NORMALIZED SOLUTIONS TO A SCHRÖDINGER-BOPP-PODOLSKY SYSTEM UNDER NEUMANN BOUNDARY CONDITIONS

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Abstract. In this paper we study a Schrödinger-Bopp-Podolsky system of partial differential equations in a bounded and smooth domain of $\mathbb{R}^3$ with a non constant coupling factor. Under a compatibility condition on the boundary data we deduce existence and multiplicity of solutions by means of the Ljusternik-Schnirelmann theory.

1. INTRODUCTION

The Schrödinger-Newton equation consists of a nonlinear coupling of the Schrödinger equation with a gravitational potential of newtonian form, representing the interaction of a particle with its own gravitational field.

In 1998, Benci and Fortunato [2] treated a similar problem, where the Schrödinger equation was coupled with Maxwell’s equations. Such coupling represents the interaction of the particle with its own electromagnetic field. The coupling factor is a constant $q \neq 0$. In their paper the authors consider standing waves solutions in the purely electrostatic field and this leads to the so-called Schrödinger-Poisson system. They impose a Dirichlet boundary condition both on the matter field $u$ and the electrostatic field $\phi$ and employed variational methods and critical point theory to develop a procedure that would become standard to treat other similar problems.

Later, Pisani and Siciliano [10] treated a Schrödinger-Poisson system with Neumann boundary conditions on the scalar field $\phi$ and considered the case in which the interaction factor responsible for the coupling of the equations is non-constant. This gives rise to important and interesting considerations regarding the geometry of the manifold on which find the solutions.

In this paper we treat a modification of the problem dealt with by Pisani and Siciliano consisting in the addition of a biharmonic term in the equation of the electrostatic potential and imposing appropriate boundary conditions. The problem studied can be interpreted as a coupling of the Schrödinger equation with the Bopp-Podolsky electrodynamics (for more details on this matter, see [3] and the references therein). However here we focus on the mathematical aspects of the problem.

We point out that in the literature there are few papers concerning Schrödinger-Bopp-Podolski systems. Beside [3] we cite here [4,8] where the authors study the problem with a critical nonlinearity, [5] where solutions with a priori given interaction energy for the Schrödinger-Bopp-Podolsky system are found, [7] where the problem has been addressed in the context of closed 3–dimensional manifolds both in the subcritical and critical case and [11] where the fiberng method has been used to prove existence results depending on a parameter and also nonexistence.

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Coming back to our problem, the aim here is to study the following system of partial differential equations in a connected, bounded, smooth open set $\Omega \subset \mathbb{R}^3$:

\begin{align}
-\Delta u + q\phi u - \kappa|u|^{p-2}u &= \omega u \quad \text{in } \Omega \tag{1.1} \\
\Delta^2 \phi - \Delta \phi &= qu^2 \quad \text{in } \Omega \tag{1.2}
\end{align}

in the unknowns $u, \phi : \Omega \to \mathbb{R}$ and $\omega \in \mathbb{R}$. Here $\kappa \in \mathbb{R}$ and $q : \Omega \to \mathbb{R}$ are given. We assume the following boundary conditions:

\begin{align}
u &= 0 \quad \text{on } \partial\Omega \tag{1.3} \\
\frac{\partial \phi}{\partial n} &= h_1 \quad \text{on } \partial\Omega \tag{1.4} \\
\frac{\partial \Delta \phi}{\partial n} &= h_2 \quad \text{on } \partial\Omega \tag{1.5}
\end{align}

and for simplicity we assume $h_1, h_2$ continuous. The symbol $n$ denotes the unit vector normal to $\partial\Omega$ pointing outwards. Since $u$ represents physically the amplitude of the wave function of a particle confined in $\Omega$, we assume the following normalizing condition:

\begin{equation}
\int_{\Omega} u^2 dx = 1. \tag{1.6}
\end{equation}

We also assume that the coupling factor $q$ is continuous on $\overline{\Omega}$:

\begin{equation}
q \in C(\overline{\Omega}). \tag{1.7}
\end{equation}

Our main result is the following:

**Theorem 1.** Let $\kappa > 0$, $p \in (2, 10/3)$ and

\begin{equation}
\alpha := \int_{\partial\Omega} h_2 ds - \int_{\partial\Omega} h_1 ds. \tag{1.8}
\end{equation}

Assume that $\inf_{\Omega} q < \alpha < \sup_{\Omega} q$ and that $|q^{-1}(\alpha)| = 0$. Then there exist infinitely many solutions $(u_n, \omega_n, \phi_n) \in H^1_0(\Omega) \times \mathbb{R} \times H^2(\Omega)$ to the problem (1.1) and (1.2) under conditions (1.3)-(1.7), with

\begin{equation}
\int_{\Omega} |\nabla u_n|^2 dx \to +\infty.
\end{equation}

Moreover the ground state solution can be assumed positive.

Our approach is variational, indeed the solutions will be found as critical points of an energy functional restricted to a suitable constraint. In this context by a ground state solution we mean the solution with minimal energy. Moreover as a byproduct of the proof, we obtain that also the energy of these solutions is divergent.

**Remark 1.** It is easy to see that if $\kappa < 0$ the result holds with $p \in (2, 6)$. For $\kappa = 0$, see [1].

The paper is organized as follows.

In Section 2 we introduce an auxiliary problem which will be useful in order to deal with homogeneous boundary conditions.

In Section 3 we give some properties of the constraint $M$ on which we will find the solution.

In Section 4 we introduce the energy functional and show that its critical points on $M$ will give solutions of the problem.

In the final Section 5 by implementing the Ljusternick-Schnirelmann theory we prove Theorem 1.

As a matter of notations, we use the letters $c, c', \ldots$ to denote positive constant whose value can change from line to line. We use $\| \cdot \|_p$ to denote the standard $L^p$–norm.
2. An auxiliary problem

Our aim is to define a functional whose critical points will be the weak solutions to the problem. In order to deal with homogeneous boundary conditions, that will permit to write the functional in a simpler form, we make a change of variable.

Consider the following auxiliary problem (where \( \alpha \) is defined in (1.8))

\[
\Delta^2 \chi - \Delta \chi = \frac{\alpha}{|\Omega|} \quad \text{in } \Omega, \tag{2.1}
\]

\[
\frac{\partial \chi}{\partial n} = h_1 \quad \text{on } \partial \Omega, \tag{2.2}
\]

\[
\frac{\partial \Delta \chi}{\partial n} = h_2 \quad \text{on } \partial \Omega, \tag{2.3}
\]

\[
\int_{\Omega} \chi \, dx = 0. \tag{2.4}
\]

It is easy to see it has a unique solution. Indeed, let \( \theta \) be the unique function satisfying

\[
\Delta \theta - \theta = \frac{\alpha}{|\Omega|} \quad \text{in } \Omega
\]

\[
\frac{\partial \theta}{\partial n} = h_2 \quad \text{on } \partial \Omega,
\]

\[
\int_{\Omega} \theta \, dx = \int_{\partial \Omega} h_1 \, ds,
\]

and then let \( \chi \) the unique function which satisfies

\[
\Delta \chi = \theta, \quad \text{in } \Omega
\]

\[
\frac{\partial \chi}{\partial n} = h_1 \quad \text{in } \partial \Omega
\]

\[
\int_{\Omega} \chi \, dx = 0,
\]

see e.g. [13]. Then it is easy to see that by construction \( \chi \) satisfies (2.1)-(2.4).

The change of variables we make is

\[
\varphi = \phi - \chi - \mu,
\]

where

\[
\mu = \frac{1}{|\Omega|} \int_{\Omega} \phi \, dx.
\]

With the new variables \((u, \omega, \varphi, \mu)\) our problem becomes

\[
-\Delta u + q(\chi + \varphi)u - \kappa |u|^{p-2}u = \omega u - \mu qu \quad \text{in } \Omega, \tag{2.5}
\]

\[
\Delta^2 \varphi - \Delta \varphi = qu^2 - \frac{\alpha}{|\Omega|} \quad \text{in } \Omega, \tag{2.6}
\]

\[
u = 0 \quad \text{on } \partial \Omega, \tag{2.7}
\]

\[
\int_{\Omega} u^2 \, dx = 1, \tag{2.8}
\]

\[
\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial \Omega, \tag{2.9}
\]

\[
\frac{\partial \Delta \varphi}{\partial n} = 0 \quad \text{on } \partial \Omega, \tag{2.10}
\]

\[
\int_{\Omega} \varphi \, dx = 0. \tag{2.11}
\]
Notice that the compatibility condition between (2.6), (2.9) and (2.10) now reads as

\[ \int_{\Omega} qu^2 \, dx = \alpha. \]

Let us define the sets

\[
S := \left\{ u \in H^1_0(\Omega) : \int_{\Omega} u^2 \, dx = 1 \right\},
\]

\[
N := \left\{ u \in H^1_0(\Omega) : \int_{\Omega} qu^2 \, dx = \alpha \right\},
\]

\[
M := S \cap N.
\]

Recall that \( \alpha \) depends on both the boundary conditions to the original problem.

If the problem has a solution, then of course \( M \neq \emptyset \). Hence,

(2.12) \quad q_{\min} \leq \alpha \leq q_{\max}

where

\[
q_{\min} = \inf_{\Omega} q \quad \text{and} \quad q_{\max} = \sup_{\Omega} q.
\]

Indeed, if \( \alpha < q_{\min} \), then

\[
\alpha = \int_{\Omega} qu^2 \, dx \geq q_{\min} > \alpha,
\]

which is a contradiction. The case \( \alpha > q_{\max} \) is analogous.

From (2.12) we deduce that \( q^{-1}(\alpha) \) is not empty, and indeed is its measure that will play a major role.

Suppose \( \alpha = q_{\min} \) and \( |q^{-1}(\alpha)| = 0 \). Then

\[
\int_{\Omega} qu^2 \, dx = \int_{\{x \in \Omega: q(x) > \alpha\}} qu^2 \, dx > \alpha,
\]

so \( M = \emptyset \). If \( \alpha = q_{\max} \) and \( |q^{-1}(\alpha)| = 0 \) we proceed in an analogous manner to conclude that \( M \) is empty and so the problem has no solutions. Therefore, we arrive at the following necessary condition for the existence of solutions: either

(2.13) \quad q_{\min} < \alpha < q_{\max}

or

(2.14) \quad |q^{-1}(\alpha)| \neq 0.

3. The manifold \( M \)

We now state some properties of the set \( M \), referring the reader to [10] for the omitted proofs.

We first note that \( M \) is symmetric with respect to the origin: if \( u \in M \) then \(-u \in M \). This follows trivially from the definition of \( M \). We also note that \( M \) is weakly closed in \( H^1_0(\Omega) \).

Now, we want to show that under condition (2.13) the set \( M \) is not empty. For this, we introduce the following notation.

Let \( A \subset \Omega \) be an open subset and define

\[
S_A := \left\{ u \in H^1_0(A) : \int_A u^2 \, dx = 1 \right\}
\]

and

\[
g_A : u \in S_A \mapsto \int_A qu^2 \, dx \in \mathbb{R}.
\]
It is immediately seen that 
\[ g_A(S_A) \subset \overline{[\inf_A q, \sup_A q]} \].

**Lemma 1.** The following inclusion holds:
\[ (\inf_A q, \sup_A q) \subset g_A(S_A). \]

We can conclude the following:

**Proposition 1.** Let \( A \subset \Omega \) be an open subset. If \( \alpha \in (\inf_A q, \sup_A q) \) then there exists \( u \in H^1_0(A) \) such that
\[ \int_A u^2 dx = 1 \quad \text{and} \quad \int_A qu^2 dx = \alpha. \]

In particular by taking \( A = \Omega \) we get

**Theorem 2.** Assume that \( \inf_\Omega q < \alpha < \sup_\Omega q \). Then \( M \) is not empty.

Let us recall the definition of genus of Krasnoselkii. Given \( A \) a closed and symmetric subset of some Banach space, with \( 0 \not\in A \), the **genus** of \( A \) is denoted as \( \gamma(A) \) and defined as the least integer \( k \) for which there exists a continuous and even map \( h : A \to \mathbb{R}^k \setminus \{0\} \). By definition it is \( \gamma(\emptyset) = 0 \) and if it is not possible to construct continuous odd maps from \( A \) to any \( \mathbb{R}^k \setminus \{0\} \), it is set \( \gamma(A) = +\infty \).

It is known that the genus is a topological invariant (by odd homeomorphism) and that the genus of the sphere in \( \mathbb{R}^N \) is \( N \).

The next result says that \( M \) has subsets of arbitrarily large genus.

**Theorem 3.** Let \( u_1, \ldots, u_k \in M \) be functions with disjoint supports and let
\[ V_k = \langle u_1, \ldots, u_k \rangle \]
be the space spanned by \( u_1, \ldots, u_k \). Then \( M \cap V_k \) is the \((k-1)\)-dimensional sphere, hence \( \gamma(M \cap V_k) = k \).

Now, it is natural if one raises the question of whether there exists such functions with disjoint supports for arbitrary \( k \). The answer is positive:

**Theorem 4.** If \((2.13)\) holds then for every \( k \geq 2 \) there exist \( k \) functions \( u_1, \ldots, u_k \in M \) with disjoint supports. Hence \( \gamma(M) = +\infty \).

Let
\[ G_1 : u \in H^1_0(\Omega) \mapsto \int_\Omega u^2 dx - 1 \in \mathbb{R}, \]
\[ G_2 : u \in H^1_0(\Omega) \mapsto \int_\Omega qu^2 dx - \alpha \in \mathbb{R} \]
and
\[ G = (G_1, G_2). \]

Then
\[ M = \left\{ u \in H^1_0(\Omega) : G_1(u) = G_2(u) = 0 \right\} = G^{-1}(0). \]

We note that \( G \) is of class \( C^1 \).

Let us show, for the reader convenience, that \( G_1'(u) \) and \( G_2'(u) \) are linearly independent, so \( G \) will be a submersion and \( M \) a submanifold of codimension 2.
**Proposition 2.** Assume $M$ is not empty. The differentials $G'_1(u)$ and $G'_2(u)$ are linearly independent for every $u \in M$ if and only if

$$|q^{-1}(\alpha)| = 0. \tag{3.1}$$

**Proof.** First, assume (3.1). We will show that $G'_1(u)$ and $G'_2(u)$ are linearly independent, for all $u \in M$. Suppose that there are $a, b \in \mathbb{R}$ such that

$$aG'_1(u) + bG'_2(u) = 0 \quad \text{in } H^{-1}(\Omega)$$

for some $u \in M$. Evaluating this expression in $u$, we find that

$$a + b\alpha = 0.$$ 

Then

$$aG'_1(u)[v] + bG'_2(u)[v] = b(-\alpha \int_{\Omega} uv dx + \int_{\Omega} quv dx) = 0 \quad \forall v \in H^1_0(\Omega),$$

that is,

$$b \int_{\Omega} (q - \alpha) uv dx = 0 \quad \forall v \in H^1_0(\Omega).$$

If $b \neq 0$ then we would have $(q - \alpha)u = 0$ a.e., and hence, in view of (3.1), $u = 0$, a contradiction. Thus $G'_1(u)$ and $G'_2(u)$ are linearly independent for all $u \in M$.

Now, suppose (3.1) is not satisfied. Then $q^{-1}(\alpha)$ has not empty interior, hence there is some test function $u$ with support in $q^{-1}(\alpha)$ such that $||u||_2^2 = 1$. It is immediately seen that $u \in M$ because $qu = \alpha u$ and hence $G'_2(u) = \alpha G'_1(u)$,

which completes the proof. \hfill \Box

4. **Variational setting**

We now proceed to study the variational framework of the problem. Our aim is to construct a functional whose critical points will be the weak solutions of the problem.

Following [6], let

$$V = \left\{ \xi \in H^2(\Omega) : \frac{\partial \xi}{\partial n} = 0 \text{ on } \partial \Omega \right\}.$$ 

We remark that $V$ is a closed subspace of $H^2(\Omega)$. Indeed, let $\{v_n\} \subset V$ such that $v_n \to v$ in $V$. Then $0 = \gamma_1(v_n) \to \gamma_1(v)$ and hence $\gamma_1(v) = 0$, where $\gamma_1$ denotes the trace operator which for smooth functions gives the directional derivative in the direction of the exterior normal on the boundary. Being a closed subspace, $V$ inherits the Hilbert space structure of $H^2(\Omega)$.

Recall that

$$\varphi = \phi - \chi - \mu$$

where

$$\mu = \frac{1}{|\Omega|} \int_{\Omega} \phi dx.$$ 

In this way, we have $\varphi = 0$, where from now on, given a function $f$, we denote with $\overline{f}$ its average in $\Omega$. Consider then the following natural decomposition of $V$:

$$V = \widetilde{V} \oplus \mathbb{R} \tag{4.1}$$

where

$$\widetilde{V} = \{ \eta \in V : \overline{\eta} = 0 \}.$$ 

On $\widetilde{V}$ we have the equivalent norm

$$||\eta||_{\widetilde{V}} = \left( ||\nabla \eta||^2 + ||\Delta \eta||^2 \right)^{1/2}.$$
Consider the functional $F : H^1_0(\Omega) \times H^2(\Omega)$ defined below:

$$F(u, \varphi) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \int_\Omega q(\varphi + \chi)u^2 dx - \kappa \int_\Omega |u|^p dx$$

$$- \frac{1}{4} \int_\Omega (\Delta \varphi)^2 dx - \frac{1}{4} \int_\Omega |\nabla \varphi|^2 dx - \frac{\alpha}{2|\Omega|} \int_\Omega \varphi dx.$$ 

It is easy to see that this functional is of class $C^1$ and that given $u \in H^1_0(\Omega)$ and $\varphi \in H^2(\Omega)$ we have

$$F'_u(u, \varphi)[v] = \int_\Omega \nabla u \nabla v dx + \int_\Omega q(\varphi + \chi)uv dx - \kappa \int_\Omega |u|^{p-2}uv dx$$

$$F'_\varphi(u, \varphi)[\xi] = \frac{1}{2} \int_\Omega q\xi^2 dx - \frac{1}{2} \int_\Omega \Delta \varphi \Delta \xi dx - \frac{1}{2} \int_\Omega \nabla \varphi \nabla \xi dx - \frac{\alpha}{2|\Omega|} \int_\Omega \xi dx$$

for every $v \in H^1_0(\Omega)$ and $\xi \in H^2(\Omega)$.

Then, $(u, \varphi, \omega, \mu) \in H^1_0(\Omega) \times H^2(\Omega) \times \mathbb{R} \times \mathbb{R}$ is a weak solution to (2.5)-(2.11) if and only if

$$(u, \varphi) \in M \times \tilde{V},$$

$$\forall v \in H^1_0(\Omega) : F'_u(u, \varphi)[v] = \omega \int_\Omega uv dx - \mu \int_\Omega quv dx,$$

$$\forall \xi \in V : F'_\varphi(u, \varphi)[\xi] = 0.$$

**Theorem 5.** Let $(u, \varphi) \in H^1_0(\Omega) \times H^2(\Omega)$. Then there exist $\omega, \mu \in \mathbb{R}$ such that $(u, \varphi, \omega, \mu)$ is a solution to (2.5)-(2.11) if and only if $(u, \varphi)$ is a critical point of $F$ constrained on $M \times \tilde{V}$, in which case the real constants $\omega, \mu$ are the two Lagrange multipliers with respect to $F'_u$.

**Proof.** Indeed $(u, \varphi)$ is a critical point of $F$ constrained on $M \times \tilde{V}$ if and only if

$$\forall v \in T_u M : F'_u(u, \varphi)[v] = 0,$$

$$\forall \xi \in \tilde{V} : F'_\varphi(u, \varphi)[\xi] = 0.$$

Note that the tangent space to $\tilde{V}$ at $\varphi$ is $\tilde{V}$ itself.

Then a weak solution, according to (4.2) and the Lagrange multipliers rule, is a constrained critical point.

Suppose on the contrary that $(u, \varphi)$ is a constrained critical point. Then, again by the Lagrange multipliers rule, we have that there exists $\omega, \mu \in \mathbb{R}$ such that

$$\forall v \in H^1_0(\Omega) : F'_u(u, \varphi)[v] = \omega \int_\Omega uv dx - \mu \int_\Omega quv dx.$$ 

It remains to prove that $F'_\varphi(u, \varphi)[\xi] = 0$ for all $\xi \in V$. But this follows by the decomposition (4.1), noticing that $F'_\varphi(u, \varphi)[v] = 0$ for every constant $r \in \mathbb{R}$. Then (4.2) is satisfied and this concludes the proof. $\square$

The functional $F$ constrained on $M \times \tilde{V}$ is unbounded from above and from below. This issue has been addressed by Benci and Fortunato [2] and in many subsequent papers. Their standard reduction argument goes as follows:

(i) For every fixed $u \in H^1_0(\Omega)$ there exists a unique $\Phi(u)$ such that $F'_\varphi(u, \Phi(u)) = 0$.

(ii) The map $u \mapsto \Phi(u)$ is of class $C^1$.

(iii) The graph of $\Phi$ is a manifold, and we are reduced to study the functional $J(u) = F(u, \Phi(u)$, possibly constrained.
However the method sketched above fails in our situation, for two reasons. First, we see that
\[ F'_\varphi(u, \varphi) = 0 \text{ with } \varphi \in \bar{V} \]
is just
\[ \Delta^2 \varphi - \Delta \varphi - qu^2 + \alpha/|\Omega| = 0 \text{ in } \Omega, \]
\[ \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial \Omega, \]
\[ \frac{\partial \Delta \varphi}{\partial n} = 0 \text{ on } \partial \Omega, \]
\[ \int_{\Omega} \varphi dx = 0. \]
The problem above has not a unique solution for any fixed \( u \): this happens, due to the compatibility condition, if and only if \( u \in N \). Moreover, since \( N \) is not a manifold (unless \( \alpha \neq 0 \)) we cannot require the map \( \Phi : u \mapsto \Phi(u) \) to be of class \( C^1 \) in \( N \). We shall then extend such a map \( \Phi \).

**Proposition 3.** For every \( w \in L^{6/5}(\Omega) \) there exists a unique \( L(w) \in \bar{V} \) solution of
\[ \Delta^2 \varphi - \Delta \varphi - w = 0 \text{ in } \Omega, \]
\[ \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial \Omega, \]
\[ \int_{\Omega} \varphi dx = 0. \]
The map \( L : L^{6/5}(\Omega) \to \bar{V} \) is linear and continuous, hence of class \( C^\infty \).

**Proof.** The weak solutions to the problem are functions \( \varphi \) in the Hilbert space \( \bar{V} \) such that
\[ \int_{\Omega} \Delta \varphi \Delta v dx + \int_{\Omega} \nabla \varphi \nabla v dx = \int_{\Omega} w v dx \quad \forall v \in \bar{V}. \]
So the result follows by applying the Riesz Theorem since the bilinear form \( b : \bar{V} \times \bar{V} \to \mathbb{R} \) given by
\[ b(\varphi, v) = \int_{\Omega} \Delta \varphi \Delta v dx + \int_{\Omega} \nabla \varphi \nabla v dx. \]
is just the scalar product in \( \bar{V} \). \( \square \)

The following proposition follows from well-known properties of Nemytsky operators.

**Proposition 4.** The map
\[ u \in L^6(\Omega) \mapsto qu^2 \in L^{6/5}(\Omega) \]
is of class \( C^1 \).

As a consequence of the previous propositions, we can define the following map:
\[ \Phi : u \in H^1_0(\Omega) \mapsto L(qu^2) \in \bar{V}. \]
It is clear that
\[ \Phi(u) = \Phi(-u) = \Phi(|u|). \]
Moreover, for every \( (u, \varphi) \in H^1_0(\Omega) \times \bar{V} \) we have that \( \varphi = \Phi(u) \) if and only if for every \( \eta \in \bar{V} \)
\[ \int_{\Omega} \Delta \varphi \Delta \eta dx + \int_{\Omega} \nabla \varphi \nabla \eta dx = \int_{\Omega} qu^2 \eta dx. \]
Taking $\eta = \Phi(u)$ we have in particular the important relation
\begin{equation}
(4.3) \quad \int_\Omega (\Delta \Phi(u))^2 \, dx + \int_\Omega |\nabla \Phi(u)|^2 \, dx = \int_\Omega q u^2 \Phi(u) \, dx.
\end{equation}

The right hand side above is the interaction energy term. Then we infer
\begin{align*}
||\Phi(u)||^2_{\tilde{V}} &\leq ||q||_\infty \int_\Omega u^2 \Phi(u) \, dx \\
&\leq c ||u||^2_2 ||\Phi(u)||_2 \\
&\leq c ||\nabla u||^2_2 ||\Phi(u)||_2 \\
&\leq c ||\nabla u||^2_2 ||\Phi(u)||_{\tilde{V}}
\end{align*}

and hence
\begin{equation}
(4.4) \quad ||\Phi(u)||_{\tilde{V}} \leq c ||\nabla u||^2_2,
\end{equation}

that is, $\Phi$ is bounded on bounded sets. We have

**Lemma 2.** If $u_n \rightharpoonup u$ in $H^1_0(\Omega)$ then
\begin{equation*}
\int_\Omega qu_n^2 \Phi(u_n) \, dx \to \int_\Omega qu^2 \Phi(u) \, dx.
\end{equation*}

Moreover the map $\Phi$ is compact.

**Proof.** Let $u_n \rightharpoonup u$ in $H^1_0(\Omega)$ and define $B_n, B : \tilde{V} \to \mathbb{R}$ by
\begin{align*}
B_n(\eta) &:= \int_\Omega qu_n^2 \eta \, dx, \\
B(\eta) &:= \int_\Omega qu^2 \eta \, dx.
\end{align*}

Such operators are continuous due to the Hölder’s inequality. For example:
\begin{equation*}
\left| \int_\Omega qu^2 \eta \, dx \right| \leq ||q||_\infty ||u||^2_2 ||\eta||_2 \leq c ||\nabla \eta||_2 \leq c ||\eta||_{\tilde{V}}
\end{equation*}

(where here $c$ depends on $u$).

Due to the compact embedding of $H^1_0(\Omega)$ into $L^p(\Omega)$ for $p \in [1, 6)$, we get $u_n^2 \to u^2$ in $L^{6/5}(\Omega)$ and then
\begin{align*}
|B_n(\eta) - B(\eta)| &\leq ||q||_\infty ||u_n^2 - u^2||_{6/5} ||\eta||_6 \\
&\leq c ||q||_\infty ||u_n^2 - u^2||_{6/5} ||\eta||_{\tilde{V}}.
\end{align*}

Hence
\begin{equation*}
||B_n - B|| \leq \sup_{\eta \neq 0} \frac{c ||u_n^2 - u^2||_{6/5} ||\eta||_{\tilde{V}}}{||\eta||_{\tilde{V}}} \to 0,
\end{equation*}

namely $B_n \to B$ as operators in $\tilde{V}$.

On the other hand, we have that $\Phi(u_n) \rightharpoonup \Phi(u)$ in $\tilde{V}$. Indeed, let $g \in \tilde{V}'$. Then there is some $v_g \in \tilde{V}$ such that
\begin{equation*}
g(\Phi(u_n)) = \int_\Omega \nabla \Phi(u_n) \nabla v_g \, dx + \int_\Omega \Delta \Phi(u_n) \Delta v_g \, dx = \int_\Omega qu_n^2 v_g \, dx.
\end{equation*}

But then
\begin{align*}
g(\Phi(u_n)) - g(\Phi(u)) &= \int_\Omega q(u_n^2 - u^2) v_g \, dx \\
&\leq ||q||_\infty ||u_n^2 - u^2||_2 ||v_g||_2 \to 0
\end{align*}

since $u_n^2 \to u^2$ in $L^2(\Omega)$ as well.
We then conclude that
\[ \int_\Omega qu_n^2 \Phi(u_n) dx \to \int_\Omega qu^2 \Phi(u) dx \]
and by (4.3) that \( \| \Phi(u_n) \|_{\tilde{V}} \to \| \Phi(u) \|_{\tilde{V}} \). Consequently \( \Phi(u_n) \to \Phi(u) \) in \( \tilde{V} \).

\[ \square \]

Note that for every \( u \in N \) we have that \( F'(u, \Phi(u)) = 0 \). Indeed, \( \Phi(u) \) is the unique solution to the problem in Proposition 3 with \( w = qu^2 \).

We now define the reduced functional of a single variable:
\[ J : H^1_0(\Omega) \to \mathbb{R} \]
\[ u \mapsto - \int_\Omega F(u, \Phi(u)) dx \]

With the notation \( \varphi_u := \Phi(u) \) the functional \( J \) is explicitly given by (recall (4.3))
\[ J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \int_\Omega q\varphi_u u^2 dx + \frac{1}{2} \int_\Omega q\chi u^2 dx - \frac{\kappa}{p} \int_\Omega |u|^p dx \]
\[ - \frac{1}{4} \int_\Omega (\Delta \varphi_u)^2 dx - \frac{1}{4} \int_\Omega |\nabla \varphi_u|^2 dx - \frac{\alpha}{2 |\Omega|} \int_\Omega \varphi_u dx \]
\[ = \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{4} \int_\Omega (\Delta \varphi_u)^2 dx + \frac{1}{4} \int_\Omega |\nabla \varphi_u|^2 dx + \int_\Omega q\chi u^2 dx \]
\[ - \frac{\kappa}{p} \int_\Omega |u|^p dx. \]

We note that \( J \) is of class \( C^1 \) on \( H^1_0(\Omega) \) and even. Moreover, for every \( u \in M \) we have that
\[ J'(u)[v] = F'_u(u, \varphi_u)[v] + F'_\varphi(u, \varphi_u)[\Phi'(u)[v]] = F'_u(u, \varphi_u)[v] \quad \forall v \in H^1_0(\Omega) \]
and hence we deduce the following

**Theorem 6.** The pair \((u, \varphi) \in M \times \tilde{V}\) is a critical point of \( F \) constrained on \( M \times \tilde{V} \) if and only if \( u \) is a critical point of \( J|_M \) and \( \varphi = \Phi(u) \).

5. PROOF OF THE MAIN RESULT

The next lemma will be useful.

**Lemma 3.** Let \( D \) be a regular domain of \( \mathbb{R}^N \) and
\[ 1 \leq s \leq N, \]
\[ s < p < s^* = \frac{Ns}{N-s} \]
and
\[ 0 < r \leq N \left( 1 - \frac{p}{s^*} \right). \]

Then there exists a constant \( C > 0 \) such that for every \( u \in W^{1,s}(D) \) it holds that
\[ ||u||_p^p \leq C||u||_{W^{1,s}}^p ||u||_r^r \]

**Proof.** See [9, Lemma 3.1]. □

**Remark 2.** If \( D \) is bounded, then the conclusion of the lemma is true also in the case \( p \in [1, s] \) with \( r < p \). Also, if \( D \) is bounded and \( u \in W^{1,p}_0(D) \), then, by Poincaré inequality,
\[ ||u||_p^p \leq C||\nabla u||_{s^*}^p ||u||_r^r. \]

The following lemma gives the existence of solutions to our modified problem.
Lemma 4. The functional $J$ on $M$ is weakly lower semicontinuous and coercive. In particular, it has a minimum $u \in M$, and it can be assumed positive.

Proof. We have

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4} \int_{\Omega} (\Delta u)^2 dx + \frac{1}{4} \int_{\Omega} |\nabla \varphi u|^2 dx + \int_{\Omega} q|u|^2 dx - \frac{\kappa}{p} \int_{\Omega} |u|^p dx$$

$$\geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \|q\|_\infty \|\chi\|_\infty - \frac{\kappa}{p} \int_{\Omega} |u|^p dx.$$ 

Finally, we apply Lemma 3 with $s = 2$ and $N = 3$. Since $p \in (2,10/3)$ it holds that

$$p - 2 < 3 \left(1 - \frac{p}{6}\right) < 2$$

and we can choose

$$p - 2 < r < 3 \left(1 - \frac{p}{6}\right),$$

so that by the Lemma it follows that

$$\frac{\kappa}{p} \int_{\Omega} |u|^p dx \leq c \|
abla u\|_2^{p-r}.$$ 

Hence,

$$J(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \|q\|_\infty \|\chi\|_\infty - c' \|
abla u\|_2^{p-r}$$

and thus $J$ is coercive and bounded from below on $M$.

Now, let $\{u_n\} \subset M$ such that $u_n \rightharpoonup u$. Since $M$ is weakly closed, $u \in M$. By Lemma 2 we know that

$$\frac{1}{4} \int_{\Omega} (\Delta \varphi u_n)^2 dx + \frac{1}{4} \int_{\Omega} |\nabla \varphi u_n|^2 dx \rightarrow \frac{1}{4} \int_{\Omega} (\Delta \varphi u)^2 dx + \frac{1}{4} \int_{\Omega} |\nabla \varphi u|^2 dx.$$ 

We also know that $u_n^2 \rightarrow u^2$ in $L^{6/5}(\Omega)$ so

$$\int_{\Omega} q\chi (u_n^2 - u^2) dx \leq c \int_{\Omega} u_n^2 - u^2 dx \leq c |u_n - u|_{6/5} \rightarrow 0.$$ 

Finally, the first and last terms are the norms of $u$ in $H^1_0(\Omega)$ and $L^p(\Omega)$ (up to constants), so they are weakly lower semicontinuous.

Thus $J$ is weakly lower semicontinuous and the existence of the minimum follows by standard results. Note that $J(u) = J(|u|)$ so the minimum may be assumed to be positive. \[ \square \]

We will use a deformation argument to show that there are infinitely many solutions. A crucial point is that the functional satisfies the Palais-Smale condition. We recall that in general, it is said that the $C^1$ functional $I$ satisfies the Palais-Smale condition on the manifold $M$, if any sequence $\{u_n\} \subset M$ such that $\{I(u_n)\}$ is bounded and $I(u_n) \rightarrow 0$ in the tangent bundle, admits a convergent subsequence to an element $u \in M$.

Proposition 5. The functional $J$ satisfies the Palais-Smale condition on $M$.

Proof. Let $\{u_n\} \subset M$ be such that

$$\{J(u_n)\} \text{ is bounded}$$

and

$$J'|_{M}(u_n) \rightarrow 0.$$ 

(5.1)
By (5.1) there exists two sequences of real numbers \( \{\lambda_n\} \), \( \{\beta_n\} \) and a sequence \( \{v_n\} \subset H^{-1} \) such that \( v_n \to 0 \) and

\[
−\Delta u_n + q(\varphi_n + \chi)u_n - \kappa|u_n|^{p-2}u_n = \lambda_n u_n + \beta_n q u_n + v_n \tag{5.2}
\]

where \( \varphi_n := \varphi_{u_n} \).

Since \( J \) is coercive and \( \{J(u_n)\} \) is bounded, we know that \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \). Hence there exists \( u \in H^1_0(\Omega) \) such that \( u_n \rightharpoonup u \), up to a subsequence. By the compact embeddings and Lemma 2 we know that

\[
u_n \to u \quad \text{in} \quad L^p(\Omega), \quad \varphi_n \to \varphi_u \quad \text{in} \quad H^2(\Omega). \tag{5.3}
\]

Also, since \( M \) is weakly closed, we know that \( u \in M \). It only remains to show that \( u_n \to u \) in \( H^1_0(\Omega) \).

By (5.2) we have that

\[
\frac{1}{2} \int_\Omega |\nabla u_n|^2 \, dx + \frac{1}{2} \int_\Omega q(\varphi_n + \chi) u_n^2 \, dx - \frac{\kappa}{p} \int_\Omega |u_n|^p \, dx - \langle v_n, u_n \rangle = \lambda_n + \alpha \beta_n. \tag{5.4}
\]

By (5.3) we infer

\[
\left| \int_\Omega \left( q(\varphi_n + \chi) u_n^2 - q(\varphi_u + \chi) u^2 \right) \, dx \right| \leq c \int_\Omega |\varphi_n + \chi| |u_n^2 - u^2| \, dx + \int_\Omega u^2 |\varphi_n - \varphi| \, dx = o_n(1)
\]

where we are denoting with \( o_n(1) \) a vanishing sequence. Then the right-hand side of (5.4) is bounded and we can assume that

\[
\lambda_n + \alpha \beta_n = \xi + o_n(1)
\]

with \( \xi \in \mathbb{R} \). Then (5.2) becomes

\[
−\Delta u_n + q(\varphi_n + \chi)u_n - \kappa|u_n|^{p-2}u_n - v_n = (\xi + o(1))u_n - \beta_n(q - \alpha)u_n. \tag{5.5}
\]

Now, since \( u \in M \) we know that \( |u|^2 = 1 \). This, together with the assumption \( |q^{-1}(\alpha)| = 0 \) implies that \( (q - \alpha)u \) is not identically zero. Then there exists a test function \( w \in C_0^\infty(\Omega) \) such that

\[
\int_\Omega (q - \alpha)uw \, dx \neq 0.
\]

Evaluating (5.5) on this \( w \) we get

\[
\int_\Omega \nabla u_n \nabla w \, dx + \int_\Omega q(\varphi_n + \chi)u_n w \, dx - \kappa \int_\Omega |u_n|^{p-2}u_n w \, dx - \langle v_n, w \rangle - (\lambda + o_n(1)) \int_\Omega u_n w \, dx = \beta_n \int_\Omega (q - \alpha)u_n w \, dx \tag{5.6}
\]

and using again (5.3) we see that every term in the left-hand side converges. Also, by the weak convergence of \( \{u_n\} \),

\[
\int_\Omega (q - \alpha)u_n \, dx \to \int_\Omega (q - \alpha)u \, dx.
\]

This implies, coming back to (5.6), that \( \{\beta_n\} \) is bounded, which in turn implies that \( \{\lambda_n\} \) is bounded.
Applying (5.5) to $u_n - u$ we get
\begin{equation}
\int_{\Omega} \nabla u_n \nabla (u_n - u) dx + \int_{\Omega} q(\varphi_n + \chi) u_n (u_n - u) dx - \kappa \int_{\Omega} |u_n|^{p-2} u_n (u_n - u) dx - \langle v_n, u_n - u \rangle = (\lambda + o(1)) \int_{\Omega} u_n (u_n - u) dx \quad (5.7)
\end{equation}
Since (again by (5.3)) we have
\begin{align*}
\int_{\Omega} q(\varphi_n + \chi) u_n (u_n - u) dx &\to 0, \quad \langle v_n, u_n - u \rangle \to 0, \\
\int_{\Omega} |u_n|^{p-2} u_n (u_n - u) dx &\to 0, \quad (\lambda + o(1)) \int_{\Omega} u_n (u_n - u) dx \to 0
\end{align*}
we conclude by (5.7) that $||\nabla u_n||_2 \to ||\nabla u||_2$ and so $u_n \to u$ in $H^1_0(\Omega)$. \hfill \square

Now we can give the proof of Theorem 1. By Theorem 3, $M$ has compact, symmetric subsets of genus $k$ for every $k \in \mathbb{N}$.

Let us recall now a classical result in critical point theory. We give the proof for the reader convenience.

**Lemma 5.** For any $b \in \mathbb{R}$ the sublevel
\[ J^b = \{ u \in M : J(u) \leq b \} \]
has finite genus.

**Proof.** We argue by contradiction. Suppose that
\[ D = \{ b \in \mathbb{R} : \gamma(J^b) = \infty \} \neq \emptyset. \]
Since $J|_{M}$ is bounded from below, then $D$ is bounded from below. Then
\[ -\infty < \underline{b} = \inf D < \infty. \]
Moreover, since $J|_{M}$ satisfies the Palais-Smale condition, the set
\[ Z = \{ u \in M : J(u) = \underline{b}, J'|_{M}(u) = 0 \} \]
is compact. Hence there exists a closed symmetric neighborhood $U_Z$ of $Z$ such that $\gamma(U_Z) < \infty$. By the Deformation Lemma, there exists an $\varepsilon > 0$ such that $J^{\underline{b} - \varepsilon}$ includes a deformation retract of $J^{\underline{b} + \varepsilon} \setminus U_Z$. Then, by the properties of the genus,
\[ \gamma(J^{\underline{b} + \varepsilon}) \leq \gamma(J^{\underline{b} + \varepsilon} \setminus U_Z) + \gamma(U_Z) \leq \gamma(J^{\underline{b} - \varepsilon}) + \gamma(U_Z) < \infty, \]
a contradiction. \hfill \square

Let $n \in \mathbb{N}$. By Lemma 5 there exists some $k \in \mathbb{N}$ depending on $n$ such that
\[ \gamma(J^n) = k. \]
Let
\[ A_{k+1} = \{ A \subset M : A = -A, \overline{A} = A, \gamma(A) = k + 1 \} \]
that we know is not empty by Theorem 3.
By the monotonicity property of the genus, any $A \in A_{k+1}$ is not contained in $J^n$, then

$$c_n = \inf_{A \in A_{k+1}} \sup_{u \in A} J(u) \geq n.$$  

Well known results (see e.g. [12]) say that $c_n$ are critical levels for $J|_M$ and then there is a sequence \{$u_n$\} of critical points such that

$$J(u_n) = c_n \rightarrow +\infty$$

The critical points give rise to Lagrange multipliers $\omega_n, \mu_n$ and then, recalling the decomposition $\varphi = \phi - \chi - \mu$, to solutions $(u_n, \omega_n, \phi_n) \in H^1_0(\Omega) \times \mathbb{R} \times H^2(\Omega)$ of the original problem.

We show that $||\nabla u_n||_2 \rightarrow +\infty$. Since

$$\int_\Omega q\chi u_n^2 dx \leq ||q\chi||_{\infty},$$

and by (4.4) it is

$$||\varphi_n||_{\tilde{V}} = \int_\Omega (\Delta \varphi_n)^2 dx + \int_\Omega |\nabla \varphi_n|^2 dx \leq c ||\nabla u_n||^2_2,$$

we see that

$$|J(u_n)| \leq (1 + c)||\nabla u_n||^2 + c'||\nabla u_n||^p_2 + ||q\chi||_{\infty}$$

and then \{$u_n$\} can not be bounded.

This concludes the proof of Theorem 1.

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