Two-dimensional noncommutative gravitational quantum well

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Abstract
In this paper we consider two kinds of noncommutative space-time commutation relations in two-dimensional configuration space and feature the absolute value of the minimal length from the generalized uncertainty relations associated to the particular commutation relations. We study the problem of the two-dimensional gravitational quantum well in new Hermitian variables and confront the experimental results for the first lowest energy state of the neutrons in the Earth’s gravitational field to estimate the upper bounds on the noncommutativity parameters. The absolute value of the minimum length is smaller than a few nanometers.

Keywords: deformed algebras, minimal length, noncommutative quantum mechanics, gravitational quantum well

1. Introduction

A new set of noncommutative space-time commutation relations in two-dimensional configuration space has been recently introduced [1]. The space–space commutation relations are deformations of the standard flat noncommutative space-time relations that have position dependent structure constants. These deformations lead to minimal lengths and it has been found that any object in this two dimensional space is string-like in the sense that having a fundamental length in one direction beyond which a resolution is impossible. Under suitable transformations and choices, new variables that are Hermitian have been set and some simple models have been solved in these new variables. Some extensions of this work have been made in [2, 3] and the model of harmonic oscillator has been solved in these new variables [4]. The aim of the present work is to solve the analog of the problem of a particle in the quantum
well of the Earth’s gravitational field in new variables obtained from deformed commutation relations.

The gravitational quantum well is obtained by considering a particle of mass \( m \) moving in the \( xy \) plane subject to the Earth’s gravitational field \( \vec{g} = -g \hat{e}_z \) and a perfectly reflecting mirror placed at the bottom (at \( x = 0 \)). In ordinary quantum mechanics this problem is well-known and studied in text books [5–9]. A possibility for measuring such states for neutrons has been discussed in [10]. The lowest quantum state of neutrons in such a system has been observed experimentally at the Institut Laue-Langevin [11, 12]. The existence of this phenomenon has been confirmed and studied in more details in [13]. These experiments gave the opportunity to confront observations and various theoretical models concerning quantum effects in gravity. Indeed various papers on the gravitational quantum well in a noncommutative geometry can be found in the literature [14–21].

In the present paper we consider the gravitational quantum well in two-dimensional quantum mechanics with deformed commutation relations and we determine how the parameters of deformation affect its energy spectrum and consider the experimental results to place upper bounds on the parameters of deformations. The paper is organized as follows. In the next section, we summarize the deformed commutation relations in two-dimensional quantum mechanics, the details being given in [1]. In section 3, we discuss the analog of the gravitational quantum well in new variables obtained from the deformed commutation relations and determine for each case the shift in the energy spectrum. In section 4, we give concluding remarks where we compare our results with the existing results in the literature and study the upper bounds on the parameters of noncommutativity.

2. Deformed commutation relations in two-dimensional quantum mechanics

A particle moving in \( d \)-dimensions is described in wave mechanics by a configuration space \( \mathbb{R}^d \) and a Hilbert space \( L^2 \) of square integrable wavefunctions \( \psi(x) \) over \( \mathbb{R}^d \). The inner product on \( L^2 \) is

\[
(\phi, \psi) = \int d^d x \phi^\dagger(x) \psi(x).
\]

(1)

The elements of this Hilbert space are labelled by \( \psi(x) \equiv |\psi\rangle \) and the elements of its dual by \( \langle \psi| \), which maps elements of \( L^2 \) onto complex numbers by \( \langle \phi|\psi\rangle = (\phi, \psi) \). One of the basic axioms of commutative quantum mechanics is that physical observables correspond to Hermitian operators \( A \) on \( L^2 \). In the two-dimensional quantum system the Heisenberg algebra is

\[
\begin{align*}
[\hat{x}_s, \hat{y}_s] &= 0; \quad [\hat{x}_s, \hat{p}_x] = i\hbar; \quad [\hat{y}_s, \hat{p}_y] = i\hbar; \\
[\hat{x}_s, \hat{p}_y] &= 0; \quad [\hat{y}_s, \hat{p}_x] = 0; \quad [\hat{p}_x, \hat{p}_y] = 0,
\end{align*}
\]

(2)

where the operators \( \hat{x}_s, \hat{y}_s, \hat{p}_x, \hat{p}_y \) are Hermitian operators acting on the space of square integrable function. A unitary representation of the algebra in (2) is the Schrödinger representation

\[
\begin{align*}
\hat{x}_s \psi(x, y) &= x \psi(x, y); \quad \hat{y}_s \psi(x, y) = y \psi(x, y); \\
\hat{p}_x \psi(x, y) &= -i\hbar \frac{\partial}{\partial x} \psi(x, y); \quad \hat{p}_y \psi(x, y) = -i\hbar \frac{\partial}{\partial y} \psi(x, y).
\end{align*}
\]

(3)

(4)

Two-dimensional non-commutative quantum mechanics (NCQM) is based on a simple modification of the commutations relations between the Hermitian position operators \( \hat{x}_0, \hat{y}_0 \) and the hermitian momentum operators \( \hat{p}_{x0}, \hat{p}_{y0} \) which satisfy
\[ [\hat{x}_0, \hat{y}_0] = i\hbar; \quad [\hat{x}_0, \hat{p}_{\hat{y}_0}] = i\hbar; \quad [\hat{y}_0, \hat{p}_{\hat{y}_0}] = i\hbar; \]
\[ [\hat{p}_{\hat{x}_0}, \hat{p}_{\hat{y}_0}] = 0; \quad [\hat{x}_0, \hat{p}_{\hat{y}_0}] = 0; \quad [\hat{y}_0, \hat{p}_{\hat{x}_0}] = 0, \]  
(5)

where \( \theta \in \mathbb{R} \) is the spatial non-commutative parameter of dimension of length square. If \( \theta \) is set to zero, we obtain the standard Heisenberg commutations relations (2). The operators \( \hat{x}_0, \hat{y}_0, \hat{p}_{\hat{x}_0}, \hat{p}_{\hat{y}_0} \) can be realized as follows

\[ \hat{x}_0 \psi(x, y) = x \psi(x, y); \quad \hat{y}_0 \psi(x, y) = y \psi(x, y); \]
\[ \hat{p}_{\hat{x}_0} \psi(x, y) = -i\hbar \frac{\partial}{\partial x} \psi(x, y); \quad \hat{p}_{\hat{y}_0} \psi(x, y) = -i\hbar \frac{\partial}{\partial y} \psi(x, y). \]  
(6)

Here \( \ast \) replaces the usual product of fields. The \( \ast \) product is defined as follows

\[ (f \ast g)(x) = \exp \left( \frac{i}{2} \theta_{ij} \partial_x^i \partial_y^j \right) f(x)g(y)|_{x=y} \]  
(7)

where \( f \) and \( g \) are two arbitrary infinitely differentiable functions on \( \mathbb{R}^{3+1} \) and \( \theta_{ij} \) is real and antisymmetric, i.e. \( \theta_{ij} = \theta_{ji} \) ( \( \epsilon_{ij} \) a completely antisymmetric tensor with \( \epsilon_{12} = 1 \)).

We can express the phase space operators \( \hat{x}_0, \hat{y}_0, \hat{p}_{\hat{x}_0}, \hat{p}_{\hat{y}_0} \) in terms of the standard phase space operators \( \hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y \) by performing the asymmetric Bopp-shift in space that can set in two different ways

\[ \hat{x}_0 = \hat{x} - \frac{\theta}{\hbar} \hat{p}_y; \quad \hat{y}_0 = \hat{y}; \]  
(8)

or

\[ \hat{x}_0 = \hat{x}; \quad \hat{y}_0 = \hat{y} + \frac{\theta}{\hbar} \hat{p}_x. \]  
(9)

There are some advantages in using the asymmetric Bopp shift such as the decoupling of the variables in some of the problems and some simplifications of expressions. The equations (8) and (9) do not always lead to the same results for the same problems. For that reason the symmetrical Bopp shift

\[ \hat{x}_0 = \hat{x} - \frac{\theta}{(2\hbar)} \hat{p}_y; \quad \hat{y}_0 = \hat{y} + \frac{\theta}{(2\hbar)} \hat{p}_x, \]  
(10)

is often used.

We consider new phase space operators \( \hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y \) that satisfy the following commutations relations

\[ [\hat{x}, \hat{y}] = i\theta(1 + \tau y^2); \quad [\hat{x}, \hat{p}_y] = i\hbar(1 + \tau y^2); \quad [\hat{y}, \hat{p}_x] = i\hbar(1 + \tau x^2); \]
\[ [\hat{p}_x, \hat{p}_y] = 0; \quad [\hat{x}, \hat{p}_y] = 2i\tau \hat{x} \hbar(\hat{y} \hat{p}_y + \hbar \hat{x}); \quad [\hat{y}, \hat{p}_x] = 0, \]  
(11)

where

\[ \hat{x} = \hat{x}_0 + i\theta \tau \hat{y}_0 + \tau \hat{y}_0^2 \hat{x}_0; \quad \hat{y} = \hat{y}_0; \quad \hat{p}_x = \hat{p}_{\hat{x}_0}; \quad \hat{p}_y = \hat{p}_{\hat{y}_0} - i\hbar \tau \hat{x}_0 + \tau \hat{y}_0^2 \hat{p}_{\hat{y}_0}, \]  
(12)

or by using the symmetric Bopp-shift in equation (10).
\[ \hat{x} = \hat{x}_t = \theta \frac{\hat{p}_y}{2\hbar} + \tau \hat{y}_0^2 \hat{x}_t + \theta \tau \left( i \hat{y}_s + \frac{1}{\hbar} \hat{x}_s \hat{p}_y - \frac{1}{2\hbar} \hat{y}_s^2 \right) + \theta^2 \tau \left( \frac{i}{2\hbar} \hat{p}_y + \frac{1}{(2\hbar)^2} \hat{p}_x^2 \hat{x}_t - \frac{1}{2\hbar^2} \hat{y}_s \hat{p}_y \hat{p}_y \right) - \frac{\tau \theta^3}{8\hbar^2} \hat{p}_s^2 \hat{p}_y; \]
\[ \hat{y} = \hat{y}_t = \theta \frac{\hat{p}_x}{2\hbar}; \]
\[ \hat{p}_x = \hat{p}_x; \]
\[ \hat{p}_y = \hat{p}_y + \tau \left( -i \hbar \hat{y}_s + \hat{y}_0^2 \hat{p}_y \right) + \theta \tau \left( -\frac{i}{2} \hat{p}_s + \frac{1}{\hbar} \hat{x}_s \hat{p}_x \hat{p}_y \right) + \frac{\tau \theta^2}{4\hbar^2} \hat{p}_s^2 \hat{p}_y. \]

(13)

It is easy to verify that the variables \( \hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y \) are Hermitian. The algebra (11) is a position dependent deformed Heisenberg algebra and satisfies the Jacobi identity. We do not provide an analog of the Schrödinger representation for the variables \( \hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y \) but by means of the equations (12) or (13) one can define the action of those operators.

The commutation relations in equations (2), (5) and (11) lead respectively to uncertainty relations.

In the case of the equation (2) we have the standard uncertainty relations
\[ \Delta \hat{x}_t \Delta \hat{p}_x \geq \frac{\hbar}{2}; \quad \Delta \hat{y}_t \Delta \hat{p}_y \geq \frac{\hbar}{2}. \]

(14)

In the situation of the equation (5), we have an additional uncertainty due to the noncommutativity of the position operators
\[ \Delta \hat{x}_0 \Delta \hat{y}_0 \geq \frac{\theta}{2}; \quad \Delta \hat{x}_0 \Delta \hat{p}_0 \geq \frac{\hbar}{2}; \quad \Delta \hat{y}_0 \Delta \hat{p}_0 \geq \frac{\hbar}{2}. \]

(15)

In the situation of the equation (11), we feature the uncertainty relations in the positions operators
\[ \Delta \hat{x} \Delta \hat{y} \geq \frac{1}{2} |\langle [\hat{x}, \hat{y}] \rangle| \]

that is equivalent to
\[ \Delta \hat{x} \Delta \hat{y} \geq \frac{\theta}{2} \left( 1 + \tau \langle \hat{y}^2 \rangle \right). \]

(17)

we consider the fact that \( \langle \hat{y}^2 \rangle = \langle \hat{y} \rangle^2 + \Delta \hat{y}^2 \) and we set the function
\[ f(\Delta \hat{x}, \Delta \hat{y}) = \Delta \hat{x} \Delta \hat{y} - \frac{\theta}{2} \left( 1 + \tau \langle \hat{y} \rangle^2 + \tau \Delta \hat{y}^2 \right) \]

(18)

and solving
\[ \partial_{\Delta \hat{x}} f(\Delta \hat{x}, \Delta \hat{y}) = 0 \quad \text{and} \quad f(\Delta \hat{x}, \Delta \hat{y}) = 0, \]

(19)

leads to a minimal length in \( \hat{x} \) in a simultaneous \( \hat{x}, \hat{y} \)-measurement
\[ \Delta \hat{x}_{\text{min}} = \theta \sqrt{\tau} \sqrt{1 + \tau \langle \hat{y} \rangle^2}, \]

(20)

the absolute minimal value \( \Delta \hat{x}_{\text{min}} \) being \( \theta \sqrt{\tau} \).
3. Deformed commutation relations and the gravitational quantum well

We consider in two-dimensional configuration space, the model of a particle of mass $m$ subjected to the Earth’s gravitational field in one direction that is the vertical taken to be described by $(-g)\mathbf{e}_z$. In the direction transverse to the gravitational field, $y$, the particle is free. The Hamiltonian of the system is given by

$$\hat{H}_0 = \frac{1}{2m} \hat{p}_x^2 + \frac{1}{2m} \hat{p}_y^2 + mg\hat{x}_z,$$

(21)

where the operators $\hat{x}_z$, $\hat{y}_z$, $\hat{p}_x$, $\hat{p}_y$ satisfy the commutation relations in equation (2). The Schrödinger equation is given by

$$\hat{H}_0 \Psi(x, y) = E \Psi(x, y).$$

(22)

As it is clearly seen, the system is decoupled and the solution to the eigenvalue equation (22) is given by

$$\Psi(x, y) = \psi_n(x)\psi(y), \quad E \equiv E_{n,k};$$

(23)

where $\psi(y)$ is the wave function in the $y$-direction and $\psi_n(x)$ the wave function in the $x$-direction. Since the particle is free in the $y$-direction, the wave function is

$$\psi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk g(k) e^{iky},$$

(24)

where $g(k)$ determines the shape of the wave packet and the energy spectrum is continuous

$$E_k = \frac{\hbar^2 k^2}{2m}. \quad (25)$$

In the $x$-direction, the eigenfunctions can be expressed in terms of the Airy function $\phi(z)$ with appropriate normalization [6] as follows

$$\psi_n(x) = \alpha_n \phi(z),$$

(26)

where the normalization factor for the $n$th eigenstate is given by:

$$\alpha_n = \left( \frac{h^2}{2m \pi g} \right)^{\frac{1}{4}} \int_{r_n}^{+\infty} dz \phi^2(z) \left( \frac{2m^2 g}{\hbar^2} \right)^{\frac{1}{4}} \left( x - \frac{E_n}{mg} \right),$$

(27)

and the eigenvalues are determined by the roots $r_n$ of the Airy function as follows

$$E_n = -\left( \frac{mg^2 \hbar^2}{2m} \right)^{\frac{1}{2}} r_n. \quad (28)$$

The spectrum of the system is then

$$\Psi(x, y) = \psi_n(x)\psi(y), \quad E_{n,k} = -\left( \frac{mg^2 \hbar^2}{2} \right)^{\frac{1}{2}} \times r_n + \frac{\hbar^2 k^2}{2m}. \quad (29)$$

In this paragraph we consider the analog of the quantum gravitational well on noncommutative flat space. The Hamiltonian of the system is given by

$$\hat{H} = \frac{1}{2m} \hat{p}_{z_0}^2 + \frac{1}{2m} \hat{p}_{x_0}^2 + mg\hat{x}_0,$$

(30)
where the operators $\hat{x}_0, \hat{y}_0, \hat{p}_x, \hat{p}_y$ satisfy the algebra (5). By mean of the symmetric Bopp shift in equation (10), we can rewrite the Hamiltonian in equation (30) as follows

$$\hat{H}_\theta = \frac{1}{2m} \hat{p}_x^2 + \frac{1}{2m} \hat{p}_y^2 + mg(\hat{x}_0 - \frac{\theta}{2\hbar} \hat{p}_y),$$

(31)

then

$$\hat{H}_\theta = \frac{1}{2m} \hat{p}_x^2 + \frac{1}{2m} \hat{p}_y^2 + mg \hat{x}_0 - \frac{mg}{2\hbar} \hat{p}_y,$$

(32)

that is equivalent to

$$\hat{H}_\theta = \hat{H}_0 - \theta \frac{mg}{2\hbar} \hat{p}_y.$$

(33)

We consider $\theta$ to be positive and very small and therefore must be a small correction at the low-energy level. We treat then, the new term appearing in (33), as a perturbation in the commutative Hamiltonian $H_0$. The shift caused by this term on the system’s energy levels is given by the expectation value of the perturbation on the system’s wave function that is

$$\Delta E_{n,k} = -\theta \frac{mg}{2\hbar} \langle \Psi(x,y) | \hat{p}_y | \Psi(x,y) \rangle;$$

$$= -\frac{\theta mg}{2},$$

(34)

and

$$E_{n,k,\theta} = -\left( \frac{mg^2 \hbar^2}{2} \right)^{\frac{1}{2}} \times r_n + \frac{\hbar^2 k^2}{2m} - \theta \frac{mg \hbar}{2}.$$  

(35)

Let us consider now the analog of the quantum gravitational well from position-dependent noncommutativity. The Hamiltonian is given by

$$H = \frac{1}{2m} \hat{p}_x^2 + \frac{1}{2m} \hat{p}_y^2 + mg \hat{x},$$

(36)

where the operators $\hat{x}, \hat{p}_x, \hat{p}_y$ satisfy the commutation relations in equation (11). In order to find the spectrum one can work directly in the position dependent noncommutative variables or by mean of the representation in equation (12) or the representation in equation (13). The problem is that in a situation of noncommutativity of the spatial operators there is an ambiguity in the meaning of the wavefunctions in the position representation. So we study the problem using the representation in equation (13) where the operators $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ are expressed in terms of the operators $\hat{x}_s, \hat{y}_s, \hat{p}_x, \hat{p}_y$. We assume that the parameters $\tau$ and $\theta$ are positive and very small and only the leading approximation in $\theta$ and $\tau$ is considered. The terms proportional to $\hat{y}_s^2$ and $\hat{p}_x^2$ do not affect the particle’s energy spectrum in the direction of the gravitational field [14], the expectation value $\langle \hat{y}_s \rangle$ is a finite value which is expected for a localized group of plane waves and will not produce any shift on the energy levels [14]. We set the expectation value $\langle \hat{y}_s \rangle$ to be zero in order to ensure the Hermiticity condition of the operator $\hat{p}_y$ in equation (13). The Hamiltonian (36) is then reduced to

$$\hat{H}_{\theta,\tau} = \hat{H}_0 - \theta \frac{mg}{2\hbar} \hat{p}_y - \tau \frac{\hbar^2}{2m}.$$  

(37)

The shift being determined by
\[ \Delta E_{n,k} = \langle \Psi(x,y) \left| -\frac{\theta mg}{2\hbar} \hat{p}_y - \frac{\hbar^2}{2m} \right| \Psi(x,y) \rangle; \quad (38) \]

\[ = -\frac{\theta mg}{2} - \frac{\hbar^2}{2m}, \quad (39) \]

so

\[ E_{n,k,\theta,\tau} = -\left( \frac{mg^2 \hbar^2}{2} \right)^{1/3} r_n + \frac{\hbar^2 k^2}{2m} - \frac{\theta mg}{2} - \frac{\hbar^2}{2m}. \quad (40) \]

4. Concluding remarks

We consider in this work two kinds of deformed commutation relations in two-dimensional quantum mechanics. The first kind being the most used in noncommutative quantum mechanics since it leads to less ambiguities. The second kind is dynamical and leads to the existence of minimal length.

For the first kind, we study the analog of the gravitational quantum well in new variables induced by the deformation parameter \( \theta \). The energy of the system is given by

\[ E_{n,k,\theta} = -\left( \frac{mg^2 \hbar^2}{2} \right)^{1/3} r_n + \frac{\hbar^2 k^2}{2m} - \frac{\theta mg}{2}. \quad (41) \]

The first remark is that the energy shift caused by this deformed commutation relations is negative. The noncommutativity parameter \( \theta \) affects the energy spectrum and that has been also stated in [20]. In a situation of noncommutativity in both configuration and momentum spaces it has been found that the leading order noncommutativity in configuration space does not affect the energy spectrum of the system [14]. Confronting our theoretical results to the experiments performed in references [11, 12], in the sense that the absolute value of the shift induced by the noncommutativity parameter \( \theta \) should be smaller than the maximum difference of the energy levels provided by the experiments, we have for the first level energy (\( n = 1 \))

\[ \frac{\theta mg}{2} \leq \Delta E^\text{exp}_1, \quad (42) \]

with

\[ \Delta E^\text{exp}_1 = 6.55 \times 10^{-32} \text{ J}. \quad (43) \]

For the neutrons \( m = 1.675 \times 10^{-27} \text{ kg} \), and in the experiments the neutrons had a mean horizontal speed \( \langle v_y \rangle = 6.5 \text{ m s}^{-1} \) so that \( k = \langle p_y \rangle / \hbar = m(\langle v_y \rangle / \hbar = 1.03 \times 10^9 \text{ m}^{-1} \). Then by considering \( g = 9.81 \text{ m s}^{-2} \) and \( \hbar = 10.59 \times 10^{-35} \text{ Js} \), we can calculate the upper bound for the parameter \( \theta \) from equation (42) as

\[ \theta \lesssim 0.755 \times 10^{-13} \text{ m}^2. \quad (44) \]

The upper bounds on the parameter of spatial noncommutativity \( \theta \) have been also established in [20, 22–24]; in [20] where the experimental context is the same with the present paper, the upper bound established is \( \theta \lesssim 0.771 \times 10^{-13} \text{ m}^2 \) for \( n = 1 \), in [22] existing experiments related to Lorentz invariance bound the parameter of the spatial noncommutativity to \( \theta \lesssim 0.388 \times 10^{-33} \text{ m}^2 \), in [23] measurements of the Lamb shift establish the upper bound to \( \theta \lesssim 0.388 \times 10^{-39} \text{ m}^2 \); in [24] the upper bound \( \theta \lesssim 0.155 \times 10^{-54} \text{ m}^2 \) is established from an
analysis of clock-comparison experiments. The upper bounds of the parameter of spatial non-
commutativity $\theta$ depend on the experiments and in order to improve the results more accurate
experimental data are needed.

In the second case, we study the dynamical deformed commutation relations that is a
deformation of the commutation relations of flat noncommutative space-time. This kind of
deformed commutation relations leads to the existence of minimal length, the absolute mini-
mal value being $\Delta \hat{x}_{\text{min}} = \theta \sqrt{\tau}$ m. The spectrum of the Hamiltonian is given by

$$E_{n,k,\theta,\tau} = -\left(\frac{m g^2 h^2}{2}\right)^{1/3} r_n + \frac{h^2}{2m} k^2 - \frac{\theta m g k}{2} - \frac{\tau h^2}{2m}.$$  (45)

Here again the energy shift caused by this deformed commutation relation is negative. In order
to confront the experimental results we compare the absolute value of the energy shift with the
experiments results at the first level, that is

$$\frac{\theta m g k}{2} + \frac{\tau h^2}{2m} \leq \Delta E_1^{\text{exp}}, \text{ with } \Delta E_1^{\text{exp}} = 6.55 \times 10^{-32} \text{ J.}$$  (46)

Here, appears the effects of both parameters of noncommutativity $\tau$ and $\theta$. In order to be con-
sistent with the experiments the parameters $\theta$ and $\tau$ should satisfy the inequality

$$8.46 \times 10^{-10} \theta + 3.34 \times 10^{-42} \tau \lesssim 6.55 \times 10^{-32}. \quad (47)$$

From equation (44) we can estimate the upper bound for the parameter $\tau$ to be

$$\tau \lesssim 6.26 \times 10^8 \text{ m}^{-2}. \quad (48)$$

In our assumption $\tau$ is considered to be very small and therefore the bound of $\tau$ in equa-
tion (48) is not a good estimation for the parameter $\tau$.

With respect to equations (44) and (48), the absolute value of the minimal length is bounded as

$$\Delta \hat{x}_{\text{min}} = \theta \sqrt{\tau} \lesssim 1.87 \times 10^{-9} \text{ m.} \quad (49)$$

The upper bound of the absolute value of the minimal length in equation (49) confirms the
conclusion in [16]. Indeed, F Brau and F Buisseret studied the dynamics of a particle in a
gravitational quantum well in the context of nonrelativistic quantum mechanics with a two-
dimensional analog of the modified Heisenberg algebra $[\hat{x}, \hat{p}] = i(1 + \beta^2 \hat{p}^2)$, $h = c = 1$.
They conclude that if $\beta$ is a quantity that depends on the energetic content of the system an
upper bound can be derived from the experimental results [11, 12] and the minimum length
scale associated to neutrons moving in a gravitational quantum well is smaller than a few
nanometers.

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References

[1] Fring A, Gouba L and Scholtz F G 2010 Strings from position-dependent noncommutativity J. Phys. A: Math. Theor. 43 345401
[2] Fring A, Gouba L and Bagchi B 2010 Minimal areas from q-deformed oscillator algebras J. Phys. A: Math. Theor. 43 425202
[3] Dey S, Fring A and Gouba L 2012 PT-symmetric noncommutative spaces with minimal volume uncertainty relations J. Phys. A: Math. Theor. 45 385302
[4] Alavi S A and Abbaspour S 2014 Dynamical noncommutative quantum mechanics J. Phys. A: Math. Theor. 47 045303
[5] Goldman I I, Krivchenkov V D, Kogan V I and Galitscii V M 1960 Problems in Quantum Mechanics (New York: Academic)
[6] Landau L D and Lifshitz E M 1976 Quantum Mechanics (Oxford: Pergamon)
[7] ter Haar D 1964 Selected Problems in Quantum Mechanics (New York: Academic)
[8] Luschikov V I and Frank A I 1978 Quantum effects occurring when ultracold neutrons are stored on a plane Pis’ma Zh. Eksp. Teor. Fiz. 28 607–9
[9] Nesvizhevsky V V et al 2002 Quantum states of neutrons in the Earth’s gravitational field Nature 415 297–9
[10] Nesvizhevsky V V et al 2003 Measurement of quantum states of neutrons in the Earth’s gravitational field Phys. Rev. D 67 102002
[11] Nesvizhevsky V V et al 2005 Study of the neutron quantum states in the gravity field Eur. Phys. J. C 40 479–91
[12] Bertolami O, Rosa J G, de Arag C M L, Castorina P and Zappalà D 2005 Noncommutative gravitational quantum well Phys. Rev. D 72 025010
[13] Bertolami O and Rosa J G 2006 The gravitational quantum well J. Phys.: Conf. Ser. 33 118–30
[14] Brau F and Buisseret F 2006 Minimal length uncertainty relation and gravitational quantum well Phys. Rev. D 74 036002
[15] Banerjee R, Roy B D and Samanta S 2006 Remarks on the noncommutative gravitational quantum well Phys. Rev. D 74 045015
[16] Saha A 2007 Time-space noncommutativity in gravitational quantum well scenario Eur. Phys. J. C 51 199–205
[17] Chang L N, Lewis Z, Minic D and Takeuchi T 2011 On the minimal length uncertainty relation and the foundations of string theory Adv. High Energy Phys. 2011 493514
[18] Castello-Branco K H C and Martins A G 2010 Free-fall in a uniform gravitational field in noncommutative quantum mechanics J. Math. Phys. 51 102106
[19] Bhat A, Dey S, Faizal M, Hou C and Zhao Q 2017 Modification of Schrödinger–Newton equation due to braneworld models with minimal length Phys. Lett. B 770 325–30
[20] Carroll S M, Harvey J A, Kostelecký V A, Lane C D and Okamoto T 2001 Noncommutative field theory and Lorentz violation Phys. Rev. Lett. 87 141601
[21] Chaichian M, Sheikh-Jabbari M M and Tureanu A 2001 Hydrogen atom spectrum and the lamb shift in noncommutative QED Phys. Rev. Lett. 86 2716
[22] Mocioiu I, Pospelov M and Roiban R 2000 Low-energy limits on the antisymmetric tensor field background on the brane and on the non-commutative scale Phys. Lett. B 489 390–6