On the Identifying Content of Instrument Monotonicity *

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Abstract

This paper studies the identifying content of the instrument monotonicity assumption of Imbens and Angrist (1994) on the distribution of potential outcomes in a model with a binary outcome, a binary treatment and an exogenous binary instrument. In the context of this setup, conclusions from previous results can be generally summarized as follows: (i) imposing instrument monotonicity can misspecify the model; and (ii) when the model is not misspecified, instrument monotonicity does not have any identifying content on the marginal distributions of the potential outcomes. In this paper, I demonstrate that instrument monotonicity can however have identifying content on features of the joint distribution of the potential outcomes when the model is not misspecified. I illustrate how this identifying content can lead to additional informative conclusions with respect to the proportion who benefit, a specific feature of the joint distribution of the potential outcomes.

KEYWORDS: Instrument monotonicity, separable treatment selection model, partial identification, distributional treatment effects.

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1 Introduction

This paper studies the identifying content of the instrument monotonicity assumption of Imbens and Angrist (1994) in a model with a binary outcome, a binary treatment and an exogenous binary instrument. More specifically, let $Y$ denote the observed binary outcome and $D$ denote an indicator for whether treatment was received or not. Further, let $Y_0$ denote the binary potential outcome if treatment was not received and $Y_1$ denote the binary potential outcome if treatment was received. The observed outcome and potential outcomes are related by the equation

$$Y = Y_1 \cdot D + Y_0 \cdot (1 - D).$$

In addition, following the framework of Imbens and Angrist (1994), let $Z$ denote the observed binary instrument, $D_0$ denote the potential value of treatment receipt if $Z = 0$ and $D_1$ denote the potential value of treatment receipt if $Z = 1$, i.e. the observed treatment receipt and potential treatment receipts are related by the equation

$$D = D_1 \cdot Z + D_0 \cdot (1 - Z).$$

Here the instrument is exogenous and hence assumed to satisfy

$$(Y_0, Y_1, D_0, D_1) \perp Z,$$

i.e. it is jointly statistically independent of all the potential variables. Under the setup introduced above, I study the identifying content on the distribution of potential outcomes, i.e.

$$\text{Prob}[(Y_0, Y_1) \in A]$$

for all $A \subseteq \{0, 1\}^2$, of the instrument monotonicity assumption of Imbens and Angrist (1994), i.e.

$$D_1 \geq D_0,$$

which states that the instrument monotonically affects the receipt of treatment for every unit in the population.

Since instrument monotonicity additionally restricts the underlying model, it is potentially possible that it provides identifying content on the distribution of potential outcomes. As further elaborated in Remark 2.3 below, Balke and Pearl (1997), Heckman and Vytlacil (2001) and Kitagawa (2009) have previously explored this possibility on various features of the distribution of potential outcomes. In the context of the setup described above, their conclusions can be generally summarized as follows. First, they find that imposing instrument monotonicity can misspecify the model, i.e. the distribution of the observed data may not be compatible with the imposed assumptions. Second, when the model is not misspecified, they find that the marginal distributions of the potential outcomes, i.e.

$$(\text{Prob}[Y_0 \in A_0], \text{Prob}[Y_0 \in A_1])$$
for all $A_0, A_1 \subseteq \{0, 1\}$, are partially identified, but that instrument monotonicity has no additional identifying content. In particular, they find that the corresponding identified sets, i.e. the set of all possible parameter values that are compatible with the observed data and the imposed assumptions, with and without instrument monotonicity are identical. This moreover implies that instrument monotonicity has no identifying content on the average treatment effect or more generally any function of the marginal distributions of the potential outcomes.

However, the second conclusion does not characterize all the possible identifying content that instrument monotonicity may have on the distribution of potential outcomes, and in particular does not capture identifying content that may be present on features of the joint distribution of the potential outcomes. In this paper, I demonstrate that instrument monotonicity can in fact have identifying content on features of the joint distribution of the potential outcomes when the model is not misspecified. In Section 2.1 below, I begin by characterizing the identified sets for the distribution of potential outcomes with and without instrument monotonicity. The analysis in particular reveals that instrument monotonicity restricts the values that the distribution of potential outcomes can take only through restrictions on features of the joint distribution. I show that these additional restrictions have identifying content in the sense that the identified set with instrument monotonicity can be a strict subset of that without instrument monotonicity for certain values of the distribution of the observed data. In Section 2.1 below, I then illustrate that this identifying content can lead to additional informative conclusions with respect to a specific feature of the joint distribution of the potential outcomes,

$$\text{Prob}[Y_1 > Y_0],$$

which evaluates the proportion of units who benefit in the sense that they have a high outcome value if treatment was received versus a low outcome value if treatment was not received - see Heckman et al. (1997) and Manski (1997) for an early exposition on the policy relevance of such parameters based on features of the joint distribution of the potential outcomes. I show that instrument monotonicity can have identifying content to affect the lower bound of the identified set to lead to further conclusions on whether the proportion who benefit is greater than a specific value.

2 Identification Analysis

I begin by introducing the formal setup and notation for the identification analysis pursued in this paper. Denote by

$$X = (Y, D, Z) \sim P$$
the random variable that summarizes the observed random variables, where \( P \) denotes the distribution of this random variable defined on a sample space \( \mathcal{X} = \{0,1\}^3 \). Analogously, denote by

\[
W = (Y_0, Y_1, D_0, D_1, Z) \sim Q \in \mathcal{Q}_W
\]

the random variable that summarizes the model random variables, where \( Q \) denotes the distribution of this random variable defined on a sample space \( \mathcal{W} = \{0,1\}^5 \) and \( \mathcal{Q}_W \) denotes the set of all probability distributions defined on \( \mathcal{W} \). The observed random variables and the model random variables are related by the equations in (1) and (2). In turn, this relationship implies that the distribution of the data and the distribution of the underlying model are related by

\[
\sum_{w \in \mathcal{W}_x} Q\{W = w\} = P\{X = x\}
\]

for all \( x = (y, d, z) \in \mathcal{X} \), where \( \mathcal{W}_x \) is the set of all \( w = (y_0, y_1, d_0, d_1, z) \in \mathcal{W} \) such that \( d = d_1 \cdot z + d_0 \cdot (1 - z) \) and \( y = y_1 \cdot d + y_0 \cdot (1 - d) \).

The identification analysis aims to study what we can learn on a pre-specified parameter of interest based on \( Q \) using the restrictions imposed on \( Q \) by a given value of \( P \) through (5) and by additional imposed assumptions. In particular, I focus on a combination of two different assumptions, namely the instrument exogeneity assumption in (3) and the instrument monotonicity assumption in (4). Note that the instrument exogeneity assumption can be restated as a restriction in terms of \( Q \) as:

**Assumption E.** For all \( w = (y_0, y_1, d_0, d_1, z) \in \mathcal{W} \):

\[
Q\{W = w\} = Q\{Y_0 = y_0, Y_1 = y_1, D_0 = d_0, D_1 = d_1\} \cdot Q\{Z = z\}
\]

Similarly, note that the instrument monotonicity assumption can be restated as a restriction in terms of \( Q \) as:

**Assumption M.** \( Q\{D_0 = 1, D_1 = 0\} = 0 \).

Using the above introduced setup, the identification analysis can be stated in terms of studying the identified sets under different assumptions, i.e. the set of feasible parameter values such that the distribution of the model satisfies the imposed restrictions. To be specific, for a pre-specified parameter of interest \( \theta : \mathcal{Q}_W \to \Theta \), I study the two following identified sets

\[
\theta(\mathcal{Q}_E) = \{\theta(Q) \in \Theta : Q \in \mathcal{Q}_E\} \quad \text{and} \quad \theta(\mathcal{Q}_{EM}) = \{\theta(Q) \in \Theta : Q \in \mathcal{Q}_{EM}\},
\]

where

\[
\mathcal{Q}_E = \{Q \in \mathcal{Q}_W : Q \text{ satisfies (5) and Assumption E}\}
\]
denotes the set of all model distributions that satisfy the restrictions imposed by the observed data and the instrument exogeneity assumption, and

\[ Q_{EM} = \{ Q \in Q_W : Q \text{ satisfies (5), Assumption E and Assumption M} \} \]

denotes the set of all model distributions that satisfy the restrictions imposed by the observed data, the instrument exogeneity assumption and the instrument monotonicity assumption. Since the only difference between \( Q_E \) and \( Q_{EM} \) is the imposition of the instrument monotonicity assumption, comparing the two identified sets in (6) will in turn reveal the identifying content of the instrument monotonicity assumption in terms of the pre-specified parameter. In Section 2.1 below, I first perform this analysis for a pre-specified parameter that corresponds to the distribution of potential outcomes. In Section 2.2 below, I then use this analysis to focus on a pre-specified parameter that corresponds to a specific feature of the joint distribution of the potential outcomes.

Before proceeding, note that if \( Q_E \) or \( Q_{EM} \) is empty then the corresponding model is said to be misspecified, i.e. the distribution of the observed data is incompatible with the imposed assumptions on the model. As previously noted in Balke and Pearl (1997) and Kitagawa (2009), the instrument monotonicity assumption has identifying content in the sense that \( Q_{EM} \) can be empty for some values of the distribution of the observed data in contrast to \( Q_E \), which implies that if \( Q_{EM} \) is empty then

\[ \theta(Q_{EM}) = \emptyset \subset \theta(Q_E) , \]

i.e. the identified set under \( Q_{EM} \) is a strict subset of the identified set under \( Q_E \). In this paper, I focus on the identifying content of the instrument monotonicity assumption in the case where \( Q_{EM} \) is non-empty. To this end, I suppose that \( P \) satisfies

\[ P\{Z = 1\} \in (0, 1) , \]

and also

\[ P\{Y = y, D = 1|Z = 1\} - P\{Y = y, D = 1|Z = 0\} \geq 0 , \]

\[ P\{Y = y, D = 0|Z = 0\} - P\{Y = y, D = 0|Z = 1\} \geq 0 \]

for \( y \in \{0, 1\} \). From Kitagawa (2015, Proposition 1), it follows that these conditions on \( P \) are necessary and sufficient to ensure that \( Q_{EM} \) is non-empty.

**Remark 2.1.** Following results in Vytlacil (2002), note that the instrument monotonicity assumption in (4) is equivalent to assuming a separable treatment selection equation, i.e.

\[ D_z = 1\{z \cdot \beta + \epsilon \geq 0\} , \]

where \( \beta \) is a non-random and non-negative real number. In turn, the identification analysis pursued in this paper equivalently studies the identifying content of imposing a separable treatment selection equation.
2.1 Identified Sets for the Joint Distribution

In this section, I study the parameter of interest that corresponds to the distribution or more specifically the joint distribution (JD) of potential outcomes. To be specific, let

\[(Y_0, Y_1) \sim Q^\dagger \in Q_{W^\dagger},\]

where \(Q^\dagger\) denotes the distribution of potential outcomes defined on the sample space \(W^\dagger = \{0, 1\}^2\) and \(Q_{W^\dagger}\) denotes the set of all distributions defined on \(W^\dagger\). Denoting by \(M_{JD} : W \rightarrow W^\dagger\) the mapping that corresponds to

\[M_{JD}(w) = (y_0, y_1),\]

for all \(w = (y_0, y_1, d_0, d_1, z) \in W\), the parameter of interest \(\theta_{JD} : Q_W \rightarrow Q_{W^\dagger}\) can be defined by

\[\theta_{JD}(Q) = QM_{JD}^{-1} = Q^\dagger.\] (9)

In order to be succinct in the statement of the proposition below, I introduce the following shorthand notation:

\[Q^\dagger \{Y_0 = i, Y_1 = j\} \equiv Q^\dagger_{i,j}\] (10)

for all \(i, j \in \{0, 1\}\), and

\[P\{Y = i, D = d|Z = z\} \equiv P_{i,d|z}\] (11)

for all \(i, d, z \in \{0, 1\}\). The proof of this proposition is presented in Appendix A.

**Proposition 2.1.** Suppose that \(P\) satisfies the conditions in (7) and (8a)-(8b) and that the parameter of interest \(\theta_{JD} : Q_W \rightarrow Q_{W^\dagger}\) is defined as in (9). Then, \(\theta_{JD}(Q_E)\) in (6) is equal to the set of all \(Q^\dagger \in Q_{W^\dagger}\) that satisfy the following nontrivial inequalities for all \(i, j \in \{0, 1\}\):

\[Q^\dagger_{i,j} \leq \min\{P_{i,0|0} + P_{j,1|0}, P_{i,0|1} + P_{j,1|1}\},\] (12a)

\[Q^\dagger_{i,0} + Q^\dagger_{i,1} \leq P_{i,0|0} + P_{0,1|0} + P_{1,1|0},\] (12b)

\[Q^\dagger_{0,i} + Q^\dagger_{1,i} \leq P_{i,1|1} + P_{0,0|1} + P_{1,0|1}.\] (12c)

Similarly, \(\theta_{JD}(Q_{EM})\) in (6) is equal to the set of all \(Q^\dagger \in Q_{W^\dagger}\) that satisfy the nontrivial inequalities in (12a)-(12c) in addition to the following for all \(i, j \in \{0, 1\}\):

\[Q^\dagger_{i,j} + Q^\dagger_{i,1-j} + Q^\dagger_{1-i,j} \leq P_{i,0|0} + P_{1-i,0|1} + P_{1-j,1|0} + P_{j,1|1}.\] (13)

Proposition 2.1 reveals that instrument monotonicity introduces additional restrictions on the distribution of potential outcomes in (13) that may not be implied by those in (12a)-(12c). To see this more clearly, note that the inequalities in (12a)-(12c) imply that all \(Q^\dagger \in Q_{W^\dagger}\) must satisfy

\[Q^\dagger_{i,j} + Q^\dagger_{i,1-j} + Q^\dagger_{1-i,j} \leq P_{i,0|0} + P_{1-i,0|1} + P_{1-j,1|0} + P_{j,1|1} + \min\{P_{i,0|1}, P_{j,1|0}\}\]
for all \(i, j \in \{0, 1\}\). These inequalities however do not imply those in (13) provided that
\[
\min \{P_{i,0|1}, P_{j,1|0}\} > 0 \quad (14)
\]
for some \(i, j \in \{0, 1\}\). It then directly follows that instrument monotonicity will provide identifying content on the distribution of potential outcomes in the sense that
\[
\theta_{JD}(Q_{EM}) \subset \theta_{JD}(Q_E)
\]
whenever two specific conditions are satisfied for some \(i, j \in \{0, 1\}\). First, the inequality in (13) is not implied by those in (12a)-(12c), i.e. \(P\) satisfies (14), and, second, it is also not implied by the fact that \(Q\) is a probability distribution, i.e. \(P\) also satisfies
\[
P_{i,0|0} + P_{1-i,0|1} + P_{1-j,1|0} + P_{j,1|1} < 1.
\]

**Remark 2.2.** In a model without potential treatment receipts, Beresteanu et al. (2012) and Mourifie et al. (2017) characterize the identified set for the distribution of potential outcomes under the assumption that the instrument is independent of the potential outcomes, i.e.
\[
(Y_0, Y_1) \perp Z.
\]
When \(P\) satisfies the conditions in (7) and (8a)-(8b), their resulting identified set is identical to \(\theta_{JD}(Q_E)\) stated in Proposition 2.1.

### 2.2 Identified Sets for the Proportion who Benefit

Proposition 2.1 in the preceding section revealed that instrument monotonicity can in fact have identifying content on the distribution of potential outcomes. In this section, I illustrate how this identifying content can lead to informative conclusions with respect to a scalar parameter of interest that corresponds to a specific feature of this distribution.

As further elaborated in Remark 2.3 below, previous studies have shown instrument monotonicity does not have any identifying content on the marginal distributions of the potential outcomes and hence any function based on it. In turn, I study a parameter of interest \(\theta_{PB} : Q_W \to \mathbb{R}\) that corresponds to a specific feature of the joint distribution of the potential outcomes defined by
\[
\theta_{PB}(Q) = Q\{Y_0 = 0, Y_1 = 1\} \equiv Q\{Y_1 > Y_0\}, \quad (15)
\]
the proportion of units who benefit (PB) in the sense that they have a high outcome value if treatment was received versus a low outcome value if treatment was not received. Using the shorthand notation introduced in (10) and (11), note that this parameter of interest can be re-written as
\[
\theta_{PB}(Q^\dagger) = Q^\dagger_{0,1} = 1 - \left[ Q^\dagger_{1,0} + Q^\dagger_{0,0} + Q^\dagger_{1,1} \right].
\]
The following result characterizing the identifying content of instrument monotonicity with respect to this parameter of interest then directly follows from the inequalities stated in Proposition 2.1.

**Corollary 2.1.** Suppose that $P$ satisfies the conditions in (7) and (8a)-(8b) and that the parameter of interest $\theta_{PB} : Q_{W} \rightarrow R$ is defined as in (15). Then, $\theta_{PB}(Q_E)$ in (6) is equal to the closed interval $[L_{E,PB}, U_{E,PB}]$, where

$$L_{E,PB} = \max \{0, 1 - P_{1,0|0} - P_{1,1|0} - P_{0,0|1} - P_{0,1|1} - \min \{P_{1,0|1}, P_{0,1|0}\}\} ,$$

$$U_{E,PB} = \min \{P_{0,0|0} + P_{1,1|0}, P_{0,0|1} + P_{1,1|1}\} .$$

Similarly, $\theta_{PB}(Q_{EM})$ in (6) is equal to the closed interval $[L_{EM,PB}, U_{EM,PB}]$, where

$$L_{EM,PB} = \max \{0, 1 - P_{1,0|0} - P_{1,1|0} - P_{0,0|1} - P_{0,1|1}\} ,$$

$$U_{EM,PB} = \min \{P_{0,0|0} + P_{1,1|0}, P_{0,0|1} + P_{1,1|1}\} .$$

Corollary 2.1 illustrates that instrument monotonicity does not affect the upper bound but can affect the lower bound of the identified set of the proportion who benefit. In particular, by inspecting the two lower bounds, the instrument monotonicity assumption will strictly narrow the bounds when $P$ is such that

$$P_{1,0|0} + P_{1,1|0} + P_{0,0|1} + P_{0,1|1} < 1 , \quad (16)$$

and

$$\min \{P_{1,0|1}, P_{0,1|0}\} > 0 . \quad (17)$$

These possibly narrower bounds can additionally identify if the proportion who benefit is strictly greater than a specific value, i.e. the sign of

$$\Delta_\tau(Q) \equiv \theta_{PB}(Q) - \tau$$

for some pre-specified $\tau \in [0, 1]$. In particular, if $P$ satisfies the conditions in (7) and (8a)-(8b) in addition to those in (16) and (17), then instrument monotonicity can additionally identify that the sign of $\Delta_\tau(Q)$ is strictly positive for all

$$\tau \in [L_{E,PB}, L_{EM,PB}] .$$

In order to better understand the strength of this additional identifying power, consider the following numerical example where $P$ is such that

$$P_{0,0|0} = 0.3 , \quad P_{0,1|0} = 0.3 , \quad P_{1,0|0} = 0.3 , \quad P_{1,1|0} = 0.1 ,$$

$$P_{0,0|1} = 0.1 , \quad P_{0,1|1} = 0.3 , \quad P_{1,0|1} = 0.3 , \quad P_{1,1|1} = 0.3 .$$
Note that this $P$ satisfies the conditions in (7) and (8a)-(8b) in addition to those in (16) and (17). Then, using Corollary 2.1, the identified sets are such that

$$\theta_{PB}(Q_E) = [0, 0.4] \quad \text{and} \quad \theta_{PB}(Q_{EM}) = [0.2, 0.4] ,$$

and hence instrument monotonicity allows us to additionally conclude that the proportion who benefit in strictly greater than $\tau \in [0, 0.2)$.

**Remark 2.3.** The marginal distributions of the potential outcomes are uniquely determined by the vector

$$(Q\{Y_0 = 1\}, Q\{Y_1 = 1\}) ,$$

where each element can be rewritten as follows

$$Q\{Y_0 = 1\} = Q^\dagger_{1,0} + Q^\dagger_{1,1} = 1 - [Q^\dagger_{0,1} + Q^\dagger_{0,0}] ,$$

$$Q\{Y_1 = 1\} = Q^\dagger_{0,1} + Q^\dagger_{1,1} = 1 - [Q^\dagger_{1,0} + Q^\dagger_{0,0}] .$$

Kitagawa (2009) shows that instrument monotonicity has no identifying content on this vector. In turn, it also directly follows that instrument monotonicity has no identifying content on any function of this vector including those for the average treatment effect

$$E_{Q}[Y_1 - Y_0] = Q\{Y_1 = 1\} - Q\{Y_0 = 1\} .$$

Balke and Pearl (1997) and Heckman and Vytlacil (2001) also provide results focusing on specific functions of this vector such as the average treatment effect. Note that Heckman and Vytlacil (2001) and Kitagawa (2009) also respectively extend their analyses to cases where the outcomes and instruments are continuous and where the outcomes are continuous.

**Remark 2.4.** In addition to the proportion who benefit, one can analogously use the inequalities stated in Proposition 2.1 to derive identified sets for alternate features of the joint distribution such as

$$Q\{Y_1 \geq Y_0\} \equiv Q\{Y_0 = 0, Y_1 = 0\} + Q\{Y_0 = 0, Y_1 = 1\} ,$$

i.e. the proportion who weakly benefit, or the converse of these parameters

$$Q\{Y_0 > Y_1\} \quad \text{and} \quad Q\{Y_0 \geq Y_1\} ,$$

i.e. the proportion who lose and the proportion who weakly lose respectively. ■
A  Proof of Proposition 2.1

For each $Q \in Q_W$, let $\pi_Q$ denote its probability mass function, i.e. the function $\pi_Q : W \to [0,1]$ satisfying

$$\sum_{w \in W} \pi_Q(w) = 1 . \quad (18)$$

Using the restrictions imposed by the data in (5), note that Assumption E is equivalent to

$$\pi_Q(w) = \pi_Q(y_0, y_1, d_0, d_1, 1) \cdot p_z + \pi_Q(y_0, y_1, d_0, d_1, 0) \cdot p_z \quad (19)$$

for all $w = (y_0, y_1, d_0, d_1, z) \in W$, where $p_z \equiv P\{Z = z\}$ is observed. Similarly, note that Assumption M is equivalent to

$$\sum_{w \in W_M} \pi_Q(w) = 0 , \quad (20)$$

where $W_M = \{(y_0, y_1, d_0, d_1, z) \in W : d_0 = 1, \ d_1 = 0\}$. Denoting by $\Pi_W$ the set of all functions from $W$ to $[0,1]$, let

$$\Pi_E = \{\pi \in \Pi_W : \pi \text{ satisfies (18) and (19)}\} \quad (21)$$

denote the set of all probability mass functions that correspond to a distribution $Q \in Q_E$, and

$$\Pi_{EM} = \{\pi \in \Pi_W : \pi \text{ satisfies (18), (19) and (20)}\} \quad (22)$$

denote the set of all probability mass functions that correspond to a distribution $Q \in Q_{EM}$. In particular, note that both $\Pi_E$ and $\Pi_{EM}$ are closed and convex sets, and, since $P$ satisfies the conditions in (7) and (8a)-(8b), it follows from Kitagawa (2015, Proposition 1) that they are also non-empty.

Next, note that the parameter of interest $Q^\dagger$ is a probability distribution defined on the sample space $\mathcal{W}^\dagger$, i.e. we are interested in $Q^\dagger(\mathcal{A}^\dagger)$ for every $\mathcal{A}^\dagger \subseteq \mathcal{W}^\dagger$. To this end, fix $\mathcal{A}^\dagger \subseteq \mathcal{W}^\dagger$ and $\mathcal{A} = M_{JD}^{-1} \mathcal{A}^\dagger \subseteq \mathcal{W}$. Since $Q^\dagger = QM_{JD}^{-1}$, it follows that

$$Q^\dagger(\mathcal{A}^\dagger) = \sum_{w \in \mathcal{A}} \pi_Q(w) . \quad (23)$$

Then, since the function in (23) is a linear function of $\pi$ and both $\Pi_E$ and $\Pi_{EM}$ are closed, convex and non-empty, it follows that the images of this function with respect to $\Pi_E$ and $\Pi_{EM}$ are closed and convex sets in $\mathbb{R}$, which implies that

$$Q^\dagger(\mathcal{A}^\dagger) \in [L_{E,\mathcal{A}^\dagger}, U_{E,\mathcal{A}^\dagger}] \quad \text{and} \quad Q^\dagger(\mathcal{A}^\dagger) \in [L_{EM,\mathcal{A}^\dagger}, U_{EM,\mathcal{A}^\dagger}]$$
for all \(Q^\dagger \in \theta_{JD}(Q_E)\) and \(Q^\dagger \in \theta_{JD}(Q_{EM})\) respectively, where the upper bounds are given by

\[
U_{E,A^\dagger} = \max_{\pi_Q \in \Pi_E} \sum_{w \in A} \pi_Q(w) \quad \text{and} \quad U_{EM,A^\dagger} = \max_{\pi_Q \in \Pi_{EM}} \sum_{w \in A} \pi_Q(w)
\]

and, since \(Q^\dagger\{A^\dagger\} = 1 - Q^\dagger\{A^{\dagger c}\}\) for \(A^{\dagger c} = W^\dagger \setminus A^\dagger\), the lower bounds are trivially obtained using the upper bounds calculated for \(A^{\dagger c} \subseteq W^\dagger\), i.e.

\[
L_{E,A^\dagger} = 1 - U_{E,A^{\dagger c}} \quad \text{and} \quad L_{EM,A^\dagger} = 1 - U_{EM,A^{\dagger c}}.
\]

To conclude the proof, note that since the maximization problems have a linear objective with linear equality and inequality constraints, the upper bounds are solutions to linear programming problems. Using the strategy in Balke and Pearl (1997) and further outlined in Balke (1995) to obtain symbolic expressions for linear programming problems in such settings, we can obtain expressions for the upper bounds \(U_{A^\dagger}\) for each \(A^\dagger \subseteq W^\dagger\) in terms of \(P\). The result then follows by collecting the nontrivial terms that determine these upper bounds.
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