Gaussian, Mean Field and Variational Approximation: the Equivalence

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Abstract

We show the equivalence between the three approximation schemes for self-interacting (1+1)-D scalar field theories. Based on rigorous results of [1, 2], we are able to prove that the Gaussian approximation is very precise for certain limits of coupling constants. The $\lambda \phi^4 + \sigma \phi^2$ model will be used as a concrete application.

I. INTRODUCTION

We start by clarifying the terms which appear in the title. For a given theory of self-interacting quantum fields, by Gaussian approximation we mean the Gaussian part of the interacting measure. By mean field approximation we understand the leading term of the expansion given in [1, 2] and by variational approximation we understand the old variational technique performed with some particular trial states [3-5].

The novelty of this paper is the method of extracting the Gaussian piece of the interacting measure. Also we hope that the equivalence between the three methods will lead to a better understanding of the self-interacting field theories. The expansion around the mean field (MF) approximation have been successfully used in [1, 2] to $\lambda^{-2} \sum_{n=1}^{N} \alpha_n (\lambda \phi)^{2n}$ interactions, for small $\lambda$. The key role of this approximation is that, after the translation to $\phi - \phi_{MF}$, it brings all the coupling constants near to zero where the cluster expansion [6] can be applied.
The results show that the mean field approximation incorporates the nonanalytic part of the Schwinger functions. Our Gaussian approximation will play the same role. For $\lambda \to 0$, $\infty$, it provides a transformation of the field which brings all the coupling constants near to zero. Actually, for $\lambda \to 0$, the Gaussian approximation reduces to the mean field approximation.

A similar procedure have been used by Glimm et al in [7] to prove the equivalence between the model: $\lambda \phi^4 + \sigma \phi^2$; with large $\lambda$, and the model: $\lambda \phi^4 + \sigma \phi^2$; with small $\lambda$ and negative $\sigma$. However, the variational approximation scheme (especially in the form of [5]) became very popular among physicists even thought that there were no estimates of the errors related to this scheme. The equivalence of the three methods allows us to use the exact results of [1, 2] to estimate these errors.

II. OUTLINE OF THE STRATEGY

We consider in this paper (1+1)-D scalar fields with self-interactions. To build such theories, one can follow the strategy presented in [8]. Consider first a space cut-off interaction:

$$U (s) = \int_{|x|<s} d^2x : V (\phi (x)) :_{m_0},$$

where the normal ordering is with respect to the vacuum of the free field of mass $m_0$. The next step is the investigation of the cut-off interacting measure:

$$d\mu_s = \frac{e^{-U(s)+\frac{m_0^2}{2}\int_{|x|<s}(\phi(x) - \xi)^2} d\mu_{0,\xi}}{\int e^{-U(s)+\frac{m_0^2}{2}\int_{|x|<s}(\phi(x) - \xi)^2} e^{-U'(s)} d\mu_{0,\xi}} = e^{-U'(s)} d\mu_{0,\xi},$$

where $\mu_{0,\xi}$ is the Gaussian measure corresponding to the covariance $C_{m_0} = \frac{1}{\Delta + m_0^2}$ and mean $\xi$. Note that the additional term to the potential $U$ cancels the mass and the mean of the measure $\mu_{0,\xi}$. The last step is the investigation of the (thermodynamic) limit of this measure: $\lim_{s \to \infty} \mu_s$. In the case when the limit is well defined, one can build the physical Minkowsky field by using one of the well known reconstruction methods. The hardest part of such a program is obvious the thermodynamic limit. Our strategy of pursuing this program...
will be as follow. The investigation of $\mu_s$ is equivalent with the investigation of the moments (Schwinger functions):

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle_s = \int \phi(x_1) \ldots \phi(x_n) \, d\mu_s. \tag{3}$$

The thermodynamic limit can be studied by considering the dynamical system:

$$\frac{d}{ds} \begin{pmatrix} \langle \phi(x_1) \rangle_s \\ \langle \phi(x_1) \phi(x_2) \rangle_s \\ \vdots \\ \langle \phi(x_1) \phi(x_n) \rangle_s \end{pmatrix} = \vec{X}, \tag{4}$$

where $\vec{X}$ is an infinite vector of whom expression can be deduced by simply taking the derivative of the right part of (3) in respect to $s$. The thermodynamic limit is reduced now to the study of the stable fixed points of the dynamical system (4): if the initial conditions for this dynamical system lie in the basin of attraction of a stable fixed point then we found the thermodynamic limit. We will show that the initial conditions can be modified by changing the boundary conditions of the field. Actually, in the Gaussian approximation we can choose the initial conditions to superimpose over the stable fixed points. This will lead to a set of self-consistency equations which will provide us an approximative value for the expectation value and the mass of the field. For interactions of the type: $\lambda^{-2} P(\lambda \phi)$, the solution of this set of equations reveals two things: in the limit $\lambda \to 0$ we recover what Glimm et al call the mean field approximation. In the limit $\lambda \to \infty$ (strong couplings), the mass increases much faster than $\lambda$ so by a rescaling of the field we are again in the small coupling constants domain. To show the equivalence between Gaussian and Variational approximations, we compute the vacuum energy in Gaussian approximation and show that it coincides with the expression given in [5].

### III. THE GAUSSIAN APPROXIMATION

Let us start by specifying our notations. In general, by $: \cdot :_\mu$ we denote the normal ordering with respect to some measure $\mu$. If the measure corresponds to a covariance $C$ and mean $\xi$,
we denote it by :·:_{C,\xi} and further, if the covariance corresponds to some mass, \( m \), then we use :·:_{m,\xi}. Instead of :·:_{\mu}, we will use :·:_{s}. We denote the expectation values with respect to some measure \( \mu \) by \( \langle \cdot \rangle_{\mu} \) and we apply the same shorthands as for the normal ordering.

A. The dynamics of the expectation values

Starting from:

\[
\langle \phi(x_1) \ldots \phi(x_n) \rangle_{s} = \frac{\int \phi(x_1) \ldots \phi(x_n) e^{-U'(s)} d\mu_0}{\int e^{-U'(s)} d\mu_0},
\]

then:

\[
\frac{d}{ds} \langle \phi(x_1) \ldots \phi(x_n) \rangle_{s} = \left\langle \int dx \left[ [V' (\phi (x))]_s - V' (\phi (x)) \right] \phi (x_1) \ldots \phi (x_n) \right\rangle_{s}
\]

or, shortly:

\[
\frac{d}{ds} \langle \phi(x_1) \ldots \phi(x_n) \rangle_{s} = \left\langle [\langle U' (\delta \Gamma) \rangle_s - U' (\delta \Gamma)] \phi (x_1) \ldots \phi (x_n) \right\rangle_{s},
\]

where \( \Gamma \) is the curve \(|x| = s\) and \( \delta \) is the Dirac delta function. As is stated now, the above equation has the initial condition:

\[
\langle \phi(x_1) \ldots \phi(x_n) \rangle_{s=0} = \langle \phi(x_1) \ldots \phi(x_n) \rangle_{m_0,\xi}.
\]

in particular: \( \langle \phi(x) \rangle_{s=0} = \xi \) and \( \langle : \phi(x) :_{s=0}; \phi(y) :_{s=0} \rangle = C_{m_0} (x - y) \).

B. The change of covariance

We can allow us a little more liberty in choosing the initial conditions. If one uses the Gaussian perturbation identity [6]:

\[
d\mu_{(C-1+v)^{-1}} = Z^{-1} \exp (-: V :_{C}) d\mu_C,
\]

where \( : V :_{C} = \frac{1}{2} \int v(x) : \phi(x)^2 :_{C} \) and \( Z \equiv \int \exp (-: V :_{C}) d\phi_C \), it follows that, up to a change of boundary conditions:

\[
d\mu_s = \frac{e^{-U'(s)} d\mu_0}{\int e^{-U'(s)} d\mu_0} = \frac{e^{-\tilde{U}(s)} d\mu_{C_{m,\xi}}}{\int e^{-U'(s)} d\mu_{C_{m,\xi}}},
\]
where:
\[
\tilde{U}(s) \equiv \int_{|x|<s} d^2 x \tilde{V}(\phi(x)) = \int_{|x|<s} d^2 x \left\{ \phi(x) \right\}_{m_0} - \frac{m^2}{2} (\phi(x) - \xi)^2 :_{m_0}
\]  
(11)

and \(d\mu_{C_m,\xi}\) is the Gaussian measure corresponding to the covariance \(C_m = \frac{1}{-\Delta + m^2}\) and mean \(\xi\). Now it is the time to discuss the boundary conditions. One can see that from very beginning some boundary conditions were imposed. For example in [2] the additional term cancels the mass and mean of the measure \(d\mu_{0,\xi}\) only for \(|x| < s\). This cut-off is equivalent with adding an interaction in the region \(|x| > s\). In the limit \(s \to \infty\), it provides us the external conditions. One can read [9] for a discussion of this boundary conditions. The same remarks for [10]: the Gaussian perturbation identity was cut off, the effect being the change of the external conditions. In the light of this remarks, one can see that all the time when \(m\) and \(\xi\) are changed, the boundary conditions are changed or, reciprocally, we can change \(m\) and \(\xi\) by a modification of the boundary conditions. As a result, we can control the initial conditions of the dynamical system [8] by imposing different boundary conditions:

\[
\begin{align*}
\frac{d}{ds} \langle \phi(x_1) ... \phi(x_n) \rangle_s &= \left\langle \left[ \langle \tilde{U}(\delta_T) \rangle_s - \tilde{U}(\delta_T) \right] \phi(x_1) ... \phi(x_n) \right\rangle_s \\
\langle \phi(x_1) ... \phi(x_n) \rangle_{s=0} &= \langle \phi(x_1) ... \phi(x_n) \rangle_{m,\xi}
\end{align*}
\]

(12)

If the system is multiphasic, then [12] will have more than one stable fixed point. The above procedure can be used in the following way to select a pure phase: enforce the thermodynamic limit to converge to a fixed stable point by choosing the initial conditions in the basin of attraction of this point. There is now a nice picture of critical phenomena in terms of the bifurcation theory. Suppose [12] has a degenerate fixed point and there is a bifurcation point where the degeneracy is lifted. Then the critical values of the parameters are given by the bifurcation point.
C. The Gaussian approximation

We consider that $\mu_s$ is approximately Gaussian and symmetric at translations, or one can think that from now we are interested only in the symmetric Gaussian part (and the mean) of the measure $\mu_s$. The major property of a Gaussian measure is that the normal ordered powers are orthogonal:

$$\langle : \phi^n : s : \phi^m : s \rangle_s = 0 \text{ if } m \neq n. \quad (13)$$

We will select the following two equations of the dynamical system which in the Gaussian approximation write:

$$\frac{d}{ds} \langle \phi(0) : s \rangle_s = -T_{1,s} \left( \tilde{V} (\phi(x)) \right) \langle : \phi(0) : s \rangle_s$$

$$\frac{d}{ds} \left( \langle : \phi(0) : s^2 : s \rangle_s - \langle : \phi(x) : s^2 : m_0 \rangle_{m_0} \right) = -T_{2,s} \left( \tilde{V} (\phi(x)) \right) \langle : \phi^2 (\delta_{T'}) : s : \phi(0) : s^2 : s \rangle_s \quad (14)$$

where $T_{n,s} (\cdot)$ means the coefficient of $: \phi^n : s$ of the given expression. The point $x$ satisfies $|x| = s$ and we have used the symmetry of the problem to separate the terms. Note that the second equation can be written in the above form because $\langle \tilde{U} (\delta_{T'}) : s \rangle_s - \tilde{U} (\delta_{T'})$ has no free term and $: \phi^2 : s - : \phi : s^2$ is a constant. We have taken the combination $\langle : \phi(0) : s^2 : s \rangle_s - \langle : \phi(x) : s^2 : m_0 \rangle_{m_0}$ because it is a finite quantity and for the reason explained in the following. After one changes the normal ordering in $\tilde{V} (\phi)$ from $: \cdot : m_0$ to $: \cdot : s$,

$$: \phi(x)^n : m_0 = \lim_{a \to 0} \sum_{k=0}^{n} C_n^k \times$$

$$\times \frac{\partial^{n-k}}{\partial a^{n-k}} \left\{ \exp \left( \frac{a^2}{2} \left( \langle : \phi(x) : s^2 : s \rangle_s - \langle : \phi(x) : s^2 : m_0 \rangle_{m_0} \right) + a \langle \phi(x) : s \rangle_s \right) \right\} : \phi(x)^k : s, \quad (15)$$

relation which is valid in the Gaussian approximation, one can see that $T_{n,s} \left( \tilde{V} (\phi(x)) \right)$ is a function only of the combination $\langle : \phi(x) : s^2 : s \rangle_s - \langle : \phi(x) : s^2 : m_0 \rangle_{m_0}$ and the expectation value $\langle \phi(x) : s \rangle_s$. Now, because we consider also the symmetric part of $\mu_s$ at translations, the above
system can be written in the form:

\[
\begin{align*}
\frac{d}{ds} \langle \phi(x) \rangle_s &= -T_{1,s} \left( \tilde{V}(\phi(x)) \right) \langle :\phi(\delta_{\Gamma}) :s :\phi(0) :s \rangle_s \\
\frac{d}{ds} \left( \langle :\phi(x) :^2_s \rangle_s - \langle :\phi(x) :^2_{m_0/m_0} \rangle_s \right) &= -T_{2,s} \left( \tilde{V}(\phi(x)) \right) \langle :\phi^2(\delta_{\Gamma}) :s :\phi(0) :^2_s \rangle_s \\
\langle \phi(x) \rangle_{s=0} &= \xi , \quad \left( \langle :\phi(x) :^2_{s=0} \rangle_{s=0} - \langle :\phi(x) :^2_{m_0/m_0} \rangle_{m_0} \right) = \frac{1}{4\pi} \ln \left( \frac{m_0^2}{\ell^2} \right).
\end{align*}
\]  

Let us rewrite this system in the following notations:

\[
\begin{align*}
X &= \langle \phi(x) \rangle_s \\
Y &= \langle :\phi(x) :^2_s \rangle_s - \langle :\phi(x) :^2_{m_0/m_0} \rangle_{m_0}.
\end{align*}
\]  

Then:

\[
\begin{align*}
\frac{dX}{ds} &= -T_{1,s} \left( X, Y \right) \langle :\phi(\delta_{\Gamma}) :s :\phi(0) :s \rangle_s \\
\frac{dY}{ds} &= -T_{2,s} \left( X, Y \right) \langle :\phi^2(\delta_{\Gamma}) :s :\phi(0) :^2_s \rangle_s \\
X(0) &= \xi , \quad Y(0) = \frac{1}{4\pi} \ln \left( \frac{m_0^2}{\ell^2} \right).
\end{align*}
\]  

It is easily now to find the fixed points of the dynamical system (more exactly, the first two coordinates of the fixed points), which are given by the conditions:

\[
\begin{align*}
T_{1,s} \left( X, Y \right) &= 0 , \quad T_{2,s} \left( X, Y \right) = 0.
\end{align*}
\]  

Because we do not have to much information about the basin of attraction of the fixed points, the best thing we can do is to superimpose the initial conditions over the stable points. This leads to our self-consistency equations:

\[
\begin{align*}
T_{1,s} \left( \xi, \frac{1}{4\pi} \ln \left( \frac{m_0^2}{\ell^2} \right) \right) &= 0 , \quad T_{2,s} \left( \xi, \frac{1}{4\pi} \ln \left( \frac{m_0^2}{\ell^2} \right) \right) = 0,
\end{align*}
\]  

which provide us the Gaussian approximation of the expectation value \( \langle \phi(x) \rangle_{s \to \infty} = \xi \) and the two point Schwinger function \( \langle :\phi(x_1) :_{s \to \infty} :\phi(x_2) :_{s \to \infty} \rangle_{s \to \infty} = C_m(x_1, x_2). \)
IV. THE EQUIVALENCE WITH THE VARIATIONAL METHOD

We calculate the energy of the vacuum state in the Gaussian approximation. Will follow that our expression is the same with that of [5]. Moreover, our self-consistency equations can be derived by minimizing this energy with respect to $\xi$ and $m$.

The term $Z$, which appears in the Gaussian perturbation identity, was unimportant for the expectation values but will be essential for the calculus of the vacuum energy. According to [8], the energy per unit length can be calculated as:

$$\varepsilon = - \lim_{s \to \infty} \frac{\ln \int e^{-U(s)} \, d\mu_{0,\xi}}{\pi s^2}. \quad (21)$$

After the change of the covariance:

$$\varepsilon = - \lim_{s \to \infty} \frac{\ln Z^{-1} \int e^{-\tilde{U}(s)} \, d\mu_{C_{m},\xi}}{\pi s^2}, \quad (22)$$

with:

$$Z = \int \exp \left( \frac{m^2 - m_0^2}{2} \int d^2 x : (\phi(x) - \xi)^2 :_{m_0} \right) \, d\mu_{0,\xi}. \quad (23)$$

which can be calculated formally [1] as:

$$Z = \exp \left( \frac{1}{2} Tr \left\{ \ln \left( 1 + \frac{m^2 - m_0^2}{-\Delta + m_0^2} \right) - \frac{m^2 - m_0^2}{-\Delta + m_0^2} \right\} \right). \quad (24)$$

In this form, the above quantity is infinite. Taking into account that when we have performed the change of covariance we have considered the integral $\int d^2 x : (\phi(x) - \xi) :_{m_0}$ only on $|x| < s$, we should do the same thing for $Z$. Considering the operator:

$$\hat{K} = \ln \left( 1 + \frac{m^2 - m_0^2}{-\Delta + m_0^2} \right) - \frac{m^2 - m_0^2}{-\Delta + m_0^2}, \quad (25)$$

with the kernel:

$$K(x, y) = \int d^2 k \left\{ \ln \left( 1 + \frac{1}{4\pi^2 k h^2 + m_0^2} \right) - \frac{1}{4\pi^2} \frac{m^2 - m_0^2}{k^2 + m_0^2} \right\} e^{-ik(x-y)}, \quad (26)$$

it follows:

$$Tr \left[ \hat{K} \right] = \int_{|x|<s} d^2 x \int d^2 k \left\{ \ln \left( 1 + \frac{1}{4\pi^2 k h^2 + m_0^2} \right) - \frac{1}{4\pi^2} \frac{m^2 - m_0^2}{k^2 + m_0^2} \right\} \quad (27)$$
if we restrict the domain of integration at $|x| < s$. Finally:

$$Tr \left[ \hat{K} \right] = \frac{\pi s^2}{4\pi} \left( m^2 - m_0^2 - m^2 \ln \frac{m^2}{m_0^2} \right).$$

(28)

In the Gaussian approximation, the other term of $\varepsilon$: 

$$- \ln \left\langle e^{-\tilde{U}(s)} \right\rangle_{m,\xi}$$

is given by $T_{0,s}(\tilde{V})$.

In consequence:

$$\varepsilon = T_{0,s}(\tilde{V}) + \frac{1}{8\pi} \left( m^2 - m_0^2 - m^2 \ln \frac{m^2}{m_0^2} \right).$$

(29)

The last expression is completely equivalent with that of [5]. Indeed, for an analytic function, $F$, it follows from [5] that:

$$T_{0,s} \left[ : F(\phi(x)) :_{m_0} \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n F(x)}{\partial x^n} \left|_{x=0} \frac{\partial^n}{\partial x^n} \left\{ \exp \left( \frac{a^2}{8\pi} \ln \frac{m_0^2}{m^2} + a\xi \right) \right\} \right|_{a=0}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} \frac{\partial^n F}{\partial x^n} \left|_{x=0} \xi^{n-k} \frac{\partial^k}{\partial a^k} \exp \left( \frac{a^2}{8\pi} \ln \frac{m_0^2}{m^2} \right) \right|_{a=0}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} \frac{\partial^n F}{\partial x^n} \left|_{x=0} \xi^{n-k} \frac{\partial^k}{\partial a^k} \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{a^2}{8\pi} \ln \frac{m_0^2}{m^2} \right)^p \right|_{a=0}$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{(n-2p)! (2p)!} \frac{\partial^n F}{\partial x^n} \left|_{x=0} \xi^{n-2p} \frac{a^{2p}}{p!} \left( \frac{1}{8\pi} \ln \frac{m_0^2}{m^2} \right)^p \right|_{a=0}$$

$$= \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{1}{8\pi} \ln \frac{m_0^2}{m^2} \right)^p \sum_{n=0}^{\infty} \frac{1}{(n-2p)!} \frac{\partial^n F}{\partial x^n} \left|_{x=0} \xi^{n-2p} = \exp \left( \frac{1}{8\pi} \ln \frac{m_0^2}{m^2} \frac{d^2}{dx^2} \right) F(x) \right|_{x=\xi}. $$

With this result we have:

$$\varepsilon (m, \xi) = \exp \left( \frac{1}{8\pi} \ln \frac{m_0^2}{m^2} \frac{d^2}{dx^2} \right) \left( V(\xi) - \frac{m^2}{2} (x - \xi)^2 \right) \left|_{x=\xi} \right.$$ 

$$+ \frac{1}{8\pi} \left( m^2 - m_0^2 - m^2 \ln \frac{m_0^2}{m^2} \right) = \exp \left( \frac{1}{8\pi} \ln \frac{m_0^2}{m^2} \frac{d^2}{d\xi^2} \right) V(\xi) + \frac{1}{8\pi} (m^2 - m_0^2).$$

(30)

We show in the following that our self-consistency equations can be also derived by mini-
mizing $\varepsilon$ in respect to $\xi$ and $m$. Indeed:

$$T_{1,s} \left[ \tilde{V} (\phi (x)) : \varepsilon_0 \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \tilde{V} (x)}{\partial x^n} \right|_{x=0} 
\cdot \left. \frac{n}{\partial a^{n-1}} \left\{ \exp \left( \frac{a^2}{8\pi} \ln \frac{m_0^2}{m^2} + a\xi \right) \right\} \right|_{a=0}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \tilde{V}}{\partial x^n} \right|_{x=0} \frac{n}{\partial a^n} \left\{ a \exp \left( \frac{a^2}{8\pi} \ln \frac{m_0^2}{m^2} + a\xi \right) \right\} \bigg|_{a=0}$$

$$= \frac{\partial}{\partial \xi} \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \tilde{V}}{\partial x^n} \right|_{x=0} \frac{n}{\partial a^n} \left\{ \exp \left( \frac{a^2}{8\pi} \ln \frac{m_0^2}{m^2} + a\xi \right) \right\} \bigg|_{a=0} = \frac{\partial \varepsilon}{\partial \xi} \quad (31)$$

Using the same notation as before, $Y = \frac{1}{8\pi} \ln \frac{m_0^2}{m^2}$, for the second equations we have:

$$T_{2,s} \left[ \tilde{V} (\phi (x)) : \varepsilon_0 \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \tilde{V}}{\partial x^n} \right|_{x=0} \cdot \left. n \left( n - 1 \right) \frac{\partial^{n-2}}{\partial a^{n-2}} \left\{ \exp \left( \frac{a^2}{8\pi} \ln \frac{m_0^2}{m^2} + a\xi \right) \right\} \right|_{a=0}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \tilde{V}}{\partial x^n} \right|_{x=0} \frac{n}{\partial a^n} \left\{ a^2 \exp (a^2 Y + a\xi) \right\} \bigg|_{a=0}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \tilde{V}}{\partial x^n} \right|_{x=0} \frac{n}{\partial a^n} \left\{ \exp (a^2 Y + a\xi) \right\} \bigg|_{a=0}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \tilde{V}}{\partial x^n} \right|_{x=0} \frac{n}{\partial a^n} \left\{ \exp (a^2 Y + a\xi) \right\} \bigg|_{a=0}$$

$$= \frac{\partial}{\partial Y} \left[ T_{0,s} \left( \tilde{V} \right) \right] + \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial}{\partial Y} \left( \frac{\partial^n \left[ \frac{m^2}{2} (x - \xi)^2 \right]}{\partial x^n} \right) \bigg|_{x=0} \frac{n}{\partial a^n} \left\{ \exp (a^2 Y + a\xi) \right\} \bigg|_{a=0} \quad (32)$$

Using that:

$$\frac{\partial}{\partial Y} \left[ T_{0,s} \left( \tilde{V} \right) \right] = \frac{\partial}{\partial Y} \left\{ \varepsilon - m^2 \left( \frac{1}{8\pi} + Y \right) \right\} \cdot \frac{\partial}{\partial Y} \left\{ m^2 \left( \frac{1}{8\pi} + Y \right) \right\} = Y \frac{\partial m^2}{\partial Y} \quad (33)$$

and expanding the second term of (32), it follows:

$$T_{2,s} \left[ \tilde{V} (\phi (x)) : \varepsilon_0 \right] = \frac{\partial \varepsilon}{\partial Y} \quad (34)$$
We have proven that the Gaussian approximation of the vacuum energy density is equal with the expression obtained in [5] and that our self-consistency equations can be also derived by minimizing the energy density with respect to $\xi$ and $m$. In consequence the two methods are completely equivalent.

V. THE $V(\phi) = \lambda \phi^4 + \sigma \phi^2$ MODEL

We apply in this section the Gaussian approximation to this particular model. The self-consistency equations are:

$$
\begin{align*}
\begin{cases}
T_{1,s} \left[ \tilde{V}(\phi) ; m_0 \right] &= \frac{\xi}{\pi} \left( 3\lambda \ln \frac{m_0^2}{m^2} + 4\pi \lambda \xi^2 + 2\pi \sigma \right) = 0 \\
T_{2,s} \left[ \tilde{V}(\phi) ; m_0 \right] &= \frac{3\lambda}{\pi} \left( \ln \frac{m_0^2}{m^2} + 4\pi \xi^2 \right) + \sigma - \frac{m^2}{2} = 0.
\end{cases}
\end{align*}
$$

(35)

Note that the first equation always has the solution $\xi = 0$. Let us analyze first this situation. The second equation leads to:

$$
m^2 = \frac{3\lambda}{\pi} W_0 \left( \frac{m_0^2}{3\lambda} \exp \left( \frac{2\pi \sigma}{3\lambda} \right) \right),
$$

(36)

where $W_0$ is the Lambert $W$ function of rank zero. For $\sigma > 0$, we can introduce the classical mass: $m_c^2 = V''(0) = 2\sigma$. One can see that, if we choose $m_0 = m_c$, then $m_c$ is a solution of the above equation. For $\sigma < 0$, the solution $\xi = 0$ becomes unstable. The other solutions must satisfy:

$$
\begin{align*}
\begin{cases}
m^2 &= m_0^2 \exp \left( \frac{2\pi}{3\lambda} (2\lambda \xi^2 + \sigma) \right) \\
m^2 &= 8\lambda \xi^2.
\end{cases}
\end{align*}
$$

(37)

For $\sigma < 0$ we can define the mean field approximation: $(\xi_c)^2 = -\frac{\sigma}{2\lambda}$ and $m_c^2 = V''(\xi_c) = 8\lambda \xi_c^2 = -4\sigma$. One can see that, if we start with $m_0 = m_c$, then the mean field approximation is a solution of our self-consistency equations. This is true for more general interactions and
it shows the link between the Gaussian and mean field approximations. Fortunately this is not the only solution. The general solution of \[37\] (for \(m_0 = m_c\)) is:

\[
\xi^2 = \frac{-3}{4\pi} W \left( \frac{2\pi \sigma}{3\lambda} \exp\left( \frac{2\pi \sigma}{3\lambda} \right) \right),
\]

where \(W\) can be the Lambert function of rank 0 or \(-1\). It follows that:

\[
\xi^2 = \begin{cases} 
    \frac{-3}{4\pi} W_{-1} \left( \frac{2\pi \sigma}{3\lambda} \exp\left( \frac{2\pi \sigma}{3\lambda} \right) \right), & 2\pi \sigma/3\lambda \leq -1 \\
    \frac{-3}{4\pi} W_0 \left( \frac{2\pi \sigma}{3\lambda} \exp\left( \frac{2\pi \sigma}{3\lambda} \right) \right), & 2\pi \sigma/3\lambda > -1
\end{cases}
\]

(38)

is the solution which coincides with \(\xi_c\). It gives the leading term of \(\langle \phi \rangle\) in the limit \(\lambda \to 0\). The other solution is:

\[
\xi^2 = \begin{cases} 
    \frac{-3}{4\pi} W_0 \left( \frac{2\pi \sigma}{3\lambda} \exp\left( \frac{2\pi \sigma}{3\lambda} \right) \right), & 2\pi \sigma/3\lambda \leq -1 \\
    \frac{-3}{4\pi} W_{-1} \left( \frac{2\pi \sigma}{3\lambda} \exp\left( \frac{2\pi \sigma}{3\lambda} \right) \right), & 2\pi \sigma/3\lambda > -1
\end{cases}
\]

(39)

which gives the leading term of \(\langle \phi \rangle\) in the limit \(\lambda \to \infty\). Moreover, even when the classical picture is lost, \(\sigma > 0\), in which case the potential does show only one minima, there is a solution which shows symmetry breaking in the limit \(\lambda \to \infty\):

\[
\xi^2 = \frac{-3}{4\pi} W_{-1} \left( -\frac{\pi m_0^2}{6\lambda} \exp\left( \frac{2\pi \sigma}{3\lambda} \right) \right).
\]

(41)

The nice behavior of these solutions relies on the fact that, for \(\lambda \to 0\), the first solution keep the mass constant (while the coupling constants goes to zero) and, for \(\lambda \to \infty\), the second solution gives a very large mass such that the ratio between the coupling constants and mass goes to zero. To see this, let us calculate the expression of the potential after we apply the Gaussian transformations. It follows from the analysis of the last sections that the interacting measure is equal to (up to our boundary conditions):

\[
d\mu_\Lambda = Z_{\Lambda}^{-1} e^{-\int_{x \in \Lambda}^{x} \lambda \phi(x)^4 + \sigma \phi(x)^2 - \frac{m^2}{2} (\phi(x) - \xi)^2} d\mu_{C_m, \xi}
\]

(42)
and after the change of the normal ordering in the exponent, with the values of \( m \) and \( \xi \) given by the self-consistency equations, the coefficients of \( :\phi^2 : \) and \( :\phi : \) cancel out, the result being:

\[
d\mu_\Lambda = Z_\Lambda^{-1} e^{-\int_{x\in\Lambda} \lambda \phi(x)^4 + 4\lambda \xi \phi(x)^3 \cdot m,\xi} \cdot \mu_{C,m,\xi}. \tag{43}
\]

Finally we rescale the mass at the unity by using the rescaling identity \footnote{\[6\]}:

\[
d\mu_{\Lambda'} = Z_{\Lambda'}^{-1} e^{-\int_{x\in\Lambda'} \frac{1}{8\xi^2} \phi(x)^4 + \frac{1}{2\xi} \phi(x)^3 \cdot 1,\xi} \cdot \mu_{C_1,\xi}, \tag{44}
\]

with \( \Lambda' = \Lambda / m^2 \). Note that we have omitted the free term of the potential, \( \varepsilon(m,\xi) \), which only shifts the energies and is unimportant for the expectations values. Thus one can see that the small coupling constant regime is achieved for large values of \( \xi \). However, even for large values of \( \xi \), the convergence of the thermodynamic limit cannot be proved by an ordinary cluster expansion. This is due to the fact that the potential \( V = \frac{1}{8\xi^2} \phi^4 + \frac{1}{2\xi} \phi^3 \) is not bounded from below uniformly in \( \xi \). When the field is localized near to \( -\xi \), it behaves as \( -\xi^2 \). Nevertheless, an expansion in phase boundaries \footnote{\[1\]} will solve the problem. We can give now the asymptotic expansion of Schwinger functions. As usual, we will apply the formula for integration by parts to the interaction \footnote{\[14\]}:

\[
\int :\phi(x)^m : R(\phi) \, d\mu_\Lambda = \int \int C_1(x - y) :\phi(x)^{m-1} : \left\{ \frac{\delta R}{\delta \phi(y)} - R \frac{\delta V}{\delta \phi(y)} \right\} \, dy \, d\mu_\Lambda. \tag{45}
\]

The formula is still true for the case when the normal ordering is with respect to a Gaussian measure with a nonzero mean value (see Appendix). For \( \langle \phi \rangle \) we have to apply this formula three times to find the first correction to \( \xi \). Redenoting \( \Lambda' \) as \( \Lambda \), it follows:

\[
\langle \phi(x) \rangle = \int \phi(x) \, d\mu_\Lambda = \int ( :\phi(x) :_{1,\xi} + \xi) \, d\mu_\Lambda = \xi + \frac{3}{2\xi^2} \left[ \int C_1(x)^3 \, dx - \frac{9}{2} \int C_1(x) C_1(y) C_1(x - y)^2 \, dxdy \right] + o(1/\xi^4). \tag{46}
\]
For the connected part of the two point Schwinger function, it follows:

\[
\langle \phi(x) \phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle = \int : \phi(x) :_{1,\xi} : \phi(y) :_{1,\xi} d\mu_\Lambda + o(1/\xi^3) \\
= C_1 (x - y) + \frac{9}{2\xi} \int \int C_1 (x - z) C_1 (y - u) C_1 (z - u)^2 dzdu + o(1/\xi^3) .
\]

(47)

If we go back to the original scale, we can conclude:

\[
\begin{cases}
\langle \phi(x) \rangle = \xi + \frac{0.021}{\xi^3} + o(1/\xi^4) \\
\langle \phi(x) \phi(y) \rangle_T = C_m (x - y) + \frac{5.6 \times 10^{-4}}{\xi^2} + o(1/\xi^3)
\end{cases}
\]

(48)

VI. CONCLUSIONS

There are two important consequences of the analysis. First, it was unclear before what is the meaning of the parameter \( m \). Second, it follows from the last section that this approximation is very precise for certain range of coupling constants. In that regime, it is easily to see that \( m \) is in fact an approximation of the self-interacting field mass, i.e. the singular eigenvalue of the mass operator: \( \hat{M} = \sqrt{\hat{P}^2} \). About the \( \lambda \phi^4 + \sigma \phi^2 \) model, one can see that it is completely determined by only one parameter, the mean value of the field, which is experimentally measurable.

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VII. APPENDIX

We prove first that the Wick powers are orthogonally if and only if the measure is Gaussian
i.e.: \( \langle e^{a\phi} \rangle = \exp \left[ \frac{a^2}{2} \langle \phi \phi \rangle + a \langle \phi \rangle \right] \). This can be seen from:

\[
\frac{d}{da} \langle e^{a\phi} \rangle = \langle \phi e^{a\phi} \rangle = \langle \phi : e^{a\phi} \rangle + \langle \phi \rangle \langle e^{a\phi} \rangle = \langle \phi : e^{a\phi} \rangle + \langle \phi \rangle \langle e^{a\phi} \rangle
\]

Using the orthogonality of the Wick powers we can continue

\[
\frac{d}{da} \langle e^{a\phi} \rangle = a \langle \phi : \phi \rangle \langle e^{a\phi} \rangle + \langle \phi \rangle \langle e^{a\phi} \rangle
\]

(50)
equation which integrated out (together with the initial condition \( \langle e^{a\phi} \rangle|_{a=0} = 1 \)) leads to
the desired expression. The other implication follows from

\[
\langle \phi : \exp (a\phi) :: \exp (b\varphi) \rangle = \exp \left[ \frac{a^2}{2} \langle \phi : \phi \rangle + a \langle \phi \rangle \right] \exp \left[ \frac{b^2}{2} \langle \varphi : \varphi \rangle + b \langle \varphi \rangle \right]
\]

(52)
so, for \( \phi \) and \( \varphi \) jointly Gaussian variables (with mean different by zero):

\[
\langle \phi : \exp (a\phi) :: \exp (b\varphi) \rangle = \frac{\exp (a\phi + b\varphi)}{\langle \exp (a\phi) \rangle \langle \exp (b\varphi) \rangle} = \frac{\langle \exp (a\phi + b\varphi) \rangle}{\langle \exp (a\phi) \rangle \langle \exp (b\varphi) \rangle}
\]

(51)

Taking derivatives in respect to \( a \) and \( b \) and then the limit \( a, b \to 0 \), we can form any Wick
power of \( \phi \) and \( \varphi \). Because of the expression on the right side, it is obvious that the Wick
powers are orthogonally.

**Integration by parts** (first formulation)

\[
\int \phi (f) \langle \phi \rangle d\mu_{C,\xi} = \int f (x) C (x - y) \frac{\delta R}{\delta \phi (y)} dxdy\mu_{C,\xi}.
\]

(53)
It is enough to prove this formula for the case: \( R(\phi) = \exp[i\phi(g)] \). In this case:

\[
\int :\phi(f) :_{C,\xi} e^{i\phi(g)} d\mu_{C,\xi} = -i \frac{d}{d\lambda} \int :e^{i\lambda\phi(f)} :_{C,\xi} e^{i\phi(g)} d\mu_{C,\xi}
\]

\[
= -i \frac{d}{d\lambda} \int \frac{e^{i\phi(g+\lambda f)}}{\langle e^{i\phi(f)} \rangle} d\mu_{C,\xi} = -i \frac{d}{d\lambda} \frac{\exp[-\frac{1}{2}C(g+\lambda f, g+\lambda f) + i(\xi, g+\lambda f)]}{\exp[-\frac{1}{2}C(\lambda f, \lambda f) + i(\xi, \lambda f)]}
\]

\[
= iC(f, g) \exp[-\frac{1}{2}C(g, g) + i(\xi, g)] = i \int f(x) C(x-y) g(y) dxdy \langle e^{i\phi(g)} \rangle \quad (54)
\]

\[
= \int f(x) C(x-y) \frac{d}{d\phi(y)} \langle e^{i\phi(g)} \rangle dxdy \mu_{C,\xi},
\]

where \((\cdot, \cdot)\) is the scalar product in \( L_2(\mathbb{R}^2) \) and \( f \) and \( g \) were considered in \( L_1(\mathbb{R}^2) \). Next we show that: \( \delta : \phi(f)^n : \delta\phi(x) = : \delta\phi(f)^n / \delta\phi(x) : \), where the normal ordering is with respect to an arbitrary measure, which is helpful when one computes \( \delta R/\delta\phi \). Indeed:

\[
\frac{\delta}{\delta\phi(x)} :\phi(f)^n : = \frac{\delta}{\delta\phi(x)} \frac{d^n}{d\lambda^n} :e^{\lambda\phi(f)} : \bigg|_{\lambda=0} = \frac{\delta}{\delta\phi(x)} \frac{d^n}{d\lambda^n} \langle e^{\lambda\phi(f)} \rangle \bigg|_{\lambda=0}
\]

\[
= \frac{d^n}{d\lambda^n} \frac{\lambda f(x) e^{\lambda\phi(f)}}{\langle e^{\lambda\phi(f)} \rangle} \bigg|_{\lambda=0} = n f(x) \frac{d^{n-1}}{d\lambda^{n-1}} \frac{e^{\lambda\phi(f)}}{\langle e^{\lambda\phi(f)} \rangle} \bigg|_{\lambda=0} = n f(x) :\phi(f)^{n-1} : \quad (55)
\]

\[
= : \frac{\delta}{\delta\phi(x)} \phi(f)^n : .
\]

**Integration by parts (second formulation)**

\[
\int :\phi(f)^n :_{C,\xi} R(\phi) d\mu_{C,\xi} = \int f(x) C(x-y) :\phi(f)^{n-1} :_{C,\xi} \delta R/\delta\phi(y) dxdy \mu_{C,\xi}. \quad (56)
\]

We start with the identity:

\[
: \phi(f)^n :_{C,\xi} = : \phi(f) :_{C,\xi} :\phi(f)^{n-1} :_{C,\xi} - \int f(x) C(x-y) \frac{\delta\phi(f)^{n-1}}{\delta\phi(y)} :_{C,\xi} dxdy, \quad (57)
\]
which can be proven as follow:

\[ \phi(f)^n :_{C,\xi} = \frac{d^n}{d\lambda^n} :_{1,\xi} = \left. \frac{d^n}{d\lambda^n} \exp[\lambda \phi(f)] \right|_{\lambda=0} \cdot \exp[\lambda \phi(f)] \left|_{\lambda=0} \right. \]

\[ = \left. \frac{d^n}{d\lambda^n} \exp \left[ \lambda \phi(f) - \frac{1}{2} C(f, f) - \lambda (\xi, f) \right] \right|_{\lambda=0} \]

\[ = \left. \frac{d^{n-1}}{d\lambda^{n-1}} \left[ (\phi(f) - (\xi, f)) \frac{\exp[\lambda \phi(f)]}{\exp[\lambda \phi(f)]}_{C,\xi} - \lambda C(f, f) \frac{\exp[\lambda \phi(f)]}{\exp[\lambda \phi(f)]}_{C,\xi} \right] \right|_{\lambda=0} \]

\[ = \left. \phi(f) :_{C,\xi} (\phi(f)^{n-1} :_{C,\xi} - \frac{d^{n-1}}{d\lambda^{n-1}} \int f(x) C(x - y) \frac{\delta}{\delta \phi(y)} \exp[\lambda \phi(f)] \left|_{\lambda=0} \right. \right) dxdy \]

\[ = \left. \phi(f) :_{C,\xi} (\phi(f)^{n-1} :_{C,\xi} \int f(x) C(x - y) \frac{\delta}{\delta \phi(y)} \phi(f)^{n-1} :_{C,\xi} dxdy. \right) \]

Then we can continue:

\[ \int :_{C,\xi} R(\phi) d\mu_{C,\xi} = \int :_{C,\xi} (\phi(f)^{n-1} :_{C,\xi} R(\phi) d\mu_{C,\xi} \]

\[ - \int \int f(x) C(x - y) \frac{\delta}{\delta \phi(y)} :_{C,\xi} \phi(f)^{n-1} :_{C,\xi} R(\phi) \right) dxdyd\mu_{C,\xi} \]

\[ = \int \int f(x) C(x - y) \frac{\delta}{\delta \phi(y)} :_{C,\xi} R(\phi) \right) dxdyd\mu_{C,\xi} \]

\[ - \int \int f(x) C(x - y) \frac{\delta}{\delta \phi(y)} :_{C,\xi} R(\phi) \right) dxdyd\mu_{C,\xi}, \]

which leads to the desired expression.
REFERENCES

[1] Glimm J, Jaffe A, Spencer T: Ann. Phys. 101, 610 (1976)

[2] Imbrie J Z, Commun. Math. Phys. 82, 261 (1981)

[3] Stevenson P M, Phys. Rev. D 30, 1712 (1984); ibid 32, 1389 (1985); ibid 33, 2305 (1986);

[4] Ingermanson R, Nucl. Phys. B266, 620 (1986)

[5] Coleman S, Aspects of Symmetry, Ch. 6, Cambridge Univ. Press, (1994-1996)

[6] Glimm J, Jaffe A, Quantum Physics, Berlin, Heilderberg, New York: Springer (1981)

[7] Glimm J, Jaffe A, Spencer T: Existence of Phase Transitions for QF, Marseille Conference (1975)

[8] Simon B: The $P(\phi)_2$ Euclidean (Quantum) Field Theory, Princeton: Princeton University Press (1974)

[9] Fröhlich J, in Invariant Wave Equation, Proc. of Eltore Majorana (1978)