Transverse conformal Killing forms and a Gallot-Meyer Theorem for foliations

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Abstract. We study transverse conformal Killing forms on foliations and prove a Gallot-Meyer theorem for foliations. Moreover, we show that on a foliation with $C$-positive normal curvature, if there is a closed basic 1-form $\phi$ such that $\Delta_B \phi = qC \phi$, then the foliation is transversally isometric to the quotient of a $q$-sphere.

1 Introduction

On Riemannian manifolds, Killing vector fields and conformal vector fields are very important geometric objects. Killing forms and conformal Killing forms are generalizations of such objects and were introduced by K. Yano [28] and T. Kashiwada [11,12]; they have been considered by many researchers [19,23,24,25]. Recently, U. Semmelmann [23] defined the conformal Killing forms by the kernel of the twistor operator $T$, which is defined on the exterior algebra, and proved many interesting results [19,23].

In this paper, we study the analogous problems on foliations. Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a compact oriented Riemannian manifold $M$ with codimension $q$. A transversal infinitesimal automorphism of $\mathcal{F}$ is an infinitesimal automorphism preserving the leaves. A transversal Killing field is a transversal infinitesimal isometry, i.e., its flow preserves the transverse metric. A transversal conformal field is a normal field with a flow preserving the conformal class of the transverse metric. Such geometric objects give some information about the leaf space $M/\mathcal{F}$. There are several known results about transversal Killing and conformal fields [7,9,21,26,27,29]. Since the space of transversal infinitesimal automorphisms can be identified with the space of the basic 1-forms, we can consider natural generalizations to differential forms. These are called

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transverse conformal Killing forms or transverse twistor forms, which are defined to be basic forms \( \phi \) such that for any vector field \( X \) normal to the foliation,

\[
\nabla_X \phi - \frac{1}{r+1} i(X) d\phi + \frac{1}{q-r+1} X^* \wedge \delta_T \phi = 0,
\]

where \( r \) is the degree of the form \( \phi \), \( X^* \) the dual 1-form of \( X \), and \( q \) is the codimension of \( \mathcal{F} \). See Section 3 for the definition of \( \delta_T \). The transverse conformal Killing forms with \( \delta_T \phi = 0 \) are called transverse Killing forms.

This paper is organized as follows. In Section 2, we review well-known facts concerning basic forms and give a generalization of Meyer’s theorem for foliations. In Section 3, we study transverse conformal Killing forms on foliations and prove some vanishing theorems. In Section 4, we study a Gallot-Meyer theorem [3] for foliations. Namely, let \( (M, g_M, \mathcal{F}) \) be a compact Riemannian manifold with a foliation \( \mathcal{F} \) (codim\( \mathcal{F} = q \)) that has \( C \)-positive normal curvature and a bundle-like metric \( g_M \). Then, for any basic \( r \)-form \( \phi \) \((1 \leq r \leq q-1)\), any eigenvalue \( \lambda_B \) of the basic Laplacian \( \Delta_B \) for \( \phi \) satisfies

\[
\lambda_B \geq \begin{cases} 
  r(q-r+1)C, & \text{if } d_B \phi = 0, \\
  (r+1)(q-r)C, & \text{if } \delta_B \phi = 0.
\end{cases}
\]

Moreover, we show that, if \( M \) admits a closed basic 1-form such that \( \Delta_B \phi = qC \phi \) and the foliation \( \mathcal{F} \) has \( C \)-positive normal curvature, then the foliation is transversally isometric to the quotient of a \( q \)-sphere. Lastly, we study special transverse Killing forms in Section 5.

## 2 Basic forms and Meyer’s Theorem

Let \( (M, g_M, \mathcal{F}) \) be a \((p+q)\)-dimensional Riemannian manifold with a foliation \( \mathcal{F} \) of codimension \( q \) and a bundle-like metric \( g_M \) with respect to \( \mathcal{F} \). Then we have an exact sequence of vector bundles

\[
0 \longrightarrow L \longrightarrow TM \overset{\pi}{\longrightarrow} Q \longrightarrow 0,
\]

where \( L \) is the tangent bundle and \( Q = TM/L \) is the normal bundle of \( \mathcal{F} \). The metric \( g_M \) determines an orthogonal decomposition \( TM = L \oplus L^\perp \), identifying \( Q \) with \( L^\perp \) and inducing a metric \( g_Q \) on \( Q \). The metric is bundle-like if and only if \( L_X g_Q = 0 \) for every \( X \in \Gamma L \), where \( L_X \) is the transverse Lie derivative. Let \( V(\mathcal{F}) \).
be the space of all vector fields $Y$ on $M$ satisfying $[Y, Z] \in \Gamma L$ for all $Z \in \Gamma L$. An element of $V(\mathcal{F})$ is called an \textit{infinitesimal automorphism} of $\mathcal{F}$. Let
\[
\bar{V}(\mathcal{F}) = \{ \bar{Y} := \pi(Y) \mid Y \in V(\mathcal{F}) \}.
\]
(2.2)
Then we have an associated exact sequence of Lie algebras
\[
0 \longrightarrow \Gamma L \longrightarrow V(\mathcal{F}) \xrightarrow{\pi} \bar{V}(\mathcal{F}) \longrightarrow 0.
\]
(2.3)

Let $\nabla$ be the transverse Levi-Civita connection on $Q$, which is torsion-free and metric with respect to $g_Q$. Let $R^\nabla, K^\nabla, \rho^\nabla$ and $\sigma^\nabla$ be the transversal curvature tensor, transversal sectional curvature, transversal Ricci operator and transversal scalar curvature with respect to $\nabla$, respectively. Let $\Omega^*_B(\mathcal{F})$ be the space of all \textit{basic forms} on $M$, i.e.,
\[
\Omega^*_B(\mathcal{F}) = \{ \omega \in \Omega^*(M) \mid i(X)\omega = 0, \ i(X)d\omega = 0, \ \forall X \in \Gamma L \}.
\]
(2.4)
Then $L^2(\Omega^*(M))$ is decomposed as [1, 22]
\[
L^2(\Omega^*(M)) = L^2(\Omega_B(\mathcal{F})) \oplus L^2(\Omega_B(\mathcal{F}))^\perp.
\]
(2.5)
We have $\Omega^*_B(\mathcal{F}) \subset \Gamma(\Lambda^*Q^*)$ and $\bar{V}(\mathcal{F}) \cong \Omega^*_B(\mathcal{F})$. Now we define the connection $\nabla$ on $\Omega^*_B(\mathcal{F})$, which is induced from the connection $\nabla$ on $Q$ and Riemannian connection $\nabla^M$ of $g_M$. This connection $\nabla$ extends the partial Bott connection $\nabla^\circ$ given by $\nabla^\circ_X \phi = L_X \phi$ for any $X \in \Gamma L$ [10]. Then the basic forms are characterized by $\Omega^*_B(\mathcal{F}) = \text{Ker} \nabla^\circ \subset \Gamma(\Lambda^*Q^*(\mathcal{F}))$. Let $P : L^2(\Omega^*(M)) \to L^2(\Omega^*_B(\mathcal{F}))$ be the orthogonal projection onto basic forms [22], which preserves smoothness in the case of Riemannian foliations. For any $r$-form $\phi$, we put the basic part of $\phi$ as $\bar{\phi}_B := P\phi$. The exterior differential on the de Rham complex $\Omega^*(M)$ restricts a differential $d_B : \Omega^*_B(\mathcal{F}) \to \Omega^{r+1}_B(\mathcal{F})$. Let $\kappa \in Q^*$ be the mean curvature form of $\mathcal{F}$. Then it is well known [1] that $\kappa_B := P\kappa$ is closed. We now recall the star operator $\bar{*} : \Omega^r(M) \to \Omega^{q-r}(M)$ given by [22]
\[
\bar{*} \phi = (-1)^{p(q-r)} \ast (\phi \wedge \chi_F), \quad \forall \phi \in \Omega^r(M),
\]
(2.6)
where $\chi_F$ is the characteristic form of $\mathcal{F}$ and $\ast$ is the Hodge star operator associated to $g_M$. The operator $\bar{*}$ maps basic forms to basic forms and has the property that [27]
\[
\bar{*} \phi = \bar{*} \phi \wedge \chi_F, \quad \forall \phi \in \Omega^r_B(\mathcal{F}).
\]
(2.7)
For any \( \phi, \psi \in \Omega_B^r(\mathcal{F}) \), \( \phi \wedge \bar{\psi} = \psi \wedge \bar{\phi} \) and also \( \bar{\phi}^2 = (-1)^r(\bar{q} - r)\phi \). Let \( \nu \) be the transversal volume form, i.e., \( \ast \nu = \chi_F \). The pointwise inner product \( \langle \ , \ \rangle \) on \( \Lambda^rQ^* \) is defined uniquely by

\[
\langle \phi, \psi \rangle \nu = \phi \wedge \bar{\psi}.
\]

The global inner product \( \langle \cdot, \cdot \rangle_B \) on \( L^2(\Omega_B^r(\mathcal{F})) \) is

\[
\langle \phi, \psi \rangle_B = \int_M \langle \phi, \psi \rangle \mu_M, \quad \forall \phi, \psi \in \Omega_B^r(\mathcal{F}),
\]

where \( \mu_M = \nu \wedge \chi_F \) is the volume form with respect to \( g_M \). With respect to this scalar product, the formal adjoint \( \delta_B : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^{r-1}(\mathcal{F}) \) of \( d_B \) is given by [22]

\[
\delta_B \phi = (-1)^{q(r+1)+1} \ast d_T \bar{\phi} = \delta_T \phi + i(\kappa_B^2)\phi,
\]

where \( d_T = d - \kappa_B \wedge \) and \( \delta_T = (-1)^{q(r+1)+1} \ast d \bar{\phi} \) is the formal adjoint operator of \( d_T \) with respect to \( L^2(\Omega_B^r(\mathcal{F})) \). Note that

\[
i(\kappa_B^2)\phi = (-1)^{q(r+1)} \bar{\kappa}_B \wedge \bar{\phi}, \quad \forall \phi \in \Omega_B^r(\mathcal{F}),
\]

since \( \kappa_B \) is basic.

**Lemma 2.1** On a Riemannian foliation \( \mathcal{F} \), for any basic \( r \)-form \( \phi \),

\[
\langle i(\kappa_B^2)\phi, \delta_T \phi \rangle \geq -\frac{1}{2}|\delta_T \phi|^2 - \frac{1}{2}|i(\kappa_B^2)\phi|^2, \quad \text{and}
\]

\[
\langle d_B \phi, \kappa_B \wedge \phi \rangle \leq \frac{1}{2}|d_B \phi|^2 + \frac{1}{2}|\kappa_B \wedge \phi|^2.
\]

**Proof.** From \( |\delta_B \phi|^2 \geq 0 \) and \( |d_T \phi|^2 \geq 0 \), the proof follows. \( \Box \)

The basic Laplacian \( \Delta_B \) is given by \( \Delta_B = d_B \delta_B + \delta_B d_B \). Let \( \mathcal{H}_B^r(\mathcal{F}) = \text{Ker} \Delta_B \) be the set of the basic-harmonic forms of degree \( r \). Then we have [10, 22]

\[
\Omega_B^r(\mathcal{F}) = \mathcal{H}_B^r(\mathcal{F}) \oplus \text{im}d_B \oplus \text{im} \delta_B
\]

with finite dimensional \( \mathcal{H}_B^r(\mathcal{F}) \). Let \( \{E_a\}(a = 1, \cdots, q) \) be a local orthonormal frame for \( Q \), chosen so that each \( E_a \in \mathcal{V}(\mathcal{F}) \). Then the dual frame \( \{\theta^a\} \) consists of local basic forms. Let \( \nabla^*_\mathcal{F} \) be a formal adjoint of \( \nabla_\mathcal{F} = \sum_a \theta^a \otimes \nabla_{E_a} : \Omega_B^r(\mathcal{F}) \rightarrow Q^* \otimes \Omega_B^r(\mathcal{F}) \). Then \( \nabla^*_\mathcal{F} = -\sum_a (i(E_a) \otimes \text{id}) \nabla_{E_a} + (i(\kappa_B^2) \otimes \text{id}) \), and so

\[
\nabla^*_\mathcal{F} \nabla_\mathcal{F} = -\sum_a \nabla_{E_a, E_a}^2 \nabla_{\kappa_B^2} : \Omega_B^r(\mathcal{F}) \rightarrow \Omega_B^r(\mathcal{F}),
\]
where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ for any $X, Y \in TM$. The operator $\nabla^*_\nabla \nabla$ is positive definite and formally self adjoint on the space of basic forms \[4\]. We define the bundle map $A_Y : \Lambda^r Q^* \rightarrow \Lambda^r Q^*$ for any $Y \in V(F)$ \[9\] by

$$A_Y \phi = L_Y \phi - \nabla_Y \phi.$$ \hspace{1cm} (2.16)

Since $L_X \phi = \nabla_X \phi$ for any $X \in \Gamma L$, $A_Y$ preserves the basic forms and depends only on $\bar{Y}$. We recall the generalized Weitzenböck formula.

**Theorem 2.2** \[6\] On a Riemannian foliation $F$, we have

$$\Delta_B \phi = \nabla^*_\nabla \nabla \phi + F(\phi) + A_{\bar{Y}}^\sharp \phi, \quad \phi \in \Omega^r_B(F),$$ \hspace{1cm} (2.17)

where $F(\phi) = \sum_a \theta^a \wedge i(E_b) R^\nabla(E_b, E_a) \phi$. If $\phi$ is a basic 1-form, then $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$.

Let $R^\nabla : \Lambda^2 Q^* \rightarrow \Lambda^2 Q^*$ be the normal curvature operator, which is defined by

$$\langle R^\nabla(\omega_1 \wedge \omega_2), \omega_3 \wedge \omega_4 \rangle = g_Q(R^\nabla(\omega_1^\sharp, \omega_2^\sharp) \omega_3^\sharp, \omega_4^\sharp),$$ \hspace{1cm} (2.18)

where $\omega_i \in Q^*(i = 1, \cdots, 4)$. Then $F$ has constant transversal sectional curvature $C$ if and only if $R^\nabla \omega = C \omega$ for any basic 2-form $\omega$.

**Definition 2.3** A Riemannian foliation $F$ is said to have $C$-positive normal curvature if there exists a positive constant $C$ such that

$$\langle R^\nabla \omega, \omega \rangle \geq C|\omega|^2$$ \hspace{1cm} (2.19)

for any 2-form $\omega \in \Gamma(\Lambda^2 Q^*)$.

Then we have a generalization of the Meyer theorem for foliations.

**Theorem 2.4** Let $F$ be a Riemannian foliation that has $C$-positive normal curvature on a compact Riemannian manifold $(M, g_M)$. Then for any arbitrary basic $r$-form $\phi$ ($1 \leq r \leq q - 1$, $q = \text{codim} F$),

$$\langle F(\phi), \phi \rangle \geq r(q - r)C|\phi|^2.$$ \hspace{1cm} (2.20)

The equality holds locally if and only if $F$ has constant transversal sectional curvature $C$. 

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Proof. The proof is similar to that in [16]. See also [18]. □

Let $\mathcal{A}$ be the O’Neill’s integrability tensor [20], which satisfies

$$\mathcal{A}_X Y = \frac{1}{2} \pi^\perp [X, Y], \quad X, Y \in \Gamma L^\perp,$$

(2.21)

where $\pi^\perp : TM \to L$ is an orthogonal projection. It is trivial that $L^\perp$ is integrable if and only if $\mathcal{A} = 0$ on $\Gamma L^\perp$. Moreover, it is well-known [20] that, for any unit normal vectors $X, Y \in \Gamma Q$, we have

$$K^\nabla (X, Y) = K^M (X, Y) + 3|\mathcal{A}_X Y|^2,$$

(2.22)

where $K^M$ is the sectional curvature of $g_M$ on $M$. Equivalently, for any basic 1-forms $\omega_1$ and $\omega_2$, \[ \langle R^\nabla (\omega_1 \wedge \omega_2), \omega_1 \wedge \omega_2 \rangle = \langle R^M (\omega_1 \wedge \omega_2), \omega_1 \wedge \omega_2 \rangle + |\mathcal{A}_{\omega_1} \omega_2|^2, \]

(2.23)

where $R^M$ is the curvature operator on $M$. From the equation (2.23), if $M$ has a $C$-positive curvature, then the foliation $\mathcal{F}$ has also a $C$-positive normal curvature. Hence we have the following corollary.

Corollary 2.5 Let $\mathcal{F}$ be a Riemannian foliation on a space $(M, g_M)$ that has $C$-positive curvature and $g_M$ a bundle-like metric. Then for any basic $r$-form $\phi$,

$$\langle F(\phi), \phi \rangle \geq r(q - r)C|\phi|^2.$$

(2.24)

If the equality holds, then $L^\perp$ is integrable and $M$ has constant curvature $C$. The converse holds if $M$ has a constant curvature $C$.

Proof. Inequality (2.24) is a consequence of Theorem 2.4, because $\mathcal{F}$ has $C$-positive normal curvature. If equality holds, then the transversal sectional curvature is the constant $C$. From (2.22), we know that $\mathcal{A}_X Y = 0$ for all $X, Y \in \Gamma Q$, which means $L^\perp$ is integrable. □

3 Transverse conformal Killing forms

For details for the non-foliation case, see [23]. Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$. Then $Q^* \otimes \Lambda^r Q^*$ of $O(q)$-representation is isomorphic to the following direct sum:

$$Q^* \otimes \Lambda^r Q^* \cong \Lambda^{r-1} Q^* \oplus \Lambda^{r+1} Q^* \oplus \Lambda^{r-1} Q^*,$$

(3.1)
where $\Lambda^{r,1}Q^*$ is the intersection of the kernels of wedge product and contraction map. Elements of $\Lambda^{r,1}Q^*$ can be considered as 1-forms on $Q$ with values in $\Lambda^rQ^*$. For any $s \in Q, \omega \in Q^*$ and $\phi \in \Lambda^rQ^*$, the projection $Pr_{\Lambda^{r,1}} : Q^* \otimes \Lambda^rQ^* \rightarrow \Lambda^{r,1}Q^* \subset Q^* \otimes \Lambda^rQ^*$ is then explicitly given by
\[
[Pr_{\Lambda^{r,1}}(\omega \otimes \phi)](s) = \omega(s)\phi - \frac{1}{r+1}i(s)(\omega \wedge \phi) - \frac{1}{q-r+1}s^* \wedge i(\omega^\natural)\phi, \quad (3.2)
\]
where $s^*$ denotes the $g_Q$-dual 1-form to $s$, i.e., $s^*(v) = g_Q(s,v)$ and $\omega^\natural$ is the vector defined by $\omega(s) = g_Q(\omega^\natural, s)$. Now, we define the operator $T_{tr} : \Gamma(\Lambda^rQ^*) \rightarrow \Gamma(\Lambda^{r,1}Q^*)$ by
\[
[T_{tr}\phi](v) : = [Pr_{\Lambda^{r,1}}(\nabla v\phi)](v)
= \nabla_v\phi - \frac{1}{r+1}i(v)d_B\phi + \frac{1}{q-r+1}v^* \wedge \delta_T\phi
\quad \text{for any } v \in Q. \quad (3.3)
\]
for any $v \in Q$. The formal adjoint operator $T_{tr}^* : \Gamma(\Lambda^{r,1}Q^*) \rightarrow \Gamma(\Lambda^rQ^*)$ of $T_{tr}$ is given by
\[
T_{tr}^* = (-\sum_a i(E_a) \otimes id)\nabla_{E_a} + (i(\kappa_B^\natural) \otimes id). \quad (3.4)
\]
For any $\phi \in \Lambda^{r,1}Q^*$ and any $v \in Q$, $v^* \wedge (T_{tr}\phi)(v) = 0 = i(v)(T_{tr}\phi)(v)$. So (3.3) is proved.

**Definition 3.1** A basic $r$-form $\phi \in \Omega^r_B(\mathcal{F})$ is called a transverse conformal Killing $r$-form if for any vector field $X \in \Gamma Q$,
\[
\nabla_X\phi = \frac{1}{r+1}i(X)d_B\phi - \frac{1}{q-r+1}X^* \wedge \delta_T\phi, \quad (3.5)
\]
where $\delta_T = \delta_B - i(\kappa_B^\natural)$. In addition, if the basic $r$-form $\phi$ satisfies $\delta_T\phi = 0$, it is called a transverse Killing $r$-form.

Note that a transverse conformal Killing 1-form is a $g_Q$-dual form of a transverse conformal Killing vector field. In fact, let $\phi$ be a transverse conformal Killing 1-form. Then for any $Y, Z \in \Gamma Q$
\[
g_Q(\nabla_Z\phi^\natural, Y) = (\nabla_Z\phi)(Y) = \frac{1}{2}(i(Z)d_B\phi)(Y) - \frac{1}{q}(\delta_T\phi)Z^*(Y) = \frac{1}{2}d_B\phi(Z, Y) - \frac{1}{q}(\delta_T\phi)g_Q(Z, Y).
\]

Since $\delta_T \phi = -\text{div} \nabla \phi^\sharp$, we have

$$(L_{\phi^\sharp} g_Q)(Y,Z) = \frac{2}{q} \text{div} \nabla (\phi^\sharp) g_Q(Y,Z) \quad \forall Y,Z \in \Gamma Q.$$ 

This means that $\phi^\sharp$ is a transverse conformal Killing vector field. Note that since $\phi$ is basic, the $g_Q$-dual vector $\phi^\sharp$ and $g_M$-dual vector $\phi^\sharp$ are the same.

**Proposition 3.2** A basic $r$-form $\phi$ is a transverse conformal Killing form if and only if, for any vector fields $X,Y,X_a \in \Gamma Q$,

$$\{i(X)\nabla_Y \phi + i(Y)\nabla_X \phi\}(X_2, \cdots, X_r)$$

$$= 2 g_Q(X,Y)\theta(X_2, \cdots, X_r) - \sum_{a=2}^r (-1)^a \{g_Q(Y,X_a)\theta(X_2, \cdots, \hat{X}_a, \cdots, X_r)$$

$$+ g_Q(X,X_a)\theta(Y, X_2, \cdots, \hat{X}_a, \cdots, X_r)\},$$

where $\theta = -\frac{1}{q-r+1}\delta_T \phi$ and $\hat{X}$ means that $X$ is deleted.

A basic $r$-form $\phi$ is a transverse Killing $r$-form if and only if for any $X \in \Gamma Q$,

$$i(X)\nabla_X \phi = 0. \quad (3.6)$$

Then we have the following proposition.

**Proposition 3.3** Let $\phi$ be a transverse Killing $r$-form and let $\gamma$ be a transversal geodesic, i.e., $\gamma' \in \Gamma Q$ and $\nabla_{\gamma'} \gamma' = 0$. Then $i(\gamma')\phi$ is parallel along the transversal geodesic $\gamma$.

**Proof.** In fact, for any transverse Killing form $\phi \in \Omega^r_B(\mathcal{F})$, we have

$$\nabla_{\gamma'} i(\gamma') \phi = i(\nabla_{\gamma'} \gamma') \phi + i(\gamma') \nabla_{\gamma'} \phi = 0. \quad \square$$

**Lemma 3.4** On a Riemannian foliation $\mathcal{F}$ of codimension $q$, any basic $r$-form $\phi$ satisfies

$$|\nabla \phi|^2_B \geq \frac{1}{r+1} |d_B \phi|^2_B + \frac{1}{q-r+1} |\delta_T \phi|^2_B. \quad (3.7)$$

The equality holds if and only if $\phi$ is a transverse conformal Killing $r$-form, i.e., $T_{\text{tr}} \phi = 0$. 8
Proof. This proof is similar to the one in [23]. □

Let \( \bar{g}_M = g_L + \bar{g}_Q(= e^{2u}g_Q) \), where \( u \) is a basic function. Then we have the following.

**Proposition 3.5** On a Riemannian foliation \( \mathcal{F} \), the transverse Levi-Civita connections \( \bar{\nabla} \) and \( \nabla \) are related by

\[
\bar{\nabla}_X \phi = \nabla_X \phi - rX(u)\phi - d_Bu \wedge i(X)\phi + X^* \wedge i(\text{grad}_\nabla(u))\phi
\]

(3.8)

for any \( X \in \Gamma Q \) and \( \phi \in \Omega^r_B(\mathcal{F}) \), where \( \text{grad}_\nabla(u) = \sum_a E_a(u)E_a \) is a transversal gradient of \( u \).

**Proof.** From the relation between \( \nabla \) and \( \bar{\nabla} \) given by

\[
\bar{\nabla}_X \pi(Y) = \nabla_X \pi(Y) + X(u)\pi(Y) - g_Q(\pi(X), \pi(Y))\text{grad}_\nabla(u),
\]

the proof follows. □

Then we have the following theorem.

**Theorem 3.6** Let \((M, g_M, \mathcal{F})\) be a Riemannian manifold with a foliation \( \mathcal{F} \) and a bundle-like metric \( g_M \). Let \( \phi \in \Omega^r_B(\mathcal{F}) \) be a transverse conformal Killing \( r \)-form. Then \( \bar{\phi} = e^{(r+1)u}\phi \) is a transverse conformal Killing \( r \)-form with respect to the metric \( \bar{g}_Q = e^{2u}g_Q \).

**Proof.** First, from (3.8) and \( \kappa_{\bar{g}} = e^{-2u}\kappa \) we have that, for any basic \( r \)-form \( \phi \),

\[
\bar{d}_B \phi = d_B \phi, \quad \bar{\delta}_B \phi = e^{-2u}\{\delta_B + (2r - q)i(\text{grad}_\nabla(u))\}\phi.
\]

(3.9)

Let \( \phi \in \Omega^r_B(\mathcal{F}) \) be a transverse conformal Killing \( r \)-form. Then, for \( \bar{\phi} = e^{(r+1)u}\phi \), we have from (3.8)

\[
i(X)\bar{d}_B \bar{\phi} = (r+1)e^{(r+1)u}\{X(u)\phi - d_Bu \wedge i(X)\phi + \frac{1}{r+1}i(X)d_B \phi\},
\]

\[
\hat{X}^* \wedge \bar{\delta}_T \bar{\phi} = e^{(r+1)u}X^* \wedge \{\delta_T \phi - (q-r+1)i(\text{grad}_\nabla(u))\phi\},
\]

where \( \hat{X}^* = e^{2u}X \) is \( \bar{g}_Q \)-dual form to \( X \). Hence from (3.1) and (3.8), we have

\[
\frac{1}{r+1}i(X)\bar{d}_B \bar{\phi} - \frac{1}{q-r+1}\hat{X}^* \wedge \bar{\delta}_T \bar{\phi} = e^{(r+1)u}\left\{\frac{1}{r+1}i(X)d_B \phi - \frac{1}{q-r+1}X^* \wedge \delta_T \phi + X(u)\phi - d_Bu \wedge i(X)\phi + X^* \wedge i(\text{grad}_\nabla(u))\phi\right\}
\]

= \( \bar{\nabla}_X \bar{\phi} \).
So $\tilde{\phi}$ is a transverse conformal Killing $r$-form with respect to $\bar{g}_Q$. □

From (3.4), we have the another generalized Weitzenböck formula.

**Proposition 3.7** On a Riemannian foliation $\mathcal{F}$, we have the following Weitzenböck formula: for any basic $r$-form $\phi$

$$T^*_{tr}T_{tr}\phi = \nabla^*_{tr}\nabla_{tr}\phi - \frac{1}{r+1}\delta_B d_B\phi - \frac{1}{q-r+1}d_T \delta_T\phi,$$  

(3.10)

where $d_T = d_B - \kappa_B \wedge$ is formal adjoint of $\delta_T$.

From (2.10), we have that on $\Omega^r_B(\mathcal{F})$

$$[T^*_{tr}T_{tr}, \star] = 0.$$  

(3.11)

Hence we have the following corollary.

**Corollary 3.8** Any basic $r$-form $\phi$ is a transverse conformal Killing $r$-form if and only if $\bar{\star}\phi$ is a transverse conformal Killing $(q-r)$-form.

Now we define $K : \Omega^r_B(\mathcal{F}) \to \Omega^r_B(\mathcal{F})$ by

$$K(\phi) = d_B i(\kappa^*_B)\phi + \kappa_B \wedge \delta_T\phi.$$  

(3.12)

Trivially $K$ is formally self adjoint operator and $K$ is identically zero when $\mathcal{F}$ is minimal. From (2.17) and (3.10), we have the following proposition.

**Proposition 3.9** On a Riemannian foliation $\mathcal{F}$, we have that, for any basic $r$-form $\phi \in \Omega^r_B(\mathcal{F})$,

$$T^*_{tr}T_{tr}\phi = \frac{r}{r+1}\delta_B d_B\phi + \frac{q-r}{q-r+1}d_B\delta_B\phi + \frac{1}{q-r+1}K(\phi) - F(\phi) - A_{\kappa^*_B}\phi.$$  

(3.13)

**Corollary 3.10** Any basic $r$-form $\phi$ is a transverse conformal Killing form if and only if

$$F(\phi) + A_{\kappa^*_B}\phi = \frac{r}{r+1}\delta_B d_B\phi + \frac{q-r}{q-r+1}d_B\delta_B\phi + \frac{1}{q-r+1}K(\phi).$$  

(3.14)

**Corollary 3.11** Any transverse Killing $r$-form $\phi$ satisfies

$$F(\phi) = \frac{r}{r+1}\delta_T d_B\phi.$$  

(3.15)
Now we choose the bundle-like metric \( g_M \) such that \( \delta_B \kappa_B = 0 \). Any bundle-like metric may be modified to such a metric without changing \( g_Q \) \[14,15\]. Then for any form \( \phi \), \( \ll \nabla_{\kappa_B^2} \phi, \phi \gg_B = 0 \). Hence any transverse conformal Killing \( r \)-form satisfies
\[
\frac{1}{r+1} \ll (\kappa_B^2) d_B \phi, \phi \gg_B = \frac{1}{q-r+1} \ll \kappa_B \wedge \delta_T \phi, \phi \gg_B .
\] (3.16)

From (3.14) and (3.16), any transverse conformal Killing \( r \)-form \( \phi \) satisfies
\[
\ll F(\phi), \phi \gg_B
= \frac{r}{r+1} \| d_B \phi \|^2_B + \frac{q-r}{q-r+1} \| \delta_T \phi \|^2_B + \frac{q-2r}{q-r+1} \ll \kappa_B \wedge \delta_T \phi, \phi \gg_B .
\] (3.17)

From (2.12) and (3.17), we have
\[
\ll F(\phi), \phi \gg_B
\geq \frac{r}{r+1} \| d_B \phi \|^2_B + \frac{q}{2(q-r+1)} \| \delta_T \phi \|^2_B + \frac{2r-q}{2(q-r+1)} \| (\kappa_B^2) \phi \|^2_B .
\] (3.18)

Hence we have the following theorem.

**Theorem 3.12** Let \((M, g_M, F)\) be a compact Riemannian manifold with a foliation \( F \) of codimension \( q \) and a bundle-like metric \( g_M \) such that \( \delta_B \kappa_B = 0 \). Suppose \( F \) is non-positive and negative at some point. Then, for any \( 1 \leq r \leq q-1 \), there are no transverse conformal Killing \( r \)-forms on \( M \).

**Proof.** From (3.18), if \( 2r \geq q \), it is trivial. For \( 2r \leq q \), it follows from the fact that \( \pi : \Omega^r_B(F) \to \Omega^{q-r}_B(F) \) is an isometry and \( \pi \phi \) is a transverse conformal Killing form when \( \phi \) is transverse conformal Killing form. So the proof is completed. \( \square \)

**Corollary 3.13** Let \((M, g_M, F)\) be as in Theorem [3.12]. Suppose \( F \) is quasi-negative. Then, for any \( 1 \leq r \leq q-1 \), there are no transverse Killing \( r \)-forms on \( M \).

Since \( F(\phi) = \rho^\nabla(\phi^\pi) \) for any basic 1-form \( \phi \), we have the following corollary.

**Corollary 3.14** Let \((M, g_M, F)\) be as in Theorem [3.12]. Suppose that \( \rho^\nabla \) is non-positive and negative at some point. Then there are no transverse conformal Killing vector fields on \( M \).
Remark. When $F$ is minimal, Corollary 3.14 was proved in [9,21].

Remark. On Riemannian manifolds, the conformal Killing forms are sometimes denoted twistor forms, because conformal Killing forms are defined similarly to the twistor spinors in spin geometry. Moreover, the conformal Killing form is directly related to the twistor spinor $\psi$, which satisfies the equation
$$\nabla_M^X \psi = -\frac{1}{n} X \cdot D \psi$$
for vector fields $X$, where $D$ is the Dirac operator. As with ordinary manifolds, on Riemannian foliations, transverse conformal Killing forms are related to transversal twistor spinors $\psi$, which satisfy the equation
$$\nabla_X \psi = -\frac{1}{q} \pi(X) \cdot D_b \psi - \frac{1}{2q} \kappa_B \cdot \psi$$
for all $X \in V(F)$, where $D_b$ is the basic Dirac operator [4,5,6]. In fact, let $\psi_1, \psi_2$ be transversal twistor spinors on a transverse spin foliation. Then the basic $r$-form $\phi_r$ defined on any vectors $X_1, \ldots, X_r \in V(F)$ by
$$\phi_r(X_1, \ldots, X_r) = \langle (\bar{X}_1^* \wedge \cdots \wedge \bar{X}_r^*) \cdot \psi_1, \psi_2 \rangle \quad (3.19)$$
is a transverse conformal Killing form. The proof is similar to the one in [23].

4 Gallot-Meyer’s theorem for foliations

Let $(M, g_M, F)$ be a compact Riemannian manifold $M$ with a foliation $F$ of codimension $q$ and a bundle-like metric $g_M$. We begin with the following Lemma.

Lemma 4.1 For any basic $r$-form $\phi$ on $M$, we have that

$$\begin{align*}
|\ll \delta_B \phi, i(\kappa_B^s)\phi \gg_B | & \leq \|\delta_B \phi\|_B \|i(\kappa_B^s)\phi\|_B, \\
|\ll d_B \phi, \kappa_B \wedge \phi \gg_B | & \leq \|d_B \phi\|_B \|\kappa_B \wedge \phi\|_B.
\end{align*}$$

(4.1) (4.2)

The equalities hold if and only if $\delta_B \phi = si(\kappa_B^s)\phi$ and $d_B \phi = t\kappa_B \wedge \phi$ for real numbers $s, t$, respectively.

Proof. The proof follows from the Cauchy-Schwartz inequality. $\square$

From Proposition 3.9 we have the following proposition.

Proposition 4.2 Let $(M, g_M, F)$ be a compact Riemannian manifold with a foliation $F$ of codimension $q$ and a bundle-like metric $g_M$. Then, for any basic
$r$-form $\phi$ ($1 \leq r \leq q - 1$),

$$
\|T_{tr}\phi\|_B^2 + \frac{1}{r + 1}\|i(\kappa_B^r)\phi\|_B^2 + \frac{r - 1}{r + 1} \ll \|\delta_B\phi, i(\kappa_B^r)\phi \gg_B + \ll i(\kappa_B^r)d_B\phi, \phi \gg_B
$$

$$
= \frac{r}{r + 1}\|d_B\phi\|_B^2 + \frac{r - 1}{r + 1}\|\delta_B\phi\|_B^2 - \ll F(\phi, \phi \gg_B + \frac{1}{2} \ll (\delta_B\kappa_B)\phi, \phi \gg_B, \quad (4.3)
$$

where $\bar{r} = q - r$.

**Proof.** By direct calculation, $\langle \nabla_{\kappa_B^r} \phi, \phi \rangle_B = \frac{1}{2}\kappa_B^r(|\phi|_B^2) = \frac{1}{2}\langle d_B|\phi|_B^2, \kappa_B \rangle_B$. By integrating, $\ll \nabla_{\kappa_B^r} \phi, \phi \gg_B = \frac{1}{2} \ll (\delta_B\kappa_B)\phi, \phi \gg_B$. So the proof is completed, using (2.10), (2.16), (3.12). \(\square\)

From Proposition 4.2, we can generalize the Gallot-Meyer’s Theorem(cf. [3]) to the foliation setting.

**Proposition 4.3** Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ (codim$\mathcal{F} = q$) that has $C$-positive normal curvature and a bundle-like metric $g_M$. Then, for any basic $r$-form $\phi$ ($1 \leq r \leq q - 1$), any eigenvalue $\lambda_B$ of $\Delta_B$ for $\phi$ satisfies

$$
\lambda_B \geq \left\{ \begin{array}{ll}
    r(q - r + 1)C + B_1, & \text{if } d_B\phi = 0, \\
    (r + 1)(q - r)C + B_2, & \text{if } \delta_B\phi = 0,
\end{array} \right. \quad (4.4)
$$

where

$$
B_1 = \frac{\alpha_1^2}{2} - \alpha_2 - \alpha_1\sqrt{r(q - r + 1)C + \frac{\alpha_1^2}{4} - \alpha_2},
$$

$$
B_2 = \frac{\beta_1^2}{2} - \beta_2 - \beta_1\sqrt{(r + 1)(q - r)C + \frac{\beta_1^2}{4} - \beta_2},
$$

$$
\alpha_1 = \frac{q - r - 1}{q - r}\max(|\kappa_B|), \quad \alpha_2 = \frac{q - r + 1}{2(q - r)}\max(\delta_B\kappa_B), \quad \beta_1 = \frac{r + 1}{r}\max(|\kappa_B|) \text{ and } \beta_2 = \frac{r + 1}{2}\max(\delta_B\kappa_B).
$$

**Proof.** Let $\Delta_B\phi = \lambda_B\phi$ and $d_B\phi = 0$. Since $\|\kappa_B \land \phi\|_B^2 + \|i(\kappa_B^r)\phi\|_B^2 = \|\kappa_B|\phi|_B^2$, from (4.1), we have

$$
| \ll i(\kappa_B^r)\phi, \delta_B\phi \gg_B | \leq \lambda_B^\frac{1}{2}\|\kappa_B|\phi|_B\|_B \|\phi\|_B \leq \lambda_B^\frac{1}{2}\max(|\kappa_B|)\|\phi|_B^2. \quad (4.5)
$$

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From (2.20), (4.3) and (4.5), we have

\[
\|T_{\text{tr}}\phi\|_B^2 + \frac{1}{q - r + 1}\|i(\kappa_B^r)\phi\|_B^2
\]

\[
\leq \int_M \left( \frac{q - r}{q - r + 1}\lambda_B + \frac{q - r - 1}{q - r + 1}\kappa_B|\lambda_B^{\frac{1}{2}} - r(q - r)C + \frac{1}{2}(\delta_B\kappa_B)|\right)^2\|\phi\|_B
\]

\[
\leq \left( \frac{q - r}{q - r + 1}\lambda_B + \frac{q - r - 1}{q - r + 1}\max(|\kappa_B|)\lambda_B^{\frac{1}{2}} - r(q - r)C + \frac{1}{2}\max(\delta_B\kappa_B)\right)\|\phi\|_B^2
\]

\[
= \frac{q - r}{q - r + 1}\{\lambda_B + \alpha_1\lambda_B^{\frac{1}{2}} - r(q - r + 1)C + \alpha_2\}\|\phi\|_B^2,
\]

where \(\alpha_1 = \frac{q - r - 1}{q - r}\max(|\kappa_B|) \geq 0\) and \(\alpha_2 = \frac{q - r + 1}{2(q - r)}\max(\delta_B\kappa_B) \geq 0\). Hence we have

\[
\lambda_B + \alpha_1\lambda_B^{\frac{1}{2}} - r(q - r + 1)C + \alpha_2 \geq 0,
\]

which yields

\[
\lambda_B^{\frac{1}{2}} \geq \sqrt{(q - r + 1)C + \frac{\alpha_1^2}{4} - \alpha_2 - \frac{\alpha_1}{2}}.
\]

Hence we have

\[
\lambda_B \geq r(q - r + 1)C + \frac{\alpha_1^2}{2} - \alpha_2 - \alpha_1\sqrt{(q - r + 1)C + \frac{\alpha_1^2}{4} - \alpha_2}.
\]

Therefore, the first inequality of (4.4) is proved. For the proof of the second inequality of (4.4), let \(\Delta_B\phi = \lambda_B\phi\) and \(\delta_B\phi = 0\). From (4.11), we get

\[
| \ll d_B\phi, \kappa_B \wedge \phi \gg_B | \leq \lambda_B^{\frac{1}{2}}\max(|\kappa_B|)\|\phi\|_B^2.
\]

From (2.20), (4.3) and (4.10), we have

\[
\|T_{\text{tr}}\phi\|_B^2 + \frac{1}{q - r + 1}\|i(\kappa_B^r)\phi\|_B^2
\]

\[
\leq \left( \frac{r}{r + 1}\lambda_B + \max(|\kappa_B|)\lambda_B^{\frac{1}{2}} - r(q - r)C + \frac{1}{2}\max(\delta_B\kappa_B)\right)\|\phi\|_B^2
\]

\[
= \frac{r}{r + 1}\{\lambda_B + \beta_1\lambda_B^{\frac{1}{2}} - (r + 1)(q - r)C + \beta_2\}\|\phi\|_B^2,
\]

where \(\beta_1 = \frac{r + 1}{2r}\max(|\kappa_B|) \geq 0\) and \(\beta_2 = \frac{r + 1}{2r}\max(\delta_B\kappa_B) \geq 0\). Hence we have

\[
\lambda_B^{\frac{1}{2}} \geq \sqrt{(r + 1)(q - r)C + \frac{\beta_1^2}{4} - \beta_2 - \frac{\beta_1}{2}},
\]
which yields
\[ \lambda_B \geq (r + 1)(q - r)C + \frac{\beta^2}{2} - \beta_2 - \beta_1 \sqrt{(r + 1)(q - r)C + \frac{\beta^2}{4} - \beta_2}. \] (4.13)

Hence (4.4) follows. \( \square \)

**Proposition 4.4** Let \((M, g_M, F)\) be as in Proposition 4.3. Then, for any basic \(r\)-form \(\phi\) \((0 < r < q)\),

(1) if \(\phi \in \text{Ker} d_B\) is an eigenform corresponding to \(\lambda_B = r(q - r + 1)C + B_1\), then \(\phi\) is a transverse conformal Killing \(r\)-form and \(\kappa_B = 0\).

(2) if \(\phi \in \text{Ker} \delta_B\) is an eigenform corresponding to \(\lambda_B = (r + 1)(q - r)C + B_2\), then \(\phi\) is a transverse Killing \(r\)-form and \(\kappa_B = 0\).

**Proof.** Let \(\phi \in \text{Ker} d_B\) be an eigenform with \(\lambda_B = r(q - r + 1)C + B_1\). From (4.6), \(T \tr \phi = 0\) and \(i(\kappa_B^\sharp)\phi = 0\). So \(\phi\) is the transverse conformal Killing form. Moreover, from the second inequality in (4.6), we have
\[ |\kappa_B| = \max(|\kappa_B|), \quad \delta_B \kappa_B = \max(\delta_B \kappa_B). \]

So \(|\kappa_B|\) and \(\delta_B \kappa_B\) are constant. Since \(d_B \kappa_B = 0\) \([1]\), we have \(\Delta_B \kappa_B = 0\), i.e., \(\kappa_B \in \mathcal{H}_B(F)\). On a foliation \(F\) that has \(C\)-positive normal curvature, \(\mathcal{H}_B(F) = 0\) \([18]\). So there exists a basic function \(h\) such that \(\kappa_B = dh\). Therefore, \(0 = \int_M \Delta h = \int_M (\delta_B \kappa_B) = (\delta_B \kappa_B) \text{Vol}(M)\), which means \(\delta_B \kappa_B = 0\). This means that \(h\) is a harmonic function on \(M\), and so \(h\) is constant, since \(M\) is compact. Hence \(\kappa_B = 0\). This proves (1). The proof of (2) is similar. \( \square \)

**Remark.** Note that \(B_1(\leq 0)\) and \(B_2(\leq 0)\) depend on the mean curvature form of \(F\). Therefore, Proposition 4.4 implies that, on a foliation \(F\) that has \(C\)-positive normal curvature, if there exists a closed basic \(r\)-form (resp. coclosed basic \(r\)-form) with eigenvalue \(\lambda_B = r(q - r + 1)C + B_1\) (resp. \((r + 1)(q - r)C + B_2\)), then \(B_1 = 0\) (resp. \(B_2 = 0\)).

Now, we recall the tautness theorem \([117]\) on a foliated Riemannian manifold.

**Theorem 4.5** *(Tautness theorem)* Let \((M, g_M, F)\) be a compact Riemannian manifold with a Riemannian foliation \(F\) of codimension \(q \geq 2\) and a bundle-like metric \(g_M\). If the transversal Ricci operator \(\rho^\nabla\) is positive definite, then \(F\) is taut, i.e., there exists a bundle-like metric \(\tilde{g}_M\) for which all leaves are minimal submanifolds.
From Proposition 4.3 and Theorem 4.5, we have the following theorem.

**Theorem 4.6** Let \((M, g_M, F)\) be as in Theorem 4.3. Then, for any basic \(r\)-form \(\phi\) \((1 \leq r \leq q - 1)\), any eigenvalue \(\lambda_B\) of \(\Delta_B\) for \(\phi\) satisfies

\[
\lambda_B \geq \begin{cases} 
  r(q - r + 1)C, & \text{if } d_B \phi = 0, \\
  (r + 1)(q - r)C, & \text{if } \delta_B \phi = 0.
\end{cases} \tag{4.14}
\]

**Proof.** From the assumption \(\nabla R \geq C \cdot \text{id}\), we have \(\rho \nabla \geq (q - 1)C \cdot \text{id}\). From Theorem 4.5, since \(F\) is taut, the basic component of the mean curvature form \(\kappa\) is exact. By the reasoning in the proof of Corollary 4.2 in [13], we may modify the bundle-like metric \(g_M\) such that the basic Laplacian is unchanged as an operator, such that the transverse metric is the same as that of the original metric, and such that \(\kappa = 0\). Hence the proof follows from Proposition 4.3. \(\square\)

**Remark.** Let \(F\) have \(C\)-positive normal curvature. Then, by the tautness theorem, we can choose a bundle-like metric \(g_M\) such that \(\kappa = 0\). So, from Theorem 4.5, any closed \(r\)-form \(\phi\) with \(\lambda_B \phi = r(q-r+1)C\) is a transverse conformal Killing \(r\)-form, and a coclosed \(r\)-form \(\phi\) with \(\lambda_B \phi = (r+1)(q-r)C\) is a transverse Killing \(r\)-form.

Now, we recall the generalized Obata theorem [13] for foliations. Let \(\rho^\nabla(X) \geq (q - 1)CX\) for any constant \(C > 0\). If the smallest eigenvalue \(\lambda_1\) of the basic Laplacian acting on functions is \(\lambda_1 = qC\), then \(F\) is minimal and transversally isometric to the action of a finite subgroup of \(O(q)\) acting on the \(q\)-sphere of constant curvature \(C\). Hence we have the following theorem.

**Theorem 4.7** Let \((M, g_M, F)\) be a compact Riemannian manifold with a foliation \(F\) that has \(C\)-positive normal curvature and a bundle-like metric \(g_M\). If \(M\) admits a closed basic 1-form \(\phi\) such that \(\Delta_B \phi = qC \phi\), then \(F\) is minimal and transversally isometric to the action of a finite subgroup of \(O(q)\) acting on the \(q\)-sphere of constant curvature \(C\).

**Proof.** Let \(\phi\) be a closed basic 1-form and \(\Delta_B \phi = qC \phi\). If we put \(f = \delta_B \phi\), then

\[
\Delta_B f = \delta_B d_B \delta_B \phi = \delta_B \Delta_B \phi = qC \delta_B \phi = qC f.
\]

Since the normal curvature operator satisfies \(\nabla R \geq C \cdot \text{id}\) on \(\Lambda^2 Q^*\), we have \(\rho^\nabla(X) \geq (q - 1)CX\). So by the generalized Obata theorem, the foliation is minimal and transversally isometric to the action of a finite subgroup of \(O(q)\) acting on the \(q\)-sphere of constant of curvature \(C\). \(\square\)
5 Special transverse Killing forms

Definition 5.1 A transverse Killing $r$-form $\phi$ is called special with $\beta$ if it satisfies
\[ \nabla_X d_B \phi = \beta X^* \wedge \phi, \quad \forall X \in \Gamma(Q), \quad (5.1) \]
where $\beta$ is a constant.

Let $\phi \in \Omega^r_B(\mathcal{F})$ be a special transverse Killing $r$-form with $\beta$. Then we have
\[ \Delta_B \phi = \delta_T d_B \phi + L_{\kappa_B} \phi. \quad (5.2) \]
Since $\delta_T = -\sum_a i(E_a) \nabla_{E_a}$ on $\Omega^r_B(\mathcal{F})$, from (5.2) we have
\[ \Delta_B \phi - L_{\kappa_B} \phi = -\beta(q-r)\phi. \quad (5.3) \]
From (3.15), (5.2) and (5.3), if $\phi$ is a special transverse Killing $r$-form with $\beta$, then
\[ F(\phi) = -\frac{r(q-r)}{r+1} \beta \phi. \quad (5.4) \]
So we have the following proposition.

Proposition 5.2 Let $\mathcal{F}$ be a Riemannian foliation that has $C$-positive normal curvature on a Riemannian manifold $(M, g_M)$. If $\phi$ is a special transverse Killing $r$-form with $\beta$, then
\[ \beta \leq -(r+1)C. \]

From (5.4) and Theorem 2.4 we have the following theorem.

Theorem 5.3 Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$. Assume that the transversal sectional curvature $K^\nabla$ is a positive constant $C$. Then any transverse Killing $r$-form is special with $\beta = -(r+1)C$.

Proof. Let $\phi$ be a transverse Killing $r$-form, i.e., $\nabla_X \phi = \frac{1}{r+1} i(X) d_B \phi$. Hence we have
\[ R^\nabla(X,Y) \phi = \frac{1}{r+1} \{ i(Y) \nabla_X - i(X) \nabla_Y \} d_B \phi. \]
Hence we have

\[ \sum_a \theta^a \wedge R^\nabla(X, E_a)\phi = \frac{r}{r+1} \nabla_X d_B \phi. \]  

(5.5)

Since \( \mathcal{F} \) has constant transversal curvature \( C \), we have from

\[ \sum_a \theta^a \wedge R^\nabla(X, E_a)\phi = -r CX^* \wedge \phi. \]  

(5.6)

From (5.5) and (5.6), the proof is completed. \( \square \)

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