Isomonodromic deformations in genus zero and one: algebrogeometric solutions and Schlesinger transformations

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1 Introduction

Here we review some recent developments in the theory of isomonodromic deformations on Riemann sphere and elliptic curve. For both cases we show how to derive Schlesinger transformations together with their action on tau-function, and construct classes of solutions in terms of multi-dimensional theta-functions.

The theory of isomonodromic deformations of ordinary matrix differential equations of the type

$$\frac{d\Psi}{d\lambda} = A(\lambda) \Psi ,$$

where \( A(\lambda) \) is a matrix-valued meromorphic function on \( \mathbb{C} \), is a classical area intimately related to the matrix Riemann-Hilbert problem on the Riemann sphere. Over the last 20 years this has become a powerful tool in areas like soliton theory, statistical mechanics, theory of random matrices, quantum field theory etc. The main object associated with the isomonodromic deformation equations is the so-called \( \tau \)-function.

After the classical work of Schlesinger the important contributions to the development of the subject were made in the papers of Jimbo, Miwa and their collaborators in the early 80’s [2, 3, 4, 5].

There are only a few cases where the matrix Riemann-Hilbert problem may be solved explicitly in terms of known special functions.

However, as was already discovered by Schlesinger himself, there exists a large class of transformations which allow to get an infinite chain of new solutions starting from the known ones. They share the characteristic feature that they shift the eigenvalues of the residues of the connection \( A(\lambda) \) in (1.1) by integer or half-integer values, thus changing the associated monodromies by sign only. These transformations – nowadays called Schlesinger transformations – were systematically studied in [4, 5]. In particular, it turns out that being written in terms of the \( \tau \)-functions the superposition laws of these transformations provide a big supply of discrete integrable systems.

Recently in papers [6, 7] it was solved a class of \( 2 \times 2 \) Riemann-Hilbert problems with arbitrary off-diagonal monodromy matrices in terms of multidimensional theta-functions. The equations for \( \tau \)-function were integrated in the paper [6] to give the following result:

$$\tau(\lambda_j) = [\det A]^{-\frac{1}{2}} \prod_{j<k}(\lambda_j - \lambda_k)^{-\frac{1}{2}} \Theta \left[ \begin{array}{c} p \\ q \end{array} \right] (0|B) .$$

where all the objects associated to auxiliary hyperelliptic curve are defined below in Sect.2.3.

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The natural question of generalizing the theory of isomonodromic deformations on the sphere to higher genus surfaces was addressed by several authors. Here, we mention the contributions of Okamoto [8, 9] and Iwasaki [10].

For the case of the torus, recently two different explicit forms of equations of isomonodromic deformations were proposed. In work of the author and Samtleben [11] it were studied isomonodromic deformations of non-singlevalued meromorphic connection on the torus whose “twists” (which determine the transformation of the connection $A(\lambda)$ with respect to tracing along basic cycles of the torus) vary with respect to the deformation parameters. The isomonodromic deformation equations for these connections hence contain transcendental dependence on the dynamical variables, which makes it difficult to analyse this system in a way analogous to the Schlesinger system on the sphere. On the other hand, Takasaki [12] considered connections on the torus whose twists remain invariant with respect to the parameters of deformation. In Takasaki’s form, the equations of isomonodromic deformations have already the same degree of non-linearity as the ordinary Schlesinger system.

In the paper [13] it were constructed transformations of Schlesinger type for elliptic isomonodromic deformations in Takasaki form, and it was derived the action of these transformation on elliptic version of $\tau$-function. Here we review these results, and, in addition, present the generalization of results of the paper [6] to elliptic case. We show how to solve certain class of Riemann-Hilbert problems on the torus in terms of Prym theta-functions. In turn, this allows to construct a class of algebro-geometric solutions of elliptic Schlesinger system.

In sect.2 we introduce Schlesinger system on the Riemann sphere. For $2 \times 2$ case we discuss elementary Schlesinger transformations together with their action on $\tau$-function and, following [1], derive class of algebro-geometric solutions of the Schlesinger system in terms of theta-functions of auxiliary hyperelliptic curve. In sect.3 we describe equations of elliptic isomonodromic deformations with constant twists [12], and, following [13], construct elliptic version of elementary Schlesinger transformations. The new result of this paper - the construction of algebro-geometric solutions of elliptic Schlesinger system in terms of Prym theta-functions - is presented in sect.3.

2 Schlesinger system on the Riemann sphere: Schlesinger transformations and algebro-geometric solutions

2.1 Schlesinger system

Consider the following ordinary linear differential equation (1.1) for a matrix-valued function $\Psi(\lambda) \in SL(2, \mathbb{C})$ and

$$A(\lambda) = \sum_{j=1}^{N} \frac{A_j}{\lambda - \lambda_j}, \quad (2.2)$$

where the residues $A_j \in sl(2, \mathbb{C})$ are independent of $\lambda$. Regularity at $\lambda = \infty$ requires

$$\sum_{j=1}^{N} A_j = 0, \quad (2.3)$$
and allows to further impose the initial condition $\Psi(\lambda=\infty) = I$. The matrix $\Psi(\lambda)$ defined in this way lives on the universal covering $X$ of $\mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_N\}$. Its asymptotical expansion near the singularities $\lambda_j$ is given by

$$\Psi(\lambda) = G_j \Psi_j \cdot (\lambda - \lambda_j)^T \psi_j,$$

(2.4)

with $G_j, C_j \in SL(2, \mathbb{C})$ constant, $\Psi_j = I + O(\lambda-\lambda_j) \in SL(2, \mathbb{C})$ holomorphic around $\lambda = \lambda_j$, and where $T_j$ is a traceless diagonal matrix with eigenvalues $\pm t_j$. The residues $A_j$ of (2.2) are encoded in the local expansion as

$$A_j = G_j T_j G_j^{-1}.$$

(2.5)

Upon analytical continuation around $\lambda = \lambda_j$, the function $\Psi(\lambda)$ in $\mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_N\}$ changes by right multiplication with some monodromy matrices $M_j$

$$\Psi(\lambda) \rightarrow \Psi(\lambda) M_j,$$

(2.6)

$$M_j = C_j^{-1} e^{2\pi i T_j} C_j.$$

In the sequel we shall consider the generic case when none of $t_j$ is integer or half-integer.

The assumption of independence of all monodromy matrices $M_i$ of the positions of the singularities $\lambda_j$: $\partial M_i / \partial \lambda_j = 0$ is called the isomonodromy condition; it implies the following dependence of $\Psi(\lambda)$ on $\lambda_j$

$$\frac{\partial \Psi}{\partial \lambda_j} = - \frac{A_j}{\lambda - \lambda_j} \Psi,$$

(2.7)

as follows from (2.4) and normalization of $\Psi(\lambda)$ at $\infty$. Compatibility of (1.1) and (2.7) then is equivalent to the classical Schlesinger system [1]:

$$\frac{\partial A_j}{\partial \lambda_i} = [A_j, A_i] \frac{1}{\lambda_j - \lambda_i}, \quad i \neq j, \quad \frac{\partial A_j}{\partial \lambda_j} = - \sum_{i \neq j} [A_j, A_i] \frac{1}{\lambda_j - \lambda_i},$$

(2.8)

describing the dependence of the residues $A_j$ on the $\lambda_i$. Obviously, the eigenvalues $t_j$ of the $A_j$ are integrals of motion of the Schlesinger system. The functions $G_j$ have the following dependence on $\lambda_j$:

$$\frac{\partial G_j}{\partial \lambda_i} = A_i G_j \frac{1}{\lambda_i - \lambda_j}, \quad i \neq j, \quad \frac{\partial G_j}{\partial \lambda_j} = - \sum_{i \neq j} A_i G_j \frac{1}{\lambda_i - \lambda_j},$$

(2.9)

which obviously implies (2.8).

To introduce the notion of the $\tau$-function for the Schlesinger system, one notes that (2.8) is a multi-time Hamiltonian system [2] with respect to the Poisson structure on the residues $A_j$

$$\{A_j^\alpha, A_j^\beta\} = \delta_{ij} \varepsilon^{\alpha \beta \gamma} A_j^\gamma,$$

(2.10)

($\alpha, \beta, \gamma$ denoting $\mathfrak{sl}(2)$ algebra indices with the completely antisymmetric structure constants $\varepsilon^{\alpha \beta \gamma}$) and Hamiltonians

$$H_i = \frac{1}{4\pi i} \int_{\lambda_i} \text{tr} A_i^2(\lambda) d\lambda = \frac{1}{2} \sum_{j \neq i} \text{tr} A_i A_j \frac{1}{\lambda_j - \lambda_i}.$$

(2.11)
Explicitly, (2.8) takes the form
\[
\frac{\partial A_j}{\partial \lambda_i} = \{H_i, A_j\},
\] (2.12)
and all the Hamiltonians $H_j$ Poisson-commute.

The $\tau$-function $\tau(\{\lambda_j\})$ of the Schlesinger system then is defined as the generating functions of the Hamiltonians
\[
\frac{\partial \ln \tau}{\partial \lambda_j} = H_j,
\] (2.13)
where compatibility of these equations follows from (2.8). This $\tau$-function is closely related to the Fredholm determinant of a certain integral operator associated to the Riemann-Hilbert problem (see [14] for details).

2.2 Schlesinger transformations on the Riemann sphere

Schlesinger transformations are symmetry transformations of the Schlesinger system (2.8) which map a given solution $\{A_j(\{\lambda_i\})\}$ to another solution $\{\hat{A}_j(\{\lambda_i\})\}$ with the same number and positions of poles $\lambda_j$ such that the related eigenvalues $t_j$ are shifted by integer or half-integer values $t_j \to t_j + n_j/2$, $n_j \in \mathbb{Z}$. The monodromy matrices $M_j$ hence remain invariant or change sign under this transformation. We shall restrict ourselves to elementary Schlesinger transformations, which change only two $t_j$'s, say, $t_k$ and $t_l$ for $k \neq l$ by $\pm 1/2$. The transformed variables will be denoted by $\hat{\Psi}, \hat{A}_j, \hat{t}_j$, etc. Without loss of generality we consider the case
\[
\hat{t}_j = \begin{cases}
t_j + \frac{1}{2} & \text{for } j = k, l \\
t_j & \text{else}
\end{cases}.
\] (2.14)

Our presentation here mainly follows [15]. For the transformed function $\hat{\Psi}$ we make the ansatz
\[
\hat{\Psi}(\lambda) = F(\lambda) \Psi(\lambda),
\] (2.15)
with
\[
F(\lambda) = \sqrt{\frac{\lambda - \lambda_k}{\lambda - \lambda_l}} S_+ + \sqrt{\frac{\lambda - \lambda_l}{\lambda - \lambda_k}} S_-,
\] (2.16)
where the matrices $S_{\pm}$ do not depend on $\lambda$ and are uniquely determined by [15]:
\[
S_{\pm}^2 = S_{\pm}, \quad S_+ + S_- = I, \quad S_+ G_1^1 = S_- G_k^1 = 0.
\] (2.17)

By $G_j^\alpha$ here we denote the $\alpha$-th column of the matrix $G_j$ ($\alpha = 1, 2$). Combining the columns $G_k^1$ and $G_l^1$ into a $2 \times 2$ matrix
\[
G = (G_k^1, G_l^1),
\] (2.18)
we can deduce from (2.17) the following simple formula for $S_{\pm}$:
\[
S_{\pm} = G P_{\pm} G^{-1},
\] (2.19)
with projection matrices
\[
P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]
It is easy to check using the local expansion of $\Psi$ at the singularities $\lambda_j$ (2.4) and the defining relations for $S_\pm$ (2.19) that the transformed function $\hat{\Psi}$ at $\lambda_j$ has a local expansion of the form (2.4) with the same matrices $G_j$ and the desired transformation (2.14) of the $t_j$. The matrices $G_j$ change to new matrices $\hat{G}_j$. Thus, $\hat{\Psi}$ satisfies the system

$$\frac{\partial \hat{\Psi}}{\partial \lambda} = \sum_{j=1}^{N} \frac{\hat{A}_j}{\lambda - \lambda_j} \hat{\Psi}, \quad \frac{\partial \hat{\Psi}}{\partial \lambda_j} = -\frac{\hat{A}_j}{\lambda - \lambda_j} \hat{\Psi},$$

(2.20)

where the functions $\hat{A}_j(\{\lambda_i\})$ build a new solution of the Schlesinger system (2.8).

On the level of the residues $A_j$, the form of the Schlesinger transformation is not very transparent; however, it turns out that the associated $\tau$-function transforms in a rather simple way. Namely, for $\hat{\Psi}$ we find

$$\text{tr} \hat{A}^2 = \text{tr} A^2 + 2 \text{tr} \left[ F^{-1} \frac{dF}{d\lambda} A \right] + \text{tr} \left[ F^{-1} \frac{dF}{d\lambda} \right]^2.$$

(2.21)

For example, the Hamiltonians $H_j$ for $j \neq k, l$ transform as follows:

$$\hat{H}_j - H_j = \left( \frac{1}{\lambda_j - \lambda_k} - \frac{1}{\lambda_j - \lambda_l} \right) \text{tr} [A_j S_+] = \frac{\text{tr} [A_j G P G^{-1}]}{\lambda_j - \lambda_k} + \frac{\text{tr} [A_j G P G^{-1}]}{\lambda_j - \lambda_l}$$

$$= \text{tr} \left[ \frac{\partial G}{\partial \lambda_j} G^{-1} \right]$$

according to (2.9). Hence the transformed $\tau$-function $\hat{\tau}$ is given by $\hat{\tau} = f(\lambda_k, \lambda_l) \det G \cdot \tau$ with some function $f(\lambda_k, \lambda_l)$ to be determined from the transformation of $H_k, H_l$. Taking into account the transformation of Hamiltonians $H_k$ and $H_l$ following from (2.21) we find the following formula describing the action of elementary Schlesinger transformation (2.14) on the $\tau$-function:

$$\hat{\tau}(\{\lambda_j\}) = \left\{ (\lambda_k - \lambda_l)^{-1/2} \det G \right\} \cdot \tau(\{\lambda_j\}).$$

(2.22)

Other elementary Schlesinger transformations like may be obtained in a similar way by building the matrix $G$ from $G_1^k$ and $G_2^l$ instead of (2.18), etc. Moreover, all such transformations with different $k$ and $l$ may be superposed to get the general Schlesinger transformation which simultaneously shifts an arbitrary number of the $t_j$ by some integer or half-integer constants. These general transformations were in detail studied in [3, 4, 5].

### 2.3 Algebro-geometric solutions of Schlesinger system

Let us take $N = 2g + 2$ and introduce the hyperelliptic curve $\mathcal{L}$ of genus $g$ by the equation

$$w^2 = \prod_{j=1}^{2g+2} (\lambda - \lambda_j)$$

(2.23)

with branch cuts $[\lambda_{2j+1}, \lambda_{2j+2}]$. Let us choose the canonical basis of cycles $(a_j, b_j)$, $j = 1, \ldots, g$ such that the cycle $a_j$ encircles the branch cut $[\lambda_{2j+1}, \lambda_{2j+2}]$. Cycle $b_j$ starts
from one bank of branch cut \([\lambda_1, \lambda_2]\), goes to the second sheet through the branch cut \([\lambda_{2j+1}, \lambda_{2j+2}]\), and comes back to another bank of the branch cut \([\lambda_1, \lambda_2]\).

The dual basis of holomorphic 1-forms on \(\mathcal{L}\) are given by \(\lambda_{k-1}d\lambda\), \(k = 1, \ldots, g\).

Let us introduce two \(g \times g\) matrices of \(a\)- and \(b\)-periods of these 1-forms:

\[
A_{kj} = \oint_{a_j} \lambda_{k-1} d\lambda, \quad B_{kj} = \oint_{b_j} \lambda_{k-1} d\lambda. \tag{2.24}
\]

The holomorphic 1-forms

\[
dU_k = \frac{1}{w} \sum_{j=1}^{g} (A^{-1})_{kj} \lambda_{j-1} d\lambda \tag{2.25}
\]

satisfy the normalization conditions \(\oint_{a_j} dU_k = \delta_{jk}\).

The matrices \(A\) and \(B\) define the symmetric \(g \times g\) matrix of \(b\)-periods of the curve \(\mathcal{L}\):

\[
B = A^{-1} B.
\]

Let us cut the curve \(\mathcal{L}\) along all basic cycles to get the fundamental polygon \(\hat{\mathcal{L}}\).

For any meromorphic 1-form \(dW\) on \(\mathcal{L}\) we can define the integral \(\int_{P}^{Q} dW\), where the integration contour lies inside of \(\hat{\mathcal{L}}\) (if \(dW\) is meromorphic, the value of this integral might also depend on the choice of integration contour inside of \(\hat{\mathcal{L}}\)). The vector of Riemann constants corresponding to our choice of the initial point of this map is given by the formula (see [16])

\[
K_j = \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{g} B_{jk}. \tag{2.26}
\]

The characteristic with components \(p \in \mathbb{C}^g/2\mathbb{C}^g\), \(q \in \mathbb{C}^g/2\mathbb{C}^g\) is called half-integer characteristic: the half-integer characteristics are in one-to-one correspondence with the half-periods \(Bp + q\). To any half-integer characteristic we can assign parity which by definition coincides with the parity of the scalar product \(4\langle p, q \rangle\).

The odd characteristics which will be of importance for us in the sequel correspond to any given subset \(S = \{\lambda_{i_1}, \ldots, \lambda_{i_g-1}\}\) of \(g - 1\) arbitrary non-coinciding branch points.

The odd half-period associated to the subset \(S\) is given by

\[
Bp^S + q^S = \sum_{j=1}^{g-1} \int_{\lambda_{i_j}}^{\lambda_{i_j}} dU - K \tag{2.26}
\]

where \(dU = (dU_1, \ldots, dU_g)^t\). Denote by \(\Omega \subset \mathbb{C}\) the neighbourhood of the infinite point \(\lambda = \infty\), such that \(\Omega\) does not overlap with projections of all basic cycles on \(\lambda\)-plane. Let the \(2 \times 2\) matrix-valued function \(\Phi(\lambda)\) be defined in the domain \(\Omega\) of the first sheet of \(\mathcal{L}\) by the following formula,

\[
\Phi(\lambda \in \Omega_\lambda) = \begin{pmatrix}
\varphi(\lambda) & \varphi(\lambda^*) \\
\psi(\lambda) & \psi(\lambda^*)
\end{pmatrix}, \tag{2.27}
\]

where functions \(\varphi\) and \(\psi\) are defined in the fundamental polygon \(\hat{\mathcal{L}}\) by the formulas:

\[
\varphi(\lambda) = \Theta [p_q] \left( \int_{\lambda_1}^{\lambda} dU + \int_{\lambda_1}^{\lambda} dU \bigg| B \right) \Theta [p^S] \left( \int_{\lambda}^{\lambda^*_q} dU \bigg| B \right), \tag{2.28}
\]

\[
\psi(\lambda) = \Theta [p_p] \left( \int_{\lambda_1}^{\lambda} dU + \int_{\lambda_1}^{\lambda} dU \bigg| B \right) \Theta [p^S] \left( \int_{\lambda}^{\lambda^*_p} dU \bigg| B \right), \tag{2.29}
\]
with two arbitrary (possibly \(\{\lambda_j\}\)-dependent) points \(\lambda_\varphi, \lambda_\psi \in \mathcal{L}\) and arbitrary constant complex characteristic \([p^S_q]\); \(*\) is the involution on \(\mathcal{L}\) interchanging the sheets. An odd theta characteristic \([p^S_q]\) corresponds to an arbitrary subset \(S\) of \(g - 1\) branch points via Eq. (2.26).

Since domain \(\Omega\) does not overlap with projections of all basic cycles of \(\mathcal{L}\) on \(\lambda\)-plane, domain \(\Omega^\ast\) does not overlap with the boundary of \(\hat{\mathcal{L}}\), and functions \(\varphi(\lambda^\ast)\) and \(\psi(\lambda^\ast)\) in (2.27) are uniquely defined by (2.28), (2.29) for \(\lambda \in \Omega\).

Now choose some sheet of the universal covering \(X\), define new function \(\Psi(\lambda)\) in subset \(\Omega\) of this sheet by the formula

\[
\Psi(\lambda \in \Omega) = \sqrt{\frac{\det \Phi(\infty) \det \Phi(\lambda)}{\Phi^{-1}(\infty)\Phi(\lambda)}}
\]

and extend on the rest of \(X\) by analytical continuation.

Function \(\Psi(\lambda)\) transforms as follows with respect to the tracing around basic cycles of \(\mathcal{L}\) (by \(T_{a \lambda}\) and \(T_{b \lambda}\) we denote corresponding operators of analytical continuation):

\[
T_{a \lambda} [\Psi(\lambda)] = \Psi(\lambda) e^{2\pi i p \sigma_3}; \quad T_{b \lambda} [\Psi(\lambda)] = \Psi(\lambda) e^{-2\pi i q \sigma_3}
\]

The following statement proved in the paper [6] claims that function \(\Psi\) satisfies condition of isomonodromy, and, therefore, provides a class of solutions of Schlesinger system:

**Theorem 2.1** Let \(p, q \in \mathbb{C}^g\) be an arbitrary set of \(2g\) constants such that characteristic \([p^S_q]\) is not half-integer. Then:

1. Function \(\Psi(Q \in X)\) defined by (2.30) is independent of \(\lambda_\varphi\) and \(\lambda_\psi\), and satisfies the linear system (2.2) with

\[
A_j \equiv \text{res}_{\lambda = \lambda_j} \{ \Psi_\lambda \Psi^{-1}\},
\]

which in turn solve the Schlesinger system (2.8).

2. Monodromies (2.7) of \(\Psi(\lambda)\) around points \(\lambda_j\) are given by

\[
M_j = \begin{pmatrix} 0 & -m_j \\ m_j^{-1} & 0 \end{pmatrix},
\]

where constants \(m_j\) may be expressed in terms of \(p\) and \(q\) as follows:

\[
m_1 = i, \quad m_2 = i \exp\{ -2\pi i \sum_{k=1}^{g} p_k \}
\]

\[
m_{2j+1} = -i \exp\{ 2\pi i q_j - 2\pi i \sum_{k=j}^{g} p_k \}
\]

\[
m_{2j+2} = i \exp\{ 2\pi i q_j - 2\pi i \sum_{k=j+1}^{g} p_k \}
\]

for \(j = 1, \ldots, g\).

3. The \(\tau\)-function, corresponding to solution (2.5) of the Schlesinger system, has the following form:

\[
\tau(\{\lambda_j\}) = [\text{det} \mathcal{A}]^{-\frac{1}{2}} \prod_{j<k} (\lambda_j - \lambda_k)^{-\frac{1}{2}} \Theta [p^S_q] (0|B).
\]
3 Elliptic isomonodromic deformations: Schlesinger transformations and algebro-geometric solutions

3.1 Isomonodromic deformations on the torus

Consider the elliptic curve $E$ with periods 1 and $\mu$ together with the canonical basis of cycles $(a, b)$. A (naive) straightforward generalization of the idea of isomonodromic deformations from the complex plane to the torus $E$ runs into difficulties related to the absence of meromorphic functions on the torus with just one simple pole. An independent variation of the simple poles of a meromorphic connection $A$ on the torus preserving the monodromies around the singularities and basic cycles is impossible for the following simple reason. Existence of such a deformation would imply a version of (2.7) with the function $\frac{A_j}{\lambda - \lambda_j}$ on the r.h.s. being substituted by a meromorphic function with only one simple pole on the torus, which gives rise to the contradiction. Therefore, one of the underlying assumptions has to be relaxed.

E.g. one may consider the case where not all the poles of the connection $A$ are varied independently. Another possibility is the assumption that some of the poles of $A$ are of order higher than one [9]. A third alternative which we shall consider here, is to relax the condition of single-valuedness of the connection $A$ on $E$ and assume that $A$ has “twists” with respect to analytical continuation along the basic cycles $a$ and $b$, i.e.

$$A(\lambda + 1) = QA(\lambda)Q^{-1}, \quad A(\lambda + \mu) = RA(\lambda)R^{-1},$$

where the matrices $Q, R$ do not depend on $\lambda$. By a gauge transformation of the form $A \rightarrow SAS^{-1} + dSS^{-1}$ with $S$ holomorphic but possibly multi-valued, one may bring the connection into a form where $Q = I$ and $R = e^{k\sigma_3}$, where $\sigma_\alpha$ denote the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

The equations of isomonodromic deformations with this choice of the twist were considered in [11] where the multi-valuedness of $A$ had a natural origin in the holomorphic gauge fixing of Chern-Simons theory on the punctured torus. The resulting equations however are rather complicated in comparison with the Schlesinger system on the sphere. This is due to the fact that the twist $\kappa$ itself becomes a dynamical variable – i.e. changes under isomonodromic deformations – and in generic situation has a highly non-trivial $\lambda_j$-dependence. Therefore, instead of being bilinear with respect to the dynamical variables, this Schlesinger system on the torus becomes highly transcendental.

An alternative form of the elliptic Schlesinger system was proposed by Takasaki [12] who considered the restriction $Q = \sigma_3, R = \sigma_1$, related to the classical limit of Etingof’s elliptic version of the Knizhnik-Zamolodchikov-Bernard system on the torus [17]. This choice of fixing the twists turns out to be compatible with the isomonodromic deformations equations, therefore essentially simplifying the dynamics as compared to [11]. It results into studying isomonodromic deformations of the system

$$\frac{d\Psi}{d\lambda} = A(\lambda) \Psi, \quad (3.34)$$

$$A(\lambda) \equiv \sum_{j=1}^{N} \sum_{\alpha=1}^{3} A^\alpha_j w_\alpha(\lambda - \lambda_j)\sigma_\alpha,$$
with $\lambda \in \mathbb{C}$. Functions $w_\alpha$ on the torus are defined in Appendix (see (A.87)). The connection $A(\lambda)$ obviously has only simple poles on $E$ and the following twist properties, cf. (A.88)

$$A(\lambda + 1) = \sigma_3 A(\lambda) \sigma_3, \quad A(\lambda + \mu) = \sigma_1 A(\lambda) \sigma_1. \quad (3.35)$$

Since the residues of all $w_\alpha$ at $\lambda = 0$ coincide, the residue of $A(\lambda)$ at $\lambda_j$ is

$$A_j \equiv \sum_\alpha A_j^\alpha \sigma_\alpha.$$

As in the case of the Riemann sphere, the function $\Psi$ has regular singularities at $\lambda = \lambda_j$ with the same local properties (2.4)–(2.6). The twist properties of $\Psi$ take the form

$$\Psi(\lambda + 1) = \sigma_3 \Psi(\lambda) M_a \quad \Psi(\lambda + \mu) = \sigma_1 \Psi(\lambda) M_b, \quad (3.36)$$

with monodromy matrices $M_a, M_b$ along the basic cycles of the torus. Moreover, as in the case of Riemann sphere, $\Psi(\lambda)$ has monodromies $M_j$ around the singularities $\lambda_j$.

The isomonodromy condition on the torus requires that all monodromies $M_j, M_a$ and $M_b$ are independent of the positions of singularities $\lambda_j$ and the module $\mu$ of the torus. As on the Riemann sphere this implies that the function $\partial \Psi / \partial \lambda_j \Psi^{-1}$ has the only simple pole at $\lambda = \lambda_j$ with residue $-A_j$. In addition, it has the following twist properties

$$\frac{\partial \Psi}{\partial \lambda_j} \Psi^{-1}(\lambda + 1) = \sigma_3 \frac{\partial \Psi}{\partial \lambda_j} \Psi^{-1}(\lambda) \sigma_3,$$

$$\frac{\partial \Psi}{\partial \lambda_j} \Psi^{-1}(\lambda + \mu) = \sigma_1 \left( \frac{\partial \Psi}{\partial \lambda_j} \Psi^{-1}(\lambda) - \frac{\partial \Psi}{\partial \lambda} \Psi^{-1}(\lambda) \right) \sigma_1.$$

Therefore,

$$\frac{\partial \Psi}{\partial \lambda_j} = -\sum_\alpha A_j^\alpha w_\alpha(\lambda - \lambda_j) \sigma_\alpha \Psi. \quad (3.37)$$

To derive the equation with respect to module $\mu$ we observe that $\partial \Psi / \partial \mu \Psi^{-1}$ is holomorphic at $\lambda = \lambda_j$ (but not at $\lambda = \lambda_j + \mu$) and has twist properties

$$\frac{\partial \Psi}{\partial \mu} \Psi^{-1}(\lambda + 1) = \sigma_3 \frac{\partial \Psi}{\partial \mu} \Psi^{-1}(\lambda) \sigma_3,$$

$$\frac{\partial \Psi}{\partial \mu} \Psi^{-1}(\lambda + \mu) = \sigma_1 \left( \frac{\partial \Psi}{\partial \mu} \Psi^{-1}(\lambda) - \frac{\partial \Psi}{\partial \lambda} \Psi^{-1}(\lambda) \right) \sigma_1.$$

Taking into account the periodicity properties of the functions $Z_\alpha$ (A.90), this hence implies

$$\frac{\partial \Psi}{\partial \mu} = \sum_{j=1}^N \sum_{\alpha=1}^3 A_j^\alpha Z_\alpha(\lambda - \lambda_j) \sigma_\alpha \Psi. \quad (3.38)$$

The compatibility conditions of the equations (3.34), (3.37) and (3.38) then yield the $\lambda_i$ and $\mu$ dependence of the residues $A_j$. The result is summarized in the following

**Theorem 3.1** [12] Isomonodromic deformations of the system (3.34) are described by the following elliptic version of the Schlesinger system:

$$\frac{dA_j}{d\lambda_i} = \left[ A_j, \sum_\alpha^3 A_i^\alpha w_\alpha(\lambda_j - \lambda_i) \sigma_\alpha \right], \quad i \neq j, \quad (3.39)$$

9
\[
\begin{align*}
\frac{dA_j}{d\lambda_j} &= -\sum_{i \neq j} \left[ A_j : \sum_{\alpha=1}^{3} A_\alpha^\alpha w_\alpha (\lambda_j - \lambda_i) \sigma_\alpha \right], \\
\frac{dA_j}{d\mu} &= -\sum_{i=1}^{N} \left[ A_j : \sum_{\alpha=1}^{3} A_\alpha^\alpha Z_\alpha (\lambda_j - \lambda_i) \sigma_\alpha \right].
\end{align*}
\]

The corresponding equations for the matrices \(G_j\) from (2.4) take a form analogous to the equations (2.9) on the Riemann sphere:

\[
\begin{align*}
\frac{\partial G_j}{\partial \lambda_i} &= \sum_{\alpha} A_\alpha^\alpha w_\alpha (\lambda_i - \lambda_j) \sigma_\alpha G_j, & \frac{\partial G_j}{\partial \lambda_j} &= -\sum_{i=1}^{N} \sum_{\alpha} A_\alpha^\alpha w_\alpha (\lambda_i - \lambda_j) \sigma_\alpha G_j. \tag{3.40}
\end{align*}
\]

The system (3.39) admits a multi-time Hamiltonian formulation with respect to the Poisson structure (2.10) on the residues

\[
\{A_i^\alpha, A_j^\beta\} = \delta_{ij} \varepsilon^{\alpha\beta\gamma} A_j^\gamma. \tag{3.41}
\]

The Hamiltonians describing deformation with respect to the variables \(\lambda_i\) and to the module \(\mu\) of the torus are respectively given by

\[
H_i = \frac{1}{4\pi i} \oint_{\lambda_i} \text{tr} A^2(\lambda) d\lambda = \sum_{j \neq i} \sum_{\alpha} A_j^\alpha A_\alpha^\alpha w_\alpha (\lambda_j - \lambda_i), \tag{3.42}
\]

\[
H_\mu = -\frac{1}{2\pi i} \oint_{a} \text{tr} A^2(\lambda) d\lambda = -\sum_{i,j} \sum_{\alpha} A_j^\alpha A_i^\alpha Z_\alpha (\lambda_i - \lambda_j). \tag{3.43}
\]

The representation of \(H_\mu\) as contour integral along the basic \(a\)-cycle in (3.43) was derived in [13]. All Hamiltonians Poisson-commute as a direct consequence of (3.41).

The \(\tau\)-function of the elliptic Schlesinger system (3.39) is defined as generating function \(\tau(\{\lambda_j\}, \mu)\) of the Hamiltonians

\[
\frac{\partial \ln \tau}{\partial \lambda_j} = H_j, \quad \frac{\partial \ln \tau}{\partial \mu} = H_\mu; \tag{3.44}
\]

it is uniquely determined up to an arbitrary \((\mu, \{\lambda_j\})\)-independent multiplicative constant. Compatibility of equations (3.44) is a corollary of the elliptic Schlesinger system.

### 3.2 Schlesinger transformations for elliptic isomonodromic deformations

The natural generalization of the notion of Schlesinger transformations on the Riemann sphere to the elliptic case was given in the paper [13]. Starting from any solution of the elliptic Schlesinger system (3.39) with associated function \(\Psi\) satisfying (3.34) and (3.36) we construct a new solution \(\hat{A}_j, \hat{\Psi}\) with eigenvalues \(\hat{t}_j\) which differ from the \(t_j\) by integer or half-integer values. In particular, we will consider the elliptic analog of the elementary Schlesinger transformation (2.14) on the Riemann sphere. The following construction was inspired by the papers [18], [19].
As an elliptic analog of the function \( F(\lambda) \) from (2.16) we shall choose the following ansatz

\[
F(\lambda) = \frac{f(\lambda)}{\sqrt{\det f(\lambda)}},
\]  

(3.45)

\[
f(\lambda) = \frac{1}{2} + \sum_{\alpha=1}^{3} J_{\alpha} w_{\alpha}(\lambda - \frac{1}{2}(\lambda_k + \lambda_l)) \sigma_{\alpha},
\]

(3.46)

where the functions \( J_{\alpha}(\lambda_j, \mu) \) depend on \( G_k \) and \( G_l \) and will be defined below. The elementary elliptic Schlesinger transformation is described by the following

**Theorem 3.2** [13] Let the functions \( \{A_j(\{\lambda_i\})\} \) satisfy the elliptic Schlesinger system (3.39) with twist properties (3.35) and let the function \( \Psi \) satisfy the associated linear system (3.34). For two arbitrary non-coinciding poles \( \lambda_k \) and \( \lambda_l \), define the new function

\[
\hat{\Psi}(\lambda) \equiv F(\lambda) \Psi(\lambda),
\]

(3.47)

where \( F(\lambda) \) is given by formula (3.45) and \( \lambda \)-independent coefficients \( J_{\alpha} \) are defined by

\[
\sum_{\alpha} J_{\alpha} w_{\alpha}\left(\frac{1}{2}(\lambda_k - \lambda_l)\right) \sigma_{\alpha} = -\frac{1}{2} G \sigma_3 G^{-1};
\]

(3.48)

as above we denote by \( G \) the matrix (2.18) containing the first columns of the matrices \( G_k \) and \( G_l \).

Then the function \( \hat{\Psi}(\lambda) \) satisfies the equations (3.34), (3.37), (3.38) and the twist conditions (3.36) with the transformed functions

\[
\hat{A}_j(\{\lambda_i\}) \equiv \text{res}_{\lambda=\lambda_j} \left\{ \frac{d\hat{\Psi}}{d\lambda} \hat{\Psi}^{-1} \right\}.
\]

(3.49)

In turn, the functions \( \hat{A}_j \) satisfy the elliptic Schlesinger system (3.39). For the eigenvalues \( t_j \) we have

\[
\hat{t}_j = \left\{ \begin{array}{ll}
t_j + \frac{1}{2} & \text{for } j = k, l \\
t_j & \text{else}
\end{array} \right.
\]

(3.50)

The monodromy matrices \( \hat{M}_j, \hat{M}_a \) and \( \hat{M}_b \) of the function \( \hat{\Psi} \) coincide with the monodromies of \( \Psi \), except for \( \hat{M}_k = -M_k \) and \( \hat{M}_l = -M_l \).

**Proof.** The proper local behaviour of function \( \hat{\Psi} \) at singularities \( \lambda_j \) is ensured by the relations

\[
S_{\pm}^2 = S_{\pm}, \quad S_+ S_- = I, \quad S_+ G_l^1 = S_- G_k^1 = 0;
\]

(3.51)

for

\[
S_{\pm} \equiv \frac{1}{2} \mp \sum_{\alpha} J_{\alpha} w_{\alpha}\left(\frac{1}{2}(\lambda_k - \lambda_l)\right) \sigma_{\alpha} = G P_\pm G^{-1},
\]

(3.52)

which in complete analogy to (2.17) describe annihilation of the vectors \( G_k^1 \) and \( G_l^1 \) by the matrices \( f(\lambda_k) \) and \( f(\lambda_l) \), respectively. Obviously, equations (3.49) are a consequence of (3.47). Similarly to the case of the sphere, it is then easy to verify that (3.49) provide the required asymptotical expansions (2.4) for the function \( \hat{\Psi} \) with parameters \( \hat{G}_j, C_j \) and \( \hat{t}_j \).
Concerning the global behavior of $\hat{\Psi}$ we note that the prefactor $(\text{det } f(\lambda))^{-1/2}$ in (3.45) provides the condition $\text{det } \hat{\Psi} = 1$ and kills the simple pole of $f(\lambda)$ at $\lambda = (\lambda_k + \lambda_l)/2$. Therefore, the only singularities of $F(\lambda)$ on $E$ are the zeros of $\text{det } f(\lambda)$. Since $\text{det } f(\lambda)$ has only one pole – this is the second order pole at $\lambda = (\lambda_k + \lambda_l)/2$ – it must have also two zeros on $E$ whose sum according to Abel’s theorem equals $\lambda_k + \lambda_l$. According to (3.49) these are precisely $\lambda_k$ and $\lambda_l$. It remains to check that $\hat{\Psi}$ satisfies conditions (3.36) with the same matrices $M_a$ and $M_b$. This follows from the twist properties

$$f(\lambda + 1) = \sigma_3 f(\lambda) \sigma_3, \quad f(\lambda + \mu) = \sigma_1 f(\lambda) \sigma_1,$$

which in turn follow from (3.45) and the periodicity properties (A.88) of the functions $w_j(\lambda)$.

As a result of rather long calculations one can prove the elliptic analog of formula (2.22) describing the transformation of the $\tau$-function under the action of elliptic Schlesinger transformations.

**Theorem 3.3** \[13\] The $\tau$-function $\hat{\tau}$ corresponding to the Schlesinger-transformed solution $\hat{A}_j$ (3.48) of the elliptic Schlesinger system is related to the $\tau$-function corresponding to the solution $A_j$ as follows

$$\hat{\tau}(\{\lambda_j\}, \mu) = \left\{ \left[ w_1 w_2 w_3 \left( \frac{\lambda_k - \lambda_l}{2} \right) \right]^{1/2} \text{det } \left[ GJ^{1/2} \right] \right\} \cdot \tau(\{\lambda_j\}, \mu), \tag{3.50}$$

where $G$ is the matrix (2.18) containing the first columns of the matrices $G_k, G_l$,

$$J \equiv \sum_{\alpha=1}^{3} J_A \sigma_A$$

and the functions $J_\alpha$ are defined in terms of $G$ via (3.47).

The natural open problem arising here is to construct elliptic generalizations of integrable chains associated to ordinary Schlesinger system \[13\].

In the next section we shall present the extension of construction of algebro-geometric solutions of Schlesinger system to the case of elliptic isomonodromic deformations.

### 3.3 Algebro-geometric solutions of elliptic Schlesinger system

To construct theta-functional solutions of elliptic Schlesinger system (3.33) let us assume that $N = 2g$ and introduce two-sheet covering $\mathcal{L}$ of torus $E$ with branch points $\lambda_1, \ldots, \lambda_{2g}$. Genus of $\mathcal{L}$ equals $g + 1$. Denote by * the involution of $\mathcal{L}$ interchanging the sheets of the covering. Let us choose the canonical basis of cycles on $\mathcal{L}$ in such a way (see figure 6.2 on p.215 of \[22\]) that

$$a_1^* = -a_{g+1}, \quad b_1^* = -b_{g+1}$$

$$a_j^* = -a_j, \quad b_j^* = -b_j, \quad j = 2, \ldots, g.$$ 

The basic holomorphic differentials $dU_1, \ldots, dU_{g+1}$ on $\mathcal{L}$ normalized by

$$\int_{a_j} dU_k = \delta_{jk}, \quad j, k = 1, \ldots, g + 1$$
transform as follows under the action of involution $*$:

$$dU_1(P^*) = -dU_{g+1}(P) \quad dU_j(P^*) = -dU_j(P), \quad j = 2, \ldots, g$$  \hfill (3.51)

Let us introduce the following Prym differentials $dV_j$, $j = 1, \ldots, g$:

$$dV_1 = \frac{1}{2}(dU_1 + dU_{g+1}) \quad dV_j = dU_j \quad j = 2, \ldots, g$$  \hfill (3.52)

and symmetric $g \times g$ matrix of their $b$-periods:

$$\Pi_{jk} = \oint_{b_j} dV_k$$  \hfill (3.53)

which has positively-defined imaginary part.

**Remark 3.1** Differentials $dV_j$ and matrix $\Pi$ were first introduced by Bobenko [22] in the studies of classical tops admitting elliptic Lax representation. These objects are related to standard Prym differentials $dW_j$ and standard Prym matrix $\Pi^{\text{Prym}}$ as follows:

$$dW_1 = 2dV_1, \quad dW_j = dV_j, \quad j = 2, \ldots, g; \quad \Pi = 2S\Pi^{\text{Prym}}S$$

where $S$ is diagonal matrix $S = \text{diag}(\frac{1}{2}, 1, \ldots, 1)$.

Denote by $\hat{\mathcal{L}}$ the universal covering of curve $\mathcal{L}$.

**Theorem 3.4** Define $2 \times 2$ matrix-valued function $\Phi(P)$ on $\hat{\mathcal{L}}$ by the formulas

$$\Phi(P) = \Phi(P) = \begin{pmatrix} \varphi(P) & \varphi(P) \\ \psi(P) & \psi(P) \end{pmatrix},$$

where

$$\varphi(P) = \Theta\left[ \begin{pmatrix} p \\ q \end{pmatrix} \right] \left( \int_{\lambda_1}^{\lambda_1} dV \right), \quad \psi(P) = \Theta\left[ \begin{pmatrix} p \\ q + \frac{1}{2}e_1 \end{pmatrix} \right] \left( \int_{\lambda_1}^{\lambda_1} dV \right);$$

$p, q \in \mathbb{C}$ are arbitrary constant vectors such that $p_1 = 0$. Then the function $\Phi$ is holomorphic and invertible on $\mathcal{L}$ outside of branch points $\lambda_j$ and transforms as follows with respect to analytical continuation along the basic cycles of $\mathcal{L}$:

$$T_{a_1}[\Phi(P)] = \sigma_1 \Phi(P)$$

$$T_{b_1}[\Phi(P)] = \sigma_3 \Phi(P) e^{-2\pi iq_1} e^{-\pi i\Pi_{11}} e^{-2\pi i \int_{\lambda_1}^{\lambda_1} dV_i}$$

$$T_{a_j}[\Phi(P)] = \Phi(P) e^{2\pi iq_j}$$

$$T_{b_j}[\Phi(P)] = \Phi(P) e^{-2\pi iq_j} e^{-\pi i\Pi_{jj}} e^{-2\pi i \int_{\lambda_1}^{\lambda_1} dV_j}$$

for $j = 2, \ldots, g$.

**Proof.** Taking into account the definition of Prym differentials $dV_j$ (3.52) we see that

$$T_{a_1}\left[ \int_{\lambda_1}^{\lambda_1} dV \right] = \int_{\lambda_1}^{\lambda_1} dV + \frac{e_1}{2}$$

$$T_{a_j}\left[ \int_{\lambda_1}^{\lambda_1} dV \right] = \int_{\lambda_1}^{\lambda_1} dV + e_1, \quad j = 2, \ldots, g$$

13
\[ T_{b_j} \left[ \int_{\lambda_1}^P dV \right] = \int_{\lambda_1}^P dV + \Pi e_j, \quad j = 1, \ldots, g \]

Substituting these expressions into the formulas for \( \varphi \) and \( \psi \) and taking into account behaviour of 1-forms \( dU_j \) under the action of involution \( \ast \), we derive the following transformation properties of functions \( \varphi \) and \( \psi \):

\[
T_{a_1}[\varphi(P)] = \psi(P) \quad T_{a_1}[\psi(P)] = \varphi(P) \tag{3.60}
\]

\[
T_{a_1}[\varphi(P^*)] = \psi(P^*) \quad T_{a_1}[\psi(P^*)] = \varphi(P^*) \tag{3.61}
\]

\[
T_{b_1}^\ast [\varphi(P)] = e^{-2\pi i q_1} e^{-\pi i \Pi j_1 - 2\pi i \int_{\lambda_1}^P \psi(P)} \tag{3.62}
\]

\[
T_{b_1}^\ast [\psi(P)] = e^{-2\pi i q_1} e^{-\pi i \Pi j_1 - 2\pi i \int_{\lambda_1}^P \psi(P)} \tag{3.63}
\]

\[
T_{b_1}^\ast [\varphi(P^*)] = e^{2\pi i q_1} e^{-\pi i \Pi j_1 - 2\pi i \int_{\lambda_1}^P \psi(P)} \tag{3.64}
\]

\[
T_{b_1}^\ast [\psi(P^*)] = e^{2\pi i q_1} e^{-\pi i \Pi j_1 - 2\pi i \int_{\lambda_1}^P \psi(P)} \tag{3.65}
\]

and

\[
T_{a_j}[\varphi(P)] = e^{2\pi i q_j} \varphi(P) \tag{3.66}
\]

\[
T_{a_j}[\varphi(P^*)] = e^{-2\pi i q_j} \varphi(P^*) \tag{3.67}
\]

\[
T_{b_1}^\ast [\varphi(P)] = e^{-2\pi i q_1} e^{-\pi i \Pi j_1 - 2\pi i \int_{\lambda_1}^P \psi(P)} \tag{3.68}
\]

\[
T_{b_1}^\ast [\varphi(P^*)] = e^{2\pi i q_1} e^{-\pi i \Pi j_1 - 2\pi i \int_{\lambda_1}^P \psi(P)} \tag{3.69}
\]

Transformation laws of \( \psi \) along cycles \( a_j, b_j \) for \( j > 1 \) coincide with transformation laws of \( \varphi \).

Combining the above relations into matrix form, we come to the transformation laws (3.56) - (3.59).

It remains to verify non-degeneracy of \( \Phi(P) \) outside of singularities \( \lambda_j \). We know that \( \det \Phi(P) \) has at least simple zeros at the points \( \lambda_j \) (at these points the columns of \( \Phi(P) \) are proportional to each other). To check that \( \det \Phi(P) \) does not vanish on \( \hat{\mathcal{L}} \) outside of \( \lambda_j \) let us first observe that

\[
T_{a_1}[\det \Phi(P)] = - \det \Phi(P) \quad T_{a_j}[\det \Phi(P)] = \det \Phi(P), \quad j = 2, \ldots, g, \tag{3.70}
\]

\[
T_{b_j}[\det \Phi(P)] = e^{-2\pi i \Pi j_1} e^{-4\pi i \int_{\lambda_1}^P \psi(P)} \det \Phi(P) \quad j = 1, \ldots, g. \tag{3.71}
\]

Now let us calculate the integral

\[
\oint_{\partial \hat{\mathcal{L}}} d \ln \det \Phi(P) \tag{3.72}
\]

\[
= \oint_{a_1} d \{ \ln \det \Phi(P) - \ln \det T_{b_1}[\Phi(P)] \} + \oint_{a_{n+1}} d \{ \ln \det \Phi(P) - \ln \det T_{b_{n+1}}[\Phi(P)] \}
\]

\[
+ \sum_{j=2}^g \oint_{a_j} d \{ \ln \det \Phi(P) - \ln \det T_{b_1}[\Phi(P)] \}. \tag{3.73}
\]

Taking into account (3.70) and (3.71) as well as normalization of the basic integrals \( dU_j \) we see that the first two terms of the r.h.s. of this expression equal \( 2\pi i \), whereas each
term in the sum equals $4\pi i$. Altogether, we get $4\pi i g$, and, therefore, $\det \Phi(P)$ has in $\tilde{\mathcal{L}}$ exactly $2g$ zeros which coincide with $\lambda_j$.

Let us also choose some domain $\Omega \subset E$ which does not overlap with projections of all basic cycles on $E$. Then domain $\Omega^*$ does not overlap with the boundary of $\tilde{\mathcal{L}}$ and functions $\varphi^*(P)$ and $\psi^*(P)$ are uniquely defined in $\tilde{\mathcal{L}}$ by (3.55). Let us now choose some sheet of universal covering $X$ of torus $E$ with punctures $\{\lambda_1, \ldots, \lambda_{2g}\}$, and define new function $\Psi(\lambda)$ in subset $\Omega$ of this sheet by the formula

$$
\Psi(\lambda \in \Omega) = \frac{1}{\sqrt{\det \Phi(\lambda) \Phi(\lambda)}}.
$$

(3.73)

Then we extend function $\Psi(\lambda)$ on the rest of $X$ by analytical continuation.

The following theorem shows that function $\Psi$ satisfies conditions of isomonodromy, and, therefore, generates a class of solutions of elliptic Schlesinger system (3.39):

**Theorem 3.5** Function $\Psi(\lambda \in X)$ defined by formulas (3.54), (3.55) and (3.73) is holomorphic and invertible on $X$ outside of the points $\lambda_j$, $j = 1, \ldots, 2g$. Moreover, it transforms as follows with respect to analytical continuation along basic cycles of $E$:

$$
T_a[\Psi(\lambda)] = i\sigma_1 \Psi(\lambda) \quad \text{and} \quad T_b[\Psi(\lambda)] = i\sigma_3 \Psi(\lambda) e^{-2\pi i q_1 \sigma_3}
$$

(3.74)

and around closed cycles surrounding points $\lambda_j$:

$$
T_{\lambda_j}[\Psi(\lambda)] = \Psi(\lambda) M_j
$$

(3.75)

where $T_a$, $T_b$ and $T_{\lambda_j}$ denote corresponding operators of analytical continuation;

$$
M_j = \begin{pmatrix}
0 & -m_j \\
m_j & 0
\end{pmatrix},
$$

(3.76)

and

$$
m_1 = i \quad m_2 = -i \exp\{-2\pi i \sum_{j=2}^{g} p_j\}
$$

(3.77)

$$
m_{2l} = -i \exp\{2\pi i q_l - 2\pi i \sum_{k=l+1}^{g} p_k\}
$$

(3.78)

$$
m_{2l-1} = i \exp\{2\pi i q_l - 2\pi i \sum_{k=l}^{g} p_k\}
$$

(3.79)

for $l = 2, \ldots, g$.

**Proof.** Holomorphy and invertibility of function $\Psi$ follows from the same statements concerning function $\Phi$ (3.54). Relations (3.74) directly follow from (3.56), (3.57). To calculate $m_j$ let us observe that, according to (3.56) - (3.59), monodromies of $\Psi$ are related to constants $p$ and $q$ as follows:

$$
M_{2j} M_{2j-1} = e^{2\pi i p_j \sigma_3}, \quad j = 2, \ldots, g
$$

(3.80)

$$
M_{2j-1} M_{2j-2} = e^{2\pi i (q_j - q_{j-1}) \sigma_3}, \quad j = 3, \ldots, g
$$

(3.81)
Moreover, we order monodromies is such a way that
\[ M_a M_b M_a^{-1} M_b^{-1} M_{2g} \ldots M_1 = I \]  
(since \( M_a = I \) first four factors in this relation drop out). Monodromy \( M_1 \) corresponding to our choice of basic cycles equals \( \iota \sigma_1 \). Altogether all these relations lead to (3.76), (3.77) - (3.79) after elementary calculations.

**Corollary 3.1 Residues**

\[ A_j(\{\lambda_j\}, \mu) \equiv \text{res}_{\lambda=\lambda_j} \Psi_{\lambda} \Psi^{-1} \]  
satisfy elliptic Schlesinger system (3.39).

### 4 Outlook

Let us mention several applications of the mathematical results described above. Recently [23] it was established close relationship between Schlesi nger system and Ernst equation of general relativity, which allows to apply to the Ernst equation all results of sect.2. In particular, one can get in this way a class of algebrogeometric solutions of Ernst equation [24], which turns out to coincide with the class of algebrogeometric solutions of Ernst equation known since 1988 [25]. It is rather satisfactory that certain subclass of genus 2 algebrogeometric solutions of Ernst equation recently found realistic physical application in the problem of description of different kinds of dust discs [26, 27]. Another application of construction of sect.2 is the theory of \( SU(2) \)-invariant gravitational instantons [28] where it allows to considerably simplify the results of Hitchin [29].

So far we don’t know about physical applications of elliptic version of Schlesinger system, and all results of sect.3 have at the moment pure mathematical significance; however we strongly believe that such applications will be found in the near future.

### A Some elliptic functions

The elliptic theta-function with characteristic \([p, q] \ (p, q \in \mathbb{C})\) on a torus \(E\) is defined by the series
\[
\vartheta[p, q](\lambda|\mu) = \sum_{m \in \mathbb{Z}} e^{\pi i (m+p)^2 + 2\pi i (m+p)(\lambda+q)}.
\]  
(A.85)

Let us introduce on the torus \(E\) the standard Jacobi theta-functions:
\[
\begin{align*}
\vartheta_1(\lambda) & \equiv -\vartheta \left[ \frac{1}{2}, 0 \right] (\lambda|\mu), \\
\vartheta_2(\lambda) & \equiv \vartheta \left[ \frac{1}{2}, 0 \right] (\lambda|\mu), \\
\vartheta_3(\lambda) & \equiv \vartheta(\lambda) \equiv \vartheta[0, 0](\lambda|\mu), \\
\vartheta_4(\lambda) & \equiv \vartheta \left[ 0, \frac{1}{2} \right] (\lambda|\mu),
\end{align*}
\]  
(A.86)

and corresponding theta-constants
\[
\vartheta_j \equiv \vartheta_j(0), \quad j = 2, 3, 4.
\]
We define the following three combinations of Jacobi theta-functions:

\[ w_1(\lambda) = \pi \vartheta_2 \vartheta_3 \vartheta_4(\lambda), \quad w_2(\lambda) = \pi \vartheta_2 \frac{\vartheta_3(\lambda)}{\vartheta_1(\lambda)}, \quad w_3(\lambda) = \pi \vartheta_3 \frac{\vartheta_2(\lambda)}{\vartheta_1(\lambda)}. \quad (A.87) \]

All these functions have simple poles at \( \lambda = 0 \) with residue 1. Moreover, they possess the following periodicity properties:

\[ w_1(\lambda + 1) = -w_1(\lambda) \quad w_1(\lambda + \mu) = w_1(\lambda), \quad (A.88) \]
\[ w_2(\lambda + 1) = -w_2(\lambda) \quad w_2(\lambda + \mu) = -w_2(\lambda), \quad (A.88) \]
\[ w_3(\lambda + 1) = w_3(\lambda) \quad w_3(\lambda + \mu) = -w_3(\lambda). \quad (A.88) \]

Let us also define the following functions \( Z_\alpha \):

\[ Z_1 = \frac{w_1}{2\pi i \vartheta_4(\lambda)}, \quad Z_2 = \frac{w_2}{2\pi i \vartheta_3(\lambda)}, \quad Z_3 = \frac{w_3}{2\pi i \vartheta_2(\lambda)}. \quad (A.89) \]

which have the periodicity properties:

\[ Z_1(\lambda + 1) = -Z_1(\lambda) \quad Z_1(\lambda + \mu) = Z_1(\lambda) - w_1, \quad (A.90) \]
\[ Z_2(\lambda + 1) = -Z_2(\lambda) \quad Z_2(\lambda + \mu) = -Z_2(\lambda) + w_2, \quad (A.90) \]
\[ Z_3(\lambda + 1) = Z_3(\lambda) \quad Z_3(\lambda + \mu) = -Z_3(\lambda) + w_3. \quad (A.90) \]

It is easy to verify the identity

\[ \frac{dw_\alpha}{d\mu}(\lambda) = \frac{dZ_\alpha}{d\lambda}(\lambda), \quad (A.91) \]

which follows from analyticity and twist properties of both sides.

Notice also that functions \( w_\alpha \) may be represented as ratios of Jacobi’s elliptic functions as follows:

\[ w_1(\lambda) = \frac{1}{\text{sn}(\lambda)}, \quad w_2(\lambda) = \frac{\text{dn}(\lambda)}{\text{sn}(\lambda)}, \quad w_3(\lambda) = \frac{\text{cn}(\lambda)}{\text{sn}(\lambda)}. \quad (A.92) \]

In calculation of transformation of elliptic \( \tau \)-function under the action of elliptic Schlesinger transformations one has to use also the summation theorem and some integral relations for functions \( w_\alpha \).

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