Fast Marginalized Block Sparse Bayesian Learning Algorithm

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Abstract—The performance of sparse signal recovery from noise corrupted, underdetermined measurements can be improved if both sparsity and correlation structure of signals are exploited. One typical correlation structure is the intra-block correlation in block sparse signals. To exploit this structure, a framework, called block sparse Bayesian learning (BSBL), has been proposed recently. Algorithms derived from this framework showed superior performance but they are not very fast, which limits their applications. This work derives an efficient algorithm from this framework, using a marginalized likelihood maximization method. The proposed algorithm significantly outperforms traditional algorithms which only exploit block structure such as Model-CoSaMP [2] and Block-OMP [9]. It has similar recovery accuracy as BSBL algorithms [3]. However, it is much faster than existing BSBL algorithms [3] and thus is more suitable for large scale problems.

Throughout the paper, Bold symbols are reserved for vectors and matrices. \( \text{Tr}(A) \) computes the trace of the matrix. \( \text{diag}(A) \) extracts the diagonal vector from a matrix A and \( \text{diag}^{-1}(a) \) builds a matrix with a as its diagonal vector. \( A^T \) denotes the transpose of matrix A.

I. INTRODUCTION

Sparse signal recovery and the associated compressed sensing [1] can recover a signal with small number of measurements with high probability of successes (or sufficient small errors), given that the signal is sparse or can be sparsely represented in some domain. It has been found that exploiting structure information [2], [3] of a signal can further improve the recovery performance. In practice, a signal generally has rich structures. One structure widely used is the block/group sparse structure [4–5], which refers to the case when nonzero entries of a signal cluster around some locations. Existing algorithms exploit such information showed improved recovery performance.

Recently, noticing intra-block correlation widely exists in real-world signals, Zhang and Rao [3], [6] proposed the block sparse Bayesian learning (BSBL) framework. A number of algorithms have been derived from this framework, and showed superior ability to recover block sparse signals or even non-sparse signals [7]. But these BSBL algorithms are not fast, and thus cannot be applied to large-scale datasets.

In this work, we propose an efficient implementation using the fast marginalized likelihood maximization (FMLM) method [8]. Thanks to the BSBL framework, it can exploit both block structure and intra-block correlation. Experiments conducted on both synthetic data and real life data showed that the proposed algorithm significantly outperforms traditional algorithms which only exploit block structure such as Model-CoSaMP [2] and Block-OMP [9]. It has similar recovery accuracy as BSBL algorithms [3]. However, it is much faster than existing BSBL algorithms [3] and thus is more suitable for large scale problems.

II. THE FRAMEWORK OF THE BLOCK SPARSE BAYESIAN LEARNING ALGORITHM

A. The basic BSBL Framework [3]

A block sparse signal \( x \) has the following structure:

\[
x = [x_1, \ldots, x_{d_1}, \ldots, x_N]^T,
\]

which means \( x \) has \( g \) blocks, and only a few blocks are nonzero. Here \( d_i \) is the block size for the \( i \)th block. Moreover, for time series data \( x \), the samples within each block are usually correlated. To model the block sparse and intra-block correlation, the BSBL framework [3] suggests to use the parameterized Gaussian distribution:

\[
p(x; \gamma_i, B_i) = \mathcal{N}(x; 0, \gamma_i B_i).
\]

with unknown deterministic parameters \( \gamma_i \) and \( B_i \). \( \gamma_i \) represents the confidence of the relevance of the \( i \)th block and \( B_i \) captures the intra-block correlation. The framework further assumes that blocks are mutually independent. Therefore, we write the signal model as,

\[
p(x; \{ \gamma_i \}, \{ B_i \}) = \mathcal{N}(x; 0, \Gamma),
\]

where \( \Gamma \) is a block diagonal matrix with the \( i \)th principal diagonal given by \( \gamma_i B_i \).

The observation \( y \) is obtained by

\[
y = y = \Phi x + n,
\]

where \( \Phi \) is a \( M \times N \) sensing matrix and \( n \) is the observation noise. The sensing matrix \( \Phi \) is an underdetermined matrix.
and the observation noise is assumed to be independent and Gaussian with zero mean and variance equal to $\beta^{-1}$. $\beta$ is also unknown. Thus the likelihood is given by

$$p(y|x; \beta) = \mathcal{N}(\Phi x, \beta^{-1}I).$$  \hspace{1cm} (5)

The main body of the BSBL algorithm algorithmically between the estimation of the posterior $p(x|y; \{\gamma_i, B_i\}, \beta) = \mathcal{N}(\mu, \Sigma)$ with $\Sigma \triangleq (\Gamma^{-1} + \Phi^T \beta \Phi)^{-1}$ and $\mu \triangleq \Sigma \Phi^T \beta y$ and maximization the likelihood $p(y|\{\gamma_i, B_i\}, \beta) = \mathcal{N}(y; 0, C)$ with $C = \beta^{-1}I + \Phi \Gamma \Phi^T$. The update rules for the parameters $\{\gamma_i, B_i\}$ and $\beta$ are derived using the Type II Maximum Likelihood [3, 10] method, which leads to the following cost function,

$$L(\{\gamma_i, B_i\}, \beta) = \log|C| + y^T C^{-1} y, \quad \hspace{1cm} (6)$$

Once all the parameters, namely $\{\gamma_i, B_i\}, \beta$, are estimated, the MAP estimate of the signal $x$ can be directly obtained from the mean of the posterior, i.e.,

$$x = \Sigma \Phi^T \beta y. \quad \hspace{1cm} (7)$$

**B. The Extension to the BSBL Framework**

In the original BSBL framework [3], $\gamma_i$ and $B_i$ together formed the covariance matrix of the $i$th block $x_i$, which can be conveniently modeled using a symmetric, positive semi-definite matrix $A_i$,

$$p(x_i; A_i) = \mathcal{N}(x_i; 0, A_i). \quad \hspace{1cm} (8)$$

The diagonal element of $A_i$, denoted by $A_{ij}^{ij}$, models the variance of the $j$th signal $x_{ij}$ in the $i$th block $x_i$. Such variance parameter $A_{ij}^{ij}$ determines the relevance [10] for the signal $x_{ij}$. In the extension to the BSBL framework, we may conveniently introduce the term block relevance, defined as the average of the estimated variance of signals in $x_i$,

$$\gamma_i \triangleq \frac{1}{d_i} \text{Tr}(A_i). \quad \hspace{1cm} (9)$$

The notation block relevance cast the Relevance Vector Machine [10] as a specialized form of one BSBL variant where we assume $A_i$ has the following structure,

$$A_i = \text{diag}^{-1}(A_1^{ij}, \cdots, A_{d_i}^{ij}). \quad \hspace{1cm} (10)$$

This variant of BSBL is for the signals with block structure and each signal in the blocks are spiky and do not correlate with each other.

The off-diagonal elements $A_{jk}^{ij}, j \neq k$ models the covariance of the block signal. It has been shown that exploit such structural information is beneficial in recovering piecewise smooth block sparse signals [3, 7]. There are rich classes of covariance matrix, namely Compound Symmetric, Auto-Regressive (AR), Moving-Average (MA) etc. These structures can be inferred during the learning process of $A_i$.

The signal model and the observation model are built up in a similar way,

$$p(x_1; \{A_i\}) = \mathcal{N}(x_1; 0, \Gamma), \quad \hspace{1cm} (11)$$

$$p(y_1; \{A_i\}, \beta) = \mathcal{N}(y_1; \Phi x_1, \beta^{-1}I). \quad \hspace{1cm} (12)$$

The parameters $\{A_i\}$ and $\beta$ can be estimated from the cost function (6) using the type II maximization method.

**III. The Fast Marginalized Block SBL Algorithm**

There are several methods to minimize the cost function (6). In [3], the author provided a bound optimization method and a hybrid $\ell_1$ method to derive the update rules for $\gamma_i$ and $B_i$. In the following we consider to extend the marginalized likelihood maximization method within the BSBL framework. This method was used by Tipping et al. [3] for their fast SBL algorithm and later by Ji et al. [11] for their Bayesian compressive sensing algorithm.

**A. The Main-body of the Algorithm**

The cost function (6) can be optimized in a block way. We denote by $\Phi_i$ the $i$th block in $\Phi$ with the column indexes corresponding to the $i$th block of the signal $x$. Then $C$ can be rewritten as:

$$C = \beta^{-1}I + \sum_{m \neq i} \Phi_m A_m \Phi_m^T + \Phi_i A_i \Phi_i^T, \quad \hspace{1cm} (13)$$

$$C_{-i} = C_{-i} - \Phi_i A_i \Phi_i^T, \quad \hspace{1cm} (14)$$

where $C_{-i} \triangleq \beta^{-1}I + \sum_{m \neq i} \Phi_m A_m \Phi_m^T$. Using the Woodbury Identity,

$$|C| = |A_i||C_{-i}| |A_i^{-1} + s_i|, \quad \hspace{1cm} (15)$$

$$C^{-1} = C_{-i}^{-1} - C_{-i}^{-1} \Phi_i (A_i^{-1} + s_i)^{-1} \Phi_i^T C_{-i}^{-1}, \quad \hspace{1cm} (16)$$

where $s_i \triangleq \Phi_i^T C_{-i}^{-1} \Phi_i$ and $q_i \triangleq \Phi_i^T C_{-i}^{-1} y$. Equation (6) can be rewritten as:

$$\mathcal{L} = \log|C_{-i}| + y^T C_{-i}^{-1} y + \log|I_{d_i} + A_i s_i| - q_i^T (A_i^{-1} + s_i)^{-1} q_i, \quad \hspace{1cm} (17)$$

where $\mathcal{L}(-i) \triangleq \log|C_{-i}| + y^T C_{-i}^{-1} y$, and

$$\mathcal{L}(i) = \log|I_{d_i} + A_i s_i| - q_i^T (A_i^{-1} + s_i)^{-1} q_i, \quad \hspace{1cm} (19)$$

which only depends on $A_i$.

Setting $\frac{\partial \mathcal{L}(i)}{\partial A_i} = 0$, we have the updating rule

$$A_i = s_i^{-1}(q_i q_i^T - s_i) s_i^{-1}. \quad \hspace{1cm} (20)$$

The block relevance $\gamma_i$ is calculated using (9) and the correlation structural is inferred from (20) by investigating a symmetric matrix $B_i$ calculated as

$$B_i = \frac{A_i}{\gamma_i}. \quad \hspace{1cm} (21)$$

The diagonal elements of $B_i$ are regularized to unity (i.e., $\text{diag}(B_i) = 1$) to maintain the spectrum of $A_i$.

**B. Imposing the Structural Regularization on $B_i$**

As noted in [3], regularization to $B_i$ is required due to limited data. It has been shown [6] that in noiseless cases the regularization does not affect the global minimum of the cost function (6), i.e., the global minimum still corresponds to the true solution; the regularization only affects the probability of the algorithm to converge to the local minima. A good regularization can largely reduce the probability of local
C. Remarks on β

The parameter β−1 is the noise variance in our model. It can be estimated by

\[ \beta = \frac{M}{\text{Tr}([\Phi^T \Phi]) + \| y - \Phi \mu \|^2_2} \]

However, the resulting updating rule is not robust due to the constructive and reconstruction nature [5, 8] of the proposed algorithm. In practice, people treat it as a regularizer and assign some specific values to it [11]. Similar to [11], we select β−1 = 10−6 in noiseless simulations, β−1 = 0.1\| y \|^2_2 in general noisy scenarios (e.g. SNR < 20 dB), and β−1 = 0.01\| y \|^2_2 in high SNR scenarios (e.g. SNR ≥ 20 dB).

D. The BSBL-FM algorithm

The proposed algorithm (denoted as BSBL-FM) is given in Fig. 1. Within each iteration, it only updates the block signal that attributes to the deepest descent of \( \mathcal{L}(i) \). The detailed procedures on re-calculation of \( \mu, \Sigma, \{ s_i \}, \{ q_i \} \) are similar to [8]. The algorithm terminates when the maximum change of the cost function is smaller than a threshold \( \eta \). In the experiments thereafter, we set \( \eta = 1e^{-4} \).

IV. EXPERIMENTS

In the experiment, we compared the proposed algorithm with the state-of-the-art block based recovery algorithms. For comparison, two BSBL algorithms, i.e., BSBL-BO and BSBL-\( \ell_1 \) [3], were used (BSBL-\( \ell_1 \) used the Group Basis Pursuit [12] in its inner loop). Besides, a variational inference based SBL algorithm (denoted by VBGS [5]) was selected. It used its default parameters. Model-CoSaMP [2] and Block-OMP [9] (given the true sparsity) were used as the benchmark in noiseless situations, while the Group Basis Pursuit [12] was used as the benchmark in noisy situations.

The performance indexes were the normalized mean square error (NMSE) in noisy situations and the success rate in noiseless situations. The NMSE was defined as \( \| \hat{x} - x_{\text{gen}} \|^2_2 / \| x_{\text{gen}} \|^2_2 \), where \( \hat{x} \) was the estimate of the true signal \( x_{\text{gen}} \). The success rate was defined as the percentage of successful trials in total experiments (A successful trial was defined the one when NMSE < 10−5).

In all the experiments except for the last one, the sensing matrix was a random Gaussian matrix, and it was generated in each trial of each experiment. The computer used in the experiments had 2.5GHz CPU and 2G RAM.
**A. Empirical Phase Transition**

In the first experiment, we studied the phase transitions of all the algorithms in exact recovery of block sparse signals in noiseless situations. The phase transition curve \([13]\) is to show how the success rate is affected by the sparsity level (defined as \(\rho = K/M\), where \(K\) is the total number of non-zero elements) and indeterminacy (defined as \(\delta = M/N\)).

The generated signal consisted of 20 blocks with the identical block size 25. The number of non-zero blocks varied from 1 to 10 while their locations were determined randomly. Each non-zero block was generated by a multivariate Gaussian distribution \(N(0, \Sigma_{\text{gen}})\) with \(\Sigma_{\text{gen}} \triangleq \text{Toeplitz}([1, r, \ldots, r^{24}])\). The parameter \(r\), which reflected the intra-block correlation level, was set to 0.95. The number of measurements varied from \(M = 50\) to \(M = 250\).

The results averaged over 100 trials are shown in Fig. 2. Both BSBL-FM and BSBL-BO showed impressive phase transition performance. We see that as a greedy method, BSBL-FM performed better than VBGS, Model-CoSaMP and Block-OMP.

**B. Performance in Noisy Environments with varying \(N\)**

This experiment was designed to show the advantage of our algorithm in speed. The signal consisted of 32 blocks with identical block size five of which were randomly located non-zero blocks. The length of the signal, \(N\), was varied from 512 to 2048 with fixed indeterminacy ratio \(M/N = 0.5\). The intra-block correlation level, i.e., \(r\), of each block (generated as in Section 4.V-A) was uniformly chosen from 0.8 to 0.99. The SNR, defined as \(\text{SNR}(\text{dB}) \triangleq 20 \log_{10} (||\Phi \delta_x||_2/||\delta_x||_2)\), was fixed to 15dB. In this experiment we also calculated the oracle result, which was the least square estimate of \(\delta_x\) given the true support.

The results (Fig. 3) show that the proposed algorithm, although the recovery performance was slightly poorer than BSBL-BO and BSBL-\(\ell_1\), had the obvious advantage in speed. This implies that the proposed algorithm may be a better choice for large-scale problems. The BSBL-FM(1), BSBL-FM(2) and BSBL-BO even outperformed the oracle estimate, this may be due to that the oracle property utilized only the true support information while ignored the structure in signals (i.e., the intra-block correlation). Also, by comparing BSBL-FM(1) and BSBL-FM(2) to BSBL-FM(0), we can see its recovery performance was improved due to the exploitation of intra-block correlation.

**C. Application to Telemonitoring of Fetal Electrocardiogram**

Fetal electrocardiogram (FECG) telemonitoring via low energy wireless body-area networks [7] is an important approach to monitor fetus health state. BSBL, as an important branch of compressed sensing, has shown great promising in this application [7]. Using BSBL, one can compress raw FECG recordings using a sparse binary matrix, i.e.,

\[
y = \Phi x
\]

where \(x\) is a raw FECG recording, \(\Phi\) is the sparse binary matrix, and \(y\) is the compressed data. It have been showed [14] that using a sparse binary matrix as the sensing matrix can greatly reduce the energy consumption while achieving competitive compression ratio. Then \(y\) is sent to a remote computer. In this computer BSBL algorithms can recover the raw FECG recordings with high accuracy such that Independent Component Analysis (ICA) decomposition [15] on the recovered recordings keeps high fidelity (and a clean FECG is presented after the ICA decomposition).
using the same dataset, the same sensing matrix (a sparse binary matrix with the size 256 × 512 and each column consisting of 12 entries of 1s with random locations), and the same block partition ($d_i = 32(\forall i)$).

We compared our algorithm BSBL-FM with VBGS, Group-BP and BSBL-BO. All the algorithms first recovered the discrete cosine transform (DCT) coefficients $\theta$ of the recordings according to

$$ y = (\Phi D) \theta $$

using $y$ and $\Phi D$, where $D$ was the basis of the DCT transform such that $x = D \theta$. Then we reconstructed the original raw FECG recordings according to $x = D \theta$ using $D$ and $\theta$.

The NMSE measured on the recovered FECG recordings is shown in Fig. 4. We can see although BSBL-FM had slightly poorer recovery accuracy than BSBL-BO, it had much faster speed. In fact, the ICA decomposition on the recovered recordings by BSBL-FM also presented a clean FECG (see Fig. 5), and the decomposition was almost the same as the ICA decomposition on the original recordings.

In this experiment we noticed that VBGS took long time to recover the FECG recordings, and had the largest NMSE. Besides, the ICA decomposition on its recovered recordings didn’t present the clean FECG. This reason may be due to the fact that the DCT coefficients of the raw FECG recordings are not sufficiently sparse, and recovering these less-sparse coefficients is very difficult for non-BSBL algorithms [7]. This experiment shows the robustness and the speed efficiency of the proposed algorithm applied to real-life applications.

V. CONCLUSION

In this paper, we proposed a fast BSBL algorithm that can exploit both the block sparsity and the intra-block correlation of the signal. Experiments showed that it significantly outperforms non-BSBL algorithms, and has close recovery performance as existing BSBL algorithms, but is the fastest among the BSBL algorithms.

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