CHARACTERISTIC NUMBER ASSOCIATED TO MASS LINEAR PAIRS

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Abstract. Let $\Delta$ be a Delzant polytope in $\mathbb{R}^n$ and $b \in \mathbb{Z}^n$. Let $E$ denote the symplectic fibration over $S^2$ determined by the pair $(\Delta, b)$. Under certain hypotheses, we prove the equivalence between the fact that $(\Delta, b)$ is a mass linear pair (D. McDuff, S. Tolman, Polytopes with mass linear functions. I. Int. Math. Res. Not. IMRN 8 (2010) 1506-1574.) and the vanishing of a characteristic number of $E$. Denoting by $\text{Ham}(M_{\Delta})$ the Hamiltonian group of the symplectic manifold defined by $\Delta$, we prove the existence of infinite cyclic subgroups in $\pi_1(\text{Ham}(M_{\Delta}))$ when $\Delta$ satisfies any of the following conditions: (i) it is the trapezium associated with a Hirzebruch surface, (ii) it is a $\Delta_p$ bundle over $\Delta_1$, (iii) $\Delta$ is the truncated simplex associated with the one point blow up of $\mathbb{C}P^n$.

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1. Introduction

Let $(N, \Omega)$ be a closed connected symplectic $2n$-manifold. By $\text{Ham}(N, \Omega)$, we denote the Hamiltonian group of $(N, \Omega)$ [6, 8]. Associated with a loop $\psi$ in $\text{Ham}(N, \Omega)$, there exist characteristic numbers which are invariant under deformation of $\psi$. These invariants are defined in terms of characteristic classes of fibre bundles and their explicit values are not easy to calculate, in general. Here, we will consider a particular invariant $I$, whose definition we will recall below. By proving the non-vanishing of $I$ for certain loops, we will deduce the existence of infinite cyclic subgroups of $\pi_1(\text{Ham}(N, \Omega))$, when $N$ is a toric manifold. The vanishing of the invariant $I$ on particular loops in $\text{Ham}(N, \Omega)$ is related with the concept of mass linear pair, which has been developed in [7]. In this introduction, we will state the main results of the paper and will give a schematic exposition of the concepts involved in these statements.

A loop $\psi$ in $\text{Ham}(N, \Omega)$ determines a Hamiltonian fibre bundle $E \to S^2$ with standard fibre $N$, via the clutching construction. Various characteristic numbers for the fibre bundle $E$ have been defined in [4]. These numbers give rise to topological invariants of the loop $\psi$. In this article, we will consider only the following characteristic number

\begin{equation}
I(\psi) := \int_E c_1(VTE) c^n,
\end{equation}

where $VTE$ is the vertical tangent bundle of $E$ and $c \in H^2(E, \mathbb{R})$ is the coupling class of the fibration $E \to S^2$ [3, 6]. $I(\psi)$ depends only on the homotopy class of

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the loop $\psi$. Moreover, the map

$$I : \psi \in \pi_1(\text{Ham}(N, \Omega)) \mapsto I(\psi) \in \mathbb{R}$$

is an $\mathbb{R}$-valued group homomorphism [4].

Our purpose is to study this characteristic number when $N$ is a toric manifold and $\psi$ is a 1-parameter subgroup of $\text{Ham}(N)$ defined by the toric action. The referred 1-parameter subgroup is determined by an element $b$ in the integer lattice of the Lie algebra of the corresponding torus. On the other hand, a toric symplectic manifold is determined by its moment polytope. For a general polytope, a mass linear function on it is a linear function “whose value on the center of mass of the polytope depends linearly on the positions of the supporting hyperplanes” [7]. In this article, we will relate the vanishing of the number $I(\psi)$ with the fact that $b$ defines a mass linear function on the polytope associated with the toric manifold. In the following paragraphs, we provide a more detailed exposition of this relation.

Let $T$ be the torus $(U(1))^n$ and $\Delta = \Delta(n, k)$ the polytope in $t^*$ with $m$ facets defined by

$$\Delta(n, k) = \bigcap_{j=1}^{m} \{ x \in t^* : \langle x, n_j \rangle \leq k_j \},$$

where $k_j \in \mathbb{R}$ and the $n_j \in t$ are the outward conormals to the facets. The facet defined by the equation $\langle x, n_j \rangle = k_j$ will be denoted $F_j$, and we put $\text{Cm}(\Delta)$ for the mass center of the polytope $\Delta$.

In [7] the chamber $C_\Delta$ of $\Delta$ is defined as the set of $k' \in \mathbb{R}^m$ such that the polytope $\Delta' := \Delta(n, k')$ is analogous to $\Delta$; that is, the intersection $\bigcap_{j \in J} F_j$ is nonempty iff $\bigcap_{j \in J} F'_j \neq \emptyset$ for any $J \subset \{1, \ldots, m\}$. When we consider only polytopes which belong to the chamber of a fixed polytope we delete the $n$ in the notation introduced in (1.3).

Further, McDuff and Tolman introduced the concept of mass linear pair: Given the polytope $\Delta$ and $b \in t$, the pair $(\Delta, b)$ is mass linear if the map

$$k \in C_\Delta \mapsto \langle \text{Cm}(\Delta(k)), b \rangle \in \mathbb{R}$$

is linear. That is,

$$\langle \text{Cm}(\Delta(k)), b \rangle = \sum_j R_j k_j + C,$$

where $R_j$ and $C$ are constant.

Let us assume that $\Delta$ is a Delzant polytope [1]. We shall denote by $(M_\Delta, \omega_\Delta, \mu_\Delta)$ the toric manifold determined by $\Delta$ ($\mu_\Delta : M \to t^*$ being the corresponding moment map). Given $b$, an element in the integer lattice of $t$, we shall write $\psi_b$ for the loop of Hamiltonian diffeomorphisms of $(M_\Delta, \omega_\Delta)$ defined by $b$ through the toric action. We will let $I(\Delta; b)$ for the characteristic number $I(\psi_b)$. When we consider only polytopes in the chamber of a given polytope, we will write $I(k; b)$ instead of $I(\Delta(k); b)$ for $k$ in this chamber.

The group $G$ of the translations defined by the elements of $t^*$ acts freely on $C_\Delta$. We put $r := m - n$ for the dimension of the quotient $C_\Delta/G$. Thus, $r$ is the number of effective parameters which characterize the polytopes in $C_\Delta$ considered as “physical bodies”.

We will prove the following theorem:
**Theorem 1.** Let \((\Delta, b)\) be a pair consisting of a Delzant polytope in \(t^*\) and an element in the integer lattice of \(t\). If \(r \leq 2\), the following statements are equivalent

(a) \(I(k; b) = 0\), for all \(k \in C_\Delta\).

(b) \((\Delta, b)\) is a mass linear pair as in (1.5), with \(\sum_j R_j = 0\).

In [12], by direct computation, we proved the equivalence between the vanishing of \(I(k; b)\) on \(C_\Delta\) and the fact that \((\Delta, b)\) is a mass linear pair, when \(\Delta\) satisfies any of the following conditions:

(i) it is the trapezium associated with a Hirzebruch surface,

(ii) it is a \(\Delta_p\) bundle over \(\Delta_1\) [7],

(iii) \(\Delta\) is the truncated simplex associated with the one point blow up of \(\mathbb{C}P^n\).

On the other hand, when \(\Delta\) is any of these polytopes (i)-(iii), the number \(r\) is equal to 2; thus, from Theorem 1 and the result of [12], it follows that condition (b) does not hold for \((\Delta, b)\). More precisely, we have the following consequence of Theorem 1:

**Theorem 2.** Given the Delzant polytope \(\Delta\) and \(b\) an element in the integer lattice of \(t\). If \(r \leq 2\) and \((\Delta, b)\) is not mass linear, then \(\psi_b\) generates an infinite cyclic subgroup in \(\pi_1(\text{Ham}(M_\Delta(k), \omega_\Delta(k)))\), for all \(k \in C_\Delta\).

In the proof of Theorem 1, a formula for the characteristic number \(I(\psi_b)\) obtained in [11] plays a crucial role. This formula gives \(I(\psi_b)\) in terms of the integrals, on the facets of the polytope, of the normalized Hamiltonian function corresponding to the loop \(\psi_b\) (see (2.9)). From this expression for \(I(\psi_b)\), we will deduce a relation between the directional derivative of map (1.4) along the vector \((1, \ldots, 1)\) of \(\mathbb{R}^m\), the Euclidean volume of \(\Delta(k)\) and \(I(k; b)\) (see (3.1)). From this relation, it is easy to complete the proof of Theorem 1.

This article is organized as follows: In Section 2, we study the characteristic number \(I(k; b)\), when \((\Delta, b)\) is a linear pair and \(k\) varies in the chamber of \(\Delta\); we prove that \(I(k; b)\) is a homogeneous polynomial of the \(k_j\) (Proposition 6).

In Section 3, we prove Theorem 1. In Proposition 11, a sufficient geometric condition for the Delzant polytope \(\Delta\) to admit a mass linear pair \((\Delta, b)\) is given. For a Delzant polytope \(\Delta\), Proposition 12 gives a necessary condition for the vanishing of \(I(k; b)\) on \(C_\Delta\). We also express \(\sum_j R_j\) in terms of the displacement of the center of mass \(\text{Cm}(\Delta(k))\) produced by the change \(k_j \rightarrow k_j + 1\) (Proposition 13).

Section 4 concerns the form which Theorem 2 adopts, when \(\Delta\) is a Delzant polytope of the particular types (i)-(iii) mentioned above (see Corollary 11, Theorems 17 and 20). We also prove that, in these particular cases, if \((\Delta, b)\) is a mass linear pair, then \(\sum_j R_j = 0\).

2. A CHARACTERISTIC NUMBER

Let us suppose that the polytope \(\Delta\) defined in [11] is a Delzant polytope in \(t^*\). Following [2], we recall some points of the construction of \((M_\Delta, \omega_\Delta, \mu_\Delta)\) from the
polytope $\Delta$. We put $\tilde{T} := (S^1)^{m-n}$. The $n_i$ determine weights $w_j \in \tilde{t}^*$, $j = 1, \ldots, m$ for a $T$-action on $C^m$. Then moment map for this action is

$$J : z \in C^m \mapsto J(z) = \pi \sum_{j=1}^{m} |z_j|^2 w_j \in \tilde{t}^*.$$  

The $k_i$ define a regular value $\sigma$ for $J$, and the manifold $M_\Delta$ is the following orbit space

$$M_\Delta = \{ z \in C^m : \pi \sum_{j=1}^{m} |z_j|^2 w_j = \sigma \}/\tilde{T},$$  

where the relation defined by $\tilde{T}$ is

$$\text{(2.2) } (z_j) \simeq (z'_j) \text{ iff there is } y \in \tilde{t} \text{ such that } z'_j = z_j e^{2\pi i \langle \nu, y \rangle} \text{ for } j = 1, \ldots, m.$$  

Identifying $\tilde{t}^*$ with $\mathbb{R}^r$, $\sigma = (\sigma_1, \ldots, \sigma_r)$ and each $\sigma_a$ is a linear combination of the $k_j$.

Given a facet $F$ of $\Delta$, we choose a vertex $p$ of $F$. After a possible change in numeration of the facets, we can assume that $F_1, \ldots, F_n$ intersect at $p$. In this numeration $F = F_j$, for some $j \in \{1, \ldots, n\}$.

If we write $z_a = \rho_a e^{i\theta_a}$, then the symplectic form can be written on $\{[z] \in M : z_a \neq 0, \forall a\}$

$$\omega_\Delta = (1/2) \sum_{i=1}^{n} d\rho_i^2 \wedge d\varphi_i,$$

with $\varphi_i$ an angular variable, linear combination of the $\theta_a$.

The action of $T = (S^1)^n$ on $M_\Delta$

$$(\alpha_1, \ldots, \alpha_n)[z_1, \ldots, z_m] := [\alpha_1 z_1, \ldots, \alpha_n z_n, z_{n+1}, \ldots, z_m]$$

endows $M_\Delta$ with a structure of toric manifold. Identifying $t^*$ with $\mathbb{R}^n$, the moment map $\mu_\Delta : M_\Delta \to t = \mathbb{R}^n$ is defined by

$$\text{(2.4) } \mu_\Delta([z]) = \pi (\rho_1^2, \ldots, \rho_n^2) + (d_1, \ldots, d_n),$$

where the constants $d_i$ are linear combinations of the $k_j$ and

$$\text{(2.5) } \text{im} \mu_\Delta = \Delta.$$  

The facet $F = F_j$ of $\Delta$ is the image by $\mu_\Delta$ of the submanifold

$${D}_j = \{ [z_1, \ldots, z_m] \in M_\Delta : z_j = 0 \}.$$  

We write $x_i := \pi \rho_i^2$, then

$$\text{(2.6) } \int_{M_\Delta} (\omega_\Delta)^n = n! \int_{\Delta} dx_1 \ldots dx_n.$$  

Let $b$ be an element in the integer lattice of $t$. The normalized Hamiltonian of the circle action generated by $b$ is the function $f$ determined by,

$$f = \langle \mu_\Delta, b \rangle + \text{constant} \text{ and } \int_{M_\Delta} f (\omega_\Delta)^n = 0.$$  

That is, $f = \langle \mu_\Delta, b \rangle - \langle Cm(\Delta), b \rangle$, where

$$\text{(2.7) } \langle Cm(\Delta), b \rangle = \frac{\int_M \langle \mu_\Delta, b \rangle (\omega_\Delta)^n}{\int_M (\omega_\Delta)^n}.$$
Moreover,

$$\int_{M_\Delta} \langle \mu_\Delta, b \rangle (\omega_\Delta)^n = n! \int_{\Delta} \sum_{i=1}^{n} b_i x_i \, dx_1 \ldots dx_n.$$  \hspace{1cm} (2.8)

An expression for the value of the invariant $I(\psi_b)$ in terms of integrals of the Hamiltonian function has been obtained in Section 4 of [11] (see also [10] and [9])

$$I(\Delta; b) := I(\psi_b) = -n \sum_{F \text{ facet}} N(F),$$  \hspace{1cm} (2.9)

where the contribution $N(F)$ of the above facet $F = F_j$ (with $j = 1, \ldots, n$) is

$$N_j := N(F) = (n-1)! \int_{F_j} f \, dx_1 \ldots \hat{dx}_j \ldots dx_n$$

with $dx_1 \ldots \hat{dx}_j \ldots dx_n := dx_1 \ldots dx_{j-1}dx_{j+1} \ldots dx_n$.

Given $\Delta = \Delta(n, k)$, we consider the polytope $\Delta' = \Delta(n, k')$ obtained from $\Delta$ by the translation defined by a vector $a$ of $t^*$. As we said, we write $I(k; b)$ and $I(k'; b)$ for the corresponding characteristic numbers. According to the construction of the respective toric manifolds,

$M_{\Delta'} = M_\Delta, \quad \omega_{\Delta'} = \omega_\Delta, \quad \mu_{\Delta'} = \mu_\Delta + a.$

But the *normalized* Hamiltonians $f$ and $f'$ corresponding to the action of $b$ on $M_\Delta$ and $M_{\Delta'}$ are equal. Thus, it follows from (2.9) that $I(k; b) = I(k'; b)$. More precisely, we have the evident proposition:

**Proposition 3.** If $a$ is an arbitrary vector of $t^*$, then $I(k; b) = I(k'; b)$, for $k'_j = k_j + \langle a, n_j \rangle$, $j = 1, \ldots, m$.

By Proposition 3, we can assume that all $d_j$ in (2.4) are zero for the determination of $I(k; b)$.

The following lemma is elementary:

**Lemma 4.** If

$$S_n(\tau) := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \left| \sum_{i=1}^{n} x_i \leq \tau, \quad 0 \leq x_j, \quad \forall j \right. \right\},$$

then

$$\int_{S_n(\tau)} f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n = \begin{cases} \frac{\tau^n}{n!}, & \text{if } f = 1 \\
^{n+c}_{(n+c)!}, & \text{if } f = x_i^c, \quad c = 1, 2 \\
^{n+2}_{(n+2)!}, & \text{if } f = x_ix_j, \quad i \neq j. 
\end{cases}$$

More general, if $c_1, \ldots, c_n \in \mathbb{R}_{> 0}$, we put

$$S_n(c, \tau) := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \left| \sum_{i=1}^{n} c_i x_i \leq \tau, \quad 0 \leq x_j, \quad \forall j \right. \right\},$$
an hyperplane \( \langle \beta \rangle \) of Euclidean motions in \( \mathbb{R}^n \) of subsets of \( \Delta \) such that:

\[
T \text{ the } \kappa \text{ vertices are the solutions to (2.14)}
\]

\[
\int_{S_n(c, \tau)} dx_1 \ldots dx_n = \frac{1}{n!} \prod_{i=1}^{n} \frac{\tau}{c_i}, \int_{S_n(c, \tau)} x_j dx_1 \ldots dx_n = \frac{1}{(n+1)!} \frac{\tau}{c_j} \prod_{i=1}^{n} \frac{\tau}{c_i}
\]

Thus, in the particular case that \( \Delta = S_n(c, \tau) \), the integral \( \int_{M_\Delta} (\omega_\Delta)^n \) is a monomial of degree \( n \) in \( \tau \), and \( \int_{M_\Delta} \langle \mu_\Delta, b \rangle (\omega_\Delta)^n \) is a monomial of degree \( n + 1 \).

We return to the general case in which \( \Delta \) is the polytope defined in (1.3). Its vertices are the solutions to (2.12), hence, the coordinates of any vertex of \( \Delta \) are linear combinations of the \( k_j \).

A hyperplane in \( \mathbb{R}^n \) through a vertex \( (x^0_1, \ldots, x^0_n) \) of \( \Delta \) is given by an equation of the form

\[
\langle x, n \rangle = \langle x^0, n \rangle =: \kappa.
\]

Thus, the independent term \( \kappa \) is a linear combination (l. c.) of the \( k_j \). Moreover, the coordinates of the common point of \( n \) hyperplanes

\[
\langle x, \tilde{n}_i \rangle = \kappa_i,
\]

with \( \kappa_i \) l. c. of the \( k_j \) are also l. c. of the \( k_j \).

By drawing hyperplanes through vertices of \( \Delta \) (or more generally, through points which are the intersection of \( n \) hyperplanes as (2.14)), we can obtain a family \( \{ \beta S \} \) of subsets of \( \Delta \) such that:

a) Each \( \beta S \) is the transformed of a simplex \( S_n(b, \tau) \) by an element of the group of Euclidean motions in \( \mathbb{R}^n \).

b) For \( \alpha \neq \beta, \alpha S \cap \beta S \) is a subset of the border of \( \alpha S \).

c) \( \bigcup_{\beta} \beta S = \Delta \).

Thus, by construction, each facet of \( \beta S \) is contained in a hyperplane \( \pi \) of the form \( \langle x, n \rangle = \kappa \), with \( \kappa \) l. c. of the \( k_j \).

On the other hand, the hyperplane \( \pi \) is transformed by an element of \( SO(n) \) in an hyperplane \( \langle x, n' \rangle = \kappa \). If \( T \) is a translation in \( \mathbb{R}^n \) which applies \( S_n(b, \tau) \) onto \( \beta S \), then this transformation maps \( (0, \ldots, 0) \) in a vertex \( a = (a_1, \ldots, a_n) \) of \( \beta S \). So, the translation \( T \) transforms \( \pi \) in \( \langle x, n \rangle = \kappa + \langle a, n \rangle =: \kappa' \). As each \( a_j \) is a l. c. of the \( k_j \), so is \( \kappa' \). Hence, any element of the group of Euclidean motions in \( \mathbb{R}^n \) which maps \( S_n(b, \tau) \) onto \( \beta S \) transforms the hyperplane \( \pi \)

\[
\langle x, n' \rangle = \kappa',
\]

with \( \kappa' \) a l. c. of the \( k_j \).

Let assume that \( (R T_a)(S(b, \tau)) = \beta S \), with \( R \in SO(n) \) and \( T_a \) the translation defined by \( a \). Then the oblique facet of \( S(b, \tau) \), contained in the hyperplane \( \sum b_i x_i = \tau \), is the image by \( T_{-a} R^{-1} \) of a facet of \( \beta S \), which in turn is contained in a hyperplane of equation (2.15) (\( \kappa' \) being a l. c. of the \( k_j \)). The argument of the preceding paragraph applied to \( R^{-1} \) and \( T_{-a} \) proves that \( \tau \) is a l. c. of the \( k_j \). Hence, by (2.11) the integral

\[
\int_{\beta S} dx_1 \ldots dx_n = \int_{S_n(b, \tau)} dx_1 \ldots dx_n
\]
is a monomial of degree \( n \) of a l. c. of the \( k_j \). Thus,

\[(2.16) \quad \int_M (\omega_\Delta)^n = \sum_\beta \int_{\beta S} dx_1 \ldots dx_n,\]

is a homogeneous polynomial of degree \( n \) of the \( k_j \).

Similarly,

\[(2.17) \quad \int_{M_\Delta} \langle \mu_\Delta, b \rangle (\omega_\Delta)^n\]

is a homogeneous polynomial of degree \( n + 1 \) of the \( k_j \). Analogous results hold for

\[\int_{F_j} dx_1 \ldots \hat{dx}_j \ldots dx_n, \quad \text{and} \quad \int_{F_j} \langle \mu_\Delta, b \rangle dx_1 \ldots \hat{dx}_j \ldots dx_n.\]

From formulas (2.6)-(2.10) together with the preceding argument, it follows the following proposition:

**Proposition 5.** Given a Delzant polytope \( \Delta \), if \( b \) belongs to the integer lattice of \( t \), then \( I(k; b) \) is a rational function of the \( k_j \), for \( k \in C_\Delta \).

Analogously, we have

**Proposition 6.** If \( (\Delta, b) \) is mass linear pair, then \( I(k; b) \) is a homogeneous polynomial in the \( k_j \) of degree \( n \), when \( k \in C_\Delta \).

We will use the following simple lemma in the proof of Theorem 1.

**Lemma 7.** If \( \hat{k}_j = sk_j \) for \( j = 1, \ldots, m \), with \( s \in \mathbb{R} \), then \( \text{Cm}(\Delta(n, \hat{k})) = s \text{Cm}(\Delta(n, k)) \).

**Proof.** The vertices of \( \Delta(n, k) \) are the solutions of (2.12) and the vertices of \( \Delta(n, \hat{k}) \) are the solutions of \( \langle x, n_j a \rangle = sk_j a \), with \( a = 1, \ldots, n \). Thus, the vertices of \( \Delta(n, \hat{k}) \) are those of \( \Delta(n, k) \) multiplied by \( s \). \( \square \)

The Lemma also follows from the fact that (2.16) and (2.17) are homogeneous polynomials of degree \( n \) and \( n + 1 \), respectively.

3. Proof of Theorem 1

Let us assume that the polytope \( \Delta \) defined by (1.3) is Delzant and let \( k \) be an element of \( C_\Delta \). We denote by \( M_\Delta(k) \), \( \omega_\Delta(k) \) and \( \mu_\Delta(k) \), the manifold, the symplectic structure and the moment map (resp.) determined by \( \Delta(k) \). The facets of \( \Delta(k) \) will be denoted by by \( F(k)_j \).

Let \( b \) be an element in the integer lattice of \( t \). We put

\[ A_\Delta(k) := \int_{M(k)} \langle \mu(k), b \rangle (\omega(k))^n, \quad B_\Delta(k) := \int_{M(k)} (\omega(k))^n. \]

By (2.6), \( \frac{1}{n!} B(k) \) is the Euclidean volume of the polytope \( \Delta(k) \). Given a facet \( F(k)_j \), we can assume that \( j \in \{1, \ldots, n\} \) (see third paragraph of Section 2). So, \( F(k)_j \) is defined by the equation \( x_j = 0 \). If we make an infinitesimal variation of the facet \( F(k)_j \), by means of the translation defined by \( k_j \to k_j + \epsilon \) (keeping unchanged the other \( k_i \)), then the volume of \( \Delta(k) \) changes according to

\[ \frac{1}{n!} B(k) \to \frac{1}{n!} B(k) + \epsilon \int_{F(k)_j} dx_1 \ldots \hat{dx}_j \ldots dx_n + O(\epsilon^2). \]
We write \( dX^j \) for \( dx_1 \ldots dx_j \ldots dx_n \). Thus,
\[
\frac{\partial B^{(k)}}{\partial k_j} = n! \int_{F_{(k_j)}} dX^j, \quad \frac{\partial A^{(k)}}{\partial k_j} = n! \int_{F_{(k_j)}} \langle \mu^{(k)}, b \rangle dX^j.
\]
So, by (2.7),
\[
\frac{\partial}{\partial k_j} \langle \text{CM}(\Delta(k)), b \rangle = \frac{n!}{(B^{(k)})^2} \left( B^{(k)} \int_{F_{(k_j)}} \langle \mu^{(k)}, b \rangle dX^j - A^{(k)} \int_{F_{(k_j)}} dX^j \right).
\]
From (2.9) and (2.10), it follows
\[
(3.1) \quad \sum_{j=1}^{m} \frac{\partial}{\partial k_j} \langle \text{CM}(\Delta(k)), b \rangle = -\frac{1}{B^{(k)}} I(k; b).
\]
Thus, we have proved the following proposition:

**Proposition 8.** \( I(k; b) = 0 \) for all \( k \in \mathcal{C}_\Delta \) iff \( \sum_{j=1}^{m} \frac{\partial}{\partial k_j} \langle \text{CM}(\Delta(k)), b \rangle = 0 \), for all \( k \in \mathcal{C}_\Delta \).

Next, we will parametrize the quotient \( \mathcal{C}_\Delta/G \) (of classes of polytopes in \( \mathcal{C}_\Delta \) modulo translation) defined in the Introduction.

After a possible renumbering, we may assume that the intersection of facets \( F_1, \ldots, F_n \) is a vertex of \( \Delta \). Thus, the conormals \( n_1, \ldots, n_n \) are linearly independent in \( \mathfrak{t} \). So, given \( k \in \mathcal{C}_\Delta \), there is a unique \( v \in \mathfrak{t}^* \), such that,
\[
(3.2) \quad \langle v, n_i \rangle + k_i = 0, \quad i = 1, \ldots, n.
\]
(Expressing the \( n_i \) in terms of a basis of \( \mathfrak{t} \) and \( v \) in the dual basis, (3.2) is a compatible and determined system of linear equations for the coordinates of \( v \).) Moreover \( v = v(k) \) depends linearly of the \( k_i \); that is, \( \langle v(k), c \rangle \) is a linear function of \( k_1, \ldots, k_n \), for all \( c \in \mathfrak{t} \).

If \( m - n = 2 \), we write
\[
\lambda = k_{n+1} + \langle v(k), n_{n+1} \rangle, \quad \tau = k_m + \langle v(k), n_m \rangle,
\]
where \( v(k) \) the element in \( \mathfrak{t}^* \) defined by (3.2). From the linearity of \( v \) with respect to the \( k_i \), it follows that \( \lambda \) and \( \tau \) are linear combinations of \( k_1, \ldots, k_m \).

The polytope in \( \mathcal{C}_\Delta \) defined by \( (k'_1 = 0, \ldots, k'_n = 0, \lambda, \tau) \) will be denoted by \( \Delta_0(\lambda, \tau) \). It is the result of the translation of \( \Delta(k) \) by the vector \( v(k) \); i. e.,
\[
(3.3) \quad \Delta_0(\lambda, \tau) = \Delta(k) + v(k).
\]
Let \( b \) an element in the integer lattice of \( \mathfrak{t} \), we define the function \( g \) by
\[
g(\lambda, \tau) := \langle \text{CM}(\Delta_0(\lambda, \tau)), b \rangle.
\]
The function \( g \) is defined on the pairs \( (\lambda, \tau) \) such that \( (0, \ldots, 0, \lambda, \tau) \in \mathcal{C}_\Delta \). By Lemma 7 it follows
\[
g(s\lambda, s\tau) = sg(\lambda, \tau),
\]
for any real number \( s \) such that \( (s\lambda, s\tau) \) belongs to the domain of \( g \). This property implies that
\[
(3.4) \quad g = \lambda \frac{\partial g}{\partial \lambda} + \tau \frac{\partial g}{\partial \tau}.
\]

**Theorem 9.** If \( I(k; b) = 0 \), for all \( k \in \mathcal{C}_\Delta \) and \( r = 2 \), then \( \langle \text{CM}(\Delta(k)), b \rangle = \sum_j R_j k_j \), with \( R_j \) constant (that is, \( (\Delta, b) \) is a mass linear pair) and \( \sum_j R_j = 0 \).
Proof. We set \( f(k_1, \ldots, k_m) := \langle \mathcal{C}_m(\Delta(k), b) \rangle \). It follows from (3.3) that
\[
(3.5) \quad f(k) = g(\lambda, \tau) - \langle v(k), b \rangle.
\]
By the hypothesis and Proposition 8,
\[
(3.6) \quad \sum_{j=1}^{m} \frac{\partial f}{\partial k_j} = 0.
\]
Since
\[
\sum_{j=1}^{m} \frac{\partial f}{\partial k_j} = \frac{\partial g}{\partial \lambda} \sum_{j=1}^{m} \frac{\partial \lambda}{\partial k_j} + \frac{\partial g}{\partial \tau} \sum_{j=1}^{m} \frac{\partial \tau}{\partial k_j} - \langle \frac{\partial v}{\partial k_j}, b \rangle,
\]
from (3.6) we deduce
\[
(3.7) \quad p \frac{\partial g}{\partial \lambda} + q \frac{\partial g}{\partial \tau} - t = 0,
\]
where \( p, q, t \) stand for the following constants
\[
p = \sum_{j=1}^{m} \frac{\partial \lambda}{\partial k_j}, \quad q = \sum_{j=1}^{m} \frac{\partial \tau}{\partial k_j}, \quad t = \langle \frac{\partial v}{\partial k_j}, b \rangle.
\]
Since \( q\lambda - p\tau \) and \( t\tau - qg \) are first integrals of (3.7), the general solution of this equation is
\[
(3.8) \quad g(\lambda, \tau) = \left( \frac{t}{q} \right) \tau + \Phi(q\lambda - p\tau),
\]
where \( \Phi \) is a derivable function of one variable.

It follows from (3.3) and (3.3) that
\[
(3.9) \quad \Phi(u) = u \Phi'(u).
\]
Thus, \( \Phi(u) = \alpha u \), with \( \alpha \) constant. We have for \( f \)
\[
f(k) = (b/q)\tau + \alpha(q\lambda - p\tau) - \langle v(k), b \rangle.
\]
In other words, \( f \) is a linear function of the \( k_j \); i.e., \( f(k) = \sum_j R_j k_j \), with \( R_j \) constant. From (3.6), it follows \( \sum_j R_j = 0 \). \( \square \)

Remark. The proof of Theorem 9 can be adapted to the simpler case when \( r = 1 \). In this case, the function \( g(\lambda) = \langle \mathcal{C}_1(\Delta_0(\lambda), b) \rangle \), satisfies \( p \frac{\partial g}{\partial \lambda} - t = 0 \) and \( g(s\lambda) = sg(\lambda) \). So, \( g(\lambda) = (t/p)\lambda + \langle v, b \rangle = \langle v(\lambda), b \rangle \) is a linear map of the variables \( k_j \).

On the other hand, the proof of this theorem does not admit an adaptation to the case \( r > 2 \). In fact, the corresponding function \( \Phi \) would be a function of \( r - 1 \) variables \( \Phi(u_1, \ldots, u_{r-1}) \). The equation which corresponds to (3.9) in this case would be
\[
\Phi = \sum_{i=1}^{r-1} u_i \frac{\partial \Phi}{\partial u_i}.
\]
But this condition does not implies the linearity of \( \Phi \).

When \( (\Delta, b) \) is a mass linear pair as in (1.3), by (3.1)
\[
(3.10) \quad I(k; b) = -B(k) \sum_j R_j,
\]
for all \( k \in C_\Delta \). From (3.10), we deduce the following proposition:
Proposition 10. Let \(\Delta, b\) be a mass linear pair. \(I(k; b) = 0\) for all \(k \in C_\Delta\) iff \(\sum_j R_j = 0\).

**Proof of Theorem** It is a direct consequence of Proposition 10, Theorem 9 and the Remark above. □

We will deduce a sufficient condition for a Delzant polytope \(\Delta\) to admit mass linear functions. We write \(\dot{\mathcal{C}}_{m}(\Delta(k)) := d \frac{d}{d\epsilon}_{\epsilon=0} \mathcal{C}_{m}(\Delta(k+\epsilon)), \) with \(\dot{\epsilon} = (\epsilon, \ldots, \epsilon)\).

Proposition 11. If all points \(\dot{\mathcal{C}}_{m}(\Delta(k))\), for \(k \in C_\Delta\), belong to a hyperplane of \((\mathbb{R}^n)^*\) with a conormal vector in \(\mathbb{Z}^n\) and \(r \leq 2\), then \(\Delta\) admits a mass linear function.

**Proof.** Let \(b \in \mathbb{Z}^n\) be a conormal vector to the hyperplane, then
\[0 = \langle \dot{\mathcal{C}}_{m}(\Delta(k)), b \rangle = \langle \sum_j \frac{\partial}{\partial k_j} \mathcal{C}_{m}(\Delta(k)), b \rangle.\]

By (3.1), \(I(k, b) = 0\); Theorem 9 applies and \((\Delta, b)\) is a mass linear pair. □

**Proposition 12.** Let \(\Delta\) be a Delzant polytope, such that \(k = 0\) belongs to the closure of \(C_\Delta\). If \(r \leq 2\), a necessary condition for the vanishing of \(I(k; b)\) on \(C_\Delta\) is
\[\langle d \frac{d}{d\epsilon}_{\epsilon=0} \mathcal{C}_{m}(\Delta(\dot{\epsilon})), b \rangle = 0.\]

**Proof.** If \(I(k; b)\) vanishes on \(C_\Delta\), then \((\Delta, b)\) is a linear pair, by Theorem 1. Thus, \(\langle \mathcal{C}_{m}(\Delta(k)), b \rangle = \sum_j R_j k_j + C\), on \(C_\Delta\). So, given \(k \in C_\Delta\) and \(\epsilon\) small enough
\[\langle \mathcal{C}_{m}(\Delta(k+\dot{\epsilon})), b \rangle = \sum_j R_j k_j + \epsilon \sum_j R_j + C.\]

By Theorem 1 \(\sum_j R_j = 0\). Thus, for any \(k \in C_\Delta\),
\[\frac{d}{d\epsilon}_{\epsilon=0} \langle \mathcal{C}_{m}(\Delta(k+\dot{\epsilon})), b \rangle = 0.\]

Taking the limit as \(k \to 0\),
\[0 = \lim_{k \to 0} \frac{d}{d\epsilon}_{\epsilon=0} \langle \mathcal{C}_{m}(\Delta(k+\dot{\epsilon})), b \rangle = \langle \frac{d}{d\epsilon}_{\epsilon=0} \mathcal{C}_{m}(\Delta(\dot{\epsilon})), b \rangle.\]

Next, we will describe a geometric interpretation of the number \(\sum_j R_j\). Given an arbitrary Delzant polytope \(\Delta\). If \(a\) is a vector of \(t^*\), then
\[\langle \mathcal{C}_{m}(\Delta(k')), \mathcal{C}_{m}(\Delta(k)) + a, \]
if \(k'_j = k_j + \langle a, n_j \rangle\).

We will denote by \(d\) the element of \(t^*\) defined by the following relation
\[\langle \mathcal{C}_{m}(\Delta(\tilde{k})), \mathcal{C}_{m}(\Delta(\tilde{k})) + d, \]
with \(\tilde{k}_j = k_j + 1\) for all \(j\).
that is, \( N \) with vertices is a toric manifold. The corresponding moment polytope \( \Delta \) is the trapezium in straights \( k \).

The vertices of \( \Delta(\tau) \) for \( (k, T) \) where the equivalence defined by the quotient \( \langle \cdot, \cdot \rangle \) is given by

\[
\langle \text{Cm}(\Delta(k_j + \langle d, \mathbf{n}_j \rangle)), \mathbf{b} \rangle = \sum R_j k_j + \sum R_j \langle d, \mathbf{n}_j \rangle + C.
\]

These formulas allow us to state the following proposition, that gives an interpretation of the sum \( \sum R_j d \) in terms of the variation of Cm(\( \Delta(k) \)) with the \( k_j \).

**Proposition 13.** Let \( (\Delta, \mathbf{b}) \) be a mass linear pair as in (1.3). Then,

\[
\sum_j R_j \langle d, \mathbf{n}_j \rangle = \langle d, \mathbf{b} \rangle = \sum_j R_j,
\]

d being the element of \( t^* \) defined by (5.12).

4. Examples

In this Section, we will deduce the particular form which adopts Theorem 2 when \( \Delta \) is a polytope of the types (i)-(iii) mentioned in the Introduction. For each case, we will determine the center of mass of the corresponding polytope \( \Delta(k) \) and the condition for \( (\Delta, \mathbf{b}) \) to be a mass linear pair. We will dedicate a subsection to each type.

4.1. Hirzebruch surfaces. Given \( r \in \mathbb{Z}_{>0} \) and \( \tau, \lambda \in \mathbb{R}_{>0} \) with \( \sigma := \tau - r \lambda > 0 \), in [12] we considered the Hirzebruch surface \( N \) determined by these numbers. \( N \) is the quotient

\[
\{ z \in \mathbb{C}^4 : |z_1|^2 + r|z_2|^2 + |z_4|^2 = \tau/\pi, |z_2|^2 + |z_3|^2 = \lambda/\pi \}/T,
\]

where the equivalence defined by \( T = (S^1)^2 \) is given by

\[
(a, b) \cdot (z_1, z_2, z_3, z_4) = (a z_1, a^* b z_2, b z_3, a z_4),
\]

for \( (a, b) \in (S^1)^2 \).

The manifold \( N \) equipped with the following \((U(1))^2\) action

\[
(\epsilon_1, \epsilon_2)[z_j] = [\epsilon_1 z_1, \epsilon_2 z_2, z_3, z_4],
\]

is a toric manifold. The corresponding moment polytope \( \Delta \) is the trapezium in \( \mathbb{R}^2 \) with vertices

\[
P_1 = (0, 0), \ P_2 = (0, \lambda), \ P_3 = (\tau, 0), \ P_4 = (\sigma, \lambda).
\]

That is, \( N \) is the toric manifold \( M_\Delta \) determined by the trapezium \( \Delta \).

As the conormals to the facets of \( \Delta \) are the vectors \( \mathbf{n}_1 = (1, 0), \mathbf{n}_2 = (0, 1) \), \( \mathbf{n}_3 = (0, 1) \) and \( \mathbf{n}_4 = (1, r) \), the facets of a generic polytope \( \Delta(k) \) in \( C_\Delta \) are on the straights

\[-x = k_1, \ -y = k_2, \ y = k_3, \ x + ry = k_4.\]

The vertices of \( \Delta(k) \) are the points

\[-(k_1, -k_2), \ (-k_1, k_3), \ (k_4 - rk_3, k_3), \ (k_4 + rk_2, k_2).\]

Thus, the translation in the plane \( x, y \) defined by \( -(k_1, -k_2) \) transforms the trapezium determined by the vertices (1.1) in \( \Delta(k) \), if

\[
\tau = k_4 + rk_2 + k_3, \ \lambda = k_3 + k_2.
\]
(4.3) \( \text{Cm}(\Delta(\phi)) = \text{Cm}(\Delta) + (-k_1, -k_2) \).

Moreover, the mass center of \( \Delta \) is

(4.4) \( \text{Cm}(\Delta) = \left( \frac{3r^2 - 3r\tau \lambda + r^2 \lambda^2}{3(2\tau - r\lambda)}, \frac{3\lambda\tau - 2r\lambda^2}{3(2\tau - r\lambda)} \right) \).

The chamber \( \mathcal{C}_\Delta \) consists of the points \((k_1, \ldots, k_4)\) such that \(\tau - r\lambda > 0\), with \(\tau\) and \(\lambda\) given by (4.3). So, the point \(k = 0\) belongs to the closure of \( \mathcal{C}_\Delta \). From (4.3), together with (4.2) and (4.4), it follows

(4.5) \( \text{Cm}(\Delta(\phi)) = \left( \frac{r^2}{12}, -\frac{r\epsilon}{6} \right), \)

where \(\epsilon = (\epsilon, \epsilon, \epsilon, \epsilon)\). By Proposition 12 if \(I(k; \mathbf{b})\) with \(\mathbf{b} = (b_1, b_2) \in \mathbb{Z}^2\) vanishes on the chamber \( \mathcal{C}_\Delta \), then \(rb_1 - 2b_2 = 0\).

On the other hand, from (4.4) and (4.3), it follows

(4.6) \( \langle \text{Cm}(\Delta(\phi)), \mathbf{b} \rangle = \frac{(3r^2 - 3r\tau \lambda + r^2 \lambda^2)b_1 + (3\lambda\tau - 2r\lambda^2)b_2}{3(2\tau - r\lambda)} - k_1b_1 - k_2b_2. \)

By (4.2), expression (4.6) is linear in the \(k_i\) iff

\[
(3r^2 - 3r\tau \lambda + r^2 \lambda^2)b_1 + (3\lambda\tau - 2r\lambda^2)b_2 = 3(2\tau - r\lambda)(A\tau + B\lambda).
\]

From this relation, it follows the above condition \(rb_1 = 2b_2\). In this case (4.6) reduces to

(4.7) \( \langle \text{Cm}(\Delta(\phi)), \mathbf{b} \rangle = -\frac{b_1}{2}k_1 + \frac{b_1}{2}k_4. \)

Comparing (4.5) with (4.7), we obtain, \(R_1 = -R_4 = \frac{1}{r}, R_2 = R_3 = 0\); so, \(\sum_j R_j = 0\). That is, the condition \(\sum_j R_j = 0\) holds for all the mass pairs \((\Delta, \mathbf{b})\) when \(\Delta\) is the polytope associated to a Hirzebruch surface. Hence,

**Proposition 14.** \((\Delta, \mathbf{b})\) is a mass linear pair iff \(rb_1 = 2b_2\). Moreover, in this case \(\sum_j R_j = 0\).

By Theorem 2 we have

**Corollary 15.** If \(rb_1 \neq 2b_2\), then \(\psi_\mathbf{b}\) generates an infinite cyclic subgroup in \(\pi_1(\text{Ham}(M\Delta, \omega_\Delta))\).

**Remark.**

We denote by \(\phi\) the following isotopy of \(M\Delta\)

\[
\phi_t[z] = [e^{2\pi it} z_1, z_2, z_3, z_4].
\]

\(\phi\) is a loop in the Hamiltonian group of \(M\Delta\). By \(\phi'\) we denote the Hamiltonian loop

\[
\phi'_t[z] = [z_1, e^{2\pi it} z_2, z_3, z_4].
\]

In Theorem 8 of [10] we proved that \(I(\phi') = (-2/r)I(\phi)\). If \(\mathbf{b} = (b_1, b_2) \in \mathbb{Z}^2\), then

\[
I(\psi_\mathbf{b}) = b_1I(\phi) + b_2I(\phi') = (b_1 - (2/r)b_2)I(\phi).
\]
4.2. $\Delta_p$ bundle over $\Delta_1$. Given the integer $p > 1$, as McDuff and Tolman in [7], we consider the following vectors in $\mathbb{R}^{p+1}$

(4.8) $n_i = -e_i$, $i = 1, \ldots, p$. $n_{p+1} = \sum_{i=1}^{p} e_i$, $n_{p+2} = -e_{p+1}$, $n_{p+3} = e_{p+1} - \sum_{i=1}^{p} a_i e_i$,

where $e_1, \ldots, e_{p+1}$ is the standard basis of $\mathbb{R}^{p+1}$ and $a_i \in \mathbb{Z}$. We write $a := (a_1, \ldots, a_p) \in \mathbb{Z}^p$, $A := \sum_{i=1}^{p} a_i$, $a \cdot a = \sum_{i=1}^{p} a_i^2$.

Let $\lambda, \tau$ be real positive numbers with $\lambda + a_i > 0$, for $i = 1, \ldots, p$. In this subsection we will consider the polytope $\Delta$ in $(\mathbb{R}^{p+1})^*$ defined by the above conormals $n_j$ and the following $k_j$

(4.9) $k_1 = \cdots = k_p = k_{p+2} = 0$, $k_{p+1} = \tau$, $k_{p+3} = \lambda$.

This polytope will be also denote by $\Delta_0(\lambda, \tau)$. It is a $\Delta_p$ bundle on $\Delta_1$ (see [7]). When $p = 2$, $\Delta = \Delta_0(\lambda, \tau)$ is the prism whose base is the triangle of vertices $(0, 0, 0), (\tau, 0, 0)$ and $(0, \tau, 0)$ and whose ceiling is the triangle determined by $(0, 0, \lambda)$, $(\tau, 0, \lambda + a_1 \tau)$ and $(0, \tau, \lambda + a_2 \tau)$.

We assume that the above polytope $\Delta$ is a Delzant polytope. The manifold (2.1) is in this case

$M_\Delta = \{ z \in \mathbb{C}^{p+3} : \sum_{i=1}^{p+1} |z_i|^2 = \tau/\pi, -\sum_{j=1}^{p} a_j |z_j|^2 + |z_{p+2}|^2 + |z_{p+3}|^2 = \lambda/\pi \} / \sim,$

where $(z_j) \simeq (z'_j)$ iff there are $\alpha, \beta \in U(1)$ such that $z'_j = \alpha \beta^{-a_j} z_j$, $j = 1, \ldots, p$; $z'_{p+1} = \alpha z_{p+1}$; $z'_{k} = \beta z_k$, $k = p + 2$, $p + 3$.

Thus, $M_\Delta$ is the total space of the fibre bundle $\mathbb{P}(L_1 \oplus \cdots \oplus L_p \oplus \mathbb{C}) \rightarrow \mathbb{C} P^1$, where $L_j$ is the holomorphic line bundle over $\mathbb{C} P^1$ with Chern number $a_j$.

The symplectic form (2.3) is

$\omega_\Delta = (1/2)(\sigma_1 + \cdots + \sigma_p + \sigma_{p+2})$,

where $\sigma_k = d\rho_k^2 \wedge d\varphi_k$.

And the moment map

(4.10) $\mu_\Delta([z]) = (x_1, \ldots, x_p, x_{p+2}),$

where $x_i := \pi \rho_i^2$.

**Proposition 16.** The coordinates $\bar{x}_j$ of $\text{Cm}(\Delta_0(\lambda, \tau))$ are given by:

(4.11) $\bar{x}_k = \frac{\tau}{p+2} \frac{\lambda(p+2) + \tau(A + a_k)}{\lambda(p+1) + \tau A}$, for $k = 1, \ldots, p$.

(4.12) $\bar{x}_{p+2} = \frac{1}{2} \frac{(p+1)(p+2)\lambda^2 + 2(p+2)A\lambda \tau + (a \cdot a + A^2)\tau^2}{(p+2)((p+1)\lambda + A\tau)}.$
Proof. Since the points \([z] \in M_\Delta\) satisfy \(|z_{p+2}|^2 \leq \lambda/\pi + \sum_{j=1}^p a_j|z_j|^2\), by (2.6) and Lemma 4 we have

\[
\int_{M_\Delta} (\omega_\Delta)^{p+1} = (p + 1)! \int_{S_p(\tau)} \left( \lambda + \sum_{j=1}^p a_j x_j \right)^2 = (p + 1)! \left( \frac{\lambda \tau^{p+1}}{p!} + \frac{\tau^{p+2}}{(p+2)!} \sum_{j \neq k} a_j + \frac{2\tau^{p+2}a_k}{(p+2)!} \right).
\]

Similarly, for \(k = 1, \ldots, p\)

\[
\int_{M_\Delta} x_k (\omega_\Delta)^{p+1} = (p + 1)! \left( \frac{\lambda \tau^{p+1}}{p!} + \frac{\tau^{p+2}}{(p+2)!} \sum_{j \neq k} a_j + \frac{2\tau^{p+2}a_k}{(p+2)!} \right).
\]

The \(k\)-th coordinate of \(\text{Cm}(\Delta)\), \(\bar{x}_k\), is the quotient of (4.14) by (4.13); that is,

\[
\bar{x}_k = \frac{\tau}{p+2} \frac{\lambda(p + 2) + \tau (A + a_k)}{\lambda(p + 1) + \tau A}.
\]

For the \(p + 2\)-coordinate of \(\text{Cm}(\Delta)\), we need to calculate \(\int_{M} x_{p+2}(\omega_\Delta)^{p+1}\). By Lemma 4

\[
\frac{1}{(p + 1)!} \int_{M} x_{p+2}(\omega_\Delta)^{p+1} = \frac{1}{2} \int_{S_p(\tau)} \left( \lambda + \sum_{j=1}^p a_j x_j \right)^2 = \frac{1}{2} \left( \lambda^2 \frac{\tau^p}{p!} + \frac{2A \lambda \tau^{p+1}}{(p+1)!} + \frac{(a \cdot a + A^2) \tau^{p+2}}{(p+2)!} \right).
\]

Formula (4.12) is a consequence of (4.13) together with (4.15). \(\square\)

The translation in \((\mathbb{R}^{p+1})^*\) defined by the vector \((-k_1, \ldots, -k_p, -k_{p+2})\) transforms the hyperplanes \(\langle x, n_{p+3}\rangle = \lambda\) and \(\langle x, n_{p+1}\rangle = \tau\) in

\[
\langle x, n_{p+3}\rangle = \lambda - k_{p+2} + \sum_{j=1}^p a_j k_j, \quad \langle x, n_{p+1}\rangle = \tau - \sum_{j=1}^p k_j,
\]

respectively.

Let \(\Delta(k)\) be a polytope with \(k = (k_1, \ldots, k_{p+3})\) generic in the chamber \(C_\Delta\). From (4.10), it follows that \(\Delta(k)\) is the image of the polytope \(\Delta_0(\lambda, \tau)\) by the translation determined by \((-k_1, \ldots, -k_p, -k_{p+2})\), whenever

\[
k_{p+2} - \sum_{j=1}^p a_j k_j + k_{p+3} = \lambda, \quad \sum_{j=1}^p k_j + k_{p+1} = \tau.
\]

In this case,

\[
\text{Cm}(\Delta(k)) = \text{Cm}(\Delta_0(\lambda, \tau)) - (k_1, \ldots, k_p, k_{p+2}).
\]

According to (4.17), the coordinates of the mass center \(\text{Cm}(\Delta(\ell))\), with \(\ell = (\epsilon, \ldots, \epsilon)\), can be obtained substituting in (4.11) and in (4.12) \(\lambda\) by

\[
\epsilon - \sum_{j=1}^p a_j \epsilon + \epsilon = (2 - A)\epsilon
\]

and \(\tau\) by \((p + 1)\epsilon\), and finally take into account (4.18). These operations give

\[
\bar{x}_j(\Delta(\ell)) = \frac{\epsilon}{2(p+2)} ((p + 1)a_j - A), \quad j = 1, \ldots, p
\]

\[
\bar{x}_{p+2}(\Delta(\ell)) = \frac{\epsilon}{4(p+2)} (-A^2 + (p + 1)(a \cdot a)).
\]
Given $\mathbf{b} = (b_1, \ldots, b_p, b)$, with $\hat{\mathbf{b}} = (b_1, \ldots, b_p, 0)$ and $\dot{\mathbf{b}} = (0, \ldots, 0, b)$.

\[
\left. \frac{d}{d\epsilon} \text{Cm}(\Delta(\epsilon)), \right|_{\epsilon=0} \mathbf{b} = \frac{1}{4(p+2)} \left( (p+1)(2\mathbf{a} \cdot \hat{\mathbf{b}} - b \mathbf{a} \cdot \mathbf{a}) - A(2B + bA) \right),
\]

where $\mathbf{a} \cdot \hat{\mathbf{b}} = \sum_{j=1}^{p} a_j b_j$ and $B = \sum_{j=1}^{p} b_j$.

By Proposition 12, we have:

**Theorem 17.** Let $\Delta$ be the $\Delta_p$ bundle over $\Delta_1$ defined by (4.8) and (4.9). Given $\mathbf{b} = (\hat{\mathbf{b}}, \dot{\mathbf{b}}) \in \mathbb{Z}^{p+1}$, if

\[
(p+1)(2\mathbf{a} \cdot \hat{\mathbf{b}} - b \mathbf{a} \cdot \mathbf{a}) - A(2B + bA) \neq 0,
\]

then $\psi_{\mathbf{b}}$ defines an infinite cyclic subgroup in the fundamental group $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$.

It is straightforward to check that (4.19) is also a sufficient condition for $(\Delta, \mathbf{b})$ to be a mass linear pair.

Since

\[
\langle \text{Cm}((\Delta_0(\lambda, \tau)), \mathbf{b}) \rangle = \langle \text{Cm}((\Delta_0(\lambda, \tau)), \hat{\mathbf{b}}) \rangle + \langle \text{Cm}((\Delta_0(\lambda, \tau)), \dot{\mathbf{b}}) \rangle,
\]

if (4.19) holds, using (4.11) and (4.12), one obtains

\[
\langle \text{Cm}((\Delta_0(\lambda, \tau)), \mathbf{b}) \rangle = \frac{b\lambda}{2} + \frac{b}{2(p+2)A} + \frac{(\mathbf{a} \cdot \hat{\mathbf{b}} + AB)}{(p+2)A} \tau.
\]

By (4.18), for $k \in C_\Delta$,

\[
\langle \text{Cm}((\Delta(k)), \mathbf{b}) \rangle = \langle \text{Cm}((\Delta_0(\lambda, \tau)), \mathbf{b}) \rangle - \sum_{j=1}^{p} b_j k_j - b k_{p+2},
\]

with $\lambda$ and $\tau$ given by (4.17).

If $\mathbf{b} = \hat{\mathbf{b}}$, the condition (4.19) reduces to $(p+1)a \cdot \hat{\mathbf{b}} = AB$ and

\[
\langle \text{Cm}((\Delta(k)), \mathbf{b}) \rangle = \sum_{j=1}^{p} R_j k_j,
\]

where

\[
R_j = \frac{(\mathbf{a} \cdot \hat{\mathbf{b}} + AB)}{(p+2)A} - b_j, \quad j = 1, \ldots, p; \quad R_{p+1} = \frac{(\mathbf{a} \cdot \hat{\mathbf{b}} + AB)}{(p+2)A}, \quad R_{p+2} = R_{p+3} = 0.
\]

So,

\[
\sum_{j=1}^{p+3} R_j = \frac{(p+1)a \cdot \hat{\mathbf{b}} - AB}{(p+2)A} = 0.
\]

A similar calculation for the case $\mathbf{b} = \dot{\mathbf{b}}$ shows that the corresponding $\sum_j R_j$ vanishes. That is,

**Proposition 18.** Let $\Delta$ be a $\Delta_p$ bundle over $\Delta_1$. If $(\Delta, \mathbf{b})$ is a mass linear pair, then $\sum_j R_j = 0$. 

For \( p = 2 \), let \( \mathbf{b} \) be the following linear combination of the conormal vectors
\[
\mathbf{b} = \gamma_1 \mathbf{n}_1 + \gamma_2 \mathbf{n}_2 + \gamma_3 \mathbf{n}_3 \quad \text{with} \quad \gamma_1 + \gamma_2 + \gamma_3 = 0.
\]
By (4.8), \( \mathbf{b} = (b_1, b_2, 0) \) with \( b_1 = \gamma_3 - \gamma_1, b_2 = \gamma_3 - \gamma_2 \). In this case condition (4.19) reduces to
\[
3(a_1 b_1 + a_2 b_2) = (a_1 + a_2)(b_1 + b_2).
\]
Or in terms of the \( \gamma_i \)
\[
(4.20) \quad a_1 \gamma_1 + a_2 \gamma_2 = 0.
\]
This is a necessary and sufficient condition for \((\Delta, \mathbf{b})\) to be mass linear. This result is the statement of Lemma 4.8 in [7].

4.3. **One point blow up of \( \mathbb{CP}^n \).** In this subsection \( \Delta \equiv \Delta_0(\lambda, \tau) \) will be
\[
(4.21) \quad \Delta = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \middle| \sum_{i=1}^{n} x_i \leq \tau, \ 0 \leq x_i, \ x_n \leq \lambda \right\},
\]
where \( \tau, \lambda \in \mathbb{R}_{>0} \) and \( \sigma := \tau - \lambda > 0 \). That is, \( \Delta \) is the polytope obtained truncating the simplex \( S_n(\tau) \), defined in Lemma 4, by a “horizontal” hyperplane through the point \( (0, \ldots, 0, \lambda) \). The manifold \( M_\Delta \) associated with \( \Delta \) is the one point blow up of \( \mathbb{CP}^n \).

The mass center of the simplex \( S_n(\tau) \) is the point
\[
(4.22) \quad \text{Cm}(S_n(\tau)) = \frac{\tau}{n+1}w,
\]
with \( w = (1, \ldots, 1) \).

As the volume of \( S_n(\tau) \) is \( \tau^n/n! \), it follows from (4.22)
\[
(\tau^n - \sigma^n) \text{Cm}(\Delta) = \tau^n \frac{\tau}{n+1}w - \sigma^n \left( \frac{\sigma}{n+1}w + \lambda e_n \right).
\]
That is,
\[
(4.23) \quad \text{Cm}(\Delta) = \frac{1}{\tau^n - \sigma^n} \left( \frac{\tau^{n+1} - \sigma^{n+1}}{n+1}w - \lambda \sigma e_n \right).
\]

Given \( k = (k_1, \ldots, k_{n+2}) \in C_\Delta \), the facets of \( \Delta(k) \) are in the following hyperplanes:
\[
(4.24) \quad -x_j = k_j, \quad j = 1, \ldots, n; \quad \sum_{i=1}^{p} x_k = k_{n+1}; \quad x_{n+1} = k_{n+2}.
\]
As in the preceding subsections,
\[
(4.25) \quad \Delta(k) = \Delta_0(\lambda, \tau) - (k_1, \ldots, k_n),
\]
provided \( \lambda = k_n + k_{n+2} \) and \( \tau = \sum_{i=1}^{n+1} k_i \).

The pair \((\Delta, \mathbf{b} = (b_1, \ldots, b_n))\) is mass linear iff there exist \( A, B, C \in \mathbb{R} \) such that
\[
\sum_{j=1}^{n-1} b_j \frac{\tau^{n+1} - \sigma^{n+1}}{n+1} + b_n \left( \frac{\tau^{n+1} - \sigma^{n+1}}{n+1} - (\tau - \sigma)\sigma^n \right) = (A\tau + B\sigma + C)(\tau^n - \sigma^n),
\]
for all \( \tau, \sigma \) “admissible”. A simple calculation proves the following proposition:

**Proposition 19.** The pair \((\Delta, \mathbf{b})\) is mass linear iff
\[
b_n = \frac{1}{n} \sum_{j=1}^{n-1} b_j.
\]
From Theorem 2 together with Proposition 19 it follows the following theorem:

**Theorem 20.** If \( b = (b_1, \ldots, b_n) \in \mathbb{Z}^n \) and \( \sum_{j=1}^{n-1} b_j \neq nb_n \), then \( \psi_b \) generates an infinite cyclic subgroup in \( \pi_1(\text{Ham}(M_\Delta, \omega_\Delta)) \).

For \( k \in C_\Delta \), by (4.26)
\[
\langle \text{Cm}(\Delta(k)), b \rangle = \langle \text{Cm}(\Delta_0(\lambda, \tau)), b \rangle - \sum_{j=1}^{n} b_j k_j.
\]
If \( (\Delta, b) \) is a mass linear pair, by (4.23) and Proposition 19 we have
\[
\langle \text{Cm}(\Delta_0(\lambda, \tau)), b \rangle = b_n \tau.
\]
Thus,
\[
\langle \text{Cm}(\Delta(k)), b \rangle = \sum_{j=1}^{n+1} R_j k_j,
\]
where \( R_j = b_n - b_j \), for \( j = 1, \ldots, n \) and \( R_{n+1} = b_n \). Hence, we have the following proposition:

**Proposition 21.** Let \( \Delta \) be the polytope obtained by truncating the standard \( n \)-simplex \( S_n(\tau) \) by a horizontal hyperplane. If \( (\Delta, b) \) is a mass linear pair, then
\[
\sum_{j=1}^{n} R_j = 0.
\]

**Remark.** When \( n = 3 \) the toric manifold \( M_\Delta \) is
\[
M_\Delta = \left\{ z \in \mathbb{C}^5 : |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = \tau / \pi, |z_3|^2 + |z_4|^2 = \lambda / \pi \right\} / T,
\]
where the action of \( T = (U(1))^2 \) is defined by
\[
(a, b)(z_1, z_2, z_3, z_4, z_5) = (az_1, az_2, abz_3, bz_4, az_5),
\]
for \( a, b \in U(1) \).

We consider the following loops in the Hamiltonian group of \( (M_\Delta, \omega_\Delta) \)
\[
\psi_t[z] = [z_1 e^{2\pi it}, z_2, z_3, z_4, z_5], \quad \psi'_t[z] = [z_1, z_2 e^{2\pi it}, z_3, z_4, z_5],
\]
\[
\tilde{\psi}_t[z] = [z_1, z_2, z_3 e^{2\pi it}, z_4, z_5].
\]
In [11] (Remark in Section 4), we gave formulas that relate the characteristic numbers associated with these loops
\[
I(\psi) = I(\psi') = (-1/3)I(\tilde{\psi}).
\]
So, for \( b = (b_1, b_2, b_3) \in \mathbb{Z}^3 \),
\[
I(\psi_b) = (b_1 + b_2 - 3b_3)I(\psi).
\]
(4.27)

By Proposition 19 the vanishing of \( I(\psi_b) \) in (4.27) is equivalent to the fact that \( (\Delta, b) \) is a mass linear pair. This equivalence is a new checking of Theorem 1.

**References**

[1] T. Delzant, *Hamiltoniens périodique et images convexes de l’application moment*. Bull. Soc. Math. France 116 (1988), 315-338.
[2] V. Guillemin, *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*. Birkhäuser, Boston, (1994).
[3] V. Guillemin, L. Lerman, S. Sternberg, *Symplectic fibrations and multiplicity diagrams*. Cambridge U.P., Cambridge, (1996).
[4] F. Lalonde, D. McDuff, L. Polterovich, *Topological rigidity of Hamiltonian loops and quantum homology*. Invent. Math. 135 (1999), 369-385.
[5] D. McDuff, *A survey of the topological properties of symplectomorphism groups*. Topology, geometry and quantum field theory, 173-193, London Math. Soc. Lecture Note Ser., 308. Cambridge U. P., Cambridge, (2004).

[6] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Clarendon Press, Oxford, (1998).

[7] D. McDuff, S. Tolman, *Polytopes with mass linear functions. I*. Int. Math. Res. Not. IMRN 8 (2010) 1506-1574.

[8] L. Polterovich, *The geometry of the group of symplectic diffeomorphisms*, Birkhäuser, Basel, (2001).

[9] E. Shelukhin, *Remarks on invariants of Hamiltonian loops*. J. Topol. Anal. 2 (2010) 277-325.

[10] A. Viña, *A characteristic number of Hamiltonian bundles over $S^2$*. J. Geom. Phys. 56 (2006), 2327-2343.

[11] A. Viña, *Hamiltonian diffeomorphisms of toric manifolds and flag manifolds*. J. Geom. Phys. 57 (2007), 943-965.

[12] A. Viña, *A characteristic number of bundles determined by mass linear pairs*. arXiv:0809.1506 [math.SG].

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