Real Interpolation of Hardy-Type Spaces and BMO-Regularity

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Abstract
Let $\Omega$ be a $\sigma$-finite measurable space. Suppose that $(X, Y)$ is a couple of quasi-Banach lattices of measurable functions on $\mathbb{T} \times \Omega$ satisfying some additional assumptions. The Hardy-type spaces $X_A$ consist of functions on $\mathbb{D} \times \Omega$ belonging to the Smirnov class $N^+$ in the first variable such that their boundary values are in $X$. Here $\mathbb{T}$ is the unit circle and $\mathbb{D}$ is the open unit disc of the complex plane. Couple $(X_A, Y_A)$ is said to be $K$-closed in $(X, Y)$ with constant $C$ if for any $f \in X$, $g \in Y$ such that $H = f + g \in X_A + Y_A$ there exist some $F \in X_A, G \in Y_A$ satisfying $H = F + G$, $\|F\|_X \leq C\|f\|_X$ and $\|G\|_Y \leq C\|g\|_Y$. This property is shown to be equivalent to the stability of the real interpolation $(X_A, Y_A)_{\theta, p} = (X_A + Y_A) \cap (X, Y)_{\theta, p}$ and to the BMO-regularity of the associated lattices $(L^1, (X^r)\prime Y^r)_{\delta, q}$ under fairly broad assumptions. The inclusion $(X^{1-\theta}Y^\theta)_A \subset (X_A, Y_A)_{\theta, \infty}$ is also characterized in these terms. New examples of couples $(X_A, Y_A)$ with this stability are given, proving that this property is strictly weaker than the usual BMO-regularity of $(X, Y)$.

Keywords Hardy-type spaces · Real interpolation · $K$-closedness · AK-stability · BMO-regularity

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1 Introduction

This work is concerned with the stability of the real interpolation for one-dimensional Hardy-type spaces and the related properties. See [13] for a comprehensive overview with the appropriate references, and e.g. [1] for the generalities on the interpolation spaces. In this rather informal introduction we do not attempt to trace the detailed history of the developments mentioned here, and the formal definitions and statements of the results will be given independently of this introduction in Sect. 2 below. For a slightly different description of the results see the announcement [35] of the present work. It also elaborates on certain details omitted here for the sake of clarity.

To give an illustrative example right away, for classical Hardy spaces the stability of the real interpolation is the formula

\[(H_p, H_q)_{\theta,r} = [H_p + H_q] \cap (L_p, L_q)_{\theta,r} = H_r\]

with \(0 < \theta < 1\), \(0 < p < q \leq \infty\), \(\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}\). That is, for the Lebesgue spaces \(X = L_p\) and \(Y = L_q\) the real interpolation functor \(\mathcal{F}((\cdot, \cdot)) = (\cdot, \cdot)_{\theta,p}\) commutes with the intersection by a suitable class of analytic functions \(S = H_p + H_q\):

\[\mathcal{F}((X \cap S, Y \cap S)) = S \cap \mathcal{F}((X, Y))\]

For weighted Hardy spaces the validity of a similar formula

\[(H_p(u), H_q(v))_{\theta,r} = [H_p(u) + H_q(v)] \cap (L_p, L_q)_{\theta,r} = H_r(u^{1-\theta}, v^\theta)\]

with \(0 < \theta < 1\), \(0 < p, q \leq \infty\), \(\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}\) is completely characterized by the condition \(\log \frac{u}{v} \in \text{BMO}\). Here \(u\) and \(v\) are some suitable weights, and the weighted spaces are defined by \(X(w) = wX\) with the corresponding definition for weighted Hardy spaces.

Many results describing the stability of the real and complex interpolation of Hardy spaces were obtained by the early 80s, and it was already rather well understood by the mid-90s, including the weighted and vector-valued cases. Moreover, for couples \((H_p(u), H_q(v))\) with \(1 \leq p, q \leq \infty\) the condition \(\log \frac{u}{v} \in \text{BMO}\) completely characterizes the existence of a partial retraction from \((L_p(u), L_q(v))\) and thus also the stability with respect to all interpolation functors (see [13,15]). We note in passing that the existence of a partial retraction and the stability with respect to general interpolation functors is still unclear for \(p < 1\).

However, interesting questions go beyond the stability for couples of weighted Hardy spaces. For spaces \(X\) of functions on the unit circle \(\mathbb{T}\) the Hardy-type spaces \(X_A\) consist of the functions from suitable spaces of analytic functions on the unit disk \(\mathbb{D}\) such that their boundary values are in \(X\). We work with rather general quasi-normed lattices of measurable functions \(X\). They are also called the ideal spaces since they are characterized by the inclusion \(L_\infty X \subset X\). Some of the other names for these spaces in wide use (up to certain details pertaining to the particular applications) are the Köthe function spaces and (simply) Banach function spaces.
At the first glance it may seem as though not much can be done in such a general setting. However, for the complex interpolation of Hardy-type spaces a so-called BMO-regularity property (see Definition 2.5 below) that generalizes the condition \( \log \frac{u}{v} \in \text{BMO} \) above turned out to be necessary as well as sufficient under rather general assumptions; see [11]. Intuitively, the stability of interpolation imposes a restriction on the spaces to be “smooth” enough to allow the appropriate decompositions in spaces of analytic functions. The BMO-regularity property for one lattice (see Definition 2.4 below) characterizes the “smoothness” of its unit ball in a rather peculiar sense, describing the situation when any given function can be dominated by a weight \( w \) satisfying \( \log w \in \text{BMO} \) with the control on the norm of the majorant and the BMO constant. These impressive results motivated a substantial amount of research attempting, on the one hand, to better understand the BMO-regularity property, and, on the other hand, to see if this property also characterizes the stability for the real interpolation and some other interesting phenomena.

For arbitrary couples \((X_A, Y_A)\) of Hardy-type spaces it was quickly realized that the BMO-regularity property is sufficient for the stability of the real interpolation, and even for the \( K \)-closedness of the corresponding couple. The \( K \)-closedness property of \((X_A, Y_A)\) in \((X, Y)\) means that arbitrary measurable decompositions of functions from \(X_A + Y_A\) into a sum of functions from \(X\) and \(Y\) can be made analytic with the appropriate control on the norm of the individual parts (see Definition 2.3 below). Following [19], we call this property the AK-stability of the couple \((X, Y)\).

The \( K \)-closedness property, along with the stability of interpolation, found many applications in analysis. We only mention briefly a couple of them here that are most familiar to the author and illustrate the impact of these results.

In [16], a simple proof was found of a rather famous result [2] that the Grothendieck theorem about 2-summing operators holds true for the disc algebra \( C_A \). See also [17] and [10, Chap. 16]. This result is derived, essentially, from the stability of interpolation for couples of weighted Hardy spaces \( (H^2(w^{-\frac{1}{2}}), H^\infty) \) with certain weights \( w \). Some related results use the stability of interpolation for couples of vector-valued Hardy spaces \( H^\infty(l^p) \) and other subtle interpolation properties.

The results mentioned above appear to be some of the earliest (and somewhat implicit) applications of the stability of interpolation for Hardy spaces, and the corresponding constructions, mostly discovered by S. V. Kislyakov and J. Bourgain in the course of the 1980s, are quite elementary. These very applications motivated the development of the theory of interpolation for Hardy-type spaces. On the other hand, we mention a recent result [20], where an old problem about the vector-valued corona theorem with data in \( H^\infty(l^1) \) was solved with the help of the AK-stability of a couple of weighted vector-valued spaces \( (L^\infty(l^\infty)(v), L^\infty(l^1)(v^{-1})) \). The weight \( v \) arises as a BMO-majorant of a function in \( L^\infty(l^2) \). Both the AK-stability of this couple and the BMO-regularity of \( L^\infty(l^2) \) are rather nontrivial properties that were only established due to gradual and systematic development of the theory, mostly during the course of the 90s culminating with [18], and, at least at present, unlike the former examples they are not easily verified by elementary constructions.

In the view of these developments it is natural to inquire whether BMO-regularity is also necessary as well as sufficient for the stability of the real interpolation, and
for the AK-stability. This problem remained open for a long time, at least beyond some special cases. The theory of the real interpolation of Hardy-type spaces does not seem to be complete without a more or less definitive answer to this question, which was elusive for the 25 years since the BMO-regularity property was first introduced in [11]. In [31] the author claimed to have proven the equivalence under some restrictions. Unfortunately, it was recently discovered that these results are flawed, and the specific mistake is the formula [31, Proposition 18]

\[(X, Y)_{α, p} \subset (X, Z)_{γ, p} \subset (Z, Y)_{β, q}\]

which is false, e.g., with \(X = L^2_{0,∞}, Y = L^1, Z = L^∞\) and \(γ = \frac{1}{2}\). However, it was established that under some assumptions the necessary condition for a couple \((X, Y)\) to be AK-stable is the BMO-regularity of a real interpolation space \((L^1, X^\prime Y^\prime)_{θ, r}\), and the flawed part is the derivation of the BMO-regularity of \(X^\prime Y^\prime\) from this property. We will see that, surprisingly, these two properties are in fact not equivalent in general, but the former property generally is equivalent to the AK-stability of the couple \((X, Y)\).

The main goal of the present work is to comprehensively describe the relationship between AK-stability, BMO-regularity and the stability of the real interpolation of Hardy-type spaces. We will show in Theorems 2.7 and 2.8 below that under some standard assumptions the stability \((X_1, Y_1)_{θ, r} = [X_1 + Y_1] \cap (X, Y)_{θ, r}\),

the AK-stability of \((X, Y)\) and the inclusion \((X^1 Y^θ)_{A} \subset (X_1, Y_1)_{θ, ∞}\) are completely characterized by a weaker BMO-regularity property. We will call it the weak-type BMO-regularity property: the BMO-regularity of a real interpolation space \((L_1, (X^r Y^r)_{θ, s}_r)\) with some \(r > 0, 0 < θ < 1\) and \(0 < s ≤ ∞\). We will see that this property is also equivalent to the BMO-regularity of the couple \(((X, Y)_{α, p}, (X, Y)_{β, q})\) with some \(α ≠ β\) (equivalently, with all \(0 < α < β < 1\)).

This result leads to previously unknown examples of couples of AK-stable lattices that are not BMO-regular. It suffices to consider lattices that coincide with the Lorentz spaces \(L^p_{-q_j}\) with different values of \(q_j\) when restricted on different sets. In turn, this indicates that even though the BMO-regularity and weak-type BMO-regularity (generally equivalent to the AK-stability by our results) are very similar, there is a crucial difference between the two: the former is stable under multiplication with BMO-regular couples, whereas the latter is not.

A surprising new observation leading to these results is that a couple \((L^∞_z, Z^α)\) is AK-stable merely if a related couple \((L^∞_z, (L^∞_z)_{β, ∞})\) is AK-stable with some \(0 < α < β < 1\) (see Proposition 5.1 below). This suggests the aforementioned examples, and it already allows us to characterize the AK-stability in terms of the weak-type BMO-regularity under the assumption that \(X^\prime Y^\prime\) is a Banach lattice.

This observation suggests that, unlike BMO-regularity, the AK-stability property may be extended to wider couples on a scale. And indeed, this turns out to be the case: at least under some rather general assumptions, if \((E, F)\) is merely an AK-stable couple of lattices of types \(C_{θ_j}(X, Y)\) with some \(0 < θ_0 < θ_1 < 1\) satisfying some technical assumptions, then \((X, Y)\) is also an AK-stable couple (see Theorem 10.1 below). Thus, the converse is true to a well known fact that couples of real interpolation spaces constructed from an AK-stable couple are also AK-stable. This result is rather involved, and unlike all the important arguments of the theory up to this point that take
advantage of a fixed point theorem, the Fan–Glicksberg–Kakutani fixed point theorem does not seem to suffice for the proof, and we instead rely on the Powers fixed point theorem for compositions of acyclic maps.

This indicates that, unlike BMO-regularity, the AK-stability property of a couple is insensitive to relatively subtle nuances that do not significantly affect at least some intermediate couples of spaces of types $C_\theta$. As a consequence, we prove that as soon as one such couple $(E, F)$ is AK-stable then all of them are. In other words, either all such couples on a single scale are simultaneously AK-stable, or all of them are simultaneously not AK-stable. Such a property seems to be rare for the stability of interpolation of subspaces in general, since it is easy to find examples of scales and one-dimensional subspaces where it fails.

These results depend in a crucial way on a stronger version of AK-stability that we call the bounded AK-stability, meaning that the respective analytic decompositions $H = F + G$ for given measurable decompositions $H = f + g \in X_A + Y_A$ can be constructed by multiplication with some bounded analytic functions $U$ and $1 - U$, i.e., $F = UH = U(f + g)$ and $G = (1 - U)H = (1 - U)(f + g)$, which is equivalent to separate control of the norms of $u = Ug$ in $X$ and $v = (1 - U)f$ in $Y$. These functions $u$ and $v$ thus belong to the intersection $X \cap Y$, so their norms can also be usefully estimated in terms of the norms of the spaces of type $C_\theta(X, Y)$.

It was noticed in [29] that the bounded AK-stability property allows us to improve the convexity of AK-stable lattices. We further develop these ideas, and prove a rather general result (see Theorem 6.6 below) showing how even countable analytic decompositions, such as those that arise in the real interpolation spaces of couples of Hardy-type spaces, can often be made bounded in a similar sense. This allows us to improve the convexity of the couples that are stable with respect to the real interpolation, and gain the nontrivial convexity that is required to apply the results [11] on the stability of the complex interpolation. We mention that there is a different approach to [11] to be published elsewhere that avoids the nontrivial convexity assumptions altogether. However, it still takes advantage of the general result about bounded analytic decompositions established in the present work.

The plan of the paper is as follows. In the next Sect. 2 we give all the necessary formal definitions and state the main results. At the end of it we present some examples of couples of lattices that are AK-stable but not BMO-regular. In Sect. 3 we generalize some the properties of the usual BMO-regularity to the case of weak-type BMO-regularity. Then, in Sect. 4 the properties of the bounded AK-stability are studied, and in Sect. 5 we prove the main results for the case of general (i.e. not necessarily discrete) additional variable. These results are much weaker compared to what we can do in the discrete case, but they are also rather short and simple, do not use fixed point arguments, and give a good idea of what is going on in the discrete case.

Most results after that require the second variable to be discrete. In Sect. 6 we briefly discuss the phenomenon of the bounded $N^+$-stability and bounded analytic decompositions in general, which also includes the notion of bounded AK-stability, and state a rather general result which will be proven in Sect. 8 with the help of a fixed point theorem. In order to do this, in Sect. 7 the topology of pointwise convergence on compact sets is introduced for Hardy-type spaces $X_A$, and we prove that the Fatou property of the lattice $X$ implies that the unit ball of $X_A$ is compact with respect to
this topology. As an application we show that the so-called strong AK-stability is equivalent to the usual one for quasi-Banach lattices.

The remaining thee sections contain the proof of the main result. In Sect. 9 the bounded AK-stability of a couple \((X, Y)\) is derived from the inclusion \((X^{1-\theta} Y^\theta)_A \subset (X_A, Y_A)_\theta,\infty\). This also provides a nice, short and essentially self-contained proof that the stability for the real interpolation functor \((\cdot, \cdot)_\theta,\infty\) is equivalent to the AK-stability for a couple of Banach lattices with the Fatou property. Moreover, the much simpler Fan–Glicksberg–Kakutani theorem can be used in this argument instead of the Powers theorem with little additional effort. Section 10 contains the proof of the crucial observation that the bounded AK-stability of a couple of spaces of type \(C_\theta, (X, Y)\) implies the bounded AK-stability of \((X, Y)\). Finally, in Sect. 11 we verify the main result.

2 Statement of the Main Results

We mostly work with spaces of measurable functions on the measurable space \(\mathbb{T} \times \Omega\), where \(\mathbb{T}\) is the unit circle with the Lebesgue measure and \((\Omega, \mu)\) is some \(\sigma\)-finite measurable space that represents an additional variable. For technical reasons we will often assume \(\Omega\) to be discrete, which means that \(\Omega\) is at most countable.

A quasi-normed lattice of measurable functions \(X\) on a \(\sigma\)-finite measurable space \(\mathcal{M}\) is a quasi-normed space of measurable functions \(X\) such that the norm is compatible with the natural order: if \(|f| \leq g\) for some function \(g \in X\) then \(f \in X\) and \(\|f\|_X \leq \|g\|_X\). For simplicity we only work with lattices \(X\) such that \(\text{supp } X = \mathcal{M}\) up to a set of measure 0. For more detail on the normed lattices and their properties see, e.g., [12, Chap. 10].

We say that \(X\) has the Fatou property if for any \(f_j \in X\), \(\|f_j\|_X \leq 1\) such that \(f_j \to f\) almost everywhere we also have \(f \in X\) and \(\|f\|_X \leq 1\). For normed lattices \(X\) the Fatou property is equivalent to the closedness of the unit ball \(B_X\) with respect to the convergence in measure on sets of finite measure, and it implies that \(X\) is a Banach lattice. \(X\) is said to have order continuous norm if for any nonincreasing sequence \(f_j \in X\) converging to 0 almost everywhere we also have \(\|f_j\|_X \to 0\). The order dual \(X'\) can be defined as a lattice with the norm \(\|g\|_{X'} = \sup_{f \in B_X} \int |fg|\). The Fatou property is equivalent to \((X')' = X\), and the order continuity is equivalent to \(X' = X^*\). For example, \(L'_p = L_p^\prime\) for all \(1 \leq p \leq \infty\).

Let \(0 < p, q \leq \infty\). A quasi-normed lattice \(X\) is said to be \(p\)-convex with a constant \(C_o\) if \(\left\langle \left(\sum_j |f_j|^p\right)^{\frac{1}{p}} \right\rangle_X \leq C_o \left(\sum_j \|f_j\|_{X}^p\right)^{\frac{1}{p}}\) for all finite collections \(\{f_j\} \subset X\) with the appropriate modification if \(p = \infty\). Similarly, lattice \(X\) is \(q\)-concave with a constant \(C_x\) if a converse inequality \(\left\langle \left(\sum_j \|f_j\|_{X}^q\right)^{\frac{1}{q}} \right\rangle_X \leq C_x \left(\sum_j |f_j|^q\right)^{\frac{1}{q}}\) is satisfied for all finite collections \(\{f_j\} \subset X\). A normed lattice is \(1\)-convex, and every lattice is \(\infty\)-concave. If \(X\) is both \(p\)-convex with a constant \(C_o\) and \(q\)-concave with a constant \(C_x\) then it can be equivalently renormed so that \(C_o = C_x = 1\) (see, e.g., [22, Volume I, Proposition 1.d.8]), thus we may often implicitly assume that these constants are equal to 1.
For quasi-normed lattices $X$ and $Y$ of measurable functions on a $\sigma$-finite measurable space $\mathcal{M}$ the pointwise product $XY$ is defined by the quasinorm $\|h\|_{XY} = \inf_{h = fg} \|f\|_X \|g\|_Y$, and the power $X^\delta$, $\delta > 0$ is defined by the quasinorm $\|f\|_{X^\delta} = \|f^{1/\delta}\|_X^\delta$. This construction is also sometimes called the $\frac{1}{\delta}$-convexification of $X$, because if $X$ is $p$-convex and $q$-concave with some $0 < p, q \leq \infty$ then $X^\delta$ is $\left(\frac{q^\delta}{p}\right)$-convex and $\left(\frac{q}{\delta}\right)$-concave. It is convenient to let $X^0 = L_\infty$. Together these definitions also yield the definition of a Calderón-Lozanovskiȋ product $Z = X^{1-\theta} Y^\theta, 0 \leq \theta \leq 1$, which naturally inherits many properties from $X$ and $Y$. If $X$ and $Y$ are Banach lattices with the Fatou property then so is the product $Z$, the dual can be computed as $Z' = X'^{-\theta} Y^{-\theta}$, and $L_1 = X'X$ by the Lozanovskiȋ factorization theorem (see [23]).

Let $N^+$ be the set of boundary values of the Smirnov class of analytic functions on the disc (see, e.g., [27], [9]). We denote by $N^+ \otimes \Omega$ the set of measurable functions $f$ on $\mathbb{T} \times \Omega$ such that $f(\cdot, \omega) \in N^+$ for almost all $\omega \in \Omega$. A Hardy-type space $X_A$ is defined for a space $X$ of measurable functions on $\mathbb{T} \times \Omega$ by $X_A = X \cap (N^+ \otimes \Omega)$. For example, from the Lebesgue spaces $L_p$, $0 < p \leq \infty$ we get the usual Hardy spaces $(L_p)_\theta = H_p$, but this definition also yields the Hardy-Lorentz spaces $H_{p,q}$, the weighted Hardy spaces $H_p(w)$, the variable exponent Hardy spaces $H_{p(\cdot)}$, the vector-valued Hardy spaces $H_p(N)$ and many others.

**Definition 2.1** Suppose that $X$ is a quasi-normed lattice of measurable functions on $\mathbb{T} \times \Omega$. We say that $X$ satisfies property $(\ast)$ with constant $C$ if for any $f \in X$, $f \neq 0$ there exists a majorant $g \geq |f|$ such that $\|g\|_X \leq C \|f\|_X$ and $\log g(\cdot, \omega) \in L_1$ for almost all $\omega \in \Omega$.

This property is often assumed to avoid degeneration. It says, essentially, that lattice $X$ has a complete set of outer functions. If it is satisfied, then by [11, Lemma 2.2] it is also satisfied with arbitrary constants $C > 1$.

Let $r > 0$. If an $r$-convex quasi-normed lattice $X$ of measurable functions on $\mathbb{T} \times \Omega$ has the Fatou property and property $(\ast)$ then $X_A$ is a closed subspace of $X$; see, e.g., [18, §1] (where $X$ is assumed to be normed, but this is easily generalized to quasi-normed lattices).

One of the interesting questions of the theory of interpolation spaces is their stability with respect to the intersection with subspaces. In the present work we only consider the stability with respect to the intersection with spaces of analytic functions, and so we only give the definitions of rather general phenomena specialized to the case of Hardy-type spaces. For the theory of interpolation spaces see, e.g., [1].

**Definition 2.2** We say that a couple $(X, Y)$ of quasi-normed lattices of measurable functions on $\mathbb{T} \times \Omega$ is $N^+$-stable with respect to an interpolation functor $\mathcal{F}$ if $\left[\mathcal{F}((X, Y))\right]_A = \mathcal{F}((X_A, Y_A))$.

For the real interpolation functors the $N^+$-stability is implied by the following property, which is on its own of considerable interest.

**Definition 2.3** A quasi-normed couple $(X, Y)$ of lattices of measurable functions on $\mathbb{T} \times \Omega$ is called AK-stable with constant $C$ if $(X_A, Y_A)$ is $K$-closed in $(X, Y)$ with constant $C$. That is, for any $H \in X_A + Y_A$ and $f \in X$, $g \in Y$ such that $H = f + g$
there exist some \( F \in X_A, G \in Y_A \) such that \( H = F + G \) and \( \|F\|_X \leq C \|f\|_X, \|G\|_Y \leq C \|g\|_Y \).

The BMO-regularity properties introduced below were found to be closely related to the above properties. For the first time they were explicitly introduced and extensively studied, apparently, in [11] in order to characterize the stability of the complex interpolation for Hardy-type spaces, and then in [13] for both the real and the complex interpolation, although they were also somewhat implicitly used before in a different form (later found to be equivalent to BMO-regularity) in various stability results such as [14].

**Definition 2.4** A quasi-normed lattice \( X \) of measurable functions on \( \mathbb{T} \times \Omega \) is called BMO-regular with constants \( (C, m) \) if for any nonzero \( f \in X \) there exists a majorant \( u \geq |f| \) such that \( \|u\|_X \leq m \|f\|_X \) and \( \|\log u(\cdot, \omega)\|_{BMO} \leq C \) for almost all \( \omega \in \Omega \).

As a quick example, we mention that all rearrangement invariant quasi-Banach lattices that are intermediate spaces for the couple \( (L_{r' \infty}, L_{\infty}) \) with some \( r > 0 \), such as the Lorentz spaces \( L_{p, q} \) with \( 0 \leq p, q \leq \infty \), are BMO-regular (see, e.g., [30, Proposition 2]). On the other hand, if \( X \) is a BMO-regular lattice then the weighted lattice \( X(w) \) is BMO-regular if and only if \( \log w(\cdot, \omega) \in BMO \) uniformly in almost all \( \omega \in \Omega \) (see, e.g., [30, Proposition 5]).

**Definition 2.5** A couple \( (X, Y) \) of quasi-normed lattices of measurable functions on \( \mathbb{T} \times \Omega \) is said to be BMO-regular with constants \( (C, m) \) if for all nonzero \( f \in X \) and \( g \in Y \) there exist some majorants \( u \geq |f| \) and \( v \geq |g| \) such that \( \|u\|_X \leq m \|f\|_X, \|v\|_Y \leq m \|g\|_Y \) and \( \|\log \frac{u(\cdot, \omega)}{v(\cdot, \omega)}\|_{BMO} \leq C \) for almost all \( \omega \in \Omega \).

It is easy to see that if both lattices \( X \) and \( Y \) are BMO-regular then couple \( (X, Y) \) is also BMO-regular. If \( X \) and \( Y \) are \( r \)-convex with some \( r > 0 \) then the BMO-regularity of \( (X, Y) \) is equivalent to the BMO-regularity of \( (X')' (Y')' \) for lattices with the Fatou property (see [30, Theorem 8]).

It is well known that BMO-regularity of a couple \( (X, Y) \) implies its AK-stability (see, e.g., [13, Theorem 3.3]). This (up to some detail) follows from the fact that couples \( (L_{\infty}(u), L_{\infty}(v)) \) are AK-stable for the corresponding BMO-majorants \( (u, v) \). The converse was long suspected to be true, i.e. that some kind of BMO-regularity is also necessary for AK-stability, and for couples of weighted Lebesgue spaces AK-stability is indeed equivalent to BMO-regularity (see [13, Theorem 3.2], [28, Theorem 1] with the original result obtained in [3, Theorem 1.8]). There are also some couples with an additional variable for which this is true (see [18, Theorem 1] and [29, Theorem 2]). However, we will show that under some natural assumptions AK-stability and even the stability with respect to the real interpolation are completely characterized in terms of a weaker property.

**Definition 2.6** Suppose that \( (X, Y) \) is a couple of quasi-Banach lattices of measurable functions on \( \mathbb{T} \times \Omega \) such that \( X \) is \( r \)-convex with some \( r > 0 \). We say that \( (X, Y) \) is weak-type BMO-regular if \( (L_1, (X')' Y')_\theta, p \) is BMO-regular with some \( 0 < \theta < 1 \) and \( 0 < p \leq \infty \).
This definition is meant to be understood in the sense that the BMO-regularity is present at least for small enough values of $\theta$. By Proposition 3.5 below, Definition 2.6 does not essentially depend on $p$ and $r$. The name of this property was chosen because it is naturally related to a certain weak-type boundedness of the harmonic analysis operators such as the Hilbert transform or the Hardy-Littlewood maximal function, similarly to how the usual BMO-regularity is related to the boundedness of these operators by the results of [30]; see [35, §5].

By [30, Theorem 8] mentioned above and Proposition 3.3 below, a BMO-regular couple of quasi-Banach lattices with the Fatou property is also weak-type BMO-regular. The converse is false in general; see examples at the end of this section. One crucial difference to note between the BMO-regularity and weak-type BMO-regularity is that, unlike the former, the latter is not stable under the multiplication of couples in its various specific forms, i.e., if $(X, Y)$ and $(E, F)$ are both weak-type BMO-regular then $(XE, YF)$ is not necessarily weak-type BMO-regular, not even for couples $(E, F) = (L^p (l^p), L^q (l^q))$ with $p \neq q$. Otherwise the main result together with [29, Theorem 2] would have given us the equivalence between the weak-type BMO-regularity and the BMO-regularity. Without the additional variable this multiplication also fails for BMO-regular couples of weighted Lebesgue spaces $(E, F) = (L^p (u), L^q (v))$ (see [29, Proposition 21]). It still is not clear, however, whether the weak-type BMO-regularity is stable under multiplication by at least a couple of unweighted Lebesgue spaces $(E, F) = (L^p, L^q)$, $p \neq q$, without the additional variable.

On the other hand, the distinction between these properties does not appear to be big. The equivalence of conditions (v) and (vi) of Theorem 2.7 below shows that under its assumptions the BMO-regularity is equivalent to the weak-type BMO-regularity for couples obtained by the real interpolation from a single couple. For example, these properties coincide for couples of weighted Lebesgue spaces. Since both of these conditions are invariant under raising the lattices to any positive power, the convexity assumptions in this equivalence may be further relaxed away to the assumption that both lattices are $r$-convex with some $r > 0$.

There are other interesting natural spaces to be investigated where one might suspect the equivalence of the weak-type BMO-regularity to the usual one, such as couples of weighted vector-valued Lebesgue spaces $L^p (l^p)$, weighted Orlicz spaces and variable exponent Lebesgue spaces $L^p (\cdot)$ to name a few.

Let $(X, Y)$ be a compatible couple of quasi-Banach spaces and $0 < \theta < 1$. By an equivalent definition (see, e.g., [1, Theorem 3.5.2]), a quasi-Banach space $Z$ is said to be of type $C^\theta (X, Y)$ if $(X, Y)_{\theta, 1} \subset Z \subset (X, Y)_{\theta, \infty}$. For example, the real and the complex interpolation spaces $(X, Y)_{\theta, p}$ and $(X, Y)_{\theta}$ as well as the Calderón-Lozanovskiĭ products $X^{1-\theta} Y^\theta$ in the case of lattices are all of type $C^\theta (X, Y)$ (see, e.g., [1, Theorem 4.7.1] and Proposition 3.4 below). This notion places space $Z$ on an interpolation scale between $X$ and $Y$ at the point $\theta$ in a rather specific sense.

We are now ready to state the main result of the present work. It establishes that under some standard assumptions the weak-type BMO-regularity completely characterizes various properties related to the stability of the real interpolation. Moreover, it also shows that these properties are closely related to one another, and both AK-stability
Theorem 2.7 Suppose that \((X, Y)\) is a couple of quasi-normed \(r\)-convex lattices of measurable functions on \(\mathbb{T} \times \Omega\) with a discrete space \(\Omega\) and some \(r > 0\) satisfying the Fatou property and property \((\ast)\) such that \(X^{1-\theta}Y^\theta\) are Banach lattices with some \(0 < \theta_0 < \theta < \theta_1 < 1\). The following conditions are equivalent.

(i) \((X, Y)\) is \(\mathbb{N}^+\)-stable with respect to \((\cdot, \cdot)_{\theta,s}\) for some (equivalently, for all) \(1 \leq s \leq \infty\).

(ii) \((X, Y)\) is AK-stable.

(iii) \((E, F)\) is AK-stable for some (equivalently, for all) quasi-normed \(r\)-convex lattices \(E\) and \(F\) of measurable functions on \(\mathbb{T} \times \Omega\) satisfying the Fatou property and property \((\ast)\) such that \(E\) is of type \(C_\alpha (X, Y)\) and \(F\) is of type \(C_\beta (X, Y)\) with some \(0 \leq \alpha < \beta < 1\) (equivalently, for all such couples with some \(0 \leq \alpha < \beta \leq 1\)).

(iv) \((X, Y)_{\alpha,p}\) is AK-stable with some \(0 < \alpha < \theta < \beta < 1\) (equivalently, with all) \(0 < \alpha < \beta < 1\) and \(0 < p, q \leq \infty\).

(v) \((X, Y)_{\alpha,p}\) is BMO-regular with some (equivalently, with all) \(0 < \alpha < \beta < 1\) and \(0 < p, q \leq \infty\).

(vi) \((X, Y)_{\alpha,p}\) is weak-type BMO-regular with some (equivalently, with all) \(0 < \alpha < \beta < 1\) and \(0 < p, q \leq \infty\).

(vii) \((X, Y)\) is weak-type BMO-regular for some (equivalently, for all) lattices \(E\) and \(F\) defined in condition (iii)

(viii) \((X, Y)\) is weak-type BMO-regular.

(ix) \((X^{1-\theta}Y^\theta)_{\alpha,p}\) is \(\mathbb{N}^+\)-stable for some (equivalently, for all) \(0 < \alpha < \beta < 1\) and \(0 < p, q \leq \infty\).

In particular, a couple \((X, Y)\) satisfying the assumptions of Theorem 2.7 may be \(\mathbb{N}^+\)-stable with respect to the real interpolation \((\cdot, \cdot)_{\theta,q}\) but not with respect to the complex interpolation \((\cdot, \cdot)_{\theta}\), since the latter stability is equivalent to the BMO-regularity of the couple \((X, Y)\) (at least under some mild convexity assumptions) by [11, Theorem 5.12].

In a certain sense, the natural convexity assumptions are that both \(X\) and \(Y\) are Banach lattices, but we are able to relax them to the 1-convexity on a segment \([\theta_0, \theta_1]\) of the scale without much complexity added to the proof. One simple case where these convexity assumptions are satisfied is when lattices \(X\) and \(Y\) are \(p\)-convex and \(q\)-convex, respectively, with some \(p, q > 0\) such that \(p \vee q > 1\).

Condition (ix) generalizes the corresponding result for couples of weighted Lebesgue spaces (see [3, Theorem 1.8], [13, Theorem 3.2], [28, Theorem 1]). As a consequence, this shows that if \(X^{1-\theta}Y^\theta\) has order continuous norm and \((X, Y)\) is \(\mathbb{N}^+\)-stable with respect to an interpolation functor \(\mathcal{F}\) of type \(C_\theta\) then \((X, Y)\) is weak-type BMO-regular. This suggests an interesting question: for which functors \(\mathcal{F}\) of type \(C_\theta\) the same holds true? The positive answer for all such \(\mathcal{F}\) is equivalent to the statement that the inclusion \([X, Y]_{\theta,1}\) implies that \((X, Y)\) is weak-type BMO-regular.

We note that although the assumptions made in Theorem 2.7 are fairly broad, the generality of these results is still not entirely satisfying. In particular, the convexity
assumptions on the lattices and the discreteness assumption on \( \Omega \) arise because they ensure crucial geometrical and topological properties of certain maps used in the proofs, and at present it is not clear how to get around these restrictions. A more specialized approach yields the equivalence between AK-stability and BMO-regularity for arbitrary \( \Omega \) but with certain restrictions on lattices.

**Theorem 2.8** Let \((X, Y)\) be a couple of quasi-normed lattices of measurable functions on \( \mathbb{T} \times \Omega \) satisfying the Fatou property and property \((*)\) such that either \(X\) is \(r\)-convex with some \(r > 0\) and \(Y = L_\infty\), or all three lattices \(X, Y\) and \(X'Y\) are Banach. Then couple \((X, Y)\) is AK-stable if and only if it is weak-type BMO-regular.

The proof of Theorem 2.8 is given in Sect. 4.2 below (see Propositions 5.3, 5.5 and Corollary 5.8), and on its own it is fairly uncomplicated. The individual transitions are obtained under broader sets of assumptions that all reduce to the bounded case \(Y = L_\infty\) for the “if” part and to the symmetric case \(Y = X'\) for the “only if” part. The “only if” part is also valid if both lattices are Banach and at least one of them has nontrivial convexity or concavity. We also obtain an “only if” result for the strong AK-stability (see Definition 4.1) in the case when \(X\) is \(p\)-convex with some \(p > 1\) and \(Y\) is merely \(r\)-convex with some \(r > 0\) (see Proposition 5.7). The “if” part is somewhat less satisfactory, requiring \((X')Yr\) to be a Banach lattice with some \(r > 0\), but this at least covers all couples of weighted Lebesgue spaces, or, more generally, couples \((X, Y)\) such that \(X\) is \(q\)-concave and \(Y\) is \(q\)-convex with some \(0 < q \leq \infty\), whereas in our “only if” results at least one of the lattices must be Banach unless the other lattice is \(L_\infty\). Thus, the “only if” results do not cover the classical case of couples of weighted Lebesgue spaces when both of them are quasi-Banach. This discrepancy will be addressed in future work.

We now give some examples of lattices \(Y\) such that couple \((L_1, Y)\) is weak-type BMO-regular and hence AK-stable but \(Y\) fails to be BMO-regular. Let \(\mu\) be a point mass, i.e. we do not consider the additional variable. For a measurable set \(E \subset \mathbb{T}\) and quasi-normed lattices \(Y_0\) and \(Y_1\) of measurable functions on \(\mathbb{T}\) we define a composite lattice

\[
Y = \chi_{\mathbb{T} \setminus E} Y_0 + \chi_E Y_1 = \{ \chi_{\mathbb{T} \setminus E} f + \chi_E g \mid f \in Y_0, g \in Y_1 \}
\]

with a norm \(\|\chi_{\mathbb{T} \setminus E} f + \chi_E g\|_Y = \|\chi_{\mathbb{T} \setminus E} f\|_{Y_0} + \|\chi_E g\|_{Y_1}\). For simplicity, we choose the half-circle \(E = [0, \pi)\) and the Lorentz spaces \(Y_j = L_{t,s_j}, f \in \{0, 1\}\) with some \(1 < t < \infty, 0 < s_0, s_1 \leq \infty\). Then it is easy to see that \((L_1, Y)_{\theta,p} = L_{q,p}\) is BMO-regular, where \(0 < \theta < 1\) and \(q = \left(\frac{1-\theta}{t} + \frac{\theta}{7}\right)^{-1}\), so \((L_1, Y)\) is weak-type BMO-regular. In fact, the same is true for any lattice \(Y\) satisfying continuous inclusions \(L_{t,r} \subset Y \subset L_{t,\infty}\) with some \(r > 0\).

However, \(Y\) is not BMO-regular if \(s_0 \neq s_1\). To see this, suppose that, more generally, \(Y_0 \subsetneqq Y_1\) are some rearrangement invariant spaces with the Fatou property and \(Y\) is BMO-regular. Then by [30, Theorem 1] the Hardy-Littlewood maximal operator \(Mf(e^{ix}) = \sup_{0 < r \leq \pi} \frac{1}{2r} \int_{x-r}^{x+r} |f(e^{is})|ds, x \in \mathbb{R}\) is bounded in \(Z = \left(L_1^{-\sigma} Y^\alpha\right)\beta\) for some \(0 < \sigma, \beta < 1\). Observe that \(Z = \chi_{\mathbb{T} \setminus E} Z_0 + \chi_E Z_1\) with \(Z_j = \left(L_1^{-\sigma} Y_j^\alpha\right)\beta\).
\( j \in \{0, 1\}, \) and \( Z_0 \subsetneq Z_1. \) Otherwise the equality \( Z_0 = Z_1 \) would have implied that \( L_1^{1-\alpha} Y_0^\alpha = L_1^{1-\alpha} Y_1^\alpha, Y_0' = (L_1^{1-\alpha} Y_0^\alpha)' = (L_1^{1-\alpha} Y_1^\alpha)' = Y_1', \) and \( Y_0 = (Y_0')' = Y_1'. \) Let \( f \in Z_1 \setminus Z_0, \) and let

\[
g(e^{ix}) = \chi_{[0,\pi)}(x) \left( t \mapsto \chi_{[0,2\pi)}(t) f(e^{it}) \right)^*(x), \quad x \in [0, 2\pi)
\]

be the nonincreasing rearrangement of \( f \) restricted to the upper half-circle. Then \( g \in Z, Mg \in Z, \) and in particular \( \chi_{T \setminus E} Mg \in Z_0. \) It is easy to see that \( g \in Z_1 \setminus Z_0 \) and \( Mg(e^{-ix}) \geq \frac{1}{2\pi} \int_0^x g(e^{it}) \, dt \geq \frac{1}{4} g(e^{ix}) \) for \( 0 < x < \frac{\pi}{2}, \) which contradicts \( \chi_{T \setminus E} Mg \in Z_0. \)

### 3 Some Properties of Weak-Type BMO-Regularity

The following formula (also appearing in [20, Lemma 1] with a short proof) seems to be rather well known; see, e.g., [36, Theorem 3.7].

**Proposition 3.1** Suppose that \( E \) and \( F \) are Banach lattices of measurable functions on the same \( \sigma \)-finite measurable space having the Fatou property such that \( EF \) is also a Banach lattice. Then \( E' = (EF)'F. \)

**Proposition 3.2** [31, Proposition 14] Let \( X \) and \( Y \) be some quasi-Banach lattices of measurable functions on some \( \sigma \)-finite measurable space. Then

\[
(X, Y)^{\alpha, \theta, p}_\phi = (X^\alpha, Y^\alpha)^{\theta, p}_\phi
\]

for all \( \alpha > 0, 0 < \theta < 1 \) and \( 0 < p \leq \infty. \)

For a moment we need a more precise \( A_p \)-regularity property (see [30, Definition 1]), which is obtained from Definition 2.4 by replacing \( \| \log u(\cdot, \omega) \|_{\text{BMO}} \leq C \) with the corresponding boundedness of the Muckenhoupt constant of the weight \( u. \) The next proposition is a simple consequence of the well-known fact that a lattice \( Z \) is BMO-regular if and only if \( Z^\delta \) is \( A_2 \)-regular with some \( \delta > 0 \) (see, e.g., the remarks after [30, Definition 1]), [31, Proposition 17] and Proposition 3.2.

**Proposition 3.3** Suppose that quasi-Banach lattices \( X \) and \( Y \) of measurable functions on \( \mathbb{T} \times \Omega \) are BMO-regular. Then the real interpolation space \((X, Y)_\theta, q) \) is also a BMO-regular lattice for all \( 0 < \theta < 1 \) and \( 0 < q \leq \infty. \)

The following observation is well known. It is an easy consequence of the definition of the Calderón–Lozanovskii product and the Young inequality (see, e.g., the proof of [19, Lemma 5]).

**Proposition 3.4** Let \( X_0 \) and \( X_1 \) be some quasi-normed lattices of measurable functions on the same measurable space. Then the Calderón–Lozanovskii product \( X_0^{1-\theta} X_1^\theta \) is a space of type \( C_\theta(X_0, X_1) \) for all \( 0 < \theta < 1. \)
Definition 2.6 of weak-type BMO-regularity does not restrict values of $r$ (within its sensible range based on the convexity of $X$) and $p$.

**Proposition 3.5** Suppose that $(X, Y)$ is a couple of quasi-normed lattices of measurable functions on $\mathbb{T} \times \Omega$ such that $X$ is $r$-convex with some $r > 0$. Then $(X, Y)$ is weak-type BMO-regular if and only if $(L_1, L_1^{1-\gamma}[(X^s)^s]^\gamma)_{\theta, q}$ is BMO-regular for some $0 < \theta < 1$ (equivalently, for all sufficiently small $\theta > 0$) and for some (equivalently, for all) $0 < s \leq r$, $0 < \gamma \leq 1$ and $0 < q \leq \infty$.

Indeed, by Proposition 3.3 we can take arbitrary $p$ in Definition 2.6 with any smaller $\theta$, since lattice $L_1$ is BMO-regular and by the reiteration theorem (see, e.g., [1, Theorem 3.5.3]) we have

$$
(L_1, (L_1, (X^r)'(Y^r)))_{\eta, q} = (L_1, (X^r)'(Y^r))_{\eta \theta, q}
$$

for arbitrary $0 < \eta < 1$ and $0 < q \leq \infty$. For the independence from $r$, observe that $(X^s)'Y^s = \left(L_1^{1-\gamma}[(X^r)^r]\right)' \eta^s = L_1^{1-\gamma}[(X^r)'Y^r]^{\gamma \gamma} = L_1^{1-\gamma}[(X^s)'Y^s]^{\gamma \gamma}$, thus by the reiteration theorem and Proposition 3.4

$$
(L_1, L_1^{1-\gamma}[(X^s)^s]^\gamma)_{\theta, q} = (L_1, (X^r)'(Y^r))_{\theta, q}.
$$

**Corollary 3.6** Suppose that $(X, Y)$ is a couple of quasi-normed lattices of measurable functions on $\mathbb{T} \times \Omega$ such that $X$ is $r$-convex with some $r > 0$. If $(X, Y)$ is weak-type BMO-regular then so is $(X^\delta, Y^\delta)$ for all $\delta > 0$.

The weak-type BMO-regularity has the natural symmetry, duality and divisibility properties.

**Proposition 3.7** Suppose that $X$, $Y$ and $Z$ are $r$-convex quasi-normed lattices of measurable functions on $\mathbb{T} \times \Omega$ with some $r > 0$ satisfying the Fatou property. The following conditions are equivalent.

(i) $(X, Y)$ is weak-type BMO-regular.
(ii) $(XZ, YZ)$ is weak-type BMO-regular.
(iii) $((Y^r)', (X^r)')$ is weak-type BMO-regular.
(iv) $(Y, X)$ is weak-type BMO-regular.

Indeed, by Corollary 3.6 condition (i) is equivalent to the weak-type BMO-regularity of the couple $(X^\delta, Y^\delta)$. By Proposition 3.1 we have

$$
(X^\delta)'Y^\delta = (X^\delta Z^\delta)'Z^\delta Y^\delta = (X^\delta Z^\delta)'Y^\delta Z^\delta,
$$

so condition (i) is equivalent to the weak-type BMO-regularity of $(XZ)^\delta, (YZ)^\delta)$, which is equivalent to condition (ii) by another application of Corollary 3.6.
With the help of the equivalence of conditions (i) and (ii), the equivalence of (i) and (iii) follows in the standard way (see, e.g., the proof of [30, Theorem 8]): condition (i) is equivalent to the weak-type BMO-regularity of \((X^r, Y^r)\), which is equivalent to the same of
\[
(X^r (X^r)' , Y^r (X^r)') = (L_1, (X^r)' (X^r)' ) = ((Y^r)' Y^r, (X^r)' Y^r ,
\]
which in turn is equivalent to condition (ii) with \(Z = Y^r\) and couple \(((Y^r)', (X^r)')\) in place of \((X, Y)\).

The symmetry of the weak-type BMO-regularity, which is trivial for the usual BMO-regularity, seems to require the self-duality of the BMO-regularity property (see [18, Theorem 2], [30, Theorem 1] and [32, Theorem 1]). By Proposition 3.5 condition (i) is equivalent to the BMO-regularity of lattice \(Z_1 = \left( L_1, L_1^{1/2} [ (X^r)' Y^r ]^{1/2} \right)_{\theta,1}\) with some \(0 < \theta < 1\), which is equivalent to the BMO-regularity of
\[
Z_1^{1/2} = \left( L_2, L_2^{1/2} [ (X^r)' (Y^r)' ]^{1/2} \right)_{\theta,2},
\]
by Proposition 3.2. Since both lattices in the latter couple have order continuous norm, their intersection is dense in each of them, and their Banach duals coincide with the order duals. Moreover, the intersection of these spaces is separable by [12, Chap. IV, §3, Theorem 3], and it is also dense in the interpolation space \(Z_1^{1/2}\) by [1, Theorem 3.4.2 (b)], so by the same theorem from [12] lattice \(Z_1^{1/2}\) has order continuous norm. Therefore, by the duality theorem for the real interpolation [1, Theorem 3.7.1] and [30, Theorem 1] the BMO-regularity of \(Z_1\) is equivalent to the BMO-regularity of the dual lattice
\[
\left( Z_1^{1/2} \right)' = \left( L_2', L_2^{1/2} [ (X^r)' (Y^r)' ]^{1/2} \right)' = \left( L_2, L_2^{1/2} [ (Y^r)' (X^r)' ]^{1/2} \right)_{\theta,2}.
\]
Raising it to the power 2 with the help of Proposition 3.2 and making use of Proposition 3.5 yields the equivalence to the weak-type BMO-regularity of \((Y, X)\), which is condition (iv).

### 4 Bounded AK-Stability

We need the following species of the AK-stability property. They were introduced in [19] and in [29] respectively, but appeared implicitly in earlier research.

**Definition 4.1** A quasi-normed couple \((X, Y)\) of lattices of measurable functions on \(\mathbb{T} \times \Omega\) is called strongly AK-stable with constant \(C\) if for any \(H \in (X + Y)_A\) and \(f \in X, g \in Y\) such that \(H = f + g\) there exist some \(F \in X_A, G \in Y_A\) such that \(H = F + G\) and \(\| F\|_X \leq C \| f\|_X, G\|_Y \leq C \| g\|_Y\).
The distinction between the AK-stability and the strong AK-stability appears to be mostly technical in nature. These properties are known to be equivalent for couples of Banach lattices satisfying the Fatou property. We will also prove the equivalence for couples of \( r \)-convex quasi-normed lattices, \( r > 0 \), with the Fatou property if \( \Omega \) is discrete; see Proposition 7.2 below and remarks before it.

**Definition 4.2** A quasi-normed couple \((X, Y)\) of lattices of measurable functions on \( \mathbb{T} \times \Omega \) is called boundedly AK-stable with constant \( C \) if for any \( f \in X \) and \( g \in Y \) there exists some \( U \in H_\infty (\mathbb{T} \times \Omega) \) such that \( \|U\|_{H_\infty} \leq C, \|Ug\|_X \leq C\|f\|_X \) and \( \|(1 - U)f\|_Y \leq C\|g\|_Y \).

Compared to the AK-stability, this property is often very convenient, because it only involves estimates for products of bounded analytic functions with arbitrary measurable functions from the lattices. The following reformulation (clarifying the discussion in [29, Sect. 1.3]) illustrates its relation to the usual AK-stability. In this form it also easily generalizes to the AK-stability of several lattices.

**Proposition 4.3** Suppose that \((X, Y)\) is a couple of quasi-normed lattices of measurable functions on \( \mathbb{T} \times \Omega \) satisfying property \((\ast)\). Then it is boundedly AK-stable if and only if for any \( H \in (X + Y)_L \) and \( f, g \in X \) such that \( H = f + g \) there exist some \( U \in H_\infty \) such that \( \|UH\|_X \leq c\|f\|_X, \|(1 - U)H\|_Y \leq c\|g\|_Y \) and \( \|U\|_{H_\infty} \leq c \) with a constant \( c \) independent of \( f, g \) and \( H \).

Indeed, it is easy to see that the bounded AK-stability of a couple \((X, Y)\) implies its strong AK-stability with decompositions of the form \( F = UH \) and \( G = (1 - U)H \) (see, e.g., the proof of [29, Proposition 4]). Conversely, suppose that we are given some \( f \in X \) and \( g \in Y \). We may assume that these functions are nonnegative and, moreover, \( \log f(\cdot, \omega), \log g(\cdot, \omega) \in L_1 \) for almost all \( \omega \in \Omega \) by property \((\ast)\). We construct an outer function \( H = \exp(\log[f + g] + i\mathcal{H}[f + g]) \) such that \( |H| = f + g \) almost everywhere. Here \( \mathcal{H} \) denotes the Hilbert transform acting in the first variable. Then function \( U \) from the assumptions also satisfies Definition 4.2, since \( |U g| \leq |U H| \) and \( |1 - U|f| \leq |(1 - U)H| \) almost everywhere.

We will see that for couples satisfying the assumptions of either Theorem 2.8 or Theorem 2.7 the AK-stability is equivalent to the bounded AK-stability. It is easy to establish this equivalence in the important case \( X = L_\infty \). In Proposition 5.2 below we will show that at least for \( r \)-convex lattices with the Fatou property, \( r > 0 \), the strong AK-stability can be replaced with the usual AK-stability in this result.

**Proposition 4.4** [29, Proposition 3] Let \( Z \) be a quasi-normed lattice of measurable functions on \( \mathbb{T} \times \Omega \) satisfying property \((\ast)\). Couple \((L_\infty, Z)\) is strongly AK-stable if and only if it is boundedly AK-stable.

The bounded AK-stability is stable with respect to multiplication with a lattice.

**Proposition 4.5** [29, Proposition 2] Let \( X, Y \) and \( Z \) be quasi-normed lattices of measurable functions on \( \mathbb{T} \times \Omega \). If \((X, Y)\) is boundedly AK-stable then \((XZ, YZ)\) is also boundedly AK-stable.
These two simple results imply that a BMO-regular couple \((X, Y)\) is boundedly AK-stable, which was already noted in [29, §1.3]. For clarity, let us spell out an argument proving this. For a weight \(w\), which for simplicity we assume to be positive almost everywhere, and for a lattice \(Z\) the weighted lattice \(Z(w)\) is defined by \(Z(w) = \{wf \mid f \in Z\}\) with norm \(\|g\|_{Z(w)} = \|gw^{-1}\|_Z\). If \(u\) and \(v\) are some BMO-majorants in the sense of Definition 2.5 then \((L_{\infty}(u), L_{\infty}(v))\) is boundedly AK-stable, which was already implicit in the proof at the end of [13, §3.4]. But for this couple the bounded AK-stability also follows from the usual one: \((L_{\infty}, L_{\infty}(u^{-1}v))\) is AK-stable, so by Proposition 4.4 it is boundedly AK-stable, and we may apply Proposition 4.5 with \(Z = L_{\infty}(u)\).

The strong AK-stability is stable under multiplication by a lattice.

**Proposition 4.6** [19, Lemma 4] Let \(X, Y\) and \(Z\) be quasi-normed lattices of measurable functions on \(\mathbb{T} \times \Omega\) such that \(Z\) has property \((*)\). If \((X, Y)\) is strongly AK-stable then \((XZ, YZ)\) is also strongly AK-stable.

The following result is a substantial improvement over [29, Proposition 6], which only yielded the bounded AK-stability of the couple \((XL_1, YL_1)\).

**Proposition 4.7** Let \(X\) and \(Y\) be Banach lattices of measurable functions on \(\mathbb{T} \times \Omega\) satisfying the Fatou property and property \((*)\). Suppose also that \(X'Y\) is a Banach space. Then \((X, Y)\) is AK-stable if and only if it is boundedly AK-stable.

Indeed, if \((X, Y)\) is AK-stable then by Proposition 4.6 so is \((X'X, X'Y) = (L_1, X'Y)\), by [18, Lemma 7] couple \((L'_1, (X'Y)') = (L_{\infty}, (X'Y)')\) is AK-stable, and by Proposition 4.4 it is boundedly AK-stable. By Proposition 4.5 and Proposition 3.1 couple \((L_{\infty}Y, (X'Y)Y) = (Y, (X'Y)') = (Y, X)\) is then boundedly AK-stable, and it remains to change the order of this couple.

The bounded AK-stability admits further useful reformulations.

**Lemma 4.8** Suppose that \(\alpha, \beta > 0\). A couple \((X, Y)\) of quasi-normed lattices of measurable functions on \(\mathbb{T} \times \Omega\) is boundedly AK-stable with some constant \(C\) if and only if for any \(f \in X\) and \(g \in Y\) there exists some \(V \in H_{\infty} (\mathbb{T} \times \Omega)\) such that \(\|V\|_{H_{\infty}} \leq C\), \(\|V\|_a g \|_{X} \leq C\|f\|_{X}\) and \(\|1 - |V|^\beta f\|_Y \leq C\|g\|_Y\) with some constant \(C'\).

To prove the “if” part, take some integer numbers \(M \geq \alpha\) and \(N \geq \beta\), let \(f \in X\) and \(g \in Y\) be some nonnegative functions, and let \(V\) be a function from the statement of the lemma. Functions \(V^M\) and \((1 - V)^N\) satisfy the assumptions of the corona theorem (see, e.g., [21, Proposition 2]), so there exist some \(U_0, U_1 \in H_{\infty}\) such that \(V^M U_0 + (1 - V)^N U_1 = 1\) and \(\|U_j\|_{H_{\infty}} \leq c_1, j \in \{0, 1\}\) with some \(c_1\) independent of \(f\) and \(g\). Let \(U = V^M U_0\). Then \(|U| \leq c_1|V|^M \leq c_1 C'^M - \alpha |V|^\alpha\), and similarly \(|1 - U| \leq c_1 C'^N - \beta |1 - V|^\beta\), which yields the claimed estimates. For the “only if” part we choose some integer numbers \(M \geq \frac{1}{\alpha}, N \geq \frac{1}{\beta}\), apply the corona theorem to find some bounded analytic functions \(V_0\) and \(V_1\) satisfying \(U^M V_0 + (1 - U)^N V_1 = 1\) with suitable estimates, and take \(V = U^M V_0\).

We mention that the use of the corona theorem can be easily avoided in the proof of Lemma 4.8, perhaps at a slight expense of clarity and generalizations to several lattices.
In the “if” part we may first take $V_1 = V^M$, which yields estimates $\|V_1g\|_X \leq C''\|f\|_X$ and $\|1 - V_1^\beta f\|_Y \leq C''\|g\|_Y$ from the assumptions with some suitable constant $C''$, since $|1 - V_1| = \left| (1 - V) \sum_{j=0}^{M-1} V^j \right| \leq MC^{1/M-1}|1 - V|$. Taking $U = 1 - (1 - V_1)^N$ then yields the bounded AK-stability of $(X, Y)$ by a similar estimate, and the “only if” part is treated in the same way.

Observe that by the homogeneity the conditions of Lemma 4.8 may be restricted to $\|g\|_Y = 1$. Further replacing $f$ with $f_1$ such that $f = sf_1$ and $s = \|f\|_X$ yields the following characterization of bounded AK-stability that will be used in the proof of Theorem 10.1 below.

**Corollary 4.9** Let $\alpha, \beta > 0$. A couple $(X, Y)$ of quasi-normed lattices of measurable functions on $\mathbb{T} \times \Omega$ is boundedly AK-stable with some constant $C$ if and only if for any $s > 0$ and $f, g \in X, Y$ such that $\|f\|_X = \|g\|_Y = 1$ there exists some $V \in H_\infty (\mathbb{T} \times \Omega)$ such that $\|V\|_{H_\infty} \leq C'$, $\|V^\alpha g\|_X \leq C's$ and $\|1 - V^\beta f\|_Y \leq C's^{-1}$ with some constant $C'$.

The following observation is a generalization of [29, Proposition 1], where the case $\delta < 1$ was trivially established. See also [19, Theorem 2] and [13, Theorem 3.6]. It is interesting to note that the latter theorem together with [29, Proposition 1] already implies that if a couple $(X, Y)$ of quasi-normed lattices is boundedly AK-stable then $(X^{\delta}, Y^{\delta})$ is AK-stable for all $\delta > 0$. However, it does not seem to say anything about the bounded AK-stability, and both [19, Theorem 2] and [13, Theorem 3.6] are rather nontrivial results.

**Proposition 4.10** Suppose that $(X, Y)$ is a couple of quasi-normed lattices of measurable functions on $\mathbb{T} \times \Omega$. If $(X, Y)$ is boundedly AK-stable then so is $(X^{\delta}, Y^{\delta})$ for all $\delta > 0$.

Indeed, if $f \in X^{\delta}$ and $g \in X^{\delta}$ are nonnegative then by the bounded AK-stability of $(X, Y)$ there exists some $V \in H_\infty$ such that

$$\|V\|_{L_\infty} \leq C, \quad \|V^{1/2}g^{1/2}\|_X \leq C \left\|\left(1 - V\right)^{1/2}f\right\|_X \quad \text{and} \quad \left\|\left(1 - V\right)^{1/2}f\right\|_Y \leq C \left\|g^{1/2}\right\|_Y.$$ 

These conditions are exactly $\|V^\delta g\|_{X^\delta} \leq C^\delta \|f\|_{X^\delta}$ and $\|1 - V^\delta f\|_{Y^\delta} \leq C^\delta \|g\|_{Y^\delta}$, so by Lemma 4.8 couple $(X^{\delta}, Y^{\delta})$ is boundedly AK-stable.

## 5 Proof of Theorem 2.8

The following observation is key for the “if” part of Theorem 2.8.

**Proposition 5.1** Suppose that $Z$ is a quasi-normed lattice of measurable functions on $\mathbb{T} \times \Omega$ such that couple $(L_\infty, L_\infty, Z)_{\theta, \infty}$ is boundedly AK-stable with some $0 < \theta < 1$. Then $(L_\infty, Z^\alpha)$ is boundedly AK-stable for all $0 < \alpha < \theta$.

Indeed, let $f \in L_\infty$ and $g \in Z^\alpha$. For simplicity we may assume that $g$ is nonnegative and $f = 1$. Then $g^{\alpha/\theta} \in Z^\theta \subset (L_\infty, Z)_{\theta, \infty}$ with norm at most $c\|g\|_{Z^\alpha}^{\theta/\alpha}$.
by Proposition 3.4, where $c$ is some constant independent of $g$. By the assumed bounded AK-stability there exists some $U \in H_\infty$ such that $\|U\|_{L_\infty} \leq C$, $\|U g^{\frac{\alpha}{\rho}}\|_{L_\infty} \leq C \|f\|_\infty = C$ and $\|(1 - U)f\|_{(L_\infty, Z)_{\theta, \infty}} \leq c C \|g\|_{Z}$. From the second estimate it follows that

$$\|U g\|_{L_\infty} = \|U|\frac{\alpha}{\rho} g^{\frac{\alpha}{\rho}}\|_{L_\infty} \leq C^{1-\frac{\alpha}{\rho}} \|U g^{\frac{\alpha}{\rho}}\|_{L_\infty} \leq C. \quad (1)$$

But we also have $\|(1 - U)f\|_{L_\infty} \leq C + 1$. By Proposition 3.4 and the reiteration theorem lattice $Z^\alpha$ is a space of type $C_{\alpha}^\rho (L_\infty, (L_\infty, Z)_{\theta, \infty})$, thus

$$\|(1 - U)f\|_{Z^\alpha} \leq c_1 \|(1 - U)f\|_{L_\infty}^{1-\frac{\alpha}{\rho}} \|(1 - U)f\|_{(L_\infty, Z)_{\theta, \infty}} \leq c_2 \|g\|_{Z} \quad (2)$$

with some constants $c_1$ and $c_2$ independent of $g$. (1) and (2) together show that couple $(L_\infty, Z^\alpha)$ is indeed boundedly AK-stable.

This result together with Proposition 4.10 allows us to relax the assumption of the strong AK-stability to the usual AK-stability in Proposition 4.4, if we additionally assume that $Z$ has the Fatou property.

**Proposition 5.2** Suppose that $Z$ is a $r$-convex quasi-normed lattice of measurable functions on $\mathbb{T} \times \Omega$ with some $r > 0$ satisfying the Fatou property and property $(\*)$. Couple $(L_\infty, Z)$ is AK-stable if and only if it is boundedly AK-stable.

By [22, Remark 3 after Volume II, Proposition 2.g.22] lattice $E = (L_\infty, Z)_{\theta, \infty}$ is Banach up to an equivalent renorming if $\theta$ is small enough, and it has the Fatou property (see, e.g., Proposition 6.9 below). Couple $(L_\infty, E)$ is AK-stable by [13, Lemma 1.1], and it is strongly AK-stable by [19, Lemma 3]. By Proposition 4.4 it is boundedly AK-stable, and by Proposition 5.1 couple $(L_\infty, Z^\alpha)$ is boundedly AK-stable for all $0 < \alpha < \theta$. By Proposition 4.10 couple $\left(\frac{1}{L_\infty}, (Z^\alpha)^{\frac{1}{\rho}}\right) = (L_\infty, Z)$ is then boundedly AK-stable as claimed.

We are now ready to prove the “if” part of Theorem 2.8.

**Proposition 5.3** Let $(X, Y)$ be a couple of quasi-Banach lattices of measurable functions on $\mathbb{T} \times \Omega$ that are $r$-convex with some $r > 0$ satisfying the Fatou property and property $(\*)$. Suppose also that $(X^r)'Y^r$ is a Banach lattice. If $(X, Y)$ is weak-type BMO-regular then it is boundedly AK-stable.

By Proposition 3.5 lattice $\left(L_1, L_1^{1-\beta} [(X^r)'(Y^r)]^{\beta}\right)_{\theta, 1}$ is BMO-regular with some $0 < \beta, \theta < 1$. Similarly to the proof of symmetry in Proposition 3.7, by the duality theorems for the real interpolation and for BMO-regularity [30, Theorem 1] lattice

$$\left(L_1, L_1^{1-\beta} [(X^r)'(Y^r)]^{\beta}\right)_{\theta, 1} = \left(L_\infty, [(X^r)'(Y^r)]^{\beta}\right)_{\theta, \infty}$$

is BMO-regular. Lattice $L_\infty$ is BMO-regular, so couple

$$\left(L_\infty, \left(\frac{1}{L_\infty}, \left[(X^r)'(Y^r)]^{\beta}\right)_{\theta, \infty}\right)$$
is BMO-regular, and hence boundedly AK-stable. By Proposition 5.1 it follows that couple \( \left( L_\infty, [\!(X^r)'(Y^r)]^\beta \alpha \right) \) is boundedly AK-stable for all \( 0 < \alpha < \theta \). Therefore, couple

\[
\left( L_\infty Y^r, \left[\!(\!(X^r)'(Y^r))'\right]' Y^r \right) = \left( Y^r Y^r, \left[\!(X^r)'\right]' Y^r \right) = \left( Y^r Y^r, X^r Y^r \right)
\]

is also boundedly AK-stable by Proposition 4.5 (applied with \( Z = Y^r Y^r \)) with some \( \gamma = \beta \alpha \). Here we have used the formula from Proposition 3.1. Finally, by Proposition 4.10 couple \((Y, X)\) is boundedly AK-stable, and it remains to change the order of the lattices.

The proof of the “only if” part of Theorem 2.8 is based on the following result, which is a fairly simple consequence of [11, Theorem 3.3].

**Proposition 5.4** [31, Corollary 13] Let \( Z \) be a Banach lattice of measurable functions on \( \mathbb{T} \times \Omega \) satisfying the Fatou property and property \((\ast)\), and assume that both \( Z \) and \( Z' \) have order continuous norm. Suppose that \((Z, Z')\) is AK-stable. Then the Riesz projection is bounded in \((L_2, Z)_{\zeta, 2}\) for all sufficiently small \( 0 < \zeta < 1 \).

We establish the “only if” part of Theorem 2.8 under a somewhat broader set of assumptions. The next proposition covers the same assumptions as in the main results of [31] and follows the corresponding details of the reductions. However, we generalize them substantially. Specifically, [31] only had assumption (ii) below with \( p = 2 \), assumption (v) with \( p \in \{1, 2, \infty\} \), and it did not have assumptions (iv) and (vi). For simplicity, we state these assumptions with some asymmetry and redundancy. Most prominently, assumption (vi) generalizes assumptions (ii), (iv) and (v), and all four of these assumptions are superseded by Corollary 5.8 below. These conditions strongly resemble the superreflexivity assumptions used in [11].

**Proposition 5.5** Suppose that \((X, Y)\) is a couple of Banach lattices of measurable functions on \( \mathbb{T} \times \Omega \) satisfying the Fatou property, property \((\ast)\) and at least one of the following conditions:

(i) \( X \) and \( Y \) have order continuous norm and \( Y = X' \);
(ii) \( X \) is \( p \)-convex and \( Y \) is \( p' \)-convex with some \( 1 \leq p \leq \infty \);
(iii) \( X'Y \) is Banach;
(iv) \( X \) is \( p \)-concave and \( Y \) is \( p \)-convex with some \( 1 \leq p \leq \infty \);
(v) \( X = L_p \) with \( 1 \leq p \leq \infty \);
(vi) \( X \) is \( p \)-convex and \( q \)-concave with some \( 1 \leq p \leq q \leq \infty \) and \( Y \) is \( \left( \frac{1}{p'} + \frac{1}{q} \right)^{-1} \)-convex.

If couple \((X, Y)\) is AK-stable then it is weak-type BMO-regular.

Suppose that under the assumptions of Proposition 5.5 couple \((X, Y)\) is AK-stable. Under assumptions (i), Proposition 5.4 applied to \( Z = Y \) shows that the Riesz projection is bounded in \( Z_1 = (L_2, Y)_{\zeta, 2} \) with some \( 0 < \zeta < 1 \), which implies by [18, Theorem 3] (see also [30]) that \( Z_1 \) is BMO-regular. This yields with the help of Proposition 3.2 the BMO-regularity of \( Z_1^2 = (L_2, Y)_{\zeta, 2}^2 = (L_1, Y^2)_{\zeta, 1} = (L_1, X'Y)_{\zeta, 1} \), which is the weak-type BMO-regularity of \((X, Y)\).
Now we will show that the conclusion holds true if the assumptions (ii) are satisfied in the special case $p = 2$. Let $X_1 = X^2$ and $Y_1 = Y^2$. Thus $(X, Y) = \left( \frac{1}{2} X^2, \frac{1}{2} Y^2 \right)$ is an AK-stable couple of Banach lattices. By Proposition 4.6 couple 
\[
\left( X_1^\frac{1}{2} (X_1 Y_1)^\frac{1}{2}, Y_1^\frac{1}{2} (X_1 Y_1)^\frac{1}{2} \right) = \left( \left[ X_1^\frac{1}{2} X_1^\frac{1}{2} \right] X_1^\frac{1}{2} Y_1^\frac{1}{2}, \left[ Y_1^\frac{1}{2} Y_1^\frac{1}{2} \right] X_1^\frac{1}{2} Y_1^\frac{1}{2} \right)
\]
\[
= \left( L_1^\frac{1}{2} X_1^\frac{1}{2} Y_1^\frac{1}{2}, L_1^\frac{1}{2} X_1^\frac{1}{2} Y_1^\frac{1}{2} \right) = \left( L_1^\frac{1}{2} X_1^\frac{1}{2} Y_1^\frac{1}{2}, L_1^\frac{1}{2} X_1^\frac{1}{2} Y_1^\frac{1}{2} \right)
\]
is also AK-stable. This couple satisfies assumptions (i), so it is weak-type BMO-regular, and therefore couple $(X, Y)$ is weak-type BMO-regular by the transition (ii) \Rightarrow (i) of Proposition 3.7 (applied with $Z = \left( X_1 Y_1 \right)^\frac{1}{2}$).

If assumptions (iii) are satisfied, by Proposition 4.7 couple $(X, Y)$ is boundedly AK-stable, and by Proposition 4.10 so is couple \( \left( \frac{1}{2} X^2, \frac{1}{2} Y^2 \right) \). This couple satisfies assumptions (ii) with $p = 2$, thus it is weak-type BMO-regular, and Proposition 3.5 again yields the weak-type BMO-regularity of the original couple $(X, Y)$.

Under assumptions (iv) lattice $X'$ is $p'$-convex, so lattice $X'Y$ is $1$-convex and the couple satisfies assumptions (iii).

Under assumptions (v) cases $p = 1$ and $p = \infty$ satisfy assumptions (iii) (in the case $p = \infty$ we need to reverse the order of the couple), so the interesting case is $1 < p < \infty$. We may further assume that $p \geq 2$, otherwise we may pass to the duals in the AK-stability by [18, Lemma 7] and then use the duality in Proposition 3.7. The AK-stability of $(L_p, Y) = \left( Y^{\frac{1}{p}} \left( Y^{\frac{p'}{p}} \right)^{\frac{1}{p'}}, Y^{\frac{1}{p'}} Y^{\frac{1}{p}} \right)$ by [19, Corollary to Lemma 4] is equivalent to the AK-stability of \( \left( Y^{\frac{1}{p'}} Y^{\frac{1}{p}} \right) = \left( Y^{\frac{1}{p'}} Y^{\frac{1}{p}} \right) \). Let $Z_2 = Y^{\frac{1}{p'}} Y^{\frac{1}{p}}$. By Proposition 4.6 the latter AK-stability implies the AK-stability of 
\[
\left( Y^{\frac{1}{p'}} Z_2, Y^{\frac{1}{p'}} Z_2 \right)
\]
\[
= \left( Y^{\frac{1}{p'}} Y^{\frac{1}{p'}} \right) L^{\frac{1}{p'}}, Y^{\frac{1}{p'}} Y^{\frac{1}{p'}} \right) Y^{\frac{1}{2}} \left( \frac{1}{p'} - \frac{1}{p} \right) \left( \frac{1}{p'} - \frac{1}{p} \right) L^{\frac{1}{p'}} \right)
\]
\[
= \left( Y^{\frac{1}{p'}} L^{\frac{1}{p'}}, Y^{\frac{1}{p'}} L^{\frac{1}{p'}} \right) = \left( Y^{\frac{1}{p'}} L^{\frac{1}{p'}}, Y^{\frac{1}{p'}} L^{\frac{1}{p'}} \right)
\]
This couple satisfies assumptions (i), so it is weak-type BMO-regular. By running the respective multiplications and divisions in reverse we get the weak-type BMO-regularity of the original couple $(L_p, Y)$ by Proposition 3.7.

Now, suppose that assumptions (vi) are satisfied. If $p = \infty$ then assumptions (iv) are satisfied with the order of $X$ and $Y$ reversed, so the interesting case is $1 \leq p < \infty$. Let $Z_3 = (X^p)^{\frac{1}{p}}$. Observe that $L_1 = X^p (X^p)'$, so $L_p = L^{\frac{1}{p}} = XZ_3$. Couple
\((XZ_3, YZ_3) = (L_p, YZ_3)\) is AK-stable by Proposition 4.6. Lattice \(X^p\) is \(\frac{q}{p}\)-concave, so \((X^p)’\) is \(\left(\frac{q}{p}\right)’\)-convex and \(Z_3\) is \(r\)-convex with \(r = p \left(\frac{q}{p}\right)’ = \frac{qp}{q-p}\). Simple computations show that \(YZ_3\) is then \(1\)-convex and hence Banach up to a renorming. Therefore, this couple satisfies assumptions (v), so it is weak-type BMO-regular, and \((X, Y)\) is weak-type BMO-regular by Proposition 3.7. Finally, assumptions (ii) with arbitrary \(p\) are exactly assumptions (vi) with \(q = \infty\).

**Corollary 5.6** Suppose that \((X, Y)\) is a couple of \(r\)-convex quasi-normed lattices of measurable functions on \(\mathbb{T} \times \Omega\) with some \(r > 0\) satisfying the Fatou property and property (\(\ast\)). If \((X, Y)\) is boundedly AK-stable then \((X, Y)\) is weak-type BMO-regular.

Indeed, by Proposition 4.10 couple \(\left(X^\frac{q}{2}, Y^\frac{q}{2}\right)\) is boundedly AK-stable, so condition (ii) of Proposition 5.5 is satisfied with \(p = 2\). Therefore, this couple is weak-type BMO-regular, and by Corollary 3.6 couple \((X, Y)\) is weak-type BMO-regular.

For couples of quasi-normed lattices it is possible to establish the necessity of the weak-type BMO-regularity for the strong AK-stability if one of the lattices has nontrivial convexity. It is not clear if the strong AK-stability is equivalent to the usual one for couples of quasi-normed lattices if \(\Omega\) is arbitrary.

**Proposition 5.7** Let \((X, Y)\) be a couple of quasi-normed lattices of measurable functions on \(\mathbb{T} \times \Omega\) satisfying the Fatou property and property (\(\ast\)). Suppose also that \(X\) is \(p\)-convex with some \(1 < p < \infty\) and \(Y\) is \(r\)-convex with some \(r > 0\). If \((X, Y)\) is strongly AK-stable then it is weak-type BMO-regular.

Similarly to the proof of Proposition 5.5, we define a Banach lattice \(Z = (X^p)^{\frac{1}{p}}\). By Proposition 4.6 couple \((ZX, ZY) = (L_p, ZY)\) is AK-stable. By [13, Lemma 1.1] couple \((L_p, E), E = (L_p, ZY)_{\theta, q}\) is also AK-stable for all \(0 < \theta < 1\) and \(0 < q \leq \infty\). By [22, Remark 3 after Volume II, Proposition 2.g.22] lattice \(E\) is Banach up to an equivalent renorming if \(\theta\) is small enough. This couple satisfies assumptions (v) of Proposition 5.5, thus it is weak-type BMO-regular. This means that lattice \(F = (L_1, L_pE)_{\theta, s}\) is BMO-regular with some \(0 < \eta < 1\) and \(0 < s \leq \infty\). Observe that \(L_pE = L_1^{1 - \frac{1}{p}} (L_1, [ZY]^p)_{\theta, s} = C^{\frac{1}{p}} (L_1, [ZY]^p)\) by Proposition 3.2, therefore this lattice is of type \(C^{\frac{1}{p}} (L_1, (L_1, [ZY]^p)_{\theta, s}) = C^{\frac{1}{p}} (L_1, [ZY]^p)\) by Proposition 3.4 and the reiteration theorem. Thus, \(F = (L_1, [ZY]^p)_{\theta, s} = (L_1, (X^p)^{\frac{q}{p}})_{\theta, s}\) is a BMO-regular lattice, which means that couple \((X, Y)\) is weak-type BMO-regular by Proposition 3.5 as claimed.

Proposition 5.7 together with the fact that the order dual lattice of a \(q\)-concave Banach lattice is \(q’\)-convex, [19, Lemma 3],[18, Lemma 7] and Proposition 3.7 imply the following.

**Corollary 5.8** Let \((X, Y)\) be a couple of Banach lattices of measurable functions on \(\mathbb{T} \times \Omega\) satisfying the Fatou property and property (\(\ast\)), and at least one of these lattices is either \(p\)-convex with some \(p > 1\) or \(q\)-concave with some \(q < \infty\). If \((X, Y)\) is AK-stable then it is weak-type BMO-regular.
These results verify the “only if” part of Theorem 2.8. In the case \( Y = L_\infty \) it follows from Proposition 5.2 and either Corollary 5.6 or Proposition 5.7. In the case when \( X'Y \) is a Banach lattice this follows from Proposition 5.5 with assumptions (iii).

### 6 Bounded N+\(^{\ast}\)-stability

Similarly to the bounded AK-stability, we may define a stronger version of N+\(^{\ast}\)-stability with respect to the real interpolation functors. Let \( \lambda > 1 \) be a base for power decompositions. We may fix the standard value \( \lambda = 2 \) in the present work. Recall that \( J(t, g; X, Y) = \|g\|_X \vee t \|g\|_Y \) for \( t > 0 \) and \( g \in X \cap Y \), and the real interpolation space \((X, Y)_{\theta, p}\) may be defined by the so-called \( J \) method as the space of functions \( f \in X + Y \) having decompositions \( f = \sum_j f_j \) with finite norm

\[
\|f_j\|_{\chi(X, Y)_{\theta, p}} = \left\{ \lambda^{-\theta j} J(\lambda_j, f_j; X, Y) \right\}_{j \in \mathbb{Z}} \|f\|_{lp},
\]

and the norm of \( f \) in \((X, Y)_{\theta, p}\) is taken to be the infimum of \( \|f_j\|_{\chi(X, Y)_{\theta, p}} \) over all such decompositions.

**Definition 6.1** Let \((X, Y)\) be a couple of quasi-Banach lattices of measurable functions on \( \mathbb{T} \times \Omega \). We say that \((X, Y)\) is boundedly N+\(^{\ast}\)-stable with respect to \((\cdot, \cdot)_{\theta, p}\) with constant \( C \) if for any \( f \in [(X, Y)_{\theta, p}]_A \) there exists some \( \varphi = \{\varphi_j\}_{j \in \mathbb{Z}} \in H_\infty(l^1) \) with norm at most \( C \) such that \( \sum_j \varphi_j = 1 \) and

\[
\left\| \left\{ \lambda^{-\theta j} J(\lambda_j, \varphi_j f; X, Y) \right\}_{j \in \mathbb{Z}} \right\|_{lp} \leq C \|f\|_{(X, Y)_{\theta, p}}.
\]

It is easy to verify that this definition does not depend on a particular choice of the base \( \lambda > 1 \). Unlike the bounded AK-stability, it is not clear if this property is stable with respect to the multiplication by a lattice as in Proposition 4.5. Also, it is not clear if there are easy and general equivalence results deriving it from the usual N+\(^{\ast}\)-stability similarly to Propositions 4.4 and 4.7, or if it is stable under raising to powers greater than 1. However, raising to powers \( 0 < \delta < 1 \) still works, and it allows us to improve the convexity of the boundedly N+\(^{\ast}\)-stable lattices.

**Proposition 6.2** Suppose that a couple \((X, Y)\) of quasi-Banach lattices of measurable functions on \( \mathbb{T} \times \Omega \) satisfying property (\( \ast \)) is boundedly N+\(^{\ast}\)-stable with respect to \((\cdot, \cdot)_{\theta, p}\). Then \((X^\delta, Y^\delta)\) is boundedly N+\(^{\ast}\)-stable with respect to \((\cdot, \cdot)_{\theta, \frac{\delta}{\mathbb{Z}}}\) for all \( 0 < \delta < 1 \).

Indeed, suppose that \( F \in [(X^\delta, Y^\delta)_{\theta, \frac{\delta}{\mathbb{Z}}}]_A \) with norm 1 under the assumptions of Propostion 6.2. Lattice \((X, Y)_{\theta, p}\) has property (\( \ast \)) by [31, Proposition 9], so there exists some \( g \geq |F|, \|g\|_{(X^\delta, Y^\delta)_{\theta, \frac{\delta}{\mathbb{Z}}}} \leq 2 \) such that \( \log g(\cdot, \omega) \in L_1 \) for almost all \( \omega \in \Omega \). We construct the corresponding outer function \( G = \exp(g + i\mathcal{H}g) \). Observe that \( G^{\frac{1}{\delta}} \in [(X, Y)_{\theta, p}]_A \) with norm at most \( 2^{\frac{1}{\delta}} \) by proposition 3.2, so there exists some \( \varphi = \{\varphi_j\}_{j \in \mathbb{Z}} \in H_\infty(l^1) \) with norm at most \( C \) satisfying Definition 6.1 with \( f = G^{\frac{1}{\delta}} \).
Then the same function $\varphi$ yields the claimed bounded $N^+$-stability for $F$, which follows from the estimate

$$|\varphi_j F| \leq |\varphi_j G| = |\varphi_j|^{1-\delta} |\varphi_j G^{1+\frac{\delta}{2}}|^{\delta} \leq C^{1-\delta} |\varphi_j G^{\frac{1}{2}}|^{\delta}.$$  

We will show that bounded stability in the sense of Definition 6.1 often naturally arises from the usual stability. We do this in a more abstract setting that will be useful elsewhere.

**Definition 6.3** Let $I \subset \mathbb{Z}$, $\mathcal{M}$ be a $\sigma$-finite measurable space. Suppose that $\mathcal{X}$ is a quasi-normed lattice of measurable functions on $\mathcal{M} \times I$ and $R$ is a quasi-normed lattice of measurable functions on $\mathcal{M}$. We say that $\mathcal{X}$ is sumvable if for any $\{f_j\}_{j \in I} \in \mathcal{X}$ the sum $\sum_{j \in I} |f_j|$ is finite almost everywhere. Let $S_n \{f_j\} = \sum_{j \in I \cap \{-n,n\}} f_j$, $n \in \mathbb{N} \cup \{\infty\}$. We say that $\mathcal{X}$ is uniformly $R$-sumvable if $\|S_\infty - S_n\|_{\mathcal{X} \to R} \to 0$.

**Definition 6.4** Let $\mathcal{M}$ be a $\sigma$-finite measurable space, and suppose that $\mathcal{X}$ is a sumable quasi-normed lattice of measurable functions on $\mathcal{M} \times I$. We define a lattice

$$J(\mathcal{X}) = \left\{ \sum_{j \in I} f_j \mid \{f_j\}_{j \in I} \in \mathcal{X} \right\}$$

with the corresponding quasi-norm

$$\|f\|_{J(\mathcal{X})} = \inf \left\{ \|\{f_j\}_{j \in I}\|_{\mathcal{X}} \mid \sum_{j \in I} f_j = f, \{f_j\}_{j \in I} \in \mathcal{X} \right\}.$$  

For example, if $X$ and $Y$ are quasi-Banach lattices, taking $I = \mathbb{Z}$ and the lattice $\mathcal{X}(X,Y)_{\theta,p}$ defined by its norm introduced at the beginning of the section yields $J(\mathcal{X}(X,Y)_{\theta,p}) = (X,Y)_{\theta,p}$. It is easy to see that $\mathcal{X}(X,Y)_{\theta,p}$ is uniformly $(X+Y)$-sumvable.

**Definition 6.5** Let $I \subset \mathbb{Z}$, and let $\mathcal{X}$ be a sumable quasi-normed lattice of measurable functions on $\mathbb{T} \times \Omega \times I$. We say that $J(\mathcal{X})$ is $N^+$-stable with constant $C$ if for any $f \in [J(\mathcal{X})]_A$ there exists some $F = \{f_j\}_{j \in I} \in \mathcal{X}_A$ such that $f = \sum_{j \in I} f_j$ and $\|F\|_{\mathcal{X}} \leq C \|f\|_{J(\mathcal{X})}$. We say that $J(\mathcal{X})$ is boundedly $N^+$-stable with constant $C$ if in the above we may take $f_j = f \varphi_j$, $j \in I$, $\{\varphi_j\}_{j \in \mathbb{Z}} \in H_\infty (t^1)$ with norm at most $C$.

In the example above, the (bounded) $N^+$-stability of $(\mathcal{X}(X,Y)_{\theta,p})$ is exactly the (bounded) $N^+$-stability of the couple $(X,Y)$ with respect to $(\cdot,\cdot)_{\theta,p}$.

**Theorem 6.6** Let $I \subset \mathbb{Z}$ and let $\mathcal{X}$ be a uniformly $R$-sumvable Banach lattice of measurable functions on $\mathbb{T} \times \Omega \times I$. Suppose that $\Omega$ is a discrete space, $\mathcal{X}$ has the Fatou property and $J(\mathcal{X})$ has property $(\ast)$. If $J(\mathcal{X})$ is $N^+$-stable then it is boundedly $N^+$-stable.
The proof of Theorem 6.6 is given in Sect. 8 below. As a consequence of Proposition 6.2, we at once get the following.

**Corollary 6.7** Suppose that \((X, Y)\) is a couple of Banach lattices of measurable functions on \(\mathbb{T} \times \Omega\) with a discrete space \(\Omega\) satisfying the Fatou property and property (\(*\)). If \((X, Y)\) is \(N^+\)-stable with respect to \((\cdot, \cdot)_{\sigma, \rho}\) then \((X^\delta, Y^\delta)\) is boundedly \(N^+\)-stable with respect to \((\cdot, \cdot)_{\sigma, \rho}\) for all \(0 < \delta \leq 1\).

The equivalence of AK-stability and the bounded AK-stability for couples of Banach lattices with the Fatou property and a discrete \(\Omega\) is also an easy consequence of Theorem 6.6. The proof of Theorem 2.7 allows us to further relax the convexity assumptions; see Corollary 11.2 below. We also mention that although we only work with couples of lattices in the present work, the proof as written also yields the boundedness of the AK-stability for an arbitrary finite family of lattices.

**Corollary 6.8** Suppose that \(X_0\) and \(X_1\) are Banach lattices of measurable functions on \(\mathbb{T} \times \Omega\) satisfying the Fatou property and property (\(*\)). Suppose also that \(\Omega\) is a discrete space. Then couple \((X_0, X_1)\) is AK-stable if and only if it is boundedly AK-stable.

Let \(I = \{0, 1\}\). Suppose that \((X, Y)\) is AK-stable. Let \(g_j \in X_j\) be some nonzero functions, and define a lattice \(\mathcal{X}\) on \(\mathbb{T} \times \Omega \times I\) with a norm \(\|\{f_j\}_{j \in I}\|_{\mathcal{X}} = \bigvee_{j \in I} \|g_j\|_{X_j}^{-1} \|f_j\|_{X_j}\). The AK-stability implies the \(N^+\)-stability of \(J(\mathcal{X})\), and incidentally is equivalent to the latter satisfied uniformly over arbitrary functions \(g_j \in X_j, j \in I\). Let \(h_j \geq |g_j|, j \in I\) be the corresponding majorants from property (\(*\)), \(h = \sum_{j \in I} |h_j|\) and \(H = \exp \left( \log h + i \mathcal{H}(\log h) \right) \in [J(\mathcal{X})]_A\) with norm at most \(c\) with some \(c\) independent of \(g_j\). By Theorem 6.6 lattice \(J(\mathcal{X})\) is boundedly \(N^+\)-stable with a constant \(C\), so there exist some \(\{\varphi_j\}_{j \in I} \in H_\infty(l^1)\) with norm at most \(C\) such that \(\sum_{j \in I} \varphi_j = 1\) and

\[
\left\| \varphi_k \sum_{j \in I} |g_j| \right\|_{X_k} \leq \|\varphi_k H\|_{X_k} \leq C \|g_k\|_{X_k}, \quad k \in I.
\]

Setting \(U = \varphi_0\) yields the bounded AK-stability of the couple \((X_0, X_1)\).

The uniform \(R\)-summability property is a convenient condition that ensures the closedness with respect to the convergence in measure (on all sets of finite measure) of the set of corresponding decompositions for lattices \(\mathcal{X}\) with the Fatou property, and also that lattices \(J(\mathcal{X})\) inherit the Fatou property from \(\mathcal{X}\).

**Proposition 6.9** Let \((\mathcal{M}, \mu)\) be a \(\sigma\)-finite measurable space, \(I \subset \mathbb{Z}\), and let \(\mathcal{X}\) be a uniformly \(R\)-summable Banach lattice of measurable functions on \(\mathcal{M} \times I\) with the Fatou property. Let \(C > 0\). Then the graph of a set-valued mapping \(D_{\mathcal{X}, C} : J(\mathcal{X}) \to 2^{\mathcal{X}}\) defined by

\[
D_{\mathcal{X}, C}(f) = \left\{ \{f_j\}_{j \in I} \mid f = \sum_{j \in I} f_j, \|\{f_j\}_{j \in I}\|_{\mathcal{X}} \leq C \right\}, \quad f \in J(\mathcal{X})
\]

is a closed graph.
is closed with respect to the convergence in measure on sets of finite measure. In particular, sets \( D_{X,C}(f) \) are closed with respect to this convergence, they are nonempty for \( C \geq \|f\|_{J(X)} \) and \( J(X) \) is a Banach lattice with the Fatou property.

Indeed, let \( f_k \in J(X) \) and \( g^{(k)} \in D_{X,C}(f_k) \) be some sequences such that \( g^{(k)} \to g \) and \( f_k \to f \) in measure on sets of finite measure. By passing to a subsequence we may assume that \( g^{(k)} \to g \) and \( f_k \to f \) almost everywhere, so \( g \in X \) with \( \|g\|_X \leq C \) by the Fatou property. Observe that

\[
f = (f - f_k) + S_\infty g^{(k)} = (f - f_k) + S_\infty g + S_n \left( g^{(k)} - g \right) + (S_\infty - S_n) \left( g^{(k)} - g \right) .
\]  

(3)

Let \( E \subset M \) be a measurable set of finite measure and \( \varepsilon > 0 \). The first and the third terms on the right-hand side of (3) converge to 0 almost everywhere in \( k \) for any \( n \). With the help of the Egoroff theorem and the diagonal process we may choose an increasing sequence \( n \mapsto k_n \) such that \( \|S_n \left( g^{(k_n)} - g \right) \| \leq 2^{-n} \) and \( |f - f_{k_n}| \leq 2^{-n} \) on a set \( F \subset E \) such that \( \mu(E \setminus F) < \varepsilon \), and in particular the first and the third terms in (3) converge to 0 almost everywhere with \( k = k_n \). By the uniform \( R \)-boundedness

\[
\left\| (S_\infty - S_n) \left( g^{(k_n)} - g \right) \right\|_R \leq \|S_\infty - S_n\|_{X \to R} \left( \left\| g^{(k_n)} \right\|_X + \|g\|_X \right) \to 0,
\]

so the fourth term in (3) with \( k = k_n \) converges to 0 in measure on sets of finite measure. Thus \( f = S_\infty g \) almost everywhere on \( F \). Since \( \varepsilon > 0 \) and \( E \) are arbitrary, it is easy to see that \( f = S_\infty g \) almost everywhere, and therefore \( g \in D_{X,C}(f) \), which shows that the graph of \( D_{X,C} \) is closed with respect to the convergence in measure on sets of finite measure.

Now, with \( C > \|f\|_{J(X)} \) sets \( D_{X,C}(f) \) are evidently nonempty, and

\[
D_{X,\|f\|_{J(X)}}(f) = \bigcap_{C > \|f\|_{J(X)}} D_{X,C}(f)
\]

is nonempty as an intersection of a centered family of nonempty convex sets that are bounded in the lattice \( X \) with the Fatou property and closed with respect to the convergence in measure on sets of finite measure by \( [12, \text{Chap. 10, §5, Theorem 3}] \).

7 Topology of Uniform Convergence on Compact Sets

Our methods for establishing certain properties of interest such as Theorem 6.6 and its corollaries are based on a fixed point theorem, and they rely on the closedness of certain maps in suitable topologies that also make certain bounded sets in lattices compact. At present, it is not clear whether it is possible to carry out these arguments for general spaces \( \Omega \). Fortunately, at least for discrete spaces \( \Omega \) there is a natural topology of uniform convergence on compact sets in \( \mathbb{D} \times \Omega \) that allows us to verify the required properties of the maps without much trouble. Here \( \mathbb{D} \) is the open unit disc of the complex plane.
Proposition 7.1 Let $X$ be an $r$-convex quasi-normed lattice of measurable functions on $\mathbb{T} \times \Omega$ with a discrete space $\Omega$ and some $r > 0$. Suppose that $X$ satisfies the Fatou property and property $(\ast)$. Then the closed unit ball $B_{X_A}$ of $X_A$ is compact in the topology $\tau$ of the uniform convergence on all compact sets of $\mathbb{D} \times \Omega$.

Since $\tau$ is metrizable, it suffices to verify that for any sequence $f_n \in B_{X_A}$ there is a subsequence converging to some $f \in B_{X_A}$ in $\tau$. We will first prove the claim for Banach lattices $X$. Observe that there exists some $g \in X'$ such that $\|g\|_{X'} = 1$ and $g \geq 0$ a.e. (see, e.g., [29, Proposition 9]). Lattice $X'$ also satisfies property $(\ast)$ (see [18, Lemma 2]), so there exists some $w \in X'$ such that $w \in X'$, $w > g > 0$ a.e. and $\log w(\cdot, \omega) \in L_1$ for almost all $\omega \in \Omega$. We may assume that $\|w\|_{X'} = 1$. Thus we can construct an outer function $W = \exp(\log w + i\mathcal{H}(\log w))$ such that $|W| = w$ almost everywhere.

Let $\Omega_N \subset \Omega$, $N \in \mathbb{N}$ be an increasing sequence of finite sets such that $\bigcup_N \Omega_N = \Omega$. We inductively construct a sequence of increasing sequences $k \mapsto s_{N,k}$ starting with $s_{0,k} = k$ such that $s_{N,k}$ is a subsequence of $s_{N-1,k}$. Sequence $f_{s_{N-1,k}} W \chi_{\mathbb{T} \times \Omega_N}$ belongs to the unit ball of the space $H_1(\mathbb{T} \times \Omega_N)$, which is dual to $C(\mathbb{T} \times \Omega_N)/C_A(\mathbb{T} \times \Omega_N)$, so there exists an increasing subsequence $s_{N,k}$ of $s_{N-1,k}$ such that sequence $f_{s_{N,k}} W \chi_{\mathbb{T} \times \Omega_N}$ converges in the $\ast$-weak topology to some $h_N \in H_1(\mathbb{T} \times \Omega_N)$, and therefore $f_{s_{N,k}} W \chi_{\mathbb{D} \times \Omega_N} \rightharpoonup h_N$ in $\tau$. Functions $h_M$ and $h_N$ coincide on $\mathbb{D} \times \Omega_M$ for all $M \leq N$, and we may define a function $h$ by $h(z, \omega) = h_N(\omega)$ for $z \in \mathbb{D}$ and some $N$ such that $\omega \in \Omega_N$. For the diagonal sequence $n': N \mapsto s_{N,N}$ we have $f_{n'} W \rightharpoonup h$ in $\tau$.

We need to verify that $f = W^{-1}h \in B_{X_A}$. Indeed, by a well-known corollary to the Fatou property (see, e.g., [29, Proposition 10] or [30, Proposition 3.3]) there exists a sequence $\varphi_j$ of finite convex combinations of $\{f_{n'}\}_{n' > j}$ such that $\varphi_j \to \varphi$ almost everywhere on $\mathbb{T} \times \Omega$ for some $\varphi \in B_{X_A}$, and we also have $\varphi_j \to f$ in $\tau$. Since $\|\varphi_j W(\cdot, \omega)\|_{H_1} \leq 1$ for all $\omega \in \Omega$, sequence $\varphi_j W$ satisfies the assumptions of the Khinchin-Ostrowski theorem ([27, Chap. II, §7.1]; see also [5, §6.1] for a modern treatment). It follows that the boundary values of $f W$ coincide with $\varphi W$, thus the boundary values of $f$ belong to $B_X$.

Now suppose that $X$ is $r$-convex with some $r \geq 1/N$, $N \geq 2$ being an integer number, and $f_n \in B_{X_A}$. $X^{1/N}$ is a Banach lattice up to an equivalent renorming because it is $1$-convex and satisfies the Fatou property, so the conclusion of Proposition 7.1 applies to it. By a similar construction to the above there exists some

$$w \in \left(X^{1/N}\right)^{\prime}, \quad \|w\|_{X^{1/N}} = 1$$

and an outer function $W$ such that $|W| = w$ almost everywhere. From the inclusion $|f_n|^{1/N} w \in L_1$ it follows that functions $F_n = f_n W^N$ belong to the unit ball of $\left([L_1]^N\right)_A = H_{1/N}$, hence they admit inner-outer factorization $F_n = I_n G_n$. Observe that both $H_{0,n} = G_n^{1/N} W^{-1}$ and $H_{1,n} = I_n H_{0,n}$ belong to the closed unit ball of $\left(X^{1/N}\right)_A$, and $f_n = H_{0,n}^{-1} H_{1,n}$. For some subsequence $n'$ we have $H_{0,n'} \to h_0$.
and \( H_{1,n'} \to h_1 \) in \( \tau \) with some \( h_0, h_1 \) in the closed unit ball of \( \left( X_1^{1/} \right)_A \), and \( f = h_0^{-1}h_1 \in B_{X_A} \) as claimed.

It is interesting to note that Proposition 7.1 allows us to generalize the equivalence of AK-stability and strong AK-stability to quasi-normed lattices with discrete space \( \Omega \), although we do not use this generalization in the present work. For the case of Banach lattices and arbitrary \( \Omega \) this was already established in [19, Lemma 3], and for \( Y = L_\infty \) and arbitrary \( \Omega \) this follows from Proposition 5.2.

**Proposition 7.2** Let \((X, Y)\) be a couple of quasi-normed lattices of measurable functions on \( T \times \Omega \) satisfying the Fatou property and property \((*)\). Suppose also that \( X \) and \( Y \) are r-convex with some \( r > 0 \) and \( \Omega \) is discrete. Then couple \((X, Y)\) is AK-stable if and only if it is strongly AK-stable.

Indeed, suppose that \( H \in (X + Y)_A \) and \( H = f + g \) with some \( f \in X \) and \( g \in Y \). Following the proof of [19, Lemma 3] we construct a sequence of outer functions \( \varphi_n \) such that \( |\varphi_n| \leq 1 \), \( \varphi_n \to 1 \) in \( \tau \) and \( \varphi_n H \in X_A + Y_A \). By the AK-stability there exist some \( F_n \in X_A \) and \( G_n \in Y_A \) such that \( \varphi_n H = F_n + G_n \), \( \|F_n\|_X \leq C\|f\|_X \) and \( \|G_n\|_Y \leq C\|g\|_Y \). By Proposition 7.1 there exists a subsequence \( n' \) such that \( F_{n'} \to F \) and \( G_{n'} \to G \) in \( \tau \) with some \( F \in X_A \), \( G \in Y_A \). But then we also have \( H = F + G \).

### 8 Proof of Theorem 6.6

We will use the following fixed point theorem, which is hard to find in the literature in this particular form.

**Theorem 8.1** Let \( K_j, 1 \leq j \leq n \) be convex compact sets in some (Hausdorff) locally convex linear topological spaces, and let \( K_0 = K_n \). Suppose that \( \Phi_j : K_j \to 2^{K_{j+1}}, 0 \leq j \leq n - 1 \) are set-valued maps taking nonempty convex closed values. Suppose also that the graphs of \( \Phi_j \) are closed. Then their composition \( \Phi = \Phi_{n-1} \circ \ldots \circ \Phi_0 \) has a fixed point. That is, \( x \in \Phi(x) \) for some \( x \in K_0 \).

Formally, this result is a particular case of a very general [25, Corollary 1.1]. In the case when the composition \( \Phi \) consists of a single map (that is, if \( n = 1 \)), Theorem 8.1 becomes the classical Fan–Glicksberg–Kakutani fixed point theorem [4], [6] that generalizes the Kakutani fixed point theorem from the finite-dimensional spaces to the locally convex spaces. The finite-dimensional case of Theorem 8.1 is contained in the Powers fixed point theorem [26] (see also [8, §19.9] for the statement in context), which generalizes the Lefschetz fixed point theorem and is true for a much more general class of compositions of maps taking acyclic values. The generalization of the Powers theorem to the locally convex spaces can also be made directly by an approximation argument similarly to the Fan–Glicksberg–Kakutani theorem.

We will carry out some arguments based on a fixed point theorem applied to maps acting on sets of some majorants in lattices. In many cases it suffices to endow the sets of logarithms of such majorants with the weak topology of a weighted
space $L_2(\omega)$; see, e.g., [33, Proposition 30]. However, it is not clear if the convergence in this topology implies the convergence in the weak topology of $L_1$ since inclusion $L_\infty \subset [L_2(\omega)]' = L_2(\omega^{-1})$ cannot be satisfied in general, which leads to further complications. Fortunately, the sets of majorants still turn out to be compact in the weak topology of $L_1$. Some care needs to be taken, however, since it is not clear whether these sets are separable in this topology.

Lemma 8.2 Let $f \in L_1$ and $f \geq 1$ almost everywhere. Then $\log f \in L_2$ and $\| \log f \|_{L_2} \leq 2 \| f \|_{L_1}^{1/2}$.

We only need to observe that $\log \left( f^{1/2} \right) \leq f^{1/2}$ and

$$\int (\log f)^2 = 4 \int \left( \log f^{1/2} \right)^2 \leq 4 \int f.$$

Proposition 8.3 Let $X$ be a Banach lattice of measurable functions on $T \times \Omega$ with discrete $\Omega$, and let $f \in X$. Suppose that $X$ satisfies the Fatou property and property ($\ast$). Suppose also that $\log f(\cdot, \omega) \in L_1$ for all $\omega \in \Omega$. Then for all $A > 0$ sets

$$V_{X,f,A} = \{ \log g \in L_1 \mid g \geq f, \| g \|_X \leq A \}$$

are compact in the topology of weak convergence in $L_1(T \times \{\omega\})$ for all $\omega \in \Omega$.

First, suppose that $\Omega$ consists of a single point, so we only have one variable. It is well known that the positive part of the unit ball of a Banach lattice is logarithmically convex, so $V_{X,f,A}$ is a convex set. By the Fatou property $V_{X,f,A}$ is closed with respect to the convergence in measure on sets of finite measure, thus $V_{X,f,A}$ is also closed in $L_1$ and therefore weakly closed. By the Dunford–Pettis theorem it suffices to prove that the set $V_{X,f,A}$ is bounded and uniformly absolutely continuous. Let $B$ be a measurable set of $T$. We take some $w \in X'$ as in the proof of Proposition 7.1. Then $\{wg \mid \log g \in V_{X,f,A}\}$ is a bounded set in $L_1$, and by Lemma 8.2 we have $\| \log^+[wg] \|_{L_2} \leq 2A^{1/2}$ for all $\log g \in V_{X,f,A}$. Thus, by the Hölder inequality

$$\int_B \log^+[wg] \leq \| \log^+[wg] \|_{L_2} |B|^{1/2} \rightarrow 0$$

as $|B| \rightarrow 0$, and this convergence is uniform in $\log g \in V_{X,f,A}$. On the other hand, we also have

$$\int_B \log^-[wg] \geq \int_B \log^-[wf] \rightarrow 0$$
as $|B| \to 0$ uniformly in $\log g \in V_{X,f,A}$. Therefore,

$$\{\log wg \mid \log g \in V_{X,f,A}\} = \log w + V_{X,f,A}$$

is bounded and uniformly absolutely continuous, which implies its relative compactness in the weak topology of $L_1$. It follows that $V_{X,f,A}$ is compact in the weak topology of $L_1$.

For arbitrary discrete $\Omega$ we consider the restricted lattices $X_\omega = \{h(\cdot,\omega) \mid h \in X\}$ with the corresponding norms $\|g\|_{X_\omega} = \|g\chi_{\mathbb{T} \times \{\omega\}}\|_X$, $\omega \in \Omega$. They also satisfy the assumptions of Proposition 8.3. By the Tychonoff $\chi_{\mathbb{T} \times \{\omega\}}$ theorem space $V = \prod_{\omega \in \Omega} V_{X_\omega, f(\cdot,\omega), A}$ is compact with the product topology. It suffices to show that $V_{X,f,A} \subset V$ is closed in $V$. Observe that the map

$$\Phi_0(\log g)(z,\omega) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log g(e^{i\theta},\omega) \, d\theta \right) \quad (4)$$

declared for all $\log g(\cdot,\omega) \in L_1$, $z \in \mathbb{D}$ and $\omega \in \Omega$ is continuous as a map from $V_{X,f,A}$ to $AB_{X_A}$ with topology $\tau$. The integral in (4) is the convolution with the Schwarz kernel, and $\Phi_0(\log g)$ is an outer function with the boundary values satisfying $|\Phi_0(\log g)| = g$ almost everywhere. If $\log g_\alpha \in V_{X,f,A}$ is a net converging to some $\log g \in V$ in $V$ then $\Phi_0(\log g_\alpha) \to \Phi_0(\log g)$, and by Proposition 7.1 $\Phi_0(\log g) \in AB_{X_A}$, so $\|g\|_X \leq A$. On the other hand, since $|\Phi_0(\log g_\alpha)| \geq |\Phi_0(\log f)|$ on $\mathbb{T}$, we also have $|\Phi_0(\log g_\alpha)(z,\omega)| \geq |\Phi_0(\log f)(z,\omega)|$ for all $z \in \mathbb{D}$ and $\omega \in \Omega$. Passing to the limit in $\alpha$ yields $|\Phi_0(\log g)(z,\omega)| \geq |\Phi_0(\log f)(z,\omega)|$, and passing to the boundary values shows that $g \geq f$. Therefore, $\log g \in V_{X,f,A}$, and $V_{X,f,A}$ is indeed closed in $V$.

We now begin the proof of Theorem 6.6. Suppose that under its assumptions $f_0 \in [J(\mathcal{X})]_A$ with norm 1, and $J(\mathcal{X})$ is $N^+$-stable with constant $C$. By property (\#) there exists some $f \geq |f_0|$, $\|f\|_{J(\mathcal{X})} \leq 2$ and $f(\cdot,\omega) \in L_1$ for all $\omega$. We define a set-valued map $\Phi_1 : 4B_{[J(\mathcal{X})]_A} \to 2^{4CB_{X_A}}$ by

$$\Phi_1(g) = \left\{\{g_j\}_{j \in I} \in \mathcal{X}_A \mid g = \sum_{j \in I} g_j, \|\{g_j\}_{j \in I}\|_\mathcal{X} \leq 4C, g \in 4B_{[J(\mathcal{X})]_A} \right\}.$$ 

It has nonempty convex values. Let us show that $\Phi_1$ has closed graph with respect to $\tau$. Indeed, suppose that $g_k \in 4B_{X_A}$ and $h_k \in \Phi_1(g_k)$ are such that $g_k \to g$ and $h_k \to h$ in $\tau$. By [29, Proposition 10] and the convexity of the graph of $\Phi_1$ we may replace these sequences with some outer convex combinations such that additionally $g_k \to g$ and $h_k \to h$ almost everywhere. By Proposition 6.9 it follows that $h \in \Phi_1(g)$.

Now we define a set-valued map $\Phi_2 : 4CB_{X_A} \to 2^{V_{J(\mathcal{X}), f,A}}$ by

$$\Phi_2(\{h_j\}_{j \in I}) = \left\{\log w \in V_{J(\mathcal{X}), f,A} \mid w \geq f + \frac{1}{2C} \sum_{j \in I} |h_j| \right\}.$$
for \( \{h_j\}_{j \in I} \in 4CB\chi_A \). This map also has nonempty convex values. To verify the closedness of the graph, suppose that \( h^{(k)} = \left\{ h^{(k)}_j \right\}_{j \in I} \in 4CB\chi_A \) and \( \log w_k \in \Phi_2 \left( \{h_j\}_{j \in I} \right) \) are such that \( h^k \to h \) in \( \tau \) and \( \log w_k(\cdot, \omega) \to \log w(\cdot, \omega) \) in the weak topology of \( L_1 \) for all \( \omega \in \Omega \). Let \( n \in \mathbb{N} \). Estimate

\[
\log \left( |\Phi_0(\log f)| + \frac{1}{2C} \sum_{j \in I \cap [-n,n]} |h_j^{(k)}| \right) \leq \log |\Phi_0(\log w_k)| \tag{5}
\]

is satisfied on \( \mathbb{T} \times \Omega \). By [7, Proposition 2.2] the function on the left-hand side of (5) is subharmonic, and the function on the right-hand side of (5) is harmonic, so (5) is also satisfied on \( \mathbb{D} \times \Omega \). Passing to the limit in \( k \), passing to the boundary values and then passing to the limit in \( n \) shows that \( \log w \in \Phi_2(h) \).

We now define the composition map \( \Phi = \Phi_2 \circ \Phi_1 \circ \Phi_0 \). It satisfies the assumptions of Theorem 8.1, so there exists some \( \log w \in V_j(\chi_A)^{f,4} \) such that \( \log w \in \Phi(\log w) \). This implies that there exist some \( \{h_j\}_{j \in I} \in 4CB\chi_A \) satisfying \( \Phi_0(\log w) = \sum_{j \in I} h_j \) and \( \sum_{j \in I} |h_j| \leq 2C|\Phi_0(\log w)| \). Therefore, functions \( \varphi_j = \frac{h_j}{\Phi_0(\log w)} \), \( j \in I \) satisfy

\[
\left\| \{\varphi_j\}_{j \in I} \right\|_{H^\infty(I)} \leq 2C, \sum_{j \in I} \varphi_j = 1, \text{ and } |f_0\varphi_j| \leq \frac{|f_0|}{\log |h_j|} \leq |h_j| \text{ almost everywhere on } \mathbb{T} \times \Omega \text{ for all } j \in I, \text{ which shows that } \{\varphi_j\}_{j \in I} \text{ provide the } N^+\text{-stability for } f_0 \text{ with constant } 4C \text{ in the sense of Definition 6.5. We note in passing that the constant can be improved to } 2C \text{ if one takes advantage of the fact that the constant in property } (*) \text{ can be made arbitrarily close to } 1 \text{ (see [11, Lemma 2.2])}.

9 Sufficiency of Condition (ix) for AK-Stability

The following result is in a certain way a natural development of Corollary 6.8. Since \( X^{1-\theta}Y^\theta \) is a space of type \( C_0(X,Y) \), it also provides a nice direct and self-contained link between \( N^+\)-stability with respect to \( (\cdot, \cdot)_\theta,\infty \) and AK-stability. Observe that the AK-stability of \( (X,Y) \) naturally implies the \( N^+\)-stability of this couple with respect to \( (\cdot, \cdot)_\theta,\infty \), which in turn implies the inclusion (6) below. Moreover, after a more direct proof based on the Powers fixed point theorem we will show how a natural modification of this technique (building on some of the ideas from [34]) allows one to prove Corollary 6.8 using only the Fan–Glicksberg–Kakutani fixed point theorem.

Proposition 9.1 Suppose that \( (X,Y) \) is a couple of Banach lattices of measurable functions on \( \mathbb{T} \times \Omega \) with a discrete space \( \Omega \) satisfying the Fatou property and property (*) Then

\[
\left( X^{1-\theta}Y^\theta \right)_A \subset (X_A,Y_A)_{\theta,\infty} \tag{6}
\]

with some \( 0 < \theta < 1 \) if and only if couple \( (X,Y) \) is boundedly AK-stable.

Indeed, suppose that (6) is true and we are given some \( f \in X \) and \( g \in Y \). The argument that follows is similar to the proof of Theorem 6.6, although the details (that we repeat for clarity) are somewhat simpler. We may assume that \( f \) and \( g \)
satisfy \( \log f(\cdot, \omega), \log g(\cdot, \omega) \in L_1 \) for all \( \omega \in \Omega \). Let \( D_X = V_{X,f,2\|f\|_X} \) and \( D_Y = V_{Y,g,2\|g\|_Y} \) be the sets defined in Proposition 8.3.

Let \( u \in D_X \) and \( v \in D_Y \). Then \( w = u^{1-\theta} v^\theta \in X^{1-\theta} Y^\theta \) with norm at most \( 2\|\hat{f}\|_{X^{1-\theta}} \|g\|_Y^\theta \). We construct an outer function \( W = \Phi_0(\log w), \|W\| = w \) on \( \mathbb{T} \times \Omega \), where \( \Phi_0 \) is the map defined by (4). Thus \( W \in (X^{1-\theta} Y^\theta)_A \) with the same estimate for the norm as \( w \), and from (6) it follows that \( W = F + G \) with some \( F \in X_A \) and \( G \in Y_A \) satisfying \( \|F\|_X \leq C \|f\|_{X^{1-\theta}} \|g\|_Y^\theta \) and \( \|G\|_Y \leq C \|f\|_{X^{1-\theta}} \|g\|_Y^\theta \) for all \( t > 0 \) with some \( C \) independent of \( f, g \) and \( t \). Choosing \( t = \frac{\|f\|_X}{\|g\|_Y} \) yields \( \|F\|_X \leq C \|f\|_X \) and \( \|G\|_Y \leq C \|g\|_Y \). Let \( C_X = C \|f\|_X \) and \( C_Y = C \|g\|_Y \). We see that a set-valued map \( \Phi_1 : D_X \times D_Y \to 2^{C_X B_{X_A} \times C_Y B_{Y_A}} \) defined by

\[
\Phi_1(u, v) = \{(F, G) \mid F \in C_X B_{X_A}, G \in C_Y B_{Y_A}, \Phi_0\left(\log \left[ u^{1-\theta} v^\theta \right] \right) = F + G \}
\]

takes nonempty convex values for all \( \log u \in D_X, \log v \in D_Y \). We also define a set-valued map \( \Phi_2 : C_X B_{X_A} \times C_Y B_{Y_A} \to 2^{D_X \times D_Y} \) by

\[
\Phi_2(F, G) = \{(u, v) \mid u_{1} \geq f \vee \frac{1}{C} |F|, \quad v_{1} \geq g \vee \frac{1}{C} |G| \}, \quad F \in C_X B_{X_A}, G \in C_Y B_{Y_A},
\]

that also takes nonempty convex values.

We endow \( D_X \) and \( D_Y \) with the topology of weak convergence in \( L_1(\mathbb{T} \times \Omega) \) for all \( \omega \in \Omega \). We also endow \( C_X B_{X_A} \) and \( C_Y B_{Y_A} \) with the topology \( \tau \) of uniform convergence on compact sets of \( \mathbb{D} \times \Omega \). These sets are thus compact and convex in their respective locally convex linear topological spaces by Proposition 8.3 and Proposition 7.1.

It is easy to see that the graph of \( \Phi_1 \) is closed. Let us show that the graph of \( \Phi_2 \) is also closed. Suppose that \( F_\alpha \in C_X B_{X_A}, G_\alpha \in C_Y B_{Y_A} \) and \( (\log u_\alpha, \log v_\alpha) \) \( \in \Phi_2(F_\alpha, G_\alpha) \) are some nets such that \( F_\alpha \to F, G_\alpha \to G, \log u_\alpha \to \log u \) and \( \log v_\alpha \to \log v \) in the respective spaces. We construct outer functions \( U_\alpha = \Phi_0(\log u_\alpha), V_\alpha = \Phi_0(\log v_\alpha), U = \Phi_0(\log u), V = \Phi_0(\log v), \phi = \Phi_0(\log f) \) and \( \psi = \Phi_0(\log g) \). Then \( |U_\alpha| \geq |\phi| \vee \frac{1}{C} |F_\alpha| \) and \( |V_\alpha| \geq |\psi| \vee \frac{1}{C} |G_\alpha| \) on \( \mathbb{D} \times \Omega \). Passing to the limit in \( \alpha \) yields \( |U| \geq |\phi| \vee \frac{1}{C} |F| \) and \( |V| \geq |\psi| \vee \frac{1}{C} |G| \) on \( \mathbb{D} \times \Omega \), and therefore also almost everywhere on \( \mathbb{T} \times \Omega \), so \( u \geq f \vee \frac{1}{C} |F|, \quad v \geq g \vee \frac{1}{C} |G| \) and \( (\log u, \log v) \in \Phi_2(F, G) \).

Now we define the composition map \( \Phi = \Phi_2 \circ \Phi_1 \). It satisfies the assumptions of Theorem 8.1, so there exist some \( \log u \in D_X, \log v \in D_Y \) such that \( (\log u, \log v) \in \Phi(\log u, \log v) \). This means that \( W = \Phi_0\left(\log \left[ u^{1-\theta} v^\theta \right] \right) = F + G \) for some \( F \in C_X B_{X_A}, G \in C_Y B_{Y_A} \) satisfying \( |F| \leq Cu \) and \( |G| \leq Cv \) on \( \mathbb{T} \times \Omega \).

Let \( V = \frac{F}{W} \) and \( U = \frac{G}{W} \). Then \( V + U = 1, \chi_{|u| \leq v} |V| \leq \chi_{|u| \leq v} \frac{|F|}{W} \leq C, \) and similarly \( \chi_{|u| > v} |U| \leq C \) almost everywhere on \( \mathbb{T} \times \Omega \). Therefore, also \( \chi_{|u| \leq v} |V| = \chi_{|u| > v} |U| \leq C + 1 \), and thus \( V \in \mathcal{H}_\infty \) with norm at most \( C + 1 \). Observe that \( |V|^\theta = \frac{|F|^\theta}{u^{1-\theta} v^\theta} \leq Cu^\theta \), and similarly \( 1 - V |f|^{1-\theta} \leq Cv^{1-\theta} \). This implies
that $V$ satisfies the conditions of Lemma 4.8 with $\alpha = \frac{1}{\theta}$ and $\beta = \frac{1}{1-\theta}$, so $(X, Y)$ is indeed boundedly AK-stable.

Now we will show how Proposition 9.1 can be derived from the Fan–Glicksberg–Kakutani fixed point theorem using a suitable approximation. As a consequence, this yields a relatively simple proof of Corollary 6.8 (see the remarks at the beginning of this section). A similar but somewhat more involved modification can also be made in the proof of Theorem 6.6 as well.

Let $\Omega_N$ be as in the proof of Proposition 7.1, $0 < r_N < 1$, $\varepsilon_N > 0$, $r_N \to 1$ and $\varepsilon_N \to 0$. It is easy to see that a set-valued map $\tilde{\Phi}_1^{(N)} : D_X \times D_Y \to 2^{C_X B_{X_A} \times C_Y B_{Y_A}}$ defined by

$$\tilde{\Phi}_1^{(N)}(\log u, \log v) = \{(F, G) \mid F \in C_X B_{X_A}, G \in C_Y B_{Y_A}, \quad \left| \Phi_0 \left( \log \left[ u^{1-\theta} v^\theta \right] \right) - (F + G) \right| < \varepsilon_N \text{ on } r_N \overline{D} \times \Omega_N \}$$

is lower semicontinuous: if $\log u_\alpha \in D_X$, $\log v_\alpha \in D_Y$ are some nets converging to some functions $\log u$, $\log v$ and $(F, G) \in \tilde{\Phi}_1^{(N)}(\log u, \log v)$ then $\Phi_0 \left( \log \left[ u_\alpha^{1-\theta} v_\alpha^\theta \right] \right)$ converges in $\tau$ to $\Phi_0 \left( \log \left[ u^{1-\theta} v^\theta \right] \right)$, so $\left| \Phi_0 \left( \log \left[ u_\alpha^{1-\theta} v_\alpha^\theta \right] \right) - (F + G) \right| < \varepsilon_N$ for all $\alpha > \beta$ with some $\beta$ on $r_N \overline{D} \times \Omega_N$. Therefore, the closure of this map

$$\overline{\tilde{\Phi}_1^{(N)}}(\log u, \log v) = \{(F, G) \mid F \in C_X B_{X_A}, G \in C_Y B_{Y_A}, \quad \left| \Phi_0 \left( \log \left[ u^{1-\theta} v^\theta \right] \right) - (F + G) \right| \leq \varepsilon_N \text{ on } r_N \overline{D} \times \Omega_N \}$$

is also lower semicontinuous, and it takes nonempty convex compact values. By the Michael selection theorem [24] there exists a continuous selection $\Phi_1^{(N)} : D_X \times D_Y \to C_X B_{X_A} \times C_Y B_{Y_A}$ of $\tilde{\Phi}_1^{(N)}$, that is, $\Phi_1^{(N)}(\log u, \log v) \in \tilde{\Phi}_1^{(N)}(\log u, \log v)$ for all $\log u \in D_X$, $\log v \in D_Y$.

We now proceed as before with a set-valued map $\Phi^{(N)} = \Phi_2 \circ \Phi_1^{(N)}$ in place of $\Phi$. This map takes convex values and has closed graph. By the Fan–Glicksberg–Kakutani theorem (which is a much less involved particular case of Theorem 8.1 with $n = 1$) maps $\Phi^{(N)}$ have some fixed points $\log u_N \in D_X$, $\log v_N \in D_Y$, $(\log u_N, \log v_N) \in \Phi^{(N)}(\log u_N, \log v_N)$. Let $(F_N, G_N) = \Phi_1^{(N)}(\log u_N, \log v_N)$. By the compactness of $D_X \times D_Y$ and $C_X B_{X_A} \times C_Y B_{Y_A}$ these sequences have some limit points $\log u \in D_X$, $\log v \in D_Y$, $F \in C_X B_{X_A}$, $G \in C_Y B_{Y_A}$ respectively, and it is easy to see that $(F, G) \in \Phi_1(\log u, \log v)$, so $(\log u, \log v)$ is a fixed point of the original map $\Phi$.

10 Bounded AK-Stability on an Interpolation Scale

The following result is a generalization of Proposition 5.1. It is also a key component for both the necessity and the sufficiency of weak-type BMO-regularity for the stability of the real interpolation in Theorem 2.7 in its full generality.
Theorem 10.1 Let \((X_0, X_1)\) and \((Y_0, Y_1)\) be two couples of \(r\)-convex quasi-normed lattices of measurable functions on \(\mathbb{T} \times \Omega\) with some \(r > 0\) and a discrete \(\Omega\) satisfying the Fatou property and property (\(\ast\)). Suppose that \(Y_j\) is of type \(\mathcal{C}_{\theta_j}(X_0, X_1), j \in \{0, 1\}\) with some \(0 < \theta_0 < \theta_1 < 1\). If \((Y_0, Y_1)\) is boundedly AK-stable then so is \((X_0, X_1)\).

First of all, we may assume that \(r \leq 1\). By Proposition 4.10, Proposition 3.2 and the inclusion \((X, Y)_{\theta_j, 1} \subset (X, Y)_{\theta_j, \frac{1}{r}}\) we may raise all lattices to the power \(r\) and thus assume that they are all Banach.

Let \(Z_j = (Y_0, Y_1)_{\delta_j, \infty}, j \in \{0, 1\}\) with some \(0 < \delta_0 < \delta_1 < 1\). Then by the reiteration theorem \(Z_j = (X_0, X_1)_{\alpha_j, \infty}\) with some \(0 < \alpha_0 < \alpha_1 < 1\), and this couple is AK-stable by [13, Lemma 1.1]. Then it is boundedly AK-stable with a constant \(C\) by Corollary 6.8.

The proof now follows essentially the same idea as the proof of Proposition 5.1. Let \(\alpha_0 < \beta_0 < \beta_1 < \alpha_1\). We will first show that couple \((E_0, E_1)\) with \(E_j = X_0^{1-\beta_j} X_1^{\beta_j}, j \in \{0, 1\}\) is boundedly AK-stable. Suppose that \(f_j \in E_j, j \in \{0, 1\}\) are some nonnegative functions such that \(\|f_j\|_{E_j} = 1\), and let \(t > 0\). By applying property (\(\ast\)) we may assume that \(\log f_j(\cdot, \omega) \in L_1\) for all \(\omega \in \Omega\).

Let \(\gamma_0 = 1 - \alpha_0 - (1 - \beta_0) \frac{\alpha_0}{\beta_0}, \gamma_1 = \alpha_1 - \frac{1 - \alpha_1}{1 - \beta_1} \beta_1, \zeta_0 = 1 - (1 - \alpha_1) \frac{1 - \beta_0}{1 - \beta_1}\) and \(\zeta_1 = \alpha_0 \frac{\beta_1}{\beta_0}\). Note that the arguments that follow are symmetric with respect to interchanging \(X_0\) and \(X_1\) and simultaneously replacing \(\alpha_j\) with \(1 - \alpha_1 - j, j \in \{0, 1\}\), and the same is true for \(\beta_j\) and \(\gamma_j\), so it suffices to verify these arguments and computations for one side only.

It is easy to see that \(0 < \gamma_0, \gamma_1 < 1\) and \(\zeta_0 = (\beta_0 - \beta_1) \frac{1 - \alpha_1}{1 - \beta_1} + \alpha_0 = \beta_0 \frac{1 - \alpha_1}{1 - \beta_1} + \gamma_1, \zeta_1 = (\beta_1 - \beta_0) \frac{\alpha_0}{\beta_0} + \alpha_0 = (1 - \beta_1) \frac{\alpha_0}{\beta_0} + 1 - \gamma_0\), so in particular \(\alpha_0 < \zeta_0, \zeta_1 < \alpha_1\) and \(\gamma_1 < \zeta_0, \zeta_0 < 1 - \zeta_1\). We take some \(\omega_j \in X_j^{\gamma_j}\) with norm 1 such that \(\omega_j > 0\) almost everywhere, \(j \in \{0, 1\}\) (see, e.g., [29, Proposition 9]). By making use of property (\(\ast\)) we may also assume that \(\log \omega_j(\cdot, \omega) \in L_1\) for all \(\omega \in \Omega\). Let \(D_j = V_{X_j^{\gamma_j}, \omega_j, 2}, j \in \{0, 1\}\) be the sets defined in Proposition 8.3. Observe that for any \(u_j \in D_j\) we have \(g_0 = f_0^{\frac{\alpha_0}{\beta_0}} u_0 \in X_0^{1 - \alpha_0} X_1^{\alpha_0} \subset Z_0\) and \(g_1 = f_1^{1 - \frac{\alpha_1}{\beta_1}} u_1 \in X_0^{1 - \alpha_1} X_1^{\alpha_1} \subset Z_1\). The norms of these functions in these spaces are at most \(c\) with some \(c\) independent of \(f_0, f_1, t\) and \(Z_0, Z_1\). By the bounded AK-stability of \((Z_0, Z_1)\) and Corollary 4.9 there exists some \(U \in H_\infty\) with norm at most \(C\) such that \(\|U g_1\|_{Z_0} \leq c C t\) and \(\|(1 - U) g_0\|_{Z_1} \leq c C t^{-1}\).

Let \(F_j = X_0^{1 - \xi_j} X_1^{\xi_j}, j \in \{0, 1\}\). Since \(F_j\) is a space of type \(\mathcal{C}_{\eta_j}(Z_0, Z_1)\) with \(\eta_j = \frac{\xi_j - \alpha_0}{\alpha_1 - \alpha_0}\), we have \(\|U g_1\|_{F_0} \leq c_1 \|U g_1\|_{Z_0} \leq c_2 t^{-1 - \eta_0}\), and similarly \(\|(1 - U) g_0\|_{F_1} \leq c_3 t^{-\eta_1}\) with some \(c_1, c_2\) and \(c_3\) independent of \(f_0, f_1, t\). Thus, a set-valued map \(\Phi_1 : D_0 \times D_1 \rightarrow 2^{D_0 \times D_1 \times C_B H_{\infty}}\) defined by

\[
\Phi_1(\log u_0, \log u_1) = \{ (\log u_0, \log u_1, U) \mid U \in H_\infty, \|U\|_{H_\infty} \leq C, \|U f_1^{1 - \frac{\alpha_1}{\beta_1}} u_1\|_{F_0} \leq c_2 t^{-\eta_0}, \|(1 - U) f_0^{\frac{\alpha_0}{\beta_0}} u_0\|_{F_1} \leq c_3 t^{-\eta_1} \}\}
\]
takes nonempty convex values.
Now suppose that $U \in \Phi_1(\log u_0, \log u_1)$. From the definition of the pointwise lattice products

$$F_0 = \left( X_0^{1-\zeta} X_1^{\zeta_0-\gamma} \right) X_1^{\gamma_1},$$

$$F_1 = \left( X_0^{1-\zeta_1} X_1^{\zeta_1} \right) X_0^{\gamma_0}$$

it follows that there exist some nonnegative functions $v_j \in B_{X^{\gamma_j}}, j \in \{0, 1\}$ such that

$$\left\| U f_1^{\frac{1-\zeta_1}{1-\gamma_i}} u_1 v_1^{-1} \right\|_{X_0^{1-\zeta_0} X_1^{\zeta_0-\gamma_i}} \leq 2c_2 t^{1-\eta_0},$$

$$\left\| (1 - U) f_0^{\frac{\alpha_0}{\eta_0}} u_0 v_0^{-1} \right\|_{X_0^{1-\zeta_0} X_1^{\zeta_1}} \leq 2c_3 t^{-\eta_1}.$$

By replacing $v_j$ with $v_j \lor \omega_j, j \in \{0, 1\}$ we may assume that $\log v_j \in D_j$ at the same time as these estimates hold true. Thus, a set-valued map $\Phi_2 : \Phi_1(D_0 \times D_1) \to 2^{D_0 \times D_1}$ defined by

$$\Phi_2(\log u_0, \log u_1, U) = \{(\log u_0, \log v_1) | W_0 = \Phi_0 \left( \log \left[ f_1^{\frac{1-\zeta_1}{1-\gamma_i}} u_1 \right] \right), W_1 = \Phi_0 \left( \log \left[ f_0^{\frac{\alpha_0}{\eta_0}} u_0 \right] \right) \}.$$
such that $\left| W_0^{(\alpha)} \right| = \frac{1}{f_1^{1-\beta_1}} u_1^{(\alpha)}$, $\left| W_1^{(\alpha)} \right| = \frac{\alpha_0}{f_0^{\beta_0}} u_0^{(\alpha)}$, $\left| W_0 \right| = f_1^{1-\beta_1} u_1$, and $\left| W_1 \right| = f_0^{\beta_0} u_0$ on $\mathbb{T} \times \Omega$. Then $W_j^{(\alpha)} \to W_j$ in $\tau$, and thus also $U_\alpha W_0^{(\alpha)} \to U W_0$ and $(1 - U_\alpha) W_1^{(\alpha)} \to (1 - U) W_1$ in $\tau$. Observe that

$$\left\| U_\alpha W_0^{(\alpha)} \right\|_{F_0} \leq c_2 t^{1-\gamma_0}, \quad \left\| (1 - U_\alpha) W_1^{(\alpha)} \right\|_{F_1} \leq c_3 t^{-\eta_1}.$$ 

By Proposition 7.1 we may pass to the limit in $\tau$ in these estimates. This shows that $(\log u_0, \log u_1, U) \in \Phi_1(\log u_0, \log u_1)$, and the graph of $\Phi_1$ is indeed closed.

Map $\Phi_2$ also has closed graph. This is verified in the same way as $\Phi_1$: if nets $\log u_j^{(\alpha)}$, $U_\alpha$ are as above and

$$\left( \log v_0^{(\alpha)}, \log v_1^{(\alpha)} \right) \in \Phi_2 \left( \log u_0^{(\alpha)}, \log u_1^{(\alpha)}, U_\alpha \right)$$

are such that $\log v_j^{(\alpha)} \to \log v_j$ in $D_j$, $j \in \{0, 1\}$, then we construct outer functions $V_j^{(\alpha)} = \Phi_0 \left( \log v_j^{(\alpha)} \right)$, $V_j = \Phi_0 \left( \log v_j \right)$ and pass to the limit in the estimates

$$\left\| U_\alpha W_0^{(\alpha)} \left[ V_0^{(\alpha)} \right]^{-1} \right\|_{X_0^{1-\gamma_0} X_1^{\gamma_0 - \gamma_1}} \leq 2c_2 t^{1-\gamma_0},$$

$$\left\| (1 - U_\alpha) W_1^{(\alpha)} \left[ V_1^{(\alpha)} \right]^{-1} \right\|_{X_0^{1-\gamma_0} X_1^{\gamma_0 - \gamma_1}} \leq 2c_3 t^{-\eta_1}$$

with the help of Proposition 7.1 to show that indeed

$$(\log v_0, \log v_1) \in \Phi_2(\log u_0, \log u_1, U).$$

Thus $\Phi$ satisfies the assumptions of Theorem 8.1, and by its conclusion there exist some $\log u_j \in D_j$, $j \in \{0, 1\}$ such that $(\log u_0, \log u_1) \in \Phi(\log u_0, \log u_1)$. This means that for some $U \in C B_{H_\infty}$ we have estimates

$$\left\| U f_1^{1-\beta_1} \right\|_{X_0^{1-\gamma_0} X_1^{\gamma_0 - \gamma_1}} = \left\| U f_1^{1-\beta_1} u_1 u_1^{-1} \right\|_{X_0^{1-\gamma_0} X_1^{\gamma_0 - \gamma_1}} \leq 2c_2 t^{1-\gamma_0},$$

$$\left\| (1 - U) f_0^{\beta_0} \right\|_{X_0^{1-\gamma_0} X_1^{\gamma_0 - \gamma_1}} = \left\| (1 - U) f_0^{\beta_0} u_0 u_0^{-1} \right\|_{X_0^{1-\gamma_0} X_1^{\gamma_0 - \gamma_1}} \leq 2c_3 t^{-\eta_1}.$$
Observe that by the choice of the parameters we have \( X_0^{1-\xi_0}X_1^{\xi_1-\gamma_1} = E_0^{1-\beta_1} \) and \( X_0^{1-\xi_1-\gamma_0}X_1^{\xi_1} = E_1^{\rho_0} \), and by simple computations \( \eta_0 = 1 - \frac{\beta_1 - \rho_0}{\alpha_1 - \alpha_0} \cdot \frac{1-\alpha_1}{1-\beta_1} \), \( \eta_1 = \frac{\beta_1 - \rho_0}{\alpha_1 - \alpha_0} \cdot \frac{\alpha_0}{\beta_0} \). The estimates above imply that

\[
\left\| U f_1 \right\|_{E_0^{1-\beta_1}} = \left\| U f_1 \right\|_{E_0^{1-\beta_1}} \leq c_4 t \left( 1 - \eta_0 \right)^{1-\beta_1} = c_4 t^{1-\alpha_1},
\]

\[
\left\| 1 - U f_0 \right\|_{E_1^{\rho_0}} \leq c_4 t \left( 1 - \eta_1 \right)^{1-\rho_0} = c_4 t^{1-\alpha_1 - \alpha_0}
\]

with a constant \( c_4 \) independent of \( f_0, f_1 \) and \( t \). Since \( s = t^{1-\alpha_1 - \alpha_0} \) takes arbitrary positive values, by Corollary 4.9 this implies that couple \((E_0, E_1)\) is indeed boundedly AK-stable.

To complete the proof of Theorem 10.1 we establish the following simple result.

**Proposition 10.2** Suppose that \((X_0, X_1)\) is a couple of \( r \)-convex quasi-normed lattices on \( \mathbb{T} \times \Omega \) with some \( r > 0 \) having the Fatou property and property \( (*) \). Suppose also that \( \Omega \) is discrete, and let \( 0 \leq \beta_0 < \beta_1 \leq 1 \). Couple \((X_0, X_1)\) is boundedly AK-stable if and only if couple \((E_0, E_1) = \left( X_0^{1-\beta_0}X_1^{\beta_0}, X_0^{1-\beta_1}X_1^{\beta_1} \right)\) is boundedly AK-stable.

Indeed, let \( F = X_0^{1-\beta_1}X_1^{\beta_0} \). If couple \((X_0, X_1)\) is boundedly AK-stable then couple \((X_0^{1-\beta_0}X_1^{\beta_0}, X_0^{1-\beta_1}X_1^{\beta_1})\) is boundedly AK-stable by Proposition 4.10, and couple \((E_0, E_1) = \left( X_0^{1-\beta_0}F, X_1^{1-\beta_0}F \right)\) is then boundedly AK-stable by Proposition 4.5.

Conversely, let \( 0 < \delta \leq \frac{\xi_1}{2} \). Couple

\[
\left( E_0^{\delta}, E_1^{\delta} \right) = \left( \left( F^\delta \right)^{\frac{1}{2}} \left[ X_0^{\delta(1-\beta_0)} \right], \left( F^\delta \right)^{\frac{1}{2}} \left[ X_1^{\delta(1-\beta_0)} \right] \right)
\]

is boundedly AK-stable by Proposition 4.10, which by \cite[Theorem 2]{19} implies that couple \((X_0^{\delta(1-\beta_0)}, X_1^{\delta(1-\beta_0)})\) is AK-stable. By Corollary 6.8 this couple is also boundedly AK-stable, and therefore by Proposition 4.10 couple \((X_0, X_1)\) is boundedly AK-stable.

A slightly different approach to the proof of Theorem 10.1 is to establish first the corresponding asymmetrical theorem with \( X_0 = Y_0 \), which leads to slightly easier computations (including the computations in Corollary 4.9, since we only need the case \( \alpha = 1 \), and then consecutively apply it twice: first in order to extend the bounded AK-stability from the couple \((Y_0, Y_1)\) to the couple \((Y_0, X_1)\) under the assumptions of Theorem 10.1, and then to extend it to the entire \((X_0, X_1)\).
11 Proof of Theorem 2.7

First, we establish the following general version of Theorem 2.8 under an additional assumption that $\Omega$ is discrete.

**Proposition 11.1** Let $(X, Y)$ be a couple of quasi-Banach lattices of measurable functions on $\mathbb{T} \times \Omega$ with discrete $\Omega$ that are $r$-convex with some $r > 0$ satisfying the Fatou property and property $(\ast)$. Couple $(X, Y)$ is boundedly AK-stable if and only if it is weak-type BMO-regular.

Indeed, the “only if” part of the proposition is done in Corollary 5.6. Conversely, suppose that $(X, Y)$ is weak-type BMO-regular. By Proposition 3.5 lattices $Y_j = (L_1, (X^\prime)^\prime Y^\prime)_\theta_j, p$ are BMO-regular with some $0 < \theta_0 < \theta_1 < 1$, thus couple $(Y_0, Y_1)$ is boundedly AK-stable. It satisfies the assumptions of Theorem 10.1 with $X_0 = L_1$ and $X_1 = (X^\prime)^\prime Y^\prime$, hence couple $(X_0, X_1)$ is boundedly AK-stable. By Proposition 4.10 couple $\left( X^{\frac{1}{2}}_0, X^{\frac{1}{2}}_1 \right)$ is AK-stable, and by [19, Theorem 2] couple $(X^\prime, Y^\prime)$ is AK-stable. By Corollary 6.8 it is boundedly AK-stable, and by Proposition 4.10 couple $(X, Y)$ is also boundedly AK-stable as claimed.

Let us show the necessity of the weak-type BMO-regularity for the $N^+$-stability. Suppose that under the assumptions of Theorem 2.7 condition (i) is satisfied, i. e. couple $(X, Y)$ is $N^+$-stable with respect to $(\cdot, \cdot)_{\theta_1, s}$. Let $0 < \alpha_1 < \theta < \beta_1 < \theta_1$. From the reiteration theorem it easily follows that $(X, Y)_{\alpha_1, p_1}, (X, Y)_{\beta_1, q_1}$ is also $N^+$-stable with respect to $(\cdot, \cdot)_{\eta_1, s}$, where $\eta_1 < 1$ determined by $\theta = (1 - \eta_1)\alpha_1 + \eta_1\beta_1$ and all $1 < p, q \leq \infty$, since

$$
\left[ (X, Y)_{\alpha_1, p} (X, Y)_{\beta_1, q} \right]_{\eta_1, s} = \left[ (X, Y)_{\alpha_1, p} \right]_{\theta_1, s} \\
= \left[ (X, Y)_{\alpha_1, p}, (X, Y)_{\beta_1, q} \right]_{\eta_1, s} \subset \left[ (X, Y)_{\alpha_1, p} \right]_{\theta_1, s} \cdot \left[ (X, Y)_{\beta_1, q} \right]_{\theta_1, s}.
$$

By the same reiteration the lattices in this couple are real interpolation spaces for the couple $(X^{1-\theta_0}Y^{\theta_0}, X^{1-\theta_1}Y^{\theta_1})$ of Banach spaces, so they are also Banach. We may assume that $r \leq 1$. Let $0 < \delta < r$, $E = (X^\delta, Y^\delta)_{\alpha_1, \eta_1, s}$ and $F = (X^\delta, Y^\delta)_{\beta_1, \eta_1, s}$.

By Proposition 3.2 and Corollary 6.7 couple $(E, F) = (X, Y)_{\alpha_1, p}, (X, Y)_{\beta_1, q}$ is also $N^+$-stable with respect to $(\cdot, \cdot)_{\eta_1, s}$. Let $0 < \gamma < \eta_1 < \zeta < 1$, let $0 < \eta < 1$ be such that $\eta_1 = (1 - \eta)\gamma + \eta\zeta$, and let $\alpha = (1 - \gamma)\alpha_1 + \gamma\beta_1$, $\beta = (1 - \zeta)\alpha_1 + \zeta\beta_1$. By [1, Theorem 4.7.2] and the reiteration theorem we have for all $r \leq s < \infty$

$$
\left[ (X^\delta, Y^\delta)_{\alpha_1, \gamma, \eta_1, s}, (X^\delta, Y^\delta)_{\beta_1, \gamma, \eta_1, s} \right]_{\eta} = \left[ (E, F)_{\gamma, \delta, \eta}, (E, F)_{\delta, \eta} \right]_{\eta} \\
= \left[ (E, F)_{\eta_1, \delta, \eta_1, s} \right]_{\eta} = \left[ (E, F)_{\eta_1, \delta, \eta_1, s} \right]_{\eta} \\
\subset \left[ (X^\delta, Y^\delta)_{\alpha_1, \gamma, \eta_1, s}, (X^\delta, Y^\delta)_{\beta_1, \gamma, \eta_1, s} \right]_{\eta} = \left[ (X^\delta, Y^\delta)_{\alpha_1, \gamma, \eta_1, s}, (X^\delta, Y^\delta)_{\beta_1, \gamma, \eta_1, s} \right]_{\eta}.
$$
That is, couple \( \left( (X^\delta, Y^\delta)_{\alpha^\delta}, (X^\delta, Y^\delta)_{\beta^\delta} \right) \) is \( N^+ \)-stable with respect to the complex interpolation \((\cdot, \cdot)_\theta\) as well. This is a couple of lattices that are \( \frac{\gamma}{\delta} \)-convex, so by [11, Theorem 5.12] and the remark after it this couple is BMO-regular. Raising it to the power \( \frac{1}{\delta} \) yields condition (v). This proves \( (i) \Rightarrow (v) \).

Now we establish the equivalence in Theorem 2.7 starting with (ii), first ignoring the “for all” parts of the conditions. Transition (ii) \( \Rightarrow (i) \) is an immediate and well-known consequence of the definitions, and transition (i) \( \Rightarrow (v) \) has been verified above.

As was already mentioned in Sect. 2, transition (v) \( \Rightarrow (vi) \) is true for any couple of quasi-Banach lattices with the Fatou property and follows from [30, Theorem 8] and Proposition 3.3. Transition (vi) \( \Rightarrow (vii) \) is trivial. Transition (vii) \( \Rightarrow (iii) \) follows from Proposition 11.1. Transition (iii) \( \Rightarrow (iv) \) easily follows from the reiteration theorem (which yields \((X, Y)_{\gamma,t} = (E, F)_{\eta,t}\) for all \(0 < t \leq \infty\) and \(\theta_0 < \gamma < \theta_1\) with some \(0 < \eta < 1\) and [13, Lemma 1.1]). By the same reasoning, if condition (iv) is satisfied for some \(0 < \alpha < \theta < \beta < 1\) then it is also satisfied for some \(\theta_0 < \alpha < \theta < \beta < \theta_1\), and by Corollary 6.8 the couple in condition (iv) with such \(\alpha\) and \(\beta\) is boundedly AK-stable. By Theorem 10.1 couple \((X, Y)\) is then boundedly AK-stable. Not only does it prove (iv) \( \Rightarrow (ii) \), which completes the chain and shows that the first 7 conditions of Theorem 2.7 are equivalent, but it also yields their equivalence to a stronger version of condition (ii) stating that \((X, Y)\) is boundedly AK-stable, which by Proposition 11.1 implies (ii) \( \Leftrightarrow (viii) \). We also get a more general version of Corollary 6.8.

**Corollary 11.2** Suppose that \(X\) and \(Y\) are quasi-normed lattices of measurable functions on \(\mathbb{T} \times \Omega\) with discrete \(\Omega\) satisfying the Fatou property and property (\(\ast\)). Suppose also that \(X^{1-\theta}Y_\theta\), \(j \in \{0, 1\}\) are Banach lattices with some \(0 < \theta_0 < \theta_1 < 1\). Then \((X, Y)\) is AK-stable if and only if it is boundedly AK-stable.

We have thus verified the equivalence of the first 8 conditions. Transition (ii) \( \Rightarrow (ix) \) was discussed directly above Proposition 9.1. To get the converse, let \(X_j = X^{1-\theta_0}Y^{\theta_0}\), \(j \in \{0, 1\}\). These lattices are Banach by the assumptions. Then \(X^{1-\theta}Y_\theta = X^{1-\eta}X^{\eta}_\theta\) with some \(0 < \eta < 1\) satisfying \(\theta = (1 - \eta)\theta_0 + \eta\theta_1\). On the other hand, \((X_A, Y_A)_{\theta,\infty} = ((X_A, Y_A)_{\theta_0,1}, (X_A, Y_A)_{\theta_1,1})_{\eta,\infty} \subset (X_0[A], (X_1[A])_{\eta,\infty}\)) by the reiteration theorem, therefore \(\left(X^{1-\eta}X^\eta_1\right)_A \subset (X_0[A], (X_1[A])_{\eta,\infty}\)) by Proposition 9.1 to couple \((X_0, X_1)\) yields its AK-stability, which proves (ix) \( \Rightarrow (iii) \).

It remains to verify the “for all” parts of the conditions of Theorem 2.7. Suppose that \((X, Y)\) is AK-stable and \((E, F)\) are as in condition (iii). Let \(\alpha < \beta_0 < \beta_1 < \beta\) and \(E_0 = X^{1-\beta_0}Y^{\beta_0}, E_1 = X^{1-\beta_1}Y^{\beta_1}\). Couple \((X, Y)\) is boundedly AK-stable by Corollary 11.2, and couple \((E_0, E_1)\) is boundedly AK-stable by Proposition 10.2. Spaces \(E_j\) are of types \(C_{\beta_j}(X, Y)\), \(j \in \{0, 1\}\), so by the reiteration theorem they are also of types \(C_{\alpha_j}(E, F)\) with \(0 < \alpha_j < 1\) satisfying \(\beta_j = (1 - \alpha_j)\alpha + \alpha_j\beta, j \in \{0, 1\}\). By Theorem 10.1 it follows that \((E, F)\) is boundedly AK-stable, which is condition (iii). The weak-type BMO-regularity of \((E, F)\), which is condition (vii), follows from Proposition 11.1. In particular, this also yields the “for all” parts of conditions (iv) and (vi).

Finally, observe that \((X^{\delta}, Y^{\delta})\) is boundedly AK-stable by Proposition 4.10, and it is a couple of Banach lattices with nontrivial convexity for \(0 < \delta < r\). We may
repeat the proof of (i) ⇒ (v) above for this couple with \( \frac{p}{\delta} \) and \( \frac{q}{\delta} \) in place of \( \frac{s}{\delta} \), respectively, which allows us to take any \( 0 < \alpha < \beta < 1 \) and yields the BMO-regularity of the couple \( \left( (X^\delta, Y^\delta)_{\alpha, \frac{p}{\delta}}, (X^\delta, Y^\delta)_{\beta, \frac{q}{\delta}} \right) \). Raising it to the power \( \frac{1}{\delta} \) by Proposition 3.2 verifies condition (v), unless \( p = q = \infty \). The latter case can be routinely reduced to \( p = q = 1 \) by duality as follows. Suppose that \( (X, Y) \) is boundedly AK-stable. Couple \( \left( (X^2)\ell^1, (Y^2)\ell^1 \right) = \left( (X^\delta)^{\frac{1}{2}}L^{-\frac{1}{2}}_1, (Y^\delta)^{\frac{1}{2}}L^{-\frac{1}{2}}_1 \right) \) is AK-stable by [18, Lemma 7]. The transition (i) ⇒ (v) may be applied to this couple with \( p = q = 1 \), and subsequent passing to the duals (see the proof of the symmetry in Proposition 3.7) and raising to the power \( \frac{2}{\delta} \) concludes the proof.

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