Riesz-type criteria for $L$-functions in the Selberg class

Shivajee Gupta and Akshaa Vatwani

Abstract. We formulate a generalization of Riesz-type criteria in the setting of $L$-functions belonging to the Selberg class. We obtain a criterion which is sufficient for the grand Riemann hypothesis (GRH) for $L$-functions satisfying axioms of the Selberg class without imposing the Ramanujan hypothesis on their coefficients. We also construct a subclass of the Selberg class and prove a necessary criterion for GRH for $L$-functions in this subclass. Identities of Ramanujan–Hardy–Littlewood type are also established in this setting, specific cases of which yield new transformation formulas involving special values of the Meijer $G$-function of the type $G_{n,0}^{0,n}$.

1 Introduction

The well-known Riemann hypothesis asserts that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ lie on the line $\text{Re}(s) = 1/2$. In 1916, Riesz [23] showed that a necessary and sufficient criterion for the Riemann hypothesis is the bound

$$\sum_{n=1}^{\infty} \mu(n) \frac{x}{n} e^{-x/n^2} = O(x^{1/4 + \delta}),$$

for any $\delta > 0$. Around the same time, Hardy and Littlewood [14] established that the bound

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-x/n^2} = O(x^{-1/4 + \delta}),$$

for any $\delta > 0$ is equivalent to the Riemann hypothesis. Various variants of Riesz-type criteria have been given in the literature, for instance, for Dirichlet $L$-functions and in the setting of primitive Hecke forms by Dixit, Roy, and Zaharescu in [13] and [12], respectively; by Dixit, Gupta, and Vatwani [9] for the Dedekind zeta function; and by Banerjee and Kumar [3] for $L$-functions associated with primitive Maass cusp forms over the congruence subgroup $\Gamma_0(N)$. In [16], Kühn, Robles, and Roy...
obtained a generalized Riesz-type criterion involving functions which are reciprocal under a certain Hankel transformation. Their result holds for $L$-functions in the reduced Selberg class of degree one. Recently, Agarwal, Maji, and Garg [1] obtained a generalization of (1.1) and (1.2) by giving criteria involving the sum \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} e^{-x/n^2} \), where \( k \geq 1 \) is a real number. In forthcoming work, they also obtain analogs of this for Dirichlet $L$-functions.

In this paper, we obtain Riesz-type criteria for the grand Riemann hypothesis (GRH) for a broad class of $L$-functions. More precisely, we obtain criteria which are sufficient for GRH and applicable to functions $F(s)$ satisfying axioms of the Selberg class $S$ without imposing the Ramanujan hypothesis for the coefficients of $F(s)$. On the other hand, we also obtain necessary criteria for GRH which apply to functions in a subclass of the Selberg class, which we denote as $S^\ast$. Essentially, elements of $S^\ast$ have a polynomial Euler product and a functional equation involving Gamma functions evaluated on shifts of rational multiples of $s$. It is expected that this subclass does not exclude any important examples. In fact, $S^\ast$ is conjectured to be $S$. This result is stated as Theorem 2.4.

Some key features of our result are as follows. The number of $\Gamma$-factors of the type $\Gamma(\alpha s + \beta)(\alpha > 0)$ in the functional equation of $F(s)$, for which $\beta = 0$, plays an important role in our results. This parameter is denoted $j_F$ later in the paper and statements of our results are different depending on whether $j_F$ is zero or nonzero. Another feature is that we incorporate the degree of the $L$-function in a new way in order to generalize the kernel of (1.2). It is clear that the natural analog of $\mu(n)$ is the coefficient appearing in the Dirichlet series of $1/F(s)$, denoted by $b_F(n)$. However, the kernel $e^{-x/n^2}$ of (1.2) should be thought of as the function $\exp(-y^2)$ evaluated on $\sqrt{x}/n$. We generalize this to a function $Z_{a,\beta}(\sqrt{x}/n)^{d_F}$ (see (2.5) and (2.6)), where $d_F$ is the degree of the $L$-function $F(s)$ and $a, \beta$ are parameters explicitly arising from the functional equation of $F(s)$. The presence of the degree $d_F$ in the exponent of the argument is a key factor which was not obvious from existing results and plays a crucial role in our theorems. The function $Z_{a,\beta}(x)$ (see (2.5)) can essentially (up to a possible residue term) be viewed as an inverse Mellin transform of the product of Gamma factors arising in the functional equation of $F(s)$. Indeed, this is a natural and apt generalization if we interpret the function $\exp(-y^2)$ appearing in (1.2) as the inverse Mellin transform of the factor $\Gamma(s/2)$ which arises in the functional equation of $\zeta(s)$. Our methods rely on obtaining nontrivial estimates (up to a residue term) for $Z_{a,\beta}(x)$ as well as its derivative by exploiting the close connection between $Z_{a,\beta}(x)$ and the Meijer G-function. These are given in Lemmas 3.5 and 3.6.

Moreover, along the lines of [13], we introduce an additional parameter $z$ by inserting a cosh term. This allows for greater flexibility, enabling us to obtain a sufficient criterion for the case when all but finitely many zeros of $F(s)$ lie on the critical line. This result (Theorem 2.5) is valid for any element $F(s)$ satisfying assumptions of the Selberg class without necessarily satisfying the Ramanujan hypothesis. Our results which do not require the Ramanujan hypothesis on the coefficients of the $L$-function are thus valid for a larger class of $L$-functions, for instance, Artin $L$-functions and automorphic $L$-functions (associated with automorphic representations of $GL(n)$ over number fields).
The intuition behind such equivalent criteria for the Riemann hypothesis lies in what are known as identities of Ramanujan–Hardy–Littlewood type. In particular, the criterion (1.2) was motivated via a striking modular transformation obtained by Ramanujan, involving infinite series of the Möbius function \[22\] (see also [4, p. 468, Entry 37]). This identity was later corrected by Hardy and Littlewood [14, p. 156, Section 2.5] to give the following.

**Theorem 1.1** (Ramanujan–Hardy–Littlewood) Let \( \alpha \) and \( \beta \) be two positive numbers such that \( \alpha \beta = \pi \). Assume that the series \( \sum_{\rho} \left( \frac{\Gamma(1-\rho)}{\zeta(\rho)} \right) x^\rho \) converges for every positive real \( x \), where \( \rho \) runs through the nontrivial zeros of \( \zeta(s) \), and that the nontrivial zeros of \( \zeta(s) \) are simple. Then

\[
\sqrt{\alpha} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-\alpha^2/n^2} - \sqrt{\beta} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-\beta^2/n^2} = - \frac{1}{2\sqrt{\beta}} \sum_{\rho} \frac{\Gamma(1-\rho)}{\zeta(\rho)} \beta^\rho.
\]

Such identities can be viewed as encoding relations between arithmetical and analytic pieces of information about the relevant \( L \)-function. In this case, the left-hand side contains arithmetical information about \( \zeta(s) \) due to the Möbius function, whereas the right-hand side pertains to analytic information involving the nontrivial zeros of \( \zeta(s) \).

Showing the convergence of the series on the right-hand side is nontrivial. While it is widely believed that the series is rapidly convergent, what is currently known is that it is convergent under a certain bracketing of the terms. More precisely, this bracketing means that for some positive constant \( c \), the terms for which

\[
|\text{Im } \rho - \text{Im } \rho'| < \exp(-c \text{ Im } \rho / \log(\text{Im } \rho)) + \exp(-c \text{ Im } \rho'/ \log(\text{Im } \rho')),
\]

are included in the same bracket (see [25, p. 220]). It also turns out that convergence under such bracketing is enough to prove (1.3) [14, p. 158], [25, p. 220].

There exist several generalizations of (1.3) in various settings. In [12], Dixit, Roy, and Zaharescu obtained such an analog for Hecke forms. In [24], Roy, Zaharescu, and Zaki obtained a result of the type (1.3) where the Möbius function is replaced by a convolution of Dirichlet characters with the Möbius function. Further results of this kind have also been obtained by Dixit [7, 8], Kühn, Robles, and Roy [16], Dixit, Roy, and Zaharescu [13], Agarwal, Garg, and Mafi [1], Dixit, Gupta, and Vatwani [9], etc.

In this paper, we prove a very general such identity for \( L \)-functions satisfying axioms of the Selberg class other than the Ramanujan hypothesis. This result is stated as Theorem 2.1. By taking specific examples of \( L \)-functions in \( S \), we not only recover many existing such identities in the literature, but also obtain elegant new transformations involving special values of Meijer \( G \)-functions of the type \( G_n^{m,0} \), where \( n \in \mathbb{N} \). Some such consequences are given in Corollaries 2.2 and 2.3.

The paper is organized as follows. We first define the Selberg class of \( L \)-functions and then proceed to state our main results in Section 2. In Section 3, we give some preliminary lemmas which will be needed in our proofs. In Section 4, we prove Theorem 2.1, which is our analog of the Ramanujan–Hardy–Littlewood identity for the Selberg class. In Section 5, we give proofs of the transformations that arise as
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special cases of Theorem 2.1. Section 6 is devoted to proving Riesz-type criteria for the GRH for our general \( L \)-functions.

2 Statements of results

In 1991, Selberg introduced a general class \( S \) of Dirichlet series satisfying certain axioms. A Dirichlet series \( F(s) \) is defined to be in \( S \) if it satisfies the following conditions:

(i) (Dirichlet series) \( F(s) \) can be expressed as a Dirichlet series

\[
F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},
\]

which is absolutely convergent in the region Re\((s) > 1\), with \( a_F(1) = 1 \).

(ii) (Analytic continuation) There exists a nonnegative integer \( m \), such that \((s - 1)^m F(s)\) is an entire function of finite order.

(iii) (Functional equation) Using the notation \( \Phi(s) = \overline{\Phi(\bar{s})} \), \( F(s) \) satisfies a functional equation of the type

\[
\Phi(s) = \omega \bar{\Phi}(1 - s),
\]

where

\[
\Phi(s) = Q^s F(s) \prod_{i=1}^{q} \Gamma(\alpha_i s + \beta_i) = \gamma_F(s) F(s) \text{ (say)},
\]

where \( Q > 0 \), \( \alpha_i > 0 \), \( q \in \mathbb{N} \) and \( \beta_i \), \( \omega \) are complex numbers with Re\((\beta_i) \geq 0 \) and \(|\omega| = 1\).

(iv) (Euler product) For Re\((s) > 1\), we have

\[
F(s) = \prod_p F_p(s),
\]

where \( F_p(s) = \exp\left(\sum_{j=1}^{\infty} \frac{g(p^j)}{p^{\beta j}}\right) \), with \( g(p^j) \ll p^{j\theta} \) for some \( \theta < \frac{1}{2} \).

(v) (Ramanujan hypothesis) For every \( \epsilon > 0 \),

\[
a_F(n) \ll \epsilon n^\epsilon.
\]

Many well-known functions such as the Riemann zeta function, Dirichlet \( L \)-functions, Dedekind zeta functions associated with algebraic number fields, etc. are elements of the Selberg class. The function \( \gamma_F(s) \) in axiom (iii) is called the \( \gamma \)-factor of \( F \). This may not be unique, for instance, by application of the duplication formula for any of the \( \Gamma \)-functions involved. However, it is known that the \( \gamma \)-factor is unique up to a constant (see [5, Theorem 2.1]). The information in (iii) can be summarized by denoting \((Q, \alpha, \beta, w)\) as the data of \( F \), where \( \alpha \) and \( \beta \) are vectors given by \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q) \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_q) \). We will also use \( \hat{\beta} \) to denote \((\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_q) \). The data of \( F \) is thus not uniquely determined in general. One defines the degree \( d_F \) of \( F \in S \) as

\[
d_F = 2 \sum_{i=1}^{q} \alpha_i.
\]
It is well known that \(d_F\) is an invariant, that is, the degree of \(F\) is uniquely determined by the function \(F(s)\). We note that due to the Euler product (axiom (iv)), any \(F \in \mathcal{S}\) does not vanish in the region \(\Re(s) > 1\) (see, for instance, [5, Lemma 2.1]). Hence, there is an arithmetic function \(b_F(n)\) such that for \(\Re(s) > 1\),
\[
\frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{b_F(n)}{n^s}.
\]

In order to understand the analog of the Riemann hypothesis for \(F \in \mathcal{S}\), some discussion of zeros of \(F\) is necessary. The zeros of \(F\) in the region \(\Re(s) < 0\) are called trivial zeros. Using the functional equation, it is easy to see that the trivial zeros of \(F(s)\) are located at the poles of the \(\gamma\)-factors \(\gamma_F\). More precisely, the trivial zeros occur at
\[
\left\{ s = \frac{-m - \beta_i}{\alpha_i} : m \in \mathbb{N} \cup \{0\} \right\} \cap \{ s : \Re(s) < 0 \}.
\]
The other zeros of \(F\) occur inside the critical strip \(0 \leq \Re(s) \leq 1\). All such zeros except at \(s = 0\) are called the nontrivial zeros of \(F\). The case \(s = 0\) is more delicate. It is often possible for \(F\) to have a zero at \(s = 0\). Let
\[
k_F = \text{order of the pole of } F(s) \text{ at } s = 1.
\]
If we put \(s = 0\) into the functional equation of \(F\), then the \(\gamma\)-factors on either side are entire except for poles coming from factors of the type \(\Gamma(\alpha_is + \beta_i)\) whenever \(\beta_i\) happens to be zero. Let \(j_F\) be the number of components of \(\beta = (\beta_1, \ldots, \beta_q)\) which are zero, that is,
\[
(2.2) \quad j_F = \#\{1 \leq i \leq q : \beta_i = 0\}.
\]
In order to ensure that \(j_F\) is well defined for a given \(F \in \mathcal{S}\), we adopt the convention that \(\gamma_F\) is of the form \(Q^s \prod_{i=1}^{q} \Gamma(\alpha_i s + \beta_i)\), with \(q\) being the least such integer. That is, if \(\gamma_F' = Q^s \prod_{i=1}^{q'} \Gamma(\alpha_i' s + \beta_i')\) is any other admissible \(\gamma\)-factor for \(F\), then we have \(q' \geq q\). Thus, \(F(s)\) has a zero of order \(j_F - k_F\) at \(s = 0\). Moreover, note that for \(F\) satisfying axiom (ii) of \(\mathcal{S}\), we must have \(j_F \geq k_F\) since \(F\) cannot have a pole at \(s = 0\). For instance, for the Dedekind zeta function \(\zeta_K(s)\), associated with a number field \(K\), one sees that \(k_F = 1\), while \(j_F = r_1 + r_2\), where \(r_1\) and \(2r_2\) are the number of real and imaginary embeddings of \(K\) into \(\mathbb{C}\), respectively. Thus, \(\zeta_K(s)\) has a zero at \(s = 0\), whenever \(r_1 + r_2 - 1 > 0\). The GRH conjectures that all the nontrivial zeros of \(F\) lie on the critical line \(\Re(s) = \frac{1}{2}\).

We now set up some notation. The line integral \(\int_{c-i\infty}^{c+i\infty} F(s) ds\) will be denoted as \(\int_{(c)}\). We also define a parameter \(c_F\) depending on the data of \(F\) as follows:
\[
(2.4) \quad c_F = \begin{cases} 
\min_{1 \leq i \leq q} \left\{ \frac{1}{\Re(\beta_i)} \right\}, & \text{if } \Re(\beta_i) = 0 \forall 1 \leq i \leq q, \\
\min_{1 \leq i \leq q} \left\{ \frac{\Re(\beta_i)}{\alpha_i} \right\}, & \text{otherwise}.
\end{cases}
\]
It is clear that there are no nontrivial zeros of \(F\) to the right of \(\Re(s) = -c_F\). Define for
\(-c_F < c < 0\) and \(x > 0\),
(2.5) \[ Z_{\alpha, \beta}(x) := \frac{1}{2\pi i} \int_{(c)} \prod_{i=1}^{q} \Gamma (\alpha_i s + \beta_i) x^{-s} ds, \]

and

(2.6) \[ \mathcal{P}_{\alpha, \beta, z}(y) := \sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{\alpha, \beta} \left( \frac{n}{y} \right) \cosh \left( \frac{\sqrt{y} z}{n} \right), \]

for \( y > 0 \) and \( z \in \mathbb{C} \). The exponent \( d_F \) introduced here in the definition of the kernel \( \mathcal{P}_{\alpha, \beta, z}(y) \) will play a key role in what follows.

Our first result is a modular relation for functions \( F \) in the Selberg class. We will denote the derivative of \( F(s) \) with respect to \( s \) by \( F'(s) \).

**Theorem 2.1** Let \( F \) satisfy axioms (i)–(iv) of the class \( \mathcal{S} \). For every positive real \( x \), assume the convergence of the series

\[ \sum_{\rho} \frac{\prod_{i=1}^{q} \Gamma (\alpha_i(1- \rho) + \beta_i)}{F'(\rho)} x^\rho, \]

where \( \rho \) runs through the nontrivial zeros of \( F(s) \). Suppose that each nontrivial zero of \( F(s) \) is simple. Let \( \eta, \nu > 0 \) be such that \( \eta \nu = \frac{1}{2\pi} \). Let \( r = j_F - k_F \), where \( k_F, j_F \) are as defined in (2.2), (2.3), respectively. Then, for \( r > 0 \), we have

(2.7) \[ \omega \sqrt{\eta} \sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{\alpha, \beta} \left( \frac{n}{\eta} \right) - \sqrt{\nu} \sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{\alpha, \beta} \left( \frac{\nu}{n} \right) = - \frac{1}{\sqrt{\nu}} \sum_{\rho} \prod_{i=1}^{q} \Gamma (\alpha_i(1- \rho) + \beta_i) \left( \frac{\eta}{\rho} \right)^r \frac{\prod_{i=1}^{q} \Gamma (\alpha_i(1-s) + \beta_i)}{F'(\tilde{s})} \left| s=1 \right. \]

\[ - \frac{1}{\sqrt{\nu}} \sum_{\rho} \prod_{i=1}^{q} \Gamma (\alpha_i(1-s) + \beta_i) \left| s=0 \right. \]

If \( r = 0 \), we have the following more simplified relation:

(2.8) \[ \omega \sqrt{\eta} \sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{\alpha, \beta} \left( \frac{n}{\eta} \right) - \sqrt{\nu} \sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{\alpha, \beta} \left( \frac{\nu}{n} \right) = - \frac{1}{\sqrt{\nu}} \sum_{\rho} \prod_{i=1}^{q} \Gamma (\alpha_i(1- \rho) + \beta_i) \left( \frac{\eta}{\rho} \right)^r \frac{\prod_{i=1}^{q} \Gamma (\alpha_i(1-s) + \beta_i)}{F'(\tilde{s})} \left| s=0 \right. \]

A straightforward consequence of Theorem 2.1 is that the classical result of Ramanujan, Hardy, and Littlewood follows as a special case since, for \( F(s) = \zeta(s) \), we have \( Z_{\alpha, \beta}(x) = Z_{(1/2), (0)}(x) \), which equals 2\((e^{-x^2} - 1)\). Moreover, when \( F \) is the Dedekind zeta function of a number field \( \mathbb{K} \), it is easy to check that

\[ \alpha = \left( \frac{1}{2}, \ldots, \frac{1}{2}, 1, \ldots, 1 \right), \quad \beta = (0, \ldots, 0) \quad \text{and} \quad Z_{\alpha, \beta}(x) = \frac{1}{\sqrt{2\pi i}} \int_{(c)} \Gamma(r_1) \left( \frac{s}{2} \right)^r_2 \Gamma(s) x^{-s} ds, \]

where \(-\frac{1}{2} < \epsilon < 0\) and \( r_1, 2r_2 \) are the number of real and complex embeddings of \( \mathbb{K} \), respectively. It thus follows that a special case of Theorem 2.1 yields Theorem 1.3 and Corollary 3.2 of [9].
We now state more nontrivial consequences of Theorem 2.1 among which are certain elegant transformations involving special values of the Meijer G-function (defined in (3.3)). In particular, special values of Meijer G-functions of the type $G^{n,0}_{0,n}$ for $n = 1, 2, 3, \ldots$ come into play. Some interesting such special values involve the modified Bessel function of the second kind, which we proceed to define below.

The Bessel function of the first kind of order $\nu$ is defined by [26, p. 40]

$$J_{\nu}(z) := \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\nu}}{m! \Gamma(m+1+\nu)} \quad (z \in \mathbb{C}).$$

The modified Bessel functions of the first and second kinds of order $\nu$ are defined by [26, pp. 77–78]

$$I_{\nu}(z) := \begin{cases} e^{-i \pi \nu} I_{\nu}(e^{i \pi} z), & \text{if } -\pi < \arg z \leq \frac{\pi}{2}, \\ e^{i \pi \nu} I_{\nu}(e^{i \pi} z), & \text{if } \frac{\pi}{2} < \arg z \leq \pi, \end{cases}$$

$$K_{\nu}(z) := \frac{\pi}{2} \frac{\Gamma(1-\nu) - I_{\nu}(z)}{\sin \nu \pi},$$
respectively, with $K_{\nu}(z)$ defined by $\lim_{\nu \to -\nu} K_{\nu}(z)$ if $\nu$ is an integer.

We state some special values of the Meijer G-function which are relevant in the context of Theorem 2.1:

$$(2.8) \quad G^{4,0}_{0,4} \left( \begin{array}{c} b + \frac{1}{4} \bar{b} + \frac{2}{4} b + \frac{3}{4} \bar{b} \\ 2 \end{array} \bigg| z \right) = \sqrt{2} \pi \frac{\pi}{4} e^{-4\pi^2/4},$$

$$G^{4,0}_{0,4} \left( \begin{array}{c} b + \frac{1}{2} b - c \bar{c} \\ 2 \end{array} \bigg| z \right) = 8 \sqrt{\pi} b K_{2\nu-2b} \left( 2 \sqrt{2} \left( -z \right)^{1/4} \right) K_{2\nu-2b} \left( \frac{2 \sqrt{2} \sqrt{-z}}{(-z)^{1/4}} \right).$$

$(2.9)$

Using the identity (2.8), one can derive the following transformation. We give the proof in Section 5.

**Corollary 2.2** Let $F \in \mathcal{S}$ with data $(Q, \alpha, \beta, \omega)$, where $\alpha = (4)$ and $\beta = (\beta_1)$. For every positive real $x$, assume the convergence of the series

$$\sum_\rho \frac{\Gamma(4(1-\hat{\rho}) + \beta_1)}{F'(\rho)} x^{\hat{\rho}}.$$ 

where $\rho$ runs through the nontrivial zeros of $F(s)$. Suppose that each nontrivial zero of $F(s)$ is simple. Let $\eta, \nu > 0$ be such that $\eta \nu = \frac{1}{\beta_1}$. For the case $\beta_1 = 0$, we let $r = 1 - k_F$, where $k_F$ is as defined in (2.2). Then, for $r > 0$, we have

$$(2.10) \quad \frac{\omega \sqrt{\eta}}{4} \sum_{n=1}^{\infty} \frac{b_F(n)}{n} \left( e^{-\left( \frac{n}{4} \right)^{1/4}} - 1 \right) - \frac{\sqrt{v}}{4} \sum_{n=1}^{\infty} \frac{b_F(n)}{n} \left( e^{-\left( \frac{n}{4} \right)^{1/4}} - 1 \right) = -\frac{1}{\sqrt{v}} \sum_\rho \frac{\Gamma(4(1-\hat{\rho}))}{F'(\rho)} \nu^{\hat{\rho}} - \frac{1}{\sqrt{v}(r-1)!} \frac{d^{r-1}}{ds^{r-1}}(s-1)^r \frac{\Gamma(4(1-s))}{F(s)} \nu^s \bigg|_{s=1}$$

and

$$(2.10) \quad -\frac{1}{\sqrt{v}(r-1)!} \frac{d^{r-1}}{ds^{r-1}} \frac{\Gamma(4(1-s))}{F(s)} \nu^s \bigg|_{s=0}.$$
When \( r = 0 \), the same identity holds without the last two residue terms appearing. When \( \beta_1 \neq 0 \), we let \( r = -k_F \). Then \( r = 0 \) is the only possibility. We then have

\[
\frac{\omega \sqrt{n}}{4} \sum_{n=1}^{\infty} \frac{b_F(n)}{n} \left( \frac{\eta}{n} \right) \beta_{1/4} e^{-\left( \frac{\eta}{n} \right)^{1/4}} - \frac{\sqrt{v}}{4} \sum_{n=1}^{\infty} \frac{b_F(n)}{n} \left( \frac{v}{n} \right) \beta_{1/4} e^{-\left( \frac{v}{n} \right)^{1/4}} = -\frac{1}{v} \sum_{\rho} \frac{\Gamma \left( 4(1-\rho) + \beta_1 \right)}{F'(\rho)} \nu^{\rho}.
\]

(2.11)

The identity (2.9) yields the following transformation involving the modified Bessel function of the second kind.

**Corollary 2.3** Let \( F \in S \) with data \((Q, (1,1,1), (0, 1/2,-C,C), \omega)\), where \( C \) is a nonzero real constant. For every positive real \( x \), assume the convergence of the series

\[
\sum_{\rho} \frac{\Gamma(1-\rho) \Gamma \left( \frac{3}{2} - \rho \right) \Gamma(1-C-\rho) \Gamma(1+C-\rho)}{F'\rho} x^{\rho},
\]

where \( \rho \) runs through the nontrivial zeros of \( F(s) \). Suppose that each nontrivial zero of \( F(s) \) is simple. Let \( \eta, \nu > 0 \) be such that \( \eta \nu = \frac{1}{Q^2} \). Let \( r = 1 - k_F \), where \( k_F \) is as defined in (2.2). Then, for \( r > 0 \), we have

\[
\frac{\omega \sqrt{n}}{4} \sum_{n=1}^{\infty} \frac{b_F(n)}{n} \left( 8\sqrt{\pi} K_{2C} \left( 2\sqrt{2} \left( -\frac{\eta}{n} \right)^{1/4} \right) K_{2C} \left( \frac{2\sqrt{2}\sqrt{\pi}}{\left( -\frac{\eta}{n} \right)^{1/4}} \right) + \frac{\pi \sqrt{\pi}}{C \sin \pi C} \right)
\]

\[
-\frac{\sqrt{v}}{4} \sum_{n=1}^{\infty} \frac{b_F(n)}{n} \left( 8\sqrt{\pi} K_{2C} \left( 2\sqrt{2} \left( -\frac{\nu}{n} \right)^{1/4} \right) K_{2C} \left( \frac{2\sqrt{2}\sqrt{\pi}}{\left( -\frac{\nu}{n} \right)^{1/4}} \right) + \frac{\pi \sqrt{\pi}}{C \sin \pi C} \right)
\]

\[
= -\frac{1}{v} \sum_{\rho} \frac{\Gamma(1-\rho) \Gamma \left( \frac{3}{2} - \rho \right) \Gamma(1-C-\rho) \Gamma(1+C-\rho)}{F'\rho} \nu^{\rho}
\]

\[
-\frac{1}{v(r-1)!} \frac{d^{r-1}}{ds^{r-1}} \left| \Gamma(1-s) \Gamma \left( \frac{3}{2} - s \right) \Gamma(1-C-s) \Gamma(1+C-s) \right|_{s=1} \nu^{s}.
\]

(2.12)

When \( r = 0 \), the same identity holds without the last two residue terms appearing.

Similarly, the identities (2.13) given below can be used to recover Corollaries 3.4 and 3.3 of [9], respectively.

\[
G_{0,1}^{1,0} \left( \frac{b}{z} \right) = e^{-z} z^b,
\]

\[
G_{0,2}^{2,0} \left( \frac{b}{z} \left( b + \frac{1}{2} \right) z \right) = \sqrt{\pi} z^b e^{-2\sqrt{z}}.
\]

(2.13)
It is also possible to obtain more such transformations involving special values of $G_{0,n}^{n,0}$ for other values of $n$, for instance, using any of the following identities:

\[
G_{0,2}^{2,0} \left( \begin{array}{c} b \\ c \end{array} \mid z \right) = 2z^{1/2}(b+c)K_{b-c}(2\sqrt{z}), \quad G_{0,3}^{3,0} \left( \begin{array}{c} b b + \frac{1}{3} b + \frac{2}{3} \end{array} \mid z \right) = \frac{2\pi}{\sqrt{3}} z^{b} e^{-3z^{1/3}},
\]

\[
G_{0,5}^{5,0} \left( \begin{array}{c} b b + \frac{1}{3} b + \frac{2}{3} b + \frac{3}{3} \end{array} \mid z \right) = \frac{4\pi^2}{\sqrt{5}} z^{b} e^{-5z^{1/5}}.
\]

Since the proofs follow analogously to those of Corollaries 2.2 and 2.3, we do not delve into them here.

Our next main result gives Riesz-type criteria for the GRH, applicable to a general class of $L$-functions. Before stating this, we make some remarks about certain additional conditions that we need to impose on our $L$-function $F(s)$.

The first such condition pertains to (iv) of the Selberg class axioms. From the perspective of known examples, in particular, automorphic $L$-functions, it is natural to restrict $\frac{1}{F(p,s)}$ to be a polynomial in $p^{-s}$ of degree independent of $p$ (see, for instance, [5]). Axiom (iv) may thus be strengthened to axiom (iv)$'$ given below. We say that $F$ has polynomial Euler product if it satisfies the following:

(iv)$'$ (Polynomial Euler product) For $\text{Re}(s) > 1$, we have

\[
F(s) = \prod_p \prod_{i=1}^m \left( 1 - \frac{y_i(p)}{p^s} \right)^{-1},
\]

where $y_i(p)$ are complex numbers. It is conjectured that all $L$-functions in the Selberg class have a polynomial Euler product. It is well known that under (iv)$'$, the axiom (v) is equivalent to the bound

\[
(2.14) \quad |y_i(p)| \leq 1,
\]

for all primes $p$ and $i = 1, \ldots, m$ (see, for instance, [2, p. 347]).

In all known examples of $L$-functions from $S$, it is possible to find a normalization such that in axiom (iii), one has all $\alpha_i$’s to be rational numbers. In fact, Conrey and Ghosh observed that one could take $\alpha_i = \frac{1}{2}$ for all $i$ (cf. [5, p. 12]). We thus formulate a subclass of functions called $S^*$ satisfying axioms (i)-(iii), (iv)$'$, (v), with the additional restriction that in the data $(Q, \alpha, \beta, w)$ of $F$, all the components of $\alpha$ are rational numbers.

We do not appear to exclude any known examples of important $L$-functions under these new hypotheses. Indeed, it is conjectured that $S^* = S$. Our next result gives Riesz-type criteria for the GRH in this setting.

**Theorem 2.4** Let $F$ satisfy axioms (i)-(iv) of the class $S$. Let $j_F$ be as defined in (2.3). Then the following hold.

(a) If $F \in S^*$ and $j_F = 0$, then the GRH for $F(s)$ implies that $P_{\alpha,\beta,\varepsilon}(y) = O_F,\delta,\varepsilon \left( y^{-\varepsilon+\delta} \right)$ as $y \to \infty$, for any $\delta > 0$. 
Let $\epsilon > 0$. If $F \in S^*$ and $j_F \neq 0$, then the GRH for $F(s)$ implies that

\[
\mathcal{P}_{\alpha, \beta, 0}(y) + \sum_{n=1}^{[y^{1+\epsilon}]-1} \frac{b_F(n)}{n} \text{Res}_{s=0} \prod_{i=1}^{\alpha} \Gamma \left( \alpha_i s + \beta_i \right) \left( \frac{\sqrt{\gamma}}{n} \right)^{-d_F s} \ll_{F, \delta} y^{-\frac{1}{4} + \delta}
\]

as $y \to \infty$ for any $\delta > 0$.

(c) The estimate $\mathcal{P}_{\alpha, \beta, 0}(y) = O_{F, \delta} \left( y^{-\frac{1}{4} + \delta} \right)$ as $y \to \infty$ for any $\delta > 0$ implies the GRH for $F(s)$.

Part (c) is valid for the larger class of functions satisfying axioms (i)–(iv) of the Selberg class. We remark that our result gives an equivalent condition for the GRH when $F \in S^*$ satisfies $j_F = 0$, namely, we show that, in this case, GRH is equivalent to the estimate $\mathcal{P}_{\alpha, \beta, 0}(y) \ll_{F, \delta} \left( y^{-\frac{1}{4} + \delta} \right)$. The role of $j_F$ (and hence the data of $F$) is a new feature of our result, and has not been recorded explicitly before.

Until this point, we have not made use of the flexibility offered by the parameter $z$ in the definition (2.6) of $\mathcal{P}_{\alpha, \beta, z}(y)$. This parameter allows us to give a criterion for an “almost” GRH phenomenon as follows. This is a generalization of Theorem 1.1(b) of [13].

**Theorem 2.5** Let $F$ satisfy axioms (i)–(iv) of the class $S$. If $z \neq 0$ and $\arg z \neq \pm \frac{\pi}{2}$, then the estimate $\mathcal{P}_{\alpha, \beta, z}(y) = O_{F, \delta} \left( y^{-\frac{1}{4} + \delta} \right)$, as $y \to \infty$ for any $\delta > 0$ implies that $F(s)$ has at most finitely many nontrivial zeros of the critical line.

**Remark** We note that the Ramanujan bound (axiom (v)) for $F(s)$ will be used only in the proofs of parts (a) and (b) of Theorem 2.4. In particular, it is used to conclude that GRH implies square root cancellation in the summatory function of $b_F(n)$, which is in turn used to derive the estimate (6.2).

### 3 Preliminary results

Throughout this section, let $F \in S$. Let us consider

\[
N_F(T) := \# \{ s : F(s) = 0, 0 < \text{Re}(s) < 1 \text{ and } |\text{Im}(s)| < T \}.
\]

It is well known that as $T \to \infty$, $N_F(T)$ satisfies the asymptotic formula (see [20, p. 264])

\[
N_F(T) = \frac{d_F}{\pi} T \log T + CT + O(\log T),
\]

where $d_F$ is the degree of $F$ and $C = C(F)$ is some constant. This, in particular, gives

\[
N_F(T + 1) - N_F(T) \ll \log T.
\]

From (3.1), one can conclude that the number of zeros of $F(s)$ in the critical strip between the horizontal lines $\text{Im}(s) = T + 1$ and $\text{Im}(s) = T - 1$ is $O(\log T)$.

Another result that will be used repeatedly in our proofs is the following well-known estimate for the Gamma function.
Lemma 3.1 (Stirling's formula, [6]) For $s = \sigma + it$ with $C \leq \sigma \leq D$, we have as $|t| \to \infty$,

\begin{equation}
|\Gamma(s)| = (2\pi)^{\frac{1}{2}} |t|^\sigma \frac{1}{2} e^{-\frac{1}{2} \pi |t|} \left(1 + O \left( \frac{1}{|t|} \right) \right).
\end{equation}

The Meijer G-function is defined as [18, p. 143]

\begin{equation}
G_{p,q}^{m,n}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds,
\end{equation}

where $L$ is a curve from $-i\infty$ to $i\infty$ which separates the poles of the factors $\Gamma(b_j - s)$ from those of the factors $\Gamma(1 - a_j + s)$. For more details, we refer the reader to [18, p. 143].

Recall the definitions (2.5) and (2.6) of $Z_{a,b}(x)$ and $P_{a,b,z}(y)$, respectively. We will prove some preliminary results about these functions.

Lemma 3.2 Let $0 < \text{Re}(s) < \frac{1}{2}$, $\alpha = (\alpha_1, \ldots, \alpha_q)$, $\beta = (\beta_1, \ldots, \beta_q)$, with $\alpha_i > 0$ and $\text{Re}(\beta_i) > 0$ for each $i$. Then

\begin{equation}
\int_0^\infty y^{-s-1} P_{a,b,z}(y) dy = \frac{2}{d_F F(2s + 1)} \sum_{i=0}^{\infty} \frac{z^{2t}}{2^t} \prod_{i=1}^q \frac{2\alpha_i}{d_F} (t - s) + \beta_i.
\end{equation}

Proof Let

$$\phi(s, \alpha, \beta, z) = \int_0^\infty y^{-s-1} P_{a,b,z}(y) dy.$$ 

Substituting $y$ by $\frac{x}{n^2}$, we have

$$n^{-2s-1} \phi(s, \alpha, \beta, z) = \int_0^\infty \frac{x^{-s-1}}{n} P_{a,b,z} \left( \frac{x}{n^2} \right) dx.$$ 

Multiplying both sides by $a_F(n)$ and summing over all $n$, we obtain from (2.6)

\begin{align*}
F(2s + 1) \phi(s, \alpha, \beta, z) &= \sum_{n=1}^{\infty} \int_0^\infty \frac{x^{-s-1} a_F(n)}{n} \sum_{m=1}^{\infty} \frac{b_F(m)}{m} Z_{a,\beta} \left( \left( \frac{\sqrt{x}}{mn} \right)^{d_F} \cosh \left( \frac{\sqrt{zx}}{mn} \right) \right) dx \\
&= \sum_{n=1}^{\infty} \int_0^\infty \frac{x^{-s-1} a_F(n)}{n} \sum_{m=1}^{\infty} \frac{b_F(m)}{m} \int_0^\infty \frac{z^{2t} x^t}{m^{2t} n^{2t} (2t)!} \frac{1}{2 \pi i} \\
&\quad \times \int_{(c)} \prod_{i=1}^{q} \Gamma \left( \alpha_i s' + \beta_i \right) \left( \frac{\sqrt{x}}{mn} \right)^{-d_F s'} \frac{d s'}{d s'} dx,
\end{align*}

using the series expansion for $\cosh(x)$. Applying the Weierstrass $M$-test and the Lebesgue dominated convergence theorem, one can show that
Lemma 3.3. \( F(2s + 1) \phi(s, \alpha, \beta, z) \)

\[
\sum_{i=0}^{\infty} \frac{z^{2i}}{(2i)!} \int_0^\infty x^{-s+t-1} \frac{1}{2\pi i} \int \prod_{i=1}^q \Gamma (\alpha_i s' + \beta_i) \sum_{n=1}^{\infty} \frac{a_F(n)}{n^{2t+1-dFs'}} (\sqrt{x})^{-dFs'} ds' dx
\]

\[
= \sum_{i=0}^{\infty} \frac{z^{2i}}{(2i)!} \int_0^\infty x^{-s+t-1} \frac{1}{2\pi i} \int \prod_{i=1}^q \Gamma (\alpha_i s' + \beta_i) (\sqrt{x})^{-dFs'} ds' dx.
\]

(3.5)

Substituting \( s' \) by \( \frac{-2w}{d_F} \), we have

\[
\frac{1}{2\pi i} \int_0^\infty x^{-s+t-1} \int \prod_{i=1}^q \Gamma (\alpha_i s' + \beta_i) x^{-\frac{dFs'}{2}} ds' dx
\]

\[
= \frac{2}{d_F 2\pi i} \int_0^\infty x^{-s+t-1} \int \prod_{i=1}^q \Gamma \left( \frac{2w}{d_F} + \beta_i \right) x^w dw dx
\]

\[
(3.6) = \frac{2}{d_F} \prod_{i=1}^q \Gamma \left( \frac{2\alpha_i}{d_F} (t-s) + \beta_i \right).
\]

upon replacing \( x \) by \( \frac{1}{2} \) and using the Mellin inversion theorem [19, pp. 341–343]. Combining (3.5) and (3.6) completes the proof.

\[\left(\square\right)\]

Lemma 3.3. For \( -c_F < c < 0 \) and \( x > 0 \), we have

\[ Z_{\alpha, \beta}(x) \ll_F x^{-c}. \]

Proof. Let us write \( \tilde{\beta}_i = a_i + ib_i \). Using the functional equation of \( \Gamma \), we have from (2.5),

\[
Z_{\alpha, \beta}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{i=1}^q \Gamma (\alpha_i s + \beta_i + 1)}{\prod_{i=1}^q \Gamma (\alpha_i s + \beta_i)} x^{-s} ds
\]

\[
= \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\prod_{i=1}^q \Gamma (\alpha_i c + a_i + 1 + i(a_i t - b_i))}{\prod_{i=1}^q \Gamma (\alpha_i c + a_i + i(a_i t - b_i))} x^{-c-it} dt.
\]

It can be seen that

\[
1 + |t| \ll_F |\alpha_i c + a_i| + |\alpha_i t - b_i| \ll_F |\alpha_i c + a_i + i(\alpha_i t - b_i)|,
\]

using the Cauchy–Schwarz inequality. Using this for the denominator and Stirling’s formula for the numerator, we get

\[
Z_{\alpha, \beta}(x) \ll_F \int_{-\infty}^{\infty} \prod_{i=1}^q \left( |\alpha_i t - b_i|^{\alpha_i c + a_i + \frac{1}{2} e^{\frac{2}{3}|\alpha_i t - b_i|}} \right) \frac{1}{(1 + |t|)^q} x^{-c} dt.
\]

Noting that the exponent \( \alpha_i c + a_i + \frac{1}{2} \) is positive in the region \( -c_F < c < 0 \) due to the choice of \( c_F \) (see (2.4)), it is easy to see that \( Z_{\alpha, \beta}(x) \ll_F x^{-c} \), as needed. \[\left(\square\right)\]
Lemma 3.4 For $-c_F < c < 0$ and $x > 0$, we have
\[ Z_{a,b}'(x) \ll_F x^{-c-1}, \]
where $Z_{a,b}'(x)$ denotes the derivative of $Z_{a,b}(x)$ with respect to its argument $x$.

Proof The proof follows in exactly the same manner as that of Lemma 3.3. \hfill \blackslug

For $x \in \mathbb{R}$ and $1 < d < 1 + c_F$, we define the following function which is closely related to the function $Z_{a,b}(x)$ defined in (2.5). Let
\[
\tilde{Z}_{a,b}(x) := \frac{1}{2\pi i} \int_{(d)} \prod_{i=1}^{q} \Gamma \left( \alpha_i s + \tilde{\beta}_i \right) x^{-s} ds.
\]

The advantage of using $\tilde{Z}_{a,b}(x)$ is that one is able to obtain better upper bounds for this function by relating it to the Meijer $G$-function. Moreover, $\tilde{Z}_{a,b}(x)$ differs from $Z_{a,b}(x)$ only by a residue term. More precisely, using Cauchy's residue theorem to shift the line of integration from $c$ to $d$, we have
\[
Z_{a,b}(x) = \tilde{Z}_{a,b}(x) - \text{Res}_{s=0} \prod_{i=1}^{q} \Gamma \left( \alpha_i s + \tilde{\beta}_i \right) x^{-s}.
\]

In what follows, we show exponential decay as $x \to \infty$ for both $\tilde{Z}_{a,b}(x)$ as well as $\tilde{Z}_{a,b}'(x)$ (its derivative with respect to $x$). These results are used when $\tilde{Z}_{a,b}$ arises from the data of a function $F$ in the class $S^*$ defined preceding Theorem 2.4.

Lemma 3.5 Let $\alpha = (\alpha_1, \ldots, \alpha_q)$, $\beta = (\beta_1, \ldots, \beta_q)$, with $\alpha_i > 0$ and $\text{Re}(\beta_i) > 0$ for all $i$. Assume that each component of $\alpha$ is rational, say $\alpha_i = \frac{m_i}{n_i}$ with $(m_i, n_i) = 1$. Let $L$ be the lcm $[n_1, \ldots, n_q]$. Denote the positive integer $\alpha_i L$ by $k_i$. Using the notation
\[
C_1 = \frac{L d_F}{2 \left( \prod_{i=1}^{q} k_i \right)^{2/d_F}}, \quad C_2 = \frac{2}{d_F}, \quad C_3 = \frac{1-q}{2} + \sum_{i=1}^{q} \tilde{\beta}_i,
\]
we have as $x \to \infty$,
\[
\tilde{Z}_{a,b}(x) \ll_F \exp \left( -C_1 x^{C_2} \right) x^{C_1 C_3}.
\]

Proof We have from the definition
\[
\tilde{Z}_{a,b}(x) = \frac{1}{2\pi i} \int_{(d)} \prod_{i=1}^{q} \Gamma \left( \alpha_i s + \tilde{\beta}_i \right) x^{-s} ds,
\]
where $d > 0$. The change of variable $s = Lz$ gives
\[
\tilde{Z}_{a,b}(x) = \frac{L}{2\pi i} \int_{(d/L)} \prod_{i=1}^{q} \Gamma \left( k_i z + \tilde{\beta}_i \right) x^{-z} dz.
\]

Using the Gauss multiplication formula, we have
\[
\Gamma \left( k_i \left( z + \frac{\tilde{\beta}_i}{k_i} \right) \right) = (2\pi)^{\frac{1}{2} - k_i} k_i \left( z + \frac{\tilde{\beta}_i}{k_i} \right)^{-\frac{1}{2} - k_i} \prod_{j=1}^{k_i} \Gamma \left( z + \frac{\tilde{\beta}_i}{k_i} + j - 1 \right) k_i.
\]
Thus, \( \tilde{f}_{\alpha,\beta}(x) \) equals

\[
\frac{L}{2\pi i} (2\pi)^{\frac{q-1}{2}} \Sigma_{i=1}^{q} k_i \left( \prod_{i=1}^{q} k_i^\beta_i \right) \int \left( \frac{x^L}{\Pi_i^{q} k_i^{k_i}} \right)^{-z} \prod_{i=1}^{q} k_i \Gamma \left( z + \frac{\beta_i}{k_i} + \frac{j-1}{k_i} \right) \frac{dz}{(d/L)}.
\]

The above expression agrees with that for a Meijer G-function. In particular, since \( d/L > 0 \), the line of integration does separate the poles as prescribed in (3.3). Putting

\[
(3.9) \quad b_{ij} = \frac{\beta_i}{k_i} + \frac{j-1}{k_i}, \quad 1 \leq j \leq k_i, \ 1 \leq i \leq q,
\]

and \( w = \frac{x^L}{\Pi_i^{q} k_i^{k_i}} \), we find that

\[
\tilde{f}_{\alpha,\beta}(x) = L(2\pi)^{\frac{q-1}{2}} \Sigma_{i=1}^{q} k_i \left( \prod_{i=1}^{q} k_i^\beta_i \right) \frac{1}{2\pi i} \int \frac{1}{w^{\frac{q}{k_i}} \prod_{i=1}^{q} k_i \Gamma \left( z + b_{ij} \right)} \frac{dz}{(d/L)} = L(2\pi)^{\frac{q-1}{2}} \Sigma_{i=1}^{q} k_i \left( \prod_{i=1}^{q} k_i^\beta_i \right) G_{\Sigma k_i, 0} \left( b_{ij} \right) w.
\]

(3.10)

We now use an asymptotic relation for the Meijer G-function from [18, p. 180]. One can compute that in our case, \( \sigma = \Sigma_{k=1}^{q} k_i \) and \( \theta = \frac{1}{2} \left( \Sigma_{i=1}^{q} \beta_i + \frac{1+q}{2} \right) \), where \( \sigma \) and \( \theta \) are defined as per the notation on page 180 of [18]. We get

\[
\tilde{f}_{\alpha,\beta}(x) \ll_F \frac{G_{\Sigma k_i, 0} \left( b_{ij} \right) w}{\sigma^{\frac{q}{k_i}}} \exp \left( -\sum_{i=1}^{q} k_i w^{1/\Sigma k_i} \right) w^\theta
\]

(3.11)

\[
\ll_F \exp \left( -\sum_{i=1}^{q} k_i \left( \frac{x^L}{\Pi_i^{q} k_i^{k_i}} \right)^{1/\Sigma k_i} \right) x^{L \left( \frac{\Sigma_{i=1}^{q} \beta_i + \frac{1+q}{2}}{\Sigma k_i} \right)}.
\]

Observing that

\[
\sum_{i=1}^{q} k_i = \sum_{i=1}^{q} \alpha_i L = \frac{L d_F}{2},
\]

we complete the proof.

\[\blacksquare\]

**Lemma 3.6** Let \( \alpha = (\alpha_1, \ldots, \alpha_q) \), \( \beta = (\beta_1, \ldots, \beta_q) \), with \( \alpha_i > 0 \) and \( \Re(\beta_i) > 0 \) for all \( i \). Assume that each component of \( \alpha \) is rational, say \( \alpha_i = \frac{m_i}{n_i} \) with \( (m_i, n_i) = 1 \). Let \( L \) be the lcm \( \left[ n_1, \ldots, n_q \right] \). Denoting the positive integer \( \alpha_i \) by \( k_i \), we define \( C_i \)’s as in (3.8). Then as \( x \to \infty \), we have

\[
\tilde{f}'_{\alpha,\beta}(x) \ll \exp \left( -C_1 x^{C_2} \right) x^{C_3 C_4} C_4,
\]

where \( C_4 = \max \{ 0, C_2 - 1 \} \).
Proof  We will use the identity (12) from Luke [18, p. 151], which gives after some simplification

\[ (3.12) \]

\[
\frac{d}{dz} \left( G_0^{\Sigma k_i, 0} \left( \frac{\cdot}{b_{i,j}} \right) \right) = \frac{b_{11}}{z} G_0^{\Sigma k_i, 0} \left( \frac{\cdot}{b_{i,j}} \right) - \frac{1}{z} G_0^{\Sigma k_i, 0} \left( b_{11} + \frac{\cdot}{b_{i,j}} \right),
\]

where the \( b_{i,j} \)'s are as defined in (3.9) and \( b^*_{i,j} \) denotes the same sequence as \( b_{i,j} \) without the first term \( b_{11} \). From (3.10), letting \( z = \frac{x^s}{\prod b_{i,j}^{k_i}} \) and \( L_1 = L(2\pi)^{\frac{3}{2}} \sum_i k_i \left( \prod_{i=1}^q k_i \right) \), we have

\[
\Delta'_{\alpha, \beta}(x) = L_1 \frac{d}{dz} \left( G_0^{\Sigma k_i, 0} \left( \frac{\cdot}{b_{i,j}} \right) \right) \frac{dz}{dx}
\]

\[
= \frac{Lx^{L-1} b_{11}}{\prod b_{i,j}^{k_i}} \frac{1}{z} L_1 G_0^{\Sigma k_i, 0} \left( \frac{\cdot}{b_{i,j}} \right) - \frac{Lx^{L-1} \frac{1}{z} L_1 G_0^{\Sigma k_i, 0} \left( b_{11} + \frac{\cdot}{b_{i,j}} \right)},
\]

using (3.12). For the first term above, we use (3.10) followed by Lemma 3.5. For the second term, following the notation of [18, p. 180], the parameter \( \sigma = \sum k_i \) while \( \theta \) takes the value \( \frac{1}{\sum b_{i,j}} \left( \sum b_{i,j} + 1 + \frac{\frac{1}{z} - \frac{\frac{1}{z}}{2}} {2} \right) \). We then apply the estimate (3.11). We have thus obtained

\[
\Delta'_{\alpha, \beta}(x) \ll_F \left( \frac{\beta_i}{k_i} \exp \left( -C_1 x C_2 \right) x C_3 C_4 + \frac{L x}{C_4} \exp \left( -C_1 x C_2 \right) x C_4 \right)
\]

\[
\ll_F \exp \left( -C_1 x C_2 \right) x C_3 C_4 + C_4.
\]

4 Ramanujan–Hardy–Littlewood-type identity for Selberg class

Before giving the proof of Theorem 2.1, we prove the following results. We will first derive an approximate formula for \( \frac{d}{dF}(s) \) in terms of zeros \( \rho \) of \( F \) which are near \( s \). This is a generalization of Theorem 9.6(A) of [25].

**Lemma 4.1** Let \( F \) satisfy axioms (i)–(iv) of the class \( \mathbb{S} \). If \( \rho = \beta + iy \) runs through zeros of \( F(s) \),

\[
(4.1) \quad \frac{F'(s)}{F(s)} = \sum_{|t-y| \leq 1} \frac{1}{s - \rho} + O_F(\log t),
\]

uniformly for \( \sigma \in [-c_F, 1 + c_F] \), where \( s = \sigma + it \) and \( c_F \) is as defined in (2.4).

**Proof** Let \( s = \sigma + it \) with \( t > 2 \) and \( \sigma \in [-c_F, 1 + c_F]. \) We will use Lemma (α) on page 56 of [25] with \( f(s) = F(s), s_0 = 2 + it \) and \( r = 4(2 + c_F) \). Then, from Lemma 3.4 (in particular, (5) and (9)) of [11], it is clear that the hypothesis

\[
\left| \frac{f(s)}{f(s_0)} \right| < e^M,
\]
holds with $M = A \log t$ for some constant $A$ depending on $F$. Lemma (a) of [25, p. 56] then yields

$$
\frac{F'(s)}{F(s)} = \sum_{\rho:|\rho-s_0| \leq 2(2+c_F)} \frac{1}{s-\rho} + O_F(\log t)
$$

(4.2)

for $|\sigma-\frac{1}{2}| \leq \frac{c}{4}$, and so, in particular, for $-c_F \leq \sigma \leq 1 + c_F$ due to the choice of $r$. We would like to replace the main term in (4.2) by

$$
\sum_{\rho:|\gamma-t| \leq 1} \frac{1}{s-\rho},
$$

(4.3)

where $\gamma$ refers to the ordinate of the zero $\rho$. Indeed, the sum in (4.2) runs over more number of terms than in (4.3), but since $|\rho - s_0| \leq 4 + 2c_F$ implies $|\gamma - t| \leq 4 + 2c_F$, this difference is at most

$$
N_F(t + 4 + 2c_F) - N_F(t - 4 - 2c_F) \ll_F \log t,
$$

by (3.1). This completes the proof. □

**Lemma 4.2** Let $F$ satisfy axioms (i)–(iv) of the class $S$. Suppose that the degree of $F$ is $d_F$. Let $A_1 > 0$ be a sufficiently small constant. Let $T \to \infty$ through values such that $|T - \gamma| > \exp(-A_1 \gamma/\log \gamma)$ for every ordinate $\gamma$ of a zero of $F(s)$. Then, for $\sigma \in [-c_F, 1 + c_F]$, where $c_F$ is as defined in (2.4), we have

$$
|F(\sigma + iT)| \geq e^{A_2 T},
$$

where $0 < A_2 < \frac{\pi d_F}{4}$.

**Proof** Our starting point for the proof is (4.1), which holds uniformly for $s = \sigma + it$ with $\sigma \in [-c_F, 1 + c_F]$. Integrating (4.1) with respect to $s$ from $1 + c_F + it$ to $z = \sigma' + it$, where $\sigma' \in [-c_F, 1 + c_F]$, and $t$ does not equal the ordinate of any zero of $F(s)$, we get

$$
\log F(z) - \log F(1 + c_F + it) = \left[ \sum_{|t-\gamma| \leq 1} \log(s-\rho) \right]_{1+c_F+it}^{z} + O_F(\log t)
$$

(4.4)

$$
= \sum_{|t-\gamma| \leq 1} \log(z-\rho) - \sum_{|t-\gamma| \leq 1} \log(1 + c_F + it - \rho) + O_F(\log t).
$$

Since $\log(1 + c_F + it - \rho)$ is bounded when $|t-\gamma| \leq 1$, and the number of terms in the second sum above is $\ll \log t$ from (3.1), we obtain

$$
\sum_{|t-\gamma| \leq 1} \log(1 + c_F + it - \rho) = O_F(\log t).
$$

Moreover,

$$
|F(1 + c_F + it)| \leq \sum_{n=1}^{\infty} \frac{|a(n)|}{n^{1+c_F}},
$$

which is bounded due to absolute convergence of $F(s)$ to the right of Re($s$) = 1. Consequently, (4.4) yields

$$
\log F(s) = \sum_{|t-\gamma| \leq 1} \log(s-\rho) + O_F(\log t).
$$
Since \( \text{Re}(s - \rho) \) is bounded, taking real parts on both sides, we obtain
\[
\log|F(s)| = \sum_{|t - y| \leq 1} \log|s - \rho| + O_F(\log t)
\geq \sum_{|t - y| \leq 1} \log|t - y| + O_F(\log t).
\]

Let \( A_1 \) be a sufficiently small constant to be chosen later. We now let \( s = \sigma + iT \), where \( T \to \infty \) through values such that \( |T - y| > \exp(-A_1 y/\log y) \) for every ordinate \( y \) of a zero of \( F(s) \). Then the above inequality yields
\[
(4.5) \quad \log|F(\sigma + iT)| \geq -\sum_{|T - y| \leq 1} A_1 y + O_F(\log T).
\]

From (3.1), we see that
\[
(4.6) \quad \sum_{|T - y| \leq 1} \frac{A_1 y}{\log y} \leq \sum_{|T - y| \leq 1} A_1 \frac{T + 1}{\log(T - 1)} \leq C_1 A_1 T,
\]
for some absolute constant \( C_1 > 0 \). Equations (4.5) and (4.6) imply that
\[
\log|F(\sigma + iT)| \geq -C_2 A_1 T,
\]
for some absolute constant \( C_2 > 0 \). Choosing \( A_1 \) sufficiently small so that \( C_2 A_1 < \frac{\pi d_F}{4} \) completes the proof. 

### 4.1 Proof of Theorem 2.1

From (2.5), we have
\[
\sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{\alpha, \beta} \left( \frac{\eta}{n} \right) = \sum_{n=1}^{\infty} \frac{b_F(n)}{n} \frac{1}{2\pi i} \int_{(c)} \prod_{i=1}^{q} \Gamma \left( \alpha_i s + \beta_i \right) \left( \frac{\eta}{n} \right)^{-s} ds
\]
\[
= \frac{1}{2\pi i} \int_{(c)} \prod_{i=1}^{q} \Gamma \left( \alpha_i s + \beta_i \right) \sum_{n=1}^{\infty} \frac{b_F(n)}{n^{1-s}} \eta^{-s} ds
\]
\[
= \frac{1}{2\pi i} \int_{(c)} \prod_{i=1}^{q} \Gamma \left( \alpha_i s + \beta_i \right) \frac{1}{F(1-s)} \eta^{-s} ds.
\]

In the second step above, we have interchanged the order of integration and summation by using Stirling’s formula (3.2). Using the functional equation (2.1), we obtain
\[
(4.7) \quad \sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{\alpha, \beta} \left( \frac{\eta}{n} \right) = \frac{\tilde{\omega}}{2\pi i} \int_{(c)} Q^{1-2s} \prod_{i=1}^{q} \Gamma \left( \alpha_i (1-s) + \beta_i \right) F(\tilde{s}) \eta^{-s} ds.
\]

We first assume that \( r = j_F - k_F > 0 \). As seen in Figure 1, we consider the rectangular contour \( \mathcal{C} \) with sides \([c - iT, d - iT], [d - iT, d + iT], [d + iT, c + iT], \) and \([c + iT, c - iT], \) where \( T > 0 \) and \( 1 < d < 1 + c_F \). From the choice of \( c_F \) as given in (2.4) and the discussion preceding it, it is clear that the only possible poles inside \( \mathcal{C} \), for the \( \Gamma \)-factors appearing in the integrand of (4.7) are at \( s = 1 \), coming from those factors
Figure 1: The contour $\mathcal{C}$.

for which $\beta_i = 0$. As there are $j_F$ such factors and $F$ has a pole of order $k_F$ at $s = 1$, this yields a pole of order $j_F - k_F$ at $s = 1$ for the integrand in (4.7). Again from the discussion preceding (2.4), $F(s)$ has a zero of order $j_F - k_F$ at $s = 0$, thereby giving a pole of same order for our integrand at $s = 0$.

Finally, we also have simple poles at conjugates of nontrivial zeros of $F(s)$. Applying Cauchy’s residue theorem gives

$$
\frac{1}{2\pi i} \left[ \int_{c-iT}^{d-iT} + \int_{d+iT}^{c+iT} + \int_{c+iT}^{c-iT} + \int_{d-iT}^{d+iT} \right] \hat{\omega} Q^{1-2s} \prod_{i=1}^{q} \Gamma \left( \alpha_i (1 - s) + \beta_i \right) \frac{\eta^{-s}}{F(\bar{s})} ds
$$

$$
= \sum_{\rho} \hat{\omega} Q^{1-2\rho} \prod_{i=1}^{q} \Gamma \left( \alpha_i (1 - \bar{\rho}) + \beta_i \right) \frac{\eta^{-\bar{\rho}}}{F'(\rho)}
$$

$$
+ \frac{1}{(r-1)!} \frac{d^{r-1}}{ds^{r-1}} \bigg|_{s=0} \hat{\omega} Q^{1-2s} \prod_{i=1}^{q} \Gamma \left( \alpha_i (1 - s) + \beta_i \right) \frac{\eta^{-s}}{F(\bar{s})}
$$

$$
+ \frac{1}{(r-1)!} \frac{d^{r-1}}{ds^{r-1}} \bigg|_{s=1} \hat{\omega} Q^{1-2s} \prod_{i=1}^{q} \Gamma \left( \alpha_i (1 - s) + \beta_i \right) \frac{\eta^{-s}}{F(\bar{s})}
$$

(4.8)

By Lemma 4.2, Stirling’s formula (3.2) and the definition of $d_F$, as $|T| \to \infty$,
where the second equality is valid as $\text{Re}(1 - \tilde{u}) = d > 1$. As $T \to \infty$, since $-\epsilon_F < c' < 0$, using (2.5), we obtain

\begin{equation}
\frac{1}{2\pi i} \int_{d-iT}^{d+iT} \tilde{\omega} Q^{1-2s} \prod_{i=1}^{q} \frac{\Gamma(\alpha_i(1-s) + \beta_i)}{F(\tilde{s})} \eta^{-s} ds = \tilde{\omega} Q \nu \sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{\alpha,\beta} \left( \frac{\nu}{n} \right).
\end{equation}

Letting $T \to \infty$ in (4.8) and using (4.7) and (4.9), we get

\begin{align*}
&- \sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{\alpha,\beta} \left( \frac{\eta}{n} \right) + \tilde{\omega} Q \nu \sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{\alpha,\beta} \left( \frac{\nu}{n} \right) \\
&= \sum_{\rho} \tilde{\omega} Q^{1-2\rho} \prod_{i=1}^{q} \frac{\Gamma(\alpha_i(1-\rho) + \beta_i)}{F'(\rho)} \eta^{-\rho} \\
&\quad + \frac{1}{(r-1)!} \int_{s=0}^{1} s^{r-1} \tilde{\omega} Q^{1-2s} \prod_{i=1}^{q} \frac{\Gamma(\alpha_i(1-s) + \beta_i)}{F(\tilde{s})} \eta^{-s} ds \\
&\quad + \frac{1}{(r-1)!} \int_{s=1}^{r} s^{r-1} \tilde{\omega} Q^{1-2s} \prod_{i=1}^{q} \frac{\Gamma(\alpha_i(1-s) + \beta_i)}{F(\tilde{s})} \eta^{-s} ds
\end{align*}

Multiplying both sides by $-\omega \sqrt{\eta}$ and using the fact $\eta \nu = \frac{1}{Q^r}$, we get the required result (2.7).

For $r \leq 0$, it is easy to see that last two terms on the right-hand side of (4.8) do not appear, but the remainder of the proof remains unchanged. This completes the proof.
5 Proofs of Corollaries 2.2 and 2.3

In this section, we prove some consequences of Theorem 2.1.

5.1 Proof of Corollary 2.2

We will use the notation set up in the statement of Lemma 3.5. From the data of $F$, we have $q = 1$, $\alpha = (4)$, $\beta = (\beta_1)$, and hence, $L = 1$ and $k_1 = 4$. The sequence $b_{1j}$ of (3.9) is given by $b_{1j} = (\tilde{\beta}_1 + j - 1)/4$ with $j = 1, \ldots, 4$. We express $Z_{(4),(\beta_1)}(x)$ in terms of $Z_{(4),(\beta_1)}(x)$ using (3.7) and then write $Z_{(4),(\beta_1)}(x)$ in terms of the Meijer $G$-function using (3.10) to obtain

$$Z_{(4),(\beta_1)}(x) = \begin{cases} (2\pi)^{-\frac{1}{2}} \frac{x^{\tilde{\beta}_1}}{2} G_{0,4}^{4,0} \left( \frac{\tilde{\beta}_1}{4} + \frac{1}{4} + \frac{2 \tilde{\beta}_1}{4} + \frac{3}{4} \right), & \text{if } \tilde{\beta}_1 \neq 0, \\ (2\pi)^{-\frac{1}{2}} \frac{x^{\bar{\beta}_1}}{2} G_{0,4}^{4,0} \left( \frac{1}{4} + \frac{2}{4} + \frac{3}{4} \right) - \frac{1}{4}, & \text{if } \tilde{\beta}_1 = 0. \end{cases}$$

For this special value of the Meijer $G$-function, (2.8) gives

$$Z_{(4),(\beta_1)}(x) = \begin{cases} \frac{1}{4} x^{\tilde{\beta}_1/4} e^{-x^{1/4}}, & \text{if } \beta_1 \neq 0, \\ \frac{1}{4} e^{-x^{1/4}} - \frac{1}{4}, & \text{if } \beta_1 = 0. \end{cases}$$

Substituting these values into Theorem 2.1, we obtain (2.10) and (2.11), respectively, as desired.

5.2 Proof of Corollary 2.3

Again, following the notation of Lemma 3.5, in this case, we have $q = 4$, $L = 1$, and $k_i = 1$ for $1 \leq i \leq 4$. Calculating the residue term of (3.7), it can be seen that

$$Z_{(1,1,1,1),(0,\frac{1}{2},-C,C)}(x) = \tilde{Z}_{(1,1,1,1),(0,\frac{1}{2},-C,C)}(x) - \Gamma(1/2) \Gamma(-C) \Gamma(C).$$

We note that the sequence $b_{ij}$ of (3.9) is now given by $\{b_{ij}\}_{i=1}^4 = \{0,1/2,-C,C\}$. Using 3.10, we have

$$\tilde{Z}_{(1,1,1,1),(0,\frac{1}{2},-C,C)}(x) = G_{0,4}^{4,0} \left( 0 \left| \frac{1}{2} - C \right| x \right).$$

Putting $b = 0$ and $c = C$ in (2.9), we get

$$G_{0,4}^{4,0} \left( 0 \left| \frac{1}{2} - C \right| z \right) = 8\sqrt{\pi} K_{2C} \left( 2\sqrt{2}(-z)^{1/4} \right) K_{2C} \left( \frac{2\sqrt{2} \sqrt{x}}{(-z)^{1/4}} \right).$$

Using the reflection formula to simplify the residue term in (5.1) and combining it with (5.2) and (5.3), we obtain

$$Z_{(1,1,1,1),(0,\frac{1}{2},-C,C)}(x) = 8\sqrt{\pi} K_{2C} \left( 2\sqrt{2}(-x)^{1/4} \right) K_{2C} \left( \frac{2\sqrt{2} \sqrt{x}}{(-x)^{1/4}} \right) + \frac{\pi \sqrt{\pi}}{C \sin \pi C}.$$ 

Substituting this into Theorem 2.1 yields (2.12).
6 Riesz-type criteria for Selberg class

6.1 Proof of Theorem 2.4

For $F \in S^*$, consider the summatory function of the coefficients of $F(s)^{-1}$, given by

$$M_F(x) := \sum_{n \leq x} b_F(n).$$

It can be checked that $F \in S^*$ is contained in the general class of $L$-functions constructed on page 94 of Iwaniec–Kowalski [15]. In particular, axiom (iv)' and the inequality (2.14) play a crucial role here. We may thus apply Proposition 5.14 of [15] to obtain that GRH implies the bound

$$M_F(x) \ll_\delta x^{\frac{1}{2}+\delta},$$

for any $\delta > 0$, as $x \to \infty$. Let us define

$$M_F(h, N) = \sum_{n=h}^{N} \frac{b_F(n)}{n}.$$

Applying partial summation to (6.1), one can deduce that as $h \to \infty$, we have under GRH

$$M_F(h, n) \ll_\delta h^{-\frac{1}{2}+\delta},$$

for any $\delta > 0$, uniformly for any $N \geq h$. If $N < h$, we interpret $M_F(h, N)$ to be zero.

We now commence the proof of Theorem 2.4.

6.1.1 Proof of Theorem 2.4(a)

Let $\varepsilon > 0$ be sufficiently small. From (2.6), we may write

$$P_{a, \beta, \varepsilon}(v^2) = \sum_{n=1}^{\infty} \frac{b_F(n)}{n} Z_{a, \beta} \left( \left( \frac{v}{n} \right)^{d_F} \right) \cosh \left( \frac{vz}{n} \right)$$

$$= \left[ \sum_{n=1}^{h-1} + \sum_{n=h}^{\infty} \right] \frac{b_F(n)}{n} Z_{a, \beta} \left( \left( \frac{v}{n} \right)^{d_F} \right) \cosh \left( \frac{vz}{n} \right)$$

$$=: P_1 + P_2, \ (\text{say}),$$

where $h$ equals the greatest integer less than or equal to $v^{1-\varepsilon}$, that is, $h = [v^{1-\varepsilon}]$. We first consider the partial sum of $P_2$, given by

$$P_2(N) := \sum_{n=h}^{N} \frac{b_F(n)}{n} Z_{a, \beta} \left( \left( \frac{v}{n} \right)^{d_F} \right) \cosh \left( \frac{vz}{n} \right)$$

$$= \sum_{n=h}^{N} (M_F(h, n) - M_F(h, n-1)) Z_{a, \beta} \left( \left( \frac{v}{n} \right)^{d_F} \right) \cosh \left( \frac{vz}{n} \right)$$

$$= \sum_{n=h}^{N-1} M_F(h, n) (f(n) - f(n+1)) + M_F(h, N) f(N),$$

(6.4)
where

\[ f(t) := Z_{\alpha, \beta} \left( \left( \frac{v}{t} \right)^{d_F} \right) \cosh \left( \frac{vz}{t} \right). \]

Using Lemma 3.3 and the bounds (6.2) and \( \cosh(xz) \ll_z x \) as \( x \to 0 \), we find that

\[ M_F(h, N) f(N) \ll_{\delta, F, z} h^{-\frac{1}{2} + \delta} \left( \frac{N}{v} \right)^{cd_F}, \]

where \(-c_F < c < 0\). Letting \( N \to \infty \), the above expression goes to zero, so that (6.4) yields

\[ P_2 = \sum_{n=h}^{\infty} M_F(h, n) (f(n) - f(n+1)) \]

\[ = \left[ \sum_{n=h}^{[Cv]} + \sum_{n=[Cv]}^{\infty} \right] M_F(h, n) (f(n) - f(n+1)) \]

\[ =: P_3 + P_4 \quad \text{(say)}. \]

Here, \( C \) is a constant to be chosen later and \([Cv]\) denotes the greatest integer less than or equal to \( Cv \). Using the mean value theorem, there exists \( \lambda_n \in (n, n+1) \) such that

\[ f(n+1) - f(n) = f'(\lambda_n) \]

\[ = -\frac{d_F v^{d_F}}{\lambda_n^{d_F+1}} Z_{\alpha, \beta}' \left( \frac{v}{\lambda_n} \right) \cosh \left( \frac{vz}{\lambda_n} \right) - \frac{v}{\lambda_n^2} Z_{\alpha, \beta} \left( \frac{v}{\lambda_n} \right) \sinh \left( \frac{vz}{\lambda_n} \right). \]

We will estimate \( f(n) - f(n+1) \) in two different ranges of \( n \), as relevant for \( P_3 \) and \( P_4 \) in what follows.

At this point, the assumption that \( j_F = 0 \) plays a crucial role. Indeed, when \( j_F = 0 \), one sees from (3.7) that \( Z_{\alpha, \beta} \) is the same as \( Z_{\alpha, \beta}' \). Thus, for \( v/\lambda_n \) sufficiently large, we can apply Lemmas 3.5 and 3.6. Since \( n \leq \lambda_n << n \), this means that in the range \( n \leq [Cv] \), for sufficiently small \( C \), we have

\[ |f(n) - f(n+1)| \ll_{F, z, C} \exp \left( -C_1 \left( \frac{v}{n} \right)^2 + \frac{v|\text{Re}(z)|}{n} \right) \left( \frac{v}{n} \right)^{2C_3 + d_F C_4} \left( \frac{\nu d_F}{n^{d_F+1}} + \frac{\nu}{n^2} \right), \]

where \( C_i \)’s are as defined in (3.8). Here, we have used the fact that \( d_F C_2 = 2 \) and that \( \cosh(xz) \) and \( \sinh(xz) \) are trivially of the order \( \exp(x \text{Re}(z)) \). Using the above estimate and (6.2) in (6.6), we obtain

\[ P_3 \ll_{\delta, F, z, C} \sum_{n=h}^{[Cv]} \frac{h^{-\frac{1}{2} + \delta}}{n} \exp \left( -C_1 \left( \frac{v}{n} \right)^2 + \frac{v|\text{Re}(z)|}{n} \right) \left( \frac{v}{n} \right)^{2C_3 + d_F C_4 + d_F}. \]

Keeping in mind that \( C_1 > 0 \), it can be seen that the argument of \( \exp \) is negative if \( n < C_1 v/|\text{Re} z| \). In the range of \( n \) applicable to \( P_3 \), it is enough to ensure that
\[ C < \frac{C_1}{|\text{Re} z|}. \]

We choose \( C \) to be sufficiently small, say \( C = C_1/(M C_1 + |\text{Re} z|) \), where \( M > 0 \) is a sufficiently large fixed constant, so that (6.8) and (6.10) are both satisfied. Writing \( 2C_3 + d_F C_4 + d_F \) as \( C_5 \), (6.9) then gives

\[ P_3 \ll_{\delta,F,z,C} h^{-\frac{1}{4} + \delta} \sum_{n=h}^{[Cv]} \frac{v C_5}{n C_5 + 1} \ll_{\delta,F,z,C} h^{-\frac{1}{4} + \delta} v C_5 \int_h^{Cv} \frac{1}{t C_5 + 1} dt. \]

Thus, we obtain

\[ P_3 \ll_{\delta,F,z,C} h^{-\frac{1}{4} + \delta} v C_5 h^{-C_5} \ll_{\delta,F,z,C} h^{-\frac{1}{4} + \delta}, \]

for any \( \delta > 0 \), keeping in mind that \( h \) was chosen to be \([v^{1-\varepsilon}]\).

We now turn to \( P_4 \). Since the range of \( n \) is now \( n > [Cv] \), we will use Lemmas 3.3 and 3.4 to bound \( Z_{\alpha,\beta} \) and \( Z'_{\alpha,\beta} \), respectively. Moreover, we have \( \cosh(xz) \ll z \) and \( \sinh(xz) \ll z \) when \( x > 0 \) is bounded above by some absolute constant. Applying all this to (6.7) gives, for \( n > [Cv] \),

\[ f(n) - f(n+1) \ll_{F,z,C} v^{d_F} \frac{1}{n^{d_F + 1}} \left( \frac{v}{n} \right)^{(-c-1)d_F} + \frac{v}{n^2} \left( \frac{v}{n} \right)^{-cd_F + 1} \ll_{F,z,C} \frac{1}{n} \left( \frac{v}{n} \right)^{-cd_F + 2}. \]

Again, from (6.6), we have

\[ P_4 \ll_{\delta,F,z,C} h^{-\frac{1}{4} + \varepsilon} \sum_{n=h}^{\infty} \frac{v^{-cd_F + 2}}{n^{-cd_F + 3}} \ll_{\delta,F,z,C} h^{-\frac{1}{4} + \delta}, \]

for any \( \delta > 0 \), in the same manner as done for \( P_3 \) in (6.11). Combining (6.6), (6.12), and (6.13), we obtain that, for any \( \delta > 0 \),

\[ P_2 \ll_{\delta,F,z,C} h^{-\frac{1}{4} + \delta}. \]

We now consider the first series of (6.3). Again, since \( j_F = 0 \), we have \( Z_{\alpha,\beta} = \tilde{Z}_{\alpha,\beta} \). Moreover, we have assumed \( F \in \mathcal{S}^* \), which means that Lemma 3.5 can be applied. This bound gives

\[ P_1 \ll \sum_{n=1}^{h-1} \frac{|b_F(n)|}{n} \exp \left( -C_1 \left( \frac{v}{n} \right)^2 + \frac{v |\text{Re} z|}{n} \right) \left( \frac{v}{n} \right)^{2C_3}. \]

Here, we have used the trivial bound \( \cosh(vz/n) \ll \exp(|v|/\text{Re} z/n) \). Since \( b_F \) is the inverse of \( a_F \) under Dirichlet convolution, using induction one can derive the following relation for any prime \( p \) and \( m \in \mathbb{N} \),

\[ b_F(p^m) = -\sum_{j=1}^{m} a_F(p^j) b_F(p^{m-j}). \]

Using the Ramanujan bound (axiom (v)) for \( a_F(n) \) and induction on \( m \), it follows that axiom (v) is also true for \( b_F \) on prime powers. As \( b_F(n) \) is multiplicative, this gives
Whenever \( j_F \neq 0 \), \( Z_{\alpha,\beta} \) and \( \tilde{Z}_{\alpha,\beta} \) differ by the residue term given in (3.7). We first write \( \mathcal{P}_{\alpha,\beta,0}(v^2) = P_1 + P_2 \) as in (6.3). Using (6.5) and (6.7), keeping in mind that \( z = 0 \), we can write

\[
P_2 = \sum_{n=h}^{\infty} \frac{d_F v^{d_F}}{\lambda_n^{d_F+1}} M_{F}(h, n) Z'_{\alpha,\beta} \left( \left( \frac{v}{\lambda_n} \right)^{d_F} \right).
\]

The primary departure here from the proof of part (a) given above is that due to the nontrivial residue term in (3.7), we no longer have an exponential decay bound for \( Z_{\alpha,\beta}' \). We resort to Lemma 3.4 instead. This along with (6.2) (which is true because of GRH) yields

\[
P_2 \ll_{\delta,F} h^{-\frac{1}{2}+\delta} \sum_{n=h}^{\infty} \frac{v^{d_F}}{h^{d_F+1}} \left( \frac{v}{n} \right)^{-(c-1)d_F} \ll_{\delta,F} h^{-\frac{1}{2}+\delta} \sum_{n=h}^{\infty} \frac{v^{-cd_F}}{n^{cd_F+1}}.
\]

Using the integral test as done in (6.11), we find that \( P_2 \ll_{\delta,F} h^{-\frac{1}{2}+\delta} \), for any \( \delta > 0 \).

Turning to \( P_1 \), we have

\[
P_1 = \sum_{n=1}^{h-1} \frac{b_F(n)}{n} Z_{\alpha,\beta} \left( \left( \frac{v}{n} \right)^{d_F} \right)
\]

(6.16) \( = \sum_{n=1}^{h-1} \frac{b_F(n)}{n} \tilde{Z}_{\alpha,\beta} \left( \left( \frac{v}{n} \right)^{d_F} \right) - \sum_{n=1}^{h-1} \frac{b_F(n)}{n} \text{Res}_{s=0} \prod_{i=1}^{q} \Gamma \left( \alpha_i s + \hat{\beta}_i \right) \left( \frac{v}{n} \right)^{-d_{F,s}} \)

The first sum of (6.16) satisfies the bound (6.15) as done in the proof of part (a), and is thus negligible compared to \( P_2 \). Note that the assumption \( F \in S^* \) is used here. Using \( h = \lfloor v^{1-\epsilon} \rfloor \) and replacing \( v^2 \) by \( y \), the second term becomes

\[
\sum_{n=1}^{[v^{1-\epsilon}]-1} \frac{b_F(n)}{n} \text{Res}_{s=0} \prod_{i=1}^{q} \Gamma \left( \alpha_i s + \hat{\beta}_i \right) \left( \frac{v}{n} \right)^{-d_{F,s}}.
\]

Hence, GRH implies the estimate (2.15).
6.1.3 Proof of Theorem 2.4(c)

As \( z \to 0 \), only the \( t = 0 \) term survives in (3.4). Thus, for \( 0 < \text{Re}(s) < 1/2 \), we have

\[
F(2s + 1) \int_0^\infty y^{-s-1} \mathcal{P}_{\alpha,\beta,0}(y) dy = \frac{2}{d_F} \prod_{i=1}^q \Gamma \left( -\frac{2\alpha_i s}{d_F} + \hat{\beta}_i \right).
\]

We will try to extend this identity to a larger region of the complex plane. The right-hand side has no poles in the region \( \text{Re}(s) \leq 0 \) except for a pole of order \( j_F \) at \( s = 0 \). Since \( F(2s + 1) \) has a pole of order \( k_F \) at \( s = 0 \), let us multiply both sides by \( s^\ell \) where \( \ell = \max[k_F, j_F] \). This gives

\[
(6.17) \quad s^\ell F(2s + 1) \int_0^\infty y^{-s-1} \mathcal{P}_{\alpha,\beta,0}(y) dy = \frac{2s^\ell}{d_F} \prod_{i=1}^q \Gamma \left( -\frac{2\alpha_i s}{d_F} + \hat{\beta}_i \right).
\]

We next partition the above integral as

\[
I := \int_0^\infty y^{-s-1} \mathcal{P}_{\alpha,\beta,0}(y) dy = \int_0^1 y^{-s-1} \mathcal{P}_{\alpha,\beta,0}(y) dy + \int_1^\infty y^{-s-1} \mathcal{P}_{\alpha,\beta,0}(y) dy =: I_1 + I_2.
\]

The assumed bound \( \mathcal{P}_{\alpha,\beta,0}(y) \ll F_{\delta,\ell} y^{-\frac{1}{2} + \delta} \), implies that \( I_2 \) is integrable in the region \( -\frac{1}{4} < \text{Re}(s) \leq 0 \). In order to analyze \( I_1 \), we apply (2.6) and the bound of Lemma 3.3 to \( Z_{\alpha,\beta} ((\sqrt{\gamma}/n)^{d_F}) \). It can then be seen that \( I_1 \) is integrable for \( \text{Re}(s) < -cd_F/2 \), where \( -c_F < c < 0 \). It thus follows that \( I \) is analytic for \( \text{Re}(s) \in (-1/4, \epsilon) \) for \( \epsilon > 0 \) sufficiently small. As both sides of the identity (6.17) are analytic in this region, by the principle of analytic continuation, (6.17) holds for an extended region of \( C \), in particular, for \(-\frac{1}{5} < \text{Re}(s) \leq 0 \).

Since the right-hand side of (6.17) does not vanish in the region \(-\frac{1}{4} < \text{Re} s < 0 \) and the integral \( I \) is analytic here, we see that \( F(2s + 1) \) does not vanish in this region. This implies the nonvanishing of \( F(s) \) in \( \frac{1}{2} < \text{Re}(s) < 1 \) as needed.

6.1.4 Proof of Theorem 2.5

Let \( z \neq 0 \), \( \text{arg} z \neq \pm \pi/2 \) be fixed. Let \( s = \sigma + iT \), and \( \hat{\beta}_i = \delta_i + iy_i \) for \( i = 1, \ldots, q \). From (3.4), we have, for \( 0 < \text{Re} s < 1/2 \),

\[
(6.18) \quad s^{k_F} F(2s + 1) \phi(s, \alpha, \beta, z) = \frac{2s^{k_F}}{d_F} \sum_{t=0}^\infty \frac{z^{2t}}{(2t)!} \prod_{i=1}^q \Gamma \left( \frac{2\alpha_i t}{d_F} - s + \hat{\beta}_i \right),
\]

where \( \phi(s, \alpha, \beta, z) = \int_0^\infty y^{-s-1} \mathcal{P}_{\alpha,\beta,z}(y) dy \), and \( k_F \) is the order of the pole of \( F(s) \) at \( s = 1 \). We will first extend this identity to the region \(-1/4 < \text{Re}(s) \leq 0 \). Using Stirling’s formula (3.2), as \( |T| \to \infty \), the product of Gamma factors above can be written as

\[
(6.19) \quad (2\pi)^{q/2} \left( 1 + O_F \left( \frac{1}{|T|} \right) \right) \prod_{i=1}^q \left( \left( \frac{2\alpha_i T}{d_F} - s \right) + \delta_i \right)^{2\alpha_i t_{s,s}} e^{-\frac{1}{2} \frac{2\alpha_i T}{d_F} - \gamma_i}.
\]

In particular, using (6.19), it is easy to check that the radius of convergence of the power series on the right-hand side of (6.18) is infinite. Using the assumed bound \( \mathcal{P}_{\alpha,\beta,z}(y) \ll F_{\delta,\ell} y^{-\frac{1}{2} + \delta} \), the integral \( \phi(s, \alpha, \beta, z) \) can be shown to be analytic in the
region $\text{Re}(s) \in (-1/4, \varepsilon)$ for $\varepsilon > 0$ sufficiently small, exactly as done in the proof of Theorem 2.4(c) above. Moreover, the factor $s^k F(2s + 1)$ is entire. From the principle of analytic continuation, we see that (6.18) holds in $-1/4 < \text{Re}(s) \leq 0$.

We now consider $s = \sigma + iT$, where $-1/4 < \sigma < 0$. Plugging (6.19) into (6.18), we obtain for $s^k F(2s + 1) \phi(s, \alpha, \beta, z)$, the main term

$$
\frac{2s^k (2\pi)^{q/2}}{d_F} \prod_{i=1}^{q} \left| \frac{2\alpha_i T}{d_F} - y_i \right|^{\frac{2\eta_i u + \delta_i - \frac{1}{2}}{d_F} e^{-\frac{1}{2} \frac{2\alpha_i T}{d_F} - y_i}} \sum_{t=0}^{\infty} \frac{z^{2t}}{(2t)!} \prod_{i=1}^{q} \left| \frac{2\alpha_i T}{d_F} - y_i \right|^{\frac{2\alpha_i T}{d_F} - y_i} (6.20)
$$

using the Taylor series for $\cosh(z)$. The error term involved is $O(1/|T|)$ times the absolute value of (6.20). Note that if $\text{arg} \ z = \pm \pi$, then the argument of $\cosh$ above is purely imaginary, which would mean that the cosh term is bounded since $\cosh z = \cos(iz)$. Thus, we see that if $z \neq 0$ and $\text{arg} \ z \neq \pm \pi$, then as $T \to \infty$, the cosh term is unbounded. This means that as $T \to \infty$, the expression (6.20) tends to infinity in absolute value. Hence, there must exist a sufficiently large value $T_{F,z}$ of $T$, such that for $T > T_{F,z}$, the expression (6.20) is nonzero. Since $s^k \phi(s, \alpha, \beta, z)$ is analytic in this region, we must have that $F(2s + 1)$ does not vanish for $s = \sigma + iT$ with $\sigma \in (-1/4, 0)$ and $T > T_{F,z}$. This means that any zeros of $F(s)$ in the critical strip which are of the line $\text{Re}(s) = 1/2$ must lie in a vertical strip of fixed height. As the number of such zeros is finite, this proves the result.

7 Concluding remarks

We make some remarks regarding potential directions of further investigation here. Theorem 2.1 and Corollaries 2.2, 2.3, each assume the convergence of a certain series. It is possible that this assumption can be weakened by assuming convergence under the kind of bracketing given by (1.4). This may be feasible along the lines of the treatment in the classical case.

Indeed, it may even be possible to prove such “bracketed convergence” as has been done for the Riemann zeta function. This is a nontrivial question, which we have not attempted to resolve here.

Another natural question is whether some of the hypotheses assumed to prove Riesz-type criteria for our $L$-functions can be eliminated. While it is known that the Euler product axiom and the Ramanujan hypothesis are crucial in order to consider GRH (see, for instance, [21, pp. 27–28]), it is worth asking whether the polynomial Euler product assumption can be dispensed with. One can also enquire whether Ramanujan–Hardy–Littlewood type identities are valid for larger classes of $L$-functions, for instance, the Lindelöf class [10, 17].

In line with Lemmas 3.5 and 3.6, it is possible to obtain nontrivial estimates for higher derivatives of $\tilde{Z}_{\alpha, \beta}(x)$, which may find useful applications in related problems.
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Discipline of Mathematics, Indian Institute of Technology Gandhinagar, Palaj, Gandhinagar, India
e-mail: shivajee.o@iitgn.ac.in akshaa.vatwani@iitgn.ac.in