Decays of a neutral particle with zero spin and arbitrary $CP$ parity into two off-mass-shell $Z$ bosons

T.V. Zagoskin$^{1,*}$ and A.Yu. Korchin$^{1,2,**}$

$^1$NSC “Kharkov Institute of Physics and Technology”, 61108 Kharkov, Ukraine
$^2$V.N. Karazin Kharkov National University, 61022 Kharkov, Ukraine

Effects are investigated of $CP$ symmetry violation in the decay of a scalar particle $X$ (the Higgs boson) into two off-mass-shell $Z$ bosons both decaying into a fermion-antifermion pair, $X \rightarrow Z_1^* Z_2^* \rightarrow f_1 \bar{f}_1 f_2 \bar{f}_2$. The most general form of the amplitude of the transition $X \rightarrow Z_1^* Z_2^*$, wherein the boson $X$ may not have definite $CP$ parity, is considered. Limits of applicability of the narrow-$Z$-width approximation used when obtaining differential widths of the decay under consideration are determined. Various observables connected with the structure of the amplitude of the decay $X \rightarrow Z_1^* Z_2^*$ are studied. These observables are analyzed in the Standard Model, as well as in models conceding indefinite $CP$ parity of the Higgs boson. An experimental measurement at the LHC of angular and invariant mass distributions of the decay $X \rightarrow Z_1^* Z_2^* \rightarrow f_1 \bar{f}_1 f_2 \bar{f}_2$ can give information about the $CP$ properties of the Higgs boson and its interaction with the $Z$ boson.

I. INTRODUCTION

In 2012 the ATLAS and CMS collaborations detected $^1$ a neutral boson $h$ with a mass of about 126 GeV. At the present time, detailed study of properties of this particle, called the Higgs boson, is an important task. The Standard Model (SM) Higgs boson is a state with $J^{CP} = 0^{++}$, and all the available experimental data about properties of the particle $h$ are close to the corresponding theoretical predictions about the SM Higgs boson (see, for example, $^2$). In particular, the spin of the boson $h$ is equal to zero or two, and many hypotheses in which the spin of $h$ is two are excluded with probability 95% or higher $^3$. At the same time, the situation may be more complicated. For example, some supersymmetric models predict existence of neutral bosons with negative or even indefinite $CP$ parity $^5$. The issue of the $CP$ parity of the Higgs boson is also related to the search for $CP$ symmetry breaking sources which are additional to the mechanism built into the Cabibbo-Kobayashi-Maskawa quark-mixing matrix. Such sources of $CP$ violation, for example, in the Higgs sector, could help in explaining the known problem of the matter-antimatter asymmetry in the Universe $^8$.

$^*$Electronic address: taras.zagoskin@gmail.com
$^{**}$Electronic address: korchin@kipt.kharkov.ua
It has been suggested [9, 10] that the \( CP \) properties of the Higgs boson be studied by investigation of decays into two photons, \( h \to \gamma\gamma \), via measurement of the polarization characteristics of the photons. In Refs. [11] the decay to the photon and the \( Z \) boson, \( h \to Z^*\gamma \to f\bar{f}\gamma \), has been examined while [12, 13] study the decay to the photon and a lepton pair, \( h \to \gamma l^+l^- \). In these papers it has been shown that the “forward-backward” escape asymmetry for the final fermions carries information about the \( CP \) properties of the \( h \) boson and physics beyond the SM.

Investigation into the decay of the Higgs boson into two \( Z \) bosons with their consequent decay to fermions is another opportunity to ascertain the \( CP \) properties of \( h \). Such a cascade decay wherein the final fermions are leptons, along with the two-photon decay channel, has allowed the determination [1] of the mass of the particle \( h \) with the highest accuracy. In Refs. [14–17] theoretical distributions of the decay \( h \to Z_1^*Z_2^* \to f_1\bar{f}_1f_2\bar{f}_2 \) have been studied at various values of the spin of \( h \) and in case of various \( CP \) properties of this boson. In [14] it has been reported what properties of experimental distributions testify about a particular spin and a particular \( CP \) parity of \( h \). In [15–17] asymmetries measurement of which allows clarification of the mentioned properties of the Higgs boson are suggested and investigated. Finally, papers [18] put forward various methodologies on getting constraints on the Higgs boson couplings to \( ZZ \), \( W^-W^+ \), \( \gamma\gamma \) and \( Z\gamma \) from experimental data.

Besides, various theories with spontaneous breaking of the conformal invariance (for example, theories of technicolor) assume the existence of one more neutral zero-spin particle which interacts with the gauge bosons – the dilaton. At present, the mass of the dilaton is not determined, but according to estimates performed in Ref. [19], in some models the mass can exceed \( 10^4 \) GeV. Along with that, in [20–22] it has been shown that the variant in which the boson \( h \) is the dilaton is not excluded.

In order to clarify the \( CP \) properties of the particle \( h \) and the hypothetical dilaton we consider a neutral particle \( X \) with zero spin and arbitrary \( CP \) parity. We examine the decay \( X \to Z_1^*Z_2^* \to f_1\bar{f}_1f_2\bar{f}_2 \) in case of the non-identical fermions, \( f_1 \neq f_2 \), and study in detail the differential width of this decay with respect to the three angles of the fermions in the helicity frame and with respect to the invariant masses of the fermion pairs \( f_1\bar{f}_1 \) and \( f_2\bar{f}_2 \). The most general \( X \to Z_1^*Z_2^* \) vertex, which generalizes the corresponding SM vertex and contains a term corresponding to the negative \( CP \) parity of the particle \( X \), is used.

We also find limits of applicability of the narrow-width approximation for the \( Z \) boson for the presented calculation of differential widths of the given decay. By means of this approximation we derive a formula for the total width of the decay \( X \to Z_1^*Z_2^* \to f_1\bar{f}_1f_2\bar{f}_2 \) (the formula is valid also in case \( f_1 = f_2 \)) and a formula for the total width of the decay \( h \to Z_1^*Z_2^* \). These formulas are more general and more precise than those obtained in Ref. [23].

Next we find observables connected with the structure of the amplitude of the decay \( X \to Z_1^*Z_2^* \). The formula for the fully differential decay width contains nine coefficients related to the amplitude \( X \to Z_1^*Z_2^* \). For each of them one or two observables linear in this coefficient are defined. Note that some of these observables, as well as different ones, have been studied in [15, 17, 24], however we also obtain new experimentally measurable quantities.
and analyze the dependences of the observables on the mass of one of the Z bosons \((Z^+_1)\) in much more detail than it has been done in the mentioned papers. This analysis is carried out within the framework of the SM as well as in certain SM extensions wherein the boson \(h\) is a mixture of a CP-even state and a CP-odd one. Measurement of the suggested observables at the LHC can yield important information about the CP properties of the Higgs boson and its interaction with the Z boson.

II. FORMALISM FOR THE DECAYS \(X \rightarrow Z^+_1Z^-_2 \rightarrow f_1\bar{f}_1f_2\bar{f}_2\)

A. The amplitude of the decay \(X \rightarrow Z^+_1Z^-_2\) and the fully differential decay width for \(X \rightarrow Z^+_1Z^-_2 \rightarrow f_1\bar{f}_1f_2\bar{f}_2\)

Let us consider the decay of a neutral spin-zero particle \(X\) with arbitrary CP parity into two off-mass-shell Z bosons \((Z^+_1\) and \(Z^-_2\)) each of which decays to a fermion-antifermion pair, \(f_1\bar{f}_1\) and \(f_2\bar{f}_2\),

\[
X \rightarrow Z^+_1Z^-_2 \rightarrow f_1\bar{f}_1f_2\bar{f}_2, \tag{1}
\]

where \(m_X > 2(m_{f_1} + m_{f_2})\) (to satisfy the law of conservation of energy in a rest frame of \(X\)), \(m_X\) is the mass of the particle \(X\), \(m_{f_j}\) is the mass of the fermion \(f_j\). We will consider this decay at tree level. If \(m_X \in (4m_b, 2m_t)\) \((m_b\) is the mass of the b quark, \(m_t\) is the mass of the t quark), which holds true if \(X = h\), then \(f_j = e^-, \mu^-, \tau^-, \nu_e, \nu_\mu, \nu_\tau, u, c, d, s, b\). If \(m_X > 4m_t\), which is possible \([19]\) if \(X\) is the dilaton, then \(f_j\) can be the top quark as well.

From the energy-momentum conservation we find that \(a_1\) and \(a_2\) (\(a_j\) is the mass squared of the boson \(Z^+_j\), i.e. the invariant mass squared of the pair \(f_j\bar{f}_j\)) lie within limits

\[
4m^2_{f_1} < a_1 < (m_X - 2m_{f_2})^2, \quad 4m^2_{f_2} < a_2 < (m_X - \sqrt{a_1})^2. \tag{2}
\]

The amplitude \(A_{X \rightarrow Z^+_1Z^-_2}(\lambda_1, \lambda_2)\) of the decay of \(X\) into \(Z^+_1\) and \(Z^-_2\) is equal to \([13, 17, 24]\)

\[
A_{X \rightarrow Z^+_1Z^-_2}(\lambda_1, \lambda_2) = 2\sqrt{2}G_F m_Z^2 \left( a_z (e_1^* \cdot e_2^*) + \frac{b_z}{m_X} (e_1^* \cdot (p_1 + p_2)) (e_2^* \cdot (p_1 + p_2)) + i \frac{c_z}{m_X} \varepsilon_{\mu\nu\rho\sigma} (p_1^\mu + p_2^\mu) (p_1^\nu - p_2^\nu) (e_1^* \cdot (e_2^*))^* \right), \tag{3}
\]

where \(\lambda_j, e_j, p_j\) are respectively the helicity, the polarization 4-vector and the 4-momentum of the boson \(Z^+_j\), \(G_F\) is the Fermi constant, \(m_Z\) is the mass of the Z boson, \(a_z, b_z, c_z\) are complex-valued dimensionless functions of \(a_1\) and \(a_2\); \(\varepsilon_{\mu\nu\rho\sigma}\) is the Levi-Civita symbol \((\varepsilon_{0123} = 1)\). Note that at tree level

- if \(X\) is the SM Higgs boson, then \(a_Z = 1, b_Z = c_Z = 0\);

- if the CP parity of \(X\) is -1, then \(a_Z = b_Z = 0\) and \(c_Z \neq 0\);

- if the CP parity of \(X\) is indefinite, then \(a_Z \neq 0, c_Z \neq 0\) and/or \(b_Z \neq 0, c_Z \neq 0\).
Calculating the Lorentz-invariant amplitude \( A_{X \rightarrow Z_1^* Z_2^*}(\lambda_1, \lambda_2) \) in a reference frame in which \( p_1 + p_2 = 0 \), we derive that

\[
\begin{align*}
A_{X \rightarrow Z_1^* Z_2^*}(-1, -1) &= 2\sqrt{2G_F m_Z^2} \left(a_Z - c_Z \frac{\lambda^2(m_X^2, a_1, a_2)}{m_X^2}\right), \\
A_{X \rightarrow Z_1^* Z_2^*}(0, 0) &= -2\sqrt{2G_F m_Z^2} \left(a_Z \frac{m_X^2 - a_1 - a_2}{2\sqrt{a_1a_2}} + b_Z \frac{\lambda(m_X^2, a_1, a_2)}{4m_X^2\sqrt{a_1a_2}}\right), \\
A_{X \rightarrow Z_1^* Z_2^*}(1, 1) &= 2\sqrt{2G_F m_Z^2} \left(a_Z + c_Z \frac{\lambda^2(m_X^2, a_1, a_2)}{m_X^2}\right), \\
A_{X \rightarrow Z_1^* Z_2^*}(\lambda_1, \lambda_2) &= 0, \quad \lambda_1 \neq \lambda_2,
\end{align*}
\]

where the function \( \lambda(x, y, z) \) is defined in the standard way: \( \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \).

![Fig. 1: The kinematics of the decay \( X \rightarrow Z_1^* Z_2^* \rightarrow f_1 \bar{f}_1 f_2 \bar{f}_2 \). The momenta of \( Z_1^* \) and \( Z_2^* \) are shown in a rest frame of \( X \), the momenta of \( f_1 \) and \( \bar{f}_1 \) \((f_2 \text{ and } \bar{f}_2)\) are shown in a rest frame of \( Z_1^* \) \((Z_2^*)\).](image)

To describe the decay \( \Box \), let us introduce the following angles (see Fig. 1): \( \theta_1 \) \((\theta_2)\) is the angle between the momentum of \( Z_1^* \) \((Z_2^*)\) in a rest frame of \( X \) and the momentum of \( f_1 \) \((f_2)\) in a rest frame of \( Z_1^* \) \((Z_2^*)\) and \( \varphi \) is the azimuthal angle between the planes of the decays \( Z_1^* \rightarrow f_1 \bar{f}_1 \) and \( Z_2^* \rightarrow f_2 \bar{f}_2 \). Further we go into the case of the non-identical fermions, \( f_1 \neq f_2 \). Using the helicity formalism (see, for example, [25]), we obtain that in the approximation of the massless fermions, \( m_{f_1} = m_{f_2} = 0 \), the differential decay width of \( \Box \) with respect to \( a_1, a_2, \theta_1, \theta_2, \varphi \) appears as follows:

\[
\frac{d^5\Gamma}{da_1 da_2 d\theta_1 d\theta_2 d\varphi} = \sqrt{2G_F m_Z^2} \frac{m_X^2}{(4\pi)^3 m_X^2} \left(a_x^2 + v_Z a_y^2 + v_Z^2 \right) \frac{\lambda^2(m_X^2, a_1, a_2)a_1 a_2}{D(a_1)D(a_2)} \times \sin \theta_1 \sin \theta_2 (|A_{||}|^2 + |A_{\perp}|^2) \left((1 + \cos^2 \theta_1)(1 + \cos^2 \theta_2) + 4A_{f_1} A_{f_2} \cos \theta_1 \cos \theta_2 \right) + 4|A_{||}|^2 \sin^2 \theta_1 \sin^2 \theta_2 - 4 \text{Re}(A_{||}^* A_{||}) (A_{f_1} \cos \theta_1 (1 + \cos^2 \theta_2) + A_{f_2} \cos \theta_2 (1 + \cos^2 \theta_1)) + 4\sqrt{2} \sin \theta_1 \sin \theta_2 \left(\text{Re}(A_{||}^* A_{\perp}) \cos \varphi - \text{Im}(A_{||}^* A_{\perp}) \sin \varphi\right) (A_{f_1} A_{f_2} \cos \theta_2 + \cos \theta_1 \cos \theta_2) - \left(\text{Re}(A_{||}^* A_{\perp}) \cos \varphi - \text{Im}(A_{||}^* A_{\perp}) \sin \varphi\right) (A_{f_1} \cos \theta_2 + A_{f_2} \cos \theta_1)) \sin^2 \theta_1 \sin^2 \theta_2 ((|A_{||}|^2 - |A_{\perp}|^2) \cos 2\varphi - 2 \text{Im}(A_{||}^* A_{\perp}) \sin 2\varphi),
\]

where \( a_f \) is the projection of the weak isospin of a fermion \( f \), \( v_f \equiv a_f - 2 \frac{q_f}{e} \sin^2 \theta_W \), \( q_f \) is the electric charge of the fermion \( f \), \( e \) is the electric charge of the positron, \( \theta_W \) is the weak mixing angle, \( D(a_{1,2}) \equiv (a_{1,2} - m_Z^2)^2 + (m_Z \Gamma_Z)^2 \),
$\Gamma_Z$ is the total width of the $Z$ boson, $A_f \equiv \frac{2a_1\nu_{r_f}}{a_1^2 + a_2^2}$.

$$A_(a_1, a_2) \equiv \frac{A_{X \rightarrow Z^1Z^2}(1, 1) + A_{X \rightarrow Z^1Z^2}(-1, -1)}{2\sqrt{G_Fm_Z^2}} = \sqrt{2}a_Z,$$

$$A_(a_1, a_2) \equiv \frac{A_{X \rightarrow Z^1Z^2}(1, 1) - A_{X \rightarrow Z^1Z^2}(-1, -1)}{2\sqrt{G_Fm_Z^2}} \lambda^2(m_X^2, a_1, a_2),$$

$$2\sqrt{G_Fm_Z^2} = \Bigg( \frac{a_2m_X^2 - a_1 - a_2}{2\sqrt{a_1a_2}} + b_Z \frac{\lambda(m_X^2, a_1, a_2)}{4m_X^2\sqrt{a_1a_2}} \Bigg).$$

Futhermore the approximation $m_{f_1} = m_{f_2} = 0$ is used. Using Eq. (5), one can connect the ratios of quantities $|A_0|^2, |A_\parallel|^2 + |A_\perp|^2, |A_\parallel|^2 - |A_\perp|^2, \text{Re}(A_0A_\parallel), \text{Re}(A_0A_\perp), \text{Im}(A_0A_\parallel), \text{Im}(A_0A_\perp), \text{Im}(A_\perp A_\parallel)$ to $|A_0|^2 + |A_\parallel|^2 + |A_\perp|^2$ with functions of $a_1, a_2$ which can be measured in experiment. We will call these ratios the helicity coefficients of the decay $X \rightarrow Z^1Z^2$.

B. A differential width $\frac{d^2\Gamma}{da_1da_2}$

The number of the decays

$$h \rightarrow Z^1Z^2 \rightarrow l_1^-l_1^+l_2^-l_2^+ \ (l_j = e, \mu),$$

detected in the ATLAS experiment \cite{2} wherein the invariant mass of the four leptons was in the interval $[120 \text{ GeV}, 130 \text{ GeV}]$, is equal to 32. The number of the decays \cite{3} detected in the CMS experiment \cite{3} in which the four-lepton invariant mass was within $[121.5 \text{ GeV}, 130.5 \text{ GeV}]$, is equal to 25. In view of the insignificant amount of data, at the present time an experimental dependence of the distribution $\frac{d^2\Gamma}{da_1da_2}$ for any of the decays \cite{4} is not available. Let us consider differential decay widths of \cite{4} with respect to four and fewer variables. Integrating Eq. (5) with respect to $\theta_1, \theta_2, \varphi$, we obtain

$$\frac{d^2\Gamma}{da_1da_2} = \frac{\sqrt{2}G_Fm_Z^2}{9(2\pi)^2m_X^2} (a_1^2 + v_1^2)(a_2^2 + v_2^2) \frac{\lambda^2(m_X^2, a_1, a_2)}{D(a_1)D(a_2)} \sum_{p=0,1,1} |A_p|^2.$$  

(8)

It follows from Eqs. (3), (6) that the dependence of the differential width $\frac{d^2\Gamma}{da_1da_2}$ on $a_Z, b_Z, c_Z$ boils down only to the dependence on $|a_Z|, |b_Z|, |c_Z|$ and on cos (arg $b_Z - arg a_Z$).

The available experimental data on properties of the particle $h$ are close to the corresponding theoretical predictions about the SM Higgs boson (see, for example, \cite{2, 3}). That is why $a_{hZ} \approx 1, b_{hZ} \approx 0, c_{hZ} \approx 0$, where

$$a_{hZ} \equiv a_Z|_{X=h}, \ b_{hZ} \equiv b_Z|_{X=h}, \ c_{hZ} \equiv c_Z|_{X=h}.$$

In Fig. \cite{2} we show the differential decay width \cite{8} for $X \rightarrow Z^1Z^2 \rightarrow l_1^-l_1^+l_2^-l_2^+ \ (l_j = e, \mu, \tau, l_1 \neq l_2)$ as a function of $\sqrt{a_1}, \sqrt{a_2}$ in the SM for $|a_Z| = 1, b_Z = c_Z = 0$ and $m_X = m_h$, where $m_h$ is the mass of the Higgs boson $h$. The range of $\sqrt{a_1}, \sqrt{a_2}$ in this plot is determined by the inequalities \cite{2} in the approximation of the massless fermions. In calculations and when plotting graphs the experimental data listed in Table \cite{10} are used, and

$$\sin^2\theta_W = 1 - \frac{m_W^2}{m_Z^2},$$

where $m_W$ is the mass of the $W$ boson.
Fig. 2: The dependence of the differential decay width \( \frac{d^2\Gamma}{da_1 da_2} \) of \( X \rightarrow Z_1^* Z_2^* \rightarrow l_1^- l_2^+ l_1^- l_2^+ \) \((l_j = e, \mu, \tau, l_1 \neq l_2)\) on \( \sqrt{a_1} \) and \( \sqrt{a_2} \) in the SM for \( m_X = m_h \).

Table I: Values of the Fermi constant, of the masses of \( h, Z, W \) and of the total width of \( Z \) [26].

| Parameter      | Value                              |
|----------------|------------------------------------|
| \( G_F \)      | \( 1.1663787(6) \times 10^{-5} \) GeV\(^{-2} \) |
| \( m_h \)      | \( 125.7(4) \) GeV                 |
| \( m_Z \)      | \( 91.1876(21) \) GeV              |
| \( m_W \)      | \( 80.385(15) \) GeV               |
| \( \Gamma_Z \) | \( 2.4952(23) \) GeV               |

As one can see from Fig. 2 in the SM the function \( \frac{d^2\Gamma}{da_1 da_2} \) has peaks at \( \sqrt{a_1} = m_Z \) and \( \sqrt{a_2} = m_Z \), resulting from the quantities \( D(a_1) \) and \( D(a_2) \) in (8).

Let us calculate the ratio of a typical value of \( \frac{d^2\Gamma}{da_1 da_2} \) in the SM on the peaks to its typical value in an area in which \( \sqrt{a_1} \) and \( \sqrt{a_2} \) significantly differ from \( m_Z \) (we will call this area “plateau”). As indicative values of \( \sqrt{a_1} \) and \( \sqrt{a_2} \) on the peaks we take \( \sqrt{a_1} = m_Z \), \( \sqrt{a_2} = \frac{1}{2}(m_h - m_Z) \) and \( \sqrt{a_1} = \frac{1}{2}(m_h - m_Z) \), \( \sqrt{a_2} = m_Z \) (see [2]), and values on the “plateau” are chosen \( \sqrt{a_1} = \sqrt{a_2} = \frac{1}{2}m_Z \). It follows from (8) that in the SM for any \( f_1, f_2 \) values of \( \frac{d^2\Gamma}{da_1 da_2} \) at \( \sqrt{a_1} = m_Z \) or \( \sqrt{a_2} = m_Z \) are approximately 100 times as great as values of this function on the “plateau”.

If \( m_X \neq m_h \) but just greater than \( m_Z \), then \( \sqrt{a_1} \) and/or \( \sqrt{a_2} \) can be equal to \( m_Z \) (according to [2]), and, consequently, in this case the behavior of the function \( \frac{d^2\Gamma}{da_1 da_2} \) in the SM is similar to that in case \( m_X = m_h \). That is why for any \( m_X > m_Z \) and for any final fermions the differential width \( \frac{d^2\Gamma}{da_1 da_2} \) in the SM has a sharp maximum at \( \sqrt{a_1} = m_Z \) or \( \sqrt{a_2} = m_Z \). Therefore, if \( |a_Z| \approx 1, b_Z \approx 0, c_Z \approx 0 \) (which is the case of a small distinction between the couplings and their SM values), \( \frac{d^2\Gamma}{da_1 da_2} \) also has a sharp maximum at \( \sqrt{a_1} = m_Z \) or \( \sqrt{a_2} = m_Z \), provided that \( m_X > m_Z \).
C. Limits of applicability of the narrow-

In Refs. \[27–29\] the accuracy of the narrow-width approximation has been studied for calculation of the total widths of various decays along with the total and differential cross sections of various processes. It is shown that in many cases (especially for processes beyond the SM) this approximation is not applicable. In this connection the question arises whether the narrow-Z-width approximation is applicable for obtaining the differential width \( \frac{d\Gamma}{da_2} \) by means of integrating \( \frac{d\Gamma}{da_1 da_2} \). In this subsection we find the interval of all the \( a_2 \)-values for which the approximate integration is valid.

We consider the \( m_X \)-values such that \( m_X > m_Z \) and the dependences of \( a_Z(1, a_2) \), \( b_Z(1, a_2) \), \( c_Z(1, a_2) \) such that for any \( f_1 \) and \( f_2 \) \( \frac{d\Gamma}{da_1 da_2} \) has a sharp maximum when \( \sqrt{a_1} = m_Z \) or \( \sqrt{a_2} = m_Z \) (an example of such dependences is \( |a_Z| \approx 1, b_Z \approx 0, c_Z \approx 0 \)). Then while calculating the differential width \( \frac{d\Gamma}{da_2} \) one may use the narrow-Z-width approximation:

\[
\frac{d\Gamma}{da_2} = \int_0^{(m_X - m_Z)^2} da_1 \frac{d\Gamma}{da_1 da_2} \approx \int_0^{(m_X - m_Z)^2} da_1 \frac{\pi}{m_Z \Gamma_Z} \delta(a_1 - m_Z^2) f(a_1, a_2)
\]

\[
= \frac{\pi}{m_Z \Gamma_Z} f(m_Z^2, a_2) \quad \forall \sqrt{a_2} \in (0, m_X - m_Z - \Delta],
\]

(9)

where \( \Delta \) is some positive quantity and

\[
f(a_1, a_2) = \frac{\sqrt{G_F^2 m_Z^8}}{9(2\pi)^4 m_X^2} (a_1^2 + v_1^2) (a_2^2 + v_2^2) \frac{\lambda^2(m_X, a_1, a_2)a_1 a_2}{D(a_2)} \sum_{p=0,\perp,\parallel} |A_p|^2.
\]

(10)

\( \Delta > 0 \) since in Eq. (9) one may use the approximation \( \frac{d\Gamma}{da_1 da_2} = \frac{\pi}{m_Z \Gamma_Z} \delta(a_1 - m_Z^2) f(a_1, a_2) \) only when \( \sqrt{a_2} < m_X - m_Z \), because if \( \sqrt{a_2} \) approaches \( m_X - m_Z \), the peak of \( \frac{d\Gamma}{da_1 da_2} \) at \( \sqrt{a_1} = m_Z \) gets less sharp and at \( \sqrt{a_2} = m_X - m_Z \) the peak disappears (see Fig. 2 and Eq. (8)). However, the derivation (9) does not allow one to estimate the accuracy of the formula \( \frac{d\Gamma}{da_2} \approx \frac{\pi}{m_Z \Gamma_Z} f(m_Z^2, a_2) \) at a given value of \( \sqrt{a_2} \), and for this reason it is not clear what value of \( \Delta \) should be chosen.

To clarify this point, let us derive the formula for \( \frac{d\Gamma}{da_2} \) in the following way:

\[
\frac{d\Gamma}{da_2} = \int_0^{(m_X - m_Z^2)} \int_{m_Z^2 - \varepsilon_2}^{m_Z^2 + \varepsilon_2} da_1 \frac{d\Gamma}{da_1 da_2} \approx \int_{m_Z^2 - \varepsilon_1}^{m_Z^2 + \varepsilon_2} \int_{m_Z^2 - \varepsilon_1}^{m_Z^2 + \varepsilon_2} da_1 \frac{d\Gamma}{da_1 da_2} = \int_{m_Z^2 - \varepsilon_1}^{m_Z^2 + \varepsilon_2} \int_{m_Z^2 - \varepsilon_1}^{m_Z^2 + \varepsilon_2} da_1 \frac{f(a_1, a_2)}{(a_1 - m_Z^2)^2 + (m_Z \Gamma_Z)^2}
\]

\[
\approx \frac{\pi}{m_Z \Gamma_Z} f(m_Z^2, a_2),
\]

(11)

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are some positive quantities such that \( m_Z \Gamma_Z \ll \varepsilon_j \ll m_Z^2 \), the variable \( a_2 \) takes values in the interval \( \left( 0, (m_X - \sqrt{m_Z^2 + \varepsilon_2})^2 \right) \).
One of the approximations used in Eq. (11) is the switch from the integration over an interval \((0, (m_X - \sqrt{a_2})^2)\) to the integration over an interval \((m_Z^2 - \varepsilon_1, m_Z^2 + \varepsilon_2)\). Thus, \(m_Z^2 - \varepsilon_1\) has to be greater than or equal to \(4m_{f_1}^2\) (which holds true since \(\varepsilon_1 \ll m_Z^2\)) and \(m_Z^2 + \varepsilon_2\) has to be less than or equal to \((m_X - \sqrt{a_2})^2\), i.e. \(a_2 \leq (m_X - \sqrt{m_Z^2 + \varepsilon_2})^2\). The latter inequality restricts the interval of all the \(a_2\)-values for which these approximations are applicable. Consequently, in order to apply them for as long an interval of \(a_2\)-values as possible, one should use the minimal \(\varepsilon_2\)-value at which the approximations are valid.

While obtaining (11), we also used an approximation \(A \approx \pi\) \(= \pi\ \arctan\ \frac{m_Z}{m_Z\Gamma_Z} + \arctan\ \frac{m_Z}{m_Z\Gamma_Z}\). Let us define \(\varepsilon_1\) as \(\varepsilon_1 \equiv m_Z\sqrt{m_Z\Gamma_Z}\) (so as \(\frac{m_Z}{m_Z\Gamma_Z} = \frac{m_Z}{\varepsilon_1}\)). The values of quantities \(A\) and \(m_h - \sqrt{m_Z^2 + \varepsilon_2}\) which are listed in Table II specify for the considered \(\varepsilon_2\)-values the accuracy of the approximation \(A \approx \pi\) and the maximal value of \(\sqrt{a_2}\) at which the narrow-\(Z\)-width approximation is applicable in case \(X = h\).

Table II: Values of \(A\) \((\varepsilon_1 \equiv m_Z\sqrt{m_Z\Gamma_Z})\) and of \(m_h - \sqrt{m_Z^2 + \varepsilon_2}\) at various values of \(\varepsilon_2\).

| \(\varepsilon_2\) | \(A\) | \(m_h - \sqrt{m_Z^2 + \varepsilon_2}\) (GeV) |
|------------------|------|-----------------------------------------|
| 0                | 0.45π | 34.51                                   |
| \(m_Z\Gamma_Z\)  | 0.70π | 33.27                                   |
| 2\(m_Z\Gamma_Z\) | 0.80π | 32.05                                   |
| 3\(m_Z\Gamma_Z\) | 0.85π | 30.84                                   |
| 4\(m_Z\Gamma_Z\) | 0.87π | 29.65                                   |

According to Table II, if \(\varepsilon_2 < 3m_Z\Gamma_Z\), then \(A < 0.85\pi\) and, in view of the big difference between \(A\) and \(\pi\), we will not apply the approximations (11) for such values of \(\varepsilon_2\). Hence we will use \(\varepsilon_2 = 3m_Z\Gamma_Z\). It follows from (11) that

\[
\frac{d\Gamma}{da_2} \approx \frac{\sqrt{3G_F} m_Z}{9.2^{\ast}\pi^4 m_X^4 \Gamma_Z} (a_{f_1}^2 + v_{f_1}^2) (a_{f_2}^2 + v_{f_2}^2) \frac{\lambda^A(m_X^2, m_Z^2, a_2)}{D(a_2)} \sum_{p = 0, \|, \perp} |A_p'|^2 \tag{12}
\]

\(\forall \sqrt{a_2} \in \left(0, m_X - \sqrt{m_Z^2 + \varepsilon_2}\right)\),

where \(A_p' \equiv A_p(m_Z^2, a_2)\) \((p = 0, \|, \perp)\).

Note that in Refs. [14, 16, 17] when plotting dependences of \(\frac{d\Gamma}{da_2}\) on \(\sqrt{a_2}\), formulas for \(\frac{d\Gamma}{da_2}\) which correspond to (12) have been used, but these graphs have been plotted for \(\sqrt{a_2} \leq m_X - m_Z\), despite the fact that Eq. (11) is not valid at \(\varepsilon_2 = 0\) (see Table II), and, therefore, the plotted dependences significantly differ from the true ones in the interval \(\sqrt{a_2} \in (m_X - \sqrt{m_Z^2 + 3m_Z\Gamma_Z}, m_X - m_Z)\).
D. An inequality constraining \( a'_{hZ}, b'_{hZ}, c'_{hZ} \) from CMS data

According to [3],

\[
\frac{\sigma(pp \to h) \Gamma(h \to Z_1^* Z_2^* \to 4l)}{\sigma_{SM}(pp \to h) \Gamma_{SM}(h \to Z_1^* Z_2^* \to 4l)} = 0.93^{+0.26}_{-0.23}(\text{stat})^{+0.13}_{-0.09}(\text{syst}),
\]

(13)

where \( \sigma(pp \to h) \) is the cross section for production of \( h \) in \( pp \) collisions,

\[
\Gamma(h \to Z_1^* Z_2^* \to 4l) \equiv \Gamma(h \to Z_1^* Z_2^* \to 4e) + \Gamma(h \to Z_1^* Z_2^* \to 4\mu) + \Gamma(h \to Z_1^* Z_2^* \to 2e2\mu) = 2 \Gamma(h \to Z_1^* Z_2^* \to 4e) + \Gamma(h \to Z_1^* Z_2^* \to 2e2\mu),
\]

(14)

\( \Gamma_h \) is the total width of the boson \( h \), \( \sigma_{SM}(pp \to h), \Gamma_{SM}(h \to Z_1^* Z_2^* \to 4l), \Gamma_{hSM} \) are the predictions of the SM for respectively \( \sigma(pp \to h) \), \( \Gamma(h \to Z_1^* Z_2^* \to 4l) \), \( \Gamma_h \) at \( m_h = 125.6 \) GeV. Obtaining (13), the CMS collaboration has combined data from \( pp \) collisions corresponding to an integrated luminosity of 5.1 fb\(^{-1} \) at a center-of-mass energy \( \sqrt{s} = 7 \) TeV and 19.7 fb\(^{-1} \) at \( \sqrt{s} = 8 \) TeV.

We consider the case in which the functions \( |a'_{hZ}|, |b'_{hZ}|, |c'_{hZ}| \), \( \cos(\arg b'_{hZ} - \arg a'_{hZ}) \) do not depend on \( a_2 \). Here we define

\[
a'_{hZ} \equiv a_{hZ}(m_Z^2, a_2), \quad b'_{hZ} \equiv b_{hZ}(m_Z^2, a_2), \quad c'_{hZ} \equiv c_{hZ}(m_Z^2, a_2).
\]

Then using the approximation

\[
\frac{\sigma(pp \to h)}{\Gamma_h} \approx \frac{\sigma_{SM}(pp \to h)}{\Gamma_{hSM}}
\]

(15)

and Eqs. (13) (within one standard deviation), (A10), (A13) (see Appendix A), we derive the relation

\[
|a'_{hZ}|^2 + 0.015 |b'_{hZ}|^2 + 0.177 \text{Re}(a'_{hZ}b'_{hZ}) + 0.037 |c'_{hZ}|^2 \in [0.68, 1.22].
\]

(16)

While obtaining (16) we plugged the central values of \( m_h, m_Z, \Gamma_Z \) listed in Table I into Eq. (A10). Note that the latter equation is derived at tree level and without allowance for the interference term connected with the permutation of the identical fermions in case \( f_1 = f_2 \). The interference contribution to \( \Gamma(h \to Z_1^* Z_2^* \to 4l) \) at tree level is expected to be negligible since in the SM at \( m_h = 140 \) GeV it amounts to 2.99% (see Table 1 in Ref. 30).

Using the data of Table I and considering two sigma errors where available, we obtain that at \( \sqrt{s} = 8 \) TeV

\[
\frac{\sigma(pp \to h)/\Gamma_h}{\sigma_{SM}(pp \to h)/\Gamma_{hSM}} \in (0.17, \infty),
\]

(17)

which means that the approximation (15) does not contradict the experimental limits.

Moreover, assuming that all the couplings of the Higgs boson except for \( a_{hZ}, b_{hZ} \) and \( c_{hZ} \) are equal to their SM values, we can verify (15). In this case the only anomalous contribution to \( \Gamma_h \) comes from \( \Gamma(h \to Z_1^* Z_2^*) \), which makes up, in the SM, only about 2.81% [34] of the total Higgs boson width, and therefore \( \Gamma_h \) is unlikely to
Table III: Experimental and theoretical results for the total production cross-section of the Higgs boson in pp collisions and for its total width.

| Expression                              | Result                                      |
|-----------------------------------------|---------------------------------------------|
| $\sigma(pp \rightarrow h)$             | $33.0 \pm 5.3\text{(stat)} \pm 1.6\text{(syst)}$ pb at $\sqrt{s} = 8\text{ TeV}$ [31] |
| $\sigma_{SM}(pp \rightarrow h)$        | $22.09$ pb (uncertainties not available) at $\sqrt{s} = 8\text{ TeV}$ [32] |
| $\Gamma_h$                            | $< 22$ MeV at 95% confidence level (CL) [33] |
| $\Gamma_{h,SM}$                        | $4.15 \pm 0.16$ MeV [34]                    |

substantially differ from its SM prediction. Besides, the inequality (16) means that $|\Gamma(h \rightarrow Z_1^*Z_2^*) - \Gamma_{SM}(h \rightarrow Z_1^*Z_2^*)|/\Gamma_{SM}(h \rightarrow Z_1^*Z_2^*) \in [0,0.32]$ because its left-hand side is

$$\frac{\Gamma(h \rightarrow Z_1^*Z_2^* \rightarrow 4l)}{\Gamma_{SM}(h \rightarrow Z_1^*Z_2^* \rightarrow 4l)} = \frac{\Gamma(h \rightarrow Z_1^*Z_2^*)}{\Gamma_{SM}(h \rightarrow Z_1^*Z_2^*)}$$

(see [13, A10]). For this reason (16) implies that the relative change of $\Gamma_h$ is less than $2.81\% \cdot 0.32 \approx 0.90\%$, and, consequently, (16) is consistent with the approximation $\Gamma_h \approx \Gamma_{h,SM}$.

The dominant contribution to the Higgs boson production cross section $\sigma_{SM}(pp \rightarrow h)$ comes from the gluon fusion process $gg \rightarrow h$, which is independent of the $hZZ$ vertex. The processes involving the $hZZ$ interaction, i.e. the Higgs-strahlung $Zh$ and the $Z$ boson fusion, constitute much less parts of $\sigma_{SM}(pp \rightarrow h)$. Specifically, at $\sqrt{s} = 8\text{ TeV}$ they can be estimated as $0.41$ pb and $0.70$ pb respectively [32]. The total production cross section at this energy is $22.09$ pb (see Table III), so the processes of interest contribute about 5% of the total cross section. That is why it seems improbable that the couplings $a_{hZ}$, $b_{hZ}$ and $c_{hZ}$ provide a significant difference between $\sigma(pp \rightarrow h)$ and $\sigma_{SM}(pp \rightarrow h)$. However, a derivation of the dependence of the total production cross section on the $hZZ$ couplings would require a separate study.

Summarizing the discussion of the approximation (15), we can infer, firstly, that it is consistent with the available data [31, 33] and, secondly, under the assumption that the only anomalous Higgs boson couplings are related to the $hZZ$ vertex, Eq. (15) is most likely to be valid due to the small contributions of the $hZZ$ vertex to $\Gamma_h$ and $\sigma(pp \rightarrow h)$.

E. Constraints on $a_{hZ}^{'}, b_{hZ}^{'}, c_{hZ}^{'}$

The inequality (16) constrains the whole six-dimensional space formed by the real and imaginary parts of the couplings $a_{hZ}^{'}, b_{hZ}^{'}, c_{hZ}^{'}$ to the set of ellipsoids allowed by (13). Note that a similar interpretation has been suggested in Ref. 35.

From (16) it follows that the variant $a_{hZ}^{'} = 0, b_{hZ}^{'} = 0, c_{hZ}^{'} = 1$ (negative $CP$ parity of the boson $h$) is excluded. Now let us find constraints on the values of $b_{hZ}^{'}$ and $c_{hZ}^{'}$, assuming that $a_{hZ}^{'}$ is taken from the SM, i.e. $|a_{hZ}^{'}| = 1$.
or \( a'_hZ = 1 \). Then
\[
|a'_hZ| = 1 \quad \text{and} \quad b'_hZ = 0 \Rightarrow |c'_hZ| \in [0, 2.44]; 
\]
\[
a'_hZ = 1 \quad \text{and} \quad c'_hZ = 0 \quad \text{and} \quad \Im b'_hZ = 0 \Rightarrow b'_hZ \in ([-12.66, -9.31] \cup [-2.22, 1.14]), 
\]
\[
a'_hZ = 1 \quad \text{and} \quad c'_hZ = 0 \quad \text{and} \quad \Re b'_hZ = 0 \Rightarrow \Im b'_hZ \in [-3.84, 3.84]. 
\]

Let us compare (15) with the \( hZZ \) coupling constraints obtained by the CMS (36) and ATLAS (37) collaborations.

For this purpose we first express our \( XZZ \) couplings in terms of the CMS ones \( \tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \) (we denote \( a_1, a_2, a_3 \) from (36) as \( \tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \) to avoid confusion):
\[
a_Z = \tilde{a} \left( \tilde{a}_1 - \exp(i\phi_{\Lambda_1}) \frac{a_1 + a_2}{\Lambda_1^2} + \frac{m_X^2 - a_1 - a_2}{m_Z^2} \tilde{a}_2 \right), 
\]
\[
b_Z = -2\tilde{a} \frac{m_X^2}{m_Z^2} \tilde{a}_2, 
\]
\[
c_Z = -i\tilde{a} \frac{m_X^2}{m_Z^2} \tilde{a}_3, 
\]
where \( \tilde{a} \equiv \alpha_{\phi}/2 \), \( \alpha_0 \) is the proportionality factor of the amplitude \( A(HZZ) \) of the transition \( X \rightarrow Z^*_1Z^*_2 \) (see Eq. (1) in (36)), \( v \equiv 1/\sqrt{2G_F} \) is the vacuum expectation value of the Higgs field, \( \Lambda_1 \) is a scale of physics beyond the SM, \( \phi_{\Lambda_1} \) is the phase in the term with \( \Lambda_1 \). In general \( \tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \) may depend on \( a_1 \) and \( a_2 \), however in (36) they are set to be constant. The ATLAS \( XZZ \) couplings \( \alpha, \kappa_{SM}, \kappa_{HZZ}, \kappa_{AZZ} \) are related to the CMS ones in the following way:
\[
\tilde{a} \left( \tilde{a}_1 - \exp(i\phi_{\Lambda_1}) \frac{a_1 + a_2}{\Lambda_1^2} + \frac{m_X^2 - a_1 - a_2}{m_Z^2} \tilde{a}_2 \right) = \kappa_{SM} \cos \alpha, \quad \tilde{a} \tilde{a}_2 = \frac{v}{4A} \kappa_{HZZ} \cos \alpha, \quad \tilde{a} \tilde{a}_3 = \frac{v}{4A} \kappa_{AZZ} \sin \alpha, 
\]
where \( A \) is the EFT energy scale. Note that comparing the Lagrangian (1) in (37) with the one describing the interaction of the SM Higgs field with \( ZZ \) and \( W^-W^+ \), one can deduce that the coupling \( g_{HZZ} \) from (1) in (37) is equal to \( 2m_Z^2/v \). In (37) the couplings \( \alpha, \kappa_{SM}, \kappa_{HZZ}, \kappa_{AZZ} \) are considered constant and real.

In Refs. (36, 37) 95% CL allowed regions for \( hZZ \) couplings are reported (see Table IV). Note that
\[
\tilde{\kappa}_{HZZ} \equiv \frac{v}{4A} \kappa_{HZZ}, \quad \tilde{\kappa}_{AZZ} \equiv \frac{v}{4A} \kappa_{AZZ}, 
\]
\[
\frac{\tilde{\kappa}_{HZZ}}{\kappa_{SM}} = -2m_Z^2 a_Z + (m_X^2 - a_1 - a_2) b_Z, \quad \frac{\tilde{\kappa}_{AZZ}}{\kappa_{SM}} \tan \alpha = 2m_Z^2 a_Z + (m_X^2 - a_1 - a_2) b_Z 
\]
and in the limit \( \Lambda_1 \rightarrow \infty \) the CMS and ATLAS ratios coincide:
\[
\frac{\tilde{a}_2}{\tilde{a}_1} = \lim_{\Lambda_1 \rightarrow \infty} \frac{\tilde{\kappa}_{HZZ}}{\kappa_{SM}} = \frac{\tilde{a}_3}{\tilde{a}_1} = \lim_{\Lambda_1 \rightarrow \infty} \left( \frac{\tilde{\kappa}_{AZZ}}{\kappa_{SM}} \tan \alpha \right). 
\]

Following (37), we assume the ATLAS \( hZZ \) couplings to be constant. Then considering the case \( \kappa_{HZZ} = 0 \), we find that our couplings \( a_hZ, b_hZ, c_hZ \) are constant as well (see (19), (20)), and using (18a) we obtain an allowed interval for \( \tilde{\kappa}_{AZZ} \tan \alpha/\kappa_{SM} \) (see Table V). However, in case \( \kappa_{HZZ} = 0 \) the results (18b) and (18c) only show that \( h \) may be the SM Higgs boson, and thus they do not constrain any \( hZZ \) couplings.
Table IV: The CMS [36] and ATLAS [37] 95% CL allowed regions for hZZ couplings. The last row shows the conditions under which these regions have been derived.

| CMS | ATLAS |
|-----|-------|
| $\tilde{\kappa}_{HZZ}$ $\kappa_{SM}$ | $\tilde{\kappa}_{AZZ}$ $\kappa_{SM}$ |
| $\tan\alpha$ |
| $[-2.28, -1.88] \cup [-0.69, \infty)$ | $[-2.05, 2.19]$ |
| Im $\tilde{\kappa}_{HZZ} = 0$, $\phi_{\Lambda_1} = 0$ or $\pi$ | $\kappa_{AZZ} = 0$ $\kappa_{HZZ} = 0$ |

If $\kappa_{HZZ} \neq 0$, then $a_{hZ}$ acquires a dependence on the invariant masses squared $a_1$ and $a_2$, and therefore the constraints [16] and [18] get invalid since they have been derived under the assumption that $|a'_{hZ}|$ is independent of $a_2$. Therefore to constrain the ATLAS couplings in case $\kappa_{HZZ} \neq 0$, we start with Eqs. [13] and [15], which demonstrate that within one standard deviation

$$\Gamma(h \rightarrow Z_1^*Z_2^* \rightarrow 4l)/\Gamma_{SM}(h \rightarrow Z_1^*Z_2^* \rightarrow 4l) \in [0.68, 1.22].$$

To obtain $\Gamma$ we have to calculate the integral (A1) for $a'_{hZ}$ depending on $a_2$. Taking into account the limits of the integration, we substitute $a_2$ with $(m_X - m_Z)^2/2$ in the expression for $a'_{hZ}$ (see [19a], [20]) and therefore derive Eq. (A10) where $a_Z$ has the expression (19a), with $a_2 = (m_X - m_Z)^2/2$. It means that if $\kappa_{HZZ}$ is not zero, we may use [16] and [18] with $a'_{hZ}$ determined by Eq. (19a) where $a_2$ is replaced by $(m_h - m_Z)^2/2$. This conclusion allows us to constrain $\tilde{\kappa}_{HZZ}/\kappa_{SM}$ and $\text{Im } \tilde{\kappa}_{HZZ}/\text{Re } \kappa_{SM}$, as one can see in Table V.

Table V: Our allowed regions for the ATLAS hZZ couplings. The last two rows show the conditions under which these regions have been derived.

| $\tilde{\kappa}_{HZZ}/\kappa_{SM}$ | $\tilde{\kappa}_{AZZ}/\kappa_{SM}$ $\tan\alpha$ | $\text{Im } \tilde{\kappa}_{HZZ}/\text{Re } \kappa_{SM}$ |
|----------------|---------------------|------------------------|
| $[-2.38, -1.89] \cup [-0.24, 1.13]$ | $[-1.28, 1.28]$ | $[-1.01, 1.01]$ |
| $a_2 = (m_h - m_Z)^2/2$ in [19a], $a'_{hZ} = 1$, $\kappa_{HZZ} = 0$, $\text{Im } \tilde{\kappa}_{HZZ}/\text{Re } \kappa_{SM} = 0$ | $a_2 = (m_h - m_Z)^2/2$ in [19a], $a'_{hZ} = 1$, $\kappa_{AZZ} \sin\alpha = 0$, $\text{Im } \kappa_{HZZ} = 0$ | $a'_{hZ} = 1$, $\kappa_{AZZ} \sin\alpha = 0$, $\text{Re } \kappa_{HZZ} = 0$ |

Note that the results [16], [18] along with the regions shown in Table V are estimated with consideration of the one sigma interval in [13], with the approximation [15], the central values of $m_h$, $m_Z$, $\Gamma_Z$ from Table I and Eq. (A10). Comparing Tables IV and V one notices significant overlaps between the constraints reported in papers [36, 37] and our ones. In addition, we present an allowed interval for the ratio $\text{Im } \tilde{\kappa}_{HZZ}/\text{Re } \kappa_{SM}$ unconstrained in Refs. [36, 37].
Fig. 3: The distribution $\frac{1}{\Gamma} \frac{d\Gamma}{da_2}$ as a function of $\sqrt{a_2}$ for the decay $h \to Z_1^* Z_2^* \to f_1 \bar{f}_1 f_2 \bar{f}_2$ in case $|a'_{hZ}| = 1, b'_{hZ} = 0, c'_{hZ} = 0$ (solid line); $|a'_{hZ}| = 1, b'_{hZ} = 0, |c'_{hZ}| = 0.5$ (dashed line); $a'_{hZ} = 1, b'_{hZ} = -0.5, c'_{hZ} = 0$ (dash-dotted line).

We choose the following sets of values of $a'_{hZ}, b'_{hZ}$ and $c'_{hZ}$:

$$|a'_{hZ}| = 1, \ b'_{hZ} = 0, \ c'_{hZ} = 0,$$
$$a'_{hZ} = 1, \ b'_{hZ} = 0, \ c'_{hZ} = 0.5,$$
$$a'_{hZ} = 1, \ b'_{hZ} = 0, \ c'_{hZ} = 0.5 i,$$
$$a'_{hZ} = 1, \ b'_{hZ} = -0.5, \ c'_{hZ} = 0$$  \hspace{1cm} (24)

and

$$a'_{hZ} = 1, \ b'_{hZ} = -0.5 i, \ c'_{hZ} = 0,$$  \hspace{1cm} (25)

which are consistent with the constraints (18). The sets (24) and (25) will be used for examination of further results.

Regarding the selected values in (24) and (25) one should mention that even in the SM the couplings $b_{hZ}$ and $c_{hZ}$ acquire small values due to electroweak radiative corrections where $\text{Im} \ b_{hZ}$ and $\text{Im} \ c_{hZ}$ come from the absorptive parts of the corresponding loop diagrams. In Eqs. (24), (25) we assume that the $hZZ$ vertex may be significantly modified by physics beyond the SM.

It is of interest to study the distribution $\frac{1}{\Gamma} \frac{d\Gamma}{da_2}$ as a function of $\sqrt{a_2}$ for various sets of $a'_{Z}, b'_{Z}, c'_{Z}$. Here $a'_{Z} \equiv a_{Z}(m_{Z}^2, a_2), b'_{Z} \equiv b_{Z}(m_{Z}^2, a_2), c'_{Z} \equiv c_{Z}(m_{Z}^2, a_2)$. In accordance with (8), the function $\frac{1}{\Gamma} \frac{d\Gamma}{da_2}$ is independent of the final fermion state. Figure 3 shows this observable in case $X = h$.

As one can see from Fig. 3, the function $\frac{1}{\Gamma} \frac{d\Gamma}{da_2}$ is sensitive to $b'_{hZ}$ and almost insensitive to $c'_{hZ}$. For this reason, having measured this distribution with sufficient accuracy, one can get significant constraints on the values of $b'_{hZ}$.

However, one should keep in mind that this conclusion is obtained for the case in which $|a'_{hZ}|, |b'_{hZ}|, |c'_{hZ}|$ and
cos(\arg b_{hZ} - \arg a'_{hZ}) are independent of \(a_2\), and their \(a_2\)-dependence can considerably modify the dependence of \(\frac{d\Gamma}{da_2}\). In Sec. 11F we develop methods of getting constraints on the dependences of \(a'_Z\), \(b'_Z\), \(c'_Z\) on \(a_2\).

**F. Connection between the helicity coefficients of the decay \(X \to Z_1^+ Z_2^-\) and observables**

Let us consider now arbitrary dependences of \(a_Z(a_1, a_2)\), \(b_Z(a_1, a_2)\), \(c_Z(a_1, a_2)\) such that the differential width \(\frac{d\Gamma}{da_1 da_2 d\theta_1 d\theta_2}\) has a sharp maximum as a function of \(a_1\) and \(a_2\) at \(\sqrt{a_1} = m_Z\) or \(\sqrt{a_2} = m_Z\) for any \(f_1\), \(f_2\).

From Eq. (5), using approximations analogous to those used when deriving the formulas (11), (12), we carry out integration over \(a_1\) and some of the angular variables. Then we obtain the following relations between observables \(O_i(a_2)\) and the helicity coefficients:

\[
O_1^{(1)}(a_2) \equiv \left( \frac{d\Gamma}{da_2} \right)^{-1} \left( \int_0^{\frac{\pi}{2}} d\theta_1 \frac{d^2\Gamma}{da_2 d\theta_1} - \int_0^{\frac{\pi}{2}} d\theta_2 \frac{d^2\Gamma}{da_2 d\theta_2} \right) = -\frac{3}{2} A_{f_1} \text{Re}(A_1^f A_1'^f) \sum_p |A_p|^2,
\]

\[
O_1^{(2)}(a_2) \equiv \left( \frac{d\Gamma}{da_2} \right)^{-1} \left( \int_0^{\frac{\pi}{2}} d\theta_1 \frac{d^2\Gamma}{da_2 d\theta_1} - \int_0^{\frac{\pi}{2}} d\theta_2 \frac{d^2\Gamma}{da_2 d\theta_2} \right) = -\frac{3}{2} A_{f_2} \text{Re}(A_1^f A_1'^f) \sum_p |A_p|^2,
\]

under the condition \(\sqrt{a_2} \in (0, m_X \sqrt{m_Z^2 + \epsilon_2}]\).

One can write these two formulas in the following way:

\[
O_1^{(1,2)}(a_2) \equiv \left( \frac{d\Gamma}{da_2} \right)^{-1} \left( \int_0^{\frac{\pi}{2}} d\theta_1 \alpha \frac{d^2\Gamma}{da_2 d\theta_1} - \int_0^{\frac{\pi}{2}} d\theta_2 \frac{d^2\Gamma}{da_2 d\theta_2} \right) = \frac{3}{2} A_{f_1,2} \text{Re}(A_1^f A_1'^f) \sum_p |A_p|^2.
\]

Then we deduce that

\[
O_2(a_2) \equiv \left( \frac{d\Gamma}{da_2} \right)^{-1} \left( \int_0^{\frac{\pi}{2}} d\theta_2 \frac{d^2\Gamma}{da_2 d\theta_2} + \int_0^{\frac{\pi}{2}} d\theta_1 \frac{d^2\Gamma}{da_2 d\theta_1} \right) = \left( \frac{d\Gamma}{da_2} \right)^{-1} \left( \int_0^{\frac{\pi}{2}} d\theta_1 \frac{d^2\Gamma}{da_2 d\theta_1} + \int_0^{\frac{\pi}{2}} d\theta_2 \frac{d^2\Gamma}{da_2 d\theta_2} \right)
\]

\[
= \frac{1}{4} \left( \sin \beta - \sin \alpha \right) (3 + \sin^2 \alpha + \sin^2 \beta - \sin \alpha \sin \beta + 3 \sum_p |A_p|^2 (\sin \beta \cos^2 \beta - \sin \alpha \cos^2 \alpha)),
\]

\[
0 \leq \alpha < \beta \leq \frac{\pi}{2},
\]

\[
O_4(a_2) \equiv \left( \frac{d\Gamma}{da_2} \right)^{-1} \left( \int_0^{\frac{\pi}{2}} d\theta_2 \left( \int_0^{\frac{\pi}{2}} d\theta_1 \frac{d^2\Gamma}{da_2 d\theta_1 d\theta_2} \right) - \int_0^{\frac{\pi}{2}} d\theta_1 \frac{d^2\Gamma}{da_2 d\theta_1 d\theta_2} \right) \quad \Rightarrow \quad \frac{9}{16} A_{f_1} A_{f_2} |A_1^f|^2 + |A_1'^f|^2 \sum_p |A_p|^2,
\]

\[
O_4(a_2) \equiv \left( \frac{d\Gamma}{da_2} \right)^{-1} \left( \int_0^{\frac{\pi}{2}} d\phi \frac{d^2\Gamma}{da_2 d\phi} - \int_0^{\frac{\pi}{2}} d\phi \frac{d^2\Gamma}{da_2 d\phi} \right) \Rightarrow \frac{1}{2\pi} \sum_p |A_p|^2.
\]
\[ O_5(a_2) = \left( \frac{d\Gamma}{da_2} \right)^{-1} \left( \int_{0}^{\pi} d\varphi \frac{d^2\Gamma}{da_2 d\varphi} - \int_{\frac{\pi}{2}}^{\pi} d\varphi \cdot \int_{\frac{\pi}{2}}^{\pi} d\varphi - \int_{\frac{-\pi}{2}}^{-\frac{\pi}{2}} d\varphi \right) = -\frac{1}{\pi} \frac{\text{Im}(A_{11}^* A_{11})}{\sum_p |A_p|^2}. \] (31)

\[ O_6(a_2) = \left( \frac{d\Gamma}{da_2} \right)^{-1} \left( \int_{0}^{\pi} d\varphi \frac{d^2\Gamma}{da_2 d\varphi} - \int_{\frac{\pi}{2}}^{\pi} d\varphi \cdot \int_{\frac{\pi}{2}}^{\pi} d\varphi - \int_{\frac{-\pi}{2}}^{-\frac{\pi}{2}} d\varphi \right) = \frac{9}{32} \sqrt{2\pi A_{11}} \frac{\text{Re}(A_{11}^* A_{11})}{\sum_p |A_p|^2}. \] (32)

\[ O_{4}^{(1,2)}(a_2) = 3 \sqrt{2 A_{11}} \text{Im}(A_{11}^* A_{11}^*) \sum_p |A_p|^2, \] (33)

\[ O_{4}^{(1,2)}(a_2) = -3 \sqrt{2 A_{11}} \text{Re}(A_{11}^* A_{11}^*) \sum_p |A_p|^2, \] (34)

\[ O_9(a_2) = \left( \frac{d\Gamma}{da_2} \right)^{-1} \left( \int_{0}^{\pi} d\varphi \frac{d^2\Gamma}{da_2 d\varphi} - \int_{\frac{\pi}{2}}^{\pi} d\varphi \cdot \int_{\frac{\pi}{2}}^{\pi} d\varphi - \int_{\frac{-\pi}{2}}^{-\frac{\pi}{2}} d\varphi \right) = \frac{9}{32} \sqrt{2\pi A_{11}} A_{11} \frac{\text{Im}(A_{11}^* A_{11}^*)}{\sum_p |A_p|^2}. \] (35)

From the measured observables \( O_i(a_2) \) one can get constraints on the dependences of the couplings \( a'_{hZ}(a_2), b'_{hZ}(a_2) \) and \( c'_{hZ}(a_2) \). As for \( O_2(a_2) \), it can be measured at a fixed value of \( \sqrt{a_2} \) and at various values of the parameters \( \beta \) and \( \alpha \). Then, after obtaining central values and uncertainties of a quantity \( |A_{11}^*|^2 / \sum_p |A_p|^2 \) from Eq. (28) at several sets of values of \( \beta, \alpha \), one can combine these central values and uncertainties and thereby get a value of \( |A_{11}^*|^2 / \sum_p |A_p|^2 \) with greater precision than in case of any particular values of \( \beta, \alpha \).

As an illustration of the behavior of these observables, in Fig. 4 we show their \( \sqrt{a_2} \)-dependence with the constant \( a'_{hZ}, b'_{hZ}, c'_{hZ} \) from the sets (44). The observable \( O_2(a_2) \) is presented for \( \beta = 90^\circ \) and \( \alpha = 70^\circ \).

As one can see, for each of the observables \( O_2(a_2), O_3(a_2), O_4(a_2), O_6(a_2) \) their dependences on \( \sqrt{a_2} \) for all the four sets (44) are very close. The observables \( O_2(a_2) \) (at \( \beta = 90^\circ, \alpha = 70^\circ \)) and \( O_4(a_2) \) are relatively large with the maximum values greater than 0.05, and thus to measure these observables a relatively small amount of data is needed, while \( O_3(a_2) \) and \( O_6(a_2) \) are smaller, which complicates their experimental observation.

Further, \( O_{4}^{(1,2)}(a_2), O_5(a_2), O_{8}^{(1,2)}(a_2), O_9(a_2) \) vanish for \( c'_{hZ}(a_2) = 0 \), according to Eqs. (27), (31), (34), and Fig. 4. Therefore, these observables can give significant constraints on the \( CP \)-odd coupling \( c'_{hZ}(a_2) \), although their moduli are relatively small.

The functions \( O_{7}^{(1,2)}(a_2) \) are proportional to \( \text{Im}(a'_{hZ} b'_{hZ}) \) (see (33), (44)), and, consequently, they are equal to zero for any set from (44). Among all the observables under consideration, \( O_{7}^{(1,2)}(a_2) \) are the only ones vanishing in case \( b'_{hZ}(a_2) = 0 \) for any \( a'_{hZ}(a_2) \) and \( c'_{hZ}(a_2) \). Therefore, knowing the dependences \( O_{7}^{(1,2)}(a_2) \) allows one to get notable constraints on the function \( b'_{hZ}(a_2) \). Although in case (25) these observables turn out to be relatively small in absolute value (see Fig. 5).
Fig. 4: The observables $O_1^{(1,2)}$, $O_2$ (at $\beta = 90^\circ$, $\alpha = 70^\circ$), $O_3$, $O_4$, $O_5$, $O_6$, $O_9^{(1,2)}$, $O_9$ for the decay $h \to Z_1^*Z_2^* \to l_1^- l_2^+ l_2^+ l_2^-$ ($l_j = e, \mu, \tau, l_1 \neq l_2$) as functions of $\sqrt{a_2}$ in case $a_{hZ}^{'} = 1, b_{hZ} = 0, c_{hZ}^{'} = 0$ (solid lines); $a_{hZ}^{'} = 1, b_{hZ} = 0, c_{hZ}^{'} = 0.5$ (dashed lines); $a_{hZ}^{'} = 1, b_{hZ} = 0, c_{hZ}^{'} = 0.5i$ (dash-dotted lines); $a_{hZ}^{'} = 1, b_{hZ} = -0.5, c_{hZ}^{'} = 0$ (dotted lines).

Fig. 5: The observables $O_7^{(1,2)}$ for the decay $h \to Z_1^*Z_2^* \to l_1^- l_1^+ l_1^+ l_2^-$ ($l_j = e, \mu, \tau, l_1 \neq l_2$) versus $\sqrt{a_2}$ in case $a_{hZ}^{'} = 1, b_{hZ} = -0.5i, c_{hZ}^{'} = 0$. 
Note that from (27)-(35), regardless of the values of the couplings \( a'_Z(a_2), b'_Z(a_2) \) and \( c'_Z(a_2) \), it follows that for any \( a_2 \)

\[
O^{(1,2)}_1 \in \left[-\frac{3}{4}A_{f_1}, \frac{3}{4}A_{f_2}\right], \quad O_2 \in [0, 1], \quad O_3 \in [0, \frac{9}{16}A_{f_1}A_{f_2}], \quad O_4, O_5 \in [-\frac{1}{2\pi}, \frac{1}{2\pi}], \quad O_6, O_7 \in [-\frac{9}{64}\sqrt{2\pi}A_{f_1}A_{f_2}, \frac{9}{64}\sqrt{2\pi}A_{f_1}A_{f_2}], \quad O^{(1,2)}_8, O^{(1,2)}_9 \in \left[-\frac{3}{16}\sqrt{2A_{f_1}}, \frac{3}{16}\sqrt{2A_{f_2}}\right].
\] (36)

Since \( A_{e-} = A_{\mu-} = A_{\tau-} \approx 0.214, A_{e+} = A_{\mu+} = A_{\nu_e} = 1, A_u = A_c = A_t \approx 0.697, A_d = A_s = A_b \approx 0.941 \), the moduli of \( O^{(1,2)}_1(a_2), O^{(1,2)}_7(a_2), O^{(1,2)}_8(a_2), O_3(a_2), O_6(a_2), O_9(a_2) \) for the decays (I) with quarks and/or neutrinos in the final states are greater than those for the decays (I) to leptons, therefore, the former processes seem more feasible for experimental study. On the other hand, detection of leptons is much simpler. That is why the study of each decay channel of the type (I) has advantages and disadvantages which strongly depend on experimental methods and parameters of detectors. Consequently, measurement of the observables \( O^{(1,2)}_1(a_2), \ldots, O_9(a_2) \) for various decay channels and for various invariant masses of the fermion pair \( \sqrt{a_2} \) may help to put constraints on the \( XZZ \) couplings \( a'_Z(a_2), b'_Z(a_2) \) and \( c'_Z(a_2) \).

III. CONCLUSIONS

In the present paper the decay of a neutral particle \( X \) with zero spin and arbitrary \( CP \) parity into two off-mass-shell \( Z \) bosons \( (Z^+_1 \) and \( Z^+_2) \) each of which decays to a fermion-antifermion pair, i.e. the decay \( X \to Z^+_1Z^+_2 \to f_1\bar{f}_1f_2\bar{f}_2 \), has been considered. The given decay has been examined at tree level for the non-identical fermions, \( f_1 \neq f_2 \). In the approximation of the massless fermions a formula for the fully differential width has been obtained. It has been established that the narrow-\( Z \)-width approximation is applicable for finding differential decay widths of \( X \to Z^+_1Z^+_2 \to f_1\bar{f}_1f_2\bar{f}_2 \) only if the invariant mass \( \sqrt{a_2} \) of the pair \( f_2\bar{f}_2 \) lies in an interval \( \left(0, m_X - \sqrt{m^2_Z + \varepsilon^2_2}\right) \).

If the parameter \( \varepsilon_2 \) gets larger, the accuracy of the used approximation increases, but the interval in which the approximation is valid reduces. As an optimal value of \( \varepsilon_2 \) we have chosen \( \varepsilon_2 = 3m_Z\Gamma_Z \).

In the narrow-\( Z \)-width approximation, but without the neglect of \( \Gamma_Z \) in the propagator of \( Z^+_2 \), a formula for the total width of the decay (I) and the total width of \( h \to Z^+_1Z^+_2 \) have been derived. The former formula is valid in case \( f_1 = f_2 \) as well. Note that in Ref. 23 within the framework of the SM the total width of the decay \( X \to ZZ^* \to Zf\bar{f} \) has been found in the approximation \( \Gamma_Z \approx 0 \) in the propagator of \( Z^* \). In an analogous way one can obtain the total width of the decay (I) in the SM after the neglect of \( \Gamma_Z \) in the propagator of \( Z^+_2 \), however the formula (A2), derived in the present paper, is more general and more precise.

Using the CMS data 3, we have found constraints on the couplings \( a'_Z, b'_Z, c'_Z \), which determine the \( hZZ \) interaction and the \( CP \) properties of the boson \( h \) detected in the experiments 1. Comparing our constraints with those reported in Refs. 35, 37, one can notice appreciable overlaps between the three results. Besides, we have derived an allowed interval for a ratio not studied in 36, 37. Taking our allowed regions into account, we have
selected several sets of values of the couplings $g'_{aZ}$ ($q = a, b, c$) and analyzed results for these sets.

The observables $O_{1}^{(1,2)}(a_2), ..., O_{9}(a_2)$, measurement of which will allow one to get constraints on the dependences of $g'_{Z}$ on $\sqrt{a_{Z}}$, are defined. It is shown that the observables $O_{1}^{(1,2)}(a_2), O_{5}(a_2), O_{8}^{(1,2)}(a_2), O_{9}(a_2)$ become zero in case $c'_{Z}(a_2) = 0$, and therefore their experimental dependences on $\sqrt{a_{Z}}$ can put significant constraints on the $CP$-odd coupling $c'_{Z}(a_2)$. The observables $O_{8}^{(1,2)}(a_2)$ vanish if $b'_{Z}(a_2) = 0$, and, therefore, their measurement is important for finding the $CP$-even coupling $b'_{Z}(a_2)$.

Note that the absolute values of $O_{1}^{(1,2)}(a_2), O_{7}^{(1,2)}(a_2), O_{8}^{(1,2)}(a_2), O_{3}(a_2), O_{6}(a_2)$ and $O_{9}(a_2)$ for the decays $|f_1$ and/or $f_2$ is a quark or a neutrino are greater than those for the processes in which the fermions are leptons. At the same time, the processes with the leptons are much more convenient from the experimental point of view.

Thus, measurement of the observables $O_{1}^{(1,2)}(a_2), ..., O_{9}(a_2)$ for the decays $|f_1, f_2$ can help to clarify the $CP$ properties of the particle $X$ and the structure of the amplitude of the decay $X \rightarrow Z_{1}^{'*}Z_{2}^{'*}$.

The authors thank Sergiy Ivashyn for useful discussions. The work is partially supported by the National Academy of Sciences of Ukraine (project 1015-1/2015) and the Ministry of Education and Science of Ukraine (project 0115U00473).

Appendix A: Calculation of the total widths of the decays $X \rightarrow Z_{1}Z_{2} \rightarrow f_{1}f_{1}, f_{2}f_{2}$ and $h \rightarrow Z_{1}^{'*}Z_{2}^{'*}$

In this Appendix we calculate the total width of the decay $X \rightarrow Z_{1}Z_{2} \rightarrow f_{1}f_{1}, f_{2}f_{2}$ for the $m_{X}$-values such that $m_{X} > m_{Z}$ and for the dependences of $a_{Z}(a_1, a_2), b_{Z}(a_1, a_2), c_{Z}(a_1, a_2)$ such that the differential width $\frac{d\Gamma}{da_{Z}}$ has a sharp maximum when $\sqrt{a_{1}} = m_{Z}$ or $\sqrt{a_{2}} = m_{Z}$ and the functions $|a'_{Z}|, |b'_{Z}|, |c'_{Z}|, \cos(\arg b'_{Z} - \arg a'_{Z})$ ($q'_{Z} \equiv q_{Z}(m_{Z}^{2}, a_2)$; $q = a, b, c$) are independent of $a_2$. For example, $|a_{Z}| \approx 1, b_{Z} \approx 0, c_{Z} \approx 0$ are such dependences (see Sec. II B). Then we calculate the total decay width of $h \rightarrow Z_{1}^{'*}Z_{2}^{'*}$ and examine the applicability of an approximation $\Gamma_{Z} \approx 0$ for derivation of the total widths.

1. The total width of the decay $X \rightarrow Z_{1}Z_{2} \rightarrow f_{1}f_{1}, f_{2}f_{2}$

Analogously to the derivation of Eq. (11), we find that

$$\Gamma \approx \frac{-2\pi}{m_{Z}^{2}\Gamma_{Z}} \frac{(m_{X} - m_{Z})^{2}}{0} \left( m_{Z}^{2}, a_{2} \right).$$  \hspace{1cm} (A1)

Let us consider the case wherein $|a'_{Z}|, |b'_{Z}|, |c'_{Z}|, \cos(\arg b'_{Z} - \arg a'_{Z})$ are independent of $a_2$. Having exactly calculated the integral in Eq. (A1) with allowance for Eqs. (I0) and (I9), we obtain:

$$\Gamma \approx \left( a'_{f_{1}}^{2} + v_{f_{1}}^{2} \right) \left( a'_{f_{2}}^{2} + v_{f_{2}}^{2} \right) f_{0}(a'_{Z}, b'_{Z}, c'_{Z}, m_{Z}, \Gamma_{Z}, s),$$ \hspace{1cm} (A2)
where

\[ f_0(a'_2, b'_2, c'_2, m_Z, \Gamma_Z, s) = \frac{\sqrt{2}G^4_f}{2m_3^{\pi+1}}Z \]

\[ \times (1 - \alpha) \left( -24(23 \alpha - 5)|a'_2|^2 + (3 \alpha^2 - 37 \alpha^2 - \alpha(235 + 6\beta^2) + 77 - 54\beta^2)|b'_2|^2 \right. \]

\[ - 16(2\alpha^2 + 26 \alpha - 13 + 3 \beta^2)|Re(a'_2 b'_2) + 64(\alpha^2 + 40 \alpha - 11 + 6\beta^2)|c'_2|^2 \right) + 6 \ln \left( \frac{1}{\alpha} \right) \]

\[ \times (4(12 \alpha^2 - 18 \alpha + 3 - \beta^2)|a'_2|^2 + (30 \alpha^2 - 10 \alpha(3 - \beta^2) + 5 - 10 \beta^2 + \beta^4)|b'_2|^2 \]

\[ + 8(6 \alpha^2 - \alpha(9 - \beta^2) + 2 - 2\beta^2)|Re(a'_2 b'_2) - 32(6 \alpha^2 - \alpha(9 - \beta^2) + 1 - 3\beta^2)|c'_2|^2 \]

\[ + 3\beta \frac{\sqrt{1}}{\beta} \left( P(\alpha, \beta, a'_2, b'_2, c'_2, r_+, -4\beta r_-) \right. \]

\[ \times \ln (1 - \alpha)^2 \sqrt{(4\alpha - 1 + \beta^2)^2 + 4\beta^2 + (3\alpha^2 - 1\beta^2 + \beta^2(\alpha + 1)^2 + s \sqrt{2}(1 - \alpha)((3\alpha - 1)r_- - \beta(\alpha + 1)r_+) \]

\[ + 2P(\alpha, \beta, a'_2, b'_2, c'_2, r_-, 4\beta r_+) \left( \pi - \arg(-\alpha(3\alpha - 1 - \beta^2) - \beta^2 + s \frac{1 - \alpha}{\sqrt{2}}(\beta r_+ - \alpha r_-) \right. \]

\[ \left. + i(1 - \alpha)(s \frac{\alpha r_+ + \beta r_-}{\sqrt{2}} - \beta(1 - \alpha))) \right) \right], \quad (A3) \]

\[ \alpha \equiv \left( \frac{m_\beta}{m_X} \right)^2, \quad \beta \equiv \frac{m_\beta}{m_X}, \]

\[ P(\alpha, \beta, a'_2, b'_2, c'_2, x, y) \equiv 2(2x(12 \alpha^2 - 4 \alpha + 1 - \beta^2) + y(6 \alpha - 1))|a'_2|^2 + (x(16 \alpha^2 - 8 \alpha(1 - \beta^2) + 1 \]

\[ - 6\beta^2 + \beta^4) + y(4\alpha - 1 + \beta^2)|b'_2|^2 + (4x(8 \alpha^2 - 2 \alpha(3 - \beta^2) + 1 - 3\beta^2) \]

\[ + y(8 \alpha - 3 + \beta^2)|Re(a'_2 b'_2) - 8\alpha(4 \alpha(\alpha^2 - \alpha(9 - \beta^2) - 2\beta^2) \]

\[ + y(6 \alpha - 1 + \beta^2)|c'_2|^2, \quad (A4) \]

\[ r_\pm \equiv \sqrt{(4\alpha - 1 + \beta^2)^2 + 4\beta^2 \pm (4\alpha - 1 + \beta^2)} \]

\[ (A5) \]

In place of s one may take 1 or -1 (\( f_0(a'_2, b'_2, c'_2, m_Z, \Gamma_Z, -1) = f_0(a'_2, b'_2, c'_2, m_Z, \Gamma_Z, 1) \)). In this article the argument \( \arg z \) of a complex number \( z \) is defined as follows:

\[ \arg z = \arctan \frac{\text{Im } z}{\text{Re } z} + \pi \text{n(Re } z, \text{ Im } z) \quad \forall z \in \mathbb{C}|\text{Re } z \neq 0, \]

\[ \arg z = \pi \left( \frac{1}{2} + \Theta(-\text{Im } z) \right) \quad \forall z \in \mathbb{C}|(\text{Re } z = 0 \text{ and Im } z \neq 0), \quad (A6) \]

where \( n(x, y) \equiv \Theta(-x) + 2 \Theta(x)\Theta(-y) \quad \forall x \neq 0, \]

\[ \Theta(x) \equiv 0 \quad \forall x \in (-\infty, 0], \quad \Theta(x) \equiv 1 \quad \forall x \in (0, +\infty). \quad (A7) \]

From the definition \( [A6] \) it follows that \( \arg z \) is the angle counted clockwise on the complex plane from the vector (Re z, Im z) towards the vector (1, 0) and \( \arg z \in [0, 2\pi) \). Sometimes in literature a different function

\[ \arg' z \equiv \arg z - 2\pi \Theta(-\text{Im } z) \quad (A8) \]
is used as the argument of $z$. From (A8) it follows that $\arg' z \in (-\pi, \pi]$. Note that we have already used $\arg z$ above in the expression $\cos(\arg bZ - \arg aZ)$, but since $\cos(\arg bZ - \arg aZ) = \cos(\arg' bZ - \arg' aZ)$, at that point the distinction between $\arg z$ and $\arg' z$ was irrelevant.

Calculating the integral over $a_2$ in Eq. (A11), one finds an antiderivative of $f(m_Z^2, a_2)$ on the interval $[0, (m_X - m_Z)^2]$. In this antiderivative the function $u_1(a_2)$ naturally appears, where $u_1(a_2)$ is a complex-valued dimensionless function such that

$$\forall a_2 \in [0, (m_X - m_Z)^2) \quad \text{Im} u_1(a_2) \neq 0,$$

$$\text{Im} u_1((m_X - m_Z)^2) = 0, \quad \text{Re} u_1((m_X - m_Z)^2) < 0. \quad \text{(A9)}$$

$\arg' u_1(a_2)$ does not emerge in place of $\arg u_1(a_2)$ since, according to (A8), the function $\arg' z$ has a discontinuity on the half-line $\text{Im} z = 0, \text{Re} z < 0$ and thus $\arg' u_1(a_2)$ has a discontinuity at the point $a_2 = (m_X - m_Z)^2$. To avoid this drawback it is convenient to use $\arg z$ in Eq. (A3).

Note that in case of the Higgs boson, i.e. $X = h$, Eq. (A3) can also be written in terms of the function $arg'$; for this one has to substitute in Eq. (A3) $\pi - arg ...$ by $s\pi - arg' ...$, since according to (A8) and to data of Table I $\pi - arg ... = s\pi - arg' ...$.

In case of the identical fermions, $f_1 = f_2$, one may neglect the interference term and then in order to obtain a formula for $\Gamma$ one has to multiply the right-hand side of the relation (A2) by $\frac{1}{2\pi^2}$ (in view of the identity of the final fermions) and by 2 (since the contribution of the diagram with the permutation of the particles to $\Gamma$ is equal to that of the diagram without the permutation), i.e. to multiply the right-hand side by $\frac{1}{4}$. Consequently, for any $f_1$ and $f_2$

$$\Gamma \approx (1 - \frac{1}{2}\delta_{f_1,f_2})(a_{f_1}^2 + v_{f_1}^2)(a_{f_2}^2 + v_{f_2}^2)f_0(a_{f_2}', b_{f_2}', c_{f_2}', m_Z, Z, s) \equiv \Gamma_X, \quad \text{(A10)}$$

where $\delta_{f_1,f_2} \equiv 0 (1)$ at $f_1 \neq f_2$ ($f_1 = f_2$). The neglected interference term seems small based on qualitative arguments of Ref. [38]. For a quantitative estimate we can use Ref. [39] (see Table 1 there), according to which the interference contribution to $\Gamma(h \rightarrow Z^*_f Z^*_f \rightarrow 4e)$ in the SM at tree level is 5.80% for $m_h = 140$ GeV.

In Ref. [22] the width of the decay $h \rightarrow ZZ^* \rightarrow Zf\bar{f}$ has been derived at tree level in the SM after the neglect of $\Gamma_Z$ in the propagator of $Z^*$. Following [23], when calculating the integral in Eq. (A11), in the expression for $f(m_Z^2, a_2)$ we may also neglect $\Gamma_Z$, and then we obtain the following approximate formula for $\Gamma$ in the SM:

$$\Gamma|_{SM} \approx (1 - \frac{1}{2}\delta_{f_1,f_2})\frac{\sqrt{2G^2(m_Z^2 - m_X^2)}}{2\cdot 2 \cdot \pi^4}\frac{(a_{f_1}^2 + v_{f_1}^2)(a_{f_2}^2 + v_{f_2}^2)}{6 - 8 \alpha + 20 \alpha^2} \arccos \left(\frac{3\alpha - 1}{2\alpha^2}\right) + \frac{1 - \alpha}{\alpha}(2 - 13\alpha + 47\alpha^2) + 3(1 - 6\alpha + 4\alpha^2) \ln \frac{1}{\alpha}$$

$$\equiv \Gamma_0|_{SM}. \quad \text{(A11)}$$

From (A11) and (A10) we obtain that at $m_X = m_h$

$$\Gamma_0|_{SM} \approx 1.001 \times \Gamma_X|_{SM}. \quad \text{(A12)}$$
Besides, $\Gamma_0 > \Gamma_{\Gamma Z}$ (for any $a'_z, b'_z, c'_z, f_1, f_2$) since when deriving the formula for $\Gamma_0$ one neglects the width $\Gamma_Z$ in $f(m^2_Z, a_2)$ and the value of the integral increases. Still according to (A12), the difference between $\Gamma_0|_{SM}$ and $\Gamma_{\Gamma Z}|_{SM}$ is about one per mille.

Finally, note that at $m_X = m_h$ we can represent the dependence of the function $f_0$ on the $XZZ$ couplings $a'_z, b'_z, c'_z$ in the convenient form:

\begin{equation}
 f_0(a'_z, b'_z, c'_z, m_Z, \Gamma_Z, s) \approx (3.359|a'_z|^2 + 0.052|b'_z|^2 + 0.594 \text{Re}(a'_z b'_z) + 0.125|c'_z|^2) \text{ keV}. \tag{A13}
\end{equation}

2. The total width of the decay $h \rightarrow Z_1^*Z_2^*$

The total decay width $\Gamma(h \rightarrow Z_1^*Z_2^*)$ is

\begin{equation}
 \Gamma(h \rightarrow Z_1^*Z_2^*) = \sum_{f_1} \sum_{f_2 \geq f_1} \Gamma|_{m_X=m_h}, \tag{A14}
\end{equation}

where the sums run over the fermions $e^-, \mu^-, \tau^-, \nu_e, \nu_\mu, \nu_\tau, u_i, d_i, c_i, s_i, b_i$ (since $m_h \in (4m_6, 2m_t)$, $i = r, g, b$ is an index of quark color. It follows from Eqs. (A14), (A12) that in the SM

\begin{equation}
 \Gamma_0(h \rightarrow Z_1^*Z_2^*) \approx 1.001 \times \Gamma_{\Gamma Z}(h \rightarrow Z_1^*Z_2^*). \tag{A15}
\end{equation}

Further we use Eq. (A10) since it is more precise than Eq. (A11) and consider the case wherein $|a'_{hZ}|, |b'_{hZ}|, |c'_{hZ}|$, $\cos(\arg b'_{hZ} - \arg a'_{hZ})$ do not depend on $a_2$. From Eqs. (A14), (A10), (A3) we derive that

\begin{equation}
 \Gamma(h \rightarrow Z_1^*Z_2^*) \approx f_0(a'_{hZ}, b'_{hZ}, c'_{hZ}, m_Z, \Gamma_Z, s)|_{m_X=m_h} \left( \frac{1}{2} \sum_{f_1} (a^2_{f1} + v^2_{f1}) + \frac{1}{2} \sum_{f_2 \neq f_1} (a^2_{f2} + v^2_{f2}) \right) \approx \frac{f_0(...)}{18} \left( \frac{103}{2} - 100 \left( \frac{m_W}{m_Z} \right)^2 + 80 \left( \frac{m_W}{m_Z} \right)^4 \right)^2. \tag{A16}
\end{equation}

Carrying out calculations, we find the total decay width for the sets $[24], [25]$

\begin{align*}
 |a'_{hZ}| = 1, b'_{hZ} = 0, c'_{hZ} = 0 \Rightarrow \Gamma(h \rightarrow Z_1^*Z_2^*) \approx 91.16^{+16.66}_{-14.50} \text{ keV}, \\
 |a'_{hZ}| = 1, b'_{hZ} = 0, |c'_{hZ}| = 0.5 \Rightarrow \Gamma(h \rightarrow Z_1^*Z_2^*) \approx 92.01^{+16.85}_{-14.67} \text{ keV}, \\
 a'_{hZ} = 1, b'_{hZ} = -0.5, c'_{hZ} = 0 \Rightarrow \Gamma(h \rightarrow Z_1^*Z_2^*) \approx 83.45^{+15.06}_{-13.14} \text{ keV}, \\
 a'_{hZ} = 1, b'_{hZ} = \pm 0.5i, c'_{hZ} = 0 \Rightarrow \Gamma(h \rightarrow Z_1^*Z_2^*) \approx 91.51^{+16.74}_{-14.57} \text{ keV}. \tag{A17}
\end{align*}

The uncertainties shown in Eqs. (A17) are calculated by finding the maximum and minimum values of the function $\Gamma(h \rightarrow Z_1^*Z_2^*)$ in the region $v \in [v_0 - 3\sigma_v, v_0 + 3\sigma_v] \ (v = G_F, m_h, m_Z, m_W, \Gamma_Z)$. Here $v_0$ is the central value of a quantity $v$, $\sigma_v$ is the 1-standard-deviation uncertainty of $v$; according to the data of Table III $G_{F0} =$
1.1663787 \times 10^{-5} \text{ GeV}^{-2}, \sigma_G = 6 \times 10^{-12} \text{ GeV}^{-2}, m_{h0} = 125.7 \text{ GeV}, \sigma_{m_h} = 0.4 \text{ GeV} \text{ etc.}

[1] G. Aad et al. (ATLAS Collaboration), Phys. Lett. B 716, 1 (2012);
S. Chatrchyan et al. (CMS Collaboration), Phys. Lett. B 716, 30 (2012).
[2] G. Aad et al. (ATLAS Collaboration), Phys. Lett. B 726, 88 (2013).
[3] S. Chatrchyan et al. (CMS Collaboration), Phys. Rev. D 89, 092007 (2014).
[4] S. Chatrchyan et al. (CMS Collaboration), Phys. Rev. Lett. 110, 081803 (2013).
[5] A. Pilaftsis, C.E.M. Wagner, Nucl. Phys. B 553, 3 (1999).
[6] V. Barger, P. Langacker, M. McCaskey et al., Phys. Rev. D 79, 015018 (2009).
[7] G.C. Branco, P.M. Ferreira, L. Lavoura et al., Phys. Rep. 516, 1 (2012).
[8] D. Bailin, A. Love, Cosmology in Gauge Field Theory and String Theory (Institute of Physics Publishing, Bristol-Philadelphia, 2004).
[9] M.B. Voloshin, Phys. Rev. D 86, 093016 (2012).
[10] F. Bishara, Y. Grossman, R. Harnik et al., JHEP 1404, 084 (2014).
[11] A.Yu. Korchin, V.A. Kovalchuk, Phys. Rev. D 88, 036009 (2013);
A.Yu. Korchin, V.A. Kovalchuk, Acta Phys. Polon. B 44, 2121 (2013).
[12] J. S. Gainer, W. Y. Keung, I. Low and P. Schwaller, Phys. Rev. D 86, 033010 (2012).
[13] A.Y. Korchin and V.A. Kovalchuk, Eur. Phys. J. C 74, 3141 (2014).
[14] S.Y. Choi, D.J. Miller, M.M. Mühlleitner et al., Phys. Lett. B 553, 61 (2003).
[15] V.A. Kovalchuk, J. Exp. Theor. Phys. 107, 774 (2008).
[16] A. Menon, T. Modak, D. Sahoo et al., Phys. Rev. D 89, 095021 (2014).
[17] Y. Sun, X.-F. Wang, and D.-N. Gao, Int. J. Mod. Phys. A 29, 1450086 (2014).
[18] A. De Rujula, J. Lykken, M. Pierini et al., Phys. Rev. D 82, 013003 (2010);
Y. Gao, A. V. Gritsan, Z. Guo et al., Phys. Rev. D 81, 075022 (2010);
S. Bolognesi, Y. Gao, A. V. Gritsan et al., Phys. Rev. D 86, 095031 (2012);
D. Stolarski and R. Vega-Morales, Phys. Rev. D 86, 117504 (2012);
P. Avery, D. Bourilkov, M. Chen et al., Phys. Rev. D 87, no. 5, 055006 (2013);
M. Chen, T. Cheng, J. S. Gainer et al., Phys. Rev. D 89, no. 3, 034002 (2014);
B. Bhattacharjee, T. Modak, S. K. Patra et al., arXiv:1503.08924 [hep-ph].
[19] M. Gasperini, Phys. Lett. B 327, 214 (1994).
[20] Z. Chacko, R. Franceschini, and R.K. Mishra, JHEP 1304, 015 (2013).
[21] B. Bellazzini, C. Csáki, J. Hubisz et al., Eur. Phys. J. C 73, 2333 (2013).
[22] J. Serra, EPJ Web Conf. 60, 17005 (2013).
[23] W.-Y. Keung and W.J. Marciano, Phys. Rev. D 30, 248 (1984).
[24] R. M. Godbole, D. J. Miller and M. M. Mühlleitner, JHEP 0712, 031 (2007).
[25] W.-Y. Keung, I. Low, and J. Shu, Phys. Rev. Lett. 101, 091802 (2008);
    T.L. Trueman, Phys. Rev. D 18, 3423 (1978).
    J. R. Dell’Aquila and C. A. Nelson, Phys. Rev. D 33, 80 (1986).
[26] K.A. Olive et al. (Particle Data Group), Chin. Phys. C 38, 090001 (2014).
[27] N.N. Achasov and V.V. Gubin, JETP Lett. 62, 191 (1995) [Pisma Zh. Eksp. Teor. Fiz. 62, 182 (1995)].
[28] D. Berdine, N. Kauer, and D. Rainwater, Phys. Rev. Lett. 99,111601 (2007).
[29] C.F. Uhlemann and N. Kauer, Nucl. Phys. B 814,195 (2009).
[30] A. Bredenstein, A. Denner, S. Dittmaier and M. M. Weber, Phys. Rev. D 74, 013004 (2006).
[31] G. Aad et al. (ATLAS Collaboration), arXiv:1504.05833 [hep-ex].
[32] LHC Higgs Cross Section Working Group, https://twiki.cern.ch/twiki/bin/view/LHCPhysics/CERNYellowReportPageAt8TeV.
[33] V. Khachatryan et al. (CMS Collaboration), Phys. Lett. B 736, 64 (2014).
[34] S. Heinemeyer et al. (LHC Higgs Cross Section Working Group), arXiv:1307.1347v2 [hep-ph].
[35] J. S. Gainer, J. Lykken, K. T. Matchev et al., Phys. Rev. Lett. 111, 041801 (2013).
[36] V. Khachatryan et al. (CMS Collaboration), Phys. Rev. D 92, no. 1, 012004 (2015).
[37] G. Aad et al. (ATLAS Collaboration), arXiv:1506.05669v1 [hep-ex].
[38] J. C. Romão and S. Andringa, Eur. Phys. J. C 7, 631 (1999).