Symmetry and block structure of the Liouvillian superoperator in partial secular approximation

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We address the structure of the Liouvillian superoperator for a broad class of bosonic and fermionic Markovian open systems interacting with stationary environments. We show that the accurate application of the partial secular approximation in the derivation of the Bloch-Redfield master equation naturally induces a symmetry on the superoperator level, which may greatly reduce the complexity of the master equation by decomposing the Liouvillian superoperator into independent blocks. Moreover, we prove that, if the steady state of the system is unique, one single block contains all the information about it, and that this imposes a constraint on the possible steady-state coherences of the unique state, ruling out some of them. To provide some examples, we show how the symmetry appears for two coupled spins interacting with separate baths, as well as for two harmonic oscillators immersed in a common environment. In both cases the standard derivation and solution of the master equation is simplified, as well as the search for the steady state. The block-diagonalization may not appear when a local master equation is chosen.

I. INTRODUCTION

Open quantum systems are nowadays a well-established framework whose theoretical aspects have been investigated in depth [1][3]. An important branch is represented by Markovian open systems [4][5] and quantum dynamical semigroups [6]. In particular, the generator of a quantum dynamical semigroup is a time-independent Liouvillian superoperator $\mathcal{L}$, such that if $\rho_S(0)$ is the initial state of the system, the state at time $t$ is given by $\rho_S(t) = \exp(\mathcal{L}t)\rho_S(0)$. The solution of the master equation providing the dynamics of the system relies on finding the Liouvillian $\mathcal{L}$.

The evolution of a dynamical semigroup is described by a master equation in the so-called Gorini–Kossakowski–Sudarshan–Lindblad (GKLS) form [7][9]. This form has been extensively studied during the recent years [10][19], with a particular attention on the steady state structure, given the importance of, for instance, steady-state coherences in quantum thermodynamics [20][22] or of information-preserving steady states [23]. The form of the steady state is also crucial to understand the process of quantum thermalization [24]. The investigation of the role of symmetry in the semigroup evolution has been very active as well [25][33]. Given the simplicity of the GKLS master equation, a thorner issue has been to characterize which microscopic physical models of systems and environments lead to a reduced system evolution described by this master equation. In 1965 Redfield derived a Markovian master equation by assuming weak coupling between system and environment and making some considerations about the relevant timescales of the evolution [34]. This derivation and the subsequent Bloch-Redfield master equation are still commonly employed nowadays [1][2][35]. A more formal derivation has been provided by Davies [36][37], showing that the semigroup evolution is perfectly recovered when the coupling between system and environment is infinitesimally small. In some situations, e.g. in the case of two slightly-detuned spins [38][39], the Davies’ limit cannot be performed, since it corresponds to applying a “full secular approximation” removing all the oscillating terms in the interaction picture dynamics without discriminating which of them are fast and which are slow, instead of a more accurate partial secular approximation. The latter was implicitly suggested by Redfield himself [34], and an extensive study about it has been performed in the very recent past [35][39][43], in particular showing that applying an accurate partial secular approximation to the microscopic derivation of the master equation allows one to recover the GKLS form [40][41].

In this work we show how a symmetry on the superoperator level arises due to the partial secular approximation. Our discussion is valid for a broad class of systems, that can be recast as $M$ non-interacting fermionic or bosonic modes weakly coupled to stationary Markovian environments. The symmetry consists in the invariance of the Liouvillian superoperator under the action of the total-number-of-particles superoperator. We stress that the symmetry is on the superoperator level, i.e. it is not a symmetry of the system Hamiltonian, but of the full master equation of the open system. Following the formalism discussed in Ref. [27], we can exploit it to block-diagonalize the Liouvillian superoperator and greatly reduce the complexity of the master equation. Complexity reduction of abstract GKLS master equations in fermionic or bosonic systems has been also addressed in some extensive works by Prosen et al. [12][14][15], while Torres exploited a symmetry of the Hamiltonian to find the solution of master equations without gain [44]. Once having exploited the symmetry to obtain the block-diagonalization, we observe that, if the steady state of the system is unique, one single block contains all the information about it. This not only helps to find it, but also imposes a constraint on the corresponding steady-state coherences. Curiously, the symmetry arises only when considering a global master equation, while it may not be valid anymore.

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when using a local one [39, 43, 46].

We review the derivation of the Bloch-Redfield master equation and subsequent partial secular approximation in Sec. III, as well as the theory of symmetries and conserved quantities in Lindblad master equations. Sec. III is devoted to the discussion of the symmetry on the superoperator level and to the block-diagonalization of the Liouvillian. In particular, Sec. III A discusses the class of system for which our analysis is valid, while Sec. III B presents the main result and Sec. III C its consequences. We provide some illustrative examples of the action of the symmetry in Sec. IV, distinguishing between fermionic and bosonic scenarios. Finally, we conclude in Sec. V with a discussion about our results.

II. FORMAL FRAMEWORK

A. Markovian master equations with partial secular approximation

Let us consider an open quantum system $S$ with associated Hilbert space $\mathcal{H}_S$ of dimension $N$, described at time $t$ by the $N \times N$ density matrix $\rho_S(t)$. $S$ is coupled to an external environment $E$ through the interaction Hamiltonian $\hat{H}_I$, and throughout the work we restrict ourselves to stationary environments. The full system-bath Hamiltonian can be written as:

$$\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_I$$

$$= \hat{H}_S + \hat{H}_E + \mu \sum_{\alpha} \hat{A}_\alpha \otimes \hat{B}_\alpha,$$ (1)

where $\hat{H}_S$ is the free Hamiltonian of the system, $\hat{H}_E$ is the free Hamiltonian of the environment, $\hat{A}_\alpha$ are system operators while $\hat{B}_\alpha$ are bath operators. $\mu$ is a coupling constant with units of energy, and in the weak-coupling limit considered here we assume $\mu$ far smaller than the other characteristic energies of the system. We set $\hbar = 1$, so that the units of measure of time are $[\text{time}] = [\text{energy}]^{-1}$.

We term $|e_n\rangle$ the eigenvectors of the free Hamiltonian of the system, which may be degenerate as well, such that $\hat{H}_S = \sum_n \epsilon_n |e_n\rangle \langle e_n|$. The jump operators of the system are defined as [11]

$$\hat{A}_\alpha(\omega) = \sum_{\epsilon_m : \epsilon_m - \epsilon_n = \omega} |e_n\rangle \langle e_n| \hat{A}_\alpha |e_m\rangle \langle e_m|.$$ (2)

We assume that the open system $S$ follows a Markovian, non-unitary evolution due to the coupling to the stationary environment $E$. The master equation describing a time-independent dynamical semigroup is written as:

$$\frac{d}{dt} \rho_S(t) = \mathcal{L}[\rho_S(t)],$$ (3)

where $\mathcal{L}$ is the Liouvillian superoperator acting on the $N^2$-dimensional Hilbert space $\mathcal{L}$ of the linear operators on $\mathcal{H}_S$, called Liouville space [27], which contains the convex subset of the density matrices. In particular, for the Bloch-Redfield master equation in partial secular approximation (PSA) [1, 2, 39]:

$$\mathcal{L} = -i[\hat{H}_S + \hat{H}_{LS}, \cdot] + \mathcal{D}[\cdot],$$ (4)

where $\hat{H}_{LS}$ is the Lamb-shift Hamiltonian given by:

$$\hat{H}_{LS} = \sum_{\alpha, \beta} \sum_{(\omega, \omega')} S_{\alpha\beta}(\omega, \omega') \hat{A}_\alpha(\omega) \hat{A}_\beta(\omega'),$$ (5)

while the dissipator reads

$$\mathcal{D}[\rho_S] = \sum_{\alpha, \beta} \sum_{(\omega, \omega')} \gamma_{\alpha\beta}(\omega, \omega') \left( \hat{A}_\beta(\omega) \rho_S \hat{A}_\alpha(\omega') - \frac{1}{2} \{ \hat{A}_\alpha(\omega') \hat{A}_\beta(\omega), \rho_S \} \right).$$ (6)

$S_{\alpha\beta}(\omega, \omega')$ and $\gamma_{\alpha\beta}(\omega, \omega')$ are functions of the autocorrelation functions of the bath operators $\hat{B}_\alpha$. The PSA removes all the terms in the summation with frequencies $\omega$ and $\omega'$ such that

$$\exists t^* \text{ such that } |\omega - \omega'|^{-1} \ll t^* \ll \tau_R,$$ (7)

where $\tau_R$ is the relaxation time of the system, i.e. the time in which $\rho_S$ approaches the dynamical equilibrium [1, 39]. We can express Eq. (7) as $\omega - \omega' \neq O_{\tau}(\tau_R^{-1})$, where for convenience we introduce the notation $O_{\tau}$, defined as:

$$x = O_{\tau}(y) \Leftrightarrow \exists t^* \text{ such that } x^{-1} \ll t^* \ll y^{-1}.$$ (8)

In the weak-coupling limit considered here we have

$$\tau_R = O(\mu^{-2}),$$ (9)

being the master equation of the second order in $\mu$ [3]. Appendix A1 discusses why Eq. (3) with Liouvillian in Eq. (4) can be recast in the GKLS form:

$$\mathcal{L}[\rho_S(t)] = -i[\hat{H}'', \rho_S(t)]$$

$$+ \sum_{l=1}^{N^2-1} \hat{F}_l \rho_S(t) \hat{F}_l^\dagger - \frac{1}{2} \{ \hat{F}_l \hat{F}_l^\dagger, \rho_S(t) \},$$ (10)

where $\hat{H}'' = \hat{H}''^\dagger$ is the effective Hamiltonian including the Lamb shift, and $\{ \hat{F}_l \}_{l=1}^{N^2-1}$ are the Lindblad operators [11].

From now on, we will use calligraphic letters (such as $\mathcal{L}$) to indicate superoperators acting on $\mathcal{L}$, while we will use capital letters, with hats when needed to avoid confusion, (such as $\hat{H}$) for operators living in $\mathcal{L}$, which for instance may act on the Hilbert space of the system $\mathcal{H}_S$. The density matrices $\rho_S$ are elements of $\mathcal{L}$ as well. Appendix A2 discusses the language of superoperators in more detail.

1 We refer the reader to Ref. [39] for their precise form.
B. Symmetries and conserved quantities in the Lindblad formalism

In this section we introduce the concepts of symmetries and conserved quantities in the Lindblad formalism following the recent work by Albert and Jiang [27]. Let us assume that the Lindblad evolution of an open system $S$ is described by the Liouvillian superoperator $\mathcal{L}$ as discussed in Sec. II A. Given an observable $\hat{J} = J^\dagger$ acting on $H_5$ and living in $L$, we have the following definitions:

- $\hat{J}$ is a conserved quantity if it is a constant of motion under the non-unitary evolution generated by the master equation, i.e. if $\mathcal{L}^t[\hat{J}(t)] = 0$ for all $t$.

We construct the one-parameter unitary group whose elements are $\hat{U}_\phi = \exp(\im \phi \hat{J})$ with $\phi \in \mathbb{R}$, and then we define the associated superoperators $\mathcal{U}_\phi$ as $\hat{U}_\phi[\hat{O}] = \hat{U}_\phi^\dagger \hat{O} \hat{U}_\phi$, with $\hat{O} \in L$. We can analogously write $\hat{U}_\phi = \exp(\im \phi \hat{J})$, where $\mathcal{J}$ is the superoperator associated to $\hat{J}$ through $\mathcal{J} = [\hat{J}, \cdot]$. In the language of the isomorphism introduced through the tensor product notation in Appendix A.2 we have $\mathcal{J} = \mathbb{I} \otimes \mathbb{I}_N - \mathbb{I}_N \otimes \hat{J}^T$.

- $\hat{J}$ generates a continuous symmetry on the superoperator level if $\mathcal{U}_\phi^L \mathcal{U}_\phi = \mathcal{L}$ for all $\phi$, or equivalently $[\mathcal{J}, \mathcal{L}] = 0$. The continuous symmetry is also called covariance [47–49], given that it corresponds to the equivalence $\hat{U}_\phi^L \mathcal{L}[\rho_S] \hat{U}_\phi = \mathcal{L}[\hat{U}_\phi^L \rho_S \hat{U}_\phi^\dagger]$ for any state of the system $\rho_S$.

If the evolution of the system were unitary and driven only by the Hamiltonian $\hat{H}_S$, according to Noether’s theorem a conserved quantity would always generate a symmetry and vice versa. In the framework of open systems this is no longer true, since for instance a symmetry on the superoperator level not always implies a symmetry on the operator level. In particular, if the master equation is in the Lindblad form as in Eq. (10), we can consider the following three propositions:

(i) $[\hat{J}, \hat{H}_S^\dagger] = [\hat{J}, \hat{F}_I] = 0 \quad \forall I$,

(ii) $\frac{d}{dt}[\hat{J}(t)] = \mathcal{L}^t[\hat{J}(t)] = 0$,

(iii) $\mathcal{U}_\phi^L \mathcal{U}_\phi = \mathcal{L} \forall \phi \in \mathbb{R}$, or equivalently $[\mathcal{J}, \mathcal{L}] = 0$.

Then, we have that (i) implies (ii) and (iii), but no other logical implications are present [27]. This tells us that in order for an observable $\hat{J}$ to both be a conserved quantity (ii) and generate a symmetry (iii), it needs to commute both with the Hamiltonian $\hat{H}_S$ driving the unitary part of the evolution and with each Lindblad operator.

For the purpose of this paper, we are interested in the observable representing the total number of particles in a system: suppose we have a system of $M$ bosonic or fermionic modes; then, the Hilbert space of the system is the tensor product of the Hilbert spaces of the $M$ modes. The total-number-of-particles operator reads

$$\hat{N} = \sum_{k=1}^{M} \hat{n}_k,$$

where $\hat{n}_k$ is the particle number operator of the $k$-th mode. $\hat{N}$ generates the one-parameter group $\hat{U}_\phi = \exp\left(\im \phi \hat{N} \right)$. If we set $\phi = \pi$, we obtain the parity operator:

$$\hat{P} = \exp\left(\im \pi \hat{N} \right).$$

The parity operator satisfies the properties $\hat{P}^2 = \mathbb{I}$ and $\hat{P}^\dagger = \hat{P}$, and as a consequence it only has two eigenvalues, $\pm 1$. Parity is an observable which can generate a discrete symmetry on the superoperator level. In analogy with the definition of a continuous symmetry, we write the parity superoperator as

$$\mathcal{P} = \exp(i\pi \hat{N}),$$

where $\hat{N}$ is defined as

$$\hat{N} = [\hat{N}, \cdot].$$

Equivalently, using the tensor product notation (see Appendix A.2) we have

$$\mathcal{N} = \hat{N} \otimes \mathbb{I} - \mathbb{I} \otimes \hat{N}^T.$$

Being different objects, symmetries and conserved quantities play a different role in the analysis of the evolution of open quantum systems [10, II, 27]. Conserved quantities are of fundamental importance to identify the structure of the space of stationary states of the system [27], related to the problem of finding decoherence-free subspaces [50]. Symmetries can help in simplifying the form of the Liouvillian superoperator, and thus in solving the master equation. Indeed, if we identify a symmetry such that $[\mathcal{J}, \mathcal{L}] = 0$, we can block-diagonalize the Liouvillian with each block labeled by a different eigenvalue of $\mathcal{J}$. As we will see in the next section, this can greatly reduce the complexity of the master equation.

III. THE BLOCK STRUCTURE OF THE LIOUVILLIAN IN PARTIAL SECULAR APPROXIMATION

In this section we will show how, for a broad class of models, the partial secular approximation naturally induces a symmetry on the superoperator level, which can be exploited to simplify the master equation. Note that we can apply the concepts of Sec. II B introduced in the Lindblad formalism, to

2 Discrete symmetries in the Lindblad formalism deserve a separate discussion, and we refer the interested reader to Ref. [27].
the Bloch-Redfield master equation in partial secular approx-
imation, since as explained in Appendix A, the latter can be
brought to the GKLS form.

We start by introducing the suitable class of Hamiltonians
in Sec. IIIA and then we focus on the identification of the
symmetry in Sec. IIIB. Section IIIC discusses a series of in-
teresting applications and consequences of the main result.

A. Delimiting the suitable class of systems

Our analysis applies to all systems that can be cast as the
sum of the free Hamiltonians of $M$ non-interacting bosonic
or fermionic modes, with

$$
\hat{H}_S = \sum_{k=1}^{M} E_k \hat{c}_k \hat{c}_k^\dagger = \sum_{k=1}^{M} E_k \hat{n}_k,
$$

and $E_k$ is the energy quantum of the $k$-th mode.

Eq. (16) describes a broad class of Hamiltonians which are
particularly relevant in the fields of condensed matter and op-
tical physics. For instance, any quadratic Hamiltonian, that is
to say any Hamiltonian of the form

$$
\hat{H}_S = \sum_{j, k=1}^{M} \left( \alpha_{jk} \hat{a}_j^\dagger \hat{a}_k + \beta_{jk} \hat{a}_j \hat{a}_k + h.c. \right),
$$

where $\hat{a}_j$ is an annihilation operator, can be rewritten as a
sum of non-interacting modes as in Eq. (16) [51, 52]. This is
just a sufficient but not necessary condition, since more com-
plex Hamiltonians may be taken into the form of Eq. (17). In
the case of bosons, all $\hat{H}_S$ preserving Gaussian states can be
recast as Eq. (16). These Hamiltonians contain linear and/or
bilinear terms and can be reduced into the form of Eq. (17)
through displacement transformations. Systems of uncoupled
spins can be trivially seen as non-interacting fermions via Jor-
dan–Wigner transformation [51, 53], and thus are suitable for
our discussion. The same holds for interacting spin chains in
which the total number of spin excitations is conserved (see Ap-
pendix C). Two coupled qubits can be transformed into free fermions as well, as discussed in Appendix D while extensions to wider systems of interacting spins (such as the
Heisenberg model) are tricky and must be considered case by
case.

We now set the relevant assumptions on the interaction
Hamiltonian $\hat{H}_I$ in Eq. (1). First of all, recalling that $\mu$ is
the system-bath coupling constant defined in Eq. (1), we set
$E_k \neq O_{t'}(\mu^2) \forall k$, where we have used the notation intro-
duced in Eq. (8). Then, the interaction Hamiltonian is suit-
able for our analysis if at least one of the following conditions
holds:

- **Condition I**. Each system operator $\hat{A}_\alpha$ in Eq. (1) in-
volves only single excitations, that is to say, each $\hat{A}_\alpha$ is a
first-degree polynomial in the creation and annihi-
lation operators $\hat{c}_k$. For instance, $\hat{A}_{\alpha'} = \hat{c}_1 + \hat{c}_2^\dagger$ is
a valid system operator, while $\hat{A}_{\alpha'} = \hat{c}_1^\dagger \hat{c}_2$ or $\hat{A}_{\alpha'} = \hat{c}_1 \hat{c}_2$ are
not.

- **Condition II**. Let us consider the set of energies $K = \{E_k\}_{k=1}^{M}$. Create two new sets by randomly select-
ing some elements of $K$, that can be repeated as well, and term them $X$ and $Y$; assume that they have different
cardinality (number of elements): $|X| \neq |Y|$. Then, we exclude situations such that $\sum_{E_m \in X} E_m = \sum_{E_l \in Y} E_l + O_{t'}(\mu^2)$. This condition can be relaxed
depending on the structure of the system operators in
the interaction Hamiltonian, as we will show in the proof in Appendix B. **Condition II relaxed as explained in Appendix B comprises Condition I together with the assumption $E_k \neq O_{t'}(\mu^2)$.**

B. The symmetry of the partial secular approximation

We will now show that, if the requirements of Sec. IIIA are
satisfied, the number superoperator $\hat{N}$ defined in Eq. (15)
commutes with the Liouvillian in partial secular approxima-
tion Eq. (16), and therefore generates a symmetry on the super-
operator level.

**Proposition 1** (Symmetry). Let $\mathcal{L}$ be the Liouvillian superop-
erator describing the Markovian evolution of a quantum sys-
tem that can be written as a collection of bosonic or fermionic
non-interacting modes. If $\mathcal{L}$ has been derived, starting from
the microscopic model of system+environment, through the
Bloch-Redfield master equation in partial secular approxima-
tion, then it commutes with the number superoperator:

$$
[N, \mathcal{L}] = 0,
$$

provided that one of the conditions I or II on the interaction
Hamiltonian $\hat{H}_I$ discussed in Sec. IIIA holds.

**Proof.** In Appendix B. \qed

Note that Proposition 1 may be considered as the extension
to systems of $M$ modes of the concept of phase-covariant master equation [49, 54–56]. By now, the latter has been
addressed as the problem in which a system of a single qubit follows an open dynamics described by the Liouvill-
ian $\mathcal{L}$ which is covariant under a phase transformation, i.e.

$$
e^{-i\phi\sigma_z} \mathcal{L}[\rho_{S}]e^{i\phi\sigma_z} = \mathcal{L}[e^{-i\phi\sigma_z}\rho_{S}e^{i\phi\sigma_z}],$$

which corresponds to Eq. (18) in the case of a single fermionic mode. Therefore, the symmetry group generated by $\hat{N}$ is isomorphic to $U(1)$. A complete characteriza-
tion of the single-qubit phase-covariant master equation can be found in the supplementary material of Ref. [54].

C. Consequences of the symmetry

In this section we discuss a list of interesting consequences of
the symmetry presented in Proposition 1. We start with a
simple corollary:

**Corollary 1** (Parity). If the conditions for Proposition 1 hold,
then the parity superoperator $\hat{P}$ is a symmetry on the superopera-
tor level as well: $[\hat{P}, \mathcal{L}] = 0$. 

Proof. If the conditions for Proposition 1 hold, then $[\hat{N}, \mathcal{L}] = 0$. But according to Eq. (15) $\mathcal{P} = \exp(i\pi \hat{N})$, thus the parity superoperator must commute with the Liouvillian as well demonstrating the assertion.

Notice that the symmetries in Proposition 1 and Corollary 1 are, in general, only on the superoperator level. Indeed, we are not imposing any further condition on the form of the interaction and on the spectral density of environment, that is to say, the result of Eq. (18) is an interesting consequence of the partial secular approximation only. This includes cases in which the parity of the number of particles (on the operator level) is modified by the interaction with the environment. For instance, the very common decay of a single mode of the electromagnetic field, described as the result of Eq. (18) is an interesting consequence of the partial secular approximation it fulfils Eq. (18).

How can we exploit Eq. (15) for the analysis of the open system? As already mentioned in Sec. [15] the symmetry generated by the number superoperator allows us to block-diagonalize the Liouvillian in a way that is particularly convenient for the solution of the master equation. Indeed, the eigenvectors of $\hat{N}$ in the representation expressed by Eq. (15) are given by the tensor product of the diagonal basis of $\hat{H}_S$ with itself. That is to say, if we rewrite the system Hamiltonian as $\hat{H}_S = \sum_n \epsilon_n \langle e_n \rangle \langle e_n \rangle$, we choose the basis of the space of superoperators $\{ |e_n\rangle \otimes |e_m\rangle \}_{n,m}$. This is exactly the basis we work with when deriving the Bloch-Redfield master equation, since it is the basis in which we write the jump operators [11,29]. Therefore, if we express $\mathcal{L}$ as a matrix in the basis $|e_n\rangle \otimes |e_m\rangle$, and we regroup all the elements of the basis which are eigenvectors of $\hat{N}$ with the same eigenvalue $d$, we naturally find the blocks of the Liouvillian in such basis. Note that $d$ is the difference between the number of particles in the state $|e_n\rangle$ and the number of particles in $|e_m\rangle$. We can express this fact in the following proposition:

**Proposition 2 (Blocks).** In a system of $M$ bosonic or fermionic modes in which Proposition 1 holds, the Liouvillian superoperator can be divided into blocks as $\mathcal{L} = \bigoplus_d \mathcal{L}_d$, where $\mathcal{L}_d$ is the block labeled by the eigenvalue $d$ of $\hat{N}$. Let us write $\mathcal{L}_d$ as a matrix in a basis $\{|e_j\rangle \otimes |e_k\rangle\}_{j,k}$ which spans its space, where $|e_j\rangle$ and $|e_k\rangle$ are eigenvectors of $\hat{H}_S$. Then, if we write $\mathcal{L}_{-d}$ as a matrix in the basis $\{|e_k\rangle \otimes |e_j\rangle\}_{j,k}$ these matrices satisfy $\mathcal{L}_d = \mathcal{L}_{-d}$.

Proof. The Liouvillian can be block-diagonalized thanks to the symmetry expressed by Eq. (18), generated by $\hat{N}$ whose eigenvalues label the blocks. $\mathcal{L}$ describes the dynamics of the density matrix of the system $\rho_S$ as in Eq. (3), but since $(\rho_S)_{jk} = (\rho_S)_{kj}^*$, we have in the chosen bases $\mathcal{L}_d = \mathcal{L}_{-d}$.

Proposition 2 tells us that the symmetry in Eq. (18) not only provides a block division for the Liouvillian, but also reduces the number of independent blocks, e.g. for fermions from $2M + 1$ to $M + 1$. This may greatly simplify the solution of the master equation, which now would live in spaces of lower dimension. We will show in Sec. IV some examples of how this block diagonalization and complexity reduction.

Each block of the Liouvillian superoperator may give us important insight about a certain physical phenomenon of interest. If we know that a given block contains all the relevant information about such phenomenon, we may indeed analyze only this block and neglect all the rest, thus working in a far smaller space than the one in which $\mathcal{L}$ lives. This happens, for instance, in Ref. [27], where two independent blocks of the Liouvillian superoperator describing the decay of two spins (corresponding to the blocks discussed in Proposition 2) contain all the information about two different physical phenomena, namely superradiance and quantum synchronization. Besides, note that all the populations of the state of the system belong to the block $\mathcal{L}_0$.

Finding the unique steady state of a relaxing Lindblad dynamics is another example of the advantages entailed by the block structure of $\mathcal{L}$: a steady state of the open dynamics is a state $\rho_{ss}$ such that $\mathcal{L}[\rho_{ss}] = 0$. It always exists at least one steady state for finite systems [11] and, if it is unique, then the semigroup is relaxing, i.e. any state is driven toward $\rho_{ss}$ for $t \to \infty$, and no oscillating coherence survives.

The unique steady state “lives” in the subspace of the block $\mathcal{L}_0$ only. Indeed, let us call $\Pi_0$ the projector over the eigenspace of $\hat{N}$ associated to the eigenvalue 0. Then, the following proposition holds:

**Proposition 3 (Steady state).** If the conditions for Proposition 1 hold and the semigroup generated by $\mathcal{L}$ is relaxing toward a unique steady state $\rho_{ss}$, then $\Pi_0[\rho_{ss}] = \rho_{ss}$. i.e. the only non-zero elements of the density matrix representing $\rho_{ss}$, in the excitation basis are the ones with equal number of excitations in the ket and in the bra.

Proof. Let us suppose that the steady state $\rho_{ss}$ has a non-zero component in a subspace projected by $\Pi_d$ with $d \neq 0$: $\Pi_d[\rho_{ss}] \neq 0$. Coming back to the space of density matrices, this means that the density matrix of the steady state in the excitation basis has some non-zero elements with different number of particles in the bra and in the ket. Therefore, there exists a block $\mathcal{L}_d$ with $d \neq 0$ having a zero eigenvalue. Furthermore, the block $\mathcal{L}_0$ must have a zero eigenvalue as well, since for $\rho_{ss}$ to be a physical state it must possess diagonal elements. We now build a new state $\rho_{ss}'$ such that $\Pi_0[\rho_{ss}'] = \Pi_0[\rho_{ss}]$ and $\Pi_k[\rho_{ss}'] = 0$ for all $k \neq 0$. $\rho_{ss}'$ is a physical state (since we have obtained it by removing coherences from $\rho_{ss}$) and is a steady state as well, since it has the same elements of $\rho_{ss}$ in the space projected by $\Pi_0$ whose evolution must be independent from the one of the elements in the space projected by $\Pi_d$. Therefore, the steady state is not unique anymore and we have proven the assertion by contradiction.

Proposition 3 implies the corollary that $\mathcal{L}_0$ is the only block having an eigenvalue equal to zero, while all the eigenvalues of the remaining blocks have negative real part. Another immediate consequence is the following:

**Corollary 2 (Steady-state coherences).** If the conditions for Proposition 1 hold and the semigroup generated by $\mathcal{L}$ is relaxing toward a unique steady state, then the only non-zero
steady-state coherences in the excitation basis must have the same number of excitations in the ket and in the bra.

Proposition 3 is telling us that, when the semigroup dynamics is relaxing toward a unique steady state as it is often the case, we only need to find the eigenvalues and eigenvectors of the block $L_0$ to characterize the stationary state. In particular, this restricts the range of possible steady-state coherences in which we may be interested, e.g. for thermodynamics tasks. Proposition 3 does not give information about scenarios with a broader space of steady states, such as in the presence of decoherence free subspaces and/or oscillating coherences. Further studies are needed toward this direction.

Finally, let us comment that Proposition 1 and Eq. (18) may not be valid if we choose the local approach to derive the master equation [39,45,46] of a system composed of interacting subsystems (that can be rewritten as non-interacting normal modes). Indeed, the local basis used to find the jump operators would not coincide anymore with the diagonal basis of the normal modes of $\hat{H}_S$ [39,58], and this may create extra-terms in the Liouvillian superoperator which would not respect the rules discussed in Sec. III B. For some particular cases, this fact may turn the global approach computationally more convenient than the local one.

IV. EXAMPLES

In this section we will propose a couple of physical examples (one for fermions, one for bosons) in which the symmetry of Eq. (18) appears, and we will show how it significantly reduces the complexity of the master equation by a block diagonalization of the Liouvillian superoperator. We choose as examples some simple low-dimensional cases, whose solution is in general already known, in order to show how to identify and employ the symmetry also in familiar scenarios. Of course, more cumbersome situations would exhibit an even more drastic dimensionality reduction. For simplicity, from now on we will drop the hat sign over the operators living in the Liouville space $\mathbf{L}$.

A. Fermions

Consider a system of $M$ non-interacting fermions, with Hamiltonian:

$$\sum_{k=1}^{M} E_k f_k^\dagger f_k.$$  \hspace{1cm} (19)

If we let the fermions interact with local and/or collective baths through an interaction Hamiltonian $H_I$ which satisfies one of the conditions discussed in Sec. III A, the Liouvillian superoperator will be block-diagonal with each block labeled by the eigenvalues of the operator $\mathcal{N}$ in Eq. (14). We will provide the explicit form of such a Liouvillian for $M = 2$ fermions in Sec. IV A 1, being this case of utmost importance in different fields such as quantum computation or quantum thermodynamics. Before that, let us establish the dimension of each block for any $M$. Let us term $d$ an (integer) eigenvalue of $\mathcal{N}$ assuming values $d = -M, \ldots, -1, 0, 1, \ldots, M$. The dimension of the block $L_d$ is given by the number of excitation-basis vectors, written in the tensor notation of Eq. (15), which have a difference between the number of excitations on the left and on the right of the tensor product equal to $d$. Taking into account all the possible combinations of suitable excitations in the vectors and all their possible permutations, the dimension of $L_d$ reads:

$$\text{dim}(L_d) = \sum_{k=|d|}^{M} \left( \binom{M}{k} \cdot \binom{M-k}{|d|} \right).$$  \hspace{1cm} (20)

1. Two interacting spins as decoupled fermions

Consider a system of two interacting spins with Hamiltonian:

$$H_S = \frac{\omega_1}{2} \sigma_1^z + \frac{\omega_2}{2} \sigma_2^z + \lambda \sigma_1^+ \sigma_2^-.$$  \hspace{1cm} (21)

By employing the Jordan-Wigner transformations, a rotation and a Bogoliubov transformation (see the discussion in Appendices C and D), we can rewrite the system Hamiltonian as:

$$H_S = E_1 \left( 2f_1^\dagger f_1 - 1 \right) + E_2 \left( 2f_2^\dagger f_2 - 1 \right),$$  \hspace{1cm} (22)

where $f_1$ and $f_2$ are fermionic operators satisfying the fermionic anticommutation rules: $\{ f_j, f_k^\dagger \} = \delta_{jk}$, while the expressions of the energies $E_1$ and $E_2$ can be found in Eq. (D10). The interaction eigenbasis of $H_S$ is $\{ |00\rangle_f, |01\rangle_f, |10\rangle_f, |11\rangle_f \}$, and its relation with the canonical spin basis can be found in Eqs. (D17) and (D18). Although this transformation may appear redundant in the simple case of two qubits, it is fundamental to diagonalize more complex chains of interacting spins [53], see for instance Appendix C.

We couple each qubit to a separate thermal bath, such that the Hamiltonian of the environment is $H_E = \sum_k \Omega_k a_k^\dagger a_k + \sum_l \Omega_l b_l^\dagger b_l$ and the interaction Hamiltonian reads:

$$H_I = \sum_k g_k \sigma_1^z (a_k^\dagger + a_k) + \sum_l g_l \sigma_2^z (b_l^\dagger + b_l),$$  \hspace{1cm} (23)

where $g_k$ and $g_l$ determine the spectral densities of the baths [11]. As mentioned before, these are not relevant for the present discussion and are assumed to display fast decaying correlation functions, inducing a Markovian evolution. We assume that the both baths are in a thermal state with temperature respectively $T_1$ and $T_2$. Such as a system is of fundamental importance e.g. for the understanding of quantum heat transport in quantum thermodynamics [40].
Using Eqs. (D11) and (D12) we can rewrite the interaction Hamiltonian as:

$$\begin{align*}
H_I = \sum_k g_k \left( \cos(\theta + \phi)(f_k^1 + f_k) + \sin(\theta + \phi)(f_k^1 + f_k) \right) (a_k^+ + a_k) \\
+ \sum_l g_l' \left( \cos(\theta - \phi)P(f_l^1 - f_l) + \sin(\theta - \phi)P(f_l^1 - f_l) \right) (b_l^+ + b_l),
\end{align*}$$

(24)

where $P$ is the parity operator, and we notice that each separate bath plays now the role of a common bath between the two fermionic modes. Note that Eq. (24) satisfies the second condition on the interaction Hamiltonian presented in Sec. III A.

The interacting Hamiltonian Eq. (24) leads to the following master equation:

$$\begin{align*}
\frac{d}{dt}\rho_S(t) &= -i[H_S + H_{LS}, \rho_S(t)] + \sum_{i,j=1,2} \gamma_{ij}^\dagger \left( f_i \rho_S(t) f_j - \frac{1}{2} \{ f_i^\dagger f_j, \rho_S(t) \} \right) \\
&+ \sum_{i,j=1,2} \gamma_{ij}^\dagger \left( f_i \rho_S(t) f_j - \frac{1}{2} \{ f_i^\dagger f_j, \rho_S(t) \} \right) + \sum_{i,j=1,2} \eta_{ij}^\dagger \left( P f_i \rho_S(t) f_j P - \frac{1}{2} \{ f_i^\dagger f_j, \rho_S(t) \} \right) \\
&+ \sum_{i,j=1,2} \eta_{ij}^\dagger \left( f_i^\dagger P \rho_S(t) P f_j - \frac{1}{2} \{ f_i^\dagger f_j, \rho_S(t) \} \right),
\end{align*}$$

(25)

where the Lamb-shift Hamiltonian reads $H_{LS} = \sum_{i,j=1,2} (s_{ij}^1 f_i^1 f_j + s_{ij}^2 f_i f_j^1)$. The coefficients $\gamma_{ij}$, $\gamma_{ij}^\dagger$, $\eta_{ij}$, $\eta_{ij}^\dagger$, $s_{ij}$, and $s_{ij}^\dagger$ depend on the spectral densities of the baths, on the temperature and on the weights of each term in the interaction Hamiltonian. We do not provide their explicit value here, and we refer the interested reader to the derivation in Refs. [39].

We now find the Liouvillian superoperator representing the master equation (25) in the tensor product notation, as in Eq. (A5). We identify five symmetry blocks of $\mathcal{L}$, associated to the following bases: $|11\rangle_f \otimes |11\rangle_f , |10\rangle_f \otimes |10\rangle_f , |11\rangle_f \otimes |00\rangle_f$ corresponding to $\mathcal{N} = 0$; $|11\rangle_f \otimes |00\rangle_f , |11\rangle_f \otimes |01\rangle_f , |10\rangle_f \otimes |00\rangle_f , |01\rangle_f \otimes |00\rangle_f$ corresponding to $\mathcal{N} = 1$; $|00\rangle_f \otimes |11\rangle_f , |01\rangle_f \otimes |11\rangle_f , |00\rangle_f \otimes |10\rangle_f , |00\rangle_f \otimes |01\rangle_f$ corresponding to $\mathcal{N} = -1$; $|11\rangle_f \otimes |00\rangle_f$ corresponding to $\mathcal{N} = 2$; $|00\rangle_f \otimes |11\rangle_f$ corresponding to $\mathcal{N} = -2$. The Liouvillian can be written as $\mathcal{L} = \bigoplus_{d=-2}^{2} \mathcal{L}_d$, where the matrices representing each block in the associated basis are:

$$\mathcal{L}_0 = \begin{pmatrix}
-\gamma_{10} - \eta_{10} & \gamma_{22} + \eta_{22} & -\gamma_{21} - \eta_{21} & -\gamma_{12} - \eta_{12} & \gamma_{11} + \eta_{11} & 0 \\
\gamma_{22} + \eta_{22} & -\xi_{11} - \xi_{11} & \gamma_{22} + \eta_{22} & -\gamma_{12} - \eta_{12} & \gamma_{11} + \eta_{11} & 0 \\
-\gamma_{21} - \eta_{21} & \gamma_{22} + \eta_{22} & -\gamma_{21} - \eta_{21} & -\gamma_{12} - \eta_{12} & \gamma_{11} + \eta_{11} & 0 \\
-\gamma_{12} - \eta_{12} & \gamma_{12} + \eta_{12} & -\gamma_{12} - \eta_{12} & -\gamma_{12} - \eta_{12} & \gamma_{11} + \eta_{11} & 0 \\
\gamma_{11} + \eta_{11} & 0 & \gamma_{11} + \eta_{11} & \gamma_{22} + \eta_{22} & -\gamma_{12} - \eta_{12} & -\gamma_{12} - \eta_{12} \\
0 & \gamma_{11} + \eta_{11} & \gamma_{22} + \eta_{22} & -\gamma_{12} - \eta_{12} & -\gamma_{12} - \eta_{12} & -\gamma_{12} - \eta_{12}
\end{pmatrix},$$

(26)

$$\mathcal{L}_1 = \begin{pmatrix}
-i\omega' - \xi_{11} - \xi_{11} + \xi_{11} + \xi_{11} & is_{12} - \xi_{12} - \xi_{12} & \eta_1 - \eta_{12} & \eta_2 - \eta_{12} & \gamma_{11} - \gamma_{12} & \gamma_{12} - \gamma_{12} \\
is_{12} - \xi_{12} - \xi_{12} & -i\omega' - \xi_{11} - \xi_{11} + \xi_{11} + \xi_{11} & \eta_2 - \eta_{12} & \eta_1 - \eta_{12} & \gamma_{11} - \gamma_{12} & \gamma_{12} - \gamma_{12} \\
\eta_1 - \eta_{12} & \eta_2 - \eta_{12} & -i\omega' - \xi_{11} - \xi_{11} & \gamma_{11} - \gamma_{12} & \gamma_{12} - \gamma_{12} & \gamma_{12} - \gamma_{12} \\
\eta_2 - \eta_{12} & \eta_1 - \eta_{12} & \gamma_{11} - \gamma_{12} & -i\omega' - \xi_{11} - \xi_{11} & \gamma_{12} - \gamma_{12} & \gamma_{12} - \gamma_{12} \\
\gamma_{11} - \gamma_{12} & \gamma_{12} - \gamma_{12} & \gamma_{11} - \gamma_{12} & \gamma_{12} - \gamma_{12} & -i\omega' - \xi_{11} - \xi_{11} & \gamma_{12} - \gamma_{12} \\
\gamma_{12} - \gamma_{12} & \gamma_{12} - \gamma_{12} & \gamma_{12} - \gamma_{12} & \gamma_{12} - \gamma_{12} & \gamma_{12} - \gamma_{12} & -i\omega' - \xi_{11} - \xi_{11}
\end{pmatrix},$$

(27)

$$\mathcal{L}_{-1} = \mathcal{L}_1^\dagger, \mathcal{L}_2 = -i(\omega' + \omega'') - \xi_{10} - \xi_{01}$$

and $\mathcal{L}_{-2} = \mathcal{L}_2^\dagger$. When convenient, we have used the abbreviations $\omega' = 2E_1 + s_{11} + s_{11}^\dagger$, $\omega'' = 2E_2 + s_{22} + s_{22}^\dagger$, $s_{ij} = s_{ij}^\dagger = s_{ij}^\dagger$, $\gamma_{11} = \gamma_{11}^\dagger + \gamma_{11}^\dagger$, $\eta_{11} = \eta_{11}^\dagger + \eta_{11}^\dagger$, $\xi_{11} = \gamma_{11}^\dagger + \eta_{11}^\dagger$ and $\xi_{01} = \gamma_{01}^\dagger + \eta_{01}^\dagger$.
\[
\rho_s = \begin{pmatrix}
\rho_{11,11} & \rho_{11,10} & \rho_{11,01} & \rho_{11,00} \\
\rho_{10,11} & \rho_{10,10} & \rho_{10,01} & \rho_{10,00} \\
\rho_{01,11} & \rho_{01,10} & \rho_{01,01} & \rho_{01,00} \\
\rho_{00,11} & \rho_{00,10} & \rho_{00,01} & \rho_{00,00}
\end{pmatrix}
\]

FIG. 1. (Color online) Density matrix of the state of the system with Hamiltonian equation (2) in the fermionic interactions basis. The master equation driven by the Liouvillian \( \mathcal{L} \) couples only elements of the density matrix with the same color. In particular, the block \( \mathcal{L}_0 \) is represented by the color red, \( \mathcal{L}_1 \) by the color green, \( \mathcal{L}_{-1} \) by the color blue, \( \mathcal{L}_2 \) by the color violet and \( \mathcal{L}_{-2} \) by the color orange.

representing an independent block).

Furthermore, if we are interested in finding the steady state of the evolution and the latter is unique, we just have to analyze the matrix \( \mathcal{L}_0 \). To be sure that the condition on the uniqueness holds, one has to check that no decoherence-free subspaces are present. Their appearance can be detected a priori using different conditions on the interaction Hamiltonian or on the master equation [50], otherwise they can be revealed by the presence of more than one null eigenvalue in the spectrum of the Liouvillian superoperator. Since we have set non-zero, unbalanced temperatures of the baths, the steady state may contain coherences as well, but only the ones corresponding to the eigenvalue 0 of \( N \), namely \( \rho_{10,01} \) and \( \rho_{01,10} \). The same steady-state coherences were found using a non-secular master equation in a couple of recent works [22, 59]. We will provide an example of the appearance of these coherences below and in the next example about harmonic oscillators.

Note that, if we had performed the full secular approximation instead of the partial one, we would have introduced a broader symmetry on the superoperator level, dividing the block \( \mathcal{L}_0 \) into two additional parts. Indeed, if the spectrum of \( H_S \) is non-degenerate, the full secular approximation decouples coherences and populations [1]. The symmetry generated by \( N \) is therefore providing us with new “selection rules” that indicate the allowed transitions between elements of the density matrix: in the partial secular regime some of the coherences may exchange “amplitude” with the diagonal elements. We can visualize this through a concrete case of the two-coupled-qubits example: let us consider a scenario with \( \omega_1 = 1, \omega_2 = 1, \lambda = 0.01, T_1 = \omega_1/k_B, T_2 = \omega_1/10k_B \) and Ohmic spectral densities (from now on for simplicity we use dimensionless units for time and energy). Using these values, the fermionic energies read \( 2E_1 = 1.01005 \) and \( 2E_2 = 0.99005 \). \( \mu \) denotes the strength of the qubit-bath coupling, and considering the weak-coupling limit we set \( \mu = 10^{-1.5} \). According to Eq. (9), \( \tau_B \approx 1000 \) and therefore the partial secular approximation must conserve the terms in the master equation associated to the frequency difference \( 2E_1 - 2E_2 = 0.02 \), which does not satisfy the condition in Eq. (7). Using the above values, we can calculate the coefficients of the master equation (25) according to the discussion in Ref. [1] and we can compute the dynamics of the two qubits. According to the suitable conditions [50], we have checked that no decoherence-free subspace is present in this scenario.

We now want to visualize how the partial secular approximation induces selection rules between different elements of the density matrix: we consider the evolution of the two qubits starting from the excited fermionic state \( |11\rangle_f \), i.e. \( \rho_S(0) = |11\rangle_f \langle 11| \), whose dynamics is driven by the block \( \mathcal{L}_0 \). We monitor the expectation value of some observables which pertain to different blocks of the Liouvillian superoperator as a function of time, in particular we choose:

\[ P_{11}(t) = \text{Tr}[\rho_S(t) |11\rangle_f \langle 11|] = \rho_{11,11}(t), \]
\[ P_{00}(t) = \text{Tr}[\rho_S(t) |00\rangle_f \langle 00|] = \rho_{00,00}(t), \]
\[ C_0(t) = \text{Tr}[\rho_S(t) f_1^† f_2 f_2^† f_1^† f_1] = 2 \text{Re}[\rho_{01,10}(t)], \]
\[ C_1(t) = \text{Tr}[\rho_S(t) f_1^† f_1 f_1^† f_1] = 2 \text{Re}[\rho_{11,01}(t)] + 2 \text{Re}[\rho_{00,10}(t)], \]
\[ C_2(t) = \text{Tr}[\rho_S(t) f_1^† f_2 f_2^† f_1 f_1] = 2 \text{Re}[\rho_{11,00}(t)]. \]

\( P_{11} \), \( P_{00} \) and \( C_0 \) depend on elements of the block \( \mathcal{L}_0 \), while \( C_1 \) of the block \( \mathcal{L}_1 \) and \( C_2 \) of the block \( \mathcal{L}_2 \). Figure 2 depicts their evolution: \( P_{11} \) is the only non-zero mean value at time 0. Therefore, the symmetry brought by the partial secular approximation \( [N, \mathcal{L}] = 0 \) denies the possibility that \( C_1 \) and \( C_2 \) may change their value during the evolution, since their dynamics is driven by blocks different from \( \mathcal{L}_0 \). On the contrary, \( C_0 \) increases and stabilizes to a non-zero value at infinite time,
since it is the mean value of the observable expressing the exchange of excitations between the fermions, whose dynamics is driven by $\mathcal{L}_0$.

Let us now briefly discuss how things would change if we applied the full secular approximation to derive the master equation (25), i.e. if we removed the terms with $i \neq j$. The full secular approximation decouples coherences and populations [11], therefore, in the scenario discussed before, not only it would inhibit any transition that may “activate” $C_1$ and $C_2$, but would also keep $C_0$ null, given that the latter depends on the density matrix element $\rho_{01,10}$. This means that, in Fig. 2, the full secular approximation would make the dotted red line ($C_0$) overlap with the green and purple lines ($C_1$ and $C_2$), thus proving itself not suitable to treat the current scenario [39].

B. Bosons

If we consider a generic bosonic system for which Eq. (18) holds, we will still have a block diagonalization of the Liouvillean superoperator which will simplify the resolution of the master equation, but each block will have infinite dimension. Here, we want to focus on a simpler case in which the symmetry expressed by Eq. (18) leads to a dimensionality reduction as well: we restrict ourselves to the space of Gaussian states [60] and we consider only a master equation conserving Gaussianity. Therefore, we only need to analyze the dynamics of the covariance matrix, neglecting any displacement which may be eliminated through a suitable transformation.

Let us consider a system of $M$ non-interacting bosons with Hamiltonian:

$$H_S = \sum_{k=1}^{M} E_k a_k^\dagger a_k.$$  

(28)

Given the presence of local or common baths leading to a Gaussian Markovian master equation, we want to study the dynamics of a Gaussian state with no displacement. For convenience, we choose to write the covariance matrix of the state using the creation and annihilation operators, i.e. a generic element of the covariance matrix may be written in one of these three forms:

$$\langle a_i^\dagger a_j \rangle \quad \text{or} \quad \langle a_i^\dagger a_j^\dagger \rangle \quad \text{or} \quad \langle a_i a_j \rangle,$$

(29)

where the average is performed on the chosen Gaussian state.

We define as $\delta$ the difference between number of creations and number of annihilations in an element of the covariance matrix Eq. (29), assuming values $2, 0, -2$, respectively.

It is easy to understand what the symmetry defined by Eq. (18) is telling us about the evolution of the covariance matrix: the dynamics of an element of the covariance matrix with value $\delta$ can only be a function of elements of the covariance matrix with the same value $\delta$ [3]. We can collect the elements of the covariance matrix Eq. (29) (which cannot be trivially obtained through commutations of the other elements) in a vector $\mathbf{x}$. The evolution of $\mathbf{x}$ as a function of time is then given by the formula

$$\frac{d\mathbf{x}}{dt} = B\mathbf{x} + \mathbf{b}.$$  

(30)

The matrix $B$ is block-diagonal labelling each block with the value $\delta$: $B = \bigoplus_{\delta=-2,0,2} B_\delta$. Furthermore, $\langle a_i a_j \rangle = \langle a_i^\dagger a_j \rangle^\dagger$ and the symmetry assures us that these two moments do not couple in the master equation, therefore the block $B_0$ is trivially obtained by the block $B_{-2}$. We can now calculate the dimension of each block $B_\delta$:

$$\dim(B_0) = M^2,$$

$$\dim(B_{\pm 2}) = \left(\frac{M+2-1}{2}\right).$$

(31)

1. Two bosons in a common bath

As an example, we consider the system of two displaced non-interacting bosons with Hamiltonian:

$$H_S = \sum_{k=1,2} (\omega_k a_k^\dagger a_k - \alpha_k a_k - \alpha_k^* a_k^\dagger).$$

(32)

The Hamiltonian can be recast in the standard form of Eq. (16) through a suitable displacement operator $D(\alpha)$ [60]:

$$D(\alpha)^\dagger a_k D(\alpha) = a_k + \alpha_k.$$  

Therefore we have:

$$H_S = \sum_{k=1,2} E_k a_k^\dagger a_k,$$

(33)

which describes two non-interacting harmonic oscillators. We couple the system to a common bosonic environment $H_E = \sum_i \Omega_i c_i^\dagger c_i$ in a thermal state with temperature $T > 0$. The system-bath interaction Hamiltonian is:

$$H_I = \sum_i g_i (a_1 + a_1^\dagger + a_2 + a_2^\dagger)(c_i + c_i^\dagger),$$

(34)

where $g_i$ determines the spectral density, which is not relevant for the present discussion. The evolution of the system coupled to the environment is given by the master equation with Liouvillean:

$$\mathcal{L}^{1}[O] = i[H_S + H_{ES}, O]$$

$$+ \sum_{ij=1,2} \gamma_{ij}^\dagger \left( a_i^\dagger O a_j - \frac{1}{2} \{ a_i^\dagger a_j, O \} \right)$$

$$+ \sum_{ij=1,2} \gamma_{ij} \left( a_i O a_j^\dagger - \frac{1}{2} \{ a_i a_j^\dagger, O \} \right),$$

(35)

where the $n$-th moment of the form $\langle a_1^\dagger a_2^\dagger \ldots a_{n-1}^\dagger a_n \rangle$ with $\delta = 2l - n$ can only be a function of $m$-th moments with the same value of $\delta$ (difference between number of creations and number of annihilations in the moment).
where $O$ is an operator acting on the Hilbert space of the system, and $\gamma_{ij}^0$ and $\gamma_{ij}^\dagger$ are respectively the coefficients describing the decay and the absorption, which depend on the spectral density and on the temperature of the environment. The Lamb-shift Hamiltonian reads:

$$H_{LS} = \sum_{ij=1,2} s_{ij} a_i a_j.$$  \hspace{1cm} (36)

The elements $\gamma_{12}$ and $\gamma_{21}$ are different from zero only if the harmonic oscillators are slightly detuned (or not detuned at all) [39]. We remind that, assuming that the initial state is harmonic oscillators are slightly detuned (or not detuned at all) [39]. We remind that, assuming that the initial state is Gaussian, then it will remain Gaussian due to the form of Eq. (30). The master equation describing the evolution of $\mathbf{x}$ has the form of Eq. (32). The vector $b$ can be written as $b = \oplus_{\delta=-2,0,2} B_{\delta}$. We have that $b_{\pm2} = 0$, while $b_0 = \left(\begin{array}{c} \gamma_1^\dagger \\ \gamma_2^\dagger \\ \gamma_{12}^\dagger \\ \gamma_{21}^\dagger \end{array}\right)$.  \hspace{1cm} (37)

The elements of the covariance matrix can be collected in a vector $\mathbf{x}$ of dimension 10. In particular, we choose to parametrize it according to the basis $\{a_1^\dagger a_2^\dagger, a_1^\dagger a_2, a_1 a_2^\dagger, a_1 a_2\}$ with $\delta = 2$, $\{a_1^\dagger a_1, a_2 a_2^\dagger, a_1 a_2^\dagger, a_1^\dagger a_2\}$ with $\delta = -2$, $\{a_1^\dagger a_1, a_2 a_2\}$ with $\delta = 0$. The master equation describing the evolution of $\mathbf{x}$ has the form of Eq. (30). The blocks are given by:

$$B_0 = \left(\begin{array}{cccc} -\gamma_{12}^\dagger & 0 & -2i\gamma_{12}^\dagger - \gamma_{21}^\dagger & -2i\gamma_{12}^\dagger + \gamma_{21}^\dagger \\ 0 & -\gamma_{21}^\dagger & -2i\gamma_{21}^\dagger - \gamma_{12}^\dagger & -2i\gamma_{21}^\dagger + \gamma_{12}^\dagger \\ -2i\gamma_{12}^\dagger + \gamma_{21}^\dagger & -2i\gamma_{21}^\dagger + \gamma_{12}^\dagger & i\Delta\omega - \gamma_{12}^\dagger - \gamma_{21}^\dagger & 0 \\ -2i\gamma_{21}^\dagger + \gamma_{12}^\dagger & -2i\gamma_{12}^\dagger + \gamma_{21}^\dagger & 0 & -i\Delta\omega - \gamma_{12}^\dagger - \gamma_{21}^\dagger \end{array}\right),$$  \hspace{1cm} (38)

$$B_{-2} = \left(\begin{array}{cccc} -2iE_1^\dagger - \gamma_{12}^\dagger & -2i\gamma_{12}^\dagger + \gamma_{21}^\dagger & 0 \\ -2i\gamma_{12}^\dagger - \gamma_{21}^\dagger & -2iE_2^\dagger + \gamma_{21}^\dagger & 0 \\ -i(E_1^\dagger + E_2^\dagger) - \gamma_{12}^\dagger + \gamma_{21}^\dagger & -i(E_2^\dagger + E_2^\dagger) - \gamma_{12}^\dagger + \gamma_{21}^\dagger & 0 \\ 0 & -2iE_2^\dagger - \gamma_{21}^\dagger & -2iE_2^\dagger - \gamma_{21}^\dagger \end{array}\right),$$  \hspace{1cm} (39)

and $B_2 = B_{-2}^\ast$. We have defined $E_1^\prime = E_1 + s_{11}$, $E_2^\prime = E_2 + s_{22}$, $\Delta\omega = E_1^\prime - E_2^\prime$, $\gamma_j^\dagger = \gamma_{jj}^\dagger - \gamma_{jj}^\dagger$.  

V. DISCUSSION AND CONCLUSIONS

In this paper we have shown how the Liouvillian superoperator $L$ of a broad class of open quantum systems can be block-diagonalized through a symmetry on the superoperator level, namely the invariance under the action of the number superoperator $N$, defined in Eq. (15), such that $[N,L] = 0$. This symmetry arises when we derive the standard Bloch-Redfield master equation of the open system applying a suitable partial secular approximation whose condition is given by Eq. (7). The requirements for the microscopic model are that the system Hamiltonian can be recast as $M$ non-interacting bosonic or fermionic modes (Eq. (16)) and that the system operators in the interaction Hamiltonian satisfy the conditions discussed at the end of Sec. [IIIA] which are usually fulfilled in the majority of physical systems of importance to quantum information or condensed matter physics. This includes, for instance, any system with Hamiltonian quadratic in the bosonic or fermionic operators, and coupled to a thermal bath through operators which are linear in the field operators, as well as some spin systems.

The existence of the symmetry is formalized and proven in Proposition 1. Corollary 1 states that such symmetry implies the invariance under the action of the parity superoperator as well. Proposition 2 shows that we can exploit Proposition 1 to decompose the Liouvillian superoperator into blocks, and that in the fermionic case only $M + 1$ of them are independent. This greatly reduces the complexity of the master equation. Furthermore, each block may be the only part of the Liouvillian we have to manipulate in order to find a certain physical quantity, for example Proposition 3 shows that, when unique, the steady state is determined only by one single block. This implies that the allowed steady-state coherences in the excitation basis of an unique steady state are only the ones with equal number of excitations in the ket and the bra, as formal-
ized in Corollary 2.

A couple of examples are also discussed. In Sec. IV A we have found the dimension of each block \( \mathcal{L}_d \) of the Liouvillian superoperator in the case of a system of \( M \) fermionic modes (Eq. (20)), and we have shown how to apply this to a system of two coupled qubits. In this scenario, an originally \( 16 \times 16 \) Liouvillian is decomposed into five blocks of dimension 6, 4, 4, 1 and 1. The information about the steady state is contained in the \( 6 \times 6 \) block only. The decomposition greatly simplifies the master equation and also allows to obtain some analytical solutions. Then, in Sec. IV B we have discussed the case of bosons, focusing in particular on Gaussian states. We have shown how to decompose the equation for the evolution of the covariance matrix employing the symmetry of the number superoperator, and we have applied it to study the case of two harmonic oscillators in a common bath. In the presence of small detuning, we have detected steady-state coherences by focusing on a system of only 4 linear equations, instead of the original system of 10 equations.

These results may be relevant in disparate fields. For instance, reducing the complexity of the master equation describing transport in quantum systems is of great importance [61], and our discussion may be especially relevant for materials which exhibit quasi-degeneracies in the Hamiltonian spectrum and thus require a master equation in partial secular approximation [62]. The latter is also important to study the heat current from two unbalanced reservoirs, since it solves any deficiency that a global master equation may display with respect to a local one [39]. The symmetry generated by \( N \) may be also relevant in the field of quantum metrology. Indeed, as discussed in Sec. III B it may be seen as a generalization of the concept of phase-covariant master equation, which plays a fundamental role in defining the limits for the frequency estimation of a single qubit [54–65]. Therefore, a protocol for the frequency estimation of multiple detuned qubits should rely on our result to distinguish between the possible noise models and their origin. Finally, Proposition 3 and Corollary 2 are very relevant in quantum thermodynamics and quantum thermalization. Indeed, they define a strict law on the possible steady-state coherences that may appear in the steady state of a Markovian process. The relation between coherences and diagonal elements is also important to improve the performance of quantum thermal machines [66].

Possible extensions of this work could address other situations where the number superoperator symmetry can arise. In particular, beyond stationary environments considered here, non-stationary autocorrelation functions of the bath would add a temporal dependence to the coefficients of the Lamb-shift Hamiltonian and of the dissipator in Eqs. (5) and (6). This would affect the way in which we perform the partial secular approximation. As a consequence, there may exist scenarios in which the symmetry is broken. Consider for instance a single-mode electromagnetic field in a squeezed bath [1]: the master equation would contain terms of the form \( a \rho_{\text{field}} a^\dagger \), where \( a \) is the annihilation operator of the field. Clearly, in this case \( \{ N, \mathcal{L} \} \neq 0 \). Note however that we would still recover the symmetry of the parity superoperator: \( \{ P, \mathcal{L} \} = 0 \). Different scenarios may arise considering different states of the environment, and further investigation is needed to extent our work to these cases.

A further direction could be exploring non-linear scenarios beyond quadratic system Hamiltonians, even if the latter include many bosonic and fermionic systems of interest to quantum information. In particular, for systems of many coupled spins where the number of spin excitations is not conserved, even if diagonalization through the Jordan-Wigner transformations is possible, the resulting fermionic Hamiltonian generally depends on a collective phase, violating the “non-interacting” condition. In these scenarios, the validity of the symmetry \( \{ N, \mathcal{L} \} = 0 \) must be checked case by case, using the physical considerations discussed in Sec. III B. On the contrary, the block decomposition holds for any excitations-preserving system of spins in common or separate thermal baths (with a final requirement on the interaction Hamiltonian).

The extension of Proposition 3 to scenarios with more than one steady state, e.g. in the presence of decoherence free subspaces or oscillating coherences, would also be interesting. In particular, some open questions not addressed here are: does the block \( \mathcal{L}_0 \) contain all the information about any steady state of the system? If not, are there particular cases in which this holds? Can we find an analogous theorem for oscillating coherences? Investigation about the same symmetry for non-Markovian master equations in the weak coupling limit may be interesting as well. In particular, we expect to find the same results for the case of a time-local non-Markovian master equation in the secular regime [67], while non-secular terms would break the symmetry. Finally, it would be useful to employ our findings to implement a fast, manageable code to solve the dynamics of the open system by exploiting its symmetry, as already done for the case of identical atoms [29].

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Appendix A: The formalism of GKLS master equations

1. From the Bloch-Redfield to the Lindblad equation

The fact that, in general, the Bloch-Redfield master equation does not preserve positivity and is not in the GKLS form (or Lindblad form) \([7, 8]\) is a very well-known issue \([68]\). The standard procedure to derive a Markovian master equation makes use of the full secular approximation \([11, 8]\), i.e.
removes all the terms with $\omega \neq \omega'$ in Eqs. (5) and (6), in order
to recover the semigroup structure of a master equation in the
Lindblad form. This, however, may lead to major mistakes
when the condition in Eq. (7) is not fulfilled [39]. Nonetheless,
some recent studies have shown that the PSA performed
through a suitable coarse-graining does lead to a GKLS master
equation [40][41], as can be also found in a previous work
which did not mention the PSA [69]. This method of applying
the PSA is analogous to the one used in the present paper,
based on the condition in Eq. (7), up to a negligible error. A
related discussion is provided in Ref. [42]. As a matter of fact,
the Bloch-Redfield master equation does follow the dynamics
of a GKLS master equation up to an error due to the approxi-
mation of the dynamics of the microscopic model to a Marko-
vian evolution. A significant deviation of the Bloch-Redfield
master equation from the Lindblad form must be considered as
a signature of the failure of the Born-Markov approximations
to describe the physical model, and not viceversa, as proven in
Ref. [43].

For the reasons explained above, we are allowed to assume
that the master equation in PSA with Liouvillian Eq. (4) can be
rewritten in the GKLS or Lindblad form as in Eq. (10). The
Lindblad operators become linear combinations of the jump
operators $\hat{A}_\alpha$, and can be obtained for each specific case by di-
agonalizing the matrix $\gamma_{\alpha\beta}(\omega,\omega')$ in Eq. (6) [40][41]. Analogously,
we can write the master equation in the Lindblad form
in the Heisenberg picture (11):

$$\frac{d}{dt} \hat{j}(t) = i[H, \hat{j}(t)] + \sum_{l=1}^{N^2-1} \hat{F}_l \hat{j}(t) \hat{F}_l - \frac{1}{2} \{\hat{F}_l \hat{F}_l, \hat{j}(t)\}, \quad (A1)$$

where $\hat{j} = \hat{j}^\dagger$ is an observable living in $\mathcal{L}$, whose expectation
value can be found as $\langle \hat{j}(t) \rangle = \text{Tr} \left[\hat{j}(t) \rho_S\right] = \text{Tr} \left[\hat{j}\rho_S(t)\right]$.  

2. Working with superoperators

It is very convenient to extend the bra-ket notation to the
Liouville space $\mathcal{L}$ [27][52]. Suppose that $\{|e_j\rangle\}_{j=1}^N$ is a basis
of the Hilbert space $H_S$. Then, any operator $\hat{O}$ (or equivalently
density matrix) in $\mathcal{L}$ can be written as:

$$\hat{O} = \sum_{j,k=1}^N O_{jk} |e_j\rangle \langle e_k| . \quad (A2)$$

We now perform the following isomorphism, passing from a
description of $\hat{O}$ as an operator acting on $H_S$ to a description
as a $N^2$-dimensional vector:

$$\hat{O} \rightarrow |O\rangle = \sum_{j,k=1}^N O_{jk} |e_j\rangle \otimes |e_k\rangle . \quad (A3)$$

Given $\hat{O}, \hat{R} \in \mathcal{L}$, the reader can verify that this $N^2$-dimensional space is furnished with the Hilbert-Schmidt
scalar product $\langle O|R\rangle = \text{Tr} \left(\hat{O}^\dagger \hat{R}\right)$, and that the following properties hold:

$$|OR\rangle = \hat{O} \otimes \mathbb{I}_N |R\rangle, \quad |RO\rangle = \mathbb{I}_N \otimes \hat{O}^T |R\rangle , \quad (A4)$$

where $\mathbb{I}_N$ is the $N \times N$ identity matrix.

Using Eq. (A4), we can now write the explicit form of the
Liouvillian superoperator starting from the Bloch-Redfield
master equation in Eqs. (3), (5) and (6):

$$\mathcal{L} = -i \left( (\hat{H}_S + \hat{H}_{LS}) \otimes \mathbb{I}_N - \mathbb{I}_N \otimes (\hat{H}_S + \hat{H}_{LS}) \right) + \sum_{\alpha,\beta} \sum_{(\omega,\omega') \in \text{PSA}} \frac{1}{2} \left( \hat{A}_\alpha(\omega') \hat{A}_\beta(\omega) \otimes \mathbb{I}_N + \mathbb{I}_N \otimes (\hat{A}_\alpha(\omega') \hat{A}_\beta(\omega))^T \right) . \quad (A5)$$

Appendix B: Proof of Proposition 1

We want to proof that $[\mathcal{N}, \mathcal{L}] = 0$. For convenience, we work using the isomorphism “flattening” matrices into vectors
(see Appendix A2, Eq. (A3)), so that $\mathcal{L}$ can be written as in
Eq. (A5) and $\mathcal{N}$ as in Eq. (15). Looking at the structure of $\mathcal{L}$, we recognize four different parts of the Liouvillant that must commute with $\mathcal{N}$; in particular, to proof the statement it is sufficient to verify the following four assertions:

1. $[\mathcal{N}, \hat{H}_S \otimes \mathbb{I}_N] = [\mathcal{N}, \mathbb{I}_N \otimes \hat{H}_S^T] = 0.$

2. $[\mathcal{N}, \hat{A}_\beta(\omega) \otimes \hat{A}_\alpha^*(\omega')] = 0 \text{ for all } \alpha, \beta \text{ and } (\omega,\omega') \in \text{PSA}.$

3. $[\mathcal{N}, \hat{A}_\alpha^*(\omega') \hat{A}_\beta(\omega) \otimes \mathbb{I}_N] = 0 \text{ for all } \alpha, \beta \text{ and } (\omega,\omega') \in \text{PSA}.$

4. $[\mathcal{N}, \mathbb{I}_N \otimes (\hat{A}_\alpha^*(\omega') \hat{A}_\beta(\omega))^T] = 0 \text{ for all } \alpha, \beta \text{ and } (\omega,\omega') \in \text{PSA}.$

This means that the value of the coefficients of the master
equation does not play a role in the appearance of the sym-
metry.

Assertion (1) is easily proven: the system Hamiltonian
Eq. (16) cannot change the number of particles in any mode,
since $[\hat{H}_S, \hat{n}_k] = 0 \forall k$, and thus $[\hat{H}_S, \mathbb{N}] = 0$, therefore $[\mathcal{N}, \hat{H}_S \otimes \mathbb{I}_N] = 0.$

Given the Hamiltonian of a system of $M$ modes, $\hat{H}_S = \sum_{k=1}^M E_k \hat{n}_k$, we write an eigenvector as $|e\rangle$, with $\hat{H}_S |e\rangle = e |e\rangle$ and $e = \sum_{k=1}^M E_k n_k^e$, where $n_k^e$ is the number of excitation in each mode of $|e\rangle = [n_1^e, \ldots, n_M^e]$. The total number of particles in $|e\rangle$ is given by $n^e = \sum_{k=1}^M n_k^e$. Using the same notation for generic eigenvectors $|\psi\rangle = [|e\rangle, |\psi\rangle]$ and $|e\rangle \langle e|$, we write the jump operators as $\hat{A}_\beta(\omega) = \sum_{e', e = \omega} \langle e| \hat{A}_\beta(\omega) |e', \psi\rangle \langle e, \psi|$ and $\hat{A}_\alpha(\omega') = \sum_{e', e = \omega'} \langle e| \hat{A}_\alpha(\omega') |e', \psi\rangle \langle e, \psi|$. Let us now consider
Assertion (2): we write the commutator as
\[
[N, \hat{A}_\alpha(\omega) \otimes \hat{A}_\alpha^\dagger(\omega')]
= \hat{N} \hat{A}_\alpha(\omega) \otimes \hat{A}_\alpha^\dagger(\omega') - \hat{A}_\alpha(\omega') \hat{N} \otimes \hat{A}_\alpha^\dagger(\omega')
+ \hat{A}_\alpha(\omega) \otimes \hat{A}_\alpha^\dagger(\omega') \hat{N}^T - \hat{A}_\alpha(\omega') \hat{N}^T \otimes \hat{A}_\alpha^\dagger(\omega')
= \sum_{\epsilon' = \epsilon = \omega'} (n^\epsilon - n^{\epsilon'}) |\epsilon\rangle \langle \epsilon| [\hat{A}_\alpha(\omega), \hat{A}_\alpha^\dagger(\omega')] (B1)
- \hat{A}_\alpha(\omega) \otimes \sum_{\epsilon' = \epsilon = \omega'} (n^\epsilon - n^{\epsilon'}) |\epsilon\rangle \langle \epsilon| [\hat{A}_\alpha^\dagger(\omega'), \hat{A}_\alpha^\dagger(\omega')].
\]
Since \((\omega, \omega') \in \text{PSA}\), according to Eq. (7) we must have \(\omega - \omega' = \mathcal{O}_t (\mu^2)\), therefore \(\sum_{k=1}^M E_k (n^\epsilon_k - n^{\epsilon'}_k + n^{\epsilon'}_k - n^\epsilon_k) = \mathcal{O}_t (\mu^2)\) or \(\sum_{k=1}^M E_k (n^\epsilon_k - n^{\epsilon'}_k) = \mathcal{O}_t (\mu^2)\). But, assuming Condition II on the interaction Hamiltonian: it is sufficient to assume it on the interaction Hamiltonian: it is sufficient to assume it on the energy difference \(\omega - \omega'\) as for Assertion (2), we find that \(n^\epsilon = n^{\epsilon'}\) and the commutator in Eq. (B1) vanishes, proving Assertion (3). Assertion (4) is verified analogously, and we have proven Proposition 1.

Appendix C: Jordan-Wigner transformations

In this appendix we briefly present the well-known Jordan-Wigner technique [51, 53] to represent spins as fermions, and we show how to employ it to recast the Hamiltonian of an excitation-preserving spin chain in a quadratic fermionic Hamiltonian. Given a system of \(M\) spins, we apply the following Jordan-Wigner transformations to write each spin operator as a function of fermionic operators:

\[
\sigma^z_k = c^\dagger_k c_k - \frac{1}{2},
\sigma^+_k = c^\dagger_k e^{i\pi \sum_{l<k} n_l},
\sigma^-_k = c_k e^{-i\pi \sum_{l<k} n_l}. \tag{C1}
\]

The reader can verify the anticommutation rules \(\{c_j, c_k^\dagger\} = \delta_{jk}\), \(\{c_j, c_k\} = 0\).

Let us now suppose that the spins are interacting in an excitation-preserving chain, with Hamiltonian:

\[
H_{SC} = \sum_{k=1}^M \omega_k \sigma^z_k + \sum_{k=1}^{M-1} J_k (\sigma^+_k \sigma^-_{k+1} + h.c.). \tag{C2}
\]

Using the Jordan-Wigner transformations in Eq. (C1), we can express it as:

\[
H_{SC} = \sum_{k=1}^M \frac{\omega_k}{2} \left( c^\dagger_k c_k - \frac{1}{2} \right) + \sum_{k=1}^{M-1} J_k (c^\dagger_{k+1} e^{i\pi n_k} c_k + h.c.), \tag{C3}
\]

where we have used \(e^{i\pi n_k} c_k = 0\) for \(k \geq l\). Noticing \(e^{i\pi n_k} c_k |0\rangle_k = 0\) and \(e^{i\pi n_k} c_k |1\rangle_k = c_k |1\rangle_k\), we observe that the phase \(e^{i\pi n_k}\) has no effects and can be removed from the Hamiltonian. Therefore, Eq. (C3) is quadratic in the fermionic operators and can be recast in the form of Eq. (16), thus being suitable for the analysis in Sec. III.

Appendix D: Two coupled spins as free fermions

In this appendix we show how to employ the Jordan-Wigner transformations to write a system of two coupled spins as non-interacting fermions (part of the discussion was already addressed in Ref. [70]). For convenience, we rewrite the Jordan-Wigner transformations Eq. (C1) for two spins as:

\[
\sigma^z_1 = 1 - 2c^\dagger_1 c_1, \quad \sigma^z_2 = 1 - 2c^\dagger_2 c_2,
\sigma^+_1 = c^\dagger_1 + c_1, \quad \sigma^+_2 = (1 - 2c^\dagger_1 c_1)(c^\dagger_2 + c_2), \tag{D1}
\]

where \(c_1\) and \(c_2\) are fermionic operators. The free Hamiltonian of the coupled qubits, given in Eq. (21), is now transformed following Eq. (D1):

\[
H_S = \frac{\omega_1}{2} (1 - 2c^\dagger_1 c_1) + \frac{\omega_2}{2} (1 - 2c^\dagger_2 c_2) + \lambda (c^\dagger_1 c_1 - c^\dagger_2 c_2). \tag{D2}
\]

In order to diagonalize \(H_S\) written in terms of fermionic operators, we first perform the Bogoliubov transformation

\[
c_1 = \cos \theta \xi_1 + \sin \theta \xi_1^*, \quad c_2 = \cos \theta \xi_2 - \sin \theta \xi_1^*, \tag{D3}
\]

and then the rotation

\[
\xi_1 = \cos \phi f^1_1 + \sin \phi f^1_2, \quad \xi_2 = \cos \phi f^2_1 - \sin \phi f^2_2, \tag{D4}
\]

where \(\xi_1\) and \(\xi_2\) are eigenstates of the Hamiltonian, and \(f^1_1\), \(f^1_2\), \(f^2_1\), \(f^2_2\) are free fermions.
Let us now write Eq. [D2] after having applied the Bogoliubov transformation:

\[
H_S = +\frac{\omega_1}{2} \left[ 1 - 2 \left( \cos^2 \theta \xi_1^1 \xi_1^1 + \sin^2 \theta \xi_2^1 \xi_2^1 + \sin \theta \cos \theta (\xi_2^1 \xi_1^1 + h.c.) \right) \right] \\
+ \frac{\omega_2}{2} \left[ 1 - 2 \left( \cos^2 \theta \xi_2^2 \xi_2^2 + \sin^2 \theta \xi_1^2 \xi_1^2 + \sin \theta \cos \theta (\xi_2^2 \xi_1^2 + h.c.) \right) \right] \\
+ \lambda \left[ 2 \cos \theta \sin \theta (\xi_1^1 \xi_1^1 + \xi_2^1 \xi_2^1) - 2 \cos \theta \sin \theta + (\cos^2 \theta - \sin^2 \theta)(\xi_2^1 \xi_1^1 + h.c.) + (\xi_1^1 \xi_2^1 + h.c.) \right].
\]

(D5)

We set \( \theta \) so as to delete all the double-excitation terms in Eq. [D5]

\[-\omega_+ \sin \theta \cos \theta + \lambda (\cos^2 \theta - \sin^2 \theta) = 0 \quad \Rightarrow \quad \tan 2\theta = \frac{2\lambda}{\omega_+}\]

(D6)

with \( \omega_+ = \omega_1 + \omega_2 \).

Using the condition in Eq. [D6] we now write \( H_S \) after having performed the rotation:

\[
H_S = +\frac{\omega_1}{2} \left[ 1 - 2 \left( \sin^2 \theta \sin^2 \phi - \cos^2 \theta \sin^2 \phi \right) f_1^1 f_1^1 + \left( \sin^2 \theta \cos^2 \phi - \cos^2 \theta \sin^2 \phi \right) f_2^1 f_2^1 \\
+ \sin \phi \cos \phi (f_1^1 f_2^1 + h.c.) + \cos^2 \theta \right) \right] \\
+ \frac{\omega_2}{2} \left[ 1 - 2 \left( \sin^2 \theta \sin^2 \phi - \cos^2 \theta \sin^2 \phi \right) f_2^2 f_2^2 + \left( \sin^2 \theta \cos^2 \phi - \cos^2 \theta \sin^2 \phi \right) f_1^2 f_1^2 \\
- \sin \phi \cos \phi (f_1^2 f_2^2 + h.c.) + \cos^2 \theta \right) \right] \\
+ \lambda \left[ \sin 2\theta (f_1^1 f_1^1 + f_2^1 f_2^1) + \sin 2\phi (f_1^1 f_1^1 - f_2^1 f_2^1) + (\cos^2 \phi - \sin^2 \phi)(f_1^1 f_2^1 + h.c.) \right].
\]

(D7)

In order to eliminate the remaining cross terms, we set the value of \( \phi \):

\[-\omega_- \sin \phi \cos \phi + \lambda (\cos^2 \phi - \sin^2 \phi) = 0 \quad \Rightarrow \quad \tan 2\phi = \frac{2\lambda}{\omega_-}\]

(D8)

with \( \omega_- = \omega_1 - \omega_2 \).

By employing the relations \( \cos^2 \alpha \cos^2 \beta - \sin^2 \alpha \sin^2 \beta = (\cos 2\alpha + \cos 2\beta)/2 \) and \( \cos^2 \alpha \sin^2 \beta - \sin^2 \alpha \cos^2 \beta = (\cos 2\alpha - \cos 2\beta)/2 \), we finally obtain the Hamiltonian

\[H_S = E_1 \left( 2f_1^1 f_1^1 - 1 \right) + E_2 \left( 2f_2^1 f_2^1 - 1 \right),\]

(D9)

where

\[E_1 = \sqrt{\lambda^2 + \omega_+^2/4 + \sqrt{\lambda^2 + \omega_-^2/4}}/2, \quad E_2 = \sqrt{\lambda^2 + \omega_+^2/4 - \sqrt{\lambda^2 + \omega_-^2/4}}/2.\]

(D10)

We can now proceed to write the spin operators in terms of the fermionic operators. Let us start with \( \sigma_z^f \):

\[\sigma_z^f = \cos(\theta + \phi) \left( f_1^1 + f_1^1 \right) + \sin(\theta + \phi) \left( f_2^1 + f_2^1 \right).\]

(D11)

By noticing that \((1-2c_1^1 c_1)(1-2c_2^1 c_2)(c_2 - c_2^1) = \sigma_z^f \), we can readily obtain

\[\sigma_z^c = \cos(\theta - \phi) P \left( f_2^1 - f_2^1 \right) + \sin(\theta - \phi) P \left( f_1^1 - f_1^1 \right),\]

(D12)

where

\[P = (1-2c_1^1 c_1)(1-2c_2^1 c_2) = (2f_1^1 f_1^1 - 1)(2f_2^1 f_2^1 - 1)\]

(D13)

is the parity operator, which tells us whether the number of excitations in the system is even or odd. The Hamiltonian \( H_S \) conserves the parity of the excitation number, i.e. \([H_S, P] = 0\), thus we are sure that \( P \) has the form presented in Eq. [D13].

The form of the operators \( \sigma_z^f \) and \( \sigma_z^c \) is more involved, since they inevitably contain the “double emission” and “double absorption” terms \( f_1 f_2 \) and \( f_1^2 f_2^2 \), which we could find in the coupling of the original Hamiltonian Eq. [21] \( \lambda \sigma_z^f \sigma_z^c \). In some particular scenarios, it is possible to perform a rotating wave approximation on such direct interaction, and to write it as \( \lambda(\sigma_z^f \sigma_z^c + \sigma_z^c \sigma_z^f) \),
which does not add excitations into the system. In this case, diagonalizing the system Hamiltonian is easier and can be done by just a single rotation [4]. Anyway, with the aim at a more complete description, we keep the counter-rotating terms in the Hamiltonian and we write the operators as:

\[
\begin{align*}
\sigma_1^z &= (\cos 2\theta + \cos 2\phi) f_1^\dagger f_1 + (\cos 2\theta - \cos 2\phi) f_2^\dagger f_2 - \cos 2\theta - 2 \left( \cos \phi \sin \phi (f_1 f_2^\dagger + h.c.) + \cos \theta \sin \theta (f_1 f_2 + h.c.) \right), \\
\sigma_2^z &= (\cos 2\theta + \cos 2\phi) f_2^\dagger f_2 + (\cos 2\theta - \cos 2\phi) f_1^\dagger f_1 - \cos 2\theta - 2 \left( \cos \phi \sin \phi (f_1 f_2^\dagger + h.c.) + \cos \theta \sin \theta (f_1 f_2 + h.c.) \right).
\end{align*}
\] (D14)

Finally, we find the new basis that diagonalizes \( H_S \) as a function of the canonical basis \{\{11\}, \{10\}, \{01\}, \{00\}\}, which corresponds respectively to both spins up, first spin up and second down, etc... To represent the excitation basis of each couple of fermionic operators, we employ a subscript indicating to which operator we are referring, while we do not use subscripts for the canonical basis; for instance from Eq. [D1] we understand that:

\[
\begin{align*}
|00\rangle_c &= |11\rangle, \\
|01\rangle_c &= |10\rangle, \\
|10\rangle_c &= |01\rangle, \\
|11\rangle_c &= |00\rangle. 
\end{align*}
\] (D15)

From Eq. [D4] we see that the vacuum state of \( f_1, f_2 \) is the fully-excited state of \( \xi_1, \xi_2 \), i.e. \( |00\rangle_f = |11\rangle_c \). In order to find \( |00\rangle_f \), i.e. the ground state of \( H_S \), we thus impose that \( \xi_1 \) and \( \xi_2 \) applied on a linear combination of the states in Eq. [D15] read 0. For instance,

\[
\xi_1^\dagger \sum_{\alpha, \beta = 0, 1} a_{\alpha \beta} |\alpha \beta\rangle_c = 0 \\
\Rightarrow \cos \theta a_{00} = -\sin \theta a_{11}, \quad a_{01} = 0.
\] (D16)

Finally we have:

\[
|00\rangle_f = + \sin \theta |11\rangle - \cos \theta |00\rangle.
\] (D17)

The remaining states are obtained by applying \( f_1^\dagger \) and \( f_2^\dagger \) on the ground state, and they read:

\[
|01\rangle_f = -\sin \phi |10\rangle + \cos \phi |01\rangle, \\
|10\rangle_f = -\cos \phi |10\rangle - \sin \phi |01\rangle, \\
|11\rangle_f = + \cos \theta |11\rangle + \sin \theta |00\rangle.
\] (D18)

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