AROUND THE HOSSZÚ-GLUSKIN THEOREM FOR $n$-ARY GROUPS

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Dedicated to Bolesław Gleichgewicht on his 85th birthday

Abstract. We survey results related to the important Hosszú-Gluskin Theorem on $n$-ary groups adding also several new results and comments. The aim of this paper is to write all such results in uniform and compressive forms. Therefore some proofs of new results are only sketched or omitted if their completing seems to be not too difficult for readers. In particular, we show as the Hosszú-Gluskin Theorem can be used for evaluation how many different $n$-ary groups (up to isomorphism) exist on some small sets. Moreover, we sketch as the mentioned theorem can be also used for investigation of $Q$-independent subsets of semiabelian $n$-ary groups for some special families $Q$ of mappings. Such investigations will be continued.

1. Introduction

The non-empty set $G$ together with an $n$-ary operation $f : G^n \to G$ is called an $n$-ary groupoid (or an $n$-ary operative – in the Gluskin terminology, cf. [37]) and is denoted by $(G; f)$. We will assume that $n > 2$.

According to the general convention similar to that introduced in the theory of $n$-ary systems by G. Čupona (cf. [7]) the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by $x^i_j$. In the case $j < i$ it is the empty symbol. If $x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x$, then instead of $x^i_{i+t}$ we write $(^t x)$. In this convention $f(x_1, \ldots, x_n) = f(x^n_1)$ and

$$f(x_1, \ldots, x_i, x, \ldots, x_{x_{i+t+1}}, \ldots, x_n) = f(x^i_1, ^t x, x^n_{i+t+1}).$$

If $m = k(n - 1) + 1$, then the $m$-ary operation $g$ of the form

$$g(x_1^{k(n-1)+1}) := f(f(\ldots, f(f(x_1^n), x_2^{2n-1}), \ldots), x_{(k-1)(n-1)+2})$$

is denoted by $f(k)$ and is called the simple iteration of the operation $f$ (cf. [52]) or an $m$-ary operation derived from $f$. In certain situations, when the arity of $g$ does not play a crucial role or when it will differ depending on additional assumptions, we write $f_{(i)}$ to mean $f(k)$ for some $k = 1, 2, \ldots$.

An $n$-ary groupoid $(G; f)$ is called $(i, j)$-associative if

$$f(x_1^{i-1}, f(x_i^{n+i-1}, x_{n+i}^{2n-1})) = f(x_1^{i-1}, f(x_j^{n+j-1}, x_{n+j}^{2n-1}))$$

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holds for all $x_1, \ldots, x_{2n-1} \in G$. If this identity holds for all $1 \leq i < j \leq n$, then we say that the operation $f$ is associative and $(G; f)$ is called an $n$-ary semigroup (or, in the Gluskin’s terminology, an $n$-ary associative). It is clear that an $n$-ary groupoid is associative iff it is $(1, j)$-associative for all $j = 2, \ldots, n$. In the binary case (i.e. for $n = 2$) it is a usual semigroup.

If for all $x_0, x_1, \ldots, x_n \in G$ and fixed $i \in \{1, \ldots, n\}$ there exists an element $z \in G$ such that

$$f(x_1^{i-1}, z, x_{i+1}^n) = x_0,$$

then we say that this equation is $i$-solvable or solvable at the place $i$. If this solution is unique, then we say that (2) is uniquely $i$-solvable.

An $n$-ary groupoid $(G; f)$ uniquely solvable for all $i = 1, \ldots, n$ is called an $n$-ary quasigroup. An associative $n$-ary quasigroup is called an $n$-ary group. For an $n$-ary quasigroup with the non-empty center to be an $n$-ary group it is sufficient to postulate the $(i, j)$-associativity for some fixed $1 \leq i < j \leq n$ (cf. [11]). It is clear that for $n = 2$ we obtain a usual group.

Note by the way that in many papers $n$-ary semigroups ($n$-ary groups) are called $n$-semigroups ($n$-groups, respectively). Moreover, in many papers, where the arity of the basic operation does not play a crucial role, we can find the term a polyadic semigroup (polyadic group) (cf. [39], [60], [66]).

Now such and similar $n$-ary systems have many applications in different branches. For example, in the theory of automata [39] $n$-ary semigroups and $n$-ary groups are used, some $n$-ary groupoids are applied in the theory of quantum groups [55]. Different applications of ternary structures in physics are described by R. Kerner in [43]. In physics there are used also such structures as $n$-ary Filippov algebras (see [56]) and $n$-Lie algebras (see [73]). Some $n$-ary structures induced by hypercubes have application in error-correcting and error-detecting coding theory, cryptology, as well as in the theory of $(t, m, s)$-nets (see for example [11] and [45]).

The idea of investigations of such groups seems to be going back to E. Kasner’s lecture [12] at the fifty-third annual meeting of the American Association for the Advancement of Science in 1904. But the first paper concerning the theory of $n$-ary groups was written (under inspiration of Emmy Noether) by W. Dörnte in 1928 (see [8]). In this paper Dörnte observed that any $n$-ary groupoid $(G; f)$ of the form $f(x_1^n) = x_1 \circ x_2 \circ \ldots \circ x_n$, where $(G; \circ)$ is a group, is an $n$-ary group but for every $n > 2$ there are $n$-ary groups which are not of this form. $n$-ary groups of the first form are called reducible to the group $(G; \circ)$ or derived from the group $(G; \circ)$, the second one are called irreducible. Moreover, in some $n$-ary groups there exists an element $e$ (called an $n$-ary identity or neutral element) such that

$$f(e^{(i-1)}, x, e^{(n-i)}) = x,$$

holds for all $x \in G$ and for all $i = 1, \ldots, n$. It is interesting that $n$-ary groups containing a neutral element are reducible (cf. [8]). Irreducible $n$-ary groups do not contain such elements. On the other hand, there are $n$-ary groups with two, three and more neutral elements. The set $\mathbb{Z}_{n-1} = \{0, 1, \ldots, n - 2\}$ with the operation $f(x_1^n) = (x_1 + x_2 + \ldots + x_n) \mod (n - 1))$ is a simple example of an $n$-ary group in which every element is neutral. All $n$-ary groups with this property are derived from the commutative group of the exponent $k/(n - 1)$.
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It is worthwhile to note that in the definition of an \(n\)-ary group, under the assumption of the associativity of \(f\), it suffices only to postulate the existence of a solution of (2) at the places \(i = 1\) and \(i = n\) or at one place \(i\) other than 1 and \(n\). Then one can prove the uniqueness of the solution of (2) for all \(i = 1, \ldots, n\) (cf. [60], p. 213[17]).

The above definition of \(n\)-ary groups is a generalization of the Weber’s and Huntington formulation of axioms of a group as a semigroup in which the equations \(xa = b, ya = b\) have solutions. Many authors used the notion of \(n\)-ary groups as a generalization of Pierpont’s definition of groups as a semigroup with neutral and inverse elements. Unfortunately, in this case we obtain only \(n\)-ary groups derived from groups.

E.I. Sokolov proved in [67] that in the case of \(n\)-ary quasigroups (i.e. in the case of the existence of a unique solution of (2) at any place \(i = 1, \ldots, n\)) it is sufficient to postulate the \((j, j + 1)\)-associativity for some fixed \(j = 1, \ldots, n - 1\).

Using the same method as Sokolov we can prove the following proposition (for details see [15]):

**Proposition 1.** An \(n\)-ary groupoid \((G; f)\) is an \(n\)-ary group if and only if (at least) one of the following conditions is satisfied:

(a) the \((1, 2)\)-associative law holds and the equation (2) is solvable for \(i = n\) and uniquely solvable for \(i = 1\),

(b) the \((n - 1, n)\)-associative law holds and the equation (2) is solvable for \(i = 1\) and uniquely solvable for \(i = n\),

(c) the \((i, i + 1)\)-associative law holds for some \(i \in \{2, \ldots, n - 2\}\) and the equation (2) is uniquely solvable for \(i\) and some \(j > i\).

In [16] (see also [5]) the following characterization of \(n\)-ary groups is given:

**Proposition 2.** An \(n\)-ary semigroup \((G; f)\) is an \(n\)-ary group if and only if for some \(k \in \{1, 2, \ldots, n - 2\}\) and all \(a^*_1 \in G\) there are elements \(x_{k+1}^{n-1}, y_{k+1}^{n-1} \in G\) such that

\[
    f (a^*_1, x_{k+1}^{n-1}, b) = f (b, y_{k+1}^{n-1}, a^*_1) = b
\]

for all \(b \in G\).

**Proposition 3.** An \(n\)-ary semigroup \((G; f)\) is an \(n\)-ary group if and only if for some \(i, j \in \{1, 2, \ldots, n - 1\}\) and all \(a, b \in G\) there are \(x, y \in G\) such that

\[
    f (x, (i-1) \cdot (n-i), y) = f (x, (j-1) \cdot (n-j), y) = b. \tag{5}
\]

Putting in the above proposition \(i = j = 1\) we obtain the following main result of [71].

**Corollary 4.** An \(n\)-ary semigroup \((G; f)\) is an \(n\)-ary group if and only if for all \(a, b \in G\) there are \(x, y \in G\) such that

\[
    f (x, (n-1) \cdot a) = f (x, (n-1) \cdot y) = b. \tag{6}
\]
From the definition of an $n$-ary group $(G; f)$ we can directly see that for every $x \in G$ there exists only one $z \in G$ satisfying the equation

\[ f\left(\frac{(n-1)}{x}, z\right) = x. \]

(6)

This element is called skew to $x$ and is denoted by $\overline{x}$. In a ternary group ($n = 3$) derived from the binary group $(G; \circ)$ the skew element coincides with the inverse element in $(G; \circ)$. Thus, in some sense, the skew element is a generalization of the inverse element in binary groups. This suggests that for $n \geq 3$ any $n$-ary group $(G; f)$ can be considered as an algebra $(G; f, -)$ with two operations: one $n$-ary $f : G^n \to G$ and one unary $\overline{\cdot} : x \to \overline{x}$. Dörnte proved (see [8]) that in ternary groups for all $x \in G$ we have $\overline{\overline{x}} = x$, but for $n > 3$ this is not true. For $n > 3$ there are $n$-ary groups in which one fixed element is skew to all elements (cf. [12]) and $n$-ary groups in which any element is skew to itself. Then, in the second case, of course the $n$-ary group operation $f$ is idempotent. An $n$-ary group in which $f\left(\frac{(n)}{x}\right) = x$ for every $x \in G$ is called idempotent.

Nevertheless, the concept of skew elements plays a crucial role in the theory of $n$-ary groups. Namely, as Dörnte proved, the following theorem is true.

**Theorem 5.** In any $n$-ary group $(G; f)$ the following identities:

\[ f\left(\frac{(i-2)}{x}, \overline{x}, \frac{(n-1)}{x}, y\right) = y, \]

(7)

\[ f\left(\frac{n-j}{x}, \overline{x}, \frac{j-2}{x}\right) = y, \]

(8)

\[ f\left(\frac{(k-1)}{x}, \overline{x}, \frac{(n-k)}{x}\right) = x \]

(9)

hold for all $x, y \in G$, $2 \leq i, j \leq n$ and $1 \leq k \leq n$.

The first two identities, called now Dörnte’s identities, are used by many authors to describe the class of $n$-ary groups. For example, in 1967 B. Gleichgewicht and K. Glazek proved in [9] (see also [53]) that for fixed $n \geq 3$ the class of all $n$-ary groups, considered as algebras of type $(n, 1)$, forms a Mal’cev variety and found the system of identities defining this variety. This means that all congruences of a given $n$-ary group commute and that the lattice of all congruences of a fixed $n$-ary group is modular. But, as was observed many years later, from the theorem on page 448 in Gluskin’s paper [37] it follows that the system of identities given by B. Gleichgewicht and K. Glazek is not independent. For similar axiom considerations, see also [3], [51] and [62] (for other systems of axioms, see, e.g., [56]). The first independent system of identities defining this variety was given in our paper [15]. Now we give the minimal system of such identities. This is the main result of [9].

**Theorem 6.** The class of $n$-ary groups coincides with the variety of $n$-ary groupoids $(G; f, -)$ with a unary operation $\overline{-} : x \to \overline{x}$ satisfying for some fixed $i, j \in \{2, \ldots, n\}$ the Dörnte identities (7), (8) and the identity

\[ f(f(x_1^i), x_{n+1}^{2n-1}) = f(x_1, f(x_2^{n+1}, x_{n+2}^{2n-1})). \]

Theorem 6 gives the minimal system of identities defining $n$-ary groups. In fact, for $n > 3$ the set $Z$ of all integers with the operation $f(x_1^i) = x_{n-1}^i + x_n$ is an example of a $(1, 2)$-associative $n$-ary groupoid in which (7) holds for $\overline{\overline{x}} = 0$ but (8) is not satisfied. Similarly, $(Z; f)$ with $f(x_1^i) = x_1$ is an example of a $(1, 2)$-associative
n-ary groupoid satisfying (5) but not (7). It is clear that the (1, 2)-associativity cannot be deleted.

Note by the way that in some papers there are investigated so-called infinitary semigroups and quasigroups, i.e. groupoids \((G; f)\), where the number of variables in the operation \(f : G^\infty \to G\) is infinite, but countable. Infinitary semigroups are the infinitary groupoids \((G; f)\), where for all natural \(i, j\) the operation \(f\) satisfies the identity
\[
f(x_1^{-1}, f(x_2^{-1}, y)) = f(x_1^{-1}, f(x_2^{-1}, y)).
\]
Infinitary quasigroups are infinitary groupoids \((G; f)\) in which the equation \(f(x_1^{k-1}, z_k, x_1^{\infty}) = x_0\) has a unique solution \(z_k\) at any place \(k\).

From the general results obtained in \([8]\) and \([10]\) one can deduce that infinitary groups have only one element. Below we present a simple proof of this fact.

If \((G; f)\) is an infinitary group, then, according to the definition, for any \(y, z \in G\) and \(u = f(\overrightarrow{y})\) there exists \(x \in G\) such that \(z = f(u, y, x, \overrightarrow{y})\). Thus
\[
f(z, \overrightarrow{y}) = f(f(u, y, x, \overrightarrow{y}, \overrightarrow{y}), \overrightarrow{y}) = f(u, y, f(x, \overrightarrow{y}, \overrightarrow{y}))
\]
\[
= f(f(\overrightarrow{y}), y, f(x, \overrightarrow{y}, \overrightarrow{y})) = f(y, f(\overrightarrow{y}), f(x, \overrightarrow{y}, \overrightarrow{y}))
\]
\[
= f(y, u, y, f(x, \overrightarrow{y}, \overrightarrow{y})) = f(y, f(u, y, x, \overrightarrow{y}, \overrightarrow{y}), \overrightarrow{y}) = f(y, z, \overrightarrow{y}),
\]
i.e. for all \(y, z \in G\) we have
\[
f(z, \overrightarrow{y}) = f(y, z, \overrightarrow{y}).
\]
Using this identity and the fact that for all \(x, y \in G\) there exists \(z \in G\) such that \(x = f(z, \overrightarrow{y})\), we obtain
\[
f(\overrightarrow{x}) = f(x, f(z, \overrightarrow{y}, \overrightarrow{y})) = f(x, f(y, z, \overrightarrow{y}, \overrightarrow{y}), \overrightarrow{y})
\]
\[
= f(x, y, f(z, \overrightarrow{y}, \overrightarrow{y})) = f(x, y, \overrightarrow{x}),
\]
which together with the existence of only one solution at the second place implies \(x = y\). Hence \(G\) has only one element.

According to Theorem \([8]\) the class of all \(n\)-groups (for \(n > 2\)) can be considered as a variety of algebras \((G; f, \cdot)\) with one \(n\)-ary operation \(f\) and one unary \(\cdot : x \to x\). The class of \(n\)-ary groups can be also considered as a variety of algebras of different types (cf. \([13]\) and \([74]\)).

Theorem \([6]\) is valid for \(n > 2\), but, as it was observed in \([9]\), this theorem can be extended to the case \(n = 2\). Namely, let \(\cdot : x \to \hat{x}\) be a unary operation, where \(\hat{x}\) is defined as a solution of the equation \(f(2)(\overrightarrow{x}, \hat{x}, x) = x\). Then using the same method as in the proof of Theorem 2 in \([15]\) we can prove:

**Theorem 7.** Let \((G; f)\) be an \(n\)-ary \((n \geq 2)\) semigroup with a unary operation \(\cdot : x \to \hat{x}\). Then \((G; f, \cdot)\) is an \(n\)-ary group if and only if for some \(i, j \in \{2, \ldots, 2n-1\}\) the following identities
\[
f(2)(y, \overrightarrow{x}, \hat{x}, f(2)(\overrightarrow{x}, \hat{x}, \overrightarrow{x}, x, y)) = y = f(2)(x, \overrightarrow{x}, x, y)
\]
hold.
From this theorem we can deduce other definitions of $n$-ary ($n \geq 2$) groups presented in [22], [24], [61] and [65].

An $n$-ary group is said to be semiabelian if the following identity

$$(10) \quad f(x_1^n) = f(x_n, x_2^{n-1}, x_1)$$

is satisfied. In this case the operation $\cdot : x \to \overline{x}$ is a homomorphism (cf. [29]). Note by the way that the class of all semiabelian $n$-ary groups coincides with the class of medial $n$-ary groups (cf. [12], [29]). (Some authors used also the name abelian instead of semiabelian (see, e.g., [65], [29]).) Such $n$-ary groups are a special case of $\sigma$-permutable $n$-ary groups (cf. [68]), i.e. $n$-ary groups in which $f(x_1^n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for fixed $\sigma \in S_n$. An $n$-ary group which is $\sigma$-permutable for every $\sigma \in S_n$ is usually called commutative.

An $n$-ary power of $x$ in an $n$-ary group $(G; f)$ is defined in the following way:

$$x^{<0>} = x, \quad x^{<1>} = f(x^n) \quad \text{and} \quad x^{<k+1>} = f(x^{<k>}, x^{<k>}) \quad \text{for all } k > 0.$$ 

In this convention $x^{<k>}$ means an element $z$ such that $f(x^{<k-1>}, x^{<k>}, z) = x^{<0>} = x$.

Then $\overline{x} = x^{<1>}$ and

$$f(x^{<k_1>}, \ldots, x^{<k_n>}) = x^{<k_1+\ldots+k_n+1>} \quad \text{and} \quad (x^{<k>})^{<t>} = x^{<kt-n+1+k+t>}$$

(cf. [60], [26] or [10]).

Now, putting $\overline{x}^{(0)} = x$ and denoting by $\overline{x}^{(m+1)}$ the skew element to $\overline{x}^{(m)}$, from the above two identities and results obtained by W. A. Dudek in [10] and [12] we deduce the following proposition.

**Proposition 8.** In any $n$-ary group $x^{(m)} = x^{<S_m>}$, where $S_m = \frac{(2-n)^m - 1}{n-1}$.

This means that for every $n > 2$ we have $\overline{x} = x^{<n-3>}$. In particular, $\overline{x} = x^{<1>}$ in all 4-ary groups, and $\overline{x} = x^{<2>}$ in all 5-ary groups (cf. [29]).

## 2. HOSSZÚ-GLUSKIN ALGEBRAS

Let $(G; f)$ be an $n$-ary group. Fixing in $f(x_1^n)$ some $m < n$ elements we obtain a new $(n-m)$-ary operation which in general is not associative. It is associative only in the case when these fixed elements are located in some special places, for example, in the case when this new operation has the form

$$(11) \quad g(x_k^r) = f(x_1, a_1^r, x_2, a_2^r, x_3, a_3^r, \ldots, a_{k-1}^r, x_k, a_k^r),$$

where $k + r(k-1) = n$ and $a_1, \ldots, a_r \in G$ are fixed. Of course, in this case $(G; g)$ is a $k$-ary group. It is denoted by $rel_G f$ and is called a $k$-ary retract of $(G; f)$ (see, e.g., [18]). For different elements $a_1, \ldots, a_r$ we obtain different $k$-ary retracts, but all these $k$-ary retracts (for a fixed $n$-ary group) are isomorphic (cf. [18]). Therefore, we can consider only retracts for $a_1 = \ldots = a_r = a$. In such retracts the element skew to $x$ has the form

$$f(x_1^{(n-r-2)}, a^r x_2, a^r x_3, a^r x_4, a^r x_5, \ldots, a^r x_k),$$

where $\overline{x}$ and $\overline{a}$ are skew in $(G; f)$. This means that the skew elements in this retract can be expressed by the operations of an $n$-ary group $(G; f)$. 

A very important role play binary retracts, especially retracts denoted by $\text{ret}_a(G; f)$, where $x \circ y = f(x, a, y)$. The identity of the group $(G; \circ)$ is $\overline{a}$. One can verify that the inverse element to $x$ has the form

$$x^{-1} = f(\overline{a}, x^{(n-3)}, x, \overline{a}).$$

Thus, in this group

$$x \circ y^{-1} = f(x, y^{(n-3)}, y, \overline{a}).$$

Binary retracts of an $n$-ary group $(G; f)$ are commutative only in the case when $(G; f)$ is semiabelian (medial). So, $(G; f)$ is semiabelian if and only if

$$f(x, a^{(n-2)}, y) = f(y, a^{(n-2)}, x)$$

holds for all $x, y \in G$ and some fixed $a \in G$.

M. Hosszú was first who observed a strong connection between $n$-ary groups and their binary retracts. He proved in [40] the following theorem:

**Theorem 9.** An $n$-ary groupoid $(G; f)$, $n > 2$, is an $n$-ary group if and only if

(i) on $G$ one can define a binary operation $\cdot$ such that $(G; \cdot)$ is a group,

(ii) there exist an automorphism $\varphi$ of $(G; \cdot)$ and $b \in G$ such that $\varphi(b) = b$,

(iii) $\varphi^{n-1}(x) = b \cdot x \cdot b^{-1}$ holds for every $x \in G$,

(iv) $f(x^n) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \varphi^3(x_4) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$ for all $x_1, \ldots, x_n \in G$.

Two years later, this theorem was proved by L. M. Gluskin (see [57]) in a more general form (for so-called positional operatives). For a generalization to $n$-ary semigroups, see also [55] and [77]. In another version this theorem was also formulated by E. L. Post (cf. [60], p. 246). An elegant short proof was given by E. I. Sokolov in [67]. His proof is based on the observation that $(G; \cdot) = r_{t_a}(G; f)$.

Then we have:

$$\varphi(x) = f(\overline{a}, x, a^{(n-2)}).$$

and

$$b = f(\overline{a}^{(n)}).$$

From (14) and (7) or (8), we can deduce that commutative $n$-ary group operations have the form $f(x^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot b$, where $(G; \cdot)$ is a commutative group.

Note that the last condition of Theorem 9 can be rewritten in the form

$$f(x^n) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \varphi^3(x_4) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n.$$

The above theorem has the following generalization (cf. [17]):

**Theorem 10.** An $n$-ary groupoid $(G; f)$, $n > 2$, is an $n$-ary group if and only if

(i) on $G$ one can define a $k$-ary operation $g$ such that $(G; g)$ is a $k$-ary group and $k - 1$ divides $n - 1$,

(ii) there exist an automorphism $\varphi$ of $(G; g)$ and elements $b_2, \ldots, b_k \in G$ such that $\varphi(b_i) = b_i$ for $i = 2, \ldots, k$,

(iii) $g(\varphi^{n-1}(x), b_k^2) = g(b_k^2, x)$ holds for every $x \in G$,

(iv) $f(x^n) = g_{(j)}(x_1, \varphi(x_2), \varphi^2(x_3), \ldots, \varphi^{n-1}(x_n), b_k^2)$ for all $x_1, \ldots, x_n \in G$. 


In this theorem \((G; g) = ret_{a_1}(G; f)\), where \(a_1 = \ldots = a_r = a\). In this case, we get:

\[
\varphi(x) = f(\varpi, \frac{(n-r-2)}{a}, x, \frac{(r)}{a}),
\]

(17)

\[
b_2 = f(\cdot(\frac{(n-r-2)n}{a}, \varpi, \frac{(n-k)(n-r-2)}{a}), b_3 = \ldots = b_k = a.
\]

(18)

Other important generalizations can be found in [41] (for heaps), [54] (for vector valued groups), [66] (for partially associative \(n\)-ary quasigroups).

Following E. L. Post (see [60], cf. [4], p. 36–40, and [28]) a binary group \(G^* = (G^*; \circ)\) is said to be a covering group for the \(n\)-ary group \((G; f)\) if there exists an embedding \(\tau: G \rightarrow G^*\) such that \(\tau(G)\) is a generating set of \(G^*\) and \(\tau(f(x_1)) = \tau(x_1) \circ \tau(x_2) \circ \ldots \circ \tau(x_n)\) for every \(x_1, \ldots, x_n \in G\). \(G^*\) is a universal covering group (or a free covering group) if for any covering group \(G^*_1\) there exists a homomorphism from \(G^*\) onto \(G^*_1\) such that the following diagram is commutative (or compatible – in another terminology):

\[
\begin{array}{c}
G \\
\downarrow \\
\downarrow \\
G^* \rightarrow \rightarrow G^*_1
\end{array}
\]

Post proved in [60] that for every \(n\)-ary group \((G; f)\) there exist a covering group \((G^*; \circ)\) and its normal subgroup \(G^*_0\) such that \(G^*/G^*_0\) is a cyclic group of order \(n - 1\) and \(f(x^*_1) = x_1 \circ x_2 \circ \ldots \circ x_n\) for all \(x_1, \ldots, x_n \in G\). So, the theory of \(n\)-ary groups is closely related to the theory of cyclic extensions of groups, but these theories are not equivalent.

Indeed, the above theorems show that for any \(n\)-ary group \((G; f)\) we have the sequence

\[
O \rightarrow (G_0; \circ) \rightarrow (G^*; \circ) \xrightarrow{\zeta} C(n) \rightarrow O,
\]

where \((G^*; \circ)\) is the free covering group of \((G; f)\) with \(G = \zeta^{-1}(1)\), and \(1\) is a generator of the cyclic (additively writing) group \(C(n) = (C_n; +_n)\).

Example 1. Consider two cyclic extensions of the cyclic group \(C(3)\) by \(C(3)\):

\[
0 \rightarrow C(3) \xrightarrow{\alpha} C(9) \xrightarrow{\beta_1} C(3) \rightarrow 0
\]

and

\[
0 \rightarrow C(3) \xrightarrow{\alpha} C(9) \xrightarrow{\beta_2} C(3) \rightarrow 0,
\]
where the homomorphisms $\alpha$, $\beta_1$ and $\beta_2$ are given by:
\[
\begin{align*}
\alpha(x) &= 3x \quad \text{for} \quad x \in C_3, \\
\beta_1(x) &\equiv x(\text{mod } 3) \quad \text{for} \quad x \in C_9, \\
\beta_2(x) &\equiv 2x(\text{mod } 3) \quad \text{for} \quad x \in C_9.
\end{align*}
\]

It is easy to verify that if $g(x, y, z, v) = (x + y + z + v)(\text{mod } 9)$, then the 4-ary groups ($\beta_1^{-1}(1); g$) and ($\beta_2^{-1}(2); g$), corresponding to those extensions are isomorphic to the 4-ary groups $(C_3; f_1)$, $(C_3; f_2)$, respectively, where

\[
f_1(x, y, z, v) \equiv (x + y + z + v + 1)(\text{mod } 3)
\]

and

\[
f_2(x, y, z, v) \equiv (x + y + z + v + 2)(\text{mod } 3).
\]

These 4-groups are isomorphic. The isomorphism $\varphi : (C_3; f_1) \to (C_3; f_2)$ has the form $\varphi(x) \equiv 2x(\text{mod } 3)$. Nevertheless, the above-mentioned extensions are not equivalent (because there is no automorphism $\lambda$ of $C(9)$ such that $\lambda \circ \alpha = \alpha$ and $\beta_2 \circ \lambda = \beta_1$).

The algebra $(G; \cdot, \varphi, b)$ of the type $(2, 1, 0)$, where $(G; \cdot)$ is a (binary) group, $b \in G$ is fixed, $\varphi \in \text{Aut}(G; \cdot)$, $\varphi(b) = b$ and $\varphi^{-1}(x) = b \cdot x \cdot b^{-1}$ for every $x \in G$ is called a Hosszú-Gluskin algebra (briefly: an HG-algebra). We say that an HG-algebra $(G; \cdot, \varphi, b)$ is associated with an n-group $(G; f)$ if the identity $\varphi(b)$ is satisfied. In this case we say also that an n-group $(G; f)$ is $(\varphi, b)$-derived from the group $(G; \cdot)$.

A k-ary HG-algebra $(G; g, \varphi, b_k)$ can be defined similarly. Binary HG-algebras are studied in [172, 173] and [174].

Theorems 9 and 10 state that every k-ary HG-algebra is associated with some n-ary group. Any n-ary group is $(\varphi, b)$-derived from some binary group and $(\varphi, b_k)$-derived from some k-ary group.

3. Calculation of n-ary groups on small sets

Results presented in the previous section give the possibility to evaluate the number of non-isomorphic n-ary groups. To calculate these groups we must use the following result proved in [18].

**Theorem 11.** Two n-ary groups $(G_1; f_1)$, $(G_2; f_2)$ are isomorphic if and only if for every $c \in G_1$ there exists an isomorphism $h : \text{ret}_c(G_1; f_1) \to \text{ret}_d(G_2; f_2)$ such that $d = h(c)$, $h(f_1(\overline{x}, \ldots, \overline{c})) = f_2(\overline{d}, \ldots, \overline{d})$ and $h(f_1(\overline{x}, x^{(n-2)})) = f_2(\overline{d}, h(x), d^{(n-2)})$.

**Corollary 12.** Two commutative n-ary groups $(G_1; f_1)$, $(G_2; f_2)$ are isomorphic if and only if for every $c \in G_1$ there exists an isomorphism $h : \text{ret}_c(G_1; f_1) \to \text{ret}_d(G_2; f_2)$ such that $d = h(c)$ and $h(f_1(\overline{x}, \ldots, \overline{c})) = f_2(\overline{d}, \ldots, \overline{d})$.

If $(G, \cdot)$ is an abelian group, then, of course, we can consider the automorphism of the form $\alpha(x) = x^{-1}$. Then $G$ with the operation
\[
f(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2^{-1} \cdot x_3 \cdot \ldots \cdot x_n^{-1} \cdot x_n
\]
is an n-ary group if $n$ is odd. Such n-ary groups are characterized by the following theorem proved in [31].

**Theorem 13.** Let $m$ be odd and let $(G; f)$ be an n-ary group. Then the operation $f$ has the form $\Box$, where $(G; \cdot)$ is an abelian group, if and only if
In this case a where (10 WIEŚLAW A. DUDEK AND KAZIMIERZ GLAZEK

operation of form (19), one can obtain the ternary term operation

Proposition 14. Let \( G \) be a group and let \( t_1, \ldots, t_n \) be fixed integers. Then \( G \) with the operation

\[
 f(x, y, z) = x - y + z
\]

which is a so-called Mal’tsev term in the group \( G \). Of course, it is idempotent and medial (entropic – in another terminology). Such ternary operations appear in several branches of mathematics. For example, they play very important role in affine geometry and the theory of modes (because of idempotency and mediality), in the theory of congruences in general algebras (because existence of a Mal’tsev term in general algebras implies permutability of congruences and then modularity of lattices of congruences) and also in the theory of clones which is important in Universal Algebra and as well in Multiple-valued Logics.

From results obtained in \(^{31}\) (cf. also \(^{70}\)) we can deduce:

**Proposition 14.** Let \((G; \cdot)\) be a group and let \( t_1, \ldots, t_n \) be fixed integers. Then \( G \) with the operation

\[
 f(x^n) = (x_1)^{t_1} \cdot (x_2)^{t_2} \cdots (x_{n-1})^{t_{n-1}} \cdot (x_n)^{t_n},
\]

is an \( n \)-ary group if and only if

1. \( x^{t_1} = x = x^{t_n} \),
2. \( t_j = k^j \) for some integer \( k \) and all \( j = 2, \ldots, n - 1 \),
3. \( (x \cdot y)^k = x^k \cdot y^k \).

In this case we say that \((G; f)\) is derived from the \( k \)-exponential group.

**Proposition 15.** An \( n \)-ary group \((G; \cdot)\) is derived from the \( k \)-exponential \((k > 0)\) group \((G; \cdot)\) if and only if there exists \( a \in G \) such that

1. \( f(a, \ldots, a) = a \),
2. \( f(k) = (a, \ldots, a) = x \).

Moreover, \((G; \cdot) = \text{ret}_a(G; f)\).

Using the above results we can describe all non-isomorphic \( n \)-ary groups with small numbers of elements.

For this let \((\mathbb{Z}_k; +)\) be the cyclic group modulo \( k \). Consider the following \( n \)-ary operation:

\[
 f_a(x^n) \equiv (x_1 + \ldots + x_n + a) \pmod{k},
 g_d(x^n) \equiv (x_1 + dx_2 + \ldots + d^{n-2}x_{n-1} + x_n) \pmod{k},
 g_{d,c}(x^n) \equiv (x_1 + dx_2 + \ldots + d^{n-2}x_{n-1} + x_n + c) \pmod{k},
\]

where \( a \in \mathbb{Z}_k, \ c, d \in \mathbb{Z}_k \setminus \{0, 1\}, \ d^{n-1} \equiv 1 \pmod{k} \). Additionally, for the operation \( g_{d,c} \) we assume that \( dc \equiv c \pmod{k} \) holds. By Theorem \(^{31}\) \((\mathbb{Z}_k; f_a), (\mathbb{Z}_k; g_d)\) and \((\mathbb{Z}_k; g_{d,c})\) are \( n \)-ary groups.

In \(^{31}\) the following theorem is proved:
Theorem 16. A k-element n-ary group \((G; f)\) is \(\langle \varphi, b \rangle\)-derived from the cyclic group of order \(k\) if and only if it is isomorphic to exactly one n-ary group of the form \((\mathbb{Z}_k; f_a), (\mathbb{Z}_k; g_a)\) or \((\mathbb{Z}_k; g_a, c)\), where \(d \mid \text{gcd}(k, n - 1)\) and \(c \mid k\).

An infinite cyclic group can be identified with the group \((\mathbb{Z}; +)\). This group has only two automorphisms: \(\varphi(x) = x\) and \(\varphi(x) = -x\). So, according to Theorem 9 n-ary groups defined on \(\mathbb{Z}\) have the form \((\mathbb{Z}; f_a)\) or \((\mathbb{Z}; g_{-1})\), where

\[
g_{-1}(x^n) = x_1 - x_2 + x_3 - x_4 + \ldots + x_n,
\]

and \(n\) is odd. Since \(\varphi_k(x) = x + k\) is an isomorphism of n-ary groups \((\mathbb{Z}; f_a)\) and \((\mathbb{Z}; f_b)\), where \(a = b + (n - 1)k\), the calculation of non-isomorphic n-ary groups of the form \((\mathbb{Z}; f_a)\) can be reduced to the case when \(a = 0, 1, \ldots, n - 2\). From Corollary 12 it follows that these n-ary groups are non-isomorphic.

So, we have proved

Theorem 17. An n-ary group \(\langle \varphi, b \rangle\)-derived from the infinite cyclic group \((\mathbb{Z}; +)\) is isomorphic to an n-ary group \((\mathbb{Z}; f_a)\), where \(0 \leq a \leq (n - 2)\), or to \((\mathbb{Z}; g_{-1})\), where \(n\) is odd.

Denote by \(\text{Inn}(G; \cdot)\) the group of all inner automorphisms of \((G; \cdot)\), by \(\text{Out}(G; \cdot)\) the factor group of \(\text{Aut}(G; \cdot)\) by \(\text{Inn}(G; \cdot)\), and by \(\text{Out}_n(G; \cdot)\) the set of all cosets \(\gamma \in \text{Out}(G; \cdot)\) containing \(\gamma\) such that \(\gamma^n \in \text{Inn}(G; \cdot)\). Then, as it is proved in [3], for centerless groups, i.e. groups for which \(\text{card} (\text{Cent}(G; \cdot)) = 1\), the following theorem is true.

Theorem 18. Let \((G; \cdot)\) be a centerless group such that \(\text{Out}_n(G; \cdot)\) is abelian, and let \((G; f)\) be \(\langle \alpha, a \rangle\)-derived, and \((G; g)\) be \(\langle \beta, b \rangle\)-derived from \((G; \cdot)\). Then \((G; f)\) is isomorphic to \((G; g)\) if and only if \(\alpha \circ \beta^{-1} \in \text{Inn}(G; \cdot)\).

The number of pairwise non-isomorphic n-ary groups \(\langle \varphi, b \rangle\)-derived from a centerless group \((G; \cdot)\) is smaller or equal to \(s = \text{card} (\text{Out}_n(G; \cdot))\). It is equal to \(s\) if and only if \(\text{Out}(G; \cdot)\) is abelian.

For every \(n\) and \(k \neq 2, 6\), there exists exactly one n-ary group which is \(\langle \varphi, b \rangle\)-derived from \(S_k\) (for \(k = 2\) and \(k = 6\) we have one or two such n-ary groups relatively to evenness of \(n\)).

Let now \((G; \cdot)\) be an arbitrary group, \(c \in G\), \(\varphi \in \text{Aut}(G; \cdot)\). Let us put

\[
f_\varphi^{(c)}(x^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot c,
\]

\[
g_\varphi^{(c)}(x^n) = x_1 \cdot \varphi(x_1) \cdot \ldots \cdot \varphi^{n-1}(x_n),
\]

\[
g_{\varphi, c}(x^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot c.
\]

For example (for details see [32]), we have the following:

Theorem 19. Let \(l = \text{gcd}(n - 1, 12)\), \((G_4; *)\) be the Klein four-group (with 0 as the neutral element), let \(\gamma, \varepsilon \in \text{Aut}(G_4; *)\), where \(\gamma\) is of order 2 and \(\varepsilon\) of order 3, and let \(c \in G_4 \{0\}\) be the fix point of \(\gamma\). Then every n-ary group \(\langle \varphi, b \rangle\)-derived from \((G_4; *)\) is isomorphic to exactly one \((G_4; f)\), where \(f\) is one of the following n-ary group operations:

(a) \(f_0^{(s)}, f_1^{(s)}, g_\gamma^{(s)}, g_{\varepsilon}^{(s)}\) or \(g_\varepsilon^{(s)}\) for \(l = 12\),

(b) \(f_0^{(s)}, f_1^{(s)}, g_\gamma^{(s)}\) or \(g_\varepsilon^{(s)}\) for \(l = 6\),

(c) \(f_0^{(s)}, f_1^{(s)}, g_\gamma^{(s)}\) or \(g_{\varepsilon}^{(s)}\) for \(l = 4\),
Comparing our results with results obtained in [32], [33] and [34] (cf. also [60] for $k = 2, 3$) we can tabularize the numbers of $n$-ary groups on $k$-element sets with $k < 8$ in the following way (we use the abbreviations: commut. = commutative, idem. = idempotent):

$k = 2$, $l = \gcd(n - 1, 2)$

| $l$ | 0 | 1 |
|-----|---|---|
| $t \equiv 0 \pmod{2}$ | 2 | 1 |
| $t \equiv 1 \pmod{2}$ | 1 | 0 |

$k = 3$, $l = \gcd(n - 1, 6)$

| $l$ | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| $t \equiv 0 \pmod{6}$ | 3 | 2 | 2 | 1 |
| $t \equiv 1 \pmod{6}$ | 1 | 1 | 0 | 0 |
| $t \equiv 2 \pmod{6}$ | 0 | 0 | 0 | 0 |

$k = 4$, $l = \gcd(n - 1, 12)$

| $l$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|---|
| $t \equiv 0 \pmod{12}$ | 10 | 8 | 9 | 3 | 7 | 2 |
| $t \equiv 1 \pmod{12}$ | 5 | 4 | 5 | 2 | 4 | 2 |
| $t \equiv 2 \pmod{12}$ | 2 | 1 | 2 | 0 | 1 | 0 |
| $t \equiv 3 \pmod{12}$ | 3 | 2 | 1 | 1 | 1 | 0 |
| $t \equiv 4 \pmod{12}$ | 1 | 0 | 0 | 0 | 0 | 0 |

$k = 5$, $l = \gcd(n - 1, 20)$

| $l$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|---|
| $t \equiv 0 \pmod{20}$ | 5 | 3 | 2 | 4 | 2 | 1 |
| $t \equiv 1 \pmod{20}$ | 2 | 2 | 2 | 1 | 1 | 1 |
| $t \equiv 2 \pmod{20}$ | 1 | 1 | 1 | 0 | 0 | 0 |
| $t \equiv 3 \pmod{20}$ | 3 | 1 | 0 | 3 | 1 | 0 |
| $t \equiv 4 \pmod{20}$ | 0 | 0 | 0 | 0 | 0 | 0 |

$k = 6$, $l = \gcd(n - 1, 6)$

| $l$ | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| $t \equiv 0 \pmod{6}$ | 7 | 3 | 5 | 2 |
| $t \equiv 1 \pmod{6}$ | 4 | 2 | 2 | 1 |
| $t \equiv 2 \pmod{6}$ | 1 | 0 | 0 | 0 |
| $t \equiv 3 \pmod{6}$ | 1 | 0 | 1 | 0 |
| $t \equiv 4 \pmod{6}$ | 1 | 1 | 1 | 1 |
4. Term equivalence of $n$-ary groups

For any general algebra $\mathfrak{A} = (A; F)$ one can define the set $T^{(n)}(\mathfrak{A})$ of all $n$-ary term operations as the smallest set of $n$-ary operations on $A$ containing $n$-ary projections (or $n$-ary trivial operations, in another terminology) and closed under compositions with fundamental operations. Then the set $T(\mathfrak{A}) = \bigcup_{n=1}^{\infty} T^{(n)}(\mathfrak{A})$ of all term operations is the smallest set of operations on the set $A$ containing the set $F$ of fundamental operations and all projections $e_i^{[n]}(x_i^n) = x_i$, ($i = 1, 2, \ldots, n$, $n = 1, 2, \ldots$), and closed under (direct) compositions. Of course, $T(\mathfrak{A})$ is a clone in the sense of Ph. Hall (see, e.g., [3]). It is worth mentioning that the term operations were also called algebraic operations by several authors (see, e.g., [51]). Two algebras $\mathfrak{A}_1 = (A; F)$ and $\mathfrak{A}_2 = (A; G)$ are called term equivalent if $T(\mathfrak{A}_1) = T(\mathfrak{A}_2)$ (see, e.g., [27], p. 32, 56). If elements from some subsets $A_1$ and $A_2$ of $A$ are treated as constant elements of algebras $\mathfrak{A}_1 = (A; F \cup A_1)$ and $\mathfrak{A}_2 = (A; G \cup A_2)$, respectively, and $T(\mathfrak{A}_1) = T(\mathfrak{A}_2)$, then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are polynomially equivalent. Two varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ of algebras (perhaps of different types) are term equivalent (polynomially equivalent, respectively) if for every algebra $\mathfrak{A}_1 \in \mathcal{V}_1$ there exists an algebra $\mathfrak{A}_2 \in \mathcal{V}_2$ term equivalent (polynomially equivalent, resp.) to $\mathfrak{A}_1$, and vice versa.

Using Theorem 10 and taking into account formulas (17) and (18), we have

**Theorem 20.** Let $\mathfrak{G} = (G; f^-, a)$ be an $n$-ary group for a fixed $n > 2$, an element $a$ belong to $G$, and let $k$ be such a natural number that $(k - 1)$ divide $(n - 1)$. Then the algebra $\mathfrak{G}_a = (G; g, \varphi, b_2^k)$, where $\varphi$ is an automorphism of a $k$-ary group $G, g$, $(k - 1)$ divides $(n - 1)$, and $b_2, \ldots, b_k$ are constant elements in $G$ such that $\varphi(b_i) = b_i$ for $i = 2, \ldots, k$ and $g(\varphi^{n-1}(x), b_2^k) = g(b_2^k, x)$ for all $x \in G$.

Indeed, $f$ is determined by $g, \varphi$ and $b_2, \ldots, b_k$ by the formula (iv) from Theorem 10. The function $\tau$ : $x \rightarrow \varphi$ can be easily expressed by the operation $g$. Namely, if $f = g(\cdot)$, then $\tau = x^{<k}\varphi$, where $x^{<k}$ is a $k$-ary power of $x$. According to Theorem 10 the element $\varphi$ also can be expressed by $g, \varphi$ and $b_2, \ldots, b_n$ as a solution $z$ of the equation

$$x = f(x^{(n-1)}, z) = g(x, \varphi(x), \varphi^2(x), \ldots, \varphi(x)^{n-2}, b_2^k, z).$$

| $k = 7$ | $l = \gcd(n - 1, 42)$ | $n \equiv t \pmod{42}$ | $t = 1$ | $t = 2$ | $t = 3$ | $t = 2$ | $t = 1$ |
|--------|-------------------------|--------------------------|--------|--------|--------|--------|--------|
| 7      | 42                     | all                      | $t = 1$ | $t = 1$ | $t = 1$ | $t = 1$ | $t = 1$ |
| 2      | 22                     | commutative              | 2      | 2      | 2      | 2      | 1      |
| 3      | 42                     | non-comm., medial., idem., | 2      | 2      | 2      | 1      | 1      | 1      |
| 6      | 42                     | commutative, idempotent   | 1      | 1      | 1      | 0      | 0      | 0      |
| 7      | 42                     | $t = t_5$                 | 1      | 1      | 1      | 0      | 0      | 0      |
| 8      | 42                     | $t = t_6$                 | 1      | 1      | 1      | 0      | 0      | 0      |
| 9      | 42                     | $t = t_7$                 | 1      | 1      | 1      | 0      | 0      | 0      |
| 10     | 42                     | $t = t_8$                 | 1      | 1      | 1      | 0      | 0      | 0      |
| 11     | 42                     | $t = t_9$                 | 1      | 1      | 1      | 0      | 0      | 0      |

$t_4 = 15, 29$, $t_5 = 8, 36$, $t_6 = 7, 13, 19, 25, 31, 37$, $t_7 = 4, 10, 16, 28, 34, 40$, $t_8 = 3, 5, 9, 11, 17, 21, 23, 33, 35, 39, 41$, $t_9 = 0, 2, 6, 12, 14, 18, 20, 24, 26, 30, 32, 38$. 
Conversely, the operations of \((G; g, \varphi, b)\) are term derived from the operations of \((G; f, -)\) by \((17)\) and \((13)\). \((G; g) = ret_a(G; f)\), where \(a_1 = \ldots = a_r = a\), which completes the proof of Theorem 20.

By Theorem 21 and formulas \((14)\) and \((15)\), we have the following corollaries.

**Corollary 21.** Let \(\mathfrak{G} = (G; f, -)\) be an \(n\)-ary group for a fixed \(n > 2\), and let an element \(a\) belong to \(G\). Then the algebra \(\mathfrak{G}_a = (G; f, -, a)\) is term equivalent to the \(HG\)-algebra \((G; \cdot, \varphi, b)\), where \((G; \cdot)\) is a group, \(\varphi \in Aut(G; \cdot)\), \(b \in G\), \(\varphi(b) = b\), \(\varphi^{n-1}(x) = b \cdot x \cdot b^{-1}\) for all \(x \in G\).

**Corollary 22.** For fixed \(n > 2\), the variety of \(n\)-ary groups (as algebras of type \((n, 1)\)) is polynomially equivalent to the variety of the corresponding \(HG\)-algebras (as algebras of type \((2, 1, 1, 0)\)).

Let now \(\mathfrak{G} = (G; f, -)\) be a semiabelian \(n\)-ary group \((n > 2)\). Then the \(HG\)-algebra associated with \(\mathfrak{G}\) has a commutative group operation denoted by +. Let \(\mathcal{H} = (G; +, \varphi, b)\) be associated with \(\mathfrak{G}\) and \(\mathcal{H}_a = (G; f, -, a)\). Then \(\mathcal{H}\) and \(\mathcal{H}_a\) are term equivalent (see Theorems 21 and 20 Corollary 21 and formulas \((12) - (18)\)). In this case we have

\[
-y = f(\bar{x}, x, \bar{x}, \bar{x}), \\
x + y = f(x, (-y), (-y), \bar{x}), \\
\varphi(x) = f(\bar{x}, x, (n-2)a), \\
\text{and} \quad b = f(\bar{x}^n).
\]

We can describe all term operations of \(\mathcal{H}_a\) by using the language of \(HG\)-algebras.

At first, we consider all unary term operations. Denote by \(g_i(x)\) the following operation

\[
g_i(x) = k_{i1}\varphi^{l_{i1}}(x) + k_{i2}\varphi^{l_{i2}}(x) + \ldots + k_{it}\varphi^{l_{it}}(x)
\]

for some \(t, l_{i1}, \ldots, l_{it}\) non-negative integers and some \(k_{i1}, \ldots, k_{it} \in \mathbb{Z}\). Then it is easily to verify

**Lemma 23.** Let \(\mathcal{H} = (G; +, \varphi, b)\) be the \(HG\)-algebra associated with a semiabelian \(n\)-ary group \(\mathfrak{G}\). Then all unary term operations of \(\mathcal{H}\) (and of \(\mathfrak{G}_a\)) are of the form

\[
g(x) = g_i(x) + k_gb
\]

for some \(g_i\) of the form \((20)\) and \(k_g \in \mathbb{Z}\).

Indeed, it is enough to observe that \(g \in T^{(1)}(\mathcal{H})\), \(\varphi(g(x))\) is again of the form \((20)\), and the set of all such operations is closed under addition.

**Theorem 24.** Let \(\mathcal{H} = (G; +, \varphi, b)\) be the \(HG\)-algebra associated with a semiabelian \(n\)-ary group \(\mathfrak{G}\). Then all \(n\)-ary term operations of \(\mathcal{H}\) (and of \(\mathfrak{G}_a\)) are of the form

\[
F(x_1, \ldots, x_m) = \sum_{i=1}^{m} g_i(x_i) + k_Fb
\]

for some \(g_i(x)\) of the form \((20)\) and \(k_F \in \mathbb{Z}\).
A verification of this theorem can be done by induction with respect to the complexity of term operations and we left it to readers.

5. $Q$-independent sets in $HG$-algebras

E. Marczewski observed at the end of the 1950s that there are common features of linear independence of vectors and set-theoretical independence, and proposed a general scheme of independence called here $M$-independence. Recall that the notion of set-theoretical independence (or, more generally, independence in Boolean algebras, see, e.g., [11, 2, 25, 50]) was introduced at the mid-1930s by G. Fichtenholtz and L. Kantorovich [21] and also, independently, by E. Marczewski himself, and this notion is very important in Measure Theory (see, e.g., [21, 47, 48, 57]), and also, independently, by E. Marczewski himself.

Let $\mathfrak{A} = (A; F)$ be an algebra $\emptyset \neq X \subseteq A$. The set $X$ is said to be $M$-independent (see [19, 21]) $(X \in Ind(\mathfrak{A}; M)$, for short) if

(a) $(\forall n \in \mathbb{N}, n \leq \text{card}(X)) \ (\forall f, g \in T^n(\mathfrak{A})) \ (\forall a_1, \ldots, a_n \in X)

\[ f(a_1^n) = g(a_1^n) \Rightarrow f = g \ (\text{in } A). \]

This condition is equivalent to each of the following ones

(b) $(\forall n \in \mathbb{N}, n \leq \text{card}(X)) \ (\forall f, g \in T^n(\mathfrak{A})) \ (\forall p : X \rightarrow A) \ (\forall a_1, \ldots, a_n \in X)

\[ f(a_1^n) = g(a_1^n) \Rightarrow f(p(a_1), \ldots, p(a_n)) = g(p(a_1), \ldots, p(a_n)). \]

(c) $(\forall p \in A^X) \ (\exists \tilde{p} \in Hom(\langle X \rangle_\mathfrak{A}, \mathfrak{A})) \ \tilde{p}|_X = p$, where $\langle X \rangle_\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$ generated by $X$.

(d) $\langle X \rangle_\mathfrak{A}$ is a $\mathbb{K}$-$\text{free algebra}$ $\mathbb{K}$-$\text{freely generated by } X$, where $\mathbb{K} = \langle \mathfrak{A} \rangle$ (or, by Birkhoff Theorem, $\mathbb{K} = HSP\{\mathfrak{A}\}$, a variety generated by $\mathfrak{A}$).

Basic properties of $M$-independence are the following ones:

- ("hereditarity") $X \in Ind(\mathfrak{A}, M)$, $Y \subseteq X \Rightarrow Y \in Ind(\mathfrak{A}, M)$,
- $(\forall X \subseteq A) \ (\forall \text{finite } Y \subseteq X) \ (Y \in Ind(\mathfrak{A}, M) \Rightarrow X \in Ind(\mathfrak{A}, M))$

(i.e. the family $J = Ind(\mathfrak{A}, M)$ is of finite character).

The notion of $M$-independence is stronger than that of independence with respect to the closure operator of such a kind $X \mapsto \langle X \rangle_\mathfrak{A}$ (for $X \subseteq A$).

There are some notions of independence which are not special cases of $M$-independence, such as:

- linear independence in abelian groups,
- independence with respect to a closure operator $\mathcal{C}$ (i.e. $\mathcal{C}$-independence),
- stochastic independence,
- “independence-in-itself” defined by J. Schmidt (in 1962),
- “weak independence” used by S. Świerczkowski (in 1964).

For this reason, a general notion of independence with respect to a family of mappings was proposed by E. Marczewski in 1966 (and studied in [53] and [25]).

This notion is general enough to cover the above-mentioned kinds of independences.

Let $\emptyset \neq X \subseteq A$ and

$Q_X \subseteq A^X = M_X = \{ p \mid p : X \rightarrow A \}$,

$Q(A) = Q = \bigcup \{ Q_X \mid X \subseteq A \}$,
For abelian groups, the notion of \( G \)-independence gives us the well-known linear independence.

Now we can able to obtain some results on \( Q \)-independence (for special families \( Q \) of mappings, e.g., for \( Q = M \) and \( G \)) in \( HG \)-algebras of type \( \mathcal{H} = (G; +, \varphi, b) \), where \((G; +)\) is an abelian group.

In this case, the equality

\[ F_1(x_1, \ldots, x_m) = F_2(x_1, \ldots, x_m) \]  

(for two term operations of the form \( \mathcal{H} \) in \( \mathcal{H} \)) is equivalent to the equality

\[ H(x_1, \ldots, x_m) = 0, \]

where \( H \in \mathcal{T}(m)(\mathcal{H}) \), i.e. \( H(x_1, \ldots, x_m) = \sum_{i=1}^{m} h_i(x_i) + k_n b \), and \( 0 \) denotes the zero of the group \((G; +)\).

Consider a subset \( X \) of \( G \). Let for \( a_1, \ldots, a_m \in X \) the equality

\[ H(a_1, \ldots, a_m) = 0, \]

hold. Taking into account the mapping \( p : X \to (X)_\mathfrak{A} \) defined by \( p(a_i) = 0 \) and \( p(x) = x \) for \( x \in X \setminus \{a_1, \ldots, a_m\} \), we get \( k_n b = 0 \). (We observe that such mapping \( p \) belongs to families \( \mathcal{M} \) and \( \mathcal{G} \).) Therefore we have

\[ \sum_{i=1}^{m} h_i(a_i) = 0. \]

Consider the mapping \( q_j : X \to (X)_\mathfrak{A} \) defined for fixed \( j \in \{1, \ldots, m\} \) as follows:

\[ q_j(x) = \begin{cases} a_j & \text{if } x = a_j, \\ 0 & \text{if } x \neq a_j. \end{cases} \]

We obtain \( h_j(a_j) = 0 \) for all \( j = 1, 2, \ldots, m \). (In the considered case all \( q_j \) belong to \( \mathcal{M} \) and \( \mathcal{G} \).)
Theorem 25. Let $X \subseteq G$ be a subset of the HG-algebra $\mathcal{H} = (G; +, \varphi, b)$. Then $X \in \text{Ind}(\mathcal{H}, G)$ if and only if for any $m \leq \text{card}(X)$ for all $a_1, \ldots, a_m \in X$ and every term operation $H(x_1, \ldots, x_m) = \sum_{i=1}^{m} h_i(x_i) + k_H b$ the equality

$$
\sum_{i=1}^{m} h_i(a_i) + k_H b = 0
$$

is equivalent with

$$
(\forall i \in \{1, \ldots, m\}) (h_i(a) = 0 \& k_H b = 0).
$$

Moreover, $X$ is $\mathcal{M}$-independent in this HG-algebra iff for all pairwise different elements $a_1, \ldots, a_m$ from $X$ equality (26) implies $h_i(x) = 0$ for all $i = 1, 2, \ldots, m$ and $k_H b = 0$.

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