STABILITY OF NORMALIZED SOLITARY WAVES FOR THREE COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this paper we establish existence and stability results concerning fully nontrivial solitary-wave solutions to 3-coupled nonlinear Schrödinger system

\[ i\partial_t u_j + \partial_{xx} u_j + \left( \sum_{k=1}^{3} a_{kj} |u_k|^p \right) |u_j|^{p-2} u_j = 0, \quad j = 1, 2, 3, \]

where \( u_j \) are complex-valued functions of \( (x, t) \in \mathbb{R}^2 \) and \( a_{kj} \) are positive constants satisfying \( a_{jk} = a_{kj} \) (symmetric attractive case). Our approach improves many of the previously known results. In all variational methods used previously to study the stability of solitary waves, which we are aware of, the constraint functionals were not independently chosen. Here we study a problem of minimizing the energy functional subject to three independent \( L^2 \) mass constraints and establish existence and stability results for a true three-parameter family of solitary waves.

1. Introduction. In recent years much attention has been given to the study of coupled nonlinear Schrödinger (CNLS) equations because of their applications in a variety of physical and biological settings. The CNLS equation models physical systems in which the field has more than one component. For example, the CNLS equations play an important role in wavelength-division multiplexing [11, 17] and multichannel bit-parallel- wavelength optical fiber networks [28], where the pulses propagate at least in two channels simultaneously. In addition, the CNLS equations arise in plasma physics [25], multispecies and spinor Bose-Einstein condensates [12, 13, 14], biophysics [24], nonlinear Rossby waves [26], to name a few.

In this paper, we consider the time-dependent 3-coupled nonlinear Schrödinger equations given by

\[
\begin{cases}
    iu_{j,t} + u_{j,xx} + \left( \sum_{k=1}^{3} a_{kj} |u_k|^p \right) |u_j|^{p-2} u_j = 0, & j = 1, 2, 3, \\
    u_j = u_j(x, t) \in \mathbb{C}, & (x, t) \in \mathbb{R}^2,
\end{cases}
\]

(1.1)

where \( u_j \) are dimensionless complex amplitude of the \( j \)-th component of the underlying physical system and \( a_{jk} \) are constants satisfying \( a_{jk} = a_{kj} \) for all \( j, k \in \{1, 2, 3\} \). The interaction matrix \( (a_{jk})_{j,k=1}^{3} \) contains information about the nature of the
interactions between the different components of the wave functions. The Landau constants $a_{jj}$ describe the self-modulation of the wave packets, and the coupling constants $a_{kj}$ ($k \neq j$) are the wave-wave interaction coefficients, which describe the cross-modulation of the wave packets. The interaction is (purely) attractive if all couplings are positive and the interaction is (purely) repulsive interaction when these couplings are negative. Here we study the symmetric attractive interactions. Throughout this paper, we shall denote by $u_{j,t}$ the partial derivative of $u_j$ with respect to $t$ and by $u_{j,xx}$ the second partial derivative with respect to $x$.

By a solitary-wave solution of (1.1) we mean a function $(\Phi_1, \Phi_2, \Phi_3)$ such that $\Phi_j$ are in $H^1$ and $(u_1, u_2, u_3)$ defined by

$$u_j(x, t) = e^{i(\omega_j - \sigma^2 t + i\sigma x + i\beta_j)} \Phi_j(x - 2\sigma t), \quad j = 1, 2, 3, \quad (1.2)$$

is a solution of (1.1) for some real numbers $\omega_j, \sigma$, and $\beta_j$. (Here $H^1 = H^1(\mathbb{R})$ denotes the usual Sobolev space consisting of complex-valued measurable functions on the line such that both $f$ and its first spatial derivative are in $L^2$. For more details on our notation, see below.) When $\sigma = 0$, solutions of the form (1.2) are usually referred to as standing-wave solutions. Inserting the ansatz (1.2) into (1.1), we see that $(\Phi_1, \Phi_2, \Phi_3)$ solves the following system of ordinary differential equations

$$- \Phi_j'' + \omega_j \Phi_j = \left( \sum_{k=1}^{3} a_{jk}|\Phi_k|^p \right) |\Phi_j|^{p-2} \Phi_j, \quad j = 1, 2, 3. \quad (1.3)$$

System (1.3) has many semi-trivial (or collapsing) solutions, i.e., solution $(u_1, u_2, u_3)$ with at least one, but not all, component being zero. In these cases the system collapses into system with fewer components. A natural question relevant for 3-coupled nonlinear systems such as (1.3) is the existence and stability results of nontrivial solutions (we will call a solution nontrivial if all three components of the solution are non-zero). In the literature these solutions are also referred to as co-existing solutions. This paper aims to address the issues of the existence of nontrivial solutions to (1.3) and the stability of corresponding solitary waves for the full equations (1.1).

In what follows we denote by $Y$ the product space $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$. The following definition of stability is used throughout the paper.

**Definition 1.1.** Let $\Sigma \subset Y$ be a set of vectors of solitary-wave profiles $\Phi = (\Phi_1, \Phi_2, \Phi_3)$; i.e., each $\Phi \in \Sigma$ corresponds to a solution $u(x, t)$ of (1.1). We say that $\Sigma$ is a stable set of solitary-wave profiles if for any $\epsilon > 0$ there exists $\delta > 0$ such that for every $\Psi$ (in a suitable space $X$ of initial data) satisfying $\inf_{w \in \Sigma} \|\Psi - w\|_Y < \delta$, the solution $u(x, t)$ of (1.1) with $u(x, 0) = \Psi(x)$ satisfies

$$\sup_{t \in \mathbb{R}} \inf_{w \in \Sigma} \|u(x, t) - w\|_Y < \epsilon.$$

Implicit in the above definition of stability is the assumption that the initial-value problem associated to (1.1) is globally well-posed in some space $X$ of ordered triples of functions of $x$. Here we adapt the standard notion of the well-posedness. More precisely, we say that the IVP for (1.1) is globally well-posed (g.w.p.) in $X$ if for a given $\Psi \in X$ there exists a unique $u(x, t)$ such that $u(x, 0) = \Psi(x)$, $u(x, t) \in X$ for all $t \in \mathbb{R}$, and $u(x, t)$ solves (1.1) in some (possibly weak) sense. Moreover, the map $t \mapsto u(\cdot, t)$ is in the space $\mathcal{C}(\mathbb{R}; X)$ of continuous maps from $\mathbb{R}$ to $X$, and the solution map $\Psi \mapsto u(x, t)$ from the initial data to the solution defines a continuous map from $X$ to $\mathcal{C}(\mathbb{R}; X)$ in the appropriate topology. For our purposes, the well-posedness
result in [18] (see also [9]) is most convenient because it is set in the energy space \( Y \) and their method works for the range \( 2 \leq p < 3 \). It has been proved in [18] that for any initial data \( u(x,0) \) lying in the space \( Y \), there exists a unique solution \( u(x,t) \) of (1.1) in \( C(\mathbb{R},Y) \) emanating from \( u(x,0) \), and \( u(x,t) \) satisfies
\[
\mathcal{H}(u(x,t)) = \mathcal{H}(u(x,0)) \quad \text{and} \quad \mathcal{Q}(u_j(x,t)) = \mathcal{Q}(u_j(x,0)),
\]
where \( \mathcal{H} \) and \( \mathcal{Q} \) are the following conserved quantities
\[
\mathcal{H}(u(x,t)) = \int_{-\infty}^{\infty} \left( \sum_{j=1}^{3} |u_{j,x}(x,t)|^2 - \frac{1}{p} \sum_{k,j=1}^{3} a_{kj} |u_k(x,t)|^p |u_j(x,t)|^p \right) \, dx \quad (1.4)
\]
and
\[
\mathcal{Q}(u_j) = \int_{-\infty}^{\infty} |u_j(x,t)|^2 \, dx, \quad 1 \leq j \leq 3. \quad (1.5)
\]

The mathematically exact theory for the nonlinear stability of solitary-wave solutions began with Benjamin’s theory [4] (see also Bona [7]) for the Korteweg-de Vries equation. After their papers on KdV and the regularized long-wave equations, there are numerous literatures that have been devoted to the study of stability of solitary-wave solutions for a variety of nonlinear dispersive equations. In particular, Cazenave and Lions [10] developed an alternate approach to proving existence and stability of solitary waves when they are minimizers of the energy functional and when a compactness condition on minimizing sequences holds. Their approach makes use of the concentration compactness principle of Lions [16] and has the advantage of requiring less detailed analysis than the local methods. The Cazenave and Lions method has since been adapted by different authors to prove existence and stability results of a variety of nonlinear dispersive and wave equations (see, for example, [1, 2, 5, 6, 19, 20, 23] and references therein).

We now summarize the known results on the stability of solitary-wave solutions of the coupled NLS systems. First, we provide some important results concerning two-component NLS solitary waves that are relevant in our work. In the case when \( p = 2 \), \( a_{11} = a_{22} = 1 \), and \( a_{21} = a_{12} = \beta > -1 \), the system (1.3) is known to have explicit semi-trivial solution \( (\Phi_1, \Phi_2, 0) \) of the form
\[
\Phi_1(x) = \Phi_2(x) = \Phi_{\Omega}(x) = \sqrt{\frac{2\Omega}{1 + \beta}} \text{sech}(\sqrt{\Omega} x), \quad \Omega > 0. \quad (1.6)
\]
In [23], Ohta proved a stability result for two-component NLS solitary waves of the form \( e^{i\Omega t} \Phi_{\Omega}(x) \) (1, 1), for some \( \Omega > 0 \). Notice that since (1.1) is invariant under the Galilean transformations
\[
u_j(x,t) \mapsto e^{-i\sigma^2 + idx} u_j(x - 2\sigma t, t), \quad d, \sigma \in \mathbb{R}, \quad j = 1, 2, 3,
\]
and the phase transformations \( u_j(x,t) \mapsto e^{i\beta_j} u_j(x,t) \), \( \beta_j \in \mathbb{R} \), one can write the solitary wave \( e^{i\Omega t} \Phi_{\Omega}(x) \) (1, 1) into the following form
\[
(u_1, u_2) = e^{i(\Omega - \sigma^2)t + idx} \Phi_{\Omega}(x - 2\sigma t) \left( e^{i\beta_1}, e^{i\beta_2} \right).
\]
When \( a_{21} = a_{12} = \beta > 0, \quad \beta < \min\{a_{11}, a_{22}\} \) or \( \beta > \max\{a_{11}, a_{22}\} \) and \( \beta^2 > a_{11} a_{22} \), Nguyen and Wang [19] proved the stability of solutions of the form
\[
e^{i\Omega t} \Phi_{\Omega}(x) \left( \frac{\beta - a_{22}}{\beta^2 - a_{11} a_{22}}, \frac{\beta - a_{22}}{\beta^2 - a_{11} a_{22}} \right),
\]
where $\phi_\Omega(x)$ is defined as in (1.6) with $\beta = 0$. We also note that the same authors (see [21]) have proved the stability of a two-parameter family of solitary waves for two-component version of (1.1) in the special case $p = 2$, using the same method as in [2]. Similar techniques have been used in [6] to prove the stability of (positive) ground-state solutions of a more general two-component coupled NLS equations with power-type nonlinearities.

For 3-coupled systems such as (1.1), there are a variety of interesting results concerning the existence of nontrivial solutions. However, to our knowledge, the only available works regarding the stability of nontrivial solutions for the full systems of type (1.1) are the papers [20, 22]. In [20], Nguyen and Wang considered (1.1) in the special case when $p = 2$, and proved the stability (in the sense defined above) of solutions, given by

$$\sqrt{2\Omega} e^{i\Omega t} \text{sech}(\sqrt{\Omega}x)(\alpha_1, \alpha_2, \alpha_3), \quad \Omega > 0, \quad (1.7)$$

under certain conditions on coefficients $a_{jk}$. More precisely, they made the following assumptions on the matrix $B = (a_{jk})$ of positive coefficients:

1. $B$ is invertible and the linear system $B\vec{a} = \vec{1}$ is solvable for $\vec{a} = (\alpha_1^2, \alpha_2^2, \alpha_3^2) \in \mathbb{R}_+^3$, where $\vec{1} = (1, 1, 1)$;
2. For all pair $j \neq k$, $a_{jk} < \min\{a_{jj}, a_{kk}\}$ and det $B_{jk}$ has the sign of $(-1)^{j+k+1}$ for $j \neq k$;
3. For all pair $j \neq k$, $a_{jk} > \max\{a_{jj}, a_{kk}\}$ and det $B_{jk}$ has the sign of $(-1)^{j+k}$ for $j \neq k$.

Then, using Lions’ concentration compactness principle, they proved that if the matrix $B$ satisfies (1) and (2) or (3), the solutions of (1.1) of the form (1.7) are stable in $Y$. The method used by Nguyen and Wang in [20] uses techniques from [19] with crucial ideas that the constraints on the $L^2$ norms of components are not independently prescribed and that the matrix of coefficients $B$ gives rise to positive numbers $\alpha_j$ such that the Euler-Lagrange equations can be rewritten as uncoupled equations.

The paper [22] is concerned with the stability of certain form of travelling-wave solutions to $m$-component version of (1.1) with $a_{ij} = a, a_{kj} = b$, and $a + 2b > 0$. Their results generalize the ones obtained in [19, 20] to include a more general case of coupled nonlinear Schrödinger equations. To state the precise statement of their stability result, for any $\omega_1 = \omega_2 = \Omega > 0$, set

$$\phi_{\Omega, a+2b}(x) = \left(\frac{\Omega}{a + 2b}\right)^{1/(2p-2)} \phi(\sqrt{\Omega}x),$$

where $\phi(x)$ is the unique positive, spherically symmetric, and decreasing solution of

$$\begin{cases}
-u_{xx} + u - |u|^{2p-2}u = 0, & x \in \mathbb{R},
\quad u \in H^1(\mathbb{R}) \setminus \{0\}.
\end{cases}$$

It has been shown in [22] that when $b > 0$ and $a + 2b > 0$, and for $2 \leq p < 3$, travelling-wave solutions to (1.1) of the form $e^{i\Omega t} \phi_{\Omega, a+2b}(x)(1, 1, 1)$ are stable in the following sense: for any $\epsilon > 0$, there exists $\delta > 0$ such that if $u_{1,0} \in Y$ with

$$\inf_{\gamma_y \in \mathbb{R}} \inf_{y \in \mathbb{R}} \left\{\sum_{j=1}^3 \|u_{j,0} - e^{i\gamma_y} \phi_{\Omega, a+2b}(\cdot + y)\|_1\right\} < \delta,$$
then the solution $u(x,t)$ of (1.1) with initial condition $u(\cdot,0) = u_{1,0}$ satisfies
\[
\inf_{\gamma_j \in \mathbb{R}} \inf_{y \in \mathbb{R}} \left\{ \sum_{j=1}^{3} \| u_j - e^{i\gamma_j} \phi_{\Omega,\alpha+26}(\cdot + y) \|_1 \right\} < \epsilon,
\]
uniformly for all $t \geq 0$.

As seen from the preceding discussion, the stability results obtained from all these papers [19, 20, 22, 23] are for one-parameter family of solitary waves, in which each component is a multiple of a hyperbolic secant function. Their stability results were obtained by purely variational methods in which the constraints were not independently chosen. In this paper we study the variational problem of finding the extremum of the energy functional $H(u_1, u_2, u_3)$ satisfying the constraints
\[
\int_{-\infty}^{\infty} |u_1|^2 \, dx = r, \quad \int_{-\infty}^{\infty} |u_2|^2 \, dx = s, \text{ and } \int_{-\infty}^{\infty} |u_3|^2 \, dx = t.
\]
Such solutions are sometimes referred to as normalized solutions and the associated solitary waves as normalized solitary waves. Our method leads to the existence of independently prescribed $L^2$-norm solutions to two-component systems.

We now describe the main results of this paper. We prove that the full equations (1.1) have a non-empty stable set of positive normalized solitary-wave solutions for all positive constants $a_k = a_{j,k}$ and all $p \in [2,3)$ (we say that a solution of (1.1) is positive if each component is in the form $e^{i\theta} \varphi(x)$, where $\theta$ is a real constant and $\varphi$ is a real-valued positive function). The existence result is obtained via a variational approach and using Cazenave-Lions method [16, 10]. The parameters $\omega_1, \omega_2, \omega_3 \in \mathbb{R}$ in the equations (1.3) appear as Lagrange multipliers. More precisely, let $H$ and $Q$ be as defined in (1.4) and (1.5), respectively; it is easy to see using the Sobolev embedding theorem that $H$ and $Q$ define continuous maps from $Y$ to $\mathbb{R}$. For $r,s,t > 0$, let
\[
\Delta_{r,s,t} = \{ f \in Y : \| f_1 \|^2 = r, \| f_2 \|^2 = s, \text{ and } \| f_3 \|^2 = t \} \tag{1.8}
\]
and define the function $\lambda(r,s,t)$ by
\[
\lambda(r,s,t) = \inf \{ H(f) : f \in \Delta_{r,s,t} \}. \tag{1.9}
\]
A minimizing sequence for $\lambda(r,s,t)$ is any sequence $\{ f_n \}$ in $Y$ such that
\[
\lim_{n \to \infty} \| f_{1,n} \|^2 = r, \quad \lim_{n \to \infty} \| f_{2,n} \|^2 = s, \quad \lim_{n \to \infty} \| f_{3,n} \|^2 = t, \quad \text{and} \quad \lim_{n \to \infty} H(f_n) = \lambda(r,s,t).
\]
To each minimizing sequence $\{ f_n \}$ of $\lambda(r,s,t)$, we associate a sequence of nondecreasing functions $P_n : [0, \infty) \to [0, r + s + t]$ defined by
\[
P_n(\eta) = \sup_{y \in \mathbb{R}} \int_{y-\eta}^{y+\eta} \rho_n(x) \, dx,
\]
where $\rho_n(x)$ is given by
\[
\rho_n(x) := |f_{1,n}(x)|^2 + |f_{2,n}(x)|^2 + |f_{3,n}(x)|^2.
\]
A standard argument shows that any uniformly bounded sequence of nondecreasing functions on $[0, \infty)$ must have a subsequence which converges pointwise to a nondecreasing limit function on $[0, \infty)$. Hence $P_n(\eta)$ has such a subsequence, which we
again denote by \( P_n \). Let \( P(\eta) : [0, \infty) \to [0, r + s + t] \) be the nondecreasing function to which \( P_n \) converges, and define

\[
γ = \lim_{n \to +\infty} P(\eta).
\]  

(1.10)

Then \( γ \) satisfies \( 0 \leq γ \leq r + s + t \). The method of Cazenave and Lions [10], as applied to this situation, consists of two observations. The first is that if \( γ = r + s + t \), then the minimizing sequence \( \{ f_n \} \) has a subsequence which, when its terms are suitably translated, converges strongly in \( Y \) to an element of the set \( \mathcal{O}_{r,s,t} \) defined by

\[
\mathcal{O}_{r,s,t} = \{ \Phi \in Y : H(\Phi) = \lambda(r, s, t), \Phi \in \Delta_{r,s,t} \}.
\]  

(1.11)

The second is that certain properties of the variational problem imply that \( γ \) must equal \( r + s + t \) for every minimizing sequence \( \{ f_n \} \). It follows that not only do minimizers exist in \( Y \), but every minimizing sequence converges in \( Y \) norm to the set \( \mathcal{O}_{r,s,t} \). Typically, one proves \( γ = r + s + t \) by ruling out other two possibilities, namely \( γ = 0 \) and \( 0 < γ < r + s + t \). A lemma (Lemma 2.10 of [2]) concerning the symmetric rearrangement of functions plays an important role in our proof. In Sections 2 and 3, we provide the details of the method.

We prove below (see Theorem 3.6) that the problem (1.9) has a solution in \( \Delta_{r,s,t} \) for the range \( 2 \leq p < 3 \). In particular, the set \( \mathcal{O}_{r,s,t} \) is nonempty. The set \( \mathcal{O}_{r,s,t} \) consists of solitary-wave profiles for (1.1). More precisely, if \( (\Phi_1, \Phi_2, \Phi_3) \in \mathcal{O}_{r,s,t} \), they satisfy the Euler-Lagrange equations

\[
-\Phi_{j,xx} + ω_j \Phi_j = \left( \sum_{k=1}^{3} a_{kj}|\Phi_k|^p \right) |\Phi_j|^{p-2}\Phi_j, \quad j = 1, 2, 3,
\]

where \( ω_j \) are the Lagrange multipliers. These equations are satisfied by \( (\Phi_1, \Phi_2, \Phi_3) \) if and only if the functions \( u_1, u_2, \) and \( u_3 \) defined by

\[
(u_1(x,t), u_2(x,t), u_3(x,t)) = (e^{iω_1t}\Phi_1(x), e^{iω_2t}\Phi_2(x), e^{iω_3t}\Phi_3(x))
\]  

(1.12)

are solutions of the 3-CNLS system (1.1), and since (1.1) is invariant under the Galilean transformations and the phase transformations, one can always write (1.12) into the form (1.2).

The question about the characterization of the set \( \mathcal{O}_{r,s,t} \) is addressed in Section 3 (see Theorem 3.7). Namely, we prove that for each \( \Phi \in \mathcal{O}_{r,s,t} \) there exists positive real-valued functions \( φ_1, φ_2, φ_3 \in H^1 \) such that

\[
Φ_j(x) = λ_jφ_j(x), \quad λ_j \in S^1, \quad j = 1, 2, 3.
\]

Also, the functions \( Φ_j \) are infinitely differentiable on \( \mathbb{R} \).

Finally, Theorem 4.1 proves that the \( \mathcal{O}_{r,s,t} \) forms a stable set for the associated initial-value problem to (1.1) in the sense of Definition 1.1.

**Notation.** We denote by \( L^p = L^p(\mathbb{R}), 1 \leq p \leq \infty \), the space of all measurable functions \( f \) on \( \mathbb{R} \) for which the norm \( |f|_p \) is finite, where

\[
|f|_p = \left( \int_{-\infty}^{\infty} |f|^p \, dx \right)^{1/p}
\]

for \( 1 \leq p < \infty \).

and \( |f|_\infty \) is the essential supremum of \( |f| \) on \( \mathbb{R} \). Whether we intend the functions in \( L^p \) to be real-valued or complex-valued will be clear from the context. We denote
by $H^1(\mathbb{R})$ the Sobolev space of all complex-valued functions defined on $\mathbb{R}$ such that $f$ and its distributional derivative $f'$ are in $L^2$. The norm $\| \cdot \|_1$ on $H^1$ is defined by

$$ \|f\|_1 = \left( \int_{-\infty}^{\infty} (|f(x)|^2 + |f'(x)|^2) \, dx \right)^{1/2}. $$

In particular, we use $\|f\|$ to denote the $L^2$ norm of a function $f$. We define $X$ to be the product space $X = H^1(\mathbb{R}) \times H^1(\mathbb{R})$ furnished with the product norm

$$ \|(f,g)\|_X^2 = \int_{-\infty}^{\infty} (|f(x)|^2 + |g(x)|^2) \, dx + \int_{-\infty}^{\infty} (|f'(x)|^2 + |g'(x)|^2) \, dx $$

and the space $Y$ to be the space $Y = H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$ equipped with the product norm, which we denote by $\| \cdot \|_Y$. That is,

$$ \|(f_1, f_2, f_3)\|_Y^2 = \sum_{j=1}^{3} \int_{-\infty}^{\infty} |f_j(x)|^2 \, dx + \sum_{j=1}^{3} \int_{-\infty}^{\infty} |f'_j(x)|^2 \, dx. $$

If $T > 0$ and $Z$ is any Banach space, we denote by $C([0,T]; Z)$ the Banach space of continuous maps $f : [0,T] \to Z$, with norms given by

$$ \|f\|_{C([0,T]; Z)} = \sup_{t \in [0,T]} \|f(t)\|_Y. $$

For notational convenience, we set $f = (f_1, f_2, f_3)$, $f_n = (f_{1,n}, f_{2,n}, f_{3,n})$, and $\Phi = (\Phi_1, \Phi_2, \Phi_3)$. We denote by $S^1$ the set of all complex numbers of the form $e^{i\theta}$, i.e.,

$$ S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}. $$

The letter $C$ will be used to denote various constants whose exact values are immaterial and which may vary from one line to the next.

2. **The variational problem.** Throughout this section, we assume that $p \in [2,3]$ and $a_{jk}$ are positive real constants satisfying $a_{kj} = a_{jk}$ for all $k,j \in \{1,2,3\}$.

We first establish some properties of the variational problem and its minimizing sequences which are independent of the value $\gamma$. The first Lemma states that the infimum must be finite and negative and that minimizing sequences are bounded uniformly.

**Lemma 2.1.** If $\{f_n\}$ is a minimizing sequence for $\lambda(r,s,t)$, then there exists a constant $B$ such that

$$ \sum_{j=1}^{3} \|f_{j,n}\|_1 \leq B. $$

Moreover, for any $r, s, t > 0$, one has $-\infty < \lambda(r,s,t) < 0$.

**Proof.** Let $\{f_n\}$ be a minimizing sequence for $\lambda(r,s,t)$. Then $\|f_{1,n}\|, \|f_{2,n}\|,$ and $\|f_{3,n}\|$ are bounded. Using the Gagliardo-Nirenberg inequality, we have

$$ \int_{-\infty}^{\infty} |f_{j,n}|^2 \, dx \leq C \left( \int_{-\infty}^{\infty} |f_{j,n,x}|^2 \, dx \right)^{(p-1)/2} \left( \int_{-\infty}^{\infty} |f_{j,n}|^2 \, dx \right)^{(p+1)/2}, \quad (2.1) $$

where $C = C(p,r,s,t).$ For $j, k = 1, 2, 3$, using the Cauchy-Schwarz inequality, we also have

$$ \int_{-\infty}^{\infty} |f_{j,n}|^p |f_{k,n}|^p \, dx \leq \frac{1}{2} \left( \int_{-\infty}^{\infty} |f_{j,n}|^{2p} \, dx + \int_{-\infty}^{\infty} |f_{k,n}|^{2p} \, dx \right), \quad (2.2) $$
Now
\[
\| \mathbf{f}_n \|_Y^2 = \| f_{1,n} \|_1^2 + \| f_{2,n} \|_1^2 + \| f_{3,n} \|_1^2
\]
\[
= \mathcal{H}(\mathbf{f}_n) + \frac{1}{p} \int_{-\infty}^{\infty} \sum_{k,j=1}^{3} a_{kj} |f_{k,n}|^p |f_{j,n}|^p + (r + s + t),
\]
and \( \mathcal{H}(\mathbf{f}_n) \) is bounded since \( \{\mathbf{f}_n\} \) is a minimizing sequence. Using (2.1) and (2.2), it follows that
\[
\| \mathbf{f}_n \|_Y^2 \leq C \left( 1 + \| \mathbf{f}_n \|_{Y^p}^{p-1} \right),
\]
and hence, the existence of the desired bound \( B \) follows.

The claim \( \lambda(r, s, t) > -\infty \) easily follows from the estimates (2.1) and (2.2). To prove \( \lambda(r, s, t) < 0 \), choose any function \( f_1 \in H^1 \) such that \( \| f_1 \|^2 = r \). Let
\[
f_2 = (s/r)^{1/2} f_1 \quad \text{and} \quad f_3 = (t/r)^{1/2} f_1.
\]
Then \( \| f_2 \|^2 = s \) and \( \| f_3 \|^2 = t \). For each \( \theta > 0 \), define \( f_{j,\theta} = \theta^{1/2} f_j(\theta x), \ j = 1, 2, 3 \). Then, for all \( \theta \), we have
\[
\| f_{1,\theta} \|^2 = \| f_1 \|^2 = r, \quad \| f_{2,\theta} \|^2 = \| f_2 \|^2 = s, \quad \text{and} \quad \| f_{3,\theta} \|^2 = \| f_3 \|^2 = t,
\]
and
\[
\mathcal{H}(\mathbf{f}_\theta) = \int_{-\infty}^{\infty} \left( \theta^2 \sum_{j=1}^{3} |f_{j,\theta}|^2 - \theta^{p-1} \sum_{k,j=1}^{3} \frac{1}{p} a_{kj} |f_k|^p |f_j|^p \right) dx.
\]
Substituting \( f_2 \) and \( f_3 \) on the right-hand side, we obtain
\[
\mathcal{H}(\mathbf{f}_\theta) = \int_{-\infty}^{\infty} \left[ \theta^2 \left( 1 + \frac{s}{r} + \frac{t}{r} \right) |f_{1,\theta}|^2 - \frac{1}{p} \theta^{p-1} A_p(a_{kj}, r, s, t) |f_1|^{2p} \right] dx,
\]
where the quantity \( A_p = A_p(a_{kj}, r, s, t) \) is given by
\[
A_p = a_{11} + a_{22} \frac{s^p}{r^p} + a_{33} \frac{t^p}{r^p} + 2a_{12} \frac{s^p}{r^p} \frac{t^p}{r^p} + 2a_{13} \frac{t^p}{r^p} \frac{t^p}{r^p} + 2a_{23} \frac{s^p}{r^p} \frac{t^p}{r^p} > 0.
\]
By taking \( \theta = \theta_0 \) sufficiently small, we get \( \mathcal{H}(\mathbf{f}_{\theta_0}) < 0 \), proving that \( \lambda(r, s, t) < 0 \). □

**Lemma 2.2.** Let \( \{\mathbf{f}_n\} \subset Y \) be a minimizing sequence for \( \lambda(r, s, t) \). Then for all sufficiently large \( n \),

(i) if \( r > 0 \) and \( s, t \geq 0 \), then there exists \( \delta_1 > 0 \) such that \( \| f_{1,nx} \| \geq \delta_1 \).

(ii) if \( s > 0 \) and \( r, t \geq 0 \), then there exists \( \delta_2 > 0 \) such that \( \| f_{2,nx} \| \geq \delta_2 \).

(iii) if \( t > 0 \) and \( r, s \geq 0 \), then there exists \( \delta_3 > 0 \) such that \( \| f_{3,nx} \| \geq \delta_3 \).

**Proof.** To prove statement (i), suppose to the contrary that no such constant \( \delta_1 \) exists. Then, by passing to a subsequence if necessary, we may assume there exists a minimizing sequence such that \( \lim_{n \to \infty} \| f_{1,nx} \| = 0 \). By Gagliardo-Nirenberg inequalities,
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} |f_{1,n}|^p |f_{j,n}|^p dx = 0, \ j = 1, 2, 3.
\]
Therefore, we have that
\[
\lambda(r, s, t) = \lim_{n \to \infty} \mathcal{H}(\mathbf{f}_n) = \lim_{n \to \infty} \left( \sum_{j=2}^{3} \| f_{j,nx} \|^2 - \frac{1}{p} \sum_{k,j=2}^{3} a_{kj} |f_{k,nx}|^p \right).
\]
Pick any \( \psi_1 \geq 0 \) such that \( \| \psi_1 \|^2 = r \). For every \( \theta > 0 \), define \( \psi_{1,\theta}(x) = \theta^{1/2} \psi_1(\theta x) \).

Then \( \| \psi_{1,\theta} \|^2 = r \), and hence, for all \( n \),
\[
\lambda(r, s, t) \leq \mathcal{H}(\psi_{1,\theta}, f_{2,nx}, f_{3,nx}).
\]
On the other hand, if we define
\[ \eta = \theta^2 \int_{-\infty}^{\infty} |\psi_{1,x}|^2 \, dx - \theta^{p-1} \int_{-\infty}^{\infty} \frac{a_{11}}{p} |\psi_1|^{2p} \, dx, \] (2.4)
then \( \eta < 0 \) for sufficiently small \( \theta \). Then, for all \( n \in \mathbb{N} \),
\[ \lambda(r, s, t) \leq H(\psi_1, \theta, f_2, n, f_3, n) \]
\[ \leq \int_{-\infty}^{\infty} \left( \sum_{j=2}^{3} |f_{j,nx}|^2 - \frac{1}{p} \sum_{k,j=2}^{3} a_{kj}|f_{j,n}|^p|f_{k,n}|^p \right) \, dx + \eta. \]
Consequently
\[ \lambda(r, s, t) \leq \lim_{n \to \infty} \int_{-\infty}^{\infty} \left( \sum_{j=2}^{3} |f_{j,nx}|^2 - \frac{1}{p} \sum_{k,j=2}^{3} a_{kj}|f_{j,n}|^p|f_{k,n}|^p \right) \, dx + \eta, \]
which contradicts (2.3) and (2.4), and hence, statement (i) follows. The statements (ii) and (iii) can be proved similarly. \( \square \)

**Lemma 2.3.** Let \( \alpha_1, \alpha_2, \beta > 0 \) and \( p \in [2, 3) \). For \( f, g \in H^1(\mathbb{R}) \), define
\[ F(f, g) = \|f_x\|^2 + \|g_x\|^2 - \frac{1}{p} \left( \alpha_1|f|^{2p} + \alpha_2|g|^{2p} + 2\beta|fg|^p \right). \] (2.5)
Then for any \( \alpha_1, \alpha_2 > 0 \) there exists a nontrivial solution to the problem
\[ m(\alpha_1, \alpha_2) = \inf \{ F(f, g) : \|f\|^2 = \alpha_1 \text{ and } \|g\|^2 = \alpha_2 \}. \] (2.6)
Furthermore, if \( (\tilde{\phi}_{a_1}, \tilde{\phi}_{a_2}) \) is a solution of (2.6), then there exists \( \theta_1, \theta_2 \in \mathbb{R} \) and positive real-valued functions \( \tilde{\phi}_{a_1} \) and \( \tilde{\phi}_{a_2} \) such that \( \tilde{\phi}_{a_1}(x) = e^{i\theta_1} \phi_{a_1}(x) \) and \( \tilde{\phi}_{a_2}(x) = e^{i\theta_2} \phi_{a_2}(x) \). In particular,
\[ F(\tilde{\phi}_{a_1}, \tilde{\phi}_{a_2}) = m(\alpha_1, \alpha_2). \] (2.7)

*Proof.* Let \( \{(f_n, g_n)\} \) be a sequence of functions in \( X \) such that \( \lim_{n \to \infty} \|f_n\|^2 = \alpha_1 \), \( \lim_{n \to \infty} \|g_n\|^2 = \alpha_2 \), and
\[ \lim_{n \to \infty} F(f_n, g_n) = m(\alpha_1, \alpha_2). \]
By the use of concentration compactness argument, it has been proved in one of our earlier papers [6] that some subsequence of \( (f_n, g_n) \) converges, after suitable translations, strongly to some function \( (\phi_{a_1}, \phi_{a_2}) \) in \( X \) norm. Then the pair \( (\tilde{\phi}_{a_1}, \tilde{\phi}_{a_2}) \) must satisfy
\[ F(\tilde{\phi}_{a_1}, \tilde{\phi}_{a_2}) = m(\alpha_1, \alpha_2) \] (2.8)
and must also be a solution of the Euler-Lagrange equation
\[ \begin{cases} -\tilde{\phi}_{a_1}'' + \mu_1 \tilde{\phi}_{a_1} = \alpha_1|\tilde{\phi}_{a_1}|^{2p-2} \tilde{\phi}_{a_1} + \beta|\tilde{\phi}_{a_2}|^p \tilde{\phi}_{a_1}, \\ -\tilde{\phi}_{a_2}'' + \mu_2 \tilde{\phi}_{a_2} = \alpha_2|\tilde{\phi}_{a_2}|^{2p-2} \tilde{\phi}_{a_2} + \beta|\tilde{\phi}_{a_1}|^p \tilde{\phi}_{a_2}, \end{cases} \]
for some real numbers \( \mu_1 \) and \( \mu_2 \). As in the proof of Theorem 3.7 below, one can show that there exists real numbers \( \theta_1, \theta_2 \), and real-valued positive functions \( \phi_{a_1} \) and \( \phi_{a_2} \) such that \( \tilde{\phi}_{a_1}(x) = e^{i\theta_1} \phi_{a_1}(x) \) and \( \tilde{\phi}_{a_2}(x) = e^{i\theta_2} \phi_{a_2}(x) \). Then, for some sequence \( \{y_n\} \) of real numbers,
\[ \{(e^{-i\theta_1} f_{nk}(\cdot + y_k), e^{-i\theta_2} g_{nk}(\cdot + y_k))\} \]
converges strongly to \((\phi_{a_1}, \phi_{a_2})\) in \(X\). Finally, since \(F(\phi_{a_1}, \phi_{a_2}) = F(\hat{\phi}_{a_1}, \hat{\phi}_{a_2})\), \(2.7\) follows from \(2.8\).

**Lemma 2.4.** Suppose \(\{f_n\}\) is a minimizing sequence for \(\lambda(r, s, t)\), where \(r, s, t > 0\). Then, for each \(j = 1, 2, 3\), there exists \(\delta_j > 0\) such that for all sufficiently large \(n\),

\[
\int_{-\infty}^{\infty} \left( |f_{j,nx}|^2 - \frac{1}{p} |f_{j,nx}|^p \sum_{k=1}^{3} a_{jk} |f_k,n|^p \right) \, dx \leq -\delta_j.
\]

**Proof.** We prove the lemma for \(j = 1\). The proofs for cases \(j = 2, 3\) are similar. Suppose the conclusion is false. Then, by passing to a subsequence if necessary, we may assume that there exists a minimizing sequence \(\{f_n\}\) for which

\[
\liminf_{n \to \infty} \int_{-\infty}^{\infty} \left( |f_{1,nx}|^2 - \frac{1}{p} |f_{1,nx}|^p \sum_{k=1}^{3} a_{1k} |f_k,n|^p \right) \, dx \geq 0, \tag{2.9}
\]

and so

\[
\lambda(r, s, t) = \lim_{n \to \infty} \mathcal{H}(f_{1,n}, f_{2,n}, f_{3,n}) \geq \liminf_{n \to \infty} \int_{-\infty}^{\infty} \left( \sum_{j=2}^{3} |f_{j,nx}|^2 - \frac{1}{p} \sum_{k,j=2}^{3} a_{jk} |f_{j,nx}|^p |f_k,n|^p \right) \, dx \tag{2.10}
\]

Let \(F, \phi_s, \text{ and } \phi_t\) as in Lemma 2.3 with \(f = f_{2,n}, g = f_{3,n}, \alpha_1 = \alpha_2, \alpha_2 = a_{33}, \beta = a_{23}, a_1 = s, \text{ and } a_2 = t\). Then \(2.10\) implies that

\[
\lambda(r, s, t) \geq F(\phi_s, \phi_t). \tag{2.11}
\]

On the other hand, take any \(f_1 \in H^4\) such that \(\|f_1\|^2 = r\) and

\[
\int_{-\infty}^{\infty} \left( |f_{1,x}|^2 - \frac{a_{11}}{p} |f_1|^2p - \frac{2a_{12}}{p} |f_1|^p |\phi_s|^p - \frac{2a_{13}}{p} |f_1|^p |\phi_t|^p \right) \, dx < 0. \tag{2.12}
\]

To construct such a function \(f_1\), take an arbitrary smooth, non-negative function \(\psi\) with compact support such that \(\psi(0) = 1\) and \(\|\psi\| = r\), and for \(\theta > 0\), define \(\psi_\theta(x) = \theta^{1/2} \psi(\theta x)\). Then, \(f_1 = \psi_\theta\) satisfies \(2.12\) for sufficiently small \(\theta\). Therefore, we have

\[
\lambda(r, s, t) \leq \mathcal{H}(f_1, \phi_s, \phi_t) \leq \int_{-\infty}^{\infty} \left( |f_{1,x}|^2 - \frac{a_{11}}{p} |f_1|^2p - \frac{2a_{12}}{p} |f_1|^p |\phi_s|^p - \frac{2a_{13}}{p} |f_1|^p |\phi_t|^p \right) \, dx + F(\phi_s, \phi_t) < F(\phi_s, \phi_t),
\]

which contradicts \(2.11\), and hence lemma follows.

Given a non-negative measurable function \(w : \mathbb{R} \to [0, \infty)\), we denote by \(w^*\) the symmetric rearrangement of \(w\) (see page 80 of [15] for the definition and a concise exposition of the basic properties of symmetric rearrangements). Notice that if \(f\) is in \(Y\), \(|f_j|\) are in \(H^1\), and hence \(|f_j|^*\) are well-defined. A basic property about rearrangements is that they preserve \(L^p\) norms (Page 81 of [15]), so that

\[
\int_{-\infty}^{\infty} (|f_j|^*)^{2p} \, dx = \int_{-\infty}^{\infty} |f_j|^{2p} \, dx. \tag{2.13}
\]

The next lemma states that \(\mathcal{H}(f)\) decreases when \(f_j\) are replaced by \(|f_j|\), and when \(|f_j|\) are symmetrically rearranged.
Lemma 2.5. For all \((f_1, f_2, f_3) \in Y\), one has
\[
\mathcal{H}(|f_1|, |f_2|, |f_3|) \leq \mathcal{H}(f_1, f_2, f_3)
\] (2.14)
and
\[
\mathcal{H}(|f_1|^*, |f_2|^*, |f_3|^*) \leq \mathcal{H}(f_1, f_2, f_3).
\] (2.15)

Proof. The proof of (2.14) follows from a standard fact of analysis that if \(g \in H^1\), then \(|g(x)|\) is in \(H^1\) and
\[
\int_{-\infty}^{\infty} ||g||^2_x \, dx \leq \int_{-\infty}^{\infty} |g_x|^2 \, dx.
\] (2.16)
A proof of (2.16) is given in Lemma 3.5 of [3]. To prove (2.15), Theorem 3.4 of [15] implies that for \(k \neq j\),
\[
\int_{-\infty}^{\infty} (|f_j|^*)^p (|f_k|^*)^p \, dx \geq \int_{-\infty}^{\infty} |f_j|^p |f_k|^p \, dx.
\] (2.17)
From Lemma 7.17 of [15], we have that
\[
\int_{-\infty}^{\infty} (|f_j|^*)^2 \, dx \leq \int_{-\infty}^{\infty} |f_j|^2 \, dx.
\]
In light of these facts and (2.13), the claim (2.15) follows from (2.14).

The next lemma is one-dimensional version of Proposition 1.4 of [8]. A proof of this lemma is given in [2].

Lemma 2.6. Suppose \(f\) and \(g\) are non-negative, even, \(C^\infty\) functions with compact support in \(\mathbb{R}\), and non-increasing on \(\{x : x \geq 0\}\). Let \(x_1\) and \(x_2\) be numbers such that \(f(x + x_1)\) and \(g(x + x_2)\) have disjoint supports, and define
\[
w(x) = f(x + x_1) + g(x + x_2).
\]
Then the distributional derivative \((w^*)^r\) of \(w^r\) is in \(L^2\), and satisfies
\[
\|(w^*)^r\|^2 \leq \|w^r\|^2 - \frac{3}{4} \min\{\|f^r\|^2, \|g^r\|^2\}.
\] (2.18)

We now prove that \(\lambda(r, s, t)\) is strictly subadditive:

Lemma 2.7. Let \(r_1, r_2, s_1, s_2, t_1, t_2 \geq 0\) be given, and suppose that \(r_1 + r_2 > 0, s_1 + s_2 > 0, t_1 + t_2 > 0, r_1 + s_1 + t_1 > 0,\) and \(r_2 + s_2 + t_2 > 0\). Then
\[
\lambda(r_1 + r_2, s_1 + s_2, t_1 + t_2) < \lambda(r_1, s_1, t_1) + \lambda(r_2, s_2, t_2).
\] (2.19)

Proof. We follow closely the arguments used in [2]. For \(i = 1, 2\), we first construct minimizing sequences \((f_{1,n}^{(i)}, f_{2,n}^{(i)}, f_{3,n}^{(i)})\) for \(\lambda(r_i, s_i, t_i)\) such that for all \(i \in \{1, 2\}, \ j \in \{1, 2, 3\}, \) and all \(n \in \mathbb{N}\), \(f_{j,n}^{(i)}\) are real-valued, non-negative, even, non-increasing, \(C^\infty\) with compact support on \(\mathbb{R}\), and satisfies
\[
\|f_{1,n}^{(i)}\|^2 = r_i, \|f_{2,n}^{(i)}\|^2 = s_i, \text{ and } \|f_{3,n}^{(i)}\|^2 = t_i.
\]
Without loss of generality, take \(i = 1\), since the case \(i = 2\) is exactly similar. We may also assume that \(r_1 > 0, s_1 > 0,\) and \(t_1 > 0\), otherwise we can just take \(f_{1,n}^{(1)}, f_{2,n}^{(1)}, \) or \(f_{3,n}^{(1)}\) to be identically zero on \(\mathbb{R}\) for all \(n\). Let \((e_{1,n}^{(1)}, p_{1,n}^{(1)}, q_{1,n}^{(1)})\) be any minimizing sequence for \(\lambda(r_1, s_1, t_1)\). By the continuity of \(\mathcal{H}\) and the density of compactly supported functions in \(H^1\), we can approximate \((e_{1,n}^{(1)}, p_{1,n}^{(1)}, q_{1,n}^{(1)})\) by
compactly supported functions \((c_n^{(2)}, p_n^{(2)}, q_n^{(2)})\). Then \((c_n^{(2)}, p_n^{(2)}, q_n^{(2)})\) also form a minimizing sequence for \(\lambda(r_1, s_1, t_1)\). Define
\[
(c_n^{(3)}, p_n^{(3)}, q_n^{(3)}) = (|c_n^{(2)}|^*, |p_n^{(2)}|^*, |q_n^{(2)}|^*)
\]
Then, by Lemma 2.5, the sequence \((c_n^{(3)}, p_n^{(3)}, q_n^{(3)})\) form a minimizing sequence for \(\lambda(r_1, s_1, t_1)\), and for each \(n\), the functions \(c_n^{(3)}, p_n^{(3)}, q_n^{(3)}\) are non-negative, even, non-increasing on \(\{x : x \geq 0\}\), belong to \(H^1\), and have compact support. Next, observe that if \(\psi\) be any non-negative, even, \(C^\infty\), decreasing function for \(x \geq 0\) with compact support, then the convolution of \(\psi\) with any function \(f\), defined as
\[
f * \psi(x) = \int_{-\infty}^{\infty} f(x - y)\psi(y) \, dy,
\]
is also non-negative, even, decreasing on \(\{x : x \geq 0\}\), and have compact support. Moreover, if we define \(\psi_\epsilon = (1/\epsilon)\psi(x/\epsilon)\) for \(\epsilon > 0\) and choose \(\psi\) such that \(\psi_\epsilon(x) \, dx = 1\), the functions \(f * \psi_\epsilon\) converge strongly to \(f\) in \(H^1\) as \(\epsilon \to 0\). Finally, if \(\psi\) is \(C^\infty\) then \(f * \psi_\epsilon\) will be also \(C^\infty\). Thus, by defining
\[
(c_n^{(4)}, p_n^{(4)}, q_n^{(4)}) = (c_n^{(3)} * \psi_n, p_n^{(3)} * \psi_n, q_n^{(3)} * \psi_n),
\]
with \(c_n\) chosen appropriately small for \(n\) large, and setting
\[
\begin{align*}
f_1^{(1)} &= (r_1)^{1/2}c_n^{(4)} / \|c_n^{(4)}\|, & f_2^{(1)} &= (s_n)^{1/2}p_n^{(4)} / \|p_n^{(4)}\|, & f_3^{(1)} &= (t_n)^{1/2}q_n^{(4)} / \|q_n^{(4)}\|,
\end{align*}
\]
we obtain the desired minimizing sequence \((f_1^{(1)}, f_2^{(1)}, f_3^{(1)})\) for \(\lambda(r_1, s_1, t_1)\).

Next, for each \(n\), choose a number \(x_n\) such that for each \(1 \leq j \leq 3\), \(f_j^{(1)}(x)\) and \(f_j^{(2)}(x + x_n)\) have disjoint support, and define
\[
f_j^{(n)} = \left(f_j^{(1)} + \tilde{f}_j^{(2)}\right)^*, \quad \text{where } \tilde{f}_j^{(2)}(x) = f_j^{(2)}(x + x_n), \quad 1 \leq j \leq 3.
\]
Then we have that \((f_1^{(1)}, f_2^{(2)}, f_3^{(1)})\) \(\in \Delta_{r_1+r_2,s_1+s_2,t_1+t_2}\) and hence,
\[
\lambda(r_1 + r_2, s_1 + s_2, t_1 + t_2) \leq H(f_1^{(1)}, f_2^{(2)}, f_3^{(1)}).
\] (2.20)

From Lemma 2.6, we have
\[
\int_{-\infty}^{\infty} \sum_{j=1}^{3} |f_j^{(n)}|^2 \, dx \leq \int_{-\infty}^{\infty} \sum_{j=1}^{3} |(f_j^{(1)} + \tilde{f}_j^{(2)})|^2 \, dx - K_n
\] (2.21)
where \(K_n\) is given by
\[
K_n = \frac{3}{4} \sum_{j=1}^{3} \min \left\{ \|f_j^{(1)}\|^2, \|f_j^{(2)}\|^2 \right\}.
\] (2.22)

Then, using (2.20), (2.21), and rearrangements properties (2.13) and (2.17), we have that for all \(n\),
\[
\lambda(r_1 + r_2, s_1 + s_2, t_1 + t_2) \leq H(f_1^{(1)}, f_2^{(2)}, f_3^{(1)})
\] (2.23)
Hence, by taking limit as $n \to \infty$, we obtain
\[
\lambda(r_1 + r_2, s_1 + s_2, t_1 + t_2) \leq \sum_{i=1}^{2} \lambda(r_i, s_i, t_i) - \liminf_{n \to \infty} K_n. 
\tag{2.24}
\]
Since $r_1 + r_2 > 0$, $s_1 + s_2 > 0$, and $t_1 + t_2 > 0$, either both of $r_1$ and $r_2$, $s_1$ and $s_2$, $t_1$ and $t_2$, are positive or one of them is zero and the other is positive. To prove (2.19), it suffices to consider the following five cases:

(i) $r_1, r_2 > 0$ and $s_1, s_2, t_1, t_2 \geq 0$;
(ii) $r_1 = 0, r_2 > 0, s_2 > 0$, and $t_1 = 0$;
(iii) $r_1 = 0, r_2 > 0, s_2 > 0$, and $t_1 > 0$;
(iv) $r_1 = 0, r_2 > 0, s_2 = 0$, and $t_1 = 0$; and
(v) $r_1 = 0, r_2 > 0, s_2 = 0$, and $t_1 > 0$.

In the case (i), i.e., when $r_1, r_2 > 0$, Lemma 2.2 guarantees that there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that for all sufficiently large $n$, $\|f^{(1)}_{1,n,x}\| \geq \delta_1$ and $\|f^{(2)}_{1,n,x}\| \geq \delta_2$. Let $\delta = \min(\delta_1, \delta_2) > 0$. Then, (2.22) gives $K_n \geq 3\delta/4$ for all sufficiently large $n$, and from (2.24), we have
\[
\lambda(r_1 + r_2, s_1 + s_2, t_1 + t_2) \leq \lambda(r_1, s_1, t_1) + \lambda(r_2, s_2, t_2) - 3\delta/4
\tag{2.25}
\]

in the case (ii), since $r_1 + s_1 + t_1 > 0$, so $s_1 > 0$ too. Then, using Lemma 2.2 again, there exist numbers $\delta_3, \delta_4 > 0$ such that for all sufficiently large $n$, $\|f^{(1)}_{2,n,x}\| \geq \delta_3$ and $\|f^{(2)}_{2,n,x}\| \geq \delta_4$. Let $\delta = \min(\delta_3, \delta_4) > 0$. Then, (2.22) gives $K_n \geq 3\delta/4$ for all sufficiently large $n$ and the claim follows from (2.24).

Next, consider the case (iii) that $r_1 = 0, r_2 > 0, s_2 > 0$, and $t_1 > 0$. If $s_1 > 0$ or $t_2 > 0$, then the proof is similar to the proof as in the case (ii) above. Thus, we may assume that $s_1 = 0$ and $t_2 = 0$. Then, we have to prove that
\[
\lambda(r_2, s_2, t_1) < \lambda(0, 0, t_1) + \lambda(r_2, s_2, 0).
\tag{2.26}
\]
It is well-known that (see, for example [9]) the equation
\[
-u'' + \sigma_3 u = a_{33}|u|^{2p-1}
\tag{2.27}
\]
has, for any $\sigma_3 > 0$, a unique positive solution $u_{a_{33}}$ in $H^1$, which is explicitly given by $u_{a_{33}}(x) = e^{i\theta}\psi(x + x_0)$, where $x_0, \theta \in \mathbb{R}$ and $\psi$ is given by
\[
\psi(x) = \left(\frac{\sigma_3 p}{a_{33}}\right)^{1/(2p-2)} \text{sech}^{2/(2p-2)} \left(\frac{\sqrt{\sigma_3}(2p-2)x}{2}\right).
\tag{2.28}
\]
For any $t_1 > 0$, let $\psi_{t_1}$ be a solution to the problem
\[
\lambda(0,0,t_1) = \inf \{ |f_x|^2 - \frac{a_{33}}{p} |f|^{2p} : f \in H^1 \text{ and } \|f\|^2 = t_1 \}.
\]
Then $\psi_{t_1}$ satisfies the Lagrange multiplier equations (2.26), in which $\sigma_3$ is the Lagrange multiplier. Therefore, $\psi_{t_1} = \psi$ up to a phase factor and a translation, where $\psi$ is as given in (2.27). Now let $\phi_{r_2}$ and $\phi_{s_2}$ be as defined in Lemma 2.3 so that $\lambda(r_2, s_2, 0) = F(\phi_{r_2}, \phi_{s_2})$. Then, clearly
\[
\int_{-\infty}^{\infty} |\phi_{r_2}|^p |\psi_{t_1}|^p > 0 \text{ and } \int_{-\infty}^{\infty} |\phi_{s_2}|^p |\psi_{t_1}|^p > 0.
\]
Thus, we have that
\[
\lambda(r_2, s_2, t_1) \leq H(\phi_{r_2}, \phi_{s_2}, \psi_{t_1}) = \int_{-\infty}^{\infty} \left( |(\psi_{t_1})_x|^2 - \frac{a_{13}}{p} |\psi_{t_1}|^{2p} \right) dx \\
+ \int_{-\infty}^{\infty} \left( |(\phi_{r_2})_x|^2 + |(\phi_{s_2})_x|^2 - \frac{a_{11}}{p} |\phi_{r_2}|^{2p} - \frac{a_{22}}{p} |\phi_{s_2}|^{2p} - \frac{2a_{12}}{p} |\phi_{r_2}| |\phi_{s_2}|^p \right) dx \\
- \frac{2a_{13}}{p} \int_{-\infty}^{\infty} |\phi_{r_2}|^{p} |\psi_{t_1}|^{p} dx - \frac{2a_{23}}{p} \int_{-\infty}^{\infty} |\phi_{s_2}|^{p} |\psi_{t_1}|^{p} dx,
\]
from which it follows that
\[
\lambda(r_2, s_2, t_1) \leq \lambda(0, 0, t_1) + \lambda(r_2, s_2, 0) - \frac{2a_{13}}{p} \int_{-\infty}^{\infty} |\phi_{r_2}|^{p} |\psi_{t_1}|^{p} dx \leq \lambda(0, 0, t_1) + \lambda(r_2, s_2, 0).
\]
This proves (2.25). In case (iv), we have to prove that
\[
\lambda(r_2, s_2, t_1) < \lambda(0, 0, t_1) + \lambda(r_2, s_2, 0),
\]
which can be proved using exactly the same argument as used in the proof of (2.25). Finally, in case (v), we may assume that \( t_2 = 0 \); otherwise the claim follows from Lemma 2.2, (2.22), and (2.24). Then, in case (v), we have to prove that
\[
\lambda(r_2, s_1, t_1) < \lambda(0, s_1, t_1) + \lambda(r_2, 0, 0).
\]
The proof of (2.29) is similar to the proof of (2.25) as well. This completes the proof of the lemma.  

3. Existence of solitary-wave solutions. We now consider separately the three possibilities \( \gamma = r + s + t, \) \( 0 < \gamma < r + s + t, \) and \( \gamma = 0. \)

**Lemma 3.1.** Suppose \( \gamma = r + s + t. \) Then there exists a sequence of real numbers \( \{y_1, y_2, y_3, \ldots\} \) such that

1. for every \( z < r + s + t \) there exists \( \eta = \eta(z) \) such that
\[
\int_{y_n - \eta}^{y_n + \eta} \left( |f_{1,n}|^2 + |f_{2,n}|^2 + |f_{3,n}|^2 \right) dx > z
\]
for all sufficiently large \( n. \)

2. the sequence \( \{w_n\} \) defined by
\[
w_{j,n}(x) = f_{j,n}(x + y_n) \text{ for } x \in \mathbb{R} \text{ and } j \in \{1, 2, 3\},
\]
has a subsequence which converges in \( Y \) norm to a function \( \Phi \in O_{r,s,t}. \) In particular, \( O_{r,s,t} \) is nonempty.

**Proof.** Statement 1 is just a consequence of Lions’ concentration compactness lemma [16]. To prove statement 2, observe first that from statement 1, there exists \( \eta_k \in \mathbb{R} \) such that, for every \( k \in \mathbb{N}, \) we have
\[
\int_{-\eta_k}^{\eta_k} \sum_{j=1}^{3} |w_{j,n}|^2 dx > (r + s + t) - \frac{1}{k},
\]
for all sufficiently large \( n. \) (In other words, the measures
\[
\mu_n = (|w_{1,n}|^2 + |w_{2,n}|^2 + |w_{3,n}|^2) dx
\]
form a “tight” family on $\mathbb{R}$, in the sense that for every $\epsilon > 0$, there exists a fixed compact set $K$ such that $\mu_n(\mathbb{R}\setminus K) < \epsilon$ for all $n \in \mathbb{N}$.) As $\|w_{1,n}\|_1 + \|w_{1,n}\|_1 + \|w_{1,n}\|_1 \leq B$, hence from the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ on bounded intervals $\Omega$, it follows that some subsequence of \{(w_{1,n}, w_{2,n}, w_{3,n})\} converges in $L^2(-\eta_k, \eta_k)$ norm to a limit function $(\Phi_1, \Phi_2, \Phi_3)$ satisfying

$$
\int_{-\eta_k}^{\eta_k} \sum_{j=1}^{3} |\Phi_j|^2 \, dx > (r + s + t) - \frac{1}{k},
$$

Using a Cantor diagonalization process, together with the fact that

$$
\int_{-\infty}^{\infty} \sum_{j=1}^{3} |w_{j,n}|^2 \, dx = r + s + t, \quad \text{for all } n,
$$

we conclude that some subsequence of \{(w_{1,n}, w_{2,n}, w_{3,n})\} converges in $L^2(\mathbb{R})$ norm to a limit $(\Phi_1, \Phi_2, \Phi_3) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$ satisfying

$$
\int_{-\infty}^{\infty} \sum_{j=1}^{3} |\Phi_j|^2 \, dx = r + s + t.
$$

Furthermore, by the weak compactness of the unit sphere and the weak lower semicontinuity of the norm in Hilbert space, \{(w_{1,n}, w_{2,n}, w_{3,n})\} converges weakly to $(\Phi_1, \Phi_2, \Phi_3)$ in $Y$, and that

$$
\|(\Phi_1, \Phi_2, \Phi_3)\|_Y \leq \liminf_{n \to \infty} \|(w_{1,n}, w_{2,n}, w_{3,n})\|_Y.
$$

Next, from the Gagliardo-Nirenberg inequality, we have

$$
|w_{j,n} - \Phi_j|^{2p} \leq C \left( \int_{-\infty}^{\infty} |w_{j,n}' - \Phi_j'|^2 \, dx \right)^{(p-1)/2} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |w_{j,n} - \Phi_j|^2 \, dx \right)^{(p+1)/2}
$$

$$
\leq C \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |w_{j,n} - \Phi_j|^2 \, dx \right)^{(p+1)/2},
$$

where $C$ denotes various constants independent of $n$. Hence $w_{j,n} \to \Phi_j$ in $L^{2p}$ norm also. It follows that

$$
\mathcal{H}(\Phi_1, \Phi_2, \Phi_3) \leq \lim_{n \to \infty} \mathcal{H}(w_{1,n}, w_{2,n}, w_{3,n}) = \lambda(r, s, t),
$$

whence $\mathcal{H}(\Phi_1, \Phi_2, \Phi_3) = \lambda(r, s, t)$ and $(\Phi_1, \Phi_2, \Phi_3) \in \mathcal{O}_{r,s,t}$. Since

$$
|\Phi_j|^{2p} = \lim_{n \to \infty} |w_{j,n}|^{2p}, \quad \|\Phi_j\| = \lim_{n \to \infty} \|w_{j,n}\|,
$$

and

$$
\mathcal{H}(\Phi_1, \Phi_2, \Phi_3) = \lim_{n \to \infty} \mathcal{H}(w_{1,n}, w_{2,n}, w_{3,n}),
$$

we conclude that

$$
\|(\Phi_1, \Phi_2, \Phi_3)\|_Y = \lim_{n \to \infty} \|(w_{1,n}, w_{2,n}, w_{3,n})\|_Y.
$$

As $Y$ is a Hilbert space, an elementary exercise in Hilbert space theory then follows that $(w_{1,n}, w_{2,n}, w_{3,n})$ converges to $(\Phi_1, \Phi_2, \Phi_3)$ in $Y$ norm. $\square$

The following result, which we state here without proof, is a special case of Lemma I.1 of [16]. For a proof, see Lemma 2.13 of [2].
Lemma 3.2. Suppose \( f_n \) is a sequence of functions which is bounded in \( H^1 \) and which satisfies, for some \( R > 0 \),
\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} f_n^2 \, dx = 0. \tag{3.3}
\]
Then for every \( k > 2 \), \( |f_n|_k \to 0 \) as \( n \to \infty \).

We can now rule out the case of vanishing:

Lemma 3.3. For any minimizing sequence \( \{f_n\} \in Y \), \( \gamma > 0 \).

**Proof.** Suppose to contrary that \( \gamma = 0 \). By Lemma 2.1, \( \{|f_1,n|\}, \{|f_2,n|\}, \) and \( \{|f_3,n|\} \) are bounded sequences in \( H^1 \). Therefore, Lemma 3.2 implies that \( |f_j,n|_{2p}^2 \, dx \to 0 \).

For \( k, j = 1, 2, 3 \) and \( j \neq k \), we have
\[
\int_{-\infty}^{\infty} |f_{jn}|^p |f_{kn}|^p \, dx \leq C \left( \int_{-\infty}^{\infty} |f_{jn}|^{2p} \, dx \right)^{1/2} \left( \int_{-\infty}^{\infty} |f_{kn}|^{2p} \, dx \right)^{1/2},
\]
and it follows also that
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} |f_{jn}|^p |f_{kn}|^p \, dx = 0.
\]
Hence
\[
\lambda(r,s,t) = \lim_{n \to \infty} \mathcal{H}(f_n) \geq \liminf_{n \to \infty} \int_{-\infty}^{\infty} \sum_{j=1}^{3} |f_{jn},n|^2 \, dx \geq 0, \tag{3.4}
\]
contradicting Lemma 2.1. This proves \( \gamma > 0 \). \( \square \)

Lemma 3.4. Suppose \( r,s,t > 0 \) and let \( \{f_n\} \) be any minimizing sequence for \( \lambda(r,s,t) \). Let \( \gamma \) be as defined in (1.10). Then there exist numbers \( r_1 \in [0, r], s_1 \in [0, s] \) and \( t_1 \in [0, t] \) such that
\[
\gamma = r_1 + s_1 + t_1 \tag{3.5}
\]
and
\[
\lambda(r_1, s_1, t_1) + \lambda(r-r_1, s-s_1, t-t_1) \leq \lambda(r,s,t). \tag{3.6}
\]

**Proof.** We shall follow the arguments in [1]. Let \( \epsilon \) be an arbitrary positive number. By the definition of \( \gamma \), it follows that \( \gamma - \epsilon < P(\eta) \leq P(2\eta) \leq \gamma \) for \( \eta \) sufficiently large. By taking \( \eta \) larger if necessary, we may also assume that \( \frac{1}{\eta} < \epsilon \). From the definition of \( P \), we can choose \( N \) so large that, for every \( n \geq N \),
\[
\gamma - \epsilon < P_n(\eta) \leq P_n(2\eta) \leq \gamma + \epsilon.
\]
Hence, for each \( n \geq N \), we can find \( y_n \) such that
\[
\int_{y_n-\eta}^{y_n+\eta} \sum_{j=1}^{3} |f_{jn}|^2 \, dx > \gamma - \epsilon \quad \text{and} \quad \int_{y_n-2\eta}^{y_n+2\eta} \sum_{j=1}^{3} |f_{jn}|^2 \, dx < \gamma + \epsilon. \tag{3.7}
\]
Now choose \( \rho \in C^\infty_{c}([-2,2]) \) such that \( \rho \equiv 1 \) on \([-1,1] \), and let \( \sigma \in C^\infty(\mathbb{R}) \) be such that \( \rho^2 + \sigma^2 \equiv 1 \) on \( \mathbb{R} \). Set, for \( \eta > 0 \), \( \rho_\eta(x) = \rho(x/\eta) \) and \( \sigma_\eta(x) = \sigma(x/\eta) \), and define the functions
\[
f_n^{(1)}(x) = \rho_\eta(x-y_n)f(x) \quad \text{and} \quad f_n^{(2)}(x) = \sigma_\eta(x-y_n)f(x).
\]
Then, for each \( j = 1, 2, 3 \), and \( k = 1, 2 \), the sequences \( \{f_{jn}^{(k)}\} \) are bounded in \( L^2 \). Thus, by passing to subsequences if necessary, we may assume that there exist \( r_1 \in [0, r], s_1 \in [0, s] \) and \( t_1 \in [0, t] \) such that
\[
||f_{1,n}^{(1)}||^2 \to r_1, \quad ||f_{2,n}^{(1)}||^2 \to s_1, \quad \text{and} \quad ||f_{3,n}^{(1)}||^2 \to t_1, \tag{3.8}
\]
whence it follows also that
\[ \|f_{1,n}^{(2)}\|^2 \to r - r_1, \|f_{2,n}^{(2)}\|^2 \to s - s_1, \text{ and } \|f_{3,n}^{(2)}\|^2 \to t - t_1. \quad (3.9) \]

Now
\[ r_1 + s_1 + t_1 = \lim_{n \to \infty} \int_{-\infty}^{\infty} \sum_{j=1}^{3} |f_{j,n}^{(1)}|^2 \, dx = \lim_{n \to \infty} \int_{-\infty}^{\infty} \rho_n^2 \sum_{j=1}^{3} |f_{j,n}|^2 \, dx. \]

From (3.7), it follows that, for every \( n \in \mathbb{N}, \)
\[ \gamma - \epsilon < \int_{-\infty}^{\infty} \rho_n^2 \sum_{j=1}^{3} |f_{j,n}|^2 \, dx < \gamma + \epsilon. \]

Hence \( |(r_1 + s_1 + t_1) - \gamma| < \epsilon. \) Next, we claim that for all \( n, \)
\[ \mathcal{H}(f_n^{(1)}) + \mathcal{H}(f_n^{(2)}) \leq \mathcal{H}(f_n) + C\epsilon \quad (3.10) \]
To see (3.10), we write
\[
\mathcal{H}(f_n^{(1)}) = \int_{-\infty}^{\infty} \rho_n^2 \left( \sum_{j=1}^{3} |f_{j,n}|^2 - \frac{1}{p} \sum_{k,j=1}^{3} a_{kj} |f_{k,n}|^p |f_{j,n}|^p \right) \, dx \\
+ \int_{-\infty}^{\infty} \left( \rho_n^p \sum_{j=1}^{3} |f_{j,n}|^2 + 2 \rho_n \rho_n^2 \sum_{j=1}^{3} f_{j,n} f_{j,n} \right) \, dx \\
+ \frac{1}{p} \int_{-\infty}^{\infty} (\rho_n^p - \rho_n^{2p}) \sum_{k,j=1}^{3} a_{kj} |f_{k,n}|^p |f_{j,n}|^p \, dx,
\]
where, for ease of notation, we have written the functions \( \rho_n(x - y_n) \) simply as \( \rho_n. \) Similar estimate holds for \( \mathcal{H}(f_n^{(2)}). \) Since \( \rho_n^2 + \sigma_n^2 \equiv 1, \) \( |\rho_n'| = |\rho'| \infty / \eta, \) and \( |\sigma'_n| = |\sigma'| \infty / \eta, \) an application of Hölder’s inequality yields
\[
\mathcal{H}(f_n^{(1)}) + \mathcal{H}(f_n^{(2)}) = \mathcal{H}(f_n) + O(1/n)
+ \frac{1}{p} \int_{-\infty}^{\infty} \left[ (\rho_n^2 - \rho_n^{2p}) + (\sigma_n^2 - \sigma_n^{2p}) \right] \sum_{k,j=1}^{3} a_{kj} |f_{k,n}|^p |f_{j,n}|^p \, dx,
\]
where \( O(1/n) \) denotes a term bounded in absolute value by \( C/n \) with \( C \) independent of \( \eta \) and \( n. \) Using (3.7), one can see that
\[
\left| \int_{-\infty}^{\infty} \left[ (\rho_n^2 - \rho_n^{2p}) + (\sigma_n^2 - \sigma_n^{2p}) \right] f_{k,n} f_{j,n}^p \, dx \right| \leq |f_{k,n}|^p \int_{\eta \leq |x - y_n| \leq 2n} 2 |f_{j,n}|^p \, dx \\
\leq C \epsilon,
\]
where again \( C \) denotes various constants independent of \( \eta \) and \( n. \) Then, (3.10) follows by choosing \( \eta \) large enough so that \( |O(1/\eta)| \leq \epsilon. \)

To prove (3.6), notice that for any given value of \( \epsilon, \) each of the terms in (3.10) is bounded independently of \( n, \) so by passing to a subsequence if necessary, we may assume that
\[ \mathcal{H}(f_n^{(1)}) \to H_1 \text{ and } \mathcal{H}(f_n^{(2)}) \to H_2. \quad (3.11) \]
Then, $H_1 + H_2 \leq \lambda(r, s, t) + C\epsilon$. Since $\epsilon$ can be taken arbitrarily small and $\eta$ arbitrarily large, combining the results of the preceding paragraphs, we can find sequences $\{f_{n,k}^{(1)}\}$ and $\{f_{n,k}^{(2)}\}$, for each $k \in \mathbb{N}$, such that

$$\|f_{1,n,k}^{(1)}\|^2 \to r_1(k), \quad \|f_{2,n,k}^{(1)}\|^2 \to r_1(k), \quad \|f_{3,n,k}^{(1)}\|^2 \to r_1(k),$$

$$\|f_{1,n,k}^{(2)}\|^2 \to r - r_1(k), \quad \|f_{2,n,k}^{(2)}\|^2 \to s - s_1(k), \quad \|f_{3,n,k}^{(2)}\|^2 \to t - t_1(k),$$

$$\mathcal{H}(f_{n,k}^{(1)}) \to H_1(k), \quad \text{and} \quad \mathcal{H}(f_{n,k}^{(2)}) \to H_2(k),$$

where $r_1(k) \in [0, r]$, $s_1(k) \in [0, s]$, $t_1(k) \in [0, t]$,

$$|r_1(k) + s_1(k) + t_1(k) - \gamma| \leq \epsilon, \quad (3.12)$$

and

$$H_1(k) + H_2(k) \leq \lambda(r, s, t) + \frac{1}{k}. \quad (3.13)$$

By passing to subsequences, we may assume that

$$r_1(k) \to r_1 \in [0, r], \quad s_1(k) \to s_1 \in [0, s], \quad t_1(k) \to t_1 \in [0, t],$$

$$H_1(k) \to H_1, \quad \text{and} \quad H_2(k) \to H_2.$$

Also, by redefining $\{f_{n,k}^{(1)}\}$ and $\{f_{n,k}^{(2)}\}$ as the diagonal subsequences

$$f_{n,k}^{(1)} = f_{n,k}^{(1,n)} \quad \text{and} \quad f_{n,k}^{(2)} = f_{n,k}^{(2,n)},$$

we may assume that (3.8), (3.9), and (3.11) hold.

By letting $k \to \infty$ in (3.12) yields (3.5). The claim (3.6) follows from (3.13) provided we can show that

$$H_1 \geq \lambda(r_1, s_1, t_1), \quad (3.14)$$

and

$$H_2 \geq \lambda(r - r_1, s - s_1, t - t_1). \quad (3.15)$$

To prove (3.14), consider first the case that $r_1, s_1, \text{ and } t_1$ are all positive. Then, for $n$ sufficiently large, $\|f_{j,n,k}^{(1)}\|$ are all positive for each $j = 1, 2, 3$, so we may define

$$\alpha_n = \frac{\sqrt{r_1}}{\|f_{1,n,k}^{(1)}\|}, \quad \beta_n = \frac{\sqrt{s_1}}{\|f_{2,n,k}^{(1)}\|}, \quad \text{and} \quad \gamma_n = \frac{\sqrt{t_1}}{\|f_{3,n,k}^{(1)}\|},$$

which gives $(\alpha_n f_{1,n,k}^{(1)}, \beta_n f_{2,n,k}^{(1)}, \gamma_n f_{3,n,k}^{(1)}) \in \Delta_{r_1, s_1, t_1}$. Consequently, we have

$$\mathcal{H}(\alpha_n f_{1,n,k}^{(1)}, \beta_n f_{2,n,k}^{(1)}, \gamma_n f_{3,n,k}^{(1)}) \geq \lambda(r_1, s_1, t_1).$$

As all scaling factors tend to 1 as $n \to \infty$, it follows that

$$\mathcal{H}(\alpha_n f_{1,n,k}^{(1)}, \beta_n f_{2,n,k}^{(1)}, \gamma_n f_{3,n,k}^{(1)}) \to H_1,$$

and hence (3.14) follows. Next, we prove (3.14) if exactly one of $r_1, s_1, \text{ or } t_1$ is zero. Consider the case that $r_1 = 0, s_1 > 0, \text{ and } t_1 > 0$. Then, using Gagliardo-Nirenberg inequality, we have that

$$\int_{-\infty}^{\infty} |f_{1,n,k}^{(1)}| |f_{j,n,k}^{(1)}|^{p} \, dx \to 0 \quad \text{for all } 1 \leq j \leq 3,$$
and hence, we deduce that

\[ H_1 = \lim_{n \to \infty} \mathcal{H}(f_n^{(1)}) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \left( |f_{1,nx}|^2 + |f_{2,nx}|^2 + |f_{3,nx}|^2 - \frac{a_{22}}{p} |f_{2,nx}|^{2p} - \frac{a_{33}}{p} |f_{3,nx}|^{2p} \right) dx \]

\[ \geq \liminf_{n \to \infty} \int_{-\infty}^{\infty} \left( \sum_{j=2}^{3} |f_{j,nx}|^2 - \frac{1}{p} \sum_{k,j=2}^{3} a_{kj} |f_{k,nx}|^{p} |f_{j,nx}|^{p} \right) dx \geq \lambda(0, s, t). \]

Finally, if \( r_1 = 0, s_1 = 0, \) and \( t_1 > 0, \) then we have

\[ H_1 = \lim_{n \to \infty} \mathcal{H}(f_n^{(1)}) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \left( \sum_{j=2}^{3} |f_{j,nx}|^2 - \frac{a_{33}}{p} |f_{3,nx}|^{2p} \right) dx \]

\[ \geq \liminf_{n \to \infty} \int_{-\infty}^{\infty} \left( |f_{1,nx}|^2 - \frac{a_{33}}{p} |f_{3,nx}|^{2p} \right) \geq \lambda(0, 0, t_1). \]

This completes the proof of (3.14). The proof of (3.15) is similar with \( r - r_1, s - s_1, \) and \( t - t_1 \) playing the roles of \( r_1, s_1, \) and \( t_1 \) respectively. \( \square \)

The following lemma rules out the possibility of dichotomy of minimizing sequences:

**Lemma 3.5.** For every minimizing sequence, \( \gamma \notin (0, r + s + t). \)

**Proof.** Suppose to the contrary that \( 0 < \gamma < r + s + t. \) Let \( r_1, s_1, \) and \( t_1 \) be as defined in Lemma 3.4, and let \( r_2 = r - r_1, s_2 = s - s_1, \) and \( t_2 = t - t_1. \) Then \( r_2 + s_2 + t_2 = (r + s + t) - \gamma > 0, \) and also \( r_1 + s_1 + t_1 = \gamma > 0. \) Moreover, \( r_1 + r_2 > 0, s_1 + s_2 = s > 0, \) and \( t_1 + t_2 = t > 0. \) Therefore Lemma 2.7 implies that that (2.19) holds. But this contradicts (3.6) and thus, lemma follows. \( \square \)

The next theorem proves the existence of minimizing pairs for (1.9), and hence, existence of three-parameter family of solitary waves for (1.1) provided that \( a_{kj} > 0 \) for all \( k, j \in \{1, 2, 3\} \) and \( 2 \leq p < 3. \)

**Theorem 3.6.** The set \( O_{r,s,t} \) is not empty. Moreover, if \( \{f_n\} \) is any minimizing sequence for \( \lambda(r, s, t), \) then

1. There exists a sequence \( \{y_n\} \subset \mathbb{R} \) and an element \( \Phi \in O_{r,s,t} \) such that \( \{f_n(\cdot + y_n)\} \) has a subsequence converging strongly in \( Y \) to \( \Phi. \)
2. Each function \( \Phi \in O_{r,s,t} \) is a solution of (1.3) for some \( \omega_1, \omega_2, \omega_3 > 0, \) and therefore when inserted into (1.2) yields a three parameter family solitary-wave solution of (1.1).
3. The following holds:

\[ \lim_{n \to \infty} \inf_{y \in \mathbb{R}} \inf_{\Phi \in O_{r,s,t}} \|f_n(\cdot + y) - \Phi\|_Y = 0. \]

4. The following holds:

\[ \lim_{n \to \infty} \inf_{\Phi \in O_{r,s,t}} \|f_n - \Phi\|_Y = 0. \]

**Proof.** From Lemmas 3.3 and 3.5, it follows that \( \gamma = r + s + t. \) Then, by Lemma 3.1, the set \( O_{r,s,t} \) is not empty and statement 1 follows.
To see the validity of statement 2, notice that $\Phi$ is in the minimizing set $\mathcal{O}_{r,s,t}$ for $\lambda(r,s,t)$, and so minimizes $\mathcal{H}(f)$ subject to $f \in \Delta_{r,s,t}$. The Lagrange multiplier principle asserts that there exist real numbers $\omega_1$, $\omega_2$, and $\omega_3$ such that
\[
\delta \mathcal{H}(\Phi_1, \Phi_2, \Phi_3) + \omega_1 \delta \mathcal{Q}(\Phi_1) + \omega_2 \delta \mathcal{Q}(\Phi_2) + \omega_3 \delta \mathcal{Q}(\Phi_3) = 0, \quad (3.16)
\]
where $\delta$ denotes the Fréchet derivative. Computing the Fréchet derivatives we see that the equations
\[
-\Phi_{1,xx} + \omega_1 \Phi_1 = a_{11} |\Phi_1|^{2p-2} \Phi_1 + (a_{12}|\Phi_2|^p + a_{13}|\Phi_3|^p) |\Phi_1|^{p-2} \Phi_1, \\
-\Phi_{2,xx} + \omega_2 \Phi_2 = a_{22} |\Phi_2|^{2p-2} \Phi_2 + (a_{12}|\Phi_1|^p + a_{23}|\Phi_3|^p) |\Phi_2|^{p-2} \Phi_2, \\
-\Phi_{3,xx} + \omega_3 \Phi_3 = a_{33} |\Phi_3|^{2p-2} \Phi_3 + (a_{13}|\Phi_1|^p + a_{23}|\Phi_2|^p) |\Phi_3|^{p-2} \Phi_3, \quad (3.17)
\]
hold, at least in the sense of distributions. A straightforward bootstrapping argument (cf. Lemma 1.3 of [27]) shows that distributional solutions are also classical solutions.

Multiplying the first equation in (3.17) by $\Phi_1$, the second equation by $\Phi_2$, and the third equation by $\Phi_3$, and integrating over $\mathbb{R}$, we obtain
\[
\int_{-\infty}^{\infty} \left( |\Phi_j'|^2 - |\Phi_j|^p \sum_{k=1}^{3} a_{jk} |\Phi_k|^p \right) dx = -\omega_j \int_{-\infty}^{\infty} |\Phi_j'|^2 dx, \quad j = 1, 2, 3. \quad (3.18)
\]
By Lemma 2.4, applied to the constant sequence $f_n = \Phi$, we have that
\[
\int_{-\infty}^{\infty} \left( |\Phi_j'|^2 - |\Phi_j|^p \sum_{k=1}^{3} a_{jk} |\Phi_k|^p \right) dx < 0, \quad j = 1, 2, 3,
\]
and hence, $\omega_1, \omega_2, \omega_3 > 0$. This then completes the proof of statement 2.

To prove statement 3, suppose that it is false. Then, there there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a number $\varepsilon > 0$ such that
\[
\liminf_{n \to \infty} \inf_{y \in \mathbb{R}} \|f_{n_k}(\cdot + y) - \Phi\|_Y \geq \varepsilon
\]
for all $k \in \mathbb{N}$. As $\{f_{n_k}\}$ itself a minimizing sequence for $\lambda(r,s,t)$, it follows from statement 1 that there exists a sequence of real numbers $\{y_k\}$ and an element $(\Phi_{1,0}, \Phi_{2,0}, \Phi_{3,0})$ of $\mathcal{O}_{r,s,t}$ such that
\[
\liminf_{k \to \infty} \|f_{n_k}(\cdot + y_k) - (\Phi_{1,0}, \Phi_{2,0}, \Phi_{3,0})\|_Y = 0.
\]
This contradiction proves statement 3.

Finally, since the functionals $\mathcal{H}$ and $\mathcal{Q}$ are invariant under translations, then $\mathcal{O}_{r,s,t}$ clearly contains any translate of $\Phi$ if it contains $\Phi$, and hence, statement 4 follows from statement 3.

We now address the question about the characterization of the set $\mathcal{O}_{r,s,t}$.

**Theorem 3.7.** For every $\Phi$ in $\mathcal{O}_{r,s,t}$, there exist numbers $\theta_j \in \mathbb{R}$ and real functions $\phi_j$ such that $\phi_j(x) > 0$, for all $x \in \mathbb{R}$, and
\[
\Phi_j(x) = e^{i\theta_j} \phi_j(x), \quad j = 1, 2, 3.
\]

Also, the functions $\Phi_j$ are infinitely differentiable on $\mathbb{R}$.

**Proof.** We write the complex-valued functions $\Phi_j$ as
\[
\Phi_j(x) = e^{i\theta_j} \phi_j(x), \quad j = 1, 2, 3, \quad (3.19)
\]
where \( \theta_j : \mathbb{R} \to \mathbb{R} \) and \( \phi_j(x) = |\Phi_j(x)|, \ j = 1, 2, 3. \) Notice that \( (\phi_1, \phi_2, \phi_3) \) is also in \( \mathcal{O}_{r,s,t} \), as follows from Lemma 2.5. Therefore, \( (\phi_1, \phi_2, \phi_3) \) satisfies the Lagrange multiplier equations
\[
-\phi_{j,xx} + \omega_j \phi_j = |\phi_j|^{p-2} \phi_j \sum_{k=1}^{3} a_{jk} |\phi_k|^p, \ j = 1, 2, 3. \tag{3.20}
\]
(That the Lagrange multipliers stay same follows from the fact that they are determined by the equation (3.18), and this equation is unchanged when \( (\Phi_1, \Phi_2, \Phi_3) \) is replaced by \( (\phi_1, \phi_2, \phi_3) \).) Using (3.19), we compute
\[
\Phi''_1 = e^{i\theta_1(x)} \left( \omega_1 \phi_1 - |\phi_1|^{p-2} \phi_1 \sum_{k=1}^{3} a_{ik} |\phi_k|^p - Z(x) \right), \tag{3.21}
\]
where
\[
Z(x) = (\theta'_1(x))^2 \phi_1(x) - 2i \theta'_1(x) \phi_1(x) - i \theta''_1(x) \phi_1(x).
\]
On the other hand, since \( (\Phi_1, \Phi_2, \Phi_3) \) satisfies the same equations (3.20) as \( (\phi_1, \phi_2, \phi_3) \), it follows that
\[
\Phi''_1 = e^{i\theta_1(x)} \left( \omega_1 \phi_1 - |\phi_1|^{p-2} \phi_1 \sum_{k=1}^{3} a_{ik} |\phi_k|^p \right). \tag{3.22}
\]
From (3.21) and (3.22), we obtain that \( Z(x) = 0 \), and by equating the real part of this equation, we conclude that \( \theta'_1(x) = 0 \), and hence \( \theta_1(x) \) is constant. Similarly, \( \theta_2(x) \) and \( \theta_3(x) \) are constants.

Next, a straightforward calculation using Fourier transform shows that for each \( j = 1, 2, 3 \), the operator \(-\partial_x^2 + \omega_j \) appearing in (3.20) is invertible on \( H^1 \), with inverse given by convolution with the function
\[
K_{\omega_j}(x) = \frac{1}{2\sqrt{\omega_j}} e^{-\sqrt{\omega_j} |x|}.
\]
The Lagrange multiplier equations associated with \( (\phi_1, \phi_2, \phi_3) \) can then be rewritten in the form
\[
\phi_1 = K_{\omega_1} \ast \left( a_{11} |\phi_1|^{2(p-2)} \phi_1 + a_{12} |\phi_2|^{p} |\phi_1|^{p-2} \phi_1 + a_{13} |\phi_3|^{p} |\phi_1|^{p-2} \phi_1 \right),
\]
\[
\phi_2 = K_{\omega_2} \ast \left( a_{22} |\phi_2|^{2(p-2)} \phi_2 + a_{12} |\phi_1|^{p} |\phi_2|^{p-2} \phi_2 + a_{23} |\phi_3|^{p} |\phi_2|^{p-2} \phi_2 \right),
\]
\[
\phi_3 = K_{\omega_3} \ast \left( a_{33} |\phi_3|^{2(p-2)} \phi_3 + a_{13} |\phi_1|^{p} |\phi_3|^{p-2} \phi_3 + a_{23} |\phi_2|^{p} |\phi_3|^{p-2} \phi_3 \right).
\]
Since the convolutions of \( K_{\omega_j} \) with functions that are everywhere non-negative and not identically zero must produce everywhere positive functions, it follows that \( \phi_j(x) > 0 \) for all \( x \in \mathbb{R} \).

4. Stability of solitary waves. Our stability result is the following.

**Theorem 4.1.** For every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if
\[
\inf_{\Phi \in \mathcal{C}_{r,s,t}} \| (f_0, g_0, h_0) - \Phi \|_Y < \delta,
\]
then the solution \( u(x,t) \) of (1.1) with \( u(x,0) = (f_0(x), g_0(x), h_0(x)) \) satisfies
\[
\sup_{t \in \mathbb{R}} \inf_{\Phi \in \mathcal{C}_{r,s,t}} \| u(\cdot, t) - \Phi \|_Y < \epsilon
\]
Proof. Suppose that $O_{r,s,t}$ is not stable. Then there exist a number $\epsilon > 0$, a sequence of times $t_n$, and a sequence $\{u_n(x, 0)\}$ in $Y$ such that for all $n$,

$$\inf\{\|u_n(x, 0) - \Phi\|_Y : \Phi \in O_{r,s,t}\} < \frac{1}{n};$$

(4.1)

and

$$\inf\{\|u_n(\cdot, t_n) - \Phi\|_Y : \Phi \in O_{r,s,t}\} \geq \epsilon,$$

(4.2)

for all $n$, where $u_n(x, t)$ solves (1.1) with initial data $u_n(x, 0)$. Since $u_n(x, 0)$ converges to an element in $O_{r,s,t}$ in $Y$ norm, and since for $\Phi \in O_{r,s,t}$, we have $Q(\Phi_1) = r$, $Q(\Phi_2) = s$, $Q(\Phi_3) = t$, and $H(\Phi) = \lambda(r,s,t)$, we therefore have

$$Q(u_{1,n}(x,0)) \to r, Q(u_{2,n}(x,0)) \to s, Q(u_{3,n}(x,0)) \to t,$$

and $H(u_n(x,0)) \to \lambda(r,s,t)$. Let us denote $u_{1,n}(\cdot, t_n)$ by $U_{1,n}$, $u_{2,n}(\cdot, t_n)$ by $U_{2,n}$, and $u_{3,n}(\cdot, t_n)$ by $U_{3,n}$. We now choose $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ such that

$$Q(\alpha_n, u_{1,n}(x,0)) = r, Q(\beta_n u_{2,n}(x,0)) = s, Q(\gamma_n u_{3,n}(x,0)) = t,$$

for all $n$. Thus, $\alpha_n \to 1$, $\beta_n \to 1$, and $\gamma_n \to 1$. Hence the sequences $f_{1,n} = \alpha_n U_{1,n}$, $f_{2,n} = \beta_n U_{2,n}$, and $f_{3,n} = \gamma_n U_{3,n}$ satisfies $Q(f_{1,n}) = r, Q(f_{2,n}) = s, Q(f_{3,n}) = t$, and

$$\lim_{n \to \infty} H(f_n) = \lim_{n \to \infty} H(u_n(\cdot, t_n)) = \lim_{n \to \infty} H(u_n(x,0)) = \lambda(r,s,t).$$

Therefore, $\{f_n\}$ is a minimizing sequence for $\lambda(r,s,t)$. From Theorem 3.6, it follows that for all $n$ sufficiently large, there exists $\Phi_n \in O_{r,s,t}$ such that $\|f_n - \Phi_n\|_Y < \epsilon/2$. But then we have

$$\epsilon \leq \|u_n(\cdot, t_n) - \Phi_n\|_Y \leq \|u_n(\cdot, t_n) - f_n\|_Y + \|f_n - \Phi_n\|_Y \leq |1 - \alpha_n| \cdot \|U_{1,n}\|_1 + |1 - \beta_n| \cdot \|U_{2,n}\|_1 + |1 - \gamma_n| \cdot \|U_{3,n}\|_1 + \frac{\epsilon}{2}$$

and by taking $n \to \infty$, we obtain that $\epsilon \leq \epsilon/2$, a contradiction, and we conclude that $O_{r,s,t}$ must in fact be stable.

REFERENCES

[1] J. Albert and J. Angulo, Existence and stability of ground-state solutions of a Schrödinger-KdV system, Proc. Royal Soc. of Edinburgh A, 133 (2003), 987–1029.
[2] J. Albert and S. Bhattarai, Existence and stability of a two-parameter family of solitary waves for an NLS-KdV system, Adv. Differential Eqns., 18 (2013), 1129–1164.
[3] J. Albert, J. Bona and J.-C. Saut, Model equations for waves in stratified fluids, Proc. Royal. Soc. of Edinburgh, Sect. A 453 (1997), 1233–1260.
[4] T. B. Benjamin, The stability of solitary waves, Proc. Roy. Soc. London Ser. A, 328 (1972), 153–183.
[5] S. Bhattarai, Solitary waves and a stability analysis for an equation of short and long dispersive waves, Nonlinear Anal., 75 (2012), 6506–6519.
[6] S. Bhattarai, Stability of solitary-wave solutions of coupled NLS equations with power-type nonlinearities, Adv. Nonlinear Anal., 4 (2015), 73–90.
[7] J. Bona, On the stability theory of solitary waves, Proc. Roy. Soc. London Ser. A, 344 (1975), 363–374.
[8] J. Byeon. Effect of symmetry to the structure of positive solutions in nonlinear elliptic problems, J. Differential Eqns., 163 (2000), 429–474.
[9] T. Cazenave, Semilinear Schrödinger Equations, 10, AMS-Courant Lect. Notes in Math., 2003.
[10] T. Cazenave and P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys., 85 (1982), 549–561.
[11] S. Chakravarty, M. J. Ablowitz, J. R. Sauer and R. B. Jenkins, Multisoliton interactions and wavelength-division multiplexing, Opt. Lett., 20 (1995), 136–138.
F. Dalfovo, S. Giorgini, L. P. Pitaevskii and S. Stringari, Theory of Bose-Einstein condensation in trapped gases, *Rev. Mod. Phys.*, **71** (1999), 463–512.

T.-L. Ho, Spinor Bose condensates in optical traps, *Phys. Rev. Lett.*, **81** (1998), p742.

Y. Kawaguchi and M. Ueda, Spinor Bose-Einstein condensates, *Phys. Reports*, **520** (2012), 253–381.

E. H. Lieb and M. Loss, *Analysis*, 2nd ed., **14**, AMS-Grad. Stud. Math., 2001.

P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, Part 1, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 109–145.

L. F. Mollenauer, S. G. Evangelides and J. P. Gordon, Wavelength division multiplexing with solitons in ultra-long transmission using lumped amplifiers, *J. Lightwave Technol.*, **9** (1991), 362–367.

N. V. Nguyen, R.-S. Tian, B. Deconinck and N. Sheils, Global existence for a system of Schrödinger equations with power-type nonlinearities, *Jour. Math. Phys.*, **54** (2013), 011503.

N. V. Nguyen and Z-Q. Wang, Orbital stability of solitary waves for a nonlinear Schrodinger system, *Adv. Differential Eqns.*, **16** (2011), 977–1000.

N. V. Nguyen and Z-Q. Wang, Orbital stability of solitary waves of a 3-coupled nonlinear Schrödinger system, *Nonlinear Anal.*, **90** (2013), 1–26.

N. V. Nguyen and Z-Q. Wang, Existence and stability of a two-parameter family of solitary waves for a 2-coupled nonlinear Schrödinger system, *Discrete and Continuous Dynamical Systems - Series A (DCDS-A)*, **36** (2016), 1005–1021.

N. V. Nguyen, R. Tian and Z-Q. Wang, Stability of traveling-wave solutions for a Schrödinger system with power-type nonlinearities, preprint.

M. Ohta, Stability of solitary waves for coupled nonlinear Schrödinger equations, *Nonlinear Anal.*, **26** (1996), 933–939.

A. C. Scott, Launching a davydov soliton: I. soliton analysis, *Phys. Scr.*, **29** (1984), p279.

B. K. Som, M. R. Gupta and B. Dasgupta, Coupled nonlinear Schrödinger equation for Langmuir and dispersive ion acoustic waves, *Phys. Lett. A*, **72** (1979), 111–114.

J. Q. Sun, Z. Q. Ma and M. Z. Qin, Simulation of envelope Rossby solitons in a pair of cubic Schrödinger equations, *Appl. Math. Comput.*, **183** (2006), 946–952.

T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, **106** AMS-CBMS, 2006.

C. Yeh and L. Bergman, Enhanced pulse compression in a nonlinear fiber by a wavelength division multiplexed optical pulse, *Phys. Rev. E*, **57** (1998), p2398.

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