On multilinear distorted multiplier estimate and its applications

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In this article, we investigate the multilinear distorted multiplier estimate (Coifman–Meyer type theorem) associated with the Schrödinger operator $H = -\Delta + V$ in the framework of the corresponding distorted Fourier transform. Our result is the “distorted” analog of the multilinear Coifman–Meyer multiplier operator theorem in Coifman and Meyer (1978), which extends the bilinear estimates of Germain et al (2015) to multilinear case for all dimensions. As applications, we give the estimate of Leibniz’s law of integer-order derivations for the multilinear distorted multiplier for the first time, and we obtain small data scattering for a kind of generalized mass-critical NLS with good potential in low dimensions $d = 1, 2$.

KEYWORDS
multilinear estimate, Schrödinger operator, distorted Fourier transform, scattering, nonlinear Schrödinger equation

MSC CLASSIFICATION
35P25; 35Q55; 42B15; 42B10

1 | INTRODUCTION

The study of multilinear pseudodifferential operators goes back to the pioneering works of Coifman and Meyer1–4; since then, there has been a large amount of work on various generalizations of their results, and we will only make a rough list here. After Lacey and Thiele’s work5,6 on the bilinear Hilbert transform, with different assumptions on the symbols, the boundedness of multilinear operators in harmonic analysis in the classical Fourier transform setting has been well studied by many authors, for example, Bényi and Torres,7 Gilbert and Nahmod,8 Grafakos and Kalton,9 Grafakos and Torres,10 Kenig and Stein,11 Muscalu et al,12 and Tomita.13

Some of the multilinear operators studied above are multilinear multipliers defined in the framework of the classical Fourier transform and the classical Fourier transform of a function can be regarded as the projection into the eigenfunctions space of the absolutely continuous spectrum of Laplacian operator $-\Delta$. For a given potential $V: \mathbb{R}^d \to \mathbb{R}$, consider the associated Schrödinger operator $H := -\Delta + V$. When $V \in L^2(\mathbb{R}^d)$, $H$ can be realized as a self-adjoint operator on $L^2(\mathbb{R}^d)$ with domain $\mathcal{D}(H) = H^2(\mathbb{R}^d)$. We can impose a compactness condition on the multiplication operator associated with $V$ so that the spectral properties of $H$ resemble those of $H_0 = -\Delta$. We say that $V$ is short-range (or of class SR) provided that

$$u \in H^2_2(\mathbb{R}^d) \leftrightarrow (1 + |x|)^{1+\varepsilon}Vu \in L^2_2(\mathbb{R}^d)$$

is a compact operator,

for some $\varepsilon > 0$. It was shown by Agmon14 that, for $V$ of class SR, $\sigma(H) = \{\lambda_j\}_{j \in \mathbb{N}} \cup [0, \infty)$; the continuous spectrum being $[0, \infty)$, and the discrete spectrum consisting of a countable set of real eigenvalues $\{\lambda_j\}$, each of finite multiplicity.
Furthermore, we have the orthogonal decomposition

\[ L^2(\mathbb{R}^d) = L^2_{ac}(\mathbb{R}^d) \oplus L^2_p(\mathbb{R}^d), \]

where \( L^2_{ac}(\mathbb{R}^d) \) is the span of the eigenfunctions corresponding to the eigenvalues \( \{ \lambda_j \} \) and \( L^2_p(\mathbb{R}^d) \) is the absolutely continuous subspace for \( H \). Then, we may try to define distorted Fourier transform on the absolutely continuous subspace for \( H \), and multilinear distorted multiplier, see Theorem 1.1 and (1.14), respectively, for details below.

We investigate the estimate of the multilinear distorted multiplier (Coifman–Meyer type theorem) associated with the Schrödinger operator \( H = -\Delta + V \) in the framework of the corresponding distorted Fourier transform. As we know, there is only a small amount of research on this topic. Germain et al\(^1\) investigated the bilinear estimates and applied it to the 3D quadratic nonlinear Schrödinger equations with a potential \( V \), but have very little regularity in \( x \). More precisely, for \( \varepsilon > 0 \) (cf. Agmon\(^1\)), Under this assumption, the distorted plane waves are relatively smooth in \( x \) but have very little regularity in \( \xi \). More precisely, for fixed \( \xi \in \mathbb{R}^d \setminus \{0\}, \)

\[ e(\cdot; \xi) \in (x)^s H^2_{\xi} \quad \text{for any} \quad s > (d + 1)/2, \tag{1.3} \]

however, the map \( (x, \xi) \mapsto e(x; \xi) \) is merely measurable. One can improve this by requiring additional decay and regularity of \( V \) (cf., e.g., Ikebe\(^1\)).
In view of the Fourier transform, we expect that the family \(\{e^\cdot;\xi\}\) forms a basis for the absolutely continuous subspace of \(H\). This is indeed true, as was first proved by Ikebe\(^{17}\) and later generalized by several authors. For consistency of presentation, we give here the version due to Agmon (cf., Agmon\(^{14}\) Theorem 6.2). Before that, let us now impose assumption H2, namely, that \(H\) has no discrete spectrum. However, we remark that many results in this paper can be directly generalized to potentials with discrete eigenvalues by simply projecting on the absolutely continuous subspace \(L^2\) throughout. That said, the result is the following.

**Theorem 1.1** (Agmon\(^{14}\) and Ikebe\(^{17}\)). Consider the Schrödinger operator \(H\) with potential \(V\) satisfying H2 and

\[
(1 + |x|^{2(1+\epsilon)}) \int_{B(x)} |V(y)|^2 |y - x|^{-d+\epsilon} dy \in L^\infty_{\infty}(\mathbb{R}^d) \text{ for some } \epsilon > 0, 0 < \theta < 4. \tag{1.4}
\]

Define the distorted Fourier transform \(F^d\) by

\[
\left( F^d f \right)(\xi) := f^d(\xi) := \frac{1}{(2\pi)^{d/2}} \lim_{R \to \infty} \int_{B_R} e(x;\xi) f(x) dx,
\]

where \(B_R\) is the ball of radius \(R\) centered at the origin in \(\mathbb{R}^d\). Then, \(F^d\) is an isometric isomorphism on \(L^2(\mathbb{R}^d)\) with inverse formula

\[
f(x) = \left( F^{d^{-1}} f^d \right)(x) := \frac{1}{(2\pi)^{d/2}} \lim_{R \to \infty} \int_{B_R} e(x;\xi) f^d(\xi) d\xi,
\]

Moreover, \(F^d\) diagonalizes \(H\) in the sense that, for all \(f \in H^2(\mathbb{R}^d)\),

\[
H f = F^{d^{-1}} M F^d f,
\]

where \(M\) is the multiplication operator \(u \mapsto |x|^2 u\).

**Remark 1.2.** We are now able to give a precise meaning to assumption H1: H1 is satisfied provided that

1. The family of eigenfunctions \(\{e^\cdot;\xi\}\) exists with the regularity stated in Equation (1.3);
2. The operator \(F^d\) defined by Equation (1.5) exists and exhibits the properties described in Theorem 1.1.

Once we have defined the distorted Fourier transform, for any function \(m : \mathbb{R}^d \to \mathbb{C}\), we define the distorted Fourier multiplier \(m(D^p)\) to be the operator,

\[
m(D^p) := F^{d^{-1}} m(\xi) F^d,
\]

this is an analog of the well-studied Fourier multiplier \(m(\nabla)\) given by \(m(\nabla) := F^{d^{-1}} m(\xi) F\). For us, the importance of \(\Omega\) lies in the intertwining relations

\[
e^{itH} = \Omega e^{it\nabla} \Omega*, \quad F^d \Omega = \Omega, \quad m(D^p) = \Omega m(\nabla) \Omega*, \tag{1.9}
\]

In other words, \(\Omega\) allows us to translate back and forth between the flat and distorted cases. Clearly, then, information about the structure and boundedness properties of \(\Omega\) is extremely valuable. We collect some results about the boundedness properties of \(\Omega\) below for different dimensions.

**Theorem 1.3** (Other studies\(^{18–23}\)).

1. \(d \geq 3\). (Previous studies\(^{18–21}\)). Let \(k \in \mathbb{N}\) and consider the Schrödinger operator \(H\) with real potential \(V : \mathbb{R}^d \to \mathbb{R}\) for \(d \geq 3\). Fix \(p_0, k_0\) as follows:

\[
\begin{cases} 
p_0 = 2, k_0 = 0 & \text{if } d = 3 \\
p_0 > d/2, k_0 := \lfloor (d - 1)/2 \rfloor & \text{if } d \geq 4
\end{cases}
\]
The main results

Assume that for some $\delta > (3d/2) + 1$,

$$
\langle x \rangle^\delta \| \partial^\alpha V \|_{L_p^p(\mathbb{R}^d)} \leq L_k^\alpha \left( \mathbb{R}^d \right), \text{ for all } \alpha \text{ with } |\alpha| \leq k + k_0.
$$

(1.10)

Then, $V$ is of class SR and so $\Omega$ and $\Omega'$ are well-defined as operators on $L^2 \left( \mathbb{R}^d \right) \cap W^{k,p} \left( \mathbb{R}^d \right)$. If we additionally assume that $V$ is of Generic-type:

$$
\text{there is } v \in \langle x \rangle^\delta L^2 \left( \mathbb{R}^d \right) \text{ solving } Hu = 0, \text{ for any } \theta > \frac{1}{2}.
$$

(1.11)

Then, $\Omega$ and $\Omega'$ may be extended to bounded operators defined on $W^{k,p} \left( \mathbb{R}^d \right)$.

2. $d = 2$ (Jensen and Yajima22). Suppose that $V(x)$ is real-valued and $|V(x)| \leq C(x)^{-\delta}, x \in \mathbb{R}^d$, for some $\delta > 6$, and 0 is neither an eigenvalue nor a resonance of $H$, namely, there are no solutions $u \in H^2_{loc} \left( \mathbb{R}^2 \right) \setminus \{0\}$ of $-\Delta u + Vu = 0$, which for some $a$, $b_1$, and $b_2$ satisfy for $|\alpha| \leq 1$,

$$
\partial^\alpha_x \left( \frac{u - a - \frac{b_1x_1 + b_2x_2}{|x|^2}}{x} \right) = O \left( |x|^{-1-\varepsilon-|\alpha|} \right), \text{ } |x| \to \infty.
$$

Then, the wave operators $\Omega$ are bounded in $L^p \left( \mathbb{R}^2 \right)$ for all $p, 1 < p < \infty$. Moreover, the boundedness of wave operator $\Omega$ in the Sobolev space $W^{k,p}(\mathbb{R}^2)$ can be obtained by applying the commutator method for any $1 < p < \infty, k = 0, \ldots, l$, if $V$ satisfies $\|D^\gamma V(x)\| \leq C_\gamma(x)^{-\delta}$ for $|\gamma| \leq 1$ and 0 is neither an eigenvalue nor a resonance of $H$.

3. $d = 1$ (Weder22). Let $f_j(x, k), j = 1, 2, 3k \geq 0$, be the Jost solutions to the following equation:

$$
-\frac{d^2}{dx^2} u + Vu = k^2 u, k \in \mathbb{C}
$$

(1.12)

let $[u, v]$ denote the Wronskian of $u$ and $v$: $[u, v] := \left( \frac{d}{dx} u \right) v - u \frac{d}{dx} v$. A potential $V$ is said to be generic if $[f_1(x, 0), f_2(x, 0)] \neq 0$ and $V$ is said to be exceptional if $[f_1(x, 0), f_2(x, 0)] = 0$. If $V$ is exceptional there is a bounded solution (a half-bound state, or a zero energy resonance) to Equation (1.12) with $k = 0$. For $l = 0, 1, \ldots$, we denote $V^{(l)} := \frac{d^l}{dx^l}V(x)$. Note that $V^{(0)} = V$. Suppose that $V \in L^1_\gamma$ with $\| V \|_{L^1_\gamma} := \int |V(x)|(1 + |x|^\gamma) dx$, where in the generic case $\gamma > 3/2$ and in the exceptional case $\gamma > 5/2$, and that for some $k = 1, 2, \ldots, V^{(l)} \in L^1, l = 0, 1, 2, \ldots, k - 1$. Then, $\Omega$ and $\Omega'$ originally defined on $W^{k,p} \cap L^2, 1 \leq p \leq \infty$ have extensions to bounded operators on $W^{k,p}, 1 < p < \infty$. Moreover, there are constants $C_p, 1 < p < \infty$, such that:

$$
\| \Omega f \|_{k,p} \leq C_p \| f \|_{k,p}, \| \Omega^* f \|_{k,p} \leq C_p \| f \|_{k,p}.
$$

(1.13)

$f \in W^{k,p} \cap L^2, 1 < p < \infty$. Furthermore, if $V$ is exceptional and $a := \lim_{x \to -\infty} f_j(x, 0) = 1, \Omega$ and $\Omega'$ have extensions to bounded operators on $W^{k,1}$ and to bounded operators on $W^{k,\infty}$, and there are constants $C_1$ and $C_\infty$ such that Equation (1.13) holds for $p = 1$ and $p = \infty$.

## 1.3 The main results

We start by considering the following multilinear distorted multiplier of the form:

$$
T(f_1, f_2, \ldots, f_k)(x) := \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} m(\xi_1, \xi_2, \ldots, \xi_k) f_1^x(\xi_1) f_2^x(\xi_2) \cdots f_k^x(\xi_k) e(x, \xi_1) e(x, \xi_2) \cdots e(x, \xi_k) d\xi_1 d\xi_2 \cdots d\xi_k.
$$

(1.14)

When $e(x, \xi_j) = e^{ix\xi_j}, f_j^x = \hat{f}_j$, the multilinear distorted multiplier defined above becomes the Coifman–Meyer multilinear multiplier. Note that the case $m = 1$ corresponds (up to a constant factor) to the product of $f_1, \ldots, f_k$. We say that the multiplier $m$ satisfies Coifman–Meyer type bounds if the following homogeneous bounds hold for sufficiently many multi-indices $\alpha_1, \alpha_2, \ldots, \alpha_k$:
\[ \left| \partial_{\xi_1}^{a_1} \partial_{\xi_2}^{a_2} \cdots \partial_{\xi_k}^{a_k} m(\xi_1, \xi_2, \ldots, \xi_k) \right| \leq C(|\xi_1| + |\xi_2| + \ldots + |\xi_k|)^{-|a_1|+|a_2|+\ldots+|a_k|} \] (1.15)

Our result is the distorted analog of the multilinear Coifman–Meyer multiplier operator theorem in Coifman and Meyer,\(^1\) but with a little integrability index destruction when we have no assumption on the \(L^p\) boundedness of the Riesz transform \(R = \nabla(-\Delta + V)^{-1/2}\).

**Theorem 1.4.** For \(d \geq 1\), let \(V \in L^{p_0}(\mathbb{R}^d)\) be a potential satisfying \(H1, H2,\) and \(H3\) for \(p_{V'} := \frac{d}{s_1+s_2}\), with \(s_1, s_2\) defined in (3.16) below. Suppose that \(m(\xi_1, \xi_2, \ldots, \xi_k)\) is a Coifman–Meyer symbol in \(k\) variables as in Equation (1.15), then

1. For \(p_j, r' \in (1, \infty), j = 1, \ldots, k\) satisfy \(\frac{1}{r'} = \sum_{j=1}^k \frac{1}{p_j}\),

\[ ||T(f_1, \ldots, f_k)||_{L^{r'}(\mathbb{R}^d)} \lesssim m(\cdot) \prod_{j=1}^k ||f_j||_{L^{p_j}(\mathbb{R}^d)} \] (1.16)

provided that the Riesz transform \(R = \nabla(-\Delta + V)^{-1/2}\) is bounded on \(L^{p_j}, j = 1, \ldots, k\) and \(L^{r'}\).

2. suppose instead that \(V\) satisfies \(H3\), for \(p_j, r, p_j \in (1, \infty), j = 1, \ldots, k\) satisfying \(\frac{1}{r'} = \sum_{j=1}^k \frac{1}{p_j} = \sum_{j=1}^k \frac{1}{p_j} - \frac{\varepsilon}{d}\) for some \(\varepsilon > 0\),

\[ ||T(f_1, \ldots, f_k)||_{L^{r'}(\mathbb{R}^d)} \lesssim m(\cdot) \prod_{j=1}^k ||f_j||_{L^{p_j}(\mathbb{R}^d)} + \prod_{j=1}^k ||f_j||_{L^{p_j}(\mathbb{R}^d)} \] (1.17)

**Remark 1.5** (More discussions of the conditions for Theorem 1.4).

1. **About assumption H1:** It follows that sufficient conditions for H1 are that \(V\) satisfies H2, Equation (1.4) and \(V = O(|x|^{-\varepsilon})\) as \(|x| \to \infty\), for some \(\varepsilon > 0\).

2. **About assumption H2:** Once Equation (1.4) is satisfied, we rule out the existence of nonnegative eigenvalues.

   Additionally, if the negative part of \(V\) is not very large, there are no negative eigenvalues (e.g., if \(d \geq 3\), Hardy’s inequality implies that the condition \(V \geq -(d-2)|x|^2/4|\) is sufficient to rule out both nonpositive eigenvalues and resonances at 0 as defined in Equation 1.11), then H2 holds.

3. **About Riesz transform:** (a) If \(V\) belongs to the class of \(B_d\), by theorem 1.2 in Auscher and Ben Ali,\(^{24}\) the Riesz transform \(R = \nabla(-\Delta + V)^{-1/2}\) is bounded on \(L^p, 1 < p < \infty\). Here, in Auscher and Ben Ali,\(^{24}\) \(B_q, 1 < q \leq \infty\), is the class of the reverse Hölder weights: \(w \in B_q\) if \(w \in L^{q,\infty}(\mathbb{R}^d)\), \(w > 0\) almost everywhere, and there exists a constant \(C\) such that for all cube \(Q\) of \(\mathbb{R}^d\),

\[ \left( \frac{1}{|Q|} \int_Q w^q(x)dx \right)^{1/q} \leq \frac{C}{|Q|} \int_Q w(x)dx \]

If \(q = \infty\), then the left-hand side is the essential supremum on \(Q\). Examples of \(B_q\) weights are the power weights \(|x|^{-a}\) for \(-\infty < a < d/q\) and positive polynomials for \(q = \infty\). (b) Let \(V\) be a real potential such that

\[ V \in C^\infty(\mathbb{R}^d), \ V = O(x^3) \text{ as } x \to 0. \]

Moreover, assume \(|V(\varepsilon z)| \leq a|z|^{-2} \text{ for } a < (d/2 - 1)^2\) or \(V_- = 0\), where \(V_-\) is the negative part of \(V\), then \(H = -\Delta + V\) does not have a zero-resonance nor nontrivial \(L^2\) kernel, by theorem 1.3 in Guillarmou and Hassell.\(^{25}\) For \(d \geq 3\), the Riesz transform \(R = \nabla(-\Delta + V)^{-1/2}\) is bounded on \(L^p, 1 < p < d\).

**Remark 1.6.** When the symbol depends on the spatial variable \(x\), that is, \(m(\xi_1, \xi_2, \ldots, \xi_k)\) becomes \(m(x; \xi_1, \xi_2, \ldots, \xi_k)\), we think that following the argument to the proof of Theorem 1.4, we can get the same result if we add appropriate condition to \(m(x; \xi_1, \xi_2, \ldots, \xi_k)\) on the spatial variable \(x\) additionally. In particular, when \(m(x; \xi_1, \xi_2, \ldots, \xi_k) = a(x)m(\xi_1, \xi_2, \ldots, \xi_k)\), we can get the same conclusion as Theorem 1.4 by just letting \(a \in L^\infty\) additionally.

Our results extend Germain et al.’s\(^{15}\) bilinear estimates to the multilinear case and hold for all dimensions \(d \geq 1\). We think the assumptions for the potential in our theorem can be weakened properly, and now the assumptions for the potential are not optimal. Note that our multilinear estimate cannot be obtained directly from bilinear estimate by
induction, because even in the framework of classical Fourier transform, it can not be obtained. In our proof, after the
distorted-frequency localization, we do not divide the distorted-frequency region of multiple summations into upper and
lower triangular regions roughly according to the symmetry, such as \( \Lambda(f_1, \ldots, f_{k+1}) = C \sum_{N_i < \ldots < N_{k+1}} \Lambda(f_1, N_i, \ldots, f_{k+1}, N_{k+1}) \).

where \( k + 1 \)-linear form \( \Lambda(f_1, \ldots , f_k, f_{k+1}) \) is defined in Equation (3.1). While we divide the distorted-frequency region
of multiple summations into high distorted-frequency and low distorted-frequency parts, the low distorted-frequency part is

\[
\Lambda_L(f_1, \ldots, f_{k+1}) := \sum_{N_i \leq 1}^{N_1 \leq \ldots \leq N_{k+1}} \Lambda(f_1, N_i, \ldots, f_{k+1}, N_{k+1}) = \Lambda(f_{1; \leq 1}, \ldots, f_{k+1; \leq 1}).
\]

For the low distorted-frequency part, we eliminate the multiple summations and estimate directly, and then obtain the
multilinear estimate without the destruction of integrability index. In this way, we will not worry too much about the
limitation of index and dimensions of Sobolev's embedding theorem in the subsequent proof. This removes the limitation
of dimension; for the high distorted-frequency part, we partly decompose the multiple summations region into the upper
and lower triangular regions, and make use of more symmetries and cancellations, so as to cut down the multiplicities
of the summations about distorted-frequency and prevent the logarithmic divergence caused by multiple summations.
Finally we obtain as desired, see Section 3 for details.

Compared with the flat case of multilinear multipliers, the difficulties we face come from the nonlinear spectral distri-
bution when we try to estimate the multilinear distorted multipliers; here, the nonlinear spectral distribution is given as
follows

\[
M(\xi_1, \ldots, \xi_k, \xi_{k+1}) = \int_{\mathbb{R}^d} e(x, \xi_1)e(x, \xi_2) \ldots e(x, \xi_{k+1})dx
\]
in flat case, \( M(\xi_1, \ldots, \xi_k, \xi_{k+1}) = \delta(\xi_1 + \ldots + \xi_k + \xi_{k+1}) \). However, in the distorted case, \( M(\xi_1, \ldots, \xi_k, \xi_{k+1}) \neq \delta(\xi_1 + \ldots + \xi_k + \xi_{k+1}) \), we don’t have convolution structure \( F^\xi(fg) = \int f^\xi(\xi - \eta)g^\xi(\eta)d\eta \) any more. There-
fore, we know little about the distorted-frequency support distribution of the multilinear distorted multipliers, which
makes it impossible for us to estimate the fractional derivatives of the multilinear distorted multipliers by Bony decom-
position. We only obtain the estimates of integer derivatives of the multilinear distorted multipliers providing that the
Riesz transform \( R = \nabla(-\Delta + V)^{-1/2} \) is bounded on \( L^p, 1 < p < \infty \), and we obtain the estimates of even integer derivatives
of the multilinear distorted multipliers if we do not have the assumption that the Riesz transform \( R = \nabla(-\Delta + V)^{-1/2} \) is
bounded on \( L^p, 1 < p < \infty \).

**Theorem 1.7** (Leibniz’s law of integer order derivations). For \( s \geq 0 \) an integer, \( V \in W^{2,\infty}(\mathbb{R}^d) \cap W^{s,\infty}(\mathbb{R}^d), d \geq 1, \) let \( V \)
satisfy H1, H2, and H3*.

1. **If the Riesz transform** \( R = \nabla(-\Delta + V)^{-1/2} \) **is bounded on** \( L^p, 1 < p < \infty \). **Then, for** \( \sum_{j=1}^{k} \frac{1}{p_j} = 1 - \frac{1}{r} \), **we have**

\[
\|T(f_1, \ldots, f_k)\|_{W^{s,r}} \lesssim \sum_{i=1}^{k} \|f_i\|_{W^{s,r}} \prod_{j=1, j \neq i}^{k} \|f_j\|_{L^{p_j}} + \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}.
\]  

(1.18)

and

\[
\|T(f_1, \ldots, f_k)\|_{W^{s,r}} \lesssim \sum_{i=1}^{k} \|f_i\|_{W^{s,r}} \prod_{j=1, j \neq i}^{k} \|f_j\|_{L^{p_j}}.
\]  

(1.19)

2. **If we do not have the assumption that the Riesz transform** \( R = \nabla(-\Delta + V)^{-1/2} \) **is bounded on** \( L^p, 1 < p < \infty \). **For** \( s = 2k \geq 0 \ **an even integer,** p_j, r_j, p_j' \in (1, \infty), j = 1, \ldots, k **satisfying** \( \frac{1}{p} = \sum_{j=1}^{k} \frac{1}{p_j} = \sum_{j=1}^{k} \frac{1}{p_j'} = \frac{s}{d} \ **for some** \ e > 0 \ **we have**

\[
\|T(f_1, \ldots, f_k)\|_{W^{s,r}} \lesssim \sum_{i=1}^{k} \|f_i\|_{W^{s,r}} \prod_{j=1, j \neq i}^{k} \|f_j\|_{L^{p_j'}} + \prod_{j=1}^{k} \|f_j\|_{L^{p_j}} + \sum_{i=1}^{k} \|f_i\|_{W^{s,r}} \prod_{j=1, j \neq i}^{k} \|f_j\|_{L^{p_j'}} + \prod_{j=1}^{k} \|f_j\|_{L^{p_j'}}.
\]  

(1.20)
and
\[
\|T(f_1, \ldots, f_k)\|_{W^m} \lesssim \sum_{i=1}^{k} \|f_i\|_{W^{m_1}} \prod_{j=1, j \neq i}^{k} \|f_j\|_{L^{m_2}} + \sum_{i=1}^{k} \|f_i\|_{W^{m_1}} \prod_{j=1, j \neq i}^{k} \|f_j\|_{L^{m_2}}. 
\]  
(1.21)

**Remark 1.8.**

1. The second term on the right-hand side of Equations (1.18) and (1.20) comes from the contribution of the term containing \( V(x) \) in Equations (4.1) and (4.2).
2. Without the assumption that the Riesz transform \( \mathcal{R} = \nabla(-\Delta + V)^{-1/2} \) is bounded on \( L^p, 1 < p < \infty \), we can only obtain the estimates for the even integer derivatives of the multilinear distorted multiplier and small data scattering for a kind of generalized potential in low dimensions \( d \).

As another application, we consider the following generalized mass-critical nonlinear Schrödinger equation with good potential in low dimensions \( d = 1, 2 \):
\[
iu_t - \Delta u + Vu = a(x)F(u), \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}^d. 
\]  
(1.22)

when \( d = 1 \), \( F(u) = T(\bar{u}, \bar{u}, u, u, u)(x) \); when \( d = 2 \), \( F(u) = T(\bar{u}, u, u)(x) \). Note that the case symbol \( m = 1 \) corresponds (up to a constant factor) to the product of the functions. Therefore, in this case, when \( V = 0 \) and \( a(x) \equiv 1 \), or \( a(x) \equiv -1 \), the Equation (1.22) becomes a classical mass-critical nonlinear Schrödinger equation in \( d = 1, 2 \).

For good potential \( V \): \( V \) satisfies H1, H2, and H3 and assume that the Riesz transform \( \mathcal{R} = \nabla(-\Delta + V)^{-1/2} \) is bounded on \( L^p, 1 < p < \infty \) when \( d = 1, 2 \), we have the scattering of the generalized mass-critical NLS with good potential for small data in low dimensions \( d = 1, 2 \).

**Theorem 1.9** (Local wellposedness and small data scattering). For \( d = 1, 2 \), \( a(x) \in L^\infty \), the Equation (1.22) has the following properties:

1. (Local wellposedness) For any \( u_0 \in L^2_{r}(\mathbb{R}^d) \), there exists \( T(u_0) > 0 \) such that Equation (1.22) is locally well posed on \([-T, T]\). The term \( T(u_0) \) depends on the profile of the initial data as well as its size. Moreover, Equation (1.22) is well posed on an open interval \( I \subseteq \mathbb{R}, 0 \in I \);

2. (Small data scattering) There exists \( \varepsilon_0(d) > 0 \), such that if
\[
\|u_0\|_{L^2_r(\mathbb{R}^d)} \leq \varepsilon_0(d), 
\]  
(1.23)

then Equation (1.22) is globally well posed and scattering, that is, there exist \( u^\pm \in L^2_r(\mathbb{R}^d) \) such that
\[
\|u(t) - e^{it\Delta} u^\pm\|_{L^2_r} \to 0, \quad \text{as } t \to \pm \infty. 
\]  
(1.24)

Finally, we organize the paper as follows: We list some notations and basic lemmas in Section 2. In the third section, we give the proof of the main results, and the fourth section are the applications, including the estimate of Leibniz’s law of integer-order derivations for the multilinear distorted multiplier and small data scattering for a kind of generalized mass-critical NLS with good potential in low dimensions \( d = 1, 2 \).

2 | PRELIMINARY

We will use the notation \( X \lesssim Y \) whenever there exists some constant \( C > 0 \) so that \( X \leq CY \). Similarly, we will use \( X \sim Y \) if \( X \lesssim Y \lesssim X \). Also, we use the Japanese bracket convention \( \langle x \rangle^2 := 1 + |x|^2 \). Let \( \psi \in C_0^\infty (\mathbb{R}^d) \) be a radial, decreasing function
\[
\psi(x) := \begin{cases} 
1, & \text{if } |x| \leq 1, \\
0, & \text{if } |x| > 2.
\end{cases} 
\]  
(2.1)

For \( N \in 2^\mathbb{Z} \), we denote
\[
\psi \left( \frac{X}{N} \right) - \psi \left( \frac{2X}{N} \right) =: \phi \left( \frac{X}{N} \right). 
\]
The Littlewood–Paley operators are then given by

\[ p_N^d = \phi \left( \frac{D^d}{N} \right) \quad \text{and} \quad p_{<N}^d = \psi \left( \frac{D^d}{N} \right) \quad N \in 2\mathbb{Z}, \]

then we have distorted-frequency decomposition, \( f = \sum_{N \in \mathbb{Z}} p_N^d f. \)

We define distorted Sobolev norm as

\[ \| f \|_{W^{k,p}_\delta} := \left( \sum_{N \in \mathbb{Z}} N^{2\delta} \| p_N^d f \|_{L^p}^2 \right)^{1/2}, \]

\[ \| f \|_{H^{k,p}_\delta} := \left( \sum_{N \in \mathbb{Z}} \langle N \rangle^{2\delta} \| p_N^d f \|_{L^p}^2 \right)^{1/2}. \]

A pair \((p, q)\) is called admissible if

\[ \frac{2}{p} = d \left( \frac{1}{2} - \frac{1}{q} \right) \]

and \(4 \leq p \leq \infty\) when \( d = 1; 2 < p \leq \infty\) when \( d = 2; \) or \(2 \leq p \leq \infty\) when \( d \geq 3.\) By intertwining relations (1.9) and the classical Strichartz estimate, suppose \((p, q)\) and \((\tilde{p}, \tilde{q})\) are admissible pairs, and \( I \subset \mathbb{R}\) is a possibly infinite time interval. Then, we have the following Strichartz estimate for the Schrödinger operator \( H = -\Delta + V\) providing that \( V\) satisfies \( H1, H2,\) and \( H3^*.\)

**Lemma 2.2** (Germain et al\(^{15}\)). For \( 1 < p < \infty \) and \( V\) satisfying \( H1, H2,\) and \( H3^*,\) we have

\[ \left\| \int_0^t e^{itH} f(t) dt \right\|_{L^p_t L^2_x(I \times \mathbb{R}^d)} \leq \tilde{p} \tilde{q} d \| f \|_{L^p(I \times \mathbb{R}^d)} \]

\[ \left\| \int \int_{\mathbb{R}^d} e^{it\Delta} F(t) dt \right\|_{L^q(I \times \mathbb{R}^d)} \]

and

\[ \left\| \int e^{it\Delta} F(t) dt \right\|_{L^p_t L^q_x(I \times \mathbb{R}^d)} \]

Before we begin to prove Theorem 1.4, though, we need the following maximal and square function estimates, which actually are stated by the Lemma 3.3 in Germain et al\(^{15}\).

**Lemma 2.3** (Lemma 3.3 in Germain et al\(^{15}\)). (a) Suppose that \( W\) is an operator that is point-wise bounded by an \( L^p \)

\[ |Wf(x)| \leq C\tilde{W}|f|(x) \quad \text{for all} \quad f \in L^p(\mathbb{R}^d), \quad x \in \mathbb{R}^d \]

for some positive operator \( \tilde{W}\) that is bounded on \( L^p(\mathbb{R}^d)\) for \( 1 \leq p \leq \infty.\) Let \( \psi \in C^0_0(\mathbb{R}^d).\) For each \( n \in \mathbb{R}^d,\) the operators

\[ f \mapsto \sup_{N \in \mathbb{Z}^d} \left| \int \frac{e^{2\pi i \langle x \rangle}}{N} \psi \left( \frac{V}{N} \right) f \right| \quad \text{and} \quad f \mapsto \sup_{N_1, N_2 \in \mathbb{Z}^d} \left| \int \frac{e^{2\pi i \langle x \rangle}}{N_1} \psi \left( \frac{V}{N_1} \right) f \right| \]

are bounded on \( L^p \) for all \( 1 < p \leq \infty \) with a bound \( \leq \langle n \rangle^d.\)
(b) For each $n \in \mathbb{R}^d$, the operators

$$f \mapsto \sup_{N \in \mathbb{Z}} \left| e^{2\pi \frac{i}{n} N} \psi \left( \frac{D_N^f}{N} \right) f \right|$$

and $f \mapsto \sup_{N_1, N_2 \in \mathbb{Z}} \left| e^{2\pi \frac{i}{n_1} N_1} \psi \left( \frac{D_{N_2}^f}{N_2} \right) f \right|$ are bounded on $L^p(\mathbb{R}^d)$ for all $1 < p \leq \infty$ with a bound $\lesssim \langle n \rangle^d$.

(c) Let $U$ be any bounded linear operator on $L^p$ for some $1 \leq p < \infty$ and suppose that $\{f_n\} \subset L^p(\mathbb{R}^d)$ is a sequence of functions. Then

$$\left\| \left( \sum_{n \in \mathbb{Z}} |U f_n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| \left( \sum_{n \in \mathbb{Z}} |f_n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)},$$

whenever the right-hand side is finite.

(d) Moreover, if $\phi$ is smooth and supported on an annulus, the operator

$$f \mapsto \left( \sum_{N_1, N_2 \in \mathbb{Z}} \sup_{N_1, N_2} \left| e^{2\pi \frac{i}{n_1} N_1} \phi \left( \frac{D_{N_2}^f}{N_2} \right) f \right|^2 \right)^{1/2}$$

is bounded on $L^p(\mathbb{R}^d)$ for all $1 < p \leq \infty$ with bound $\lesssim \langle n \rangle^d$.

(e) we have the following point-wise inequality

$$\left| \psi \left( \frac{N}{N} \right) f(x - y) \right| \lesssim \langle N |y| \rangle^d M f(x),$$

where $Mf$ is the Hardy–Littlewood maximal function.

### 3 The Proof of Theorem 1.4

**Proof of Theorem 1.4.** Taking one test function $f_{k+1} \in L^r(\mathbb{R}^d)$, we define $k + 1$-linear form $\Lambda(f_1, \ldots, f_k, f_{k+1})$ which is associated to the $k$-linear operator $T(f_1, \ldots, f_k)$ in the framework of the distorted Fourier transform as follows,

$$\Lambda(f_1, \ldots, f_k, f_{k+1}) := \int_{\mathbb{R}^d} T(f_1, \ldots, f_k, f_{k+1})(x) \, dx$$

$$= \int \cdots \int m(\xi_1, \ldots, \xi_k) f_1^*(\xi_1) \cdots f_k^*(\xi_k) e(x, \xi_1) \cdots e(x, \xi_k) f_{k+1}(x) d\xi_1 \cdots d\xi_k dx$$

$$= \int \cdots \int m(\xi_1, \ldots, \xi_k) f_1^*(\xi_1) \cdots f_k^*(\xi_k) f_{k+1}^*(\xi_{k+1}) M(\xi_1, \ldots, \xi_k, \xi_{k+1}) d\xi_1 \cdots d\xi_k d\xi_{k+1}$$

(3.1)

where $M(\xi_1, \ldots, \xi_k, \xi_{k+1}) = \int_{\mathbb{R}^d} e(x, \xi_1) e(x, \xi_2) \cdots e(x, \xi_{k+1}) dx$ is nonlinear spectral distribution. By duality, for $\frac{1}{r} + \frac{1}{p} = 1$,

$$\|T(f_1, \ldots, f_k)\|_{L^p} \lesssim \sup_{\|f_{k+1}\|_{L^r} \leq 1} |\Lambda(f_1, \ldots, f_k, f_{k+1})|.$$

We generalize the multiplier $m(\xi_1, \ldots, \xi_k)$ as $k + 1$ variables multiplier $m(\xi_1, \ldots, \xi_k, \xi_{k+1})$,

$$\Lambda(f_1, \ldots, f_k, f_{k+1})$$

$$= \int \cdots \int m(\xi_1, \ldots, \xi_k, \xi_{k+1}) f_1^*(\xi_1) \cdots f_k^*(\xi_k) f_{k+1}^*(\xi_{k+1}) M(\xi_1, \ldots, \xi_k, \xi_{k+1}) d\xi_1 \cdots d\xi_k d\xi_{k+1}$$
Step 1. Decomposition of $\Lambda$. We start by Littlewood–Paley decomposition of $f_j$ with respect to distorted Fourier transform.

$$f_j = \sum_{N_j \in \mathbb{Z}} P_{N_j}^d f_j = \sum_{N_j \in \mathbb{Z}} f_j \cdot \mathbb{1}_{N_j}, \quad j = 1, \ldots, k+1.$$  

As a result, we obtain that

$$\Lambda(f_1, \ldots, f_{k+1}) = \sum_{N_j \in \mathbb{Z}} \Lambda(f_1 \cdot \mathbb{1}_{N_j}, \ldots, f_{k+1} \cdot \mathbb{1}_{N_{k+1}})$$

$$= \Lambda_L(f_1, \ldots, f_{k+1}) + \Lambda_H(f_1, \ldots, f_{k+1}).$$

where

$$\Lambda_L(f_1, \ldots, f_{k+1}) := \sum_{N_j \leq 1, \ldots, N_{k+1} \leq 1} \Lambda(f_1 \cdot \mathbb{1}_{N_j}, \ldots, f_{k+1} \cdot \mathbb{1}_{N_{k+1}})$$

and

$$\Lambda_H(f_1, \ldots, f_{k+1}) := \sum_{\max\{|N_1|, \ldots, |N_{k+1}|\} \geq 1} \Lambda(f_1 \cdot \mathbb{1}_{N_1}, \ldots, f_{k+1} \cdot \mathbb{1}_{N_{k+1}})$$

$$+ \sum_{N_2 \geq \max\{|N_1|, \ldots, |N_{k+1}|\}} \Lambda(f_1 \cdot \mathbb{1}_{N_1}, \ldots, f_{k+1} \cdot \mathbb{1}_{N_{k+1}})$$

$$+ \ldots$$

$$+ \sum_{N_{k+1} \geq \max\{|N_1|, \ldots, |N_k|\}} \Lambda(f_1 \cdot \mathbb{1}_{N_1}, \ldots, f_{k+1} \cdot \mathbb{1}_{N_{k+1}})$$

$$=: I_1 + \ldots + I_{k+1}.$$

We first take $\Lambda_L(f_1, \ldots, f_{k+1})$ into consideration. Let $\tilde{\psi} \in C_0^\infty(\mathbb{R}^d)$ be given such that $\tilde{\psi} \psi = \psi$. Define $\tilde{m}$ by

$$\tilde{m}(\xi_1, \xi_2, \ldots, \xi_{k+1}) := m(\xi_1, \xi_2, \ldots, \xi_{k+1}) \tilde{\psi}(\xi_1) \tilde{\psi}(\xi_2) \ldots \tilde{\psi}(\xi_{k+1}),$$

then we expand $\tilde{m}$ in a Fourier series, if $(\xi_1, \ldots, \xi_{k+1}) \in [-K/2, K/2]^{(k+1)d},$

$$\tilde{m}(\xi_1, \ldots, \xi_{k+1}) = \sum_{n_1, \ldots, n_{k+1} \in \mathbb{Z}^d} a(n_1, \ldots, n_{k+1}) \tilde{\psi}(\xi_1) \tilde{\psi}(\xi_2) \ldots \tilde{\psi}(\xi_{k+1}).$$

Using the stationary phase method, we have the bound for $a(n_1, \ldots, n_{k+1})$:

$$|a(n_1, \ldots, n_{k+1})| \lesssim (1 + |n_1| + \ldots + |n_{k+1}|)^{-3(k+1)d},$$

thus by H"{o}lder’s inequality and Lemma 2.3 (b), we have

$$|\Lambda_L(f_1, \ldots, f_{k+1})| = |\Lambda(f_1 \cdot \mathbb{1}_{N_1}, \ldots, f_{k+1} \cdot \mathbb{1}_{N_{k+1}})|$$

$$= \left| \sum_{n_1, n_2, \ldots, n_{k+1}} a(n_1, \ldots, n_{k+1}) \int f_{1 \cdot \mathbb{1}_{N_1}}(x) f_{2 \cdot \mathbb{1}_{N_2}}(x) \ldots f_{k+1 \cdot \mathbb{1}_{N_{k+1}}}(x) dx \right|$$

$$\lesssim \sum_{n_1, n_2, \ldots, n_{k+1}} (1 + |n_1| + \ldots + |n_{k+1}|)^{-3(k+1)d} \prod_{j=1}^{k+1} \| f_{j \cdot \mathbb{1}_{N_j}} \|_{L^p}$$

$$\lesssim \prod_{j=1}^{k+1} \| f_j \|_{L^p},$$

where $f_{j \cdot \mathbb{1}_{N_j}}(x) := P_{f_{j \cdot \mathbb{1}_{N_j}}}^d (\mathcal{F}^{-1}_j n_j \xi_j f_j \mathbb{1}_{j \cdot \mathbb{1}_{N_j}}(\xi_j)).$
For $\Lambda_H(f_1, \ldots, f_{k+1})$ part, we can just treat with $I_1$, because the other terms can be controlled in the same way,

$$I_1 = \sum_{N_1 \geq \text{max}(N_2, \ldots, N_{k+1})} \Lambda(f_1, N_1, \ldots, f_{k+1}, N_{k+1})$$

$$= \sum_{N_1 \geq \text{max}(N_2, N_3, \ldots, N_{k+1})} \sum_{N_1 \geq N_2, N_3, \ldots, N_{k+1}} \Lambda(f_1, N_1, \ldots, f_{k+1}, N_{k+1})$$

$$+ \sum_{N_1 \geq \text{max}(N_2, N_3, \ldots, N_{k+1})} \sum_{N_1 \geq N_2, N_3, \ldots, N_{k+1}} \Lambda(f_1, N_1, \ldots, f_{k+1}, N_{k+1})$$

$$+ \ldots$$

$$\vdots$$

$$= I_{1,2} + \ldots + I_{1,k+1}.$$  \hfill (3.7)

In the following, due to the similarities, we still only estimate the first term $I_{1,2}$, and our default summation range about $N_1$ is $N_1 \geq 1$, if not necessary, we will not mention it again.

Let $\phi \in C_0^\infty(\mathbb{R})$ be given such that $\phi\phi = \phi$. Define $\tilde{m}^{N_1}$ by

$$\tilde{m}^{N_1} \left( \frac{\xi_1}{N_1}, \frac{\xi_2}{N_1}, \ldots, \frac{\xi_{k+1}}{N_1} \right) := m(\xi_1, \xi_2, \ldots, \xi_{k+1})\phi \left( \frac{\xi_1}{N_1} \right) \phi \left( \frac{\xi_2}{N_1} \right) \cdots \phi \left( \frac{\xi_{k+1}}{N_1} \right).$$  \hfill (3.8)

then we expand $\tilde{m}^{N_1}$ in a Fourier series, if $(\xi_1, \ldots, \xi_{k+1}) \in [-K/2, K/2]^{k+1}$,

$$\tilde{m}^{N_1} (\xi_1, \ldots, \xi_{k+1}) = \sum_{n_1, \ldots, n_{k+1} \in \mathbb{Z}^d} a^{N_1} (n_1, \ldots, n_{k+1}) e^{2\pi i \sum_{j=1}^{k+1} n_j \xi_j}.$$  \hfill (3.9)

Using the stationary phase method, we have the bound for $a^{N_1} (n_1, \ldots, n_{k+1})$:

$$|a^{N_1} (n_1, \ldots, n_{k+1})| \lesssim (1 + |n_1| + \ldots + |n_{k+1}|)^{-3(k+1)d}$$  \hfill (3.10)

meanwhile,

$$I_{1,2} = \sum_{N_1 \geq N_2, N_3, \ldots, N_{k+1}} \sum_{(\xi_1, \ldots, \xi_{k+1}) \in \mathbb{Z}^d} \Lambda(f_1, N_1, \ldots, f_{k+1}, N_{k+1})$$

$$= \sum_{N_1 \geq N_2, N_3, \ldots, N_{k+1}} \sum_{n_1, \ldots, n_{k+1} \in \mathbb{Z}^d} a^{N_1} (n_1, \ldots, n_{k+1}) \Xi_{N_1, \ldots, N_{k+1}}.$$  \hfill (3.11)

where

$$\Xi_{N_1, \ldots, N_{k+1}} := \int \cdots \int (e^{2\pi i n_1 \xi_1} f^{\sharp}_{1, N_1} (\xi_1)) \cdots (e^{2\pi i n_{k+1} \xi_{k+1}} f^{\sharp}_{k+1, N_{k+1}} (\xi_{k+1}))$$

$$M(\xi_1, \ldots, \xi_{k+1}) d\xi_1 \ldots d\xi_{k+1}$$

$$= \int \cdots \int f^{\sharp}_{1, N_1} (\xi_1) f^{\sharp}_{2, N_2} (\xi_2) \cdots f^{\sharp}_{k+1, N_{k+1}} (\xi_{k+1})$$

$$M(\xi_1, \ldots, \xi_{k+1}) d\xi_1 \ldots d\xi_{k+1}$$

$$= \int f_{1, N_1} (x) f_{2, N_2} (x) \cdots f_{k+1, N_{k+1}} (x) dx.$$
Here, we have denoted $f^p_{1,N_1,n_1}(\xi_1) := e^{2\pi i n_1 \xi_1} f^p_{1,N_1}(\xi_1)$, and $f^p_{j,N_2,n_2}(\xi_2) := e^{2\pi i n_2 \xi_2} f^p_{j,N_2}(\xi_2)$, $j = 2, \ldots, k + 1$. Therefore,

$$
|I_{1,2}| = \left| \sum_{N_1} \sum_{N_2 \leq N_1} \sum_{N_3 \leq N_2} \sum_{N_4 \leq N_3} \Lambda(f_{1:N_1}, \ldots, f_{k+1:N_{k+1}}) \right| \\
= \left| \sum_{N_1} \sum_{N_2 \leq N_1} \sum_{N_3 \leq N_2} \sum_{N_4 \leq N_3} a^{N_1}(n_1, \ldots, n_k) \Xi_{N_1, \ldots, N_{k+1}}^{n_1, \ldots, n_{k+1}} \right| \\
\leq \sum_{N_1 \geq 1} \sum_{N_2} \sum_{N_3 \leq N_2} \sum_{N_4 \leq N_3} (1 + |n_1| + \ldots + |n_{k+1}|)^{-3(k+1)d} \sum_{N_1} \sum_{N_2} \sum_{N_3} \sum_{N_4} \Xi_{N_1, \ldots, N_{k+1}}^{n_1, \ldots, n_{k+1}}.
$$

As a result, we are reduced to proving the following estimates:

1. For $p_j \in (0, \infty)$ s.t. $\sum_{j=1}^{k+1} \frac{1}{p_j} = 1$, assume that the Riesz transform $R = \nabla(-\Delta + V)^{-1/2}$ is bounded on $L^{p_j}$, $j = 1, \ldots, k + 1$, then we have

$$
\left| \sum_{N_1 \geq 1} \sum_{N_2} \sum_{N_3 \leq N_2} \sum_{N_4 \leq N_3} \Xi_{N_1, \ldots, N_{k+1}}^{n_1, \ldots, n_{k+1}} \right| \leq \prod_{j=1}^{k+1} \langle n_j \rangle^d \| f_j \|_{L^{p_j}}.
$$

2. Suppose instead that $V$ satisfies $H^{3*}$ and we do not have the assumption of Riesz transform $R = \nabla(-\Delta + V)^{-1/2}$ being bounded on $L^{p_j}$, $j = 1, \ldots, k + 1$ any more, then for $p_j, \tilde{p}_j \in (0, \infty)$ s.t. $\sum_{j=1}^{k+1} \frac{1}{p_j} = 1$, $\sum_{j=1}^{k+1} \frac{1}{\tilde{p}_j} = 1 + \frac{c}{d}$, we have

$$
\left| \sum_{N_1 \geq 1} \sum_{N_2} \sum_{N_3 \leq N_2} \sum_{N_4 \leq N_3} \Xi_{N_1, \ldots, N_{k+1}}^{n_1, \ldots, n_{k+1}} \right| \leq \prod_{j=1}^{k+1} \langle n_j \rangle^d \| f_j \|_{L^{p_j}} + \prod_{j=1}^{k+1} \langle n_j \rangle^d \| f_j \|_{L^{\tilde{p}_j}}.
$$

**Step 2.** Recall

$$
M(\xi_1, \ldots, \xi_k, \xi_{k+1}) = \int_{\mathbb{R}^d} e(x, \xi_1)e(x, \xi_2) \ldots e(x, \xi_{k+1}) dx.
$$
Using Green's formula and the definition of distorted plane wave functions, formally, we have

\[ |\xi_1|^2 M(\xi_1, \ldots, \xi_k, \xi_{k+1}) \]

\[ = \int_{\mathbb{R}^d} \text{He}(x, \xi_1) e(x, \xi_2) \ldots e(x, \xi_{k+1}) dx \]

\[ = \int_{\mathbb{R}^d} V(x) e(x, \xi_1) e(x, \xi_2) \ldots e(x, \xi_{k+1}) dx \]

\[ - \int_{\mathbb{R}^d} e(x, \xi_1) \Delta [e(x, \xi_2) \ldots e(x, \xi_{k+1})] dx \]

\[ = \sum_{j=2}^{k+1} |\xi_j|^2 M(\xi_1, \ldots, \xi_k, \xi_{k+1}) \]

\[ - (k - 1) \int_{\mathbb{R}^d} V(x) e(x, \xi_1) e(x, \xi_2) \ldots e(x, \xi_{k+1}) dx \]

\[ + 2 \sum_{2 \leq j < l \leq k+1} \int_{\mathbb{R}^d} e(x, \xi_1) e(x, \xi_2) \ldots \nabla e(x, \xi_j) \cdot \nabla e(x, \xi_l) \ldots e(x, \xi_{k+1}) dx \]

which holds in the sense of distribution. Thus,

\[ \Xi^{n_1, \ldots, n_{k+1}}_{N_1, \ldots, N_{k+1}} \]

\[ = \int \cdots \int f^g_{1;N_1,n_1}(\xi_1) f^g_{2;N_2,n_2,N_1}(\xi_2) \ldots f^g_{k+1;N_{k+1},n_{k+1},N_1}(\xi_{k+1}) \]

\[ \times M(\xi_1, \ldots, \xi_{k+1}) d\xi_1 \ldots d\xi_{k+1} \]

\[ = \sum_{j=2}^{k+1} \int \cdots \int \frac{|\xi_j|^2}{|\xi_1|^2} f^g_{1;N_1,n_1}(\xi_1) f^g_{2;N_2,n_2,N_1}(\xi_2) \ldots f^g_{k+1;N_{k+1},n_{k+1},N_1}(\xi_{k+1}) \]

\[ \times M(\xi_1, \ldots, \xi_{k+1}) d\xi_1 \ldots d\xi_{k+1} \]

\[ - (k - 1) \frac{1}{N_1^2} \int f_{1;N_1,n_1}(x) f_{2;N_2,n_2,N_1}(x) \ldots f_{k+1;N_{k+1},n_{k+1},N_1}(x) V(x) dx \]

\[ + 2 \sum_{2 \leq j < l \leq k+1} \frac{1}{N_1^2} \int f_{1;N_1,n_1}(x) f_{2;N_2,n_2,N_1}(x) \]

\[ \times \ldots \times \nabla f_{j;N_j,n_j,N_1}(x) \cdot \nabla f_{l;N_l,n_l,N_1}(x) \ldots f_{k+1;N_{k+1},n_{k+1},N_1}(x) dx \]

\[ =: I + II + III. \]

Here, we denote

\[ f_{1;N_1,n_1} := P^{-1} \frac{N_1}{|x|^2} P^g f_{1;N_1,n_1} = P^{-1} e^{2\pi i \frac{n_1}{N_1}} \phi \left( \frac{x}{N_1} \right) P^g f_1, \text{ with } \phi(\xi) := |\xi|^2 \phi(\xi). \]
We start with the contribution of $N_1^j\cdot N_j$, $j = 2, \ldots, k + 1$, and $\hat{f}_{j,N_j,N_j} := F^{-1}\frac{|\xi|^2}{N_j^2}f_{j,N_j,N_j} = F^{-1}e^{2\pi\mu j/N_j}\varphi\left(\frac{\xi}{N_j}\right)F^2f_j$, $j = 2, \ldots, k + 1$, with $\varphi(\xi) := |\xi|^2\varphi(\xi)$. By Hölder’s inequality and Lemma 2.3(e), (b), and (d), for $1 = \sum_{i=1}^{k+1} \frac{1}{p_i}$, we have

$$\sum_{N_1} \left| \sum_{N_2 \leq N_1} \sum_{(\leq N_1)} I \right|$$

$$\leq \sum_{j=2}^{k+1} \sum_{N_1} \left| \sum_{N_2 \leq N_1} \sum_{(\leq N_1)} \int \frac{|\xi_j|^2}{|\xi_1|^2}f_{1,N_1,N_1}(\xi_1)f_{2,N_2,N_1}(\xi_2) \right|$$

$$\cdots \int f_{k+1,\leq N_k+1}(\xi_{k+1})M(\xi_1, \ldots, \xi_{k+1})d\xi_1 \cdots d\xi_{k+1}$$

$$= \sum_{j=2}^{k+1} \sum_{N_1} \left| \sum_{N_2 \leq N_1} \sum_{(\leq N_1)} \int f_{1,N_1,N_1}(\xi_1)f_{2,N_2,N_1}(\xi_2)f_{3,\leq N_3,N_1}(\xi_3) \right|$$

$$\cdots \int f_{k+1,\leq N_k+1}(\xi_{k+1}) \cdots \int f_{k+1,\leq N_k+1}(\xi_{k+1})M(\xi_1, \ldots, \xi_{k+1})d\xi_1 \cdots d\xi_{k+1}$$

$$= \sum_{j=2}^{k+1} \sum_{N_1} \left| \sum_{N_2 \leq N_1} \sum_{(\leq N_1)} \int f_{1,N_1,N_1}(\xi_1)f_{2,N_2,N_1}(\xi_2)f_{3,\leq N_3,N_1}(\xi_3) \right|$$

$$\cdots \int f_{j-1,\leq N_{j-1},N_1}(\xi_{j-1})f_{j+1,\leq N_j+1,N_1}(\xi_{j+1})d\xi_1 \cdots d\xi_{k+1}$$

$$= \sum_{j=2}^{k+1} \left| \sum_{N_2 \leq N_1} \sum_{(\leq N_1)} \int f_{1,N_1,N_1}(\xi_1)f_{2,N_2,N_1}(\xi_2)f_{3,\leq N_3,N_1}(\xi_3) \right|$$

$$\cdots \int f_{j-1,\leq N_{j-1},N_1}(\xi_{j-1})f_{j+1,\leq N_j+1,N_1}(\xi_{j+1})d\xi_1 \cdots d\xi_{k+1}$$

$$\leq \sum_{j=2}^{k+1} \left| \sum_{N_2 \leq N_1} \sum_{(\leq N_1)} \int f_{1,N_1,N_1}(\xi_1)f_{2,N_2,N_1}(\xi_2)f_{3,\leq N_3,N_1}(\xi_3) \right|$$

$$\cdots \int f_{j-1,\leq N_{j-1},N_1}(\xi_{j-1})f_{j+1,\leq N_j+1,N_1}(\xi_{j+1})d\xi_1 \cdots d\xi_{k+1}$$

$$\leq \left\| \left( \sum_{N_1} \int f_{1,N_1,N_1}(\xi_1)^2 \right)^{1/2} \right\|_{L^p_1} \left\| \left( \sum_{N_1} \sup_{N_2 \leq N_1} \left| f_{2,N_2,N_1}(\xi_2) \right| \right)^{1/2} \right\|_{L^p_2} \left\| \sup_{N_1,N_j} \left| f_{j,N_j,N_j}(\xi_j) \right| \right\|_{L^p} \left\| \prod_{i=3}^{k+1} \sup_{N_i,N_j} \left| f_{i,\leq N_i,N_i}(\xi_i) \right| \right\|_{L^p} \left\| f \right\|_{L^p_1}$$

$$\leq \prod_{i=1}^{k+1} \langle n_i \rangle^d \| f \|_{L^p_1}.$$
For II, we use the Sobolev embedding in the distorted Fourier transform setting. Let \( s_j > 0 \) satisfy \( \sum_{j=1}^{2} s_j < 1, s_j \leq \frac{d}{p_j} \), and we denote \( \frac{1}{q_i} = \frac{1}{p_j} - \frac{s_j}{d} \), \( j = 1, 2 \). By Hölder inequality, Lemma 2.3(e), (b), (d), and Sobolev embedding, we have

\[
\sum_{N_1 \geq 1} \left| \sum_{N_2} \sum_{\ldots} \right| II \leq \sum_{N_1 \geq 1} \left| \sum_{N_2} \sum_{\ldots} \frac{1}{N_1} \int f_{1; N_1, n_1} (x) f_{2; N_2, n_2, N_1} (x) \ldots f_{k+1; N_{k+1}, n_{k+1}, N_1} (x) V(x) dx \right|
\]

\[
\leq \sum_{N_1, N_2} \left| \sum_{\ldots} \frac{N_2^{s_2}}{N_1^{1-s_1}} \left| \int \left( N_1^{-s_1} f_{1; N_1, n_1} \right) \left( N_2^{-s_2} f_{2; N_2, n_2, N_1} \right) \left( \prod_{j=3}^{k+1} f_{j; N_j, n_j, N_1} \right) V(x) dx \right| \right|
\]

(3.16)

\[
\leq \int \left( \sum_{N_1 \geq 1} N_1^{2s_1} \left| f_{1; N_1, n_1} \right| \right)^{1/2} \left( \sum_{N_2} N_2^{-2s_2} \sup_{N_1} \left| f_{2; N_2, n_2, N_1} \right| \right)^{1/2} \prod_{j=3}^{k+1} \sup_{N_j, n_j, N_1} \left| f_{j; N_j, n_j, N_1} \right| |V(x)| dx
\]

\[
\leq \|V\|_{L^{p_j}} \prod_{j=1}^{2} \langle n_j \rangle^d \| f_j \|_{\dot{W}^{-s_j, q_j}} \prod_{l=3}^{k+1} \langle n_l \rangle^d \| f_l \|_{L^p} \leq \prod_{j=1}^{k+1} \langle n_j \rangle^d \| f_j \|_{L^p}
\]

Finally, we estimate the contribution of the term III, which is also the one that causes us to complete the proofs separately according to the assumption that the Riesz transform is bounded or unbounded. We first bound the contribution of III under the assumption that \( \mathcal{R} \) is bounded on \( L^p (\mathbb{R}^d) \), \( j = 1, \ldots, k + 1 \).
\[
\leq \int \left( \sum_{N_2} \sup_{N_1 \geq N_2} \left| f_{2,N_2,n_2,N_2} \right|^2 \right)^{1/2} \left( \sum_{N_1 \geq 1} \left| f_{1,N_1,n_1} \right|^2 \right)^{1/2} \\
\times \prod_{m=3, m \neq j, i}^{k+1} \sup_{N_m \leq N_i} \left| f_{m;N_m,n_m,N_m} \right| \sup_{N_i \geq N_j} \left| \mathfrak{B} \tilde{f}_{j,N_j,n_j,N_j} \right| \ dx
\]
\[
\leq \prod_{m=1}^{k+1} \langle n_m \rangle^d \| f_m \|_{L^m}.
\]

This finishes the proof of Equation (3.13).

**Step 3.** Proof of Equation (3.14). We assume that the potential \( V \) satisfies assumption \( H3' \), which implies the \( L^p(\mathbb{R}^d) \) boundedness of the operator \( \mathfrak{B} : f \mapsto (I - \Delta + V)^{-1/2} f = \mathcal{V}(D^\omega)^{-1} f \). This follows directly by noting that \( (D^\omega)^{-1} = \Omega(\mathcal{V})^{-1} \Omega^* \), using assumption \( H3' \) and the boundedness of \( \mathcal{V}(\mathcal{V})^{-1} \). We denote \( \mathfrak{B} f_{j,N_j,n_j,N_j} \). We split the analysis into three cases, depending on the size of \( N_i \) and \( N_j \), by symmetry, we may further assume \( N_i \leq N_j \).

Case 1. \( N_i \leq N_j < 1 \). In this case, applying Lemma 2.3(e), (b), and (d), \( l^2 \leq l^\infty \) and Sobolev embedding, for \( \sum_{j=1}^{k+1} \frac{1}{P_j} = 1 + \frac{1}{d} \), and \( 2 < j \leq l < k + 1 \), we bound as follows:

\[
\sum_{N_1 \geq 1} \sum_{N_2 \leq N_1} \sum_{N_3 \leq N_2} \cdots \sum_{N_{k+1} \leq N_r} |III| \\
\leq \int \sum_{1 \leq N_1} \sum_{N_2 \leq N_1} \sum_{N_3 \leq N_2} \cdots \sum_{N_{k+1} \leq N_r} \left| f_{1,N_1,n_1} \right| \left| f_{2,N_2,n_2,N_2} \right| \left| \mathfrak{B} \tilde{f}_{j,N_j,n_j,N_j} \right| \\
\times \left( N_i^{-c} \mathfrak{B} \tilde{f}_{j,N_j,n_j,N_j} \right) \prod_{m=3, m \neq j, i}^{k+1} \left| f_{m;N_m,n_m,N_m} \right| \ dx
\]
\[
\leq \int \left( \sum_{N_2} \sup_{N_1 \geq N_2} \left| f_{2,N_2,n_2,N_2} \right|^2 \right)^{1/2} \left( \sum_{N_1 \geq 1} \left| f_{1,N_1,n_1} \right|^2 \right)^{1/2} \\
\times \prod_{m=3, m \neq j, i}^{k+1} \sup_{N_m \leq N_i} \left| f_{m;N_m,n_m,N_m} \right| \sup_{N_i \geq N_j} \left| \mathfrak{B} \tilde{f}_{j,N_j,n_j,N_j} \right| \ dx
\]
\[
\leq \prod_{m=1, m \neq l}^{k+1} \langle n_m \rangle^d \| f_m \|_{L^m} \times \langle n_l \rangle^d \| f_l \|_{H^{\infty -1}}
\]
\[
\leq \prod_{m=1}^{k+1} \langle n_m \rangle^d \| f_m \|_{L^m}.
\]
Case 2. \( N_1 \leq 1 \leq N_j \). Similarly to Case 1, for \( \sum_{j=1}^{k+1} \frac{1}{p_j} = 1 + \frac{\varepsilon}{d} \), we can obtain

\[
\sum_{N_1 \geq 1} \left| \sum_{N_2 \leq N_1} \sum_{N_3, \ldots, N_{k+1} \leq (N_2, \ldots, N_{k+1})} \right| III \left| \sum_{N_2 \leq N_1, N_3, \ldots, N_{k+1} \leq (N_2, \ldots, N_{k+1})} \frac{N_j N_k}{N_1^{1+\varepsilon}} \left| f_{1:N_1,n_1} \left| f_{2:N_2,n_2,N_1} \left| \mathcal{B} f_{j:N_j,n_j,N_1} \right| \right. \right. \left. \right. \right.
\]
\[
\times N_1^{-\varepsilon} |\mathcal{B} f_{1:N_j,n_j,N_1}| \prod_{m=3,m\neq j,l}^{k+1} |f_{m:N_2,n_m,N_1}| dx
\]
\[
\lesssim \int \left( \sum_{N_2 \geq (N_2)} \sup_{N_1 \geq N_2} \left| f_{2:N_2,n_2,N_1} \right|^2 \right) \left( \sum_{N_1 \geq 1} \left| f_{1:N_1,n_1} \right|^2 \right)^{1/2}
\]
\[
\times \prod_{m=3,m\neq j,l}^{k+1} \sup_{m=3,m\neq j,l} |f_{m:N_2,n_m,N_1}| \sup_{N_1 \geq N_j} \left| \mathcal{B} f_{1:N_j,n_j,N_1} \right| \sum_{N_1 \geq N_j} \left| \mathcal{B} f_{j:N_j,n_j,N_1} \right| dx
\]
\[
\lesssim \prod_{m=1}^{k+1} \langle n_m \rangle^d \| f_m \|_{L^p} \times \langle n_l \rangle^d \| f_l \|_{L^q^{\varepsilon}}
\]

Case 3. \( 1 \leq N_i \leq N_j \). Similarly, for \( \sum_{j=1}^{k+1} \frac{1}{p_j} = 1 \), we have

\[
\sum_{N_1 \geq 1} \left| \sum_{N_2 \leq N_1} \sum_{N_3, \ldots, N_{k+1} \leq (N_2, \ldots, N_{k+1})} \right| III \left| \sum_{N_2 \leq N_1, N_3, \ldots, N_{k+1} \leq (N_2, \ldots, N_{k+1})} \frac{N_j N_k}{N_1^{1+\varepsilon}} \left| f_{1:N_1,n_1} \left| f_{2:N_2,n_2,N_1} \left| \mathcal{B} f_{j:N_j,n_j,N_1} \right| \right. \right. \left. \right. \right.
\]
\[
\times \prod_{m=3,m\neq j,l}^{k+1} |f_{m:N_2,n_m,N_1}| dx
\]
\[
\lesssim \int \left( \sum_{N_2 \geq (N_2)} \sup_{N_1 \geq N_2} \left| f_{2:N_2,n_2,N_1} \right|^2 \right) \left( \sum_{N_1 \geq 1} \left| f_{1:N_1,n_1} \right|^2 \right)^{1/2}
\]
\[
\times \prod_{m=3,m\neq j,l}^{k+1} \sup_{m=3,m\neq j,l} |f_{m:N_2,n_m,N_1}| \sup_{N_1 \geq N_j} \left| \mathcal{B} f_{1:N_j,n_j,N_1} \right| \sum_{N_1 \geq N_j} \left| \mathcal{B} f_{j:N_j,n_j,N_1} \right| dx
\]
\[
\lesssim \prod_{m=1}^{k+1} \langle n_m \rangle^d \| f_m \|_{L^p}
\]
4 | Application

4.1 | Application 1: Leibniz’s law of integer-order derivations

Taking one test function $f_{k+1} \in L^r(\mathbb{R}^d)$, for $s \geq 0$, since the operator $H = -\nabla + V$ is self-adjoint, and Plancherel’s theorem still holds for distorted Fourier transform, we can get

\[
\int_{\mathbb{R}^d} H T(f_1, \ldots, f_k) f_{k+1}(x) \, dx
= \int_{\mathbb{R}^d} T(f_1, \ldots, f_k) H f_{k+1}(x) \, dx
= \int \cdots \int m(\xi_1, \ldots, \xi_k) f_1(\xi_1) \ldots f_k(\xi_k) e(x, \xi_1) \ldots e(x, \xi_k) H^s f_{k+1}(x) \, dx \, d\xi_1 \ldots d\xi_k \, dx
\]

\[
= \int \cdots \int m(\xi_1, \ldots, \xi_k) f_1(\xi_1) \ldots f_k(\xi_k) f_{k+1}(\xi_k+1)
\times |\xi_{k+1}|^2 M(\xi_1, \ldots, \xi_k, \xi_{k+1}) \, dx \, d\xi_1 \ldots d\xi_{k+1}
\]

where $M(\xi_1, \ldots, \xi_k, \xi_{k+1}) = \int_{\mathbb{R}^d} e(x, \xi_1) e(x, \xi_2) \ldots e(x, \xi_{k+1}) \, dx$ is nonlinear spectral distribution. Let $s = 1$ and $s = \frac{1}{2}$, respectively, for example, using Green’s formula and the definition of distorted plane wave functions, formally in the sense of distribution, we have

Case $s = 1$

\[
|\xi_{k+1}|^2 M(\xi_1, \ldots, \xi_k, \xi_{k+1})
= \int_{\mathbb{R}^d} e(x, \xi_1) e(x, \xi_2) \ldots H e(x, \xi_{k+1}) \, dx
= \int_{\mathbb{R}^d} V(x) e(x, \xi_1) e(x, \xi_2) \ldots e(x, \xi_{k+1}) \, dx
- \int_{\mathbb{R}^d} e(x, \xi_{k+1}) \Delta [e(x, \xi_1) \ldots e(x, \xi_k)] \, dx
= \sum_{j=1}^k |\xi_j|^2 M(\xi_1, \ldots, \xi_k, \xi_{k+1})
- (k - 1) \int_{\mathbb{R}^d} V(x) e(x, \xi_1) e(x, \xi_2) \ldots e(x, \xi_{k+1}) \, dx
+ 2 \sum_{1 \leq j < k} \int_{\mathbb{R}^d} e(x, \xi_1) e(x, \xi_2) \ldots \nabla e(x, \xi_j) \cdot \nabla e(x, \xi_l) \ldots e(x, \xi_{k+1}) \, dx
\]

Therefore,

\[
\int_{\mathbb{R}^d} H T(f_1, \ldots, f_k) f_{k+1}(x) \, dx
= \int_{\mathbb{R}^d} T(H f_1, \ldots, f_k) f_{k+1}(x) \, dx + \ldots + \int_{\mathbb{R}^d} T(f_1, \ldots, H f_k) f_{k+1}(x) \, dx
- (k - 1) \int_{\mathbb{R}^d} T(f_1, \ldots, f_k) V(x) f_{k+1}(x) \, dx \tag{4.1}
+ 2 \sum_{1 \leq j < k} \int \cdots \int m(\xi_1, \ldots, \xi_k) f_1(\xi_1) \ldots f_k(\xi_k) f_{k+1}(\xi_k+1)
\times M_{ij}(\xi_1, \ldots, \xi_k, \xi_{k+1}) \, dx \, d\xi_1 \ldots d\xi_{k+1} d\xi_{k+1}
\]
Case $s = \frac{1}{2}$:

\[
|\xi_{k+1}|^2 M(\xi_1, \ldots, \xi_k, \xi_{k+1}) = \int_{\mathbb{R}^d} e(x, \xi_1) e(x, \xi_2) \ldots He(x, \xi_{k+1}) dx
\]

\[
= \int_{\mathbb{R}^d} V(x) e(x, \xi_1) e(x, \xi_2) \ldots e(x, \xi_{k+1}) dx
\]

\[
- \int_{\mathbb{R}^d} e(x, \xi_{k+1}) \Delta [e(x, \xi_1) \ldots e(x, \xi_k)] dx
\]

\[
= \int_{\mathbb{R}^d} V(x) e(x, \xi_1) e(x, \xi_2) \ldots e(x, \xi_{k+1}) dx
\]

\[
+ \sum_{1 \leq j \leq k} \int_{\mathbb{R}^d} e(x, \xi_1) e(x, \xi_2) \ldots \nabla e(x, \xi_j) e(x, \xi_{j+1}) \ldots \nabla e(x, \xi_{k+1}) dx
\]

Therefore,

\[
\int_{\mathbb{R}^d} H^\frac{1}{2} T(f_1, \ldots, f_k) f_{k+1}(x) dx
\]

\[
= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} m(\xi_1, \ldots, \xi_k) f_1^2(\xi_1) \ldots f_k^2(\xi_k)(|D^2|^{-1} f_{k+1})^2(\xi_{k+1})
\]

\[
\times |\xi_{k+1}|^2 M(\xi_1, \ldots, \xi_k, \xi_{k+1}) d\xi_1 \ldots d\xi_k d\xi_{k+1}
\]

\[
= \int_{\mathbb{R}^d} T(f_1, \ldots, f_k) V(x)|D^2|^{-1} f_{k+1}(x) dx
\]

\[
+ \sum_{1 \leq j \leq k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} m(\xi_1, \ldots, \xi_k) f_1^2(\xi_1) \ldots f_k^2(\xi_k)(|D^2|^{-1} f_{k+1})^2(\xi_{k+1})
\]

\[
\times \bar{M}_{j,k+1}(\xi_1, \ldots, \xi_k, \xi_{k+1}) d\xi_1 \ldots d\xi_k d\xi_{k+1}
\]

where $\bar{M}_{j,k}(\xi_1, \ldots, \xi_k, \xi_{k+1}) = \int_{\mathbb{R}^d} e(x, \xi_1) e(x, \xi_2) \ldots \nabla e(x, \xi_j) \cdot \nabla e(x, \xi_k) \ldots e(x, \xi_{k+1}) dx$. To the term involving new nonlinear spectral distribution $\bar{M}_{j,k}(\xi_1, \ldots, \xi_k, \xi_{k+1})$, following the arguments in the proof of Theorem 1.4 and using interpolation, while to the rest terms in Equations (4.1) and (4.2), applying Theorem 1.4 and Sobolev embedding, we derive Theorem 1.7.

### 4.2 Application 2: Scattering of the generalized mass-critical NLS with good potential for small data in low dimensions

We consider the following generalized mass-critical nonlinear Schrödinger equation with good potential in low dimensions $d = 1, 2$:

\[
u_t - \Delta u + Vu = a(x) F(u), \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}^d. \tag{4.3}
\]

when $d = 1$, $F(u) = T(\bar{u}, \bar{u}, u, u, u)(x)$; when $d = 2$, $F(u) = T(\bar{u}, u, u)(x)$. where recalling that

\[
T(f_1, f_2, \ldots, f_k)(x) := \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} m(\xi_1, \xi_2, \ldots, \xi_k) f_1^2(\xi_1) f_2^2(\xi_2) \ldots f_k^2(\xi_k)
\]

\[
eq (x, \xi_1) e(x, \xi_2) \ldots e(x, \xi_k) d\xi_1 d\xi_2 \ldots d\xi_k.
\]

and $m$ is a Coifman–Meyer multiplier satisfying Equation (1.15). Note that the case $m = 1$ corresponds (up to a constant factor) to the product of $f_1, \ldots, f_k$. Therefore, in this case, when $V = 0$ and $a(x) \equiv 1$, or $a(x) \equiv -1$, Equation (4.3) becomes a classical mass-critical nonlinear Schrödinger equation in $d = 1, 2$. For good potential $V$, $V$ satisfies $H1, H2, \text{and } H3$, and assume that the Riesz transform $R = \nabla(-\Delta + V)^{-1/2}$ is bounded on $L^p, 1 < p < \infty$, we have the scattering of the generalized mass-critical NLS with good potential for small data in low dimensions $d = 1, 2$. 
Theorem 4.1 (Local wellposedness and small data scattering). For \( d = 1, 2, a(x) \in L^\infty \), Equation (4.3) has the following properties:

1. (Local wellposedness) For any \( u_0 \in L^2_2(\mathbb{R}^d) \), there exists \( T(u_0) > 0 \) such that Equation (4.3) is locally well posed on \([-T, T]\). The term \( T(u_0) \) depends on the profile of the initial data as well as its size. Moreover, Equation (4.3) is well posed on an open interval \( I \subset \mathbb{R}, 0 \in I \).

2. (Small data scattering) There exists \( \epsilon_0(d) > 0 \), such that

\[
\|u_0\|_{L^2(\mathbb{R}^d)} \leq \epsilon_0(d),
\]

then Equation (4.3) is globally well posed and scattering, that is, there exist \( u^\pm \in L^2_2(\mathbb{R}^d) \) such that

\[
\|u(t) - e^{it\Delta}u^\pm\|_{L^2_2} \to 0, \; \text{as} \; t \to \pm \infty.
\]

Proof. With the Strichartz estimate for the Schrödinger operator \( H = -\Delta + V \) in Section 2, we can obtain local well-posedness and small data scattering for Equation (4.3) by the contraction mapping principle and bootstrap argument. Those steps are standard. We recommend to refer to section 1.3 in Dodson for more details. Due to the different nonlinear terms, we give the nonlinear estimates that may be used below:

\[
\left\| \int_0^t e^{itH}a(x)F(u(\tau))d\tau \right\|_{L^{2d+1}_2(\mathbb{R}\times \mathbb{R}^d)} \lesssim \|F(u)\|_{L^{\frac{2d+2}{d+1}}_2(\mathbb{R}\times \mathbb{R}^d)} \lesssim \|u\|_{L^{\frac{2d+2}{d+1}}_2(\mathbb{R}\times \mathbb{R}^d)}^{1+\frac{1}{d}},
\]

and for example, when \( d = 2 \), \( T(\bar{u}, u)(x) - T(\bar{v}, u)(x) = T(\bar{u} - \bar{v}, u)(x) + T(\bar{v}, u - v)(x) + T(\bar{v}, v)(x) + T(\bar{v}, u - v)(x), \) therefore

\[
\|a(x)(F(u) - F(v))\|_{L^{\frac{2d+2}{d+1}}_2(\mathbb{R}\times \mathbb{R}^d)} \lesssim \left( \|u\|_{L^{\frac{2d+2}{d+1}}_2(\mathbb{R}\times \mathbb{R}^d)}^{\frac{2d+2}{d+1}} + \|v\|_{L^{\frac{2d+2}{d+1}}_2(\mathbb{R}\times \mathbb{R}^d)}^{\frac{2d+2}{d+1}} \right) \|u - v\|_{L^{\frac{2d+2}{d+1}}_2(\mathbb{R}\times \mathbb{R}^d)}^{\frac{2d+2}{d+1}}
\]

Following the standard argument, we first get scattering for Equation (4.3) with respect to the Schrödinger operator \( H = -\Delta + V \) as follows,

\[
\|u(t) - e^{itH}u^\pm\|_{L^2_2} \to 0, \; \text{as} \; t \to \pm \infty.
\]

In addition, since wave operator \( \Omega \) exists and is complete, we finally get scattering for Equation (4.3) in the sense of Equation (4.6).

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CONFLICT OF INTEREST
This work does not have any conflicts of interest.

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