PARTIAL DATA CALDERÓN PROBLEMS FOR $L^{n/2}$ POTENTIALS ON ADMISSIBLE MANIFOLDS

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Abstract. We solve the partial data Calderón problem on conformally transversally anisotropic (CTA) manifolds with $L^{n/2}$ potentials - on par with sharp unique continuation result of Jerison-Kenig [10]. A trivial consequence of this is the sharp regularity improvement to the result of Kenig-Sjöstrand-Uhlmann [14]. This is done by constructing a "Green’s function" which possesses both desirable boundary conditions and satisfies semiclassical type estimates in the suitable $L^p$ spaces. No Carleman estimates were used in the writing of this article which makes it starkly different from the traditional approaches based on Bukhgeim-Uhlmann [2] and Kenig-Sjöstrand-Uhlmann [14].

1. Introduction

The pioneering works of Bukhgeim-Uhlmann [2] and Kenig-Sjöstrand-Uhlmann [14] on partial data Calderón problems for the Schrödinger operator $\Delta + q$ have inspired many works on the subject (see review article [13] and the references therein). Except cases where the domain geometry is trivial (e.g. flat/spherical boundary), all of them are based on the $L^2$ Carleman estimate approach developed by [14], [2], and [11].

In order to use the Carleman-based approach one must assume a-priori that $q \in L^{\infty}$ - an unsatisfactory assumption since we know that unique continuation holds even for potentials $q$ which are in $L^{n/2}$ (see [10]). Various full data Calderón problems in the $L^{n/2}$ limit were studied (see [9] and references therein) without the use of Carleman estimates. However, the techniques in [9] do not translate immediately to the more challenging partial data problems.

We propose a different method to solve partial data problems which bypasses the traditional Carleman approach. This allows us to minimize the assumption on the potential $q$ to $L^{n/2}$ - on par with the sharp assumption for unique continuations [10].

Turns out that it is more convenient to apply this new approach in the more general geometric setting of "conformally transversally anisotropic" (CTA) manifold first introduced by [8]. These are manifolds $M = \mathbb{R} \times M_0$ endowed with metrics conformal to $dy_1^2 \oplus g_0$ where $g_0$ is the metric on the closed compact manifold $M_0$. Suppose $\Omega \subset M$ is a smooth bounded domain is compactly contained in $\mathbb{R} \times \Omega_0$ where $\Omega_0 \subset M_0$ is a simple domain. If $\Gamma_\pm \subset \partial \Omega$ are compactly contained in the sets

$$\{ y \in \partial \Omega \mid \pm g(\partial y_1, \nu(y)) > 0 \}$$

where $\nu$ denotes the inward pointing normal, let $\Gamma_D := \partial M \setminus \Gamma_+$ and $\Gamma_N := \partial M \setminus \Gamma_-$. Let $q \in L^{n/2}(\Omega)$ and assume well-posedness of the Dirichlet BVP for $\Delta_g + q$, denote by $\Lambda_q : H^{\frac{n}{4}}(\partial \Omega) \to H^{-\frac{n}{4}}(\partial \Omega)$
the Dirichlet-Neumann map. (We refer the reader to the appendix of [9] for the definition of the Dirichlet-Neumann map for \( q \in L^{n/2}(\Omega) \).) We have the following theorem:

**Theorem 1.1.** Let \( q_1, q_2 \in L^{n/2}(\Omega) \) be such that \( \Lambda_{q_1} f \mid_{\Gamma_N} = \Lambda_{q_2} f \mid_{\Gamma_N} \) for all \( f \in C_0^\infty(\Gamma_D) \). Then \( q_1 = q_2 \).

Kenig-Salo [11] was the first to consider partial data type problems on CTA manifolds. The result for [11] is for sufficiently regular potentials whereas the focus of this article is on potentials which are unbounded. Partial data for unbounded potentials was also studied in [4] in the Euclidean setting for data on roughly half of the boundary.

This result is new even in the Euclidean setting and can be seen as a sharp regularity (i.e. \( q \in L^{n/2} \)) version of the main theorem by Kenig-Sjöstrand-Uhlmann in [14]. Indeed, consider a bounded smooth domain \( \Omega \subset \mathbb{R}^n \), \( z_0 \in \mathbb{R}^n \) a point not in the closure of the convex hull of \( \Omega \), and let \( \Gamma_\pm \subset \partial \Omega \) be an open subsets compactly contained in

\[ \{ z \in \partial \Omega \mid \pm \nu(z) \cdot (z - z_0) > 0 \}. \]

Define \( \Gamma_D \) and \( \Gamma_N \) as before, we have after a change of coordinates \( y_1 = \log |z - z_0| \),

**Corollary 1.2.** Let \( q_1, q_2 \in L^{n/2}(\Omega) \) be such that \( \Lambda_{q_1} f \mid_{\Gamma_N} = \Lambda_{q_2} f \mid_{\Gamma_N} \) for all \( f \in C_0^\infty(\Gamma_D) \). Then \( q_1 = q_2 \).

Observe that geometrizing the problem from the Euclidean case to the more general setting of Theorem [14] "linearizes" the log variable which allows us to adapt the parabolic flow construction of [3] to our setting. The price we pay, of course, is that the underlying geometry becomes more involved and the Fourier multiplier Green’s function constructed by [21] is no longer suitable. To remedy this difficulty we use instead the Green’s function constructed via Fourier series. These were first used in [12] and later in [9].

We will take the Green’s function of [12] and transform it into a Dirichlet Green’s function which will be the key to solving our problem. To simplify notations fix throughout this article \( p := \frac{2n}{n-2} \) and \( p' := \frac{2n}{n+2} \).

**Proposition 1.3.** There exists a Green’s function

\[ G_{\Gamma, \pm} : L^2(\Omega) \to _{h^{-1}}L^2(\Omega), \quad G_{\Gamma, \pm} : L^{p'}(\Omega) \to _{h^{-2}}L^p(\Omega), \]

which resolves the conjugated Laplacian \( h^2 e^{\pm y_1/h} \Delta_g e^{\pm y_1/h} G_{\Gamma, \pm} = \text{Id} \) in \( \Omega \). Furthermore, \( G_{\Gamma, \pm} v \in H^1(\Omega) \) for all \( v \in L^{p'} \) and \( G_{\Gamma, \pm} v \mid_{\Gamma_N} = 0 \).

Throughout this article the notation \( T : X \to _{h^m} Y \) indicates that the operator norm of the operator \( T \) from \( X \) to \( Y \) is bounded by \( O(h^m) \).

The theory and methods developed in this article along with earlier work in this direction [4] has a central theme: Given any Green’s function for the conjugated Laplacian with suitable semiclassical \( L^p \) estimates, there is a systematic way to "upgrade" it to one which obeys the Dirichlet boundary condition while simultaneously preserving the same estimates. One can, of course, adapt similar methods of this article and [4] to obtain different types of boundary conditions (e.g. Neumann, Robin, etc.) and different types of conjugation with elliptic operators (Dirac, bi-Laplace, etc). As these Dirichlet Greens functions are the pivotal piece in many inverse problems [14] [15] [8], unique continuations and Carleman estimates [16] [11] [17] [18] [19], we anticipate that their scope of application extends beyond Calderón problems. The explicit nature of their construction, bypassing the traditional
route of abstract functional analysis machinery (see [14]), also gives hope for the possibility of CGO based numerical reconstruction algorithms with partial data in the spirit of [20, 6, 7].

This article is organized in the following way. In Section 2 we outline a \( \Psi \)DO calculus which is compatible with our symbol class. In Section 3 we construct the parabolic flow in the context of this \( \Psi \)DO calculus and solve the Dirichlet problem for the flow. In Section 4 we use this flow to construct the Green’s function which is the key piece for proving Theorem 1.1. In Section 5 we construct the CGO using these Green’s functions and finally employ them in Section 6 to prove Theorem 1.1.

2. Elementary Semiclassical \( \Psi \)DO theory on CTA Manifolds

We collect a set of facts about semiclassical pseudodifferential operators and also use this opportunity to establish some notations and conventions which we will use throughout. Throughout this article we will use the Weyl quantization to produce operators acting on sections of the half-density bundle \( \Omega^{1/2} (M) \), which we identify with the trivial line bundle via the volume form. This has the advantage that symbols of semiclassical operators in \( \Psi (M) \) are defined up to \( h^2 S^{k-2} (M) \). Proofs of the results in this section are contained as [4, 22, and 23].

2.1. Semiclassical Sobolev Spaces. We use semiclassical Sobolev spaces with the norm
\[
\| u \|_{W^{k,r}(M)} := \| (hD)^k u \|_{L^r},
\]
which is equivalent to the one involving derivatives
\[
\sum_{|\alpha| \leq k} \| (hD)^\alpha u \|_{L^r}.
\]
Let \( M = \mathbb{R} \times M_0 \) where \( M_0 \) is a closed compact manifold with metric \( g_0 \) and consider the metric \( dx_1^2 \oplus g_0 \) which makes \( M \) a transversally anisotropic manifold ([8]). Denote the elements of \( M \) by \( (x_1, x') \). The cotangent bundle of \( M \) has a natural splitting \( T^* M = T^* \mathbb{R} \oplus T^* M_0 \) whose elements we write as \( (\xi_1, \xi') \). We define the mixed Sobolev norms for \( u \in C_c^\infty (M) \) by
\[
\| u \|_{W^{k,r}(M)} := \| (hD')^k (hD')^\ell u \|_{L^r},
\]
and the space \( W^{k,r}(M) W^{\ell,r}(M) \) by completion. For convenience we will drop the \( M_0 \) and \( M \) in this notation and use the convention that the first \( W^{k,r} \) denotes multiplication by \( (hD')^k \) and the second \( W^{\ell,r} \) denotes multiplication by \( (hD')^\ell \). Note that with this definition we have that for \( k \geq 0 \),
\[
W^{-k,r} W^{\ell,r} \subset W^{\ell-k,r}(M).
\]
In addition to Hörmander symbols \( S^0_1 (M) \), we will also consider symbols in the class \( S^0_0 (M) \) which do not decay when differentiated with respect to \( \xi \):
\[
| \partial_x^\alpha \partial_\xi^\beta a (x, \xi) | \leq C_{\alpha, \beta} |\xi|^k
\]
We denote by the symbol space \( S^k_1 (M_0) S^\ell_j (M) \) by product symbols of the form \( ba (x', \xi) \) where \( b (x', \xi') \in S^k_1 (M_0) \) and \( a (x', \xi) \in S^\ell_j (M) \) for \( j = 0, 1 \). Again, to simplify notation we will drop the \( M_0 \) and \( M \) and just write \( S^k_1 S^\ell_j \).

Boundedness of quantization of symbols \( a \in S^0_0 (M) \) acting on Sobolev spaces are given by semiclassical version of Calderón-Vaillancourt: for all \( 1 < r < \infty \) and \( h > 0 \) sufficiently
small,
\[
(2.2) \quad \|a(x, hD)u\|_{L^r} \leq \sum_{|\alpha|,|\beta|\leq k(n)} p_{\alpha,\beta}(a)\|u\|_{L^r} + C\sqrt{h}\|u\|_{L^r}.
\]

Here \(p_{\alpha,\beta}\) is the semi-norm defined by \(p_{\alpha,\beta}(a) := \sup_{x,\xi} |\partial^{\alpha}_x \partial^{\beta}_\xi a(x, \xi)| |\langle \xi \rangle|^{|\beta|}\) and \(k(n)\) depends on dimension only. We shall henceforth denote by \(k(n)\) to be the smallest integer for which (2.2) holds.

For symbols in \(S^k_1 S^{-\ell}_1 \cup S^k_1 S^{-k(n) - \ell}_0\), we have the following mapping properties derived in [4].

**Proposition 2.1.** If \(b(x', \xi') \in S^k_1\) and \(a(x', \xi) \in S^k_1 \cup S^{-k(n) + \ell}_0\) then
\[
b(a(x', hD)) : W^{m,r} \rightarrow W^{m-k, r} W^{1, r}.
\]

In addition, we have the following compositional calculus result.

**Proposition 2.2.** If \(a \in S^k_1 S^{\ell_1}_1 \cup S^k_1 S^{-k(n) + \ell_1}_0\) and \(b \in S^k_2 S^{\ell_2}_1 \cup S^k_2 S^{-k(n) + \ell_2}_0\) then
\[
b(x'hD)a(x', hD) = ab(x', hD) + \frac{h}{2i}\{a, b\}(x', hD) + h^2 m(x', hD)
\]
where \(m(x', hD) : W^{k, r} W^{\ell, r} \rightarrow W^{k-k_1-k_2, r} W^{\ell-\ell_1-\ell_2, r}\).

3. **Parabolic Equation**

Denote by \(M_+ := (0, \infty) \times M_0 \subset M\) and \(M_- := (-\infty, 0) \times M_0 \subset M\). Let \(B(x', \xi') \in S^1_1(M_0)\), and define the semiclassical pseudodifferential operator
\[
j(x', hD) = h\partial_{x_1} + B(x', hD)
\]
on \(M\). It follows by considering the \(M_0\) and \(R\) direction separately and applying the semiclassical Calderón-Vaillancourt theorem that \(j(x', hD)\) is a bounded operator \(j(x', hD) : W^{1, r}(M) \rightarrow L^{r}(M)\) for \(1 < r < \infty\). In this section we follow [4] and derive some properties of its inverse.

We assume that the real part of \(B(x', \xi')\) obeys the ellipticity condition
\[
c|\xi'| \leq \text{Re}B(x', \xi') \leq C|\xi'|
\]
uniformly in \(x'\), for some constants \(c, C > 0\) which ensures the ellipticity of \(j(x, \xi) := i\xi_1 + B(x', \xi')\).

Unfortunately even with ellipticity the symbol \((i\xi_1 + B)^{-1}\) is not in general in the class \(S^{-1}_1(M)\) (but rather in \(S^{-1}_0(M)\)). To remedy this we assume that there exists a first order symbol \(i\xi_1 + B_-(x', \xi')\) with compact characteristic set, such that
\[
(i\xi_1 + B)(i\xi_1 + B_-) = \mathcal{P}(x', \xi) + a_0
\]
where \(\mathcal{P}(x', \xi)\) is a second order polynomial in \(\xi\) whose zeros stay within \(|\xi'| \leq C\) for some large \(C > 0\) and \(a_0 \in S^{-\infty}(M_0)\). It was shown in [4] that (3.3) implies
\[
(j^{-1}(x', \xi) = (i\xi_1 + B)^{-1} \in S^0 S^{-1}_1 + S^{-\infty} S^{-1+k(n)}_0 + S^1_1 S^{-2}_1
\]
which then implies
\[
j^{-1}(x', hD) : L^r \rightarrow W^{1, r}, \quad 1 < r < \infty.
\]
The operator $j^{-1}(x', hD)$ turns out to be equivalent to solving the Cauchy problem for the parabolic flow with initial condition on $x_1 = 0$. Indeed, let $U \subset M_0$ be a coordinate chart and $u$ be a smooth function which is compactly supported in the (infinite) strip $\mathbb{R} \times U$. Identifying $u$ with its pull-back by the coordinate map we can write

$$j^{-1}(x', hD)u(x_1, x') = h^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{rac{1}{h} (x' - y') \cdot \xi'} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{rac{1}{h} (x_1 - s) t} u(s, y')}{i t + B(\frac{x' + y'}{2}, \xi')} \, dt ds \, d\xi' d\gamma'. $$

The inner integral can be computed using residue theorem to obtain

$$j^{-1}(x', hD)u(x_1, x') = h^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{rac{1}{h} (x' - y') \cdot \xi'} \int_{-\infty}^{x_1} e^{-\frac{x_1}{h} B(\frac{x' + y'}{2}, \xi')} \, ds d\xi' d\gamma'. $$

Therefore, $j^{-1}(x', hD)u(x_1, x') \mid_{x_1 \leq 0} = 0$. A partition of unity argument on the compact manifold $M_0$ shows that this holds for all $u \in C_c^\infty(M_+)$. A density argument allows us to conclude that

$$(3.6) \quad j^{-1}(x', hD)u \in W^{1,r}(M), \quad j^{-1}(x', hD)u \mid_{x_1 \leq 0} = 0 $$

if $u \in L^r(M)$ and $u \mid_{M_0} = 0$.

Henceforth we will refer to the support property given by (3.6) as “preserving support in $M_+$."

Standard semiclassical calculus allows us to turn $j^{-1}(x', hD)$ into an inverse of $j(x', hD)$. First observe that such if $a(x', \xi') \in S^1_1(M_0)$ (3.4) and Proposition 2.2 yields

$$(3.7) \quad a(x', hD')j^{-1}(x', hD) = (aj^{-1})(x', hD) + \frac{h}{2i} (j^{-2}\{a, B\})(x', hD) + h^2 m(x', hD)$$

where $m(x', hD)$ and $(j^{-2}\{a, B\})(x', hD)$ map $L^r \to h_0 L^r$. Using this composition formula and Proposition 2.2 we invert $J := j(x', hD) + hB_0(x', hD')$:

**Proposition 3.1.** Let $J := j(x', hD) + hB_0(x', hD)$ for some $B_0(x', \xi') \in S^0_1(M_0)$. For $h > 0$ sufficiently small there exists an $M_+$ support preserving inverse $J^{-1} : L^r \to W^{1,r}$ of the form

$$J^{-1} = j^{-1}(x', hD)(1 + h(j^{-1}B_0)(x', hD) + h^2 m_2(x', hD))$$

where $m_2(x', hD) : L^r \to h_0 L^r$.

One final consequence of the structure of $J^{-1}$ is the following disjoint support property of [3]. We sketch the proof for the convenience of the reader:

**Lemma 3.2.** Let $1_{M_{-}} \in L^\infty(M)$ be the indicator function for $x_1 \leq 0$ and $\epsilon > 0$. Then for all $f \in L^r(M)$,

$$\|J^{-1}1_{M_{-}}f\|_{W^{1,r}(\{x_1 \geq \epsilon\})} \leq C_\epsilon h^2 \|f\|_{L^r}(M).$$

**Proof.** Let $\zeta_\epsilon(x_1)$ be a smooth cutoff function which is identically one on $\{x_1 \geq \epsilon\}$ and identically zero on an open set containing $\{x_1 \leq 0\}$. Then

$$\|J^{-1}1_{M_{-}}f\|_{W^{1,r}(\{x_1 \geq \epsilon\})} \leq \|\zeta_\epsilon J^{-1}1_{M_{-}}f\|_{W^{1,r}(M)}. $$

Therefore it suffices to show that

$$\|\zeta_\epsilon J^{-1}1_{M_{-}}f\|_{W^{1,r}(M)} \leq C_\epsilon h^2 \|f\|_{L^r}(M). $$

From Proposition 3.1 we have that

$$J^{-1} = j^{-1}(x', hD)(1 + h(j^{-1}B_0)(x', hD) + h^2 m_2(x', hD))$$
We will show this for the principal part $\zeta \rho^{-1}(x', hD)1_{M_+}$ and leave the lower order term, which can be written out explicitly using the terms in $\mathcal{G}_\mathcal{L}$ to the reader. Writing $\rho^{-1}(x', \xi)$ using expansion $[\mathcal{G}_\mathcal{L}]$ we see that the desired estimate is a special case of disjoint support property for operators of the type $\zeta \rho^{-1}(x', hD)1_{M_+}$ for symbols $a$ and $b$ in the suitable symbol class.

\[\square\]

4. Dirichlet Green’s Function

In this section we assume that the metric $g$ on $\mathbb{R} \times M_0$ takes the form $dy_1^2 \oplus g_0$ (i.e. no conformal factor). Let $\Omega \subset M = \mathbb{R} \times M_0$ be a smooth bounded domain contained in $I \times M_0$ for some compact interval $I \subset \mathbb{R}$. If $\Gamma_+ \subset \partial \Omega$ is an open subset of the boundary compactly contained in $\{y \in \partial \Omega \mid \pm g(\partial_{y_1}, \nu(y)) > 0\}$, we would like to invert

$$h^2 \Delta := h^2 e^{\pm y_1/h} \Delta e^{\pm y_1/h},$$

with Dirichlet boundary condition on $\Gamma_+$ and have the inverse satisfy good $L^p \to L^p$ estimates. Note that in this geometric setting every connected component of $\Gamma_\pm$ can be expressed as a portion of the graph of a smooth function $\{y_1 = f(y')\}$ with $f \in C^\infty(M_0)$. For the purpose of simplifying notation we will only work with the "+" sign and set $\Gamma := \Gamma_+$. The theory we develop here works equally well for the ",-" sign.

We begin with the result of [9] and [12]. Let $g = dy_1^2 \oplus g_0$ be the metric on the product manifold $M = \mathbb{R} \times M_0$. Kenig-Salo-Uhlmann in [12] constructed a Green’s function $G^M_\pm$ solving $e^{\pm y_1/h} h^2 \Delta g e^{\pm y_1/h} G^M_\pm = 1d$ of the form

$$G^M_\pm f = \sum \int_{-\infty}^{\infty} \frac{e^{i \xi_1 \xi_1 \hat{f}_l(\xi_1, y')}{2i h \xi_1 - 1 + h^2 \lambda_l} d\xi_1$$

where $f_l(y_1, y') := e_l(y') \int_{M_0} e_l f(y_1, \cdot) dy_1', \Delta g_0 e_l = \lambda_l e_l$, and $\hat{f}_l$ denotes the Fourier transform in the $y_1$ direction. By [9] for any compactly supported functions $\chi, \tilde{\chi} \in C^\infty(\mathbb{R})$ one has

$$\tilde{\chi}(y_1) G^M_\pm \chi(\cdot) : L^2 \to h^{-1} L^2, \quad \tilde{\chi}(y_1) G^M_\pm \chi(\cdot) : L^p \to h^{-2} L^p.$$  

This operator does not satisfy the desired boundary conditions along $\Gamma$ so more work will be needed. To this end we need to derive some finer properties for this operator. In particular, we would like to show that away from the characteristic set of $e^{\pm y_1/h} h^2 \Delta g e^{\pm y_1/h}$ the operator $G^M_\pm$ behaves more or less like a $\Psi DO$:

**Lemma 4.1.** Let $\rho(y', \xi) \in S^{-\infty}$ be any symbol such that $\rho = 1$ in a neighbourhood of the set $\{\xi_1 = 0, |\xi'|_{y_1(y)} = 1\}$ then

$$G^M_\pm = \text{Op}(S^{-2}_1) + \text{Op}(S^0_1) \rho(hD) G^M_\pm = \text{Op}(S^{-2}_1) + G^M_\pm \rho(hD) \text{Op}(S^0_1)$$

Furthermore, one can write

$$\tilde{\chi}(y_1) G^M_\pm \chi(\cdot) = (\tilde{\chi}(y_1) G^M_\pm \chi(\cdot))^c + \text{Op}(S^{-2}_1)$$

where

$$(\tilde{\chi}(y_1) G^M_\pm \chi(\cdot))^c : L^2 \to h^{-1} H^k, \quad (\tilde{\chi}(y_1) G^M_\pm \chi(\cdot))^c : L^p \to h^{-2} W^{k,p}$$
Proof. The first statement comes from ellipticity of \( e^{\pm y_1/h^2} \Delta_y e^{\pm y_1/h} \) away from the support of the symbol \( \rho \) and one can construct both left and right semi-classical parametrix.

For the last statement, choose \( \rho, \tilde{\rho} \in S^{-\infty}(M) \) compactly supported on each fiber such that \( \rho = 1 \) is supported in a compact neighbourhood of the characteristic set and \( \tilde{\rho} = 1 \) on \( \text{supp}(\rho) \). We can write

\[
\tilde{\chi} G^M_\pm \chi = \tilde{\rho}(hD) \tilde{\chi} G^M_\pm \chi + (1 - \tilde{\rho}(hD)) \tilde{\chi} G^M_\pm \chi.
\]

For the first term, setting \( (\tilde{\chi} G^M_\pm t \chi)^c := \tilde{\rho}(hD) \tilde{\chi} G^M_\pm t \chi \) we have by (4.1)

\[
(\chi(y_1) G^M_\pm \chi(\cdot))^c : L^2 \rightarrow_{h^{-1}} H^k, \quad (\chi(y_1) G^M_\pm \chi(\cdot))^c : L^{p'} \rightarrow_{h^{-2}} W^{k,p}.
\]

Substituting \( G^M_\pm = \text{Op}(S_1^{-2}) + \text{Op}(S_0^0) \rho(hD) G^M_\pm \rho(hD) \text{Op}(S_1^0) \) into the second term completes the proof. \( \square \)

Another useful statement about the microlocal support of the Green’s function is the following

**Lemma 4.2.** If the support of \( a \in S^1_0(M) \) is disjoint from the characteristic set \( \{ \xi_1 = 0, |\xi'| = 1 \} \) then

\[
\|a(y, hD) \tilde{\chi} (y_1) G^M_\pm \chi(\cdot)\|_{L^2 \rightarrow L^2} \leq C, \quad \|a(y, hD) \tilde{\chi} (y_1) G^M_\pm \chi(\cdot)\|_{L^{p'} \rightarrow L^2} \leq Ch^{-1}
\]

for any compactly supported function \( \chi, \tilde{\chi} \in C_0^\infty(\mathbb{R}) \).

**Proof.** In the proof of Lemma 4 choose \( \rho, \tilde{\rho} \in S^{-\infty}(M) \) so that their supports are disjoint from \( a(y, \xi) \). The second statement comes directly from using the first statement to write

\[
a(y, hD) \tilde{\rho}(y', hD) \tilde{\chi} G^M_\pm \chi = a(y, hD) \tilde{\rho}(y', hD) \tilde{\chi} \left( \text{Op}(S_1^{-2}) + G^M_\pm \text{Op}(S_1^0) \rho(y', hD) \right) \chi.
\]

This clearly maps \( H^{-k}(M) \rightarrow H^k(M) \) for all \( k \) with decay \( h^\infty \). Therefore it suffices to analyze the mapping properties of \( a(x, hD)(1 - \tilde{\rho}(x', hD)) \tilde{\chi} G^M_\pm \chi \). Substituting

\[
G^M_\pm = \text{Op}(S_1^{-2}) + \text{Op}(S_0^0) \rho(hD) G^M_\pm \rho(hD) \text{Op}(S_1^0)
\]

into this term completes the proof. \( \square \)

In order to deal with the fact that portions of \( \partial \Omega \) which are described by graphs of smooth functions \( y_1 = f(y') \), we will consider portions of the boundary which can be straightened by a coordinate change of the type

\[
(4.2) \quad \gamma : (y_1, y') \mapsto (x_1, x') = (y_1 - f(y'), y').
\]

Under this change of variables, the push-forward of the conjugated Laplacian \( h^2 \tilde{\Delta}_+ := \gamma_*(e^{-y_1/h^2} \Delta_y e^{y_1/h}) \) is

\[
(4.3) h^2 \tilde{\Delta}_+ = (1 + |df|^2) \text{Op} \left( \xi_1^2 - 2 \xi_1 \frac{i - g_0(\xi', df) + hF}{1 + |df|^2} - \frac{1 - |\xi'|^2 + h\xi'(K)}{1 + |df|^2} \right) + h^2 \text{Op}(S_1^0(M_0))
\]

for some real valued \( F \in C^\infty(M_0) \) and \( K \in C^\infty(M_0; T M_0) \). Observe that \( \tilde{G}^M_+ := \gamma_* G^M_+ \) is a Green’s function for \( h^2 \tilde{\Delta}_+ \). If \( I \subset \mathbb{R} \) is a compact interval, choose \( \tilde{\chi}, \chi \in C_0^\infty(M) \) which is equal to \( 1 \) on \( I \times M_0 \) with \( \tilde{\chi} \chi = \chi \). We can also require that \( \gamma_* \tilde{\chi} \) is a function of \( x_1 \) only so that

\[
[\text{Op}(S_0^k(M_0), (\gamma_* \tilde{\chi}))] = [\text{Op}(S_0^k(M_0), (\gamma_* \chi))] = 0.
\]
Define
\[ G^I_+ := \tilde{\chi} G^M_+ \chi, \quad \tilde{G}^I_+ := \gamma_+ G^I_+ \]
As a consequence of Lemma 4.1 we have that
\[ \tilde{G}^I_+ = (G^I_+)^c + \text{Op}(S^{-2}_1), \quad G^I_+ = (G^I_+)^c + \text{Op}(S^{-2}_1) \]
where
\[ (G^I_+)^c, (\tilde{G}^I_+)^c : L^2 \to h^{-1} H^k, \quad (G^I_+)^c, (\tilde{G}^I_+)^c : L^p \to h^{-2} W^{k,p}. \]

4.1. **Decomposition of** $\tilde{\Delta}_+$. It was observed in [3] that the principal symbol of $\frac{1}{1+|df|^2} h^2 \tilde{\Delta}_+$ factors formally as
\[ \xi^2 - 2\xi_1 \frac{(i - g_0(df, \xi') - ihF)}{1 + |df|^2} - \frac{(1 - |\xi'|^2 - ih\xi'(K))}{1 + |df|^2} = \left( \xi_1 - i \frac{(1 + ig_0(\xi', df) + hF) - r_0}{1 + |df|^2} \right) \times \left( \xi_1 - i \frac{(1 + ig_0(\xi', df) + hF) + r_0}{1 + |df|^2} \right) \]
where
\[ r_0 := \left( (1 + ig_0(\xi', df) + hF)^2 - (1 - |\xi'|^2 - ih\xi'(K))(1 + |df|^2) \right)^{1/2} \]
is the standard branch of the square root. To avoid the discontinuity of the square root we make the following modification:
From examination of the square root, we see that the (standard) branch cut occurs on the set
\[ \{2(1 + hF)g_0(df, \xi') + h(1 + |df|^2)\xi'(K) = 0\} \cap \{(1 + hF)^2 \leq (1 - |\xi'|^2)(1 + |df|^2) + h^2 \left( \frac{1 + |df|^2}{2(1 + hF)} \right)^2 \} \]
To avoid this set, observe that for all $\delta > 0$ there exists an $\epsilon > 0$ and $h_0 > 0$ such that if $2(1 + hF)g_0(df, \xi') + h(1 + |df|^2)\xi'(K) = 0$ and $|\xi'|^2 \geq \sup_{x \in M_0} \frac{|df|^2}{1 + |df|^2} + \delta$ then
\[ (1 + hF)^2 \geq \epsilon + (1 - |\xi'|^2)(1 + |df|^2) + h^2 \left( \frac{1 + |df|^2}{2(1 + hF)} \right)^2, \quad \forall h < h_0 \]
Thus, let $0 < c < c' < 1$ be a constant such that $\frac{|df|^2}{1 + |df|^2} < c$ for all $x'$ and let $\tilde{\rho}_0(\xi')$ be a compactly supported symbol on $M_0$ such that $\tilde{\rho}_0 = 1$ for $|\xi'|^2 \leq c$ and supp$(\tilde{\rho}_0) \subset B_{\sqrt{c'}}$. Introduce a second cutoff $\rho$ such that it is identically 1 on $|\xi'|^2 \leq c'$ but supp$(\rho) \subset B_1$.
Observe that
\[ \inf_{\xi \in \text{supp} \rho, \ x' \in M_0} \left| \xi_1^2 - 2\xi_1 \frac{(i - g_0(df, \xi') - ihF)}{1 + |df|^2} - \frac{(1 - |\xi'|^2 - ih\xi'(K))}{1 + |df|^2} \right| > 0. \]
Since the discontinuity of the square root occurs within $|\xi'|^2 \leq |df|^2(1 + |df|^2)^{-1}$, it follows that for $\xi'$ in the support of $1 - \tilde{\rho}_0$, the function
\[ (1 + ig_0(\xi', df) + hF)^2 - (1 - |\xi'|^2 - ih\xi'(K))(1 + |df|^2) \]
stays uniformly away from the discontinuity. This means that
\[ r := (1 - \tilde{\rho}_0)r_0 \]
is a smooth symbol. We can now decompose $\xi_1^2 - 2\xi_1 \frac{(i - g_0(df, \xi') - ihF)}{1 + |df|^2} - \frac{(1 - |\xi'|^2 - ih\xi'(K))}{1 + |df|^2}$ as
\[ (\xi_1 - \tilde{a}_- + hm_0)(\xi_1 - \tilde{a}_+ - hm_0) + \tilde{a}_0 + h\{\tilde{a}_-, \tilde{a}_+\} - hm_0\tilde{a}_- + h^2 m_0^2 \]
with \( m_0(x', \xi') := -\tilde{a}_+^{-1} \{\tilde{a}_-, \tilde{a}_+\} \). Here the \( \tilde{a}_\pm \in S_1^1(M_0) \) and \( \tilde{a}_0 \in S^{-\infty}(M_0) \) are defined by
\[
\tilde{a}_0(x', \xi') := \tilde{\rho}_0(-2 + \rho_0) \left( 1 + ig_0(df, \xi') + hF \right)^2 - (1 - |\xi'|^2 - ih\xi'(K))(1 + |df|^2) \\
\tilde{a}_\pm(x', \xi') := \frac{i(1 + ig_0(df, \xi') + hF) \pm r}{2}.
\]

We also denote by \( \tilde{A}_0 \) and \( \tilde{A}_\pm \) their respective quantizations. Observe that the support of \( \tilde{a}_0 \) is compactly contained in the interior of the set where \( \tilde{\rho} = 1 \).

We now quantify (4.9) to see that
\[
\frac{1}{1 + |df|^2} h^2 \tilde{\Delta}_+ = QJ + \tilde{A}_0 - h\tilde{E}_1 + h^2 \tilde{E}_0 + h^2 \text{Op}(\mathcal{S}^0_1(M_0))
\]
where \( \tilde{e}_1 = m_0 \tilde{a}_- \in S_1^1(M_0) \), \( \tilde{e}_0 \in \mathcal{S}^0_1(M_0) \), and \( Q \) and \( J \) are the operators obtained by quantizing \( \xi_n - \tilde{a}_- + hm_0 \) and \( \xi_n - \tilde{a}_+ + hm_0 \) respectively. Again, here we use capitalization to denote the quantization of a symbol. We have the following estimate:

**Lemma 4.3.** If \( \tilde{G}_+^I \) is the Green’s function defined by (4.11), then there is an operator \( (\tilde{E}_1 \tilde{G}_+^I)^c \) with bounds
\[
(\tilde{E}_1 \tilde{G}_+^I)^c : L^2 \to_{h^0} H^k, \quad (\tilde{E}_1 \tilde{G}_+^I)^c : L^r \to_{h^{-1}} H^k \, \forall k \in \mathbb{N}.
\]

**Proof.** First write
\[
(4.12) \tilde{E}_1 \tilde{G}_+^I = \text{Op}(\tilde{\alpha}_+^{-1} m_0 \tilde{a}_- \tilde{a}_+) \tilde{G}_+^I = \text{Op}(\tilde{\alpha}_+^{-1} m_0) \text{Op}(\tilde{a}_- \tilde{a}_+) \tilde{G}_+^I + h \tilde{e}_1(x', hD') \tilde{G}_+^I
\]
for some \( \tilde{e}_1 \in S_1^0(M_0) \). Note that
\[
\text{Op}(\tilde{a}_- \tilde{a}_+) \tilde{G}_+^I = \text{Op}(\tilde{a}_- \tilde{a}_+) \gamma_s(\tilde{x}) \gamma_s G^M_+ \gamma_s \chi = \gamma_s(\tilde{x}) \gamma_s \chi. 
\]

We were able to commute multiplication by \( \gamma_s \tilde{x} \) and \( \text{Op}(\tilde{a}_- \tilde{a}_+) \) thanks to the fact that in (4.4) we have chosen \( \tilde{x} \) so that \( \gamma_s \tilde{x} \) is a function of \( x_1 \) only. Expanding \( \tilde{a}_- \tilde{a}_+ \) we see using (4.3) that
\[
\gamma_s(\tilde{x}) \text{Op}(\tilde{a}_- \tilde{a}_+) \gamma_s G^M_+ \gamma_s \chi = \gamma_s(\tilde{x}) \left( 1 + |df|^2 \right) - \gamma_s(\tilde{x}) \left( h^2 \partial_{\tilde{y}}^2 G^M_+ \right) - 2 \text{Op}(\frac{i - g_0(\tilde{\xi}', df) + hF}{1 + |df|^2}) \gamma_s(\tilde{h} \partial_{\tilde{y}} G^M_+)
\]
\[
+ \tilde{\rho}_0(x', hD') \text{Op}(\mathcal{S}_1^1(M_0)) \gamma_s(\tilde{x}) \gamma_s \chi.
\]

We first show that the first term of (4.12) is a sum of an operator in \( \text{Op}(\mathcal{S}_1^{r-1}(M_0)) \text{Op}(\mathcal{S}^0_1(M)) \) with an operator mapping \( L^2 \to_{h^0} H^k \) and \( L^r \to_{h^{-1}} H^k \). To this end it suffices to show that (4.13) is the sum of an operator in \( \mathcal{P}^0_1(M) \) and an operator mapping \( L^2 \to_{h^0} H^k \) and \( L^r \to_{h^{-1}} H^k \).

Using Lemma 4.1 we see that the last term is of the form
\[
h^2 \gamma_s(\tilde{x}) \left( \text{Op}(\mathcal{S}_1^{r-2}(M)) + \text{Op}(\mathcal{S}^{-\infty}(M)) \right) \gamma_s \chi.
\]
This is the sum of a \( \Psi \text{DO} \) and a term which takes \( L^2 \to_{h^0} H^k \) and \( L^r \to_{h^{-1}} H^k \) due to Proposition 4.1 and Sobolev embedding. For the second last term, since \( \tilde{\rho} \) is microlocally supported away from the characteristic set, it is of the form \( \text{Op}(\mathcal{S}_1^{r-2}(M)) + h^\infty \text{Op}(\mathcal{S}^{-\infty}(M)) \) by Lemma 4.1.
We analyze the term involving $h\partial_{y_1}G^M_+$ in (4.13). Using Lemma 4.1 we can write

$$h\partial_{y_1}G^M_+ = \text{Op}(S^{-1}_1(M)) + h\text{Op}(S^{-\infty}(M))G^M_+ \text{Op}(S^{-\infty}(M))h\partial_{y_1}G^M_+ \text{Op}(S^{-\infty}(M)).$$

The first term is a $\Psi$DO. The second term takes $L^2 \to_{h^0} H^k$ and $L^{p'} \to_{h^{-1}} H^k$ due to (4.11) and Sobolev embedding. The operator $h\partial_{y_1}G^M_+ : L^2 \to_{h^0} L^2$ since the Fourier multiplier $h^{1/2}v_{\xi_{11}+h\lambda_{11}}^{2}$ is now uniformly bounded. Therefore, the term in (4.13) involving $h\partial_{y_1}G^M_+$ can be written as an operator in $\Psi^0(M)$ plus a term which takes $L^2 \to_{h^0} H^k$ and $L^{p'} \to_{h^{-1}} H^k$. Same argument shows that the term in (4.13) involving $h^{2}\partial_{y_1}G^M_+$ can be written as the sum of a $\Psi^0(M)$ operator and a term mapping $L^2 \to_{h^0} H^k$ and $L^{p'} \to_{h^{-1}} H^k$.

We have thus shown that (4.13) is the sum of an operator in $\Psi^0(M)$ with an operator mapping $L^2 \to_{h^0} H^k$ and $L^{p'} \to_{h^{-1}} H^k$. The second term of (4.12) can be treated analogously to see that it is the sum of an operator in $\text{Op}(S^0_1(M_0))\text{Op}(S^{-2}_1(M))$ with an operator mapping $L^2 \to_{h^0} H^k$ and $L^{p'} \to_{h^{-1}} H^k$. This completes the proof. □

4.2. Approximate Semiclassical Inverse. Let $\tilde{\Omega} \subset M$ be a smooth bounded open subset contained in $M_+$ with a portion of the boundary intersecting $x_1 = 0$. Choose a bounded open interval $I \subset \mathbb{R}$ such that $\tilde{\Omega} \subset \gamma(I \times M_0)$. Let $\tilde{G}^I_+$ be the Green’s function defined by (4.4), and $J^+ := J^{-1}1_{M_+}$ where $J^{-1}$ is defined as Proposition 3.1. We first show that the operator

$$E_\ell := (1 - \tilde{\rho}(x', hD'))J^+J\tilde{G}^I_+$$

is a suitable parametrix for the operator $h^2\tilde{\Delta}_+$ in $\tilde{\Omega}$ at large frequencies. We see first using (4.14) and Proposition 3.1 that

$$E_\ell : L^2 \to_{h^{-1}} H^1, \quad E_\ell : L^{p'} \to_{h^{-2}} H^1, \quad E_\ell : L^{p'} \to_{h^{-1}} L^p.$$  

We now state the parametrix property for $E_\ell$. In the following statement we denote $1_\tilde{\Omega}$ to be the indicator function of $\tilde{\Omega}$. If $v \in L^p(\tilde{\Omega})$ we also use $1_\tilde{\Omega}v$ to denote its trivial extension to a function in $L^p(M)$.

**Proposition 4.4.** The operator $E_\ell$ is a Dirichlet parametrix in the sense that for all $v \in L^{p'},$

$$h^21_\tilde{\Omega}\tilde{\Delta}_+E_\ell1_\tilde{\Omega}v = ((1 + |df|^2)(1 - \tilde{\rho}(hD'))(1 + |df|^2)^{-1} + R_1 + R_1')v, \quad E_\ell v |_{M_+} = E_\ell v |_{x_1=0} = 0$$

as distribution on $\tilde{\Omega}$ with

$$R_1 : L^2 \to hL^2, \quad R_1 : L^{p'} \to_{h^0} L^2, \quad R_1 : L^{p'} \to_{h^0} L^p.$$  

Furthermore, if supp$(v) \subset M_+$ then supp$(R_1v) \subset \overline{M}_+$.

**Proof.** We compute in the sense of distributions acting on $C^\infty_c(\tilde{\Omega})$ and express $h^2\tilde{\Delta}_+$ using (4.11) to get

$$h^2(1 + |df|^2)^{-1}\tilde{\Delta}_+E_\ell = (I - \tilde{\rho}(x', hD'))(1 + |df|^2)^{-1}h^2\tilde{\Delta}_+J^+J\tilde{G}^I_+ + [h^2\tilde{\Delta}_+, \tilde{\rho}]J^+J\tilde{G}^I_+$$

$$+ h\tilde{E}_1(I - J^+J)\tilde{G}^I_+ + R.$$

(4.16)
where
\[ R = (I - \tilde{\rho}(x', hD'))(1 + |df|^2)\left( \tilde{A}_0(I - J+J) - h^2\tilde{E}_0(I - J+J) + h^2\Psi^0_1(M_0) + h^2\Psi^0_1(M_0)J+J \right)G^I_+ . \]

Using (4.3), Sobolev embedding, and the fact that \( I - \tilde{\rho}(x', hD') \) is microlocally disjoint from \( \tilde{A}_0 \) by the choice of \( \tilde{\rho} \) in (4.10), we see that every term in \( R \) takes \( L^2 \rightarrow hL^2 \), \( L^{p'} \rightarrow h^0 L^2 \).

Directly by using Lemma 4.3 the term \( h\tilde{E}_1\tilde{G}^I_+ \) can be written as \( R_t + R'_t \) where \( R_t \) and \( R'_t \) satisfies the estimates of (4.15). Writing explicitly the term
\[ h\tilde{E}_1\tilde{G}^I_+ = h\tilde{E}_1J^{-1}1_{M_+}J\tilde{G}^I_+ \]
we can commute \( \tilde{E}_1 \) with all the pseudodifferential operators by using standard calculus. Estimate the terms involving commutators using (4.5) to see that they are of the form (4.15). This is done in (4.16) and we can use Lemma 4.3 again to show that it is of the form (4.15).

The only remaining term to treat in (4.16) is the \((1 + |df|^2)^{-1}h^2\tilde{\Delta}_+, \tilde{\rho})J^+J\tilde{G}^I_+\) term. This is done in

**Lemma 4.5.** The commutator term
\[ [(1 + |df|^2)^{-1}h^2\tilde{\Delta}_+, \tilde{\rho}(x', hD')]J^+J\tilde{G}^I_+ \]
maps \( L^2 \rightarrow hL^2 \) and \( L^{p'} \rightarrow h^0 L^2 \).

and the proof is complete. \( \square \)

**Proof of Lemma 4.5.** Since \([hD_1, \tilde{\rho}(x', hD')] = 0 \) we have, using the expression (4.3)
\[ [(1 + |df|^2)^{-1}h^2\tilde{\Delta}_+, \tilde{\rho}]J^+J\tilde{G}^I_+ \equiv \left[ \text{Op}\left( -2i\frac{\xi_0(x', df) + hiF}{1 + |df|^2} \right), \tilde{\rho}(x', hD') \right]hD_1J^+J\tilde{G}^I_+ \]
\[ + \left[ \text{Op}\left( \frac{1 - |\xi|^2 + h\xi_0(K)}{1 + |df|^2} \right), \tilde{\rho}(x', hD') \right]J^+J\tilde{G}^I_+ , \]
\[ \equiv \left[ \text{Op}\left( -2i\frac{\xi_0(x', df) + hiF}{1 + |df|^2} \right), \tilde{\rho}(x', hD') \right] \left( I + \tilde{A}_+ J^{-1} \right)1_{M_+}J\tilde{G}^I_+ \]
\[ + \left[ \text{Op}\left( \frac{1 - |\xi|^2 + h\xi_0(K)}{1 + |df|^2} \right), \tilde{\rho}(x', hD') \right] J^+J\tilde{G}^I_+ . \]

Here "\( \equiv \)" denotes equivalence modulo a map taking \( L^2 \rightarrow hL^2 \) and \( L^{p'} \rightarrow h^0 L^2 \). Observe that since both \( \left[ \text{Op}\left( -2i\frac{\xi_0(x', df) + hiF}{1 + |df|^2} \right), \tilde{\rho} \right] \) and \( \left[ \text{Op}\left( \frac{1 - |\xi|^2 + h\xi_0(K)}{1 + |df|^2} \right), \tilde{\rho}(x', hD') \right] \) are \( \Psi DO \) on \( M_0 \) they commute with the indicator function \( 1_{M_+} \). Using this and the fact that \( \tilde{\rho}(x', hD')(x', \xi') \) is supported away from the characteristic set of \( \tilde{\Delta}_+ \) we have
\[ [(1 + |df|^2)^{-1}h^2\tilde{\Delta}_+, \tilde{\rho}]J^+J\tilde{G}^I_+ \equiv h(I + \tilde{A}_+J^{-1})1_{M_+}a(x, hD)\tilde{G}^I_+ + hJ^+b(x, hD)\tilde{G}^I_+ \]
for some \( a, b \in S^0_1(M) \) supported away from the characteristic set of \( \tilde{\Delta}_+ \). We now apply Lemma 4.2 to see that this operator takes \( L^2 \rightarrow hL^2 \) and \( L^{p'} \rightarrow h^0 L^2 \). \( \square \)

The following Lemma says that \( E_\varepsilon \) is almost like \( G^I_+ \) on compact subsets of the open set \( M_+ \):
Lemma 4.6. Let \( a(x, hD) \) be a first order differential operator whose coefficients are compactly supported in the region \( \{ x_1 \geq \epsilon \} \) for some \( \epsilon > 0 \). Then
\[
1_\omega ha(x, hD)(\tilde{G}^I_+ - E_\ell)1_\omega : L^2(\tilde{\Omega}) \to_h L^2(\tilde{\Omega}),
\]
\[
1_\omega ha(x, hD)(\tilde{G}^I_+ - E_\ell)1_\omega : L^p(\tilde{\Omega}) \to_h L^p(\tilde{\Omega}).
\]

Proof. We have by definition
\[
1_\omega ha(x, hD)(\tilde{G}^I_+ - E_\ell)1_\omega = 1_\omega ha(x, hD)(\tilde{G}^I_+ - (1 - \tilde{\rho}(x', hD'))J^+ J\tilde{G}^I_+)1_\omega.
\]
For the \( 1_\omega ha(x, hD)(\tilde{G}^I_+ - J^+ J\tilde{G}^I_+)1_\omega \) portion we have
\[
1_\omega ha(x, hD)(I - J^+ J)\tilde{G}^I_+1_\omega = 1_\omega ha(x, hD)J^{-1}1_MJ\tilde{G}^I_+1_\omega.
\]
By assumption \( a(x, hD) \) is a first order differential operator whose coefficients are supported in \( \{ x_1 \geq \epsilon > 0 \} \). The proof then follows from Lemma 3.2 and the mapping properties of (4.5).

Moving on to the \( 1_\omega ha(x, hD)(\tilde{\rho}(x', hD')J^{-1}1_MJ\tilde{G}^I_+)1_\omega \) portion we have
\[
1_\omega ha(x, hD)(\tilde{\rho}(x', hD')J^{-1}1_MJ\tilde{G}^I_+)1_\omega \equiv 1_\omega ha(x, hD)(J^{-1}1_MJ\tilde{\rho}(x', hD')\tilde{G}^I_+)1_\omega
\]
where \( \equiv \) denotes equality up to a map taking \( L^2(\tilde{\Omega}) \to_h L^2(\tilde{\Omega}) \) and \( L^p(\tilde{\Omega}) \to_h L^p(\tilde{\Omega}) \). Note that we were able to commute \( \tilde{\rho}(x', hD') \) with \( 1_M \) because the indicator function is constant along each fiber \( \{ x_1 = \text{const} \} \) and \( \tilde{\rho}(x', hD') \) acts in the \( x' \) direction only. The proof is complete by observing that (4.7) says that \( \tilde{\rho}(x', \xi') \) is supported away from the characteristic set of \( \Delta_+ \) and apply Lemma 4.2. \( \square \)

At small \( \xi' \) on the support of \( \tilde{\rho}(x', \xi') \) the square root defined in (4.6) is discontinuous so we cannot factor \( \Delta_+ \) as in (4.9). Here we are saved by the fact that \( \Delta_+ \) is actually elliptic thanks to (4.7). The parametrix in this region can then be constructed via straightforward elliptic calculus. To this end define
\[
P(x', \xi) := \xi_1^2 - 2\xi_1 \frac{(i - g_0(df(\xi', i \hbar F))}{1 + |df|^2} - \frac{(1 - |\xi'|^2 - i\hbar \xi'(K))}{1 + |df|^2}
\]
and
\[
E_s := \tilde{\rho}^{-1}P(x', hD) \circ (1 + |df|^2)^{-1}.
\]
The following proposition says that \( E_s \) inverts \( h^2\Delta_+ \) at small frequencies, up to an \( O(h) \) error.

Proposition 4.7. We have that \( E_s : L^r \to W^{2,r} \) for all \( r \in (1, \infty) \). Moreover for all \( r \in (1, \infty) \)
\[
h^2\Delta_+ E_s = (1 + |df|^2)\tilde{\rho}(x', hD')(1 + |df|^2)^{-1} + R_s
\]
for some \( R_s : L^r \to hL^r \).

Proof. Standard symbol calculus defined in Section 2 does not apply as \( 1/P(x', \xi) \) is not a proper symbol, due to the zeros of \( P(x', \xi) \).

We can remedy this by writing
\[
\tilde{\rho}(\xi')/P(x', \xi) = (1 - \chi_3(\xi))\tilde{\rho}(\xi')/P(x', \xi) + \chi_3(\xi)\tilde{\rho}(\xi')/P(x', \xi)
\]
where $\chi_3(\xi) \in S^{-\infty}(M)$ is a smooth symbol supported only for $|\xi| < 3$, and identically one in the ball $|\xi| \leq 2$.

Now note that by (1.7), $\mathcal{P}(x', \xi)$ is properly elliptic on the support of $\tilde{\rho}(\xi')$, and therefore $\chi_3(\xi)\tilde{\rho}(\xi')/\mathcal{P}(x', \xi) \in S^{-\infty}(M)$. Moreover, since the characteristic set of $\mathcal{P}(x', \xi)$ lies well inside the set where $\chi_3 \equiv 1$, we have that $(1 - \chi_3(\xi))/\mathcal{P}(x', \xi) \in S_1^{-2}(M)$.

Therefore $\tilde{\xi}^j(x', hD)$ is defined as the sum of two operators, one in the symbol class $S^{-\infty}(M)$ and the other of which is in the symbol class $S^{-\infty}S_1^{-2}(M)$. Then Proposition 2.1 asserts that $E_s : L^r \to W^{2,r}$ is a bounded operator and Proposition 2.2 asserts that

$$h^2(1 + |df|^2)^{-1} \Delta_+ \text{Op} \left( \frac{\tilde{\rho}}{\mathcal{P}} \right) = \text{Op}((1 - \chi_3)\tilde{\rho}) + \text{Op}(\chi_3\tilde{\rho}) + hR_{-1} = \text{Op}(\tilde{\rho}) + R_s$$

as we wanted. □

It turns out that $E_s$ preserves support in $M_+$.

**Proposition 4.8.** Suppose $v \in L^r(M)$, with $1 < r < \infty$, and supp($v$) is contained in the closure of $M_+$. Then both supp($E_s v$) and supp($R_s v$) are contained in $M_+$, where $R_s$ is the operator from Proposition 4.7. In particular, $E_s v |_{x_1 = 0} = 0$ if supp($v$) $\subseteq M_+$.

**Proof.** Let $U \subset M_0$ be a coordinate patch. It suffices to prove this statement for compactly supported smooth functions $v$ in the (infinite) strip $\mathbb{R} \times U = \{(x_1, x') \mid x' \in U\}$. Let $v(x_1, x')$ also denote the pullback function by the coordinate map then $\text{Op}(\tilde{\rho}/\mathcal{P}) v(x_1, x')$ is

$$(4.17) \quad h^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\xi_1(x_1-y_1)/h} h \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\tilde{\rho}(x'+y', \xi')}{\mathcal{P}(x'+y', \xi')} v(y_1, y') e^{i\xi_1(x_1-y_1)/h} \, d\xi_1 \, dy_1 \, d\xi' \, dy'. $$

We want to evaluate the inner most integral in $\xi_1$ using the residue calculus. Since $e^{i\xi_1(x_1-y_1)/h}$ is analytic, we need to understand the zeros of $\mathcal{P}(x'+y', \xi)$ as a polynomial in $\xi_1$. Factoring and suppressing the dependence on the spacial variable, we have

$$\mathcal{P}(\frac{x'+y'}{2}, \xi) = - (\xi_1 - a_+) (\xi_1 - a_-)$$

where $a_{\pm} = \frac{i(1 + |g_0(df')|^2 + hF) \pm r_0}{1 + |df|^2}$ with $r_0$ the square root give by (1.6).

Therefore $\mathcal{P}(x', \xi)$, viewed as a polynomial in $\xi_1$, has two roots: $a_+$ and $a_-$. The symbol $a_+$ has positive imaginary part because $r_0$ is defined using the branch of the $\sqrt{\cdot}$ with cut along the negative real axis.

We want to ensure that on the support of $\tilde{\rho}(\frac{x'+y'}{2}, \xi')$ the imaginary part of $a_- (\frac{x'+y'}{2}, \xi')$ is strictly positive for all $h > 0$ small, $x'$, $y'$, and $\xi'$. First, by (1.7) the polynomial $\mathcal{P}(\frac{x'+y'}{2}, \xi)$ never vanishes on the support of $\tilde{\rho}(\frac{x'+y'}{2}, \xi')$ so the imaginary part of $a_-$ is bounded away from zero on the support of $\tilde{\rho}(\frac{x'+y'}{2}, \xi')$ (otherwise $\xi_1$ can be chosen to make $\mathcal{P}(\frac{x'+y'}{2}, \xi)$ close to zero, contravening (1.7)). This means that on the support of $\tilde{\rho}(\frac{x'+y'}{2}, \xi')$, the real-valued function $1 + hF - \text{Re}(r_0)$ stays uniformly away from zero. Note that the standard branch of $\sqrt{\cdot}$ defined on $\mathbb{C} \setminus \{z \in \mathbb{C} \mid \text{Re}(z) \leq 0\}$ has a continuous extension onto the closed blowup manifold $[\mathbb{C}; \{z \in \mathbb{C} \mid \text{Re}(z) \leq 0\}] \to \mathbb{C}$. This means that for each fixed $h > 0$ small, $x' \in M_0$, and $y' \in M_0$, the function $1 + hF - \text{Re}(r_0)$ is either uniformly positive or uniformly negative for all $\xi'$. Choosing $\xi' = 0$, we see that $1 + hF - \text{Re}(r_0)$ is uniformly positive. Therefore we have that $\text{Im}(a_-) > 0$ on the support of $\tilde{\rho}(\frac{x'+y'}{2}, \xi')$ as well.
Therefore to evaluate the \( \xi_1 \) integral of \((4.17)\) we must use the contour integral in the upper-half of \( \mathbb{C} \). Doing so we get
\[
2\pi i \hat{\rho}\left(\frac{x' + y'}{2}, \xi'\right) \int_{-\infty}^{x_1} \frac{v(y_1, y')(e^{i\alpha_-(x_1-y_1)/h} - e^{i\alpha_+(x_1-y_1)/h})}{(a_+ - a_-)} \, dy_1
\]
for the case when \( a_+ \neq a_- \). So \((4.17)\) can be written as
\[
(4.18) \quad 2\pi i h^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\xi'(x'-y')/h} \int_{-\infty}^{x_1} \hat{\rho}(\xi')v(y_1, y')(e^{i\alpha_-(x_1-y_1)/h} - e^{i\alpha_+(x_1-y_1)/h}) \, ds \, d\xi' \, dy'.
\]

We now treat the case when \( a_+ \) is close to \( a_- \). On the set where \( a_+ = a_- \), the residue calculus tells us that the integral vanishes, and near this set we have
\[
\lim_{a_+ - a_- \to 0} \frac{e^{i\alpha_-(x_1-y_1)/h} - e^{i\alpha_+(x_1-y_1)/h}}{(a_+ - a_-)} = \frac{i(x_1 - y_1)}{h} e^{i\alpha_-(x_1-y_1)/h}.
\]
Therefore the integral on the right side of \((4.18)\) converges, and so if \( v \in C_c^\infty(\mathbb{R} \times U) \) is supported only in \( \tilde{\Omega}_+ \), it is clear that
\[
(4.19) \quad \text{Op}(\hat{\rho}/\mathcal{P})v(x_1, x') = 0 \quad \text{for} \quad x_1 \leq 0.
\]

Using Proposition \((4.7)\) and boundedness of the trace operator we see that if \( v \in L^r(M) \) is supported in \( \tilde{\Omega}_+ \) then \( \text{Op}(\hat{\rho}/\mathcal{P})v(x_1, \cdot) = 0 \) for \( x_1 \leq 0 \).

The support property for \( R_s \) then follows from writing
\[
(1 + |df|^2)^{-1/2} \Delta h^2 \Delta E_s - h^2 \Delta E_s = (1 + |df|^2)^{-1} = h R_s
\]
and noting that every operator on the left hand side of this equation has the desired support property.

**4.3. Proof of Proposition \((1.3)\)** We now turn the semiclassical parametrix constructed in Subsection \((1.2)\) into a proper inverse for \( h^2 \Delta_+ \). By Propositions \((4.4)\) and \((4.7)\) see that \( 1_{\Omega}(E_s + E_d)1_{\Omega} \) is a parametrix for the operator \( h^2 \Delta_+ \) in the domain \( \Omega \). As one expects, this parametrix can be modified into a Green’s function.

We begin with the case where \( \Gamma \) is a component of the boundary which coincides with the graph of a function. In particular, let \( \Omega \) be a bounded domain in \( M \), and suppose \( f \in C^\infty(M_0) \) such that \( \Omega \) lies in the set \( \{ y_1 > f(y') \} \), with a portion of the boundary \( \Gamma \subset \partial \Omega \) lying on the graph \( \{ y_1 = f(y') \} \). Denote by \( \gamma \) the change of variable \( (y_1, y') \mapsto (x_1 = y_1 - f(y'), x' = y') \) and \( \tilde{\Omega} = \gamma(\Omega) \). In this setting \( \tilde{\Omega} \subset M_+ \) and \( \gamma(\Gamma) \subset \{ x_1 = 0 \} \).

**Proposition 4.9.** There exists a Green’s function
\[
G_\Gamma : L^2(\tilde{\Omega}) \to_{h^{-1}} L^2(\Omega), \quad G_\Gamma : L^{p'}(\tilde{\Omega}) \to_{h^{-2}} L^p(\Omega).
\]
which satisfies the relation \( h^2 \Delta_+ G_\Gamma = I_d \) as distributions on \( \Omega \). It has the explicit representation
\[
G_\Gamma = \gamma^*\left(1_{\tilde{\Omega}}E_s + E_d\right)1_{\tilde{\Omega}}(I + R)
\]
with \( R \) obeying the estimates
\[
R : L^{p'}(\tilde{\Omega}) \to_{h^{0}} L^2(\tilde{\Omega}), \quad R : L^2(\tilde{\Omega}) \to_{h} L^2(\tilde{\Omega}).
\]
Furthermore, \( G_\Gamma v \in H^1(\Omega) \) for all \( v \in L^{p'} \) and \( G_\Gamma v \mid_{\Omega} = 0 \).
Proof. Change coordinates \((y_1, y') \mapsto (x_1, x')\) so that \(\tilde{\Gamma} := \gamma(\Gamma) \subset \{x_1 = 0\}\) and let \(\tilde{\Delta}_+\) be the pulled-back conjugated Laplacian. By Proposition 4.4 and Proposition 4.7,
\[
h^2\tilde{\Delta}_+ 1_{\tilde{\Omega}}(E_s + E_\ell) 1_{\tilde{\Omega}} = I + R_s + R_\ell + R'_\ell
\]
with \(R_s + R'_\ell\) mapping \(L^r(\tilde{\Omega}) \to h \ L^r(\tilde{\Omega})\). Let \(S : L^r(\tilde{\Omega}) \to L^r(\tilde{\Omega})\) denote the inverse of \((1 + R_s + R'_\ell)\) by Neumann series. Then in \(\tilde{\Omega}\) we have
\[
h^2\tilde{\Delta}_+ 1_{\tilde{\Omega}}(E_s + E_\ell) 1_{\tilde{\Omega}} S = I + R_\ell S.
\]
By Proposition 4.4 we have \(R_\ell S : L^2(\tilde{\Omega}) \to h \ L^2(\tilde{\Omega})\) while \(R_\ell S : L^{p'}(\tilde{\Omega}) \to h^{p'} L^2(\tilde{\Omega})\). Therefore we can invert by Neumann series again to obtain a right inverse for \(h^2\tilde{\Delta}_+\) of the form
\[
1_{\tilde{\Omega}}(E_s + E_\ell) 1_{\tilde{\Omega}} S(I + R_\ell S)^{-1}
\]
and
\[
(I + R_\ell S)^{-1} : L^2 \to h^{p'} L^2, \quad (I + R_\ell S)^{-1} : L^{p'} \to h^p L^2 + L^{p'}.
\]
Changing variables we see that \(h^2\Delta_+ G_\Gamma = Id\) by setting
\[
G_\Gamma := \gamma^* \left( 1_{\tilde{\Omega}}(E_s + E_\ell) 1_{\tilde{\Omega}} S(I + R_\ell S)^{-1} \right).
\]
The mapping properties and Dirichlet boundary condition follows then from the analogous properties for \(E_\ell\) and \(E_s\) outlined in Propositions 4.1-4.4 and (4.14). \(\Box\)

To prove Proposition 4.3 in the general case, we patch together Green’s functions as \([4]\). Let \(\Gamma\) be a closed and connected component of \(\partial\Omega\) contained in an open set \(\Omega\) such that there exists \(f \in C^\infty(M_0)\) for which \(\Omega_G \cap \partial M \subset \{y_1 \geq f(y')\}\) and \(\partial M \cap \Omega_G \cap \{y_1 = f(y')\} = \Gamma\). We may choose \(\Omega_G\) small enough such that \(\Omega_G \cap \Omega\) is contained in the epigraph of \(f\). If a compact connected component of \(\partial\Omega\) satisfies this condition, we say that \(\Gamma\) is compatible with a smooth function \(f\).

Choose \(\chi \in C^\infty_c(\Omega_G)\) such that \(\chi = 1\) on \(\Gamma\) and define \(\Omega := \Omega_G \cap \{y_1 > f(y')\}\). By the fact that \((\Omega_G \cap \partial M) \setminus \Gamma\) lies strictly above the graph \(y_1 = f(y')\), we can arrange \(\chi\) so that
\[
(4.20) \quad \exists \epsilon > 0 \ | \ \text{supp}(1_{\Omega} D \chi) \subset \{(y_1, y') \ | \ y_1 \geq f(y') + \epsilon\}.
\]
Choose \(I \subset \mathbb{R}\) such that \(\Omega\) is contained in \(I \times M_0\) and let \(G_\Gamma^I\) be the Green’s function defined via \((4.4)\). Let \(G_\Gamma\) now the Green’s function constructed in Proposition 4.9 for the domain \(\Omega\) with vanishing Dirichlet condition along the portion \(\{y_1 = f(y')\}\).

Now define
\[
(4.21) \quad \Pi_\Gamma : L^{p'}(\Omega) \to h^{-2} L^{p}(\Omega), \quad \Pi_\Gamma : L^{2}(\Omega) \to h^{-1} H^1(\Omega)
\]
by \(\Pi_\Gamma := \chi 1_{\Omega}(G_\Gamma^I - G_\Gamma) 1_{\Omega}\).

Proposition 4.9 yields the boundary condition
\[
(4.22) \quad \Pi_\Gamma v \in H^1(\Omega), \quad (\Pi_\Gamma v) |\Gamma = (G_\Gamma^I v) |\Gamma, \quad \forall v \in L^{p'}(\Omega).
\]

Lemma 4.10. One has the estimates
\[
h^2\Delta_+ 1_{\Omega} \Pi_\Gamma : L^{p'}(\Omega) \to h^{p'} L^2(\Omega), \quad h^2\Delta_+ 1_{\Omega} \Pi_\Gamma : L^{2}(\Omega) \to h^1 L^2(\Omega).
\]

With this lemma we are in a position to construct a general Green’s function for the \(h^2\Delta_+\) on a general domain \(\Omega\). Let \(\Gamma \subset \partial \Omega\) be a compact set contained in \(\{y \in \partial \Omega \ | \ g(\partial \Omega, v(y)) > 0\}\). Since \(\Omega \subset I \times M_0\) for some simple manifold \(M_0\) we may write \(\Gamma\) as the disjoint union \(\bigcup \Gamma_j\) of connected compact components \(\Gamma_j\) each of which is compatible with a smooth function \(f_j\).
For each $\Gamma_j$ construct $\chi_j$ and $\Pi_{\Gamma_j}$ as earlier. One then, by (4.22), has that
\[
\left( G^I_+ v - \sum_{j=1}^k \Pi_{\Gamma_j} v \right) |_{\Gamma} = 0, \quad \forall v \in L^{p'}(\Omega).
\]
Furthermore by Lemma 4.10 $h^2 \Delta_+ 1_\Omega(G^I_+ - \sum_{j=1}^k \Pi_{\Gamma_j}) 1_\Omega = I + R'$ with
\[
R' : L^2(\Omega) \to_{h} L^2(\Omega), \quad R' : L^{p'}(\Omega) \to_{h^0} L^2(\Omega).
\]
Note that we can as before find an inverse $(1 + R')^{-1} : L^2 \cup L^{p'} \to_{h^0} L^2$. Proposition 1.3 is now complete by the estimates of (4.21) and (4.1). All that remains is to give a proof of Lemma 4.10.

**Proof of Lemma 4.10.** By Proposition 1.3 $G_{\Gamma}$ is by construction a right inverse for $h^2 \Delta_+$ in $\mathcal{D}$, and $\chi 1_\Omega$ is supported only on $\mathcal{D}$, so $\chi h^2 \Delta_+ 1_\Omega G_{\Gamma} v(y) = v(y)$ as distributions on $\Omega$. Meanwhile $G^I_+$ is an honest right inverse for $h^2 \Delta_+$ on $\Omega$, so $h^2 \Delta_+ 1_\Omega G^I_+ v = v$ as distributions on $\Omega$. Therefore as distributions on $\Omega$, the only term in $h^2 \Delta_+ \Pi_{\Gamma_j} v(y)$ is $[h^2 \Delta_+, \chi_j(y)]1_\Omega(G^I_+ - G_{\Gamma_j})1_\Omega v(y)$. Since the sum is finite in the definition of $\Pi_{\Gamma}$ we may assume without loss of generality that there is only one term in the sum and write $\chi_j = \chi$.

To analyze this term we will change coordinates by $(y_1, y') \mapsto (x_1 = y_1 - f'(y'), x' = y')$ and mark the pushed forward domains, functions and operators with a tilde. Then by the push-forward expression for the operator $G_{\Gamma}$ stated in Proposition 4.9 the operator in our term becomes
\[
[h^2 \tilde{\Delta}_+, \tilde{\chi}(x)]1_{\tilde{\Omega}}(G^I_+ - (E_s + E_\ell))1_{\tilde{\Omega}}(I + R))1_{\tilde{\Omega}}
\]
where
\[
R : L^{p'}(\tilde{\Omega}) \to_{h^0} L^2(\tilde{\Omega}), \quad R : L^2(\tilde{\Omega}) \to_{h} L^2(\tilde{\Omega}).
\]
Computing the commutator $[h^2 \tilde{\Delta}_+, \tilde{\chi}]$ explicitly in conjunction with the operator estimates in Proposition 4.7 and (4.14) we have that
\[
(4.23) \quad [h^2 \tilde{\Delta}_+, \tilde{\chi}]1_{\tilde{\Omega}}(G^I_+ - (E_s + E_\ell))1_{\tilde{\Omega}}S(1 + R_i S)^{-1})1_{\tilde{\Omega}} = [h^2 \tilde{\Delta}_+, \tilde{\chi}]1_{\tilde{\Omega}}(G^I_+ - (E_s + E_\ell))1_{\tilde{\Omega}}S1_{\mathcal{D}} + E
\]
where
\[
E : L^{p'}(\tilde{\Omega}) \to_{h^0} L^2(\tilde{\Omega}), \quad E : L^2(\tilde{\Omega}) \to_{h} L^2(\tilde{\Omega}).
\]
We see that the $E$ term in (4.23) has the correct boundedness properties, so it remains only to analyze the first term of (4.23):
\[
[h^2 \tilde{\Delta}_+, \tilde{\chi}]1_{\tilde{\Omega}}(G^I_+ - (E_s + E_\ell))1_{\tilde{\Omega}}S1_{\mathcal{D}}.
\]
Since we are only doing the computation as distributions acting on $C^\infty(\tilde{\Omega})$, the first order differential operator $[h^2 \tilde{\Delta}_+ \tilde{\chi}]$ commutes with the indicator function $1_{\tilde{\Omega}}$, and we have
\[
[h^2 \tilde{\Delta}_+, \tilde{\chi}]1_{\tilde{\Omega}}(G^I_+ - (E_s + E_\ell))1_{\tilde{\Omega}} = 1_{\tilde{\Omega}}[h^2 \tilde{\Delta}_+, \tilde{\chi}](G^I_+ - E_{\ell})1_{\tilde{\Omega}} - 1_{\tilde{\Omega}}[h^2 \tilde{\Delta}_+, \tilde{\chi}]E_{\ell}1_{\tilde{\Omega}}.
\]
Now $E_{\ell}$ maps $L^2 \to_{h^0} H^2$ and $L^{p'} \to W^{2,p'} \to_{h^{-1}} H^1$. Meanwhile the commutator $[h^2 \tilde{\Delta}_+, \tilde{\chi}]$ maps $H^4$ to $L^2$ with the gain of $h$, so the term involving $E_{\ell}$ has the desired behaviour. Therefore the only term of difficulty is $1_{\tilde{\Omega}}[h^2 \tilde{\Delta}_+, \tilde{\chi}](G^I_+ - E_{\ell})1_{\tilde{\Omega}}$. But by (4.20) the term $1_{\tilde{\Omega}}[h^2 \tilde{\Delta}_+, \tilde{\chi}]$ is a first order differential whose coefficients are supported in $\{x_1 \geq \epsilon > 0\}$. The proof then follows from Lemma 4.6. \qed
5. Complex Geometrical Optics

Let $M = \mathbb{R} \times M_0$ and $g = dy_1^2 + g_0$ be a metric on $M$. Consider the bounded domain $\Omega \subset M$ and let $\Gamma \subset \partial \Omega$ be an open subset of the boundary compactly contained in $\{y \in \partial \Omega \mid g(\nu(x), \partial y) > 0\}$ where $\nu_n$ denotes the normal vector. By Proposition 1.3 there exists a Green’s function $G_T$ for $h^2\Delta_\phi$ with vanishing trace on $\Gamma$ and

$$G_T : L^2(\Omega) \to h^{-1} L^2(\Omega), \quad G_T : L^p(\Omega) \to h^{-2} L^p(\Omega).$$

5.1. Application of Green’s Function to Solvability. In the geometric setting described above, we can use the same argument as in [9] to prove the following

**Proposition 5.1.** Let $L \in L^2(\Omega)$ with $\|L\|_{L^2} \leq Ch^2$, and let $q \in L^{n/2}(\Omega)$. For all $a = a_h \in L^\infty$ with $\|a_h\|_{L^\infty} \leq C$, there exists a solution of

$$e^{-y_1/h^2}(\Delta_y + q)e^{y_1/h}r = h^2qa + L \quad r|_\Gamma = 0$$

with estimates $\|r\|_{L^2} \leq o(1)$ and $\|r\|_{L^p} \leq O(1)$.

Observe that we can generalize this to metrics which are conformal to $dy_1^2 \oplus g_0$ (i.e. CTA metrics). Indeed, if $c^{-1}q$ is a metric conformal to $g = dy_1^2 \oplus g_0$ then one can write as in [8] the Schrödinger operator for $c^{-1}g$ as

$$c^{n+2}c^{-1}q)u = (\Delta c^{-1}q + q(c^{n+2}u)$$

where $q_c := cq + c^{n+2}\Delta_g c^{-n+2} \in L^{n/2}(\Omega)$. Therefore Proposition 5.1 immediately generalizes to metrics which are conformal to $dy_1^2 \oplus g_0$.

**Corollary 5.2.** Let $g$ be a CTA metric on $M = \mathbb{R} \times M_0$. Let $\Omega \subset M$ be a bounded open subset and $\Gamma \subset \{y \in \partial M \mid g(\partial y_1, \nu(y)) > 0\}$. For all $L \in L^2(\Omega)$ with $\|L\|_{L^2} \leq Ch^2$, $q \in L^{n/2}(\Omega)$, and $a = a_h \in L^\infty$ with $\|a_h\|_{L^\infty} \leq C$, there exists a solution of

$$e^{-y_1/h^2}(\Delta_y + q)e^{y_1/h}r = h^2qa + L \quad r|_\Gamma = 0$$

with estimates $\|r\|_{L^2} \leq o(1)$ and $\|r\|_{L^p} \leq O(1)$.

5.2. CGO In Conformally Transversally Anisotropic Manifold. We first construct the CGO ansatz following the method of [14]. Assume that $\Omega \subset \mathbb{R} \times M_0$ where $\Omega_0 \subset M_0$ is a simple manifold compactly contained in a slightly larger simple manifold $\Omega_0$. Let $\omega \in \Omega_0 \setminus \Omega$ and set $(t, \theta)$ to be spherical coordinate around this point. The metric in these coordinates is $g = c(dy_1^2 \oplus dt^2 \oplus g_0)$ so $\phi + i\psi = y_1 + it$ solves the eikonal equation

$$g(d\phi + id\psi, d\phi + id\psi) = 0.$$

We can solve the transport equation $g(d\phi + i\psi, da) + \Delta_g (\phi + i\psi) = 0$, by setting

$$a = |g|^{-1/4} e^{i(y_1 + it)\beta(\theta)}$$

where $\beta$ is any smooth function on $S^{n-2}$. With $\phi$, $\psi$, and $a$ chosen as such we have

$$e^{-(\phi + i\psi)/h} \Delta_g e^{(\phi + i\psi)/h}a = O_L(h^2).$$

We now need to construct a reflection term $e^{\ell/h} b$ which kills $e^{(\phi + i\psi)/h}a$ on $\Gamma$. As before we will construct $\ell$ supported near $\Gamma$ solving the approximate equation

$$g(d\ell, d\ell)(z) = \text{dist}(z, \Gamma)\infty$$
with boundary condition $\ell \mid_{\Gamma} = (\phi + i\psi) \mid_{\Gamma}$ and $\partial_y\ell \mid_{\Gamma} = -\partial_y(\phi + i\psi) \mid_{\Gamma}$. The construction will be localized so we may assume without loss of generality that $\Gamma$ is compactly contained in single connected component of $\{ y \in \partial\Omega \mid g(\nu(y), \partial_y) > 0 \}$. Using boundary normal coordinates $(s, z') \in \mathbb{R}_+ \times \partial M$ near $\Gamma$, we may express the metric as $ds^2 \oplus g'$ for some symmetric two tensor $g'$ which annihilates $\partial_s$ where $s$ is the distance away from the boundary. Note that in a small neighbourhood of $\Gamma$, $g(ds, d\phi) \geq \epsilon$ which allows us to solve for individual terms of the formal expansion

$$\ell(s, z') = \sum_{j=0}^{\infty} s^j \ell_j(z'), \quad \ell_0(z') = \phi(0, z') + i\psi(0, z'), \quad \ell_1(z') = -\partial_y(\phi + i\psi)(0, z')$$

so that (5.4) is satisfied. Borel Lemma can then be employed to construct $\ell(s, z')$ solving (5.4). Similarly, the approximate transport equation

$$g(d\ell, db) = d(z, \partial M)^\infty \mid_{\Gamma} = -a \mid_{\Gamma}$$

can also be solved iteratively using formal power series and the fact $|g(ds, d\ell)| \geq \epsilon$ near $\Gamma$. Since we are only interested in the behaviour of $b$ at and near $\Gamma$, we may construct it to be supported in a small neighbourhood of $\Gamma$.

The ansatz given by $e^{\ell/h}b$ satisfies $e^{-\phi/h}h^2 \Delta_g(e^{\ell/h}b) = e^{(\ell-\phi)/h}(O(dist(z, \Gamma)^\infty + O_L^\infty(h^2))$. Note that by the boundary condition for $\partial \ell$ and the fact that $\Gamma \subset \subset \{ y \in \partial\Omega \mid g(\nu(y), \partial_y) \}$, we have the comparison $\frac{1}{h}d(z, \Gamma) \leq \phi(z) - \ell(z) \leq Cd(z, \Gamma)$ for $z$ on the support of $b$. So by analyzing $d(z, \Gamma) \leq \sqrt{h}$ and $dist(z, \Gamma) \geq \sqrt{h}$ we have that

$$e^{-\phi/h}h^2 \Delta_g(e^{\ell/h}b) = e^{(\ell-\phi)/h}O_L^\infty(h^2), \quad e^{\ell/h}b \mid_{\Gamma} = -e^{(\phi+i\psi)/h}a \mid_{\Gamma}.$$

We have constructed an ansatz of the form

$$u_{\text{ans}} = e^{(\phi+i\psi)/h}a + e^{\ell/h}b = e^{(\phi+i\psi)/h}(a + a_h)$$

where $a$ is of the form (5.3) and $a_h$ satisfies $\|a_h\|_{L^\infty} \leq C$ and $a_h \to 0$ pointwise such that

$$h^2(\Delta_g + q)u_{\text{ans}} = h^2e^{(\phi+i\psi)/h}(q(a + a_h) + L), \quad u_{\text{ans}} \mid_{\Gamma} = 0$$

with $\|L\|_{L^2} \leq Ch^2$.

Following precisely the argument of Prop. 3.4 in [9], the ansatz $u_{\text{ans}}$ combined with Corollary 5.1 allows us to construct the suitable CGO for solving our inverse problem. The only difference is that thanks to the fact that $G_\Gamma$ satisfies the Dirichlet condition on $\Gamma$ our CGO has vanishing trace on $\Gamma$. Note that by switching the sign the technique we have developed applies to $\Gamma_\pm$ compactly contained in $\{ z \in \partial\Omega \mid \pm g(\partial_y, \nu(z)) > 0 \}$ if we consider ansatz in (5.6) of the form $e^{\pm(\phi+i\psi)/h}(a + a^\pm_h)$. Therefore, we are able to construct CGOs $u_\pm$ vanishing on $\Gamma_\pm$ respectively:

**Proposition 5.3.** Let $\Omega$ be a bounded smooth domain in a CTA manifold $(M, g)$ and $\Gamma_\pm \subset \partial\Omega$ be an open subset compactly contained in

$$\{ z \in \partial\Omega \mid \pm g(\partial_y, \nu(z)) > 0 \}.$$ 

For all $q \in L^{n/2}$ there exists solutions to

$$(\Delta_g + q)u_\pm = 0, \quad u_\pm \in H^1(\Omega), \quad u_\pm \mid_{\Gamma_\pm} = 0$$

of the form

$$u_\pm = e^{\pm(\phi+i\psi)/h}(a + a^\pm_h + r).$$
where \( a \) is of the form (5.3), \( \| a_h^\pm \|_{L^\infty} \leq C, a_h^\pm \to 0 \) pointwise in \( \Omega \) as \( h \to 0 \). The remainder \( r \in L^p \) satisfies the estimates \( \| r \|_{L^2} = o(1) \) and \( \| r \|_p \leq C \) as \( h \to 0 \).

6. Proof of Theorem 1

We prove Theorem 1 using the ideas of [9]. Let \( \Gamma_\pm \) be open sets such that
\[
\partial\Omega \setminus B \subset \Gamma_+ \subset \{ z \in \partial\Omega \mid g(\partial_{y_1}, nu(z) > 0) \}, \quad \partial\Omega \setminus F \subset \Gamma_- \subset \{ z \in \partial\Omega \mid g(\partial_{y_1}, \nu(z) < 0) \}
\]

By Proposition 5.3, there exists solutions \( u_\pm \in H^1(\Omega) \) solving
\[
(\Delta + q_1)u_+ = 0, \quad u_+ |_{\Gamma_+} = 0, \quad (\Delta + q_2)u_- = 0, \quad u_- |_{\Gamma_-} = 0
\]
of the form
\[
u_i = e^{\frac{i(\phi+i\nu)}{\kappa}}(a_\pm + a_h^\pm + r_\pm), \quad \| r_\pm \|_{L^2} = o(1), \quad \| r_\pm \|_p = O(1).
\]

where \( a_\pm \) are of the form (5.3).

Since \( u_\pm \) are solutions belonging to \( H^1(\Omega) \) and vanish on \( \partial\Omega \setminus B \) and \( \partial\Omega \setminus F \) respectively, we have the following boundary integral identity (see Lemma A.1 of [9])
\[
\int_\Omega u_-(q_1 - q_2)u_+ = 0.
\]

Inserting the expressions for \( u_\pm \) gives
\[
(6.1) \quad 0 = \int_\Omega q(a_+ a_- + a_h^- a_h^+ + \bar{a}_h^- + \bar{a}_h^+ a_- + a_h^- r_+ + a_h^+ r_- + a_- r_+ + a_- r_+)\]

where \( q = q_1 - q_2 \). The function \( q \in L^{n/2} \subset L^1 \) and
\[
\| a_h^\pm \|_{L^\infty} \leq C, \quad \lim_{h \to 0} a_h^\pm(x) = 0 \quad \forall x \in \Omega
\]

by (5.6). Therefore, terms \( \lim_{h \to 0} \int_\Omega q(a_+ a_- + a_h^- a_h^+ + \bar{a}_h^- + \bar{a}_h^+ a_-) = 0 \). For the terms involving \( \int_\Omega |q a_h^\pm r_\pm| \), we note that for all \( \epsilon > 0 \) we may split \( q = q^\delta + q^\phi \) where \( q^\phi \in L^\infty \) while \( \| q^\delta \|_{L^{n/2}} \leq \epsilon \). Then, using the fact that \( \| a_h^\pm \|_{L^\infty} \leq C \),
\[
\int_\Omega |q a_h^\pm r_\pm| \leq C(\| q^\phi \|_{L^\infty} \| r_\pm \|_{L^2} + \| q^\delta \|_{L^{n/2}} \| r_\pm \|_p).
\]

By the estimates on \( r_\pm \) given in Proposition 5.3, we have that \( \lim_{h \to 0} \| r_\pm \|_{L^2} = 0 \) and \( \| r_\pm \|_p \leq C \). Therefore, the limit
\[
\lim_{h \to 0} \int_\Omega |q a_h^\pm r_\pm| \leq C \epsilon
\]

for all \( \epsilon > 0 \) and therefore the limit vanishes. The terms \( \int_\Omega e^{2i\xi} q(r_- a_+ + a_- r_+) \) can be estimated the same way. For the last term, we again decompose, for all \( \epsilon > 0 \), \( q = q^\phi + q^\phi \).

The integral \( \int_\Omega e^{2i\xi} x q r_- r_+ \) is then estimated by
\[
\int_\Omega |q^\delta r_- r_+| + \int_\Omega |q^\phi r_- r_+| \leq \| q^\delta \|_{L^\infty} \| r_- \|_{L^2} \| r_+ \|_{L^2} + \| q^\phi \|_{L^{n/2}} \| r_- \|_p \| r_+ \|_p.
\]

The \( L^p \) norms of \( r_\pm \) stay uniformly bounded while the \( L^2 \) norms vanish when \( h \to 0 \). Therefore the limit
\[
\lim_{h \to 0} \int_\Omega |q r_- r_+| \leq C \| q^\phi \|_{L^{n/2}} \leq C \epsilon
\]

for all \( \epsilon > 0 \) and therefore vanishes.
These limits show that the only surviving term out of (6.1) is
\[
0 = \int_{\Omega} qa + a_d V_g = \int_0^\infty \int_{S^{n-2}} \int_{-\infty}^{\infty} q(y_1, t, \theta) e^{i(y_1 \lambda + i\lambda t)} \beta(\theta) dy_1 d\theta dr.
\]
Where the function $\beta$, the coordinates $\theta$ and $t$ are chosen as in the definition of [9]. The proof now follows precisely as in [9] to show $q = 0$. 

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