Separable states and positive maps

Erling Størmer

11-10-2007

Abstract

Using the natural duality between linear functionals on tensor products of C*-algebras with the trace class operators on a Hilbert space $H$ and linear maps of the C*-algebra into $B(H)$, we study the relationship between separability, entanglement and the Peres condition of states and positivity properties of the linear maps.

Introduction

In an earlier paper [14] we studied the duality between linear functionals $\tilde{\phi}$ on a tensor product $A \hat{\otimes} T$ of an operator system $A$ and the trace class operators $T$ on a Hilbert space $H$, and bounded linear maps $\phi : A \to B(H)$ given by the formula $\tilde{\phi}(a \otimes b) = Tr(\phi(a)b^t)$. The main emphasis was on positivity properties of $\phi$ on cones in $A \hat{\otimes} T$ obtained by classes of positive maps. In the present paper we shall see how this study yields a natural framework for the study of separable states of $A \hat{\otimes} T$, for example we recover results of Horodecki et.al [9] and Horodecki, Shor and Ruskai [11] on characterizations of separable states. In addition we shall obtain characterizations of states on $A \hat{\otimes} T$ satisfying the Peres condition, viz $\rho \circ (\iota \otimes t)$ is positive, where $t$ is the transpose map and $\iota$ the identity map. In particular we see that nondecomposable maps yield natural examples of entangled states which satisfy the Peres condition; for this see also [7], [8]. In the last section we study the definite set of a positive map $\phi$ on a C*-algebra $A$, i.e. the set of self-adjoint operators in $A$ such that $\phi(a^2) = \phi(a)^2$, and show that if $\tilde{\phi}$ is a separable state, then the image of the definite set is a C*-subalgebra of the center of the C*-algebra generated by $\phi(A)$. As a corollary we obtain a decomposition result for separable states in the finite dimensional case.

The author is indebted to E. Alfsen for his careful reading of the manuscript and several useful comments.

1 Cones and states

In this section we recall notation and concepts from [14] and show a general characterization of separable states close to that in [11]. For more details on the following see [14].
By an operator system we shall mean a norm closed self-adjoint set $A$ of bounded operators on a Hilbert space containing the identity. We denote by $A \otimes B(H)$ the algebraic tensor product of $A$ and $B(H)$ and by $A \hat{\otimes} B(H)$ the closure of $A \otimes B(H)$ in the operator norm. If $T$ denotes the trace class operators on $H$, then $A \hat{\otimes} T$ denotes the projective tensor product of $A$ and $T$. We denote by $B(A, H)$, (resp. $B(A, H)^+$) the bounded (resp. positive) linear maps of $A$ into $B(H)$. The $BW$-topology on $B(A, H)$ is the topology of bounded pointwise weak convergence, i.e. a net $(\phi_\nu)$ converges to $\phi$ if it is uniformly bounded, and $\phi_\nu(a) \to \phi(a)$ weakly for all $a \in A$. We denote by $t$ the transpose map on $B(H)$ with respect to some orthonormal basis for $H$. Then by abuse of notation we get that the transpose map on $B(K) \otimes B(H)$ is $t \otimes t$. We shall also denote by $Tr$ the usual trace on $B(H)$ which takes the value 1 on minimal projections. We recall Lemma 2.1 in [14].

**Lemma 1** With the above notation there is an isometric isomorphism $\phi \to \tilde{\phi}$ between $B(A, H)$ and $(A \hat{\otimes} T)^*$ given by

$$
\tilde{\phi}(a \otimes b) = Tr(\phi(a)b^T), \quad a \in A, \ b \in T.
$$

Furthermore, $\phi \in B(A, H)^+$ if and only if $\tilde{\phi}$ is positive on the cone $A^+ \hat{\otimes} T^+$ generated by operators of the form $a \otimes b$ with $a$ and $b$ positive.

We recall Definition 2.3 in [14]. It says that a BW-closed subcone $K \neq 0$ of $B(B(H), H)^+$ is a mapping cone if it has a BW-dense subset of ultra weakly continuous maps and is invariant in the sense that if $\alpha \in K$, and $a, b \in B(H)$ then the map $x \to \alpha(axb^*)a^*$ belongs to $K$. Three mapping cones will be of special interest in the following, namely $B(B(H), H)^+, CP(H)$ - the set of completely positive maps in $B(B(H), H)$, and $S(H)$ - the BW-closed cone generated by maps of the form

$$
x \to \sum_{i=1}^{n} \omega_i(x)a_i,
$$

where $\omega_i$ is a normal state on $B(H)$ and $a_i \in B(H)^+$. The latter maps are said to be of "Holevo form" in [11]. By Lemma 2.4 in [14], $S(H)$ is the minimal mapping cone and $B(B(H), H)^+$ the maximal one.

If $K$ is a mapping cone and $A$ an operator system as before, we denote by $P(A, K)$ the cone

$$
P(A, K) = \{ x \in A \hat{\otimes} T : \iota \otimes \alpha(x) \geq 0 \ \forall \alpha \in K \}.
$$

By Lemma 2.8 in [14], $P(A, K)$ is a proper norm closed convex cone in $A \hat{\otimes} T$ containing the cone $A^+ \hat{\otimes} T^+$. A map $\phi \in B(A, H)$ is said to be $K$-positive if

$$
\tilde{\phi}(\sum a_i \otimes b_i) = \sum Tr(\phi(a_i)b_i^T) \geq 0 \ \text{whenever } \sum a_i \otimes b_i \in P(A, K).
$$

By Theorem 3.2 in [14], $\phi$ is completely positive if and only if $\tilde{\phi}$ is positive on the cone $(A \hat{\otimes} T)^+$, the closure of the positive operators in $A \hat{\otimes} T$, if and only if $\phi$ is $CP(H)$ - positive.
If $C \subseteq V$ and $D \subseteq W$ are closed convex cones in two real locally convex vector spaces in duality, we denote by $C^*$ (resp. $D^*$) the set of $w \in W$ such that $<v, w> \geq 0 \ \forall v \in C$, (resp. $v \in V$ such that $<v, w> \geq 0 \ \forall w \in D$). Thus $\phi$ is K-positive if and only if $\hat{\phi} \in P(A, K)^*$. By a straightforward application of the Hahn-Banach Theorem for closed convex cones, see e.g. [1], Prop. 1.32, we have

$$P(A, K) = P(A, K)^{**}.$$ 

We say a positive linear functional $\rho$ on $A \otimes B(H)$ is separable if it belongs to the norm closure of positive sums of states of the form $\sigma \otimes \omega$, where $\sigma$ is a state of $A$ and $\omega$ a normal state of $B(H)$. Otherwise $\rho$ is called entangled. We denote the set of separable states by $S(A, H)$. It is a norm closed cone in $(A \otimes T)^*$. As for $P$ above $S(A, H) = S(A, H)^{**}$. Our next result is closely related to Theorem 2 in [11].

**Theorem 2** Let $A$ be an operator system and $\phi \in B(A, H)$. Then the following conditions are equivalent:

(i) $\tilde{\phi}$ is a separable positive linear functional.

(ii) $\phi$ is $S(H)$-positive.

(iii) $\tilde{\phi}$ is a BW-limit of maps of the form $x \rightarrow \sum_{i=1}^{n} \omega_i(x)b_i$ with $\omega_i$ a state of $A$, and $b_i \in B(H)^+$.

**Proof.** (i) $\Leftrightarrow$ (ii). Let $S_n$ denote the positive normal linear functionals on $B(H)$, and let $x = \sum x_i \otimes y_i \in A \otimes B(H)$. Then

$$x \in P(A, S(H))$$

$\Leftrightarrow (i \otimes b)(x) \geq 0 \ \forall \omega \in S_n, b \geq 0$

$\Leftrightarrow \sum x_i \omega(y_i) \otimes b = \sum x_i \otimes \omega(y_i)b \geq 0 \ \forall \omega \in S_n, b \geq 0$

$\Leftrightarrow \sum x_i \omega(y_i) \geq 0 \ \forall \omega \in S_n$

$\Leftrightarrow \rho \otimes \omega(x) = \sum \rho(x_i)\omega(y_i) = \rho(\sum x_i \omega(y_i)) \geq 0 \ \forall \omega \in S_n, \rho \in A^{**}$

$\Leftrightarrow \eta(x) \geq 0 \ \forall \eta \in S(A, H)$

$\Leftrightarrow x \in S(A, H)^{**}$.

Thus $\phi$ is $S(H)$-positive if and only if $\tilde{\phi} \in P(A, S(H))^* = S(A, H)^{**} = S(A, H)$, proving that (i) $\Leftrightarrow$ (ii). The equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 3.6 in [13], since a map $\alpha \in S(H)$ if and only if $t \circ \alpha \circ t \in S(H)$. The proof is complete.

In [11] maps like $x \rightarrow \sum \omega_i(x)b_i$ are called "entanglement breaking".

It is possible to give a direct proof of a less general form of the equivalence (i) $\Leftrightarrow$ (iii) above. Suppose $\phi(a) = \sum \omega_i(a)b_i$ for $a \in A, b_i \in B(H)^+, \omega_i$ state of $A$. Then

$$\tilde{\phi}(a \otimes b) = Tr(\phi(a)b^i) = \sum Tr(\omega_i(a)b_ib^i) = \sum \omega_i(a)Tr(b_ib^i) = \sum \omega_i(a)\rho_i(b),$$
where \( \rho_i(b) = Tr(b_i b^i) \) is a positive linear functional. Thus \( \tilde{\phi} \) is separable.

Conversely, if \( \phi = \sum \omega_i \otimes \rho_i \), let \( \tilde{\rho}_i(b) = \rho_i(b^i) = Tr(b_i b) \). Then we have

\[
Tr(\phi(a) b^i) = \tilde{\phi}(a \otimes b) = \sum \omega_i(a) \rho_i(b) = \sum \omega_i(a) \tilde{\rho}_i(b^i) = \sum Tr(\omega_i(a) b_i b^i).
\]

This holds for all \( b \in \mathcal{T} \), hence \( \phi(a) = \sum \omega_i(a) b_i \).

**Corollary 3** Let \( H \) be separable and \( \phi \in B(A,H)^+ \). Suppose \( \phi(A) \) is contained in an abelian \( C^* \)-algebra. Then \( \tilde{\phi} \) is separable.

**Proof.** By hypothesis there is an abelian von Neumann algebra \( B \subseteq B(H) \) such that \( \phi : A \to B \). Let \( (B_n) \) be an increasing sequence of finite dimensional von Neumann subalgebras of \( B \) such that \( \bigcup_n B_n \) is weakly dense in \( B \). Let \( E_n : B \to B_n \) be normal conditional expectations such that \( E_{n-1}|_{B_n} \circ E_n = E_{n-1} \). Then \( \phi(x) = \lim_{n} E_n \circ \phi(x) \) for all \( x \in A \). Since \( B_n \) is finite dimensional, \( E_n \circ \phi(x) = \sum \omega_i^n(x) e_i^n \), where \( \omega_i^n \) are positive linear functionals on \( A \) and \( e_i^n \) are minimal projections in \( B_n \). Since \( \phi \) is a BW-limit of the sequence \( E_n \circ \phi \), \( \phi \) is separable by Theorem 2. The proof is complete.

A celebrated necessary condition for a state \( \rho \) on \( A \hat{\otimes} \mathcal{T} \) to be separable is the **Peres condition**, i.e. \( \rho \circ (t \otimes t) \geq 0 \). A map \( \phi \in B(A,H) \) is said to be **copositive** if \( t \circ \phi \) is completely positive.

**Proposition 4** Let \( \phi \in B(A,H) \). Then \( \tilde{\phi} \) satisfies the Peres condition if and only if \( \phi \) is both completely positive and copositive.

**Proof.** If \( a \in A \) and \( b \in \mathcal{T} \) we have, since the trace is invariant under transposition,

\[
\tilde{\phi}(a \otimes b^i) = Tr(\phi(a) b) = Tr(t \circ \phi(a) b^i) = (t \circ \phi)(a \otimes b).
\]

Thus \( \tilde{\phi} \) satisfies the Peres condition if and only if both \( t \circ \phi \) and \( \phi \) are positive. Using Theorem 3.2 in [14] this holds if and only if \( t \circ \phi \) and \( \phi \) are completely positive, hence if and only if \( \phi \) is both completely positive and copositive.

## 2 States on \( B(K) \otimes B(H) \)

In this section we study the case when the operator system \( A \) equals \( B(K) \) for a Hilbert space \( K \). But first we consider the finite dimensional case. Let \( M_n = M_n(\mathbb{C}) \) denote the complex \( n \times n \) matrices, and let \( \phi : M_n \to M_m \), so \( \phi \in B(A, \mathbb{C}^m) \), where \( A = M_n \) and \( H = \mathbb{C}^m \). Let \( (e_{ij}) \) be a complete set of matrix units in \( M_n \). Then the **Choi matrix** for \( \phi \) is

\[
C_\phi = \sum e_{ij} \otimes \phi(e_{ij}) = t \otimes \phi(P) \in M_n \otimes M_m,
\]

where \( \frac{1}{n} P \) is the 1-dimensional projection \( \frac{1}{n} \sum e_{ij} \otimes e_{ij} \), - the so-called maximally entangled state, see [3]. Denote by \( \phi^t \) the map \( t \circ \phi \circ t \), where \( t \) denotes the
transpose map in either $M_n$ or in $M_m$. Then $\phi$ is completely positive if and only if $\phi^t$ is completely positive. It was shown by Choi [3] that $\phi$ is completely positive if and only if $C_\phi$ is positive. We use the convention that the density matrix for a state $\rho$ is the positive matrix $h$ such that $\rho(x) = Tr(hx)$.

**Lemma 5** $C_{\phi^t}$ is the density matrix for $\tilde{\phi}$.

*Proof.* Let $a \in M_n, b \in M_m$. Since the transpose $t$ on $M_n \otimes M_m$ is the tensor product of the transpose operators on $M_n$ and $M_m$, we have

$$
Tr(C_{\phi^t}a \otimes b) = \sum Tr(e_{ij} \otimes \phi^t(e_{ij})(a \otimes b)) = \sum Tr(e_{ji} \otimes \phi^t(e_{ji})(a^t \otimes b^t)) = \sum Tr(e_{ji}a^t)Tr(\phi(e_{ji})b^t) = \sum a_{ji}Tr(e_{ji}\phi^t(b^t)) = Tr(a\phi^t(b^t)) = \tilde{\phi}(a \otimes b).
$$

In the above computation $\phi^*$ is the adjoint of $\phi$ as an operator between $M_n$ and $M_m$ considered as the trace class operators on $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. The proof is complete.

**Lemma 6** Let $H = \mathbb{C}^m$ and $\phi \in B(M_n, H)$. Then $\phi$ is positive if and only if $C_{\phi^t} \in P(M_n, S(H))$, if and only if $C_\phi \in P(M_n, S(H))$. Hence $P(M_n, S(H)) = \{C_\phi : \phi \geq 0\}$.

*Proof.* By Theorem [2] or rather the proof of the equivalence $(i) \iff (ii)$,

$$C_{\phi^t} \in P(M_n, S(H)) = S(M_n, H)^*$$

$$\iff Tr(C_{\phi^t}a \otimes b) \geq 0 \forall a \in M_n^+, b \in M_m^+$$

$$\iff \phi \geq 0$$

by Lemma [4] proving the first statement. Since $\phi \geq 0$ if and only if $\phi^t \geq 0$, the above is equivalent to $C_\phi$ being in $P(M_n, S(H))$.

Each element $x \in P(M_n, S(H))$ defines a linear functional $\rho$ on $M_n \otimes M_m$ by $\rho(y) = Tr(xy)$. By Lemma [1] there is $\phi \in B(M_n, \mathbb{C}^m)$ such that $\rho(a \otimes b) = Tr(\phi(a)b^t)$, hence by Lemma [5] and the first part of the proof, $x = C_{\phi^t}$ with $\phi \geq 0$. Thus the last statement follows, completing the proof.

We shall now apply the finite dimensional results to study states on $B(K) \otimes B(H)$ and to prove an infinite dimensional extension of the Horodecki Theorem [9]. Recall that a state and a positive linear map on a Von Neumann algebra are said to be normal if they are weakly continuous on bounded sets.
**Theorem 7** Let \( \rho \) be a normal state on \( B(K) \otimes B(H) \) with \( K \) and \( H \) Hilbert spaces and with density operator \( h \). Then \( \rho \) is separable if and only if \( \iota \otimes \psi(h) \geq 0 \) for all normal positive maps \( \psi : B(H) \to B(K) \).

*Proof.* Suppose \( \rho \) is separable and normal. Then \( \rho \circ (\iota \otimes \phi) \) is a normal state for all unital normal positive maps \( \phi : B(K) \to B(H) \). Let \( \psi \) be as in the statement of the theorem. Then the adjoint map \( \psi^* \) is a positive map of the trace class operators on \( K \) into those on \( H \). Thus if \( x \geq 0 \) is of finite rank in \( B(K) \otimes K = B(K) \otimes B(K) \), then

\[
\text{Tr}((\iota \otimes \psi)(hx)) = \text{Tr}(h(\iota \otimes \psi^*)(x)) = \rho(\iota \otimes \psi^*(x)) \geq 0,
\]
hence \( \iota \otimes \psi(h) \geq 0 \).

To show the converse we first assume \( K \) and \( H \) are finite dimensional. Then by Lemma 4 \( P(M_n, S(H)) = \{ C_\phi : \phi \geq 0 \} \). Thus by Theorem 2 and Lemma 5 \( \rho \) is separable if and only if for all positive \( \phi : B(K) \to B(H) \)

\[
\text{Tr}(\iota \otimes \phi^*)(hP) = \text{Tr}(h(\iota \otimes \phi)(P)) = \text{Tr}(hC_\phi) \geq 0,
\]
where \( P \) is the rank one matrix such that \( C_\phi = \iota \otimes \phi(P) \). Since \( P \geq 0 \), and by assumption \( \iota \otimes \phi^*(h) \geq 0 \), it follows that \( \rho \) is separable.

We next consider the general case when \( K \) and \( H \) may be infinite dimensional. Assume \( \iota \otimes \psi(h) \geq 0 \) for all normal \( \psi : B(H) \to B(K) \). Since the maps \( \psi_f(x) = \psi(fxf) \) are positive for all finite dimensional projections \( f \), it is clear that \( \iota \otimes \psi((e \otimes f)h(e \otimes f)) \geq 0 \) for all normal positive maps \( \psi : B(H) \to B(K) \) with \( e \) a finite dimensional projection in \( B(K) \). Let

\[
\psi_{e \otimes f}(y) = e\psi(fy)f, \quad y \in B(H).
\]

Then \( \iota \otimes \psi_{e \otimes f}(h) \geq 0 \). Now every normal positive map \( \phi : B(fH) \to B(eK) \) is of the form \( \psi_{e \otimes f} \) with \( \psi \) as above, because we can define \( \phi : B(H) \to B(K) \) by \( \psi(x) = \phi(fxf) \). Thus by the part of the proof on the finite dimensional case, the positive linear functional \( \omega(x) = \rho((e \otimes f)x(e \otimes f)) \) is separable on \( B(eK) \otimes B(fH) \). Since this holds for all finite dimensional projections \( e \) and \( f \) and \( \rho \) is normal, it follows that \( \rho \) is separable. The proof is complete.

We expect that the above theorem can be generalized to Von Neumann algebras other than \( B(K) \). If \( A \) is a semi-finite Von Neumann algebra then so is \( A \otimes B(H) \), hence each normal state on \( A \otimes B(H) \) has a density operator with respect to a trace, and the formulation of the theorem has a natural generalization. In the type III case a formulation in terms of modular theory ought to be possible.

We next restate the Peres condition in terms of the density matrix of the normal state \( \rho \).

**Theorem 8** Let \( \rho \) be a normal state on \( B(K) \otimes B(H) \) with density operator \( h \), and let \( \iota \) denote the transpose map of either \( B(K) \) or \( B(H) \). Then the following
conditions are equivalent:
(i) $\rho$ satisfies the Peres condition.
(ii) $\iota \otimes t(h) \geq 0$.
(iii) $t \otimes \iota(h) \geq 0$.
(iv) $h \in P(B(K), C P(H)) \cap P(B(K), \text{copos}(H))$, where $\text{copos}(H)$ denotes the copositive maps of $B(H)$ into itself.

Proof. Assume (i). Since the trace on $B(K) \otimes B(H)$ is invariant under $\iota \otimes t$, we have

$$\rho \circ (\iota \otimes t)(a \otimes b) = Tr(h(\iota \otimes t)(a \otimes b)) = Tr(\iota \otimes t(h)(a \otimes b)).$$

Since $\rho \circ (\iota \otimes t) \geq 0$ it follows that $\iota \otimes t(h) \geq 0$.

Conversely, if (ii) holds then by the above computation $\rho \circ (\iota \otimes t) \geq 0$, hence (i) holds. The equivalence of (ii) and (iii) follows since $t \otimes \iota(h) = t \otimes (\iota \otimes t(h))$, and the fact that $t \otimes t$ is an order-automorphism.

We have

$$P(B(K), \text{copos}(H)) = \{ x \in B(K) \otimes B(H) : \iota \otimes \phi(x) \geq 0 \ \forall \operatorname{copositive} \phi \}$$

$$= \{ x \in B(K) \otimes B(H) : \iota \otimes t(x) \geq 0 \},$$

because a copositive map is the composition of a completely positive map and the transpose map. Thus (ii) is equivalent to (iv), completing the proof.

Let $A$ be a $C^∗$-algebra. Then a map $\phi \in B(A, H)$ is called decomposable if it is the sum of a completely positive map and a copositive map. Otherwise $\phi$ is called nondecomposable. Since a map $\phi \in B(A, C^n)$ is completely positive if and only if $\iota \otimes \phi : M_n \otimes A \to M_n \otimes M_n$ is positive [9], Lemma 5.1.3, it follows from [13] that $\phi \in B(A, C^n)$ is decomposable if and only if whenever $h$ and $t \otimes \iota(h)$ belong to $(M_n \otimes A)^+$ then $\iota \otimes \phi(h) \geq 0$. Thus $\phi$ is nondecomposable if and only if there exists $h \in (M_n \otimes A)^+$ such that $t \otimes \iota(h) \geq 0$ while $\iota \otimes \phi(h)$ is not positive. Suppose that $A = B(H)$, $\phi$ normal, and $h$ as above. Then there exists by normality of $\phi$ a finite dimensional projection $f \in B(H)$ such that $\iota \otimes \phi((1 \otimes f)h(1 \otimes f))$ is not positive. We can thus assume $h$ is of finite rank. Normalizing $h$ we thus have by Theorem [8] that the state $\rho(x) = Tr(hx)$ satisfies the Peres condition, while by Theorem [4] $\rho$ is entangled. We have thus proved

**Theorem 9** Let $\phi : B(H) \to M_n$ be normal positive and nondecomposable. Then there exists a normal state $\rho$ on $B(H) \otimes M_n$ with density operator $h$ such that $t \otimes \iota(h) \geq 0$, while $\iota \otimes \phi(h)$ is not positive. Hence $\rho$ is entangled but satisfies the Peres condition.

An explicit example of the situation in the above theorem is given in [13] and [5]. Then $\dim H = n = 3$, and $\phi : M_3 \to M_3$ is the nondecomposable Choi map [4]. Other examples can be found in [7] and [8]. A large class of nondecomposable maps are the projections onto spin factors of dimension greater than 6,
If $A$ and $B$ are $C^*$-algebras, and $\phi: A \to B$ is a positive map of norm $\leq 1$ then the (self-adjoint) *definite set* $D_\phi$ of $\phi$ is the set of self-adjoint operators in $A$ such that $\phi(a^2) = \phi(a)^2$. If $a \in D_\phi$ and $b \in A$ then $\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ and $\phi(aba) = \phi(a)\phi(b)\phi(a)$, see [12]. We show in the present section that if $\phi$ is of the form $\phi(x) = \sum \omega_i(x)b_i$ as in Theorem [2] then $\phi(D_\phi)$ is contained in the center of the $C^*$-algebra generated by $\phi(A)$. In particular, if $\phi$ is faithful, then $D_\phi$ is abelian. As a consequence we get a decomposition result for separable states.

**Theorem 10** Let $A$ be a unital $C^*$-algebra and $\phi \in B(A, H)^+$ with $\phi(1) = 1$. Suppose $\phi$ is of the form $\phi(x) = \sum_{i=1}^n \omega_i(x)b_i$ with $\omega_i$ states of $A$ and $b_i \in B(H)^+$. Let $e$ be a projection in the definite set $D_\phi$ of $\phi$, and put $f = 1 - e$. Then $\phi(e)$ and $\phi(f)$ are projections in $B(H)$ and satisfy

$$
\phi(x) = \phi(exe) + \phi(fxf) = \phi(e)\phi(x)\phi(e) + \phi(f)\phi(x)\phi(f)
$$

for all $x \in A$. Hence $\phi(D_\phi)$ is an abelian $C^*$-algebra contained in the center of the von Neumann algebra generated by $\phi(A)$. In particular, if $\phi$ is faithful then $D_\phi$ is an abelian $C^*$-algebra.

**Proof.** Since $e \in D_\phi$, $\phi(e)$ and $\phi(f)$ are mutually orthogonal projections. Thus

$$
0 = \text{Tr}(\phi(e)\phi(f)) = \text{Tr}(\sum \omega_i(e)b_i\omega_j(f)b_j) = \sum \omega_i(e)\omega_j(f)\text{Tr}(b_i b_j).
$$

Since each summand is positive we have

$$
\omega_i(e)\omega_j(f)\text{Tr}(b_i b_j) = 0 \forall i, j.
$$

In particular

$$
\omega_i(e)\omega_j(f)\text{Tr}(b_i^2) = 0 \forall i.
$$

Since $b_i \neq 0$ either $\omega_i(e) = 0$ or $\omega_i(f) = 0$ for all $i$. In particular, $e$ or $f$ belongs to the left and right kernel of $\omega_i$, hence $\omega_i(exf) = \omega_i(fxe) = 0$ for all $x$. Thus $\omega_i(x) = \omega_i(exe) + \omega_i(fxf)$ for all $x$, so that

$$
\phi(x) = \phi(exe) + \phi(fxf) = \phi(e)\phi(x)\phi(e) + \phi(f)\phi(x)\phi(f),
$$

where the last equality follows since $e, f \in D_\phi$.

To show the last statement in the theorem we consider the ultra-weakly continuous extension $\phi^{**}$ of $\phi$ to the second dual $A^{**}$ of $A$. If $a \in D_{\phi}$ the abelian von Neumann algebra generated by $a$ in $A^{**}$ is contained in $D_{\phi^{**}}$ and

or more generally, positive projections onto nonreversible Jordan algebras, see [12]. See [14] for another class of nondecomposable maps. Another result close to the above theorem can be found in [2]. Previous examples of entangled states which satisfy the Peres condition have been exhibited by P. Horodecki [10].
is generated by its projections. It thus suffices to show that for each projection $e \in D_\phi$, $\phi(e)$ belongs to the commutant of $\phi(A)$. But this is immediate from the above equation.

If $\phi$ is faithful then the restriction of $\phi$ to $D_\phi$ is an isomorphism, hence is abelian, since $\phi(D_\phi)$ is abelian. The proof is complete.

**Corollary 11** Let $A \subseteq B \subseteq B(H)$ be unital C*-algebras with $H$ separable. Suppose $\phi : B \to A$ is a conditional expectation. Then $\phi$ is separable if and only if $A$ is abelian.

**Proof.** By Corollary 3 if $A$ is abelian then $\tilde{\phi}$ is separable. Since $\phi$ is a conditional expectation, the self-adjoint part of $A$ equals the definite set $D_\phi$, hence by Theorem 10 $A$ is abelian if $\tilde{\phi}$ is separable, completing the proof.

Let $\tilde{\phi} = \sum \lambda_i \omega_i \otimes \rho_i$ be a faithful separable state on $M_n \otimes M_m$, which is a convex sum of states $\omega_i$ on $M_n$ and $\rho_i$ on $M_m$. By symmetry in $M_n$ and $M_m$ in Lemma 1 there exists a completely positive map $\psi : M_n \to M_n$ such that $\tilde{\phi}(a \otimes b) = Tr(a^t \psi(b))$. Then by Theorem 11 and the faithfulness of $\tilde{\phi}$, $D_\phi$ and $D_\psi$ are abelian C*-algebras. Let $(e_j)_{j=1}^p$ be minimal projections in $D_\phi$ and $(f_k)_{k=1}^q$ be minimal projections in $D_\psi$. From the proof of Theorem 11 the values of $\omega_i(e_j)$ and $\rho_i(f_k)$ are 0 or 1. In particular, the supports of $\omega_i$ and $\rho_i$ are contained in some $e_j$ and $f_k$ respectively. Hence $e_j \otimes f_k$ are mutually orthogonal projections with sum 1 such that

$$\tilde{\phi}(x) = \sum_{i,j} \tilde{\phi}(e_j \otimes f_k x e_j \otimes f_k),$$

for all $x \in M_n \otimes M_m$

We say $\tilde{\phi}$ is irreducible if $D_\phi = D_\psi = R$ when we have cut down by the support of $\tilde{\phi}$, and we say a family $(\eta_k)$ of states are orthogonal if their supports are mutually orthogonal. Summing up we have shown

**Corollary 12** Every separable state on $M_n \otimes M_m$ is a convex sum of orthogonal irreducible separable states.

**References**

[1] E.M. Alfsen and F.W. Shultz, *State spaces of operator algebras*, Mathematics: Theory and Applications, Birkhauser, Boston (2001).

[2] F. Benatti, F. Floreanini, and M. Piani, *Non-decomposable quantum semigroups and bound entangled states*, quant-ph/0411095.

[3] M-D. Choi, *Completely positive maps on complex matrices*, Linear Algebra and Appl. 10 (1975), 285-290.

[4] M-D. Choi, *Positive definite biquadratic forms*, Linear Algebra and Appl. 12 (1975), 95-100.
[5] M-D. Choi, *Positive linear maps*, AMS. Proc. Sympo. Pure Math. 1982.

[6] E.G. Effros and Z-J. Ruan, *Operator spaces*, London Math. Soc. Monographs, New Series, 23 (2000), Oxford Pess - Oxford.

[7] K-C. Ha, S-H. Kye, and Y.S. Park, *Entangled states with positive partial transposes arising from indecomposable positive linear maps*, Phys. Lett. A 313 (2003), 163-174.

[8] K-C. Ha and S-H. Kye, *Construction of entangled states with positive partial transposes based on indecomposable positive linear maps*, Phys. Lett. A 325 (2004), 315-323.

[9] M. Horodicki, P.Horodicki, and R.Horodicki, *Separability of mixed states: necessary and sufficient conditions*, Physics Letters. A 223, (1996), 1-8.

[10] P. Horodicki, *Separability criterion and inseparable mixed states with positive partial transposition*, Physics Letters, A 232, (1997), 333-339.

[11] P. Horodicki, P.W.Shor, and M.B.Ruskai, *Entanglement breaking channels*, quant-ph /0302031.

[12] E. Størmer, *Decomposition of positive projections on C*-algebras*, Math. Ann. 247 (1980).

[13] E. Størmer, *Decomposable positive maps on C*-algebras*, Proc.A.M.S. 86, No.3 (1982), 402-404.

[14] E. Størmer, *Extension of positive maps into B(H)*, Jour. Funct. Anal. 66, No.2 (1986), 235-254.

[15] B.M.Terhal, *A family of indecomposable positive linear maps based on entangled quantum states*, quant-ph/9810091

Department of Mathematics, University of Oslo, 0316 Oslo, Norway.
e-mail: erlings@math.uio.no