Abstract. We present a complete suite of algorithms for finding isotropic vectors of quadratic forms (of any dimension) over an arbitrary global field of characteristic different from 2.

1. Introduction

The notion of isotropy is fundamental for the algebraic theory of quadratic forms. Given a quadratic form \( q \), a vector \( v \) is called isotropic when \( q(v) = 0 \). The problem of constructing an isotropic vector of a given form has been discussed in literature since the 18th century. The oldest approach, the author is aware of, is often accredited to Lagrange, and is based on the descent method and deals with ternary forms over \( \mathbb{Z} \). Recently, with the growing interest in computational algebra, many new algorithms have been devised, see e.g., [3, 9, 15, 22, 24, 25]. Unfortunately, all these algorithms are designed for forms with either integer or polynomial coefficient, and heavily depend on the idiosyncrasies of arithmetic of the (Euclidean) rings of integers and polynomials. Consequently, they do not allow for any generalization to other fields, like for example number fields. Here the problem of constructing an isotropic vector is equivalent to solving a degree 2 homogeneous multivariate Diophantine equation over a ring of algebraic integers, where the arithmetic of such ring can be quite... weird. That is probably the reason, why the problem of finding isotropic vectors of forms over number fields has—to the best of our knowledge—remained unsolved. The sole purpose of the present paper is to remedy this situation to a certain degree. We present explicit algorithms that construct isotopic vectors for quadratic forms over arbitrary global fields of characteristic \( \neq 2 \), either number fields or global function fields of odd characteristic. All algorithms described in this paper have been implemented in Magma (see [1]) and will be incorporated into the next major update of the Magma package CQF (see [16]) for computations on quadratic forms.

2. Notation

Throughout this paper, we use the following notation and terminology. Since the theory of quadratic forms in characteristic 2 is very different from all other characteristics, we exclude the case of characteristic 2 from all our considerations. For this reason, in the rest of the paper, the term “global field” actually means “global field of characteristic \( \neq 2 \)”. By \( K \), we will denote our base field, which is either a number field or an algebraic function field over some finite field \( \mathbb{F}_q \), where \( q \) is a power of an odd prime. If \( L/K \) is a finite field extension, then \( N_{L/K} : L \rightarrow K \) is the associated norm. A diagonal quadratic form \( a_1x_1^2 + \cdots + a_nx_n^2 \) is denoted \( \langle a_1, \ldots, a_n \rangle \). Since for degenerate forms the problem of constructing isotropic vectors is trivial and so completely uninteresting, all forms are always assumed to be non-degenerate, which means that every \( a_i \) is nonzero.

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A place of $K$ is an equivalence class of valuations. We denote the places using fraktur letters $p$, $q$, $r$. A place of a number field can be either archimedean or non-archimedean. Over global function fields all places are non-archimedean. We write $K_p$ for the completion of $K$ at $p$. In addition, if $p$ is non-archimedean we write $\text{ord}_p : K^\times \to \mathbb{Z}$ for the associated normalized discrete valuation. The non-archimedean places correspond to prime ideals in a maximal order of $K$. For this reason, we will use the terms “non-archimedean places” and “primes” interchangeably. We will sometimes write “finite primes” to emphasize the fact that they are non-archimedean. A prime $p$ of a number field is called dyadic if $\text{ord}_p 2 \neq 0$. Otherwise, it is called non-dyadic. In global function fields all primes are non-dyadic.

If $p$ is a finite prime and $a, b \in K^\times$, then $(a, b)_p$ denotes the Hilbert symbol of $a$ and $b$ at $p$. If further $q = \langle a_1, \ldots, a_n \rangle$ is a quadratic form, then following [10] we define the Hasse invariant of $q$ at $p$ by the formula

$$s_p(q) := \prod_{i<j}(a_i, a_j)_p.$$  

With a quadratic form $q = \langle a_1, \ldots, a_n \rangle$ we associate a finite set of primes $\mathfrak{P}(q)$ consisting of:

- all dyadic primes (provided that $K$ is a number field),
- all these primes $p$ such that $\text{ord}_p a_i$ is odd for at least one $i \leq n$.

Recall that a finite set $S$ of places of a global field is called a Hasse set if it contains all archimedean places. In particular, in a global function field, every finite set of places is a Hasse set. Let $S$ be a Hasse set, an element $a \in K^\times$ is said to be $S$-singular if $\text{ord}_p a$ is even for every prime $p$ not in $S$. It is called an $S$-unit if $\text{ord}_p a = 0$ for every $p \not\in S$. The group of all $S$-singular elements is denoted $E_S(K)$ and the group of all $S$-units by $U_S(K)$. Observe that $K^\times$ is a subgroup of $E_S(K)$, hence we may consider the quotient group $\mathbb{E}_S(K) := E_S(K)/K^\times$ of $S$-singular square classes. It is a subgroup of the square class group $\kappa^\times/K^\times$ of $K$. Moreover, we will denote $\mathbb{U}_S(K) := U_S(K)/U_S(K) \cap K^\times$ the group of $S$-units modulo squares. It can be treated as a subgroup of $E_S(K)$. Observe that $\mathbb{E}_S(K)$ is an elementary 2-group, hence it is an $\mathbb{F}_2$-linear space.

The ideal class group of $K$ is denoted by $\text{Cl}(K)$. If $S$ is a Hasse set then $\mathcal{I}_S(K)$ is the subgroup of the group of fractional ideals of $K$, generated by the primes not in $S$ and $\text{Cl}_S(K) := \mathcal{I}_S(K)/(\text{principal ideals})$ is the associated $S$-class group. Finally, if $m = m_0 \cdot m_\infty$ is a modulus, then $\text{Cl}_m(K)$ denotes the ray-class group. If $\alpha$ is a fractional ideal its classes in the respective class groups are denoted $[\alpha]_m$, $[\alpha]_S$ and $[\alpha]_\infty$. Let $L/K$ be a finite extension of global fields. A fractional ideal $\mathfrak{A}$ of $L$ is called pseudo-principal if $\mathfrak{A} = \alpha \cdot a \cdot \mathbb{Z}_L$ for some element $\alpha \in L$ and some fractional ideal $\alpha$ of $K$. The group of fractional ideals of $L$ modulo the pseudo-principal ones is called (see e.g., [23]) the relative pseudo-class group and denoted $\text{Cl}(L/K)$.

If the field is clear from context, we write $U_S$, $E_S$, $\mathbb{E}_S$, $\text{Cl}$, $\text{Cl}_S$ and $\text{Cl}_m$ instead of $U_S(K)$, $E_S(K)$, $\mathbb{E}_S(K)$, $\text{Cl}(K)$, $\text{Cl}_S(K)$ and $\text{Cl}_m(K)$, respectively.

3. Algorithmic prerequisites

The algorithms presented in this paper depend on the number of already existing tools. First of all, we need routines for solving systems of linear equations (basically over $\mathbb{F}_2$) and for basic arithmetic in a global field (see e.g., [1]). As far as more advanced tools are concerned, we need:

- (1) Factorization of ideals in global fields (see e.g., [7] [14]).
- (2) Solution to a norm equation of a form $N_{L/K}(x) = b$, where $b \in K^\times$ and $L/K$ is a finite field extension. This problem is discussed in [8] [11] [12] [13] [23].
(3) Construction of ideal class groups, ray-class groups and relative pseudo-class groups (see e.g., [4][5][8][8]).

(4) Construction of a basis (over $E_2$) of the group $E_S$ of $S$-singular square classes. The author is aware of three existing methods. One is described in [17]. Another one is outlined in Magma manual [2]. In this method, one constructs a set $T$ containing $S$ and such that the $T$-class number is odd. Then, it is known that the groups $E_T$ and $U_T$ coincide. Hence the sought group $E_S$ can be constructed as a subspace of $U_T$. A third method, due to A. Czogała, was recently described in [18].

4. Algorithms

In this section, we present the main results of the paper, namely the algorithms for constructing an isotropic vector of a given quadratic form. Our general strategy is as follows: for forms of dimensions 2 or 3, the corresponding vector is constructed directly. On the other hand, if $q$ is a quadratic form of dimension $\geq 4$, we express it as an orthogonal sum $q = q_1 \perp q_2$ of two forms, whose dimensions are roughly half of $\dim q$. Next, we search for an element $c \in K^*$ such that $c$ is represented by $q_1$ and $-c$ by $q_2$. We then recursively construct isotropic vectors for $(c) \perp q_1$ and $(c) \perp q_2$ and use these two vectors to construct the sought isotropic vector of the original form $q$. Unfortunately, the details differ significantly from dimension to dimension. Consequently, over non-real global fields, we need dedicated algorithms for dimensions 4 and 5, and a general algorithm that works only for forms of dimension 6 or higher. For formally real fields, the situation is even more subtle. As a result, we need dedicated algorithms for dimensions 4 through 7 and a general algorithm that deals with forms of dimension 8 and above.

4.1. Dimension 3 and below. Let us begin with forms of the lowest dimensions.Unary forms cannot be isotropic, and so we ignore them. Further, the construction of an isotropic vector of a binary form is trivial and boils down to computing the square root of its discriminant. Hence the first dimension that we need to consider is dimension three.

Let $q = \langle a_1, a_2, a_3 \rangle$ be an isotropic ternary form and $v = (v_1, v_2, v_3) \in K^3$ be its isotropic vector, we are looking for. Without loss of generality we may assume that all three binary subforms $\langle a_i, a_j \rangle$, with $1 \leq i < j \leq 3$ are anisotropic. In particular, this means that $-a_j/a_i$ is not a square in $K$ and that $v_3$ is non-zero. We have $a_1v_1^2 + a_2v_2^2 + a_3v_3^2 = 0$. Rearranging the terms we write

\[-\frac{a_3}{a_1} = \left(\frac{v_1}{v_3}\right)^2 + \frac{a_2}{a_1} \cdot \left(\frac{v_2}{v_3}\right)^2.\]

Now, the right-hand side is the norm of an element

\[\xi = \frac{v_1}{v_3} + \frac{v_2}{v_3} \cdot \sqrt{-\frac{a_2}{a_1}}\]

in the quadratic field extension $K(\sqrt{-a_j/a_i})/K$. Therefore, the problem of constructing an isotropic vector of a ternary form is equivalent to the one of solving the norm equations. The latter problem is well known (see Section 3). Summarizing this discussion, we obtain the following algorithm.

Algorithm 1. Given an isotropic ternary form $q = \langle a_1, a_2, a_3 \rangle$ over a global field $K$, this algorithm constructs a vector $v \in K^3$ such that $q(v) = 0$.

1. If $-a_ia_j$ is a square for some $1 \leq i < j \leq 3$, say $-a_ia_j = d^2$, then output $v = (v_1, v_2, v_3)$, where

\[v_i = a_j, \quad v_j = d\quad \text{and} \quad v_k = 0\]

for $k \notin \{i, j\}$, and quit.
Lemma 4.1. Let \( S \) be a Hasse set of \( K \). If the \( S \)-class number is odd, then the groups \( U_S \) and \( E_S \) coincide.

Proof. One inclusion, namely \( U_S \subseteq E_S \) holds always. Hence, we need to show only the opposite inclusion. To this end, we treat \( U_S \) and \( E_S \) as \( \mathbb{F}_2 \)-vector spaces. It suffices to show that their dimensions are equal. By [10, p. 607] we have

\[
\dim_{\mathbb{F}_2} E_S = \dim_{\mathbb{F}_2} U_S + \dim_{\mathbb{F}_2} \text{Cl}_S/\text{Cl}_E^2.
\]

Now, the last term is zero since the \( S \)-class number is odd, by assumption. Therefore, the dimension of the subspace \( U_S \) coincides with the dimension of the whole vector space \( E_S \) and so the two groups are equal. \( \square \)

4.2. Dimension 4. The case of 4-dimensional quadratic forms requires some preparation. We begin with the following lemma.

Algorithm 2. Given an isotropic quadratic form \( q = (a_1, a_2, a_3, a_4) \) of dimension 4 over a global field \( K \), this algorithm outputs a vector \( v \in K^4 \) such that \( q(v) = 0 \).

(1) If there are indices \( 1 \leq i < j < k \leq 4 \) such that \( (a_i, a_j, a_k) \) is isotropic, then execute Algorithm 1 to obtain its isotropic vector \( v \in K^3 \). Output \( v \) with zero inserted at the “missing spot” and quit.

(2) Take \( L := K(\sqrt{-a_2/a_1}) \) and \( M := K(\sqrt{-a_4/a_3}) \).

(3) Construct a set \( S \) of finite primes of \( K \) such that:

(a) \( S \) contains \( \mathcal{P}(q) \).

(b) The \( (S \cup \{\text{archimedean places}\}) \)-class number is odd.

(c) Primes of \( L \) above \( S \) generate the relative pseudo-class group \( \text{Cl}_1(L/K) \).

(d) Primes of \( M \) above \( S \) generate the relative pseudo-class group \( \text{Cl}_1(M/K) \).

(4) Denote \( T := S \cup \{\text{archimedean places}\} \).

(5) Let \( \hat{S}(L), \hat{S}(M), \hat{T}(L) \) and \( \hat{T}(M) \) be the sets of places of \( L \) and \( M \) that lie above the places in \( S \) and \( T \), respectively.

(6) Repeat the following steps:

(a) Construct bases (over \( \mathbb{F}_2 \))

\[ E_K = \{\kappa_1, \ldots, \kappa_k\}, \quad E_L = \{\lambda_1, \ldots, \lambda_l\}, \quad E_M = \{\mu_1, \ldots, \mu_m\}. \]

of \( E_T \), \( E_{\hat{T}(L)} \) and \( E_{\hat{T}(M)} \), respectively.

(b) Find the coordinates \( \varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\} \) of \( a_1 \) and \( e_1, \ldots, e_k \in \{0, 1\} \) of \( -a_3 \) with respect to \( E_K \).

(c) For every \( \lambda_j \in E_L \) find the coordinates \( \varepsilon_j \in \{0, 1\} \) of the norm \( N_{L/K}(\lambda_j) \) with respect to \( E_K \), where \( j \leq l \) and \( i \leq k \).

(d) For every \( \mu_j \in E_M \) find the coordinates \( \varepsilon_j \in \{0, 1\} \) of the norm \( N_{L/K}(\mu_j) \) with respect to \( E_K \), where \( j \leq m \) and \( i \leq k \).
(e) Check if the following system of $\mathbb{F}_2$-linear equations is solvable:

\[
\begin{pmatrix}
\varepsilon_{L,1} & \cdots & \varepsilon_{L,k} \\
\vdots & & \vdots \\
\varepsilon_{L,k} & \cdots & \varepsilon_{L,1}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{M,1} & \cdots & \varepsilon_{M,k} \\
\vdots & & \vdots \\
\varepsilon_{M,k} & \cdots & \varepsilon_{M,1}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
\vdots \\
x_k
\end{pmatrix}
= \begin{pmatrix}
\varepsilon_1 + e_1 \\
\vdots \\
\varepsilon_k + e_k
\end{pmatrix}.
\]

If it is, denote a solution by $(\xi_1, \ldots, \xi_l; \zeta_1, \ldots, \zeta_m)$ and exit the loop.

(f) If there is no solution, find a new prime $q$ of $K$ such that $q \not\in S$.

Append $q$ to $S$, append the primes of $L$ above $q$ to $\hat{S}(L)$ and the primes
of $M$ above $q$ to $\hat{S}(M)$. Reiterate the loop.

(7) Let $v_1, \ldots, v_4$ be elements of $K$ such that

\[
\alpha := \lambda_1^1 \cdots \lambda_l^1 = v_1 + v_2 \sqrt{-\frac{a_2}{a_1}} \in L^\times,
\]

\[
\beta := \mu_1^1 \cdots \mu_m^1 = v_3 + v_4 \sqrt{-\frac{a_4}{a_3}} \in M^\times.
\]

(8) Set

\[
u := \sqrt{-\frac{a_1 \cdot N_{L/K}(\alpha)}{a_3 \cdot N_{M/K}(\beta)}},
\]

(9) Output $v = (v_1, v_2, uv_3, uv_4)$ and quit.

Proof of correctness. If the algorithm terminates in step (1), then its output is
evidently correct. Hence, without loss of generality we may assume that no 3-
dimensional subform $(a_i, a_j, a_k)$ of $q$ is isotropic. This implies that neither $-a_2/a_1$
or $-a_4/a_3$ is a square in $K$. In particular, $L$ and $M$ are proper quadratic extensions
of $K$.

We will first prove that if the algorithm terminates then its output is correct. Only then we will show that the algorithm indeed terminates. Thus, assume that
the algorithm terminates and let $\alpha = \lambda_1^1 \cdots \lambda_l^1$ and $\beta = \mu_1^1 \cdots \mu_m^1$ be the two
square classes constructed in step (7). The exponents $\xi_1, \ldots, \xi_l$ and $\zeta_1, \ldots, \zeta_m$ form
a solution to system (4). Observe that, since we are working over $\mathbb{F}_2$, the system
is equivalent to the following one

\[
\begin{cases}
\varepsilon_1 + \sum_{j \leq l} x_j \cdot \varepsilon_{j,1} = e_1 + \sum_{j \leq m} y_j \cdot \varepsilon_{j,1} \\
\vdots \\
\varepsilon_k + \sum_{j \leq l} x_j \cdot \varepsilon_{j,k} = e_k + \sum_{j \leq m} y_j \cdot \varepsilon_{j,k}.
\end{cases}
\]

Therefore, for every $i \leq k$ we have

\[
k_i^{\varepsilon_1} \cdot \prod_{j \leq l} k_j^{\varepsilon_{j,i}} = k_i^{\varepsilon_k} \cdot \prod_{j \leq m} k_j^{\varepsilon_{j,i}}.
\]

Taking the products of both sides over all $i \leq k$ and using (6b) we obtain

\[
a_1 \cdot N_{L/K}(\alpha) \equiv (-a_3) \cdot N_{M/K}(\beta) \pmod{K^\times 2}.
\]

This means that $(-a_1 \cdot N_{L/K}(\alpha))/(a_3 \cdot N_{M/K}(\beta))$ is a square in $K$. As in step (8)
of the algorithm, denote the square root by $u$ and let $v_1, v_2$ and $v_3, v_4$ have the same
meaning as in step (7). Thus we have
\[ a_1 \cdot N_L/K \left( v_1 + v_2 \sqrt{-\frac{a_2}{a_1}} \right) = (-a_3) \cdot u^2 \cdot N_M/K \left( v_3 + v_4 \sqrt{\frac{a_4}{a_3}} \right). \]
This yields \( a_1 v_1^2 + a_2 v_2^2 = -a_3 (v_3)^2 - a_4 (v_4)^2 \) and so \( q(v) = 0 \), as desired.

In order to finish the proof, we must show that the algorithm terminates. In other words, that after appending sufficiently many primes to the set \( S(K) \), system \( \mathcal{A} \) will eventually have a solution.

The form \( q \) is isotropic by assumption. Hence, there is some vector \( w = (w_1, w_2, w_3, w_4) \in K^4 \) such that \( q(w) = 0 \). Without loss of generality, we may assume that all four coordinates of \( w \) are non-zero, since otherwise \( q \) would have an isotropic subform \( \langle a_1, a_2, a_3, a_4 \rangle \) and the algorithm would terminate already in step (4).

Denote \( a := a_1 w_1^2 + a_2 w_2^2 = (-a_3) w_3^2 + (-a_4) w_4^2 \).

Further, if \( K \) is a number field let \( S_\infty \) be the set of all archimedean places of \( K \) and \( T := S \cup S_\infty \). Otherwise, if \( K \) is a function field, set \( T := S \). Now, [20, Lemma 2.1] says that there is a prime \( q \notin T \) and a \((T \cup \{q\})\)-singular element \( c \) such that
\[ c \equiv a \pmod{\mathfrak{p}^{1+ord_\mathfrak{p} 4}}. \]
for every prime \( \mathfrak{p} \in S \) and \( \text{sgn}_\mathfrak{p} c = \text{sgn}_\mathfrak{p} a \) for all real places \( \mathfrak{r} \) of \( K \). It follows from the local square theorem (see e.g., [19, Theorem VI.2.19]) that the local square classes \( a \cdot K^{x}_\mathfrak{p} \) and \( c \cdot K^{x}_\mathfrak{p} \) coincide for every place \( \mathfrak{p} \in T \). We claim that \( c \) is represented over \( K \) by the forms \( \langle a_1, a_2 \rangle \) and \( \langle -a_3, -a_4 \rangle \).

Denote \( \hat{q}_1 := \langle -c, a_1, a_2 \rangle \) and \( \hat{q}_2 := \langle c, a_3, a_4 \rangle \). Clearly, the forms \( \hat{q}_1 \) and \( \hat{q}_2 \) are locally isotropic at all complex places of \( K \). Fix any place \( \mathfrak{p} \in T \), be it archimedean or non-archimedean. We know that the forms \( \langle -a, a_1, a_2 \rangle \) and \( \langle a, a_3, a_4 \rangle \) are isotropic, so are their localizations \( \langle -a, a_1, a_2 \rangle \otimes K_\mathfrak{p} \) and \( \langle a, a_3, a_4 \rangle \otimes K_\mathfrak{p} \). Now, \( a \cdot c \in K^{x}_\mathfrak{p} \) hence \( \langle -a, a_1, a_2 \rangle \otimes K_\mathfrak{p} \cong \hat{q}_1 \otimes K_\mathfrak{p} \) and \( \langle a, a_3, a_4 \rangle \otimes K_\mathfrak{p} \cong \hat{q}_2 \otimes K_\mathfrak{p} \).

This means that \( \hat{q}_1 \otimes K_\mathfrak{p} \) and \( \hat{q}_2 \otimes K_\mathfrak{p} \) are both isotropic, as desired.

Next, take a finite prime \( \mathfrak{p} \) not in \( S \) but distinct from \( q \). In particular, \( \mathfrak{p} \) is non-dyadic since all the dyadic primes are in \( \mathfrak{P}(q) \subseteq S \). Then all the elements \( a_1, \ldots, a_4 \) and \( c \) have even valuations at \( \mathfrak{p} \), consequently the forms \( \hat{q}_1 \otimes K_\mathfrak{p} \) and \( \hat{q}_2 \otimes K_\mathfrak{p} \) are isotropic by [19, Corollary VI.1.51].

Finally, consider the singled-out prime \( q \). By the previous paragraph and [19, Proposition V.3.22] for every prime \( \mathfrak{p} \neq q \) we have
\[ 1 = s_\mathfrak{p}(\hat{q}_1) \cdot (-1 - \det \hat{q}_1)_\mathfrak{p} = (c, -a_1 a_2)_\mathfrak{p} \cdot (a_1, a_2)_\mathfrak{p}. \]
Therefore, using Hilbert reciprocity law (see e.g., [21, Chapter VIII]) we can write
\[ 1 = (c, -a_1 a_2)_q \cdot (a_1, a_2)_q \cdot \prod_{\mathfrak{p} \neq q} (c, -a_1 a_2)_\mathfrak{p} \cdot \prod_{\mathfrak{p} \neq q} (a_1, a_2)_\mathfrak{p} \]
\[ = (c, -a_1 a_2)_q \cdot (a_1, a_2)_q \]
\[ = s_q(\hat{q}_1) \cdot (1, -\det \hat{q}_1)_q. \]
Hence, again by [19, Proposition V.3.22] we obtain that \( \hat{q}_1 \otimes K_q \) is isotropic. A similar argument shows that \( \hat{q}_2 \otimes K_q \) is isotropic, as well. We have shown that the forms \( \hat{q}_1 \) and \( \hat{q}_2 \) are isotropic at every completion of \( K \). Therefore, by the local-global principle, they are isotropic over \( K \), as well. This proves the claim.

It follows from our claim that there is a \((T \cup \{q\})\)-singular element \( c \in K^x \) such that
\[ c \in a_1 \cdot N_L/K (L^x) \cap (-a_3) \cdot N_M/K (M^x). \]
Now, step 1 of the algorithm ensures that the $T$-class number is odd and so $\mathbb{E}_{T∪\{q\}}(K) = U_{T∪\{q\}}(K)$, by Lemma 4.1. Moreover, the primes above $S$ generate the relative pseudo-class groups $\text{Cl}_1(L/K)$ and $\text{Cl}_1(M/K)$. Thus, [20, Theorem 4.2] asserts that

$$N_{L/K} \left( U_{T(L)∪Ω_L}(L) \right) = N_{L/K} \left( L^\times \right) \cap U_{T∪\{q\}}(K)$$

and likewise

$$N_{M/K} \left( U_{T(M)∪Ω_M}(M) \right) = N_{M/K} \left( M^\times \right) \cap U_{T∪\{q\}}(K).$$

Here $Ω_L$ (respectively $Ω_M$) is the set of primes of $L$ (respectively $M$) laying above $q$. It follows that

$$c \cdot K^{\times 2} \in a_1 \cdot N_{L/K} \left( U_{T(L)∪Ω_L}(L) \right) \cap (−a_3) \cdot N_{M/K} \left( U_{T(M)∪Ω_M}(M) \right)$$

$$\subseteq a_1 \cdot E_{T∪\{q\}}(K) \cap (−a_3) \cdot E_{T∪\{q\}}(K) = E_{T∪\{q\}}(K).$$

This implies that system 4 has a solution. Let $Q$ be the set of primes of $K$ such that after appending any $q ∈ Q$ to $S$, system 4 becomes solvable. We have already observed that this set is non-empty. In fact it has positive density. Although this fact is not explicit in the statement of [20 Lemma 2.1], it follows from its proof, where the existence of $q$ is obtained from the generalized Dirichlet density theorem. This way we prove that the algorithm exits the loop (and consequently terminates) in finite time (see Remark 2 below).

**Remark 1.** To ensure in step 3 that the $S$-class number is odd, it suffices to start from $S = \Psi(q)$ and append to it primes whose classes generate the quotient group $\text{Cl}_3(K)/\text{Cl}_2(K)^2$.

**Remark 2.** In order to rigorously prove that after finitely many iterations, we will come across a prime $q$ from a positive-density set, we should iterate over the primes in an exhaustive manner. If $K$ is a number field, we may for example start with a prime number $p = 3$ and loop over all the primes of $K$ dividing $p$. Then take $p = 5$ and again loop over all the primes of $K$ over $p$, and so on. A similar exhaustive search can be applied for global function fields, as well. In practice, a much more efficient solution is just to pick $q$ at random. However, in the worst-case scenario, this might theoretically result in an infinite loop if, by bad luck, we keep selecting “bad primes”.

### 4.3. Dimension 5

We may now turn our attention to forms of dimension 5. Somehow surprisingly, this is the most challenging case of all.

**Algorithm 3.** Given an isotropic quadratic form $q = (a_1, \ldots, a_5)$ of dimension 5 over a global field $K$, this algorithm outputs a vector $v ∈ K^5$ such that $q(v) = 0$.  

1. If the 4-dimensional form $q_i$, obtained from $q$ by omitting $a_i$ for some $i ≤ 5$, is isotropic, then execute Algorithm 3 to construct its isotropic vector $w$. Expand $w$ to a 5-dimensional vector $v$, by inserting 0 at $i$-th coordinate. Output $v$ and quit.

2. Denote $q_1 := (a_1, a_2, a_3)$ and $q_2 := (a_4, a_5)$.

3. Let $S_∞ = \{τ_1, \ldots, τ_r\}$ be the set of all the real places of $K$ (if any) at which either $q_1$ or $q_2$ is locally anisotropic.

4. For every $τ_i ∈ S_∞$, if $q_1 ⊗ K_{τ_i}$ is anisotropic, set

$$a_i := \begin{cases} 1 & \text{if } \text{sgn}_{τ_i} a_1 = −1, \\ 0 & \text{if } \text{sgn}_{τ_i} a_1 = 1. \end{cases}$$
Otherwise, if \( q_1 \otimes K_{v_1} \) is isotropic and \( q_2 \otimes K_{v_1} \) is not, set

\[
\alpha_i := \begin{cases} 
1 & \text{if } \text{sgn}_{v_1} \alpha_i = 1, \\
0 & \text{if } \text{sgn}_{v_1} \alpha_i = -1.
\end{cases}
\]

Build a vector \( \alpha := (\alpha_1, \ldots, \alpha_r)^T \).

(5) Denote \( T := \mathfrak{P}(q) \cup \{ \text{archimedean places of } K \} \).

(6) Repeat the following steps:

(a) Construct a basis \( \mathcal{B} = \{ \kappa_1, \ldots, \kappa_k \} \) of the group \( \mathbb{E}_T \) of \( T \)-singular square classes.

(b) Let \( S = \{ p_1, \ldots, p_s \} \) be the subset of \( T \) consisting of these finite primes, where either \( q_1 \) or \( q_2 \) is locally anisotropic.

(c) For every \( p_i \in S \) find a set \( H(p_i) = \{ h_{i,l} \mid l \leq l_{p_i} \} \) of generators of the local square-class group \( K_{p_i}^*/K_{p_i}^{*2} \). Here \( l_{p_i} = 2 \) for non-dyadic \( p_i \) and \( l_{p_i} = 2 + (K_{p_i} \setminus \mathbb{Z}_{p_i}) \) for dyadic \( p_i \).

(d) For every \( p_i \in S \) find an element \( c_i = h_{i,1}^{\varepsilon_{i,1}} \cdots h_{i,p_i}^{\varepsilon_{i,p_i}} \) for some \( \varepsilon_1, \ldots, \varepsilon_{l_{p_i}} \in \{0,1\} \), such that the forms

\[
(\langle -c_i \rangle \perp q_1) \otimes K_{p_i}, \quad \text{and} \quad (\langle c_i \rangle \perp q_2) \otimes K_{p_i}
\]

are both isotropic.

(e) For every \( p_i \in S \) and \( l \leq l_{p_i} \), set

\[
\beta_{il} := \begin{cases} 
1 & \text{if } (c_i, h_{il})_{p_i} = -1, \\
0 & \text{if } (c_i, h_{il})_{p_i} = 1.
\end{cases}
\]

Build a vector \( \beta := (\beta_{11}, \ldots, \beta_{1l_{p_1}}, \beta_{21}, \ldots, \beta_{s l_{p_s}})^T \) of length \( \sum_{p_i \in S} l_{p_i} \).

(f) Construct a matrix \( A = (a_{ij}) \) with \( k \) columns (indexed by the elements in \( \mathcal{B} \)) and \( r \) rows (indexed by the places in \( S_{\infty} \)), setting

\[
a_{ij} := \begin{cases} 
1 & \text{if } \text{sgn}_{v_1} \kappa_j = -1, \\
0 & \text{if } \text{sgn}_{v_1} \kappa_j = 1.
\end{cases}
\]

(g) Construct a block matrix

\[
B = \begin{pmatrix} 
B_1 \\
\vdots \\
B_s
\end{pmatrix}
\]

where the blocks \( B_1, \ldots, B_s \) are indexed by the primes \( p_i \in S \). For every \( p_i \), let the block \( B_i = (b_{ij}) \) have \( l_{p_i} \) rows (indexed by the elements of \( H(p_i) \)) and \( k \) columns, with

\[
b_{ij} := \begin{cases} 
1 & \text{if } (h_{i,1}, \kappa_j)_{p_i} = -1, \\
0 & \text{if } (h_{i,1}, \kappa_j)_{p_i} = 1.
\end{cases}
\]

(h) Check if the following system of \( \mathbb{F}_2 \)-linear equation solvable:

\[
\begin{pmatrix} 
A \\
B
\end{pmatrix} \begin{pmatrix} 
x_1 \\
\vdots \\
x_k
\end{pmatrix} = \begin{pmatrix} 
\alpha \\
\beta
\end{pmatrix}.
\]

If it is, denote a solution by \( (\xi_1, \ldots, \xi_k) \) and exit the loop.

(i) Otherwise, when system \( (\heartsuit) \) has no solution, find a new prime \( q \notin T \); append \( q \) to \( T \) and reiterate the loop.

(7) Set \( c := \kappa_1^{\xi_1} \cdots \kappa_k^{\xi_k} \).

(8) Execute Algorithm \( \spadesuit \) and find an isotropic vector \((v_0, v_1, v_2, v_3)\) of the form \( \langle -c \rangle \perp q_1 \).
(9) Execute Algorithm \([1]\) and find an isotropic vector \((w_0, w_1, w_2)\) of the form \(\langle c \rangle \perp \mathbb{Q}_2\).

(10) Output \(v := (w_1 / w_0, w_2 / w_0, w_3 / w_0, w_4 / w_0, w_5 / w_0)\).

**Proof of correctness.** The correctness of the value outputted in step \(1\) follows from the correctness of Algorithm \([2]\). Hence, without loss of generality, we may assume that the forms \(\langle a_i, a_j, a_k, a_l \rangle\) are anisotropic for all \(1 \leq i < j < k < l \leq 5\) and so the algorithm proceeds beyond step \(1\). The form \(q\) is isotropic by assumption. Thus, there is some \(b \in K^\times\) such that \(b\) is represented by \(q_1\), and \((-b)\) by \(q_2\). By the local-global principle the forms \((-b) \perp q_1\) and \((b) \perp q_2\) are locally isotropic in every completion of \(K\). This justifies the existence of the corresponding elements \(c_1, \ldots, c_5\) found in step \(6a\). Hence assume that for every prime \(p_i \in S\) we have constructed the set \(H(p_i)\) of representatives of the local square class group \(K_{p_i}/K_{p_i}^\times\) and fixed an element \(c_i = h_{1,1}^{c_{i,1}} \cdots h_{i,j}^{c_{i,j}}\) such that the corresponding forms are locally isotropic at \(p_i\). As in the proof of correctness of Algorithm \([2]\) by \([20]\) Lemma 2.1, we obtain a prime \(q \notin \mathcal{S}\) and an \((\mathcal{S} \cup \{q\})\)-singular element \(c \in K^\times\) such that:

- \(\text{sgn}_{r_i} c = \text{sgn}_{r_i} b\) for every real place \(r_i\) of \(K\),
- \(c \equiv c_i \mod p_i^{1+\text{ord}_{p_i}1}\) for every \(p_i \in \mathcal{S}\).

In particular, the local square classes \(c \cdot K_{p_i}^\times\) and \(c_i \cdot K_{p_i}^\times\) coincide by the local square theorem. We will show that once the prime \(q\) is appended to the set \(T\), system \([4]\) becomes solvable and every solution lead to an isotropic vector.

Let \(\mathcal{B} = \{\kappa_1, \ldots, \kappa_k\}\) be a basis of the group \(\mathbb{E}_T\) and \(\xi_1, \ldots, \xi_k \in \{0, 1\}\) be the coordinates of \(c\) with respect to \(\mathcal{B}\), hence \(c = \kappa_1^{\xi_1} \cdots \kappa_k^{\xi_k}\). Denote

\[q_1 := \langle c \rangle \perp q_1 = \langle -c, a_1, a_2, a_3 \rangle\quad \text{and}\quad q_2 := \langle c \rangle \perp q_2 = \langle c, a_4, a_5 \rangle\]

The two forms are trivially isotropic at every complex place of \(K\). Further, by the definition of \(S_{\mathcal{S}}\), for every real place \(r_i\) not in \(S_{\mathcal{S}}\), already the forms \(q_1 \otimes K_{r_i}\) and \(q_2 \otimes K_{r_i}\) are isotropic and so are \(q_1 \otimes K_{r_i}\) and \(q_2 \otimes K_{r_i}\). Conversely, fix a real place \(r_i \in S_{\mathcal{S}}\). Then at least one of the forms \(q_1 \otimes K_{r_i}\) or \(q_2 \otimes K_{r_i}\) must be anisotropic. Suppose that it is the first one of them. Therefore we have

\[(-1)^{a_i} = \text{sgn}_{r_i} a_1 = \text{sgn}_{r_i} a_2 = \text{sgn}_{r_i} a_3\]

Now, \((b) \perp q_1) \otimes K_{r_i}\) is isotropic, hence \(\text{sgn}_{r_i} b = \text{sgn}_{r_i} a_1\) and so we have:

\[(-1)^{a_i} = \text{sgn}_{r_i} b = \text{sgn}_{r_i} c = \text{sgn}_{r_i} \left(\prod_{j \leq k} \kappa_j^{\xi_j}\right) = \prod_{j \leq k} (-1)^{a_j} \xi_j\]

It follows that \(q_1 \otimes K_{r_i}\) is isotropic and the coordinates of \(c\) form a solution to the “\(A\)-part” of \([4]\). Moreover, we know that the original form \(q\) is isotropic. Hence either \(\text{sgn}_{r_i} c = \text{sgn}_{r_i} a_1 \neq \text{sgn}_{r_i} a_4\) or \(\text{sgn}_{r_i} c = \text{sgn}_{r_i} a_1 \neq \text{sgn}_{r_i} a_5\). In either case, the form \(q_2 \otimes K_{r_i}\) is also isotropic.

Now, suppose that \(q_1 \otimes K_{r_i}\) is isotropic but \(q_2 \otimes K_{r_i}\) is not. Then clearly, \(q_1 \otimes K_{r_i}\) is also isotropic, and we have

\[(-1)^{a_i} = -\text{sgn}_{r_i} a_4 = -\text{sgn}_{r_i} a_5 = \text{sgn}_{r_i} b = \text{sgn}_{r_i} c = \prod_{j \leq k} (-1)^{a_j} \xi_j\]

It follows that the form \(q_2 \otimes K_{r_i}\) is isotropic and the coordinates of \(c\) again form a solution to the “\(A\)-part” of \([4]\).

We may now focus on non-archimedean places. First take a prime \(p \notin \mathcal{T}\) but distinct from \(q\). Then \(p\) is non-dyadic and all the coefficients \(a_1, \ldots, a_5\) and \(c\) have even valuations at \(p\), hence \(q_1 \otimes K_p\) and \(q_2 \otimes K_p\) are isotropic by \([13]\) Corollary VI.2.5. In turn, suppose that \(p \in T \setminus \mathcal{S}\). Then the forms \(q_1 \otimes K_p\) and \(q_2 \otimes K_p\) are again isotropic since their sub forms \(q_1 \otimes K_p\) and \(q_2 \otimes K_p\) are isotropic. Now, take a prime
Let $p_i \in S$, be it dyadic or non-dyadic. Then the square classes $c \cdot K_{p_i}^{x^2}$ and $c_i \cdot K_{p_i}^{x^2}$ coincide, hence for every $h_{i,l} \in H(p_i)$ we have

$$(-1)^{b_{i,1}} = (c_i, h_{i,l})_{p_i} = (c, h_{i,l})_{p_i} = \prod_{j \leq k} (c_j, h_{i,l})_{p_i} = \prod_{j \leq k} (-1)^{b_{i,j}^2}.$$  

This means that the coordinates of $c$ satisfy the "B-part" of system (4). Conversely, if $c' = \kappa_1 \cdots \kappa_k$, for another solution $\zeta_1, \ldots, \zeta_k \in \{0, 1\}$ to (4), then we infer from the above equalities that $(c', h_{i,l})_{p_i} = (c, h_{i,l})_{p_i}$ for every $h_{i,l} \in H(p_i)$. It follows from non-degeneracy of Hilbert symbol (see e.g., [19, Theorem VI.2.16]) that the local square classes $c' \cdot K_{p_i}^{x^2}$ and $c_i \cdot K_{p_i}^{x^2}$ coincide. Now, the element $c_i$ was selected in such a way that the forms

$$((-c_i) q_1) \otimes K_{p_i} \quad \text{and} \quad ((c_i) q_2) \otimes K_{p_i}$$

were isotropic. Consequently, the forms $q_1 \otimes K_{p_i}$ and $q_2 \otimes K_{p_i}$ are isotropic, as well.

Finally, we arrive at the prime $q$. The coefficients $a_1, a_2$ and $a_3$ have even valuations at $q$, hence the form $q_1 \otimes K_{q}$ is isotropic by [19, Corollary VI.2.5]. Moreover, for every $p \neq q$, we have proved above that $q_2 \otimes K_{p}$ is isotropic. By [19, Proposition V.3.22] we have

$$1 = s_p(q_2) \cdot (-1, -d_2)_p = (c, -a_4 a_5)_p \cdot (-a_4, -a_5)_p.$$  

Using Hilbert reciprocity law we obtain

$$1 = (c, -a_4 a_5)_q \cdot (-a_4, -a_5)_q \cdot \prod_{p \neq q} (c, -a_4 a_5)_p \cdot \prod_{p \neq q} (-a_4, -a_5)_p$$

$$= (c, -a_4 a_5)_q \cdot (-a_4, -a_5)_q$$

$$= s_q(q_2) \cdot (-1, -d_2)_q.$$  

Therefore, [19, Proposition V.3.22] says that $q_2 \otimes K_{q}$ is isotropic.

This way we have proved that the coordinates of $c$ form a solution to system (4), and the forms $q_1$ and $q_2$ are locally isotropic at every place of $K$, hence they are isotropic over $K$ by the local-global principle. Consequently there are vectors $(w_0, v_0, v_1, v_2, v_3)$ and $(w_0, w_1, w_2)$ such that

$$-b_0^2 + a_1 v_1^2 + a_2 v_2^2 + a_3 v_3^2 = 0 = b w_0^2 + a_4 w_1^2 + a_5 w_2^2.$$  

A direct calculation shows that the vector $v := (v_1/w_0, v_2/v_0, v_3/v_0, w_1/w_0, w_2/w_0)$ is an isotropic vector of $q$, we have been looking for.

As at the end of the proof of correctness of Algorithm 2 let $Q$ be the set of primes $q$ such that, after appending any of them to $T$, system (4) becomes solvable. The same arguments, that we used for Algorithm 2 show that $Q$ has positive density and the algorithm terminates in finite time (see Remark 4).

4.4. Dimension 6 and above.

Algorithm 4. Given an isotropic quadratic form $q = (a_1, \ldots, a_6)$ of dimension 6 over a global field $K$, this algorithm constructs a vector $v \in K^6$ such that $q(v) = 0$.

1. Denote $q_1 := (a_1, a_2, a_3)$ and $q_2 := (-a_4, -a_5, -a_6)$.

2. If $q_1$ is isotropic, then execute Algorithm 7 to construct an isotropic vector $(v_1, v_2, v_3)$ of $q_1$. Output $v = (v_1, v_2, v_3, 0, 0, 0)$ and quit.

3. Likewise, if $q_2$ is isotropic, find its isotropic vector $(w_1, w_2, w_3)$, output $v = (0, 0, 0, w_1, w_2, w_3)$ and quit.

4. Let $S \subseteq \mathcal{P}(q)$ be the set of all the dyadic primes of $K$ and these non-dyadic ones at which either $q_1$ or $q_2$ is locally anisotropic.

5. For every $p_i \in S$ find some $c_i \in K^x$ such that $-c_i \cdot \det q_1 \notin K_{p_i}^{x^2}$ and $-c_i \cdot \det q_2 \notin K_{p_i}^{x^2}$. 


(6) If $K$ is non-real, then use Chinese remainder theorem to construct $c \in K^\times$ such that

$$c \equiv c_i \pmod{p_i^{1+\ord_{p_i} q}}$$

for every $p_i \in S$.

(7) If $K$ is formally real then:

(a) Using [15] Algorithm 3 to construct a totally positive element $a \in K^\times$ satisfying $q$ for all $p_i \in S$.

(b) Let $S_\infty$ be the set of all real places $\tau$ of $K$ such that either $q_1 \otimes K_\tau$ or $q_2 \otimes K_\tau$ is anisotropic.

(c) For every $\tau \in S_\infty$, if $q_1 \otimes K_\tau$ is anisotropic, set $\varepsilon_\tau := \sgn_\tau(a_1)$, otherwise, when $q_2 \otimes K_\tau$ is anisotropic, set $\varepsilon_\tau := -\sgn(a_2)$.

(d) Using [15] Algorithm 4 to construct an element $b \in K^\times$ such that:

- $\sgn_\tau b = \varepsilon_\tau$ for every $\tau \in S_\infty$,
- $b$ is a local square at every $p_i \in S$.

(e) Set $c := a \cdot b$.

(8) Execute Algorithm 5 to find isotropic vectors:

- $(v_0, v_1, v_2, v_3)$ of $\langle -c \rangle \perp q_1$,
- $(w_0, w_1, w_2, w_3)$ of $\langle -c \rangle \perp q_2$.

(9) Output $v = (v_0/w_0, v_2/v_0, v_3/v_0, w_1/w_0, w_2/w_0, w_3/w_0)$.

Proof of correctness. If the algorithm terminates either in step (6) or (7), then the correctness of its output follows from the correctness of the previously discussed algorithms. Hence, without loss of generality, we may assume that the forms $q_1$ and $q_2$ are both anisotropic. If $p$ is a finite prime of $K$, then the local square class group $K_p^\times/K_p^{\times_2}$ consists of at least four classes. Hence, for every $p_i \in S$, the corresponding $c_i$ exists.

We claim that $c \in K^\times$ constructed either in step (6) or (7) is represented by both $q_1$ and $q_2$. Denote

$$\hat{q}_1 := \langle -c \rangle \perp q_1 \quad \text{and} \quad \hat{q}_2 := \langle -c \rangle \perp q_2.$$

Both forms are trivially isotropic at every complex place of $K$. Pick any real place $\tau$ of $K$. If $\tau \notin S_\infty$, then already the forms $q_1 \otimes K_\tau$ and $q_2 \otimes K_\tau$ are isotropic and so are $\hat{q}_1 \otimes K_\tau$ and $\hat{q}_2 \otimes K_\tau$. Conversely, assume that $\tau$ is in $S_\infty$. Therefore, at least one of the forms $q_1 \otimes K_\tau$, $q_2 \otimes K_\tau$ is anisotropic. Suppose it is the first one, then all its coefficients must have the same sign, and so

$$\varepsilon_\tau = \sgn_\tau a_1 = \sgn_\tau a_2 = \sgn_\tau a_3 = \sgn_\tau b = \sgn_\tau c.$$

It follows that $q_1 \otimes K_\tau$ is isotropic. Now, $q_2 \otimes K_\tau$ is either isotropic (then so is $\hat{q}_2 \otimes K_\tau$) or it is anisotropic. In the latter case, we have

$$\sgn_\tau a_4 = \sgn_\tau a_5 = \sgn_\tau a_6 = -\varepsilon_\tau,$$

since $q$ itself is isotropic, by assumption. It follows that $\sgn_\tau(-a_4) = \sgn_\tau c$, hence $\hat{q}_2 \otimes K_\tau$ is also isotropic. The case, when $q_1 \otimes K_\tau$ is isotropic and $q_2 \otimes K_\tau$ is not, is fully analogous.

We can now focus on non-Archimedean places. Fix some finite prime $p$ of $K$. If $p \notin S$, then both forms $q_1 \otimes K_p$ and $q_2 \otimes K_p$ are isotropic thus, $\hat{q}_1 \otimes K_p$ and $\hat{q}_2 \otimes K_p$ are isotropic, too. Conversely assume that $p = p_i \in S$. It follows from the local square theorem (see e.g., [19] Theorem VI.2.19) and Eq. (7) that $c \cdot c_i \in K_p^{\times 2}$.

Now, it is known (see e.g., [19] Corollary VI.2.15) that over a local field, there is only one (up to isometry) anisotropic form of dimension 4 and its determinant is necessarily a square. However, in our case

$$\det \hat{q}_1 \cdot K_p^{\times 2} = (-c) \cdot \det q_1 \cdot K_p^{\times 2} = (-c_i) \cdot \det q_1 \cdot K_p^{\times 2} \neq K_p^{\times 2}.$$
Algorithm 5. Given a form $q = \langle a_1, \ldots, a_d \rangle$ of dimension $d \geq 6$ over a non-real global field $K$, this algorithm construct a vector $v \in K^d$ such that $q(v) = 0$.

1. Execute Algorithm 4 with the input $q = \langle a_1, \ldots, a_d \rangle$ and denote its output by $(v_1, \ldots, v_d)$.
2. Return $v := (v_1, \ldots, v_d, 0, \ldots, 0)$.

For real fields, there is one more special case that we need to consider, namely dimension 7. Fortunately, the corresponding algorithm is very similar to the previous one.

Algorithm 6. Given an isotropic form $q = \langle q_1, \ldots, q_7 \rangle$ of dimension 7 over a real number field $K$, this algorithm outputs its isotropic vector.

1. Denote $q_1 := \langle a_1, a_2, a_3 \rangle$ and $q_2 := \langle -a_4, -a_5, -a_6, -a_7 \rangle$.
2. If $q_1$ is isotropic, then execute Algorithm 4 to construct an isotropic vector $(v_1, v_2, v_3)$ and set $v := (v_1, v_2, v_3, 0, 0, 0, 0)$ and quit.
3. Likewise, if $q_2$ is isotropic, then call Algorithm 5 to obtain an isotropic vector $(w_1, w_2, w_3, w_4)$. Output $v := (0, 0, 0, w_1, w_2, w_3, w_4)$ and quit.
4. Let $S$ be the set of all the non-archimedean places of $K$ at which $q_1$ is locally anisotropic.
5. Let $S_\infty$ be the set of all real places of $K$, at which either $q_1$ or $q_2$ is locally anisotropic.
6. For every $p \in S$ find $c_p \in K^\times$ such that $-c_p \cdot \det q_1 \notin K_p^{\times 2}$.
7. For every $p \in S_\infty$, if $q_1 \otimes K_p$ is anisotropic, set $\varepsilon_p := \text{sgn}_p(a_1)$, otherwise if it is $q_2 \otimes K_p$ that is anisotropic, set $\varepsilon_p := \text{sgn}(a_4)$.
8. Find a totally positive element $a \in K^\times$ such that $a \equiv c_p \pmod{p^{1+\text{ord}_p 4}}$ for every $p \in S$.
9. Find an element $b \in K^\times$ such that:
   - $\text{sgn}_p(b) = \varepsilon_p$ for every $p \in S_\infty$,
   - $b$ is a local square of every $p \in S$.
10. Set $c := a \cdot b$. 
(11) Execute Algorithm 2 and find an isotropic vector \((v_0, \ldots, v_3)\) of the form \(\langle -c \rangle \perp q_1\).

(12) Using Algorithm 3 construct an isotropic vector \((w_0, \ldots, w_4)\) of the form \(\langle -c \rangle \perp q_2\).

(13) Output \(v := (v_1/v_0, v_2/v_0, v_3/v_0, w_1/w_0, w_2/w_0, w_3/w_0, w_4/w_0)\).

The correctness of this algorithm is proved in the very same way as of Algorithm 4, except that now \(q_2 := \langle -c \rangle \perp q_2\) is trivially isotropic at every non-archimedean place, since its dimension equals 5. The next (and last) algorithm deals with forms of dimensions 8 and above, over real number fields. Again it is very similar to Algorithms 4 and 6.

**Algorithm 7.** Given a quadratic form \(q = (a_1, \ldots, a_d)\) of dimension \(d \geq 8\) over a real number field \(K\), this algorithm returns a vector \(v \in K^d\) such that \(q(v) = 0\).

1. Denote \(k := \lfloor d/2 \rfloor\) and 
   \[ q_1 := (a_1, \ldots, a_k), \quad q_2 := (-a_{k+1}, \ldots, -a_d). \]

2. If \(q_1\) is isotropic, then construct its isotropic vector \((v_1, \ldots, v_k)\). Output \(v := (v_1, \ldots, v_k, 0, \ldots, 0)\) and quit.

3. If \(q_2\) is isotropic, then construct its isotropic vector \((w_1, \ldots, w_{d-k})\). Output \(v := (0, \ldots, 0, w_1, \ldots, w_{d-k})\) and quit.

4. Let \(S_\infty\) be the set of all real places of \(K\), at which either \(q_1\) or \(q_2\) is locally anisotropic.

5. For every \(r \in S_\infty\), if \(q_1 \otimes K_r\) is anisotropic, set \(\varepsilon_r := \text{sgn}_r(a_1)\), otherwise if \(q_2 \otimes K_r\) is anisotropic, set \(\varepsilon_r := -\text{sgn}_r(a_k+1)\).

6. Find an element \(c \in K^\times\) such that 
   \[ \text{sgn}_r(c) = \varepsilon_r \]
   for every \(r \in S\).

7. Find isotropic vectors \((v_0, v_1, \ldots, v_k)\) and \((w_0, w_1, \ldots, w_{d-k})\) of \(\langle -c \rangle \perp q_1\) and \(\langle -c \rangle \perp q_2\), respectively.

8. Output \(v := (v_1/v_0, v_2/v_0, v_3/v_0, w_1/w_0, w_2/w_0, w_3/w_0, w_4/w_0)\).

Notice that the forms \(\langle -c \rangle \perp q_1\) and \(\langle -c \rangle \perp q_2\) have dimensions strictly greater than 4, since \(d \geq 8\). Consequently, both forms are locally isotropic at every non-archimedean place of \(K\) by [19, Theorem VI.2.12]. The proof of correctness of Algorithm 7 is thus analogous to those of Algorithms 4 and 6.

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Email address: przemyslaw.koprowski@us.edu.pl

Institute of Mathematics, University of Silesia, ul. Bankowa 14, Katowice, Poland