ASYMPTOTICS OF DEGENERATING EISENSTEIN SERIES

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Dedicated to Takahide Kurokawa and Kimio Miyajima
on the occasion of their 60th birthdays

Abstract. We give some estimates for the asymptotic orders of degenerating Eisenstein series for certain families of degenerating punctured Riemann surfaces, motivated by the question of identifying $L_2$-cohomology of the Takhtajan-Zograf metric that is originally asked by To and Weng.

1. Introduction

We consider the Teichmüller space $T_{g,n}$ and the associated Teichmüller curve $T_{g,n}$ of Riemann surfaces of type $(g, n)$ (i.e., Riemann surfaces of genus $g$ and with $n > 0$ punctures). We will assume that $2g - 2 + n > 0$, so that each fiber of the holomorphic projection map $\pi : T_{g,n} \rightarrow T_{g,n}$ is stable or equivalently, it admits the complete hyperbolic metric of constant sectional curvature $-1$. The kernel of the differential $TT_{g,n} \rightarrow T^vT_{g,n}$ forms the so-called vertical tangent bundle over $T_{g,n}$, which is denoted by $T^vT_{g,n}$. The hyperbolic metrics on the fibers induce naturally a Hermitian metric on $T^vT_{g,n}$.

In the study of the family of $\bar{\partial}$-operators acting on the $k$-differentials on Riemann surfaces (i.e., cross-sections of $(T^vT_{g,n})^{-k}|_{\pi^{-1}(s)} \rightarrow \pi^{-1}(s)$, $s \in T_{g,n}$), Takhtajan and Zograf introduced in [11] a Kähler metric on $T_{g,n}$, which is known as the Takhtajan-Zograf metric. In [11], they showed that the Takhtajan-Zograf metric is invariant under the natural action of the Teichmüller modular group $\Mod_{g,n}$ and it satisfies the following remarkable identity on $T_{g,n}$:

$$c_1(\lambda_k, \|\cdot\|_{Q,k}) = \frac{6k^2 - 6k + 1}{12\pi^2} \omega_{WP} - \frac{1}{9} \omega_{TZ}. $$

Here $\lambda_k = \det(\text{ind} \; \bar{\partial}_k) = \bigwedge^{\text{max}} \text{Ker} \; \bar{\partial}_k \otimes (\bigwedge^{\text{max}} \text{Coker} \; \bar{\partial}_k)^{-1}$ denotes the determinant line bundle on $T_{g,n}$, $\|\cdot\|_{Q,k}$ denotes the Quillen metric on $\lambda_k$, and $\omega_{WP}$, $\omega_{TZ}$ denote the Kähler form of the Weil-Petersson metric, the Takhtajan-Zograf metric on $T_{g,n}$ respectively. In [13], Weng studied the Takhtajan-Zograf metric in terms of Arakelov intersection, and he proved that $\frac{1}{9} \omega_{TZ}$ coincides with the first Chern form of an associated metrized
Takhtajan-Zograf line bundle over the moduli space $\mathcal{M}_{g,n} = T_{g,n}/\text{Mod}_{g,n}$. Recently, Wolpert [16] gave a natural definition of a Hermitian metric on the Takhtajan-Zograf line bundle whose first Chern form gives $\frac{3}{2}\omega_{TZ}$. Furthermore, we can observe that in the second term of the asymptotic expansion of the Weil-Petersson metric near the boundary of $\mathcal{M}_{g,n}$, the Takhtajan-Zograf metrics on the boundary moduli spaces could appear (see [7]).

We propose a program of identifying $L^2$-cohomology of $\mathcal{M}_{g,n}$ with respect to the Takhtajan-Zograf metric $H^*(\mathcal{M}_{g,n}, \omega_{TZ})$. Originally, Saper ([8]) applied Masur’s formula ([4]) to show that $L^2$-cohomology of $\mathcal{M}_{g,0}$ with respect to the Weil-Petersson metric $H^*(\mathcal{M}_{g,0}, \omega_{WP})$ is naturally isomorphic to $H^*(\mathcal{M}_{g,0}, \mathbb{R})$. However, it is disappointing that the results for the asymptotics of the Takhtajan-Zograf metrics in [6] are not sufficient for us to determine $H^*(\mathcal{M}_{g,n}, \omega_{TZ})$.

In the present paper, we prove some estimates for the degenerating orders of Eisenstein series for certain families of degenerating punctured Riemann surfaces, which may be an important step for calculating $H^*(\mathcal{M}_{g,n}, \omega_{TZ})$. It should be noted that there are already some results for the behaviors of degenerating Eisenstein series ([2], [3], [5], [9]).

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2. Main Theorems

2.1. Settings and notation. For simplicity of exposition, we consider a degenerating family $\{S_l\}$ of Riemann surfaces of type $(g,1)$ with two zero-homologous pinching geodesics $\gamma_1$ and $\gamma_2$ which divide the surface $S_l$ into three components $S^1_l, S^2_l, S^3_l$: the geodesic $\gamma_1$ divides $S^1_l$ from $S^2_l$, the geodesic $\gamma_2$ divides $S^2_l$ from $S^3_l$, and $S^1_l$ has the unique puncture. (It should be noted that all claims in propositions, theorems, etc. are easily generalized to the case of any degenerating family of hyperbolic surfaces of finite type with at least one puncture. In some of the statements, we will give remarks for the general case.) The vector-valued parameter $l$ varies around the origin in the Euclidian space $\mathbb{R}^{6g-4}$, where $l = 0$ represents the unique degenerate surface $S_0$ in the family. The limit surface $S_0$ consists of three components $S^1_0, S^2_0, S^3_0$ which are the limits of $S^1_l, S^2_l, S^3_l$ respectively as $l \to 0$. Let $q_j$ be the node shared by $S^j_0$ and $S^{j+1}_0$ ($j = 1, 2$). A puncture the smooth surface in the degenerate family originally has will be called an old puncture, for simplicity. It should be noted that the degenerating family can be described by the modified infinite-energy harmonic maps $f^l : S_0 \to S_l \setminus \{\gamma_1, \gamma_2\}$, which are introduced by S. Wolpert ([15]).
Let $L_l(\gamma)$ be the hyperbolic length of a simple closed geodesic $\gamma$ on $S_l$. For $0 \leq k \leq 1$ and $j = 1, 2$, set
\[
N_{\gamma_j}(k) = \left\{ p \in S_l \left| d_l(p, \gamma_j) \leq k \sinh^{-1}\left(1/\sinh\frac{L_l(\gamma_j)}{2}\right) \right\},
\]
the collar neighborhood around $\gamma_j$ in $S_l$, where $d_l(\cdot, \cdot)$ denotes the hyperbolic distance on $S_l$. Here we remark that
\[
\sinh^{-1}\left(1/\sinh\frac{x}{2}\right) = -\log x + 2 \log 2 + O(x^2), \quad x \to 0,
\]
which will be essentially used in the proofs of Lemma 2.3 and Lemma 2.6.

For $a \geq 1$, the $a$-cusp region $C_j(a) \subset S_0^j \cup S_0^{j+1}$ around the node $q_j$ is the union of two copies of $(z \mapsto z + 1) \setminus \{ z \in H \mid \text{Im} z \geq a \}$, equipped with the metric $ds^2 = (dy^2 + dx^2)/y^2$, where $H := \{ z \in \mathbb{C} \mid \text{Im} z > 0 \}$ is the upper half plane.

Let $(f^l)^*\Delta_l$ denote the pull-back of the negative hyperbolic Laplacian $\Delta_l$ on $S_l$ by $f^l$, that is, for a $C^2$-function $h$ on $S_0$,
\[
(f^l)^*\Delta_l \ (h) = \Delta_l \ (h \circ (f^l)^{-1}) \circ f^l.
\]
Let $\Delta_0$ denote the negative hyperbolic Laplacian on $S_0$. Then, it is known that $(f^l)^*\Delta_l$ converges to $\Delta_0$ uniformly on any compact subset of $S_0$ in the $C^3$-norm (see [13]). And, for a function $g$ on $S_l$, the pull-back of $g$ by $f^l$ is defined as
\[
(f^l)^*g = g \circ f^l.
\]
It should be noted that a $C^2$-function $g$ on $S_l$ satisfies
\[
(f^l)^*\Delta_l \ ( (f^l)^*g ) = \Delta_l(g) \circ f^l,
\]
which will be used in the proof of Lemma 2.7.

2.2. The counting function of orbits. Let $\Gamma_l$ be a Fuchsian group uniformizing $S_l$ such that $S_l \simeq H/\Gamma_l$. We normalize it such that $\Gamma_l$ contains a parabolic element $z \mapsto z + 1$. A cyclic group generated by the parabolic element is denoted by $\Gamma_{\infty}$. Then the Eisenstein series for $\Gamma_l$ associated to the unique puncture is expressed as
\[
E^l(z, s) = \sum_{\delta \in \Gamma_{\infty}\setminus\Gamma_l} (\text{Im} \delta z)^s, \quad z \in H, \text{ Re } s > 1.
\]
Here for any $z$ in $H$ and any equivalent class $[\delta]$ in $\Gamma_{\infty}\setminus\Gamma_l$, we can select the unique representative $\hat{\delta}$ for $[\delta]$ such that $-\frac{1}{2} \leq \text{Re} \hat{\delta} z < \frac{1}{2}$. Such $\hat{\delta} = \hat{\delta}(z, [\delta])$ will be called the canonical representative.

$E^l(z, s)$ is invariant under the action of $\Gamma_l$. Thus it can be considered as a function on $S_l$. Moreover, it is well known that the Eisenstein series satisfies
\[
(\Delta - s(s - 1)) E^l(z, s) = 0, \quad z \in H, \text{ Re } s > 1,
\]
which will play a crucial role in the proof of Lemma 2.7. Here $\Delta := 4 (\text{Im} z)^2 \frac{\partial^2}{\partial s^2}$ is the negative hyperbolic Laplacian on $H$, invariant under $\Gamma_l$, and thus it naturally descends to $\Delta_l$ on $S_l$. 
Now we are ready to present a new way to study the asymptotics of the Eisenstein series. When \( \text{Im} z < 1 \) and \( z \) is not equivalent to any point of \( \{w \in H \mid \text{Im} w > 1\} \) under the action of \( \Gamma_\infty \backslash \Gamma_t \), it is easy to see that for \( \delta \) in \( \Gamma_\infty \backslash \Gamma_t \), \( \text{Im} \hat{\delta}(z) = e^{-d(h, \hat{\delta}z)} \), where \( d(\cdot, \cdot) \) denotes the hyperbolic distance in \( H \) and \( h = \{w \in H \mid -\frac{1}{2} \leq \text{Re} w < \frac{1}{2}, \text{Im} w = 1\} \).

We introduce two counting functions of orbits of \( z \) with \( \text{Im} z < 1 \):

\[
\Pi_l(h, z, t) := \#\{[\delta] \in \Gamma_\infty \backslash \Gamma_t \mid d(h, \hat{\delta}z) \leq t\},
\]
\[
\Pi_l(z, t) := \#\{[\delta] \in \Gamma_\infty \backslash \Gamma_t \mid d(i, \hat{\delta}z) \leq t\},
\]

where \( \hat{\delta} \) is the canonical representative. Here we should remark that \( d(i, \hat{\delta}z) = \min_{\delta \in [\delta]} d(i, \delta z) \).

For \( z \) with \( \text{Im} z < 1 \) not equivalent to any point of \( \{w \in H \mid \text{Im} w > 1\} \) under the action of \( \Gamma_\infty \backslash \Gamma_t \), we can observe

\[
E_l(z, s) = \int_0^\infty e^{-st} d\Pi_l(h, z, t).
\]

We will state a famous property of \( \Pi_l(z, t) \) as in the form suited to our purpose.

**Proposition 2.1.** There exists an absolute constant \( U \) such that for \( z \in H \) with \( \text{Im} z < 1 \), the following estimate holds:

\[
\Pi_l(z, t) \leq U e^t \quad \text{for any } t \geq 0 \text{ and any } \Gamma_t.
\]

**Proof.** Our proof is based on the discussion in [12] p.516. Let \( B(p, r) \) denote a hyperbolic ball centered at \( p \) with radius \( r \) in \( H \). Now the collar lemma assures us that in any hyperbolic surface with at least one puncture, each puncture has a horocyclic neighborhood with area 2 (see [10]). Then we can find a universal constant \( \varepsilon > 0 \) such that orbits \( B(\delta i, \varepsilon) \) for \( \delta \in \Gamma_t \) are mutually disjoint for any \( \Gamma_t \). Because if \( d(\delta i, z) \leq t \) then \( B(\delta i, \varepsilon) \subset B(z, t + \varepsilon) \), we have

\[
\Pi_l(z, t) = \#\{[\delta] \in \Gamma_\infty \backslash \Gamma_t \mid d((\hat{\delta})^{-1}i, z) \leq t\}
\]
\[
\leq \#\{[\delta] \in \Gamma_\infty \backslash \Gamma_t \mid (\hat{\delta})^{-1}(B(i, \varepsilon)) \subset B(z, t + \varepsilon)\}
\]
\[
\leq \frac{|B(z, t + \varepsilon)|}{|B(i, \varepsilon)|} = \frac{\sinh^2 \left(\frac{t + \varepsilon}{2}\right)}{\sinh^2 \frac{\varepsilon}{2}}
\]
\[
\leq \frac{e^{\varepsilon}}{2 \sinh^2 \frac{\varepsilon}{2}} e^t \quad \text{for } t \geq 0.
\]

Here \( |\cdot| \) denotes the hyperbolic area in \( H \). \( \square \)

**Proposition 2.2.** Let \( s > 1 \). Let \( z \in H \) with \( \text{Im} z < 1 \) be not equivalent to any point of \( \{w \in H \mid \text{Im} w > 1\} \) under the action of \( \Gamma_\infty \backslash \Gamma_t \). Then we obtain

\[
\Pi_l(z, t) \leq \Pi_l(h, z, t) \leq \Pi_l(z, t + 1),
\]
\[ E^l(z, s) = s \int_0^\infty e^{-st} \Pi_l(h, z, t) \, dt, \]
\[ s \int_0^\infty e^{-st} \Pi_l(z, t) \, dt \leq E^l(z, s) \leq s \int_0^\infty e^{-st} \Pi_l(z, t + 1) \, dt. \]

**Proof.** Because \( d(i, \delta z) \leq d(h, \delta z) + 1 \), it follows from Proposition 2.1 that
\[ \Pi_l(h, z, t) \leq \Pi_l(z, t + 1) \leq eU e^t. \]
Then, integrations by parts and Proposition 2.1 provide
\[ E^l(z, s) = \int_0^\infty e^{-st} d\Pi_l(h, z, t) = \left[ e^{-st} \Pi_l(h, z, t) \right]_0^\infty + s \int_0^\infty e^{-st} \Pi_l(h, z, t) \, dt \]
\[ = s \int_0^\infty e^{-st} \Pi_l(h, z, t) \, dt \leq s \int_0^\infty e^{-st} \Pi_l(z, t + 1) \, dt. \]
This is the right-hand inequality in the statement.

Next we will prove the left-hand inequality. Since \( d(h, \delta z) \leq d(i, \delta z) \), it is easy to see that
\[ \Pi_l(h, z, t) \geq \Pi_l(z, t), \]
\[ E^l(z, s) = s \int_0^\infty e^{-st} \Pi_l(h, z, t) \, dt \geq s \int_0^\infty e^{-st} \Pi_l(z, t) \, dt. \]

\[ \square \]

### 2.3. Upper bounds for degenerating Eisenstein series.

We are going to present upper bounds for Eisenstein series on the components \( S^2_l \) and \( S^3_l \).

**Lemma 2.3.** Assume \( \text{Re } s > 1 \). There exists an absolute constant \( M_2(\text{Re } s) \) depending only on \( \text{Re } s \) such that for \( L_l(\gamma_1), L_l(\gamma_2) < 2 \sinh^{-1} 1 \) and \( 0 \leq k \leq 1 \), then
\[ |E^l(z, s)| \leq M_2(\text{Re } s) L_l(\gamma_1)^{(1+k)(\text{Re } s-1)} \] on \( \partial N_{\gamma_1}(k) \cap S^2_l \),
\[ |E^l(z, s)| \leq M_2(\text{Re } s) L_l(\gamma_1)^{2(\text{Re } s-1)}L_l(\gamma_2)^{(1+k)(\text{Re } s-1)} \] on \( \partial N_{\gamma_2}(k) \cap S^3_l \).

**Proof.** Because \( |E^l(z, s)| \leq E^l(z, \text{Re } s) \) holds, it is enough to show in the case \( s > 1 \). For \( z \) in \( H \), \( [z] \) denotes the corresponding point of \( S_l \). (2.1) implies easily that the distance of any curve connecting \( [z] \) on \( \partial N_{\gamma_1}(k) \cap S^2_l \) and the horocycle \( [h] \) is greater than \((1 + k)[\text{the width of half collar}]\). Therefore we see
\[ \Pi_l(h, z, -(1 + k) \log L_l(\gamma_1)) = 0. \]
Then Proposition 2.2 yields
\[ E^l(z, s) = s \int_{-(1+k)\log L_l(\gamma_1)}^\infty e^{-st} \Pi_l(h, z, t) \, dt. \]
By Propositions 2.1 and 2.2, it concludes that
\[
E^l(z, s) \leq s \int_{-\infty}^{\infty} e^{-st} \Pi_l(z, t + 1) \, dt \\
\leq s \int_{-\infty}^{\infty} e^{-st} e^{Ue^t} \, dt \\
= eUs \int_{-\infty}^{\infty} e^{-(s-1)t} \, dt \\
= \frac{eUs}{s-1} L_l(\gamma_1)^{(1+k)(s-1)}.
\]
The second case is similar. Just replace \(-(1+k) \log L_l(\gamma_1)\) with \(-2 \log L_l(\gamma_1) - (1 + k) \log L_l(\gamma_2)\).

\[\square\]

**Corollary 2.4.** Assume as in Lemma 2.3. Then for all \(L_l(\gamma_1), L_l(\gamma_2) < 2 \sinh^{-1} 1\) and all \(k\) with \(0 \leq k \leq 1\), it holds that
\[
|E^l(z, s)| \leq M_2(\text{Re } s) L_l(\gamma_1)^{(1+k)(\text{Re } s-1)} \quad \text{on } S^2_l - N_{\gamma_1}(k),
\]
\[
|E^l(z, s)| \leq M_2(\text{Re } s) L_l(\gamma_1)^{2(\text{Re } s-1)} L_l(\gamma_2)^{(1+k)(\text{Re } s-1)} \quad \text{on } S^3_l - N_{\gamma_2}(k).
\]
Here \(M_2(\text{Re } s)\) is the constant appearing in Lemma 2.3.

**Proof.** Because \(|E^l(z, s)| \leq E^l(z, \text{Re } s)\) holds, it is enough to show the statements for \(s > 1\). By (2.3), it is easy to see that \(E^l(z, s)\) is subharmonic. The maximal principle for subharmonic functions provides
\[
\sup_{z \in S^2_l - N_{\gamma_1}(k)} E^l(z, s) \leq \sup_{z \in S^2_l \cup S^2_l - N_{\gamma_1}(k)} E^l(z, s) \\
= \sup_{z \in \partial N_{\gamma_1}(k) \cap S^2_l} E^l(z, s) \\
\leq M_2(s) L_l(\gamma_1)^{(1+k)(s-1)}.
\]
(Remark: even in the case \(S^3_l \cup S^2_l\) has other old punctures, our discussion remains valid because \(E^l(z, s)\) assumes 0 at the old punctures.) The second case is similar. Just use the second inequality in Lemma 2.3. \(\square\)

We will summarize the special case for \(k = 0, 1\) in Corollary 2.4 as follows.

**Theorem 2.5.** Assume \(\text{Re } s > 1\). Then for all \(L_l(\gamma_1), L_l(\gamma_2) < 2 \sinh^{-1} 1\), it holds that
\[
|E^l(z, s)| \leq M_2(\text{Re } s) L_l(\gamma_1)^{(\text{Re } s-1)} \quad \text{on } S^2_l,
\]
\[
|E^l(z, s)| \leq M_2(\text{Re } s) L_l(\gamma_1)^{2(\text{Re } s-1)} \quad \text{on } S^2_l - N_{\gamma_1}(1),
\]
\[
|E^l(z, s)| \leq M_2(\text{Re } s) L_l(\gamma_1)^{2(\text{Re } s-1)} L_l(\gamma_2)^{(\text{Re } s-1)} \quad \text{on } S^3_l,
\]
\[
|E^l(z, s)| \leq M_2(\text{Re } s) L_l(\gamma_1)^{2(\text{Re } s-1)} L_l(\gamma_2)^{2(\text{Re } s-1)} \quad \text{on } S^3_l - N_{\gamma_2}(1).
\]
Here \(M_2(\text{Re } s)\) is the constant appearing in Lemma 2.3.
Remark 1. Corollary 2.4 and Theorem 2.5 have essentially improved the order estimates for the degenerating Eisenstein series in [5] Theorem 1 (2).

2.4. Lower bounds for degenerating Eisenstein series. Now we are ready to present lower bounds for Eisenstein series on the components $S_l^2$ and $S_l^3$. Henceforth, the set of points in $S_l$ the injectivity radii of which are greater than $\sinh^{-1} 1$ will be called the thick part of $S_l$.

Lemma 2.6. Let $s > 1$. There exist positive constants $K_i = K_i(s, \{S_l\})$ ($i = 1, 2, 3$) depending only on $s$ and the degenerating family $\{S_l\}$ such that for $L_l(\gamma_1), L_l(\gamma_2) < 2 \sinh^{-1} 1$ and $0 \leq k \leq 1$, then

\[
\begin{align*}
E_l^1(z, s) &\geq K_1 L_l(\gamma_1)^{(1+k)s} \quad \text{on } \partial N_{\gamma_1}(k) \cap S_l^2, \\
E_l^1(z, s) &\geq K_2 L_l(\gamma_1)^{2s} L_l(\gamma_2)^{(1-k)s} \quad \text{on } \partial N_{\gamma_2}(k) \cap S_l^2, \\
E_l^1(z, s) &\geq K_3 L_l(\gamma_1)^{2s} L_l(\gamma_2)^{(1+k)s} \quad \text{on } \partial N_{\gamma_2}(k) \cap S_l^3.
\end{align*}
\]

Proof. We mimic the proof of Lemma 4.2 in [14] p.84. For $z \in H$ with $\text{Im } z < 1$,

\[(\text{Im } z)^s \geq e^{-sd(z, h)}.
\]

Since $E_l^1(z, s) = \sum_{\delta \in \Gamma_\infty \backslash \Gamma} (\text{Im } \delta z)^s$ is a sum of positive terms over $\Gamma_\infty \backslash \Gamma$-orbits of $z$, we obtain

\[E_l^1(z, s) \geq e^{-sd(z, h)},
\]

where $d(z, h)$ denotes the distance from $h$ to the $\Gamma$-orbits of $z$. We should recall two facts here. The first one is (2.1). The second one is that the diameters of the thick parts of $S_l$ are bounded by a positive constant $D$ for all small $L_l(\gamma_1), L_l(\gamma_2)$, where $D$ depends only on the degenerating family $\{S_l\}$ (For example, by using the Bers constant we can easily see the second fact. Refer to Theorem 5.2.6 in [1] p.130.) Then, for $z \in \partial N_{\gamma_1}(k) \cap S_l^2$, we can observe that $d(z, h) \leq -(1 + k) \log L_l(\gamma_1) + D'$. Here $D'$ is a constant depending only on the degenerating family. Then we have

\[E_l^1(z, s) \geq e^{-sd(z, h)} \geq e^{-sD'(1+k)s}.
\]

The remaining two cases are similar. \hfill \square

Lemma 2.7. Let $s > 1$. For $i = 1, 2$, let $\Omega_i$ be any region ($\in S_0^{i+1}$) containing $\partial C_i(1) \cap S_0^{i+1}$. There exist positive constants $P_i = P_i(s, \Omega_i, \{S_l\})$ depending only on $s$ and $\Omega_i$ and the degenerating family $\{S_l\}$ such that for any sufficiently small $L_l(\gamma_1)$, then

\[
\begin{align*}
(f^i)^* E_l^1(z, s) &\geq P_1 L_l(\gamma_1)^{2s} \quad \text{on } \Omega_1, \\
(f^i)^* E_l^1(z, s) &\geq P_2 L_l(\gamma_1)^{2s} L_l(\gamma_1)^{2s} \quad \text{on } \Omega_2.
\end{align*}
\]

Proof. We will show only the first case. The second case is similar. We set

\[P_i = \inf_{z \in \Omega_i} L_l(\gamma_1)^{-2s} (f^i)^* E_l^1(z, s).
\]
Suppose that there exists a subsequence \( l_j \to 0 \) such that \( \lim_{j \to \infty} P_{l_j} = 0 \). Consider the function \( P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} (f^{l_j})^* E^{l_j} (z, s) \). By (2.2) and (2.3), we can observe that
\[
((f^{l_j})^* \Delta_{l_j} - s(s - 1)) P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} (f^{l_j})^* E^{l_j} (z, s) = 0
\]
and
\[
\inf_{z \in \Omega_1} P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} (f^{l_j})^* E^{l_j} (z, s) = 1.
\]
We choose another region \( \Omega'_1 \) such that \( \Omega_1 \subset \Omega'_1 \subset S_0^2 \). Because \( ((f^{l_j})^* \Delta_{l_j} - s(s - 1)) \) are uniformly non-degenerate on \( \Omega' \), the Harnack inequality provides
\[
\sup_{z \in \Omega_1} P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} (f^{l_j})^* E^{l_j} (z, s) \leq c(\Omega_1, \Omega'_1) \inf_{z \in \Omega_1} P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} (f^{l_j})^* E^{l_j} (z, s)
\]
\[
= c(\Omega_1, \Omega'_1) < \infty.
\]
Then using the interior Schauder estimate and the diagonal method as in the proof of Theorem 1 in [3], we can have a further subsequence which will be denoted by the same notation such that \( P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} (f^{l_j})^* E^{l_j} (z, s) \) and its first and second derivatives converge uniformly on any compact subset of \( \Omega_1 \) to a nonnegative function \( G(z, s) \) and its derivatives respectively. Then \( G(z, s) \) satisfies
\[
(\Delta_0 - s(s - 1)) G(z, s) = 0
\]
and
\[
\sup_{z \in \Omega_1} G(z, s) \leq \lim_{j \to \infty} \sup_{z \in \Omega_1} P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} (f^{l_j})^* E^{l_j} (z, s) \leq c(\Omega_1, \Omega'_1) < \infty.
\]
Now it should be noted that \( \Omega_1 \supset (f^{l})^{-1} (\partial N_{\gamma_1}(1) \cap S_1^2) \) for any sufficiently small \( l \) because \( (f^{l})^{-1} (\partial N_{\gamma_1}(1) \cap S_1^2) \) converges to \( \partial C_1(1) \cap S_0^2 \) as \( l \to 0 \). We choose another region \( \Omega''_1 \subset \Omega_1 \) such that \( \Omega''_1 \supset (f^{l})^{-1} (\partial N_{\gamma_1}(1) \cap S_1^2) \) for any sufficiently small \( l \). Then we have
\[
\sup_{z \in \Omega_1} G(z, s) \geq \sup_{z \in \Omega''_1} G(z, s)
\]
\[
= \lim_{j \to \infty} \sup_{z \in \Omega''_1} P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} (f^{l_j})^* E^{l_j} (z, s)
\]
\[
\geq \lim_{j \to \infty} \sup_{z \in (f^{l_j})^{-1} (\partial N_{\gamma_1}(1) \cap S_1^2)} P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} (f^{l_j})^* E^{l_j} (z, s)
\]
\[
\geq \lim_{j \to \infty} \inf_{z \in (f^{l_j})^{-1} (\partial N_{\gamma_1}(1) \cap S_1^2)} P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} (f^{l_j})^* E^{l_j} (z, s)
\]
\[
= \lim_{j \to \infty} \inf_{w \in \partial N_{\gamma_1}(1) \cap S_1^2} P_{l_j}^{-1} L_{l_j} (\gamma_1)^{-2s} E^{l_j} (w, s)
\]
\[
\geq \lim_{j \to \infty} P_{l_j}^{-1} K_1 = +\infty.
\]
Here we used the first inequality in Lemma 2.6. This is a contradiction. \( \square \)
Remark 2. All claims in Lemmas 2.6, 2.7 remain valid even in the case where the components $S_i^2, S_3^0$ have other old punctures. However, care for such additional old punctures is needed in the proof of the following theorem.

Theorem 2.8. Let $s > 1$. For all sufficiently small $L_i(\gamma_1), L_i(\gamma_2)$, it holds that

$$E^s(z, s) \geq Q_1 L_i(\gamma_1)^{2s} \quad \text{on } S_i^2 - f^i(C_2(a)),$$

$$E^s(z, s) \geq Q_2 L_i(\gamma_1)^{2s} L_i(\gamma_2)^s \quad \text{on } S_i^2 \cap N_{\gamma_2}(1),$$

$$E^s(z, s) \geq Q_3 L_i(\gamma_1)^{2s} L_i(\gamma_2)^{2s} \quad \text{on } S_i^3.$$

Here $Q_1 = Q_1(s, a, \{S_i\})$ is a positive constant depending only on $s, a$ and the degenerating family $\{S_i\}$, $Q_i = Q_i(s, \{S_i\})$ ($i = 2, 3$) are positive constants depending only on $s$ and the degenerating family $\{S_i\}$.

Remark 3. In the case where $S_i^2$ has additional old punctures ($i = 2, 3$), we have to replace $S_i^2$ with $S_i^2 - f^i(t)$ (the union of all neighborhoods of old punctures), and all $Q_i$'s depend on all the removed neighborhoods.

Proof. First, we will show the first inequality. Set

$$Q_l = \inf_{z \in \{f^i\}^{-1}(N_{\gamma_1}(1) \cap S_0^2)} L_l(\gamma_1)^{-2s} E_l^s(z, s) = \inf_{w \in N_{\gamma_1}(1) \cap S_0^2} L_l(\gamma_1)^{-2s} E_l^s(w, s).$$

Due to the first inequality in Lemma 2.7, all we have to prove is that $Q_l$ is larger than a positive constant for all small $l$. Assume that there exists a subsequence $l_j \to 0$ such that $\lim_{j \to \infty} Q_{l_j} = 0$. For each $j$, we can find a point $w_j \in N_{\gamma_1}(1) \cap S_0^2$ such that

$$L_{l_j}(\gamma_1)^{-2s} E_{l_j}(w_j, s) = \inf_{w \in N_{\gamma_1}(1) \cap S_0^2} L_{l_j}(\gamma_1)^{-2s} E_{l_j}(w, s).$$

If $w_j$ is not on the geodesic $\gamma_1$, set $z_j = (f^{l_j})^{-1}(w_j)$. Divide our situation into three cases. (If necessary, we will take a subsequence which is denoted by the same symbol, for simplicity.)

I. infinitely many $w_j$ are on the geodesic $\gamma_1$.

II. there exists $b \geq 1$ such that all but finitely many $z_j$ are outside of $C_1(b) \cap S_0^2$.

III. there exists a subsequence such that $\lim_{j \to \infty} z_j = q_1$.

In case I, due to the first inequality with $k = 0$ in Lemma 2.6,

$$Q_{l_j} = L_{l_j}(\gamma_1)^{-2s} E_{l_j}(w_j, s) \geq K_1 L_{l_j}(\gamma_1)^{-s} \geq K_1 > 0 \quad \text{for all large } j.$$

This is a contradiction.

In case II, we choose a region $\Omega_1 (\subseteq S_0^2)$ which contains $\partial C_1(1) \cap S_0^2$ and $z_j$. Due to the first inequality in Lemma 2.7,

$$Q_{l_j} = L_{l_j}(\gamma_1)^{-2s} (f^{l_j})^* E_{l_j}(z_j, s) \geq \inf_{z \in \Omega_1} L_{l_j}(\gamma_1)^{-2s} (f^{l_j})^* E_{l_j}(z, s) \geq P_1(\Omega_1) > 0.$$

This is a contradiction.
In case III, there exists $0 \leq k_j \leq 1$ such that $f^{l_j}(z_j) \in \partial N_{\gamma_1}(k_j) \cap S^2_{l_j}$.

Then due to the first inequality in Lemma 2.6, we have

$$Q_{l_j} = L_{l_j}(\gamma_1)^{-2s}(f^{l_j})^s E^{l_j}(z_j, s) \geq \inf_{w \in \partial N_{\gamma_1}(k_j) \cap S^2_{l_j}} L_{l_j}(\gamma_1)^{-2s} E^{l_j}(w, s) \geq K_1 L_{l_j}(\gamma_1)(k_j - 1)^s \geq K_1 > 0$$

for all large $j$. This is a contradiction. We have proved the first inequality.

Next, we will show the second inequality in a similar method. We set

$$Q'_{l_j} = \inf_{w \in N_{\gamma_2}(1) \cap S^2_{l_j}} L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(w, s).$$

Assume that there exists a subsequence $l_j \to 0$ such that $\lim_{j \to \infty} Q'_{l_j} = 0$. For each $j$, we can find a point $w_j \in N_{\gamma_2}(1) \cap \tilde{S}^2_l$ such that

$$L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(w_j, s) = \inf_{w \in N_{\gamma_2}(1) \cap S^2_{l_j}} L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(w, s).$$

If $w_j$ is not on the geodesic $\gamma_2$, set $z_j = (f^{l_j})^{-1}(w_j)$. Divide our situation into three cases. (If necessary, we will take a subsequence which is denoted by the same symbol, for simplicity.)

I’. infinitely many $w_j$ are on the geodesic $\gamma_2$, if $z_j$ exists $b \geq 1$ such that all but finitely many $z_j$ are outside of $C_2(b) \cap S^2_0$,

III’. there exists a subsequence such that $\lim_{j \to \infty} z_j = q_2$.

In case I’, due to the second inequality with $k = 0$ in Lemma 2.6,

$$Q'_{l_j} = L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(w_j, s) \geq K_2 > 0 \quad \text{for all large } j.$$ 

This is a contradiction.

In case II’, we choose a region $\Omega'_{l} (\in S^2_0)$ which contains $\partial C_1(1) \cap S^2_0$ and $z_j$. Due to the first inequality in Lemma 2.7,

$$Q'_{l_j} = L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} (f^{l_j})^s E^{l_j}(z_j, s) \geq \inf_{z \in \Omega'_{l}} L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} (f^{l_j})^s E^{l_j}(z, s) \geq P_1(\Omega'_{l}) L_{l_j}(\gamma_2)^{-s} \geq P_1(\Omega'_{l}) > 0$$

for all large $j$. This is a contradiction.

In case III’, there exists $0 \leq k_j \leq 1$ such that $f^{l_j}(z_j) \in \partial N_{\gamma_2}(k_j) \cap S^2_{l_j}$.

Then due to the second inequality in Lemma 2.6, we have

$$Q'_{l_j} = L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(f^{l_j}(z_j), s) \geq \inf_{w \in \partial N_{\gamma_2}(k_j) \cap S^2_{l_j}} L_{l_j}(\gamma_1)^{-2s} L_{l_j}(\gamma_2)^{-s} E^{l_j}(w, s) \geq K_2 L_{l_j}(\gamma_1)^{-k_j s} \geq K_2 > 0$$

for all large $j$. This is a contradiction. We have proved the second inequality. We can prove the third inequality in the same way as the first inequality, using the third inequality in Lemma 2.6 and the second inequality in Lemma 2.7. \[\square\]
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