Relative State Quantum Logic

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Abstract
A projective quantum logic in terms of relative states is developed, emphasizing the importance of information transfer between a system under study and its environment. The need for accounting for the historical evolution of system is highlighted and it is found that the conjunction of observations involving conjugate variables can be consistently defined but is found to be non-commutative. It is shown that the Birkhoff and von Neumann approach to quantum logic is unable to deal with such conjunctions. It is found that whilst the proposed scheme is still not distributive in general, the discrepancy is directly related to interference effects that may disappear when information is transferred from the system to its environment. It is argued that the probabilities associated with projections be mapped to an orthocomplemented ternary logic, in which it is shown that the law of the excluded middle still holds.

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1 Introduction

In their seminal work of 1936, Birkhoff and von Neumann [1] proposed a scheme of quantum logic in which propositions about physical observables are mapped to subspaces of the total Hilbert space of a physical system. This new logic contained a particular feature that distinguished it from orthodox sentential calculus based on Boolean logic: namely that the new logic was non-distributive. This feature arises from the way that, logical disjunction becomes equated with the sum of subspaces associated with propositions, whilst logical conjunction is equated to their intersection.

A key aspect of this is that the intersection of any two subspaces defined by two distinct vectors $|\phi\rangle$ and $|\chi\rangle$ is always the null space. This may be interpreted as saying that the system cannot simultaneously be in the states $|\phi\rangle$ and $|\chi\rangle$. We might express this formally in terms of the probability of this conjunction by writing $P(\phi \land \chi) = 0$. Now, if $|\phi\rangle$ and $|\chi\rangle$ are orthogonal to one another, this statement is unproblematic. However, if they are not orthogonal, then there will be a non-zero conditional probability $P(\phi|\chi) = |\langle\phi|\chi\rangle|^2$ that we will find the system in state $|\phi\rangle$ given that it is initially in state $|\chi\rangle$. This would then be inconsistent with the classical definition of conditional probability

$$P(\phi|\chi) = \frac{P(\phi \land \chi)}{P(\chi)}.$$  \hspace{1cm} (1)

We may also note that since $|\langle\phi|\chi\rangle|^2 = |\langle\chi|\phi\rangle|^2$, for non-orthogonal vectors on the same Hilbert space we have $P(\phi|\chi) = P(\chi|\phi)$, meaning
that for Eq. (1) to hold we must have $P(\phi \land \chi) \neq P(\chi \land \phi)$. Indeed, we argue in this paper that for non-orthogonal vectors on the same Hilbert space this is actually the case.

It might be argued against this that such conjunctions are simply undefined. To counter this, we point out that such non-orthogonal vectors are associated with conjugate variables $\phi$ and $\chi$ and that non-commutativity of operations for such is well known in quantum mechanics. Specifically, such conjunctions may arise in the analysis of measurements occurring at different times. For example, we might associate $P(\phi \land \chi)$ with the probability $|\langle \phi | \chi \rangle|^2 |\langle \chi | \Psi \rangle|^2$ of projecting an arbitrary state $|\Psi\rangle$ onto the state $|\chi\rangle$ followed by a subsequent projection onto $|\phi\rangle$. Note that this does indeed yield a probability different from that obtained by performing these projections in the opposite order. We therefore argue that the probabilities $P(\phi \land \chi)$ and $P(\chi \land \phi)$ are defined but are not treated correctly by the Birkhoff/von Neumann (BvN) approach.

Related to the issue of subsequent measurements is the question of how historical information about a system is recorded. It is generally assumed as a postulate of quantum mechanics that when we measure a system, it ‘collapses’ to an eigenstate of the observable we were measuring. However, this eigenstate can store no information about the history of the system (i.e. what state it was in before we measured it) so if we, as observers, are to have any knowledge of this (or indeed knowledge of the new state of the system) information must be transferred from the system to us in some way. That is, we must also consider the larger Hilbert space of the system and its environment. We shall argue that in doing this, we gain a new perspective on the nature of quantum logic and its fundamental relationship to information transfer.

### 1.1 Birkhoff/von Neumann logic

Propositions in classical logic may be described as assertions about possible ‘states of affairs’. These are then ascribed a ‘truth value’ that in binary logic may be either ‘true’ ($T$) or ‘false’ ($F$). In quantum mechanics, a ‘state of affairs’ may be associated with the outcome of a measurement on a system for some observable $O$ (although our own approach would be more that a ‘state of affairs’ corresponds to a physical state). The original motivation for a quantum logic then lay in providing a framework for ‘experimental questions’ about the outcome of such measurements. Here we give a brief overview of the BvN approach. For a more comprehensive review of BvN logic and its development by others, see Refs. [2–5].
If we let \( \{ R_i \} \) be a set of possible results for an observable \( O \), an experimental question \( X_i \) may then be ‘given the system is the state \( \Psi \), will an observation of \( O \) produce the result \( R_i \)?’ To this, we may associate a probability \( P(X_i|\Psi) \) that the answer is ‘yes’. We may then set up a one-to-one mapping between a set of questions \( \{ X_i \} \) and a set of results \( \{ R_i \} \). Here, the questions serve as the analogues of classical propositions but the set of results rely on the rules of quantum mechanics, meaning that the logic relating the \( \{ X_i \} \) must reflect this.

In the BvN approach, the questions \( \{ X_i \} \) are assumed to define a ‘lattice’, which requires that elements of the set may be related by a partial ordering operator ‘\( \subseteq \)’. Here, we have used the symbol for ‘is a subset of’ but the relation is an abstract one of greater generality. The relation that Birkhoff and von Neumann actually use is ‘is a subspace of’, associating each question \( X_i \) with a subspace of the total Hilbert space of the system. This relation satisfies the requirement that two elements are related by \( X \subseteq Y \) if and only if \( P(X|\Psi) \leq P(Y|\Psi) \) for any state \( \Psi \).

Given the existence of the partial ordering operator, one may then define the ‘meet’ of two elements \( X \land Y \) as the the greatest lower bound (if it exists) of both \( X \) and \( Y \) with respect to the ordering relation \( \subseteq \). This plays the role of the logical conjunction of \( X \) and \( Y \). The ‘join’ of two elements is then as the least upper bound (if it exists) of both \( X \) and \( Y \) with respect to \( \subseteq \). This plays the role of the logical disjunction of \( X \) and \( Y \).

The lattice then has a maximum element with a probability of unity and a minimum element with a probability of zero. In BvN logic, these correspond to the entire Hilbert space and the null space respectively.

Being based on subspaces, the BvN lattice has the important property that each element \( X \) has an orthogonal complement \( X^\perp \) such that \( P(X^\perp|\Psi) = 1 - P(X|\Psi) \) for all \( \Psi \), which we may intuitively interpret as relating to ‘not \( X \)’. Moreover, the complement operator \( ^\perp \) satisfies the relation \( (X^\perp)^\perp = X \). The logic is then said to be ‘orthocomplemented’ meaning that \( X \subseteq Y \) implies \( Y^\perp \subseteq X^\perp \). This also implies that De Morgan’s laws hold.

Where BvN logic differs from a Boolean algebra (also defined as a lattice) in that it is non-distributive, meaning that the distributive laws of conjunction over disjunction (and vice versa) do not hold. This turns out to be due to the fact that each element is associated with a subspace of the total Hilbert space \( \mathcal{H} \), with the conjunction of two elements being the intersection of the subspaces and disjunction being their sum.

As an illustration, consider a simple two level system, such as the
spin of an electron or the polarisation of a photon. Such a system may be described in terms of spin-states

$$|Ψ⟩ = |↑⟩_z ⟨↑|ψ⟩ + |↓⟩_z ⟨↓|ψ⟩,$$

where $|↑⟩_z$ and $|↓⟩_z$ are the eigenvectors of the Pauli $σ_z$ matrix. We note straight-away that this is not the only possible representation we could choose. The same system could be written as

$$|Ψ⟩ = |↑⟩_x ⟨↑|ψ⟩ + |↓⟩_x ⟨↓|ψ⟩,$$

where $|↑⟩_x$ and $|↓⟩_x$ are eigenvectors of the Pauli $σ_x$ matrix. Here, each of the vectors for either basis lies in its own subspace, so the intersection of any two of them will be the null subspace whilst the sum of any two spans the entire Hilbert space. In the following, we shall denote the minimum and maximum elements of the lattice by ‘0’ and ‘1’ respectively.

Let us consider the proposition ‘$↑_z \land (↑_x \lor ↓_x)$’. Now $↑_x \lor ↓_x = 1$, since it corresponds to the whole Hilbert space. Meanwhile the intersection of any subspace with 1 is itself, so we have

$$↑_z \land (↑_x \lor ↓_x) = ↑_z \land 1,$$

$$= ↑_z.$$ (2)

On the other hand, the proposition ‘$(↑_z \land ↑_x) \lor (↑_z \land ↓_x)$’ involves the intersections of vectors, which are zero, so

$$(↑_z \land ↑_x) \lor (↑_z \land ↓_x) = 0 \lor 0,$$

$$= 0.$$ (3)

Comparing (2) and (3), we have

$$↑_z \land (↑_x \lor ↓_x) \neq (↑_z \land ↑_x) \lor (↑_z \land ↓_x),$$

and so the distributive law fails.

One feature that we may note immediately about the above analysis is that the amplitudes of the vectors, constituting the information content of the state, do not feature at all. In contrast, in the approach we develop in this paper the probabilities that these amplitudes produce are of crucial importance. We also find that whilst the distributive law still does not hold in general, its failure is for quite different reasons and its re-emergence in what we may call the ‘classical limit’ is related to information transfer between the system and its
environment. Moreover, the discrepancy between the quantum and
classical logical cases is found to be related to quantum interference
terms, which become suppressed when information is transferred from
the system to its environment.

1.2 Overview of paper

The approach we develop in this paper rests heavily on the relative
state formulation [6] for bipartite and multipartite systems. The es-
sentials of this are reviewed in Section 2.1, where we also introduce
the novel concept of ‘partial relative states’, which we advocate as a
possible mechanism for storing historical information about a system
in its environment.

Our emphasis will be on developing a ‘projective quantum log-}

ic’, for which the basic tool is the ‘projection valued measure’ (PVM),
discussed in Section 2.2. Whilst such measures may be used to both
define subspaces of a Hilbert space, a particular PVM, which may be
associated with the eigenvectors of a particular observable, may only
be mapped to a distributive lattice. We argue that for deal with the
conjunction of conjugate variables, the more general formulation of
the ‘positive operator-valued measure’ (POVM) is required. However,
as discussed in Section 2.3 on the basis of Naimark’s theorem [7], a
POVM may be mapped to a PVM on a larger Hilbert space. We take
this larger Hilbert space to be that of the environment of a system and
on the basis of this develop our relative state formulation in Section 3.

In Section 3 we focus specifically on the problem of logical con-
junction and disjunction when the ‘propositions’ involved relate to
measurements of conjugate variables. Rather that finding such con-
junctions to be undefined, we argue that meaning must still be given to
subsequent measurements (requiring knowledge of the historical evolu-
tion of the system). In particular, in Section 3.1, we find that that we
can define the logical conjunction of conjugate variables on the same
Hilbert space but that such conjunction is non-commutative. Defining
the disjunction of conjugate variables is also problematic, although we
find a general projector to serve this purpose in Section 3.2.

The distributive law is re-examined in Section 3.4 in the light of
our approach, where we find that it still continues to fail in general.
However, we find that the discrepancy is given by the interference
terms which become suppressed when information is transferred from
the system to its environment. Here we see the analogous emergence
of classicality described by the suppression of interference terms in
decoherence theory [8,9].

The exposition given up to and including Section 3 may be con-
sidered as being of more relevance to information theory that to a description of a formal logic. The mapping of the results of projective measurements to ‘truth values’ is given in Section 4. Here, however, we argue for a mapping to a ternary logic, involving truth values ‘true’, ‘false’ and ‘uncertain’. Note, however, that the ternary logic argued for is distinct from that of Kleene [10] or Łukasiewicz [11] in that the law of the excluded middle (and other tautologies) still holds. This is because the resulting logic is still orthocomplemented.

Finally, we discuss our conclusions in Section 5.

2 Preliminaries

2.1 Representation of physical systems

In formal quantum mechanics, a physical system $\mathcal{S}$ is represented by a vector $|\psi_0\rangle$ on a Hilbert space $\mathcal{H}_S$. If, for instance, $\{ |\phi_i\rangle \}$ is a complete, orthonormal basis set of vectors spanning $\mathcal{H}_S$, we may then write $|\psi_0\rangle$ as the superposition

$$|\psi_0\rangle = \sum_i |\phi_i\rangle \langle \phi_i | \psi_0 \rangle.$$  \hspace{1cm} (4)

According to the Born rule [12], the squared modulus $|\langle \phi_i | \psi_0 \rangle|^2$ is interpreted as the probability of finding the system in state $|\phi_i\rangle$ on measurement, given that the system is originally in the state $|\psi_0\rangle$. It is then generally accepted as a fundamental postulate that after measurement, the system will have discontinuously ‘collapsed’ to this eigenstate. Whilst we do not adopt any particular interpretational framework in this paper, we note that the state $|\phi_i\rangle$ can contain no information about the previous state of the system. If we, as observers, are to have any knowledge of the history of the system, that information must be communicated to its environment in some way.

To model the interaction between systems, we consider the total Hilbert space encompassing both system and environment to be the tensor product of separable subspaces $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$. For simplicity, we shall just consider a bipartite system consisting of a system $\mathcal{S}$ interacting with its environment $\mathcal{E}$. If $\{ |\phi_i\rangle \}$ is now a basis set spanning the subsystem Hilbert space $\mathcal{H}_S$ and $\{ |\epsilon_j\rangle \}$ is a basis set spanning the environmental Hilbert space $\mathcal{H}_E$, a general state of the total system may be written down as

$$|\Psi(t)\rangle = \sum_{ij} a_{ij}(t) \phi_i \epsilon_j,$$  \hspace{1cm} (5)
where \( a_{ij} = \langle \phi_i \epsilon_j | \Psi \rangle \) and its squared modulus \( |a_{ij}|^2 \) is interpreted as the joint probability of finding both the system in state \( |\phi_i\rangle \) and the environment in state \( |\epsilon_j\rangle \).

We shall find it useful to rewrite Eq. (5) as a single summation using the relative state formulation [6]

\[
|\Psi(t)\rangle = \sum_i a_i(t) |\phi_i\rangle |R_i(t)\rangle,
\]

where the relative state of \( |\phi_i\rangle \) may be expressed as

\[
|R_i(t)\rangle = \sum_j \frac{a_{ij}(t)}{a_i(t)} |\epsilon_j\rangle.
\]

Note that, whilst the \( |R_i\rangle \) are normalized so that \( \langle R_i | R_i \rangle = 1 \), in general these states are not orthogonal. If the \( |R_i\rangle \) do become orthogonal, the \( |\phi_i\rangle \) may be called ‘pointer states’ [13] and we will refer to the corresponding \( |R_i\rangle \) as ‘relative pointer states’.

Let us suppose that the total system is started in the tensor product

\[
|\Psi_0\rangle = |\psi_0\rangle |R_0\rangle,
\]

where we may refer to \( |R_0\rangle \) as the ‘ready state’. This would be the case if the system had been prepared in the particular eigenstate \( |\psi_0\rangle \) via a measurement of the corresponding observable. The total system then evolves according to

\[
|\Psi(t)\rangle = U(t) |\psi_0\rangle |R_0\rangle,
\]

where \( U(t) = \exp(-iHt/\hbar) \) and \( H \) is the total Hamiltonian (including the interaction Hamiltonian coupling the system and environment). If the initial subsystem state \( |\psi_0\rangle \) is given by Eq. (4), this then evolves into Eq. (6) describing the joint system. Hence, we may describe the action of \( U(t) \) via

\[
U(t) \sum_i |R_0\rangle |\phi_i\rangle \langle \phi_i | = \sum_i |R_i\rangle |\phi_i\rangle \langle \phi_i |.
\]

To appreciate the role the relative states play in modeling information transfer between the system and environment, we can construct the reduced density matrix \( \rho_S \) of the system via

8
\[ \rho_S(t) = \text{Tr}_E[\rho(t)], \]

where \( \rho(t) = |\Psi(t)\rangle \langle \Psi(t)| \) is the pure state density matrix for the total system and \( \text{Tr}_E \) is the partial trace operator taken over environmental states (we may find the reduced density matrix for the environment \( \rho_E \) in an analogous manner). Using the \(|\epsilon_j\rangle\) to trace over the environmental degrees of freedom, we find

\[
\rho_S(t) = \sum_j \langle \epsilon_j | \rho(t) | \epsilon_j \rangle,
= \sum_{i,i'} |\phi_i\rangle a_i^\dagger(t) a_i(t) \langle \mathcal{R}_i'(t)|\mathcal{R}_i(t)\rangle \langle \phi_{i'}| .
\] (11)

We may now use \( \rho_S(t) \) to find the entropy of entanglement \( S(\rho_S) \) via the von Neumann entropy

\[ S(\rho_S) = -\sum_i \lambda_i \log \lambda_i, \] (12)

where the \( \lambda_i \) are the eigenvalues of the density matrix \( \rho_S \). Although not obvious, we may find the same measure of entanglement by calculating \( S(\rho_E) \) using the reduced density matrix for the environment.

Now, if we start the system in the tensor product of Eq. (8), we will have \(|\mathcal{R}_i\rangle = |\mathcal{R}_0\rangle\), for all \( i \) with \( \langle \mathcal{R}_i'|\mathcal{R}_i\rangle = 1 \). The reduced density matrix will reduce to the expression for the pure state density matrix on \( \mathcal{H}_S \), for which the von Neumann entropy is identically zero. We may interpret this as saying that no information has been communicated between the two systems.

At the opposite extreme, if the relative states become orthogonal to one another so that \( \langle \mathcal{R}_{i'}|\mathcal{R}_i\rangle = \delta_{i'i} \) for all \( i,i' \), the off-diagonal elements will disappear and \( \rho_S \) will take the form of a ‘mixed state’ density matrix with eigenvalues \( |a_i|^2 \). This corresponds to the situation in which \( S \) has become entangled\(^1\) with its environment \( \mathcal{E} \) and the maximum possible information has been communicated. Thus, the orthogonality of the relative states provides a direct indicator of the degree of entanglement in the total system and, hence, the information communicated between the two components.

\(^1\)By the definition used by most authors, if the diagonal elements are all equal, then the joint system has become ‘maximally entangled’ when the off-diagonal elements of \( \rho_S \) disappear.
2.1.1 Partial relative states

In practice, the Hilbert space of the environment $E$ will be much larger than that of $S$ and so may be able to encode the kind of historical information about the system that $S$ is not able to record itself. To explore this possibility, we first assume that at time $t = t_1$, the relative states of Eq. (6) have all become orthogonal to one another, writing the state down as

$$|\Psi(t_1)\rangle = \sum_i |R_i(t_1)\rangle |\phi_i\rangle \langle \phi_i| \psi_0\rangle,$$

(13)

where we have put $a_i = \langle \phi_i|\psi_0\rangle$, absorbing any time dependent phase into $|R_i(t)\rangle$. At this point, Eq. (13) represents the 'pre-measurement' state of von Neumann’s scheme of ideal quantum measurement [15]. The question of what happens to reduce this to just one possibility is still an open question and the subject of much contention. Let us assume that the system does in fact collapse to just one of these possibilities

$$|\Psi(t_1)\rangle \rightarrow |R_i(t_1)\rangle |\phi_i\rangle \equiv |\Psi_i(t_1)\rangle,$$

(14)

corresponding to a measurement of the system being in the state $|\phi_i\rangle$.

We shall call this the ‘objective collapse’ picture since it implies an actual discontinuous change in the total state vector.

Let the subsequent evolution of $|\Psi_i(t_1)\rangle$ state be given by

$$|\Psi_i(t_2)\rangle = U(t_2 - t_1) |\Psi_i(t_1)\rangle = U(t_2 - t_1) |R_i(t_1)\rangle |\phi_i\rangle.$$

We now introduce a set of basis vectors $\{|\chi_j\rangle\}$ on the system Hilbert space $\mathcal{H}_S$ that are, in general, conjugate to the original $|\phi_i\rangle$. That is, $\langle \chi_j|\phi_i\rangle$ is neither zero nor unity. Let us then suppose that the evolution above leads to

$$U(t_2 - t_1) |R_i(t_1)\rangle |\phi_i\rangle = \sum_j |R_{ji}(t_2)\rangle |\chi_j\rangle \langle \chi_j|\phi_i\rangle$$

(15)

and that the $|R_{ji}(t_2)\rangle$ have become orthogonal. Each of these components is now associated with a probability $|\langle \chi_j|\phi_i\rangle|^2$ of finding the system in the state $|\chi_j\rangle$ given that it was originally in $|\phi_i\rangle$.

So long as both the $|R_i(t_1)\rangle$ and $|R_{ji}(t_2)\rangle$ do actually become orthogonal, there is no penalty to retaining the full superposition of
Eq. (13) and summing Eq. (15) over the $|\phi_i\rangle$ weighted by their amplitudes at $t = t_1$ (which will be constants) to obtain

$$|\Psi(t_2)\rangle = \sum_{ij} |R_{ji}(t_2)\rangle |\chi_j\rangle \langle \chi_j|\phi_i\rangle \langle \phi_i|\psi_0\rangle .$$  \hspace{1cm} (16)$$

Here, we may refer to the $|R_{ji}(t_2)\rangle$ as ‘partial’ relative states, since a summation of them corresponds to a particular relative state. Each of these partial relative states may be thought of as encoding information about both measurements of $\phi$ and $\chi$ at times $t_1$ and $t_2$ respectively.

Note that Eq. (16) has the same form as we would under the ‘subjective collapse’ picture, according to which there is no collapse of the total state vector. In this no-collapse picture, an ‘effective’ collapse may occur if the relative states no longer interact with each other after becoming orthogonal - a condition that is mathematically possible but by no means necessary.

We shall retain the formalism of Eq. (16) in the present work, although we must allow for the possibility that the partial relative states do not become orthogonal (this may just be due to the choice of system states and may not be taken as saying anything about the validity of the objective collapse picture). Suppose, then, that the states $|R_{ji}(t_2)\rangle$ retained no information about the former $|\phi_i\rangle$ states so that we could put $|R_{ji}(t_2)\rangle \rightarrow |R_{j}(t_2)\rangle$. In this case, Eq. (16) reduces to

$$|\Psi(t_2)\rangle \rightarrow \sum_j |R_{j}(t_2)\rangle |\chi_j\rangle \langle \chi_j|\psi_0\rangle ,$$  \hspace{1cm} (17)$$

which is again in the same form as Eq. (13).

In summary, then, we argue that the relative state formalism provides a mechanism by which information about the past evolution of the system may be stored. This is essential for any system of logic that involves reasoning about events occurring at different points in time.

### 2.2 Projection valued measures

Projective quantum logic makes heavy use of the concept of a ‘projection valued measure’ (PVM). Before defining this, it will be helpful to first establish the concept of a ‘measurable space’.

**Definition:** If $\Lambda$ is a set and $\mathcal{P}(\Lambda)$ is the power set of $\Lambda$, then a subset $\Sigma \subseteq \mathcal{P}(\Lambda)$ is a $\sigma$-algebra if it satisfies the following properties:
(1) \( \Lambda \in \Sigma \).
(2) \( \Sigma \) is closed under complementation.
(3) \( \Sigma \) is closed under countable unions.

Taken together, the pair \((\Lambda, \Sigma)\) is referred to as measurable or Borel space.

**Definition:** A projection valued measure, \( \hat{\Phi} \), on a measurable space \((\Lambda, \Sigma)\) is a mapping from the \(\sigma\)-algebra \(\Sigma\) to a set of self-adjoint, orthogonal projectors on a Hilbert space \(\mathcal{H}\) such that

\[
\hat{\Phi}_\Lambda = I_{\mathcal{H}},
\]

where \(I_{\mathcal{H}}\) is the identity operator on \(\mathcal{H}\) and for every \(|\xi\rangle, |\eta\rangle \in \mathcal{H}\) we may define a function from \(\Sigma\) to the complex numbers \(\mathbb{C}\)

\[
X \rightarrow \langle \xi | \hat{\Phi}_X | \eta \rangle,
\]

where \(X \in \Sigma\).

For our purposes we shall define \(\Lambda\) to be the index set of a basis set of \(\mathcal{H}\) and \(\Sigma = \mathcal{P}(\Lambda)\) (the power set of \(\Lambda\)). Using, for example, the basis set \(\{|\phi_i\rangle\}\) and defining an element of \(\hat{\Phi}_\Lambda\) to be \(\hat{\Phi}_i = |\phi_i\rangle \langle \phi_i|\), we would then have

\[
\hat{\Phi}_\Lambda = \sum_{i \in \Lambda} |\phi_i\rangle \langle \phi_i| = I_{\mathcal{H}}.
\]

Extending this notation to a general index set \(X \in \Sigma\), we may write a general element of \(\hat{\Phi}\) as

\[
\hat{\Phi}_X = \sum_{i \in X} \hat{\Phi}_i = \sum_{i \in X} |\phi_i\rangle \langle \phi_i|.
\]

**2.2.1 The Born rule**

It is of particular note that for any normalized vector \(|\Psi\rangle\) of \(\mathcal{H}\), such as Eq. (4), the mapping of Eq. (18) gives

\[
\langle \Psi | \hat{\Phi}_i | \Psi \rangle = \langle \psi | \phi_i \rangle \langle \phi_i | \psi \rangle = |\langle \phi_i | \psi \rangle|^2.
\]

This is a non-negative, real number between 0 and unity. Hence, Eq. (21) may provide the basis of a probability measure. According to
Gleason’s theorem [16], any such measure on a Hilbert space with a dimension greater than two takes the form of the Born rule [12]

\[ P(X) = \text{Tr} \left[ \rho \hat{\Phi}_X \right], \quad (22) \]

where \( \rho \) is a positive semi-definite operator, here taken to be the density matrix. In the case of a pure state density matrix \( \rho = |\Psi\rangle \langle \Psi| \), it is straightforward to show that this becomes

\[ P(X|\Psi) = \langle \Psi | \hat{\Phi}_X | \Psi \rangle, \quad (23) \]

where we have written \( P \) as a conditional probability given that the system is in the state \( |\Psi\rangle \). So long as these measures are between 0 and 1 and that the sum of the probabilities for all disjunct sets \( X \) is equal to 1, the forms of Eqs. (22) and (23) continue to hold for the more general positive operator-valued measures discussed later in Section 2.3.

The probability measures found in this way are central to the approach to the quantum logic developed in this paper. The fact that the projectors \( \hat{\Phi}_X \) have eigenvalues of just unity or zero has been taken by some to be indicative of binary truth values [2]. However, we take the view that the continuous values for the probability measures they yield should instead be mapped to ternary truth values, as described in detail in Section 4.

### 2.2.2 Properties of a PVM

A distinctive feature of an element of a PVM is that it is idempotent, i.e.

\[ \hat{\Phi}_X^2 = \hat{\Phi}_X, \quad (24) \]

as can be clearly seen from inspection of Eq. (20). Also easy to show is the product rule for elements of a PVM

\[ \hat{\Phi}_X \hat{\Phi}_Y = \hat{\Phi}_Y \hat{\Phi}_X = \sum_{i \in X \cap Y} |\phi_i\rangle \langle \phi_i|, \quad \equiv \hat{\Phi}_{X \cap Y}. \quad (25) \]

Similarly, we may define the element for the union of sets \( X \) and \( Y \)
$$\hat{\Phi}_{X \cup Y} = \sum_{i \in X \cup Y} |\phi_i\rangle \langle \phi_i|,$$

$$= \hat{\Phi}_X + \hat{\Phi}_Y - \hat{\Phi}_{X \cap Y}. \quad (26)$$

As shown above in Eq. (19), every PVM has an identity element by definition. In addition, since $\Sigma$ is a $\sigma$-algebra, every PVM will also have a ‘null’ or ‘zero’ element, that we may denote by

$$\hat{\Phi}_\emptyset = \sum_{i \in \emptyset} |\phi_i\rangle \langle \phi_i| = 0. \quad (27)$$

Moreover, since $\Sigma$ is closed under complementation, if $\hat{\Phi}_X$ is an element of a projection valued measure, then its complement

$$\hat{\Phi}_X^\perp \equiv I_\mathcal{H} - \hat{\Phi}_X \quad (28)$$

is also an element. It is straight-forward to show that these are orthogonal to one another. That is

$$\hat{\Phi}_X^\perp \hat{\Phi}_X = 0. \quad (29)$$

Equation (28) implies that a logic based on a PVM will be orthocomplemented.

### 2.2.3 Connection to the Birkhoff/von Neumann approach

Using these results, it we can show how the use of a PVM may be partially mapped to the BvN approach of identifying propositions about a system with subspaces of the total system Hilbert space $\mathcal{H}$. Firstly, it may be shown that for PVM $\Phi$ acting on a Hilbert space $\mathcal{H}$, the set of vectors $|\Psi\rangle$ satisfying

$$\hat{\Phi}_X |\Psi\rangle = |\Psi\rangle, \quad (30)$$

defines a subspace $[X] \subseteq \mathcal{H}$. Moreover, we may easily show that the orthogonal complement of $[X]$ may be defined in the same way using the element $\hat{\Phi}_X^\perp$. This is identified with the logical negation of the proposition $X$.

Furthermore, we can show that the element $\hat{\Phi}_{X \cap Y}$ defines the intersection of subspaces $[X] \cap [Y]$ whilst $\hat{\Phi}_{X \cup Y}$ defines the sum of subspaces $[X] + [Y]$. In the BvN approach, these subspaces are identified with the logical operations of conjunction and disjunction respectively.
However, we have taken $\Sigma$ to be the power set of the index set $\Lambda$ of a given basis set spanning the Hilbert space. This means that $\Sigma$ may be mapped to a distributive lattice (and therefore constitutes a Boolean algebra) with the set theoretic operations of intersection and union serving as the meet and join of two subsets respectively. In contrast to this, the quantum logic emerging from the BvN approach is famously non-distributive.

The problem here is that the power set of all possible subspaces cannot be mapped to a single PVM based on one particular basis set. If we try to combine operations with a different PVM based on a conjugate variable $\chi$, say, we encounter severe problems.

Suppose we have one PVM $\hat{\Phi}$ based on the elements of the basis set $\{|\phi_i\rangle\}$ and a second PVM $\hat{X}$ based on the conjugate set $\{|\chi_i\rangle\}$, such that, in general $\langle \chi_j | \phi_i \rangle \neq \delta_{ij}$. Attempting to construct a projector representing the conjunction of $\chi_j$ and $\phi_i$, we might try the product

$$|\chi_j\rangle \langle \chi_j | \phi_i \rangle \langle \phi_i |.$$

(31)

However, this is neither idempotent nor self-adjoint. Moreover, taken as a member of a set covering all $i$ and $j$, these elements would clearly not be orthogonal. Evidently, as it stands, Eq. (31) is not an element of PVM.

One might still use PVMs to provide a family of overlapping Boolean algebras to span the quantum logic as a whole, as in the approach of Kochen and Specker [17]. These authors then refer to such a family as a ‘partial Boolean algebra’. The general question of taking the conjunction of conjugate variables still remains though.

According to the BvN picture, the conjunction of two eigenvectors representing conjugate variables corresponds to the intersection of their subspaces, which is the null space $\emptyset$. This is interpreted as saying that a physical system cannot be simultaneously in the eigenstates of different conjugate variables. Whilst this may be true, we argue that there is still be a non-zero probability for finding the system in the state $|\chi_j\rangle$ given that it is initially in the state $|\phi_i\rangle$ and vice versa.

There are many cases where we may wish to to use the concept of logical conjunction to describe sequences of events, each of which may be associated with conditional probabilities of the form $|\langle \chi_j | \phi_i \rangle|^2$, where $\chi$ and $\phi$ are conjugate variables (for instance, passing a photon prepared in state $|\phi_i\rangle$ though a polarizer aligned with the $|\chi\rangle$ basis). Hence, we need a language to describe this. Clearly, however, modeling this entirely in terms of PVMs will be problematic. We therefore generalize our discussion of probability measures to be the result of
positive operator-valued measures, introduced in the next subsection. The practical task of modeling the conjunction of conjugate variables is deferred until Section 3.

2.3 Positive operator-valued measures

A positive operator-valued measure (POVM) is the generalization of a PVM and is defined very similarly. An essential difference is that the elements of a POVM are not necessarily orthogonal to one another.

Definition: A positive operator-valued measure \( \hat{F} \) on a measurable space \((\Lambda, \Sigma)\) is a mapping from \(\Sigma\) to the set of positive semi-definite operators \(\{\hat{F}_k\}\) on a Hilbert space \(\mathcal{H}\) such that

\[
\hat{F}_\Lambda = \sum_{k \in \Lambda} \hat{F}_k = I_{\mathcal{H}},
\]

where we have again taken \(\Lambda\) to be an index set of some kind but no longer necessarily that of a basis set. Here the phrase ‘positive semi-definite operator’ means that, for any \(|\psi\rangle \in \mathcal{H}\),

\[
\langle \psi | \hat{F}_k | \psi \rangle \geq 0.
\]

Clearly, if \(\Lambda\) is the index set a set of self-adjoint, orthogonal projectors, then \(\hat{F}\) will reduce to a PVM. Hence PVMs belong to the larger class of POVMs.

2.3.1 Naimark’s theorem

Of particular importance for the use of POVMs is the application of Naimark’s dilation theorem [7]. This says that for any POVM \(\hat{F}\) defined on a Hilbert space \(\mathcal{H}_S\), there exists a larger Hilbert space \(\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E\) and a transformation operator \(V\) that maps a PVM \(\hat{\Theta}\) defined on \(\mathcal{H}\) to \(\hat{F}\)

\[
\hat{F}_X = V^\dagger \hat{\Theta}_X V. \tag{32}
\]

For example, an element of the PVM on the higher Hilbert space may be defined via

\[
\hat{\Phi}_X = I_S \otimes \hat{\Phi}_X. \tag{33}
\]
where $I_S$ is the identity operator on $H_S$ and $\hat{\Phi}_X^E$ is a PVM acting on $H_E$. One may then construct $V$ via the general prescription

$$V = U(t)I_S \otimes |R_0\rangle.$$  

(34)

giving

$$\hat{F}_X(t) = \langle R_0 | \otimes I_S U^\dagger(t) \hat{\Phi}_X U(t) I_S \otimes |R_0\rangle.$$  

(35)

Assuming a pure density matrix for the total system $\rho = |\Psi\rangle \langle \Psi|$, we can calculate the probability associated with the PVM on the higher Hilbert space via

$$P(X|\Psi;t) = \langle \Psi(t) | \hat{\Phi}_X | \Psi(t) \rangle.$$  

(36)

Using Eq. (9) to substitute for $|\Psi(t)\rangle$, we have

$$P(X|\Psi;t) = \langle \psi_0 | R_0 | U^\dagger(t) \hat{\Phi}_X U(t) | R_0 \psi_0 \rangle.$$  

Comparing this to Eq. (35), we recognize that this is equal to the POVM $\hat{F}_X(t)$ acting on $H_S$

$$P(X|\Psi;t) = \langle \psi_0 | \hat{F}_X(t) | \psi_0 \rangle.$$  

Finally, using the expansion of Eq. (6), we obtain the expression

$$P(X|\Psi;t) = \sum_i |a_i(t)|^2 \langle \mathcal{R}_i(t) | \hat{\Phi}_X^E | \mathcal{R}_i(t) \rangle,$$  

(37)

giving the probability measure in terms of relative states.

3 The relative state formulation

3.1 Conjunction of conjugate variables

In Birkhoff/von Neumann quantum logic, the intersection of subspaces representing conjugate variables is the null space, meaning that the logical conjunction of such variables evaluates to 'false' in all cases. Based on the arguments above, we insist that this is not the case. The problem remains, however, how to represent such a conjunction
in terms of projection operators. Clearly, we cannot accomplish this
naively with the product of PVM elements since, for instance, the
product $|\chi_j\rangle \langle \chi_j| \phi_i \rangle \langle \phi_i|$ is neither self-adjoint nor idempotent.
Consider, however, the relative state expansion of Eq. (16), in
which the factors $|\chi_j\rangle \langle \chi_j| \phi_i \rangle \langle \phi_i|$ do appear. We shall now construct
a PVM on the larger Hilbert space using one of the partial relative
states $|R_{ji}\rangle$ by defining

$$\hat{\Phi}_{ji} = I_S \otimes |R_{ji}\rangle \langle R_{ji}|. \quad (38)$$

We can circumvent the problem that, in general, the partial relative
states are not orthogonal by defining the PVM itself to be

$$\hat{\Phi} = \{\hat{\Phi}_{ji}, I_{H} - \hat{\Phi}_{ji}\},$$

where $I_{H}$ is the identity operator on the total Hilbert space $H = H_S \otimes H_E$.

From inspection, we then define the transformation operator

$$V = \sum_{ij} |R_{ji}\rangle |\chi_j\rangle \langle \chi_j| \phi_i \rangle \langle \phi_i|. \quad (39)$$

Operating on Eq. (38) with this, we have

$$V^\dagger \hat{\Phi}_{ji} V = \sum_{i'j'} |\phi_{i'}\rangle \langle \phi_{i'}| \chi_{j'} \rangle \langle \chi_{j'}| R_{j'i'} \rangle \langle R_{j'i'}| R_{ji} \rangle \langle R_{ji}| R_{j'i'} \rangle \langle \chi_{j'}| \phi_{i'} \rangle \langle \phi_{i'}|,$n

$$\equiv \hat{F}_{ji}. \quad (40)$$

So far, we have not made any assumptions about the orthogonality
of the $|R_{ji}\rangle$. If we now assume that they have become orthogonal, then
the expression above reduces to

$$\hat{F}_{ji} \rightarrow |\phi_i \rangle \langle \phi_i| \chi_j \rangle \langle \chi_j| \phi_i \rangle \langle \phi_i|, \quad (41)$$

which is evidently a positive operator on $H_S$. Summing $\hat{F}_{ji}$ over both
indices, we find

$$\sum_{ij} \hat{F}_{ji} = I_S,$$
demonstrating that \( \hat{F}_{ji} \) is an element of a POVM.

Calculating the probability associated with this POVM, we find

\[
\langle \psi_0 | \hat{F}_{ji} | \psi_0 \rangle = |\langle \chi_j | \phi_i \rangle|^2 |\langle \phi_i | \psi_0 \rangle|^2. \tag{42}
\]

(Note that we would have obtained the same value from \( \langle \Psi | \hat{\Phi}_{ji} | \Psi \rangle \)).

This is then the probability of finding the system in the state \( |\chi_j\rangle \) after previously finding it in the state \( |\phi_i\rangle \) (given that it was initially in the state \( |\psi_0\rangle \)).

Formally, we can identify \( |\langle \chi_j | \phi_i \rangle|^2 \) as the conditional probability \( P(\chi_j | \phi_i) \), whilst, given that \( |\psi_0\rangle \) is arbitrary, we may simply call \( |\langle \phi_i | \psi_0 \rangle|^2 \) the probability \( P(\phi_i) \) for finding the system in \( |\phi_i\rangle \). Applying the classical definition of conditional probability Eq. (1), this would then give us

\[
|\langle \chi_j | \phi_i \rangle|^2 |\langle \phi_i | \psi_0 \rangle|^2 = P(\chi_j | \phi_i)P(\phi_i) = P(\phi_i \land \chi_j). \tag{43}
\]

That is, the joint probability of \( \phi_i \) and \( \chi_j \). However, this is not equal to value we would have obtained taking these measurements in the reverse order.

It is important to note that the conditional probabilities for conjugate variables on the same Hilbert space are commutative, in the sense that

\[
P(\chi_j | \phi_i) = |\langle \chi_j | \phi_i \rangle|^2,
= |\langle \phi_i | \chi_j \rangle|^2 = P(\phi_i | \chi_j). \tag{43}
\]

On the face of it, this would appear to be in contradiction with Eq. (1) for classical probabilities. However, since the physical operations we are modeling are generally non-commutative, we argue that the same non-commutativity should also apply to the conjunction of conjugate variables. Applying Eq. (43) to Eq. (1), we conclude that for conjugate variables of the same Hilbert space

\[
P(\phi_i \land \chi_j)P(\chi_j) = P(\chi_j \land \phi_i)P(\phi_i). \tag{44}
\]

Here, we have adopted the convention that the variable on the left of the conjunction operator ‘\( \land \)’ is projected first.

Following through the relative state analysis above but with the variables in reverse order, we would obtain the POVM
\[ \hat{F}_{ij} = |\chi_j\rangle \langle \chi_j| \langle \phi_i| \langle \phi_i| \chi_j\rangle \langle \chi_j|. \]

From this, we have

\[
\langle \psi_0| \hat{F}_{ij} |\psi_0\rangle = |\langle \phi_i| \chi_j\rangle|^2 |\langle \chi_j| \psi_0\rangle|^2 = P(\phi_i|\chi_j) P(\chi_j) = P(\chi_j \land \phi_i),
\]

which is consistent with Eq. (44).

### 3.1.1 Non-orthogonal partial relative states

The POVM obtained in Eq. (41) depended on the assumption that the partial relative states \(|R_{ji}\rangle\) had become orthogonal to one another. Let us now relax this assumption. Suppose instead that information about the \(|\chi_j\rangle\) states had been recorded in the environment but that all information about the \(|\phi_i\rangle\) states had been lost. We can model this by saying that \(\langle \mathcal{R}_{j'j'}|\mathcal{R}_{ji}\rangle \to \delta_{j'j}\) for all \(i\) and \(j\).

In this case, Eq. (40) reduces to

\[ \hat{F}_{ji} \to |\chi_j\rangle \langle \chi_j|, \]

which is just the projector onto the state \(|\chi_j\rangle\). The expectation value of this is just \(|\langle \chi_j| \psi_0\rangle|^2 = P(\chi_j|\psi_0)\), the probability of finding the system in the state \(|\chi_j\rangle\). Hence, the POVMs obtained via this method (and the probabilities they produce) depend crucially on the information transferred to the environment.

Consider now the extreme situation in which no information is transmitted to the environment. This is the situation in which the total system is in a tensor product of system and environmental states, so we can model it by putting \(\langle \mathcal{R}_{j'j'}|\mathcal{R}_{ji}\rangle \to 1\) for all \(i\) and \(j\). In this case, we find

\[ \hat{F}_{ji} \to \sum_i |\phi_i\rangle \langle \phi_i| = I_S. \]

We may interpret this as the disjunction of all possible states of \(\phi\). In accordance with the mapping scheme that we shall introduce in Section 4, we will ascribe this a truth value of ‘true’ since the corresponding probability is now unity. This then effectively says that a measurement of \(\phi\) will yield one of the eigenstates \(|\phi_i\rangle\), which we may intuitively accept as a ‘true’ statement.
Note that this ‘true’ result occurs in the case of no information being passed to the environment. One may then find this reminiscent of Wittgenstein’s comment [18] that a tautology (necessarily ‘true’ statement) tells us nothing about the world.

### 3.2 Disjunction

In setting up the PVM of Eq. (38), we limited the elements to a complimentary pair to avoid the problem of non-orthogonal relative states. In the case of orthogonal relative states, there is no longer a problem and we could construct a PVM out of elements such as $|\mathcal{R}_i\rangle \langle \mathcal{R}_i|$. We could then deal with the disjunction of different possibilities via the union of these elements, as indicated by Eq. (26) in the section covering PVMs.

Unfortunately, this is not an option in the general case where we are dealing with non-orthogonal states. To deal with this, we first consider the problem of constructing a self-adjoint, idempotent projector $\hat{\Theta}_X$ satisfying

$$\hat{\Theta}_X |\psi\rangle = |\psi\rangle,$$

for

$$|\psi\rangle = \sum_{j \in X} a_j |\phi_j\rangle,$$

where, in general, the vectors $|\phi_j\rangle$ are non-orthogonal (and so $a_j \neq \langle \phi_j | \psi \rangle$). A projector satisfying Eq. (45) will then also satisfy

$$\hat{\Theta}_X |\phi_j\rangle = |\phi_j\rangle,$$

for each vector in the set $X$. Hence, each vector $|\phi_j\rangle$ belongs to the subspace defined by Eq. (45) and $\hat{\Theta}_X$ can serve as a projector for defining the disjunction of these elements. In Appendix A.1 it is shown that the required projector is given by

$$\hat{\Theta}_X = \sum_{i,j \in X} |\phi_i\rangle (A^{-1})_{ij} \langle \phi_j|,$$

where $A_{ij} = \langle \phi_i | \phi_j \rangle$.

We may now choose any set $\{ |\mathcal{R}_i\rangle \}$ of relative states and construct the projector
\[ \hat{\Phi}_X = I_S \otimes \sum_{i,j \in X} |\mathcal{R}_i\rangle (R^{-1})_{ij} \langle \mathcal{R}_j|, \]

where \( R_{ij} = \langle \mathcal{R}_i|\mathcal{R}_j \rangle \) and define the PVM

\[ \hat{\Phi} = \{ \hat{\Phi}_X, I_{\mathcal{H}} - \hat{\Phi}_X \}, \]

as before. The only remaining problem occurs in the case of a tensor product of system and environmental states, in which case we have \( R_{ij} = 1 \) for all elements and the matrix becomes singular. In this case the PVM should be replaced with \( \{ I_{\mathcal{H}}, 0 \} \).

Applying this PVM to a relative state expansion of the form of Eq. (6), we have from Eq. (37)

\[ P(X|\Psi; t) = \sum_{i \in X} |a_i(t)|^2 + \sum_{i \not\in X} |a_i(t)|^2 \langle \mathcal{R}_i(t) | \hat{\Phi}^c_X | \mathcal{R}_i(t) \rangle. \]

As \( \langle \mathcal{R}_i' | \mathcal{R}_i \rangle \to \delta_{i'i} \), the second term will tend to zero, leaving the result for the union of PVM elements \( |\phi_i\rangle \langle \phi_i| \) on \( \mathcal{H}_S \). Note that in the case of a tensor product, we have \( P(X|\Psi; t) = 1 \).

### 3.3 Conditional states

It has been emphasised that the probabilities we obtain through projective measures are conditional probabilities, depending on the initial state of the system. It is often useful to assume that the system is in a particular state to start with. To this end, we may construct projected states corresponding to a projector \( \hat{\Phi}_Y \) via the prescription

\[ |\Psi_Y \rangle = \lim_{|\epsilon| \to 0} \frac{\hat{\Phi}_Y |\Psi\rangle}{\sqrt{\langle \Psi | \hat{\Phi}_Y |\Psi\rangle + |\epsilon|}}, \tag{48} \]

where the positive infinitesimal \( |\epsilon| \) is included to ensure the correct convergence in the case \( \hat{\Phi}_Y |\Psi\rangle = 0 \). Dropping the explicit reference to \( \Psi \), we may then define a general conditional probability via

\[ P(X|Y) \equiv \langle \Psi_Y | \hat{\Phi}_X | \Psi_Y \rangle. \tag{49} \]

Here ‘\( Y \)’ may be taken as denoting the state of affairs corresponding to the state \( |\Psi_Y\rangle \). Hence, Eq. (48) constitutes a procedure for constructing a state representing any particular state of affairs.
As a particular example, let us consider the state in which the environment is in the relative state $|R_i(t)\rangle$ (where $|R_i(t)\rangle$ is the relative state of system state $|\phi_i\rangle$, as in Eq. (6)). Here we project onto $|\Psi\rangle$ with

$$\hat{\Phi}_{R_i} = I_S \times |R_i\rangle \langle R_i|.$$  

(dropping the explicit $t$ dependence for brevity). Applying Eq. (48), we obtain

$$|\Psi_{R_i}\rangle = \sum_{i'} a_{i'} |\phi_{i'}\rangle |R_i\rangle \langle R_i| R_{i'}\rangle \sqrt{\sum_{i'} |a_{i'}|^2 |\langle R_i| R_{i'}\rangle|^2}.$$ 

Using this, we may obtain the probability $P(\phi_k| R_i) = |\langle \phi_k| \Psi_{R_i}\rangle|^2$ that the system is found in state $|\phi_k\rangle$ given that the environment is in $|R_i\rangle$,

$$|\langle \phi_k| \Psi_{R_i}\rangle|^2 = \frac{|a_k|^2 |\langle R_i| R_k\rangle|^2}{\sqrt{\sum_{i'} |a_{i'}|^2 |\langle R_i| R_{i'}\rangle|^2}}.$$ 

In the limit of complete information transfer when $\langle R_i| R_{i'}\rangle = \delta_{i,i'}$ for all $i$, this gives $|\langle \phi_k| \Psi_{R_i}\rangle|^2 = \delta_{k,i}$. That is, it is certainly the case for $k = i$ and certainly not the case for $k \neq i$. In the opposite limit of no information transfer in which $\langle R_i| R_{i'}\rangle = 1$ for all $i'$, we have $|\langle \phi_k| \Psi_{R_i}\rangle|^2 = |a_k|^2$, the probability for finding $|\phi_k\rangle$ in the isolated system.

### 3.4 The distributive law

In this section, we will use Eq. (48) to investigate disjunction and conjunction of operators on just the system Hilbert space $H_S$. We then compare the results to the more general relative states approach.

Let us assume an element of a PVM on $H_S$ describing the disjunction of two orthogonal states

$$\hat{\Phi}_{\phi_1 \lor \phi_2} = |\phi_1\rangle \langle \phi_1| + |\phi_2\rangle \langle \phi_2|.$$ 

Applying Eq. (48), we obtain the conditional state

$$|\psi_{\phi_1 \lor \phi_2}\rangle = \frac{|\phi_1\rangle \langle \phi_1| \psi_0\rangle + |\phi_2\rangle \langle \phi_2| \psi_0\rangle}{\sqrt{P(\phi_1 \lor \phi_2| \psi_0)}}.$$
where the probability $P(\phi_1 \lor \phi_2 | \psi_0) = \langle \psi_0 | \hat{\Phi}_{\phi_1 \lor \phi_2} | \psi_0 \rangle$ is

$$P(\phi_1 \lor \phi_2 | \psi_0) = |\langle \phi_1 | \psi_0 \rangle|^2 + |\langle \phi_2 | \psi_0 \rangle|^2,$$

$$= P(\phi_1 | \psi_0) + P(\phi_2 | \psi_0).$$

That is, this is the sum of the probabilities for the individual states. Since $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal to one another, this corresponds to the result of classical probability for the union of independent events.

We now define the projector for finding the system in state $|\chi_j\rangle$

$$\hat{\Phi}_{\chi_j} = |\chi_j\rangle \langle \chi_j|,$$

where $\chi$ and $\phi$ are conjugate variables. The conditional probability for finding the system in this state, given that it was in either of the states $|\phi_1\rangle$ or $|\phi_2\rangle$ is then found from

$$P(\chi_j | \phi_1 \lor \phi_2) = \langle \psi_{\phi_1 \lor \phi_2} | \chi_j \rangle \langle \chi_j | \psi_{\phi_1 \lor \phi_2} \rangle,$$

Applying Eq. (1), we can find the probability

$$P([\phi_1 \lor \phi_2] \land \chi_j | \psi_0) = P(\chi_j | \phi_1 \lor \phi_2) P(\phi_1 \lor \phi_2 | \psi_0).$$

This is found to be

$$P([\phi_1 \lor \phi_2] \land \chi_j | \psi_0) = P(\phi_1 \land \chi_j | \psi_0) + P(\phi_2 \land \chi_j | \psi_0) + 2 |Z| \cos \theta,$$

(50)

where

$$P(\phi_i \land \chi_j | \psi_0) = |\langle \chi_j | \phi_i \rangle|^2 |\langle \phi_i | \psi_0 \rangle|^2$$

and the interference term $2 |Z| \cos \theta$ arises from the cross products

$$Z = |Z| e^{i\theta} = \langle \psi_0 | \phi_1 \rangle \langle \phi_1 | \chi_j \rangle \langle \chi_j | \phi_2 \rangle \langle \phi_2 | \psi_0 \rangle.$$  

(51)

Here we see both the interference effects characteristic of alternative possibilities in quantum mechanics and also the failure of the distributive law. That is, if the distributive law were to hold, we would expect the interference terms to be zero and we would just have the sum of probabilities for $\phi_1 \land \chi_j$ and $\phi_2 \land \chi_j$.  

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Let us now investigate the same situation using the relative state formulation. We take the state of the system to be given by Eq. (6). Applying Eq. (48), we now have

$$|\Psi_{\phi_1 \lor \phi_2} \rangle = \frac{|\phi_1 \rangle \langle \psi_0| R_1 + |\phi_2 \rangle \langle \psi_0| R_2}{\sqrt{P(\phi_1 \lor \phi_2 | \Psi)}} ,$$

where we find $P(\phi_1 \lor \phi_2 | \Psi) = P(\phi_1 \lor \phi_2 | \psi_0)$, as earlier.

Constructing an expression for the probability $P((\phi_1 \lor \phi_2) \land \chi_j | \Psi)$, we obtain a result with the same form as Eq. (50), except that now the interference terms are multiplied by the inner product of the relative states

$$Z = \langle \psi_0 | \phi_1 \rangle \langle \chi_j | \phi_2 \rangle \langle \phi_2 | \psi_0 \rangle \langle R_1 | R_2 \rangle .$$

If the total system stores no information about the $|\phi_i\rangle$ states, then we have $\langle R_1 | R_2 \rangle = 1$ and we regain the full interference terms of Eq. (51). On the other hand, if the maximum information is stored, then we would have $\langle R_1 | R_2 \rangle = 0$ and the interference terms disappear (analogous to the well-known suppression of interference terms in decoherence theory [8, 9]). Moreover, in this case, the distributive law now holds. That is, this rule of classical logic re-emerges when we have the maximum information transfer between the system and environment. Note that we could have also obtained the classical result using the PVM on the total Hilbert space

$$\hat{\Phi}_X = I_S \otimes (|R_{j1}\rangle \langle R_{j1}| + |R_{j2}\rangle \langle R_{j2}|) ,$$

in the limit $\langle R_{j'i'} | R_{ji} \rangle = \delta_{i'i'} \delta_{j'j}$ and calculating the probability as usual.

We may assert then that the well-known interference terms of quantum mechanics emerge when there is a lack of information transfer between us as observers embedded in the environment of a system and the system itself. Moreover, it is these quantum interference effects that lead to the non-classicality of quantum logic and that classicality becomes restored as we obtain information about the system.

4 Probability and truth values

4.1 Unary truth operators

In binary propositional logic, one associates one of two ‘truth values’, ‘true’ ($T$) or ‘false’ ($F$), with an assertion about an alleged state of
affairs. Here, we specify that a ‘state of affairs’ relates to a possible state of a physical system. However, the notion of associating such binary values in any absolute way to a physical system is dubious on two counts. Firstly, on the basis of the Born rule interpretation, the ontology of a physical state is associated with many-valued probabilities, not binary choices. Secondly, and perhaps more significantly, ‘facts’ about the world correspond to encoded information, which will be different for different relative states. That is, what may be ‘true’ given one relative state may not be ‘true’ for another.

Rather than trying to get quantum theory to fit within the framework of binary logic, we argue instead that a three-valued ternary logic of ‘true’, ‘false’ and ‘uncertain’ (U) is more appropriate and propose a scheme for mapping probabilities to these values. To facilitate this, we shall define unary logical operators $T(X)$, $F(X)$ and $U(X)$, which return only binary logical values $T$ or $F$, and which may be read as ‘X is true’, ‘X is false’ and ‘X is uncertain’ respectively. We shall take it as a given that one may associate a proposition $X$ with a probability $P(X)$ that it describes an actual state of affairs.

The operators $T(X)$, $F(X)$ and $U(X)$ are then mapped to $P(X)$ via the following definitions:

$$T(X) = \begin{cases} T, & \text{if } P(X) = 1, \\ F, & \text{otherwise}, \end{cases}$$

$$F(X) = \begin{cases} T, & \text{if } P(X) = 0, \\ F, & \text{otherwise}, \end{cases}$$

$$U(X) = \begin{cases} T, & \text{if } 0 < P(X) < 1, \\ F, & \text{otherwise}, \end{cases}$$

These definitions encapsulate the intuitive notion that anything that is certain to be the case, i.e. with a probability of one, must be ‘true’, whilst anything which is certainly not the case (probability of zero) must be false. We use the term ‘uncertain’ to categorize all other cases, although the terms ‘undefined’ or ‘unknown’ may have served just as well. We shall refer to $T(X)$, $F(X)$ and $U(X)$ as ‘truth operators’.

Since we have shown in Section 2.3 that a POVM may always be mapped to a PVM, we discuss here only how the definitions of Eq. (52) to (54) may be mapped to a projection valued measure $\hat{\Phi}_X$, simply using ‘X’ to refer to the state of affairs associated with the element $\Phi_X$. Using the results of Sections 2.2.1 and 2.2.2, we may derive the usual rules of classical probability theory. In particular,
using $\overline{X}$ to denote ‘not $X$’, we find $P(\overline{X}) = 1 - P(X)$. This leads to the tautology

$$T(X \lor \overline{X}).$$  \hfill (55)

That is, the law of the excluded middle. Note that this expression will be ‘true’ ($T$) even when $X$ (and hence $\overline{X}$) are ‘uncertain’ ($U(X) = T$). This distinguishes this scheme of logic from other ternary logics such as Kleene [10] or Łukasiewicz [11] logic in which the law of the excluded middle does not hold.

### 4.2 Conditional probability and implication

As a further mapping of probability to ternary logic, we assert that the truth of a material implication $X \Rightarrow Y$ is related to the conditional probability $P(Y|X)$ via

$$T(X \Rightarrow Y) = \{ P(Y|X) = 1 \},$$

$$F(X \Rightarrow Y) = \{ P(Y|X) = 0 \},$$

$$U(X \Rightarrow Y) = \{ 0 < P(Y|X) < 1 \},$$  \hfill (56)

where the expressions on the right-hand-side are either $T$ or $F$. That is, if it is certain that $Y$ is the case when $X$ is given ($P(Y|X) = 1$), then it is true that $X$ implies $Y$. Similarly, $X \Rightarrow Y$ is false when $P(Y|X) = 0$ and uncertain for probabilities between 0 and 1.

Whilst perhaps in accordance with an intuitive concept of implication, the prescription of Eq. (56) becomes problematic in the case of $P(X) = 0$, when the calculational procedure

$$P(Y|X) = \frac{P(XY)}{P(X)}$$  \hfill (57)

(where $P(XY)$ is the joint probability of $X$ and $Y$) is not defined. This corresponds to the case of a false antecedent (i.e. ‘$X$ is false’).

Now, in binary sentential calculus, it is permissible for a false proposition to imply any proposition. This has long been seen as an undesirable feature of the calculus and gives rise to one of the so-called ‘paradoxes of material implication’. In particular, the theorem $A \Rightarrow (\sim A \Rightarrow B)$ (‘$A$ implies that $A$ being false implies $B$’) appears to contradict the intuitive meaning of material implication.

From a set theoretic point of view $P(X) = 0$ implies $X = \emptyset$, the empty set. Now the material implication $X \Rightarrow Y$ means that $Y$ must be entailed by $X$. However, by definition, nothing is entailed by the
empty set. In fact, the subset of \( Y \) entailed by \( X \) according to the Eq. (57) is its intersection with \( X \) (i.e. \( P(XY) = P(X \cap Y) \)), which for \( X = \emptyset \) is \( \emptyset \cap Y = \emptyset \).

In other words, whilst ‘nothing implies nothing’ may be strictly true, if \( X = \emptyset \) then no part of \( Y \) is actually implied by \( X \Rightarrow Y \), so no material implication is contained in the residual logical form. One strategy that we might then adopt is to impose this conclusion on Eq. (56) by modifying (57) to read

\[
P(Y|X) = \lim_{\varepsilon \to 0} \frac{P(XY)}{P(X) + \varepsilon},
\]

in analogy with our construction of conditional states in Eq. (48). This is now well-defined in the case that \( P(X) = 0 \) and imposes the result that, since in this case, \( P(XY) = P(\emptyset \cap Y) = 0 \), we have \( P(Y|X) = 0 \).

With this condition in place, we may now reconsider the proposition \( A \Rightarrow (\sim A \Rightarrow B) \). Firstly, we note that, due to the logical equivalence of the propositions \( A \Rightarrow (B \Rightarrow C) \) and \( AB \Rightarrow C \) (where \( AB \) is the logical conjunction of \( A \) and \( B \)), according to Eq. (56), we have

\[
T(AB \Rightarrow C) \equiv \{ P(C|\overline{A}) = 1 \}.
\]

Hence, for \( A \Rightarrow (\sim A \Rightarrow B) \), we have

\[
P(B|\overline{A}) = \lim_{\varepsilon \to 0} \frac{P(BA\overline{A})}{P(\overline{A}) + \varepsilon} = 0,
\]

via the normal rules of probability theory. According to the mappings (52) to (54) then, this proposition actually turns out to be ‘false’.

5 Conclusions

Although the discussion of the last section proposes a method of connecting probability with ‘truth values’, a more natural mapping would be between probability and information. Throughout this work, we have emphasized the role of information transfer between systems and the notion that it is the informational content of a system that defines ‘facts about the world’. In particular, our emphasis has been on the information we have about the historical evolution of a system. Specifically, the past history of a system may only be known through information transfer to a larger environment with sufficient degrees of freedom to store that information.
In Section 3.4, we illustrated how the distributive law of classical logic, whilst not generally holding, may re-emerge through information transfer between a system and its environment. The lack of information we have about a system was also shown to be connected with the quintessentially quantum mechanical phenomenon of interference. This strongly suggests a picture in which we may associate interference effects with our lack of knowledge of a system. As our knowledge is supplemented, such quantum mechanical effects disappear and we see the re-emergence of a classical world to which we may apply (to some extent) classical reasoning.

However, we have argued that the logic we apply to the world should not that of binary logic but rather a ternary logic mapped from probability theory. The need and argument for this is not new - indeed it may be traced back to antiquity and Aristotle’s famous ‘Sea Battle’, raising the problem of future contingents. Aristotle himself may have wanted to say that whilst neither a proposition $P$ about the future (e.g. ‘there will be a sea battle tomorrow’) nor its negation $\sim P$ is either ‘true’ or ‘false’, the disjunction $P \lor \sim P$ is true. However, at the time, he did not have the formal logic to justify this. Nor did the ternary logics of Kleene [10] or Łukasiewicz [11] fix this problem, since the disjunction of two ‘undefined’ propositions in these is still ‘undefined’. We would argue that any statement about the outcome of an experiment (or the resolution of any physical state in a superposition of possibilities) is of the same nature as the problem of future contingents and that an orthocomplemented ternary logic such as described here is the correct approach to use.

In this paper, we have started a preliminary description of a projective quantum logic in terms of relative states, arguing that - when applied to a system in isolation - the mathematical treatment should be in terms of positive operator valued measures. However, it has also been argued on the basis of Naimark’s theorem that when applied to a larger Hilbert space (specifically including the environment of a system, including us as observers) the use of projection valued measures is sufficient.

We have also introduced the concept of ‘partial relative states’ for describing the storage of information about the history of a system within a larger environment. Any general logical system must be applicable to events occurring at different times, so some system of analyzing historical data is essential. In particular, there is a need for the logical conjunction of conjugate variables to describe events at different times. We have shown that, for conjugate variables, such conjunction is non-commutative and may be described using the relative state formulation.
Should the methods and concepts describe here survive critical analysis, there is considerable scope for development. We have been concerned with the description and justification of underlying concepts rather than the formal axiomatization of the logic. Moreover, although we have highlighted that the law of the excluded middle holds, we have refrained from deriving further theorems. Such questions as to how and when the system of logic converges to classical logic are also of interest. More generally, we would advocate exploration of the connection between logic (of any kind) and information theory.

A Appendix

A.1 Generalized projector

We consider the problem of constructing a projector $\hat{\Theta}_X$ satisfying

$$
\hat{\Theta}_X |\psi\rangle = |\psi\rangle,
$$

for

$$
|\psi\rangle = \sum_{j \in X} a_j |\phi_j\rangle,
$$

where, in general, the vectors $|\phi_j\rangle$ are non-orthogonal.

Multiplying Eq. (59) on the left by $\langle \phi_i |$, we have

$$
\langle \phi_i | \psi \rangle = \sum_j a_j \langle \phi_i | \phi_j \rangle.
$$

Defining a matrix $A$ with elements $A_{ij} = \langle \phi_i | \phi_j \rangle$, we may invert this to give

$$
a_i = \sum_j \left(A^{-1}\right)_{ij} \langle \phi_j | \psi \rangle.
$$

Multiplying this on both sides by $|\phi_i\rangle$ then gives

$$
a_i |\phi_i\rangle = |\phi_i\rangle \sum_j \left(A^{-1}\right)_{ij} \langle \phi_j | \psi \rangle.
$$

In other words,
\[
\hat{\Theta}_i = |\phi_i\rangle \sum_j (A^{-1})_{ij} \langle \phi_j |
\]

is the projector we must apply to \( |\psi\rangle \) to obtain \( a_i |\phi_i\rangle \).
To check this, we note that \( \hat{\Theta}_i \) should satisfy
\[
\hat{\Theta}_i |\phi_j\rangle = \delta_{ij} |\phi_j\rangle .
\]
Now,
\[
\hat{\Theta}_i |\phi_j\rangle = |\phi_i\rangle \sum_k (A^{-1})_{ik} \langle \phi_k | \phi_j \rangle ,
\]
\[
= |\phi_i\rangle \sum_k (A^{-1})_{ik} A_{kj} ,
\]
\[
= |\phi_i\rangle \delta_{ij} = \delta_{ij} |\phi_j\rangle ,
\]
as required.
The total required projector satisfying \( \hat{\Theta} |\psi\rangle = |\psi\rangle \) is then given by \( \hat{\Theta}_X = \sum_i \hat{\Theta}_i \), i.e.
\[
\hat{\Theta}_X = \sum_{ij} |\phi_i\rangle \langle A^{-1} \rangle_{ij} |\phi_j\rangle .
\] (60)

Note that in the limiting case of all vectors being orthogonal, we would have \( A_{ij} = \delta_{ij} \) and hence \( (A^{-1})_{ij} = \delta_{ij} \). In this case, the projector reduces to
\[
\hat{\Theta}_X = \sum_{i \in X} |\phi_i\rangle \langle \phi_i | .
\]
Taking the conjugate transpose of (60), we have
\[
\hat{\Theta}_X^\dagger = \sum_{ij} |\phi_j\rangle \langle A^{-1} \rangle_{ij} |\phi_i\rangle .
\]
However, since \( A \) is Hermitian, then so will be \( A^{-1} \), meaning \( (A^{-1})_{ij}^* = (A^{-1})_{ji} \), so
\[
\hat{\Theta}_X^\dagger = \sum_{ji} |\phi_j\rangle \langle A^{-1} \rangle_{ji} |\phi_i\rangle = \hat{\Theta}_X ,
\]
proving that $\hat{\Theta}_X$ is self-adjoint.

Operating on $\hat{\Theta}_X$ with itself,

$$\hat{\Theta}_X^2 = \sum_{ijj'} |\phi_i\rangle \langle A^{-1}\rangle_{ij} \langle \phi_{j'}| (A^{-1})_{i'j'} \langle \phi_{j'}|,$$

$$= \sum_{ii'j'} |\phi_i\rangle \delta_{ii'} (A^{-1})_{i'j'} \langle \phi_{j'}|,$$

$$= \sum_{ij} |\phi_i\rangle (A^{-1})_{ij} \langle \phi_j| = \hat{\Theta}_X,$$

showing that $\hat{\Theta}_X$ is idempotent.

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