Autonomous Dynamical System Description of de Sitter Evolution in Scalar Assisted $f(R) - \phi$ Gravity

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In this letter we will study the cosmological dynamical system of an $f(R)$ gravity in the presence of a canonical scalar field $\phi$ with an exponential potential, by constructing the dynamical system in a way that it is render autonomous. This feature is controlled by a single variable $m$, which when it is constant, the dynamical system is autonomous. We focus on the $m = 0$ case which, as we demonstrate by using a numerical analysis approach, leads to an unstable de Sitter attractor, which occurs after $N \sim 60$ e-foldings. This instability can be viewed as a graceful exit from inflation, which is inherent to the dynamics of de Sitter attractors.

PACS numbers: 95.35.+d, 98.80.-k, 98.80.Cq, 95.36.+x

I. INTRODUCTION

The phase space structure of a cosmological dynamical system may reveal important information regarding the existence and stability of the fixed points. Also the existence of a fixed point with physical significance may can also be revealed by the study of the cosmological dynamical system. In the literature, the dynamical systems approach has been adopted in various theoretical contexts [1–32], see also [33], and all the studies aimed to reveal the stability structure and the interconnection of fixed points by using various phase space trajectories. In this letter we aim to study the behavior of the de Sitter attractors of an $f(R)$ gravity [35–38], in the presence of a canonical scalar field with potential $V(\phi)$, which we denote for simplicity as $f(R) - \phi$ gravity. For a recent inflationary realization in the context of $f(R) - \phi$ gravity, see Ref. [39]. We will construct an autonomous dynamical system by appropriately manipulating the cosmological equations, and we will study the specific behavior of de Sitter attractors. This work is in the spirit of Refs. [31, 32], where an autonomous dynamical system approach was also considered. As we will show, the dynamical system is rendered autonomous only when the scalar potential is an exponential [40], so by making this assumption, we will investigate the behavior of the de Sitter attractors in this particular case. The importance of having an autonomous dynamical system can be revealed by the following example, which can be found in Ref. [41], $\dot{x} = -x + t$, which can be solved and the solution is $x(t) = t - 1 + e^{-t}(x_0 + 1)$. As it is obvious, the solution approaches $t - 1$ asymptotically, and also the fixed point is $x = t$. However the fixed point $x = t$ is not a solution of the dynamical system, and if the standard theorems for fixed point are applied, the wrong conclusion of having the solutions approaching $x = t$ is obtained. In view of the above example, it is conceivable that an autonomous cosmological dynamical system may reveal important information with regard the existence and stability of fixed points. As we will demonstrate in this letter, the only parameter that contains a time-dependence in the dynamical system is $m = -\frac{\dot{H}}{H}$, with $H$ being the Hubble rate of the Universe. Hence we will assume that this parameter is equal to zero, without specifying the cosmological evolution for which this is possible. As it turns out, there exists a fixed point which is actually a de Sitter fixed point. By carefully examining the phase structure, we will demonstrate by using a numerical analysis, that this fixed point is reached for $N \sim 50 - 60$ e-foldings, however at $N \sim 70$ it is rendered unstable due to an existing instability in some variables. Therefore, this actually describes a de Sitter attractor which becomes internally unstable, a feature that can be viewed as an exit from the inflationary attractor solution.

This letter is organized as follows: In section II, we present the general structure of the autonomous dynamical system, and we demonstrate why the exponential potential is needed in order to render the dynamical system autonomous. In section III we perform an in depth numerical analysis, in which we investigate the existence and the stability of the de Sitter attractor. Also we examine the qualitative features of the phase space, which reveals whether the de Sitter fixed point is stable or not. Finally, the conclusions follow in the end of the letter.

Before we proceed, let us fix the geometric background, which is a flat Friedmann-Robertson-Walker (FRW) metric, with line element,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2 ,$$ (1)
with $a(t)$ being the scale factor. Also for the FRW metric, the Ricci scalar is equal to,

$$R = 6 \left( \dot{H} + 2H^2 \right),$$

with $H = \frac{\dot{a}}{a}$, being as usual the Hubble rate.

II. THE AUTONOMOUS DYNAMICAL SYSTEM OF $f(R, \phi)$ GRAVITY WITH EXPONENTIAL POTENTIAL

As we briefly mentioned in the introduction, having the cosmological evolution expressed in terms of an autonomous dynamical system, may be vital for the correct description of the dynamical evolution. So in this section, we shall present the general structure of an $f(R)$-φ scalar-tensor gravity and we shall use the equations of motion in order to obtain an autonomous dynamical system. A general $f(R)$-φ gravity action, with scalar potential $V(\phi)$ has the following form,

$$S = \int d^4x \sqrt{-g} \left( \frac{f(R)}{2\kappa^2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right),$$

where $\kappa^2 = 8\pi G = \frac{1}{M_p^2}$ and $M_p$ stands for the Planck mass scale. By varying the action with respect to the metric, we obtain,

$$F(R)R_{\mu\nu}(g) - \frac{1}{2} f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f(R) + g_{\mu\nu} \Box F(R) = \kappa^2 (\nabla_\mu \phi \nabla^\mu \phi) - \frac{1}{2} g_{\mu\nu} \nabla^\rho \nabla_\rho - V g_{\mu\nu},$$

and also the scalar field satisfies,

$$\Box \phi = V'(\phi),$$

where the prime indicates differentiation with respect to the scalar field $\phi$. For the metric (1), and by assuming that the scalar field depends solely on the cosmic time, the cosmological equations can be written in the following form,

$$3FH^2 + \frac{1}{2} (f - RF) + 3HF = \kappa^2 \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),$$

$$-2FH - \ddot{F} + HF = \kappa^2 \ddot{\phi},$$

$$\ddot{\phi} + 3H \dot{\phi} + V' = 0,$$

where the “dot” indicates differentiation with respect to the cosmic time. The autonomous dynamical system of $f(R)$ - φ gravity can be obtained if we use the following variables,

$$x_1 = -\frac{\dot{F}(R)}{F(R)H},$$

$$x_2 = -\frac{f(R)}{6F(R)H^2},$$

$$x_3 = \frac{R}{6H^2},$$

$$x_4 = \frac{V(\kappa^2 \phi)}{6FH^2},$$

$$x_5 = \frac{\kappa^2 \ddot{\phi}}{6F},$$

$$x_6 = \frac{\kappa^2}{6F}.$$

The case with a general potential $V(\phi)$ is difficult to tackle, since in all cases an non-autonomous system is constructed, so we focus on the case that the potential is equal to $V(\phi) = e^{-\lambda \phi}$.

A convenient variable that may quantify in an optimal way the duration of de Sitter phase is the $e$-foldings number $N$, so by using the variables (7) and also the equations (10), we get the following dynamical system,

$$\frac{dx_1}{dN} = -x_1^2 - x_2 x_1 - 3x_1 + 2x_3 + 6x_5 - 4,$$

$$\frac{dx_2}{dN} = x_2 + x_1 x_2 - 2x_2 x_3 + 4x_2 - 4x_3 + 8,$$

$$\frac{dx_3}{dN} = -x_2 x_3 + 8x_3 - 8,$$

$$\frac{dx_4}{dN} = x_1 x_4 - \frac{x_3 x_4}{3} + \frac{\lambda x_4 \sqrt{x_5}}{\sqrt{x_6}} + \frac{2x_4}{3},$$

$$\frac{dx_5}{dN} = x_1 x_4 - 2x_3 x_5 + \frac{2\lambda x_4 \sqrt{x_5}}{\sqrt{x_6}} + 4x_5,$$

$$\frac{dx_6}{dN} = x_1 x_6.$$
where the parameter $m$ stands for,

$$m = -\frac{\dddot{H}}{H^3}.$$  \hspace{1cm} (9)

Since we are interested in de Sitter or quasi-de Sitter attractors, the variable $m$ for a quasi-de Sitter attractor is zero. So in order to reveal the structure of the quasi de Sitter attractor, we shall focus on the case of $m = 0$ and we shall perform an analysis of the behavior of the dynamical system (9). The effective equation of state (EoS) for a general $f(R, \phi)$ theory is equal to,

$$w_{eff} = -1 - \frac{2\dot{H}}{3H^2},$$  \hspace{1cm} (10)

and by using the variable $x_3$, it can be expressed in terms of $x_3$ as follows,

$$w_{eff} = -\frac{1}{3}(2x_3 - 1).$$  \hspace{1cm} (11)

In the following we shall focus on the behavior of the de Sitter and of the quasi-de Sitter attractors.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Numerical solutions $x_1(N), x_2(N), x_3(N), x_4(N), x_5(N)$ and $x_6(N)$ for the dynamical system (9), for the initial conditions $x_1(0) = -0.01, x_2(0) = 0, x_3(0) = 2.05, x_4(0) = 0, x_5(0) = 7, x_6(0) = -2$, and for $m = 0, \lambda = 100$. In the upper left, the black curve corresponds to the variable $x_5$, the red curve to the parameter $x_3$ and the blue curve to the parameter $x_1$.}
\end{figure}

A. Study of the Phase Space and the de Sitter Attractor

As we already stressed earlier, the only source of non-autonomous structure in the dynamical system is contained in the parameter $m$, and for certain cosmological evolutions this parameter is constant. For example, if we choose $m = 0$, this can occur for a quasi-de Sitter evolution with scale factor $a(t) = e^{H_0t - H_0t^2}$, but also the symmetric bounce yields $m = 0$, in which case the scale factor is $a(t) = e^{At^2}$. Also the case $m = -9/2$ can occur for a matter dominated...
cosmological evolution with $a(t) \sim t^{2/3}$. In this work we shall focus on the case $m = 0$, however without specifying the specific choice of the Hubble rate, and also note that the general form of the $f(R)$ gravity is not specified. The fixed points of the dynamical system \[\mathcal{S}\] can be found by solving the system of equations $f_i = 0$, $i = 1, \ldots, 6$, and the only solution is $\phi^*_1 = (x_1, x_3, x_5) = (0, 2, 0)$. The rest of the variables are not fixed, and in the rest of this section we shall investigate the behavior all the variables. Firstly, let us note that the case $x_3 = 2$ corresponds to $w_{eff} = -1$, as can be verified by Eq. \[\text{(1)}\], so the case $m = 0$ yields a fixed point with $w_{eff} = -1$, and therefore this is a de Sitter fixed point. The stability of this point cannot be addressed in a conventional way, as it can be shown, since the linearization matrix contains entries which are infinite. So we shall numerically solve the system of differential equations \[\mathcal{S}\] by using various initial conditions, and for various values of the $e$-foldings number $N$. As we now demonstrate, the results are particularly interesting, since once the de Sitter attractor is reached around $N \sim 60$ $e$-foldings, the de Sitter attractor becomes unstable. We have solved numerically the system of differential equations, for the initial conditions $x_1(0) = -0.01$, $x_2(0) = 0$, $x_3(0) = 2.05$, $x_4(0) = 0$, $x_5(0) = 7$, $x_6(0) = -2$, and in Fig. \[\text{(1)}\] we plot the behavior of the parameters $x_i$ as functions of the $e$-foldings number $N$. In the upper left, the black curve corresponds to the variable $x_3$, the red curve to the parameter $x_4$, and the blue curve to the parameter $x_1$. As it can be seen, the variable $x_3$ tends to $x_3 = 2$ quite fast, and for $N < 60$, however the fixed point values for $x_1$ and $x_5$ are not reached at all. Therefore, this shows that although the de Sitter attractor is reached, it is unstable due to the behavior of the variables $x_1$ and $x_5$. The instability in the variables $x_1$ and $x_5$ of the fixed point values $(x_1, x_5) = (0, 0)$ can also be seen in the $x_1 - x_3$ plane, and we present the plot in Fig. \[\text{(2)}\] for various initial conditions. As it can be seen, the fixed point values are approached around $N \sim 60$, and the three curves meet each other, however, around $N \sim 80 - 90$, the three curves split. This clearly indicates that the de Sitter attractor $x_3 = 2$, becomes eventually unstable after $N \sim 70$ $e$-foldings. The same can also be seen in the $x_1 - x_3$ plane, which we plot in Fig. \[\text{(3)}\] where although in all the curves, the value $x_3$ is reached, the curves split and the attractor $x_3 = 2$ becomes eventually unstable.

Let us briefly discuss the physical interpretation of the present numerical analysis. As it seems from Figs. \[\text{(1)}\] \[\text{(2)}\] and \[\text{(3)}\] the de Sitter attractor $x_3 = 2$ is reached for $N \sim 60$, however, due to the fact that the variables $x_1$ and $x_5$ become unstable, this de Sitter attractor is rendered unstable. Hence, if the de Sitter attractor is an inflationary attractor, then although the attractor $x_3 = 2$ is reached for $N \sim 60$ and even earlier as it can be seen in the upper left plot of Fig. \[\text{(1)}\] the de Sitter attractor becomes unstable, and this can be seen as a graceful exit from the inflationary era.

Hence the $f(R) - \phi$ theory with exponential scalar potential, can describe accurately an inflationary era which comes to an end eventually, due to inherent instabilities. We have also checked the case $m = -9/2$ and we found that the matter dominated era cannot be consistently described by a $f(R) - \phi$, so perfect fluids are needed to describe this era, however we defer from going into details for brevity.
FIG. 3: Parametric plot in the plane $x_1 - x_3$ for $N = (0, 80)$ for the dynamical system $\phi$, with $m = 0$, $\lambda = 100$, and for various initial conditions.

III. CONCLUSIONS

In this letter we investigated the dynamical evolution of the autonomous system which is constructed from an exponential $f(R) - \phi$ gravity. By appropriately choosing the variables related to the physical quantities entering the equations of motion, we were able to construct a dynamical system, in which the only time dependence is solely contained in the variable $m = -\frac{H}{H^3}$. We focused on the specific case $m = 0$, without specifying the Hubble rate and we investigated the behavior of the corresponding autonomous dynamical system. As we demonstrated, the case $m = 0$ leads to a de Sitter attractor solution, and by employing a numerical analysis investigation, we demonstrated that the de Sitter fixed point is reached at $N \sim 50 - 60$ e-foldings. After $N \sim 60$ some variables of the dynamical system become unstable, and as a result, the dynamical system is rendered unstable and the de Sitter attractor ceases to be the final attractor of the theory. As we argued, this feature resembles the graceful exit from the inflationary de Sitter attractor, so this seems to be the case. In effect, the exponential $f(R) - \phi$ gravity has a de Sitter attractor which is reached around $N \sim 50 - 60$ and after that it becomes unstable. This instability could be viewed as a graceful exit from the inflationary de Sitter attractor, and this graceful exit is inherent to the dynamics of de Sitter attractors.

We need to note that the instability of the de Sitter attractor is not guaranteed for all theories of modified gravity which are studied in the way we just presented. Actually, a pure $f(R)$ gravity theory, even in the presence of matter and radiation perfect fluids, has a stable de Sitter attractor corresponding to the case $m = 0$. This study will be reported elsewhere.

Acknowledgments

Financial support by the Research Committee of the Technological Education Institute of Central Macedonia, Serres, under grant SAT/ME/230518-126/15, is gratefully acknowledged.

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