T-product Tensor Expander Chernoff Bound

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Abstract

In probability theory, the Chernoff bound gives exponentially decreasing bounds on tail distributions for sums of independent random variables and such bound is applied at different fields in science and engineering. In this work, we generalize the conventional Chernoff bound from the summation of independent random variables to the summation of dependent random T-product tensors. Our main tool used at this work is majorization technique. We first apply majorization method to establish norm inequalities for T-product tensors and these norm inequalities are used to derive T-product tensor expander Chernoff bound. Compared with the matrix expander Chernoff bound obtained by Garg et al., the T-product tensor expander Chernoff bound proved at this work contributes following aspects: (1) the random objects dimensions are increased from matrices (two-dimensional data array) to T-product tensors (three-dimensional data array); (2) this bound generalizes the identity map of the random objects summation to any polynomial function of the random objects summation; (3) Ky Fan norm, instead only the maximum or the minimum eigenvalues, for the function of the random T-product tensors summation is considered; (4) we remove the restriction about the summation of all mapped random objects is zero, which is required in the matrix expander Chernoff bound derivation.

Index terms— Random Tensors, Tail Bound, Ky Fan Norm, Log-Majorization, T-product Tensor, Graph

1 Introduction

The Chernoff bound provides the exponential decreasing inequality on tail distribution of sums of independent random variables and such bound is used extensively at different fields of science and engineering. For instance, the Chernoff bound is used to estimate approximation error of statistical machine learning algorithms [1]. In communication networking system, the Chernoff bound is utilized to establish bounds for packet routing problems which are used to design congestion reduction routing protocol in sparse networks [2]. It is a tighter bound than the known first- or second-moment-based tail bounds such as Markov’s inequality or Chebyshev’s inequality, which only yield power-law bounds on tail distribution. However, neither Markov’s inequality nor Chebyshev’s inequality requires that the variates are independent, which is necessary by the Chernoff bound [3].

There are several directions to generalize the Chernoff bound. One major direction is to increase the dimension of random objects from random variables to random matrices. The works of Rudelson [4], Ahlswede-Winter [5] and Tropp [6] demonstrated that a similar concentration bound is also valid for matrix-valued random variables. If $X_1, X_2, \cdots, X_n$ are independent $m \times m$ Hermitian complex random matrices

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with \( \|X_i\| \leq 1 \) for \( 1 \leq i \leq n \), where \( \| \cdot \| \) is the spectral norm, we have following Chernoff bound for the version of \( n \) i.i.d. random matrices:

\[
\Pr\left( \left\| \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}[X] \right\| \geq \vartheta \right) \leq m \exp(-\Omega(n\vartheta^2)),
\]

(1)

where \( \Omega \) is a constant related to the matrix norm. This is also called “Matrix Chernoff Bound” and is applied to many fields, e.g., spectral graph theory, numerical linear algebra, machine learning and information theory \[7\]. Recently, the author generalized matrix bounds to various tensors bounds under Einstein product, e.g., Chernoff, Bennett, and Bernstein inequalities associated with tensors under Einstein product in \[8\].

Another direction to extend from the basic Chernoff bound is to consider non-independent assumptions for random variables. By Gillman \[9\] and its refinement works \[10, 11\], they changed the independence assumption to Markov dependence and we summarize their works as follows. We are given for random variables. By Gillman \[9\] and its refinement works \[10, 11\], they changed the independence assumption to Markov dependence and we summarize their works as follows. We are given \( \mathcal{G} \) as a regular \( \lambda \)-expander graph with vertex set \( \mathcal{V} \), and \( g : \mathcal{V} \to \mathbb{C} \) as a bounded function. Suppose \( v_1, v_2, \ldots, v_\kappa \) is a stationary random walk of length \( \kappa \) on \( \mathcal{G} \), it is shown that:

\[
\Pr\left( \left\| \frac{1}{\kappa} \sum_{j=1}^{\kappa} g(v_j) - \mathbb{E}[g] \right\| \geq \vartheta \right) \leq 2 \exp(-\Omega(1 - \lambda)\kappa\vartheta^2).
\]

(2)

The value of \( \lambda \) is also the second largest eigenvalue of the transition matrix of the underlying graph \( \mathcal{G} \). The bound given in Eq. (2) is named as “Expander Chernoff Bound”. It is natural to generalize Eq. (2) to “Matrix Expander Chernoff Bound”. Wigderson and Xiao in \[12\] began first attempt to obtain partial results of “Matrix Expander Chernoff Bound” and the complete solution is given later by Garg et al. \[13\].

Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) be a regular graph whose transition matrix has second largest eigenvalue as \( \lambda \), and let \( g : \mathcal{V} \to \mathbb{C}^{m \times m} \) be a function satisfy following:

1. For each \( v \in \mathcal{V} \), \( g(v) \) is a Hermitian matrix with \( \|g(v)\| \leq 1 \);

2. \( \sum_{v \in \mathcal{V}} g(v) = 0 \).

Then, for a stationary random walk \( v_1, \ldots, v_\kappa \) with \( \epsilon \in (0, 1) \), matrix expander Chernoff bound derived by Garg et al. \[13\] is expressed as:

\[
P\left( \lambda_{g,\text{max}} \left( \frac{1}{\kappa} \sum_{j=1}^{\kappa} g(v_j) \right) \geq \epsilon \right) \leq m \exp(-\Omega(1 - \lambda)\kappa\epsilon^2),
\]

\[
P\left( \lambda_{g,\text{min}} \left( \frac{1}{\kappa} \sum_{j=1}^{\kappa} g(v_j) \right) \leq -\epsilon \right) \leq m \exp(-\Omega(1 - \lambda)\kappa\epsilon^2),
\]

(3)

where \( \lambda_{g,\text{max}}, \lambda_{g,\text{min}} \) is the largest (smallest) eigenvalue of the summation of \( \kappa \) matrices obtained by the mapping \( g \).

The T-product operation between two three order tensors was invented by Kilmer and her collaborators in \[14, 15\] to generalize the traditional matrix product. T-product operation has been shown as an important linear algebra tool in many domains: multilinear algebra \[16-19\], numerical linear algebra \[20\], signal processing \[21, 22\], machine learning \[23\], image processing \[24\], computer vision \[25, 26\], low-rank tensor approximation \[27, 29\] etc. However, all these applications assume that systems modelled by T-product tensors are deterministic and such assumption is not true and practical in solving T-product tensors associated issues. In recent years, there are more works begin to study random tensors, see \[8, 30, 31\] and
references therein. Chernoff bound for independent sum of random T-product tensors are considered by the
author previous work [32]. In this work, we will remove the independent assumption of random T-product
tensors by considering T-product tensor expander Chernoff bound. We first transform our majorization
technique used in [33] for random tensors under Einstein product to random tensors under T-product by
deriving norm inequalities of T-product tensors, then we apply these inequalities to build our T-product ten-
sor expander Chernoff bound. Although the author also utilizes norm inequalities of T-product tensors in
his recent work [34], the application of these norm inequalities of T-product tensors is different at this work
by considering dependent, instead independent, T-product tensors summation. For self-contained presenta-
tions purposes, the portion about majorization techniques and norm inequalities of T-product tensors are also
presented in this work. The main result of this paper is summa-

**Theorem 1.1 (T-product Tensor Expander Chernoff Bound)** Let \( G = (\mathcal{V}, E) \) be a regular undirected
graph whose transition matrix has second eigenvalue \( \lambda \), and let \( g : \mathcal{V} \rightarrow \mathbb{R}^{m \times m \times p} \) be a function. We assume following:

1. A nonnegative coefficients polynomial raised by the power \( s \geq 1 \) as \( f : x \rightarrow (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)^s \) satisfying \( f \left( \exp \left( t \sum_{j=1}^{\kappa} g(v_j) \right) \right) \geq \exp \left( t f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right) \) almost surely;

2. For each \( v \in \mathcal{V} \), \( g(v) \) is a symmetric T-product tensor with \( f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \) as TPD T-product tensor;

3. \( \|g(v)\| \leq r \);

4. For \( \tau \in [\infty, \infty] \), we have constants \( C \) and \( \sigma \) such that \( \beta_0(\tau) \leq \frac{C}{\sigma \sqrt{2\pi}} \exp \left( -\frac{\tau^2}{2\sigma^2} \right) \).

Then, we have

\[
\Pr \left( \left\| f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right\|_{(k)} \geq \vartheta \right) \leq \min_{t>0} \left\{(n+1)^{(s-1)} e^{-\vartheta t} \left[ a_0 k + C \left( mp + \sqrt{\frac{mp - k}{k}} \right) \right] + \sum_{l=1}^{n} a_l \exp \left( 8\kappa \lambda + 2(\kappa + \delta \lambda)lsrt + 2(\sigma(\kappa + \delta \lambda)lsr)^2 t^2 \right) \right\},
\]

where \( \lambda = 1 - \lambda \).

The rest of this paper is organized as follows. In Section 2, we review T-product tensors basic con-
cepts and introduce a powerful scheme about antisymmetric Kronecker product for T-product tensors. In
Section 3, we apply a majorization technique to prove T-product tensor norm inequalities. Our main result
about the T-product tensor expander Chernoff bound is provided in Section 4. Finally, concluding remarks
are given by Section 5.

## 2 T-product Tensors

In this section, we will introduce fundamental facts about T-product tensors in Section 2.1. Several unitarily
invariant norms about a T-product tensor are defined in Section 2.2. A powerful scheme about antisymmetric
Kronecker product for T-product tensors will be provided by Section 2.3.
2.1 T-product Tensor Fundamental Facts

For a third order tensor $C \in \mathbb{R}^{m \times n \times p}$, we define bcirc operation to the tensor $C$ as:

$$\text{bcirc}(C) \overset{\text{def}}{=} \begin{bmatrix} C^{(1)} & C^{(p)} & C^{(p-1)} & \cdots & C^{(2)} \\ C^{(2)} & C^{(1)} & C^{(p)} & \cdots & C^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C^{(p)} & C^{(p-1)} & C^{(p-2)} & \cdots & C^{(1)} \end{bmatrix},$$

where $C^{(1)}, \ldots, C^{(p)} \in \mathbb{C}^{m \times n}$ are frontal slices of tensor $C$. The inverse operation of bcirc is denoted as $\text{bcirc}^{-1}$ with relation $\text{bcirc}^{-1}(\text{bcirc}(C)) \overset{\text{def}}{=} C$. Another operation to the tensor $C$ is unfolding, denoted as $\text{unfold}(C)$, which is defined as:

$$\text{unfold}(C) \overset{\text{def}}{=} \begin{bmatrix} C^{(1)} \\ C^{(2)} \\ \vdots \\ C^{(p)} \end{bmatrix}.$$  

(6)

The inverse operation of unfold is denoted as fold with relation $\text{fold}(\text{unfold}(C)) \overset{\text{def}}{=} C$.

The multiplication between two third order tensors, $C \in \mathbb{R}^{m \times n \times p}$ and $D \in \mathbb{R}^{n \times l \times p}$, is via T-product and this multiplication is defined as:

$$C \circ D \overset{\text{def}}{=} \text{fold}(\text{bcirc}(C) \cdot \text{unfold}(D)),$$

where $\cdot$ is the standard matrix multiplication. For given third order tensors, if we apply T-product to multiply them, we call them T-product tensors. A T-product tensor $C \in \mathbb{R}^{m \times n \times p}$ will be named as square T-product tensor if $m = n$.

For a symmetric T-product tensor $C \in \mathbb{R}^{m \times m \times p}$, we define Hermitian transpose of $C$, denoted by $C^H$, as

$$C^H = \text{bcirc}^{-1}((\text{bcirc}(C))^H).$$

(8)

And a tensor $D \in \mathbb{C}^{m \times m \times p}$ is called a Hermitian T-product tensor if $D^H = D$. Similarly, for a symmetric T-product tensor $C \in \mathbb{R}^{m \times m \times p}$, we define transpose of $C$, denoted by $C^T$, as

$$C^T = \text{bcirc}^{-1}((\text{bcirc}(C))^T).$$

(9)

And a tensor $D \in \mathbb{R}^{m \times m \times p}$ is called a symmetric T-product tensor if $D^T = D$.

The identity tensor $I_{m,m,p} \in \mathbb{R}^{m \times m \times p}$ can be defined as:

$$I_{m,m,p} = \text{bcirc}^{-1}(I_{mp}),$$

(10)

where $I_{mp}$ is the identity matrix in $\mathbb{R}^{mp \times mp}$. For a square T-product tensor, $C \in \mathbb{R}^{m \times m \times p}$, we say that $C$ is nonsingular if it has an inverse tensor $D \in \mathbb{R}^{m \times m \times p}$ such that

$$C \circ D = D \circ C = I_{m,m,p}.$$  

(11)

A zero tensor, denoted as $O_{mnp} \in \mathbb{C}^{m \times n \times p}$, is a tensor that all elements inside the tensor as 0.
For any circular matrix \( C \in \mathbb{R}^{m \times m} \), it can be diagonalized with the normalized Discrete Fourier Transform (DFT) matrix, i.e., \( C = F_m^H D F_m \), where \( F_m \) is the Fourier matrix of size \( m \times m \) defined as

\[
F_m \overset{\text{def}}{=} \frac{1}{(m \times p)} \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{(m-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{(m-1)} & \omega^{2(m-1)} & \cdots & \omega^{(m-1)(m-1)}
\end{bmatrix},
\]

where \( \omega = \exp \left( \frac{2\pi i}{m} \right) \) with \( i^2 = -1 \). This DFT matrix can also be used to diagonalize a T-product tensor as \[15\]

\[
\text{bcirc}(C) = (F_m^H \otimes I_m) \text{Diag} \left( C_i : i \in \{1, \cdots, m\} \right) (F_m \otimes I_m),
\]

where \( \otimes \) is Kronecker Product and \( \text{Diag} \left( C_i : i \in \{1, \cdots, m\} \right) \in \mathbb{C}^{mp \times mp} \) is a diagonal block matrix with the \( i \)-th diagonal block as the matrix \( A_i \).

The inner product between two T-product tensors \( C \in \mathbb{C}^{m \times n \times p} \) and \( D \in \mathbb{C}^{m \times n \times p} \) is defined as:

\[
\langle C, D \rangle = \sum_{i,j,k} c_{i,j,k}^* d_{i,j,k},
\]

where \( * \) is the complex conjugate operation.

We say that a symmetric T-product tensor \( C \in \mathbb{R}^{m \times m \times p} \) is a T-positive definite (TPD) tensor if we have

\[
\langle \lambda, C \ast \lambda \rangle > 0,
\]

holds for any non-zero T-product tensor \( \lambda \in \mathbb{R}^{m \times 1 \times p} \). Also, we said that a symmetric T-product tensor is a T-positive semidefinite (TPSD) tensor if we have

\[
\langle \lambda, C \lambda \rangle \geq 0,
\]

holds for any non-zero T-product tensor \( \lambda \in \mathbb{R}^{m \times 1 \times p} \). Given two T-product tensors \( C, D \), we use \( C \succ (\succeq) D \) if \( (C \succ D) \) is a TPSD (TPD) T-product tensor.

We have the following theorem from Theorem 5 in \[17\].

**Theorem 1** If a T-product tensor \( C \in \mathbb{R}^{m \times m \times p} \) can be diagonalized as

\[
\text{bcirc}(C) = (F_m^H \otimes I_m) \text{Diag} \left( C_i : i \in \{1, \cdots, m\} \right) (F_m \otimes I_m),
\]

where \( F \) is the DFT matrix defined by Eq. (12); then \( C \) is symmetric, TPD (TPSD) if and only if all matrices \( C_i \) are Hermitian, positive definite (positive semidefinite).

Let \( C \in \mathbb{R}^{m \times m \times p} \) can be block diagonalized as Eq. (17). Then, a real number \( \lambda \) is said to be a T-eigenvalue of \( C \), denoted as \( \lambda(C) \), if it is an eigenvalue of some \( C_i \) for \( i \in \{1, \cdots, m\} \). The largest and smallest T-eigenvalue of \( C \) are represented by \( \lambda_{\max}(C) \) and \( \lambda_{\min}(C) \), respectively. We use \( \lambda_{i,j} \) for the \( j \)-th largest T-eigenvalue of the matrix \( C_i \). We also use \( \sigma_{i,j} \), named as T-singular values, for the \( j \)-th largest singular values of the matrix \( C_i \).

We define the T-product tensor trace for a tensor \( C \in \mathbb{C}^{m \times m \times p} \), denoted by \( \text{Tr}(C) \), as following

\[
\text{Tr}(C) \overset{\text{def}}{=} \sum_{i=1}^{m} \sum_{k=1}^{p} c_{iik},
\]

which is the summation of all entries in j-diagonal components. Then, we have the following lemma about trace properties.
**Lemma 1** For any tensors $C, D \in \mathbb{C}^{m \times m \times p}$, we have

$$\text{Tr}(cC + dD) = c\text{Tr}(C) + d\text{Tr}(D),$$

(19)

where $c, d$ are two constants. And, the transpose operation will keep the same trace value, i.e.,

$$\text{Tr}(C) = \text{Tr}(C^T).$$

(20)

Finally, we have

$$\text{Tr}(C \ast D) = \text{Tr}(D \ast C).$$

(21)

**Proof:** Eqs. (19) and (20) are true from trace definition directly.

From T-product definition, the $i$-th frontal slice matrix of $D \ast C$ is

$$D^{(i)}C^{(1)} + D^{(i-1)}C^{(2)} + \cdots + D^{(1)}C^{(i)} + D^{(m)}C^{(i+1)} + \cdots + D^{(i+1)}C^{(m)},$$

(22)

similarly, the $i$-th frontal slice matrix of $C \ast D$ is

$$C^{(i)}D^{(1)} + C^{(i-1)}D^{(2)} + \cdots + C^{(1)}D^{(i)} + C^{(m)}D^{(i+1)} + \cdots + C^{(i+1)}D^{(m)}.$$  

(23)

Because the matrix trace of Eq. (22) and the matrix trace of Eq. (23) are same for each slice $i$ due to linearity and invariant under cyclic permutations of matrix trace, we have Eq. (21) by summing over all frontal matrix slices. □

Below, we will define the determinant of a T-product tensor $C \in \mathbb{R}^{m \times m \times p}$, represented by $\det(C)$, as

$$\det(C) = \prod_{i=1, j=1}^{i=m, j=p} \lambda_{i,j}. $$

(24)

We have the following theorem from Theorem 6 in [17] about symmetric T-product tensor decomposition.

**Theorem 2** Every symmetric T-product tensor $C \in \mathbb{R}^{m \times m \times p}$ can be factored as

$$C = U^T \ast D \ast U,$$

(25)

where $U$ is an orthogonal tensor, i.e., $U^T \ast U = I_{m \times m \times p}$, and $D$ is a $F$-diagonal tensor, i.e., each frontal slice of $D$ is a diagonal matrix, such that diagonal entries of $(F_m \otimes I_m) \circ \text{birc}(D) \left( I_{m}^H \otimes F_m \right)$ are T-eigenvalues of $C$. If $C$ is a TPD (TPSD) tensor, then all of its T-eigenvalues are positive (nonnegative).

From Theorem 2 and Lemma 1 we have the fact that

$$\text{Tr}(C) = \sum_i \lambda_i(C).$$

(26)

If a symmetric T-product tensor $C \in \mathbb{R}^{m \times m \times p}$ can be expressed as the format shown by Eq. (17), the T-eigenvalues of $C$ with respect to the matrix $C_i$ are denoted as $\lambda_{i,k_i}$, where $1 \leq k_i \leq m$, and we assume that $\lambda_{i,1} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,m}$ (including multiplicities). Then, $\lambda_{i,k_i}$ is the $k_i$-th largest T-eigenvalue associated to the matrix $C_i$. If we sort all T-eigenvalues of $C$ from the largest one to the smallest one, we use $\hat{k}$, a smallest integer between $1$ to $m \times p$ (inclusive) associated with $p$ given positive integers $k_1, k_2, \cdots, k_p$ that satisfies

$$\lambda_{\hat{k}} = \min_{1 \leq i \leq m} \lambda_{i,k_i}.$$ 

(27)
and
\[
\lambda_k \geq \lambda_{i,k_i+1},
\]
for all \(1 \leq i \leq p\). Moreover, we set \(\tilde{i}\) from \(\lambda_k\) as
\[
\tilde{i} = \arg \min_i \{\lambda_k = \lambda_{i,k_i}\}.
\]
Then, we will have the following Courant-Fischer theorem for T-product tensors.

**Theorem 3** Given a symmetric T-product tensor \(C \in \mathbb{R}^{m \times m \times p}\) and \(p\) positive integers \(k_1, k_2, \cdots, k_p\) with \(1 \leq k_i \leq m\), then we have
\[
\lambda_k = \max_{S \in \mathbb{R}^{m \times 1 \times p}} \min_{\dim(S) = \{k_1, \cdots, k_p\}} \frac{\langle X, C^* X \rangle}{\langle X, X \rangle}
\]
\[
= \min_{T \in \mathbb{R}^{m \times 1 \times p}} \max_{\dim(T) = \{n-k_1, \cdots, n-k_1+1, \cdots, n-k_p\}} \frac{\langle X, C^* X \rangle}{\langle X, X \rangle}
\]
where \(\lambda_k\) and \(\tilde{i}\) are defined by Eqs. (27), (28) and (29).

**Proof:**
First, we have to express \(\langle X, C^* X \rangle\) by matrices of \(C_i\) and \(X_i\) through the representation shown by Eq. (17). It is
\[
\langle X, C^* X \rangle = \frac{1}{p} \langle \text{bcirc}(X), \text{bcirc}(C)\text{bcirc}(X) \rangle
\]
\[
= \frac{1}{p} \text{Tr} \left( \text{bcirc}(X) H \text{bcirc}(C)\text{bcirc}(X) \right)
\]
\[
= \frac{1}{p} \text{Tr} \left( F^H \text{Diag} \left( x_i^H A_i x_i : i \in \{1, \cdots, p\} \right) F_p \right)
\]
\[
= \frac{1}{p} \text{Tr} \left( \text{Diag} \left( x_i^H A_i x_i : i \in \{1, \cdots, p\} \right) \right) = \frac{1}{p} \sum_{i=1}^p x_i^H A_i x_i
\]
(31)
We will just verify the first characterization of \(\lambda_k\). The other is similar. Let \(S_i\) be the projection of \(S\) to the space with dimension \(k_i\) spanned by \(v_{i,1}, \cdots, v_{i,k_i}\), for every \(x_i \in S_i\), we can write \(x_i = \sum_{j=1}^{k_i} c_{i,j} v_{i,j}\). To show that the value \(\lambda_k\) is achievable, note that
\[
\langle X, C^* X \rangle
\]
\[
\langle X, A \rangle
\]
\[
\geq \frac{1}{p} \sum_{i=1}^p x_i^H A_i x_i
\]
\[
= \frac{1}{p} \sum_{i=1}^p \lambda_k c_{i,j}^* c_{i,j}
\]
\[
= \lambda_{k_i,\tilde{i}}
\]
(32)
To verify that this is the maximum, let $T_i$ be the projection of $T$ to the space with dimension $n - k_i + 1$, then the intersection of $S$ and $T_i$ is not empty. We have

$$\min_{X \in S} \frac{\langle X, C \star X \rangle}{\langle X, X \rangle} \leq \min_{X \in S \cap T_i} \frac{\langle X, C \star X \rangle}{\langle X, X \rangle}. \quad (33)$$

Any such $x_i \in S \cap T_i$ can be expressed as $x_i = \sum_{j=k_i}^m c_{i,j}v_{i,j}$, and any $i$ for $i \neq \tilde{i}$, we have $x_i \in S \cap T_i$ expressed as $x_i = \sum_{j=k_i+1}^m c_{i,j}v_{i,j}$. Then, we have

$$\frac{\langle X, C \star X \rangle}{\langle X, X \rangle} = \frac{1}{p} \sum_{i=1}^p x_i^H A_ix_i = \frac{1}{p} \sum_{i=1}^p x_i^H A_i x_i = \frac{\sum_{i=1}^p \sum_{j=k_i+1}^m \lambda_{i,j} c_{i,j}^* c_{i,j}}{\sum_{i=1}^p \sum_{j=k_i+1}^m c_{i,j}^* c_{i,j}} \leq \lambda_{\tilde{k}}. \quad (34)$$

Therefore, for all subspaces $S$ of dimensions $\{k_1, \ldots, k_p\}$, we have $\min_{X \in S} \frac{\langle X, C \star X \rangle}{\langle X, X \rangle} \leq \lambda_{\tilde{k}} \quad \square$

### 2.2 Unitarily Invariant T-product Tensor Norms

Let us represent the $T$-eigenvalues of a symmetric $T$-product tensor $H \in \mathbb{R}^{m \times m \times p}$ in decreasing order by the vector $\vec{\lambda}(H) = (\lambda_1(H), \ldots, \lambda_{m \times m \times p}(H))$, where $m \times p$ is the total number of $T$-eigenvalues. We use $\mathbb{R}_{\geq 0}$ to represent a set of nonnegative (positive) real numbers. Let $\| \cdot \|_\rho$ be a unitarily invariant tensor norm, i.e., $\|H \star U\|_\rho = \|U \star H\|_\rho = \|U\|_\rho$, where $U$ is any unitary tensor. Let $\rho : \mathbb{R}_{\geq 0}^{m \times p} \rightarrow \mathbb{R}_{\geq 0}$ be the corresponding gauge function that satisfies Hölder’s inequality so that

$$\|H\|_\rho = \|H\|_\rho = \rho(\vec{\lambda}(\|H\|)), \quad (35)$$

where $|H| \overset{\text{def}}{=} \sqrt{H^H \star H}$. The bijective correspondence between symmetric gauge functions on $\mathbb{R}_{\geq 0}^{m \times p}$ and unitarily invariant norms is due to von Neumann \[35\].

Several popular norms can be treated as special cases of unitarily invariant tensor norm. The first one is Ky Fan like $k$-norm \[35\] for tensors. For $k \in \{1, 2, \ldots, m \times p\}$, the Ky Fan $k$-norm \[35\] for tensors $H \in \mathbb{R}^{m \times m \times p}$, denoted as $\|H\|_{(k)}$, is defined as:

$$\|H\|_{(k)} \overset{\text{def}}{=} \sum_{i=1}^k \lambda_i(|H|). \quad (36)$$

If $k = 1$, the Ky Fan $k$-norm for tensors is the tensor operator norm, denoted as $\|H\|$. The second one is Schatten $p$-norm for tensors, denoted as $\|H\|_p$, is defined as:

$$\|H\|_p \overset{\text{def}}{=} (\text{Tr} |H|^p)^{\frac{1}{p}}, \quad (37)$$
For each $k$, Lemma 2 to prove majorization relations. With IV.1.6 from [36]. This proof is based on mathematical induction. The base case for $n = 2$ has been shown by Theorem IV.1.6 from [36].

We assume that Eq. (38) is true for $n = m$, where $m > 2$. Let $\odot$ be the component-wise product (Hadamard product) between two vectors. Then, we have

$$
\rho \left( \prod_{i=1}^{m+1} b_i^{a_i} \right) = \rho \left( \odot_{i=1}^{m+1} b_i^{a_i} \right),
$$

where $\odot_{i=1}^{m+1} b_i^{a_i}$ is defined as $(\prod_{i=1}^{m+1} b_i^{a_i})$.

Under such notations, Eq. (39) can be bounded as

$$
\rho \left( \odot_{i=1}^{m+1} b_i^{a_i} \right) = \rho \left( \left( \odot_{i=1}^{m} b_i^{a_i} \right)^{\frac{\sum \alpha_j}{\sum_j \alpha_j}} \odot b_{m+1}^{\sum \alpha_j} \right) \leq \rho \left( \prod_{i=1}^{m+1} b_i^{a_i} \right) \leq \prod_{i=1}^{m+1} \rho(b_i)^{a_i}.
$$

By mathematical induction, this lemma is proved.

\[\square\]

### 2.3 Antisymmetric Kronecker Product for T-product Tensors

In this section, we will discuss a machinery of antisymmetric Kronecker product for T-product tensors and this scheme will be used later for log-majorization results. Let $\mathcal{H}$ be an $m \times p$-dimensional Hilbert space. For each $k \in \mathbb{N}$, let $\mathcal{H}^{\otimes k}$ denote the $k$-fold Kronecker product of $\mathcal{H}$, which is the $(m \times p)^k$-dimensional Hilbert space with respect to the inner product defined by

$$
(X_1 \otimes \cdots \otimes X_k, Y_1 \otimes \cdots \otimes Y_k) = \prod_{i=1}^{k} (X_i, Y_i).
$$

For $X_1, \cdots, X_k \in \mathcal{H}$, we define $X_1 \wedge \cdots \wedge X_k \in \mathcal{H}^{\otimes k}$ by

$$
X_1 \wedge \cdots \wedge X_k = \frac{1}{\sqrt{k!}} \sum_{\sigma} (\text{sgn}\sigma) X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)}.
$$

where $p \geq 1$. If $p = 1$, it is the trace norm.

Following inequality is the extension of Hölder inequality to gauge function $\rho$ which will be used later to prove majorization relations.

**Lemma 2** For $n$ nonnegative real vectors with the dimension $r$, i.e., $b_i = (b_{i1}, \cdots, b_{ir}) \in \mathbb{R}_{\geq 0}^r$, and $\alpha > 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, we have

$$
\rho \left( \prod_{i=1}^{n} b_i^{\alpha_i} \right) \leq \prod_{i=1}^{n} \rho(b_i)^{\alpha_i}
$$

**Proof:** This proof is based on mathematical induction. The base case for $n = 2$ has been shown by Theorem IV.1.6 from [36].

We assume that Eq. (38) is true for $n = m$, where $m > 2$. Let $\odot$ be the component-wise product (Hadamard product) between two vectors. Then, we have

$$
\rho \left( \prod_{i=1}^{m+1} b_i^{a_i} \right) = \rho \left( \odot_{i=1}^{m+1} b_i^{a_i} \right),
$$

where $\odot_{i=1}^{m+1} b_i^{a_i}$ is defined as $(\prod_{i=1}^{m+1} b_i^{a_i})$.

Under such notations, Eq. (39) can be bounded as

$$
\rho \left( \odot_{i=1}^{m+1} b_i^{a_i} \right) = \rho \left( \left( \odot_{i=1}^{m} b_i^{a_i} \right)^{\frac{\sum \alpha_j}{\sum_j \alpha_j}} \odot b_{m+1}^{\sum \alpha_j} \right) \leq \rho \left( \prod_{i=1}^{m+1} b_i^{a_i} \right) \leq \prod_{i=1}^{m+1} \rho(b_i)^{a_i}.
$$

By mathematical induction, this lemma is proved.

\[\square\]
where \( \sigma \) runs over all permutations on \( \{1, 2, \ldots, k\} \) and \( \text{sgn} \sigma = \pm 1 \) depending on \( \sigma \) is even or odd. The subspace of \( \mathcal{S}_k \) spanned by \( \{X_1 \wedge \cdots \wedge X_k\} \), where \( X_i \in \mathcal{S}_k \), is named as \( k \)-fold antisymmetric Kronecker product of \( \mathcal{S}_k \) and represented by \( \mathcal{S}_k^\wedge k \).

For each \( \mathcal{C} \in \mathbb{R}^{m \times m \times p} \) and \( k \in \mathbb{N} \), the \( k \)-fold Kronecker product \( \mathcal{C}^\otimes k \in \mathbb{R}^{mk \times mk \times pk} \) is given by

\[
\mathcal{C}^\otimes k \star (X_1 \otimes \cdots \otimes X_k) \overset{\text{def}}{=} (\mathcal{C} \star X_1) \otimes \cdots \otimes (\mathcal{C} \star X_k).
\]

Because \( \mathcal{S}_k^\wedge k \) is invariant for \( \mathcal{C}^\otimes k \), the antisymmetric Kronecker product of \( \mathcal{C}^\wedge k \) of \( \mathcal{C} \) can be defined as \( \mathcal{C}^\wedge k = \mathcal{C}^\otimes |\mathcal{S}_k^\wedge k| \), then we have

\[
\mathcal{C}^\wedge k \star (X_1 \wedge \cdots \wedge X_k) = (\mathcal{C} \star X_1) \wedge \cdots \wedge (\mathcal{C} \star X_k).
\]

We will provide the following lemmas about antisymmetric Kronecker product.

**Lemma 3** Let \( A, B, C, E \in \mathbb{R}^{m \times m \times p} \) be T-product tensors, for any \( k \in \{1, 2, \ldots, m \times p\} \), we have

1. \( (A^\wedge k)^T = (A^T)^\wedge k \).
2. \( (A^\wedge k) \star (B^\wedge k) = (A \star B)^\wedge k \).
3. If \( \lim_{i \to \infty} \|A_i - A\| \to 0 \), then \( \lim_{i \to \infty} \|A_i^\wedge k - A^\wedge k\| \to 0 \).
4. If \( C \succeq \mathcal{O} \) (zero tensor), then \( C^\wedge k \succeq \mathcal{O} \) and \( (C^p)^\wedge k = (C^\wedge k)^p \) for all \( p \in \mathbb{R}_{>0} \).
5. \( |A|^\wedge k = |A^\wedge k| \).
6. If \( E \succeq \mathcal{O} \) and \( E \) is invertible, \( (E^{z})^\wedge k = (E^\wedge k)^z \) for all \( z \in \mathbb{E} \).
7. \( \|E^\wedge k\| = \prod_{i=1}^{k} \lambda_i(|E|) \).

**Proof:** Items 1 and 2 are the restrictions of the associated relations \( (A^H)^\otimes k = (A^\otimes k)^H \) and \( (A \star B)^\otimes k = (A^\otimes k) \star (B^\otimes k) \) to \( \mathcal{S}_k^\wedge k \). The item 3 is true since, if \( \lim_{i \to \infty} \|A_i - A\| \to 0 \), we have \( \lim_{i \to \infty} \|A_i^\otimes k - A^\otimes k\| \to 0 \) and the associated restrictions of \( A_i^\otimes k, A^\otimes k \) to the antisymmetric subspace \( \mathcal{S}_k^\wedge k \).

For the item 4, if \( C \succeq \mathcal{O} \), then we have \( C^\wedge k = ((C^{1/2})^\wedge k)^H \star ((C^{1/2})^\wedge k) \succeq \mathcal{O} \) from items 1 and 2. If \( p \) is rational, we have \( (C^p)^\wedge k = (C^\wedge k)^p \) from the item 2, and the equality \( (C^p)^\wedge k = (C^\wedge k)^p \) is also true for any \( p > 0 \) if we apply the item 3 to approximate any irrational numbers by rational numbers.

Because we have

\[
|A|^\wedge k = \left(\sqrt{A^H A}\right)^\wedge k = \sqrt{(A^\wedge k)^H A^\wedge k} = |A^\wedge k|,
\]

from items 1, 2 and 4, so the item 5 is valid.

For item 6, if \( z < 0 \), item 6 is true for all \( z \in \mathbb{R} \) by applying the item 4 to \( E^{-1} \). Since we can apply the definition \( E^z \overset{\text{def}}{=} \exp(z \ln E) \) to have

\[
C^p = E^z \iff C = \exp\left(\frac{z}{p} \ln E\right),
\]

where \( C \succeq \mathcal{O} \). The general case of any \( z \in \mathbb{C} \) is also true by applying the item 4 to \( C = \exp\left(\frac{\tilde{z}}{p} \ln E\right) \).

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For the item 7 proof, it is enough to prove the case that $E \succeq O$ due to the item 5. Then, from Theorem 2, there exists a set of orthogonal tensors $\{U_1, \cdots, U_r\}$ such that $|E| \ast U_i = \lambda_i U_i$ for $1 \leq i \leq m \times p$. We then have

\[
|E|^{\land k} (U_{i_1} \land \cdots \land U_{i_k}) = |E| \ast U_{i_1} \land \cdots \land |E| \ast U_{i_k} = \left( \prod_{i=1}^{k} \lambda_i(|E|) \right) U_{i_1} \land \cdots \land U_{i_k},
\]

where $1 \leq i_1 < i_2 < \cdots < i_k \leq m \times p$. Hence, $\| |E|^{\land k} \| = \prod_{i=1}^{k} \lambda_i(|E|)$. □

3 Multivariate T-product Tensor Norm Inequalities

In this section, we will begin with the introduction of majorization techniques in Section 3.1. Then, the majorization with integral average and log-majorization with integral average will be introduced by Section 3.2 and Section 3.3. These majorization results will be used to prove T-product tensor norm inequalities in Section 3.4.

3.1 Majorization Basis

In this subsection, we will discuss majorization and several lemmas about majorization which will be used at later proofs.

Let $x = [x_1, \cdots, x_r] \in \mathbb{R}^{m \times p}$, $y = [y_1, \cdots, y_r] \in \mathbb{R}^{m \times p}$ be two vectors with following orders among entries $x_1 \geq \cdots \geq x_r$ and $y_1 \geq \cdots \geq y_r$, weak majorization between vectors $x, y$, represented by $x \prec_w y$, requires following relation for vectors $x, y$:

\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i,
\]

where $k \in \{1, 2, \cdots, r\}$. Majorization between vectors $x, y$, indicated by $x \prec y$, requires following relation for vectors $x, y$:

\[
\begin{align*}
\sum_{i=1}^{k} x_i & \leq \sum_{i=1}^{k} y_i, \quad \text{for } 1 \leq k < r; \\
\sum_{i=1}^{m \times p} x_i & = \sum_{i=1}^{m \times p} y_i, \quad \text{for } k = r.
\end{align*}
\]

For $x, y \in \mathbb{R}_{\geq 0}^{m \times p}$ such that $x_1 \geq \cdots \geq x_r$ and $y_1 \geq \cdots \geq y_r$, weak log majorization between vectors $x, y$, represented by $x \prec_{w \log} y$, requires following relation for vectors $x, y$:

\[
\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i,
\]

where $k \in \{1, 2, \cdots, r\}$, and log majorization between vectors $x, y$, represented by $x \prec_{\log} y$, requires equality for $k = r$ in Eq. (50). If $f$ is a single variable function, $f(x)$ represents a vector of $[f(x_1), \cdots, f(x_r)]$. From Lemma 1 in [37], we have
Lemma 4 (1) For any convex function $f : [0, \infty) \to [0, \infty)$, if we have $x \prec y$, then $f(x) \prec_w f(y)$.

(2) For any convex function and non-decreasing $f : [0, \infty) \to [0, \infty)$, if we have $x \prec_w y$, then $f(x) \prec_w f(y)$.

Another lemma is from Lemma 12 in [37], we have

Lemma 5 Let $x, y \in \mathbb{R}^{m \times p}_{\geq 0}$ such that $x_1 \geq \cdots \geq x_r$ and $y_1 \geq \cdots \geq y_r$ with $x \prec \log y$. Also let $y_i = [y_{i;1}, \cdots, y_{i;r}] \in \mathbb{R}^{m \times p}_{\geq 0}$ be a sequence of vectors such that $y_{i;1} \geq \cdots \geq y_{i;r} > 0$ and $y_i \to y$ as $i \to \infty$. Then, there exists $i_0 \in \mathbb{N}$ and $x_i = [x_{i;1}, \cdots, x_{i;r}] \in \mathbb{R}^{m \times p}_{\geq 0}$ for $i \geq i_0$ such that $x_{i;1} \geq \cdots \geq x_{i;r} > 0$, $x_i \to x$ as $i \to \infty$, and

$$x_i \prec \log y_i \text{ for } i \geq i_0.$$  

For any function $f$ on $\mathbb{R}_{\geq 0}$, the term $f(x)$ is defined as $f(x) \overset{\text{def}}{=} (f(x_1), \cdots, f(x_r))$ with conventions $e^{-\infty} = 0$ and $\log 0 = -\infty$.

3.2 Majorization with Integral Average

Let $\Omega$ be a $\sigma$-compact metric space and $\nu$ a probability measure on the Borel $\sigma$-field of $\Omega$. Let $C, D_\tau \in \mathbb{R}^{m \times m \times p}$ be symmetric T-product tensors. We further assume that tensors $C, D_\tau$ are uniformly bounded in their norm for $\tau \in \Omega$. Let $\tau \in \Omega \to D_\tau$ be a continuous function such that $\sup\{\|D_\tau\| : \tau \in \Omega\} < \infty$. For notational convenience, we define the following relation:

$$\mathbb{E}[\int_{\Omega} \lambda_1(D_\tau) d\nu(\tau), \cdots, \int_{\Omega} \lambda_{m \times p}(D_\tau) d\nu(\tau)] \overset{\text{def}}{=} \int_{\Omega^{m \times p}} \lambda(D_\tau) d\nu^{m \times p}(\tau).$$  

(52)

If $f$ is a single variable function, the notation $f(C)$ represents a tensor function with respect to the tensor $C$.

Theorem 4 Let $\Omega, \nu, C, D_\tau$ be defined as the beginning part of Section 3.2 and $f : \mathbb{R} \to [0, \infty)$ be a non-decreasing convex function, we have following two equivalent statements:

$$\mathbb{E}[f(C)] \overset{\text{w}}{\leq} \int_{\Omega^{m \times p}} f(D_\tau) d\nu^{m \times p}(\tau) \iff \|f(C)\|_\rho \leq \int_{\Omega} \|f(D_\tau)\|_\rho d\nu(\tau),$$  

(53)

where $\|\cdot\|_\rho$ is the unitarily invariant norm defined in Eq. (55).

Proof: We assume that the left statement of Eq. (53) is true and the function $f$ is a non-decreasing convex function. From Lemma 4 we have

$$\lambda(f(C)) = f(\lambda(C)) \overset{\text{w}}{\leq} f\left(\int_{\Omega^{m \times p}} \lambda(D_\tau) d\nu^{m \times p}(\tau)\right).$$  

(54)

From the convexity of $f$, we also have

$$f\left(\int_{\Omega^{m \times p}} \lambda(D_\tau) d\nu^{m \times p}(\tau)\right) \leq \int_{\Omega^{m \times p}} f(\lambda(D_\tau)) d\nu^{m \times p}(\tau) = \int_{\Omega^{m \times p}} \lambda(f(D_\tau)) d\nu^{m \times p}(\tau).$$  

(55)

Then, we obtain $\lambda(f(C)) \overset{\text{w}}{=} \int_{\Omega^{m \times p}} \lambda(f(D_\tau)) d\nu^{m \times p}(\tau)$. By applying Lemma 4.4.2 in [38] to both sides of $\lambda(f(C)) \overset{\text{w}}{=} \int_{\Omega^{m \times p}} \lambda(f(D_\tau)) d\nu^{m \times p}(\tau)$ with gauge function $\rho$, we obtain

$$\|f(C)\|_\rho \leq \rho\left(\int_{\Omega^{m \times p}} \lambda(f(D_\tau)) d\nu^{m \times p}(\tau)\right) \leq \int_{\Omega} \rho(\lambda(f(D_\tau))) d\nu(\tau) = \int_{\Omega} \|f(D_\tau)\|_\rho d\nu(\tau).$$  

(56)
Therefore, the right statement of Eq. (53) is true from the left statement.

On the other hand, if the right statement of Eq. (53) is true, we select a function \( f \) defined by \( f \equiv \max \{ x + c, 0 \} \), where \( c \) is a positive real constant satisfying \( C + cI \geq O, \) \( D_\tau + cI \geq O \) for all \( \tau \in \Omega \), and tensors \( C + cI, D_\tau + cI \). If the Ky Fan \( k \)-norm at the right statement of Eq. (53) is applied, we have

\[
\sum_{i=1}^{k} (\lambda_i(C) + c) \leq \sum_{i=1}^{k} \int_{\Omega} (\lambda_i(D_\tau) + c) d\nu(\tau). \tag{57}
\]

Hence, \( \sum_{i=1}^{k} \lambda_i(C) \leq \sum_{i=1}^{k} \int_{\Omega} \lambda_i(D_\tau) d\nu(\tau) \), this is the left statement of Eq. (53). \( \square \)

Next theorem will provide a stronger version of Theorem 4 by removing weak majorization conditions.

**Theorem 5** Let \( \Omega, \nu, C, D_\tau \) be defined as the beginning part of Section 3.2 and \( f : \mathbb{R} \rightarrow [0, \infty) \) be a convex function, we have following two equivalent statements:

\[
\bar{\lambda}(C) \prec \int_{\Omega^{m \times p}} \bar{\lambda}(D_\tau) d\nu^{m \times p}(\tau) \iff \| f(C) \|_\rho \leq \int_{\Omega} \| f(D_\tau) \|_\rho d\nu(\tau), \tag{58}
\]

where \( \| \cdot \|_\rho \) is the unitarily invariant norm defined in Eq. (55).

**Proof:** We assume that the left statement of Eq. (58) is true and the function \( f \) is a convex function. Again, from Lemma 4 we have

\[
\bar{\lambda}(f(A)) = f(\bar{\lambda}(A)) \prec_w f \left( \int_{\Omega^{m \times p}} \bar{\lambda}(D_\tau) d\nu^{m \times p}(\tau) \right) \leq \int_{\Omega^{m \times p}} f(\bar{\lambda}(D_\tau)) d\nu^{m \times p}(\tau), \tag{59}
\]

then,

\[
\| f(A) \|_\rho \leq \rho \left( \int_{\Omega^{m \times p}} f(\bar{\lambda}(D_\tau)) d\nu^{m \times p}(\tau) \right) \\
\leq \int_{\Omega} \rho \left( f(\bar{\lambda}(D_\tau)) \right) d\nu(\tau) = \int_{\Omega} \| f(D_\tau) \|_\rho d\nu(\tau). \tag{60}
\]

This proves the right statement of Eq. (58).

Now, we assume that the right statement of Eq. (58) is true. From Theorem 4 we already have \( \bar{\lambda}(C) \prec_w \int_{\Omega^{m \times p}} \bar{\lambda}(D_\tau) d\nu^{m \times p}(\tau) \). It is enough to prove \( \sum_{i=1}^{m \times p} \lambda_i(C) \geq \int_{\Omega} \sum_{i=1}^{m \times p} \lambda_i(D_\tau) d\nu(\tau) \). We define a function \( f \equiv \max \{ c - x, 0 \} \), where \( c \) is a positive real constant satisfying \( C \leq cI, D_\tau \leq cI \) for all \( \tau \in \Omega \) and tensors \( cI - C, cI - D_\tau \). If the trace norm is applied, i.e., the sum of the absolute value of all eigenvalues of a symmetric T-product tensor, then the right statement of Eq. (58) becomes

\[
\sum_{i=1}^{m \times p} \lambda_i(cI - C) \leq \int_{\Omega} \sum_{i=1}^{m \times p} \lambda_i(cI - D_\tau) d\nu(\tau). \tag{61}
\]

The desired inequality \( \sum_{i=1}^{m \times p} \lambda_i(C) \geq \int_{\Omega} \sum_{i=1}^{m \times p} \lambda_i(D_\tau) d\nu(\tau) \) is established. \( \square \)
3.3 Log-Majorization with Integral Average

The purpose of this section is to consider log-majorization issues for unitarily invariant norms of TPSD T-product tensors. In this section, let $C, D_\tau \in \mathbb{R}^{m \times m \times p}$ be TPSD T-product tensors with $m \times p$ nonnegative T-eigenvalues by keeping notations with the same definitions as at the beginning of the Section 3.2. For notational convenience, we define the following relation for logarithm vector:

$$
\left[ \int_{\Omega} \log \lambda_1(D_\tau)d\nu(\tau), \cdots, \int_{\Omega} \log \lambda_{m \times p}(D_\tau)d\nu(\tau) \right] \overset{\text{def}}{=} \int_{\Omega^{m \times p}} \log \tilde{\lambda}(D_\tau)d\nu^{m \times p}(\tau). \quad (62)
$$

**Theorem 6** Let $C, D_\tau$ be TPSD T-product tensors, $f : (0, \infty) \to [0, \infty)$ be a continuous function such that the mapping $x \to \log f(e^x)$ is convex on $\mathbb{R}$, and $g : (0, \infty) \to [0, \infty)$ be a continuous function such that the mapping $x \to g(e^x)$ is convex on $\mathbb{R}$, then we have following three equivalent statements:

$$
\tilde{\lambda}(C) \prec_w \log \exp \int_{\Omega^{m \times p}} \log \tilde{\lambda}(D_\tau)d\nu^{m \times p}(\tau); \quad (63)
$$

$$
\|f(C)\|_\rho \leq \exp \int_{\Omega} \log \|f(D_\tau)\|_\rho d\nu(\tau); \quad (64)
$$

$$
\|g(C)\|_\rho \leq \int_{\Omega} \|g(D_\tau)\|_\rho d\nu(\tau). \quad (65)
$$

**Proof:** The roadmap of this proof is to prove equivalent statements between Eq. (63) and Eq. (64) first, followed by equivalent statements between Eq. (63) and Eq. (65).

**Eq. (63) $\implies$ Eq. (64)**

There are two cases to be discussed in this part of proof: $C, D_\tau$ are TPD tensors, and $C, D_\tau$ are TPSD T-product tensors. At the beginning, we consider the case that $C, D_\tau$ are TPD tensors.

Since $D_\tau$ are positive, we can find $\varepsilon > 0$ such that $D_\tau \geq \varepsilon I$ for all $\tau \in \Omega$. From Eq. (63), the convexity of $\log f(e^x)$ and Lemma 4 we have

$$
\tilde{\lambda}(f(C)) = f \left( \exp \left( \log \tilde{\lambda}(C) \right) \right) \prec_w f \left( \exp \int_{\Omega^{m \times p}} \tilde{\lambda}(D_\tau)d\nu^{m \times p}(\tau) \right) \leq \exp \left( \int_{\Omega^{m \times p}} \log f \left( \tilde{\lambda}(D_\tau) \right) d\nu^{m \times p}(\tau) \right). \quad (66)
$$

Then, from Eq. (65), we obtain

$$
\|f(C)\|_\rho \leq \left( \exp \left( \int_{\Omega^{m \times p}} \log f \left( \tilde{\lambda}(D_\tau) \right) d\nu^{m \times p}(\tau) \right) \right). \quad (67)
$$

From the function $f$ properties, we can assume that $f(x) > 0$ for any $x > 0$. Then, we have following bounded and continuous maps on $\Omega$: $\tau \to \log f(\lambda_i(D_\tau))$ for $i \in \{1, 2, \cdots, m \times p\}$, and $\tau \to \log \|f(D_\tau)\|_\rho$. Because we have $\nu(\Omega) = 1$ and $\sigma$-compactness of $\Omega$, we have $\tau_k^{(n)} \in \Omega$ and $\alpha_k^{(n)}$ for $k \in \{1, 2, \cdots, n\}$ and $n \in \mathbb{N}$ with $\sum_{k=1}^n \alpha_k^{(n)} = 1$ such that

$$
\int_{\Omega} \log f(\lambda_i(D_\tau))d\nu(\tau) = \lim_{n \to \infty} \sum_{k=1}^n \alpha_k^{(n)} \log f(\lambda_i(D_{\tau_k^{(n)}})), \text{ for } i \in \{1, 2, \cdots, m \times p\}; \quad (68)
$$
and
\[ \int_{\Omega} \log \| f(D_\tau) \|_\rho \, d\nu(\tau) = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k(\rho) \log \| f(D_{\tau_k}^{(n)}) \|_\rho. \] (69)

By taking the exponential at both sides of Eq. (68) and apply the gauge function \( \rho \), we have
\[ \rho \left( \exp \int_{\Omega^{m \times p}} \log f(\tilde{\lambda}(D_\tau)) \, d\nu^{m \times p}(\tau) \right) = \lim_{n \to \infty} \rho \left( \prod_{k=1}^{n} f(\tilde{\lambda}(D_{\tau_k}^{(n)}))^{\alpha_k(\rho)} \right). \] (70)

Similarly, by taking the exponential at both sides of Eq. (69), we have
\[ \exp \left( \int_{\Omega} \log \| f(D_\tau) \|_\rho \, d\nu(\tau) \right) = \lim_{n \to \infty} \prod_{k=1}^{n} \| f(D_{\tau_k}^{(n)}) \|_{\rho}^{\alpha_k(\rho)}. \] (71)

From Lemma 2, we have
\[ \rho \left( \prod_{k=1}^{n} f(\tilde{\lambda}(D_{\tau_k}^{(n)}))^{\alpha_k(\rho)} \right) \leq \prod_{k=1}^{n} \rho^{\alpha_k(\rho)} \left( f(\tilde{\lambda}(D_{\tau_k}^{(n)}))) \right) \]
\[ = \prod_{k=1}^{n} \rho^{\alpha_k(\rho)} \left( \tilde{\lambda}(f(D_{\tau_k}^{(n)}))) \right) \]
\[ = \prod_{k=1}^{n} \| f(D_{\tau_k}^{(n)}) \|_{\rho}^{\alpha_k(\rho)}. \] (72)

From Eqs. (70), (71) and (72), we have
\[ \rho \left( \exp \int_{\Omega^{m \times p}} \log f(\tilde{\lambda}(D_\tau)) \, d\nu^{m \times p}(\tau) \right) \leq \exp \int_{\Omega} \log \| f(D_\tau) \|_\rho \, d\nu(\tau). \] (73)

Then, Eq. (64) is proved from Eqs. (67) and (73).

Next, we consider that \( C, D_\tau \) are TPSD T-product tensors. For any \( \delta > 0 \), we have following log-majorization relation:
\[ \prod_{i=1}^{k} (\lambda_i(C) + \epsilon_\delta) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log (\lambda_i(D_\tau) + \delta) \, d\nu(\tau), \] (74)

where \( \epsilon_\delta > 0 \) and \( \delta \in \{1, 2, \cdots r\} \). Then, we can apply the previous case result about TPD tensors to TPD tensors \( C + \epsilon_\delta I \) and \( D_\tau + \delta I \), and get
\[ \| f(C) + \epsilon_\delta I \|_\rho \leq \exp \int_{\Omega} \log \| f(D_\tau) + \delta I \|_\rho \, d\nu(\tau) \] (75)

As \( \delta \to 0 \), Eq. (75) will give us Eq. (64) for TPSD T-product tensors.

Eq. (64) \iff Eq. (64)

We consider TPD tensors at first phase by assuming that \( D_\tau \) are TPD T-product tensors for all \( \tau \in \Omega \). We may also assume that the tensor \( C \) is a TPD T-product tensor. Since if this is a TPSD T-product tensor, i.e., some \( \lambda_i = 0 \), we always have following inequality valid:
\[ \prod_{i=1}^{k} \lambda_i(C) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log \lambda_i(D_\tau) \, d\nu(\tau) \] (76)
If we apply $f(x) = x^p$ for $p > 0$ and $\|\cdot\|_{\rho}$ as Ky Fan $k$-norm in Eq. (64), we have
\[
\log \sum_{i=1}^{k} \lambda_i^p (C) \leq \int_{\Omega} \log \sum_{i=1}^{k} \lambda_i^p (D_{\tau}) \, d\nu(\tau). \tag{77}
\]
If we add $\log \frac{1}{k}$ and multiply $\frac{1}{p}$ at both sides of Eq. (77), we have
\[
\frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i^p (C) \right) \leq \frac{1}{p} \int_{\Omega} \log \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i^p (D_{\tau}) \right) \, d\nu(\tau). \tag{78}
\]
From L’Hopital’s Rule, if $p \to 0$, we have
\[
\frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i^p (C) \right) \to \frac{1}{k} \sum_{i=1}^{k} \log \lambda_i (C), \tag{79}
\]
and
\[
\frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i^p (D_{\tau}) \right) \to \frac{1}{k} \sum_{i=1}^{k} \log \lambda_i (D_{\tau}), \tag{80}
\]
where $\tau \in \Omega$. Applying Eqs. (79) and (80) into Eq. (78) and taking $p \to 0$, we have
\[
\sum_{i=1}^{k} \lambda_i (C) \leq \int_{\Omega} \sum_{i=1}^{k} \log \lambda_i (D_{\tau}) \, d\nu(\tau). \tag{81}
\]
Therefore, Eq. (63) is true for TPD tensors.

For TPSD T-product tensors $D_{\tau}$, since Eq. (64) is valid for $D_{\tau} + \delta I$ for any $\delta > 0$, we can apply the previous case result about TPD tensors to $D_{\tau} + \delta I$ and obtain
\[
\prod_{i=1}^{k} \lambda_i (C) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log (\lambda_i (D_{\tau}) + \delta) \, d\nu(\tau), \tag{82}
\]
where $k \in \{1, 2, \cdots, r\}$. Eq. (63) is still true for TPSD T-product tensors as $\delta \to 0$.

**Eq. 63 $\implies$ Eq. 65**

If $C, D_{\tau}$ are TPD tensors, and $D_{\tau} \geq \delta I$ for all $\tau \in \Omega$. From Eq. (63), we have
\[
\tilde{\lambda}(\log C) = \log \tilde{\lambda}(C) \prec_{w} \int_{\Omega^{m \times p}} \log \tilde{\lambda}(D_{\tau}) \, d\nu^{m \times p}(\tau) = \int_{\Omega^{m \times p}} \tilde{\lambda}(\log D_{\tau}) \, d\nu^{m \times p}(\tau). \tag{83}
\]
If we apply Theorem 4 to $\log C, \log D_{\tau}$ with function $f(x) = g(e^x)$, where $g$ is used in Eq. (65), Eq. (65) is implied.

If $C, D_{\tau}$ are TPSD T-product tensors and any $\delta > 0$, we can find $\epsilon_\delta \in (0, \delta)$ to satisfy following:
\[
\prod_{i=1}^{k} (\lambda_i (C) + \epsilon_\delta) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log (\lambda_i (D_{\tau}) + \delta) \, d\nu(\tau). \tag{84}
\]
Then, from TPD T-product tensor case, we have
\[
\|g(C + \epsilon_\delta I)\|_{\rho} \leq \int_{\Omega} \|g(D_{\tau} + \delta I)\|_{\rho} \, d\nu(\tau). \tag{85}
\]
Eq. (65) is obtained by taking $\delta \to 0$ in Eq. (85).

**Eq. (63) $\iff$ Eq. (65)**

For $k \in \{1, 2, \cdots, r\}$, if we apply $g(x) = \log(\delta + x)$, where $\delta > 0$, and Ky Fan $k$-norm in Eq. (65), we have

$$
\sum_{i=1}^{k} \log (\delta + \lambda_i (C)) \leq \sum_{i=1}^{k} \int_{\Omega} \log (\delta + \lambda_i(D_\tau)) \, d\nu(\tau).
$$

(86)

Then, we have following relation as $\delta \to 0$:

$$
\sum_{i=1}^{k} \log \lambda_i (C) \leq \sum_{i=1}^{k} \int_{\Omega} \log \lambda_i(D_\tau) \, d\nu(\tau).
$$

(87)

Therefore, Eq. (63) can be derived from Eq. (65). □

Next theorem will extend Theorem 6 to non-weak version.

**Theorem 7** Let $C$, $D_\tau$ be TPSD T-product tensors with $\int_{\Omega} \|D_\tau\|_p^p \, d\nu(\tau) < \infty$ for any $p > 0$, $f : (0, \infty) \to [0, \infty)$ be a continuous function such that the mapping $x \to \log f(e^x)$ is convex on $\mathbb{R}$, and $g : (0, \infty) \to [0, \infty)$ be a continuous function such that the mapping $x \to g(e^x)$ is convex on $\mathbb{R}$, then we have following three equivalent statements:

$$
\bar{\lambda}(C) \prec_{\text{log}} \exp \int_{\Omega^m \times p} \log \bar{\lambda}(D_\tau) \, d\nu^{m \times p}(\tau);
$$

(88)

$$
\|f(C)\|_\rho \leq \exp \int_{\Omega} \log \|f(D_\tau)\|_\rho \, d\nu(\tau);
$$

(89)

$$
\|g(C)\|_\rho \leq \int_{\Omega} \|g(D_\tau)\|_\rho \, d\nu(\tau).
$$

(90)

**Proof:**

The proof plan is similar to the proof in Theorem 6. We prove the equivalence between Eq. (88) and Eq. (89) first, then prove the equivalence between Eq. (88) and Eq. (90).

**Eq. (88) $\iff$ Eq. (89)**

First, we assume that $C, D_\tau$ are TPD T-product tensors with $D_\tau \geq \delta I$ for all $\tau \in \Omega$. The corresponding part of the proof in Theorem 6 about TPD tensors $C, D_\tau$ can be applied here.

For case that $C, D_\tau$ are TPSD T-product tensors, we have

$$
\prod_{i=1}^{k} \lambda_i(C) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log (\lambda_i(D_\tau) + \delta_n) \, d\nu(\tau),
$$

(91)

where $\delta_n > 0$ and $\delta_n \to 0$. Because $\int_{\Omega^m \times p} \log \left(\bar{\lambda}(D_\tau) + \delta_n\right) \, d\nu^{m \times p}(\tau) \to \int_{\Omega^m \times p} \log \bar{\lambda}(D_\tau) \, d\nu^{m \times p}(\tau)$ as $n \to \infty$, from Lemma 5 we can find $a^{(n)}$ with $n \geq n_0$ such that $a_1^{(n)} \geq \cdots \geq a_r^{(n)} > 0$, $a^{(n)} \to e^{(n)}(C)$ and $a^{(n)} \prec_{\text{log}} \exp \int_{\Omega^m \times p} \log \bar{\lambda}(D_\tau + \delta_n I) \, d\nu^{m \times p}(\tau)$

Selecting $C^{(n)}$ with $\bar{\lambda}(C^{(n)}) = a^{(n)}$ and applying TPD tensors case to $C^{(n)}$ and $D_\tau + \delta_n I$, we obtain

$$
\|f(C^{(n)})\|_\rho \leq \exp \int_{\Omega} \log \|f(D_\tau + \delta_n I)\|_\rho \, d\nu(\tau)
$$

(92)
where \( n \geq n_0 \).

There are two situations for the function \( f \) near 0: \( f(0^+) < \infty \) and \( f(0^+) = \infty \). For the case with \( f(0^+) < \infty \), we have

\[
\left\| f(\mathbf{C}(n)) \right\|_\rho = \rho(f(\mathbf{a}(n))) \rightarrow \rho(f(\overline{\mathbf{C}})) = \left\| f(\mathbf{C}) \right\|_\rho,
\]

and

\[
\left\| f(\mathcal{D}_\tau + \delta_n \mathcal{I}) \right\|_\rho \rightarrow \left\| f(\mathcal{D}_\tau) \right\|_\rho,
\]

where \( \tau \in \Omega \) and \( n \rightarrow \infty \). From Fatou–Lebesgue theorem, we then have

\[
\limsup_{n \rightarrow \infty} \int\!\!\int_{\Omega} \log \left\| f(\mathcal{D}_\tau + \delta_n \mathcal{I}) \right\|_\rho \, d\nu(\tau) \leq \int\!\!\int_{\Omega} \log \left\| f(\mathcal{D}_\tau) \right\|_\rho \, d\nu(\tau).
\]

By taking \( n \rightarrow \infty \) in Eq. (92) and using Eqs. (93), (94), (95), we have Eq. (89) for case that \( f(0^+) < \infty \).

For the case with \( f(0^+) = \infty \), we assume that \( \int\!\!\int_{\Omega} \log \left\| f(\mathcal{D}_\tau) \right\|_\rho \, d\nu(\tau) < \infty \) (since the inequality in Eq. (89) is always true for \( \int\!\!\int_{\Omega} \log \left\| f(\mathcal{D}_\tau) \right\|_\rho \, d\nu(\tau) = \infty \)). Since \( f \) is decreasing on \((0, \epsilon)\) for some \( \epsilon > 0 \). We claim that the following relation is valid: there are two constants \( a, b > 0 \) such that

\[
a \leq \left\| f(\mathcal{D}_\tau + \delta_n \mathcal{I}) \right\|_\rho \leq \left\| f(\mathcal{D}_\tau) \right\|_\rho + b;
\]

for all \( \tau \in \Omega \) and \( n \geq n_0 \). If Eq. (96) is valid and \( \int\!\!\int_{\Omega} \log \left\| f(\mathcal{D}_\tau) \right\|_\rho \, d\nu(\tau) < \infty \), from Lebesgue’s dominated convergence theorem, we also have Eq. (89) for case that \( f(0^+) = \infty \) by taking \( n \rightarrow \infty \) in Eq. (92).

Below, we will prove the claim stated by Eq. (96). By the uniform boundedness of tensors \( \mathcal{D}_\tau \), there is a constant \( \kappa > 0 \) such that

\[
0 < \mathcal{D}_\tau + \delta_n \mathcal{I} \leq \kappa \mathcal{I},
\]

where \( \tau \in \Omega \) and \( n \geq n_0 \). We may assume that \( \mathcal{D}_\tau \) is TPD tensors because \( \left\| f(\mathcal{D}_\tau) \right\|_\rho = \infty \), i.e., Eq. (96) being true automatically, when \( \mathcal{D}_\tau \) is TPSD T-product tensors. From Theorem 2 we have

\[
f(\mathcal{D}_\tau + \delta_n \mathcal{I}) = \sum_{\lambda_j^*(\mathcal{D}_\tau) + \delta_n \leq \epsilon} f(\lambda_j^*(\mathcal{D}_\tau) + \delta_n) \mathcal{U}_j^* \mathcal{U}_j^H + \sum_{\lambda_j^*(\mathcal{D}_\tau) + \delta_n \geq \epsilon} f(\lambda_j^*(\mathcal{D}_\tau) + \delta_n) \mathcal{U}_j^* \mathcal{U}_j^H
\]

\[
\leq \sum_{\lambda_j^*(\mathcal{D}_\tau) + \delta_n \leq \epsilon} f(\lambda_j^*(\mathcal{D}_\tau)) \mathcal{U}_j^* \mathcal{U}_j^H + \sum_{\lambda_j^*(\mathcal{D}_\tau) + \delta_n \geq \epsilon} f(\lambda_j^*(\mathcal{D}_\tau)) \mathcal{U}_j^* \mathcal{U}_j^H
\]

\[
\leq f(\mathcal{D}_\tau) + \sum_{\lambda_j^*(\mathcal{D}_\tau) + \delta_n \geq \epsilon} f(\lambda_j^*(\mathcal{D}_\tau) + \delta_n) \mathcal{U}_j^* \mathcal{U}_j^H.
\]

Therefore, the claim in Eq. (96) follows by the triangle inequality for \( \left\| \cdot \right\|_\rho \) and \( f(\lambda_j^*(\mathcal{D}_\tau) + \delta_n) < \infty \) for \( \lambda_j^*(\mathcal{D}_\tau) + \delta_n \geq \epsilon \).

\[\text{Eq. (88) } \iff \text{Eq. (89)}\]
The weak majorization relation
\[
\prod_{i=1}^{k} \lambda_i(C) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log \lambda_i(D_\tau) d\nu(\tau),
\]
(99)
is valid for \( k < m \times p \) from Eq. (63) \( \implies \) Eq. (64) in Theorem 6. We wish to prove that Eq. (99) becomes equal for \( k = m \times p \). It is equivalent to prove that
\[
\log \det(C) \geq \int_{\Omega} \log \det(D_\tau) d\nu(\tau),
\]
(100)
where \( \det(\cdot) \) is defined by Eq. (24). We can assume that \( \int_{\Omega} \log \det(D_\tau) d\nu(\tau) \geq -\infty \) since Eq. (100) is true for \( \int_{\Omega} \log \det(D_\tau) d\nu(\tau) = -\infty \). Then, \( D_\tau \) are TPD tensors.

If we scale tensors \( C, D_\tau \) as \( aC, aD_\tau \) by some \( a > 0 \), we can assume \( D_\tau \leq I \) and \( \lambda_i(D_\tau) \leq 1 \) for all \( \tau \in \Omega \) and \( i \in \{1, 2, \cdots, m \times p\} \). Then for any \( p > 0 \), we have
\[
\frac{1}{m \times p} \left\| D_\tau^{-\varrho} \right\|_1 \leq \lambda_r^{-\varrho}(D_\tau) \leq (\det(D_\tau))^{-\varrho},
\]
(101)
and
\[
\frac{1}{\varrho} \log \left( \frac{\left\| D_\tau^{-\varrho} \right\|}{m \times p} \right) \leq -\log \det(D_\tau).
\]
(102)

If we use tensor trace norm, represented by \( \| \cdot \|_1 \), as unitarily invariant tensor norm and \( f(x) = x^{-\varrho} \) for any \( \varrho > 0 \) in Eq. (89), we obtain
\[
\log \left\| C^{-\varrho} \right\|_1 \leq \int_{\Omega} \log \left\| D_\tau^{-\varrho} \right\|_1 d\nu(\tau).
\]
(103)
By adding \( \log \frac{1}{m \times p} \) and multiplying \( \frac{1}{\varrho} \) for both sides of Eq. (103), we have
\[
\frac{1}{\varrho} \log \left( \frac{\| C^{-\varrho} \|}{m \times p} \right) \leq \frac{1}{\varrho} \int_{\Omega} \log \left( \frac{\| D_\tau^{-\varrho} \|}{m \times p} \right) d\nu(\tau)
\]
(104)
Similar to Eqs. (79) and (80), we have following two relations as \( \varrho \to 0 \):
\[
\frac{1}{\varrho} \log \left( \frac{\| C^{-\varrho} \|}{m \times p} \right) \to -\frac{1}{m \times p} \log \det(C),
\]
(105)
and
\[
\frac{1}{\varrho} \log \left( \frac{\| D_\tau^{-\varrho} \|}{m \times p} \right) \to -\frac{1}{m \times p} \log \det(D_\tau).
\]
(106)
From Eq. (102) and Lebesgue’s dominated convergence theorem, we have
\[
\lim_{\varrho \to 0} \int_{\Omega} \frac{1}{\varrho} \log \left( \frac{\| D_\tau^{-\varrho} \|}{m \times p} \right) d\nu(\tau) = -\frac{1}{m \times p} \int_{\Omega} \log \det(D_\tau) d\nu(\tau)
\]
(107)
Finally, we have Eq. (100) from Eqs. (104) and (107).

\textbf{Eq. (88) \implies Eq. (90)}

First, we assume that \( \mathcal{C}, \mathcal{D}_\tau \) are TPD tensors and \( \mathcal{D}_\tau \geq \delta \mathcal{I} \) for \( \tau \in \Omega \). From Eq. (88), we can apply Theorem 5 to \( \log \mathcal{C}, \log \mathcal{D}_\tau \) and \( f(x) = g(e^x) \) to obtain Eq. (90).

For \( \mathcal{C}, \mathcal{D}_\tau \) are TPSD T-product tensors, we can choose \( \mathcal{C}^{(n)} \) and corresponding \( \mathcal{C}^{(n)} \) for \( n \geq n_0 \) given \( \delta_n \to 0 \) with \( \delta_n \to 0 \) as the proof in Eq. (88) \implies Eq. (89). Since tensors \( \mathcal{C}^{(n)}, \mathcal{D}_\tau + \delta_n \mathcal{I} \) are TPD T-product tensors, we then have

\[
\left\| g(\mathcal{C}^{(n)}) \right\|_\rho \leq \int_\Omega \left\| g(\mathcal{D}_\tau + \delta_n \mathcal{I}) \right\|_\rho d\nu(\tau). \tag{108}
\]

If \( g(0^+) < \infty \), Eq. (90) is obtained from Eq. (108) by taking \( n \to \infty \). On the other hand, if \( g(0^+) = \infty \), we can apply the argument similar to the portion about \( f(0^+) = \infty \) in the proof for Eq. (88) \implies Eq. (89) to get \( a, b > 0 \) such that

\[
a \leq \left\| g(\mathcal{D}_\tau + \delta_n \mathcal{I}) \right\|_\rho \leq \left\| g(\mathcal{D}_\tau) \right\|_\rho + b, \tag{109}
\]

for all \( \tau \in \Omega \) and \( n \geq n_0 \). Since the case that \( \int_\Omega \left\| g(\mathcal{D}_\tau) \right\|_\rho d\nu(\tau) = \infty \) will have Eq. (90), we only consider the case that \( \int_\Omega \left\| g(\mathcal{D}_\tau) \right\|_\rho d\nu(\tau) < \infty \). Then, we have Eq. (90) from Eqs. (108), (109) and Lebesgue’s dominated convergence theorem.

\textbf{Eq. (88) \iff Eq. (90)}

The weak majorization relation

\[
\sum_{i=1}^k \log \lambda_i(\mathcal{C}) \leq \sum_{i=1}^k \int_\Omega \log \lambda_i(\mathcal{D}_\tau) d\nu(\tau) \tag{110}
\]

is true from the implication from Eq. (65) to Eq. (66) in Theorem 6. We have to show that this relation becomes identity for \( k = m \times p \). If we apply \( \| \cdot \|_\rho = \| \cdot \|_1 \) and \( g(x) = x^{-\rho} \) for any \( \rho > 0 \) in Eq. (90), we have

\[
\frac{1}{\rho} \log \left( \frac{\| \mathcal{C}^{-\rho} \|_1}{m \times p} \right) \leq \frac{1}{\rho} \log \left( \int_\Omega \frac{\| \mathcal{D}_\tau^{-\rho} \|_1}{m \times p} d\nu(\tau) \right). \tag{111}
\]

Then, we will get

\[
\frac{-\log \det(\mathcal{C})}{m \times p} = \lim_{\rho \to 0} \frac{1}{\rho} \log \left( \frac{\| \mathcal{C}^{-\rho} \|_1}{m \times p} \right) \leq \lim_{\rho \to 0} \frac{1}{\rho} \log \left( \int_\Omega \frac{\| \mathcal{D}_\tau^{-\rho} \|_1}{m \times p} d\nu(\tau) \right) = \frac{1}{m \times p} \int_\Omega \log \det(\mathcal{D}_\tau) d\nu(\tau), \tag{112}
\]

which will prove the identity for Eq. (110) when \( k = m \times p \). The equality in \( = 1 \) will be proved by the following Lemma 6.

\textbf{Lemma 6 Let} \( \mathcal{D}_\tau \) \textbf{be TPSD T-product tensors with} \( \int_\Omega \| \mathcal{D}_\tau^{-p} \|_\rho d\nu(\tau) < \infty \) for any \( p > 0 \), \textbf{then we have}

\[
\lim_{p \to 0} \left( \frac{1}{p} \log \int_\Omega \frac{\| \mathcal{D}_\tau^{-p} \|_1}{m \times p} d\nu(\tau) \right) = -\frac{1}{m \times p} \int_\Omega \log \det(\mathcal{D}_\tau) d\nu(\tau) \tag{113}
\]
Theorem 1.12 Let \( L \in \mathbb{R}^{m \times m \times p} \) be a finite sequence of bounded T-product tensors with dimensions \( \mathcal{L}_k \in \mathbb{R}^{m \times m \times p} \), then we have

\[
\lim_{n \to \infty} \left( \prod_{k=1}^{m} \exp \left( \frac{L_k}{n} \right) \right)^n = \exp \left( \sum_{k=1}^{m} \mathcal{L}_k \right) \tag{115}
\]

Proof:

We will prove the case for \( m = 2 \), and the general value of \( m \) can be obtained by mathematical induction. Let \( \mathcal{L}_1, \mathcal{L}_2 \) be bounded tensors act on some Hilbert space. Define \( \mathcal{C} \triangleq \exp((\mathcal{L}_1 + \mathcal{L}_2)/n) \) and \( \mathcal{D} \triangleq \exp(\mathcal{L}_1/n) \ast \exp(\mathcal{L}_2/n). \) Note we have following estimates for the norm of tensors \( \mathcal{C}, \mathcal{D} \):

\[
\|\mathcal{C}\|, \|\mathcal{D}\| \leq \exp \left( \frac{\|\mathcal{L}_1\| + \|\mathcal{L}_2\|}{n} \right) = [\exp (\|\mathcal{L}_1\| + \|\mathcal{L}_2\|)]^{1/n}. \tag{116}
\]

From the Cauchy-Product formula, the tensor \( \mathcal{D} \) can be expressed as:

\[
\mathcal{D} = \exp(\mathcal{L}_1/n) \ast \exp(\mathcal{L}_2/n) = \sum_{i=0}^{\infty} \frac{(\mathcal{L}_1/n)^i}{i!} \ast \sum_{j=0}^{\infty} \frac{(\mathcal{L}_2/n)^j}{j!} = \sum_{l=0}^{\infty} n^{-l} \sum_{i=0}^{l} \frac{\mathcal{L}_1^i}{i!} \ast \frac{\mathcal{L}_2^{l-i}}{(l-i)!}. \tag{117}
\]
then we can bound the norm of $C - D$ as

$$
\|C - D\| = \left| \sum_{i=0}^{\infty} \frac{((L_1 + L_2)^i)}{i!} - \sum_{i=0}^{\infty} \frac{\left(\frac{\rho}{\|L_1\|} + \frac{\rho}{\|L_2\|}\right)^i}{i!} \right|
$$

$$
= \left| \sum_{i=2}^{\infty} \frac{k^{-i}((L_1 + L_2)^i)}{i!} - \sum_{i=2}^{\infty} \frac{\left(\frac{\rho}{\|L_1\|} + \frac{\rho}{\|L_2\|}\right)^i}{i!} \right|
$$

$$
\leq \frac{1}{k^2} \left[ \exp(\|L_1\| + \|L_2\|) + \sum_{i=2}^{\infty} \frac{n^{-i} \left(\|L_1\| + \|L_2\|\right)^i}{i!} \right]
$$

$$
= \frac{1}{n^2} \left[ \exp(\|L_1\| + \|L_2\|) + \sum_{i=2}^{\infty} \frac{n^{-i} \left(\|L_1\| + \|L_2\|\right)^i}{i!} \right]
$$

$$
\leq \frac{2 \exp (\|L_1\| + \|L_2\|)}{n^2}. \tag{118}
$$

For the difference between the higher power of $C$ and $D$, we can bound them as

$$
\|C^m - D^m\| = \left| \sum_{m=0}^{n-1} C^m (C - D) C^{n-1} \right|
$$

$$
\leq \exp(\|L_1\| + \|L_2\|) \cdot n \cdot \|L_1 - L_2\|, \tag{119}
$$

where the inequality $\leq$ uses the following fact

$$
\|C\| \cdot \|D\|^{n-1} \leq \exp (\|L_1\| + \|L_2\|) \leq \exp (\|L_1\| + \|L_2\|), \tag{120}
$$

based on Eq. (116). By combining with Eq. (118), we have the following bound

$$
\|C^m - D^m\| \leq \frac{2 \exp (2 \|L_1\| + 2 \|L_2\|)}{n}. \tag{121}
$$

Then this lemma is proved when $n$ goes to infinity. \hfill \Box

Below, new multivariate norm inequalities for T-product tensors are provided according to previous majorization theorems.

**Theorem 8** Let $C_i \in \mathbb{R}^{m \times m \times p}$ be TPD tensors, where $1 \leq i \leq n$, $\|\cdot\|_{\rho}$ be a unitarily invariant norm with corresponding gauge function $\rho$. For any continuous function $f : (0, \infty) \to [0, \infty)$ such that $x \to \log f(e^x)$ is convex on $\mathbb{R}$, we have

$$
\left\| f \left( \exp \left( \sum_{j=1}^{n} \log C_i \right) \right) \right\|_{\rho} \leq \exp \int_{-\infty}^{\infty} \log \left\| f \left( \prod_{i=1}^{n} C_1^{1+it} \right) \right\|_{\rho} \beta_0(t) dt, \tag{122}
$$

where $\beta_0(t) = \frac{\pi}{2(\cosh(\pi t) + 1)}$.

For any continuous function $g(0, \infty) \to [0, \infty)$ such that $x \to g(e^x)$ is convex on $\mathbb{R}$, we have

$$
\left\| g \left( \exp \left( \sum_{j=1}^{n} \log C_i \right) \right) \right\|_{\rho} \leq \int_{-\infty}^{\infty} \left\| g \left( \prod_{i=1}^{n} C_1^{1+it} \right) \right\|_{\rho} \beta_0(t) dt. \tag{123}
$$

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Proof: From Hirschman interpolation theorem [39] and \( \theta \in [0, 1] \), we have
\[
\log |h(\theta)| \leq \int_{-\infty}^{\infty} \log |h(\theta)|^{1-\theta} \beta_{1-\theta}(t) dt + \int_{-\infty}^{\infty} \log |h(1+\theta)|^{\theta} \beta_{\theta}(t) dt, \tag{124}
\]
where \( h(z) \) be uniformly bounded on \( S \defeq \{ z \in \mathbb{C} : 0 \leq \Re(z) \leq 1 \} \) and holomorphic on \( S \). The term \( d\beta_{\theta}(t) \) is defined as:
\[
\beta_{\theta}(t) \defeq \frac{\sin(\pi \theta)}{2\theta(\cos(\pi t) + \cos(\pi \theta))}. \tag{125}
\]
Let \( H(z) \) be a uniformly bounded holomorphic function with values in \( \mathbb{C}^{m \times m \times p} \). Fix some \( \theta \in [0, 1] \) and let \( \mathcal{U}, \mathcal{V} \in \mathbb{C}^{m \times m \times p} \) be normalized tensors such that \( \langle \mathcal{U}, \mathcal{H}(\theta) \ast \mathcal{V} \rangle = \|H(\theta)\| \). If we define \( h(z) \) as \( h(z) \defeq \langle \mathcal{U}, \mathcal{H}(z) \ast \mathcal{V} \rangle \), we have following bound: \( |h(z)| \leq \|H(z)\| \) for all \( z \in S \). From Hirschman interpolation theorem, we then have following interpolation theorem for tensor-valued function:
\[
\log \|H(\theta)\| \leq \int_{-\infty}^{\infty} \log \|H(\theta)\|^{1-\theta} \beta_{1-\theta}(t) dt + \int_{-\infty}^{\infty} \log \|H(1+\theta)\|^{\theta} \beta_{\theta}(t) dt. \tag{126}
\]
Let \( H(z) = \prod_{i=1}^{n} C_{i}^{z} \). Then the first term in the R.H.S. of Eq. (126) is zero since \( H(it) \) is a product of unitary tensors. Then we have
\[
\log \left\| \prod_{i=1}^{n} C_{i}^{\theta} \right\|^{\frac{1}{n}} \leq \int_{-\infty}^{\infty} \log \left\| \prod_{i=1}^{n} C_{i}^{1+it} \right\|^{\frac{1}{n}} \beta_{\theta}(t) dt. \tag{127}
\]
From Lemma 19 we have following relations:
\[
\left( \prod_{i=1}^{n} (\wedge^{k} C_{i})^{\theta} \right)^{\frac{1}{n}} = \wedge^{k} \left( \prod_{i=1}^{n} C_{i}^{\theta} \right)^{\frac{1}{n}}, \tag{128}
\]
and
\[
\left( \prod_{i=1}^{n} (\wedge^{k} C_{i})^{1+it} \right)^{\frac{1}{n}} = \wedge^{k} \left( \prod_{i=1}^{n} C_{i}^{1+it} \right)^{\frac{1}{n}}. \tag{129}
\]
If Eq. (127) is applied to \( \wedge^{k} C_{i} \) for \( 1 \leq k \leq r \), we have following log-majorization relation from Eqs. (128) and (129):
\[
\log \lambda \left( \prod_{i=1}^{n} C_{i}^{\theta} \right)^{\frac{1}{n}} < \int_{-\infty}^{\infty} \log \lambda \left( \prod_{i=1}^{n} C_{i}^{1+it} \right)^{\frac{1}{n}} \beta_{\theta}(t) dt. \tag{130}
\]
Moreover, we have the equality condition in Eq. (130) for \( k = r \) due to following identities:
\[
\det \left[ \prod_{i=1}^{n} C_{i}^{\theta} \right]^{\frac{1}{n}} = \det \left[ \prod_{i=1}^{n} C_{i}^{1+it} \right]^{\frac{1}{n}} = \prod_{i=1}^{n} \det C_{i}. \tag{131}
\]
At this stage, we are ready to apply Theorem 7 for the log-majorization provided by Eq. (130) to get following facts:

\[ \| f \left( \prod_{i=1}^{n} C_i^\theta \right) \|_\rho \leq \exp \int_{-\infty}^{\infty} \log \| f \left( \prod_{i=1}^{n} C_i^{1+ut} \right) \|_\rho \beta_\theta(t) dt, \] (132)

and

\[ \| g \left( \prod_{i=1}^{n} C_i^\theta \right) \|_\rho \leq \int_{-\infty}^{\infty} \| g \left( \prod_{i=1}^{n} C_i^{1+ut} \right) \|_\rho \beta_\theta(t) dt. \] (133)

From Lie product formula for tensors given by Lemma 7, we have

\[ \left| \prod_{i=1}^{n} C_i^\theta \right| \to \exp \left( \sum_{i=1}^{n} \log C_i \right). \] (134)

By setting \( \theta \to 0 \) in Eqs. (132), (133) and using Lie product formula given by Eq. (134), we will get Eqs. (122) and (123).

\[ \square \]

### 4 T-product Tensor Expander Chernoff Bound

In this section, we will begin with the derivation for the expectation bound of Ky Fan \( k \)-norm for the product of TPD tensors in Section 4.1. This bound will be used in the next Section 4.2 for preparing T-product tensor expander Chernoff bounds.

#### 4.1 Expectation Estimation for Product of T-product Tensors

In this section, we will generalize techniques used in scalar valued expander Chernoff bound proof in [40] and matrix valued expander Chernoff bound proof in [13] to build the inequality for the expectation of Ky Fan \( k \)-norm for the sequential multiplications of TPD T-product tensors. Note that our proof can remove the restriction that the summation of all mapped T-product tensors should be zero tensor, i.e., \( \sum_{v \in \mathcal{G}} g(v) = 0 \).

Let \( A \) be the normalized adjacency matrix of the underlying graph \( \mathcal{G} \) and let \( \tilde{A} = A \otimes I_{m^2 \times m^2 \times p^2} \), where the identity tensor \( I_{m^2 \times m^2 \times p^2} \) has dimensions as \( m^2 \times m^2 \times p^2 \). We use \( F \in \mathbb{R}^{(n \times m^2) \times (n \times m^2) \times p^2} \) to represent block diagonal T-product tensor valued matrix where the \( v \)-th diagonal block is the T-product tensor

\[ \mathcal{T}_v = \exp \left( \frac{tg(v)(a + ib)}{2} \right) \otimes \exp \left( \frac{tg(v)(a - ib)}{2} \right) \in \mathbb{R}^{m^2 \times m^2 \times p^2}. \] (135)

The T-product tensor \( F \) can also be expressed as

\[ F = \begin{bmatrix} \mathcal{T}_v_1 & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_v_2 & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{T}_v_n \end{bmatrix}. \] (136)
Then the T-product tensor \((\mathcal{F} \star \tilde{A})^\kappa\) is a T-product block tensor valued matrix whose \((u, v)\)-block is a tensor with dimensions as \(m^2 \times m^2 \times p^2\) expressed as:

\[
\sum_{v_1, \ldots, v_{\kappa-1} \in \mathcal{V}} A_{u, v_1} \left(\prod_{j=1}^{\kappa-2} A_{v_j, v_{j+1}}\right) A_{v_{\kappa-1}, v} \left(T_u \star T_{v_1} \star \cdots \star T_{v_{\kappa-1}}\right)
\]  

(137)

We define a \(p^2 \times p^2\) square matrix with first column as all one column, represented by \(\mathbb{I} \mathbb{O}_{p^2}\). If \(p = 2\), we have

\[
\mathbb{I} \mathbb{O}_{2^2} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]  

(138)

Given a \(m \times n\) matrix \(X = [x_1, x_2, \ldots, x_n]\) where \(x_i\) are \(m \times 1\) vectors, we define Vec\((X)\) as:

\[
\text{Vec}(X) = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}.
\]  

(139)

Then, for T-product tensor \(\mathcal{C}, \mathcal{B} \in \mathbb{R}^{m \times m \times p}\), we have following relation:

\[
\langle \mathbb{I} \mathbb{O}_{p^2} \otimes \text{Vec}^T(I_m), \text{bcirc}(\mathcal{B} \otimes \mathcal{C}) \cdot (\mathbb{I} \mathbb{O}_{p^2} \otimes \text{Vec}(I_m)) \rangle = \text{Tr}\left(\mathcal{C} \star \mathcal{B}^T\right),
\]  

where \(\cdot\) is the standard matrix multiplication.

Let \(U_0 \in \mathbb{R}^{(n \times m^2 \times p^2) \times p^2}\) be a matrix with \((n \times m^2 \times p^2)\) rows and \(p^2\) columns obtained by \(\frac{1}{\sqrt{n}} \otimes (\mathbb{I} \mathbb{O}_{p^2} \otimes \text{Vec}(I_m))\), where \(1\) is the all ones vector with size \(n\). Then, we will have following expectation of \(\kappa\) steps transition of symmetric T-product tensors from the vertex \(v_1\) to the vertex \(v_\kappa\),

\[
\mathbb{E}\left[\text{Tr}\left(\prod_{i=1}^{\kappa} \frac{t g(v_i)(a+ib)}{2} \right) \cdot \prod_{i=1}^{\kappa} \frac{t g(v_i)(a-ib)}{2}\right] = \langle U_0^T, \text{bcirc}\left((\mathcal{F} \star \tilde{A})^\kappa\right) \cdot U_0 \rangle.
\]  

(141)

If we define \(\text{bcirc}\left((\mathcal{F} \star \tilde{A})^\kappa\right) \cdot U_0\) as \(U_\kappa\), the goal of this section is to estimate \(\langle U_0^T, U_\kappa \rangle\).

The idea is to separate the space of \(U\) as the subspace spanned by the \(m^2 \times p^2\) matrices \(1 \otimes e_i\) which is denoted by \(\mathbb{U}^\parallel\), where \(1 \leq i \leq (m^2 \times p^2)\) and \(e_i \in \mathbb{R}^{(m^2 \times p^2) \times p^2}\) is the matrix with 1 in at the position \(i\) of the first column and 0 elsewhere, and its orthogonal complement space, denoted by \(\mathbb{U}^\perp\). Following lemma is required to bound how the tensor norm is changed in terms of aforementioned subspace and its orthogonal space after acting by the T-product tensor \(\mathcal{F} \star \tilde{A}\).

**Lemma 8** Given parameters \(\lambda \in (0, 1), a \geq 0, r > 0, 0, \) and \(t > 0\). Let \(G = (\mathcal{V}, \mathcal{E})\) be a regular \(\lambda\)-expander graph on the vertices set \(\mathcal{V}\) and \(\|g(v_i)\| \leq r\) for all \(v_i \in \mathcal{V}\). Each vertex \(v \in \mathcal{V}\) will be assigned a tensor \(\mathcal{T}_v\), where \(\mathcal{T}_v \equiv \frac{g(v)(a+ib)}{2} \otimes I_{m,m,p} + I_{m,m,p} \otimes \frac{g(v)(a-ib)}{2} \in \mathbb{R}^{m^2 \times m^2 \times p^2}\). Let \(\mathcal{F} \in \mathbb{R}^{(n \times m^2) \times (n \times m^2) \times p^2}\) to represent block diagonal T-product tensor valued matrix where the \(v\)-th diagonal block is the T-product tensor \(\exp(t \mathcal{T}_v) = \mathcal{T}_v\). For any matrix \(U \in \mathbb{R}^{(n \times m^2 \times p^2) \times p^2}\), we have following bounds for the spectral norm:
1. \( \left\| \left( \circ \left( \mathcal{F} \ast \lambda \right) \cdot U^\perp \right) \right\| \leq \gamma_1 \| U^\perp \| \), where \( \gamma_1 = \exp(tr\sqrt{a^2 + b^2}) \);

2. \( \left\| \left( \circ \left( \mathcal{F} \ast \lambda \right) \cdot U_{\perp}^\perp \right) \right\| \leq \gamma_2 \| U_{\perp}^\perp \| \), where \( \gamma_2 = \lambda(\exp(tr\sqrt{a^2 + b^2}) - 1) \);

3. \( \left\| \left( \circ \left( \mathcal{F} \ast \lambda \right) \cdot U_{\perp}^\perp \right) \right\| \leq \gamma_3 \| U^\perp \| \), where \( \gamma_3 = \exp(tr\sqrt{a^2 + b^2}) - 1 \);

4. \( \left\| \left( \circ \left( \mathcal{F} \ast \lambda \right) \cdot U_{\perp}^\perp \right) \right\| \leq \gamma_4 \| U^\perp \| \), where \( \gamma_4 = \lambda \exp(tr\sqrt{a^2 + b^2}) \).

**Proof:**

For Part 1, let \( 1 \in \mathbb{R}^n \) be all ones vector, and let \( U^\perp = 1 \otimes v \) for some \( v \in \mathbb{R}^{(m^2 \times p^2) \times p^2} \). Then, we have

\[
\left( \circ \left( \mathcal{F} \ast \lambda \right) \cdot U^\perp \right) = \left( \circ \left( \mathcal{F} \right) \cdot U^\perp \right) = 1 \otimes \left( \frac{1}{n} \sum_{v \in \mathbb{Z}} \circ \left( \exp(t \mathcal{F}_v) \right) \cdot v \right)
\]

and we can bound \( \frac{1}{n} \sum_{v \in \mathbb{Z}} \circ \left( \exp(t \mathcal{F}_v) \right) \) further as

\[
\left\| \frac{1}{n} \sum_{v \in \mathbb{Z}} \circ \left( \exp(t \mathcal{F}_v) \right) \right\| = \left\| \frac{1}{n} \sum_{v \in \mathbb{Z}} \sum_{i=0}^{\infty} \circ \left( \frac{t^i \mathcal{F}_v}{i!} \right) \right\|
\]

\[
= \left\| 1 + \frac{1}{n} \sum_{v \in \mathbb{Z}} \sum_{i=1}^{\infty} \circ \left( \frac{t^i \mathcal{F}_v}{i!} \right) \right\|
\]

\[
\leq 1 + \frac{1}{n} \sum_{v \in \mathbb{Z}} \sum_{i=1}^{\infty} \left\| \circ \left( \mathcal{F}_v \right) \right\|^i
\]

\[
\leq 1 + \sum_{i=1}^{\infty} \frac{(tr\sqrt{a^2 + b^2})^i}{i!} = \exp(tr\sqrt{a^2 + b^2}),
\]

where the last inequality is due to the fact that \( \left\| \frac{tg(v)\left(a+ib\right)}{2} \otimes I_{m_m,p} + I_{m,m,p} \otimes \frac{tg(v)(a-ib)}{2} \right\| \leq 2tr \times \sqrt{a^2 + b^2} \).

Then Part 1. of this lemma is established due to

\[
\left\| \left( \circ \left( \mathcal{F} \ast \lambda \right) \cdot U^\perp \right) \right\| = \sqrt{n} \left\| \frac{1}{n} \sum_{v \in \mathbb{Z}} \circ \left( \exp(t \mathcal{F}_v) \right) \cdot v \right\|
\]

\[
\leq \sqrt{n} \|v\| \exp(tr\sqrt{a^2 + b^2}) = \exp(tr\sqrt{a^2 + b^2}) \| U^\perp \|.
\]

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For Part 2, since \((bcirc(\hat{A}) \cdot U^\perp)\| = 0\), we have
\[
\left\| \left( bcirc \left( F \star \hat{A} \right) \cdot U^\perp \right) \right\| = \left\| \left( bcirc \left( (F - I) \star \hat{A} \right) \cdot U^\perp \right) \right\|
\leq \left\| bcirc \left( (F - I) \star \hat{A} \right) \cdot U^\perp \right\|
\leq \max_{v \in \mathbb{V}} \left\| bcirc \left( \exp(t \tilde{T}_v) - I \right) \right\| \left\| bcirc \left( \hat{A} \right) \cdot U^\perp \right\|
\leq \max_{v \in \mathbb{V}} \left\| \sum_{i=1}^{\infty} bcirc \left( \frac{t^i \tilde{T}_v}{i!} \right) \right\| \left\| bcirc \left( \hat{A} \right) \cdot U^\perp \right\|
\leq (\exp(tr \sqrt{a^2 + b^2}) - 1) \lambda \left\| U^\perp \right\|, 
\]
where the last inequality uses that the underlying graph \(\mathcal{G}\) is a \(\lambda\)-expander graph, i.e., \(\|Ax\| \leq \lambda \cdot \|x\|\). Therefore, Part 2 is also valid.

For Part 3, because \((U^\perp)^\perp = 0\), we have \(\left( bcirc \left( F \star \hat{A} \right) \cdot U^\perp \right)^\perp = \left( bcirc \left( F \right) \cdot U^\perp \right)^\perp = \left( bcirc \left( F - I \right) \cdot U^\perp \right)^\perp\). Then, we can upper bound as
\[
\left\| \left( bcirc \left( F - I \right) \cdot U^\perp \right) \right\| \leq \left\| bcirc \left( F - I \right) \cdot U^\perp \right\|
= \max_{v \in \mathbb{V}} \left\| bcirc \left( \exp(t \tilde{T}_v) - I \right) \right\| \cdot \left\| U^\perp \right\|
\leq \max_{v \in \mathbb{V}} \left\| \sum_{i=1}^{\infty} bcirc \left( \frac{t^i \tilde{T}_v}{i!} \right) \right\| \cdot \left\| U^\perp \right\|
\leq (\exp(tr \sqrt{a^2 + b^2}) - 1) \left\| U^\perp \right\|, 
\]
hence, Part 3 is also proved.

Finally, for Part 4, we have
\[
\left\| \left( bcirc \left( F \star \hat{A} \right) \cdot U^\perp \right)^\perp \right\| \leq \left\| bcirc \left( F \star \hat{A} \right) \cdot U^\perp \right\|
\leq \|F\| \left\| bcirc \left( \hat{A} \right) \cdot U^\perp \right\| \leq \exp(tr \sqrt{a^2 + b^2}) \lambda \left\| U^\perp \right\|, 
\]
where we use \(\|F\| \leq \exp(tr \sqrt{a^2 + b^2})\) (shown at previous part) and the underlying graph \(\mathcal{G}\) is a \(\lambda\)-expander graph.

In the following, we will apply Lemma 8 to bound the following term provided by Eq. (141)
\[
\left\langle U_0^T \cdot bcirc \left( \left( F \star \hat{A} \right)^\kappa \right) \right. \cdot U_0 \right. \right. \right. \right. \right.
\]
This bound is formulated by the following Lemma 9

**Lemma 9** Let \(\mathcal{G}\) be a regular \(\lambda\)-expander graph on the vertex set \(\mathbb{V}\), \(g : \mathbb{V} \rightarrow \mathbb{R}^{m \times m \times p}\), and let \(v_1, \cdots, v_\kappa\) be a stationary random walk on \(\mathcal{G}\). If \(tr \sqrt{a^2 + b^2} < 1\) and \(\lambda(2 \exp(tr \sqrt{a^2 + b^2}) - 1) \leq 1\), we have:
\[
\mathbb{E} \left[ \left( \prod_{i=1}^{\kappa} \exp \left( \frac{tg(v_i)(a + ib)}{2} \right) \right) \cdot \left( \prod_{i=\kappa}^{1} \exp \left( \frac{tg(v_i)(a - ib)}{2} \right) \right) \right] \leq (m^2 \times p^2) \exp \left[ \kappa \left( 2tr \sqrt{a^2 + b^2} + \frac{8}{1 - \lambda} + \frac{16tr \sqrt{a^2 + b^2}}{1 - \lambda} \right) \right].
\]
Proof: There are two phases for this proof. The first phase is to bound the evolution of tensor norms $\|U_i\|$ and $\|U_i\|$, respectively. The second phase is to bound $\gamma_i$ for $1 \leq i \leq 4$ in Lemma 8. We begin with the derivation for the bound $\|U_i\|$, where $U_i$ is the output tensor after acting by the tensor $F \star \tilde{A}$ for $i$ times. It is

$$
\|U_i\| = \left\| \left( \text{bcirc} \left( F \star \tilde{A} \right) \cdot U_{i-1} \right)^{\dagger} \right\|
\leq \left\| \left( \text{bcirc} \left( F \star \tilde{A} \right) \cdot U_{i-1}^{\dagger} \right)^{\dagger} \right\| + \left\| \left( \text{bcirc} \left( F \star \tilde{A} \right) \cdot U_{i-1} \right)^{\dagger} \right\|
\leq \gamma_3 \|U_{i-1}\| + \gamma_4 \|U_{i-1}\|
\leq 2 \left( \gamma_3 + \gamma_3 \gamma_4 + \gamma_3 \gamma_4^2 + \cdots \right) \max_{j<i} \|U_j\| \leq \frac{\gamma_3}{1-\gamma_4} \max_{j<i} \|U_j\|, \tag{150}
$$

where $\leq_1$ is obtained from Lemma 8. $\leq_2$ is obtained by applying the inequality at $\leq_1$ repeatedly. The next task is to bound $\|U_i\|$, we have

$$
\|U_i\| = \left\| \left( \text{bcirc} \left( F \star \tilde{A} \right) \cdot U_{i-1} \right)^{\dagger} \right\|
\leq \left\| \left( \text{bcirc} \left( F \star \tilde{A} \right) \cdot U_{i-1}^{\dagger} \right)^{\dagger} \right\| + \left\| \left( \text{bcirc} \left( F \star \tilde{A} \right) \cdot U_{i-1} \right)^{\dagger} \right\|
\leq \gamma_1 \|U_{i-1}\| + \gamma_2 \|U_{i-1}\|
\leq 2 \left( \gamma_1 + \frac{\gamma_3}{1-\gamma_4} \right) \max_{j<i} \|U_j\|, \tag{151}
$$

where $\leq_1$ is obtained from Lemma 8. $\leq_2$ is obtained from Eq. (150). From Eqs (141), (150) and (151), we have

$$
\mathbb{E} \left[ \text{Tr} \left( \prod_{i=1}^{\kappa} \exp \left( \frac{tg(v_i) (a+ib)}{2} \right) \right) \right]
= \langle U_0^T, U_{\kappa} \rangle \leq \|U_0\| \cdot \|U_{\kappa}\| = (m \times p) \cdot \|U_{\kappa}\|
\leq (m \times p) \left( \gamma_1 + \frac{\gamma_2 \gamma_3}{1-\gamma_4} \right)^\kappa \|U_0\| \leq (m^2 \times p^2) \left( \gamma_1 + \frac{\gamma_2 \gamma_3}{1-\gamma_4} \right)^\kappa. \tag{152}
$$

The second phase of this proof requires us to bound following four terms: $\gamma_i$ for $1 \leq i \leq 4$. Since $tr \sqrt{a^2 + b^2} < 1$, we can bound $\gamma_1$ as following:

$$
\gamma_1 = \exp(tr \sqrt{a^2 + b^2}) \leq 1 + 2tr \sqrt{a^2 + b^2}; \tag{153}
$$

$$
\gamma_2 = \lambda (\exp(tr \sqrt{a^2 + b^2}) - 1) \leq 2\lambda tr \sqrt{a^2 + b^2}; \tag{154}
$$

$$
\gamma_3 = \exp(tr \sqrt{a^2 + b^2}) - 1 \leq 2tr \sqrt{a^2 + b^2}; \tag{155}
$$

and the condition $\lambda (2 \exp(tr \sqrt{a^2 + b^2}) - 1) \leq 1$, we have

$$
1 - \gamma_4 = 1 - \lambda \exp(tr \sqrt{a^2 + b^2}) \geq \frac{1-\lambda}{2}. \tag{156}
$$
By applying Eqs. (153), (154), (155) and (156) to the upper bound in Eq. (152), we also have

\[
(m^2 \times p^2) \left( \gamma_1 + \frac{\gamma_2^2 \gamma_3}{1 - \gamma_4} \right)^\kappa \leq (m^2 \times p^2) \left[ 1 + 2(\text{tr} \sqrt{a^2 + b^2} + \frac{8\lambda_t r_\gamma(a^2 + b^2)}{1 - \lambda} \right] \leq (m^2 \times p^2) \left[ (1 + 2\text{tr} \sqrt{a^2 + b^2}) \left( 1 + \frac{8}{1 - \lambda} \right) \right] \leq (m^2 \times p^2) \exp \left[ \kappa \left( 2\text{tr} \sqrt{a^2 + b^2} + \frac{8}{1 - \lambda} + \frac{16\text{tr} \sqrt{a^2 + b^2}}{1 - \lambda} \right) \right] \quad (157)
\]

This lemma is proved.

\[ □ \]

4.2 Derivation of T-product Tensor Expander Chernoff Bound

We begin with a lemma about a Ky Fan $k$-norm inequality for the sum of T-product tensors before deriving our main result about T-product tensor expander Chernoff bound.

**Lemma 10** Let $C_i \in \mathbb{C}^{m \times m \times p}$ be symmetric T-product tensors, then we have

\[
\left\| \sum_{i=1}^m C_i \right\|^s_{(k)} \leq m^{s-1} \sum_{i=1}^m \|C_i\|^s_{(k)}
\]

where $s \geq 1$ and $k \in \{1, 2, \cdots, m \times p \}$.

**Proof:** Since we have

\[
\left\| \sum_{i=1}^m C_i \right\|^s_{(k)} = \sum_{j=1}^k \lambda_j \left( \left| \sum_{i=1}^m C_i \right|^s \right) = \sum_{j=1}^k \lambda_j^s \left( \left| \sum_{i=1}^m C_i \right|^s \right) = \sum_{j=1}^k \sigma_j^s \left( \sum_{i=1}^m C_i \right),
\]

where we have orders for eigenvalues as $\lambda_1 \geq \lambda_2 \geq \cdots$, and singular values as $\sigma_1 \geq \sigma_2 \geq \cdots$.

From Lemma 9 in [34] about majorization relation between T-product tensors sum, we have

\[
\sum_{j=1}^k \sigma_j^s \left( \sum_{i=1}^m C_i \right) \leq \sum_{j=1}^k \left( \sum_{i=1}^m \sigma_j(C_i) \right),
\]

where $k \in \{1, 2, \cdots, m \times p \}$. Then, we have

\[
\sum_{j=1}^k \sigma_j^s \left( \sum_{i=1}^m C_i \right) \leq \sum_{j=1}^k \left( \sum_{i=1}^m \sigma_j(C_i) \right)^s \leq m^{s-1} \sum_{j=1}^k \left( \sum_{i=1}^m \sigma_j^s(C_i) \right) = m^{s-1} \sum_{j=1}^k \left( \sum_{i=1}^m \sigma_j(|C_i|) \right) = m^{s-1} \sum_{j=1}^k \left( \sum_{i=1}^m \sigma_j(|C_i|^s) \right) = m^{s-1} \sum_{i=1}^m \|C_i|^s_{(k)}
\]

We are ready to present our main theorem about the T-product tensor expander bound for Ky Fan $k$-norm.
Theorem 1.1 (T-product Tensor Expander Chernoff Bound) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a regular undirected graph whose transition matrix has second eigenvalue $\lambda$, and let $g : \mathcal{V} \to \mathbb{R}^{m \times n \times p}$ be a function. We assume following:

1. A nonnegative coefficients polynomial raised by the power $s \geq 1$ as $f : x \to (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)^s$ satisfying $f \left( \exp \left( t \sum_{j=1}^{\kappa} g(v_j) \right) \right) \geq \exp \left( tf \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right)$ almost surely;

2. For each $v \in \mathcal{V}$, $g(v)$ is a symmetric T-product tensor with $f \left( \sum_{j=1}^{\kappa} g(v_j) \right)$ as TPD T-product tensor;

3. $\|g(v)\| \leq r$;

4. For $\tau \in [\infty, \infty]$, we have constants $C$ and $\sigma$ such that $\beta_0(\tau) \leq \frac{C}{\sigma \sqrt{2\pi}} \exp \left( -\frac{\tau^2}{2\sigma^2} \right)$.

Then, we have

$$\Pr \left( \left\| f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right\|_{(k)} \geq \vartheta \right) \leq \min_{t>0} \left\{ (n+1)(s-1)e^{-\vartheta t} \left[ a_0k + C \left( mp + \sqrt{(mp-k)mp} \right) \right], \right.$$

$$\left. \sum_{l=1}^{n} a_l \exp \left( 8\kappa \lambda + 2(\kappa + 8\lambda)lsr t + 2(\sigma(\kappa + 8\lambda)lsr)^2 t^2 \right) \right\},$$

(4)

where $\lambda = 1 - \lambda$.

**Proof:** Let $t > 0$ be a parameter to be chosen later, then we have

$$\Pr \left( \left\| f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right\|_{(k)} \geq \vartheta \right) = \Pr \left( \exp \left( \left\| f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right\|_{(k)} \right) \geq \exp (\vartheta t) \right)$$

$$= \Pr \left( \left\| \exp \left( tf \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right) \right\|_{(k)} \geq \exp (\vartheta t) \right)$$

$$\leq \exp (-\vartheta t) \mathbb{E} \left( \left\| \exp \left( tf \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right) \right\|_{(k)} \right)$$

$$\leq \exp (-\vartheta t) \mathbb{E} \left( \left\| f \left( \exp \left( tf \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right) \right) \right\|_{(k)} \right),$$

(162)

where equality $=1$ comes from spectral mapping theorem and TPD of $f$, inequality $\leq 2$ is obtained from Markov inequality, and the last inequality $\leq 3$ is based on our function $f$ assumption (first assumption).
From Eq. (123) in Theorem 8 we can further bound the expectation term in Eq. (162) as

\[
E \left( \left\| f \left( \exp \left( t \sum_{j=1}^{\kappa} g(v_j) \right) \right) \right\|_{(k)} \right)
\]

\[
\leq E \left( \int_{-\infty}^{\infty} f \left( \prod_{j=1}^{\kappa} \exp \left( t g(v_j) (1 + i \tau) \right) \right) \right) \left\| \beta_0(\tau) \right\|_{(k)} d\tau
\]

\[
= E \left( \int_{-\infty}^{\infty} \sum_{l=0}^{n} a_l \prod_{j=1}^{\kappa} \exp \left( t g(v_j) (1 + i \tau) \right) \right) \left\| \beta_0(\tau) \right\|_{(k)} d\tau
\]

\[
\leq 2 (n + 1)^{(s-1)} E \left( \int_{-\infty}^{\infty} \sum_{l=0}^{n} a_l \prod_{j=1}^{\kappa} \exp \left( t g(v_j) (1 + i \tau) \right) \right) \left\| \beta_0(\tau) \right\|_{(k)} d\tau
\]

\[
= (n + 1)^{(s-1)}. \quad (163)
\]

where equality \( =_1 \) comes from the function \( f \) definition, inequality \( \leq_2 \) is based on Lemma 10. Each summand for \( l \geq 1 \) in Eq. (163) can further be bounded as

\[
E \left( \left\| \prod_{j=1}^{\kappa} \exp \left( t g(v_j) (1 + i \tau) \right) \right\|_{(k)} \right) = E \left( \left\| \prod_{j=1}^{\kappa} \exp \left( t g(v_j) (1 + i \tau) \right) \right\|_{(k)} \right)
\]

\[
\leq 1 \frac{1}{mp} E \left( \text{Tr} \left( \prod_{j=1}^{\kappa} \exp \left( t g(v_j) (1 + i \tau) \right) \right) \right) +
\]

\[
E \left( \left( \frac{mp - k}{kmp} \right) \text{Tr} \left( \prod_{j=1}^{\kappa} \exp \left( t g(v_j) (1 + i \tau) \right) \right)^{2l} \right) \right)
\]

\[
\leq 2 mp \exp \left[ \kappa \left( 2 l tr \sqrt{1 + \tau^2 + \frac{8}{1 - \lambda} + \frac{16 l tr \sqrt{1 + \tau^2}}{1 - \lambda} \right) \right] +
\]

\[
\left\{ \frac{(mp - k)mp}{k} \exp \left[ \kappa \left( 4 l tr \sqrt{1 + \tau^2 + \frac{8}{1 - \lambda} + \frac{32 l tr \sqrt{1 + \tau^2}}{1 - \lambda} \right) \right] \right\}^{1/2}
\]

\[
\leq 3 \left( mp + \sqrt{\frac{(mp - k)mp}{k}} \right) \cdot \exp \left[ \kappa \left( 2 l tr (1 + \tau) + \frac{8}{1 - \lambda} + \frac{16 l tr (1 + \tau)}{1 - \lambda} \right) \right]. \quad (164)
\]

where \( \leq_1 \) comes from Theorem 2.2 in [41] and our T-product tensor trace definition provided by Eq. (18), and \( \leq_2 \) comes from Lemma 9 and the last inequality \( \leq_3 \) is obtained by bounding \( \sqrt{1 + \tau^2} \) as \( 1 + \tau \).
From Eqs. (162), (163), and (164), we have

\[
\Pr \left( \left\| f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right\| \geq \vartheta \right) \leq \min_{t>0} \left( n + 1 \right) \left( \sum_{j=1}^{\kappa} g(v_j) \right) \left( a_0k + \left( mp + \sqrt{k(mp - k)mp} \right) \right) \cdot \exp \left( -\frac{\vartheta^2}{8\sigma^2v^2} + \frac{\vartheta}{2\sigma^2v^2} - \frac{1}{2\sigma^2} + 8\kappa \lambda \right). \tag{166}
\]

where inequality \( \leq 1 \) is obtained by the distribution bound for \( \beta_0(\tau) \) via another distribution function \( \frac{C \exp(\frac{\tau^2}{2\sigma^2})}{\sigma \sqrt{2\pi}} \), and the last equality comes from Gaussian integral with respect to the variable \( \tau \) by setting \( 1 - \lambda = \bar{\lambda} \). \( \square \)

Following corollary is about a tensor expander bound with identity function \( f \).

**Corollary 1** If we consider the special case of Theorem [11] by assuming that the function \( f : x \to x \) is an identity map, then we have

\[
\Pr \left( \left\| \sum_{j=1}^{\kappa} g(v_j) \right\| \geq \vartheta \right) \leq C \left( mp + \sqrt{k(mp - k)mp} \right) \cdot \exp \left( -\frac{\vartheta^2}{8\sigma^2v^2} + \frac{\vartheta}{2\sigma^2v^2} - \frac{1}{2\sigma^2} + 8\kappa \lambda \right). \tag{166}
\]

**Proof:** From Theorem [11] since the exponent is a quadratic function of \( t \), the minimum of this quadratic function is achieved by selecting \( t \) as

\[
t = \frac{\vartheta - 2(\kappa + \bar{\lambda})r}{4\sigma^2r^2(\kappa + \bar{\lambda})^2}, \tag{167}
\]

then, we have the desired bound after some algebra by applying Eq. (167) in Eq. (4) and setting \( l = s = 1 \), all \( a_i = 0 \) for \( 1 \leq i \leq n \) except \( a_1 = 1 \). \( \square \)

## 5 Conclusions

In this work, we first build tensor norm inequalities for T-product tensors based on the concept of log-majorization, and apply these new tensor norm inequalities to derive the T-product tensor expander Chernoff bound which generalizes the matrix expander Chernoff bound by adopting more general norm for tensors,
There are several future directions that can be explored based on the current work. The first is to consider other types of T-product tensor expander Chernoff bound under other non-independent assumptions among random T-product tensors. Another direction is to characterize random behaviors of other T-product tensor related quantities besides norms or eigenvalues, for example, what is the T-product tensor rank behavior for the summation of random T-product tensors.

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