A coset-type construction for the deformed Virasoro algebra

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Abstract

An analog of the minimal unitary series representations for the deformed Virasoro algebra is constructed using vertex operators of the quantum affine algebra $U_q(\widehat{sl}_2)$. A similar construction is proposed for the elliptic algebra $A_{q,p}(\widehat{sl}_2)$.

1 Introduction

The deformed Virasoro algebra (DVA) introduced in [1] is presented in terms of a generating series $T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n}$ as follows.

$$f \left( \frac{z_2}{z_1} \right) T(z_1) T(z_2) - T(z_2) T(z_1) f \left( \frac{z_1}{z_2} \right) = \frac{(x^{r-1} - x^{-r+1})(x^r - x^{-r})}{x - x^{-1}} \left( \delta \left( x^{-2} \frac{z_1}{z_2} \right) - \delta \left( x^2 \frac{z_2}{z_1} \right) \right), \quad (1)$$

$$f(z) = \frac{1}{1 - z} \frac{(x^{2r} z; x^4 \infty)(x^{-2r+2} z; x^4 \infty)}{(x^{2r+2} z; x^4 \infty)(x^{-2r+4} z; x^4 \infty)}. \quad (2)$$

Here $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ and

$$(z; p_1, \cdots, p_k)_{\infty} = \prod_{n_1, \cdots, n_k = 0}^{\infty} (1 - z p_1^{n_1} \cdots p_k^{n_k}).$$

In the limit $x \to 1$, the Virasoro algebra is recovered with the central charge

$$c = 1 - \frac{6}{r(r - 1)}.$$
See also [2] for a formulation from the viewpoint of chiral vertex algebras.

A bosonic realization of DVA was presented in [1]. An analog of the minimal unitary series representations was constructed in the work [3] using the BRST complex. (To our knowledge, however, a complete proof about its cohomological structure is yet unavailable.) Apart from these, little is known about representations of DVA. In this note we give an alternative construction of the latter representations based on vertex operators (VO’s) of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$. The method is elementary as explained below.

Recall the analogs of the simplest chiral primary fields $\phi_{12}(z), \phi_{21}(z)$ introduced in [3]: $\phi_{21}(z)$ is the type I VO’s (half transfer matrices), while $\phi_{12}(z)$ is the type II VO’s (particle creation/annihilation operators). (In the body of the text, we will employ the notation $\psi(z)$ for $\phi_{12}(z)$.) In the conformal case, we have the well known fusion rule

$$\phi_{12}\phi_{12} \sim I + \phi_{13}. \quad (3)$$

In the deformed case, the singularity of $\rho(z_1/z_2)^{-1}\phi_{12}(z_1)\phi_{12}(z_2)$ (with an appropriate scalar factor $\rho(z)$ [20]) consists of a series of poles

$$\frac{z_2}{z_1} = x^2, x^{-2r+4}, x^{-4r+6}, \ldots$$

As an analog of (3), we have

$$\frac{1}{\rho(z_1/z_2)}\phi_{12}(z_1)\phi_{12}(z_2) = \frac{id}{1 - x^{-2}z_2/z_1} + O(1) \quad (z_1 \to x^{-2}z_2). \quad (4)$$

The DVA current $T(z)$ appears at the second pole $z_2/z_1 = x^{-2r+4}$ [4]:

$$\frac{1}{\rho(z_1/z_2)}\phi_{12}(z_1)\phi_{12}(z_2) = \text{const.} \cdot \frac{T(z)}{1 - x^{-2r-4}z_2/z_1} + O(1) \quad (z_1 \to x^{-2r+4}z_2). \quad (5)$$

This fact is naturally expected from (4), since in the conformal case the descendant of the identity operator is the Virasoro current itself.

The formula (3) is easy to establish directly in the bosonic realization. Alternatively, we can regard (3) as defining the DVA current in terms of $\phi_{12}(z)$. As we will show, the commutation relations (4) can be extracted from those of $\phi_{12}(z)$ and the structure of its poles, without invoking the explicit bosonic expression of $T(z)$. In [3] was given a construction of $\phi_{12}(z)$ which utilizes the $q$-VO’s for the quantized affine algebra $U_q(\hat{\mathfrak{sl}}_2)$.

The above procedure then gives rise to a representation of DVA on the tensor product of integrable modules of $U_q(\hat{\mathfrak{sl}}_2)$ (see Proposition 3.3). In the simplest case, this affords $c = 1/2$ representations of DVA in terms of a quadratic expression of free fermions. The present method is similar in spirit to the coset construction of the Virasoro algebra. It
should be mentioned, however, that the analog of the Sugawara operators for DVA is unknown.

The same recipe can be applied to the VO’s for the elliptic algebra $\mathcal{A}_{q,p}(\hat{\mathfrak{sl}}_2)$, assuming their existence and expected analyticity properties. In particular, at the Ising point, this leads to a fermionic representation of DVA at $c = -2$. We will comment on these in the last section.

2 Vertex operators for $U_q(\hat{\mathfrak{sl}}_2)$

2.1 Notation

Let $\Lambda_0, \Lambda_1$ be the fundamental weights for $U_q(\hat{\mathfrak{sl}}_2)$, and set $\rho = \Lambda_0 + \Lambda_1$. A dominant integral weight of level $k$ has the form

$$\lambda = \lambda_l = (k + 2 - l)\Lambda_0 + l\Lambda_1 - \rho, \quad 1 \leq l \leq k + 1. \quad (6)$$

The corresponding integrable highest weight module is denoted by $V(\lambda)$. Let further $V_z$ denote the evaluation module based on a finite dimensional module $V$.

The VO’s of type I and type II are intertwiners of $U_q(\hat{\mathfrak{sl}}_2)$ modules

$$\tilde{\Phi}^{(\mu,\lambda)}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z, \quad (7)$$

$$\tilde{\Psi}^{*(\eta,\xi)}(z) : V_z \otimes V(\xi) \longrightarrow V(\eta). \quad (8)$$

We set

$$\Phi^{(\mu,\lambda)}(z) = \tilde{\Phi}^{(\mu,\lambda)}(z)z^{\Delta_\mu - \Delta_\lambda}, \quad \Psi^{*(\eta,\xi)}(z) = \tilde{\Psi}^{*(\eta,\xi)}(z)z^{\Delta_\eta - \Delta_\xi}.$$

For a weight $\lambda$, $\Delta_\lambda$ is given by

$$\Delta_\lambda = \frac{(\lambda_l, \lambda_l + 2\rho)}{2(k + 2)} = \frac{l^2 - 1}{4(k + 2)}.$$

We will mostly follow the conventions of [1], wherein $\Phi^{(\mu,\lambda)}(z)$ and $\Psi^{*(\eta,\xi)}(z)$ are written as $\Phi^{\mu V}_V(z)$ and $\Phi^{\eta \xi}_V(z)$, respectively. Sometimes we drop the superscripts and write $\Phi(z), \Psi^{*}(z)$, regarding them as operators on the direct sum $\oplus_{l=1}^{k+1}V(\lambda_l)$.

In what follows we shall focus attention to the case

$$V = \mathbb{C}^2, \quad \xi, \eta \text{ has level one}.$$

The components of VO’s with respect to the natural basis $v_+, v_- \in V$ are defined by $\Phi(z) = \sum_\varepsilon \Phi_\varepsilon(z) \otimes v_\varepsilon$ and $\Psi^{*}_\varepsilon(z) = \Psi^{*}(z) (v_\varepsilon \otimes \cdot)$. We set $Pv \otimes v' = v' \otimes v$, and denote by $\tilde{R}(z)$ the standard $R$-matrix for the two-dimensional module (e.g. p.70 in [1]).
2.2 Properties of $\Phi(z), \Psi^*(z)$

We set

$$r = k + 3, \quad x = -q, \quad p^* = x^{2(r-1)}, \quad \{z\} = (z; x^4, p^*)_\infty,$$

$$\eta_I(z) = \frac{\{p^* x^2 z\}^2}{\{p^* x^4 z\} \{p^* z\}}, \quad \eta_{II}(z) = \frac{(x^2 z; x^4)_\infty}{(z; x^4)_\infty},$$

$$\rho_I(z) = z^{\frac{1}{2}} \frac{\eta_I(z)}{\eta_I(z_{-1})}, \quad \rho_{II}(z) = z^{\frac{1}{2}} \frac{\eta_{II}(z)}{\eta_{II}(z_{-1})}.$$ 

From the theory of $q$-KZ equation we know the following facts.

(a1) The product

$$\prod_{i<j} \eta_I(z_j/z_i) \cdot \Phi(z_1) \cdots \Phi(z_N)$$

is meromorphic, the only singularities being simple poles at $z_j/z_i = p^{s-i}x^2$, $i < j$, $s = 1, 2, \ldots$. Likewise

$$\prod_{i<j} \eta_{II}(z_j/z_i) \cdot \Psi^*(z_1) \cdots \Psi^*(z_N)$$

is meromorphic, the only singularities being simple poles at $z_j/z_i = x^2$, $i < j$.

(a2)

$$P R(z_1/z_2) \Phi^{(\nu, \mu)}(z_1) \Phi^{(\mu, \lambda)}(z_2)$$

$$= \rho_I \left( \frac{z_1}{z_2} \right) \sum_{\mu'} \Phi^{(\nu, \mu')}(z_2) \Phi^{(\mu', \lambda)}(z_1) W \left( \frac{\lambda}{\mu'} \left| \frac{z_1}{z_2} \right. \right),$$

$$\Psi^*(z_1) \Psi^*(z_2) \left( P R(z_1/z_2) \right)^{-1} = \rho_{II} \left( \frac{z_1}{z_2} \right) \Psi^*(z_2) \Psi^*(z_1).$$

(a3)

$$\sum_{\epsilon} x^{-\epsilon/2} \Phi^{(\nu, \mu)}(x^{-2} z) \Phi^{(\mu, \lambda)}(z) = x^{2(\Delta_\mu - \Delta_\lambda) - 1/2} g^\mu_\lambda \delta_{\nu\lambda} \times \text{id},$$

$$\text{res}_{z_1=x^{-2}z_2} \Psi^*_{\epsilon_1}(z_1) \Psi^*_{\epsilon_2}(z_2) \frac{dz_1}{z_1} = g x^{\epsilon_2/2} \delta_{\epsilon_{1}+\epsilon_{2},0} \times \text{id},$$

where $g = (x^2; x^4)_\infty/(x^4; x^4)_\infty$ and $g^\mu_\lambda$ are constants given in eq.(B3), $\sum$ (there $r_\pm$ is misprinted as $r_\mp$).

\[\text{By abuse of language, we say a function is ‘meromorphic’ if it is a product of a meromorphic function and power functions in the coordinates } z_1, \cdots, z_N.\]
In (11),
\[
W\left(\begin{array}{ccc}
\lambda & \mu & z \\
\mu' & \nu &
\end{array}\right) = \overline{W}_k\left(\begin{array}{ccc}
\lambda & \mu & z \\
\mu' & \nu &
\end{array}\right) \times z^{\Delta_N + \Delta_N' - \Delta_{\mu' - \Delta_{\mu} - 1/2(k+2)}},
\]
(15)
denotes the Boltzmann weights for the RSOS model (see Appendix B, eq. (B.2) in [6]). We shall be concerned with the following properties rather than their explicit formulas.

(i) \(W\left(\begin{array}{ccc}
\lambda & \mu & z \\
\mu' & \nu &
\end{array}\right)\) is meromorphic on \(\mathbb{C}\setminus\{0\}\) with simple poles at \(z = x^{-2}p^s (s \in \mathbb{Z})\),

(ii) We have the periodicity
\[
W\left(\begin{array}{ccc}
\lambda & \mu & p^s z \\
\mu' & \nu &
\end{array}\right) = W\left(\begin{array}{ccc}
\lambda & \mu & z \\
\mu' & \nu &
\end{array}\right),
\]
(16)

(iii) At \(z = x^{-2}\) we have
\[
\text{res}_{z=x^{-2}} W\left(\begin{array}{ccc}
\lambda & \mu & z \\
\mu' & \nu &
\end{array}\right) \frac{dz}{z} = \delta_{\nu\lambda}a_{\mu\lambda}b_{\mu'\lambda},
\]
(17)
with some constants \(a_{\mu\lambda}, b_{\mu'\lambda}\).

Notice that the \(W\) factor for \(\Psi^*(z)\) is simply a scalar. This is a reflection of the fact that the two-dimensional module \(V\) is ‘perfect’ (in the sense of crystal base theory) for level one representations.

Property (a3) follows from the fact that \(\text{res}_{z=x^{-2}} P\overline{R}(z)dz/z\) is proportional to the projector onto the trivial module
\[
V_{x^{-2}z} \otimes V_z \to \mathbb{C}.
\]

3 Coset-type construction

3.1 Vertex operators of SOS type

Following [8], let us introduce the operator \(\psi^{(\mu,\lambda;1-i,i)}(z)\) by the composition
\[
V(\lambda) \otimes V(\Lambda_i) \longrightarrow V(\mu) \otimes V_z \otimes V(\Lambda_i) \longrightarrow V(\mu) \otimes V(\Lambda_{1-i}).
\]
Namely we set
\[
\psi^{(\mu,\lambda;1-i,i)}(z) = (\text{id} \otimes \Psi^{*(\Lambda_{1-i},\Lambda_i)}(z)) \circ (\Phi^{(\mu,\lambda)}(z) \otimes \text{id})
\]
\[
= \sum_{\epsilon = \pm} \Phi_{\epsilon}^{(\mu,\lambda)}(z) \otimes \Psi_{\epsilon}^{*(\Lambda_{1-i},\Lambda_i)}(z).
\]
(18)
Dropping superscripts we shall often write \( \psi^{(\mu,\lambda)}(z) \) or \( \psi(z) \). Clearly \( \psi^{(\mu,\lambda)}(z) \) commutes with the diagonal action of \( U_q(\widehat{sl}_2) \).

Set
\[
\eta(z) = \eta_I(z)\eta_{II}(z) = \frac{\{x^2z\} \{p^*x^2z\}}{\{z\} \{p^*x^4z\}},
\]
\( \rho(z) = \rho_I(z)\rho_{II}(z) = z^{\frac{r-s}{2}}\frac{\eta(z)}{\eta(z^{-1})}. \)

Then the properties (a1)–(a3) entail the following.

**Proposition 3.1** (b1) The product
\[
\prod_{i<j} \eta(z_j/z_i) \cdot \psi(z_1) \cdot \psi(z_N)
\]
is meromorphic, with at most simple poles at \( z_j/z_i = p^{s-r}x^2 \) \( (i < j, s \geq 0) \).

(b2) As meromorphic functions we have
\[
\psi^{(\nu,\mu)}(z_1)\psi^{(\mu,\lambda)}(z_2) = \rho(z_1)\rho(z_2) W\left( \begin{array}{c} \lambda \\ \mu' \\ \nu \end{array} \mid \begin{array}{c} z_1 \\ z_2 \end{array} \right).
\]

(b3)
\[
\text{res}_{z_1 = x^{-2}z_2} \psi^{(\nu,\mu)}(z_1)\psi^{(\mu,\lambda)}(z_2) \frac{dz_1}{z_1} = \delta_{\nu,\lambda}c_{\mu,\lambda} \times \text{id},
\]
with \( c_{\mu,\lambda} = x^{2(\Delta_{\mu} - \Delta_{\lambda}) - 1/2}gg_{\lambda}^\mu \).

Comparing (23) with (22) and (17) we find
\[
c_{\lambda_+,\lambda} : c_{\lambda_-,\lambda} = a_{\lambda_+,\lambda} : a_{\lambda_-,\lambda}
\]
where \( \lambda_\pm = \lambda \pm (\Lambda_1 - \Lambda_0) \).

### 3.2 DVA generators

Property (b3) describes the behavior of \( \psi(z_1)\psi(z_2) \) at the ‘first’ pole \( z_2/z_1 = x^2 \). We now look at the next pole. Define
\[
T^{(\lambda,i)}_{\pm}(z) = \text{res}_{z' = z} \frac{1}{\rho(p^*x^{-2}z')} \psi^{(\lambda,\lambda; i,1-i)}(x^{-2}z')\psi^{(\mu,\lambda; 0,1)\pm}(x^{-r+2}z) \frac{dz'}{z'}
\]
\( = a_{\lambda,\lambda} \sum_{\mu'} \psi^{(\lambda,\mu'; i,1-i)}(x^{-r+2}z')\psi^{(\mu',\lambda; 0,1)\pm}(x^{-r+2}z) b_{\mu',\lambda}. \)
In the second line we used the periodicity (16) and (17).

From eq.(24) we obtain
\[ T^{(\lambda; i)}_{-}(z) = \frac{C_{\lambda, -\lambda} f^{(\lambda; i)}_{-}}{C_{\lambda, +}} T^{(\lambda; i)}_{+}(z). \] (27)

We claim that the properties (b1)-(b3) imply the following.

**Proposition 3.2** (c1) As meromorphic functions,
\[ f \left( \frac{z_2}{z_1} \right) T^{(\lambda; i)}_{-}(z_1) T^{(\lambda; i)}_{+}(z_2) = f \left( \frac{z_1}{z_2} \right) T^{(\lambda; i)}_{+}(z_2) T^{(\lambda; i)}_{-}(z_1), \] (28)

where \( f(z) \) denotes the structure function (3) of DVA.

(c2) The left hand side of (28) is holomorphic in the neighborhood of \( |z_2/z_1| \leq x^{-2} \) except for simple poles at \( z_2/z_1 = x^{-2}, x^{2} \).

(c3) The residues are given by
\[ \text{res}_{z_1=x^{-2}z_2} \left( \left( \frac{z_2}{z_1} \right) T^{(\lambda; i)}_{-}(z_1) T^{(\lambda; i)}_{+}(z_2) \right) \frac{dz_1}{z_1} = -\text{res}_{z_2=x^{2}z_1} \left( \left( \frac{z_1}{z_2} \right) T^{(\lambda; i)}_{+}(z_1) T^{(\lambda; i)}_{-}(z_2) \right) \frac{dz_2}{z_2} = \left( x^{r-1} - x^{r+1} \right) \left( x^r - x^{-r} \right) (C^{(\lambda; i)}_{\pm})^2 \times \text{id}, \]

where \( C^{(\lambda; i)}_{\pm} \) is a constant related to the matrix element \( \langle T^{(\lambda; i)}_{\pm} \rangle \) with respect to the highest weight vector \( |\lambda\rangle \otimes |\Lambda_i\rangle \) by (see (32) below)
\[ \langle T^{(\lambda; i)}_{\pm} \rangle = C^{(\lambda; i)}_{\pm} \times (x^{l_1} + x^{-l_1}) \]

with \( \lambda = \lambda_l, l_0 = l \) and \( l_1 = r - 1 - l \).

The proof of Proposition 3.2 will be given in the next section.

Let us introduce the normalized operator
\[ T(z) = (C^{(\lambda; i)}_{\pm})^{-1} T^{(\lambda; i)}_{\pm}(z). \] (29)

Consider the irreducible decomposition
\[ V(\lambda) \otimes V(\Lambda_i) = \bigoplus_{\nu} V(\nu) \otimes \Omega_{\lambda, \Lambda_i; \nu}. \]

In the right hand side, \( \nu \) runs over dominant integral weights of level \( k + 1 \), and \( \Omega_{\lambda, \Lambda_i; \nu} \) signifies the space of highest weight vectors of that weight. Clearly \( T(z) \) acts on \( \Omega_{\lambda, \Lambda_i; \nu} \).

From the assertions (c1)–(c3) we conclude that

**Proposition 3.3** The operator \( T(z) \) (29), (23) affords a representation of DVA on \( \Omega_{\lambda, \Lambda_i; \nu} \).
3.3 Example

As an example, let us take the case $k = 1, r = 4$. In this case $\rho(z)$ and the $W$ weight simplify to

$$
\rho(z)W \left( \begin{array}{cc}
\Lambda_i & \Lambda_{1-i} \\
\Lambda_{1-i} & \Lambda_i
\end{array} \right) |z| = -1,
$$

and (b2) becomes

$$
\psi(z_1)\psi(z_2) = -\psi(z_2)\psi(z_1).
$$

Moreover the only poles of both sides are $z_2/z_1 = x^{-2}, x^2$, with residues proportional to the identity. Writing

$$
x^i\psi^{(\Lambda_{1-i},\Lambda_i;1-j,j)}(z) = \begin{cases}
\psi^{NS}(z) & i \equiv j \mod 2 \\
\psi^R(z) & i \not\equiv j \mod 2
\end{cases}
$$

we see that the Fourier expansion takes the form

$$
\psi^{NS}(z) = \sum_{n \in \mathbb{Z}+1/2} \psi_n z^{-n}, \quad \psi^R(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n}.
$$

In both cases $\psi(z) = \psi^{NS}(z), \psi^R(z)$ is a free fermion, whose Fourier components satisfy

$$
\psi_m \psi_n + \psi_n \psi_m = \delta_{m+n,0} (x^{2m} + x^{-2m}).
$$

The DVA current reads

$$
T(z) = \frac{(1 - x^6)}{x^2(1 + x^2)} \psi(x^{-2}z)\psi(x^2z).
$$

This is a deformation of the well known representation of the Virasoro algebra at $c = 1/2$ in terms of free fermions.

4 Proof of Proposition 3.2

4.1 Proof of (c1)

Set

$$
F^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4)}(z_1, z_2, z_3, z_4)
= \psi^{(\lambda_0,\lambda_1)}(z_1)\psi^{(\lambda_1,\lambda_2)}(z_2)\psi^{(\lambda_2,\lambda_3)}(z_3)\psi^{(\lambda_3,\lambda_4)}(z_4).
$$
We have the commutation relation
\[
F^{(\lambda_0, \lambda_1, \lambda_2, \lambda_3)}(z_1, z_2, z_3, z_4) = \prod_{i<j} \rho(z_i/z_j) \sum_{\mu_1, \mu_2, \mu_3} F^{(\lambda_{\mu_1}, \lambda_{\mu_2}, \lambda_{\mu_3})}(z_3, z_4, z_1, z_2) \times W \left( \frac{\lambda_0}{\mu_0} \frac{\lambda_1}{\mu_1} \frac{\lambda_2}{\mu_2} \frac{\lambda_3}{\mu_3} \frac{z_1}{z_3} \frac{z_2}{z_4} \frac{z_1}{z_3} \frac{z_2}{z_4} \right),
\]
where the last factor means (see the figure below)
\[
\sum_{\mu} W \left( \frac{\lambda_4}{\mu_4} \frac{\lambda_3}{\mu_3} \frac{z_2}{z_4} \right) W \left( \frac{\lambda_3}{\mu_3} \frac{\lambda_2}{\mu_2} \frac{z_2}{z_4} \right) W \left( \frac{\lambda_2}{\mu_2} \frac{\lambda_1}{\mu_1} \frac{z_1}{z_3} \right) W \left( \frac{\lambda_1}{\mu_1} \frac{\lambda_0}{\mu_0} \frac{z_1}{z_3} \right).
\]

Choosing \((\lambda_0, \cdots, \lambda_4) = (\lambda, \lambda_\pm, \lambda, \lambda_\pm, \lambda)\), taking the residue at \(z_1 = x^{-2}z_2\), \(z_3 = x^{-2}z_4\) and writing \(z = z_2/z_4\), we find
\[
c^2_{\lambda, \lambda} \times \text{id} = \rho(z)^2 \rho(x^2 z) \rho(x^{-2} z) \times \sum_{\mu, \nu} c_{\mu, \lambda} c_{\nu, \lambda} W \left( \frac{\lambda}{\mu} \frac{\lambda_\pm}{\lambda} \frac{\lambda_\pm}{\lambda} \frac{\lambda}{\nu} \frac{z, x^2 z, x^{-2} z, z} \right). \tag{31}
\]

Next we take the residue at \(z_1 = p^* x^{-2}z_2\), \(z_3 = p^* x^{-2}z_4\). Renaming the variables and using the periodicity \([10]\) of the Boltzmann weights, we obtain
\[
T^{(\lambda)}_{\pm}(z_1) T^{(\lambda)}_{\pm}(z_2) = \rho(z)^2 \rho(p^* x^{-2} z) \rho(p^* x^{-2} z) \times \sum_{\varepsilon_1, \varepsilon_2} T^{(\lambda)}_{\varepsilon_1}(z_1) T^{(\lambda)}_{\varepsilon_2}(z_2) W \left( \frac{\lambda}{\mu} \frac{\lambda_\pm}{\lambda} \frac{\lambda_\pm}{\lambda} \frac{\lambda}{\nu} \frac{z, x^2 z, x^{-2} z, z} \right),
\]
with \(z = z_1/z_2\). (Here and after we suppress the index \(i\) for brevity.) By \((27)\) and \((31)\) the right hand side simplifies to
\[
\frac{\rho(p^* x^{-2} z) \rho(p^* x^{-2} z)}{\rho(x^2 z) \rho(x^{-2} z)} T^{(\lambda)}_{\pm}(z_2) T^{(\lambda)}_{\pm}(z_1) = \frac{f(z_1/z_2)}{f(z_2/z_1)} T^{(\lambda)}_{\pm}(z_2) T^{(\lambda)}_{\pm}(z_1),
\]
where we have used the relation
\[
\frac{\eta(p^* x^{-2} z) \eta(p^* x^{-2} z)}{\eta(x^{-2} z) \eta(x^2 z)} = \frac{(1 - x^{-2} z)(1 - x^2 z)}{(1 - p^* x^{-2} z)(1 - p^* x z)} f(z). \tag{32}
\]
The proof is over.
4.2 Proof of (c2)

We shall show that, in the neighborhood of $|z_2/z_1| \leq 1$, (i) $f(z_2/z_1)T^{(\Lambda)}_{\pm}(z_1)T^{(\Lambda)}_{\pm}(z_2)$ is meromorphic, (ii) the only pole is $z_2/z_1 = x^2$, (iii) the pole is simple with residue proportional to the identity. The rest of the assertion is then a consequence of the commutation relation (28).

Recall that

$$G^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4)}(z_1, z_2, z_3, z_4) = \prod_{i<j} \eta(z_j/z_i)F^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4)}(z_1, z_2, z_3, z_4)$$

has poles only at $z_j/z_i = x^2 p^{*-s}$ ($i < j$, $s = 0, 1, 2, \ldots$) and that they are all simple. From the definition (24) we see that $T^{(\lambda)}_{\pm}(z_1)T^{(\lambda)}_{\pm}(z_2)$ is a linear combination of the expression

$$\eta\left(\frac{z_2}{z_1}\right)^{-2} \eta\left(p^s x^{-2} \frac{z_2}{z_1}\right) \eta\left(p^{s-1} x^2 \frac{z_2}{z_1}\right)^{-1} \times G^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4)}(x^{-r+2} z_1, x^{r-2} z_1, x^{r-2} z_2, x^{r-2} z_2).$$

In the neighborhood of $|z_2/z_1| \leq 1$, the last factor can have poles only at $z_2/z_1 = x^2, p^*, 1$, whose multiplicities are at most 2, 1, 1 respectively. Multiplying $f(z_2/z_1)$, using (32) and

$$\eta(z) \eta(x^2 z) = \frac{(p^* x^2 z; p^*)_\infty}{(z; p^*)_\infty},$$

we find (setting $z = z_2/z_1$) that

$$f\left(\frac{z_2}{z_1}\right)T^{(\lambda)}_{\pm}(z_1)T^{(\lambda)}_{\pm}(z_2) = \frac{(p^* x^{-2} z; p^*)_\infty}{(x^2 z; p^*)_\infty}(1 - p^{*-1} z)(1 - z)(1 - p^* z)$$

$$\times \text{sum of terms } G^{(\mu_0,\mu_1,\mu_2,\mu_3,\mu_4)}(x^{-r+2} z_1, x^{r-2} z_1, x^{r-2} z_2, x^{r-2} z_2),$$

which is regular at $z = p^*, 1$. The pole $z = x^2$ is apparently double, but the leading term is proportional to

$$\frac{1}{(1 - x^{-2} z)^2} \sum_{\mu,\nu} c_{\mu,\lambda} c_{\nu,\lambda} W\left(\nu, \lambda \left| \begin{array}{c} \lambda \mu \\ \mu \nu \end{array} \right. x^{-4}\right).$$

Taking the residue of (33) at $z = x^{-2}$ we find that the sum is actually 0. This shows that the pole is simple.

4.3 Proof of (c3)

The residue can be calculated in the following manner.

$$\text{res}_{z_1=x^{-2} z_2} f\left(\frac{z_2}{z_1}\right)T^{(\lambda)}_{\pm}(z_1)T^{(\lambda)}_{\pm}(z_2) \frac{dz_1}{z_1}$$
Since (15) is \( \delta_{\mu,\mu'} \) at \( z = 1 \), the right hand side becomes

\[
\text{res}_{z_1 = x^{-2} z_2} \frac{f \left( \frac{z_2}{z_1} \right)}{\rho \left( p^* x^{-2} z_1 \right) \rho \left( p^* x^{-2} z_2 \right)} \times F^{(\lambda, \lambda_{\pm}, \lambda_{\pm}, \lambda)}(x^{-r} z_1, x^{-r-2} z_2, x^{-r} z_2, x^{-r-2} z_1, x^{-r} z_2) \frac{dz_1'}{z_1} \otimes \frac{dz_2'}{z_2} \otimes \frac{dz_1}{z_1}
\]

\[
= \text{res}_{z_1 = x^{-2} z_2} \text{res}_{z_1 = x^{-2} z_2'} \frac{f \left( \frac{z_2}{z_1} \right)}{\rho \left( p^* x^{-2} z_1 \right) \rho \left( p^* x^{-2} z_2 \right)} \times \sum_{\mu} F^{(\lambda, \lambda_{\pm}, \mu, \lambda_{\pm}, \lambda)}(x^{-r} z_1, x^{-r-2} z_2, x^{-r} z_2, x^{-r} z_1, x^{-r} z_2) \frac{dz_1'}{z_1} \otimes \frac{dz_2'}{z_2} \otimes \frac{dz_1}{z_1}
\]

\[
= - A B c_{\lambda_{\pm}, \lambda}^2
\]

where we have set

\[
A = \text{res}_{z_1 = 1} \frac{\rho(p^* z_1)}{\rho(p^* x^{-2} z_1)} \frac{dz_1}{z_1}, \quad B = \left. \frac{f(x^2 z)}{\rho(p^* x^{-2} z)} \right|_{z = 1}.
\]

In order to compare this with the matrix element \( \langle T^{(\lambda, \lambda)}_{\pm}(z) \rangle \), we need the knowledge about the two point functions of \( \Phi(z), \Psi(z) \). A formula for the latter can be found e.g. in [7]. For \( \Phi(z) \), we use the \( q \)-KZ equation

\[
\langle \Phi^{(\lambda, \mu)}(p^* z_1) \Phi^{(\mu, \lambda)}(z_2) \rangle = x^{\frac{1}{2}} \frac{\eta(z_2/z_1)}{\eta(p^* - 2z_1/z_2)} (p^* - \phi \otimes 1) \overline{r}(z_1/z_2) \langle \Phi^{(\lambda, \mu)}(z_1) \Phi^{(\mu, \lambda)}(z_2) \rangle
\]

and we reduce the computation to (13) (here \( \phi = (\overline{\lambda} + \overline{\mu})/(r - 1) \) and \( \overline{\lambda} \) denotes the classical part of \( \lambda \)). Omitting the details we only give the result.

\[
\langle T^{(\lambda)}_{\pm}(z) \rangle = (x^{l_1} + x^{-l_1}) \times x^{\frac{1}{2} + \frac{1}{r-1} - \frac{3}{2} x^{2} (\Delta_{\lambda_{\pm}}(z) \Delta_{\lambda_{\pm}})} g_{\lambda_{\pm}}^{\lambda_{\pm}}
\]

\[
\times \frac{(p^* x^2; x_1^4) \infty (x^2; p^*) \infty (p^* x^2; p^*) \infty}{(p^* x^2; x_1^4) \infty (p^* x^2; p^*) \infty}. \tag{33}
\]

Here \( \lambda = \lambda_l \) and \( l_0 = l, l_1 = r - 1 - l \).
5 Elliptic algebra \( A_{q,p}(\hat{sl}_2) \)

We briefly comment on a similar construction of DVA current from the level-one VO’s of the elliptic algebra \( A_{q,p}(\hat{sl}_2) \). (For the notation, see [8]. Here we also write the parameters of \( A_{q,p}(\hat{sl}_2) \) as \( q = -x \), \( p = x^{2r} \) and \( p^* = x^{2r-2} \).) Basically, this complies with Lukyanov’s observation [4] but with some modification as explained below.

The missing knowledge which is necessary for the construction of DVA current from VO’s is the analyticity condition of \( n \)-point functions governed by the \( q \)-KZ type equation for \( A_{q,p}(\hat{sl}_2) \). (See [9] as for the recent progress). Since quite little is known about this \( q \)-KZ equation at present, the best we can do is to start from a suitable ansatz for the analyticity. Once we assume that, we are able to construct a DVA current from the type II VO’s as

\[
T(z) = C_{II} \text{res}_{\zeta' = \zeta} \sum_{\varepsilon} \frac{\xi(p^{*1-1}x^2 \zeta^2/\zeta'^2; p, q)}{\xi(p^*, x^{-2} \zeta^2/\zeta'^2; p^*, q)} \Psi_{\varepsilon}(-x^{(r-2)/2} \zeta') \Psi_{\varepsilon}^*(x^{(r-2)/2} \zeta) \frac{d \zeta'}{\zeta'},
\]

satisfying the commutation relation

\[
f \left( \zeta_2^2/\zeta_1^2 \right) T(\zeta_1)T(\zeta_2) - T(\zeta_2)T(\zeta_1) f \left( \zeta_1^2/\zeta_2^2 \right) = \frac{c}{2} \left( \delta(-x^{-1} \zeta_2/\zeta_1) - \delta(-x^{-1} \zeta_2/\zeta_1) - \delta(-x \zeta_2/\zeta_1) + \delta(x \zeta_2/\zeta_1) \right),
\]

where the constant \( c \) is chosen as before

\[
c = \frac{(x^{r-1} - x^{-r+1})(x^r - x^{-r})}{x - x^{-1}},
\]

and \( C_{II} \) is some constant. Note that the current \( T(\zeta) \) given by (34) is odd in \( \zeta \), i.e. \( T(-\zeta) = -T(\zeta) \), and the RHS of (35) is odd in \( \zeta_1 \) in accordance with the parity of \( T(\zeta) \).

At the Ising point \( p^{1/2} = x^2 \), the type-II VO is realized by the fermion (30) as

\[
\sum_{\varepsilon} \Psi_{\varepsilon}^*(\zeta) = \psi^R(\zeta^2) + \psi^{NS}(\zeta^2),
\]

and this gives a fermionic realization of DVA with \( c = -2 \).

\[
T(\zeta) = (x - x^{-1}) \psi^{NS}(\zeta^2) \psi^R(\zeta^2).
\]

We also note that the same can be done with type I VO’s and observe the composition

\[
C_{I} \text{res}_{\zeta' = \zeta} \sum_{\varepsilon} \frac{\xi(px^2 \zeta^2/\zeta'^2; p, q)}{\xi(p^{*1-1}x^{-2} \zeta^2/\zeta'^2; p^*, q)} \Phi_{\varepsilon}(-x^{(r+1)/2} \zeta') \Phi_{\varepsilon}^*(x^{-(r+1)/2} \zeta) \frac{d \zeta'}{\zeta'}. \]

\[
= -C_{I} x^{r+1} \frac{\Theta_p(x^{-2})}{(p; p)^3} \sum_{\varepsilon} \Phi_{\varepsilon}(-x^{(r+1)/2} \zeta) \Phi_{\varepsilon}^*(x^{(r+1)/2} \zeta).
\]
enjoyes the DVA relation with a suitable constant $C_f$.

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Errata to “Remarks on the deformed Virasoro algebra”

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In the paper [1], subsection 4.3, the proof of (c3) contains a gap concerning the commutability of the operations of taking residues. We give here a corrected proof.

Let

\[ G(\lambda_0, \ldots, \lambda_n)(z_1, \ldots, z_n) = \prod_{i<j} \eta(z_j/z_i) \psi(\lambda_0, \lambda_1)(z_1) \cdots \psi(\lambda_{n-1}, \lambda_n)(z_n). \]  

(1)

The following are direct consequences of Proposition 3.1.

Lemma 5.1 (1) \( G(\lambda_0, \ldots, \lambda_n)(z_1, \ldots, z_n) \) is holomorphic except for simple poles at \( z_j/z_i = p^s x^2 \) (i < j, s ≥ 0).

(2) We have

\[ G^{(\lambda_0, \ldots, \lambda_n)}(\lambda_{k-1}, \lambda_k, \lambda_{k+1}) \cdots (z_k, z_{k+1}, \ldots) = \left( \frac{z_k}{z_{k+1}} \right)^{2(r-1)} \sum_{\mu} G^{(\lambda_{k-1}, \lambda_k, \lambda_{k+1})} \cdots (z_{k+1}, z_k, \ldots) W \left( \begin{array}{c} \lambda_{k+1} \\ \mu \\ \lambda_{k-1} \end{array} \right| z_k/z_{k+1} \right). \]

(3) As \( z_k \to x^{-2} z_{k+1} \) we have

\[ G(\lambda_0, \ldots, \lambda_n)(z_1, \ldots, z_n) = \frac{\delta_{\lambda_0=\lambda_{k+1}} \delta_{\lambda_{k+1}=\lambda_{k-1}}}{1 - x^{-2} z_k/z_{k+1}} \prod_{i<k} \left( \frac{p^* x^2 z_i}{z_{k+1}} ; p^* \right)_\infty \prod_{j>k} \left( \frac{p^* x^2 z_j}{z_{k+1}} ; p^* \right)_\infty \times \eta(x^2) G^{(\lambda_0, \ldots, \lambda_k, \lambda_{k+2}, \ldots)}(z_1, \ldots, z_{k-1}, z_{k+2}, \ldots, z_n) + O(1). \]
Lemma 5.2 We have
\[
X = \operatorname{res}_{z_1=x^{-2}z_2} \varphi(z_2/z_1) \frac{d\tilde{z}_1}{z_1} \left( \operatorname{res}_{z'_1=z_1} H(z'_1, z_1, z_2) \frac{d\tilde{z}'_1}{z'_1} \right),
\]
where
\[
H(z'_1, z_1, z_2) = \operatorname{res}_{z_1=x^{-2}z_2} G^{(\lambda, \lambda, \lambda, \lambda)}(x^{-r-2}z'_1, x^{-r+2}z_1, x^{-r-2}z'_2, x^{-r+2}z_2) \frac{d\tilde{z}'_2}{z'_2},
\]
\[
\varphi(z) = \frac{x^{-r-2} + \frac{2\pi i}{x^{r+2} - z_2}}{(p^*x^{-2}z; p^*)^\infty} (1 - p^*z)(1-z)(1-p^*z).
\]

Lemma 5.3 In the neighborhood of \( z'_1 = z_1 = x^{-2}z_2 \), \( H(z'_1, z_1, z_2) \) is holomorphic except for simple poles at \( z'_1 = z_1 = x^{-2}z_2 \). As \( z_1 \to x^{-2}z_2 \), we have
\[
H(z'_1, z_1, z_2) = -x^{-2} \frac{c_{\lambda, \lambda} \eta(x^2)}{1 - x^{-2}z_2/z_1} G^{(\lambda, \lambda, \lambda, \lambda)}(x^{-r-2}z'_1, x^{-r-2}z_1, x^{-r-2}z'_2, x^{-r+2}z_2)
\]
\[
\times \frac{1}{1 - p^*(p^*-1 \frac{x^{2}z_1}{z_2}; p^*)^\infty} (x^2; p^*)^\infty (p^*-1 \frac{x^{2}z_1}{z_2}; p^*)^\infty.
\]

Proof. Consider \( G = G^{(\lambda, \lambda, \lambda, \lambda)}(x^{-r-2}z'_1, x^{-r+2}z_1, x^{-r-2}z'_2, x^{-r+2}z_2) \). From Lemma 5.1, the only poles of \( G \) in the neighborhood of \( z'_1 = z_1 = x^{-2}z_2 \), \( z'_2 = z_2 \) are
\[
z'_1 = z_1, \quad z'_2 = z_2, \quad z_1 = x^{-2}z_2, \quad z'_1 = x^{-2}z'_2.
\]
As \( z_1 \to x^{-2}z_2 \) we have
\[
G = \left( p^{*-1} - \frac{x^{2}z_2}{z_2} \right)^{\frac{r}{2}} \frac{c_{\lambda, \lambda} \eta(x^2)}{1 - x^{-2}z_2/z_1} G^{(\lambda, \lambda, \lambda, \lambda)}(x^{-r-2}z'_1, x^{-r-2}z_1) W \left( \frac{\lambda \lambda}{x^{-2}z_2} \right)
\]
\[
\times \eta(x^2) \frac{(x^{2}z_1^{2}; p^*)^\infty (x^{2}z_2^{2}; p^*)^\infty}{(p^*-1 \frac{x^{2}z_1}{z_2}; p^*)^\infty (p^*-1 \frac{x^{2}z_2}{z_2}; p^*)^\infty} + O(1).
\]

Likewise, as \( z'_1 \to x^{-2}z'_2 \) we have
\[
G = \left( p^{*-1} - \frac{x^{2}z_1}{z'_2} \right)^{\frac{r}{2}} \frac{c_{\lambda, \lambda} \eta(x^2)}{1 - x^{-2}z'_2/z'_1} G^{(\lambda, \lambda, \lambda, \lambda)}(x^{-r+2}z_1, x^{-r+2}z_2) W \left( \frac{\lambda \lambda}{x^{-2}z_2} \right)
\]
\[
\times \eta(x^2) \frac{(x^{2}z_1^{2}; p^*)^\infty (x^{4}z_2^{2}; p^*)^\infty}{(p^*-1 \frac{x^{2}z_1}{z_2}; p^*)^\infty (p^*-1 \frac{x^{2}z_2}{z_2}; p^*)^\infty} + O(1).
\]
Taking the residue of (3) at \( z'_2 = z_2 \), we find that \( H(z'_1, z_1, z_2) \) is regular at \( z'_1 = x^{-2}z_2 \).

The behavior (3) is a consequence of (4).

**Proof of (c3).** From Lemma 5.2 and Lemma 5.3 we can change the order of the residues as

\[
X = \text{res}_{z'_1 = x^{-2}z_2} \frac{dz'_1}{z'_1} \left( \text{res}_{z_1 = x^{-2}z_2} \varphi(z_2/z_1) H(z'_1, z_1, z_2) \frac{dz_1}{z_1} \right).
\]

Using (3) we find

\[
X = -x^{r-2} \varphi(x^2) \text{res}_{z'_1 = x^{-2}z_2} \frac{dz'_1}{z'_1} c_{\lambda \pm, \lambda} \eta(x^2)
\times G^{(\lambda, \lambda \pm, \lambda)}(x^{r-2}z'_1, x^{r-2}z_2) \left( x^{2-z'_2/z'_1} p^* \right) \infty \left( x^{2-z_2/z_1} p^* \right) \infty
\times \left( p^{*-1} \right) \infty \left( p^* \right) \infty
\]
\[
= -x^{r-2} x^{2-z_2} c_{\lambda \pm, \lambda} \left( \frac{\eta(x^2)}{\eta(p^*x^{-2})} \right)^2 (1-x^2)(1-p^*x^2).
\]

After simplification, we obtain the result stated in subsection 4.3.

There are also the following corrections.

(i) The right hand side of Eq.(13) should read

\[
x^{2(\Delta_\mu - \Delta_\lambda) + 1/2} g^\mu_\lambda \delta_{\nu \lambda} \times \text{id} \quad \text{for } \mu = \lambda \pm.
\]

(ii) Line next to Eq.(23), the formula for \( c_{\mu \lambda} \) should read

\[
c_{\mu \lambda} = x^{2(\Delta_\mu - \Delta_\lambda) + 1/2} g^\mu_\lambda \quad \text{for } \mu = \lambda \pm.
\]

(iii) Third line above subsection 4.3: this equation should read

\[
\frac{1}{(1-x^{-2}z)^2} \sum_{\mu, \nu} b_{\mu, \lambda} b_{\nu, \lambda} c_{\mu, \lambda} c_{\nu, \lambda} W \left( \frac{\nu \lambda}{\mu} \left| x^{-4} \right. \right).
\]

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