On R-matrix approaches to knot invariants

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Abstract

We present an elementary introduction to several knot theory approaches, which originate from various physical models, and which turned out to be extremely fruitful recently.

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1 Introduction

The most intriguing paradox of the up to day science is that a knot theory turns from a chapter of mathematics to a chapter of physics. The present text is intended to be a pedagogical introduction into several topics which might illuminate this phenomenon. Although there is a huge amount of literature on the subject, many gaps are still to be filled. In one hand, classical text-books on a knot theory, like [1], are written in a pure mathematical tradition. On the other hand, reviews of the “physical” approaches, e.g. [2], remain too schematic in many points. This drawback is inevitable since the “physical” ideas about knots involve pieces of very different sciences. Moreover, many of these ideas should be treated as a source of inspiration and intuition but not as more or less consistent approaches, at least not so far. Some mathematical textbooks, e.g., [3], comment on this subject, but rather briefly and with a certain skepticism. In the present text, we concentrate on certain class of knot invariants, which may be evaluated in the framework of the so called vertex-operator, or R-matrix approach. This approach has several different versions, and each one is related to certain physical topics. We try to keep the rather elementary level of presentation, and mostly restrict ourselves by the simplest examples, but we try to explain various details and subtleties. In each case, we intentionally do not start from presenting a completed construction, doing what a naive person would do and indicating one pitfall after another instead.

In the remaining of the section, we list the subjects of the subsequent sections, attempting to explain why these topics attract attention of physicists.

1.1 Monodromy group

The main source of the correspondence between knot polynomials and a plenty of other quantities is that a symmetric (permutation) group action may be naturally visualized with a knot. A permutation may be realized as sequence of transpositions (permutations of the neighboring elements), and one may plate a related to the permutation braid, labeling the strands with the numbers from 1 to \( n \) and intertwining a strand \( i \) with the strand \( i + 1 \) for a transposition of the same elements. The knot that is closure of the braid is hence related to an element of a permutation group. Moreover, the equivalent, i.e., producing the same position of elements, permutations turn to correspond to the topologically...
equivalent knots. Despite that the inverse is literally wrong, a one to one correspondence may set is one considers a certain deformation of a symmetric group.

Such a deformation naturally arises in the following problem. Consider a space with a set of special points and an arbitrary quantity that continuously depends on positions of these points (in some region) and naively should be symmetric w.r.t. to their permutations. Such a quantity, however, may change when one moves the points continually so that they are permuted in result. Such an operation is called a monodromy, and if a quantity is then changed, it is said to possess a non-trivial monodromy; such quantities are often considered in complex analysis \[6, 9\]. In case of several special points, one may consider a monodromy group, which is generated by pairwise interchanging of the points, with a composition as a group multiplication (in particular, if a group generator corresponds to counter-clockwise moving of the points, the inverse generator corresponds to the clockwise one). The generators of the monodromy group satisfy then \(2.2, 2.3\), but not \(2.5\), at least not generally. In other words, a twice applied monodromy, although recovering the original positions of points, still may not recover the original value of the quantity \[6, 9\]. An element of a monodromy group corresponds to a braid in a similar way as an element of the symmetric group. But distinct monodromies correspond to distinct braids if not generally then much more often then distinct permutations.

Suppose then, that there is a whole set of quantities possessing a monodromies under permutations of the same set of points, and, in addition, these quantities are known to form a basis in a linear space. Then the corresponding monodromies are linear transformations of the space, and one may talk about a monodromy matrices. In this case, one has a representation of the monodromy group, which at the same time is a representation of a braid group, since the relations \(2.2, 2.3\) are satisfied. Monodromy matrices may satisfy non only the group relations but also some additional ones. In particular, a certain linear combination of monodromy matrices may vanish. As we discuss further in details, this turns out to be essential when one uses these matrices to construct a knot invariant, which is invariant of the braid closure.

We discuss the construction for knot invariants based on the braid representation of knot and on the representation theory of the symmetric groups in sec.2.2.

1.2 Exactly solvable planar models

The above discussed approach to knot invariants is based on the observation that a braid corresponds, at the same time, to a knot and to an element of a symmetric group’s extension. There is also another approach, which originally comes from studies of the exactly solvable planar lattice models \[70\]. In this kind of models, vertex operators satisfy a certain relation, the Yang-Baxter equation, as a consequence of an operator algebra. On the other hand, a knot may be projected on a plane. As a result, one obtains a self-intersecting curve, which may be seen as four-valent graph. If one selects one of two directions on the original curve, then the graph is directed too. If one also specifies, which line was the upper one w.r.t. the projection plain in each self-crossing, one obtains a knot planar diagram (e.g., fig.5), with the vertices are called crossings. Up to an isomorphism of the projection plane, a crossing is equivalent to one of shown in fig.3.1, which are referred to as direct and inverse crossings, respectively. A knot may be unambiguously recovered from such a diagram \[3\], but, similarly to the case of a braid representations, many different diagrams may represent the same knot. The case is that continuous transformations of a knot may correspond to non-trivial deformations of the knot diagram. Hence, if a knot invariant is introduced as a quantity related to the knot planar diagram, one should take care that the introduced quantity is invariant under continuous deformation of the diagram along with certain additional transformations, which also correspond to continuous transformations of the knot. If a quantity has a form of the product of operators placed in the vertices, contracted along the edges (see fig.5 and 3.49), it is automatically invariant under to the smooth transformations of the planar diagram, while the invariance under smooth deformations of the knot is still to be imposed.

As we discuss in what follows, the corresponding constraints on the considered quantities are reduced (modulo certain subtleties) to a system of equations on vertex operators. One of these equations turns
out to be nothing but the Yang-Baxter equation. In this sense, a knot invariant is presented in this approach as an observable in an exactly solvable planar model. Let us emphasize once more, that it the Yang-Baxter equation, which relates these two subjects.

Solutions of the Yang-Baxter equation are well studied in a theory of integrable models [70]. A particular one-parametric family of solutions, presented in [65], makes an interest in the context of knot theory. A member of this family is called quantum $R$-matrix. A Lie group along with it’s representation corresponds to a quantum $R$-matrix, which may be presented explicitly as a series in group generators, with the coefficients depending on a formal parameter $q$. We will consider in details the simplest solution, which correspond to the fundamental representation of $sl_N$. This solution was used in [63] to evaluate a knot invariant called a HOMFLY polynomial in the framework of the vertex-operator approach. The resulting expression may be analytically continued to arbitrary values of $A$, yielding a rational function of two formal variables, $q$ and $A = q^N$. A particular case of $N = 2$ corresponds to the Jones polynomial, and substituting $A = 1$ one gets Alexander polynomial [7]. If one takes a quantum $R$-matrix for the fundamental representation of the group $so_N$ instead, one obtains a Kauffman polynomial. In turn, $R$-matrices in higher representations give rise to the quantities called colored knot or link invariants [64]. All explicit computations in the present text are relevant to the (uncolored) HOMFLY polynomials. Nevertheless, most of the statements holds for arbitrary link invariants that may be calculated with help of the $R$-matrices. For this reason, will do not specify, what particular invariant is meant when unnecessarily.

We should make a reservation here, that the construction under discussion originally was formulated in terms of braids [63]. As we demonstrated in the previous section, Yang-Baxter equation arises there as one of the braid group relations. The braid-group version of the approach is completely sufficient from a computation standpoint. Moreover, this is perhaps the only well-defined version of the construction of interest [72]. In view of these remarks, an attempt of a more general presentation may be seen just as muddying the waters. Nevertheless, these kind of efforts is not devoid of grounds. First, there certain families of knots, whose structure is not enjoyed by a braid representation [41, 46, 47], i.e., alike knots are not, in general, closes of alike braids. It would be nice then to extend the available formalism to representations of knots adequate for such cases (what is partially done in the cited works). Second, various issues of a knot theory, which attract even more researchers’ attention than the subject of the present text, involve just a general (not braid) version of the vertex operator construction. The relevant extension is known in a particular case, where it reproduces a Jones polynomial, under the name of Kauffman formalism [13, 24]. The latter one is an essential ingredient of such an object as a Khovanov polynomial [20, 21, 24, 25]. The most intriguing issue of the up to day knot theory, the superpolynomials of knots [28, 29], are defined (as far as they are) in terms of Khovanov-Rozanski theory [22], whose exceptional both theoretical and computational complexity sharply diverges with relatively simple final answers. An extension of the Kauffman formalism to a more general case of HOMFLY polynomial might open a way to an alternative construction, more nice and useful in the context, reproducing the same knot invariants [48, 49]. Recently introduced polynomials of virtual knots [27] also require for Kauffman formalism, so that proper generalizations are necessary for further advances in the subject [50, 51]. Finally, a representation of knot polynomial [81, 82, 52–54, 84, 85] closely related to the presented one and yet different, also involves the Yang-Baxter equation and it’s solutions, this time arising as constraints on Wesso-Zumino-Witten conformal blocks [66]. The issue was originally insighted by an idea of meaning knot polynomials as observables in a quantum field theory [11, 69], and, in turn, insights further attempts of this kind [71, 72]. Although the approach treated here involves solutions of the Yang-Baxter equation in a strictly different way, a version valid for arbitrary planar diagrams would give a new breath to the “knots in QFT” idea (see the last of the cited works, and also sec 4 for more comments on the issue).

The vertex-operator approach is discussed in details in sec 3.
1.3 Non-perturbative Chern-Simons theory

We have already discussed two sources of a physicist’s enthusiasm around knots, namely, the interpretation of braids and knots as elements of a symmetric group’s extension, and the construction for knot invariants in terms of vertex operators of exactly solvable planar models. Nevertheless, a subject was left beyond our scope subject, probably, the most intriguing and obscure one. This an idea of relating knot invariants to quantum observables in a topological field theory. Since the states of arts in the latter subject is at least very far from deriving answers explicitly from rigorous definitions, non of this sort of inspiring guesses has a status even of clearly stated, verifiable and falsifiable conjecture. A whole science was developed from a there arising intuitive picture yet.

The first idea of that kind, which was a subject of numerous discussions in the late 80-s, in particular, of an unpublished work by Shwarz [11], and become extremely popular after the work of Witten [69], claims that “Jones polynomial is a Wilson average in Chern-Simons theory”. The form of the conjecture admitted an immediate extension to other knots invariants, and, the most intriguing, to knots in topologically non-trivial (other than $R^3$ or $S^3$) three-dimensional spaces.

In sec.4 we partially reproduce a picture, brought in [69] as a motivation for this conjecture, keeping aside the exact formulation of the mathematical statement made and proved in the paper.

1.4 Perturbative Chern-Simons theory

Apart from various constructions inspired by [69], which are aimed to be exact, but which are implicit from a QFT standpoint, there is a huge number of attempts to put the idea of knot invariants as observables in a quantum field theory in a more standard framework. This would imply a perturbative evaluation of Chern-Simons Wilson average, carried out in a QFT textbook manner. An output of numerous works on this subject [82, 73, 74, 81, 75, 76] is that the evaluation may be successfully completed, modulo certain ambiguities, especially in what concerns the gauge fixing. In any given order of a perturbation theory, one obtains the so called Kontsevich integral representation for Vasiliev invariants [77, 82, 79, 80, 78]. This is indeed a strong argument in favor of the correspondence between the HOMFLY polynomial and the CS Wilson average, since the expansion of HOMFLY in the logarithm of the formal variable $q$ may be verified to give exactly the same series. Yet, no fundamental reason for assembling of the perturbative series in the expression for HOMFLY comes from this approach. There is a conjecture [71, 72], that some other gauge fixing would reproduce literally the HOMFLY polynomial, written in the form discussed in sec.3, i.e. as a contraction of certain vertex operators. However, a naive definition of the the proper gauge fixing turns out to be inconsistent, and no way to save it is yet known. Hence, the correspondence of knot invariants to CS Wilson averages, although often seen as common knowledge, remains very far from being exhaustively explored.

We concern the subject a little in sec.5.
2 Knot polynomial as an invariant of braid or monodromy group

2.1 From permutation symmetry to braid group

2.1.1 Braid group as a deformation of symmetric group

Equivalence transformations of braids

\[
\begin{align*}
\sigma_1\sigma_2\sigma_1 &= \sigma_2\sigma_1\sigma_2 \\
\sigma_1\sigma_3 &= \sigma_3\sigma_1 \\
\sigma_1\tilde{\sigma}_1 &= 1
\end{align*}
\] (2.1)

In this section, the discuss in details a relation of knots and braids to a symmetric group, as well to a certain extension of it. We start from recalling the definitions of the symmetric and braid groups. Generator \(\sigma_i\) of the symmetric group permutes elements \(i\) and \(i+1\). Taking product of two permutations is merely doing them successively. The unity is the trivial permutation (which changes nothing), and it remains to define what the inverse permutation is. As already said, generators of the symmetric group are transposition \(\sigma_i\), which interchange an element \(i\) with the neighboring element \(i+1\), in other words, a permutation \(P\) may be presented as a sequence \(\sigma_{i_1}\ldots\sigma_{i_k}\) of transpositions. The inverse permutation is then the reversed sequence \(\sigma_{i_k}\ldots\sigma_{i_1}\), since a transposition applied twice gives the trivial permutation.

In turn, a braid group generator \(\sigma_i\) corresponds to an intertwining of the strands \(i\) and \(i+1\). Taking product of two braids is plating one after the other on the same strands. The unity is the trivial braid consisting of unintertwined strands. A group generator \(\sigma_i\) and the inverse element \(\tilde{\sigma}_i\) are braidings of the adjacent strands \(i\) and \(i+1\), with one and the other orientations, respectively, (2.1-II). Indeed, two stands intertwined successively in one and in the other way may be separated in the space. Canceling of the consecutive \(\sigma_i\) and \(\tilde{\sigma}_i\) is one of the further discussed equivalence transformations of braids. Since a braid is by definition a product \(\sigma_{i_1}\ldots\sigma_{i_k}\) of the generators, the inverse braid is the product \(\tilde{\sigma}_{i_k}\ldots\tilde{\sigma}_{i_1}\) of the inverse elements in the reversed order.

The above mentioned correspondence between symmetric and braid groups is based on the fact that both group’s generators satisfy the two series of constraints,

\[
\begin{align*}
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, & i &= 1,\ldots,n-1, \\
\sigma_i\sigma_j &= \sigma_j\sigma_i, & i,j &= 1,\ldots,n, & |i-j| &\neq 1,
\end{align*}
\] (2.2, 2.3)

In the above formulas, \(\sigma_i\) is a group generator, corresponding to a crossing of the strands \(i\) and \(i+1\) in case of the braid group, or to a permutation of the elements \(i\) and \(i+1\) in case of the permutation group. A group element is by definition a product of the group generators and visa versa; two products of generators produce the same group element if and only if they can be transformed into each other by applying the group relations. Two products of the symmetric group generators are equivalent if and only if they realize the same permutation, e.g.,

\[
\sigma_1\sigma_3\text{xyzt} = \sigma_3\sigma_1\text{xyzt} = \text{yxtz}, \quad \text{or} \quad \sigma_1\sigma_2\sigma_1\text{xyz} = \sigma_2\sigma_1\sigma_2\text{xyz} = \text{zyx}.
\] (2.4)

In turn, two products of braid group generators yield the same braid group element if and only if the corresponding braids are isotopic, i.e. if one may transform continuously one into the other in the three-dimensional space keeping the ends fixed and never intersecting the strands. It is straightforward to see, that the constraints (2.2, 2.3) reflect the transformations of this kind, (2.1). Moreover, the Artin
Theorem 3 claims that any two equivalent, in the above sense, braids are transformed into each other by a sequence of transformations (2.1). Equality (2.1-III) just determines the inverse of a braid group generators, and (2.1-I,II) are specific for the braid group relations, which are named as invariance under the first and the second Artin transformations, respectively. Hence, all isotopic braids realize one and the same permutation (the inverse is not true, as we discuss in what follows).

To summarize, a braid corresponds to a permutation, in a way respecting the group structure. Moreover, the correspondence may be set explicit; if the strands are labeled by their positions in the first braid’s section, then their positions in the last braid’s section yield the permutation associated with the braid.

Although a permutation may be related to a braid in the way just described, there is a major difference between these two objects. Namely, the braid keeps information about the entire sequence of transpositions made, not just about the resulting permutation. The reason is that generators of the symmetric group satisfy, apart from (2.2) and (2.3), one more relation

\[ \sigma_i^2 = 1. \]  

(2.5)

Hence, a braid, and the knot being its closure, in fact represent not just a symmetric group element, but an element of this group’s deformation. In particular, one may consider some monodromy group as such an extension, as we discussed in the introduction.

2.1.2 Braids closures and knots

Equivalence transformations of braids closures.

\[
\begin{align*}
\text{I} & \quad \sim \quad \text{II} \quad \sim \quad \text{II}' \\
\text{III} & \quad \sim \quad \text{IV} \quad \sim \quad \text{IV}'
\end{align*}
\]  

(2.6)

So far we have considered the braids and their equivalence transformations. However, a knot is the closure of a braid \( \mathcal{B} \). Hence, to construct a knot invariant, one should treat braids closures instead. Their equivalence transformations include those of the braids, as well as two more transformations. The first one is pulling a sequence of crossings \( \sigma_1\sigma_2\ldots \sigma_k \) from the beginning of the braid to it’s end through the closure; the reversed sequence, \( \sigma_k \ldots \sigma_2\sigma_1 \), appears then at the braid’s end, (2.6-I). Hence, for a given element \( \mathcal{B} \) of the braid group, all the elements from the same conjugation class, i.e., those of the form \( \mathcal{B}\mathcal{B}\mathcal{B}^{-1} \) for a group element \( \mathcal{B} \), correspond to the isotopic braid closures. The transformation is called the first Markov transform. If one has a representation of a braid group, then each braid is related to a linear operator, the same one for all equivalent braids. Traces of the operators, in turn, coincide for all braids from the same conjugation class. However, the obtained quantity is not yet a knot invariant, since there is the second equivalence transformation of braid closures that is absent for the braids. The one reduces to contracting of a unknotted loop, (2.6-II or 2.6-II'), and is referred to the second Markov transformation. An invariant of a braid closure should be also invariant under this transform. As we illustrate in the forthcoming examples, this may be achieved by taking a certain linear combination of the group elements traces evaluated in various irreducible representations, in other words, by selecting a certain reducible representation of the braid group.

According to the Markov theorem 3, any two braid closures that represent topologically equivalent knots are transformed into each other by a sequence transformations, which include group
multiplication law (2.1-III), Artin transformations (2.1-I,II), and Markov transformations, (2.6-I,II). In particular, transformation (2.6-II') may be done this way, and hence should not be included in the list of elementary ones. Hence, a monodromy group of anything may be used to construct a knot invariant, with the invariance under (2.1) being provided by the monodromy group properties, (2.6-I) achieved by taking the trace, and only (2.6-II) being imposed separately. Let us emphasize, that the only fact that matters here is that the braid group extends a monodromy group. A knot invariant may be expressed via monodromies of these or those quantities, but there is no reason to think that the very quantities are related to knots or braids in a deeper sense, at least not generally.

The thus far considered braid representation may be seen as a one-dimensional description of a knot, as the knot is encoded by a sequence of permutation or monodromy operators. There are other knot representations of a similar kind [17]. In particular, a knot may be related to a braid, whose strands are oriented differently. A “closure” of such a braid includes an auxiliary element. [41]. A conventional notation for Pretzel knots [17],[46],[47] is this type one. All such braid-like representations may be used to represent a knot invariant as an invariant of some monodromy group, in a similar way and in the same sense as described above. The invariance under analogues of braid transformations (2.2, 2.3) may be provided in a universal manner, by putting into crossings monodromy matrices of anything; the first transformation of the braid closures is accounted for taking the trace. The only but major difference will be in treating the second (strand number changing) transformation of the braid closures, i.e., in searching for a linear combination of monodromy matrices traces that would be a braid closure invariant, and thus, due to the Markov theorem [3], a knot invariant.

2.1.3 Unlinked knots and multiplication property

One degenerated case was left thus far beyond the scope of our consideration. Namely, if a strand of braid is never intertwined with the other ones (it may be the first or the last strand in a braid section, see fig.1), then the closure of the strand, which is just a circle, is separated from the closure of the braid. More generally, the first \(k\) strands may never cross the \(m-k\) last ones, or the braid may be reduced to one of the type by the equivalence transformations; the closure is a disjoint union of two or more knots or links. From the formal standpoint, an invariant of a disjoint union a quantity, independent of the invariants of the components. However, many link invariants, among them Alexander, Jones, Kaufman, and HOMFLY polynomials, by definition possess a multiplication property [3]. Namely, an invariant of a disjoint union of links equal to the product of the invariants of the components. We set the value of the link invariant for the closure of the braid in fig.1 (we write \(B \otimes 1\) for it, meaning the \(B\) is the braid without the last strand) according to the multiplication property, i.e.,

\[
H(B \otimes 1) \equiv H(B)H(1).
\] (2.7)

It may be shown that the invariant we will come to in the end possesses the multiplication property generally. Moreover, (2.7) in fact sets the co-algebra structure on the braids, which turns to be essential for definition of the knot invariants of interest [63]. We partially illustrate this in the examples below.

2.2 From braid group invariant to knot invariant

2.2.1 Knot polynomials as symmetric group invariants

The idea is to construct a knot invariant as a trace of a group element. An auxiliary step is to note that [3]

\[ \text{The named property applies to not normalized knot invariants, but a similar claim holds for the normalized ones.} \]
Step 2.1 A knot is the closure of a braid.

A braid $B$ is associated with a matrix $B(B)$; a composite braid $B_1 \# B_2$ that consists of two successive braids $B_1$ and $B_2$ plaited on the same strands corresponds to a product of the associated matrices,

$$B(B_1 \# B_2) = B(B_1)B(B_2).$$

(2.8)

The matrices $B$ hence lie in a representation of the braid group. The next idea is that

Step 2.2 A representation of the braid group is a representation of the symmetric group with the same number of generators.

Indeed, generators of both groups satisfy relations (2.2,2.3), which are also satisfied for generators of a symmetric group, in addition satisfying (2.5). and hence a representation of the symmetric group is not necessarily a representation of a braid group. However, we are interested in finding not all matrix representations of a braid group, but at least one, which provides knot invariants. Therefore we set that $B$ lie in a representation of the symmetric group, whose arbitrary finite-dimensional irreducible representations are known [4, 9]. The made anzatz dictates that

Step 2.3 Dimensions of the matrices of crossing operators are dimension of the symmetric group irreducible representations.

E.g., for the one-strand braid they are $1 \times 1$ since the only (irreducible) representation of symmetric group one generator is the one-dimensional representation. As we demonstrate in sec.2.2.4 matrices for the two-strand braid are of the same size, since both (irreducible) representations of the symmetric group with two elements are one-dimensional. In turn, the symmetric group with three elements has one- and two-dimensional irreducible representations, hence the three-strand matrices may be either one-, or two-dimensional (see sec.2.2.5), etc.

Once a representation of the symmetric group is chosen, each generator $\sigma_i$ corresponds to a linear operator $B_i$; one takes the product of these linear operators along the braid, putting the operator $B_i$ for the crossing of $i$-th and $i+1$-st strands. The trace of this product is a braid invariant for any representation of the symmetric group; this is guaranteed by (2.2), (2.3), which define the braid group, and by (2.5), which is responsible for cancelation of two consecutive mutually inverse crossings. Unfortunately, this trace is just a number (not a polynomial), and, much worse, is not sensitive to the orientation of crossing due to the same (2.5). In particular, this number will be the same for the whole series of $[2,n]$ torus knots and links, which are closures of various two-stand braids (fig.7).

2.2.2 Knot polynomials as Hecke algebra invariants

Luckily, a symmetric group admits a one-parametric deformation, which is called Hecke algebra [62]. The Hecke algebra generators satisfy the braid group defining relations, and the deformation of (2.5):

$$\sigma_i^2 = (q - q^{-1})\sigma_i + 1,$$

(2.9)

where $q$ is a new formal variable. An important here theorem is that the finite-dimensional irreducible representations of Hecke algebra are in one to one correspondence with those of the symmetric group, the latter recovering from the former ones for $q = 1$. Taking an arbitrary representation of the Hecke algebra, one constructs the product linear operators along the braid just as before; the only difference is that the operator $B_i$ stands now for the direct (of any one out of two possible orientations) crossing of $i$-th and $i+1$-st strands, and the operator $B_i^{-1}$ stands for the corresponding inverse crossing (invertibility of operators is guaranteed by (2.9)). In principle, one may just take explicit expressions for Hecke algebra operators in some representation (the ones are known [62]), take their product along a braid as described above and obtain a braid group invariant, what is guaranteed by (2.2) and (2.3).

We, however, go an other way, which seems us more illustrative.
Step 2.4 Writing down matrices $B_i$ of size defined on Step 2.3 with undefined matrix elements, we impose on them the constraints (2.2) and (2.3).

By that, we suppose the $B_1$ to be diagonal (what may be achieved by choosing a proper basis) and

Step 2.5 Solve the obtained system of equations w.r.t. to the elements of the matrices $B_i$, $i > 1$, expressing them via the eigenvalues of the matrix $B_1$.

In case of success, it remains to

Step 2.6 Take a product of the obtained matrices, passing along the braid and putting for a crossing of $i$-th and $i+1$-st strands $B_i$, for one orientation of the crossing, and $R_i^{-1}$ for the inverse orientation (fig. 3.1).

The obtained quantity will be a matrix, whose elements depend on a set of the formal variables ($B_1$ eigenvalues), and which will turn the same for various braids that reduce to each other by the equivalence transformations encoded in (2.2,2.3). The trace of the same expression, which is also a function of several formal variables, will be also the same for all braids of the same conjugation class (i.e., for those related by a conjugation, $B \to \tilde{B}B\tilde{B}^{-1}$, by an element $\tilde{B}$ of the braid group, what is pictorially presented as drawing a pair of mutually inverse braids on the ends of the initial one). Yet, to construct a knot invariant, one more and rather non-trivial step is still needed.

2.2.3 From braid group invariant to knot invariant

As already said, topologically the same knot may be presented as the closure of infinitely many various braids. These braids are transformed into each other with help of (2.2,2.3), as long as they are of the same width. The last thing to account for is that the same knot may be also represented by braids of a different width. To achieved this, one must

Step 2.7 Sum up the constructed braid group conjugation classes invariants with certain weight coefficients,

or, in other words, to carry out the presented program in a certain reducible representation of a symmetric group.

We interrupt the general description at this point, sending a reader to [62] for the sequel, and proceed with carrying out the formulated steps explicitly in the concrete examples.

2.2.4 Two-strand braids

Step 1. A two strand braid has the form $\sigma_1^n$, where $\sigma_1$ is the only braid group generator. The closures of various two-strand braids (fig. 7) represent the so named series $T^{2,n}$ of torus knots (for $n$ odd) and links (for $n$ even). In particular, $T[2,0]$ is a pair of unlinked unknots, and $T[2,1]$ is the once intertwined unknot (fig. 4), $T[2,2]$ is the Hopf link, $T[2,3]$ is a trefoil knot [5], $T[2,4]$ is the Solomon’s link, etc (see figures at [17]).

Step 2. There are $2! = 2$ permutations of two elements, $xy$ and $yx$. The space of their formal linear combinations is the two-dimensional, a basis may be chosen as

$$X_S = \frac{1}{2}(xy + yx), \quad X_A = \frac{1}{2}(xy - yx).$$

(2.10)

One refers to the formal expressions $X_S$ and $X_A$ as to one-dimensional representations of the symmetric group $S_2$, implying that are conserved, up to factor, subjected to the symmetric group generator $\sigma_1$ that permutes $x$ and $y$,

$$\sigma_1 X_S = X_S, \quad \sigma_1 X_A = -X_A.$$  

(2.11)
Step 3. According to the anzatz of Step 2.3 one should take two matrices $1 \times 1$, denote them $\lambda$ and $\mu$. No constraints are imposed on the single generator of the two-strand braid group, hence Steps 2.4-2.5 are omitted, and we go to

Step 6. According to the ansatz of Step 2.6 a two-strand braid is associated with a matrix product

$$B (\sigma^n_1) = (B^I_1)^n, \quad I = S \text{ or } A. \quad (2.12)$$

where $n$ is the number of crossings, positive or negative depending if the crossings are of direct or inverse orientation, respectively (see fig 3.1).

Finally, the indeed non-trivial step, already in the case concerned, is

Step 7. Following the ansatz of Step 2.7 we write the invariant of a braid in the form

$$\lambda^n \chi_S + \mu^n \chi_A, \quad (2.13)$$

where $n$ is the number of crossings, positive or negative depending on their orientation (see 2.1-II), the eigenvalues $\lambda$ and $\mu$ are considered as formal variables, while $\chi_S$ and $\chi_A$ are (weight) coefficients to define.

To obtain a knot (or link), not just a braid invariant, one has to impose some extra conditions on (2.13). E.g., one may observe that the closure of the two parallel strands reproduces a pair of unlinked unknots, while the closure of the two once intersected strands reproduces the single unknot with a contractible loop. The value of invariant of the unknot is not defined at the moment, and we denote it just $\chi$; due to multiplication property (2.7) for braids like the one in (2.1-II), the invariant of a pair of unlinked unknots equals $\chi^2$. The resulting constraints read:

$$1^{(2)} \sim 1^{(1)} \otimes 1^{(1)} \Rightarrow \chi^2 = \chi_S + \chi_A,$$

$$B^{(2)}_1 \sim 1^{(1)} \Rightarrow \chi = \lambda \chi_S + \mu \chi_A, \quad (2.14)$$

System (2.14) enables one to express the coefficients of irreducible representations in terms of the eigenvalues of the corresponding operators and value of the invariant for the unknot:

$$\chi_S = \frac{(\mu \chi - 1) \chi}{\mu - \lambda}, \quad \chi_A = \frac{(\lambda \chi - 1) \chi}{\lambda - \mu}. \quad (2.15)$$

The resulting expression for invariant of the knot represented as the closure of a two-strand braid with $n$ crossings oriented the upper one in 2.1-II, or with $-n$ crossings oriented as the lower one, is

$$H(\mathbb{B}; \lambda, \mu) = \lambda^n \frac{(\mu \chi - 1) \chi}{\mu - \lambda} + \mu^n \frac{(\lambda \chi - 1) \chi}{\lambda - \mu}, \quad (2.16)$$

where $\lambda$ and $\mu$ may be substituted by arbitrary numbers or just left as formal parameters.

### 2.2.5 Three-strand braids

Step 1. A three-strand braid has the form $\sigma_1^{a_1} \sigma_2^{b_1} \sigma_1^{a_2} \sigma_2^{b_2} \ldots$, where $\sigma_1$ and $\sigma_2$ are the braid group generators, while $a_1$, $a_2$, $\ldots$ and $b_1$, $b_2$, $\ldots$ are integer numbers, positive or negative. Not all braids with various $a$ and $b$ are inequivalent; many of them are isotopic, and hence must be equivalent as braid group elements. The corresponding formal expressions are brought to each other with help of the braid group relation

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \quad (2.17)$$
which is a particular case of (2.2) for \( m = 3 \). The smallest values of \( a \) and \( b \) correspond to unknots, there unlinked ones for the trivial braid \( 1 \) (all \( a \) and \( b \) equal zero), two unlinked ones (one once intertwined) for the braids \( \sigma_1^{e-1} \) and \( \sigma_2^{e-1} \), and one twice intertwined for the braids \( \sigma_1^{e-1} \sigma_2^{e-1} \sim \sigma_2^{e-1} \sigma_1^{e-1} \), where \( s_1 \) and \( s_2 \) equal 1 or \(-1\) independently of each other. The first non-trivial knot is obtained for \( a_1 = a_2 = 1 \), and \( b_1 = b_2 = 1 \); this is the trefoil knot with an additional contractible loop on a wire. Setting all \( a \) and \( b \) equal to one (2\( n \) of them non-zero), one obtains a braid of the form \((\sigma_1\sigma_2)^n\), whose close is a torus knot (for \( n \) not multiplies 3) or link (otherwise) out of the series named \( T^{3,n} \), which starts form the twice intertwined unknot (\( n = 1 \)), the trefoil knot with a contractible loop (\( n = 2 \)), and the Borromean rings link (\( n = 3 \), \( L6a4 \) or \( 6_2^3 \) in [17]). The simplest so called twist knots are represented by three-stand braids, namely, the figure-eight, or two half-twist knot \((4_1 \text{ in } [17])\) is the closure of the braid \( \sigma_1(\sigma_2)^{-1}\sigma_1(\sigma_2)^{-1} \), and the next, three half-twist knot \((5_2 \text{ in } [17])\) is the closure of the braid \( \sigma_1^3\sigma_2\sigma_1^{-1}\sigma_2 \). An infinitely many topologically different non-torus and non-twist knots are among the closures of three-strand braids, see [17].

**Step 2.** There are \( 3! = 6 \) permutations of six elements, \( \{xyz, xzy, yxz, yzx, zxy, zyx\} \). In analogy with the symmetric group of two elements, two linear combinations are conserved up to a factor by all permutations,

\[
X_S = xyz + yxz + xzy + yzx + zyx + zxy \equiv (xyz), \quad \sigma_1 X = \sigma_2 X = X,
\]

and

\[
X_A \equiv xyz - yxz - xzy + yzx + zxy - zyx \equiv [xyz]. \quad \sigma_1 X = \sigma_2 X = -X,
\]

Hereafter, we use parentheses for a full symmetrization, i.e., for the sum over all permutations, and we use square brackets for a full antisymmetrization, i.e., for the sum over all even permutations minus the sum over all odd permutations.

The linear space spanned by the permutations of three elements has the dimension \( 3! = 6 \), a generic vector being of the form:\footnote{Subscripts of the coefficients reflect the sequence of permutations that yields a given element, i.e., \( a_{121} \) is the coefficient of \( zyx = \sigma_1\sigma_2\sigma_1 xyz \).}

\[
X = axyz + a_{1}yxz + a_{2}xyz + a_{12}yxz + a_{21}zxy + a_{121}zxy.
\]

The cases, (2.18) with \( a = a_1, a_2 = a_{12}, a_{21} = a_{121} \), and (2.19) with \( a = a_2, a_1 = a_{21}, a_{12} = a_{121} \), correspond to the two common eigenvectors of the group generators, and thus to the two one-dimensional eigenspaces of the symmetric group. Apart from that, the generators have a common four-dimensional eigenspace (complementary to the subspace spanned by the two common eigenvectors), given by the system of constraints:

\[
\begin{align*}
{a + a_1 + a_2 + a_{12} + a_{21} + a_{121} = 0,} \\
{a - a_1 - a_2 + a_{12} + a_{21} - a_{121} = 0} \\
\end{align*}
\]

i.e. coefficients of the permutations mutually related by a cyclic shift must sum up to zero. It is straightforward to verify that this property is covariant under the transpositions, both of the first two and of the last two elements. Hence, (2.21) indeed defines a subspace of (2.20) invariant under the action of the symmetric group.

The four-dimensional space given by (2.21) is by definition a representation of the symmetric group, but not an irreducible one. It turns out to decompose into two two-dimensional irreducible representations. More precisely, it includes the two-dimensional subspace

\[
X_{AS} = a(xy)z + b(yz)x + c(zx)y \equiv axyz + ayxz + byzx + bzyx + cxyz + cxyz,
\]

\[
a + b + c = 0.
\]
each vector of which generates a two-dimensional irreducible representation of the symmetric group.

Indeed, $X_{AS}$ is, by construction, an eigenvector of the first generator:

$$\sigma_1 X_{AS} = X_{AS}; \quad (2.23)$$

in addition, it satisfies the identity

$$X_{AS} + \sigma_2 X_{AS} + \sigma_1 \sigma_2 X_{AS} = (a + b + c)(xyz) = 0. \quad (2.24)$$

To verify it, let us notice, that (2.24) contains $3 \cdot 6 = 18$ summands and includes each of 6 monomials exactly 3 times, with different coefficients, as they arise from different brackets in (2.22), e.g.,

$$(a + b + c)xyz = (axzy + \sigma_2 czxy + \sigma_1 \sigma_2 byzx),$$

$$(a + b + c)yxz = (ayxz + \sigma_2 byzx + \sigma_1 \sigma_2 czxy), \quad (2.25)$$

and similarly for the other monomials. Relations (2.23) and (2.24) guarantee that the representation in question is two-dimensional, since any $X_{AS}$ from subspace (2.22) and the corresponding $\sigma_2 X_{AS}$ are turned by the group generators into linear combinations of each other,

$$
\begin{array}{c|cc}
\sigma_1 & X & \sigma_2 X \\
\hline
X & -X - \sigma_2 X \\
\sigma_2 & X & \sigma_2 X \\
\end{array}
$$

(2.26)

In fact, it is not necessary to start exactly from (2.22) to construct a two-dimensional irreducible representation of the symmetric group. Subspace (2.21) is a direct sum of two two-dimensional subspaces of the first generator’s eigenvectors, with the eigenvalues 1 and $-1$, respectively. On the other hand, same subspace (2.21) is a similar direct sum w.r.t. the second generator. A plane containing simultaneously two first generator’s eigenvectors and two second generator’s eigenvectors (the ones with distinct eigenvalues, otherwise there would be more common eigenvectors, while their is non) is a two-dimensional common eigenspace of the generators. One has to parametrize all these planes in one or another way. In the what follows, we checked and used, that each first generator’s eigenvector with the eigenvalue 1 from subspace (2.21) generates, together with it’s image under the second generator, a two-dimensional common eigenspace of the generators; hence, these eigenspaces are in one to one correspondence with the first generators eigenvectors. One can use the first generators eigenvectors with the eigenvalue $-1$ equally well. The latter ones also lie in subspace (2.21) and have the form

$$X_{SA} = a[xy]z + b[yz]x + c[zx]y \equiv axyz - ayxz + byzx - bzyx + czxy - cxzy, \quad a + b + c = 0, \quad (2.27)$$

and satisfy the identities, similar to (2.23,2.24):

$$\sigma_1 X_{SA} = -X_{SA}, \quad (2.28)$$

and

$$X_{SA} - \sigma_2 X_{SA} + \sigma_1 \sigma_2 X_{SA} = (a + b + c)(xyz) = 0, \quad (2.29)$$

the last one being verified with help of the equalities

$$(a + b + c)xyz = (axzy - \sigma_2(-cxzy) + \sigma_1 \sigma_2 byzx),$$

$$-(a + b + c)yxz = (-ayxz - \sigma_2 byzx - \sigma_1 \sigma_2 czxy), \quad (2.30)$$

e tc. One could also start from eigenvectors of the second generator instead. Of course, in all cases one will obtain the same set of two-dimensional planes, just differently parametrized. Each of these planes is a common eigenspace of the two symmetric group generators, thus being an invariant subspace (in other words, a space of a representation) of the entire symmetric group. The group generators
acting on such a subspace may be presented as $2 \times 2$ matrices. In particular, if one selects a basis as \{\(X_{AS}, \sigma_2 X_{AS}\)}, for any \(X_{AS}\) from (2.21), the matrices are read from table (2.26),

\[
\sigma_1 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma_2 X \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\] (2.31)

For practical purposes, a basis of eigenvectors of \(\sigma_1\) (or \(\sigma_2\)) is more convenient:

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad X \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{3}}(X + 2\sigma_2 X) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\] (2.32)

where the normalization factor in front of the second eigenvector is chosen so that the matrix of \(\sigma_2\) is symmetric. Independence of these matrices of \(a, b,\) and \(c\), entering expression (2.22) for \(X\), means that all the representations with various \(a, b,\) and \(c\) are isomorphic.

In explicit calculations, it is convenient to take a certain \(X_{AS}\) from (2.22), the conventional choice is

\[
a = -b = 1 \text{ in (2.22)} \Rightarrow X = (xy)z - (zx)y, \quad X + 2\sigma_2 X = 2[yz]x - [xy]z - [zx]y, \quad (2.33)
\]
\[
a = b = 1, \quad c = -2 \text{ in (2.22)} \Rightarrow X = (xy)z + (zx)y - 2(yz)x, \quad X + 2\sigma_2 X = 3[yz]x - 3[zx]y.
\]

Finally, we have constructed three distinct representations of the symmetric group with three elements, two one-dimensional ones, (2.18), (2.19), and a two-dimensional one, generated by any vector of form (2.22). In fact, there is an entire two-dimensional space of such vectors, given by (2.22). Each of these vectors generates a two-dimensional representation, altogether filling a four-dimensional space, and, together with the two one-dimensional representations, covering the entire six-dimensional space spanned by the \(3! = 6\) permutations of six elements.

**Step 3.** We have obtained three distinct irreducible representations of the symmetric group with three elements, one-dimensional ones (2.18), (2.19), and a two-dimensional one, generated by any vector of form (2.22). A known theorem claims that there no other ones [4]. The anzatz of Step 2.3 dictates then to take three pairs of matrices, \(B_i^{SS}\) and \(B_i^{AA}\) of the size \(1 \times 1\), and \(B_i^{AS}\) of the size \(2 \times 2\), with \(i = 1, 2\). The next step is to determine the explicit form of the matrices from the braid group relations.

**Step 4.** Matrices \(B_1\) and \(B_2\) must satisfy group relations (2.17). For the one-dimensional representations, it gives merely

\[
B_1^{SS} = B_2^{SS} = \alpha, \quad B_1^{AA} = B_2^{AA} = \delta,
\] (2.34)

while for the two dimensional representation a non-trivial equations is obtained,

\[
B_1^{SA} B_2^{SA} B_3^{SA} = B_2^{SA} B_1^{SA} B_2^{SA},
\]

\[
\begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\] (2.35)

**Step 5.** Equality (2.35) is satisfied if \(B_2 = B_1\). Apart from that, there is the non-trivial solution for \(B_2:\)

\[
a = \frac{\gamma^2}{\gamma - \beta}, \quad bc = \frac{\beta \gamma (\beta \gamma - \beta^2 - \gamma^2)}{(\beta - \gamma)^2}, \quad d = \frac{\beta^2}{\beta - \gamma}.
\] (2.36)
and we proceed with determining the standing weight coefficients from matching the invariant of a braid group conjugation class. At this step, we have obtained an invariant of a braid group conjugation class. The ansatz of

\[ B^1_{SA} = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \quad (B^1_{SA})^{-1} = \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \]

which may be chosen, e.g., as

\[ \text{Id, } B_1, \ B_2, \ B_1B_2, \ B_2B_1, \ B_2B_1B_2. \] (2.38)

This is easy to verify in each particular case, e.g. (the omitted superscript SA is assumed),

\[ \begin{pmatrix} B_2B_1B_2^{-1} B_2B_1B_2 \end{pmatrix} = B_1 B_2B_1B_2 B_1 = B_1 B_2B_1B_2 = (\beta + \gamma) B_1 - \beta \gamma \]

is assumed),

\[ \begin{pmatrix} (\beta + \gamma) B_1 - \beta \gamma \end{pmatrix} = (\beta + \gamma) \left( B_1 B_2 + B_2 B_1 \right) + (\beta + \gamma)^2 B_1 B_2 B_1. \]

A trace of a braid group element, which enters the knot invariant definitions, therefore, expands over the basis traces as well; the latter ones are independent of \( b \),

\[ \text{Tr } 1 = 2, \quad \text{Tr } B_1 = \text{Tr } B_2 = \beta + \gamma, \quad \text{Tr } B_1 B_2 = \text{Tr } B_2 B_1 = a\beta + d\gamma, \quad \text{Tr } B_1 B_2 B_1 = a\beta^2 + d\gamma^2, \]

and, hence, so does the knot invariant. When discussing the matrices themselves, we will fix this arbitrariness so that the matrices turn symmetric; the result then is

\[ B^1_{SA} = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \quad (B^1_{SA})^{-1} = \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \]

\[ B^2_{SA} = \begin{pmatrix} \frac{\gamma^2}{\gamma-\beta} & \sqrt{\frac{\beta\gamma-\beta^2-\gamma^2}{\beta-\gamma}} & \frac{\beta^2}{\gamma-\beta} \\ \sqrt{\frac{\beta\gamma-\beta^2-\gamma^2}{\beta-\gamma}} & \frac{\beta\gamma-\beta^2-\gamma^2}{\beta-\gamma} & \frac{\beta\gamma-\beta^2-\gamma^2}{\beta-\gamma} \\ \frac{\beta}{\gamma(\gamma-\beta)} & \frac{\gamma}{\beta(\gamma-\beta)} & \frac{\beta\gamma}{\beta(\gamma-\beta)} \end{pmatrix}, \]

\[ (B^2_{SA})^{-1} = \begin{pmatrix} \frac{\beta}{\beta-\gamma} \sqrt{\frac{\beta\gamma-\beta^2-\gamma^2}{\beta\gamma}} & \frac{1}{\beta-\gamma} \sqrt{\frac{\beta\gamma-\beta^2-\gamma^2}{\beta\gamma}} & \frac{\beta\gamma}{\beta(\gamma-\beta)} \\ \frac{1}{\beta-\gamma} \sqrt{\frac{\beta\gamma-\beta^2-\gamma^2}{\beta\gamma}} & \frac{1}{\beta-\gamma} \sqrt{\frac{\beta\gamma-\beta^2-\gamma^2}{\beta\gamma}} & \frac{\beta\gamma}{\beta(\gamma-\beta)} \\ \frac{\beta\gamma}{\beta(\gamma-\beta)} & \frac{\beta\gamma}{\beta(\gamma-\beta)} & \frac{\beta\gamma}{\beta(\gamma-\beta)} \end{pmatrix}. \] (2.39)

**Step 6.** Following the ansatz of **Step 2.6**, we write the invariant of a three-strand braid in the form

\[ B^I \left( \sigma_1^{a_1} \sigma_2^{a_2} \ldots \sigma_1^{a_k} \sigma_2^{b_k} \right) = \prod_{i=1}^{k} (B^I)_{a_i}^{b_i}, \quad I = SS, \ AS, \ AA. \] (2.40)

**Step 7.** At this step, we have obtained an invariant of a braid group conjugation class. The ansatz of **Step 2.7** dictates then to search for invariant of a knot presented as the closure of a three-strand braid \( \sigma_1^{a_1} \sigma_2^{a_2} \ldots \sigma_1^{a_k} \sigma_2^{b_k} \) in the form

\[ \sum_{I=S,A,SA} \chi_I \text{Tr} \left( B^I \right)^{a_k} \left( B^I \right)^{b_k} = \chi_{SS} \alpha^n + \chi_{SA} \text{Tr} \left( (B^I_{SA})^{a_1} (B^I_{SA})^{b_1} \right) + \chi_{AA} \delta^n, \] (2.41)

and we proceed with determining the there standing weight coefficients from matching the invariant values for braids with different numbers of strands representing same knots. For three-strand braids, this step is even less trivial compared to that for two-stand braid, so we split it into two substeps.
Step 7-a: co-product for operators and weight coefficients. This step is based on the multiplication property \((2.7)\) [\footnote{fig}1], which implies that
\[
\left(\sigma_1^{(3)}\right)^n \sim \left(\sigma_1^{(2)}\right)^n \otimes \mathbb{1}^{(1)} \Rightarrow \lambda^n \chi_{\text{SS}} + \mu^n \chi_{\text{AA}} = \alpha^n \chi_{\text{SS}} + (\beta^n + \gamma^n) \chi_{\text{SA}} + \delta^n \chi_{\text{AA}}
\]
for an arbitrary \(n\). Relation \((2.42)\) may be seen as a system of infinitely many homogeneous linear equations. The only solution is
\[
\chi_{\text{SS}} = \chi_{\text{SA}} = \chi_{\text{AA}} = 0,
\]
unless the equations are linearly dependent. A principle minor occupying in the lines with \(n = 0, \ldots, 5\) is a Wademonde determinant,
\[
\det \begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
\lambda & \mu & \alpha & \beta & \gamma & \delta \\
\lambda^2 & \mu^2 & \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\
\lambda^3 & \mu^3 & \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \\
\lambda^4 & \mu^4 & \alpha^4 & \beta^4 & \gamma^4 & \delta^4 \\
\lambda^5 & \mu^5 & \alpha^5 & \beta^5 & \gamma^5 & \delta^5
\end{vmatrix}
= (\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)(\gamma - \delta) \cdot \lambda(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)(\gamma - \delta) - \mu(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)(\gamma - \delta);
\]
any other one differs just by a factor of type \(\lambda^{m_1} \mu^{m_2} \alpha^{m_3} \beta^{m_4} \gamma^{m_5} \delta^{m_6}\). Analysis of smaller sized minors shows, that rank of the matrix in the l.h.s. of \((2.44)\), which is the number of linearly independent solutions of \((2.42)\), equals to the number of distinct eigenvalues among \(\lambda, \mu, \alpha, \beta, \gamma, \delta\), the coefficient before each one becoming one of the relations generating the space of solutions of \((2.42)\).

There is a distinguished solution, which has a profound group theory sense:
\[
\lambda = \alpha = \beta, \quad \mu = \gamma = \delta
\]
\[
\chi_{\text{SS}} = \chi_{\text{SA}} + \chi_{\text{AA}}, \quad \chi_{\text{SA}} = \chi_{\text{AA}}.
\]
Relations \((2.45)\) are produced by a coproduct structure on the braid group \([64]\).

Note, that \((2.42)\) is satisfied as long as \((2.45)\) does; in particular, the coefficients are thus far completely independent of eigenvalues. The last step consists in relating them two.

Step 7-b. It remaining step relies on transformation \((2.6)\) II). As already mentioned, it is the corresponding constraint on the coefficients and eigenvalues that turn a braid invariant to a knot invariant. There is an infinite set on constraints of the form
\[
\left(\sigma_1^{(3)}\right)^{n-1} \sigma_2^{(3)} \sim \left(\sigma_1^{(2)}\right)^{n-1} \Rightarrow \lambda^{n-1} \chi_{\text{SS}} + \mu^{n-1} \chi_{\text{AA}} = \lambda^n \chi_{\text{SS}} + \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda - \mu} \chi_{\text{SA}} + \mu^n \chi_{\text{AA}},
\]
for all integer \(n\). Any three of these equations are linearly dependent since
\[
\lambda \begin{vmatrix}
\lambda^n \\
\lambda^m \\
\lambda^k
\end{vmatrix}
- \mu \begin{vmatrix}
\mu^n \\
\mu^m \\
\mu^k
\end{vmatrix}
= \begin{vmatrix}
\lambda^{n+1} - \mu^{n+1} \\
\lambda^{m+1} - \mu^{m+1} \\
\lambda^{k+1} - \mu^{k+1}
\end{vmatrix}.
\]

Hence, one may determine all three three-strand coefficients, selecting any two of constrains \((2.46)\), e.g. with \(n = 1, 2\) and completing them by any one of \((2.45)\), e.g. with \(n = 0:\)
\[
\begin{align*}
\mathbb{1}^{(3)} & \sim \mathbb{1}^{(2)} \otimes \mathbb{1}^{(1)} \Rightarrow \chi_{\text{SS}} + \chi_{\text{AA}} = \chi_{\text{SS}} + 2 \chi_{\text{SA}} + \chi_{\text{AA}}, \\
\sigma_2^{(3)} \sim \mathbb{1}^{(2)} \Rightarrow \chi_{\text{SS}} + \chi_{\text{AA}} = \lambda \chi_{\text{SS}} + (\lambda + \mu) \chi_{\text{SA}} + \mu \chi_{\text{AA}}, \\
\sigma_1^{(3)} \sigma_2^{(3)} \sim B_1^{(2)} \Rightarrow \chi_{\text{SS}} + \mu \chi_{\text{AA}} = \lambda^2 \chi_{\text{SS}} + \lambda \mu \chi_{\text{SA}} + \mu^2 \chi_{\text{AA}}.
\end{align*}
\]
wherefrom, taking into account (2.15), one gets
\[ \chi_{SS} = \frac{\chi(\mu \chi - 1)(\lambda^2 \chi + \mu - \lambda)}{(\mu - \lambda)\lambda\mu}, \quad \chi_{AA} = \frac{\chi(\lambda \chi - 1)(\mu^2 \chi + \lambda - \mu)}{(\lambda - \mu)\lambda\mu}, \]
(2.49)
and
\[ \chi_{SA} = \frac{\chi(\mu \chi - 1)(\lambda \chi - 1)}{\lambda\mu}. \]

It may be checked that selecting any other three linearly independent constrains from of (2.46) and (2.45) provides the same answer.

**The result.** Finally, the invariant of the knot presented as the closure of a three-strand braid is expressed via the two-strand eigenvalues as
\[
H\left(\sigma_1^{a_1} \sigma_2^{b_1} \ldots \sigma_1^{a_k} \sigma_2^{b_k}\right) = \lambda^n \frac{\chi(\mu \chi - 1)(\lambda^2 \chi + \mu - \lambda)}{(\mu - \lambda)\lambda\mu} + \mu^n \frac{\chi(\lambda \chi - 1)(\mu^2 \chi + \lambda - \mu)}{(\lambda - \mu)\lambda\mu} + \operatorname{Tr} \prod_k \left(B_{1A}^{SA} \right)^{a_k} \left(B_{2A}^{SA} \right)^{b_k} \chi(\mu \chi - 1)(\lambda \chi - 1) \frac{1}{\lambda\mu},
\]
(2.50)
where \(B_{1A}^{SA}\) and \(B_{2A}^{SA}\), as well as the inverse matrices, are given by (2.39) with \(\beta = \lambda\) and \(\gamma = \mu\). Let us emphasize, that \(\lambda\) and \(\mu\) enter (2.50) just as formal variables; substituting for them arbitrary numbers, or, equivalently, taking the coefficient of their any powers provides a numeric knot invariant.

### 2.2.6 Four-strand braids

**Step 1.** A four-strand braid has the form
\[ \mathcal{B}^{(4)} = \sigma_1^{a_1} \sigma_2^{b_1} \sigma_3^{c_1} \sigma_1^{a_1} \sigma_2^{b_1} \sigma_3^{c_1} \ldots \sigma_k^{a_1} \sigma_2^{b_1} \sigma_3^{c_1}. \]
(2.51)
The particular case of \(\mathcal{B}^{(4)} = (\sigma_1 \sigma_2 \sigma_3)^k\) corresponds to the torus knots (even \(k\)) and links (odd \(k\)) of the so named series \(T^{4,k}\). The four and five half-twist knots (6_1 and 7_2 in [17], respectively), as well as all knots with no more then 7 crossings may be presented as the closures of four-strand braids [17].

**Step 2.** As usual, we begin from listing the irreducible representations of the symmetric group, this time acting on four elements. Similarly to the previous cases, there are two one-dimensional representations, fully symmetric one,
\[ X_{SSS} = (xyzt), \quad \sigma_1 X_{SSS} = \sigma_2 X_{SSS} = \sigma_3 X_{SSS} = X_{SSS}, \]
(2.52)
where all the permutations of \(xyzt\) enter with the same coefficient, and fully antisymmetric one,
\[ X_{AAA} = [xyzt], \quad \sigma_1 X_{AAA} = \sigma_2 X_{AAA} = \sigma_3 X_{AAA} = -X_{AAA}, \]
(2.53)
where coefficients of all the permutations are equal up to a sign, being plus for even permutations, and minus for odd permutations (we recall that in all formulas parentheses and the square brackets stand for stand, correspondingly, for the fully symmetric and antisymmetric combinations constructed from the embarked elements). Apart from the one-dimensional representations, the symmetric group with four elements has several more complicated ones. These representations are constructed even less trivial than a two dimensional representation of the symmetric group with three elements. For this reason, we sketch briefly the common representation theory approach to the problem, before listing the representations in question explicitly.
An important here theorem states that the irreducible representations of a symmetric group with $k$ elements are in one-to one correspondence with the partitions of $k$

$$k = k_1 + k_2 + \ldots + k_m, \quad k_1 \geq k_2 \geq \ldots \geq k_m, \quad k_1, k_2, \ldots, k_m \in \mathbb{N} \quad (2.54)$$

Moreover, an irreducible representation corresponding to a given partition of $k$ is constructed within the approach explicitly, as a formal linear combination of permutations. We start from revising the cases of two and three elements on this language. A partition of $k$ defined as $[2.54]$ we denote by $[k_1k_2 \ldots k_m]$.

There two partitions of 2, 2 and $1 + 1$ (we write [2] and [11] for them). They correspond to two irreducible representations $\begin{pmatrix} 2 \end{pmatrix}$ of the symmetric group with two elements, where the elements are distributed over the round brackets as 2, $X_S \equiv X_2 = (xy)$, and as $1 + 1$, $X_A \equiv X_{11} = a(x)(y) + b(y)(x) = axy + byx$ with $a + b = 0$. Similarly, three elements are distributed over the three round brackets in there irreducible representations $\begin{pmatrix} 2 \end{pmatrix}, \begin{pmatrix} 2 \end{pmatrix}, \begin{pmatrix} 2 \end{pmatrix}$ of the symmetric group according to one of three partitions of 3, $X_S \equiv X_3, X_SA \equiv X_{21}, X_A \equiv X_{111}$.

In turn, all the irreducible representations of the symmetric group with four elements the are enumerated by partitions of 4 $\begin{pmatrix} 4 \end{pmatrix}$ $\begin{pmatrix} 4 \end{pmatrix}$. The representation corresponding to a partition $[k_1 \geq k_2 \geq k_3 \geq k_4]$ ($k_1 + k_2 + k_3 + k_4 = 4$) is constructed in the two following steps. First, one writes $xyzt$, putting the parentheses around the first $k_1$, next $k_2$, next $k_3$, and last $k_4$ elements; the parentheses (symmetrization sign) are separated by a group product, e.g.,

$$(xy)(zt) \equiv (xy + yx)(zt + tz) = xyzt + yxzt + xytz + yxtz. \quad (2.55)$$

Then, one lists all permutations of $xyzt$ that remained inequivalent provided the parentheses standing as described. These permutations are assembled then to a linear combination, with the coefficient satisfying a certain system of linear equations. We do not describe here neither a general form of this system, nor rules for constructing it, restricting ourselves by listing it explicitly in the particular cases. Roughly speaking, a linear combination associated with a given partition is to be, first, an eigenspace of the entire symmetric group, second, linearly independent of the linear combinations for the previous partitions (in the inverse lexicographic order). Such a system is in the most cases overdefined, and we also verify, for the cases considered, that each solution of this system gives rise to an irreducible representation of the symmetric group. However, it may be shown, that all the representations corresponding to a given partition are isomorphic $\begin{pmatrix} 4 \end{pmatrix}$. One may equivalently start from putting the square brackets (antisymmetrization sign) in accordance with each partition, and then impose some other (constructed in a similar manner as for expressions with the parentheses) systems of linear equations on the coefficients of the corresponding linear combinations. The representation with parentheses for a certain partition will be then isomorphic to the representation with the square brackets for the dual partition (see fig.??). We use the easiest way of these two in each case, in one case demonstrating their equivalence.

The simplest representations correspond to the partitions $\begin{pmatrix} 4 \end{pmatrix}$ and $\begin{pmatrix} 1111 \end{pmatrix}$; they are already listed fully symmetric and antisymmetric representations correspondingly, $X_4 \equiv X_{SSS}$ and $X_{1111} \equiv X_{AAA}$.

Next comes the partition $\begin{pmatrix} 31 \end{pmatrix}$. The corresponding irreducible representation of the symmetric group with four elements is a straightforward analog of $\begin{pmatrix} 21 \end{pmatrix}$-type representation of the symmetric group with three elements:

$$X_{31} = a(xyzt)t + b(yzt)x + c(ztx)y + d(txy)z, \quad a + b + c + d = 0 \quad (2.56)$$

The general form of the linear combination is dictated by the anzatz for a $\begin{pmatrix} 31 \end{pmatrix}$-type representation, while the constraint on the coefficients arises as a condition of linear independence of $X_{31}$ and $X_4$ (which is obtained for $a = b = c = d$), being formulated in a symmetric (i.e., invariant under entire symmetric group) form. The constructed linear combination is fully symmetric under the permutations of the first three elements, i.e.,

$$\sigma_1 X_{31} = \sigma_2 X_{31} = X_{31}, \quad (2.57)$$
and also satisfies the identity

\[(1 + \sigma_3 + \sigma_2\sigma_3 + \sigma_1\sigma_2\sigma_3)X_{31} = (a + b + c + d)(xyzt) = 0.\] (2.58)

Identity (2.58) is similar to identity (2.24) for 21-type representation of a symmetric group. Each permutation of \(xyzt\) appears in the resulting expression for the four times, picked up from one of the four summands in (2.56) by one of the four group elements entering (2.58), e.g.,

\[a_{xyzt} + d\sigma_3\sigma_1\sigma_2\sigma_3 xyzt + b\sigma_1\sigma_2\sigma_3 \sigma_2\sigma_3 xyzt = (a + b + c + d)xyzt,\] (2.59)

and similarly for other monomials. The constructed representation turns out to be three-dimensional, with a basis may be chosen as \(\{X, \sigma_3X, \sigma_2\sigma_3X\}\). To verify that, it suffices to check that the action of the symmetric group generators on the basis elements expands, in account for (2.57, 2.58), over the same basis; the corresponding expressions are summarized in the table:

|   | \(X\) | \(\sigma_3X\) | \(\sigma_2\sigma_3X\) |
|---|-------|---------------|---------------------|
| \(\sigma_1\) | \(X\) | \(\sigma_3X\) | \(-X - \sigma_3X - \sigma_2\sigma_3X\) |
| \(\sigma_2\) | \(X\) | \(\sigma_2\sigma_3X\) | \(\sigma_3X\) |
| \(\sigma_3\) | \(\sigma_3X\) | \(X\) | \(\sigma_2\sigma_3\sigma_2X = \sigma_2\sigma_3X\) |

(2.60)

Note that any linear combination of type (2.56), there is a three-dimensional space of them, gives rise to a three-dimensional representation of the symmetric group; all these representations are isomorphic, since the action of the group generators on the basis vectors is given in all cases by the same table (2.60). According the above mentioned rule, the representation for the transposed partition, which is for the case is [211], is obtained by substituting all parentheses with the square brackets, and demanding (in a symmetric manner) that all the obtained expression is linearly independent of \(X_{1111}\) (which corresponds to \(a = -b = c = -d = 1\)).

\[X_{211} = a[xyz]t + b[zyt]x + c[xt]y + d[txy]z, \quad a - b + c - d = 0\] (2.61)

Similarly to \(X_{31}\), their are the identities valid for this representation

\[\sigma_1X_{211} = \sigma_2X_{211} = -X_{211},\] (2.62)

\[(1 - \sigma_3 + \sigma_2\sigma_3 - \sigma_1\sigma_2\sigma_3)X_{211} = (a + b + c + d)(xyzt) = 0,\]

the latter one being verified for each permutation, e.g., for \(xyzt\), as

\[(a + b + c + d)xyzt = a_{xyzt} - \sigma_3(-dxzyt) + \sigma_2\sigma_3 czty - \sigma_1\sigma_2\sigma_3(-byzt)\] (2.63)

As a result, a [211]-type representation of the symmetric group is of the same dimension three, as a [31]-type one. Basises in may be chosen similarly in both cases; in case of [211], the group generators act on the basis vectors as:

|   | \(X\) | \(\sigma_3X\) | \(\sigma_2\sigma_3X\) |
|---|-------|---------------|---------------------|
| \(\sigma_1\) | \(-X\) | \(\sigma_3X\) | \(-X + \sigma_3X - \sigma_2\sigma_3X\) |
| \(\sigma_2\) | \(-X\) | \(\sigma_2\sigma_3X\) | \(\sigma_3X\) |
| \(\sigma_3\) | \(\sigma_3X\) | \(X\) | \(\sigma_2\sigma_3\sigma_2X = -\sigma_2\sigma_3X\) |

(2.64)

The above table confirm that the presented \(X_{211}\) indeed gives rise to a three-dimensional representation of the symmetric group. Again, (2.61) sets an entire three-dimensional set of isomorphic [211]-type representations, each one may be alternatively defined via action (2.64) of the group generators on the basic elements.

One more partition remains, namely [22]. The corresponding irreducible representation of the symmetric group has a slightly more involved structure, than the above listed ones. A general linear combination fitting the [22]-type anzatz includes six summands:

\[X_{22} = a_1(xy)(zt) + a_2(zt)(xy) + a_3(xz)(yt) + a_4(yt)(xz) + a_5(yz)(xt) + a_6(xt)(yz)\] (2.65)
The very anzatz implies that

\[ X_{22} = \sigma_1 X_{22} = \sigma_3 X_{22}. \]  

(2.66)

A linear space spanned by all permutations of 4 elements has the dimension 4! = 24. We have already established, that there are two one-dimensional representations ([4]- and [1111]-type ones), and two tree-dimensional spaces of three-dimensional representations ([31]- and [211]-type ones). According the mentioned theorem [4], each vector of the remained subspace belongs to in a [22]-type representation. Hence, the dimension of this subspace, which is 4! − 2 · 1 · 1 − 2 · 3 · 3 = 4, equals to the number of linearly independent [2.65]-type expressions allowed by the corresponding constraints. Actually, the representation in question has the dimension two and the multiplicity two, as we demonstrate in the below.

Four constraints on six coefficients are imposed in accordance with that. These constraints may be presented in one of two equivalent forms:

\[
\begin{align*}
 a_1 + a_2 + a_6 &= 0, \\
 a_1 + a_3 + a_5 &= 0, \\
 a_2 + a_3 + a_4 &= 0, \\
 a_4 + a_5 + a_6 &= 0,
\end{align*}
\]

\(\text{and in the similar ones for other permutations. Hence, a combination of form (2.65) constrained by (2.67) satisfies, in addition to (2.66), the identities}

(2.67)

\[
\begin{align*}
 (1 + \sigma_2 + \sigma_1 \sigma_2) X_{22} &= (a_1 + a_2 + a_6)(xyz)t + (a_1 + a_3 + a_5)(txy)z + (a_2 + a_3 + a_4)(zt)xy + (a_4 + a_5 + a_6)(yzt)x \\
 &= 0, \\
 (1 + \sigma_2 + \sigma_3 \sigma_2) X_{22} &= (a_3 + a_4 + a_5)t(xyz) + (a_2 + a_4 + a_6)(txy) + (a_1 + a_5 + a_6)(zt)x + (a_1 + a_2 + a_3)x(yzt) \\
 &= 0.
\end{align*}
\]

Dimension of the examined representation equals two, what is checked by acting by the group generators on the basic elements, which may be chosen as \{X, \sigma_2X\}:

| \(X\) | \(\sigma_2X\) |
|---|---|
| \(\sigma_1\) | \(-X - \sigma_2X\) |
| \(\sigma_2\) | \(\sigma_2X\) |
| \(\sigma_3\) | \(-X - \sigma_2X\) |

(2.71)

To summarize, we have constructed formal linear combinations of permutations of four elements, corresponding to two one-dimensional representations ([4]- and [1111]-type ones), two three-dimensional
spaces of vectors, each one giving rise to a three-dimensional representation of the symmetric group (a \([31]-\) and \([211]-\)type one, correspondingly), and a two dimensional space of vectors, each one giving rise to a two-dimensional representation of the symmetric group (a \([22]-\)type one). The described spaces do not intersect by construction and form altogether a linear space of the dimension \(2 \cdot 1 \cdot 1 + 2 \cdot 3 \cdot 3 + 1 \cdot 2 \cdot 2 = 24 = 4!\). Hence, any linear combination of the \(4!\) permutations of four elements is expanded over vectors from the spaces of the listed irreducible representations.

**Step 3.** We have constructed five distinct irreducible representations of the symmetric group with four elements, one-dimensional ones \((2.52)\) and \((2.53)\), two-dimensional one \((2.65)\), and three-dimensional ones \((2.56)\) and \((2.61)\). It is known that there are no other, inequivalent ones \([4]\). **Step 2.3** then tells one to take five triples matrices of the corresponding sizes. After the general scheme of constructing the irreducible representation was discussed, it is natural to label these matrices with the partitions. There are \(1 \times 1\) matrices \(B_4^i\) and \(B_{1111}^i\), three-dimensional ones \(B_{31}^i\) and \(B_{211}^i\), and two-dimensional ones \(B_{22}^i\), with \(i = 1, 2, 3\). We omit the superscripts unless the size of matrices is essential.

**Step 4.** The next step is to impose on the matrices the group relations, which for the case take the form

\[
\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \quad \sigma_3 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_2, \quad \sigma_1 \sigma_3 = \sigma_3 \sigma_1. \tag{2.72}
\]

**Step 5.** One must then solve the constraints w.r.t. to the matrix elements, expressing them all via a necessary number of free parameters. Similarly to the three-strand case, one’s of the matrices eigenvalues may be taken as such parameters. We approach to the problem at several steps.

**Step 5-a: coinciding of eigenvalues from Yang-Baxter equation.** One conclusion may be done for an arbitrary braid. Namely, the matrices \(B_{i,Y}^Y\), a partition \(Y\) given, a position \(i\) in a braid section vary, have the same eigenvalues. This follows from the Yang-Baxter equation, which may be brought to the form:

\[
(B_i - \lambda)B_{i+1} = B_{i+1}(B_i - \lambda), \tag{2.73}
\]

Taking the determinants from both sides, one obtains that the characteristic polynomials of the matrices coincide,

\[
det(B_i - \lambda) = det(B_{i+1} - \lambda), \tag{2.74}
\]

what is equivalent to the coincidence of all the eigenvalues up to a permutation; it is natural then to enumerate the basis vectors \(e^{(k)}\) so that \(\lambda_i^{(k)} = \lambda_i^{(k)}\).

**Step 5-b: commuting of non-adjacent operators.** Other steps in solving \((2.72)\) are not so straightforward. The next in simplicity one concerns the third the constraints, which reflects the commutation of the non-adjacent crossings. For one-dimensional matrices the property is held trivially. The corresponding two-dimensional matrices must be both diagonal, their eigenvalues coincide, as already established,

\[
B_{122}^{22} = \begin{pmatrix} \lambda_{22,1} \\ \lambda_{22,2} \end{pmatrix} \Rightarrow B_{322}^{22} = \begin{pmatrix} \lambda_{22,1} \\ \lambda_{22,2} \end{pmatrix}, \text{ or } B_{322}^{22} = \begin{pmatrix} \lambda_{22,1} \\ \lambda_{22,2} \end{pmatrix}. \tag{2.75}
\]

The same might be true for the three-dimensional matrices. A case is more involved if two of three eigenvalues coincide; the corresponding matrices then commute provided that they have a block structure:

\[
B_{131}^{31} = \begin{pmatrix} \lambda_{31,1} \\ \lambda_{31,1} \\ \lambda_{31,2} \end{pmatrix} \Rightarrow B_{331}^{31} = \begin{pmatrix} a & b \\ b & c \\ \lambda_{31,2} \end{pmatrix}, \tag{2.76}
\]
and similarly for the other three-dimensional representations
\[
B_1^{211} = \begin{pmatrix}
\lambda_{211,1} & \lambda_{211,1} \\
\lambda_{211,1} & \lambda_{211,2}
\end{pmatrix} \Rightarrow B_3^{211} = \begin{pmatrix}
\tilde{a} & \tilde{b} \\
\tilde{b} & \tilde{c}
\end{pmatrix}
\begin{pmatrix}
\lambda_{211,1} & \lambda_{211,2}
\end{pmatrix}.
\]

(2.77)

We examine this possibility in the next paragraph, postponing the question of whether this is the case until Step 2.7.

**Step 5-b: form of blocks from the Yang-Baxter equation.** We return now to the first two of constraints (2.72), this time using them to express the non-diagonal matrix elements via the eigenvalues; for three-dimensional matrices we take ansätze (2.76, 2.77).

The first of equations (2.72) reduces to (2.35), both for the two-dimensional matrices, and for the two-by blocks in the three-dimensional matrices. The only difference is that \(\beta\) and \(\gamma\) are substituted by the corresponding eigenvalues. Since (2.35) has the only (modulo the above mentioned subtleties) solution, one just writes
\[
B_2^{22} = B_2^{SA} (\beta = \lambda_{22,1}, \gamma = \lambda_{22,2}),
\]
\[
B_2^{31} = \begin{pmatrix}
\lambda_{31,1} \\
\lambda_{31,2}
\end{pmatrix},
\]
\[
B_2^{31} = \begin{pmatrix}
\lambda_{31,1} \\
\lambda_{31,2}
\end{pmatrix},
\]
\[
B_2^{22} = \begin{pmatrix}
\lambda_{211,1} \\
\lambda_{211,2}
\end{pmatrix},
\]
\[
B_2^{31} = \begin{pmatrix}
\lambda_{211,1} \\
\lambda_{211,2}
\end{pmatrix},
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B_2^{31} = \begin{pmatrix}
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B_2^{31} = \begin{pmatrix}
\lambda_{211,1} \\
\lambda_{211,2}
\end{pmatrix},
\]
\[
B_2^{31} = \begin{pmatrix}
\lambda_{211,1} \\
\lambda_{211,2
}
with the corresponding constraints on knot invariants being

\[
\begin{align*}
\lambda_4^n \chi_4 + (\lambda_{31,1}^n + \lambda_{31,2}^n + \lambda_{31,3}^n) \chi_{31} + (\lambda_{22,1}^n + \lambda_{22,2}^n) \chi_{22} + \\
+ (\lambda_{211,1}^n + \lambda_{211,2}^n + \lambda_{211,3}^n) \chi_{211} + \lambda_{1111}^n \chi_{1111} = \lambda^n \chi_2 \chi_3^2 + \mu^n \chi_{11} \chi_1^2.
\end{align*}
\]

Similarly to that in the three-strand case, a homogenous linear system of constraints on \(\chi_2 \chi_1^2, \chi_{11} \chi_1^2, \) and \(\chi_{Y,i} \) (with \(Y\) running over the partitions of 4, and \(i\) running from 1 to the number of corresponding eigenvalues) was obtained. It is straightforward to verify that a principle minor of this system is proportional to the Wandermond determinant composed of the eigenvalues. Hence, non-trivial solution for characters are their for some of the eigenvalues coincide. In particular, the system is satisfied if

\[
\begin{align*}
\chi_2 \chi_1^2 &= (\chi_3 + \chi_{21}) \chi_1 = \chi_4 + 2 \chi_{31} + \chi_{22} + \chi_{211}, \\
\chi_{11} \chi_1^2 &= (\chi_{21} + \chi_{111}) \chi_1 = \chi_{31} + \chi_{22} + 2 \chi_{211} + \chi_{1111},
\end{align*}
\]

and

\[
\begin{align*}
\lambda_4 &= \lambda, \quad \lambda_{31,1} = \lambda_{31,2} = \lambda, \quad \lambda_{31,3} = \mu, \quad \lambda_{22,1} = \lambda, \quad \lambda_{22,2} = \mu, \\
\lambda_{211,1} &= \lambda, \quad \lambda_{211,2} = \lambda_{211,3} = \mu, \quad \lambda_{1111} = \mu.
\end{align*}
\]

This is the solution, which enters in the definition of the knot invariant of the interest. The case is that rules (2.84) and (2.85) reflect a co-algebra structure on the braid group, similarly to (2.45) in case of three-strand braids. Although any other solution might give rise to a knot invariant, only the named one is systematically studied. The reason is, probably, in that the additional structure enables to re-define the knot invariant in an explicit and concise manner [63, 64, 72], not just as a solution named one is systematically studied. The reason is, probably, in that the additional structure enables to re-define the knot invariant in an explicit and concise manner [63, 64, 72], not just as a solution

Step 7-b: weight coefficients via eigenvalues. In remains to express the weight coefficients via eigenvalues. Just as in the previous cases, some of the necessary relations follow from the second of the equivalence transformations, specific for braid closures, (???-I,II). To determine the five characters, one needs overall five independent equations, e.g.,

\[
I^{(4)} \sim I^{(3)} \otimes I \Rightarrow \chi_4 + \chi_{31} + \chi_{22} + \chi_{211} + \chi_{1111} = \chi_{11} \chi_3 + \chi_1 \chi_{21} + \chi_1 \chi_{111},
\]

\[
B_3^{(4)} \sim I^{(3)} \Rightarrow \lambda \chi_4 + (2 \lambda + \mu) \chi_{31} + (\lambda + \mu) \chi_{22} + (\lambda + 2 \mu) \chi_{211} + \mu \chi_{1111} = \lambda \chi_3 + (\lambda + \mu) \chi_{21} + \mu \chi_{111},
\]

\[
B_2^{(4)} B_3^{(4)} \sim B_2^{(3)} \Rightarrow \lambda^2 \chi_4 + (\lambda^2 + \lambda \mu) \chi_{31} + \lambda \mu \chi_{22} + (\lambda \mu + \mu^2) \chi_{211} + \mu^2 \chi_{1111} = \lambda^2 \chi_3 + \lambda \mu \chi_{21} + \mu^2 \chi_{111},
\]

\[
B_1^{(4)} B_2^{(4)} B_3^{(4)} \sim B_1^{(3)} B_2^{(3)} \Rightarrow \lambda^3 \chi_4 + \lambda \mu \lambda^2 \chi_{31} + \mu^2 \lambda \chi_{211} + \mu^3 \chi_{1111} = \lambda^2 \chi_3 + \lambda \mu \chi_{21} + \mu^2 \chi_{111},
\]

\[
B_1^{(4)} B_2^{(4)} B_3^{(4)} B_2^{(3)} \sim B_1^{(3)} B_2^{(3)} B_2^{(3)} B_3^{(3)} \Rightarrow \lambda^7 \chi_4 - \lambda^4 \mu^3 \chi_{31} - \lambda^3 \mu^3 (\lambda + \mu) \chi_{22} - \lambda^2 \mu^4 \chi_{211} + \mu^7 \chi_{1111} = \lambda^6 \chi_3 - 2 \lambda^5 \mu^3 \chi_{21} + \mu^6 \chi_{111}.
\]
Note, that to produce five linearly independent equations, one has to take at least one braid with a strand entering no crossings, and at least one three-stand braid not reducible to a two-stand one. The solution reads,

\[ \chi_4 = \frac{\chi_1(\mu \chi_1 - 1)(\mu^2 \chi_1 + \lambda - \mu)(\mu^3 \chi_1 + \lambda^2 - \lambda \mu + \mu^2)}{(\lambda - \mu)^2(\lambda^2 + \mu^2)(\lambda^2 + \mu^2 - \lambda \mu)}, \quad (2.91) \]

\[ \chi_{31} = -\frac{\chi_1(\mu \chi_1 - 1)(\mu^2 \chi_1 + \lambda - \mu)}{(\lambda - \mu)^2(\lambda^2 + \mu^2)}, \]

\[ \chi_{22} = \frac{\lambda \mu \chi_1^2((\mu \chi_1 - 1)^2(\lambda \chi_1 - 1)}{(\lambda - \mu)^2(\lambda^2 + \mu^2)} \]

\[ \chi_{211} = -\frac{\chi_1(\mu \chi_1 - 1)(\mu^2 \chi_1 + \lambda - \mu)}{(\lambda - \mu)^2(\lambda^2 + \mu^2)}, \]

\[ \chi_{1111} = \frac{\chi_1(\mu \chi_1 - 1)(\lambda^2 \chi_1 + \mu - \lambda)(\lambda^3 \chi_1 + \lambda^2 - \lambda \mu + \mu^2)}{(\lambda - \mu)^2(\lambda^2 + \mu^2)(\lambda^2 + \mu^2 - \lambda \mu)}. \quad (2.92) \]

**The result.** Putting everything together, we obtain that the invariant of the knot presented as the closure of a four-strand braid is computed as

\[ H \left( \sigma_1^{a_1} \sigma_2^{b_1} \sigma_3^{c_1} \cdots \sigma_1^{a_k} \sigma_2^{b_2} \sigma_3^{c_k} \right) = \sum_{Y \vdash 4} \chi_Y \sum_{i=1}^k \left( B_1^i \right)^{a_i} \left( B_2^i \right)^{b_i} \left( B_3^i \right)^{c_i}, \quad (2.93) \]

where \( Y \vdash 4 \) means that \( Y \) runs over partitions of 4 (\( Y = [4], [31], [22], [211], [1111] \)), the matrices \( B^Y \) are

\[ B_1^i = B_2^i = \lambda, \quad B_1^{1111} = B_2^{1111} = \mu, \]

\[ B_2^{22} = \lambda \mu \sigma_2^A(\lambda, \mu) = \begin{pmatrix} \mu^2 & \sqrt[2]{\lambda \mu(\lambda^2 - \mu^2) \\ \mu - \lambda & \lambda - \mu \end{pmatrix}, \]

\[ B_2^{31} = \lambda \mu \sigma_1^A(\lambda, \mu), \quad B_2^{211} = \begin{pmatrix} \mu & \lambda \mu \sigma_2^A(\lambda, \mu) \end{pmatrix}, \quad (2.94) \]

and the weight coefficients \( \chi_Y \) are given by (2.92).

**2.2.7 Weight coefficients as SU(\( N \)) characters**

The obtained expressions for the weight coefficients have a remarkable property, which reveals a deeper underlying structure. Namely, the crossing matrices arise as a deformation of the symmetric group generators. If one substitutes now the two-strand eigenvalues in (2.15, 2.49, 2.92) by the eigenvalues of the rank two symmetric group generator on the symmetric and antisymmetric representations \( \sigma_1^A, [21] \), respectively, i.e., \( \lambda = 1 \) and \( \mu = -1 \), the weight coefficients turn into the quotients of the gamma function resembling expressions\(^3\)

\[ \chi_2 = \frac{\chi(\chi + 1)}{2}, \quad \chi_{11} = \frac{\chi(\chi - 1)}{2}, \quad (2.95) \]

\[ \chi_3 = \frac{\chi(\chi + 1)(\chi + 2)}{6}, \quad \chi_{21} = \frac{\chi(\chi + 1)(\chi - 1)}{3}, \quad \chi_{111} = \frac{\chi(\chi - 1)(\chi - 2)}{6} \quad (2.96) \]

\(^3\)We label all the coefficients by partitions this time; in particular, \( \chi_2 \equiv \chi_S, \chi_{11} \equiv \chi_A, \chi_{SS} \equiv \chi_3, \chi_{SA} \equiv \chi_21, \text{ and } \chi_{AA} \equiv \chi_{111}. \)
3.2 How to construct a knot invariant

3.1 Knot polynomial as a contraction of vertex operators

Knot polynomial as an observable in an exactly solvable planar model

In particular, for $\chi$ being equal to any integer $N$, a coefficient $\chi_Y$ for a Young diagram $Y$ equals to the dimension of $su(N)$ representation corresponding to the Young diagram $Y$; as well as symmetric group irreducible representations, $su(N)$ irreducible representations are in one-to-one correspondence with the Young diagrams [4, 9].

The observed property admits a generalization. As already mentioned in sec.2.2.2 the symmetric group admires a deformation called Hecke algebra, or $q$-symmetric group. As we will see in sec.3.4 the generator of the $q$-symmetric group acts on it’s two irreducible representation with the eigenvalues $\lambda = q$ and $\mu = -q^{-1}$, with $q$ begin a formal parameter entering group relations. Substituting the so deformed eigenvalues in (2.95-2.100), one obtains that (2.95-2.97) is substituted by

$$\chi_2 = \frac{[N][N+1]}{2}, \quad \chi_{11} = \frac{[N][N-1]}{2}, \quad (2.98)$$

$$\chi_3 = \frac{[N][N+1][N+2]}{[2][3]}, \quad \chi_{21} = \frac{[N][N+1][N-1]}{[3]}, \quad \chi_{111} = \frac{[N][N-1][N-2]}{[2][3]}, \quad (2.99)$$

$$\chi_4 = \frac{[N][N+1][N+2][N+3]}{[2][3][4]}, \quad \chi_{31} = \frac{[N][N+1][N+2][N-1]}{[2][4]}, \quad \chi_{22} = \frac{[N][N+1][N-1]}{[3]}, \quad (2.98)$$

$$\chi_{211} = \frac{[N][N+1][N-1][N-2]}{[2][4]}, \quad \chi_{1111} = \frac{[N][N+1][N+2][N+3]}{[2][3][4]}.$$  

with the standard notation $[N] = \frac{q^N - q^{-N}}{q - q^{-1}} = q^{N-1} + q^{N-3} + \ldots + q^{-N-1}$ used. For $q = 1$, $[N] = N$, and (2.98-2.100) reduce to (2.95-2.97) with $\chi_1 = N$. Quantities (2.98-2.100) are referred to as quantum dimensions of quantum group $U_q(SU(N))$ irreducible representations [63]. They may be seen as traces of the exponentiated quadratic Casimir operator over the corresponding $SU(N)$ representations [73].

3 Knot polynomial as an observable in an exactly solvable planar model

3.1 Knot polynomial as a contraction of vertex operators

3.2 How to construct a knot invariant

In the previous section, we have discussed in how knot invariants may be evaluated with help of the symmetric group representations. This section is devoted to another approach to knot invariants, which also arises in a number of physical problems. Namely, a knot invariant may be seen as a certain contraction of vertex operators in a planar model, provided that a model has some special properties. Let us start with a brief sketch of the construction, to proceed then with explicit presentation of details.

First,

**Step 3.1 Cut a curve that is a planar projection of a knot in such a way that non of the pieces has self-crossings any longer.**

Such a cutting may be carried out as follows. The first cut is made elsewhere on the curve but the self-intersection points. Then, one proceeds in the direction selected on the curve from the cut to the first point, where the curve intersects the already passed piece for the first time. The
next cut is done on the curve just before this point. After that the same procedure is repeated, this time one starting from the second cut. In case of a knot, one terminates at the first cut by the end. In case of a link, the procedure is carried out for each connection component separately. An example of the resulting cut diagram is presented in fig. 5. After the procedure is completed,

**Step 3.2** *Each of the obtained self-intersection-free pieces acquires its own number.*

As a result, one obtains a collection of enumerated directed curves. Different curves intersect now in the self-intersection points of the original curve. The obtained figure may be also seen as a four-valent planar graph, with each vertex labeled by a pair of numbers, which were associated on **Step 3.2** with the pieces of the curve entering the vertex. In addition, the graph has \( n \) incoming and \( n \) outgoing vertices, each one also labeled by a number (see fig. 5). A knot is encoded by the obtained cut diagram together with the closing rule, which matches the incoming vertices with the outgoing ones. In some cases, one a braid representation of the knot, but not in general. On the next step,

**Step 3.3** *A self-intersection-free curve number \( p \) corresponds to a vector space \( L_p \).*

Then,

**Step 3.4** *A vertex where curve number \( p \) intersects the curve number \( q \) corresponds to a linear operator \( V^{(p,q)} \), which maps the space \( L_p \otimes L_q \) into istef.*

This operator has, apart from \( p \) and \( q \) indicating the crossing, four tensor indices, and is labeled as \( V_{ijkl}^{(p,q)} \), where the superscripts run in the space of origin, \( i \in V_p, j \in V_q \) and are related to the incoming strands, while and the subscripts run in the space of image, \( k \in V_p, l \in V_q \) and are related to the outgoing strands, the first index in each pair being related to the upper (in the space) strand, fig. 3.1. To reflect this fact, we label each edge by an index running in the corresponding \( L \). Now,

**Step 3.5** *A would be topological invariant is a product of the vertex operators over all vertices, contracted along all intrinsic edges.*

If a vertex operator \( V^{(p,q)} \) depends not on the position of the vertex in a two-dimensional plane but only on the numbers \( p \) and \( q \) of the crossing line, then the obtained contraction is automatically invariant under smooth deformations of the planar graph. Apart from them, a graph admits certain non-smooth deformations, corresponding to smooths moves of the projected curves in the three-dimensional space; they are analogues of the equivalence transformations of braids. In analogy continued, a theorem \([?]\) claims that any of such transformation is a combination of two elementary ones, deletion of the inverse crossings, (3.10), and pulling of a line through the crossing of another one, (3.1). The first one is an analog of the rule “inverse operators for the inverse crossings” for constructing the braid invariants, while the second one is the Yang-Baxter equation, presented as constraint (2.2) on braid group generators earlier, taking form of an equation on the vertex operators now. The next step is hence to

**Step 3.6** *Impose constraints (3.10) and (3.1) on the obtained contraction.*

The vertex operators hence satisfy two out of there constrains on the permutation group generators and may be seen as operators of a representation of a symmetric group extension. In particular, one may substitute the vertex operators by some monodromy operators, with the operator \( V_{ijkl}^{(p,q)} \) being responsible for the permutation of special points number \( p \) and number \( q \). Hence, the observation, already made for braids, extends to arbitrary planar diagrams, just a number of strand is substituted by a number of non-self-intersecting piece. Again, a knot invariant may be constructed from monodromy matrices, this time with help of an arbitrary knot diagram. However, we deviate from this path here, proceeding with introducing the construction of our main interest.

The next step is in solving the obtained constraints. The most simple one is (3.10). If a vertex operator \( V^{(p,q)} \) is seen as an automorphism of the linear space \( V_p \otimes V_q \), the inverse operator \( V_{ij}^{(p,q)} \) may be introduced; then,
Step 3.7 The mutually inverse crossings of the same lines correspond to mutually inverse operators that by definition satisfy (3.10).

Next, there comes a main deviation from the above presented braid group method. It is Yang-Baxter equation (3.1), which is taken as a starting point now. As we already mentioned in the introduction, solutions of the equation are explicitly known from theory of integrable models. As a next step, we are going to

Step 3.8 Substitute one of the explicitly known solutions of the Yang-Baxter equations for vertex operators, by definition satisfying (3.1).

As an output of Steps 3.1–3.8 one obtains an operator corresponding to the entire diagram with one superscript for one incoming edge and one subscript for one outgoing edge. This operator enjoys (3.10) and (3.1), thus being an invariant of a planar diagram in the above specified sense. One might expect then the this operator’s full contraction dictated by the closing rule to be a knot invariant. In particular, this contraction must be one and the same for all cuttings of the same knot projection. The real situation turns out to be more involved. Apart from transformations (3.10) and (3.1), a planar diagram admits a loop-contracting transformation (3.14), which, unlike (3.10) and (3.1), changes an algebraic number of crossings, and, hence, not be formulated as a transformation of an unclosed diagram. Transformation (3.14) is an analog of the strand-changing transformation of a braid (2.6–II). Similarly that case, one should now

Step 3.9 Search for a linear combination of several planar-diagram invariants that would be invariant under (3.14) as well.

This step completes the construction of interest; according to the theorem explicitly formulated in the next section, the obtained combination, if any, is indeed a knot invariant.

3.3 Explicit form of vertex operators from a topological invariance

In the present section, we complete presentation of the vertex operator approach to the knot invariants by writing out explicitly the simplest form of these operators and by verifying for them the constraints providing a topological invariance of the contraction. According to a known theorem, three constraints are enough to impose; they are illustrated in (3.10), (3.1), and (3.14) and referred to as Reidemeister moves. Each of these constraints is associated with a singular transformation of a planar diagram corresponding to non-singular transformation of the knot. The theorem states that any two diagrams of the (topologically) same knot may be transformed into each other by a combination of three Reidemeister moves. Hence, any quantity that satisfy the three corresponding constraints would be a knot invariant.

3.3.1 Yang-Baxter equation

The next constraint corresponds to the third Reidemeister move (in the next coming evaluation, these are delta-symbols, not the $R$-matrices, stand in all crossings; the crossings where $R$-matrices still stand

4In this case the obtained knot invariant is an (uncolored) HOMFLY polynomial.
This constraint is called a Yang-Baxter equation, and it's solutions are studied in the integrable models theory under the name of $R$-matrices [70]. In particular, a one-parametric family of solutions is constructed explicitly for a finite-dimensional representation of a regular Lie group [65]. Here we restrict ourselves by verifying the Yang-Baxter equation for the simplest of the solution, which corresponds to the fundamental representation of the group $su(N)$. Non-zero elements of the examined solution are:

$$R^{ii}_{ii} = q; \quad R^{ij}_{ij} = 1, \quad i \neq j; \quad R^{ji}_{ji} = q - q^{-1}, \quad 1 \leq i < j \leq N,$$

(3.2)

where $N$ is an integer number, and $q$ is a formal parameter of the family. This solution gives rise to (uncolored) HOMFLY polynomials (see examples below).

Let us verify that (3.2) satisfies the Yang-Baxter equation explicitly. Since we write the solution in a certain basis, the notations are not covariant any longer; in particular, no sum after repeated indices is assumed default. Solution (3.2) may be presented as

$$R^{ij}_{kl} = a_{ij}\delta^i_k \delta^j_l + b_{ij}\delta^i_l \delta^j_k, \quad a_{ij} = 1 + (q - 1)\delta_{ij}, \quad b_{ij} = (q - q^{-1})\theta_{ij}$$

(3.3)

with $\theta_{ij} \equiv 1$ for $i < j$, and $\theta_{ij} \equiv 0$ otherwise. Verifying of the YB equation may be then carried out pictorially. Four out of eight pairs of diagrams merely coincide:

$$R^{ii}_{ii} R^{bb}_{cm} R^{ac}_{nl} = R^{ik}_{ac} R^{jc}_{ba} R^{la}_{ml}$$

(3.1)

One more pair coincides unless $i = j < k$ or $j < k = i$:
\( a_{ij}a_{ji}b_{jk} = a_{ik}a_{ki}b_{jk} \)

and the three remaining ones groups to coincide

\[
\begin{align*}
\frac{j}{i} & \frac{k}{m} \quad \text{if not } i = j < k \\
\frac{n}{l} & \quad = \quad \frac{i}{k} & \quad \text{if not } i = j < k
\end{align*}
\]

\[
(3.6)
\]

\[
\begin{align*}
a_{ij}b_{ji}b_{ik} + a_{ji}b_{ji}b_{jk} = a_{ji}b_{ik}b_{jk}
\end{align*}
\]

and

\[
\begin{align*}
a_{ik}b_{ji}b_{jk} & = a_{ik}b_{ji}b_{ik} + a_{jk}b_{ji}b_{ki}
\end{align*}
\]

due to the identities \( \theta_{ji}\theta_{ik} + \theta_{ij}\theta_{jk} = \theta_{ik}\theta_{jk} \) and \( \theta_{ji}\theta_{ik} + \theta_{jk}\theta_{ki} = \theta_{ji}\theta_{jk} \), which hold unless \( i = j < k \), or \( j < i = k \), correspondingly. The cases \( i = j < k \) and \( j < k = i \) need for a separate treatment; one has to take into account that not all diagrams (3.6-3.8) are independent for some of \( i, j, k \) coinciding. For \( i = j < k \), the diagrams (3.6) and (3.7) contribute to the same component \( (iik)_{lnm} \); adding the equalities termwise, one gets correct in the case in case identity,

\[
\begin{align*}
a_{ii}b_{ik} & = a_{ik}a_{ki}b_{ik} + a_{ij}b_{ik}b_{ki}
\end{align*}
\]

due to the identity \( q^2 = 1 + q(q - q^{-1}) \). Similarly, for \( i = j < k \), equality takes place for the sums of
diagrams (3.7) and (3.8) contributing to the component \((i j i)_{lmn}\),

\[
\begin{align*}
    &a_{ij}a_{ij}b_{ji} + a_{ii}b_{ji}^2 = b_{ji}a_{ii}, \\
    &\quad j < i = k
\end{align*}
\]  

(3.9)

One may verify that solution (3.3), together with the corresponding solution for the inverse crossing operators (see sec.3.3.2) enjoys analogues of (3.1) with the various directions of arrows and mutual placements of lines in the space (which one is the “upper” one in each crossing on the projection) for the corresponding allowed moves [70].

### 3.3.2 Inverse crossings

The next (in fact, the most simple out of the three ones) constraint reflects the invariance under the second Reidemeister move, (3.10-I).

\[
\sum_{a,b} R^{ij}_{ab} \tilde{R}^{ab}_{kl} = \delta^i_k \delta^j_l
\]

\[
\sum_{a,b} \tilde{R}^{ab}_{ij} \tilde{R}^{al}_{kl} = \delta^i_k \delta^j_l
\]

The condition gives rise to the constraints, which relate the operators corresponding to the crossings of the mutually inverse orientations. Because (3.10-I) is a system of \(N^2\) linear equations on \(N^2\) variables (which may be verified to non-degenerate), matrix elements of the inverse crossing operators are determined therefrom explicitly and unambiguously. The most simple is to write for them the anzatz, similar to expression (3.3),

\[
\tilde{R}^{ij}_{kl} = \tilde{a}_{ij} \delta^i_k \delta^j_l + \tilde{b}_{ij} \delta^i_j \delta^j_k.
\]  

(3.10)

Equations (3.10-I) then take the form

\[
\sum_{p,q} R^{ij}_{pq} \tilde{R}^{pq}_{kl} = (\tilde{a}_{ij}a_{ij} + \tilde{b}_{ij}b_{ji}) \delta^i_k \delta^j_l + (\tilde{a}_{ij}b_{ij} + \tilde{b}_{ij}a_{ji}) \delta^i_l \delta^j_k = \delta^i_k \delta^j_l.
\]  

(3.11)

With \(a_{ij}\) and \(b_{ij}\) from (3.3), there are three distinct non-trivial cases, which give (note that \(\tilde{b}_{ii} = 0\) by definition)

\[
\begin{align*}
    &i = k \leq j = l \quad \Rightarrow \quad \tilde{a}_{ij}a_{ij} = 1 \quad \Rightarrow \quad \tilde{a}_{ii} = q^{-1}, \quad \tilde{a}_{ij} = 1, \quad i > j \\\n    &i = k > j = l \quad \Rightarrow \quad \tilde{a}_{ij}a_{ij} + b_{ij}b_{ji} = 1 \quad \Rightarrow \quad \tilde{b}_{ij} = 0, \quad i > j \\\n    &i = l < j = k \quad \Rightarrow \quad \tilde{a}_{ij}b_{ij} + \tilde{b}_{ij}a_{ji} = 0 \quad \Rightarrow \quad \tilde{b}_{ij} = q^{-1} - q.
\end{align*}
\]  

(3.12)

Yang-Baxter equation (3.1) holds for the obtained solution automatically, since the matrix elements for the inverse crossing operators appeared to be related with that for the direct ones by a plain change
of parameter, \(\tilde{a}_{ij}(q) = a_{ij}(q^{-1})\) and \(\tilde{b}_{ij}(q) = b_{ij}(q^{-1})\), while (3.3) satisfies (3.1) for an arbitrary value of \(q\).

One may consider a similar transform with one of the stands’ orientation reversed, (3.1-II), and verify that the corresponding system of equation has the same solution (3.12).

Thereby, if one puts operators (3.3) in the direct crossings, one should must put the operators (recall that \(\theta_{ij} = 1\) for \(i < j\), and \(\theta_{ij} = 0\) for \(i \geq j\))

\[
\tilde{R}^{ij}_{kl} = \tilde{a}_{ij}\delta^i_k \delta^j_l + \tilde{b}_{ij}\delta^i_k \delta^j_l, \quad \tilde{a}_{ij} = 1 + (q^{-1} - 1)\delta_{ij}, \quad \tilde{b}_{ij} = (q^{-1} - q)\theta_{ij}
\]

(3.13)
in the inverse crossings.

### 3.3.3 Contractible loops and turn-over operators

One more transformation of the knot, which is smooth though looking singular at a planar projection, is contracting of a loop:

\[
\begin{array}{c}
\overset{i}{\downarrow}a \\
\bigcirc
\end{array}
\quad =
\begin{array}{c}
\overset{i}{\downarrow} \\
k
\end{array}
\]

\[
\sum_{a,b} M^b_a R^{ia}_{bk} = \delta^i_k
\]

(3.14)

This operation is called the first Reidemeister move. Unlike two other moves, it can not be formulated as a constrain on vertex operators, since it changes the algebraic number of crossings at the planar diagram, similarly to the strand number changing transformation of a braid (in fact, these two transformations are equivalent [3]). If one constructs a knot invariant purely in terms of braids, this move invariance is achieved by summing of the braid invariants corresponding to the various symmetric group irreducible representations with the proper weights (see sec.2.2). Searching for a construction relevant to an arbitrary planar diagram, one may introduce, apart from four-script crossing operators, a two-script turn-over operator. One turn-over operator must be inserted for one turn-over of the tangent vector to the knot planar projection.

#### Explicit expression for the turn-over operators

To determine the exact form of the turn-over operators, one should first impose, and then solve the corresponding constrains, which arise from topological invariance of the entire construction. The simplest relation involving a turn-over operator arises from the first Reidemeister move, (3.14). We are going to write down and to solve the corresponding equations for the turn-over operators \(\mathcal{M}\) supposing that the crossing operators \(R\) are given by (3.3). Let us start from the particular values of \(N\). For \(N = 2\), constraints (3.14) take the explicit form

\[
\begin{cases}
 m_1 R_{11}^{12} + m_2 R_{21}^{12} = 1, \\
m_2 R_{22}^{22} = 1
\end{cases}
\Rightarrow
\begin{cases}
 qm_1 + (q - q^{-1})m_2 = 1, \\
qm_2 = 1
\end{cases}
\]

(3.15)

and have the solution

\[
m_1 = q^{-3}, \quad m_2 = q^{-1}
\]

(3.16)

For \(N = 3\), one has the system

\[
\begin{cases}
 m_1 R_{11}^{11} + m_2 R_{22}^{12} + m_3 R_{31}^{13} = 1, \\
m_2 R_{22}^{22} + m_3 R_{32}^{23} = 1, \\
m_3 R_{33}^{33} = 1
\end{cases}
\Rightarrow
\begin{cases}
 qm_1 + (q - q^{-1}) (m_2 + m_3) = 1, \\
qm_2 + (q - q^{-1})m_3 = 1, \\
qm_3 = 1
\end{cases}
\]

(3.17)
whose solution is

\[ m_1 = q^{-5}, \ m_2 = q^{-3}, \ m_3 = q^{-1}. \]  

(3.18)

Now it is easy to write down both the constraints and their monomial solutions for generic \( N \). Equations (3.14) may be rewritten as

\[ m_i R_{ii}^{ij} + \sum_{j=i+1}^{N} m_j R_{ji}^{ij} = 1 \ \Rightarrow \ \ qm_i + (q - q^{-1}) \sum_{j=i+1}^{N} m_j = 1, \]  

(3.19)

wherefrom one expresses explicitly the elements of the turn-over operator,

\[ m_i = q^{2i-2N-1}, \quad i = 1, \ldots, N. \]  

(3.20)

A contraction of vertex operator in the other two indices, in turn, is carried out with help of the conjugate operator, \( \mathcal{M}' \).

\[
\begin{array}{cccc}
\mathcal{M}R & \mathcal{M}'R & \tilde{\mathcal{M}}\tilde{R} & \tilde{\mathcal{M}}'\tilde{R} \\
I & II & III & IV
\end{array}
\]

(3.21)

Contracting loop (3.21-II) gives rise to a system of constraints on matrix elements of the operator, a one similar to (3.19). E.g., for \( N = 2 \) the system is

\[
\left\{ \begin{array}{c}
m_1' R_{11}^{11} = 1, \\
m_1' R_{21}^{12} + m_2' R_{22}^{22} = 1,
\end{array} \right. \ \Rightarrow \ \left\{ \begin{array}{c}
qm_1' = 1, \\
(q-q^{-1})m_1' + qm_2' = 1,
\end{array} \right.
\]

(3.22)

and it has the solution

\[ m_1' = q^{-1}, \ m_2' = q^{-3}. \]  

(3.23)

For \( N = 3 \),

\[
\left\{ \begin{array}{c}
m_1' R_{11}^{11} = 1, \\
m_1' R_{21}^{12} + m_2' R_{22}^{22} = 1, \\
m_1' R_{31}^{13} + m_2' R_{32}^{23} + m_3' R_{33}^{33} = 1,
\end{array} \right. \ \Rightarrow \ \left\{ \begin{array}{c}
qm_1' = 1, \\
(q-q^{-1})m_1' + qm_2' = 1, \\
(q-q^{-1})(m_1' + m_2') + qm_3' = 1,
\end{array} \right.
\]

(3.24)

what gives

\[ m_1' = q^{-1}, \ m_2' = q^{-3}, \ m_3' = q^{-5}. \]  

(3.25)

Finally, the constraints and their solutions for generic \( N \) are, respectively,

\[
\sum_{j=1}^{i-1} m_j' R_{ij}^{ij} + m_i' R_{ii}^{ii} = 1 \ \Rightarrow \ \ (q-q^{-1}) \sum_{j=1}^{i-1} m_j' + qm_i' = 1,
\]

(3.26)

and

\[ m_j' = q^{-2i+1}, \quad i = 1, \ldots, N, \]  

(3.27)
Loops attached to the inverse vertices. One also should introduce the operators \( \tilde{\mathcal{M}} \) and \( \tilde{\mathcal{M}}' \) contracting the inverse \( R \)-matrix in one and the other pairs of indices. The matrix elements of the operators must satisfy the system of constraints, which, according to (3.13), are obtained from (3.19) and (3.26), correspondingly, by changing \( q \) for \( q^{-1} \). Therefore, the same change of parameters relates the unique solutions of the systems,

\[
\tilde{m}_i = q^{2N-2i+1}, \quad i = 1, \ldots, N, \tag{3.28}
\]

and

\[
\tilde{m}'_i = q^{2i-1}, \quad i = 1, \ldots, N. \tag{3.29}
\]

Non-commuting of crossing and turn-over operators.

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics{crossing_diagram} \\
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\includegraphics{turnover_diagram}
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\includegraphics{crossing_diagram} \\
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\includegraphics{turnover_diagram}
\end{array}
\end{array}
\tag{3.30}
\]

The operator contraction corresponding to a planar diagram must include the turn-over operator each time, when the tangent vector makes a complete turnover. In particular, one may suggest to insert a turn-over operator in each cut made on the knot diagram. This procedure is self-consistent, if the result does not depend on the way of cutting. One might expect that the turn-over operator \( \mathcal{M} \) must just commute with the crossing operator \( \mathcal{R} \),

\[
\sum_a \mathcal{M}^a_i R^b_{kl} = \sum_a \mathcal{M}^a_i R^b_{kl} \Rightarrow m_i m_j R^{ij}_{kl} = m_k R^{ij}_{kl}, \tag{3.31}
\]

However, this condition is obviously not satisfied, at least for solutions (3.3) and (3.20,3.27) for \( R \) and \( \mathcal{M} \). On the other hand, one may not just pull a cut through a given crossing. Indeed, one must make a cut before one completes a next turn-over. Since each crossing joins four vertices, one returns there twice; moreover, if one leaves a crossing by an outgoing edge, one enters the same vertex by the incoming edge connected to the other outgoing edge (unless this is a crossing of different link components, but we postpone this case at the moment). Before returning to the same point, one completes at least one turn-over. Hence, two closed loops are adjacent to each vertex; if one just moves a joined to the vertex cut (suppose that there is a one) on the other side of vertex, one removes a cut either from one, or from the other closed loop, what is forbidden.

Instead (3.31), the calculated \( \mathcal{M} \) and \( \mathcal{M}' \) satisfy the bilinear relations

\[
\sum_{a,b} \mathcal{M}^a_i \mathcal{M}^b_j R^{ab}_{kl} = \sum_{a,b} \mathcal{M}^a_i \mathcal{M}^b_j R^{ab}_{kl} \Rightarrow m_i m_j R^{ij}_{kl} = m_k m_l R^{ij}_{kl}, \tag{3.32}
\]

\[
\sum_{a,b} \mathcal{M}^a_i \mathcal{M}^b_j R^{ab}_{kl} = \sum_{a,b} \mathcal{M}^a_i \mathcal{M}^b_j R^{ab}_{kl} \Rightarrow m_i m_j R^{ij}_{kl} = m_i m_j' R^{ij}_{kl}, \tag{3.32}
\]

which correspond to pairwise moves of cuts, as in (3.30). Indeed, solution (3.3) assumes that there are four cases to verify,

\[
\begin{align*}
m_i m_i R^{ii}_{ij} &= m_i m_i R^{ii}_{ij}, & m_i m_i R^{ii}_{ii} &= m_i m_i R^{ii}_{ii}, \\
m_i m_j R^{ij}_{ij} &= m_i m_j R^{ij}_{ij}, & m_j m_i R^{ij}_{ij} &= m_j m_i R^{ij}_{ij}, & i \neq j, \\
m_j m_i R^{ji}_{ji} &= R^{ji}_{ji} m_i m_j, & m_j m_i R^{ji}_{ji} &= m_j m_i R^{ji}_{ji}, & Si \leq j.
\end{align*} \tag{3.33}
\]
Most of these equalities are trivial; the only non-trivial one holds for \( m \) and \( m' \) from (3.20) and (3.27),

\[
m_i m'_i = q^{2i-2N-1} q^{-2i+1} = q^{-2N} = m_j m'_j.
\] (3.34)

In other words, (3.32) follows from the two facts. First, all the non-zero elements \( R^{ij}_{kl} \) of the crossing operator under consideration, which is given by (3.3), have the scripts satisfying \( i + j = k + l \iff i - l = k - j \). Second, all the non-zero matrix elements of the corresponding turn-over operators, which are placed on the diagonal, and depend on the occupied position, according to (3.20) and (3.27), exponentially, the powers are opposite for the conjugate operators. As a result, product of the matrix elements depend only on the sum of the scripts for similar operators, \( m_i m_j = q^{4N-2-2i-2j} \), and on the difference of the scripts the conjugate operators, \( m_i m'_j = q^{2(l-i)-2N+1} \).

A similar observation may be done for the inverse crossing operators and the corresponding turn-over operators.

### 3.3.4 Framing of a knot

**Twisted circles as full contractions of \( R \)-matrices** As the first application of the \( R \)-matrix formalism, consider an invariant of the twisted circle (fig.3I). Following the algorithm of sec 3.2 one makes two cuts as in the figure. The diagram contains the single crossing, and the corresponding invariant equals to the contraction of the \( R \)-matrix with two \( M \)-matrices inserted in the two pairs of indices,

\[
\sum_{i,j,a,b} R^{ij}_{ab} M_i^a M_j^b = \sum_{i,j} R^{ij}_{i,j} m_i m'_j = \sum_{i=1}^{N} m_i = \sum_{j=1}^{N} m'_j = \sum_{i=1}^{N} q^{-2i+1} = q^{-N} q^{-N} = q^{-N} [N].
\] (3.35)

In turn, a similar diagram with the inverse crossing (fig.3II) gives the value of the invariant

\[
\sum_{i,j,a,b} \tilde{R}^{ij}_{ab} M_i^a \tilde{M}_j^b = \sum_{i,j} \tilde{R}^{ij}_{i,j} \tilde{m}_i \tilde{m}'_j = \sum_{i=1}^{N} \tilde{m}_i = \sum_{j=1}^{N} \tilde{m}'_j = \sum_{i=1}^{N} 2q^{2i-1} = q^{N} [N].
\] (3.36)

On the other hand, (3.35) must coincide with (3.36) for a properly defined knot invariant, since both diagrams in fig 3 are projections of the curves that may be continuously transformed into a plain circle, i.e. to the unknot. One may cure this problem by putting one more element into the \( R \)-matrix construction.

**Topological normalization** To formulate the problem more clearly, let us consider a plain circle. Following the program of sec 3.2 one should treat it’s planar projection as a graph with no vertices and with one cut (fig 3). Such a graph may be associated with trace of a turn-over operator. One of the four turn-over operators \( M, M', \tilde{M}, \tilde{M}' \) may be taken equally well in the case. A problem is their traces differ, as we already verified above,

\[
\sum_{i=1}^{N} M_i^i = \sum_{i=1}^{N} M_i^{i} = q^{-N} [N], \quad \sum_{i=1}^{N} \tilde{M}_i^i = \sum_{i=1}^{N} \tilde{M}_i^i = q^{N} [N].
\] (3.37)

**Invariants of framed knots in \( R \)-matrix construction** A clue is an observation that expressions (3.35) and (3.36) are obtained from the knot diagrams with the different values of a relative invariant, a writhe number, which by definition equals to the number of direct crossings minus number...
of inverse crossings (see fig 3.1). A writhe number is unchanged by the second (3.10) and the third (3.1) Reidemeister moves, but increases or decreases by one by the first move (3.14). Answers (3.35) and (3.36) coincide provided that we multiply them by factors $q^{wN}$ with the corresponding writhe numbers $w = 1$ and $w = -1$ for figs. 4-I and II, respectively. This observation, however, does not solve the problem with plain circle, for which $w = 0$. This problem may be avoided by rescaling of the crossing operators as

$$\mathcal{M} \rightarrow \mathfrak{m} \equiv q^{-N} \mathcal{M}, \quad \text{and} \quad \mathcal{M}' \rightarrow \mathfrak{m}' \equiv q^{-N} \mathcal{M'},$$

(3.38)

The corresponding matrix elements are

$$m_i = q^{2i-N-1}, \quad m_i' = q^{N-2i+1}, \quad m_j = q^{N-2i+1}, \quad m_j' = q^{2i-N-1}, \quad i = 1, \ldots, N.$$  

(3.39)

All the four traces in (3.37) are then equal $[N]$; (3.35) then matches (3.36) and the answer for the plain circle after each expression is multiplied on the factor of $q^{-wN}$ with it’s own $w$ (the factor is inverse to the previously suggested ones). Motivated by this consideration, one may introduce then two more rules,

**Step 3.10** Rescale the the turn-over operators so that traces of all the four operators equal, and

**Step 3.11** Multiply the result of Steps 3.1-3.9 on $q^{-wN}$, with $w$ being the writhe number of knot diagram involved in the evaluation.

If one follows the rule, both diagrams in fig [4] as well as the plain circle in fig [3] yield one and the same value of the invariant for the unknot, $[N]$. Moreover, it may be shown, that Steps 3.10,3.11 together with Step 3.10 guarantee that the $R$-matrix contractions read off from equivalent knot diagrams with different writhes numbers yield the same value of the knot invariant [64]. However, one has to check whether the suggested rescaling does not break already imposed constraints (3.14,3.10,3.11). Equalities (3.10) and (3.1) which do not involve turn-over operators, indeed, remain unaffected. Unlike them, condition (3.14) gains an extra factor of $q^N$ for loops in fig 3.21-I,II, and $q^{-N}$ for loops in fig 3.21-III,IV. Deformed this way, relation (3.14) is called a $q$-Reidemeister-I move. Since a loop in (3.14) may be contracted in the space, a supposed deformation of the constraint is inept, unless one endows a knot with an additional structure. The needed structure may be represented graphically, by substituting a knot by a knotted ribbon. The first Reidemeister move is not an equivalence transformation of such knots, since it causes a twist of the ribbon. Formally speaking, a knot, or each component of a link should be associated now with an integer number, which equals to a number of ribbon intertwining and is changed by one under the first Reidemeister move. The obtained object is called a framed knot or link [3]. We have demonstrated in the elementary example that a knot invariant of the studied type is, in fact, an invariant not just of a knot, but of a framed knot [64]. Somewhat surprisingly, a $R$-matrix contractions read off from equivalent knot diagrams with different writhes numbers yield the same value of the knot invariant [64]. However, one has to check whether the suggested rescaling does not break already imposed constraints (3.14,3.10,3.11). Equalities (3.10) and (3.1) which do not involve turn-over operators, indeed, remain unaffected. Unlike them, condition (3.14) gains an extra factor of $q^N$ for loops in fig 3.21-I,II, and $q^{-N}$ for loops in fig 3.21-III,IV. Deformed this way, relation (3.14) is called a $q$-Reidemeister-I move. Since a loop in (3.14) may be contracted in the space, a supposed deformation of the constraint is inept, unless one endows a knot with an additional structure. The needed structure may be represented graphically, by substituting a knot by a knotted ribbon. The first Reidemeister move is not an equivalence transformation of such knots, since it causes a twist of the ribbon. Formally speaking, a knot, or each component of a link should be associated now with an integer number, which equals to a number of ribbon intertwining and is changed by one under the first Reidemeister move. The obtained object is called a framed knot or link [3]. We have demonstrated in the elementary example that a knot invariant of the studied type is, in fact, an invariant not just of a knot, but of a framed knot [64]. Somewhat surprisingly, attempt of interpreting the same invariant as a QFT observable lead one to the same conclusion (see sec 5.2), and even associates the introduced integer number with one more structure related to a knot (see sec 5.3).

Although required in the definition of the knot invariant of the studied type, choice of framing does not affect the answer, provided that one follows Step 3.11 (the corresponding factor is called framing factor). The framing independence becomes even more explicit if the multiplying of the entire answer on a framing factor is substituted be rescaling each crossing operator as $R \rightarrow q^{-N} R$, and each inverse crossing operator as $\tilde{R} \rightarrow q^{N} \tilde{R}$. Both constraints (3.10), (3.1) are preserved by such a change, and (3.14) holds again without extra factors. However, the framing independence property breaks both for knots in topologically non-trivial (other than $R_3$ or $S_3$) spaces [3] and for some other (yet conjectured) generalizations of the studied quantities [26].
3.3.5 Closures of braids

A major problem of the $R$-matrix formalism is to determine an explicit form of the turn-over operators for an arbitrary cut diagram of a knot. As a first illustration of the issue, note that (3.36) may be equally well changed for the contraction (from now on, we write the shifted turn-over operators 3.38 and the corresponding matrix elements 3.39 in all formulas, and we put the framing factor introduced at Step 3.11 in front of the entire expression from the very beginning),

$$q^{-N} \sum_{i,j,a,b} R^{ij}_{ab} m^b_i m^a_j = q^{-N} \sum_{i,j} R^{ij}_{ij} m_i m_j = q^{-N} \sum_{i=1}^N m_i = [N].$$

Indeed, after the contacted over one pair of indices is completed, one obtains just a $\delta$-symbol, which do not distinguish the operators $\mathcal{M}$ and $\mathcal{M}'$. Generally speaking, rule (3.14) is not sufficient to determine all needed turn-over operators, since the contracted scripts do not belong to the same $R$-matrix for a general cut. E.g., let us search for the turn-over operators needed for closing a three-strand braid, and let us impose then constrain (3.14) not just on the answer but on operator products associated with unclosed braids. In particular, if the third strand enters a single crossing (as in the l.h.s. of (2.6-II)), one obtains a two-strand braid with a contractible loop after closing the braid. The corresponding operator identity holds, if one inserts the operator $\mathcal{M}$ in a closure of the third strand,

$$B^{ij}_{kl} = \sum_{a,b,c} B^{ij}_{ac} R_{bc}^{ca} \mathcal{M}^b_a.$$  \hspace{1cm} (3.41)

On the other hand, a similar condition for the case of the first strand entering a single crossing (as in l.h.s. of (2.6-II')) is satisfied if the first strand is closed with help of the transposed operator $\mathcal{M}^b_a$,

$$B^{ij}_{kl} = \sum_{a,b,c} B^{ij}_{a} R_{kb}^{ac} \mathcal{M}^b_a.$$  \hspace{1cm} (3.42)

A known theorem (see [3] or see 2.2 for details) claims that all equivalence transformations that reduce a closure of a three-strand braid to a two-strand one are exhausted by (3.41) and (3.42), each one being sufficient. However, the conditions do not fix the form of an operator that closes the second strand. In particular, there is no reason to take neither $\mathcal{M}$, nor $\mathcal{M}'$. Moreover, one may replace crossings deleted in transformations (2.6-II,II') with the corresponding inverse crossing. One should then take the different operators, $\mathcal{M}$ and $\mathcal{M}'$. Old conditions (3.42 3.41) will be no longer satisfied then.

A conventional cure is to close all strands in the braid with help of the same operator $\mathcal{M}$. Relation 3.41 holds then as an operator identity, while (3.42) is satisfied for the entire contraction only. As an illustration, let us consider the case of adding a free loop to the two-strand braid with three crossings. For simplicity, we substitute for crossing operators solution (3.3) with $N = 2$,

$$R_{11}^{11} = R_{22}^{22} = q, \quad R_{12}^{12} = R_{21}^{21} = 1, \quad R_{21}^{12} = q - q^{-1}. $$  \hspace{1cm} (3.43)

The initial two-strand braid is then associated with the operator

$$B^{ij}_{pq} = \sum_{k,l,m,n} R^{ij}_{kl} M_{mn} R_{pq}^{mn}, $$  \hspace{1cm} (3.44)

whose non-zero elements may be evaluated straightforwardly,

$$B_{11}^{11} = B_{22}^{22} = q^3, \quad B_{12}^{12} = B_{21}^{21} = q^2 - 1 + q^{-2}, \quad B_{12}^{12} = q^3 - q + q^{-1} - q^{-3}, \quad B_{21}^{21} = q - q^{-1}. $$  \hspace{1cm} (3.45)

Now, if one adds the third strand that enters a single crossing, and then closes the braid inserting the operator $\mathcal{M}$ in each strand, one obtains

$$B_{11}^{ij} \mathcal{M}_{11} + B_{21}^{ij} \mathcal{M}_{21} + (B_{11}^{ij} \mathcal{M}_{11} + B_{21}^{ij} \mathcal{M}_{21}) \mathcal{M}_{21} + (B_{22}^{ij} \mathcal{M}_{22} + B_{22}^{ij} \mathcal{M}_{22}) \mathcal{M}_{22} =$$

$$= (qq^{-1} + (q + q^{-1})q) (q^3 q^{-1} + (q - q^{-1})q) q^{-1} +$$

$$+ (q^{-1}) (q^3 - q + q^{-1} - q^{-3}) q^{-1} + q^{-1} q =$$

$$= q^5 + q^3 + q - q^{-3},$$  \hspace{1cm} (3.46)
if the strand is added from the right, and

\[
\left( R_{11} \right) \left( B_{11} + B_{12} \right) m_1 + \left( R_{21} \right) \left( B_{21} + B_{22} \right) m_2 = \\
= \left( q q^{-1} \right) \left( q^3 q^{-1} + \left( q^3 - q + q^{-1} - q^{-3} \right) q \right) q^{-1} + \\
+ \left( q - q^{-1} \right) q^{-1} + q \left( (q - q^{-1}) q^{-1} + q^3 q \right) q = \\
= q^5 + q^3 + q - q^{-3},
\]

(3.47)

if the strand is added from the left. Both answers are the same as for the closure of the initial two-strand braid,

\[
B_{11}^2 \left( m_1 \right) + \left( B_{12} + B_{21} \right) m_1 m_2 + \left( m_2 \right)^2 = \\
= q^3 \left( q^{-1} \right)^2 + \left( q - q^{-1} \right) + \left( q^3 - q + q^{-1} - q^{-3} \right) q^{-1} + q^3 q^2 = \\
= q^5 + q^3 + q^{-1} - q^{-3}.
\]

(3.48)

The answer multiplied by the framing factor of \( q^{-wN} = q^{-6} \) equals \( (q + q^{-1})(q^{-2} + q^{-6} - q^{-8}) \), where the value of the invariant for the unknot, \( [N] = [2] \) stands in the first bracket, and the Jones polynomial for the trefoil knot \([17]\) \((q \text{ therein is our } q^2)\) in the second one. Nevertheless, it is straightforward to verify that \textit{partial contractions} \((3.41,3.42)\), which may be associated with closing just the added strand (as in \(2.6\text{-II and -II}'\)), reproduces the initial two-strand operator if the strand is added from the right, but not if the strand was added from the left.

The case is opposite for contractible loops with the inverse crossings; invariance holds for operators if the strand is added from the left, and only for full contractions if the strand is added from the right (this follows from the relations \( \tilde{M} = (\tilde{M}')^{-1} \) and \( \tilde{M}' = (\tilde{M})^{-1} \)).

### 3.3.6 Explicit evaluation of knot invariant for a trefoil

After all discussions, we are finally ready to complete an explicit evaluation of the studied invariant for the simplest knot, which is the trefoil knot. The \( R \) matrix construction dictates, with all conclusions and assumptions of sec.3.2 and 3.3 taken into account, to related with the knot diagram in \(5\) the expression

\[
q^{-wN} \sum_{i,j,k,l} R_{ai}^{kc} R_{bj}^{ia} R_{dj}^{cb} m_i m_j m_k = q^{-3N} \sum_{i,k,l} R_{ai}^{kc} R_{bj}^{ia} R_{ck}^{cb} m_i m_k.
\]

(3.49)
The formula enables an analytic continuation to arbitrary values of \( N \), merely by substituting \( q^N \) for a new independent variable \( A \). Divided by the value of the invariant for the unknot \( [N] = \frac{A-A^{-1}}{q-q^{-1}} \), the resulting expression yields the HOMFLY polynomial for the trefoil knot [17] (with \( a = A^{-1} \) and \( z = q - q^{-1} \)),

\[
\mathcal{H}_N(A, q) = \frac{A^{-3}}{2} \left\{ q^3 \frac{Aq - A^{-1}q^{-1}}{q - q^{-1}} - q^{-3} \frac{Aq^{-1} - A^{-1}q}{q - q^{-1}} \right\} = A^{-2}(q^2 + q^{-2}) - A^{-4}.
\]
3.3.7 Mirror symmetry

If one changes all crossings in fig.5-I for the inverse ones, one obtains the knot diagram in fig.5-II. Evaluation of the associated knot invariant repeats the above calculation almost literally, with only difference that the operators $R$ and $\mathfrak{M}$ are substituted by the corresponding inverse operators $\tilde{R}$ and $\tilde{\mathfrak{M}}$. The matrix elements of the latter ones are given by (3.41), and (3.28), respectively, and they are obtained from the corresponding elements of $R$ and $\mathfrak{M}$ by substituting $q \rightarrow q^{-1}$. In addition, the writhe number in the framing factor changes for the opposite one. Hence, the answer for the reflected diagram is

$$H_{N}^{\tilde{3}_{1}}(N,q^{-1}) = q^{-3N} \left\{ q^{-3} \frac{[N][N+1]}{2} - q^{3} \frac{[N][N-1]}{2} \right\}.$$  \hspace{1cm} (3.56)

Expressions (3.54) and (3.56) are different, they are not even proportional to each other. This agrees with the inequivalence of knots represented by diagrams in figs.5-I and II, which are named as left- and right-hand trefoils, respectively. The above reasoning applies to an arbitrary knot diagram, and one obtains that the HOMFLY polynomials of the knot $\tilde{K}$ and its mirror image $\tilde{K}$ are related as $H_{\tilde{K}}(q) = H_{K}(q^{-1})$. Some other knot invariants, such as colored HOMFLY polynomials [64] and superpolynomials [28, 29] enjoy their analogues of this symmetry.

In contrast to the considered transformation, reversing the orientation of a knot diagram (i.e., reflecting the directions of arrows) does not affect the answer at all, since all the direct crossings remain direct ones, and the inverse crossings remain the inverse ones (fig.5-III). The result is a diagram of the same knot, just projected on the “ceiling” instead of the “floor”.

3.4 From vertex operators to symmetric/monodromy group representations

In this section, we discuss how the vertex operator approach is related to the braid group approach (see sec.2.1). A naive guess might be that the braid crossing operators $B$ are just particular cases of the vertex operators $R$, written down in a certain basis. The real case is very close to that, but is still somewhat different. In the current section, we discuss the relation between the vertex-operators and the braid crossing matrices, comparing them explicitly in the simplest examples.

As we already mentioned, the history went just the opposite way. The $R$-matrices approach was originally formulated applied to braids only [63, 64]. The method was then developed into a working computational tool [30, 32, 33, 34, 37, 38, 39, 36, 40] in the same terms, and was extended to wider class of knot knot representations only recently [41, 46, 47]. Constructing a consistent $R$-matrix formalism valid for arbitrary knot diagrams is a separate problem [?], and it is still far from being solved exhaustively.

3.4.1 Reduction of four-indices operators to matrices. Twisted $R$-matrices

A serial connection of crossings, as in fig.6, is associated with a contraction of the $R$-matrices in a pair of indices, as we already have seen when we considered the second Reidemeister move. Such a combination may be seen as a matrix product. A common trick is to introduce a twisted $R$-matrix, which differs by a permutation of the subscripts

$$R_{ij}^{kl} \equiv R_{ik}^{lj}.$$  \hspace{1cm} (3.57)

In both matrices the superscripts are placed at the incoming and the lowers ones at the outgoing strands; however, while the two lest scripts of $R$ are related to the upper line w.r.t. to the projection plane, while the two left scripts of $\mathcal{R}$ are related to the left strand in the braid. Contraction in fig.6 is rewritten by this trick as a matrix product

$$\sum_{a,b} R_{ba}^{ab} R_{ik}^{ab} = \sum_{a,b} R_{ab}^{ab} R_{kl}^{ab} \equiv \sum_{j} R_{j}^{l} R_{K}^{j},$$  \hspace{1cm} (3.58)
with the multi-indices $I$, $J$, and $K$ standing for the pairs of indices $(i, j)$, $(a, b)$, and $(k, l)$ correspondingly. The constraint due to the second Reidemeister move is rewritten then as

$$\sum_J R^I_J \tilde{R}^J_K = \delta^I_K \Rightarrow \tilde{R}^J_J = (R^{-1})^I_J,$$  \hspace{1cm} (3.59)

i.e. successive crossings of mutually inverse orientations correspond to the mutually inverse matrices. Finally, the full contraction, associated with the braid closure, would correspond to taking the trace, were not there the turn-over operators. Before proceeding with writing out the explicit expression, we discuss one more difference between regular and twisted $R$-matrices.

### 3.4.2 Eigenvectors of regular and twisted crossing operators in a two-strand braid

Generally, $R$ (and hence $\mathcal{R}$) maps a tensor product of vector spaces, associated with the incoming strands to that of the ones associated with the outgoing strands. Since all the four spaces are of the same dimension, either $R$ or $\mathcal{R}$ (but not they both at the same time) may be looked at as an automorphism. In particular, one may find the eigenvalues and eigenvectors of these operators. As we will see in the next section, this will notably simplify the computation in case of several operators $R$ contacted as in fig.7. From the practical point of view, these are eigenvalues of $R$ (not of $\mathcal{R}$), which are needed, due to the multiplication rule (3.58). Yet, we explore the eigenvalue problem for both matrices to better demonstrate a difference in their properties.

The very form of (3.3) supposes that the entire $N^2$-dimensional space spanned by $\xi_i \eta_j$ ($i, j = 1, \ldots, n$) decomposes into a sum of $N$ one-dimensional eigenspaces spanned by $\xi_i \eta_i$ ($i = 1, \ldots, n$) and $\frac{N(N-1)}{2}$ two-dimensional spaces spanned by $\xi_i \eta_j$ and $\eta_j \xi_i$ ($i, j = 1, \ldots, n$ and $i < j$). The same statement is valid for $\mathcal{R}$. Hence, one already has $N$ coinciding eigenvalues $R^{ii}_{ii} = R^{ii}_{ii} = q$ with the corresponding eigenvectors, both of $R$ and $\mathcal{R}$:

$$\lambda_i = q, \quad X_i = \xi_i \eta_i, \quad i = 1, \ldots, N.$$  \hspace{1cm} (3.60)

It remains to examine one of $\frac{N(N-1)}{2}$ identical two-dimensional spaces. Making use of expression (3.3) for $i < j$, one gets the eigenvalue problem for $R$

$$\sum_{i,j} R^{ij}_{kl} (\alpha \xi_i \eta_j + \beta \xi_j \eta_i) = \alpha \xi_k \eta_l + (q - q^{-1}) \alpha + \beta) \xi_l \eta_k = \lambda (\alpha \xi_k \eta_l + \beta \xi_l \eta_k),$$  \hspace{1cm} (3.61)

whence $\lambda = 1$, $\alpha = 0$, and the only (for a given pair $k, l$) eigenvector is $\xi_k \eta_l$; there is also the adjoint vector $\xi_k \eta_l - \xi_l \eta_k$, i.e., $R$ is a Jordan cell w.r.t. to the stated eigenvalue problem. Unlike that, $\mathcal{R}$, whose eigenvalue problem differs from (3.61) by a permutation of indices $k$ and $l$ in the last term of the equality, has two distinct eigenvalues; they are the two roots of the characteristic equation

$$\lambda^2 - (q - q^{-1}) \lambda - 1 = 0,$$  \hspace{1cm} (3.62)

which, in turn is obtained as consistency condition of the system

$$\lambda \beta = \alpha,$$  \hspace{1cm} (3.63)

$$\lambda \alpha = (q - q^{-1}) \alpha + \beta,$$  \hspace{1cm} (3.64)

on the eigenvectors components. Hence, the eigenvalues and the corresponding eigenvectors are

$$\lambda_+ = q, \quad x^+_{ij} = q \xi_i \eta_j + \xi_j \eta_i,$$  \hspace{1cm} (3.65)

$$\lambda_- = -1/q, \quad x^-_{ij} = \xi_i \eta_j - q \xi_j \eta_i.$$  \hspace{1cm} (3.66)
The observed difference between \( R \) and \( \mathcal{R} \) is seen especially well from the matrix form of their corresponding cells; say, one has for \( i, j \) running the values 1, 2

\[
R_{(12)}^{(12)} \equiv \begin{pmatrix} R_{12}^{12} & R_{21}^{12} \\ R_{12}^{21} & R_{21}^{21} \end{pmatrix} = \begin{pmatrix} 1 & q-q^{-1} \\ 0 & 1 \end{pmatrix},
\]

\[
\mathcal{R}_{(12)}^{(12)} \equiv \begin{pmatrix} R_{21}^{12} & R_{12}^{12} \\ R_{21}^{21} & R_{12}^{21} \end{pmatrix} = \begin{pmatrix} q-q^{-1} & 1 \\ 1 & 0 \end{pmatrix}.
\] (3.67)

Following the general rule, one obtains the eigenvalues and the eigenvectors of the inverse \( R \)- and \( \mathcal{R} \)-matrices by the substituting \( q \to q^{-1} \). It is straightforward to check that the eigenvectors of the straight and inverse operators coincide.

### 3.4.3 Turn-over operators in a two-stand braid and character decomposition

As discussed above, an operator contraction yielding a knot invariant contains not only the crossing operators, but also the turn-over operators, their explicit form determined in sec.3.3.3 for crossing operators of form (3.3). Two turn-over operators are needed in the case, since a two strand braid is obtained from it’s closure by making two cuts. Modulo discussion of sec.3.3.3, for crossing operators of form (3.3). Two turn-over operators are needed in the

For the vertex operators from the braid crossing operators is that the first one has degenerate eigenvalues, \( q \) (the “symmetric” one) with the multiplicity \( \frac{N(N+1)}{2} \), \( q^{-1} \) (the “antisymmetric” one) with the multiplicity \( \frac{N(N-1)}{2} \), for the vertex operator of form (3.3). When one evaluates trace (3.68), each degenerate eigenvalue is multiplied on...
the trace of the squared turn-over operator over the corresponding “symmetric” and “antisymmetric” subspaces. As a result, one obtains just the weight coefficients, which were determined in (2.16) as solutions of the topological invariance constraints (although (3.70) reproduces just a particular case of (2.16) with \( \lambda = q, \mu = -q^{-1}, \chi = [N] \)). Although we solved the topological invariance constraints to determine the matrix elements of the turn-over operators, an explicit expression for the very operators is available in group theory, at least in the particular case of braid representations [64], [72]. It is clear from the form of this expression, that the coefficients of the \( R \)-matrix eigenvalues \( q \) and \( q^{-1} \) are nothing but the quantum dimensions of the first symmetric and antisymmetric \( \text{su}(N) \) representations, respectively. We came to the same result twice, in (2.16) and in (3.70), with help of straightforward computations.

3.4.4 Common eigenspaces of twisted crossing operators in a three-stand braid

One may derive a generalization of (3.68) for an arbitrary braid [64, 72, 32]. An additional difficulty one encounters with is a presence of several kinds of crossings, each one bringing a contribution to (3.68) of its own form. E.g., there will be two distinct operators for two kinds of crossings in a three-strand braid (fig.8). The operators do not commute, and, hence, may not have a basis of common eigenvectors. Nevertheless, they do have a number of common eigenvectors, while the complimentary subspace decomposes into a sum of two-dimensional common eigenspaces. We demonstrate this explicitly in what follows.

An adaptation of the \( R \)-matrix approach to braids is heavily involves the fact that twisted \( R \)-matrices form the group, which extends the permutation group. Namely, solution (3.3) for the operator \( R_{ij}^{kl} \) may be seen as so referred to \( q \)-permutation operator, whose action is defined on a tensor product of vector spaces \( V \otimes V \) as

\[
R(q)\xi_i\eta_j = \begin{cases} 
\xi_i\eta_j, & i > j \\
q\xi_i\eta_j, & i = j \\
(q + q^{-1})\xi_i\eta_j + \xi_j\eta_i, & i < j.
\end{cases}
\] (3.71)

where \( \xi \) and \( \eta \) are vectors of \( V \), the subscripts run from 1 to \( \dim V \). In particular, common eigenspaces of braid \( R \)-matrices are constructed similarly to irreducible representations of the symmetric group (see sec.2.2). To make this analogy explicit, we briefly review the formulas from Step 2.2 of sec.2.2.5, this time presenting them in a form generalizable for \( R \)-matrices case.

First, let us notice that two irreducible representations (2.10) of rank two symmetric group may be rewritten by introducing two linear operators, which are called, correspondingly, symmetrizer and antisymmetrizer:

\[ (xy) = \frac{1}{2}(1 + \sigma_1) \equiv Sxy, \quad [xy] - \frac{1}{2}(1 - \sigma_1) \equiv Axy, \] (3.72)

and satisfy

\[ S^2 = S, \quad A^2 = A, \quad SA = AS = 0, \quad S + A = 1, \] (3.73)

thus being the orthogonal projectors. Relations (3.73) follow straight from definitions (3.72) and from the property \( \sigma_1^2 = 1 \) of the symmetric group generator. The irreducible representations of the symmetric group may be determined just in terms of the introduced operators. Since

\[ \sigma_1 S = S, \quad \sigma_1 A = -A, \] (3.74)

any vectors of type \( SX \) and \( AX \) are eigenvectors of \( \sigma_1 \) with the eigenvalues 1 and \(-1\) correspondingly. Similarly, one may introduce three pairwise orthogonal projectors on the common eigenspaces of the
symmetric group with three elements,

\[ X_S = (xyz) = \frac{1}{6} (\mathbb{1} + \sigma_1 + \sigma_2 + \sigma_1 \sigma_2 + \sigma_2 \sigma_1 + \sigma_1 \sigma_2 \sigma_1) \equiv SSxyz, \quad (3.75) \]

\[ X_{SA} = \frac{1}{2} (1 + \sigma_1)(a + b \sigma_1 \sigma_2 + c \sigma_2 \sigma_1) xyz \equiv SAxyz, \]

\[ X_A = [xyz] = -\frac{1}{6} (1 - \sigma_1 + \sigma_2 + \sigma_1 \sigma_2 + \sigma_2 \sigma_1 - \sigma_1 \sigma_2 \sigma_1) \equiv AAxyz, \]

so that

\[ AA \cdot SS = SS \cdot AA = SA \cdot SS = SS \cdot SA = SA \cdot AA = AA \cdot SA = 0, \quad (3.76) \]

\[ SS^2 = SS, \quad SA^2 = SA, \quad AA^2 = AA. \]

One may verify then, that operators \( SS_q, \ AA_q, \) and \( S_q S_a \) give the projectors on three distinct irreducible representations of the symmetric group. In case of the one-dimensional representations, one has to check that \( \sigma_1 SS = \sigma_2 SS = SS \) and \( \sigma_1 AA = \sigma_2 AA = -AA \); this is done by substituting for \( SS \) and \( AA \) their explicit expressions \((3.75)\) and using that \( \sigma_1^2 = \sigma_2^2 = 1 \). For the two-dimensional representation, one gets \( \sigma_1 S_1 A = S_1 A S \), and the check reduces then to ensuring that the expressions \( S_q S_a, \ \sigma_2 S_q S_a \), and \( \sigma_1 \sigma_2 S_q S_a \) are linearly dependent, treated as formal polynomials in group generators. This indeed follows form \((3.75)\) given that the squared group generators are the unities:

\[
\begin{align*}
S \cdot SA &= 2 + 2\sigma_1 - \sigma_1 \sigma_2 - \sigma_2 \sigma_1 - \sigma_1 \sigma_2 \sigma_1, \\
\sigma_2 S \cdot SA &= -1 - \sigma_1 + 2\sigma_2 - \sigma_1 \sigma_2 + 2\sigma_2 \sigma_1 - \sigma_1 \sigma_2 \sigma_1, \\
\sigma_1 \sigma_2 \sigma_2 S \cdot A S &= -1 - \sigma_1 - \sigma_2 + 2\sigma_1 \sigma_2 - \sigma_2 \sigma_1 + 2\sigma_1 \sigma_2 \sigma_1 \\
\end{align*}
\]

\[ \Downarrow \]

\[ (\mathbb{1} + \sigma_2 + \sigma_1 \sigma_2) SA = 0. \quad (3.77) \]

Identity \((3.77)\) implies that any expression of the form \( S \cdot SAX \), where \( X \) is a formal polynomial in permutations, generates a two-dimensional representation of the symmetric group with three elements. Moreover, matrix expressions for group generators in this representations, e.g., \((2.32)\), may derived from the operator identities

\[
\begin{align*}
\sigma_1 \cdot \mathbb{1} &= \sigma_1, \\
\sigma_1 \cdot \frac{1 + 2\sigma_2}{\sqrt{3}} &= -\frac{1 + 2\sigma_2}{\sqrt{3}}, \\
\sigma_2 \cdot \frac{1 + 2\sigma_2}{\sqrt{3}} &= \frac{\sqrt{3}}{2} \cdot \mathbb{1} + \frac{1}{2} \cdot \frac{1 + 2\sigma_2}{\sqrt{3}}. \quad (3.78) \\
\end{align*}
\]

Similar formulas may be derived for symmetric groups with more elements. Then, there is a one-to-one correspondence of the symmetric group irreducible representations and common eigenspaces of the braid \( R \)-matrices \([63, 32]\). Moreover, the partition-based approach discussed a little in sec.2.2.6 may be extended to determine the common eigenspaces of the braid \( R \)-matrices explicitly \([32, 37]\), in principle, for arbitrary braids. The only reservation should be done here. Practically, the common eigenspaces are found not as irreducible representations of \( q \)-symmetric group, but as that of the quantum group \( U_q(SU(N)) \), the two ones are related by the analog of Shur-Weyl duality \([9, 4, 36]\). An output is an explicit expression for knot polynomials in terms of eigenvalues of quantum \( R \)-matrices and quantum Racah coefficients \([64, 32]\). The first ones are known in full generality, hence the entire problem is concentrated in evaluating the second ones. While the case of \( SU(2) \) group is a text-book subject \(?? \), not much is known beyond it (see \([35, 60, 7]\) for rare exceptions). In the particular case of \( R \)-matrices we discuss here, which are related to the fundamental representation of the quantum group \( SU(N) \) and give rise to the HOMFLY polynomials, all needed ingredients are determined explicitly, and a concise and computationally effective procedure is available \([39]\). The same tools enables to calculate colored HOMFLY polynomials as well, by means of the cabling procedure \([40]\). Finally, an
attempt of applying the approach to studies of superpolynomials was recently performed [49], in the framework of elaborating a modified Khovanov formalism [48].

As usual, we illustrate the approach discussed on the simplest relevant example, that of braid $R$-matrices for a three-strand braid. We start from recalling the even simpler case of the two-strands. The generalizations of (3.72) by definition satisfy

$$S_2^q = S_q, \quad A_2^q = A_q, \quad S_q A_q = A_q S_q = 0,$$

and

$$R_1 S_q = q S_q, \quad R_1 A_q = - q^{-1} A_q.$$

The operators $S_q$ and $A_q$ are referred to as, correspondingly, $q$-symmetrizer and $q$-antisymmetrizer; explicit expressions for them are

$$S_q \equiv \frac{1}{q^{|2|}_q} (1 + q R_1),$$

and

$$A_q \equiv - \frac{q}{|2|_q} (1 - q^{-1} R_1),$$

and properties (3.79, 3.80) are verified with help of generalization of the identity $\sigma_1^2 = 1$ for the symmetric group generator:

$$R_1^2 - (q - q^{-1}) R_1 - 1 = 0$$

Relation (3.61) is a characteristic equation for the braid $R$-matrix, whose eigenvalues were determined in sec.3.4.2.

In case of three-strand braid, the relevant extension of formulas for the symmetric group are less straightforward. Yet, they remain rather simple for projectors on fully symmetric and anti-symmetric representations. These projectors must satisfy

$$SS_q^2 = SS_q, \quad AA_q^2 = AA_q, \quad SS_q A_q A_q = AA_q S_q S_q = 0,$$

and

$$R_1 S_q S_q = R_2 S_q S_q = q S_q S_q, \quad R_1 A_q A_q = R_2 A_q A_q = - q^{-1} A_q.$$

Explicit expressions for $SS_q$ and $AA_q$ are straightforward analogues of formulas (3.75) for $SS$ and $AA$; symmetric group generators are substituted by the corresponding $R$-matrices, a factor of $q^k$ is put before a product of $k$ $R$-matrices, and the normalization factors are changed so that the first and second of equalities (3.84) are satisfied:

$$SS_q \equiv \frac{1}{q^{|2|}_q^2 |3|_q} \left(1 + q R_1 + q R_2 + q^2 R_1 R_2 + q^2 R_2 R_1 + q^3 R_1 R_2 R_1\right),$$

and

$$AA_q \equiv - \frac{q^3}{|2|_q^3 |3|_q} \left(1 - q^{-1} R_1 - q^{-1} R_2 + q^{-2} R_1 R_2 + q^{-2} R_2 R_1 - q^{-3} R_1 R_2 R_1\right).$$

Satisfying of (3.84) and (3.85) is checked with help of (3.61), a similar identity for $R_2$, and the Yang-Baxter equation, which in the case takes the form $R_1 R_2 R_1 = R_2 R_1 R_2$. An expression for the remaining projector (denote it $AS_q$) is more involved; the easiest way to obtain it is:

$$AS_q = 1 - SS_q - AA_q = \frac{1}{|3|_q} (R_1 - R_2)^2;$$

the equalities

$$AS_q = AS_q, \quad AS_q S_q S_q = SS_q A_q = AA_q S_q = A_q A_q S_q = 0,$$
follow then just from (3.84).

Relations (3.85) already imply that $SS_q$ and $AA_q$ are projectors on the common eigenvectors of $R_1$ and $R_2$, i.e., that for $X$ being any formal polynomial in $R$-matrices, $SS_qX$ and $AA_qX$ are the eigenvectors, with the values $q$ and $-q^{-1}$, correspondingly. It remains to verify that, in analogy with the symmetric group case, the expression $S_qAS_q$ yields a projector on a two-dimensional common eigenspace. One immediately gets that

$$R_1S_qAS_q = S_qAS_q. \tag{3.90}$$

An analog of (3.77) is also derived straightforwardly, though a bit lengthy. Using the eigenvalue equations for $R_1$ and $R_2$, and the Yang-Baxter equation, one obtains that the operators $S_qAS_q$, $R_2S_qAS_q$, and $R_1R_2S_qAS_q$ are expanded over the six basis products of $R$-matrices as

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
 & 1 & R_1 & R_2 & R_1R_2 & R_2R_1 \\
\hline
S_qAS_q & q^2 + 1 & q^3 + q^{-1} & -1 & -1 & -q \\
R_2S_qAS_q & -q^{-1} & -1 & q^2 + q^{-2} & -q & q^3 + q^{-1} & -q^2 \\
R_1R_2S_qAS_q & -1 & -q & -q & 1 + q^{-2} & -q^2 & q + q^{-1} \\
\hline
\end{array} \tag{3.91}
$$

wherefrom one derives the identity:

$$(1 + qR_2 + q^2R_1R_2) S_qAS_q = 0. \tag{3.92}$$

Relations (3.90, 3.92) imply that a basis in a two-dimensional common eigenspace of the $R$-matrices is obtained from an arbitrary formal polynomial in $R$-matrices $X$ as

$$S_qAS_qX \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad R_2S_qAS_qX \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.93}$$

Alternatively, one may construct a basic of $R_1$ eigenvectors in the same space. Writing down the corresponding condition

$$R_1(\alpha + \beta R_2) S_qAS_q = \lambda (\alpha + \beta R_2) S_qAS_q \tag{3.94}$$

and solving the system

$$\begin{cases} q\alpha - \beta q^{-2} = \lambda \alpha, \\
\beta q^{-1} = \lambda \beta, \end{cases} \quad \iff \begin{cases} \lambda = q, \\
\lambda = -q^{-1}, \quad \beta = 0, \quad \alpha = q^2[2]q^3 \beta, \end{cases} \tag{3.95}$$

one obtains the corresponding expressions for the basis vectors,

$$SS_qAS_qX \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{[3]_q}} (1 + q^2[2]_q R_2X) SS_qAS_qX \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.96}$$

and for acting on them $R$-matrices,

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -\frac{1}{q^2[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ \frac{\sqrt{[3]_q}}{[2]_q} & \frac{1}{[2]_q} \end{pmatrix}. \tag{3.97}$$

The result coincides with the formulas for the braid-crossing matrices, derived in the framework of the symmetric group approach in sec 2.2.5.
The summary  Content of the present section may be reviewed as follows. The contraction of the vertex operators \((R\)-matrices\) associated with a closure of a three-strand braid may be presented as a composition of linear operators of two types, \(R_1 \equiv R \otimes I\) and \(R_2 \equiv I \otimes R\). Both operators acts on the space \(V \otimes V \otimes V\), where one may chose a basis of tensor monomials \(\xi_i \otimes \xi_j \otimes \xi_k \in V\) composed of basis vectors \(\xi\) of the space \(V\), the subscripts running from 1 to \(N\). Operators \(R_1\) and \(R_2\) may be seen then as rank six tensors \((R_a)^{ijk}_{lmn}\) for \(a = 1, 2\). If we substitute (3.2) for \(R\), all the non-zero elements have the form \((R_a)^{ijk}_{\sigma(ijk)}\), with the subscripts being a permutation of the superscripts. In other words, a linear space \(V_{ijk}\) spanned by all the monomials \((\xi_i \otimes \xi_j \otimes \xi_k)^{\sigma(ijk)}\), where \(\sigma\) runs over all distinct permutations of \(ijk\), is a common eigenspace of \(R_1\) and \(R_2\). The dimension of this space equals one for \(i = j = k\), two for \(i = j \neq k\), and six for \(i \neq j \neq k\). As we verified above, the subspace \(V_{ijk}\) further decomposes into irreducible common eigenspaces of \(R_1\) and \(R_2\), which are hence the common eigenspaces of all the three-strand operators. The irreducible common eigenspaces are either one- or two-dimensional, so that the operators \(R_1\) and \(R_2\) acting on these spaces are represented either by \(1 \times 1\) or by \(2 \times 2\) matrices. In a certain basis, the matrices reproduce the braid crossing matrices of sec. 2.2.5.

The smallest eigenspaces turn out to be nothing but the spaces of the symmetric group irreducible representations, as the \(R\)-matrices eigenvalues \(q\) and \(-q^{-1}\) tend to 1 and \(-1\), respectively. The same spaces for a generic \(q\) are the spaces of Hecke algebra (or \(q\)-symmetric group) irreducible representations.

4  From integral representation for a linking number to QFT description of knot invariants

4.1 Naive sketch of the construction

Unlike two above sections, where explicit and more or less rigorously defined constructions for knot invariants were discussed, we turn now to a subject, where each claim should be treated rather than a source of an intuition then as an exact statement. We mean the very inspiring and picturesque description of link invariants drawn in [69] and kept in mind in the numerous subsequent papers (see, e.g., [81], [82], [84], the series of very fruitful works [52]-[61], and also review [85]) includes the following elements.

1. A keystone of the construction is the claim that only classical Chern-Simons fields should be considered, i.e., the space solution of the classical equations of motions modulo the gauge transformations is considered.

2. It is observed, that SC equation are turned into linear ones by a gauge fixing, at least locally; hence, gauge inequivalent solutions in a “small enough” region form a linear space.

3. Three-dimensional space is cut with a collection of two-dimensional surfaces into “small enough” regions, each one containing an “enough small” piece of the original knot or link.

4. It is stated, that classical CS fields inside a “small” region are in one-to-one correspondence, with the classical CS fields tangent to the bordering it surface; by that, if the former ones had a one-dimensional singularity along a piece of the link, the latter ones have a point-like singularities at where the link penetrates the surface.

5. A space of tangent fields on a certain surface with certain point-like singularities is a space of states in an other quantum field theory, namely of conformal blocks is Wess-Zumino-Witten theory [66], which are also known to form a linear space; to set a “one-to-one correspondence” means then to map the corresponding linear spaces onto each other.

6. WZW conformal blocks are known to have monodromies, i.e. to change under continues permutation of special points. Since such a permutation by the end recovers the original pattern of
the special points, a result of the monodromy lies in the original linear space and is expanded
over a basis in it, yielding by that a so called monodromy matrix. The latter one is rigorously
defined and may be explicitly evaluated in the framework of WZW theory.

7. On the other hand, a knot may be untied by certain transformations, which are seen from the
standpoint of the two-dimensional surface as some monodromies. A number of link invariants,
among them (uncolored) Jones and HOMFLY polynomials, may be determined recursively, from
a system of equations relating the values of the invariant for links related by a monodromy (in the
above sense). Monodromy matrices of WZW conformal blocks turn to satisfy these equations.

8. As a consequence of this observation, it is supposed that a knot invariant may be somehow as-
ssembled from the monodromy matrices of WZW conformal blocks. The result is should coincide
with an already known knot invariant, Jones or HOMFLY polynomial, since both quantities
satisfy the same system of equations, and this system is known to have a single solution.

The idea formulated in the last item turns out to be an indeed fruitful one, both for theoretical
analysis and for explicit computations (see the above cited works). Nevertheless, the rigorously defined
construction for knot invariants obtained on this way far deviates from the original great ambition,
being much more in the spirit of vertex operator approach discussed in the previous section. In partic-
ular, it looses the automatic topological invariance of the conjectured three-dimensional construction.
An extension of the working approach to the topologically non-trivial spaces is completely unclear
either. These considerations may motivate a ones again revising of the original idea, searching for a
more explicit realization of it. The task itself is definitely beyond the scope of the present text. We
would like make some comments on how this problem might be approached to. Much of these section’s
content may look like cracking a nut with a steamhammer; we believe yet, that this is what one can start with.

In the next two sections, we discuss how the sketched here program might be realized for the
simplest link invariant, which is the linking number. In the next section, we explain why and in what
sense this invariant equals to the Chern-Simons Wilson average; in the section after it, we attempt to
reduce calculation of the same invariant to the sequence of the above listed items, indicating various
pitfalls already in this trivial case.

4.2 Abelian Chern-Simons as a static magnetic field

In the present section, we briefly discuss a notion of Wilson average in the Abelian Chern-Simons
theory. This quantity is easy to imagine since it has just the form of the magnetostatic field circulation,
with the Chern-Simons potential standing for the magnetic field; we use $H$ for this quantity in the
present section to draw on the physical intuition. The Abelian Chern-Simons action in the three-
dimensional euclidian space \[\text{CS}\] reads ($\kappa$ is an arbitrary integer, which we set to be equal one for
simplicity, and hereafter a dot stands for a scalar product),

$$S_{\text{CS}}^{ab}[A_0 = H_z, \ A_x = H_x, \ A_y = H_y] \equiv \frac{\kappa}{4\pi} \int d^3 x e^{ijk} A_i \partial_j A_k \equiv \frac{1}{4\pi} \int dV \ \bar{H} \cdot \text{rot} \bar{H},$$

(4.1)

and sources of the field are introduced with help of the interaction term

$$S_{\text{int}}^{ab}[A_0 = H_z, \ A_x = H_x, \ A_y = H_y] \equiv \int d^3 x A_k j^k = \int dV \ \bar{H} \cdot \bar{j}.$$  

(4.2)

The classical equations of motion have then the form of Maxwell’s fourth equation for a static field,

$$\text{rot} \bar{H}(\bar{x}) = 4\pi \bar{j}(\bar{x}).$$  

(4.3)

Identity (4.3) implies that $H$ may be treated as a static magnetic field provided that the condition
(Maxwell’s second equation for the static field)

$$\text{div} \bar{H} = 0$$  

(4.4)
is also satisfied. This may be always achieved by a gradient transformation \( A_k \to A_k + \partial_k f \) \[73\], which leaves (4.1) unchanged. After fixing a gauge by the condition \( \partial_k A_k = 0 \), one still may add to the field a gradient of a harmonic function (such that \( \partial^2 f = 0 \)). The remaining gauge invariance means that we determine the magnetic field up to an arbitrary source-free (rotor-vanishing) summand. The ambiguity may be removed by imposing the boundary conditions. It suffices to demand that \( A_k \) is regular everywhere but the area occupied by sources and and vanishes at the infinity; this just corresponds to identifying \( A_k \) with a physical magnetic field \( \vec{H} \), so that one may draw an intuition on the subject from a school-level text-book (e.g., \[8\]). Recall that fixing a gauge and boundary conditions is necessary both to determine a form of the classical field explicitly and to make a sense of the functional integration.

Consider now a line-like source \( \vec{j}(\vec{x}) = I \delta(\vec{l} - \vec{x}) \). Solving (4.3) with the relevant r.h.s. and substituting the obtained field back to the action, one gets

\[
S_{CS}^{ab}[\vec{H}] = \int dV \vec{H} \cdot \vec{j} = \int_{\gamma} I d\vec{l} \cdot \vec{H} \equiv \Gamma_{\gamma} \left[ \vec{H} \right],
\]

(4.5)
i.e., a circulation of a magnetic field along the current circuit \( \gamma \) that produces the field. The quantity itself is, of course, undefined. Yet, it might arise in a physically sensible problem, indicating that the made infinitely thin wire approximation is meaningless in the case, and a certain thick wire model must be introduced instead. However, some kind of contributions to (4.5) make sense and may be evaluated already in the current approximation. Namely, suppose that there are two non-intersecting, but perhaps intertwined wires, i.e., \( \vec{j}(\vec{x}) = I_1 \delta(\vec{l}_1 - \vec{x}) + I_2 \delta(\vec{l}_2 - \vec{x}) \). The Abelian action (the circulation), which is bi-linear in the field, decomposes then into four contributions,

\[
S_{CS}^{ab} \left[ \vec{H}^{cl} = \vec{H}_{1}^{cl} + \vec{H}_{2}^{cl} \right] = \Gamma_{11} + \Gamma_{12} + \Gamma_{21} + \Gamma_{22},
\]

(4.6)
and

\[
\text{rot} \vec{H}_i(\vec{x}) = I_i \delta(\vec{l}_i - \vec{x}), \quad \Gamma_{ij} \equiv \oint_{\gamma_i} I_i d\vec{l}_i \cdot \vec{H}_j.
\]

(4.7)
two of them, \( \Gamma_{12} \) and \( \Gamma_{21} \), being well-defined even for a line-like source. These terms give a circulation of the first wire magnetic field along the second wire. We will concentrate on these off-diagonal, well-defined terms, postponing the treatment of the diagonal, singular ones till sec.5.1.

An Abelian Wilson loop is by definition an exponentiated circulation of the Chern-Simons field over a closed circuit

\[
W(\gamma) [A_0 = H_z, \ A_x = H_x, \ A_y = H_y] \equiv \exp \left( \Gamma_{\gamma} \right).
\]

(4.8)
A Wilson average is a quantity introduced in QFT, which naively might be seen as “an average of a Wilson loop over all field configurations”, and is written symbolically as

\[
\langle W(\gamma) \rangle_S = \frac{\int [DH] \exp \left\{ I \oint_{\gamma} d\vec{l} \cdot \vec{H} + \frac{1}{4\pi} \int dV \vec{H} \cdot \text{rot} \vec{H} \right\}}{\int [DH] \exp \left\{ \frac{1}{4\pi} \int dV \vec{H} \cdot \text{rot} \vec{H} \right\}}.
\]

(4.9)
One may give a rigorous meaning to expression (4.9) for the action quadratic in the field. In the case one has a so called Gaussian functional integral. The definition of this quantity relies on various rules of plain Gaussian integration, since the infinite-dimensional analogues of the corresponding formulas are often straightforward and consistent with each other. The most straightforward way is to define a Gaussian average of a quantity as the value of the quantity at the critical point of the integration measure (up to a certain numeric factor). In case of (4.10), this implies evaluating the integrand for the classical field configuration, which by definition corresponds to the vanishing first variation of the action, and hence of the integration measure,

\[
\langle W_{\gamma}[A] \rangle_S[A] \equiv W_{\gamma}[A = A_{cl}], \quad \left. \frac{\delta S}{\delta A} \right|_{A = A_{cl}} = 0, \quad \text{for } S \text{ being quadratic form in } A.
\]

(4.10)
A non-Gaussian “integral over fields” may be defined as a sum of asymptotic perturbation series, mod-
ulo all the arising problems \[10\]; we concern the subject a little in sec.5. There are also various con-
structions claiming to provide a non-perturbative definition of a Chern-Simons Wilson average (apart
from \[69\], one may name an approach \[82, 81\], \[52]-\[54\], \[84\], \[85\] based on Kniznik-Zamolodchikov
equation \[83\] and attempts of relating the discussed in sec.3.2 \(R\)-matrix construction to Chern-Simons
theory in the temporal gauge \[71, 72\]), but no uniform theory is developed yet. Anyway, the action is
Gaussian for the Abelian Chern-Simons theory, hence the
Abelian Chern-Simons Wilson average is
just a Wilson loop evaluated for the classical Chern-Simons field
given by (4.10). This is the quantity
we are using as a toy model of would be a QFT description of a knot invariant.

Before proceeding, we remark that (4.10) for
\(S[A_{cl} = \Gamma_{12}]\) is itself a knot, or, more precisely, link,
invariant. Indeed, the circulation of the one circuit’s magnetic field along another circuit equals, due
to the magnetic field circulation theorem (the integral form of Maxwell’s fourth equation for a static
field), to the number of crossings of a surface spanned by the second circuit by the first circuit; the
number is, in turn, the linking number of the circuits.

### 4.3 Explicating of the three-dimensional construction in Abelian case

Now we are going to reformulate evaluation of integral (4.5) in spirit of the conjectured three-
dimensional construction. First, we suggest, how a contribution to this integral from a small region
may be rewritten as a surface integral of a tangent field on a punctured sphere. Next, we try to first
determine this field in purely internal for the punctured sphere terms, and then to invent a rule for
assembling thus obtained separate contributions into a linking number. We argue that the supposed
procedure would be consistent only under a rigorous restriction on a form of the “elementary pieces”,
which actually brings us back to the vertex operator approach. Finally, we carry out the above listed
program head-on for certain geometric realization of the Hopf link, revealing an unjustified technical
complexities, at the same time verifying that it still may be realized.

#### 4.3.1 Elementary pieces of link and the corresponding operators

For the Abelian theory, items 1-2 of sec.4.1 are omit, since the theory is originally Gaussian, and the
Wilson average is expressed via the classical field from the very beginning (see sec.4.2). As an item
3, we select a surface intersecting both circuits, being, for simplicity, a topological sphere. A theorem
mentioned in \[69\] claims that a link in the plain three-dimensional space (or in the three-dimensional
sphere) may be cut into “small enough” (see further) discs by topological spheres only. Items 4-7
reduce to introducing a collection of tangent to the surface fields that carry all relevant information
about the interior region, whatever that means. To carry out items 4-7, we start from the answer,
attempting to bring linking-number integral (4.7) to the desired form. The resulting construction
would apply for an explicit definition of the quantum Abelian Wesso-Zumino-Witten theory \[66\],
to which one should pass from the Abelian Chern-Simons theory in the framework of the approach
discussed.

We transform the internal pieces of the circuits continuously, keeping the endpoints unmoved,
and so that one contour never intersects the other one, into a pair of non-intersecting curves on the
sphere. The sphere encircles “small enough” region wherein means that the described procedure may
be completed. Consider a part of linking-number integral (4.7) over the internal pieces of the circuits.
An important observation is that this integral may be interpreted as an integral over an unclosed
circuit on the selected sphere of a complex function, holomorphic on the sphere without two points.
Indeed, an integral in a single-integral form of (4.7) may be taken not only over the second circuit, but
over an arbitrary curve homotopic to it (i.e., may be continuously transformed in the second circuit
never intersecting the first circuit). The same applies to the integral over any piece of the circuit,
as long as the transformation keeps the endpoints unmoved; this is guaranteed by the magnetic field
being rotor-free in a current-free area. Using this fact, transform one of the circuits, continuously and
with the endpoints unmoved, into an unclosed curve \(\gamma_{AB}\) on the sphere \(S\) connecting the endpoints...
A and B of the circuits internal piece. The “internal” part of (4.7) then rewrites as

$$\Gamma_{12}^{AB} = \int_{\gamma_{AB} \subset S} d\vec{l} \cdot \vec{H}^1(\vec{l}).$$

(4.11)

Note that it is still the magnetic field $\vec{H}^2(x)$ of the whole first circuit that enters (4.11), not the magnetic field of its internal piece! Only the tangent to the sphere component contributes to (4.11) the selected integration circuit. On the other hand, a circulation of this field along an arbitrary closed curve on the sphere equals to the circulation of its tangent component; this circulation must be zero unless the curve encircles one of the first circuit endpoints. The integrand in (4.11) is then a component of the first circuit magnetic field $H^2$ tangent to the selected sphere, which is a smooth tangent rotor-free vector field on a sphere without two points; moreover, the circulation of the field over the curve the a thrown away point inside is known to be $4\pi I_1$. It is known from the complex analysis that such a vector field may be presented as a complex function holomorphic on the Riemann sphere everywhere except for two points, where it has plain poles with the certain residues (equal to $4\pi I_1$). Indeed, for

$$\vec{H} \equiv (H_R, H_\theta, H_\phi), \ z \equiv \theta - i\phi, \ H_z \equiv H_\theta + iH_\phi,$$

(4.12)

one has

$$\oint_{\gamma \subset S} d\vec{l} \cdot \vec{H} = \int_{\cup \gamma_\theta} d\phi H_\phi + \int_{\cup \gamma_\theta} d\theta H_\theta \equiv \oint_{\gamma} dz H_z,$$

(4.13)

where and $\cup \gamma_\phi$ and $\cup \gamma_\theta$ denote a partition of the closed circuit $\gamma$ into parts, where, correspondingly, $\theta$ and $\phi$ may be used as integration variables.

What was said above is summarized in the following formulas:

$$\Gamma_{AB}^{CD} = \int_{\gamma_{AB} \sim \gamma_{CD}} dz H^C_D(z),$$

(4.14)

$$\oint_{\gamma = \partial G} dz H^C_D(z) = 4\pi I \delta_{C \wedge D \in G}$$

(4.15)

A form of the function $H_z$ is strongly restricted by (4.14)(4.15); according to the Riemann-Roch theorem [6], the only solution to this constrain is

$$H^C_D = 2I \left\{ \frac{1}{z - d} + \frac{1}{z - c} \right\},$$

(4.16)

where $c$ and $d$ are the values of $z$ at the first circuits points $C$ and $D$, correspondingly. This solution is defined up to addition of an analytic (of $z$ but not of $\bar{z}$) function, which has no plain poles or branching points; in particular, it may have singularities of other type, say, second-order poles. In other words, any function of type

$$f(z) = \sum_{k=2}^{\infty} \sum_{i=1}^{\infty} \frac{c_{i,k}}{(z - a_i)^k} + b_0 + \sum_{k=1}^{\infty} b_k z^k$$

(4.17)

may be added to (4.16). The grate arbitrariness reflects the dependence of the tangent magnetic field on the shape of the first circuit; only the the plane-pole contribution (4.16) is fully defined by the penetration points positions and by the magnitude of current together with its direction near these points.

Substituting (4.16) into (4.14), one must specify an integration circuit. A pair of circuits with the $a$ and $b$ as the endpoints give the same value of the integral as long as the loop they enclose encircles either non of the penetration points $c$ and $d$, or both of them (these two cases are equivalent
on a sphere). A proper circuit is the one homotopic to the original placement of the second circuit. To impose this constrain without referring to the three-dimensional picture, one should make a cut connecting the points \(c\) and \(d\). The selection rule is then to take a circuit either intersecting the cut or not, depending on what the second circuit’s image does. After the integral was taken, one obtains that

\[
\Gamma_{AB}^{CD} = 2 \log \frac{(a-c)(b-d)}{(a-d)(b-c)}, \tag{4.18}
\]

with \(a\) and \(b\) being the values of \(z\) at the second circuits endpoints \(A\) and \(B\), correspondingly. The answer is a multivalued complex function \([6, 9]\). Treated as a function, of the variable \(a\) of \(b\), \(\Gamma_{AB}^{CD}\) may be analytically continued everywhere in Riemann sphere with a cut from \(c\) to \(d\) (such a one was already made). When analytically continued along a closed circuit encircling just one of the points \(c\) and \(d\), and thus intersecting the cut, the function acquires the corresponding increment,

\[
\log(e^{2\pi z}) = \int_{\text{arg} w=0}^{\text{arg} w=2\pi} \frac{dw}{w} = 2\pi i. \tag{4.19}
\]

If one treats \(\Gamma_{AB}^{CD}\) as a function of the variable \(c\) or \(d\) instead, all is the same provided that the integration circuit is seen as a cut from \(a\) to \(b\).

As an output of this step, a “small enough” (see the definition above) piece of a link is substituted by a sphere with four marked points, pairwise connected by the two cuts, with a logarithmic function on it. The rest of the link is attached to the marked points.

### 4.3.2 Cutting and gluing of a link

**Item 7** is intended to match the *implicitly* defined tangent fields with the *implicit* definition of a link invariant. It is hence not needed in the our attempt of constructing the fields *explicitly*. We will return to the item after getting the result, carrying out the supposed procedure as a consistency check. Now we are to follow **item 8**, which consists in “gluing” the introduced elementary pieces of a link, i.e. in assembling the associated quantities into the link invariant.

In the case considered the Wilson average is merely the exponentiated linking number, so that all the contributions of elementary pieces are just complex functions, whose product yields a the sought quantity. The problem yet turns out far non-trivial, once the contributions of separate pieces are given as in sec 4.3.1 i.e., purely in terms of the surface data. Nevertheless, it is a such kind of gluing that is impled in \([69]\). Hence, we are to try this way in our toy model. Since there is no much difference in the Abelian case, we will calculate just a linking number (not its exponent) for simplicity, keeping in mind that the elementary contributions must be summed up.

Attempting to “glue” two contributions associated with two elementary surfaces as in sec 4.3.1 one encounters at least two major problems. First, the marked points (the penetration points and the endpoints of an integration circuit) enter \(4.18\) via their coordinates, somehow introduced on the sphere. Second, \(4.18\) is defined only up to and integral of \(4.17\), which is a function of type \(F(a) - F(b)\) for \(F(z)\) being an arbitrary function of the marked points coordinates, analytic and uniquely defined w.r.t. to each one (no branching points), but possibly having any number of point-like singularities of any other kind. Of course, \(4.14\) is insensitive to a *globally* defined \(4.17\)-type summand (which corresponds to shifting the magnetic field in \(4.7\) by a circulation-free magnetic field), but at the level of solving \(4.15\), such an addition is permitted for each sphere *separately*. All these problems are indeed essential for a determination of a link invariant in the framework of the three-dimensional construction discussed. In the present text, we restrict ourselves by studying the phenomenon for our toy model in several simplest cases, sending the reader to works \([82, 81]\), \([52]-[54], [84], [85]\), and references therein for a generic construction.
A seeming paradox. In the rest of the section we comment a bit more on the gluing problem, demonstrating that it is indeed a very subtle point. We are going to illustrate a failure of a possible naive picture and to reveal the complexity of the actual one. It suffices treat a quite simple case. Consider a circle and a sphere, penetrated by the circle in two points. Then, cut the obtained solid sphere into two solid half-spheres with a disc bordered at the equator; the “angles” of the half-spheres may be done as smooth as one wishes. The circle penetrates now the upper half-sphere at two points $A$ and $B$ and the lower half-sphere at four points $A'$, $B'$, $A''$, $B''$. Suppose that we have somehow defined the tangent magnetic field of this circle on the sphere; on the first half-sphere it has the form

$$
\mathcal{H}_1 = \frac{I_a}{z-a} + \frac{I_b}{z-b} - \frac{I_a}{z-c_1} - \frac{I}{z-d_1} \text{ no-pole part},
$$

and on the second one

$$
\mathcal{H}_2 = -\frac{I_a}{z-a} - \frac{I_b}{z-b} + \frac{I_a}{z-c_2} + \frac{I}{z-d_2} \text{ no-pole part},
$$

where $u$ and $v$ are holomorphic coordinates on the corresponding spheres, and $u(A) = a_u$, $u(B) = b_u$, $v(A) = a_v$, $v(B) = b_v$, $v(A') = v(A) = b_v$, $v(A') = a'_v$ are positions of the corresponding penetration points in these coordinates. On the disc these fields must coincide as vector fields; in some coordinate $z$ they have the form

$$
\mathcal{H} = \frac{I_a}{z-a} + \frac{I_b}{z-b} - \frac{I_a}{z-c_1} - \frac{I}{z-d_1} \text{ no-pole part}.
$$

Now we run into a contradiction. Assume that, first, both functions (4.20, 4.21) are everywhere analytic, i.e., a value $\mathcal{H}_1(u)$ or $\mathcal{H}_2(v)$ at any point of the corresponding half-sphere is obtained as a sum of the Laurent series, calculated in any other point, and, second, there are everywhere analytic change of variables $v = v(z)$, $u = u(z)$, then the functions $\mathcal{H}_1(v(z))$ and $\mathcal{H}_2(u(z))$ must coincide for all values of $z$. Yet, they can not, since $\mathcal{H}_1(v(z))$ has two poles, while $\mathcal{H}_2(v(z))$ has four ones.

The source of the contradiction is, of course, in the analyticity assumptions we made. If the second of them (about change of variables) is indeed valid in a large number of cases (see [6] for the corresponding examples and theorems), the first one turns out to be a fatal error. This is easily demonstrated with the following text-book exercise.

Failure of a naive speculation: an example. Consider a curve $\gamma$ given by the formula ($n$ is a positive integer)

$$
\gamma : \begin{cases}
  x^{\frac{1}{n}} + y^{\frac{1}{n}} = 1, & 0 \leq x \leq 1, \\
  y = 0, & 1 \leq x
\end{cases},
$$

and a constant vector field $\vec{\xi} = (1, 0)$. A number of derivatives at the stapling point equals $n - 1$ and may be set as large as possible. Introduce a function $\phi$ of the curve $\phi(x, y) = \xi \cdot \vec{\tau}(x, y)$ for $(x, y) \in \gamma$, where $\vec{\tau}$ is the unit tangent vector. The function is given then by an explicit expression

$$
\phi(x(t), y(t)) = \tau_{\gamma,x} = \frac{1}{\sqrt{1 + (y')^2}} = \begin{cases}
  \frac{1}{\sqrt{1 + t^2}}, & t \leq 0, \\
  0, & 0 \leq t
\end{cases},
$$

where we introduced a coordinate on the curve as $t = -\frac{x}{y}$ for $0 \leq x \leq 1$ and $t = x$ for $1 \leq x$, and used the differential form of (4.23) for $x \leq 1$, $\frac{dx}{x} + \frac{dy}{y} = 0$, to express $y'$ via $\tau$. Though the vector field $\vec{\xi}$ is everywhere analytic, i.e., equals to the sum of it’s series consisting of the single term, the function $\phi(t)$ is analytic just on each of the segments $0 \leq t$ and $t \leq 0$ separately, since the series

$$
1 + \sum_{k=1}^{\infty} \frac{(2k - 1)!!}{k!} (-1)^k t^{2k} = 1 - \frac{1}{2} t^2 + \frac{3}{8} t^4 - \frac{5}{16} t^6 + \ldots,
$$

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converge to $\phi(t)$ only for $0 < t$ and $-1 \leq t < 0$, respectively, i.e. up to the stapling point, never mind how smooth the curve behaves there. A similar phenomenon may occur even for an infinitely smooth stapling; indeed, the same vector projected field on the curve $\tilde{\gamma}$,

$$\tilde{\gamma} : \left\{ \begin{array}{ll} y = e^{-\frac{x^2}{2}}, & x \leq 0 \\ 0, & 0 \leq x \end{array} \right. ,$$

(4.26)
gives rise the function $\tilde{\phi}(x) = \tilde{\phi}\left(e^{-\frac{x^2}{2}}\right)$, whose series is identically zero at the stapling point $x = 0$.

**Summary.** A similar phenomenon occurs in a generic case; a projection of a everywhere analytic vector field on a surface may be everywhere analytic on the surface only if the entire surface is given by an analytic function [6]. A textbook complex analysis thus crashes a naive interpretation of item 8 from sec.4.1, requiring for a serious revision of the implied procedure. At least two ways come immediately.

First, one wish the tangent vectors fields be not just everywhere holomorphic but everywhere analytic. A shape of surface is strongly restricted then, e.g., a geometric (not just topological) sphere should be considered. At least two problems arise then. First, a link may not be cut, in general, into a collection of discs with desired properties (see sec.4.3.1) by a geometric spheres only, even in the plain three-dimensional space. Second (as already demonstrated), two geometric spheres, as well as any two everywhere analytic surfaces, are not required to have a common (two- or one-dimensional) areas. As a result, one has loses an obvious way of gluing the solutions for the tangent field, obtained on different surfaces.

To avoid the gluing problem, one may solve the equation (4.3) head-on, in the entire space at once, and then project the solution on the chosen surfaces; one obtains the desired collection of surfaces, each one being enabled with a collection of marked points pairwise connected with circuits and with a meromorphic vector field (no matter, where is also analytic, once it is already determined). Starting from this point, an approach sketched in sec.4.1 proceeds working; an operator, which is an integral of the vector field over a suitable circuit, corresponds to each sphere, and in case of the linking number their contributions are merely summed up. A very complexity of this approach already in the simplest example, which is presented in the next section, explains, why all more or less explicit constructions prefer another approach.

The second way consists in permitting non-analytic surfaces, but deforming the link in such a way all the surfaces contain just closed circuits (in fact, “almost” closed). Only the pole part of the field giving function contributes then to the integral (4.11), and one is not required to take care about the no-pole part; apart from the real magnetic field of the second circuit, any holomorphic (analytic or not) function with all the poles at the penetration points passes. To “glue” the obtained answers, one just substitutes the positions of poles by their global coordinates. A completely unambiguous and very practical method developed in [81],[82],[52]–[61],[84],[85] is based on the described idea. In fact, the approach returns us to the two-dimensional description; a link is again reduced to a planar graph, with the so selected spheres as vertex operators.

### 4.3.3 Head-on presentation of the construction for a Hopf link

The last thing we would like to do in the present section is writing out explicitly all elements of the toy model under discussion for in a simple case, namely, for a Hopf link of a specific shape, see fig.10. We rely on the explained in sec.4.2 magnetostatic analogy to draw on a physical intuition.

The field produced by the first circuit in the space The most straightforward way to carry out of sec.4.3.1’s programm starts from explicit determining the first contour’s magnetic field. The ring-shaped circuit produces the field that is a solution of the static Maxwell’s fourth equation

$$\text{rot } \vec{H}(\vec{r}) = 4\pi I \delta(z) \delta(r - a),$$

(4.27)
with vanishing of the field at the infinity as a boundary condition. Here, where \( a \) is the ring’s radius, \( I \) is the flowing current, and the center of the polar coordinates coincides with the center of the ring. One may write down the solution explicitly, making use of the axial symmetry. Indeed, vanishing of the axial component \( H_\phi \) and the angular independence of the remaining two ones, \( H_z = H_z(z, r) \) and \( H_r = H_r(z, r) \), follow from the rotational symmetry and from a magnetic induction being an axial vector; after that, two out of three equations in (4.27) are satisfied trivially:

\[
\partial_z H_\phi - \partial_\phi H_z = \partial_r H_\phi - \partial_\phi H_r.
\] (4.28)

The two remaining components, \( H_z \) and \( H_r \) satisfy the equation

\[
\partial_z H_r - \partial_r H_z = 4\pi \delta(z) \delta(r - a),
\] (4.29)

which is formally equivalent to

\[
\partial_z H_x - \partial_x H_z = 4\pi \delta(z) \left( \delta(x - a) + \delta(x + a) \right),
\] (4.30)

in every plane passing through the ring’s diameter orthogonally to the ring’s plane. Thanks to the linearity of (4.30) (the superposition principle), its solution is a sum of two planar rotor Green functions (magnetic fields of an infinite long wires), \( G(x - a) \) and \( G(x + a) \),

\[
\vec{H} = (H_x, H_r, H_\phi) = \left( \frac{r - a}{z^2 + (r - a)^2} + \frac{r - a}{z^2 + (r + a)^2}, \frac{z}{z^2 + (r - a)^2} - \frac{z}{z^2 + (r + a)^2}, 0 \right).
\] (4.31)

**Tangent fields.** Following the algorithm of sec.4.3.1 we select now two non-intersecting spheres of a radius \( b \) with the centers on the ring (fig.10); we will suppose the spheres to be “small enough” whenever necessary. It is the most easy to visualize tangent fields on the spheres, if one introduces local coordinates on spheres as follows (fig.12; arcs \( AA'_\perp \) and \( BB'_\perp \) are projections of integration contour on the figure’s plane). A plane of a constant polar angle intersects a sphere by a circle. An angle \( \alpha \) that labels a point on the circle together with the polar angle \( \phi \), which labels the circle, may be chosen as local coordinates on a sphere. For convenience, we will use a *relative* polar angle \( \varphi \), i.e., the one measured from the center of a sphere (\( \varphi = \phi - \phi_0 \)), as a local coordinate on the sphere. Since the circles \( \varphi = const \) lie in the plane of the system’s symmetry, as well as the field \( H \), the projection of the field on the sphere is a tangent field to the circles. In global coordinates, a circle \( \varphi = const \) is given by the equation

\[
\gamma_\varphi : \quad \phi = const, \quad (r - q_\varphi)^2 + z^2 = \rho_\varphi^2,
\] (4.32)

where parameters \( q \) and \( \rho \) are expressed via the polar angle \( \varphi \) (we omit the subscript in what follows) and parameters of the configuration as (see fig.11).

\[
\rho_\varphi = (b^2 \cos^2 \varphi - a^2 \sin^2 \varphi) \cos^2 \varphi, \quad q = a \cos^2 \varphi, \quad -\pi \leq \alpha \leq \pi.
\] (4.33)

The global coordinates are then expressed via the local ones as

\[
r = q + \rho \cos \alpha, \quad z = \rho \sin \alpha, \quad -\frac{b}{a} \leq \sin \varphi \leq \frac{b}{a},
\] (4.34)

so that the one-form to be integrated becomes in the local coordinates

\[
d\vec{l} \cdot \vec{H}^\pm = \frac{\rho \sin^2 \alpha + (\rho \cos \alpha + c) \cos \alpha}{(\rho \cos \alpha + c)^2 + \rho^2 \sin^2 \alpha} \rho d\alpha = \frac{\rho + c \cos \alpha}{\rho^2 + c^2 + 2pc \cos \alpha} \rho d\alpha.
\] (4.35)
Note that the ring may pass inside as well as outside of the sphere depending on value of $\varphi$ (see fig.11).

As a consistency check, one may evaluate the contour integral $\oint d\bar{l} \cdot \vec{H}^\pm,$

$$\int_0^{\pi} d\alpha \frac{d\vec{H}^\pm}{d\alpha} = \int_\infty (P + Q)(Pu^2 + Q) \frac{du}{(1 + u^2)(P^2u^2 + Q^2)} = \int_\infty \left( \frac{1}{1 + u^2} - \frac{P^2}{P^2u^2 + Q^2} \right) \frac{(Pu^2 + Q)}{Q - P} du =$$

$$= \int_{-\infty}^\infty \left( \frac{PQ}{P^2u^2 + Q^2} - \frac{1}{1 + u^2} \right) du = (\operatorname{sgn}(PQ) - 1)\pi, \quad (4.36)$$

where we introduced the parameters (we omit the superscripts the above formula, implying that one takes parameters with plus and minus for $H^+$ and $H^-$, respectively)

$$c^\pm = q \pm a, \quad P^\pm = \rho - c^\pm = \rho - q \mp a, \quad Q^\pm = \rho + c^\pm = \rho + q \pm a \quad (4.37)$$

and we made the standard change of variables $u = \tan \frac{\alpha}{2}$, using that

$$\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1, \quad \frac{1}{\cos^2 \frac{\alpha}{2}} = 1 + \tan^2 \frac{\alpha}{2}, \quad d\tan \frac{\alpha}{2} = \frac{d\alpha}{2} \cos^2 \frac{\alpha}{2}. \quad (4.38)$$

The quantity $PQ = \rho^2 - (q \pm a)^2$ is positive or negative if the point $(0, \pm a)$ is placed inside or outside the circle $z^2 + (q - r)^2 = \rho^2$, respectively. Hence, $\oint d\bar{l} \cdot \vec{H}^+ = 0$, and $\oint d\bar{l} \cdot \vec{H}^-$ is either 0 or $2\pi$, depending on whether the integration contour is intertwined with the ring, just as required by the circulation theorem.

**Elementary contributions.** Suppose now that the second contour is partially placed on the spheres, being parallel to the ring elsewhere in the space, as in fig.10. Only the pieces of the contour that lie on the sphere contribute then to the linking-number integral. The program of sec.4.3.1 supposes now to associate with each sphere its elementary contribution, which is given by an integral of the tangent field over an unclosed contour,

$$\Omega_{AB} \equiv \int_A^B d\bar{l} \cdot \vec{H}. \quad (4.39)$$

The value of the quantity depends on the integration contour, since the corresponding integral over the closed loop equals not zero but quantity $(4.36)$. If the corresponding arc penetrates the ring’s plane in the ring’s interior, then

$$\Omega^+_{AB} \equiv \int_{\gamma^+_{AB}} \vec{H} \cdot d\bar{l} = \int_{\alpha_1}^{\alpha_2} d\alpha \frac{\vec{H} \cdot d\bar{l}}{d\alpha} = \arctan \frac{Pu}{Q} - \arctan u \begin{array}{l} u \left|_{u_B}^{u_A} \right. \end{array} = \frac{\alpha}{2} - \arctan \frac{z}{r} - \frac{\alpha}{2}, \quad (4.40)$$

If the penetration point is in the ring’s exterior instead, then

$$\Omega^-_{AB} \equiv \int_{\gamma^-_{AB}} \vec{H} \cdot d\bar{l} = \int_{-\pi}^{\alpha_1} d\alpha \frac{\vec{H} \cdot d\bar{l}}{d\alpha} + \int_{\alpha_2}^{\pi} d\alpha \frac{\vec{H} \cdot d\bar{l}}{d\alpha} = 2\pi - \Omega^+_{AB}, \quad (4.41)$$

where we supposed for simplicity that $PQ > 0$. Hence, the “elementary contribution” $\Omega_{AB}$ depends on the global coordinates of points $A$ and $B$ as

$$\Omega^s_{AB} = \arctan \frac{z_A}{r_A} - \arctan \frac{z_B}{r_B} + 2\pi \epsilon_s, \quad (4.42)$$

where $s = \pm$, $\epsilon_+ = 0$, and $\epsilon_- = 1$. Now notice, that if the points $A$ and $A'$ lying on the different spheres are connected by an arc that is parallel the ring, whose form is given by the equations $z = 0$, $PQ = 0,$
then coordinates \( z \) and \( r \) of the points coincide, i.e., \( z_A = z_{A'} \), \( r_A = r_{A'} \). The same is true for the second pair of points, \( B \) and \( B' \). Since the “elementary contribution” \( \Omega_{AB} \) depends only on coordinates of \( r \) and \( z \) of points \( A \) and \( B \), and on the discreet parameter \( \epsilon \), which is responsible for selecting an integration contour, then the entire configuration is associated with the quantity

\[
\Omega_{AB} + \Omega_{B'A'} = \Omega_{AB} - \Omega_{AB}' = 2\pi\delta_{s,s'} ,
\]

where \( \delta_{s,s'} = 1 \) for \( s = s' \) (both + or both −), and \( \delta_{s,s'} = 0 \) otherwise. In the first case the contour is linked with the ring, in the second case it does not. Hence, the entire construction under consideration at least reproduces the correct answer for the case.

5 Perturbative evaluation of Wilson average and expansion of HOM-FLY polynomial

In the present section we proceed with the discussion on how the considered knot invariants may be related to QFT observables. This time, we consider a perturbative evaluation of the CS Wilson average, carrying out the computation explicitly in the simplest examples. In details it’s simplest steps. Unlike the previous section, we consider a non-Abelian theory as well as the Abelian one. We would send the reader to papers \[82, 73, 74, 81, 75, 76\] for the systematic treatment of the subject.

5.1 Linking number as a perturbative Wilson average

We start from presenting a perturbative QFT interpretations of the simplest link invariant, which is the linking number. As already said in sec.4 the invariant is associated with the Abelian Chern-Simons theory, with a quadratic action. As a definition of a Gaussian functional integral one may take, apart from (4.10), the rule

\[
\langle A_i(x)A_j(y) \rangle \equiv G_{ij}(x-y),
\]

which extends a similar formula for finite-dimensional integration [10],

\[
\langle A_i A_j \rangle = \frac{\int \prod_{i=1}^{N} dA_i A_k A_l \exp(-\frac{1}{2} \sum_{i,j=1}^{N} K_{ij} A_i A_l)}{\int \prod_{i=1}^{N} dA_i \exp(-\frac{1}{2} \sum_{i,j=1}^{N} K_{ij} A_i A_l)} = K_{ij}^{-1}.
\]

In (5.1), \( G(x - y) \) stands for the Green function, which is seen as the inverse of the kinetic operator (see explicit formulas for the Abelian Chern-Simons in sec.5.1.1). A definition of the Green function is ambiguous unless the boundary conditions are specified; this reflects a similar ambiguity in definition of the functional integral. Formula (5.1) should be coupled with the Wick theorem [10], which claims that the average of even number of multiplied fields \( \langle A(x_1)A(x_2) \ldots A(x_{2k}) \rangle \) equals to a sum over all pairings, i.e. products \( \prod_{i,j} \langle A(x_i)A(x_j) \rangle \) where each \( x \) enters once. The last thing to require is the vanishing of the average of an odd number of multiplied fields, \( \langle A(x_1)A(x_2) \ldots A(x_{2k-1}) \rangle = 0 \). A similar finite-dimensional average is identically zero from symmetry considerations, as contraction of an odd function with an even one: this is not so trivial in an infinite-dimensional case, since it is impossible in many cases to define a functional integral that enjoys all the seemingly present symmetries.

The earlier presented definition of a Gaussian functional integral via (4.10) is equivalent to the definition via (5.1), the Wick theorem and vanishing of odd averages [10], but the latter one turns to be much more adequate for practical purposes.
5.1.1 Second order

For the sake of convenience, we write down once again the action of the Abelian Chern-Simons theory
\[ S_{CS}^{Abelian} = \frac{\kappa}{4\pi} \int d^3x e^{kij} A_k \partial_i A_j. \] (5.3)

Following [74], we rescaled the Chern-Simons field as \( A \rightarrow gA \), compared to sec.4.2 and [69, 73], so that the Wilson loop
\[ W(C) = \exp \left\{ g \int dx^i A_i \right\}. \] (5.4)

includes explicitly the constant \( g \equiv \sqrt{\frac{4\pi}{\kappa}} \), which may be used as a formal expansion parameter. We assume the same gauge fixing (\( \partial_k A^k = 0 \)), and the same boundary conditions (regularity and vanishing at the infinity in the source-free area) as in sec.4.2.

The exponent expands in perturbation theory series as
\[ \langle W(C) \rangle = (1) + g \left\langle \int dx^i A_i \right\rangle + g^2 \left\langle \left( \int dx^i A_i \right)^2 \right\rangle + \ldots, \] (5.5)

The \( O(1) \) term by definition equals one, the term \( O(g^2) \) vanishes again by definition, as an average of the odd function, and the first non-trivial \( O(g^2) \) terms is singular. Indeed, the second order term of (5.5) is, according to (5.1),
\[ \left\langle \int_C dx^i \int_C dy^j A_i(x)A_j(y) \right\rangle = \int_C dx^i \int_C dy^j \langle A_i(x)A_j(y) \rangle = \int_C dx^i \int_C dy^j G_{ij}(z). \] (5.6)

A double integral of a Green function typically appears divergent, and so it does in the case. To verify that, let us write down the integrand explicitly.

The Green function by definition appears in the integral form of the classical equations of motions,
\[ \epsilon^{kji} \partial_j A_i(x) = 4\pi J^k(x) + \text{boundary conditions} \] (5.7)
\[ \Downarrow \]
\[ A_i(x) = \int d^3y G_{ij}(x - y) J_j(y) = \int d^3y J^j(y) G_{ji}(y - x). \] (5.8)

The gauge and boundary conditions hold for \( A_i(x) \), if \( G_{ij}(x - y) \) satisfy similar conditions, i.e., \( \partial^j G_{ij}(x - y) = \partial^j G_{ij}(x - y) = 0 \), and \( G_{ij} \) is regular in the whole space but the field sources and vanishes at the infinity. To derive an explicit expression for \( G_{ij}(x - y) \), one may rewrite (5.7) as \( \partial_i A_j - \partial_j A_i = 4\pi \epsilon_{ijk} J^k \) and take the divergency of both parts to obtain \( \partial^2 A_i = 4\pi \epsilon_{ijk} \partial^j J^k \) provided that \( \partial_i A^i = 0 \). The same follows from (5.8) if the Green function satisfy \( \partial^2 G_{ij}(x - y) = 4\pi \epsilon_{ijk} \delta^k \delta(x - y) \) \( (J^k(y) \) is unaffected by \( \partial^2 \) since \( \partial \equiv \frac{\partial}{dx} \). The latter condition together with the gauge and boundary conditions define \( G_{ij}(x - y) \) unambiguously, hence the solution
\[ G_{ij}(x - y) = 4\pi \epsilon_{ijk} \partial^k \frac{1}{|x - y|}, \] (5.9)

which satisfies all those conditions, should satisfy (5.8) as well. Note that the Green function is not required to be a solution of the classical equation with the delta-function in the r.h.s.; definition (5.8) requires only that
\[ \epsilon^{kji} \partial_j G_{il}(x - y) = 4\pi \delta^k_l \delta(x - y) + \partial^k f_l \] (5.10)
for a function \( f \) vanishing fast enough, since \( \partial_k J^k = 0 \) due to the classical equations of motion.

Substitution of (5.9) to (5.6) indeed yields a divergent integral. The singularity we come across is conventionally resolved by shifting the second integration contour relative to the first (and the original) one [82].

\[
\oint_C dy^i \rightarrow \oint_{C'} dy^i. \tag{5.11}
\]

The maximum (or somehow averaged) distance \( \varepsilon \) between the contours \( C \) and \( C' \) is a regularization parameter, which will be send to zero in the end of the calculation. As usual, the answer does not have a definite limit as \( \varepsilon \) tends to 0; instead, all possible values of this limit are parameterized by a new independent variable. This new variable usually is treated as an energy scale in case of an UV regularization of a perturbative QFT. The regularization parameter in the problem considered has its own meaning, which, in particular, inspires viewing on certain knot invariants as QFT observables.

To make a precise statement, let us note that (5.6) with the Green function from (5.9) and regularized as (5.11) takes the explicit form

\[
\oint_{C'} dy^i \oint_C dx^j \varepsilon_{ijk} \frac{(x^k - y^k)}{|\vec{x} - \vec{y}|^3}. \tag{5.12}
\]

The obtained expression is an integral expression for the linking number \( L(C', C) \) of the original and shifted contours \( C \) and \( C' \) [3]. Recall that we have already obtained the result while discussing the magnetostatic analogy in sec.4.2, where the same integral appeared in form (4.7) and was evaluated by the circulation theorem. Integral (5.12) equals \( L(C', C) \) independently of \( \varepsilon \). Hence, its value in the limit \( \varepsilon \rightarrow 0 \), or \( C' \rightarrow C \), is defined up to an arbitrary integer number \( L(C', C) \). A possible interpretation of the observed phenomenon is that CS Wilson average is related not to a closed contour, but to a closed ribbon. The quantity thus depends not only on the shape of contour, but on the number of the ribbon intertwinnings as well. As we demonstrated in sec.3.3.4, a similar property of certain knot invariants arise in the framework of their completely different definition.

As an output of the section, we have the following expansion for a framed Wilson average:

\[
\langle W(C, C') \rangle = 1 + gL(C, C') + O(g^2). \tag{5.13}
\]

Some steps in derivation of (5.13) indeed may be carried out, the remaining ones are merely taken as a definition.

5.1.2 Higher orders

At a first glance, it seems that taking higher orders into account requires for introducing more and more new contours, \( C'', C''', \ldots \). Actually, the situation is different. According to the Wick theorem, which is included in the definition of functional integral [10], each order of perturbation series for (5.5) equals to a sum over pairings, i.e., products of pairwise averages. E.g., the fourth order term in
pairings, each one being equal to $n_k$ all terms of odd order vanish in (5.5), and each term of even order 2 $C_H$ Hence, it is enough to substitute exactly half of the contours appearing in each pairing by the contour $C'$. After that all terms of expansion (5.5) assemble into the exponent of the linking number. Indeed, all terms of odd order vanish in (5.5), and each term of even order $2k$ yields the sum over $(2k - 1)!!$ pairings, each one being equal to $n_k$. Then, in somewhat symbolic notations,

$$\langle \exp(W) \rangle = \sum_{k=0}^{\infty} \frac{(W^k)}{k!} = \sum_{k=0}^{\infty} (2k - 1)!! \frac{n_{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{n_{2k}}{2k!} = \exp \left( \frac{n^2}{2} \right).$$

(5.15)

5.2 Perturbative Wilson average in non-Abelian Chern-Simons

5.2.1 Formulating the problem

In this section, we recall some notions related to the perturbative Wilson average in non-Abelian Chern-Simons theory. A more accurate and detailed description may be found, e.g., in [73].

Non-Abelian Chern-Simons action. A non-Abelian Chern-Simons theory in three-dimensional euclidean space is given by the action

$$S = \int d^3x \text{Tr}_{\text{adj}} \left\{ \epsilon^{ijk} (A_i \partial_j A_k + \frac{2g}{3} A_i A_j A_k) \right\},$$

(5.16)

where the generators $T$ of the algebra $\mathfrak{g}$, related to a gauge group $G$, are taken in the adjoint representation, and we use the same normalization for the Chern-Simons field as in sec. 5.1

Wilson lines. Similarly to Abelian theory, one introduces a Wilson line. However, definition [4.8] is ambiguous in the case, since Chern-Simons fields at different space points no longer commute with each other. The correction goes down to substituting a plain exponent by a path ordered exponential, which is defined as a formal series, where the fields in each term are ordered along the path,

$$U_\gamma(A) = \exp \left( g \int_{\gamma} dx_i A_i \right) \equiv 1 + \sum_{k=0}^{\infty} \frac{g^k}{k!} \int_{\gamma} dx_k \ldots \int_{\gamma} dx_1 A_k(x_k) \ldots A_i(x_1) =$$

$$= 1 + \sum_{k=0}^{\infty} g^k \int_{\gamma} dx_k \int_{\gamma} dx_1 \ldots \int_{\gamma} dx_{k-1} A_k(x_k) \ldots A_i(x_2) A_i(x_1),$$

(5.17)
As a result, even the Gaussian average \( \langle \dots \rangle \) goes not very far due to the numerous technical difficulties, see, e.g., [74, 75].

Sec. 5.1, are also studied, but the attempts of explicit calculations of knot invariants with help of these gauges, among them the covariant Lorentz gauge \( \partial_{\mu}A_{\mu} = 0 \), considered in [73] and in fact used in sec. 5.1, are also studied, but the attempts of explicit calculations of knot invariants with help of these gauges go not go very far due to the numerous technical difficulties, see, e.g., [74, 75].

Regularization and framing. Unlike the case of the Abelian theory, action (5.16) is cubic so the functional integral is defined only as a perturbation series in the coupling parameter, and one should interpret the terms as Gaussian averages defined as in sec. 5.1. However, correlators defined in this way remain finite in each order of perturbation theory. Similarly to the Abelian case, it is the Wilson loop, which needs a regularization. We recall that an ambiguity arises from multiple integrals of the Green function, formally entering the perturbative expansion for the Wilson loop. The non-Abelian regularization of the quantity is defined, just as an Abelian one, by shifting half of the integration contours w.r.t. the original one, so that the Green function is nowhere integrated twice over the same loop (see sec. 5.3.2). One more problem arises at this point in the non-Abelian theory. To recover a path ordering in the regularized series, one has to somehow match the points of the original and shifted contours. For this reason, the non-Abelian regularization consists not just in shifting the contour, but in introducing a normal vector field on it. Merely speaking, each vector connects a point \( X \) on the original contour \( \gamma \) with its image \( X' \) on the shifted contour \( \gamma' \), and this dictates the rule of the path ordering. Introducing such a vector field on the knot is referred to framing of the knot [2]. By historical reasons, the same term is for used in [3], [64], and in sec. 3.3.4 for a certain integer number associated with a knot. These two notions are indeed related with each other. The integer number is an invariant of the vector field, namely, a number of compete turn-overs the vector does as once passing the entire knot.

Abelian gauges. As a gauge field, Chern-Simons field is defined up to a certain transformation,

\[ A_i(x) \rightarrow \Omega^{-1}(x)A_i(x)\Omega(x) - i\Omega(x)\partial_i\Omega(x), \quad \Omega \in G. \]  

As a result, even the Gaussian average \( \langle A_i(x)A_j(y) \rangle \), which by definition equals to the Green function of the kinetic operator \( \delta_{ab}\epsilon^{ijk}\partial_i \), is determined only modulo a similar transformation. To determine the Green function unambiguously, one must impose an additional condition the fields [10]. Such a procedure is called fixing a gauge. A remarkable property of the Chern-Simons theory is that action (5.16) becomes quadratic in certain gauges, which are called Abelian gauges. The most thoroughly studied Abelian gauges are the temporal gauge [71, 72], where \( A_0 = 0 \), and the light-cone gauge [76], where a certain linear combination of the field components vanishes. A particular case of a light-cone gauge, which is called a holomorphic gauge, is discussed a little in the next section. Non-Abelian gauges, among them the covariant Lorentz gauge \( \partial_{\mu}A_{\mu} = 0 \), considered in [73] and in fact used in sec. 5.1, are also studied, but the attempts of explicit calculations of knot invariants with help of these gauges go not go very far due to the numerous technical difficulties, see, e.g., [74, 75].

\[ U^{(2)}_{\gamma}(A) = U^{(2)}_{\gamma,I}(A) + U^{(2)}_{\gamma,II}(A), \quad (5.18) \]

\[ U^{(2)}_{\gamma,I}(A) = \frac{1}{2} \int_{\bar{\mathcal{D}}} dy^i \int_{\bar{\mathcal{D}}} dy^j A_j(y)A_i(x), \]

\[ U^{(2)}_{\gamma,II}(A) = \frac{1}{2} \int_{\bar{\mathcal{D}}} dy^i \int_{\bar{\mathcal{D}}} dy^j A_i(x)A_j(y) = \frac{1}{2} \int_{\bar{\mathcal{D}}} dy^i \int_{\bar{\mathcal{D}}} dy^j A_j(y)A_i(x) = U^{(2)}_I(A). \]

To construct an observable, i.e., a gauge invariant quantity, one should consider a closed contour \( \gamma \) and take the trace

\[ W_{\gamma,Q}(A) = \text{Tr}_Q \text{pexp} \left( g \oint dx^i A_i \right), \quad (5.19) \]

substituting the generators with there matrices in a representation \( Q \) of the algebra \( \mathfrak{g} \).
Form of the Gaussian perturbation series. To summarize, the perturbative series (5.19) for the Wilson average (5.19) in Abelian gauge may be interpreted according to the rules listed in sec.5.1. Modulo all of the subtleties mentioned above, it may be written as

\[ \langle W_\gamma(A) \rangle = \sum_{k=0}^{\infty} g^{2k} \text{Tr} T_{a_1} \cdots T_{a_{2k}} \sum_{\sigma} \prod_{i=1}^{k} \oint_{\gamma} dx^i \int_{\gamma'} dx^i \sigma(i) G_{a_i a_x(i)} (x_i - x_{\sigma(i)}) . \tag{5.21} \]

where the second sum is taken over all pairings \( \sigma \) of the elements \( \{i_1, \ldots, i_{2k}\} \), and the second integral is taken over the shifted contour \( \gamma' \), and the integration should be performed from the reference point to the image of the point \( x \) on \( \gamma' \), which is dictated by the choice of framing.

5.3 Konzevich integral as perturbation series in the light-cone gauge

5.3.1 Peculiarities of the holomorphic gauge

Wick rotation and complexification of the Chern-Simons field. The construction under discussion relies on the following trick [76],[78]. Let us introduce complex coordinates in the real three-dimensional space,

\[ (x, y, t) \rightarrow (x + iy, x - iy, t) \equiv (z, \bar{z}, t). \tag{5.22} \]

The components of the one form \( A_k \) in the selected complex coordinates then turn out to be

\[ A_k dx^k = A_x dz + A_z d\bar{z} + A_t dt = A_x (dx + idy) + A_z (dx - idy) + A_t dt = A_x dx + A_y dy + A_t dt \]

\[ \Rightarrow A_x = A_z + A_{\bar{z}}, \quad A_y = iA_z - iA_{\bar{z}} \Rightarrow A_z = \frac{1}{2} (A_x - iA_y), \quad A_{\bar{z}} = \frac{1}{2} (A_x + iA_y). \tag{5.23} \]

A term of order \( k \) in the perturbation series for path ordered exponential (5.21) then takes the form of a multiple integral over \((dzd\bar{z}dt)^k\) of a function depending on the three variables \((z, \bar{z}, t)\). One makes then an observation that if the components \( A_z, A_{\bar{z}}, \) and \( A_0 \) satisfy a certain condition, then the perturbative series for the Wilson average takes a very specific form. First, the action turns out to be quadratic, and one may treat it as a Gaussian average, i.e., evaluate the terms of the perturbation series for the Wilson average takes a very specific form. First, the action turns out to be quadratic, and one may treat it as a Gaussian average, i.e., evaluate the terms of the perturbation series by the Gaussian integration rules listed in sec.5.1. Second, the resulting integrands, which depends on \((z_k, \bar{z}_k, t)\) via the classical equations’ Green function, appear to be meromorphic functions of \( z \) (see (5.27) in the next paragraph). In other words, the integrand may be reduced to an expression of the form \( f_k(z) \delta^{k}(\bar{z}) \), with \( f(z) \) being a holomorphic (to be more precise, meromorphic) function of \( z \). The integral over \( dz^k \) can then be taken trivially, and contours for integrations over \( dz \) may be deformed in \((z, \bar{z})\) plane in an arbitrary smooth way. The described properties are achieved if

\[ A_{\bar{z}} = 0. \tag{5.24} \]

Of course, this condition is never satisfied for real-valued \((A_x, A_y, A_z)\). Conversely, if \((A_x, A_y, A_z)\) take arbitrary complex values, (5.24) may be imposed as a gauge-fixing condition, but after that \( A \) has \( (3 - 1) \cdot 2 = 4 \) independent real components, instead of desired 2. One says then, that \((A_x, A_y, A_z)\) take values in a certain three-dimensional subspace in the entire six-dimensional complex space, which does not coincide with the real subspace, but admits imposing (5.24) and thus allows reducing the number of independent components of the field by one instead.
Form of the Green function. Substituting (5.24) into (5.16), one obtains
\[ A_0 (\partial_x A_y - \partial_y A_x) + A_x (\partial_y A_0 - \partial_0 A_y) + A_y (\partial_0 A_x - \partial_x A_0) + \]
\[ + \frac{2}{3} \left( A_0 (A_x A_y - A_y A_x) + A_x (A_y A_0 - A_0 A_y) + A_y (A_0 A_x - A_x A_0) \right) = \]
\[ = A_0 (-i \partial_x A_x - \partial_y A_y) + A_x (\partial_y A_0 + i \partial_0 A_y) - i A_x (\partial_0 A_x - \partial_x A_0) + \]
\[ + \frac{2}{3} \left( -i A_x (A_x A_0 - A_0 A_x) - i A_x (A_0 A_x - A_x A_0) \right) = \]
\[ = -A_0 \partial_x A_x + A_x \partial_x A_0 = -A_0 \partial_x A_x + A_x \partial_x A_0, \]  
(5.25)
i.e., first, the cubic term vanishes as expected, second, all derivatives w.r.t. \( \bar{z} \) vanish from the answer. The classical equations of motion then become
\[ \partial_z A_z(z, \bar{z}, t) = 0, \quad \partial_z A_0(z, \bar{z}, t) = 0, \]  
(5.26)
and the Green function (under the corresponding boundary conditions) is
\[ G_{\bar{z}0} = G_{0\bar{z}} = G_{zz} = G_{\bar{z}\bar{z}} = 0, \]
\[ G_{00} = G_{zz} = G_{\bar{z}0} = 0, G_{0\bar{z}}(t, z) = -G_{\bar{z}0}(t, \bar{z}) = \frac{\delta(t)}{z}. \]  
(5.27)
Indeed, one may verify that
\[ \partial_z G_{0z}(t, z, \bar{z}) = \delta(t) \delta(z) \delta(\bar{z}), \]  
(5.28)
either using the definition
\[ \partial_z f(z, \bar{z}) \equiv \lim_{|C| \to 0} \frac{1}{|C|} \oint_{z \in C} f(z, \bar{z}) \, dz, \]  
(5.29)
or passing to the real vector fields:
\[ \partial_z u \equiv (\partial_x + i \partial_y)(u_x - i u_y) = \text{div} \bar{u} + i \text{rot} \bar{u}, \]
\[ \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - i y}{x^2 + y^2} = \frac{\bar{r}}{r^2} \]
\[ \Gamma_{\partial G} \frac{\bar{r}}{r^2} = 0, \quad \Phi_{\partial G} \frac{\bar{r}}{r^2} = \delta_0 \in G. \]  
(5.30)
In addition, the Green function is identity in the group indices since (5.26) does not contain them.

Vanishing of the vertical arcs contributions. In the holomorphic gauge
\[ A_\bar{z} = 0, \quad A_\mu dx^\mu = A_\bar{z} dz + A_0 dt = (A_\bar{z} \dot{z} + A_0) dt, \]  
(5.31)
and, as we verified, the only non-vanishing component in Green function is \( G_{0\bar{z}} = -G_{\bar{z}0}. \) Therefore, all non-zero pairings in (5.21) arise from the terms with the same number of \( A_0 \) and \( A_\bar{z} \) and hence include at least one \( \bar{z} \) (the zeroes term, which equals 1, is the only exception). Hence, vertical (places along \( t \) axis) pieces of the integration contour, on which \( \dot{z} \) vanishes, do not contribute to the integrals (5.21). Evaluation of the integral is hence much simplified if one (making use of the holomorphic property) deforms the contour into an “as vertical as possible” one. Only two kinds of intervals necessarily remain non-vertical for any position of contour. They are neighborhoods of turning points (fig.9) and of would be crossing points after projecting the contour on \( z \) plane (see fig.9).
Decoupling of distant pieces. The Green function $G(z - z')$ given by (5.30) is a decreasing function of the argument’s absolute value; it tends to zero, when the points $z'$ and $z$ are moved apart. At the same time, the holomorphic property allows one to locate all the non-trivial pieces arbitrary far from each other. The cross terms, which contain $G(z - z')$ with $z$ and $z'$ from different non-trivial pieces (see fig.9), then vanish. However, the parts of contour that connect different non-trivial pieces appear to bring their own non-trivial contributions. Explicit determination of such contributions turns out to be the most involved part of perturbative evaluation of the Chern-Simons Wilson average in the holomorphic gauge. These contributions have even a special name of Drinfeld associators [5, 6, 57, 58]. It is especially interesting, that the Drinfeld associators are (modulo certain subtleties [83]) solutions of the Knizhnik-Zamolodchikov equation [59], which was originally derived for WZW conformal blocks. This is a separate subject, and we do not go into any further details here, sending to the above cited works for a systematic treatment.

Group structure as a consequence of the path ordering. Each term of the perturbation series in a holomorphic gauge has one remarkable property. Suppose that one selects a piece of the integration contour, the piece is not necessarily connected. Each order contribution to the Wilson average may be presented then as contraction of two operators associated with the selected piece and with the remaining part of the contour, respectively. In more details, one should label each of $n$ connection components of the selected piece $C_0$ with a pair of scripts $(i, j)$; a term of the perturbation series may be written then as a contraction $T^i_1j_2...j_nT^i_1j_2...j_n$, where the operators $T$ and $T$ are associated with the selected piece, and with the remaining part of the contour $C$, respectively. Indeed, each term in perturbation series (5.19) is a (averaged and integrated) path ordered product of the (associated with the gauge group) algebra elements, which may be split into pieces in the desired way. E.g., one has for a connected selected piece

$$\langle A^{a_1}(z_1, t_1) ... A^{a_k}(z_k, t_k)A^{b_1}(w_1, s_1) ... A^{b_m}(w_k, s_k) \rangle (T_{a_1} ... T_{a_k})^j (T_{b_1} ... T_{b_m})^j = \quad (5.32)$$

where $(z, t)$ are points of the selected piece $C_0$, and $(w, s)$ are points of the remaining part $\bar{C}_0$ of the contour $C$, and $f$ is a function. Similarly, if the selected piece has two connection components, then

$$\langle A^{a_1}(z_1, t_1) ... A^{a_k}(z_k, t_k)A^{c_1}(u_1, p_1) ... A^{c_l}(u_k, p_k)A^{b_1}(w_1, s_1) ... A^{b_m}(w_k, s_k)A^{d_1}(v_1, q_1) ... A^{d_n}(v_k, q_k) \rangle \cdot \quad (5.33)$$

where the operator $F_0 \otimes F_0'$ associated with the selected piece now has a form of a tensor product, the components related to the one and to the other arcs composing the piece, and the operator $G_0 \otimes G_0'$ related to the remaining part of the contour has a similar structure. The observed property, together with the holomorphic property, which leads to the vanishing of the cross contributions, implies that the perturbative Wilson average may be presented as a contraction of some “non-trivial piece operators”. This relates the considered perturbative approach to the vertex-operator approach discussed in sect.3 which yielded a similar representation for the knot invariant.

Summary. As a result of the above listed properties, terms of perturbation series are expressed via three kinds of contributions illustrated in fig.9. This brings us once again to the vertex-contraction representation of knot invariants a’la the one discussed in sect.3 however providing a very different realization of it.
5.3.2 Explicit perturbative evaluation of crossing operators

In this section, we carry out an explicit evaluation of the Wilson average in the holomorphic gauge, meaning this object as described above, for the simplest “non-trivial piece”, in the first two non-vanishing orders.

Consider an average of a single crossing:

\[
\langle W(C) \rangle = \left\langle \exp \left( \int_{\gamma_1} A_\mu dx^\mu \right) \otimes \exp \left( \int_{\gamma_2} A_\mu dx^\mu \right) \right\rangle.
\] (5.34)

Second order

\[
W^{(2)} = \frac{1}{2!} \left\langle p \left( \int_0^1 dt_1 (A_0^1 + \dot{z}^1 A_z^1) \right)^2 \otimes I \right\rangle + \frac{1}{2!} \left\langle I \otimes p \left( \int_0^1 dt_2 (A_0^2 + \dot{z}^2 A_z^2) \right)^2 \right\rangle + \left\langle \int_0^1 dt_1 (A_0^1 + \dot{z}^1 A_z^1) \otimes \int_0^1 dt_2 (A_0^2 + \dot{z}^2 A_z^2) \right\rangle \equiv \frac{1}{2} W^{(2,0)} + W^{(1,1)} + \frac{1}{2} W^{(0,2)}. \tag{5.35}
\]

The terms \(W^{(2,0)}\) and \(W^{(0,2)}\) are singular when treated literally. According to the general prescription (see sec. 5.2), they are regularized by introducing a normal vector field on the contour; the field sets a shifted contour and an identification of points of both contours. In the selected coordinates, the field may be treated as a complex function on the original contour \(n(z(t), t)\), and the image of its point \((z(t), z)\) on the shifted contour is then \(z' \equiv z + \epsilon n(t)\), where \(\epsilon\) is parameter will be send to zero. We write

\[
p \left( \int_0^1 dt (A_0 + \dot{z} A_z) \right)^2 \to p \left( \int_0^1 dt (A_0 + \dot{z} A_z) \int_0^1 dt' (A_0' + \dot{z}' A_z') \right).
\] (5.36)

We assume that a point on the original contour is identified with a point with a larger value of \(t\) on the second contour, i.e.,

\[
p A_0(t) A_z'(t) = p A_z(t) A_0'(t) = A_0(t) A_z'(t); \tag{5.37}
\]

Then, substituting (5.27) in (5.36), one obtains a group factor multiplied by an integral of the simultaneous Green function (say, for \(W^{(2,0)}\)):

\[
W^{(2,0)} = \sum_a (T_a \otimes T_a) J^{(2,0)},
\]

\[
J^{(2,0)} = \int_0^1 dt \int_0^1 dt' \left( z^{1'} \langle A_0^1 A_z^1 \rangle + z^1 \langle A_z^1 A_0^1 \rangle \right) = \int_0^1 dt \frac{\dot{z}^1 - \dot{z}'^{1'}}{z^1 - z'^1} \equiv \alpha_{11}, \tag{5.38}
\]

the latter being equal to \(\int dt \frac{\hat{n}(t)}{n(t)} \equiv \alpha\), what is the rotation angle of vector \(n(t)\) along the corresponding line \(\gamma\). The term \(W^{(0,2)}\) is evaluated similarly.

The remaining term \(W^{(1,1)}\) does not need a regularization\(^5\) note also that there is no path-ordering of operators placed on different lines since they are separated by tensor product in (5.36). One obtains

\[
W^{(1,1)} = \sum_a (T_a \otimes T_a) J,
\]

\[
J = \int_0^1 dt \int_0^1 dt' \left( z^{2'} \langle A_0^1 A_z^2 \rangle + z^2 \langle A_z^1 A_0^2 \rangle \right) = \int_0^1 dt \frac{\dot{z}^1 - \dot{z}^2}{z^1 - z^2} \equiv \alpha_{12}, \tag{5.39}
\]

\(^5\)More accurately, one assumes that a shifted contour intertwines only its counterparts, not the other line in the selected area.
which is an angle on which rotates the horizontal vector pointing from the first line to the second one, when one follows along the lines from down to up; if the lines matching the origins and the ends of the lines are parallel, then \( \alpha_{12} = 2\pi n_{12} \), where \( n_{12} \) is an integer or half-integer number of mutual intertwists of the lines around the point corresponding to a crossing on a planar diagram; \( n_{12} = \frac{1}{2} \) in case of isolated crossing with single intertwaving. Putting everything together, one gets (a sum over repeated group indices is assumed)

\[
W^{(2)} = \frac{\alpha_{11}}{2} T_a^2 \otimes I + \frac{\alpha_{22}}{2} I \otimes T_a^2 + \pi T_a \otimes T_a. 
\]

(5.40)

**Fourth order**

\[
W_{12}^{(4)} = \frac{1}{4!} \left\langle p \left( \int_0^1 dt \left( A_0^1 + \dot{z}^1 A_z^1 \right) \right)^4 \otimes I \right\rangle + \frac{1}{4!} \left\langle I \otimes p \left( \int_0^1 dt \left( A_0^2 + \dot{z}^2 A_z^2 \right) \right)^4 \right\rangle + \\
+ \frac{1}{3!} \left\langle p \left( \int_0^1 dt \left( A_0^1 + \dot{z}^1 A_z^1 \right) \right)^3 \otimes p \left( \int_0^1 dt \left( A_0^2 + \dot{z}^2 A_z^2 \right) \right) \right\rangle + \\
+ \frac{1}{3!} \left\langle p \left( \int_0^1 dt \left( A_0^1 + \dot{z}^1 A_z^1 \right) \right) \otimes p \left( \int_0^1 dt \left( A_0^2 + \dot{z}^2 A_z^2 \right) \right)^2 \right\rangle + \\
+ \frac{1}{2!} \left\langle p \left( \int_0^1 dt \left( A_0^1 + \dot{z}^1 A_z^1 \right) \right)^2 \otimes p \left( \int_0^1 dt \left( A_0^2 + \dot{z}^2 A_z^2 \right) \right)^2 \right\rangle \equiv \\
\equiv \frac{1}{24} W^{(4,0)} + \frac{1}{6} W^{(3,1)} + \frac{1}{4} W^{(2,2)} + \frac{1}{6} W^{(1,3)} + \frac{1}{24} W^{(0,4)}. 
\]

(5.41)

The term \( W^{(4,0)} \) picks up contributions from six terms, and each of these terms gives two non-vanishing contractions:

\[
W^{(4,0)} = \iiint_0^1 (dt)^4 \dot{z}^2 \left\{ \langle p (A_0 A_0 A_z A_z) \rangle + \langle p (A_0 A_z A_0 A_z) \rangle + \langle p (A_0 A_z A_z A_0) \rangle + \langle p (A_z A_0 A_0 A_z) \rangle + \langle p (A_z A_0 A_z A_0) \rangle + \langle p (A_z A_z A_0 A_0) \rangle \right\} \otimes I = \\
= \iiint_0^1 (dt)^4 \dot{z}^2 \left\{ G_{0z} G_{0z} T_a T_a T_b T_b + G_{0z} G_{0z} T_a T_b T_b T_a + G_{0z} G_{0z} T_a T_a T_b T_b + G_{0z} G_{0z} T_a T_b T_a T_b + \right. \\
+ G_{0z} G_{0z} T_a T_a T_a T_b + G_{0z} G_{0z} T_a T_b T_a T_b + G_{0z} G_{0z} T_a T_a T_b T_b + G_{0z} G_{0z} T_a T_b T_a T_b + \\
\left. + G_{0z} G_{0z} T_a T_b T_b T_b + G_{0z} G_{0z} T_b T_b T_b T_a + G_{0z} G_{0z} T_b T_b T_a T_a + G_{0z} G_{0z} T_b T_a T_a T_b + G_{0z} G_{0z} T_b T_b T_b T_b \right\} \otimes I = \\
= \iiint_0^1 (dt)^4 \dot{z}^2 (G_{0z} + G_{z0})^2 (T_a T_a T_b T_b + T_a T_b T_a + T_b T_b T_a) \otimes I. 
\]

(5.42)

The integral again splits into two factors, one depending only on coordinates, the second one only on group generators. It is a non-trivial property of \( W^{(4)} \) unlike the similar property of \( W^{(2)} \); while the latter one contained only one combination of group generators and was necessarily proportional to it, the former one contains three different combinations of generators\(^6\) with *a priori* different coefficients. The quadruple integral of the coordinate dependent factor is just the square of the integral in \( (5.38) \);

under supposed prescriptions it equals \( \alpha_{11}^2 \). The term \( W^{(0,4)} \) is evaluated similarly.

\(^6\) Unless a certain group structure is assumed, see sec. ??.
For the term $W^{(3,1)}$ one has

$$W^{(3,1)} = \iiint (dt^1)^2 (dt^2)^2 \left\{ \left( \dot{z}^1 \right)^2 \langle p \left( A_0^1 A_1^1 A_2^1 + A_1^1 A_0^1 A_2^1 + A_2^1 A_0^1 A_1^1 \right) \otimes A_0^2 \rangle \\
+ \dot{z}^1 \dot{z}^2 \langle p \left( A_0^1 A_1^1 A_2^1 + A_1^1 A_0^1 A_2^1 + A_2^1 A_0^1 A_1^1 \right) \otimes A_2^2 \rangle \right\} = \\
\iiint (dt^1)^2 (dt^2)^2 \left\{ \left( \dot{z}^1 \right)^2 \left( G_{02}^{11} G_{70}^{12} T_a T_b + G_{02}^{11} G_{70}^{12} T_b T_a + \right. \right.
+ G_{z0}^{11} G_{z0}^{12} T_a T_b + G_{02}^{11} G_{z0}^{12} T_a T_b + G_{z0}^{11} G_{z0}^{12} T_b T_a + \\
\left. \left. + G_{z0}^{11} G_{z0}^{12} T_b T_a + G_{z0}^{11} G_{z0}^{12} T_a T_b + G_{z0}^{11} G_{z0}^{12} T_a T_b + G_{z0}^{11} G_{z0}^{12} T_b T_a \right) \otimes T_b \right\} \\
\iiint (dt^1)^3 (dt^2) \dot{z}^1 \left( G_{02}^{11} + G_{z0}^{11} \right) \left( \dot{z}^1 G_{02}^{11} + \dot{z}^2 G_{z0}^{11} \right) \left( T_a T_a T_b + T_a T_b T_a + T_b T_a T_a \right) \otimes T_b. \quad (5.43)$$

Similarly to the case of $W^{(4,0)}$, the coordinate dependent factor is separated from the coordinate independent combination of group generators, the former one, in turn, splits into two multipliers. An evaluation of $W^{(1,3)}$ is carried out similarly. For the remaining term one has

$$W^{(2,2)} = \iiint (dt^1)^2 (dt^2)^2 \left\{ \left( \dot{z}^1 \right)^2 \langle p \left( A_0^1 A_1^1 \right) \otimes p \left( A_0^2 A_2^2 \right) \rangle + \\
+ \dot{z}^1 \dot{z}^2 \langle p \left( A_0^1 A_1^1 + A_2^1 A_0^1 \right) \otimes p \left( A_0^2 A_2^2 + A_2^1 A_0^2 \right) \rangle + \\
+ \left( \dot{z}^2 \right)^2 \langle p \left( A_0^1 A_1^1 \right) \otimes p \left( A_2^1 A_0^2 \right) \rangle \right\}. \quad (5.45)$$

The average multiplied by $\left( \dot{z}^1 \right)^2$ yields two contractions and equals to

$$G_{z0}^{11} G_{z0}^{12} T_a T_b \otimes T_a T_b + G_{z0}^{11} G_{z0}^{12} T_b T_a \otimes T_b T_a. \quad (5.46)$$

Similarly, the average multiplied by $\left( \dot{z}^2 \right)^2$ is

$$G_{02}^{11} G_{02}^{12} T_a T_b \otimes T_a T_b + G_{02}^{11} G_{02}^{12} T_b T_a \otimes T_b T_a. \quad (5.47)$$

Finally, the average multiplied by $\dot{z}^1 \dot{z}^2$ yields eight contractions arising from the four summands:

$$G_{02}^{11} G_{02}^{12} T_a T_a \otimes T_b T_b + G_{02}^{11} G_{02}^{12} T_b T_b \otimes T_a T_a + G_{z0}^{11} G_{02}^{12} T_a T_a \otimes T_b T_b + G_{z0}^{11} G_{02}^{12} T_b T_b \otimes T_a T_a + \\
+ G_{02}^{11} G_{z0}^{12} T_a T_a \otimes T_b T_b + G_{02}^{11} G_{z0}^{12} T_b T_b \otimes T_a T_a + G_{z0}^{11} G_{z0}^{12} T_a T_a \otimes T_b T_b + G_{z0}^{11} G_{z0}^{12} T_b T_b \otimes T_a T_a. \quad (5.48)$$

Collecting all coefficients before each tensor structure, one obtains for the integrand in (5.45)

$$\dot{z}^1 \dot{z}^2 \left( G_{02}^{11} + G_{z0}^{11} \right) \left( G_{02}^{12} + G_{z0}^{12} \right) T_a T_a \otimes T_b T_b + \left( \dot{z}^1 G_{02}^{11} + \dot{z}^2 G_{z0}^{11} \right)^2 \left( T_a T_b \otimes T_a T_b + T_b T_a \otimes T_a T_b \right). \quad (5.49)$$

Unlike all other contributions up to the level four, $W^{(2,2)}$ is not a product of coordinate dependent function and coordinate independent combination of group generators; it is a sum of two such products instead.

Putting everything together, one obtains that the fourth-order contribution to a single crossing $W^{(4)}$ is equal to

$$\frac{(a_{22})^2}{24} \left( T_a T_a T_a T_b + T_a T_b T_a T_b + T_a T_b T_b T_a \right) \otimes I + \frac{(a_{11})^2}{24} I \otimes \left( T_a T_a T_b T_b + T_a T_b T_a T_b + T_b T_b T_a T_b \right) + \\
+ \frac{a_{11} a_{12}}{6} \left( T_a T_a T_a T_b + T_a T_b T_a T_b + T_a T_b T_b T_a \right) \otimes T_b + \frac{a_{22} a_{12}}{6} T_b \otimes \left( T_a T_a T_b T_b + T_a T_b T_a T_b + T_b T_b T_a T_b \right) + \\
+ \frac{a_{11} a_{22}}{4} T_a T_a \otimes T_b T_b + \frac{(a_{12})^2}{4} \left( T_a T_b \otimes T_a T_b + T_b T_a \otimes T_b T_a \right). \quad (5.50)$$
Two parallel lines In this case $\alpha_{12} = 0$ and Wilson average $W(||)$ up to the fourth order is

$$I \otimes I + \frac{\alpha_{11}}{2} T_a T_a \otimes I + \frac{\alpha_{22}}{2} I \otimes T_b T_b + \frac{\alpha_{11} \alpha_{22}}{4} T_a T_a \otimes T_b T_b +$$

$$\frac{(\alpha_{22})^2}{24} (T_a T_a T_b T_b + T_a T_b T_a T_b + T_b T_a T_b T_a) \otimes I + \frac{(\alpha_{11})^2}{24} I \otimes (T_a T_a T_b T_b + T_a T_b T_a T_b + T_b T_a T_b T_a)$$

$$+ \ldots =$$

$$= \left( I + \frac{\alpha_{11}}{2} T_a T_a + \frac{(\alpha_{11})^2}{24} (T_a T_a T_b T_b + T_a T_b T_a T_b + T_b T_a T_b T_a) + \ldots \right) \otimes$$

$$\otimes \left( I + \frac{\alpha_{22}}{2} T_a T_a + \frac{(\alpha_{22})^2}{24} (T_a T_a T_b T_b + T_a T_b T_a T_b + T_b T_a T_b T_a) + \ldots \right),$$

where dots stand for the subsequent orders. Our accuracy allows to verify that the factorization takes place in the lowest non-trivial (second) order.

Isolated crossing in the trivial framing This case is defined by $\alpha_{11} = \alpha_{22} = 0$, $\alpha_{12} = \pi$. Up to the fourth order

$$W(\times) = I \otimes I + \pi T_a \otimes T_a + \frac{\pi^2}{4} \left( T_a T_b \otimes T_a T_b + T_b T_a \otimes T_a T_b \right) + \ldots,$$

where dots stand for the subsequent orders.

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Figure 10: Hopf link of specific geometric shape: bulk view
Figure 11: Hopf link of specific geometric shape: top view
Figure 12: Hopf link of specific geometric shape: section of symmetry plain