Robust portfolio optimization with multi-factor stochastic volatility

Ben-Zhang Yang, Xiaoping Lu, Guiyuan Ma, and Song-Ping Zhu

1Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China
2School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, Australia
3Department of Statistics, The Chinese University of Hong Kong, Hong Kong, China

Abstract: This paper studies a robust portfolio optimization problem under the multi-factor volatility model introduced by Christoffersen et al. (2009). The optimal strategy is derived analytically under the worst-case scenario with or without derivative trading. To illustrate the effects of ambiguity, we compare our optimal robust strategy with some strategies that ignore the information of uncertainty, and provide the corresponding welfare analysis. The effects of derivative trading to the optimal portfolio selection are also discussed by considering alternative strategies. Our study is further extended to the cases with jump risks in asset price and correlated volatility factors, respectively. Numerical experiments are provided to demonstrate the behavior of the optimal portfolio and utility loss.

Key Words: Robust portfolio selection; multi-factor volatility; jump risks; non-affine stochastic volatility; ambiguity effect.

1 Introduction

In the classical Black-Scholes-Merton framework of work, volatility of the stock is assumed to be constant, which does not fully reflect the market dynamics as shown by the observations of volatility smile (or skew) and term structure. Stochastic volatility models have been adopted to address the issue by allowing volatility to vary over time (for example, [10, 11, 26, 28, 30, 31, 37, 39]). Stochastic volatility models also address term structure effects by modeling the mean reversion in variance dynamics. There are several generalized models which introduce one-factor stochastic volatility, such as, Stein & Stein [32], Heston [22], and Schöbel & Zhu [33]. However, the improvement resulting from one-factor stochastic volatility diffusion models may still not capture satisfactorily the term structure of implied volatility in practice (see, for example, [9, 17, 8, 13]). As a result, many researchers attempt to develop multi-factor stochastic volatility models.

*Corresponding author: xplu@uow.edu.au
Such an attempt can be traced back to Christoffersen et al. [8] who proposed a multi-factor model, which shows much more flexibility in controlling the level and slope of the smile. An additional advantage is that the multi-factor model also provides more flexibility in modeling the volatility term structure. By replacing the CIR variance process with a Wishart process, Da Fonseca et al. [17] developed a multiple-factor stochastic volatility model which offers more consistent option prices than one-factor models. By introducing another volatility process into the Heston model, Christoffersen et al. [8] proposed a multi-factor volatility model which provides a better fit for the term structure of volatility and the empirical European options prices than Heston’s model. Li & Zhang [25] have verified the conclusion in Christoffersen et al. [8] by analyzing an index option data set. Moreover, the multi-factor Heston model provides efficient asset pricing for various financial derivatives while keeping the analytical tractability of the original model.

Multi-factor volatility models have also been applied recently in portfolio selection problems. Recently, Escobar et al. [13] considered an optimal investment problem under one multi-factor stochastic volatility. In their model, the eigenvalues of the covariance matrix of asset returns follow independent square-root stochastic processes. Optimal investment strategies under this model are subsequently derived in closed form. However, in practice, investors quite often are uncertain about the probability distribution of the underlying or a precise physical measure to choose for a particular model used in the portfolio optimization. As a result, one may consider a robust alternative model for the stock and its volatilities to avoid miss-specification when making investment decisions. The investor should identify the worst-case measure and specify the optimal portfolio under the worst-case scenario. The recent work of Bergen et al. [4] studied the robust multivariate portfolio selection problem with stochastic covariance in the presence of ambiguity and provided the optimal multivariate intertemporal portfolio.

In this paper, we study the robust optimal portfolio problem under the multi-factor model introduced by Christoffersen et al. [8]. With an assumption that there may exist a certain degree of ambiguity in an investor’s mind in term of the dynamics of a stock and its volatilities, we analytically derive the robust optimal investment strategy and study the ambiguity effects [14, 34, 40] in complete and incomplete markets. Moreover, the empirical phenomena concerned with jump risks in the asset price have received much attention recently and thus different models are proposed to incorporate jump risks (see, for example, [10, 27, 31, 36]). In this article, we also provide compensative studies for the robust portfolio selection by introducing the asset price with multi-factor volatilities in the presence of jump risks. In addition, we extend our analysis to include the correlation between asset volatilities, which has been studied to capture and predict the evolution of correlation between asset returns or multivariate volatilities (see, for example, [7, 20, 21]). By considering the specific correlation relationship in the various volatility components of asset price, we derive the corresponding robust optimal portfolio selection.

The contributions of this paper are as follows: i) the optimal investment strategy with the worst-case measure is explicitly derived for a two-factor volatility model for complete and incomplete markets, respectively; ii) some other strategies that ignore the information of uncertainty are compared with our robust optimal strategy and the corresponding utility loss are also discussed, thus demonstrating the effects of ambiguity; iii) an extension with additional jump risk in asset price is studied comprehensively for the first time to the best of
our knowledge; iv) another extension with correlation between asset volatilities is also considered with an approximation being provided for solving robust optimal investment on the moment-information context.

The rest of the paper is structured as follows. Our model formulation is introduced in Section 2. Section 3 presents the optimal investment strategies for the worst-case measure in complete and incomplete markets, as well as the sub-optimal strategies. In this section, we also provide a comprehensive analysis for two extensions. Section 4 shows experimental results, and concluding remarks are given in Section 5. Some detailed proofs are provided in the appendices.

2 Basic Model

We propose a portfolio selection problem under the well-known framework proposed in [26] and extended in [29, 15], where an investor has access to stock and derivatives markets. Suppose the money market account follows

$$dM(t) = rM(t)dt,$$

(1)

where $r$ is a constant risk-free interest rate. In this study, we assume that the stock price and its volatilities follow the multi-factor volatility model proposed by Christoffersen et al. [8]. Without loss of generality, we consider a two-factor stochastic volatility model in this paper. Specifically, the stock price satisfies:

$$dS(t) = (r + \lambda_1 V_1(t) + \lambda_2 V_2(t)) S(t)dt + \sqrt{V_1(t)} S(t) dW_1(t) + \sqrt{V_2(t)} S(t) dW_2(t),$$

(2)

where $W_j = W_j(t)_{t \in [0,T]} (j = 1, 2)$ are two Brownian motions and $\lambda_j V_j(t)$ represent the premia associated with $W_j$.

For simplicity, we started with the hypothesis that two volatility components are independent here, i.e. $W_1$ and $W_2$ are independent, and we shall extend to the correlated volatilities scenario afterwards. It is assumed that the market prices of the risks associated with $W_1$ and $W_2$ are given by $\lambda_1 \sqrt{V_1(t)}$ and $\lambda_2 \sqrt{V_2(t)}$, respectively, where $\lambda_1$ and $\lambda_2$ are constants. The volatilities, $V_1(t)$ and $V_2(t)$, are assumed to follow

$$dV_1(t) = \kappa_1 (\theta_1 - V_1(t)) dt + \sigma_1 \sqrt{V_1(t)} \left( \rho_1 dW_1(t) + \sqrt{1 - \rho_1^2} dZ_1(t) \right),$$

(3)

$$dV_2(t) = \kappa_2 (\theta_2 - V_2(t)) dt + \sigma_2 \sqrt{V_2(t)} \left( \rho_2 dW_2(t) + \sqrt{1 - \rho_2^2} dZ_2(t) \right),$$

(4)

where $Z_j = Z_j(t)_{t \in [0,T]}$ are another two independent Brownian motions; $\rho_j$ are the correlation parameters; $\kappa_j$, $\theta_j$ and $\sigma_j$ are the mean-reverting speed, the long-term mean and the volatility of volatility in the $V_j(t)$, respectively. The Feller condition is assumed to be satisfied, i.e. $2\kappa_j \theta_j \geq \sigma_j^2$ holds (see [22]). Further, it is assumed that the market price of risk associated with $Z_1$ and $Z_2$ are given by $\mu_1 \sqrt{V_1(t)}$ and $\mu_2 \sqrt{V_2(t)}$, respectively. Here the constants $\mu_1$ and $\mu_2$ are interpreted as volatility risk premium parameters.

Remark 2.1. The proposed model, driven by (1), (2), (3) and (4), differs from the models established in the existing literature. In fact, the models studied in [26, 24] only consider the single factor volatility and have not involved the special volatility structure in the asset price. Our model is fundamentally different from the...
model in [4] [12]: we study the portfolio problem with one risky asset with multi-factor volatility components having no independent constraints, while they study the multi-dimensional portfolio problem with multiple independently assets and each asset only has one risk factor.

As pointed out in [8], the variance is the sum of two uncorrelated factors that may be individually correlated with stock returns. Due to the special structure of the variance, we further assume that there are two derivative products to separately hedge each component of the volatility risks and another one derivative to hedge stock. To complete the market with respect to different volatility risks, we assume that the investor can trade three stock options with price $O^{(i)}(t)$, $i = 1, 2, 3$. We denote the option price as $O^{(i)}(t) = g^{(i)}(S, V_1, V_2, t)$ for some twice continuously differentiable function $g^{(i)}$. Using Itô’s lemma, we obtain the following dynamics for the option price

$$dO^{(i)}(t) = rO^{(i)}(t)dt + \sum_{j=1}^{2}\left(g^{(i)}S + \sigma_j g^{(i)}_j\right)\left(\lambda_j V_j dt + \sqrt{\nu_j}dW_j(t)\right) + \sum_{j=1}^{2}\sigma_j \sqrt{1 - \rho_j^2} g^{(i)}_j \left(\mu_j V_j dt + \sqrt{\nu_j}dZ_j(t)\right),$$

where $g^{(i)}_S$ and $g^{(i)}_V_j$ denote the partial derivatives of $g^{(i)}$ with respect to $S$ and $V_j$, respectively. The equations (2)-(5) therefore form our reference model.

In reality, an investor always faces uncertainty about the probability distribution for the reference model and then considers a set of possible alternative models when making investment decisions. We assume that the investor is uncertain about the distribution of noises $W_1, W_2, Z_1$ and $Z_2$ in the asset price and its volatility processes and that $\mathcal{F}_t$ is the filtration generated by four Brownian motions $W_1, W_2, Z_1$ and $Z_2$. Let perturbation process $\mathbf{e} = (e^S(t), e^V_1(t), e^V_2(t))$ be a $\mathbb{R}^3$-valued $\mathcal{F}_t$-progressively measurable process and $\mathcal{E}[0, T]$ be the space of all $\mathcal{F}_t$-measurable processes such that $\mathbf{e}$ is a well-defined Radon-Nikodým derivative process:

$$Z^e_t = E\left[\frac{d\mathbb{P}^e}{d\mathbb{P}} | \mathcal{F}_t\right] = \exp\left(-\int_0^t \sum_{j=1}^{2}\left[\frac{1}{2} \left((e^S_j(\tau))^2 + (e^V_j(\tau))^2\right) d\tau + e^S_j(\tau)dW_j(\tau) + e^V_j(\tau)dZ_j(\tau)\right]\right).$$

According to Girsanov’s theorem, the processes defined as

$$\tilde{W}_j(t) = \int_0^t e^S_j(\tau)d\tau + W_j(t), \quad \tilde{Z}_j(t) = \int_0^t e^V_j(\tau)d\tau + Z_j(t),$$

are Brownian motions under the probability measure $\mathbb{P}^e$. Due to the difficulty identifying the reference model from the available market data, for each perturbation process $\mathbf{e}$, the investor considers an alternative model: the stock price is observed by

$$dS(t) = \left(r + \sum_{j=1}^{2}\lambda_j V_j(t) - \sqrt{\nu_j(t)}e^S_j(t)\right)S(t)dt + \sum_{j=1}^{2}\sqrt{\nu_j(t)}S(t)d\tilde{W}_j(t),$$

and at the same time, its variance processes, $V_1$ and $V_2$ are observed by

$$dV_i(t) = \left[\kappa_i(\theta_i - V_i(t)) - \rho_{ii}\sigma_i \sqrt{V_i}e^S_i - \sqrt{1 - \rho_{ii}^2}\sigma_i \sqrt{V_i}e^V_i\right]dt + \sigma_i \sqrt{V_i(t)}\left(\rho_{ii}\tilde{W}_1(t) + \sqrt{1 - \rho_{ii}^2}\tilde{Z}_j(t)\right).$$
while the option prices satisfies
\[
\begin{align*}
dO^{(i)}(t) &= rO^{(i)}(t)dt + \sum_{j=1}^{2} \left(g^{(i)}S + \sigma_j \beta_j g^{(i)}_V \right) \left((\mu_j V_j - e_j^S)\sqrt{V_j}dt + \sqrt{V_j}d\hat{W}_j(t) \right) \\
&\quad + \sum_{j=1}^{2} \sigma_j \sqrt{1-\rho_j^2}g^{(i)}_V \left((\mu_j V_j - e_j^V)\sqrt{V_j}dt + \sqrt{V_j}d\hat{Z}_j(t) \right).
\end{align*}
\]

Under the actual trading scenario, different investors can obtain the information about the probability distributions of the stock price and the volatility processes from different sources. Equivalently, each investor chooses one certain measure, which is determined by perturbation process \(\mathbf{e}\). For, each perturbation process, the investor will consider the corresponding alternative model. Because the perturbation process can be chosen arbitrarily, the alternative model therefore allows for different levels of ambiguity about the stock and its volatility. Naturally an investor will attempt to find a robust decision based on the sources that are more reliable and useful.

Assume that the ambiguous investor wishes to derive an optimal strategy which maximizes the expected utility of terminal wealth \(X_T\). Similar to [35][16][12], we assume the investor’s preferences are assumed to be described by a CRRA (Constant Relative Risk Aversion) utility function with parameter \(\gamma > 1\). Denote a new process \(Y(s) = (X(s), V_1(s), V_2(s))\) and let \(y = (x, v_1, v_2)\) be the values of \(Y(s)\) at time \(t\), then the expected utility achieved by a trading strategy \(\Pi\) is given by
\[
w^\mathbf{e}(t, y; \Pi) = \frac{1}{1-\gamma} \mathbb{E}_{\mathcal{F}_T}^{\mathbf{e}}[(X_T)^{1-\gamma}]
\]
where \(\Pi\) satisfies the following conditions within the space \(\mathcal{U}(0, T)\): (i) \(\Pi\) is a \(\mathcal{F}_t\)-progressively measurable process; (ii) Under \(\Pi\), the wealth process \(X_t\) of the investor is non-negative for \(t \in [0, T]\); (iii) The integrability conditions necessary for the expectation operator in (10) to be well-defined are satisfied. Thus, the indirect utility function of the investor is
\[
J(t, y) = \sup_{\Pi \in \mathcal{U}(0, T)} \inf_{\mathbf{e} \in \mathcal{E}(0, T)} \left( w^\mathbf{e}(t, y; \Pi) + \mathbb{E}_{\mathcal{F}_T}^{\mathbf{e}} \left[ \int_t^T \sum_{j=1}^{2} \left( \frac{(e_j^S(s))^2}{2\Psi_j^S(s, Y(s))} + \frac{(e_j^V(s))^2}{2\Psi_j^V(s, Y(s))} \right) ds \right] \right),
\]
here the expectation is taken with respect to the distribution from the alternative model, and the integral term in (11) is the penalty incurred by deviating from the reference model. As pointed out by Anderson et al. [2], the penalty term is an expected log-likelihood ratio on the basis of the relative entropy. The state-dependent scaling functions \(\Psi_j^S\) and \(\Psi_j^V\) in (11) are defined below for analytical tractability as in [16][12][13]
\[
\Psi_j^S = \frac{\phi_j^S}{(1-\gamma)J(t, y)}, \quad \Psi_j^V = \frac{\phi_j^V}{(1-\gamma)J(t, y)},
\]
where the positive constants \(\phi_j^S\) and \(\phi_j^V\) are called ambiguity aversion parameters which describe the ambiguity aversion level about stock price and its volatilities, respectively. \(\Psi_j^S\) and \(\Psi_j^V\) represent the strength of an investor’s preference for robustness. In other words, the values of \(\Psi_j^S\) and \(\Psi_j^V\) are greater for the investor who has less faith for the reference model.
3 Robust optimal investment strategies

With all the necessary assumptions presented in the previous section, we are now ready to solve the optimization problem (11). Closed-form solutions for the optimal investment problems are derived in both complete and incomplete markets. For practical applications, we extend the model to general setting with jump risk scenario and correlated volatilities scenario, respectively.

3.1 The complete market case

Let $\pi^S$ be the fraction of wealth invested in the stock, $\pi^i (i = 1, 2, 3)$ be the fraction of wealth invested in $i$-th derivative, and the rest of the wealth be used for investments in money market account. Then the wealth $X_t$ of the investor is given by

$$
\frac{dX(t)}{X(t)} = \pi^S(t)\frac{dS(t)}{S(t)} + \pi^1 dO^1(t) + \pi^2 dO^2(t) + \pi^3 dO^3(t) + (1 - \pi^S - \pi^1 - \pi^2 - \pi^3) rdt
$$

$$
= rdt + \sum_{j=1}^{2} (\beta_j^S V_j - \beta_j^V \sqrt{\rho_j} \sqrt{V_j}) dt + \beta_j^S \sqrt{\rho_j} \sqrt{V_j} \tilde{W}_j(t) + \beta_j^V \sqrt{\rho_j} \sqrt{V_j} Z_j(t),
$$

where

$$
\beta_j^S = \pi_j^S + \sum_{i=1}^{3} \frac{g_i^j + \sigma_j^i g_i^j}{O^i(t)} \pi^i, \quad \beta_j^V = \sum_{i=1}^{3} \frac{\sigma_j^i \sqrt{1 - \rho_j^2 g_i^j}}{O^i(t)} \pi^i
$$

for $j = 1, 2$.

It follows from (14) that $\beta_1^S$, $\beta_2^S$, $\beta_1^V$ and $\beta_2^V$ represent the investor’s wealth exposure to the fundamental risk factors $W_1$, $W_2$, $Z_1$ and $Z_2$, respectively, and in matrix form

$$
\begin{bmatrix}
\beta_1^S \\
\beta_2^S \\
\beta_1^V \\
\beta_2^V
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\pi^S \\
\pi^1 \\
\pi^2 \\
\pi^3
\end{bmatrix} =: A \begin{bmatrix}
\pi^S \\
\pi^1 \\
\pi^2 \\
\pi^3
\end{bmatrix}.
$$

The matrix $A$ in (15) is assumed to be of full rank to keep the completeness of market with respect to the chosen derivative securities, the risky stock, and the money market account (see also [26, 12]). Thus, any exposure can be achieved by taking an appropriate position in a complete market. To build our analysis independent of the derivative security chosen, we consider the exposures instead of the portfolio weights.

Remark 3.1. It should be noted that the relation between completeness of market with the rank property of $A$ always keeps under both independent or correlated multi-factor volatility components, which differs from the results in [26, 12, 13].
The value function $J(t,x,v_1,v_2)$ satisfies the robust Hamilton-Jacobi-Bellman (HJB) PDE:

$$
\sup_{\beta^S_j,\beta^V_j,\epsilon^S_j,\epsilon^V_j} \left\{ J_t + x(r + \sum_{j=1}^{2} (\beta^S_j \lambda_j v_j - \beta^S_j \sqrt{\epsilon^S_j} + \beta^V_j \mu_j v_j - \beta^V_j \sqrt{\epsilon^V_j}))J_x + \frac{1}{2} \sum_{j=1}^{2} \left[(\beta^S_j)^2 + (\beta^V_j)^2\right]v_j J_{xx} + \sum_{j=1}^{2} (\kappa_j (\theta_j - v_j) - \rho_j \sigma_j \sqrt{\epsilon^S_j} - \sqrt{1 - \rho^2_j} \sigma_j \sqrt{\epsilon^V_j}) J_{v_j} \right\} = 0.
$$

The solution to (16) is provided in the following proposition.

**Proposition 3.1.** In a complete market, the indirect utility function of an ambiguity and risk averse investor is

$$
J(t,x,v_1,v_2) = \frac{x^{1-\gamma}}{1-\gamma} \exp \left[H_1(\tau)V_1 + H_2(\tau)V_2 + h(\tau)\right],
$$

where $\tau = T - t$ and functions $H_j, j = 1, 2$ and $h$ are given by

$$
H_j(\tau) = \frac{2c_j(1 - e^{-d_j \tau})}{2d_j + (a_j + d_j)(e^{-d_j \tau} - 1)},
$$

$$
h(\tau) = (1 - \gamma) \rho \tau - \sum_{j=1}^{2} \kappa_j \theta_j \left(\frac{a_j + d_j}{2b_j} \tau + \frac{1}{b_j} \ln \left(\frac{e^{-d_j \tau} a_j + d_j}{a_j + d_j} \right) \right).
$$

The optimal exposures to the risk factors $W_i$ and $Z_i$ ($i=1,2$) are

$$
\beta^S_j = \frac{\lambda_j}{\gamma + \phi^S_j} + \frac{(1 - \gamma - \phi^S_j) \sigma_j \rho_j}{(1 - \gamma)(\gamma + \phi^S_j)} H_j(\tau),
\beta^V_j = \frac{\mu_j}{\gamma + \phi^V_j} + \frac{(1 - \gamma - \phi^V_j) \sigma_j \sqrt{1 - \rho^2_j}}{(1 - \gamma)(\gamma + \phi^V_j)} H_j(\tau).
$$

The worst case measure are given by

$$
e^S_j = \left(\frac{\lambda_j}{\gamma + \phi^S_j} + \frac{\sigma_j \rho_j H_j(\tau)}{(1 - \gamma)(\gamma + \phi^S_j)}\right) \phi^S_j \sqrt{\epsilon^S_j},
\epsilon^V_j = \left(\frac{\mu_j}{\gamma + \phi^V_j} + \frac{\sigma_j \sqrt{1 - \rho^2_j} H_j(\tau)}{(1 - \gamma)(\gamma + \phi^V_j)}\right) \phi^V_j \sqrt{\epsilon^V_j}.
$$

**Proof.** See Appendix A1.

**Remark 3.2.** It is well-known that there exists a unique equivalent risk-neutral measure $\mathbb{P}$ in the complete market. Because the optimization problem (11) under measure $\mathbb{P}$ is equivalent to

$$
\max_{\mathbb{P} \in U(T)} \frac{1}{1-\gamma} E^\mathbb{P} \left[(X_T)^{1-\gamma}\right],
$$

under measure $\mathbb{P}$, the complete market condition determines the uniqueness of the worst-case measure and ensures the consistency of optimal solution under the worst-case measure with the risk-neutral measure.

Before exploring the impacts of ambiguity on the optimal exposures and the worst-case probability measure, we should mention the sign restrictions of the parameters, that is, $\lambda_j > 0$ and $\mu_j < 0$ and $\rho_j < 0$. It follows immediately from (19) that the optimal stock risk exposures $\beta^S_j$ are positive, whereas the
optimal volatility risk exposures $\beta_j^V$ are negative. It is obvious that each of the optimal exposures consists of a myopic component and a hedge component. The myopic components are constant in time and decrease as the corresponding ambiguity aversion parameters increase, whereas the hedge components are time-dependent and vanish as the investment horizon reaches the terminal time.

As shown in [20], the worst-case measure $e_j^S$ and $e_j^V$ vary linearly with respect to the $j$-th factor of the volatility, $\sqrt{V_j}$, however, they depend on both ambiguity aversion parameters $\phi_j^S$ and $\phi_j^V$ through function $H_j$. Thus, the model uncertainty can be thought of as the uncertainty about parameters $\Lambda_j$ and $\mu_j$, which control the market prices of all risks. Due to the combined effects of ambiguity aversion parameters and risk aversion parameters, the signs of the measures cannot be determined definitely, similar conclusions were also reported in [24, 26].

It is well known that for Heston-type processes, optimal portfolio solutions may not be well-behaved (see, for example, Proposition 5.2 in Kraft [23]). Therefore, it is necessary to enforce certain conditions on the parameter values to ensure that the solution is finite and real-valued. In particular, we ensure that the optimal change of measure and the function $J(t, x, v_1, v_2)$ are both well defined. The financial implication is that portfolio managers need to know whether a basket of parameter estimates is applicable for their investments.

**Proposition 3.2.** The Random-Nilodým derivative process under worst case measure

$$((e_j^S)^*, (e_j^S)^*, (e_j^V)^*, (e_j^V)^*))$$

is well-defined, thus the optimal portfolio is well-behaved, if

$$\left(\phi_j^S\right)^2 \frac{\nu_j^2 \sigma_j^2}{\gamma + \phi_j^S} + \left(\phi_j^V\right)^2 \frac{\mu_j^2 \sigma_j^2}{\gamma + \phi_j^V} \leq \kappa_1, \quad \left(\phi_j^S\right)^2 \frac{\nu_j^2 \sigma_j^2}{\gamma + \phi_j^S} + \left(\phi_j^V\right)^2 \frac{\mu_j^2 \sigma_j^2}{\gamma + \phi_j^V} \leq \kappa_2.$$

**Proof.** It is sufficient to show that Novikov’s condition is satisfied for the worst-case probability measure, that is,

$$E^\mathbb{P}\left[\exp\left\{\frac{1}{2} \int_0^T ((e_j^S)^*)^2 + ((e_j^S)^*)^2 + ((e_j^V)^*)^2 + ((e_j^V)^*)^2 dt\right\}\right] < \infty.$$

Applying (20) in Proposition 3.1 one obtains

$$E^\mathbb{P}\left[\exp\left\{\frac{1}{2} \int_0^T (K_1(T - t)V_1(t) + K_2(T - t)V_2(t)) dt\right\}\right] < \infty,$$

where

$$K_j(T - t) = \left(\frac{\Lambda_j}{\gamma + \phi_j} + \frac{\sigma_j \beta_j H_j(T - t)}{(1 - \gamma)(\gamma + \phi_j)}\right)^2 (\phi_j^S)^2 + \left(\frac{\mu_j}{\gamma + \phi_j} + \frac{\sigma_j \sqrt{1 - \rho_j^2} H_j(T - t)}{(1 - \gamma)(\gamma + \phi_j)}\right)^2 (\phi_j^V)^2$$

for $j = 1, 2$.

Define $k_1 = \sup_{t \in [0, T]} K_1(T - t)$ and $k_2 = \sup_{t \in [0, T]} K_2(T - t)$, then

$$E^\mathbb{P}\left[\exp\left\{\frac{1}{2} \int_0^T (K_1(T - t)V_1(t) + K_2(T - t)V_2(t)) dt\right\}\right] < E^\mathbb{P}\left[\exp\left\{\frac{k_1}{2} \int_0^T V_1(t) dt\right\}\right] + E^\mathbb{P}\left[\exp\left\{\frac{k_2}{2} \int_0^T V_2(t) dt\right\}\right] < \infty.$$
The first inequity in (21) holds because $V_1$ and $V_2$ are independent, while the second holds if $k_1 \leq \frac{\kappa_1^2}{\sigma_1^2}$ and $k_2 \leq \frac{\kappa_2^2}{\sigma_2^2}$ (see [23]).

Recall that $\gamma > 1$, $\lambda_j > 0$, $\rho_j < 0$ and $\mu_j < 0$. Thus, $K_j$ reaches its maximum value at $t = T$ because $H_j$ is maximum at $t = T$. Therefore,

$$k_1 = (\phi^{S}_1)^2 \frac{\lambda_1^2}{(\gamma + \phi^{S}_1)^2} + (\phi^{V}_1)^2 \frac{\mu_1^2}{(\gamma + \phi^{V}_1)^2}, \quad k_2 = (\phi^{S}_2)^2 \frac{\lambda_2^2}{(\gamma + \phi^{S}_2)^2} + (\phi^{V}_2)^2 \frac{\mu_2^2}{(\gamma + \phi^{V}_2)^2},$$

which yield the desired results immediately. \(\square\)

## 3.2 The incomplete market case

After studying the optimal portfolio in complete case, we now turn to consider the case when the investor can not trade in derivatives. This condition implies an incomplete market because the volatility risks can not be hedged directly. If the investor has no access to derivative securities, then we can obtain the indirect utility function of the investor by setting $\pi_1 = \pi_2 = \pi_3 = 0$ in (16). We obtain a proposition below to report the results.

**Proposition 3.3.** In an incomplete market, the indirect function of an ambiguity and risk averse investor is given by

$$J(t, x, v_1, v_2) = \frac{x^{1-\gamma}}{1-\gamma} \exp \left( H_1(\tau)v_1 + H_2(\tau)v_2 + \bar{h}(\tau) \right),$$

where the functions $H_1$, $H_2$ and $\bar{h}$ are obtained from (62).

The optimal exposures to the stock risk factors $W_i$ ($j = 1, 2$) are

$$\beta^{S}_j = \frac{\lambda_j}{\gamma + \phi^{S}_j} + \frac{(1 - \gamma - \phi^{S}_j)\gamma_j \rho_j}{(1 - \gamma)(\gamma + \phi^{S}_j)} H_j(\tau),$$

(23)

The worst case measure are

$$e_j^{S} = \left( \frac{\lambda_j}{\gamma + \phi^{S}_j} + \frac{\sigma_j \rho_j H_j(\tau)}{(1 - \gamma)(\gamma + \phi^{S}_j)} \right) \phi^{S}_j \sqrt{V_j}, \quad e_j^{V} = \frac{\sigma_j \sqrt{1 - \rho_j^2 H_j(\tau)}}{1 - \gamma} \phi^{V}_j \sqrt{V_j}.$$

(24)

**Proof.** See Appendix A2. \(\square\)

The derived optimal investments $\beta^{S}_1$ and $\beta^{S}_2$ in the risky asset are time-dependent. It is emphasized that the optimal exposures $\beta^{S}_1$ and $\beta^{S}_2$ depends on their volatility ambiguity parameters $\phi^{V}_1$ and only through the functions $H_1$ and $H_2$, which vanishes as the investment horizon decreases. Thus, the investment strategy of short-term investors in an incomplete market is relatively insensitive to volatility ambiguity. This feature is in sharp contrast to complete markets given in Proposition 3.1. Indeed, in the complete market case, the volatility ambiguity parameter has an additional impact on the optimal strategy through exposure $\beta^{V}_1$, which is absent in an incomplete market. One of the principal differences from the complete market case is that the perturbations $e^{V}_1$ and $e^{V}_2$ vanish as the investment horizon reduces. The intuitive explanation is that since the myopic investor does not care about future volatility, he/she also ignores any uncertainty associated with volatility.
3.3 Suboptimal strategies and utility losses

We start with the indirect utility function of the investor who follows an admissible suboptimal strategy \( \Pi \) given by

\[
J^{\Pi}(t, y) = \inf_{e \in \mathcal{E}[t, T]} \left[ w^e_t(t, y; \Pi) + \mathbb{E}_y^{\mathcal{E}^y} \left[ \int_t^T \sum_{j=1}^2 \frac{(e^e_j(s))^2}{2 \Psi^e_j(s, Y(s))} + \frac{(e^V_j(s))^2}{2 \Psi^V_j(s, Y(s))} ds \right] \right].
\]

(25)

The value function (25) thus satisfies the robust HJB PDE:

\[
\inf_{e^e_j, e^V_j} \left\{ J^{\Pi}_t + x(r + \sum_{j=1}^2 (\beta^e_j \lambda_j v_j - \beta^e_j \sqrt{v_j} e^e_j + \beta^V_j \mu_j v_j - \beta^V_j \sqrt{v_j} e^V_j)) J^{\Pi}_t \right. \\
+ \frac{1}{2} \lambda^2 \sum_{j=1}^2 (\beta^e_j)^2 + (\beta^V_j)^2) v_j J^{\Pi}_t + \sum_{j=1}^2 (\kappa_j (\theta_j - v_j) - \rho_j \sigma_j \sqrt{v_j} e^e_j - \sqrt{1 - \rho^2 \sigma_j \sqrt{v_j} e^V_j}) J^{\Pi}_v \\
+ \frac{1}{2} \lambda^2 v_j J^{\Pi}_v + \sum_{j=1}^2 \sigma_j v_j x(\beta^e_j \rho_j + \beta^V_j \sqrt{1 - \rho^2 \sigma_j \sqrt{v_j}}) J^{\Pi}_{x,v} + \sum_{j=1}^2 \frac{(e^e_j)^2}{2 \Psi^e_j} + \frac{(e^V_j)^2}{2 \Psi^V_j} = 0.
\]

(26)

Solving the optimal problem, we obtain the general worst case measures for suboptimal strategy as follows:

\[
(e^e_j)^* = \Psi^e_j(x \theta^e_j J_x + \rho_j \sigma_j J_{x,v}) \sqrt{v_j}, \quad (e^V_j)^* = \Psi^V_j(x \theta^V_j J_x + \sqrt{1 - \rho^2 \sigma_j J_{x,v}}) \sqrt{v_j}.
\]

(27)

We now consider three types of suboptimal strategies: \( \Pi_1 \), the investor ignores the uncertainty about the second volatility component; \( \Pi_2 \), the investor ignores the uncertainty about the first volatility component, and \( \Pi_3 \), the investor cannot trade derivatives. Similar to the studies in [12, 13, 4], the wealth-equivalent utility loss \( L^{\Pi} \) for a strategy \( \Pi \in \{\Pi_1, \Pi_2, \Pi_3\} \) is defined as the solution to

\[
J(t, x(1 - L^{\Pi}), v_1, v_2) = J^{\Pi}(t, x, v_1, v_2),
\]

where \( J^{\Pi} \) is the indirect utility function of the investor who follows a suboptimal strategy. We only consider the indirect utility functions which are of exponentially affine form. Thus,

\[
L^{\Pi} = 1 - \exp \left\{ \frac{1}{1 - \gamma} \left[ (H_1^{\Pi} - H_1) v_1 + (H_2^{\Pi} - H_2) v_2 + (h^{\Pi} - h) \right] \right\},
\]

where \( H_1, H_1^{\Pi}, H_2, H_2^{\Pi}, h \) and \( h^{\Pi} \) are some functions which will be discussed below. Next, we evaluate the indirect utility functions in the complete and incomplete markets when an investor follows the sub-optimal strategies. Because of the symmetry of \( \Pi_1 \) and \( \Pi_2 \), we will only discuss the cases associated with \( \Pi_1 \) and \( \Pi_3 \).

If an investor takes the strategy \( \Pi_1 \), all parameters describing the uncertainties of the stock risk and volatility risk associated with the first component of the volatility will disappear.

**Proposition 3.4.** The indirect utility function of an investor who adopts strategy \( \Pi_1 \) is given by

\[
J^{\Pi_1}(t, x, v_1, v_2) = \frac{\lambda^1}{1 - \gamma} \exp \left( H_1^{\Pi_1}(\tau) v_1 + H_2^{\Pi_1}(\tau) v_2 + h^{\Pi_1}(\tau) \right),
\]

(28)

where functions \( H_1^{\Pi_1}, H_2^{\Pi_1} \) and \( h^{\Pi_1} \) can be solved from (30) by setting \( \bar{\phi}^N = \bar{\phi}^V = 0. \)
Proof. Without loss of generality, we assume the indirect utility function of all suboptimal strategy \( \Pi \) as

\[
J^{\Pi} = \frac{x^{1-\gamma}}{1-\gamma} \exp \left( H_1(\tau)v_1 + H_2(\tau)v_2 + h(\tau) \right).
\]

For mathematical tractability, we choose

\[
\psi_j^s = \frac{\phi_j^s}{(1-\gamma)J(t, x, v_1, v_2)} \quad \text{and} \quad \psi_j^v = \frac{\phi_j^v}{(1-\gamma)J(t, x, v_1, v_2)},
\]

where the ambiguity aversion parameters \( \phi_j^s, \phi_j^v > 0 \).

The functions \( H_1^{\Pi}, H_2^{\Pi} \) and \( h_1^{\Pi} \) should satisfy the HJB PDE which leads to

\[
0 = - (h^{\Pi})' - \sum_{j=1}^{2} (H_j^{\Pi} v_j' + (1-\gamma)(r + \sum_{j=1}^{2} (\beta_j^s \lambda_j v_j + \beta_j^v \mu_j v_j) - \frac{\gamma(1-\gamma)}{2} \sum_{j=1}^{2} (\beta_j^s)^2 + (\beta_j^v)^2)v_j
\]

\[+ \sum_{j=1}^{2} (k_j (\theta_j - v_j)) H_j^{\Pi} + \frac{1}{2} \sum_{j=1}^{2} \sigma_j^2 v_j (H_j^{\Pi})^2 + \frac{1}{2} \sum_{j=1}^{2} \sigma_j v_j x (\beta_j^s \rho_j + \beta_j^v \sqrt{1-\rho_j^2}) H_j^{\Pi}
\]

\[- \sum_{j=1}^{2} \frac{1}{2} (1-\gamma) \phi_j^s v_j \left[ (\beta_j^s)^2 + 2\beta_j^s \rho_j \sigma_j \frac{H_j^{\Pi}}{1-\gamma} + \rho_j^2 \sigma_j^2 \frac{(H_j^{\Pi})^2}{(1-\gamma)^2} \right]
\]

\[- \sum_{j=1}^{2} \frac{1}{2} (1-\gamma) \phi_j^v v_j \left[ (\beta_j^v)^2 + 2\beta_j^v \sqrt{1-\rho_j^2} \sigma_j \frac{H_j^{\Pi}}{1-\gamma} + (1-\rho_j^2) \sigma_j^2 \frac{(H_j^{\Pi})^2}{(1-\gamma)^2} \right].
\]

Comparing the terms with and without multiplier \( v_j \), one can certainly obtain the following Riccati equations

\[
\begin{align*}
(H_j^{\Pi})' &= a_j H_j^{\Pi} + b_j (H_j^{\Pi})^2 + c_j, & H_j^{\Pi}(0) &= 0, \\
(H_1^{\Pi})' &= a_1 H_1^{\Pi} + b_1 (H_1^{\Pi})^2 + c_1, & H_1^{\Pi}(0) &= 0, \\
(H_2^{\Pi})' &= a_2 H_2^{\Pi} + b_2 (H_2^{\Pi})^2 + c_2, & H_2^{\Pi}(0) &= 0, \\
(h^{\Pi})' &= k_1 \theta_1 H_1^{\Pi} + k_2 \theta_2 H_2^{\Pi} + (1-\gamma)r, & h^{\Pi}(0) &= 0,
\end{align*}
\]

where

\[
a_j = -k_j + \sigma_j (1-\gamma) (\beta_j^s \rho_j + \beta_j^v \sqrt{1-\rho_j^2} - \phi_j^s \beta_j^s \rho_j \sigma_j - \phi_j^v \beta_j^v \sqrt{1-\rho_j^2} \sigma_j),
\]

\[
b_j = \frac{\sigma_j^2}{2} - \frac{\phi_j^s \rho_j^2 \sigma_j^2}{2(1-\gamma)} - \frac{\phi_j^v (1-\rho_j^2) \sigma_j^2}{2(1-\gamma)},
\]

\[
c_j = (1-\gamma)(\beta_j^s \lambda_1 + \beta_j^v \lambda_2) - \frac{1}{2} (1-\gamma) (\beta_j^s)^2 + (\beta_j^v)^2 - \frac{(1-\gamma)(\beta_j^s)^2 \phi_j^s}{2} - \frac{(1-\gamma)(\beta_j^v)^2 \phi_j^v}{2}.
\]

\[\Box\]

Remark 3.3. For an investor who adopts strategy \( \Pi_2 \) in complete markets and incomplete markets, the indirect function \( J^{\Pi_2} \) can be obtained by setting \( \phi_2^s = \phi_2^v = 0 \) in (30).

We are now ready to consider the strategy \( \Pi_3 \) that the investor cannot trade derivatives. As pointed out by Liu & Pan [26], a risk-averse investor benefits from derivative trading without ambiguity. Recently, Escobar et al. [12] showed that the loss from ignoring derivative trading depends on the ambiguity aversion parameters under a stochastic volatility. Here we report the utility loss under the multi-factor volatility model.
Proposition 3.5. In complete markets, the welfare loss from ignoring the information about the ambiguity is strictly positive, that is, \( L^{\Pi_1} > 0 \) and \( L^{\Pi_2} > 0 \). In addition, the welfare loss from no derivative trading is also strictly positive so that \( L^{\Pi_3} > 0 \).

Proof. Recall the utility loss is calculated as

\[
L^\Pi = 1 - \exp \left\{ \frac{1}{1 - \gamma} \left[ (H_1^\Pi - H_1)^{v_1} + (H_2^\Pi - H_2)^{v_2} + (h^\Pi - h) \right] \right\}.
\]

If the investor chooses strategy \( \Pi_1 \), then \( H_2^{\Pi_1} = H_2 \) since the strategy does not affect on the uncertainty about the second component of the ambiguity. Assume that \( \beta_1^1 = p_1^S + p_2^S \tilde{H}_1 \) and \( \beta_1^V = p_1^V + p_2^V \tilde{H}_1 \), where \( p_j^S \) and \( p_j^V \) (\( j = 1, 2 \)) are constants. Let \( \tilde{H}_1 \) be the function satisfying the equation \( (\tilde{H}_1)' = \tilde{a}\tilde{H}_1 + \tilde{b}(\tilde{H}_1)^2 + \tilde{c} \), where \( \tilde{a}, \tilde{b} \) and \( \tilde{c} \) are all constants, we obtain the following system of equations in terms of \( H_1^{\Pi_1} \) and \( \tilde{H}_1 \)

\[
\begin{cases}
(H_1^{\Pi_1})' = P_1 + P_2\tilde{H}_1 + P_3(H_1^{\Pi_1})^2 + P_4H_1^{\Pi_1} + P_5\tilde{H}_1H_1^{\Pi_1} + P_6(H_1^{\Pi_1})^2, \quad H_1^{\Pi_1}(0) = 0 \\
(\tilde{H}_1)' = \tilde{a}\tilde{H}_1 + \tilde{b}(\tilde{H}_1)^2 + \tilde{c}, \quad \tilde{H}_1(0) = 0,
\end{cases}
\]

where

\[
\begin{align*}
P_1 &= (1 - \gamma)(\lambda_1 p_1^S + \lambda_2 p_2^V) - \frac{1}{2}(\gamma + \tilde{\phi}_1^S)(p_1^S)^2 - \frac{1}{2}(\gamma + \tilde{\phi}_1^V)(p_1^V)^2, \\
P_2 &= (1 - \gamma)(\lambda_1 p_1^S + \lambda_2 p_2^V - p_1^S p_2^V(\gamma + \tilde{\phi}_1^S) - p_1^V p_2^V(\gamma + \tilde{\phi}_1^V)), \\
P_3 &= -\frac{1}{2}(1 - \gamma)((1 + \tilde{\phi}_1^S)(p_2^S)^2 + (1 + \tilde{\phi}_1^V)(p_2^V)^2), \\
P_4 &= -\kappa_1 + \sigma_1(\rho_1(1 - \gamma - \tilde{\phi}_1^S)p_1^S + \sqrt{1 - \rho_1^2(1 - \gamma - \tilde{\phi}_1^V)p_1^V}), \\
P_5 &= \sigma_1(\rho_1(1 - \gamma - \tilde{\phi}_1^V)p_2^S + \sqrt{1 - \rho_1^2(1 - \gamma - \tilde{\phi}_1^V)p_2^V}), \\
P_6 &= \frac{1}{2(1 - \gamma)}\sigma_1^2(1 - \gamma - \rho_1^2\tilde{\phi}_1^S - (1 - \rho_1^2)\tilde{\phi}_1^V).
\end{align*}
\]

It follows from (60) and (31) that

\[
(H_1^{\Pi_1} - H_1)' + E(t)(H_1^{\Pi_1} - H_1)' = F(t),
\]

where

\[
E = -P_4 - P_6(H_1^{\Pi_1} + H_1) - P_5\tilde{H}_1
\]

and

\[
F = P_1 - c + (P_4 - a)H_1 + (K_6 - b)H_1^2 + P_2\tilde{H}_1 + P_3\tilde{H}_1^2 + P_5\tilde{H}_1H_1.
\]

After solving equation (33), we have

\[
H_1^{\Pi_1}(t) - H_1(t) = e^{-\int_t^T E(s)ds} \int_t^T e^{\int_s^T E(r)dr} F(s)ds.
\]

Because of the assumption that the suboptimal strategies are admissible, the integral \( e^{-\int_t^T E(s)ds} \) is bounded and positive. Furthermore, the choice of strategy \( \Pi_1 \) means that \( \tilde{\phi}_1^S = 0 \) and \( \tilde{\phi}_1^V = 0 \). Thus,

\[
F = \frac{\gamma - 1}{2\gamma^2(\phi_1^S + \gamma)} \left( \lambda_1(1 + \phi_1^S + (\gamma + \phi_1^S)\rho_1\sigma_1(\tilde{H}_1 - \frac{\gamma(\gamma + \phi_1^S - 1)}{(\gamma - 1)(\gamma + \phi_1^S)}H_1) \right)^2 \\
+ \frac{\gamma - 1}{2\gamma^2(\phi_1^V + \gamma)} \left( \lambda_1(1 + \phi_1^V) \sqrt{1 - \rho_1^2\sigma_1(\tilde{H}_1 - \frac{\gamma(\gamma + \phi_1^V - 1)}{(\gamma - 1)(\gamma + \phi_1^V)}H_1}) \right)^2 > 0.
\]
This leads to \( h_{1}^{I} - H_{1} > 0 \). Following (50), we obtain

\[
\begin{align*}
 h_{1}^{I} - h = \kappa_{1} \theta_{1} \int_{0}^{\ell} \left( H_{1}^{I}(s) - H_{1}(s) \right) ds > 0.
\end{align*}
\]

Therefore,

\[
\begin{align*}
 L_{1}^{I} = 1 - \exp \left\{ \frac{1}{1 - \gamma} \left[ (H_{1}^{I} - H_{1}) v_{1} + (H_{1}^{I} - h) \right] \right\} > 0.
\end{align*}
\]

Due to symmetry \( L_{2}^{I} > 0 \) can be proved in a similar way. Now If \( L_{1}^{I} \) is chosen, from Propositions 3.1 and 3.3 we can obtain, for \( j = 1, 2 \),

\[
\begin{align*}
 H_{j}^{I} (t) - H_{j}(t) &= e^{- \int_{t}^{\ell} E_{j}(s) ds} \int_{t}^{\ell} e^{\int_{s}^{\ell} E_{j}(\tau) d\tau} F_{j}(s) ds
\end{align*}
\]

where

\[
\begin{align*}
 F_{j} = \frac{\gamma - 1}{2(\gamma + \phi_{j})} \left( \lambda_{2} + \frac{\sigma_{j} H_{j}^{I} \sqrt{1 - \rho_{j}^{2}(\gamma + \phi_{j}) - 1}}{\gamma - 1} \right).
\end{align*}
\]

It is straightforward to see that \( F_{1} > 0 \) and \( F_{2} > 0 \), thus we have \( L_{1}^{I} > 0 \). This completes the proof. \( \Box \)

### 3.4 Asset price with jump risks

Now we extend our robust optimal portfolio selection to the cases when there are jump risks in asset price. Based on the analysis in the last section, we deliver a series of results for the extensions. With jump risks, the asset price follows the dynamics under multi-factor volatility is as follows

\[
\begin{align*}
 \Delta S(t) = \left( r + \lambda_{1} V_{1}(t) + \lambda_{2} V_{2}(t) + J^{(\tilde{\varphi} - \tilde{\varrho})} (V_{1}(t) + V_{2}(t)) \right) dt + \sqrt{V_{1}(t)} dW_{1}(t)
\end{align*}
\]

\[
\begin{align*}
 + \sqrt{V_{2}(t)} dW_{2}(t) + J^{(\tilde{\varphi} - \tilde{\varrho})} dN(t) - \tilde{\varrho} (V_{1}(t) + V_{2}(t)) dt,
\end{align*}
\]

where \( N(t) \) is a Poisson process independent of all Brownian motions with stochastic arrival intensity \( \tilde{\varrho}(V_{1}(t) + V_{2}(t)) \). Here the jump size of the Poisson process, \( J^{\tilde{\varphi}} > -1 \), is set to be constant. Then the option price dynamics under multi-factor volatility is as follows

\[
\begin{align*}
 dO(t) = rO(t) dt + \sum_{j=1}^{2} \left( \left( g_{S}^{(i)} S + \sigma_{j} \rho_{j} g_{V_{j}}^{(i)} \right) (\lambda_{1} V_{j} dt + \sqrt{V_{j}} dW_{j}(t)) + \sum_{j=1}^{2} \sigma_{j} \sqrt{1 - \rho_{j}^{2} g_{V_{j}}^{(i)} (\mu_{j} V_{j} dt + \sqrt{V_{j}} dZ_{j}(t))} \right)
\end{align*}
\]

\[
\begin{align*}
 + \Delta g^{(i)} (\tilde{\varphi} - \tilde{\varrho}) (V_{1}(t) + V_{2}(t)) + dN(t) - \tilde{\varrho} (V_{1}(t) + V_{2}(t)) dt,
\end{align*}
\]

where \( \Delta g^{(i)} = g^{(i)}((1 + j) S, V_{1}, V_{2}) - g^{(i)}(S, V_{1}, V_{2}) \).

We thus obtain a new reference model which combines multi-factor stock volatility with jump risks in asset price. To make the model more tractable, we still assume that the investor is uncertain only about the distribution of Brownian motions. In other words, there is no ambiguity about \( N_{t} \). One additional derivative is needed to hedge the added jump risks.
For each perturbation process \( e \), the investor will consider the following alternative model: the stock price is observed by
\[
\frac{dS(t)}{S(t-)} = \left( r + \sum_{j=1}^{2} [\lambda_j V_j(t) - \sqrt{V_j(t)}e_j(t)] + f^S(\nu^p - \nu^Q)(V_1(t) + V_2(t)) \right) dt + \sum_{j=1}^{2} \sqrt{V_j(t)}d\tilde{W}_j(t) + f^S(dN(t) - \nu^p(V_1(t) + V_2(t))dt),
\]
and at the same time, its variances still follow the dynamic \( \tilde{\sigma}_j(t) \) while the option prices of the stock satisfy
\[
dO^{(i)}(t) = rO^{(i)}(t)dt + \sum_{j=1}^{2} \left( (g_j^{(1)}S + \sigma_j \rho_j g_j^{(1)}) \right) \left( (\lambda_j V_j - e_j \sqrt{V_j})dt + \sqrt{V_j}d\tilde{Z}_j(t) \right) + \Delta g^{(i)}(\nu^p - \nu^Q)(V_1(t) + V_2(t)) + dN(t) - \nu^p(V_1(t) + V_2(t))dt),
\]
where \( \Delta g^{(i)} = g^{(i)}((1 + j)S, V_1, V_2) - g^{(i)}(S, V_1, V_2) \), \( i = 1, \ldots, 4 \).

The value function \( V \) satisfies the robust HJB PDE:
\[
0 = \sup_{\beta_j, \theta_j, \rho_j, \varphi_j} \inf \left\{ J(t) + \int_0^t \left( r + \sum_{j=1}^{2} (\beta_j^S \lambda_j V_j - \beta_j^S J_{j,xx} + \beta_j^V \mu_j V_j - \beta_j^V J_{j,x}) dt + \sqrt{V_j} \sigma_j \nu_1 \right) \right\},
\]
where \( J(t, x(1 + \beta^N_j, v_1, v_2)) = J(t, x, v_1, v_2) \).

**Proposition 3.6.** In a complete market, the indirect utility function of an ambiguity and risk averse investor is given by
\[
J(t, x, v_1, v_2) = \frac{x^{1-\gamma}}{1-\gamma} \exp \left( C_1(\tau)v_1 + C_2(\tau)v_2 + c(\tau) \right),
\]
where the functions \( C_1, C_2 \) and \( c \) are given by
\[
C_1(\tau) = \frac{2x_1(1 - e^{-d_1 \tau})}{(a_1 + d_1)(e^{-d_1 \tau} - 1)}, \quad C_2(\tau) = \frac{2x_2(1 - e^{-d_2 \tau})}{(a_2 + 2d_2)(e^{-d_2 \tau} - 1)},
\]
\[
c(\tau) = (1 - \gamma)\tau \sum_{j=1}^{2} \left( a_j + d_j \right) \frac{1}{2b_j} \ln \left( \frac{e^{-d_j \tau(a_j + d_j)} - a_j - d_j}{1} \right),
\]
with \( d_1 = \sqrt{a_1^2 - 4b_1s_1}, \quad d_2 = \sqrt{a_2^2 - 4b_2s_2} \) and
\[
a_j = -\kappa_j + \frac{\lambda_1(1 - \gamma - \phi_j^S)\sigma_j}{\gamma + \phi_j^V} + \frac{\lambda_2(1 - \gamma - \phi_2^V)\sigma_j}{\gamma + \phi_2^V} \sqrt{1 - \rho_j^2},
\]
\[
b_j = \frac{\sigma_j^2 - \phi_j^S \rho_j^2 \sigma_j^2}{2(1 - \gamma)} - \frac{\phi_j^V(1 - \rho_j^2) \sigma_j^2}{2(1 - \gamma)} + \frac{(1 - \gamma - \phi_j^V)\sigma_j^2}{2(1 - \gamma)(\gamma + \phi_j^V)} + \frac{(1 - \gamma - \phi_j^V)^2 \sigma_j^2}{2(1 - \gamma)(\gamma + \phi_j^V)} + \frac{(1 - \gamma - \phi_j^S)^2 \sigma_j^2}{2(1 - \gamma)(\gamma + \phi_j^S)},
\]
\[
s_j = \frac{(1 - \gamma)A_1^2}{2(\gamma + \phi_j^S)} + \frac{(1 - \gamma)A_2^2}{2(\gamma + \phi_j^V)} + \left( \frac{\nu^p}{\nu^Q} \frac{1}{\gamma \nu^Q} - (\gamma - 1)\nu^Q - \nu^p \right).
\]
The optimal exposures to the risk factors $W_i, Z_i (j = 1, 2)$ and $N(t)$ are

\[
\beta^S_j = \frac{\lambda_j}{\gamma + \phi^S_j} + \frac{(1 - \gamma - \phi^S_j)\sigma_j \rho_j}{(1 - \gamma)(\gamma + \phi^S_j)} C_j(\tau), \quad \beta^V_j = \frac{\mu_j}{\gamma + \phi^V_j} + \frac{(1 - \gamma - \phi^V_j)\sigma_j \sqrt{1 - \rho^2_j}}{(1 - \gamma)(\gamma + \phi^V_j)} C_j(\tau). \tag{43}
\]

and

\[
\beta^N = \frac{1}{j} \left( \left( \frac{\nu^P}{\nu^Q} \right)^{1/y} - 1 \right).
\]

The worst case measure are

\[
e^S_j = \left( \frac{\lambda_j}{\gamma + \phi^S_j} + \frac{\sigma_j \rho_j C_j(\tau)}{(1 - \gamma)(\gamma + \phi^S_j)} \right) \phi^S_j \sqrt{\nu_j}, \quad e^V_j = \left( \frac{\mu_j}{\gamma + \phi^V_j} + \frac{\sigma_j \sqrt{1 - \rho^2_j} C_j(\tau)}{(1 - \gamma)(\gamma + \phi^V_j)} \right) \phi^V_j \sqrt{\nu_j}. \tag{44}
\]

**Proof.** The proof is similar to the one in Proposition 3.1, so we omit it here. \qed

**Proposition 3.7.** In an incomplete market, the indirect function of an ambiguity and risk averse investor is given by

\[
J = \frac{\nu^{1-y}}{1 - \gamma} \exp \left( C_1(\tau)v_1 + C_2(\tau)v_2 + \bar{c}(\tau) \right), \tag{45}
\]

where functions $C_1, C_2$ and $\bar{c}$ can be derived can be derived as similar forms to those in Proposition 3.3 and the optimal exposures to the stock risk factors $W_i (j = 1, 2)$ are

\[
\beta^S_j = \frac{\lambda_j}{\gamma + \phi^S_j} + \frac{(1 - \gamma - \phi^S_j)\sigma_j \rho_j}{(1 - \gamma)(\gamma + \phi^S_j)} \bar{C}_j(\tau). \tag{46}
\]

The worst case measure are

\[
e^S_j = \left( \beta^S_j + \frac{\sigma_j \rho_j \bar{C}_j(\tau)}{1 - \gamma} \right) \phi^S_j \sqrt{\nu_j}, \quad e^V_j = \frac{\sigma_j \sqrt{1 - \rho^2_j} \bar{C}_j(\tau)}{1 - \gamma} \phi^V_j \sqrt{\nu_j}. \tag{47}
\]

**Proof.** The proof is similar to that for Proposition 3.3, so we omit it here. \qed

**Corollary 3.1.** The utility loss from ignoring jump risk is strictly positive if $\nu^P \neq \nu^Q$ and is zero otherwise. In other words, $\nu^P = \nu^Q$ implies that the suboptimal strategy of ignoring jump risk becomes optimal and this nullifies the welfare loss.

**Proof.** Let $\Pi_4$ denote this strategy, we prove that if $\nu^Q \neq \nu^P$,

\[
L^{\Pi_4} = 1 - \exp \left( \frac{1}{1 - \gamma} \left\{ (C_1^{\Pi_4} - C_1)v_1 + (C_2^{\Pi_4} - C_2)v_2 + (c^{\Pi_4} - c) \right\} \right) > 0.
\]

Indeed, we have

\[
C_1^{\Pi_4}(t) - C_1(t) = e^{-\int_t^T E_4(s)ds} \int_t^T e^{\int_t^y E_4(t)dt} F_4(s)ds,
\]

and

\[
C_2^{\Pi_4}(t) - C_2(t) = e^{-\int_t^T E_5(s)ds} \int_t^T e^{\int_t^y E_5(t)dt} F_5(s)ds,
\]
where
\[
\begin{align*}
E_4 &= -(a + b(C_1 + C_1^4)) \\
E_5 &= -(a + b(C_2 + C_2^4)) \\
F_4 &= F_5 = -\left(\frac{\nu^2}{\nu^2 - \nu^2}\right)^{1/\gamma} \gamma \nu^2 - (\gamma - 1) \nu^2 - \nu^2.
\end{align*}
\]

Using standard calculus techniques it can be shown that \(F_4 = F_5 > 0\) if \(\nu^2 \neq \nu^2\). Therefore, we can certainly obtain that \(\Pi^* > 0\). This completes the proof. 

\[\square\]

3.5 Correlated volatility processes

In this subsection, we extend our model to the case where two volatilities are correlated in the asset price. This is an often occurring phenomenon which has been verified in the literature \[19, 20, 21, 37\]. Assuming \(\langle dW_1(t), dW_2(t) \rangle = \rho dt\), we obtain a new robust PDE with respect to \(X, V_1\) and \(V_2\) after solving the robust optimization problem \[11\]. However, we encounter difficulty in solving the new robust PDE, as the correlation between \(W_1\) and \(W_2\) adds a non-affine term \(\sqrt{V_1}V_2\):

\[
\begin{align*}
\sup_{\beta_1, \beta_2, \epsilon_1, \epsilon_2} &\inf_{\beta_1, \beta_2, \epsilon_1, \epsilon_2} \left\{ J_t + x(r + \frac{1}{2} \sum_{j=1}^{2} \left( \beta_j^S \lambda_j \mu_j - \beta_j^S \sqrt{\nu_j} \mu_j - \beta_j^S \sqrt{\nu_j} \epsilon_j \right) J_{x} + \frac{1}{2} \alpha \sum_{j=1}^{2} \left( \epsilon_j^2 \Gamma_j + \beta_j^2 \right) V_j J_{xx} \\
&+ \sum_{j=1}^{2} \left( \kappa_j (\theta_j - \nu_j) - \rho_j \sigma_j \sqrt{\nu_j} \epsilon_j \right) J_{v_j} + \frac{1}{2} \sum_{j=1}^{2} \sigma_j^2 V_j J_{v_j v_j} \right\} \quad (48)
\end{align*}
\]

Inspired by the ideas of Grzelak et al. \[19, 20, 21\], we develop an approximation method to solve the PDE and, therefore, the optimal problem. For the mean-reverting variance processes \(3\) and \(4\) when the Feller conditions are satisfied, we have

\[
V_1(t) = c_1(t) \chi(d_1, \lambda_1(t)), \quad V_2(t) = c_2(t) \chi(d_2, \lambda_2(t)),
\]

where for \(j = 1, 2\)

\[
c_j(t) = \frac{1}{4 \kappa_j} \sigma_j^2 (1 - e^{-\kappa_j t}), \quad d_j = \frac{4 \kappa_j \theta_j}{\sigma_j^2}, \quad \lambda_j(t) = \frac{4 \kappa_j e^{-\kappa_j t} V_j(0)}{\sigma_j^2 (1 - e^{-\kappa_j t})}.
\]

and \(\chi(d_j, \lambda_j(t))\) is a non-central chi-squared random variable with the degree of freedom parameter \(d_j\) and non-centrality parameter \(\lambda_j(t)\). Then the expectation and variance of the volatility \(\sqrt{V_j}\) can be found as

\[
\begin{align*}
E \left( \sqrt{V_j(t)} \right) &= \sqrt{2 c_j(t) e^{-\lambda_j(t)/2}} \sum_{k=0}^{\infty} k! \left( \frac{\lambda_j(t)}{2} \right)^k \frac{\Gamma(d_j/2 + k)}{\Gamma(d_j/2 + k)} \Gamma(d_j/2 + k) \\
\text{Var} \left( \sqrt{V_j(t)} \right) &= c_j(t)(d_j + \lambda_j(t)) - 2 c_j(t) e^{-\lambda_j(t)} \left( \sum_{k=0}^{\infty} k! \left( \frac{\lambda_j(t)}{2} \right)^k \frac{\Gamma(d_j/2 + k)}{\Gamma(d_j/2 + k)} \right)^2.
\end{align*}
\]

(49)

where \(\Gamma(z) = \int_0^{\infty} t^{-1} e^{-t} dt\).
Thus, on the context of moments, an approximation method for the dynamics with the types of $\sqrt{V_1}$ and $\sqrt{V_2}$ is introduced in \[19\]. Having the same first two moments as $\sqrt{V_1}$ and $\sqrt{V_2}$ respectively, the following processes can be regarded as the estimates for $\sqrt{V_j}$

$$dU_j(t) = \mu_j^U(t)dt + \psi_j^U(t)\left(\rho_jdW_j(t) + \sqrt{1 - \rho_j^2}dZ_j(t)\right), \quad U_j(0) = \sqrt{V_j(0)} > 0$$

for $j = 1, 2$ with deterministic time-dependent drift $\mu_j^U(t)$ and volatility $\psi_j^U(t)$ given by

$$\mu_j^U(t) = \frac{1}{2\sqrt{\nu}} \frac{\Gamma(d_{j-1}^+ + 1)}{\Gamma(d_{j}^+)}\left(\frac{1}{2}\sigma_j^2e^{-\kappa_j t}\tilde{F}\left(\frac{1}{2}, \frac{d_{j}^+}{2}, -\frac{\lambda_j(t)}{2}\right) - \frac{4c_j(t)\kappa_j(\epsilon_j/t)V_j(0)}{\sigma_j^2(e^{\epsilon_j} - 1)^2}\tilde{F}\left(\frac{1}{2}, \frac{d_{j}^+ + 1}{2}, -\frac{\lambda_j(t)}{2}\right)\right)$$

and

$$\psi_j^U(t) = \left(\frac{\sigma_j^2e^{-\kappa_j t}}{4}(d_{j}^+ + \lambda_j(t)) - \frac{4c_j(t)\kappa_j(\epsilon_j/t)V_j(0)}{\sigma_j^2(e^{\epsilon_j} - 1)^2} - 2\tilde{E}(V_j(t)\mu_j^U(t))\right)^{1/2},$$

where $\tilde{E}(V_j(t))$ is given by \[49\] and the regularized hypergeometric function $\tilde{F}(a; b; c) = F(a; b; c)/\Gamma(b)$.

We introduce an additional variable $Y(t) = U_1(t)U_2(t) \approx \sqrt{V_1(t)V_2(t)}$ and specify

$$dY(t) = \mu^Y dt + U_1(t)\psi_2^U(t)\left(\rho_2dW_2(t) + \sqrt{1 - \rho_2^2}dZ_2(t)\right) + U_2(t)\psi_1^U(t)\left(\rho_1dW_1(t) + \sqrt{1 - \rho_1^2}dZ_1(t)\right),$$

where $\mu^Y = \mu_1^U(t)U_2(t) + \mu_2^U(t)U_1(t) + \rho\psi_1^U(t)\psi_2^U(t)$.

Equations \[21\]-\[25\], \[30\] and \[31\] form the new reference model. Allowing the perturbation process $e$, the corresponding alternative model consists of \[7\], \[8\] and \[9\] with the following affinity correction processes

$$dU_j(t) = \mu_j^U(t)dt + \psi_j^U(t)\left(\rho_j(d\tilde{W}_j(t) - e_j^X(t)) + \sqrt{1 - \rho_j^2}(d\tilde{Z}_j(t) - e_j^Y(t))\right), \quad U_j(0) = \sqrt{V_j(0)} > 0$$

and

$$dY(t) = \mu^Y dt + U_1(t)\psi_2^U(t)\left[\rho_2(d\tilde{W}_2(t) - e_2^X(t)) + \sqrt{1 - \rho_2^2}(d\tilde{Z}_2(t) - e_2^Y(t))\right]$$

$$+ U_2(t)\psi_1^U(t)\left[\rho_1(d\tilde{W}_1(t) - e_1^X(t)) + \sqrt{1 - \rho_1^2}(d\tilde{Z}_1(t) - e_1^Y(t))\right], \quad Y(0) = \sqrt{V_1(0)V_2(0)} > 0.$$

We now introduce a new process $R(s) = (X(s), V_1(s), V_2(s), U_1(s), U_2(s), Y(s))$. The expected utility achieved by a trading strategy $\Pi$ is given by

$$w^e(t, r; \Pi) = \frac{1}{1 - \gamma} E^r_{t, x}[\left(X_T\right)^{1 - \gamma}],$$

where $r = (x, v_1, v_2, u_1, u_2, y)$ denotes the value of $R(s)$ at time $t$.

The indirect utility function of the investor is

$$J(t, r) = \sup_{\Pi \in \Pi(t, T)} \inf_{\epsilon \in \mathcal{E}(t, T)} \left(w^e(t, r; \Pi) + E^r_{t, x}\left[\int_t^T \sum_{j=1}^2 \frac{(e_j^X(s))^2}{2\psi_j^X(s, Y(s))} + \frac{(e_j^Y(s))^2}{2\psi_j^Y(s, Y(s))}ds\right]\right).$$
Then in the complete market, the value function \((52)\) satisfies the robust HJB PDE:

\[
\begin{align*}
\sup_{\beta^1_t, \beta^2_t} \inf_y \left( J_x + x(r + \sum_{j=1}^2 (\beta^j_t \lambda_j v_j - \beta^j_t u_j e^j) + \beta^V_t \mu_j v_j - \beta^V_t u_j e^V_j))J_x \right.
\end{align*}
\]

\[
+ \sum_{j=1}^2 (\kappa_j(\theta_j - v_j) - \rho_j \sigma_j u_j e^j)J_{x} - \sum_{j=1}^2 (\mu^j Y_j - \rho_j \psi^j Y e^j) J_{y} - \sum_{j=1}^2 (\rho_j Y_j - \rho_j \psi^j Y e^j) J_{u_j}
\]

\[
+ \left( \mu^V - \rho_2 \psi^V e^2 u_1 - \frac{1 - \rho^2_2 \psi^2 Y^2}{2} u_1 - \rho_1 \psi^1 U e^1 u_2 - \frac{1 - \rho^2_1 \psi^1 U^2}{2} u_2 \right) J_{y}
\]

\[
+ \frac{1}{2} \sigma^2 \sum_{j=1}^2 (\beta^j)^2 \sigma_j v_j x + \sum_{j=1}^2 \sigma_j v_j x (\beta^j \rho_j + \beta^V \rho_1 \psi^j Y e^j) \sqrt{1 - \rho^2_1 \psi^1 U^2} J_{x} + \rho \psi_1 U \beta^j \psi^j U^2 u_2 x J_{x u_1}
\]

\[
+ \rho \psi_2^V \beta^2 \psi^2 U u_1 x J_{x u_2} + \frac{1}{2} \sum_{j=1}^2 (\psi^2 U^2 - \beta^j \beta^V U^2 \psi^j Y^2) J_{y} + \frac{1}{2} \sum_{j=1}^2 \Sigma_{jj} J_{y j} + \Sigma_{12} J_{y 12}
\]

\[
+ \sum_{j=1}^2 \Sigma_{j j + 2} J_{y j u j} + \Sigma_{14} J_{y u 1} + \Sigma_{23} J_{y u 2} + \Sigma_{15} J_{y 1} + \Sigma_{25} J_{y 2} + \frac{1}{2} \sum_{j=3}^4 \Sigma_{j j} J_{j u j}
\]

\[
+ \Sigma_{34} J_{u 1 u 2} + \Sigma_{35} J_{u 1 u 2} + \Sigma_{45} J_{u 1 u 2} + \Sigma_{55} J_{y} + \frac{1}{2} \sum_{j=1}^2 \left( \frac{(e^j)^2}{2} + \frac{(e^V)^2}{2} \right) = 0,
\]

where the symmetrical matrix \(\Sigma(t)\) is given as follows:

\[
\begin{bmatrix}
\frac{1}{2} \sigma_1^2 v_1 & \rho \rho_1 \rho_2 \sigma_1 \sigma_2 & \sigma_1 u_1 \psi^U_1 & \rho \rho_1 \sigma_1 \rho_1 \psi^U_1 u_1 & \rho \rho_1 (1 - \rho^2) \sigma_1 \psi^U_1 \\
\frac{1}{2} \sigma_2^2 v_2 & \rho \rho_1 \rho_2 \sigma_2 \psi^U_1 u_2 & \sigma_2 \psi^U_1 u_2 & \rho \rho_1 \rho_2 \sigma_2 \psi^U_1 u_2 & (\psi^U)^2 u_2 + \rho \rho_1 \rho_2 \psi^U_1 \psi^U_1 u_1 \\
\frac{1}{2} (\psi^U)^2 & \rho \rho_1 \rho_2 \psi^U_1 \psi^U_1 & \frac{1}{2} (\psi^U)^2 & \rho \rho_1 \rho_2 \psi^U_1 \psi^U_1 & \frac{1}{2} (\psi^U)^2 + u_2^2 (\psi^U)^2 + \rho \rho_1 \rho_2 \psi^U_1 \psi^U_1
\end{bmatrix}
\]

We further assume that the value function \(J\) is of the affine-form

\[
J(t, x, v_1, v_2, u_1, u_2, y) = \frac{x^{1-\gamma}}{1-\gamma} \exp(H^V_1(t-T)v_1 + H^V_2(t-T)v_2 + H^U_1(t-T)u_1 + H^U_2(t-T)u_2 + H^Y(t-T)y + h),
\]

where \(H^V_1, H^V_2, H^U_1, H^U_2, H^Y\) and \(h\) can be solved from \((69)\) in Appendix A3. Thus, the corresponding optimal exposures and worst case measures can be derived analytically.

It is worth to note that is that the correlation between two volatility processes can affect both worst-case measures and optimal exposures. As mentioned before the optimal exposures to the stock and additional multi-factor volatility risks consist of myopic and hedge components. Each hedge component is dependent on the correlated volatility structure. The worst-case measures to the stock price and volatility are also dependent of the correlated volatility structure which is different from the results in \((31)\) for the independent multi-factor volatility model.
4 Numerical experiments

In this section, numerical experiments are carried out to examine the behavior of the optimal portfolio and the utility losses. We assume that the investment horizon is $T = 10$ (years), the risk-free interest rate is $r = 0.05$ and the risk aversion parameters is $\gamma = 4$. This standard assumption can be also found in Flor & Larsen [16] and Escobar et al. [12]. $j^S = -15\%$, $\nu^P = 0.1$ and $\nu^Q = 0.3$ are used as in [6] and [12]. For the referee model, we use the same set of parameters as Liu & Pan [26], in addition to the parameters associated with the multi-factor volatility, that is, $V_1(0) = 0.2^2$, $\kappa_1 = 3$, $\theta_1 = 0.1^2$, $\rho_1 = -0.7$, $\sigma_1 = 0.25$, $\lambda_1 = 3$, $\mu_1 = -3$, $V_2(0) = 0.01^2$, $\kappa_2 = 3.5$, $\theta_2 = 0.2^2$, $\mu_2 = -0.3$, $\sigma_2 = 0.01$, $\lambda_2 = 2$, $\mu_2 = -3$.

As explained in Section 2, the parameters $\phi_1^S$, $\phi_2^S$, $\phi_1^V$ and $\phi_2^V$ describe the preference for robustness. Anderson et al. [2] argue that these parameters should be chosen in such a way that it should be difficult to distinguish the reference model from the worst case model based on a time series of final length. The worst case models considered by the robust investor are given by Propositions 3.1 and 3.3 for the complete market and incomplete market, respectively. Propositions 3.6 and 3.7 derive the worst case models under the general case models considered by the robust investor are given by Propositions 3.1 and 3.3 for the complete market and incomplete market, respectively. The worst case models under the general jump diffusion cases. Define two Radon-Nikodým derivatives $Z_1(t) = E^P \left[ \frac{dP}{d\mathbb{F}_t} | \mathbb{F}_t \right]$ and $Z_2(t) = E^P \left[ \frac{dP}{d\mathbb{F}_t} | \mathbb{F}_t \right]$ and consider their logarithms:

$$
\xi_1(t) = \ln Z_1(t) = - \int_0^t \sum_{j=1}^2 \left[ \frac{1}{2} \left( (e_j^S(t))^2 + (e_j^V(t))^2 \right) dt + e_j^S(t) dW_j(t) + e_j^V(t) dZ_j(t) \right],
$$

$$
\xi_2(t) = \ln Z_2(t) = \int_0^t \sum_{j=1}^2 \left[ \frac{1}{2} \left( (e_j^S(t))^2 + (e_j^V(t))^2 \right) dt + e_j^S(t) dW_j(t) + e_j^V(t) dZ_j(t) \right].
$$

Based on a sample of finite length $T$, it is clear that the decision maker will discard the reference model mistakenly for the worst case model if $\xi_1(T) > 0$. On the other hand, if the worst case model is true, then it will be rejected erroneously if $\xi_2(T) > 0$ (or $\xi_1(T) < 0$). We accordingly define the detection-error probability

$$
e_T(\phi_1^S, \phi_2^S, \phi_1^V, \phi_2^V) = \frac{1}{2} P(\xi_1(T) > 0 | P, \mathbb{F}_0) + \frac{1}{2} P(\xi_1(T) < 0 | P, \mathbb{F}_0),
$$

which can be calculated explicitly by using Fourier inversion method, and further details are given in Appendix A4.

Figures 1(a) - 1(d) show the effects of different ambiguity-aversion parameters for the detection-error probability in complete markets and incomplete markets, respectively. On the whole, the detection-error are different for the volatilities $V_1$ and $V_2$ both in complete and incomplete markets. It can be seen from the figures that the larger the ambiguity-aversion parameters, the smaller the probability of making mistakes. Furthermore, the ambiguity-aversion level for asset price and that for volatility exhibit different sensitivity for the detection-error probability.

Figures 2(a) and 2(b) show adjustments $e_2^S / \sqrt{V_2}$ and $e_2^V / \sqrt{V_2}$ to $\lambda_1$ and $\lambda_2$ for second component uncertainty in the complete market. The figure demonstrates that adjustments are functions of ambiguity-reversion parameters. Adjustment $e_2^S / \sqrt{V_2}$ is affected heavily by $\phi_2^S$, but only slightly by $\phi_2^V$, while adjustment $e_2^V / \sqrt{V_2}$ is affected heavily by $\phi_2^V$ and slightly by $\phi_2^S$. This result is consistent with that of the single volatility model in [12].
Figures 3 and 4 show the optimal exposures in complete markets and incomplete markets, respectively. Firstly, stock risk exposure is more sensitive to ambiguity about the stock risk than the volatility risk and the volatility risk exposure is more sensitive to ambiguity about the volatility risk than the stock risk. Second, the ambiguity in one component of the volatility has no effect on the optimal exposure of the other component of the volatility. It should be pointed out that the optimal stock exposure decreases as the stock ambiguity parameter increases, whereas the optimal volatility exposure increases as the volatility ambiguity parameter increases. The differences between the results for the two markets are also large, indicating that derivative trading is of great importance for hedging risk and thus for making better portfolio choices.

Figure 5 shows that the welfare loss in the complete market is significantly larger than that in the incomplete market. It is seen that the loss increases substantially with ambiguity about the stock price and its volatility in the complete market. As for the incomplete market, the loss does not seem to change significantly with the ambiguity about the volatility process. We also observe that the two volatility processes in the asset price have different effects on utility loss, thus one cannot neglect the contribution of the multi-factor volatility.
Adjustment to $\lambda_1$ and $\lambda_2$ for second component uncertainty in the complete market.

Figure 2: Adjustment to $\lambda_1$ and $\lambda_2$ for second component uncertainty in the complete market.

structure.

Figure 6 depicts that the variation of the welfare loss with ambiguity about the volatility and stock risks. Not trading derivatives causes lower ambiguity-aversion and the larger losses. It is further demonstrated that the utility loss from ignoring ambiguity in a complete market can be almost as much as the loss from not trading the derivatives. The ambiguity-aversion parameters for two-factor volatility risks play a more important role on utility loss than the ones for the asset price. Therefore, including derivative trading is necessary for an investor when making investment decisions.

5 Concluding remarks

We have studied the robust optimal portfolio selection problem for a risk-averse and ambiguity-averse investor who can invest in a money market account, a stock whose price follows the multi-factor volatility model, and several correlated derivatives. The robust optimal investment strategy is determined for the cases with/without derivative trading. Then, sub-optimal investment strategies and their utility losses were discussed and therefore the effects of ambiguity and market-completeness are examined. Two extension cases, jump risks in asset price and correlated multi-factor volatilities, are also studied comprehensively and thus the corresponding optimal investment strategies are derived.

We have demonstrated that the ambiguity about asset price and volatility could influence investment decisions in both complete markets and incomplete markets, and therefore one should consider searching for a robust optimal portfolio. The contributions from jump risk and the correlated volatilities were proved to be significant in robust optimal portfolio selection problems.
Appendix

A1. The proof of Proposition 3.1

Proof. Solving optimal problem (16) with respect to \( e^S_j \) and \( e^V_j \), we have

\[
(e^S_j)^* = \Psi^S_j \left( x \theta^S_j J_s + \rho_j \sigma_j J_{v_j} \right) \sqrt{\nu_j},
\]

\[
(e^V_j)^* = \Psi^V_j \left( x \theta^V_j J_s + \sqrt{1 - \rho_j^2 \sigma_j J_{v_j}} \right) \sqrt{\nu_j}.
\]

(55)
Substituting (55) into (16), we obtain the following equation

\[
\sup_{\beta^S, \beta^V} \left\{ J_t + x(r + \sum_{j=1}^{2}(\beta^S_j \lambda_j + \beta^V_j \mu_j)J_x) + \frac{1}{2} \sum_{j=1}^{2} [(\beta^S_j)^2 + (\beta^V_j)^2]v_j J_{xj} + \frac{1}{2} \sum_{j=1}^{2} (\kappa_j(\theta_j - v_j))J_{vj} 
\right. 
+ \frac{1}{2} \sum_{j=1}^{2} \sigma^2_j v_j J_{vj} + \sum_{j=1}^{2} \sigma_j v_j x(\beta^S_j \rho_j + \beta^V_j \sqrt{1 - \rho_j^2})J_{vjx} 
\left. 
- \sum_{j=1}^{2} \frac{1}{2} \psi_j^S v_j \left[ x^2(\beta^S_j)^2 J_x^2 + 2x\beta_j^S \rho_j x_j v_j + \rho_j^2 \sigma_j^2 J_{vj}^2 \right] 
\right. 
- \sum_{j=1}^{2} \frac{1}{2} \psi_j^V v_j \left[ x^2(\beta^V_j)^2 J_x^2 + 2x\beta_j^V \sqrt{1 - \rho_j^2} \sigma_j J_{xj} v_j + (1 - \rho_j^2) \sigma_j^2 J_{vj} \right] \right\} = 0. 
\]

Assuming that the solution \( J \) is of the form

\[
J(t, x, v_1, v_2) = \frac{x^{1-\gamma}}{1-\gamma} \exp(H_1(T-t)v_1 + H_2(T-t)v_2 + h(T-t)) 
\]

and choosing

\[
\psi_j^S = \frac{\phi_j^S}{(1-\gamma)J(t, x, v_1, v_2)}, \quad \psi_j^V = \frac{\phi_j^V}{(1-\gamma)J(t, x, v_1, v_2)}, \quad \phi_j^S, \phi_j^V > 0, 
\]

equation (56) yields

\[
\sup_{\beta^S, \beta^V} \left\{ -h' - \sum_{j=1}^{2} (H_j'\gamma) + (1-\gamma)(r + \sum_{j=1}^{2} (\beta^S_j \lambda_j + \beta^V_j \mu_j)) - \frac{\gamma(1-\gamma)}{2} \sum_{j=1}^{2} [(\beta^S_j)^2 + (\beta^V_j)^2]v_j 
\right. 
+ \frac{1}{2} \sum_{j=1}^{2} (\kappa_j(\theta_j - v_j))H_j + \frac{1}{2} \sum_{j=1}^{2} \sigma_j^2 v_j (H_j^2) + \sum_{j=1}^{2} \sigma_j v_j x(\beta^S_j \rho_j + \beta^V_j \sqrt{1 - \rho_j^2})H_{jx} 
\left. 
- \sum_{j=1}^{2} \frac{1}{2} (1 - \gamma) \phi_j^S \left[ (\beta^S_j)^2 + 2\beta_j^S \rho_j \frac{H_j}{1-\gamma} + \rho_j^2 \sigma_j^2 \frac{(H_j)^2}{(1-\gamma)^2} \right] 
\right. 
- \sum_{j=1}^{2} \frac{1}{2} (1 - \gamma) \phi_j^V \left[ (\beta^V_j)^2 + 2\beta_j^V \sqrt{1 - \rho_j^2} \sigma_j \frac{H_j}{1-\gamma} + (1 - \rho_j^2) \sigma_j^2 \frac{(H_j)^2}{(1-\gamma)^2} \right] \right\} = 0, 
\]
where $h'$ represents the time derivative of function $h(t)$. From (57), we derive the optimal exposures as

\[
\begin{align*}
(\beta_{j}^S)^* &= \frac{\lambda_j}{\gamma + \phi_j^S} + \frac{(1 - \gamma - \phi_j^S)\sigma_j \rho_j}{(1 - \gamma)(\gamma + \phi_j^S)} H_j(T - t), \\
(\beta_{j}^V)^* &= \frac{\mu_j}{\gamma + \phi_j^V} + \frac{(1 - \gamma - \phi_j^V)\sigma_j \sqrt{1 - \rho_j^2}}{(1 - \gamma)(\gamma + \phi_j^V)} H_j(T - t). 
\end{align*}
\]
where can be obtained from (60). This completes the proof. □

To determine $H_1$ and $H_2$ and $h$, we substitute (58) into (57) to obtain

$$
-h' - \sum_{j=1}^{2} (H_j)' v_j + (1 - \gamma)(r + \sum_{j=1}^{2} (\kappa_j(\theta_j - v_j))H_j) + \frac{1}{2} \sum_{j=1}^{2} \sigma_j^2 v_j (H_j)^2 - \sum_{j=1}^{2} \frac{1}{2} \phi_j v_j \rho_j^2 \sigma_j^2 (H_j)^2 \frac{(H_j)^2}{1 - \gamma} - \sum_{j=1}^{2} \frac{1}{2} \phi_j v_j (1 - \rho_j^2) \sigma_j^2 (H_j)^2 \frac{(H_j)^2}{1 - \gamma} + \sum_{j=1}^{2} \frac{(1 - \gamma) \lambda_j^2}{2(\gamma + \phi_j^2)} + \frac{\lambda_1(1 - \gamma - \phi_j^S)\sigma_j \rho_j H_j}{\gamma + \phi_j^2} \frac{(1 - \gamma - \phi_j^S)^2 \sigma_j^2 (1 - \rho_j^2)(H_j)^2}{2(1 - \gamma)(\gamma + \phi_j^2)} v_j
$$

$$
+ \sum_{j=1}^{2} \left[ \frac{(1 - \gamma) \lambda_j^2}{2(\gamma + \phi_j^2)} + \frac{\lambda_2(1 - \gamma - \phi_j^V)\sigma_j \sqrt{1 - \rho_j^2} H_j}{\gamma + \phi_j^2} + \frac{(1 - \gamma - \phi_j^V)^2 \sigma_j^2 (1 - \rho_j^2)(H_j)^2}{2(1 - \gamma)(\gamma + \phi_j^2)} \right] v_j = 0.
$$

Comparing the terms concerned with $v_j$, we get the following Riccati equations

$$
H_1' = a_1 H_1 + b_1 (H_1)^2 + c_1, \quad H_1(0) = 0,
$$
$$
H_2' = a_2 H_2 + b_2 (H_2)^2 + c_2, \quad H_2(0) = 0,
$$
$$
h' = \kappa_1 \theta_1 H_1 + \kappa_2 \theta_2 H_2 + (1 - \gamma) r, \quad h(0) = 0,
$$

where

$$
a_j = -\kappa_j + \frac{\lambda_1(1 - \gamma - \phi_j^S)\sigma_j \rho_j}{\gamma + \phi_j^2} + \frac{\lambda_2(1 - \gamma - \phi_j^V)\sigma_j \sqrt{1 - \rho_j^2}}{\gamma + \phi_j^2},
$$
$$
b_j = \frac{\sigma_j^2}{2} - \frac{\phi_j^S \rho_j^2 \sigma_j^2}{2(1 - \gamma)} - \frac{\phi_j^V (1 - \rho_j^2) \sigma_j^2}{2(1 - \gamma)} + \frac{(1 - \gamma - \phi_j^S)^2 \sigma_j^2 (1 - \rho_j^2)}{2(1 - \gamma)(\gamma + \phi_j^2)} - \frac{(1 - \gamma - \phi_j^V)^2 \sigma_j^2 (1 - \rho_j^2)}{2(1 - \gamma)(\gamma + \phi_j^2)},
$$
$$
c_j = \frac{(1 - \gamma) \lambda_j^2}{2(\gamma + \phi_j^2)} + \frac{(1 - \gamma) \lambda_j^2}{2(\gamma + \phi_j^2)}. $$

For simplicity, let $d_1 = \sqrt{a_1^2 - 4b_1c_1}$ and $d_2 = \sqrt{a_2^2 - 4b_2c_2}$, the solution to $H_1$, $H_2$ and $h$ as shown in (18) can be obtained from (60). This completes the proof. □
A2. The proof of Proposition 3.3

**Proof.** By setting \( \pi_1 = \pi_2 = \pi_3 = 0 \) in (15), we have \( \pi^S = \beta^S_1 = \beta^S_2 \) and \( \beta^V_1 = \beta^V_2 = 0 \). Then, we obtain

\[
\pi^S = \frac{\lambda_1}{\gamma + \phi_1^S} + \frac{(1 - \gamma - \phi_1^S) \sigma_1 \rho_1}{(1 - \gamma)(\gamma + \phi_1^S)} \tilde{H}_1(\tau)
\]

and

\[
\pi^S = \frac{\lambda_2}{\gamma + \phi_2^S} + \frac{(1 - \gamma - \phi_2^S) \sigma_2 \rho_2}{(1 - \gamma)(\gamma + \phi_2^S)} \tilde{H}_2(\tau).
\]

Note that \( \pi^S \) is a function of either \( \tilde{H}_1 \) or \( \tilde{H}_2 \). Writing \( \pi^S = \pi_1(\tilde{H}_1) \) and \( \pi^S = \pi_2(\tilde{H}_2) \) and then substituting them in to (39), we obtain the following HJB equation

\[
0 = -h' + \sum_{j=1}^{2} (\tilde{H}_j)' v_j + (1 - \gamma)(r + \sum_{j=1}^{2} \pi_j(\tilde{H}_j) \lambda_j v_j) - \frac{\gamma(1 - \gamma)}{2} \sum_{j=1}^{2} \pi_j^2(\tilde{H}_j) v_j
\]

\[
+ \sum_{j=1}^{2} (\kappa_j(\theta_j - v_j)) \tilde{H}_j + \frac{1}{2} \sum_{j=1}^{2} \sigma_j^2 v_j(\tilde{H}_j)^2 + \sum_{j=1}^{2} \sigma_j v_j \rho_j \tilde{H}_j \pi_j(\tilde{H}_j)
\]

\[
- \frac{2}{2} (1 - \gamma) \phi_j^S v_j \left[ (\pi_j(\tilde{H}_j))^2 + 2 \pi_j(\tilde{H}_j) \rho_j \sigma_j \frac{\tilde{H}_j}{1 - \gamma} + \rho_j^2 \sigma_j^2 \frac{(\tilde{H}_j)^2}{(1 - \gamma)^2} \right]
\]

\[
- \frac{2}{2} (1 - \gamma) \phi_j^V v_j (1 - \rho_j^2) \sigma_j^2 \frac{(\tilde{H}_j)^2}{(1 - \gamma)^2}.
\]

Comparing the terms, the solution of \( \tilde{H}_1, \tilde{H}_2, \tilde{h} \) are solved by

\[
\begin{cases}
\tilde{H}_j' = (1 - \gamma) \pi_j(\tilde{H}_j) \lambda_j - \frac{\gamma(1 - \gamma)}{2} \pi_j^2(\tilde{H}_j) \lambda_j + \frac{1}{2} \sigma_j^2(\tilde{H}_j)^2 + (1 - \gamma) \sigma_j \rho_j \tilde{H}_j \pi_j(\tilde{H}_j)
\end{cases}
\]

\[
- \frac{1}{2} (1 - \gamma) \phi_j^S \left[ (\pi_j(\tilde{H}_j))^2 + 2 \pi_j(\tilde{H}_j) \rho_j \sigma_j \frac{\tilde{H}_j}{1 - \gamma} + \rho_j^2 \sigma_j^2 \frac{(\tilde{H}_j)^2}{(1 - \gamma)^2} \right]
\]

\[
- \frac{1}{2} (1 - \gamma) \phi_j^V (1 - \rho_j^2) \sigma_j^2 \frac{(\tilde{H}_j)^2}{(1 - \gamma)^2},
\]

\[
\tilde{h}' = \kappa_1 \theta_1 \tilde{H}_1 + \kappa_2 \theta_2 \tilde{H}_2 + (1 - \gamma) r,
\]

\[
\tilde{h}(0) = 0.
\]

The worst-case measures can be solved directly by (24) and thus the proof is complete. \( \Box \)

A3. The solution of robust HJB PDE (54)

Solving optimal problem (54) with respect to \( e_1^S, e_2^S, e_1^V, e_2^V \), we obtain

\[
\begin{cases}
(e_1^S)' = \Psi_1^S (x \phi_1^S u_1 J_x + \rho_1 \sigma_1 u_1 J_{v_1} + \rho_1 \psi_1^U J_{u_1} - \rho_1 \psi_1^U u_2 J_y),
\end{cases}
\]

\[
\begin{cases}
(e_2^S)' = \Psi_2^S (x \phi_2^S u_2 J_x + \rho_2 \sigma_2 u_2 J_{v_2} + \rho_1 \psi_2^U J_{u_2} - \rho_2 \psi_2^U u_1 J_y),
\end{cases}
\]

\[
\begin{cases}
(e_1^V)' = \Psi_1^V (x \phi_1^V u_1 J_x + \sqrt{1 - \rho_1^2} \sigma_1 u_1 J_{v_1} - \sqrt{1 - \rho_1^2} \psi_1^U J_{u_1} - \sqrt{1 - \rho_1^2} \psi_1^U u_2 J_y),
\end{cases}
\]

\[
\begin{cases}
(e_2^V)' = \Psi_2^V (x \phi_2^V u_2 J_x + \sqrt{1 - \rho_2^2} \sigma_2 u_2 J_{v_2} - \sqrt{1 - \rho_2^2} \psi_2^U J_{u_2} + \sqrt{1 - \rho_2^2} \psi_2^U u_1 J_y).
\end{cases}
\]
Substituting (63) into (54) and choosing

\[ \psi_j^S = \frac{\phi_j^S}{(1-\gamma)J}, \quad \psi_j^V = \frac{\phi_j^V}{(1-\gamma)J}, \tag{64} \]

we further derive

\[
\begin{align*}
(\beta_1^S)^* &= \frac{1}{\phi_2^S - \gamma} \left[ \lambda_1 + \sigma_1 \rho_1 H_{V_1}^U + \frac{1}{u_1} \rho \rho_2 \psi_2^U H_{U_2}^U + (\rho \rho_2 \psi_2^U + \rho_1 \frac{u_2}{u_1} \psi_1^U) H_{Y}^V \right. \\
&\quad \left. - \frac{\psi_1^S}{1-\gamma} (\rho \sigma_1 H_{V_1}^U + \frac{1}{u_1} \rho \psi_1^U H_{U_1}^U - \frac{u_2}{u_1} \rho \psi_1^U H_{Y}^V) \right] = a_1^S(t) + a_1^U(t) \frac{1}{u_1} + a_2^U(t) \frac{u_2}{u_1}, \\
(\beta_2^S)^* &= \frac{1}{\phi_2^S - \gamma} \left[ \lambda_2 + \sigma_2 \rho_2 H_{V_2}^U + \frac{1}{u_2} \rho \rho_1 \psi_1^U H_{U_1}^U + (\rho \rho_1 \psi_1^U + \rho_2 \frac{u_1}{u_2} \psi_2^U) H_{Y}^V \right. \\
&\quad \left. - \frac{\psi_2^S}{1-\gamma} (\rho \sigma_2 H_{V_2}^U + \frac{1}{u_2} \rho \psi_2^U H_{U_2}^U - \frac{u_1}{u_2} \rho \psi_2^U H_{Y}^V) \right] = a_2^S(t) + a_2^U(t) \frac{1}{u_2} + a_3^U(t) \frac{u_1}{u_2}, \\
(\beta_1^V)^* &= \frac{1}{\phi_1^V - \gamma} \left[ \mu_1 + \sqrt{1 - \rho_1^2} \psi_1^U H_{Y}^V - \frac{\psi_1^V}{1-\gamma} \left( \mu_1 - \rho_1 H_{V_1}^U - \psi_1^U \frac{u_2}{u_1} H_{Y}^V \right) \right] \\
&\quad = b_1^V(t) + b_1^U(t) \frac{1}{u_1} + b_2^U(t) \frac{u_2}{u_1}, \\
(\beta_2^V)^* &= \frac{1}{\phi_2^V - \gamma} \left[ \mu_2 + \sqrt{1 - \rho_2^2} \psi_2^U H_{Y}^V - \frac{\psi_2^V}{1-\gamma} \left( \mu_2 - \rho_2 H_{V_2}^U - \psi_2^U \frac{u_1}{u_2} H_{Y}^V \right) \right] \\
&\quad = b_2^V(t) + b_2^U(t) \frac{1}{u_2} + b_3^U(t) \frac{u_1}{u_2}.
\end{align*}
\tag{65} \]

Substituting (65) and (64) into (63), we certainly have

\[
\begin{align*}
(\varepsilon_1^S)^* &= g_1^U(t) u_1 + g_2^U(t) u_2 + g_3^U(t), \\
(\varepsilon_2^S)^* &= g_1^U(t) u_1 + g_2^U(t) u_2 + g_3^U(t), \\
(\varepsilon_1^V)^* &= k_1^U(t) u_1 + k_2^U(t) u_2 + k_3^U(t), \\
(\varepsilon_2^V)^* &= k_1^U(t) u_1 + k_2^U(t) u_2 + k_3^U(t).
\end{align*}
\tag{66} \]
Collecting terms with respect to \( v_j \), \( u_j \), \( u_1 \), \( u_2 \) and \( y \), we then rewrite (67) by a simpler form:

\[
0 = J_t + x(r + \sum_{j=1}^2 (\lambda_j v_j a_1' + a_2' \frac{1}{u_j} + a_3' \frac{u_{3-j}}{u_j}) + \mu_j v_j (b_1' + b_2' \frac{1}{u_j} + b_3' \frac{u_{3-j}}{u_j})))J_x
\]

\[
+ \sum_{j=1}^2 \left( (\kappa(j\theta - v_j))J_{v_j} - \sum_{j=1}^2 \mu_j u_j + \frac{\mu_j u_2 + \mu_2 u_1 + \rho \psi_1' \psi_1^U}{u_j} J_y \right)
\]

\[
+ \frac{1}{2} x^2 \sum_{j=1}^2 [(a_1' + a_2' \frac{1}{u_j} + a_3' \frac{u_{3-j}}{u_j})^2 + (b_1' + b_2' \frac{1}{u_j} + b_3' \frac{u_{3-j}}{u_j})^2]v_j J_{v_j}
\]

\[
+ \sum_{j=1}^2 \sigma_j v_j x(\rho_j a_1' + a_2' \frac{1}{u_j} + a_3' \frac{u_{3-j}}{u_j}) + \sqrt{1 - \rho_j^2} (b_1' + b_2' \frac{1}{u_j} + b_3' \frac{u_{3-j}}{u_j}) J_{v_j}
\]

\[
+ \rho \psi_1' \psi_1^U (a_1' + a_2' \frac{1}{u_2} + a_3' \frac{u_{3-j}}{u_2}) u_2 x J_{u_2} + \rho \psi_2' \psi_2^U (a_1' + a_2' \frac{1}{u_1} + a_3' \frac{u_{3-j}}{u_1}) u_1 x J_{u_2}
\]

\[
+ x \left( (a_1' + a_2' \frac{1}{u_1} + a_3' \frac{u_{3-j}}{u_1}) u_1 (\rho_2 u_2 \psi_2' + \rho_1 u_1 \psi_1') + (a_1^2 + a_2^2 \frac{1}{u_2} + a_3^2 \frac{u_{3-j}}{u_2}) u_2 (\rho_2 u_2 \psi_2' + \rho_1 u_1 \psi_1') \right)
\]

\[
+ \sqrt{1 - \rho_1^2} (b_1' + b_2' \frac{1}{u_1} + b_3' \frac{u_{3-j}}{u_1}) \psi_1' \psi_1 + \sqrt{1 - \rho_2^2} (b_1' + b_2' \frac{1}{u_2} + b_3' \frac{u_{3-j}}{u_2}) \psi_2' \psi_2)
\]

\[
+ \sum_{j=1}^2 \Sigma_{j,j} J_{v_j v_j} + \Sigma_{12} J_{v_1 v_2} + \sum_{j=1}^2 \Sigma_{j,j+2} J_{v_j u_j} + \Sigma_{14} J_{u_1 u_2} + \Sigma_{23} J_{u_1 u_2} + \Sigma_{15} J_{v_1 y} + \Sigma_{25} J_{v_2 y}
\]

\[
+ \sum_{j=3}^4 \Sigma_{j,j-3} u_j u_j + \Sigma_{34} J_{u_1 u_2} + \Sigma_{35} J_{u_1 y} + \Sigma_{45} J_{u_2 y} + \Sigma_{55} J_{v_2 y}
\]

\[
- (1 - \gamma) J \sum_{j=1}^2 \frac{1}{\phi_j} (g_1^j u_1 + g_2^j u_2 + g_3^j) + \frac{1}{\phi_j} (k_1^j u_1 + k_2^j u_2 + k_3^j)
\]

Collecting terms with respect to \( v_1, v_2, u_1, u_2 \) and \( y \), we then rewrite (67) by a simpler form:

\[
0 = J_t + p_1^V(t) v_1 J + p_2^V(t) v_2 J + p_1^U(t) u_1 J + p_2^U(t) u_2 J + p^Y(t) y J + C(t),
\]

which implies that \( H_1^V, H_2^V, H_1^U, H_2^U, H^Y \) and \( h \) can be obtained by following ODEs:

\[
(H_1^V(t))' = -p_1^V(t), \quad (H_2^V(t))' = -p_2^V(t), \quad (H_1^U(t))' = -p_1^U(t), \quad (H_2^U(t))' = -p_2^U(t), \quad h'(t) = -C(t).
\]

**A4. Detection-error probabilities**

Define the conditional characteristic functions

\[
f_1(\omega, t, T) = E^\mathbb{P} [\exp(\iota \omega \xi_1(T))|F_t^{S,V_1,V_2}] = E^\mathbb{P} [(Z_1(T))^{i\omega} | F_t^{S,V_1,V_2}]
\]

and

\[
f_2(\omega, t, T) = E^\mathbb{P} [\exp(\iota \omega \xi_1(T))|F_t^{S,V_1,V_2}] = E^\mathbb{P} [(Z_1(T))^{i\omega+1} | F_t^{S,V_1,V_2}],
\]

where \( \iota^2 = -1 \) and \( \omega \) is the transform variable. Denote \( e_t = [e_1^S, e_2^S, e_1^V, e_2^V] \) and

\[
\sigma = \left[ \sigma_1 \sqrt{V_1 \rho_1}, \sigma_2 \sqrt{V_2 \rho_2}, \sigma_1 \sqrt{V_1} \sqrt{1 - \rho_1^2}, \sigma_2 \sqrt{V_2} \sqrt{1 - \rho_2^2} \right].
\]
Since the conditional characteristic functions are martingales, the Feynman-Kac theorem implies that \( f_1 \) and \( f_2 \) satisfy the same PDE

\[
\frac{\partial f}{\partial t} + \sum_{j=1}^{2} \kappa_j (\theta_j - V_j) \frac{\partial f}{\partial V_j} + \frac{1}{2} \sum_{j=1}^{2} \sigma_j^2 \theta_j^2 \frac{\partial^2 f}{\partial V_j^2} + \frac{1}{2} \sum_{j=1}^{2} \sigma_j^2 \frac{\partial^2 f}{\partial V_j^2} = 0
\]  

with different terminal conditions \( f_1(\omega, T, T) = Z_{11}^{\omega}(T) \) and \( f_2(\omega, T, T) = Z_{12}^{\omega+1}(T) \), respectively. Conjecturing the solution in the form \( f = Z_{11}^{\omega}(t) \exp(C(t)V_1 + C(t)V_2 + D(t)) \) and substituting it into (70), we obtain

\[
\frac{dC_j(t)}{dt} - \kappa_j C_j(t) + \frac{1}{2} \omega(\omega - 1)(q_j^S + q_j^V)^2 + \frac{1}{2} \sigma_j^2 C_j(t) - i\omega C_j(t) \sigma_i(q_j^S \rho_j + q_j^V \sqrt{1 - \rho_j^2}) = 0
\]

for \( j = 1, 2 \) and

\[
\frac{dD(t)}{dt} + \kappa_1 \theta_1 C_1(t) + \kappa_2 \theta_2 C_2(t) = 0,
\]

subject to \( C_1(T) = C_2(T) = D(T) = 0 \).

Similarly, we can obtain the solution for \( f_2 \) by replicating \( i\omega \) with \( i\omega + 1 \) in the above equations. Thus, we have the detection-error probability

\[
\varepsilon_T(\phi_1^S, \phi_2^S, \phi_1^V, \phi_2^V) = \frac{1}{2} - \frac{1}{2\pi} \int_0^{\infty} \left[ \text{Re} \left( \frac{f_1(\omega, 0, T)}{i\omega} \right) - \text{Re} \left( \frac{f_2(\omega, 0, T)}{i\omega} \right) \right] d\omega.
\]

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