A NEW APPROACH TO THE EQUIVARIANT TOPOLOGICAL COMPLEXITY

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ABSTRACT. We present a new approach to equivariant version of the topological complexity, called a symmetric topological complexity. It seems that the presented approach is more adequate for the analysis of an impact of symmetry on the motion planning algorithm than the one introduced and studied by Colman and Grant. We show many bounds for the symmetric topological complexity comparing it with already known invariants and prove that in the case of a free action it is equal to the Farber’s topological complexity of the orbit space. We define the Whitehead version of it.

1. Introduction

A topological invariant introduced by Farber in [6, 7], and called the topological complexity, was the first to estimate a complexity of motion planning algorithm. With the configuration space $X$ of a mechanical robot he associated a natural number $TC(X)$ called topological complexity of $X$. To be more precise he considered the natural fibration

$$\pi: PX \to X \times X$$

from the free path space in $X$ which assigns to a path $\gamma$ defined on the unit interval its ends $(\gamma(0), \gamma(1))$. The topological complexity is the least $n$ such that $X \times X$ can be covered by $n$ open sets $U_1, \ldots, U_n$ such that for each $i$ there is a homotopical section $s_i: U_i \to PX$ to $\pi$. This invariant is a special case of the well known Lusternik-Schnirelmann (or LS for short) category of $X \times X$ (cf [3] for more detailed exposition of this notion and other references).

In this paper we discuss the following question: If the mechanical robot admits a symmetry with respect to a compact Lie group (and therefore the configuration space $X$ admits it too) what is an appropriate definition of the topological complexity that takes into account that symmetry? An answer

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is not that simple as it may look like and it is not unique. We define an invariant, different than the equivariant topological complexity introduced by H. Colman and A. Grant in [4], called the symmetric topological complexity. By showing its properties we would like to demonstrate that in many situations it is better than that of [4].

Let $G$ be a compact Lie group. Let us assume that $X$ is a $G$ space, i.e. $G$ acts continuously on $X$ (therefore we assume that $G$ is the "symmetry group" that appears in $X$). The formulae for topological complexity uses the natural fibration $1.1$. If the space $X$ is a $G$ space then $PX$ is a $G$ space in a natural way, and so does $X \times X$ by the diagonal action. It would be natural to define the equivariant complexity by assuming that all maps are $G$ maps. Actually this approach has been studied in [4]. We will use the notation introduced there $TC_G(X)$ to denote this invariant. In spite of its mathematical naturalness this approach has some disadvantages that we present below.

![Picture 1. A symmetric robot arm with an action of $\tau$](image1)

Let us consider a mechanical robot arm that admits a symmetry. For simplicity let us assume that $G = \mathbb{Z}/2 = \{1, \tau\}$ as showed in the picture [1]. The element $\tau$ acts by interchanging the part $A$ of the arm with $B$. Assume we are given a path $\xi$ between points $x$ and $y$ in the configuration space $X$, as noted in picture [2].

![Picture 2. A path in configuration space](image2)

Note that although points $x$ and $\tau x \in X$ are distinct in the configuration space there is no physical difference between these two states of a mechanical
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Therefore it is natural to require that the path $\xi$ determines a path between $\tau x$ and $\tau y$ – namely $\tau \xi$. This natural requirement leads us to a definition of equivariant topological complexity $TC_G(X)$. On the other hand if the task the mechanical arm is supposed to perform is symmetric we would like the path $\xi$ to determine the following four paths – between $x$ and $y$, $\tau x$ and $\tau y$ as well as between $x$ and $\tau y$, $\tau x$ and $y$. In other words we would like to exploit the $G \times G$ structure of the space $X \times X$. The main problem is that usually $PX$ is not a $G \times G$ space. We will show in section 3 how to deal with this problem by defining so called symmetric topological complexity, $STC_G(X)$.

Unlike many other equivariant versions of numerical invariants the equivariant topological complexity does not have the required mathematical properties – for example when the group $G$ acts freely on $X$ then in general $TC_G(X) \neq TC(X/G)$ where $X/G$ is the orbit space and $TC(X/G)$ is the topological complexity of $X/G$. We will show that in our case $STC_G(X) = TC(X/G)$. A bridge to apply advanced homotopy theory in the theory of Lusternik-Schnirelmann category is the Whitehead version of it (cf. [18]). We will show that for the symmetric topological complexity we can define a Whitehead version of it and for a finite group $G$ it gives the original symmetric topological complexity. We conjecture that the same holds for any compact Lie group. Finally we provide examples which distinguish the equivariant topological complexity and the symmetric topological complexity and calculate the latter in several cases.

2. LUSTERNIK-SCHNIRELMANN CATEGORY

2.1. Basic definitions. In this section we define and give some basic properties of a version of an equivariant Lusternik-Schnirelmann category for topological spaces that we will use later on in our considerations. We shall the standard notations of the theory of compact Lie group transformations of [2].

Let $G$ be a Lie group and let $A$ be a closed $G$ subset of a $G$ space $X$.

**Definition 2.1.** A metrizable $G$-space $X$ we call a $G$-ANR if for every equivariant imbedding $\iota : X \rightarrow Y$ as a closed $G$-subset of a metrizable $G$-space $Y$ there exists a $G$ neighborhood $U$ of $\iota(X)$ in $Y$ and a continuous equivariant retraction $U \rightarrow \iota(X)$.

Throughout this paper we assume that $X$ is a compact $G$-ANR (see [16] for the properties of $G$-ANRs). The class of $G$-ANRs includes $G$-ENRs.
(cf. [11] for the definition), countable $G$-CW complexes thus smooth $G$-manifolds with smooth action of $G$.

**Definition 2.2.** We call on open $G$ set $U \subseteq X$ *$G$-compressable* into $A$ whenever the inclusion map $\iota_U: U \subseteq X$ is $G$ homotopic to $c: U \to X$ such that $c(U) \subseteq A$.

This allows us to define our main tool

**Definition 2.3.** An $A$-Lusternik-Schnirelmann $G$-category of a $G$ space $X$ is the least $n$ such that $X$ can be covered by $U_1, \ldots, U_n$ open $G$ subsets of $X$ each $G$-compressable into $A$. We denote in by $A \text{cat}_G(X)$.

Remind, we say that a $G$-space $X$ is $G$-path-connected if for every closed subgroup $H \subset G$ the space $X^H$ is path-connected.

Note that we have a relation to the standard Lusternik-Schnirelmann category. By $*$ we denote a fixed one point subset of $X$ provided it is invariant, i.e $* \in X^G$.

**Remark 2.4.** If $X$ is path connected and $G$ is the trivial group then we have

$$*_\text{cat}_G(X) = \text{cat}(X).$$

If $* \in X^G$ and $X^H$ is path connected for all closed subgroups $H \subseteq G$ then

$$*_\text{cat}_G(X) = \text{cat}_G(X)$$

where $\text{cat}_G(X)$ denote the equivariant Lusternik-Schnirelmann category of $X$ (cf. [14]).

This version of the LS category has many similarities to the standard category. We say that $(X, A)$ $G$-dominates $(Y, B)$ if there are $G$-maps

$$f: (X, A) \to (Y, B) \text{ and } g: (Y, B) \to (X, A)$$

such that $fg \simeq \text{id}_{(Y, B)}$ in the equivariant topological category of pairs of spaces.

**Theorem 2.5.** If $(X, A)$ $G$-dominates $(Y, B)$, then

$$A \text{cat}_G(X) \geq_B \text{cat}_G(Y).$$

**Proof.** The proof is similar to that of Lemma 1.29 in [3] after a suitable change of categories. \qed
2.2. **The Whitehead definition of the category.** As the classical LS category, the defined above notion of a category (Definition 2.3) has its Whitehead counterpart.

We recall that a pair of $G$-spaces $A \subset X$ is called a $G$-cofibration (or the Borsuk pair) if it has the equivariant homotopy extension property, i.e. for any $G$-space $Y$ and $G$-homotopy $h : A \times I \rightarrow Y$ there exists an equivariant homotopy $H : X \times I \rightarrow Y$ extending $h$.

**Definition 2.6.** Let $A \subseteq X$ be a closed $G$-cofibration with $A$ invariant. By a fat $A$-sum we mean for every $n \in \mathbb{N}$ a $G$-space $F^n_A(X) \subset X^n := X \times \ldots \times X$ defined as follows:

- $F^1_A := A$
- $F^n_A(X)$ is the colimit in the category of $G$-spaces of the following diagram:

\[
\begin{array}{ccc}
A \times F^{n-1}_A(X) & \longrightarrow & X \times F^{n-1}_A(X) \\
\downarrow & & \downarrow \\
A \times X^{n-1} & \longrightarrow & F^n_A(X)
\end{array}
\]

**Definition 2.7.** We say that the $G$-Whitehead $A$-category, denoted by $\text{Acat}^\text{Wh}_G(X)$, is less or equal $n$ if and only if there is a $G$-mapping $\xi_n : X \rightarrow F^n_A(X)$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_n} & F^n_A(X) \\
\downarrow & & \downarrow \\
X^n & \Rightarrow & X^n
\end{array}
\]

where $\Delta_n : X \rightarrow X^n$ is the diagonal mapping.

**Theorem 2.8.** Let $X$ be a $G$-space and $\iota : A \subseteq X$ closed $G$-cofibration. Then $\text{Acat}_G(X) =_A \text{cat}^\text{Wh}_G(X)$

Before giving the proof of the theorem (which follows the proof of theorem 1.55 in [3]) we need a technical lemma:

**Lemma 2.9.** Under the assumptions of the theorem, if $\{U_i\}_{i=1}^n$ is an invariant open covering of $X$ such that for each $i$ there exists $G$-map $s_i : U_i \rightarrow A$ such that $G_i : \iota \circ s_i$ is $G$-homotopic to $(U_i \subseteq X)$ then there exist an open and invariant covering $\{V_i\}_{i=1}^n \subseteq \{U_i\}_{i=1}^n$ such that for each $i$ there exists a $G$-homotopy $H_i : X \times I \rightarrow X$ with $H_i(x,0) = x$ for each $x \in X$ and $H_i(x,1) = \iota \circ s_i$ for $x \in V_i$. 
Proof. Here \( \{V_i\}_{i=1}^n \leq \{U_i\}_{i=1}^n \) means that for every \( 1 \leq i \leq n \), \( V_i \subset U_i \). By a direct argument, we can find invariant coverings \( \{V_i\} \) and \( \{W_i\} \) of \( X \) such that

\[
V_i \subset \bar{V}_i \subset W_i \subset \bar{W}_i \subset U_i.
\]

Since \( X \) is a \( G \)-ANR, \( X/G \) is normal. Moreover \( (\bar{V}_i/G) \cap ((X \setminus W_i)/G) = \emptyset \) in \( X/G \). Consequently, by normality of \( X \) there exists a \( G \)-invariant continuous function \( \lambda: X \to I \) be such that \( \lambda(\bar{V}_i) = 1 \) and \( \lambda(X \setminus W_i) = 0 \). For each \( i \) we define the \( G \)-homotopy by:

\[
H_i: X \times I \ni (x,t) \mapsto \begin{cases} x, & x \in X \setminus W_i \\ G_i(x, t \cdot \lambda(x)), & x \in \bar{W}_i \end{cases}
\]

□

Remark 2.10. Of course the converse implication in the lemma above also holds. Given a family of \( G \)-homotopies \( H_i \) with an invariant covering \( \{V_i\} \) of \( X \) it is sufficient to set \( s_i := H_i(\cdot, 1)|_{V_i} \).

Proof of the theorem 2.8. If \( \text{cat}_G(X) \leq n \) then we have \( n \) \( G \)-homotopies \( H_i: X \times I \to X \) satisfying conditions of the lemma 2.9. Now to show that \( \text{cat}_G^{Wh}(X) \leq n \) it is sufficient to put \( \xi_n: X \ni x \mapsto (H_i(x, 1))_{i=1}^n \in F^n_X \).

Conversely, if \( \text{cat}_G^{Wh}(X) \leq n \) then we are given \( \xi_n: X \to F^n_X \) such that \( \Delta \) and \( (F^n_A(X) \subseteq X^n) \circ \xi_n \) are \( G \)-homotopic by a homotopy \( \zeta \). We denote by \( \zeta^i \) the \( i \)-th coordinate of \( \zeta \). Since \( A \subseteq X \) is a \( G \)-cofibration then there exists \( N = N(A) \) an invariant and open neighborhood of \( A \) in \( X \) such that \( A \) is a \( G \)-deformation retract of \( N \). Let us denote this equivariant deformation retraction by \( R \). Then \( R(x, 0) = x \) and \( R(x, 1) \in A \) for \( x \in N \). Set \( U_i := H^{-1}(N, 1) \). It is easy to see that \( \{U_i\} \) is an invariant open covering of \( X \). Moreover setting

\[
H_i: X \times I \ni (x,t) \mapsto \begin{cases} \zeta^i(x, 2t), & 0 \leq t \leq 1/2 \\ R(\zeta^i(x, 1), 2t - 1), & 1/2 \leq t \leq 1. \end{cases}
\]

we obtain the required family of \( G \)-homotopies with \( s_i: U_i \to A \) equal to \( H_i(\cdot, 1)|_{U_i} \). □

For the case without symmetry, i.e. \( G = e \), we have (comp. [10, Chapter 4]) that a space \( X \) is \( n \)-connected if \( \pi_i(X) = 0 \) for all \( i \leq n \). Likewise, a pair \( (X; A) \) is \( n \)-connected if \( \pi_i(X; A) = 0 \) for all \( i \leq n \). A natural analog of the definition in the equivariant case is the following:
Definition 2.11 (comp. [13], definition I.2.1). We call a $G$-space $X$ $G$-n-connected if $X^G \neq \emptyset$ and $\pi_i(X^H) = 0$ for all $i \leq n$ and all closed subgroups $H \subset G$. Likewise, a $G$-pair $(X, A)$ is $n$-connected if $A^G \neq \emptyset$ and $\pi_i(X^H, A^H) = 0$ for all $i \leq n$ and all closed subgroups $H \subset G$.

The following fact is well-known in the non-equivariant case, e.g. [10, Proposition 4.13 and Corollary 4.16]. Since we could not find any direct reference of the equivariant case we reprove the CW approximation theorem as stated in [13, theorem XI.3.6] making some minor changes.

Proposition 2.12. If $(X, A)$ is an $n$-connected $G$-CW pair for a discrete $G$ then there exists a $G$-CW pair $(Z; A) \sim (X; A)$ rel $A$ such that all cells of $Z \setminus A$ have dimension greater than $n$.

Proof. We construct a family of $G$-CW complexes $A \subseteq Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$ together with maps $\gamma_i : (Y_i, A) \rightarrow (X, A)$ such that $\pi_q(\gamma_i)$ is a surjection for $q > i + n$ and an isomorphism for $q \leq i + n$. Let $b \in A^G$. We choose a representative map

$$s^q_H : (I^{q+1}, \partial I^{q+1}) \rightarrow (X^H, A^H)$$

for each element of $\pi_q(X^H, A^H, b)$ where $q > 0$ and $H$ runs over conjugacy classes of closed subgroups of $G$. Let $Y_0$ be the disjoint union of spaces $G/H \times I^q$, one for each chosen $s^q_H$ and of $A$. Let $\gamma_0$ be a map induced by all $s^q_H$. For each $s^q_H : (I^q, \partial I^q) \rightarrow (X^H, A^H)$ we identify each $x \in \partial I^q$ with $s^q_H(x) \in A$ in $Y_0$ hence obtaining

$$\gamma_0 : Y_0 := (\bigsqcup_{s^q_H} G/H \times I^q) \cup_{\bigsqcup_{s^q_H} \partial I^q} A.$$

Note that $\gamma_0^H$ is an isomorphisms on $\pi_i$ for $i \leq n$ and a surjection for $i > n$. Indeed, surjectivity is obvious for all $i$, for injectivity for $i \leq n$ let $H$ be a closed subgroup of $G$. Note that $(Y_0)^H$ is of the form $(\bigsqcup_{s^q_H} I^q) \cup_{\bigcup_{s^q_H} \partial I^q} A^H$ where the sum is taken over all $s^q_H$, such that there is a $G$-map $G/H \rightarrow G/H'$. We have a cofibration

$$A^H \rightarrow (Y_0)^H \rightarrow \bigvee_{s^q_H} S^q$$

which shows that $H_i((Y_0)^H, A^H)$ for $i \leq n$. Therefore by [19, corollary 7.10] we get that $\pi_i((Y_0)^H, A^H) = 0$.

Now assume that $\gamma_i : (Y_i, A) \rightarrow (X, A)$ has been constructed. We choose representative maps $(f, g)$ for each pair of elements in $\pi_q((Y_i)^H, A^H, b)$ that are equalized by $\pi_q(\gamma_i)$ (note that then $q > i + n$). Using the cellular approximation theorem [13, theorem 3.4] we assume that $f$ and $g$ have image...
in the $q$ skeleton of $Y_i$. Let $(Y_{i+1}, A)$ be the homotopy coequalizer of the disjoint union of all such maps – i.e. $(Y_{i+1}, A)$ is obtained by attaching $G/H_+ \wedge (I^q, \partial I^q) \times I_+$ via each chosen pair. Note that such operation does not affect $\pi_*(Y_i, A)$ for $* \leq d+i$ and kills the kernel of $\pi_{i+d+1}(\gamma_i)$. We define $\gamma_{i+1}$ with the use of homotopies $\gamma_i f \simeq \gamma_i g$ based at $b$. Now it is enough to triangulate $Y_{i+1}$ as a $G$-CW complex containing $Y_i$. We set $(Z, A) = \cup_i (Y_i, A)$ and $\gamma = \cup \gamma_i : (Z, A) \to (Y, A)$.

For a compact, i.e. finite $G$-CW complex $X$ by $\dim_G X$ we mean the maximal dimension of $G$-cells of the form $G/H \times I^n$ that appear in the construction of $X$. Consequently, $\dim_G X = \dim X/G$ where the dimension on the right hand side is the cell dimension equals to the covering dimension of $X/G$. Furthermore, if $G$ is discrete then $\dim_G X = \dim X = \dim X/G$.

**Theorem 2.13.** Let $A \subseteq X$ be a pair of $G$-CW-complexes for a discrete $G$. pair $(X, A)$ is $G$ $n$-connected then $\text{Acat}(X) \leq \dim_G X/(n+1) + 1$.

**Proof.** Since $(X, A)$ is $G$ $n$-connected we may assume that $X \setminus A$ has no $k$ dimensional $G$-cells for $k \leq n$. Then $F^n_A(X)$ and $X$ have similar $s(n+1) - 1$ skeleton. Let $s$ satisfy $(s-1)(n+1) \leq \dim_G X \leq s(n+1)$ and using the equivariant cellular approximation (comp. [13, theorem 3.4]) theorem we get that the diagonal map $\Delta_s : X \to X^s$ is $G$ homotopic to a $G$ cellular map $\xi : X \to F^s_A(X)$.

At the end of this subsection we prove a technical lemma that will be used later on to prove the product formula for the category (theorem 2.16).

**Lemma 2.14.** Let $X$ and $Y$ be $G$ spaces and $A \subseteq X$ and $B \subseteq Y$ its closed $G$ subsets. There is a commutative diagram

$$
\begin{array}{ccc}
F^n_A(X) \times F^m_B(Y) & \longrightarrow & F^{n+m-1}_A(X \times Y) \\
\downarrow & & \downarrow \\
X^n \times Y^m & \xrightarrow{\omega^{n,m}} & (X \times Y)^{n+m-1}
\end{array}
$$

such that $\omega^{n,m} \circ (\Delta_n(X) \times \Delta_m(Y)) = \Delta_{n+m-1}(X \times Y)$.

**Proof.** We prove the theorem inductively. If $(n, m) = (n, 1)$ then our diagram is of the form

$$
\begin{array}{ccc}
F^n_A(X) \times B & \longrightarrow & F^n_{A \times B}(X \times Y) \\
\downarrow & & \downarrow \\
X^n \times Y & \xrightarrow{n \omega} & (X \times Y)^n
\end{array}
$$
and it is easy to see that it is commutative whenever we set
\[ \alpha(x_1, \ldots, x_n, b) = (x_1, \ldots, x_n, b, \ldots, b). \]
The condition on the diagonal is also satisfied. Similar argument prove the statement for \((n, m) = (1, m)\). Now let us assume that \(n, m \geq 2\). Since in the category of \(G\) CW complexes the product of two pushouts is a pushout of products therefore we have a pushout
\[
\begin{array}{c}
A \times B \times F^n_A(X) \times F^m_B(Y) \rightarrow X \times Y \times F^n_A(X) \times F^m_B(Y).
\end{array}
\]
We get a commutative diagram
\[
\begin{array}{ccc}
A \times B \times F^n_A(X) \times F^m_B(Y) & \rightarrow & A \times B \times F^{n+m-1}_A(X \times Y) \\
\downarrow & & \downarrow \\
X \times Y \times F^n_A(X) \times F^m_B(Y) & \rightarrow & X \times Y \times F^{n+m-1}_A(X \times Y) \\
\downarrow & & \downarrow \\
A \times B \times X^n \times Y^m & \rightarrow & A \times B \times (X \times Y)^{n+m-1} \\
\downarrow & & \\
F^{n+1}_A(\times) \times F^{m+1}_B(\times) & \rightarrow & F^{n+1}_A(\times) \times F^{m+1}_B(\times)
\end{array}
\]
where \(\eta\) is the universal map between two pushouts. The whole diagram is over the map
\[
\omega^{n+1, m+1} := \text{id}_{X \times Y} \times \omega^{n, m} : X \times Y \times X^n \times Y^m \rightarrow X \times Y \times (X \times Y)^{n+m-1}
\]
and the assertion follows. \(\square\)

2.3. Bounds for the category.

**Lemma 2.15.** Let \(X\) be a \(G\) set, \(H \subseteq G\) closed subgroup and assume that \(A \subseteq B\) are its closed invariant subsets. Then:

1) \(b\text{cat}_G(X) \leq_A \text{cat}_G(X);\)
2) \(a\text{cat}_G(X) \leq_B \text{cat}_G(X) \cdot_A \text{cat}_G(B);\)
3) \(A/G\text{cat}(X/G) \leq_A \text{cat}_G(X);\)
4) \(A\text{cat}_H(X) \leq_A \text{cat}_G(X);\)

**Proof.** For the proof of 2 let us assume that \(b\text{cat}_G(X) \leq n\) and \(a\text{cat}_G(B) \leq m\). Let \(U_1, \ldots, U_n\) be open invariant subsets of \(X\), each compressible by \(s_i\) into \(B\) and \(V_1, \ldots, V_m\) open invariant subsets of \(B\), each compressible by \(t_j\) into \(A\). Let
\[
W^j_i := s_i^{-1}(V_j).
\]
We know that \( \{ W_j^i \} \) is an invariant open covering of \( X \). We define \( r_j^i := t_j \circ s_i|_{W_j^i} \) then it can be readily seen that \( W_j^i \) is compressable into \( A \) by \( r_j^i \). Since the cardinality of \( \{ W_j^i \} \) is \( n \cdot m \) the inequality follows.

The rest of the points are obvious (or were mentioned before). \( \Box \)

The category behaves well (i.e. similar to the standard LS category) under taking products.

**Theorem 2.16.** Let \( X \) and \( Y \) be two \( G \) spaces, \( A \subseteq X \), \( B \subseteq Y \) their closed \( G \) subsets. Then

\[
A \times B \text{cat}_G(X \times Y) \leq A \text{cat}_G(X) + B \text{cat}_G(Y) - 1
\]

where \( X \times Y \) is given the diagonal action.

**Proof.** We prove the theorem using lemma 2.14. Note that whenever we have a commutative diagrams

\[
\begin{align*}
X & \xrightarrow{\xi_n} F_A^n(X) \xrightarrow{\Delta_n(X)} X^n \\
Y & \xrightarrow{\xi_m} F_B^m(Y) \xrightarrow{\Delta_m(Y)} Y^m
\end{align*}
\]

Then commutative is also the following diagram

\[
\begin{align*}
X \times Y & \xrightarrow{\xi_n \times \xi_m} F_A^n(X) \times F_B^m(Y) \xrightarrow{\Delta_n(X) \times \Delta_m(Y)} (X \times Y)^{n+m} \\
& \downarrow \subseteq \downarrow \subseteq \\
X^n \times Y^m & \xrightarrow{\omega^{n,m}} (X \times Y)^{n+m-1}
\end{align*}
\]

Now \( \omega^{n,m} \circ (\Delta_n(X) \times \Delta_m(Y)) = \Delta_{n+m-1}(X \times Y) \) which ends the proof. \( \Box \)

**Remark 2.17.** Note that we do not need any additional assumption for the action of \( G \) on \( X \) and \( Y \). In case \( A \) and \( B \) consists of a single point and \( X \) and \( Y \) are \( G \) connected our result is equivalent to that obtained in [4, theorem 3.15] nevertheless our approach is much more general.

**Theorem 2.18.** Let \( X \) be a \( G \) space and \( Y \) be a \( H \) space for Lie groups \( G \) and \( H \). Then for a closed \( G \) subspace \( A \subseteq X \) and a closed \( H \) subset \( B \subseteq Y \) we have (where we consider \( X \times Y \) as a standard \( G \times H \) space)

\[
A \times B \text{cat}_{G \times H}(X \times Y) \leq A \text{cat}_G(X) + B \text{cat}_H(Y) - 1.
\]

**Proof.** Follows directly from 2.16 since we can consider \( X \) as a \( G \times H \) space with trivial \( H \) action and \( Y \) as a \( G \times H \) space with trivial \( G \) action. \( \Box \)

We end this section with a remark concerning the category of \( H \) invariant elements for a closed subgroup \( H \subseteq G \).
Theorem 2.19. Let $X$ be a $G$ space, $A \subseteq X$ its closed $G$ subset, $H$ closed subgroups of $G$ then

$$\mathcal{A}^H \text{cat}_H(X^H) \leq \text{cat}_G(X).$$

Proof. If $U \subseteq X$ is $G$ compressable into $A$ then $V := U \cap X^H$ is $H$ compressable into $A^H$ (which follows from the equivariant condition for the homotopy). $\Box$

3. Topological robotics in presence of a symmetry

3.1. Basic classical concepts. Let $X$ be a topological space with an action of a compact Lie group $G$. Consider the space of all continuous paths $s: I \to X$ with compact open topology denoted by $PX$. $PX$ posses a natural action of $G$.

Observe that the natural projection

$$p: PX \ni s \mapsto (s(0), s(1)) \in X \times X$$

is a continuous, $G$-equivariant $G$-fibration. Whenever we talk about $X \times X$ we consider it as a $G$-space (via the diagonal action) unless explicitly stated.

Definition 3.1. By a motion planning algorithm on an open set $U \subseteq X \times X$ we mean a section

$$s: U \to PX$$

of the fibration $p$, i.e. $p \circ s = (U \subseteq X \times X)$.

Definition 3.2. An equivariant motion planning algorithm on an open set $U \subseteq X \times X$ is a $G$-equivariant section

$$s: U \to PX$$

of the $G$-fibration $p$, i.e. $p \circ s = (U \subseteq X \times X)$.

An invariant motion planning algorithm is a motion planning algorithm of the orbit space.

3.2. Farber’s topological complexity.

Definition 3.3. Topological complexity of $X$, denoted by $TC(X)$, is the smallest $n$ such that $X \times X$ can be covered by $U_1, \ldots, U_n$ – open subsets such that for each $i$ there exists $s_i: U_i \to PX$ a motion planning algorithm on $U_i$.

Similarly, equivariant topological complexity, denoted by $TC_G(X)$, of a $G$-space $X$ is the smallest $n$ such that $X \times X$ can be covered by $U_1, \ldots, U_n$
– invariant open subsets such that for each \( i \) there exists \( s_i: U_i \to PX \) an equivariant motion planning algorithm on \( U_i \).

Note that equivariant motion planning algorithm does not have to induce an invariant one – free path space is not \( G \times G \)-space unlike \( X \times X \).

**Example 3.4.** Let \( X = G = S^1 \) with \( G \) acting by multiplication from the left. Note that \( X/G \) is trivial so that \( TC(X/G) = 1 \) whereas \( p: (S^1)^I \to S^1 \times S^1 \) cannot have a section, in particular cannot have an equivariant one, so \( TC_G(X) \geq 2 \). This shows that topological complexity of an orbit space and equivariant topological complexity does not have to coincide, even in the simplest examples of a free action.

Our aim is to give a suitable definition of a motion planning algorithm in an equivariant setting which induces an invariant motion planning algorithm and as mentioned in the introduction have a reasonable geometric meaning. Moreover we want this motion planning algorithms to give a topological complexity which coincides with the topological complexity of an orbit space for free \( G \) spaces. In order to do so we will translate it into the language of the Lusternik-Schnirelmann category.

Let \( \Delta: X \to X^n \) be the diagonal. We denote the image of \( \Delta_2 \) in \( X \times X \) by \( \Delta(X) \).

**Remark 3.5.** Let \( X \) be a \( G \) space. The map

\[
\pi: PX \to X \times X
\]

is a \( G \)-fibration (satisfies the homotopy lifting property for \( G \)-maps).

**Proof.** Since \( \{0,1\} \subseteq I \) is a closed \( G \) cofibration (satisfies \( G \) equivariant version of homotopy extension property) where we consider \( I \) as a trivial \( G \) space and functors \( map_G(I,-) \) and \( - \times I \) as well as \( map_G(\{0,1\},-) \) and \( - \times \{0,1\} \) are conjugate the proof follows from [19] theorem 7.8. \( \square \)

**Lemma 3.6.** For a \( G \)-space \( X \) the following statements are equivalent:

1) \( TC_G(X) \leq n \);
2) there exist \( n \) invariant open sets \( U_1, \ldots, U_n \) which cover \( X \times X \) and \( s_i: U_i \to PX \) such that \( p \circ s_i \) is \( G \)-homotopic to \( id \) (as mappings \( U_i \to X \times X \));
3) \( \Delta(X)cat_G(X \times X) \leq n \).

**Proof.** 1)\( \Rightarrow \)2) is obvious.

2)\( \Rightarrow \)1). Let \( s: U \to PX \) be such that \( H: p \circ s \simeq (\cdot U \subseteq X \times X) \) (as \( G \)-maps), where \( p: PX \to X \times X \). Then from the equivariant homotopy
lifting property there exists a $G$-homotopy $\hat{H}: U \times I \rightarrow PX$ such that the following diagram is commutative:

$$
\begin{array}{c}
U \times \{0\} \xrightarrow{s} PX \\
\downarrow \subseteq \downarrow \hat{H} \\
U \times I \xrightarrow{H} X \times X
\end{array}
$$

now it is sufficient to set $\bar{s}(a, b) := \hat{H}(a, b; 1)$.

2)⇔3). Let $H: PX \times I \rightarrow PX$ be given as:

$$
H(\omega; t)(s) = \omega(s(1 - t)) \text{ for } \omega \in PX, s, t \in I.
$$

It is a $G$-deformation retraction between $PX$ and $\iota(X) \subseteq PX$, where $\iota(x) \equiv x$ assigns to every point $x \in X$ the constant map defined by it; $\iota$ in this case is a $G$-homeomorphism onto the image. Now given $\bar{s}: U \rightarrow PX$ we can compose it with $p \circ H_1$ to get $\hat{s}: U \rightarrow \Delta(X) G$-homotopic in $X$ to the inclusion $U \subseteq X \times X$. On the other hand given $\hat{s}: U \rightarrow X$ we see that $\hat{s} = \Delta_2 \circ \hat{s}'$, where $\hat{s}': U \rightarrow X$ is a $G$-map. We can compose it with $\iota$ to get $\bar{s}: U \rightarrow PX$ such that $p \circ \bar{s}$ is homotopic in $X \times X$ to the inclusion $U \subseteq X \times X$. Note that $\Delta_2 = p \circ \iota$. These processes are mutually inverse up to $G$-homotopy so that we proved the equivalence.

Hence we obtained a characterization of topological complexity in terms of a suitable version of LS-category.

3.3. **Symmetric topological complexity.** The main problem arising from geometric interpretation, as mentioned in the introduction, is that $PX$ is not a $G \times G$-space – which is equivalent to the problem that $\Delta(X)$ is not a $G \times G$-space. But the latter can be easily fixed.

For a given $G$-space $X$ by $\mathcal{T}(X)$ we denote the saturation $\mathcal{T}(X) := (G \times G) \cdot \Delta(X) \subseteq X \times X$ of the diagonal with respect to the group $G \times G$.

Now instead of $\Delta(X)$ in the definition of equivariant topological complexity we consider $\mathcal{T}(X) \subseteq X \times X$ and instead of considering open subsets $G$-compressable into $\Delta(X)$ we consider open subsets of $X \times X$ which are $G \times G$-compressable into $\mathcal{T}(X)$.

**Definition 3.7.** For a $G$-space $X$ we define symmetric topological complexity as

$$
STC_G(X) =_{\mathcal{T}(X)} cat_{G \times G}(X \times X).
$$
One should distinguish this notion with the symmetric motion planning algorithms studied in [8] where a natural symmetry (action) of the group $\mathbb{Z}/2$ comes from the time reverse of the motion.

Let us state a lemma similar to 3.6 but formulated for the symmetric topological complexity. For a $G$ space $X$ we consider $PX \times_{\gamma(X)} PX := \{ (\gamma, \delta) \in PX \times PX : G \cdot \gamma_1 = G \cdot \delta_0 \}$ as a $G \times G$ space with the obvious multiplication $(g_1, g_2) \cdot (\gamma, \delta) = (g_1 \gamma, g_2 \delta)$. Note that we have a natural $G \times G$ map $p: PX \times_{\gamma(X)} PX \to X \times X$ given by $p(\gamma, \delta) = (\gamma(0), \delta(1))$.

**Remark 3.8.** Let $X$ be a $G$ space. The map $p: PX \times_{\gamma(X)} PX \to X \times X$ is a $G \times G$-fibration (satisfies the homotopy lifting property for $G$-maps).

**Proof.** Note that $\{ (0, 0), (0, 1), (1, 0), (1, 1) \} \subseteq I \times I$ is a closed $G \times G$-cofibration where we consider $\{ (0, 0), (0, 1), (1, 0), (1, 1) \}$ and $I \times I$ as trivial $G \times G$ spaces. Therefore from [19] theorem 7.8 the following map $p: (X \times X)^{I \times I} \to X \times X \times X \times X$ where $p(f) = (f(0, 0), f(0, 1), f(1, 0), f(1, 1))$ is a $G \times G$ fibration. We know also that any projection from the product of two $G \times G$ spaces is $G \times G$ fibration and that the composition of two $g \times G$ fibrations is a $G \times G$ fibration. Now it is enough to note that $p = p^{1,4} \circ \overline{p}_{\gamma(X) \times X}: \overline{p}^{-1}(X \times \neg(X) \times X) \to X \times X$ which ends the proof. \qed

**Lemma 3.9.** For a $G$-space $X$ the following statements are equivalent:

1) $STC_G(X) \leq n$;

2) there exist $n$ $G \times G$ invariant open sets $U_1, \ldots, U_n$ which cover $X \times X$ and $G \times G$ maps $\bar{s}_i: U_i \to PX \times_{\gamma(X)} PX$ such that $p \circ \bar{s}_i$ is $G \times G$-homotopic to id (as mappings $U_i \to X \times X$);

3) $\gamma(X) \text{cat}_{G \times G}(X \times X) \leq n$.

**Proof.** 1) $\iff$ 3) by the definition.

3) $\Rightarrow$ 2). Let $\iota_U: U \subseteq X \times X$ be an open $G \times G$ invariant subset that in $G \times G$ compressible into $\neg(X)$. Let $H: \iota_U \simeq e$ be a $G \times G$ homotopy where
c(U) ⊆ Ⅎ(X). From the equivariant homotopy lifting property we get that
\[ \begin{array}{ccc}
U \times \{0\} & \xrightarrow{s} & PX \times \gamma(X) \times PX \\
\subseteq & \xrightarrow{\hat{H}} & \downarrow \downarrow \\
U \times I & \xrightarrow{H} & X \times X \\
\end{array} \]

where \( s(u_1, u_2) = (c_{u_1}, c_{u_2}) \) and \( c_u \) is the constant path equal to \( u \). Now it is enough to set \( s_i := \hat{H}(-, 1) \).

2) \( \Rightarrow \) 3). Let \( H: PX \times \gamma(X) \times PX \times I \rightarrow PX \times \gamma(X) \times PX \) be given as:

\[ H(\gamma, \delta, t)(\tau, \tau') = (\gamma(\tau + t(1 - \tau)), \delta((1 - t)\tau)). \]

It is a \( G \times G \) deformation retraction between \( PX \times \gamma(X) \times PX \) and \( \iota(\gamma(X)) \subseteq PX \times \gamma(X) \times PX \) where \( \iota \) assigns to \( (u_1, u_2) \) the constant maps defined by it. Then if \( s_i: U \rightarrow X \) is a \( G \times G \) map such that \( F: p \circ s_i \simeq id_U \) for a \( G \times G \) homotopy \( F \) then \( U \) is \( G \times G \) compressible into \( \gamma(X) \) as

\[ id_U \simeq p \circ s_i \sim p \circ id_{PX \times \gamma(X) \times PX} \circ s_1 \simeq p \circ H(-, 1) \circ s_i \]

and \( H(-, 1) \circ s_i: U \rightarrow \gamma(X) \). □

**Remark 3.10.** For a \( G \)-space \( X \) we have inequality \( TC(X/G) \leq STC_G(X) \).

One of our main requirements was that our version of equivariant topological complexity of \( X \) should be equal to the topological complexity of the orbit space \( X/G \). The symmetric topological complexity satisfies this condition.

**Theorem 3.11.** Let \( X \) be a free \( G \)-space. Then \( TC(X/G) = STC_G(X) \).

Let us recall the Covering Homotopy Theorem of Palais:

**Theorem 3.12.** Let \( G \) be a compact Lie group, \( X, Y \) \( G \)-spaces, \( f: X \rightarrow Y \) a \( G \)-equivariant map. Denote by \( f': X/G \rightarrow Y/G \) the map induced by \( f \). Let \( F': X/G \times I \rightarrow Y/G \) be a homotopy which preserves the orbit structure and starts at \( f' \). Then there exists an equivariant homotopy \( X \times I \rightarrow Y \) covering \( F' \) starting at \( f \).

**Proof of theorem 3.11.** Assume that \( TC(X/G) \leq n \) then there exists a \( G \times G \) invariant covering \( U_1, \ldots, U_n \) of \( X \times X \) and \( s_i: U_i \rightarrow \Delta(X/G) \) such that \( s_i \) is homotopic to \( U_i \subseteq X \) via the homotopy \( H: U_i \times I \rightarrow X \times X \) (we assume it starts at the identity). Since the action of \( G \times G \) is free on \( X \times X \), the homotopy \( H \) preserves the orbit structure. Hence from the Covering Homotopy Theorem we get a \( G \times G \)-equivariant homotopy \( \tilde{H}: U_i \times I \rightarrow X \times X \) starting at \( U_i \subseteq X \times X \). For the orbit map \( \pi: X \times X \rightarrow X/G \times X/G \)
we have $\pi^{-1}(\Delta(X/G)) = \mathfrak{T}(X)$ hence $G \times G$-map $\tilde{s}_i: U_i \to \mathfrak{T}(X)$ can be defined by the formula $\tilde{s}_i(z) = \tilde{H}(z, 1)$. 

As a direct consequence of computation of the topological complexity of real projective space by M. Farber, S. Tabachnikov and S. Yuzvinsky ([9]) we get the following

**Corollary 3.13.** If $n \neq 1, 3, 7$, then $STC_{Z_2}(S^n)$ is equal to the smallest $k$ for which $\mathbb{R}P^n$ admits an immersion in $\mathbb{R}^{k-1}$.

3.4. **Whitehead symmetric topological complexity.** From the classical theory (comp. [5] for the non-equivariant case, [12] for short explanation how to pass to the equivariant one) we get that the map

$$\Delta(X) \subseteq X \times X$$

for a $G$-CW complex $X$ is a closed $G$-cofibration; nevertheless the case of

$$\mathfrak{T}(X) \subseteq X \times X$$

and a question if it is a $G \times G$ cofibration is much more complicated and we do not know the answer for a general Lie group $G$. Here we will prove it for a finite $G$.

**Corollary 3.14.** From the theorem 2.8, lemma 3.6 and the remark above we get that

$$TC_G(X) = \Delta(X) \text{ cat}^W_G(X \times X)$$

In particular for the classical topological complexity we get that

$$TC(X) = \Delta(X) \text{ cat}(X \times X) = \Delta(X) \text{ cat}^Wh(X \times X).$$

Moreover if $\mathfrak{T}(X) \subseteq X \times X$ is a closed $G \times G$-cofibration then

$$STC_G(X) = \mathfrak{T}(X) \text{ cat}_G(X \times X) = \mathfrak{T}(X) \text{ cat}^W_{G \times G}(X \times X).$$

Let us investigate closely the question if $\mathfrak{T}(X) \subseteq X \times X$ is a $G \times G$ cofibration. Since we assume that $X$ is a compact $G$-CW-complex we have that $X$ is a compact $G$-ANR, i.e.

**Definition 3.15.** A metrizable $G$-space $X$ we call a $G$-ANR if for every equivariant imbedding $\iota: X \to Y$ as a closed $G$-subset of a metrizable $G$-space $Y$ there exists a $G$ neighborhood $U$ of $\iota(X)$ in $Y$ and a continuous equivariant retraction $U \to \iota(X)$.

It is known that a countable $G$-CW complex is a $G$-ANR.
**Definition 3.16.** A pair \((Y, X)\) of \(G\)-spaces \(X \overset{G}{\rightarrow} Y\) is called \(G\)-cofibration if for any topological space \(Z\), any \(G\)-map \(f : Y \rightarrow Z\) every \(G\)-homotopy \(f : X \times I \rightarrow Z\), \(f_0 = f|_X\) extends to a \(G\)-homotopy \(F_t : Y \times I \rightarrow Z\).

We can use as well an equivalent formulation

**Theorem 3.17.** Let \(G\) be a compact Lie group. A compact metrizable space \(G\)-space \(X\) is a \(G\)-ANR if and only if every pair \((Y, X)\) is a closed \(G\)-cofibration for \(Y\) and \(X\) metrizable, \(X\) closed and invariant in \(Y\).

Jaworowski proved the following theorem

**Theorem 3.18.** Let \(G\) be a compact Lie group. A compact \(G\)-space is a \(G\)-ANR if and only if for every closed subgroup \(H \subset G\) the fixed point space \(X^H\) is an ANR.

Denote by \(\tilde{X}\) the Cartesian product \(X \times X\). Recall that \(\tilde{X}\) possesses a natural action of the group \(\tilde{G} := G \times G\) induced by the action of \(G\) on \(X\).

We will show that \(\overline{\Delta}(X)\) is a \(\tilde{G}\)-ANR and consequently \(\overline{\Delta}(X) \subseteq \tilde{X}\) is a \(\tilde{G}\)-cofibration as follows from Proposition 3.17

**Theorem 3.19.** Let \(G\) be a compact Lie group and \(X\) a compact \(G\)-ANR. Then \(\overline{\Delta}(X)\) is a \(\tilde{G}\)-ANR.

We are able to show the statement under an additional assumption that \(G\) is finite.

**Proof.** We assume that \(G\) is finite. First observe that \(\overline{\Delta}(X)\) can be represented as the saturation of \(\Delta(X) \subseteq \tilde{X}\) with respect to the action of group \(G_1 := G \times \{e\} \subseteq \tilde{G}\), i.e.

\[
\overline{\Delta}(X) = \{(gx, x) : g \in G, x \in X\}
\]

Indeed, since \(G_1 \subseteq G\), \(\{(gx, x) : g \in G, x \in X\} \subseteq \overline{\Delta}(X)\). On the other hand any \(z = (x_1, x_2) = (g_1x_1, g_2x_2)\) can be represented as \((\tilde{g}x, x)\) where \(y = g_2x\) and \(\tilde{g} = g_1g_2^{-1}\). This shows that \(\overline{\Delta}(X) \subseteq \{(gx, x)\}\).

Of course \(\overline{\Delta}(X)\) is \(\tilde{G}\)-invariant closed subset of \(\tilde{X}\) as a continuous image of the compact space \(\tilde{G} \times \Delta(X)\). In view of the Jaworowski theorem (3.18) it is sufficient to show that for every closed subgroup \(\tilde{H} \subseteq \tilde{G}\) the space \(\overline{\Delta}(X)^{\tilde{H}}\) is an ANR.

Let \(h = (h_1, h_2) \in \tilde{H}\). A point \((gx, x)\) belongs to \(\overline{\Delta}(X)^{\tilde{H}}\) (or equivalently to \(X^{(h)}\), \(\{h\}\) the cyclic group generated by \(h\)) if and only if \(h(gx, x) = (gx, x)\). The latter is equivalent to \(h_2x = x\) and \(h_1gx = gx\). The first equality gives \(x \in X^{h_2}\), and the second \(gx \in X^{h_1}\). Since \(G_{gx} = gG_xg^{-1}\) the latter
means that \( x \in X^{g^{-1}h_1g^{-1}} \). Consequently \( (gx, x) \in \mathcal{T}(X)^h \) if and only if \( x \in X^{h_2} \cap X^{g^{-1}h_1g^{-1}} \).

Next note that \( X^h \cap X^{h'} = X^{\{h, h'\}} \), where \( \{h, h'\} \) is a subgroup generated by \( h \) and \( h' \). Indeed since \( h \subseteq \{h, h'\}, h' \subseteq \{h, h'\} \), \( X^{\{h, h'\}} \subseteq X^h \) and \( X^{\{h, h'\}} \subseteq X^{h'} \), thus \( X^{\{h, h'\}} \subseteq X^h \cap X^{h'} \).

Conversely, if \( x \in X^h \cap X^{h'} \) then \( x \in X^{h_1} h_1^{i_1} \cdots h_1^{i_1} h_1^{i_2} \) for any word \( h_1^{i_1} h_1^{i_2} \cdots h_1^{i_1} h_1^{i_2} \). This means that \( x \in X^{\{h, h'\}} \), thus \( X^h \cap X^{h'} \subseteq X^{\{h, h'\}} \).

From it follows that given \( g \in G \) for \( \mathcal{T}(X) \) := \{\( (gx, x) \), \( x \in X \) and \( h = (h_1 h_2) \in H \) the fixed point set \( \mathcal{T}(X)^h \) is equal to

\[
\mathcal{T}(X)^h = X^{\{h', h_2\}}
\]

where \( h' = gh_1g^{-1} \). But such a set is an ANR, because \( X \) is a \( G \)-ANR.

Observe next that

\[
(3.1) \quad \mathcal{T}(X)^h_{g_1} \cap \mathcal{T}(X)^h_{g_2} = X^{h_2} \cap X^{h_2} \cap X^{g_1^{-1}h_1g_1} \cap X^{g_2^{-1}h_1g_2} = X^{\{h', h'', h_2\}},
\]

with \( h' = g_1^{-1}h_1g_1 \) and \( h'' = g_2^{-1}h_1g_2 \). Consequently, it is an ANR.

Since \( \mathcal{T}(X) = \bigcup_{g \in G} \mathcal{T}(X)_g \), \( G \) is finite, we know that \( \mathcal{T}(X)^h \) is a finite union of ANRs such that the intersections of each two of them are ANRs. From the well known property of ANRs:

"\( X, Y, X \cap Y \) are ANRs implies that \( X \cup Y \) is an ANR "

it follows that \( \mathcal{T}(X)^h \) is an ANR.

Now lets take \( h = (h_1, h_2) \), and \( h' = (h_1', h_2') \). For a given \( g \in G \)

\[
(3.2) \quad \mathcal{T}(X)^h_g \cap \mathcal{T}(X)^h_g' = X^{h_2} \cap X^{h_2} \cap X^{g_1h_1g_1} \cap X^{g_1h_1g_1^{-1}} = X^{\{h_2, h_2', gh_1g_1^{-1}, gh_1g_1^{-1} \}}
\]

Observe that for any \( \tilde{G} \)-subset \( A \) of \( \tilde{X} \) we have

\[
A^H = \bigcap_{h \in H} A^h
\]

Now let \( h_1 = (h_1', h_2'), \ldots, h_s = (h_s', h_s) \) be all elements of \( H \subseteq \tilde{G} \). For a given \( g \in G \)

\[
(\mathcal{T}(X)_g)^H = \bigcap_{h \in H} (\mathcal{T}(X)_g)^h = X^{\{h_1', h_2', \ldots, h_s', gh_1g_1^{-1}, \ldots, gh_1g_1^{-1} \}}
\]

and consequently this set is an ANR, since \( X \) is a \( G \)-ANR.

Finally, by the same argument \( \mathcal{T}(X)^H = \bigcup_{g \in G} (\mathcal{T}(X)_g)^H \) is an ANR, because for two \( g_1, g_2 \in G \) we have

\[
(\mathcal{T}(X)_{g_1})^H \cap (\mathcal{T}(X)_{g_2})^H = X^{\{h_1', h_2', \ldots, h_s', g_1h_1g_1^{-1}, \ldots, g_1h_1g_1^{-1}, g_2h_1g_1^{-1}, \ldots, g_2h_1g_1^{-1} \}}
\]

This shows that \( (\mathcal{T}(X)_{g_1})^H \cap (\mathcal{T}(X)_{g_2})^H \) is an ANR and consequently so is the union \( \bigcup_{g \in G} (\mathcal{T}(X)_g)^H = \mathcal{T}(X)^H \). \( \square \)
Remark 3.20. Let $H \subset \hat{G} = G \times G$, $p_1 : G \times G \to G$, $p_2 : G \times G \to G$ the projections onto the corresponding coordinates. Put $H_1 = p_1(H)$, $H_2 = p_2(H)$. Let $\hat{H} = H_1 \times H_2 \subset \hat{G}$.

Since the subgroup $\hat{H}$ contains $H$ we have the corresponding inclusion

(3.3) $A^{\hat{H}} \subset A^H$

for any $\hat{G}$ subset $A$ of $\tilde{X}$. However, since the inclusion $H \subset \hat{H}$ is proper in general, also the inclusion (3.3) is proper in general!

Open problem 3.21. Is it true that $\mathcal{N}(X) \subseteq X \times X$ is always a closed $G \times G$-cofibration for a compact $G$-CW complex $X$ and a compact Lie group $G$?

The above formulated problem seems be difficult in general.

3.5. Bounds for the symmetric topological complexity. We start with a product formula for the symmetric and equivariant topological complexity.

Theorem 3.22. Let $X$ and $Y$ be any $G$ spaces. Then for $X \times Y$ treated as a $G$ space via the diagonal action we have

$$TC_G(X \times Y) \leq TC_G(X) + TC_G(Y)$$

moreover if $\mathcal{N}(X) \subseteq X \times X$ and $\mathcal{N}(Y) \subseteq Y \times Y$ are $G \times G$ cofibrations then

$$STC_G(X \times Y) \leq TC_G() + STC_G(Y)$$

Proof. Since $\Delta(X \times Y) = \Delta(X) \times \Delta(Y)$ and $\mathcal{N}(X \times Y) \subseteq \mathcal{N}(X) \times \mathcal{N}(Y)$, where we consider $X \times Y$ to be a $G$ space via the diagonal action, this is a direct consequence of theorem [2.16]

Corollary 3.23. Let $X$ be a $G$ space and $Y$ a $H$ space. We consider $X \times Y$ as a $G \times H$ space. Then

$$TC_{G \times H}(X \times Y) \leq TC_G(X) + TC_H(Y)$$

moreover if $\mathcal{N}(X) \subseteq X \times X$ is a $G \times G$ cofibration and $\mathcal{N}(Y) \subseteq Y \times Y$ is a $H \times H$ cofibration then

$$STC_{G \times H}(X \times Y) \leq TC_G() + STC_H(Y)$$

Proof. Since $\Delta(X \times Y) = \Delta(X) \times \Delta(Y)$ and $\mathcal{N}(X \times Y) = \mathcal{N}(X) \times \mathcal{N}(Y)$ this is a direct consequence of corollary [2.18]

□
Lemma 3.24. Let \( X \) be a \( G \) set, \( H \subseteq G \) closed subgroup and assume that \( A \subseteq B \) are its closed invariant subsets. Then:

1) \( \text{STC}_G(X) \leq \text{Ocat}_{G \times G}(X \times X) \) for any \( G \times G \) orbit \( O \subseteq \mathcal{O}(X) \);
2) \( \text{oCat}_{G \times G}(X \times X) \leq \text{STC}_G(X) \cdot \text{Ocat}_{G \times G}(\mathcal{O}(X)) \) for any \( G \times G \) orbit \( O \subseteq \mathcal{O}(X) \);
3) \( \text{TC}(X/G) \leq \text{STC}_G(X) \);

Proof. This is a direct consequence of lemma 2.15. □

From our point of view one of the most important properties of the symmetric topological complexity is that it is indeed finite for a large family of \( G \) spaces \( X \).

We have an obvious inequality

\[ \text{TC}(X) \leq \text{cat}(X \times X) \]

we will show that it passes to the equivariant case. For completeness let us first recall

Theorem 3.25 ([4], proposition 5.6). If \( X \) is \( G \) connected then

\[ \text{TC}_G(X) \leq \text{cat}_G X \times X \]

We give a similar result concerning the symmetric topological complexity.

Theorem 3.26. If \( X \) is \( G \)-path-connected then

\[ \text{STC}_G(X) \leq \text{Ocat}_{G \times G} X \times X \]

where \( O = G \cdot x_0 \) for some \( x_0 \in X \).

Proof. Let \( U \) be a set \( G \times G \) compressible into \( O \times O \). We have a \( G \times G \) homotopy \( F: U \times I \to X \times X \) such that \( F: \text{id}_U \simeq c \) where \( c(U) \subseteq O \times O \). Let \( H = G_{x_0} \times G_{x_0} \). We know that \( p((PX \times_{\gamma(X)} PX)^H) = (X \times X)^H \) which follows from the \( G \) connectivity of \( X \) hence \( p(\gamma, \delta) = (x_0, x_0) \). Then we define \( s: U \to PX \times_{\gamma(X)} PX \) by \( s(y_0, y_1) = (g_0, g_1) \cdot (\gamma, \delta) \) whenever \( c(y_0, y_1) = (g_0, g_1) \cdot (x_0, x_0) \). Now note that \( p \circ s \simeq c \simeq \text{id}_U \). □

Remark 3.27. The above theorem allows us to show that \( \text{STC}_G(X) \) is in many cases finite – for example if \( x_0 \in X^G \) and \( X \) is \( G \) connected then we have \( x_0 \times x_0 \text{cat}_{G \times G}(X \times X) \leq 2x_0 \text{cat}_G(X \times X) - 1 = 2 \text{cat}_G - 1 \) by theorem 2.18.

Equivariant and symmetric topological complexity share some basic homotopical properties:
Theorem 3.28. Let $X$ $G$-dominates $Y$, that is there are $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq id_Y$ are $G$-homotopic. Then

$$TC_G(X) \geq TC_G(Y), \ STC_G(X) \geq STC_G(Y).$$

Proof. The part concerning $TC_G(X)$ can be found in [4], theorem 5.2.

For the proof for $STC_G(X)$ let $H : fg \simeq id_Y$ be the $G$ homotopy. Note that then

$$H : (X \times X, \tilde{\gamma}(X)) \times I \to (Y \times Y, \tilde{\gamma}(Y))$$

is the required homotopy between $(f \times f) \circ (g \times g)$ and $id_{(Y \times Y, \tilde{\gamma}(Y))}$. Now the assertion follows from 2.5. □

Corollary 3.29. For a $G$ set $X$ and closed subgroup $H$ of $G$ we have

$$TC_H(X^H) \leq TC_G(X), \ STC_H(X^H) \leq STC_G(X).$$

Proof. The part concerning $TC_G(X)$ follows from [4], proposition 5.3.

For the proof for $STC_G(X)$ let $\tilde{H} = H \times H$. Note that

$$\tilde{\gamma}(X)^{\tilde{H}} = \tilde{\gamma}(X^H) \supseteq (H \times H)\Delta(X^H)$$

therefore from theorem 2.19 we get that

$$TSC_H(X^H) =_{(H \times H)\Delta(X^H)} cat_{H \times H}(X^H \times X^H) \leq \gamma(X) cat_{G \times G}(X \times X) = TSC_G(X).$$

□

Corollary 3.30. For a $G$ space $X$ such that $X^G \neq \emptyset$ we have that

$$STC_G(X) \geq TC_G(X^G) = TC(X^G).$$

4. Examples of calculations.

We end this article with calculations of $TSC_G(X)$ in some basic examples.

Example 4.1. Let $G$ act on itself by left translations. The action of $G$ is free and therefore from theorem 3.11 we get that

$$STC_G(G) = TC(G/G) = TC(\ast) = 1$$

which is in contrast to the case of equivariant topological complexity where we have that $TC_G(G) = cat(G)$ (comp. [4], theorem 5.11).
Example 4.2. Let $\mathbb{Z}/2 = \{1, \tau\}$ act on $S^n$, $n \geq 1$ by reflecting the last coordinate. Note that for $n = 1$ the set $(S^1)^{\mathbb{Z}/2}$ is disconnected so that $TG_{\mathbb{Z}/2}(S^1) = STC_{\mathbb{Z}/2}(S^1) = \infty$. If $n > 1$ then $S^n$ is $\mathbb{Z}/2$ connected so that

$$STC_{\mathbb{Z}/2}(S^n) \leq cat_{\mathbb{Z}/2 \times \mathbb{Z}/2}(S^n \times S^n) = 2cat_{\mathbb{Z}/2}(S^n) - 1 = 3$$

by theorem 3.26. On the other hand, since $(S^n)^{\mathbb{Z}/2} \cong S^{n-1}$, we have that (comp. 3.29) $STC_{\mathbb{Z}/2}(S^n) \geq TC(S^{n-1}) = 3$ for $n$ odd.

For an even $n$ let $U_1 \subseteq S^n \times S^n$ be defined as follows

$$U_1 = \{(x, y) \in (S^n)^2 : x \neq -y \text{ if } x, y \in S^{n-1}\}.$$ 

Then for each $(x_1, x_2) \in U_1$ there is a unique shortest path $s'(x_1, x_2)$ joining $x_1$ or $\tau x_1$ and $x_2$ or $\tau x_2$ in the upper hemisphere. Let $s_1(x_1, x_2) = (\alpha_1s_{[0, 1]}, \alpha_2s_{[1, 1]})$ in case we were joining $\alpha_1x_1$ with $\alpha_2x_2$ for $\alpha_i \in \mathbb{Z}/2$.

Let $U_2 \subseteq S^n \times S^n$ be defined as follows

$$V_2 = \{(x, y) \in (S^n)^2 : x, y \in S^{n-1}, x \neq y\}.$$ 

Now $V_2$ has a small $\mathbb{Z}/2$ invariant open neighborhood $U_2$ in $S^n$ such that the projection $\pi : U_2 \rightarrow V_2$ into the equator $S^{n-1}$ is $\mathbb{Z}/2$ equivariant deformation retraction. We define for each $(x_1, x_2)$ a path from $x_1$ to $x_2$ as follows. First choose a non vanishing vector field $\nu$ on $S^{n-1}$. The path $s_2'(x_1, x_2)$ consists of four parts. First by the shortest path we move $x_1$ to $\pi(x_1)$, then using the shortest path we move $\pi(x_1)$ to $-\pi(x_2)$ and using the vector field $\nu$ we move $-\pi(x_2)$ to $\pi(x_2)$ using a spherical arch defined by $\nu(\pi(x_2))$ and we end by moving through the shortest path $\pi(x_2)$ to $x_2$. We obtain $s$ from $s'$ by cutting it into two parts.

As it can be easily checked these two sets satisfy the definition of the symmetric topological complexity and prove that $STC_{\mathbb{Z}/2}(S^n) = 2$ for $n$ even.

Note that we have $TC_{\mathbb{Z}/2}(S^n) = 3$ for $n > 1$ as shown in [4], example 5.9.

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