Gravitational dynamics of large stellar systems

Stephen L W McMillan
Department of Physics, Drexel University, Philadelphia, PA 19104, USA
E-mail: steve@physics.drexel.edu

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Abstract
Internal dynamical evolution can drive stellar systems into states of high central
density. For many star clusters and galactic nuclei, the time scale on which
this occurs is significantly less than the age of the universe. As a result, such
systems are expected to be sites of frequent interactions among stars, binary
systems and stellar remnants, making them efficient factories for the production
of compact binaries, intermediate-mass black holes and other interesting and
eminently observable astrophysical exotica. We describe some elements of the
competition among stellar dynamics, stellar evolution and other mechanisms
to control the dynamics of stellar systems, and discuss briefly the techniques
by which these systems are modeled and studied. Particular emphasis is placed
on pathways leading to massive black holes in present-day globular clusters
and other potentially detectable sources of gravitational radiation.

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(Some figures in this article are in colour only in the electronic version)

1. Dynamical evolution in a nutshell

1.1. Evolutionary time scales
The dynamics of a self-gravitating system is defined by two fundamental time scales. The
dynamical time, $t_d$, is the time required for a typical star to cross the system. It is also the
time scale on which the system establishes virial equilibrium, $2T + U = 0$, where $T$ and $U$ are
the kinetic and gravitational potential energies of the system, respectively. A convenient formal
definition in terms of conserved quantities is

$$t_d = \frac{GM^{5/2}}{(-4E)^{3/2}}.$$  (1)
where $M$ is the total mass of the system and $E = T + U$ is the total energy. In virial equilibrium, this expression assumes the more familiar form

$$t_d = \left(\frac{GM}{R^3}\right)^{-1/2} = 4.7 \times 10^5 \text{ yr} \left(\frac{M}{10^3 M_{\odot}}\right)^{-1/2} \left(\frac{R}{1 \text{ pc}}\right)^{3/2}, \quad (2)$$

where $R = -GM^2/2U$ is the virial radius and we have added for reference some astrophysically relevant scales. In this paper, we confine our attention to systems in virial equilibrium (neglecting such interesting cases as merging star clusters or colliding galaxies) and take equation (2) as our working definition of the dynamical time scale.

The second fundamental time scale is the relaxation time, $t_r$, on which two-body encounters transfer energy between individual stars and cause the system to approach thermal equilibrium. Spitzer (1987) derives the local expression

$$t_r = 0.03 \langle v^2 \rangle^{3/2} G^2 \langle m \rangle \rho \log \Lambda, \quad (3)$$

where $\langle v^2 \rangle$ is the velocity dispersion, $\langle m \rangle$ is the mean stellar mass, $\rho$ is the density, $N = M / \langle m \rangle$ and $\Lambda \sim 0.4N$ for a system in virial equilibrium. Replacing all quantities by their cluster-wide ('half-mass') averages, we obtain the half-mass relaxation time

$$t_{rh} = 0.08 M^{1/2} R^{3/2} G^{-1/2} \langle m \rangle \log \Lambda \approx \frac{N}{12 \log \Lambda} t_d. \quad (4)$$

A natural distinction can be drawn between ‘collisionless’ systems, whose long relaxation times mean that they will not undergo significant internal dynamical evolution during the age of the universe, and ‘collisional’ systems, which evolve significantly in less than a Hubble time. Galaxies fall into the former category—a typical relaxation time in a modest galactic spheroid, with mass $10^{11} M_{\odot}$ and radius 10 kpc, is $\sim 3 \times 10^{16}$ yr. Star clusters and some galactic nuclei fall into the latter category—a globular cluster, for example, with $M \sim 10^6 M_{\odot}$ and $R \sim 10$ pc, has $t_r \sim 10^{10}$ yr. The median relaxation time of the Galactic globular cluster system is $\sim 1.2$ Gyr (Harris 1996). We further limit our discussion here to collisional systems, in which substantial internal evolution is expected on time scales less than the age of the universe.

1.2. Evolution of collisional stellar systems

Self-gravitating systems are inherently unstable, and no final equilibrium state exists for a star cluster. The escape of high-velocity stars (evaporation) and the internal effects of two-body relaxation, which transfers energy from the inner to the outer regions of the cluster, precipitate the phenomenon known as core collapse (Antonov 1962, Cohn 1980). During this phase, the central portions of the cluster accelerate toward infinite density while the outer regions expand. The process is readily understood by recognizing that, according to the virial theorem, a self-gravitating system has negative specific heat—reducing its energy causes it to heat up. Hence, as relaxation transports energy from the (dynamically) warmer central core to the cooler outer regions, the core contracts and heats up as it loses energy. The time scale for the process to go to completion (i.e. a core of zero size and formally infinite density) is $t_{cc} \sim 15 t_{rh}$ for a system of identical masses and significantly less in the case of a broad spectrum of masses (Inagaki 1985, Portegies Zwart and McMillan 2002). Figure 1 compares the appearance of pre- and post-collapse systems.

In systems with a mass spectrum, two-body interactions accelerate the dynamical evolution by driving the system toward energy equipartition, in which the velocity dispersions of stars of different masses would have $m \langle v^2 \rangle \sim$ constant. The result is mass segregation,
where more massive stars slow down and sink toward the center of the cluster on a time scale (Spitzer 1969):

\[ t_s \sim \frac{\langle m \rangle}{m} t_{\text{rh}}. \]  

(5)

Thus, a collisional stellar system inevitably evolves toward a state in which the most massive objects become concentrated in the high-density central core and this process can occur in less than a Hubble time. From an astrophysical point of view, these ‘most massive objects’ are the most interesting objects in the system—black holes, neutron stars, massive white dwarfs and binary star systems. Dynamical evolution provides a natural and effective mechanism for concentrating astrophysically interesting objects in regions of high stellar density, thereby promoting interactions among them.

These interactions not only create potentially observable objects, such as x-ray binaries, millisecond pulsars and (perhaps) sources of gravitational radiation, but in many cases can also play important roles in the dynamics of their parent system. Binary stars, for example, act as heat sources to the stellar system through superelastic encounters with other cluster members (Heggie 1975), ultimately halting and even reversing core collapse (McMillan et al 1991, McMillan and Hut 1994). At the same time, they greatly increase the effective cross sections for close encounters and collisions between stars and remnants. The pursuit of a full understanding of the dynamical evolution of collisional stellar systems therefore drives us to study not just the gravitational dynamics of the stars themselves, but also the physics of stellar interactions and collisions, and the processes driving the stellar and binary evolution of the resulting objects.

2. Modeling dense stellar systems

The past decade has seen groundbreaking advances in both hardware design and software development and integration, all of which have been crucial in advancing our understanding of this field. Here we describe a few of the computational techniques currently in use.

2.1. The state of the art

Consistent with our growing understanding of the importance of stellar and binary interactions in collisional stellar systems, the leading programs in this area are the
various ‘kitchen sink’ packages that combine treatments of dynamics, stellar and binary evolution, and stellar hydrodynamics within a single simulation. Of these, the most widely used are the N-body codes \texttt{NBODY} (Aarseth 2003) and \texttt{kira} (e.g. Portegies Zwart \textit{et al} (2001)) and the Monte Carlo codes developed by Giersz (1998; also see Giersz and Heggie (2008)), Freitag \textit{et al} (2006) and Rasio and coworkers (e.g. Fregeau \textit{et al} (2003)).

These codes differ principally in their handling of the large-scale dynamics, employing conceptually similar approaches to stellar and binary evolution and collisions. All use approximate descriptions of stellar evolution, generally derived from look-up tables based on the detailed evolutionary models of Eggleton \textit{et al} (1989) or Hurley \textit{et al} (2000). They also rely on semi-analytic or heuristic rule-based treatments of binary evolution, conceptually similar from code to code, but significantly different in detail. In most cases, collisions are implemented in the simple ‘sticky-sphere’ approximation, where stars are taken to collide (and merge) if they approach within the sum of their effective radii. The effective radii may be calibrated using hydrodynamical simulations, and mass loss may be included in some approximate way. Freitag’s Monte Carlo code uses a more sophisticated approach, interpolating encounter outcomes from a pre-computed grid of SPH simulations (Freitag and Benz 2005). Small-scale dynamics of multiple stellar encounters, such as binary and higher-order encounters, may be handled by look-up from precomputed cross sections or—more commonly—by direct integration, either in isolation or as part of a larger N-body calculation.

Codes employing direct integration may also include post-Newtonian terms in the interactions between compact objects.

N-body codes incorporate detailed descriptions of stellar dynamics at all levels, using direct integration of the individual (Newtonian) stellar equations of motion for all stars. Monte Carlo methods are designed for efficient computation of relaxation effects in collisional stellar systems, a task which they accomplish by reducing stellar orbits to their orbital elements—energy and angular momentum—effectively orbit averaging the motion of each star. Relaxation is modeled by randomly selecting pairs of stars and applying interactions between them in such a way that, on average, the correct rate is obtained. This may be implemented in a number of ways, but interactions are generally realized on time scales comparable to the orbit-averaged relaxation time. As a result, Monte Carlo schemes can be orders of magnitude faster than direct N-body codes. Joshi \textit{et al} (2000) report an empirical CPU time scaling of $O(N^{1.4})$ for core-collapse problems (compared with $N^3$ for N-body methods, as discussed below). To achieve these speeds, however, the geometry of the system must be simple enough that the orbital integrals can be computed from a star’s instantaneous energy and angular momentum. In practice, this limits the approach to spherically symmetric systems in virial equilibrium and global dynamical processes occurring on relaxation (or longer) time scales.

The major attraction of N-body methods is that they are assumption-free, in the sense that all stellar interactions are automatically included in all orders, without the need for any simplifying approximations or the inclusion of additional reaction rates to model particular physical processes of interest. Thus, problems inherent to Monte Carlo methods, related to departures from virial equilibrium, spherical symmetry, statistical fluctuations, the form of (and indeed the existence of) phase-space distribution functions and the possibility of interactions not explicitly coded in advance, simply do not arise. The price of this is computational expense. Each of the $N$ particles must interact with every other particle a few hundred times over the course of every orbit, each interaction requires $O(N)$ force calculations and a typical (relaxation time) run spans $O(N)$ orbits (see equation (4)). The resulting $O(N^3)$ scaling of the total CPU time means that, even with the best time-step algorithms, integrating even a fairly
small system of, say, $N \sim 10^5$ stars requires sustained teraflop speeds for several months (Hut et al. 1988).

The technological solution now in widespread use is the ‘GRAPE’ (short for ‘GRAvity PipE’) series of machines developed by Sugimoto and co-workers at Tokyo University (Ebisuzaki et al. 1993). Abandoning algorithmic sophistication in favor of simplicity and raw computing power, these machines achieve high speed by mating a fourth-order Hermite integration scheme (Makino and Aarseth 1992) with special-purpose hardware in the form of highly parallel, pipelined ‘Newtonian force accelerators’ which implement the computation of all interparticle forces entirely in hardware. Operationally, the hardware is simple to program, as it merely replaces the function that computes the force on a particle by a call to the hardware interface libraries; the remainder of the user’s $N$-body code is unchanged. The effect of GRAPE on simulations of stellar systems has been nothing short of revolutionary. Today, GRAPEs lie at the heart of almost all detailed $N$-body simulations of star clusters and dense stellar systems.

Recently, graphics processing units (GPUs) have begun to achieve speeds and price/performance levels previously attainable only by GRAPE systems (see Belleman et al. (2008) for a recent GPU implementation of the GRAPE interface). It appears that commodity components may be poised to outpace special-purpose computers in this specialized area of computational science, just as they have already done in general-purpose computing.

2.2. The MODEST initiative

The realization that the evolution of dense stellar systems results in interactions not only among stars, but also among stellar modelers and their programs, has been a major motivating factor in the ‘MODEST’ program (http://www.manybody.org/modest). Short for MOdeling DEnse STellar systems, MODEST is a loosely knit collection of groups working on all aspects of the theory and observations of star clusters, including stellar dynamics, stellar evolution, stellar hydrodynamics and cluster formation. MODEST has hosted some 20 meetings over the past 5 years, providing an invaluable forum for discussion and collaboration among researchers in this field.

A key goal of MODEST is to provide a software framework for large-scale simulations of dense stellar systems, within which existing programs for dynamics, stellar evolution and hydrodynamics can be easily coupled. An important aspect of this is the incorporation of ‘live’ treatments of stellar and binary evolution and ultimately stellar hydrodynamics directly into kitchen-sink $N$-body simulations. Such an undertaking is essential if one wishes to model the evolution of a dense stellar system, in which stellar collisions may be commonplace events, creating wholly new channels for stars to evolve and allowing the formation of stellar species completely inaccessible by standard stellar and binary evolutionary pathways.

A pioneering effort to couple stellar dynamics and stellar evolution was reported by Church (2006), who combined Aarseth’s NBODY6 with a version of Eggleton’s EV (Eggleton 2006). This heroic accomplishment hard coded the two programs into a single application, providing proof of concept that two such disparate modules could in fact be successfully merged. The MODEST approach to code integration, called ‘MUSE’, for MUltiscale MUlitiphysics Scientific Environment (http://muse.11), adopts a rather different approach, providing instead a modular python framework within which programs written by many different authors, and in many different languages, can interoperate with as little intrusion as possible into the internal operation of each (Portegies Zwart et al. 2008).

This model has many advantages. By providing well-defined interfaces between modules, it allows researchers (or students) to quickly build real scientific applications using state-of-
the-art techniques, without first having to become experts in the many details of each module. Equally important, the MUSE structure allows users to write generic scripts that do not depend on the details of the algorithms used, providing, for the first time, ‘plug and play’ functionality that allows different combinations of modules to be combined and compared.

3. Black holes in star clusters

Rather than attempting to describe the many astrophysical consequences of collisional stellar dynamics, we turn to a specific problem of particular interest to the relativity community, the possible presence of black holes in these systems.

Black holes (BHs) are natural products of stellar evolution in massive stars, and may also result from dynamical interactions in dense stellar environments. They can significantly influence the dynamics of their parent cluster, and may also have important observational consequences via their x-ray emission, the production of gravitational waves and their effect on the structural properties of the system in which they reside. Globular clusters offer particularly rich environments for the production of black holes in statistically significant numbers, yet direct evidence for black holes in globulars is scarce, although several independent lines of investigation now hint at their presence. Despite the lack of firm observational support, the past three decades have seen many theoretical studies of the formation and dynamics of stellar- and intermediate-mass black holes in star clusters.

3.1. Observations of stellar-mass black holes

Given the numerous theoretical studies of the formation and dynamical consequences of BHs in star clusters, and the overwhelming weight of opinion on the inevitability of BHs as consequences of stellar evolution, there is remarkably little observational evidence for stellar-mass (i.e. less than a few tens of solar masses) BHs in globular clusters. There appears to be no evidence for such low-mass BHs in the Galactic globular cluster population and only one firm observation of a stellar-mass BH in an extra-Galactic globular cluster.

The sole extra-Galactic BH was reported by Maccarone et al (2007), who observed a bright x-ray source in a cluster in NGC 4472, a bright elliptical galaxy in the Virgo cluster. Its x-ray luminosity of $\sim 4 \times 10^{39}$ erg s$^{-1}$ is the Eddington luminosity for a $35M_\odot$ object, and the source shows substantial variability on a time scale of hours, perhaps indicating a considerably larger mass, so even this candidate may actually fall into the intermediate-mass black hole (IMBH) range. Interestingly, the parent cluster is itself quite bright ($V = 21$, or $L \sim 7.5 \times 10^5 L_\odot$ at a distance of 16 Mpc) and lies far (30 kpc) from the center of the host galaxy, perhaps placing this system in the same category as the leading candidates for IMBHs in the Milky Way (ω Centauri: $L \approx 1.1 \times 10^6 L_\odot$; Harris (1996)) and M31 (G1: $L \sim 2.1 \times 10^6 L_\odot$; Meylan et al (2001)), as discussed in section 3.2.

3.2. Intermediate-mass black holes

Although there is considerable theoretical uncertainty about how such objects might form, the observational evidence for IMBHs may actually be stronger than that for stellar-mass BHs. Several lines of reasoning support the assertion that IMBHs exist in globular clusters: (1) observations of ultraluminous x-ray sources (ULXs), (2) dynamical modeling and (3) studies of a cluster structure.
3.2.1. Ultraluminous x-ray sources. The bright x-ray source M82 X-1 (Matsumoto and Tsuru 1999, Matsumoto et al 2001, Kaaret et al 2001) is the strongest ULX candidate for an IMBH. With a peak luminosity of more than $10^{41}$ erg s$^{-1}$, it is too bright to be an ordinary x-ray binary, while its location 200 pc from the center of M82 argues against a supermassive black hole. This luminosity is consistent with an accreting compact object of at least 350 solar masses, possibly an IMBH. The discovery of $54.4 \pm 0.9$ mHz quasi-periodic oscillations (Strohmayer and Mushotsky 2003) supports this view.

An intriguing aspect of M82 X-1 is its apparent association with the young dense cluster MGG-11 (McCrady et al 2003). Figure 2 (from Portegies Zwart et al 2004) shows superimposed near-IR (HST) and x-ray (Chandra) images of the region containing M82 X-1 and MGG-11. (The offset between the infrared cluster and the x-ray source is consistent with the absolute pointing accuracies of the two telescopes.) The cluster age is between 7 and 12 Myr. Portegies Zwart et al 2004) found that such an association would be consistent with the scenario described in section 4 for runaway stellar growth in a dense cluster.

Numerous authors have noted that high x-ray luminosity is by no means conclusive evidence of an IMBH (see Miller and Colbert (2004) for a review of some alternative possibilities). Soria (2006, 2007) points out that most ULXs may in fact be consistent with the high-luminosity tail of the x-ray binary luminosity function and that ULXs are generally not associated with young star clusters. Nevertheless, the runaway collision scenario has become the ‘standard’ mechanism for IMBH formation in clusters, against which others are assessed.
3.2.2. Dynamical modeling of the cluster velocity structure. Currently, the most definitive statements about IMBHs in globular clusters have come from Gebhardt and collaborators (Gerssen et al 2002, 2003, Noyola et al 2006, Noyola 2008), based on axisymmetric, three-integral dynamical models of cluster potentials, using full line-of-sight velocity information to constrain the model fits. IMBH masses have been reported for the Galactic globular clusters M15 and ω Centauri (figure 1), and for the cluster G1 in M31, although these results have not been without controversy.

Gerssen et al (2002) reported dynamical evidence for a $4 \pm 2 \times 10^3 M_\odot$ IMBH in M15. This result was criticized by Baumgardt et al (2003a), who pointed out that a ‘standard’ dynamically evolved cluster model would yield similar results, and was marred by a crucially mislabeled figure in an earlier study of M15’s dynamics. Subsequently, the estimate of the IMBH mass was reduced to $2 \pm 2 \times 10^3 M_\odot$ (Gerssen et al 2003). It now seems clear, from a variety of different arguments, that the core of this highly centrally concentrated cluster does contain on the order of $1000 M_\odot$ of non-luminous matter (Dull et al 1997, Baumgardt et al 2003a, Phinney 1993). However, given the age and likely history of M15, the dark material most plausibly consists of stellar remnants—neutron stars and/or heavy white dwarfs—providing a much more natural explanation of the invisible mass. The models do not rule out an IMBH at the $\sim 500$–$1000 M_\odot$ level, but present no compelling reason to conclude that one exists.

Much firmer evidence for IMBHs is found in G1 and ω Centauri, the largest globular clusters in M31 and the Milky Way, respectively. Both clusters are large enough that their relaxation times are long, making it unlikely that the dynamics of stellar evolution products could mimic the effect of a central IMBH, as in M15.

In G1, a cluster with mass $\sim 1.5 \times 10^7 M_\odot$, situated some 40 kpc from the center of M31 (Meylan et al 2001), the dynamical models yield a black hole mass of $1.8 \pm 0.5 \times 10^4 M_\odot$ (Gebhardt et al 2002, 2005). The initial results were also questioned by Baumgardt et al (2003b), in part not only because the $0.013''$-radius sphere of influence of the supposed IMBH lies well inside the central pixel of the HST image, but also because plausible N-body models without black holes could reproduce the cluster’s observed surface density and velocity dispersion profiles quite well. However, Gebhardt et al (2005) have argued that the use of the full velocity distribution, and not just its lowest moments, is essential in order to properly define the cluster potential and that the N-body simulations simply lack the resolution necessary for them to be meaningfully compared with the observational data.

Support for the Gebhardt et al result has come from the recent detection of x-ray emission from G1 (Pooley and Rappaport 2006). The observed luminosity $L_X \sim 2 \times 10^{36}$ erg s$^{-1}$ is consistent with the Bondi–Hoyle accretion of intracluster gas onto an IMBH in the relevant mass range. Further support comes from radio observations of G1 (Ulvestad et al 2007), which imply a radio to x-ray flux ratio consistent with a $2 \times 10^4 M_\odot$ black hole (Merloni et al 2003).

Noyola et al (2006) have reported an IMBH mass of $4 \pm 1 \times 10^4 M_\odot$ in ω Centauri, a globular cluster with mass $\sim 5 \times 10^6 M_\odot$, lying $\sim 6$ kpc from the center of the Milky Way. Interestingly, both G1 and ω Centauri lie near the low-mass extension of the ‘$M$–$\sigma$’ relation for active galactic nuclei (Merritt and Ferrarese 2001, Tremaine et al 2002), reinforcing the suspicion that G1 and ω Centauri are in fact not ‘real’ globular clusters, but rather are the cores of dwarf spheroidal systems stripped by the tidal fields of M31 and the Milky Way (Meylan et al 2001, Freeman 1993). This possibility does not explain the origin of the IMBHs, but it obviously moves the question into a very different arena.

3.2.3. Indirect evidence for central black holes. Noyola and Gebhardt (2006) find that a surprisingly large fraction (≈25%) of globular clusters hitherto thought to have ‘classical’
cores in fact have shallow power-law surface brightness profiles in their central regions. The
detection of these weak cusps is due in large part to high-resolution HST observations of the
innermost arcsecond of these systems. These observations remain somewhat controversial,
but are in good agreement with theoretical simulations by Baumgardt et al. (2005) of clusters
containing central black holes comprising ~0.1–1% of the total cluster mass, and have been
interpreted as indirect indicators of IMBHs in these clusters. Ongoing detailed dynamical
studies by Noyola et al. (2008) of selected clusters from their earlier study, including NGC
2808, 47 Tucanae and the ‘weak cusp’ systems M54 and M80, should shed much further light
on the dynamics of these intriguing systems.

Curiously, although dense stellar systems are among the most promising environments
for the formation of IMBHs (see section 4), they may not be the best place to look for evidence
of massive black holes. Baumgardt et al. (2005) and Heggie et al. (2007) have pointed out
that core-collapse clusters such as M15 are probably least likely to harbor IMBHs. Rather,
dynamical heating by even a modest IMBH is likely to lead to a cluster containing a fairly
extended core. Comparing the outward energy flux from stars relaxing inward in the Bahcall–
Wolf (1976) cusp surrounding the IMBH (of mass $M_{\text{BH}}$) to the outward flux implied by
two-body relaxation at the cluster half-mass radius, Heggie et al estimate the equilibrium ratio
of the half-mass ($R_h$) to the core ($R_c$) radius. Calibrating to simulations, they conclude that

\[ \frac{R_h}{R_c} \sim 0.23 \left( \frac{M}{M_{\text{BH}}} \right)^{3/4}. \]  

(6)

Trenti (2008) has suggested that the imprint of this process can be seen in his ‘isolated and
relaxed’ sample of Galactic clusters having relaxation times less than 1 Gyr, a half-mass to
tidal radius ratio $R_h/R_t < 0.1$ and an orbital ellipticity of less than 0.1. Roughly half of the
clusters in this sample have core radii substantially larger than would be expected on the basis
of simple stellar dynamics and binary heating. However, Hurley (2007) has argued out that
such anomalously large core to half-mass ratios may also be explained by the presence of
stellar-mass BH binaries heating the cores of these clusters (also see Mackey et al. (2007)).

4. Formation of intermediate-mass black holes

Accepting without further debate the still sketchy observational evidence for IMBHs in
globular clusters, we now turn to the question of how such black holes might have formed.
The leading possibilities are that (1) they are primordial, the result of stellar evolution in
supermassive population III stars; (2) they are the result of runaway stellar collisions in young
dense star clusters and (3) they are the result of mergers of BHs over the lifetime of a cluster.
We focus here on the latter two possibilities, since the formation mechanisms involved are
arguably much more relevant to the physics of dense stellar environments.

The likelihood of multiple stellar collisions in dense stellar systems was first demonstrated
in N-body simulations of dense clusters (Portegies Zwart et al. 1999, 2004, Portegies Zwart
and McMillan 2002, McMillan and Portegies Zwart 2004) and later also in Monte Carlo
models (Freitag et al. 2006). Hydrodynamic simulations indicate that, when massive stars
collide in clusters, they are very likely to merge. The question then becomes ‘What next?’
Unfortunately, while the dynamical processes leading to collision runaways are simple and
well known, there are several prominent ‘missing links’ in the chain of reasoning starting from
a young stellar system and ending with a massive black hole, all involving key aspects of the
physics of massive stars.
4.1. Mass segregation and runaway mergers

The dynamics of runaway mergers is straightforward. Massive stars sink to the cluster core on a time scale \( t_s \) (equation (5)). The result, for a realistic (Kroupa 2001) stellar-mass spectrum, is the formation of a dense sub-core of massive stars in a time \( t_{cc} \sim 0.2 t_s \). Here, \( t_s \) can be the half-mass relaxation time for a small system, or the core relaxation time for a larger one, as discussed below. High densities in the core lead to collisions and mergers, which naturally involve the most massive stars, and the process runs away, with one collision product growing rapidly in mass and radius and outstripping the competition. The resultant merger (and, we assume, black hole) mass is in the IMBH range. For a runaway to occur, the collision process must complete before the first supernovae occur, that is, within 3–5 Myr. This requires short relaxation times, or, equivalently, high cluster densities.

Figure 3 shows a typical set of results, obtained by Portegies Zwart et al. (2004) in their \( N \)-body study of the M82 clusters discussed earlier. The different runs represent a broad range of initial conditions, with and without initial binaries and with both Salpeter (1955) and Kroupa (2001) mass functions, for systems of 128k–585k stars. The runaway masses in \( \text{NBODY4} \) are generally larger than those in \( \text{Starlab} \), since \( \text{NBODY4} \) adopts systematically larger stellar radii and hence collision cross sections, for the most massive stars. Comparable results (in both time scale and total runaway mass) have been obtained in Monte Carlo simulations (e.g. Freitag et al. 2006).

According to the dynamical models, a runaway always occurs if a central core of massive stars (\( m \gtrsim 20M_\odot \)) can form by mass segregation before the first supernova occurs. In small systems, having \( t_{th} \lesssim 25 \) Myr, essentially all the massive stars can reach the center in the time available. The merger mass fraction, for a typical (Kroupa 2001) initial mass function, is \( \sim 0.01 \) (Portegies Zwart and McMillan 2002). In large systems, only a fraction of massive stars can reach the center before exploding as supernovae. Freitag et al. (2006) report a somewhat smaller mass fraction for the IMBH in their Monte Carlo models. McMillan and Portegies Zwart (2004) find that, for \( t_{th} \gtrsim 25 \), the merger fraction is expected to scale as \( t_{th}^{-1/2} \).
4.2. Evolution of the merger product

Freitag et al (2006) find that direct mass loss due to the collision itself is generally unimportant, amounting to less than ~10% of the total mass in most cases. However, the problem of determining the evolution of a possibly rapidly rotating collision product, from its initial non-equilibrium state back to the anomalous main sequence and beyond, presents significant computational challenges (e.g. Sills et al 2003).

Perhaps the greatest uncertainty in the runaway process has to do with mass loss from supermassive stars. Massive main sequence stars are well known to have very strong winds, and these compete with collisions in setting the mass of the final object (e.g. Vanbeveren et al 2007, Belkus et al 2007). Portegies Zwart et al (1999) recognized that the specifics of the mass loss prescription completely control the outcome of the collision runaway. Applying most of the mass loss late, at the terminal-age main sequence, permits the process to build massive stars, and this prescription has (unfortunately) been adopted in most dynamical models to date. A more realistic treatment of mass loss (based on Langer et al (1994)) can significantly reduce the mass of the final collision product, while (probably unrealistic) early mass loss can shut down the process completely.

By way of calibration, we note that dynamical models typically predict the accretion of ~10^3 M⊙ of material in ~10^6 yr, for a net accretion rate of ~10^{-3} M⊙/yr. This is comparable to the mass loss rates reported for several massive stars, but substantially less than the largest rates known, e.g. ~0.1 M⊙/yr inferred during the decade around the ‘great outburst’ of Eta Carinae in 1843 (Morris et al 1999). Belkus et al (2007) find that massive (300–1000 M⊙) solar-metallicity stars end their lives as relatively low-mass (40–50 M⊙) black holes, but comparably massive stars in low-metallicity clusters may give rise to IMBHs in the 150–200 M⊙ range. Yungelson et al (2008) find a final black hole mass of ~150 M⊙ for a star with the initial mass 1000 M⊙.

These studies suggest that it may be difficult to retain enough mass to form an IMBH, although recently, Suzuki et al (2007) have performed simulations of collisionally merged stars, using a more realistic mass-loss model than in most earlier dynamical studies, and find that, because of the extended envelopes of the merged systems, most collisions are expected to occur early, during the Kelvin–Helmholtz contraction phase, before mass loss by stellar winds become significant. They conclude that stellar-mass loss does not prevent the formation of massive stars with masses up to ~1000 M⊙. However, this in turn contrasts sharply with the work of Glebbeek (2008; also see Gaburov et al (2008), Glebbeek et al (2008)), who follows the evolution of a series of collision products drawn from N-body simulations, computing in detail the return to the main sequence and the subsequent evolution of each merged star. He concludes that, at least in stars of solar or near-solar metallicity, strong line-driven winds will expel much of the accumulated envelope, resulting in a Wolf–Rayet star of mass considerably less than 100 M⊙ by the time a supernova occurs. Obviously, the final word on this important but poorly understood process has yet to be written.

Finally, even if mass loss were not a factor, we remind the reader that the assumption that a 1000 M⊙ star will ultimately form a 1000 M⊙ black hole is also largely a matter of conjecture (see Heger (2008)).

4.3. Connection with globular clusters

Consider a compact, centrally concentrated young star cluster, with a central density high enough for the runaway collision scenario to operate. For a ~10^6 M⊙ system, the constraint on the relaxation time presented in section 4.1 requires a mean density of ~5 × 10^5 M⊙ pc^{-3}, for a
half-mass radius of $\sim 0.2$ pc. One might reasonably wonder what such a highly concentrated, dense system has to do with the relatively low-density globular clusters seen today. Can runaway collisional processes shortly after formation account for the IMBHs that may exist in the Galactic globular cluster system? Based on simulations of many aspects of the problem, we can construct a plausible scenario in which collisionally formed IMBHs might now reside in globular clusters.

After the IMBH has formed, numerous evolutionary processes combine both to expand the cluster half-mass radius $R_h$ and to reduce its central concentration, as measured by the ratio $R_h/R_c$ (equation (6)). If the cluster is not mass segregated at birth, the dynamical effects of segregation by massive stars and their remnants will cause the concentration to decrease significantly, and will also result in a modest overall expansion of the cluster (Merritt et al. 2004, Mackey et al. 2007). At the same time, mass loss due to stellar evolution drives further overall expansion by a factor of 2–3 (Takahashi and Portegies Zwart 2000, Mackey et al. 2007). Vesperini et al. (2008) find that mass loss from the cores of initially mass-segregated clusters is particularly effective in reducing the central concentration. Finally, heating due to the central IMBH (section 3.2.3) again reduces the concentration and causes significant expansion of $R_h$ over the lifetime of the cluster; the half-mass radii of the model clusters considered by Baumgardt et al. (2005) expanded by factors of 5–7.

Taken together, these figures suggest that the collision runaway scenario could indeed account for an IMBH in a system the size of a typical globular cluster—if one is ever confirmed!

4.4. Systems of black holes

We now briefly consider the opposite extreme, in which the relaxation time is long enough such that the massive stars form stellar-mass black holes ‘in place’, before significant dynamical evolution can occur.

The evolution of black hole systems in low-density clusters has been studied by a number of authors. Kulkarni et al. (1993) and Sigurdsson and Hernquist (1993) considered the dynamics of a population of black holes in an idealized star cluster. They found that mass segregation rapidly transports the black holes to the cluster core, where the black hole subsystem evolves rapidly, forming binaries that ultimately eject most of the black holes from the cluster. Both papers concluded that few observable stellar-mass black holes would be expected in the Galactic globular cluster system today. Subsequently, Portegies Zwart and McMillan (2000) performed $N$-body simulations, and generally verified these findings, but also concluded that the ejected black hole binaries could be significant sources of gravitational waves (GWs) in the LIGO band.

These studies were idealized in several important ways. They were not dynamically self-consistent (the $N$-body simulations combined many small-$N$ simulations to achieve better statistics), did not include a spectrum of black hole masses and did not include post-Newtonian relativistic effects (all used the Peters and Mathews (1963) formula for GW energy losses). Miller and Hamilton (2002) pointed out that a sufficiently massive black hole, at the top end of the stellar black hole mass spectrum, or perhaps the result of a ‘failed’ runaway merger, could survive ejection by dynamical interactions, and proposed a mechanism whereby black holes in binary systems would merge by the emission of GW to successively form more massive objects, ultimately resulting in an IMBH. However, it now seems unlikely that this process can operate efficiently in the face of the large GW recoil velocities found in recent numerical simulations of black hole mergers (see Centrella (2008) for a review).

O’Leary et al. (2006) carried out Monte Carlo simulations of a population of black holes (roughly 50% binaries) drawn from a population synthesis calculation starting from realistic
distributions of stellar and binary properties. They included approximate post-Newtonian effects in their model and found that, although modest IMBHs ($\sim 100 M_\odot$, up to a maximum of $\sim 600 M_\odot$) did form, in most cases the recoil speed substantially exceeded the escape speed from the cluster and that the most likely outcome was evaporation of the entire black hole subsystem. More recent studies, incorporating semi-analytic treatments of GW recoil calibrated to numerical relativity calculations, greatly strengthen this latter result (Amaro-Seoane et al 2008). The recoil speeds in these simulations are substantially higher than are obtained in the post-Newtonian (up to 3.5PN) approximation, with the result that virtually none of the BH merger products are expected to remain bound to the cluster. This finding effectively closed the BH merger avenue for IMBH formation and growth, unless an already very massive ($>100 M_\odot$) seed BH can form by some other means.

Although this scenario does not appear to offer a viable channel for IMBH production, the many BH binary mergers present potentially rich targets for the enhanced and advanced LIGO detectors. O’Leary et al (2007) infer an advanced LIGO detection rate of up to hundreds of events per year, broadly consistent with the range given previously by Portegies Zwart and McMillan (2000). However, these estimates are plagued by large uncertainties, due principally to the spread in structural properties of the parent clusters, amplified by the scaling of the GW emission process. Specifically, using the distributions of BH binary energy and eccentricity reported by Portegies Zwart and McMillan (2000) and the Peters and Matthews (1963) radiation formula, it is readily shown that the characteristic time scale at which the distribution of GW merger times peaks is proportional to $m_{bh}^5 (M/R)^{-4}$, where $m_{bh}$ is a typical stellar BH mass and $M$ and $R$ are, respectively, the mass and radius of the cluster. The uncertainty is compounded when these numbers are scaled up from individual cluster simulations to integrated populations of clusters across the universe. Nevertheless, the very fact that the current-best estimates predict many detections by upcoming ground-based instruments means that, whether or not the predicted events are actually seen, the observations should place strong constraints on the cluster formation history of the universe.

Recently, Mackey et al (2007) have reported a set of fully self-consistent (Newtonian) dynamical simulations of young dense clusters. They find that the combination of stellar-mass loss and dynamical heating has a significant dynamical impact on the cluster, causing the core (and, to a lesser extent, the cluster as a whole) to expand in a manner strikingly similar to the core radius–age relation observed in the young clusters in the LMC (Mackey and Gilmore 2003). However, because of the expansion, the black hole interaction rate drops sharply at late times, and a substantial population of black holes remains after 10 Gyr. If, as seems plausible, this general dynamical result scales to globular-cluster-sized systems, it suggests that the absence of evidence of stellar-mass black holes is not evidence of absence, but simply reflects their low current interaction rate with other cluster members.

5. Discussion

There is growing, but arguably still inconclusive, evidence that some globular star clusters may harbor massive black holes in their cores. Perhaps the strongest cases come from dynamical studies of G1 in M31 and $\omega$ Centauri in the Milky Way, but even here the conclusions are tainted to some degree by the suspicion that these massive clusters may actually be the stripped cores of dwarf spheroidal galaxies, in which case they shed little light on the physics of young star clusters or the collisional processes that may give rise to IMBHs in stellar systems. Indirect evidence based on cluster structural parameters and central density and velocity profiles is almost as compelling, and follow-up studies of the central velocity structure of ‘real’ globular clusters have the potential to revolutionize the debate on this subject.
From a dynamical modeler’s perspective, at least, the possibility that collisions in sufficiently dense young clusters might lead to runaway mergers offers the most interesting path to IMBHs in globular clusters. Extensive dynamical simulations leave no doubt that stellar collisions occur in sufficiently dense systems, and simple arguments lead to the inevitable conclusion that there is easily enough mass potentially available in these systems to produce objects with masses exceeding $1000M_\odot$. In many ways, this process provides a ‘natural’ mechanism for IMBH formation, but critical aspects of the evolution of the merger product—specifically, the role of stellar winds and the end result of the evolution of very massive stars—remain poorly understood. At present, it appears that stars with metallicities comparable to those of the Sun may be unable to retain enough mass to form an IMBH, but that collisions in low-metallicity systems at early times may conceivably have led to runaways in the progenitors of today’s globular clusters.

The role of dense stellar systems as potential gravitational wave (GW) source factories has been recognized for many years, and dynamical interactions in these systems may produce a wide variety of interesting GW sources. We confine ourselves here to systems containing IMBHs formed by dynamical means, again emphasizing that all of the following estimates should be treated with caution (if not outright skepticism), combining as they do the major uncertainties inherent in both the formation mechanism of the IMBHs themselves and the spread in properties of their parent clusters.

1. **BH–IMBH inspirals.** Intermediate-mass-ratio inspirals (IMRIs) involving IMBHs formed at the centers of dense star clusters may be important sources of low-frequency GWs for LISA (Barack and Cutler 2004, Gair et al 2004, Will 2004, Amaro-Seoane et al 2007). These could plausibly arise through a variety of mechanisms in star clusters, including hardening of an IMBH binary, exchange interactions between a BH binary and an IMBH, tidal capture of the progenitor star, three-body resonances or even direct capture via the emission of GW during a close encounter (see Mandel et al (2007) for a review). With the best current estimates for merger rates and signal processing performance, LISA could detect $\sim 10$ such IMBH IMRIs during a 3 year run (Miller 2002, Mandel et al 2007). Advanced LIGO may also be able to detect IMRIs of stellar-mass compact objects into low-mass ($400M_\odot$) IMBHs in globular clusters, at a rate of $\sim 10$ events per year (Brown et al 2007, Mandel et al 2007).

The weakness of the gravitational radiation from a typical BH inspiral into an IMBH means that the signal from tens of thousands of orbits must be integrated in order to detect the source. Therefore, knowledge of the waveform, and especially the probable distribution of orbital eccentricities as determined by simulations, is essential to their detection. Mandel et al (2007) conclude that IMRIs in clusters are unlikely to exhibit significant eccentricity by the time they enter the LISA band. However, even for low-eccentricity systems, the theoretical IMRI waveforms are not well known, and this uncertainty may still lead to difficulties in determining the parameters of observed systems.

2. **IMBH–IMBH binaries.** There are also hints that under some circumstances, more than one IMBH might form in such a cluster (Gürkan et al 2006). Two IMBHs in a cluster core are expected to form a binary and merge rapidly by GW emission, in as little as a few million years. Such IMBH–IMBH mergers would be powerful and unique probes of the star formation history of the universe. Fregeau et al (2006) estimate that LISA could detect tens of IMBH–IMBH inspirals per year, while advanced LIGO might observe ten merger and ringdown events per year. However, they find that most such mergers occur at redshift $z \sim 1$, presumably from relatively high-metallicity progenitors, which may provide the least favorable environment for IMBH production (see section 4.2).
(3) IMBH–SMBH mergers. On larger scales, IMBHs in clusters close enough to the center of a galaxy may spiral in and merge with the central supermassive black hole (Miller 2005, Matsubayashi et al 2007). Taking the simple dynamical estimates presented in section 4.1 at face value and integrating over a cluster population representative of the inner Galactic bulge, Portegies Zwart et al (2006) infer a IMBH–SMBH merger rate in the Galactic center of a few per 100 Myr. The scaling of these rates to larger galactic nuclei, and hence the net merger rate potentially observable by LISA, is not well known, although Matsubayashi et al (2007) estimate a net LISA event rate of ∼10 per year.

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References

Aarseth S J 1999 Publ. Astron. Soc. Pac. 111 1333
Aarseth S J 2003 The Gravitational N-Body Problem (Cambridge: Cambridge University Press)
Amaro-Seoane P, Gair J R, Freitag M, Miller M C, Mandel I, Cutler C J and Babak S 2007 Class. Quantum Grav. 24 113
Amaro-Seoane P, Kupi G and Fregeau J M 2008 in preparation
Antonov V A 1962 Vestn. Leningr. Gos. Unis. 7 135
Bahcall J N and Wolf R A 1976 Astrophys. J. 209 214
Barack L and Cutler C 2004 Phys. Rev. D 70 12
Baumgardt H and Makino J 2003 Mon. Not. R. Astron. Soc. 340 227
Baumgardt H, Makino J and Ebisuzaki T 2004 Astrophys. J. 613 1133
Baumgardt H, Makino J and Hut P 2005 Astrophys. J. 620 238
Baumgardt H, Hut P, Makino J, McMillan S L W and Portegies Zwart S F 2003a Astrophys. J. 582 L21
Baumgardt H, Makino J, Hut P, McMillan S L W and Portegies Zwart S F 2003b Astrophys. J. 589 295
Belkus H, Van Bever J and Vanbeveren D 2007 Astrophys. J. 659 1576
Bellemare R G, Bédorf J and Portegies Zwart S F 2008 New Astron. 13 103
Brown D A, Brink J, Fang H, Gair J R, Li C, Lovelace G, Mandel I and Thorne K S 2007 Phys. Rev. Lett. 99 201102
Centrella J 2008 STSci Spring Symp. Black holes (Apr. 2007) eds M Livio and A M Koehemoer (Cambridge: Cambridge University Press)
Church R 2006 PhD Thesis Cambridge University
Cohn H N 1980 Astrophys. J. 242 765
Dull J D, Cohn H N, Lugger P M, Murphy B W, Seitzer P O, Callanan P J, Rutten R G M and Charles P A 1997 Astrophys. J. 481 267
Ebisuzaki T, Makino J, Taiji M, Fukushima T, Sugimoto D, Ito T and Okumura S K 1993 Publ. Astron. Soc. Japan 45 269
Eggleton P P 2006 Evolutionary Processes in Binary and Multiple Stars (Cambridge: Cambridge University Press)
Eggleton P P, Fitchett M J and Tout C A 1989 Astrophys. J. 347 998
Freeman K C 1993 Galactic bulges IAU Symp. vol 153 ed H Dejonghe and H J Habing (Dordrecht: Kluwer) p 263
Fregeau J M, Larson S L, Miller M C, O’Shaughnessy R and Rasio F A 2006 Astrophys. J. 646 L135
Freitag M and Benz W 2005 Mon. Not. R. Astron. Soc. 358 1133
Freitag M, Güürkan M A and Rasio F A 2006 Mon. Not. R. Astron. Soc. 368 141
Freitag M, Rasio F A and Baumgardt H 2006 Mon. Not. R. Astron. Soc. 368 121
Gaburov E, Guenther A and Portegies Zwart S F 2008 Mon. Not. R. Astron. Soc. 384 376
Gair J R, Barack L, Cutler C, Creighton T, Larson S L, Phinney E S and Vallisneri M 2004 Class. Quantum Grav. 21 S1595
Gebhardt K, Rich R M R and Ho L C 2002 Astrophys. J. 578 L41
Gebhardt K, Rich R M R and Ho L C 2005 Astrophys. J. 634 1093
Giersen J, van der Marel R P, Gebhardt K, Guhathakurta P, Peterson R and Pryor C 2002 Astron. J. 124 327
Giersen J, van der Marel R P, Gebhardt K, Guhathakurta P, Peterson R and Pryor C 2003 Astron. J. 125 376
Giersz M 1998 Mon. Not. R. Astron. Soc. 298 1239
Giersz M and Heggie D C 2008 Preprint 0801.3968
Glebbeek E 2009 PhD Thesis Utrecht University
Glebbeek E, Gaburov E, de Mink S E, Pols O and Portegies Zwart S F 2008 in preparation
Gükan M A, Fregeau J M and Rasio F A 2006 Astrophys. J. 640 39
Harris W E 1996 Astron. J. 112 1487 (2003 revision of the catalog)
Heger A 2008 STScI Spring Symp. Black holes (Apr. 2007) eds M Livio and A M Koehemomoer (Cambridge: Cambridge University Press)
Heggie D C 1975 Mon. Not. R. Astron. Soc. 173 729
Heggie D C, Hut P, Minishige S, Makino J and Baumgardt H 2007 Publ. Astron. Soc. Japan 59 507
Heggie D C 2008 Mon. Not. R. Astron. Soc. 379 93
Heggie D C, Hut P and Tout C A 2000 Mon. Not. R. Astron. Soc. 315 543
Inagaki S 1985 Dynamics of star clusters IAU Symp. vol 113 ed J Goodman and P Hut (Dordrecht: Reidel)
Joshi K J, Rasio F A and Portegies Zwart S F 2000 Astrophys. J. 540 969
Kaaret P, Prestwich A H, Zezas A, Murray S S, Kim D-W, Kilgard R E, Schlegel E M and Ward M J 2001 Mon. Not. R. Astron. Soc. 321 L29
Kroupa P 2001 Mon. Not. R. Astron. Soc. 322 231
Kulkarni S R, Hut P and McMillan S L W 1993 Nature 364 421
Langer N, Hamann W-R, Lennon M, Najarro F, Pauldrach A W A and Puls J 1994 Astron. Astrophys. 290 819
Lombardi J C 2007 MODEST-8 Workshop (Bonn, Dec. 2007)
Maccarone T J, Kundu A, Zepf S E and Rhode K L 2007 Nature 445 183
Makino J and Aarseth S J 1992 Publ. Astron. Soc. Japan 44 141
Makino J, Taiji M, Ebisuzaki T and Sugimoto D 1997 Astrophys. J. 480 432
Mandel I, Brown D A, Gair J R and Miller M C 2007 Preprint 0705.0285
Matsumoto H, Tsuru T G 1999 Publ. Astron. Soc. Japan 51 321
Matsumoto H, Tsuru T G, Koyama H, Awaki H, Canizares C R, Kawai N, Matsushita S and Kawabe R 2001 Astrophys. J. 547 L25
McCready N, Gilbert A M and Graham J R 2003 Astrophys. J. 596 240
McMillan S L W, Hut P and McMillan S L W 1993 Nature 364 421
McMillan S L W, Hut P and Makino J 1991 Astrophys. J. 372 111
McMillan S L W and Portegies Zwart S F 2004 Massive Stars in Interactive Binaries (Astronomical Society of the Pacific Conference Series vol 367) ed N St-Louis and A F J Moffat (San Francisco: Astronomical Society of the Pacific) p 697
Merloni A, Heinz S and di Matteo T 2003 Mon. Not. R. Astron. Soc. 345 1057
Merritt D and Ferrarese L 2001 Astrophys. J. 547 140
Merritt D, Piatak S, Portegies Zwart S F and Hensendorf M 2004 Astrophys. J. 608 25
Meylan G, Sarajedini A, Jablonka P, Djorgovski S G, Bridges T and Rich R M 2001 Astron. J. 122 830
Miller M C 2002 Astrophys. J. 581 438
Miller M C 2005 Astrophys. J. 618 426
Miller M C and Colbert E J M 2004 Int. J. Mod. Phys. D 13 1
Miller M C and Hamilton D P 2002 Mon. Not. R. Astron. Soc. 330 232
Morris P W et al 1999 Nature 402 502
Noyola E 2008 STScI Spring Symp. Black holes (Apr. 2007) eds M Livio and A M Koehemomoer (Cambridge: Cambridge University Press)
Noyola E and Gebhardt K 2006 Astron. J. 132 447
Noyola E, Gebhardt K and Kissler-Patig M 2008 in preparation
Noyola E, Gebhardt K J and Bergmann M 2006 New Horizons in Astronomy: Frank N Bash Symposium (ASP Conference Series vol 352) ed S J Kannappan et al (San Francisco: Astronomical Society of the Pacific) p 269
O’Leary R M, Rasio F A, Fregeau J M, Ivanova N and O’Shaughnessy R 2006 Astrophys. J. 637 937
Peters P C and Mathews J 1963 Phys. Rev. 131 345
Phinney E S 1993 Structure and Dynamics of Globular Clusters ed S Djorgovski and G Meylan (San Francisco: Astronomical Society of the Pacific) p 141
Pooley D and Rappaport S 2006 Astrophys. J. 644 45
Portegies Zwart S F et al 2008 Simulation of multiphysics multiscale systems Int. Conf. on Computational Science ed R Bellman
Portegies Zwart S F, Baumgardt H, Hut P, Makino J and McMillan S L W 2004 Nature 428 724

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