Adaptive Convolutions

Ilja Klebanov*

May 8, 2018

Abstract

When smoothing a function $f$ via convolution with some kernel, it is often desirable to adapt the amount of smoothing locally to the variation of $f$. For this purpose, the constant smoothing coefficient of regular convolutions needs to be replaced by an adaptation function $\mu$. This function is matrix-valued which allows for different degrees of smoothing in different directions. The aim of this paper is twofold. The first is to provide a theoretical framework for such adaptive convolutions. The second purpose is to derive a formula for the automatic choice of the adaptation function $\mu = \mu_f$ in dependence of the function $f$ to be smoothed. This requires the notion of the local variation of $f$, the quantification of which relies on certain phase space transformations of $f$. The derivation is guided by meaningful axioms which, among other things, guarantee invariance of adaptive convolutions under shifting and scaling of $f$.

Keywords: adaptive kernel smoothing, convolution, Young’s inequality, continuity equation, phase space transformation, windowed Fourier transform, Wigner transform, local variation, axiomatic approach, invariance

2010 MSC: 44A35, 65D10, 42B10

1 Motivation

The convolution of two integrable functions $f, g: \mathbb{R}^d \to \mathbb{R}$,

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x - y) \, dy,$$

is a basic mathematical tool with applications in probability theory, image processing, optics, acoustics and many other areas. If $g$ is a probability density, say a Gaussian density

$$g(x) = (2\pi \sigma^2)^{-d/2} \exp \left( -\frac{\|x\|^2}{2\sigma^2} \right),$$

the convolution $(f * g)(x)$ evaluated at a point $x$ can be viewed as the average over all values $f(y), \ y \in \mathbb{R}^d$, weighted by $g(x - y)$, i.e. the contribution of $f(y)$ to $(f * g)(x)$ decreases as the distance between $x$ and $y$ increases. Convolutions are therefore often used to ‘flatten’ or ‘smooth’ a function $f$ and $g$ is called a smoothing kernel in this case.

*Zuse Institute Berlin (ZIB), Takustraße 7, 14195 Berlin, Germany (klebanov@zib.de).
A natural question arising here is how to choose the standard deviation \( \sigma \) of \( g \), i.e., how strongly we want to smooth the function \( f \). Roughly speaking, the aim is usually to flatten out the bumps and edges without losing the shape of the function entirely.

**Example 1.** Assume that we decided that some \( \sigma > 0 \) is adequate to smooth a function \( f_1 : \mathbb{R}^d \to \mathbb{R} \), and \( f_2(x) = f_1(\alpha x) \) is a scaled version of \( f_1 \) by some factor \( \alpha > 0 \). In order to achieve a similar smoothing effect for \( f_2 \), the density \( g \) has to be scaled accordingly, \( \tilde{g} = \alpha^d g(\alpha x) \) (the prefactor \( \alpha^d \) is just a normalization factor), as visualized in Figure 1:

\[
(f_2 \ast \tilde{g})(x) = \alpha^d \int_{\mathbb{R}^d} f_1(\alpha y) g(\alpha(x - y)) \, dy = \int_{\mathbb{R}^d} f_1(y) g(\alpha x - y) \, dy = (f_1 \ast g)(\alpha x).
\]

Figure 1: Choosing proper standard deviations of the density \( g \) to smooth differently scaled versions of the function \( f_1 \).

In other words, once the ‘degree’ or ‘extent’ of smoothing is agreed upon, the width of the smoothing kernel \( g \) has to be adapted to the ‘variation’ of the function \( f \). However, if the variation of the function changes considerably in space, no single suitable width \( \sigma \) can be found and one is forced to adapt it locally,

\[
(f \ast \mu \, g)(x) := \int f(y) |\det \mu(y)| g(\mu(y)(x - y)) \, dy,
\]

where \( \mu : \mathbb{R}^d \to \text{GL}(d, \mathbb{R}) \) is a measurable (matrix-valued) function, which scales the density \( g \) locally by different factors \( \mu(y) \).

**Example 2.** Taking up our functions \( f_1 \) and \( f_2 \) from Example 1, we build up a new function

\[
f(x) = f_1(x) + f_2(x - a),
\]

separating \( f_1 \) and \( f_2 \) in space by a shift \( a \in \mathbb{R}^d \). Choosing \( g \) as a smoothing kernel would be inappropriate for one ‘part’ of the function (oversmoothing), as choosing \( \tilde{g} \) would be for the other (undersmoothing), see Figure 2 (a) and (b). For the shift \( a = 8 \) and scaling factor \( \alpha = 6 \), the adaptive convolution \( f \ast \mu \, g \) with

\[
\mu(y) = \begin{cases} 
1 & \text{if } x < 7, \\
6 & \text{if } x \geq 7,
\end{cases}
\]

provides a suitable solution for both parts (see Figure 2c).
While in this example the adaptation function \( \mu \) was chosen manually, the question arises on how this choice can be automatized – what is a good adaptation function \( \mu = \mu_f \) in dependence of \( f \)? We wish to address this issue by first imposing proper axioms concerning the behavior of \( \mu_f \) under shifting and scaling of \( f \), which guarantee invariance of the adaptive convolution under such transformations. In order to derive a formula for \( \mu_f \) that fulfills these axioms, we will introduce a measure for the local variation of \( f \) which relies on certain phase space transformations of \( f \).

One possible application of the adaptive convolution framework is variable kernel density estimation, which is discussed in a companion paper \[6\].

This paper is structured as follows. In Section \[2\] the framework and theory of adaptive convolutions is presented in a slightly more general setup. Two versions of Young’s inequality as well as a differentiation rule and a continuity equation for adaptive convolutions are discussed. In Section \[3\] we address the second main issue of this paper – the automatic choice of the adaptation function \( \mu \). In Appendix \[A\] further approaches to adaptive smoothing are discussed. The proofs are provided in Appendix \[B\].
2 Theoretical Properties of Adaptive Convolutions

Definition 3 (adaptive convolutions, adaptation function). We define the generalized convolution \( f \ast G \) of two measurable functions \( f: \mathbb{R}^d \to \mathbb{R} \) and \( G: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) as the integral operator with kernel \( G \):

\[
(f \ast G)(x) := \int f(y) G(x, y) \, dy,
\]

whenever the integral is well-defined. Let \( f \in L^1(\mathbb{R}^d) \), \( g \in L^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \), \( \mu: \mathbb{R}^d \to \text{GL}(d, \mathbb{R}) \) be a measurable function and

\[
g_{\mu,p}(x, y) := |\det (\mu(x))|^{1/p} g(\mu(x)(x - y)),
\]

where \( 1/p := 0 \) for \( p = \infty \). We define the \( \mu \)-adaptive convolution of \( f \) with \( g \) by

\[
f \ast_{\mu} g := f \ast g_{\mu,p}.
\]

\( \mu \) will be called adaptation function. In the case \( p = 1 \), the definition coincides with \(^2\) and we will omit the index \( p \): \( f \ast_{\mu} g := f \ast g_\mu \) and \( g_\mu := g_{\mu,1} \).

Remark 4. (a) We will allow \( \mu(x) \notin \text{GL}(d, \mathbb{R}) \) and even attain infinite values in the nodes of \( f \), i.e. for \( x \notin \text{supp}(f) \), since this does not influence the integral (using the convention \( 0 \cdot \infty := 0 \)).

(b) The adaptive convolution is not symmetric and the notation \( f \ast_{\mu} g \) indicates that \( g \) is scaled by \( \mu(y) \), while \( f \ast g \) can be used, if \( f \) is to be scaled (we will not need the second notation).

(c) The adaptive convolution is not associative.

(d) The adaptive convolution is linear in both arguments.

(e) The \( \mu \)-adaptive convolution reduces to the common convolution \( f \ast g \) for \( \mu \equiv \text{Id} \).

(f) In Proposition \(^{10}\) we will slightly abuse the notation of adaptive convolutions by applying it to matrix-valued functions \( f \in L^1(\mathbb{R}^d, \mathbb{R}^{m \times n}) \), \( g \in L^p(\mathbb{R}^d, \mathbb{R}^{n \times \ell}) \) or to the case where \( f \) is an operator acting on \( g \). The definitions go analogously.

2.1 Young’s Inequality for Adaptive Convolutions

In the following, we will discuss a weak and a strong version of Young’s inequality \(^7\), \(^1\) Theorem 3.9.4 and the conditions under which they hold. This will result in the generalization of Young’s inequality for convolutions to the case of adaptive convolutions.

Theorem 5. Let \( f \in L^1(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \) and \( G: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be measurable such that \( \|G(\cdot, y)\|_p \leq \Gamma \) for some \( \Gamma \geq 0 \) independent of \( y \in \mathbb{R}^d \). Then

\[
\|f \ast G\|_p \leq \|f\|_1 \Gamma.
\]

This suffices to prove the weak version of Young’s inequality for \( \mu \)-adaptive convolutions:

Corollary 6 (Young’s inequality). Let \( f \in L^1(\mathbb{R}^d) \), \( g \in L^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \) and \( \mu: \mathbb{R}^d \to \text{GL}(d, \mathbb{R}) \) be a measurable function. Then \( f \ast_{\mu} g \in L^p(\mathbb{R}^d) \) and

\[
\|f \ast_{\mu} g\|_p \leq \|f\|_1 \|g\|_p.
\]
We will use the standard multi-index notation for differentiation of adaptive convolutions. This will require some notation:

Similar to common convolutions, there are slightly modified (but non-symmetric!) rules for the differentiation of adaptive convolutions. In particular, if

\[ G(x,y) \]

be measurable such that \( \|G(\cdot,y)\|_p \leq \Gamma \) for all \( y \in \mathbb{R}^d \) and \( \|G(x,\cdot)\|_p \leq \Gamma \) for all \( x \in \mathbb{R}^d \) for some \( \Gamma \geq 0 \). Then

\[ \|f * G\|_r \leq \|f\|_q \Gamma. \]

**Remark 8.** In the particular case \( p = 1 \) we also have

\[ \int_{\mathbb{R}^d} (f * \mu g) (x) \, dx = \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} g_\mu(x,y) \, dx \, dy = \left( \int_{\mathbb{R}^d} f(y) \, dy \right) \left( \int_{\mathbb{R}^d} g(x) \, dx \right) \]

and, if \( g: \mathbb{R}^d \to \mathbb{R} \) is a probability density function,

\[ \|f * \mu g\|_1 = \int_{\mathbb{R}^d} |f(y)| \int_{\mathbb{R}^d} g_\mu(x,y) \, dx \, dy = \|f\|_1. \]

### 2.2 Derivatives of Adaptive Convolutions

Similar to common convolutions, there are slightly modified (but non-symmetric!) rules for the differentiation of adaptive convolutions. This will require some notation:

**Notation 9.** We will use the standard multi-index notation for \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \) and \( x \in \mathbb{R}^d \),

\[
|\alpha| := \alpha_1 + \cdots + \alpha_d, \\
\partial_\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d}, \\
x_\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}.
\]

If \( g \in C^m(\mathbb{R}^d) \), \( m \in \mathbb{N} \), then its \( k \)-th derivative \( (k \leq m) \) can be viewed as a symmetric multilinear map \( D^k g: (\mathbb{R}^d)^k \to \mathbb{R} \). Further, \( k \) vectors \( v_1, \ldots, v_k \in \mathbb{R}^d \) give rise to a linear form on the space \( \mathcal{SM}_k(\mathbb{R}^d) \) of \( k \)-fold symmetric multilinear forms on \( \mathbb{R}^d \),

\[ [v_1, \ldots, v_k]: \mathcal{SM}_k(\mathbb{R}^d) \to \mathbb{R}, \quad \phi \mapsto \phi(v_1, \ldots, v_k), \]

which is just the identification of the Banach space \( (\mathbb{R}^d)^k \) with its double dual. Finally, for \( \alpha \in \mathbb{N}_0^d \) and a matrix \( M \in \mathbb{R}^{d \times d} \) with columns \( M_{\cdot,1}, \ldots, M_{\cdot,d} \), \( \alpha(M) \) will denote the following \( |\alpha| \)-tuple of vectors in \( \mathbb{R}^d \):

\[
\alpha(M) := \left[ M_{\cdot,1}, \ldots, M_{\cdot,1}, M_{\cdot,2}, \ldots, M_{\cdot,2}, \ldots, M_{\cdot,d}, \ldots, M_{\cdot,d} \right] \in \mathcal{L} \left( \mathcal{SM}_k^{(|\alpha|)}(\mathbb{R}^d), \mathbb{R} \right).
\]

Since it will be viewed as a linear map acting on symmetric multilinear forms as described above, the order of the vectors does not matter.

**Proposition 10.** Let \( f \in L^1(\mathbb{R}^d) \), \( g \in C^m(\mathbb{R}^d) \) for some \( m \in \mathbb{N} \), \( \mu: \mathbb{R}^d \to GL(d, \mathbb{R}) \) be a bounded measurable function and \( 1 \leq p \leq \infty \) such that \( \partial_\alpha g \in L^p(\mathbb{R}^d) \) for all \( \alpha \leq m \).

Then \( f * \mu^p g \in C^m(\mathbb{R}^d) \) and for all \( \alpha \in \mathbb{N}^d \) with \( |\alpha| \leq m \), the derivative \( \partial_\alpha (f * \mu^p g) \in L^p(\mathbb{R}^d) \) is given by (we slightly abuse the notation as mentioned in Remark 4 (f))

\[ \partial_\alpha (f * \mu^p g) = (f \cdot \alpha(\mu)) * \mu^p D^{[\alpha]} g. \]
2.3 Continuity Equation for Convolutions and Adaptive Convolutions

Assume that \((\rho_t)_{t \geq 0}\) is a time-dependent probability density, which fulfills the continuity equation

\[
\partial_t \rho_t = -\text{div}(\rho_t v_t) = -\text{div}(j_t)
\]

for some velocity field \(v_t : \mathbb{R}^d \to \mathbb{R}^d\) (or current \(j_t : \mathbb{R}^d \to \mathbb{R}^d\)). Assume further that we want to smooth \(\rho_t\) by considering an adaptive convolution \(\rho_{g,t} = \rho_t * \mu_t g\) with a time-dependent adaptation function \((\mu_t)_{t \geq 0}\) (since we are dealing with probability density functions, \(p = 1\) here). How does the continuity equation have to be modified in order to describe the evolution of \(\rho_{g,t}\)? We were surprised to find an explicit formula for the modified continuity equation:

**Proposition 11.** Let \(\rho_t \in L^1(\mathbb{R}^d)\) be a time-dependent probability density function, which fulfills the continuity equation

\[
\partial_t \rho_t + \text{div} j_t = 0 \quad (t \in \mathbb{R})
\]

for some current \(j_t \in L^1(\mathbb{R}^d, \mathbb{R}^d)\), such that \((\rho_t, j_t)_{t \in \mathbb{R}} \in C^1(\mathbb{R}^{1+d}, \mathbb{R}^{1+d})\).

Further, let \(g \in L^p \cap C^1(\mathbb{R}^d)\) for some \(1 \leq p < \infty\),

\[
\gamma(x) := x g(x), \quad N_t(x) = \begin{pmatrix}
    j_t(x) & (D_x (\mu_t)_1^\top) (x) \\
    \vdots & \vdots \\
    j_t(x) & (D_x (\mu_t)_d^\top) (x)
\end{pmatrix} \in \mathbb{R}^{d \times d} \quad (x \in \mathbb{R}^d),
\]

such that \(g, \gamma\) and all their first derivatives are bounded: \(g, \gamma \in W^{1,\infty}\).

Finally, let \((\mu_t)_{t \in \mathbb{R}} \in C^2(\mathbb{R}^{1+d}, \text{GL}(d, \mathbb{R}))\), such that for each \(i, j = 1, \ldots, d, \ t \in \mathbb{R}\)

\[
(\mu_t)_{i,j}, \nabla (\mu_t)_{i,j}, \partial_t (\mu_t)_{i,j}, (\mu_t^{-1})_{i,j} \in L^\infty.
\]

Then

\[
\rho_{g,t} := \rho_t * \mu_t g \quad (t \in \mathbb{R})
\]

is a probability density function, which fulfills the continuity equation

\[
\partial_t \rho_{g,t} = -\text{div} j_{g,t} \quad \text{for} \quad j_{g,t} = j_t * \mu_t g - [\mu_t^{-1}(N_t + \rho_t \partial_t (\mu_t) \mu_t^{-1})] * \mu_t \gamma.
\]

Further, \((\rho_{g,t}, j_{g,t})_{t \in [0,\infty)} \in C^1(\mathbb{R}^{1+d}, \mathbb{R}^{1+d})\).

**Corollary 12.** Under the assumptions of Proposition 11, if \(\rho_{g,t} = \rho_t * g_{A_t}\) is the common convolution, but with time-dependent scaling matrix \((A_t)_{t \in \mathbb{R}} \in C^2(\mathbb{R}, \text{GL}(d, \mathbb{R}))\) of the smoothing kernel,

\[
ge_{A_t}(x) = |\det A_t| g(A_t x), \quad \gamma_{A_t}(x) = g_{A_t}(x) A_t x,
\]

then the probability density function \(\rho_{g,t}\) solves the continuity equation

\[
\partial_t \rho_{g,t} = -\text{div} j_{g,t} \quad \text{for} \quad j_{g,t} = j_t * g_{A_t} - [\rho_t A_t^{-1} \partial_t (A_t^{-1})] * \gamma_{A_t}.
\]

Further, \((\rho_{g,t}, j_{g,t})_{t \in [0,\infty)} \in C^1(\mathbb{R}^{1+d}, \mathbb{R}^{1+d})\).
3 Automatic Choice of the Adaptation Function $\mu$

The adaptation function $\mu$ in Example 2 was chosen manually for the adaptive smoothing of a function $f \in L^1(\mathbb{R}^d)$. Let us now discuss how this choice can be performed automatically in dependence of the function $f$ that we want to smooth. To this end, we will have to get a grip on the local variation of $f$, an issue that we will address by means of certain phase space transforms introduced in the following subsection.

Finding a good dependence for the adaptation function $\mu = \mu_f$ on the function $f$ is a difficult task and will only be partially answered here. We will justify the implicit formula

$$\mu^2_f(x) = \frac{\left( \nabla f \nabla^\top f - f D^2 f \right) * G^2_{(\lambda \mu_f)^{-1}}(x)}{(2 - \lambda^2) f^2 * G^2_{(\lambda \mu_f)^{-1}}(x)},$$

where $0 < \lambda < \frac{1}{\sqrt{2}}$, but neither prove uniqueness, nor any kind of optimality. Also, we will assume that $f \neq 0$ lies in the Sobolev space $W^{2,2}(\mathbb{R}^d, \mathbb{R})$, and restrict ourselves to radially symmetric smoothing kernels $g$, i.e. $g(x) = \gamma(\|x\|^2)$ for some function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}$.

**Remark 13.** The square root $M^{1/2} = \sqrt{M}$ of a symmetric and positive definite matrix $M \in \mathbb{R}^{d \times d}$ will denote the unique symmetric and positive definite matrix $\tilde{N} \in \mathbb{R}^{d \times d}$ such that $\tilde{N}^2 = M$ (see [3, Theorem 7.2.6]).

Let us first gather some conditions, which we would like our adaptation function $\mu_f$ to fulfill (see also the motivation in Section 1):

**Axiom 14 (Adaptation Axioms).** Let $\mathcal{M} = \{\mu : \mathbb{R}^d \to \text{GL}(d, \mathbb{R}) : \mu \text{ measurable}\}$. We say that a mapping

$$m : W^{2,2}(\mathbb{R}^d, \mathbb{R}) \to \mathcal{M}, \quad f \mapsto \mu_f,$$

fulfills the Adaptation Axioms, if for any $a \in \mathbb{R}^d$, $\alpha \in \mathbb{R} \setminus \{0\}$, $A \in \text{GL}(d, \mathbb{R})$, any parametrized function $f(t) = \sum_{k=1}^K f_k(\alpha - a_k^{(t)})$, $t \geq 0$, with $f_k \in W^{2,2}(\mathbb{R}^d, \mathbb{R})$, $a_k^{(t)} \in \mathbb{R}^d$, such that $\|a_k^{(t)} - a_j^{(t)}\| \to \infty$ for all $k \neq j$, and any $x \in \mathbb{R}^d$,

(A1) $\mu_{f(\alpha \cdot)}(x) = \mu_f(x - \alpha)$  (invariance under shifting),

(A2) $\mu_{\alpha \cdot f} = \mu_f$  (invariance under scalar multiplication),

(A3) $\mu_{f(A \cdot)}^T(x) \mu_f(A \cdot) = A^\top \mu_f(Ax) A$  (invariance under scaling),

(A4) $\mu_{f(t)}(x + a_k^{(t)}) \to \infty \mu_{f_k}(x)$ for all $k = 1, \ldots, K$  (locality),

(A5) $\mu_f(x)$ should describe some kind of variation of $f$ locally around $x \in \mathbb{R}^d$.

While axiom (A5) is rather subjective, axioms (A1)–(A3) are chosen such that the $\mu_f$-adaptive convolution behaves nicely under shifting and scaling of $f$. Axiom (A4) guarantees that a sum $f = \sum_{k=1}^K f_k$ of several functions $f_1, \ldots, f_K$ with ‘far apart’ supports is smoothed in approximately the same way as these functions would have been smoothed separately, $f * \mu_f g \approx \sum_{k=1}^K f_k * \mu_{f_k} g$, see also the motivating Example 2. These consequences are stated in the following theorem:

**Theorem 15.** Assuming Adaptation Axioms [14] (A1)–(A4) and adopting that notation, we have for any $f \in W^{2,2}(\mathbb{R}^d, \mathbb{R})$, radially symmetric $g \in L^p$ and $x \in \mathbb{R}^d$:
Fourier transform, Wigner transform) \( \) (iii) the arbitrary covariance matrices allows for different window sizes in different directions.

\[ \text{(Gaussian, Fourier transform, windowed Fourier transform, adaptive windowed} \]

Definition 17 \( \) space of rapidly decreasing functions.

\[ \text{(ii) the} \]

One possible choice that fulfills the Adaptation Axioms (A1)–(A4) is \( \) (stretched function \( \Rightarrow \) stretched convolution),

Remark 16. Adaptation Axiom (A3) is a necessary detour around the more appealing condition \( \mu_{f(A,x)}(x) = f(Ax)A, \)

since the product of a symmetric and positive definite matrix with an invertible matrix is in general no longer symmetric and positive definite. This is also the reason why we have to restrict ourselves to radially symmetric smoothing kernels \( g \). Roughly speaking, covariance matrices are easier to treat than their square roots.

One possible choice that fulfills the Adaptation Axioms (A1)–(A4) is

\[ \mu_f^{(a)} = \sqrt{\nabla f \nabla f^\top} / f^2. \]

However, \( \nabla f \nabla f^\top \) is only positive semi-definite and therefore \( \mu_f^{(a)} \in \text{GL}(d, \mathbb{R}) \) could be violated. Also, it is unclear in how far the Adaptation Axiom (A5) is fulfilled. Obviously, this last axiom is not rigorous and we will discuss it now. In order to get a grasp on it, we will make use of three transformations explained in the following subsection.

3.1 Phase Space Transformations

In this section, we will discuss four transformations, which will allow us to quantify the (local) variation of a function. All four transformations can be viewed from various perspectives and we will focus on the time-frequency point of view. In the following, \( \mathcal{S}(\mathbb{R}^d, \mathbb{C}) \) will denote the Schwartz space of rapidly decreasing functions.

Definition 17 (Gaussian, Fourier transform, windowed Fourier transform, adaptive windowed Fourier transform, Wigner transform). \( \) We define

(i) the Gaussian function \( G[a, \Sigma]: \mathbb{R}^d \to \mathbb{R} \) with mean \( a \in \mathbb{R}^d \) and symmetric and positive definite covariance matrix \( \Sigma \in \text{GL}(d, \mathbb{R}) \) as well as the abbreviation \( G_{\Sigma} := G[0, \Sigma], \)

(ii) the Fourier transform \( \mathcal{F}: \mathcal{S}(\mathbb{R}^d, \mathbb{C}) \to \mathcal{S}(\mathbb{R}^d, \mathbb{C}), \)

(iii) the windowed Fourier transform \( \mathcal{F}_Q: \mathcal{S}(\mathbb{R}^d, \mathbb{C}) \to \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}) \) with window width \( Q \in \text{GL}(d, \mathbb{R}), \) \( Q^\top = Q, \)

(iv) the adaptive windowed Fourier transform \( \mathcal{F}_Q: \mathcal{S}(\mathbb{R}^d, \mathbb{C}) \to \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}) \) with variable window width \( Q: \mathbb{R}^d \to \text{GL}(d, \mathbb{R}), \) \( Q^\top = Q, \)

\[ \text{(i) the Gaussian function} \]

Typically, windowed Fourier transforms are defined for isotropic covariance matrices, i.e. \( Q = \sigma \text{Id.} \) Using arbitrary covariance matrices allows for different window sizes in different directions.
(v) the Wigner transform $W: \mathcal{S}(\mathbb{R}^d, \mathbb{C}) \to \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ by $^2$

$$G[a, \Sigma](x) = (2\pi)^{-d/2} |\det \Sigma|^{-1/2} \exp \left[ -\frac{1}{2} (x - a)^\top \Sigma^{-1} (x - a) \right], \tag{4}$$

$$\mathcal{F} f(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) e^{-iy\xi} \, dy, \tag{5}$$

$$\mathcal{F}_Q f(x, \xi) = \pi^{-d/4} \int_{\mathbb{R}^d} f(y) G_2(x - y) e^{-iy\xi} \, dy, \tag{6}$$

$$\mathcal{F}_Q f(x, \xi) = \pi^{-d/4} \int_{\mathbb{R}^d} f(y) G_2(x) e^{-iy\xi} \, dy, \tag{7}$$

$$W f(x, \xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} f \left( x + \frac{y}{2} \right) f \left( x - \frac{y}{2} \right) e^{iy\xi} \, dy. \tag{8}$$

We will restrict ourselves to presenting only a few properties of these transforms, further properties can be found in e.g. [2].

**Proposition 18** (Plancherel Theorem and Fourier Inversion Formula). The Fourier transform is an isometric isomorphism on $\mathcal{S}(\mathbb{R}^d, \mathbb{C})$ with inverse given by

$$\mathcal{F}^{-1} f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\xi) e^{ix\xi} \, d\xi.$$

*Proof.* See [2]. \hfill \square

For $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{C})$, the Fourier inversion formula implies

$$ (2\pi)^{-d} \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} e^{-ix\xi} \, d\xi \, dx = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathcal{F} f(\xi) e^{i0\xi} \, d\xi = \mathcal{F}^{-1} \mathcal{F} f(0) = f(0). $$

The technique of using $(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\xi} \, d\xi$ as a $\delta$-distribution will be used in the proofs of Propositions 20, 21, 22 and 25.

From the point of view of time-frequency analysis, the Fourier transform yields a decomposition of the signal $f$ into its frequencies: for each frequency $\xi$, it indicates the extent of its occurrence in $f$. However, one is often interested in the local frequencies of $f$, meaning, which frequencies of $f$ occur at (or close to) a specific point in time $x$. This can be analyzed using a windowed Fourier transform, also called Gabor transform, which does not ‘see’ the values and frequencies of $f$ far from $x$. Applied to each point in time $x$, this defines the mapping (6) on the phase space.

The width $Q$ of the Gaussian function $G_2$ is a double-edged sword: The smaller it is chosen, the more accurate the time-frequency description becomes in the $x$-direction, since only values that are very close to the considered time $x$ contribute to $\mathcal{F}_Q f(x, \bullet)$. However, the smaller the window, the more ‘difficult’ it becomes to determine the frequencies in this small time period and the more ‘blurred’ the frequency decomposition $\mathcal{F}_Q f(x, \bullet)$ becomes in $\xi$-direction. This issue is a manifestation of the so-called uncertainty principle, see the discussion in [3, Chapter 2].

---

$^2$Usually, the Wigner transform is defined by $W(f, g)(x, \xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} f \left( x + \frac{y}{2} \right) g \left( x - \frac{y}{2} \right) e^{iy\xi} \, dy$. We will only use its definition on the diagonal, $W f := W(f, f)$, where it is real-valued, which can be seen by applying the transformation $y \mapsto -y$ to the defining integral.
Figure 3: Visualization of how different frequencies of a function $f$ are represented by the Fourier transform $\mathcal{F}f$, the windowed Fourier transform $\mathcal{F}_Qf$ and the Wigner transform $Wf$ (only the moduli squared of $\mathcal{F}f$ and $\mathcal{F}_Qf$ are plotted).

The adaptive windowed Fourier transform \( \mathcal{F}_Q \) allows to perform this trade-off differently in different regions of $\mathbb{R}^d$ by choosing the width to be $x$-dependent.

At the expense of losing positivity, the Wigner transform $Wf$ of $f$ provides a way to ‘deblurr’ $|\mathcal{F}_Qf|^2$, resulting in the ‘correct’ marginal densities, as stated by the following proposition:

**Proposition 19.** Let $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{C})$ and $G_\sigma(x, \xi) = (2\pi\sigma^2)^{-d/2} \exp\left( -\frac{\|x\|^2 + \|\xi\|^2}{2\sigma^2} \right)$ denote a Gaussian in phase space. Then the Wigner transform $Wf$ of $f$ fulfills

$$\int_{\mathbb{R}^d} Wf(x, \xi) \, d\xi = |f(x)|^2, \quad \int_{\mathbb{R}^d} Wf(x, \xi) \, dx = |\mathcal{F}f(\xi)|^2,$$

and

$$Wf * G_{\sqrt{1/2}} = |\mathcal{F}_1 f|^2.$$  

**Proof.** See \[4, (3.10)\].

3.2 Transformations and the Adaptation Function $\mu$

We are now ready to construct an adaptation function $\mu_f$ that fulfills the Adaptation Axioms \[14\]. Let us start with a global version. A natural way to describe the global variation of a function
\( f \in L^1(\mathbb{R}^d, \mathbb{R}) \) is to consider its spectral density \( \rho \times |\mathcal{F}f|^2 \), since functions with high oscillations tend to have high values of \( |\mathcal{F}f|^2 \) away from the origin.

The spectral density also has the proper behaviour under scaling of \( f \) – if \( f \) is scaled by some factor \( \alpha \neq 0 \), \( \tilde{f}(x) = f(\alpha x) \), the (global) variation is scaled by \( \alpha^{-1} \) and in fact the Fourier transform (and thereby the spectral density) is scaled accordingly:

\[
\mathcal{F}\tilde{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\alpha y) e^{-iy\cdot\xi} dy = \frac{(2\pi)^{-d/2}}{\alpha^d} \int_{\mathbb{R}^d} f(y) e^{-iy\cdot\xi/\alpha} dy = \alpha^{-d} \mathcal{F}f(\alpha^{-1}\xi). \tag{9}
\]

In order to assign a value for the variation to a function \( f \), we will therefore consider the expectation value and covariance defined in the following proposition:

**Proposition 20.** Let \( f \in W^{2,2}(\mathbb{R}^d, \mathbb{R}) \setminus \{0\} \). The expectation value and covariance matrix of the probability distribution \( \mathbb{P}_\rho \) given by the spectral density

\[
\rho = \frac{|\mathcal{F}f|^2}{\|\mathcal{F}f\|_{L^2}^2} = \frac{|\mathcal{F}f|^2}{\|f\|_{L^2}^2}
\]

(\textit{here we used the Plancherel theorem} \cite{18}) are:

\[
\mathbb{E}_\rho = 0 \quad \text{and} \quad \text{Cov}_\rho = \frac{\int_{\mathbb{R}^d} (\nabla f\nabla f^\top)(z) dz}{\|f\|_{L^2}^2}.
\]

The adaptation function \( \mu_f \), which in this global setting is just a constant adaptation matrix \( \mu_f \in \text{GL}(d, \mathbb{R}) \), can now be assigned the square root of the covariance matrix,

\[
\mu_f^{(b)} := \sqrt{\text{Cov}_\rho} = \frac{\sqrt{\int_{\mathbb{R}^d} (\nabla f\nabla f^\top)(z) dz}}{\|f\|_{L^2}},
\]

and the Adaptation Axioms \cite{14}(A1)--(A3) can easily be verified (in a global, \( x \)-independent sense). However, we are not interested in a global, but in a local adaptation. Therefore, we will study the ‘local frequencies’ of \( f \) by taking its windowed Fourier transform \( \mathcal{F}Qf(x, \xi) \) instead of its Fourier transform.

Again, let us consider the expectation value and covariance of the corresponding probability density in \( \xi \):

**Proposition 21.** Let \( f \in W^{2,2}(\mathbb{R}^d, \mathbb{R}) \setminus \{0\} \) and \( Q \in \text{GL}(d, \mathbb{R}) \), \( Q^\top = Q \). Then, for each \( x \in \mathbb{R}^d \), the expectation value and covariance matrix of the probability distribution \( \mathbb{P}_{\rho_x} \) given by the density

\[
\rho_x(\xi) = \frac{|\mathcal{F}Qf(x, \xi)|^2}{\|\mathcal{F}Qf(x, \bullet\|_{L^2}^2}
\]

are:

\[
\mathbb{E}_{\rho_x} = 0 \quad \text{and} \quad \text{Cov}_{\rho_x} = \frac{Q^{-2}}{2} + \frac{(\nabla f\nabla f^\top - f D^2 f) * G_{Q^2}^2(x)}{2f^2}.
\]
Again, we can set

$$
\mu_f^{(c)}(x) := \sqrt{\text{Cov}_{\rho_x}} = \sqrt{\frac{Q^{-2}}{2} + \frac{(\nabla f \nabla f^\top - f D^2 f) \ast G^2_{Q^2}(x)}{2 f^2 \ast G_{Q^2}(x)}}.
$$

(10)

However, while the Adaptation Axioms 14 (A1), (A2) and (A4) are fulfilled, the scale invariance (A3) is violated. The reason for this is that the window width $Q$ does not scale with the local variation of $f$ — a formula analogous to (9) does not hold for windowed Fourier transforms. One might try to adapt $Q$ locally by using the adaptive windowed Fourier transform (7) but this would require a priori knowledge of the local variation of $f$, which we are trying to find in the first place.

We will discuss two different solutions for this problem:

(A) One might avoid the circular reasoning described above by a fixed point approach (and a fixed point iteration in practical applications):

- Derive a formula analogous to (10) for adaptive windowed Fourier transforms.
- Choose the local width $Q(x) \in \text{GL}(d, \mathbb{R})$ proportional to $\mu_f^{-1}(x)$, resulting in an implicit formula for $\mu_f$.

(B) Since the lack of scale invariance is caused by the (constant) width of the window, or, in other words, by the blurry way we look at the function, we will ‘deblur’ it by replacing the term $|F_Q f(x, \xi)|^2$ in the probability density $\rho_x$ from Proposition 21 with the Wigner transform $W f(x, \xi)$. This replacement is motivated by the discussion in Section 3.1 and by Proposition 19 in particular. Since $W f$ can take negative values, we will consider $|W f|^2$ instead of $W f$, which is a probability density function, if properly normalized.

Let us start with approach (A).

**Proposition 22.** Let $f \in W^{2,2}(\mathbb{R}^d, \mathbb{R}) \setminus \{0\}$ and $Q: \mathbb{R}^d \to \text{GL}(d, \mathbb{R})$. Then, for each $x \in \mathbb{R}^d$, the expectation value and covariance matrix of the probability distribution $\mathbb{P}_{\rho_x}$ given by the density

$$
\rho_x(\xi) = \frac{|F_Q f(x, \xi)|^2}{\|F_Q f(x, \bullet)\|_{L^2}^2}
$$

are:

$$
\mathbb{E}_{\rho_x} = 0 \quad \text{and} \quad \text{Cov}_{\rho_x} = \frac{(Q^\top Q)^{-1}(x)}{2} + \frac{(\nabla f \nabla f^\top - f D^2 f) \ast G^2_{(Q^\top Q)(x)}(x)}{2 f^2 \ast G^2_{(Q^\top Q)(x)}}.
$$

Setting $\mu_f(x) := \sqrt{\text{Cov}_{\rho_x}}$ as before and choosing $Q(x) = (\lambda \mu_f)^{-1}(x), 0 < \lambda < \sqrt{2}$, as discussed in approach (A) above, we arrive at an implicit formula for $\mu_f = \mu_f^{(d)}$ (without loss of generality we assume $\mu_f$ to be symmetric and positive definite),

$$
\mu_f^{(d)}(x) = \frac{\lambda^2 \mu_f^2(x)}{2} + \frac{(\nabla f \nabla f^\top - f D^2 f) \ast G^2_{(\lambda \mu_f)^{-2}(x)}(x)}{2 f^2 \ast G^2_{(\lambda \mu_f)^{-2}(x)}}.
$$

(11)

This formula can be simplified by combining the term on the left-hand side with the first term on the right-hand side, but we prefer this form, because it guarantees that the right-hand side is positive definite by Proposition 22 which is essential for the fixed point iteration (12).
Remark 23. Here and in the following we will assume that the function $f$ is such that (11) has a unique symmetric and positive definite solution $\mu_f$ and that the corresponding fixed point iteration

$$\mu_f^{(n+1)}(x) = \sqrt{\frac{(\lambda \mu_f^{(n)}(x))^2}{2} + \frac{(\nabla f \nabla f^\top - f D^2 f) * G^2}{2 f^2 * G^2 (\lambda \mu_f^{(n)})^{-2}(x)}}(x)$$

(12)

converges to that solution for every symmetric and positive definite choice $\mu_f^{(0)}$. The class of functions $f$ for which this is the case remains an open problem.

The crucial advantage of the choice (11) is that it fulfills the Adaptation Axioms 14:

Theorem 24. Let $\mu_f^{(d)}$ be the solution of the implicit equation (11). Then it fulfills the Adaptation Axioms (A1)–(A4).

Now let us discuss approach (B), which suggests to replace the windowed Fourier transform in Proposition 21 by the modulus squared $|Wf|^2$ of the Wigner transform $Wf$. In contrast to the windowed Fourier transform, a formula analogous to (9) holds for the Wigner transform $Wf$ and thereby for $|Wf|^2$, which is a promising property for scale invariance (again, $f(x) := f(\alpha x)$ for some $\alpha \neq 0$):

$$W\hat{f}(x, \xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} f(\alpha x + \frac{\alpha y}{2}) f\left(\alpha x - \frac{\alpha y}{2}\right) e^{i\xi y} dy$$

$$= \frac{(2\pi)^{-d}}{\alpha^d} \int_{\mathbb{R}^d} f\left(\frac{\alpha x + y}{2}\right) f\left(\alpha x - \frac{y}{2}\right) e^{i\xi y} dy$$

$$= \alpha^{-d} Wf(\alpha x, \xi/\alpha),$$

$$\Rightarrow |W\hat{f}|^2(x, \xi) = \alpha^{-2d} |Wf|^2(\alpha x, \xi/\alpha).$$

As before, let us compute the expectation value and covariance of the corresponding probability density in $\xi$:

Proposition 25. Let $f \in W^{2,2}(\mathbb{R}^d, \mathbb{R}) \setminus \{0\}$. Then, for each $x \in \mathbb{R}^d$, the expectation value and covariance matrix of the probability distribution $\mathbb{P}_{\rho_x}$ given by the density

$$\rho_x(\xi) = \frac{|Wf|^2(x, \xi)}{||Wf(x, \cdot)||^2_{L^2}}$$

are:

$$E_{\rho_x} = 0 \quad \text{and} \quad \text{Cov}_{\rho_x} = \frac{\left(\nabla f \nabla f^\top - f D^2 f\right) * f^2}{4 (f^2) * (f^2)}(2x).$$

This time, if we choose

$$\mu_f^{(e)}(x) := \sqrt{\text{Cov}_{\rho_x}} = \sqrt{\frac{\left(\nabla f \nabla f^\top - f D^2 f\right) * f^2}{4 (f^2) * (f^2)}(2x)},$$

(13)

the Adaptation Axioms (A1)–(A3) are fulfilled, as stated by the following proposition, but not (A4), as demonstrated by Example 30.
Theorem 26. $\mu_f^{(e)}$ as defined by (13) fulfills the Adaptation Axioms (A1)–(A3).

We will now state one more result, which suggests a possibility to calibrate the original width of the smoothing kernel $g$:

**Corollary 27.** If $f = G[a, \Sigma]$ is a Gaussian density given by (4), then we have

$$
\mu_f^{(d)} = \frac{1}{\sqrt{2-\lambda^2}} \Sigma^{-1/2}, \quad \mu_f^{(e)} = \frac{1}{2} \Sigma^{-1/2}.
$$

Therefore, in order to gauge the overall extent of the smoothing process, Gaussian functions $f$ (or just the standard Gaussian) are well suited for the calibration of the original width of the smoothing kernel $g$.

### 3.3 Examples

**Example 28.** The application of (10), (11) and (13) to the function $f(x) = f_1(x) + f_2(x-a)$ from Example 2 yields the results presented in Figure 4. The scale invariance (Adaptation Axiom (A3)) of the choices (11) and (13) is clearly visible: $f_2(x) = f_1(\alpha x)$ and, accordingly, $\mu_f$ is $\alpha = 6$ times higher in the ‘right’ domain than in the ‘left’ one.

![Figure 4: $\mu_f^{(e)}$, $\mu_f^{(d)}$, and $\mu_f^{(e)}$ as given by the formulas (10), (11), and (13) describe the local variation of $f$. Choosing them as adaptation functions yields proper local scaling of $g$ and thereby an adequate smoothing of $f$ everywhere (the width $\sigma$ of the Gaussian kernel $g$ as well as $Q$ in (10) and $\lambda$ in (11) were chosen manually).](image-url)
**Example 29.** We also present a 2-dimensional example, where $f$ is chosen as the following highly curved density (a strongly deformed Gaussian):

$$f(x) = \frac{1}{2\pi \sigma} \exp \left( -\frac{1}{2} \left( \frac{x_1}{\sigma} \right)^2 + \left( x_2 - \alpha \left( \frac{x_1}{\sigma} \right)^2 \right)^2 \right), \quad \alpha = 4, \sigma = 5. \quad (14)$$

![Figure 5: The curved density (14) smoothed by standard convolution and adaptive convolution using rules (11) and (13) (the width $\sigma$ of the Gaussian kernel $g$ as well as $\lambda$ in (11) were chosen manually). It is evident how the last two adapt to the local behavior of $f$. The red ellipses show 80% contours of the kernels $g(\bullet - y)$ in (b) and the (stretched) kernels $|\det \mu f(y)| g(\mu_f(y)(\bullet - y))$ in (c) and (d) for several centers $y$.](image)

Our last example demonstrates, how the Adaptation Axiom [14](A4) is fulfilled by (11) but violated by (13). The latter fails to capture solely local properties of $f$ because it makes use of the convolution of $f^2$ with itself (and with its derivatives).

**Example 30.** Consider the function

$$f(x) = f_1(x) + f_2(x - ta) + f_3(x + ta),$$

where $f_1, f_2, f_3$ are three functions 'located at the origin', $t > 0$ and $a \in \mathbb{R}^d \setminus \{0\}$. For $t \to \infty$ the three parts will 'drift apart'. However, the convolution of $f_2(\bullet - ta)$ and $f_3(\bullet + ta)$ (and of their derivatives) remain unchanged as $t$ grows and, since (13) depends on these convolutions, $f_2$ and $f_3$ will have an influence on the smoothing of $f_1$ no matter how large $t$ becomes.
Figure 6: Adaptive smoothing of the function $f$ from Example 30 using the rules (11) and (13). While the two Gaussians $f_2(\bullet - ta)$ and $f_3(\bullet + ta)$ are smoothed nicely in both cases (in fact, $\mu_f$ is constant there as stated in Corollary 27), the smoothing of $f_1$ via (13) is highly influenced by the other two parts, resulting in undesirable oversmoothing in $x_2$-direction at its center. This effect remains unchanged even if $t$ is increased. The red ellipses show 80% contours of the (stretched) kernels $|\det \mu_f(y)|g(\mu_f(y)(\bullet - y))$ for several centers $y$. The width $\sigma$ of the Gaussian kernel $g$ as well as $\lambda$ in (11) were chosen manually.

4 Conclusion

After defining adaptive convolutions (for other types of adaptive convolutions see Appendix A) and analyzing their theoretical properties, we have derived a formula for the adaptation function $\mu_f$, which allows automatic adjustment of the local smoothing of a function $f$. The requirements for such a formula were reasonable axioms on how the adaptation function $\mu_f$ should behave under transformations (shifting and scaling) of $f$. Its derivation relied on the notion of the local variation of $f$, which we argued can be quantified by means of certain phase space transforms. Several suggestions for the mapping $f \mapsto \mu_f$ were made, but only (11) succeeds in fulfilling all Adaptation Axioms 14.

The choice (13) looked promising, but failed to capture solely local properties of $f$ and therefore, as demonstrated in Example 30, could not realize a key property required for adaptive convolutions: If the function $f = \sum_{k=1}^{K} f_k$ is the sum of several well-separated functions $f_1, \ldots, f_K$, then its adaptive convolution should also be (approximately) the sum of the adaptive convolutions of $f_1, \ldots, f_K$ (Theorem 15(iv)).
The overall extent of the smoothing effect can be calibrated by choosing the width $\sigma$ of the smoothing kernel $g$ and applying the adaptive convolution to Gaussian functions $G[a, \Sigma]$, see Corollary 27. The choice of the parameter $\lambda$ remains an open problem. In our examples the values in the interval $[0.6, 1.2]$ provided favorable results.

As a byproduct, we obtain an adaptive window selection method for time-frequency representations, which is invariant under linear transformations of the signal, allows different window sizes in different directions and adapts locally to the signal’s (mean squared) frequency.

The considerations in this paper are mainly theoretical and the computation of the adaptation functions (11) and (13) appears intricate and costly (except for simple examples like the ones presented here). An application of adaptive convolutions to variable kernel density estimation is discussed in a companion paper [6], where also a numerical scheme for the computation was developed in the case of Gaussian kernels.

A Other Types of Adaptive Convolutions

In the case of the common convolution $f \ast g$, the contribution of $f(y)$ to $(f \ast g)(x)$ depends, roughly speaking, on the distance between $x$ and $y$. In the following, we will introduce two further types of adaptive convolutions, for which the contribution of $f(y)$ to the convolution evaluated in $x$ depends on the distance between $h(x)$ and $h(y)$, where $h$ is a function which controls the adaptation.

**Definition 31** (adaptive convolutions of types two and three). Let $1 \leq p \leq \infty$, $f \in L^1(\mathbb{R}^d)$, $g_1 \in L^1(\mathbb{R}^n)$, $h: \mathbb{R}^d \rightarrow \mathbb{R}^n$ be a measurable function and $g_2, g \in L^p(\mathbb{R}^n)$ such that

$$0 < \|g(h(\bullet) - z)\|_p < \infty \quad \text{and} \quad 0 < \|g_2(h(\bullet) - z)\|_p < \infty \quad \text{for almost all } z \in \mathbb{R}^d.$$ 

We define the $h$-adaptive convolutions of types two and three by

$$(f \ast^p [g \mid h])(x) = f \ast \hat{G}_p, \quad (f \ast^p [g_1, g_2 \mid h])(x) = f \ast \tilde{G}_p,$$

where

$$G_p(x, y) = \|g\|_p \frac{g(h(x) - h(y))}{\|g(h(\bullet) - h(y))\|_p},$$

$$\tilde{G}_p(x, y) = \|g_2\|_p \int g_1(z - h(y)) \frac{g_2(z - h(x))}{\|g_2(z - h(\bullet))\|_p} \, dz.$$ 

Again, we will omit the index $p$ in the case $p = 1$.

**Remark 32.** $f \ast^p [g \mid h]$ is homogeneous in $g$ and $f \ast^p [g_1, g_2 \mid h]$ is homogeneous in $g_2$ and linear in $g_1$.

**Proposition 33** (Young’s inequality). Under the conditions of Definition 31, we have:

$$\|f \ast^p [g \mid h]\|_p \leq \|f\|_1 \|g\|_p \quad \text{and} \quad \|f \ast^p [g_1, g_2 \mid h]\|_p \leq \|f\|_1 \|g_1\|_1 \|g_2\|_p.$$ 

17
Figure 7: Adaptive Convolutions of type two and three for Gaussian $g$, $g_1$, $g_2$, a cubic polynomial $h(x) = x^3 + x/3$ and $p = 1$. Note that $\hat{G}$ is symmetric, while $G$ is not (see also Proposition 34). Both convolutions provide a strong smoothing close to zero, and nearly no smoothing away from zero, where $G$ and $\hat{G}$ act nearly like Dirac $\delta$-distributions.

Figure 8: Adaptive Convolutions of type two and three for Gaussian $g$, $g_1$, $g_2$, a quadratic $h(x) = x^2$ and $p = 1$. Observe how the function $f$ is ‘smoothed over to the right side’ due to the non-injectivity of $h$. Again, the smoothing is stronger close to zero.
If \( g_1 = g_2 =: g \), the \( h \)-adaptive convolution of type three fulfills the general Young’s inequality:

**Proposition 34.** Let \( 1 \leq p, q, r \leq \infty \) such that \( 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). Let \( f \in L^q (\mathbb{R}^d) \), \( g_1 = g_2 =: g \in L^1 \cap L^p (\mathbb{R}^n) \) and \( h: \mathbb{R}^d \rightarrow \mathbb{R}^n \) be a measurable function such that \( 0 < \|g(h(\cdot) - z)\|_p < \infty \) for almost all \( z \in \mathbb{R}^d \). Then \( \hat{G}_p \) is symmetric and

\[
\|f \ast^p [g, g \, | \, h]\|_r \leq \|f\|_q \|g\|_p .
\]

**Remark 35.** Several generalizations to adaptive convolutions of types two and three are possible:

1. Nothing changes, if we replace \( h \) by two different functions \( h_1, h_2: \mathbb{R}^d \rightarrow \mathbb{R}^n \) in the following way:

\[
(f \ast^p [h_1, h_2])(x) = f \ast G_p, \quad (f \ast^p [g_1, g_2 \, | \, h_1, h_2])(x) = f \ast \hat{G}_p,
\]

where

\[
G_p(x, y) = \|g\|_p \frac{g(h_1(x) - h_2(y))}{\|g(h_1(\cdot) - h_2(\cdot))\|_p},
\]

\[
\hat{G}_p(x, y) = \|g_2\|_p \int g_1(z - h_1(y)) \frac{g_2(z - h_2(x))}{\|g_2(z - h_2(\cdot))\|_p} \, dz.
\]

2. If \( g_1 \in L^1 (\mathbb{R}^d \times \mathbb{R}^n) \), \( g_2, g \in L^p (\mathbb{R}^d \times \mathbb{R}^n) \) depend on an additional parameter in \( \mathbb{R}^d \) and

- \( \int_{\mathbb{R}^d} |g_1(y, z - h(y))| \, dy \leq \Gamma_1 \) for some constant \( \Gamma_1 > 0 \) (independent of \( z \)),
- for almost all \( z \in \mathbb{R}^d \),

\[
0 < \int |g(\tilde{x}, z - h(\tilde{x}))|^p \, d\tilde{x} < \infty \quad \text{and} \quad 0 < \int |g_2(\tilde{x}, z - h(\tilde{x}))|^p \, d\tilde{x} < \infty,
\]

we can generalize

\[
G_p(x, y) = \|g\|_p \frac{g(x, h(x) - h(y))}{(\int |g(\tilde{x}, z - h(\tilde{x}))|^p \, d\tilde{x})^{1/p}},
\]

\[
\hat{G}_p(x, y) = \|g_2\|_p \int g_1(y, z - h(y)) \frac{g_2(x, z - h(x))}{(\int |g_2(\tilde{x}, z - h(\tilde{x}))|^p \, d\tilde{x})^{1/p}} \, dz.
\]

In this case, Young’s inequality takes the forms

\[
\|f \ast^p [g \, | \, h]\|_p \leq \|f\|_1 \|g\|_p \quad \text{and} \quad \|f \ast^p [g_1, g_2 \, | \, h]\|_p \leq \Gamma_1 \|f\|_1 \|g_2\|_p .
\]

**Remark 36.** If \( g, g_1, g_2 \) are Gaussian functions or similar (in the sense that they attain their maximum in the origin and decay monotonically to zero as \( x \rightarrow \pm \infty \)) and \( Dh(x) \) is invertible for each \( x \in \mathbb{R}^d \), both convolutions approximate the \( \mu \)-adaptive convolution from Section 3 for \( \mu = Dh \).
More precisely, in this case the linearization of \( h \) at \( y \), \( h(x) - h(y) \approx Dh(y)(x - y) \), is meaningful and yields

\[
G_p(x, y) = \|g\|_p \frac{g(h(x) - h(y))}{g(h(\bullet) - h(y))} \approx \|g\|_p \frac{g(Dh(y)(x - y))}{g(Dh(\bullet)(\bullet - y))} = \det Dh(y)^{1/p} \frac{g(Dh(y)(x - y))}{g(Dh(\bullet)(\bullet - y))},
\]

\[
\tilde{G}_p(x, y) = \|g\|_p \int g_1(z) \frac{g_2(z - h(x) + h(y))}{g_2(z - h(\bullet) + h(y))} \, dz \approx \|g\|_p \int g_1(z) \frac{\tau g_2(Dh(y)(x - y) - z)}{g_2(Dh(\bullet)(\bullet - y) - z)} \, dz = \det Dh(y)^{1/p} (g_1 \ast \tau g_2)(Dh(y)(x - y)),
\]

where \( \tau g_2(x) := g_2(-x) \) denotes the reflection of \( g_2 \). This implies

\[
f \ast^p [g | h] \approx f \ast^p Dh \, g \quad \text{and} \quad f \ast^p [g_1, g_2 | h] \approx f \ast^p Dh (g_1 \ast \tau g_2).
\]

This observation is clear on an intuitive level, since \( G_p(x, y) \) is the magnitude of the contribution of \( f(y) \) in the term \( (f \ast^p [g | h])(x) \). In the case of the common convolution (and Gaussian \( g \) or similar), this magnitude depends on the distance between \( x \) and \( y \). Here, it depends on the distance between \( h(x) \) and \( h(y) \), hence the convolution is weighted by the ‘slope’ of \( h \), see Figure [7].

However, if \( h \) is not injective, the new convolutions \( f \ast^p [g | h] \) and \( f \ast^p [g_1, g_2 | h] \) yield further possibilities. For example, we can let a value \( f(y) \) contribute strongly to \( f \ast^p [g | h](x) \), even though \( x \) is far away from \( y \) (without contributing strongly to most values in between) \(^1\) by choosing \( h \) such that \( h(y) \approx h(x) \), see Figure [8].

### B Proofs

**Proof of Theorem [3]** The cases \( p = 1 \) and \( p = \infty \) are straightforward:

\[
\|f \ast G\|_1 \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)| |G(x, y)| \, dy \, dx = \int_{\mathbb{R}^d} |f(y)| \int_{\mathbb{R}^d} |G(x, y)| \, dx \, dy \leq \|f\|_1 \Gamma,
\]

\[
\|f \ast G\|_{\infty} \leq \operatorname{ess \, sup}_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)| |G(x, y)| \, dy \leq \int_{\mathbb{R}^d} |f(y)| \, dy \Gamma = \|f\|_1 \Gamma.
\]

Now, let \( 1 < p < \infty \) and \( p' \) denote its conjugate exponent, i.e. \( \frac{1}{p} + \frac{1}{p'} = 1 \). Hölder’s inequality yields

\[
|f \ast G|(x) \leq \int_{\mathbb{R}^d} |f(y)|^\frac{1}{p'} |f(y)|^\frac{1}{p} |G(x, y)| \, dy \leq \left\| |f|^{\frac{1}{p'}} \right\|_{p'} \left\| |f|^{\frac{1}{p}} \right\|_p |G(x, \bullet)|_p,
\]

which implies

\[
\left\| f \ast G \right\|_p^p \leq \left\| f \right\|_1^{\frac{1}{p'}} \left\| f \right\|_p \left\| |G(\bullet, \bullet)|_p \right\|_p^p \leq \left\| f \right\|_1^{\frac{1}{p'}} \int_{\mathbb{R}^d} |f(y)| |G(x, y)|^p \, dy \, dx = \left\| f \right\|_1^{\frac{1}{p'}} \int_{\mathbb{R}^d} |f(y)| |G(x, y)|^p \, dy \, dx \leq \left\| f \right\|_1^{1+p/p'} \Gamma^p \leq \left\| f \right\|_1^p \Gamma^p.
\]

\[\square\]
Proof of Corollary 6. First note that for $p < \infty$ and $y \in \mathbb{R}^d$ the change of variables formula implies:

\[
\|g_{\mu,p}(\cdot, y)\|^p_p = \int_{\mathbb{R}^d} |\det \mu(y)| |g(\mu(y)(x - y))|^p \, dx = \int_{\mathbb{R}^d} |g(x)|^p \, dx = \|g\|^p_p.
\] (15)

For $p = \infty$, the statement $\|g_{\mu,p}(\cdot, y)\| \leq \|g\|_p$ is trivial. Theorem 5 proves the claim.

Proof of Theorem 7. The case $r = \infty$ was treated in Theorem 5 and $r < \infty$ implies $p, q < \infty$. Since

\[
\frac{r - q}{qr} + \frac{r - p}{pr} + 1 = \frac{1}{q} - \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{p} - 1 = 1,
\]

the generalized Hölder’s inequality yields

\[
|f \ast G|(x) \leq \int_{\mathbb{R}^d} |f(y)|^{1 - \frac{r}{q}} |f(y)|^{\frac{r}{q}} |G(x, y)|^{1 - \frac{p}{r}} |G(x, y)|^{\frac{p}{r}} \, dy
\]

\[
\leq \left[ |f|^{\frac{r}{r-q}} \right]_{A} \left[ |G(x, \cdot)|^{\frac{r}{r-p}} \right]_{B(x)} \left[ |f|^\frac{r}{q} |G(x, \cdot)|^{\frac{r}{p}} \right]_{C(x)}
\]

and

\[
A^r = \left( \int |f(y)|^q \, dy \right)^{\frac{r-q}{q}} = \|f\|_q^{r-q},
\]

\[
B(x)^r = \left( \int |G(x, y)|^p \, dy \right)^{\frac{r-p}{p}} = \|G(x, \cdot)\|_p^{r-p} \leq \Gamma^{r-p},
\]

\[
C(x)^r = \int |f(y)|^q |G(x, y)|^p \, dy.
\]

This implies

\[
\|f \ast G\|_r \leq \int A^r B(x)^r C(x)^r \, dx \leq \|f\|_q^{r-q} \Gamma^{r-p} \int |f(y)|^q \, \int_{\mathbb{R}^d} |G(x, y)|^p \, dx \, dy \leq \|f\|_q^r \Gamma^r.
\]

Proof of Proposition 10. For all $j = 1, \ldots, d$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| < m$, we have by induction:

\[
\partial_{x_j} \partial^\alpha (f *_{\mu}^p g)(x) = \partial_{x_j} \left( \int_{\mathbb{R}^d} f(y) |\det(\mu(y))|^{1/p} \alpha(\mu(y)) D^{[\alpha]} g(\mu(y)(x - y)) \, dy \right)
\]

\[
= \int_{\mathbb{R}^d} f(y) |\det(\mu(y))|^{1/p} \left[ \alpha(\mu(y)), \mu(y)_{\cdot,j} \right] D^{[\alpha]+1} g(\mu(y)(x - y)) \, dy
\]

\[
= \int_{\mathbb{R}^d} f(y) |\det(\mu(y))|^{1/p} (\alpha + e_j)(\mu(y)) D^{[\alpha]+1} g(\mu(y)(x - y)) \, dy
\]

\[
= \left[ (f \cdot (\alpha + e_j)(\mu)) *_{\mu}^p D^{[\alpha]+1} g \right](x).
\]
Proof of Proposition 11] Without loss of generality, we may assume that \( \det \mu_t(x) > 0 \) for all \( x \in \mathbb{R}^d \) and \( t \in \mathbb{R} \). We will use the abbreviations \( \delta_t := (\det \mu_t)^{1/p} \), \( f_\mu(x, y) := f(\mu(y)(x - y)) \) for functions \( \mu : \mathbb{R}^d \to \text{GL}(d, \mathbb{R}), f : \mathbb{R}^d \to \mathbb{R} \) (note that latter notation differs by a prefactor from the one used in Definition 3) and \( A_{k,t}^\top \), will denote the transpose of the \( k \)-th row of the matrix \( A \). The observations

\[
(tr[A_{k,t}^{-1}(y)\partial_k\mu_t(y)])_{k=1}^d j_t(y) = \sum_{k=1}^d tr[A_{k,t}^{-1}(y)\partial_k\mu_t(y)] j_{k,t}(y) = tr[\mu_t^{-1}(y) \sum_{k=1}^d \partial_k\mu_t(y) j_{k,t}(y)] \\
= tr[\mu_t^{-1}(y) \left( \sum_{k=1}^d j_{k,t}(y) \left( \partial_k(\mu_t)^\top \right)(y) \right)] = tr[\mu_t^{-1}(y) N_t(y)],
\]

\[
D_y[\mu_t(y)(x - y)] = \begin{pmatrix} D_y[\mu_t(y_1)(x - y)] \\ \vdots \\ D_y[\mu_t(y_d)(x - y)] \end{pmatrix} = \begin{pmatrix} (x - y)^\top (D_y(\mu_t)^\top)(y) \\ \vdots \\ (x - y)^\top (D_y(\mu_t)^\top)(y) \end{pmatrix} - \mu_t(y),
\]

\[
M_t(x, y) j_t(y) = \begin{pmatrix} j_t(y)^\top (D_y(\mu_t)^\top)(y) (x - y) \\ \vdots \\ j_t(y)^\top (D_y(\mu_t)^\top)(y) (x - y) \end{pmatrix} = N_t(y)(x - y)
\]

lead to

\[
\left( \nabla_y \left[ \delta_t(y) g_\mu_t(x, y) \right] \right)^\top j_t(y) \\
= \delta_t(y) \left( g_\mu_t(x, y) \left( tr[A_{k,t}^{-1}(y)\partial_k\mu_t(y)] \right)_{k=1}^d + (\nabla g_\mu_t(x, y)^\top [M_t(x, y) - \mu_t(y)]) \right) j_t(y) \\
= \delta_t(y) \left( g_\mu_t(x, y) tr[A_{k,t}^{-1}(y) N_t(y)] + (\nabla g_\mu_t(x, y)^\top [N_t(y)(x - y) - \mu_t(y) j_t(y)]) \right) \\
= \text{div}_x \left( \delta_t(y) g_\mu_t(x, y) \left[ \mu_t^{-1}(y) N_t(y)(x - y) - j_t(y) \right] \right)
\]

and

\[
\hat{\partial}_t \left[ \delta_t(y) g_\mu_t(x, y) \right] = \delta_t(y) \left( tr[A_{k,t}^{-1}(y)\partial_k\mu_t(y)] \right) g_\mu_t(x, y) + (\nabla g_\mu_t(x, y)^\top \partial_t(\mu_t)^\top \partial_t(\mu_t)(x - y)) \\
= \text{div}_x \left( \delta_t(y) g_\mu_t(x, y) \mu_t^{-1}(y) \partial_t(\mu_t)(x - y) \right).
\]

Combining these two, we get:

\[
\hat{\partial}_t \rho_{g_t}(x) = \int_{\mathbb{R}^d} \hat{\partial}_t(\rho_t(y)) \rho_t(y) dy + \delta_t \left[ \delta_t(y) g_\mu_t(x, y) \right] \rho_t(y) dy + \text{div}_x \left( \delta_t(y) g_\mu_t(x, y) \mu_t^{-1}(y) \partial_t(\mu_t)(x - y) \right) dy \\
= -\text{div} j_t(x).
\]
The existence of all integrals follows directly from the assumptions and Corollary 6, while \((j_{g,t})_{t \in [0,\infty)} \in C^1([0,\infty) \times \mathbb{R}^d, \mathbb{R}^d)\) follows from Proposition 10.

**Proof of Theorem 23.** Again, let \(g(x) = \gamma(\|x\|^2)\) for some function \(\gamma: \mathbb{R} \to \mathbb{R}\). We have:

\[
(f(\bullet - a) * \mu_f(\bullet - a)) g(x) = \int f(y - a) |\det (\mu_f(y - a))| g(\mu_f(y - a)(x - y)) \, dy \\
= \int f(y) |\det (\mu_f(y))| g(\mu_f(y)(x - a - y)) \, dy = (f * \mu_f g)(x - a),
\]

\[
(\alpha f * \mu_{\alpha f} g)(x) = \int \alpha f(y) |\det (\mu_f(y))| g(\mu_f(y)(x - y)) \, dy = \alpha (f * \mu_f g)(x),
\]

\[
(f(A \cdot \bullet) * \mu_{f(A \cdot \bullet)} g)(x) = \int f(Ay) |\det (\mu_{f(A \cdot \bullet)}(y))| g(\mu_{f(A \cdot \bullet)}(y)(x - y)) \, dy \\
= \int f(Ay) |\det (\mu_f(Ay) A)| \gamma((x - y)^T A^T \mu_f(Ay)^T \mu_f(Ay) A(x - y)) \, dy \\
= \int f(y) |\det (\mu_f(y))| \gamma((Ax - y)^T \mu_f(y)^T \mu_f(y)(Ax - y)) \, dy \\
= \int f(y) |\det (\mu_f(y))| g(\mu_f(y)(Ax - y)) \, dy = (f * \mu_f g)(Ax).
\]

\[
(f^{(t)} * \mu_{f^{(t)}} g)(x + a_k^{(t)}) = \int f^{(t)}(y) |\det \mu_{f^{(t)}}(y)| g(\mu_{f^{(t)}}(y)(x + a_k^{(t)} - y)) \, dy \\
= \sum_{j=1}^{K} \int f_j(y + a_k^{(t)} - a_j^{(t)}) |\det \mu_{f^{(t)}}(y + a_k^{(t)})| g(\mu_{f^{(t)}}(y + a_k^{(t)})(x - y)) \, dy \\
\xrightarrow{t \to \infty} \int f_k(y) |\det \mu_{f_k}(y)| g(\mu_{f_k}(y)(x - y)) \, dy = (f_k * \mu_{f_k} g)(x).
\]

**Proof of Proposition 20.** The proof is analogous to the one of Proposition 25.

**Proof of Proposition 27.** The proof is analogous to the one of Proposition 25.

**Proof of Proposition 22.** The proof is analogous to the one of Proposition 25. Alternatively, it follows from Proposition 21.

**Proof of Theorem 24.** Since \((\phi(\bullet - a) \ast \psi)(x) = (\phi \ast \psi)(x - a)\), the right hand side of (11) is translation-invariant for any choice of \(\mu_f\), which proves (A1). If \(\mu_f\) solves (11), then it also solves (11) for \(\tilde{f} = \alpha f\) \((\alpha > 0)\) in place of \(f\), proving (A2). For (A3) let \(\mu_f\) be the solution of (11), \(f(x) = f(Ax)\) for some \(A \in \text{GL}(d, \mathbb{R})\) and \(\mu_f(x) := \sqrt{\det \mu_f(Ax)^T \mu_f(Ax)} A\). Then we have for \(x, y \in \mathbb{R}^d\)

\[
(\nabla \tilde{f} \nabla \tilde{f}^T - \tilde{f} D^2 \tilde{f})(x) = A^T (\nabla f \nabla f^T - f D^2 f)(Ax) A, \quad G_{\mu_f^{-2}}(x) = |\det A| G_{\mu_{\tilde{f}}^{-2}(Ax)}(Ay),
\]

23
and, since for any functions \( \phi, \psi \), for which the convolution \( \phi \ast \psi \) exists, we have
\[
(\phi(A \cdot \cdot) \ast (\psi(A \cdot \cdot)) (x) = |\det A|^{-1} (\phi \ast \psi)(Ax),
\]
\( \mu_f \) solves \( (11) \) for \( \tilde{f} \) in place of \( f \). To prove \( (A4) \), let \( R[f, \mu](x) \) denote the right-hand side of \( (11) \) with an arbitrary \( \mu \) in place of \( \mu_f \). Adopting the notation of the Adaptation Axioms \( (14) \) we have
\[
R[f(t) \ast \delta(t) + a_k^{(t)}], \mu](x) \xrightarrow{t \to \infty} R[f_k, \mu](x)
\]
since each \( f_k \in W^{2,2}(\mathbb{R}^{d}, \mathbb{R}) \). Hence, for \( \tilde{f}(t) = f(t) \ast \delta(t) + a_k^{(t)} \), the solution of the the implicit formula \( \mu^2 = R[\tilde{f}(t), \mu] \) is asymptotically given by \( \mu = \mu_{\tilde{f}(t)} = \mu_{f_k} \). Therefore,
\[
\mu_{f(t) \ast \delta(t) + a_k^{(t)}} (x) \xrightarrow{t \to \infty} \mu_{f_k}(x).
\]

**Proof of Proposition \( (25) \)** As the Wigner transform is real-valued, we get for real-valued functions \( f \):
\[
Wf(x, -\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} f\left(x + \frac{y}{2}\right) f\left(x - \frac{y}{2}\right) e^{-iy\xi} dy
\]
\[
= (2\pi)^{-d} \int_{\mathbb{R}^d} f\left(x + \frac{y}{2}\right) f\left(x - \frac{y}{2}\right) e^{iy\xi} dy = Wf(x, \xi) = Wf(x, \xi),
\]
and therefore the expectation value of \( \mathbb{E}_{\rho_x} \) vanishes. For the covariance matrix, we use the transformation
\[
z_1 = y_1 - y_2, \quad z_2 = y_1 + y_2
\]
and the function
\[
F(z_1, z_2) = f\left(x + \frac{z_2 + z_1}{4}\right) f\left(x - \frac{z_2 + z_1}{4}\right) f\left(x + \frac{z_2 - z_1}{4}\right) f\left(x - \frac{z_2 - z_1}{4}\right)
\]
to compute:
\[
\int_{\mathbb{R}^d} |Wf|^2(x, \xi) d\xi = (2\pi)^{-2d} \int_{\mathbb{R}^d} f\left(x + \frac{y}{2}\right) f\left(x - \frac{y}{2}\right) e^{i(y_1 - y_2)\xi} dy_1 dy_2\xi
\]
\[
= \frac{(2\pi)^{-2d}}{2^d} \int_{\mathbb{R}^d} F(z_1, z_2) e^{i\xi_1 \xi_2} dz_1 dz_2 = \frac{(2\pi)^{-d}}{2^d} \int_{\mathbb{R}^d} F(0, z_2) dz_2
\]
\[
= \frac{(2\pi)^{-d}}{2^d} \int_{\mathbb{R}^d} f^2(x + \frac{z_2}{2}) f^2(x - \frac{z_2}{2}) dz_2 = 2^d (2\pi)^{-d} \int_{\mathbb{R}^d} f^2(z) f^2(2x - z)^2 dz
\]
\[
= 2^d (2\pi)^{-d} (f^2 \ast (f^2)(2x)),
\]
\[
\int_{\mathbb{R}^d} |Wf|^2(x, \xi) \xi \xi^T d\xi = (2\pi)^{-2d} \int_{\mathbb{R}^d} f\left(x + \frac{y}{2}\right) f\left(x - \frac{y}{2}\right) e^{i(y_1 - y_2)\xi} \xi \xi^T dy_1 dy_2\xi
\]
\[
= \frac{(2\pi)^{-2d}}{2^d} \int_{\mathbb{R}^d} F(z_1, z_2) e^{i\xi_1 \xi_2} dz_1 dz_2 = -\frac{(2\pi)^{-d}}{2^d} \int_{\mathbb{R}^d} D_{z_1}^2 \xi F(0, z_2) dz_2
\]
\[
= \frac{(2\pi)^{-d}}{2^{d+3}} \int_{\mathbb{R}^d} f^2(x + \frac{z_2}{2}) \left[\nabla_f \nabla f^T - f D^2 f\right] (x - \frac{z_2}{2}) + f^2(x - \frac{z_2}{2}) \left[\nabla_f \nabla f^T - f D^2 f\right] (x + \frac{z_2}{2}) dz_2
\]
\[
= \frac{(2\pi)^{-d}}{2^{d+3}} \int_{\mathbb{R}^d} f^2(z) \left[\nabla_f \nabla f^T - f D^2 f\right] (2x - z) + f^2(z) \left[\nabla_f \nabla f^T - f D^2 f\right] (z) dz_2
\]
\[
= 2^{d-2} (2\pi)^{-d} (f^2 \ast \left[\nabla_f \nabla f^T - f D^2 f\right])(2x).
\]
Taking the quotient proves the formula for the covariance matrix.
\[
\square
\]
Proof of Theorem 26. Adaptation Axiom 14 (A1) follows from

\[
(f(\star - a) \ast g(\star - b))(x) = \int f(y - a)g(x - y - b)\,dy = \int f(y)g(x - (a + b) - y)\,dy
\]

\[
= (f \ast g)(x - (a + b)).
\]

(A2) is straightforward and (A3) follows from

\[
(f(A \star) \ast g(A \star))(x) = \int f(Ay)g(A(x - y))\,dy = |\det A|^{-1} \int f(y)g(Ax - y)\,dy = \frac{(f \ast g)(Ax)}{|\det A|}
\]

in the following way:

\[
\mu_{f(A \star)}^{(d)}(x) \mu_{f(A \star)}^{(d)}(x) = \frac{f^2(A \star) \ast \left[A^\top \nabla f(A \star) \nabla f^\top(A \star)A - A^\top f D^2 f(A \star)A\right]}{f^2(A \star) \ast f^2(A \star)} (2x)
\]

\[
= \frac{A^\top \left[f^2 \ast \left(\nabla f \nabla f^\top - f D^2 f\right)\right]A}{f^2 \ast f^2} (2Ax) = A^\top \mu_{f}^{(d)}(Ax) \mu_{f}^{(d)}(Ax) A.
\]

Proof of Corollary 27. A simple computation shows:

\[
Df(x) = -f(x)(x - a)^\top \Sigma^{-1}
\]

\[
D^2 f(x) = f(x) \left[\Sigma^{-1}(x - a)(x - a)^\top \Sigma^{-1} - \Sigma^{-1}\right]
\]

\[
(\nabla f \nabla f^\top - f D^2 f)(x) = f^2(x) \Sigma^{-1}.
\]

The claim follows from the definitions (11) of \(\mu_f^{(d)}\) and (13) of \(\mu_f^{(c)}\).

Proof of Proposition 33. For the h-adaptive convolution of type two the property \(\|G(\bullet,y)\|_p = \|g\|_p\) is straightforward for all \(y \in \mathbb{R}^d, 1 \leq p \leq \infty\) and Theorem 5 proves the claim. For the h-adaptive convolution of type three, we denote

\[
\gamma(x,z) := \frac{g_2(z - h(x))}{\|g_2(z - h(\bullet))\|_p}
\]

and observe for \(y \in \mathbb{R}^d\) and \(p = 1, \infty\):

\[
\left\|\hat{G}(\bullet,y)\right\|_1 \leq \|g_2\|_1 \int \int |g_1(z - h(y)) \gamma(x,z)|\,dz\,dx = \|g_1\|_1 \int \int |g_1(z - h(y))| \gamma(x,z)\,dx\,dz = \|g_1\|_1 \|g_2\|_1,
\]

\[
\left\|\hat{G}(\bullet,y)\right\|_\infty = \|g_2\|_\infty \sup_{x \in \mathbb{R}^d} \int |g_1(z - h(y)) \gamma(x,z)|\,dz \leq \|g_2\|_\infty \int |g_1(z - h(y))|\,dz = \|g_1\|_1 \|g_2\|_\infty.
\]

For \(1 < p < \infty\), let \(p'\) denote the conjugate of \(p\) (i.e. \(1/p + 1/p' = 1\)). Hölder’s inequality yields

\[
|\hat{G}(x,y)| \leq \|g_2\|_p \int |g_1(z - h(y))|^{1/p'} |g_1(z - h(y))|^{1/p} \gamma(x,z)\,dz
\]

\[
\leq \|g_2\|_p \|g_1(\bullet - h(y))^{1/p'}\|_{p'} \|g_1(\bullet - h(y))^{1/p} \gamma(x,\bullet)\|_p,
\]

25
which implies for each $y \in \mathbb{R}^d$,
\[
\|\tilde{G}(\bullet, y)\|_p^p \leq \|g_2\|_p^p \|g_1\|^{1/p'} \left\| \int g_1(\bullet - h(y))^{1/p} \gamma(x, \bullet) \right\|_p^p \int |g_1(z - h(y)) \gamma(x, z)|^p \, dz \, dx \\
= \|g_2\|_p^p \|g_1\|^{1/p'} \left\| \int |\gamma(x, z)|^p \, dz \right\|_1 = \|g_1\|_1 \|g_2\|_p^p.
\]

Therefore $\|\tilde{G}(\bullet, y)\|_p \leq \|g_1\|_1 \|g_2\|_p$ (for all $y \in \mathbb{R}^d$ and $1 \leq p \leq \infty$) also holds for type three and again Theorem 5 proves the claim.

\begin{proof}
Since in this case $g_1 = g_2 = g$, the symmetry of $\tilde{G}$ follows from
\[
\tilde{G}(x, y) = \|g\|_p \int g(z - h(y)) \frac{g(z - h(x))}{\|g(z - h(\bullet))\|_p} \, dz = \|g\|_p \int \frac{g(z - h(y))}{\|g(z - h(\bullet))\|_p} g(z - h(x)) \, dz = \tilde{G}(y, x).
\]

Following the proof of Proposition 33, we conclude that for each $1 \leq p \leq \infty$ both $\|\tilde{G}(\bullet, y)\|_p \leq \|g\|_1 \|g\|_p$ for each $y \in \mathbb{R}^d$ and $\|\tilde{G}(x, \bullet)\|_p \leq \|g\|_1 \|g\|_p$ for each $x \in \mathbb{R}^d$. Theorem 7 proves the claim.
\end{proof}

Acknowledgements.
I thank Caroline Lasser for many insightful and motivating discussions.

References

[1] V. I. Bogachev. Measure theory. Vol. I and II. Berlin: Springer, 2007.
[2] G. B. Folland. Harmonic analysis in phase space. Number 122. Princeton University Press, 1989.
[3] K. Gröchenig. Foundations of time-frequency analysis. Springer Science & Business Media, 2001.
[4] M. Hillery, R. O’Connell, M. Scully, and E. P. Wigner. Distribution functions in physics: fundamentals. Springer, 1997.
[5] R. A. Horn and C. R. Johnson. Matrix analysis. Cambridge University Press, 2012.
[6] I. Klebanov. Axiomatic approach to variable kernel density estimation. ArXiv e-prints, 2018.
[7] W. Young. On the multiplication of successions of fourier constants. Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, 87(596):331–339, 1912.