Hamilton-Jacobi approach for linearly acceleration-dependent Lagrangians

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We develop a constructive procedure for arriving at the Hamilton-Jacobi framework for the so-called affine in acceleration theories by analysing the canonical constraint structure. We find two scenarios in dependence of the order of the emerging equations of motion. By properly defining generalized brackets, the non-involutive constraints that originally arose, in both scenarios, may be removed so that the resulting involutive Hamiltonian constraints ensure integrability of the theories and, at the same time, lead to the right dynamics in the reduced phase space. In particular, when we have second-order in derivatives equations of motion we are able to detect the gauge invariant sector of the theory by using a suitable approach based on the projection of the Hamiltonians onto the tangential and normal directions of the congruence of curves in the configuration space. Regarding this, we also explore the generators of canonical and gauge transformations of these theories. Further, we briefly outline how to determine the Hamilton principal function $S$ for some particular setups. We apply our findings to some representative theories: a Chern-Simons-like theory in $(2 + 1)$-dim, an harmonic oscillator in $2D$ and, the geodetic brane cosmology emerging in the context of extra dimensions.

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I. INTRODUCTION

There is a broad class of relevant Lagrangian functions that are not built from the standard prescription $L = T - V$. Closely related to this set are the second-order derivative theories that in many cases are considered extensions, or corrections, to the usual physical theories, possessing an intricate gauge freedom which is responsible for not allowing the highest derivatives to be solved algebraically in the equations of motion. In this sense, particular attention has been focused on the so-called affine in acceleration theories [1–3], mainly motivated by model building associated with some relativistic acceleration phenomena including general relativity, electromagnetism, or modified gravity theories. These are characterized by a linear dependence on the accelerations of the systems and are necessarily singular giving rise in general to both gauge ambiguities and gauge symmetries. In fact, these theories have been widely studied by means of the Lagrangian and Hamiltonian formalism. However, such analyses can be supplemented and, in many cases improved, by another scheme defined for singular systems as it is the case of the Hamilton-Jacobi formalism.

In this paper we construct in a consistent manner a Hamilton-Jacobi (HJ) framework for linearly acceleration-dependent Lagrangians. In providing this scheme we also furnish with a great alternative to analyze the phase space constraint structure of this type of theories. The HJ framework we are interested in, introduced by Carathéodory for regular systems [4] and further strengthened from the physical point of view for singular systems by Güler and many authors [5–11], is essentially based on the variational principle named equivalent Lagrangians method and the first-order partial differential equations theory. This scheme is obtained directly from the Lagrangian formalism without going through the usual approximation of the canonical transformations of the Hamiltonian formalism, even in the case of singular theories, so it becomes an adequate shortcut for arriving at the HJ framework. For regular systems we can arrive straightforwardly at the commonly known HJ partial differential equation. On the contrary, for singular systems constraints arise as necessary conditions to ensure the existence of extremes for a given action but, without ceasing to be partial differential equations from which the characteristic system of equations turn out to depend on several independent variables, named parameters, which may be related with the gauge information of the systems [12].

The complete set of HJ partial differential equations must obey geometric conditions in order to guarantee their integrability. These conditions are equivalent to the consistency conditions developed in the Dirac-Bergmann approach for constrained systems [13, 14]. In consequence, the HJ integrability analysis separates the constraints in involutive and non-involutive under an extended Poisson bracket [12, 16]. The existence of non-involutive constraints signals a dependence between the parameters of the theory which leads to a redefinition of the symplectic structure of the phase space in order to have control of the right dynamics of the theory.

In contrast to the Dirac-Bergmann approach, this HJ framework possesses robust geometric foundations and
does not need support from the split of first- and second-class constraints to build the right evolution of physical constrained systems as well as to obtain the gauge symmetry information. For theories being linear on the accelerations non-involutive constraints are always present. To unravel this situation within this geometric framework it is mandatory to introduce a generalized bracket (GB) structure that leaves the complete set of constraints in involution since the so-called Frobenius integrability conditions are fulfilled \[12\], thus solving the problem of integrability. This framework also links the complete set of involutive HJ equations with the canonical and gauge symmetries. Regarding this, we find that theories that only have non-involutive constraints are characterized by symmetries. Regarding this, we find that theories that integrability. This framework also links the complete set of involutive HJ equations with the canonical and gauge symmetries. For theories being linear on the accelerations non-involutive constraints are always present. To unravel this situation within this geometric framework it is mandatory to introduce a generalized bracket (GB) structure that leaves the complete set of constraints in involution since the so-called Frobenius integrability conditions are fulfilled \[12\], thus solving the problem of integrability. This framework also links the complete set of involutive HJ equations with the canonical and gauge symmetries.

II. HAMILTON-JACOBI FORMALISM FOR AFFINE IN ACCELERATION THEORIES

We are interested in physical systems with a finite number of degrees of freedom described by Lagrangian functions, including only terms linear in accelerations, and described by the action

\[ S[q^\mu] = \int dt \, L(q^\mu, \dot{q}^\mu, \ddot{q}^\mu, t) \quad \mu = 1, 2, \ldots, N; \quad (1) \]

where

\[ L(q^\mu, \dot{q}^\mu, \ddot{q}^\mu, t) = K_\mu(q^\rho, \dot{q}^\rho, t) \ddot{q}^\mu - V(q^\rho, \dot{q}^\rho, t). \quad (2) \]

Summation over repeated indices is henceforth understood. According to the higher-order derivative theories viewpoint the configuration space \( C_{2N} \) is spanned by the \( N \) coordinates \( q^\mu \) and their \( N \) velocities \( \dot{q}^\mu \). An overdot stands for the derivative with respect to the time parameter \( t \) so that \( \ddot{q}^\mu = dq^\mu/dt \) and so on. Here, \( K_\mu \) and \( V \) are assumed to be smooth functions defined on \( C_{2N} \).

Guided by the Hamilton’s principle adapted to actions of the form \( \int \), the optimal trajectory \( q^\mu = q^\mu(t) \) parametrized by \( t \), is obtained by solving the Euler-Lagrange (EL) equations of motion (eom)

\[ \frac{\partial L}{\partial q^\mu} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^\mu} \right) = 0. \]

Explicitly, these are given by

\[ N_{\mu \nu} \ddot{q}^\nu - M_{\mu \nu} \dot{q}^\nu + F_\mu = 0, \quad (3) \]

where

\[ N_{\mu \nu} := \frac{\partial K_\nu}{\partial \dot{q}^\mu} - \frac{\partial K_\mu}{\partial \dot{q}^\nu} = -N_{\nu \mu}, \quad (4) \]

\[ M_{\mu \nu} := \frac{\partial K_\nu}{\partial q^\mu} - \frac{\partial}{\partial q^\mu} \left( -\frac{\partial V}{\partial q^\nu} - \frac{\partial K_\mu}{\partial q^\rho} \dot{q}^\rho + \frac{\partial M_{\rho \nu}}{\partial q^\rho} \dot{q}^\rho \right), \quad (5) \]

\[ F_\mu := \frac{\partial}{\partial q^\mu} \left( -\frac{\partial V}{\partial q^\mu} - \frac{\partial K_\mu}{\partial q^\rho} \dot{q}^\rho + \frac{\partial M_{\rho \mu}}{\partial q^\rho} \dot{q}^\rho + \frac{\partial V}{\partial q^\mu} \right) \dot{q}^\mu + \frac{\partial V}{\partial q^\mu}. \quad (6) \]

It is worth noting that \( N_{\mu \nu} = N_{\mu \nu}(q^\rho, \dot{q}^\rho, t) \), \( M_{\mu \nu} = M_{\mu \nu}(q^\rho, \dot{q}^\rho, \ddot{q}^\rho, t) \) and \( F_\mu = F_\mu(q^\rho, \dot{q}^\rho, \ddot{q}^\rho, t) \). The form \( \int \) for the eom proves to be fairly useful and allows the theory to be better understood in the HJ scheme to be developed.

Following the original Carathéodory’s equivalent Lagrangians approach \[4\], \[6\] later extended to second-order in derivatives theories \[4, 8\], in order to have an extreme configuration of the action \( \int \) the necessary conditions are associated to the existence of a family of surfaces defined by a generating function, \( S(q^\mu, \dot{q}^\mu, t) \), such that it
satisfies
\[ \frac{\partial S}{\partial q^\mu} = \frac{\partial L}{\partial \dot{q}^\mu} =: P_\mu, \]  
(7)
\[ \frac{\partial S}{\partial q^\mu} = \frac{\partial L}{\partial \dot{q}^\mu} \frac{\partial \dot{L}}{\partial L} =: p_\mu, \]  
(8)
\[ \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^\mu} \dot{q}^\mu + \frac{\partial S}{\partial \dot{q}^\mu} \ddot{q}^\mu - L = 0, \]  
(9)
where, on physical grounds, \( P_\mu \) denotes the conjugate momenta to the velocities \( \dot{q}^\mu \) while \( p_\mu \) are the conjugate momenta to the coordinates \( q^\mu \).

The HJ framework in which we are interested in, emerges from (7), (8) and (9) considered as partial differential equations (PDE) for \( S \). Indeed, for non-singular systems it is straightforward to convert (9) into a PDE for \( S \) by solving for \( \dot{q}^\mu \) in terms of \( q^\mu, \ \ddot{q}^\mu \) and partial derivatives of \( S \) which is obtained by appropriately inverting (7). However, for singular physical systems this is not possible as the Hessian matrix with elements \( H_{\mu\nu} \) is not invertible. In addition, the momenta \( P_\mu \) to the accelerations \( \ddot{q}^\mu \) are not independent of the momenta \( P_\mu \) to the velocities \( \dot{q}^\mu \).

This fact defines what is known as an affine in acceleration theory [1–3]. The rank of the Hessian matrix is zero which causes that the manifold \( \mathcal{M} \) is not fully spanned by \( R = N - 0 = N \) variables, \( \dot{q}^\mu \), all of them related to the kernel of \( H_{\mu\nu} \). In this sense, within the HJ scheme we are going to discuss below we have to treat all the generalised velocities as free parameters [3, 7]. Clearly, we can not invert any of the accelerations \( \dot{q}^\mu \) in favour of the coordinates, the velocities \( \dot{q}^\mu \) and the momenta \( P_\mu \) through the partial derivative of the function \( S \), fact that entails the presence of \( N \) constraints given by the definition of \( P_\mu \) itself. As one can infer from (7), the constraints form a set of PDE of first-order for \( S \) which, in order to be integrable, must obey the so-called Frobenius integrability conditions, (11, 12), (see (29) below). On the other hand, relying on the structure of the momenta \( p_\mu \), (see (12) below) we could or not solve for the accelerations \( \ddot{q}^\mu \) in favour of the remaining variables of the phase space which could or not give rise to more contraints. Indeed, such a dependence rests heavily on the nature of the quantities \( \mathcal{K}_\mu \) which in turn promote two possible scenarios characterized by the order of the com. As a matter of fact, the momenta \( P_\mu \) and \( p_\mu \) written out in full are
\[ P_\mu = \mathcal{K}_\mu(t, \dot{q}^\nu, \ddot{q}^\nu), \]  
(11)
\[ p_\mu = -\frac{\partial \mathcal{V}}{\partial q^\mu} - \frac{\partial \mathcal{K}_\mu}{\partial q^\nu} \dot{q}^\nu + N_{\mu\nu} \ddot{q}^\nu. \]  
(12)
Note the linear dependence on the accelerations in (12).

Thecom in this framework are written as total differential equations known as characteristic equations, (11), (12), whose solutions are trajectories in dependence of independent variables, \( t^\nu \), in a reduced phase space, as we will see below. For the further analysis, due to the existence of several parameters in the development, in due course it will necessary to relabel the indexes associated with the main geometric quantities. For that reason, it is mandatory to discuss systematically the two feasible scenarios.

### III. Theories with Third-order Equations of Motion

According to the Jacobi’s theorem in matrix algebra, the determinant of any antisymmetric matrix of odd order has determinant equal to zero. Certainly, by observing (3), not necessarily all coordinates \( q^\mu \) will obey third-order eom in some sector of the configuration space. In order to ensure the existence of independent third-order eom we confine ourselves to consider an even number of variables \( q^\mu \). Hence, we set \( N = 2n \) implying thus that \( N_{\mu\nu} \) is non-vanishing and that \( \det(N_{\mu\nu}) \neq 0 \) in this scenario. Under these conditions, to determine the time evolution it is needed to consider 6n initial conditions for the quantities \( \dot{q}^\mu, \ddot{q}^\mu, \mathcal{V} \) with \( \mu = 1, 2, \ldots, 2n \) at initial time \( t = t_0 \). Additionally, from (12) it should be noted that accelerations can be solved in favour of the rest of the phase space variables.

From the result (10), all the generalized velocities have the status of parameters so, it is reasonable to introduce the notation: \( t^0 := t \)
\[ t^\mu := \dot{q}^\mu \quad \text{and} \quad H_\mu^P := \frac{\partial L}{\partial \dot{q}^\mu} = -\mathcal{K}_\mu(t, \dot{q}^\nu, \ddot{q}^\nu), \]  
(13)
as well as the set of coordinates \( t^I \) in the following order
\[ t^I := q^I := (t^0, t^{1\mu}), \quad I = 0, 1, 2, \ldots, 2n. \]  
(14)
As a result, relationship (11) reads
\[ \frac{\partial S}{\partial t^I} + H_\mu^P(t^I, \dot{q}^\nu, \ddot{q}^\nu, \frac{\partial S}{\partial \dot{q}^\nu}) = 0. \]  
(15)
In the same spirit, by introducing the Hamilton function
\[ H_0 := \frac{\partial S}{\partial \dot{q}^\mu} t^\mu + \frac{\partial S}{\partial \dot{q}^\mu} \ddot{t}^\mu - L(t^0, q^\mu, t^{1\mu}, \dot{t}^{1\mu}), \]  
(16)
which does not depend explicitly on \( \dot{t}^{1\mu} \), as it may be checked straightforwardly, one finds that the expression (9) becomes
\[ \frac{\partial S}{\partial \dot{q}^0} + H_0(t^0, t^{1\mu}, q^\mu, \frac{\partial S}{\partial \dot{q}^\mu}, \frac{\partial S}{\partial \ddot{t}^{1\mu}}, \frac{\partial S}{\partial \dot{t}^{1\mu}}) = 0, \]  
(17)
that is the common expression of the Hamilton-Jacobi equation. In terms of the original notation, the Hamilton function reads
\[ H_0 = p_\mu \dot{q}^\mu + \mathcal{V}(q^\nu, \dot{q}^\nu, t). \]  
(18)

Note the linear dependence on the accelerations in (12).
We can express (15) and (17) as a unified set of PDE for the generating function $S$. To do this, it is useful to assume that $P_0 := \frac{\partial S}{\partial t}$ is conjugate canonical momentum to $t^0$. We then find

$$\frac{\partial S}{\partial t} + H_I\left(t^I, q^\mu, \frac{\partial S}{\partial q^\mu}, \frac{\partial S}{\partial t^J}\right) = 0, \quad I,J = 0,1,2,\ldots,2n, \quad (19)$$

where $H_I := \langle H_0, H_\mu^P \rangle$. In the following, the $2n + 1$ relations (19) will be referred to as the Hamilton-Jacobi partial differential equations (HJPDE). Bearing in mind (7), we can also write (15) and (17) in the form

$$H_0 := P_0 + H_0(t^0, t^\nu, q^\mu, P_\mu, p_\mu) = 0, \quad (20)$$

$$H_\mu^P := P_\mu + H_\mu^P(t^0, t^\nu, q^\mu) = 0, \quad (21)$$

which acquire the compact constrained Hamiltonian fashion

$$H_I(t^I, q^\nu, P_J, p_\nu) := P_I + H_I(t^I, q^\nu, P_J, p_\nu) = 0, \quad (22)$$

where $H_I := \langle H_0, H_\mu^P \rangle$ and $P_I := \langle P_0, P_\mu \rangle$. These expressions have thus acquired the well-known form of canonical Dirac constraints. The constraints written in the form (23) are also referred to as Hamiltonians in this HJ scheme. In a sense, this HJ approach replaces the analysis of the $2n$ canonical constraints, $H_\mu^P = 0$, with the analysis of the $(2n + 1)$ HJPDE given by relations (19).

Some remarks are in order. We do not have a further HJPDE relating the momenta $p_\mu$ because from (12) we observe the linear dependence on $\dot{q}^\mu$ which allows us to write the accelerations in favour of the momenta that can be inserted in the Legendre transformation defining the canonical Hamiltonian. We shall prove this in short; in fact, within the Dirac-Bergmann approach for constrained systems, this feature signals the presence of second-class constraints. On the other hand, the Hamiltonian $H_0$, (20), is said to be associated with the time parameter $t^0$ while the Hamiltonians $H_\mu^P$, (21), are associated with the remaining parameters $t^\nu$ related to the velocities of the system.

The equations of motion, known as characteristic equations (CE), associated to the Hamiltonian set (22), are given as total differential equations [3; 6]. At this initial stage these are given by

$$dq^\mu = \frac{\partial H_I}{\partial P_\mu} dt^I \quad dq^I = \frac{\partial H_I}{\partial P_I} dt^J, \quad (23)$$

$$dp_\mu = -\frac{\partial H_I}{\partial q^\mu} dt^I \quad dp_I = -\frac{\partial H_I}{\partial q^I} dt^J. \quad (24)$$

We would like to emphasize that $t^\nu = \dot{q}^\mu$ have a status of independent evolution parameters, on an equal footing to $t$. To prove this it is enough to evaluate (23) for $\dot{q}^\mu$; indeed, $\dot{q}^\mu = \frac{\partial H_0}{\partial P_\mu} dt^0 + \frac{\partial H_\mu^P}{\partial P_\mu} dt^\nu = dt^\mu$.

On mathematical grounds, within this HJ formalism it is said that $t^I$ are the independent variables or parameters of the theory. In fact, the number of parameters is determined not only by the rank of the Hessian matrix but also by the integrability conditions. On physical grounds, the parameters encode the local symmetries and gauge transformations (see below for details). The solution of the first equations in (23) and (24) leads to a congruence of parametrized curves in the configuration space $C_{2N+1}$, given by $q^\mu = q^\mu(t^I)$. In a like manner, the generating function $S(t^I, q^\mu, \dot{q}^\mu)$ is satisfying

$$ds = \frac{\partial S}{\partial t^I} dt^I + \frac{\partial S}{\partial q^\mu} dq^\mu = -H_I dt^I + P_\mu dq^\mu, \quad (25)$$

where $S$ and $H_I$ have been considered.

For two arbitrary functions $F, G \in \Gamma_{2N+1} := T^*C_{2N+1}$, that is, functions in the extended phase space spanned by the variables $(t^I, q^\mu)$ and their conjugate momenta $(P_I, p_\mu)$, we introduce the extended Poisson bracket (PB)

$$\{F, G\} = \frac{\partial F}{\partial t^I} \frac{\partial G}{\partial P_I} - \frac{\partial F}{\partial q^\mu} \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial P_I} \frac{\partial G}{\partial t^I} + \frac{\partial F}{\partial P_\mu} \frac{\partial G}{\partial q^\mu}. \quad (26)$$

We may therefore express evolution in $\Gamma_{2N+1}$ as follows

$$dF = \{F, H_I\} dt^I, \quad (27)$$

where the $t^I$ play the role as parameters of the Hamiltonian flows generated by the Hamiltonians $H_I$. In passing, the CE (23) and (24) may be obtained from (27) by evaluating $F$ for any of the phase space variables. In this HJ framework, the dynamical evolution is provided by (27) which is referred to as the fundamental differential.

### A. Integrability conditions

With the intention of integrating the HJPDE (19), it is convenient to rely in the method of characteristics [4]. On physical grounds, it is completely unclear whether or not all coordinates are relevant parameters of the theory, so it is crucial to find a subspace among the parameters $t^I$ where the system becomes integrable. Regarding this, the matrix occurring in (3)

$$N_{\mu\nu} = \frac{\partial K_\nu}{\partial q^\mu} - \frac{\partial K_\mu}{\partial q^\nu} = \{H_\mu^P, H_\nu^P\}, \quad (28)$$

plays an important role to unravel under what conditions the eom associated with the action (1) will be integrable.

The complete solution of (19) (or (22)) is given by a family of parametrized curves in the configuration space $\Gamma_{2N+1}$. This means that the Hamiltonians must close as an algebra. Accordingly, it ensures the existence of such a family where $C_{2N+1}$ is said to be associated with the action (1) will be integrable.

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In this sense, the fulfillment of the Frobenius integrability conditions [4; 12]

$$\{H_I, H_J\} = C_{IJ}^K H_K, \quad (29)$$

ensures the existence of such a family where $C_{IJ}^K$ are the structure coefficients of the theory. This means that the Hamiltonians must close as an algebra. Accordingly, it must be imposed that $dH_0$ and $dH_\mu^P$ are vanishing identically

$$dH_I = 0. \quad (30)$$
Guided by the aforementioned Jacobi theorem, we have non-involutive constraints since we have $N_{\mu
u} \neq 0$ and det$(N_{\mu
u}) \neq 0$, that is, a regular case. Hence, no new Hamiltonians arise from the realization of $dH_I = 0$ but a relation of dependence between the parameters of the theory. Indeed, we have that $dt^{\mu} = -(N^{-1})^{\mu\nu} \{H^P_{\nu}, H_0\} dt^{0}$ where $(N^{-1})^{\mu\nu}$ denotes the inverse matrix of $N_{\mu
u}$ such that $N_{\mu
u}(N^{-1})^{\mu\nu} = \delta^{\mu}_{\nu}$ or $(N^{-1})^{\mu\nu} N_{\mu\nu} = \delta^{\mu}_{\nu}$. In such a case it is often enough to consider that $t^0$ is the independent parameter of the theory. From (27) we infer now that the evolution of $F \in \Gamma_{2N+1}$ is provided by

$$dF = \{F, H_0\}^* dt^0,$$  \hspace{1cm} (31)

where

$$\{F, G\}^* := \{F, G\} - \{F, H^P_{\nu}\} (N^{-1})^{\mu\nu}\{H^P_{\nu}, G\}. \hspace{1cm} (32)$$

In this HJ spirit, the remaining variables, $q^{\mu}$, are referred to as dependent variables. Note that the $N$ independent Hamiltonians (20) and (21) will fix the dynamics on the phase space in a unique way.

The bracket structure introduced in (32) is referred to as the generalized bracket (GB) which has all the known properties of the standard Poisson bracket. In the present case, this redefines the dynamics by eliminating the parameters $t^{\mu}$ with exception of $t^0$. Accordingly, the non-involutive Hamiltonians have been absorbed in the GB. As a matter of fact, the GB is closely related to the Dirac bracket arising in the Dirac-Bergmann Hamiltonian approach for constrained systems [12 13 14]. Therefore, under the fundamental differential (30), the Hamiltonians preserve the condition (31) in the reduced phase space defined by $H^P_{\mu} = 0$, and we have as a result an integrable set of Hamiltonians.

**B. Characteristic equations**

The characteristic equations may now be computed from (27). First,

$$dq^{\mu} = \{q^{\mu}, H_0\}^* dt^0 = \dot{q}^{\mu} dt^0,$$  \hspace{1cm} (33)

which is a trivial identity. Second,

$$dt^{\mu} = \dot{q}^{\mu} = \{q^{\mu}, H_0\}^* dt^0,$$

$$= (N^{-1})^{\mu\nu} \left( p^{\nu} + \frac{\partial V}{\partial q^{\nu}} + \frac{\partial K^p_{\nu}}{\partial q^{\nu}} \dot{q}^{p} \right) dt^0,$$  \hspace{1cm} (34)

provides the accelerations of the mechanical system as we observe from (12). Third,

$$dp^{\mu} = \{p^{\mu}, H_0\}^* dt^0,$$

$$= -\frac{\partial V}{\partial q^{\mu}} dt^0 + \frac{\partial K^p_{\nu}}{\partial q^{\mu}} dt^v,$$  \hspace{1cm} (35)

represents the equations of motion provided by (3) by direct substitution of the previous characteristic equation and (12). Finally,

$$dP^{\mu} = \{P^{\mu}, H_0\}^* dt^0,$$

$$= \left( -p_\mu - \frac{\partial V}{\partial q^{\mu}} \right) dt^0 + \frac{\partial K^p_{\nu}}{\partial q^{\mu}} dt^v.$$  \hspace{1cm} (36)

This is nothing but the definition of the momenta $p^{\mu}$, namely $p^{\mu} = \partial L/\partial \dot{q}^{\mu} - dp^{\mu}/dt$, once we insert (34), in agreement with (5).

Regarding the Hamilton principal function we get

$$dS = \{S, H_0\}^* dt^0,$$

$$= P_0 dt^0 + p^{\mu} dq^{\mu} + P^{\mu} dt^{\mu}.$$  \hspace{1cm} (37)

In other words,

$$dS = p^{\mu} dq^{\mu} - H_I dt^I,$$  \hspace{1cm} (38)

where the Hamiltonians (22) have been introduced. This expression is in agreement with (25). On the other hand, bearing in mind the nature of the matrix (4), we must recall that the only independent parameter is $t^0$ so that

$$dS = p^{\mu} dq^{\mu} + P^{\mu} dq^{\mu} - H_0 dt^0.$$  \hspace{1cm} (39)

Some comments are in order. First, it has been inferred that the solutions to the characteristic equations, in the complete phase space are given by expressions of the form $q^{\mu} = q^{\mu}(t^I)$, which represent congruences of curves in the $(N + 1)$ parametric space, where the $t^I$ play the role of coordinates. From a physical point of view, the actual dynamics of the system is achieved in a reduced phase space and dictated by the fundamental differential (30). Hence, the solutions to the characteristic equations in the physical sector of phase space are given by $q^{\mu} = q^{\mu}(t^0)$, representing congruences of one-parameter curves. In comparison with the Ostrogradski-Hamilton analysis for this type of theories [20 21], we note that we have a second-class system as observed from (28) since $N_{\mu\nu} \neq 0$.

**IV. THEORIES WITH SECOND-ORDER EQUATIONS OF MOTION**

It is readily inferred from (3) that to obtain second-order equations of motion the matrix $N_{\mu\nu}$ must vanish identically. Unlike the previous case, the matrix $M_{\mu\nu}$ becomes symmetric. This fact can be proved from (5) with support with the property $\partial K^p_{\mu}/\partial q^{\nu} = \partial K^p_{\nu}/\partial q^{\mu}$. From the facts that now $F_{\mu} = F_{\mu}(q^{\nu}, \dot{q}^{\nu}, t)$ and $M_{\mu\nu} = M_{\mu\nu}(q^{\nu}, \dot{q}^{\nu}, t)$, the eqns (3) specialize to Newton-like equations of the form

$$M_{\mu\nu} \ddot{q}^{\nu} = F_{\mu}, \hspace{1cm} \mu, \nu = 1, 2, \ldots, N; \hspace{1cm} (39)$$

where $M_{\mu\nu}$ can be interpreted as the components of a mass-like matrix while the term $F_{\mu}$ may be interpreted as a force vector. In fact, the matrix $M_{\mu\nu}$ corresponds to the Hessian matrix of a first-order equivalent Lagrangian $L_d$
where we have introduced the antisymmetric matrix \( \bar{A} \). For this particular case, the Hamiltonian constraints (20) and (21) hold. Further, at the initial stage, the evolution in phase space is dictated by the fundamental differential (27). The next step, under the new setup, is to test the integrability conditions for the Hamiltonians (20) and (21) as we will discuss in short.

A. Integrability conditions

This particular setting determines a fully constrained system. Indeed, to prove this statement we must first mention that the expression for the PB (26) holds and then we must proceed to test the integrability condition for \( H^p_\mu \) by using (27)

\[
dH^p_\mu = \{H^p_\mu, H_0\} dt^0 + \{H^p_\mu, H^p_\nu\} dt^\nu, = - \left(p_\mu + \frac{\partial \nu}{\partial \mu} + \frac{\partial}{\partial q^\nu} \right) dt^0,
\]

where we have considered \( N_{\mu\nu} = 0 \) as we observe from (29). Then, we identify new Hamiltonian constraints given by

\[
H^p_\mu := p_\mu + \frac{\partial \nu}{\partial \mu} + \frac{\partial}{\partial q^\nu} = 0.
\]

The integrability conditions have to be tested with these Hamiltonians as well. As before, when separating \( t^0 \) from the remaining parameters \( t^\nu \) we find

\[
dH^p_\mu = \{H^p_\mu, H_0\} dt^0 + \{H^p_\mu, H^p_\nu\} dt^\nu, = - F_\mu dt^0 + C_{\mu\nu} dt^\nu = 0,
\]

where we have introduced the antisymmetric matrix

\[
C_{\mu\nu} := \{H^p_\mu, H^p_\nu\} = - C_{\nu\mu},
\]

and recognize that \( \{H_0, H^p_\mu\} = F_\mu \) as defined in (3). Clearly, the variations (41) do not vanish identically, and in consequence, the idea to promote them as new constraints of the theory is deceptive. These are mere dependence relationships between the parameters of the theory. We then have a complete set of Hamiltonians

\[
H_0 = P_0 + H_0(t^0, t^\mu, q^\mu, p_\mu) = P_0 + p_\mu t^\mu + V,
\]

\[
H^p_\mu = P_\mu + H^p_\mu (t^0, t^\nu, q^\nu) = P_\mu - K^\mu,
\]

\[
H^p_\mu = p_\mu + H^p_\mu (t^0, t^\nu, q^\nu, p_\nu) = p_\mu + \frac{\partial \nu}{\partial \mu} + \frac{\partial}{\partial q^\nu} t(t^\nu)
\]

Certainly, the Hamiltonians (43), (44) and (45) are non-involutive constraints. To satisfy the integrability condition it is required to remove the non-involutive constraints by redefining the fundamental differential trough a generalised bracket structure. To perform this, the constraints (40) must enter the game accompanied of new parameters. In order to further analyse the integrability conditions it is necessary to introduce a convenient notation and a relabeling of the indices. In agreement with the scheme previously outlined in (13), introducing the quantities

\[
\bar{t}^\mu := q^\mu \quad \text{and} \quad H^P_\mu := - \frac{\partial L}{\partial q^\mu} + \frac{d}{dt} \left( \frac{\partial L}{\partial q^\mu} \right),
\]

with the understanding that \( \bar{\mu} = N + 1, N + 2, \ldots \). The \( \bar{t}^\mu \) are expected to be in relation to the generalized coordinates \( q^\mu \). Additionally, we introduce the complete set of parameters \( t^I \) in the order

\[
I = 0, 1, \ldots, N, N + 1, \ldots, 2N,
\]

as well as the notation \( H_I := \{H_0, H^P_\bar{\mu}, H^P_\bar{\nu}\}, \ H_I = (H_0, H^P_\mu, H^P_\nu) \) and \( \mathcal{P}_I = (P_0, P_\mu, P_\nu) \), respecting that order, with \( I, J = 0, 1, 2, \ldots, N, N + 1, \ldots, 2N \). Surely, we can also express the Hamiltonians in a constrained Hamiltonian fashion according to (22)

\[
H_I(t^I, \mathcal{P}_I) = \mathcal{P}_I + H_I(t^I, \mathcal{P}_I) = 0.
\]

As before, the \( 2N + 1 \) relations (45) represent HJPDE. Similarly, as in previous section, it will be said that the Hamiltonians \( H^p_\mu, (40) \), are associated with the parameters \( \bar{t}^\mu \) that are in relation with the coordinates of the physical system.

Now, the evolution of the theory is derived from the fundamental differential

\[
dF = \{F, H_0\} dt^0 + \{F, H^p_\mu\} dt^\mu + \{F, H^p_\nu\} dt^\nu,
\]

where the space of parameters has been expanded. At this point it is worthwhile to remark that the integrability of the system now should be tested by using (49). At this stage, we should be able to elucidate if the system is integrable either in a complete or in a partial manner. By an appropriate relabeling of the indices, when using the fundamental differential (49) as well as separating the time parameter from the remaining ones, the condition

\[
dH_I = \{H_I, H_J\} dt^J = 0 \quad \text{on the Hamiltonians (48)}
\]

is written as

\[
dH_I = \{H_I, H_0\} dt^0 + \{H_I, H_A\} dt^A.
\]

Here, \( I, J = 0, A \) with \( A = \mu, \bar{\mu}, 1, 2, \ldots, N + 1, \ldots, 2N \) and \( \mathcal{H}_A = (H^P_\mu, H^p_\nu) \) being the Hamiltonians organized in that suitable order. Explicitly

\[
dH_0 = \{H_0, H_A\} dt^A = 0,
\]

\[
dH_A = \{H_A, H_0\} dt^0 + \mathcal{M}_{AB} dt^B = 0,
\]

where we have introduced the antisymmetric matrix components \( \mathcal{M}_{AB} := \{H_A, H_B\} \). This is a \( 2N \times 2N \) partitioned matrix

\[
\mathcal{M} = \begin{pmatrix}
0 & - M_{\mu\bar{\nu}} \\
M_{\bar{\mu}\bar{\nu}} & Q_{\bar{\mu}\bar{\nu}}
\end{pmatrix}
\]

decomposed in terms of the symmetric matrix \( M_{\mu\bar{\nu}} \) and the antisymmetric one \( Q_{\bar{\mu}\bar{\nu}} \) defined as

\[
\{H^p_\mu, H^p_\nu\} := - M_{\mu\bar{\nu}} = - M_{\bar{\nu}\mu},
\]

\[
\{H^p_\mu, H^p_\nu\} := Q_{\bar{\mu}\bar{\nu}} = - Q_{\bar{\nu}\bar{\mu}}.
\]
Notice that (54) agrees with the expression defining the matrix \( M \). If all the parameters are independent, then \( \{ H, \dot{H} \} = 0 \) and \( M_{AB} = 0 \) are required. These conditions lead to the only possible solution of the equations of motion and the system becomes integrable. On the contrary, it could happen that the Hamiltonians do not obey (51) and (52) so that the fulfillment of the integrability conditions leads to assume a linear dependence on the parameters \( t^A \), leading us to define generalized brackets. For that reason, the discussion must be addressed on the case where \( M_{AB} \) is different from zero. Certainly, from (52) we get

\[
M_{AB} dt^B = -\{H, \dot{H}_0\} dt^0. \tag{56}
\]

Therefore, at this new stage, the integrability analysis bifurcates.

- If \( M_{AB} \) is non-singular then \( \det(M_{AB}) \neq 0 \). In such a case the inverse matrix \( (M^{-1})^{AB} \) exists so that \( (M^{-1})^{AC} M_{CB} = \delta^A_B \) or \( M_{AC}(M^{-1})^{CB} = \delta^A_B \). Indeed, we will have that

\[
M^{-1} = \begin{pmatrix}
(M^{-1})^{\mu\nu} & (M^{-1})^{\mu\nu} \\
(M^{-1})^{\mu\nu} & 0
\end{pmatrix}. \tag{57}
\]

The realization of \( dH = 0 \) leads to consider that \( dt^0 \) and \( dt^A \) are dependent. We infer from (56) that \( t^0 = t \) is the independent parameter of the theory

\[
dt^A = - (M^{-1})^{AB} \{H_B, \dot{H}_0\} dt^0. \tag{58}
\]

In this manner, when substituting (58) into (50), we observe that the evolution of \( F \in \Gamma_{2N+1} \) is provided by

\[
dF = \{ F, \dot{H}_0 \}^* dt^0, \tag{59}
\]

where

\[
\{ F, G \}^* := \{ F, G \} - \{ F, H_A \} (M^{-1})^{AB} \{ H_B, G \}. \tag{60}
\]

This structure helps to redefine the dynamics by eliminating the parameters \( t^A \) with exception of \( t^0 \).

In passing, we would like to mention that the complete set of Hamiltonians \( H_i \) are in involution with the bracket structure (60). In a sense, this set up is similar to one outlined in the previous section.

- If \( M_{AB} \) is singular then \( \det(M_{AB}) = 0 \). This property is related to the existence of a gauge symmetry in the Lagrangian (2). The rank of \( M_{AB} \) being, say \( R = 2N - r \), implies the existence of \( r \) left (or right) null eigenvectors \( \lambda_\alpha^A \), or zero-modes, of \( M_{AB} \) such that \( M_{AB} \lambda_\alpha^A = 0 \) where \( \alpha \) labels the independent zero-modes with \( \alpha = 1, 2, \ldots, r \). In such a case the original configuration manifold \( C_{2N} \), not including the time parameter, becomes splitted in two submanifolds: \( C_r \) spanned by \( r \) coordinates, \( t^\alpha \), related to the kernel of \( M_{AB} \), and \( C_{2N-r} \) spanned by \( R \) coordinates \( t^{A'} \), with \( A' = r+1, r+2, \ldots, 2N \), associated with the regular part of \( M_{AB} \). Under these conditions, we ensure the existence of a \( R \times R \) submatrix, say \( M_{A'B'} \), such that \( \det(M_{A'B'}) \neq 0 \) so that we have an inverse matrix \( (M^{-1})^{A'B'} \) satisfying \( (M^{-1})^{A'C'} M_{C'B'} = \delta^{B'}_{B} \) or \( \lambda^A_{AC'} (M^{-1})^{C'A'} = \delta^A_{B'} \). According to this splitting we set \( A = \alpha, A' = 1, 2, \ldots, r, r+1, \ldots, 2N \).

The expansion of the condition (50) reads

\[
-\{H, \dot{H}_0\} dt^0 = \delta^A_B M_{A'B'} dt^{B'} + \delta^A_B M_{A\alpha} dt^{\alpha'} + \delta^A_B M_{A\beta} dt^{\beta}. \tag{61}
\]

We must inspect two cases

a) If \( A = A' \) it follows that \( dt^{A'} \) can be expressed in terms of \( dt^\alpha \) and \( dt^0 \). Indeed, from (61) we get

\[
M_{A'B'} dt^{B'} = -\{H_{A'}, \dot{H}_0\} dt^\alpha', \tag{62}
\]

where we have introduced the label \( \alpha' = 0, 1, 2, \ldots, r \). Thence,

\[
dt^{A'} = -(M^{-1})^{A'B'} \{H_{B'}, \dot{H}_0\} dt^\alpha'. \tag{62}
\]

We have therefore a relation of dependence between the parameters \( dt^{A'} \) and \( dt^\alpha' \).

b) If \( A = \alpha \), from (61) again, it is inferred that

\[
-\{H, \dot{H}_0\} dt^0 - \{H, \dot{H}_\beta\} dt^\beta - \{H, H_{A'}\} (M^{-1})^{A'B'} \{H_{B'}, \dot{H}_0\} dt^0 - \{H, H_{A'}\} (M^{-1})^{A'B'} \{H_{B'}, \dot{H}_\beta\} dt^\beta \tag{63}
\]

where we have inserted (52). Based on the linear independence between \( dt^0 \) and \( dt^\alpha \), it follows that

\[
\{H, \dot{H}_0\} = \{H, H_{A'}\} (M^{-1})^{A'B'} \{H_{B'}, \dot{H}_0\}, \tag{64}
\]

\[
\{H, \dot{H}_\beta\} = \{H, H_{A'}\} (M^{-1})^{A'B'} \{H_{B'}, \dot{H}_\beta\}. \tag{65}
\]

should be considered as conditions that fix the
subspace of parameters where the system becomes integrable.

The aforementioned dependence on the variables, (62), when inserted into (56), allows to determine the evolution of $F \in \Gamma_{2N+1}$ as follows

$$dF = \{F, \mathcal{H}_\alpha\}^\ast d\tau^\alpha,$$

(66)

where

$$\{F, G\}^\ast := \{F, G\} - \{F, \mathcal{H}_I\}\mathcal{M}^{-1}\mathcal{A}'B' \{\mathcal{H}_B', G\}. \quad (67)$$

As already mentioned, the variables $t^\alpha$ are the independent parameters of the theory whereas the remaining variables $t^{A'}$ are the dependent variables. Clearly, under this split of the variables $t^A$ into $t^\alpha$ and $t^{A'}$, each of them will be in relation to both coordinates and velocities of the system.

The bracket structure introduced either in (60) or in (67), is also referred to as the generalized bracket. As a matter of fact, this structure is closely related to the Dirac bracket arising in the Dirac-Bergmann approach for constrained systems [13–15]. Therefore, the dynamical evolution of the theory depends on $t^\alpha$ parameters, $t^\alpha$.

On the other hand, it may happens that the constraints $\mathcal{H}_I = 0$ do not satisfy $d\mathcal{H}_I = 0$ identically when (59) or (60) are considered as fundamental differentials. In such a case, the integrability condition leads us to obtain equations of the form $f(q^\mu, \dot{q}^\mu, p_\mu, P_\mu) = 0$ which should also be considered as constraints of the system. In a like manner, the integrability conditions must be tested for $f$, which could also generate more Hamiltonians. Once we have found the complete set of involutive Hamiltonians, it is mandatory to incorporate them within the HJ framework where some of them must be considered as generators of the dynamics. This incorporation must be accompanied by the introduction of more parameters to the theory, these related to the new constraints that generate dynamics, derived from the integrability analysis. Thereupon, the space of parameters has been increased where, every arbitrary parameter is in relation to the generators of the dynamics [11, 12]. As a result, in this new scenario, the final form of the fundamental differential reads

$$dF = \{F, \mathcal{H}_\alpha\}^\ast d\tau^\alpha,$$

(68)

with the understanding that $\tau^\alpha$ denotes the complete set of independent parameters where the index $\tau$ spans over the entire set of these parameters. Hence, the fundamental differential (68) must be used to obtain the right evolution in the reduced phase space through the GB.

1. Integrability analysis based on zero-modes

A useful procedure to identify the irreversible submatrix $\mathcal{M}_{A'B'}$ as well as for the search of gauge information of the physical systems, under the present conditions, is based on the zero-modes of $\mathcal{M}_{AB}$. To develop this, consider a non-singular transformation $G$ acting on the differentials of the parameters, except $t^0$, given by $dt^A = G^A_B dt^B$. By expanding the indices covering the ker$(\mathcal{M}_{AB})$ and its complement subspace, we have $dt^A = G^A_{\alpha} dt^\alpha + G^A_{A'} dt^{A'}$. We can choose now as a suitable basis of the linear transformation the zero-modes $\lambda^A_{(\alpha)}$ with $\alpha = 1, 2, \ldots, r$, and a set of $R$ vectors $\lambda^A_{A'}$, with $A' = r + 1, r + 2, \ldots, 2N$, chosen in a way such that they do not depend on the zero-modes or on one another and on the condition that det$(G^A_B) \neq 0$. Relative to this basis we have that $G^A_{\alpha} = \lambda^A_{(\alpha)}$ and $G^A_{A'} = \lambda^A_{A'}$, so that we can build the evolution of the system by using the fundamental differential

$$dF = \{F, \mathcal{H}_0\} dt^0 + \{F, \mathcal{H}_\alpha\} dt^\alpha + \{F, \mathcal{H}_{A'}\} dt^{A'}, \quad (69)$$

where

$$\mathcal{H}_\alpha := \mathcal{H}_A \lambda^A_{(\alpha)} = 0, \quad (70)$$

$$\mathcal{H}_{A'} := \mathcal{H}_A \lambda^A_{A'} = 0. \quad (71)$$

From this viewpoint we must now work with the equivalent set of $2N + 1$ Hamiltonians given by (20), (70) and (71). According to this, the integrability of the system should be tested by considering (69). The integrability condition applied to the projected Hamiltonians become

$$d\mathcal{H}_0 = \{\mathcal{H}_0, \mathcal{H}_\alpha\} dt^\alpha + \{\mathcal{H}_0, \mathcal{H}_{A'}\} dt^{A'} = 0, \quad (72)$$

$$d\mathcal{H}_\alpha = \{\mathcal{H}_\alpha, \mathcal{H}_0\} dt^0 = 0, \quad (73)$$

$$d\mathcal{H}_{A'} = \{\mathcal{H}_{A'}, \mathcal{H}_0\} dt^0 + \mathcal{M}_{A'B'} dt^{B'} = 0, \quad (74)$$

where we have defined

$$\mathcal{M}_{A'B'} := \mathcal{M}_{AB} \lambda^A_{A'} \lambda^B_{B'} \lambda^B_{B'}. \quad (75)$$

This is an antisymmetric non-singular matrix that is everywhere invertible on the reduced phase space so the existence of an inverse matrix, say $(\mathcal{M}^{-1})^{A'B'}$, is ensured. From (74) we identify a dependence among some of the original parameters. Indeed, we have that $dt^{A'} = -(\mathcal{M}^{-1})^{A'B'} \{\mathcal{H}_{B'}, \mathcal{H}_0\} dt^0$ so that when inserted into (69) it follows straightforwardly that the evolution must be given by

$$dF = \{F, \mathcal{H}_\alpha\}^\ast dt^\alpha,$$

(76)

where

$$\{F, G\}^\ast := \{F, G\} - \{F, \mathcal{H}_{A'}\}(\mathcal{M}^{-1})^{A'B'} \{\mathcal{H}_B', G\}. \quad (77)$$

with the understanding that $\mathcal{H}_\alpha := \{\mathcal{H}_0, \mathcal{H}_\alpha\}$ and $\alpha' = 0, 1, 2, \ldots, r$. In arriving to the new fundamental differential (70) we have considered that $\mathcal{H}_{A'}, \mathcal{H}_\alpha = 0$. On the other hand, from (72) and by inserting $dt^{A'}$, deduced from (74), into (72) we have that

$$d\mathcal{H}_0 = \{\mathcal{H}_0, \mathcal{H}_\alpha\} dt^\alpha = 0, \quad (78)$$

$$d\mathcal{H}_\alpha = \{\mathcal{H}_\alpha, \mathcal{H}_0\} dt^0 = 0. \quad (79)$$
are identically vanishing. To prove this, observe that \( \{H_0, H_\alpha\}dt^0 = \{H_0, H_\lambda\} \lambda^A(\alpha) dt^0 \); now, from (50) we find that \( \{H_0, H_\alpha\}dt^0 = -M_B A^A(\alpha) dt^0 = 0 \) so that it is inferred that \( \{H_0, H_\alpha\} = 0 \). Therefore, the fundamental differential (70) provides the \((r + 1)\)-parameter evolution in the phase space in which the Hamiltonians \( H_\alpha \) are the generators. It should be noted that Hamiltonians (20), (70) and (71) become involutive under the GB given by (77) so that the theory becomes integrable. We would like to mention that this alternative approach is suitable for systems exhibiting a neatly geometric structure as it is the case, for instance, for extended objects evolving in an ambient spacetime (17).

### B. Characteristic equations

The characteristic equations that arise in the regular case governed by the fundamental differential (59), are similar to those obtained in the previous section. On the other hand, for the singular case, having at our disposal the GB (77), the dynamics of the system is provided by the fundamental differential (70) by evaluating \( F \) for the phase space variables. In this spirit, it will be convenient to write the characteristic equations in terms of the components of the zero-modes \( \lambda^A(\alpha) \). First, for \( F = \dot{q}^\mu \) we have

\[
dq^\mu = \{q^\mu, H_0\} dt^0 + \{q^\mu, H_\lambda\} dt^\alpha,
\]

where we have considered that \( \{H_B, H_\lambda\} = 0 \). Second, for \( F = \dot{p}_{\bar{\mu}} \) we obtain

\[
d\dot{q}^\mu = -(M^{-1})^A_{A'} \frac{\partial H_A}{\partial p_{\bar{\mu}}} \{H_{A'}, H_0\} dt^0 + \lambda^\mu(\alpha) dt^\alpha.
\]

For \( F = P_\mu \), we get

\[
dP_\mu = \left[ -p_\mu - \frac{\partial V}{\partial q^{\bar{\mu}}} + (M^{-1})^A_{A'} \frac{\partial H_A}{\partial q^{\bar{\mu}}} \{H_{A'}, H_0\} \right] dt^0 + \frac{\partial H_\alpha}{\partial q^{\bar{\mu}}} dt^\alpha.
\]

Finally, for \( F = p_{\bar{\mu}} \)

\[
dp_{\bar{\mu}} = \left[ -\frac{\partial V}{\partial q^{\mu}} + (M^{-1})^A_{A'} \frac{\partial H_A}{\partial q^{\mu}} \{H_{A'}, H_0\} \right] dt^0 + \frac{\partial H_\alpha}{\partial q^{\mu}} dt^\alpha.
\]

From this viewpoint, the characteristic equations provide on the one hand the time evolution whereas, on the other hand, provide the canonical and gauge transformations by analysing the evolution of the system at fixed time \( dt^0 \) along the remaining parameters. To correctly reproduce the equations of motion, it will be necessary to choose appropriate parameters \( t^\alpha \). This is so since when calculating the characteristic equations from the original form (19), we can observe that the differentials \( dt^\alpha \) are arbitrary so, under the conditions in this scenario, the right dynamics in the physical phase space is fixed by choosing some values for the parameters.

Regarding the Hamilton principal function, \( S = S(q^\mu, \dot{q}^\mu, t^0) \), we have that

\[
dS = \{S, H_0\} dt^0 + \{S, H_\lambda\} dt^\alpha,
\]

\[
= \frac{\partial S}{\partial q^{\bar{\mu}}} \dot{q}^{\bar{\mu}} dt^0 + (\frac{\partial S}{\partial q^{\mu}} \lambda^\mu(\alpha) + \frac{\partial S}{\partial \dot{q}^\mu} \lambda^\mu(\alpha)) dt^A'
\]

\[
\quad + \left( \frac{\partial S}{\partial q^{\bar{\mu}}} \lambda^{\bar{\mu}}(\alpha) + \frac{\partial S}{\partial \dot{q}^{\mu}} \lambda^{\mu}(\alpha) \right) dt^\alpha.
\]

Taking into account (7), (8) and (9) projected along the \( \lambda^A(\alpha) \) vectors, and collected into \( P_I = (P_0, p_\alpha, p_{A'}, P_\alpha, P_{A'}) \) we have

\[
dS = -H_{A'} dt^A + P_{A'} dt^A + P_{A'} dt^A'
\]

where we have considered (48). In summary, (81) define a reduced phase space with coordinates \( (q^A, \dot{q}^A, p_{A'}, P_{A'}) \). As we already mentioned, given some initial conditions, the solution to the characteristic equations will be dynamical trajectories restricted to the variables \( q^A \) whose parametric equations will be of the form \( q^A = q^A(t, t^\alpha) \).

### C. Generator of gauge symmetries

Once the complete set of involutive Hamiltonians, \( H_\pi \), satisfying \( \{H_\pi, H_\pi\} = C_\pi^\alpha H_\pi \), has been obtained, these must be considered as generators of infinitesimal canonical transformations of the form, (12)

\[
\delta z^A = \{z^A, H_\pi\} \delta t^\pi,
\]

where \( z^A = (q^A, p_{A'}) \). These are referred to as the characteristic flows of the system. Here, \( \delta t^\pi := t^\pi - t^\pi = \delta t^\pi(z^A) \). In particular, when these transformations are taken at constant time, \( \delta t^0 = 0 \), the expression (85) defines a special class of transformations

\[
\delta z^A = \{z^\tilde{A}, H_\alpha\} \delta t^\alpha,
\]

which, by observing that they remain in the reduced phase space, \( T^* \mathcal{C}_P \), form the so-called infinitesimal contact transformations in the spirit of the constrained Hamiltonian framework by Dirac (13). In this sense, the transformations (86) do not alter the physical states of the system. Thus, \( t^\alpha \) denotes the set of all independent parameters where \( t^0 \) is excluded. Clearly, transformations (86) are generated by

\[
G := H_\pi \delta t^\alpha.
\]
so that \( \delta_G z^\Lambda = \{z^\Lambda, G\}^* \).

(88)

Thus, \( \delta_G z^\Lambda \) is the specialization of (88) to \( T^*C_r \) where \( G \) is the generating function of the infinitesimal canonical transformation. In the spirit of the theory of gauge fields, transformations (88) stand for the gauge transformations of the theory. \[ \text{[12]} \]

V. APPLICATIONS

Having disposal of the general aspects of our development we can now turn to consider some examples that illustrate how the above schemes work.

A. Galilean invariant (2 + 1)-dim model with a Chern-Simons-like term

Consider the effective Lagrangian of a non-relativistic system \[ 18 \]

\[
L(q^\mu, \dot{q}^\mu, \ddot{q}^\mu, t) = -k\epsilon_{\mu\nu}\dot{q}^\mu\dot{q}^\nu + \frac{m}{2}\dot{q}^2 \quad \mu, \nu = 1, 2, \]

where \( k \) is a parameter of the theory, \( m \) is a constant and \( \epsilon_{\mu\nu} \) is the Levi-Civita antisymmetric metric with \( \epsilon_{12} = 1 \). This one-particle model with second-order derivatives describes a free motion in the \( D = 2 \) space with non-commuting coordinates and internal structure described by oscillator modes with negative energies. Once we identify the basic structures \( K_{\mu} = k\epsilon_{\mu\nu}\dot{q}^\nu \) and \( \nu = -(m/2)\ddot{q}^2 \), from (11), (12) and (13), it is straightforward to compute

\[
N_{\mu\nu} = -2k\epsilon_{\mu\nu}, \quad M_{\mu\nu} = -m\delta_{\mu\nu} \quad \text{and} \quad F_{\mu} = 0, \]

respectively. Hence, from (13) the eom are given by

\[
2k\epsilon_{\mu\nu}\dot{q}^\nu - m\delta_{\mu\nu}\dot{q}^\nu = 0, \]

which are of third-order in the derivatives. On the other hand, from (11) and (12), the momenta for this theory reads

\[
P_{\mu} = k\epsilon_{\mu\nu}\dot{q}^\nu, \quad \text{and} \quad p_{\mu} = m\dot{q}^\mu - 2k\epsilon_{\mu\nu}\ddot{q}^\nu. \]

The canonical Hamiltonian \( H_0 = P_{\mu}\dot{q}^\mu + P_{\nu}\dot{q}^\nu - L \) can be readily obtained

\[
H_0 = p_{\mu}\dot{q}^\mu - \frac{m}{2}\dot{q}^2. \]

Now, from (19) (or (22)), we have the HJPDE

\[
\mathcal{H}_0 = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^\mu}\dot{q}^\mu - \frac{m}{2}\delta_{\mu\nu}\dot{t}^\mu\dot{t}^\nu, \]

(90)

\[
\mathcal{H}_\mu^P = \frac{\partial S}{\partial \dot{q}^\mu} - k\epsilon_{\mu\nu}\dot{t}^\nu = 0, \]

(91)

or,

\[
\mathcal{H}_0 = P_0 + p_{\mu}\dot{q}^\mu - \frac{m}{2}\delta_{\mu\nu}t^\mu t^\nu, \]

\[
\mathcal{H}_\mu^P = P_\mu - kt_\mu t^\nu = 0, \]

(92)

in a constrained Hamiltonian fashion. By noticing that the inverse matrix of (11), for the present case, is given by \((N^{-1})_{\mu\nu} = (1/2k)e^{\mu\nu}\), the GB (12) reads

\[
\{F, G\}^* = \{F, G\} - \frac{1}{2k}\{F, \mathcal{H}_\mu^P\}e^{\mu\nu}\{H_\nu^P, G\}, \]

(93)

where \( F \) and \( G \) are phase space functions. Defining \( Q^\nu := 2\dot{q}^\nu \) and \( Q := 2/2k \) and \( K := k/2 \), we are able to find the non-vanishing fundamental generalized brackets of the theory

\[
\{q^\mu, p_{\nu}\}^* = \delta_{\mu\nu}, \quad \{Q^\mu, Q^\nu\}^* = Qe^{\mu\nu}, \]

\[
\{Q^\mu, P_{\nu}\}^* = K\epsilon_{\mu\nu}. \]

(94)

We thus find that \((q^\mu, p_{\nu})\) and \((Q^\mu, P_{\nu})\) are the canonical pairs of the theory under the GB structure. In passing, we observe that under the GB (99) the coordinates \( Q^\mu \) are non-commutative. Regarding the characteristic equations, by considering the results of (11) we have that (13) is a mere identity. In the same spirit, (34) leads to \( dt^\mu = \dot{q}^\mu = (1/2k)e^{\mu\nu}(p_{\nu} - m\dot{q}_{\nu})dt^0 \) which is in agreement with (98). On the other hand (33) yields to \( dp_{\mu} = -(p_{\mu} - m\dot{q}_{\mu})dt^0 - k\epsilon_{\mu\nu}dt^\nu \) which is also in agreement with (99). Finally, (30) leads to \( d\dot{q}_{\mu} = 0 \). This last fact plays a double duty. On one hand, this leads to the equation of motion (61) once we insert the expression (93). On the other hand, this yields the fact that the momenta \( p_{\mu} \) are constant.

It is worthwhile to mention that in (18) this system has been analysed using the equivalent canonical Hamiltonian given by

\[
H_0 = -\frac{m}{2k^2}P^2 + \frac{1}{k}e^{\mu\nu}P_{\mu}p_{\nu}, \]

(95)

where the authors focused on the quantum properties of the system.

B. Harmonic oscillator in 2D

A case where the matrix (53) is regular is provided by the Lagrangian \[ 19 \]

\[
L(q^\mu, \dot{q}^\mu, \ddot{q}^\mu, t) = -\frac{m}{2}q^\mu\ddot{q}^\mu - \frac{k}{2}\dot{q}^2 \quad \mu, \nu = 1, 2. \]

(96)

\[
L(q^\mu, \dot{q}^\mu, \ddot{q}^\mu, t) = -\frac{m}{2}q^\mu\ddot{q}^\mu - \frac{k}{2}\dot{q}^2 \quad \mu, \nu = 1, 2. \]

(97)

where \( m \) and \( k \) are constants. This is a non-standard way to study an isotropic harmonic oscillator as we will see shortly. We readily identify that \( K_{\mu} = -(m/2)\delta_{\mu\nu}\ddot{q}^\nu \) and \( \nu = (k/2)\delta_{\mu\nu}\dot{q}^\nu \). From (51), (52) and (53) we find that

\[
N_{\mu\nu} = 0 \quad M_{\mu\nu} = -m\delta_{\mu\nu} \quad \text{and} \quad F_{\mu} = kq_{\mu}. \]

(98)
Consequently, from (3) we have the second-order equations of motion

\[ m \ddot{q}^\mu + k q^\mu = 0, \]

which is nothing but the well known Hooke’s law. Regarding the momenta, from (11) and (12), we get

\[ P_\mu = -\frac{m}{2} q_\mu \quad \text{and} \quad p_\mu = \frac{m}{2} q_\mu. \tag{105} \]

For this pedagogical case, the corresponding Legendre transformation yields

\[ H_0 = p_\mu \dot{q}^\mu + \frac{k}{2} q^2. \tag{106} \]

On account of the expressions (43), (44) and (45), the Hamiltonians of the theory are

\[ \mathcal{H}_0 = P_\mu + p_\mu \dot{q}^\mu + \frac{k}{2} q^2 = 0, \tag{107} \]

\[ \mathcal{H}_P^\mu = P_\mu + \frac{m}{2} q_\mu = 0, \tag{108} \]

\[ \mathcal{H}_P^\mu = p_\mu - \frac{m}{2} q_\mu = 0. \tag{109} \]

These expressions, together with (26), allow us to quickly determine that the matrix \( Q_{\mu\nu} \), (55), vanishes. Thus, the partitioned matrix (53) turns out to be non-singular. This, and its inverse, are given by

\[ \mathcal{M} = m \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{M}^{-1} = \frac{1}{m} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{110} \]

respectively. Under these conditions, \( t^0 = \tau \) is the only independent parameter of the model and the evolution in the phase space is provided by the fundamental differential given by \( Q_{\mu\nu} \) where, on account of (60), the GB is given by

\[ \{ F, G \}^* = \{ F, G \} + \frac{1}{m} \{ F, H_\mu^P \} \delta^{\mu\nu} \{ H_\nu^P, G \} \]

\[ -\frac{1}{m} \{ F, H_\mu^P \} \delta^{\mu\nu} \{ H_\nu^P, G \}. \tag{111} \]

Having at our disposal this GB and by defining \( \pi_\mu := 2p_\mu \) and \( \Pi_\mu := 2P_\mu \), we find that the non-zero fundamental generalized brackets are

\[ \{ q^\mu, \pi_\nu \}^* = \delta^{\mu\nu}, \quad \{ q^\mu, \dot{q}^\nu \}^* = \frac{1}{m} \delta^{\mu\nu}, \quad \{ \pi_\mu, \Pi_\nu \}^* = m \delta_{\mu\nu}. \tag{112} \]

Applying (\( q^\mu, \pi_\nu \)) and \( (\dot{q}^\mu, \Pi_\nu) \) are the canonical pairs of the theory but this fact is deceptive. This is definitely an example of the fact that the theory described by (102) needs a boundary term in order to restore the original properties of a harmonic oscillator without the need to describe it in terms of second-order terms. Regarding the characteristic equations, when using (59) and (112), all the dynamical and geometrical information is correctly reproduced, as stated at the beginning of IV B.

C. Geodetic brane cosmology with a cosmological constant

Consider the cosmological effective Lagrangian for a geodetic brane-like universe governed by the Regge-Teitelboim (RT) model \( ^{36} \)

\[ L(a, t, \dot{a}, \ddot{a}, \tau) = \frac{a^2}{N^5} (\ddot{a} - \dot{a} \dot{t}) + \frac{a}{N} \Upsilon, \tag{113} \]

where \( \Upsilon := t^2 - N^2 a^2 \Lambda^2 \). Here, and in what follows, \( \Lambda^2 := \Lambda_3/3 \) where \( \Lambda \) and \( \alpha \) are constants and \( N := \sqrt{t^2 - \dot{a}^2} \) represents the lapse function that commonly appears when we perform an ADM decomposition of the RT model. Further, a dot stands for derivative with respect to the time parameter \( \tau \) where \( q^\mu = (t(\tau), a(\tau)) \). This model is invariant under reparameterizations of the coordinates (for more details see \( ^{22, 28} \)).

For this theory we readily identify \( K_\mu = a^2 \dot{t}(-\dot{a}, \dot{t})/N^3 \) and \( V = -(a/N) \Upsilon \) with \( \mu = 1, 2 \) and \( \bar{\mu} = 3, 4 \). From these, we compute the basic structures \( 4, 5 \) and (3):

\[ N_{\mu\nu} = 0, \quad (M_{\mu\bar{\nu}}) = \frac{a \Phi}{N^5} \begin{pmatrix} \dot{a}^2 - i \dot{a} \\ -i \dot{a} t \end{pmatrix}^2, \quad (Q_{\mu\bar{\nu}}) = \frac{i \Theta}{N^3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{114} \]

and

\[ (F_\mu) = -\frac{\dot{a} \Theta}{N^3} (-\dot{a} t). \tag{115} \]

where

\[ \Theta := t^2 - 3N^2a^2 \Lambda^2, \tag{116} \]

\[ \Phi := 3t^2 - N^2a^2 \Lambda^2. \tag{117} \]

Note that \( \text{det}(M_{\mu\bar{\nu}}) = 0 \) where its rank is \( r = 1 \). Contrary to this fact, we have that \( \text{det}(Q_{\mu\bar{\nu}}) \neq 0 \). By inserting (114) and (115) into (3) we find a solely equation of motion

\[ \frac{d}{dt} \begin{pmatrix} \dot{a} \\ \dot{t} \end{pmatrix} = \frac{N^2 \Theta}{a \Phi} \frac{\dot{a} \Phi}{N^5}, \tag{118} \]

which is of second-order in derivatives in the variables \( t(\tau) \) and \( a(\tau) \).

The momenta of the theory, from (11) and (12), are

\[ (P_\mu) = \frac{a^2}{N^3} (-\dot{a} \dot{t}) \quad \text{and} \quad (p_\mu) = -\frac{a \Upsilon}{N^3} (-\dot{t} \dot{a}). \tag{119} \]

In this spirit, the corresponding Legendre transformation yields

\[ H_0 = p_\mu \dot{t} + p_\mu \dot{a} - \frac{a}{N} \Upsilon. \tag{120} \]

On physical grounds, the HJ analysis gets more convenient when using the projector approach based on zero-modes. The partitioned matrix (53) is given by

\[ \mathcal{M} = \frac{\Phi}{N^4} \begin{pmatrix} 0 & 0 & -\dot{a}^2 & \dot{t} \dot{a} \\ 0 & 0 & \dot{t} \dot{a} & -\dot{t}^2 \\ -i \dot{a} t^2 & 0 & -N^2 \Upsilon \Theta \end{pmatrix}. \tag{121} \]
where $\tilde{\Theta} := (i/N)\Theta$ and $\tilde{\Phi} := (a/N)\Phi$ with $\Theta$ and $\Phi$ defined in (116) and (117), respectively. The rank of this matrix, being $R = 2$, signals the presence of two zero-modes. Indeed, guided by the notation introduced in Sect. IV A 1 we have that the vectors $\lambda^A_{(\alpha, A^\prime)}$ as depicted in IV A 1 we have that the vectors

$$
\lambda(\alpha) = \begin{cases} 
\lambda(1) = \frac{1}{2} \begin{pmatrix} i \\ \dot{\alpha} \\ 0 \\ 0 \end{pmatrix}, \\
\lambda(2) = \begin{pmatrix} \Theta \dot{\alpha} \\ \Theta i \\ -\Phi i \\ -\Phi \dot{\alpha} \end{pmatrix}
\end{cases},
$$

and

$$
\lambda_{A^\prime} = \begin{cases}
\lambda(3) = \begin{pmatrix} i \\ \dot{\alpha} \\ 0 \\ 0 \end{pmatrix}, \\
\lambda(4) = \begin{pmatrix} i \\ \dot{\alpha} \\ 0 \\ i \end{pmatrix}
\end{cases},
$$

span the kernel of (121) and its complement subspace, respectively, where $\alpha = 1, 2$ and $A^\prime = 3, 4$. Bearing in mind that $i^2 = \tau$, the original HJPDE for the present case are given by

$$
\mathcal{H}_0 = \frac{\partial S}{\partial \tau} + i^\mu \frac{\partial S}{\partial \dot{\mu}} - \frac{a}{N} \tau = 0,
$$

$$
\mathcal{H}_\mu^\nu = \frac{\partial S}{\partial \mu} - \frac{a^2}{N^2} n_\mu = 0,
$$

$$
\mathcal{H}_\mu^\nu = \frac{\partial S}{\partial \mu} + \frac{a}{N^2} \dot{q}_\mu = 0,
$$

where we have introduced the vectors

$$
\dot{q}^\mu = \begin{pmatrix} i \\ \dot{\alpha} \end{pmatrix} \quad \text{and} \quad n^\mu = \frac{1}{N} \begin{pmatrix} i \\ \dot{\alpha} \end{pmatrix},
$$

which are orthogonal in the sense that $\eta_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = 0$ where $(\eta_{\mu\nu}) = \text{diag}(-1, 1)$. In this sense note that $\eta_{\mu\nu} n^\mu n^\nu = 1$. In fact, these represent both the time vector field and the normal vector to the brane-like universe. By projecting the original Hamiltonians along (122), as dictated by (70) and (71), we get

$$
\mathcal{H}_\alpha = \begin{cases} 
\mathcal{H}_1 = \hat{C}_1, \\
\mathcal{H}_2 = \tilde{\Theta} C_2 - \tilde{\Phi} C_3,
\end{cases} \quad \mathcal{H}_{A^\prime} = \begin{cases} 
\mathcal{H}_3 = C_2, \\
\mathcal{H}_4 = C_4,
\end{cases}
$$

(124)

where

$$
C_1 := P_i \dot{i} + P_\alpha \dot{\alpha} = 0,
$$

$$
C_2 := P_i \dot{\alpha} + P_\alpha i - \frac{a^2 i}{N} = 0,
$$

$$
C_3 := P_i \dot{i} + P_\alpha \dot{\alpha} - \frac{a}{N} \tau = 0,
$$

$$
C_4 := P_\alpha \dot{i} + P_i \dot{\alpha} = 0.
$$

(125–128)

Note that $C_4 = \mathcal{H}_0$ which identically vanishes as a consequence of the reparameterization invariance of the model. In the representation (124) the Hamiltonians split into involutive, $\mathcal{H}_\alpha$, and non-involutive, $\mathcal{H}_{A^\prime}$, ones. The extended Poisson algebra among the $C_i$, with $i = 1, 2, 3, 4$ is as follows

$$
\begin{align*}
\{C_1, C_2\} &= 0 \quad \{C_2, C_3\} = -C_4 \\
\{C_1, C_3\} &= -C_3 \quad \{C_2, C_4\} = -C_3 - \tilde{\Phi}, \\
\{C_1, C_4\} &= -C_4 \quad \{C_3, C_4\} = -\tilde{\Phi}.
\end{align*}
$$

(129)

Clearly, the $C_i$ determine a non-involutive set of phase space functions. In this sense, the Hamiltonians (124) obey the extended Poisson algebra

$$
\begin{align*}
\{H_1, H_2\} &= -\bar{H}_2 \quad \{H_2, H_3\} = \frac{6a^2}{N^2} \bar{H}_2 - \tilde{\Phi} \bar{H}_4 \\
\{H_1, H_3\} &= 0 \quad \{H_2, H_4\} = -\frac{3a^2}{N^2} \bar{H}_2 \\
\{H_1, H_4\} &= -\frac{1}{2} \bar{H}_4 \quad \{H_3, H_4\} = \frac{1}{\tilde{\Phi}} \bar{H}_2 - \tilde{\Phi} \bar{H}_3 - \bar{\Phi}, \quad (130)
\end{align*}
$$

what verifies that $\mathcal{H}_{A^\prime} = \{H_3, H_4\}$ are non-involutive Hamiltonians. From (76) and (124) we can find the regular submatrix embedded in the partitioned matrix (121) as well as its inverse

$$
M = \tilde{\Phi} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \text{M}^{-1} = \frac{1}{\tilde{\Phi}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

(131)

respectively. Under these conditions the complete dynamics of the theory is dictated by the fundamental differential (76) where the corresponding GB is given by

$$
\{F, G\}^* = \{F, G\} - \frac{1}{\tilde{\Phi}} \{F, H_3\} \{H_4, G\} - \frac{1}{\tilde{\Phi}} \{F, H_4\} \{H_3, G\}.
$$

(132)

Once we have reduced the phase space by constructing the appropriate GB, (132), we can extract physical information of the theory. In this spirit, the non-vanishing fundamental GB between the phase space variables are
Clearly, the pair \((t, p_i)\) is the unique canonical one so we have the presence of only a physical degree of freedom in this theory. It is worthwhile to mention that the Hamiltonians \(H_\alpha\) obey a truncated Virasoro algebra as discussed in [22, 26, 27].

1. Characteristic equations

Guided by \(120, 132\) and \(80, 83\) the evolution along the complete set of parameters is given as follows. First,
\[
dt = \dot{t} d\tau - \Phi \dot{t} dt^2 \\
\da = \dot{a} d\tau - \Phi \dot{a} dt^2
\]
where these transformations are taken at constant time, \(\delta t^2\). It is therefore inferred, from \(87\), that the gauge transformations are given by
\[
\delta_G z^A = \{z^A, G\} = \{z^A, H_\alpha\} \delta t^\alpha.
\]
From \(124\) we have that
\[
\delta_G t = -\Phi \dot{t} dt^2 \\
\delta_G a = -\Phi \dot{a} dt^2
\]
Similarly, regarding the momenta, we obtain
\[
\delta_G p_t = 0 \\
\delta_G p_a = -\frac{\alpha T \Theta}{N} \delta t^2,
\]
and
\[
\delta_G P_t = -\frac{1}{2} P_t \delta t^1 - \left[ 2 \Theta P_a - 2 \Phi p_t - \frac{a^2}{N} \Theta \right] \delta t^2 \\
\delta_G P_a = -\frac{1}{2} P_a \delta t^1 - 2 \left( \Theta P_t - \Phi p_a + \frac{a\ddot{a}}{N} \Phi a^2 \Lambda^2 \right) \delta t^2
\]
Therefore, by considering the definitions \(\epsilon_2 := \Phi \delta t^2\) and \(2 \epsilon_1 := \delta t^1\), one is able to show that \(\delta L = 0\) whenever \(\epsilon_2(\tau)\) is vanishing at the extrema located at \(\tau = \tau_1\) and \(\tau = \tau_2\). This brane model has recently been studied also using an HJ approach but, using auxiliary variables where the number of Hamiltonians grows enormously. 22. Regarding the gauge symmetries, it was shown in 22 that, by considering \(\epsilon_1\) and \(\epsilon_2\), \(\delta_G t\) and \(\delta_G a\) reflect the presence of the invariance under reparameterizations of the model \(113\) while \(\delta_G t\) and \(\delta_G a\) reflect the presence of an inverse Lorentz-like transformation in the velocities.

VI. CONCLUDING REMARKS

In this paper we have analyzed the integrability of the linearly acceleration-dependent theories within the Hamilton-Jacobi framework for second-order singular...
systems. Our construction entails mainly two different scenarios according to the nature of the equations of motion. In both cases it was shown the presence of a GB structure as a consequence that the original theory, from the beginning, contains non-involutive constraints. Unlike the Dirac-Bergmann approach for constrained systems, within the HJ framework it is not mandatory to split the constraints into first- and second-class although they are closely related.

On the other hand, when the partitioned matrix turns out to be singular and when the involutive constraints can be identified, these determine the so-called characteristic flows which are in connection with the gauge transformations that leave the action invariant. Along this line, in order to illustrate our HJ development some theories were considered where the obtained results are in agreement with previous analyzes. Although is out of the scope of the paper, the obtaining of the Hamilton principal function is an element that need to be explored in detail. Referring to this, despite that the abstract procedure is clear regarding the calculation process, one should always be cautious. On the one hand, one may to use the PDE techniques in order to solve the HJPDE or, on the other hand, when one knows the form of the coordinates and momenta as functions of the parameters it is possible to perform the integration of the corresponding characteristic equation which in general is rather involved. Attempts in this direction have been done in . The analysis here achieved has been carried out, for simplicity, for systems with a finite number of degrees of freedom. For field theories we believe that it is possible to extend our analysis as long as we are careful with the functional analysis. In a sense, our development pave the way to be applied to relativistic second-order geometric systems characterized by a linear dependence on the coordinates and momenta as functions of the parameters it turns out to be singular and when the involutive constraints can be identified, these determine the so-called characteristic flows which are in connection with the gauge transformations that leave the action (14) invariant.

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**Appendix A: On the matrix $M_{\mu \nu}$**

The matrix $M_{\mu \nu}$ contains important geometric information so it is not just a notational resource. This is nothing but the Hessian matrix associated to a first-order Lagrangian, $L_d$, where a surface term, $L_s$, in the Lagrangian 2 is identified. To prove this, suppose that we are able to write the Lagrangian 2 in terms of a dynamical Lagrangian and a surface Lagrangian as follows

$$L(q^\mu, \dot{q}^\mu, \ddot{q}^\mu, t) = L_d(q^\mu, \dot{q}^\mu, t) + L_s(q^\mu, \dot{q}^\mu, \ddot{q}^\mu, t), \quad (A1)$$

where

$$L_s := \frac{dh(q^\mu, \dot{q}^\mu, t)}{dt}, \quad (A2)$$

for a smooth function $h(q^\mu, \dot{q}^\mu, t)$. From the Ostrogradski-Hamilton viewpoint, the momenta are defined by the Hessian matrix associated to a first-order dynamical Lagrangian and a surface Lagrangian as follows

$$\mu = K_{\mu},$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}^\mu} - \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{q}^\mu} \right) = \frac{\partial L_d}{\partial \dot{q}^\mu} + \frac{\partial L_s}{\partial \dot{q}^\mu} - \frac{d}{dt} \left( \frac{\partial L_s}{\partial \ddot{q}^\mu} \right) = -\frac{\partial V}{\partial \dot{q}^\mu} - \frac{\partial K_{\mu \nu}}{\partial \dot{q}^\nu} \dot{q}^\nu. \quad (A4)$$

Clearly, the momenta conjugate to the coordinates can be written in terms of the corresponding momenta arising from $L_d$ and $L_s$.

$$p_\mu = p_{d \mu} + p_{s \mu}.$$ 

On the one hand we have that

$$\frac{\partial p_{d \mu}}{\partial \dot{q}^\nu} \frac{\partial p_{d \mu}}{\partial \dot{q}^\nu} + \frac{\partial p_{s \mu}}{\partial \dot{q}^\nu} = W_{\mu \nu} + \frac{\partial p_{s \mu}}{\partial \dot{q}^\nu}, \quad (A5)$$

where $W_{\mu \nu} := \frac{\partial p_{d \mu}}{\partial \dot{q}^\nu}$ is the Hessian matrix associated with the Lagrangian $L_d$. On the other hand, we have that

$$\frac{\partial p_{s \mu}}{\partial \dot{q}^\nu} = \frac{\partial}{\partial \dot{q}^\nu} \left( -\frac{\partial V}{\partial \dot{q}^\nu} - \frac{\partial K_{\mu \nu}}{\partial \dot{q}^\nu} \dot{q}^\nu \right) = \frac{\partial K_{\nu \mu}}{\partial \dot{q}^\nu} - M_{\mu \nu}, \quad (A6)$$

where we have substituted defining the matrix $M_{\mu \nu}$. Hence, by considering \(A3\), from \(A5\) and \(A6\) it follows that $M_{\mu \nu} = -W_{\mu \nu}$, provided that

$$\frac{\partial K_{\nu \mu}}{\partial \dot{q}^\nu} = \frac{\partial p_{s \mu}}{\partial \dot{q}^\nu}. \quad (A7)$$

We have therefore proved that, when a surface term is existing in the Lagrangian, the matrix $M_{\mu \nu}$ is nothing but the Hessian matrix associated to the dynamical Lagrangian $L_d$ whenever the relationship \(A7\) holds.
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