Unification approach to the separation axioms between $T_0$ and completely Hausdorff

Francisco G. Arenas†, Julian Dontchev‡ and Maria Luz Puertas§

March 30, 2022

Abstract

The aim of this paper is to introduce a new weak separation axiom that generalizes the separation properties between $T_1$ and completely Hausdorff. We call a topological space $(X, \tau)$ a $T_{\kappa,\xi}$-space if every compact subset of $X$ with cardinality $\leq \kappa$ is $\xi$-closed, where $\xi$ is a general closure operator. We concentrate our attention mostly on two new concepts: kd-spaces and $T_{1\frac{1}{2}}$-spaces.

1 Introduction

The definitions of most (if not all) weak separation axioms are deceptively simple. However, the structure and the properties of those spaces are not always that easy to comprehend.

In this paper we try to unify the separation axioms between $T_0$ and completely Hausdorff by introducing the concept of $T_{\kappa,\xi}$-spaces. We call a topological space $(X, \tau)$ a $T_{\kappa,\xi}$-space if every compact subset of $X$ with cardinality $\leq \kappa$ is $\xi$-closed where $\xi$ is a given closure operator. With different settings on $\kappa$ and $\xi$ we derive most of the well-known separation properties ‘in the semi-closed interval $[T_0, T_3]$’. We are going to consider not only Kuratowski closure operators but more general closure operators, such as the $\lambda$-closure operator for...
example ($\xi$ is a general closure operator on $X$ if $\xi : \exp X \to \exp X$ and $\xi \emptyset = \emptyset$, $A \subseteq \xi A$, $A \subseteq B \implies \xi A \subseteq \xi B, \xi \xi A = \xi A$).

A subset $A$ of a topological space $(X, \tau)$ is called $\lambda$-closed [1] if it is the intersection of a closed set and a $\Lambda$-set (a $\Lambda$-set is a set that is the intersection of a family of open sets [7]). Complements of $\lambda$-closed sets are called $\lambda$-open. The family of all $\lambda$-open sets is a topology on $X$ if and only if $X$ is (by definition) a $\lambda$-space [1]. This topology is coarser than $\tau$ and is denoted by $\tau_\lambda$.

For a topological space $(X, \tau)$, the family of all regular open sets forms a base for a new topology $\tau_s$, coarser than $\tau$, which is often called the semi-regularization of $\tau$. A point $x \in X$ is called a $\delta$-cluster point of a subset $A$ of $X$ [13] if $A \cap U \neq \emptyset$ for every regular open set $U$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$ and is denoted by $\Cl_\delta(A)$. The set $A$ is called $\delta$-closed [13] if $A = \Cl_\delta(A)$. Complements of $\delta$-closed sets are called $\delta$-open and the family of all $\delta$-open sets is a topology on $X$, coarser than $\tau$, denoted by $\tau_\delta$.

The first part of the following lemma is well-known, while the second part unites [8, Lemma 4] and [11, Lemma 1.1]. Recall first that a subset $A$ of a space $(X, \tau)$ is called locally dense [3] (= preopen) if $A \subseteq \Int(\Cl(A))$. Note that every open and every dense set is locally dense. The family of all regular open subsets of a topological space $(X, \tau)$ will be denoted by $RO(X, \tau)$.

**Lemma 1.1** (i) $\tau_s = \tau_\delta$.

(ii) If $A$ is a locally dense subset of a topological space $(X, \tau)$, then:

(a) $RO(A, \tau|A) = \{ R \cap A : R \in RO(X, \tau) \}$.

(b) $(\tau|A)_s = \tau_s|A$. $\square$

Back in 1968, Veličko [13] introduced the concept of $\theta$-open sets. Recall that set $A$ is called $\theta$-open [13] if every point of $A$ has an open neighborhood whose closure is contained in $A$. The $\theta$-interior [13] of $A$ in $X$ is the union of all $\theta$-open subsets of $A$ and is usually denoted by $\text{Int}_\theta(A)$. Complement of a $\theta$-open set is called $\theta$-closed. It is equivalent to stipulate that $\Cl_\theta(A) = \{ x \in X : \Cl(U) \cap A \neq \emptyset, U \in \tau \text{ and } x \in U \}$ and a set $A$ is $\theta$-closed.
if and only if $A = \text{Cl}_\theta(A)$. All $\theta$-open sets form a topology on $X$, coarser than $\tau$, usually denoted by $\tau_\theta$. Note that a space $(X, \tau)$ is regular if and only if $\tau = \tau_\theta$. Note also that the $\theta$-closure of a given set need not be a $\theta$-closed set – however it is always $\delta$-closed. Moreover, always $\tau_\theta \subseteq \tau_\delta \subseteq \tau$.

Recall that a topological space $(X, \tau)$ is called:

1. a **unique sequence space** (= $US$-space) if the limit point of every converging sequence is uniquely determined,
2. a **kc-space** \cite{14} if every compact set of $X$ is closed.
3. **weakly Hausdorff** \cite{11} if its semi-regularization is $T_1$, i.e. if each singleton is $\delta$-closed.
4. an **$hT^R_1$ space** \cite{12} if every subspace of $(X, \tau)$ is weakly Hausdorff.

A subset $A$ of a topological space $(X, \tau)$ is called **zero-open** if for each $x \in A$ there exists a zero-set $Z$ (in $X$) and a cozero-set $C$ (in $X$) such that $x \in C \subseteq Z \subseteq A$. Complements of zero-open sets are called **zero-closed** and the family of all zero-open sets is a topology on $X$, coarser than $\tau$, denoted by $\tau_z$. Recall that a topological space $(X, \tau)$ is called **completely Hausdorff** if each two different points have disjoint cozero neighborhoods.

A set $A$ is called **Urysohn-open** if for each $x \in A$ there exist two open sets $U$ and $V$ in $X$ such that $x \in U \subseteq \text{cl}(U) \subseteq V \subseteq \text{cl}(V) \subseteq A$. Complements of Urysohn-open sets are called **Urysohn-closed** and the family of all Urysohn-open sets is a topology on $X$, coarser than $\tau$ and finer than the quasi-topology $\tau_q$. The **quasi-topology** $\tau_q$ on $X$ is the topology having as base the clopen subsets of $(X, \tau)$.

2 **$T_{\kappa,\xi}$-spaces**

**Definition 1** A topological space $(X, \tau)$ is called a **$T_{\kappa,\xi}$-space** if every compact subset of $X$ with cardinality $\leq \kappa$ is $\xi$-closed, where $\xi$ is a general closure operator. When $\xi$ is the usual closure operator, we use the notation $T_{\kappa,c}$-space.

The following theorem shows that most of the well-known definitions of separation axioms placed between $T_1$ and $T_2$ can be derived from the definition of $T_{\kappa,\xi}$-spaces.
Theorem 2.1 Let $(X, \tau)$ be a topological space with $|X| = \kappa$. Then:

(i) $X$ is completely Hausdorff if and only if $X$ is a $T_{\kappa, x}$-space.

(ii) $X$ is Urysohn if and only if $X$ is a $T_{\kappa, U}$-space.

(iii) $X$ is Hausdorff if and only if $X$ is a $T_{\kappa, \emptyset}$-space.

(iv) $X$ is a kc-space if and only if $X$ is a $T_{\kappa, c}$-space.

(v) $X$ is weakly Hausdorff if and only if $X$ is a $T_{1, \delta}$-space.

(vi) $X$ is $T_1$ if and only if $X$ is a $T_{1, c}$-space.

(vii) $X$ is $T_0$ if and only if $X$ is a $T_{1, \lambda}$-space.

Proof. (i) Since $\{x\}$ is compact in $X$, it is zero-closed, so $X \setminus \{x\}$ is zero-open. Hence, if $y$ is a point of $X$ distinct from $x$ we have that $y \in X \setminus \{x\}$; that is, there exist a zero set $Z$ and a cozero set $C$ in $X$ such that $y \in C \subseteq Z \subseteq X \setminus \{x\}$. Then $C$ and $X \setminus Z$ are disjoint cozero neighborhoods of $x$ and $y$ respectively; hence the space is completely Hausdorff.

Let $A$ be a compact subset of the completely Hausdorff space $X$. Given a point $x \notin A$ and a point $y \in A$, there exists two cozero sets $C_y$ and $D_y$ with $x \in C_y$, $y \in D_y$ and $C_y \cap D_y = \emptyset$. Clearly, $\{D_y : y \in A\}$ is an open (cozero, in fact) covering of the compact set $A$, so there exist a finite subcovering $\{D_y : i = 1, \ldots, n\}$. Set $D = \bigcup_{i=1}^n D_y$. Note that $D$ is a cozero set containing $A$. Now set $C = \bigcap_{i=1}^n C_y$; $C$ is a cozero set containing $x$ and $C \cap D = \emptyset$; so we have $x \in C \subseteq X \setminus D \subseteq X \setminus A$. As the point $x$ was arbitrary and $X \setminus D$ is a zero set, we have that $X \setminus A$ is zero-open, so $A$ is zero-closed, as desired.

(ii) Since $\{x\}$ is compact in $X$, it is Urysohn-closed, so $X \setminus \{x\}$ is Urysohn-open. Hence, if $y$ is a point of $X$ distinct from $x$, we have that $y \in X \setminus \{x\}$, that is, there exist two open sets $U$ and $V$ in $X$ such that $y \in U \subseteq \text{cl}(U) \subset V \subset \text{Cl}(V) \subseteq X \setminus \{x\}$. Then, $U$ and $X \setminus \text{Cl}(V)$ are open neighborhoods of $x$ and $y$ respectively with disjoint closures; hence the space is Urysohn.

Let $A$ be a compact subset of the Urysohn space $X$. Given a point $x \notin A$ and a point $y \in A$, there exists open sets $U_y$ and $V_y$ with $x \in U_y$, $y \in V_y$ and $\text{Cl}(U_y) \cap \text{Cl}(V_y) = \emptyset$. $\{V_y : y \in A\}$ is an open covering of the compact set $A$, so there exist a finite subcovering $\{V_y : i = 1, \ldots, n\}$. Take $V = \bigcup_{i=1}^n V_y$; $V$ is an open set containing $A$ and $\text{Cl}(V) = \bigcup_{i=1}^n \text{Cl}(V_y)$. Now set $U = \bigcap_{i=1}^n U_y$. Clearly, $U$ is an open set containing $x$ and $\text{Cl}(U) = \bigcap_{i=1}^n \text{Cl}(U_y)$. Note that
Cl(U) ∩ Cl(V) = ∅; so we have x ∈ U ⊆ Cl(U) ⊆ X \ Cl(V) ⊆ X \ V ⊆ X \ A. As the point x was arbitrary, we have that X \ A is Urysohn-open, so A is Urysohn-closed, as desired.

(iii) By Theorem 4.3 from [6], a space is Hausdorff if and only if every compact set is θ-closed.

(iv) By definition a space is kc if and only if every compact set is closed.

(v) A space is weakly Hausdorff if and only if its semi-regularization is T₁, i.e., if every singleton is δ-closed [11].

(vi) is obvious.

(vii) This is Theorem 2.5 from [1]. ✷

Of particular interest is probably the class of Tκ,δ-spaces, i.e., the spaces in which compact sets are δ-closed, since they have not been considered in the literature until now. In order to be consistent with the definition of kc-space we will call this class of spaces kd-spaces. The relations between the spaces mentioned above are given in the following diagram:

\[
\text{Hausdorff} \quad \longrightarrow \quad \text{kd-space} \quad \longrightarrow \quad \text{kc-space} \quad \longrightarrow \quad \text{US-space} \\
\text{h}T^R_1 \quad \text{space} \quad \longrightarrow \quad \text{weakly Hausdorff} \quad \longrightarrow \quad \text{T}_1\text{-space}
\]

**Example 2.2** Example of a weakly Hausdorff space which is not a kd-space, not even a US-space: Let \( \mathbb{R} \) be the real line and let \( X = [0, 1] \cup [2, 3] \cup S \), where \( S = \{4, 5, 6, \ldots\} \). For each \( x \in X \) we define a neighborhood filter in the following way. If \( x \in [0, 1] \cup (2, 3] \cup S \), then the neighborhoods of \( x \) are the ones inherited by the usual topology on the real line. A neighborhood of 1 (resp. of 2) are the sets containing some interval \( (a, 1] \) (resp. \( [2, b) \)) along with all but finitely many points of \( S \). It is easily observed that we have a topology on \( X \). Since the sequence \( (4, 5, 6, \ldots) \) converges both to 1 and to 2, then \( X \) is not an US-space. In particular \( X \) is not a kd-space. Observe that both 1 and 2 can be represented as intersection of regular closed sets, hence they are both δ-closed as is every other point of \( X \). Hence \( X \) is weakly Hausdorff.
Example 2.3 Example of a kd-space which is not Hausdorff. Consider Example 3 from [9]. Let $X$ be the interval $[0, 1]$ of real numbers with the following topology: all points from $(0, 1)$ are clopen; the basic neighbourhoods of 0 are of the form $[0, x)$ for $x > 0$; the neighbourhoods of 1 are of the form $X \setminus F$, where $F \subseteq [0, 1)$ is either a finite set or a sequence that converges to 0 with respect to the standard topology. It is shown in [9] that this space is not Hausdorff. It can be easily observed that all compact sets are closed, and hence δ-closed, since the space is semi-regular [9].

Recall that a topological space $(X, \tau)$ is called a $C'$-space if every compact set is compactly closed. A set $A \subseteq X$ is called compactly closed if $A \cap K$ is compact for any compact subset $K$ of $X$ or equivalently if the canonical injection $i: A \rightarrow X$ is compact.

Proposition 2.4 Every kc and hence every kd-space is a C'-space.

Example 2.5 The real line with the cofinite topology is an example of a C'-space which is not a kc-space.

In 1979, Bankston [2] introduced the anti operator on a given topological space. Recall that a space $(X, \tau)$ is called anti-compact if every compact subset of $X$ is finite. Anti-compact spaces are sometimes called pseudo-finite. The class of topological spaces where compact sets are finite was also considered by in 1981 by Sharma [10].

Theorem 2.6 For an anti-compact topological space $(X, \tau)$ the following conditions are equivalent:

1. $X$ is a kd-space.
2. $X$ weakly Hausdorff.

Proof. Since (1) $\Rightarrow$ (2) is valid for any topological space, then we only need to verify (2) $\Rightarrow$ (1). Let $K$ be a compact subset of $X$. Since $X$ is anti-compact, then $K$ is finite. By (2), each point of $K$ is δ-closed and since the family of δ-closed sets is closed under finite additivity, then $K$ is δ-closed. Hence $X$ is a kd-space. □
Example 2.7 There is a simple example of a kc-space which is not a kd-space. Let $\mathbb{R}$ be the real line with the cocountable topology $\tau_{cc}$. Since $(\mathbb{R}, \tau_{cc})$ is anti-compact and $T_1$, it is clear that $(\mathbb{R}, \tau_{cc})$ is a kc-space. But the semi-regularization topology is the indiscrete one. Thus $(\mathbb{R}, \tau_{cc})$ is not a kd-space.

Proposition 2.8 (i) For a first countable topological space $(X, \tau)$ the following conditions are equivalent:

1. $X$ Hausdorff.
2. $X$ is a kd-space.
3. $X$ is a kc-space.
4. $X$ is an US-space.

(ii) A semi-regular topological space $(X, \tau)$ is a kd-space if and only if it is a kc-space. □

Remark 2.9 It is well-known that in the class of sequential spaces the concepts of kc-spaces and US-spaces coincide. However, we do not know if there exists an easy example of a sequential US-space which is not a kd-space.

Theorem 2.10 Locally dense subspaces of kd-spaces are kd-spaces.

Proof. Follows easily from Lemma [13] (ii). □

Corollary 2.11 Let $(X_\alpha, \tau_\alpha)_{\alpha \in \Omega}$ be a family of topological spaces. For the topological sum $X = \sum_{\alpha \in \Omega} X_\alpha$ the following conditions are equivalent:

1. $X$ is a kd-space.
2. Each $X_\alpha$ is a kd-space. □

According to Mršević, Reilly and Vamanamurthy a function $f: (X, \tau) \to (Y, \sigma)$ is called super-closed (resp. super-continuous) if the image of every closed subset of $X$ is $\delta$-closed in $Y$ (resp. the preimage of every open set is $\delta$-open). A bijection $f: (X, \tau) \to (Y, \sigma)$ is called a super-homeomorphism if $f$ is both super-closed and super-continuous.

Proposition 2.12 If $(X, \tau)$ is compact and $(Y, \sigma)$ is a kd-space, then every continuous function $f: (X, \tau) \to (Y, \sigma)$ is perfect and super-closed and hence every (super-)continuous bijection $f: (X, \tau) \to (Y, \sigma)$ is a (super-)homeomorphism.
Proof. Let \( K \) be a closed subset of \( X \). Clearly \( K \) is compact. Since \( f \) is continuous, then \( f(K) \) is compact. Since \( Y \) is a kd-space, then \( f(K) \) is \( \delta \)-closed and hence \( f \) is super-closed. For any \( y \in Y \), \( \{y\} \) is \( \delta \)-closed, since \( Y \) is a kd-space. Since \( f \) is continuous, then \( f^{-1}\{y\} \) is closed and moreover compact, since \( X \) is compact. Thus \( f \) is perfect. The rest of the claim is obvious. \( \square \)

3 \( T_{\kappa,\lambda} \)-spaces

Yet another class of particular interest is probably the class of \( T_{\kappa,\lambda} \)-spaces, i.e. the spaces in which compact sets are \( \lambda \)-closed, as they have not been also considered in the literature until now. In order to be consistent with the fact that they are placed between \( T_{\frac{1}{2}} \) (spaces where every set is \( \lambda \)-closed, see Theorem 2.6 of [1]) and \( T_{\frac{1}{3}} \) (spaces where every finite set is \( \lambda \)-closed, see Theorem 3.1 of [1]), we call them \( T_{\frac{1}{3}} \) spaces. Are they properly placed between the other two separation axioms? The following examples shows that they are.

Example 3.1 Let \( X \) be the set of non-negative integers with the topology whose open sets are those which contain 0 and have finite complement (so closed sets are the finite sets that do not contain 0). Every point is closed except 0, which is neither open nor closed nor even locally closed. This space is neither \( T_{\frac{1}{2}} \) nor \( T_D \) (= singletons are locally closed), although it is \( T_0 \). However it is \( T_{\frac{1}{4}} \). To see that it is not \( T_{\frac{1}{3}} \), just note that every subset of \( X \) is compact, so if it is \( T_{\frac{1}{2}} \), it is also \( T_{\frac{1}{3}} \), but we have proved it is not.

Example 3.2 Next we present an example of a \( T_{\frac{1}{4}} \)-space that is not \( T_{\frac{1}{3}} \). Let \( X \) be an uncountable set and let \( p \) a fixed point in \( X \). Then the family \( \tau_p = \{\emptyset\} \cup \{U \subset X : p \in U \text{ and } X \setminus U \text{ is countable}\} \) is called the cocountable topology in \( X \) generated by \( p \) and any space \( X \) equipped with such a topology is called cocountably point generated space. Using Theorem 3.5, we first show that this space is \( T_{\frac{1}{4}} \). Let \( K \) be compact and \( y \notin K \). If \( y \neq p \), then we are done, since \( \{y\} \) is closed. If \( y = p \), then \( K \) does not contain \( y \). We will show that \( K \) is finite. If \( K \) is infinite, let \( K = A \cup B \), where \( A \) and \( B \) are disjoint and \( A \) is denumerable. Now, \( \{B \cup \{x\} : x \in A\} \) is an infinite open cover of \((K, \tau_p|K)\), which has no finite subcover.
By contradiction, \( K \) is finite and thus closed. However, \((X, \tau_p)\) is not a \( T_{\frac{3}{2}} \)-space, since \( \{p\} \) is neither open nor closed.

The proof of the following result is easy and hence omitted.

**Proposition 3.3** For an anti-compact topological space \((X, \tau)\) the following conditions are equivalent:

1. \( X \) is \( T_{\frac{3}{2}} \).
2. \( X \) is \( T_{\frac{3}{4}} \).

**Question 3.4** The only compact subsets of the space in Example 3.2 are the finite ones. Is it true that if a topological space \( X \) is \( T_{\frac{3}{2}} \) and is not \( T_{\frac{3}{4}} \), then \( X \) is anti-compact?

We have the following characterization of \( T_{\frac{3}{2}} \)-spaces.

**Theorem 3.5** For a topological space \((X, \tau)\) the following conditions are equivalent:

1. For every compact subset \( F \) of \( X \) and every \( y \notin F \) there exists a set \( A_y \) containing \( F \) and disjoint from \( \{y\} \) such that \( A_y \) is either open or closed.
2. \( X \) is \( T_{\frac{3}{2}} \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( F \subseteq X \) be a compact subset of \( X \). Then for every point \( y \notin F \) there exists a set \( A_y \) containing \( F \) and disjoint from \( \{y\} \) such that \( A_y \) is either open or closed. Let \( L \) be the intersection of all open sets \( A_y \) and let \( C \) be the intersection of all closed sets \( A_y \). Clearly, \( L \) is a \( \Lambda \)-set and \( C \) is closed. Note that \( F = L \cap C \). This shows that \( F \) is \( \lambda \)-closed, and hence is \( T_{\frac{3}{2}} \).

(2) \( \Rightarrow \) (1) Let \( F \) be a compact subset of \( X \) and \( y \) be a point of \( X \setminus F \). Since \( X \) is \( T_{\frac{3}{2}} \) \( F \) is \( \lambda \)-closed, \( F = L \cap C \), where \( L \) is a \( \Lambda \)-set and \( C \) is closed. If \( C \) does not contain \( y \), then \( X \setminus C \) is an open set containing \( y \) and we are done. If \( C \) contains \( y \), then \( y \notin L \) and thus for some open set \( U \), containing \( F \), we have \( y \notin U \), and we are also done. \( \square \)

Recall that topological space \((X, \tau)\) is called a \emph{weak} \( R_0 \)-space [2] if every \( \lambda \)-closed singleton is a \( \Lambda \)-set. Note that \( T_{\frac{3}{2}} \) neither implies weak \( R_0 \) nor \( R_0 \), since a space is \( T_1 \) if and only if is \( T_0 \) and \( R_0 \) if and only if is \( T_0 \) and weak \( R_0 \) (Theorem 2.9 of [3]), and we have that \( T_{\frac{3}{2}} \) implies \( T_0 \) and there are \( T_{\frac{3}{2}} \) spaces that are not \( T_1 \).
Question 3.6 Is there a (nice) characterization for semiregular spaces in terms of $T_{\kappa,\xi}$-spaces?

Acknowledgement. The authors thank the referee for his help in improving the quality of this paper.

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Area of Geometry and Topology  
Faculty of Science  
Universidad de Almería  
04071 Almería  
Spain  
e-mail: farenas@ualm.es

Department of Mathematics  
University of Helsinki  
PL 4, Yliopistonkatu 15  
00014 Helsinki  
Finland  
e-mail: dontchev@cc.helsinki.fi, dontchev@e-math.ams.org  
[http://www.helsinki.fi/~dontchev/](http://www.helsinki.fi/~dontchev/)

Area of Geometry and Topology  
Faculty of Science  
Universidad de Almería  
04071 Almería  
Spain  
e-mail: mpuertas@ualm.es