Non-critical dimensions for critical problems involving fractional Laplacians

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Abstract

We study the Brezis–Nirenberg effect in two families of noncompact boundary value problems involving Dirichlet-Laplacian of arbitrary real order $m > 0$.

Keywords: Fractional Laplace operators, Sobolev inequality, Hardy inequality, critical dimensions.

1 Introduction

Let $m, s$ be two given real numbers, with $0 \leq s < m < \frac{n}{2}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain in $\mathbb{R}^n$ and put

$$2^*_m = \frac{2n}{n - 2m}.$$ 

We study equations

\begin{align*}
(-\Delta)^m u &= \lambda (-\Delta)^s u + |u|^{2^*_m - 2} u \quad \text{in} \ \Omega, \tag{1.1}
\end{align*}

\begin{align*}
(-\Delta)^m u &= \lambda |x|^{-2s} u + |u|^{2^*_m - 2} u \quad \text{in} \ \Omega, \tag{1.2}
\end{align*}

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under suitably defined Dirichlet boundary conditions. In dealing with equation (1.2) we always assume that \( \Omega \) contains the origin. For the definition of fractional Dirichlet–Laplace operators \((-\Delta)^m, (-\Delta)^s\) and for the variational approach to (1.1), (1.2) we refer to the next section.

The celebrated paper \([3]\) by Brezis and Nirenberg was the inspiration for a fruitful line of research about the effect of lower order perturbations in noncompact variational problems. They took as model the case \( n > 2, m = 1, s = 0 \), that is,

\[
-\Delta u = \lambda u + |u|^{\frac{4}{n-2}} u \quad \text{in} \; \Omega, \quad u = 0 \quad \text{on} \; \partial \Omega.
\]

(1.3)

Brezis and Nirenberg pointed out a remarkable phenomenon that appears for positive values of the parameter \( \lambda \): they proved existence of a nontrivial solution for any small \( \lambda > 0 \) if \( n \geq 4 \); in contrast, in the lowest dimension \( n = 3 \) non-existence phenomena for sufficiently small \( \lambda > 0 \) can be observed. For this reason, the dimension \( n = 3 \) has been named critical for problem (1.3).

Clearly, as larger \( s \) is, as stronger the effects of the lower order perturbations are expected in equations (1.1), (1.2). We are interested in the following question: Given \( m < \frac{n}{2} \), how large must be \( s \) in order to have the existence of a ground state solution, for any arbitrarily small \( \lambda > 0 \)? In case of an affirmative answer, we say that \( n \) is not a critical dimension.

We present our main result, that holds for any dimension \( n \geq 1 \) (see Section 4 for a more precise statement).

**THEOREM.** If \( s \geq 2m - \frac{n}{2} \) then \( n \) is not a critical dimension for the Dirichlet boundary value problems associated to equations (1.1) and (1.2).

We point out some particular cases that are included in this result.

- If \( m \) is an integer and \( s = m - 1 \), then at most the lowest dimension \( n = 2m + 1 \) is critical.

- For any \( n > 2m \) there always exist lower order perturbations of the type \(|x|^{-2s} u\) and of the type \((-\Delta)^s u\) such that \( n \) is not a critical dimension.

- If \( m < 1/4 \) then no dimension is critical, for any choice of \( s \in [0, m) \).

\(^1\) compare with [13], [8].
After [3], a large number of papers have been focussed on studying the effect of linear perturbations in noncompact variational problems of the type (1.1). Most of these papers deal with \( s = 0 \), when the problems (1.1) and (1.2) coincide. Moreover, as far as we know, all of them consider either polyharmonic case \( 2 \leq m \in \mathbb{N} \), see for instance [13], [6], [2], [10], [7], or the case \( m \in (0,1) \), see [14], [15]. We cite also [4], where equation (1.1) is studied in case \( m = 2, s = 1 \). Thus, our Theorem 4.2 covers all earlier existence results.

Finally, we mention [1] (see also [16]) where equation (1.1) for the so-called Navier-Laplacian is studied in case \( m \in (0,1), s = 0 \). For a comparison between the Dirichlet and Navier Laplacians we refer to [12].

The paper is organized as follows. After introducing some notation and preliminary facts in Section 2 we provide the main estimates in Section 3. In Section 4 we prove Theorem 1 and point out an existence result for the case \( s < 2m - \frac{n}{2} \).

## 2 Preliminaries

The fractional Laplacian \((-\Delta)^m u\) of a function \( u \in C_0^\infty(\mathbb{R}^n)\) is defined via the Fourier transform

\[
\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, dx
\]

by the identity

\[
\mathcal{F}[(-\Delta)^m u](\xi) = |\xi|^{2m} \mathcal{F}[u](\xi).
\] (2.1)

In particular, Parseval’s formula gives

\[
\int_{\mathbb{R}^n} (-\Delta)^m u \cdot u \, dx = \int_{\mathbb{R}^n} |(-\Delta)^m u|^2 \, dx = \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[u]|^2 \, d\xi.
\]

We recall the well known Sobolev inequality

\[
\int_{\mathbb{R}^n} |(-\Delta)^m u|^2 \, dx \geq \mathcal{S}_m \left( \int_{\mathbb{R}^n} |u|^{2m} \, dx \right)^{2/2m},
\] (2.2)

that holds for any \( u \in C_0^\infty(\mathbb{R}^n) \) and \( m < \frac{n}{2} \), see for example [17] 2.8.1/15.
Let $\mathcal{D}^m(\mathbb{R}^n)$ be the Hilbert space obtained by completing $C_0^\infty(\mathbb{R}^n)$ with respect to the Gagliardo norm

$$
\|u\|_m^2 = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} u|^2 \, dx.
$$

(2.3)

Thanks to (2.2), the space $\mathcal{D}^m(\mathbb{R}^n)$ is continuously embedded into $L_2^{2m}(\mathbb{R}^n)$. The best Sobolev constant $S_m$ was explicitly computed in [5]. Moreover, it has been proved in [5] that $S_m$ is attained in $\mathcal{D}^m(\mathbb{R}^n)$ by a unique family of functions, all of them being obtained from

$$
\phi(x) = (1 + |x|^2)^{\frac{2m-n}{2}}
$$

(2.4)

by translations, dilations in $\mathbb{R}^n$ and multiplication by constants.

Dilations play a crucial role in the problems under consideration. Notice that for any $\omega \in C_0^\infty(\mathbb{R}^n)$, $R > 0$ it turns out that

$$
\int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\omega](\xi)|^2 \, d\xi = R^{n-2m} \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\omega(R\cdot)](\xi)|^2 \, d\xi
$$

(2.5)

$$
\int_{\mathbb{R}^n} |\omega|^{2m} \, dx = R^n \int_{\mathbb{R}^n} |\omega(R\cdot)|^{2m} \, dx.
$$

Finally, we point out that the Hardy inequality

$$
\int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} u|^2 \, dx \geq H_m \int_{\mathbb{R}^n} |x|^{-2m} |u|^2 \, dx
$$

(2.6)

holds for any function $u \in \mathcal{D}^m(\mathbb{R}^n)$. The best Hardy constant $H_m$ was explicitly computed in [11].

The natural ambient space to study the Dirichlet boundary value problems for (1.1), (1.2) is $\tilde{\mathcal{H}}^m(\Omega) = \{u \in \mathcal{D}^m(\mathbb{R}^n) : \text{supp } u \subset \overline{\Omega}\}$, endowed with the norm $\|u\|_m$. By Theorem 4.3.2/1 [11], for $m + \frac{1}{2} \notin \mathbb{N}$ this space coincides with $H_0^m(\Omega)$ (that is the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$), while for $m + \frac{1}{2} \in \mathbb{N}$ one has $\tilde{\mathcal{H}}^m(\Omega) \subsetneq H_0^m(\Omega)$. Moreover, $C_0^\infty(\Omega)$ is dense in $\tilde{\mathcal{H}}^m(\Omega)$. Clearly, if $m$ is an integer then $\tilde{\mathcal{H}}^m(\Omega)$ is the standard Sobolev space of functions $u \in H^m(\Omega)$ such that $D^\alpha u = 0$ for every multiindex $\alpha \in \mathbb{N}^n$ with $0 \leq |\alpha| < m$. 

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We agree that $(-\Delta)^0 u = u$, $\tilde{H}^0(\Omega) = L^2(\Omega)$, since it reduces to the standard $L^2$ norm in case $m = 0$.

We define (weak) solutions of the Dirichlet problems for (1.1), (1.2) as suitably normalized critical points of the functionals

$$
\mathcal{R}_{\lambda,m,s}[u] = \frac{\int_\Omega |(-\Delta)^m u|^2 \, dx - \lambda \int_\Omega |(-\Delta)^s u|^2 \, dx}{\left( \int_\Omega |u|^{2m} \, dx \right)^{2/2m}}
$$

and

$$
\tilde{\mathcal{R}}_{\lambda,m,s}[u] = \frac{\int_\Omega |(-\Delta)^m u|^2 \, dx - \lambda \int_\Omega |x|^{-2s} |u|^2 \, dx}{\left( \int_\Omega |u|^{2m} \, dx \right)^{2/2m}}
$$

respectively. It is easy to see that both functionals (2.7), (2.8) are well defined on $\tilde{H}^m(\Omega) \setminus \{0\}$.

We conclude this preliminary section with some embedding results.

**Proposition 2.1** Let $m, s$ be given, with $0 \leq s < m < n/2$.

i) The space $\tilde{H}^m(\Omega)$ is compactly embedded into $\tilde{H}^s(\Omega)$. In particular the infima

$$
\Lambda_1(m,s) := \inf_{u \in \tilde{H}^m(\Omega), u \not= 0} \frac{\|u\|_{2m}^2}{\|u\|_s^2}, \quad \tilde{\Lambda}_1(m,s) := \inf_{u \in \tilde{H}^m(\Omega), u \not= 0} \frac{\|u\|_{2m}^2}{\|x|^{-s}u\|_0^2}
$$

are positive and achieved.

ii) $\inf_{u \in \tilde{H}^m(\Omega), u \not= 0} \frac{\|u\|_{2m}^2}{\|u\|_{L^{2m}}^2} = S_m$.

Statement i) is well known for $\Lambda_1(m,s)$ and follows from (2.6) for $\tilde{\Lambda}_1(m,s)$. To check ii), use the inclusion $\tilde{H}^m(\Omega) \hookrightarrow D^m(\mathbb{R}^n)$ and a rescaling argument. Clearly, the Sobolev constant $S_m$ is never achieved on $\tilde{H}^m(\Omega)$. 

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3 Main estimates

Let $\phi$ be the extremal of the Sobolev inequality \eqref{e:2.2} given by \eqref{e:2.4}. In particular, it holds that
\begin{equation}
M := \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} \phi|^2 \, dx = S_m \left( \int_{\mathbb{R}^n} |\phi|^{2^*_m} \, dx \right)^{2/2^*_m}.
\end{equation}

Fix $\delta > 0$ and a cutoff function $\varphi \in C_0^\infty(\Omega)$, such that $\varphi \equiv 1$ on the ball $\{ |x| < \delta \}$ and $\varphi \equiv 0$ outside $\{ |x| < 2\delta \}$. If $\delta$ is sufficiently small, the function
\[ u_\varepsilon(x) := \varepsilon^{2m-n} \varphi(x) \phi \left( \frac{x}{\varepsilon} \right) = \varphi(x) \left( \varepsilon^2 + |x|^2 \right)^{\frac{2m-n}{2}} \]
has compact support in $\Omega$. Next we define
\[ A_\varepsilon := \int_{\Omega} |(-\Delta)^{\frac{m}{2}} u_\varepsilon|^2 \, dx \quad \tilde{A}_\varepsilon := \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \, dx \]
and we denote by $c$ any universal positive constant.

Lemma 3.1 It holds that
\[ \begin{cases} 
A_\varepsilon \leq \varepsilon^{2m-n} \left( M + c\varepsilon^{n-2m} \right) & \text{if } s > 2m - \frac{n}{2} \\
A_\varepsilon \tilde{A}_\varepsilon \geq c\varepsilon^{4m-n-2s} & \text{if } s > 2m - \frac{n}{2} \\
\tilde{A}_\varepsilon \tilde{A}_\varepsilon \geq c |\log \varepsilon| & \text{if } s = 2m - \frac{n}{2} \\
B_\varepsilon \geq \varepsilon^{-n} \left( (M S_m^{-1})^{2m/2} - c\varepsilon^n \right). & \end{cases} \]

Proof of \eqref{3.2a}. First of all, from \eqref{e:2.5} we get
\begin{equation}
A_\varepsilon = \varepsilon^{2m-n} \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\varphi(\varepsilon \cdot)\phi]|^2 \, d\xi.
\end{equation}

Thus
\[ \Gamma_\varepsilon := \varepsilon^{n-2m} A_\varepsilon - M = \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\varphi(\varepsilon \cdot)\phi]|^2 \, d\xi - \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\phi]|^2 \, d\xi. \]
We need to prove that

\[ |\Gamma_m^\varepsilon| \leq c\varepsilon^{n-2m}. \quad (3.4) \]

If \( m \in \mathbb{N} \), the proof of (3.3) has been carried out in [3], [7]. Here we limit ourselves to the more difficult case, namely, when \( m \) is not an integer. We denote by \( k := \lfloor m \rfloor \geq 0 \) the integer part of \( m \), so that \( m - k > 0 \). Then

\[
\Gamma_m^\varepsilon = \int_{\mathbb{R}^n} |\xi|^{2k} \mathcal{F}[U_-] \cdot (|\xi|^{2(m-k)} \mathcal{F}[U_+] d\xi = 2^{2(m-k)+\frac{n}{2}} \frac{\Gamma(m-k + \frac{n}{2})}{\Gamma(-(m-k))} \int_{\mathbb{R}^n} (-\Delta)^k U_-(x) \cdot \nabla P. \int_{\mathbb{R}^n} \frac{U_+(x) - U_+(y)}{|x-y|^{n+2(m-k)}} dy \, dx,
\]

where \( U_\pm = \varphi(\varepsilon \cdot )\phi \pm \phi \) (the last equality follows from [9, Ch. 2, Sec. 3]).

We split the interior integral as follows:

\[
V.P. \int_{\mathbb{R}^n} \Psi dy = V.P. \int_{|y-x| \leq \frac{|x|}{2}} \Psi dy + \int_{|y-x| \geq \frac{|x|}{2}} \Psi dy + \int_{|y-x| \geq |x|} \Psi dy.
\]

We claim that \( |I_j| \leq c|x|^{2k-n} \) for \( j = 1, 2, 3 \). Indeed, the Lagrange formula gives

\[
|I_1| \leq \max_{|y-x| \leq \frac{|x|}{2}} |D^2 U_+(y)| \cdot \int_{|z| \leq \frac{|x|}{2}} \frac{dz}{|z|^{n+2(m-k)-2}} \leq c|x|^{-(n-2m+2)} \cdot |x|^{2-2(m-k)} = c|x|^{2k-n}.
\]

As concerns the last two integrals we estimate

\[
|I_2| \leq \int_{|y-x| \geq \frac{|x|}{2}} \frac{c|x|^{-(n-2m)}}{|x-y|^{n+2(m-k)}} dy \leq |x|^{-(n+2(m-k))} \cdot c|x|^{2m} = c|x|^{2k-n}
\]

and finally

\[
|I_3| \leq \int_{|y-x| \geq \frac{|x|}{2}} \frac{c|x|^{-(n-2m)}}{|x-y|^{n+2(m-k)}} dy \leq c|x|^{-(n-2m)} \cdot \int_{|z| \geq \frac{|x|}{2}} \frac{dz}{|z|^{n+2(m-k)}} \leq c|x|^{-(n-2m)} \cdot |x|^{-2(m-k)} = c|x|^{2k-n},
\]

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and the claim follows. Now, since
\[
|(-\Delta)^k U_-(x)| \leq \frac{c}{|x|^{n-2(m-k)}} \chi\{|x|\geq \delta/\varepsilon\} + \frac{c\varepsilon^{2k}}{|x|^{n-2m}} \chi\{|\delta/\varepsilon|\leq |x| \leq 2\delta/\varepsilon\},
\]
we obtain
\[
|\Gamma_m^\varepsilon| \leq c \int_{|x|\geq \delta/\varepsilon} \frac{dx}{|x|^{2n-2m}} + c \int_{\delta/\varepsilon \leq |x| \leq 2\delta/\varepsilon} \varepsilon^{2k} \frac{dx}{|x|^{2n-2(m+k)}} \leq c\varepsilon^{n-2m},
\]
that completes the proof of (3.4) and of (3.2a).

**Proof of (3.2b) and (3.2c).** We use the Hardy inequality (2.6) to get
\[
A_\varepsilon^s \geq cA_\varepsilon^s \geq c\varepsilon^{4m-2s-n} \int_{\mathbb{R}^n} |x|^{-2s} |\varphi(\varepsilon \cdot)\phi|^2 \, dx
\]
\[
\geq c\varepsilon^{4m-2s-n} \int_{|x|<\delta/\varepsilon} \frac{dx}{|x|^{2s}(1+|x|^2)^{n-2m}}.
\]
The last integral converges as \(\varepsilon \to 0\) if \(s > 2m - \frac{n}{2}\), and diverges with speed \(|\log \varepsilon|\) if \(s = 2m - \frac{n}{2}\).

**Proof of (3.2d).** For \(\varepsilon\) small enough we estimate by below
\[
\int_{\mathbb{R}^n} |u_\varepsilon^{2m}| = \varepsilon^{-n} \int_{\mathbb{R}^n} |\varphi(\varepsilon \cdot)\phi|^{2m} \, dx = \varepsilon^{-n} \left( \int_{\mathbb{R}^n} |\phi|^{2m} \, dx - \int_{|x|>\delta/\varepsilon} |\varphi(\varepsilon \cdot)\phi|^{2m} \, dx \right)
\]
\[
\geq \varepsilon^{-n} \left( (M S_m^{-1})^{2m/2} - c \right) \int_{|x|>\delta/\varepsilon} |x|^{-2n} \, dx
\]
\[
= \varepsilon^{-n} \left( (M S_m^{-1})^{2m/2} - c\varepsilon^n \right)
\]
and the Lemma is completely proved. \(\square\)

### 4 Two noncompact minimization problems

In this section we deal with the minimization problems
\[
S^\Omega_\lambda(m,s) = \inf_{u \in H^m(\Omega) \setminus \{0\}} \mathcal{R}^\Omega_{\lambda,m,s}[u]; \quad \overline{S}^\Omega_\lambda(m,s) = \inf_{u \in H^m(\Omega) \setminus \{0\}} \overline{\mathcal{R}}^\Omega_{\lambda,m,s}[u],
\]
where the functionals \(\mathcal{R}\) and \(\overline{\mathcal{R}}\) are introduced in (2.7) and (2.8), respectively.
Lemma 4.1 The following facts hold for any $\lambda \in \mathbb{R}$:

i) $S^\Omega_\lambda(m, s) \leq S_m$;

ii) If $\lambda \leq 0$ then $S^\Omega_\lambda(m, s) = S_m$ and it is not achieved;

iii) If $0 < S^\Omega_\lambda(m, s) < S_m$, then $S^\Omega_\lambda(m, s)$ is achieved.

The same statements hold for $\tilde{S}^\Omega_\lambda(m, s)$ instead of $S^\Omega_\lambda(m, s)$.

Proof. The proof is nowadays standard, and is essentially due to Brezis and Nirenberg [3]. We sketch it for the infimum $S^\Omega_\lambda(m, s)$, for the convenience of the reader.

Fix $\varepsilon > 0$ and take $u \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$ such that

$$
(S_m + \varepsilon) \left( \int_{\mathbb{R}^n} |u|^{2m} dx \right)^{1/m} \geq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{2}{2m}} u|^2 dx.
$$

Let $R > 0$ be large enough, so that $u_R(\cdot) := u(R\cdot) \in C_0^\infty(\Omega)$. Using (2.5) we get

$$
S^\Omega_\lambda(m, s) \leq \frac{\|u\|^{2m} - \lambda R^{2(s-m)} \|u\|^2_{L^s_m}}{\|u\|^2_{L^s_m}} \leq (S_m + \varepsilon) \left( 1 + cR^{2(s-m)} \right),
$$

where $c$ depends only on $u$ and $\lambda$. Letting $R \to \infty$ we get $S^\Omega_\lambda(m, s) \leq (S_m + \varepsilon)$ for any $\varepsilon > 0$, and $i)$ is proved.

Next, if $\lambda \leq 0$ then clearly $S^\Omega_\lambda(m, s) = S_m$. If $\lambda = 0$ then $S_m$ is not achieved. The more it is not achieved for $\lambda < 0$, and $ii)$ holds.

Finally, to prove $iii)$ take a minimizing sequence $u_h$. It is convenient to normalize $u_h$ with respect to the $L^{2m}_m$-norm, so that

$$
\int_\Omega |(-\Delta)^{\frac{2}{2m}} u_h|^2 dx - \lambda \int_\Omega |(-\Delta)^{\frac{2}{2m}} u_h|^2 dx = S^\Omega_\lambda(m, s) + o(1).
$$

We can assume that $u_h \to u$ weakly in $\tilde{H}^{m}(\Omega)$ and strongly in $\tilde{H}^s(\Omega)$ by Proposition [2.1]. Since

$$
\lambda \int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 dx = \lambda \int_\Omega |(-\Delta)^{\frac{s}{2}} u_h|^2 dx + o(1)
$$

$$
= \int_\Omega |(-\Delta)^{\frac{s}{2}} u_h|^2 dx - S^\Omega_\lambda(m, s) + o(1)
$$

$$
\geq (S_m - S^\Omega_\lambda(m, s)) + o(1),
$$
then $u \neq 0$. By the Brezis–Lieb lemma we get

$$1 = \|u_h\|_{L^{2\ast}_m}^{2\ast} = \|u_h - u\|_{L^{2\ast}_m}^{2\ast} + \|u\|_{L^{2\ast}_m}^{2\ast} + o(1).$$

Thus

$$S^\Omega_\lambda(m, s) = \|u_h\|_m^2 - \lambda \|u_h\|^s_s + o(1)$$

$$= \left(\|u_h - u\|_m^2 + \|u\|_m^2\right) - \lambda \left(\|u_h - u\|^2_s + \|u\|^2_s\right) + o(1)$$

$$= \frac{\left(\|u_h - u\|_{L^{2\ast}_m}^{2\ast} - \lambda \|u_h - u\|_s^{2\ast}\right) + \left(\|u\|_m^{2\ast} - \lambda \|u\|_s^{2\ast}\right)}{\left(\|u_h - u\|_{L^{2\ast}_m}^{2\ast} + \|u\|_{L^{2\ast}_m}^{2\ast}\right)^{2/2\ast}} + o(1)$$

$$\geq S^\Omega_\lambda(m, s) \cdot \frac{\xi_h^2 + 1}{(\xi_h^{2\ast} + 1)^{2/2\ast}} + o(1),$$

where we have set

$$\xi_h := \frac{\|u_h - u\|_{L^{2\ast}_m}}{\|u\|_{L^{2\ast}_m}}.$$

Since $2^\ast > 2$, this implies that $\xi_h \to 0$, that is, $u_h \to u$ in $L^{2\ast}_m$ and hence $u$ achieves $S^\Omega_\lambda(m, s)$.

\[ \square \]

We are in position to prove our existence result, that includes the theorem already stated in the introduction.

**Theorem 4.2** Assume $s \geq 2m - \frac{n}{2}$.

i) If $0 < \lambda < \Lambda_1(m, s)$ then $S^\Omega_\lambda(m, s)$ is achieved and \textbf{(1.1)} has a nontrivial solution in $H^m(\Omega)$.

ii) If $0 < \lambda < \bar{\Lambda}_1(m, s)$ then $\bar{S}^\Omega_\lambda(m, s)$ is achieved and \textbf{(1.2)} has a nontrivial solution in $H^m(\Omega)$.

**Proof.** Since $0 < \lambda < \Lambda_1(m, s)$ then $S^\Omega_\lambda(m, s)$ is positive, by Proposition \textbf{2.1}. The main estimates in Lemma \textbf{3.1} readily imply $S^\Omega_\lambda(m, s) < S_m$. By Lemma \textbf{4.1}, $S^\Omega_\lambda(m, s)$ is achieved by a nontrivial $u \in H^m(\Omega)$, that solves \textbf{(1.1)} after multiplication by a suitable constant. Thus \textbf{i)} is proved. For \textbf{ii)} argue in the same way. \[ \square \]
In the case $s < 2m - \frac{n}{2}$ the situation is more complicated. We limit ourselves to point out the next simple existence result.

**Theorem 4.3** Assume $s < 2m - \frac{n}{2}$.

i) There exists $\lambda^* \in [0, \Lambda_1(m,s))$ such that the infimum $S_\lambda^0(m,s)$ is attained for any $\lambda \in (\Lambda^*, \Lambda_1(m,s))$, and hence (1.1) has a nontrivial solution.

ii) There exists $\tilde{\lambda}^* \in [0, \tilde{\Lambda}_1(m,s))$ such that the infimum $\tilde{S}_\lambda^0(m,s)$ is attained for any $\lambda \in (\tilde{\Lambda}^*, \tilde{\Lambda}_1(m,s))$, and hence (1.2) has a nontrivial solution.

**Proof.** Use Proposition 2.1 to find $\varphi_1 \in \overline{H}^m(\Omega)$, $\varphi_1 \neq 0$, such that

$$\int_\Omega |(-\Delta)^\frac{s}{2} \varphi_1|^2 dx = \Lambda_1(m,s) \int_\Omega |(-\Delta)^\frac{s}{2} \varphi_1|^2 dx.$$ 

Then test $S_\lambda^0(m,s)$ with $\varphi_1$ to get the strict inequality $S_\lambda^0(m,s) < S_m$. The first conclusion follows by Proposition 2.1 and Lemma 4.1. For (1.2) argue similarly. □

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