Covering convex bodies by cylinders and lattice points by flats *

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Abstract

In connection with an unsolved problem of Bang (1951) we give a lower bound for the sum of the base volumes of cylinders covering a d-dimensional convex body in terms of the relevant basic measures of the given convex body. As an application we establish lower bounds on the number of k-dimensional flats (i.e. translates of k-dimensional linear subspaces) needed to cover all the integer points of a given convex body in d-dimensional Euclidean space for $1 \leq k \leq d - 1$.

1 Introduction

In a remarkable paper [Ba] Bang has given an elegant proof of the plank conjecture of Tarski showing that if a convex body is covered by finitely many planks in d-dimensional Euclidean space, then the sum of the widths of the planks is at least as large as the minimal width of the body. A celebrated extension of Bang’s theorem to d-dimensional normed spaces has been given by Ball in [B3]. In his paper Bang raises also the important related question whether the sum of the base areas of finitely many cylinders covering a 3-dimensional convex body is at least half of the minimum area of a 2-dimensional projection of the body. If true, then Bang’s estimate is

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sharp due to a covering of a regular tetrahedron by two cylinders described in [Ba]. We investigate this challenging problem of Bang in $d$-dimensional Euclidean space. Our main result is Theorem 3.1 presented and proved in Section 3. As a special case, we get that the sum of the base areas of finitely many cylinders covering a 3-dimensional convex body is always at least one third of the minimum area 2-dimensional projection of the body.

In [BeH] Bezdek and Hausel has established a discrete version of Tarski’s plank problem by asking for the minimum number of hyperplanes that can cover the integer points within a convex body in $d$-dimensional Euclidean space. Theorem 5.1 of Section 5 gives an improvement of their result, which under some conditions improves also the corresponding estimate of Talata [Ta]. A related but different problem of covering the lattice points within a convex body by linear subspaces was investigated in [BarHPT]. Last but not least, Theorem 3.1 combined with some additional ideas leads to a lower bound on the number of $k$-dimensional flats (i.e. translates of $k$-dimensional linear subspaces) needed to cover all the integer points of a given convex body in $d$-dimensional Euclidean space for $1 \leq k \leq d - 1$. This is the topic of Section 4 and its main result, Theorem 4.1, actually improves the corresponding estimate of Talata [Ta].

2 Notation

In this paper we identify a $d$-dimensional affine space with $\mathbb{R}^d$. By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^d$. The canonical Euclidean ball and sphere in $\mathbb{R}^d$ are denoted by $B_2^d$ and $S^{d-1}$. By a subspace we always mean a linear subspace.

By a convex body in $\mathbb{R}^d$ we always mean a compact convex set with non-empty interior. The interior of $K$ is denoted by $\text{int} K$. Let $K \subset \mathbb{R}^d$ be a convex body with the origin 0 in its interior. We denote by $K^o$ the polar of $K$, i.e.

$$K^o = \{ x \mid \langle x, y \rangle \leq 1 \text{ for every } y \in K \}.$$  

The Minkowski functional of $K$ (or the gauge of $K$) is

$$\|x\|_K = \inf \{ \lambda > 0 \mid x \in \lambda K \}. $$

If $K$ is a centrally symmetric convex body with its center of symmetry at the origin, then $\|x\|_K$ defines a norm on $\mathbb{R}^d$ with the unit ball $K$. 

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The *Banach-Mazur distance* between two convex bodies $K$ and $L$ in $\mathbb{R}^d$ is defined by
\[
d(K, L) = \inf \{ \lambda > 0 \mid a \in L, b \in K, L - a \subset T(K - b) \subset \lambda (L - a) \},\]
where the infimum is taken over all linear operators $T : \mathbb{R}^d \to \mathbb{R}^d$. The Banach-Mazur distance between $K$ and the closed Euclidean ball $B_{d/2}$ (say, of unit radius) we denote by $d_K$. As it is well-known, John’s Theorem ([J]) implies that for every $K$, $d_K$ is bounded by $d$, while for centrally-symmetric convex body $K$, $d_K \leq \sqrt{d}$ (see e.g. [B1]).

Given a convex body $K$ in $\mathbb{R}^d$ we denote its distance to symmetric bodies by
\[
 sd_K := \inf \{ \lambda > 0 \mid a \in \mathbb{R}^d, -(K - a) \subset \lambda (K - a) \}. \tag{1} \]
Clearly, $sd_K \leq d_K \leq d$. In fact, $sd_K$ is one of the ways to measure the asymmetry of the convex body $K$. We refer to [Gr] for the related discussion.

Let $K$ be a convex body in $\mathbb{R}^d$. We denote its volume by $\text{vol}(K)$. When we would like to emphasize that we take $d$-dimensional volume of a body in $\mathbb{R}^d$ we write $\text{vol}_d(K)$.

Given a linear subspace (in short, a subspace) $E \subset \mathbb{R}^d$ we denote the orthogonal projection on $E$ by $P_E$ and the orthogonal complement of $E$ by $E^\perp$. We will use the following theorem, proved by Rogers and Shephard ([RS], see also [C] and Lemma 8.8 in [Pi1]).

**Theorem 2.1** Let $1 \leq k \leq d - 1$. Let $K$ be a convex body in $\mathbb{R}^d$ and $E$ be a $k$-dimensional subspace of $\mathbb{R}^d$. Then
\[
 \max_{x \in \mathbb{R}^d} \text{vol}_{d-k} (K \cap (x + E^\perp)) \text{vol}_k(P_EK) \leq \binom{d}{k} \text{vol}_d(K).
\]

**Remark.** Note that the reverse estimate
\[
 \max_{x \in \mathbb{R}^d} \text{vol}_{d-k} (K \cap (x + E^\perp)) \text{vol}_k(P_EK) \geq \text{vol}_d(K)
\]
is a simple application of the Fubini Theorem and is correct for any measurable set $K$ in $\mathbb{R}^d$.

We will be using the following parameters of a convex body $K$ with 0 in its interior
\[
 M(K) := \int_{S^{d-1}} \|x\|_K \ d\sigma(x),
\]
where \( \sigma \) denotes the normalized Lebesgue measure on \( S^{d-1} \), \( M^*(K) := M(K^c) \), and

\[
M M^*(K) := \inf M(T(K - a)) M^*(T(K - a)),
\]

where the infimum is taken over all invertible linear maps \( T : \mathbb{R}^d \to \mathbb{R}^d \) and all \( a \) in the interior of \( K \). Note that \( M^*(K) \) is the half of mean width of \( K \). Below we need the following theorem.

**Theorem 2.2** There exist absolute positive constants \( C \) and \( \alpha \) such that for every \( d \geq 1 \) and every convex body \( K \) in \( \mathbb{R}^d \) one has

\[
M M^*(K) \leq C d^{1/3} \ln^\alpha (d + 1).
\]

Moreover, if \( K \) is centrally symmetric then

\[
M M^*(K) \leq C \ln (d + 1).
\]

The second estimate in this theorem is a well-known fact from Asymptotic Theory of finite dimensional normed spaces (see, e.g., [Pi1, To]). In fact, it is a combination of results by Lewis ([L]), by Figiel and Tomczak-Jaegermann ([FT]) with a deep theorem by Pisier on the so-called Rademacher projection ([Pi2]). The result in the general case is due to Rudelson ([Rud]). The both estimates of the theorem plays an essential role in the Asymptotic Theory.

The lattice width of a convex body \( K \) in \( \mathbb{R}^d \) is defined as

\[
w(K, \mathbb{Z}^d) = \min \left\{ \max_{x \in K} \langle x, y \rangle - \min_{x \in K} \langle x, y \rangle \mid y \in \mathbb{Z}^d, y \neq 0 \right\}.
\]

Note that, if the origin is in the interior of \( K \), then

\[
w(K, \mathbb{Z}^d) = \min \left\{ \|y\|_{K^c} + \|- y\|_{K^c} \mid y \in \mathbb{Z}^d, y \neq 0 \right\}.
\]

The flatness parameter of \( K \) is defined as

\[
\text{Flt}(K) = \sup w(TK, \mathbb{Z}^d),
\]

where the supremum is taken over all invertible affine maps \( \mathbb{R}^d \to \mathbb{R}^d \) satisfying \( TK \cap \mathbb{Z}^d = \emptyset \). The following theorem was proved in [Ban] for the centrally symmetric case and the case of an ellipsoid, and in [BanLPS] for the general case. It improves the previous bound by Kannan and Lovász ([KL]), who showed \( \text{Flt}(K) \leq Cd^2 \).
Theorem 2.3 There exist absolute positive constants $C$ and $c$ such that for every $d \geq 1$ and every convex body $K$ in $\mathbb{R}^d$ one has

$$cd \leq \text{Flt}(K) \leq CdM^*(K).$$

Moreover, $\text{Flt}(K) \leq d$ if $K$ is an ellipsoid.

3 Covering by cylinders

In this section we introduce a volumetric parameter related to covering by cylinders and provide corresponding estimates.

By a cylinder in $\mathbb{R}^d$ we always mean a 1-codimensional cylinder, that is, a set $C \subset \mathbb{R}^d$ that can be presented as $C = \ell + B$, where $\ell$ is a line containing 0 in $\mathbb{R}^d$ and $B$ is a measurable set in $E := \ell ^\perp$. Let $K \subset \mathbb{R}^d$ be a convex body and $C \subset \mathbb{R}^d$ be a cylinder. The cross-sectional volume of $C$ with respect to $K$ we denote by

$$\text{crv}_K(C) := \frac{\text{vol}_{d-1}(C \cap E)}{\text{vol}_{d-1}(P_E K)}.$$ 

It is easy to see that for every $(d - 1)$-dimensional subspace $H \subset \mathbb{R}^d$ not containing $\ell$ one has

$$\text{crv}_K(C) = \frac{\text{vol}_{d-1}(C \cap H)}{\text{vol}_{d-1}(PK)},$$

where $P$ is the projection on $H$ with the kernel $\ell$. We would also like to notice that for every invertible affine map $T : \mathbb{R}^d \to \mathbb{R}^d$ one has $\text{crv}_K(C) = \text{crv}_{TK}(TC)$.

Theorem 3.1 Let $K$ be a convex body in $\mathbb{R}^d$. Let $C_1, \ldots, C_N$ be cylinders in $\mathbb{R}^d$ such that

$$K \subset \bigcup_{i=1}^N C_i.$$ 

Then

$$\sum_{i=1}^N \text{crv}_K(C_i) \geq \frac{1}{d}.$$
Moreover, if $K$ is an ellipsoid then

$$\sum_{i=1}^{N} \text{crv}_K(C_i) \geq 1.$$  

**Proof:** In this proof we denote $v_n := \text{vol}_n(B^n_2)$. Every $C_i$ can be presented as $C_i = \ell_i + B_i$, where $\ell_i$ is a line containing 0 in $\mathbb{R}^d$ and $B_i$ is a body in $E_i := \ell_i^\perp$.

We first prove the theorem for ellipsoids. Since $\text{crv}_K(C) = \text{crv}_{TK}(TC)$ for every invertible affine map $T : \mathbb{R}^d \to \mathbb{R}^d$, we may assume that $K = B^d_2$. Then

$$\text{crv}_K(C_i) = \frac{\text{vol}_{d-1}(B_i)}{v_{d-1}}.$$

Consider the following (density) function on $\mathbb{R}^d$

$$p(x) = 1/\sqrt{1 - |x|^2}$$

for $|x| < 1$ and $p(x) = 0$ otherwise. The corresponding measure on $\mathbb{R}^d$ we denote by $\mu$, that is $d\mu(x) = p(x)dx$. Let $\ell$ be a line containing 0 in $\mathbb{R}^d$ and $E = \ell^\perp$. It follows from direct calculations that for every $z \in E$ with $|z| < 1$

$$\int_{\ell+z} p(x) \, dx = \pi.$$

Thus, we have

$$\mu(B^d_2) = \int_{B^d_2} p(x) \, dx = \int_{B^d_2 \cap E} \int_{\ell+z} p(x) \, dx \, dz = \pi \cdot v_{d-1}$$

and for every $i \leq N$

$$\mu(C_i) = \int_{C_i} p(x) \, dx = \int_{B_i} \int_{\ell_i+z} p(x) \, dx \, dz = \pi \cdot \text{vol}_{d-1}(B_i).$$

Since $B^d_2 \subset \bigcup_{i=1}^{N} C_i$, we obtain

$$\pi \cdot v_{d-1} = \mu(B^d_2) \leq \mu\left(\bigcup_{i=1}^{N} C_i\right) \leq \sum_{i=1}^{N} \mu(C_i) = \sum_{i=1}^{N} \pi \cdot \text{vol}_{d-1}(B_i).$$
It implies
\[ \sum_{i=1}^{N} \text{crv}_{B^2_d}(C_i) = \sum_{i=1}^{N} \frac{\text{vol}_{d-1}(B_i)}{v_{d-1}} \geq 1. \] (2)

Now, we show the general case. For \( i \leq N \) denote \( \bar{C}_i = C_i \cap \mathbf{K} \) and note that
\[ \mathbf{K} \subset \bigcup_{i=1}^{N} \bar{C}_i \quad \text{and} \quad P_E \bar{C}_i = B_i \cap P_E \mathbf{K}. \]
Since \( \bar{C}_i \subset \mathbf{K} \) we have also
\[ \max_{x \in \mathbb{R}^d} \text{vol}_1 \left( \bar{C}_i \cap (x + \ell_i) \right) \leq \max_{x \in \mathbb{R}^d} \text{vol}_1 \left( \mathbf{K} \cap (x + \ell_i) \right). \]
Therefore, applying Theorem 2.1 (and Remark after it, saying that we don’t need convexity of \( \bar{C}_i \)) we obtain for every \( i \leq N \)
\[ \text{crv}_{\mathbf{K}}(C_i) = \frac{\text{vol}_{d-1}(B_i)}{\text{vol}_{d-1}(P_E \bar{C}_i)} \geq \frac{\text{vol}_{d-1}(P_E \bar{C}_i)}{\text{vol}_{d-1}(P_E \mathbf{K})} \]
\[ \geq \frac{\text{vol}_d(C_i)}{\max_{x \in \mathbb{R}^d} \text{vol}_1 \left( C_i \cap (x + \ell_i) \right)} \frac{\text{vol}_1 \left( K \cap (x + \ell_i) \right)}{d \text{vol}_d(K)} \geq \frac{\text{vol}_d(C_i)}{d \text{vol}_d(K)}. \]
Using that \( \bar{C}_i \)'s covers \( \mathbf{K} \), we observe
\[ \sum_{i=1}^{N} \text{crv}_{\mathbf{K}}(C_i) \geq \frac{1}{d}, \]
which completes the proof. \( \square \)

**Remark 1.** If \( \mathbf{K} \) is close to the Euclidean ball (and \( d \) is not very big), then the following estimate can be better than the general one
\[ \sum_{i=1}^{N} \text{crv}_{\mathbf{K}}(C_i) \geq \frac{1}{d^{d-1}}. \]
It can be obtained as follows: Using that \( \text{crv}_{\mathbf{K}}(C) = \text{crv}_{T \mathbf{K}}(TC) \) for an invertible affine transformation, we may assume that \( B^d_2 \) is a distance ellipsoid for \( \mathbf{K} \), namely assume that \( B^d_2 \subset \mathbf{K} \subset d_{\mathbf{K}} B^d_2 \). Then
\[ \sum_{i=1}^{N} \text{crv}_{\mathbf{K}}(C_i) = \sum_{i=1}^{N} \frac{\text{vol}_{d-1}(B_i)}{\text{vol}_{d-1}(P_E \mathbf{K})} \geq \sum_{i=1}^{N} \frac{\text{vol}_{d-1}(B_i)}{\text{vol}_{d-1}(P_E, d_{\mathbf{K}} B^d_2) \geq \frac{1}{d^{d-1}}}. \]
\[ \geq d_K^{d+1} \sum_{i=1}^{N} \text{crv}_{B_2}(C_i) \geq d_K^{d+1} \]

(in the last inequality we used “moreover” part of Theorem 3.1). Recall that 
\( d_K \leq \sqrt{d} \) for any centrally symmetric convex body \( K \) in \( \mathbb{R}^d \) and \( d_K \leq d \) in general. Thus, if \( d = 3 \) and \( K \) is a centrally-symmetric convex body close to the Euclidean ball, then this estimate is better than the general one given by Theorem 3.1.

**Remark 2.** Note that the proof of Theorem 3.1 can be extended to the case of cylinders of other dimensions. Indeed, given \( k < d \) define a \( k \)-codimensional cylinder \( C \) as a set which can be presented in the form \( C = H + B \), where \( H \) is a \( k \)-dimensional subspace of \( \mathbb{R}^d \) and \( B \) is a measurable set in \( E := H^\perp \). As before, given a convex body \( K \) and a \( k \)-codimensional cylinder \( C = H + B \) denote

\[ \text{crv}_K(C) := \frac{\text{vol}_{d-k}(C \cap E)}{\text{vol}_{d-k}(P_EK)} = \frac{\text{vol}_{d-k}(P_EC)}{\text{vol}_{d-k}(P_EK)} = \frac{\text{vol}_{d-k}(B)}{\text{vol}_{d-k}(P_EK)}. \]

Repeating the proof of Theorem 3.1 (the general case), we obtain that if a convex body \( K \) is covered by \( k \)-codimensional cylinders \( C_1, \ldots, C_n \), then

\[ \sum_{i=1}^{N} \text{crv}_K(C_i) \geq \frac{1}{\binom{d}{k}}. \]

As was noted by Bang ([Ba]), the case \( k = d - 1 \) here corresponds to the “plank problem”, indeed, in this case we have the sum of relative widths of the body. As we mentioned in the introduction, Ball ([B3]) proved that such sum should exceed 1 in the case of centrally symmetric body \( K \), while the general case is still open. Our estimate implies the lower bound \( 1/d \). Of course, Ball’s Theorem implies the estimate \( 1/sd_K \).

## 4 Covering lattice points by lines and flats

**Theorem 4.1** Let \( K \) be a convex body in \( \mathbb{R}^d \) containing the origin in its interior. Let \( \ell_1, \ldots, \ell_N \) be lines in \( \mathbb{R}^d \) such that

\[ K \cap \mathbb{Z}^d \subset \bigcup_{i=1}^{N} \ell_i. \]
Then
\[ N \geq \left( \frac{w(K \cap -K, Z^d)}{Cd MM^* (K \cap -K)} \right)^{d-1} \geq \left( \frac{w(K \cap -K, Z^d)}{C_0 d \ln(d+1)} \right)^{d-1}, \]
where \( C \) and \( C_0 \) are absolute positive constants. If, in addition, \(-K \subset sd_K K\) (that is, if infimum in (1) attains at \( a = 0 \)), then
\[ N \geq \left( \frac{w(K, Z^d)}{Csd_{K} d MM^* (K)} \right)^{d-1} \geq \left( \frac{w(K, Z^d)}{C_0 d \ln^a(d+1)} \right)^{d-1}, \]
where \( C, C_0, \) and \( \alpha \) are absolute positive constants.

Moreover, if \( K \) is an ellipsoid centered at the origin, then
\[ N \geq C_{d} \frac{d}{2d} \]

Proof: Let \( \lambda > 0 \) be such that
\[ K \subset \bigcup_{i=1}^{N} (\ell_i + \lambda K) \quad \text{and} \quad K \not\subset \bigcup_{i=1}^{N} (\ell_i + \lambda \text{int}(K)). \]

Since \( 0 \in K \), we have \( 0 \in l_i \) for some \( i \), which clearly implies that \( \lambda \leq 1 \).

For \( i \leq N \) let \( H_i \) denote the \((d-1)\)-dimensional subspace orthogonal to \( \ell_i \) and let \( P_i \) denote the orthogonal projection on \( H_i \). We define
\[ C_i := \ell_i + \lambda K = \ell_i + \lambda P_i K. \]

Then crv\(_K\)(\( C_i \)) = \( \lambda^{d-1} \). Theorem 3.1 implies \( N \geq c^d \lambda^{-d+1} \), where \( c \) is a positive absolute constant.

Now, \( K \not\subset \bigcup_{i=1}^{N} (\ell_i + \lambda \text{int}(K)) \) if and only if there exists \( x \in K \) such that for every \( i \leq N \) one has \( x \not\in \ell_i + \lambda \text{int}(K) \), i.e. \( (x - \lambda \text{int}(K)) \cap \ell_i = \emptyset \). Let \( y = (1 - \lambda/2)x \). By convexity of \( K \) we have
\[ \left( y + \frac{\lambda}{2} (K \cap -K) \right) \subset K \cap (x - \lambda \text{int}(K)). \]

Since \( K \cap Z^d \subset \bigcup_{i=1}^{N} \ell_i \), we obtain
\[ \left( y + \frac{\lambda}{2} (K \cap -K) \right) \cap Z^d = \emptyset. \]
Using Theorem 2.3 (and, if needed, approximating $\lambda$ by $\lambda - \varepsilon$ with small enough $\varepsilon$), we observe

$$\frac{\lambda}{2} \left( w(K \cap -K, \mathbb{Z}^d) - y + \frac{\lambda}{2} (K \cap -K), \mathbb{Z}^d \right)$$

$$\leq \text{Flt}(K \cap -K) \leq C d MM^* (K \cap -K),$$

where $C$ is an absolute constant. Thus,

$$N \geq c^d \lambda^{-d+1} \geq c^d \left( \frac{w(K \cap -K, \mathbb{Z}^d)}{2Cd MM^* (K \cap -K)} \right)^{d-1}.$$

This shows the left-hand side of the first estimate. The right-hand side follows by Theorem 2.2. Note that in the case of ellipsoid we have $C = c = 1$, $MM^* (K \cap -K) = 1$, which implies the “moreover” part of the theorem.

The second estimate follows the same lines. For the sake of completeness we sketch it. Let $0 < \lambda \leq sd_K$ be such that

$$K \subset \bigcup_{i=1}^N (\ell_i - 2\lambda K) \quad \text{and} \quad K \notin \bigcup_{i=1}^N (\ell_i - \lambda \text{int}K).$$

Repeating arguments of the first part we obtain that $N \geq c^d \lambda^{-d+1}$ and $(x + \lambda \text{int}K) \cap \ell_i = \emptyset$ for every $i \leq N$. Convexity of $K$ and the inclusion $-K \subset sd_KK$ yields for $y = (1 - \lambda/(sd_K + 1))x$

$$\left( y + \frac{\lambda}{sd_K + 1} \text{intK} \right) \subset K \cap (x + \lambda \text{int}K).$$

It implies

$$\left( y + \frac{\lambda}{sd_K + 1} \text{intK} \right) \cap \mathbb{Z}^d = \emptyset$$

and, by Theorem 2.3,

$$\frac{\lambda}{sd_K + 1} w(K, \mathbb{Z}^d) \leq C_1 d MM^* (K).$$

Therefore,

$$N \geq c^d \lambda^{-d+1} \geq c^d \left( \frac{w(K, \mathbb{Z}^d)}{C_1 (sd_K + 1) d MM^* (K)} \right)^{d-1},$$

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which proves the left-hand estimate (with $C = 2C_1$). Since $sd_K \leq d$, Theorem 2.2 implies the right-hand side inequality. □

**Remark.** It is not difficult to see that the proof above can be extended almost without changes to the case of $k$-dimensional flats instead of lines (one needs to use Remark 2 following Theorem 3.1). In particular, for a centrally symmetric body $K = -K$, whose integer points are covered by the $k$-dimensional flats $H_1, \ldots, H_N$ we have

$$N \geq \left( \frac{w(K, \mathbb{Z}^d) (d - k)}{C d^2 \ln(d + 1)} \right)^{d-k}.$$ 

We omit the details and precise estimates in the non-symmetric case.

## 5 Covering lattice points by hyperplanes

The following theorem improves the estimate of the Remark after Theorem 4.1 in the case $k = d - 1$, extending a Bezdek-Hausel result from [BeH].

**Theorem 5.1** Let $K$ be a centrally symmetric (with respect to the origin) convex body in $\mathbb{R}^d$. Let $H_1, \ldots, H_N$ be hyperplanes in $\mathbb{R}^d$ such that

$$K \cap \mathbb{Z}^d \subset \bigcup_{i=1}^N H_i.$$

Then

$$N \geq c \frac{w(K, \mathbb{Z}^d)}{d \text{ MM}^*(K)} \geq c_0 \frac{w(K, \mathbb{Z}^d)}{d \ln(d + 1)},$$

where $c, c_0$ are absolute positive constants.

**Proof:** The proof is based on the Ball’s solution of the plank problem. Namely, we use that given a centrally symmetric body $K \subset \mathbb{R}^d$ and $N$ hyperplanes $H_1, \ldots, H_N$ in $\mathbb{R}^d$ there exists $x \in \mathbb{R}^d$ such that

$$L := x + \frac{1}{N+1} K \subset K$$

and the interior of $L$ is not met by any $H_i$ (see Corollary or abstract in [B3]).
Since all integer points of $K$ are covered by $H_i$’s, we observe that
\[ \text{int} L \cap \mathbb{Z}^d = \emptyset. \]

Applying Theorem 2.3, we obtain
\[
\frac{1}{N+1} \, w \left( K, \mathbb{Z}^d \right) = w \left( L, \mathbb{Z}^d \right) \leq \text{Ft}(K) \leq Cd \, MM^*(K),
\]
where $C$ is an absolute constant. Together with Theorem 2.2 it implies the desired result. □

References

[B1] K. Ball, *Flavors of geometry* in *An elementary introduction to modern convex geometry*, Levy, Silvio (ed.), Cambridge: Cambridge University Press. Math. Sci. Res. Inst. Publ. 31, 1–58 (1997).

[B2] K. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. Lond. Math. Soc., II. Ser. 44 (1991), 351–359.

[B3] K. Ball, *The plank problem for symmetric bodies*, Invent. Math. 104 (1991), 535–543.

[Ba] T. Bang, *A solution of the ”plank problem”*, Proc. Amer. Math. Soc. 2 (1951), 990–993.

[Ban] Banaszczyk, W. *Inequalities for convex bodies and polar reciprocal lattices in $\mathbb{R}^n$ II. Application of $K$-convexity*, Discrete Comput. Geom. 16 (1996), no. 3, 305–311.

[BanLPS] W. Banaszczyk, A. E. Litvak, A. Pajor, S. J. Szarek, *The flatness theorem for nonsymmetric convex bodies via the local theory of Banach spaces*, Math. Oper. Res. 24 (1999), no. 3, 728–750.

[BarHPT] I. Bárány, G. Harcos, J. Pach, G. Tardos, *Covering lattice points by subspaces*, Period. Math. Hung. 43 (2001), 93–103.

[BeH] K. Bezdek, T. Hausel, *On the number of lattice hyperplanes which are needed to cover the lattice points of a convex body*, Böröczky, K. (ed.)
et al., Intuitive geometry. Proceedings of the 3rd international conference held in Szeged, Hungary, 1991. Amsterdam: North-Holland. Colloq. Math. Soc. János Bolyai. 63 (1994), 27–31.

[C] G. D. Chakerian, *Inequalities for the difference body of a convex body*, Proc. Amer. Math. Soc. 18 (1967), 879–884.

[FT] T. Figiel, N. Tomczak-Jaegermann, *Projections onto Hilbertian subspaces of Banach spaces*, Isr. J. Math. 33 (1979), 155–171.

[Gr] B. Grünbaum, *Measures of symmetry for convex sets*. 1963 Proc. Sympos. Pure Math., Vol. VII pp. 233–270 Amer. Math. Soc., Providence, R.I.

[J] F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, 187–204. Interscience Publishers, Inc., New York, N. Y., 1948.

[KL] R. Kannan, L. Lovász, *Covering minima and lattice-point-free convex bodies*, Ann. of Math. 128 (1988), 577–602.

[L] D. R. Lewis, *Ellipsoids defined by Banach ideal norms*, Mathematika 26 (1979), 18–29.

[Pi1] G. Pisier, *Holomorphic semi-groups and the geometry of Banach spaces*, Ann. Math. 115 (1982), 375–392.

[Pi2] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge University Press 1989.

[RS] C. A. Rogers, G. C. Shephard, *Convex bodies associated with a given convex body*, J. London Math. Soc. 33 (1958), 270–281.

[Rud] M. Rudelson, *Distances between non-symmetric convex bodies and the $MM^*$-estimate*, Positivity 4 (2000), 161–178.

[Ta] I. Talata, *Covering the lattice points of a convex body with affine subspaces*, Bolyai Soc. Math. Stud. 6 (1997), 429–440.
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