On hyperbolized nonlinear Schredinger type equations

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Abstract. A nonlinear Schredinger type equation in the case of an unbounded in space operator having a non-divergent form is considered. A new approach based on the hyperbolization procedure is suggested. The possibilities of the method proposed are tested.

1. Introduction
A nonlinear Schredinger equation (NLS) is actively applied to describe a lot of problems in physics, e.g., in the theory of surface waves, in plasma oscillations, in nonlinear optics, in biophysics etc. Hyperbolization of NLS equation is provided by an adding of the second derivative of an unknown function in time with a small parameter. At the same time the hyperbolization method [1, 2] is intensively developed. This idea was first proposed for electrodynamics by A. Melani [3]. For the problems of hydrodynamics, the fundamental work in this direction is the work of B.N. Chetverushkin [1]. Due to its suitable adaptation for a high-performance parallel computing this method is widely used nowadays (in hydro and gas dynamics especially) [4]. This procedure makes it possible to construct explicit schemes possessing the better stability condition in comparison with classic schemes (both for parabolic equations and for non-stationary Schredinger type ones [5]). This fact is very important in the case of small mesh size. Moreover, hyperbolization permits to avoid the phenomenon of random phases taking place for NLS [5].

2. Relaxation method for NLS
Dynamics of a plasma density jump in a given field is governed by the nonlinear Schredinger type equation [6],

\[ -2i \frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} + (-x + |E|^2)E - B = 0, \]

where \( \frac{\partial^2 E}{\partial x^2} - xE \) is the unbounded operator, the continuous spectrum of which covers the entire real axis. Its eigenfunctions are expressed in terms of the Airy functions. Consequently, Fourier components of E in (17) are varied according to the \( \exp(ik^3t) \) rule in time. This fact yields a severe restriction for the time step (\( \Delta t \sim \Delta x^3 \)).

The heuristic substitution

\[ E = U e^{i((t+T)(x/2+(t+T)^2)/24)} \]
leads to a semi-bounded operator in the spatial coordinate for $U$:

$$-2i \frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} + i(t + T) \frac{\partial U}{\partial x} + NU = Be^{-i(t+T)(x/2+(t+T)^2/24)}.$$  \hspace{1cm} (3)

Here $N = |E|^2$. The meaning of the newly introduced parameter $T$ will be clear later on.

Calculations for $U$ are possible in a small time interval due to random phases appearing in high frequency components. To avoid this effect, it is necessary to return to $E$ function after few time steps considering it as the new initial data for $t = 0$. Then the previous procedure is repeated again and again.

Further regularization for (3) is a hyperbolization procedure: adding of the second derivative in time with a small parameter:

$$\mu \frac{\partial^2 U}{\partial t^2} = -2i \frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} + i(t + T) \frac{\partial U}{\partial x} + NU - Be^{-i(t+T)(x/2+(t+T)^2/24)}.$$  \hspace{1cm} (4)

We introduce two additional functions $H$ and $G$ and represent the hyperbolized equation as a system of three equations with three unknown functions $U$, $H$ and $G$:

$$\frac{\partial U}{\partial t} = i \frac{\partial H}{\partial x} + iG, \quad i\mu \frac{\partial H}{\partial t} = 2H + \frac{\partial U}{\partial x} + i(t + T)U.$$  \hspace{1cm} (5)

For the function $G$, we obtain the equation, reducing this system to a hyperbolized equation (4). To do this, we differentiate the first of the equations (5) with $t$, multiply the result by $\mu$ and make a substitution under the sign of the derivative with respect to $x$ using the second equation of the system (5).

$$\mu \frac{\partial^2 U}{\partial t^2} = i\mu \frac{\partial H}{\partial x} + i\mu \frac{\partial G}{\partial t} = 2\frac{\partial H}{\partial x} + \frac{\partial^2 U}{\partial x^2} + i(t + T)\frac{\partial U}{\partial x} + \mu i \frac{\partial G}{\partial t}.$$  \hspace{1cm} (6)

Replacing $\frac{\partial H}{\partial x}$ by virtue of the first equation of the system (5), we get

$$\mu \frac{\partial^2 U}{\partial t^2} = -2i \frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} + i(t + T)\frac{\partial U}{\partial x} - 2G + i\mu \frac{\partial G}{\partial t}.$$  \hspace{1cm} (7)

This equation coincides with the hyperbolized equation (4) if the function satisfies the equation

$$i\mu \frac{\partial G}{\partial t} - 2G = NU - Be^{-i(t+T)(x/2+(t+T)^2/24)}.$$  \hspace{1cm} (8)

The equation for the function $H$, and the function $U$ itself, have the meaning of stream relaxation with the relaxation parameter $\mu$. The equation for the function $G$ sets the relaxation of the part of the original operator, which is responsible for stabilizing the spreading of the wave packet due to the nonlinear dispersion law in accordance with the expansion of the initial dispersion equation in a series in powers of the amplitude $E$.

We introduce in the space $(x,t)$ uniform in each direction calculation grids with a chess arrangement of knots: with integer indices for conservative variables $U$ and $G$

$$\omega_c = \{x_j = jh_x, -N_x \leq j \leq N_x, t_k = kh_t, 0 \leq k \leq M\}$$

and with fractional indices of coordinates of nodes

$$\omega_f = \{x_{j+1/2} = (j + 1/2)h_x, -N_x \leq j \leq N_x, t_{k+1/2} = (k + 1/2)h_t, 0 \leq k \leq M\}$$
for stream variable.

After each step in $t$ we will return to the original values of $E$, i.e. we will recalculate the initial data for $U$, $G$ and (somewhat more complicated) for $H$ at the point with fractional coordinates.

In order not to clutter up the formulas, we will use superscripts $k$ to denote points by $t$, and lower $j$ by $x$ only where it is difficult or impossible to understand the results.

The equation for the function $G$ is a linear equation of the first order. We can integrate it at the step of $t$, replacing the function $NU$ with a linear combination of the extreme points.

$$G(t) = G(0)e^{-i2t/\mu} - \frac{i}{\mu} \int_0^t \left( \frac{\tau}{t} NU(t) - \frac{\tau - t}{t} NU(0) \right) e^{i2(\tau-t)/\mu} d\tau - \hat{B} =$$

$$= G(0)D - NU(t)P + NU(0)Q - \hat{B}.$$

Here $D = e^{-i2t/\mu}$, $P = \frac{1}{2} \left( 1 - \frac{\mu}{2}(1 - D) \right)$, $Q = \frac{1}{2} \left( D + \frac{\mu}{2}(1 - D) \right)$,

$$\hat{B} = \frac{B}{\mu} \int_0^t e^{(2(\tau-t)/\mu - (\tau + T)(x/2 + (\tau + T)^2/24))} d\tau. \quad (10)$$

$$H(t/2) = H(-t/2)D - \frac{i}{\mu} \int_{-t/2}^{t/2} \left( \frac{\partial}{\partial x} U(0) + iTU(0) \right) e^{i2(t-\tau)/\mu} d\tau \quad (11)$$

$$H(t/2) = H(-t/2)D + \frac{1}{2} \left( \frac{\partial}{\partial x} U(0) + iTU(0) \right) (1 - D)$$

Here it becomes clear the meaning of the introduction of the parameter $T$ - we more accurately consider the relaxation of the flows.

$$U(t) = U(0) + \frac{i}{h_x} \int_0^t \left( H_{j+\frac{1}{2}}^{k+\frac{1}{2}} - H_{j-\frac{1}{2}}^{k+\frac{1}{2}} \right) dr - \frac{i}{2} \int_0^t (G(t)u + G(0)u) dt =$$

$$= U(0) + \frac{it}{h_x} \left( H_{j+\frac{1}{2}}^{k+\frac{1}{2}} - H_{j+\frac{1}{2}}^{k-\frac{1}{2}} \right) - \frac{it}{2} G(0)(1 - D) - \frac{it}{2} NU(t) + \frac{it}{2} NU(0) - \frac{it}{2} \hat{B}. \quad (12)$$

We rewrite the last relation, dropping the duplicate index $j$ on $U$ and putting all the terms with $U(t)$ to the left side

$$U(t) \left( 1 + \frac{itP}{2} N(U(t)) \right) = R =$$

$$= U(0) + \frac{it}{h_x} \left( H_{j+\frac{1}{2}}^{k+\frac{1}{2}} - H_{j+\frac{1}{2}}^{k-\frac{1}{2}} \right) - \frac{it}{2} G(0)(1 - D) + \frac{it}{2} NU(0) - \frac{it}{2} \hat{B}. \quad (13)$$

Hence we obtain the equation for determination of $|U(t)|^2$

$$|U(t)|^2 \left( 1 + \frac{tImP}{2} N(U(t)) \right)^2 + \frac{t^2(ReP)^2}{4} N(U(t))^2 = |R|^2. \quad (14)$$

Since hyperbolization is a method of finding an approximate solution of the original Schredinger equation for the function $E$, after each step of finding the function $U(t)$ we return to the function $E(t)$. We consider that $E_{new}(0) = E(t)$. It remains for us to go to $U_{new}$, and set the initial data
for auxiliary functions, based on the condition \( \frac{\partial H}{\partial t} \) at the point \( t = 0 \) is determined from the original equation. Just ask

\[
U_{\text{new}}(t) = E(0)e^{i(\alpha-t)(x/\alpha^2-(\alpha-t)/\alpha^4)/24}) = U(0)e^{i(\alpha-x/\alpha^2-3i\alpha-T^2-3\alpha^2Y)/24},
\]
\[
U_{\text{new}}(0) = E(t)e^{-it(x/\alpha^2+3\alpha^2T^2)/244} = U(t)e^{i(itx/\alpha^2+3i\alpha^2T^2)/244},
\]

The new value of \( H(-t/2) \) is obtained as the half of sum of the corresponding expressions on the upper and lower layers

\[
H_{\text{new}}(\frac{-t}{2}) = \frac{1}{4}x(U_{\text{new}}(0)_{j+1} - U_{\text{new}}(0)_{j} + U_{\text{new}}(-t)_{j+1} - U_{\text{new}}(-t)_{j}) +
\]
\[
+ \frac{i}{2}(T(U_{\text{new}}(0)_{j+1} + U_{\text{new}}(0)_{j}) + (-t + T)(U_{\text{new}}(-t)_{j+1} + U_{\text{new}}(-t)_{j})),
\]

Respectively

\[
G_{\text{new}}(0) = (N_{\text{new}}U_{\text{new}}(0) - B)e^{-iT(x/\alpha^2+3\alpha^2T^2)/244)/2}.
\]

In fact, we assume that at each step for the auxiliary functions \( H \) and \( G \) we set new initial data based on the condition that their time derivatives are equal to zero. This is equivalent to specifying the second initial given \( \frac{\partial U}{\partial t} \) for equation (4) from the original equation (3).

The described technique is simple, easily parallelized and gives hope for an increase in the time step due to the explicit solution of the equation for nonlinear dispersion. However, it makes it impossible to determine the maximum allowable values of the relaxation parameter \( \mu \) and the time step. To find them, we construct an approximate solution of the original equation.

### 3. Slip-step Fourier method for hyperbolized Schrodinger equation

In [5] the method of operator exponents constructed a scheme of the SSFM type for the NLS. Application of this method to the hyperbolized Schrödinger equation also allows us to find the optimal ratio between the time step and the parameter \( \mu \).

Consider a hyperbolized equation in the form more suited to the operator exponent method, adding an additional attenuation with the \( b \) parameter

\[
a^2\mu^2 \left( \frac{\partial^2 U}{\partial t^2} + b\frac{\partial U}{\partial t} \right) + 2i\mu\frac{\partial U}{\partial x} - \frac{\partial^2 U}{\partial x^2} - it\frac{\partial U}{\partial x} + \left( \frac{t^2}{4} - N - 2d \right)U + Be^{-it(x/\alpha^2+d)} = 0.
\]

Previously we introduce a new unknown function, taking into account a small parameter, so that the resulting hyperbolic operator has an explicitly computed Green function. Introduce a new time

\[
t = t_{\text{old}}/(a\mu).
\]

Let’s make a replacement

\[
U = Ve^{-(b\mu^2+2i/\alpha\mu)t/2}
\]

\[
\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} - ia\mu\frac{\partial V}{\partial x} + \left( \frac{1}{a^2\mu^2} - \frac{b^2a^2\mu^2}{4} - 2d - N \right)V - ibV + Be^{it(a\mu x/\alpha^2+\eta)} = 0.
\]

To simplify further calculations, we fix the parameters

\[
b = a\mu/2, \quad d = \frac{a^4\mu^4}{32}, \quad \alpha = \frac{a\mu}{2}, \quad \hat{D} = De^{-it\eta}\delta(k-a\mu t/2), \quad \eta = \frac{1}{2a\mu} - \frac{a^5\delta/32 + i a^2\mu^2}{4}.
\]

Therefore, the application of operator methods for solving the Cauchy problem is simpler and at the same time uniformly carried out by making the Fourier transform with respect to the spatial variable\( s \).

\[
\frac{\partial^2 FV}{\partial t^2} + \left( \frac{a{\mu}}{2} \right)^2 \left( t + \frac{2k}{a\mu} \right)^2 FV - i\frac{a\mu}{2} FV - F(NV) + Be^{-it\eta}\delta(k-a\mu t/2).
\]
This equation can be written in the form of a system
\[
\frac{\partial}{\partial t} \begin{pmatrix} Fv \\ Fw \end{pmatrix} = \begin{pmatrix} -\alpha^2 (t + 2k / \alpha \mu)^2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Fv \\ Fw \end{pmatrix} + i\alpha \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Fv \\ Fw \end{pmatrix} + F \begin{pmatrix} 0 \\ N \end{pmatrix} F^{-1} \begin{pmatrix} Fv \\ Fw \end{pmatrix} + \tilde{B} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

The fundamental solution looks rather complicated even in the one-dimensional space case
\[
G(t) = e^{i\alpha (t^2 + 4kt / \alpha \mu) / 2} \int_0^t e^{-i\alpha(s^2 + 4ks / \alpha \mu)} ds,
\]

We reduce our system to diagonal form by means of the transformation
\[
\begin{pmatrix} Fv \\ Fw \end{pmatrix} = \begin{pmatrix} G_1 & i(\alpha t + 2k / \mu)G_1 \\ i(\alpha t + 2k / \mu)G + \bar{G}_1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = Q(t) \begin{pmatrix} A \\ B \end{pmatrix},
\]

We introduce a new vector function
\[
V = \begin{pmatrix} A \\ B \end{pmatrix}, \quad \tilde{B}(t) = Q^{-1}(t) \begin{pmatrix} 0 \\ B \end{pmatrix}.
\]

We obtain the equation for the function \( V \) with a bounded operator \( C(t) \):
\[
\frac{dV}{dt} = C(t)V + \tilde{D}(t) = Q^{-1}(t)F \begin{pmatrix} 0 \\ N(y) \end{pmatrix} F^{-1} Q(t)V + \tilde{B}(t).
\]

Further replacements are completely analogous to those described in [5] and give us the approximate solution on the interval \( 0 \leq \tau \leq t \): At the point \( \tau = t \) we obtain:
\[
\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} F^{-1} Q(t) Q^{-1}(0) F \begin{pmatrix} 1 \\ -tN(0)/2 \\ 1 \end{pmatrix} F^{-1} Q(0) F \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} + \left( t e^{i(\eta + \alpha \mu x/2)}/2 \right) \alpha \mu x/2 \right) Fw(t) = \frac{\partial w(t)}{\partial t}
\]

since the initial data for \( Fw(t) = \frac{\partial w(t)}{\partial t} \) are found from the condition that at the point \( t \) the original Schrödinger equation is satisfied.

In order to carry out stable calculations, excluding education random phase by repeated multiplication by \( G_1 = e^{-i\alpha (t^2 + 4kt / \alpha \mu)} \), it is necessary to satisfy that condition
\[
\alpha (t^2 + 4kt / \alpha \mu) < \pi/2.
\]

The calculations take place in a bounded spectral region \( |K| \leq K_{max} \), therefore, returning to the original time, we get
\[
\frac{t_{old}^2}{\mu} + \frac{8K_{max}^2}{t_{old}^2} \leq \pi/2.
\]

From the equality we find
\[
t_{old} \approx \pi \mu^2 / 8K_{max}.
\]

Taking, for example, \( \mu^2 = \frac{1}{2K_{max}^{1/2}} \) we get for the hyperbolized nonlinear Schrödinger equation (HNSE) restriction \( t_{old} \leq \frac{\pi^{1/2}}{16K_{max}^{1/2}} \). Recall that for for the NSE this restriction has the form
\[
t_{NSE} \leq \frac{2}{2K_{max}^{1/2}}.
\]

At \( K_{max} = 2^7 = 128 \) the step can be increased by 4 times. At \( K_{max} = 2^9 = 10248 \) in 8 times.

Note that operator replacements have automatic parallelism and are simply and practically error-free programmable.
Figure 1. Amplitude decay of the field $|E|$ in time. Both formation and evolution of travelling waves of the Airy type is observed.

4. Conclusion
Typical dynamics of the numerical solution of Eq. (17) obtained by the relaxation method for HNLS (4) is presented in Fig. 1. The initial data are zero, the small external parameter $B = 0.1$. The principal result is the essential difference of wave flow $U$ relaxation for the HNLS, achieved by adjusting the phase of the oscillations in comparison with a heat flow obtained by amplitude constancy.

Initially, a field growing due to external influences quickly falls apart into scattering waves, such as Airy functions, but with a growing frequency. It is characteristic that smooth leading wave fronts are directed in different directions. This is due to the fact that the waves generated at $x > 0$ and $x < 0$ are solutions of the equation $\frac{\partial^2 \varphi}{\partial x^2} \pm x \varphi$, i.e. the Airy functions $Ai(\pm x)$. Thus we get the wave $Ai(-x)$ in the region of negative $x$. The figure shows that it is the solution of the Schrödinger equation that is found: there is no characteristic sign of the solution of the hyperbolic equation two identical waves $\varphi(x - at)$ and $\varphi(x + at)$, traveling in opposite directions.

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