ON THE EXTENSION OF AXIALLY SYMMETRIC VOLUME FLOW AND MEAN CURVATURE FLOW

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ABSTRACT. We investigate conditions of singularity formation of mean curvature flow and volume preserving mean curvature flow in an axially symmetric setting. We prove that no singularities can develop during a finite time interval, if the mean curvature is bounded within that time interval on the entire surface. We prove this for volume preserving mean curvature flow as well as for mean curvature flow.

1. INTRODUCTION

Consider \(n\)-dimensional hypersurfaces \(M_t\), defined by a one parameter family of smooth immersions \(x_t : M^n \to \mathbb{R}^{n+1}\). The hypersurfaces \(M_t\) are said to move by mean curvature, if \(x_t = x(\cdot, t)\) satisfies

\[
\frac{d}{dt} x(l, t) = -H(l, t) \nu(l, t), \quad l \in M^n, t > 0.
\]

By \(\nu(l, t)\) we denote a smooth choice of unit normal on \(M_t\) at \(x(l, t)\) (outer normal in case of compact surfaces without boundary), and by \(H(l, t)\) the mean curvature with respect to this normal.

If the evolving compact surfaces \(M_t\) are assumed to enclose a prescribed volume \(V\) the evolution equation changes as follows:

\[
\frac{d}{dt} x(l, t) = - \left( H(l, t) - h(t) \right) \nu(l, t), \quad l \in M^n, t > 0,
\]

where \(h(t)\) is the average of the mean curvature,

\[
h(t) = \frac{\int_{M_t} H \, dg_t}{\int_{M_t} dg_t},
\]

and \(g_t\) denotes the metric on \(M_t\). As the initial surface we choose a compact \(n\)-dimensional hypersurface \(M_0\), with boundary \(\partial M_0 \neq \emptyset\). We assume \(M_0\) to be smoothly embedded in the domain

\[
G = \{ x \in \mathbb{R}^{n+1} : a < x_1 < b \}, \quad a, b > 0,
\]

and \(\partial M_0 \subset \partial G\). Here we have a free boundary. We consider an axially symmetric surface contained in the region \(G\) between the two parallel planes \(x_1 = a\) and \(x_1 = b\). Motivated by the fact that the stationary solution to the associated Euler Lagrange equation satisfies a Neumann boundary condition, we also assume the surface to meet the planes at right angles along its boundary. We consider the general question whether a singularity can develop if the mean curvature of a surface is bounded for a given time interval. Le and Sesum investigated related problems in [12]. In particular they proved that for mean curvature flow, if all singularities are of type I, the mean curvature blows

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up at the first singular time $T$.

This paper is organised as follows. In sections 3 and 4, we study volume preserving mean curvature flow and investigate whether a singularity could develop if $H$ is bounded. In sections 5 and 6 we study the same question for mean curvature flow. We prove the following two theorems.

**Theorem 1.1.** For $x(l, t) \in M_t$ satisfying (1.2), if $|H(l, t)| \leq c$ for all $l \in M^2$ and for all $t \in [0, T)$ then, there exists a constant $c'$, such that $|A|(l, t) \leq c'$ for $t \in [0, T)$, i.e. the flow can be extended past time $T$.

**Theorem 1.2.** For $x(l, t) \in M_t$ satisfying (1.1), if $|H(l, t)| \leq c$ for all $l \in M^2$ and for all $t \in [0, T)$ then, there exists a constant $c'$, such that $|A|(l, t) \leq c'$ for $t \in [0, T)$, i.e. the flow can be extended past time $T$.

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2. **Notation**

We use the same notation as in [11], which was based upon Huisken’s [9] and Athanassenas’ [1] notation in describing the 2-dimensional axially symmetric hypersurface. Let $\rho_0 : [a, b] \to \mathbb{R}$ be a smooth, positive function on the bounded interval $[a, b]$ with $\rho_0(a) = \rho_0(b) = 0$. Consider the 2-dimensional hypersurface $M_0$ in $\mathbb{R}^3$ generated by rotating the graph of $\rho_0$ about the $x_1$-axis. We evolve $M_0$ along its mean curvature vector keeping its enclosed volume constant subject to Neumann boundary conditions at $x_1 = a$ and $x_1 = b$. Equivalently, we could consider the evolution of a periodic surface defined along the whole $x_1$ axis. By definition the evolution preserves axial symmetry. The position vector $x$ of the hypersurface satisfies the evolution equation

$$\frac{d}{dt}x = -(H - h)\nu + H + h\nu,$$

where $H$ is the mean curvature vector. Since $\Delta x = H$, where $\Delta$ denotes the Laplacian on the surface, we obtain,

(2.1) $$\frac{d}{dt}x = \Delta x + h\nu.$$

Let $i_1, i_2, i_3$ be the standard basis in $\mathbb{R}^3$, corresponding to $x_1, x_2, x_3$ axes and $\tau_1(t), \tau_2(t)$ be a local orthonormal frame on $M_t$ such that

$$\langle \tau_2(t), i_1 \rangle = 0, \quad \text{and} \quad \langle \tau_1(t), i_1 \rangle > 0.$$

Let $\omega = \frac{x}{|x|} \in \mathbb{R}^3$ denote the unit outward normal to the cylinder intersecting $M_t$ at the point $x(l, t)$, where $\dot{x} = x - \langle x, i_1 \rangle i_1$. Let

$$y = \langle x, \omega \rangle \quad \text{and} \quad v = \langle \omega, \nu \rangle^{-1}.$$
Here $y$ is the height function. We call $v$ the gradient function. We note that $v$ is a geometric quantity, related to the inclination angle; in particular $v$ corresponds to $\sqrt{1 + \rho'^2}$ in the axially symmetric setting. The quantity $v$ has facilitated results such as gradient estimates in graphical situations (see for example [5, 6]).

We introduce the quantities (see also [9])

$$p = \langle \tau_1, i_1 \rangle y^{-1}, \quad q = \langle \nu, i_1 \rangle y^{-1},$$

so that

$$p^2 + q^2 = y^{-2}.$$  

The second fundamental form has one eigenvalue equal to $p = \frac{1}{\rho}\sqrt{1 + \rho'^2}$ and one eigenvalue equal to

$$k = \langle \nabla_1 \nu, \tau_1 \rangle = \frac{-\rho''}{(1 + \rho'^2)^{3/2}}.$$  

We note that $\rho(x_1, t)$ is the radius function such that $\rho: [a, b] \times [0, T) \to \mathbb{R}$, whereas $y(l, t)$ is the height function and $y: M^2 \times [0, T) \to \mathbb{R}$.

3. EVOLUTION EQUATIONS AND PRELIMINARY RESULTS - Volume Flow

**Lemma 3.1.** We have the following evolution equations:

(i) $\frac{d}{dt} y = \Delta y - \frac{1}{y} + hpy$;

(ii) $\frac{d}{dt} v = \Delta v - |A|^2 v + \frac{v}{y} - 2\frac{v}{y} |\nabla v|^2$;

(iii) $\frac{d}{dt} k = \Delta k + |A|^2 k - 2q^2 (k - p) - hk^2$;

(iv) $\frac{d}{dt} p = \Delta p + |A|^2 p + 2q^2 (k - p) - hp^2$;

(v) $\frac{d}{dt} H = \Delta H + (H - h)|A|^2$;

**Proof.** Evolution equations (i), (ii), (iii) and (iv) are proved in [3] Lemma 5.1 and (v) is proved in [8].

**Bounds on $h$.** Getting bounds for $h(t)$ is very important as $h(t)$ is a global term. This enables us to use the maximum principles. Athanassenas ([2] Proposition 1.4) has obtained bounds for $h(t)$ for an axially symmetric hypersurface between two parallel planes having orthonormal Neumann boundary data. The proof is of a geometric nature and works because of the axial symmetry. We state it here.

**Proposition 3.2. (Athanassenas)** Assume $\{M_t\}$ to be a family of smooth, rotationally symmetric surfaces, solving (1.2) for $t \in [0, T)$. Then the mean value $h$ of the mean curvature satisfies

$$0 < c_2 \leq h \leq c_3,$$

with $c_2$ and $c_3$ constants depending on the initial hypersurface $M_0$.

Next we prove a height dependent gradient estimate.

**Lemma 3.3.** There exists a constant $c_4$ depending only on the initial hypersurface, such that $vy < c_4$, independent of time.
Proof. We calculate from Lemma 3.1
\[
\frac{d}{dt}(yv - c_3 t) = \Delta(yv) - \frac{2}{v} \langle \nabla v, \nabla(yv) \rangle - yv|A|^2 + h - c_3.
\]
As \( h \leq c_3 \) we get by the parabolic maximum principle
\[
yv - c_3 t \leq \max_{M_0} yv,
\]
\[
yv \leq \max_{M_0} yv + c_3 T =: c_4.
\]
\( \square \)

**Proposition 3.4.** Assume \( M_t \) to be axially symmetric surfaces as described in Section 2 that evolve by (1.2). Then there is a constant \( c_1 \) depending only on the initial hypersurface, such that the principal curvatures \( k \) and \( p \) satisfy \( k/p < c_1 \), independent of time.

**Proof.** Similar to equation (19) of [9] we calculate from Lemma 3.1
\[
\frac{d}{dt} \left( \frac{k}{p} \right) = \Delta \left( \frac{k}{p} \right) + \frac{2}{p} \left\langle \nabla p, \nabla \left( \frac{k}{p} \right) \right\rangle + 2q^2 \frac{p}{p^2} (p - k) (p + k) + \frac{hk}{p} (p - k).
\]
If \( k/p \geq 1 \) then \( (p - k) < 0 \). By the parabolic maximum principle we obtain
\[
\frac{k}{p} \leq \max \left( 1, \max_{M_0} \frac{k}{p} \right) =: c_1.
\]
\( \square \)

In the next Proposition we will get bounds for \( \frac{|k|}{p} \) if the mean curvature is bounded for all \( t < T \).

**Proposition 3.5.** If there exists a constant \( C \) such that \( |H|(l, t) \leq C \) for all \( l \in M^2 \) and for all \( t < T \), then there exists another constant \( c_0 \) such that \( \frac{|k|}{p}(l, t) \leq c_0 \) for all \( l \in M^2 \) and for all \( t < T \).

**Proof.** As
\[
-C \leq H = k + p \leq C,
\]
and \( \frac{1}{p} = vy \leq c_4 \) from Lemma 3.3
\[
\frac{|k|}{p} \leq 1 + Cc_4 =: c_0.
\]
\( \square \)

We note that the above proposition holds for a hypersurface evolving by mean curvature or by volume preserving mean curvature.

4. RESCALED SURFACE - VOLUME FLOW

4.1. Rescaling procedure. We follow a rescaling technique similar to that of Huisken and Sirestrari used in [10]. Consider time intervals \( [0, T - \frac{1}{i}] \), for \( i \geq 1, i \in \mathbb{N} \), and determine the point \( l_i \in M^2 \) and the (latest in that interval) time \( t_i \), such that
\[
|A|(l_i, t_i) = \max_{t \leq T - \frac{1}{i}} |A|(l, t);
\]
\[
|A|(l, t) = \max_{l \in M^2} |A|(l, t).\]
as it is an axially symmetric hypersurface we choose \( l_i \in M^2 \), such that \( x(l_i, t_i) \) is on the \((x_1, x_3)\) plane. Let \( \alpha_i = |A|(l_i, t_i) \) and \( x_i = x(l_i, t_i) \). We consider the family of rescaled surfaces \( \mathcal{M}_{i, \tau} \) defined by the following immersions:

\[
\mathcal{M}_{i, \tau} = \left\{ \mathcal{M}_{i, \tau} \right\}_{\tau \in [0, T]}
\]

\[ \tilde{x}_i(\cdot, \tau) = \alpha_i \left( x(\cdot, \alpha_i^2 \tau + t_i) - (x_i, i_1) i_1 \right), \]

where \( \tau \in [-\alpha_i^2 t_i, \alpha_i^2 (T - t_i - \frac{1}{T})] \). Here \( t = \alpha_i^{-2} \tau + t_i \) where \( t \in [0, T - \frac{1}{T}] \).

**Figure 1.** The rescaling

Note that we rescale from a point on the axis of rotation corresponding to the maximum curvature \( |A|^2 \). Thus the rescaled surfaces \( \mathcal{M}_{i, \tau} \) satisfy axial symmetry and we denote by \( \tilde{\rho}_{i, \tau} \) their generating curves - the cross section of \( \mathcal{M}_{i, \tau} \) with the \((x_1, x_3)\) plane. We look at a sequence of rescaled surfaces at different times \( t_i \) where \( t_i \to T \). We denote by \( |\tilde{A}_i| \) and \( \tilde{H}_i \) the second fundamental form and the mean curvature associated with the immersion \( \tilde{x}_i \). By the definition of \( \tilde{x}_i \) we have

\[ \tilde{H}_i(\cdot, \tau) = \alpha_i^{-1} H(\cdot, \alpha_i^{-2} \tau + t_i), \quad \text{and} \quad |\tilde{A}_i|(\cdot, \tau) = \alpha_i^{-1} |A|(\cdot, \alpha_i^{-2} \tau + t_i). \]

For \( t \leq T - \frac{1}{T} \) from the definition of \( \alpha_i \) we have

\[ |A|(\cdot, \alpha_i^{-2} \tau + t_i) \leq |A|(l_i, t_i), \] that is \( \alpha_i^{-1} |A|(\cdot, \alpha_i^{-2} \tau + t_i) \leq 1 \).

We observe that \( \mathcal{M}_{i, \tau} \) flow by mean curvature, preserving the enclosed volume of the rescaled surface: To see this, let

\[ \tilde{h}_i = \frac{\int_{\mathcal{M}_{i, \tau}} \tilde{H}_i(\cdot, \tau) d\tilde{g}_\tau}{\int_{\mathcal{M}_{i, \tau}} d\tilde{g}_\tau}, \]

and note that the metric satisfies

\[ \sqrt{\det \tilde{g}_{ij}(\tau)} = \alpha_i^n \sqrt{\det g_{ij}(t)}, \]

where (with a slight abuse of notation) we denoted by \( \tilde{g}_\tau \) the metric on \( \mathcal{M}_{i, \tau} \), and by \( \tilde{g}_{ij}(\tau) \) the components of \( \tilde{g}_\tau \). Therefore

\[ \tilde{h}_i(\tau) = \frac{\int_{\mathcal{M}_{i, \tau}} \alpha_i^{-1} H(\cdot, \alpha_i^{-2} \tau + t_i) \alpha_i^n dt}{\int_{\mathcal{M}_{i, \tau}} \alpha_i^n dt}, \]

and finally

\[ \frac{d}{d\tau} \tilde{x}_i = \alpha_i \frac{dx}{dt} \frac{dt}{d\tau} = -\alpha_i^{-1} (H - h) \nu = -(\tilde{H}_i - \tilde{h}_i) \nu. \]
Now we prove Theorem 1.1.

Proof of Theorem 1.1. Assume a singularity develops at \( t = T \), i.e. \( |A|^2 \to \infty \) as \( t \to T \). We rescale the surface as discussed above in (4.2). As the rescaled flow satisfies (4.4), the curvature bound \( |\tilde{A}_i|^2 \leq 1 \) for any \( i \) implies analogous bounds on all its covariant derivatives as in Theorem 4.1 in [8]. On the rescaled surfaces we have the maximum curvature achieved at \( (l_i, 0) \) where \( l_i \in M^n \), which means \( \tilde{A}_i(l_i, 0) = 1 \) for all \( i \). As \( |H|(l, t) \) is bounded for all \( t < T \), we have \( \frac{|H|}{p} \leq c_0 \) on \( M_t \) by Lemma 3.5. In regards to the curvature we have:

\[
|A| = \sqrt{k^2 + p^2} \leq c_5 p \leq c_5 y^{-1},
\]

(4.5)

\[
|\tilde{A}_i| = \alpha_i^{-1} |A| \leq \alpha_i^{-1} c_5 y^{-1} = c_5 (\alpha_i y)^{-1} = c_5 \tilde{y}^{-1}.
\]

We will show that for a fixed \( \tau_0 \), a subsequence of the rescaled surfaces converges as \( i \) goes to infinity. We call the limiting hypersurface \( \mathcal{M}_{\tau_0} \). We will explain this process in detail.

Converging sequence of points. Without loss of generality let us assume that the singularity develops at the origin. We translate each generating curve to have \( \langle x(l_i, t_i), i_1 \rangle = 0 \). We rename them again with the same \( \rho \) and work with them for the rest of the section. We choose \( \bar{l} \in M^2 \) such that \( x(\bar{l}(t), t) \) is on the \((x_1, x_3)\) plane, and

\[
\tilde{l}(t) = \{ l \in M^2 : \langle x(l(t), t), i_1 \rangle = 0 \},
\]

We note that \( \bar{l}(t_i) = l_i \). For a fixed \( \tau_0 \) as \( i \) goes to infinity, \( t \) goes to \( T \) making the \( \max_{M_t} |A| \) on the corresponding original hypersurfaces go to infinity. Therefore on the rescaled hypersurfaces for a fixed \( \tau_0 \) we can find \( N_0 \in \mathbb{N} \) such that for \( i > N_0 \)

\[
|\tilde{A}_i|(\bar{l}(\alpha_i^{-2} \tau_0 + t_i), \tau) \geq \frac{1}{2}.
\]
As
\[ |\tilde{A}_i(\cdot, \tau_0)\tilde{y}(\cdot, \tau_0) \leq c_5, \text{ from (4.5)}, \]
we have
\[ \tilde{y}(\tilde{l}(\alpha_i^{-2}\tau_0 + t_i), \tau_0) \leq 2c_5 \text{ for } i > N_0. \]

For \( \tau = \tau_0 \) and \( i > N_0 \) we look at the points on the rescaled surface generating curves \( \tilde{\rho}_{i,\tau_0} \) which are on the \( x_3 \) axis. As
\[ 0 \leq \tilde{\rho}_i(\cdot, \tau_0)\bigg|_{x_1=0} \leq 2c_5, \]
we find a subsequence of points that converge. We call the subsequence of corresponding rescaled generating curves \( \tilde{\rho}_{1,\tau_0} \) and the limiting point on the \( x_3 \) axis \( c_* := \lim_{i \to \infty} \tilde{\rho}_{1,\tau_0} \bigg|_{x_1=0}. \)

**Converging sequence of tangent vectors.** Now we look at the unit tangent vectors of \( \tilde{\rho}_{1,\tau_0} \bigg|_{x_1=0}. \)

![Figure 3. The convergence of points.](image)

![Figure 4. The unit tangent vectors.](image)

We translate the unit tangent vectors to the origin (see figure 4). By identifying these unit vectors with points on the sphere, we observe that a subsequence of these points converges and so obtain a subsequence of tangent vectors that converges. By translating back each tangent vector to its original position, we find the corresponding subsequence of rescaled generating curves; we call it \( \tilde{\rho}_{2,\tau_0}^{2}. \) Now we have found a subsequence of rescaled generating curves, where the points and the tangent vectors converge at \( x_1 = 0. \)

**Convergence in a small ball.** As \( |\tilde{A}_i| \leq 1 \) for every rescaled hypersurface, we can roll a ball of radius 1 over the entire hypersurface in such a way that when it touches any point, it does not
Figure 5. The unit ball sitting on the rescaled hypersurface intersect the curve anywhere else. Thus we have a gradient bound in a small ball of radius $\delta < 1$ on the hypersurface. Now we look at the sequence of curves $\tilde{\rho}^2_{i,\tau_0}$ near $x_1 = 0$.

Figure 6. Convergence in a small ball

We rotate each curve around the point on the curve where $x_1 = 0$, so that the tangent vector at $x_1 = 0$ is parallel to the $x_1$ axis (see figure 6). For $|x_1| < \delta$, each of these rotated curves has its gradient bounded by the same constant $\left(\tilde{\rho}' \leq \frac{\delta}{\sqrt{1 - \delta^2}}\right)$. This gives equicontinuity. Also at $x_1 = 0$ the height of each curve is bounded by $2c_5$. Hence the height of all the rotated curves is bounded by $2c_5 + (1 - \sqrt{1 - \delta^2})$ for $|x_1| < \delta$ (see figure 5). Therefore our curves are uniformly bounded. By Arzela-Ascoli, there exists a subsequence that converges uniformly. Let us call this subsequence of rotated curves $g^3_{i,\tau_0}$.

Thus we have $C^0$ convergence for the sequence $g^3_{i,\tau_0}$ in the small ball $B_{(0,c_*)}(\delta)$, which is the ball of radius $\delta$ centred at $(0, c_*)$. We call the corresponding subsequence before rotation $\tilde{g}^3_{i,\tau_0}$. We differentiate the sequence $g^3_{i,\tau_0}$ and call it $g'^3_{i,\tau_0}$. We note that $|g'^3_{i,\tau_0}|$ is bounded when $|x_1| < \delta$. As the curvature of the rescaled surfaces is bounded we have $\frac{|g''^3_{i,\tau_0}|}{\left(1 + (g'^3)^2\right)^{\frac{3}{2}}} \leq 1$, giving uniform bounds on $g''^3$ for $|x_1| < \delta$. Thus we have uniform boundedness and equicontinuity for the sequence $g^3_{i,\tau_0}$. From Arzela-Ascoli there exists a subsequence that converges uniformly. Let us call it $g'^4_{i,\tau_0}$. We call the corresponding subsequence before differentiation $g^4_{i,\tau_0}$. From the Theorem of uniform
convergence and differentiability we have $C^1$ convergence for the sequence $g^4_{i,\tau_0}$ in the small ball $B(0,c_\ast)(\delta)$. We call the corresponding subsequence before rotation $\tilde{\rho}^4_{i,\tau_0}$.

We note that the subsequence before rotation $\tilde{\rho}^4_{i,\tau_0}$ converges as well, because the tangent vectors and the points at $x_1 = 0$ converge. Therefore the unrotated curves converge in the ball $B(0,c_\ast)(\delta)$ as well.

**Convergence in a compact set.** On the subsequence of rotated curves $g^4_{i,\tau_0}$ we find another set of points as follows. We mark the points on the rotated curves $g^4_{i,\tau_0}$ $\mid x_1 = \pm \epsilon$ where $\epsilon < \delta$ such that only a finite number of points are outside the ball $B(0,c_\ast)(\delta)$. Let us first look at the points $g^4_{i,\tau_0} \mid x_1 = -\epsilon$.

These new points and their tangent vectors converge on the unrotated curves $\tilde{\rho}^4_{i,\tau_0}$ as shown previously. We call these new points $p^5_i$ (see figure 7). Now we translate each curve $\tilde{\rho}^4_{i,\tau_0}$ along the $x_1$ such that the point $p^5_i$ is on the $x_3$ axis. Then we rotate each curve as before, around $p^5_i$ such that the tangent vector of each curve is parallel to the $x_1$ axis. For $|x_1| < \delta$, by Arzela-Ascoli, we find a subsequence of $g^4_{i,\tau_0}$ that converges in $C^0$. We call this subsequence $g^5_{i,\tau_0}$. As previously by considering the differentiated sequence $g^5_{i,\tau_0}$ we find another subsequence $g^6_{i,\tau_0}$ that convergence in $C^1$. We call the corresponding subsequence before rotation $\tilde{\rho}^6_{i,\tau_0}$. We note that the original subsequence $\tilde{\rho}^6_{i,\tau_0}$ converges.

To summarise, we first found the limiting curve in a small ball $B(0,c_\ast)(\delta)$. Then by considering a ball centred on the limiting curve in $B(0,c_\ast)(\delta)$, we found another subsequence that converged in both these balls (see figure 8). In this manner, by considering a diagonal subsequence, we can extend our limiting curve along small balls centred on the limiting curve. Therefore given any compact set $[\alpha, \beta] \times [0, \gamma]$ where $\gamma > 2c_5$, we repeat the above process, and find a subsequence uniformly converging in a union of small balls (see figure 9), which is a subset of the given compact set. This subsequence may not be unique. We get the associated hypersurface by rotating this curve around the $x_1$ axis.

We note that $\alpha_i$ goes to $\infty$ as $i$ goes to $\infty$. As $\tilde{H}_i = \alpha_i^{-1}H$ and as $|H(l,t)| \leq c$ for this theorem for all $l \in M^2$ and for all $t < T$, $\tilde{H}_i$ goes to 0, as $i$ goes to $\infty$. As $0 < c_2 \leq h \leq c_3$ and $\tilde{h}_i = \alpha_i^{-1}h$, we have $\tilde{h}_i(\tau_0)$ going to 0 as well. As $\frac{d}{dt} \tilde{x}_i = -(\tilde{H}_i - \tilde{h}_i)\tilde{\nu}_i$, we observe that
\[ \lim_{i \to \infty} \frac{d}{d\tau} \tilde{x}_i = 0. \] Thus \( M_{\tau_0} \) is a stationary solution and therefore independent of \( \tau \). We rename this stationary solution \( M \).

\textit{The contradiction.} The limiting solution \( M \) is a catenoid as it is the only axially symmetric minimal surface with zero mean curvature. However for large \( i \) if we rescale back \( M_{i,\tau_0} \), to get an understanding of the original surface, then we see that the estimate \( vy \leq c \) does not hold on that surface. Therefore as we will show in the next paragraph we get a contradiction.

We denote the quantities associated to the catenoid \( M \) by a hat \( \hat{\cdot} \). We obtain the catenoid \( M \) by rotating \( \hat{y} = c_5 \cosh(c_5^{-1} \hat{x}_1) \), around the \( x_1 \) axis where \( \hat{x}_1 \) is the \( x_1 \) coordinate of the limiting surface \( M \). As the convergence is \( C^1 \), for any \( \epsilon_1 > 0 \) and for any \( l_0 \in M^2 \) and for any fixed \( \tau_0 \) we have

\[
|\hat{v}(l_0)\hat{y}(l_0) - \tilde{v}_i(l_0, \tau_0)\hat{y}_i(l_0, \tau_0)| \leq \epsilon_1 \quad \text{for large} \; i,
\]

\[
\hat{v}(l_0)\hat{y}(l_0) - \epsilon_1 \leq \tilde{v}_i(l_0, \tau_0)\hat{y}_i(l_0, \tau_0) \quad \text{for} \; i > I_0, \; \text{for some} \; I_0 \in \mathbb{N}
\]

For the catenoid \( \hat{v} = \sqrt{1 + \hat{y}^2} = \cosh(c_5^{-1} \hat{x}_1) \). Therefore
\[ c_5 \cosh^2(c_5^{-1} \hat{x}_1(l_0)) - \epsilon_1 \leq \alpha_i v(l_0, \alpha_i^{-2} \tau + t_i) y(l_0, \alpha_i^{-2} \tau + t_i), \]

\[ \frac{c_5}{2\alpha_i} (\cosh(2c_5^{-1} \hat{x}_1) + 1) - \frac{\epsilon_1}{\alpha_i} \leq vy \quad \text{for } i > I_0. \]

As

\[ \hat{x}_1(l_0) = (\hat{x}(l_0), \hat{t}_1) = \lim_{j \to \infty} \langle \alpha_j \left( x(l_0, \alpha_j^{-2} \tau + t_j) - (x_j, t_1) \right) \rangle, \]

\[ \hat{x}_1(l_0) = \lim_{j \to \infty} \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right), \]

where \( x_{1j} = \langle x_j, t_1 \rangle \), for any \( \epsilon_2 > 0 \) we have

\[ \left| \hat{x}_1(l_0) - \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right) \right| \leq \epsilon_2 \quad \text{for large } j, \]

giving us

\[ \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right) - \epsilon_2 < \hat{x}_1(l_0) \leq \epsilon_2 + \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right) \quad \text{for } j > J_0. \]

For given \( \epsilon_1, \epsilon_2 \) and for a fixed \( i > I_0 \), we pick large \( j > J_0 \), \( j \gg i \), such that

\[ \frac{1}{\alpha_j} \left( \epsilon_2 + 2c_5 \log \left( \frac{4c_i}{c_5} \left( c_4 + \frac{\epsilon_1}{\alpha_i} \right) - 1 \right) \right) < \epsilon, \]

with \( \epsilon \) sufficiently small and with \( c_4 \) as in Lemma 3.3. Then we choose \( l_0 \in M^2 \) such that

\[ x_1 \left( l_0, \alpha_j^{-2} \tau_0 + t_j \right) > x_{1j} + \epsilon \]

holds. Considering \( M_t \) as a periodic surface, we can find points on the hypersurface which lie an \( \epsilon \) distance away from \( x_{1j} \). Therefore

\[ x_1 \left( l_0, \alpha_j^{-2} \tau_0 + t_j \right) > x_{1j} + \frac{1}{\alpha_j} \left( \epsilon_2 + 2c_5 \log \left( \frac{4c_i}{c_5} \left( c_4 + \frac{\epsilon_1}{\alpha_i} \right) - 1 \right) \right). \]

We will use the above inequality to arrive at the contradiction. By (4.7) and (4.9)

\[ \hat{x}_1(l_0) > \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right) - \epsilon_2 > 2c_5 \log \left( \frac{4c_i}{c_5} \left( c_4 + \frac{\epsilon_1}{\alpha_i} \right) - 1 \right) > 0. \]

Therefore

\[ \cosh \left( \frac{1}{2c_5} \left( \alpha_j (x_1(l_0, \alpha_i^{-2} \tau + t_j) - x_{1j}) - \epsilon_2 \right) \right) < \cosh \left( \frac{1}{2c_5} \hat{x}_1(l_0) \right) \quad \text{for } j > J_0, \]

By (4.6)

\[ \frac{c_5}{2\alpha_i} \left( \cosh \left( \frac{1}{2c_5} (\alpha_j (x_1 - x_{1j}) - \epsilon_2) \right) + 1 \right) - \frac{\epsilon_1}{\alpha_i} < \frac{c_5}{2\alpha_i} \left( \cosh \left( \frac{1}{2c_5} \hat{x}_1(l_0) \right) + 1 \right) - \frac{\epsilon_1}{\alpha_i} < vy, \]

\[ \frac{c_5}{2\alpha_i} \left( \cosh \left( \frac{1}{2c_5} (\alpha_j (x_1 - x_{1j}) - \epsilon_2) \right) + 1 \right) - \frac{\epsilon_1}{\alpha_i} < vy, \]

\[ \frac{c_5}{4\alpha_i} \left( e^{\frac{1}{c_5} (\alpha_j (x_1 - x_{1j}) - \epsilon_2)} - e^{-\frac{1}{c_5} (\alpha_j (x_1 - x_{1j}) - \epsilon_2)} + 2 \right) - \frac{\epsilon_1}{\alpha_i} < vy, \]
for $\epsilon_1, \epsilon_2, i, j$ and $l_0$ as previously chosen. As $\alpha_j \left(x_1(l_0, \alpha_j^{-2}t + t_j) - x_{1j}\right) - \epsilon_2 > 0$ by the choice of $l_0$
\[\frac{c_5}{4\alpha_i} \left(e^{\frac{1}{2c_5}(\alpha_j(x_1-x_{1j})-\epsilon_2)} - 1 + 2\right) - \frac{\epsilon_1}{\alpha_i} \leq \frac{c_5}{4\alpha_i} \left(e^{\frac{1}{2c_5}(\alpha_j(x_1-x_{1j})-\epsilon_2)} - e^{-\frac{1}{2c_5}(\alpha_j(x_1-x_{1j})-\epsilon_2)} + 2\right) - \frac{\epsilon_1}{\alpha_i},\]
giving us
\[\frac{c_5}{4\alpha_i} \left(e^{\frac{1}{2c_5}(\alpha_j(x_1-x_{1j})-\epsilon_2)} + 1\right) - \frac{\epsilon_1}{\alpha_i} \leq vy.\]
But from (4.9), we have
\[x_1 \left(l_0, \alpha_j^{-2}t_0 + t_j\right) > x_{1j} + \frac{1}{\alpha_j} \left(\epsilon_2 + 2c_5 \log \left(\frac{4\alpha_i}{c_5} \left(c_4 + \frac{\epsilon_1}{\alpha_i}\right) - 1\right)\right),\]
\[\alpha_j \left(x_1 \left(l_0, \alpha_j^{-2}t_0 + t_j\right) - x_{1j}\right) - \epsilon_2 > 2c_5 \log \left(\frac{4\alpha_i}{c_5} \left(c_4 + \frac{\epsilon_1}{\alpha_i}\right) - 1\right),\]
\[\frac{1}{2c_5}(\alpha_j(x_1(l_0, \alpha_j^{-2}t_0+t_j)-x_{1j})-\epsilon_2) > \frac{4\alpha_i}{c_5} \left(c_4 + \frac{\epsilon_1}{\alpha_i}\right) - 1,\]
\[\frac{c_5}{4\alpha_i} \left(e^{\frac{1}{2c_5}(\alpha_j(x_1(x_l_0, \alpha_j^{-2}t_0+t_j)-x_{1j})-\epsilon_2)} + 1\right) - \frac{\epsilon_1}{\alpha_i} > c_4.\]
From Lemma 3.3 we know that $vy \leq c_4$. Therefore (4.10) and (4.11) contradict Lemma 3.3: that means if we examine the rescaled surfaces, we can see that the estimate $vy \leq c_4$ does not hold on the corresponding original, non-rescaled hypersurfaces, near the singular time $T$.
Therefore our original assumption is wrong. Hence, there exists a constant $c'$ such that $|A|(l, t) \leq c'$ for $t \in [0, T)$. As in Theorem 4.1 in [8] we get bounds for the covariant derivatives of $|A|$ as well. Thus the flow can be extended past time $T$.

5. Evolution Equations and Preliminary Results - Mean Curvature Flow

We consider an axially symmetric, $n$ dimensional hypersurface in $\mathbb{R}^{n+1}$ with Neumann boundary data and use the same notation as in the volume preserving case. In essence we consider the same initial hypersurface, but evolving by mean curvature defined by
\[\frac{d}{dt}x(l, t) = -H(l, t)\nu(l, t).\]

We consider the possibility of a singularity developing in the hypersurface $M_t$ in which $|H(l, t)| \leq c$, for all $t < T$, without imposing any additional conditions on $H$. In this section we prove that a singularity cannot develop in this case. If $H(l, t) > 0$ on all of $M_t$, then from the evolution equation for $\frac{|A|^2}{H^2}$ one can see that $|A|^2 \leq c' H^2$ (see [9]); thus no singularities develop if $H(l, t)$ is bounded. However we cannot use this evolution equation when $H = 0$. As we do not impose the condition $H > 0$ on the initial hypersurface, we repeat the blow up technique used in volume preserving mean curvature flow in the last section.

First we will prove the required preliminary results for mean curvature flow.

Lemma 5.1. We have the following evolution equations:
(i) \( \frac{d}{dt} y = \Delta y - \frac{1}{y} \);
(ii) \( \frac{d}{dt} v = \Delta v - |A|^2 v + \frac{v}{y} - \frac{2}{v} |\nabla v|^2 \);
(iii) \( \frac{d}{dt} k = \Delta k + |A|^2 k - 2q^2 (k - p) \);
(iv) \( \frac{d}{dt} p = \Delta p + |A|^2 p + 2q^2 (k - p) \);
(v) \( \frac{d}{dt} H = \Delta H + (H - h)|A|^2 \);

Proof. Evolution equations (i), (iii) and (iv) are proved in [9], (ii) is proved in [4] and (v) is proved in [7].

The next Lemma corresponds to Lemma 3.3 in the volume preserving case.

**Lemma 5.2.** There exists a constant \( c'_4 \) depending only on the initial hypersurface, such that \( vy < c'_4 \), independent of time.

Proof. Similarly to Lemma 3.3, we obtain
\[
\frac{d}{dt} (yv) = \Delta (yv) - \frac{2}{v} \langle \nabla v, \nabla (yv) \rangle - yv |A|^2.
\]
From the parabolic maximum principle
\[
yv \leq \max_{M_0} yv =: c'_4.
\]

The following corresponds to Proposition 3.4 in the volume preserving setting.

**Proposition 5.3.** There is a constant \( c_1 \) depending only on the initial hypersurface, such that \( \frac{|k|}{p} < c_1 \) in \( M_t \) for all \( t < T \).

Proof. As in [9]
\[
\frac{d}{dt} \left( \frac{k}{p} \right) = \Delta \left( \frac{k}{p} \right) + \frac{2}{p} \langle \nabla p, \nabla \left( \frac{k}{p} \right) \rangle + \frac{2q^2}{p^2} (p - k) (p + k).
\]
For \( \frac{k}{p} > 1 \), i.e. \( k > 0 \), we get the last term to be negative on \( M_t \). Thus
\[
\frac{k}{p} \leq \max \left( 1, \max_{M_0} \frac{k}{p} \right) =: c_1.
\]

6. Rescaled Surface - Mean Curvature Flow

Now we have our prerequisites. To show the equivalent of Theorem 1.1, we employ the same rescaling that proceeds (4.2). The only difference being
\[
\frac{d}{d\tau} \tilde{x}_i = \alpha_i \frac{dx_i}{dt} \frac{dt}{d\tau} = -\alpha_i^{-1} H \nu = -\tilde{H}_i \nu.
\]

Proof of Theorem 1.2. This is the same proof as in Theorem 1.1, the only exceptions being that we use Lemma 5.2 instead of Lemma 3.3 and we refer to Proposition 2.3 in [9] for the analogous bounds on covariant derivatives of \( \tilde{A} \).
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