Coherent states associated with tridiagonal Hamiltonians

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Abstract

It has been shown that a positive semi-definite Hamiltonian $H$, that has a tridiagonal matrix representation in a given basis, can be represented in the form $H = A^\dagger A$, where $A$ is a forward shift operator playing the role of an annihilation operator. Such representation endows $H$ with rich supersymmetric properties yielding results analogous to those obtained by studying the Hamiltonian as a differential operator. Here, we study the coherent states which we define as being the eigenstates of the operator $A$. We explicitly find the expansion coefficients of these states in the given basis. We further identify a complete set of special coherent states which themselves can be used as basis. In terms of these special coherent states, we show that a general coherent state has the expansion form of a Lagrange interpolation scheme. As application of the developed formalism, we work out examples of systems having pure discrete, pure continuous, or mixed energy spectrum.

1 Introduction

Coherent states (CS) have been introduced by Schrödinger as states which behave in many respects like classical states [1]. They acquired this name when Glauber [2] realized that they were particularly convenient to describe optical coherence. In particular, the electromagnetic radiation generated by a classical current is a multimode coherent state. So is the light produced by a laser in certain regimes [3]. Thereafter, CS became cornerstones of modern quantum optics [4] and more recently, CS found applications in quantum information experiments [5].

CS also are mathematical tools which provide a close connection between classical and quantum formalisms so as to play a central role in the semiclassical analysis [6, 7]. In general, CS are overcomplete family of normalized ket vectors $|\zeta\rangle$ which are labeled by points $\zeta$ of a phase-space
domain $X$, belonging to a Hilbert space $\mathcal{H}$ that corresponds to a specific quantum model and provide $\mathcal{H}$ with a resolution of its identity operator as

$$1_H = \int_X |\zeta\rangle \langle \zeta| d\mu(\zeta).$$

with respect to a suitable integration measure $d\mu(\zeta)$ on $X$. These CS are constructed for that space with $\mathcal{H}$ having either a discrete or continuous basis in different ways [8]: “à la Glauber” as eigenfunctions of an annihilation operator; as states minimizing some uncertainty principle or they can be obtained “à la Gilmore-Perelomov” as orbits of a unitary operator acting on a specific or fiducial state by appealing to the representation theory of Lie groups [9]. Their number states expansion over the eigenstates basis of the Hamiltonian has also lead to a generalization which is known as the Hilbertian probabilistic scheme [7].

These states have long been known for the Hamiltonian of the harmonic oscillator, whose properties have been used as a model for more quantum Hamiltonian systems. Most of theoretical or mathematical research treats the system’s Hamiltonian as a differential operator. Here we examine its matrix representation in a space spanned by complete square integrable basis functions. In particular, we focus on Hamiltonians whose matrix representations are of the tridiagonal form. This in not a restriction since tridiagonality can always be achieved by a complete new basis constructed from a linear combination of the original basis. In fact, using an algorithm such as that of Lanczos [10, 11], one can construct such a basis from only one seed function.

The advantage of tridiagonality is that it makes the coefficients of expansion of the eigenvector in the basis satisfy recursion relations similar to those satisfied by orthogonal polynomials. In fact, the rich tools of the field of orthogonal polynomials and the associated spectral theory have been brought to bear to mathematically underpin the properties of tridiagonal Hamiltonians, making this approach an active field of research. Having investigated the supersymmetry of such Hamiltonians [12], we now investigate a closely related topic of the coherent states associated with these Hamiltonians.

The paper is organized as follows. In section 2, we review the needed properties of tridiagonal Hamiltonians and define the coherent states associated with them. In section 3, we explicitly solve for the coefficients of the coherent state vector expansion in the basis, and then clarify how to compute the time development of the resulting state. In section 4, we examine and characterize the properties of a special set of coherent states. In particular, we show that the set is complete. This property qualifies the special set of coherent states to be used as a basis. We then proceed to find the expansion coefficients of a general coherent states in term of the special basis. We conclude by showing that the resulting expansion is completely analogous to the known Lagrange interpolation scheme. Borrowing from known results regarding this scheme, we immediately write down several different but equivalent forms for such expansion. In section 5, we work out several examples illustrating the results in the previous sections. We conclude in section 6 with some comments.
2 Review of the properties of tridiagonal Hamiltonians

We assume that we are given a positive semi-definite Hamiltonian $H$ (with zero as the value of the lowest energy in its spectrum) acting on a Hilbert space $\mathcal{H}$, that has a tridiagonal matrix representation in the orthonormal basis $\{|\phi_n\rangle\}_{n=0}^{\infty}$ of $\mathcal{H}$ with known coefficients $\{a_n, b_n\}_{n=0}^{\infty}$

$$H_{n,m} = \langle \phi_n | H | \phi_m \rangle = b_{n-1} \delta_{n,m+1} + a_n \delta_{n,m} + b_n \delta_{n,m-1}. \quad (2.1)$$

We solve the energy eigenvalue equation $H|E\rangle = E|E\rangle$ by expanding the eigenvector $|E\rangle$ in the basis $|\phi_n\rangle$ as $|E\rangle = \sum_{n=0}^{\infty} f_n(E)|\phi_n\rangle$. Making use of the tridiagonality of $H$, we readily obtain the following recurrence relations for the expansion coefficients

$$E f_0(E) = a_0 f_0(E) + b_0 f_1(E)$$
$$E f_n(E) = b_{n-1} f_{n-1}(E) + a_n f_n(E) + b_n f_{n+1}(E), \quad n = 1, 2, \ldots, \quad (2.2)$$

The operator $H$ may admit, in addition to the continuous spectrum $\sigma_c$, a discrete part $\{E_\mu\}_\mu$ both of which lead to the following form of the resolution of the identity operator on $\mathcal{H}$

$$\sum_\mu |E_\mu\rangle \langle E_\mu| + \int_{\sigma_c} |E\rangle \langle E| \, dE = 1_{\mathcal{H}}. \quad (2.3)$$

This translates into the following orthogonality relation for the expansion coefficients

$$\sum_\mu f_n(E_\mu) (f_m(E_\mu))^* + \int_{\sigma_c} f_n(E) (f_m(E))^* \, dE = \delta_{n,m}. \quad (2.4)$$

If we now define $p_n(E) = \frac{f_n(E)}{f_0(E)}$, then $\{p_n(E)\}$ is a set of polynomials that satisfy the three-term recursion relation

$$E p_n(E) = b_{n-1} p_{n-1}(E) + a_n p_n(E) + b_n p_{n+1}(E), \quad n = 1, 2, \ldots, \quad (2.5)$$

with the initial conditions $p_0(E) = 1$ and $p_1(E) = (E - a_0)b_0^{-1}$. If we further define $\Omega(E) = |f_0(E)|^2$ and $\Omega_\mu = |f_0(E_\mu)|^2$, then the relation (2.4) now translates into the following orthogonality relation for the polynomial $p_n$

$$\sum_\mu \Omega_\mu p_n(E_\mu) (p_m(E_\mu))^* + \int_{\sigma_c} \Omega(E) p_n(E) (p_m(E))^* \, dE = \delta_{n,m}. \quad (2.6)$$

We have shown [12] that we can write the Hamiltonian $H$ in the form $H = A^\dagger A$ where the forward-shift operator $H$ is defined by its action on the basis vector as

$$A |\phi_n\rangle = c_n |\phi_n\rangle + d_n |\phi_{n-1}\rangle \quad (2.7)$$

where the coefficients $\{c_n, d_n\}_{n=0}^{\infty}$ are related to the coefficients $\{a_n, b_n\}_{n=0}^{\infty}$ and the polynomials $\{p_n\}_{n=0}^{\infty}$

$$d_0 = 0, \quad c_n^2 = -b_n \frac{p_{n+1}(0)}{p_n(0)}, \quad d_{n+1}^2 = -b_n \frac{p_n(0)}{p_{n+1}(0)}, \quad a_n = c_n^2 + d_n^2, \quad b_n = c_n d_n. \quad (2.8)$$

We define the coherent states associated with the tridiagonal Hamiltonian as the eigenstates of the operator $A$. 



3
3 General solution of the coherent state

3.1 General solution at $t = 0$

We write the coherent state $|z\rangle$ as the normalized solution to the eigenvalue equation

$$A|z\rangle = z|z\rangle. \tag{3.1}$$

We expand the state in terms of the basis as

$$|z\rangle = \sum_{n=0}^{\infty} \Lambda_n(z) |\phi_n\rangle. \tag{3.2}$$

Using Eq. (2.7), this equation gives

$$\sum_{n=0}^{\infty} z\Lambda_n(z) |\phi_n\rangle = \sum_{n=0}^{\infty} [c_n\Lambda_n(z) + d_{n+1}\Lambda_{n+1}(z)] |\phi_n\rangle. \tag{3.3}$$

This yields the recursion solution for the coefficients $\{\Lambda_n(z)\}_{n=0}^{\infty}$, namely,

$$\Lambda_0(z) = \left(\sum_{n=0}^{\infty} |Q_n(z)|^2\right)^{-\frac{1}{2}} \text{ and } \Lambda_n(z) = \Lambda_0(z) Q_n(z), \ n \geq 1 \tag{3.4}$$

where

$$Q_0(z) = 1 \text{ and } Q_n(z) = \prod_{j=0}^{n-1} \left(1 + \frac{z - c_j}{d_{j+1}}\right), \ n \geq 1. \tag{3.5}$$

This general representation of the system’s coherent state has two special cases. The first one pertains to the fact that the ground state $|E = 0\rangle$ is itself a coherent state $|z\rangle$ with $z = 0$. To see this, we use the expansion of the ground state $|E = 0\rangle = \sum_{n=0}^{\infty} f_n(0) |\phi_n\rangle$. Hence,

$$A|E = 0\rangle = \sum_{n=0}^{\infty} [c_nf_n(0) + d_{n+1}f_{n+1}(0)] |\phi_n\rangle. \tag{3.6}$$

But we know from Eq. (2.8) that

$$\frac{f_{n+1}(0)}{f_n(0)} = \frac{p_{n+1}(0)}{p_n(0)} = -\frac{c_n}{d_{n+1}}. \tag{3.7}$$

Thus, $A|E = 0\rangle = 0$. In section 5, we give two examples of how to use the general form of the solution in Eq. (3.2) to find the explicit form of the ground states. The second special case is when the Hamiltonian has a known pure discrete energy spectrum. The Hamiltonian, of course, has a diagonal matrix representation in the basis composed of the system’s energy eigenvectors. Since diagonal matrices are special tridiagonal ones, the above conditions leading to Eq. (3.2) apply. In fact, all the $c_n$ vanish and $d_n = \sqrt{E_n}$. Then,

$$Q_n(z) = \frac{z^n}{\sqrt{(E_n)!}}. \tag{3.8}$$
where \((E_0)! = 1\) and \((E_n)! = E_1...E_n\) for \(n \geq 1\) denotes the generalized factorial. This simplifies the expansion sum Eq. (3.2) and may renders it doable. Even more, because we know how the energy eigenstates develop in time, each term contributing to sum in the expansion picks up a factor \(e^{-iE_n t}\). In section 6 we illustrate this by considering the case of the one-dimensional harmonic oscillator.

### 3.2 Time development of the coherent states

If we know the details of the spectrum of the Hamiltonian, we can then describe the time-development of the coherent state \(|z\rangle\). We can invert the expansion of the energy eigenvector in term of the basis function to write the basis function in term of the complete energy eigenstates using the resolution of the identity Eq. (2.3) resulting in

\[
|\phi_n\rangle = \sum_{\mu} \sqrt{\Omega(\mu)} p_{m}(E_{\mu}) |E_{\mu}\rangle + \int_{\sigma_c} \sqrt{\Omega(E)} p_{m}(E) |E\rangle dE.
\]

(3.9)

Substituting back in the expansion of the coherent states, we obtain the coherent eigenstate in term of the energy eigenstate of the Hamiltonian,

\[
|z\rangle = \Lambda_0(z) \sum_{n=0}^{\infty} Q_n(z) \left[ \sum_{\mu} \sqrt{\Omega(\mu)} p_{m}(E_{\mu}) |E_{\mu}\rangle + \int_{\sigma_c} \sqrt{\Omega(E)} p_{m}(E) |E\rangle dE \right].
\]

(3.10)

Because we know the time-development of the energy eigenstates, we can immediately write down the time development of the coherent state as

\[
|z, t\rangle = \Lambda_0(z) \sum_{n=0}^{\infty} Q_n(z) \left[ \sum_{\mu} \sqrt{\Omega(\mu)} p_{m}(E_{\mu}) e^{-iE_n t} |E_{\mu}\rangle + \int_{\sigma_c} \sqrt{\Omega(E)} p_{m}(E) e^{-iEt} |E\rangle dE \right].
\]

(3.11)

We will see in section 5 an application of this relation to specific systems.

### 4 Special coherent states and their use a basis

When we examine the general solution of the coherent states as given by the infinite series expansion of Eq. (3.2) and (3.4), we can identify some special states having interesting properties. These are the ones where the parameter \(z\) coincides with one of the coefficients \(\{c_n\}_{n=0}^{\infty}\). When \(z = c_\alpha\), the function \(Q_n(c_\alpha)\) vanishes identically for all \(n \geq \alpha + 1\). If we define a matrix \(\Lambda\) whose elements are \(\Lambda_{n,\alpha} = \Lambda_n(c_\alpha)\), then it is an upper triangular matrix. In terms of this matrix, we can write the form of the special coherent state \(|c_\alpha\rangle\) simply as

\[
|c_\alpha\rangle = \sum_{n=0}^{\alpha} |\phi_n\rangle \Lambda_{n,\alpha}.
\]

(4.1)

Because of the finiteness of the sum, finding the state \(|c_\alpha\rangle\) is grossly easier than finding \(|z\rangle\) for general \(z\). As examples, the first two of these special states are \(|c_0\rangle = |\phi_0\rangle\) and \(|c_1\rangle = |\phi_1\rangle + |\phi_2\rangle\).
|φ₀⟩Λ₀₁ + |φ₁⟩Λ₁₁. Additionally, it is clear that |cₐ⟩ can be written as a linear combination of the first (α + 1) members of the basis set. Conversely, it is then possible to write each member of the basis |φₙ⟩ as a linear combination of the first (n + 1) members of the special coherent states. In fact, we can put the set \{|cₐ⟩\}₀⁻→∞ into one-to-one correspondence with the set \{|φₙ⟩\}₀⁻→∞. Therefore, similar to the basis, the set \{|cₐ⟩\}₀⁻→∞ is complete. This is sufficient condition to qualify this set to be designated as basis. We know that it is normalized by construction but not orthogonal.

The matrix Λ allows us to construct |cₐ⟩ from the set \{|φₙ⟩\}₀⁻→∞ via Eq. (4.1). For convenience, we need a relation that enables us to construct |φₙ⟩ from the set \{|cₐ⟩\}₀⁻→∞. For that purpose we use the inverse matrix \(\bar{Λ}\) to the matrix Λ. Similar to Λ, the matrix \(\bar{Λ}\) is also upper triangular satisfying

\[
\sum_{n=0}^{∞} Λ₀ₙ \bar{Λ}ₙ₟ = δₙₚ, \quad \sum_{n=0}^{∞} \bar{Λ}ₙₙ Λₙ₟ = δₘₚ.
\] (4.2)

In order to move conveniently between the two bases, we use Eq. (4.1) and the above orthogonality relations to conclude that

\[
|φₙ⟩ = \sum_{α=0}^{n} |cₐ⟩ \bar{Λ}ₙ₟.
\] (4.3)

The expansion of a general coherent state |z⟩ in terms of the basis \{|φₙ⟩\}₀⁻→∞, as in Eq. (3.2), can now be written in terms of the basis \{|cₐ⟩\}₀⁻→∞ using the above relation as follows

\[
|z⟩ = \sum_{n=0}^{∞} \sum_{α=0}^{n} |cₐ⟩ \bar{Λ}ₙ₟ Λₙ(z).
\] (4.4)

We decouple the indices in the above double sums by using the relation [13]

\[
\sum_{n=0}^{∞} \sum_{α=0}^{n} F(α, n - α) = \sum_{n=0}^{∞} \sum_{α=0}^{∞} F(α, n).
\] (4.5)

This leads to the following desired expansion in terms of the special basis,

\[
|z⟩ = \sum_{α=0}^{∞} |cₐ⟩ Kₐ(z), \quad Kₐ(z) = \sum_{n=0}^{∞} \bar{Λ}ₙ₟ Λₙ(z).
\] (4.6)

It is necessary to check that this formula reproduces the special coherent states. For suppose that z = cₛ, then Λₙ₋₄(z) = Λₙ₋₄ₜ and

\[
Kₐ(c₄) = \sum_{n=0}^{∞} \bar{Λ}ₙ₟ Λₙ₋₄ₜ = \sum_{j=α}^{β} \bar{Λ}ₙ₟ Λₙ₋₄ₜ = δₘₜ.
\] (4.7)

The last sum vanishes identically for β ≤ α - 1, while for β ≥ α, Kₐ(c₄) = δₘₜ by the orthogonality relation Eq. (4.2). Hence |z⟩ = |c₄⟩. We now have to find explicit the explicit values of Kₐ(z) in terms of the basic parameters of the system. We first claim that the matrix elements of \(\bar{Λ}\) are given explicitly by

\[
\bar{Λ}ₘₚ = \begin{cases} \frac{1}{Λₙ₟} \prod_{j=α+1}^{m} \left(\frac{d_j}{cₜ - cₙ} \right), & \text{if } m ≥ α + 1 \\ \frac{1}{Λₙ₟}, & \text{if } m = α \\ 0, & \text{if } m ≤ α - 1. \end{cases}
\] (4.8)
We give the proof in Appendix A. We further state the following summation formula which will be helpful in finding the explicit form of the function $K_\alpha(z)$.

$$1 + \sum_{n=1}^{\gamma} \prod_{j=\alpha}^{n+a-1} \left( \frac{z - c_j}{c_\alpha - c_{j+1}} \right) = \prod_{k=\alpha+1}^{\gamma+a} \left( \frac{z - c_k}{c_\alpha - c_k} \right), \quad \gamma \geq \alpha + 1. \quad (4.9)$$

For the proof of (4.9) see Appendix B. Now we examine in detail the quantity $K_\alpha(z)$.

$$K_\alpha(z) = \sum_{n=0}^{\infty} \tilde{\Lambda}_{\alpha,n+\alpha} \Lambda_{n+\alpha}(z)$$

$$= \tilde{\Lambda}_{\alpha,\alpha} \Lambda_\alpha(z) \sum_{n=0}^{\infty} \frac{\tilde{\Lambda}_{\alpha,n+\alpha} \Lambda_{n+\alpha}(z)}{\Lambda_\alpha(z)}$$

$$= \tilde{\Lambda}_{\alpha,\alpha} \Lambda_\alpha(z) \sum_{n=0}^{\infty} \prod_{j=\alpha+1}^{n+a} \left( \frac{z - c_j}{c_\alpha - c_{j+1}} \right) \prod_{k=\alpha+1}^{n+a-1} \frac{z - c_k}{d_{k+1}}$$

$$= \tilde{\Lambda}_{\alpha,\alpha} \Lambda_\alpha(z) \left[ 1 + \sum_{n=1}^{\infty} \prod_{j=\alpha}^{n+a-1} \left( \frac{z - c_j}{c_\alpha - c_{j+1}} \right) \right]. \quad (4.10)$$

By Eq. (4.9), the quantity in curly brackets is just $\prod_{k=\alpha+1}^{\infty} \left( \frac{z - c_k}{c_\alpha - c_k} \right)$. Thus, we have the major result

$$K_\alpha(z) = \frac{1}{\Lambda_\alpha(z)} \sum_{\alpha=0}^{\infty} \Lambda_\alpha(z) \prod_{k=\alpha+1}^{\infty} \left( \frac{z - c_k}{c_\alpha - c_k} \right). \quad (4.11)$$

We now use the facts that

$$\tilde{\Lambda}_{\alpha,\alpha} = \frac{1}{\Lambda_{\alpha,\alpha}} = \frac{1}{\Lambda_{0,\alpha}} \prod_{j=0}^{a-1} \left( \frac{d_{j+1}}{c_\alpha - c_j} \right); \quad \Lambda_\alpha(z) = \Lambda_0(z) \prod_{j=0}^{a-1} \left( \frac{z - c_j}{d_{j+1}} \right). \quad (4.12)$$

Thus, $\tilde{\Lambda}_{\alpha,\alpha} \Lambda_\alpha(z) = \frac{\Lambda_0(z)}{\Lambda_{0,\alpha}} \prod_{j=0}^{a-1} \left( \frac{z - c_j}{c_\alpha - c_j} \right)$. Therefore, we have another version for $K_\alpha(z)$, namely

$$K_\alpha(z) = \frac{\Lambda_0(z)}{\Lambda_{0,\alpha}} \prod_{j=0}^{a-1} \left( \frac{z - c_j}{c_\alpha - c_j} \right) \prod_{k=\alpha+1}^{\infty} \left( \frac{z - c_k}{c_\alpha - c_k} \right) = \prod_{k=\alpha}^{\infty} \frac{z - c_k}{c_\alpha - c_k}. \quad (4.13)$$

This leads to the following revealing form of the expansion of a general coherent state $|z\rangle$ in terms of the special set of coherent states $\{|c_\alpha\rangle\}_{\alpha=0}^{\infty}$,

$$\frac{1}{\Lambda_0(z)} |z\rangle = \sum_{\alpha=0}^{\infty} L_\alpha(z) \frac{|c_\alpha\rangle}{\Lambda_{0,\alpha}}, \quad L_\alpha(z) = \prod_{k=\alpha+1}^{\infty} \frac{z - c_k}{c_\alpha - c_k}. \quad (4.14)$$

This is just the realization of the Lagrange’s interpolation scheme of the set $\{|c_\alpha\rangle\}_{\alpha=0}^{\infty}$ of values of the vector function $\frac{1}{\Lambda_0(z)} |z\rangle$ of the variable $z$ at the set of points $\{z = c_\alpha\}_{\alpha=0}^{\infty}$. We recognize that the quantity $L_\alpha(z)$ is just the Lagrange kernel [14]. From the rich literature on Lagrange interpolation scheme, we know that there are other equivalent versions to the one depicted in Eq. (4.14). We just quote them below without further justification,

$$L_\alpha(z) = \prod_{k=\alpha}^{\infty} \frac{z - c_k}{c_\alpha - c_k}; \quad L(z) \equiv \prod_{\beta=0}^{\infty} (z - c_\beta); \quad \sigma_\alpha = \frac{1}{\prod_{k\neq\alpha}^{\infty} (c_\alpha - c_\beta)} \quad (4.15)$$
\[ (i) \quad \frac{1}{\Lambda_0(z)} |z\rangle = \sum_{\alpha=0}^{\infty} L_\alpha(z) \frac{|c_\alpha\rangle}{\Lambda_{0,\alpha}}, \quad (ii) \quad \frac{1}{\Lambda_0(z)} |z\rangle = L(z) \sum_{\alpha=0}^{\infty} \frac{\sigma_\alpha}{z-c_\alpha} \frac{|c_\alpha\rangle}{\Lambda_{0,\alpha}} \]

\[ (iii) \quad \frac{1}{\Lambda_0(z)} |z\rangle = \sum_{\alpha=0}^{\infty} \frac{\sigma_\alpha}{z-c_\alpha} \frac{|c_\alpha\rangle}{\Lambda_{0,\alpha}}. \quad (4.16) \]

We must now show that form (i), and equivalently the others, satisfies two necessary properties. First, when \( z = c_\beta \) it yields the coherent state \(|c_\beta\rangle\). This easily follows from the fact that \( L_\alpha(c_\beta) = \delta_{\alpha,\beta} \). Second, the form has to satisfy the defining equation for the coherent state, namely, \( A |z\rangle = z |z\rangle \). This is demonstrated in Appendix C. Later in section 5, we contrast these forms with the original representation of the coherent state \(|z\rangle\) in the basis \( \{|\phi_n\rangle\}_{n=0}^{\infty} \) by comparing the associated space density \( \rho(z;x) = |\langle x |z\rangle|^2 \) of each form.

5 Examples

5.1 The ground state as a coherent state.

The ground state \(|E = 0\rangle\) is a coherent state \(|z\rangle\) with \( z = 0 \). We illustrate this with two examples.

5.1.1 The Morse oscillator

It is known that the one-dimensional Morse Hamiltonian

\[ -\frac{1}{2} \frac{d^2}{dx^2} + V_0(e^{-2\alpha x} - 2e^{-\alpha x}) + \frac{1}{2} \alpha^2 D^2 \]  

has a tridiagonal matrix representation in the complete orthonormal basis

\[ \phi_n(x) = \sqrt{\frac{n! \alpha}{\Gamma(n + 2\gamma + 1)}} y^{\gamma + \frac{1}{2}} e^{-\frac{y}{2}} L_n^{(2\gamma)}(y) \]  

where the variable \( y \) is \( y(x) = \frac{\sqrt{8V_0}}{\alpha} e^{-\alpha x} \), where \( V_0 \) and \( \alpha \) are nonnegative given parameters of the oscillator, \( D = \frac{\sqrt{2V_0}}{\alpha} - \frac{1}{2} \) and \( \gamma \) is a free scale parameter [15]. Here, \( L_n^{(\eta)}(\cdot) \) denotes the Laguerre polynomial ([16], p.239). The associated coefficients \( \{c_n, d_n\}_{n=0}^{\infty} \) with this Hamiltonian are given explicitly as

\[ c_n(\gamma) = \frac{\alpha}{\sqrt{2}} (n + \gamma + \frac{1}{2} - D) \), \quad d_n(\gamma) = -\frac{\alpha}{\sqrt{2}} \sqrt{n(n + 2\gamma)}. \]  

Since \( \gamma \) is a free scale parameter, and if the potential supports one or more bound states, then we may choose \( \gamma \) to have the value \( \gamma_0 = D - \frac{1}{2} \). In that case \( c_0(\gamma_0) = 0 \) and hence the coherent state \(|z = 0\rangle = |c_0(\gamma_0)\rangle\). Therefore the ground state simply has one term in the expansion of Eq.\((3.2)\), namely, \( |z = 0\rangle = |\phi_0\rangle \). In that case, we have by Eq.\((5.2)\)

\[ \phi_0(x) = \sqrt{\frac{\alpha}{\Gamma(2D)}} y^D e^{-\frac{y}{2}} \]  

\[ (5.4) \]
which is indeed the ground state wavefunction. It is interesting to note that if there are more than one bound state, we can alternatively choose the free scale parameter to have the value $\gamma_0 = D - \frac{3}{2}$. With this choice, $c_1(\gamma_1) = 0$ and hence $|z = 0) = |c_1(\gamma_1))$. We know that

$$|c_1(\gamma_1)) = |\phi_0(\gamma_1))\Lambda_{0,1}(\gamma_1) + |\phi_1(\gamma_1))\Lambda_{1,1}(\gamma_1). \tag{5.5}$$

In this case,

$$c_0(\gamma_1) = -\frac{\alpha}{\sqrt{2}}, \quad d_1(\gamma_1) = -\frac{\alpha}{\sqrt{2}} \sqrt{(2D - 2)}. \tag{5.6}$$

We also have

$$Q_1(\gamma_1) = \frac{c_1(\gamma_1) - c_0(\gamma_1)}{d_1(\gamma_1)} = \frac{-1}{\sqrt{(2D - 2)}}. \tag{5.7}$$

Thus, we have explicitly for the special coherent state,

$$|c_1(\gamma_1)) = \Lambda_{0,1}(\gamma_1) \left( |\phi_0(\gamma_1)) + |\phi_1(\gamma_1)) Q_1(\gamma_1)) \right) \tag{5.8}$$

$$= \Lambda_{0,1}(\gamma_1) \left( |\phi_0(\gamma_1)) - \frac{1}{\sqrt{2D - 2}} |\phi_1(\gamma_1)) \right). \tag{5.9}$$

If we now carefully substitute the needed parameters in the above equation, we find again that the coherent state wavefunction

$$\langle x | z = 0) = \langle x | c_1(\gamma_1)) = \sqrt{\frac{\alpha}{\Gamma(2D)}} y^\nu e^{-\frac{y^2}{2}} \tag{5.10}$$

is the ground state.

### 5.1.2 The radial harmonic oscillator

The $\ell - th$ partial wave harmonic oscillator Hamiltonian,

$$-\frac{1}{2} \frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{2r^2} + \frac{1}{2} \omega^2 r^2 - (\ell + \frac{3}{2})\omega \tag{5.11}$$

has a tridiagonal matrix representation in the orthonormal basis

$$\phi_n(x) = \sqrt{\frac{n! (2\lambda)}{\Gamma(n + \nu + 1)}} y^{\nu + \frac{1}{2}} e^{-\frac{y^2}{2}} L_n^{(\nu)}(y^2) \tag{5.12}$$

where $y(r) = \lambda r, \nu = \ell + \frac{1}{2}$, and $\lambda$ is a free scale parameter [12]. The Hamiltonian has a pure discrete spectrum with energies $E_n = 2n\omega$. The associated coefficients $\{c_n, d_n\}_{n=0}^\infty$ with this Hamiltonian are given explicitly as

$$c_n = \frac{(\lambda - \omega)}{\sqrt{2}} \sqrt{n + \ell + \frac{3}{2}}, \quad d_n = \frac{(\lambda + \omega)}{\sqrt{2}} \sqrt{n}. \tag{5.13}$$

If we choose the free scale parameter to have the value $\lambda_0 = \sqrt{\omega}$, then all the $c_n$ vanish. Hence in the expansion (3.2) for the coherent state $|z = 0)$ only the first term survives. Therefore, $|z = 0) = |\phi_0(\lambda_0))$. In that case, we have from Eq. (5.12)

$$\phi_0(r) = \sqrt{\frac{2\sqrt{\omega}}{\Gamma(\lambda + \frac{3}{2})}} (\sqrt{\omega} r)^{\ell + 1} e^{-\frac{\omega r^2}{2}} \tag{5.14}$$
which is indeed the ground state wavefunction of the radial oscillator. It is curious to note that, because all of $c_\alpha(\lambda_0)$ vanish, all the special states $|c_\alpha(\lambda_0)\rangle$ become the ground state. This can also be confirmed by the fact that all the matrix elements $\Lambda_{0,\alpha}$ for $\alpha \geq 1$ vanish because all of the functions $Q_n$ vanish for $n \geq 1$. What remains are $\Lambda_{0,0} = 1$ and $Q_0 = 1$, yielding $\langle r|c_\alpha(\lambda_0)\rangle = \phi_0(r)$, the ground state.

5.2 The coherent state of systems with known pure discrete energy spectrum

We consider the familiar case of the one-dimensional harmonic oscillator with frequency $\omega$. For this system $E_n = n\omega$, and hence $Q_n(z) = \frac{1}{\sqrt{n!}} \left( \frac{z}{\sqrt{\omega}} \right)^n$. We then have $\Lambda_0(z) = e^{-|z|^2/2\omega}$. Therefore, the coherent state function at time $t$ is given by

$$Z(x,t) \equiv \langle x | z, t \rangle = \Lambda_0(z) \sum_{n=0}^{\infty} Q_n(z) e^{-iE_n t} \phi_n(x) = \sqrt{\frac{\omega}{\pi}} \left( e^{-\frac{|z|^2}{2\omega}} \right) \sum_{n=0}^{\infty} \left( \frac{ze^{-i\omega t}/\sqrt{2\omega}}{n!} \right)^n H_n(\sqrt{\omega} x),$$

where $H_n(\cdot)$ denotes the Hermite polynomial ([16], p.249). Here, we have used the fact that the orthonormal energy eigenfunctions are the known

$$\phi_n(x) = \sqrt{\frac{\omega}{\sqrt{\pi}} \frac{e^{-\frac{|z|^2}{2\omega}}}{\sqrt{2^n n!}}} H_n(\sqrt{\omega} x).$$

The sum in Eq.(5.15) is doable [16], and yields

$$Z(x,t) \equiv \sqrt{\frac{\omega}{\pi}} e^{-\frac{|z|^2}{2\omega}} e^{-\frac{|z|^2}{2\omega}} \exp \left( 2\sqrt{\omega} x \left( \frac{ze^{-i\omega t}/\sqrt{2\omega}}{\sqrt{2\omega}} \right) - \left( \frac{ze^{-i\omega t}/\sqrt{2\omega}}{\sqrt{2\omega}} \right)^2 \right).$$

If we now write $z = |z| e^{i\phi}$, and define the density of the coherent state $\rho(z; x, t) \equiv |Z(x, t)|^2$, we finally have

$$\rho(z; x, t) \equiv \frac{\omega}{\pi} \exp \left( -\omega(x - \frac{\sqrt{2}}{\omega} |z| \cos(\omega t - \varphi))^2 \right).$$

This is just a Gaussian density function of constant width and with center $\bar{x}$ oscillating harmonically as

$$\bar{x}(z, t) = \frac{\sqrt{2}}{\omega} |z| \cos(\omega t - \varphi).$$

5.3 Time development of special coherent state

Even with the availability of Eq. (3.11) to describe the time development of a general coherent state, it is a considerable challenge to find the answer in close form. However, the difficulty is diminished substantially if we able to utilize the special coherent state to act in place of the
general coherent state. For, if the basis is endowed with a free scale parameter \( \lambda \), it may be possible to choose \( \lambda \) to have the value \( \lambda_\alpha \) such that \( z = c_\alpha(\lambda_\alpha) \). In that case, \(|z| = |c_\alpha(\lambda_\alpha)| \) and the calculation difficulty is considerably reduced. We illustrate this case, by examining the time development of coherent state associated with radial harmonic oscillator and another associated with a radial free particle.

5.3.1 Time development of the coherent state of the radial harmonic oscillator

For this system, \( c_n(\lambda) = \frac{(\lambda - \omega)}{\sqrt{2}} \sqrt{n + \ell + \frac{3}{2}} \). Suppose the value of \( z \) is compatible with the restriction on \( \lambda \) (which is \( Re(\lambda) > 0 \)) so that we can choose \( \lambda \) to have the value

\[
\lambda = \lambda_0 \equiv \frac{z}{\sqrt{2\ell + 3}} + \sqrt{\frac{z^2}{2\ell + 3} + \omega}.
\]

This immediately makes \(|z| = |c_0(\lambda_0)| \). This, in turns, means that \(|z| = |\phi_0(\lambda_0)| \). From Eq. (3.11), and since the spectrum is purely discrete, we have

\[
|z, t\rangle = \sum_{\mu=0}^{\infty} \sqrt{\Omega_\mu(\lambda_0)} p_n^{(\lambda_0)}(E_\mu) e^{-iE_\mu t}|E_\mu\rangle
\]

where we have shown the explicit the dependence on \( \lambda_0(z) \). Noting that the discrete density is [12]

\[
\Omega_\mu(\lambda_0) = \frac{\Gamma(\mu + \nu + 1)}{\mu! \Gamma(\mu + \nu + 1)} \tau^\mu (1 - \tau)^{\nu+1}, \quad \tau = \left( \frac{\lambda_0^2 - \omega}{\lambda_0^2 + \omega} \right)^2,
\]

we have

\[
\psi(z; r, t) = \langle r | z, t\rangle = \left( \frac{2\sqrt{\omega}}{\Gamma(\nu + 1)} \right)^{\frac{1}{2}} \frac{[1 - \tau]^{(1+\nu)/2}}{[1 - y(t)]^{(1+\nu)}} \frac{[\sqrt{\omega} r]^{(\ell+1)}}{[\sqrt{\omega} r]^{(\ell+1)}} e^{-\omega r^2} e^{\frac{1+y(t)}{1-y(t)}},
\]

\[
y(t) = \sqrt{\tau} e^{-2i\omega t}, \quad \nu = \ell + \frac{1}{2}.
\]

As a necessary quality check on this result, we note that \( \psi(z; r, 0) = \phi_0(r, \lambda_0) \). The space density of coherent state is just \( \rho(z; r, t) = |\psi(z; r, t)|^2 \),

\[
\rho(z; r, t) = |\psi(z; r, t)|^2 = \frac{2[\omega G(t)]^{\nu+1}}{\Gamma(\nu + 1)} r^{2\ell+2} e^{-\omega r^2 G(t)}, \quad G(t) = \frac{1 - \tau}{1 + \tau - 2\sqrt{\tau} \cos(2\omega t)}.
\]

The average position \( \bar{r}(t) \) of the coherent state is computed via the relation

\[
\bar{r}(t) = \int_0^{\infty} r \rho(z; r, t) \, dr.
\]

The result is

\[
\bar{r}(t) = \frac{\Gamma(\nu + \frac{3}{2})}{\Gamma(\nu + 1)} \sqrt{\frac{\lambda_0^2 \sin^2(\omega t) + \omega^2 \cos^2(\omega t)}{\omega^2 \lambda_0^2}}.
\]

We show in Fig.1 the sinusoidal nature of the time development of \( \bar{r}(t) \) and in Fig.2 a phase diagram of \( \frac{d}{dt} \bar{r}(t) \) versus \( \bar{r}(t) \).
5.3.2 Time development of the coherent state of the radial free particle

In contrast to the above example of the radial harmonic oscillator where the energy spectrum is purely discrete, we work out the same problem but for a radial free particle whose energy spectrum is purely continuous. The Hamiltonian associated with this problem is tridiagonal in the same orthonormal basis as the previous one, but now yielding the quantity $c_n(\lambda) = \frac{\lambda}{\sqrt{2}} \sqrt{n + \ell + \frac{3}{2}}$. If we now choose the free scale parameter $\lambda$ to have the value $\lambda_0 = \frac{2z}{\sqrt{2\ell + 3}}$, then $|z) = |c_0(\lambda_0))$.

From Eq. (3.11), we have

$$|z, t) = \int_0^\infty \sqrt{\Omega(E)} e^{-iEt} |E\rangle dE. \quad (5.28)$$

Noting that the continuous density is [12]:

$$\Omega(E, \lambda_0) = \frac{2 \lambda_0^2}{\Gamma(\nu + 1)} (\frac{2E}{\lambda_0^2})^\nu e^{-2E/\lambda_0^2} \quad (5.29)$$

and $\langle r|E\rangle = \sqrt{r} J_\nu(kr)$ with $k^2/r = E$, we get

$$\psi(z; r; t) = [\beta(t) \lambda]^{\nu + 1} \sqrt{\frac{2 \beta(t)}{\Gamma(\nu + 1)}} [\beta(t) r]^{t+1} e^{-\frac{[\beta(t) r]^2}{2}}, \quad \beta(t) = \frac{\lambda}{\sqrt{1 + i\lambda^4 t^2}} \quad (5.30)$$

The space density of the coherent state becomes

$$\rho(r, t) = \frac{2}{\Gamma(\nu + 1)} \left( \frac{\beta \lambda^2}{\lambda} \right)^{\nu + 2} r^{2\ell + 2} e^{-\left( \frac{\beta^2 + \beta^* r^2}{2} \right)} \quad (5.31)$$

This translates into

$$\rho(z; r, t) = \frac{2 \lambda_0}{\Gamma(\nu + 1) (1 + \lambda_0^4 t^2)^{\nu + 1}} (\lambda_0 r)^{2\ell + 2} e^{-\frac{\lambda_0^2 + \lambda_0^* r^2}{1 + \lambda_0^4 t^2}} \quad (5.32)$$

The average position of the coherent state is

$$\bar{r}(t) = \frac{\Gamma(\nu + \frac{3}{2})}{\Gamma(\nu + 1)} \sqrt{\frac{1 + \lambda_0^4 t^2}{\lambda_0^2}} \quad (5.33)$$

Asymptotically, for large time, the average position progresses linearly with time,

$$\lim_{t \to \infty} \bar{r}(t) = \frac{\Gamma(\nu + \frac{3}{2})}{\Gamma(\nu + 1)} \lambda_0 t \quad (5.34)$$

as expected. We show in Fig.3 and Fig.4 the time dependence of $\bar{r}(t)$ and $\frac{d}{dt}\bar{r}(t)$ respectively. We note in passing that the results of the previous example on the radial harmonic oscillator reduce to the above results in the limit of vanishing frequency $\omega$.

5.4 Performance of the Lagrange form of the coherent states

Here we consider, as a reference, the coherent state $(z = 0.83)$ of the Morse oscillator as represented by a converged finite sum of order $N$ of Eq. (3.2). We also construct a converged finite sum of the
three Lagrange forms of the coherent state. For each, we find the real space density and compare its graph with that of the reference case in Fig.5. We note that all methods give practically identical results. This is not surprising since for the same scale parameter $\gamma$ and order of approximation $N$, the finite basis $\{|\phi_n\rangle\}_{n=0}^{N-1}$ span the same subspace as that spanned by the finite basis $\{|c_\alpha\rangle\}_{\alpha=0}^{N-1}$. Had we chosen to find a scale parameter $\gamma_0$ such that $z = c_0(\gamma_0)$, the coherent state can also be represented as $|c_0(\gamma_0)\rangle$. In that case $\gamma_0 = 4.646$. The reader can easily verify that any of the four densities displayed in Fig.5 is practically identical to the density $|\langle x|c_0(\gamma_0)\rangle|^2$.

6 Comments

The over-completeness of the coherent states is usually handled through a demonstration of the resolution of the identity as in Eq. (1.1). In this paper, we tackle this issue by exhibiting a proper subset of coherent states, namely, $\{|c_\alpha\rangle\}_{\alpha=0}^{\infty}$, which is complete.

Also, it is known that the canonical coherent state $\{|c_\alpha\rangle\}_{\alpha=0}^{\infty}$ can be generated from the fiducial state $|0\rangle$, that satisfies $A|0\rangle = 0$, through the relation $|z\rangle = e^{(zA^\dagger-z^*A)}|0\rangle$. In our approach, because the forward shift and backward shift operators, $A$ and $A^\dagger$, are not true lowering and raising operators, the terms $(A^\dagger)^n|0\rangle$ exhibit no particular simplicity. Instead, we use the general basis $\{|\phi_n\rangle\}_{n=0}^{\infty}$ which tridiagonalizes the Hamiltonian. Often times, this basis allows the freedom of choosing a scale parameter flexibly. This is a great advantage which we have shown how to exploit effectively. Furthermore, we introduced the complete subset of coherent states, $\{|c_\alpha\rangle\}_{\alpha=0}^{\infty}$ to be used as an indigenous basis to describe any general coherent state $|z\rangle$. In this basis $|z\rangle$ is really just the result of interpolating “à la Lagrange” the special coherent states $\{|c_\alpha\rangle\}_{\alpha=0}^{\infty}$.

Appendix A

We show here that the matrix $\bar{A}$ whose elements are given by Eq. (4.8), is actually the inverse of the matrix $A$. We first show that $\bar{A} A = 1$. Because both matrices are upper triangular, we have

$$
(\bar{A} A)_{\alpha,\beta} = \sum_{n=0}^{\infty} \bar{A}_{\alpha,n} A_{n,\beta} = \sum_{n=\alpha}^{\beta} \bar{A}_{\alpha,n} A_{n,\beta}.
$$

(A.1)

Since the matrix $\bar{A} A$ is also upper triangular, the element $(\bar{A} A)_{\alpha,\beta}$ must vanish for $\alpha \geq \beta + 1$. If $\alpha = \beta$, then $(\bar{A} A)_{\beta,\beta} = \bar{A}_{\beta,\beta} A_{\beta,\beta} = 1$. Finally, if $\beta \geq \alpha + 1$, we have

$$
(\bar{A} A)_{\alpha,\beta} = \sum_{n=\alpha}^{\beta} \bar{A}_{\alpha,n} A_{n,\beta} = \bar{A}_{\alpha,\alpha} A_{\alpha,\beta} + \bar{A}_{\alpha,\alpha+1} A_{\alpha+1,\beta} + \ldots + \bar{A}_{\alpha,\beta-1} A_{\beta-1,\beta} + \bar{A}_{\alpha,\beta} A_{\beta,\beta}.
$$

(A.2)

This means that

$$
(\bar{A} A)_{\alpha,\beta} = \bar{A}_{\alpha,\alpha} A_{\alpha,\beta} \left[ 1 + \frac{\bar{A}_{\alpha,\alpha+1} A_{\alpha+1,\beta}}{\bar{A}_{\alpha,\alpha} A_{\alpha,\beta}} + \ldots + \frac{\bar{A}_{\alpha,\beta-1} A_{\beta-1,\beta}}{\bar{A}_{\alpha,\alpha} A_{\alpha,\beta}} + \frac{\bar{A}_{\alpha,\beta} A_{\beta,\beta}}{\bar{A}_{\alpha,\alpha} A_{\alpha,\beta}} \right].
$$
The equation for the coherent state, namely, \( A_f \) is exact. Therefore, we have extended the summation by adding terms with zero contribution. Now, by appealing to Eq. (4.9) while putting \( \gamma = \beta - \alpha \) and substituting \( z = c_\beta \), we see that the last expression vanishes.

Appendix B

With \( \gamma \geq \alpha + 1 \), we verify that the left-hand side (LHS) of Eq. (4.9) of the Lemma is the same as the right-hand side (RHS).

\[
LHS = 1 + \left( \frac{z - c_\alpha}{c_\alpha - c_{\alpha + 1}} \right) + \left( \frac{z - c_\alpha}{c_\alpha - c_{\alpha + 1}} \right) \left( \frac{z - c_{\alpha + 1}}{c_\alpha - c_{\alpha + 2}} \right) + \cdots + \prod_{j=\alpha}^{\gamma+\alpha-1} \left( \frac{z - c_j}{c_\alpha - c_{j+1}} \right)
\]

\[
= \left( \frac{z - c_{\alpha + 1}}{c_\alpha - c_{\alpha + 1}} \right) + \left( \frac{z - c_\alpha}{c_\alpha - c_{\alpha + 1}} \right) \left( \frac{z - c_{\alpha + 1}}{c_\alpha - c_{\alpha + 2}} \right) + \cdots + \prod_{j=\alpha}^{\gamma+\alpha-1} \left( \frac{z - c_j}{c_\alpha - c_{j+1}} \right)
\]

\[
= \left( \frac{z - c_{\alpha + 1}}{c_\alpha - c_{\alpha + 1}} \right) \left( \frac{z - c_{\alpha + 2}}{c_\alpha - c_{\alpha + 2}} \right) \cdots \left( 1 + \left( \frac{z - c_\alpha}{c_\alpha - c_{\alpha + 2}} \right) \right) = \prod_{j=\alpha+1}^{\gamma+\alpha} \left( \frac{z - c_j}{c_\alpha - c_j} \right) = RHS. \quad (B.1)
\]

Appendix C

Here, we demonstrate that the Lagrange expansion of a coherent state satisfies the defining equation for the coherent state, namely, \( A \mid z \rangle = z \mid z \rangle \). The key to the proof is that if a function \( f(z) \) is any polynomial in \( z \) of order \( n \), then the Lagrange interpolation \( f(z) = \sum_{\alpha=0}^{n} f(c_\alpha) L_\alpha(z) \) is exact. Therefore,

\[
A \mid z \rangle = \Lambda_0(z) \sum_{\alpha=0}^{\infty} c_\alpha \frac{L_\alpha(z)}{\Lambda_{0,\alpha}} \mid \phi_n \rangle
\]

\[
= \Lambda_0(z) \sum_{\alpha=0}^{\infty} c_\alpha L_\alpha(z) \frac{\Lambda_{n,\alpha}}{\Lambda_{0,\alpha}} \mid \phi_n \rangle
\]

\[
= \Lambda_0(z) \sum_{\alpha=0}^{\infty} \sum_{n=0}^{\infty} c_{\alpha + n} L_{n+\alpha}(z) \frac{\Lambda_{n,n+\alpha}}{\Lambda_{0,n+\alpha}} \mid \phi_n \rangle
\]

\[
= \Lambda_0(z) \sum_{n=0}^{\infty} \mid \phi_n \rangle \sum_{\alpha=0}^{\infty} c_{\alpha + n} L_{n+\alpha}(z) \frac{\Lambda_{n,n+\alpha}}{\Lambda_{0,n+\alpha}}. \quad (C.1)
\]

We decoupled the connection between the two indices according to the rule of Eq. (4.5). Here, we have extended the summation by adding terms with zero contribution. Now,

\[
A \mid z \rangle = \Lambda_0(z) \sum_{n=0}^{\infty} \mid \phi_n \rangle \sum_{k=n}^{\infty} c_k L_k(z) \frac{\Lambda_{n,k}}{\Lambda_{0,k}}
\]
\[
\Lambda_0(z) \sum_{n=0}^{\infty} |\phi_n\rangle \sum_{k=0}^{\infty} c_k L_k(z) \frac{\Lambda_{n,k}}{\Lambda_{0,k}} \\
= \Lambda_0(z) \sum_{n=0}^{\infty} |\phi_n\rangle \sum_{k=0}^{\infty} c_k L_k(z) \prod_{j=0}^{n-1} \left( \frac{c_k - c_j}{d_{j+1}} \right). \tag{C.2}
\]

The inner summation is the Lagrange interpolation of polynomial functions. Hence,

\[
A |z\rangle = \Lambda_0(z) \sum_{n=0}^{\infty} |\phi_n\rangle z \prod_{j=0}^{n-1} \left( \frac{z - c_j}{d_{j+1}} \right) = z \sum_{n=0}^{\infty} |\phi_n\rangle \Lambda_n(z) = z |z\rangle. \tag{C.3}
\]

Data availability

The data that support the findings of this study are available within the article.

FIGURE CAPTIONS:

Fig. 1: The sinusoidal behavior of the average position of the coherent state of a radial harmonic oscillator with frequency $\omega = 2$, angular momentum $L = 1$ and $z = 3$. The value of the spacial scale parameter is $\gamma_0 = 3.291$.

Fig. 2: The average position vs its time development of the coherent state of a radial harmonic oscillator with frequency $\omega = 2$, angular momentum $L = 1$ and $z = 3$.

Fig. 3: The average position of the coherent state of a radial free particle with angular momentum $L = 1$ and $z = 3$. The value of the special scale parameter is $\gamma_0 = 2.863$.

Fig. 4: The time development of the velocity of the average position of the coherent state of a radial free particle with angular momentum $L = 1$ and $z = 3$.

Fig. 5: The coherent state space density for Morse oscillator with $V_0 = 10$ and $\alpha = 1$ which supports four bound states. The graphs show the comparison of the various methods with the reference case $g\psi = |\langle x|z\rangle|^2$ with $|z\rangle$ resulting from the use of the expansion of Eq. (3.2) with $\gamma = 3$ and order of approximation $N = 10$. The graphs $g_i(x)$, $g_{II}(x)$ and $g_{III}(x)$ are for the three forms of the Lagrange interpolation schemas $i$, $ii$ and $iii$ respectively in Eq. (4.16), for the same choices of scale parameter and order of approximation.
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