Bayesian Estimation of Fault Slip Distribution for Slow Slip Events Based on an Efficient Hybrid Optimal Directional Gibbs Sampler and Its Application to the Guerrero 2006 Event

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Abstract
An efficient Bayesian approach is proposed to infer fault slip from geodetic data in a Slow Slip Event (SSE). The physical model of the slip process reduces to a multiple linear regression with constraints. Assuming a Gaussian model for the geodetic data and considering a multivariate truncated normal prior distribution for the unknown fault slip, the resulting posterior distribution is also a multivariate truncated normal. A prior slip distribution having a detailed correlation structure to impose natural coherence in the fault slip is proposed. Regarding the posterior, an ad hoc algorithm based on a Hybrid Optimal Directional Gibbs sampler is proposed that allows to sample efficiently from the resulting high-dimensional posterior slip distribution without supercomputing resources. A synthetic fault slip example illustrates the flexibility and accuracy of the proposed approach. This methodology is also applied to a real data set for the 2006 Guerrero, Mexico, SSE, where the objective is to recover the fault slip on a known interface that produces displacements observed at ground geodetic stations. As a by-product, our approach further allows us to estimate the Moment Magnitude for the 2006 Guerrero SSE with uncertainty quantification.

Keywords Multiple linear regression model · Slow slip events · Mexico subduction zone · Multivariate truncated normal distribution · Bayesian uncertainty quantification · Markov chain Monte Carlo

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1 Introduction

A major task of geophysics is to make quantitative statements about Earth’s interior in terms of surface measurements. One fundamental element of earthquake investigations is estimating the slip’s magnitude and distribution along a fault plane. Fault slips may consist of complex and heterogeneous source processes, while limited geodetic data typically leads to an ill-posed inverse problem (IP). Conventionally, regularization is used to transform such IPs into a well-posed optimization problem for a single-source model. The most common approach is adding Tikhonov regularization terms to smooth the solution (Wallace and Beavan 2010; Radiguet et al. 2011), as well as including positivity constraints and reducing the solution space (Tago et al. 2021). All these regularization terms try to be justified by the modeled physical processes. These strategies lead to the solution of a well-posed approximate and computationally feasible problem. However, regularization schemes are not suited for formal uncertainty quantification. These schemes produce only limited point-wise solution estimates. Moreover, the lack of sufficient physical interpretation of some critical regularization terms may introduce bias in the solutions. For more robust and informative IP solutions and to quantify their uncertainty formally, a different approach is required.

The Bayesian statistical approach provides a more rigorous framework to handle constraints and uncertainty quantification (UQ) of IPs. A model for observations is assumed (leading to a likelihood), and prior information is incorporated using probability density functions (PDFs) to determine the posterior distribution through the Bayes theorem, which quantifies our inference’s uncertainty. A prior PDF is established through its interpretation as a modeling device of the probabilistic prior knowledge available on source slips and imposing model restrictions using truncated PDFs (Minson et al. 2013; Nocquet 2018). In modern Bayesian analyses, Markov Chain Monte Carlo (MCMC) algorithms are standard tools to sample from the posterior distribution. Many versions of the MCMC method have been proposed in the literature, but the Metropolis-Hastings and the Gibbs sampler algorithms are the most common (Robert and Casella 2013).

Bayesian techniques have been less frequently used for slip inversions, mainly because the simulation from the posterior distribution is not straightforward. This problem typically occurs when the number of parameters to estimate is large (of order $10^3$ or higher), such as in SSEs. In Minson et al. (2013), the authors developed a framework for Bayesian inversion of finite fault earthquake models. They combined a Metropolis algorithm with simulated annealing and genetic algorithms to sample high-dimensional problems in a parallel computing framework. The method remains computationally expensive despite parallelization. Finite-dimensional Gaussian process approximation to allow for inequality constraints in the entire domain is proposed in Maatouk and Bay (2017). Their problem is equivalent to simulating a Gaussian vector restricted to convex sets, and they use an improved rejection sampling in which only the random coefficients in the convex set are selected. The authors mention that these kinds of methods work well when the acceptance rate is reasonably high. However, they are inefficient when the acceptance rate is low, as in the case of high-dimension and/or when the support is narrowly bounded. Maatouk and Bay (2017) also mention
that MCMC or Gibbs sampling methods can accelerate the multivariate truncated normal (MTN) simulation. There are many methods available for simulating the MTN distribution (Kotecha and Djuric 1999; Robert and Casella 2013). Most of these are based on the Gibbs sampler, which is simple to use and has the advantage of accepting all proposals generated and, therefore, is not affected by poor acceptance rates, such as rejection sampling. These methods work well in many situations but may become very slow in the presence of high correlation and/or high-dimensionality.

Nocquet (2018) shows that a MTN prior can be applied to achieve positivity or bound constraints. The author employs recent findings in MTN probability calculations to derive relevant posterior statistics (e.g., posterior marginal PDF, mean, and covariances) without performing MCMC sampling. However, evaluating these quantities requires complex numerical integration over a hyper-rectangle, while quantifying the uncertainty of a function of the parameters (e.g., Moment Magnitude $M_w$) is not straightforward. In contrast, when Monte Carlo samples from the posterior are available, the posterior uncertainty of the quantities of interest may be directly obtained. In Michalak (2008), the author provides a statistically rigorous methodology for geostatistical interpolation and inverse modeling, subject to multiple and spatially-variable inequality constraints. Using a Gibbs sampler to characterize the marginal probability distribution at each estimation point, the approach uses a MTN prior probability distribution. These types of algorithms are systematic Gibbs samplers for which CPU time increases linearly with dimension (Christen et al. 2017). An optimality criterion for the MCMC direction Gibbs algorithm sampling from a MTN distribution was proposed in Christen et al. (2017). The latter criterion consists of minimizing the Mutual Information between two consecutive steps of the Markov chain, resulting in an algorithm well-suited for high correlation and dimensionality. This algorithm’s main advantage is that the number of iterations to obtain an independent sample increases linearly with dimensionality and is typically close to theoretical limits.

This work proposes a Bayesian approach for estimating the parameters in a constrained multiple linear regression model. A prior slip distribution with a detailed correlation structure is proposed to impose natural coherence in the fault slip. Combining the algorithm given in Christen et al. (2017) and in Montesinos-López (2016), this paper proposes a Hybrid Optimal Directional Gibbs algorithm that allows us to sample from high-dimensional problems efficiently when the posterior is a MTN distribution. With the methodology from Christen et al. (2017), the burn-in period is reduced, while with the Montesinos-López (2016) approach, the chain’s correlation is reduced. Besides presenting a synthetic example, our method is applied to quantify the uncertainty in the IP of seismic slip along the subduction interface in the 2006 Guerrero, Mexico, SSE. Moreover, with our method, it is possible to provide the posterior distribution of the Moment Magnitude for this event in a straightforward manner.

A SSE is a slip produced at a fault that does not generate seismic waves. However, the induced deformation may be registered at the surface from SSEs, lasting several weeks to a few months. SSEs have been observed in different fault configurations around the world, and the role they play in the seismic cycle is an active research topic (Radiguet et al. 2011; Cruz-Atienza et al. 2021).

In Mexico, SSEs have been identified in different segments along the subduction region on the Pacific coast, where the Cocos Plate and the North American Plate collide.
In the so-called Guerrero GAP (GGap), before the great Mw8.2 Tehuantepec event on 8 September 2017, SSEs showed a periodicity of approximately four years and a duration from six to twelve months (Cruz-Atienza et al. 2021). The SSE occurring within Guerrero state in 2006, one of the most studied globally, was recorded at 15 continuous GPS stations (Radiguet et al. 2011; Cavalié et al. 2013; Tago et al. 2021) as millimetric displacements on the surface. This event offers the opportunity to analyze the slip’s spatial evolution and delimit the characteristics of a typical SSE in the GGap. In this case, the IP consists of recovering the slip along a known interface that produces displacements at the surface observed at those 15 continuous GPS geodetic stations.

2 Methodology

2.1 Bayes Inference

A wide range of applications are concerned with the solution of an IP (Kaipio and Somersalo 2006): given some observations of an output, \( y = (y_1, \ldots, y_n) \), determine the corresponding inputs \( \theta \).

IPs appear in many branches of science and mathematics, mainly in situations where quantities of interest are different from those that can be measured. In IPs, model parameter values must be estimated from the observed data. Historically, IPs have been studied as deterministic problems in analysis and only, relatively recently, have been correctly treated as problems in statistical inference. In deterministic inversion from noisy data, the IPs may have no solution, or the solution may not be unique, or it may depend sensitively on measurements \( y_i \) (Kaipio and Somersalo 2006). A way to approach these difficulties is to formulate the IP in the Bayesian framework. Stuart (2010) studied conditions for the well-posedness of one particular Bayesian formulation of IPs. In this framework, a noise model is assumed for the observations (e.g., an additive Gaussian noise model)

\[
y_i = F(\theta) + \varepsilon_i,
\]

where the errors, \( \varepsilon_i \), follow a normal distribution with mean zero and variance \( \sigma^2 \). Evaluations of \( F \) are referred to as solutions of the forward problem and, consequently, \( F \) is called the Forward Map (FM). In general, the FM is a complex non-linear map with input parameters \( \theta \), defined by an initial/boundary value problem for a system of ordinary or partial differential equations.

The observational model (1) generates a PDF of \( y \) given the parameters \( \theta \), namely \( P_{Y|\theta} (y|\theta) \). For fixed data \( y \), and as a function of \( \theta \), this is called the likelihood function. Based on the available information, a prior model \( P_{\theta}(\cdot) \) is stated for \( \theta \), and a posterior distribution is obtained through the Bayes theorem

\[
P_{\theta|Y}(\theta|y) = \frac{P_{Y|\theta}(y|\theta) P_{\theta}(\theta)}{P_Y(y)}.
\]
The denominator is the normalization constant, also called the marginal likelihood of the observations or model evidence.

In a frequentist (classical) statistical paradigm, the likelihood is usually maximized to obtain a single estimate of the parameter of interest. Uncertainty is defined by the sampling distribution based on the idea of infinite repeated sampling. In contrast, the goal of Bayesian inversion is not only to obtain a single estimate for the unknowns but to quantify their uncertainty consistently with the observed data. Therefore, the unknowns are described by probability distributions. Before any observation is available, there exists a large uncertainty in the unknown. After making the measurements, the uncertainty is reduced, and the task is to quantify it and provide probabilistic answers to questions of interest. Previous information regarding the physics of the problem, which is not specific enough to be incorporated into the direct problem, may be incorporated into the prior probability distribution. The posterior distribution of the unknown parameter quantifies the uncertainty of the possible values of these parameters consistent with the observed data.

2.2 Forward Map

In our particular setting, the direct problem begins with the elastostatic equations representation theorem that models the displacement $u(x)$, at the coordinates $x$ of the GPS station, due to a slip $d(\xi)$, produced at fault $\Gamma$, as

$$u_j (x) = \int_\Gamma T_k \left( S_{ij} (\xi; x), \hat{n} (\xi) \right) d_k (\xi) dS(\xi), \quad j \in \{x, y, z\},$$

where $T_k (\cdot, \cdot)$ is the $k$-component of the traction on the fault computed through the Somigliana tensor, $S_{ij} (\xi; x)$, $i$-component of the displacement due to an impulse force acting in the $j$-direction, and the fault normal vector $\hat{n} (\xi)$ (Udías et al. 2014). Equation (2) can be obtained by imposing a discontinuity in the displacements over a focal region and the representation theorem applied to the elastostatics equations. Moreover, Eq. (2) can also be obtained from Eq. (3.70) of Udías et al. (2014) by taking the limit $t \to \infty$. This formulation was also used in Tago et al. (2021), but others have probably used it before. For instance, the Okada model (Okada 1985), which is considered a homogeneous medium, is the most popular approach. If the traction and the slip are projected along the dip component, $d$-direction, and along the strike direction, $s$-direction, Eq. (2) can be written in matrix form as

$$\begin{bmatrix} u_x (x) \\ u_y (x) \\ u_z (x) \end{bmatrix} = \int_\Gamma \begin{bmatrix} T_s (S_{ix} (\xi; x), \hat{n} (\xi)) \\ T_s (S_{iy} (\xi; x), \hat{n} (\xi)) \\ T_s (S_{iz} (\xi; x), \hat{n} (\xi)) \end{bmatrix} \begin{bmatrix} d_s (\xi) \\ d_d (\xi) \end{bmatrix} dS(\xi),$$

or in a more compact vector notation as

$$u (x) = \int_\Gamma T (\xi; x) d (\xi) dS(\xi).$$
Under the assumption that the fault’s geometry is known, it is possible to discretize it into \( M \) subfaults. Denoting the subfaults centroid coordinates by \( \{\xi^1, \xi^2, \ldots, \xi^M\} \), the integral can be approximated as

\[
\mathbf{u}(\mathbf{x}) \simeq \sum_{i=1}^{M} A^i \mathbf{T} \left( \xi^i; \mathbf{x} \right) \mathbf{d} \left( \xi^i \right),
\]

where \( A^i \) is the \( i \)-subfault area. Finally, to compute the displacement for \( N \) receivers, the displacements can be arranged in a single vector such that the entire computation is reduced to a simple matrix–vector product as

\[
\begin{bmatrix}
\mathbf{u}(\mathbf{x}^1) \\
\mathbf{u}(\mathbf{x}^2) \\
\vdots \\
\mathbf{u}(\mathbf{x}^N)
\end{bmatrix} =
\begin{bmatrix}
A^1 \mathbf{T} \left( \xi^1; \mathbf{x}^1 \right) & A^2 \mathbf{T} \left( \xi^2; \mathbf{x}^1 \right) & \cdots & A^M \mathbf{T} \left( \xi^M; \mathbf{x}^1 \right) \\
A^1 \mathbf{T} \left( \xi^1; \mathbf{x}^2 \right) & A^2 \mathbf{T} \left( \xi^2; \mathbf{x}^2 \right) & \cdots & A^M \mathbf{T} \left( \xi^M; \mathbf{x}^2 \right) \\
\vdots & \vdots & \ddots & \vdots \\
A^1 \mathbf{T} \left( \xi^1; \mathbf{x}^N \right) & A^2 \mathbf{T} \left( \xi^2; \mathbf{x}^N \right) & \cdots & A^M \mathbf{T} \left( \xi^M; \mathbf{x}^N \right)
\end{bmatrix}
\begin{bmatrix}
\mathbf{d}(\xi^1) \\
\mathbf{d}(\xi^2) \\
\vdots \\
\mathbf{d}(\xi^M)
\end{bmatrix},
\]

or more compactly as

\[
\mathbf{U} = \mathbf{XD},
\]

where \( \mathbf{U} \in \mathbb{R}^{3N} \) is the vector of surface displacements, North, East, and Vertical, for the \( N \) stations stored in a single ordered vector, \( \mathbf{X} \in \mathbb{R}^{3N \times 2M} \) is the discretized FM operator built with the subfault tractions and areas, and \( \mathbf{D} \in \mathbb{R}^{2M} \) is the unknown vector of subfault slips, along strike (\( d_s \)) and dip (\( d_d \)) for each of the \( M \) subfaults stored in a single ordered vector \( \mathbf{D} := (d_{s1}, d_{d1}, d_{s2}, d_{d2}, \ldots, d_{sM}, d_{dM})^T \).

Finally, note that the Somigliana tensor required for the tractions may be computed for heterogeneous media. In fact, the Somigliana tensor used in this manuscript was computed for a regional non-homogeneous velocity model of Mexico, which is a 4-layer one-dimensional structure (Campillo et al. 1996).

### 2.3 Data Likelihood

The IP consists of recovering the slip at each subfault of a known interface that produces displacements observed at geodetic stations. Due to the linearity of the FM, in Eq. (3), the Bayesian inversion is solved as a multiple linear regression model with constraints on the coefficients.

A simple representation of the observation and modeling errors is used, assuming a multiple linear Gaussian model of the form

\[
\mathbf{Y} = \mathbf{XD} + \mathbf{\varepsilon},
\]

where \( \mathbf{\varepsilon} \) follows a Gaussian distribution, \( \mathbf{\varepsilon} \sim \mathcal{N}_3(\mathbf{0}, \Sigma) \), and \( \Sigma = \mathbf{I} \otimes \gamma \) is a known covariance matrix associated with the data errors, with \( \gamma = \text{diag} \left( \left[ \sigma_x^2, \sigma_y^2, \sigma_z^2 \right] \right) \) being...
the North, East, and Vertical standard deviations of the measurements process, \( \mathbf{I} \) is an identity matrix of order \( N \), and \( \otimes \) denote the Kronecker product. That is

\[
\mathbf{Y}\mid \mathbf{D} \sim N_{3N} (\mathbf{XD}, \Sigma),
\]

where \( \mathbf{Y} \in \mathbb{R}^{3N} \) are the observed displacements (3 components for each station) at the \( N \) geodetic stations stored in a single ordered vector, as in Eq. (3). That is, the errors are conditionally independent given \( \mathbf{D} \). Therefore, the likelihood is given by

\[
\pi (\mathbf{Y}\mid \mathbf{D}) = (2\pi)^{-3N/2} |\mathbf{A}|^{1/2} \exp \left\{ -\frac{1}{2} (\mathbf{Y} - \mathbf{XD})^T \mathbf{A} (\mathbf{Y} - \mathbf{XD}) \right\},
\]

where \( \mathbf{A} = \Sigma^{-1} \) is the precision matrix.

2.4 Prior Elicitation

Bayesian formulation of IPs requires specifying a prior distribution for each model parameter. The proposal of the prior density is an essential step of Bayesian analyses and is often the most challenging and critical part of the approach. Usually, the major problem while proposing an adequate prior density lies in the nature of the prior information. The likelihood typically dominates the posterior distribution for larger sample sizes, rendering prior specification less critical. In contrast, the prior distribution plays a much more crucial role in small samples because the posterior distribution represents a compromise of the prior knowledge and the observed evidence. On the one hand, for the 2006 Guerrero SSE, there are only 45 observations (\( N = 15 \) GPS stations) and 1178 parameters (\( M = 589 \) subfaults) to estimate (for the proposed model). On the other hand, the problem is so ill-posed that the inversion becomes useless unless prior knowledge is put into the problem in terms of at least simple restrictions on the possible solutions for \( \mathbf{D} \). Thus, defining an adequate prior distribution is crucial for the inversion within the framework of a formal Bayesian approach. A Gaussian process prior is used for the joint PDF of the fault slip displacements \( \mathbf{D} \).

In statistics, a Gaussian process is a stochastic process (i.e., a collection of random variables indexed by time or space) such that every finite collection of those random variables has a multivariate normal distribution. The most commonly used probability densities in statistical IPs are undoubtedly Gaussian because they are easy to construct. However, they form a much more versatile class of densities than is usually believed (Kaipio and Somersalo 2006).

For the slip vector \( \mathbf{D} \), a GP prior distribution is considered, that is, \( \mathbf{D} \sim N (\mathbf{0}, \Sigma_0) \), but with truncated support, along strike \( d_s^i \in (a_s, b_s) \) and along dip \( d_d^i \in (a_d, b_d), i = 1, \ldots, M, \) where \( \Sigma_0 = \sigma^2_\beta \mathbf{A}_0^{-1} \) denote the covariance matrix, with \( \mathbf{A}_0 = \beta \mathbf{W} \mathbf{C}^{-1} \mathbf{W} \beta, \) and \( \sigma^2_\beta \) is an unknown scaling factor that characterizes the variability of \( \mathbf{D} \) (see Appendix B with further considerations about how to establish the hyperparameter \( \sigma_\beta \)), and \( \beta = \mathbf{I} \otimes \beta, \) with \( \beta = \text{diag} ([\beta_s, \beta_d]) \), where different precisions, \( \beta_s \) and \( \beta_d \), are considered for the along strike and along dip components, respectively. The proposal is to consider an along dip variance five times greater than the along strike.
variance (i.e., $\beta_s = 1$ and $\beta_d = 1/5$); because it is expected that most of the slip occurs along the opposite of the subduction direction. In the Mexican subduction region, no SSEs have been inverted (Bekaert et al. 2015; Radiguet et al. 2011) at depths greater than 50 km. Therefore, the matrix $W$ of weights is included to penalize slips at depths greater than $z_{\text{lim}} = 50$ km as follows

$$W(i, j) = \begin{cases} 1 + 0.5 (\text{depth}(i, j) - z_{\text{lim}}) / 1e3 & \text{depth}(i, j) > z_{\text{lim}}, \\ 1 & \text{depth}(i, j) \leq z_{\text{lim}}. \end{cases} \quad (6)$$

Note that these weights lead to more realistic modeling than truncating the region, which would lead to an abrupt slip depth restriction.

The correlation matrix $C$ is used to introduce correlation between parameters of nearby subfaults, and it is constructed using the Matérn covariance function, explained in Appendix A. Different correlation lengths used in the Matérn covariance function, $\lambda_s$ and $\lambda_d$, are considered for the along-strike and along-dip components, respectively. Also, it is considered that the strike component of subfault $i$ has zero correlation with the dip component of the subfault $j$. However, estimating $\lambda_s$ and $\lambda_d$ is commonly very difficult, and in this case, a more pragmatic approach is taken by performing a sensitivity analysis along a set of reasonable values for these correlation lengths $\lambda_s$ and $\lambda_d$; see Sect 2.7 for more details.

Thus, the prior density is

$$\pi(D) = \frac{1}{Z_{\text{prior}}} \exp \left\{ -\frac{1}{2\sigma^2} D^T A_0 D \right\} \mathbb{I}_{(a, b)}(D), \quad (7)$$

where $\mathbb{I}_{(a, b)}$ is the indicator function, $a = 1_M \otimes [a_s, a_d]$, $b = 1_M \otimes [b_s, b_d]$, with $1_M$ the all-ones vector of length $M$, and $Z_{\text{prior}}$ is an unknown normalization constant of this MTN distribution. The constraints $a_s \leq d_s \leq b_s$ and $a_d \leq d_d \leq b_d$, $i = 1, \ldots, M$, imposed on $D$, are based on available information on the physical processes being modeled. If it is assumed that the coupling has been removed from the GPS data, then only slips due to an SSE should be considered. However, a slightly negative slip for the dip component ($a_d < 0$) is permitted because, on the one hand, no slips are expected in some subfaults (If a random variable is positive, then its expected value is positive. In this way, if $a_d > 0$ is considered, it would force small slips in all subfaults.), and on the other hand, the coupling removal is not precise. Together with the constraints on the support, the density function Eq. (7) is indeed a MTN distribution.

### 2.5 Posterior Distribution

The likelihood function given in Eq. (5) and the prior distribution given in Eq. (7) are combined via Bayes’ theorem to form the so-called posterior distribution, namely
\[ \pi (D|Y) \propto \pi (Y|D) \pi (D), \]
\[ \propto \exp \left\{ -\frac{1}{2} (Y - XD)^T A (Y - XD) \right\} \exp \left\{ -\frac{1}{2\sigma^2_D} D^T A_0 D \right\} \mathbb{I}_{(a,b)}(D), \]
\[ = \exp \left\{ -\frac{1}{2} \left[ D^T X^T A X D - 2Y^T A X D + \frac{1}{\sigma_D^2} D^T A_0 D \right] \right\} \mathbb{I}_{(a,b)}(D), \]
\[ = \exp \left\{ -\frac{1}{2} \left[ D^T \left( X^T A X + \frac{1}{\sigma_D^2} A_0 \right) D - 2D^T X^T A Y \right] \right\} \mathbb{I}_{(a,b)}(D), \]
\[ = \exp \left\{ -\frac{1}{2} \left[ (D - \mu_p)^T A_p (D - \mu_p) \right] \right\} \mathbb{I}_{(a,b)}(D), \]
where \( A_p = X^T A X + \frac{1}{\sigma_D^2} A_0, \) \( A_0 = \beta WC^{-1} W \beta \) and \( \mu_p = A_p^{-1} X^T A Y. \) Thus
\[ \pi (D|Y) = \frac{1}{Z_{post}} \exp \left\{ -\frac{1}{2} \left[ (D - \mu_p)^T A_p (D - \mu_p) \right] \right\} \mathbb{I}_{(a,b)}(D), \]
where \( Z_{post} \) is an unknown normalization constant. Therefore, \( D|Y \) has a MTN distribution.

To obtain information from \( D|Y, \) one needs to calculate relevant posterior statistics (e.g., marginals of the subfaults slips, expected values, quantifying the uncertainty of a function of \( D, \) etc.). This is not trivial, and accordingly, the use of a MCMC sampler is proposed. In the next section, an Algorithm to simulate from the MTN distribution is introduced.

### 2.6 Posterior Exploration and MCMC

The MCMC simulation methods are algorithms used to produce samples from a distribution \( \pi, \) which is usually complex, without simulating such distribution directly. These methods are based on constructing an ergodic Markov chain \( X^{(t)} \) for which its stationary distribution is precise \( \pi. \) These methods have proven to be very useful in several areas, particularly in Bayesian statistics (Minson et al. 2013; Michalak 2008). There are several methods available to sample from complex posterior distributions, such as the hit-and-run sampler (Kaufman and Smith 1998) or the Hamiltonian Monte Carlo methods (Girolami and Calderhead 2011). However, the Optimal Direction Gibbs (ODG) sampler is especially suited to sample from a MTN posterior because the full conditional is tractable and easy to simulate for each direction, and the Hessian is explicit, from which the optimal directions can be obtained. Moreover, as shown in Christen et al. (2017), the efficiency in MTN canonical examples reaches the minimum theoretical Integrated Autocorrelation Time.

The Gibbs sampler (Gelfand and Smith 1990) is an MCMC algorithm that systematically or randomly simulates conditional distributions on a set of directions. A
The general case of the Gibbs sampler is the ODG sampling, which chooses an arbitrary direction \( e \in \mathbb{R}^n \) such that \( \|e\| = 1 \), and samples from the conditional distribution along such direction; by choosing the directions systematically or randomly from the canonical base, the standard Gibbs sampler or the Random Scan Gibbs is obtained, respectively. This can be written as

\[
X^{(t+1)} = X^{(t)} + re,
\]

where the length \( r \in \mathbb{R} \) has distribution proportional to \( \pi(X^{(t)} + re) \).

The authors in Christen et al. (2017) propose using mutual information as a measure of dependence between two consecutive iterations of the Gibbs sampler. The mutual information of two random variables \( X \) and \( Y \), \( I(X, Y) \), is the Kullback–Leibler divergence between the joint model \( f_{Y,X} \) and the independent alternative \( f_Y(y) f_X(x) \), namely

\[
I(X, Y) = \int \int f_{Y,X}(y, x) \log \frac{f_{Y,X}(y, x)}{f_Y(y) f_X(x)} \, dx \, dy.
\]

From the properties inherited from the Kullback–Leibler divergence \( I \geq 0 \) and from the Jensen inequality, it is easy to prove that \( I = 0 \) if and only if \( f_{Y,X} = f_Y(y) f_X(x) \) (i.e., if and only if \( X \) and \( Y \) are independent).

The authors in Christen et al. (2017) explore an optimality criterion for the Direction Gibbs algorithm from mutual information. This criterion minimizes the mutual information between two consecutive steps, \( X^{(t)} \) and \( X^{(t+1)} \), of the Markov chain generated by the algorithm. In a heuristic way, they also propose a direction distribution for the case where the target distribution is the MTN distribution. They take the directions, \( e \), as the eigenvectors of the precision matrix \( \mathbf{A} \), so \( e = \{e_1, e_2, \ldots, e_n\} \). The \( i \)-th direction will be selected with probability proportional to \( \lambda_i^{-b} \), where \( \lambda_i \) is the eigenvalue corresponding to the \( i \)-th eigenvector, \( i = 1, 2, \ldots, n \), and \( b \) is a random variable with Beta distribution. Then, the probability of selecting the \( i \)-th direction is given by

\[
h_1(e_i) = \lambda_i^{-b} / k_1,
\]

where \( k_1 = \sum_{i=1}^n \lambda_i^{-b} \). See Christen et al. (2017) for more details.

Now, let \( X = X^{(t)} \) and \( Y = X^{(t)} + re \) be two consecutive steps and denote by \( X_i, Y_i \), \( i = 1, \ldots, n \), the elements of \( X \) and \( Y \), respectively. In Montesinos-López (2016), the authors propose the Mutual Information as a dependence measure, but now no longer on the complete vectors \( X \) and \( Y \). Instead, only on \( Y_i \) and the full vector \( X \); they call it the marginal mutual information and is written as \( I_e(Y_i, X^{(t)}) \), that is

\[
I_e(Y_i, X) = \int \int f_{Y_i,X}(y, x) \log \frac{f_{Y_i,X}(y, x)}{f_{Y_i}(y) f_X(x)} \, dx \, dy.
\]
The idea is to choose directions for which \( I_e(Y_i, X), \forall i = 1, \ldots, n \), is minimized. In this way, the dependency of each entry of the new generated vector \( Y \) with the current state \( X \) is reduced.

Assume a multivariate normal distribution is given, with the precision matrix \( A_{n \times n} \) and mean vector \( \mu_{n \times 1} \), but with truncated support, \( x_i \in (a_i, b_i), -\infty \leq a_i < b_i \leq \infty, i = 1, \ldots, n \). The PDF of this MTN can be written as

\[
\pi(x; \mu, A, a, b) = \frac{\exp \left\{ -\frac{1}{2} (x - \mu)^T A (x - \mu) \right\}}{\int_a^b \exp \left\{ -\frac{1}{2} (x - \mu)^T A (x - \mu) \right\} dx},
\]

To generate samples from the MTN distribution, in Montesinos-López (2016), the authors take the directions \( e \) as the standardized columns of the covariance matrix \( A^{-1} \), so \( e = \{e_1, e_2, \ldots, e_n\} \). The \( i \)-th direction will be selected with probability \( (h_2(e_i)) \) proportional to \( I^{-1}_i \), with

\[
I_i := \sum_{j=1}^n I_{e_i}(Y_j, X) = -\frac{1}{2} \sum_{j=1}^n \log(\rho_{ij}^2),
\]

where \( \rho_{ij} \) is the correlation between the variables \( Z_i \) and \( Z_j \), with \( Z \sim \pi \). Then, the probability of selecting the \( i \)-th direction is given by

\[
h_2(e_i) = I^{-1}_i / k_2,
\]

where \( k_2 = \sum_{i=1}^n I^{-1}_i \). See Montesinos-López (2016) for more details of its derivation. Thus, they give more weight to the directions that make the \( I_i \)'s small.

In this article, the probability of selecting the \( i \)-th direction is slightly modified as follows

\[
h_3(e_i) = I^{-b}_i / k_3,
\]

where \( k_3 = \sum_{i=1}^n I^{-b}_i \) and \( b \) is a random variable with Beta distribution. This modification slightly reduces the time to obtain a pseudo-independent sample (by 3%) and also reduces the initial burn-in length.

These algorithms are referred to as ODG1 or ODG3 when the direction distribution is \( h_1 \) or \( h_3 \), respectively. This paper combines the ODG1 and ODG3 algorithms in a hybrid Gibbs. The ODG1 algorithm provides a faster convergence to the target distribution reducing the burn-in, while with the ODG3 algorithm, the chain’s correlations are reduced, saving 60% of the time to obtain a pseudo-independent sample compared to the ODG1 algorithm. Combining both algorithms reduces the chain’s correlations, and the support is better explored. The resulting algorithm is called the HODG algorithm.
2.6.1 Algorithmic Framework

The proposed method to simulate the MTN distribution will be divided into two algorithms. Algorithm 1 presents the preprocessing steps in which the directions ($e_i$’s) and weights needed to create the distribution of the directions $h(\cdot)$ are computed. In Algorithm 2, the iterative process to generate the Markov chain is presented. With probability $p = 0.5$ it is simulated from the ODG1 algorithm, and with probability $1 - p = 0.5$, it is simulated from the ODG3 algorithm. Parameter $b$ is considered random to select directions with very low selection probabilities, thus allowing a better mix and avoiding directions that get trapped in some density areas. Finally, a step length $r$ of a univariate truncated normal distribution is proposed to guarantee that the new state is in the bounded support.

**Algorithm 1: Preprocessing**

**input:** The mean vector $\mu_{n \times 1}$ and the precision matrix $A_{n \times n}$.

Step 1. Compute the eigenvectors and eigenvalue of the precision matrix $A$, \( \{e_1, e_2, \ldots, e_n\} \) and $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively;

Step 2. Normalize the columns of the covariance matrix $\Sigma = A^{-1}$, these will be \( \{e_{c1}, e_{c2}, \ldots, e_{cn}\} \);

Step 3. Compute the correlation matrix ($\rho$) corresponding to $\Sigma$;

Step 4. Compute the weights $I_i$

$$I_i = -\frac{1}{2} \sum_{j=1}^{n} \log \left( \rho_{ij}^2 \right),$$

where $\rho_{ij}$ is the correlation between the variables $X_i$ and $X_j$;

2.7 Pragmatic Approach to Correlation Lengths Selection

Note that if the full conditional distribution for the slip vector $D$ had a known tractable form, then full MCMC sampling could be performed on the low-dimensional marginal posterior distribution over hyperparameters (Norton et al. 2018; Fox and Norton 2016), namely $\lambda_s$ and $\lambda_d$. In Fox and Norton (2016), the authors propose sampling the low-dimensional marginal posterior distribution over hyperparameters in a linear-Gaussian inverse problem. Then, the full conditional for the latent variable is sampled using only the approximately independent samples of the hyperparameters obtained from a MCMC run over its marginal distribution. However, in our case, sampling from the marginal distribution of $\lambda_s$ and $\lambda_d$ can be challenging because the way these parameters interact in the correlation matrix $C$ makes it impossible to factorize them. In addition, marginalizing these hyperparameters would require evaluating the normalization constant of the MTN posterior, which requires a very high computational cost (Nocquet 2018).
Algorithm 2: HODG: Multivariate Truncated Normal

**input**: The mean vector $\mu_{n \times 1}$, the precision matrix $A_{n \times n}$, the directions $\{e_1, e_2, \ldots, e_n\}$ and $\{e'_1, e'_2, \ldots, e'_n\}$, the weights $l_i$, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, the support $(a, b)$, a initial value $X(0)$, and the number of simulations $M$.

**output**: A sample of size $M$ from $X \sim \text{MTN}(\mu_{n \times 1}, A_{n \times n}, a, b)$, with the truncated support $x_i \in (a_i, b_i)$, $i = 1, \ldots, n$.

for $t \leftarrow 1$ to $M$

Set $x = X(t-1)$;

Simulate $b \sim B(2, 9)$, where $B$ is the Beta distribution;

Simulate $p \sim U(0, 1)$, where $U$ is the uniform distribution;

if $p < 0.5$ then

$h(e_i) = \lambda_i^{-b_i} / k_1$, where $k_1 = \sum_{i=1}^{n} \lambda_i^{-b_i}$;

else

$h(e'_i) = I_i^{-b_i} / k_2$, where $k_2 = \sum_{i=1}^{n} I_i^{-b_i}$;

Propose a direction $e$ from the direction distribution $h(\cdot)$;

Simulate $r \sim TN(\mu_r, \tau_r, c, d)$, where $TN$ is the univariate truncated normal distribution, $\mu_r = -e^T A(x - \mu)$ is the mean, $\tau_r = e^T Ae$ is the precision, and $c = \max_{i \in \{1, \ldots, n\}} \left\{ \frac{a_i - x_i}{e_i} : e_i > 0 \right\} \cup \left\{ \frac{b_i - x_i}{e_i} : e_i < 0 \right\}$,

d = \min_{i \in \{1, \ldots, n\}} \left\{ \frac{a_i - x_i}{e_i} : e_i < 0 \right\} \cup \left\{ \frac{b_i - x_i}{e_i} : e_i > 0 \right\}$.

Set $X(t) = x + re$;

Setting a prior distribution for $\lambda_s$ and $\lambda_d$ and including these parameters in the posterior would be the theoretically way to proceed, but it is computationally unfeasible, as just noted.

The correlation length selection for the MTN prior model must be carefully done since each correlation length defines a different matrix $C$, possibly leading to different results. Several criteria have been proposed to select between competing correlation lengths.

In the maximum-likelihood framework, the most well-known criterion for model comparison is the Akaike Information Criterion (AIC), which involves the marginal likelihood. The deviance information criterion (DIC) has been proposed as a Bayesian alternative to the AIC (Spiegelhalter et al. 2002) to select the model that better fits the data between a pool of competing models. The DIC is particularly useful in Bayesian model selection problems where the model’s posterior distributions have been obtained by MCMC simulation. The DIC’s advantage is that it reduces each model to a single number summary and that the models to be compared do not need to be nested. However, in high-dimensional settings like ours, the Monte Carlo estimation of the DIC may become especially difficult. The DIC experiments were performed in this research to find its estimation very un-robust (results not shown).

Instead, a more pragmatic approach is considered. A sensitivity analysis was performed using prior information on the values of these hyperparameters by calculating the posterior distribution for some reasonable set of values for the correlation lengths.
\( \lambda_s \) and \( \lambda_d \). Indeed, changing the correlation lengths drastically will change the resulting posterior. Fortunately, it is able to stipulate reasonable values for these correlation lengths \( \lambda_s \) and \( \lambda_d \) using our prior knowledge about the fault slip. For a grid of reasonable combinations of these parameters, only minor differences are observed in the resulting posterior, and therefore a proxy value may be set for all conclusions. Several considerations were used in constructing the prior for \( D \) (e.g., different variances and zero correlation between along dip and along strike displacements, etc.), and it is within those constraints and within reasonable correlation lengths that the selection of \( \lambda_s \) and \( \lambda_d \) is not critical; indeed, for the examples here analyzed.

### 3 Synthetic Example

To verify how precisely a fault slip is recoverable, using the actual data collecting design for the real data presented in Sect. 4, it proceeds with the following synthetic data analysis. A synthetic data set is generated based on the same fault geometry and geodetic station configuration as the 2006 Guerrero SSE. For constructing this data set, it is assumed a priori that the slips (\( D \)) have a MTN with zero mean vector, restricting the support according to the available information on the GGap. That is

\[
D \sim N(0, \sigma_\beta^{-2}A_0)
\]

subject to \(-0.1 \text{ m} \leq d_i^s \leq 0.1 \text{ m}, -0.08 \text{ m} \leq d_i^d \leq 0.4 \text{ m}, \)

\(i = 1, 2, \ldots, M\), where \( A_0 = \beta WC^{-1}W \beta \), \( C \) is the correlation matrix given by Eq. (8), and \( W \) is the matrix of weights computed with Eq. (6). For the remaining parameters, the following is considered: \( \lambda_s = 40 \text{ km}, \lambda_d = 45 \text{ km}, \sigma_\beta^2 = 0.0002 \text{ m}, \)

\( \beta_s = 1 \) and \( \beta_d = 0.2 \). For the generation of the synthetic example, a slip is fixed at a subfault \( D_i = (d_i^s, d_i^d) \) (approximately at the same location where the maximum slip is suspected to occur in the real 2006 GGap SSE), and the remaining subfault slips are simulated from the conditional distribution \( D_{-i} | D_i = d_i \) (also a MTN), where the notation \( D_{-i} = (d_i^s, d_i^d, d_{i+1}^s, d_{i+1}^d, \ldots, d_M^s, d_M^d)\) represents the \((2M-2)\)-dimensional vector created by deleting the entries of the i-th subfault from the \(2M\)-dimensional vector \( D \).

Once the slip vector \( D \) is generated, the forward problem (\( U = XD \)) is solved, and a Gaussian noise is added to obtain our synthetic observations. Figure 1a shows how the true slip \( D \) and the synthetic measurements. Following this strategy of simulation of the synthetic data and considering the statistical model given in Sect. 2.2, simulations of the MTN distribution are obtained using the HODG algorithm. The hyperparameter \( \sigma_\beta^2 \) is set as explained in the Appendix B.

Different correlation lengths are considered for \( \lambda_s \) and \( \lambda_d \) in the corresponding a priori distributions. The posterior distribution is computed in a grid of reasonable values for the correlation lengths \( \lambda_s : [20, 90] \text{ km} \times \lambda_d : [20, 90] \text{ km} \), with 5 km steps. The resulting posteriors estimated for the fault slip are similar in all cases, as illustrated in Figs. 1b, 2a, and 3a (similar results were obtained with the rest of the correlations lengths in the above-mentioned grid, results not shown). Therefore, \( \lambda_s = 40 \text{ km} \) and \( \lambda_d = 45 \text{ km} \) were selected as proxy neutral values for these parameters.
Fig. 1 Slip models on the plate interface (heat colors) and the associated model surface displacement predictions (arrows): 

(a) True synthetic displacements

(b) Bayesian inversion: $\lambda_s = 40$ km, $\lambda_d = 45$ km

**Fig. 1** Slip models on the plate interface (heat colors) and the associated model surface displacement predictions (arrows): a true synthetic slips and b the median of the posterior samples using correlation lengths $\lambda_s = 40$ km and $\lambda_d = 45$ km
Fig. 2 Slip models on the plate interface (heat colors) and the associated model surface displacement predictions (arrows): a the median of the posterior samples for the slip inversion using correlation lengths $\lambda_s = 35$ km and $\lambda_d = 50$ km and b a MLE estimation
Fig. 3 Slip posterior median (a) and its CV expressed as a percentage (b), resulting from the Bayesian inversion of synthetic GPS cumulative displacements for the correlation lengths $\lambda_s = 40$ km and $\lambda_d = 55$ km. Higher (darker) CV values imply more uncertainty in the inferred slip.
3.1 Posterior Distribution and Uncertainty Representations

The median of the posterior samples of each subfault was plotted for some chosen correlation lengths in Figs. 1b, 2a, and 3a, with a heat map. GPS station locations are also plotted with green triangles and their corresponding data using arrows. Blue-solid and red-solid arrows show the observed (synthetic) surface slips, while dashed arrows show the predictions. The black contours at the arrowheads represent the posterior error calculated for the inversion in the data. Black lines represent the isodepth contours (in km) of the subducted oceanic slab. It can be seen that the fit of the data is excellent in all cases, and the slip solution is almost perfect concerning the true slip seen in Fig. 1a.

Because one has access to the conditional posterior distribution of $D$, conditioned on estimates of the covariance parameters, one can look at point estimators such as the posterior mean, the posterior median, and the maximum a posteriori (MAP). However, these point estimators may be unrepresentative of the actual posterior. The mean and median may be misleading for long-tailed asymmetric PDFs, and the MAP may be unrepresentative in the presence of skewness. In this synthetic case, far better results are obtained with the median of the posterior samples. For comparisons and to observe the crucial importance of the inclusion of correctly modeled prior information, a Maximum Likelihood Estimation (MLE) estimation in Fig. 2b is included.

An advantage of the Bayesian approach is that it produces not only one optimal model, but the sampling yields a large ensemble of probable models sampled from the posterior distribution. In Fig. 3a, b, the median and the uncertainty of the slips considering $\lambda_s = 40$ km and $\lambda_d = 55$ km, respectively, are shown.

Now, the representation of uncertainty is considered. This task is not a straightforward one for a posterior distribution in a high-dimensional vector field. This article employs the posterior coefficient of variation (CV) at each subfault. The CV is a statistical measure of the dispersion of a probability distribution around its mean. The CV represents the ratio of the standard deviation $\sigma$ to the mean $\mu$ ($CV = \frac{\sigma}{\mu}$), showing a relative quantity of the degree of variation, independent of the scale of the variable. This metric provides a tool to compare the data dispersion between different data series. MCMC simulations are used to estimate the CV at each subfault. Because the mean displacement in some regions may be zero, a variation of the CV is used to represent the uncertainty; namely, $CV = \frac{\sigma}{\mu + \delta}$, where $\delta = 0.09$ m. The value of $\delta$ was chosen so that the denominator in CV is always positive, owing to the restrictions on the prior’s support. In Fig. 3b, the CV is plotted to compare the posterior uncertainty in the inferred (inverted) slips in each subfault. It can be seen that the lowest relative uncertainty is located in the areas where the greatest slips were found. That is, the median is more representative. In addition, note that there is an apparent decrease in uncertainty where the GPS stations are located. These regions of lower uncertainty are consistent with the regions with maximum restitution index computed by Tago et al. (2021) through a mobile checkerboard strategy. In this paper, the slip median and its corresponding CVs maps represent the posterior distribution and the UQ of the inversion (e.g., Fig. 3). The following section uses the same strategy to study the 2006 GGap SSE inversion.
3.2 Comments on Method Performance

Our algorithm uses an ad-hoc method to sample from MTN distributions. In the class of MTN sampling methods, the proposed algorithm is better than the one reported in Christen et al. (2017) as explained in Sect. 2.6. Christen et al. (2017) shows that the latter algorithm reaches almost minimal theoretical Integrated Autocorrelation Time among the canonical MTN examples. Comparison between different MTN sampling algorithms can be found in Christen et al. (2017).

Regarding computational cost, only a relatively small number of FM evaluations is needed to assess the MCMC convergence. Therefore, our method does not need a CPU cluster to run the algorithm. The whole sampling is run on a personal computer for about 40 min in a single processor. Moreover, since our method can handle a MTN distribution with correlation structures, coherence among subfaults can be included as an essential modeling feature to recover the fault slips in the context of very limited data.

4 Real Case: 2006 Guerrero Slow Slip Event

This section presents a real data application to illustrate the performance of our approach. The 2006 Guerrero SSE is studied with data collected by the Instituto de Geofísica, Universidad Nacional Autónoma de México (UNAM), and the Servicio Sismológico Nacional. In 2006, a SSE in Guerrero was recorded by \( N = 15 \) GPS stations. The stations are located, mainly, along the coast and on a transect perpendicular to the trench, between Acapulco and the north of Mexico City (Radiguet et al. 2011). These same locations were used in the synthetic analysis presented in the previous section.

4.1 Observations and Data Preprocessing

The GPS data must be preprocessed, considering the time-varying climate phenomena. In addition, the inter-SSE steady-state motion is subtracted to isolate the GPS data related to an SSE event. That is, the tectonic coupling is removed. For the actual GPS data in the GGap 2006 event, this paper uses the data processed by Radiguet et al. (2011) with their proposed standard deviations, \( \sigma_x = 0.0021 \) m, \( \sigma_y = 0.0025 \) m, and \( \sigma_z = 0.0051 \) m in the north, east and vertical directions, respectively. All these quantities are measured in meters. The time window considered to compute the displacements was from January 2, 2006, to May 15, 2007.

The Bayesian inversion is solved as a multiple linear regression model with constraints on the coefficients, considering the statistical model (4) as explained in Sect. 2.2. As in the synthetic case, the regularization parameter, \( \sigma^2_\beta \), is obtained by minimizing Eq. (12) given in Appendix B.

The HODG sampler explained in Algorithm 2 is used to sample from the resulting MTN posterior distribution. Regarding the correlation lengths, the posterior distribution was computed in a grid search along the hyperparameter space \( \lambda_s \).
Fig. 4 Bayesian inversion (posterior median) of fault slips of the 2006 Guerrero SSE for correlation lengths $\lambda_s = 40$ km and $\lambda_d = 45$ km

$[20, 90]$ km $\times \lambda_d : [20, 90]$ km, with 5 km steps. Similar results are observed as in the synthetic example, and the correlation lengths of $\lambda_s = 40$ km and $\lambda_d = 45$ km were finally selected as proxy values.

The median of the posterior samples for the static inversion was plotted (heat colors) in Fig. 4. The black contours at the arrowheads represent the data uncertainty, and the horizontal lines in the vertical component represent the quantiles 0.025, 0.5, and 0.975, respectively. The CV is plotted in Fig. 5.

All GPS data is well recovered by the method within the estimated uncertainty bounds. The posterior median shows a compact region where most of the slip took place. It is consistent with the most recent inversions, where the region of maximum slip is located from 30 to 40 km depth and with a slight updip penetration in the north-west section (Bekaert et al. 2015; Tago et al. 2021). Recent offshore observations showed that the mechanical properties in that segment of the subduction slab are different, and it may explain the inferred updip slip (Plata-Martínez et al. 2021). However, since the CV is high in that region, > 50%, it should be taken with caution. A better instrumentation deployment should be considered to avoid any misinterpretation. Despite the similarities with previous studies, it is important to mention that most of the previous works are supported on a constrained optimization framework, a different point-wise estimate than the one presented here (e.g., Radiguet et al. 2011; Tago et al. 2021). As explained in the introduction, these estimates may be biased, and the comparison with our results should be analyzed carefully.
Figure 5 shows the posterior CV, which is used as a measure of the uncertainty in our solution. Uncertainty is low in the region where the fault’s slip is concentrated and nearby coastline GPS stations. The former is a consequence of the solution to the IP; the latter is expected since the GPS station illuminates the nearby faults. The uncertainty is also low on the upper part of the color map, where the Cocos plate dives into the mantle. This is a consequence of the prior information built into our prior distribution. Specifically, the weight matrix $W$ represents our knowledge that the Cocos and North American plates are not coupled at such depths. Therefore, very low or no uncertainty in the slip is observed in this region.

### 4.2 Uncertainty Quantification of the Moment Magnitude

Given a particular slip vector $D$, the Moment Magnitude ($M_w$) is computed as

$$M_w(D) = \frac{2}{3} \left( \log_{10} M_0(D) - 9.1 \right),$$

where $M_0(D)$ is the seismic moment in N·m which is computed as $M_0(D) = \mu \sum_{i \in E} d_i A_i$, where $\mu = 32 \cdot 10^9$ Pa is the crustal rigidity, $d_i = \sqrt{(d_{is}^i)^2 + (d_{is}^j)^2}$ is the slip in the $i$-subfault, $A_i$ is the $i$-subfault area in m², and $E$ clusters the subfaults within the 1 cm slip contour (Stein and Wysession 2009).
Using the Bayesian approach has a further advantage; estimates and UQ of the inferred parameters can be consistently produced. That is, the posterior distribution of $M_w(D)$ is well defined as the transformation of the random vector $D|Y$. Moreover, since a Monte Carlo sample $D^{(1)}, D^{(2)}, \ldots, D^{(T)}$, of the posterior $D|Y$, is already available $M_w^{(i)} = M_w(D^{(i)}), i = 1, 2, \ldots, T$, is a Monte Carlo sample from the posterior distribution of the Moment Magnitude.

In Fig. 6, the posterior distributions of $M_w$ for three synthetic inversions with different correlation lengths and the 2006 Guerrero SSE inversion are presented. The vertical black lines locate the $M_w$ of the posterior median, and the blue lines represent the actual moment of magnitude for the synthetic cases. Note that these posterior distributions are skewed towards lower values than the actual true $M_w$, but the $M_w$ computed for the posterior median fault slips (i.e., Figs. 1b, 2a, and 3a) are closer to the true $M_w$.

The gap between the $M_w$ computed with the posterior median and the location of the peak with the highest frequency peak and the skewness of the posterior distribution of $M_w$ is a result of the logarithmic scale of the Moment Magnitude.
For the 2006 Guerrero SSE, the point estimate $M_w^\text{median} = 7.31$ can be extracted from the posterior distribution of $M_w$. This value is consistent with the study by Bekaert et al. (2015). Whereas, the values $M_w = 7.4$ and $M_w = 7.5$ computed by Tago et al. (2021) and Radiguet et al. (2011), respectively, fall within $M_w$’s posterior support. The obtained point estimate is consistent with previous studies. However, the skewed posterior distribution has not been shown before. The synthetic data shows that the $M_w$ of the posterior median is a better point estimate than other options. Note that similar posterior estimates for earthquakes can be found in Gombert et al. (2018) and in Ragon et al. (2021).

5 Closing Remarks

Solving inverse problems that include uncertainty quantification (UQ) in geophysics are challenging problem. Computationally feasible methods such as Tikhonov regularization may introduce biases due to non-physically interpretable regularization terms, and solutions may misrepresented part of the phenomena. Moreover, different considerations in the regularization terms may produce dissimilar solutions. Bayesian methods provide a natural alternative to explore the posterior distribution of inverse problems and provide formal UQ.

This work developed an efficient Bayesian approach to estimate fault slip induced by a Slow Slip Event (SSE). The fault slip posterior is calculated through a constrained multiple linear regression model and geodetic observations. A Gaussian model is postulated for geodetic data, and a model of the prior information as multivariate truncated normal (MTN) distribution for the unknown slip. The resulting fault slip posterior PDF is also a MTN. Regarding the posterior, an efficient ODG sampler algorithm is proposed to sample from this high-dimensional MTN. The computations can be carried out on a personal computer as opposed to other MCMC samplers that require high-performance computing. A further advantage of our MCMC algorithm is that no parameter has to be adapted or tuned.

Prior elicitation is a fundamental part of the modeling process regarding the particular physical problem under study. This paper uses the Matérn covariance function to control the subfault autocovariance and impose physically interpretable slip restrictions (on the prior distribution). Different correlation lengths within physically reasonable values are considered in the prior distribution showing low sensitivity on the posterior distributions. This low sensitivity is to be expected since the prior distribution was constructed under several considerations, featuring substantial structure. It is within this structure and some reasonable correlation lengths that specific values of the correlation are not critical in the reconstruction.

In many applications, the MAP estimator is chosen as a representative solution to the inverse problem due to its computational feasibility by usual regularization schemes. Our results in the synthetic data case show that the posterior median is an accurate fault slips’s point-wise estimate, whereas the MLE is biased. Moreover, UQ is represented by the coefficient of variation. This paper compares variability between subfaults and shows areas with the most certainty. Since Monte Carlo samples are obtained of the full conditional posterior distribution, conditioned on estimates of the
covariance parameters, both these quantities are readily available, in sharp contrast to regularization methods, where these quantities can not be recovered.

For the 2006 Guerrero SSE, the median of the posterior distribution of the slip shows a compact slip patch where most of the slip is located from 30 to 40 km depth with a slight updip penetration in the north-west section. Both of these main characteristics are consistent with the most recent studies of Bekaert et al. (2015) and Tago et al. (2021). Besides those coincidences, it is possible to evaluate the uncertainty through the coefficient of variation, which is lower in regions with large slips. The computed posterior distribution of the Moment Magnitude $M_w$ is skewed, and all the Moment Magnitude values obtained in the abovementioned studies fall within the $M_w$ posterior distribution’s support.

The Bayesian framework provides for consideration of different representations of uncertainty. It is clear that more SSE’s should be analyzed for the Guerrero Gap, and elsewhere. A further improvement would be to learn the particular fault slip parameters from multiple SSE’s analyses, such as the correlation length. Moreover, border effects should be formally included in the covariance matrix by improving the covariance operator (Daon and Stadler 2018). For now, border effects do not seem evident in the maps produced by the median, as seen in the examples presented here. However, these ideas require future research.

Computationally efficient Bayesian methods are being developed for many inverse problems in geophysics. In many cases, they provide access to full posterior distributions, which provide better and more informative estimates for the solutions and their UQ. Our proposed methodology falls within this category and is applied to an actual data set for the 2006 Guerrero SSE. The objective was to recover the slip on a known interface and its formal uncertainty quantification from observations at a few geodetic stations.

Two directions are considered regarding extending the methods presented in this paper. First, multiple faults can be considered. The forward map is identical, but a careful construction of the prior’s correlation matrix is required to ensure independence between faults. Second, there is the difficult problem of uncertainty in the Greens function, such as due to the uncertainty of the velocity model. In Ragon and Simons (2020) and Hallo and Gallovič (2016), a first-order approximation is used to stipulate a distribution for the Greens function, which leads to an MTN posterior. This approach seems directly applicable in our case. In Agata et al. (2021), ensemble modeling is proposed, although quite computationally intensive. The different ensembles may be calculated in parallel, and the approach of Agata et al. (2021) also seems applicable in principle. Such considerations are left for further research.

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Declarations

Conflict of interest The authors declare no conflicts of interest regarding this research.

Appendix A: Matérn Covariance

The Matérn covariance is a covariance function widely used in spatial statistics to define the covariance between measurements made at two points separated by $d$ distance units.

In a Gaussian process, the essential ingredient is the covariance function, which introduces a correlation between nearby points. To construct the correlation matrix $C$, this paper uses the most simplified version form of the Matérn covariance that remains smooth at 0 and analytic, namely

$$C(i, j) = \left(1 + \sqrt{3} \frac{d(i, j)}{\lambda}\right) \exp\left\{-\sqrt{3} \frac{d(i, j)}{\lambda}\right\},$$

where $d(i, j)$ is the distance between the subfault $i$ and the subfault $j$, and $\lambda$ is the correlation length. Note that as $\lambda$ increases, more coefficients of the matrix $C$ become relevant (i.e., more subfaults are correlated).

Appendix B: Determining the Variance $\sigma^2_\beta$

Note that the variance $\sigma^2_\beta$ could be considered as a random variable. The full conditional $\sigma^2_\beta$ also has a known form, namely $\sigma^2_\beta|\cdots \sim \text{Inv-Gamma}(\frac{d}{2} + a_\beta, \frac{1}{2} D^T A_0 D + b_\beta)$, and could, in principle, be included in the Gibbs sampling. However, this requires a computationally expensive reevaluation and inversion of the precision matrix $A_\beta$ at each MCMC step. For this reason, simulations are made only from the conditional distribution of $D$, conditioned on the value of $\sigma^2_\beta$ that maximizes the marginal posterior distribution of this parameter, as follows.

Akaike’s Bayesian Information Criterion (ABIC) has been widely applied in geophysical inversions to determine the regularization parameters. Following ABIC, a criterion by maximizing the marginal posterior distribution of this parameter is proposed, that is

$$\max : f(\sigma^2_\beta|Y) = \int f(D, \sigma^2_\beta|Y) dD = \frac{1}{f(Y)} \int f(Y, D, \sigma^2_\beta) dD.$$ 

(9)

For this, the following hierarchical model is considered

$$Y|D \sim N(XD, A),$$

$$D|\sigma^2_\beta \sim N_d\left(0, \frac{1}{\sigma^2_\beta} A_0\right); \quad A_0 = \beta W C^{-1} W \beta,$$
\[ \sigma_{\beta}^2 \sim \text{Inv-Gamma}(a_{\beta}, b_{\beta}), \]  

(10)

where \( A = \Sigma^{-1} \) is the precision matrix, and Inv-Gamma \((\alpha, \beta)\) denote a inverse Gamma distribution with shape parameter \(\alpha\) and scale parameter \(\beta\). Note that

\[
\int f(Y, D, \sigma_{\beta}^2) \, dD = \int f(Y, D|\sigma_{\beta}^2) \pi(\sigma_{\beta}^2) \, dD = \pi(\sigma_{\beta}^2) \int f(Y, D|\sigma_{\beta}^2) \, dD = \pi(\sigma_{\beta}^2) m(Y|\sigma_{\beta}^2),
\]

where \( \pi(\sigma_{\beta}^2) \) is the prior distribution for \(\sigma_{\beta}^2\), and

\[
m(Y|\sigma_{\beta}^2) := \int f(Y, D|\sigma_{\beta}^2) \, dD.
\]

(11)

So, maximizing (9) is equivalent to minimizing

\[
\min: \ell(\sigma_{\beta}^2) = -\log \left( m \left( Y|\sigma_{\beta}^2 \right) \right) - \log \left( \pi \left( \sigma_{\beta}^2 \right) \right).
\]

Now, with the hierarchical model (10) and using the derivation of (11) given in Xu (2019)

\[
m \left( Y|\sigma_{\beta}^2 \right) = \frac{1}{(2\pi)^{n/2} \sqrt{|\Sigma_{py}|}} \exp \left\{ -\frac{1}{2} (Y - X\mu_0)^T \Sigma_{py}^{-1} (Y - X\mu_0) \right\},
\]

where \( \mu_0 = 0 \) is the prior mean of \(D\), and \( \Sigma_{py} = A^{-1} + XA_0^{-1}X^T\sigma_{\beta}^2 \), it follows that the optimal prior variance is obtained by minimizing

\[
\min: \ell(\sigma_{\beta}^2) = \ln \left\{ \det \left( \Sigma_{py} \right) \right\} + Y^T \Sigma_{py}^{-1}Y + (a_{\beta} + 1) \log \left( \sigma_{\beta}^2 \right) + \frac{b_{\beta}}{\sigma_{\beta}^2}.
\]

(12)

Minimizing the above expressions is straightforward because it is a function defined in \( \mathbb{R}^1 \).

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