THE DISTINGUISHABILITY OPERATION
ON REGULAR LANGUAGES

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Abstract
In this paper we study the language of the words that, for a given language \(L\), distinguish between pairs of different left-quotients of \(L\). We characterize this distinguishability operation, show that its iteration has always a fixed point, and we generalize this result to operations derived from closure operators and Boolean operators. We give an upper bound for the state complexity of the distinguishability, and prove its tightness. We show that the set of minimal words that can be used to distinguish between different quotients of a language \(L\) has at most \(n - 1\) elements, where \(n\) is the state complexity of \(L\), and we also study the properties of its iteration.

1 Introduction

Regular languages and operations over them have been extensively studied during the last sixty years, the applications of automata being continuously extended in different areas. As a practical example, we can use automata to model various electronic circuits. The testing of the circuits can be done by applying several inputs to various pins of a circuit, and checking the output produced. Because in many cases the circuits emulate automata, it is useful to develop general tools for testing various properties of automata, such as testing the relation between the response of the circuit for the same signal, applied to different gates. To answer if an automaton is minimal requires to test if two states are equivalent or not. The easiest way to do this is to use as input different words, and see if for both states, we reach states with the same finality, thus, in case of a circuit, for both input gates we will get the same value of the output bit. However, checking every possible word is a tedious task, and it would be useful to limit the testing only to the words that can distinguish between states.

\(^{(A)}\)Authors partially funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT under projects PEst-C/MAT/UI0144/2013 and FCOMP-01-0124-FEDER-020486.
Therefore, it is worth studying the language that distinguish between all non-equivalent states of a given deterministic finite automaton. Given an automaton $A$ we denote the distinguishing language by $D(A)$.

The idea of studying word or state distinguishability is not new. In 1958, Ginsburg studied the length of the smallest uniform experiment which distinguishes the terminal states of a machine [10], and with Spanier in [11], he studies whether or not an arbitrary semigroup can serve as an input for a machine that distinguishes between the elements of the input semigroup. A comparable work was done for terminal distinguishability by Semper e [16], where terminal segments of automata are studied to characterize language families that can be identified in the limit from positive data. Indeed, knowing that an automaton $A$ has at most $n$ states, and having the language $D(A)$, together with the words of length at most $n + 1$ that are in the language $L(A)$, we can recover the initial automaton $A$. Note that without the language $D(A)$, any learning procedure will only approximate the language $L(A)$. For example, in case we know $M$ to be the set of all the words of a language $L$ with length at most $n + 1$, we can infer that $L$ is a cover language for $M$, but we cannot determine which one of these cover languages is $L$. Thus, any learning procedure would only be able to guess $L$ from $M$, and the guess would not be accurate, as the number of cover automata for a finite language can be staggeringly high [7, 8]. In [14], Restivo and Vaglica proposed a graph-theoretical approach to test automata minimality. For a given automaton $A$ they associate a digraph, called pair graph, where vertices are pairs $\{p, q\}$ of states of $A$, and edges connect vertices for which the states have a transition from the same symbol in $A$. Then, two states $p$ and $q$ of $A$ are distinguishable if and only if there is a path from the vertex $\{p, q\}$ to a vertex $\{p', q'\}$, where $p'$ is final and $q'$ is non-final, i.e., there exists a word that distinguishes between them. A related research topic is the problem of finding a minimal DFA that distinguishes between two words by accepting one and rejecting the other. It was studied by Blumer et al. in [2], and recently Demaine et al. in [9] reviewed several attempts to solve the problem and presented new results.

In the present paper we do not separate two words by a language, instead, we distinguish between non-equivalent quotients of the same language. We use many powerful tools such as language quotients, atoms, and universal witnesses, that hide proof complexity, helping us to produce a presentation easier to follow. We introduce the notation in Section 2 define and prove general properties of the distinguishability operation in Section 3 and prove some state complexity results in Section 4. In Section 5 we analyze the set of minimal words with respect to quasi-lexicographical order that distinguishes different quotients of a regular language. In Section 6 we present a class of operands using closure operations and Boolean operations, that have a fixed point under iteration. The conclusion, together with open problems and future work, are included in Section 7.

2 Notation and Definitions

For a set $T$, its cardinality is denoted by $|T|$. An alphabet $\Sigma$ is a finite non-empty set, and the free monoid generated by $\Sigma$ is $\Sigma^*$. A language is a subset of $\Sigma^*$. The complement of a language $L$ is $\overline{L} = \Sigma^* \setminus L$. The length of a word $w \in \Sigma^*$, $w = w_1w_2\ldots w_n$, $w_i \in \Sigma$, $1 \leq i \leq n$, with
A deterministic finite automaton (DFA) is a quintuple \( \mathcal{A} = (Q, \Sigma, q_0, \delta, F) \), where \( Q \) is a finite non-empty set, the set of states, \( \Sigma \) is the alphabet, \( q_0 \in Q \) is the initial state, \( F \subseteq Q \) is the set of final states, and \( \delta : Q \times \Sigma \rightarrow Q \) is the transition function. This function defines for each symbol of the alphabet a transformation of the set \( Q \) of states (i.e. a map from \( Q \) to \( Q \)). The transition semigroup of a DFA \( \mathcal{A} \), \([1],[2]\), is the semigroup of transformations of \( Q \) generated by the transformations induced by the symbols of \( \Sigma \).

A reduced DFA is a DFA with all states reachable from the initial state (accessible), and all states can reach a final state (useful), except at most one that is a sink state or dead state, i.e., a state where all output transitions are self loops.

The transition function \( \delta \) can be extended to \( \delta : Q \times \Sigma^* \rightarrow Q \) by \( \delta(q, \varepsilon) = q \), and \( \delta(q, wa) = \delta(\delta(q, w), a) \). The language recognized by a DFA \( \mathcal{A} \) is \( \mathcal{L}(\mathcal{A}) = \{ w \in \Sigma^* \mid (q_0, w) \in F \} \). We denote by \( L_q \) and \( R_q \) the left and right languages of \( q \), respectively, i.e., \( L_q = \{ w \mid \delta(q_0, w) = q \} \), and \( R_q = \{ w \mid \delta(q, w) \in F \} \).

A regular language is a language recognized by a DFA. A regular language \( L \) induces on \( \Sigma^* \) the Myhill-Nerode equivalence relation: \( x \equiv_L y \) if, for all \( w \in \Sigma^* \), we have that \( xw \in L \) if and only if \( yw \in L \). If \( \mathcal{A} = (Q, \Sigma, \delta, q_0, F) \) is a DFA recognizing the language \( L \) and \( R_q = R_p \), then we say that \( p \) and \( q \) are equivalent, and write \( p \equiv_L q \). A DFA is minimal if it has no equivalent states. The left quotient, or simply quotient, of a regular language \( L \) by a word \( w \) is the language \( w^{-1}L = \{ x \mid wx \in L \} \). A quotient corresponds to a equivalence class of \( \equiv_L \). If a language \( L \) is regular, the number of distinct quotients is finite, and it is exactly the number of states in the minimal DFA recognizing \( L \). This number is called the state complexity of \( L \), and is denoted by \( sc(L) \). In a minimal DFA, for each \( q \in Q \), \( R_q \) is exactly a quotient. If some quotient of a language \( L \) is \( \emptyset \), this means that the minimal DFA of \( L \) has a dead state.

A nondeterministic finite automata (NFA) is a quintuple \( \mathcal{N} = (Q, \Sigma, I, \delta, F) \), where \( Q, \Sigma, \) and \( F \) are the same as in the DFA definition, \( I \subseteq Q \) is the set of initial states, and \( \delta : Q \times \Sigma \rightarrow 2^Q \) is the transition function. The transition function can also be extended to subsets of \( Q \) instead of states, and to words instead of symbols of \( \Sigma \). The language recognized by an NFA \( \mathcal{N} \) is \( \mathcal{L}(\mathcal{N}) = \{ w \mid \delta(I, w) \cap F \neq \emptyset \} \). It is obvious that a DFA is also an NFA. Any NFA can be converted in an equivalent DFA by the well known subset construction.

More notation and definitions related to formal languages can be consulted in \([15],[17]\).

### 3 The Distinguishability Operation

Let \( L \) be a regular language. For every pair of words, \( x, y \in \Sigma^* \), with \( x \not\equiv_L y \), there exists at least one word \( w \) such that either \( xw \in L \) or \( yw \in L \). Let \( \mathcal{A} = (Q, \Sigma, \delta, q_0, F) \) be a DFA such that \( L = \mathcal{L}(\mathcal{A}) \). If two states \( p, q \in Q \), \( p \not\equiv q \), then there exists at least one word \( w \) such that \( \delta(p, w) \in F \Leftrightarrow \delta(q, w) \in F \). We say that \( w \) distinguishes between the words \( x \) and \( y \), in the
Given \( x, y \in \Sigma^* \), the language that distinguishes \( x \) from \( y \) w.r.t. \( L \) is
\[
D_L(x, y) = \{ w \mid xw \in L \nleftrightarrow yw \in L \}.
\]
(1)

Naturally, we define the distinguishability language of \( L \) by
\[
D(L) = \{ w \mid \exists x, y \in \Sigma^* \ (xw \in L \text{ and } yw \notin L) \}.
\]
(2)

It is immediate that
\[
D = \bigcup_{x,y \in \Sigma^*} D_L(x, y).
\]

In the same way, for the DFA \( A \), we define \( D_L(p, q) \) for \( p, q \in Q \), and
\[
D(A) = \{ w \mid \exists p, q \in Q \ (\delta(p, w) \in F \text{ and } \delta(q, w) \notin F) \}.
\]
(3)

**Lemma 3.1** Let \( A_1, A_2 \) be two reduced DFAs such that \( L(A_1) = L(A_2) = L \). Then \( D(A_1) = D(A_2) = D(L) \).

**Proof.** Let \( A_1 = (Q, \Sigma, q_0, \delta, F) \) and \( L = L(A_1) = L(A_2) \). It is enough to prove that \( D(L) = D(A_1) \). If \( w \in D(L) \), then we have two words \( x, y \in \Sigma^* \) such that \( xw \in L \) and \( yw \notin L \).

Let \( p = \delta(q_0, x) \) and \( q = \delta(q_0, y) \). Then, \( \delta(q_0, xw) \in F \) and \( \delta(q_0, yw) \notin F \), so \( w \in D(A_1) \). If \( w \in D(A_1) \), then there exist \( p, q \in Q \) such that \( \delta(p, w) \in F \) and \( \delta(q, w) \notin F \); as \( A_1 \) is reduced, there must exist \( x, y \) such that \( \delta(q_0, x) = p \) and \( \delta(q_0, y) = q \), therefore \( \delta(q_0, xw) = \delta(p, w) \in F \) and \( \delta(q_0, yw) = \delta(q, w) \notin F \), hence, \( xw \in L \) and \( yw \notin L \), i.e., we conclude \( w \in D(L) \).

This shows that the operator \( D \) is independent of the automata we choose to represent the language.

In what follows, we present some characterization results for the distinguishability operation, and show that iterating the operation leads always to a fixed point.

The distinguishability operation can be expressed directly by means of the language quotients, as it can be seen in the following result.

**Theorem 3.2** Let \( L \) be a regular language with \( \{ w^{-1}L \mid w \in \Sigma^* \} \) its set of quotients. Then we have the equality
\[
D(L) = \bigcup_{x \in \Sigma^*} x^{-1}L \setminus \bigcap_{x \in \Sigma^*} x^{-1}L.
\]

**Proof.** Let \( w \in D(L) \), i.e., \( \exists x, y \in \Sigma^* \), \( xw \in L \) \( \wedge yw \notin L \). Then \( w \in x^{-1}L \wedge w \notin y^{-1}L \), therefore \( D(L) \subseteq \bigcup_{x \in \Sigma^*} x^{-1}L \setminus \bigcap_{x \in \Sigma^*} x^{-1}L \).

Let \( w \in \bigcup_{x \in \Sigma^*} x^{-1}L \setminus \bigcap_{x \in \Sigma^*} x^{-1}L \), then let \( x \) and \( y \) be such that \( w \in x^{-1}L \) and \( w \notin y^{-1}L \). Thus, \( w \in D(L) \). Hence, we conclude
\[
D(L) = \bigcup_{x \in \Sigma^*} x^{-1}L \setminus \bigcap_{x \in \Sigma^*} x^{-1}L.
\]
Corollary 3.3 For a regular language \( L \) and \( x, y \in \Sigma^* \), \( D_L(x, y) \) is the symmetric difference of the correspondent quotients, \( D_L(x, y) = (x^{-1}L) \Delta (y^{-1}L) \).

To help the reader better understand how these languages look like, we present some examples.

Example 3.4 If the language \( L \) has only one quotient, i.e., \( L = \emptyset \) or \( L = \Sigma^* \), then \( D(L) = \emptyset \), as there are no different quotients to distinguish.

Example 3.5 If \( D(L) = \{ \varepsilon \} \), then we can only distinguish between final and non-final states, thus the minimal DFA of \( L \) has exactly two states corresponding to its two quotients.

Example 3.6 We consider a family of languages \( L_n \) for which \( D(L_n) = \Sigma^* \). Let \( L_n = \mathcal{L}(A_n) \) for \( 3 \leq n \), with \( A_n = \langle Q_n, \{0, 1\}, \delta_n, 0, \{0\} \rangle \), where \( Q_n = \{0, \ldots, n-1\} \), \( \delta_n(i, 0) = i + 1 \) mod \( n - 1 \), for \( 0 \leq i \leq n - 1 \) and \( \delta_n(1, 1) = 0 \), \( \delta_n(0, 1) = 1 \), \( \delta_n(i, 1) = i \), for \( 2 \leq i \leq n - 1 \). In Figure 1, we present \( A_5 \). Both symbols of the alphabet induce permutations on \( Q_n \): 1 induces a transposition (2-cycle), and 0, an \( n \) cyclic permutation. It follows that, the transition semigroup of \( A_n \) is the symmetric group \( S_n \) of degree \( n \), i.e., the set of all permutations of \( Q_n \). We always have \( \varepsilon \in D(L) \), and every \( w \in \Sigma^+ \) induces a permutation on the states, for \( 0 \leq i, j \leq n - 1 \), \( \delta(i, w) = i_w \) and \( \delta(j, w) = j_w \), with \( i_w \neq j_w \). Then, there must exist at least a pair \((i, j)\), such that \( w \in R_i \) and \( w \notin R_j \), i.e., \( w \in D(L_n) \), thus \( D(L_n) = \Sigma^* \). Bell et al. [1] studied those families of automata, and in particular, proved that they are uniformly minimal, i.e., minimal for every non-trivial choice of final states [14].

Example 3.7 Consider the automaton \( A \) in Figure 2. We have that \( \mathcal{L}(A) \neq D(\mathcal{L}(A)) \), but \( D(D(\mathcal{L}(A))) = D(\mathcal{L}(A)) \). The minimal automaton for \( D(\mathcal{L}(A)) \) is presented in Figure 3.

Example 3.8 Considering the language \( L = ((0 + 1)(0 + 1))^*(\varepsilon + 1) \), in Figure 4 one can find, from left to right, a DFA that accepts \( L \), one that accepts \( D(L) = (0 + 1^*10)^* \), and one for \( D^n(L) = \varepsilon \), for \( n \geq 2 \).

From the last example, we can see that the language \( D(L) \) contains the word 0110, but also the words 110, 10, and 0, which are all suffixes of 0110. This observation suggests that \( D(L) \) is suffix closed, which is proved in the following theorem.
Theorem 3.9 If $L$ is a regular language, then the language $D(L)$ is suffix closed, i.e.,

$$(\forall w \in D(L))(\forall x, y \in \Sigma^*)(w = xy \implies y \in D(L)).$$

Proof. Let $w \in D(L)$, i.e., there exist $x, y \in \Sigma^*$ such that $xw \in L$ and $yw \notin L$. If $v$ is a suffix of $w$, i.e., $w = uv$, for an $u \in \Sigma^*$, then we can write $xuv \in L$ and $yuv \notin L$, which means that $v \in D(L)$. \hfill \qed

Using Theorem 3.2, if $w \in D(L)$, then $w$ is a suffix of a word in $L$, and a suffix of the complement of $L$, because

$$\bigcup_{x \in \Sigma^*} x^{-1}L \setminus \bigcap_{x \in \Sigma^*} x^{-1}L = \bigcup_{x \in \Sigma^*} x^{-1}L \bigcap \bigcup_{x \in \Sigma^*} x^{-1}L.$$
Accordingly, $D(L) \subseteq \text{suff}(L) \cap \text{suff}(\overline{L})$, where $\text{suff}(L)$ is the language of all suffixes of $L$. If $w \in \text{suff}(L) \cap \text{suff}(\overline{L})$, then we can find $x$ and $y$ such that $xw \in L$ and $yw \in \overline{L}$, thus $w \in D(L)$. Therefore, we just found a new way to express the distinguishability language of $L$:

**Theorem 3.10** If $L$ is a regular language, then

$$D(L) = \text{suff}(L) \cap \text{suff}(\overline{L}).$$

Because $D(L)$ is suffix closed, $D(L) = \text{suff}(D(L))$, hence $\text{suff}(D(L)) \subseteq D(L) \subseteq \text{suff}(L)$ and $D^2(L) \subseteq D(L) \subseteq \text{suff}(L)$. In general, we have for every $n \geq 1$, the following inclusion

$$D^{n+1}(L) \subseteq D^n(L).$$

Consequently, we may ask if this hierarchy is infinite or not, in other words, we may ask if for any language $L$, there exists $n \geq 0$ such that $D^{n+1}(L) = D^n(L)$.

**Example 3.11** Consider the language $L = \mathcal{L}(A)$, where $A$ is given below, on the left. For the language $L$, we have that $L \neq D(L)$ and $D(L) \neq D^2(L) = D^n(L)$, for $n \geq 2$. The minimal automaton for $D^2(L)$ is depicted on the right. The minimal automaton for $D(L)$ has 7 states.

The following lemma will be useful for the rest of the section.

**Lemma 3.12** If $L, M \subseteq \Sigma^*$ are suffix-closed languages, then $\text{suff}(L) \cap \text{suff}(M) = \text{suff}(L \cap M)$, and $\text{suff}(L) \cup \text{suff}(M) = \text{suff}(L \cup M)$.

**Proof.** It is obvious that the equality holds for reunion, and the inclusion $\text{suff}(L \cap M) \subseteq \text{suff}(L) \cap \text{suff}(M)$, for intersection is true. If $w \in \text{suff}(L) \cap \text{suff}(M)$, then there exist $x, y \in \Sigma^*$ such that $xw \in L$ and $yw \in M$. Because $L$ and $M$ are suffix closed, then $w \in L \cap M \subseteq \text{suff}(L \cap M)$.

In the following result, we prove that the iteration of $D$ operations always reaches a fixed point.

**Theorem 3.13** Let $L \subseteq \Sigma^*$ be a regular language. Then we have that $D^3(L) = D^2(L)$. 

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**Proof.** It is obvious that the equality holds for reunion, and the inclusion $\text{suff}(L \cap M) \subseteq \text{suff}(L) \cap \text{suff}(M)$, for intersection is true. If $w \in \text{suff}(L) \cap \text{suff}(M)$, then there exist $x, y \in \Sigma^*$ such that $xw \in L$ and $yw \in M$. Because $L$ and $M$ are suffix closed, then $w \in L \cap M \subseteq \text{suff}(L \cap M)$.

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**Theorem 3.13** Let $L \subseteq \Sigma^*$ be a regular language. Then we have that $D^3(L) = D^2(L)$.
Proof. We have the following equalities:
\[ D^2(L) = D(D(L)) = \text{suff}(D(L)) \cap \text{suff}(\overline{D(L)}) = D(L) \cap \text{suff}(\overline{D(L)}). \quad (6) \]

Now, computing the next iteration of \( D \), we get \( D^3(L) = \text{suff}(D^2(L)) \cap \text{suff}(\overline{D^2(L)}) \).

Using (6) and Lemma 3.12 we obtain the equalities
\[ \text{suff}(D^2(L)) = \text{suff}(D(L) \cap \text{suff}(\overline{D(L)})) = \text{suff}(\overline{D(L)}) \cup \text{suff}(D(L)). \]

Because \( D^2(L) \) is a suffix-closed language, it follows that
\[
\begin{align*}
D^3(L) &= D^2(L) \cap (\text{suff}(\overline{D(L)}) \cup \text{suff}(\overline{D(L)})) \\
&= (D^2(L) \cap \text{suff}(\overline{D(L)})) \cup (D^2(L) \cap \text{suff}(\overline{D(L)})) \\
&= D^2(L) \cup (D^2(L) \cap \text{suff}(\overline{D(L)})) = D^2(L).
\end{align*}
\]

The following results give some characterization for the languages that are fixed points for \( D \).

Theorem 3.14 If \( L \) is a suffix-closed regular language with \( \emptyset \) as one of the quotients, then \( L \) is a fixed point for \( D \), i.e., \( D(L) = L \).

Proof. We know that \( D(L) = \text{suff}(L) \cap \text{suff}(\overline{L}) \). Because \( z^{-1}L = \emptyset \), for some word \( z \), we have that \( \Sigma^* = \text{suff}(\overline{L}) \). Thus, \( D(L) = \text{suff}(L) \cap \Sigma^* = \text{suff}(L) = L \). \( \square \)

Corollary 3.15 Let \( L \) be a regular language. If \( D(L) \) has \( \emptyset \) as a quotient, then \( D^2(L) = D(L) \).

Note that suffix-closeness of \( L \) is not sufficient to ensure that \( L \) has \( \emptyset \) as quotient. For that, it is enough to consider the language given by \( 0^* + 0^*1(1 + 00^*1)^* \). However, if \( L \) is a \( D \) fixed point, the implication yields.

Theorem 3.16 Let \( L \) be a regular language. If \( D(L) = L \), then \( L \) has \( \emptyset \) as a quotient.

Proof. Let \( L \) be a regular language that is fixed point for \( D \), thus \( L \) is suffix closed and
\[ (\forall w \in L)(\exists u \in \Sigma^*)(uw \notin L). \quad (7) \]

By absurd, suppose that \( L \) does not have \( \emptyset \) as quotient, i.e.,
\[ (\forall w \in \Sigma^*)(\exists v \in \Sigma^*)(wv \in L). \quad (8) \]

Let \( w \notin L \) (\( \Sigma^* \) is not a fixed point for \( D \)). Thus by (8) there exists a \( v_0 \in \Sigma^* \) such that \( wv_0 \in L \) and because \( L \) is suffix closed we know that \( v_0 \in L \). Using (7) there exists \( u_0 \) such that \( u_0wv_0 \notin L \). Using the same reasoning we can find \( u_1 \in \Sigma^* \) and \( v_1 \in L \) such that
\[ u_0wv_0v_1 \in L \text{ and } u_0u_0wv_0v_1 \notin L. \]
The word $wv_0v_1$ distinguishes $u_0$ from $u_1u_0$, thus these words cannot belong to the same quotient.

Suppose that we have iterated $n$ times this process having $u_n \cdots u_0wv_0 \cdots v_n \in L$ and $u_n \cdots u_0wv_0 \cdots v_n \notin L$, with all $u_i \cdots u_0$ belonging to distinct quotients. We can apply this process one more time, obtaining

$$u_n \cdots u_0wv_0 \cdots v_{n+1} \in L \text{ and } u_{n+1} \cdots u_0wv_0 \cdots v_{n+1} \notin L.$$ 

It is easy to see that the word $wv_0 \cdots v_{n+1}$ distinguishes $u_{n+1} \cdots u_0$ from any of the previous words $u_i \cdots u_0$ (with $i \leq n$) because $u_i \cdots u_0wv_0 \cdots v_{n+1}$ is a suffix of $u_n \cdots u_0wv_0 \cdots v_{n+1} \in L$. Thus, the number of $L$ quotients cannot be finite, a contradiction.

By contraposition over the last result, we get that a language $L$ with all quotients non-empty cannot be a fixed point for $D$. Using Theorem 3.14, we know that, if a language $L$ is such that $\Sigma^* = \text{suff}(L)$, then $\emptyset$ must be one of the quotients of $L$. In Examples 3.6–3.8 and Example 3.11, we have languages $L$ with all quotients non-empty and $L \neq D(L)$.

For a finite language $L$, the following corollary shows that the distinguishability language of $L$ coincides with the set of all suffixes of $L$, therefore, $L$ is a fixed point of the $D$ operator.

**Corollary 3.17** If $L$ is a finite language, then $D(L) = \text{suff}(L)$. 

*Proof.* We only have to notice that $\text{suff}(L) = \Sigma^*$, thus $\text{suff}(L) \subseteq \text{suff}(L)$. 

The minimal DFA that represents the set of suffixes of a finite language $L$ is called the *suffix automaton*, and several optimized algorithms for its construction were studied in the literature. Thus, we can use an algorithm for building the suffix automaton in order to obtain $D(L)$. Recently, Mohri et al. [13] gave new upper bounds on the number of states of the suffix automaton as a function of the size of the minimal DFA of $L$, as well as other measures of $L$. In Section 4, we study the state complexity of $D(L)$ as a function of the state complexity of $L$, for any general regular language $L$. In Section 6, we generalize the results on the characterization of the distinguishability operation, hence, we can define more operations on regular languages.

## 4 State Complexity

By Theorem 3.10, we know that $D(L)$ can be obtained using the suffix operator, complement and intersection, therefore, it is a result of combining three operations, two unary and one binary. We would like to estimate the state complexity of the $D$ operation and check if the upper bound is tight. The following theorem shows the construction for $D(L)$, in case $L$ is recognized by a DFA.

**Theorem 4.1** Let $A = (Q, \Sigma, \delta, i, F)$ be a reduced DFA recognizing a language $L$. Then $A_d = (Q_d, \Sigma, \delta_d, Q, F_d)$ is a DFA that accepts $D(L)$, where

- $Q_d \subseteq 2^Q$, 

• for \( a \in \Sigma \) and \( S \subseteq Q \) and \( |S| > 1 \), \( \delta_d(S, a) = \{ \delta(q, a) \mid q \in S \} \),
• \( F_d = \{ S \mid S \cap F \neq \emptyset \text{ and } S \cap (Q \setminus F) \neq \emptyset \} \).

Proof. Considering that \( D(L) = \text{suff}(L) \cap \text{suff}(L) \), we can use the usual subset construction for \( \text{suff}(L) \) to build an NFA with the same transition function as \( A \), and all its states being initial. For \( \text{suff}(L) \), the corresponding NFA will be the same, but flipping the finality to all the states. Because both operands share the same structure, the DFA corresponding to the intersection will be the DFA resulting from the subset construction considering a suitable set of final states (they must contain at least one final state and a non-final one). As all states \( S \subseteq 2^Q \) with \( |S| = 1 \) are either final or non-final, they cannot be useful, therefore they can be ignored. \( \square \)

Let \( A = (Q, \Sigma, \delta, i, F) \) be the minimal DFA recognizing \( L \) with \( |Q| = n \). Let \( Q = \{0, \ldots, n\} \) and \( R_i, 0 \leq i \leq n \), be the left-quotients of \( L \) (possibly including the empty set). From Theorem 3.2 we have:

\[
D(L) = \bigcup_{i \in Q} R_i \setminus \bigcap_{i \in Q} \left( \bigcup_{i \in Q} R_i \right) = \bigcap_{i \in Q} \left( \bigcup_{i \in Q} R_i \right) \cup \left( \bigcap_{i \in Q} R_i \right).
\]

In the following we identify the states of \( A \) with the corresponding left-quotients. Instead of using traditional techniques to prove the correctness of tight upper bounds of operational state complexity, here we consider a method based on the atoms of regular expressions. Using this approach, we aim to provide yet another piece of evidence for their broad applicability.

Brzozowski and Tamm introduced the notion of atoms of regular languages in \([5]\) and studied their state complexity in \([9]\). An atom of a regular language \( L \) with \( n \) quotients \( R_0, \ldots, R_{n-1} \) is a non-empty intersection \( K_0 \cap \cdots \cap K_{n-1} \), where each \( K_i \) is a quotient \( R_i \), or its complement \( \overline{R_i} \). In particular, \( A_Q = \bigcap_{i \in Q} R_i \) (\( A_Q = \bigcap_{i \in Q} \overline{R_i} \) is an atom with zero complemented (uncomplemented) quotients. In \([8]\) it was proved that the state complexity of both those atoms is \( 2^n - 1 \). Using similar arguments, we prove the following theorem.

**Theorem 4.2** If \( A \) is a minimal DFA with \( n \) states recognizing \( L \), then \( \text{sc}(D(L)) \leq 2^n - n \).

Proof. Let \( A = (Q, \Sigma, \delta, i, F) \) be the minimal DFA recognizing \( L \) with \( |Q| = n \). Then \( Q = \{0, \ldots, n\} \), and let \( R_i, 0 \leq i \leq n - 1 \) be the (left-)quotients of \( L \). Using Equation (9), every quotient \( w^{-1}D(L) \) of \( D(L) \), for \( w \in \Sigma^* \), is given by:

\[
w^{-1}D(L) = \left( \bigcup_{i \in Q} w^{-1}R_i \right) \cap \left( \bigcap_{i \in Q} w^{-1}R_i \right),
\]

where all \( w^{-1}R_i, 0 \leq i \leq n - 1 \), are also quotients of \( L \), and they may not be distinct. Considering all non-empty subsets of quotients of \( L \), there would be at most \( 2^n \) quotients of \( D(L) \). However, all subsets, \( R_j \), with exactly one element will lead to the empty quotient. Thus, \( \text{sc}(D(L)) \leq 2^n - n \). \( \square \)

Brzozowski \([3]\) presented a family of languages \( U_n \) which provides witnesses for the state complexity of several individual and combined operations over regular languages. Brzozowski and Tamm \([6]\) proved that \( U_n \) was also a witness for the worst-case state complexity of atoms. This family is defined as follows. For each \( n \geq 2 \), we construct the DFAs
\[ D_n = (\{0, \ldots, n-1\}, \{0, 1, 2\}, \delta, 0, \{n-1\}) \], where \( \delta(i, 0) = i + 1 \mod n, \delta(0, 1) = 1, \delta(1, 1) = 0, \delta(i, 1) = i \) for \( i > 1 \), \( \delta(i, 2) = i \) for \( 0 \leq i \leq n-2 \), and \( \delta(n-1, 2) = 0 \). We denote by \( U_n \) the language accepted by \( D_n \), i.e.,

\[ U_n = \mathcal{L}(D_n). \] (10)

We show that \( U_n \) is also a witness for the lower-bound of the state complexity of \( D(L) \). First, observe that automata \( D_n, n \geq 2 \) are minimal. Next, we give the lower bound for the number of states of a DFA accepting \( D(U_n) \).

**Theorem 4.3** For \( n \geq 2 \), the minimal DFA accepting \( D(U_n) \) has \( 2^n - n \) states.

**Proof.** Let \( A_n = (R_0 \cap \ldots \cap R_n) \), and \( A_\emptyset = (R_0 \cap \ldots \cap R_n) \), be the two atoms of \( U_n \) as above, where \( R_i \) are its quotients \( 0 \leq i \leq n-1 \). Then \( D(U_n) = A_n \cup A_\emptyset \). Brzozowski and Tamm proved that \( sc(A_n) = sc(A_\emptyset) = 2^n - 1 \). Applying the construction given in Theorem 4.1 to \( D(U_n) \), and noting that a regular language and its complement have the same state complexity, we obtain the upper bound.

If \( sc(L) = 1 \), by Example 3.4 we have that \( sc(D(L)) = 1 \).

Having considered some properties of the distinguishability language, we would like to select only the set of minimal words that distinguishes between distinct quotients. Obviously, this is a subset of \( D(L) \), and in the following section we study its properties.

## 5 Minimal Distinguishable Words

An even more succinct language distinguishing all different quotients of a regular language, in fact a finite one, can be obtained if we consider only the shortest word that distinguishes each pair of quotients.

**Definition 5.1** Let \( L \) be a regular language, and assume we have an order over the alphabet \( \Sigma \). If \( x, y \in \Sigma^* \) and \( x \not\equiv_L y \), we define

\[ D_L(x, y) = \min \{ w \mid w \in D_L(x, y) \}, \]

where minimum is considered with respect to the quasi-lexicographical order. In case \( x \equiv_L y \), \( D_L(x, y) \) is undefined. The set of minimal words distinguishing quotients of a language \( L \) is

\[ D(L) = \{ D_L(x, y) \mid x, y \in \Sigma^*, x \not\equiv_L y \}. \]

We can observe that \( D_L(x, y) = \min(x^{-1}L \Delta y^{-1}L) \), where \( \Delta \) is the symmetric difference of two sets.

**Example 5.2** We present a few simple cases. Similar to the \( D \) operator, we have the equalities: \( D(\Sigma^*) = D(\emptyset) = D(\{\varepsilon\}) = \emptyset \). In case \( a \in \Sigma \), \( D(a) = D(\{a\}) = \{a, \varepsilon\} \), and \( D(\{a^n\}) = D(\{a^n\}) = \{a^i \mid 0 \leq i \leq n\} \).
Example 5.3 Consider the language $L$ of Example 3.11. We have the following equalities $D(L) = \{\varepsilon, 0, 1, 01, 11\}$, $D(D(L)) = \{\varepsilon, 1, 01, 11\}$, and $D(D^2(L)) = \{\varepsilon, 1, 11\}$.

The previous example suggests that $D(L)$ is also suffix closed.

**Theorem 5.4** If $L$ is a regular language, then $D(L)$ is suffix closed.

*Proof.* Let $w \in D(L)$, and let $w = uv$, with $u, v \in \Sigma^*$. Because $w \in D(L)$, we can find two other words, $x, y \in \Sigma^*$, such that $xw \in L$ and $yw \notin L$, i.e., $xuv \in L$ and $yuv \notin L$. It follows that $v \in D_L(xu, yu)$. Since $v \in D_L(xu, yu)$, there exists $v' \in D_L(xu, yu)$ and $v' \leq v$. Hence, $uv' \leq uw$ and $wv' \in D_L(x, y)$, which implies that $w = uv \leq uv'$. Then we must have $uv' = uv$, which implies that $v = v' \in D_L(xu, yu) \subseteq D(L)$.

The next result gives an upper-bound for the number of elements of $D(L)$.

**Theorem 5.5** If $L$ is a regular language with state complexity $n \geq 2$, then $|D(L)| \leq n - 1$.

*Proof.* For any three sets $A, B$ and $C$ we have the equality $(A \Delta B) \Delta (B \Delta C) = A \Delta C$. Therefore, we can distinguish any pair from $n$ distinct sets with at most $n - 1$ elements. To prove the theorem it is enough to choose the minimal words satisfying the above conditions, since the $n$ quotients of $L$ are all distinct (their symmetric difference is non-empty).

Now, we prove that the upper-bound is reached.

**Theorem 5.6** The bound $n - 1$ for the minimum size of $D(L)$, for a regular language $L$ with state complexity $n \geq 2$, is tight.

*Proof.* Consider again the family of languages $U_n$, described by Equation (10). For each state $0 \leq i \leq n - 1$ of $D_n$, let $R_i$ be the corresponding quotient. It is easy to see that the minimal words for each quotient $R_i$ are $0^n - i - 1$, and we can disregard the largest one.

We now consider the iteration of the $D$ operator. Because $D(L) \subseteq D(L), D(L) \subseteq \text{suff}(L)$, and $D(L)$ is suffix closed, it follows that $D^2(L) \subseteq D(L)$, and, in general,

$$D^{n+1}(L) \subseteq D^n(L), \text{ for all } n \geq 1. \tag{11}$$

By the finiteness of $D(L)$, it follows that there exists $n \geq 0$ such that $D^{n+1}(L) = D^n(L)$. For instance, considering the family of languages $U_n$ defined by equation (10), we have that $D^2(U_n) = D(U_n)$.

Contrary to the hierarchy for $D(L)$, where the fixed point is reached for $n = 2$, in the case of $D(L)$ we have that for any $n \geq 0$, there is a language for which the fixed point is reached after $n$ iterations of $D$.

**Theorem 5.7** Given a regular language $L$ with state complexity $n$, the fixed point of $D^n(L)$, is reached for $0 \leq i \leq n - 2$. 
THE DISTINGUISHABILITY OPERATION ON REGULAR LANGUAGES

Proof. Because $D(L)$ is suffix closed, $\varepsilon \in D^i(L)$ for all every $i \geq 1$, thus any automaton recognizing $D^i(L)$ has at least 2 states. By Theorem 5.5 $|D(L)| \leq n - 1$. Using Equation (11) we either have the same set, or a smaller set, thus we may lose at least one element at each iteration. Hence, $i \leq n - 2$. 

If in the previous theorem we have established an upper-bound for the number of iterations of the $D$ operator necessary to reach a fixed point, in the next one we show that the upper-bound can be reached.

Theorem 5.8 For all $n \geq 3$, there exists a regular language $L_n$, with $sc(L_n) = n$, such that

1. $D^{m-1}(L_n) \neq D^m(L_n)$, for all $m < n - 2$, and
2. $D^{n-2}(L_n) = D^{n-1}(L_n)$.

Proof. Consider the family of languages $W_m = \text{suff}(0^m1) = \{0^i1 \mid 0 \leq i \leq m\} \cup \{\varepsilon\}$, $m \geq 0$. Then $sc(W_m) = m+3$ and $D(W_m) = \{0^i1 \mid 0 \leq i \leq m - 1\} \cup \{\varepsilon\} = W_{m-1}$. Because $W_0 = \{1, \varepsilon\}$ is a fixed point for $D$, it follows that

1. $D^{m}(L_{m-3}) \neq D^{m-1}(L_{m-3})$, for all $m < n - 2$, and
2. $D^{n-2}(L_{m-3}) = D^{n-1}(L_{m-3})$.

Hence, we can just take $L_n = W_{n-3}$. 

In the next section we check under what conditions the results obtained so far can be generalized.

6 Boolean Operations and Closure

In Section 3 we used Boolean operations and the suffix operation to compute the distinguishability language. The suffix operation has the following properties:

1. $\text{suff}(\emptyset) = \emptyset$;
2. $L \subseteq \text{suff}(L)$;
3. $\text{suff}(\text{suff}(L)) = \text{suff}(L)$;
4. $\text{suff}(L_1 \cup L_2) = \text{suff}(L_1) \cup \text{suff}(L_2)$,

thus, it is a closure operator. If we consider distinguishability operation as a unary operation on regular languages, we can see that it is obtained by applying finitely many times a closure operator and Boolean operations. The Closure-Complement Kuratowski Theorem [12] says that using one set, one can obtain at most 14 distinct sets using finitely many times one closure operator and the complement operation. For the case of regular languages, Brzozowski et al. [4] determine the number of languages that can be obtained by applying finitely many times the Kleene closure and complement. However, the corresponding property [4] is not satisfied by the Kleene closure.

In this section we analyze the case of closure operators and Boolean operations, and ask if we apply them finitely many times we can still obtain only finitely many sets, or what is a necessary condition to obtain only finitely many sets.
In general, not all closure operators commute, for example, let \( A \) be elements in the set \( N \). Applying the closure operators repeatedly to a set and we first apply the closure operator to the set and its successors of elements in the set \( N \) adds all the odd numbers that are successors of \( 1 \).

**Lemma 6.1** Let \( A_1, \ldots, A_m \) be subsets in a super set \( M \). Then the free algebra \( (M, \cup, \cap, \cdot, \emptyset, M) \) has a finite number of elements.

**Proof.** All expressions can be reduced to the disjunctive normal form, and we only have finitely many such formulae.

**Lemma 6.2** Let \( M \) be a non-empty set and \( c_1, c_2 : 2^M \to 2^M \), two closure operators such that \( c_1 \) and \( c_2 \) commute, i.e., \( c_1 \circ c_2 = c_2 \circ c_1 \). Then \( c = c_2 \circ c_1 \) is also a closure operator.

**Proof.** Let us verify the properties of a closure operator, thus if \( L, L_1, L_2 \in 2^M \), we have:

1. \( c(\emptyset) = (c_2 \circ c_1)(\emptyset) = c_2(c_1(\emptyset)) = \emptyset \);
2. \( L \subseteq c_2(L) \subseteq c_1(c_2(L)) = (c_2 \circ c_1)(L) = c(L) \);
3. \( c(c(L)) = (c_2 \circ c_1)((c_2 \circ c_1)(L)) = (c_2 \circ c_1)((c_1 \circ c_2)(L)) = (c_2 \circ c_2 \circ c_1)(L) = (c_2 \circ c_1)(L) = c(L) \);
4. \( c(L_1 \cup L_2) = (c_2 \circ c_1)(L_1 \cup L_2) = c_2(c_1(L_1 \cup L_2)) = c_2(c_1(L_1) \cup c_1(L_2)) = c_2(c_1(L_1)) \cup c_2(c_1(L_2)) = c(L_1) \cup c(L_2) \).

In general, not all closure operators commute, for example, \( N_0, N_1 : 2^N \to 2^N \) defined by \( N_0(A) = A \cup \{ x \in N \mid \exists k > 0, x = 2k \} \), \( N_1(A) = A \cup \{ x \in N \mid \exists k \geq 0, x = 2k+1 \) if \( 2k \in A \} \), do not commute one with each other, as \( N_0 \) adds all the even numbers that are successors of elements in the set \( A \), and \( N_1 \) adds all the odd numbers that are successors of elements in the set \( A \). Applying the closure operators alternatively to a finite set \( A \), we always obtain a new set.

The next result is well known for the behaviour of closure operators when applied to a intersection of two other sets.

**Lemma 6.3** Let \( c \) be a closure operator on \( 2^M \). If \( L_1, L_2 \subseteq 2^M \) are closed subsets, then \( c(L_1) \cap c(L_2) = c(L_1 \cap L_2) \).

Assume we have a finite number of sets \( L_1, \ldots, L_m \). Using closure and complement for each set \( L_i \), \( 1 \leq i \leq m \), we can obtain a finite number of sets \([12] \), say \( M_1, \ldots, M_l \). Now consider a Boolean expression using \( M_1, \ldots, M_l \). Because we can transform all these Boolean expressions in disjunctive normal form, the number of Boolean expressions over \( M_1, \ldots, M_l \) is finite. Applying the closure operator to such an expression will commute with union, and the other sets are in the form \( c(M_1) \cap \cdots \cap M_k) \). If all \( M_{ij} \) \( 1 \leq j \leq k \) are closed sets, then \( c(M_1) \cap \cdots \cap M_k) \) is a conjunction of some other sets \( M_{j_1}, \ldots, M_{j_l}, 1 \leq j_i \leq l \). Otherwise, if a set \( M_{ij} \) \( 1 \leq j \leq k \) is not closed, we may obtain new sets, as we can see from the following example: if \( L_1 = \{aaaa, abaabbaa, b\} \), \( L_2 = \{bbb, baaab, aa\} \), where \( c = \text{ suff}(\cdot) \), then \( c(L_1 \cap c(L_2)) \cap L_2 \neq c(L_1) \cap L_2 \).

This suggests that if a unary operation that combines Boolean operations and closure operators is repeatedly applied to a set and we first apply the closure operator to the set and its
complement, then we use other Boolean operations or the closure operator finitely many times, we will always obtain finitely many sets. It follows that we have just proved the following lemma:

**Lemma 6.4** If $\circ : 2^M \rightarrow 2^M$ is defined as a reunion over intersections over $c(A)$ and $c(A)$, then

1. any iteration of $\circ$ will produce a finite number of sets;
2. if $A^c \subset A$, for all $A$, then $\circ$ has a fixed point.

Of course, if we have more than one closure operator, and we want to obtain finitely many sets, we must first apply one closure operator to the collection of sets and their complements, then all the other Boolean operators and closure operator again. In this way, we have guaranteed that we can only obtain finitely many sets. In case the operation defined this way is monotone and bounded, it will have a fixed point.

A natural extension of $D$ is to consider prefix operator and infix operator, thus, $E(L) = \text{pref}(L) \cap \text{pref}(L^c)$, or $F(L) = \text{infix}(L) \cap \text{infix}(L^c)$. Because both are closure operators, they will have the exact same properties as the $D$ operator. However, the $D$ operator is not a closure operator, because $L \not\subseteq D(L)$, thus the constructions studied in this section will not apply to either $D$ or to other derived operations like:

- $E(L) = \{E_L(x, y) \mid x \not\sim_L y\},$ where $E_L(x, y) = \min \{w \mid w \in E_L(x, y)\}$, or
- $F(L) = \{F_L((x, y)) \mid x \not\sim_L y\},$ where $F_L((x, y)) = \min \{w \mid w \in F_L((x, y))\}.$

Here minimum is considered with respect to the quasi-lexicographical order, and for the newly introduced symbols we have $x \sim_L y$ iff $(\forall u)u \in \Sigma^*, ux \in L \Leftrightarrow uy \in L$ and $x \not\sim_L y$ iff $(\forall u)(\forall v)u, v \in \Sigma^*, uxv \in L \Leftrightarrow uyv \in L$.

### 7 Conclusion

In this paper we have introduced two new operations on regular languages that help us distinguish non-equivalent words under Myhill-Nerode equivalence. The first one $D$ finds all these words and the second one $D$ produces only the minimal ones, where minimum is considered with respect to the quasi-lexicographical order. Both have fixed points under iteration. The number of iterations until a fixed point is reached is bounded by 2 for the case of $D$, and it is bounded by the state complexity of the starting language for $D$. A full characterization of the fixed points of $D$ is provided. Brzozowski’s universal witness $U_n$ reaches the upper-bound of $2^n - n$, for the state complexity of $D$. In the case of $D$ operation, the maximum number of words in the language is $n - 1$, where $n$ is the state complexity of the original language. We have generalized some results for these type of operations with arbitrary closures and Boolean operations. As open problems and future work we can consider the state complexity of combined $D$ with other language operations, in particular with reversal, because, in this last case the semantics of these operations is interesting.
References

[1] J. BELL, J. A. BRZOZOWSKI, N. MOREIRA, R. REIS, Symmetric Groups and Quotient Complexity of Boolean Operations. In: J. ESPARZA, P. FRAIGNIAUD, T. HUSFELDT, E. KOUTSOUPIAS (eds.), Proc. 41st ICALP. LNCS 8573, Springer, 2014, 1–12.

[2] A. BLUMER, J. BLUMER, D. HAUSSLER, A. EHRENFEUCHT, M. T. CHEN, J. I. SEIFERAS, The Smallest Automaton Recognizing the Subwords of a Text. Theor. Comput. Sci. 40 (1985), 31–55.

[3] J. A. BRZOZOWSKI, In Search of Most Complex Regular Languages. Int. J. Found. Comput. Sci. 24 (2013) 6, 691–708.

[4] J. A. BRZOZOWSKI, E. GRANT, J. SHALLIT, Closures in Formal Languages and Kuratowski’s Theorem. Int. J. Found. Comput. Sci. 22 (2011) 2, 301–321.

[5] J. A. BRZOZOWSKI, H. TAMM, Theory of ´Atomata. In: G. MAURI, A. LEPORATI (eds.), Proc. 15th DLT. LNCS 6795, Springer, 2011, 105–116.

[6] J. A. BRZOZOWSKI, H. TAMM, Complexity of Atoms of Regular Languages. Int. J. Found. Comput. Sci. 24 (2013) 7, 1009–1028.

[7] C. CĂMPEANU, Cover Languages and Implementations. In: S. KONSTANTINIDIS (ed.), Proc. 18th CIAA. LNCS 7982, Springer, 2013, 1.

[8] C. CĂMPEANU, A. PĂUN, Counting the Number of Minimal DFCA Obtained by Merging States. Int. J. Found. Comput. Sci. 14 (2003) 6, 995–1006.

[9] E. D. DEMAINE, S. EISENSTAT, J. SHALLIT, D. A. WILSON, Remarks on Separating Words. In: M. HOLZER, M. KUTRIB, G. PIGHIZZINI (eds.), Proc. 13th DCFS. LNCS 6808, Springer, 2011, 147–157.

[10] S. GINSBURG, On the Length of the Smallest Uniform Experiment which Distinguishes the Terminal States of a Machine. J. ACM 5 (1958) 3, 266–280.

[11] S. GINSBURG, E. SPANIER, Distinguishability of a Semi-group by a Machine. Proceedings of the American Mathematical Society 12 (1961) 4, 661–668.

[12] C. KURATOWSKI, Sur l’opération $\overline{A}$ de l’Analysis Situs. Fund. Math. 3 (1922), 182–199.

[13] M. MOHRI, P. MORENO, E. WEINSTEIN, General Suffix Automaton Construction Algorithm and Space Bounds. Theor. Comput. Sci. 410 (2009) 37, 3553–3562.

[14] A. RESTIVO, R. VAGLICA, A Graph Theoretic Approach to Automata Minimality. Theor. Comput. Sci. 429 (2012), 282–291.

[15] J. SAKAROVITCH, Elements of Automata Theory. Cambridge University Press, 2009.

[16] J. M. SEMPERE, Learning Reversible Languages with Terminal Distinguishability, Springer, 2006, 354–355.

[17] S. YU, Regular languages. In: G. ROZENBERG, A. SALOMAA (eds.), Handbook of Formal Languages. 1, Springer, 1997, 41–110.