ON DONALDSON AND SEIBERG-WITTEN INVARIANTS

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Abstract. We sketch a proof of Witten’s formula relating the Donaldson and Seiberg-Witten series modulo powers of degree \( c + 2 \), with \( c = -\frac{1}{4}(7\chi + 11\sigma) \), for four-manifolds obeying some mild conditions, where \( \chi \) and \( \sigma \) are their Euler characteristic and signature. We use the moduli space of SO(3) monopoles as a cobordism between a link of the Donaldson moduli space of anti-self-dual SO(3) connections and links of the moduli spaces of Seiberg-Witten monopoles. Gluing techniques allow us to compute contributions from Seiberg-Witten moduli spaces lying in the first (or ‘one-bubble’) level of the Uhlenbeck compactification of the moduli space of SO(3) monopoles.

1. Introduction

This article consists of lightly edited notes for a lecture by the first author at the International Georgia Topology Conference 2001. Although we shall only briefly mention technical details and qualifications appropriate for more complete accounts published elsewhere [10], [11], [15], we hope that these notes provide a convenient survey of our recent work on the SO(3)-monopole program.

1.1. Witten’s conjecture. Two kinds of invariants can be used to explore the classification problem for compact, smooth 4-manifolds:

- **Donaldson invariants**, defined using an SO(3) Yang-Mills gauge theory (discovered in 1986).
- **Seiberg-Witten invariants**, defined using a U(1) monopole gauge theory (1994).

We shall restrict our attention throughout to the case of closed, oriented 4-manifolds with \( b_1 = 0 \) and odd \( b_2^+ > 1 \). The conjectured relationship between these gauge theory invariants is described below:

**Conjecture 1.1.** A 4-manifold \( X \) has KM-simple type if and only if it has SW-simple type. If \( X \) has simple type, the KM and SW basic classes coincide, and the Donaldson and Seiberg-Witten series obey

\[
D_X^w(h) = 2^{2-c(X)}e^{\frac{1}{2}Q_X(h,h)}SW_X^w(h), \quad h \in H_2(X;\mathbb{R}).
\]

Here, the 4-manifold \( X \) has intersection form

\[
Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z},
\]

Euler characteristic \( \chi \), signature \( \sigma \) and

\[
c(X) = -\frac{1}{4}(7\chi + 11\sigma).
\]

We shall recall the definitions of the Donaldson and Seiberg-Witten series shortly.

Date: This version: August 7, 2002. arXiv:math.DG/0106221

PMNF was supported in part by NSF grants DMS-9704174 and DMS-0125170.

TGL was supported in part by NSF grants DMS-0103677.
1.2. Remarks on the problem. Before proceeding to discuss our work on Witten’s conjecture, it is interesting to compare the mathematicians’ and physicists’ approaches to establishing (1).

Witten employs a certain $N = 2$ supersymmetric quantum Yang-Mills theory. He uses rescaling, $g_t = t^2 g$, of the Riemannian metric $g$ on $X$ and metric independence of the correlation functions to relate the Donaldson invariants ($t \to 0$) with the Seiberg-Witten invariants ($t \to \infty$).

The mathematical approach to a proof of Witten’s formula proposed by Pidstrigatch and Tyurin [27] instead employs an SO(3) monopole gauge theory which generalizes both the instanton and U(1) monopole gauge theories. All three gauge theories are classical field theories and their solutions are invariant under metric rescaling, whereas Witten’s quantum field theory is sensitive to metric rescaling.

Apparently, the SO(3) monopole gauge theory provides a purely classical field theory alternative to Witten’s quantum field theory method. The problem of determining the relationship between these two approaches is surely an important one worth exploring further.

2. SO(3) monopoles

2.1. Clifford modules and spin structures. Given a Riemannian metric $g$ on $X$, let $V \to X$ be a Hermitian bundle with a linear Clifford map $\rho : T^*X \to \text{End}_\mathbb{C}(V)$,

$$\rho(\alpha)^\dagger = -\rho(\alpha), \quad \text{and} \quad \rho(\alpha)^2 = -g(\alpha, \alpha), \quad \alpha \in \Omega^1(X, \mathbb{R}).$$

Then $(\rho, V)$ defines a Clifford or $\mathbb{C}\ell(T^*X)$ module structure on $V$. If $W \to X$ is a complex-rank four Hermitian bundle, then $s = (\rho, W)$ is a spin$^c$ structure, familiar from Seiberg-Witten theory [28]. If $V \to X$ is a complex-rank eight Hermitian bundle, then we call $t = (\rho, V)$ a spin$^u$ structure.

2.2. From spin$^u$ structures to SO(3) bundles. Given $t = (\rho, V)$ on $X$, one obtains

- An SO(3) subbundle, $\mathfrak{g}_t \subset \mathfrak{su}(V)$, characterized as the span of the sections $\xi$ of $\mathfrak{su}(V)$ such that $[\xi, \rho(\omega)] = 0$, for all $\omega \in \Omega^1(X, \mathbb{R})$.
- A complex line bundle, $\det^{\frac{1}{2}}(V^+)$, where $V = V^+ \oplus V^-$ and $V^\pm$ are the $\pm 1$ eigenspaces of $\rho(\text{vol})$ on $V$.
- Splittings, if $\Lambda^2(T^*X) = \Lambda^+ \oplus \Lambda^-$ and $\Lambda^\pm$ are the $\pm 1$ eigenspaces of $*_g$ on $\Lambda^2(T^*X)$, and $\rho : \Lambda^\pm \cong \mathfrak{su}(W^\pm)$ are the usual isomorphisms of SO(3) bundles, $\mathfrak{su}(V^\pm) \cong \rho(\Lambda^\pm) \oplus \rho(\Lambda^\pm) \otimes \mathfrak{g}_t \oplus \mathfrak{g}_t$.

Moreover, for any choice of spin$^c$ structure $s = (\rho, W)$, one further obtains

- A complex-rank two Hermitian bundle, $E = \text{Hom}_{\mathbb{C}\ell(T^*X)}(W, V)$.
- A Clifford module isomorphism, $V \cong W \otimes E$.
- An isomorphism of SO(3) bundles $\mathfrak{su}(E) \cong \mathfrak{g}_t$. 
• An isomorphism of complex line bundles,
\[ \det(V^+) \cong \det(W^+) \otimes \det(E). \]

2.3. **SO(3)-monopole equations.** We call a pair \((A, \Phi)\) an SO(3) *monopole* if
\[ \text{ad}^{-1}(F^+_A) - \rho^{-1}(\Phi \otimes \Phi^*)_00 = 0, \]
\[ D_A\Phi = 0, \]
where \(A\) is a spin connection on \(V\), inducing a fixed connection \(A|_{\det(V^+)} = 2A_\Lambda\) on \(\det(V^+)\) and \(\Phi\) is a section of \(V^+\); \(\hat{A}\) is the induced connection on the SO(3) bundle \(g_t \subset \mathfrak{su}(V)\); \(F^+_A\) is the self-dual component of the curvature of \(\hat{A}\); the term \((\Phi \otimes \Phi^*)_00\) is the component of \(\Phi \otimes \Phi^*\) lying in \(\rho(\Lambda^+) \otimes g_t\); \(D_A : C^\infty(V^+) \to C^\infty(V^-)\) is the Dirac operator. We let \(\mathcal{M}_t\) be the space of SO(3) monopoles for \(t = (\rho, V)\), modulo gauge transformations.

2.4. **Singularities in SO(3)-monopole space.** We now classify the fixed points of the circle action on \(\mathcal{M}_t\) given by complex multiplication on the spinor components.

An SO(3) monopole \((A, \Phi)\) is a *Yang-Mills* or *instanton* solution if \(\Phi \equiv 0\) and \(F^+_A = 0\). Hence, there is a moduli subspace of SO(3) instantons,
\[ M^\kappa_w \hookrightarrow \mathcal{M}_t, \]
where \(\kappa = -\frac{1}{2}p_1(g_t)\) and \(w \in H^2(X; \mathbb{Z})\) lifts \(w_2(g_t) \in H^2(X; \mathbb{Z}/2\mathbb{Z})\).

An SO(3) monopole \((A, \Phi)\) is a *Seiberg-Witten* or *reducible* solution if
\[ A = B \oplus B \otimes A_L \quad \text{on} \quad V = W \oplus W \otimes L, \]
for some Hermitian line bundle \(L\), a unitary connection \(A_L\) on \(L\), and \(\Phi = \Psi \oplus 0\) with \(\Psi\) a section of \(W^+\) obeying
\[ \text{Tr}(F^+_B) - \rho^{-1}(\Psi \otimes \Psi^*)_0 - F^+_A\Lambda = 0, \]
\[ D_B\Psi = 0. \]

Hence, there are moduli subspaces of Seiberg-Witten monopoles for \(s = (\rho, W)\),
\[ M_s \hookrightarrow \mathcal{M}_t, \]
whenever \(V = W \oplus W \otimes L\).

3. **Invariants of smooth 4-manifolds**

We sketch definitions of the Donaldson series \([22]\) and Seiberg-Witten series \([30]\).

3.1. **Donaldson invariants.** Set \(A(X) = \text{Sym}(H_0(X; \mathbb{R}) \oplus H_2(X; \mathbb{R}))\), so \(z \in A(X)\) is a linear combination of monomials
\[ x^m \beta_1 \beta_2 \cdots \beta_{k-2m}, \]
with \(x \in H_0(X; \mathbb{Z})\) being the positive generator and \(\beta_i \in H_2(X; \mathbb{R})\). Cohomology classes on \(M^w_\kappa\) can be defined via a map \([3], [3]\),
\[ \mu_p : H_i(X; \mathbb{R}) \to H^{4-i}(M^w_\kappa; \mathbb{R}). \]

The Donaldson invariant is then a linear function
\[ D^w_X : A(X) \to \mathbb{R}, \]
where, for a monomial \( z \) with \( \deg(z) = 2\delta \),
\[
D_X^w(z) = \langle \mu_p(z), [M^w]\rangle
\]
with \( \mu_p(z) = \mu_p(x)^m \sim \mu_p(\beta_1) \sim \cdots \sim \mu_p(\beta_{8-2m}) \).

3.2. Kronheimer-Mrowka structure theorem. One says that a 4-manifold \( X \) has KM-simple type if for some \( w \) and all \( z \in A(X) \),
\[
D_X^w(x^2z) = 4D_X^w(z).
\]
One defines the Donaldson series by setting
\[
(2)
D_X^w(h) = D_X^w((1 + \frac{1}{2}x)e^h), \quad h \in H_2(X; \mathbb{R}).
\]
We recall the celebrated

**Theorem 3.1.** If \( X \) has KM-simple type, then there exist \( a_r \in \mathbb{Q} \) and \( K_r \in H^2(X; \mathbb{Z}) \), the KM-basic classes, such that
\[
(3)
D_X^w(h) = e^{\frac{1}{2}q_X(h,h)} \sum_{r=1}^s (-1)^{\frac{1}{2}(w^2+K_r)} a_re^{K_r,h}, \quad h \in H_2(X; \mathbb{R}).
\]
See also [17], for a similar result and independent proof by different methods.

3.3. Seiberg-Witten invariants. The Seiberg-Witten invariants comprise a function,
\[
SW_X : \text{Spin}^c(X) \to \mathbb{Z},
\]
where
\[
SW_X(s) = \langle \mu_s(x)^{\max}, [M_s]\rangle,
\]
and \( \mu_s(x) \in H^2(M_s; \mathbb{Z}) \) is a cohomology class associated to a circle action. One says that a 4-manifold \( X \) has SW-simple type if for all \( s \) for which \( M_s \) has positive dimension one has that
\[
SW_X(s) = 0,
\]
and calls \( c_1(s) = c_1(W^+) \) an SW-basic class if \( SW_X(s) \neq 0 \). We define the Seiberg-Witten series by
\[
(4)
SW_X^w(h) = \sum_{s \in \text{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(s)^{-1})} SW_X(s)e^{c_1(s),h},
\]
for all \( h \in H_2(X; \mathbb{R}) \). Witten’s prediction then takes the form stated in Conjecture [11].

4. SO(3) monopole cobordism

4.1. Bubbling and Uhlenbeck compactness. In order to apply the moduli space \( \mathcal{M}_t \) of SO(3) monopoles as a cobordism, we must use a compactification. If \( \{[A_\alpha, \Phi_\alpha]\}_{\alpha \in \mathbb{N}} \subset \mathcal{M}_t \), then the sequence converges to an ideal SO(3) monopole \( ([A_\infty, \Phi_\infty], x) \) in \( M_t \times \text{Sym}^t(X) \) if
- \( (A_\alpha, \Phi_\alpha) \to (A_\infty, \Phi_\infty) \) in \( C^\infty \) on \( X \setminus \{x\} \), modulo gauge transformations,
- \( |F_{A_\alpha}|^2 \to |F_{A_\infty}|^2 + 8\pi^2 \sum_{x \in X} \delta(x) \), as measures, where \( \delta(x) \) denotes the Dirac measure centered at \( x \).

We let \( \bar{\mathcal{M}}_t \) be the closure of \( \mathcal{M}_t \) with respect to the Uhlenbeck topology, implicit above, in the space of ideal SO(3) monopoles,
\[
\bigcup_{t=0}^N \left( M_t \times \text{Sym}^t(X) \right),
\]
where \( t_\ell = (\rho, V_\ell) \) is a spin\(^w\) structure with characteristic classes

\[ p_1(g, t) = p_1(g) + 4\ell, \quad w_2(g, t) = w_2(g), \text{ and } c_1(t_\ell) = c_1(t), \]

and \( c_1(t) := c_1(\det^{1/2}(V^+)) \). The space \( \tilde{M}_t \) is smoothly stratified, with top or zeroth level \( M_t \), and lower levels \( M_\ell \times \text{Sym}^\ell(X), \ell \geq 1 \).

4.2. Stratification of the space of SO(3) monopoles. For the top level, one has a stratification

\[ \mathcal{M}_t = M^{w}_\kappa \sqcup M^{s,0}_t \sqcup \bigcup_{t=s' \in \mathbb{S}} M_s. \tag{5} \]

The complement \( M^{s,0}_t \) in \( M_t \) of the Yang-Mills and Seiberg-Witten solutions is a smooth manifold, cut out transversely by the SO(3) monopole equations \( \tilde{\mathcal{L}} \).

A stratification of the form \( \tilde{\mathcal{L}} \) arises in each level, \( M_\ell \times \text{Sym}^\ell(X) \), of the compactification \( \mathcal{M}_t \). Though dimension-counting arguments rule out contributions from the instanton moduli subspace \( M_\ell^{w,s} \) of \( M_\ell \) to pairings with the cohomology classes appearing in equation \( \tilde{\mathcal{L}} \). Seiberg-Witten moduli subspaces of \( M_\ell \) can contribute to Donaldson invariants computed using \( M^{w}_\kappa \subset \mathcal{M}_t \). To apply the cobordism, we

- define a link \( \tilde{L}^{w}_{t_\kappa} \subset \tilde{\mathcal{M}}_t/S^1 \) of the instanton moduli subspace,

\[ \tilde{M}_\kappa^{w} \subset \tilde{\mathcal{M}}_t, \]

by restricting to spinors with \( L^2 \) norm equal to a small positive constant, and

- use gluing theory \( \mathcal{L}_t \), \( \mathcal{L}_s \) to construct links \( L^{w}_{t_s} \subset \tilde{\mathcal{M}}_t/S^1 \) of ideal Seiberg-Witten moduli subspaces,

\[ M_s \times \text{Sym}^\ell(X) \subset \tilde{\mathcal{M}}_t. \]

The links \( L^{w}_{t_s} \) are considerably more difficult to construct than \( L^{w}_{t_\kappa} \), especially when \( \ell \) is large.

4.3. SO(3)-monopole cobordism formula. The cobordism \( \tilde{\mathcal{M}}_t/S^1 \) now yields the raw identity,

\[ \left\langle \mu_p(z) \sim \mu_c^{\delta_c^{-1}}, [\tilde{L}^{w}_{t_\kappa}] \right\rangle = - \sum_{s \in \text{Spin}^c(X)} \left\langle \mu_p(z) \sim \mu_c^{\delta_c^{-1}}, [L^{w}_{t_s}] \right\rangle, \tag{6} \]

where \( \mu_c \in H^2(M^{s,0}_t, \mathbb{Z}) \) is a class associated to a circle action on \( M^{s,0}_t \). One finds that

\[ \left\langle \mu_p(z) \sim \mu_c^{\delta_c^{-1}}, [L^{w}_{t_\kappa}] \right\rangle \]

is a multiple of the Donaldson invariant, \( D_X^w(z) \). The sum in \( \tilde{\mathcal{L}} \) is over all \( s \in \text{Spin}^c(X) \), with \( L^{w}_{t_s} \) empty unless \( M_s \times \text{Sym}^\ell(X) \subset \tilde{\mathcal{M}}_t \), for some \( \ell(t, s) \geq 0 \).

The difficult aspect of using \( \tilde{\mathcal{L}} \) to derive Witten’s formula \( \tilde{\mathcal{L}} \) is to show that

\[ \left\langle \mu_p(z) \sim \mu_c^{\delta_c^{-1}}, [L^{w}_{t_s}] \right\rangle \tag{7} \]

is the correct multiple of the Seiberg-Witten invariant \( SW_X(s) \); the degree of difficulty grows rapidly with \( \ell \geq 0 \). The assertion that the pairing \( \tilde{\mathcal{L}} \) is a multiple of \( SW_X(s) \) is referred to as the multiplicity conjecture. As this conjecture follows from the work in \([14]\), we shall assume it for the rest of this note.
5. Application of the cobordism

We may consider following situations, arranged in increasing order of complexity:

- There are no Seiberg-Witten moduli spaces with non-zero invariants in $\tilde{M}_t$, so the intersection $(M_b \times \text{Sym}^\ell(X)) \cap \tilde{M}_t$ is empty for all $\ell \geq 0$ and $\text{SW}_X(s) \neq 0$. The Donaldson invariants defined by $M^w_\kappa \subset \tilde{M}_t$ are then zero and, eventually, this observation leads to a vanishing result:

$$D_X^w(h) \equiv 0 \equiv \text{SW}_X^w(h) \quad (\text{mod } h^{c(X)-2}), \quad h \in H_2(X;\mathbb{R}).$$

- Calculation of contributions from $M_b \subset \tilde{M}_t$ leads to Witten’s formula, mod $h^{c(X)}$.
- Calculation of contributions from $M_b \times \text{Sym}^\ell(X) \subset \tilde{M}_t$ for $\ell = 0, 1$ leads to Witten’s formula, mod $h^{c(X)+2}$.
- Calculation of contributions from $M_b \times \text{Sym}^\ell(X) \subset \tilde{M}_t$ for $\ell = 0, 1, 2$ leads to Witten’s formula, mod $h^{c(X)+4}$.
- Calculation of contributions from $M_b \times \text{Sym}^\ell(X) \subset \tilde{M}_t$ for $\ell \geq 3$ should lead to a verification of Witten’s formula $[11]$.

We have considered the cases $\ell = 0$ and $\ell = 1$ in detail $[10], [11], [15]$ and we would expect the case $\ell = 2$ to follow in a similar manner, by exploiting work of Leness $[23]$ on the wall-crossing formula for Donaldson invariants. At present, we can compute the general shape of the contributions for $\ell \geq 3$ (see $[14]$); complete, direct computations of those contributions appear to be difficult, though we expect indirect methods will yield the desired result $[10]$.

5.1. Level-zero Seiberg-Witten contributions. Let $B \subset H^2(X;\mathbb{Z})$ be the set of Seiberg-Witten basic classes and let $B^\perp \subset H^2(X;\mathbb{Z})$ be the $Q_X$-orthogonal complement of $B$. We call a 4-manifold $X$ abundant if $Q_X|_{B^\perp}$ has a hyperbolic sublattice. Every compact, complex algebraic, simply-connected surface with $b_2^+ \geq 3$ is abundant $[10]$.

**Theorem 5.1.** $[11]$ Assume $X$ is abundant, has $b_1 = 0$, odd $b_2^+ \geq 3$, and SW-simple type. Suppose $\Lambda \in B^\perp$ exists with $\Lambda^2 = 2 - (\chi + \sigma)$. For such $\Lambda$ and $w \in H^2(X;\mathbb{Z})$ with $w - \Lambda \equiv w_2(X) \pmod{2}$ one has, for all $h \in H_2(X;\mathbb{R})$,

$$D_X^w(h) \equiv 0 \equiv \text{SW}_X^w(h) \quad (\text{mod } h^{c(X)-2}),$$

$$D_X^w(h) \equiv 2^{-c(X)}e^{\frac{1}{2}Q_X(h,h)}\text{SW}_X^w(h) \quad (\text{mod } h^{c(X)}).$$

The vanishing assertion (8) for the Seiberg-Witten series is a statement that the Moore-Mariño-Peradze conjecture holds for (abundant) 4-manifolds of SW-simple type $[11], [24], [25]$.

5.2. Level-one Seiberg-Witten contributions. With more sophisticated analytical tools, specifically gluing theory, we can compute contributions from $M_b \times X \subset \tilde{M}_t$, and these computations lead to the

**Theorem 5.2.** $[13]$ Same hypotheses as Theorem 5.1, but now suppose $\Lambda \in B^\perp$ exists with $\Lambda^2 = 4 - (\chi + \sigma)$. For such $\Lambda$ and $w \in H^2(X;\mathbb{Z})$ with $w - \Lambda \equiv w_2(X) \pmod{2}$ one has, for all $h \in H_2(X;\mathbb{R})$,

$$D_X^w(h) \equiv 0 \equiv \text{SW}_X^w(h) \quad (\text{mod } h^{c(X)-2}),$$

$$D_X^w(h) \equiv 2^{2-c(X)}e^{\frac{1}{2}Q_X(h,h)}\text{SW}_X^w(h) \quad (\text{mod } h^{c(X)+2}).$$
We expect an identity similar to (11), but mod $h^{c(X)+4}$, by computing contributions for $\ell = 2$, when $\Lambda^2 = 6 - (\chi + \sigma)$. The restrictive hypotheses on existence of classes $\Lambda$ with prescribed even squares can be dropped if one can consider contributions for arbitrary $\ell \geq 0$.

5.3. Seiberg-Witten contributions from arbitrary levels. More generally, we establish the following in [14]:

**Theorem 5.3.** Let $X$ be a closed, connected, oriented smooth four-manifold with $b_1(X) = 0$ and odd $b_2^+(X) > 1$. Let $\Lambda, w \in H^2(X; \mathbb{Z})$ obey $w - \Lambda \equiv w_2(X) \pmod{2}$. Let $\delta, m$ be non-negative integers for which $m \leq \lfloor \delta/2 \rfloor$, where $\lfloor \cdot \rceil$ denotes the greatest integer function, and

$$\delta \equiv -w^2 - \frac{3}{4}(\chi + \sigma) \pmod{4},$$

with $\Lambda$ and $\delta$ obeying $\delta < i(\Lambda)$, where $i(\Lambda) = \Lambda^2 - \frac{1}{3}(\chi + \sigma)$. Then for any $h \in H_2(X; \mathbb{R})$ and generator $x \in H_0(X; \mathbb{Z})$, we have the following expression for the Donaldson invariant:

$$D^w_X(h^{\delta-2m}, x^m) = \sum_{s \in \text{Spin}^c(X)) \min(\ell, \lfloor \delta/2 \rfloor - m)} (-1)^{\frac{1}{2}(w^2 + w \cdot c_1(s))} SW_X(s) \times \sum_{i=0}^{\ell} (p_{\delta, \ell, m, i}(c_1(s) - \Lambda, \Lambda)Q_X^i(h),$$

(12)

where $Q_X$ is the intersection form on $H_2(X; \mathbb{R})$, $\ell = \frac{1}{2}(\delta + (c_1(s) - \Lambda)^2 + \frac{3}{4}(\chi + \sigma))$ and $p_{\delta, \ell, m, i}(\cdot, \cdot)$ is a homogeneous polynomial of degree $\delta - 2m - 2i$ with coefficients which are universal functions of $\chi, \sigma, c_1(s)^2, \Lambda^2, c_1(s) \cdot \Lambda, \delta, m, \ell$.

Although Theorem 5.3 does not immediately yield Witten’s formula (1), it is still powerful enough to prove that the Seiberg-Witten invariants determine the Donaldson invariants. Furthermore, Witten’s formula (1) itself should follow from Theorem 5.3 by indirect calculations of the remaining unknown coefficients [10].

6. OUTLINE OF THE PROOFS OF THEOREMS 5.1, 5.2, AND 5.3

We shall first sketch how to compute the rough form of the pairings,

$$\langle \mu_p(z) \rangle \sim \mu_{\mathcal{L}_{t,s}}^{\delta, -1}, [\mathcal{L}_{t,s}],$$

or, at least why these pairings have the form

$$SW_X(s) \times (\text{Factors depending only on topology}).$$

We use our gluing theory [12], [13] to construct a topological model for a neighborhood in $\mathcal{M}_t$ and hence a link, $\mathcal{L}_{t,s} \subset \mathcal{M}_t/S^1$, of the ‘stratum’

$$M_s \times \text{Sym}^\ell(X) \subset \mathcal{M}_t.$$ 

Given this topological model for $\mathcal{L}_{t,s}$, we can then apply intersection theory methods to partly compute the pairings [13].

This suffices to prove the ‘rough version’ [12] of Witten’s formula (1), which is enough to show that the Seiberg-Witten invariants determine the Donaldson invariants. We shall illustrate the method below, often assuming $\ell = 1$ for the sake of simplicity [13].

The passage from this stage to Witten’s formula (1) requires us to compute the many universal, but unknown coefficients in the rough version [12] of Witten’s formula.
6.1. Neighborhood of a Seiberg-Witten stratum. Our gluing theory \[12\], \[13\] allows us to construct a model for a neighborhood of the level \( M_s \times \text{Sym}^\ell(X) \) in a local, ‘virtual’ moduli space,
\[
\mathcal{M}^\text{vir}_{t,s} := \mathcal{N}_{t,s}(\varepsilon) \times \bar{G}_s \bar{G}_t(\delta),
\]
where \( \bar{G}_t(\delta) \) is a space of instanton gluing data and \( \mathcal{N}_{t,s}(\varepsilon) \) is a radius-\( \varepsilon \) disk subbundle of the vector bundle \( \bar{G}_s \).

A neighborhood of \( M_s \times \text{Sym}^\ell(X) \) in the true moduli space \( \mathcal{M}_t \) then takes the shape
\[
\gamma(\chi^{-1}(0) \cap \mathcal{M}^\text{vir}_{t,s}) = \mathcal{M}_t \cap \gamma(\mathcal{M}^\text{vir}_{t,s}),
\]
where \( \gamma \) and \( \chi \) are described below. The stratum \( M_s \) in \( \mathcal{M}_t \) has a ‘virtual’ normal bundle,
\[
N_{t,s} \rightarrow M_s,
\]
and an obstruction bundle with section \( \chi \),
\[
\Xi_{t,s} \rightarrow \mathcal{M}_t,
\]
while the gluing map,
\[
\gamma : \mathcal{M}^\text{vir}_{t,s} \rightarrow \text{Configuration space of ideal pairs containing } \mathcal{M}_t,
\]
gives a homeomorphism from \( \chi^{-1}(0) \cap \mathcal{M}^\text{vir}_{t,s} \) onto a neighborhood of \( M_s \times \text{Sym}^\ell(X) \) in \( \mathcal{M}_t \).

6.2. Instanton component of the topological model. When \( \ell = 1 \), the instanton gluing-data component is given by
\[
\bar{G}_t(\delta) = \left( \text{Fr}(g_t) \times X \text{Fr}(T^*X) \times \mathcal{M}_t^{s,\natural}(S^4,\delta) \right)/(\text{SO}(3) \times \text{SO}(4)).
\]

Here, \( M_k^{s,\natural}(S^4,\delta) \) is the moduli space of \( k \)-instantons on \( S^4 \), framed at the south pole \( s \), with mass center at the north pole, and scale \( \leq \delta \). For \( k = 1 \) there are homeomorphisms,
\[
\mathcal{M}_t^{s,\natural}(S^4,\delta) \cong (0,\delta] \times \text{SO}(3),
\]
\[
\mathcal{M}_t^{s,\natural}(S^4,\delta) \cong c(\text{SO}(3)),
\]
where \( c(\text{SO}(3)) \) is the cone on \( \text{SO}(3) \).

Our gluing-model \[14\] admits an Uhlenbeck stratification, given below when \( \ell = 1 \):
\[
\mathcal{M}^\text{vir}_{t,s} = \mathcal{M}^\text{vir}_{t,s} \cup (N_{t,s}(\varepsilon) - M_s) \times X \cup M_s \times X.
\]

When \( \ell \geq 2 \), the symmetric product has its usual stratification and one can also construct bundles \( \bar{G}_t(\delta,\Sigma) \rightarrow \Sigma \) with instanton moduli space fibers, for each stratum \( \Sigma \subset \text{Sym}^\ell(X) \), following the prescription of Friedman and Morgan \[15\] and developing the idea of Kotschick and Morgan \[21\] and Mrowka \[26\]. The difficult part is to assemble these local gluing data bundles into a space of global gluing data, \( \bar{G}_t(\delta) \). One essentially has,
\[
\bar{G}_t(\delta) = \bigcup_{\Sigma \subset \text{Sym}^\ell(X)} \bar{G}_t(\delta,\Sigma),
\]
but some modifications of the spaces \( \bar{G}_t(\delta,\Sigma) \) are needed to carry out this construction. Unlike the analogous problem for the anti-self-dual moduli space described in \[21\], the gluing maps do not define transition maps because the images of the gluing maps for the \( \text{SO}(3) \) monopole equations intersect only at the zero-locus of the obstruction map \( \chi \). In \[14\], we define a deformation of the moduli space of anti-self-dual connections on \( S^4 \) and deformations of the splicing maps so that the intersections of the images of the deformed
Fiber bundle structure in Seiberg-Witten link pairings. The space $\tilde{Gl}_t(\delta)$ is defined to be the union of the images of these deformed splicing maps.

6.3. Link of the Seiberg-Witten stratum. We define the link in two steps, first considering the Seiberg-Witten component of the link of $M_s \times \text{Sym}^\ell(X) \subset \tilde{M}_{t,s}^{\text{vir}}/S^1$,

$$L_{t,s}^{\text{vir},s} := \left( \partial \tilde{N}_{t,s}(\varepsilon) \times_{\mathcal{G}_s} \tilde{G}l_t(\delta) \right) / S^1,$$

and, second, defining the instanton component of the link of $M_s \times \text{Sym}^\ell(X) \subset \tilde{M}_{t,s}^{\text{vir}}/S^1$,

$$\bar{L}_{t,s}^{\text{vir},i} := \left( \tilde{N}_{t,i}(\varepsilon) \times_{\mathcal{G}_s} \partial \tilde{G}l_t(\delta) \right) / S^1,$$

a complex disk bundle over $M_s \times \partial \tilde{G}l_t(\delta)/S^1$, when $\ell = 1$, and a more general fiber bundle (that is, not a product bundle) when $\ell > 1$. When $\ell = 1$, the boundary $\partial \tilde{G}l_t(\delta)$ is defined by restricting to instantons on $S^4$ with scale $\delta$. We then define the link of $M_s \times \text{Sym}^\ell(X)$ in the virtual moduli space, $\tilde{M}_{t,s}^{\text{vir}}/S^1$,

$$\bar{L}_{t,s}^{\text{vir}} := L_{t,s}^{\text{vir},s} \cup \bar{L}_{t,s}^{\text{vir},i}.$$

Finally, we obtain

$$L_{t,s}^{\text{vir}} := \gamma \left( \chi^{-1}(0) \cap \bar{L}_{t,s}^{\text{vir}} \right) = (\tilde{M}_t/S^1) \cap \gamma(\bar{L}_{t,s}),$$

the link of $M_s \times \text{Sym}^\ell(X)$ in the true moduli space $\tilde{M}_t/S^1$.

6.4. Fiber bundle structure in Seiberg-Witten link pairings. The virtual moduli space method reduces the computation of the link pairing (13) to that of the pairing on the right-hand side below,

$$\langle \mu_p(z) \sim \mu_c^\delta \circ \chi, \bar{L}_{t,s}^{\text{vir}} \rangle = \langle \mu_p(z) \sim \mu_c^\delta \sim \varepsilon, [\bar{L}_{t,s}^{\text{vir}}] \rangle,$$

where $\varepsilon$ is Euler class of the total obstruction bundle over $\tilde{M}_{t,s}^{\text{vir}}$, with section $\chi$, and $\bar{L}_{t,s}^{\text{vir}} \cong \chi^{-1}(0) \cap \bar{L}_{t,s}^{\text{vir}}$.

When $\ell = 1$, we show in [13] that the pairing (17) with $[\bar{L}_{t,s}^{\text{vir}}]$ is expressible in terms of pairings with

$$[M_s \times \partial \tilde{G}l_t(\delta)/S^1].$$

In particular, we can show that the latter pairings are in turn products of

- Pairings with $[M_s]$, giving multiples of the Seiberg-Witten invariant, $\text{SW}_X(s)$, and
- Pairings with $[\partial \tilde{G}l_t(\delta)/S^1]$.

The pairings with $[\partial \tilde{G}l_t(\delta)/S^1]$ depend only the topology of the 4-manifold, $X$, and universal data.

When $\ell > 1$, the construction of the space

$$\tilde{N}_{t,s}(\delta) \times_{\mathcal{G}_s \times S^1} \tilde{G}l_t(\delta),$$

shows that the virtual link $\bar{L}_{t,s}^{\text{vir}}$ admits a fiber bundle structure

$$\bar{L}_{t,s}^{\text{vir}} \rightarrow M_s,$$

with fiber given by $\mathbb{C}^n \times_{S^1} \tilde{G}l_t(\delta)$ and structure group given by $\text{Map}(X, S^1)$. Hence, the pairing (17) can be written as products of

- Pairings with $[M_s]$, giving multiples of the Seiberg-Witten invariant, $\text{SW}_X(s)$,
- Pairings with the fiber $\mathbb{C}^n \times_{S^1} \tilde{G}l_t(\delta)$. 
In [14], we show how the pairings with the fiber $\mathbb{C}^n \times S^1 \text{Gl}_k(\delta)$ can be qualitatively understood in terms of homotopy data of the spin$^c$ structure $t_\ell$, of the spin$^c$ structure $s$, and of the manifold $X$.

The passage from the rough version (12) to Witten’s formula (1) to the exact formula is discussed in [16].

7. Witten’s conjecture and symplectic 4-manifolds

7.1. Gauge theory and Lefschetz fibrations. Away from finitely many critical points, a Lefschetz fibration $\pi : X \to S$ is smooth fiber bundle over a connected base, whose fibers are closed Riemann surfaces, $\Sigma$, of given genus. Possibly after blowing up, all symplectic 4-manifolds admit Lefschetz fibrations [1], [3], [4].

One can ask what is the relationship between gauge theoretic invariants and the Lefschetz fibration structure. As a first step, one could use the product $X = \Sigma \times S$, as a toy model. In this situation, the relationship between the gauge theory moduli spaces and holomorphic maps can be explored via the following techniques:

- Adiabatic limit analysis, by the scaling metric $g = g_\Sigma \oplus g_S$ as $g_\epsilon = \epsilon^2 g_\Sigma \oplus g_S$, with $\epsilon \to 0$, or
- Restriction of stable, holomorphic bundles over $\Sigma \times S$ to $\Sigma \times \{z\}$, as $z \in S$ varies.

This leads to the following identifications:

1. SO(3)-instantons over $\Sigma \times S$ are identified with holomorphic maps, $S \to M_\Sigma$, where $M_\Sigma$ is the space of flat SO(3) connections over $\Sigma$. This identification goes back to Dostoglou and Salamon [7].
2. Seiberg-Witten U(1)-monopoles over $\Sigma \times S$ are identified with holomorphic maps, $S \to M_{\Sigma,d}$, where $M_{\Sigma,d}$ is the space of vortices on the line bundle $L \to \Sigma$ with $d = \langle c_1(L), [\Sigma] \rangle$. This identification has been outlined by Salamon [29].
3. SO(3)-monopoles over $\Sigma \times S$ are identified with holomorphic maps from $S$ to a space of non-abelian vortices over $\Sigma$. Non-abelian vortices of this kind have been studied by Bradlow, Garcia-Prada, and others.

Of course, there are many unanswered questions:

- What is the relationship between the gauge theory compactifications and the compactifications of spaces of holomorphic maps?
- What is the relationship between the gauge theory invariants and Gromov-Witten invariants of spaces of flat connections or vortices over Riemann surfaces?
- Can one extend the analysis for the toy model, $\Sigma \times S$, to the case of non-trivial surface bundles or Lefschetz fibrations?
- Can symplectic 4-manifolds provide additional data we can use to help determine Witten’s formula?

We note that the relationship between Seiberg-Witten invariants and Lefschetz fibration data bears on a closely related program due to Simon Donaldson and Ivan Smith whose aims include giving Lefschetz-fibration proofs of results for symplectic 4-manifolds derived via Seiberg-Witten theory [3].

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