Nonminimal spin-field interaction of the classical electron and quantization of spin.

Alexei A. Deriglazov

Depto. de Matemática, ICE, Universidade Federal de Juiz de Fora, MG, Brazil, and Department of Physics, Tomsk State University, Lenin Prospekt 36, 634050, Tomsk, Russia

I shortly describe classical models of spinning electron and list a number of theoretical issues where these models turn out to be useful. Then I discuss the possibility to extend the range of applicability of these models by introducing an interaction, that forces the spin to align up or down relative to its precession axis.

The notion of spin of an elementary particle [1–4] was developed in attempts to explain the energy levels of atomic spectra. This analysis culminated in quantum-mechanical expression for the energy of an electron, known as the Pauli Hamiltonian (E is the Coulomb electric field and B is a constant magnetic field)

\[ H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + eA^0 - \frac{e}{mc} \left( \mathbf{S}_0 \cdot \mathbf{B} + \frac{1}{2mc} (\mathbf{S} \cdot [\mathbf{E}, \mathbf{p}]) \right). \]  

Besides the position and momentum operators, the Hamiltonian contains operators proportional to the Pauli matrices, and describe a circle around the constraint \( S^2 = \frac{1}{2} \sigma^2 \). They have discrete spectrum of eigenvalues, so their contribution to the energy turns out to be quantized. For instance, the operator \( (\hat{S}_i \cdot \mathbf{B}) \) has eigenvalues \( \lambda = \pm \frac{n}{2} |\mathbf{B}| \), that is the measurement of spin in the direction of vector \( \mathbf{B} \) always gives one of the values \( \pm \frac{n}{2} \). The extra degrees of freedom contribute to the energy levels of an electron in a good agreement with experiments. The fine structure of hydrogen-like atoms with one valent electron fixes the factor in front of S-E-p term, while Zeeman effect fixes the factor in front of S-B term.

According the canonical quantization paradigm, the classical analogy of the theory [1] could be a point particle \((\mathbf{x}, \mathbf{p})\) which carries a vector \( \mathbf{S} \) attached to it. The commutator \([\hat{S}_i, \hat{S}_j] = i\hbar \epsilon^{ijk} S^k\) implies the use of classical-mechanics bracket \( \{S^i, S^j\} = \epsilon^{ijk} S^k \). Then the Hamiltonian equations for spin are \( \dot{\mathbf{S}} = \{\mathbf{S}, H\} \), or (we use the notation \([\mathbf{A}, \mathbf{B}]\) and \((\mathbf{A}, \mathbf{B})\) for the vector and scalar product of three-dimensional vectors)

\[ \frac{d\mathbf{S}}{dt} = [\mathbf{R}, \mathbf{S}], \text{ where } \mathbf{R} = -\frac{e}{mc} \left\{ \mathbf{B} - \frac{1}{2mc} [\mathbf{p}, \mathbf{E}] \right\}. \]  

When \( \mathbf{R} \) is a constant vector, the spin precesses around \( \mathbf{R} \): the end of \( \mathbf{S} \) always lies in a plane orthogonal to \( \mathbf{R} \), and describe a circle around \( \mathbf{R} \) with an angular velocity equal to the magnitude \( |\mathbf{R}| \) of this vector. This, in essence, represents the classical model of non relativistic spin. It can be constructed on the base of a variational problem that implies both dynamical equations and the magnitude-of-spin constraint \( S^2 = \hbar^2 s(s+1) = \frac{mc^2}{2} \), see [5].

The relativistic generalization and systematic construction of the resulting model on the base of a variational problem presents an issue with almost a centenary of history, see the works [5–7] and references therein. In the pioneer work [2], Frenkel identified the components \( S_i \) of three-dimensional spin with spatial part of four-dimensional antisymmetric spin-tensor \( S_{\mu\nu} = -S_{\nu\mu} \) as follows: \( S_i = \frac{1}{2} \epsilon^{ijk} S^{jk} \), and assumed that at each instant of motion \( S^\mu \dot{x}_\nu = 0 \), which guarantees the equal number of spin degrees of freedom in relativistic and Pauli theories. Concerning the corresponding variational problem, we recall that the classical-mechanics formalism always implies the canonical Poisson brackets, so the variables with brackets \( \{S^i, S^j\} = \epsilon^{ijk} S^k \) are not appropriate to this aim. One possibility to avoid the problem consist in formulating the variational problem in terms of a vector-like basic variable \( \omega^\mu \) for the description of spin. Then spatial components of the phase-space quantity \( S^{\mu\nu} = 2(\omega^\mu \pi^\nu - \omega^\nu \pi^\mu) \), where \( \pi_\mu = \frac{\partial L}{\partial \dot{x}_\mu} \), obey the desired brackets as a consequence of canonical brackets \( \{\omega^\mu, \pi^\nu\} = \eta^{\mu\nu} \) for the basic variables. This leads to the vector model of a relativistic spin: using \( \omega^\mu \) and the square root construction \(^2\) first discovered by Hanson and Regge [7], we can write the following Lagrangian [8]

\[ S = -\frac{1}{\sqrt{2}} \int dt \left[ m^2 c^2 - \frac{\alpha}{\omega^2} \right]^\frac{1}{2} \sqrt{-\dot{x}N \dot{x} - \dot{\omega} N \dot{\omega} + \sqrt{[\dot{x}N \dot{x} + \dot{\omega} N \dot{\omega}]^2 - 4(\dot{x}N \omega)^2}}, \]  

\(^a\)Electronic address: alexei.deriglazov@ufjf.edu.br

\(^1\) Consider either frozen particle in constant magnetic field, or on circular trajectory in the Coulomb electric and constant magnetic fields.

\(^2\) The construction can be resumed as follows: variational problem with the Lagrangian \( L = \dot{x}^2 + \dot{\omega}^2 - \sqrt{[\dot{x}^2 + \dot{\omega}^2]^2 - 4(\dot{x}\dot{\omega})^2} \) implies the constraint \( (\mathbf{p}, \pi) = (\frac{\partial L}{\partial \dot{x}}, \frac{\partial L}{\partial \dot{\omega}}) = 0 \) as one of the extreme conditions.
where \( N_{\mu\nu} = \eta_{\mu\nu} - \frac{\omega_{\mu\nu}}{c^2} \) is a projector on the plane orthogonal to \( \omega \). The parameter \( \alpha \) determines the value of spin, in particular, \( \alpha = \frac{3h^2}{c^2} \) corresponds to the spin one-half particle. In the spinless limit, \( \omega^\mu = 0 \) and \( \alpha = 0 \), Eq. (3) reduces to the standard Lagrangian of a point particle, \(-mc\sqrt{-\dot{x}^2}\).

The variables \( \omega \) and \( \pi \) are affected by local symmetry transformations presented in the theory, so they are not an observable quantities. The observable quantities of spin-sector are contained among components of spin-tensor. It is accompanied by higher nonlinear Poisson brackets, that is the most part of this function with Hamiltonian, so we can write

\[
\{ x^\mu, x^\nu \} = \frac{1}{(mc)^2} \epsilon^{ijk} S^k, \quad i, j = 1, 2, 3.
\]

The r.h.s. vanishes as \( c \to \infty \), that is the spin-induced noncommutativity is a relativistic effect. It should be noted that spinning particles represent an exceptional example of intrinsically noncommutative and relativistic-invariant theory, with the spin-induced noncommutativity that manifests itself already at the Compton scale. One of immediate consequences of Eq. (4) is that in \( 1/c^2 \)-approximation the position of a particle in the Pauli (and Dirac) quantum mechanics should be described by the Pryce (d) operator (see [10 11] for the details)

\[
\hat{x}^i = x^i - \frac{\hbar}{4(mc)^2} \epsilon^{ijk} \sigma^k,
\]

instead of \( \hat{x}^i = x^i \).

2. The problem of covariant formalism and the Thomas precession. The Lagrangian (3) admits interaction with an arbitrary electromagnetic field, and thus gives the relativistic generalization of an approximate Frenkel equations. The Hamiltonian of interacting theory has a simple and expected form

\[
H = \frac{1}{2m} \left[ (p^\mu - eA^\mu)^2 - \frac{e\mu}{2c} F_{\mu\nu} S^{\mu\nu} + (mc)^2 \right] \approx mc^2 + \frac{1}{2m} (p - eA)^2 + eA^0 - \frac{e}{mc} \left[ (S, B) + \frac{1}{mc} (S, [E, p]) \right],
\]

where in the last line we put the magnetic moment \( \mu = 1 \), and left \( 1/c^2 \) approximation of the complete Hamiltonian in the physical-time parametrization. It is accompanied by higher nonlinear Poisson brackets, that is the most part of interaction in the vector model is encoded in the noncommutative phase-space geometry. The relativistic [7] and Pauli [1] Hamiltonians differ by the famous \( 1/2 \) factor in front of the last terms. This is the problem of covariant formalism, that was raised already in the pioneer works [1 3] and remains under debates to date. The spin-induced noncommutativity of vector model provides a natural solution to this problem: the Hamiltonian (4) is accompanied by non canonical brackets, and this should be taken into account during the quantization. Detailed computations shows [9] that this results in Pauli’s quantum mechanics.

It should be noted that this result is obtained without any appeal to the Thomas precession effect. The role of the Thomas spin-vector in this scheme was clarified in the recent work [12].

3. Modified Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations. The vector model admits the minimal interaction with an arbitrary gravitational field, it is sufficient to replace \( \eta_{\mu\nu} \) on \( g_{\mu\nu} \), and \( \omega^\mu \) on \( \nabla \omega^\mu = \dot{\omega}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \omega^\beta \) in the Lagrangian (3). Remarkably, this leads to equations of motion that coincide with MPTD

\[ \text{In the language of Poisson geometry this equation means that } S^{\mu\nu} p_\nu \text{ are Casimir functions of the Poisson structure.} \]
equations in the form studied by Dixon [13]. So the vector model gives an alternative description of a rotating body in gravitational field in the dipole approximation. In particular, using the vector model, it is possible to compute and analyse the three-dimensional acceleration of MPTD body. This analysis shows that MPTD equations lead to a wrong dependence of acceleration on speed of the body.

Without going into technical details [5], this can be explained as follows. In special and general relativity, the acceleration of a particle inevitably depends on its velocity in such a way that longitudinal acceleration vanishes as \( v \to c \). In electrodynamics we have

\[
\frac{dv^\mu}{ds} = F^\mu_{\nu\nu'}, \quad \text{implies} \quad a_\parallel \sim (c^2 - v^2)^{3/2} \frac{v^\nu}{c^2} 0, \quad (8)
\]

while for the geodesic equation

\[
\frac{dv^\mu}{ds} = -\Gamma^\mu_{\nu\alpha} v^\nu v^\alpha \quad \text{implies} \quad a_\parallel \sim (c^2 - v^2) \frac{v^\nu}{c} 0. \quad (9)
\]

These examples show that a covariant and reparametrization invariant equations of motion can not contain too many velocities \( v^\mu \) (four or more). Unfortunately, MPTD equations belong to this last case [5]. In the result, acceleration grows with speed and diverges in the ultra-relativistic limit \( v \to c \).

Therefore, MPTD equations do not seem to be a promising candidate for describing a rotating body. The particles/bodies with spin are now under intensive investigation and represent an important tool in the study of near horizon physics of black holes, see for example [14-16]. So, it is interesting to find a generalization of MPTD equations with an improved dependence of acceleration on speed. This can be achieved by adding a nonminimal spin-gravity interaction [5]. In the Hamiltonian formulation, this reduces to the replacement of original minimally-interacting Hamiltonian \( P^2 + (mc)^2 \) on the following:

\[
P^2 + \kappa R_{\mu\nu\alpha\beta} S_{\mu\nu} S_{\alpha\beta} + (mc)^2. \quad (10)
\]

The modified Hamiltonian strongly resembles to that of spinning particle with a magnetic moment [6], so the coupling constant \( \kappa \) is called gravimagnetic moment [17]. \( \kappa = 0 \) corresponds to the MPTD equations. The most interesting case turns out to be \( \kappa = 1 \) (gravimagnetic body). Keeping only the terms, which may contribute in the leading post-Newtonian approximation, this gives the modified equations

\[
\nabla P_\mu = -\frac{1}{4} \theta_{\mu\nu} x^\nu - \frac{\sqrt{-x^2}}{32mc} (\nabla_\mu \theta_{\nu\lambda}) S^{\sigma\lambda}, \quad \nabla S_{\mu\nu} = \frac{\sqrt{-x^2}}{4mc} \theta_{\mu\nu} S^{\sigma\lambda}, \quad (11)
\]

where \( \theta_{\mu
u} = R_{\mu\nu\alpha\beta} S^{\alpha\beta} \) is gravitational analogy of the electromagnetic field strength \( F_{\mu\nu} \). They can be compared with MPTD equations in the same approximation

\[
\nabla P_\mu = -\frac{1}{4} \theta_{\mu\nu} x^\nu, \quad \nabla S_{\mu\nu} = 0. \quad (11)
\]

We see that unit gravimagnetic moment yields quadratic in spin corrections to MPTD equations in the \( 1/c^2 \)-approximation.

Both acceleration and spin torque of gravimagnetic body have a reasonable behavior in the ultra-relativistic limit [5]. In Schwarzschild and Kerr space-times, the modified equations predict a number of qualitatively new effects [18], that could be used to test experimentally, whether a rotating body in general relativity has null or unit gravimagnetic moment.

4. Dirac equation and unobservability of Zitterbewegung. Consider the Dirac equation \( i\hbar \partial_t \Psi = \hat{H} \Psi \), where \( \hat{H} = c \alpha^i \hat{p}_i + mc^2 \beta \), and \( \alpha^i = \gamma^0 \gamma^i \) and \( \beta = \gamma^0 \) are Dirac matrices. Passing from the Schrödinger to Heisenberg picture, the time derivative of an operator \( a \) is \( i\hbar \dot{a} = [a, \hat{H}] \), and for the expectation values of basic operators of the Dirac theory we obtain the equations

\[
\dot{x}^i = c \alpha_i, \quad i\hbar \dot{a}_i = 2(c p_i - H \alpha_i), \quad i\hbar \dot{\beta} = -2c \alpha_i p_i \beta + mc^2. \quad (12)
\]

They can be solved, with the result for \( x^i(t) \) being \( x^i = a^i + dp^i t + c^i \exp(-\frac{2iHt}{\hbar}) \). The first and second terms are expected, and describe a motion along the straight line. The last term on the r.h.s. of this expression states that the free electron experiences rapid oscillations with higher frequency \( \frac{2H}{\hbar} \sim \frac{2mc^2}{\hbar} \), called the Zitterbewegung.

Although it is widely believed that Zitterbewegung could be a physically observable phenomenon, a detailed analysis does not support this belief [21-22]. The exact relation between the variable of position of vector model and the Dirac operators was computed in [11]. It is different from the naive expressions implied by Eq. (12), and without going into technical details can be described as follows. A long time ago Feynman noticed (see p. 48 in [19]), that the basic Dirac operators can be used to construct an operator that do not experiences Zitterbewegung. It is just Pryce (d) operator. But as we saw above, in the vector model namely the Pryce (d) operator represents the classical variable
of position (the latter moves along a straight line according to classical equations). The same conclusion follows from the analysis of one-half of the Dirac equation in the Foldy-Wouthuysen representation \[22, 23\].

5. Relativistic quantum mechanics with positive energies and the Dirac equation. Canonical quantization of the vector model \([3]\) leads to the Schrödinger equation

\[ i\hbar \frac{d\Psi}{dt} = c\sqrt{\mathbf{p}^2 + (mc)^2}\Psi, \quad \hat{p}^i = -i\hbar \frac{\partial}{\partial x^i}, \]  

(13)
on the space of wave functions which are two-component Weyl spinors \(\Psi(t, \mathbf{x}) = (\Psi_1, \Psi_2)\), in the representation of the operator \(\hat{x}^i = x^i\) conjugated to \(\hat{p}^i = -i\hbar \frac{\partial}{\partial x^i}\). The scalar product is

\[ \langle \Psi, \Phi \rangle = \int d^3x \Psi^\dagger \Phi, \]  

(14)
and \(P = \Psi^\dagger \Psi\), is a probability density of \(\hat{x}^i\). All solutions to the Schrödinger equation form the subspace of positive-energy solutions to the (again two-component) Klein-Gordon equation

\[ (\hat{p}^2 + m^2 c^2)\Psi \equiv \sigma^\mu \hat{p}_\mu \sigma^\nu \hat{p}_\nu \psi + m^2 c^2 \psi = 0. \]  

(15)

The novel point is that the operators \(\hat{x}^i = x^i\) and \(\sigma^i\) do not represent operators of position and spin in the vector model. As we saw above, the classical variables \(x^i\) and \(S^i\) obey noncanonical brackets, so the corresponding operators

\[ x^i \to \hat{X}^i = x^i + \frac{\hbar}{2mc(\hat{p}^0 + mc)} \epsilon^{ijk} \sigma_j \hat{p}_k, \]  

(16)

\[ \hat{S}^i = \frac{\hbar}{2mc} \left( \hat{p}^0 \sigma^i - \frac{1}{(\hat{p}^0 + mc)} (\hat{p} \sigma^0) \hat{p}^i \right), \]  

(17)
turn out to be Pryce (d) operators.

To complete the construction, it remains to show the relativistic covariance of the quantum-mechanical formalism. Here we only outline the proof of relativistic invariance of the scalar product \([14]\). Concerning the covariant rules for computation of transition probabilities and mean values of operators, see \([5, 11]\). Introduce the following operator (we denote \(\sigma^\mu = (1, \sigma^i)\), and \(\check{\sigma}^\mu = (-1, \sigma^i)\)):

\[ V = \frac{1}{mc} \sqrt{\frac{\hat{p}^0}{\hat{p}^0 + mc}}[(\check{\sigma} \hat{p}) + mc], \quad V^{-1} = \frac{1}{2\sqrt{\hat{p}^0(\hat{p}^0 + mc)}}[mc - \sigma \hat{p}], \]  

(18)
commuting with the Schrödinger operator \([13]\), then the vector \(\psi = V^{-1} \Psi\) obeys \([13]\) and \([15]\) together with \(\Psi\). Then we can write

\[ \langle \Psi, \Phi \rangle = \langle V \psi, V \phi \rangle = \int d^3x \frac{1}{mc^2} (\check{\sigma} \hat{p} \psi) \check{\sigma} \hat{p} \phi + \psi \check{\phi} \equiv \int d^3x T^0. \]  

(19)
The integrand is the null component of a four-vector

\[ I^\mu[\psi, \phi] = \frac{1}{mc^2} (\check{\sigma} \hat{p} \psi) \check{\sigma} \hat{p} \phi - \check{\psi} \check{\sigma} \hat{p} \phi, \]  

(20)
that represents a conserved current of Eq. \([15]\), that is \(\partial_\mu I^\mu = 0\), when \(\psi\) and \(\phi\) satisfy to Eq. \([15]\). So we can construct a scalar product using the invariant integral: \(\langle \psi, \phi \rangle \equiv \int d\Omega I^0\). \(\Omega\), \(d\Omega = -\frac{1}{8} \epsilon_{\mu\nu\alpha\beta} dx^\mu dx^\nu dx^\alpha dx^\beta\), computed over a space-like three-surface \(\Omega\). Using the Gauss theorem for the four-volume contained between the surfaces \(\Omega_1\) and \(\Omega_2\), we conclude that the scalar product does not depend on the choice of the surface, \(\int_{\Omega_2} = \int_{\Omega_1}\). In particular, it does not depend on time. So we can restrict ourselves to the hyperplane defined by the equation \(x^0 = \text{const}\), this reduces \(\langle \psi, \phi \rangle\) to the expression written in Eq. \([19]\): \(\langle \psi, \phi \rangle = \int d^3x T^0\). Besides, the scalar product is positive-defined, since \(I^0[\psi, \psi] > 0\). In the result, Eq. \([14]\) represents the relativistic invariant scalar product.

The operator \([18]\) is closely related with the Foldy-Wouthuysen transformation of the Dirac equation. To see this, we use the equivalence between Klein-Gordon and Dirac equations contained in the map

\[ \Psi_D[\psi] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\psi}{mc} \check{\sigma} \hat{p} \psi \end{pmatrix} = \frac{1}{\sqrt{2mc}} \begin{pmatrix} ((\check{\sigma} \hat{p}) + mc)\psi \\ ((\check{\sigma} \hat{p}) - mc)\psi \end{pmatrix}, \]  

(21)
Figura 1: Vector of force $F_i \sim -N(S_i)N(B)S_i$ always is directed towards the straight line of the vector $B$.

that relates solutions $\psi$ of Klein-Gordon equation [15] with the solutions $\Psi_D$ of the Dirac equation $(\gamma^\mu \tilde{p}_\mu + mc)\Psi_D = 0$, where $\gamma^\mu$ are $\gamma$-matrices in the Dirac representation. Applying the Foldy-Wouthuysen transformation

$$U_{FW} = \frac{\omega_p + mc + (\gamma^0 \tilde{p})}{\sqrt{2(\omega_p + mc)\omega_p}},$$

(22)
to the Dirac spinor $\Psi_D[\psi]$, we obtain

$$U_{FW}\Psi_D[\psi] = \begin{pmatrix} V\psi \\ 0 \end{pmatrix} = \begin{pmatrix} \Psi \\ 0 \end{pmatrix},$$

(23)

that is the operator $V$ is a restriction of $U_{FW}$ to the space of positive-energy right Weyl spinors $\psi$.

We emphasize that in this paragraph we did not try to give a physical interpretation of manifestly covariant Klein-Gordon and Dirac equations (the interpretation of negative-energy states and so on). Vector model leads to the relativistic quantum mechanics with positive-energy states, which is defined by the expressions (13)-(17). The covariant Klein-Gordon and Dirac formalisms were used as an auxiliary tool to test the relativistic invariance of this theory.

**Modified equation for precession of spin.** As we saw above, the classical models of spinning electron clarify a number of theoretical issues. However, due to the absence of a classical mechanism similar to the quantization of angular momentum and spin and the corresponding rules for the addition of moments, the range of applicability of these models for practical calculations is rather limited. For instance, as the basic motion of spin in magnetic field is a precession, its contribution into the classical energy is $(S, B) \sim \cos \theta_0$, and can be an arbitrary number depending only on the initial value of the angle $-\pi < \theta_0 < \pi$ between magnetic field and spin. Let us discuss a possibility to modify the basic motion, introducing a nonminimal interaction that causes the spin to align up or down relative to its precession axis.

Introduce the matrices $N^{ij}(S) \equiv \delta^{ij} - \frac{S_i S_j}{S^2}$ and $N^{ij}(B) \equiv \delta^{ij} - \frac{B^i B^j}{B^2}$, which are projectors on the plane orthogonal to the unit vectors $\hat{S} = \frac{S}{|S|}$ and $\hat{B} = \frac{B}{|B|}$. Using them, we consider the following double projection of $S$ (see Fig. 1):

$$-|B|N(S)N(B)S = (B, S)N(S)\hat{B} = (B, S)[\hat{S}, [\hat{B}, \hat{S}]].$$

(24)

This vector lies on the plane of $B$ and $S$, is tangent to the circle of the radius $|S|$, and, regardless of the angle between $B$ and $S$, is always directed towards the straight line of the vector $B$. Assuming that a nonminimal interaction of spin with magnetic field is proportional to this vector, we replace the precession equation on the following one:

$$\frac{dS}{dt} = -\frac{e}{mc}[B, S] + \beta(B, S)[\hat{S}, [\hat{B}, \hat{S}]],$$

(25)

where, on dimensional grounds, $\beta = \gamma |e|/mc$, and $\gamma > 0$ is a dimensionless constant. As we have a first-order differential equation, the end point of $S$ moves along the integral lines of the vector field written on r.h.s. of this equation. So, the evolution of $S$ consist of two motions: precession around $B$ caused by first term, plus circular motion on the plane of precession caused by second term. Due to the circular motion, a vector of spin that originally had an acute angle
with $\mathbf{B}$ lines up in the direction of $\mathbf{B}$, while a vector that had an obtuse angle lines up in the opposite to $\mathbf{B}$ direction. Angular velocity of the circular motion is $\frac{d\theta}{dt} = \frac{\gamma |\mathbf{e} B \sin 2\theta|}{2mc}$.

Acknowledgments. The work of A. A. D. has been supported by the Brazilian foundation CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brasil), and by Tomsk State University Competitiveness Improvement Program.

[1] Uhlenbeck G. E., Goudsmit G. E. Spinning Electrons and structure of Spectra // Nature. 1926. V. 117. P. 264-265.
[2] Frenkel J. Spinning Electrons // Nature. 1926. V. 117. P. 653-654.
[3] Thomas L. H. The kinematics of an Electron with an Axis // Philosophical Magazine and Journal of Science. 1927. V. 3. P. 1.
[4] Pauli W. On the Quantum Mechanics of Magnetic Electrons // Zeit. f. Phys. 1927. V. 43. P. 601-623.
[5] Deriglazov A. A., Ramírez W. G. Recent Progress on the Description of Relativistic Spin: Vector Model of Spinning Particle and Rotating Body with Gravimagnetic Moment in General Relativity // Advances in Mathematical Physics. 2017. V. 2017. Article ID 7397159; arXiv:1710.07135
[6] H. C. Corben H. C., Classical and Quantum Theories of Spinning Particles. Holden-Day, San Francisco, 1968.
[7] Hanson A. J., Regge T. The Relativistic Spherical Top // Annals of Physics. 1974. V. 87. P. 498-566.
[8] Deriglazov A. A. Lagrangian for the Frenkel Electron // Phys. Lett. B. 2014. V. 736. P. 278-282; arXiv:1406.6715
[9] Deriglazov A. A., Pupasov-Maksimov A. M. Relativistic Corrections to the Algebra of Position Variables and Spin-Orbital Interaction // Phys. Lett. B. 2016. V. 761. P. 207-212.
[10] M. H. L. Pryce M. H. The Mass-Centre in the Restricted Theory of Relativity and its Connexion with the Quantum Theory of Elementary Particles // Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences. 1948. V. 195. P. 62.