BCS-BEC crossover induced by a synthetic non-Abelian gauge field

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We investigate the ground state of interacting spin-$\frac{1}{2}$ fermions (3D) at a finite density ($\rho \sim k_F^3$) in the presence of a uniform non-Abelian gauge field. The gauge field configuration (GFC) described by a vector $\lambda \equiv (\lambda_x, \lambda_y, \lambda_z)$, whose magnitude $\lambda$ determines the gauge coupling strength, generates a generalized Rashba spin-orbit interaction. For a weak attractive interaction in the singlet channel described by a small negative scattering length ($k_F |a_s| \lesssim 1$), the ground state in the absence of the gauge field ($\lambda = 0$) is a BCS (Bardeen-Cooper-Schrieffer) superfluid with large overlapping pairs. With increasing gauge coupling strength, a non-Abelian gauge field engenders a crossover of this BCS ground state to a BEC (Bose-Einstein condensate) ground state of bosons even with a weak attractive interaction that fails to produce a two-body bound state in free vacuum. For large gauge couplings ($\lambda/k_F \gg 1$), the BEC attained is a condensate of bosons whose properties are solely determined by the gauge field (and not by the scattering length so long as it is non-zero) – we call these bosons “rashbons”. In the absence of interactions ($a_s = 0^-$), the shape of the Fermi surface of the system undergoes a topological transition at a critical gauge coupling $\lambda_F$. For high symmetry gauge field configurations we show that the crossover from the BCS superfluid to the rashbon BEC occurs in the regime of $\lambda$ near $\lambda_F$. In the context of cold atomic systems, this work makes an interesting suggestion of obtaining BCS-BEC crossover through a route other than tuning the interaction between the fermions.

I. INTRODUCTION

Recent experimental progress in the generation of synthetic gauge fields has enhanced the possibilities of controlled experimental studies of outstanding problems of quantum condensed matter and even of high energy physics using cold atomic systems. Many theoretical works have explored the possibilities of generating both Abelian and non-Abelian gauge fields. The experimental work with synthetic gauge fields have been with bosonic $^{87}$Rb atoms. In a recent commentary, the investigation of fermions in synthetic non-Abelian gauge fields has been identified as a key research direction.

The study of interacting fermions in 3D space with controlled interactions has been one of the key successes of cold atoms research. A particular example is the problem of the crossover from a BCS ground state with large overlapping pairs to a BEC of tightly bound bosonic pairs with increasing strength of attractive interactions – a phenomenon that was suggested many years earlier. The superfluid transition temperature on the BCS side is determined by the superfluid energy gap, while on the BEC side the transition temperature is determined by the condensation temperature of the tightly bound bosonic pairs of fermions. A review of BCS-BEC crossover that is particularly useful in the context of this paper may be found in reference.

An interesting question is regarding the fate of interacting fermions in the presence of a non-Abelian gauge field. Motivated by the fact that even a spatially uniform non-Abelian gauge field produces interesting physical effects for bosons, we focus on interacting fermions in uniform non-Abelian gauge fields.

In a recent paper two of us investigated how a uniform non-Abelian gauge field influences the bound state of two spin-$\frac{1}{2}$ fermions interacting via a contact attraction in the singlet channel characterized by a $s$-wave scattering length $a_s$. The type of uniform non-Abelian gauge field considered in that work leads to a generalized Rashba spin orbit interaction. A key finding of that work is that for high symmetry gauge field configurations (more precisely defined in the next section), a two-body bound state exists for any scattering length however small and negative. The study suggested that the BCS-BEC crossover is drastically affected by the presence of a non-Abelian gauge field.

Here we pursue the ideas of reference through a study of the ground state of a finite density of interacting fermions in a non-Abelian gauge field by means of mean field theory. We show that increasing the strength of a non-Abelian gauge field produces a crossover from a BCS superfluid (which is the ground state in the absence of the gauge field) to a BEC of bosons at a fixed interaction (fixed scattering length $a_s$) however small and negative. Further, the bosons that condense to form the BEC at large gauge couplings are tightly bound pairs of fermions whose properties are determined solely by the non-Abelian gauge field – we have ventured to call these bosons “rashbons”. For a given attractive interaction (fixed $a_s$), therefore, the crossover takes place the standard BCS superfluid state to a rashbon BEC. There is an additional feature of the crossover that is particularly noteworthy. The Fermi surface of the non-interacting system ($a_s = 0^-$) undergoes a transition in its topol-
ogy with increasing gauge coupling strength. We show that for high symmetry gauge field configurations, the crossover regime of gauge couplings in the presence of interactions overlaps with the regime of topological transition of the non-interacting Fermi surface. In a sense this provides a “geometrical” view of the crossover.

The statement of the problem we address and a detailed summary of our results are given in Section II. The mean field formulation is detailed in Section III. Section IV describes the results. The paper is concluded with a discussion in Section V. We recommend the reading of Section II and Section V to obtain a physical picture of our results.

II. QUESTION ADDRESSED AND SUMMARY OF RESULTS

In units where the mass of the fermions and Planck’s constant are set to unity, the Hamiltonian of the fermions moving in a uniform non-Abelian gauge field is

\[ H_{GF} = \int d^3r \Psi^\dagger(r) \left[ \frac{1}{2} \left( p^2 - A^\mu \tau^\mu \right) - V_{ext} \right] \Psi(r), \]

where \( \Psi(r) = \{ \psi_\alpha(r) \}, \sigma = \uparrow, \downarrow \) are fermion operators, \( p \) is the momentum, \( A^\mu \equiv A^\mu_i e_i \), are uniform gauge fields, \( \tau^\mu (\mu = x, y, z) \) are Pauli matrices and \( e_i \)'s are the unit vectors in the \( i \)-th direction, \( i = x, y, z \). As in [20], we specialize to \( A^\mu = \lambda \delta^\mu_i \) leading to a Hamiltonian with a generalized Rashba spin-orbit interaction

\[ H_R = \int d^3r \left( \frac{p_i^2}{2} - p_\lambda \cdot \tau \right) \Psi(r), \]

where \( p_\lambda = \sum_i p_i \lambda_i e_i \). The vector \( \lambda = \lambda \hat{\lambda} = \sum_i \lambda_i e_i \) describes a gauge field configuration (GFC) space as depicted in Fig. 1. We call \( \lambda = |\lambda| \) as the gauge coupling strength. High symmetry GFCs are important and have been classified in [20] as prolate, spherical and oblate. Of particular interest are the configurations shown in Fig. 1 called extreme prolate (EP), spherical (S) and extreme oblate (EO).

The one-particle states of \( H_R \) are

\[ |k_\lambda\rangle = |k\rangle \otimes |\alpha k_\lambda\rangle \]

that disperse as

\[ \varepsilon_{k_\lambda} = \frac{k^2}{2} - \alpha |k_\lambda| \]

where \( k \)-the momentum, and \( \alpha = \pm1 \)-the eigenvalues of the helicity operator \( p_\lambda \cdot \tau \), are the good quantum numbers. The quantity \( k_\lambda \) is defined analogously with \( p_\lambda \) in eqn. (2). For any \( \lambda \), the two helicity states for a given \( k \) are degenerate only at \( k = 0 \).

The interaction between the fermions is described by contact attraction in the singlet channel

\[ H_v = v \int d^3r \, \psi_\lambda^\dagger(r) \psi_\lambda^\dagger(r) \psi_\lambda(r) \psi_\lambda(r). \]

\[ \text{FIG. 1. (Color online) Gauge field configuration (GFC) space.} \]

The non-Abelian gauge field of eqn. (2) is described by a vector \( \lambda = (\lambda_x, \lambda_y, \lambda_z) = \lambda \hat{\lambda} \), where \( \lambda = |\lambda| \) is the gauge coupling strength and \( \hat{\lambda} \) is a unit vector. High symmetry GFCs such as extreme prolate (EP, \( \lambda = (0, 0, 1) \)), spherical (S, \( \hat{\lambda} = \frac{1}{\sqrt{3}}(1, 1, 1) \)) and extreme oblate (EO, \( \hat{\lambda} = \frac{1}{\sqrt{2}}(1, 1, 0) \)) are as shown.

The endemic ultraviolet divergence of the theory described by the Hamiltonian

\[ H = H_R + H_v \]

is handled by introducing an ultraviolet momentum cutoff \( \Lambda \). This entails characterization of the attraction by a physical parameter that describes the low energy scattering properties while making the parameter \( v \) depend on \( \Lambda \). More precisely,

\[ \frac{1}{v} + \Lambda = \frac{1}{4\pi a_s}, \]

where \( a_s \) is the s-wave scattering length in free vacuum, i.e., when the gauge field is absent (\( \lambda = 0 \)). In free vacuum (3D) only an attraction larger than a critical strength can produce a two-particle bound state. This is embodied in the fact that for \( a_s < 0 \) (BCS side) there is no two-body bound state; a bound state develops only as \( a_s \to -\infty \), or \( \frac{1}{\sigma^2} \to 0^+ \) (resonance). For \( a_s > 0 \) (BEC side) a bound state is obtained with a binding energy \( E_b = \frac{1}{a_s^2} \).

A uniform non-Abelian gauge field brings about remarkable changes in the two-body problem as shown in reference [20]. Most vividly, for high symmetry GFCs
such as S and EO, the critical scattering length $a_{sc}$ required for the formation of a bound state vanishes, i.e., there is a two-body bound state for any scattering length however small and negative (deep BCS side). The size of the binding energy of the bound state depends on the scattering length and $\lambda$ for the EO GFC while for the S GFC this dependence is algebraic. Another interesting aspect that emerges is the symmetry of the bound-state wave function. In a non-Abelian gauge field the normalized bound-state wave function is made up of spatially symmetric singlet and spatially antisymmetric triplet pieces

$$|\psi_s\rangle = |\psi_s\rangle + |\psi_t\rangle.$$  (8)

Time reversal symmetry of the Hamiltonian is preserved and this two-body-bound-state wave function picks up a nematic spin structure consistent with the symmetry of the GFC. The triplet content of the wave function which also measures the “amount of nematicity” is characterized by a parameter which we call the triplet fraction

$$\eta_t = \langle \psi_t | \psi_t \rangle.$$  (9)

The binding energy $E_b$, the triplet fraction $\eta_t$ and the spin symmetry of the two-body wave function for different GFCs are summarized in Table I. The gist of reference 20 is that high symmetry GFCs induce high degeneracy in the low energy (infrared) one particle density of states and this promotes bound state formation. Colloquially, high symmetry GFCs are “attractive interaction amplifiers”. An aspect of the two body problem that is important in the discussion below is that the physics of the two body problem in the presence of the gauge field ($\lambda > 0$) is completely determined by the dimensionless parameter $\lambda a_s$. All aspects of the solution depend only on $\lambda a_s$ when length and energy are respectively measured in units of $\lambda^{-1}$ and $\lambda^2$. This is true for any GFC except EP GFC as is evident from Table.

In this paper we investigate the system described by the Hamiltonian of eqn. (3) at finite density $\rho$ of the fermions. The finite density of particles introduces an additional energy scale which can be conveniently taken to be the Fermi energy $E_F$ (and an associated Fermi wave vector $k_F$) in the absence of the gauge field ($\lambda = 0$)

$$E_F = \frac{k_F^2}{2} = \frac{1}{2} (3\pi^2 \rho)^{2/3}.$$  (10)

At finite densities of fermions, therefore, the ground state of the system (zero temperature) of eqn. (9) is determined by the dimensionless parameters $k_F a_s$, $\lambda/k_F$, and the direction $\lambda$ in GFC space. In this paper we work at fixed density and study the evolution of the ground state of the system with $\lambda/k_F$ and $k_F a_s$ for various $\lambda$s corresponding to high symmetry GFCs.

The possibility of interesting physics in this system is suggested by the following observation which provides

| GFC | $a_{sc}$ | $a_s < 0$ | $a_s > 0$ |
|-----|---------|---------|---------|
|     | $\lambda |a_s| \ll 1$ | $1/|\lambda a_s| = 0$ | $\lambda a_s \ll 1$ |
|     | $E_b$ | $\eta_t$ | Spin Structure | $E_b$ | $\eta_t$ | Spin Structure | $E_b$ | $\eta_t$ | Spin Structure |
| EP | $-\infty$ | No bound state | 0 | $\frac{1}{2}$ | Bi-axial nematic (BW) | $\frac{1}{a_s^2}$ | 0 | singlet |
| S | 0$^+$ | $\frac{\lambda^2 a_s^2}{3}$ | $\frac{1}{2}$ | Spherical | $\frac{\lambda^2}{3}$ | $\frac{1}{4}$ | Spherical | $\frac{1}{a_s^2} + \frac{2\lambda^2}{3}$ | 0 | singlet |
| EO | 0$^+$ | $\frac{2}{e^2} e^{-\frac{2\pi^2}{\lambda a_s^2}}$ | $\frac{1}{2}$ | Uni-axial nematic (ABM) | 0.22$\lambda^2$ | 0.28 | Uni-axial nematic (ABM) | $\frac{1}{a_s^2} + \frac{\lambda^2}{2}$ | 0 | singlet |

TABLE I. Summary of the two-body problem in high symmetry GFCs ($\lambda > 0$). $E_b$ is the binding energy and $\eta_t$ is the triplet fraction (not reported in reference 20). The bi-axial spin nematic structure is similar to the BW (Balian-Werthamer) or B-phase of $^3$He, and the uni-axial nematic structure to that of the ABM (Anderson-Brinkman-Morel) or A-phase of $^3$He. The values of $E_b$ and $\eta_t$ at resonance, which correspond to the properties of the rashbon (for S and EO cases), are exact results, while others are asymptotic values.
the motivation for this work. Consider a system of non-interacting (NI) fermions ($\mathcal{H}_e = 0$ in eqn. (3)). In the absence of a gauge field ($\lambda = 0$), the ground state has a chemical potential $\mu_{\text{F}}$ and is described by two identical filled Fermi seas bounded by spherical Fermi surfaces of radius $k_F$ – one each for $\uparrow$ and $\downarrow$ spins. In the presence of the gauge field ($\lambda \neq 0$), the helicity $\alpha$ is the good quantum number along with momentum (see eqn. (3)), and hence the ground state will be two Fermi seas, one for each helicity. The chemical potential now depends on $\lambda$ through a function $\mu_{\text{NI}}(\lambda)$ that is determined by $\lambda$. Both of the Fermi seas are generally non-spherical with a shape determined by $\lambda$. Since the one particle states with opposite helicities but with same momentum are non-degenerate (see eqn. (3)), the Fermi surfaces of different helicities are not identical and evolve differently with increasing $\lambda$ (at a fixed density $\rho$). The most interesting aspect is that since the $+\text{ helicity}$ state is lower in energy than the $-\text{ helicity}$ one for all momenta, upon increasing $\lambda$, the volume enclosed by the $+\text{ helicity}$ Fermi surface increases at the expense of that of the $-\text{ helicity}$ Fermi surface. Matters come to a head at a critical gauge coupling $\lambda_T$ (which depends on $\lambda$) where the $-\text{ helicity}$ Fermi sea ceases to exist since the chemical potential $\mu_{\text{NI}}(\lambda)$ falls below the bottom of the $-\text{ helicity}$ band. Thus, for $\lambda \geq \lambda_T$ the ground state is a Fermi sea of only $+\text{ helicity}$. This is illustrated for the EO GFC in fig. 2. The values of $\lambda_T$ determined by the density of particles for various high symmetry GFCs are given in Table II; it is to be noted that in all cases $\lambda_T$ is of order $k_F$. Another aspect to be noted is that there is a change in the topology of the $+\text{ helicity}$ Fermi surface at $\lambda_T$. We call this the Fermi surface topology transition (FSTT) – hence the subscript $T$ in $\lambda_T$. For example, in the EO case, the genus of the $+\text{ helicity}$ Fermi surface changes from zero (homeomorphic to a sphere) to unity (homeomorphic to a torus) at $\lambda_T$ as illustrated in fig. 2.

What happens when the interaction $\mathcal{H}_e$ (eqn. (3)) is turned on? Consider an interaction with a small negative scattering length ($k_F|a_s| \ll 0$, deep BCS side). For $\lambda \ll \lambda_T$, the ground state is a superfluid with overlapping pairs with an exponentially small excitation gap. The chemical potential of this state is nearly unaffected and is $\mu_{\text{NI}}(\lambda)$, that of the non-interacting system in a gauge field. The only qualitative difference from the usual s-wave BCS state is that the pair wave function now has a small triplet content and associated spin nematicity induced by the gauge field. This picture changes drastically in the case of high symmetry GFCs (such as S and EO) when the gauge coupling strength $\lambda$ is tuned past $\lambda_T$. The key finding of this paper is that for high-symmetry GFCs, a BEC of tightly bound pairs is obtained for $\lambda \gg \lambda_T$ even with a small negative scattering length. In other words, one can engineer a BCS-BEC crossover with a high-symmetry GFC by increasing the gauge coupling strength even with a very weak attractive interaction that is unable to produce a two-body bound-state in free vacuum. This result arises from the fact that for $\lambda \gg \lambda_T$, the size of the two-body bound-state wave function (see Table II) becomes smaller than the inter-particle spacing. The fermions therefore form tightly bound pairs which then Bose condense in the zero center of mass momentum state. As is evident from the discussion, the physics of these results owes to the character of high-symmetry GFCs to act as attractive-interaction-amplifiers. Indeed, as $\lambda/\lambda_T \rightarrow \infty$ the chemical potential tends to that determined by the two-particle bound-state energy. Since $\lambda|a_s| \rightarrow \infty$ (fixed $a_s$), the nature of the two body bound state obtained is identical to that obtained with a resonant scattering length in the presence of the gauge field as tabulated in Table II. The properties of this bosonic bound state of two fermions is determined solely

FIG. 2. (Color online) Fermi surface topology transition (FSTT) with increasing gauge coupling strength for EO GFC. (a) Two overlapping spherical Fermi surfaces in the absence of a gauge field (b) A gauge coupling smaller than $\lambda_T$: The union of the $+\text{ and }-\text{ Fermi surfaces}$ forms a spindle torus. The apple of the spindle torus is the $+\text{ helicity}$ Fermi surface which is shown with a blue border, and the lemon of the spindle torus is the $-\text{ helicity}$ Fermi surface which is shown with a red border. (c) The gauge coupling that obtains the FSTT. The $+\text{ helicity}$ Fermi surface is a horn torus, while the $-\text{ helicity}$ Fermi surface vanishes. (d) For a gauge coupling larger than $\lambda_T$, there is only the $+\text{ helicity}$ Fermi surface which is a ring torus. Note that all the figures show only a sectioned half of the Fermi surfaces.
by the Rashba gauge field: We call these emergent bosons as “rashbons” (see Sec. IV B and second para of Sec. V for details). The BEC that is obtained for $\lambda \gg \lambda_T$ is a rashbon condensate. The results for various GFCs are tabulated in Table II which is a summary of this paper.

In the remaining sections we illustrate these conclusions by a mean field theory of the superfluid ground state of this interacting fermion system. Mean field theory is known to give a qualitatively correct description for the superfluid ground state.\(^{23}\)

### III. MEAN FIELD THEORY

We now describe the details of the mean field analysis of the superfluid ground state of fermions in a non-Abelian gauge field. This analysis involves certain straightforward manipulations beyond the standard formulation,\(^{23}\) and hence presented in detail. We note that the present analysis can treat any GFC. Results for specific GFCs of interest will be presented in the next section.

To perform a mean-field analysis of the superfluid ground state we recast the interaction term $\mathcal{H}_v$ in a convenient form

$$\mathcal{H}_v = \frac{\hbar v}{2V} \int d^3r \, S^\dagger(r) S(r),$$

where $S^\dagger(r)$ is the singlet creation operator

$$S^\dagger(r) = \frac{1}{\sqrt{2}} \left( \psi^\dagger_r(r) \psi^\dagger_\uparrow(r) - \psi^\dagger_r(r) \psi^\dagger_\downarrow(r) \right).$$

The interaction in terms of the singlet operators can be cast in momentum space

$$\mathcal{H}_v = \frac{\hbar v}{2V} \sum_q S^\dagger(q) S(q),$$

where $V$ is the volume of the system and the mean-field ansatz corresponds to taking

$$\langle S(q) \rangle = \langle S \rangle \delta_{q,0},$$

where $\langle S \rangle = \langle S(0) \rangle$ with

$$S^\dagger(0) = \frac{1}{\sqrt{2}} \sum_k \left( c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} - c^\dagger_{k\downarrow} c^\dagger_{-k\uparrow} \right).$$

The fermion operators $c^\dagger_{k\sigma}$ are defined by

$$c^\dagger_{k\sigma} = \frac{1}{\sqrt{V}} \int d^3r \, e^{-ik \cdot r} \psi^\dagger_\sigma(r).$$

It is now convenient to sum only over half of the allowed $k$ values in eqn. (15)

$$S(0) = \sqrt{2} \sum_k \left( c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} - c^\dagger_{k\downarrow} c^\dagger_{-k\uparrow} \right)$$

as is here and henceforth indicated by the prime over the summation symbol. The advantage of this exercise is that the operator $S(0)$ can be written in the helicity basis as

$$S(0) = \sqrt{2} \sum_{k\alpha} \alpha e^\dagger_{k\alpha} c^\dagger_{-k\alpha}$$
Introducing a chemical potential \( \mu \), we obtain the mean-field Hamiltonian as

\[
\mathcal{H}_{MF} = \sum_{k\alpha} \xi_{k\alpha} c_{k\alpha}^\dagger c_{k\alpha} + \Delta \sum_{k\alpha} \alpha c_{k\alpha}^\dagger c_{-k\alpha}^\dagger + \frac{\Delta}{\alpha} \sum_{k\alpha} \alpha c_{-k\alpha} c_{k\alpha} - \frac{V \Delta^2}{u} \tag{19}
\]

where \( c_{k\alpha} \) are electron operators associated with the one-particle helicity eigenstate (eqn. [4]), \( \Delta = \frac{\nu(S)}{\sqrt{2}V} \) is the order parameter (taken to be real), and \( \xi_{k\alpha} = \tilde{\xi}_{k\alpha} - \mu \). Here \( \tilde{\xi}_{k\alpha} \) is \( \xi_{k\alpha} \) referred to the bottom of the + helicity band. Noting inversion symmetry, \( \xi_{-k\alpha} = \xi_{k\alpha} \), eqn. [19] can now be recast as

\[
\mathcal{H}_{MF} = \sum_{k\alpha} \left( \xi_{k\alpha} c_{k\alpha}^\dagger c_{-k\alpha}^\dagger + \frac{\Delta}{\alpha} \sum_{k\alpha} \alpha c_{k\alpha}^\dagger c_{-k\alpha}^\dagger \right) + \alpha \sum_{k\alpha} \xi_{k\alpha} - \frac{V \Delta^2}{u} \tag{20}
\]

which now has the standard form except for the fact that the summation over \( k \) is carried out only over half of the momentum space and a sum over the two helicities is taken. The hamiltonian in eqn. [20] can now be diagonalized in terms of the Bogoliubov quasiparticle operators as

\[
\mathcal{H}_{MF} = \sum_{k\alpha} E_{k\alpha} \left( \gamma_{k\alpha 1}^\dagger \gamma_{k\alpha 1} + \gamma_{k\alpha 2}^\dagger \gamma_{k\alpha 2} \right) + \sum_{k\alpha} \left( \xi_{k\alpha} - E_{k\alpha} \right) - \frac{V \Delta^2}{u} \tag{21}
\]

where \( E_{k\alpha} = \sqrt{\xi_{k\alpha}^2 + \Delta^2} \) (\( \Delta \) is also the excitation gap), and

\[
\gamma_{k\alpha 1} = u_{k\alpha} c_{k\alpha} - \alpha v_{k\alpha} c_{-k\alpha}^\dagger \tag{22}
\]

\[
\gamma_{k\alpha 2} = \alpha v_{k\alpha} c_{k\alpha} + u_{k\alpha} c_{-k\alpha}^\dagger
\]

with

\[
u_{k\alpha}^2 = \frac{1}{2} \left( 1 + \frac{\xi_{k\alpha}}{E_{k\alpha}} \right), \quad \nu_{k\alpha}^2 = \frac{1}{2} \left( 1 - \frac{\xi_{k\alpha}}{E_{k\alpha}} \right). \tag{23}
\]

A standard analysis now leads to the gap equation

\[
-\frac{1}{u} = \sum_{k\alpha} \frac{1}{2E_{k\alpha}}. \tag{24}
\]

Noting the inversion symmetry of the problem and using the renormalization of the interaction, the gap equation becomes

\[
-\frac{1}{4\pi a_s} = \frac{1}{2V} \sum_{k\alpha} \left( \frac{1}{2E_{k\alpha}} - \frac{1}{k^2} \right). \tag{25}
\]

The number equation is

\[
\rho = \frac{1}{V} \sum_{k\alpha} \frac{1}{2} \left( 1 - \frac{\xi_{k\alpha}}{E_{k\alpha}} \right). \tag{26}
\]

The solution of eqn. [25] along with the number equation eqn. [26], determines the chemical potential \( \mu \) and the gap parameter \( \Delta \) in the ground state. The ground state \( |\Psi_G\rangle \) of the system is given by

\[
|\Psi_G\rangle = \prod_{k\alpha} (u_{k\alpha} + \alpha v_{k\alpha} c_{k\alpha} c_{-k\alpha}^\dagger) |0\rangle \tag{27}
\]

where \( |0\rangle \) is the fermion vacuum. This can be (up to a normalization) be re-written as

\[
|\Psi_G\rangle = e^{P^\dagger} |0\rangle \tag{28}
\]

where \( P^\dagger \) is the pair creation operator given by

\[
P^\dagger = \sum_{k\alpha} \alpha \phi_{k\alpha} c_{k\alpha}^\dagger c_{-k\alpha}^\dagger \tag{29}
\]

where \( \phi_{k\alpha} = \frac{\nu_{k\alpha}}{u_{k\alpha}} \). The singlet and triplet parts of the pair can be extracted by noting that

\[
P^\dagger = \sum_{k\alpha} \phi_s(k) \left( c_{k+}^\dagger c_{-k+} - c_{k-}^\dagger c_{-k-} \right) \tag{30}
\]

with

\[
\phi_s(k) = \frac{1}{2} (\phi_{k+} + \phi_{k-}) \tag{31}
\]

This analysis sheds light on how an attraction in the singlet channel in presence of a non-Abelian gauge field can produce a triplet piece in the pair wave function. The triplet content \( \eta_t \) is now defined as the weight of the triplet piece of the pair creation operator in eqn. [30]. One can also characterize this by an expectation value of the quadrupole operator of reference [22]. However, this definition for the triplet content \( \eta_t \) given above provides a physically transparent and a simple measure of the quantity of interest.

A remark about the Bogoliubov quasiparticles obtained in eqn. [22] is in order. It appears that for each helicity there are two branches of quasi-particle excitations labeled 1 and 2. Ostensibly, therefore, there are four branches of quasiparticles which at the first sight is
surprising. Note, however, that these four branches are defined only in half of the momentum space. If the Bogoliubov excitation were defined for all \( k \), they will not be independent, for example, \( \gamma_{k2} \equiv \gamma_{-k1} \). This is the motivation behind introduction of the sum over one half of the momentum space in eqn. (17). It is now clear that the formulation recovers the correct counting of excitation, i.e., within the present formulation, two excitations for each \( k \) in momentum space is recovered as four excitations for each \( k \) in half the momentum space.

IV. RESULTS FOR SPECIFIC GAUGE FIELD CONFIGURATIONS

In this section, we shall present results of how the ground state of the system evolves with \( \lambda \) for various high symmetry GFCs. We shall be concerned only with negative scattering lengths \( (a_s < 0) \) since this is the regime which has the most interesting physics. In the absence of the gauge field \( (\lambda = 0) \) there is no two body bound state, and for \( a_s < 0 \) the usual BCS superfluid ground state (BCS0) is obtained. For small \( \lambda \), i.e., \( \lambda \ll \lambda_T \), we expect and find the ground state \( |\Psi_G\rangle \) to be qualitatively close to \( |BCS0\rangle \) state with an exponentially small excitation gap and a chemical potential essentially unaltered from that of the non-interacting problem \( \mu_{NI}(\lambda) \). When \( \lambda \) is increased beyond \( \lambda_T \), we find, in some cases (S and EO GFCs), that the chemical potential \( \mu \) begins to fall and approaches \(-E_b/2\), the value set by energy of the two body bound state. This signals the crossover to the BEC state. Additionally, the pair wave function defined by eqn. (29) approaches the wave function of the two body bound state.

A summary of the results for various GFCs discussed below is given in Table. [11]

FIG. 3. (Color online) Evolution of the triplet content \( \eta_t \) of the pair wave function as a function of the gauge coupling strength \( \lambda \) for an EP GFC with \( k_F a_s = -1 \). The evolution of the same quantity of the non interacting system \( (a_s = 0^-) \) is also shown for comparison.

FIG. 4. (Color online) Evolution of the ground state of a collection of interacting fermions \( (k_F a_s = -\frac{1}{3}) \) with gauge coupling strength \( \lambda \) for the S GFC. (a) Chemical potential obtained from a numerical solution of mean field theory (MFT) is compared with the chemical potential of the non interacting system (NI) and that set by the binding energy of the two body problem \((-E_b/2\)). For \( \lambda \ll \lambda_T \) the chemical potential is indistinguishable from the that of the non-interacting system. For \( \lambda \gtrsim \lambda_T \) the chemical potential approaches the two body value indicating a crossover to a BEC. (b) Evolution of the numerically obtained mean field energy gap \( \Delta \) with the gauge coupling strength \( \lambda \). The analytical result eqn. (36) is also shown and is indistinguishable from the numerical result. (c) The dependence of the triplet content \( (\eta_t) \) of the pair wave function defined in eqn. (29) on the gauge coupling strength. This is compared with the same quantity of the non-interacting system (NI) and with that of the wave function of the two-body bound state. It is seen that the pair wave function evolves to two-body bound-state wave function.

A. Extreme prolate (EP) GFC

This GFC with \( \lambda = (0,0,\lambda) \) has an FSTT at \( \lambda_T = k_F \). Before FSTT \((\lambda < \lambda_T)\), the + helicity Fermi sea consists of the volume enclosed by two intersecting spheres of radius \( k_F \) centered around \((0,0,\pm \lambda)\), while the − helicity Fermi sea is the lens shaped region formed by the volume
common to both spheres. When $\lambda$ exceeds $\lambda_T$, the − helicity Fermi surface vanishes, and the + helicity Fermi sea is made of two disjoint spheres centered at $(0,0,\pm \lambda)$. The chemical potential $\mu_{NI}(\lambda) = E_F$, i. e., is unaffected by the EP gauge field.

For $k_F|a_s| \ll 1$, the standard result\[ for the excitation gap is

$$\frac{\Delta}{E_F} \approx \frac{8}{\pi^2} e^{-\frac{\pi}{W}} \tag{32}$$

and the chemical potential is

$$\mu \approx E_F. \tag{33}$$

Not unexpectedly, the excitation gap $\Delta$ and the chemical potential are unaltered with increasing $\lambda$. The ground state for any $\lambda$ is a superfluid state with large overlapping pairs, and there is no BCS-BEC crossover for the EP GFC. There is, however, a qualitative change in the spin structure of the pair wave function. With increasing $\lambda$, the pair wave function develops a triplet content $\eta_t$ (see Fig. [3]) which attains a value close to $\frac{1}{2}$ at $\lambda = \lambda_T$ and stays so with further increase of $\lambda$.

The physics behind this result can be traced to the fact that for the EP GFC the kinetic energy content inside the non-interacting Fermi sea is unaltered by the increase of $\lambda$. Therefore, the gauge coupling $\lambda$ stays neutral in the competition between kinetic energy and the attractive interaction. This, again, is the reason why the energetics of the two-body problem is unaffected by the presence of an EP gauge field (see Table. [I]). This is a feature specific only to EP GFCs.

It must be noted that the non-interacting ground state also has a triplet content (see Fig. [3]). As is evident (see eqn. (20)), this arises from the fact that the + helicity Fermi sea is different (and larger) than the − helicity Fermi sea. The triplet content of the non-interacting system increases monotonically with $\lambda$ and attains a value of $\frac{1}{2}$ at $\lambda = \lambda_T$ and remains at this value for any larger $\lambda$.

As expected, in the presence of an attractive interaction in the singlet channel ($a_s < 0$), the pairs have a triplet content less than that of the non-interacting system.

We note that the qualitative nature of the results for negative scattering lengths ($a_s < 0$) of larger magnitude are similar to those for $k_F|a_s| \ll 1$.

**B. Spherical (S) GFC**

When $\lambda = \frac{1}{\sqrt{3}}(1,1,1)$ a spherical (S) GFC is obtained. Starting from two identical overlapping spheres at $\lambda = 0$, the non-interacting Fermi surfaces of the two helicities continue to be spheres with their centers at the origin of the momentum space for $0 < \lambda < \lambda_T$. Here $\lambda_T = \frac{\sqrt{3}}{2^{1/3}} k_F$. When $\lambda \ll \lambda_T$, the chemical potential of the non-interacting system depends on $\lambda$ as

$$\frac{\mu_{NI}(\lambda)}{E_F} = 1 - \frac{1}{2^\frac{1}{3}} \left( \frac{\lambda}{\lambda_T} \right)^2 \left( \lambda \ll \lambda_T \right) \tag{34}$$

In this regime, the radius of the + helicity Fermi surface is larger than that of the − helicity Fermi surface. At the
FSTT, the $-$ helicity Fermi surface vanishes and ceases to exist for all $\lambda \geq \lambda_f$. After the FSTT, the $+$ helicity Fermi sea is “a sphere with a hole”, i.e., the region bounded by two concentric spherical Fermi surfaces. For $\lambda \gg \lambda_f$ the chemical potential of the non-interacting system goes as

$$\mu_{NI}(\lambda) = \frac{2 \pi^2}{9} \left( \frac{\lambda_f}{\lambda} \right)^4 (\lambda \gg \lambda_f). \tag{35}$$

Consider now the situation when $k_F|a_s| \ll 1$. When $\lambda = 0$, the usual BCS state with properties given by eqn. (32) and eqn. (33) is the ground state. For $\lambda \ll \lambda_f$, $\mu$ is very nearly equal to that given by eqn. (34); the gap equation can be solved analytically in this regime to obtain

$$\Delta = \frac{8 \mu_{NI}(\lambda)}{\exp \left( \frac{12 \mu_{NI}(\lambda)}{6 \mu_{NI}(\lambda) + \lambda^2} \right)} \exp \left( -\frac{3 \pi \sqrt{\mu_{NI}(\lambda)}}{\sqrt{2|a_s|(6 \mu_{NI}(\lambda) + \lambda^2)}} \right). \tag{36}$$

Fig. 4(a) and (b) show, respectively, the numerical solutions of the chemical potential and gap as a function of $\lambda$. Fig. 4(a) also shows the non-interacting chemical potential, and the two-body energy $-E_b/2$ (which depends on $\lambda$ and $a_s$ only). As is evident the chemical potential $\mu$ is identical to the non-interacting value $\mu_{NI}(\lambda)$ for $\lambda \ll \lambda_f$. There is also excellent agreement for the gaps obtained from the numerical solution with the analytical result given in eqn. (36). When $\lambda$ reaches $\lambda_f$ the chemical potential begins fall below $\mu_{NI}$, and on further increase of $\lambda$ ($\lambda \gtrsim \lambda_f$), the chemical potential tends to that set by the two body problem. This clearly signals a crossover from the BCS like state for $\lambda \ll \lambda_f$ to a BEC state where the fermions from tightly bound bosonic pairs which then condense in the zero center of mass momentum state.

Further corroboration of the crossover to the BEC like state with increasing $\lambda$ can be obtained by a study of the triplet fraction $\eta_t$ which is shown in Fig. 4. Again, $\eta_t$ corresponding to the non-interacting system monotonically increases and attains a value of $\frac{1}{3}$ at $\lambda_f$. The triplet content of the superfluid pair, as expected, is less than that of the non-interacting system, but has a similar qualitative behavior as the NI case in the regime $\lambda \ll \lambda_f$. The triplet fraction attains a maximum at a $\lambda$ close to $\lambda_f$ and then begins to fall. On further increase of $\lambda$, $\eta_t$ approaches that of the two-body bound-state wave function, demonstrating again that the pair wave function tends to the two-body bound-state wave function. We also see that $\lambda = \lambda_f$ marks the crossover point, i.e., the crossover regime is precisely the regime of $\lambda$ where change in the topology of the non-interacting Fermi sea takes place.

It is particularly interesting to study the BEC state that is attained when $\lambda \to \infty$. The key point as noted in section II is that the physics of the two-body bound state is determined by the dimensionless parameter $\lambda a_s$ (see Table I). Therefore, as $\lambda \to \infty$, the parameter $\frac{1}{\lambda a_s} \to 0$. Thus the state that is obtained is same as that obtained for the two-body bound state with a resonant scattering length in the presence of the gauge field ($\lambda > 0$) (Table I)! Therefore the properties of the BEC for $\lambda \to \infty$ are completely determined by $\lambda$, independent of the scattering length (as long as it is non vanishing), i.e., the system is a collection of Bosons whose properties are determined solely by the Rashba interaction. Hence we call this tightly bound bosonic state of two fermions as “rashbon”. Rashbon is a bound state of two fermions in a Rashba gauge field ($\lambda > 0$) at resonant scattering length ($\frac{1}{\lambda a_s} = 0$).

Again, for scattering lengths of larger magnitude, the qualitative physics remains identical. We shall illustrate this point in the next section by considering the EO case with a scattering length of larger magnitude.

### C. Extreme oblate (EO) GFC

The evolution of the non interacting Fermi surfaces for this GFC ($\lambda = \frac{\lambda_f}{\lambda} (1,1,0)$) is shown in fig. 2. The non-interacting chemical potential in the regime $\lambda \ll \lambda_f$ is

$$\frac{\mu_{NI}(\lambda)}{E_F} = 1 - \left( \frac{4}{3 \pi} \right)^\frac{2}{3} \left( \frac{\lambda}{\lambda_f} \right)^2 (\lambda \ll \lambda_f) \tag{37}$$

and that in the regime $\lambda \gg \lambda_f$ is

$$\frac{\mu_{NI}(\lambda)}{E_F} = \left( \frac{4}{3 \pi} \right)^\frac{2}{3} \frac{\lambda_f}{\lambda} (\lambda \gg \lambda_f). \tag{38}$$

In the regime when $\lambda \ll \lambda_f$ and for $k_F|a_s| \ll 1$, the chemical potential is well approximated by the non-interacting value. Further, we can obtain a rather lengthy analytical expression for the gap (not shown).

To illustrate that the qualitative nature of the transition is unaltered by the size of the scattering length, we study this GFC with $k_F|a_s| = 1$. The results are shown in fig. 5. These results clearly illustrate a crossover from the BCS like state to a BEC state of rashbons. Note, in particular, that $\eta_t$ of the many body pair wave function tends to that of the two body bound state wave function with a resonant scattering length (rashbon) given in Table I.

### V. DISCUSSION

We conclude the paper with further discussion of our results. On the BCS side $k_F|a_s| \ll 1$ and $\lambda \ll \lambda_f$, the transition temperature will be determined by the zero temperature gap which we have calculated in this paper. On the rashbon BEC side, the transition temperature will be determined by the mass of these emergent bosons which will be renormalized from the value of twice the fermion mass due to the gauge field.

As noted earlier, the rashbon is a bound state of two fermions in a Rashba gauge field ($\lambda > 0$) when the s-wave scattering length is infinity, i.e., at resonance. This
two-fermion bound state exists for all GFCs except the EP GFC and has a spin structure determined by $\lambda$ of the GFC (see the “resonance” column of Table. I). As is evident this state is not rotationally symmetric — it is an “anisotropic particle” that emerges. It is also interesting to contrast the rashbon state obtained in a Rashba gauge field ($\lambda \gg 0$) with the two-body quasi-bound state obtained in free vacuum ($\lambda = 0$) at resonance. In the latter case, the binding energy is zero, and the state is scale free with a singlet spin structure. This is to be contrasted with the rashbon state whose binding energy is $\lambda^2$ times a dimensionless number that depends on $\lambda$. Indeed, the state is not scale free — the wave function in the relative coordinate of the two fermions dies exponentially with a scale $\lambda^{-1}$ as noted in reference [20].

For a generic GFC, it is known that the critical scattering length $a_{sc}$ required to induce a bound state is negative and finite [20] and is given by $a_{sc} = \frac{\mathcal{F}(\lambda)}{\lambda}$ where $\mathcal{F}$ is a dimensionless function. For a given $a_{sc} < 0$, this corresponds to a critical gauge coupling strength $\lambda_c = \frac{\mathcal{F}(\lambda)}{a_{sc}}$. The crossover with increasing $\lambda$ is then governed by the relative magnitudes of $\lambda_T$ and $\lambda_c$. If $\lambda_c \lesssim \lambda_T$, the crossover regime coincides with the regime of the FSTT. On the other hand if $\lambda_c \gg \lambda_T$, the crossover regime is centered around $\lambda \approx \lambda_c$. In any case, for $\lambda \gg \max(\lambda_T, \lambda_c)$ the ground state will be a condensate of rashbons determined by the GFC in question. It is evident that except for the EP GFC, every other GFC will support a BCS-rashbon BEC crossover.

We now discuss the situation with a small positive scattering length with $k_F a_{sc} \ll 1$. In absence of a gauge field, the ground state is BEC of bosonic pairs of fermions with mass twice that of the fermion mass. In the presence of the gauge field, this BEC will evolve to the rashbon BEC as $\lambda \to \infty$, i.e, there is a BEC-rashbon BEC crossover.

The authors are not aware of any experimental realization of synthetic gauge field in fermionic systems. The natural question that arises is if the parameter regime of $\lambda \gtrsim \lambda_T$ with a high symmetry GFC can be realized in experiments. We do hope that our paper provides the motivation for this direction of experimental research.

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22 The quadrupole operator is defined as $Q_{ab} = \frac{1}{2}(S^a S^b + S^b S^a - \frac{3}{2} \delta_{ab})$, where $S^a$ are spin operators. A nematic state has $\langle S^a \rangle = 0$, while $\langle Q_{ab} \rangle \neq 0$. In the present context, the singlet piece of the two body wave function does not contribute to the quadrupole moment, while the triplet wave function has $\langle S^a \rangle = 0$, but $\langle Q_{ab} \rangle \neq 0$.
23 It is to be noted that the result for the EO case presented in [20] has a minor error of a factor of 2. The correct
results are quoted here; see also Section III B of e-print arXiv:1101.0411v2.

Throughout the paper, the chemical potential is referred to the bottom of the + helicity band.

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