Collapse of Social Engagement and its Prevention by Local Recruitments

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Social networks sometimes collapse. The threshold model of collective actions has been widely adopted as a theoretical paradigm to understand this phenomenon, but previous investigations focused on the irreversible $K$-core pruning process starting from random initial activity patterns. Here we study the vulnerability and resilience of social engagement as a network alliance problem, and find that equilibrium alliance configurations (but not the out-of-equilibrium ones obtained from irreversible pruning) may experience two distinct dynamical transitions as the number of active nodes gradually shrinks. In the intermediate phase bounded by the weak and strong tipping points, an equilibrium alliance is highly vulnerable to single-node-triggered cascading failures, yet all these global collapse events can be successfully suppressed by a simple least-effort local recruitment mechanism which flips a few flippable inactive nodes.

The proper functioning of online and offline social networks requires the active engagement of its members. But social engagement is largely a collective phenomenon because individual agents influence and are influenced by their network neighbors \cite{1,2}. Small variations of environmental parameters and localized state perturbations sometimes trigger global disruptions of social engagement, such as the rapid decline of online platforms \cite{3}, breakdown of social trust in vaccination \cite{4}, military mutiny and regime shift \cite{5}, and many others \cite{6–11}. Understanding the collapse of social engagement and exploring effective intervention mechanisms are research issues of great practical relevance \cite{12–14}.

Previous theoretical studies modeled the disruption of social engagement as an irreversible threshold dynamics \cite{15–26}. Starting from a random initial activity pattern in which the nodes are active with probability $p$, the system goes through a damage cascading process essentially identical to $K$-core pruning, with active nodes decaying to inactive if they have too few active neighbors \cite{15,16}. The final configurations were found to be extremely sensitive to $p$ if it is close to certain critical value, at which an extensive drop in the network’s activity level may occur \cite{15,20}. However, this kinetic framework neglects a crucial aspect of social engagement, namely the activity configurations are far from being random and irreversible but are the results of complicated interactions among the individual agents and are adaptive \cite{3,27}.

In the present work we study social engagement from the perspective of equilibrium statistical mechanics. We consider microscopic alliance configurations whose active nodes are supported by many active neighbors (Fig. 1) \cite{28,30}, and present a mean field theory combining the dynamics of cascading propagation with the static equilibrium of alliance configurations. We find that equilibrium alliances are quite different from kinetic alliances obtained through the $K$-core pruning process, and they can be classified into three dynamical phases depending on the abundance of active nodes (Fig. 2). In the intermediate phase (bounded by the weak and strong tipping points) the equilibrium alliances are highly vulnerable to cascading failures but all the global collapses are suppressible by flipping a small number of inactive nodes during the cascading process. These results bring new insight on the vulnerability and resilience of social engagement, and our theoretical framework may be applicable to other threshold dynamics as well.

Alliance and its collapse.—Consider a network $G$ formed by $N$ nodes and some undirected links. Nodes

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{An alliance configuration with thirty active (filled circles) and thirty inactive (open circles) nodes for a regular random network of size $N = 60$ and degree $D = 6$, with uniform threshold $K = 4$. The links having two, one, and zero active incident nodes are drawn respectively as solid, dashed, and dotted lines. The inactive node 38 has four active neighbors, so it can switch to be active (it is persuadable).
}
\end{figure}
$i$ and $j$ are neighbors if there is a link $(i, j)$ in between, and \( \partial i \equiv \{ (i, j) : (i, j) \in G \} \) is the neighborhood of $i$. At any time node $i$ either actively engages in the network (state $c_i = 1$) or is inactive ($c_i = 0$), and it may switch between these states. The engaging benefit for $i$ increases with its number $a_i = \sum_{j \in \partial i} c_j$ of active neighbors and when $a_i$ reaches a threshold $\theta_i$ the benefit outweighs the engaging cost \[13, 27]. When $a_i < \theta_i$ node $i$ is always inactive, so the network's configurations $c \equiv (c_1, c_2, \ldots, c_N)$ are those which satisfy the alliance condition $a_i \geq \theta_i$ for every active node $j$. The active nodes of $c$ are directly or indirectly supporting each other and are collectively referred to as an alliance, $A(c) \equiv \{ j : c_j = 1 \} \[28\]. A node $i$ with many active neighbors ($a_i \geq \theta_i$) may still be inactive in network $G$ (e.g., if it is engaging in a competing network \[3\]), and we consider such a node to be persuadable because it can be flipped to $c_i = 1$ (Fig. 1).

We define the energy of configuration $c$ as $E(c) = \sum_i c_i$, which is simply the size of $A(c)$. At a given energy density (or relative size) $\rho \equiv E/N$ the total number of alliances is proportional to $e^{\rho N}$ in leading order of $N$, with $s(\rho)$ being the entropy density function \[29\]. We pick a configuration $c$ uniformly at random from this exponential subspace and examine its sensitivity to local perturbations. If an active node $i$ drops out of $A(c)$ a damage cascading process may be triggered, during which some initially active nodes $j$ are forced to be inactive when the alliance conditions $a_j \geq \theta_j$ are violated \[1, 17\]. After this process finally stops the alliance might only shrink slightly (a small avalanche) or it might be extensively damaged (a collapse) \[51\]. We classify an avalanche of $A(c)$ as a collapse if the final energy density $\rho'$ is much smaller than the initial value $\rho$, and correspondingly we regard the trigger node $i$ as a break node of $c$. (The somewhat arbitrary criterion $\rho' < 0.2 \rho$ is set to justify a collapse, but if a collapse does occur it is usually a complete one, $\rho' = 0$.) The fraction $\phi$ of break nodes is computed by checking every active node of $c$. If $\phi$ is positive then the alliance $A(c)$ is highly vulnerable to single-node perturbations.

We notice that every node in the reservoir set of persuadable inactive nodes has a stabilizing effect to the alliance $A(c)$. A simple least-effort local recruitment mechanism then goes as follows: If an active node $j$ becomes unstable ($a_j$ falls below $\theta_j$) it flips to $c_j = 0$ only if it has no persuadable neighbor, otherwise it remains active and a persuadable neighbor (say $k$) is flipped to $c_k = 1$ instead. We can study the effect of this intervention mechanism by damage cascading analysis. If the quit of a single active node $i$ still leads to the collapse of $A(c)$, node $i$ is said to be a break point of $c$ under this locally protected cascading process. The fraction of break nodes of $c$ is denoted as $\psi$. Obviously $\psi \leq \phi$ for any given alliance configuration $c$. For example, all the thirty active nodes in Fig. 1 are break nodes in the unprotected dynamics ($\phi = 0.50$), but nodes 19, 26, 58 are no longer break nodes in the protected dynamics which recruits node 38 ($\psi = 0.45$).

**Simulation method and collapse theory.**—We adopt the demon algorithm of microcanonical Monte Carlo (MMC) simulation to sample alliance configurations with equal weight \[32, 33\]. At each elementary MMC step a new alliance configuration $c'$ is proposed by flipping under detailed balance a single node or a chain of same-state nodes of the incumbent $c$ \[29\]. If the energy of $c'$ does not exceed an objective value $E_o \equiv \rho N$ then $c'$ is accepted as the next configuration of the network, otherwise the network adheres to $c$. One unit time of this MMC dynamics corresponds to $N$ consecutive trials of configuration transitions. At each energy density $\rho$ we typically collect $3.2 \times 10^4$ configurations at unit time interval, and then the MMC simulation repeats at a slightly decreased $\rho$ value \[34\].

We develop a mean field theory to analyze the vulnerability of alliances. Each node $i$ contributes a term to $E(c)$ and it also imposes an alliance constraint to itself and all its neighbors. Consider a link $(i, j)$ and let us for the moment neglect the energy and constraint of node $i$, so its state $(c_i, c_j)$ is affected by the energy and constraint of node $j$ only. We denote the corresponding probability distribution as $q_{j \rightarrow i}^{c_i, c_j}$, and in addition denote by $t_{j \rightarrow i}^{1, c_i}$ the probability that (1) $c_i = c_j = 1$ and (2) if node $i$ now flips to $c_i = 0$ the damage cascading process relayed through link $(i, j)$ will only cause a tree-formed small avalanche \[34\]. Because a large random network is locally tree-like, if node $j$ is deleted from the network its neighbors will be distantly separated and be mutually independent \[35\]. Assuming this Bethe-Peierls factorization property, we get the following belief-propagation (BP) equation for $q_{j \rightarrow i}^{c_i, c_j}$:

\[
0_{c_i, c_j} = \frac{1}{z_{j \rightarrow i}} \prod_{k \in \partial j \setminus i}(0_{k, 0}^{c_i, c_j} + 1_{0, 0}^{c_i, c_j}),
\]

\[
1_{c_i, c_j} = e^{-\beta} \frac{\Theta(\sum_{k \in \partial j \setminus i} c_k + c_i - \theta_j)}{z_{j \rightarrow i}} \prod_{k \in \partial j \setminus i} q_{k \rightarrow j}^{c_k, c_j},
\]

where $z_{j \rightarrow i}$ is the normalization constant; set $\partial j \setminus i$ contains all the neighbors of node $j$ except for $i$ and $c_{\partial j \setminus i} \equiv \{ c_k : k \in \partial j \setminus i \}$: $\beta$ is the inverse temperature parameter; $\Theta(x) = 0$ if $x < 0$ and $= 1$ if $x \geq 0$ \[29\]. Similarly the probability $u_{j \rightarrow i}^{0, 1}$ of node $j$ being unpersuadable ($c_j = 0$ and $a_j < \theta_j$) and $c_i = 1$ is

\[
u_{j \rightarrow i}^{0, 1} = \frac{1}{z_{j \rightarrow i}} \prod_{k \in \partial j \setminus i} \Theta(\theta_j - 2 - \sum_{k \in \partial j \setminus i} c_k) \prod_{k \in \partial j \setminus i} q_{k \rightarrow j}^{c_k, 0}.
\]

For damage cascading without local recruitments, the
self-consistent expression for \( t_{j ightarrow i}^{1,1} \) is

\[
t_{j ightarrow i}^{1,1} = e^{-\beta} \sum_{\mathcal{e} \in \partial i} \left[ \sum_{\mathcal{k} \in \partial j} \theta(\sum_{\mathcal{k} \in \partial j} c_{k} - \theta_{j}) \prod_{k \in \partial j \backslash i} q_{k \rightarrow j}^{c_{k}+1} + \delta_{\sum_{k \in \partial j \backslash i} c_{k}} \prod_{k \in \partial j \backslash i} (\delta_{c_{k}}^{0} q_{k \rightarrow j}^{0,1} + \delta_{c_{k}}^{1} t_{j ightarrow k}^{1,1}) \right],
\]

where \( \delta_{m}^{n} = 1 \) if \( m = n \) and \( 0 \) if otherwise. The damage of node \( i \) will not propagate to \( j \) if initially \( a_{j} > \theta_{j} \) (the first term of Eq. (3)), otherwise node \( j \) will flip to \( c_{j} = 0 \) and this flip may then induce further damages to the alliance (the second term of Eq. (3)). For damage cascading with local recruitments we incorporate the blocking effect of a persuaded neighbor into the second term of Eq. (3) to get

\[
t_{j ightarrow i}^{1,1} = e^{-\beta} \sum_{\mathcal{e} \in \partial i} \left[ \sum_{\mathcal{k} \in \partial j} \theta(\sum_{\mathcal{k} \in \partial j} c_{k} - \theta_{j}) \prod_{k \in \partial j \backslash i} q_{k \rightarrow j}^{c_{k}+1} + \delta_{\sum_{k \in \partial j \backslash i} c_{k}} \prod_{k \in \partial j \backslash i} (\delta_{c_{k}}^{0} q_{k \rightarrow j}^{0,1} + \delta_{c_{k}}^{1} t_{j ightarrow k}^{1,1}) \right].
\]

To appreciate this modification, notice that node \( j \) will not flip if it has a persuasible inactive neighbor [31]. With these preparations we can now express the marginal probability \( t_{i} \) of node \( i \) being active but not being a break node as

\[
t_{i} = e^{-\beta} \sum_{\mathcal{e} \in \partial i} \Theta(\sum_{\mathcal{j} \in \partial i} c_{j} - \theta_{i}) \prod_{\mathcal{j} \in \partial i} q_{j \rightarrow i}^{c_{j}} + e^{-\beta} \sum_{\mathcal{e} \in \partial i} \Theta(\sum_{\mathcal{j} \in \partial i} c_{j} - \theta_{i}) \prod_{\mathcal{j} \in \partial i} q_{j \rightarrow i}^{c_{j}},
\]

where \( \mathcal{c}_{\partial i} \equiv \{c_{j} : j \in \partial i\} \). The marginal probability \( q_{i} \) of node \( i \) being active has the same expression as Eq. (5) but with \( t_{j ightarrow i}^{1,1} \) replaced by \( q_{j ightarrow i}^{1,1} [29] \). The average fractions of break nodes in the damage cascading process without and with local recruitments are computed by the same expression \( \phi(\psi) = \sum_{i=1}^{N} (q_{i} - t_{i})/N \), with \( t_{j ightarrow i}^{1,1} \) in Eq. (5) fixed by Eq. (3) and Eq. (4), respectively. We work on the microcanonical ensemble of fixed energy density \( \rho \), so the inverse temperature \( \beta \) is determined by the energy constraint \( \rho = \sum_{i=1}^{N} q_{i}/N \).

**Numerical results.**—If \( \theta_{i} \leq 2 \) for all the nodes \( i \) the damage cascading process would never exponentially proliferate and catastrophic collapses would never occur [31]. We therefore consider the nontrivial situations of \( \theta_{i} \geq 3 \), and for simplicity assume uniform threshold \( \theta_{i} \equiv K \) with \( K \) being an integer. An alliance is then equivalent to a K-core [13] [15] [18]. We first consider regular random (RR) networks in which every node has exactly \( D \) neighbors [36], and for which the mean-field equations are much simplified because \( q_{j ightarrow i}^{1,1} \) and \( t_{j ightarrow i}^{1,1} \) are the same for all the links [29] [31].

The theoretical results for \( D = 6 \) and \( K \in \{2, 3, 4\} \) are shown in Fig. 2. The entropy density \( s(\rho) \) is convex for \( \rho < \rho_{x} \) and concave for \( \rho > \rho_{x} \), where \( \rho_{x} \) is the inflection point [29]. The convexity of \( s(\rho) \) indicates that alliances of relative sizes \( \rho < \rho_{x} \) may be unstable and difficult to construct in the canonical statistical ensemble [29]. For \( K \geq 3 \) and without local protective recruitments, we indeed find that the fraction \( \phi \) of break nodes becomes positive as \( \rho \) decreases below a critical value \( \rho_{sw} \) (the weak tipping point) which is considerably larger than \( \rho_{x} \). With local recruitments, however, the break-node fraction \( \psi \) is zero as long as \( \rho > \rho_{sw} \) with \( \rho_{sw} \) (the strong tipping point) being much smaller than \( \rho_{sw} \). At \( D = 6 \) we have \( \rho_{sw} = 0.3885 \), \( \rho_{st} = 0.1968 \) for \( K = 3 \) and \( \rho_{sw} = 0.7585 \), \( \rho_{st} = 0.5719 \) for \( K = 4 \), with \( \rho_{sw} < \rho_{x} \) for \( K = 3 \) but \( \rho_{st} > \rho_{x} \) for \( K = 4 \). On the other hand if the alliance configurations are obtained through the irreversible \( K\)-
core pruning process starting from a completely random initial pattern, their relative sizes are greater than the kinetic threshold value \( \rho_k \), with \( \rho_k = 0.2942 \) for \( K = 3 \) and \( \rho_k = 0.6574 \) for \( K = 4 \) \( (D = 6) \) \cite{15 16 18}. All such kinetic alliance configurations are robust against single-node perturbations \( (\phi = 0) \), which is reasonable because otherwise they can not survive the pruning process. Notice that \( \rho_k < \rho_{\text{wt}} \), further illustrating that equilibrium alliances and kinetic out-of-equilibrium alliances have distinct properties.

The predicted \( \phi, \psi \) values are confirmed by our MMC results when the ratio \( \phi/\rho \) is not too close to unity (Fig. 2). (When \( \phi \) approaches \( \rho \) it becomes exceedingly hard to equilibrate the MMC dynamics.) If a catastrophic avalanche is suppressible by local recruitments, we find that the number of actually recruited nodes during the whole cascading process is only of order unity even for very large systems \cite{31}. Therefore the local recruitment mechanism is a minimum-cost intervention strategy.

We also study several other types of networks with narrowly distributed degree profiles \cite{31}, including cubic lattices, small-world networks \cite{37}, Erdös-Rényi random networks \cite{39}, and a peer-to-peer computer server network \cite{38}. The local recruitment mechanism also greatly enhances the resilience of these networks, may even to the extent of suppressing all the possible collapses (Fig. 3). If the network has a broad degree profile (e.g., scale-free-like as in a collaboration network \cite{38} \cite{39}) while the thresholds \( \theta_i \) still remain uniform, we find that its alliance configurations are robust to random perturbations and \( \phi = 0 \) \cite{31}. This robustness is attributed to the hub nodes which are highly likely to be active in the equilibrium alliance configurations and which have a strong stabilizing effect \cite{18}.

**Conclusion.**—The network equilibrium alliance problem may experience two dynamical phase transitions as the relative alliance size \( \rho \) decreases through the weak and strong tipping points \( \rho_{\text{wt}} \) and \( \rho_{\text{st}} \). In the intermediate range \( \rho \in (\rho_{\text{st}}, \rho_{\text{wt}}) \), catastrophic avalanches are triggered by single-node perturbations but they can be successfully blocked with negligible efforts through a local recruitment mechanism; when \( \rho \leq \rho_{\text{st}} \) this local mechanism still reduces the collapse risk from \( \phi/\rho \) to \( \psi/\rho \). The demonstrated huge effect of small protection efforts may partially explain why the burst of a particular collapse event is so difficult to forecast \cite{5}. The theoretical insight gained in this work helps us to better understand the resilience and collapse of social engagement in complex networks.

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