STRONG IMBALANCED TURBULENCE
A. BERESNYAK AND A. LAZARIAN
Department of Astronomy, University of Wisconsin, Madison, WI 53706;
andrey@astro.wisc.edu, lazarian@astro.wisc.edu
Received 2007 September 4; accepted 2008 April 15

ABSTRACT
We consider stationary, forced, imbalanced, or cross helical MHD Alfvénic turbulence where the waves traveling in one direction have higher amplitudes than the opposite waves. This paper is dedicated to so-called strong turbulence, which cannot be treated perturbatively. Our main result is that the anisotropy of the weak waves is stronger than the anisotropy of strong waves. We propose that critical balance, which was originally conceived as a causality argument, has to be amended by what we call a propagation argument. This revised formulation of critical balance is able to handle the imbalanced case and reduces to the old formulation in the balanced case. We also provide a phenomenological model of energy cascading and discuss the possibility of self-similar solutions in a realistic setup of driven turbulence.

Subject headings: ISM: kinematics and dynamics — MHD — turbulence

1. INTRODUCTION

MHD turbulence appears in the dynamics of conductive fluid in a generalized setting with large Reynolds numbers, or low physical dissipation. It is ubiquitous in the interstellar and intracluster medium, Earth magnetosphere, solar wind, accretion disks, etc. In fact, it is laminar flows that constitute an exception in astrophysics, while, generically, astrophysical fluids are turbulent.

The study of MHD turbulence has been an old challenge. First attempts to address it were classical papers by Iroshnikov (1963) and Kraichnan (1965) (henceforth IK model). A good account for the state of the field could be found in Biskamp (2003). Usually turbulence is subdivided into weak and strong, depending on the strength of nonlinear interaction. While weak MHD turbulence allows analytical perturbative treatment (Ng & Bhattacharjee 1996; Galtier et al. 2002; Chandran 2005), the progress in understanding strong turbulence came primarily from phenomenological and closure models that were tested by comparison with results of numerical simulations.

Important theoretical works on strong MHD turbulence include Montgomery & Turner (1981), Matthaeus et al. (1983), Shebalin et al. (1983), and Higdon (1984). Those clarified the anisotropic nature of the energy cascade and paved the way for further advancement in the field. The study by Goldreich & Sridhar (1995, hereafter GS95) identified the balance between perturbations parallel and perpendicular to the local direction of magnetic field, i.e., “critical balance,” as the key component of dynamics in strong magnetic turbulence. Although it dealt with incompressible MHD turbulence, GS95 also influenced further studies of compressible turbulence (e.g., Lithwick & Goldreich 2001). In particular, it identified the dominant role of Alfvénic perturbations for cascading of slow modes, the role that was later confirmed with numerical simulations in both weakly and strongly compressive media (Cho & Lazarian 2002, 2003).

Being a mean field model in the spirit of Kolmogorov (1941), GS95 predicts the velocity and magnetic field fluctuation strengths and their anisotropies, in terms of the local dissipation rate. Even though recently the simple dynamical model of GS95 came under criticism (see, e.g., Boldyrev 2006; Gogoberidze 2007) with the motivation to explain the deviations from GS95’s $-5/3$ spectrum in numerical simulations (see, e.g., Muller et al. 2003), we feel that it does provide a good insight into MHD turbulence. The model that we present in this paper is similar to Kolmogorov or GS95, a mean field model, which does not account for any local dynamical effects or intermittencies (see Beresnyak & Lazarian 2006). These effects are beyond the scope of this paper and will be addressed elsewhere.

While balanced MHD turbulence enjoyed much attention, the opposite regime, i.e., imbalanced turbulence, was less developed. The analytical results were obtained for weak imbalanced turbulence (Galtier et al. 2002; Lithwick & Goldreich 2003) although they are applicable in a rather narrow range of imbalances. The closure model for the imbalanced isotropic MHD turbulence in the spirit of IK model was presented by Grappin et al. (1983; see also Pouquet et al. 1976). It has a nonlinear interaction timescale much larger than Alfvénic time; i.e., it presents weak turbulence. They obtained a theoretical prediction that the sum of slopes for the inertial ranges of the oppositely moving waves should be $-3$. The numerical testing of these theoretical predictions has been successfully performed in the two-dimensional (2D) case in Pouquet et al. (1988).

Note that direct numerical studies of imbalanced MHD turbulence in three dimensions (3D) are extremely challenging. Thus, so far, the simulations of strong imbalanced turbulence were limited to rather idealized setups (Maron & Goldreich 2001; Cho et al. 2002), which did not allow making definitive conclusions about its properties. In this paper, in order to test our analytical model, we provide simulations that go a step further compared to the aforementioned studies. We hope that future higher resolution simulations will provide the definitive test for our model, as well as alternative models, of imbalanced turbulence.

We think that the best data on the imbalanced regime are currently available from observations of solar wind turbulence e.g., Horbury 1999). These data, collected by spacecraft, are consistent with the Kolmogorov $-5/3$ spectrum but do not provide sufficient insight into the anisotropy with respect to the local magnetic field. The imbalanced turbulence is not a rare exception; on the contrary, such processes as preferential decay of a weaker wave and the escape from the regions that generate perturbations make the imbalanced turbulence ubiquitous. It goes without saying that models of such turbulence are much needed in astrophysics.
There are good reasons to expect that the imbalanced turbulence is more complex than its balanced counterpart. While there was a wide agreement that a freely decaying imbalanced MHD turbulence tends toward pure Alfvénic states (see, e.g., Dobrowolny et al. 1980; Biskamp 2003; Maron & Goldreich 2001), due to the shorter nonlinear cascading time of weaker waves, the subject definitely requires more dedicated studies, and this paper should be considered in this context. In particular, in this paper we consider stationary, forced, imbalanced turbulence, where, unlike the decaying case, the imbalance stays constant. We attempt to construct a model of homogeneous imbalanced turbulence that describes small scales of decaying turbulence at any given time with larger scales providing outer scale conditions. Naturally, the turbulence in the actual astrophysical settings, e.g., solar wind turbulence, has a degree of inhomogeneity and requires considering additional processes, e.g., the degree of imbalance can be affected by parametric instabilities, reflections from the preexisting density fluctuations, shear, etc. (see Roberts et al. 1987). Those are beyond the simplified treatment of our paper.

An attempt to construct the model of stationary strong imbalanced turbulence was done in Lithwick et al. (2007, hereafter LGS07). This model assumes the strong GS95-type cascading of both large- and small-amplitude oppositely moving modes. In what follows, we propose a different model of imbalanced turbulence. In § 2 we revisit the critical balance argument and discuss a new process of cascading, which we relate to the process of propagating of Alfvénic perturbations in the field wandering that is induced by the oppositely moving wave. Using this “propagation cascading” in the case of the balanced turbulence, we recover GS95 relations. However, for the pronounced imbalance between the oppositely moving waves, this process results in a picture that is different from that in GS95. In fact, in our model, the high-amplitude Alfvénic perturbations do not follow the usual strong cascade. In this respect, our model has features that can be attributed to the generalization of the weak turbulence cascade (Grappin et al. 1983; Lithwick & Goldreich 2003). In § 3 we discuss the scaling relations for the turbulent cascade that follow from our model, and in § 4 we compare our predictions with our 3D numerical simulations of the imbalanced cascade. We discuss the tentative nature of the correspondence that we obtain within our testing and specify the issues to be addressed by further numerical studies. In § 5 we compare our results with other studies.

2. CRITICAL BALANCE REVISED

The “perpendicular cascade,” a concept that was rigorously developed in the theory of weak Alfvénic turbulence (Ng & Bhattacharjee 1996; Galtier et al. 2002), was a theory of nonlinear interacting Alfvénic waves that, due to the particular dispersion relation of the waves, conserved wave frequencies \( \omega = k_T v_A \). It deeply contrasted with earlier Iroshnikov-Kraichnan models, where, due to the assumed isotropization, the parallel wavenumber will be of the order of the total wavenumber and the frequency \( \omega = k_A v_A \approx k_T v_A \) changes with \( k \). The perpendicular cascade, however, makes the cascade stronger and not weaker, while the energy goes downscale. This raised a question of what happens when the perturbation theory breaks down and the turbulence becomes strong. GS95 argued that the turbulence will stay on the edge of being strong because of the uncertainty relation between the cascading timescale and the wave frequency \( \tau_{\text{casc}} \), which allowed for the increase of frequencies of the interacting wave packets. Since this process of increase of \( k_T \) comes essentially from the irreversibility of energy cascading downscale, we call it the “causality effect.” This way nonlinear interaction stays marginally strong by controlling the anisotropy. But is this the only way to increase frequency or \( k_T \)? In this paper we advocate supplementing the causality effect with a different mechanism that works when the cascading of the eddy is done by the countereddy, which is tilted with respect to the mean field due to the different definition of the local mean field at different scales. The details are given in the following.

Indeed, the parallel wavenumber for an Alfvénic eddy in strong turbulence actually depends on how the local mean field is defined. Maron & Goldreich (2001) have shown that the Alfvénic eddy propagates along the field lines that are defined by the average field plus the field of the counterwave.

Let us try to calculate the characteristic uncertainty in \( k_T \) that comes from this effect in the balanced case. In general, it will depend on the eddy’s polarization, but at maximum it will be around \( k \sin \theta \), where \( \theta \) is the angle of the field wandering, or \( k_A \delta h(1/k_A)/v_A \) (\( h \) is the magnetic field perturbation in Alfvénic units). What is the characteristic scale \( l \) that enters this expression? We can argue that in the case of a local balanced turbulence there is only one designated scale, the one we consider cascading from. Therefore, in the case of balanced turbulence we expect \( \delta k_T = k_A \delta h(1/k_A)/v_A \). This coincides with the well-known critical balance, derived from causality (indeed, in GS95 the cascading rate is of the order of the eddy turnover rate \( k \omega ) \). We were able to reproduce causal critical balance of strong MHD turbulence (in the balanced case) by argumentation that involved field wandering. We call this effect, which comes from field wandering, the “propagation” critical balance and argue below that it will provide a qualitatively new picture in the imbalanced case.

Let us now consider the imbalanced case. We adhere to the same “eddy” Ansatz as GS95 in that for every transverse scale of a wave there is a characteristic longitudinal scale. This is the physical meaning behind the terms “eddies” and “wave packets.” We digress from GS95 in a natural way, postulating that both waves have different anisotropies, i.e., the dependence of longitudinal scale \( \lambda \) to transverse scale \( \lambda \) is different for each kind of wave. This situation is presented in Figure 1, where some arbitrary longitudinal scale \( \lambda \) corresponds to the two different transverse scales, \( \lambda_1 \) for weak wave \( w^- \) and \( \lambda_2 \) for strong wave \( w^+ \). \( \lambda^- \) is a longitudinal scale of \( w^- \) wave having transverse scale \( \lambda_1 \). Since we originally decided to consider strong turbulence, let us assume that at least the \( w^- \) is being strongly cascaded by \( w^- \).

In this case the most effective mixing of \( w^- \) on scale \( \lambda_1 \) will be obtained through \( w^+ \) motions that are on the same scale. The longitudinal scale for \( w^- \) will be provided by causal critical balance, since its cascading is fast.

The cascading of \( w^+ \) is somewhat more complicated. Since the amplitude of \( w^- \) is not large enough to provide strong perturbations in \( w^+ \), the \( w^- \) will be perturbed weakly, and the cascading timescale will be diminished according to the “strength” of the \( w^- \), just like it does in weak turbulence. Moreover, now the \( w^- \) eddies will be cascading \( w^+ \) eddies with similar longitudinal scales, which is the generic feature of weak cascading.

\(^1\) In this paper we use wavevectors \( k, k_T \) and length scales \( \lambda = 1/k, \lambda = 1/k_A \) interchangeably. While wavevector representation highlights geometry and conservation laws, the length scales are measured quantities obtained from statistical averaging of numerical or observational data. We also use “waves,” “wave packets,” and “eddies” interchangeably.

\(^2\) This is the assumption that is often employed in a strong turbulence, such as the Kolmogorov model of incompressible hydroturbulence, or GS95.
The perturbations provided by \( w^- \) will have a transverse scale of \( \lambda_1 \). In other words, the energy of \( w^- \) will be transferred between \( \lambda_2 \) and \( \lambda_1 \). What determines the longitudinal scale for cascaded \( w^+ \)? This is the central question of this paper. We argue that this longitudinal scale will be determined by the propagation critical balance, in the following way. The wave packets of \( w^- \) are strongly aligned to the mean field on scale \( \lambda_1 \); therefore, they are randomly oriented with respect to the mean field at a larger scale \( \lambda_2 \). The rms angle of wavevector of \( w^- \)-eddies with respect to mean field on \( \lambda_2 \) will be around \( \theta \approx \beta b^+ (\lambda_2)/v_A \). This slant of \( w^- \) wave packets will determine the increase of \( k_1 \) for newly cascaded \( w^+ \) packets at \( \lambda_1 \) (see Fig. 1).

Finally, one can verify that the increase of \( k_1 \) of \( w^+ \) due to causality is smaller (due to the fact that cascading is rather slow) and can be neglected. Also, the increase of \( k_1 \) in \( w^- \)-eddies due to the propagation effect will be negligible due to very weak wandering provided by the \( w^- \) field. In this picture we expect the \( w^- \) wave packets on scale \( \lambda_1 \) to closely follow mean field lines on scale \( \lambda_1 \). Essentially, this will require \( \Lambda^- > \Lambda^+ \). Since the amplitude of the \( w^+ \) is much larger than that of the \( w^- \), it provides a much stronger increase of \( k_1 \) by propagation, effectively making the \( w^+ \) eddy less anisotropic than the \( w^- \) eddy and validating this assumption.

Summarizing the result of this section, the new interpretation of critical balance in the strongly imbalanced case is that the \( k_1 \) of the weak wave increases due to the finite lifetime of the wave packet, while in the strong wave it increases due to the field wandering of the strong wave itself on larger scales. This effect does not contradict the exact MHD solution of the wave propagating in one direction because it requires the oppositely propagating wave as an intermediary. The dealignment of the cascaded strong wave is possible because the weak wave, acting as a cascading agent, is strongly aligned with the field lines on a scale that is different (smaller) than the scale of the strong wave it is acting on.

3. PHENOMENOLOGY

We proceed with phenomenological treatment of imbalanced turbulence using energy cascading rules for each wave and the new rules for anisotropies developed in the previous section. We denote Elsässer wave amplitudes in a sense of rms values as \( w^+ \), \( w^- \), where “+” corresponds to the strong wave and “−” to the weak wave. Longitudinal and transverse scales are \( \Lambda \) and \( \lambda \), respectively, and \( \epsilon^\pm \) are energy fluxes for both waves.

Since we have different anisotropies for both waves, we are dealing with four quantities, which are \( w^+ (\lambda) \), \( w^- (\lambda) \), \( \Lambda^+ (\lambda) \), and \( \Lambda^- (\lambda) \), so we need four equations to solve for these functions. There are two equations for the constant flow of energy in \( k \)-space and another two equations that determine frequency uncertainties, or \( \Lambda \)-values.

As was explained in §2, the longitudinal scale for the weak wave will be defined by causality, or

\[
\Lambda^- = v_A \left[ \frac{w^+(\lambda_1)}{\lambda_1} \right]^{-1}.
\]

(1)

On the other hand, the strong wave’s \( \Lambda \) will be defined by propagation, as

\[
\left( \frac{\Lambda^+}{\lambda_1} \right)^{-1} = \frac{w^+(\lambda_2)}{v_A} \frac{1}{\lambda_1}.
\]

(2)

The weak wave energy flux will be determined by the strong shearing of the strong wave as

\[
\epsilon^- = \frac{w^- (\lambda_1)^2 w^+(\lambda_1)}{\lambda_1},
\]

(3)

while for the strong wave the energy flux will be decreased by the wave perturbation strength as

\[
\epsilon^+ = \frac{[w^+(\lambda_2)]^2 w^- (\lambda_1) \Lambda^- (\lambda_1)}{v_A \lambda_1} \frac{f(\lambda_1/\lambda_2)}{\lambda_2}. \]

(4)

Here \( f(\lambda_1/\lambda_2) \) is a factor that will account for the fact that our weak cascading is somewhat nonlocal and could also include the dependence on spectral slopes (see Galtier et al. 2002), although, for the sake of simplicity, in this section we assume \( f = 1 \).

Let us now discuss a self-similar solution of equations (1)–(4), which we seek in the form of

\[
w^\pm (\lambda) = w_0^\pm (\lambda/L)^\alpha^\pm, \quad \Lambda^\pm (\lambda) = \Lambda_0^\pm (\lambda/L)^\beta^\pm.
\]

(5)

The scaling exponents are calculated as \( \alpha_+ = \beta_+ = \beta \), \( \alpha_- = 1 - \beta \), and \( \alpha_- = (1 - \alpha_+)/2 \); here \( \beta \) is arbitrary. The correspondence with GS95 is obtained with \( \beta = \frac{1}{2} \).

Let us discuss the applicability of these solutions to the realistic imbalanced turbulence. Quite surprisingly, the self-similar solution cannot be applied to the imbalanced turbulence driven isotropically on the outer scale. This will require \( \Lambda_0^- (L) = \Lambda_0^+ (L) \), which is not possible because equations (3), (4), and (1) will give \( \epsilon^+_\epsilon^- = 1 \). This situation is exacerbated by the fact that equations (1)–(4) are nonlinear, so it is not possible to combine several known solutions to satisfy boundary conditions.

Suppose, however, that self-similar solutions are realizable with some outer scale boundary conditions. The way to uphold the correspondence with GS95 \( \beta = \frac{1}{2} \) is to take critical balance (1) and (2) at the outer scale for weak and strong waves, respectively, and modify (2) such that both conditions are in agreement. This is achieved, e.g., by replacing \( \Lambda^+ \) with the geometrical average of \( \Lambda^+ \) and \( \Lambda^- \). Note that \( \beta = \frac{1}{2} \) is also the one that has corresponding spectral slopes \( \alpha^\pm \) equal to each other and a Kolmogorov value of \( \frac{1}{2} \). If \( \beta = \frac{1}{2} \), the spectra are meeting and the self-similarity is broken.
With the solution, described above, we have $\Lambda_0^w/\Lambda_0^w = \epsilon^+ / \epsilon^-$ and $\lambda_2/\lambda_1 = (\epsilon^+ / \epsilon^-)^{3/2}$. Note that this solution is realized not necessarily due to a different anisotropy of $w^+$ and $w^-$ driving, but in a quite more general setting where $\Lambda_0^w(L)$ and $\Lambda_0^w(L)$ play the role of parameters of the asymptotic power-law solution. Indeed, if we drive turbulence with the same anisotropy on the outer scale, there will be a non-power-law section of the solution that will ensure that $\lambda_2/\lambda_1 > 1$, to fulfill $\epsilon^+ / \epsilon^- > 1$, which given enough inertial range, will transit into the asymptotic power-law solution. We believe that this situation is realized, e.g., in the solar wind, where the counterwaves are generated by reflection on density inhomogeneities on large scales and thus have the same outer scale anisotropy as the direct waves.

4. FACTORS, RELAXATION TO THE STEADY STATE, VISCOUS SCALE, AND LIMITING CASES

Equation (4) is written assuming that cascading is weakened in a way similar to how it is weakened in the weak Alfvénic turbulence. This assertion does not come from any analytical argument, since our turbulence could not be treated perturbatively. Due to the fact that we have two length scales, $\lambda_1$ and $\lambda_2$, and two amplitudes, $w^+$ and $w^-$, there are several dimensionally correct ways to write dissipation proportional to the fourth order of amplitude. We stuck with equation (4), as it seemed most plausible. On the second note, the cascading in weak turbulence is nonlocal and in a sense that the dissipation rate depends not only on the local values of perturbation but on the slope as well (Galtier et al. 2002; Lithwick & Goldreich 2003). Furthermore, in our model, the cascading of $w^+$ is somewhat nonlocal, if $\lambda_1 \neq \lambda_2$. The energy is being transferred not in the local vicinity of one particular scale, but between two different scales. In principle, this would require a logarithmic correction factor such as $f = 1 + f_0 \log (\lambda_2/\lambda_1)$.

Strong local driven turbulence has steady state relaxation timescales that are usually equal to the dissipation timescale. Not only is the $w^+$ cascading time much larger (see eq. [4]), but there is some nonlocality and slope dependence that, in principle, could carry information upscale and make the establishment of the steady state even longer. With the nonlinear dissipation timescale being very large, the linear damping or viscosity will become important earlier, diminishing the inertial range. These factors make numerical study of imbalanced turbulence extremely challenging. Also, in a realistic compressible turbulence the parametric instability or steepening becomes relatively important, as the turbulent cascading rates decrease.

Let us now discuss the onset of viscosity at small scales. If our turbulence would be completely local both in the sense that the same scales of "+" and "-" waves cascade each other and in a Kolmogorov sense that energy is transferred locally, that would mean an equal viscous scale for both waves and that the ratio of amplitudes must satisfy $w^+/w^- = (\epsilon^+ / \epsilon^-)^{3/2}$. Our model, however, is somewhat nonlocal, and, according to our picture of cascading, presented in Figure 1, strong waves $w^+$ on scale $\lambda_2$ are cascaded by weak waves $w^-$ on a smaller scale $\lambda_1$. Once the viscous dissipation becomes important for $w^-$ on scale $\lambda_1$, the amount of $w^+$ waves, cascaded from $\lambda_2$ to $\lambda_1$, will strongly decrease. Therefore, the $w^+$ will have a shorter inertial interval than $w^-$. This effect is observed on numerical spectra in Figure 2. Since the viscous scales are different for $w^+$ and $w^-$, the amplitude ratio mentioned above will be offset by some power of $\lambda_1/\lambda_2$. Designating the viscous scale as $\lambda_v$ and assuming $\lambda_v^+ = \lambda_v^-(\lambda_2/\lambda_1)^{3/2}$, we obtain $w^+/w^- = (\epsilon^+ / \epsilon^-)^{3/2}(\lambda_2/\lambda_1)^{3/2}$. If we use the self-similar, GS95-like solution from § 3, i.e., $\lambda_2/\lambda_1 = (\epsilon^+ / \epsilon^-)^{3/2}$, we get $w^+/w^- = (\epsilon^+ / \epsilon^-)^{3/2}$ (there is no guarantee, however, that a self-similar solution will be established in a limited inertial range).

The limit of a vanishing weak wave is understood as follows. When $\epsilon^-$ goes to zero, the steady state relaxation time goes to infinity, in other words, the steady state is never established and the amplitude of the strong wave grows indefinitely from driving. Also, as $\lambda_v^+$ increases and approaches the outer scale, the viscous dissipation becomes more important than the nonlinear dissipation and the cascading stops, with the strong wave being present only on the outer scale and being dissipated viscously.

Finally, we would like to present our preliminary numerical results of imbalanced driven turbulence. Both waves were driven independently with the same isotropy on the outer scale and different amplitudes. The energy input ratio $\epsilon^+ / \epsilon^-$ was about 4. We evolved the simulation for more than 30 Alfvén times unless energies of both species, their spectra, and anisotropies had become stationary. The existence of the stationary state of imbalanced forced turbulence is one of the main assumptions in this paper, as well as in LGS07. The resulting spectra and anisotropies are shown in Figures 2 and 3. The ratio of the total $|w^+|^2$ to $|w^-|^2$ "energies" was around 100. In the middle of the inertial
interval the ratio of the power spectra was around 70 \cite{LGS07} is predicting this ratio as \((\epsilon^+/\epsilon^-)^2 \approx 16\). The anisotropy curves (Fig. 3) diverge from the outer scale, as we suggested in § 3. The \(w^-\) perturbations exhibit extreme anisotropy on small scales, while \(w^+\) anisotropy is less pronounced. The inertial range of \(w^-\) is somewhat larger than that of \(w^+\) (Fig. 2), which is consistent with our model above. One may notice that the spectral slopes are slightly different for \(|w^+|^2\) and \(|w^-|^2\). Incidentally, this is in conflict with LGS07, which predicts the same slope. The effect of \(\text{pinning}\), i.e., the equality \(|w^+|^2 = |w^-|^2\) at the dissipation scale, predicted in isotropic imbalanced turbulence by Grappin et al. \cite{1983A&A...126...51G} and observed in 2D numerical simulation by Pouquet et al. \cite{1988PhR...163....1P}, as well as discussed for the weak imbalanced turbulence by Lithwick \& Goldreich \cite{2003ApJ...618..449L}, is not observed in our simulations either.\(^3\) One also may see eq. (1) that if the anisotropy slope for \(w^-\) is shallower than GS95’s \(\gamma\), the spectral slope of \(w^+\) should be steeper than \(-5/3\). If this is true, equation (3) gives the spectral slope of \(w^-\) that is shallower than \(-5/3\), which can explain the difference in spectral slopes.

The actual slopes observed on Figure 2 are approximately \(-1.5\) for the strong wave and \(-1.2\) for the weak wave. Note that our asymptotic power-law solution from § 3 suggests that both slopes are \(-5/3 \approx -1.67\). Also, the anisotropy slopes from Figure 3 are approximately 0.73 for the strong wave and 0.49 for the weak wave, while our asymptotic power-law solution requires equal slopes of \(\frac{\gamma}{2} \approx 0.67\). At this point it is not entirely clear whether this disagreement is entirely due to the fact that the asymptotic power-law solution is not yet established in our simulation with limited inertial range and isotropic driving on the outer scale (see the discussion in § 3) or it is partially due to the numerical effects such as the bottleneck effect. Indeed, these imbalanced runs were performed with sixth-order hyperviscosity, and in a similar balanced simulation the observed slope was around \(-1.45\).\(^4\)

At present, we think that different anisotropies of \(w^+\) and \(w^-\) perturbations and, most notably, the weaker anisotropy of \(w^-\) are the best available evidence favoring our model. Note that the weaker anisotropy of \(w^+\) is hard to explain in the framework of GS95, where the parallel scale is equal to the product of the cascading timescale and the Alfven speed.

We will present a more extensive set of imbalanced turbulence simulations and a careful comparison with existing models in a future paper.

5. DISCUSSION

The first treatment of the imbalanced case of strong anisotropic turbulence, LGS07, uses subtle argumentation in order to support the notion that anisotropies for both waves will be the same. The main argument was that even though the strain rate of the weak wave will be smaller, it will be applied coherently, thus resulting in large perturbations in the strong wave. Therefore, as LGS07 argued, the old critical balance will give the same longitudinal scale for both waves. We feel that this argument is somewhat forced. Indeed, the coherent application of the strain does not necessarily mean cascading. While in the case of the strong shearing there is only one longitudinal scale, in the case of weak shearing there are two: the correlation scale and the coherence scale. This is clearly seen in the case of weak Alfvenic turbulence when the correlation scale is essentially determined by the outer scale and is not changed by cascading, while the coherence scale is determined by nonlinear cascading and is typically much larger than the correlation scale (in fact, this inequality is the condition of applicability of weak turbulence). In weak turbulence the large coherence scale does not lead to any unusual increase in cascading. It is not at all obvious why this situation should change in the case of strong turbulence, considered in LGS07. Another shortcoming of the LGS07 model is that, while being completely local in both \(k_\parallel\) and \(k_\perp\), it dictates different nonlinear cascading timescales for “+” and “−” waves, which leads to inconsistency near the dissipation scale. Indeed, the LGS07 model has \(w^+/w^- = (\epsilon^+/\epsilon^-)^{1/2}\), while the local model must have \(w^+/w^- = (\epsilon^+/\epsilon^-)^{1/2}\) to ensure transition into viscous/resistive dissipation (see § 4).

We note that earlier closure models for isotropic MHD turbulence used phenomenological “relaxation of triple correlations” \citep[see, e.g.,][]{1976ApJ...206..227P} with the Alfvenic timescale, which effectively led to the weakening of interaction and the IK \(\sim 3/2\) spectrum. This approach introduced the idea that the large-scale field can lead to wave decorrelation and the increase of wave frequency and \(k_\parallel\). In this paper the increase of \(k_\parallel\) is obtained in a different way, namely, it is not dictated by the full background field \(B\) and the Alfvenic timescale \((k_B v_A)^{-1}\) that is derived from it, but by a perturbation \(\delta B\), which describes a difference between the local field averaged on two different scales, \(\lambda_1\) and \(\lambda_2\) (see Fig. 1).

The influence of the strong turbulence field wandering on the \(k\) vector of the wave was proposed in Farmer \& Goldreich \citeyear{2004ApJ...618..449L} in the context of the cascading of quasi-parallel modes, where it was necessary to obtain nonzero \(k_\perp\) in order for the wave to dissipate \citep[see also][]{2008ApJ...687..961B}. In our problem the situation is, in a sense, reversed, as we need to obtain the increase of \(k_\parallel\). In both cases the decorrelation seems to be a feature of strong turbulence that does not directly appear in perturbative calculations.

A. B. thanks the IceCube project for support of his research. A. L. acknowledges the NSF grant AST 03-07869 and the support from the Center for Magnetic Self-Organization in Laboratory and Astrophysical Plasma.

REFERENCES

Biskamp, D. 2003, Magnetohydrodynamic Turbulence (Cambridge: Cambridge Univ. Press).
Boldyrev, S. 2006, Ph. Rev. Lett., 96, 115002
Chandran, B. D. G. 2005, Ph. Rev. Lett., 95, 265004
Cho, J., \& Lazarian, A. 2002, Ph. Rev. Lett., 88, 245001
———. 2003, MNRAS, 345, 325
Cho, J., Lazarian, A., \& Vishniac, E. T. 2002, ApJ, 564, 291
Dobrowolny, M., Mangeney, A., \& Veltri, P. 1980, Ph. Rev. Lett., 45, 144
Farmer, A. J., \& Goldreich, P. 2004, ApJ, 604, 671
Galtier, S., Nazarenko, S. V., Newell, A. C., \& Pouquet, A. 2002, ApJ, 564, L49
Gogoberidze, G. 2007, Phys. Plasmas, 14, 022304
Goldreich, P., \& Sridhar, S. 1995, ApJ, 438, 763 (GS95)
Grappin, R., Pouquet, A., \& Leorat, J. 1983, A&A, 126, 51
Higdon, J. C. 1984, ApJ, 285, 109
Higdon, J. C. 1984, ApJ, 285, 109
Horbury, T. S. 1999, in Plasma Turbulence and Energetic Particles in Astrophysics, ed. M. Ostrowski \& R. Schlickeiser (Kraków: Uniwersytet Jagiellonski), 115
Iofridshnikov, P. S. 1963, Astron. Zh., 40, 742
Kolmogorov, A. A., 1941, Dokl. Akad. Nauk SSSR, 31, 538
Kraichnan, R. H. 1965, Phys. Fluids, 8, 1385
Kritsuk, A. G., Norman, M. L., Padoan, P., & Wagner, R. 2007, ApJ, 665, 416
Lithwick, Y., & Goldreich, P. 2001, ApJ, 562, 279
———. 2003, ApJ, 582, 1220
Lithwick, Y., Goldreich, P., & Sridhar, S. 2007, ApJ, 655, 269 (LGS07)
Maron, J., & Goldreich, P. 2001, ApJ, 554, 1175
Matthaeus, W. M., Goldstein, M. L., & Montgomery, D. C. 1983, Phys. Rev. Lett., 51, 1484
Montgomery, D. C., & Turner, L. 1981, Phys. Fluids, 24, 825
Müller, W.-C., Biskamp, D., & Grappin, R. 2003, Phys. Rev. E, 67, 066302
Ng, C. S., & Bhattacharjee, A. 1996, ApJ, 465, 845
Pouquet, A., Frisch, U., & Leorat, J. 1976, J. Fluid Mech., 77, 321
Pouquet, A., Sulem, P. L., & Meneguzzi, M. 1988, Phys. Fluids, 31, 2635
Roberts, D. A., Goldstein, M. L., Klein, L. W., & Matthaeus, W. H. 1987, J. Geophys. Res., 92, 12023
Shebalin, J. V., Matthaeus, W. H., & Montgomery, D. 1983, J. Plasma Phys., 29, 525