A new approach to the evolving 4+1 spacetime metric

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Abstract. We recently proposed [1,2] field equations that prescribe a metric $g_{\alpha\beta}(x, \tau)$ that is local in the spacetime coordinates $x$ and evolves with the external “worldtime” $\tau$ of the Stueckelberg Horwitz Piron (SHP) framework. As in SHP electrodynamics, these field equations exhibit a formal 5D symmetry ($\alpha, \beta = 0, 1, 2, 3, 5$), that is strategically broken to 4+1 representations of the Lorentz group. The resulting canonical formalism for this metric embodies a natural foliation of a 5D pseudo-manifold (encompassing both geometry and dynamics) into the $\tau$-parameterized 4D spacetime posed in SHP theory. In this paper, we consider the linearized equations for weak gravitation in this 4+1 formalism, leading to a more straightforward and intuitive derivation of the coupled first-order evolution equations for the metric.

1. Introduction

The Stueckelberg Horwitz Piron (SHP) approach to relativity [3–10] formalizes the distinction between two aspects of time [11]: coordinate and chronology. Particles and fields are defined locally with respect to a spacetime manifold, but they evolve according to a chronological time $\tau$ external to the manifold. In any coordinate system, the temporal ordering of spacetime events is characterized through the functional dependence of classical particle configurations $x^\mu(\tau)$ or quantum states $\psi(x, \tau)$ as $\tau$ grows monotonically. The indices $\mu, \nu, \lambda, \sigma = 0, 1, 2, 3$ signify a coordinatization of the manifold, including the coordinate time $x^0 = ct$ measured by laboratory clocks. Fields and potentials, such as the electromagnetic field $f_{\alpha\beta}(x, \tau)$ and the metric $g_{\alpha\beta}(x, \tau)$, are defined to be invariant under gauge transformations that depend on both $x$ and $\tau$ and so carry indices $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3, 5$. The equations of SHP field theory may thus exhibit an apparent 5D symmetry containing O(3,1) — possibly O(4,1) or O(3,2) — but any higher symmetry must be broken to 4+1 representations of Lorentz symmetry in the presence of matter.

Because the evolving spacetime metric $g_{\alpha\beta}(x, \tau)$ describes both the instantaneous structure of a 4D manifold and the evolution of that structure as $\tau$ advances, it is natural to conflate geometry with evolution into a 5D manifold [1], and then strategically break the symmetry of the associated 5D Einstein equations to 4+1. The geometrical features of the 5D manifold embodied in $g_{\alpha\beta}(x, \tau)$ are expressed in the 4D hypersurface through the induced 4D metric $\gamma_{\mu\nu}(x, \tau)$ and the extrinsic curvature $K_{\mu\nu}(x, \tau)$. In a recent paper [2] we extended the 3+1 ADM formalism [12] in general relativity (GR) to 4+1 and obtained a pair of coupled, nonlinear, differential equations of first order in $\tau$-derivatives of $\gamma_{\mu\nu}(x, \tau)$ and $K_{\mu\nu}(x, \tau)$. As in the 3+1 ADM formalism, this system can be put into canonical form (a certain algebraic combination...
of $\gamma_{\mu\nu}(x, \tau)$ and $K_{\mu\nu}(x, \tau)$ forms a momentum conjugate to the metric), although practical computations are more convenient in the Lagrangian form. Because the evolution of the spacetime metric is not parameterized by the time coordinate $x^0$, which is itself determined by the metric, a principal aspect of the “problem of time” in GR is eliminated. In this paper, we consider the linearized equations for weak gravitation in the 4+1 formalism and present a more straightforward derivation of the 4+1 differential equations, that offers more intuitive insight into their meaning.

In Section 2 we provide a brief overview of the SHP framework for electrodynamics and gravitation. Section 3 discusses the weak field approximation and shows that consistency with standard GR and gravitational phenomenology requires a modification of the field equations, breaking the 5D symmetry to 4+1. We emphasize that the geometrical structures retain 5D symmetry, and the broken symmetry obtains for the physical law that poses a relationship between geometry and sources of mass-energy. In Section 4 we summarize the 4+1 formalism for GR obtained by projection onto the spacetime as a 4D hypersurface of a 5D pseudo-spacetime. A new derivation of the 4+1 formalism for weak fields is given in section 5. In Section 6 we present examples of how this formalism can be applied to numerical computation. In particular, we show that a Schwarzschild metric with variable mass $m(\tau)$ parameter cannot be obtained by a perturbation induced by the motion of a localized source particle.

2. Elements of SHP theory

In suggesting that antiparticles be treated as particles traveling backward in time, Stueckelberg introduced a number of closely related innovations. Pair processes are observed when a worldline reverses $t$-direction at moment $t^*$, and so in pair annihilation, laboratory apparatus registers two events for $t < t^*$ but none for $t > t^*$. Classically, the worldline evolves continuously with $\dot{x}^0(\tau) > 0$ until some $\tau^*$ after which $\dot{x}^0(\tau) < 0$. Although this $\tau$ is often called proper time, Stueckelberg observed that in pair processes the squared interval

$$c^2 ds^2(\tau) = -\eta_{\mu\nu}dx^\mu dx^\nu = -\dot{x}^2(\tau) d\tau^2 \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \quad (1)$$

undergoes two sign reversals as the particle crosses the spacelike region separating future-oriented evolution to past-oriented evolution. The parameter $\tau$ must therefore be an irreducible chronological time, independent of the space and time coordinates, and similar to the external time $t$ in nonrelativistic Newtonian mechanics \[11\]. Stueckelberg also recognized that the standard Maxwell field $F^{\mu\nu}(x)$ would not permit $ds^2(\tau)$ to change sign and proposed a modified Lorentz force

$$\frac{D}{D\tau} \dot{x}^\mu = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = \frac{e}{M} \left[ F^{\mu\nu}(x) g_{\nu\rho} \frac{dx^\rho}{d\tau} + G^\mu(x) \right] \quad (2)$$

in which the vector field $G^\mu(x)$ is required to overcome conservation of $\dot{x}^2$

$$\frac{D}{D\tau} \left( \frac{1}{2} Mx^2 \right) = M \dot{x}_\mu \frac{D\dot{x}^\mu}{D\tau} = e\dot{x}_\mu G^\mu(x) \xrightarrow{G^\mu \to 0} 0. \quad (3)$$

Stueckelberg sought to derive \(2\) from a covariant Hamiltonian $K$ with the unconstrained symplectic equations

$$\frac{dx^\mu}{d\tau} = \dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \quad \frac{dp^\mu}{d\tau} = p^\mu = -\frac{\partial K}{\partial x_\mu} \quad (4)$$
but instead continued his program in quantum mechanics, where an event may tunnel probabilistically across the spacelike region, leading to spacetime diagrams of the type used by Feynman.

Horwitz and Piron [5] returned to these questions in constructing their canonical relativistic mechanics for the two-body problem, leading to solutions for relativistic generalizations of the standard central force problems, including quantum mechanical potential scattering and bound states [13–18]. To account for known phenomenology in the radiative transitions [19–21] of these bound states, the electromagnetic field $F_{\mu\nu}$ must be supplemented by a scalar interaction that in the classical context produces the vector field $eG_{\mu} = -\partial K_{\text{scalar}}/\partial x_{\mu}$ proposed by Stueckelberg through (4). The underlying origin of $K_{\text{scalar}}$ was found by Sa’ad, Horwitz, and Arshansky [6] in the gauge invariance associated with the canonical system. As shown by Fock [22], the Lorentz force in covariant form can be found by acting on the action

$$S_{\text{Maxwell}} = \int d\tau \left[ \frac{1}{2} M \dot{x}_\mu \dot{x}_\mu + \frac{e}{c} \dot{x}_\mu A_\mu (x) \right]$$

(5)

with the Euler-Lagrange equations associated with the canonical system (4). However, the Maxwell current

$$J^\mu (x) = \int d\tau \dot{X}^\mu (\tau) \delta^4 (x - X(\tau))$$

(6)

depends on a trajectory $X^\mu (\tau)$ that is only given after the equations of motion have been solved, and so the system may not be well-posed. To address this defect, one may add $\tau$-dependence to the vector potential, along with a new scalar potential as

$$S_{\text{Maxwell}} \rightarrow S_{\text{SHP}} = \int d\tau \left[ \frac{1}{2} M \dot{x}_\mu \dot{x}_\mu + \frac{e}{c} \dot{x}_\mu a_\mu (x^\lambda, \tau) + \frac{e}{c} c_5 a_5 (x^\lambda, \tau) \right]$$

(7)

$$= \int d\tau \left[ \frac{1}{2} M \dot{x}_\mu \dot{x}_\mu + \frac{e}{c} \dot{x}_\beta a_\beta (x^\lambda, \tau) \right]$$

(8)

where $\alpha, \beta, \gamma = 0, 1, 2, 3, 5$, and $x^5 = c_5 \tau$ in analogy to $x^0 = ct$. Compatibility of SHP electrodynamics with Maxwell theory requires $c_5 \ll c$ and we will neglect $(c_5/c)^2$ where appropriate. By taking the potential to be pure gauge, through $a_\alpha = \partial_\alpha \Lambda (x, \tau)$, the interaction term becomes a total $\tau$-derivative, showing that this theory is the most general U(1) gauge theory on the unconstrained phase space (see also [23]). Variation with respect to $x^\mu$ provides the Lorentz force [24] in the form

$$M \ddot{x}_\mu = \frac{e}{c} \left( \dot{x}^\nu f_{\mu\nu} + c_5 f_{5\mu} \right) = \frac{e}{c} \dot{x}^\beta f_{\mu\beta}$$

(9)

$$\frac{d}{d\tau} \left( -\frac{1}{2} M \dot{x}_\mu \dot{x}_\mu \right) = c_5 \frac{e}{c} \dot{x}^\beta f_{5\beta}$$

(10)

where the field strength

$$f_{\alpha\beta} (x, \tau) = \partial_{\alpha} a_\beta - \partial_{\beta} a_\alpha$$

(11)

becomes a dynamical quantity by introducing a kinetic term such as

$$S_{\text{field}} = \int d\tau d^4 x f^{\alpha\beta}(x, \tau) f_{a\beta}(x, \tau)$$

(12)

to the total action. The apparent 5D symmetry of the interaction term $\dot{x}^\beta a_\beta (x, \tau)$ in (8) must broken to 4+1 in (7), and so SHP electrodynamics differs in significant ways from 5D Maxwell theory. In particular, (10) permits the exchange of mass between particles and fields, and
indicates the condition for non-conservation of proper time. It has been shown [24] that the total mass, energy, and momentum of particles and fields are conserved.

In the SHP framework, \( x^\mu(\tau) \) is an irreversible physical event, occurring at time \( \tau \) with spacetime coordinates \( x^\mu \). Thus, the 4D block universe \( \mathcal{M}(\tau) \), representing the 4D manifold of general relativity and comprising all of space and coordinate time \( x^0 \), occurs at \( \tau \). A covariant Hamiltonian \( K \) generates evolution of \( \mathcal{M}(\tau) \) occurring at \( \tau \) to an infinitesimally close 4D block universe \( \mathcal{M}(\tau + d\tau) \) occurring at \( \tau + d\tau \). The configuration of spacetime, including the past and future of \( x^0 = ct \), may thus change infinitesimally from chronological moment to moment in \( \tau \), and therefore the metric structure \( \gamma_{\mu\nu}(x, \tau) \) of \( \mathcal{M}(\tau) \) must be \( \tau \)-dependent. In this formalism, a 4D metric given for all \( \tau \) would have the character of an absolute background field, in violation of the goals of general relativity.

The kinetic term \([12]\) requires that we formally raise the five-index of \( f_{\alpha\beta} \) although we may view the Lagrangian density as

\[
f^{\alpha\beta}(x, \tau)f_{\alpha\beta}(x, \tau) = f^{\mu\nu}(x, \tau)f_{\mu\nu}(x, \tau) + 2\sigma f_5^{\mu}(x, \tau)f_{\mu5}(x, \tau)
\]

(13)

in which \( \sigma = \pm 1 \) is merely the choice of sign for the vector-vector term. We emphasize that notation \( \beta = 5 \) index is a formal convenience, indicating an \( O(3,1) \) scalar quantity, not an element of a 5D tensor, and not a timelike coordinate. In particular, \( \dot{x}^5 = c_5 \) is constrained to be a constant scalar, identical in all reference frames, and \( x^5 = c_5\tau \) must not be treated as a dynamical variable. Nevertheless, in contracting on indices \( \alpha, \beta \) we write for convenience

\[
\eta_{\alpha\beta} = \text{diag}(-1,1,1,1,\sigma)
\]

(14)

in the form of a 5D flat space metric. In the SHP approach to general relativity we similarly exploit this notation while respecting the \( a \ priori \) non-dynamical character of \( x^5 \).

While the 3+1 formalism begins with a 4D block universe \( \mathcal{M} \) and defines a foliation into embedded spacelike hypersurfaces of equal coordinate time \( t \), the 4+1 formalism begins with a parameterized family of 4D spacetimes \( \mathcal{M}(\tau) \) embedded as hypersurfaces into a 5D pseudo-spacetime \( \mathcal{M}_5 \) with coordinates \( (x, \tau) \). We refer to \( \mathcal{M}_5 \) as a pseudo-spacetime to emphasize that despite the formal manifold structure, when specifying the physics we treat \( \tau \) as a parameter and not a coordinate. Moreover, \( \mathcal{M}_5 \) represents an admixture of symmetries: 4D spacetime geometry within each \( \mathcal{M}(\tau) \), and canonical dynamics between any pair \( \mathcal{M}(\tau_1), \mathcal{M}(\tau_2) \). This structure permits a 4+1 foliation by choosing \( \tau \) as the unambiguously preferred time direction (see [25][26] for a discussion of general 5D spacetime with preferred foliation). We expect no general 5D diffeomorphism invariance for \( \mathcal{M}_5 \). Because the evolution of spacetime is determined by an \( O(3,1) \) scalar Hamiltonian \( K \), with \( \tau \) as an external parameter (Poincaré invariant by definition), there is no conflict with the diffeomorphism invariance required by GR for each \( \mathcal{M}(\tau) \).

The squared interval in standard GR is invariant for a pair of instantaneously displaced points on the spacetime manifold \( \mathcal{M}(\tau) \). Instead we consider the interval

\[
dx^\alpha = \mathcal{S}^\alpha(\tau + \delta\tau) - x^\alpha(\tau)
\]

(15)

between an event \( x^\mu(\tau) \in \mathcal{M}(\tau) \) and an event \( \mathcal{S}^\mu(\tau + \delta\tau) \in \mathcal{M}(\tau + \delta\tau) \) occurring at a displaced spacetime location at a subsequent time, and expand as

\[
dx^2 = g_{\mu\nu}\dx^\mu\dx^\nu + g_{5\nu}\dx^\nu\dx^5 + g_{55}\dx^5\dx^5 = g_{\alpha\beta}(x, \tau)\dx^\alpha\dx^\beta
\]

(16)
referred to the coordinates of $x$. Regarding this quantity as the squared interval in the pseudo-spacetime $\mathcal{M}_5$, we formally combine the geometrical distance $\delta x^\mu$ between two neighboring points in one manifold, and the dynamical distance $\delta x^5 = c_5 \delta \tau$ between events occurring at two sequential times (at equal $x^\mu$). Writing the free particle Lagrangian as

$$L = \frac{1}{2} M g_{a\beta}(x, \tau) \dot{x}^a \dot{x}^\beta$$

we obtain equations of motion

$$0 = \frac{D \dot{x}^\mu}{D \tau} = \ddot{x}^\mu + \Gamma^\mu_{a\beta} \dot{x}^a \dot{x}^\beta \quad 0 = \frac{D \dot{x}^5}{D \tau} = \ddot{x}^5 + \Gamma^5_{a\beta} \dot{x}^a \dot{x}^\beta$$

where $\Gamma^\gamma_{a\beta}$ is the standard Christoffel symbol in 5D, calculated from $g_{a\beta}(x, \tau)$. But as in electrodynamics, we must not treat $x^5(\tau) = c_5 \tau$ as a dynamical variable — the 5-index denotes a scalar quantity and not an elements of a 5D tensor. We break the apparent 5D symmetry to a 4+1 representation of O(3,1) by noting that $\ddot{x}^5(\tau) = 0$ and asserting that the Christoffel symbol $\Gamma^5_{a\beta}$ plays no role in geodesic evolution of the event. Classical and quantum SHP particle mechanics in a spacetime with a $\tau$-independent local metric $\gamma_{\mu\nu}(x)$ has been studied extensively by Horwitz [9,10] and will not be discussed at length here.

Considering now a non-thermodynamic (zero-pressure) dust of geodesically evolving events, we define $n(x, \tau)$ to be the number of such events per spacetime volume, so that

$$j^a(x, \tau) = \rho(x, \tau) \dot{x}^a(\tau) = M n(x, \tau) \dot{x}^a(\tau)$$

is the 5-component event current, and

$$\nabla_a j^a = \frac{\partial j^a}{\partial x^a} + j^\gamma \Gamma^a_{\gamma a} = \frac{\partial \rho}{\partial \tau} + \nabla_\mu j^\mu = 0$$

is the continuity equation. Generalizing the 4D energy-momentum tensor to 5D, the mass-energy-momentum tensor [6,27] is

$$T^{a\beta} = M n \dot{x}^a \dot{x}^\beta = \rho \dot{x}^a \dot{x}^\beta \quad \rightarrow \quad \left\{ \begin{array}{l} T^{\mu\nu} = M n \dot{x}^\mu \dot{x}^\nu = \rho \dot{x}^\mu \dot{x}^\nu \\ T^{5\beta} = x^5 \dot{x}^\beta \rho = c_5 j^\beta \end{array} \right.$$  

combining $T^{\mu\nu}$ with $j^a$, and is conserved by virtue of the continuity and geodesic equations. As in 4D, the Bianchi identity suggests extending the Einstein equations to

$$R_{a\beta} - \frac{1}{2} g_{a\beta} R = \frac{8 \pi G}{c^4} T_{a\beta}$$

where the 5D Ricci tensor $R_{a\beta}$ and scalar $R$ are obtained by contracting indices of the 5D curvature tensor $R^{\gamma}_{a\beta}$. We will see below that the 5D symmetry of this physical law must similarly be broken to 4+1, in order to maintain consistency with standard phenomenology.

### 3. Linearized field equations

#### 3.1. Weak field approximation and 5D wave equation

The weak field approximation in 5D poses the metric as a small perturbation of the flat metric

$$g_{a\beta} = \eta_{a\beta} + h_{a\beta} \quad \rightarrow \quad \partial_\gamma g_{a\beta} = \partial_\gamma h_{a\beta} \quad (h_{a\beta})^2 \approx 0$$


\[
\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1, \sigma) \tag{24}
\]

with inverse
\[
\delta^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} \tag{25}
\]

as seen from
\[
\delta^{\alpha\gamma} g_{\beta\gamma} = \left(\eta^{\alpha\beta} - h^{\alpha\beta}\right) \left(\eta_{\beta\gamma} + h_{\beta\gamma}\right) \approx \eta^{\alpha\gamma} + h^{\alpha\gamma} - h^{\gamma}_{\gamma} = \eta^{\alpha}_{\gamma}. \tag{26}
\]

Under a coordinate translation
\[
x'^{\alpha} = x^{\alpha} + \Lambda^{\alpha}(x) \quad \rightarrow \quad \frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + \partial^{\alpha}_{\beta} \Lambda^{\alpha} \tag{27}
\]

the metric perturbation transforms as
\[
h'_{\alpha\beta} = h_{\alpha\beta} - \partial_{\alpha} \Lambda_{\beta} - \partial_{\beta} \Lambda_{\alpha} \tag{28}
\]

and so denoting the trace as
\[
h' = \eta^{\alpha\beta} h'_{\alpha\beta} = h - 2\partial^{\beta} \Lambda_{\beta} \tag{29}
\]

leads to
\[
h'_{\alpha\beta} = \left(h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h'\right) = \left(h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h\right) - \partial_{\alpha} \Lambda_{\beta} - \partial_{\beta} \Lambda_{\alpha} + \eta_{\alpha\beta} \partial^{\beta} \Lambda_{\alpha} \tag{30}
\]

with divergence
\[
\partial^{\beta}\left(h'_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h'\right) = \partial^{\beta}\left(h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h\right) - \partial^{\beta} \partial^{\beta} \Lambda_{\alpha}. \tag{31}
\]

For any function \(\partial^{\beta}\left(h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h\right)\) we can find a solution to the wave equation
\[
\partial^{\beta} \partial_{\beta} \Lambda_{\alpha} = \partial^{\beta}\left(h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h\right) \tag{32}
\]

and so choose the Lorenz gauge condition
\[
\partial^{\beta} h'_{\alpha\beta} = \partial^{\beta}\left(h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h\right) = 0 \quad \rightarrow \quad \partial^{\beta} h_{\alpha\beta} = \frac{1}{2} \partial_{\alpha} h. \tag{33}
\]

In this approximation, the Ricci tensor found by contraction on the Riemann curvature reduces to
\[
R_{\alpha\beta} \approx \frac{1}{2} \left(\partial^{\gamma} \partial^{\gamma} h^{\gamma}_{\alpha\beta} + \partial_{\alpha} \partial^{\gamma} h^{\gamma}_{\beta} - \partial^{\gamma} \partial^{\gamma} h_{\alpha\beta} - \partial_{\alpha} \partial_{\beta} h\right) \tag{34}
\]

so that
\[
R_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} R = -\frac{1}{2} \partial_{\gamma} \partial_{\gamma} h_{\alpha\beta} = \frac{1}{2} \partial^{\gamma} \partial_{\gamma} \tilde{h}_{\alpha\beta} \tag{35}
\]

in Lorenz gauge and the 5D field equation takes the form of the wave equation
\[
-\partial_{\gamma} \partial_{\gamma} h_{\alpha\beta} = \frac{16 \pi G}{c^4} T_{\alpha\beta}. \tag{36}
\]

The Green’s function \([28]\) for this equation is
\[
G(x, \tau) = \frac{1}{2\pi} \delta(x^2)\delta(\tau) + \frac{c_5}{2\pi^2} \frac{\partial}{\partial x^2} \theta(-\eta_{55} g_{\alpha\beta} x^\alpha x^\beta) \frac{1}{\sqrt{-\eta_{55} g_{\alpha\beta} x^\alpha x^\beta}} \tag{37}
\]
and using the first term, which is instantaneous in $\tau$ and dominates at long distance for many problems, leads to

$$\tilde{H}_{\alpha\beta}(x, \tau) = \frac{4G}{c^4} \int d^3x' T_{\alpha\beta}\left(t - \frac{|x - x'|}{c}, x', \tau\right) \frac{|x - x'|}{|x - x'|}$$

(38)
as a generic solution. Nevertheless, integration of both terms of (37) with a source of the type (21) will generally require numerical simulation.

### 3.2. Localized gravitational source

We consider an evolving spacetime event with 5D coordinates

$$X^a(\tau) = (X^\mu(\tau), c_5 \tau)$$

(39)

where $X^\mu$ refers to the spacetime components in $\mathcal{M}$. We introduce the notation

$$\xi^a(\tau) = \frac{1}{c} u^a(\tau) = \frac{1}{c} \frac{dX^a}{d\tau}$$

(40)

and treat this source as evolving (freely falling) under some metric $g_{\alpha\beta}(x, \tau)$ so that its geodesic motion

$$\frac{D u^a}{d\tau} = \frac{du^a}{d\tau} + \Gamma^a_{\beta\gamma}(g) u^\beta u^\gamma = 0$$

(41)

can be written

$$0 = \frac{1}{c} \xi^a + \Gamma^a_{\beta\lambda}(\xi^\lambda \xi^\sigma) + 2 \xi^5 \Gamma^a_{5\lambda} \xi^\lambda + \xi^2 \Gamma^a_{55}$$

0 = $\ddot{u}^5 = \dot{c}_5$

(42)

where

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} \eta^{\mu\lambda} \left( \partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda} \right)$$

(43)

$$\Gamma^\mu_{5\nu} = \frac{1}{2} \eta^{\mu\lambda} \left( \partial_\nu g_{5\lambda} + \partial_5 g_{\lambda\nu} - \partial_\lambda g_{5\nu} \right)$$

(44)

$$\Gamma^\mu_{55} = \frac{1}{2} \eta^{\mu\lambda} \left( 2 \partial_5 g_{5\lambda} - \partial_\lambda g_{55} \right)$$

(45)

are components of the standard 5D Christoffel connection. The 5D interval is conserved

$$\frac{d}{d\tau} u^2 = 2 \mu_a \frac{Du^a}{d\tau} = 0 \rightarrow u^2 = c^2 \xi^2 = \text{constant}$$

(46)

which for a uniformly moving particle in its rest frame in flat space becomes

$$u^2 = c^2 \eta_{\alpha\beta} \xi^\alpha \xi^\beta = (c, 0, 0, 0, c_5)^2 = -c^2 + c^2 c_5^2 \approx -1$$

(47)

where we neglect $\xi^2_5 = c^2_5 / c^2 \ll 1$, and so we take $u^2 = -1$ as the constant in any curved space. Thus,

$$-1 = g_{\alpha\beta} \xi^\alpha \xi^\beta = g_{\mu\nu} \xi^\mu \xi^\nu + 2 \xi^5 g_{\nu5} \xi^\nu + \xi^2_5 g_{55}$$

(49)

while the 4D mass

$$mc^2 \xi^2 = mc^2 g_{\mu\nu} \xi^\mu \xi^\nu$$
may vary dynamically under the influence of the 5-components of the metric

\[ mc^2 \xi^2 = mc^2 g_{\mu\nu} \xi^\mu \xi^\nu = \text{constant} - mc^2 \left( 2 \xi^5 g_{5\xi} \xi^5 + \xi^2 g_{55} \right) . \]  

(50)

With the spacetime event density

\[ \rho (x, \tau) = \rho (x - X (\tau)) \]  

(51)

the mass-energy-momentum tensor is

\[ T^a{}^b = m \rho (x, \tau) X^a X^b = m \rho (x, \tau) u^a u^b = mc^2 \rho (x, \tau) \xi^a \xi^b \]  

(52)

which is seen explicitly to be conserved by noting that \( \partial_\tau \rho (x, \tau) = -\xi^\mu \partial_\mu \rho (x, \tau) \).

3.3. Modified wave equation

From (38) and (52) we may write a first order solution to the wave equation as

\[ h_{a\beta} (x, \tau) = \mathcal{G} [T_{a\beta}] = \frac{4Gm}{c^2} \int d^3 x' \rho \left( \frac{t - |x - x'|}{c}, x', \tau \right) \xi_a \xi_\beta \]  

(53)

which we will find unsatisfactory in two ways. Taking the trace of \( h_{a\beta} \)

\[ \eta^{a\beta} h_{a\beta} = \eta^{a\beta} \left( h_{a\beta} - \frac{1}{2} \eta_{a\beta} h \right) = \frac{2 - D}{2} h \quad \rightarrow \quad h_{a\beta} = \tilde{h}_{a\beta} = \frac{1}{D - 2} \eta_{a\beta} \tilde{h} \]  

(54)

so that in \( D = 5 \) an event evolving uniformly in its rest frame with \( \xi = (1, 0, c_5/c) \) will give rise to the metric

\[ h_{00} = \tilde{h}_{00} - \frac{1}{D - 2} \eta_{00} \tilde{h} = \frac{2}{3} \mathcal{G} [T_{00}] \quad \quad h_{05} = \tilde{h}_{05} - \frac{1}{D - 2} \eta_{05} \tilde{h} = \frac{2}{3} \mathcal{G} [T_{00}] \]  

(55)

\[ h_{ij} = \tilde{h}_{ij} - \frac{1}{D - 2} \eta_{ij} \tilde{h} = \frac{1}{3} \delta_{ij} \mathcal{G} [T_{00}] \quad \quad h_{55} = \tilde{h}_{55} - \frac{1}{D - 2} \eta_{55} \tilde{h} = \frac{1}{3} \mathcal{G} [T_{00}] \]

where \( i, j = 1, 2, 3 \) and we have neglected terms in \( \xi_5^2 \approx 0 \). Qualitatively, this metric describes a structure with \( h_{00} = 2h_{ij} \) and \( h_{00} = \pm 2h_{55} \), contradicting standard GR and gravitational phenomenology which requires \( |h_{55}| \ll h_{00} \).

To address this discrepancy, we introduce a modified \( \eta_{a\beta} \) that explicitly breaks the 5D symmetry in the relationship between the 5D Einstein tensor and the source term:

\[ R_{a\beta} - \frac{1}{2} \tilde{\eta}_{a\beta} R = \frac{8 \pi G}{c^4} T_{a\beta} \]  

(56)

\[ \tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} \]  

\[ \tilde{\eta}_{5\xi} = 0 . \]

Taking the trace produces

\[ R - \frac{4}{2} R = -\frac{1}{2} \partial^\gamma \partial_\gamma \left( h - \frac{4}{2} h \right) = \frac{8 \pi G}{c^4} \tilde{\eta}_{a\beta} T_{a\beta} = \frac{8 \pi G}{c^4} \eta_{\mu\nu} T_{\mu\nu} = \frac{8 \pi G}{c^4} \tilde{T} \]  

(57)

allowing us to write the modified field equations in the form

\[ R_{\mu\nu} = -\frac{1}{2} \partial^\gamma \partial_\gamma h_{\mu\nu} = \frac{8 \pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} \tilde{\eta}_{\mu\nu} \tilde{T} \right) \]  

(58)

\[ R_{5\xi} = -\frac{1}{2} \partial^\gamma \partial_\gamma h_{5\xi} = \frac{8 \pi G}{c^4} T_{5\xi} \]
and the modified solution (55)

\[ h_{00} = \tilde{h}_{00} - \frac{1}{2} \eta_{00} \tilde{h} = \frac{1}{2} G [T_{00}] \quad h_{05} = \tilde{h}_{05} = \sigma G [T_{00}] = \sigma \tilde{\xi} G [T_{00}] \]

\[ h_{ij} = \tilde{h}_{ij} - \frac{1}{2} \eta_{ij} \tilde{h} = \frac{1}{2} \delta_{ij} G [T_{00}] \quad h_{55} = \tilde{h}_{55} = \sigma G [T_{55}] = \sigma \tilde{\xi} G [T_{00}] \]

so that taking the source as static \((X(\tau) = 0)\) and \(\varphi = 1\) the spacetime part of the metric is

\[ g_{\mu\nu} = \text{diag} \left( -1 + \frac{2Gm}{c^2r}, \left( 1 + \frac{2Gm}{c^2r} \right) \delta_{ij} \right) \approx \text{diag} \left( - \left( 1 - \frac{2Gm}{c^2r} \right)^{-1}, \left( 1 - \frac{2Gm}{c^2r} \right)^{-1} \delta_{ij} \right) \]

consistent with the spherically symmetric Schwarzschild metric. Moreover, we find

\[ g_{55} = \sigma \left( 1 + \sigma \tilde{\xi} (2Gm/c^2r) \right) \] which is suitably close to \(\sigma\). As when we obtained the geodesic equations (18), this approach preserves the 5D symmetry of the Ricci tensor \(R_{\alpha\beta}\) (geometry), but breaks 5D symmetry to 4+1 in posing the relationship between the Einstein tensor and the mass-energy-momentum tensor (physics).

### 3.4. Cosmological term

If we include a cosmological term in the modified field equation (56) we have

\[ R_{\alpha\beta} - \frac{1}{2} \tilde{\eta}_{\alpha\beta} R + \tilde{\eta}_{\alpha\beta} \Lambda = \frac{8\pi G}{c^4} T_{\alpha\beta} \]

so that taking the trace with \(\tilde{\eta}^{\alpha\beta}\) results in

\[ R - \frac{4}{2} R + 4 \Lambda = \frac{8\pi G}{c^4} \eta^{\mu\nu} T_{\mu\nu} \longrightarrow R = 4 \Lambda - \frac{8\pi G}{c^4} (T - \sigma T_{55}) . \]

The field equation can now be rewritten as

\[ R_{\alpha\beta} = \frac{8\pi G}{c^4} \left( T_{\alpha\beta} - \frac{1}{2} \tilde{\eta}_{\alpha\beta} T \right) + \frac{8\pi G}{c^4} \tilde{\eta}_{\alpha\beta} \sigma T_{55} + \tilde{\eta}_{\alpha\beta} \Lambda \]

so that identifying

\[ \Lambda = - \frac{8\pi G}{c^4} \sigma T_{55} = - \frac{8\pi G}{c^4} \sigma \kappa \]

recovers (58) and produces a field equation in \(R_{\alpha\beta}, T_{\alpha\beta},\) and \(\tilde{T} = \eta^{\mu\nu} T_{\mu\nu},\) for which the mass density \(\kappa,\) a scalar source independent of \(T_{\mu\nu}\) but not necessarily constant, plays the role of a cosmological term.

### 4. Overview of 4+1 formalism

In a 3+1 formalism such as ADM [12], the 4D spacetime \(\mathcal{M}\) is taken as the fundamental geometrical structure, in which one may choose a time direction and foliate spacetime into a collection of spacelike 3D hypersurfaces. In the SHP approach, the time \(\tau\) is considered external to spacetime, and so taken together, \(\{\mathcal{M}(\tau), \tau\}\) represents an inherently 4+1 geometrical/dynamical structure. Still, as a guide to the formulation of reasonable field equations for \(g_{\alpha\beta}(x, \tau)\) we apply the method for deriving the 3+1 formalism to derive a 4+1 model from a 5D structure. This section provides a brief summary of work presented in [2].
4.1. Pseudo-spacetime and its foliation

We construct a 5D pseudo-spacetime by defining the injective mapping

$$\Phi : \mathcal{M} \rightarrow \mathcal{M}_5 = \mathcal{M} \times \mathbb{R}$$

with coordinates $$X^a \in \mathcal{M}_5$$, for $$a = 0, 1, 2, 3, 5$$, and a natural foliation defined by level surfaces of the scalar field $$\tau(X) = \tau$$

$$\Sigma(\tau_0) = \{ X \in \mathcal{M}_5 \mid \tau(X) = X^5/c_5 = \tau_0 \}.$$  

Since $$\Sigma(\tau_0)$$ is homeomorphic to $$\mathcal{M}(\tau_0)$$ for any $$\tau_0$$ we drop reference to $$\tau_0$$ in referring to the hypersurfaces. Notating by $$\mathcal{T}(V)$$ the tangent space to a space $$V$$, the four basis elements $$E_\mu = \partial_\mu$$ for $$\mathcal{T}(\Sigma)$$ are

$$E_\mu^a = \left( \frac{\partial X^a(x, \tau)}{\partial x^\mu} \right)_{\tau_0} \quad \mu = 0, 1, 2, 3$$

and the squared interval restricted to $$X \in \Sigma$$ is

$$dX^2|_\Sigma = g_{\alpha\beta}dX^\alpha dX^\beta|_\Sigma = g_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^\mu} \frac{\partial X^\beta}{\partial x^\nu} dx^\mu dx^\nu = \gamma_{\mu\nu}dx^\mu dx^\nu$$

where we identify $$\gamma_{\mu\nu} = g_{\alpha\beta}E_\mu^\alpha E_\mu^\beta$$ as the induced 4D metric on $$\Sigma$$. The vector

$$\partial_\alpha \tau(X) = \delta_\alpha^5 \partial_5 \tau(X) = \delta_\alpha^5 \frac{1}{c_5} \partial_\tau \tau(X)$$

points in the direction of time evolution in $$\mathcal{T}(\mathcal{M}_5)$$, normal to $$\mathcal{T}(\mathcal{M})$$ in the sense that $$\tau(X) = \tau_0$$ is constant throughout $$\Sigma(\tau_0)$$. The unit time normal $$n_\alpha$$ is

$$n = \sigma \frac{1}{\sqrt{|g^{55}|}} E_5 \rightarrow n^2 = \frac{1}{|g^{55}|} g^{\alpha\beta} (E_5)_\alpha (E_5)_\beta = \frac{1}{|g^{55}|} g^{55} = \sigma$$

and

$$n^\alpha = g^{\alpha\beta} n_\beta = g^{\alpha\beta} \sigma \frac{1}{\sqrt{|g^{55}|}} \delta_5^\beta = \sigma g^{55} \frac{1}{\sqrt{|g^{55}|}} .$$

The tangent projection operator onto $$\mathcal{T}(\mathcal{M})$$ is

$$P_{\alpha\beta} = g_{\alpha\beta} - \sigma n_\alpha n_\beta \quad P^{\alpha\beta} = g^{\alpha\beta} - \sigma n^\alpha n^\beta \quad P_{\alpha\beta} P^{\gamma\beta} = P^\alpha_\beta = \delta^\beta_\alpha - \sigma n_\alpha n^\beta$$

providing the completeness relation

$$g_{\alpha\beta} = P_{\alpha\beta} + \sigma n_\alpha n_\beta \quad \delta_\beta^\alpha = P^\alpha_\beta + \sigma n^\alpha n_\beta .$$

For any $$V \in \mathcal{T}(\mathcal{M}_5)$$, the vector $$V^\alpha_\perp = P^\alpha_\beta V^\beta \in \mathcal{T}(\Sigma)$$ and so there is some vector $$v \in \mathcal{T}(\mathcal{M})$$, such that

$$V^\alpha_\perp = v^\mu E^a_\mu \rightarrow v_\mu = E^\beta_\mu V^\beta$$

since $$E^a_\mu \in \mathcal{T}(\Sigma)$$. Expressing the metric in terms of (73) we have

$$\gamma_{\mu\nu} = g_{\alpha\beta} E^a_\mu E^\beta_\nu = (P_{\alpha\beta} + \sigma n_\alpha n_\beta) E^a_\mu E^\beta_\nu = P_{\alpha\beta} E^a_\mu E^\beta_\nu = P_{\mu\nu}$$
so that the projector $P_{\alpha\beta}$ when restricted to $\Sigma$ acts precisely as the 4D metric $\gamma_{\mu\nu}$.

Adopting the interval (16), we write

$$X_2 - X_1 = (\delta x^\mu + N^\mu \delta x^5, N \delta x^5)$$

$$\delta X^\alpha = (\delta x^\mu + N^\mu \delta x^5) E^\alpha_\mu + N n^\alpha \delta x^5$$  \hspace{1cm} (76)

where $N$ is a lapse function and $N^\mu$ is a shift four-vector. The 5D squared invariant interval takes the form

$$dX^2 = g_{\alpha\beta}(x, \tau) \delta X^\alpha \delta X^\beta$$

$$= \gamma_{\mu\nu}(x, \tau) \delta x^\mu \delta x^\nu + 2 \gamma_{\mu\nu}(x, \tau) N^\mu \delta x^\nu + (\gamma_{\mu\nu}(x, \tau) N^\nu N^\sigma + \sigma^2 N^2)(\delta x^5)^2$$  \hspace{1cm} (77)

decomposing the 5D metric

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} N^\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \\ N^\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix}$$

$$g^{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} + \sigma N^\mu N^\nu - \sigma \frac{1}{N^2} N^\mu \\ -\sigma \frac{1}{N^2} N^\mu & \sigma \frac{1}{N^2} \end{bmatrix}$$  \hspace{1cm} (78)

into the spacetime and $\tau$ sectors, and unit normal $n_\alpha$ is

$$n_\alpha = \sigma \frac{1}{\sqrt{|g^{55}|}} \partial_\alpha \tau(X) = \sigma N \delta_5^\alpha.$$  \hspace{1cm} (80)

4.2. Intrinsic and Extrinsic Geometry

With Christoffel connection $\Gamma^\gamma_{\delta\beta}$, the 5D covariant derivative on $\mathcal{M}_5$ obeys $\nabla_\gamma g_{\alpha\beta} = 0$, and the curvature formula

$$[\nabla_\beta, \nabla_\alpha] X_\gamma = X_\rho R^\rho_{\gamma\delta\alpha}$$  \hspace{1cm} (81)

with Riemann tensor

$$R^\rho_{\gamma\delta\alpha} = \frac{\partial}{\partial x^\delta} \Gamma^\gamma_{\delta\alpha} - \frac{\partial}{\partial x^\alpha} \Gamma^\gamma_{\delta\beta} + \Gamma^\gamma_{\rho\alpha} \Gamma^\rho_{\delta\beta} - \Gamma^\gamma_{\rho\beta} \Gamma^\rho_{\delta\alpha}$$  \hspace{1cm} (82)

and associated Bianchi relations. The 4+1 field equations are found by projecting these relations and the field equations (22) onto $\mathcal{T}(\Sigma)$. For a vector $V = V^\perp \in \mathcal{T}(\Sigma)$ the projected covariant derivative $\nabla_a$ acts on the projected vector as

$$\nabla_a V^\perp_\beta = \nabla_a \left( P^\alpha_\beta V_\alpha \right) = P^\gamma_\beta P^\alpha_\gamma \nabla_\alpha V_\perp \longrightarrow \nabla_a P^\gamma_\beta = 0$$  \hspace{1cm} (83)

which justifies regarding $\nabla_a$ as the intrinsic covariant derivative on $\mathcal{T}(\Sigma)$, denoted as

$$D_a = \nabla_a = P^\gamma_\alpha \nabla_\gamma \hspace{1cm} D_\mu = E_\mu^a D_a = E_\mu^a P^\gamma_\alpha \nabla_\gamma = E_\mu^\gamma \nabla_\gamma$$  \hspace{1cm} (84)

and satisfying $D^\rho_\gamma \gamma_{\lambda\rho} = 0$. The projected curvature $\bar{R}^\rho_{\lambda\mu\nu}$ defined through

$$[D_\gamma, D_\mu] X_\lambda = X_\rho \bar{R}^\rho_{\lambda\mu\nu}$$  \hspace{1cm} (85)

is the intrinsic curvature tensor on the target spacetime $\mathcal{M}$. The extrinsic curvature characterizes the evolution of the unit normal to $\mathcal{T}(\mathcal{M})$ and is defined as

$$K_{\alpha\beta} = -P^\gamma_\alpha P^\delta_\beta \nabla_\delta n_\gamma$$  \hspace{1cm} (86)
where we recall that $\nabla \delta n_\gamma$ may have both normal and tangent components with respect to $\mathcal{T}(\Sigma)$. Because the projection is idempotent, we obtain the identity

$$K_{\alpha\beta} = -\nabla_\alpha n_\beta + \sigma n_\alpha (n^\gamma \nabla_\gamma n_\beta)$$

which using (80) for the unit normal $n_\alpha$ leads to

$$K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha \frac{1}{N} D_\beta N .$$

4.3. Evolution of the Hypersurface $\Sigma$

The variation of $X \in \Sigma$ for a small variation $\delta x^5$ at a given point $x_0 \in \mathcal{M}$ is found from (76) as

$$\delta X^\alpha = \left( \frac{\partial X^\alpha}{\partial x^5} \right)_{x_0} \delta x^5 \quad \delta \tau \longrightarrow E_5^\alpha = (\partial_5)^\alpha = Nn^\alpha + N^\mu E_\mu^\alpha$$

Defining $m^\alpha = Nn^\alpha$ we express $E_5$ as $\partial_5 = m + N$ and characterize time evolution through the Lie derivative in the time direction

$$L_5 = L_m + L_N .$$

For the metric $\gamma_{\alpha\beta}$, the Lie derivative is

$$L_m \gamma_{\alpha\beta} = m^\gamma \nabla_\gamma \gamma_{\alpha\beta} + \gamma_{\gamma\beta} \nabla_\alpha m^\gamma + \gamma_{\alpha\gamma} \nabla_\beta m^\gamma$$

which may be evaluated using (72) for $P_{\alpha\beta} = \gamma_{\alpha\beta}$ in the first term and using (88) to obtain

$$\nabla_\beta m_\alpha = N \nabla_\beta n_\alpha + n_\alpha \nabla_\beta N = -NK_{\alpha\beta} - n_\beta D_\alpha N + n_\alpha \nabla_\beta Nu$$

in the remaining terms. Since $L_m P_{\alpha\beta}^\gamma$ is the derivative in the normal direction of the projector onto the tangent space, direct calculation using (88) and (92) provides

$$L_m P_{\alpha\beta}^\gamma = m^\gamma \nabla_\gamma (\delta_{\alpha\beta}^\gamma - \sigma n^\gamma n_\beta) - \gamma_{\beta}^\gamma \nabla_\gamma m^\alpha + \gamma_{\alpha}^\gamma \nabla_\beta m^\gamma = 0$$

expressing compatibility of $L_m$ with $P_{\alpha\beta}^\gamma$. As a result, if $V \in \mathcal{T}(\mathcal{M}_5)$ is tangent to $\Sigma$, its Lie derivative in the time direction is tangent to $\Sigma$, and so tangent vectors propagate as tangent vectors as $\tau$ advances monotonically. Thus, (91) simplifies to

$$L_m \gamma_{\alpha\beta} = -2NK_{\alpha\beta}$$

leading to

$$L_5 \gamma_{\alpha\beta} - L_N \gamma_{\alpha\beta} = -2NK_{\alpha\beta} \quad \longrightarrow \quad L_5 \gamma_{\mu\nu} - L_N \gamma_{\mu\nu} = -2NK_{\mu\nu}$$

as the evolution equation for the metric.
4.4. Decomposition of the Riemann Tensor

Using the completeness relation (73) to write

\[ R^\gamma_{\alpha\beta} = \left( P^\gamma_\alpha + \sigma n_\alpha n^{\alpha'} \right) \left( P^\beta_\beta + \sigma n_{\beta} n^{\beta'} \right) \left( P^\gamma_\gamma + \sigma n_\gamma n^{\gamma'} \right) \left( P^\delta_\delta + \sigma n_\delta n^{\delta'} \right) R^\gamma_{\delta'\alpha'\beta'} \]  

(96)

we obtain products of the type

\[ R^\gamma_{\alpha\beta} = \delta^\alpha_{\alpha'} \delta^\beta_{\beta'} \delta^\gamma_{\gamma'} \delta^\delta_{\delta'} R^\gamma_{\delta'\alpha'\beta'} \rightarrow \sum_{n^\alpha} \left( E^\gamma_\alpha E^\beta_\beta E^\gamma_\gamma E^\delta_\delta R^\gamma_{\delta'\alpha'\beta'} \right) = E^\gamma_\alpha E^\beta_\beta E^\gamma_\gamma E^\delta_\delta R^\gamma_{\delta'\alpha'\beta'} = \rho_n \]  

(97)

expressing zero, one, or two projections onto the direction \( n^\alpha \). Notice that \( R^\gamma_{\alpha\beta} n^\alpha n^{\beta} = 0 \), because of the symmetries of the Riemann tensor. Expanding the factors of \( P_{a\beta} \) in the projected curvature formula (85) leads to an expression containing four tangent projections of \( R^\gamma_{\alpha\beta} \)

\[ p^a_{a'} p^\beta_{\beta'} R^\gamma_{\gamma' \delta' \alpha'} = \bar{R}_{\alpha\beta} - \sigma \left( K^\gamma_{\alpha\beta} - K_{\alpha\beta} \right) \]  

(98)

known as the Gauss relation. Acting on both sides with \( E^\mu_\alpha E^\nu_\beta E^\lambda_\gamma E^\delta_\delta \) extracts

\[ R^\mu_{\nu\lambda\rho} = \bar{R}_{\nu\lambda\rho} - \sigma \left( K^\mu_{\nu\lambda} - K^\mu_{\nu\lambda} \right) \]  

(99)

providing an expression for \( R^\mu_{\nu\lambda\rho} \) (the spacetime components of the intrinsic curvature on \( M_5 \)) in terms of the curvature \( \bar{R}_{\nu\lambda\rho} \) and intrinsic curvature \( K_{\nu\lambda\rho} \) of \( M \). The contracted and scalar Gauss relations

\[ p^a_{a'} p^\beta_{\beta'} R^\gamma_{\gamma' \delta' \alpha'} - \sigma p_{a\alpha} n^{\alpha'} p^\beta_{\beta'} n^{\beta'} R^\gamma_{\gamma' \delta' \alpha'} = \bar{R}_{\alpha\beta} - \sigma \left( K_{\alpha\beta} - K_{\alpha\beta} \right) \]  

(100)

\[ R - 2\sigma R_{a\beta} n^a n^\beta = \bar{R} - \sigma \left( K^2 - K_{\alpha\beta} K_{\alpha\beta} \right) \]  

(101)

are found by contracting on first on \( a, \gamma \) in (98) and then on \( a \) and \( \beta \). Applying (81) to the unit normal \( n \) as

\[ (\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) n^{\alpha'} \gamma = R^\gamma_{\alpha' \alpha \beta} n^{\alpha'} \]  

(102)

and projecting onto \( \Sigma \) leads to

\[ D_\beta \tilde{K}_\alpha = D_\alpha \tilde{K}_\beta = p^\gamma_{\gamma'} p^\beta_{\beta'} n^{\beta'} R^\gamma_{\gamma' \delta' \alpha'} \]  

(103)

which is called the Codazzi relation. Acting on \( p^a_{a'} n^{\gamma'} p^\beta_{\beta'} \) and applying (88) and (92) results in

\[ \frac{1}{N} \mathcal{L}_m K_{\alpha\beta} + \frac{1}{N} D_\alpha D_\beta N + K_{\alpha\gamma} K^\gamma_\beta = p_{a\alpha} n^{\gamma'} p^\beta_{\beta'} R^\gamma_{\gamma' \delta' \alpha'} \]  

(104)

providing an evolution equation for \( K_{\alpha\beta} \). Using the contracted Gauss relation (100) we can put (104) into the form

\[ p^a_{a'} p^\beta_{\beta'} R^\gamma_{\gamma' \delta' \alpha'} = \frac{1}{N} \mathcal{L}_m K_{\alpha\beta} + \frac{1}{N} D_\alpha D_\beta N + \bar{K}_{a\beta} - \sigma KK_{a\beta} + \sigma K_{a\beta} \]  

(105)

in which only the term \( p^a_{a'} p^\beta_{\beta'} R^\gamma_{\gamma' \delta' \alpha'} \) on the LHS refers to the 5D geometry of \( M_5 \).
4.5. Decomposition of the Einstein Equation

Taking the trace of (22) in D dimensions results in

$$g^{a\beta} \left( R_{a\beta} - \frac{1}{2} g_{a\beta} R \right) = \frac{8\pi G}{c^4} g^{a\beta} T_{a\beta} \rightarrow \frac{2 - D}{2} R = \frac{8\pi G}{c^4} T$$

(106)

and so the Einstein equation can be put into the form

$$R_{a\beta} = \frac{8\pi G}{c^4} \left( T_{a\beta} - \frac{1}{D - 2} g_{a\beta} T \right)$$

(107)

where $T = g^{a\beta} T_{a\beta}$. We decompose the source term by projecting onto $\Sigma$ and $n$ as

$$T_{a\beta} = T_{a'\beta'} \left( p_a^{\alpha'} + \sigma n^\alpha n_\alpha \right) \left( p_{\beta'}^{\beta'} + \sigma n^\beta n_{\beta'} \right) = S_{a\beta} + 2\sigma n_\alpha p_\alpha + n_\alpha n_\beta \kappa$$

(108)

where

$$S_{a\beta} = p_a^{\alpha'} R_{a'\beta'} p_{\beta'}^{\beta'} - n^\alpha T_{a\beta} = -n^\alpha p_{\beta'} \left( \frac{1}{D - 2} g_{a'\beta'} T \right)$$

(109)

so that $S_{\mu\nu}$ corresponds to the 4D energy-momentum tensor $T_{\mu\nu}$, $p_\mu$ corresponds to the mass current into the $\mu$ direction $T_{\mu\nu}$, and $\kappa$ corresponds to the mass density $T_{\Sigma\Sigma}$. It is useful in this context to regard mass as simply expressing the dynamical independence of energy and momentum, providing a variable relation between them. The trace is

$$T = g^{a\beta} T_{a\beta} = g^{a\beta} \left( S_{a\beta} - 2\sigma n_\alpha p_\alpha + n_\alpha n_\beta \kappa \right) = S - 2\sigma g^{a\beta} n_\alpha p_\alpha + g^{a\beta} n_\alpha n_\beta \kappa = S + \sigma \kappa$$

(110)

where we used

$$g^{a\beta} n_\alpha p_\beta = n \cdot p = 0$$

(111)

which follows from

$$p_\beta = -p_\beta^{\alpha'} \left( n^\alpha T_{a\beta} \right) \in T(\Sigma).$$

(112)

So, projecting the field equations (107) onto $\Sigma$ with $p_a^{\alpha'} R_{a'\beta'}$ leads to

$$p_a^{\alpha'} p_{\beta'}^{\beta'} \left( T_{a'\beta'} - \frac{1}{D - 2} g_{a'\beta'} T \right) = S_{a\beta} - \frac{1}{D - 2} \gamma_{a\beta} (S + \sigma \kappa)$$

(113)

on the RHS, while on the LHS we use (105) for $p_a^{\alpha'} p_{\beta'}^{\beta'} R_{a'\beta'}$ and find

$$\left( \frac{1}{c^5} L_T - L_N \right) K_{\mu\nu} = -D_\mu D_\nu N + N \left\{ -\sigma R_{\mu\nu} + KK_{\mu\nu} - 2K_{\mu}^{\lambda} K_{\nu\lambda} + \sigma \frac{8\pi G}{c^4} \left[ S_{\mu\nu} - \frac{1}{D - 2} \gamma_{\mu\nu} (S + \sigma \kappa) \right] \right\}$$

(114)

as the evolution equation for $K_{\mu\nu}$. The double projection onto the time direction $n$ is

$$\left( R_{a\beta} - \frac{1}{2} g_{a\beta} R \right) n^\alpha n_\beta = \frac{8\pi G}{c^4} T_{a\beta} n^\alpha n_\beta \rightarrow R_{a\beta} n^\alpha n_\beta - \frac{1}{2} \sigma R = \frac{8\pi G}{c^4} \kappa$$

(115)

which with the scalar Gauss relation (99) provides

$$\bar{R} - \sigma \left( K^2 - K_{\mu\nu} K_{\mu\nu} \right) = -\sigma \frac{16\pi G}{c^4} \kappa$$

(116)
This expression, called the Hamiltonian constraint, applies to the mass density of the gravitational field, not the energy density as in 4D GR. The mixed projection with $P^\beta_\rho n^\alpha$

$$n^\alpha P^\beta_\rho \left( R^\rho_\sigma - \frac{1}{2} \delta^\rho_\sigma R \right) = n^\alpha P^\beta_\rho \frac{8\pi G}{c^4} T^\rho_\sigma \rightarrow P^\beta_\rho n^\alpha R^\rho_\sigma = - \frac{8\pi G}{c^4} p_\rho \ (117)$$

is combined with the contracted Codazzi relation (103) and $g^\alpha_\beta n^\alpha P^\beta_\rho = n^\alpha P^\beta_\rho = 0$ to obtain

$$D_\mu K^\mu - D_\nu K^\nu = \frac{8\pi G}{c^4} p_\nu \ (118)$$

which is called the momentum constraint, referring to the flow of mass into the field. We notice that the evolution equations and constraints contain only objects defined on $\Sigma$. Unlike the evolution equations, the constraints contain no $\tau$-derivatives. If they are satisfied by the initial conditions, they will be satisfied at all times. Although the constrained quantities are $\tau$-dependent, the constraining relationship does not evolve, and so are said to propagate rather than evolve.

4.6. The 4+1 decomposition for the linearized theory

Under 4+1 decomposition, the metric is written

$$\|g_{\alpha\beta}\| = \begin{bmatrix} g_{\mu\nu} & g_{\mu 5} \\ g_{5 \mu} & g_{55} \end{bmatrix} = \begin{bmatrix} \eta_{\mu\nu} + h_{\mu\nu} & h_{\mu 5} \\ h_{5 \mu} & \eta_{55} + h_{55} \end{bmatrix} \ (119)$$

allowing us to identify

$$\|g_{\alpha\beta}\| = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix} = \begin{bmatrix} \eta_{\mu\nu} + h_{\mu\nu} & h_{\mu 5} \\ h_{5 \mu} & \eta_{55} + h_{55} \end{bmatrix} \ (120)$$

from which

$$\sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu = \sigma N^2 + \left( \eta_{\mu\nu} + h_{\mu\nu} \right) h_{5 \mu} h_{5 \nu} \approx \sigma N^2 \rightarrow \sigma N^2 = \sigma + h_{55} \ (121)$$

so

$$N = \sqrt{1 + \sigma h_{55}} \approx 1 + \frac{1}{2} \sigma h_{55} . \ (122)$$

Then

$$\|g^{\alpha\beta}\| = \begin{bmatrix} \gamma^{\mu\nu} + \sigma \frac{1}{N^2} N^\mu N^\nu & -\sigma \frac{1}{N^2} N^\mu \\ -\sigma \frac{1}{N^2} N^\mu & \frac{\sigma}{N^2} \end{bmatrix} \approx \begin{bmatrix} \eta^{\lambda\nu} - h^{\lambda\nu} & -h_{5}^{\mu} \\ -h_{5}^{\mu} & \sigma \left( 1 - \sigma h_{55} \right) \end{bmatrix} \ (123)$$

and the unit normal is

$$n^\alpha = \sigma N \delta^\alpha_5 = \sigma \sqrt{1 + \sigma h_{55}} \delta^\alpha_5 = \sigma \left( 1 + \frac{1}{2} \sigma h_{55} \right) \delta^\alpha_5 \ (124)$$

$$n^\alpha = -h_{5}^{\mu} \delta^\alpha_\mu + \left( 1 - \frac{1}{2} \sigma h_{55} \right) \delta^\alpha_5 . \ (125)$$
In light of (58) the decomposed metric field equations are the evolution equations

\[
\left( \frac{1}{\ell_5} \partial_\tau - L_N \right) \gamma_{\mu\nu} = -2N K_{\mu\nu}
\]

(126)

and

\[
\left( \frac{1}{\ell_5} \partial_\tau - L_N \right) K_{\mu\nu} = -D_\mu D_\nu N
\]

+ \frac{N}{8\pi G c^4} \left( -\sigma R_{\mu\nu} + KK_{\mu\nu} - 2K_{\mu}^{\lambda} K_{\nu\lambda} + \frac{\sigma}{c^4} \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{T} \right) \right)
\]

(127)

with Lie derivatives

\[
L_N \gamma_{\mu\nu} = D_\mu N_\nu + D_\nu N_\mu \quad L_N K_{\mu\nu} = N^\lambda \partial_\lambda K_{\mu\nu} + K_{\lambda\nu} \partial_\mu N^\lambda + K_{\mu\lambda} \partial_\nu N^\lambda
\]

(128)

and constraints

\[
\bar{R} - \sigma \left( K^2 - K^{\mu\nu} K_{\mu\nu} \right) = -\frac{\sigma}{c^4} \left( 16 \pi G \right) K
\]

(129)

\[
D_\mu K^\mu - D_\nu K = \frac{8\pi G}{c^4} p_\nu
\]

(130)

Discarding terms of the order \((h_{a\beta})^2 \approx 0\), the Lie derivative of the metric reduces to

\[
L_N \gamma_{\mu\nu} = D_\mu N_\nu + D_\nu N_\mu \approx \partial_\mu N_\nu + \partial_\nu N_\mu = \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu}
\]

(131)

and so (126) can be rewritten as

\[
\frac{1}{2N} \left( -\partial_5 \gamma_{\mu\nu} + \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu} \right) \approx \left( -\partial_5 \gamma_{\mu\nu} + \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu} \right) = K_{\mu\nu}
\]

(132)

From the definition (86) of extrinsic curvature and using (124) and (125) for \(n_a\) and \(n^a\) we find

\[
K_{a\beta} = -\left( \delta^5_\mu - \sigma n^\gamma n_\gamma \right) \left( \delta^5_\beta - \sigma n^\gamma n_\gamma \right) \left( \frac{1}{2} \partial_5 h_{55} \delta_\gamma^5 - \sigma \Gamma^5_\gamma \right)
\]

(133)

\[
= -\frac{1}{2} \partial_5 h_{55} \delta_\gamma^5 + \sigma n^\gamma n_\gamma \frac{1}{2} \partial_5 h_{55} + \sigma n^\gamma n_\gamma \frac{1}{2} \partial_5 h_{55} \delta_\gamma^5 - n^\gamma n_\gamma \frac{1}{2} \partial_5 h_{55}
\]

(134)

\[
+ \sigma \Gamma^5_\gamma - n^\gamma n_\gamma \sigma \Gamma^5_\gamma - n^\gamma n_\gamma \sigma \Gamma^5_\gamma + n^\gamma n_\gamma \sigma \Gamma^5_\gamma
\]

with spacetime components

\[
K_{\mu\nu} = -\frac{1}{2} \partial_5 h_{55} \delta_\mu^5 + \sigma n^\gamma n_\gamma \frac{1}{2} \partial_5 h_{55} + \sigma n^\gamma n_\gamma \frac{1}{2} \partial_5 h_{55} \delta_\mu^5 - n^\gamma n_\gamma \frac{1}{2} \partial_5 h_{55}
\]

\[
+ \sigma \Gamma^5_\mu - n^\gamma n_\gamma \sigma \Gamma^5_\gamma - n^\gamma n_\gamma \sigma \Gamma^5_\gamma + n^\gamma n_\gamma \sigma \Gamma^5_\gamma
\]

(135)

\[
= \sigma \Gamma^5_{\mu\nu}
\]

(136)

where we used

\[
\delta_\mu^5 = 0 \quad n_\mu = 0
\]

(137)

Comparing

\[
K_{\mu\nu} = \sigma \Gamma^5_{\mu\nu} = \frac{1}{2} \sigma \left( \partial_5 h^5_\mu + \partial_5 h^5_\nu - \partial_5 h_{\mu\nu} \right) = \frac{1}{2} \left( \partial_5 h_{5\mu} + \partial_5 h_{5\nu} - \partial_5 h_{\mu\nu} \right)
\]

(138)
with (132) we see that the evolution equation for the metric is satisfied automatically, because we have assumed we can evaluate $\partial_5 h_{\mu\nu}$ in determining $K_{\mu\nu}$. Neglecting the Lie derivative

$$L_N K_{\mu\nu} = N^\lambda \partial_\lambda K_{\mu\nu} + K_{\lambda\nu} \partial_\mu N^\lambda + K_{\mu\lambda} \partial_\nu N^\lambda \propto (h_{\alpha\beta})^2 \approx 0$$

and the quadratic terms in $K_{\mu\nu}$, equation (127) reduces to

$$\partial_5 K_{\mu\nu} = -\frac{1}{2} \sigma \partial_\mu \partial_\nu h_{55} - \sigma \bar{R}_{\mu\nu} + \sigma \frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} \bar{\eta}_{\mu\nu} T \right)$$

where used

$$D_\mu D_\nu N = \partial_\mu (\partial_\nu N) \approx \partial_\mu \partial_\nu h_{55} = \frac{1}{2} \sigma \partial_\mu \partial_\nu h_{55}$$

Similarly, the constraints reduce to

$$\bar{R} - \sigma (K^2 - K_{\mu\nu} K_{\mu\nu}) \approx \bar{R} = -\sigma \frac{16\pi G}{c^4} \kappa$$

$$D_\mu K_\nu - D_\nu K_\mu \approx \partial_\mu K_\nu - \partial_\nu K_\mu = \frac{8\pi G}{c^4} p_\nu$$

Using (125) and $P_\mu^\nu = \bar{g}_\mu^\nu \approx \bar{\eta}_\mu^\nu$ in (109), we can write the source terms as

$$S_{\mu\nu} \approx T_{\mu\nu} \quad p_\nu \approx -T_{5\nu} \quad \kappa \approx T_{55}$$

5. Modified 4+1 field equations

We now provide a relatively straightforward derivation of the 4+1 field equations in the weak field approximation, modified by the breaking of 5D symmetry through $\eta_{\alpha\beta} \rightarrow \bar{\eta}_{\alpha\beta}$. Because the external time $\tau$ provides a natural foliation of $M_5$ into spacetimes $M(\tau)$, each with metric $\gamma_{\mu\nu}(\tau)$, we similarly decompose the 5D field equations (56) into the spacetime part and the time part. To see that this makes sense, we consider the Bianchi identity for the linearized theory

$$\nabla_\alpha G^{\alpha\beta} = \nabla_\alpha \left( R^{\alpha\beta} - \frac{1}{2} \bar{\eta}^{\alpha\beta} R \right) = \partial_\alpha \left( R^{\alpha\beta} - \frac{1}{2} \bar{\eta}^{\alpha\beta} R \right) + o \left( h_{\alpha\beta}^2 \right) = 0$$

which can be rearranged as

$$\frac{1}{c_5} \partial_\tau G^{5\beta} = -\partial_\mu G^{\mu\beta} + o \left( h_{\alpha\beta}^2 \right).$$

Since the RHS must contain terms in $\bar{g}_{\alpha\beta}, \partial_\tau \bar{g}_{\alpha\beta}$, and $\partial_\tau^2 \bar{g}_{\alpha\beta}$, the components of $G^{5\beta}$ may contain at most first order $\tau$-derivatives. Therefore, the field equations

$$G_{5\beta} = R_{5\beta} - \frac{1}{2} \bar{\eta}_{5\beta} R = \frac{8\pi G}{c^4} T_{5\beta}$$

provide a set of relationships among the metric, its first-order $\tau$-derivative, and the source current $j_\beta = T_{5\beta}$, which together comprise the initial conditions for the second order field equations. This decomposition conveniently splits the 15 independent components of the 5D Ricci tensor $R_{\alpha\beta}$ into $R_{\mu\nu}$, representing 10 unconstrained equations including second $\tau$-derivatives of the metric, and $R_{5\beta}$, representing the 5 constraints on initial conditions.
Writing the Ricci tensor as
\[
R_{\alpha\beta} = \frac{1}{2} \left( \partial_\alpha \partial^\gamma h_{\beta\gamma} + \partial_\beta \partial^\gamma h_{\alpha\gamma} - \partial^\gamma \partial_\alpha h_{\beta\gamma} + \partial_\alpha \partial_\beta \eta^{\gamma\rho} h_{\rho\gamma} \right)
\]
(148)

\[
= \frac{1}{2} \left( \partial_\alpha \partial^\gamma h_{\beta\gamma} + \partial_\beta \partial^\gamma h_{\alpha\gamma} - \partial^\gamma \partial_\alpha h_{\beta\gamma} + \partial_\alpha \partial_\beta \eta^{\gamma\rho} h_{\rho\gamma} \right)
+ \partial_\alpha \partial^5 h_{5\beta} + \partial_\beta \partial^5 h_{5\alpha} - \partial^5 \partial_\alpha h_{5\beta} - \partial_\alpha \partial_\beta \eta^{5\rho} h_{5\rho} \right)
\]
(149)

the spacetime components are
\[
R_{\mu\nu} = \frac{1}{2} \left( \partial_\mu \partial^\lambda h_{\nu\lambda} + \partial_\nu \partial^\lambda h_{\mu\lambda} - \partial^\lambda \partial_\mu h_{\nu\lambda} + \partial_\mu \partial_\nu \eta^{\lambda\sigma} h_{\sigma\lambda} \right)
+ \partial_\mu \partial^5 h_{5\nu} + \partial_\nu \partial^5 h_{5\mu} - \partial^5 \partial_\mu h_{5\nu} - \partial_\mu \partial_\nu \eta^{5\rho} h_{5\rho} \right)
\]
(150)

\[
= \bar{R}_{\mu\nu} + \frac{1}{2} \sigma \left( \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu} - \partial_5 h_{\mu\nu} \right) - \frac{1}{2} \partial_\mu \partial_\nu \eta^{5\rho} h_{5\rho} \right)
\]
(151)

where we designate
\[
\bar{R}_{\mu\nu} = \frac{1}{2} \left( \partial_\mu \partial^\lambda h_{\nu\lambda} + \partial_\nu \partial^\lambda h_{\mu\lambda} - \partial^\lambda \partial_\mu h_{\nu\lambda} + \partial_\mu \partial_\nu \eta^{\lambda\sigma} h_{\sigma\lambda} \right)
\]
(152)

as the Ricci tensor on the projected 4D hypersurface. We recognize the extrinsic curvature in the second term of (151)
\[
K_{\mu\nu} = \sigma T^{5}_{\mu\nu} = \frac{1}{2} \sigma \left( \partial_\mu h^5_{\nu} + \partial_\nu h^5_{\mu} - \partial^5 h_{\mu\nu} \right) = \frac{1}{2} \left( \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu} - \partial_5 h_{\mu\nu} \right)
\]
(153)

and so we have
\[
R_{\mu\nu} = -\frac{1}{2} \partial_\mu \partial_\nu \eta^{5\rho} h_{5\rho} + \bar{R}_{\mu\nu} + \sigma \partial_5 K_{\mu\nu} .
\]
(154)

Using the spacetime components of the symmetry-broken field equation (58) for the LHS of (154) we are led to
\[
\partial_5 K_{\mu\nu} = \frac{1}{2} \partial_\mu \partial_\nu \eta^{5\rho} h_{5\rho} \right) - \sigma \bar{R}_{\mu\nu} + \sigma \frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{T} \right) .
\]
(155)

Comparing this with equation (140) obtained for $\partial_5 K_{\mu\nu}$ by projections of the 5D Riemann tensor for the linearized field, we see that (155) differs only in the replacement $\sigma = \eta^{5\rho} \rightarrow \tilde{\eta}^{5\rho} = 0$ in the first term on the RHS and the equivalent replacement $\bar{T} + \eta_{5\rho\kappa} \rightarrow \tilde{T}$ in the source. These replacements express the breaking of 5D symmetry to 4+1 in the modified field equations. Moreover, taking the trace of (58) we obtain
\[
\bar{R} = -\frac{8\pi G}{c^4} \bar{T} .
\]
(156)

In the unbroken 4+1 formalism, the Hamiltonian constraint (116) relates terms of order $\partial_5^2 \gamma_{\mu\nu}$ to the mass density $\kappa$ (which is not a source for $\bar{R}$) and $K_{\mu\nu}$ which contains only first-order $\tau$-derivatives of the metric. In the modified field theory, the 4D Ricci scalar $\bar{R}$ relates a sum of terms in $\partial_5^2 \gamma_{\mu\nu}$ to a sum of its source terms, and introduces no additional constraint.

Now considering the 5-component of (149)
\[
R_{5\beta} = \frac{1}{2} \left( \partial_5 \partial^\lambda h_{\lambda\beta} + \partial_\beta \partial^\gamma h_{5\gamma} - \partial^\lambda \partial_5 h_{\lambda\beta} - \partial_5 \partial_\beta \eta^{\lambda\rho} h_{\rho\lambda} + \partial_\beta \partial^5 h_{5\beta} - \partial_5 \partial_\beta \eta^{5\rho} h_{5\rho} \right)
\]
(157)
we apply the 5D Lorenz gauge condition (33), expanded as
\[
\partial^\lambda h_{\alpha\lambda} = \frac{1}{2} \partial_\alpha \eta^{\lambda\sigma} h_{\lambda\sigma} + \frac{1}{2} \partial_\alpha \eta^{55} h_{55} - \partial^5 h_{\alpha 5} \tag{158}
\]
leading to
\[
R_{5\beta} = \frac{1}{2} \left( -\partial_5 \partial^\gamma h_{\beta 5} - \partial^\lambda \partial_\lambda h_{5\beta} \right) = -\frac{1}{2} \partial^\gamma \partial_\gamma h_{5\beta} . \tag{159}
\]
Comparing with the time part of the symmetry-broken field equation (58) this expression provides
\[
R_{5\mu} = -\frac{1}{2} \partial^\gamma \partial_\gamma h_{5\mu} = \frac{8\pi G}{c^4} T_{5\mu} = -\frac{8\pi G}{c^4} p_\mu \tag{160}
\]
\[
R_{55} = -\frac{1}{2} \partial^\gamma \partial_\gamma h_{55} = \frac{8\pi G}{c^4} T_{55} = \frac{8\pi G}{c^4} \kappa \tag{161}
\]
which for a source of the type (52) are of order \(\xi_5\) and \(\xi_5^2\) respectively. Taking the 4-gradient of (153) and applying the Lorenz gauge condition (158) we find
\[
\partial^{\mu} K_{\mu\nu} = \frac{1}{2} \left( \partial^\mu \partial_\mu h_{5\nu} + \partial^\nu \partial_\nu h_{5\mu} - \partial_5 \partial^\mu h_{\mu\nu} \right) = \frac{1}{2} \partial^\gamma \partial_\gamma h_{5\nu} \tag{162}
\]
so that using the first of (161) puts this into the form
\[
\partial^{\mu} K_{\mu\nu} = \frac{8\pi G}{c^4} p_\nu \tag{163}
\]
which replaces the momentum constraint in the symmetry-broken field theory. In place of the Hamiltonian constraint (116), \(\bar{\eta}_{55} = 0\) reduces the scalar constraint associated with \(G_{55}\) to equation (161), now decoupled from the extrinsic curvature and the unconstrained spacetime Ricci scalar.

6. Application of the 4+1 field equations

Using the notation of Section 3.2, the source for an isolated evolving event is
\[
T_{\alpha\beta} = mc^2 \rho \left( x, \tau \right) \xi_\alpha \xi_\beta \tag{164}
\]
\[
\bar{T} = \bar{\eta}^{\mu\nu} T_{\mu\nu} = mc^2 \rho \left( x, \tau \right) \bar{\eta}^{\mu\nu} \xi_\mu \xi_\nu = mc^2 \rho \left( x, \tau \right) \bar{\xi}^2 \tag{165}
\]
\[
T_{\alpha\beta} - \frac{1}{2} \bar{\eta}_{\alpha\beta} \bar{T} = mc^2 \rho \left( x, \tau \right) Z_{\alpha\beta} \tag{166}
\]
where
\[
Z_{\alpha\beta} = \bar{\xi}_\alpha \bar{\xi}_\beta - \frac{1}{2} \bar{\eta}_{\alpha\beta} \bar{\xi}^2 . \tag{167}
\]
Using the Green’s function (38) the general expression the wave equation in the linearized approximation can be written
\[
h_{\alpha\beta} \left( x, \tau \right) = 4Gm \frac{c^2}{c^2} \int d^4 x' d\tau' G \left( x - x', \tau - \tau' \right) Z_{\alpha\beta} \left( \tau' \right) \rho \left( x', \tau' \right) \tag{168}
\]
with first-order solution
\[
h_{\alpha\beta} \left( x, \tau \right) = 4Gm \frac{c^2}{c^2} Z_{\alpha\beta} \int d^3 x' \rho \left( \frac{t - |x - x'|}{c}, x', \tau \right) \tag{169}
\]
in which the velocity factors \(Z_{\alpha\beta} \left( \tau \right)\) are taken outside integral.
6.1. “Static” source

We first consider an event evolving uniformly in its rest frame as

\[ X(\tau) = (c\tau, 0) \longrightarrow \xi(\tau) = \left(1, 0, \frac{c\delta_5}{c}\right) \longrightarrow \xi^\alpha = \delta_0^\alpha + \delta^5_5 \]  

(170)

and approximate a localized event density as \( \rho(x, \tau) = \delta^5(x) \). Neglecting \( \xi^5_5 \approx 0 \) in additive terms, we have

\[ h_{\mu\nu} = \frac{4Gm}{c^2r} \left( \phi^0_\mu \phi^0_\nu + \frac{1}{2} \eta_{\mu\nu} \right) \quad \bar{h} = \frac{4Gm}{c^2r} \left( \eta^{00} + 2 \right) = \frac{4Gm}{c^2r} \]  

(171)

so that

\[ h_{00} = \frac{4Gm}{c^2r} \left( 1 + \frac{1}{2} \eta_{00} \right) = \frac{2Gm}{c^2r} \quad h_{5\mu} = \frac{4Gm}{c^2r} \xi_5 \]

\[ h_{ij} = \frac{4Gm}{c^2r} \left( \frac{1}{2} \eta_{ij} \right) = \frac{2Gm}{c^2r} \eta_{ij} \quad h_{55} = \frac{4Gm}{c^2r} \xi_5^2 \]  

(172)

which provides the metric \( (60) \) and so recovers Newtonian gravity if we take \( \delta^5_5 = 0 \) and solve the geodesic equations with \( X = 0 \) for the test event. We may now calculate \( \partial_\tau h_{5\mu} = 0 \) and find that the evolution equation for the metric is automatically satisfied. Using \( (132) \) to evaluate the extrinsic curvature we obtain

\[ K_{\mu\nu} = \frac{1}{2} \left( \partial_\nu \left( \frac{4Gm}{c^2r} \xi_5 \phi^0_\mu \right) + \partial_\mu \left( \frac{4Gm}{c^2r} \xi_5 \phi^0_\nu \right) \right) = \frac{2Gm}{c^2r} \xi_5 \left( \phi^0_\mu \partial_\nu + \phi^0_\nu \partial_\mu \right) \frac{1}{r} \]  

(173)

so that \( \partial_\tau K_{\mu\nu} = 0 \) and the evolution equation for \( K_{\mu\nu} \) now reduces to

\[ \sigma \partial_5 K_{\mu\nu} = 0 = -\bar{R}_{\mu\nu} + \frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{T} \right) . \]

The Ricci tensor is

\[ \bar{R}_{\mu\nu} = \frac{1}{2} \left( \partial_\mu \partial^\lambda h_{\lambda\nu} + \partial_\nu \partial^\sigma h_{\mu\sigma} - \partial^\lambda \partial_\lambda h_{\mu\nu} - \partial_\mu \partial_\nu \phi^\lambda_\sigma \phi_\lambda^\sigma \right) \]  

(175)

and evaluating term-by-term produces

\[ \partial_\mu \partial^\lambda h_{\lambda\nu} = \partial_\mu \partial^\lambda \frac{4Gm}{c^2r} \left[ \phi^0_\mu \phi^0_\nu + \frac{1}{2} \eta_{\mu\nu} \right] = \partial_\mu \partial^\lambda \frac{4Gm}{c^2r} \left[ \phi^0_\mu \phi^0_\nu + \frac{1}{2} \eta_{\mu\nu} \right] = \frac{2Gm}{c^2r} \partial_\mu \partial_\nu \frac{1}{r} \]  

(176)

\[ \partial_\nu \partial^\sigma h_{\mu\sigma} = \partial_\nu \partial^\sigma \frac{4Gm}{c^2r} \frac{1}{r} \]

\[ -\partial^\lambda \partial_\lambda h_{\mu\nu} = -\partial^\lambda \partial_\lambda \frac{4Gm}{c^2r} \left[ \phi^0_\mu \phi^0_\nu + \frac{1}{2} \eta_{\mu\nu} \right] = \frac{16\pi Gm}{c^2} \left[ \phi^0_\mu \phi^0_\nu + \frac{1}{2} \eta_{\mu\nu} \right] \delta^3(x) \]  

(178)

\[ -\partial_\mu \partial_\nu \phi^\lambda_\sigma \phi_\lambda^\sigma h_{\lambda\sigma} = -\partial_\mu \partial_\nu \phi^\lambda_\sigma \phi_\lambda^\sigma \frac{4Gm}{c^2r} = -\frac{4Gm}{c^2r} \partial_\mu \partial_\nu \frac{1}{r} \]

(179)

and we obtain

\[ \bar{R}_{\mu\nu} = \frac{8\pi Gm}{c^2} \left[ \phi^0_\mu \phi^0_\nu + \frac{1}{2} \eta_{\mu\nu} \right] \delta^3(x) = \frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{T} \right) \]  

(180)

so that \( (174) \) is satisfied.
6.2. Source event in general motion

We now consider a single event in general motion, represented by the first-order solution and write the ansatz

\[ \gamma_{\mu \nu} (x, \tau) = \eta_{\mu \nu} + A (x, \tau) Z_{\mu \nu} \]

\[ N_\mu = B (x, \tau) \xi_\mu \]

\[ N = 1 \quad (181) \]

from which we have

\[ \partial_5 \gamma_{\mu \nu} = Z_{\mu \nu} \partial_5 A (x, \tau) + \frac{1}{c_s} \dot{Z}_{\mu \nu} A (x, \tau) \]

(182)

where

\[ \dot{Z}_{\mu \nu} = \left( \xi_\mu \xi_\nu + \xi_\nu \xi_\mu - \eta_{\mu \nu} \xi_\lambda \xi_\lambda \right) \]

includes acceleration of the source. Calculating the extrinsic curvature leads to

\[ K_{\mu \nu} = \frac{1}{2} \left( \partial_\mu N_\nu + \partial_\nu N_\mu - \partial_5 \gamma_{\mu \nu} \right) \]

\[ = \frac{1}{2} \left[ (\xi_\mu \partial_\nu + \xi_\nu \partial_\mu) B (x, \tau) - Z_{\mu \nu} \partial_5 A (x, \tau) - \frac{1}{c_s} \dot{Z}_{\mu \nu} A (x, \tau) \right] \]

(185)

and from

\[ \dot{R}_{\mu \nu} = \frac{1}{2} \left( \xi_\nu \xi_\lambda \partial^\lambda \partial_\mu + \xi_\mu \xi_\lambda \partial^\lambda \partial_\nu - Z_{\mu \nu} \partial^\lambda \partial_\lambda \right) A \]

(186)

so that the evolution equation for the extrinsic curvature is

\[ \partial_5 K_{\mu \nu} = -\sigma \dot{R}_{\mu \nu} + \sigma \frac{8 \pi G}{c^4} \left( T_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} \bar{T} \right) \]

\[ = -\sigma \frac{1}{2} \left[ (\xi_\nu \xi_\lambda \partial^\lambda \partial_\mu + \xi_\mu \xi_\lambda \partial^\lambda \partial_\nu - Z_{\mu \nu} \partial^\lambda \partial_\lambda ) A + \sigma \frac{8 \pi G}{c^4} mc^2 \rho (x, \tau) Z_{\mu \nu} \right] . \]

(187)

(188)

Using on the LHS leads to an equation for \( A(x, \tau) \) and \( B(x, \tau) \), but because we used the evolution equation for \( \gamma_{\mu \nu} \) in evaluating \( K_{\mu \nu} \) the combination contains \( \partial_5^2 A(x, \tau) \). If we wish to maintain first order field equations for computational purposes, we may introduce the auxiliary variable \( \chi \) to obtain

\[ \partial_5 A (x, \tau) = \chi (x, \tau) \]

\[ Z_{\mu \nu} \partial_5 \chi (x, \tau) = - \left( \xi_\mu \partial_\nu + \xi_\nu \partial_\mu \right) \partial_5 B (x, \tau) + \frac{1}{c_s} \dot{Z}_{\mu \nu} \chi (x, \tau) \]

\[ - \sigma \left( \xi_\nu \xi_\lambda \partial^\lambda \partial_\mu + \xi_\mu \xi_\lambda \partial^\lambda \partial_\nu - Z_{\mu \nu} \partial^\lambda \partial_\lambda \right) A + \sigma \frac{16 \pi G}{c^4} mc^2 \rho (x, \tau) Z_{\mu \nu} . \]

(189)

(190)

The momentum constraint is

\[ \partial^\mu K_{\mu \nu} = \frac{8 \pi G}{c^4} p_\nu \]

(191)

and takes the form

\[ \left( \xi_\mu \partial_\nu + \xi_\nu \partial_\mu \right) \partial^\mu B (x, \tau) - Z_{\mu \nu} \partial^\mu \chi (x, \tau) - \frac{1}{c_s} \dot{Z}_{\mu \nu} \partial^\mu A (x, \tau) = \frac{18 \pi G}{c^4} \rho (x, \tau) \xi_5 \xi_5 \]

(192)

providing a relation between \( A (x, \tau) \) and \( B (x, \tau) \) and thus a coupled set of first order equations for \( A (x, \tau) \) and \( \chi (x, \tau) \). Thus, the momentum constraint can be satisfied for a source of the form and expressions for \( A (x, \tau) \) and \( \rho (x, \tau) \) can be found by numerical integration of the first order evolution equations.
6.3. Mass transfer

As a final example, we consider mass transfer across spacetime induced through the motion of one localized particle in the gravitational field produced by another, and show that this cannot occur as the type of perturbation we have considered. As noted in \[1\], the dynamical mass \( M(\tau) = -M g_{\mu\nu} \xi^\mu \xi^\nu \) associated with the evolution of a test event is no longer conserved when \( g_{\mu\nu}(\tau) \) is found by substituting \( m \rightarrow m(\tau) \) in the standard Schwarzschild metric. We now seek a source particle motion that produces just this substitution: a perturbation of the static metric, such that (172) retains its basic form but with \( m(\tau) \). Using the first term of the Green’s function, we find such a metric by choosing the source event

\[
X(\tau) = (X^0(\tau), 0) \rightarrow \xi(\tau) = \left( x^0(\tau), 0, \frac{c_5}{c} \right) \rightarrow \xi^a = x^0 \delta^a_0 + x^5 \delta^a_5
\]

(193)

so that

\[
T_{\alpha\beta} = mc^2 \rho(\tau) \xi^\alpha \xi^\beta = mc^2 \rho(\tau) \left[ \delta^0_0 \delta^0_0 + \xi_0 \xi_5 \left( \delta^5_5 + \delta^5_0 \right) \right]
\]

(194)

where \( \rho(\tau) = \delta^3(x) \) and we neglect \( \xi^2_5 \). Then, from (169) we readily write the metric perturbation

\[
h_{\mu\nu} = \frac{4Gm}{c^2 r} \xi_0^2(\tau) \left( \delta^0_0 \delta^0_0 + 1 \right) - \frac{4Gm}{c^2 r} \xi_0^2(\tau) U_{\mu\nu}
\]

(195)

and for the spacetime components we denote \( m(\tau) = m^2(\tau) \). Because the expression for \( \dot{R}_{\mu\nu} \) contains no \( \tau \)-derivatives, (180) still holds, and so again the RHS of the evolution equation for \( K_{\mu\nu} \) vanishes. But this contradicts direct calculation,

\[
K_{\mu\nu} = \frac{1}{2} \left( \partial_\nu h_{5\mu} + \partial_\mu h_{5\nu} - \partial_5 h_{\mu\nu} \right)
\]

(196)

\[
= \frac{1}{2} \left( \partial_\nu \left( \frac{4Gm}{c^2 r} \xi_0(\tau) \xi_5 \delta^0_0 \right) \right) + \partial_\mu \left( \frac{4Gm}{c^2 r} \xi_0(\tau) \xi_5 \delta^0_0 \right) - \partial_5 \left( \frac{4Gm}{c^2 r} \xi_0^2(\tau) U_{\mu\nu} \right)
\]

(197)

\[
= \frac{2Gm}{c^2} \xi_0(\tau) \xi_5 \left( \delta^0_0 \partial_\nu + \delta^0_0 \partial_\mu \right) \frac{1}{r} - \frac{4Gm}{c^2} \partial_5 \xi_0^2(\tau) \delta^0_0 \delta^0_0
\]

(198)

which shows that \( \partial_5 K_{\mu\nu} \neq 0 \) for the intended scenario. We note that the momentum constraint

\[
\partial^\mu K_{\mu\nu} = \frac{8\pi G}{c^4} p_\nu
\]

(199)

which takes the form

\[
\frac{8\pi G}{c^4} p_\nu = \frac{2Gm}{c^2} \xi_0(\tau) \xi_5 \delta^0_0 \left( \partial^\mu \partial_\mu \frac{1}{r} \right) - \frac{8\pi Gm}{c^2} \xi_0(\tau) \xi_5 \delta^0_0 \delta^3(\tau) x = -\frac{8\pi G}{c^4} T_{5\nu}
\]

(200)

nevertheless holds, because \( \delta^0_0 \partial^\mu r = \delta^0_0 r = 0 \). To understand the inconsistency, we recall that the linearized 5D field equation leads to the 5D wave equation (56) and that the perturbation (195) was found from the integral (169) which uses only the first term in the Green’s function. Although this approximation is often valid, it cannot be applied to this case, because this solution satisfies the 4D wave equation, leading to the replacement

\[
\dot{R}_{\mu\nu} = - \left( \partial^\lambda \partial_\lambda + \delta^5_5 \partial_5 \right) h_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \rightarrow \partial^\lambda \partial_\lambda h_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}
\]

(201)

cauing the RHS of the evolution equation for \( K_{\mu\nu} \) to vanish in error.
Because we cannot integrate the varying mass source \[ (194) \] with the full Green's function without detailed knowledge of \( \xi_0(\tau) \), we reverse direction and ask whether a source can be found that produces the desired perturbed metric. In light of the static metric \[ (172) \] and the more general metric \[ (181) \], we choose the ansatz

\[
\gamma_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{and} \quad h_{\mu\nu} = \frac{4Gm}{c^2r}\alpha(\tau) U_{\mu\nu}
\]

where

\[
N_\mu = h5_\mu = \frac{4Gm}{c^2r}\beta(\tau) \xi_5 \delta^0_\mu 
\]

and

\[
N = 1 + \frac{1}{2}\sigma h_{55} = 1 + \sigma \frac{2Gm}{c^2r}\xi_5^2 \approx 1
\]

suggesting the source \( \rho(\tau) \) with

\[
\rho(\tau) = \frac{1}{2}\sigma \frac{1}{2}\sigma h_{55} \frac{2Gm}{c^2r}\xi_5^2 \approx 1
\]

Now rearranging the evolution equation for \( K_{\mu\nu} \) we may evaluate the source as

\[
\partial_5 K_{\mu\nu} = \frac{1}{2}\partial_5 \left( \partial_5 h_{5\mu} + \partial_\mu h_{55} - \partial_5 h_{\mu\nu} \right)
\]

\[
= \frac{1}{2}\partial_5 \left( \partial_5 \left( \frac{4Gm}{c^2r}\beta(\tau) \xi_5 \delta^0_\mu \right) + \partial_\mu \left( \frac{4Gm}{c^2r}\beta(\tau) \xi_5 \delta^0_\nu \right) \right) - \frac{1}{2}\partial_5 \left[ \frac{4G}{c^2r}\alpha(\tau) U_{\mu\nu} \right]
\]

\[
= \frac{2Gm}{c^2r}\partial_5 \beta(\tau) \xi_5 \left( \delta^0_\mu \partial_\nu + \delta^0_\nu \partial_\mu \right) \frac{1}{r} - \frac{4Gm}{c^2r}\partial_5^2 \alpha(\tau) U_{\mu\nu}.
\]

Similarly, from the definition of the extrinsic curvature, we obtain

\[
\partial_5 K_{\mu\nu} = \frac{1}{2}\partial_5 \left( \partial_5 h_{5\mu} + \partial_\mu h_{55} - \partial_5 h_{\mu\nu} \right)
\]

\[
= \frac{1}{2}\partial_5 \left( \partial_5 \left( \frac{4Gm}{c^2r}\beta(\tau) \xi_5 \delta^0_\mu \right) + \partial_\mu \left( \frac{4Gm}{c^2r}\beta(\tau) \xi_5 \delta^0_\nu \right) \right) - \frac{1}{2}\partial_5 \left[ \frac{4G}{c^2r}\alpha(\tau) U_{\mu\nu} \right]
\]

\[
= \frac{2Gm}{c^2r}\partial_5 \beta(\tau) \xi_5 \left( \delta^0_\mu \partial_\nu + \delta^0_\nu \partial_\mu \right) \frac{1}{r} - \frac{4Gm}{c^2r}\partial_5^2 \alpha(\tau) U_{\mu\nu}.
\]

Now rearranging the evolution equation for \( K_{\mu\nu} \) we may evaluate the source as

\[
\frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right) = \sigma \partial_5 K_{\mu\nu} + \bar{R}_{\mu\nu}
\]

\[
= \sigma \frac{2Gm}{c^2r}\partial_5 \beta(\tau) \xi_5 \left( \delta^0_\mu \partial_\nu + \delta^0_\nu \partial_\mu \right) \frac{1}{r} - \frac{4Gm}{c^2r}\partial_5 \alpha(\tau) U_{\mu\nu}
\]

\[
+ \frac{8\pi G}{c^2} \alpha(\tau) \delta^3(\tau) u_{\mu}u_{\nu}
\]

which may be put into the general form for a localized particle by taking \( \partial_5 \beta = 0 \). We then arrive at

\[
\frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right) = \left( \frac{8\pi Gm}{c^2} \alpha(\tau) \delta^3(\tau) - \sigma \frac{2Gm}{c^2r}\partial_5^2 \alpha(\tau) \right) U_{\mu\nu}
\]

suggesting the source

\[
T_{\mu\nu} = mc^2 \rho(x,\tau) \xi_\mu \xi_\nu = mc^2 \rho(x,\tau) \alpha(\tau) u_{\mu}u_{\nu}
\]

with \( \tau \)-dependent event density

\[
\rho(x,\tau) = \delta^3(\tau) - \sigma \frac{1}{\alpha(\tau)} \frac{1}{2\pi r} \partial_5^2 \alpha(\tau)
\]
which differs significantly from the spacetime distribution of a point particle. Checking the momentum constraint for this metric we obtain

\[
\frac{8\pi G}{c^4} p_\nu = \partial^\mu K_{\mu \nu} = \frac{2Gm}{c^2} \beta \xi_5 \left( \partial^0 \partial_v + \delta^0 \partial^\mu \partial_\mu \right) \frac{1}{r} - \frac{4Gm}{c^2} \partial_5 \alpha(\tau) U_{\mu \nu} \partial^\mu \frac{1}{r} \tag{213}
\]

which in time and space components is

\[
\frac{8\pi G}{c^4} p_0 = -\frac{8\pi Gm}{c^2} \beta \xi_5 \delta^3(x) + \frac{4Gm}{c^2} \partial_5 \alpha(\tau) u_0 \frac{u \cdot x}{r^2} \tag{214}
\]

\[
\frac{8\pi G}{c^4} p_k = \frac{4Gm}{c^2} \partial_5 \alpha(\tau) u_k \frac{u \cdot x}{r^2} \tag{215}
\]

using \( \partial^0 r = 0 \). The source \( T_{5\nu} = -p_\nu \) consistent with the source (211) is

\[
T_{5\nu} = mc^2 \rho(x, \tau) \alpha(\tau) u_5 u_\nu = mc^2 \rho(x, \tau) \alpha(\tau) \xi_5 u_\nu \tag{216}
\]

with the event density (212). We see that \( p_\nu \) cannot be brought into this form, even taking \( u = 0 \) to reduce the momentum to

\[
p_0 = -mc^2 \beta \xi_5 \delta^3(x) \quad p_k = 0 . \tag{217}
\]

Therefore, the simple replacement \( m \rightarrow m(\tau) \) in the static metric cannot be accomplished by a perturbation induced by the motion of a localized particle.

7. Summary

In the SHP approach to relativity, the constellation of spacetime events \( x^\mu(\tau) \) occurring at worldtime \( \tau \) are points in a block universe \( M(\tau) \) whose dynamical \( \tau \)-evolution is generated by a scalar Hamiltonian in a canonical formalism [2]. This structure naturally assumes the character of a 4+1 formalism, which poses the description of spacetime as an initial value problem. In much the way that Stueckelberg described particle worldlines traced out by the \( \tau \)-evolution of events, we specify the spacetime metric \( \gamma_{\mu \nu} \), extrinsic curvature \( K_{\mu \nu} \), and the matter/energy \( T_{\alpha \beta} \) at some time \( \tau \), insure consistency with constraints, and integrate forward a pair of first order differential equations to trace out spacetime structure at future times. As was accomplished in SHP electrodynamics, the field equations are found by formulating a 5D theory and strategically breaking the 5D symmetry to 4+1 representations of the Lorentz group. For the full, nonlinear Einstein equation in 5D, the formulation of 4+1 equations was found through a foliation of a 5D pseudo-spacetime into 4D hypersurfaces, and projecting the field equations onto this foliation.

In this paper, we return to the Einstein equation and argue that the 5D symmetry must be broken when setting the equality between the Ricci tensor and the matter/energy source \( T_{\alpha \beta} \).

For the linearized field equations, the 4+1 formalism follows in a straightforward manner by separating the 5D Ricci tensor into spacetime components \( R_{\mu \nu} \) leading to 10 unconstrained equations for the metric, and \( R_5 \) providing 5 constraints on the initial conditions. The significance of this separation is confirmed through the Bianchi identity. As examples, we recover the metric for a “static” event in its rest frame evolving uniformly in coordinate time, set up the evolution equations for the metric induced by a particle in general motion along some worldline, and finally show that the simple replacement \( m \rightarrow m(\tau) \) for a Schwarzschild
metric cannot be accomplished by a perturbation induced by the motion of a localized particle. We expect that solutions for more complex scenarios will require numerical integration.

An advantage to this 4+1 formalism is that by using the external time $\tau$ as evolution parameter, the symmetries of 4D spacetime are manifestly preserved at each step. Moreover, spacetime geometries may be found from sources described by specific spacetime trajectories, including reversals of the coordinate time $x^0(\tau)$, conventionally posed as closed timelike curves. A formulation of the metric induced when a pair of point-like sources cross their respective Schwarzschild radii — black hole collision — will be given in a forthcoming paper.

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