On Holomorphic Curvature of Complex Finsler Square Metric

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Abstract
The notion of the holomorphic curvature for a Complex Finsler space \((M, F)\) is defined with respect to the Chern complex linear connection on the pull-back tangent bundle. This paper is about the fundamental metric tensor, inverse tensor and as a special approach of the pull-back bundle is devoted to obtaining the holomorphic curvature of Complex Finsler Square metrics. Further, it proved that it is not a weakly Kähler.

Keywords
Complex Square Metric, Holomorphic Flag Curvature, Riemannian Curvature

1. Introduction
The notion of holomorphic curvature of a complex Finsler space is defined with respect to the Chern complex linear connection in briefly Chern (c.l.c) as a connection in the holomorphic pull back tangent bundle \(\pi^*(T'M)\) (here \(\pi\) represented as projection). In [1], Nicolta Aldea has obtained the characterization of the holomorphic bisectional curvature and gave the generalization of the holomorphic curvature of the complex Finsler spaces which are called holomorphic flag curvature. After that in (2006) he devoted to obtaining the characterization of holomorphic flag curvature.

In complex Finsler geometry, it is systematically used the concept of holomorphic curvature in direction \(\eta\). But, the holomorphic curvature is not an analogue of the flag curvature from real Finsler geometry.

This problem sets up the subject of the present paper. Our goal is to determine the conditions in which complex Finsler spaces with square metric of holomorphic curvature. As per our claim, we shall use the holomorphic curvature of
complex Finsler spaces, with respect to Chern (c.l.c) on $\pi^*(TM)$ (definition (2.4) and (2.5)). We shall see that the fundamental metric tensor $g_\pi$ and its inverse are obtained (see in Section-3). Moreover, we determine the holomorphic curvature of complex square metric (theorem (4.3)) and some special properties of holomorphic curvature are obtained (proposition (4.4)).

2. Preliminaries

This section, includes the basic notions of Complex Finsler spaces.

An $\mathbb{R}$-Complex Finsler metric on $M$ is continuous function $F: TM \to \mathbb{R}$ satisfying:

1) $L = F^2$ is a smooth on $T'M / 0$;
2) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;
3) $F(z, \lambda \eta, \bar{\zeta}, \lambda \bar{\eta}) = |\lambda|F(z, \eta, \bar{\zeta}, \bar{\eta})$, for all $\lambda \in \mathbb{R}$.

Let $M$ be a complex manifold, $\dim M = n$ and $TM$ the holomorphic tangent bundle in which as a complex manifold the local coordinates will be denoted by $(z^k, \eta^k)$. The complexified tangent bundle of $TM$ is decomposed in $T(TM) = T'(TM) \oplus T''(TM)$, where operator $\oplus$ becomes direct sum.

Considering the restriction of the projection to $T'M = T'M / 0$, for pulling back of the holomorphic tangent bundle $TM$ then it obtain a holomorphic tangent bundle $\pi^*: \pi^*(TM) \to \tilde{T'M}$, called the pull-back tangent bundle over the slit $T'M$. We denote by $\partial z^k / \partial \eta^j$, the local frame and by $\{dz^k, d\bar{z}^k\}$ the local frame and its dual.

Let $V(TM) = \ker \pi_* \subset T'(TM)$ be the vertical bundle, spanned locally by $\partial / \partial \eta^k$. A complex nonlinear connection, briefly (c.n.c), determines a supplementary complex subbundle to $V(TM)$ in $T'(TM)$, that is $T'(TM) = H(TM) \oplus V(TM)$. The adapted frames $\delta z^k / \delta z^j = \partial / \partial \eta^j - N^j_i(z, \eta)$, where $N^j_i(z, \eta)$ are the coefficients of the (c.n.c). Further we shall use the abbreviations $\delta_i = \delta / \delta z^i$, $\partial_i = \partial / \partial \eta^i$, $\delta_\tau = \delta / \delta \tau$, $\partial_\tau = \partial / \partial \tau$, and their conjugates $[2]$ $[3]$ $[4]$.

On $TM$, let $g_\pi = \delta^i_\tau \delta^j_\tau \partial_{\tau} \partial_{\tau}$ be the fundamental metric tensor of a complex Finsler space $(M, L = F^2)$.

The isomorphism between $\pi^*(TM)$ and $TM$ induces an isomorphism of $\pi^*(T_cM)$ and $T_cM$. Thus, $g_\pi$ defines an Hermitian metric structure $G(z, \eta) = g_\pi d\bar{z}^k \otimes d\bar{z}^k$ on $\pi^*(TM)$, with respect to the natural complex structure. Further, the Hermitian metric structure $G$ on $\pi^*(TM)$ induces a Hermitian inner product $h(z, \gamma) := ReG(\chi, \bar{\gamma})$ and the angle
\[
\cos(\chi_r) = \frac{ReG(\chi, \overline{\chi})}{\|\chi\|\|\overline{\chi}\|},
\]
for any \( \chi, \gamma \) the sections on \( \pi^*(TM) \), where \( \|\chi\| = \|\overline{\chi}\| = G(\chi, \overline{\chi}) \) (for details see in [5]).

On the other hand, \( H(TM) \) and \( \pi^*(TM) \) are isomorphic. Therefore, the structures on \( \pi^*(T_cM) \) can be pulled-back to \( H(TM) \oplus H(TM) \). By this isomorphism the natural co-basis \( dz^i \) is identified with \( dz^i \). In view of this constructions the pull-back tangent bundle \( \pi^*(TM) \) admits a unique complex linear connection \( \nabla \), called the Chern (c.l.c), which is metric with respect to \( G \) and of \((0,1)\)-type.

\[
\omega^i_j (z, \eta) = L^i_{jk} (z, \eta) dz^k + C^i_{jk} (z, \eta) \delta \eta^k; \tag{2.1}
\]
The Chern (c.l.c) on \( \pi^*(TM) \) determines the Chern-Finsler (c.n.c) on \( (TM) \), with the coefficients \( N^i_k = g^{ik} \frac{\delta g_{\mu \nu}}{\delta z^k} \eta^\mu \), and its local coefficients of torsion and curvature are

\[
R^i_{jk} := L^i_{jk} - L^i_{kj}; \tag{2.2}
\]

\[
R^i_{jk} := -\delta^i - C^i_{jk} - \delta_\eta N^i_k; \quad \sigma^i_{jk} := -\delta^i - C^i_{jk} = \sigma^i_{kj};
\]

\[
L^i_{jk} = g^{i\mu} \frac{\delta g_{\mu \nu}}{\delta z^k} = g^{i\mu} \frac{\delta g_{\mu \nu}}{\delta \eta_j}; \quad C^i_{jk} = g^{i\mu} \frac{\delta g_{\mu \nu}}{\delta \eta_j};
\]

\[
P^i_{jk} := -\delta^i L^i_{jk} - \delta_\eta \big( N^i_k \big) C^i_j; \quad S^i_{jk} := -\delta^i C^i_{jk} = S^i_{kj}.
\]

The Riemann type tensor

\[
R(W, z, X, Y) := G \big( R(X, \overline{Y}, W) \big),
\]
has properties:

\[
R(W, Z, X, Y) = W^i Z^j X^k Y^l R_{ijkl} = R_{ijkl} g_{ij}; \tag{2.3}
\]

\[
R^i_{jk} = R_{kj}^i = R_{jk}^i = R_{ji}^k; \quad R^i_{jk} = R_{kj}^i = R_{jk}^i = R_{ji}^k;
\]

then \( R^i_{jk} = R^i_{kj} \) if and only if \( T^i_{jk} = T^i_{kj} \) Kähler if and only if \( T^i_{jk} \eta^j = 0 \) weakly Kähler if and only if \( g_{\overline{\eta}} T^i_{jk} \eta^j \overline{\eta} = 0 \). Note that for a complex Finsler metric which comes from a Hermitian metric on \( M \), so-called purely Hermitian metric. That is \( g_{\overline{\eta}} = g_{\overline{\eta}} (z) \), the three nuances of Kähler spaces consider, in [6].

The holomorphic curvature of \( F \) in direction \( \eta \), with respect to the Chern (c.l.c) is,

\[
\kappa_x (z, \eta) := \frac{2R(\eta, \overline{\eta}, \eta, \overline{\eta})}{G^2(\eta, \overline{\eta})} = \frac{2\overline{\eta}^i \eta^k R^i_{jk}}{E^2 (z, \eta)}, \tag{2.4}
\]

where \( \eta \) is viewed as local section of \( \pi^*(TM) \), that is \( \eta := \eta^i \frac{\partial}{\partial z^i} \). Further on,
we shall simply call it holomorphic curvature. It depends both on the position \( z \in M \) and the direction \( \eta \).

**Definition 2.1.** [7] The complex Finsler space \((M, F)\) is called generalized Einstein if \( R_{\overline{z}} \) is proportional to \( t_{\overline{z}} \), that is if there exists a real valued function \( K(z, \eta) \), such that

\[
R_{\overline{z}} = K(z, \eta)t_{\overline{z}},
\]

where \( R_{\overline{z}} := R_{\overline{z}j}^{\overline{h}} \eta^{k} = -g_{\overline{h}j} \delta_{\overline{k}}^{i}(N_{i}^{k}) \eta^{k} \), \( t_{\overline{z}} := L(z, \eta)g_{\overline{z}} + \eta_{i} \overline{\eta}_{j} \), \( \eta_{i} := \frac{\partial L}{\partial \eta^{i}} \), \( \overline{\eta}_{j} := \frac{\partial L}{\partial \overline{\eta}^{j}} \).

By finding the Chern (c.l.c) on \((\pi_{*}(TM))\) determines the Chern-Finsler on \( TM' \), with the coefficient \( N_{k}^{i} = g_{\overline{m}l} \frac{\partial g_{\overline{m}l}}{\partial \overline{\eta}^{k}} \eta^{l} \) determines, we need to find the fundamental metric tensor followed by the invariants are given below:

Now, from definition of Complex Finsler metric follows that \( L \) is \((2, 0)\)-homogeneous with respect to the real scalar \( \lambda \) and is proved that the following identities are fulfilled in [8].

\[
\frac{\partial L}{\partial \eta^{i}} - \frac{\partial L}{\partial \overline{\eta}^{i}} = 2L; \quad g_{\overline{h}j}^{i} + g_{\overline{h}j}^{i} \overline{\eta}^{i} = \frac{\partial L}{\partial \eta^{i}},
\]

\[
\frac{\partial g_{\overline{h}j}^{i}}{\partial \eta^{i}} - \frac{\partial g_{\overline{h}j}^{i}}{\partial \overline{\eta}^{i}} = 0; \quad \frac{\partial g_{\overline{h}j}^{i}}{\partial \eta^{i}} \eta^{l} + \frac{\partial g_{\overline{h}j}^{i}}{\partial \overline{\eta}^{i}} \overline{\eta}^{l} = 0,
\]

\[2Lg_{\overline{h}j}^{i} \eta^{i} + g_{\overline{h}j}^{i} \overline{\eta}^{i} + 2g_{\overline{h}j}^{i} \eta^{i} \overline{\eta}^{i},\]

where,

\[g_{\overline{h}} = \frac{\partial^{2}L}{\eta^{i} \eta^{j}}; \quad g_{\overline{h}j} = \frac{\partial^{2}L}{\eta^{i} \overline{\eta}^{j}}; \quad g_{\overline{h}j} = \frac{\partial^{2}L}{\eta^{i} \eta^{j}}.\]

Here, to find the inverse of fundamental metric tensor \( g_{\overline{h}} \) we use the following proposition.

**Proposition 2.1.** Suppose:

- \((Q_{\overline{h}})\) is a non-singular \( n \times n \) complex matrix with inverse \( Q'^{i} \);
- \( C_{i} \) and \( C'_{i} = \overline{C_{i}}, i = 1, \ldots, n \) are complex numbers;
- \( C' := Q'^{i} \overline{C_{j}} \) and its conjugates; \( C'^{2} := C'^{i} \overline{C_{j}} = \overline{C'^{i}} \overline{C^{j}} \); \( H_{\overline{h}} := Q_{\overline{h}} \pm C_{i} C_{j} \).

Then,

1) \( det(H_{\overline{h}}) = (1 \pm C^{2}) \) \( det(Q_{\overline{h}}) \) (Here, \( det \) indicates determinant),
2) whenever \( (1 \pm C^{2}) \neq 0 \), the matrix \( (H_{\overline{h}}) \) is invertible and in this case its inverse is \( H'^{i} = Q'^{i} \pm \frac{1}{1 \pm C^{2}} C'^{i} C'^{j} \).

### 3. Notation of Complex Square Metrics

The \( \mathbb{R} \)-complex Finsler space produce the tensor fields \( g_{\overline{h}} \) and \( g_{\overline{h}j} \). The tensor field must \( g_{\overline{h}} \) be invertible in Hermitian geometry. These problems are about to Hermitian \( \mathbb{R} \)-complex Finsler spaces, if \( det(g_{\overline{h}} \neq 0) \) and non-Hermitian
\textbf{\textit{\(\mathbb{R}\)-complex Finsler spaces}}, if \(\det(g_{ij}) \neq 0\). In this section, we determine the fundamental tensor of complex square metric and inverse also.

Consider \(\mathbb{R}\)-complex Finsler space with square metric,

\[
L(\alpha, \beta) = \left(\frac{\alpha + |\beta|}{\alpha}\right)^2
\]

(3.1)

then it follows that

\[
F = \left(\frac{\alpha + |\beta|}{\alpha}\right).
\]

Now, we find the following quantities of \(F\).

From the equalities (2.6) and (2.7) with metric (3.1), we have

\[
\alpha L_{\alpha} + \beta L_{\beta} = 2L, \quad \alpha L_{\alpha} + \beta L_{\alpha} = L_{\alpha},
\]

(3.2)

\[
\alpha L_{\alpha} + \beta L_{\alpha} = L_{\beta}, \quad \alpha^2 L_{\alpha} + 2\alpha \beta L_{\alpha} + \beta^2 L_{\alpha} = 2L,
\]

where

\[
L_{\alpha} = \frac{\partial L}{\partial \alpha}, \quad L_{\beta} = \frac{\partial L}{\partial \beta}, \quad L_{\alpha \beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta}, \quad L_{\alpha \alpha} = \frac{\partial^2 L}{\partial \alpha^2}, \quad L_{\beta \beta} = \frac{\partial^2 L}{\partial \beta^2}.
\]

(3.3)

\[
L_{\alpha} = \frac{4(\alpha + |\beta|)^3}{\alpha^2} - \frac{2(\alpha + |\beta|)^5}{\alpha^3},
\]

(3.4)

\[
L_{\beta} = \frac{2\alpha^4}{(\alpha - \beta)^3},
\]

(3.5)

\[
L_{\alpha \alpha} = 2\left(1 + \frac{|\beta|}{\alpha}\right)^2 \left(3 \frac{|\beta|^2}{\alpha^2} - \frac{2|\beta|}{\alpha} + 4\right),
\]

(3.6)

\[
L_{\beta \beta} = 12 \left(1 + \frac{|\beta|}{\alpha}\right)^2,
\]

(3.7)

\[
L_{\alpha \alpha} = 4 \left(1 + \frac{|\beta|}{\alpha}\right)^2 \left(1 - \frac{2|\beta|}{\alpha}\right),
\]

(3.8)

\[
\alpha L_{\alpha} + \beta L_{\beta} = \frac{2F}{\alpha} \left[\alpha - |\beta| + 2\alpha |\beta|\right],
\]

(3.9)

\[
\alpha L_{\alpha} + \beta L_{\alpha} = \frac{2F}{\alpha^2} \left[4\alpha^2 + 8|\beta|^3 - \alpha |\beta|(2\alpha + 3|\beta|) + 4\alpha |\beta|\right].
\]

(3.10)

We propose to determine the metric tensors of an \(\mathbb{R}\)-complex Finsler space using the following equalities:

\[
g_{ij} = \frac{\partial^2 L(z, \eta, \bar{\eta}, \lambda \tilde{\eta})}{\partial \eta^i \partial \eta^j}, \quad g_{\bar{\eta} i} = \frac{\partial^2 L(z, \eta, \bar{\eta}, \lambda \tilde{\eta})}{\partial \eta^i \partial \bar{\eta}^j}.
\]

Each of these being of interest in the following:

Consider,

\[
\frac{\partial \alpha}{\partial \eta^i} = \frac{1}{2\alpha} (a_i \eta^i + a_i \bar{\eta}^i), \quad \frac{\partial \beta}{\partial \eta^i} = \frac{1}{2} b_i.
\]
\[
\frac{\partial \alpha}{\partial \eta^i} = \frac{1}{2\alpha} \left( a_{\eta^i} + a_{\eta^i} \right) = \frac{\partial \beta}{\partial \eta^i} = \frac{1}{2} h_i,
\]

where,
\[
l_i = (a_{\eta^i} + a_{\eta^i}^2), \quad l_\tau = a_{\eta^i} + a_{\eta^i}.
\]

Then, we can find,
\[
l_i \eta^i + l_\tau \eta^i = 2\alpha^2.
\]

We denote:
\[
\eta^i = \frac{\partial L}{\partial \eta^i} = \frac{\partial}{\partial \eta^i} F^2 = 2F \frac{\partial}{\partial \eta^i} \left( \frac{\alpha^2}{\alpha - \beta} \right),
\]
\[
\eta_i = \rho_0^i + \rho_1^i,
\]

where
\[
\rho_0 = \frac{1}{2} \alpha^{-1} L_\alpha, \quad (3.11)
\]

and
\[
\rho_1 = \frac{1}{2} L_\beta. \quad (3.12)
\]

Differentiating \(\rho_0\) and \(\rho_1\) with respect to \(\eta^i\) and \(\eta^i\) respectively, which yields:
\[
\frac{\partial \rho_0}{\partial \eta^i} = \rho_2^i \eta_i + \rho_3^i b_i,
\]

and
\[
\frac{\partial \rho_0}{\partial \eta^i} = \rho_2^i \eta_i + \rho_3^i b_i.
\]

Similarly,
\[
\frac{\partial \rho_0}{\partial \eta^i} = \eta_i l_i + \mu_0 b_i, \quad \frac{\partial \rho_0}{\partial \eta^i} = \rho_2^i \eta_i + \mu_0 b_i,
\]

where
\[
\rho_2 = \frac{4L_{\eta^i}}{4\alpha^2}, \quad \rho_3 = \frac{L_{\beta^i}}{4\alpha}, \quad \mu_0 = \frac{L_{\mu^i}}{4}.
\]

By direct computation using (3.11), (3.12), (3.13), we obtain the invariants of \(\mathbb{R}\)-complex Finsler space with Square metric: \(\rho_0, \rho_1, \rho_2, \rho_3\) are given below:
\[
\rho_0 = \frac{1}{2\alpha} \left( 4(\alpha + |\beta|)^3 - \frac{2(\alpha + |\beta|)^4}{\alpha^2} \right), \quad (3.14)
\]
\[
\rho_1 = \frac{2(\alpha + |\beta|)^3}{\alpha^2}, \quad (3.15)
\]
\[
\rho_2 = 2\alpha \left( 1 + \left| \beta \right|^2 \right) \left( \frac{3|\beta|^2}{\alpha^2} - \frac{2|\beta|}{\alpha} + 4 \right) - \frac{4(\alpha + |\beta|)^3}{\alpha^3} + \frac{2(\alpha + |\beta|)^4}{\alpha^4},
\]
(3.16)

\[
\rho_1 = \left( 1 + \left| \beta \right|^2 \right) \left( 1 - \frac{2|\beta|}{\alpha} \right),
\]
(3.17)

\[
\mu_0 = 3 \left( 1 + \left| \beta \right|^2 \right).
\]
(3.18)

**Fundamental Metric Tensor of \( \mathbb{R} \)-Complex Finsler Space with Square Metric**

The fundamental metric tensors of \( \mathbb{R} \)-complex Finsler space with \((\alpha, \beta)\) metric are given by [9]:

\[
g_\sigma = \rho_0 a_\sigma + \rho_2 l_\sigma + \mu_0 b_\sigma + \rho_1 \left( b_\sigma l_\sigma + b_l l_\sigma \right)
\]
(3.19)

By using the Equations (3.14) to (3.18) in (3.19) we have

\[
g_{ij} = \frac{F}{\alpha} \left( 2 \left( 1 + \frac{|\beta|}{\alpha} \right) - F \right) a_{ij}
\]

\[+ \alpha F \left\{ 8\alpha^3 - 4\alpha^2 + 2\alpha |\beta| (3|\beta| + \alpha) - 4\alpha^3 |\beta| - 9\alpha |\beta| + \alpha F \right\} l_i l_j \]

\[+ \frac{3F^2 - \alpha^2 - 2\alpha |\beta|}{\alpha(\alpha - F)} b_i b_j + \frac{\alpha - 2|\beta|}{2F(\alpha^2 + |\beta|^2)(2 - \alpha^2 + |\beta|)} \eta_i \eta_j, \]
(3.20)

\[
g_{ij} = \frac{F}{\alpha} \left( 2 \left( 1 + \frac{|\beta|}{\alpha} \right) - F \right) a_{ij}
\]

\[+ \alpha F \left\{ 8\alpha^3 - 4\alpha^2 + 2\alpha |\beta| (3|\beta| + \alpha) - 4\alpha^3 |\beta| - 9\alpha |\beta| + \alpha F \right\} l_i l_j \]

\[+ \frac{3F^2 - \alpha^2 - 2\alpha |\beta|}{\alpha(\alpha - F)} b_i b_j + \frac{\alpha - 2|\beta|}{2F(\alpha^2 + |\beta|^2)(2 - \alpha^2 + |\beta|)} \eta_i \eta_j \]
(3.21)

Or, equivalently,

\[
g_{ij} = A a_{ij} + B l_i l_j + C b_i b_j + D \eta_i \eta_j,
\]
(3.22)

\[
g_{ij} = A a_{ij} + B l_i l_j + C b_i b_j + D \eta_i \eta_j,
\]
(3.23)

where,

\[
A = \frac{F}{\alpha} \left( 2 \left( 1 + \frac{|\beta|}{\alpha} \right) - F \right),
\]
(3.24)

\[
B = \alpha F \left\{ 8\alpha^3 - 4\alpha^2 + 2\alpha |\beta| (3|\beta| + \alpha) - 4\alpha^3 |\beta| - 9\alpha |\beta| + \alpha F \right\},
\]
(3.25)

\[
C = \frac{3F^2 - \alpha^2 - 2\alpha |\beta|}{\alpha(\alpha - F)},
\]
(3.26)

\[
D = \frac{\alpha - 2|\beta|}{2F(\alpha^2 + |\beta|^2)(2 - \alpha^2 + |\beta|)}.
\]
(3.27)
Next to determine the determinant and inverse of the tensor field \( g_\gamma \) through the theorem below by using Proposition (2.1). The solution of the non-Hermitian metric \( Q_\gamma \) as follows.

**Theorem 3.2.** For a non-Hermitian \( \mathbb{R} \)-Complex Finsler space with Square metric \( F = \left( \frac{\alpha + |\beta|}{\alpha} \right)^2 \), then they have the following:

1) The contravariant tensor \( g_{ij}^g \) of the fundamental tensor \( g_{ij} \) is:

\[
\begin{align*}
g_{ij}^g &= \frac{\alpha^3}{F \left( 2 \left( \alpha + |\beta| \right) \alpha - \alpha \left( |\beta| - 2 \alpha \right) \right)} \left[ Aa^a \left( \frac{\alpha FB}{\alpha^3 + \alpha FB} \right) \right. \\
& \quad + \frac{c (\alpha FBc)^2}{\delta (\alpha^3 + \alpha FB\mu)^2} \left( \eta^i \eta^j + \frac{C}{\delta} b^i b^j + \frac{FBc}{\delta (1 + B\mu)} \left( b^i \eta^j + \eta^i b^j \right) \right) \\
& \quad \left. + \frac{M^2 \eta^i \eta^j + MN \left( \eta^i b^j + b^i \eta^j \right) + N^2 b^i b^j}{L} \right]
\end{align*}
\]

where

\[
X = \left[ 1 + \left( \frac{B}{1 + B\mu} + \frac{CB^2 c^2}{\delta (1 + B\mu)} \right) \right] \mu + \frac{BCc}{\delta (1 + B\mu)^2},
\]

and

\[
Y = \frac{C}{\delta} + \frac{BCc\mu}{\delta (1 + B\mu)}.
\]

\[
det \left( a_{ij} + pl_j l_i + qb_j b_i + r\eta^i \eta^j \right)
\]

\[
= \left[ 1 + \left( X \mu - Yc \right) \sqrt{D} \right] \left[ 1 + \omega + \frac{Bc^2}{1 + B\mu} \right] \left( 1 + B\mu \right) A\det \left( a_\gamma \right)
\]

where, \( D = \frac{\alpha - 2 |\beta|}{2F \left( \alpha^2 + |\beta|^2 \right) \left( 2 - \alpha^2 + |\beta| \right)} \).

**Proof.** We prove this theorem by following three steps:

**Step 1:** We write \( g_\gamma \) from (3.21) in the form.

\[
g_\gamma = \left[ Aa_\gamma + Bl_\gamma + Cb_\gamma + D\eta^i \eta^j \right].
\]

We take \( Q_\gamma = a_\gamma \) and \( C_\gamma = \sqrt{Bl} \). By applying the proposition 2.1 we obtain \( Q_\gamma^2 = a_\gamma^2 \), \( C_\gamma^2 = C_\gamma C_\gamma \). So, the matrix \( H_\gamma = Aa_\gamma - Bl_\gamma \), is invertible with

\[
H_\gamma^2 = Aa^2 + \frac{1}{1 + AB\mu} \eta^i \eta^j,
\]

\[
det \left( Aa_\gamma + Bl_\gamma \right) = \left( 1 + B\mu \right) = A\det \left( a_\gamma \right).
\]

**Step 2:** Now, we consider

\[
Q_\gamma = a_\gamma + Bl_\gamma, \quad \text{and} \quad C_\gamma = \sqrt{C_\gamma},
\]

By applying the proposition 2.1 we have
\[ Q^\eta = Aa^\eta + \frac{B\eta'\eta^7}{1 + AB\mu}, \]

\[ C^2 = C_i C^\eta = Q^\eta \times C_7 = \sqrt{C_7} \left[ Aa^\eta + \frac{B\eta'\eta^7}{1 + AB\mu} \sqrt{C^7} \right], \]

\[ c^2 = C \left[ A\omega + \frac{Bc^2}{1 + AB\mu} \right]. \]

Therefore,

\[ 1 + C^2 = 1 + C \left[ A\omega + \frac{Bc^2}{1 + AB\mu} \right] \neq 0, \]

where, \( \epsilon = b_j\eta' \), \( \omega = b_jb'. \)

It results that the inverse of \( H_y = Aa_{\eta'} + Bl_{l_{\eta}} + Cb_{b_{\eta}} \) exists and it is

\[ H^\eta = Q^\eta + \frac{1}{1 + C^2} C^\eta C_7, \]

\[ H^\eta = Aa^\eta + \frac{B\eta'\eta^7}{1 + AB\mu} + \frac{B}{1 + AB\mu + \frac{B^2 C^2}{\tau(1 + AB\mu)^2}} \eta' \eta^7 \]

\[ + \frac{BCe}{1 + AB\mu} \left( b'\eta^7 + b^7\eta' \right) + \frac{C}{\tau} b' b^7, \]

where,

\[ \delta = 1 + C \left[ A\omega + \frac{Bc^2}{1 + AB\mu} \right], \]

and,

\[ \det \left[ Aa_{\eta'} + Bl_{l_{\eta}} + Cb_{b_{\eta}} \right] = \left[ 1 + C \left( A\omega + \frac{Bc^2}{1 + AB\mu} \right) \right] (1 + AB\mu) \det (a_{\eta'}). \]

Step 3: We put

\[ Q_{\eta'} = Aa_{\eta'} + Bl_{l_{\eta}} + Cb_{b_{\eta}}, \]

and

\[ C_{j} = \sqrt{D_{\eta}}, \]

clearly observe that and obtain

\[ Q^\eta = Aa^\eta + \left( \frac{B}{1 + AB\mu + \frac{B^2 C^2}{\tau(1 + AB\mu)^2}} \right) \eta' \eta^7 \]

\[ + \frac{BCe}{1 + AB\mu} \left( b'\eta^7 + b^7\eta' \right) + \frac{C}{1 + AB\mu} b' b^7, \]

and \( C_{j} = X\eta' + Yb^7 \), where
\( X = \left[ 1 + \left( \frac{B}{1 + B\mu} + \frac{B^2 C c^2}{\delta(1 + p \mu)^2} \right) \right]^{\mu} + \frac{pq e}{\delta(1 + B\mu)}, \) \hspace{1cm} (3.36)\\
\( Y = \frac{C}{\delta} + \frac{BC c\mu}{\delta(1 + B\mu)}. \) \hspace{1cm} (3.37)

And
\[
C^2 = (X\mu + Ye)\sqrt{D},
\]
\[
1 + C^2 = \left[ Aa^7 + \left( \frac{B}{1 + b\mu} + \frac{CB^2 c^2}{\delta(1 + B\mu)} + \frac{BC e}{1 + B\mu} + \frac{C}{1 + B\mu} \right) \right] \sqrt{D} \neq 0,
\]
clearly, the matrix \( H^\tau \) is invertible.

\[
C^i = Aa^7 + \left( \frac{B\eta^i\eta^7}{1 + AB\mu} \right) + \frac{C\left( b^i + \frac{B\eta^i e\eta^7}{1 + B\mu} \right)}{\delta} \eta^7,
\]
and
\[
C^7 = Aa^7 + \left( \frac{B\eta^7\eta^i}{1 + AB\mu} \right) + \frac{C\left( b^7 + \frac{B\eta^7 e\eta^i}{1 + B\mu} \right)}{\delta} \eta^i,
\]
where
\[
C^i C^7 = X^i \eta^7 + XY \left( \eta^i b^7 + \eta^7 b^i \right) + Y^2 b^i b^7.
\]
Again by applying Proposition (2.1) we obtain the inverse of \( H^\tau \) as:
\[
H^\tau = Aa^7 + \left( \frac{\alpha FB}{\alpha^2 + F\alpha B\mu} \right) + \frac{C(\alpha FBe)^2}{\delta(\alpha^2 + F\alpha B\mu)^2} \eta^i \eta^7 + \frac{C}{\delta} b^i b^7
\]
\[
+ \frac{FBCe}{\delta(1 + B\mu)} \left( b^i \eta^7 + b^7 \eta^i \right) + \frac{X^2 \eta^i \eta^7 + XYZ \eta^7 + X^2 b^i b^7 + Y^2 b^i b^7}{L}.
\]
\[
det \left( Aa^\tau + Bl^i, + Cb^7 + Dn^7 \right) = \left[ 1 + (X\mu + Ye)\sqrt{D} \right] \left[ 1 + A\omega + \frac{Bc^2}{1 + B\mu} \right] (1 + B\mu) \det (a^\tau).
\]
But \( g^\tau = AH^\tau \), with \( H^\tau \) from last step. Thus
\[
g^\tau = \frac{1}{A} H^\tau.
\]
Therefore, from Equation (3.38) in Equation (3.40) and the Equation (3.39), then we obtained claims 1) and 2) are desired. \( \square \)

4. Holomorphic Curvature of Complex Square Metric
The holomorphic curvature is the correspondent of the holomorphic sectional
curvature in Complex Finsler geometry. Our goal is to find a notation of Complex Finsler spaces with square metric. By analogy with the naming from the real case \[10\], we shall call it the holomorphic flag curvature and we shall introduce it with respect to Chern-Finsler connection (c.n.c).

The holomorphic curvature \( K_{w}(z, \eta) \) depends on the position \( z \in M \) alone. In view of definition (2.1) we obtain the holomorphic curvature of Complex Finsler space with square metric if \( R_{\eta} = -g_{\eta}\delta_{\eta}\left(N_{l}^{i}\right)^{\eta} \), where, \( N_{l}^{i} \) is the Chern-Finsler connection coefficients.

To find Riemannian curvature \( R_{\eta} \), we need the Chern Finsler connection (c.n.c) coefficients. Now, by direct computations, we get the Chern-Finsler (c.n.c) connection coefficients;

\[
N_{k}^{i} = \frac{\alpha^{4}}{(\alpha + |\beta|)^{2}} \left[ a^{7} + \left( \frac{\alpha FB}{\alpha^{3} + F \alpha B \mu} + \frac{C(\alpha FB \epsilon)^{2}}{\delta(\alpha^{3} + \alpha FB \mu)^{2}} \right) \eta \eta^{7} + \frac{C}{\delta} b^{i} b^{7} \right.
\]

\[
+ \frac{FB \epsilon}{\delta(1 + B \mu)} \left( b^{i} \eta^{7} + b^{\eta} \eta^{7} \right) + \frac{M^{2} \eta \eta^{7} + MN \left( \eta b^{7} + b^{i} \eta^{7} + N^{2} b^{b^{7}} \right)}{L}
\]

\[
\times \left\{ \frac{a^{8}}{\alpha^{2}} - F\left( A_{2}\right)_{ij} + 2 \left( \frac{\alpha^{4} - \alpha^{4} \cdot F}{\alpha} \right) A_{i} b^{j} b^{7} \right. \right.
\]

\[
+ \left( \frac{\alpha^{4} - 2\alpha^{4} |\beta|}{\alpha^{2}} - a_{i}(A_{4}) \right)_{ij} \eta \eta^{7} \right\}, \quad (4.1)
\]

where

\[
A = 2\alpha^{2} + 2|\beta|^{2} + 2\alpha^{2} |\beta|^{2} + 8\alpha^{2} |\beta| + 8\alpha |\beta|^{2} - \alpha^{2} - 3\alpha |\beta|^{2}
\]

\[
- 2\alpha^{2} |\beta|^{2} - 3\alpha^{2} |\beta| - 4\alpha^{2} |\beta| + \alpha^{2}
\]

\[
A_{i} = \left( 4\alpha^{2} + 2|\beta|^{2} + 12\alpha |\beta| + \frac{4|\beta|^{2}}{\alpha} - \frac{5\alpha^{3}}{2} + \frac{3|\beta|^{2}}{2} - 3\alpha |\beta|^{2} \right.
\]

\[
+ 2\alpha^{2} |\beta| - |\beta| \left. \right) \eta \eta^{7} + (8|\beta| + 4\alpha^{2} |\beta| + 8\alpha^{2} + 24\alpha^{2} |\beta|^{2}
\]

\[
+ 2\alpha |\beta|^{2} - 4\alpha^{3} |\beta| - 3\alpha^{2} - 4\alpha^{2} + 5|\beta|^{2} \right) \eta^{7}
\]

\[
A_{4} = \left( 12\alpha^{2} - 4\alpha + 3|\beta|^{2} + \alpha |\beta| - 6\alpha^{2} |\beta| - 9|\beta| + \alpha + 2|\beta| \right) \eta \eta^{7}
\]

\[
+ \left( 12 - 2\alpha^{2} - 4\alpha^{3} - 4|\beta| \right) \eta^{7}
\]

\[
A_{5} = \left( 3\alpha^{2} + \alpha |\beta|^{2} + 3|\beta| \right) \eta \eta^{7} + 12 \left( |\beta|^{3} + 3\alpha^{2} |\beta| + 3\epsilon |\beta| \right) \eta^{7}
\]

\[
A_{i} = 4\alpha^{2} - 2\alpha^{4} + 4\alpha^{2} |\beta|^{2} + 12\alpha^{4} |\beta| + 16\epsilon^{4} |\beta|^{2}
\]

\[
+ 2\alpha^{2} |\beta|^{4} + 4\alpha^{2} |\beta| - 2\alpha^{4} |\beta| + 8\alpha |\beta|^{2}
\]
\[
A'_i = \left(24\alpha^6 + 14\alpha^5 - 4\alpha^2 + 2|\beta|^4 + 4\left|\frac{\beta}{\alpha}\right| + 10\alpha|\beta|^2 \right)
+ 18\alpha|\beta|^3 \left(2\alpha^2|\beta|^2 + 20\alpha^3|\beta| + 4\alpha^2|\beta|ight)\eta^i \eta^7
+ \left(8\alpha^6 + 12\alpha^5 + 4\alpha + 2\alpha^4 + 32|\beta|^4 + 10|\beta|^3 + 36\alpha^3|\beta|^2 \right)\eta^i
\]

\[A_i = 2\alpha^3 \eta^i + 3\alpha \beta|\beta|^2 \eta^7.\]

Observed that the Equation (4.1) can be expressed by the identity as:

\[N^i_i = \text{Re}\left(N^i_i\right) + \text{Img}\left(N^i_i\right),\]

where,

\[
\text{Re}\left(N^i_i\right) = \frac{\alpha^4}{(\alpha + |\beta|)^3} \left[ \frac{\alpha \text{FB}}{\alpha^3 + F \alpha B \eta} + \frac{C(\alpha \text{FB})^2}{\delta(\alpha^3 + F \alpha B \mu)} \right] \eta^i \eta^7
+ \frac{M^2}{L} \left[ \alpha^3 A_i - A_i \left( A_i \left( \frac{3}{2} \alpha \eta \eta^7 \right) \right) \right] \eta^i \eta^7
+ \frac{M^2}{L} \eta^i \eta^7 \left[ \alpha^4 - 2\alpha^3 |\beta| A_i - A_i \left( A_i \right) \right] \eta^i \eta^7
\]

\[
\text{Img}\left(N^i_i\right) = \frac{\alpha^4}{(\alpha + |\beta|)^3} \left[ \frac{\alpha \text{FB}}{\alpha^3 + F \alpha B \eta} + \frac{C(\alpha \text{FB})^2}{\delta(1 + B \mu)} \right] \eta^i \eta^7
+ \frac{M^N}{L} \left( \eta^i b^7 + b^i \eta^7 + N^2 b^i b^7 \right) - F(A_i) l_i^7
+ 2 \left( \frac{\alpha^4 - \alpha^3 F}{\alpha} \right) A_i b_i^7.
\]

Now using Equation (4.1) and \((g^7)\) on \(R_{\alpha}\) (see definition (2.1)) we get the Riemann curvature tensor \(R_{\alpha}\) as,

\[
R_{\alpha} = \frac{\alpha^4}{(\alpha + |\beta|)^3} \left( D_{17} + D_{18} + D_{19} + D_{20} \right) b^i b^7 l_i l_7 \eta^i \eta^7
+ \left(D_{17} + D_{18} + D_{19} + D_{20}\right) b^i b^7 l_i l_7 \eta^i \eta^7
+ \left(D_{21} + D_{95} + D_{73}\right) \eta^i \eta^7 l_i l_7 \eta^i \eta^7
+ \left \{D_{120} + D_{121} + D_{231} + D_{232} - D_{233}\right \} \eta^i \eta^7 l_i l_7 \eta^i \eta^7
\]

\[\text{where} \]

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\[ D_{17} = CFA_2, \quad D_{18} = N^2FA_2, \quad D_{19} = \frac{F^2A_2BCe}{\alpha (1 + B\mu)}, \quad D_{20} = \frac{XYFA_2}{L}, \]

\[ D_{41} = \frac{3\lambda^2\epsilon x^2}{2x^4 (\lambda^2 + B\mu x^2 + FB\mu + B^2 x^4)}, \quad D_{42} = \frac{\alpha^2 A_2^2 F}{L (\alpha^2 + F\alpha B\mu)}, \]

\[ D_{43} = \frac{3X^2AB}{2(\alpha^2 + F\alpha B\mu)}, \quad D_{44} = \frac{3XYAB}{2(\alpha^2 + F\alpha B\mu)}, \]

\[ D_{45} = \frac{A^2 (\alpha^2 - 2\alpha^3 |\beta|) - A_1 A_2}{A^2 (\alpha^2 + F\alpha B\mu)}, \quad D_{46} = \frac{3Y^2ABF}{2(\alpha^2 + F\alpha B\mu)}, \]

\[ D_{47} = \frac{XYAFB}{\alpha^2 + F\alpha + F\alpha B\mu}, \quad D_{48} = \frac{ACBF}{\alpha^2 + F\alpha B\mu}, \]

\[ D_{53} = \frac{CF^2\alpha A_2}{\alpha^2 + F\alpha B\mu}, \quad D_{54} = \frac{N^2F^2A_2B}{\alpha^2 + F\alpha B\mu}, \quad D_{55} = \frac{XYAF^2A_2B}{L (\alpha^2 + F\alpha B\mu)}, \]

\[ D_{58} = \frac{2Ca^2F^2B^2\alpha (\alpha - F)}{(\alpha^2 + F\alpha B\mu)^3}, \quad D_{59} = \frac{2\left(\alpha^4F^2B^2 - \alpha^3B^3F^3\right)}{(\alpha^2 + F\alpha B\mu)^3}, \]

\[ D_{60} = 2A_2\alpha^2 (\alpha - F), \quad D_{61} = \frac{C^2FA_2}{\delta^2}, \quad D_{62} = \frac{Y^2FA_2C}{\delta}, \]

\[ D_{63} = \frac{F^2A_2BC^2\epsilon}{\delta (1 + B\mu)}, \quad D_{67} = \frac{C (\alpha F\beta X^2)^2 FA_2}{\delta L (\alpha^2 + F\alpha B\mu)^2}, \quad D_{68} = \frac{X^2FA_2}{L}, \]

\[ D_{69} = \frac{Y^2FA_2}{\delta L}, \quad D_{70} = \frac{X^2Y^2FA_2}{L}, \quad D_{66} = \frac{2\epsilon^2a^2 (\alpha F\beta)^2}{\delta^2 (\alpha^2 + F\alpha B\mu)^2} A_2, \]

\[ D_{81} = \frac{XYFA_2}{L}, \quad D_{95} = \frac{XYFC_2}{L^2}, \quad D_{73} = \frac{X^3YFA_2}{L^2}, \quad D_{20} = \frac{3XYAFBc}{2\alpha \delta (1 + B\mu)}, \]

\[ D_{121} = \frac{(\alpha^4 - 2\alpha^3 |\beta|) A_2^2 - A_1 A_2}{\delta (A_1^2) (1 + B\mu)}, \quad D_{237} = \frac{A_1 (FBXY)}{L \alpha^2 (\alpha^2 + F\alpha B\mu)}, \]

\[ D_{232} = \frac{3ACXY (F\beta c)^2}{2\alpha \delta L (\alpha^2 + f\alpha B\mu)^2}, \quad D_{233} = \frac{3CAXY}{2a^2 L}, \quad D_{239} = \frac{3BCXYAe}{2\alpha \delta L (1 + B\mu) 2a^2}, \]

\[ D_{235} = \frac{\alpha^2 A_X Y}{L^2}, \quad D_{236} = \frac{3X^3YA}{2a L}, \quad D_{237} = \frac{3X^2Y^2 A}{2a L}, \]

\[ D_{238} = \frac{XY (\alpha^4 - 2\alpha^3 |\beta|) A_2^2 - A_1 A_2}{(A_2)^2}, \quad D_{241} = \frac{ACXY}{\alpha L}, \]

\[ D_{242} = \frac{XY^2A}{L}, \quad D_{243} = \frac{a F^2BA_2 XY}{\alpha^2 + F\alpha B\mu}, \quad D_{244} = \frac{CXY (\alpha F\beta c)^2 A_2}{\delta L (\alpha^2 + F\alpha B\mu)^2}, \]

\[ D_{245} = \frac{M^2NFA_2}{L^2}, \quad D_{246} = \frac{X^2Y^2FA_2}{L^2}, \quad D_{247} = \frac{2CXY (\alpha F\beta c)^2 (\alpha - F)}{\delta L (\alpha^2 + F\alpha B\mu)^2}, \]

\[ D_{253} = \frac{2X^2Y^2 (\alpha - F)}{L^2}, \quad D_{254} = \frac{(\alpha^4 - 2\alpha^3 |\beta|) A_2^2 - A_1 A_2}{L (A_2)^2}, \]

\[ D_{255} = \frac{X^2Y^2 (\alpha - F)}{L^2}, \quad D_{256} = \frac{X^2Y^2FA_2}{L^2}, \quad D_{257} = \frac{2CXY (\alpha F\beta c)^2 (\alpha - F)}{\delta L (\alpha^2 + F\alpha B\mu)^2}, \]
\[ D_{255} = \frac{CX(YFαBc)^2}{\delta L(a^3 + FaB\mu)} (A_i)^2 \left(\alpha^4 - 2\alpha^3 |β| \right) \left( A_i^\prime - A_i A_i \right), \]

\[ D_{256} = \frac{FBcXY}{\delta (a^3 + FaB\mu)} (a_i)^2 \left(\alpha^4 - 2\alpha^3 |β| \right) \left( A_i^\prime - A_i A_i \right), \]

\[ D_{257} = \frac{X^2Y}{L^2(A_i)} \left(\alpha^4 - 2\alpha^3 |β| \right) \left( A_i^\prime - A_i A_i \right). \]

Notice that, on contracting with \((\eta)^j\) in \(g^7\). We get the above coefficients D-tensor.

Again, by using Chern-Finsler connection coefficients, we get the coefficients of torsion.

\[ T_{jk} = D_{1i}a_{ij} + \{ D_{2k} + D_{29} \} a_i^7 b^j \eta^j + \{ D_{30} + D_{31} \} a_i^7 b^j \eta^j \delta^j \]

\[ + \{ D_{32} + D_{33} \} a_i^7 b^j \eta^j \delta^j + \{ D_{55} + D_{57} \} a_i^7 b^j \eta^j \delta^j + \{ D_{60} \} \eta^j b^j l_j \eta^j \eta \eta^j \]

\[ + \{ D_{73} + D_{74} + D_{185} + D_{186} \} \delta^j b^j \eta^j + \{ D_{92} \} \eta^j b^j l_j \eta^j \]

\[ + \{ D_{90} \} \eta^j b^j l_j \eta^j + \{ D_{113} + D_{117} \} b^j \eta^j \]

\[ + \{ D_{126} \} \eta^j b^j b^j \eta^j + \{ D_{127} \} \eta^j b^j b^j \eta^j + \{ D_{128} \} b^j b^j a_i^7 b \eta^j \]

\[ + \{ D_{129} \} b^j b^j a_i^7 b \eta^j + \{ D_{133} + D_{134} \} \left( \eta^j b^j b \eta^j + \eta^j b^j b \eta^j \right) \delta^j \]

\[ + \{ D_{135} \} \left( \eta^j b^j l_j b^j \eta^j + \eta^j b^j l_j b^j \eta^j \right) + \{ D_{142} \} \left( \eta^j b^j b \eta^j + \eta^j b^j \eta^j \delta^j \right) \]

\[ + \{ D_{145} \} \delta^j \left( b^j \eta^j + b^j b^j \eta^j \right) + \{ D_{92} + D_{93} \} \delta^j \left( b^j b^j \eta^j + b^j b^j \eta^j \right) \]

\[ + \{ D_{90} + D_{91} \} \eta^j b^j \eta^j b^j + \{ D_{231} \} b^j \eta^j \]

\[ + \{ D_{248} \} \eta^j b^j l_j \eta^j \left( \eta^j b^j + b^j \eta^j \right) + \{ D_{251} \} a_i^7 b \eta^j \left( \eta^j b^j + b^j \eta^j \right) \]

\[ + \{ D_{260} + D_{261} \} \delta^j \left( b^j \eta^j b^j + b^j \eta^j \right), \] (4.4)

where,

\[ D_1 = \frac{3A^2}{2\alpha^3}, \quad D_{28} = D_{29} = M^2 \left( \frac{\alpha^4 - 2\alpha |\beta| a_i - A_i A_i}{A_i^2} \right), \]

\[ D_{30} = \frac{FBc}{\delta (1 + B\mu)} \left( \frac{\alpha^4 - 2\alpha |\beta| a_i - A_i A_i}{A_i^2} \right), \]

\[ D_{32} = D_{33} = \frac{MN}{L} \left( \frac{\alpha^4 - 2\alpha |\beta| a_i - A_i A_i}{A_i^2} \right), \]

\[ D_{56} = D_{57} = \frac{MNF^2 A_i aB}{L^3 (a^3 + FaB\mu)}, \quad D_{60} = 2A_i \left( \frac{\alpha^4 - \alpha^3 F}{a} \right), \]

\[ D_{73} = \frac{N^2 aFB \left( \alpha^3 - 2\alpha |\beta| a_i - A_i A_i \right)}{L (a^3 + FaB\mu) A_i^2}, \quad D_{74} = \frac{A_i C}{\delta a}, \]

\[ D_{185} = \frac{M^2 N}{L^2} \left( \frac{\alpha^4 a_i - 2\alpha^3 |\beta| a_i - A_i A_i}{(a_i)^2} \right), \]

\[ D_{186} = \frac{CM^2}{\delta L} \left( \frac{\alpha^4 a_i - 2\alpha^3 |\beta| a_i - A_i A_i}{(a_i)^2} \right), \quad D_{88} = \frac{aF^2 BCA_i}{\delta (a^3 + FaB\mu)}. \]
\[ D_{96} = \frac{2C^2 (\alpha F B c) \left( \alpha^2 - \alpha^3 F \right)}{\delta^2 \alpha \left( \alpha^2 + FA B \mu \right)^2}, \quad D_{98} = \frac{2C \left( \alpha^2 - \alpha^3 F \right)}{\delta \alpha} A_1, \]

\[ D_{97} = \frac{2 \left( \alpha^4 - \alpha^3 F \right) FB}{\delta \alpha \left( \alpha^2 + FA B \mu \right)} A_1, \quad D_{116} = \frac{3AF BC e}{2 \delta \alpha \delta \left( 1 + B \mu \right)}, \quad D_{229} = \frac{3AMN}{2 \alpha^3} L, \]

\[ D_{113} = \frac{3ABC e F}{2 \delta \alpha \delta \left( 1 + B \mu \right)}, \quad D_{117} = \frac{A F BC e}{2 \delta \alpha \left( 1 + B \mu \right)}, \quad D_{126} = \frac{3N^2 AF BC e}{2 \delta \alpha \delta \left( 1 + B \mu \right)}, \]

\[ D_{127} = \frac{M N A F BC L}{\delta \alpha \delta \left( 1 + B \mu \right)}, \quad D_{129} = \frac{A C^2 F B c}{\delta \alpha \left( 1 + B \mu \right)}, \quad D_{133} = \frac{N^2 F^2 A B C e}{\delta \left( 1 + B \mu \right)}, \]

\[ D_{134} = \frac{F^2 A_1 \left( BC e \right)^2}{\delta \left( 1 + B \mu \right)}, \quad D_{135} = \frac{MN F^2 A B C e}{L \delta \left( 1 + B \mu \right)}, \]

\[ D_{143} = \frac{2 \left( F B c \right)^2 \left( \alpha^2 - \alpha^3 F \right)}{\delta \alpha \left( 1 + B \mu \right)} A_1, \quad D_{149} = \frac{F B c \left( \alpha^4 - 2 \alpha^3 \left| \beta \right| \right) A_1 - A_1 A_4}{\delta \left( 1 + B \mu \right)} \left( A_1 \right)^2, \]

\[ D_{150} = \frac{M N F B C e \left( \alpha^4 A_4 - 2 \alpha^3 \left| \beta \right| \right) A_1 - A_1 A_4}{L \delta \left( 1 + B \mu \right) \left( A_1 \right)^2}, \quad D_{230} = \frac{A M N}{L \alpha}, \]

\[ D_{211} = \frac{3AMN}{2 \alpha^2 L}, \quad D_{215} = \frac{A_1 \left( F B M N \right)}{L \alpha \left( \alpha^2 + FB \mu \right)}, \]

\[ D_{218} = \frac{2CM N \left( \alpha F B c \right)^2 \left( \alpha^2 - \alpha^3 F \right)}{\delta \alpha \left( \alpha^2 + FA B \mu \right)} A_1, \quad D_{251} = \frac{2CM N \left( \alpha^4 - \alpha^3 F \right)}{\delta \alpha^3} A_1, \]

\[ D_{260} = \frac{M^2 N^2 \left( \alpha^4 - 2 \alpha^3 \left| \beta \right| \right) A_1 - A_1 A_4}{\delta L \left( 1 + B \mu \right) \left( A_1 \right)^2}, \quad D_{261} = \frac{F B c \left( \alpha^4 - 2 \alpha^3 \left| \beta \right| \right) A_1 - A_1 A_4}{\delta L \left( 1 + B \mu \right) \left( A_1 \right)^2}. \]

**Theorem 4.3.** The holomorphic flag curvature of Complex Square metric

\[ F = \left( \frac{\left( \alpha + \left| \beta \right| \right)^2}{\alpha} \right) \] is given by,

\[ K_F \left( Z, \eta \right) = \alpha^2 \left( \alpha + \left| \beta \right| \right)^2 \bar{\eta} \eta^4 Re \left( N_1^i \right), \quad (4.5) \]

where,

\[ N_1^i = \frac{\alpha^4}{\left( \alpha + \left| \beta \right| \right)^2} \left[ \alpha F B c \left( \frac{3}{2} \alpha \eta \eta^7 \right) \right] \left( \frac{\alpha F B c}{\delta \left( \alpha^2 + \alpha F B \mu \right)} \right) \eta^4 \eta^7 \]

\[ + \frac{M^2}{L} \left\{ \frac{\alpha^3 A_1 - A_1 \left( \frac{3}{2} \alpha \eta \eta^7 \right)}{\alpha^3} \right\} \eta^4 \eta^7 \]

\[ + \frac{M^2}{L} \eta \eta^7 \left[ \frac{\alpha^4 - 2 \alpha^3 \left| \beta \right| A_1 - A_1 A_4}{\left( A_1 \right)^2} \right] \eta^4 \eta^7 \]
Proof. From Equation (4.2) plugging into (2.1), it yields

\[ K_f(z, \eta) = \frac{2}{L^2}(z, \eta) g_L \left[ \delta_z \left( N_i^i \right) \eta^i \eta^i \right], \tag{4.6} \]

where, \( N_i^i \) is in Equation (4.1).

Then, comparing (4.6) with (4.2) we get (4.5) as desired. \( \square \)

**Proposition 4.4.** If \( F = \left( \frac{\alpha + |\beta|^2}{\alpha} \right) \) be Complex Square metric of dimension \( n \geq 2 \) with non-zero \( K_f(z, \eta) \), then it is not a Kähler and not a weakly Kähler.

Proof. Observing Equation (4.3) \( T_{j^k}^{i} \) is non zero and since by definition it is not a Kähler. Further, on contracting \( T_{j^k}^{i} \) by \( \eta^i \), it yields

\[ T_{j^k}^{i} \eta^i \neq 0. \]

Therefore, it is not a weakly Kähler. \( \square \)

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**Conflicts of Interest**

The authors declare no conflicts of interest regarding the publication of this paper.

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