Non-Critical String Models as Topological Coset models.

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ABSTRACT

The topological coset model approach to non-critical string models is summarized. The action of a topological twisted $G/H$ coset model ($\text{rank } H = \text{rank } G$) is written down. A “topological coset algebra” is derived and compared with the algebraic structure of the $N = 2$ twisted models. The cohomology on a free field Fock space as well as on the space of irreducible representation of the “matter” affine Lie algebra are extracted. We compare the results of the $A_1^{(N-1)}$ at level $k = \frac{p}{q} - N$ with those of $(p, q)$ $W_N$ strings.

A $\frac{SL(2,R)}{SL(2,R)}$ model which corresponds to the $c = 1$ is written down. A similarity transformation on the BRST charge enables us to extract the full BRST cohomology. One to one correspondence between the physical states of the $c = 1$ and the corresponding coset model is found.

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1. Introduction

Topological quantum field theories\(^1\) (TQFTs) were recently found to be a very useful tool in the study of string theories. Non-critical string models or 2D gravitational models serve as a laboratory to explore new domains of string theories such as the non-perturbative behaviour. In this talk we summarize the attempt to analyze the \(c \leq 1\) models and their \(W_N\) generalizations as TQFTs.

A TQFT is a QFT in which all the observables, namely, all correlators of “physical operators”, are invariant under any arbitrary deformation of \(g_{\alpha\beta}\) the metric of the underlying space-time. Given a set of physical operators \(F_i[\Phi^a(x_i)]\) which are functional of the fields \(\Phi^a(x), a = 1,...,p\) of the theory and which are invariant under the symmetries of the theory, then the theory is topological iff

\[
\delta g_{\alpha\beta} < \prod_i F_i[\Phi^a(x)] > = 0. \tag{1}
\]

In particular this definition implies that all correlators are independent of distances between the operators. In a theory where the energy-momentum tensor \(T_{\alpha\beta}\) is exact under a BRST-like symmetry operator \(Q\), namely, \(T_{\alpha\beta} = \{Q, G_{\alpha\beta}\}\), all physical operators are in the cohomology of \(Q\) property (1) is obeyed. Thus, it is a TQFT.

A two-dimensional theory which is a TQFT as well as conformal, is a topological conformal field theory (TCFT). The algebraic structure of these models is characterized by the fact that \(T\) and the \(Q\) are both BRST exact. However, there is no unique structure which is common to all TCFTs. In fact it is shown here that the “topological coset algebra” differ from the algebra of the twisted \(N = 2\) models.\(^2\)

Every conformal field theory coupled to 2-D gravity is obviously a TCFT after integrating over all metric degrees of freedom. We now define the concept of a “topological model” which is a TQFT without the introduction and integration over the metric. In more than two dimensions examples of such models are the Chern-Simons theory and the four dimensional theory which corresponds to the Donaldson invariants.\(^1\) In two dimensions an example is the theory of flat gauge connections.\(^3\) We are now ready to introduce the notion of “topological coset models” which are gauged WZW models that are also topological models.\(^4,5\)

The paper is organized as follows. In section 2 we write down the quantum action of a topological \(\frac{G}{G}\) and twisted \(\frac{G}{\Pi}\) coset models as a sum of “decoupled”
matter, gauge and ghost sectors. The algebraic structure is derived in section 3 and it is found that a larger algebra than that of the TCFT\(^2\) algebra. Section 4. is devoted to a brief reminder of the cohomology on a free field Fock space as well as on an irreducible representation of the “matter” affine Lie algebra. In section 5 a comparison between the physical states of topological coset models and the corresponding string models is made. We discuss the relation between gauge holonomies and the twist needed to achieve the correspondence with the gravitational model. We then write down a topological coset model which describes the \(c = 1\) string model. A special treatment of the BRST charge using a double similarity transformation is used to extract the space of physical states. The latter are then compared with the ground ring and other states of the two dimensional string theory.

2. The quantum action

The topological coset models are constructed by gauging an anomaly free diagonal subgroup \(H \subset G\) group of a level \(k\) WZW model. The \(\frac{G}{H}\) models are defined by \(H = G\) whereas the general twisted \(\frac{G}{H}\) models require that \(rank\, H = rank\, G\). In the latter case the gauged action is a twisted supersymmetric \(G\)-WZW model, namely, the usual \(\frac{G}{H}\) model with an extra set of \((1, 0)\) anti-commuting ghosts where the dimension one fields take their values in the positive roots of \(\frac{G}{H}\) and the dimension zero fields in the negative ones. A great part of the discussion that follows applies to general compact groups, however, we will mainly concentrate on \(G = SU(N)\) and the non-compact \(SL(N)\) groups. The latter are needed since we will be interested also in fractional levels. We proceed to derive the quantum action of the \(\frac{G}{G}\) and the twisted \(\frac{G}{H}\) models.

The \(\frac{G}{G}\) model

The classical action of the \(\frac{G}{G}\) model takes the form

\[
S_k(g, A, \bar{A}) = S_k(g) + i \frac{k}{2\pi} \int_{\Sigma} d^2z Tr(g^{-1} \partial g \bar{A} + g \bar{\partial} g^{-1} A - \bar{A} g^{-1} A g + A \bar{A})
\]  

(2)

where \(g \in G\) and \(S_k(g)\) is the WZW action at level \(k\). An essential step in the analysis of this action is a “decoupling” of the matter and gauge degrees of freedom. This can be achieved by rewriting the gauge fields in terms of group elements. In
the case of a topologically trivial $\Sigma$ the gauge field can be parametrized as follows

$$A = ih^{-1} \partial h, \ A = i\bar{h} \partial \bar{h}^{-1}$$

where $h(z) \in G^c$. The action then reads\cite{6,7,8}

$$S_k(g, A) = S_k(g) - S_k(h\bar{h})$$

The Jacobian of the change of variables introduces a dimension $(1, 0)$ system of anticommuting ghosts $\chi$ and $\rho$ in the adjoint representation of the group. The quantum action thus takes the form of

$$S_k(g, h, \rho, \chi) = S_k(g) - S_k(h\bar{h}) - i \int d^2z Tr[\rho D\bar{\chi} + c.c]$$

where $D\chi = \partial \chi - i[A, \chi]$. This action involves an interaction term of the form $Tr_H(\bar{\rho}[h^{-1}\partial h, \bar{\chi}])$ and a similar term for $\rho, \chi$. By performing a chiral rotation $\bar{\rho} \rightarrow h^{-1}\bar{\rho}h$ and $\bar{\chi} \rightarrow h^{-1}\bar{\chi}h$ with $\rho \rightarrow \bar{h}\rho\bar{h}^{-1}$ and $\chi \rightarrow \bar{h}\chi\bar{h}^{-1}$, one achieves a decoupling of the whole ghost system. The price of that is an additional $S_{-2C_G}(h\bar{h})$ term in the action resulting form the corresponding anomaly. $C_G$ is the second Casimir of the adjoint representation. This result can be derived by using a non-abelian bosonization of the ghost system.\cite{9} The final quantum action after gauge fixing $\bar{h} = 1$

$$S_k(g, h, \rho, \chi) = S_k(g) - S_{-(k+2C_G)}(h) - i \int d^2z Tr[\rho D\bar{\chi} + c.c],$$

indeed, has the structure of three “decoupled” affine Lie algebra actions.

The twisted $G_{\vec{t}}$ model

The classical action of the twisted $G_{\vec{t}}$ model is given by

$$S_{(tKS)} = S_k(g, A, \bar{A}) + S_{(gh)}^G$$

$$S_k(g, A, \bar{A}) = S_k(g) - \frac{k}{2\pi} \int d^2z Tr_G[g^{-1} \partial g \bar{A}z + g \bar{\partial} g^{-1} A - \bar{A} g^{-1} A g + AA]$$

$$S_{gh}^G = \frac{i}{2\pi} \int d^2z \sum_{\alpha \in \vec{t}} [\rho^{+\alpha}(\bar{D}\chi)^{-\alpha} + \bar{\rho}^{+\alpha}(D\bar{\chi})^{-\alpha}]$$

Using the same parametrization as for the $G_{\vec{G}}$ case, one finds after inserting the
Jacobian and gauge fixing the following action

\[ S_k(g, A) = S_k(hg\bar{h}) - S_k(h\bar{h}) + \frac{i}{2\pi} \int d^2 z Tr_H[\rho \bar{D}\chi + \bar{\rho}D\bar{\chi}]. \quad (7) \]

The twisted Kazama-Suzuki\textsuperscript{[10]} action is given by

\[ S_{(tKS)}(g) = S_k(g) - \sum_{I=1}^n S_k(h^{(I)}) - \frac{k}{4\pi} \int d^2 z \sum_{s=1}^r \partial \mathcal{H}_s \bar{\partial} \mathcal{H}_s + \frac{i}{2\pi} \int d^2 z \sum_{s=1}^r \partial \mathcal{H}[\rho \bar{D}\chi + \bar{\rho}D\bar{\chi}] + \frac{i}{2\pi} \int d^2 z \sum_{\alpha \in \mathcal{H}} \partial \mathcal{H}_s \bar{\partial} \mathcal{H}_s + \frac{i}{2\pi} \int d^2 z \sum_{\alpha \in \mathcal{H}} \partial \mathcal{H}_s \bar{\partial} \mathcal{H}_s + \frac{i}{2\pi} \int d^2 z \sum_{\alpha \in \mathcal{H}} \partial \mathcal{H}_s \bar{\partial} \mathcal{H}_s + \frac{i}{2\pi} \int d^2 z \sum_{\alpha \in \mathcal{H}} \partial \mathcal{H}_s \bar{\partial} \mathcal{H}_s + \frac{i}{2\pi} \int d^2 z \sum_{\alpha \in \mathcal{H}} \partial \mathcal{H}_s \bar{\partial} \mathcal{H}_s \]  

for $G = SU(N)$ and $H = SU(N_1) \times \ldots \times SU(N_n) \times U(1)^r$ with $r = N - 1 - \sum_{l=1}^n N_l + n$, where the gauge fields $A$ take the form of $A = i \sum_{I=1}^n h^{(I)} \partial h^{(I)} + i \sum_{s=1}^r \partial \mathcal{H}_s$.

After chiral rotating the ghost fields the action takes the following form

\[ S_k = S_k(g) + \sum_{I=1}^n S_{-(k+C_G+C_H^{(I)})}(h^{(I)}) + \frac{1}{2\pi} \int d^2 z \sum_{s=1}^r \partial \mathcal{H}_s \bar{\partial} \mathcal{H}_s + \frac{2}{k+C_G} (\bar{\rho}_G - \bar{\rho}_H) \cdot \bar{\mathcal{H}} \bar{R} + \frac{i}{2\pi} \int d^2 z [\rho \bar{\partial} \chi + \bar{\rho} \partial \bar{\chi}] + \frac{i}{2\pi} \int d^2 z \sum_{\alpha \in \mathcal{H}} [\rho^{+\alpha}(\bar{\partial} \chi)^{-\alpha} + \bar{\rho}^{+\alpha}(\partial \bar{\chi})^{-\alpha}] \]  

where we have normalized the $\bar{\mathcal{H}}$ fields to be free bosons, and $\bar{\rho}_G$ and $\bar{\rho}_H$ are half the sums of the positive roots of $G$ and $H$ respectively. The action is composed of three decoupled sectors: the matter sector, the gauge sector and the ghost sector involving ghosts in $H$ and $\mathcal{G}_H$.  

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3. The Topological coset algebra

An important class of TCFT is the twisted $N = 2$ superconformal field theories. These models are characterized by the “TCFT algebra” given by

\begin{equation}
T(z) = \{Q, G(z)\} \quad Q(z) = [Q, j^\#(z)] \\
\{Q, Q(z)\} = 0 \quad \{G, G(z)\} = 0
\end{equation}

where the holomorphic currents, $T(z)$, $j^\#(z)$, $Q(z)$ and $G(z)$ are the energy-momentum tensor, a $U(1)$ current, a BRST-like current and an anti-commuting dimension two current, respectively. Obviously these holomorphic currents have also their anti-holomorphic partners with the same algebra. Let us now examine whether the topological coset model share the same algebraic structure.

The $G$ models

Let us first examine the $G$ affine Lie algebra. There are three sets of holomorphic $G$ transformations which leave (7) invariant $\delta J_g = i[\epsilon(z), g]$ $\delta h = i[\epsilon(z), h]$ and $\delta J_{(gh)} \chi^a = iff_{bc}^e \chi^c$ ; $\delta J_{(gh)} \rho^a = -iff_{bc}^e \rho^c$ with $\epsilon$ in the algebra of $G$. The corresponding currents $J^a$, $I^a$ and $J_{(gh)}^a = iff_{bc}^e \chi^c \rho^c$ satisfy the $G$ affine Lie algebra at levels $k, -(k+2c_G)$ and $2c_G$ respectively. Out of all possible linear combinations of these currents the following one is very useful,

\begin{equation}
J^{(tot)}_a = J^a + I^a + J_{(gh)}^a = J^a + I^a + iff_{bc}^e \chi^c \rho^c.
\end{equation}

It obeys an affine Lie algebra at level

\begin{equation}
k^{(tot)} = k - (k + 2c_G) + 2c_G = 0.
\end{equation}

The energy-momentum tensor $T(z)$ is a sum of Sugawara terms of the $J^a$ and $I^a$ currents and the usual contribution of a $(1, 0)$ ghost system, namely\cite{6, 7, 8}

\begin{equation}
T(z) = \frac{1}{k + c_G} J^a J^a : - \frac{1}{k + c_G} : I^a I^a : + \rho^a \partial \chi^a.
\end{equation}

The corresponding Virasoro central charge vanishes

\begin{equation}
\ell^{(tot)} = \frac{k d_G}{k + c_G} - \frac{(k + 2c_G) d_G}{-(k + 2c_G) + c_G} - 2d_G = 0
\end{equation}

This last property is an indication that the $G$ model is a TCFT.
The $Q(z)$ current has an obvious realization. It is just the BRST current which emerges from the gauge fixing of the original gauged action. It is then also easy to find its dimension two partner $G$

\[ Q(z) = \chi_a [J^a + I^a + \frac{1}{2} J^{(gh) a}] \]

\[ G = \frac{1}{k + c G} \rho_a [J^a - I^a]. \tag{15} \]

It is straightforward to check that $T$ is BRST exact, $T(z) = \{Q, G(z)\}$. The BRST current itself is also BRST exact with respect to the $U(1)$ ghost number current $j^\# = \chi^a \rho_a$, $Q(z) = \{Q, j^\#(z)\}$. By its construction the BRST charge is nilpotent so one may conclude that indeed the topological coset model share the TCFT algebra. In fact the algebraic structure is different since $G$ defined above is not a nilpotent operator (apart from the case of $U(1)$). Instead one finds $\{G, G(z)\} \equiv W(z) = \frac{1}{4 C_G} f_{abc} J^{(tot) a} \rho^a \rho^b + \partial \rho^a \rho_a$. To close the algebra \cite{111} one has to introduce one additional anticommuting current of dimension three $U = \frac{1}{12 C_G} f_{abc} J^{(tot)} \rho^a \rho^b$ such that the full “topological coset algebra” is given by

\[ \tilde{T}(z) = \{Q, G(z)\} \quad Q(z) = \{Q, j^\#(z)\} \]

\[ \{Q, Q(z)\} = 0 \quad W(z) \equiv \{G, G(z)\} \]

\[ \tilde{W}(z) = \{Q, U(z)\} \quad [W, W(z)] = 0. \tag{16} \]

The twisted $\frac{G}{H}$ models

There are two twisted $N = 2$ symmetry algebras in the twisted $\frac{G}{H}$ models. One of them which emerges from the $H$ gauge fixing part, is of the “topological coset algebra” type whereas the other, which corresponds to the fact that the original action is a twisted $N = 2$ WZW model, is a “TCFT algebra”. For the derivation of the “physical states” the relevant algebra is a “topological coset algebra” which is a direct sum of the the two. Let us start with the $H$ sector. The following set of currents obey the “topological coset model” of eqn. (16):

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\[ T^H(z) = \frac{1}{2(k + c_G)}g_{ab} : (J^a + \frac{J_G}{\pi})(J^b + \frac{J_G}{\pi}) : - \frac{1}{2(k + c_G)}g_{ab} : I^a I^b : \]

\[ - \sqrt{\frac{2}{k + C_G}}(\rho_G \rho_H) \partial(\tilde{J} + \tilde{J}^G_{G}) + g_{ab} \rho^a \partial \chi^b \]

\[ J^\#(z) = \rho^a \chi_a \]

\[ Q(z) = g_{ab} \chi^a [J^b + I^b + J^b_G + \frac{1}{2} J^a \chi^b] \]

\[ = g_{ab} \chi^a [J^b + I^b + \frac{i}{2} f^{cd}_{\gamma = \beta} \rho^c \chi^d + i f^{b}_{\gamma = \beta} \rho^\gamma \chi^{-\beta}] \]

\[ G^H = \frac{g_{ab}}{2(k + c_G)} \rho^a [J^b - I^b + i f^{b}_{\gamma = \beta} \rho^\gamma \chi^{-\beta}] - \sqrt{\frac{2}{k + C_G}}(\rho_G \rho_H) \partial \rho^a \partial \chi^b \]

\[ \text{where } \tilde{J}, \tilde{I}, \text{ and } \tilde{J}^G_G \text{ are the Cartan-subalgebra currents given in the basis in which } \]

\[ [J^i_n, J^j_m] = k \delta^{ij} \delta_{m+n}. \]

The \( G^H \) algebra on the other hand involves the following currents

\[ T^G(z) = \frac{1}{2(k + c_G)}g_{\tilde{a} \tilde{b}} : J^{\tilde{a}} J^{\tilde{b}} : - \frac{1}{2(k + c_G)}g_{ab} : (J^a + \frac{J_G}{\pi})(J^b + \frac{J_G}{\pi}) : \]

\[ + \frac{\sqrt{2}}{k + C_G}(\rho_G - \rho_H) \partial(\tilde{J} + \tilde{J}^G_{G}) + \sum_{\alpha \in \tilde{G}} \rho^\alpha (\partial \chi)^{-\alpha} \]

\[ J^\#(z) = \rho^{+\alpha} \chi^{-\alpha} \]

\[ Q^G = \sum_{\alpha \beta \gamma \in \tilde{G}} \chi^{-\alpha} (J^\alpha + \frac{i}{2} f^{\alpha}_{\gamma = \beta} \rho^\gamma \chi^{-\beta}) \]

\[ G^G = \frac{1}{k + C_G} \sum_{\alpha \beta \gamma \in \tilde{G}} \rho^\alpha (J^{-\alpha} + \frac{i}{2} f^{-\alpha}_{\gamma = \beta} \rho^\gamma \chi^{-\beta}). \]

where \( \tilde{a} \) and \( \tilde{b} \) go over the adjoint of \( G \). The combined algebra is based on the set of generators which have the form \( A(z) = A^H(z) + A^G_{G}(z) \). The combined energy momentum tensor acquires now the simplified form

\[ T(z) = \frac{1}{2(k + c_G)}g_{\tilde{a} \tilde{b}} : J^{\tilde{a}} J^{\tilde{b}} : - \frac{1}{2(k + c_G)}g_{ab} : I^a I^b : + g_{ab} \rho^a \partial \chi^b \]

\[ - \frac{\sqrt{2}}{k + C_G}(\rho_G - \rho_H) \partial \tilde{I} + \sum_{\alpha \in \tilde{G}} \rho^{+\alpha} (\partial \chi)^{-\alpha} \]
with a total Virasoro central charge is found to be

$$c = \frac{k d_G}{k + C_G} + \sum_{I=1}^{n} \frac{(k + C_G + C_{H^I}) d_{H^I}}{k + C_G} + r - 2d_H - (d_G - d_H) + 6 \sqrt{\frac{2}{k + C_G}} (\tilde{\rho}_G - \tilde{\rho}_H)^2 = 0$$

(20)

where we have used, assuming $G$ is a simply laced group, the relations $12 \tilde{\rho}_G^2 = d_G C_G$, and $\tilde{\rho}_H \cdot (\tilde{\rho}_G - \tilde{\rho}_H) = 0$.

4. BRST cohomology and Physical states

Next we proceed to extract the space of physical states of the model. We take as our definition of a physical state a state in the cohomology of $Q = Q^{(BRST)} + Q^G_H$, namely, $|\text{phys} \rangle \in H^*(Q)$. The computation of the cohomology is based on a spectral sequence decomposition approach. The extraction of the physical states was worked out in detail in refs. [9,13,14]. Here we will mention one feature of this method and the final results. The method is based on “Wakimoto bosonization” of the matter $(J)$ and “gauge” $(I)$ currents. There are two possible bosonizations which are related by an automorphism, for instance in the $SL(2,R)$ case, $J^+ \leftrightarrow J^-, J^0 \leftrightarrow -J^0$. Denoting the two parametrizations by $+$ and $-$ one has the two options $(+, +)$ and $(+, -)$ for the $(J, I)$ system. In [9,13,14] we have used a $(+, -)$ scheme since only in this way a convenient grading decomposition is possible. Even though this bosonization lacks an $SL(2,R)$ invariant vacuum, after projecting to the space of irreducible representations of affine Lie algebra the appropriate vacuum invariance is restored. The full cohomology on the Fock space of free fields was found to be

$$H(Q) \simeq H^{rel}(Q) \oplus \sum_{\{k_1, ..., k_l\}} \chi_{0}^{k_1} ... \chi_{0}^{k_l} H^{rel}(Q)$$

(21)

where the sum is over $k_1, ..., k_l$ namely all possible subsets of $1, ..., N - 1$.

The relative cohomology $H^{rel}(Q)$ is the cohomology on the space of vanishing zero modes of the components of $\rho$ in the Cartan sub-algebra of $H$ which is given by

$$H^{rel}(Q) = \{ \prod_{\alpha \in H, \alpha > 0} \chi_0^\alpha |\vec{J}, \vec{I} >; \quad \vec{J} + \vec{I} + 2\tilde{\rho}_H = 0 \}.$$ 

(22)

where $\tilde{\rho}_H$ is half the sum of the positive roots of $H$. 

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The topological coset models, as will be shown below, are intimately related to gravitational (and $W_N$) models. This correspondence occurs once the cohomology of the theory is taken in the space of irreducible representations of the $J$ sector affine Lie algebra. To simplify the picture we present here only the results of the $A(1)_1$ model. The results for the general topological coset model cases are given in refs. [14,9]. The space of physical states is composed of states built on $J = J_{r,s}$ where $r$ and $s$ are integers with either $r, s \geq 1$ or $r < 0, s \leq 0$ and $2J_{r,s} + 1 = r - (s - 1)(k + 2)$. For each such $J_{r,s}$ there is an infinite set of states with $I = I_{r-2lp,s}$ $G = -2l$ and $I = I_{r-2lp,s}$ $G = 1 - 2l$ for every positive integer $l$. For integer $k$ we have $J = 0, \ldots, \frac{k}{2}$. Let us now examine the index interpretation of the torus partition function. One has to insert the values of $\hat{L}_0 = L_0 - \frac{1}{2}J(J+1) - I(I+1)$ and $\hat{J}_0^{(tot)} = J^{(tot)}_0 - (J + I + 1)$ into $Tr[(-)^G q^{\hat{L}_0} e^{i\pi\theta \hat{J}_0^{(tot)}}]$. The end result is

$$Tr[(-)^G q^{\hat{L}_0} e^{i\pi\theta \hat{J}_0^{(tot)}}] = 2i \overline{q}^\frac{1}{2(k+2)} e^{-i\pi \theta} M_{k,j}(\tau, \theta).$$

(23)

where

$$M_{k,j}(\tau, \theta) = \sum_{l=-\infty}^{\infty} q^{(k+2)(l+\frac{1}{2})^2} \sin\{\pi\theta[(k+2)l + j + \frac{1}{2}]\}$$

(24)

is the numerator of the character which corresponds to the highest weight state $J$. We have, thus, rederived using the BRST cohomology the path integral results of ref. [7], for the torus partition function.

5. Comparison with string models

The main motivation to study the topological coset model is the idea that they are closely related to non-critical string models. More specifically, we expect a correspondence between the $A^{(1)}_{N-1}$ twisted $G$ models and $W_N$ strings and, in particular, between the case of $G = SL(2)$ and minimal models coupled to gravity. Therefore, we would like to examine now whether one can map the topological coset models into string models. In fact, for reasons that will be clarified shortly, the comparison with the gravitational models should be done with the topological coset models only after twisting their energy-momentum tensor. For $G = SL(N, R)$
the latter is given by

\[ T(z) \rightarrow \tilde{T}(z) = T(z) + \sum_{i=1}^{N-1} \partial J^{(tot)}_i(z) \]  

For the \( N = 2 \) case this type of twist in \( T(z) \) corresponds to an addition of the term proportional to \( \omega \tilde{J}_0^{tot} + \bar{\omega} J^{(tot)}_0 \) to the action of eqn. (7), where we use the following expression for the curvature \( R^2 = \partial \omega + \partial \bar{\omega} \). It is easy to realize that a similar modification of the action arises when one introduces an holonomy \( \theta_0 \) in the parametrization of the gauge fields namely

\[ A_0 = Tr[T_0 h^{-1} \partial h] + \theta_0 \]

and identifies \( \theta_0 \) with \( \omega \). In the general \( SL(N, R) \) case the holonomies are in the Kartan sub-algebra and they give rise to a \( \theta_i \tilde{J}_i^{(tot)} + \bar{\theta}_i J_i^{(tot)} \) term.

Obviously, since \( T(z) \) and \( \partial J^{(tot)}_i(z) \) are BRST exact so is \( \tilde{T}(z) \). Thus, the total Virasoro anomaly is unchanged. However, the contribution of each sector to \( c \) is modified as follows

\[ c_J \rightarrow \tilde{c}_J = c_J - d_G C_G k \quad c_{H^{(i)}} \rightarrow \tilde{c}_{H^{(i)}} = c_{H^{(i)}} + d_{H^{(i)}} C_{H^{(i)}} (k + C_G + C_{H^{(i)}}), \]

and the shift in the ghost contribution is given by a similar expression which can be found from the fact that the sum of the shifts vanishes. In what follows we consider, for simplicity, the case of \( G = SL(N, R) \). The twisted ghost sector includes the ghosts of a \( W_N \) gravity, namely, a sequence of ghosts with dimensions \((i, 1 - i)\) for \( i = 2, ..., N \) contributing \( \tilde{c}_{W_{gh}} = -2(N - 1)((N + 1)^2 + N^2) \) to \( \tilde{c} \). The rest of the ghosts are paired with commuting fields of the same conformal structure coming from the \( J \) and \( I \) sectors. For \( N = 2 \) one finds \( \tilde{c}_{W_{gh}} = -26 - 2\#_{pairs} \) where there are two pairs in the \( SL(2,R)_{\omega L(2,R)} \) model and one in the \( SL(2)_{\nu(1)} \) case. The net matter degrees of freedom have the following Virasoro anomaly

\[ c = \tilde{c}_J - \frac{1}{2}[\tilde{c}_{(gh)} - \tilde{c}_{W_{gh}}] = (N - 1)((2N^2 + 2N + 1) - N(N + 1)(t + \frac{1}{t})) \]

which is exactly that of a \((p, q)\) minimal \( W_N \) matter sector\(^{[16]}\) provided \( t \equiv k + N = \frac{p}{q} \). This was explicitly verified by analyzing the dimensions and contributions to \( \tilde{c} \) of the various free fields in the \( J \) sector\(^{[14]} \). The expression for \( c \) reduces to that of the \((p, q)\) minimal model for \( N = 2 \).

Next we want to compare the partition function of the topological coset model to that of the corresponding \((W)\) string models. For simplicity we concentrate on the relation between the \((p, q)\) minimal models and the \( G_{\nu} \) for \( G = SL(2) \) at level \( k = \frac{p}{q} - 2 \). The character of the minimal models coupled to gravity is given by
Comparing eqn. (23) to the numerator of the character of the minimal model, it is clear that a correspondence may be achieved only provided one takes \( \tau = -\frac{1}{2} \theta \). Recall that in the topological coset models we integrate in the path-integral only over \( \theta \) (and not over \( \tau \)) and the result is \( \tau \) independent. In this case the numerator of the character in the minimal model which is proportional to \( \text{Tr} \left[ (-1)^G q \hat{L}_0 \right] \) is mapped into \( \text{Tr} \left[ (-1)^G u \hat{L}_0 - \hat{J}_{(\text{tot})} \right] \) where \( u = e^{2i\pi \theta} \) in the \( G \) model. The integration over the moduli parameter of the torus is therefore replaced by the integration over the moduli of flat gauge connection. To establish this mapping we compare the number of states at a given level and ghost number in the minimal models with the corresponding numbers at the same ghost number and “twisted level” of the \( SL(2,\mathbb{R})/SL(2,\mathbb{R}) \) model. The latter are given for \( J = J_{r,s} \) by

\[
I = I_{r-2l,p,s} \quad G = -2l \quad \hat{L}_0 - \hat{J}_{(\text{tot})} = l^2pq + l(qr - sp) \\
I = I_{r-2l,p,s} \quad G = 1 - 2l \quad \hat{L}_0 - \hat{J}_{(\text{tot})} = l^2pq - l(qr + sp) + rs
\]

In the minimal models we have states built on vacua labeled by the pair \( r, s \) with \( 1 \leq r \leq p - 1 \) and \( 1 \leq s \leq q - 1 \) with \( ps > qr \) which have dimension \( h_{r,s} = \frac{(qr-sp)^2-(p-q)^2}{4pq} \). The levels of the excitations are \( \hat{L}_0 = \Delta - h_{r,s} \).

For \( G = 2l + 1 \) one has \( \Delta = A(l) = \frac{[2pql+qr+sp]^2-(q-p)^2}{4pq} \) and for \( G = 2l \)
\[
\Delta = B(l) = \frac{[2pql-qr+sp]^2-(q-p)^2}{4pq}. \]

Hence, the the contribution of the various levels to the partition function are identical to those of \( \hat{L}_0 - \hat{J}_{(\text{tot})} \) in eqn. (27) for the same ghost numbers. The respective vacua satisfy \( J = \sqrt{\frac{2l}{2q}} p_m \) and \( I = -\sqrt{\frac{2l}{2q}} p_L \) where \( p_m \) and \( p_L \) are the matter and Liouville momenta respectively. It is thus clear that for a given \( r, s \) we get the same number of states with the same ghost number parity in the two models and the two partition functions on the torus are in fact identical.
Now that the connection between the fractional level twisted $G/H$ models and the $W_N$ string models has been established, we would like to describe the two dimensional string theory, the $c = 1$ model, as a topological coset model. It is straightforward to realize that $\text{SL}(2,R)/\text{SL}(2,R)$ model with $k = -1$ or $\text{SL}(2,R)/\text{U}(1)$ with $k = -3$ contains a matter sector with $c = 1$. There is however an important difference between the $c < 1$ and $c = 1$ cases. In the latter there is no background charge for the matter sector and thus no double complex in the BRST structure. Recall that the latter enabled us to use the $(+, -)$ bosonization inspite of the lack of an explicit $\text{SL}(2,R)$ invariant vacuum. In the $(+, +)$ scheme there is no apparent way to define degrees such that the spectral sequence machinery can be applied. A direct computation of the BRST cohomology seems also intractable due to the cubic and quartic terms in $Q$. The new idea with which enables us to bypass all these obstacles, is to similarity transform $Q$ into an operator with an isomorphic cohomology. The new operator is a sum of terms acting on different sectors so that its cohomology is a direct sum of simpler cohomologies. A detailed analysis of this approach is presented in ref. [19]. Here we give a brief description of the method and its results. In the $(+, +)$ parametrization $T$ and $J^{(\text{tot})}$ are given by

$$T^{(\text{total})} = -\partial \phi^+ \partial \phi^- - i \partial^2 \phi^+ - \beta^+ \partial \gamma^+ - \beta^- \partial \gamma^- - \rho^+ \partial \chi^- - \rho^- \partial \chi^+ - 2 \rho_0 \rho_0 \chi_0$$

$$J^{(\text{tot})^+} = \beta^+ + \chi^+ \rho^0 - \chi^0 \rho^+$$

$$J^{(0(\text{total})} = \beta^+ \gamma^+ + \beta^- \gamma^- + i \partial \phi^- + \chi^+ \rho^- - \chi^- \rho^+$$

$$J^{(\text{tot})^-} = \beta^+ (\gamma^+ + \gamma^-) + \beta^- \gamma^+ \gamma^- - 2i (\gamma^+ \phi^- + t \gamma^- \phi^+) + 2(2 \partial \gamma^+ - t \partial \gamma^-) + \chi^- \rho^0 - \chi^0 \rho^-.$$  

(29)

where $t = k + 2$. The $(\beta^+, \gamma^+)$ and $(\beta^-, \gamma^-)$ are $(1, 0)$ commuting system and $\phi^+$ and $\phi^-$ are scalars.

Let us now define the dimension $(0,0)$ operators of zero ghost number

$$R = \oint \frac{dz}{2\pi i} (\chi^+ \rho^0 \gamma^- \gamma^- + 2 \chi^+ \rho^0 \gamma^+ - \chi^+ \rho^+ \gamma^+ \gamma^+)$$

$$P = -\oint \frac{dz'}{2\pi i} (i \phi^+ (\beta^+ \gamma^+ + \beta^- \gamma^- + \chi^+ \rho^- - \chi^- \rho^+))',$$

(30)

where $\oint \frac{dz'}{2\pi i}$ means that the zero modes of $\phi^+$ are excluded. We then use these
operators to transform $Q_{BRST}^{(rel)}$ to the desired form in the following way

$$e^{-P}e^RQ_{BRST}^{(rel)}e^{-R}e^P = Q_{tr}^{(rel)}$$  \hspace{1cm} (31)$$

with

$$Q_{tr}^{(rel)} = \oint \frac{dz}{2\pi i} [\chi^- \beta^+ + 2i\chi^0 \partial \phi^- - 2t\chi^+ \partial \gamma^- - 2t\phi_0^+ \chi^+ \gamma^-]$$

$$= 2 \sum_{n \neq 0} \chi_{-n}^0 \phi_n^- + \sum_n (\chi_{-n}^- \beta_n^+ - 2t(\phi_0^+ - n - 1)\chi_{-n}^+ \gamma^-).$$  \hspace{1cm} (32)$$

The mode expansions are relative to the vacuum of the twisted theory (i.e. $\gamma(z) = \sum_n \gamma_n z^{-(n+1)}$). From (31) it follows that the cohomologies of $Q_{BRST}^{(rel)}$ and of $Q_{tr}^{(rel)}$ are isomorphic, namely, for every state $|\Phi_0 >$ in the cohomology of $Q_{tr}^{(rel)}$, the state $|\Psi > = e^{-R}e^P|\Phi_0 >$ is in the cohomology of $Q_{BRST}^{(rel)}$ and vice versa.

On the following direct sum of Fock spaces

$$\bigoplus_{n \neq 0} F(\chi_{-n}^0, \rho_n^0, \phi_n^-, \phi_n^+) \bigoplus_{n} F(\chi_{-n}^-, \rho_n^+, \gamma_n^+, \beta_n^+) \bigoplus_{n} F(\chi_{-n}^+, \rho_n^-, \beta_n^-, \gamma_n^-)$$  \hspace{1cm} (33)$$

the first term is subjected to the action of the first term in eqn. (32), and similarly for the second and third terms. It is thus apparent that $Q_{tr}^{(rel)}$ indeed decomposes into a sum of anti-commuting terms which act on separate Fock spaces and, therefore, that the cohomology ring is a direct sum of smaller ones.

The nontrivial $Q_{tr}^{(rel)}$-cohomology states are spanned by

$$|\Phi_0 >= \rho_{-n}^- \gamma_{-n}^- r |\phi_0^+ = -n + 1, n > 0, \phi_0^- = r >$$

$$|\Phi_0 >= \gamma_{-n}^- r |\phi_0^+ = -n + 1, n > 0, \phi_0^- = r - 1 >$$

$$|\Phi_0 >= \chi_{-n}^- \beta_{-n}^- r |\phi_0^+ = n + 1 > 0, \phi_0^- = -r - 2 >$$

$$|\Phi_0 >= \beta_{-n}^- r |\phi_0^+ = n + 1 > 0, \phi_0^- = -r - 1 >$$

for $r = 0, 1, 2, ...$, and

$$|\Phi_0 >= |any\phi_0^+, \phi_0^- = -1 >.$$

We can now insert $|\Phi_0 >$ into the expressions for the states in the cohomology
of $Q_{\text{BRST}}^{(\text{rel})}$ as follows

$$|\Psi| = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} R^n \sum_{m=0}^{\infty} \frac{P_0^m}{m!} |\Phi_0| = e^{-R} e^{P_0} |\Phi_0|$$

7. **Physical states of the $SL(2,R)$ models versus those of the $c \leq 1$ models.**

The use of the similarity transformation method was motivated by the $k = -1$ $SL(2,R)$ case which corresponds to the $c = 1$ string model. In fact this approach is also adequate for any rational value of $t$ and thus produces a unified description of the topological coset models which are the counterparts of the $c \leq 1$ Liouville models. At ghost number $N_G = -1$, we expect that the discrete states found above would correspond to elements of the ground ring $^{[20]}$ (recall the shift in the ghost number when moving from states to operators because $|0 >_{\text{phys}} = \chi_1^+ |0 >_{SL(2,C)}$).

The lowest level state is simply $\rho_{-1}^+ |\phi_0^+ = \delta_0^- = 0$ which corresponds to the identity operator. The next two states of the cohomology of $Q_{\text{tr}}^{(\text{rel})}$ which are at level 2 translate into operators in the cohomology of $Q_{\text{BRST}}^{(\text{rel})}$ as follows:

$$\rho_{-2}^+ |\phi_0^+ = -1, \phi_0^- = 0 \rightarrow \tilde{y} = [-i\partial \phi^+ + \chi^+ (\rho^+ + 2 \rho^0 \gamma^+ + \rho^+ [(\gamma^-)^2 - (\gamma^+)^2])] e^{-i\phi^-}$$

These states are (with the appropriate identification) at the same momenta as those of the ground ring generators in the $c \leq 1$ models. In fact $\tilde{y}$ is equal to $y$ of ref. [20] with some additions from the “topological sectors”. One can also change the form of $\tilde{x}$ so it resembles that of the ground ring $x$ by adding a $Q_{\text{BRST}}^{(\text{rel})}$ exact term as follows

$$\tilde{x} = \{Q_{\text{BRST}}^{(\text{rel})}, \frac{1}{2} \rho^0 (\beta^-)^{-1} e^{i\phi^+} \} + (\beta^-)^{-1} (\chi^+ \rho^+ + i \partial \phi^- + \beta^+ \gamma^+ - \chi^- \rho^+) e^{i\phi^+}$$

(35)

The ground ring cohomology is now generated by $\tilde{x}^n \tilde{y}^m$. As in the ground ring of ref. [20], it is easy to realize that area preserving diffeomorphisms leave the ground ring invariant. These $W_\infty$ transformations are generated by currents constructed by acting on the $N_G = 1$ cohomology operators with $G_{-1}$. Recall that $G =
\[ \rho^-(J^+ - I^+) + 2\rho^0 (J^0 - I^0) + \rho^+(J^- - I^-) + \partial \rho^0. \]

For instance the generators \( \partial_{\tilde{x}} \) and \( \partial_{\tilde{y}} \) take the following form

\[
\begin{align*}
\partial_{\tilde{x}} &= G^{-1}_1 (\chi e^{-i\phi^+}) = \beta^- e^{-i\phi^+} \\
\partial_{\tilde{y}} &= G^{-1}_1 (\chi (\beta^-)^{-1} e^{i\phi^-}) = e^{i\phi^-}
\end{align*}
\]

It is easy to check that indeed, as is hinted by the notations,

\[
\partial_{\tilde{x}} \tilde{x} = \partial_{\tilde{y}} \tilde{y} = \partial_{\tilde{z}} \tilde{y} = \partial_{\tilde{y}} \tilde{x} = 0.
\]

One may wonder about the operator \((\beta^-)^{-1}\) which does not seem to be an appropriate operator to use since \(\beta^- = J^+ - I^+\). Without the inclusion of arbitrary powers of \(\beta^-\) the space of physical states of the \(\text{SL}(2,R)\) model does not recover that of the \(c \leq 1\) models. A similar situation is facing us also in the tachyonic sector. One possible prescription for regaining a full equivalence in the states is to implement an idea of ref. [21] where a further bosonization is invoked for the \((\beta^-, \gamma^-)\) system. In this bosonization \(\beta^- \equiv e^{u-iw}\) and \(\gamma^- \equiv -i\partial v e^{-u+iv}\), where \(u, v\) are free bosons with a background charge of \(-\frac{1}{2}\) and \(\frac{1}{2}\) respectively. In terms of the latter bosons, one is entitled to take any arbitrary power of \(\beta^-\) and hence we complete the missing states in the comparison with the gravitational models. For another prescription see ref. [19]. One branch of the tachyons of the \(c \leq 1\) model can be easily identified with a sector of the cohomology of the \(\text{SL}(2,R)\) model: this is the vacuum of the latter, \(|\phi_0^+ = p^+, \phi_0^- = -1 >\) which corresponds to the operator \(\chi^+ e^{i\phi^+ + \phi^- - i\phi^+}\). If one identifies \(\phi_I\) with the matter field \(X\), \(\phi_I\) with the Liouville field \(\phi\) and \(\chi^+\) with \(c\), the tachyonic states of one branch are indeed found. However, the other branch \(\chi^+ e^{i\phi^- \phi^+ + i\phi^-}\), is missing in the cohomology of \(Q_{BRST}^{(rel)}\). There are, however, additional states with no excitations at \(N_G = 0\). These are the states \((\beta_0^-)^r |\phi_0^+ = 1, \phi_0^- = -r - 1 >\) corresponding to the operators \(\chi^+ (\beta^-)^r e^{i\phi_0^- \phi^+ + i\phi^-}\). Apart from the appearance of the operator \(\beta^-\) these states are identical to a discrete series of the other branch of the tachyons. If again we bosonize \(\beta^-\) then \(r\) can take any real number and thus one finds states which correspond to the full missing branch. For \(k = -1\), restricting the values of \(r\) to the integers would correspond to the \(c = 1\) model at the self-dual radius.

The states of other ghost number are also in one to one correspondence with those of the \(c \leq 1\) Fock space relative cohomology. The only exception is that our second branch of the tachyon appears in both \(N_G = 1\) and \(N_G = 2\) whereas in the Liouville model it appears only in the former. A similar situation is revealed in the \(\text{SL}(2,R) / U(1)\) analysis of ref. [21].
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