CONTINUITY OF EXTREMAL TRANSITIONS AND FLOPS FOR CALABI–YAU MANIFOLDS

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Abstract

In this paper, we study the behavior of Ricci-flat Kähler metrics on Calabi–Yau manifolds under algebraic geometric surgeries: extremal transitions or flops. We prove a version of Candelas and de la Ossa’s conjecture: Ricci-flat Calabi–Yau manifolds related by extremal transitions and flops can be connected by a path consisting of continuous families of Ricci-flat Calabi–Yau manifolds and a compact metric space in the Gromov–Hausdorff topology. In an essential step of the proof of our main result, the convergence of Ricci-flat Kähler metrics on Calabi–Yau manifolds along a smoothing is established, which can be of independent interest.

1. Introduction

A Calabi–Yau manifold $M$ is a simply connected projective manifold with trivial canonical bundle $K_M \cong \mathcal{O}_M$. In the 1970s, S.T. Yau proved Calabi’s conjecture in [61], which says that, for any Kähler class $\alpha \in H^{1,1}(M, \mathbb{R})$, there exists a unique Ricci-flat Kähler metric $g$ on $M$ with Kähler form $\omega \in \alpha$. The study of Calabi–Yau manifolds became very interesting in the last three decades (cf. [63]). The convergence of Ricci-flat Calabi–Yau manifolds was studied from various perspectives (cf. [2], [8], [9], [13], [30], [37], [41], [56], [57], [58], [49], [60], [64]). The goal of the present paper is to study the metric behavior of Calabi–Yau manifolds under some algebraic geometric surgeries.

Let $M_0$ be a singular projective normal variety with singular set $S$. Usually there are two types of desingularizations: one is a resolution $(\tilde{M}, \tilde{\pi})$, i.e., $\tilde{M}$ is a projective manifold, and $\tilde{\pi}$ is a morphism such that $\tilde{\pi} : \tilde{M} \setminus \tilde{\pi}^{-1}(S) \to M_0 \setminus S$ is bi-holomorphic. The other is a smoothing $(\mathcal{M}, \pi)$ over the unit disc $\Delta \subset \mathbb{C}$, i.e., $\mathcal{M}$ is an $(n+1)$-dimensional variety, $\pi$ is a proper flat morphism, $M_0 = \pi^{-1}(0)$, and $M_t = \pi^{-1}(t)$ is a smooth projective $n$-dimensional manifold for any $t \in \Delta \setminus \{0\}$. If $M_0$ admits a resolution $(\tilde{M}, \tilde{\pi})$ and a smoothing $(\mathcal{M}, \pi)$, the process of going from $\tilde{M}$ to $M_t$, $t \neq 0$, is called an extremal transition, denoted by $\tilde{M} \to M_0 \leadsto M_t$. We call this process a conifold transition if $M_0$ is a

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conifold, which is a normal variety $M_0$ with only finite ordinary double points as singularities, i.e., any singular point is locally given by
\[ z_0^2 + \cdots + z_n^2 = 0, \quad \text{where} \quad \dim \mathbb{C} M_0 = n. \]
If $M_0$ admits two different resolutions $(\bar{M}_1, \bar{\pi}_1)$ and $(\bar{M}_2, \bar{\pi}_2)$ with both exceptional subvarieties of codimension at least 2, the process of going from $\bar{M}_1$ to $\bar{M}_2$ is called a flop, denoted by $\bar{M}_1 \to M_0 \to \bar{M}_2$.

Extremal transitions and flops are algebraic geometric surgeries providing ways to connect two topologically distinct projective manifolds, which are interesting in both mathematics and physics. In the minimal model program, all smooth minimal models of dimension 3 in a birational equivalence class are connected by a sequence of flops (cf. [38], [39], [35]). The famous Reid’s fantasy conjectures that all Calabi–Yau threefolds are connected to each other by extremal transitions, possibly including non-Kähler Calabi–Yau threefolds, so as to form a huge connected web (cf. [46], [48]). There is also a projective version of this conjecture, the connectedness conjecture for moduli spaces for Calabi–Yau threefolds (cf. [28], [29], [48]). Furthermore, in physics, flops and extremal transitions are related to the topological change of the space-time in string theory (cf. [7], [16], [27], [24], [48]). Readers are referred to the survey article [48] for topology, algebraic geometry, and even physics properties of extremal transitions.

In [6], physicists P. Candelas and X.C. de la Ossa conjectured that extremal transitions and flops should be “continuous in the space of Ricci-flat Kähler metrics,” even though these processes involve topologically distinct Calabi–Yau manifolds. This conjecture was verified in [6] for the non-compact quadric cone $M_0 = \{(z_0, \cdots, z_3) \in \mathbb{C}^4 | z_0^2 + \cdots + z_3^2 = 0\}$.

In the 1980s, Gromov introduced the notion of Gromov–Hausdorff distance $d_{GH}$ on the space $\mathcal{X}$ of isometric classes of all compact metric spaces (cf. [22]), such that $(\mathcal{X}, d_{GH})$ is a complete metric space (cf. [22] and Appendix A). This notion provides a framework to study the continuity of a family of compact metric spaces with possibly different topologies. The Gromov–Hausdorff topology provides a natural mathematical formulation of Candelas and de la Ossa’s conjecture as follows:

i) If $\bar{M} \to M_0 \sim M_t$, $t \in \Delta \setminus \{0\} \subset \mathbb{C}$, is an extremal transition among Calabi–Yau manifolds, then there exists a family of Ricci-flat Kähler metrics $\bar{g}_s$, $s \in (0, 1)$, on $\bar{M}$, and a family of Ricci-flat Kähler metrics $\bar{g}_t$ on $M_t$ satisfying that $\{(\bar{M}, \bar{g}_s)\}$ and $\{(M_t, \bar{g}_t)\}$ converge to a single compact metric space $(\mathcal{X}, d_{\mathcal{X}})$ in the Gromov–Hausdorff topology,
\[ (M_t, \bar{g}_t) \xrightarrow{d_{GH}} (\mathcal{X}, d_{\mathcal{X}}) \xrightarrow{d_{GH}} (\bar{M}, \bar{g}_s), \quad s \to 0, \ t \to 0. \]

ii) If $\bar{M}_1 \to M_0 \sim \bar{M}_2$ is a flop between two Calabi–Yau manifolds, then there are families of Ricci-flat Kähler metrics $\bar{g}_{i,s}$, $s \in (0, 1)$
on $\bar{M}_i$ ($i = 1, 2$) such that
\[(\bar{M}_1, \bar{g}_{1,s}) \xrightarrow{\text{d}_{GH}} (X, d_X) \xleftarrow{\text{d}_{GH}} (\bar{M}_2, \bar{g}_{2,s}), \quad s \to 0,\]
for a single compact metric space $(X, d_X)$.

Furthermore, in both cases $X$ is homeomorphic to $M_0$ and $d_X$ is induced by a Ricci-flat Kähler metric on $M_0 \setminus S$. In the present paper, we shall prove i) and ii) of the above version of Candelas and de la Ossa’s conjecture.

Let $M_0$ be a projective normal Cohen-Macaulay $n$-dimensional variety with singular set $S$, and let $\mathcal{K}_{M_0}$ be the canonical sheaf of $M_0$ ([33]). In this paper, all varieties are assumed to be Cohen-Macaulay. We call $M_0$ Gorenstein if $\mathcal{K}_{M_0}$ is a rank one locally free sheaf. Assume that $M_0$ has only canonical singularities, i.e., $M_0$ is Gorenstein, and for any resolution $(\bar{M}, \bar{\pi})$,
\[\mathcal{K}_{\bar{M}} = \bar{\pi}^* \mathcal{K}_{M_0} + \sum a_E E, \quad a_E \geq 0,\]
where $E$ are effective exceptional divisors. Consider a resolution $(\bar{M}, \bar{\pi})$ over a unit disc $\Delta \subset \mathbb{C}$ with an ample line bundle $\mathcal{L}$ on $M$, then there is an embedding $M \hookrightarrow \mathbb{P}^N \times \Delta$ such that $\mathcal{L}^m = \mathcal{O}_\Delta(1)|_M$ for some $m \geq 1$, $\pi$ is a proper surjection given by the restriction of the projection from $\mathbb{P}^N \times \Delta$ to $\Delta$, and the rank of $\pi_*$ is 1 on $M \setminus S$. This implies that $M_t, t \in \Delta \setminus \{0\}$, have the same underlying differential manifold $\bar{M}$. Moreover, if $\mathcal{L}$ is a line bundle on $M$ such that the restriction of $\mathcal{L}$ on $M_0$ is ample, then by proposition 1.41 in [39], $\mathcal{L}$ is ample on $\pi^{-1}(\Delta')$ where $\Delta' \subset \Delta$ is a neighborhood of 0.

A Calabi–Yau variety is a simply connected projective normal variety $M_0$ with trivial canonical sheaf $\mathcal{K}_{M_0} \cong \mathcal{O}_{M_0}$ and only canonical singularities. If a Calabi–Yau variety $M_0$ admits a crepant resolution $(\bar{M}, \bar{\pi})$, then $\bar{M}$ is a Calabi–Yau manifold. Our first result proves i) in the above version of Candelas and de la Ossa’s conjecture.

**Theorem 1.1.** Let $M_0$ be a Calabi–Yau $n$-variety with singular set $S$. Assume that

i) $M_0$ admits a smoothing $\pi : M \to \Delta$ over the unit disc $\Delta \subset \mathbb{C}$ such that the relative canonical bundle $\mathcal{K}_{M/\Delta}$ is trivial, i.e., $\mathcal{K}_{M/\Delta} \cong \mathcal{O}_M$ and $M$ admits an ample line bundle $\mathcal{L}$. For any $t \in \Delta \setminus \{0\}$, let $\tilde{g}_t$ be the unique Ricci-flat Kähler metric on $M_t = \pi^{-1}(t)$ with Kähler form $\tilde{\omega}_t \in c_1(\mathcal{L})|_{M_t}$. 


ii) $M_0$ admits a crepant resolution $(\tilde{M}, \tilde{\pi})$. Let $\{\tilde{g}_s\}$ $(s \in (0, 1])$ be a family of Ricci-flat Kähler metrics with Kähler classes
\[ \tilde{\pi}^*c_1(\mathcal{L})|_{M_0} \in H^{1,1}(\tilde{M}, \mathbb{R}), \]
where $\tilde{\omega}_s$ denotes the corresponding Kähler form of $\tilde{g}_s$.

Then there exists a compact length metric space $(X, d_X)$ such that
\[ \lim_{s \to 0} d_{GH}((M_t, \tilde{g}_t), (X, d_X)) = \lim_{s \to 0} d_{GH}((M, \tilde{g}_s), (X, d_X)) = 0. \]
Furthermore, $(X, d_X)$ is isometric to the metric completion $(M_0\backslash S, d_g)$ where $g$ is a Ricci-flat Kähler metric on $M_0\backslash S$, and $d_g$ is the Riemannian distance function of $g$.

The following is a simple example from [27] for which Theorem 1.1 can apply. Let $M$ be the complete intersection in $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by
\[ g_0(z_0, \ldots, z_4) = 0, \quad g_1(z_0, \ldots, z_4) = 0, \]
where $z_0, \ldots, z_4$ are homogeneous coordinates of $\mathbb{CP}^4$, $g_0, g_1$ are homogeneous coordinates of $\mathbb{CP}^1$, and $g$ and $h$ are generic homogeneous polynomials of degree 4. Then $\tilde{M}$ is a crepant resolution of the quintic conifold $M_0$ given by
\[ z_3g(z_0, \ldots, z_4) + z_4h(z_0, \ldots, z_4) = 0 \]
(cf. [48]). Hence there is a conifold transition $\tilde{M} \to M_0 \sim M$ for any smooth quintic $\tilde{M}$ in $\mathbb{CP}^4$. Theorem 1.1 implies that there is a family of Ricci-flat Kähler metrics $\tilde{g}_s$ $(s \in (0, 1])$ on $\tilde{M}$ and a family of Ricci-flat smooth quintic $(M_t, \tilde{g}_t)$ $(t \in \Delta \backslash \{0\})$ such that $M_1 = \tilde{M}$, and
\[ (M_t, \tilde{g}_t) \xrightarrow{d_{GH}} (X, d_X) \xleftarrow{d_{GH}} (M, \tilde{g}_s), \]
for a compact metric space $(X, d_X)$.

Our second result proves ii) in the above version of Candelas and de la Ossa’s conjecture.

**Theorem 1.2.** Let $M_0$ be an $n$-dimensional Calabi–Yau variety with singular set $S$, and $\mathcal{L}$ be an ample line bundle. Assume that $M_0$ admits two crepant resolutions $(\tilde{M}_1, \tilde{\pi}_1)$ and $(\tilde{M}_2, \tilde{\pi}_2)$. Let $\{\tilde{g}_{1,s}\}$ (resp. $\{\tilde{g}_{2,s}\}$ $s \in (0, 1]$) be a family of Ricci-flat Kähler metrics on $\tilde{M}_1$ (resp. $\tilde{M}_2$) with Kähler classes
\[ \lim_{s \to 0} [\tilde{\omega}_{1,s}] = \tilde{\pi}_1^*c_1(\mathcal{L}), \quad \alpha = 1, 2. \]
Then there exists a compact length metric space $(X, d_X)$ such that
\[ \lim_{s \to 0} d_{GH}((\tilde{M}_1, \tilde{g}_{1,s}), (X, d_X)) = \lim_{s \to 0} d_{GH}((\tilde{M}_2, \tilde{g}_{2,s}), (X, d_X)) = 0. \]
Furthermore, $(X, d_X)$ is isometric to the metric completion $(M_0\backslash S, d_g)$ where $g$ is a Ricci-flat Kähler metric on $M_0\backslash S$, and $d_g$ is the Riemannian distance function of $g$. 
Remark 1.3. The present arguments are inadequate to prove that $X$ is homeomorphic to $M_0$ in both Theorem 1.1 and Theorem 1.2. Additional work is required. However, if $M_0$ has only orbifold singularities, and $c_1(L)$ can be represented by an orbifold Kähler metric on $M_0$, then $X$ is homeomorphic to $M_0$ by corollary 1.1 in [49].

We now begin to describe our approach to Theorem 1.1 and Theorem 1.2. Let $M_0$ be a normal $n$-dimensional projective variety with singular set $S$. For any $p \in S$ and a small neighborhood $U_p \subset M_0$ of $p$, a pluri-subharmonic function $v$ (resp. strongly pluri-subharmonic, and pluri-harmonic) on $U_p$ is an upper semi-continuous function with value in $\mathbb{R} \cup \{-\infty\}$ ($v$ is not locally $-\infty$) such that $v$ extends to a pluri-subharmonic function $\tilde{v}$ (resp. strongly pluri-subharmonic, and pluri-harmonic) on a neighborhood of the image of some local embedding $U_p \hookrightarrow \mathbb{C}^m$. We call $v$ smooth if $\tilde{v}$ is smooth. A form $\omega$ on $M_0$ is called a Kähler form, if $\omega$ is a smooth Kähler form in the usual sense on $M_0 \setminus S$ and, for any $p \in S$, there is a neighborhood $U_p$ and a continuous strongly pluri-subharmonic function $v$ on $U_p$ such that $\omega = \sqrt{-1} \partial \bar{\partial} v$ on $U_p \cap (M_0 \setminus S)$. We call $\omega$ smooth if $v$ is smooth in the above sense. Otherwise, we call $\omega$ a singular Kähler form. If $PH_{M_0}$ denotes the sheaf of pluri-harmonic functions on $M_0$, then any Kähler form $\omega$ represents a class $[\omega]$ in $H^1(M_0, PH_{M_0})$ (cf. section 5.2 in [18]). We also have an analogue of Chern-Weil theory for line bundles on $M_0$ (see [18] for details). If $L_0$ is an ample line bundle on $M_0$, then there is an embedding $M_0 \hookrightarrow \mathbb{CP}^N$ such that $L_0^m = \mathcal{O}(1)|_{M_0}$, and the first Chern class $c_1(L_0)$ can be presented by a smooth Kähler form: $c_1(L_0) = \frac{1}{m}[\omega_{FS}|_{M_0}] \in H^1(M_0, PH_{M_0})$, where $\omega_{FS}$ denotes the standard Fubini-Study Kähler form on $\mathbb{CP}^N$.

In [18] (see also [66]), a generalized Calabi–Yau theorem was obtained, which says that if $M_0$ is a Calabi–Yau variety, then for any ample line bundle $L_0$ there is a unique Ricci-flat Kähler form $\omega \in c_1(L_0)$. We denote by $g$ the corresponding Kähler metric of $\omega$ on $M_0 \setminus S$. If $M_0$ admits a crepant resolution $(\tilde{M}, \tilde{\pi})$, and $\alpha_s \in H^{1,1}(M, \mathbb{R})$, $s \in (0, 1)$, is a family of Kähler classes with $\lim_{s \to 0} \alpha_s = \tilde{\pi}^* c_1(L_0)$, [56] proved that

$$
\bar{g}_s \longrightarrow \tilde{\pi}^* g, \quad \bar{\omega}_s \longrightarrow \tilde{\pi}^* \omega, \quad \text{when} \quad s \to 0
$$

in the $C^\infty$-sense on any compact subset $K$ of $\tilde{M} \setminus \tilde{\pi}^{-1}(S)$, where $\bar{g}_s$ is the unique Ricci-flat Kähler metric with Kähler form $\bar{\omega}_s \in \alpha_s$. Assume that $M_0$ is a Calabi–Yau conifold and $M_0$ admits a smoothing $(M, \pi)$ satisfying that the relative canonical bundle $K_{M/\Delta}$ is trivial and that $M$ admits an ample line bundle $L$ such that $L|_{M_0} = L_0$. For any $t \in \Delta \setminus \{0\}$, if $\bar{g}_t$ denotes the unique Ricci-flat Kähler metric on $M_t = \pi^{-1}(t)$ with Kähler form $\bar{\omega}_t \in c_1(L)|_{M_t}$, [49] proved that

$$
F^* \bar{g}_t \longrightarrow g, \quad F^* \bar{\omega}_t \longrightarrow \omega, \quad \text{when} \quad t \to 0,
$$

where $F: M_0 \to M_t$ is the natural projection.
in the $C^\infty$-sense on any compact subset $K \subset M_0 \setminus S$, where $F_t : M_0 \setminus S \to M_t$ is a family of embeddings. If $M_0$ is a Calabi–Yau variety (not necessarily a conifold) and $\mathcal{M}$ is smooth, then a subsequence-$C^\infty$ convergence theorem for the Ricci-flat Kähler metric $\tilde{g}_t$ on $M_t$ was obtained in [49], i.e., there is a sequence $t_k \in \Delta \setminus \{0\}$ such that $t_k \to 0$, and $F_{t_k}^* \tilde{g}_{t_k}$ converges to $g$ ($k \to \infty$) in the $C^\infty$-sense on any compact subset $K \subset M_0 \setminus S$.

In the proof of Theorem 1.1, the following generalization of the convergence results in [49] plays a significant role.

**Theorem 1.4.** Let $M_0$ be a Calabi–Yau $n$-variety ($n \geq 2$) with singular set $S$. Assume that $M_0$ admits a smoothing $\pi : \mathcal{M} \to \Delta$ such that $\mathcal{M}$ admits an ample line bundle $\mathcal{L}$ and the relative canonical bundle is trivial, i.e., $K_{\mathcal{M}/\Delta} \cong \mathcal{O}_\mathcal{M}$. If $\tilde{g}_t$ denotes the unique Ricci-flat Kähler metric with Kähler form $\tilde{\omega}_t \in c_1(\mathcal{L})|_{M_t} \in H^{1,1}(M_t, \mathbb{R})$ ($t \in \Delta \setminus \{0\}$), and $\omega$ denotes the unique singular Ricci-flat Kähler form on $M_0$ with $\omega \in c_1(\mathcal{L})|_{M_0} \in H^1(M_0, \mathcal{P}\mathcal{H}_{M_0})$, then

$$F_t^* \tilde{g}_t \to g, \quad F_t^* \tilde{\omega}_t \to \omega,$$

in the $C^\infty$-sense on any compact subset $K \subset M_0 \setminus S$, where $F_t : M_0 \setminus S \to M_t$ is a smooth family of embeddings and $g$ is the corresponding Kähler metric of $\omega$ on $M_0 \setminus S$. Furthermore, the diameter of $(M_t, \tilde{g}_t)$ ($t \in \Delta \setminus \{0\}$) satisfies

$$\text{diam}_{\tilde{g}_t}(M_t) \leq D,$$

where $D > 0$ is a constant independent of $t$.

Our proof of Theorem 1.1 is to show that $(M_0 \setminus S, g)$ has a metric completion $(X, d_X)$ satisfying the property that both $\{(M_t, \tilde{g}_t)\}$ and $\{(M_t, \tilde{g}_t)\}$ converge to $(X, d_X)$ in the Gromov–Hausdorff topology when $s \to 0$ and $t \to 0$. The same method also proves Theorem 1.2.

As an application of Theorem 1.1 and Theorem 1.2, we shall explore the path connectedness properties of a certain class of Ricci-flat Calabi–Yau threefolds. Inspired by string theory in physics, some physicists made a projective version of Reid’s fantasy (cf. [7], [24], and [48]), the so-called connectedness conjecture, which is formulated more precisely in [28] (see also [29]). This conjecture says that there is a huge connected web $\Gamma$ such that nodes of $\Gamma$ consist of all deformation classes of Calabi–Yau threefolds, and two nodes are connected $\mathcal{D}_1 - \mathcal{D}_2$ if $\mathcal{D}_1$ and $\mathcal{D}_2$ are related by an extremal transition, i.e., there is a Calabi–Yau 3-variety $M_0$ that admits a crepant resolution $\bar{M} \in \mathcal{D}_1$ and a smoothing $(\mathcal{M}, \pi)$ satisfying $\pi^{-1}(t) = M_t \in \mathcal{D}_2$ for any $t \in \Delta \setminus \{0\}$. It was shown in [24], [16], [5], and [28] that many Calabi–Yau threefolds are connected to each other in the above sense. By combining the connectedness conjecture and Theorem 1.1 and Theorem 1.2, we reach a metric version.
of connectedness conjecture as follows: if $\mathcal{CY}_3$ denotes the set of Ricci-flat Calabi–Yau threefolds $(M, g)$ with volume 1, then the closure $\overline{\mathcal{CY}}_3$ of $\mathcal{CY}_3$ in $(\mathcal{X}, d_{GH})$ is path connected, i.e., for any two points $p_1$ and $p_2 \in \mathcal{CY}_3$, there is a path $\gamma : [0, 1] \rightarrow \overline{\mathcal{CY}}_3 \subset (\mathcal{X}, d_{GH})$ such that $p_1 = \gamma(0)$ and $p_2 = \gamma(1)$.

Given a class of Calabi–Yau 3-manifolds known to be connected by extremal transitions and flops in algebraic geometry, Theorem 1.1 and Theorem 1.2 can be used to show that the closure of the class of Calabi–Yau 3-manifolds is path connected in $(\mathcal{X}, d_{GH})$. In the minimal model program, it was proved that for any two Calabi–Yau 3-manifolds $M$ and $M'$ birational to each other, there is a sequence of flops connecting $M$ and $M'$ (cf. [38], [39], [35]). In [24], it was shown that all complete intersection Calabi–Yau manifolds (CICY) of dimension 3 in products of projective spaces are connected by conifold transitions. Furthermore, in [5] and [16] a large number of complete intersection Calabi–Yau 3-manifolds in toric varieties were verified to be connected by extremal transitions, which include Calabi–Yau hypersurfaces in all toric manifolds obtained by resolving weighted projective 4-spaces. As a corollary of Theorem 1.1 and Theorem 1.2, we obtain the following result.

**Corollary 1.5.** For any Calabi–Yau manifold $M$, let

$$\mathcal{M}_M = \{(M, g) \in \mathcal{X} \mid g \text{ is a Ricci flat Kähler metric on } M \text{ with } \text{Vol}_g(M) = 1\}. $$

i) If $M$ is a three-dimensional Calabi–Yau manifold, and

$$\mathcal{BM}_M = \bigcup_{\text{all Calabi–Yau manifolds } M' \text{ birational to } M} \mathcal{M}_{M'},$$

then the closure $\overline{\mathcal{BM}}_M$ of $\mathcal{BM}_M$ in $(\mathcal{X}, d_{GH})$ is path connected.

ii) Let

$$\mathcal{CP} = \bigcup_{\text{all CICY 3-manifolds } M' \text{ in products of projective spaces}} \mathcal{M}_{M'},$$

Then the closure $\overline{\mathcal{CP}}$ of $\mathcal{CP}$ in $(\mathcal{X}, d_{GH})$ is path connected.

iii) There is a path connected component $\overline{\mathcal{E}}$ of $\overline{\mathcal{CY}}_3 \subset (\mathcal{X}, d_{GH})$ such that $\mathcal{CP} \subset \overline{\mathcal{E}}$, and $\overline{\mathcal{E}}$ contains all $(M, g)$, where $M$ is a Calabi–Yau hypersurface in a toric 4-manifold obtained by resolving a weighted projective 4-space, and $g$ is a Ricci-flat Kähler metric of volume 1 on $M$.

The study of metric behaviors under some algebraic geometric surgeries also arises from other perspectives, such as Kähler–Ricci flow (cf.
and balanced metrics on non-Kähler Calabi–Yau threefolds (cf. [20]).

The rest of the paper is organized as follows: In Section 2, we bound from above the diameters of Ricci-flat Calabi–Yau manifolds along a smoothing. In Section 3, we prove Theorem 1.4. In Section 4, we establish a link between point-wise $C^\infty$-convergence of Riemannian metrics on a ‘big’ open subset and global Gromov–Hausdorff convergence. In Section 5, we prove Theorem 1.1, Theorem 1.2, and Corollary 1.5. In Appendix A, we supply basic properties on Gromov–Hausdorff convergence used in Section 4. In Appendix B (written by Mark Gross), some bounds for volumes of Calabi–Yau manifolds along a smoothing are provided which are used in the proof of Theorem 1.4.

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2. A Priori Estimate

In this section, we obtain an estimate for diameters of Ricci-flat Calabi–Yau manifolds along a smoothing, which plays a key role in our $C^0$-estimate in the proof of Theorem 1.4.

**Theorem 2.1.** Let $M_0$ be a projective $n$-dimensional variety with singular set $S$. Assume that $M_0$ admits a smoothing $\pi : M \to \Delta$ over the unit disc $\Delta \subset \C$ such that $M$ admits an ample line bundle $\mathcal{L}$, and the relative canonical bundle is trivial, i.e., $K_{M/\Delta} \cong \mathcal{O}_M$. Let $\Omega_t$ be a relative holomorphic volume form, i.e., a nowhere vanishing section of $K_{M/\Delta}$, and let $\tilde{g}_t$ be the unique Ricci-flat Kähler metric with Kähler form $\tilde{\omega}_t \in c_1(\mathcal{L})|_{M_t} \in H^{1,1}(M_t, \mathbb{R})$, for $t \in \Delta \setminus \{0\}$. Then the diameter of $(M_t, \tilde{g}_t)$ satisfies that

$$\text{diam}_{\tilde{g}_t}(M_t) \leq 2 + D(-1)^{\frac{n}{2}} \int_{M_t} \Omega_t \wedge \bar{\Omega}_t,$$

where $D$ is a constant independent of $t$. 

Proof. Recall that $\mathcal{M}$ is an $(n + 1)$-dimensional variety with an embedding $\mathcal{M} \hookrightarrow \mathbb{CP}^N \times \Delta$ such that $\mathcal{L}^m = \mathcal{O}_\Delta(1)|_\mathcal{M}$ for an $m \geq 1$, $\pi$ is the restriction to $\mathcal{M}$ of the projection from $\mathbb{CP}^N \times \Delta$ to $\Delta$, which is a proper surjection such that the rank of $\pi_*$ is 1 on $\mathcal{M}\setminus S$. Then $M_t = \pi^{-1}(t)$ is a smooth Calabi–Yau manifold for any $t \in \Delta\setminus\{0\}$. Denote

$$\omega_t = \frac{1}{m} \omega_{FS}|_{M_t},$$

where $\omega_{FS}$ is the standard Fubini-Study metric on $\mathbb{CP}^N$, and $g_t$ is the corresponding Kähler metric of $\omega_t$. Note that $\tilde{\omega}_t$ satisfies the Monge-Ampère equation

(2.1) $$\tilde{\omega}_t^n = (-1)^{\frac{n^2}{2}} e^{\sigma_t} \Omega_t \wedge \overline{\Omega}_t,$$

where $e^{\sigma_t} = V \left( (-1)^{\frac{n^2}{2}} \int_{M_t} \Omega_t \wedge \overline{\Omega}_t \right)^{-1}$, where $V = n! \text{Vol}_{g_t}(M_t)$ is a constant independent of $t$.

For $p \in M_0\setminus S$, there are coordinates $z_0, \ldots, z_n$ on a neighborhood $U$ of $p$ in $\mathcal{M}$ such that $t = \pi(z_0, \ldots, z_n) = z_0$, $p = (0, \ldots, 0)$, and the closure $\overline{U}$ of $U$ is a compact subset of $\mathcal{M}\setminus S$. There is an $r_0 > 0$ such that $\Delta^1 \times \Delta^n \subset U$, where $\Delta^1 = \{|t| < r_0\} \subset \Delta$, $\Delta^n = \{|z_j| < r_0, j = 1, \ldots, n\} \subset \mathbb{C}^n$, and $\{t\} \times \Delta^n \subset M_t$. Note that locally $\omega_t$ and $\tilde{\omega}_t$ are families of Kähler forms on $\Delta^n \subset \mathbb{C}^n$, and there is a constant $C_1$ independent of $t$ such that

(2.2) $$C_1^{-1} \omega_E \leq \omega_t \leq C_1 \omega_E,$$

where $\omega_E = \sqrt{-1} \partial \overline{\partial} \sum_{i=1}^n |z_i|^2$ is the standard Euclidean Kähler form on $\Delta^n$, and $g_E$ denotes the corresponding Euclidean Kähler metric. We need the following fact, which is a simplified version of lemma 1.3 in [17]. For completeness, we shall sketch a proof.

**Lemma 2.2** (lemma 1.3 in [17]). For any $\delta > 0$, and any $t \in \Delta^1\setminus\{0\}$, there is an open subset $U_{t,\delta}$ of $\Delta^n$ such that

$$\text{Vol}_{g_t}(U_{t,\delta}) \geq \text{Vol}_{g_t}(\Delta^n) - \delta, \quad \text{diam}_{g_t}(U_{t,\delta}) \leq \hat{C} \delta^{-\frac{1}{2}},$$

where $\hat{C}$ is a constant independent of $t$.

**Proof.** Let $dv_E = (-1)^{\frac{n^2}{2}} dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n$ be the standard Euclidean volume form on $\Delta^n$ and for any $x_1, x_2 \in \Delta^n$, let $[x_1, x_2] \subset \Delta^n$
be the segment connecting \(x_1\) and \(x_2\). By Fubini’s Theorem, the Cauchy-Schwarz inequality, and (2.2), we have
\[
\int_{\Delta^n \times \Delta^n} \text{length}_{g_t}([x_1, x_2])^2 dv_E(x_1) dv_E(x_2)
\]
\[
\leq ||x_2 - x_1||^2_E \int_0^1 ds \int_{\Delta^n \times \Delta^n} \text{tr}_{g_E} \tilde{\omega}_t((1-s)x_1 + sx_2) dv_E(x_1) dv_E(x_2)
\]
\[
\leq 2^n \text{diam}^2_{g_E}(\Delta^n) \text{Vol}_{g_E}(\Delta^n) \int_{\Delta^n} \tilde{\omega}_t \wedge \omega_E^{n-1}
\]
\[
\leq C_2 \int_{\Delta^n} \tilde{\omega}_t \wedge \omega_t^{n-1}
\]
\[
\leq C_2 \int_{M_t} \tilde{\omega}_t \wedge \omega_t^{n-1} = \tilde{C},
\]
where \(\tilde{C}\) is a constant independent of \(t\). The second inequality is obtained by integrating first with respect to \(y = (1-s)x_1\) when \(s \leq \frac{1}{2}\), then with respect to \(y = sx_2\) when \(s \geq \frac{1}{2}\), since \(dv_E(x_i) \leq 2^n dv_E(y)\). If \(S_t = \{(x_1, x_2) \in \Delta^n \times \Delta^n | \text{length}_{g_t}([x_1, x_2]) > \tilde{C}\delta^{-1}\}\), then \(\text{Vol}_{g_E \times g_E}(S_t) < \delta\). Let \(S_t(x_1) = \{x_2 \in \Delta^n | (x_1, x_2) \in S_t\}\), and let \(Q_t = \{x_1 \in \Delta^n | \text{Vol}_{g_E}(S_t(x_1)) \geq \frac{1}{2} \text{Vol}_{g_E}(\Delta^n)\}\). By Fubini’s Theorem,
\[
\text{Vol}_{g_E}(Q_t) < 2\delta \text{Vol}_{g_E}^{-1}(\Delta^n), \quad \text{Vol}_{g_E}(S_t(x_j)) < \frac{1}{2} \text{Vol}_{g_E}(\Delta^n),
\]
for any \(x_1, x_2 \in \Delta^n \setminus Q_t\). Thus \((\Delta^n \setminus S_t(x_1)) \cap (\Delta^n \setminus S_t(x_2))\) is not empty. If \(y \in (\Delta^n \setminus S_t(x_1)) \cap (\Delta^n \setminus S_t(x_2))\), then \((x_1, y), (x_2, y) \in (\Delta^n \times \Delta^n) \setminus S_t\), and
\[
\text{length}^2_{g_t}([x_1, y] \cup [y, x_2]) \leq 2\tilde{C}\delta^{-1},
\]
and therefore
\[
\text{diam}^2_{g_t}(\Delta^n \setminus Q_t) \leq 2\tilde{C}\delta^{-1}.
\]
If we denote \(U_{t, \delta} = \Delta^n \setminus Q_t\), then by (2.2) we derive
\[
\text{Vol}_{g_t}(\Delta^n \setminus U_{t, \delta}) = \text{Vol}_{g_t}(Q_t) \leq C_3 \text{Vol}_{g_E}(Q_t) < 2C_3 \text{Vol}_{g_E}^{-1}(\Delta^n)\delta,
\]
where \(C_3 > 0\) is a constant independent of \(t\). By replacing \(\delta\) with \((2C_3)^{-1} \text{Vol}_{g_E}(\Delta^n)\delta\), we obtain the desired conclusion. q.e.d.

We return to the proof of Theorem 2.1. Let \(\delta_t = \frac{1}{2} \text{Vol}_{g_t}(\Delta^n)\), and let \(p_t \in U_{t, \delta_t}\). By (2.2), we get
\[
\delta_t \geq \frac{C_4}{2} \text{Vol}_{g_E}(\Delta^n) = \tilde{\delta}
\]
and thus \(U_{t, \delta_t} \subset B_{g_t}(p_t, r)\), where \(r = \max\{1, 2\tilde{C}\delta^{-\frac{1}{2}}\}\) and \(\tilde{C}\) is the constant in Lemma 2.2. Since \(U \subset M \setminus S\), there is a constant \(\kappa_U > 0\) such that
\[
(-1)^{n^2} \Omega_t \wedge \overline{\Omega}_t \geq \kappa_U \omega_t^n
\]
on $U \cap M_t$. Note that $\Delta^1 \times \Delta^n \subset U$, and \{t\} $\times \Delta^n \subset U \cap M_t$. Since $U_{t,\delta} \subset \{t\} \times \Delta^n$, we have $U_{t,\delta} \subset U \cap M_t$. By (2.1), we derive

\[
\text{Vol}_{\tilde{g}_t}(B_{\tilde{g}_t}(pt, r)) \geq \text{Vol}_{\tilde{g}_t}(U_{t,\delta_t}) = \frac{(-1)^2}{n!} e^{\sigma_t} \int_{U_{t,\delta_t}} \Omega_t \wedge \nabla_t \geq \frac{\kappa_U e^{\sigma_t}}{n!} \int_{U_{t,\delta_t}} \omega^n_t = \kappa_U e^{\sigma_t} \text{Vol}_{g_t}(U_{t,\delta_t}) \geq \frac{\kappa_U e^{\sigma_t}}{2} \text{Vol}_{g_t}(\Delta^n) \geq C_5 e^{\sigma_t} \text{Vol}_{g_E}(\Delta^n) = C_6 e^{\sigma_t},
\]

where $C_6$ is a constant independent of $t$. By Bishop-Gromov relative volume comparison, we obtain

\[
\text{Vol}_{\tilde{g}_t}(B_{\tilde{g}_t}(pt, 1)) \geq \frac{1}{r^{2n}} \text{Vol}_{\tilde{g}_t}(B_{\tilde{g}_t}(pt, r)) \geq C_6 \frac{e^{\sigma_t}}{r^{2n}}.
\]

In the rest of the proof, we need the following lemma.

**Lemma 2.3** (Theorem 4.1 of Chapter 1 in [50] and Lemma 2.3 in [44]). Let $(M, g)$ be a 2n-dimensional compact Riemannian manifold with nonnegative Ricci curvature. Then for any $p \in M$ and any $1 < R < \text{diam}_g(M)$, we have

\[
\frac{\text{Vol}_g(B_g(p, 2R + 2))}{\text{Vol}_g(B_g(p, 1))} \geq \frac{R - 1}{2n}.
\]

By letting $R = \frac{1}{2} \text{diam}_{\tilde{g}_t}(M_t)$, we obtain

\[
\text{diam}_{\tilde{g}_t}(M_t) \leq 2 + 8n \frac{\text{Vol}_{\tilde{g}_t}(M_t)}{\text{Vol}_{\tilde{g}_t}(B_{\tilde{g}_t}(pt, 1))} \leq 2 + De^{-\sigma_t},
\]

where $D$ is a constant independent of $t$. We conclude the proof by (2.1).

q.e.d.

The following is a consequence of Theorem 2.1 and Theorem B.1.

**Corollary 2.4.** Let $M_0$, $M$, $\mathcal{L}$, $\Omega_t$, and $\tilde{g}_t$ be as in Theorem 2.1. If in addition we assume that $M_0$ is a Calabi–Yau n-variety, then the diameter of $(M_t, \tilde{g}_t)$ has a uniform bound

\[
\text{diam}_{\tilde{g}_t}(M_t) \leq D,
\]

where $D$ is a constant independent of $t$. 
3. Proof of Theorem 1.4

Let $M_0$ be an $n$-dimensional Calabi–Yau variety with singular set $S$. Assume that $M_0$ admits a smoothing $\pi : M \to \Delta$ over the unit disc $\Delta \subset \mathbb{C}$ such that $M$ admits an ample line bundle $\mathcal{L}$, and the relative canonical bundle is trivial, i.e., $K_{M/\Delta} \cong \mathcal{O}_M$. Following the discussion at the beginning of the proof of Theorem 2.1, let

$$\omega_t = \omega_h|_{M_t} = \frac{1}{m} \omega_{FS}|_{M_t}, \quad \text{and} \quad \omega_h = \frac{1}{m} \omega_{FS},$$

for any $t \in \Delta$, where $\omega_{FS}$ is the standard Fubini-Study metric on $\mathbb{CP}^N$, and $g_t$ is the corresponding Kähler metric of $\omega_t$. Let $\Omega_t$ be a relative holomorphic volume form, i.e., a nowhere vanishing section of $K_{M/\Delta}$.

Yau’s proof of Calabi’s conjecture ([61]) asserts that there is a unique Ricci-flat Kähler metric $\tilde{g}_t$ with Kähler form $\tilde{\omega}_t \in [\omega_t] = c_1(\mathcal{L})|_{M_t} \in H^{1,1}(M_t, \mathbb{R})$ for $t \in \Delta \setminus \{0\}$, i.e., there is a unique function $\varphi_t$ on $M_t$ satisfying that $\tilde{\omega}_t = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi_t$, and

$$\frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n}{n!} = (-1)^{\frac{n^2}{2}} \chi_t \Omega_t \wedge \bar{\Omega}_t,$$

where

$$\chi_t = \log \left( n! V((-1)^{\frac{n^2}{2}} \int_{M_t} \Omega_t \wedge \bar{\Omega}_t)^{-1} \right)$$

and $V = \text{Vol}_{\tilde{g}_t}(M_t)$.

By Theorem B.1, on $M_t$ we have

$$\frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n}{n!} \geq \kappa \omega_t^n,$$

and

$$\int_{M_t} (-1)^{\frac{n^2}{2}} \chi_t \Omega_t \wedge \bar{\Omega}_t \leq \Lambda,$$

where $\kappa > 0$ and $\Lambda > 0$ are constants independent of $t \in \Delta \setminus \{0\}$. Thus there is a constant $C_1 > 0$ independent of $t$ such that

$$-C_1 \leq \chi_t \leq C_1.$$

Note that $(M_t, \tilde{g}_t)$ satisfies that

$$\text{Ric}_{\tilde{g}_t} \equiv 0, \quad \text{Vol}_{\tilde{g}_t}(M_t) \equiv V, \quad \text{and} \quad \text{diam}_{\tilde{g}_t}(M_t) \leq D,$$

where the upper bound of diameters is from Corollary 2.4. By [15], [21], and [40], $(M_t, \tilde{g}_t)$ has uniform Sobolev constants, i.e., constants $\bar{C}_{S,1} > 0$ and $\bar{C}_{S,2} > 0$ independent of $t$ such that for any $t \neq 0$ and any smooth function $\chi$ on $M_t$,

$$\|\chi\|^2_{L^{\frac{4n}{2n-2}}(\tilde{g}_t)} \leq \bar{C}_{S,1}(\|d\chi\|^2_{L^2(\tilde{g}_t)} + \|\chi\|^2_{L^2(\tilde{g}_t)}),$$

and

$$\|\chi\|^2_{L^{\frac{4n}{2n-2}}(\tilde{g}_t)} \leq \bar{C}_{S,2}(\|d\chi\|^2_{L^2(\tilde{g}_t)} + \|\chi\|^2_{L^2(\tilde{g}_t)}),$$

which concludes the proof of Theorem 1.4.
and if $\int_{M_t} \chi dv_{\bar{g}_t} = 0$,
\begin{equation}
\|\chi\|_{L^{4n-2}(\bar{g}_t)}^2 \leq \tilde{C}_{S,2} \|d\chi\|^2_{L^2(\bar{g}_t)}.
\end{equation}

Now following the standard Moser iteration argument in [61] with a trick inspired by [56], we are able to get a uniform $C^0$-estimate of the potential function $\varphi_t$.

**Lemma 3.1.** There is a constant $C > 0$ independent of $t \in \Delta \{0\}$ such that
\[\|\varphi_t\|_{C^0(M_t)} \leq C.\]

**Proof.** Let
\[f_t = \log \left( \frac{(-1)^{\frac{n^2}{2}} e^{\sigma_t} \Omega_t \wedge \overline{\Omega}_t}{\omega^n_t} \right), \quad \tilde{\varphi}_t = \int_{M_t} \varphi_t dv_{\bar{g}_t} - \varphi_t.\]

Then (3.1) shows that
\[
\omega^n_t = e^{-f_t} \tilde{\omega}_t^n = (\tilde{\omega}_t + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_t)^n, \quad \text{with} \quad \int_{M_t} \tilde{\varphi}_t dv_{\bar{g}_t} = 0.
\]

By (3.2) and (3.4), there is a constant $C_2 > 0$ independent of $t$ such that
\[e^{-f_t} = \left( \frac{(-1)^{\frac{n^2}{2}} e^{\sigma_t} \Omega_t \wedge \overline{\Omega}_t}{\omega^n_t} \right)^{-1} \leq C_2.
\]

Now we follow the standard Moser iteration argument in [61] (cf. [4]).

A direct calculation shows that (3.7)
\[\int_{M_t} |d\tilde{\varphi}_t|^2 dv_{\bar{g}_t} \leq \frac{np^2}{4(p-1)} \int_{M_t} |1-e^{-f_t}|\tilde{\varphi}_t|^{p-1} dv_{\bar{g}_t} \leq Ap \int_{M_t} |\tilde{\varphi}_t|^{p-1} dv_{\bar{g}_t},
\]
for any $p \geq 2$ (cf. (15) in chapter 7 of [4]), where $A > 0$ is a constant independent of $t$. For $p = 2$, by (3.6), (3.7), and Hölder’s inequality we see that
\[\|\tilde{\varphi}_t\|_{L^{4n-2}(\bar{g}_t)}^2 \leq \tilde{C}_{S,2} \|d\tilde{\varphi}_t\|^2_{L^2(\bar{g}_t)} \leq 2\tilde{C}_{S,2} \int_{M_t} |\tilde{\varphi}_t| dv_{\bar{g}_t} \leq 2\tilde{C}_{S,2} V^{\frac{2n}{n+2}} \|\tilde{\varphi}_t\|_{L^{4n-2}(\bar{g}_t)}^2,
\]
and thus
\[\|\tilde{\varphi}_t\|_{L^{4n-2}(\bar{g}_t)} \leq \hat{C},\]
where $\hat{C}$ is a constant independent of $t$. For $p > 2$, by (3.5), (3.7), and Hölder’s inequality we see that

$$\left\| \hat{\varphi}_t \right\|_{L^{\frac{2n}{n-2}}(\hat{g}_t)}^p \leq \hat{C}_{S,1} \left( \left\| d\hat{\varphi}_t \right\|_{L^2(\hat{g}_t)}^2 + \left\| \hat{\varphi}_t \right\|_{L^2(\hat{g}_t)}^2 \right) \leq \hat{C}_{S,1} \left( pAV^\frac{p}{2} + \left\| \hat{\varphi}_t \right\|_{L^p(\hat{g}_t)} \right) \left\| \hat{\varphi}_t \right\|_{L^p(\hat{g}_t)}^{p-1}.$$  

Let $p_0 = \frac{4n}{2n-2}$, $p_{k+1} = \frac{2n}{2n-2}p_k$ ($k \geq 0$), let $\hat{C}_0 = \hat{C}$ and let $\hat{C}_{k+1} = \hat{C}_{S,1}^{-1} (p_k AV^\frac{p_k}{2} + 1) \hat{C}_k$ if $\hat{C}_k > 1$. Otherwise, let $\hat{C}_{k+1} = \hat{C}_{S,1}^{-1} (p_k AV^\frac{p_k}{2} + 1) \hat{C}_k$. Then $\left\| \hat{\varphi}_t \right\|_{L^{p_k}(\hat{g}_t)} \leq \hat{C}_k < C_3$, a constant $C_3 > 0$ independent of $k$ and $t$. By letting $k \to \infty$, we have

$$\left\| \hat{\varphi}_t \right\|_{C^0(M_t)} \leq C_3.$$  

Since there is a $p_t \in M_t$ such that $\varphi_t(p_t) = 0$, we have

$$\left| \int_{M_t} \varphi_t dv_{\hat{g}_t} \right| \leq C_3,$$  

where $C > 0$ is a constant independent of $t$. \[ \text{q.e.d.} \]

The $C^2$-estimate for $\varphi_t$ is obtained by the same arguments as in the proof of lemma 5.2 in [49]. For completeness, we present it here.

**Lemma 3.2.** For any compact subset $K \subset M \setminus S$, there exists a constant $C_K > 0$ independent of $t$ such that on $K \cap M_t$,

$$C \omega_t \leq \omega_t \leq C_K \omega_t,$$

where $C > 0$ is a constant independent of $t$ and $K$.

**Proof.** Let $\psi_t : (M_t, \omega_t) \to (\mathbb{CP}^N, \frac{1}{m} \omega_{FS})$ be the inclusion map induced by $M \subset \mathbb{CP}^N \times \Delta$. The Chern-Lu inequality says

$$\Delta_{\omega_t} \log |\partial \psi_t|^2 \geq \frac{\text{Ric}_{\omega_t}(\partial \psi_t, \partial \psi_t)}{|\partial \psi_t|^2} - \frac{\text{Sec}(\partial \psi_t, \partial \psi_t, \partial \psi_t, \partial \psi_t)}{|\partial \psi_t|^2},$$

where Sec denotes the holomorphic bi-sectional curvature of $\frac{1}{m} \omega_{FS}$ (cf. [62]). Note that $\frac{1}{m} \psi_t^* \omega_{FS} = \omega_t$, $|\partial \psi_t|^2 = \frac{1}{m} \text{tr}_{\omega_t} \psi_t^* \omega_{FS} = \text{tr}_{\omega_t} \omega_t = n - \Delta_{\omega_t} \varphi_t$ and $\text{Ric}_{\omega_t} = 0$. Thus we have that

$$\Delta_{\omega_t} (\log \text{tr}_{\omega_t} \omega_t - 2 \mathcal{P}(\varphi_t)) \geq -2 \mathcal{P} n + \mathcal{P} \text{tr}_{\omega_t} \omega_t,$$

where $\mathcal{P}$ is a constant depending only on the upper bound of Sec. By the maximum principle and Lemma 3.1, there is an $x \in M_t$ such that $\text{tr}_{\omega_t} \omega_t(x) \leq 2n$,

$$\text{tr}_{\omega_t} \omega_t \leq 2ne^{2 \mathcal{P}(\varphi_t - \varphi_t(x))} \leq C \quad \text{and} \quad \omega_t \leq C \omega_t,$$
where $C > 0$ is a constant independent of $t$. Note that for any compact subset $K \subset \mathcal{M} \setminus S$, by (3.4) and the compactness of $K$ there exists a constant $C'_K > 0$ independent of $t$ such that on $K \cap M_t$

$$\tilde{\omega}^n_t = e^{\sigma_t} (-1)^{\frac{n^2}{2}} \Omega_t \wedge \overline{\Omega}_t \leq C'_K \omega^0_t.$$ 

Then we obtain that

$$C \omega_t \leq \tilde{\omega}_t \leq C_K \omega_t.$$ 

q.e.d.

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** In [18], it is proved that there is a unique bounded function $\hat{\varphi}_0$ on $M_0$ such that $\hat{\varphi}_0$ is smooth on $M_0 \setminus S$ and satisfies

$$(\omega_0 + \sqrt{-1} \partial \overline{\partial} \hat{\varphi}_0)^n = (-1)^{\frac{n^2}{2}} e^{\hat{\sigma}_0} \Omega_0 \wedge \overline{\Omega}_0,$$

in the distribution sense, where $\hat{\sigma}_0$ is a constant. Note that $\omega = \omega_0 + \sqrt{-1} \partial \overline{\partial} \hat{\varphi}_0$ is the unique singular Ricci-flat Kähler form with $\|\hat{\varphi}_0\|_{L^\infty} \leq C$. Let $F : (M_0 \setminus S) \times \Delta \rightarrow \mathcal{M}$ be a smooth embedding such that $F((M_0 \setminus S) \times \{t\}) \subset M_t$ and $F|(M_0 \setminus S) \times \{0\} : M_0 \setminus S \rightarrow M_0 \setminus S$ is the identity map. Let $K_1 \subset \cdots \subset K_i \subset \cdots \subset M_0 \setminus S$ be a sequence of compact subsets such that $M_0 \setminus S = \bigcup_i K_i$. On a fixed $K_i$, the embedding map

$$F_{K_i,t} = F|_{K_i \times \{t\}} : K_i \rightarrow M_t$$

satisfies that $F_{K_i,t}^* \omega_t$ $C^\infty$-converges to $\omega_0$, and $dF_{K_i,t}^{-1} J_t dF_{K_i,t}$ $C^\infty$-converges to $J_0$, where $J_t$ (resp. $J_0$) is the complex structure on $M_t$ (resp. $M_0$).

For a fixed $K_i$, let $K$ be a compact subset of $\mathcal{M} \setminus S$ such that $F_{K_i,t_k}(K_i) \subset K$ for $|t_k| \ll 1$. By (3.4), Lemma 3.1, and Lemma 3.2, there exist constants $C > 0$ and $C_K > 0$ independent of $t$ such that $C^{-1} \leq \sigma_t \leq C$, $\|\varphi_t\|_{C^\alpha(M_t)} \leq C$, and $C^{-1} \omega_t \leq \omega_t + \sqrt{-1} \partial \overline{\partial} \varphi_t \leq C_K \omega_t$ on $K$. By theorem 17.14 in [31], we have that $\|\varphi_t\|_{C^{2,\alpha}(M_t \cap K)} \leq C_K''$ for a constant $C_K'' > 0$. Furthermore, by the standard bootstrapping argument we have that for any $l > 0$, $\|\varphi_t\|_{C^{1,\alpha}(M_t \cap K)} \leq C_{K,l}$ for constants $C_{K,l} > 0$ independent of $t$. By the standard diagonal arguments and passing to a subsequence, we see that $F_{K_i,t_k}^* \varphi_{t_k}$ $C^\infty$-converges to a smooth function $\varphi_0$ on $M_0 \setminus S$ with $\|\varphi_0\|_{L^\infty} < C$ and that $\sigma_{t_k}$ converges to a $\sigma_0$, which satisfies

$$(\omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi_0)^n = (-1)^{\frac{n^2}{2}} e^{\sigma_0} \Omega_0 \wedge \overline{\Omega}_0.$$ 

Hence $\tilde{\omega}_0 = \omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi_0$ is a Ricci-flat Kähler form on $M_0 \setminus S$ with $\|\varphi_0\|_{L^\infty} < C$. By the uniqueness of the solution of (3.8), $\varphi_0 = \hat{\varphi}_0$ and $\sigma_0 = \hat{\sigma}_0$. The uniqueness of $\omega$ and the standard compactness argument
imply that $F|_{M_0 \setminus S \times \{t\}} \wedge_t$ (resp. $F|_{M_0 \setminus S \times \{t\}} \wedge_t$) $C^\infty$-converges to $\omega$ (resp. $g$) when $t \to 0$.

The diameter estimate is obtained by Corollary 2.4. q.e.d.

4. An Almost Gauge Fixing Theorem

Let $M$ be a compact $n$-manifold, and let $g_k$ be a sequence of Riemannian metrics on $M$. Assume that the Ricci curvature, volume, and diameter of $g_k$ satisfy

i) $|\text{Ric}(g_k)| \leq 1$, $\text{Vol}_{g_k}(M) \geq V > 0$ and $\text{diam}_{g_k}(M) \leq D$.

By the Gromov’s pre-compactness theorem, we may assume

ii) $(M, g_k) \xrightarrow{d_{GH}} (X, d_X)$, where $(X, d_X)$ is a compact metric space.

Suppose, in addition,

iii) $E$ is a closed subset of Hausdorff dimension $\leq n - 2$, and there is a (non-complete) Riemannian metric $g_\infty$ on $M \setminus E$ such that $g_k$ converges to $g_\infty$ in the $C^\infty$-sense on any compact subset $K \subset M \setminus E$.

Because $M \setminus E$ is path connected, $g_\infty$ induces the Riemannian distance structure defined by

$$d_{g_\infty}(x, y) = \inf_{\gamma \text{ continuous}} \{\text{length}_{g_\infty}(\gamma), \, \gamma : [0, 1] \to M \setminus E, \, \gamma(0) = x, \gamma(1) = y\}.$$

Let $(M \setminus E, g_\infty)$ denote the metric completion of $(M \setminus E, d_{g_\infty})$. Let $S_X \subset X$ denote the subset consisting of points $x \in X$ such that there is a sequence $x_k \in E \subset (M, g_k)$ and $x_k \to x$ (see comments at the end of Appendix A). It is clear that $S_X \subset X$ is a closed subset and thus $S_X$ is compact.

The main effort of this section is to prove the following result.

**Theorem 4.1.** Let $M, g_k, g_\infty, d_{g_\infty}, E, (X, d_X),$ and $S_X$ be as above. Then there is a continuous surjection $f : (M \setminus E, d_{g_\infty}) \to (X, d_X)$ such that $f : (M \setminus E, d_{g_\infty}) \to (X \setminus S_X, d_X)$ is a homeomorphism and a local isometry, i.e., for any $x \in M \setminus E$, there is an open neighborhood of $x$, $U \subset M \setminus E$, such that $f : (U, d_{g_\infty}|_U) \to (f(U), d_X|_{f(U)})$ is an isometry.

**Proof.** We first construct a dense subset $A \subseteq X \setminus S_X$ and define a local isometric embedding $h : (A, d_X) \to (M \setminus E, d_{g_\infty})$ such that $f(A)$ is dense. Then we will show that $f = h^{-1} : h(A) \to X \setminus S_X$ extends uniquely to a continuous surjection $f : (M \setminus E, g_\infty) \to (X, d_X)$ such that $f$ is a homeomorphism and a local isometric embedding on $(M \setminus E, d_{g_\infty})$.

Without loss of generality, we may assume that for all $k \geq j$, $d_{GH}((M, g_k), (M, g_j)) < 2^{-j}$. Let $\phi_j : (M, g_{j+1}) \to (M, g_j)$ denote an $2^{-j}$-Gromov–Hausdorff approximation. Then $\phi_j^{j+s} = \phi_j \circ \cdots \circ \phi_{j+s-1} : (M, g_{j+s}) \to (M, g_{j+s-1}) \to \cdots \to (M, g_j)$ is an $2^{-j+1}$-Gromov–
Hausdorff approximation. Recall that there is an admissible metric \( d_Z \) on the disjoint union \( Z = (\coprod_{k=1}^{\infty} (M, g_k)) \coprod (X, d_X) \) such that \((M, g_k) \xrightarrow{\text{d}, H} (X, d_X)\) (see Appendix A, Proposition A.1).

Let \( \epsilon_j = j^{-1} \), \( j = 1, 2, \ldots \). For \( \epsilon_1 \) and each \( g_k \), take a finite \( \epsilon_1 \)-net \( \{x_{i_1}^j\} \subset (M \setminus E, g_k) \) such that

\[
\tag{4.1}
\|\{x_{i_1}^j\}\| \leq c_1', \quad \text{and}
\]

\[
d_{g_k}(\{x_{i_1}^j\}, E) \geq \frac{\epsilon_1}{2}, \quad \text{where } d_{g_k}(\{x_{i_1}^j\}, E) = \min\{d_{g_k}(x_{i_1}^j, y), x_j \in \{x_{i_1}^j\}, y \in E\}.
\]

We may assume, passing to a subsequence if necessary, that \( \{x_{i_1}^j\} \xrightarrow{\text{d}, H, Z} \{x_{i_1}^j\}_{i_1=1}^{c_1} \subset (X, d_X) \), where \( c_1 = \|\{x_{i_1}^j\}\| \). Then by (4.2), \( \{x_{i_1}^j\}_{i_1=1}^{c_1} \subset X \setminus S_X \). We claim that there is \( k_1 > 0 \) such that for all \( k \geq k_1 \), \( \{\phi_{k_1}^k(x_{i_1}^j)\} \subset K_1 = M \setminus B_{g_{k_1}}(E, \frac{\epsilon_1}{4}) \), a compact subset. Here \( B_{g_{k_1}}(E, \frac{\epsilon_1}{4}) = \{y \in M | d_{g_{k_1}}(y, E) < \frac{\epsilon_1}{4}\} \). Assuming the claim, by iii) we may assume that passing to a subsequence \( \{\phi_{k_1}^k(x_{i_1}^j)\} \rightarrow \{y_{i_1}^j\}_{i_1=1}^{c_1} \subset (M \setminus E, g_\infty) \) point-wise, and we denote the corresponding subsequence by \( \{g_{k_1}\} \subset \{g_k\} \).

To verify the claim, we may assume \( k_1 \) large so that for all \( k \geq k_1 \),

\[
d_{Z,H}(\{x_{i_1}^j\}, \{x_{i_1}^j\}) < \frac{\epsilon_1}{9}, \quad 2^{-k_1} \ll \epsilon_1.
\]

For the sake of distinction, let \( E_0 = E \subset (M, g_{k_1}) \). Then

\[
d_{g_{k_1}}(\{\phi_{k_1}^k(x_{i_1}^j)\}, E_0) = d_{Z,H}(\{\phi_{k_1}^k(x_{i_1}^j)\}, E_0)
\geq d_{Z,H}(\{x_{i_1}^j\}, E_0) - d_{Z,H}(\{x_{i_1}^j\}, \{\phi_{k_1}^k(x_{i_1}^j)\})
\geq \frac{\epsilon_1}{2} - [d_{Z,H}(\{x_{i_1}^j\}, \{x_{i_1}^j\}) + d_{Z,H}(\{\phi_{k_1}^k(x_{i_1}^j)\}, \{x_{i_1}^j\})]
\geq \frac{\epsilon_1}{2} - \left[ \frac{\epsilon_1}{9} + d_{Z,H}(\{\phi_{k_1}^k(x_{i_1}^j)\}, \{x_{i_1}^j\}) + d_{Z,H}(\{x_{i_1}^j\}, \{x_{i_1}^j\}) \right]
\geq \frac{\epsilon_1}{2} - \left( \frac{\epsilon_1}{9} + \frac{\epsilon_1}{9} + 2^{-k_1} \right) \geq \frac{\epsilon_1}{4}.
\]

For \( \epsilon_2 \) and each \( g_{k_1} \), extend \( \{x_{i_1}^{k_1}\} \) to an \( \epsilon_2 \)-dense subset of \((M \setminus E, g_{k_1})\), \( \{x_{i_2}^{k_1}\} \subset \{x_{i_2}^{k_1}\} \), such that for all \( g_{k_1} \),

\[
\tag{4.3}
d_{g_{k_1}}(x_{i_2}^{k_1}, x_{i_2}^{k_1}) \geq \frac{\epsilon_2}{4}, \quad \|\{x_{i_2}^{k_1}\}\| \leq c'_2, \quad \text{and}
\]

\[
\tag{4.4}
d_{g_{k_1}}(\{x_{i_2}^{k_1}\}, E) \geq \frac{\epsilon_2}{2}.
\]

Similarly, by (4.3) and (4.4), passing to a subsequence we may assume that \( \{x_{i_2}^{k_1}\} \xrightarrow{\text{d}, H, Z} \{x_{i_2}\} \subset (X \setminus S_X, d_X) \). Clearly, \( \{x_{i_1}^j\} \subset \{x_{i_2}^j\}_{i_2=1}^{c_2}, \)
where $c_2 = |\{x_{i_2}\}|$. By the argument as in the above, we may assume large $k_2 > k_1$ such that for all $k \geq k_2$, $\{\phi_{k_2}^k(x_{i_2})\} \subset K_2 = (M \setminus B_{g_{k_1}}(E, \frac{c_2}{4}))$. By the compactness of $K_2$ and iii), we may assume that $\{\phi_{k_2}^k(x_{i_2})\} \rightarrow \{y_{i_2}\}_{i_2=1}^c \subset (M \setminus E, d_{g_{\infty}})$ point-wise. The natural identification $\phi_{k_1}^k(x_{i_1}) \leftrightarrow \phi_{k_2}^k(x_{i_1})$ induces an injective map, 

$\{y_{i_1}\} \rightarrow \{y_{i_2}\}$.

Repeating this process and together with a standard diagonal argument, we obtain a sequence of finite subsets of $(X \setminus S_X, d_X)$:

$\{x_{i_1}\}_{i_1=1}^c \subset \cdots \subset \{x_{i_s}\}_{i_s=1}^c \subset \cdots$,

and a sequence of finite subsets of $(M \setminus E, g_{\infty})$:

(4.5) $\{y_{i_1}\}_{i_1=1}^c \rightarrow \cdots \rightarrow \{y_{i_s}\}_{i_s=1}^c \rightarrow \cdots$.

Let $A = \bigcup_{s=1}^\infty \{x_{i_s}\}_{i_s=1}^c$, and $A_{k_l} = \bigcup_{s=1}^\infty \{x_{i_s}\}_{i_s=1}^c$. Since $A_{k_l}$ is dense in $(M \setminus E, g_{k_l})$ for all $k_l$, $A \subset (X \setminus S_X, d_X)$ is a dense subset. Let $Y$ denote the direct limit of (4.5). Then $Y \subset M \setminus E$. We now define a map, $f : A \rightarrow (M \setminus E, g_{\infty})$,

$\begin{aligned}
  f(x_{i_s}) &= [y_{i_s}] = \{y_{i_s} \rightarrow \cdots \rightarrow \cdots \}.
\end{aligned}$

It is clear that $f$ is injective since $f$ is injective on each $\{x_{i_s}\}_{i_s=1}^c$, and $f(A)$ is dense in $(M \setminus E, g_{\infty})$. From the construction of $f$, we see that $f$ is a local isometric embedding: for $x \in A$ we may assume that $x = x_{i_s}$. Since $x_{i_s} \notin S_X$ which is a compact subset of $X$, there is an $r > 0$ such that $B_{d_X}(x_{i_s}, r) \cap S_X = \emptyset$. Recall that we may assume $k_v$ large and $\phi_{k_v}(x_{i_s}) \subset K$ and $\phi_{k_v}(x_{i_s}) \rightarrow y_{i_s}$ point-wise with respect to $d_{g_{\infty}}$, where $K \subset M$ is compact such that $K \cap E = \emptyset$. Clearly, we may assume that $r$ small and a compact subset $K' \supseteq K$ such that $B_{g_{\infty}}([y_{i_s}], r) \subset K'$ and $K' \cap E = \emptyset$. By iii), $(K', g_{k_l}) \rightarrow (K', g_{\infty})$ in the $C^\infty$-sense. Observe the following two facts:

(4.5) For $z, z' \in B_{g_{\infty}}([y_{i_s}], \frac{r}{2})$, any $g_{\infty}$-minimal geodesic from $z$ to $z'$ is contained in $B_{g_{\infty}}([y_{i_s}], r)$.

(4.6) $d_{g_{\infty}}|_{B_{g_{\infty}}([y_{i_s}], \frac{r}{2})}$ (resp. $d_X|_{B(x, \frac{r}{2})}$) is determined by the lengths of curves in $B_{g_{\infty}}([y_{i_s}], r)$ (resp. $B_{d_X}(x_{i_s}, r)$). The two length structures coincide, because $(K', g_{k_l}) \rightarrow (K', g_{\infty})$ in the $C^\infty$ sense. As a consequence of (4.5) and (4.6), we conclude that

$f : \left(B_{d_X}(x_{i_s}, \frac{r}{2}), d_X|_{B_{d_X}(x_{i_s}, \frac{r}{2})}\right) \rightarrow \left(B_{g_{\infty}}([y_{i_s}], \frac{r}{2}), d_{g_{\infty}}|_{B_{g_{\infty}}([y_{i_s}], \frac{r}{2})}\right)$

is an isometry.

To uniquely extend $f : A \rightarrow (M \setminus E, g_{\infty})$ to a continuous surjection $f : (X, d_X) \rightarrow (M \setminus E, d_{g_{\infty}})$, one needs to show that $\{x_j\}, \{y_j\} \subset A$ such that $d_X(x_j, y_j) \rightarrow 0$ implies that $d_{g_{\infty}}(f(x_j), f(y_j)) \rightarrow 0$, which may require that $S_X \subset X$ has codimension at least 2. Because we do
not know whether \( \dim_H(S_X) \leq \dim_H(X) - 2 \), we will instead extend \( f^{-1} : f(A) \rightarrow A \) to a continuous map, \( f^{-1} : (M \setminus E, d_{g_{\infty}}) \rightarrow (X, d_X) \). So, we may assume that \( \{x_j\}, \{y_j\} \subset f(A) \) such that \( d_{g_{\infty}}(x_j, y_j) \rightarrow 0 \). Since \( d_{g_{\infty}} \) is a length metric, there is a path \( \gamma_i \subset M \setminus E \) from \( x_j \) to \( y_j \) such that \( \text{length}_{d_{g_{\infty}}}(\gamma_i) = d_{g_{\infty}}(x_j, y_j) + \delta_j \) and \( \delta_j \rightarrow 0 \). Since \( f^{-1} : (M \setminus E, d_{g_{\infty}}) \rightarrow (X \setminus S_X, d_X) \) is a local isometric embedding,
\[
\text{d}_X(f^{-1}(x_j), f^{-1}(y_j)) \leq \text{length}_{d_X}(f(\gamma_i)) = \text{length}_{d_{g_{\infty}}}(\gamma_i) = d_{g_{\infty}}(x_j, y_j) + \delta_j \rightarrow 0.
\]
q.e.d.

5. Proofs of Theorem 1.1, Theorem 1.2, and Corollary 1.5

Let \( M_0 \) be a Calabi–Yau \( n \)-variety with singular set \( S \) that admits a crepant resolution \((\tilde{M}, \tilde{\pi})\), and let \( \mathcal{L}_0 \) be an ample line bundle on \( M_0 \). Note that there is an embedding \( M_0 \rightarrow \mathbb{C}P^N \) such that \( \mathcal{L}_0^m = \mathcal{O}(1)|_{M_0} \) for an \( m \geq 1 \), and that the restriction of the Fubini-Study metric \( \omega_{FS}|_{M_0} \) represents \( m\mathcal{C}_1(\mathcal{L}_0) \in H^1(M_0, \mathcal{P}_H_{M_0}) \). By theorem 7.5 of [18], there is a unique Ricci-flat Kähler metric \( g \) on \( M_0 \) with Kähler form \( \omega \in \omega_{c1}(\mathcal{L}_0) \). Let \( \{g_s\} (s \in (0, 1]) \) be a family of Ricci-flat Kähler metrics with Kähler classes \( \lim_{s \rightarrow 0} [\omega_s] = \pi^*\omega_{c1}(\mathcal{L}_0) \) in \( H^{1,1}(M, \mathbb{R}) \), where \( \omega_s \) denotes the corresponding Kähler form of \( g_s \). Then
\[
\lim_{s \rightarrow 0} \text{Vol}_{g_s}(\tilde{M}) = \frac{1}{n!} c^n_1(\mathcal{L}_0) = \frac{1}{m^n n!} \int_{M_0} \omega_{FS}^n > 0.
\]
Furthermore, it is proved in [56] that
\[
g_s \rightarrow \pi^*g, \quad \text{and} \quad \omega_s \rightarrow \pi^*\omega, \quad \text{when} \; s \rightarrow 0,
\]
in the \( C^\infty \)-sense on any compact subset \( K \subset M \setminus \pi^{-1}(S) \). By [49] and [56], the diameter of \((\tilde{M}, g_s)\) has a uniform bound, i.e.,
\[
\text{diam}_{g_s}(\tilde{M}) \leq C
\]
where \( C \) is a constant independent of \( s \). By the Bishop-Gromov relative volume comparison and (5.1), \((\tilde{M}, g_s)\) is non-collapsed, i.e., there is a constant \( \kappa > 0 \) independent of \( s \) such that
\[
\text{Vol}_{g_s}(B_{g_s}(p, r)) \geq \kappa r^{2n},
\]
for any metric ball \( B_{g_s}(p, r) \subset (M, g_s) \). Gromov’s pre-compactness theorem (cf. [22]) implies that, for any sequence \( s_k \rightarrow 0 \), a subsequence of \( (\tilde{M}, g_{s_k}) \) converges to a compact length metric space \((X, d_X)\) in the Gromov–Hausdorff topology. First, we explore some metric properties of \((X, d_X)\).

**Lemma 5.1.** Let \((X, d_X)\) be as in the above. Then the following properties hold:
i) There is a closed subset $S_X \subset X$ of Hausdorff dimension $\dim_H S_X \leq 2n - 4$, and $(X \setminus S_X, d_X|_{X \setminus S_X})$ is a path metric space, i.e., for any $\delta > 0$ and any two points $x_1, x_2 \in X \setminus S_X$, there is a curve $\gamma_\delta \subset X \setminus S_X$ connecting $x_1$ and $x_2$ satisfying

$$\text{length}_{d_X}(\gamma_\delta) \leq d_X(x_1, x_2) + \delta.$$ 

ii) There is a homeomorphic local isometry $f : (X \setminus S_X, d_X) \to (M_0 \setminus S, d_0)$, i.e., for $x \in X \setminus S_X$, there is an open subset $U_x \subset \subset X \setminus S_X$ such that for any $x_1, x_2 \in U_x$, $d_X(x_1, x_2) = d_0(f(x_1), f(x_2))$.

iii) $(X, d_X)$ is isometric to the metric completion $\overline{M_0 \setminus S, d_0}$.

**Proof.** Applying general theorems in [10], [13], and [12] to our situation, i.e., $(\bar{M}, \bar{g}_{s_k}) \rightarrow (X, d_X)$, we see the following properties:

a) There is a closed subset $S' \subset X$ of Hausdorff dimension $\dim_H S' \leq 2n - 4$ such that for any $x \in S'$, there is a tangent cone $T_x X$ that is not isometric to $\mathbb{R}^{2n}$.

b) $X \setminus S'$ is a smooth open complex manifold, and $d_X|_{X \setminus S'}$ is induced by a Ricci-flat Kähler metric $g_\infty$ on $X \setminus S'$.

From section 3 of [11], we see that for any $x_1, x_2 \in X \setminus S'$, and any $\delta > 0$, there is a curve $\gamma_\delta$ connecting $x_1$ and $x_2$ in $X \setminus S'$ such that

$$\text{length}_{d_X}(\gamma_\delta) \leq \delta + d_X(x_1, x_2).$$

Note that $\pi^{-1}(S)$ is a finite disjoint union of complex subvarieties $E_i$, i.e., $\pi^{-1}(S) = \bigsqcup E_i$. If $S_X \subset X$ denotes the subset consisting of points $x \in X$ such that for each $k$ there is an $\bar{x}_k$ in the smooth part of $\pi^{-1}(S) \subset (\bar{M}, \bar{g}_{s_k})$ and $\bar{x}_k \to x$ under the Gromov–Hausdorff convergence of $\{(\bar{M}, \bar{g}_{s_k})\}$ to $(X, d_X)$, then by Theorem 4.1 there is a homeomorphic local isometry $f : (X \setminus S_X, d_X) \to (M_0 \setminus S, g)$. Thus, for any $x \in X \setminus S_X$, the tangent cone $T_x X$ is unique and isometric to $\mathbb{R}^{2n}$, which implies that $X \setminus S_X \subseteq X \setminus S'$, i.e., $S' \subseteq S_X$.

We claim that $S_X = S'$. If false, there is an $x \in S_X \setminus S'$ and there is a $\sigma > 0$ such that the metric ball $B_{g_\infty}(x, \sigma) \subset X \setminus S'$. By the volume convergence theorem due to Cheeger and Colding (cf. [9], [10]) and from $\bar{x}_k \to x$, we derive that for any $0 < \rho \leq \sigma$,

$$\lim_{k \to \infty} \text{Vol}_{\bar{g}_{s_k}}(B_{\bar{g}_{s_k}}(\bar{x}_k, \rho)) = \text{Vol}_{g_\infty}(B_{g_\infty}(x, \rho)).$$

Since $g_\infty$ is a smooth metric, $\lim_{\rho \to 0} \frac{\text{Vol}_{g_\infty}(B_{g_\infty}(x, \rho))}{\omega_{2n}\rho^{2n}} - 1 = 0$, where $\omega_{2n}$ denotes the volume of the metric 1-ball in the Euclidean space $\mathbb{R}^{2n}$. Thus for any $\varepsilon > 0$ we can find a $\rho \ll 1$ and a $k(\rho) \gg 1$ such that for any $k \geq k(\rho)$ we have

$$\left| \frac{\text{Vol}_{\bar{g}_{s_k}}(B_{\bar{g}_{s_k}}(\bar{x}_k, \rho))}{\omega_{2n}\rho^{2n}} - 1 \right| \leq \varepsilon.$$
By the proof of theorem 3.2 in [3], we see that there is a uniform lower bound $0 < \rho_h < \rho$ (independent of $s_k$) for the harmonic radius of $\bar{g}_{s_k}$ at $\bar{x}_k$, i.e., there are harmonic coordinates $h^1, \ldots, h^{2n}$ on $B_{\bar{g}_{s_k}}(\bar{x}_k, \rho_h)$ such that $\bar{g}_{s_k} = \sum_{ij} \bar{g}_{s_k, ij} dh^i dh^j$,

$$2^{-1}(\delta_{ij}) \leq (\bar{g}_{s_k, ij}) \leq 2(\delta_{ij}), \quad \rho^{1+\alpha}\|\bar{g}_{s_k, ij}\|_{C^{1,\alpha}} \leq 2,$$

where $\alpha \in (0, 1)$. Furthermore, by Ricci flatness there are constants $C_l > 0$ independent of $k$ such that

$$\|\bar{g}_{s_k, ij}\|_{C^{1,\alpha}} \leq C_l$$

on $B_{\bar{g}_{s_k}}(\bar{x}_k, \rho_h/2)$ (cf. section 4 in [1]). Hence the sectional curvature $\text{Sec}_{\bar{g}_{s_k}}$ of $\bar{g}_{s_k}$ on $B_{\bar{g}_{s_k}}(\bar{x}_k, \rho_h/2)$ and the injectivity radius $\bar{h}_{s_k}(\bar{x}_k)$ have uniform bounds,

$$\sup_{B_{\bar{g}_{s_k}}(\bar{x}_k, \rho_h/2)} |\text{Sec}_{\bar{g}_{s_k}}| \leq \Lambda, \quad \bar{h}_{s_k}(\bar{x}_k) > \iota,$$

where $\Lambda$ and $\iota$ are two constants independent of $k$.

In the rest of the proof of Lemma 5.1, we need the following theorem.

**Theorem 5.2.** Let $(M, g, \omega)$ be a complete Kähler $n$-manifold, and $p \in M$. Assume that the sectional curvature $\text{Sec}_g$ satisfies

$$\sup_{B_g(p, \frac{\rho_h}{\sqrt{n}})} \text{Sec}_g \leq \Lambda, \quad \Lambda > 0,$$

and there is a complex subvariety $E$ of dimension $m \leq n$ such that $p$ belongs to the regular part of $E$. Then

$$\text{Vol}_g(B_g(p, r) \cap E) \geq \omega r^{2m},$$

for any $r \leq \min\{i_g(p), \frac{\pi}{2\sqrt{\Lambda}}\}$, where $i_g(p)$ denotes the injectivity radius of $g$ at $p$, and $\omega = \omega(m, \Lambda)$ is a constant depending only on $m$ and $\Lambda$.

Note that similar volume comparison results were obtained for smooth minimal submanifolds in [42] and [23], for complex subvarieties of $\mathbb{C}^n$ in [25], and for minimal currents in $\mathbb{R}^n$ (cf. [43]). Since the authors could not find a proof of Theorem 5.2 in the literature, we shall present a proof at the end of this section.

By Theorem 5.2 and taking $r = \min\{\iota, \frac{\pi}{2\sqrt{\Lambda}}, \frac{\rho_h}{\sqrt{n}}\}$, we obtain

$$\text{Vol}_{\bar{g}_{s_k}}(\bar{\pi}^{-1}(S)) \geq \text{Vol}_{\bar{g}_{s_k}}(\bar{\pi}^{-1}(S) \cap B_{\bar{g}_{s_k}}(\bar{x}_k, r)) > C,$$

where $C > 0$ is a constant independent of $k$. On the other hand, since $\lim_{s \to 0} \bar{\omega}_s = \bar{\pi}^* c_1(L)|_{M_0}$ in $H^{1,1}(M, \mathbb{R})$, we have

$$\lim_{k \to \infty} \text{Vol}_{\bar{g}_{s_k}}(\bar{\pi}^{-1}(S)) = \sum_i \lim_{k \to \infty} \frac{1}{\dim E_i} \int_{E_i} \bar{\omega}_s^{\dim E_i} E_i = 0,$$

which is a contradiction.
Note that by i) \((X \setminus S_X, d_X)\) coincides with the length metric structure \((X \setminus S_X, d_{d_X})\), i.e., for any two points \(x_1, x_2 \in X \setminus S_X\),
\[
d_X(x_1, x_2) = \inf \{ \text{length}_{d_X}(\gamma) \mid \text{all curves } \gamma \subset X \setminus S_X \text{ connecting } x_1 \text{ and } x_2 \}.
\]
Consequently, \(f : (X \setminus S_X, d_X) \to (M_0 \setminus S, d_g)\) is an isometry, and \((X, d_X)\) is the unique metric completion of \((X \setminus S_X, d_X)\) since \(X \setminus S_X\) is dense in \((X, d_X)\). We obtain iii). q.e.d.

**Lemma 5.3.** Let \((X, d_X)\) be as in Lemma 5.1, and let \((M_k, g_k, \omega_k)\) be any family of Ricci-flat Kähler \(n\)-dimensional manifolds satisfying
\[
i) \quad \lim_{k \to \infty} \text{Vol}_{g_k}(M_k) = \frac{1}{n!} c_1^n(\mathcal{L}_0),
\]
\[
ii) \quad \text{There is a family of embeddings } F_k : M_0 \setminus S \to M_k \text{ such that } F_k^* g_k \to g, \quad \text{and } F_k^* \omega_k \to \omega, \quad \text{when } k \to \infty,
\]

in the \(C^\infty\)-sense on any compact subset \(K \subset \subset M_0 \setminus S\). Then
\[
(M_k, g_k) \overset{d_{GH}}{\to} (X, d_X) \overset{d_{GH}}{\to} (\bar{M}, \bar{g}_{\theta_k}).
\]

**Proof.** By Lemma 5.1, there is a homeomorphic local isometry \(f : (X \setminus S_X, d_X) \to (M_0 \setminus S, g)\). For an \(x \in X \setminus S_X\), ii) of Lemma 5.3 implies that \(\text{Vol}_{g_k}(B_{g_k}(F_k(f(x)), 1)) \geq v\) for a constant \(v > 0\) independent of \(k\). For any \(1 < R < \text{diam}_{g_k}(M_k)\), Lemma 2.3 (lemma 2.3 in [44]) shows that
\[
R \leq 1 + 2n \frac{\text{Vol}_{g_k}(B_{g_k}(F_k(f(x)), 2R + 2))}{\text{Vol}_{g_k}(B_{g_k}(F_k(f(x)), 1))}.
\]
By taking \(R = \frac{1}{2} \text{diam}_{g_k}(M_k)\), we obtain that
\[
\text{diam}_{g_k}(M_k) < 2 + 4nv^{-1} \frac{1}{n!} c_1^n(\mathcal{L}_0).
\]
By the Bishop-Gromov relative volume comparison, \((M_k, g_k)\) is non-collapsed, i.e., there is a constant \(\kappa > 0\) independent of \(k\) such that for any metric ball \(B_{g_k}(p, r) \subset M_k\),
\[
(5.4) \quad \text{Vol}_{g_k}(B_{g_k}(p, r)) \geq \kappa r^{2n}.
\]
Gromov’s pre-compactness theorem implies that a subsequence of \(\{(M_k, g_k)\}\) \(d_{GH}\)-converges to a compact length metric space \((Y, d_Y)\).
Following the proof of theorem 4.1 in [49] with a minor modification will prove that \((Y, d_Y)\) is isometric to \((X, d_X)\). Because of this, we only present a sketch of the proof. (There is an analog proof for the case of Ricci-solitons in [67].)

First, the same arguments as in the proof of lemma 4.1 in [49] imply that there exists an embedding \(\psi' : (M_0 \setminus S, g) \to (Y, d_Y)\) which is a
local isometry. Hence \( \psi = \psi' \circ f : (X \setminus S_X, d_X) \to (Y, d_Y) \) is a local isometric embedding. Thus, if \( \gamma \) is a geodesic in \( (X \setminus S_X, d_X) \), then \( \psi(\gamma) \) is a geodesic in \( (\psi(X \setminus S_X), d_Y) \). For any \( x_1, x_2 \in X \), and two sequences \( \{x_{1,j}\}, \{x_{2,j}\} \subset X \setminus S_X \) converging to \( x_1 \) and \( x_2 \) respectively, there are curves \( \gamma_j \) connecting \( x_{1,j} \) and \( x_{2,j} \) in \( X \setminus S_X \) with length_{d_Y}(\gamma_j) \leq d_X(x_{1,j}, x_{2,j}) + \frac{1}{j} \) by Lemma 5.1, which implies

\[
\text{(5.5) } d_Y(\psi(x_{1,j}), \psi(x_{2,j})) \leq \text{length}_{d_Y}(\psi(\gamma_j)) = \text{length}_{g}(\gamma_j) \leq d_X(x_{1,j}, x_{2,j}) + \frac{1}{j}.
\]

If \( x_1 = x_2 = x \), both \( \{\psi(x_j)\} \) and \( \{\psi(x'_j)\} \) are Cauchy sequences, and converge to the same limit \( y \) in \( Y \). By defining \( \tilde{\psi}(x) = y \), \( \psi \) extends to a continuous map \( \tilde{\psi} : X \to Y \) such that \( \tilde{\psi}(X) \subseteq Y \) is closed.

If \( \tilde{\psi}(X) \not\subseteq Y \), then there is a metric ball \( B_{d_Y}(y, \delta) \subset Y \setminus \tilde{\psi}(X) \) for a \( \delta > 0 \). By (5.1), (5.4), and the volume convergence theorem due to Cheeger and Colding (cf. [9], [10]), we derive

\[
\mathcal{H}^{2n}(Y) = \mathcal{H}^{2n}(X) = \text{Vol}_g(X \setminus S_X) \quad \text{and} \quad \mathcal{H}^{2n}(B_{d_Y}(y, \delta)) \geq \kappa \delta^{2n},
\]

where \( \mathcal{H}^{2n} \) denotes the 2n-dimensional Hausdorff measure. Thus

\[
\mathcal{H}^{2n}(Y) \geq \mathcal{H}^{2n}(\psi(X \setminus S_X)) + \mathcal{H}^{2n}(B_{d_Y}(y, \delta)) \geq \text{Vol}_g(X \setminus S_X) + \kappa \delta^{2n} > \mathcal{H}^{2n}(Y),
\]
a contradiction.

To show that \( \tilde{\psi} \) is an isometry, we first check that \( \tilde{\psi} \) is 1-Lipschitz. For any \( x_1 \neq x_2 \in X \), there are sequences of points \( \{x_{i,j}\} \subset X \setminus S_X \), \( i = 1, 2 \), such that \( d_X(x_{i,j}, x_i) \to 0 \) when \( j \to \infty \). Thus \( d_Y(\psi(x_{1,j}), \tilde{\psi}(x_j)) \to 0 \), \( i = 1, 2 \), when \( j \to \infty \). By (5.5) and letting \( j \to \infty \), we obtain that

\[
\text{(5.6) } d_Y(\tilde{\psi}(x_1), \tilde{\psi}(x_2)) \leq d_X(x_1, x_2)
\]
i.e., \( \tilde{\psi} \) is a 1-Lipschitz map.

Because \( \tilde{\psi}(S_X) \cup \psi(X \setminus S_X) = Y \), \( \tilde{\psi}(S_X) \supseteq Y \setminus \psi(X \setminus S_X) \). Since \( \dim_{\mathcal{H}} S_X \leq 2n - 2 \),

\[
\mathcal{H}^{2n-1}(Y \setminus \psi(X \setminus S_X)) \leq \mathcal{H}^{2n-1}(\tilde{\psi}(S_X)) \leq \mathcal{H}^{2n-1}(S_X) = 0.
\]

If there is a \( j \) such that \( g = d_X(x_{1,j}, x_{2,j}) - d_Y(\psi(x_{1,j}), \psi(x_{2,j})) > 0 \), then by section 3 of [11] we see that there is a curve \( \tilde{\gamma} \) connecting \( \psi(x_{1,j}), \psi(x_{2,j}) \) in \( \psi(X \setminus S_X) \) and

\[
d_X(x_{1,j}, x_{2,j}) \leq \text{length}_{d_X}(\psi^{-1}(\tilde{\gamma})) = \text{length}_{d_Y}(\tilde{\gamma}) \leq d_Y(\psi(x_{1,j}), \psi(x_{2,j})) + \frac{1}{2} g,
\]
a contradiction. Then \( d_X(x_{1,j}, x_{2,j}) = d_Y(\psi(x_{1,j}), \psi(x_{2,j})) \) and thus by letting \( j \to \infty \), we obtain that

\[
d_Y(\tilde{\psi}(x_1), \tilde{\psi}(x_2)) = d_X(x_1, x_2).
\]
By now, we have proved that \( \tilde{\psi} : (X, d_X) \rightarrow (Y, d_Y) \) is an isometry.

q.e.d.

After the above preparation, we are ready to prove Theorem 1.1, Theorem 1.2 and Corollary 1.5.

**Proof of Theorem 1.1.** Let \( M_0, \pi : \mathcal{M} \rightarrow \Delta, \mathcal{K}_{\mathcal{M}/\Delta}, \mathcal{L}, M_t, \tilde{g}_t, \tilde{\omega}_t, (\bar{M}, \bar{\pi}), \bar{g}_s, \bar{\omega}_s \) be as in Theorem 1.1.

Note that
\[
\lim_{s \to 0} \text{Vol}_{\bar{g}_s}(\bar{M}) = \frac{1}{n!} c^n_1(\mathcal{L})|_{M_0} \equiv \frac{1}{n!} c^n_1(\mathcal{L})|_{M_t} = \text{Vol}_{\tilde{g}_t}(M_t).
\]

By [56],
\[
\bar{g}_s \rightarrow \bar{\pi}^*g, \quad \text{and} \quad \bar{\omega}_s \rightarrow \bar{\pi}^*\omega, \quad \text{when} \quad s \to 0,
\]
in the \( C^\infty \)-sense on any compact subset \( K \subset \subset \bar{M} \setminus \bar{\pi}^{-1}(S) \). By (5.2) and Ricci flatness, we apply Gromov’s compactness theorem to conclude that for any sequence \( s_k \to 0 \), a subsequence of \((\bar{M}, \bar{g}_{s_k})\) \( d_{GH} \)-converges to a compact path metric space \((X, d_X)\), which satisfies the conclusion of Lemma 5.1. By Lemma 5.3, for any other sequence \( s'_k \to 0 \), \((\bar{M}, \bar{g}_{s'_k})\) \( d_{GH} \)-converges to \((X, d_X)\) too. Thus
\[
\lim_{s \to 0} d_{GH}((\bar{M}, \bar{g}_s), (X, d_X)) = 0.
\]

By Theorem 1.4,
\[
F_t^*\tilde{g}_t \rightarrow g, \quad \text{and} \quad F_t^*\tilde{\omega}_t \rightarrow \omega, \quad \text{when} \quad t \to 0,
\]
in the \( C^\infty \)-sense on any compact subset \( K \subset \subset M_0 \setminus S \), where \( F_t : M_0 \setminus S \rightarrow M_t \) is a family of embeddings. By Lemma 5.3 and the fact that the limit is independent of convergent subsequences, we obtain the conclusion,
\[
\lim_{t \to 0} d_{GH}((M_t, \tilde{g}_t), (X, d_X)) = 0.
\]

q.e.d.

The same argument in the proof of Theorem 1.1 also gives a proof of Theorem 1.2.

**Proof of Corollary 1.5.** Let \( M \) be a Calabi–Yau manifold, and let \( \omega_s \) (\( s \in [0, 1] \)) be a family of Ricci-flat Kähler forms. It is clear that \( \omega_s \) converges to \( \omega_0 \) when \( s \to 0 \) in the \( C^\infty \)-sense, which implies that \( \mathcal{M}_M \) is path connected in \((X, d_{GH})\). Let \( \pi' : \mathcal{M} \rightarrow \Delta \) be a smooth family of Calabi–Yau manifolds over the unit disc \( \Delta \subset \mathbb{C} \) with an ample line bundle \( \mathcal{L} \) on \( \mathcal{M} \), and let \( \tilde{\omega}_t \) be the unique Ricci-flat Kähler form on \( \pi'^{-1}(t) = M_t \) with \( \tilde{\omega}_t \in c_1(\mathcal{L})|_{M_t} \). It is standard that \( F_t^*\tilde{\omega}_t \) converges to \( \tilde{\omega}_0 \) in the \( C^\infty \)-sense, when \( t \to 0 \), where \( F_t : M_0 \rightarrow M_t \) is a smooth
family of diffeomorphisms. Thus, if \( \pi' : \bar{\mathcal{M}} \to \mathcal{D} \) is a smooth family of Calabi–Yau manifolds over connected complex manifold \( \mathcal{D} \), then

\[
\bigcup_{M_t = \pi'^{-1}(t), t \in \mathcal{D}} \mathcal{M}_t \subset (\mathfrak{X}, d_{GH})
\]

is path connected.

Note that Calabi–Yau manifolds are minimal models. If \( M \) and \( M' \) are two birationally equivalent three-dimensional Calabi–Yau manifolds, then \( M \) and \( M' \) are related by a sequence of flops (cf. [38] [39] [35]), i.e., there is a sequence of varieties \( M_1, \ldots, M_k \) such that \( M = M_1, M' = M_k, \) and \( M_{j+1} \) is obtained by a flop from \( M_j \). Consequently, there are normal projective varieties \( M_{0,1}, \ldots, M_{0,k-1} \), and small resolutions \( \bar{\pi}_j : M_j \to M_{0,j} \) and \( \bar{\pi}_j^+ : M_j \to M_{0,j-1} \). By [38], \( M_j \) has the same singularities as \( M \), and thus \( M_j \) is smooth. Since the exceptional loci of \( \bar{\pi}_j \) and \( \bar{\pi}_j^+ \) are of co-dimension at least 2, \( M_{0,j} \) has only canonical singularities, and the canonical bundle of \( M_{0,j} \) is trivial (cf. corollary 1.5 in [34]). Therefore \( M_{0,j} \) is a three-dimensional Calabi–Yau variety, and \( M_j \) is a three-dimensional Calabi–Yau manifold. By Theorem 1.2, for any \( j > 0 \),

\[
\overline{\mathcal{M}}_{M_j} \bigcup \overline{\mathcal{M}}_{M_{j+1}}
\]

is path connected, where \( \overline{\mathcal{M}}_{M_j} \) denotes the closure of \( \mathcal{M}_j \subset (\mathfrak{X}, d_{GH}) \).

By now we have proved i) of Corollary 1.5.

Let \( M_0 \) be a three-dimensional complete intersection Calabi–Yau conifold in \( \mathbb{C}P^{m_1} \times \cdots \times \mathbb{C}P^{m_l} \), and \( \bar{M} \) be a small resolution of \( M_0 \), which is a three-dimensional complete intersection Calabi–Yau (CICY) manifold in products of projective spaces. By Theorem 1.1, we see that

\[
\bigcup_{\bar{M} \in \mathcal{D}(M_0)} \overline{\mathcal{M}}_{\bar{M}} \subset (\mathfrak{X}, d_{GH})
\]

is path connected, where \( \mathcal{D}(M_0) \) denotes the set of three-dimensional CICY manifolds in \( \mathbb{C}P^{m_1} \times \cdots \times \mathbb{C}P^{m_l} \) obtained by a smoothing of \( M_0 \). If \( M \) and \( M' \) are two three-dimensional CICY manifolds in products of projective spaces, then by [24] \( M \) and \( M' \) are related by a sequence of conifold transitions, or inverse conifold transitions. Precisely, there is a sequence of three-dimensional CICY manifolds \( M_1, \ldots, M_k \) with \( M = M_1 \) and \( M' = M_k \) such that for any \( 1 \leq j \leq k \), there is a three-dimensional CICY conifold \( M_{0,j} \) in some \( \mathbb{C}P^{m_1} \times \cdots \times \mathbb{C}P^{m_l} \), \( M_j \) is a small resolution of \( M_{0,j} \) and \( M_{j+1} \) is obtained by a conifold transition, or vice versa. Thus ii) of Corollary 1.5 is followed, i.e., \( \mathcal{C} \mathfrak{X}_0 \) is path connected.

In [5] and [16], many complete intersection Calabi–Yau 3-manifolds in toric varieties were verified to be connected by extremal transitions, which include Calabi–Yau hypersurfaces in all toric 4-manifolds obtained by resolving weighted projective 4-spaces. Let \( \mathcal{C} \mathfrak{X}_0 \) be the set
of the above Calabi–Yau 3-manifolds with Ricci-flat Kähler metrics of volume 1. By Theorem 1.1, the closure $\tilde{\Sigma}_0$ is path connected. Note that a quintic in $\mathbb{CP}^4$ with Ricci-flat Kähler metric of volume 1 is in $\mathcal{C}_1 \cap \mathcal{C}_S$. Thus iii) of Corollary 1.5 is obtained.

Now we give a proof of Theorem 5.2, which can be viewed as a combination of the proof of theorem 2.0.1 in [23] and the proof of theorem 9.3 in [43].

**Proof of Theorem 5.2.** For any $r \leq \min\{i_2(p), \frac{r}{2\sqrt{\Lambda}}\}$, there are normal coordinates such that $g = d\rho^2 + g_\rho$ on $B_g(p, r)$ where $g_\rho$ is a Riemannian metric on $\partial B_g(p, r)$, $0 < \rho \leq r$. Let $f(q, \rho) = \exp_p \frac{e}{\rho} \exp_p^{-1} q$ for any $q \in \partial B_g(p, r)$. Then $f : \partial B_g(p, r) \times (0, r] \to B_g(p, r) \nabla \{p\}$ is a diffeomorphism. For any $w \in T_q(\partial B_g(p, r))$, it is clear that $J(\rho) = df|_{\rho, q} w$ is a normal Jacobi field along the geodesic $\gamma(\rho) = f(q, \rho)$ with $J(0) = 0$ and $J(r) = w$. A standard Rauch comparison argument shows that

$$|J(\rho)|_{g_\rho} \leq \frac{\sin \sqrt{\Lambda} \rho}{\sin \sqrt{\Lambda} r} |J(r)|_{g_\rho} = \frac{\sin \sqrt{\Lambda} \rho}{\sin \sqrt{\Lambda} r} |w|_{g_\rho},$$

(cf. lemma 2.0.1 in [23]). Thus the norm of the differential $df|_{\partial B_g(p, r) \times \{\rho\}}$ corresponding to the metric $g_\rho$ on $\partial B_g(p, r) \times \{\rho\}$ and $g_\rho$ on $\partial B_g(p, r)$ satisfies

$$|df|_{\partial B_g(p, r) \times \{\rho\}} \leq \frac{\sin \sqrt{\Lambda} \rho}{\sin \sqrt{\Lambda} r},$$

which implies

$$(5.7) \quad g_\rho \leq \frac{\sin^2 \sqrt{\Lambda} \rho}{\sin^2 \sqrt{\Lambda} r} g_r.$$

Denote

$$\Theta(r) = \operatorname{Vol}_{g_\rho}(B_g(p, r) \cap E).$$

Since $\Theta(r)$ is monotonically increasing, $\Theta'(r)$ exists for almost all $r$. By 4.11(3) in [43], we have

$$H_{g_r}^{2m-1}(\partial B_g(p, r) \cap E) \leq \Theta'(r).$$

Let $C = (\partial B_g(p, r) \cap E) \times (0, r] \subset \partial B_g(p, r) \times (0, r] = B_g(p, r) \nabla \{p\}$. By Fubini’s Theorem and (5.7), we see that

$$H_{g_r}^{2m}(C) = \int_0^r H_{g_r}^{2m-1}(\partial B_g(p, r) \cap E) d\rho \leq H_{g_r}^{2m-1}(\partial B_g(p, r) \cap E) \int_0^r \frac{\sin^{2m-1} \sqrt{\Lambda} \rho}{\sin^{2m-1} \sqrt{\Lambda} r} d\rho.$$

Since $E$ is a complex subvariety, $E$ is a volume minimizer and thus

$$\Theta(r) \leq H_{g_r}^{2m}(C) \leq \int_0^r \frac{\sin^{2m-1} \sqrt{\Lambda} \rho d\rho}{\sin^{2m-1} \sqrt{\Lambda} r} \Theta'(r).$$
Therefore
\[ \frac{d}{dr} \left( \Theta(r) \int_0^r \sin^{2m-1} \sqrt{\Lambda} \rho d\rho \right) \geq 0. \]

Since \( p \) is a smooth point of \( E \),
\[ \lim_{r \to 0} \frac{\Theta(r)}{\int_0^r \sin^{2m-1} \sqrt{\Lambda} \rho d\rho} = C, \]
where \( C \) is a constant depending only on \( \Lambda \) and \( m \). Thus
\[ \Theta(r) \geq C \int_0^r \sin^{2m-1} \sqrt{\Lambda} \rho d\rho \geq \pi r^{2m}. \]

q.e.d.

Appendix A. Gromov–Hausdorff Convergence of Compact Metric Spaces

In the proof of Theorem 4.1, we freely used some basic properties of the Gromov–Hausdorff convergence of compact metric spaces. For the convenience of readers, we will briefly recall related notions and proofs of these properties (cf. [47]).

Let \( (Z, d) \) be a metric space, and let \( C_Z \) denote the set of all compact subsets of \( Z \). For \( A, B \in C_Z \), the Hausdorff distance of \( A \) and \( B \) is
\[ d_H(A, B) = \inf \{ \epsilon, U_\epsilon(A) \supseteq B, U_\epsilon(B) \supseteq A \}, \]
where \( U_\epsilon(S) \) denotes the \( \epsilon \)-neighborhood of \( S \). Then \( (C_Z, d_H) \) is a complete metric space. The Gromov–Hausdorff distance can be viewed as an abstract extension of \( d_H \) on \( X \): the space of isometry classes of all compact metric spaces. For \( X, Y \in X \), the Gromov–Hausdorff distance of \( X \) and \( Y \) is
\[ d_{GH}(X, Y) = \inf_Z \{ d_Z^2(X, Y), \exists \text{ isometric embeddings}, X, Y \hookrightarrow Z, \text{ a metric space} \}. \]

In the above definition, one can consider the disjoint union that \( Z = X \coprod Y \) with an admissible metric \( d \), i.e., a metric on \( Z \) such that the restriction on \( X \) (resp. \( Y \)) is the metric on \( X \) (resp. \( Y \)).

It is not hard to check that \( d_{GH}(X, Y) = 0 \) if and only if \( X \) is isometric to \( Y \) and \( d_{GH} \) satisfies the triangle inequality. Hence, \( (X, d_{GH}) \) is a metric space.

In the proof of Theorem 4.1, the following proposition is used.

Proposition A.1. Given \( \{X_i\} \) in \( \mathcal{X} \) such that \( d_{GH}(X_i, X_{i+k}) < 2^{-i} \) for all \( i \) and \( k \), let \( Y = \coprod X_i \).

i) There is a metric \( d_Y \) on \( Y \) such that the restriction of \( d_Y \) on each \( X_i \) is the metric on \( X_i \) and \( \{X_i\} \) is a Cauchy sequence with respect to \( d_{Y,H} \).
ii) Let $X$ be the collection of equivalent Cauchy sequences, $\{\{x_i\}, x_i \in X_i\}$, equipped with the metric $\hat{d}(\{x_i\}, \{y_i\}) = \lim_{i \to \infty} d_Y(x_i, y_i)$. Then $Y \coprod X$ has an admissible metric defined by $d(x, \{x_i\}) = \lim_{i \to \infty} d(x, x_i)$.  

iii) For all $\epsilon > 0$, $X$ has a finite $\epsilon$-dense subset (thus the completion of $X$ is compact).  

iv) $d_H(X_i, X) \to 0$ as $i \to \infty$.

Proof. i) We first take, for each $i$, an admissible metric $d_{i,i+1}$ on $X_i \coprod X_{i+1}$ such that $d_{i,i+1}(X_i, X_{i+1}) < d_{GH}(X_i, X_{i+1}) + 2^{-i} < 2^{-i+1}$.

We then extend $\{d_{i,i+1}\}$ to an admissible metric on $Y$ by defining, for each pair $(i,j)$, an admissible metric $d_{i,i+j}$ on $X_i \coprod X_{i+j}$, as follows:

$$d_Y(x_i, x_{i+j}) = \inf_{x_{i+k} \in X_{i+k}} \left\{ \sum_{k=0}^{j-1} d_{i+k,i+k+1}(x_{i+k}, x_{i+k+1}) \right\}.$$  

It is straightforward to check that $d_Y$ satisfies the triangle inequality. Then $\{X_i\}$ is a Cauchy sequence with respect to $d_{Y,H}$, because for all $j$,

$$d_{Y,H}(X_i, X_{i+j}) \leq d_{Y,H}(X_i, X_{i+1}) + \cdots + d_{Y,H}(X_{i+j-1}, X_{i+j})$$

$$\leq 2^{-i+1} + 2^{-i} + \cdots + 2^{-i+j+2}$$

$$\leq 2^{-i+2}.$$

Note that $(Y, d_Y)$ may not be complete, and if not, the unique limit point is the desired limit space $X$.

ii) Consider a subset of Cauchy sequences in $Y$,  

$$\hat{X} = \{\{x_i\} : x_i \in X_i \text{ is a Cauchy sequence in } Y\},$$

and define a pseudo-metric on $\hat{X}$,

$$\hat{d}(\{x_i\}, \{y_i\}) = \lim_{i \to \infty} d_Y(x_i, y_i),$$

where the existence of the limit is from

$$|d_Y(x_i, y_i) - d_Y(x_j, y_j)| \leq d_Y(x_i, x_j) + d_Y(y_i, y_j) \to 0 \quad \text{as } i, j \to \infty.$$

Then $\hat{d}$ yields a metric on the quotient space $X = \hat{X}/\sim$, where

$$\{x_i\} \sim \{y_i\} \quad \text{iff} \quad \hat{d}(\{x_i\}, \{y_i\}) = 0.$$

We now define an admissible metric on $X \coprod Y$ by declaring

$$d(\{x_i\}, y) = \lim_{i \to \infty} d_Y(x_i, y).$$

(Because $|d_Y(x_i, y) - d_Y(x_j, y)| \leq d_Y(x_i, x_j)$, $d_Y(x_i, y)$ is a Cauchy sequence.) Since $d(\{x_i\}, y) \geq d_Y(x_k, y)$ for some $x_k \in \{x_i\}$, $d$ is indeed a metric (because $d_Y(x_{k+j}, y) > d_Y(x_k, y) > 0$).

iii) Given $\epsilon > 0$, we will construct a finite $\epsilon$-dense subset of $X$ as follows: choose $i$ so that $2^{-i} < \frac{\epsilon}{3}$. Because $X_i$ is compact, we may assume a finite $\frac{\epsilon}{3}$-net, $\{x_i^1, \ldots, x_i^\ell\}$, of $X_i$. Let $x_{i+1}^1, \ldots, x_{i+1}^\ell \in X_{i+1}$.
$X_{i+1}$ such that $d(x_i^j, x_{i+1}^j) < 2^{-i}$. Let $x_{i+2}^1, \ldots, x_{i+2}^\ell \in X_{i+2}$ such that $d(x_{i+1}^j, x_{i+2}^j) < 2^{-i-1}$. Repeating this, we obtain, for each $k$, $x_{i+k}^1, \ldots, x_{i+k}^\ell \in X_{i+k}$ such that $d(x_{i+k-1}^j, x_{i+k}^j) < 2^{-i+k+1}$. For each $1 \leq j \leq \ell$, it is clear that $\{x_{i+k}^j\}_{k=1}^\infty$ is a Cauchy sequence, that is, $\{x_{i+k}^j\}_{k=1}^\infty \in X$. Moreover, for each $1 \leq k < \infty$, $x_{i+k}^1, \ldots, x_{i+k}^\ell$ is $\frac{3\epsilon}{5}$-dense in $X_{i+k}$. This is because for any $x \in X_{i+k}$, we can choose $x' \in X_i$ such that $d(x, x') < 2^{-i}$, and let $x_i^j \in \{x_i^j\}$ such that $d(x', x_i^j) < \frac{\epsilon}{5}$. Then $d(x, x_{i+k}^j) \leq d(x, x') + d(x', x_i^j) + d(x_i^j, x_{i+k}^j) < \frac{\epsilon}{5} + 2 \cdot 2^{-i} < \frac{3\epsilon}{5}$. Finally, we check that $\{\{x_{i+k}^1\}_{k=1}^\infty, \ldots, \{x_{i+k}^\ell\}_{k=1}^\infty\}$ is an $\epsilon$-dense subset in $X$. Given any $y_k \in X$, we may assume that for a large $k$, $d(y_k, y_{k+j}) < \frac{\epsilon}{5}$. Since $\{x_k^1, \ldots, x_k^\ell\}$ is a $\frac{3\epsilon}{5}$-net for $X_k$, we may assume that $d(y_k, x_k^j) < \frac{3\epsilon}{5}$. Then

$$d(y_{k+j}, x_{k+j}^j) \leq d(y_k, y_{k+j}) + d(y_k, x_k^j) + d(x_k^j, x_{k+j}^j) < \frac{\epsilon}{5} + \frac{3\epsilon}{5} + \frac{\epsilon}{5} = \epsilon,$$

and thus $d(y_k, x_k^j) < \epsilon$.

iv) We shall show that for any $\epsilon > 0$, $B_\epsilon(X) \supseteq X_i$ and $B_\epsilon(X_i) \supseteq X$ for all large $i$.

For any $\epsilon > 0$, let $2^{-i+1} < \epsilon$. For $x_i \in X_i$, from the condition that $d_{i,i+1,H}(X_i, X_{i+1}) < 2^{-i+1}$, we define a sequence $y_k \in X_k$ such that $d(y_k, y_{k+j}) < 2^{-k+1}$ and $y_i = x_i$ (we can choose $y_1, \ldots, y_{i-1}$ arbitrarily). Clearly, $\{y_k\}$ is a Cauchy sequence and $d(x_i, \{y_k\}) < 2^{-i+1} < \epsilon$. This shows that $X_i \subseteq B_\epsilon(X)$ for $i \geq \frac{\ln \epsilon}{\ln 2} + 1$.

For any $\{x_i\} \in X$, for $i$ large, we can assume that $d(x_i, \{x_j\}) < \epsilon$. Note that this does not give $B_\epsilon(X_i) \supseteq X$, because how large $i$ is may depend on $\{x_i\}$ in $X$. To overcome this trouble, by iii), we can assume a finite $\frac{\epsilon}{5}$-dense subset, $\{y_1\}_{i=1}^{\infty}, \ldots, \{y_\ell\}_{i=1}^{\infty}$, for $X$. For each $1 \leq j \leq \ell$, we may assume some $N_j$ such that for $i \geq N_j$, $d(y_i^j, \{y_i^j\}) < \frac{\epsilon}{5}$ and $2^{-i+1} < \frac{\epsilon}{5}$. Let $N = \max\{N_1, \ldots, N_\ell\}$. For any $\{x_i\} \in X$, we may assume some $1 \leq j \leq \ell$ such that $d(\{x_i\}, \{y_i^j\}) < \frac{\epsilon}{5}$. From the above, for each $i \geq N$,

$$d(y_i^j, \{x_i\}) \leq d(y_i^j, \{y_i^j\}) + d(\{y_i^j\}, \{x_i\}) < \epsilon,$$

and thus $X \subseteq B_\epsilon(X_i)$. q.e.d.

A direct consequence of Proposition A.1 is that $(X, d_{GH})$ is a complete metric space.

A by-product of the above proof is that an abstract convergent sequence, $X, d_{GH}^X$, can be realized as a concrete Hausdorff convergence, $d_H(X_i, X) \to 0$, in $\prod X_i \prod X$ with an admissible metric $d$. In particular, it makes sense to say that $x_i \in X_i, x_i \to x \in X$ because $d(x, x_i) \to 0$. 


Appendix B. Estimates for Volume Forms

by MARK GROSS

Theorem B.1. Let \( \pi : M \to \Delta \) be a flat projective family of \( n \)-dimensional Calabi–Yau varieties, with \( M_t = \pi^{-1}(t) \) non-singular for \( t \neq 0 \) and \( M_0 = \pi^{-1}(0) \) a variety with canonical singularities. After embedding the family \( M \) in \( \mathbb{C}P^N \times \Delta \), let \( \omega_t \) denote the restriction of the Fubini-Study metric on \( \mathbb{C}P^N \) to \( M_t \). Furthermore, let \( \Omega \) be a nowhere vanishing holomorphic section of the relative canonical bundle \( K_{M/\Delta} \), and set \( \Omega_t = \Omega|_{M_t} \). Then

i) There is a \( \kappa \) independent of \( t \) such that
\[
(-1)^{\frac{n^2}{2}} \Omega_t \wedge \bar{\Omega}_t > \kappa \omega_t^n.
\]

ii) There is a constant \( \Lambda \) independent of \( t \) such that
\[
(-1)^{\frac{n^2}{2}} \int_{M_t} \Omega_t \wedge \bar{\Omega}_t < \Lambda.
\]

Proof. For i), we use an argument similar to that in [18], lemma 6.4. Let \( M^{sm} \) denote the set of points of \( M \) where \( \pi \) is smooth, i.e., the set of points where \( M \) is non-singular and \( \pi_* \) is surjective. Let \( p \in M \) be a point, and consider an open neighborhood \( U_p \) of \( p \) which embeds into \( \mathbb{C}P^{N+1} \) via \( \iota : U_p \to \mathbb{C}P^{N+1} \), with coordinates \( t, z_1, \ldots, z_N \).

The Fubini-Study form is comparable to \( \omega = \sqrt{-1} \sum_{i=1}^{N} dz_i \wedge d\bar{z}_i \), so we can assume that locally \( \omega_t \) is the restriction of \( \omega \) to \( U_p \cap M_t \). Now \( \omega^n = n!(-1)^{\frac{n^2}{2}} \sum_I dz_I \wedge d\bar{z}_I \), where the sum is over all index sets \( I \subseteq \{1, \ldots, N\} \) with \( \#I = n \). Now as \( \iota^*(dz_I) \) is a relative holomorphic \( n \)-form on \( U_p \cap M^{sm} \), there is a holomorphic function \( f_I \) on \( U_p \cap M^{sm} \) such that \( \iota^*(dz_I) = f_I \Omega \). Note that since \( M \) is necessarily normal and \( M \setminus M^{sm} \) is codimension \( \geq 2 \), we can apply Hartogs’ theorem for normal analytic spaces to extend \( f_I \) to a holomorphic function on \( U_p \). Thus
\[
\iota^* \omega^n = C(-1)^{\frac{n^2}{2}} \left( \sum_I |f_I|^2 \right) \Omega \wedge \bar{\Omega}.
\]

On an open neighborhood \( V_p \subset U_p \) of \( p \), \( |f_I| \) is bounded. This gives the desired result.

For ii), we need to apply some standard results from Hodge theory. After making a base-change \( \Delta \to \Delta \) given by \( t \mapsto t^k \) for some \( k \), we can assume that the monodromy operator \( T \) about the origin acting on

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$H^n(M_{t_0}, \mathbb{C})$ is unipotent, i.e., $(T-I)^m = 0$ for some $m$. Here $t_0 \in \Delta^* = \Delta \setminus \{0\}$ is a fixed basepoint. Let

$$N = \log(T - I);$$

this makes sense via the power series expansion. By the stable reduction theorem \cite{36}, one has a diagram

$$\tilde{M} \xrightarrow{\eta} M \xrightarrow{\pi} \Delta$$

in which $\eta$ is an isomorphism outside the central fibre and $\tilde{\pi}$ is normal crossings, i.e., locally around points of $\tilde{M}_0 = \tilde{\pi}^{-1}(0)$ there are coordinates $z_1, \ldots, z_{n+1}$ on $\tilde{M}$ such that $t = z_1 \cdots z_p$ for some $p \leq n + 1$. One has the sheaf $\Omega^1_{\tilde{M}/\Delta}(\log \tilde{M}_0)$ of logarithmic differentials on $\tilde{M}$ locally generated by $dz_1/z_1, \ldots, dz_p/z_p, dz_{p+1}, \ldots, dz_{n+1}$, and the sheaf of relative logarithmic differentials $\Omega^1_{\tilde{M}/\Delta}(\log \tilde{M}_0)$ is obtained by dividing out by the relation $dt/t = 0$. It is standard (see for example the book \cite{45} for the full background used here) that $\Omega^1_{\tilde{M}/\Delta}(\log \tilde{M}_0)$ is a rank $n$ vector bundle, and if $X$ is an irreducible component of $\tilde{M}_0$, then $\Omega^1_{\tilde{M}/\Delta}(\log \tilde{M}_0)|_X = \Omega^1_X(\log \partial X)$, where $\partial X = X \cap \hat{S}$, and $\hat{S}$ is the singular set of $\tilde{M}_0$. One then obtains the logarithmic de Rham complex $\Omega^n_{\tilde{M}/\Delta}(\log \tilde{M}_0)$, with $\Omega^p_{\tilde{M}/\Delta}(\log \tilde{M}_0)$ the $p$-th exterior power of the sheaf of relative log differentials, and the differential $d$ is the ordinary exterior derivative. In particular, we have the line bundle $\Omega^n_{\tilde{M}/\Delta}(\log \tilde{M}_0)$.

By \cite{55}, theorem 2.11, $\tilde{\pi}_* \Omega^n_{\tilde{M}/\Delta}(\log \tilde{M}_0)$ is a vector bundle whose fibre over $t \neq 0$ is $H^0(M_t, K_{M_t})$. Hence this is a line bundle. On the other hand, by assumption on $M$, $K_{M/\Delta} \cong \mathcal{O}_M$ and so $\pi_* K_{M/\Delta}$ is also a line bundle. Let $M^\circ$ be the largest open set so that $\eta^{-1}(M^\circ) \rightarrow M^\circ$ is an isomorphism, and let $i : M^\circ \rightarrow M$ be the inclusion. Then the codimension of $M \setminus M^\circ$ in $M$ is at least two. Since $M \setminus M^\mathrm{sm}$ has codimension at least two, $M \setminus (M^\mathrm{sm} \cap M^\circ)$ has codimension at least two. We have a composition of canonical sheaf homomorphisms

$$\eta_* \Omega^n_{\tilde{M}/\Delta}(\log \tilde{M}_0) \rightarrow i_* i^* \eta_* \Omega^n_{\tilde{M}/\Delta}(\log \tilde{M}_0) \cong K_{M/\Delta},$$

the latter isomorphism by Hartogs’ theorem and the fact that the isomorphism holds over $M^\mathrm{sm} \cap M^\circ$. Applying $\pi_*$ then gives a map

$$(B.1) \quad \tilde{\pi}_* \Omega^n_{\tilde{M}/\Delta}(\log \tilde{M}_0) \rightarrow \pi_* K_{M/\Delta}. $$
This map is an isomorphism over $\Delta^*$, and hence is necessarily an inclusion of sheaves. To show it is in fact an isomorphism, we need to show that any section of $\mathcal{K}_{\tilde{M}/\Delta}|_{M_0} = \mathcal{K}_{M_0}$ comes from a section of $\Omega^n_{\tilde{M}/\Delta}(\log \tilde{M}_0)|_{\tilde{M}_0}$. To see this, let $X_0$ be the proper transform of $M_0$ in $\tilde{M}_0$. Then $\eta_0 : X_0 \to M_0$ is a resolution of singularities, and since $M_0$ has canonical singularities, we have

$$\mathcal{K}_{X_0} = \eta_0^* \mathcal{K}_{M_0} + \sum_E a_E E,$$

where the sum is over all exceptional divisors $E$ of $\eta_0$ and $a_E \geq 0$. (Note $a_E$ is an integer since $M_0$ is Gorenstein.) On the other hand, $\Omega^n_{X_0}(\log \partial X_0)$ is $\mathcal{K}_{X_0} + \sum E$, where the sum is again over all exceptional divisors of $\eta_0$. So $\eta_0^* \Omega_0$, viewed as a section of $\Omega^n_{X_0}(\log \partial X_0)$, has a zero of order at least 1 along each exceptional divisor $E$. Thus $\eta_0^* \Omega_0$ extends by zero to a section of $\Omega^n_{\tilde{M}/\Delta}(\log \tilde{M}_0)|_{\tilde{M}_0}$. Thus (B.1) is surjective, hence an isomorphism.

We now recall some standard material concerning the limiting mixed Hodge structure and the nilpotent orbit theorem. Denote by $H^n$ the vector bundle $\mathbb{R}^n \tilde{\pi}_* \Omega^*_{\tilde{M}/\Delta}(\log \tilde{M}_0)$. The fibre of this bundle at $t$ is isomorphic to $H^0(M_t, \mathbb{C})$. This bundle comes along with the Gauss-Manin connection, which is flat with a regular singular point at $0 \in \Delta$.

Let $j : H \to \Delta^*$ be the universal cover, with $H$ the upper half-plane, with coordinate $w = \frac{1}{2\pi \sqrt{-1}} \log t$. Then $j^* H^n$ is now canonically identified with the trivial bundle $H \times H^n(M_{t_0}, \mathbb{C})$ by parallel transport by the Gauss-Manin connection. If $e \in H^n(M_{t_0}, \mathbb{C})$, one obtains a constant section $\sigma_e$ of $j^* H^n$ by parallel transport, and then $e^{-wN} \sigma_e$ descends to a single-valued section of $\mathcal{H}^n$ over $\Delta^*$. The bundle $\mathcal{H}^n$ is then the canonical extension of $\mathcal{H}^n|_{\Delta^*}$, i.e., the extension in which, for a basis $e_1, \ldots, e_s$ of $H^0(M_{t_0}, \mathbb{C})$, $e^{-wN} \sigma_{e_1}, \ldots, e^{-wN} \sigma_{e_s}$ form a holomorphic frame. In particular, there is an isomorphism of the fibre $\mathcal{H}^n_0 = \mathbb{H}^n(\tilde{M}_0, \Omega^*_{\tilde{M}/\Delta}(\log \tilde{M}_0)|_{\tilde{M}_0})$ with $H^0(M_{t_0}, \mathbb{C})$, isomorphic to the space of flat sections of $j^* \mathcal{H}^n$.

We also have an inclusion

$$\mathcal{F}^n := \tilde{\pi}_* \Omega^*_{\tilde{M}/\Delta}(\log \tilde{M}_0) \hookrightarrow \mathcal{H}^n.$$

The fibre of $\mathcal{F}^n$ over $0 \in \Delta$ is $\mathcal{F}^n_0 \subseteq H^n(M_{t_0}, \mathbb{C})$ under the above isomorphism, a piece of the limiting mixed Hodge structure. In particular, the value of the holomorphic section $\Omega$ of $\pi_* \mathcal{K}_{\tilde{M}/\Delta}$ at 0 under the isomorphism (B.1) defines a class $\Omega_{\text{lim}} \in \mathcal{F}^n_0$.

We now apply the nilpotent orbit theorem (see e.g., [26], chapter IV, for an exposition of this material). Let $\phi : H \to \mathbb{P}(H^n(M_{t_0}, \mathbb{C}))$ be the period map, with, for $w \in H$, $\phi(w)$ being the one-dimensional
subspace \( (j^*F^n)_w \subseteq (j^*H^n)_w \cong H^n(M_{t_0}, \mathbb{C}) \), the latter identification via the Gauss-Manin connection. Then \( e^{-wN} \phi : H \to \mathbb{P}(H^n(M_{t_0}, \mathbb{C})) \) descends to a map \( \psi : \Delta^* \to \mathbb{P}(H^n(M_{t_0}, \mathbb{C})) \) which in turn extends across the origin, with \( \psi(0) = [\Omega_{\text{lim}}] \). The nilpotent orbit is the map \( \phi^{\text{nil}} : H \to \mathbb{P}(H^n(M_{t_0}, \mathbb{C})) \) given by

\[
\phi^{\text{nil}}(w) = e^{wN}\psi(0) = e^{wN}[\Omega_{\text{lim}}].
\]

The nilpotent orbit theorem states that \( \phi^{\text{nil}} \) is a good approximation to \( \phi \), i.e., with a suitable metric on \( \mathbb{P}(H^n(M_{t_0}, \mathbb{C})) \) inducing a distance function \( \rho \), we have constants \( A \) and \( B \) such that for \( \text{Im} w \geq A > 0 \),

\[
\rho(\phi(w), \phi^{\text{nil}}(w)) \leq (\text{Im} w)^B e^{-2\pi \text{Im} w}.
\]

This implies that \( \int_{M_t} \Omega_t \wedge \overline{\Omega}_t \) is bounded independently of \( t \) near 0 provided that \( \int_{M_{t_0}} e^{wN}\Omega_{\text{lim}} \wedge e^{wN}\overline{\Omega}_{\text{lim}} \) is bounded for \( \text{Im} w \geq A \).

Now we apply the argument of proposition 2.3 and theorem 2.1 of [59]. The argument of proposition 2.3 tells us that \( \tilde{M}_0 \) has an irreducible component (in fact \( X_0 \)) with \( H^{n,0}(X_0, \mathbb{C}) \neq 0 \). Thus, by the first line of the proof of theorem 2.1, \( N_{\mathbb{C}^n} = 0 \). So in particular, \( e^{wN}\Omega_{\text{lim}} = \Omega_{\text{lim}} \), giving the desired boundedness. q.e.d.

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