EXTENSION PROPERTIES OF PLANAR UNIFORM DOMAINS

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Abstract. The classical bi-Lipschitz and quasisymmetric Schönhflies theorems in the plane [Tuk80] [BA56] are generalized in this paper for all planar uniform domains. Specifically, we show that if \( U \subseteq \mathbb{R}^2 \) is a uniform domain then it has the following two extension properties: (1) every bi-Lipschitz map \( f : \partial U \to \mathbb{R}^2 \) that can be extended homeomorphically on \( U \) can also be extended bi-Lipschitz on \( U \) and (2) if \( \partial U \) is relatively connected then every quasisymmetric map \( f : \partial U \to \mathbb{R}^2 \) that can be extended homeomorphically on \( U \) can also be extended quasisymmetrically on \( U \). In higher dimensions, we show that if \( U \) is the exterior of a uniformly disconnected set in \( \mathbb{R}^n \) then every bi-Lipschitz embedding \( f : \partial U \to \mathbb{R}^n \) extends to a bi-Lipschitz homeomorphism of \( \mathbb{R}^n \). The same is also true for quasisymmetric embeddings under the additional assumption that \( \partial U \) is relatively connected.

1. Introduction

Let \( X \) be a metric space, \( E \subseteq X \) and \( f : E \to X \) be a map in a class \( \mathcal{F} \). When can \( f \) be extended to a mapping \( F : X \to X \) in the same class? We are interested in the above extension question for the classes of bi-Lipschitz maps and quasisymmetric maps. Questions related to quasisymmetric extensions have been considered by Beurling and Ahlfors [BA56], Ahlfors [Ahl63, Ahl64], Carleson [Car74], Tukia and Väisälä [TV82, TV81, TV84], Väisälä [Väi86], Kovalev and Onninen [KO11] and Fujino [Fuj16]. Results related to bi-Lipschitz extension appear in the work of Tukia [Tuk80, Tuk81], David and Semmes [DS91], MacManus [Mac95] and Alestalo and Väisälä [AV97].

Tukia and Väisälä [TV84] showed that for \( M = \mathbb{R}^p, S^p \) and \( n > p \), any quasisymmetric mapping \( f : M \to \mathbb{R}^n \) extends to a quasisymmetric homeomorphism of \( \mathbb{R}^n \) when \( f \) is locally close to being a similarity and every bi-Lipschitz mapping \( f : M \to \mathbb{R}^n \) extends to a bi-Lipschitz mapping of \( \mathbb{R}^n \) when \( f \) is close to being an isometry. Later, Väisälä [Väi86] extended these results to all compact, \( C^1 \) or piecewise linear \((n-1)\)-manifolds \( M \) in \( \mathbb{R}^n \). Similar results appeared recently in the work of Azzam, Badger and Toro [ABT15]. The requirements on the embedding \( f \) in these three papers, ensured the homeomorphic extension of \( f \) on \( \mathbb{R}^n \).

In this article we work in a different direction; assuming that there is a homeomorphic extension, when can we extend the mapping in question to a quasisymmetric or bi-Lipschitz homeomorphism? Given a metric space \( X \) we say that \( E \subseteq X \) has the quasisymmetric extension property (resp. bi-Lipschitz extension property) in \( X \) or QSEP in short (resp. BLEP) if every quasisymmetric (resp. bi-Lipschitz)
embedding \( f : E \to X \) that can be extended as a homeomorphism of \( X \) can also be extended as a quasisymmetric (resp. bi-Lipschitz) homeomorphism of \( X \).

When \( X = \mathbb{R} \) or \( X = S^1 \), trivially every subset of \( X \) has the BLEP in \( X \) but the same is not true in the quasisymmetric class. Indeed, if \( E = \{0\} \cup \{e^{-n!}\}_{n \geq 2} \) then \( f : E \to \mathbb{R} \) with \( f(x) = (-\log x)^{-1} \) is monotone and quasisymmetric but can not be extended quasisymmetrically in any open set containing the point 0 \cite[Hei01, p. 89]{Heinonen}. Thus, more regularity for sets \( E \) should be assumed. Trotsenko and Väisälä \cite{TV99} introduced the notion of relative connectedness, a weak version of uniform perfectness, and as a corollary of their main theorem, if \( E \subset \mathbb{R}^n \) is not relatively connected, then there exists a quasisymmetric embedding \( f : E \to \mathbb{R}^n \) that can be extended homeomorphically on \( \mathbb{R}^n \) but not quasisymmetrically; see Section \ref{sec:rel_conn}. Conversely, we showed in \cite{Vel16} that if \( E \subset \mathbb{R} \) is relatively connected then it has the QSEP in \( \mathbb{R} \).

On the other hand, for each \( n \geq 2 \) there exists a relatively connected, compact and countable set \( E_n \subset \mathbb{R}^n \) and a bi-Lipschitz embedding \( f : E_n \to \mathbb{R}^n \) that admits a homeomorphic extension on \( \mathbb{R}^n \) but not a quasisymmetric extension \cite[Theorem 5.1]{Vel16}. These examples show that in dimensions \( n \geq 2 \) relative connectedness does not suffice for either the QSEP or the BLEP and the geometry of the complement of \( E \) comes into play. It follows from the celebrated work of Ahlfors \cite{Ahlfors63}, Beurling and Ahlfors \cite{BA56} and Tukia \cite{Tukia80} that \( \mathbb{R} \) and \( S^1 \) have both extension properties in \( \mathbb{R}^2 \). In this paper we extend their results to boundaries of planar uniform domains, a broad family of domains in \( \mathbb{R}^2 \) whose local geometry resembles that of the disk and of the upper half-plane.

**Theorem 1.1.** Let \( U \subset \mathbb{R}^2 \) be a \( c \)-uniform domain and \( f : \partial U \to \mathbb{R}^2 \) be an embedding that can be extended homeomorphically on \( U \):

1. If \( f \) is \( L \)-bi-Lipschitz then \( f \) extends \( L' \)-bi-Lipschitz on \( U \) with \( L' > 1 \) depending only on \( L \) and \( c \).
2. If \( \partial U \) is \( C \)-relatively connected and \( f \) is \( \eta \)-quasisymmetric then \( f \) extends \( \eta' \)-quasisymmetrically on \( U \) with \( \eta' \) depending only on \( \eta, c \) and \( C \).

The second part of Theorem 1.1 can be viewed as the converse of the following boundary quasiconformal extension result of Väisälä \cite{Vaisala85}: if \( U, U' \subset \mathbb{R}^2 \) are uniform domains and \( f : U \to U' \) is a quasiconformal homeomorphism that extends homeomorphically on \( \partial U \) then \( f \) extends quasisymmetrically on \( \partial U' \); see Lemma \ref{lem:qc_ext}. Roughly speaking, uniformity is a combination of two other notions: a domain is uniform if every pair of points can be joined by a curve whose length is comparable to the distance of the points (quasiconvexity) and it does not go too close to the boundary of the domain (John property); see Section \ref{sec:uniformity} for precise definition. The assumption of uniformity of \( U \) is somewhat necessary for both extensions as neither quasiconvexity nor John property alone are sufficient; see Section \ref{sec:nec_conds}

In \( \mathbb{R}^3 \), Theorem 1.1 fails in both cases as there exists a bi-Lipschitz embedding \( f : S^2 \to \mathbb{R}^3 \) that can be extended homeomorphically on \( \mathbb{R}^3 \) but not quasisymmetrically \cite[Section 15]{Tukia80}.

As a corollary, we obtain a sufficient condition for sets \( E \) to satisfy the QSEP and the BLEP in \( \mathbb{R}^2 \). The arguments apply verbatim in the case that \( X \) is the unit sphere \( S^2 \) and \( E \subset S^2 \).
Corollary 1.2. If $E \subset \mathbb{R}^2$ is such that each component of $\mathbb{R}^2 \setminus E$ is $c$-uniform then $E$ has the BLEP in $\mathbb{R}^2$. If moreover $E$ is $c$-relatively connected then it has the QSEP in $\mathbb{R}^2$.

The tameness of Cantor sets in $\mathbb{R}^2$ implies that in Theorem 1.1 the assumption of homeomorphic extension of $f$ on $\mathbb{R}^2 \setminus E$ can be dropped when $E$ is totally disconnected. However, in higher dimensions, due to the existence of wild Cantor sets [Dav07], an increase in dimension is needed. Moreover, in the plane, the complement of a closed set $E \subset \mathbb{R}^2$ with empty interior is uniform if and only if $E$ is uniformly disconnected [Mac99] but that is not true in $\mathbb{R}^n$ when $n \geq 3$. Uniform disconnectedness is in a sense the opposite of uniform perfectness: for each point $x$ there exists an “isolated island” $E' \subset E$ of practically any diameter whose distance from the rest of $E$ is at least a fixed multiple of its diameter. Thus, in higher dimensions uniform disconnectedness of $E$ can be used as a natural analogue of uniformity of $\mathbb{R}^n \setminus E$.

Theorem 1.3. Let $n \geq 3$, $E \subset \mathbb{R}^n$ be $c$-uniformly disconnected and $f : E \to \mathbb{R}^n$.

1. If $f$ is $L$-bi-Lipschitz then it extends to an $L'$-bi-Lipschitz homeomorphism $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $L' > 1$ depending only on $L$, $c$ and $n$.
2. If $E$ is $\eta$-quasisymmetric then $f$ extends to an $\eta'$-quasisymmetric homeomorphism $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $\eta'$ depending only on $\eta$, $c$, $C$ and $n$.

The proof of Theorem 1.3 relies on a uniformization result for Cantor sets with bounded geometry that generalizes a result of MacManus [Mac99]. Namely, in Section 6 we show that a compact set $E \subset \mathbb{R}^n$ is uniformly perfect and uniformly disconnected if and only if there exists a quasiconformal homeomorphism of $\mathbb{R}^{n+1}$ mapping $E$ onto the standard middle-third Cantor set $C \subset \mathbb{R}$.

In Section 4 we reduce the proof of Theorem 1.3 to the case that $U$ is unbounded with bounded and perfect boundary and the proof of Theorem 1.3 to the case and $E$ is compact and perfect.

The proof of Theorem 1.1 follows Carleson’s method [Car74]. The main idea is the construction of two combinatorially equivalent Whitney-type decompositions $\mathcal{D}$ and $\mathcal{D}'$ for $U$ and $U'$ respectively. That is, $\mathcal{D}$ (resp. $\mathcal{D}'$) is a family of mutually disjoint open subsets of $U$ (resp. $U'$) such that the union of their closures is the whole $U$ (resp. $U'$), the diameter of each element of $\mathcal{D}$ (resp. $\mathcal{D}'$) is comparable to its distance to $\partial U$ (resp. $\partial U'$) and there exists a homeomorphism of $\overline{U}$ onto $\mathcal{D}$ that maps each element of $\mathcal{D}$ onto exactly one element of $\mathcal{D}'$. Moreover, the boundary of every domain in $\mathcal{D}$ and $\mathcal{D}'$ is a finite union of $L$-bi-Lipschitz circles whose mutual distances and diameters are bounded below by a constant $d > 0$. We show in Section 8 that such domains possess both BLEP and QSEP.

The construction of the two decompositions makes use of the uniformity of $U$ and $U'$. In Section 5 we show that the boundary of a uniform domain satisfies a weak form of uniform connectedness: given a point $x \in \partial U$ and some $r > 0$, there exists a closed set $A \subset \partial U$ containing $x$ whose distance from $\partial U \setminus A$ is at most a multiple of $r$.

In Section 7 using the results of Section 5 we construct the decompositions $\mathcal{D}$ and $\mathcal{D}'$ and show Theorem 1.1. Towards the construction we distinguish two cases: one for the part of $U$ around non-degenerate components of $\partial U$ which we treat in Section 6 and another for the rest of $U$. In the first case the decomposition
resembles that of the exterior of a quasidisk while in the second $U$ resembles the exterior of a uniformly disconnected set.

**Acknowledgements.** We wish to thank David Herron, Pekka Koskela, Kai Rajala and Jang-Mei Wu for various discussions on this subject and Jussi Väisälä for bringing Lemma 2.10 to our attention.

2. Preliminaries

A Jordan curve (resp. Jordan line) is a subset of $\mathbb{R}^2$ homeomorphic to $S^1$ (resp. $\mathbb{R}$). A Jordan domain is a domain whose boundary is a Jordan curve. A closed set $E$ with one point is called a degenerate set. A non-degenerate compact connected set is called a continuum.

2.1. Mappings. A homeomorphism $f: D \to D'$ between two domains in $\mathbb{R}^n$ is called $K$-quasiconformal if it is orientation-preserving, belongs to $W^{1,n}_{loc}(D)$, and satisfies the distortion inequality

$$|Df(x)|^n \leq KJ_f(x) \quad \text{a.e. } x \in D,$$

where $Df$ is the formal differential matrix and $J_f$ is the Jacobian.

An embedding $f$ of a metric space $(X, d_X)$ into a metric space $(Y, d_Y)$ is said to be $\eta$-quasisymmetric if there exists a homeomorphism $\eta: [0, \infty) \to [0, \infty)$ such that for all $x, a, b \in X$ and $t > 0$

$$d_X(x, a) \leq td_X(x, b) \quad \text{implies} \quad d_Y(f(x), f(a)) \leq \eta(t)d_Y(f(x), f(b)).$$

An $\eta$-quasisymmetric map with $\eta(t) = C \max\{t^\alpha, t^{1/\alpha}\}$ for some $C > 1$ and $\alpha > 1$ is known in literature as power quasisymmetric map.

A quasisymmetric mapping between two domains in $\mathbb{R}^n$ is quasiconformal. On the other hand, a quasiconformal homeomorphism between uniform domains of $\mathbb{R}^n$ is quasisymmetric quantitatively.

**Lemma 2.1** (Väisälä, Theorem 5.6). Let $U, U'$ be $c$-uniform domains in $\mathbb{R}^n$ and $f: U \to U'$ be a $K$-quasiconformal homeomorphism. Then $f$ is $\eta$-quasisymmetric with $\eta$ depending only on $K$, $c$ and $n$.

For a systematic treatment of quasiconformal mappings see Väisälä. A map $f: X \to Y$ between metric spaces is $L$-bi-Lipschitz for some $L \geq 1$ if

$$L^{-1}d_X(x, y) \leq d_Y(f(x), f(y)) \leq Ld_X(x, y)$$

for all $x, y \in X$. Note that an $L$-bi-Lipschitz mapping is $L^2t$-quasisymmetric.

A weaker notion of bi-Lipschitz mappings is that of bounded length distortion (BLD) mappings. A mapping $f: X \to Y$ between metric spaces is $L$-BLD for some $L \geq 1$ if

$$L^{-1}\ell(\gamma) \leq \ell(f(\gamma)) \leq \ell(\gamma)$$

for all paths $\gamma: [0, 1] \to X$. Here and for the rest, $\ell$ denotes the length of a path. Clearly, $L$-bi-Lipschitz mappings are $L$-BLD mappings but BLD mappings need not be bi-Lipschitz even if they are homeomorphisms. However, BLD homeomorphisms between quasiconvex spaces are bi-Lipschitz.

**Lemma 2.2.** Let $f: X \to Y$ be an $L$-BLD homeomorphism between two $c$-quasiconvex metric spaces. Then $f$ is $Lc$-bi-Lipschitz.
A mapping $f : X \to Y$ between metric spaces is a $(\lambda, L)$-quasisimilarity for some $\lambda > 0$ and $L \geq 1$ if $L^{-1}\lambda d_X(x, y) \leq d_Y(f(x), f(y)) \leq L\lambda d_X(x, y)$ for all $x, y \in X$. Note that $(\lambda, 1)$-quasisimilarities are similarities, $(1, L)$-quasisimilarities are $L$-bi-Lipschitz and $(1, 1)$-quasisimilarities are isometries.

A curve $\Gamma \subset \mathbb{R}^2$ is a $K$-quasicircle (resp. $K$-quasiline) with $K \geq 1$ if $\Gamma = f(S^1)$ (resp. $\Gamma = f(\mathbb{R})$) for some $K$-quasiconformal $f : \mathbb{R}^2 \to \mathbb{R}^2$. A topological disk $D \subset \mathbb{R}^2$ is called a $K$-quasidisk if $\partial D$ is a $K$-quasicircle. A geometric characterization of quasicircles was given by Ahlfors [Ahl63] in terms of the bounded turning property; see Section 2.3.

A curve $\Gamma \subset \mathbb{R}^2$ is called an $L$-bi-Lipschitz circle with $L \geq 1$ if $\Gamma = f(S^1)$ for some $(\lambda, L)$-quasisimilarity $f : \mathbb{R}^2 \to \mathbb{R}^2$. A topological disk $D \subset \mathbb{R}^2$ is called an $L$-bi-Lipschitz disk if $\partial D$ is an $L$-bi-Lipschitz circle.

2.2. Relative distance. For two non-degenerate closed sets $E, E' \subset \mathbb{R}^n$ define the relative distance

$$\text{dist}^*(E, E') = \frac{\text{dist}(E, E')}{\min\{\text{diam } E, \text{diam } E'\}}$$

where $\text{dist}(E', E') = \min\{|x - y| : x \in E, y \in E'\}$. If both $E$ and $E'$ have infinite diameter we set $\text{dist}^*(E, E') = 0$.

If $E, E' \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity then $d^*(f(E), f(E')) = d^*(E, E')$. In general, if $f : E \cup E' \to Y$ is $\eta$-quasisymmetric, then

$$\frac{1}{2} \phi(d^*(E, E')) \leq d^*(f(E), f(E')) \leq \eta(2d^*(E, E'))$$

where $\phi(t) = (\eta(t^{-1}))^{-1}$; see for example [Tys98, p. 532].

2.3. Relatively connected sets. Relatively connected sets were first introduced by Trotsenko and Väisälä [TV99] in the study of spaces for which every quasisymmetric mapping is power quasisymmetric. A metric space $(X, d)$ is called $c$-relatively connected for some $c \geq 1$ if for any $x \in X$ and any $r > 0$ either $B(x, r) = \{x\}$ or $\overline{B}(x, r) = X$ or $B(x, r) \setminus B(x, r/c) \neq \emptyset$. The definition given in [TV99] is equivalent to the one above quantitatively [TV99, Theorem 4.11].

A connected space is $c$-relatively connected for any $c > 1$. Relative connectedness is a weak form of the well known notion of uniform perfectness. A metric space $X$ is $c$-uniformly perfect for some $c > 1$ if for all $x \in X$, $\overline{B}(x, r) \neq X$ implies $\overline{B}(x, r) \setminus B(x, r/c) \neq \emptyset$. The difference between the two notions is that relatively connected sets allow isolated points. In particular, if $E$ is $c$-uniformly perfect, then it is $c'$-relatively connected for all $c' > c$, and if $E$ is $c$-relatively connected and perfect, then it is $(2c + 1)$-uniformly perfect [TV99, Theorem 4.13].

The connection between relative connectedness and quasisymmetric mappings is illustrated in the following theorem from [TV99].

**Lemma 2.3** ([TV99, Theorem 6.20]). A subset $E$ of a metric space $X$ is relatively connected if and only if every quasisymmetric map $f : E \to X$ is power quasisymmetric.

It easily follows from its definition that the image of a relatively connected (resp. uniformly perfect) space under a quasisymmetric mapping is relatively connected (resp. uniformly perfect) quantitatively. We conclude the discussion on relatively connected sets with the following remark.
Remark 2.4. Suppose that $X$ is a $c$-uniformly perfect metric space and $E \subset X$ is compact with $E \cap (X \setminus E) = \emptyset$. Then, there exists $M > 1$ depending only on $c$ such that $\text{dist}(E, E \setminus X) \leq M \text{diam } E$.

2.4. Uniformly disconnected sets. In [DS97], David and Semmes introduced a scale-invariant version of total disconnectedness towards a uniformization of all metric spaces that are quasisymmetric to the standard middle-third Cantor set. A space $(X,d)$ is $c$-uniformly disconnected for some $c \geq 1$ if for all $x \in X$ and all positive $r < \frac{1}{2} \text{diam } X$, there exists $E \subset X$ containing $x$ such that $\text{diam } E \leq r$ and $\text{dist}(E, X \setminus E) \geq r/c$.

With this terminology, David and Semmes showed that a metric space is quasisymmetric to $C$ if and only if it is compact, doubling, uniformly disconnected and uniformly perfect. This result was later improved by MacManus [Mac99] for Cantor sets in $\mathbb{R}^2$; see Section 3.

In the same article, MacManus found an elegant connection between planar uniform domains and uniformly disconnected sets: a set $E \subset \mathbb{R}^2$ with empty interior is uniformly disconnected if and only if its complement is uniform [Mac99, Theorem 1.1]. In higher dimensions only the necessity is true.

It is easy to check that if $X$ is a $c$-uniformly disconnected space and $f : X \to Y$ is $\eta$-quasisymmetric then $f(X)$ is $c'$-uniformly disconnected with $c'$ depending only on $\eta$ and $c$.

2.5. Uniform domains. A domain $U \subset \mathbb{R}^n$ is said to be $c$-uniform for some $c \geq 1$ if for all $x, y \in U$, there exists a curve $\gamma \subset U$ joining $x$ with $y$ such that

$(1)$ $\ell(\gamma) \leq c|x - y|$ and

$(2)$ for all $z \in \gamma$, $\text{dist}(z, \partial U) \geq c^{-1} \min\{|x - z|, |y - z|\}$.

A curve $\gamma$ as in the above definition is called a $c$-cigar curve.

Metric spaces for which, for every two points there exists curve satisfying the first property of uniformity are called $c$-quasiconvex. If in the definition of quasiconvexity the length of curves is replaced by diameter then the space is called $c$-bounded turning. Metric spaces for which, for every two points there exists curve satisfying the second condition are called $c$-John spaces.

A curve $\Gamma \subset \mathbb{R}^2$ is a $K$-quasicircle if and only if it is $c$-bounded turning with $c$ and $K$ being related quantitatively [Ahl63]. A Jordan curve $\Gamma \subset \mathbb{R}^2$ is an $L$-bi-Lipschitz circle if and only if it is $c$-quasiconvex with $c$ and $L$ being related quantitatively [JK82]. Finally, a simply connected domain $D \subset \mathbb{R}^2$ is uniform if and only if it is a quasidisk (or quasiplane) and $D$ is a John domain if and only if its complement is bounded turning [NV91].

Two remarks are in order.

Remark 2.5. It is easy to check that all curves in the definition of uniform domains can be chosen to be simple. For the rest of the paper, all cigar curves are assumed to be simple.

Remark 2.6. Let $U \subset \mathbb{R}^n$ be a $c$-uniform domain, $x, y \in U$ and $\gamma$ be a $c$-cigar curve joining $x, y$. Then,

$$\text{dist}(\gamma, \partial U) \geq (2c)^{-1} \min\{\text{dist}(x, \partial U), \text{dist}(y, \partial U)\}.$$ 

Indeed, set $\epsilon = \min\{\text{dist}(x, \partial U), \text{dist}(y, \partial U)\}$ and let $z \in \gamma$. If $z \in \overline{B}(x, \epsilon/2) \cup \overline{B}(y, \epsilon/2)$ then $\text{dist}(z, \partial U) \geq \epsilon/2$. If $z$ is in the exterior of these balls then $\text{dist}(z, \partial U) \geq e^{-1} \min\{|z - x|, |z - y|\} \geq (2c)^{-1}\epsilon$. 

The following proposition describes the geometry of uniform domains. For the proof see Corollary 2.33 in [MS79], Theorem 2 and Lemma 3 in [Geh82] and Theorem 1.1 in [Her87].

**Proposition 2.7.** Let $U$ be a $c$-uniform domain.

1. **(Boundary components)** Each component of $\partial U$ is either a point or a $K$-quasicircle or a $K$-quasiline with $K > 1$ depending only on $c$.
2. **(Relative distance)** If $A_1, A_2$ are non-degenerate components of $\partial U$ then $\text{dist}^*(A_1, A_2) \geq (2c)^{-2}$.
3. **(Porosity)** For every $x \in \overline{U}$ and every $0 < r < \frac{1}{4} \text{diam} U$, there exists $x' \in \partial B^2(x, r)$ such that $B^2(x, r/c) \subset U$.

The porosity of $\partial U$ implies that if $U$ is bounded, there exists a point $x \in U$ such that $B^2(x, \frac{1}{4} \text{diam} U) \subset U$.

Although Proposition 2.7 provides a lot of information about the boundaries of uniform domains, it fails to characterize them. Namely, if $E \subset \mathbb{R}$ is a Cantor set with $\mathcal{H}^1(E) > 0$ then $\mathbb{R}^2 \setminus E$ trivially satisfies all three properties of Proposition 2.7 but it is not uniform. If $U$ is finitely connected and satisfies properties (1) and (2) then it is uniform. We record this observation as a remark.

**Remark 2.8.** Let $U \subset \mathbb{R}^2$ be a $c$-uniform domain and $D \subset U$ be a $K$-quasidisk such that $\text{dist}^*(D, \partial U) \geq d > 0$. Then $U \setminus D$ is $c'$-uniform with $c'$ depending only on $c, K$ and $d$.

We conclude the discussion on uniform domains with two results on the invariance of uniformity under quasisymmetric mappings. The first result says that uniform domains are preserved under quasisymmetric mappings while the second result roughly says that complement of uniform domains are preserved under quasisymmetric mappings.

**Lemma 2.9** ([GO79, Corollary 3]). If $U \subset \mathbb{R}^2$ is $c$-uniform and $f : U \to \mathbb{R}^2$ is $\eta$-quasisymmetric then $f(U)$ is $c'$-uniform with $c'$ depending only on $c$ and $\eta$.

**Lemma 2.10** ([Väisänen Theorem 5.6]). If $E \subset \mathbb{R}^2$ is closed, $\mathbb{R}^2 \setminus E$ is $c$-uniform and $f : E \to \mathbb{R}^2$ is $\eta$-quasisymmetric then $\mathbb{R}^2 \setminus f(E)$ is $c'$-uniform with $c'$ depending only on $c$ and $\eta$.

### 3. Quasisymmetric and bi-Lipschitz extension for a class of finitely connected domains

In this section we extend a well known quasisymmetric extension result of Beurling and Ahlfors [BA56] and a bi-Lipschitz extension result of Tukia [Tuk80] for a class of finitely connected uniform domains.

**Theorem 3.1** ([BA56, Tuk80]). If $f : \mathbb{S}^1 \to \mathbb{R}^2$ is $\eta_1$-quasisymmetric (resp. $L_1$-bi-Lipschitz) then it extends $\eta_2$-quasisymmetrically (resp. $L_2$-bi-Lipschitz) on $\mathbb{B}^2$ with $\eta_2$ depending only on $\eta_1$ (resp. $L_2$ depending only on $L_1$).

Let $L \geq 1$, $K \geq 1$ and $d \geq 1$. Denote by $\mathcal{Z}(K, d)$ (resp. $\mathcal{B}(L, d)$) the collection of planar bounded domains $U \subset \mathbb{R}^2$ whose boundary consists of mutually disjoint $K$-quasicircles (resp. $L$-bi-Lipschitz circles) with mutual distances and diameters bounded below by $d^{-1} \text{diam} U$. Let also $\mathcal{B}(L, d)$ be the collection...
of bounded domains $U \subset \mathbb{R}^2$ whose boundary consists of mutually disjoint $L$-bi-Lipschitz circles with mutual distances bounded below by $d^{-1} \text{diam} U$. Note that $\mathcal{B} \mathcal{L}^n(K, d) \subset \mathcal{B} \mathcal{L}(L, d)$ and $\mathcal{B} \mathcal{L}^n(L, d) \subset \mathcal{B} \mathcal{L}(L^2, d)$.

The following proposition, which is the main result of this section, generalizes Theorem 3.1 and is a special case of Theorem 1.1.

**Proposition 3.2.** Let $U \subset \mathbb{R}^2$ be a bounded domain and $f : \partial U \to \mathbb{R}^2$ be an embedding that can be extended homeomorphically on $U$.

1. If $U \in \mathcal{B} \mathcal{L}(K, d)$ and $f$ is $\eta_1$-quasisymmetric then it extends as an $\eta_2$-quasisymmetric embedding of $U$ with $\eta_2$ depending only on $\eta_1, K$ and $d$.

2. If $U \in \mathcal{B} \mathcal{L}(L, d)$ and $f$ is $L_1$-bi-Lipschitz then it extends as an $L_2$-bi-Lipschitz embedding of $U$ with $L_2$ depending only on $L_1, K$ and $d$.

We first show that domains in $\mathcal{B} \mathcal{L}(K, d)$ and $\mathcal{B} \mathcal{L}(L, d)$ are finitely connected quantitatively. Although this result follows almost immediately from the doubling property, with a little more effort, one can show the following stronger statement.

**Lemma 3.3.** For each $K, d > 1$ there exists $N \in \mathbb{N}$ such that if $D_1, \ldots, D_n \subset \mathbb{R}^2$ are disjoint $K$-quasidisks of mutual relative distances at most $d$ then $n \leq N$.

**Proof.** We use the following three well known facts. By the doubling property of $\mathbb{R}^2$, there exists universal $c_0 > 1$ such that if $B^2(z_1, r_1), \ldots, B^2(z_n, r_n)$ are mutually disjoint disks contained in the unit disk $\mathbb{B}^2$, then there exists $r_i$ such that $r_i \leq c_0/n$. Moreover, given a $K$-quasicircle $\Gamma$ there exists $c > 1$ depending only on $K$ and there exists a point $x$ in the domain $D$ enclosed by $\Gamma$ such that $B^2(x, c^{-1} \text{diam} \Gamma) \subset D \subset B^2(x, c \text{diam} \Gamma)$. Combined together, we have that if $D_1, \ldots, D_n$ are disjoint $K$-quasidisks in $B^2(x, r)$ then there exists $i = 1, \ldots, n$ such that $\text{diam} D_i \leq c c_0 r$.

Let $D_1, \ldots, D_n$ be disjoint $K$-quasidisks of mutual relative distances at most $d$. Assume that $D_n$ has the smallest diameter among them and applying a dilation we may further assume that $\text{diam} D_n = 1$. Note also that each $D_i$ is $C$-uniform for some $C > 1$ depending only on $K$.

Fix a point $x_0 \in \Gamma_n$. Since $\text{dist}^*(D_i, D_n) \leq d$, every quasiskisk $D_i$ intersects $B^2(x_0, 2d)$ for each $i = 1, \ldots, n - 1$. Note that there are at most $4 c c_0 d$ elements of $\{D_1, \ldots, D_{n-1}\}$ contained entirely in $B^2(x_0, 4d)$. Let $D_{i_1}, \ldots, D_{i_m}$ be quasiskisks in the collection $\{D_1, \ldots, D_n\}$ that have points at the exterior of $B^2(x_0, 6d)$. For each one of them, fix points $x_j \in D_{i_j} \cap B^2(x_0, 2d)$, $y_j \in D_{i_j} \setminus \overline{B^2(x_0, 2d)}$ and $\gamma_j$ a $C$-cigar curve in $D_{i_j}$ joining $x_j$ with $y_j$. Let $z_j \in \gamma_j \cap \partial B^2(x_0, 3d)$. Then, $B^2(z_j, C^{-1} d) \subset D_{i_j}$ and since $z_1, \ldots, z_m \in \partial B^2(x_0, 3d)$, necessarily $m \leq 6 \pi C$. Hence, $n \leq 4 c c_0 d + 6 \pi C$. \hfill $\square$

**Corollary 3.4.** For each $c > 1$ and $d > 1$ there exists $N > 1$ depending only on $L$ and $d$ such that every domain in $\mathcal{B} \mathcal{L}(c, d) \cup \mathcal{B} \mathcal{L}(c, d)$ has at most $N$ boundary components.

For the rest, $U_0$ denotes the square $(-1, 1) \times (-1, 1)$ and for each $m \in \mathbb{N}$ and $k \in \{1, \ldots, m\}$ we set

$$S_{m, k} = \left[\frac{4k - 2m - 3}{2m + 1}, \frac{4k - 2m - 1}{2m + 1}\right] \times \left[\frac{-1}{2m + 1}, \frac{1}{2m + 1}\right]$$

and

$$U_m = U_m \setminus \bigcup_{k=1}^{m} S_{m, k}.$$
Lemma 3.5. Each $U \in \mathcal{K}(K,d)$ (resp. $\mathcal{KL}^*(L,d)$) is $\eta$-quasisymmetric (resp. $(L',\text{diam}U)$-quasisimilar) to $U_m$ for some $0 \leq m \leq N$ with $\eta$ and $N$ depending only on $K$ and $d$ (resp. $N$ and $L'$ depending only on $L$ and $d$).

For the proof of the lemma recall that a dyadic $n$-cube $D \subset \mathbb{R}^n$ is an $n$-cube of the form $D = [i_12^k, (i_1 + 1)2^k] \times \cdots \times [i_n2^k, (i_n + 1)2^k]$ where $k, i_1, \ldots, i_n \in \mathbb{Z}$. If $n = 2$, $D$ is called a dyadic square.

Proof of Lemma.\footnote{7} The lemma is trivial if $U$ is simply connected. Suppose now that $U = D_0 \setminus (D_1 \cup \cdots \cup D_m)$ of diameters and mutual distances bounded below by $d^{-1}\text{diam}U$. By Corollary 3.4, $m \leq N$ for some $N \in \mathbb{N}$ depending only on $L$ and $d$.

Assume first that $U \in \mathcal{KL}^*(L,d)$. Applying a $(\text{diam}U, L_1)$-quasisimilarity, with $L_1$ depending only on $L$, we may assume that $D_0 = U_0$. By Lemma 5.1 there exists $L_2 > 1$ depending only on $L$ and $d$ such that for each $i = 1, \ldots, m$ there exists an $L_2$-bi-Lipschitz disk $D_i'$ containing $D_i$ such that $(2d_i^{-1}) \leq \text{dist}(z, \partial D_i) \leq (3d_i^{-1})$. Moreover, each $D_i$ contains a dyadic square $S_i$ with side-length $2^{-m_0}$ where $m_0$ is the smallest integer such that $2^{-m_0} \leq \min\{(4Ld)^{-1}, \log(2N-1)/\log 2\}$. Note that both $\text{dist}(\partial D_i, S_i)$ and $\text{diam}S_i$ are bounded below by $\delta$ for some $\delta \in (0, 1)$ depending only on $L$ and $d$.

There exists $L_3$ depending only on $L$, $d$ such that for each $i = 1, \ldots, m$ there exists an $L_3$-bi-Lipschitz mapping $f_i : \partial D_i' \cup \partial D_i \to \partial D_i' \cup \partial S_i$ with $f_i|\partial D_i' = \text{Id}$ and $f_i(\partial D_i) = \partial S_i$. By the Annulus Theorem in the LIP category [Väi77, Theorem 3.4], each $f_i$ can be extended to an $L_4$-bi-Lipschitz mapping $f_i : \overline{D_i'} \setminus D_i \to \overline{D_i'} \setminus S_i$ with $L_4$ depending only on $L$ and $d$. Moving the squares $S_i$ around and properly dialating them, we can map $D_0 \setminus (\bigcup_{i=1}^m S_i)$ bi-Lipschitz onto $U_0$. Since there are at most $(2^{2m_0+2})!$ different configurations for the position of the squares $S_1, \ldots, S_n$ inside $D_0$, the bi-Lipschitz constant of the latter map depends at most on $L$ and $d$.

The proof of the quasisymmetric case is almost identical. The only notable difference is that here we use the Annulus Theorem in the LQC category [TV81, Theorem 3.12] to obtain $\eta$-quasisymmetric extensions of the mappings $f_i$ with $\eta$ depending only on $K$ and $d$. □

Proof of Proposition.\footnote{8} Since the embedding $f$ in both cases can be extended homeomorphically on $U$, there exists a domain $U' \subset \mathbb{R}^2$ such that $\partial U' = f(\partial U)$ and $f$ can be extended to a homeomorphism of $U$ onto $U'$.

Suppose first that $U \in \mathcal{K}(K,d)$ and that $f$ is $\eta_1$-quasisymmetric. By Lemma 5.1 we may assume that $U = U_m$ where $m \leq N$ for some $N$ depending only on $K$ and $d$. Moreover, applying a $\lambda$-bi-Lipschitz homeomorphism of $U_m$ onto itself with $\lambda > 1$ depending only on $N$, we may assume that $f$ maps $\partial S_m$ onto $\partial S_{m,k}$.

If $m = 0$, the claim follows from Theorem 3.1 while if $m = 1$, it follows from the Annulus theorem in the LQC category. Assume for the rest that $m \geq 2$.

Let $S'_0 = \left[ \frac{1}{2m+1}, 1, \frac{3}{2m+1} \right]^2$ and for each $k = 1, \ldots, m$ let

$$S'_{m,k} = \left[ \frac{4k - 2m - 7/2}{2m+1}, \frac{4k - 2m - 1/2}{2m+1} \right] \times \left[ \frac{3/2}{2m+1}, \frac{3}{2m+1} \right]$$

so that $S_{k,m} \subset S'_{k,m} \subset S'_0 \subset (1,1)^2$ for each $k = 1, \ldots, m$. Extend $f$ on $\partial S'_0$ and on each $S'_{m,k}$ with identity and note that the new embedding, which we still denote by $f$, is $\eta'_1$-quasisymmetric with $\eta'_1$ depending only on $\eta_1$ and $d$. Applying the Annulus theorem in the LQC category on the interior of each $S'_{k,m} \setminus S_{k,m}$ we

\[ \text{the proof continues...} \]
obtain an \( \eta_2\)-quasisymmetric extension \( F : U \to U' \) with \( \eta_2 \) depending only on \( K \), \( d \) and \( \eta \).

Suppose now that \( U \in \mathcal{B}(L,d) \) and \( f : \partial U \to \mathbb{R}^2 \) is \( L_1 \)-bi-Lipschitz and can be extended homeomorphically on \( U \). If \( U \) is simply connected then the claim follows from Theorem 3.1. Assume now that \( U \) is not simply connected. As before, there exists \( N \in \mathbb{N} \) depending only on \( L \) and \( d \) such that \( \mathbb{R}^2 \setminus U \) has at most \( N \) bounded components \( D_1, \ldots, D_m \). Moreover, there exists \( U' \in \mathcal{B}(L,d) \) such that \( \partial U' = f(\partial U) \) and \( f \) extends to a homeomorphism from \( U \) onto \( U' \).

Let \( D_i \) be a bounded component of \( \mathbb{R}^2 \setminus U \) and set \( D'_i \) be the bounded component of \( \mathbb{R}^2 \setminus U' \) such that \( \partial D'_i = f(\partial D_i) \). Let \( j \) be the maximal integer such that \( \text{diam } D_i \leq 2^{-j-3}(L_1d)^{-1} \text{diam } U \). If \( j \leq 1 \) then \( \text{diam } D_i \geq \frac{1}{32L_1d} \text{diam } U \) and set \( \hat{D}_i = D_i \).

Suppose that \( j \geq 2 \). Fix a point \( x \in \partial D_i \) and let \( x' = f(x) \in \partial D_i \). Let \( B_i = B_2(x', 2L_1 \text{diam } \partial D_i) \), \( \hat{D}_i = B_2(x', 2L_1 \text{diam } \partial D_i) \), and \( \hat{D}'_i = B_2(x', 2L_1 \text{diam } \partial D_i) \). Note that \( D_1 \subset B_1 \subset \hat{D}_i \) and \( D_i \cap (\partial U \setminus \partial D_i) = \emptyset \) and similarly for \( D'_i \). Moreover, \( \text{diam } D_i \geq \frac{1}{32L_1d} \text{diam } U \) and

\[
\min(\text{dist}(\hat{D}_i, \partial U \setminus \partial D_i), \text{dist}(\hat{D}'_i, \partial U' \setminus \partial D'_i)) \geq (4Ld)^{-1} \text{diam } U.
\]

Therefore, \( \hat{U} = U \setminus \bigcup_{i=1}^{m} \hat{D}_i \in \mathcal{B}(L,d') \) for some \( d' \) depending only on \( d \) and \( L_1 \). For each \( i = 1, \ldots, m \) define \( f|_{\hat{D}_i \setminus \partial B_i} \) by translation and apply the Annulus Theorem in the LIP category \( \mathcal{N}\mathcal{A}1\mathcal{T} \) to extend \( f \) on \( B_i \setminus \hat{D}_i \) \( L_1 \)-bi-Lipschitz with \( L'_1 \) depending only on \( L \), \( d \) and \( L_1 \).

The extension of \( f \) on \( \hat{U} \) is similar as with with the first part of Proposition 3.1 applying Lemma 3.5 and the Annulus Theorem in the LIP category.

\[\square\]

### 3.1. A higher dimensional extension

It is well known that both cases of Theorem 3.1 are false in \( \mathbb{R}^3 \) due to the existence of Lipschitz embeddings of \( S^2 \) into \( \mathbb{R}^3 \) that can be extended homeomorphically on \( \mathbb{R}^3 \) but not quasisymmetrically; see for example \( [\text{Tuk}80] \) Section 15.

In this section we work with a much simpler setting. For \( d > 1 \) denote by \( \mathcal{C}_n(d) \) the collection of domains \( U \subset \mathbb{R}^n \) whose boundary components are boundaries of \( n \)-cubes of mutual distances bounded below by \( d^{-1} \text{diam } U \).

**Proposition 3.6.** Let \( U \in \mathcal{C}_n(d) \) and \( f : \partial U \to \mathbb{R}^n \) be \( L \)-bi-Lipschitz that is a similarity on each component of \( \partial U \) and extends homeomorphically on \( U \). Then \( f \) extends \( L' \)-bi-Lipschitz on \( U \) with \( L' \) depending only on \( L \), \( d \) and \( n \).

Before the proof of Proposition 3.6 recall that given a set \( A \subset \mathbb{R}^n \) and \( \delta > 0 \), the \( \delta \)-neighborhood of \( A \) in \( \mathbb{R}^n \) is defined as \( N_\delta(A) = \bigcup_{x \in A} B_\delta(x, \delta) \).

**Proof of Proposition 3.6.** We only give a sketch of the proof as it is similar to that of Proposition 3.1. Since \( f \) extends on \( U \), there exists a domain \( U' \subset \mathbb{R}^n \) whose boundary is a union of disjoint cubes such that \( f \) maps \( \partial U \) on \( \partial U' \) and any homeomorphic extension on \( U \) maps \( U \) on \( U' \).

Firstly, by the doubling property of \( \mathbb{R}^n \), there exists \( N \in \mathbb{N} \) depending only on \( n \) and \( d \) such that \( \partial U \) has at most \( N \) components. In particular, \( U = D_0 \setminus \bigcup_{i=1}^{m} \overline{D}_i \) where \( D_i \) are open \( n \)-cubes and \( m \leq N \).

Secondly, applying the Annulus Theorem in the LIP category, we obtain a small \( \delta > 0 \) and an \( L_1 \)-bi-Lipschitz map \( F : \mathbb{R}^n \to \mathbb{R}^n \) which is identity in \( U \setminus N_\delta(\partial U) \), maps \( \partial D_0 \) on a dyadic cube \( D'_0 \) of side-length \( 2^{k_0+1} \) and each cube
$D_i$ is mapped to a dyadic cube of side-length $2^{k_i+1}$ where $k_0$ is minimal integer such that $\text{diam } D_0 \leq 2^{k_0-1}$ and $k_1, \ldots, k_m \in \mathbb{Z}$ are maximal integers such that $2^{k_i} \leq \min\{\frac{1}{4}2^{k_0}, \frac{1}{2} \text{ diam } D_i\}$. Here, $\delta$ and $L_1$ depend only on $n$, $L$ and $d$.

If $\text{diam } D'_i$ is very small compared to $\text{diam } D'_0$ then, as in the proof of Proposition 3.2 we can replace $D'_i$ with a new dyadic cube which we still denote by $D'_i$ whose side-length is comparable to that of $D_0$ but no more than $\frac{1}{2} 2^{k_0}$.

Applying a uniformization result like Lemma 3.3 we may assume that $U = U'$. The rest of the proof follows from applying the Annulus Theorem in the LIP category $m$ times. □

4. TWO REDUCTIONS

Towards the proofs of Theorem 1.1 and Theorem 1.3 we apply some reductions. In Section 4.1 we show that $\partial U$ in Theorem 1.1 and $E$ in Theorem 1.3 can be assumed perfect. In Section 4.2 we show that $U$ can be assumed to be unbounded with compact boundary and $E$ can be assumed compact.

4.1. Perfect boundary. Let $E \subset \mathbb{R}^n$ be a closed set. For each isolated point $x \in E$ let $\pi(x) \in E$ be a point of smallest distance to $x$ and $E_x$ be the image of the standard middle-third Cantor set $\mathcal{C}$ under a similarity with scaling factor $\frac{1}{3}|x - \pi(x)|$ such that $E_x$ contains $x$. If $x \in E$ is not isolated then set $E_x = \{x\}$. Let $\hat{E} = \bigcup_{x \in E} E_x$. Note that $\hat{E}$ is closed.

Lemma 4.1. For each $c \geq 1$ there exists $c' \geq 1$ depending only on $c$ satisfying the following properties.

1. If $E \subset \mathbb{R}^n$ is $c$-relatively connected then $\hat{E}$ is $c'$-uniformly perfect.
2. If $E \subset \mathbb{R}^n$ is $c$-uniformly disconnected then $\hat{E}$ is $c'$-uniformly disconnected.
3. If $U \subset \mathbb{R}^2$ is $c$-uniform and $U' \subset U$ is the domain with $\partial U' = \partial U$ then $U'$ is $c'$-uniform.

Proof. The proof of the first claim is similar to that of Lemma 3.3 in [Vel16]. Let $x \in \hat{E}$ and $r > 0$. From the fact that $\hat{E}$ is perfect, we have $\{x\} \subset \overline{B}(x, r) \cap \hat{E}$. Suppose that $\hat{E} \setminus \overline{B}(x, r) \neq 0$. If $x \in E$ and is not isolated in $E$,

$$\emptyset \neq E \cap (\overline{B}(x, r) \setminus B^n(x, r/c)) \subset \hat{E} \cap (\overline{B}(x, r) \setminus B^n(x, r/c)).$$

Suppose $x \in E_z$ for some isolated point $z \in E$. If $r > 2c \text{ dist}(z, E \setminus \{z\})$ then $\emptyset \neq (E \setminus \{z\}) \cap \overline{B}^n(z, r/2) \subset \hat{E} \cap \overline{B}^n(x, r)$. Therefore,

$$\emptyset \neq E \cap (\overline{B}^n(z, r/2) \setminus B^n(z, (2c)^{-1}r)) \subset \hat{E} \cap (\overline{B}^n(x, r) \setminus B^n(x, (4c)^{-1}r)).$$

If $r \leq 2c \text{ dist}(z, E \setminus \{z\})$ then $(2c)^{-1}r \leq \frac{1}{10} \text{ dist}(z, E \setminus \{z\})$. The relative connectedness of $\mathcal{C}$ gives

$$\emptyset \neq E_z \cap (\overline{B}^n(z, (2c_0)^{-1}r)) \subset \hat{E} \cap (\overline{B}^n(x, r) \setminus B^n(x, (20c)^{-1}r)).$$

for some $c_0 > 1$ depending only on $c$.

To show the second claim, let $x \in \hat{E}$ and $0 < r < \frac{1}{4} \text{ diam } \hat{E}$ and let $z \in E$ be the unique point of $E$ such that $x \in E_z$. If $z$ is an accumulation point then $z = x$. Let $E'$ be the subset of $\hat{E}$ containing $x$ with $\text{diam } E' \leq r$ and $\text{dist}(E', E \setminus E') \geq c^{-1}r$. Then $\text{diam } \hat{E}' \leq \frac{1}{4}r$ and $\text{dist}(\hat{E} \setminus \hat{E}') \geq \frac{1}{4}c^{-1}r$.

Assume now that $z$ is isolated point. Since $\mathcal{C}$ is $c_0$-uniformly disconnected, the claim of the lemma follows with $c' = c_0$ if $r < \frac{1}{8} \text{ diam } E_z$. Also, by uniform disconnectedness of $\mathcal{C}$, if $r < 100 \text{ diam } E_z$, then then the claim of the lemma is true.
for $c' = c_0/400$. If $100 \operatorname{diam} E_z \geq \frac{1}{4} \operatorname{diam} \hat{E}$ then we are done. Assume the opposite and let $r > 100 \operatorname{diam} E_z$. By uniform disconnectedness of $E$, there exists $E' \subset \overline{E}$ containing $z$ such that $\operatorname{diam} E' \leq r/2$ and $\operatorname{dist}(E', \overline{E'}) \geq (2r)^{-1}r$. Then $x \in \hat{E}'$, $\operatorname{diam} \hat{E}' \leq r$ and $\operatorname{dist}(\hat{E}'', \hat{E} \setminus \hat{E}') \geq (4c)^{-1}r$.

For the third claim let $E$ be the set of isolated points of $\partial U$. Then $U' = U \setminus \hat{E}$. The uniformity of $U'$ follows from the fact that $E$ is uniformly disconnected and therefore an NUD in the sense of Väisälä [Väi88b]; see Theorem 1 and Corollary 2 in [Mac99].

Let $E \subset \mathbb{R}^n$ and a mapping $f : E \to \mathbb{R}^n$. If $f$ is $\eta$-quasisymmetric, define $\hat{f} : \hat{E} \to \mathbb{R}^n$ with $\hat{f}|E = f$ and for every isolated point $x \in E$,

$$\hat{f}|E_x(y) = f(x) + \frac{1}{\eta(1)} \frac{|f(x) - f(\pi(x))|}{|x - \pi(x)|} (y - x).$$

If $f$ is $L$-bi-Lipschitz, define $\hat{f} : \hat{E} \to \mathbb{R}^n$ with $\hat{f}|E = f$ and for every isolated point $x \in E$,

$$\hat{f}|E_x(y) = f(x) + \frac{1}{L^2} \frac{|f(x) - f(\pi(x))|}{|x - \pi(x)|} (y - x).$$

**Lemma 4.2.** Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^n$ be $\eta$-quasisymmetric (resp. $L$-bi-Lipschitz). Then $\hat{f}$ is $\eta'$-quasisymmetric (resp. $(5L)$-bi-Lipschitz) with $\eta'$ depending only on $\eta$.

**Proof.** We first show the claim for bi-Lipschitz mappings. Given two distinct points $x, y \in \hat{E}$, there exist unique $x', y' \in E$ such that $x \in E_{x'}$ and $y \in E_{y'}$. If $x' = y'$ there is nothing to prove as $\hat{f}$ is affine on $E_{x'}$. Suppose that $x' \neq y'$ and note that

$$|x - y| \geq \max\left\{ \frac{9}{10}|x' - \pi(x')|, \frac{9}{10}|y' - \pi(y')|, |x - x'|, |y - y'| \right\}.$$  

Then, $|f(x) - f(y)| \leq |f(x') - f(y')| + L^{-1}(|x - x'| + |y - y'|) \leq 5L|x - y|$ and $|f(x) - f(y)| \geq L^{-1}|x' - y'| - L^{-1}(|x - x'| + |y - y'|) \geq (2L)^{-1}|x - y|$.

The proof in the case that $f$ is quasisymmetric is similar to that of Lemma 3.3 in [Vel16]. Let $x, y, z \in E^*$ be three distinct points with $x \in E_{x'}$, $y \in E_{y'}$ and $z \in E_{z'}$ for some $x', y', z' \in E$. If $x' = y' = z'$ then $x, y, z$ are in the same $E_{x'}$ where $\hat{f}$ is affine. If $x' \neq z'$ and $x' = y'$ then the prerequisites of Lemma 2.29 in [Sem90] are satisfied (see also Remark 3.2 in [Vel16]) for $A = E \setminus \{x'\}$, $A^* = E \cup E_{x'}$ and $H = \hat{f}|A^*$. Thus, $\hat{f}|E \cup E_{x'}$ is $\eta'$-quasisymmetric for some $\eta'$ depending only on $\eta$. Hence,

$$\frac{|\hat{f}(x) - \hat{f}(y)|}{|\hat{f}(x) - \hat{f}(z)|} \leq C_1 \frac{|\hat{f}(x') - \hat{f}(y')|}{|\hat{f}(x') - \hat{f}(z')|} \leq C_1 \eta' \frac{|x - y|}{|x - z|} \leq C_1 \eta' \left( C_2 \frac{|x - y|}{|x - z|} \right)$$

for some $C_1, C_2 > 1$ depending only on $\eta$. Similarly for $x' = z' \neq y'$. If $x', y', z'$ are distinct then by Remark 3.2 in [Vel16],

$$\frac{|\hat{f}(x) - \hat{f}(y)|}{|\hat{f}(x) - \hat{f}(z)|} \leq C_3 \frac{|\hat{f}(x') - \hat{f}(y')|}{|\hat{f}(x') - \hat{f}(z')|} \leq C_3 \eta' \left( C_4 \frac{|x - y|}{|x - z|} \right)$$

for some $C_3, C_4 > 1$ depending only on $\eta$. Therefore, $\hat{f}$ is quasisymmetric. 

\[\square\]
4.2. Bounded boundary. We reduce the proof of Theorem 1.1 to the case that $U$ is the complement of a compact set, and the proof of Theorem 1.3 to the case that $E$ is compact.

For the bi-Lipschitz case we make the following observation that is used repeatedly in this section. The proof is simple and is left to the reader.

**Remark 4.3.** Let $E \subset \mathbb{R}^n$ be non-degenerate, $x_0 \in E$ and $f : E \to \mathbb{R}^n$ be $L$-bi-Lipschitz. Then, $g : \{(x - x_0)|x - x_0|^2 : x \in E \setminus \{x_0\}\} \to \mathbb{R}^n$ with

$$g(x) = \frac{f(x|x|^2 + x_0) - f(x_0)}{|f(x|x|^2 + x_0) - f(x_0)|^2}$$

is $L'$-bi-Lipschitz with $L'$ depending only on $L$.

4.2.1. Uniform domains. For this section assume that $U \subset \mathbb{R}^2$ is $c$-uniform and that $f$ is $L$-bi-Lipschitz (resp. $f$ is $\eta$-quasisymmetric and $\partial U$ is $c$-relatively connected) which admits a homeomorphic extension on $U$. Assume moreover that Theorem 1.1 holds when $U$ is unbounded and $\partial U$ is bounded. To simplify the exposition, we use complex coordinates for the rest of Section 4.2.1.

By Lemma 2.10 there exist $c' > 1$ depending only on $c$ and $L$ (resp. $\eta$) and a bounded $c'$-uniform domain $U'$ such that $f$ extends to a homeomorphism between $U$ and $U'$. There are three cases to consider.

Case 1. Suppose that $U$ is bounded.

By the porosity of $\partial U$ and $\partial U'$, there exist points $x_0 \in U$ and $x_0' \in U'$ such that $B^2(x_0, (4c)^{-1} \text{diam} U) \subset U$ and $B^2(x_0', (4c')^{-1} \text{diam} U') \subset U'$. Applying similarity mappings we may assume that $x_0 = x_0' = 0$ and $\text{diam} U = \text{diam} U' = 1$.

Assume first that $f$ is $L$-bi-Lipschitz. The domain $U \{x_0\}$ is $c_1$-uniform for some $c_1$ depending only on $c$ and the map $f_1 : \partial U \cup \{x_0\} \to \{x_0\}$ with $f_1|\partial U = f$ and $f_1(x_0) = x_0'$ is $L_1$-bi-Lipschitz for some $L_1 > 1$ depending only on $L$ and $c$.

Moreover, $f_1$ admits a homeomorphic extension on $U$. Set $V = T(U), V' = T(U')$, $g = T \circ f \circ T$. Then, $V$ is an unbounded $c_1$-uniform domain, $\partial V$ is bounded. By Remark 4.3, $g = T \circ f_1 \circ T$ is $\eta$-bi-Lipschitz defined on the boundary of an unbounded uniform domain with bounded boundary and the extension of $g$ follows by our assumption. Taking inversions again, we obtain an $L'$-bi-Lipschitz extension for $f$ with $L'$ depending only on $L$ and $c$.

Assume now that $f$ is $\eta$-quasisymmetric. Note that the inversion map $T : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ with $T(x) = x|x|^{-2}$ is 1-quasiconformal while $T|B^2(0, 2) \setminus B^2(0, (8c)^{-1})$ and $T|B^2(0, 2) \setminus B^2(0, (8c')^{-1})$ are $L_2$-bi-Lipschitz for some $L_2$ depending only on $c$ and $\eta$. Again, note that $V$ is an unbounded $c_1$-uniform domain, $\partial V$ is a bounded and $c_1$-relatively connected, and $\Phi$ is $\eta_1$-quasisymmetric that can be extended as a homeomorphism of $V$. Here, $c_1$ and $\eta_1$ depending only on $c$ and $\eta$. By our assumption, there exists an $\eta_1'$-quasisymmetric extension $G : V \to V'$. Let $F : U \to U'$ with $F = T \circ G \circ T$. Then, $F$ is $K$-quasiconformal for some $K$ depending only on $c$ and $\eta$ and by Lemma 2.7 $F$ is $\eta'$-quasisymmetric with $\eta'$ depending only on $c$ and $\eta$.

Case 2. Suppose that $U$ is unbounded and $\partial U$ contains an unbounded component.

By Proposition 2.7 $\partial U$ contains a quasiline $\Gamma$, all other components of $\partial U$ are bounded and $U$ is contained in one of the two components of $\mathbb{R}^2 \setminus \Gamma$. Fix $z_0 \in \Gamma$. 

The bi-Lipschitz case is similar to Case 1. Let \( z_0' = f(z_0) \) and \( r > 0 \). Define \( x_0 \) be the point on \( \partial B^2(z_0, r) \) such that \( B^2(x_0, r/c) \subset U \). Similarly we obtain a point \( x_0' \in \partial B^2(z_0', r) \). Extend \( f \) on \( \{x_0\} \) with \( f(x_0) = x_0' \) and the rest is as in Case 1.

Assume now that \( f \) is \( \eta \)-quasisymmetric. Applying an \( \eta_0 \)-quasisymmetric homeomorphism of \( \mathbb{R}^2 \) we may assume that \( \Gamma = f(\Gamma) = \mathbb{R}\{0\} \), \( z_0 = 0 \) and that \( U \) and \( U' \) are subsets of the upper half-plane. Here \( \eta_0 \) depends only on \( c \) and \( \eta \). For each \( k \in \mathbb{N} \) let \( z_k = (12c^2)^{n-1} \) and let \( \gamma_k \) be a \( c \)-cigar curve in \( U \) joining \( z_k \) with \( -z_k \). Note that for \( k \geq 2 \), \( \gamma_k \subset B^2(z_0, 3c(12c^2)^{k-1}) \setminus B^2(z_0, (2c)^{-1}(12c^2)^{k-1}) \) and, therefore, \( \text{dist}(\gamma_k, \gamma_{k+1}) \geq (4c)^{-1}(12c^2)^k \).

For each \( k \in \mathbb{N} \) let \( U_k \) be the domain bounded by \( \gamma_k \) and \( \Gamma \) and set \( E_k = \partial U \cap U_k \), \( E_k' = f(E_k) \). Each \( U_k \) is bounded and it is easy to check that each \( U_k \) is \( \epsilon' \)-uniform with \( \epsilon' \)-relatively connected boundary for some \( \epsilon' > 1 \) depending only on \( c \). Note also that \( \text{diam} E_k' \leq C|f(z_k) - f(-z_k)| \) for some \( C \) depending only on \( c \) and \( \eta \). Define

\[
\gamma_k' = [f(z_k), f(z_k) - 2i \text{diam } E_k'] \cup [f(-z_k), f(-z_k) - 2i \text{diam } E_k'] \cup C_1 \cup C_2
\]

where \( \sigma_1, \sigma_2 \) are circular arcs of \( \partial B^2(f(z_k), \text{diam } E_k) \), \( \partial B^2(f(-z_k), \text{diam } E_k) \) respectively so that \( \gamma_k \cup f([-z_k, z_k]) \) is the boundary of a \( K \)-quasidisk \( D_k \) which contains \( E_k \) in its closure. Here \( K \) depends only on \( c \) and \( \eta \).

Applying the quasitemetry extension property of relatively connected subsets of quasicircles \[ \text{Ve10}, \] we extend \( f \) to an \( \eta_1 \)-quasisymmetric \( f_k : \partial U_k \to \mathbb{R}^2 \) with \( f_k(\gamma_k') = \gamma_k' \). Here \( \eta_1 \) depends only on \( \eta \) and \( c \). The extension \( F_k \) of \( f_k \) on each \( U_k \) follows from Case 1. As \( U = \bigcap_{k \in \mathbb{N}} U_k \), by standard converging arguments \[ \text{Hei01}, \] Corollary 10.30, \( \{F_k\} \) subconverges to a mapping \( F : U \to \mathbb{R}^2 \) with \( F|\partial U = f \) that is \( \eta' \)-quasisymmetric.

**Case 3.** Suppose that \( U \) is unbounded and all components of \( \partial U \) are bounded.

Fix \( x \in \partial U \) and let \( A_x \) be the component of \( \partial U \) containing \( x \). Let \( r_1 > 8 \text{diam } A_x \). By Proposition 5.10, \( A_x \) is contained in a neighborhood \( N_{i_1}(A_1, r_1) \) where \( i_1 \in \{1, 2\} \) and \( A_1 \) is a component of \( \partial U \). Let \( U_1 \) be the subset of \( U \) with boundary \( N_{i_1}(A_1, r_1) \). Let \( r_2 > 8 \text{diam } N_{i_1}(A_1, r_1) \). Inductively, having defined \( r_k \) and \( N_{i_k}(A_k, r_k) \), let \( r_{k+1} > 8 \text{diam } N_{i_k}(A_k, r_k) \) and arguments similar to that of Proposition 5.10 show that \( N_{i_k+1}(A_{k+1}, r_{k+1}) \) is contained in a neighborhood \( N_{i_{k+1}}(A_{k+1}, r_{k+1}) \) where \( i_{k+1} \in \{1, 2\} \) and \( A_{k+1} \) is a component of \( \partial U \).

For each \( k \in \mathbb{N} \) let \( U_k \subset \mathbb{R}^2 \) be the unbounded domain with \( \partial U_k = N_{i_k}(A_k, r_k) \). By Lemma 5.10, each \( U_k \) is \( \epsilon' \)-uniform and each \( \partial U_k \) is \( \epsilon' \)-relatively connected for some \( \epsilon' > 1 \) depending only on \( c \). By Theorem 1.14, there exists a mapping \( F_k : U_k \to \mathbb{R}^2 \) that extends \( f|\partial U_k \) which is \( \eta' \)-quasisymmetric (resp. \( L'-\text{bi-Lipschitz} \)) with \( \eta' \) depending only on \( c \) and \( \eta \) (resp. \( L' \geq 1 \) depending only on \( c \) and \( L \)). As in Case 2, \( \{F_k\} \) subconverges to a mapping \( F : U \to \mathbb{R}^2 \) with \( F|\partial U = f \) that is \( \eta' \)-quasisymmetric (resp. \( L'-\text{bi-Lipschitz} \)).

**4.2.2. Unbounded sets.** Assume that \( E \subset \mathbb{R}^n \) \((n \geq 3)\) is unbounded and \( c \)-uniformly disconnected. Assume also that \( f : E \to \mathbb{R}^n \) is \( L \)-bi-Lipschitz (resp. \( E \) is \( c \)-uniformly perfect and \( f \) is \( \eta \)-quasisymmetric) and that Theorem 1.13 holds for bounded sets.

Fix \( x \in E \). For each \( k \in \mathbb{N} \) let \( E_k \) be a subset of \( E \) containing \( x \) such that \( \text{diam } E_k \leq 2^k \) and \( \text{dist}(E_k, E \setminus E_k) \geq c^{-12^k} \). Note that each \( E_k \) is \( c \)-uniformly disconnected as the property is preserved on subsets.

Suppose that \( E \) is uniformly perfect and that \( f \) is \( \eta \)-quasisymmetric: the bi-Lipschitz case is identical. By \( c \)-uniform perfectness, \( \text{diam } E_k \geq c^{-2^k} \). We show
that each $E_k$ is $c^2$-uniformly perfect. Let $y \in E_k$ and $r > 0$. Since $E_k$ is perfect, either $B_r^c(y, r) \cap E_k = E_k$ or $E_k \cap B_r^c(y, r) \neq \emptyset$. Assume the latter. Then, $r \leq 2^k$ and by uniform discontinuity, $B_r^c(y, c^{-1}r) \cap E_k = B_r^c(y, c^{-1}r) \cap E$. By $c$-uniform perfectness of $E$, $(B_r^c(y, c^{-1}r) \setminus B^n(y, c^{-2}r)) \cap E_k = (B_r^c(y, c^{-1}r) \setminus B^n(y, c^{-2}r)) \cap E \neq \emptyset$.

Therefore, each $E_k$ is $c^2$-uniformly disconnected and $c^2$-uniformly perfect. Assuming Theorem 1.3 for bounded sets, each $f|E_k$ extends to a mapping $F_k : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that is $\eta'$-quasisymmetric with $\eta'$ depending only on $c$ and $\eta$. As in Section 4.2.1, $\{F_k\}$ subconverges to a mapping $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $F|E = f$ that is $\eta'$-quasisymmetric.

5. Separation of boundary components for planar uniform domains

For this section we fix an unbounded $c$-uniform domain $U \subset \mathbb{R}^2$ with bounded boundary. The goal of this section is to break the boundary of $\partial U$ into sets that are contained in bi-Lipschitz disks and are far from the boundary of those disks.

5.1. Cubic thickening. In this section we show that given a continuum $E \subset \mathbb{R}^2$ and some $\epsilon > 0$ there exists a bi-Lipschitz disk $D$ containing $E$ so that each point of $\partial D$ is of distance roughly $\epsilon > 0$ from $E$.

We first review some notation from [Mac99]. Let $\epsilon > 0$. We denote by $G_\epsilon$ the square grid whose vertices are the points $(me, ne)$ where $m, n \in \mathbb{Z}$, and by $\Sigma_\epsilon$ the associated family of closed squares. Given a bounded set $W \subset \mathbb{R}^2$ let $W^\epsilon$ be the union of the elements of $\Sigma_\epsilon$ that intersect $W$. Let $T_\epsilon(W) = (W^\epsilon)^\epsilon$.

Lemma 5.1. There exists a decreasing homeomorphism $L : (0, +\infty) \to (1, +\infty)$ with the following property. If $E \subset \mathbb{R}^2$ is a continuum and $\epsilon > 0$, there exists an $L(\epsilon)$-bi-Lipschitz disk $D \subset \mathbb{R}^2$ containing $E$ such that for all $x \in \partial D$,

$$\epsilon \operatorname{diam} E \leq \operatorname{dist}(x, E) \leq 8\epsilon \operatorname{diam} E.$$

Proof. If $\epsilon \geq 3$ then the set $\gamma_E = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, E) = \epsilon \operatorname{diam} E\}$ is the boundary of an $L_0(\partial)$-bi-Lipschitz disk with $L_0$ being a universal constant [Bro72, Lemma 1].

We assume for the rest that $\epsilon < 3$. Fix $\delta = \epsilon \operatorname{diam} E$. By Lemma 2.1 in [Mac99], $T_\delta(E)$ is a closed domain whose boundary consists of at most $N_2/\epsilon$ disjoint Jordan curves, each of which is a subset of $G_\delta$. Here, the number $N_2 > 1$ is a universal constant. Moreover, the distance from any boundary point of $T_\delta(E)$ to $E$ is less than $8\delta$ and greater than $\delta$. Let $D$ be the domain bounded by the outermost component of $\partial T_\delta(E)$, that is, $D$ is the exterior of the unbounded component of $\mathbb{R}^2 \setminus T_\delta(E)$. Then, $\delta \leq \operatorname{dist}(x, E) \leq 8\delta$ for all $x \in \partial D$.

Finally, notice that there are at most $N_3/\epsilon$ squares of $G_\delta$ intersecting an $8\delta$-neighborhood $N(E, 8\delta)$ for some universal $N_3 > 1$. Therefore, there are at most $(N_3/\epsilon)^{N_3/\epsilon}$ different ways to form $D$. As each Jordan curve consisting of edges of $G_\delta$ is a bi-Lipschitz circle, $\partial D$ is $L$-bi-Lipschitz for some $L$ depending on $\epsilon$. □

In the case that $E$ is a $K$-quasidisk, $L(\epsilon)$ may increase without control as $\epsilon \to \infty$ but the disk $D_\epsilon$ given in Lemma 5.1 is always a $K'$-quasidisk with $K'$ depending only on $K$.

Lemma 5.2. Suppose that $E \subset \mathbb{R}^2$ is a $K$-quasidisk and let $D_\epsilon$ be the $L(\epsilon)$-bi-Lipschitz disk of Lemma 5.1. Then, for all $\epsilon > 0$, $D_\epsilon$ is a $K'$-quasidisk with $K'$ depending only on $K$. 

Proof. We may assume that $\epsilon$ is sufficiently small. Fix $\epsilon > 0$ and let $\delta = \epsilon \text{diam } E$, $D = D_\epsilon$, $\Gamma = \partial D$ and $\gamma = \partial E$. Since $E$ is a $K$-quasidisk, $\gamma$ satisfies the $c$-bounded turning property for some $c > 1$ depending only on $K$. We show that $\Gamma$ is $136c$-bounded turning and the lemma follows.

Let $x_1, x_2 \in \Gamma$. Since, $\Gamma$ is a polygonal curve in $\mathcal{G}_4$, it is enough if we assume that $x, y$ are non-adjacent vertices of $\mathcal{G}_4$. Therefore, $|x_1 - x_2| \geq \delta$. Contrary to our claim assume that there exist $x_3, x_4 \in \Gamma$ such that $x_3$ and $x_4$ are in different components of $\Gamma \setminus \{x_1, x_2\}$ and $\min \{|x_3 - x_1|, |x_4 - x_1|\} \geq 68c|x_1 - x_2|$. For each $i \in \{1, 2, 3, 4\}$ let $x'_i$ be the point in $\gamma$ closest to $x_i$. Then, $|x_i - x'_i| \in (\delta, 8\delta)$ for each $i = 1, 2, 3, 4$ and $x'_1$ and $x'_4$ are in different components of $\gamma \setminus \{x'_1, x'_4\}$. Therefore, for $i = 3, 4$, $|x'_1 - x'_i| \geq |x_1 - x_i| - 16\delta \geq 34c|x_1 - x_2|$ while $2c|x_1 - x'_2| \leq 2c|x_1 - x_2| + 32c\delta \leq 34c|x_1 - x_2|$. Therefore, $\min \{|x'_1 - x'_3|, |x'_1 - x'_4|\} > c|x'_1 - x'_2|$ and the $c$-bounded turning property is violated.

$\square$

Remark 5.3. In addition to Lemma 5.2 note that for each $M > 0$ there exists $L(M) > 1$ such that any subarc $\gamma$ of $\partial D_\epsilon$ whose endpoints $x, y$ satisfy $|x - y| \leq M\epsilon$, is $L(M)$-bi-Lipschitz arc.

5.2. Local separation in the boundary of $U$. In this section, given a compact $A \subset U$ that is disjoint from $\partial U \setminus A$ and a small $\epsilon > 0$, we construct a bi-Lipschitz circle that separates the two sets $A$ and $\partial U \setminus A$ and its distance from $\partial U$ is at least a fixed multiple of $\epsilon$.

Lemma 5.4. Suppose that $A \subset \partial U$ is compact and disjoint from $\partial U \setminus A$ and let $\epsilon \leq (32c)^{-1}\text{dist}(A, \partial U \setminus A)$. There exists $C, L > 1$ depending only on $c$ and $\epsilon(\text{diam } A)^{-1}$ and there exists an $L$-bi-Lipschitz disk $\Delta$ such that $\partial \Delta \subset U$, $\Delta \setminus U = A$, $\text{dist}(z, \partial \Delta) \leq 8\epsilon$ for all $z \in A$ and $C^{-1}\text{diam } A \leq \text{dist}(z', \partial U)$ for all $z' \in \partial \Delta$.

Proof. As in the proof of Lemma 5.1 consider the thickening $\mathcal{T}_\epsilon(A)$. Then, $\partial \mathcal{T}_\epsilon(A)$ consists of at most $N_0$ components in $U$ each being an $L'$-bi-Lipschitz circle with $N_0$ and $L'$ depending only on $c$ and $\epsilon(\text{diam } A)^{-1}$. The choice of $\epsilon$ implies that $\mathcal{T}_\epsilon(A) \cap \partial U = A$. Let $D$ be the closure of $\mathbb{R}^2 \setminus V$ where $V$ is the unbounded component of $\mathbb{R}^2 \setminus \mathcal{T}_\epsilon(A)$. Observe that $\text{dist}(x, A) \leq 8\epsilon$ for all $x \in \partial D$.

We claim that $\partial \mathcal{T}_\epsilon(A)$ is in $\mathbb{R}^2 \setminus D$. Contrary to the claim, assume that there exists $x \in (\partial \mathcal{T}_\epsilon(A) \cap D$. Note that dist$(x, \partial D) > \frac{1}{2}\text{dist}(A, \partial U \setminus A)$ as otherwise $\text{dist}(x, A) < \text{dist}(A, \partial U \setminus A)$. There exists $y \in \overline{U}$, exterior to $D$ such that $\text{dist}(y, A) \leq \text{dist}(A, \partial U \setminus A)$. Let $\gamma$ be a c-cigar curve joining $x$ and $y$ in $\overline{U}$ and let $z \in \partial D \cap \gamma$. Then,

$$\text{dist}(A, \partial U \setminus A) > 16\epsilon \geq 2c\text{dist}(z, A) \geq 2\min\{|x - z|, |y - z|\} \geq 2\min\{|x - z|, |y - z|\}$$

which is a contradiction and the claim follows.

If $D$ has only one component then set $\mathcal{D} = D$ and the proof is complete.

Let $D_1, \ldots, D_n$ be the components of $D$. Note that $n \leq N_0$ and that $\text{dist}(D_i, D_j) \geq \delta_0$ depending only on $c$ and $\epsilon(\text{diam } A)^{-1}$. Set $D^{(0)} = D$ and $U^{(0)} = U \setminus \mathbf{D}$. By Remark 2.8 $U^{(0)}$ is $c_0$-uniform for some $c_0 > 1$ depending only on $c$ and $\epsilon(\text{diam } A)^{-1}$.

Inductively, suppose that for some $0 \leq i < n - 1$, $D^{(i)}$ is a union of $n - i$ many $L_i$-bi-Lipschitz disks $D^{(i)}_1, \ldots, D^{(i)}_n$ such that $\partial D^{(i)} \subset U$, $\text{dist}(\partial D^{(i)}, \partial U) \geq C_i$ and $\text{dist}(D^{(i)}_j, D^{(i)}_{j'}) \geq \delta_i$ for some $C_i, \delta_i > 0$ and $c_i, L_i > 1$ depending only on $c, \epsilon(\text{diam } A)^{-1}$ and $i$. Let $x \in \partial D^{(i)}_1$,
y ∈ ∂D^0 \text{ and } \gamma \subset U^0 \text{ be a simple } c_i\text{-cigar curve joining } x_1 \text{ with } x_2. \text{ Applying Lemma 5.1 with } E = \gamma \text{ and } \epsilon = (32c_i)^{-1} \min\{|\delta_i, C_i|\} \text{ we have an } L'\text{-bi-Lipschitz disk } D' \text{ containing } \gamma. \text{ Set } D^{(i+1)} = D^{(i)} \cup D' \text{ and } U^{(i+1)} = U \setminus D^{(i+1)}. \text{ Note that } D^{(i)} \text{ is a union of } n - i - 1 \text{ many } L_{i+1}\text{-bi-Lipschitz disks } D_i^{(i+1)}, \ldots, D_{n-i-1}^{(i+1)} \text{ such that } \partial D^{(i+1)} \subset U, \text{ dist}(\partial D^{(i+1)}, \partial U) \geq d_{i+1} \text{ diam } A, \text{ dist}(D_j^{(i+1)}, D_j^{(i+1)}) \geq \delta_{i+1} \text{ diam } A \text{ and } U^{(i+1)} \text{ is } c_{i+1}\text{-uniform for some } d_{i+1}, \delta_{i+1} > 0 \text{ and } c_{i+1}, L_{i+1} > 1 \text{ depending only on } d_i, \delta_i, c_i \text{ and } L_i.

Set } \Delta = D^{(n-1)} \text{ and note that } \Delta \text{ satisfies the desired properties with constants depending only on } c \text{ and } \epsilon(\text{diam } A)^{-1}. \square

The bi-Lipschitz disk \( \Delta \) constructed in the proof of Lemma 5.4 is denoted by \( V(A, U, \epsilon) \).

**Remark 5.5.** Notice that the construction of \( V(A, U, \epsilon) \) involves creating curves in a neighborhood of \( A \). Therefore, if \( A \) and \( A' \) are mutually disjoint compact subsets of \( \partial U \) such that \( A \cap (\partial U \setminus A) = A' \cap (\partial U \setminus A') = \emptyset \) and dist\((A, A')\) is sufficiently big then \( V(A, U, \epsilon) \cap V(A', U, \epsilon') = \emptyset \) for all \( \epsilon \leq (32c)^{-1} \text{ dist}(A, \partial U \setminus A) \) and \( \epsilon' \leq (32c)^{-1} \text{ dist}(A', \partial U \setminus A') \).

### 5.3. A weak form of uniform disconnectedness

In this section we consider a different separation than that of Section 5.2 that resembles uniform disconnectedness. Given \( x \in \partial U \) and \( r > 0 \) we find an \( L\text{-bi-Lipschitz disk } \Delta \) that contains \( x \) such that

1. either \( \text{diam } \Delta \) is comparable to \( r \) and every point of \( \partial \Delta \) has distance from \( \partial U \) at least a fixed multiple of \( r \),
2. or \( \Delta \) contains a component of \( \partial U \) whose diameter is at least a fixed multiple of \( \text{diam } \Delta \) and every point of \( \partial \Delta \) has distance from \( \partial U \) at least a fixed multiple of \( r \).

Note that if only the first condition was satisfied for all \( x \) and \( r \) then \( \partial U \) would be uniformly disconnected.

**Lemma 5.6.** There exists \( C > 1 \) depending only on \( c \) such that for every non-degenerate component \( A \) of \( \partial U \) and for every positive \( r \leq C^{-2} \text{ diam } A \) there exists \( A' \subset \partial U \) containing \( A \) and a simple closed curve \( \gamma \subset U \) separating \( A' \) from \( \partial U \setminus A' \) such that

\[
C^{-1} \text{ dist}(z, A) \leq r \leq C \text{ dist}(z, \partial U) \text{ for all } z \in \gamma.
\]

**Proof.** By Proposition 5.7 \( A \) is a \( K\)-quasicircle with \( K \) depending only on \( c \). Therefore, \( A \) satisfies the \( c_1\)-bounded turning property for some \( c_1 > 1 \) depending only on \( c \). Set \( c_2 = \max\{c, c_1\} \).

Fix now \( r \leq (4c_2)^{-2} \text{ diam } A \). Find ordered points \( x_1, \ldots, x_n \) on \( A \) such that \( r/2 \leq |x_i - x_{i+1}| \leq r \) for all \( i = 1, \ldots, n \) with the convention \( x_{n+1} = x_1 \). For each \( i = 1, \ldots, n \), join \( x_i \) to \( x_{i+1} \) with a \( c\text{-cigar curve } \gamma_i \). On each \( \gamma_i \), \( i = 1, \ldots, n \), let \( z_i \in \gamma_i \) be a point such that \( \min\{|z_i - x_i|, |z_i - x_{i+1}|\} \geq |x_i - x_{i+1}|/2 \geq r/4 \). Join each \( z_i \) to \( z_{i+1} \) with a \( c\text{-cigar curve } \gamma'_i \). As before, we conventionally set \( z_{n+1} = z_1 \).

Then \( |z_i - z_{i+1}| \leq 2c_2 \) and \( \text{diam } \gamma_i \leq 2(c_2)^2r \). The assumption on \( r \) implies that \( \gamma_i \cup \gamma'_i \cup \gamma_{i+1} \cup A(x_i, x_{i+1}) \) is contractible in the exterior of \( A \). In particular, \( \gamma' = \bigcup \gamma'_i \) separates \( A \) from \( \infty \). Let \( \gamma \subset \gamma' \) be a simple closed curve homotopic to \( \gamma \) in \( U \).
For the proof of \([5.1]\), fix \(z \in \gamma\) and \(i \in \{1, \ldots, n\}\) such that \(z \in \gamma_i\). Then, \(\text{dist}(z, A) \leq |z - x_i| \leq \text{diam} \gamma_i + \text{diam} \gamma_i \leq (2c_2)^2 r\). On the other hand, by Remark \([2.4]\), \(\text{dist}(z, \partial U) \geq (c_2)^{-1} \min\{\text{dist}(z_i, \partial U), \text{dist}(z_i, \partial U)\} \geq (c_2)^{-2} r/2\). Thus, the lemma holds with \(C = (4c_2)^2\).

**Remark 5.7.** Note that \(\text{diam} \gamma_1 \cup \gamma_i \cup \gamma_{i+1} \cup A(x_i, x_{i+1}) \leq Cr\). Therefore, if \(A_1 \subset A'\) and \(A_1 \neq A\) then \(\text{diam} A_1 \leq Cr \leq C^{-1} \text{diam} A\).

Given a non-degenerate component \(A\) of \(\partial U\) and \(r < C^{-2} \text{diam} A\) we set \(N_1(A, r)\) to be a set \(A' \subset \partial U\) as in the statement of Lemma \([5.6]\). Moreover, if \(\gamma\) is a simple closed curve as in Lemma \([5.6]\) associated to \(A' = N_1(A, r)\), let \(D_1(A, r)\) be the \(L(r)\)-Lipschitz disk containing \(\gamma\) as in Lemma \([5.7]\) with \(\epsilon = r/24\).

The next lemma provides us with a different kind of a neighborhood where the radius \(r\) is big compared to the diameters of the components of \(\partial U\) in a \(r\)-neighborhood of \(A\).

**Lemma 5.8.** Let \(A\) be a component of \(\partial U\), \(x \in A\) and \(r > 2 \text{diam} A\) is such that every component \(A'\) of \(\mathbb{R}^2 \setminus U\) intersecting \(B^2(x, r)\) satisfies \(\text{diam} A' \leq c' r\) for some \(c' > 1\). Then, there exists \(C' > 1\) depending only on \(c, c'\) and there exists a simple closed curve \(\gamma\) separating \(A\) from \(\infty\) satisfying

\[
\frac{r}{2(2c')^2} \leq \frac{\text{dist}(z, \partial U)}{2c'} \leq \frac{r}{2} \leq \text{diam} \gamma \leq C' r \quad \text{for all } z \in \gamma.
\]

**Proof.** The proof follows closely that of Lemma 2.2 in \([Mac99]\).

Let \(A_1, \ldots, A_n\) be the components of \(\partial U \setminus A\) intersecting \(\partial B^2(x, r)\) such that \(\text{diam} A_i \geq (16c)^{-1} r\). By Lemma \([5.3]\), \(n\) is bounded above by a constant depending only on \(c, c'\). For each \(i = 1, \ldots, n\) let \(\gamma_i\) be a simple closed curve as in Lemma \([5.6]\) corresponding to \(A_i\) and \(r_i = (2C)^{-1} \min\{\text{diam} A_i, r\}\). Note that \((32Cc)^{-1} r \leq r_i \leq (2C)^{-1} c'r\). Let \(D_i\) be the Jordan domain enclosed by \(\gamma_i\).

By the uniformity of \(U\) and the choice of \(r_i\), there exists at least one nontrivial component of \(\partial B^2(x, r) \setminus \bigcup_{i=1}^n D_i\). Let \(V\) be the component of \(B^2(x, r) \setminus \bigcup D_i\) that contains \(x\). Suppose \(V \cap \partial B^2(x, r) = \Gamma_1 \cup \Gamma_2 \cup \cdots\) where \(\Gamma_i, i \in \mathbb{N}\), is an open subarc of \(\partial B^2(x, r)\). If \(\text{diam} \Gamma_i < (2c')^{-1} r\) then join the endpoints of \(\Gamma_i\) with a \(c_1\)-cigar curve.

Assume now that \(\text{diam} \Gamma_i \geq (2c')^{-1} r\). Let \(y_1, \ldots, y_n\) be consecutive points on \(\Gamma_i\) such that \(y_1\) and \(y_n\) are the endpoints of \(\Gamma_i\) and \((8c)^{-1} r \leq |y_j - y_{j+1}| \leq (4c)^{-1} r\). Set \(w_1 = y_1, w_n = y_n\) and if \(\text{dist}(y_j, \partial U) > (32c)^{-1} r\) for some \(j = 2, \ldots, n-1\) set \(w_j = y_j\). Otherwise, take \(z_j \in \partial U\) such that \(|y_j - z_j| = \text{dist}(y_j, \partial U)\). By the porosity of \(\partial U\) there exists \(w_j \in U \cap \partial B^2(z_j, (16c)^{-1} r)\) satisfying the conclusion of Proposition \([2.7]\). Then, \(|w_j - w_{j+1}| \leq 6(16c)^{-1} r\). For each \(j = 2, \ldots, n\) let \(\gamma_j\) be a \(c_1\)-cigar curve in \(U\) joining \(w_{j-1}\) with \(w_j\) and let \(\Gamma_i' = \bigcup_{j=1}^n \gamma_j\). The distance estimates above imply that \(\Gamma_i'\) is homotopic to \(\Gamma_i\) in \(\mathbb{R}^2 \setminus \{x\}\).

Replace \(\Gamma_i\) with \(\Gamma_i'\). Applying the same procedure to all subarcs \(\Gamma_i\), we obtain a closed curve \(\Gamma\) that is homotopic to \(\partial B^2(x, r)\) in \(\mathbb{R}^2 \setminus \{x\}\). Take \(\gamma \subset \Gamma\) to be a simple closed curve that is homotopic to \(\partial B^2(x, r)\) in \(\mathbb{R}^2 \setminus \{x\}\). \(\square\)

For the rest of the paper, Lemma \([5.8]\) is applied with \(c' = C^2\) where \(C\) is as in Lemma \([5.6]\). Given a component \(A\) of \(\partial U\) and \(r > 2 \text{diam} A\), if \(\gamma\) is as in Lemma \([5.8]\) then we denote by \(N_2(A, r)\) the subset of \(\partial U\) that is enclosed by \(\gamma\). Moreover, applying Lemma \([5.1]\) for \(E = \gamma\) and \(\epsilon = (3C')^{-1}\) \((C'\) is as in Lemma \([5.8]\), there
exists an $L$-bi-Lipschitz disk $D_2(A, r)$ that contains $N_2(A, r)$ with $L$ depending only on $c$.

The two lemmas above combined yield the next proposition.

**Proposition 5.9.** Let $x \in \partial U$, $A_x$ be the component of $\partial U$ that contains $x$ and $r > 0$.

1. If $\overline{B}(x, r)$ intersects a non-degenerate component $A$ of $\partial U$ with diameter at least $C^2 r$ then $x$ is contained in a set $N_1(A, r)$.
2. If $r \leq 8 \text{diam } A_x$ then $x$ is contained in a set $N_1(A_x, \frac{r}{8})$.
3. If $r > 8 \text{diam } A_x$ and $\overline{B}(x, r)$ intersects only components of $\partial U$ with diameter less than $C^2 r$ then $x$ is contained in a set $N_2(A_x, r)$.

Note that given a non-degenerate component $A$ of $\partial U$, $N_1(A, r)$ is always defined when $r$ is sufficiently small compared to diam $A$. On the other hand, $N_2(A, r)$ is not defined for small $r$ compared to diam $A$ even when $r$ is large, it still may not be defined.

The properties of the sets $N_i(A, r)$ and $D_i(A, r)$ are summarized in the next lemma.

**Lemma 5.10.** Suppose that $A$ is a component of $\partial U$ and $r > 0$. There exists $c' > 1$ depending only on $c$ and $c''$ depending only on $c$ and $r$ with the following properties.

1. Every component of $\mathbb{R}^2 \setminus N_1(A, r)$ is $c'$-uniform. If $\partial U$ is $c$-relatively connected then each component of $\mathbb{R}^2 \setminus N_1(A, r)$ has $c'$-relatively connected boundary.
2. Every component of $D_1(A, r) \setminus N_1(A, r)$ is $c''$-uniform. If $\partial U$ is $c$-relatively connected then each component of $D_1(A, r) \setminus N_1(A, r)$ has $c'$-relatively connected boundary.
3. If $N_2(A, r)$ is defined, then all the components of $\mathbb{R}^2 \setminus N_2(A, r)$ are $c'$-uniform. If $\partial U$ is $c$-relatively connected then each component of $\mathbb{R}^2 \setminus N_2(A, r)$ has $c'$-relatively connected boundary.
4. If $N_2(A, r)$ is defined, then all the components of $D_2(A, r) \setminus N_2(A, r)$ are $c'$-uniform. If $\partial U$ is $c$-relatively connected then each component of $D_2(A, r) \setminus N_2(A, r)$ has $c'$-relatively connected boundary.

**Proof.** We show the first two claims. The proof of the other two is similar. As every quasidisk is uniform with relatively connected boundary, it is enough to show the first for the unbounded component and the second claim for the component of $D_1(A, r) \setminus N_1(A, r)$ whose boundary contains $\partial D_1(A, r)$.

Let $U'$ be the unbounded component of $\mathbb{R}^2 \setminus N_1(A, r)$. To show uniformity of $U'$, let $x, y \in \mathbb{R}^2 \setminus N_1(A, r)$ and fix some $D_1(A, r)$. Note that if $x, y \in \mathbb{R}^2 \setminus D_1(A, r)$ then uniformity follows from the fact that $D_1(A, r)$ is a $K'$-quasidisk for some $K'$ depending only on $c$. If $x \in D_1(A, r)$ and $y \in \mathbb{R}^2 \setminus D_1(A, r)$ then join $x$ to a point $z \in \partial D_1(A, r)$ with a $c$-cigar curve $\gamma_1 \subset D_1(A, r)$ using uniformity of $U$ and then $z$ to $y$ with a $c'$-cigar curve $\gamma_2 \subset \mathbb{R}^2 \setminus D_1(A, r)$ using uniformity of $\mathbb{R}^2 \setminus D_1(A, r)$. Since $\text{dist}(z, N_1(A, r)) > d|x - z|$ for some depending only on $c$, $\gamma = \gamma_1 \cup \gamma_2$ is $c''$-cigar for some $c''$ depending only on $c$. Finally, if $x, y \in D_1(A, r)$ then $x, y \in U$ and we use uniformity of $U$.

Suppose that $\partial U$ is $c$-relatively connected. Let $x \in \partial U'$, $R > 0$ and assume that $B^2(x, R) \cap \partial U' \setminus \{x\} \neq \emptyset$ and $\partial U' \setminus B^2(x, R) \neq \emptyset$. The second assumption implies that $R < 2 \text{diam } A$. If $R \leq 8Cr$ then $\overline{B}^2(x, (8C^2)^{-1} R) \cap \partial U' = \overline{B}^2(x, (8C^2)^{-1} R) \cap$
\[ \partial U \text{ and relative connectedness is satisfied with } c' = 8c^2.c. \] Suppose now that \( 8rC < R < 2 \text{diam } A \). Then, if \( \overline{B^2}(x, R/8) \) intersects \( A \) we have \( A \setminus \overline{B^2}(x, R/8) \neq \emptyset \) and relative connectedness is satisfied with \( c' = 8 \).

Let now \( U'' \) be the bounded domain with boundary \( \partial D_1(A, r) \cup N_1(A, r) \). The uniformity of \( U'' \) follows from Remark 2.8. If \( \partial U \) is relatively connected, for the relative connectedness of \( \partial U'' \) we work as above.

5.4. **Total separation of \( \partial U \).** Fix \( \epsilon \in (0, \text{diam } \partial U) \). For each point \( x \in \partial U \) let \( D_i(x, r_x) \) be as in Section 5.3 where \( i_x \in \{1, 2\} \), \( r_x \in \{80c\epsilon, 10c\epsilon\} \) and \( A_x \) is a component of \( \partial U \). Note that \( \text{dist}(\gamma_x, \partial U) \geq 10\epsilon \).

Define \( G = \bigcup_{x \in \partial U} \gamma_x \). Then, \( \text{dist}(G, \partial U) \geq 10\epsilon \). The boundary of \( T_c(G) \) is a finite disjoint union of polygonal Jordan curves each of which lies in \( G \) and is at least distance \( \epsilon \) from \( \partial U \). Define

\[ \mathcal{G} = \{D \colon D \text{ is a bounded Jordan domain whose boundary is a component of } T_c(G)\}. \]

Note that two elements of \( \mathcal{G} \) are either disjoint or one is contained in the other. An element \( D \) of \( \mathcal{G} \) is called *minimal* if for all \( D' \in \mathcal{G}, D' \subset D \) implies \( D' = D \).

For each component \( A \) of \( \partial U \) let \( D_A \) be the minimal element of \( \mathcal{G} \) that contains \( A \) and \( D'_A \) be the *maximal element* in \( \{D_A \colon A \text{ is a component of } \partial U\} \) that contains \( A \), that is, if \( D'_A \subset D_A \) then \( D_A = D'_A \). Let \( D_1, \ldots, D_n \) be the elements of the set \( \{D'_A \colon A \text{ is a component of } \partial U\} \) and let \( A_i = \partial U \cap D_i \). Note the following.

1. \( n \leq c^{-2}(\text{diam } \partial U)^2 \).
2. Each \( D_i \) has diameter at least \( \epsilon \) and by the relative connectedness of \( \partial U \), each \( A_i \) has size at least \( \epsilon/M \) for some \( M > 1 \) depending only on \( c \).
3. For all \( i \neq j \), \( \text{dist}(D_i, D_j) \geq \epsilon \).
4. \( \text{diam } (U \setminus D) \setminus \bigcup_{i=1}^n D_i \geq \text{diam } \partial U \).

The size of each \( D_i \) can be estimated from the following lemma.

**Lemma 5.11.** Let \( \epsilon \in (0, \text{diam } \partial U) \) and \( D_1, \ldots, D_n \) be as above.

1. For all \( i = 1, \ldots, n \)

\[ \frac{1}{2}(\epsilon + \sup_A \text{diam } A) \leq \text{diam } D_i \leq 80c^2(\epsilon + \sup_A \text{diam } A) \]

where the supremum is taken over all components \( A \) of \( A_i \).
2. Each \( D_i \) is an \( L \)-bi-Lipschitz disk for some \( L \) depending only on \( c \) and \( \epsilon^{-1} \text{diam } D_i \).

**Proof.** The lower bound of the first claim follows from the fact that for each \( z \in \partial D_i \), \( \text{dist}(z, \partial U) \geq \epsilon \). For the upper bound note that for each component \( A \) of \( A_i \) we have \( \text{diam } D_i \leq \text{diam } D_i(A, r_A) \leq c((80c)\epsilon + \text{diam } A) \).

The second claim follows from the first claim and Lemma 5.11.

6. **Whitney-type decompositions around quasidisks**

Let \( D, D' \subset \mathbb{R}^2 \) be Jordan domains, \( A, A' \) be unions of disjoint closed quasidisks in \( D, D' \) respectively and \( A_0, A'_0 \) be closed quasidisks contained in \( A, A' \) respectively. Let also \( f : A \to A' \) be an \( \eta \)-quasisymmetric homeomorphism with \( f(A_0) = A'_0 \). For the rest we assume that there exist \( c > 1 \) and \( C > 1 \) with the following properties.

1. \( A \) and \( A' \) are compact with \( c \)-uniform complement.
(II) $\partial A_0$ and $\partial A_0'$ are c-bounded turning with
\[(8c)^3 \leq \text{diam } A_0 \leq C \quad \text{and} \quad C^{-1} \leq \text{diam } A_0 \leq C.
\]
(III) For all $z \in \partial D$ and all $z' \in \partial D'$
\[C^{-1} \leq \text{dist}(z, A) \leq \text{dist}(z, A_0) \leq 1 \quad \text{and} \quad C^{-1} \leq \text{dist}(z', A') \leq \text{dist}(z', A_0') \leq 1.
\]
For some $L > 1$ and $c_1 > 1$ depending only on $c$, $C$ and $\eta$, we construct in this section two families $\mathcal{D}, \mathcal{D}'$ of $L$-bi-Lipschitz disks with the following properties.

1. The elements of $\mathcal{D}$ are mutually disjoint and $D \subset \bigcup_{Q \in \mathcal{D}} \widetilde{Q}$. Similarly for $\mathcal{D}'$.
2. For all $Q \in \mathcal{D}$, $c_1^{-1} \text{diam } Q \leq \text{dist}(Q, A) \leq c_1 \text{diam } Q$. Similarly for $\mathcal{D}'$.
3. For each $Q \in \mathcal{D}$, there are at most $c_1$ elements in $\mathcal{D}$ whose boundary intersects that of $Q$. Moreover, if $\Gamma = \partial Q_1 \cap \partial Q_2 \neq \emptyset$ for $Q_1, Q_2 \in \mathcal{D}$ then $\Gamma$ is an $L$-bi-Lipschitz arc and $\text{diam } \Gamma \geq c_1^{-1} \max\{\text{diam } Q_1, \text{diam } Q_2\}$. Similarly for $\mathcal{D}'$.
4. There exists homeomorphism $g : \overline{D} \to \overline{D'}$ such that $f|\partial A = g|\partial A$ and for each $Q \in \mathcal{D}$, $g(Q) \in \mathcal{D}'$.

The construction of the two decompositions is very similar to that of Carleson [7] for the quasisymmetric embedding of $S^1$ into $\mathbb{R}^2$ and of Tukia [Tuk80] for the quasisymmetric embedding of $S^1$ into $\mathbb{R}^2$. However, the main novelty in our setting is that one should make sure that the boundaries of Whitney squares properly avoid all the components of $A \setminus A_0'$ around $A_0'$.

To simplify the exposition, we use the following notation for the rest of Section 6

For two positive quantities $a, b$ we write $a \lesssim b$ if there exists constant $C_0$ depending at most on $c$, $C$ and $\eta$ such that $a \leq C_0 b$. We write $a \simeq b$ if $a \lesssim b$ and $a \gtrsim b$.

6.1. Decomposition around the preimage. Fix an orientation on $\partial A_0$. Through $f$, an orientation on $\partial A_0'$ is also defined. Given a set of points $\{p_1, \ldots, p_n\} \subset \partial A_0$ we say that $p_i$ and $p_j$ are neighbors in the set $\{p_1, \ldots, p_n\}$ if one of the two subarcs of $\partial A_0 \setminus \{p_i, p_j\}$ contains no point from $\{p_1, \ldots, p_n\}$. Such a subarc is denoted by $\partial A_0(p_i, p_j)$. Moreover, we say that $p_i$ is on the right of $p_j$ in the set $\{p_1, \ldots, p_n\}$ if $p_i$ and $p_j$ are neighbors and under the orientation of $\partial A_0$, $p_j$ and $p_i$ are the starting and ending, respectively, points of $\partial A_0(p_i, p_j)$. In opposite case, we say that $p_i$ is on the left of $p_j$.

Since $\partial A_0$ is c-bounded turning, by assumption (III) we have that $C^{-1} \leq \text{dist}(w, \partial D) \leq 3c \leq (32c^2)^{-1} \text{diam } A_0$ for all $w \in A_0$ [VW10 Lemma 3.4].

Set $D_1 = D$ and for each $m = 2, 3, \ldots$ use Lemma 5.6 on $A$ with $\epsilon_m = (2c)^{-2m}C^{-1}$ and obtain a bi-Lipschitz disk $D_m$. Let $l_1 \in \mathbb{N}$ be the smallest integer such that $2^{-l_1} \leq 2^{-6}C^{-1}$ and let $\Delta_1$ be the bi-Lipschitz disk containing $D$ with $\epsilon = 2^{-l_1}C^{-1}$. Note that $\Delta \cap A = D \cap A$. The integer $l_1$ will be chosen in Section 6.2 towards the proof of Theorem 6.1 but in any case, it is bounded below by a constant depending only on $c$ and $\eta$.

For each $m = 2, 3, \ldots$ let $l_m \in \mathbb{N}$ be the smallest integer such that $2^{-l_m} \leq \frac{1}{16}(2c)^{-2m-1}C^{-1}$ and $\Delta_m$ be as in Lemma 5.1 where $E = \partial D_m$ and $\epsilon = 2^{-l_m}$.

Choose points $x_1, \ldots, x_n \in A_0$ following the orientation of $\partial A_0$ such that
\[16c^3 \text{diam } A_0 \leq |x_i - x_{i+1}| \leq 32c^3\]
with the convention $x_{n+1} = x_1$. Note that $k \leq N_0$ for some $N \in \mathbb{N}$ depending only on $c$ and $C$. 
For each \(i \in \{1, \ldots, k\}\) let \(\hat{y}_i \in \partial \Delta_1\) be a point closest to \(x_i\) and join \(x_i\) to \(\hat{y}_i\) with a c-cigar curve \(\sigma_i\). For each \(\sigma_i\) we construct a broken line \(\gamma_i\) as follows. For each \(z \in \gamma_i\) let \(\Sigma(z)\) be the union of all squares in \(\Sigma_{2-i(z)}\) that contain \(z\) where \(l(z)\) is the smallest integer such that \(l(z) \geq l_1\) and \(2^{-l(z)} \leq \frac{1}{6}\ \text{dist}(z, A)\). Let \(\gamma_i\) be a subarc in the boundary of \(\bigcup_{z \in \sigma_i} \Sigma(z)\) that connects \(x_i\) with \(\Delta_1\) and is entirely contained (except for its endpoints) in \(\Delta_1 \setminus A_0\). Denote by \(y_i\) the endpoint of \(\gamma_i\) which is on \(\Delta_1\).

Next, for each \(k \in \{2, 3, \ldots\}\), we modify \(\gamma_i\) close to its intersection points with \(\partial \Delta_k\). We start with \(\Delta_2\).

Let \(T_2\) be the union of all squares in \(\Sigma_{2-i(z)}\) that intersect with \(\Delta_2\). Note that \(\partial T_2\) consists of exactly two Jordan curves; one contained in \(\Delta_2\), the other contained in \(\Delta_1 \setminus \Delta_2\). Let \(p_i\) and \(q_i\) be the points of \(\gamma_i \cap \partial T_2\) such that the part of \(\gamma_i\) joining \(x_i\) with \(p_i\) is entirely in \(\Delta_2\) while the part of \(\gamma_i\) joining \(y_i\) with \(q_i\) is entirely in \(\Delta_1 \setminus \Delta_2\).

Let \(\hat{q}_i\) be the flat vertex (i.e., \(\hat{q}_i\) is the common vertex of two co-linear edges) on the component of \(\partial T_2\) containing \(q_i\) that is closest to \(p_i\) and let \(\gamma_i\) be the shorter in diameter subarc of \(T_2\) joining \(q_i\) with \(\hat{q}_i\). Let \(t_i\) be the flat vertex of \(\partial \Delta_1\) closest to \(\hat{q}_i\) and \(\tau_i\) be the line segment \([t_i, \hat{q}_i]\). Let \(\hat{p}_i\) be the flat vertex on the component of \(\partial T_2\) containing \(p_i\) that is closest to \(t_i\) and let \(\tau_3\) be the line segment \([t_i, \hat{p}_i]\). Finally, let \(\tau_4\) be the shorter in diameter subarc of \(T_2\) joining \(q_i\) with \(\hat{q}_i\). Replace \(\gamma_i(\hat{p}_i, q_i)\) with \(\bigcup \tau_i\) and note that \(\gamma_i\) intersects \(\partial \Delta_2\). Note that the two curves \(\gamma_i\) and \(\partial \Delta_i\) intersect orthogonally and their intersection is only one point \(t_i\) which we denote for the rest by \(y_{1i}\).

Similarly, we modify \(\gamma_i\) close to its intersection points with \(\partial \Delta_k\). We denote by \(y_{1ik}\) the unique intersection point of \(\partial \Delta_k\) and \(\gamma_i\).

We proceed inductively. Assume that for some \(m \in \mathbb{N}\), we have defined points \(x_w \in \partial A_0\), curves \(\gamma_w\) and points \(y_{w, l}\) where \(w \in \mathbb{N}^m\) is a finite word formed from \(m\) letters in \(\mathbb{N}\) and \(l \in \mathbb{N} \cup \{0\}\). We denote by \(|w|\) the number of letters the word \(w\) has. Conventionally, \(|\emptyset| = 0\).

Fix \(w, u \in \mathbb{N}^m\) such that \(x_w\) and \(x_u\) have been defined and in the collection \(\{x_v : |v| = m\}\), \(x_w\) is on the left of \(x_u\). In \(\partial A_0(x_w, x_u)\) choose points \(x_{wi}\) with \(i = 1, \ldots, N_w+1\) following the orientation of \(\partial A_0\) such that \(x_{wi} = x_{w(0)}\) for each \(i = 1, \ldots, N_w\).

\[
4c(2e)^{3-2m}C^{-1} \leq |x_{wi} - x_{w(i+1)}| \leq 8c(2e)^{3-2m}C^{-1}.
\]

Note that \(N_w \leq N_0\) where \(N_0\) depends only on \(c\). Without loss of generality, we assume for the rest that \(N = N_0\).

For \(i = \{1, \ldots, N_{wi}\}\) and \(m \geq |w| + 2\), let \(\hat{y}_{wi}\) be a point of \(\partial \Delta_{|w|+1}\) close to \(x_{wi}\) and let \(\sigma_{wi}\) be a c-cigar curve joining \(x_{wi}\) with \(\hat{y}_{wi}\). Construct \(\gamma_{wi}\) as in Step 1 and let \(y_{wi}\) be the point on \(\partial \Delta_{|w|+1}\) such that the part of \(\gamma_{wi}\) connecting \(x_{wi}\) with \(y_{wi}\) is entirely in \(\Delta_{|w|+1}\). As in Step 1, for each \(k \in \mathbb{N}\), we modify \(\gamma_{wi}\) close to its intersection points with \(\partial \Delta_{|w|+1+k}\) and denote with \(y_{wi1k}\) the unique intersection point of \(\gamma_{wi}\) and \(\partial \Delta_{|w|+1+k}\).

Let \(\mathcal{W}\) be the set of finite words \(w\) formed by letters \(\{1, \ldots, N\}\) for which \(x_w\) has been defined. Let also \(\mathcal{W}_k\) be the set of words \(w \in \mathcal{W}\) whose length is \(k\).

The number \((8c)^3\) in assumption (II) has been chosen so that for all \(w, u \in \mathcal{W}\),

\[
dist(\gamma_w, \gamma_u) \gtrsim \min\{\text{diam } \gamma_w, \gamma_u\}.
\]

Suppose that \(w \in \mathcal{W}\) and let \(x_w\) be on the left of \(x_w\) in the collection \(\{x_v : |v| = |w|\}\). Define \(Q_w\) to be the Jordan domain bounded by \(\gamma_w, \gamma_u, \Delta_{|w|}\) and
Note that for each $Q_w$, there exists $l_w \in \mathbb{N}$ such that $\partial Q_w$ is in $G_{2^{-l_w}}$ and $\text{diam } Q_w \simeq 2^{-l_w}$. Moreover, the distance of each $Q_w$ from $A$ is comparable to its distance from $x_w$. We collect all these observations in the following remark.

**Remark 6.1.** For each $w \in \mathcal{W}$,

1. $Q_w$ defined as above is an $L_1$-bi-Lipschitz disk with $L_1 \simeq 1$,
2. $\text{dist}(Q_w, x_w) \simeq \text{dist}(Q_w, \partial U) \simeq \text{diam } Q_w$.

If $A \cap Q_w \neq \emptyset$ set $A_w = A \cap Q_w$. Otherwise, let $z_w \in Q_w$ be a point such that $\text{dist}(z_w, \partial Q_w) \geq \frac{1}{L} \text{diam } Q_w$ and set $A_w = B^2(z_w, \frac{1}{L} \text{diam } Q_w)$.

**Remark 6.2.** If $A$ is $c$-uniformly perfect, then $\text{diam } Q_w \lesssim \text{diam } A_w$ for all $w \in \mathcal{W}$.

### 6.2. Decomposition around the image.

We construct a decomposition $\mathcal{D}' = \{Q_w'\}$ around $A_0$ that is combinatorially equivalent to $\mathcal{D} = \{Q_w\}$.

As with $D$, we first define $\Delta_1'$. Let $\epsilon = 2^{-l_1'-3}$, $l_1' \in \mathbb{N}$ and let $\Delta_1'$ be as in Lemma 6.3 with $E = \overline{D}$. As with the integer $l_1$, the number $l_1'$ can be chosen small enough in the proof of Theorem 6.1.

For each $w \in \mathcal{W}$ let $x_w' = f(x_w)$. The notion of neighbor points follows from the orientation of $\partial A_0$ induced by $f$. For the rest, two words $w, u \in \mathcal{W}$ with $|w| = |u|$ are called neighbors if $x_w$ and $x_u$ are neighbors. Similarly, if $w, u \in \mathcal{W}_k$ we say that $w$ is at the left (resp. right) of $u$ if $x_w$ is at the left (resp. right) of $u$.

For those $w \in \mathcal{W}$ for which $A_w \subset A$ let $A_w' = f(A_w)$. Next, we define $A_w'$ in the case $A_w \cap A = \emptyset$. There exists $n_0 \in \mathbb{N}$ with $n_0 \simeq 1$ such that

$$B^2(x_w', 2|x_w' - x_{2n_0}|) \subset D'$$

for all $w \in \mathcal{W}$.

Since $\mathbb{R}^2 \setminus A'$ is $c$-uniform, $A'$ is porous and, for each $w \in \mathcal{W}$ there exists $z_w' \in U$ such that $|x_w' - z_w'| = |x_w' - x_{2n_0}|$ and $B_w = B^2(z_w', c^{-1}|x_w' - x_{2n_0}|) \subset U$. Note that there exists $m_0 \in \mathbb{N}$, $m_0 \simeq 1$ such that if $w \in \mathcal{W}$ and $w_1, \ldots, w_k \in \mathcal{W}$ are such that $B_{w_i} \cap B_{w_j} \neq \emptyset$ for all $i \in \{1, \ldots, k\}$ then $k \leq m_0$. For each $w \in \mathcal{W}$ there exist $m_0$ balls $B_{w_1}^i, \ldots, B_{w_0}^i$ contained in $B_w$ such that for all $i \neq j$, $\text{dist}(B_{w_i}^i, B_{w_j}^i) \geq (4m_0)^{-1} \text{diam } B_w$, $\text{dist}(B_{w_i}^i, \partial B_w) \geq (4m_0)^{-1} \text{diam } B_w$ and $\text{diam } B_{w_i}^i \geq (2m_0)^{-1} \text{diam } B_w$. For each $w \in \mathcal{W}$ let $i_w \in \{1, \ldots, m_0\}$ so that

$$\text{dist}(B_{w_i}^i, B_{w_j}^i) \geq (4m_0)^{-1} \max\{\text{diam } B_w, \text{diam } B_w\}.$$ 

If $A_w \cap A = \emptyset$, set $A_w' = B_{w_i}^i$.

We show in the next lemma, that if we remove the extra sets $A_w'$ from $\mathbb{R}^2 \setminus A'$ we still get a uniform domain.

**Lemma 6.3.** There exists $c' \simeq 1$ such that $Y = \mathbb{R}^2 \setminus A_0' \cup_{w \in \mathcal{W}} A_w'$ is $c'$-uniform.

**Proof.** Let $x, y \in Y$. For each $w \in \mathcal{W}$ with $A_w' \cap A' = \emptyset$ define $V_w = T_{\epsilon_w}(A_w')$ with $\epsilon_w = \frac{1}{20} \text{dist}(A_w', \partial Y \setminus A_w')$. If both $x$ and $y$ are contained in some $V_w$ then the claim follows as $V_w \setminus A_w'$ is $c_1$-uniform. If $x$ is contained in some $V_w$ and $y \notin V_w$ then let $x' \in \partial V_w$ be the point of $\partial V_w$ closest to $x$ and we replace $x$ with $x'$.

Thus, we may assume that both $x$ and $y$ are outside of each $V_w$ as above. Let $\gamma$ be a $c$-cigar arc joining $x$ with $y$. Suppose that $\gamma$ does not intersect any $V_w$. Let $z \in \gamma$ and $A'_w$ be such that $\text{dist}(z, A'_w) \leq \text{dist}(z, A_{w'})$ for all $w, w' \in \mathcal{W}$. If $A'_w \subset A'$ then $\text{dist}(z, \partial Y) = \text{dist}(z, A') \geq c^{-1} \min\{|x - z|, |y - z|\}$. If $A'_w \cap A' = \emptyset$ then there exists $\delta$ depending only on $c$ and $\eta$ such that $\text{dist}(z, A'_w) \geq \delta \text{dist}(z, A')$. In either case, $\gamma$ is $c_2$-cigar in $V$ for some $c_2 \simeq 1$. 

Suppose now that \( \gamma \) intersects some \( V_w \). Let \( x', y' \in \gamma \cap \partial V_w \) be such that \( \gamma(x, x') \) and \( \gamma(y, y') \) are entirely outside of \( V_w \). As \( \gamma(x', y') \) is c-quasiconvex, \( \text{dist}(z, A') \leq c(\text{dist}(x', A') + |x' - y'|) \) for all \( z \in \gamma(x', y') \). Replace \( \gamma(x', y') \) with the shortest arc \( \sigma \) of \( \partial V_w \) joining \( x' \) and \( y' \) and note that \( \ell(\sigma) \leq \pi \ell(\gamma(x', y')) \). Moreover, as before, \( \text{dist}(x', \partial Y) \geq \delta \text{dist}(x', A') \) and similarly for \( y' \). Fix \( z \in \sigma \). Assuming \( |z - x| \leq |z - y| \), we have \( |x' - x| \leq M|x' - y| \) and \( |x - z'| \leq M(|z' - x'| + |x' - x|) \) for some \( M \) depending only on \( c \). Hence, for some \( \delta_1 \) depending only on \( c \) and \( \eta \)

\[
\text{dist}(z, A') = \text{dist}(x', A'_w) \geq \delta_1(\text{dist}(x', A') + |z' - x'|) \geq \delta_1(cM)^{-1}|z - x|.
\]

Similarly we modify the new curve \( \gamma \) at its intersections with other \( V_{w'} \).

6.2.1. Domains. For each \( x, y \in \partial A'_{0} \) fix a c-cigar curve \( \tau_{x,y} \) that joins \( x \) with \( y \). As in Section 5.1 for each \( z \in \tau_{x,y} \) let \( \Sigma(z) \) be the union of all squares in \( \Sigma(\gamma) \) that contain \( z \) where \( l(z) \) is the smallest integer such that \( 2^{-l(z)} \leq (32c')^{-1} \text{dist}(z, \partial U'_{0}) \). Let \( \tau_{x,y} \) be an arc on the boundary of \( \bigcup_{z \in \tau_{x,y} \backslash \{x, y\}} \Sigma(z) \) that connects \( x \) with \( y \) such that \( \tau_{x,y} \backslash \{x, y\} \) is contained in the Jordan domain bounded by \( \partial A'_{0} \) and \( \tau_{x,y} \).

Fix \( k \in \mathbb{N} \) and \( w \in \mathcal{W} \). Suppose that \( w_{1}, w_{2} \) are the neighbors of \( w^{1}k \) and define \( R_{w}^{(k)} \) to be the Jordan domain bounded by \( \tau_{x_{w}^{1}, x_{w}^{0}} \) and \( \partial A_{0} \).

Note that for each \( k, l \in \mathbb{N} \) and \( w \in \mathcal{W} \), \( R_{w}^{(k+l)} = R_{w}^{(k)} \cap \partial U'_{l} \). Moreover, each \( \partial R_{w}^{(k)} \) is a \( c' \)-cigar curve for some \( c' \approx 1 \). Thus, for each \( w \in \mathcal{W} \) and \( k \in \mathbb{N} \) there exists a point \( p_{w}^{(k)} \in \partial R_{w}^{(k)} \) which is a midpoint of an edge of \( \partial R_{w}^{(k)} \) such that \( \text{dist}(p_{w}^{(k)}, \partial V) \geq \text{diam}(R_{w}^{(k)}) \). Subdividing each edge of \( \partial R_{w}^{(k)} \) into 2 edges we may assume that \( p_{w}^{(k)} \) is a flat vertex of \( \partial R_{w}^{(k)} \). Recall that \( z \) is a flat vertex of a polygon \( P \) if the two edges of \( P \) with \( z \) as their common point are co-linear.

The following remark follows immediately from the quasisymmetry of \( f \) and uniformity of \( V \).

**Lemma 6.5.** Given a small positive number \( \delta_{0} \in (0, 1) \), there exists \( k_{0} \) depending only on \( c, C, \eta \) and \( \delta_{0} \) such that if \( k \geq k_{0} \) then, for all \( w \in \mathcal{W} \), the following hold.

1. \( \text{diam}(R_{w}^{(k)}) \leq \delta_{0} \) and \( x_{w} \in \partial R_{w}^{(k)} \).
2. If \( u \in \mathcal{W} \) with \( |u| = |w| \) then \( \text{dist}(R_{w}^{(k)}, R_{u}^{(k)}) \geq (1 - \delta_{0})|x_{w}' - x_{u}'| \).
3. If \( u \in \mathcal{W} \) with \( |u| < |w| \) then \( \text{dist}(R_{w}^{(k)}, V_{u}) \geq (1 - \delta_{0}) \text{dist}(x_{w}', V_{u}) \).
4. If \( uv \in \mathcal{W} \), \( |u| = |w| \) and \( u \) is not a neighbor of \( w \) then \( \text{dist}(R_{w}^{(k)}, V_{uv}) \geq (1 - \delta_{0}) \text{dist}(x_{w}', V_{uv}) \).
(5) If \( l \in \mathbb{N} \), then \( \text{diam } R_{w}^{(k+l)} \leq \delta_{0} \text{diam } R_{w}^{(k)} \).

(6) If \( l \in \mathbb{N} \) and \( u \in \mathcal{W} \) with \( |u| = |w| + k \), then

\[
\text{diam } R_{u}^{(l)} \leq \delta_{0} \text{dist}(p_{w}^{(l)}, R_{u}^{(l)}).
\]

We specify \( \delta_{0} \) in Section 6.2.2 and Section 6.2.3 and Section 6.2.4. For each \( w \in \mathcal{W} \) set \( R_{w} = R_{w}^{(k_{0})} \) and \( p_{w} = p_{w}^{(k_{0})} \). We call the domain \( R_{w} \) dome and the point \( p_{w} \), the peak of \( R_{w} \). To simplify the notation, we write \( \tau_{w} = \tau_{w}^{x_{w_{1}},x_{w_{2}}} \) where \( x_{w_{1}}^{'} \) (resp. \( x_{w_{2}}^{'} \)) is the neighbor of \( x_{w_{1}} \) at its left (resp. right) in \( \mathcal{W}_{|w|+k_{0}} \). We call the points \( x_{w_{1}}^{'} \) and \( x_{w_{2}}^{'} \) the left and right, respectively, endpoints of \( \tau_{w} \).

Before proceeding to the construction of \( \mathcal{D}' \), we make one final modification to the domes \( R_{w} \). Given a word \( w \in \mathcal{W} \), we write \( \partial R_{w} = \tau_{w} \cup \partial A_{w}^{0}(x_{w_{1}}', x_{w_{2}}') \) where \( w_{1}, w_{2} \) are the neighbors of \( w^{1k_{0}} \) in \( \mathcal{W}_{|w|+k_{0}} \). Similarly, given \( l \in \mathbb{N} \) and \( u = w^{1k_{0}} \), we have \( R_{u} = \tau_{u} \cup \partial A_{u}^{0}(x_{u_{1}}', x_{u_{2}}') \). Note first that \( \tau_{w} \) intersects \( \tau_{u} \). By modifying \( \tau_{w} \), as in Section 6.1 we may assume that the two polygonal arcs \( \tau_{w} \) and \( \tau_{u} \) intersect only at \( p_{u} \).

6.2.2. Construction of \( \mathcal{D}' \): Step 0. Given \( i \in \{1, \ldots, N\} \) we define a simple polygonal path \( \bar{s}_{i,i+1} \) that joins \( R_{i} \) with \( R_{i+1} \) as follows. Apply Lemma 5.1 on \( A_{0}^{i} \) with

\[
r = (cC)^{-2} \min_{i=1,\ldots,N} \text{dist}(p_{i}, A_{0}^{i})
\]

and obtain a Jordan domain \( \tilde{D}_{1} \) containing \( A_{0}^{i} \). Applying Lemma 5.1 on \( \tilde{D}_{1} \) with \( \epsilon = 2^{-l(1)} \), where \( l(1) \) is the smallest positive integer such that

\[
2^{-l(1)} \leq \frac{1}{16} \text{dist}(\partial \tilde{D}_{1}, \partial V \cup \{p_{1}, \ldots, p_{N}\}),
\]

we obtain a bi-Lipschitz disk \( D_{1}' \) containing \( \tilde{D}_{1} \). For each \( i = 1, \ldots, N \), there exists a subarc \( \bar{s}_{i,i+1} \) of \( \partial D_{i}' \) such that

(1) except of its endpoints, \( \bar{s}_{i,i+1} \) is in \( \Delta_{i} \setminus \bigcup R_{i} \);

(2) one of its endpoints is on \( \tau_{i} \) between the peak of \( R_{i} \) and the right endpoint of \( \tau_{i} \) and the other endpoint is on \( \tau_{i+1} \) between the peak of \( R_{i+1} \) and the left endpoint of \( \tau_{i+1} \).

Choosing \( \delta_{0} \) sufficiently small in Lemma 6.3 we may assume that \( \bigcup_{i=1}^{N} V_{i} \) is contained in the open annulus \( \tilde{T}_{0} \) whose boundary is \( \partial \Delta_{0}^{i} \) and a polygonal Jordan curve which is the union of the curves \( \bar{s}_{i,i+1} \) and subarcs of \( \tau_{i} \).

Note that \( \tilde{T}_{0} \) contains \( V_{i} \) and at most \( C_{1} \) components of \( \bigcup_{w \in \mathcal{W}_{|w| \geq 2}} V_{w} \) for some \( C_{1} > 1 \) depending only on \( c, C \) and \( \eta \). Fix \( i \in \{1, \ldots, N\} \) and suppose that \( H_{1}, \ldots, H_{m} \) are components of \( \tilde{T}_{0} \cap \bigcup_{w \in \mathcal{W}_{|w| \geq 2}} V_{w} \). There exists \( m_{1} > m_{1} + 4, m_{1} \simeq 1 \), and polygonal curves \( s_{j} \subset \mathcal{G}_{2,m_{1}}, j = 1, \ldots, m \) joining \( H_{j} \) with \( \Delta_{i}' \setminus \tilde{T}_{0} \) such that

(1) except for its endpoints, each \( s_{j} \) is entirely in \( \tilde{T}_{0} \);

(2) \( \text{dist}(s_{j}, \partial \tilde{T}_{0} \setminus \bar{s}_{i,i+1}) \gtrsim 1 \) and \( \text{dist}(s_{j}, s_{j'}) \gtrsim 1 \) when \( j \neq j' \);

(3) if \( H_{j} \) is a component of \( V_{w} \) and \( x_{w}^{'} \) is on \( \partial A_{w}^{0} \cap \partial R_{i} \) then \( s_{j} \) joins \( V_{j} \) with a point on \( \tau_{j} \) other than \( y_{j} \);

(4) if \( H_{j} \) is a component of \( V_{w} \) and \( x_{w}^{'} \) is on \( \partial A_{w}^{0} \) between the right endpoint of \( \tau_{i} \) and the left endpoint of \( \tau_{i+1} \) then \( s_{j} \) joins \( V_{j} \) with a point on \( \bar{s}_{i,i+1} \).
Let $T_0 = \tilde{T}_0 \setminus \mathcal{T}_{2^{-m_1}}(\bigcup H_j \cup \bigcup s_j)$. Given $i = 1, \ldots, N$, if $H_{j_1}, \ldots, H_{j_i}$ are all the components of $\bigcup_{w \in \mathcal{W}_i,w \geq 2} \tilde{V}_w \cap \mathcal{A}$ connected to $R_i$ as above then set $\mathcal{R}_i = R_i \cup \mathcal{T}_{2^{-m_1}}(\bigcup_{n=1}^l (H_{j_n} \cup s_{j_n}))$.

Subdividing $T_0$ we obtain Jordan domains $\mathcal{Q}'_i$ such that

1. $\mathcal{Q}'_i$ have mutually disjoint interiors and their union is all of $T_0$,
2. each $\mathcal{Q}'_i$ is in $\mathcal{G}_{2^{-m_0}(0)}$ for some positive integer $m(0) \simeq 1$,
3. $\{\mathcal{Q}'_i\} \cup \{p_i\}$ are combinatorially equivalent to $\{\mathcal{Q}_i\} \cup \{y_{i1}\}$, that is, there exists a homeomorphism $g_0 : \Delta_0 \to \Delta'_0$ extending $f$ such that $g_0(\mathcal{Q}_i) = \mathcal{Q}'_i$ and $g_0(y_{i1}) = p_i$.

Note that

$$\Delta'_0 = A'_0 \cup \bigcup_{i=1}^N \mathcal{Q}'_i \cup \bigcup_{i=1}^N \mathcal{R}_i \cup \bigcup_{i=1}^N T_{i,i+1},$$

where $T_{i,i+1}$ is a Jordan domain bounded by a subarc of $\partial A'_0$, a subarc of $\partial \mathcal{R}_i$, a subarc of $\partial \mathcal{R}_{i+1}$ and a simple polygonal arc $\sigma_{i,i+1}$.

For the induction step, we consider the following two possible cases.

6.2.3. Construction of $\mathcal{Q}'$: Decomposition in $\mathcal{R}_w$. Let $w \in \mathcal{W}_{k_0+1}$ with $l$ being a nonnegative integer. Let also $w_1$ and $w_2$ be the left and right, respectively, endpoints of $\mathcal{T}_w$. We work as in Section 6.2.2 to obtain a polygonal path $\tilde{\sigma}_{w_1,w_2}$ joining $R_{w_1}$ and $R_{w_2}$. Let $r = (cC)^{-2} \min\{\text{dist}(p_{w_1}, A'_0), \text{dist}(p_{w_1k_0}, A'_0)\}$, let $\tilde{D}$ be the domain obtained by Lemma 5.3 applied on $A_0$, $r$ and let $\tilde{D}$ be the bi-Lipschitz disk obtained by Lemma 5.1 for $\tilde{D}$ and $\epsilon = 2^{-l}$ where $l$ is the smallest integer such that

$$\epsilon \leq \frac{1}{16} \text{dist}(\partial \tilde{D}, (R_w \cap \partial V)) \cup \{p_{w_1}, p_{w_1k_0}, p_{w_2}\}.$$

Let now $\tilde{\sigma}$ be a subarc of $\partial \tilde{D}$ such that its first endpoint is on $\tau_{w_1}$ between its peak and its right endpoint, and its second endpoint is on $\tau_{w_1k_0}$ between its peak and its left endpoint. Modifying the curve on its intersection points with $\tau_w$ we may assume that the curve is contained in $R_w$. Similarly we obtain a curve $\tilde{\sigma}_{w_1,w_2}$ that does not intersect with $\tilde{\sigma}_{w_1,w_2}$.

Choosing $\delta_0$ sufficiently small in Lemma 5.3 we may assume that if $w \in \mathcal{W}$ with $|u| \leq |w| + k_0$ and $V_u \subset \mathcal{R}_w$ then $V_u$ is contained in the Jordan domain $Q_w$ bounded by a subarc of $R_{w_1}$, a subarc of $R_{w_2}$, a subarc of $\mathcal{R}_w$, $\tilde{\sigma}_{w_1,w_1k_0}$ and $\tilde{\sigma}_{w_1k_0,w_2}$.

As in Section 6.2.2 we may subdivide $Q_w$ into Jordan domains $Q'_u$. In particular, $Q_{w_1}$ is a Jordan domain with polygonal boundary in $\mathcal{G}_{2^{-m(w_1)}}$ where $m(w_1)$ is an integer such that $2^{m(w_1)} \simeq \text{diam} Q_{w_1}$. Moreover, $Q_{w_2}$ contains at most $N_1 \simeq 1$ components of $\partial V$. Therefore, there exists $m_1 \simeq 1$ such that $Q_{w_1}$ can be subdivided into Jordan domains $Q'_u$ with polygonal boundary in $\mathcal{G}_{2^{-m(w_2)}-m_1}$ where $k_0l + 1 < |u| \leq k_0(l + 1) + 1$, $x'_w$ is between $x_{w_1}'$ and $x_{w_2}'$, and

1. $\mathcal{Q}'_u$ have mutually disjoint interiors and their union is all of $Q_w$,
2. each $\partial \mathcal{Q}'_u$ is in $\mathcal{G}_{2^{-m(w)}}$ for some positive integer $m(w)$ with $\text{diam} \mathcal{Q}'_u \simeq 2^{-m(w)}$,
3. $\{\mathcal{Q}'_u\} \cup \{p_u\}$ are combinatorially equivalent to $\{Q_u\} \cup \{y_{u1}\}$, that is, there exists a homeomorphism $g_w : \Delta_0 \to \Delta'_0$ extending $f$ such that $g_w(Q_u) = \mathcal{Q}'_u$ and $g_w(y_{u1}) = p_u$. 

Note that
\[
\mathcal{R}_w \setminus \mathcal{R}_{w_1} \setminus \mathcal{R}_{w_1}^{k_0} \setminus \mathcal{R}_{w_2} = T_{w_1, w_1}^{k_0} \cup \bigcup_{l k_0 + 1 < |u| \leq (l + 1) k_0 + 10} T_{u, w_1}^{k_0},
\]
where \( T_{w_1, w_1}^{k_0} \) is a Jordan domain bounded by a subarc of \( \partial A'_0 \), a subarc of \( \partial \mathcal{R}_{w_1} \), a subarc of \( \partial \mathcal{R}_{w_1}^{k_0} \), and a simple polygonal arc \( \sigma_{w_1, w_1}^{k_0} \). The domain \( T_{w_1, w_1}^{k_0} \) is defined in similar fashion.

6.2.4. Construction of \( \mathcal{L}' \): Decomposition in \( T_{w_1, w_2} \). Let \( l \) be a nonnegative integer and \( w_1, w_2 \in \mathcal{W}_{l k_0 + 1} \) be neighbors in \( \mathcal{W}_{l k_0 + 1} \) so that \( T_{w_1, w_2}, \mathcal{R}_{w_1} \), and \( \mathcal{R}_{w_2} \) are already defined by the previous steps. The construction in this case is similar to that of Section 6.2.3 and we only sketch the steps. Consider words \( u_1, \ldots, u_k \in \mathcal{W}_{l k_0 + 1} \) such that \( u_1 \) is at the right of \( w_1 \), \( u_k \) is at the left of \( w_2 \) for \( i = 1, \ldots, k \). Each \( \mathcal{R}_{u_i} \) is contained in \( T_{w_1, w_2} \) and \( \text{diam} \mathcal{R}_{u_i} \leq \frac{1}{2} \text{dist}(\mathcal{R}_{w_1}, \partial T_{w_1, w_2}) \) when \( i = 2, \ldots, k - 1 \). As in Section 6.2.2 and Section 6.2.3, we join each \( \mathcal{R}_{u_i} \), with \( \mathcal{R}_{u_{i+1}} \), \( i = 1, \ldots, k - 1 \), with polygonal curves \( \sigma_{u_i, u_{i+1}} \) that, except for their points, do not intersect \( \partial V \) or \( \partial T_{w_1, w_2}, \partial \mathcal{R}_{u_i} \) or each other.

Let now \( \tilde{T}_{w_1, w_2} \) be the Jordan domain bounded by \( \sigma_{w_1, w_2}, \partial \mathcal{R}_{w_1}, \partial \mathcal{R}_{w_2}, \) curves \( \sigma_{u_i, u_{i+1}} \) and subarcs of \( \partial \mathcal{R}_{u_i} \). Then, as in Section 6.2.2, we construct a polygonal curve \( s_H \) joining \( H \) with the appropriate \( \mathcal{R}_{u_i} \) or the appropriate \( \sigma_{u_i, u_{i+1}} \) and remove a thickening of \( s_H \) and \( H \) from \( \tilde{T}_{w_1, w_2} \). Denote by \( \mathcal{Q}_{w_1, w_2} \) the new domain after these removals. The definitions of \( \mathcal{R}_{u_i} \) are as in Section 6.2.2 and Section 6.2.3.

As in previous sections, we subdivide \( \mathcal{Q}_{w_1, w_2} \) into domains \( \mathcal{Q}'_w \) as follows. Note that \( \partial \mathcal{Q}'_{w_1, w_2} \) is a curve in \( 
abla_{2-2m(w_1, w_2)} \) with \( m(w_1, w_2) \) being a positive integer and \( 2m(w_1, w_2) \approx \text{diam} \mathcal{Q}'_{w_1, w_2} \). Moreover, \( \mathcal{Q}_{w_1, w_2} \) contains at most \( N_2 \approx 1 \) components of \( \partial \mathcal{Q}'_{w_1, w_2} \). Therefore, there exists \( m_2 \approx 1 \) such that \( \mathcal{Q}_{w_1, w_2} \) can be subdivided to Jordan domains \( \mathcal{Q}'_w \) with polygonal boundaries in \( 
abla_{2-2m(w_1, w_2) - m} \), where \( l k_0 + 1 < |w| \leq (l + 1) k_0 + 10, x'_w \) is between \( x_{u_i} \) and \( x'_{u_i} \), and

1. \( \mathcal{Q}'_w \) have mutually disjoint interiors and their union is all of \( \mathcal{Q}_{w_1, w_2} \),
2. \( \partial \mathcal{Q}'_w \) is in \( \nabla_{2-2m(w_1, w_2)} \) for some positive integer \( m(w_1, w_2) \) with \( \text{diam} \mathcal{Q}'_w \approx 2^{-m(w_1, w_2)} \),
3. \( \{ \mathcal{Q}'_w \} \cup \{ \mathcal{Q}_w \} \) are combinatorially equivalent to \( \{ \mathcal{Q}_w \} \cup \{ y_{w_1} \} \), that is, there exists a homeomorphism \( g_{w_1, w_2} : \Delta_0 \rightarrow \Delta'_0 \) extending \( f \) such that \( g_{w_1, w_2}(\mathcal{Q}_w) = \mathcal{Q}'_w \) and \( g_{w_1, w_2}(y_{w_1}) = y_{w_2} \).

Note that
\[
T_{w_1, w_2} \setminus \bigcup_{i=1}^{n} \mathcal{R}_{w_i} = \bigcup_{i=1}^{n-1} T_{u_i, u_{i+1}} \cup \bigcup_{l k_0 + 1 < |u| \leq (l + 1) k_0 + 10} \mathcal{Q}'_w, \quad x'_w \text{ between } x'_{u_i}, x'_{u_{i+1}},
\]
where \( T_{u_i, u_{i+1}} \) is a Jordan domain bounded by a subarc of \( \partial A'_0 \), a subarc of \( \partial \mathcal{R}_{u_i} \), a subarc of \( \partial \mathcal{R}_{u_{i+1}} \) and a simple polygonal arc \( \sigma_{u_i, u_{i+1}} \).

7. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. By Section 4, we may assume for the rest that \( U \subset \mathbb{R}^2 \) is an unbounded \( c \)-uniform domain and that \( \partial U \)
is compact and perfect. and \( \partial U \) is \( c \)-uniformly perfect and that \( f : \partial U \to \mathbb{R}^2 \) is \( \eta \)-quasisymmetric that can be extended homeomorphically on \( U \). The proof in the bi-Lipschitz case is similar; see Section 6.2.1.

Since \( f \) extends homeomorphically on \( U \), there exists a unique unbounded domain \( U' \) whose boundary is \( f(\partial U) \) and \( f \) extends as homeomorphism from \( U \) onto \( U' \). By Lemma 2.10 and the quasinvariance of uniformly perfect sets, \( U \) is \( c' \)-uniform and \( \partial U' \) is \( c' \)-relatively connected with \( c' \) depending only on \( c \) and \( \eta \). For simplicity, we assume for the rest that \( c' = c \).

Fix \( x_0 \in \partial U \) and \( x_0' \in \partial U' \). Let \( D_0 = \mathbb{R}^2 \setminus \overline{B}(x_0, 2c \text{diam } \partial U) \) and \( D_0' = \mathbb{R}^2 \setminus \overline{B}(x_0', 2c \text{diam } \partial U') \). In Section 7.1 we apply the results of Section 5 and Section 6 to construct two Whitney-type decompositions, one in \( U \setminus D_0 \) and another in \( U' \setminus D_0' \) that are combinatorially equivalent. Specifically, we show the following proposition.

**Proposition 7.1.** There exist \( K > 1 \), \( d > 1 \), \( C > 1 \) depending only on \( c \) and \( \eta \) and two families of domains \( \mathcal{D}, \mathcal{D}' \subset \mathcal{P}(K, d) \) with the following properties.

1. The domains in \( \mathcal{D} \) are mutually disjoint and \( U \setminus D_0 = \bigcup_{D \in \mathcal{D}} \overline{D} \). Similarly for \( \mathcal{D}' \).

2. For all \( D \in \mathcal{D} \), \( C^{-1} \text{diam } D \leq \text{dist}(D, \partial U) \leq C \text{diam } D \). Similarly for \( \mathcal{D}' \).

3. For each \( D \in \mathcal{D} \), there are at most \( C \) elements in \( \mathcal{D} \) whose boundary intersects that of \( D \). Similarly for \( \mathcal{D}' \).

4. If \( \Gamma = \partial D \cap \partial D' \neq \emptyset \) for \( D, D' \in \mathcal{D} \) then \( \Gamma \) is an \( L \)-bi-Lipschitz arc and \( \text{diam } \Gamma \geq C^{-1} \max\{ \text{diam } D, \text{diam } D' \} \). Similarly for \( \mathcal{D}' \).

5. There exists homeomorphism \( g : U \setminus \mathcal{D} \to U' \setminus \mathcal{D}' \) such that \( f|\partial U = g|\partial U \) and for each \( D \in \mathcal{D} \), \( g(D) \in \mathcal{D}' \).

In Section 7.1 we construct the families \( \mathcal{D} \) and \( \mathcal{D}' \) while in Section 7.2 we give the proof of Theorem 1.1.

### 7.1. Whitney-type decompositions for planar uniform domains.

We describe the steps in the construction of the families \( \mathcal{D} \) and \( \mathcal{D}' \). Here and for the rest of this section we write \( E' = \partial U \) and \( E' = \partial U' \). To reduce the use of constants and simplify the exposition, we assume that all constants in the lemmas and propositions of Section 5 are equal to \( c \) for both \( U \) and \( U' \).

**Step 1.** We apply the construction of Section 5.3 on \( \partial U \) inside \( U \setminus D_0 \) with \( \epsilon = (8c)^{-3} \text{diam } E \) and obtain Jordan domains \( D_1, \ldots, D_n \). Set \( D_0 = (U \setminus D_0) \setminus \bigcup_{i=1}^n \overline{D}_i \) and note that \( \text{diam } D_0 \geq \text{diam } \partial U \).

For each \( i = 1, \ldots, N \) let \( E_i = D_i \cap \partial U \) and \( E'_i = f(A_i) \). Applying Lemma 5.3 repeatedly on each set \( E'_i \) we obtain Jordan domains \( D'_i = V(E'_1, (U' \setminus D'_0), r_1) \) and \( D'_i = V(E'_1, (U' \setminus D'_0), r_1) \) for \( i = 2, \ldots, n \) such that

\[
  r_i = (32c)^{-1} \min\{ \text{diam } E'_i, \text{dist}(E'_i, E' \setminus E'_i) \}
\]

for each \( i = 1, 2, \ldots, n \). Set \( D_0' = (U' \setminus D'_0) \setminus \bigcup_{i=1}^n \overline{D}'_i \) and note that \( \text{diam } D_0' \geq \text{diam } \partial U' \). Note that, since \( E \) is relatively connected, by Remark 2.4, in the definition of \( r_i \) and \( r_w, \epsilon_w \) below, we can replace the minimum with the distance of the two sets.

**Step 2.** Fix \( i \in \{1, \ldots, n\} \) and let

\[
  \epsilon'_i = (8c)^{-3} \min\{ \text{dist}(E'_i, \partial D'_i), \text{diam } E'_i \}
\]

Note that although \( U \cap D'_i \) may not be \( c \)-uniform like \( U \), the condition of \( c \)-uniformity for \( E' \) holds true in the scale of \( \epsilon'_i \). That is, for all \( x, y \in E'_i \) with \( |x - y| \leq \epsilon'_i \), there
exists $c$-cigar curve $\gamma$ joining them in $U \cap D_i'$. In fact, $\epsilon_i'$ has been chosen in such a way that both Lemma 5.6 and Lemma 5.8 can be applied as if $U \cap D_i'$ was $c$-uniform. Therefore, we can apply on each $E_i'$ inside $D_i'$ the construction of Section 5.4 with $\epsilon_i'$ and obtain Jordan domains $D_{i1}', \ldots, D_{in_i}'$. Set $D_i' = D_i' \setminus \bigcup_{j=1}^{n_i} D_{ij}'$.

In each $D_i$ we apply the second part of Step 1. Fix $i \in \{1, \ldots, n\}$ and let $E_{ij} = E' \cap D_{ij}'$ and $E_j = f^{-1}(A_{ij})$. By Lemma 5.10 $D_i \cap U$ is $c'$-uniform for some $c'$ depending only on $c$. Applying Lemma 5.4 repeatedly on each set $E_{ij}$ we obtain Jordan domains $D_{i1} = V(E_{i1}, U \cap D_i, (32c)^{-1})$ and $D_{ij} = V(E_{ij}, (U \cap D_i) \setminus \bigcup_{k=1}^{j-1} D_{ik}, (32c)^{-1})$ for $j = 2, \ldots, n_i$. Set $D_i = (U \cap D_i) \setminus \bigcup_{j=1}^{n_i} \partial D_{ij}$ and note that $\text{diam } D_i' \geq \epsilon_i' \text{diam } D_i$ with $\epsilon_i'$ depending only on $c$.

In general we work as follows.

**Step 2k + 1.** From previous steps we have obtained mutually disjoint Jordan domains $D_w \subset U$, mutually disjoint Jordan domains $D_w' \subset U'$, boundary sets $E_w = D_w \cap E$ and boundary sets $E_w' = D_w' \cap E' = f(E_w)$ such that, for some $M > 1$ depending only on $c$ and $\eta$,

\[
\frac{\text{diam } E_w}{M} \leq \text{dist}(E_w, E \setminus E_w) \leq M \text{ dist}(D_w, E \setminus E_w)
\]

and for all $z \in \partial D_w$,

\[
\frac{\text{dist}(z, E_w)}{M} \leq \min\{\text{dist}(E_w, E \setminus E_w), \text{diam } E_w\} \leq M \text{ dist}(z, E_w).
\]

Similarly for $E', E_w'$ and $D_w'$. As noted above, since $E$ is uniformly perfect we also have that $\text{dist}(E_w, E \setminus E_w) \leq M_0 \text{ diam } E_w$ for some $M_0 > 1$ depending only on $c$.

Fix now a a Jordan domain $D_w, w = i_1 \cdots i_{2k}$. We distinguish two cases.

**7.1.1. Case 1.** For all components $A$ of $E_w$, there exists $z \in \partial D_w$ such that $\text{diam } A \leq (8c)^3 \text{ dist}(A, z)$.

**Remark 7.2.** In Case 1, note that $\text{diam } A \leq c_1 \text{ dist}(A, E \setminus E_w)$ for each component $A$ of $E_w$ and for some $c_1 > 1$ depending only on $c$. By quasisymmetry of $f$,

\[
\text{diam } f(A) \leq c_1' \text{ dist}(f(A), E' \setminus E_w')
\]

for some $c_1'$ depending only on $c$ and $\eta$. By Lemma 5.11 $\text{diam } E_w' \leq \text{ diam } D_w' \leq c_2' \text{ dist}(\text{diam } E_w, E' \setminus E_w')$. By quasisymmetry, $\text{diam } E_w \leq c_2 \text{ dist}(\text{diam } E_w, E \setminus E_w)$ with $c_2$ depending only on $c$ and $\eta$.

Applying the construction of Section 5.4 on $E_w$ with

\[
\epsilon_w = (80c)^{-3} \min\{\text{dist}(E_w, \partial D_w), \text{diam } E_w\}
\]

we obtain Jordan domains $D_{w_i} \subset D_w$ with $i \leq n_w$. Define $E_w = D_{w_i} \cap E$, $E_w' = f(E_{w_i})$ and $D_w = D_w \setminus \bigcup_i D_{w_i}$. In each $D_{w_i}$ we apply the second part of Step 1 and obtain Jordan domains $D_{w_i}'$. Set $D_w' = D_w' \setminus \bigcup_i D_{w_i}'$.

**Step 2k + 2.** In each $D_{w_i}'$ apply the construction of Section 5.4 on $E_w'$ with

\[
\epsilon_{w_i}' = (80c)^{-3} \min\{\text{dist}(E_{w_i}', \partial D_{w_i}'), \text{diam } E_{w_i}'\}.
\]

Again, $\epsilon_{w_i}'$ has been chosen in such a way that both Lemma 5.6 and Lemma 5.8 can be applied as if $U \cap D_{w_i}'$ was $c$-uniform. Thus, we obtain Jordan domains $D_{w_iw_j} \subset D_{w_i}'$. Set $E_{w_iw_j} = D_{w_iw_j} \cap E'$, $E_{w_iw_j} = f^{-1}(E_{w_iw_j})$ and $D_{w_iw_j} = D_{w_iw_j}' \setminus \bigcup_j D_{w_iw_j}'$.

In each $D_{w_i}$ we apply the second part of Step 1. Applying Lemma 5.4 repeatedly on each set $E_{w_iw_j}$ we obtain Jordan domains $D_{w_iw_j}$. Set $D_{w_iw_j} = D_{w_iw_j} \setminus \bigcup_j D_{w_iw_iw_j}$. 

Combining Lemma \[5.10\] Lemma \[5.11\] Remark \[7.2\] and using the fact that \( E \) is uniformly perfect, we obtain the next corollary which completes the induction step.

**Corollary 7.3.** There exists \( N_2 > 1, d > 1 \) and \( K > 1 \) depending only on \( c \) and \( \eta \) with the following properties.

1. \( n_w \leq N_2 \).
2. Each \( D_{wi} \) is an \( K \)-quasidisk.
3. For all \( i \in \{1, \ldots, n_w\} \), \( \text{diam} D_{wi} \geq d^{-1} \text{diam} D_w \).
4. For all \( i \in \{1, \ldots, n_w\} \), \( \text{dist}(\partial D_w, \partial D_{wi}) \geq d^{-1} \text{diam} D_w \).
5. For all \( i, i' \in \{1, \ldots, n_w\} \), \( \text{dist}(\partial D_{wi}, \partial D_{wi'}) \geq d^{-1} \text{diam} D_w \).
6. Equations \[7.1\] and \[7.2\] hold for \( E \), \( E_{wi} \) and \( D_{wi} \) with constant \( d \).

By Remark \[7.2\] and the fact that \( n \leq N \), the same is true for domains \( D'_{wi} \).

Similarly, the properties (1)-(5) hold true for domains \( D_{wij} \) and \( D'_{wij} \).

### 7.1.2. Case 2

There exists a component \( A_0 \) of \( E_w \) such that

\[
\text{dist}(z, A_0) \leq (8c)^{-3} \text{diam} A_0 \quad \text{for all } z \in \partial D_w.
\]

Note that \( \min_{z \in \partial D_w} \text{dist}(z, A_0) \geq \delta_0 \text{diam} A_0 \) for some \( \delta_0 \in (0, 1) \) depending only on \( c \) and \( \eta \). Set \( A'_0 = f(A_0) \) and \( E'_w = f(E_w) \). We show that that \( \text{dist}(z, A'_0) \) is comparable to \( \text{diam} A'_0 \) for each \( z \in \partial D'_{w} \).

**Lemma 7.4.** There exists \( C > 1 \) depending only on \( c \) and \( \eta \) such that

\[
C^{-1} \text{diam} A'_0 \leq \text{dist}(A'_0, z) \leq C \text{diam} A'_0 \quad \text{for all } z \in \partial D'_w.
\]

**Proof.** Note that \( \text{diam} E'_w \leq \text{diam} D_w \leq (2(8c)^{-3} + 1) \text{diam} A_0 \). By quasisymmetry of \( f \), \( \text{diam} E'_w \leq c_1 \text{diam} A'_0 \) for some \( c_1 > 1 \) depending only on \( c \) and \( \eta \). By Remark \[7.2\] and the right inequality follows from equations \[7.1\] and \[7.2\] for \( E', E'_w, D'_w \). The left inequality follows from equations \[7.1\] and \[7.2\] for \( E', E'_w, D'_w \) and the fact that \( \text{diam} A'_0 \leq \text{diam} E'_w \).

By Lemma \[7.4\] all assumptions of Section \[6\] are satisfied and by setting \( D = D_w \), \( D' = D'_w \), \( A = E_w \) and \( A' = E'_w \) we proceed as follows. First we replace \( D_w \) with \( \Delta_1 \) where \( l_1 \) is chosen so that \( \text{dist}(\Delta_1, D_w) \geq \frac{1}{2} \text{dist}(D_w, D'_w) \) for all \( w' = j_1 \cdots j_{2k} \neq w \). Define \( \{Q_u\}_{u \in W}, \{Q'_u\}_{u \in W}, \{A_u\}_{u \in W} \) and \( \{A'_u\}_{u \in W} \) as in Section \[6\].

Set \( D_{wu} = Q_u \) and \( D'_{wu} = Q'_u \). If \( D_{wu} \cap \partial U = \emptyset \) then set \( D_{wu} = D_wu \) and \( D'_{wu} = D'_{wu} \). Suppose now that \( D_{wu} \cap \partial U \neq \emptyset \). If \( D_{wu} \) satisfies the assumptions of Case 1, then apply Step 1 and Step 2 and proceed as in Case 1. Otherwise, apply the decomposition of Section \[6\].

### 7.2. Proof of Theorem 1.1

Suppose that \( U \subset \mathbb{R}^2 \) is an unbounded \( c \)-uniform domain with bounded \( c \)-uniformly perfect boundary.

Set \( \mathcal{D} = \{D_w\} \) and \( \mathcal{D}' = \{D'_w\} \). Given \( D_w, D'_w \in \mathcal{D} \) whose boundaries intersect, let \( \Gamma_{w,u} = D_w \cap D_u \) and \( \Gamma'_{w,u} = D'_w \cap D'_u \). By Proposition \[7.1\] \( \Gamma_{w,u}, \Gamma'_{w,u} \) are \( L \)-bi-Lipschitz arcs and \( \text{diam} \Gamma_{w,u} \geq C^{-1} \text{diam} \partial D_w \) and \( \text{diam} \Gamma'_{w,u} \geq C^{-1} \text{diam} \partial D'_w \) for some \( C > 1 \) depending only on \( c \) and \( \eta \).

Define a homeomorphism \( g : \bigcup_{D_w \in \mathcal{D}} \partial D_w \to \bigcup_{D'_w \in \mathcal{D}'} \partial D'_w \) so that \( g|_{\Gamma_{w,u}} \) maps \( \Gamma_{w,u} \) onto \( \Gamma'_{w,u} \) by arc-length parametrization and can be homeomorphically extended on each \( D_w \). Note that \( f_u = g|_{\partial D_w} \) is a \((\lambda_w, \Lambda)\)-quasisimilarity for some \( \Lambda \) depending only on \( L_0 \) and \( d_0 \), (thus only on \( c \) and \( \eta \)) and with \( \lambda_w = \frac{\text{diam} D_w}{\text{diam} D'_w} \). By
Proposition $3.2$ each $f_w$ extends to a $(\lambda_w, \Lambda')$-quasisimilarity $F_w : D_w \to D'_w$ for some $\Lambda'$ depending only on $c$ and $\eta$.

Define $F : U \to U'$ with $F|\partial D_w = F_w$. Then, by a theorem of Väisälä on removability of singularities [Väi71 Theorem 35.1], $F$ is $K$-quasiconformal with $K$ depending only on $\Lambda'$, thus only on $c$ and $\eta$. By Lemma $4.1$ $F$ is $\eta'$ quasisymmetric for some $\eta'$ depending only on $c$ and $\eta$.

7.2.1. Proof of Theorem $1.1$ in the bi-Lipschitz case. The proof of Theorem $1.1$ in the quasisymmetric case. Since $\partial U$ may not be uniformly perfect, in Proposition $7.1$ property (4) holds with $\min$ instead of $\max$ and the families $\mathcal{D}$ and $\mathcal{\mathcal{D}}' \subset \mathcal{B}(\mathcal{L}(L_0, d))$ for some $L_0 > 1$ and $d > 1$ depending only on $L$ and $c$. Moreover, in Corollary $7.3$ the domains $D_w$ are $L'_0$-bi-Lipschitz disks and property (3) may not hold.

Since $f$ is $L$-bi-Lipschitz, there exists $c_0 > 1$ depending only on $c$ and $L$ such that $c_0^{-1} \text{diam } D'_w \leq \text{diam } D_w \leq c_0 \text{diam } D'_w$ for all $D_w \in \mathcal{\mathcal{D}}$. Therefore, applying Proposition $8.2$ and defining $F$ as above we note that $F|\partial D_w$ is $L_1$-bi-Lipschitz for all $D_w \in \mathcal{\mathcal{D}}$ with $L_1$ depending only on $c$ and $L$. Thus, $F$ is $L_1$-BLD and by Lemma $2.2$ $F$ is $L'$-bi-Lipschitz with $L'$ depending only on $c$ and $L$.

8. The assumptions of Theorem $1.1$

We discuss the assumptions of Theorem $1.1$ and how necessary they are. In Section $8.1$ applying a result of Trotsenko and Väisälä [TV99] we show that relative connectedness is necessary for the QSEP in all dimensions. In Section $8.2$ we show that uniformity is somewhat necessary for the QSEP or the BLEP on the plane as neither John property nor quasiconvexity of $U$ in Theorem $1.1$ is sufficient.

8.1. Relative connectedness. Let $E \subset \mathbb{R}^n$ be a closed set that is not relatively connected. In the proof of Theorem $6.6$ in [TV99], a quasisymmetric map $f : E \to \mathbb{R}^n$ is constructed that is not power quasisymmetric. It follows then from Lemma $2.3$ that $f$ can not be extended quasisymmetrically on $\mathbb{R}^n$. We present the construction here again to illustrate why the map $f$ can be extended homeomorphically in $\mathbb{R}^n$.

Since $E$ is not relatively connected, for each $i \in \mathbb{N}$ there exists $E_i$ containing at least two points so that $\text{dist}^*(E_i, E \setminus E_i) \geq i$. We assume for the rest that $\text{diam } E_i \leq \text{diam } E \setminus E_i$. Set $d_i = \text{dist}^*(E_i, E \setminus E_i)$. We may assume that $4 < d_1 < d_2 < \cdots$. The conditions on $E_i$ imply one of the following three cases.

Case 1. There exists subsequence $i_1 < i_2 < \cdots$ with $E_{i_1} \supset E_{i_2} \supset \cdots$. For simplicity write $E_{i_k} = E_k$ and $d_{i_k} = d_k$. Then $\{x_0\} = \bigcap_{k \in \mathbb{N}} E_k$ for some $x_0 \in E$. Set $E^0 = E \setminus E_1$ and for each $k \in \mathbb{N}$ set $E^k = E_k \setminus E_{k+1}$. Note that $E$ is a disjoint union of the sets $E^k$. Define $f : E \to \mathbb{R}^n$ with $f(x_0) = x_0$, $f|E^0(x) = x$ and for each $k \in \mathbb{N}$, $f|E^k(x) = s_k x$ with

$$s_k = \frac{e^{d_1 + \cdots + d_k}}{(1 + d_1) \cdots (1 + d_k)}.$$

Case 2. There exists subsequence $i_1 < i_2 < \cdots$ with $E_{i_1} \subset E_{i_2} \subset \cdots$. For simplicity write $E_{i_k} = E_k$ and $d_{i_k} = d_k$. Set $E^0 = E_1$ and for each $i \in \mathbb{N}$ set $E^k = E_{k+1} \setminus E_k$. Note that $E$ is a disjoint union of the sets $E^k$. Define $f : E \to \mathbb{R}^n$ with $f|E^0(x) = x$ and for each $k \in \mathbb{N}$, $f|E^k(x) = s_k x$ with

$$s_k = \frac{e^{d_1 + \cdots + d_k}}{e^{d_1 + \cdots + d_k}}.$$
Case 3. There exists subsequence $i_1 < i_2 < \cdots$ with $E_{i_1}, E_{i_2}, \ldots$ being mutually disjoint. For simplicity write $E_k = E_k, d_k = d_k$ and $x_k = x_k$. Set $E_0 = E \setminus \bigcup_{k \in \mathbb{N}} E_k$ and for each $k \in \mathbb{N}$ set $E_k \cap E$. Note that $E$ is a disjoint union of the sets $E_k$. Define $f : E \to \mathbb{R}^n$ with $f|E_0(x) = x$ and for each $k \in \mathbb{N}$, $f|E_k(x) = x_k + s_k(x - x_k)$ with

$$s_k = \frac{e^{d_k}}{(1 + d_k)}.$$

Following the proof of [TV99, Theorem 6.6], in each case $f$ is quasisymmetric but not power quasisymmetric. Thus, it only remains to show that, in each case, $f$ extends to a self homeomorphism of $\mathbb{R}^n$. Assume the first case. For each $i \in \mathbb{N}$ set $B_i = B^n(x_i, 2 \text{diam } E_i)$ and $B'_i = B^n(x_i, \frac{1}{2} d_i \text{diam } E_i)$ and $A_i = B_i \setminus B'_{i+1}$. On $\mathbb{R}^n \setminus B_1$ set $F(x) = x$ and for each $i \in \mathbb{N}$ set $F|A_i(x) = s_i x$. By [TV99 (6.8)], $F(\partial B'_{i+1})$ is contained in a ball with boundary $F(\partial B_i)$, $F(B_{i+1})$ is contained in a ball with boundary $F(\partial B'_{i+1})$ and both $\text{dist}(F(\partial B_i), F(\partial B'_{i+1}))$ and $\text{dist}(F(\partial B_{i+1}), F(\partial B'_{i+1}))$ are nonzero for each $i \in \mathbb{N}$. Therefore, $F$ can be extended homeomorphically on each $B_i \setminus B_i$.

The other two cases are similar; see [TV99 (6.14)] and [TV99 (6.18)].

8.2. Uniformity. We construct a compact, countable, relatively connected set $E \subset \mathbb{R}^2$ whose exterior is both John and quasiconvex and $E$ fails both the QSEP and the BLEP.

For each $n \in \mathbb{N}$ divide the interval $[0, 2^{-n(n-1)/2}]$ into $2^n$ intervals of equal length and let $A$ be the set of endpoints of all the intervals produced. For each $i = 1, \ldots, 4$ let $\phi^{(i)}$ is the composition of $\phi(x, y) = (−y, x)$ with itself $i$ times and set

$$E = A \cup \phi(A) \cup \phi^{(2)}(A) \cup \phi^{(3)}(A).$$

Note that $E$ is a relatively connected, compact and countable. Moreover, since $\mathbb{R}^2 \setminus (\{−1, 1\} \times \{0\}) \cup (\{0\} \times [−1, 1])$ is a John domain, $\mathbb{R}^2 \setminus E$ is also a John domain. Furthermore, we claim that $\mathbb{R}^2 \setminus E$ is $c$-quasiconvex for any $c > 1$. Indeed, let $x, y \in \mathbb{R}^2 \setminus E$ and $\gamma = [x, y]$. We may assume that $\gamma \cap E = \emptyset$ otherwise there is nothing to prove. Fix $\epsilon > 0$. We first treat the case that both of $x, y$ lie on the same axis. Without loss of generality assume that $x = (x_1, 0), y = (y_1, 0)$. There exists a point $z = (0, z_2) \in \mathbb{R}^2 \setminus E$ such that $0 < z_2 < \frac{1}{2}\epsilon|x − y|$. Setting $\gamma' = [x, (x_1, z_2)] \cup [(x_1, z_2), (y_1, z_2)] \cup [(y_1, z_2), y]$ we have that $\gamma' \subset \mathbb{R}^2 \setminus E$ and that $\ell(\gamma') \leq (1 + \epsilon)|x − y|$. Suppose now that $x, y$ are not on the same axis. Then the line segment $[x, y] \cap E$ contains at most 2 points $z_1, z_2$. For each point $z_j$, let $x_j, y_j \in [x, y]$ such that $[x_j, y_j] \cap E = \{z_j\}$ and $|x_j - z_j| = |y_j - z| \leq \frac{1}{2}\epsilon|x − y|$. Let $z'_j \in \mathbb{R}^2 \setminus E$ be on the same axis as $z_j$ and $|z'_j - z_j| \leq \frac{1}{2}\epsilon|x − y|$. Replacing each $[x_j, y_j]$ with $[x_j, z'_j] \cup [z'_j, y_j]$ we obtain a new polygonal curve $\gamma' \subset \mathbb{R}^2 \setminus E$ that joins $x$ and $y$ satisfying $\ell(\gamma') \leq (1 + \epsilon)|x − y|$. Define now $f : E \to E$ with

$$f|A \cup \phi(A)(x, y) = (y, x) \quad \text{and} \quad f|\phi^{(2)}(A) \cup \phi^{(3)}(A) = \text{Id}.$$ 

Clearly, $f$ is 2-bi-Lipschitz and as $E$ is totally disconnected, $f$ extends to a homeomorphism of $\mathbb{R}^2$. We show that $f$ can not be extended quasisymmetrically on $\mathbb{R}^2$. Suppose that there exists an $\eta$-quasisymmetric extension $F : \mathbb{R}^2 \to \mathbb{R}^2$ extending $f$. For each point $a \in A$ let $x_a = (−a, 0), y_a = (0, a)$ and $\gamma_a = \{(-at, a(1 − t)) : t \in [0, 1]\}$. Note that, for all $a \in A$, $\gamma_a$ is 2-bounded turning and 2-cigar in $\mathbb{R}^2 \setminus E$;
that is, diam $\gamma_a \leq 2|x_r - y_r|$ and dist$(z, E) \geq \frac{1}{2} \min\{|x_r - z|, |y_r - z|\}$ for all $z \in \gamma_a$. Since $F$ is $\eta$-quasisymmetric, there exists $C > 1$ depending only on $\eta$ such that $F(\gamma_a)$ is $C$-bounded and $C$-cigar in $\mathbb{R}^2 \setminus E$ for all $a \in A$.

Let $n \in \mathbb{N}$ such that $2^{-n} \leq (3C)^{-2}$. Let $a' = 2^{-n(n+1)/2} \in A$ and $a = m2^{-n}a' \in A$ with $m$ being the smallest integer bigger than $2C$. The $C$-bounded turning condition of $F(\gamma_a)$ implies that $F(\gamma_a) \cap (\{0\} \times \mathbb{R}) = F(\gamma_a) \cap (\{0\} \times [-a', a'])$. However, for any point $F(\gamma_a) \cap (\{0\} \times \mathbb{R})$ we have $\text{dist}(z, E) \leq 2^{-n}a' \leq (3C)^{-2}a' \leq (2C)^{-1}a \leq (2C)^{-1} \min\{|z - F(x_a)|, |z - F(y_a)|\}$ and the John property of $F(\gamma_a)$ is violated.

**Remark 8.1.** It turns out that John property and quasiconvexity of $\mathbb{R}^2 \setminus E$ are also not necessary for bi-Lipschitz or quasisymmetric extensions on $\mathbb{R}^2 \setminus E$. For example, let $E = (0, +\infty) \times (-1, 1) \subset \mathbb{R}^2$ and note that $E$ is not John while $\mathbb{R}^2 \setminus \overline{E}$ is not quasiconvex. Nevertheless, any $\eta$-quasisymmetric (resp. $L$-bi-Lipschitz) embedding of $E$ or $\mathbb{R}^2 \setminus \overline{E}$ into $\mathbb{R}^2$ extends to an $\eta'$-quasisymmetric (resp. $L'$-bi-Lipschitz) homeomorphism of $\mathbb{R}^2$. The proof is left to the reader.

9. **Uniformization of Cantor sets with bounded geometry**

In [DS97], David and Semmes characterized the metric spaces that are quasisymmetrically homeomorphic to the standard middle-third Cantor set $C \subset \mathbb{R}$: a metric space $(X, d)$ is $\eta$-quasisymmetric to $C$ if and only if it is compact, doubling, $c$-uniformly disconnected and $c$-uniformly perfect [DS97 Proposition 15.11].

Later, MacManus [Mac99] improved this characterization in 2 dimensions: for a compact set $E \subset \mathbb{R}^2$ there exists a quasisymmetric mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ with $F(E) = C$ if and only if $E$ is uniformly perfect and uniformly disconnected. The same however is not true in dimensions $n \geq 3$ as there exist uniformly perfect and uniformly disconnected wild Cantor sets such as the classical Antoine’s necklace [Dav07] pp. 70–75]. By increasing the dimension by 1, MacManus’ result generalizes in dimensions $n \geq 3$.

**Theorem 9.1.** For a compact set $E \subset \mathbb{R}^n$ there exists a quasisymmetric map $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $F(E) = C$ if and only if $E$ is uniformly perfect and uniformly disconnected.

One direction of Theorem 9.1 is clear. Namely, if there exists quasisymmetric homeomorphism $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ mapping $C$ onto $E$ then $E$ is $c$-uniformly perfect and $c$-uniformly disconnected with $c$ depending only on $\eta$. For the converse, we use the fact that there exists a quasisymmetric homeomorphism $f : C \to E$. Our goal is to extend this mapping quasisymmetrically to $\mathbb{R}^{n+1}$.

Consider the set of finite words $W$ formed from the letters $\{1, 2\}$ and denote by $\emptyset$ the empty word. The length of a word $|w|$ is the number of letters that the word contains with $|\emptyset| = 0$. Define $W^N$ to be the set of words in $W$ whose length is exactly $N$. Let $I_0 = [0, 1]$ and given $I_w = [a, b]$ let $I_{w_1} = [a, a + \frac{1}{2}(b - a)]$, $I_{w_2} = [b - \frac{1}{2}(b - a), b]$. For each $w \in W$ let $C_w = I_w \cap C$ and $E_w = f(C_w)$.

**Lemma 9.2.** There exists $k \in \mathbb{N}$ depending only on $\eta$ with the following property. For any $m \in \mathbb{N}$ there exist sets $\mathcal{E}_1, \ldots, \mathcal{E}_k$ whose elements are sets $E_w$ with $w \in W^m$ such that

1. $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ when $i \neq j$ and $\bigcup_{i=1}^{k} \mathcal{E}_i = \{E_w : w \in W^m\}$.
(2) for any \( i \in \{1, \ldots, k\} \) and any \( E_w, E_w' \in \mathcal{E}_i \) with \( w \neq w' \) we have
\[
\text{dist}(E_w, E_w') \geq 5 \max\{\text{diam } E_w, \text{diam } E_{w'}\}.
\]

**Proof.** By quasisymmetry of \( f \), property (2) is satisfied if \( \text{dist}(C_w, C_{w'}) \geq d3^{-m} \) where \( d = (\eta^{-1}(1/5))^{-1} \). Note that for each \( C_w \) with \( w \in \mathcal{W}^m \) there exists at most \( k \) sets \( C_{w1}, \ldots, C_{wk} \) such that \( \text{dist}(C_w, C_{w'}) < d3^{-m} \). Let \( C'_1, \ldots, C'_m \) be an enumeration of \( \{C_w : w \in \mathcal{W}^m\} \) such that for all \( 1 \leq i < j \leq 2^m \), \( C'_i \) lies at the left of \( C'_j \). For each \( i = 1, \ldots, k \) define \( A_i \) to be the integers in \( \{1, \ldots, 2^m\} \) that are of the form \( i + rk \) with \( r \in \mathbb{N} \cup \{0\} \) and set \( \mathcal{E}_i = \{f(C'_i) : j \in A_i\} \).  \( \Box \)

We are now ready to show Theorem 9.1.

**Proof of Theorem 9.1.** Assume as we may that \( \text{diam } E = 1 \). The first step of the proof is the construction of a bi-Lipschitz mapping \( \Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) that unlinks \( E \). The second step is the construction of a quasiconformal mapping \( G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) that maps the unlinked image \( \Phi(E) \) onto \( C \). The composition \( F = G \circ \Phi \) is the mapping in question.

Let \( k \) be as in Lemma 9.2 and let \( N \) be the smallest positive integer such that \( 3^{-N} \leq \min\{\eta^{-1}(1/16), \eta^{-1}((5k)^{-1})\} \). Then, for any \( w, w' \in \mathcal{W} \) with \( E_w \cap E_{w'} = \emptyset \) and any \( u \in \mathcal{W}^N \) we have
\[
(9.1) \quad \delta' \text{diam } E_w \leq \text{diam } E_w u \leq \delta \text{diam } E_w
\]
\[
(9.2) \quad \text{dist}(E_w, E_{w'}) \geq (\eta(1))^{-1} \max\{\text{diam } E_w, \text{diam } E_{w'}\}
\]
with \( \delta = \min\{\frac{1}{10}, ((5k)^{-1})\} \) and \( \delta' = (2\eta(3^N))^{-1} \).

Let \( \mathcal{E}_{i0}, \ldots, \mathcal{E}_{ik} \) be the sets of Lemma 9.2 corresponding to \( m = N \). Define \( \phi_i : E \to \mathbb{R} \) such that
\[
\phi_i|_{E_w(x)} = \frac{5(i-1)}{\delta} \quad \text{for all } x \in E_w
\]
where \( w \in \mathcal{W}^N, i = 1, \ldots, k \) and \( E_w \in \mathcal{E}_{i0} \). Inductively, suppose that we have defined \( \phi_j : E \to \mathbb{R} \) such that \( \phi_j|_{E_w} \) is constant whenever \( w \in \mathcal{W}^N \). For each \( w \in \mathcal{W}^N \) let \( \mathcal{E}_w^1, \ldots, \mathcal{E}_w^k \) be the sets of Lemma 9.2 corresponding to \( E = E_w \) and \( m = N \). Define \( \phi_j+1 : E \to \mathbb{R} \) such that
\[
\phi_j+1|_{E_{wu}(x)} = \phi_j|_{E_w(x)} + 5(i-1)\delta \text{diam } E_w \quad \text{for all } x \in E_{wu}
\]
where \( w \in \mathcal{W}^N, u \in \mathcal{W}^N, i = 1, \ldots, k \) and \( E_{wu} \in \mathcal{E}_w^i \). Note that for all \( x \in E, |\phi_i(x) - \phi_j(x)| \leq \delta \max\{i,j\} \). Therefore, the mappings \( \phi_j \) converge uniformly to a mapping \( \phi : E \to \mathbb{R} \).

We claim that \( \phi \) is Lipschitz. Indeed, let \( x, y \in E \) and let \( m_0 \in \mathbb{N} \) be the greatest integer \( m \) such that \( x, y \in E_w \) with \( w \in \mathcal{W}_m \). In particular, suppose that \( x, y \in E_{w_0} \) with \( w_0 \in \mathcal{W}_m \). Then, by (9.1) and (9.2)
\[
|\phi(x) - \phi(y)| \leq \text{diam } E_{w_0} \leq \eta(1)(\delta')^{-1}|x - y|
\]
and the claim follows.

Fix \( x_0 \in E, B_0 = B^n(x_0, 5\text{ diam } E) \) and set \( \phi|\mathbb{R}^n \setminus B_0 \equiv 0 \). Then \( \phi : (\mathbb{R}^n \setminus B_0) \cup E \to \mathbb{R} \) is \( L \)-Lipschitz for some \( L \) depending only on \( \eta \) and by Kirszbraun Theorem there exists an \( L \)-Lipschitz extension of \( \phi \) on \( \mathbb{R}^n \) which we also denote by \( \phi \). Then the mapping \( \Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) with \( \Phi(x, z) = (x, \phi(x) + z) \) is \( L' \)-bi-Lipschitz with \( L' \) depending only on \( L \).

For each \( m = 0, 1, \ldots \) and each \( w \in \mathcal{W}_m \) let \( x_w \in E_w \) and
\[
K_w = x_w + [-2 \text{ diam } E_w, 2 \text{ diam } E_w]^n.
\]
Note that if \( w \in \mathcal{W}^{mN} \) and \( u \in \mathcal{W}^N \) then \( K_{wu} \subset K_n \) and \( \text{dist}(K_{wu}, \partial K_n) \geq \frac{1}{2} \text{diam } E_u \). Note however that if \( w, w' \in \mathcal{W}^{mN} \) with \( w \neq w' \) then \( K_{wu} \) may intersect \( K_{wu}' \) which is why we lift different sets to different heights. For each \( m = 0, 1, \ldots \) and each \( w \in \mathcal{W}^{mN} \) define
\[
K_w = K_w \times [\phi_m(x_w) - 2 \text{diam } E_w, \phi_m(x_w) + 2 \text{diam } E_w].
\]
From the definition of the functions \( \phi_w \) it follows that for all \( m \in \mathbb{N} \), all distinct \( w, w' \in \mathcal{W}^{mN} \) and all \( u \in \mathcal{W}^N \)

\begin{align*}
(9.3) \quad &\text{dist}(K_w, K_{w'}) \geq \max\{\text{diam } E_w, \text{diam } E_{w'}\}; \\
(9.4) \quad &K_{wu} \subset K_w \text{ and } \text{dist}(K_{wu}, \partial K_w) \geq \frac{1}{2} \text{diam } E_w; \\
(9.5) \quad &K_w \cap \Phi(E) = \Phi(E_w) \text{ and } \text{dist}(\Phi(E_w), \partial K_w) \geq \frac{1}{2} \text{diam } E_w.
\end{align*}

For each \( m = 0, 1, \ldots \) and \( w \in \mathcal{W}^{mN} \) let \( z_w \) be the centre of \( I_w \) and define
\[
Q_w = [z_w - \frac{5}{6}3^{-mN}, z_w + \frac{5}{6}3^{-mN}] \times [-\frac{5}{6}3^{-mN}, \frac{5}{6}3^{-mN}].
\]
For each \( w \in \mathcal{W}^{mN} \) let \( g_w : \partial K_w \to \partial Q_w \) be a sense-preserving similarity mapping. By Proposition 3.6 there exists \( G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) that extends all mappings \( g_w \) so that \( G \) is the identity in a neighborhood of infinity, and for all \( w \in \mathcal{W}^{mN} \),
\[
G|K_w \setminus \bigcup_{u \in \mathcal{W}^N} K_{wu} : K_w \setminus \bigcup_{u \in \mathcal{W}^N} K_{wu} \to K_w' \setminus \bigcup_{u \in \mathcal{W}^N} K_{wu}'
\]
is a \((\frac{\text{diam } Q_w}{\text{diam } K_w}, \Lambda)\)-quasisimilarity with \( \Lambda \) depending only on \( \eta \). Therefore, by a theorem of Väisälä on removability of singularities [Väis71 Theorem 35.1], \( G \) is \( K \)-quasiconformal with \( K \) depending only on \( \eta \) and \( n \). Set \( F = G \circ \Phi \) and note that \( F \) extends \( f \) and \( F(E_w) = C_w \).

9.1. **Proof of Theorem 1.3.** Let \( E \subset \mathbb{R}^n \) be closed and \( c \)-uniformly disconnected. By Section 4 the proof is reduced to the case that that \( E \) is compact and perfect. Hence, by Brouwer’s Theorem we may assume that \( E \) is a Cantor set.

Assume first that \( E \) is \( \eta \)-perfect and \( f : E \to \mathbb{R}^n \) is \( \eta \)-quasisymmetric. By Theorem 9.1 we may assume that \( f : C \to C \) and by Theorem 1.1 and the tameness of planar totally disconnected sets [Mois77, Section 10] \( f \) extends to an \( \eta_1 \)-quasisymmetric homeomorphism \( F_1 : \mathbb{R}^2 \to \mathbb{R}^2 \) with \( \eta_1 \) depending only on \( \eta \). By the Tukia-Väisälä extension theorem [TV82], \( F_1 \) extends to an \( \eta' \)-quasisymmetric \( F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) with \( \eta' \) depending only on \( \eta \) and \( n \).

Assume now that \( f : E \to \mathbb{R}^n \) is \( L \)-bi-Lipschitz. Denote \( E' = f(E) \) and \( E'_w = f(E_w) \). In the proof of Theorem 3.1 by choosing \( N \) sufficiently large, we may assume that the right inequality of (9.1) and the inequality (9.2) are satisfied for both \( E \) and \( E' \). Then, following the construction of \( \Phi \) we can construct cubes \( K'_w \) corresponding to the sets \( E'_w \) with \( w \in \mathcal{W}^{mN} \), \( m \in \mathbb{N} \) and an \( L_2 \)-bi-Lipschitz mapping \( \Phi' : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) that satisfy properties (9.3), (9.4) and (9.5).

For each \( w \in \mathcal{W}^{mN} \) let \( g_w : \partial K_w \to \partial K'_w \) be a similarity mapping. By Proposition 3.6 there exists \( G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) that extends all mappings \( g_w \) so that \( G \) is the identity in a neighborhood of infinity, and for all \( w \in \mathcal{W}^{mN} \),
\[
G|K_w \setminus \bigcup_{u \in \mathcal{W}^N} K_{wu} : K_w \setminus \bigcup_{u \in \mathcal{W}^N} K_{wu} \to K'_w \setminus \bigcup_{u \in \mathcal{W}^N} K'_{wu}
\]
is a \((\lambda, \Lambda)\)-quasisimilarity with \(\lambda\) and \(\Lambda\) depending only on \(c\) and \(\eta\). Therefore \(G\) is BLD and by Lemma 2.2, \(G\) is \(L_3\)-bi-Lipschitz for some \(L_3\) depending only on \(L\). Thus, \(F = (\Phi')^{-1} \circ G \circ \Phi\) is \(L'\)-bi-Lipschitz for some \(L'\) depending only on \(L\) and \(c\) and extends \(f\).

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