-k}(j) B_k^{(r)}(x).$$

### 3. Further Remarks

For $n, m \in \mathbb{Z}_+$ with $n - m \geq 0$, we have

$$B_n^{(r)}(x)B_{n-m}^{(r)}(x)$$

$$= \left( \sum_{l=0}^{n} \frac{1}{l!} B_l^{(r-1)}(x) \right) \left( \sum_{p=0}^{n-m} \frac{1}{p!} B_p^{(r-1)}(x) \right) B_{n-m-p}^{(r-1)}(x) \right)$$

$$= \sum_{k=0}^{2n-m} \sum_{p=0}^{n-m} \binom{n-m}{p} \binom{n}{k-p} B_k^{(r-1)}(x) B_{n-m-p}^{(r-1)}(x) B_{n-k-p}^{(r-1)}(x). \quad (51)$$

Let us consider $p(x) = E_n(x)$ with

$$E_n(x) = p(x) = \sum_{k=0}^{n} b_k B_k(x). \quad (52)$$

Then we have

$$p^{(k)}(x) = k! \binom{n}{k} E_{n-k}(x), \quad (53)$$

and

$$b_k = \frac{1}{k!} \left< \frac{e^t - 1}{t} \mid p^{(k)}(x) \right> = \binom{n}{k} \left< \frac{e^t - 1}{t} \mid E_{n-k}(x) \right>$$

$$= \binom{n}{k} \frac{E_{n-k+1}(1) - E_{n-k+1}}{n-k+1} = -2 \binom{n}{k} \frac{E_{n-k+1}}{n-k+1}. \quad (54)$$

By (52) and (54), we get

$$E_n(x) = -2 \sum_{k=0}^{n} \binom{n}{k} \frac{E_{n-k+1}}{n-k+1} B_k(x). \quad (55)$$
From (55), we can derive the following equation.

\[
E_n(x)E_{n-m}(x) = 4 \sum_{l=0}^{n} \binom{n}{l} \frac{E_{n-l+1}}{n-l+1} B_l(x) \sum_{p=0}^{m} \binom{n-m}{n-m-p+1} E_{n-m-p+1} B_p(x)
\]

\[
= 4 \sum_{k=0}^{2n-m} \sum_{l=0}^{k} \binom{n-m}{l} \binom{n}{k-l} \frac{E_{n-m-l+1}E_{n-k-l+1}}{(n-m-l+1)(n-k+l+1)} B_l(x)B_{k-l}(x)
\]

where \(n, m \in \mathbb{Z}_+\) with \(n-m \geq 0\).

**Acknowledgements**

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology 2012R1A1A2003786.

**References**

[1] S. Araci, D. Erdal, J. J. Seo, *A study on the fermionic \(p\)-adic q-integral on \(\mathbb{Z}_p\) associated with weighted \(q\)-Bernstein and \(q\)-Genocchi polynomials*, Abstract and Applied Analysis 2011(2011), Article ID 649248, 10 pages.

[2] L. Carlitz, *A note on Bernoulli numbers and polynomials of higher order*, Proc. Amer. Math. Soc. 3, (1952), 608–613.

[3] L. Carlitz, *Note on the integral of the product of several Bernoulli polynomials*, J. London Math. Soc. 34 (1959), 361–363.

[4] J. Choi, D. S. Kim, T. Kim, Y. H. Kim, *Some arithmetic identities on Bernoulli and Euler polynomials in three variables (I)*, Adv. Stud. Contemp. Math. 18(2009), no. 1, 41–48.

[5] D. Ding, J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials*, Adv. Stud. Contemp. Math. 20(2010), no. 1, 7–21.

[6] D. S. Kim, T. Kim, *Applications of umbral calculus associated with \(p\)-adic invariant integrals on \(\mathbb{Z}_p\)*, Abstr. Appl. Anal. Appl. 2012(2012), Article ID. 86572, 12 pp.

[7] D. S. Kim, N. Lee, J. Na, H. K. Park, *Identities of symmetry for higher-order Euler polynomials in three variables (I)*, Adv. Stud. Contemp. Math. 22(2012), no. 1, 51–74.

[8] G. Kim, B. Kim, J. Choi, *The DC algorithm for computing sums of powers of consecutive integers and Bernoulli numbers*, Adv. Stud. Contemp. Math. 17 (2008), 137–145.

[9] T. Kim, *Symmetry \(p\)-adic invariant integral on \(\mathbb{Z}_p\) for Bernoulli and Euler polynomials*, J. Difference Equ. Appl. 14 (2008), 1267–1277.

[10] T. Kim, *\(q\)-Volkenborn integration*, Russ. J. Math. Phys. 9 (2002), 288–299.

[11] H. Y. Lee, N. S. Jung, C. S. Ryoo, *A numerical investigation of the roots of the second kind \(A\)-Bernoulli polynomials*, Neural Parallel Sci. Comput. 19(2011), no. 3–4, 295–306.

[12] H. Ozden, I. N. Cangul, Y. Simsek, *Remarks on \(q\)-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. 18(2009), no. 1, 41–48.

[13] S.-H. Rim, J. Jeong, *On the modified \(q\)-Euler numbers of higher order with weight*, Adv. Stud. Contemp. Math. 22(2012), no. 1, 93–98.

[14] S.-H. Rim, J. Lee, *Some identities on the twisted \((h, q)\)-Genocchi numbers and polynomials associated with \(q\)-Bernstein polynomials*, Int. J. Math. Math. Sci. 2011(2011), Art. ID 482840, 8pp.

[15] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.

[16] C. S. Ryoo, *on the generalized Barnes type multiple \(q\)-Euler polynomials twisted by ramified roots of unity*, Proc. Jangjeon Math. Soc. 13(2010), no. 2, 255–263.

[17] C. S. Ryoo, T. Kim, *A new identities on the \(q\)-Bernoulli numbers and polynomials*, Adv. Stud. Contemp. Math. 21(2011), no. 2, 161–169.

[18] Y. Simsek, *Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation function*, Adv. Stud. Contemp. Math. 16 (2008), 281–278.

[19] Y. Simsek, *Special functions related to Dedekind type DC-sums and their applications*, Russ. J. Math. Phys. 17 (2010), 495–508.
[20] K. Shiratani, S. Yokoyama, *An application of p-adic convolutions*, Mem. Fac. Sci. Kyushu Univ. Ser. A 36(1982), no. 1, 73–83.

[21] Z. Zhang, H. Yang, *Some closed formulas for generalized Bernoulli-Euler numbers and polynomials*, Proc. Jangjeon Math. Soc. 11(2008), no. 2, 191–198.

1 Department of Mathematics, Sogang University, Seoul 121-741, Republic of Korea, E-mail address: dskim@sogang.ac.kr

2 Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea, E-mail address: tkkim@kw.ac.kr