On the closed-form expected NPVs of the double barrier strategy for regular diffusions under the bail-out setting

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Abstract

The core of the research is to provide the explicit expression for the expected net present values (NPVs) of the double barrier strategy for regular diffusions. Under the so-called bail-out setting, the value of the expected NPVs of an insurance company varies according to the choice of a pair of policies, which consist of dividend payments paid out and capital injections received. In the case of the double barrier strategy, the closed-form expected NPVs are given via the bivariate \( q \)-scale function. This is accomplished by making use of a perturbation technique in [CP14], which could lead to the linear equation system. The expression obtained here shall be conducive to addressing the associated dividends control problems.

1 Introduction

In recent years, of much significance to the dividends control community are applications of fluctuation theory for spectrally negative Lévy processes (SNLPs). To be more precise, the so-called \( q \)-scale function, as the eigen-function of SNLPs, plays an increasingly crucial part in giving the analytical form of the expected NPVs in various problem settings. A range of the representative literature on this topic can be found in [LR10, BKY14, APWY16, NPYY18] and references therein. The question arising out of the studies aforesaid naturally is whether the equivalent of the \( q \)-scale function exists regarding differing classes of stochastic processes, and additionally whether the idea of utilizing that to give the closed-form expected NPVs is plausible or not. In response to the question, the case of the double barrier strategy for regular diffusions is under consideration in this paper.

The risk surplus process is the regular diffusion whose drift and volatility parameters depend on the surplus level itself. The so-called bail-out setting initiated in [APP07], which is the offshoot of classical de Finetti setting, modifies the controlled risk reserve with the capital perpetually injected so that it could never hit below zero level. Under this setting, over an infinite time horizon the cost term concerning the cumulative discounted volume of capital injected is incurred in our expected NPVs which contain the cumulative discounted dividend payments.

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The treatment of two-sided exit problem for regular diffusions in [Leh77, Zha15] offers an enlightenment for suggesting the analytical expression for the expected NPVs. A specific bivariate function represented by the two fundamental solutions to the Sturm-Liouville equation characterizes the solution to the two-sided exit problem. The double barrier strategy concerning both a positive barrier at \( a > 0 \) and the zero barrier would bring the surplus being out of the interval \([0, a]\) back to the adjacent barrier level. Making use of the inextricable connection between the double barrier strategy and two-sided exit problem, the expected NPVs are expressed explicitly with the help of the bivariate function. Here we name this function by \textit{bivariate \( q \)-scale functions for regular diffusions}, which could be thought of as the analogue of the \( q \)-scale function for SNLPs in our case.

The realization of the above-mentioned expression seems to be unwarranted for regular diffusions not least because of the lack of the spatial homogeneity property, which, as well as the strong Markov property, proves to be the necessary tool for establishing explicit expected NPVs based upon the research by [AKP04, Pis04] and is expounded as one of the fundamental characteristics for general Lévy processes in [Kl99]. To specify, for a given scale-valued stochastic process, the spatial homogeneity states that \( Z = \{ Z_t, t \geq 0 \} \), we have

\[
\{ Z_t, t \geq 0; Z_0 = x \} \text{ is equal in law to } \{ Z_t + x, t \geq 0; Z_0 = 0 \},
\]

for \( x \in \mathbb{R} \). The exact "fine" properties as to the trajectory of regular diffusions that we could possibly rely on in obtaining closed-form manifestation are just the strong Markov property and continuity of sample path. Nevertheless, in view of the peculiar construction of the double barrier strategy in conjunction with the aforementioned property, the values of the expected NPVs at two barrier levels actually fulfill the linear equations innately, which almost yields the desired. It is also noteworthy that excursion theory would not be involved in this paper.

The rest of the paper is structured as follows: in Section 2 we introduce the bivariate \( q \)-scale function for regular diffusions and give the mathematical definition of the expected NPVs and the construction of the double barrier strategy. In section 3 we present our main results of the closed-form expected NPVs of the double barrier strategy.

2 Preliminary

2.1 Bivariate \( q \)-scale functions for regular diffusions

The risk surplus process \( X = \{ X_t : t \geq 0 \} \), supposed to be the regular diffusion and given by

\[
X_t = x + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dB_s, \quad t \geq 0,
\]

(2.1)

where \( \mu \) and \( \sigma \) are continuous functions on \(( -\infty, +\infty )\) satisfying the common condition that ensures the existence and uniqueness of (2.1) holds and \( \sigma \) is strictly positive, is defined on the filtered probability space \(( \Omega, \mathcal{F}, \mathbb{P} = ( \mathcal{F}_t )_{t \geq 0}, \mathbb{P} )\) satisfying the common assumption. The probability law for the
process $X$ issued from $x$ is written as $P_x$ and the corresponding mathematical expectation is denoted by $E_x$. The infinitesimal generator of the process $X$ is the operator $\mathcal{G}$ on $C^2(\mathbb{R})$ with

$$\mathcal{G} f(x) = \frac{1}{2} \sigma^2(x) f''(x) + \mu(x) f'(x), \quad x \in \mathbb{R},$$

and the associated Sturm-Liouville equation is given by

$$(\mathcal{G} - q) f(x) = 0, \quad x \in \mathbb{R},$$

(2.2)

whose solution would be called the eigen-function of $X$. The equation (2.2) admits two fundamental positive solutions: $\phi_q^+$ (strictly increasing) and $\phi_q^-$ (strictly decreasing) if $q > 0$. Moreover, the functions $\phi_q^+$ and $\phi_q^-$ can be fixed as

$$\phi_q^+(x) = \begin{cases} E_x \left[ e^{-q \tau_x} \right], & x \in (-\infty, p] \\ (E_p \left[ e^{-q \tau_x} \right])^{-1}, & x \in (p, \infty), \end{cases} \quad \phi_q^-(x) = \begin{cases} (E_p \left[ e^{-q \tau_x} \right])^{-1}, & x \in (-\infty, p] \\ E_x \left[ e^{-q \tau_x} \right], & x \in (p, \infty), \end{cases}$$

for arbitrarily given $p \in \mathbb{R}$, where the first hitting times of $X$ is taken as

$$\tau_x := \inf\{ t > 0 : X_t = x \}, \quad x \in \mathbb{R}.$$ 

The function $s$ named as the scale function for regular diffusion $X$ satisfies (2.1) when $q = 0$. Furthermore, its derivative $s'$ can be written as

$$s'(x) = e^{-\int_k^x \frac{2u(s)}{s'(s)} ds} = \frac{(\phi_q^+)'(x)\phi_q^-(x) - (\phi_q^-)'(x)\phi_q^+(x)}{c_q},$$

(2.3)

for some $k \in (-\infty, +\infty)$ and the constant $c_q > 0$ which is independent of $x$. Subsequently, define the function $W_{(q)} : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$W_{(q)}(x, y) := \frac{\phi_q^+(x)\phi_q^-(y) - \phi_q^-(x)\phi_q^+(y)}{c_q}, \quad \text{for } q > 0.$$ 

(2.4)

It is worth mentioning that following easy-to-check property of function $W_{(q)}$ in Lemma I would be used throughout the paper without being referred to specifically.

**Lemma 1.** For all $x, y, u, z \in \mathbb{R}$, we have

$$W_{(q)}(x, y) = -W_{(q)}(y, x), \quad \frac{\partial^2 W_{(q)}(u, z)}{\partial z \partial u} = -\frac{\partial^2 W_{(q)}(z, u)}{\partial z \partial u}.$$ 

Especially, $W_{(q)}(x, x) = \frac{\partial^2 W_{(q)}(u, z)}{\partial z \partial u} |_{z=u=x} = 0$ for all $x \in \mathbb{R}$.

The result regarding two-sided exit problem in [Leh77] is restated here:

**Lemma 2.** If $(y - x)(y - z) < 0$ and $q \geq 0$, it holds that

$$E_y [e^{-q \tau_x} 1_{\{\tau_x < \tau_y\}}] = \frac{W_{(q)}(y, x)}{W_{(q)}(z, x)} = \frac{W_{(q)}(x, y)}{W_{(q)}(x, z)},$$

(2.5)
Remark 1. An extensive set of specific examples of two fundamental solutions to (2.2), i.e. $\phi_q^+$ and $\phi_q^-$, is available in [BST15]. In those examples, the corresponding scale function $s$ and constant $c_q$ are also given. The way of defining the function $W(q)$ here could also be found in [Zha15].

In the sequel, we shall make the convention that

$$W^1(u, z) = \frac{\partial W(u, z)}{\partial u} = \frac{(\phi_q^+)'(u)(\phi_q^-)(z) - (\phi_q^-)'(u)(\phi_q^+)(z)}{c_q},$$

$$W^{12}(u, z) = \frac{\partial^2 W(u, z)}{\partial z \partial u} = \frac{(\phi_q^+)'(u)(\phi_q^-)'(z) - (\phi_q^-)'(u)(\phi_q^+)'(z)}{c_q},$$

$$\mathbb{E}_y [e^{-\tau_x} 1_{\{\tau_x < \tau_y\}}] = \overline{\psi}_{x,z}(y), \quad \mathbb{E}_y [e^{-\tau_x} 1_{\{\tau_x < \tau_z\}}] = \psi_{x,z}(y),$$

for all $u, z \in \mathbb{R}$ and $x < y < z$.

2.2 The definition of the expected NPVs under the bail-out setting

The process $U^\pi$ controlled by the policy pair $\pi = \{\{D_t^\pi, R_t^\pi\} : t \geq 0\}$, which consists of dividend payments $D^\pi = \{D_t^\pi : t \geq 0\}$ and capital injections $R^\pi = \{R_t^\pi : t \geq 0\}$, is formulated as

$$U^\pi_t = X_t - D_t^\pi + R_t^\pi, \quad t \geq 0,$$

where $D^\pi$ and $R^\pi$ is non-decreasing and $\mathbb{F}$-adapted. It is also to be noted that here both the cumulative dividend payments $D^\pi$ and the volume of capital injected $R^\pi$ are right-continuous processes, starting from 0. To illustrate in more detail, $\Pi$ is the admissible class that consists of the dividend policies $\pi$ such that

$$U^\pi_t \geq 0, \quad \text{for all } t \geq 0, \quad \text{and } V^\pi_R(x) < \infty, \quad \text{a.s.}$$

In the control literature, the aim is to identify the strategy $\pi \in \Pi$ that is able to maximize

$$V^\pi(x) = V^\pi_D(x) - \varphi V^\pi_R(x), \quad x \geq 0,$$

(2.6)

where $\varphi > 1$ is the unit cost for the capital injection, $V^\pi_D$ and $V^\pi_R$ are formulated as

$$V^\pi_D(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-qt} dD_t^\pi \right], \quad V^\pi_R(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-qt} dR_t^\pi \right], \quad x \geq 0,$$

(2.7)

in which $q > 0$ is the discounting factor. Nevertheless, here we restrict ourselves to computing the value of (2.6) when the admissible policy is the double barrier strategy.

2.3 Construction of the double barrier strategy

The exact formulation of the double barrier strategy $\pi_a = (D^a, R^a)$ is given as follows:

1. To begin with, let $\sigma_0 = \tau_0$, $\sigma_a = \tau_a$ and $D_t^a = R_t^a = 0$ for $t < \sigma_0 \wedge \sigma_a$. If $X|\sigma_0 \wedge \sigma_a = 0$, then go to step 2. Otherwise, go to step 3.
2. Define
\[ R_t = - \inf_{\sigma_o \leq s \leq t} (X_s \land 0), \quad U_t = X_t - \inf_{\sigma_o \leq s \leq t} (X_s \land 0), \]
for \( t \geq \sigma_o \). Set \( \sigma_a = \inf\{t > 0 : U_t = a\} \). If \( \sigma_0 \leq t < \sigma_a \), let \( D_t^a = D_{\sigma_0}^a \), \( R_t^a = R_{\sigma_0}^a + R_t^a \) and \( U_t^a = U_t \). Then go to step 3.

3. Define
\[ D_t^a = \sup_{\sigma_a \leq s \leq t} [(X_s - a) \lor 0], \quad U_t = X_t - \sup_{\sigma_a \leq s \leq t} [(X_s - a) \lor 0], \]
for \( t \geq \sigma_a \). Set \( \sigma_0 = \inf\{t > 0 : U_t = 0\} \). If \( \sigma_a \leq t < \sigma_0 \), let \( D_t^a = D_{\sigma_0}^a + D_t^a \), \( R_t^a = R_{\sigma_0}^a + R_t^a \) and \( U_t^a = U_t \). Then go to step 2.

The strategy \( \pi_a \) is also such that the support of measure \( dD_t^a \) and \( dR_t^a \) is included in the set of \( \{U_t^a = a\} \) and \( \{U_t^a = 0\} \), respectively. The definition of the strategy \( \pi_a \) here is rephrased from [APP07]. Next, \( V^{\pi_a} \), the expected NPVs function, would be abbreviated to \( V^a \).

### 3 Main results: the closed-form expected NPVs

In order to give the explicit expression of \( V^a \) in Theorem 3, we prove an associated specific result of the function \( W_{(q)} \) here firstly.

**Lemma 3.** For all \( a > 0 \), we have
\[ W_{(q)}^{12} (0, a) > 0. \]  \hfill (3.1)

**Proof.** Since the functions \( \phi_q^+ \) and \( \phi_q^- \) are the eigen-functions of \( X \), i.e. \( \phi_q^+ \) and \( \phi_q^- \) satisfy (2.2), for \( x \geq 0 \) we have
\[
(\phi_q^+)''(x) + \frac{2\mu (x)}{\sigma^2 (x)} (\phi_q^+)'(x) = \frac{2q}{\sigma^2 (x)} \phi_q^+(x),
\]
\[
(\phi_q^-)''(x) + \frac{2\mu (x)}{\sigma^2 (x)} (\phi_q^-)'(x) = \frac{2q}{\sigma^2 (x)} \phi_q^-(x),
\]
as \( \sigma \) is strictly positive function on \( (-\infty, \infty) \). Easily, we deduce that
\[
[ (\phi_q^+)'(t)e^{\int_0^t \frac{2\mu (s)}{\sigma^2 (s)} ds} ]' \bigg|_{t=x} = \frac{2q}{\sigma^2 (x)} e^{\int_0^x \frac{2\mu (s)}{\sigma^2 (s)} ds} \phi_q^+(x), \quad (3.2)
\]
\[
[ (\phi_q^-)'(t)e^{\int_0^t \frac{2\mu (s)}{\sigma^2 (s)} ds} ]' \bigg|_{t=x} = \frac{2q}{\sigma^2 (x)} e^{\int_0^x \frac{2\mu (s)}{\sigma^2 (s)} ds} \phi_q^-(x). \quad (3.3)
\]
Integrating both sides of (3.2) and (3.3) on \([0, a]\) respectively gives
\[
(\phi_q^+)'(a)e^{\int_0^a \frac{2\mu (s)}{\sigma^2 (s)} ds} = (\phi_q^+)'(0) + \int_0^a \frac{2q}{\sigma^2 (x)} e^{\int_0^x \frac{2\mu (s)}{\sigma^2 (s)} ds} \phi_q^+(x) dx, \quad (3.4)
\]
\[
(\phi_q^-)'(a)e^{\int_0^a \frac{2\mu (s)}{\sigma^2 (s)} ds} = (\phi_q^-)'(0) + \int_0^a \frac{2q}{\sigma^2 (x)} e^{\int_0^x \frac{2\mu (s)}{\sigma^2 (s)} ds} \phi_q^-(x) dx. \quad (3.5)
\]
where $W_{(q)}^1(0, x) = \frac{(\phi_q^+(0)\phi_q^-(x)) - (\phi_q^-(0)\phi_q^+(x))}{c_q}$ for all $x \in \mathbb{R}$, since functions $\phi_q^+$ and $\phi_q^-$ are both positive, $(\phi_q^+)' > 0$ and $(\phi_q^-)' < 0$. Thus we finalize the proof.

**Theorem 4.** The function defined in (2.6) for the double barrier strategy at positive level $a > 0$ is such that

$$V^a(x) = \begin{cases} V^a(0) + \varphi x, & x \in (-\infty, 0), \\ \frac{W_{(q)}^1(0, x) - \varphi W_{(q)}^1(a, x)}{W_{(q)}^{12}(0, a)}, & x \in [0, a], \\ V^a(a) + x - a, & x \in (a, \infty). \end{cases} \quad (3.6)$$

**Proof.** Usually, the establishment of the identities which are similar to (3.6) heavily relies on the excursion theory. Nevertheless, we show that it can actually be eschewed on account of the perturbation technique in [CP14].

Let $\varepsilon > 0$. In view of the construction of the double barrier strategy and strong Markov property of $X$ at $\tau_{-\varepsilon} \wedge \tau_a$ and $\tau_0 \wedge \tau_{a+\varepsilon}$, correspondingly, we could deduce that

$$V^a(0) = -\varphi \mathbb{E}_0 \left[ \int_{\tau_{-\varepsilon} \wedge \tau_a} e^{-qt} dR^a_t \right] + \mathbb{E}_0 \left[ \int_{\tau_{-\varepsilon} \wedge \tau_a} e^{-qt} dD^a_t \right] - \varphi \mathbb{E}_0 \left[ \int_{\tau_{-\varepsilon} \wedge \tau_a} e^{-qt} dR^a_t \right]$$

$$= -\varphi \mathbb{E}_0 \left[ \int_{\tau_{-\varepsilon} \wedge \tau_a} e^{-qt} dR^a_t \right] + \overline{\psi}_{a-\varepsilon}(0)V^a(a) + \overline{\psi}_{a-\varepsilon}(0)V^a(-\varepsilon)$$

$$= -\varphi \mathbb{E}_0 \left[ \int_{\tau_{-\varepsilon} \wedge \tau_a} e^{-qt} dR^a_t \right] + \overline{\psi}_{a-\varepsilon}(0)V^a(a) + \overline{\psi}_{a-\varepsilon}(0) [V^a(0) - \varphi \varepsilon],$$

and

$$V^a(a) = \mathbb{E}_a \left[ \int_{\tau_{0} \wedge \tau_{a+\varepsilon}} e^{-qt} dD^a_t \right] + \mathbb{E}_a \left[ \int_{\tau_{0} \wedge \tau_{a+\varepsilon}} e^{-qt} dD^a_t \right] - \varphi \mathbb{E}_a \left[ \int_{\tau_{0} \wedge \tau_{a+\varepsilon}} e^{-qt} dR^a_t \right]$$

$$= \mathbb{E}_a \left[ \int_{\tau_{0} \wedge \tau_{a+\varepsilon}} e^{-qt} dD^a_t \right] + \overline{\psi}_{a+\varepsilon}(a)V^a(a + \varepsilon) + \overline{\psi}_{a+\varepsilon, 0}(a)V^a(0)$$

$$= \mathbb{E}_a \left[ \int_{\tau_{0} \wedge \tau_{a+\varepsilon}} e^{-qt} dD^a_t \right] + \overline{\psi}_{a+\varepsilon}(a) [V^a(a) + \varepsilon] + \overline{\psi}_{a+\varepsilon, 0}(a)V^a(0).$$
Furthermore, we claim that \( \mathbb{E}_0 \left[ \int_0^{\tau_{-\varepsilon} \wedge \tau_0} e^{-q t} dR_t^a \right] = o(\varepsilon) \) and \( \mathbb{E}_a \left[ \int_{\tau_{-\varepsilon} \wedge \tau_0} e^{-q t} dD_t^a \right] = o(\varepsilon) \) by

\[
0 \leq \mathbb{E}_0 \left[ \int_0^{\tau_{-\varepsilon} \wedge \tau_0} e^{-q t} dR_t^a \right] \leq \varepsilon \mathbb{E}_0 \left[ \int_0^{\tau_{-\varepsilon} \wedge \tau_0} e^{-q t} \, dt \right] \leq \varepsilon \mathbb{E}_0 \left[ \int_0^{\tau_{-\varepsilon}} e^{-q t} \, dt \right] \quad (3.7)
\]

\[
0 \leq \mathbb{E}_a \left[ \int_0^{\tau_{0} \wedge \tau_{a+\varepsilon}} e^{-q t} dD_t^a \right] \leq \varepsilon \mathbb{E}_a \left[ \int_0^{\tau_{0} \wedge \tau_{a+\varepsilon}} e^{-q t} \, dt \right] \leq \varepsilon \mathbb{E}_a \left[ \int_0^{\tau_{a+\varepsilon}} e^{-q t} \, dt \right] \quad (3.8)
\]

since the increment of \( R_a \) under \( \mathbb{P}_0 \) would not be over \( \varepsilon \) before the epoch \( \tau_{-\varepsilon} \wedge \tau_0 \) and \( D_a \) under \( \mathbb{P}_a \) could only increase up to \( \varepsilon \) before the moment \( \tau_0 \wedge \tau_{a+\varepsilon} \). Dividing both sides of (3.7) and (3.8) by \( \varepsilon > 0 \) and letting \( \varepsilon \rightarrow 0^+ \) respectively yields the linear equation system that

\[
V^a(0) = \frac{\varphi W_q(0,a) + W_1^1(0) V^a(a)}{W_1^1(0,a)}, \quad V^a(a) = \frac{W_1^q(a,0) + W_1^1(a,a) V^a(0)}{W_1^q(a,0)}.
\]

After careful calculations involving the identity given by

\[
W_1^q(a,0) W_1^{12}(0,a) + W_1^1(a,a) W_1^q(0,0) = W_1^q(0,a) W_1^q(0,a),
\]

which is shown by invoking the definition of the function \( W_q \), we have

\[
V^a(0) = \frac{W_1^q(0,0) - \varphi W_1^q(0,a)}{W_1^{12}(0,a)}, \quad V^a(a) = \frac{W_1^q(0,a) - \varphi W_1^q(a,a)}{W_1^{12}(0,a)},
\]

where \( W_1^{12}(0,a) \) is strictly positive owing to Lemma 3. By the construction of the double barrier strategy \( \pi_a \) and the strong Markov property of \( X \) at \( \tau_0 \wedge \tau_a \), for \( x \in (0,a) \) we infer that

\[
V^a(x) = \mathbb{E}_x \left[ \int_0^{\tau_0 \wedge \tau_a} e^{-q t} dD_t^a \right] = \varphi x V^a(a) + \varphi W_1^q(x,0) V^a(0) - \varphi W_1^q(0,0) W_1^1(a,a) + W_1^q(a,x) W_1^q(a,0)
\]

\[
= \frac{W_1^q(a,0) W_1^{12}(0,a) - \varphi W_1^q(a,0) W_1^q(a,x)}{W_1^q(a,0) W_1^{12}(0,a) - \varphi W_1^q(a,0) W_1^q(a,x)}
\]

where the penultimate equality is due to the calculation based upon the definition of \( W_q \) in (2.4).

The value of function \( V^a \) on \( (-\infty, 0) \cap (a, \infty) \) is obtained by the construction of strategy \( \pi_a \).  \( \square \)
Proposition 1. It holds that
\[(V^a)'(a-)=1, \quad (V^a)'(0+)=\varphi.\]

Proof. Using the value of function $V^a$ in $(0,a)$ given in Theorem 4 will lead to the result. \[\square\]

References

[AKP04] Florin Avram, Andreas E Kyprianou, and Martijn R Pistorius. Exit problems for spectrally negative lévy processes and applications to (canadized) russian options. *The Annals of Applied Probability*, 14(1):215–238, 2004.

[APP07] Florin Avram, Zbigniew Palmowski, and Martijn R Pistorius. On the optimal dividend problem for a spectrally negative lévy process. *The Annals of Applied Probability*, 17(1):156–180, 2007.

[APWY16] Benjamin Avanzi, José-Luis Pérez, Bernard Wong, and Kazutoshi Yamazaki. On optimal joint reflective and refractive dividend strategies in spectrally positive lévy processes. *UNSW Business School Research Paper*, (2016ACTL05), 2016.

[BKY14] Erhan Bayraktar, Andreas E Kyprianou, and Kazutoshi Yamazaki. Optimal dividends in the dual model under transaction costs. *Insurance: Mathematics and Economics*, 54:133–143, 2014.

[BS15] Andrei N Borodin and Paavo Salminen. *Handbook of Brownian motion-facts and formulae*. Springer Science & Business Media, 2015.

[CP14] Irmina Czarna and Zbigniew Palmowski. Dividend problem with parisian delay for a spectrally negative lévy risk process. *Journal of Optimization Theory and Applications*, 161(1):239–256, 2014.

[KI99] Sato Ken-Iti. *Lévy processes and infinitely divisible distributions*. Cambridge university press, 1999.

[Leh77] John P Lehoczky. Formulas for stopped diffusion processes with stopping times based on the maximum. *The Annals of Probability*, pages 601–607, 1977.

[LR10] Ronnie L Loeffen and Jean-François Renaud. De finetti’s optimal dividends problem with an affine penalty function at ruin. *Insurance: Mathematics and Economics*, 46(1):98–108, 2010.

[NPYY18] Kei Noba, José-Luis Pérez, Kazutoshi Yamazaki, and Kouji Yano. On optimal periodic dividend strategies for lévy risk processes. *Insurance: Mathematics and Economics*, 80:29–44, 2018.
[Pis04] Martijn R Pistorius. On exit and ergodicity of the spectrally one-sided lévy process reflected at its infimum. *Journal of Theoretical Probability*, 17(1):183–220, 2004.

[Zha15] Hongzhong Zhang. Occupation times, drawdowns, and drawups for one-dimensional regular diffusions. *Advances in Applied Probability*, 47(1):210–230, 2015.