Parameter scaling in a novel measure of quantum-classical difference for decohering chaotic systems

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In this paper we introduce a diagnostic for measuring the quantum-classical difference for open quantum systems, which is the normalized size of the quantum terms in the Master equation for Wigner function evolution. For a driven Duffing oscillator, this measure shows remarkably precise scaling over long time-scales with the parameter \( \zeta_0 = \hbar^2/D \). We also see that, independent of \( \zeta_0 \) the dynamics follows a similar pattern. For small \( \zeta_0 \) all of our curves collapse to essentially a single curve when scaled by the maximum value of the quantum-classical difference. In both limits of large and small \( \zeta_0 \) we see a saturation effect in the size of the quantum-classical difference; that is, the instantaneous difference between quantum and classical evolutions cannot be either too small or too large.

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The quantum–classical transition for open quantum systems is important to understand for fundamental and practical reasons, including the design of quantum computers. For quantum systems with classical analogues permit chaos this transition is particularly interesting because these systems often display unusual quantum effects, such as rapid entanglement generation\textsuperscript{(1)} and hyper-sensitivity to perturbation\textsuperscript{(2)}, and it would be useful to understand when such effects appear. It is an inherently multi-parameter transition depending on the relative size of \( \hbar \) compared to the characteristic action and on the strength of the interaction between the system and its environment as measured by some parameter \( D \). The chaotic behavior of the classical limit is also crucial, as seen in their decoherence dynamics argued to be related to the Lyapunov exponents of the system. An efficient approach that allows for broader conclusions is to use\textsuperscript{(b)} scaling in composite parameters, which indicates\textsuperscript{(b)} that the quantum-classical difference as measured by some quantity \( QC_D(\hbar, D, \lambda) \) is a function \( QC_D(\zeta) \) of a single composite parameter \( \zeta = \hbar^2 D^2 \lambda^2 \).

A good measure \( QC_D \) is intuitive, global, and easy to compute. A Kullback-Liebler-like distance between classically propagated and quantally propagated distributions picks out scaling properties clearly\textsuperscript{(b)} when computed for a 2-phase-space dimension map. Overlaps functions are however, difficult to compute in greater dimensions, or for flows. Studying the quantum dynamics entropy \( S_2 = \ln(P) \), where \( P = \text{Tr}(\rho^2) \) also yielded useful insights. However, the scaling did not last very long in that instance, and the direct relationship to quantum-classical difference is not clear; the use of the entropy has also been otherwise critiqued\textsuperscript{(b)}. We report here on a measure that is (a) time-dependent, (b) explicitly measures the difference between quantum and classical evolution, and (c) is calculated from the evolution of a single distribution. This measure shows remarkable scaling, in parameter space and over long time-scales, and we consequently uncover interesting insights into unexpected behavior in the dynamics. Specifically, in the classical limit, the relative size of the quantum and classical terms saturates, so that quantum-classical differences continue to be propagated, rather than decreasing with time as might be naively expected. Conversely, at the near-quantal limit of small \( D \) and large \( \hbar \) the quantum terms remain comparable to the classical instead of dominating. Taken together, this quantifies the smoothness of the quantum-classical transition for open systems compared to closed systems.

We start by considering the Master equation for a Wigner function \( \rho_W \), or quantum quasi-probability, evolved under Hamiltonian flow with potential \( V(q) \) while coupled to an external environment\textsuperscript{(4)}:

\[
\frac{\partial \rho_W}{\partial t} = L_c + L_q + T
\]

\[
= \{H, \rho_W\} + \sum_{n \geq 1} \frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!} \frac{\partial^{2n+1} V(q)}{\partial q^{2n+1}} \frac{\partial^{2n+1} \rho_W}{\partial p^{2n+1}} + D \nabla^2 \rho_W
\]  \hspace{1cm} (1)

The Poisson bracket \( L_c \) generates the classical evolution for \( \rho_W \), the quantal \( \hbar \) terms are denoted by \( L_q \) and the environmental coupling \( T \) is modeled by a diffusive term with coefficient \( D \). The computational results presented...
below use coupling only to the momentum variables. For analytical simplicity, we assume coupling to all phase-space variables, justified since the dynamical chaos mixes the various phase-space directions.

The measure we propose in this paper is a normalized average of the square of $L_q$ (see Eq. (1) for the precise definition). The square before the averaging is necessary to compensate for the arbitrary sign of $L_q$ in different regions of phase-space. For example, in the driven Duffing problem that we study below and which is given by $H = p^2/2m - Bx^2 + (C/2)x^4 + Ax \cos(\omega t)$ the quantity $\text{Tr}[L_q^2]$ is identically 0. The normalization makes the measure dimensionless, yielding

$$G(t) = \frac{\text{Tr}[L_q^2]}{\text{Tr}[(\partial_t \rho_w(t))^2]}$$

Physically $G$ is the relative size of the quantum part of the evolution for the Wigner function, and it can exceed unity through partial cancelation between $L_q$ and the Poisson bracket in the evolution equation for $\rho_w$. Note that $G$ does not measure the classicality of a state, but rather the classicality of its evolution and is analogous to the time-derivative of previous-used measures.

In Fig. 1 we plot $G(t)$ for various combinations of $\hbar$ and $D$ for a Duffing oscillator with the parameters $(m = 1, B = 10, C = 1, A = 1, \omega = 5.35)$. These results are calculated starting with a minimum uncertainty Gaussian Wigner function that is well localized in the chaotic region and propagating it using Eq. (1) as has been previously done. The figure shows remarkable scaling behavior in the measure $G(t)$. That is, the dynamics of the quantum-classical difference are seen to depend only on the composite parameter $\zeta_0 = h^2/D$ over a factor of 800 in its value. Although such behavior has been seen earlier for short times with the entropy, in this case the scaling dynamics lasts for the full time-scale monitored by us.

The absolute size of $G$ grows with $\zeta_0$, and is much greater for $\zeta_0 \approx 40$ compared to $\zeta_0 \approx 0.05$, which is reassuringly physically intuitive. Specifically, the classical limit is the regime when $\zeta_0 \leq 0.2$, where $G$ is always small that the differences between the evolution of the Wigner function $\rho_W$ and its classical counterpart $\rho_C$ are negligible.

Independent of the value of $\zeta_0$, we see similar dynamical behavior with identical stages: We start with (i) a rapid increase in $G$, which can be understood as essentially the behavior of a closed quantum system since the gradients of the distribution have not increased enough for the diffusive term to become relevant. This is followed by (ii) a turnover and an exponential decrease in $G$ as the distribution starts filling the phase-space and the diffusive terms kick in – the overall distribution continues to evolve, but the relative size of the quantal terms is now decreasing as a result. Finally, (iii) at long times $G$ saturates; this happens because the distribution has almost relaxed to its final state. At this point final stage, both $\text{Tr}[L_q(t)^2]$ and $\text{Tr}[\rho_w(t)^2]$ decrease exponentially with time. That is, at longer times, the quantal and classical contributions to the evolution of the Wigner function reach a steady-state ratio.

To examine the functional dependence of $G(t)$ on $\zeta_0$, we start at $\zeta_0 \rightarrow 0$, the classical limit for open quantum systems. Fig. 2 shows that in the classical limit the greatest value of $G(t)$ for small $\zeta_0$ is approximately

$$\max G(t) \approx 25 \left(1 - \exp(-0.08\zeta_0^2)\right).$$

FIG. 1: Quantum-classical difference as measured by $(G(t))$, which is $G(t)$ averaged over one driving period, for various initial parameters. The $y$-axis is logarithmic. Notice the remarkable scaling with the composite parameter $\zeta_0 = h^2/D$. For the initial Gaussian to be well-localized in the chaotic region, $\sigma_q^2 \leq 0.05$. Hence for our calculations, we set $\sigma_q^2 = 0.05$ and determine $\sigma_p^2$ from the constraint, $\sigma_q \sigma_p = \hbar/2$, imposed by minimum-uncertainty condition.
FIG. 2: This plot shows in when we re-scale \( G(t) \) by dividing the data for \( \zeta \leq 2 \) in Figure 1 by the maximum value of \( G(t) \), the data collapses to approximately a single curve. Also this data shows that the maximum value of \( G(t) \) as a function is described very well by a Gaussian in \( \zeta_0 \).

That is, the maximum value of \( G(t) \) scales approximately quadratically with \( \zeta_0 \) in the limit of small \( \zeta_0 \) and is only a function of \( \zeta_0 \). This is a remarkably time-independent relationship, and in Fig. 2 we show that for \( \zeta_0 \leq 2 \), \( G(t) \) collapses to essentially a single function of \( \zeta_0 \) when we divide through by the maximum value given in Eq. (3).

The transition out of the near-classical regime starts at \( \zeta_0 \approx 2 \). As \( \zeta_0 \) increases further, we expect quantum effects to increase and might naively predict that \( \text{Tr} L^2_t \approx \text{Tr}(\partial_t \rho_{w}(t))^2 \) before exceeding it and then becoming the dominant term. However, the saturation effects in Fig. 4 indicate otherwise. Further, plots (Fig. 4) of the absolute value of the Wigner function for the Duffing Oscillator evaluated at the times \( t = 5 \) and 20 for \( \zeta_0 = 0.2 \) and \( \zeta_0 = 10 \), clearly show that as quantum interference becomes more substantial, the amplitude of interference fringes approaches that of the classical phase space structure underlying it. However, these classical structures (in this case the noise-broadened homoclinic tangle of the stable and unstable manifolds) always retain a substantial contribution, again pointing to the wisdom of considering the classical dynamics when thinking about quantum chaotic systems.

This saturation effect is examined in greater detail in Fig. 4 in which it is shown that the observed saturation effect occurs because the magnitude of \( \text{Tr} L^2_t \) becomes comparable to \( \text{Tr} L^2_q \), while the diffusion term remains negligible. This saturation can be understood in the asymptotic regime by considering the influence of quantum interference effects on \( \rho_w \).

Quantum mechanical effects often appear in physical systems as interference patterns with characteristic wavenumbers that scale with \( \hbar^{-1} \). Therefore we expect that if \( \hbar^2/D \) is large we expect that \( \partial_p \rho_w \sim \hbar^{-1} \) and \( \partial_x \rho_w \sim \partial_p \rho_w \sim \hbar^{-1} \), \( \hbar^2 \partial^3_p \rho_w \sim \hbar^{-1} \) and \( D \partial^2_p \rho_w \sim D \hbar^{-1} \) for a chaotic system. Hence the quantal terms should be comparable to the classical in the limit of large \( \zeta_0 \) if \( \rho_w \) is dominated by interference effects, which is reasonable for our \( \zeta_0 = 10 \) data according to Fig. 4.

To summarize, we considered the normalized size of the quantum terms in the Master equation for Wigner function evolution in an open quantum system. For a driven Duffing oscillator, this measure shows remarkably precise scaling over long time-scales with the parameter \( \zeta_0 = \hbar^2/D \). We also see that, independent of \( \zeta_0 \) the dynamics follows a similar pattern. For small \( \zeta_0 \) all of our curves collapses to essentially a single curve when scaled by the maximum value of the quantum-classical difference. In both limits of large and small \( \zeta_0 \) we see a saturation effect in the size of the quantum-classical difference; that is, the instantaneous difference between quantum and classical evolutions cannot be either too small or too large. This further confirms the growing intuition that decoherence softens the quantum-classical transition for nonlinear systems. Open questions include whether this remarkable scaling of \( G(t) \) with \( \zeta_0 \) is exhibited for other Hamiltonians, such as the driven rotor, whose Moyal series does not terminate.

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FIG. 3: This plot shows the absolute value of $\rho_w$ for $\zeta_0 = 0.2$ ($\hbar = 0.01, D = 5 \times 10^{-4}$) and $\zeta_0 = 10$ ($\hbar = 0.125, D = 1.5625 \times 10^{-3}$) evaluated at times ranging from $t = 5$ to $t = 20$. The Wigner function for $\zeta_0 = 10$ shows clear signs of quantum interference, but the data for $\zeta_0 = 0.2$ shows no visible signs of quantum behavior at the times considered. This reflects the observation that $G(t)$ is a slowly decreasing function for $\zeta_0 = 10$ because the quantum effects plotted are also persistent with time.

FIG. 4: This plot shows that for $\zeta_0 = 10$ with ($\hbar = 0.125, D = 1.5625 \times 10^{-3}$) the classical terms are roughly comparable to the quantal terms. For $\zeta_0 = 0.2$ with ($\hbar = 0.01, D = 5 \times 10^{-4}$) the classical terms are substantially larger than the quantum terms. The diffusion term is negligible compared to the classical in both cases.

[1] S. Ghose, R. Stock, P. Jessen, L. Roshan, and A. Silberfarb, Phys. Rev. A 78, 042318 (2008); C.M. Trail, V. Madhok, and I.H. Deutsch Phys. Rev. E 78, 046211 (2008).
[2] F. Haake, “Quantum Signatures of Chaos” (Springer-Verlag, Berlin, 1991).
[3] D. Montecilva and J.P. Paz, Phys. Rev. Lett. 85, 3373 (2000).
[4] W.H. Zurek and J.P. Paz, Phys. Rev. Lett. 72, 2508 (1994); Physica 83 D, 300 (1995).
[5] A.K. Pattanayak, B. Sundaram, and B. D. Greenbaum, Phys. Rev. Lett. 90, 014103 (2003).
[6] N. Wiebe and L. Ballentine, Phys. Rev. A 72, 022109 (2005); F. Toscano et al., Phys. Rev. A 71, 010101 (R) (2005); A.R.R. Carvalho et al., Phys. Rev. E 70, 026211 (2004).
[7] A. Gammal and A. K. Pattanayak, Phys. Rev. E 75, 036221 (2007).
[8] D. A. Wisniacki and F. Toscano, Phys. Rev. E 79, 025203(R) (2009).