Interpolation inequalities between the deviation of curvature and the isoperimetric ratio with applications to geometric flows

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November 27, 2018

Abstract

Several inequalities for the isoperimetric ratio for plane curves are derived. In particular, we obtain interpolation inequalities between the deviation of curvature and the isoperimetric ratio. As applications, we study the large-time behavior of some geometric flows of closed plane curves without a convexity assumption.

Keywords: isoperimetric ratio, curvature, interpolation inequalities, geometric flow

1 Introduction

Let $f = (f_1, f_2) : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^2$ be a function such that Im$f$ is a closed plane curve with rotation number 1 and the variable of $f$ is the arc-length parameter. The unit tangent vector is $\tau = (f_1', f_2')$. Let $\nu = (-f_2', f_1')$ be

*2010 Mathematics Subject Classification: 53A04, 53C44, 35B40, 35K55
the inward unit normal vector, and let $\kappa = f''$ be the curvature vector. The (signed) area $A$ is given by

$$A = -\frac{1}{2} \int_0^L f \cdot \nu \, ds.$$  

The curvature $\kappa = \kappa \cdot \nu$ is positive when $\text{Im} f$ is convex. Since the curve has rotation number 1, the deviation of curvature is

$$\tilde{\kappa} = \kappa - \frac{1}{L} \int_0^L \kappa \, ds = \kappa - \frac{2\pi}{L}.$$  

For a non-negative integer $\ell$, we set

$$I_{\ell} = L^{2\ell+1} \int_0^L |\tilde{\kappa}^{(\ell)}|^2 \, ds,$$

which is a scale invariant quantity (cf. [1]). For $\ell_1 \leq \ell_2 \leq \ell_3$, $I_{\ell_2}$ can be interpolated by $I_{\ell_1}$ and $I_{\ell_3}$ using the Gagliardo-Nirenberg inequality. In this paper, we show that $I_{\ell}$ satisfies interpolation inequalities by use of the isoperimetric ratio and $I_m$ with $\ell \leq m$. Hereafter we call $\frac{4\pi A}{L^2}$ the isoperimetric ratio, not $\frac{L^2}{4\pi A}$. We set

$$I_{-1} = 1 - \frac{4\pi A}{L^2},$$

which is also scale invariant, and is non-negative by the isoperimetric inequality. Of course, $\tilde{\kappa} \equiv 0$ implies $\text{Im} f$ is a round circle, which attains the minimum $I_{-1} = 0$. This suggests that $I_{-1}$ can be dominated by some quantities of $\tilde{\kappa}$. Indeed, we have

$$I_{-1} = \frac{L^2 - 4\pi A}{L^2} = \frac{1}{L^2} \int_0^L (-L f \cdot \kappa + 2\pi f \cdot \nu) \, ds$$

$$= -\frac{1}{L} \int_0^L \tilde{\kappa} (f \cdot \nu) \, ds$$

$$= -\frac{1}{L} \int_0^L \tilde{\kappa} \left( f \cdot \nu - \frac{1}{L} \int_0^L f \cdot \nu \, ds \right) \, ds$$

and

$$\left| f \cdot \nu - \frac{1}{L} \int_0^L f \cdot \nu \, ds \right| \leq L.$$  

Thus it holds that

$$0 \leq I_{-1} \leq I_0^2.$$  

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However, since \( f \cdot \nu - \frac{1}{L} \int_0^L f \cdot \nu \, ds = 0 \) when \( \tilde{\kappa} \equiv 0 \), it seems that the above inequality can be improved. In Section 2, we will show an improved version
\[
0 \leq I_{-1} \leq \frac{I_0}{8\pi^2}
\]
in Theorem 2.1.

The converse inequality seems not to hold; the reason will be clarified in Section 2. However, \( I_0 \) can be estimated by use of \( I_{-1} \) with the help of \( \kappa \) and its derivative
\[
I_0 \leq I_{-1}^{\frac{1}{2}} \left[ L^3 \int_0^L \left\{ \kappa^3 \tilde{\kappa} + (\tilde{\kappa}')^2 \right\} \, ds \right]^{\frac{1}{2}}
\]
(see Theorem 2.2). Combining this inequality and the Gagliardo-Nirenberg inequality, in Theorem 3.1 we will show interpolation inequalities satisfied by \( I_{-1}, I_\ell \) and \( I_m \) for \( 0 \leq \ell \leq m \):
\[
I_\ell \leq C \left( \frac{m-\ell}{m} I_m + \frac{m-\ell}{m+1} I_m^{\frac{m+1}{m}} \right).
\]
Here the constant \( C \) depends on \( \ell \) and \( m \) but not on \( \tilde{\kappa} \) nor \( L \).

In the final section we give applications of our inequalities to the analysis of the large-time behavior of some geometric flows of closed plane curves. Using our inequalities, we can study the analysis of behavior quite easily.

## 2 Preliminaries

For the vector-valued function \( f = (f_1, f_2) : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^2 \), we define a complex-valued function by
\[
f = f_1 + if_2.
\]
We expand \( f \) by the Fourier series
\[
f = \sum_{k \in \mathbb{Z}} \hat{f}(k) \varphi_k,
\]
where
\[
\varphi_k(s) = \frac{1}{\sqrt{L}} \exp\left( \frac{2\pi i ks}{L} \right), \quad \hat{f}(k) = \int_0^L f \overline{\varphi_k} \, ds.
\]
The series \( \sum_{k \in \mathbb{Z}} k^\ell |\hat{f}(k)|^2 \) is related to \( \left( \frac{L}{2\pi} \right)^\ell \int_0^L \kappa^\ell \, ds \). To see this we need some expression of \( f^{(\ell-1)} \overline{f} \) in terms of \( \kappa \) and its derivatives. Set
\[
F_\ell = f^{(\ell-1)} \overline{f}.
\]
Lemma 2.1 It holds that
\[ F_1 = f \cdot \tau + i f \cdot \nu, \quad F_\ell = i \kappa F_{\ell-1} + F'_{\ell-1} \quad \text{for} \quad \ell \geq 2. \quad (2.1) \]

Proof. Firstly, since \( \tau = (f'_1, f'_2) \) and \( \nu = (-f'_2, f'_1) \), we have
\[ F_1 = (f_1 + if_2)(f'_1 - if'_2) = f \cdot \tau + i f \cdot \nu. \]
The recurrence relation is derived from
\[ F_\ell = f^{(\ell-1)} F = (f^{(\ell-2)} F')' - f^{(\ell-2)} F'' = F'_{\ell-1} - f^{(\ell-2)} F'F'' = F'_{\ell-1} - F_{\ell-1} F_3 \]
and
\[ F_3 = f'' F = (f'' + if''_2)(f'_1 - if'_2) = \frac{1}{2} (|f'|^2)' + i(-f''_1 f'_2 + f''_2 f'_1) = i f''. \cdot \nu = i \kappa. \]
\[ \square \]

Proposition 2.1 For \( \ell \geq 2 \), it holds that
\[ \sum_{k \in \mathbb{Z}} k^\ell |\hat{f}(k)|^2 = -i^{1-\ell} \left( \frac{L}{2\pi} \right)^\ell \int_0^L \kappa F_{\ell-1} ds. \quad (2.2) \]

Proof. It follows from the recurrence relation in Lemma 2.1 that
\[ \int_0^L F_\ell ds = i \int_0^L \kappa F_{\ell-1} ds. \]
On the other hand,
\[ \int_0^L F_\ell ds = \langle f^{(\ell-1)}, f' \rangle_{L^2} = -i^{\ell} \left( \frac{2\pi}{L} \right)^\ell \sum_{k \in \mathbb{Z}} k^\ell |\hat{f}(k)|^2. \]
\[ \square \]
Corollary 2.1 We have
\[
\sum_{k \in \mathbb{Z}} k |\hat{f}(k)|^2 = \frac{LA}{\pi}, \tag{2.3}
\]
\[
\sum_{k \in \mathbb{Z}} k^2 |\hat{f}(k)|^2 = \left(\frac{L}{2\pi}\right)^2 \int_0^L \kappa^0 ds = \frac{L^3}{4\pi^2}, \tag{2.4}
\]
\[
\sum_{k \in \mathbb{Z}} k^3 |\hat{f}(k)|^2 = \left(\frac{L}{2\pi}\right)^3 \int_0^L \kappa ds = \frac{L^3}{4\pi^2}, \tag{2.5}
\]
\[
\sum_{k \in \mathbb{Z}} k^4 |\hat{f}(k)|^2 = \left(\frac{L}{2\pi}\right)^4 \int_0^L \kappa^2 ds, \tag{2.6}
\]
\[
\sum_{k \in \mathbb{Z}} k^5 |\hat{f}(k)|^2 = \left(\frac{L}{2\pi}\right)^5 \int_0^L \kappa^3 ds, \tag{2.7}
\]
\[
\sum_{k \in \mathbb{Z}} k^6 |\hat{f}(k)|^2 = \left(\frac{L}{2\pi}\right)^6 \int_0^L \{\kappa^4 + (\kappa')^2\} ds. \tag{2.8}
\]

Remark 2.1 If \(\kappa > 0\) everywhere, it holds that
\[
\sum_{k \in \mathbb{Z}} k |\hat{f}(k)|^2 = \frac{L}{2\pi} \int_0^L \frac{1}{\kappa} \left\{1 - (f' \bar{f})'\right\} ds.
\]
In particular if \(\kappa\) is a constant, then
\[
\sum_{k \in \mathbb{Z}} k |\hat{f}(k)|^2 = \frac{L}{2\pi} \int_0^L \frac{ds}{\kappa}.
\]

Proof of Corollary 2.1 Since
\[
\int_0^L F_1 ds = i \int_0^L f \cdot \nu ds = -2iA,
\]
we obtain
\[
\sum_{k \in \mathbb{Z}} k |\hat{f}(k)|^2 = \frac{L}{2\pi} i \int_0^L f' \bar{f} ds = \frac{L}{2\pi} i \int_0^L F_1 ds = \frac{LA}{\pi}.
\]
Thus (2.3) follows. The relations (2.4)–(2.8) are consequence of (2.2) and (2.1). Indeed,
\[
\sum_{k \in \mathbb{Z}} k^2 |\hat{f}(k)|^2 = -i^{-1} \left(\frac{L}{2\pi}\right)^2 \int_0^L \kappa F_1 ds = i \left(\frac{L}{2\pi}\right)^2 \int_0^L \kappa (f \cdot \tau + i f \cdot \nu) ds,
\]
and

\[
\int_0^L \kappa f \cdot \tau ds = - \int_0^L f \cdot \nu' ds = \int_0^L \nu \cdot \tau ds = 0,
\]

\[
\int_0^L \kappa f \cdot \nu ds = \int_0^L f \cdot \tau' ds = - \int_0^L \tau \cdot \nu ds = -L.
\]

Thus (2.4) holds. Since \( f \) is parametrized by the arc-length, we have

\[
F_2 = |f'|^2 = \|f'\|^2 = 1.
\]

It follows from (2.1) that

\[
F_3 = i\kappa, \quad F_4 = -\kappa^2 + i\kappa', \quad F_5 = -3\kappa\kappa' + i(-\kappa^3 + \kappa'').
\]

Hence we obtain

\[
\int_0^L \kappa F_2 ds = 2\pi, \quad \int_0^L \kappa F_3 ds = i \int_0^L \kappa^2 ds,
\]

\[
\int_0^L \kappa F_4 ds = - \int_0^L \kappa^3 ds, \quad \int_0^L \kappa F_5 ds = -i \int_0^L \{\kappa^4 + (\kappa')^2\} ds.
\]

Consequently (2.5)–(2.8) are obtained from (2.2).

**Corollary 2.2**

\[
I_{-1} = \frac{4\pi^2}{L^3} \sum_{k \in \mathbb{Z}} k(k-1)|\hat{f}(k)|^2.
\]

**Proof.** We obtain

\[
I_{-1} = 1 - \frac{4\pi A}{L^2} = \frac{4\pi^2}{L^3} \left( \frac{L^3}{4\pi^2} - \frac{LA}{\pi} \right) = \frac{4\pi^2}{L^3} \sum_{k \in \mathbb{Z}} k(k-1)|\hat{f}(k)|^2
\]

from (2.4) and (2.3). \( \square \)

Since \( k(k-1) \geq 0 \) for \( k \in \mathbb{Z} \), we obtain the isoperimetric inequality \( I_{-1} \geq 0 \) from this corollary, which is essentially the proof by Hurwitz [1].

**Corollary 2.3**

\[
I_0 = \frac{16\pi^4}{L^3} \sum_{k \in \mathbb{Z}} k^3(k-1)|\hat{f}(k)|^2.
\]

**Proof.** The assertion is derived as

\[
I_0 = L \int_0^L \tilde{\kappa}^2 ds = L \int_0^L \kappa \tilde{\kappa} ds = L \left( \int_0^L \kappa^2 ds - \frac{2\pi}{L} \int_0^L \kappa ds \right)
\]

\[
= \frac{16\pi^4}{L^3} \sum_{k \in \mathbb{Z}} k^3(k-1)|\hat{f}(k)|^2,
\]

using (2.6) and (2.5). \( \square \)
Since $k(k - 1) \leq k^3(k - 1)$ for $k \in \mathbb{Z}$, we have

$$I_{-1} \leq \frac{4\pi^2}{L^3} \sum_{k \in \mathbb{Z}} k^3(k - 1)|\hat{f}(k)|^2 = \frac{I_0}{4\pi^2}.$$  

(2.9)

We set

$$g = \sum_{k \in \mathbb{Z}} \sqrt{k(k - 1)} \hat{f}(k) \varphi_k,$$

and then (2.9) is

$$\|g\|_{L^2}^2 \leq \frac{L^2}{4\pi^2} \|g'\|_{L^2}^2. $$

This is Wirtinger's inequality with the best constant. Therefore it is reasonable to think that (2.9) cannot be sharpened. However, the function $f$ is not an arbitrary one, but satisfies $|f'| \equiv 1$, and this suggests that we may be able to improve the constant in (2.9). Indeed, we can show an improved version by use of (2.4) and (2.5).

**Theorem 2.1** We have

$$I_{-1} \leq \frac{I_0}{8 \pi^2}. $$

Equality never holds except the trivial case $\tilde{k} \equiv 0$.

**Proof.** First observe that (2.4) and (2.5) imply

$$\sum_{k \in \mathbb{Z}} k^2|\hat{f}(k)|^2 = \sum_{k \in \mathbb{Z}} k^3|\hat{f}(k)|^2.$$

Hence

$$I_{-1} = \frac{4\pi^2}{L^3} \sum_{k \in \mathbb{Z}} k(k^2 - 1)|\hat{f}(k)|^2 = \frac{4\pi^2}{L^3} \sum_{k \in \mathbb{Z}} k(k - 1)(k + 1)|\hat{f}(k)|^2,$$

$$I_0 = \frac{16\pi^4}{L^3} \sum_{k \in \mathbb{Z}} k^2(k^2 - 1)|\hat{f}(k)|^2 = \frac{16\pi^4}{L^3} \sum_{k \in \mathbb{Z}} k^2(k - 1)(k + 1)|\hat{f}(k)|^2.$$ 

Consequently we obtain

$$\frac{I_0}{8\pi^2} - I_{-1} = \frac{2\pi^2}{L^3} \sum_{k \in \mathbb{Z}} k(k - 2)(k - 1)(k + 1)|\hat{f}(k)|^2 \geq 0,$$

because $k(k - 2)(k - 1)(k + 1) \geq 0$ for $k \in \mathbb{Z}$. 

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For the equality case, assume that \( f \) satisfies \( I_{-1} = \frac{I_0}{8\pi^2} \). It follows from the previous paragraph that

\[
f = \sum_{k=-1}^{2} \hat{f}(k) \varphi_k.
\]

Since \( |f'|^2 \equiv 1 \) and since \( \varphi_l \varphi_l = L^2 \varphi_{k-l} \), we have

\[
1 = \left| \sum_{k=-1}^{2} \frac{2\pi ik}{L} \hat{f}(k) \varphi_k \right|^2 = \frac{4\pi^2}{L^2} \sum_{k,-1}^{2} k\ell \hat{f}(k) \overline{f(\ell)} \varphi_k \varphi_\ell
\]

\[
= \frac{4\pi^2}{L^2} \sum_{m=-3}^{3} \sum_{k,-1,\ell,m} k\ell \hat{f}(k) \overline{f(\ell)} \varphi_m.
\]

From this we find that, in particular,

\[
0 = \sum_{k,-1,-1,\ell,-1} k\ell \hat{f}(k) \overline{f(\ell)} = -2 \hat{f}_2 \hat{f}_{-1}.
\]

On the other hand, it follows from (2.4) and (2.6) that

\[
0 = \sum_{k=-1}^{2} k^2(k-1) |\hat{f}(k)|^2 = -2 |\hat{f}_{-1}|^2 + 4 |\hat{f}_2|^2.
\]

Therefore \( \hat{f}_{-1} = \hat{f}_2 = 0 \). Consequently \( f = \hat{f}_0 \varphi_0 + \hat{f}_1 \varphi_1 \), which implies \( \text{Im} f \) is a round circle. Hence \( \kappa \equiv 0 \).

We remark that it is impossible to show

\[ k^2(k-1) \leq Ck(k-1) \quad \text{for} \quad k \in \mathbb{Z}, \]

and this implies that there is no hope to see \( I_0 \leq CI_{-1} \). Thereupon we give an estimate of \( I_0 \) in terms of \( I_{-1} \) with the help of \( \kappa \) and its derivative.

**Theorem 2.2** The integral \( \int_0^L \{ \kappa^3 \kappa + (\kappa')^2 \} \, ds \) is non-negative, and it holds that

\[
I_0 \leq I_{-1}^2 \left[ L^3 \int_0^L \{ \kappa^3 \kappa + (\kappa')^2 \} \, ds \right]^{\frac{1}{2}}.
\]

Equality never holds except the trivial case \( \kappa \equiv 0 \).
Proof. By Cauchy’s inequality we have
\[
I_0 \leq \frac{16\pi^4}{L^3} \left\{ \sum_{k \in \mathbb{Z}} k(k-1)|\hat{f}(k)|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{k \in \mathbb{Z}} k^5(k-1)|\hat{f}(k)|^2 \right\}^{\frac{1}{2}} = \frac{8\pi^3}{L^3} L^4 \left\{ \sum_{k \in \mathbb{Z}} k^5(k-1)|\hat{f}(k)|^2 \right\}^{\frac{1}{2}},
\]
and (2.7) and (2.8) show that
\[
\sum_{k \in \mathbb{Z}} k^5(k-1)|\hat{f}(k)|^2 = \left( \frac{L}{2\pi} \right)^6 \int_0^L \left\{ \kappa^4 + (\kappa')^2 - \frac{2\pi}{L} \kappa^3 \right\} ds
\]
\[
= \left( \frac{L}{2\pi} \right)^6 \int_0^L \left\{ \kappa^3 \tilde{\kappa} + (\tilde{\kappa}')^2 \right\} ds.
\]
Assume that \( f \) satisfies the equality case in the assertion. It follows from the equality condition of Cauchy’s inequality that \( \hat{f}_k = 0 \) except \( k = 0 \) and \( 1 \). Consequently \( \text{Im} f \) is a round circle, and \( \tilde{\kappa} \equiv 0 \). \( \square \)

3 Interpolation inequalities

In this section we derive several interpolation inequalities from Theorem 2.2.

Theorem 3.1 Let \( 0 \leq \ell \leq m \). There exists a positive constant \( C = C(\ell, m) \) independent of \( L \) such that
\[
I_\ell \leq C \left( I_{m-\ell}^{m-\ell} I_m + I_{m-\ell}^{m-\ell} I_{m+1}^{m+1} \right)
\]
holds.

Proof. When \( \ell = m \), the assertion is clear.

Let \( \ell < m \). Then \( m \geq 1 \). The Gagliardo-Nirenberg inequality shows
\[
\left( \int_0^L |\kappa^{(j)}|^p ds \right)^{\frac{1}{p}} \leq C(j, m, p) I_m^{\frac{1}{m}} (j-\frac{1}{p}+\frac{1}{2}) I_0^{\frac{1}{2}} \left\{ 1 - \frac{1}{m} (j-\frac{1}{p}+\frac{1}{2}) \right\} \]
for \( p \geq 2 \) and \( j \leq m \). Here \( C(j, m, p) \) is independent of \( L \). Combining this and Wirtinger’s inequality \( I_0 \leq \frac{L^3}{4\pi^2} \), we have
\[
L^3 \int_0^L |\tilde{\kappa}|^3 ds \leq C I_1^\frac{1}{2} I_0^\frac{3}{2} \leq C I_1^\frac{1}{2}, \quad L^2 \int_0^L |\kappa|^3 ds \leq C I_1^\frac{1}{2} I_0^\frac{5}{2} \leq C I_1^\frac{3}{2}.
\]
Therefore
\[ L^3 \int_0^L \kappa^3 \tilde{\kappa} ds = L^3 \int_0^L \left( \kappa + \frac{2\pi}{L} \right)^3 \tilde{\kappa} ds \]
\[ \leq C \left( L^3 \int_0^L \kappa^4 ds + L^2 \int_0^L |\tilde{\kappa}|^3 ds + L \int_0^L \tilde{\kappa}^2 ds \right) \]
\[ \leq C \left( I_1^2 + I_2^2 + I_1 \right). \]
Consequently the inequality in Theorem 2.2 implies
\[ I_0 \leq CI_1^2 \left( I_1 + I_1^2 \right), \]
which is the assertion with \( \ell = 0, m = 1 \).

Putting \( p = 2 \) in (3.1), we have
\[ I_j \leq C(j, m) I_m^j I_0^{1 - \frac{j}{m}}. \]  
(3.2)
Combining these with \( j = 1 \) and Young’s inequality, we have
\[ I_0 \leq C I_{-1}^2 \left( I_m^1 I_0^{1 - \frac{1}{m}} + I_m^{1/2} I_0^{3/2} (1 - \frac{1}{m}) \right) \leq \epsilon I_0 + C \epsilon \left( I_m^1 I_0 + I_m^{1/2} I_0^{1/2} \right), \]
where \( \epsilon \) is an arbitrary positive number. Consequently we obtain
\[ I_0 \leq C \left( I_{-1}^m I_0 + I_{m+1}^{m+1} I_0^{1/2} \right), \]
which is the assertion with \( \ell = 0, m \geq 2 \).

Let \( \ell \geq 1 \). Using the above inequality and (3.2) with \( j = \ell \), we obtain
\[ I_\ell \leq CI_m^\ell \left( I_{-1}^m I_0 + I_{m+1}^{m+1} I_0^{1/2} \right)^{1 - \frac{\ell}{m}} \leq C \left( I_{-1}^{m+\ell} I_0 + I_{m+1}^{m+1} I_0^{m+1} \right). \]

\[ \square \]

4 Applications to geometric flows

We give applications of our inequalities to the asymptotic analysis of geometric flows of closed plane curves. One of the flows is a curvature flow with a non-local term firstly studied by Jiang-Pan [5], and another is the area-preserving curvature flow considered by Gage [3]. If the initial curve
is convex, then the flows exist for all time keeping the convexity, and the curve approaches a round circle, this was shown in [5, 3]. The local existence of flows without a convexity assumption was shown by Ševčovič-Yazaki [7]. However, the large-time behavior for this case is still open, and seems to may occur finite-time blow-up for some non-convex initial curve [6], and the global existence for another initial non-convex curve [7]. Escher-Simonett [2] showed the global existence and investigated the large-time behavior of the area-preserving curvature flow for initial data close to a circle and without a convexity assumption. In this section, we investigate the large-time behavior of the flow without a convexity assumption assuming the global existence.

4.1 A curvature flow with a non-local term

Firstly we consider the large-time behavior of geometric flow

$$\partial_t f = \kappa - \frac{L}{2A} \nu$$

of closed plane curves. Assume that $f : \bigcup_{t \geq 0} (\mathbb{R} / L(t) \mathbb{Z} \times \{t\}) \to \mathbb{R}^2$ is a global solution with initial rotation number 1. Along the flow, the (signed) area $A$ and the isoperimetric ratio $\frac{4\pi A}{L^2}$ are non-decreasing, if the initial (signed) area is positive. Indeed,

$$\frac{dA}{dt} = -\int_0^L \partial_t f \cdot \nu \, ds = \int_0^L \left(-\kappa + \frac{L}{2A}\right) \, ds = -\frac{4\pi A + L^2}{2A} \geq 0,$$  \hspace{1cm} (4.2)

$$\frac{d}{dt} \frac{4\pi A}{L^2} = \frac{4\pi}{L^2} \frac{dA}{dt} - \frac{8\pi A}{L^3} \frac{dL}{dt} = \frac{8\pi A}{L^3} \int_0^L \partial_t f \cdot \left(-\frac{L}{2A} \nu + \kappa\right) \, ds$$

$$= \frac{8\pi A}{L^3} \int_0^L \left(\kappa - \frac{L}{2A}\right)^2 \, ds \geq 0.$$  \hspace{1cm} (4.3)

Since

$$\frac{dL}{dt} = -\int_0^L \partial_t f \cdot \kappa \, ds = -\int_0^L \kappa^2 \, ds + \frac{\pi L}{A},$$  \hspace{1cm} (4.4)

we find that $L$ is non-increasing by Gage’s inequality if $\text{Im} f$ is convex. Here we do not assume convexity.

**Theorem 4.1** Assume that $f$ is a global solution of (4.1) such that the initial rotation number is 1 and the initial (signed) area is positive. Then $\text{Im} f$
converges to a circle exponentially as \( t \to \infty \) in the following sense. There exist \( C > 0 \) and \( \lambda > 0 \) independent of \( t \) such that

\[
0 \leq I_{-1}(t) \leq Ce^{-\lambda t}.
\]

Furthermore, \( A(t) \) and \( L(t) \) converge to positive constants, say \( A_\infty \) and \( L_\infty \) respectively, as \( t \to \infty \). They satisfy \( 4\pi A_\infty = L_\infty \), and

\[
0 \leq A_\infty - A(t) \leq Ce^{-\lambda t}, \quad |L(t) - L_\infty| \leq Ce^{-\lambda t}.
\]

**Proof.** It follows from (4.2) that

\[
\frac{d}{dt} A^2 = L^2 - 4\pi A \geq 0.
\]

The second derivative is

\[
\frac{d^2}{dt^2} A^2 = \frac{d}{dt} (L^2 - 4\pi A) = 2L \frac{dL}{dt} - 4\pi \frac{dA}{dt} = \int_0^L \partial_t f \cdot (-2L \kappa + 4\pi \nu) ds = \int_0^L \left( \kappa - \frac{L}{2A} \right) (-2L \tilde{\kappa}) ds = -2L \int_0^L \kappa^2 ds \leq 0.
\]

Therefore

\[
0 \leq \frac{d}{dt} A^2 \leq \frac{d}{dt} A^2 \bigg|_{t=0}.
\]

We put \( C_0 = \left. \frac{d}{dt} A^2 \right|_{t=0} \). Since the initial (signed) area is positive, so is \( A \), and

\[
A^2 \leq C_0 t + A(0)^2.
\]

Hence

\[
\int_0^t \frac{dt}{A} \geq \int_0^t \frac{dt}{\sqrt{C_0 t + A(0)^2}} \to \infty \text{ as } t \to \infty.
\]

It follows from (4.3) that

\[
\frac{d}{dt} I_{-1} = -\frac{8\pi A}{L^3} \int_0^L \left( \tilde{\kappa} + \frac{2\pi}{L} - \frac{L}{2A} \right)^2 ds = -\frac{2\pi}{A} I_{-1}^2 - \frac{8\pi A}{L^3} \int_0^L \kappa^2 ds.
\]
Solving the differential inequality \( \frac{d}{dt} I_{-1} \leq -\frac{2\pi}{A} I_{-1}^2 \), we have

\[
0 \leq I_{-1} \leq \frac{I_{-1}(0)}{1 + 2\pi I_{-1}(0) \int_0^t \frac{dt}{A}} \to 0 \quad (t \to \infty).
\]

Also, (4.5) and Theorem 2.1 give us

\[
\frac{d}{dt} (L^2 - 4\pi A) + \frac{16\pi^2}{L^2} (L^2 - 4\pi A) \leq \frac{d}{dt} (L^2 - 4\pi A) + 2L \int_0^t \tilde{\kappa}^2 ds = 0.
\]

Using \( I_{-1} \to 0 \) as \( t \to \infty \) and (4.6), we obtain

\[
0 \leq L^2 - 4\pi A \leq (L(0)^2 - 4\pi A(0)) \exp \left( -16\pi^2 \int_0^t \frac{dt}{L^2} \right) \to 0 \quad \text{as} \quad t \to \infty.
\]

Dividing both sides by \( L^2 \), we have

\[
0 \leq I_{-1} \leq C \frac{16\pi^2}{L^2} \exp \left( -16\pi^2 \int_0^t \frac{dt}{L^2} \right) = -C \frac{d}{dt} \exp \left( -16\pi^2 \int_0^t \frac{dt}{L^2} \right).
\]

Integrating this with respect to \( t \), we obtain

\[
0 \leq \int_0^t I_{-1} dt \leq C \left\{ 1 - \exp \left( -16\pi^2 \int_0^t \frac{dt}{L^2} \right) \right\} \leq C.
\]

Integrating (4.2), we have

\[
A = A(0) + \int_0^t \frac{L^2}{2A} I_{-1} dt \leq C.
\]

Since \( A \) is non-decreasing and bounded from above, there exists a finite limit \( A_\infty = \lim_{t \to \infty} A(0, \infty) \). Hence \( \lim_{t \to \infty} L = 2\sqrt{\pi A_\infty} \in (0, \infty) \) exists, say \( L_\infty \). Consequently, we obtain

\[
0 \leq L^2 - 4\pi A \leq (L(0)^2 - 4\pi A(0)) \exp \left( -16\pi^2 \int_0^t \frac{dt}{L^2} \right) \leq Ce^{-\lambda t},
\]

\[
0 \leq \frac{L^2 - 4\pi A}{L^2} \leq Ce^{-\lambda t},
\]

\[
0 \leq A_\infty - A \leq \frac{1}{A_\infty + A} \int_0^\infty \frac{dt}{A^2} \leq \frac{1}{A_\infty + A} \int_0^\infty (L^2 - 4\pi A) dt \leq Ce^{-\lambda t},
\]

\[
|L - L_\infty| = \frac{|L^2 - 4\pi A + 4\pi(A - A_\infty)|}{L + L_\infty} \leq Ce^{-\lambda t}
\]

for some \( C > 0 \) and \( \lambda > 0 \). 

\[\square\]
Observe that the equation which $f$ satisfies is

$$\partial_t f = \partial_s^2 f - \frac{L}{2A} R \partial_s f,$$

where

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since this is a parabolic equation with a non-local term, $f$ is smooth for $t > 0$ as long as the solution exists. Hence by shifting the initial time, we may assume that the initial data is smooth. Next we consider the decay of $I_\ell$ as $t \to \infty$ along this flow.

**Theorem 4.2** Let $f$ be as in Theorem 4.1. For each $\ell \in \mathbb{N} \cup \{0\}$, there exist $C_\ell > 0$ and $\lambda_\ell > 0$ such that

$$I_\ell(t) \leq C_\ell e^{-\lambda_\ell t}.$$

**Proof.** We initially consider the behavior of $I_0$. Since

$$I_0 = L \int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 ds = L \int_0^L \kappa^2 ds - 4\pi^2,$$

we have

$$\frac{d}{dt} I_0 = \frac{dL}{dt} \int_0^L \kappa^2 ds + L \frac{d}{dt} \int_0^L \kappa^2 ds$$

$$= \int_0^L \left\{-\kappa \int_0^L \kappa^2 ds + L \left(2\nabla^2 \kappa + \|\kappa\|^2_{L^2} \right) \right\} \cdot \partial_t f ds$$

$$= \int_0^L \left\{-\kappa \int_0^L \kappa^2 ds + L \left(2\partial_s^2 \kappa + \kappa^3 \right) \right\} \left( \kappa - \frac{L}{2A} \right) ds$$

$$= -\left( \int_0^L \kappa^2 ds \right)^2 + \frac{\pi L}{A} \int_0^L \kappa^2 ds + L \int_0^L \left\{-2(\partial_s \kappa)^2 + \kappa^4 - \frac{L}{2A} \kappa^3 \right\} ds$$

$$= -2L \int_0^L (\partial_s \kappa)^2 ds + L \left\{ \int_0^L \kappa^4 ds - \frac{1}{L} \left( \int_0^L \kappa^2 ds \right)^2 \right\} - \frac{L^2}{2A} \int_0^L \kappa^2 \kappa ds$$

$$= -2L \int_0^L (\partial_s \kappa)^2 ds - \left( \int_0^L \kappa^2 ds \right)^2$$

$$+ \int_0^L \left\{ L\kappa^4 + \left( 8\pi - \frac{L^2}{2A} \right) \kappa^3 + \left( \frac{16\pi^2}{L} - \frac{2\pi L}{A} \right) \kappa^2 \right\} ds.$$
By virtue of Gagliardo-Nirenberg’s inequality, we have
\[
\frac{d}{dt} I_0 + \frac{1}{L^2} I_0^2 + \frac{2}{L^2} I_1 \leq \frac{C}{L^2} \int_0^L \left( L^3 \kappa^4 + L^2 |\kappa| |^3 + L \kappa^2 \right) ds \\
\leq \frac{C}{L^2} \left( I_1^\frac{3}{2} I_0^\frac{1}{2} + I_1^\frac{1}{2} I_0^\frac{5}{2} + I_0 \right).
\] (4.7)

Applying Young’s and Wirtinger’s inequalities, Theorem 2.1 and Theorem 3.1, we have
\[
I_1^\frac{1}{2} I_0^\frac{1}{2} \leq \epsilon I_1 + C \epsilon I_0^3,
\]
\[
I_1^\frac{1}{2} I_0^\frac{1}{2} \leq \epsilon I_1 + C \epsilon I_0^3 \leq \epsilon(I_1 + I_0) + C \epsilon I_0^3 \leq C \epsilon I_1 + C \epsilon I_0^3,
\]
\[
I_0 \leq I_1^2 \left(I_1 + I_0^2\right) \leq \left(I_1^2 + \epsilon\right) I_1 + C \epsilon I_1
\]
\[
\leq \left(I_1^2 + \epsilon\right) I_1 + C e^{-\lambda t}
\]
for any \(\epsilon > 0\). By Theorem 4.1 for each \(\epsilon > 0\) there exists \(T_0 > 0\) such that \(I_{-1} \leq \epsilon\) for \(t \geq T_0\). Therefore taking \(\epsilon\) sufficiently small, we find that there exist some constants \(C_1, C_2,\) and \(C_3\) such that
\[
\frac{d}{dt} I_0 + \frac{1}{L^2} I_0^2 + \frac{C_1}{L^2} I_1 \leq \frac{C_2}{L^2} I_0^3 + \frac{C_3}{L^2} e^{-\lambda t}
\] (4.8)
for \(t \geq T_0\). It is obvious that there exists \(T_1 \geq T_0\) satisfying
\[
\int_{T_1}^\infty \frac{C_3}{L^2} e^{-\lambda t} dt < \frac{1}{2C_2}.
\]
Furthermore, there exists \(T_2 \geq T_1\) such that \(I_0(T_2) < \frac{1}{2C_2}\), because we have
\[
\int_0^\infty I_0 dt \leq C
\]
by integrating (4.5). We would like to show \(I_0(t) < \frac{1}{C_2}\) for \(t \geq T_2\). To do this, we argue by contradiction. Then there exists \(T_3 > T_2\) such that
\[
I_0(t) < \frac{1}{C_2} \text{ for } t \in [T_2, T_3), \text{ and } I_0(T_3) = \frac{1}{C_2}.
\]
It follow from (4.8) that
\[
\frac{d}{dt} I_0 \leq \frac{C_3}{L^2} e^{-\lambda t}
\]
for $t \in [T_2, T_3]$. Hence

$$I_0(T_3) = I_0(T_2) + \int_{T_2}^{T_3} \frac{d}{dt} I_0 dt < \frac{1}{2C_2} + \int_{T_1}^{\infty} \frac{C_3}{L^2} e^{-\lambda t} dt < \frac{1}{C_2}. $$

This contradicts $I_0(T_3) = \frac{1}{C_2}$. Consequently $I_0$ is uniformly bounded, and (4.7) implies

$$\frac{d}{dt} I_0 + \frac{1}{L^2} I_0^2 + \frac{2}{L^2} I_1 \leq \frac{C}{L^2} \left( I_1^2 I_0^2 + I_1^4 + I_0 \right).$$

By Theorem 3.1 we have

$$I_1^2 I_0^2 + I_1^4 + I_0 \leq \epsilon I_1 + C \epsilon I_0 \leq \epsilon I_1 + C \epsilon I_1 \left( I_1 + I_1^2 \right) \leq \left( 2 \epsilon + C \epsilon I_1^2 \right) I_1 + C \epsilon I_1,$$

where $\epsilon$ is an arbitrary positive number. Taking $\epsilon$ sufficiently small, and $T_0$ larger if necessary, we have $\epsilon + C \epsilon I_1^2 < 1$ for $t \geq T_0$. With help of Theorem 2.1 and Theorem 4.1 we find

$$\frac{d}{dt} I_0 + C \frac{I_0}{L^2} \leq \frac{C_5}{L^2} I_{-1} \leq C_6 e^{-\lambda t}$$

for large $t$. Thus the assertion for $\ell = 0$ with some positive $\lambda_0$ has been proved.

Next we show the exponential decay of $I_\ell$ for $\ell \in \mathbb{N}$. Set

$$J_{k,p} = \left\{ L^{(1+k)p-1} \int_0^L |\partial_s^k \tilde{\kappa}|^p ds \right\}^{\frac{1}{p}}. $$

By Gagliardo-Nirenberg’s inequality we have

$$J_{k,p} \leq C J_{m,2}^{\theta} J_{0,2}^{1-\theta} = C I_0^{\theta} I_0^{1-\theta}$$

for $k \in \{0, 1, \cdots, m\}$, $p \geq 2$. Here $C$ is independent of $L$, and $\theta = \frac{1}{m} \left( k - \frac{1}{p} + \frac{1}{2} \right) \in [0, 1]$.

Now observe that

$$\frac{d}{dt} I_\ell = (2\ell + 1) L^{2\ell} \frac{dL}{dt} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 ds + L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 \partial_s (ds) + 2L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) (\partial_s \partial_s^\ell \tilde{\kappa}) ds.$$
It follows from (4.4) and Theorem 4.1 that

\[(2\ell+1)L^{2\ell} \frac{dL}{dt} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 ds = (2\ell+1)\int_0^L \kappa^2 ds + \frac{\pi L}{A} \int_0^L (\partial_s^\ell \kappa)^2 ds \leq \frac{C}{L^2} I_\ell.\]

Since \(\partial_t(ds) = -\partial_t f \cdot \kappa ds\), we have

\[L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 \partial_t(ds) = L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 \left\{-\kappa \left(\kappa - \frac{L}{2A}\right)\right\} ds \]

\[\leq \frac{L^{2\ell+2}}{2A} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 \left(\tilde{\kappa} + \frac{2\pi}{L}\right) ds \]

\[\leq \frac{L^{2\ell+2}}{2A} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 \tilde{\kappa} ds + \frac{C}{L^2} I_\ell. \quad (4.10)\]

For \(k \in \mathbb{N} \cup \{0\}\) and \(m \in \mathbb{N}\), let \(P_m^k(\tilde{\kappa})\) be any linear combination of the type

\[P_m^k(\tilde{\kappa}) = \sum_{i_1 + \cdots + i_m = k} c_{i_1, \ldots, i_m} \partial_s^{i_1} \tilde{\kappa} \cdots \partial_s^{i_m} \tilde{\kappa}\]

with universal, constant coefficients \(c_{i_1, \ldots, i_m}\). Similarly we define \(P_0^k\) as a universal constant. We can show

\[\partial_s \partial_s^k \tilde{\kappa} = \partial_s^{k+2} \tilde{\kappa} + \sum_{m=0}^3 L^{-(3-m)} P_m^k(\tilde{\kappa}) + \frac{L}{A} \sum_{m=0}^2 L^{-(2-m)} P_m^k(\tilde{\kappa})\]

by induction on \(k\). Therefore we have

\[2L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) (\partial_s \partial_s^\ell \tilde{\kappa}) ds \]

\[= -\frac{2}{L^2} I_{\ell+1} + L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) \left(\sum_{m=0}^3 L^{-(3-m)} P_m^\ell(\tilde{\kappa}) + \frac{L}{A} \sum_{m=0}^2 L^{-(2-m)} P_m^\ell(\tilde{\kappa})\right) ds.\]

Since \(P_0^\ell(\tilde{\kappa})\) is a constant,

\[\int_0^L (\partial_s^\ell \tilde{\kappa}) P_0^\ell(\tilde{\kappa}) ds = 0.\]

Since \(P_1^\ell(\tilde{\kappa}) = c \partial_s^\ell \tilde{\kappa}\), we have

\[L^{2\ell+1} \left| \int_0^L (\partial_s^\ell \tilde{\kappa}) L^{-2} P_1^\ell(\tilde{\kappa}) ds \right| = \left| c \right| \frac{1}{L^2} I_\ell,\]

\[L^{2\ell+1} \left| \int_0^L (\partial_s^\ell \tilde{\kappa}) \frac{L}{A} L^{-1} P_1^\ell(\tilde{\kappa}) ds \right| \leq \frac{C}{L^2} I_\ell.\]
Hence we obtain
\[
\frac{d}{dt} I_\ell + \frac{2}{L^2} I_{\ell+1} \leq \frac{C}{L^2} I_\ell + L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) \left( \sum_{m=2}^3 L^{-(3-m)} P_m^{\ell}(\tilde{\kappa}) + \frac{L^\ell}{A^2} P_2^\ell(\tilde{\kappa}) \right) ds. \tag{4.11}
\]
Here the first term on the last line of (4.10) is included into \( L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) \frac{L^\ell}{A} P_2^\ell(\tilde{\kappa}) \) ds.

Now we estimate each term on the right-hand side of (4.11). Firstly we have from Theorem 3.1
\[
\frac{C}{L^2} I_\ell \leq \frac{C}{L^2} \left( I_{\ell-1}^2 I_{\ell+1} + I_{\ell-1} I_{\ell+1}^2 \right) \leq \frac{C}{L^2} \left\{ \left( I_{\ell-1}^2 + \epsilon \right) I_{\ell+1} + C_\epsilon I_{\ell-1} \right\}
\]
for any \( \epsilon > 0 \). Taking \( \epsilon \) small and \( t \) large, by Theorem 4.1, the term \( \frac{C}{L^2} \left\{ \left( I_{\ell-1}^2 + \epsilon \right) I_{\ell+1} \right\} \) can be absorbed into the left-hand side of (4.11). The second term is a function decaying exponentially. We will estimate the integral of \( L^{2\ell+2-m} (\partial_s^\ell \tilde{\kappa}) P_m^\ell(\tilde{\kappa}) \) for \( m = 2 \) and 3.

We first consider the case \( m = 2 \). \( P_2^\ell(\tilde{\kappa}) \) is a linear combination of \((\partial_s^k \tilde{\kappa})(\partial_s^{\ell-k} \tilde{\kappa})\) with \( k = 0, \ldots, \ell \). By Hölder’s inequality, we have
\[
\left| L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) L^{-1} P_2^\ell(\tilde{\kappa}) \right| \leq \sum_{k=0}^\ell \frac{C}{L^2} J_{\ell,3} J_{k,3} J_{\ell-k,3},
\]
and (4.9) yields
\[
J_{j,3} \leq C I_{\ell+1}^{\theta(j,3)} I_0^{1-\theta(j,3)}, \quad \theta(j,3) = \frac{j + \frac{1}{2}}{\ell + 1}.
\]
Hence applying Young’s inequality, we obtain
\[
\left| L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) L^{-1} P_2^\ell(\tilde{\kappa}) \right| \leq \frac{C}{L^2} I_{\ell+1}^{2+\frac{1}{2}} I_0^{\frac{1}{2}+\frac{3}{2}} \leq \frac{\epsilon}{L^2} I_{\ell+1} + \frac{C_\epsilon}{L^2} I_0^{2+\frac{3}{2}}
\]
for any \( \epsilon > 0 \). \( \frac{\epsilon}{L^2} I_{\ell+1} \) can be absorbed into the left-hand side of (4.11), and the second one is a function decaying exponentially.

We can estimate for the case \( m = 3 \) similarly. Indeed, Since \( P_3^\ell(\tilde{\kappa}) \) is a linear combination of \( \partial_s^\ell \tilde{\kappa}, \partial_s^{\ell-k} \tilde{\kappa}, \) and \( \partial_s^{\ell-j-k} \tilde{\kappa} \), we have
\[
\left| L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) P_3^\ell(\tilde{\kappa}) \right| \leq \sum_{m=0}^\ell \sum_{j+k=m} J_{\ell,4} J_{j,4} J_{k,4} J_{\ell-j-k,4}
\]
\[
\leq \frac{C}{L^2} I_{\ell+1}^{2+\frac{1}{4}} I_0^{\frac{1}{4}+\frac{3}{4}} \leq \frac{\epsilon}{L^2} I_{\ell+1} + \frac{C_\epsilon}{L^2} I_0^{2+\frac{3}{2}}
\]
for any $\epsilon > 0$.

Since $\frac{L^2}{A}$ is uniformly bounded by Theorem 4.1, we have

$$|L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) \frac{L}{A} P^\ell_2(\tilde{\kappa}) \, ds| \leq C \left| L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa}) L^{-1} P^\ell_2(\tilde{\kappa}) \, ds \right| \leq \frac{\epsilon}{L^2} I_{\ell+1} + \frac{C}{L^2} I_0^{2\ell+3}.$$ 

Therefore (4.11) and Wirtinger’s inequality imply

$$\frac{d}{dt} I_{\ell+1} + \frac{1}{CL^2} I_{\ell} \leq Ce^{-\mu t},$$

which shows the exponential decay of $I_{\ell}$.

**Theorem 4.3** Let $f$ be as in Theorem 4.1, and let $f(s, t) = \sum_{k \in \mathbb{Z}} \hat{f}(k)(t) \varphi_k(s)$ be the Fourier expansion for any fixed $t > 0$. Set

$$c(t) = \frac{1}{\sqrt{L(t)}} (\Re \hat{f}(0)(t), \Im \hat{f}(0)(t)), $$

and define $r(t) \geq 0$ and $\sigma(t) \in \mathbb{R}/2\pi \mathbb{Z}$ by

$$\hat{f}(1)(t) = \sqrt{L(t)} r(t) \exp \left( \frac{2\pi i \sigma(t)}{L(t)} \right).$$

Furthermore we set

$$\hat{f}(\theta, t) = f(L(t) \theta - \sigma(t), t), \quad \text{for} \quad (\theta, t) \in \mathbb{R}/\mathbb{Z} \times [0, \infty).$$

Then the following claims hold.

1. There exists $c_{\infty} \in \mathbb{R}^2$ such that

$$\|c(t) - c_{\infty}\| \leq C e^{-\gamma t}.$$

2. The function $r(t)$ converges to the constant $\frac{L_\infty}{2\pi}$ exponentially as $t \to \infty$:

$$\left| r(t) - \frac{L_\infty}{2\pi} \right| \leq C e^{-\gamma t}.$$

3. There exists $\sigma_{\infty} \in \mathbb{R}/2\pi \mathbb{Z}$ such that

$$|\sigma(t) - \sigma_{\infty}| \leq C e^{-\gamma t}.$$
(4) For any $k \in \{0\} \cup \mathbb{N}$ there exist $C_k > 0$ and $\gamma_k > 0$ such that
\[
\|\tilde{f}(\cdot, t) - \tilde{f}_\infty\|_{C^k(\mathbb{R}/2)} \leq C_k e^{-\gamma_k t},
\]
where
\[
\tilde{f}_\infty(\theta) = c_\infty + \frac{L_\infty}{2\pi} (\cos 2\pi \theta, \sin 2\pi \theta).
\]

(5) For sufficiently large $t$, $\text{Im}\tilde{f}(\cdot, t)$ is the boundary of a bounded domain $\Omega(t)$. Furthermore, there exists $T_* \geq 0$ such that $\Omega(t)$ is strictly convex for $t \geq T_*$. 

(6) Let $D_{r_\infty}(c_\infty)$ be the closed disk with center $c_\infty$ and radius $r_\infty$. Then we have
\[
d_H(\Omega(t), D_{r_\infty}(c_\infty)) \leq Ce^{-\gamma t},
\]
where $d_H$ is the Hausdorff distance.

(7) Let $b(t) = \frac{1}{A(t)} \int \int_{\Omega(t)} x \, dx$ be the barycenter of $\Omega(t)$. Then we have
\[
\|A(t)(b(t) - c(t))\| \leq Ce^{-\gamma t}.
\]

Proof. (1) First observe that
\[
c = \frac{1}{L} \int_0^L f \, ds.
\]
Since
\[
\partial_t f = \partial_s \left( \partial_s f - \frac{L}{2A} Rf \right),
\]
we have
\[
\int_0^L \partial_t f \, ds = 0.
\]
Therefore the time-derivative of $c$ is
\[
\frac{d}{dt} c = \frac{1}{L} \int_0^L f \partial_t (ds) - \frac{1}{L^2} \frac{dL}{dt} \int_0^L f \, ds
\]
\[
= -\frac{1}{L} \int_0^L \left\{ \partial_t (f \cdot \kappa) - \frac{1}{L} \int_0^L (\partial_t f \cdot \kappa) \, ds \right\} f \, ds
\]
\[
= -\frac{1}{L} \int_0^L \left\{ \partial_t (f \cdot \kappa) - \frac{1}{L} \int_0^L (\partial_t f \cdot \kappa) \, ds \right\} \left( f - \frac{1}{L} \int_0^L f \, ds \right) \, ds.
\]
Since
\[ \partial_t f \cdot \kappa = \kappa^2 - \frac{L}{2A} \kappa = \tilde{\kappa}^2 - \frac{L}{2A} (2I_{-1} - 1) \tilde{\kappa} - \frac{\pi}{A} I_{-1} \]
decays exponentially as \( t \to \infty \), and because
\[ \left\| f - \frac{1}{L} \int_0^L f \, ds \right\|_{\mathbb{R}^2} \leq L \leq C, \]
we find that \( c \) converges exponentially to a vector, say \( c_\infty \), as \( t \to \infty \). Consequently \( \text{Im} f \) converges to a circle with center at \( c_\infty \).

(2) It follows from Proposition 2.1 and Lemma 2.1 that
\[ \sum_{k \in \mathbb{Z}} k^\ell (k - 1)|\hat{f}(k)|^2 = -i^{-\ell} \left( \frac{L}{2\pi} \right)^{\ell+1} \int_0^L \kappa F_{\ell} ds + i^{\ell-1} \left( \frac{L}{2\pi} \right)^{\ell} \int_0^L \kappa F_{\ell-1} ds \]
\[ = -i^{-\ell} \left( \frac{L}{2\pi} \right)^{\ell+1} \int_0^L \kappa F_{\ell} ds + i^{-\ell} \left( \frac{L}{2\pi} \right)^{\ell} \int_0^L F_{\ell} ds \]
\[ = -i^{-\ell} \left( \frac{L}{2\pi} \right)^{\ell+1} \int_0^L \kappa F_{\ell} ds \]
for \( \ell \geq 2 \). Since \( F_2 \) is a constant, and since \( F_\ell \) with \( \ell \geq 3 \) is a polynomial function of \( \kappa \) and its derivatives up to the \( (\ell - 3) \)rd order, they are bounded functions of \( (s,t) \). Also \( L \) is bounded and \( \tilde{\kappa} \) decays exponentially as \( t \to \infty \). Therefore when \( \ell \) is odd,
\[ \left| \sum_{k \neq 0,1} k^{\ell+1} |\hat{f}(k)|^2 \right| \leq C \left| \sum_{k \in \mathbb{Z}} k^\ell (k - 1)|\hat{f}(k)|^2 \right| \leq C e^{-\gamma t}. \]

By the Parseval identity and the Sobolev embedding theorem we have
\[ \left\| f(\cdot,t) - \hat{f}(0)(\cdot)\varphi_0(\cdot) - \hat{f}(1)(\cdot)\varphi_1(\cdot) \right\|_{C^k(\mathbb{R}/L(t)\mathbb{Z})} \leq C_k e^{-\gamma' t} \]
for any \( k \). Using the expression of \( \mathbb{R}^2 \)-valued functions, we have
\[ f(s,t) = c(t) + r(t) \left( \cos \frac{2\pi(s + \sigma(t))}{L(t)}, \sin \frac{2\pi(s + \sigma(t))}{L(t)} \right) + \rho(s,t), \]
\[ \| \rho(\cdot,t) \|_{C^k(\mathbb{R}/L(t)\mathbb{Z})} \leq C_k e^{-\gamma' t}. \]

Since
\[ \partial_s f(s,t) = \frac{2\pi r(t)}{L(t)} \left( -\sin \frac{2\pi(s + \sigma(t))}{L(t)}, \cos \frac{2\pi(s + \sigma(t))}{L(t)} \right) + \partial_s \rho(s,t), \]
we have
\[ \left| r(t) - \frac{L(t)}{2\pi} \right| = \frac{L(t)}{2\pi} \left| \frac{2\pi r(t)}{L(t)} - 1 \right| = \frac{L(t)}{2\pi} \left| \| \partial_s f(s, t) - \partial_s \rho(s, t) \| - 1 \right| \leq C e^{-\gamma t}. \]

Therefore, \( r(t) \) converges to \( r_\infty = \frac{L_\infty}{2\pi} \) exponentially as \( t \to \infty \).

(3) First we clarify the meaning of \( \partial_t f \), it is not \( \lim_{h \to 0} \frac{f(s, t + h) - f(s, t)}{h} \) as one might expect. The variable \( s \) in \( f(s, t) \) is an element of \( \mathbb{R}/L(t)\mathbb{Z} \), on the other hand, the \( s \) in \( f(s, t + h) \) is in \( \mathbb{R}/L(t+h)\mathbb{Z} \). Hence the above quotient is not well-defined. To address this, let us introduce a function \( \tilde{f} \) on \( \mathbb{R}/2\pi\mathbb{Z} \times [0, \infty) \) given by

\[ \tilde{f}(u, t) = f \left( \frac{L(t)u}{2\pi}, t \right). \]

Then the variable \( u \) is independent of \( t \), and \( \partial_t f \) is given by

\[ \partial_t f = \lim_{h \to 0} \frac{\tilde{f}(u, t + h) - \tilde{f}(u, t)}{h}. \]

We define a complex-valued function \( \tilde{f} \) by

\[ \tilde{f}(u, t) = (\Re \tilde{f}(u, t), \Im \tilde{f}(u, t)), \]

and note that the Fourier expansion of \( \tilde{f} \) is

\[ \hat{\tilde{f}}(u, t) = \sum_{k \in \mathbb{Z}} e^{iku} \tilde{f}(u, t), \]

where

\[ \hat{\tilde{f}}(u, t) = \frac{1}{\sqrt{2\pi}} e^{iku}. \]

Therefore we have

\[ \partial_t \tilde{f} = \sqrt{2\pi} \sum_{k \in \mathbb{Z}} dt \frac{d \hat{f}(k)(t)}{\sqrt{L(t)}} \phi_k(u), \]

and

\[ \int_0^{2\pi} |\partial_t \tilde{f}|^2 du = 2\pi \sum_{k \in \mathbb{Z}} \left| \frac{d \hat{f}(k)(t)}{\sqrt{L(t)}} \right|^2. \]
by the Parseval identity. Since

\[ |\partial_t \hat{f}|^2 = \|\partial_t \hat{f}\|^2 = \|\partial_t \bar{f}\|^2 = \|\kappa - \frac{L}{2A} \nu\|^2 = |\tilde{\kappa} - \frac{L}{2A} \nu_1|^2 \]

decays exponentially and uniformly in spatial variable as \( t \to \infty \), we have

\[ \sum_{k \in \mathbb{Z}} \left| \frac{d \hat{f}(k)(t)}{dt} \right|^2 \leq C e^{-2\gamma t} \]

for some \( C > 0 \) and \( \gamma > 0 \). In particular,

\[ \left| \frac{d \hat{f}(1)(t)}{dt} \right|^2 \leq C e^{-2\gamma t}. \]

Also, it is not difficult to see that

\[ \left| \frac{d \hat{f}(1)(t)}{dt} \right|^2 = \left| \frac{d}{dt} r(t) \right|^2 + 4\pi^2 r(t)^2 \left| \frac{d}{dt} \left( \sigma(t) / L(t) \right) \right|^2. \]

Since \( r(t) \) and \( L(t) \) converge exponentially to positive constants, so does \( \sigma(t) \) to some \( \sigma_\infty \in \mathbb{R}/2\pi \mathbb{Z} \).

(4) We have

\[ \hat{f}(\theta, t) = c(t) + r(t) (\cos 2\pi \theta, \sin 2\pi \theta) + \tilde{\rho}(\theta, t) \]
\[ = \hat{f}_\infty(\theta) + c(t) - c_\infty + (r(t) - r_\infty) (\cos 2\pi \theta, \sin 2\pi \theta) + \tilde{\rho}(\theta, t), \]

where

\[ \tilde{\rho}(\theta, t) = \rho(L(t)\theta - \sigma(t), t). \]

Therefore the estimates for \( c(t) \), \( r(t) \), and \( \rho(\cdot, t) \) yield

\[ \| \hat{f}(\cdot, t) - \hat{f}_\infty \|_{C^1(\mathbb{R}/2\pi \mathbb{Z})} \leq C_k e^{-\tilde{\gamma} t}. \]

(5) The above estimate implies that \( \text{Im} \hat{f}(\cdot, t) \) is the boundary of a bounded domain \( \Omega(t) \) when \( t \) is sufficiently large. Since \( \tilde{\kappa} \) converges to 0 uniformly, and since \( L \) goes to a positive constant \( L_\infty \) as \( t \to \infty \) uniformly in \( s \),

\[ \kappa = \frac{2\pi}{L} + \tilde{\kappa} \]

is strictly positive for large \( t \). Consequently \( \partial \Omega(t) \) is a strictly convex curve.
(6) Let $D_r(c)$ be the closed disk with center $c$ and the radius $r$. When $t$ is sufficiently large,
\[
\begin{align*}
d_H(\Omega(t), D_{r_\infty}(c_\infty)) & \leq d_H(\Omega(t), D_{r(t)}(c(t))) + d_H(D_{r(t)}(c(t)), D_{r_\infty}(c_\infty)) + d_H(D_{r(t)}(c_\infty), B_{r_\infty}(c_\infty)) \\
& \leq C \left( \|\hat{\rho}(\cdot, t)\|_{C^0(\mathbb{R}/\mathbb{Z})} + \|c(t) - c_\infty\| + |r(t) - r_\infty| \right) \\
& \leq Ce^{-\gamma t}
\end{align*}
\]
for some $C > 0$ and $\gamma > 0$.

(7) Clearly we have
\[
A(b - c) = \iint_{\Omega(t)} (x - c) \, dx.
\]

We define
\[
b = (b_1, b_2), \quad c = (c_1, c_2), \quad b = b_1 + ib_2, \quad c = c_1 + ic_2.
\]

From the divergence theorem, we have
\[
A(b - c) = \iint_{\Omega(t)} \{ (x_1 - c_1) + i(x_2 - c_2) \} \, dx
\]
\[
= \frac{1}{2} \iint_{\Omega(t)} \text{div}((x_1 - c_1)^2, i(x_2 - c_2)^2) \, dx
\]
\[
= -\frac{1}{2} \int_0^L \left( (f_1 - c_1)^2, i(f_2 - c_2)^2 \right) \cdot \nu \, ds
\]
\[
= -\frac{1}{2} \int_0^L \left\{ (f_1 - c_1)^2(-\partial_s f_2) + i(f_2 - c_2)^2\partial_s f_1 \right\} ds.
\]

Since, for $j = 1, 2$,
\[
\int_0^L (f_j - c_j)^2\partial_s f_j \, ds = 0,
\]
we obtain
\[
A(b - c) = -\frac{i}{2} \int_0^L \left\{ (f_1 - c_1)^2(\partial_s f_1 + i\partial_s f_2) + (f_2 - c_2)^2(\partial_s f_1 + i\partial_s f_2) \right\} ds
\]
\[
= -\frac{i}{2} \int_0^L |f - c|^2\partial_s f \, ds.
\]

It holds that
\[
|f - c|^2 = \left| re^{\frac{2\pi i(c_1+c_2)}{L}} + \rho \right|^2 = r^2 + 2r\Re\rho e^{\frac{2\pi i(c_1+c_2)}{L}} + |\rho|^2
\]

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and \( r \) is independent of \( s \). Hence we show

\[
A(b - c) = -i \frac{1}{2} \int_0^L \left\{ 2r \Re \rho e^{2\pi i (s + \sigma)} + |\rho|^2 \right\} \partial_s f \, ds.
\]

Therefore we have

\[
\|A(b - c)\| \leq |A(b - c)| \leq \frac{1}{2} \int_0^L \left( 2r|\rho| + |\rho|^2 \right) \, ds \leq Ce^{-t}.
\]

Thus we have shown each of the claims in the theorem. \( \square \)

### 4.2 The area-preserving curvature flow

It is well-known that if the initial curve is convex, then any solution of the area-preserving curvature flow

\[
\partial_t f = \tilde{\kappa} \nu
\]

(4.12)

converges to a round circle as \( t \to \infty \) as proved by Gage [3]. In this subsection, we give a proof of this fact without the convexity assumption assuming the global existence.

If the initial curve is convex, then the convexity remains for all \( t > 0 \) by the maximal principle. In this case the exponential decay of \( I_{-1}(t) \) is easily derived from Gage’s inequality. First we show the decay without the convexity assumption.

**Theorem 4.4** Assume that \( f \) is a global solution of (4.12) such that the initial rotation number is 1. Then \( \text{Im} f \) converges to a circle exponentially as \( t \to \infty \) in the sense that

\[
0 \leq L(t)^2 - 4\pi A(t) \leq (L(0)^2 - 4\pi A(0)) \exp \left( -\frac{16\pi^2}{L(0)^2} t \right),
\]

(4.13)

\[
\left| L(t) - 2\sqrt{\pi A(0)} \right| \leq \frac{L(0)^2 - 4\pi A(0)}{4\sqrt{\pi A(0)}} \exp \left( -\frac{16\pi^2}{L(0)^2} t \right).
\]

(4.14)

**Proof.** Since

\[
\frac{dL}{dt} = -\int_0^L \partial_s f \cdot \kappa \, ds = -\int_0^L \tilde{\kappa}^2 \, ds,
\]

we have

\[
L \leq L(0).
\]
From this, the area-preserving property, and Theorem 2.2 we have

\[
\frac{d}{dt}(L^2 - 4\pi A) = 2L \frac{dL}{dt} = -2L \int_0^L \tilde{\kappa}^2 ds = -2I_0
\]

\[
\leq -\frac{16\pi^2}{L^2} (L^2 - 4\pi A) \leq -\frac{16\pi^2}{L(0)^2} (L^2 - 4\pi A).
\]

Therefore

\[
L^2 - 4\pi A \leq (L(0)^2 - 4\pi A(0)) \exp \left( -\frac{16\pi^2}{L(0)^2} t \right).
\]

Hence we have \( \lim_{t \to \infty} L = 2\sqrt{\pi A(0)} \), and

\[
\left| L - 2\sqrt{\pi A(0)} \right| \leq \frac{L^2 - 4\pi A}{L + 2\sqrt{\pi A(0)}} \leq \frac{L(0)^2 - 4\pi A(0)}{4\sqrt{\pi A(0)}} \exp \left( -\frac{16\pi^2}{L(0)^2} t \right).
\]

\( \square \)

For this flow the limit value of \( L \) and the decay rate are given explicitly from the initial data.

We can prove the following theorem similarly to the proofs of Theorems 4.2–4.3 using Theorem 4.4 instead of Theorem 4.1.

**Theorem 4.5** The claims (1)–(7) in Theorem 4.3 hold also for global solutions of the area-preserving flow.

**Acknowledgment.** The first author was partly supported by Grant-in-Aid for Scientific Research (C) (17K05310), Japan Society for the Promotion Science. The authors express their appreciation to Professor Shigetoshi Yazaki and Professor Tetsuya Ishiwata for sharing information of related articles, and for discussions. The authors also would like to express their gratitude to Professor Neal Bez for English language editing.

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