Inertia groups and smooth structures of 
(n − 1)-connected 2n-manifolds

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Abstract
Let $M^{2n}$ denote a closed $(n−1)$-connected smoothable topological 2n-manifold. We show that the group $\mathcal{C}(M^{2n})$ of concordance classes of smoothings of $M^{2n}$ is isomorphic to the group of smooth homotopy spheres $\Theta_{2n}$ for $n = 4$ or 5, the concordance inertia group $I_c(M^{2n}) = 0$ for $n = 3, 4, 5$ or 11 and the homotopy inertia group $I_h(M^{2n}) = 0$ for $n = 4$. On the way, following Wall’s approach [16] we present a new proof of the main result in [9], namely, for $n = 4, 8$ and $H^n(M^{2n}; \mathbb{Z}) \cong \mathbb{Z}$, the inertia group $I(M^{2n}) \cong \mathbb{Z}_2$. We also show that, up to orientation-preserving diffeomorphism, $M^8$ has at most two distinct smooth structures; $M^{10}$ has exactly six distinct smooth structures and then show that if $M^{14}$ is a π-manifold, $M^{14}$ has exactly two distinct smooth structures.

Keywords. $(n − 1)$-connected 2n-manifold, smooth structures, the stable tangential invariant, inertia groups, concordance and homotopy inertia groups.

Classification. 57R55; 57R60; 57R50; 57R65.

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1 Introduction
We work in the categories of closed, oriented, simply-connected $Cat$-manifolds $M$ and $N$ and orientation preserving maps, where $Cat = Diff$ for smooth manifolds or $Cat = Top$ for topological manifolds. Let $\Theta_m$ be the group of smooth homotopy spheres defined by M. Kervaire and J. Milnor in [6]. Recall that the collection of homotopy spheres $\Sigma$ which admit a diffeomorphism $M \to M \# \Sigma$ form a subgroup $I(M)$ of $\Theta_m$, called the inertia group of $M$, where we regard the connected sum $M \# \Sigma^m$ as a smooth manifold with the same underlying topological space as $M$ and with smooth structure differing from that of $M$ only on an $n$-disc. The homotopy inertia group $I_h(M)$ of $M^m$ is a subset of the inertia group consisting
of homotopy spheres Σ for which the identity map id : M → M#ΣΜ is homotopic to a diffeomorphism. Similarly, the concordance inertia group of M^m, I_c(M^m) ⊆ Θ_m, consists of those homotopy spheres Σ^m such that M and M#Σ^m are concordant.

The paper is organized as following. Let M^{2n} denote a closed (n − 1)-connected smoothable topological 2n-manifold. In section 2, we show that the group C(M^{2n}) of concordance classes of smoothings of M^{2n} is isomorphic to the group of smooth homotopy spheres Θ_{2n} for n = 4 or 5, the concordance inertia group I_c(M^{2n}) = 0 for n = 3, 4, 5 or 11 and the homotopy inertia group I_h(M^{2n}) = 0 for n = 4.

In section 3, we present a new proof of the following result in [9].

**Theorem 1.1.** Let M^{2n} be an (n − 1)-connected closed smooth manifold of dimension 2n ≠ 4 such that H^n(M; Z) ≅ Z. Then the inertia group I(M^{2n}) ≅ Z_2.

In section 4, we show that, up to orientation-preserving diffeomorphism, M^8 has at most two distinct smooth structures; M^{10} has exactly six distinct smooth structures and if M^{14} is a π-manifold, then M^{14} has exactly two distinct smooth structures.

## 2 Concordance inertia groups of (n − 1)-connected 2n-manifolds

We recall some terminology from [6]:

**Definition 2.1.**

(a) A homotopy m-sphere Σ^m is a closed oriented smooth manifold homotopy equivalent to the standard unit sphere S^m in R^{m+1}.

(b) A homotopy m-sphere Σ^m is said to be exotic if it is not diffeomorphic to S^m.

**Definition 2.2.** Define the m-th group of smooth homotopy spheres Θ_m as follows. Elements are oriented h-cobordism classes [Σ] of homotopy m-spheres Σ, where Σ and Σ′ are called (oriented) h-cobordant if there is an oriented h-cobordism (W, ∂_0 W, ∂_1 W) together with orientation preserving diffeomorphisms Σ → ∂_0 W and (Σ′)^− → ∂_1 W. The addition is given by the connected sum. The zero element is represented by S^m. The inverse of [Σ] is given by [Σ^-], where Σ^- is obtained from Σ by reversing the orientation. M. Kervaire and J. Milnor [6] showed that each Θ_m is a finite abelian group (m ≥ 1).

**Definition 2.3.** Two homotopy m-spheres Σ_1^m and Σ_2^m are said to be equivalent if there exists an orientation preserving diffeomorphism f : Σ_1^m → Σ_2^m.

The set of equivalence classes of homotopy m-spheres is denoted by Θ_m. The Kervaire-Milnor [6] paper worked rather with the group Θ_m of smooth homotopy spheres up to h-cobordism. This makes a difference only for m = 4, since it is known, using the h-cobordism theorem of Smale [12], that Θ_4 ≅ Θ_4 for m ≠ 4. However the difference is important in the four dimensional case, since Θ_4 is trivial, while the structure of Θ_4 is a great unsolved problem.
**Definition 2.4.** Let $M$ be a closed topological manifold. Let $(N, f)$ be a pair consisting of a smooth manifold $N$ together with a homeomorphism $f : N \to M$. Two such pairs $(N_1, f_1)$ and $(N_2, f_2)$ are concordant provided there exists a diffeomorphism $g : N_1 \to N_2$ such that the composition $f_2 \circ g$ is topologically concordant to $f_1$, i.e., there exists a homeomorphism $F : N_1 \times [0, 1] \to M \times [0, 1]$ such that $F|_{N_1 \times 0} = f_1$ and $F|_{N_1 \times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $C(M)$.

We will denote the class in $C(M)$ of $(M^n \# \Sigma^n, \text{id})$ by $[M^n \# \Sigma^n]$. (Note that $[M^n \# \Sigma^n]$ is the class of $(M^n, \text{id})$.)

**Definition 2.5.** Let $M^m$ be a closed smooth $m$-dimensional manifold. The inertia group $I(M) \subset \Theta_m$ is defined as the set of $\Sigma \in \Theta_m$ for which there exists a diffeomorphism $\phi : M \to M \# \Sigma$.

Define the homotopy inertia group $I_h(M)$ to be the set of all $\Sigma \in I(M)$ such that there exists a diffeomorphism $M \to M \# \Sigma$ which is homotopic to $\text{id} : M \to M \# \Sigma$.

Define the concordance inertia group $I_c(M)$ to be the set of all $\Sigma \in I_h(M)$ such that $M \# \Sigma$ is concordant to $M$.

**Remark 2.6.**

(1) Clearly, $I_c(M) \subseteq I_h(M) \subseteq I(M)$.

(2) For $M = S^n$, $I_c(M) = I_h(M) = I(M) = 0$.

Now we have the following:

**Theorem 2.7.** Let $M^{2n}$ be a closed smooth $(n - 1)$-connected $2n$-manifold with $n \geq 3$.

(i) If $n$ is any integer such that $\Theta_{n+1}$ is trivial, then $I_c(M^{2n}) = 0$.

(ii) If $n$ is any integer greater than 3 such that $\Theta_n$ and $\Theta_{n+1}$ are trivial, then

$$C(M^{2n}) = \{[M^{2n} \# \Sigma] \mid \Sigma \in \Theta_{2n}\} \cong \overline{\Theta}_{2n}.$$ 

(iii) If $n = 8$ and $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$, then $M^{2n} \# \Sigma^{2n}$ is not concordant to $M^{2n}$, where $\Sigma^{2n} \in \Theta_{2n}$ is the exotic sphere. In particular, $C(M^{2n})$ has at least two elements.

(iv) If $n$ is any even integer such that $\Theta_n$ and $\Theta_{n+1}$ are trivial, then $I_h(M) = 0$.

**Proof.** Let $Cat = Top$ or $G$, where $Top$ and $G$ are the stable spaces of self homeomorphisms of $\mathbb{R}^n$ and self homotopy equivalences of $S^{n-1}$ respectively. For any degree one map $f_M : M \to S^{2n}$, we have a homomorphism

$$f_M^* : [S^{2n}, Cat/O] \to [M, Cat/O].$$

By Wall [15], $M$ has the homotopy type of $X = (\bigvee_{i=1}^k S^n_i) \cup g \mathbb{D}^{2n}$, where $k$ is the $n$-th Betti number of $M$, $\bigvee_{i=1}^k S^n_i$ is the wedge sum of $n$-spheres and $g : S^{2n-1} \to \bigvee_{i=1}^k S^n_i$ is the attaching
map of $\mathbb{D}^{2n}$. Let $\phi : M \to X$ be a homotopy equivalence of degree one and $q : X \to S^{2n}$ be the collapsing map obtained by identifying $S^{2n}$ with $X/\bigvee_{i=1}^{k} S^{n}_i$ in an orientation preserving way. Let $f_M = q \circ \phi : M \to S^{2n}$ be the degree one map.

Consider the following Puppe’s exact sequence for the inclusion $i : \bigvee_{i=1}^{k} S^{n}_i \hookrightarrow X$ along $Cat/O$:

\[ \ldots \to \bigvee_{i=1}^{k} S^{n}_i, \text{Cat/O} \xrightarrow{(S(g))^*} \bigvee_{i=1}^{k} S^{n+1}_i, \text{Cat/O} \to [X, \text{Cat/O}] \to \bigvee_{i=1}^{k} S^{n}_i, \text{Cat/O}, \]

(2.1)

where $S(g)$ is the suspension of the map $g : S^{2n-1} \to \bigvee_{i=1}^{k} S^{n}_i$.

Using the fact that

\[ \bigvee_{i=1}^{k} S^{n}_i, \text{Cat/O} \cong \prod_{i=1}^{k} S^{n+1}_i, \text{Cat/O} \]

and

\[ \bigvee_{i=1}^{k} S^{n}_i, \text{Cat/O} \cong \prod_{i=1}^{k} S^{n}_i, \text{Cat/O}, \]

the above exact sequence (2.1) becomes

\[ \ldots \to \prod_{i=1}^{k} S^{n+1}_i, \text{Cat/O} \xrightarrow{(S(g))^*} \bigvee_{i=1}^{k} S^{n+1}_i, \text{Cat/O} \to [X, \text{Cat/O}] \to \prod_{i=1}^{k} S^{n}_i, \text{Cat/O}. \]

(i): If $n$ is any integer such that $\Theta_{n+1}$ is trivial and $Cat = Top$ in the above exact sequence (2.1), by using the fact that

\[ [S^m, Top/O] = \overline{\Theta}_m \] (m $\neq$ 3, 4)

and $[S^4, Top/O] = 0$ ([10] pp. 200-201]), we have $q^* : [S^{2n}, Top/O] \to [X, Top/O]$ is injective. Hence $f_M^* = \phi^* \circ q^* : \overline{\Theta}_{2n} \to [M, Top/O]$ is injective. By using the identifications $C(M^{2n}) = [M, Top/O]$ given by [10] pp. 194-196], $f_M^* : \overline{\Theta}_{2n} \to C(M^{2n})$ becomes $[\Sigma^{2n}] \to [M\#\Sigma^{2n}]$. $I_c(M)$ is exactly the kernel of $f_M^*$, and so $I_c(M) = 0$. This proves (i).

(ii): If $n > 3$, $\Theta_n$ and $\Theta_{n+1}$ are trivial, and $Cat = Top$ then, from the above exact sequence (2.1) we have $q^* : [S^{2n}, Top/O] \to [X, Top/O]$ is an isomorphism. This shows that $f_M^* = \phi^* \circ q^* : \overline{\Theta}_{2n} \to C(M^{2n})$ is an isomorphism and hence

\[ C(M^{2n}) = \{ [M^{2n} \# \Sigma] \mid \Sigma \in \overline{\Theta}_{2n} \}. \]

This proves (ii).

(iii): If $n = 8$ and $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$, then $M^{2n}$ has the homotopy type of $X = S^n \bigcup_g \mathbb{D}^{2n}$, where $g : S^{2n-1} \to S^n$ is the attaching map. In order to prove $M^{2n} \# \Sigma^{2n}$ is not concordant to $M^{2n}$, by the above exact sequence (2.1) for $Cat = Top$, it suffices to prove
\( q^* : [S^{16}, Top/O] \to [X, Top/O] \) is monic, which is equivalent to saying that \( (S(g))^* : [SS^8, Top/O] \to [S^{16}, Top/O] \) is the zero homomorphism. For the case \( g = p \), where \( p : S^{15} \to S^8 \) is the Hopf map, \( (S(g))^* \) is the zero homomorphism, which was proved in the course of proof of lemma 1 in [2] pp. 58-59. This proof works verbatim for any map \( g : S^{2n-1} \to S^n \) as well. This proves (iii).

(iv): If \( n \) is any even integer such that \( \Theta_n \) and \( \Theta_{n+1} \) are trivial, then \( \pi_{n+1}(G/O) = 0 \). This shows that from the above exact sequence (2.1) for \( \text{Cat} = G \), \( q^* : [S^{2n}, G/O] \to [X, G/O] \) is injective. Then \( f^*_M = \phi^* \circ q^* : [S^{2n}, G/O] \to [M, G/O] \) is injective. From the surgery exact sequences of \( M \) and \( S^{2n} \), we get the following commutative diagram ([3, Lemma 3.4]):

\[
\begin{array}{cccccc}
L_{2n+1}(e) & \longrightarrow & \overline{\Theta}_{2n} & \overset{\eta_{2n}}{\longrightarrow} & \pi_{2n}(G/O) & \longrightarrow & L_{2n}(e) \\
\downarrow & & \downarrow f^*_M & & \downarrow f^*_M & & \downarrow \\
L_{2n+1}(e) & \longrightarrow & S^{Diff}(M) & \overset{\eta_M}{\longrightarrow} & [M, G/O] & \longrightarrow & L_{2n}(e)
\end{array}
\] (2.2)

By using the facts that \( L_{2n+1}(e) = 0 \), injectivity of \( \eta_{S^{2n}} \) and \( \eta_M \) follow from the diagram, and combine with the injectivity of \( f^*_M \) to show that \( f^*_M : \overline{\Theta}_{2n} \to S^{Diff}(M) \) is injective. \( I_h(M) \) is exactly the kernel of \( f^*_M \), and so \( I_h(M) = 0 \). This proves (iv).

Remark 2.8.

(i) By M. Kervaire and J. Milnor [3], \( \Theta_m = 0 \) for \( m = 1, 2, 3, 4, 5, 6 \) or 12. If \( M^{2n} \) is a closed smooth \((n - 1)\)-connected \(2n\)-manifold, by Theorem 2.7(i) and (ii), \( I_c(M^{2n}) = 0 \) for \( n = 3, 4, 5 \) or 11 and \( C(M^{2n}) \cong \Theta_{2n} \) for \( n = 4 \) or 5.

(ii) If \( M \) has the homotopy type of \( \mathbb{O} \mathbb{P}^2 \), by Theorem 1.1 and Theorem 2.7(iii), we have \( I_c(M) = 0 \neq I(M) \).

(iii) By Theorem 2.7(iv), if \( M \) has the homotopy type of \( \mathbb{H} \mathbb{P}^2 \), then \( I_h(M) = 0 \).

Definition 2.9. Let \( M \) and \( N \) are smooth manifolds. A smooth map \( f : M \to N \) is called tangential if for some integers \( k, l \), \( f^*(T(N)) \oplus e_k \cong T(M) \oplus e^l_M \).

Definition 2.10. Let \( M \) be a topological manifold. Let \( (N, f) \) be a pair consisting of a smooth manifold \( N \) together with a tangential homotopy equivalence of degree one \( f : N \to M \). Two such pairs \( (N_1, f_1) \) and \( (N_2, f_2) \) are equivalent provided there exists a diffeomorphism \( g : N_1 \to N_2 \) such that \( f_2 \circ g \) is homotopic to \( f_1 \). The set of all such equivalence classes is denoted by \( \theta(M) \).

For \( M = \mathbb{H} \mathbb{P}^2 \), [3, Theorem 4] shows \( \theta(\mathbb{H} \mathbb{P}^2) \) contains at most two elements. Now by Remark 2.8(iii), we have the following:

Corollary 2.11. \( \theta(\mathbb{H} \mathbb{P}^2) \) contains exactly two elements, with representatives given by \( (\mathbb{H} \mathbb{P}^2, \text{id}) \) and \( (\mathbb{H} \mathbb{P}^2 \# \Sigma^8, \text{id}) \), where \( \Sigma^8 \) is the exotic \( 8 \)-sphere.
3 Inertia groups of projective plane-like manifolds

In [15], C.T.C. Wall assigned to each closed oriented \((n−1)\)-connected \(2n\)-dimensional smooth manifold \(M^{2n}\) with \(n \geq 3\), a system of invariants as follows:

1. \(H = H^n(M;\mathbb{Z}) \cong \text{Hom}(H_n(M;\mathbb{Z}),\mathbb{Z}) \cong \oplus_{j=1}^{k}\mathbb{Z}\), the cohomology group of \(M\), with \(k\) the \(n\)-th Betti number of \(M\).

2. \(I : H \times H \to \mathbb{Z}\), the intersection form of \(M\) which is unimodular and \(n\)-symmetric, defined by
   \[I(x, y) = \langle x \cup y, [M] \rangle,\]
   where the homology class \([M]\) is the orientation class of \(M\).

3. A map \(\alpha : H^n(M;\mathbb{Z}) \to \pi_{n-1}(SO_n)\) that assigns each element \(x \in H^n(M;\mathbb{Z})\) to the characteristic map \(\alpha(x)\) for the normal bundle of the embedded \(n\)-sphere \(S^n\) representing \(x\).

Denote by \(\chi = S \circ \alpha : H^n(M;\mathbb{Z}) \to \pi_{n-1}(SO_{n+1}) \cong \tilde{KO}(S^n)\), where \(S : \pi_{n-1}(SO_n) \to \pi_{n-1}(SO_{n+1})\) is the suspension map. Then

\[\chi = S \circ \alpha \in H^n(M;\tilde{KO}(S^n)) = \text{Hom}(H^n(M;\mathbb{Z});\tilde{KO}(S^n))\]

can be viewed as an \(n\)-dimensional cohomology class of \(M\), with coefficients in \(\tilde{KO}(S^n)\). The obstruction to triviality of the tangent bundle over the \(n\)-skeleton is the element \(\chi \in H^n(M;\tilde{KO}(S^n))\) \[15\]. By \[15\] pp. 179-180, the Pontrjagin class of \(M^{2n}\) is given by

\[p_m(M^{2n}) = \pm a_m(2m - 1)!\chi,\]  

where \(n = 4m\) and

\[a_m = \begin{cases} 1 & \text{if } 4m \equiv 0 \pmod{8}, \\ 2 & \text{if } 4m \equiv 4 \pmod{8}. \end{cases}\]

Define \(\Theta_n(k)\) to be the subgroup of \(\overline{\Theta}_n\) consisting of those homotopy \(n\)-sphere \(\Sigma^n\) which are the boundaries of \(k\)-connected \((n+1)\)-dimensional compact manifolds, \(1 \leq k < [n/2]\). Thus, \(\Theta_n(k)\) is the kernel of the natural map \(i_k : \overline{\Theta}_n \to \Omega_n(k)\), where \(\Omega_n(k)\) is the \(n\)-dimensional group in \(k\)-connective cobordism theory \[13\] and \(i_k\) sends \(\Sigma^n\) to its cobordism class. Using surgery, we see \(\Omega_1(1)\) is the usual oriented cobordism group. So \(\overline{\Theta}_n = \Theta_n(1)\). Similarly, \(\Omega_n(2) \cong \Omega_n^{Spin}\) \((n \geq 7)\); since \(BSpin\) is, in fact, 3-connected, for \(n \geq 8\), \(\Omega_n(2) \cong \Omega_n(3)\) and \(\Theta_n(2) = \Theta_n(3) = bSpin_n\). Here \(bSpin_n\) consists of homotopy \(n\)-sphere which bound spin manifolds.

In [16], C.T.C. Wall defined the Grothendieck group \(G^{2n+1}_n\), a homomorphism \(\vartheta : G^{2n+1}_n \to \overline{\Theta}_{2n}\) such that \(\vartheta(G^{2n+1}_n) = \Theta_{2n}(n - 1)\) and proved the following theorem:
Theorem 3.1. (Wall) Let $M^{2n}$ be a $(n - 1)$-connected $2n$-manifold and $\Sigma^{2n}$ be a homotopy sphere in $\mathcal{S}_{2n}$. Then $M \# \Sigma^{2n}$ is an orientation-preserving diffeomorphic to $M$ if and only if

(i) $\Sigma^{2n} = 0$ in $\mathcal{S}_{2n}$ or

(ii) $\chi \not\equiv 0 \pmod{2}$ and $\Sigma^{2n} \in \vartheta(\mathcal{G}^{2n+1}) = \Theta_{2n}(n - 1)$

We also need the following result from [1] :

Theorem 3.2. (Anderson, Brown, Peterson) Let $\eta_n : \mathcal{S}_n \to \Omega^{Spin}_n$ be the homomorphism such that $\eta_n$ sends $\Sigma^n$ to its spin cobordism class. Then $\eta_n \neq 0$ if and only if $n = 8k + 1$ or $8k + 2$.

Proof of Theorem 1.1: Let $\xi$ be a generator of $H^n(M^{2n}; \mathbb{Z})$. Consider the case $n = 4$. Then by Itiro Tamura [14] and (3.1), the Pontrjagin class of $M^{2n}$ is given by

$$p_1(M^{2n}) = 2(2h + 1)\xi = \pm 2\chi,$$

where $h \in \mathbb{Z}$. This implies that

$$\chi = \pm (2h + 1)\xi.$$

Likewise, for $n = 8$, we have

$$p_2(M^{2n}) = 6(2k + 1)\xi = \pm 6\chi,$$

where $k \in \mathbb{Z}$. This implies that

$$\chi = \pm (2k + 1)\xi.$$

Therefore in either case, $\chi \not\equiv 0 \pmod{2}$. Now by Theorem 3.1, it follows that

$$I(M^{2n}) = \Theta_{2n}(n - 1).$$

Since $\Theta_{2n}(n - 1)$ is the kernel of the natural map $i_{n-1} : \mathcal{S}_{2n} \to \Omega_{2n}(n - 1)$, where $\Omega_{2n}(n - 1) \cong \Omega_{8}^{Spin}$ for $n = 4$ and $\Omega_{2n}(n - 1) \cong \Omega_{16}^{String} \cong \mathbb{Z} \oplus \mathbb{Z}$ for $n = 8$ [1]. Now by Theorem 3.2 and using the fact that $\mathcal{S}_{16} \cong \mathbb{Z}_2$ [6], we have $i_{n-1} = 0$ for $n = 4$ and 8. This shows that $\Theta_{2n}(n - 1) = \mathcal{S}_{2n}$. This implies that

$$I(M^{2n}) \cong \mathbb{Z}_2.$$

This completes the proof of Theorem 1.1.
4 Smooth structures of \((n - 1)\)-connected \(2n\)-manifolds

**Definition 4.1.** (\(\text{Cat} = \text{Diff}\) or \(\text{Top}\)-structure sets) Let \(M\) be a closed \(\text{Cat}\)-manifold. We define the \(\text{Cat}\)-structure set \(\mathcal{S}^{\text{Cat}}(M)\) to be the set of equivalence classes of pairs \((N, f)\) where \(N\) is a closed \(\text{Cat}\)-manifold and \(f : N \to M\) is a homotopy equivalence. And the equivalence relation is defined as follows:

\[(N_1, f_1) \sim (N_2, f_2)\] if there is a \(\text{Cat}\)-isomorphism \(\phi : N_1 \to N_2\) such that \(f_2 \circ h\) is homotopic to \(f_1\).

We will denote the class in \(\mathcal{S}^{\text{Cat}}(M)\) of \((N, f)\) by \([N, f]\). The base point of \(\mathcal{S}^{\text{Cat}}(M)\) is the equivalence class \([\langle M, \text{id}\rangle]\) of \(\text{id} : M \to M\).

The forgetful maps \(F_{\text{Diff}} : \mathcal{S}^{\text{Diff}}(M) \to \mathcal{S}^{\text{Top}}(M)\) and \(F_{\text{Con}} : \mathcal{C}(M) \to \mathcal{S}^{\text{Diff}}(M)\) fit into a short exact sequence of pointed sets [3]:

\[\mathcal{C}(M) \xrightarrow{F_{\text{Con}}} \mathcal{S}^{\text{Diff}}(M) \xrightarrow{F_{\text{Diff}}} \mathcal{S}^{\text{Top}}(M).\]

**Theorem 4.2.** Let \(n\) be any integer greater than 3 such that \(\Theta_n\) and \(\Theta_{n+1}\) are trivial and \(M^{2n}\) be a closed smooth \((n - 1)\)-connected \(2n\)-manifold. Let \(f : N \to M\) be a homeomorphism where \(N\) is a closed smooth manifold. Then

(i) there exists a diffeomorphism \(\phi : N \to M \# \Sigma^{2n}\), where \(\Sigma^{2n} \in \overline{\Theta}_{2n}\) such that the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
N & \xrightarrow{\phi} & M \# \Sigma^{2n} \\
\downarrow f & & \downarrow \text{id} \\
M & & M
\end{array}
\]

(ii) If \(I_h(M) = \overline{\Theta}_{2n}\), then \(f : N \to M\) is homotopic to a diffeomorphism.

**Proof.** Consider the short exact sequence of pointed sets

\[\mathcal{C}(M) \xrightarrow{F_{\text{Con}}} \mathcal{S}^{\text{Diff}}(M) \xrightarrow{F_{\text{Diff}}} \mathcal{S}^{\text{Top}}(M).\]

By Theorem 2.7(ii), we have

\[\mathcal{C}(M) = \{[M \# \Sigma] \mid \Sigma \in \overline{\Theta}_{2n}\} \cong \overline{\Theta}_{2n}.\]

Since \([(N, f)] \in F_{\text{Diff}}^{-1}([\langle M, \text{id}\rangle])\), we obtain

\[\text{Im}(F_{\text{Con}}) = \{[M \# \Sigma] \mid \Sigma \in \overline{\Theta}_{2n}\}.\]

This implies that there exists a homotopy sphere \(\Sigma^{2n} \in \overline{\Theta}_{2n}\) such that \((N, f) \sim (M \# \Sigma^{2n}, \text{id})\) in \(\mathcal{S}^{\text{Diff}}(M)\). This implies that there exists a diffeomorphism \(\phi : N \to M \# \Sigma^{2n}\) such that \(f\) is homotopic to \(\text{id} \circ \phi\). This proves (i).

If \(I_h(M) = \overline{\Theta}_{2n}\), then \(\text{Im}(F_{\text{Con}}) = \{[\langle M, \text{id}\rangle]\}\) and hence \((N, f) \sim (M, \text{id})\) in \(\mathcal{S}^{\text{Diff}}(M)\). This shows that \(f : N \to M\) is homotopic to a diffeomorphism \(N \to M\). This proves (ii). \(\square\)
Theorem 4.3. Let $n$ be any integer greater than 3 such that $\Theta_n$ and $\Theta_{n+1}$ are trivial and $M^{2n}$ be a closed smooth $(n-1)$-connected $2n$-manifold. Then the number of distinct smooth structures on $M^{2n}$ up to diffeomorphism is less than or equal to the cardinality of $\Theta_{2n}$. In particular, the set of diffeomorphism classes of smooth structures on $M^{2n}$ is $\{[M\#\Sigma] \mid \Sigma \in \Theta_{2n}\}$.

Proof. By Theorem 4.2(i), if $N$ is a closed smooth manifold homeomorphic to $M$, then $N$ is diffeomorphic to $M\#\Sigma^{2n}$ for some homotopy $2n$-sphere $\Sigma^{2n}$. This implies that the set of diffeomorphism classes of smooth structures on $M^{2n}$ is $\{[M\#\Sigma] \mid \Sigma \in \Theta_{2n}\}$. This shows that the number of distinct smooth structures on $M^{2n}$ up to diffeomorphism is less than or equal to the cardinality of $\Theta_{2n}$.

Remark 4.4.

(1) By Theorem 4.3 every closed smooth 3-connected 8-manifold has at most two distinct smooth structures up to diffeomorphism.

(2) If $M^8$ is a closed smooth 3-connected 8-manifold such that $H^4(M;\mathbb{Z}) \cong \mathbb{Z}$, then by Theorem 1.1 $I(M) \cong \mathbb{Z}_2$. Now by Theorem 4.3 $M$ has a unique smooth structure up to diffeomorphism.

(3) If $M = \mathbb{S}^4 \times \mathbb{S}^4$, then by Theorem 4.3 $\mathbb{S}^4 \times \mathbb{S}^4$ has at most two distinct smooth structures up to diffeomorphism, namely, $\{[\mathbb{S}^4 \times \mathbb{S}^4], [\mathbb{S}^4 \times \mathbb{S}^4\#\Sigma]\}$, where $\Sigma$ is the exotic 8-sphere. However, by [11, Theorem A], $I(\mathbb{S}^4 \times \mathbb{S}^4) = 0$. This implies that $\mathbb{S}^4 \times \mathbb{S}^4$ has exactly two distinct smooth structures.

Theorem 4.5. Let $M$ be a closed smooth 3-connected 8-manifold with stable tangential invariant $\chi = S\alpha : H^4(M) \to \pi_3(SO) = \mathbb{Z}$. Then $M$ has exactly two distinct smooth structures up to diffeomorphism if and only if $\text{Im}(S\alpha) \subseteq 2\mathbb{Z}$.

Proof. Suppose $M$ has exactly two distinct smooth structures up to diffeomorphism. Then by Theorem 4.3 $M$ and $M\#\Sigma$ are not diffeomorphic, where $\Sigma$ is the exotic 8-sphere. Since $\Theta_8 = \Theta_8(3)$, by Theorem 3.1 the stable tangential invariant $\chi$ is zero (mod 2) and hence $\text{Im}(S\alpha) \subseteq 2\mathbb{Z}$. Conversely, suppose $\text{Im}(S\alpha) \subseteq 2\mathbb{Z}$. Now by Theorem 3.1 $M$ cannot be diffeomorphic to $M\#\Sigma$, where $\Sigma$ is the exotic 8-sphere. Now by Theorem 4.3 $M$ has exactly two distinct smooth structures up to diffeomorphism.

Remark 4.6. If $n = 2, 3, 5, 6, 7$ (mod 8) or the stable tangential invariant $\chi$ of $M^{2n}$ is zero (mod 2), then by [16, Corollary, pp. 289] and Theorem 3.1 we have $I(M^{2n}) = 0$. So, by Theorem 4.3 we have the following:

Theorem 4.7. Let $n$ be any integer greater than 3 such that $\Theta_n$ and $\Theta_{n+1}$ are trivial and $M^{2n}$ be a closed smooth $(n-1)$-connected $2n$-manifold. If $n = 2, 3, 5, 6, 7$ (mod 8) or the stable tangential invariant $\chi$ of $M^{2n}$ is zero (mod 2), then the set of diffeomorphism classes of smooth structures on $M^{2n}$ is in one-to-one correspondence with group $\Theta_{2n}$. 

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Remark 4.8.

(1) By Theorem 4.7, every closed smooth 4-connected 10-manifold has exactly six distinct smooth structures, namely, $\{[M\#\Sigma] \mid \Sigma \in \Theta_{10} \cong \mathbb{Z}_6\}$.

(2) If $M^{2n}$ is $n$-parallelisable, almost parallelisable or $\pi$-manifold, then the stable tangential invariant $\chi$ of $M$ is zero [15]. Then by Theorem 4.7, we have the following:

Corollary 4.9. Let $n$ be any integer greater than 3 such that $\Theta_n$ and $\Theta_{n+1}$ are trivial and $M^{2n}$ be a closed smooth $(n-1)$-connected 2n-manifold. If $M^{2n}$ is $n$-parallelisable, almost parallelisable or $\pi$-manifold, then the set of diffeomorphism classes of smooth structures on $M^{2n}$ is in one-to-one correspondence with group $\Theta_{2n}$.

Definition 4.10. [8] The normal $k$-type of a closed smooth manifold $M$ is the fibre homotopy type of a fibration $p : B \to BO$ such that the fibre of the map $p$ is connected and its homotopy groups vanish in dimension $\geq k+1$, admitting a lift of the normal Gauss map $\nu_M : M \to BO$ to a map $\bar{\nu}_M : M \to B$ such that $\bar{\nu}_M : M \to B$ is a $(k+1)$-equivalence, i.e., the induced homomorphism $\bar{\nu}_M : \pi_i(M) \to \pi_i(B)$ is an isomorphism for $i \leq k$ and surjective for $i = k+1$. We call such a lift a normal $k$-smoothing.

Theorem 4.11. Let $n = 5, 7$ and let $M_0$ and $M_1$ be closed smooth $(n-1)$-connected 2n-manifolds with the same Euler characteristic. Then

(i) There is a homotopy sphere $\Sigma^{2n} \in \Theta_{2n}$ such that $M_0$ and $M_1\#\Sigma^{2n}$ are diffeomorphic.

(ii) Let $M^{2n}$ be a $(n-1)$-connected 2n-manifold such that $[M] = 0 \in \Omega^\text{String}_{2n}$ and let $\Sigma$ be any exotic 2n-sphere in $\Theta_{2n}$. Then $M$ and $M\#\Sigma$ are not diffeomorphic.

Proof. (i): $M_0$ and $M_1$ are $(n-1)$-connected, and $n$ is 5 or 7; therefore, $\frac{2}{\pi}$ and the Stiefel-Whitney classes $\omega_2$ vanish. So, $M_0$ and $M_1$ are $B\text{String}$-manifolds. Let $\nu_{M_j} : M_j \to B\text{String}$ be a lift of the normal gauss map $\nu_{M_j} : M_j \to BO$ in the fibration $p : B\text{String} = BO \langle 8 \rangle \to BO$, where $j = 0$ and 1. Since $B\text{String}$ is 7-connected, $p_\# : \pi_i(B\text{String}) \to \pi_i(BO)$ is an isomorphism for all $i \geq 8$. This shows that $\nu_{M_j} : M_j \to B\text{String}$ is an $n$-equivalence and hence the normal $(n-1)$-type of $M_0$ and $M_1$ is $p : B\text{String} \to BO$. We know that $\Omega^\text{String}_{2n} \cong \Theta_{2n}$, where the group structure is given by connected sum [4]. This implies that there always exists $\Sigma^{2n} \in \Theta_{2n}$ such that $M_0$ and $M_1\#\Sigma^{2n}$ are $B\text{String}$-bordant. Since $M_0$ and $M_1\#\Sigma^{2n}$ have the same Euler characteristic, by [8, Corollary 4], $M_0$ and $M_1\#\Sigma^{2n}$ are diffeomorphic.

(ii): Since the image of the standard sphere under the isomorphism $\Theta_{2n} \cong \Omega^\text{String}_{2n}$ represents the trivial element in $\Omega^\text{String}_{2n}$, we have $[M^{2n}] \neq [M\#\Sigma]$ in $\Omega^\text{String}_{2n}$. This implies that $M$ and $M\#\Sigma$ are not $B\text{String}$-bordant. By obstruction theory, $M^{2n}$ has a unique string structure. This implies that $M$ and $M\#\Sigma$ are not diffeomorphic. \qed
Theorem 4.12. Let $M$ be a closed smooth 6-connected 14-dimensional $\pi$-manifold and $\Sigma$ is the exotic 14-sphere. Then $M\#\Sigma$ is not diffeomorphic to $M$. Thus, $I(M) = 0$. Moreover, if $N$ is a closed smooth manifold homeomorphic to $M$, then $N$ is diffeomorphic to either $M$ or $M\#\Sigma$.

Proof. It follows from results of Anderson, Brown and Peterson on spin cobordism [1] that the image of the natural homomorphism $\Omega^{framed}_{14} \to \Omega^{Spin}_{14}$ is 0 and $\Omega^{String}_{14} \cong \Omega^{Spin}_{14} \cong \mathbb{Z}_2$ [4]. This shows that $[M] = 0 \in \Omega_{14}^{String}$. By Theorem 4.11(ii), $M\#\Sigma$ is not diffeomorphic to $M$. If $N$ is a closed smooth manifold homeomorphic to $M$, then $N$ and $M$ have the same Euler characteristic. Then by Theorem 4.11(i), $N$ is diffeomorphic to either $M$ or $M\#\Sigma$. □

Remark 4.13. By the above Theorem 4.12, the set of diffeomorphism classes of smooth structures on a closed smooth 6-connected 14-dimensional $\pi$-manifold $M$ is

$$\{[M], [M\#\Sigma]\} \cong \mathbb{Z}_2,$$

where $\Sigma$ is the exotic 14-sphere. So, the number of distinct smooth structures on $M$ is 2.

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