CARLEMAN INEQUALITY FOR A LINEAR DEGENERATE PARABOLIC PROBLEM

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Abstract

In this work, we prove a Carleman estimate for a parabolic problem which has a dissipative degenerate term. The prove relies on choose a suitable weight function that change of sign inside the control domain.

Keywords: Degenerate parabolic equations, Controllability, Carleman Inequality

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1. Introduction

Let us consider the degenerate parabolic problem

\[
\begin{aligned}
&u_t - (a(x) u_x)_x + c(t, x) u = h\chi_{\omega}, \quad (t, x) \in Q; \\
&u(t, 0) = 0, \\
&\text{or} \\
&(au_x)(t, 0) = 0, \\
&u(0, x) = u_0(x),
\end{aligned}
\tag{1.1}
\]

where \(T > 0\) is given, \(Q := (0, T) \times (0, 1)\), \(\omega = (\alpha, \beta) \subset \subset (0, 1)\), \(u_0 \in L^2(0, 1)\) and \(h \in L^2(Q_\omega)\) is a control that acts on the system through \(Q_\omega := (0, T) \times \omega\). We also specify some properties of \(a\).

A.1. Let \(a \in C([0, 1]) \cap C^1([0, 1])\) be a nondecreasing function satisfying \(a(0) = 0\) and \(a > 0\) on \((0, 1]\).

Additionally, we suppose that there exist a \(K \in \mathbb{R}\) such that

\[
xa'(x) \leq Ka(x), \quad \forall x \in [0, 1],
\tag{1.2}
\]

where \(K \in [0, 1)\), for the \textbf{Weak Degeneracy Case (WDC)}, and \(K \in [1, 2)\), for the \textbf{Strong Degeneracy one (SDC)}. Only for the (SDC), we also assume that

\[
\begin{cases}
\exists \theta \in (1, K] \text{ such that } \theta a \leq xa' \text{ near zero, if } K > 1; \\
\exists \theta \in (0, 1) \text{ such that } \theta a \leq xa' \text{ near zero, if } K = 1.
\end{cases}
\tag{1.3}
\]

Under these notations, we provide some examples and comments about Hypothesis A.1.

Example.
(a) Take $\gamma \in (0,1)$ and $\alpha \geq 0$. Putting $\beta = \arctan(\alpha)$, the function $a_1(x) = x^\gamma \cos(\beta x)$ fulfills \[1\] for the (WDC). On the other hand, if $\gamma \in (1,2)$, then $a_1$ becomes an example for the (SDC);

(b) For each $\theta \in (0,1)$, the function $a_2(x) = x^\theta - x$ satisfies \[1\] for the (WDC). However, if $\theta \in (1,2)$, then $a_3(x) = x^\theta + x$ satisfies \[1\] for the (SDC).

It is well known that the null-controllability for \[(1.1)\] is a consequence of an Observability inequality which in turn is a consequence of a Carleman Estimate for the following adjoint system associated to \[(1.1)\]

\[
\begin{cases}
  v_t + (a(x)v_x)_x + c(t,x)v = F, & (t,x) \in Q, \\
  v(t,0) = 0, & t \in (0,T), \\
  (av_x)(t,0) = 0, & t \in (0,T), \\
  v(T,x) = v_T(x), & x \in (0,1),
\end{cases}
\]

where $F \in L^2(Q)$ and $v_T \in L^2(0,1)$.

Alabau-Boussouira et al. obtained a Carleman inequality to \[(1.4)\] in \[3\] and proved null-controllability results to the linear and semilinear problems. However, their Carleman inequality can not be used to proved other controllability results with the same kind of degeneracy, for instance Stalkelberg-Nash null controllability or null-controllability for nonlinear problems.

Araruna et al. \[2\] proved a new Carleman estimate to \[(1.4)\] when $a(x) = x^\alpha$, $\alpha \in (0,2)$ and proved a Stalkelberg-Nash null controllability result. In order to do that they choose a suitable weight function that change of sign inside the control domain. Following this ideas, in \[3,4\], the authors extended their Carleman Inequality for a general $a = a(x)$ satisfying hypotheses \[(5.3)\], but just for the weak case, and proved a null-controllability result for a degenerate problem with nonlocal nonlinearities. The aim of the present work is extend the Carleman inequality proved in \[3\] to the strong case.

In order to state our main result let us consider $\omega' = (\alpha', \beta') \subset \omega$ and $\psi \in C^2([0,1]; \mathbb{R})$ satisfying

\[
\psi(x) := \begin{cases} 
\int_0^x \frac{y}{a(y)} \, dy, & x \in [0, \alpha') \\
\int_{\beta'}^x \frac{y}{a(y)} \, dy, & x \in [\beta', 1].
\end{cases}
\]

Setting

\[
\theta(t) := \frac{1}{t(T-t)^{\alpha}}, \quad \eta(x) := e^{\lambda |\psi|_\infty + \psi}, \quad \sigma(x,t) := \theta(t) \eta(x) \quad \text{and} \quad \varphi(x,t) := \theta(t)(e^{\lambda |\psi|_\infty + \psi} - e^{\lambda |\psi|_\infty}),
\]

where $(t,x) \in (0,T) \times [0,1]$ and $\lambda > 0$.

**Theorem 1.1** (Carleman Inequality). There exist $C > 0$ and $\lambda_0, s_0 > 0$ such that every solution $v$ of \[(1.4)\] satisfies, for all $s \geq s_0$ and $\lambda \geq \lambda_0$,

\[
\int_0^T \int_0^1 e^{2s\varphi} \left((s\lambda)\sigma v_x^2 + (s\lambda)^{5/3} \sigma^{5/3} v^2\right) \leq C \left(\int_0^T \int_0^1 e^{2s\varphi} |F|^2 + \lambda s^3 \int_0^T \int_0^1 e^{2s\varphi} \sigma^{3/2} v^2\right),
\]

where the constants $C, \lambda_0, s_0$ only depend on $\omega, a, \|c\|_{L^\infty(Q)}$ and $T$.

2. Preliminary Results

In this section we will state some notations and results which are necessary to prove Theorem 1.1. At first, we need to introduce some weighted spaces related to the function $a$, namely
Definition 2.1 (Weighted Sobolev spaces). Let us consider a real function \( a = a(x) \) as in Hypotheses A.

(I) For the (WDC), we set
\[
H_1^a := \{ u \in L^2(0,1); \ u \text{ is absolutely continuous in } [0,1], \ \sqrt{a}u_x \in L^2(0,1) \text{ and } u(1) = u(0) = 0 \},
\]
equipped with the natural norm
\[
\|u\|_{H_1^a} := \left( \|u\|_{L^2(0,1)}^2 + \|\sqrt{a}u_x\|_{L^2(0,1)}^2 \right)^{1/2}.
\]

(II) For the (SDC),
\[
H_1^a := \{ u \in L^2(0,1); \ u \text{ is absolutely continuous in } (0,1), \ \sqrt{a}u_x \in L^2(0,1) \text{ and } u(1) = 0 \},
\]
and the norm keeps the same;

(III) In both situations, the (WDC) and the (SDC),
\[
H_2^a := \{ u \in H_1^a; \ au_x \in H^1(0,1) \}
\]
with the norm \( \|u\|_{H_2^a} := \left( \|u\|_{H_1^a}^2 + \|(au_x)_x\|_{L^2(0,1)}^2 \right)^{1/2} \).

Alabau-Boussouira at al. in [1] introduced and studied some of the main properties of these spaces. Now, we will state a Hardy-Poincaré type inequality, whose proof can be found in [1]. It represents a powerful estimate in order to hand the degeneracy of the function \( a \), related to (1.1).

Proposition 2.2 (Hardy-Poincaré Inequality). Let \( \tilde{a} : [0,1] \longrightarrow \mathbb{R} \) be a continuous function such that \( \tilde{a}(0) = 0 \) and \( \tilde{a} > 0 \) in \( (0,1] \). The following statements hold:

(a) If there exists \( \theta \in (0,1) \) such that the function \( x \mapsto a(x)/x^\theta \) is nonincreasing in \( (0,1] \), then there exists a constant \( C_H > 0 \) such that
\[
\int_0^1 \frac{a(x)}{x^2} w^2(x) \leq C_H \int_0^1 a(x)|w'(x)|^2,
\]
for any real function \( w \) that is locally absolutely continuous on \( (0,1] \), continuous at 0, and satisfies
\[
w(0) = 0 \text{ and } \int_0^1 a(x)|w'(x)|^2 < +\infty.
\]

(b) If there exists \( \theta \in (1,2) \) such that the function \( x \mapsto a(x)/x^\theta \) is nondecreasing in a neighborhood of \( x = 0 \), then there exists a constant \( C_H > 0 \) such that (2.1) is valid for any function \( w \) that is locally absolutely continuous in \( (0,1] \), and satisfies
\[
w(1) = 0 \text{ and } \int_0^1 a(x)|w'|^2 < +\infty.
\]

Remark 2.3. Notice that Hypothesis A.1 implies some other useful conditions:
Then, from (3.1), we obtain the same. However, for the sake of convenience, we will reproduce the entire proof here. Proposition 3.1. In the Strong case, the proof is essentially the same, we just pay attention to the case in Proposition 2.4. Particularly for the (SDC), the assumption means that the function \( x \mapsto \frac{a(x)}{\sqrt{a(x)}} \) is nondecreasing. The wellposedness of (1.1), established in [1], is the following:

\[ u \in \mathcal{U} := \mathcal{H}^1(0,T; L^2(0,1)) \cap L^2(0,T; H^2_a) \cap C^0([0,T]; H^1_a), \]

and there exists a positive constant \( C_T \) such that

\[ \sup_{t \in [0,T]} \left( \|u(t)\|_{H^1_a}^2 \right) + \int_0^T \left( \|u(t)\|_{L^2(Q)}^2 + \|(au_x)_x\|_{L^2(Q)}^2 \right) \leq C_T \left( \|u_0\|_{H^1_a}^2 + \|h\|_{L^2(Q)}^2 \right) \]  

(2.2)

3. Proof of Theorem 1.1

We start proving a Carleman inequality for the following problem

\[
\begin{cases}
   v_t + (a(x)v)_x = h(t,x), & (t,x) \in Q, \\
   v(t,1) = 0, & t \in (0,T), \\
   v(t,0) = 0, & (Weak), t \in (0,T), \\
   (av_x)(t,0) = 0, & (Strong)
\end{cases}
\]  

(3.1)

Proposition 3.1. There exist \( C > 0 \) and \( \lambda_0 > 0 \) such that every solution \( v \) of (3.1) satisfies, for all \( s \geq s_0 \) and \( \lambda \geq \lambda_0 \),

\[
\int_0^T \int_0^1 e^{2\sigma \varphi} \left( (s\lambda)\sigma a v_x^2 + (s\lambda)^{5/3} \sigma^{5/3} v^2 \right) \leq C \left( \int_0^T \int_0^1 e^{2\sigma \varphi} |h|^2 + (s\lambda)^3 \int_0^T \int_0^1 e^{2\sigma \varphi} |\sigma v|^2 \right)
\]  

(3.2)

We just need to prove this proposition for the Strong case, since the inequality (3.2) is a consequence of that proved in [3]. In the Strong case, the proof is essentially the same, we just pay attention to the case in which \( K = 1 \). Actually, we just have to present a new proof of Lemma A.9 of [3], the other lemmas remain the same. However, for the sake of convenience, we will reproduce the entire proof here.

The proof of Proposition 3.1 relies on the change of variables \( w = e^{s\varphi} v \). Notice that

\[
\begin{align*}
   v_t &= e^{-s\varphi} (-s\varphi_tw + w), \\
   (av_x)_x &= e^{-s\varphi} (s^2\varphi_x^2 aw - s(aw_x)_x w - 2swaw_xw_x + (aw_x)_x).
\end{align*}
\]

Then, from (3.1), we obtain

\[
\begin{cases}
   L^+ w + L^- w = e^{s\varphi} h, & (t,x) \in Q, \\
   w(t,1) = 0, & t \in (0,T), \\
   w(t,0) = 0, & (Weak), t \in (0,T), \\
   (aw_x)(t,0) = 0, & (Strong) \\
   w(x,0) = w(x,T) = 0, & x \in (0,1),
\end{cases}
\]

where

\[
L^+ w := -s\varphi_tw + s^2\varphi_x^2 aw + (aw_x)_x,
\]
\[ L^-w := w_t - s(a \varphi_x)xw - 2sa \varphi_xw_x. \]

In this way,
\[ \|L^+w\|^2 + \|L^-w\|^2 + 2(L^+w, L^-w) = \|e^{x \varphi h}\|^2, \]
where \( \| \cdot \| \) and \((\cdot, \cdot)\) denote the norm and the inner product in \(L^2(Q)\), respectively.

From now on, we will prove Lemmas 3.3–3.11. The proof of Proposition 3.11 will be a consequence of these lemmas.

**Lemma 3.2.**
\[
(L^+w, L^-w) = \frac{s}{2} \int_0^T \int_0^1 \varphi_{tt}w^2 - 2s^2 \int_0^T \int_0^1 \varphi_{tx}a\varphi_xw^2 + s^3 \int_0^T \int_0^1 a\varphi_x(a\varphi_x^2)xw^2
\]
\[
+ s \int_0^T \int_0^1 (a\varphi_x)_{xx}aww_x + 2s \int_0^T \int_0^1 (a\varphi_x)_{x}aw_x^2
\]
\[
- s \int_0^T \int_0^1 a\varphi_xaw_x^2 - s \int_0^T (a^2\varphi_x^2w_x^2)|_{x=0}^{x=1}
\]

**Proof.** From the definition of \(L^+w\) and \(L^-w\) we have
\[
(L^+w, L^-w) = \int_0^T \int_0^1 (-s\varphi_{tt}w + s^2\varphi_{x}^2aw + (aw)_xw_t + s^2 \int_0^1 \varphi_{tx}(a\varphi_x)xw + 2a\varphi_xw_x)
\]
\[
- s \int_0^T \int_0^1 \varphi_{tx}a\varphi_xw^2 + 2s \int_0^T \int_0^1 (a\varphi_x)_{x}aw_x^2
\]
\[
= I_1 + I_2 + I_3 + I_4.
\]

Integrating by parts, we obtain
\[
I_1 = \frac{s}{2} \int_0^T \int_0^1 (\varphi_{tt} - 2sa\varphi_x\varphi_{xt})w^2,
\]
\[
I_2 = -s^2 \int_0^T \int_0^1 \varphi_{tx}a\varphi_xw^2,
\]
\[
I_3 = s^3 \int_0^T \int_0^1 a\varphi_x(a\varphi_x^2)xw^2
\]
and
\[
I_4 = s \int_0^T \int_0^1 (a\varphi_x)_{xx}aww_x + 2s \int_0^T \int_0^1 (a\varphi_x)_xaw_x^2 - s \int_0^T \int_0^1 (a\varphi_x)_xaw_x^2 - s \int_0^T (a^2\varphi_x^2w_x^2)|_{x=0}^{x=1},
\]
which imply the desired result.

**Lemma 3.3.** \(-s \int_0^T a^2\varphi_x^2w_x^2|_{x=0}^{x=1} \geq 0\)

**Proof.** Since \(\psi'(x) = x/a\), if \(x \in [0, \alpha']\) and \(\psi'(x) = -x/a\), if \(x \in (\beta', 1]\), we have
\[
- s \int_0^T a^2\varphi_x^2w_x^2|_{x=0}^{x=1} = -s\lambda \int_0^T a^2\psi'\sigma w_x^2|_{x=0}^{x=1} \geq 0.
\]
Lemma 3.4.

\[ s^3 \int_0^T \int_0^1 a_x(a_x^2) \, x \, w^2 \geq C \lambda^4 s^3 \int_0^T \int_0^1 a^2 |\psi'|^4 \sigma^3 w^2 - C s^3 \lambda^3 \int_0^T \int_{\omega'} \sigma^3 w^2 + C s^3 \lambda^3 \int_0^T \int_0^1 \sigma^3 w^2. \]

**Proof.** Firstly, we observe that

\[ s^3 \int_0^T \int_0^1 a_x(a_x^2) \, x \, w^2 = s^3 \lambda^3 \int_0^T \int_0^1 a \psi'(a \psi')^2 \, x \, \sigma^3 w^2 + 2 s^3 \lambda^4 \int_0^T \int_0^1 a^2 \psi'^4 \sigma^3 w^2 \]

We can see that

\[ a \psi'(a \psi')^2 \, x = \left\{ \begin{array}{ll} \frac{a^2}{2}(2a - xa'), & x \in (0, a') \\ -\frac{a^2}{2}(2a - xa'), & x \in (b', 1), \end{array} \right. \]

and \[(1.2) \] implies \( 2a - xa' \geq (2 - K)a. \) Hence,

\[ I_1 = s^3 \lambda^3 \int_0^T \int_0^{a'} a \psi'(a \psi')^2 \, x \, \sigma^3 w^2 + s^3 \lambda^3 \int_0^T \int_{\omega'} a \psi'(a \psi')^2 \, x \, \sigma^3 w^2 + s^3 \lambda^3 \left( \int_0^T \int_{\omega'} a \psi'(a \psi')^2 \, x \, \sigma^3 w^2 - C(2 - K)s^3 \lambda^3 \right) \int_0^T \int_0^1 a^2 |\psi'|^4 \sigma^3 w^2 \]

We just sum \( I_1 \) and \( I_2, \) and take \( \lambda_0 \) large enough to obtain the desired inequality.

\[ \square \]

Lemma 3.5.

\[ 2s \int_0^T \int_0^1 (a_x^2) \, a w^2 \geq -C \int_0^T \int_{\omega'} \sigma w^2 + C s \lambda^2 \int_0^T \int_0^1 a^2 (\psi')^2 \sigma w^2 + 2s \lambda \int_0^T \int_0^1 a \sigma w^2. \]

**Proof.** Observe that

\[ 2s \int_0^T \int_0^1 (a_x^2) \, a w^2 = 2s \int_0^T \int_0^1 a \sigma (a \psi') \, a \sigma w^2 + 2s \lambda \int_0^T \int_0^1 a^2 (\psi')^2 \sigma w^2 \]

Proceeding as in lemma before, we split the first integral over the intervals \([0, a'] \omega' \) and \([b', 1]. \) Since \( a^2 (\psi')^2 \geq Ca \) in \([b', 1] \) we can add the integral over \([b', 1] \) to the last integral of \( (3.2), \) which gives us the result.

\[ \square \]

Lemma 3.6.

\[ -2s^2 \int_0^T \int_0^1 \varphi_{xx} \varphi_{x} w^2 \geq -C s^2 \lambda^2 \left( \int_0^T \int_0^{a'} \frac{x^2}{a} \sigma^3 w^2 + \int_0^T \int_{\omega'} \sigma^3 w^2 + \int_0^T \int_0^1 a^2 |\psi'|^4 \sigma^3 w^2 \right) \]

**Proof.** First of all,

\[ \left| 2s^2 \int_0^T \int_0^1 \varphi_{xx} \varphi_{x} w^2 \right| \leq 2s^2 \lambda^2 \int_0^T \int_0^1 a |\psi'|^2 |\theta'| |\eta|^2 w^2 \leq C s^2 \lambda^2 \int_0^T \int_0^1 a |\psi'|^2 \sigma^3 w^2 \]

As before, we split the last integral over the intervals \([0, a'], \omega' \) and \([b', 1]. \) The result comes from the boundedness of \( a |\psi'|^2 \) in \( \omega' \) and from relations \( \psi' = x/a \) in \([0, a'] \) and \( a |\psi'|^2 \leq Ca^2 |\psi'|^4 \) in \([b', 1]. \)

\[ \square \]
Lemma 3.7.

$$-s \int_0^T \int_0^1 a \varphi_x a_x w_x^2 \geq -K \lambda s \int_0^T \int_0^1 a \sigma w_x^2 - c \lambda s \int_0^T \int_\omega \sigma w_x^2$$

Proof. In fact, from the definition of \( \psi \), we obtain

$$-s \int_0^T \int_0^1 a \varphi_x a_x w_x^2 = -s \lambda \int_0^T \int_0^1 a a_x \psi \sigma w_x^2$$

$$\geq -K s \lambda \int_0^T \int_\omega' a \sigma w_x^2 - C \lambda \int_0^T \int_\omega \sigma w_x^2,$$

where we proceeded as in the proof of Lemma 3.6.

Lemma 3.8.

$$s \int_0^T \int_0^1 (a \varphi_x)_{xx} a w_x w \geq -C s^2 \lambda^3 \int_0^T \int_0^1 a^2 |\psi'|^4 \sigma^3 w^2 - C \lambda^2 \int_0^T \int_0^1 a^2|\psi'|^2 \sigma w_x^2$$

$$- C s^2 \lambda^3 \int_0^T \int_\omega' \sigma^3 w^2 - C \lambda \int_0^T \int_\omega a \sigma w_x^2$$

$$- C s^2 \lambda^3 \int_0^T \int_\omega a \sigma w_x^2$$

Proof.

$$s \int_0^T \int_0^1 (a \varphi_x)_{xx} a w_x w = s \lambda \int_0^T \int_\omega (a \psi')_{xx} a \sigma w_x w + 2s \lambda^2 \int_0^T \int_0^1 (a \psi')_{xx} a \sigma w_x w$$

$$+ s \lambda^2 \int_0^T \int_0^1 a^2 \psi'' \psi \sigma w_x w + s \lambda^3 \int_0^T \int_0^1 a^2 (\psi')^3 \sigma w_x w$$

$$= I_1 + I_2 + I_3 + I_4.$$

The inequality will be obtained by estimating each one of these four integrals. For \( I_1 \), we have

$$|I_1| = \left| s \lambda \int_0^T \int_\omega (a \psi')_{xx} a \sigma w_x w \right| \leq C s \lambda \int_0^T \int_\omega \sigma^2 |w_x w|$$

$$= C s \lambda \int_0^T \int_\omega \sigma^{3/2} |w| |w^{1/2}| |w_x| \leq C s \lambda \int_0^T \int_\omega \sigma^3 w^2 + C s \lambda \int_0^T \int_\omega \sigma w_x^2.$$
For $I_3$, since $a' \geq 0$, for $x \in [0, a'] \cup [\beta', 1]$, we observe that
\[
|a^2 \psi''| = |x \left( \frac{a - xa'}{a} \right) | \leq |1 - \frac{xa'}{a}| \leq x \left( 1 + \frac{xa'}{a} \right) \leq x(1 + k).
\]
Hence, using again that $\sigma \leq C\sigma^2$, we get
\[
|I_3| \leq s\lambda^2 \int_0^T \int_0^{\alpha'} x \left( \frac{a - xa'}{a} \right) |\sigma w_x| + C s\lambda^2 \int_0^T \int_{\alpha'}^{x} \sigma^2 |w_x| \\
+ s\lambda^2 \int_0^T \int_{\beta'}^{1} x \left( \frac{a - xa'}{a} \right) |\sigma w_x| \\
\leq C s\lambda^2 \int_0^T \int_0^{\alpha'} x \sigma^2 |w_x| + C s\lambda^2 \int_0^T \int_{\alpha'}^{x} \sigma^2 |w_x| + C s\lambda^2 \int_0^T \int_{\beta'}^{1} x \sigma |w_x|.
\]
So, we get the same estimate for $I_2$. Finally,
\[
|I_4| \leq \int_0^T \int_0^1 s\lambda^2 a(x')^2 \sigma^{3/2} w |\lambda a \psi' \sigma^{1/2} w_x| \leq C s^2 \lambda^4 \int_0^T \int_0^1 a^2 |\psi'|^4 \sigma^3 w^2 + C \lambda^2 \int_0^T \int_0^1 a^2 |\psi'| \sigma w_x^2,
\]
and the proof is complete.

**Lemma 3.9.**
\[
\frac{s}{2} \int_0^T \int_0^1 \varphi_{tt} w^2 \leq - C s \int_0^T \int_0^{\alpha'} \sigma a w_x^2 - C s \lambda^2 \int_0^T \int_0^{\alpha'} \frac{x^2}{a} \sigma^3 w^2 - C s \int_0^T \int_{\alpha'}^{x} \sigma w_x^2 \\
- C \lambda^2 s^2 \int_0^T \int_{\alpha'}^{x} \sigma^3 w^2 - C s \int_0^T \int_0^{1} a^2 |\psi'|^2 \sigma w_x^2 - C s \lambda^2 \int_0^T \int_0^{1} a^2 |\psi'|^4 \sigma^3 w^2
\]

**Proof.** Firstly, since $|\varphi_{tt}| \leq C\sigma^{3/2}$, we have that
\[
\left| \frac{s}{2} \int_0^T \int_0^1 \varphi_{tt} w^2 \right| \leq C s \int_0^T \int_0^1 \sigma^{3/2} w^2.
\]
Therefore, we just need to bound this last integral. To do that, we will treat two separately cases, $K \neq 1$ and $K = 1$.

For $K \neq 1$, we apply Hardy-Poincaré inequality, to take
\[
\int_0^T \int_0^1 \sigma^{3/2} w^2 \leq \int_0^T \int_0^1 \left( \sigma^{1/2} \frac{\sqrt{a}}{x} w \right) \left( \sigma \frac{\sqrt{a}}{x} w \right) \leq \int_0^T \int_0^1 \sigma a w_x^2 + \int_0^T \int_0^1 \sigma \frac{x^2}{a} w^2 \\
\leq \int_0^T \int_0^1 \sigma a w_x^2 + \int_0^T \int_0^1 \frac{x^2}{a} w^2.
\]
Again, the two last intervals can be decomposed in $[0, a']$, $\omega$ and $[\beta', 1]$. At this point, relations
\[
a \leq C a^2 |\psi'|^2 \quad \text{and} \quad \frac{x^2}{a} \leq C a^2 |\psi'|^4, \quad \text{in} \quad [\beta', 1],
\]
give us the result.

For $K = 1$, Hardy-Poincaré inequality is not valid, since assumption (13) does not give us $\theta \in (1, 2)$ required in hypothesis in Proposition 2.2. Therefore, we will define a function $p = p(x)$ which the Hardy-Poincaré inequality holds.

Indeed, define $p(x) := (a(x) x^4)^{1/3}$ and let $\theta \in (0, 1)$ given by (13). If we take $q = \frac{4+\theta}{2}$, we can see that $q \in (1, 2)$ and the function $x \mapsto (p(x)/x^q)$ is nondecreasing in a neighborhood of $x = 0$, hence $p = p(x)$ satisfies the conditions of Proposition 2.2.
Let \( \eta^* = \max_{x \in [0,1]} \eta(x) \), since \( \sigma(t, x) = \theta(t) \eta(x) \), \( p(x) \leq Ca(x) \) and \( \eta(x) \geq 1 \) for all \( x \in [0,1] \), we have that

\[
\int_0^1 \left( \frac{a}{x^2} \right)^{1/3} \sigma w^2 \leq \eta^* \theta(t) \int_0^1 \left( \frac{a}{x^2} \right)^{1/3} w^2 = \eta^* \theta(t) \int_0^1 \frac{p}{x^2} w^2 \leq C \eta^* \theta(t) \int_0^1 p w^2 \leq C \int_0^1 \sigma aw^2).
\]

From this inequality and using Hölder and Young inequalities for \( p = 4/3 \) and \( q = 4 \), we finally obtain that

\[
\int_0^T \int_0^1 \sigma^{3/2} w^2 = \int_0^T \int_0^1 \left( \frac{a^{1/3}}{x^{2/3}} \sigma^{3/4} w^{3/2} \right) \left( \frac{x^{1/2}}{a^{1/4}} \sigma^{3/4} w^{1/2} \right)
\leq \left( \int_0^T \int_0^1 \left( \frac{a^{1/3}}{x^{2/3}} \sigma w^2 \right)^{3/4} \right)^{3/4}
\leq \left( \int_0^T \int_0^1 \frac{a}{x^2} \sigma w^2 \right)^{3/4} \left( \int_0^T \int_0^1 \frac{x^2}{a} \sigma w^2 \right)^{1/4}
\leq C \left( \int_0^T \int_0^1 \sigma aw^2 + \int_0^T \int_0^1 \frac{x^2}{a} \sigma w^2 \right),
\]

where this last two integral are the same obtained in the case \( K \neq 1 \). \( \square \)

**Lemma 3.10.**

\[
s^3 \lambda^3 \int_0^T \int_0^{\frac{C}{a} \sigma^3 w^2} + \lambda \int_0^T \int_0^{\sigma w^2} + s^3 \lambda^2 \int_0^T \int_0^T a^2 |\psi|^4 \sigma w^2 + s \lambda \int_0^T \int_0^T a^2 |\psi|^2 \sigma w^2 \leq C \left( \int_0^T \int_0^T e^{2s\varphi} |h|^2 + s^3 \lambda^3 \int_0^T \int_0^T \sigma w^2 + \lambda \int_0^T \int_0^T \sigma w^2 \right)
\]

**Proof.** From Lemmas 3.2, 3.9 we have

\[
(L^+ w, L^- w) \geq C \left( s^3 \lambda^3 \int_0^T \int_0^{\frac{C}{a} \sigma^3 w^2} + \lambda \int_0^T \int_0^{\sigma w^2} + \lambda^4 s^3 \int_0^T \int_0^T a^2 |\psi|^4 \sigma w^2 + s \lambda^2 \int_0^T \int_0^T a^2 |\psi|^2 \sigma w^2 \right).
\]

Hence,

\[
C \left( s^3 \lambda^3 \int_0^T \int_0^{\frac{C}{a} \sigma^3 w^2} + \lambda \int_0^T \int_0^{\sigma w^2} + \lambda^4 s^3 \int_0^T \int_0^T a^2 |\psi|^4 \sigma w^2 + s \lambda^2 \int_0^T \int_0^T a^2 |\psi|^2 \sigma w^2 \right) \leq \|L^+ w\|^2 + \|L^- w\|^2 + 2(L^+ w, L^- w) \leq \|e^{s\varphi} h\|^2,
\]

following the result. \( \square \)

Now, we intend to prove a suitable inequality which will imply Proposition 3.1. In order to do that, we recall that \( v = e^{-s\varphi} w \).
Lemma 3.11.

\[ s^3 \lambda^3 \int_0^T \int_0^{\sigma'} e^{2\sigma v x^2 / a} \sigma^3 a^2 + s \lambda \int_0^T \int_0^{\sigma'} e^{2\sigma v \sigma^2} + s^3 \lambda^4 \int_0^T \int_0^{\sigma'} e^{2\sigma v a^2} \sigma^3 v^2 + s \lambda^2 \int_0^T \int_0^{\sigma'} e^{2\sigma v a^2} |\psi'|^4 \sigma^2 v^2 \]

\[ + s^3 \lambda^4 \int_0^T \int_0^{\sigma'} e^{2\sigma v a^2} |\psi'|^4 \sigma^2 v^2 + s \lambda^2 \int_0^T \int_0^{\sigma'} e^{2\sigma v a^2} |\psi'|^4 \sigma^2 v^2 \]

\[ \leq C \left( \int_0^T \int_0^{\sigma'} e^{2\sigma v |h|^2} + \lambda s^3 \lambda^3 \int_0^T \int_0^{\sigma'} e^{2\sigma v |h|^2} + \lambda s^2 \int_0^T \int_0^{\sigma'} \sigma^2 x^2 \right) \]

Proof. Since \( v = e^{-\kappa v} w \), we have

\[ e^{\kappa v} v_x = -s \lambda |\psi| \sigma x + w_x \]

which implies

\[ e^{2\kappa v} s \lambda^2 |\psi'|^2 \sigma^2 v_x^2 = (s \lambda^2 |\psi'|^2 \sigma^2) e^{2\kappa v} v_x^2 \leq C(s \lambda^2 |\psi'|^2 \sigma^2) (s \lambda^2 |\psi'|^2 \sigma^2 w^2 + w_x^2) \]

\[ \leq C(s \lambda^4 |\psi'|^4 \sigma^2 a^2 w^2 + s \lambda^2 |\psi'|^2 \sigma^2 w_x^2) \]

Besides that,

\[ w_x = s \phi v \sigma v + e^{2\kappa v} v_x \Rightarrow w_x^2 \leq C(s \lambda^2 |\psi'|^2 \sigma^2 e^{2\kappa v} v_x^2 + e^{2\kappa v} v_x^2) \]

\[ \Rightarrow w_x^2 \leq C(s \lambda^2 |\psi'|^2 \sigma^2 e^{2\kappa v} v_x^2 + e^{2\kappa v} v_x^2), \text{ in } \omega' \]

Hence, from Lemma 3.11 we get

\[ s^3 \lambda^3 \int_0^T \int_0^{\sigma'} e^{2\sigma v x^2 / a} \sigma^3 v^2 + s \lambda \int_0^T \int_0^{\sigma'} e^{2\sigma v \sigma^2} + s^3 \lambda^4 \int_0^T \int_0^{\sigma'} e^{2\sigma v a^2} \sigma^3 v^2 \]

\[ + s \lambda^2 \int_0^T \int_0^{\sigma'} e^{2\sigma v a^2} |\psi'|^4 \sigma^2 v^2 \]

\[ \leq C \left( s^3 \lambda^3 \int_0^T \int_0^{\sigma'} e^{2\sigma v x^2 / a} \sigma^3 v^2 + s \lambda \int_0^T \int_0^{\sigma'} e^{2\sigma v \sigma^2} + s^3 \lambda^4 \int_0^T \int_0^{\sigma'} e^{2\sigma v a^2} \sigma^3 v^2 \right) \]

\[ \leq C \left( \int_0^T \int_0^{\sigma'} e^{2\sigma v |h|^2} + s \lambda^2 \int_0^T \int_0^{\sigma'} e^{2\sigma v a^2} \sigma^2 x^2 \right) \]

\[ \leq C \left( \int_0^T \int_0^{\sigma'} e^{2\sigma v |h|^2} + s \lambda^2 \int_0^T \int_0^{\sigma'} e^{2\sigma v a^2} \sigma^2 x^2 \right) \]

To complete the proof we will estimate the last integral of (3.4). Firstly, let us take \( \chi \in C_0^\infty(\omega) \) such that \( 0 \leq \chi \leq 1 \) and \( \chi \equiv 1 \) in \( \omega' \). Multiplying equation in (3.1) by \( \lambda s e^{2\sigma v \sigma^2 v^2} \) and integrating over \( Q \), we obtain

\[ \lambda s \int_0^T \int_0^{\sigma} e^{2\sigma v \sigma^2 v^2} + s \lambda s \int_0^T \int_0^{\sigma} e^{2\sigma v \sigma^2 v^2} + s \lambda s \int_0^T \int_0^{\sigma} e^{2\sigma v \sigma^2 v^2} + s \lambda s \int_0^T \int_0^{\sigma} e^{2\sigma v \sigma^2 v^2} \]

We can see that

\[ \left| \int_0^T \int_0^{\sigma} e^{2\sigma v \sigma^2 v^2} \right| = \left| \frac{1}{2} \int_0^T \int_0^{\sigma} e^{2\sigma v \sigma^2 v^2} \sigma v^2 \right| = \left| \frac{1}{2} \int_0^T \int_0^{\sigma} e^{2\sigma v \sigma^2 v^2} \sigma v^2 \right| \]

\[ \leq \left| \frac{1}{2} \int_0^T \int_0^{\sigma} e^{2\sigma v \sigma^2 v^2} \sigma v^2 \right| \leq C s \int_0^T \int_{\omega} e^{2\sigma v \sigma^2 v^2}. \]
And, analogously,
\[
\int_0^T \int_0^1 e^{2s\varphi} \sigma(\alpha v_x) v \chi = - \int_0^T \int_0^1 e^{2s\varphi} \sigma(\alpha v_x) \varphi - \int_0^T \int_0^1 (e^{2s\varphi} \sigma \chi) \alpha v_x v.
\]

Since \( \varphi_x \leq C \sigma \) and \( \sigma_x \leq C \sigma \) in \( Q_\omega \), we get
\[
\left| \int_0^T \int_0^1 (e^{2s\varphi} \sigma \chi) \alpha v_x v \right| \leq C \int_0^T \int_\omega e^{2s\varphi} \sigma^2 |\alpha v_x| |v|.
\] (3.7)

Now, from (3.6a)-(3.6d) we obtain
\[
\lambda s \int_0^T \int_\omega e^{2s\varphi} \sigma \alpha v_x^2 \leq \lambda s \int_0^T \int_0^1 e^{2s\varphi} \sigma \alpha v_x^2 \chi
\]
\[
\leq - \lambda s \int_0^T \int_0^1 e^{2s\varphi} \sigma (\alpha v_x) v \chi - \lambda s \int_0^T \int_0^1 (e^{2s\varphi} \sigma \chi) \alpha v_x v
\]
\[
\leq \lambda s \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma |v v_x | \chi + \lambda s \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma |h v | \chi + \lambda s \int_\omega \int_0^T \int_0^1 |(e^{2s\varphi} \sigma \chi) | |v v_x |
\]
\[
\leq C \lambda s^2 \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma^3 v^2 + \lambda s \int_\omega \int_0^T \int_0^1 (e^{s\varphi} h)(e^{s\varphi} \sigma v) + C \lambda s \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma^2 |v v_x |
\]
\[
\leq C \lambda^3 s^3 \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma^3 v + \frac{1}{2} \lambda s \int_\omega \int_0^T \int_0^1 e^{2s\varphi} h^2 + \frac{1}{2} \lambda s \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma^2 v^2
\]
\[
+ C \lambda s \int_\omega \int_0^T \int_0^1 (e^{s\varphi} a^{1/2} |v v_x |)(e^{s\varphi} a^{1/2} |\sigma^{3/2} |)
\]
\[
\leq C \lambda^3 s^3 \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma^3 v^2 + C \int_\omega \int_0^T \int_0^1 e^{2s\varphi} h^2
\]
\[
+ \varepsilon C \lambda s \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma a v_x^2 + C \varepsilon \int_\omega \int_0^T \int_0^1 e^{2s\varphi} a \sigma^3 v^2
\]
\[
\leq C \lambda^3 s^3 \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma^3 v^2 + C \int_\omega \int_0^T \int_0^1 e^{2s\varphi} h^2 + \varepsilon C \lambda s \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma a v_x^2.
\]

Hence, taking \( \varepsilon = 1/2C \), we get
\[
\lambda s \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma a v_x^2 \leq C \left( \int_\omega \int_0^T \int_0^1 e^{2s\varphi} h^2 + \lambda^3 s^3 \int_\omega \int_0^T \int_0^1 e^{2s\varphi} \sigma^3 v^2 \right).
\]

It last inequality combined with (3.4) completes the proof.

Now we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1.**

Let \( p, q > 1 \) and \( \beta := \frac{1}{p} + \frac{3}{q} \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). These numbers will be precise later depending on the case \( K \neq 1 \) or \( K = 1 \).

Using Hölder and Young inequalities, we have that
Now, let us estimate $I_1$ and $I_p$ taking into account the terms of the inequality given by Lemma 3.10. Splitting $I_1$ over the intervals $[0, a'], [\omega']$ and $[\beta', 1]$, and taking into account that $x^2/a$ is bounded in $\omega'$ and $x^2/a \leq a'\psi'/|a|$ in $[\beta', 1]$, we use Lemma 3.10 to obtain that

$$I_1 \leq s^3\lambda^3 \int_0^T \int_0^\alpha a\sigma^2 a^{-1/2}w^2 + C s^3\lambda^3 \int_0^T \int_{\alpha}^s \sigma^2 w^2 + C s^3\lambda^3 \int_0^T \int_0^s \sigma^2 a^2 |\psi'|^4 w^2$$

$$\leq C \left( \int_0^T \int_0^1 e^{2s^2} h^2 + s^3\lambda^3 \int_0^T \int_{\alpha}^s \sigma^2 w^2 + \lambda s \int_0^T \int_{\alpha}^s \sigma w_x^2 \right)$$

In order to estimate $I_p$, we will consider two cases $K \neq 1$ and $K = 1$. If $K \neq 1$, we choose $p = 2$ and we can apply Hardy-Poincaré inequality as following

$$I_p = s\lambda \int_0^T \int_0^1 a\sigma^2 (\sigma^2 w_x^2)^2 \leq Cs\lambda \int_0^T \int_0^1 a(\sigma^2 w_x^2)^2$$

$$= Cs\lambda \int_0^T \int_0^1 a \left( \frac{1}{2} \sigma^{-1/2} \sigma_x w + a^{1/2} w_x \right)^2$$

$$\leq Cs\lambda \int_0^T \int_0^1 a^{1/2} \sigma_x w^2 + Cs\lambda \int_0^T \int_0^1 a^{1/2} \sigma w_x^2$$

$$\leq Cs\lambda^3 \int_0^T \int_0^1 a^{1/2} \sigma^2 w^2 + Cs\lambda \int_0^T \int_0^1 a^{1/2} \sigma w_x^2$$

$$\leq Cs\lambda^3 \int_0^T \int_0^\alpha a\sigma w_x^2 + C s^3\lambda^3 \int_0^T \int_{\alpha}^s \sigma w_x^2 + C s^3\lambda^3 \int_0^T \int_0^s \sigma a^{1/2} |\psi'|^4 \sigma^2 w_x^2$$

$$+ C s\lambda \int_0^T \int_0^\alpha a^{1/2} \sigma w_x^2 + Cs\lambda \int_0^T \int_{\alpha}^s \sigma w_x^2 + Cs\lambda \int_0^T \int_0^s a^{1/2} |\psi'|^4 \sigma w_x^2$$

$$\leq C \left( \int_0^T \int_0^1 e^{2s^2} h^2 + s^3\lambda^3 \int_0^T \int_{\alpha}^s \sigma^2 w^2 + \lambda s \int_0^T \int_{\alpha}^s \sigma w_x^2 \right)$$

If $K = 1$, we will proceed as in Lemma 3.9 where have had to define a suitable function in order to apply Hardy-Poincaré inequality.

In this case, let us choose $p = 3/2$ and define $b(x) := \sqrt{a(x)}x$. Let $\theta \in (0, 1)$ given by (1.3). If we take $q = \frac{2}{3} + 1$, we can see that $q \in (1, 3/2)$ and the function $x \mapsto (b(x)/x^3)$ is nondecreasing in a neighborhood of $x = 0$, hence $b$ satisfies the conditions of Proposition 2.2.

Recalling that $\eta' = \max_{x \in [0, 1]} \eta(x)$, since $\sigma(t, x) = \theta(t)\eta(x)$, $b(x) \leq Ca(x)$ and $\eta(x) \geq 1$ for all $x \in [0, 1]$, we
have that
\[
\int_0^1 \left( \frac{a}{x^2} \right)^{p/q} |\sigma w|^2 \leq \int_0^1 \left( \frac{a}{x^2} \right)^{1/2} |\sigma w|^2 \leq \eta^* \theta(t) \int_0^1 \left( \frac{a}{x^2} \right)^{1/3} w^2 = \eta^* \theta(t) \int_0^1 \frac{b}{x^2} w^2 \leq C \eta^* \theta(t) \int_0^1 b w^2 \leq C \int_0^1 \sigma w_\beta^2.
\]
Hence,
\[
I_p = s\lambda \int_0^T \int_0^1 \sigma \left( \frac{a}{x^2} \right)^{1/2} w^2 \leq C \int_0^T \int_0^1 \sigma w_\beta^2,
\]
which is one of the integral obtained in the case \( p = 2 \).

Thus, note that for \( p = 2, \beta = 2 \) and for \( p = 3/2, \beta = 5/3 \). Therefore, in both cases, we have that
\[
(\lambda s)^{5/3} \int_0^T \int_0^1 e^{2s\varphi} \sigma^{5/3} v^{5/3} \leq (\lambda s)^{\beta} \int_0^T \int_0^1 e^{2s\varphi} \sigma^3 v^3
\]
\[
\leq I_1 + I_p \leq C \left( \int_0^T \int_\omega e^{2s\varphi} h^2 + \beta^3 \lambda s \int_0^T \int_\omega \sigma^3 w^2 + \lambda s \int_0^T \int_\omega \sigma w^2 \right).
\]
Proceeding exactly as in the proof of Lemma 3.11, we achieve
\[
(\lambda s)^{5/3} \int_0^T \int_0^1 e^{2s\varphi} \sigma^{5/3} v^{5/3} \leq C \left( \int_0^T \int_\omega e^{2s\varphi} h^2 + \lambda s^3 \int_0^T \int_\omega e^{2s\varphi} \sigma^3 v^2 \right),
\]
and the result given by Lemma 3.11 gives us
\[
s\lambda \int_0^T \int_0^1 e^{2s\varphi} a \sigma v_x^2 \leq s\lambda \int_0^T \int_0^1 e^{2s\varphi} a \sigma v_x^2 + s\lambda \int_0^T \int_\omega e^{2s\varphi} a \sigma v_x^2 + s\lambda \int_0^T \int_\beta' e^{2s\varphi} a |v'|^2 \sigma v^2_x
\]
\[
\leq C \left( \int_0^T \int_\omega e^{2s\varphi} h^2 + \lambda s^3 \int_0^T \int_\omega e^{2s\varphi} \sigma^3 v^2 \right).
\]
Therefore, this last two estimates conclude the proof of Proposition 3.1.

\[\square\]

**Proof of Theorem 7.4** If \( v \) is a solution of (1.4), then \( v \) is also a solution of (3.1) with \( h = F + cv \). In this case, applying Proposition 3.1 there exist \( C > 0, \lambda_0 > 0 \) and \( s_0 > 0 \) such that \( v \) satisfies, for all \( s \geq s_0 \) and \( \lambda \geq \lambda_0 \),
\[
\int_0^T \int_0^1 e^{2s\varphi} \left( (s\lambda) \sigma v_c^2 + (s\lambda)^{5/3} \sigma^{5/3} v^3 \right) \leq C \left( \int_0^T \int_0^1 e^{2s\varphi} |h|^2 + (s\lambda)^3 \int_0^T \int_\omega e^{2s\varphi} \sigma^3 v^2 \right).
\]
Recalling that \( c \in L^\infty(Q) \) and \( \sigma \geq C > 0 \), we can see that
\[
\int_0^T \int_0^1 e^{2s\varphi} |h|^2 = \int_0^T \int_0^1 e^{2s\varphi} |F + cv|^2
\]
\[
\leq C \int_0^T \int_0^1 e^{2s\varphi} |F|^2 + C \|c\|^2 \int_0^T \int_0^1 e^{2s\varphi} |v|^2
\]
\[
\leq C \int_0^T \int_0^1 e^{2s\varphi} |F|^2 + C \int_0^T \int_0^1 e^{2s\varphi} \sigma^{5/3} |v|^2.
\]
Therefore, taking \( \lambda_0 \) and \( s_0 \) large enough, the last integral can be absorbed by the left-hand side of (3.8), which complete the proof. \[\square\]
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