Protocorks and monopole Floer homology

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Abstract. We introduce and study a class of compact 4-manifolds with boundary that we call protocorks. Any exotic pair of simply connected closed 4-manifolds is related by a protocork twist, moreover, any cork is supported by a protocork. We prove a theorem on the relative Seiberg-Witten invariants of a protocork before and after twisting and a splitting theorem on the Floer homology of protocork boundaries. As a corollary we improve a theorem by Morgan and Szabó regarding the variation of Seiberg-Witten invariants with an upper bound which depends only on the topology of the data. Moreover we show that for any cork, only the reduced Floer homology of its boundary contributes to the variation of the Seiberg-Witten invariants after a cork twist.

1 Introduction

Corks [Akb91, Akb16] are a class of contractible 4-manifolds with boundary endowed with an involution of their boundaries that can be used to produce any simply connected exotic pair [CFHS96, Mat95]. In this paper, inspired by [CFHS96, Mat95, MS99], we define and study a class of 4-manifolds with boundary that we call protocorks which also relate any simply connected exotic pair of 4-manifolds and which support any cork. This informally means that the protocork determines the effect of the cork twist. Consequently, protocorks may be used as a device to prove general theorems about all corks, or directly to investigate exotica.

Main result. Let \( X_0 \) be a closed 4-manifold decomposed as \( X_0 = M \cup_Y N_0 \) where \( N_0 \) is a protocork with boundary \( Y \) and \( M \) is a 4-manifold with \( \partial M = -Y \). Performing a protocork twist we obtain \( X_1 = M \cup_Y N_1 \) where \( N_1 \) is another manifold with \( \partial N_1 = Y \) which we call reflection of \( N_0 \). The difference between the Seiberg-Witten invariants of \( X_0 \) and \( X_1 \) is governed by a difference element \( \Delta \), depending only on \((N_0,N_1)\), which belongs to the monopole Floer homology of \( Y, HM_\bullet(Y) \) [KM07], a module over \( \mathbb{Z}[[U]] \). Our main result regards this element.
**Theorem 1.1.** Let $N_0$ be a protocork with boundary $Y$, let $N_1$ be its reflection and let $\hat{\tau} \in \overline{HM}_\bullet(S^3)$ be the standard generator of $\overline{HM}_\bullet(S^3)$. Remove a ball from $N_i$, $i = 0, 1$ and regard the result as a cobordism $N_i \setminus \mathbb{B}^4 : S^3 \rightarrow Y$. Choose $\mu_0$, an homology orientation for $N_0 \setminus \mathbb{B}^4$. Then:

(a) there is an homology orientation $\mu_1$ on $N_1 \setminus \mathbb{B}^4$ such that, setting

$$x_i := \overline{HM}_\bullet(N_i \setminus \mathbb{B}^4, \mu_i)(\hat{\tau}) \in \overline{HM}_\bullet(Y)$$

for $i = 0, 1$, the difference element $\Delta := x_0 - x_1$ belongs to the reduced homology $HM_{\text{red}}(Y)$. In particular $\Delta$ is $U$-torsion.

(b) Let $d_{\Delta} \geq 0$ be the smallest natural number such that $U^{d_{\Delta}}$ annihilates $\Delta$. Fix a metric and perturbation on $Y$ as in Subsection 3.3 and define the Morgan-Szabó number $n_{MS}$ as $2$ plus the maximum relative degree between two irreducible generators of the chain complex relative to the trivial spin$^c$ structure. Then $n_{MS} \geq 2d_{\Delta}$.

(c) If $N_0$ is a symmetric protocork, then $\mu_1$ realizing the first item is the pushforward of $\mu_0$ via the diffeomorphism $\hat{\tau} : N_0 \rightarrow N_1$ defined in Subsection 2.5.

The proof is given in Section 5. The inequality $n_{MS} \geq 2d_{\Delta}$ of item (b) leads to the following improvement of the main technical result of Morgan-Szabó in [MS99, Theorem 1.2] regarding the variation of Seiberg-Witten invariants.

**Corollary 1.2.** Let $d_{\Delta}$ be the $U$-torsion order of the class $\Delta$, i.e. the least $k \geq 0$, such that $U^k \Delta = 0$, and let $X_0$ be an oriented, closed, simply connected 4-manifold with $b_2(X_0) > 1$ decomposed as $X_0 = M \cup_Y N_0$ where $M$ is an oriented 4-manifold with boundary $-Y$. Fix $\mathfrak{s}_0$, a spin$^c$ structure on $X_0$ and an homology orientation $\mu_0$ for $X_0$. Let $X_1 = M \cup_Y N_1$ be the result of the protocork twist operation on $X_0$ and denote by $\mathfrak{s}_1$ the spin$^c$ structure induced by $\mathfrak{s}_0$ (unique up to isomorphism). Suppose that

$$d(\mathfrak{s}_0) = \frac{1}{4}(c_1^2(\mathfrak{s}_0) - 2\chi(X_0) - 3\sigma(X_0)) \geq 2d_{\Delta}$$

then

$$SW(X_0, \mathfrak{s}_0, \mu_0) = SW(X_1, \mathfrak{s}_1, \mu_1)$$

for an homology orientation $\mu_1$ on $X_1$.

Theorem 1.2. of [MS99], rephrased in the protocork terminology, corresponds to Corollary 1.2 with $n_{MS}$, as defined in Theorem 1.1 (b), in place of $2d_{\Delta}$. Our
result improves [MS99, Theorem 1.2 ] because by item (b) our upper bound $d_\Delta$
is at least as good as Morgan-Szabó’s and has the advantage of being a purely
topological quantity, whilst $n_{MS}$ depends on the choice of Riemannian metric and
perturbation. Moreover, it can be easily expressed in terms of $\overline{HM}_*(Y)$ and has
an upper bound, given by the least $k \geq 0$ such that $U^k$ annihilates $HM^{\text{red}}(Y)$,
that can be computed without knowing $\Delta$.

Proof of Corollary 1.2. Since the result of gluing a spin$^c$ structure over $M$ to one
over $N_0$ is unique up to isomorphism (see Proposition 2.4), the gluing formula
[KM07, Proposition 3.6.1] applies to each spin$^c$ structure of $X_i$ separately and we
have that

$$SW(X_i, s_i) = \langle \overline{HM}_*(U^{d(s_i)/2} | M \setminus \mathbb{B}^4, s_M) \circ \overline{HM}_*(N_i \setminus \mathbb{B}^4)(i), \hat{i} \rangle$$

where $s_M$ is the restriction of $s_0$ to $M \setminus \mathbb{B}^4$, the spin$^c$ structure over $N_i \setminus \mathbb{B}^4$ is the
trivial one, the homology orientations of $M \setminus \mathbb{B}^4$ and $N_0 \setminus \mathbb{B}^4$ are such that their
composition is $\mu_0$ and $N_1 \setminus \mathbb{B}^4$ has the homology orientation given by Theorem 1.1.
In particular we have that

$$SW(X_0, s_0) - SW(X_1, s_1) = \langle \overline{HM}_*(M \setminus \mathbb{B}^4, s_M) U^{d(s_0)/2} \Delta, \hat{i} \rangle$$

from which the thesis follows because $U^{d(s_0)/2} \Delta = 0$ as $d(s_0) \geq 2d_\Delta$. \qed

The next corollary is an example of how protocorks can be used to prove
statements about all corks even though we are far from a classification of corks.
It roughly says that only the reduced Floer homology of the cork boundary con-
tributes to the variation in Seiberg-Witten invariants due to a cork twist. This
was known to be true for the specific case of the Akbulut cork as shown in the
proof of [LRS18, Theorem 8.1], and thus also for corks supported by it.

Corollary 1.3. Let $C$ be a cork with boundary $Y$ and involution $f : Y \to Y$. Remove
a ball from the interior of $C$ obtaining a cobordism $C \setminus \mathbb{B}^4 : \mathbb{S}^3 \to Y$ and set

$$x := \overline{HM}_*(C \setminus \mathbb{B}^4)(\hat{i}) \in \overline{HM}_*(Y).$$

Then either $x - f_* x$ or $x + f_* x$ belongs to $HM^{\text{red}}(Y)$. 

Proof of Corollary 1.3. Proposition 2.14 ensures the existence of a symmetric protocork $N_0 \subset \text{int}(C)$ supporting $(C, f)$. This means that if $T = C \setminus \text{int}(N_0)$ and we denote by $(C)_T : \emptyset \to Y$ the cobordism induced by $C$ using $f$ to identify the outcoming boundary with $Y$, then as a cobordism

$$ (C)_T \simeq T \circ (N_0)_r, \tag{7} $$

where $\tau : \partial N_0 \to \partial N_0$ is the twisting map of Subsection 2.5. Fixing an homology orientation $\mu_C$ for $C$, we have that

$$ x = \overline{HM}_*(C \setminus \mathbb{B}^4, \mu_C)(\hat{1}) = \overline{HM}_*(T, \mu_T) \circ \overline{HM}_*(N_0 \setminus \mathbb{B}^4, \mu_{N_0})(\hat{1}) \tag{8} $$

for some homology orientations $\mu_T, \mu_{N_0}$ on $T, N_0$ and by (7)

$$ f_*x = \overline{HM}_*((C \setminus \mathbb{B}^4)_f, \mu_C)(\hat{1}) = \pm \overline{HM}_*(T, \mu_T) \circ \tau_*\overline{HM}_*(N_0 \setminus \mathbb{B}^4, \mu_{N_0})(\hat{1}) \tag{9} $$

where the $\pm$ is a plus if the composition of the two homology orientations is $\mu_C$ and minus otherwise. Now, $\overline{HM}_*(N_0 \setminus \mathbb{B}^4)(\hat{1}) = \tau_*\overline{HM}_*(N_0 \setminus \mathbb{B}^4)(\hat{1})$ is the difference element $\Delta$ of Theorem 1.1 in the case of a symmetric protocork. Thus the thesis follows from $\Delta \in \overline{HM}^{\text{red}}(\partial N_0)$ and the fact that cobordism maps send reduced homology to reduced homology. \hfill \Box

Second result. In order to prove the main theorem we prove the following splitting theorem for the monopole Floer homology of protocork boundaries which determines it up to $U$-torsion elements.

It roughly says that the Floer homology is a direct sum of the Floer homology of a connected sum of $S^1 \times S^2$ and the reduced homology $\overline{HM}^{\text{red}}$ [KM07, Definition 3.6.3]. The former is well known [KM07, Proposition 36.1.3] and the latter has finite rank.

Theorem 1.4. Let $Y$ be the boundary of a protocork, then as $\mathbb{Z}[[U]]$-modules

$$ \overline{HM}_*)(Y) \simeq \overline{HM}_*((S^1 \times S^2) \oplus \overline{HM}^{\text{red}}(Y) $$

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$$ \overline{HM}_*)(Y) \simeq \overline{HM}_*((S^1 \times S^2). \tag{10} $$

In this splitting, the Floer homology of $\#_{i=1}^{b_1(Y)}(S^1 \times S^2)$ is supported in the (unique up to isomorphism) torsion spin$^c$ structure of $Y$ and the isomorphism preserves the absolute $\mathbb{Q}$-grading. Moreover, with respect to this splitting, the long exact sequence

$$ \cdots \to \overline{HM}_*(Y) \xrightarrow{f_*} \overline{HM}_*(Y) \xrightarrow{p_*} \overline{HM}_*(Y) \to \cdots \tag{11} $$
becomes the direct sum of the long exact sequence of \(#{^1_{1=1}}(\mathbb{S}^1 \times \mathbb{S}^2)\) and

\[
\ldots \rightarrow HM^{\text{red}}(Y) \xrightarrow{id} HM^{\text{red}}(Y) \rightarrow 0 \rightarrow \ldots \tag{12}
\]

We prove this theorem in Subsection 3.4, the proof relies on the functoriality of Floer homology and on the construction of some special cobordisms.

**Organization.** The paper is organized as follows: in Section 2 we define protocorks (Definition 2.3) and the protocork twist operation (Definition 2.11) and we obtain some basic properties that will be used in the other sections. The proof that any exotic pair is related by a protocork twist and that any cork has a supporting protocork is in Subsection 2.6. The first part of Section 3 provides the framework for the definition of the chain complex relative to the Floer homology of protocork boundaries that we will use in Section 4 while the second part Section 3 contains the proof of Theorem 1.4. Section 4 uses the framework of Section 3 to study the moduli spaces of reducible monopoles over a protocork. In Section 5 we prove Theorem 1.1. The most difficult part in proving the main theorem is item (b), which relates \(d_\Delta\) with the Morgan-Szabó number. This is the only item where we will need the results of Section 4. The proof of item (a) and (c) instead relies on the construction of some cobordisms. The paper also includes two appendices with some technical proofs that are probably known to experts but that we could not trace in the literature. In Appendix A we prove a density result about a certain class of perturbations, this can be applied to general 3 manifolds, not necessarily protocork boundaries. In Appendix B we prove a characterization of reducible moduli spaces that will be fundamental Section 4. More information and proof sketches may be found in the outlines at the beginning of each section.

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2 Protocorks

Outline. In Subsection 2.1 we review plumbings and their realizations. This is important to us because we will need to define diffeomorphisms of plumbings, thus to our aims is not enough to work with the diffeomorphism type of a plumbing. In Subsection 2.2 we define protocorks. In Subsection 2.3 we describe Kirby diagrams of protocorks, these will be used throughout the paper in particular to construct cobordisms that we will exploit in the proof of our theorems. In Subsection 2.4 we infer some basic algebraic-topological properties of protocorks and their boundary that will be used throughout the paper. In Subsection 2.5 we define some involutions on protocork boundaries and some diffeomorphisms of manifolds associated to protocorks and we prove some properties about them. In Subsection 2.6 we define the protocork twist operation, we prove that any simply connected exotic pair can be related by such operation and that any cork admits a supporting protocork.

Inspiration and motivation. Protocorks are a natural object to consider when dealing with exotic 4-manifolds and h-cobordisms, indeed they are implicit in the proofs of [CFHS96, Mat95] and in Morgan-Szabó’s definition of complexity of an h-cobordism [MS99]. Protocorks with sphere-number 1 appear also in the unpublished thesis of Sunukjian [Sun10] under the name $D(n, 1)$ where the author also computes $HM_*$ of the boundary using the vanishing of the triple cup product. However, to the author’s best knowledge this is the first paper to attempt a systematic study of them.

Our intention is to use protocorks to prove general statements regarding corks and exotic pairs. This is motivated by two key features: first of all we can enumerate protocorks, as they are in bijection with certain bipartite graphs. This is in stark contrast with the cork’s situation: there is no classification of corks and it is not easy to produce corks that can potentially yield new exotic pairs. Secondly, protocorks and their boundaries possess interesting symmetries (see Subsection 2.5) and other geometrical features that come in handy when doing gauge theory (see the discussion at the beginning of Section 4).

2.1 Review of plumbings and their realizations.

Plumbings were firstly introduced in the context of surgery theory by Milnor in [Mil56]. In this subsection we review them in the specific case of 4-manifolds and
establish our notation, for examples and further discussion we refer the reader to [GS99, Section 6.1].

**Plumbing graphs.** A plumbing graph is a finite, undirected multigraph where each vertex is labelled with a pair \((g, e) \in \mathbb{N}_0 \times \mathbb{Z}\) and each edge is decorated with a sign + or −. Such graphs model an intersection of surfaces in an ambient 4-manifold: each vertex represents a Riemann surface of genus \(g\) embedded with normal bundle of Euler number \(e\), and each \((−)\) edge represents a positive (negative) intersection between the surfaces. Given a plumbing graph \(\Gamma\) with set of vertices \(V\), for \(i, j \in V\), we will denote by \(r_\Gamma(i, j) \in \mathbb{N}_0\) the geometric intersection number i.e. the number of edges between \(i\) and \(j\) and by \(a_\Gamma(i, j) \in \mathbb{Z}\) the algebraic intersection number i.e. the signed count of the edges between \(i\) and \(j\). The edges constitute a multiset therefore we will adopt the following convention: we will write \(\varepsilon \simeq (v_1, v_2, \pm)\) to denote an edge \(\varepsilon\) between the vertices \(v_1, v_2\) of sign ±, of course we can have \(\varepsilon, \varepsilon’ \neq \varepsilon\).

**Realization datum.** We will denote by \(\mathbb{D}^n(r) \subset \mathbb{R}^n\), the disk of radius \(r > 0\) so that \(\mathbb{D}^n = \mathbb{D}^n(1)\). To a plumbing graph \(\Gamma\) there is associated a diffeomorphism class of 4-manifold called plumbing on \(\Gamma\). If we want to associate a specific 4-manifold we need to add some data. A realization datum for \(\Gamma\) consists of the following choices:

(a) for each vertex \(v\) of \(\Gamma\) with label \((g, e)\), is given a preferred oriented surface of genus \(g\), \(S_v\), with an oriented \(\mathbb{D}^2\)-bundle \(N_v \to S_v\) of Euler class \(e\) and

(b) denoting by \(d_v\) the degree of \(v\), is given a preferred embedding \(\varphi_v : \bigsqcup_{i=1}^{d_v} \mathbb{D}^2(2) \to S_v\) together with a preferred trivialization of \(N_v\) restricted to the image of \(\varphi_v\).

(c) Let \(v_1\) and \(v_2\) be two vertices of \(\Gamma\). Then for each edge of \(\Gamma\), \(\varepsilon \simeq (v_1, v_2, \pm)\) are chosen two unit disks \(D_{v,1}\) and \(D_{v,2}\), from the image of

\[
\bigsqcup_{i=1}^{d_{v_1}} \mathbb{D}^2(1) \subset \bigsqcup_{i=1}^{d_{v_1}} \mathbb{D}^2(2) \xrightarrow{\varphi_{v_1}} S_{v_1} \hookrightarrow S_{v_1} \times \{v_1\}
\]

and

\[
\bigsqcup_{i=1}^{d_{v_2}} \mathbb{D}^2(1) \subset \bigsqcup_{i=1}^{d_{v_2}} \mathbb{D}^2(2) \xrightarrow{\varphi_{v_2}} S_{v_2} \hookrightarrow S_{v_1} \times \{v_2\}
\]
respectively. This choice must respect the following global condition: each disk in the image of (13) and (14) is associated to \textit{exactly one} edge.

(d) Denote by $M_{\pm}$ the manifold with corners obtained by gluing $\text{int}(D^2(2)) \times D^2 \times \{0\}$ and $\text{int}(D^2(2)) \times D^2 \times \{1\}$ via the identification

$$\text{inv}_{\pm} : D^2(1) \times D^2 \times \{0\} \rightarrow D^2(1) \times D^2 \times \{1\} = (b, f, 0) \mapsto \pm (f, b, 1).$$

(15)

Then for each edge $\epsilon \approx (v_1, v_2, \pm)$ is chosen a smoothing of the corners of $M_{\pm}$ [Wal16, Section 2].

Calling $M_{\pm}$ a manifold with corners is a slight abuse of language because a neighbourhood of a singularity is diffeomorphic to

$$\mathbb{R}^2 \times \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0 \text{ or } x_2 \geq 0 \},$$

(16)
i.e. is a \textit{concave} corner, instead of being diffeomorphic to $\mathbb{R}^2 \times [0, +\infty)^2$, i.e. a \textit{convex} corner, as required by the usual definition corner point [Wal16, pg. 30]. Nevertheless the two are homeomorphic and the corner straightening theory [Wal16, Section 2] is carried out with little modifications.

\textbf{Realization.} Let $\Gamma$ be a plumbing graph and let $\mathcal{R}$ be a realization datum for $\Gamma$. Then we can construct an oriented smooth 4-manifold with boundary $P(\Gamma, \mathcal{R})$, \textit{the realization of the plumbing on} $(\Gamma, \mathcal{R})$ constructed in the following way: let

$$E(\Gamma, \mathcal{R}) = \bigsqcup_v N_v \times \{v\}$$

(17)

where the $v$ varies among vertices of $\Gamma$. Then $P(\Gamma, \mathcal{R})$ is obtained from $E(\Gamma, \mathcal{R})$ by identifying some codimension zero submanifolds: for each edge $\epsilon$ we identify $E(\Gamma, \mathcal{R})|_{D_{\epsilon,1}}$ and $E(\Gamma, \mathcal{R})|_{D_{\epsilon,2}}$ by making this diagram commutative

$$\begin{align*}
E(\Gamma, \mathcal{R})|_{D_{\epsilon,1}} & \xrightarrow{\sim} E(\Gamma, \mathcal{R})|_{D_{\epsilon,2}} \\
D^2 \times D^2 & \xrightarrow{\text{inv}_{\pm}} D^2 \times D^2
\end{align*}$$

(18)

where the vertical arrows are given by the local trivializations (where the second factor is the fiber) chosen with $\mathcal{R}$ and $\text{inv}_{\pm}$ is the involution defined by (15) that
exchanges the base with the fiber. $P(\Gamma, R)$ is naturally a smooth manifold with (concave) corners along the plumbing tori (see below). Exploiting $R$, we define a smooth structure on $P(\Gamma, R)$ by straightening the corners using the data of item (d) in the definition of realization datum.

**Remark 1.** It can be shown that the (oriented) diffeomorphism type of $P(\Gamma, R)$ is independent of the realization datum $R$ and depends on $\Gamma$ only up to isomorphism of plumbing graphs (which is an isomorphism of graphs that preserves the labels of the vertices and the sign of the edges). Thus we write $P(\Gamma)$ to denote the diffeomorphism type of the plumbing on $\Gamma$.

**Plumbing tori.** We recall a definition that we will need in Proposition (2.4) and Proposition (2.6). The torus $\partial \mathbb{D}^2 \times \partial \mathbb{D}^2 \subset \mathbb{D}^2 \times \mathbb{D}^2$ is preserved by $\text{inv}_+$. Thus via the diagram (18), each edge of $\Gamma$ gives rise to a torus lying in the boundary $Y := \partial P(\Gamma, R)$. Such tori are called *plumbing tori*. Notice that the union of plumbing tori separate $Y$ in a collection of $\mathbb{S}^1$-bundles over surfaces with boundary thus giving $Y$ the structure of a graph manifold.

**Plumbing structures and associated surfaces.** A plumbing structure on a 4-manifold $X$ is a diffeomorphism $f : X \to P(\Gamma, R)$ for some pair $(\Gamma, R)$. Given such a structure, the surface in $X$ associated to a vertex $v$ of $\Gamma$, is the preimage of the surface associated to $v$ in $P(\Gamma, R)$ (i.e. the image of the zero section of $N_v \times \{v\}$ in (17)). Notice that a tubular neighbourhood of an intersection of surfaces intersecting according to a plumbing graph $\Gamma$, has a an obvious plumbing structure.

### 2.2 The definition of protocork.

**Outline.** The aim of this subsection is to define protocorks (Definition 2.3), these can be thought as the incoming end of certain 5-dimensional h-cobordisms consisting merely of 5-dimensional 2-handles and 3-handles. Such a cobordism is specified by the intersection pattern of the belt spheres of the 2-handles with the attaching spheres of the 3-handles. We record this information in the form of a protocork plumbing graph (Definition 2.1). In order to define the protocork twist operation later, it is convenient to keep track also of the outcoming end and of the middle level of the above-mentioned cobordism, we will call the former reflection of the protocork and the latter the plumbing associated to the graph. Hence in Definition 2.3 we will define three objects: plumbing, protocork and reflection.
**Definition 2.1 (Protocork plumbing graph).** A protocork plumbing graph is a bipartite plumbing graph $\Gamma$ with $2n \geq 0$ vertices and parts labelled $A$ and $B$ satisfying the following conditions:

(a) $A$ and $B$ have cardinality $n$ called the sphere-number and their elements are denoted as $\{v_i^A\}_{i=1}^n$, $\{v_i^B\}_{i=1}^n$.

(b) the $\mathbb{D}^2$-bundle associated to each vertex is the trivial bundle over $\mathbb{S}^2$.

(c) $r_\Gamma(v_i^A, v_j^A) = r_\Gamma(v_i^B, v_j^B) = 0$ for each $i, j = 1, \ldots, n$.

(d) $a_\Gamma(v_i^A, v_j^B) = \delta_{i,j}$ for each $i, j = 1, \ldots, n$, $\delta_{i,j}$ being the Kronecker delta.

Since the only edges occur between $A$-vertices and $B$-vertices, we will write $\epsilon \simeq (i, j, \pm)$ to denote a positive/negative edge between $v_i^A$ and $v_j^B$.

A protocork plumbing graph $\Gamma$ is called symmetric if $r_\Gamma(v_i^A, v_j^B) = r_\Gamma(v_j^A, v_i^B)$ for each $i, j = 1, \ldots, n$. In particular every $\Gamma$ with sphere-number 1 is symmetric. Given a protocork plumbing graph $\Gamma$ we can form another protocork plumbing graph called the reflection of $\Gamma$ denoted as $\overline{\Gamma}$ by swapping the labels $A$ and $B$. More precisely, the part $A$ of $\overline{\Gamma}$ is the part $B$ of $\Gamma$ and the part $B$ of $\overline{\Gamma}$ is the part $A$ of $\Gamma$; the multiset of edges of $\overline{\Gamma}$ is the same multiset of edges of $\Gamma$.

Two protocork plumbing graphs are isomorphic if there is an isomorphism of unoriented graphs that preserves the labelling of the parts and the sign of the edges. In particular $\Gamma$ and $\overline{\Gamma}$ need not be isomorphic but if $\Gamma$ is symmetric they are. We will adopt the convention of drawing the $A$-vertices on the left and the $B$-vertices on the right. Figure 1 shows some examples of protocork plumbing graphs.

**Definition 2.2.** A realization datum $\mathcal{R}$ for a protocork plumbing graph $\Gamma$ is a realization datum for the plumbing graph $\Gamma$ with an extra information: to each vertex $v$ is associated a preferred trivialization of the bundle $N_v \to S_v$.

**Natural framing.** Consider a plumbing graph $\Gamma$ with set of vertices $V$ where each vertex is labelled with the trivial bundle over $\mathbb{S}^2$. Let $\mathcal{R}$ be a realization datum for $\Gamma$ where the model disk bundles are $\mathbb{D}^2 \times \mathbb{S}^2$. In this case, the plumbing realization $P := P(\Gamma, \mathcal{R})$ is a quotient of $\bigcup_{v \in V} \mathbb{D}^2 \times \mathbb{S}^2 \times \{v\}$. Let $v \in V$, then the quotient map restricts to a smooth embedding over $\text{int}(\mathbb{D}^2) \times \mathbb{S}^2 \times \{v\}$. Notice that this extends only to a topological embedding of $\mathbb{D}^2 \times \mathbb{S}^2 \times \{v\}$ because the corner straightening procedure introduces some corners in the image of this embedding.
Figure 1: From left to right: a symmetric protocork plumbing graph with sphere-number 1, an asymmetric protocork plumbing graph with sphere-number 3, the reflection of the previous example.

Define $S_v \subset P$ to be the image of $\{0\} \times S^2 \times \{v\} \subset \text{int}(\mathbb{D}^2) \times S^2 \times \{v\}$ under the quotient map. We call $S_v$ the sphere induced by $v$ in $P$. $S_v$ has a natural framing because is image of the zero section of $\text{int}(\mathbb{D}^2) \times S^2 \times \{v\}$. It is thus possible to perform surgery on the sphere induced by $v$ using its natural framing (see [GS99, Definition 5.2.1] pg. 154 for a definition of surgery) by embedding $S^2 \times \mathbb{D}^2$ as $\mathbb{D}^2(\frac{1}{2}) \times S^2 \times \{v\}$ via the radial contraction $\mathbb{D}^2 \to \mathbb{D}^2(\frac{1}{2})$.

In particular, by using the trivializations of Definition 2.2, it follows that if $\mathcal{R}$ is a realization datum for a protocork plumbing graph $\Gamma$, there is a well defined natural framing for each surface $S_v$.

We remark that in general, given a manifold $M^n$ and a framed sphere $f : S^k \times D^{n-k} \to M$, the surgered manifold $M_f$ has a well defined smooth structure except that in a neighbourhood of $f(S^k \times \partial D^{n-k})$ where it is defined up to the choice of a collar of $f(S^k \times \partial D^{n-k})$ in $M \setminus f(S^k \times \text{int}(D^{n-k}))$ and a collar of the boundary of $D^{k+1} \times S^{n-k-1}$. In our case, we have a preferred choice because the framed sphere is embedded as

$$\mathbb{D}^2(\frac{1}{2}) \times S^2 \times \{v\} \subset \text{int}(\mathbb{D}^2) \times S^2 \times \{v\}$$

therefore, using polar coordinates to produce the two collars we end up with a well defined smooth structure on the surgered manifold.

**Definition 2.3 (Protocork).** Let $\Gamma$ be a protocork plumbing graph of sphere-number $n$ and $\mathcal{R}$ be a realization datum for $\Gamma$. We associate to $(\Gamma, \mathcal{R})$ the following oriented 4-manifolds with boundary:

$$\mathbb{D}^2(\frac{1}{2}) \times S^2 \times \{v\} \subset \text{int}(\mathbb{D}^2) \times S^2 \times \{v\}$$
• The manifold $P_{1/2}(\Gamma, \mathcal{R})$ obtained by plumbing on $(\Gamma, \mathcal{R})$. We call the spheres induced by vertices in the $A$-part $A$-spheres and those induced by vertices in the $B$-part $B$-spheres.

• The manifold $P_0(\Gamma, \mathcal{R})$ obtained from $P_{1/2}(\Gamma, \mathcal{R})$ by surgering out the $B$-spheres with their natural framing. $P_0(\Gamma, \mathcal{R})$ is called the protocork associated to $(\Gamma, \mathcal{R})$.

• The manifold $P_1(\Gamma, \mathcal{R})$ obtained from $P_{1/2}(\Gamma, \mathcal{R})$ by surgering out the $A$-spheres with their natural framing. $P_1(\Gamma, \mathcal{R})$ is called the reflection of $P_0(\Gamma, \mathcal{R})$.

We remark that these three 4-manifolds come with a natural identification of their boundaries.

We will use $P_{1/2}(\Gamma)\mid \Gamma\neq 2$ $\Gamma\cdot P_0(\Gamma)\cdot P_1(\Gamma)$ to denote the diffeomorphism type of these manifolds, these depends just on the isomorphism class of $\Gamma$. With an abuse of language we will also write $N \simeq P_i(\Gamma)$ instead of $N \in P_i(\Gamma)$.

$P_0(\Gamma)$ is said to be a symmetric protocork if $\Gamma$ is a symmetric valid graph. In this case $P_1(\Gamma)$ is diffeomorphic to $P_0(\Gamma)$ as will follow from Proposition 2.6. Notice that $P_1(\Gamma, \mathcal{R})$ is also diffeomorphic to the protocork associated to $\Gamma$ with the obvious reflected realization $\overline{\mathcal{R}}$, this is why it is called reflection of $P_0(\Gamma, \mathcal{R})$.

### 2.3 Handle decomposition of protocorks.

**Outline.** We will describe a handle decomposition of $P_{1/2}(\Gamma), P_0(\Gamma)$ and $P_1(\Gamma)$ that will be used in the following sections. This decomposition will be given in terms of Kirby diagrams ([Kir78], also see [GS99, Section 5.4] for the dotted circle notation originally introduced by Akbulut [Akb77]).

If $\Gamma$ is not connected, then $P_i(\Gamma), t \in \{0, 1/2, 1\}$ is the boundary connected sum $P_t(\Gamma_1)\# \ldots \# P_t(\Gamma_{|\pi_0(\Gamma)|})$ where $\Gamma_i, i = 1, \ldots, |\pi_0(\Gamma)|$ are the connected components of $\Gamma$. The Kirby diagram of a boundary connected sum is obtained by drawing the two diagrams next to each other, therefore we will assume $\Gamma$ is connected with sphere-number $n$.

**Kirby diagram for $P_{1/2}(\Gamma)$.** An algorithm to draw a Kirby diagram for a plumbing is given in [GS99, Section 6.1], we will apply it to the specific case of protocork plumbing graphs. Fix an embedding of $\Gamma$ in $\mathbb{R}^3$ where all the vertices and a neighbourhood of the endpoints of each edge lie in $\{0\} \times \mathbb{R}^2$. Notice that protocork plumbing graphs do not have to be planar in general. Since $\Gamma$ is connected
Figure 2: From the left: embedding of $\Gamma$, the result of removal of small balls centered at the vertices $\Gamma'$, introduction of 0-framed, oriented circles.

we can choose a spanning tree $T \subseteq \Gamma$. For each vertex $v$, we remove a small ball centered at $v$ from $\Gamma \subseteq \mathbb{R}^3$ obtaining $\Gamma'$. Notice that this will also remove a portion of the edges meeting $v$, we introduce a 0-framed circle $K_v$ in $\{0\} \times \mathbb{R}^2$ meeting $\Gamma'$ along these edges and fix an orientation for $K_v$; see Figure 2 for an example.

Now for each edge not in $T$, add a dotted circle, meridional to that edge. Next we replace each positive (negative) edge of the graph with a positive (negative) clasp between the circles incident to that edge as in Figure 3. Notice that this step depends on the orientation of the knots. The clasp follows the path of the edge, thus in the end, the clasps relative to edges not in $T$ will pass through a dotted circle. In this way we obtained a Kirby diagram with $2n$ 0-framed unknots and, denoting by $E$ the multiset of edges, $k = |E| - |T| = |E| - (2n - 1)$ dotted circles. See Figure 4 for an example, recall that by convention we draw the $A$-spheres on the left and the $B$-spheres on the right.

** Kirby diagram for $P_0(\Gamma)$ and $P_1(\Gamma)$.** Let $K$ be a 0-framed unknot in a Kirby diagram and let $D$ be its spanning disk. Assume that $D$ does not meet any dotted circles. Then in the interior of the 4-manifold given by the Kirby diagram, lies a framed sphere $S$ obtained by capping $D$ with the core of the 2-handle associated to $K$. In this case it can be shown [GS99, Section 5.4] that a Kirby diagram for the surgery along $S$ is obtained by replacing the 0-framed knot $K$ with a dotted circle (isotopic to $K$). Consequently, to draw a diagram for $P_0(\Gamma)$, we exchange each 0-framed knot relative to $v_j^B$, $j = 1, \ldots, n$ with a dotted circle in the diagram for $P_{1/2}(\Gamma)$ described above. Similarly to draw the diagram for $P_1(\Gamma)$, we replace the knots relative to $v_i^A$ with dotted circles. See Figure 4 for an example. Even though we will not make use of this fact, we mention that, as it is clear from the Kirby diagram, protocorks are diffeomorphic to the complement of $b_1(P_0(\Gamma))$ ribbon disks in $\mathbb{D}^4$. 
Figure 3: a) two oriented circles joined by an edge. b) replacement of a positive edge in the spanning tree $T$ with a clasp. c) replacement of a negative edge in $T$ with a clasp. d) replacement of a positive edge not in $T$.

Figure 4: A protocork plumbing graph with its plumbing (top-right), its protocork (bottom-left) and the reflection of the protocork (bottom-right).
2.4 Homological properties of protocorks.

The next proposition (Proposition 2.4) summarizes the basic homological properties of protocorks and their boundary that we will need in the second part of the article. Some of these are used also in [MS99], without a proof. All the homologies below use $\mathbb{Z}$-coefficients when not specified otherwise.

**Proposition 2.4.** Let $\Gamma$ be a connected protocork plumbing graph with sphere-number $n$ and multiset of edges $E$. Let $N$ be a protocork associated to $\Gamma$ and denote by $Y := \partial N$ its oriented boundary. Let $X$ be a a closed, oriented smooth 4-manifold such that $X = M \cup_Y N$ where $M$ is a compact, oriented 4-manifold with $\partial M = -Y$. Then:

(a) $H_1(N) \cong \mathbb{Z}^{|E|-2n+1}$, $H_2(N) \cong H_3(N) \cong 0$, in particular $b^+(N) = 0$.

(b) $H_1(Y)$ is free and generated by cycles in the graph $\Gamma$, thus $b_1(Y) = |E|-2n+1$. The inclusion map $i : Y = \partial N \to N$ induces isomorphisms $H_1(Y) \to H_1(N)$ and $H^1(N) \to H^1(Y)$.

(c) We can find a basis of $H_2(Y)$ given by plumbing tori and $H^1(Y) \otimes H^1(Y) \to H^2(Y) = \alpha \otimes \beta \to \alpha \cup \beta$ is the zero map. A fortiori, the triple cup product of $Y$ is trivial.

(d) $b^+(M) = b^+(X)$.

If in addition $H_1(X) = 0$, then

(e) $H_1(M) = 0$ in particular $H_1(Y) \to H_1(M)$ is trivial.

(f) $H^2(M) \to H^2(Y)$ is surjective.

(g) $H^2(M) \cong H^2(X) \oplus H^2(Y)$ not canonically.

(h) Denote by $\text{Spin}^c(W)$ the isomorphism classes of spin$^c$ structures for a manifold $W$. The restriction map

$$r : \text{Spin}^c(X) \to \text{Spin}^c(M) \times \text{Spin}^c(N) = [\mathfrak{s}] \mapsto ([\mathfrak{s}|_M], [\mathfrak{s}|_N])$$ (20)

is an injection. The only spin$^c$ structure on $N$ is the trivial one up to isomorphism, hence $r$ establishes a correspondence between $\text{Spin}^c(X)$ and $\text{Spin}^c(M)$. 
(a) We consider the Kirby diagram of subsection 2.3. Generators of \( H_1(N) \) are identified with meridians of the dotted circles. Relations are given by the linking number of the 0-framed knots with the dotted circles. The latter is governed by \( \alpha_i(v^A_i, v^B_j) = \delta_{ij} \) which forces all the generators that are not induced by the edges to be trivial. This shows that \( H_1(N) \cong \mathbb{Z}^{\lfloor |E| \rfloor - 2n+1} \). Inspecting the chain complex induced by the handle decomposition we see that the boundary operator \( \partial_2 : C_2(N) \to C_1(N) \) is injective, indeed it sends the generator induced by \( v^A_i \) to the generator induced by \( v^B_i \). Hence \( H_2(N) = (0) \). Moreover there are no 3-handles, therefore \( H_3(N) = (0) \).

(b) The Kirby diagram of \( N \) induces a surgery presentation for \( Y \) where all the knots are 0-framed. Denote the meridian of the knots induced by \( v^A_i, v^B_j, e \) for \( i, j = 1, \ldots, n, e \in E \setminus T \) respectively by \( \mu_{A_i}, \mu_{B_j}, \mu_e \). From the diagram we see that the linking matrix is given by \( \ell k(\mu_{A_i}, \mu_{B_j}) = \alpha_i(v^A_i, v^B_j) = \delta_{ij} \) and zero otherwise. By [GS99, Proposition 5.3.11], \( H_1(Y) \) is generated by the meridians of the knots quotiented out by relations of the form \( 0 = p_i \mu_i + \sum q_i \ell k(\mu_i, \mu_j) \mu_j \) where \( p_i/q_i \) is the framing coefficient of the meridian \( \mu_i \). In our case \( p_i/q_i = 0 \) and the special form of the linking matrix implies that \( \mu_{A_i} = 0 = \mu_{B_j} \) while the \( \mu_e \) are unrelated. This with the previous item shows that the inclusion map \( H_1(Y) \to H_1(N) \) is an isomorphism from which follows also the assertion in cohomology.

(c) Notice that the same argument used to compute \( H_1(N) \), applies to \( P_{1/2}(\Gamma) \) and allows us to identify (not canonically) the generators of \( H_1(P_{1/2}(\Gamma)) \) (and thus of \( H_1(Y) \)) with cycles in \( \Gamma \) thanks to (b). If we consider a basis for \( H_1(Y) \) made up of these loops, then we can use as geometric duals the plumbing tori [DH14]. The plumbing tori do not intersect, hence the map \( H_2(Y) \otimes H_2(Y) \to H_1(Y) = [S_1] \otimes [S_2] \mapsto [S_1 \cap S_2] \) is trivial. This map is Poincaré dual to \( H^1(Y) \otimes H^1(Y) \to H^2(Y) = \alpha \otimes \beta \mapsto \alpha \cup \beta \).

(d) \( b^+(X) = b^+(M) + b^+(N) + \text{rank}\{ \partial : H_2(X) \to H_1(Y) \} \) [KM07, pg. 76] where \( \partial \) is the connecting morphism in the Mayer-Vietoris long exact sequence. However \( \text{rank}(\partial) = \dim \text{Ker}\{ i_1 + i_2 : H_1(Y) \to H_1(M) \oplus H_1(N) \} = 0 \) since \( i_2 : H_1(Y) \to H_1(N) \) is an isomorphism.

(e) Since \( H_1(X) = 0 \), continuing the long exact sequence we have \( 0 \to H_1(Y) \to H_1(M) \oplus H_1(N) \to 0 \), since \( H_1(Y) \cong H_1(N) \) then \( H_1(M) = 0 \). Notice also that this implies that \( H_1(Y) \to H_1(M) \) is trivial.
(f) $H^2(M) \to H^2(Y)$ is surjective indeed the triviality of $H_1(Y) \to H_1(M)$ implies by Poincaré duality that $H^2(Y) \to H^3(M, \partial M)$ is trivial and that map fits into the long exact sequence of the pair $(M, \partial M)$: $\cdots \to H^2(M) \to H^2(Y) \to H^3(M, \partial M) \to \cdots$.

(g) $H^2(M) \approx H^2(X) \oplus H^2(Y)$ not canonically. Mayer-Vietoris implies that $0 \to H^2(X) \to H^2(M) \oplus 0 \to H^2(Y) \to 0$ (the surjectivity comes from the above item while the injectivity from the surjectivity of $H^1(N) \to H^1(Y)$).

(h) Since $H^1(X) = 0$, spin$^c$ structures over $X$ are classified by the first Chern class of the determinant bundle. The claim follows from the injectivity of the restriction map $H^2(X) \to H^2(M) \oplus H^2(N)$ which can be seen from the exact sequence $H^1(M) \oplus H^1(N) \sim H^1(Y) \to H^2(X) \to H^2(M) \oplus H^2(N)$ since $H^1(N) \sim H^1(Y)$ is an isomorphism.

$\square$

2.5 The involutions $\tau$ and $\rho$.

Outline. We will show that if $(\Gamma, \mathcal{R})$ satisfies some assumptions we can construct orientation reversing involutions $\hat{\rho}_B, \hat{\rho}_A : P_{1/2}(\Gamma, \mathcal{R}) \to P_{1/2}(\Gamma, \mathcal{R})$ and $\rho_B = \hat{\rho}_B|_{\partial P_{1/2}(\Gamma, \mathcal{R})}$, $\rho_A = \hat{\rho}_A|_{\partial P_{1/2}(\Gamma, \mathcal{R})}$. Informally, $\hat{\rho}_B$ ($\hat{\rho}_A$) reflects the $B$-spheres ($A$-spheres) while acting trivially on their normal disk and fixes the $A$-spheres ($B$-spheres) while reflecting their normal disks. Moreover if $\Gamma$ and $\mathcal{R}$ are symmetric (Definition (2.5)) we will construct an orientation preserving involution $\hat{\tau} : P_{1/2}(\Gamma, \mathcal{R}) \to P_{1/2}(\Gamma, \mathcal{R})$ restricting to some involution $\tau$ on the boundary. $\hat{\tau}$ also induces a diffeomorphism $\hat{\tau} : P_0(\Gamma, \mathcal{R}) \to P_1(\Gamma, \mathcal{R})$ between the protocork and its reflection restricting to $\tau$ on the boundary. Informally $\hat{\tau}$ exchanges the $A$-spheres with the $B$-spheres. We remark that these diffeomorphisms depend on the specific realization, even though it is not reflected in the notation. We prove some properties about these involutions in Proposition 2.6. The fact that $\rho_A, \rho_B$ act trivially on the first cohomology group will allow us to compute the index of the relevant operators later (see Proposition 4.2). The involution $\tau$ is relevant to our theory because plays a role analogous to the twisting map of corks in a sense that will be made precise in in Subsection 2.6. For the sake of this subsection we limit ourself to Proposition 2.8 which says that in analogy to what happens to corks’ involutions, $\tau$ does not extend to the protocork (unless the protocork is trivial).
Assumptions on \((\Gamma, \mathcal{R})\). Denote by \(S_0\) the one-point compactification of \(\mathbb{R}^2\) equipped with the smooth structure coming from the homeomorphism with \(S^2\) induced by the stereographic projection \(S^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}^2\).

Let \(\Gamma\) be a protocork plumbing graph. We consider realization data \(\mathcal{R}\) where the model surfaces \(S_0\) are given by \(\mathbb{S}^2\), the bundles \(N_v\) are the trivial \(\mathbb{D}^2\) bundles over \(S_0\) and the embedding of the disks \(\varphi_v\) sends \(\bigcup_{i=1}^{d_v} \mathbb{D}^2(2)\) to the disks of radius 2 centered in \((5k, 0) \in \mathbb{R}^2, k = 0, \ldots, d_v - 1\) in the obvious way using translations. Let \(r : S_0 \to S_0\) be the self-diffeomorphism induced by the reflection with respect to \(\mathbb{R} \times \{0\}\) and denote by \(r_{\mathbb{D}^2} : \mathbb{D}^2 \to \mathbb{D}^2\) the restriction of \(r\) to \(\mathbb{D}^2 = \overline{B}(0, 1) \subset \mathbb{R}^2\).

We also make an assumption about the smoothing datum, item (d) in the definition of realization datum. For each vertex \(v\), and \(i = 1, \ldots, d_v\), choose \(c_{v,i} : \mathbb{T}^2 \times (-1, 1) \times [0, 1) \to S_0 \times \mathbb{D}^2\) a collar of the torus \(\varphi_v(\partial \mathbb{D}(1) \times \{i\}) \times \partial \mathbb{D}^2\) obtained using the natural polar coordinates so that the \((-1, 1)\) factor comes from the radial coordinate of \(\varphi_v(\partial \mathbb{D}(1) \times \{i\}) \subset \varphi_v(\mathbb{D}(2) \times \{i\})\) and the \([0, 1)\) factor comes from the radial coordinate of \(\partial \mathbb{D}^2 \subset \mathbb{D}^2\). These collars induce a smooth structure on \(P_{1/2}(\Gamma, \mathcal{R})\) as explained in the paragraph below about the smoothness of \(\hat{\rho}_B\), indeed given the collars we can introduce corners along the tori and perform gluing in a unique way [Wal16, Section 2.6, 2.7]. We will assume that the smoothings of the manifolds \(M_\pm\) prescribed by the realization datum \(\mathcal{R}\) arise from a choice of the collars \(c_{v,i}\)s as above.

Definition of \(\hat{\rho}_B\) and \(\rho_B\). We define an auxiliary involution \(\psi\) on the collection of bundles \(E(\Gamma, \mathcal{R})\),

\[
\psi : E(\Gamma, \mathcal{R}) \to E(\Gamma, \mathcal{R})
\]

by defining

\[
\psi(b, f, v_i^A) = (b, r_{\mathbb{D}^2}(f), v_i^A) \quad \text{for} \quad (b, f, v_i^A) \in S_0 \times \mathbb{D}^2 \times \{v_i^A\}
\]

and

\[
\psi(b, f, v_j^B) = (r(b), f, v_j^B) \quad \text{for} \quad (b, f, v_j^B) \in S_0 \times \mathbb{D}^2 \times \{v_j^B\}.
\]

So in particular \(\psi\) fixes the \(A\)-spheres. It is not difficult to check that \(\psi\) passes to the quotient \(P_{1/2}(\Gamma, \mathcal{R})\) defining an involution \(\hat{\rho}_B : P_{1/2}(\Gamma, \mathcal{R}) \to P_{1/2}(\Gamma, \mathcal{R})\). We denote the induced involution of the boundary as \(\rho_B\).
Smoothness of $\hat{\rho}_B$. The map $\hat{\rho}_B$ is clearly smooth outside of the plumbing tori, proving that is smooth over the tori is not immediate because of the smoothing of the corners used to define the plumbing. To prove it we argue as follows.

The smooth manifold $P_{1/2}(\Gamma, \mathcal{R})$ can be described as the manifold obtained by the following procedure. Firstly, for every vertex $v_i^B$ in the $B$-part set $\tilde{N}_{v_i^B} := \tilde{S}_0 \times \mathbb{D}^2$ where

$$\tilde{S}_0 = S_0 \setminus \varphi_{v_i^B}(\bigcup_{i=1}^{d_v} \text{int}(\mathbb{D}^2(1))).$$

(25)

Secondly, for every vertex $v_i^A$ in the $A$-part, let $\tilde{N}_{v_i^A}$ be the manifold obtained from $N_{v_i^A}$ by introducing a corner [Wal16, pg. 61] along the tori $\varphi_{v_i^A}(\bigcup_{j=1}^{d_v^A} \partial \mathbb{D}^2(1)) \times \partial \mathbb{D}^2$. In general this operation yields a structure of manifold with corners only up to diffeomorphism but since we have chosen collars $c_{v_i^A}$ along the tori, we obtain a precise structure.

Thirdly, glue $E_B := \bigcup_{i=1}^n \tilde{N}_{v_i^B}$ to $E_A := \bigcup_{i=1}^n \tilde{N}_{v_i^A}$ as prescribed by the edges using again the collars $c_{v_i,j}$ to obtain a smooth structure over the glued region [Wal16, Section 2.7]. The result is a smooth manifold with a natural diffeomorphism to $P_{1/2}(\Gamma, \mathcal{R})$.

We can think of the map $\psi$ as a map $E_B \bigcup E_A \to E_B \bigcup E_A$ preserving $E_B$ and $E_A$. A priori, the smoothness of the action $\psi$ over a neighbourhood of a plumbing torus in $E_A$ is not clear because we introduced corners in passing from $N_{v_i^A}$ to $\tilde{N}_{v_i^A}$. However, using our choice of collars $c_{v_i,j}$, we can see that the action of $\psi$ over such neighbourhood is conjugated to

$$\psi' \times id : \mathbb{T}^2 \times [0, 1)^2 \longrightarrow \mathbb{T}^2 \times [0, 1)^2$$

(26)

where $\psi' : \mathbb{T}^2 \to \mathbb{T}^2$, $\psi(e^{i\theta_1}, e^{i\theta_2}) = (e^{i\theta_1}, e^{-i\theta_2})$; in particular is smooth and restricts to the identity on the normal cones. Similarly the action of $\psi$ on a neighbourhood of a plumbing torus in $E_B$ is conjugated to

$$\psi'' \times id : \mathbb{T}^2 \times [0, 1)^2 \longrightarrow \mathbb{T}^2 \times [0, 1)^2$$

(27)

where $\psi(e^{i\theta_1}, e^{i\theta_2}) = (e^{-i\theta_1}, e^{i\theta_2})$. These two maps glue proving smoothness over a neighbourhood of the plumbing torus $\mathbb{T}^2 \times (-1, 1) \times [0, 1) \hookrightarrow P_{1/2}(\Gamma, \mathcal{R})$.

Definition of $\hat{\rho}_A$ and $\rho_A$. To define $\hat{\rho}_A$ and $\rho_A$ we follow the same recipe, but exchanging the role of $A$ and $B$, so that $\hat{\rho}_A$ fixes the $B$ spheres and $\hat{\rho}_B$ fixes the $A$ spheres.
Definition 2.5 (Symmetric realization datum). A realization datum $\mathcal{R}$ for a protocork plumbing graph $\Gamma$ is said to be symmetric if satisfies the hypothesis stated for $\hat{\rho}_B$ and in addition satisfies the following. For any vertex $v$ and $n \in \{1, \ldots, d_v\}$, let $D^2_n \subset S_0 \times \{v\}$ be the image of the $n$-th disk $\mathbb{D}^2 \times \{n\} \subset \bigsqcup_{i=1}^{n} \mathbb{D}^2$ under $\varphi_v$, then $\mathcal{R}$ satisfies

(a) if an edge of the form $\epsilon \simeq (i, i, \pm)$ identifies $D^{v^A}_{\epsilon,1}$ with $D^{v^B}_{n,1}$ and $D^{v^B}_{\epsilon,2}$ with $D^{v^A}_{k,2}$ then $k = n$ and

(b) if an edge of the form $\epsilon \simeq (i, j, \pm)$ with $i \neq j$ identifies $D^{v^A}_{\epsilon,1}$ with $D^{v^B}_{n,1}$ and $D^{v^B}_{\epsilon,2}$ with $D^{v^A}_{k,2}$ then exists an edge $\epsilon' = (j, i, \pm)$ identifying $D^{v^A}_{k,1}$ with $D^{v^B}_{n,1}$.

(c) For any pair of edges $\epsilon, \epsilon'$ as in item (b), the smoothings of $M_\pm$ (see item (d) in the definition of realization datum) are the same.

Notice that a protocork plumbing graph admits this kind of realization data if and only if is symmetric.

Definition of $\hat{\tau}$ and $\tau$. Now suppose that $\Gamma$ is symmetric and let $\mathcal{R}$ be a symmetric realization datum for $\Gamma$. We define an auxiliary involution on the collection of bundles $E(\Gamma, \mathcal{R})$,

$$\psi : E(\Gamma, \mathcal{R}) \to E(\Gamma, \mathcal{R})$$

by setting

$$\psi(b, f, v^A_i) = (b, f, v^B_i) \quad \text{for } (b, f, v^A_i) \in S_0 \times \mathbb{D}^2 \times \{v^A_i\}$$

and

$$\psi(b, f, v^B_j) = (b, f, v^A_j) \quad \text{for } (b, f, v^B_j) \in S_0 \times \mathbb{D}^2 \times \{v^B_j\}. $$

So in particular $\psi$ exchanges the $A$-spheres with the $B$-spheres. It is not difficult to check, using our assumptions on $\mathcal{R}$, that $\psi$ passes to the quotient $P_{1/2}(\Gamma, \mathcal{R})$ defining a smooth involution $\hat{\tau}$. We will denote the induced involution on the boundary as $\tau$.

Since $\hat{\tau}$ swaps the tubular neighbourhoods of the $A$-spheres and $B$-spheres respecting their framings and the sugery operation modifies the manifold only in that neighbourhood, the involution $\hat{\tau}$ gives also a diffeomorphism $\hat{\tau} : P_0(\Gamma, \mathcal{R}) \to P_1(\Gamma, \mathcal{R})$ that restricts to $\tau$ on the boundary.
**Proposition 2.6.** Let \( \Gamma \) be a protocork plumbing graph of sphere-number \( n \) and \( \mathcal{R} \) a realization datum for \( \Gamma \) satisfying the hypothesis stated at the beginning of this subsection. Set \( Y := \partial P_{1/2}(\Gamma, \mathcal{R}) \). Then

(a) \( \hat{\rho}_B, \hat{\rho}_A : P_{1/2}(\Gamma, \mathcal{R}) \to P_{1/2}(\Gamma, \mathcal{R}) \) and \( \rho_B, \rho_A : Y \to Y \) are orientation reversing,

(b) \( \rho_B \) and \( \rho_A \) fix \( H^1(Y; \mathbb{Z}) \) and commute.

If in addition \( \Gamma \) and \( \mathcal{R} \) are symmetric, then

(c) \( \hat{\tau} : P_{1/2}(\Gamma, \mathcal{R}) \to P_{1/2}(\Gamma, \mathcal{R}) \), \( \hat{\tau} : P_0(\Gamma, \mathcal{R}) \to P_1(\Gamma, \mathcal{R}) \) and \( \tau : Y \to Y \) are orientation preserving,

(d) If \( \Gamma \) is symmetric and non-trivial, i.e. exists \( i \in \{1, \ldots, n\} \) such that \( a_i^\Gamma(v^A_i, v^B_i) \neq r_1(v^A_i, v^B_i) \), then \( \tau \) acts non-trivially on \( H^1(Y; \mathbb{Z}) \).

(e) \( \rho_B, \rho_A \) and \( \tau \) generate an action of \( D_8 \), the dihedral group of 8-elements.

**Proof.**

(a) The auxiliary involution \( \psi \) defined in (22) is orientation reversing, indeed on a tubular neighbourhood of the \( A \)-spheres acts as a product of the identity (on the base) and a reflection (on the normal fibers). Consequently \( \hat{\rho}_B \) is orientation reversing too. The proof for \( \hat{\rho}_A \) and the induced map on the boundary is analogous.

(b) By Proposition (2.4) it is sufficient to understand the behaviour \( \rho \in \{ \rho_A, \rho_B \} \) on the plumbing tori. \( \rho \) preserve each plumbing torus \( \mathbb{T} \) but changes its orientation, indeed we can see from (23) and (24) that \( (e^{i0_b}, e^{i0_f}) \in \mathbb{T} \mapsto (e^{i0_b}, e^{-i0_f}) \in \mathbb{T} \), therefore \( \rho_\ast([\mathbb{T}]) = -[\mathbb{T}] \in H_2(Y) \). Since \( \rho \) is orientation reversing, we obtain by Poincaré duality that \( \rho^\ast : H^1(Y) \to H^1(Y) \) is the identity. Moreover since \( H_2(Y) \) has no torsion, \( \rho^\ast : H^2(Y) \to H^2(Y) \) is minus the identity and \( \rho_\ast : H_1(Y) \to H_1(Y) \) is the identity.

(c) Follows from the fact that the auxiliary involution in (28) is orientation preserving.

(d) Since \( \Gamma \) is not trivial, there is an edge \( \epsilon \simeq (i, i, +) \) that does not belong to a spanning tree of \( \Gamma \). Hence by Proposition 2.4, the plumbing torus \( \mathbb{T} \) associated to \( \epsilon \) is non-trivial in \( H^2(Y) \). On the other hand, we see from the definition of \( \tau \) that \( \tau(b, f, v^A_i) = (b, f, v^B_i) \sim_\epsilon (f, b, v^A_i) \), hence \( \tau_\ast[\mathbb{T}] = -[\mathbb{T}] \). Since \( \tau \) preserves the orientation the thesis follows.
(e) An explicit computation shows that the following relations are in place:

\[
\begin{align*}
\rho_A^2 &= \rho_B^2 = \tau^2 = 1 \\
\tau^{-1}\rho_B\tau &= \rho_A \\
\rho_A\rho_B &= \rho_B\rho_A.
\end{align*}
\]

We can therefore define a group isomorphism from \( G := \langle \rho_A, \rho_B, \tau \rangle = \langle \tau\rho_A, \tau \rangle \) to

\[
D_8 = \langle r, s \mid r^4 = 1, s^2 = 1, srs = r^{-1} \rangle
\]

by sending \( \tau\rho_A \mapsto r \in D_8 \) and \( \tau \mapsto s \in D_8 \).

Regarding item (d) in the above proposition, we also point out that it is not difficult to construct examples with sphere-number larger than one where \( \tau \) does not even preserve the plumbing tori.

We conclude this subsection by showing that, similarly to what happens to the cork’s involutions, \( \tau \) does not extend to a diffeomorphism of \( P_0(\Gamma, \mathcal{R}) \).

**Lemma 2.7.** Let \( \Gamma \) be a protocork plumbing graph with sphere-number \( n \). Let \( i \in \{1, \ldots, n\} \) be such that \( r_\Gamma(v_i^A, v_i^B) > 1 \) and let \( \gamma_i^B \subset \partial P_0(\Gamma) \) be a circle meridional to the sphere \( B_i \) in \( P_{1/2}(\Gamma) \). Then \( \gamma_i^B \) is not slice in \( P_0(\Gamma) \), i.e. does not bound a disk in the protocork \( P_0(\Gamma) \).

**Proof.** We will add some 2-handles to \( P_0(\Gamma) \) and show that the image of \( \gamma_i^B \) in this new manifold is not slice. Since adding 2-handles can only improve sliceness this will imply the thesis. With reference to the Kirby diagram constructed in Subsection 2.3, let \( C_1, \ldots, C_N, N \in \mathbb{N} \) be the collection of dotted circles. Since \( r_\Gamma(v_i^A, v_i^B) > 1 \), we can suppose without loss of generality that \( C_1, C_2 \) are dotted circles introduced by a pair of \( +, - \) edges between \( v_i^A \) and \( v_i^B \). We proceed by cancelling \( C_3, \ldots, C_N \) by adding 0-framed 2-handles meridional to each of them. Now we have reduced to the case where \( \Gamma \) is the graph showed in Figure 4 with \( d_i = 3 \). In this case we can add a pair of 2-handles to \( P_0(\Gamma) \) so that \( (P_0(\Gamma), \gamma_i^B) \) is diffeomorphic to the pair \( (W, \gamma) \) shown in Figure 5e where \( \gamma \) is the black dashed curve. This is [GS99, Exercise 9.3.5], since it is particularly relevant to us, we give a proof in Figure 5 for completeness. Now \( W \) is the Akbulut cork [Akb91] and \( \gamma \) is a meridional circle to the its dotted circle. \( \gamma \) is not slice in \( W \) ([Akb91], also [Akb16, Theorem 9.3]).
Proposition 2.8. Let $\Gamma$ be a symmetric protocork plumbing graph of sphere-number $n$ and $\mathcal{R}$ be a symmetric realization datum for $\Gamma$. Suppose that for some $i \in \{1, \ldots, n\}$ $r_i^G(v_i^A, v_i^B) > 1$ and let $Y := \partial P_0(\Gamma, \mathcal{R})$. Then the involution $\tau : Y \to Y$ does not extend to a diffeomorphism of $P_0(\Gamma, \mathcal{R})$.

Proof. Let $y_i^A \subset Y$ and $y_i^B \subset Y$ be circles meridional to the spheres $A_i$ and $B_i$ in $P_{1/2}(\Gamma, \mathcal{R})$ respectively. Then $\tau$ exchanges $y_i^A$ and $y_i^B$. $y_i^A$ is slice in $P_0(\Gamma, \mathcal{R})$ while $y_i^B$ is not by Lemma 2.7. Therefore $\tau$ cannot extend to a diffeomorphism of the protocork. \qed

2.6 Protocorks and exotic 4-manifolds.

Outline. In this subsection we define, in analogy to the case of corks, the operation of protocork twist. Firstly we define it for general protocorks, in this case the operation consists in cutting out the protocork and gluing back its reflection. In case the protocork is symmetric we obtain the same result (up to diffeomorphism) gluing back the protocork via the involution of the boundary $\tau$. Then we show that any exotic pair is related by a protocork twist (Proposition (2.13)), this follows directly from the definition of protocork without much surprise as the definition of protocork is forged exactly to this end. Next we show that any cork admits a supporting protocork (Proposition (2.14)).

Corks and exotic pairs. A pair $(X_0, X_1)$ of oriented, closed, smooth, 4-manifolds is said to be an exotic pair if $X_0$ is (orientation preservingly) homeomorphic but not (orientation preservingly) diffeomorphic to $X_1$. It is a result of of Curtis, Freedman, Hsiang and Stong [CFHS96] and Matveyev [Mat95] that if the pair is simply connected then $X_1$ can be obtained from $X_0$ by an operation called cork twist which we now review.

Definition 2.9. An (abstract) cork is a pair $(W, \tau)$ where $W$ is a compact, contractible, oriented, smooth 4-manifold with $\partial W \neq \emptyset$ and $\tau : \partial W \to \partial W$ is an orientation preserving smooth involution such that $\tau$ does not extend to an orientation preserving diffeomorphism of $W$.

We remark that by [Fre82] the involution of a cork always extend to an homeomorphism of the full manifold thanks to the contractibility assumption.

The study of corks was pioneered by Akbulut who found the first cork in [Akb91]. This cork, which we call Akbulut cork, is shown in Figure 5e where the
Figure 5: The picture shows how we obtain the Akbulut cork (5e) by adding two 2-handles to the protocork (5a). The dashed circles represent the meridian to the $A$-sphere (red) and to the $B$-sphere (black). In (5b) we add the 2-handles (blue). To obtain (5c) we cancel the 0-framed blue with the dotted circle, and similarly after sliding the red to the blue we cancel the $-1$-framed blue with the remaining dotted circle. To obtain (5d) we apply the Lemma shown in Figure 6. (5d) is clearly isotopic to (5e).
Figure 6: Lemma used to simplify the diagram in Figure 5. The diagram on the right is obtained by isotopying the red (performing an half rotation of the exterior fork).

dashed curves are exchanged by the involution $\tau$. The action of the involution $\tau$ is best seen by isotopying the diagram so to obtain the symmetric link [Akb16, Figure 10.4], then $\tau$ is induced by a rotation which preserves the link. We point out that this is not the only nomenclature in the literature as several authors call Akbulut corks what we call corks and Akbulut-Mazur cork the Akbulut cork.

Now let $X$ be a closed oriented smooth 4-manifold $X$, $(W, \tau)$ an almost cork and let $e : W \to X$ be an orientation preserving embedding. Then we can form another 4-manifold

$$X(W, \tau, e) := (X \setminus \text{int}(e(W))) \bigcup_{e \circ \tau} W$$

by cutting out $e(W)$ and gluing it back via the map $e \circ \tau : \partial W \to \partial(X \setminus e(W))$. We say that $X(W, \tau, e)$ is obtained from $X$ by a cork twist. If $X$ and $X(W, \tau, e)$ are not diffeomorphic we say that $(W, \tau, e)$ is effective [AKMR17].

**Protocork twist.** Given $M_0, M_1$ oriented manifolds we will denote by $\text{Diffeo}_+(M_0, M_1)$ the set of orientation preserving diffeomorphism $M_0 \to M_1$.

We call twisting triple a triple $(M_0, M_1, \alpha)$ where $M_0, M_1$ are oriented, compact 4-manifolds with boundary and $\alpha \in \text{Diffeo}_+(\partial M_0, \partial M_1)$. Two twisting triples $(M_0, M_1, \alpha), (M_0', M_1', \alpha')$ are said to be isomorphic if exist $\phi_i \in \text{Diffeo}_+(M_i, M_i')$ for $i = 0, 1$, such that $\alpha = \phi_1^{-1}|_{\partial M_1'} \circ \alpha' \circ \phi_0|_{\partial M_0}$.

**Definition 2.10 (Abstract protocork).** An abstract protocork is a twisting triple $(P_0, P_1, \alpha)$ isomorphic to $(P_0(\Gamma, \mathcal{R}), P_1(\Gamma, \mathcal{R}), id)$ for some $\Gamma, \mathcal{R}$ protocork plumbing graph and realization.
Given an abstract protocork \((P_0, P_1, \alpha)\) and an embedding of \(P_0\) in a closed 4-manifold, the protocork twist operation consists in cutting out \(P_0\) and gluing in \(P_1\) using the identification \(\alpha\). This is made precise by the following definition.

**Definition 2.11 (Protocork twist).** Let \((P_0, P_1, \alpha)\) be an abstract protocork, let \(X\) be a closed, oriented 4-manifold and let \(e : P_0 \to X\) be a smooth, orientation preserving embedding. The manifold

\[
X(P_0, P_1, \alpha, e) := (X \setminus \text{int}(e(P_0))) \bigcup_{e \circ \alpha} P_1
\]

is said to be obtained from \(X\) by twisting \((P_0, P_1, \alpha)\) via \(e\).

In the symmetric case we can give more economical definitions resembling those of cork and cork twist. We define a *twisting pair* to be a pair \((M_0, i)\) where \(M_0\) is a compact, oriented 4-manifold with boundary and \(i \in \text{Diffeo}_+ (\partial M_0)\). Two twisting pairs \((M_0, i)\) and \((M'_0, i')\) are *isomorphic* if exists \(\phi_0 \in \text{Diffeo}_+(M_0, M'_0)\) such that \(i = \phi^{-1}_0|_{\partial M'_0} \circ i' \circ \phi_0|_{\partial M_0}\).

**Definition 2.12 (Abstract symmetric protocork).** An *abstract symmetric protocork* is a twisting pair \((P_0, i)\) isomorphic to \((P_0(\Gamma, \mathcal{R}), \tau)\) for some symmetric protocork plumbing graph \(\Gamma\) with symmetric realization datum \(\mathcal{R}\) and involution \(\tau\) induced by \((\Gamma, \mathcal{R})\) as in Subsection 2.5.

Given an abstract symmetric protocork \((P_0, i)\) and an orientation preserving embedding \(e : P_0 \to X\) in a smooth, oriented 4-manifold \(X\), we define the manifold obtained from \(X\) by twisting \((P_0, i)\) via \(e\) to be

\[
X(P_0, i, e) := X(P_0, P_0, i, e) = (X \setminus \text{int}(e(P_0))) \bigcup_{e \circ i} P_0.
\]

The analogy between (35) and (33) motivates the name protocork twist.

**Remark 2.** Since protocork twists are defined by gluing two manifolds along their boundaries, the smooth structure on the resulting manifold is well defined only up to the choice of the collars of the boundaries and different choices lead to diffeomorphic manifolds. Therefore, if this choice is not made explicit we have to interpret the result just as a diffeomorphism type. This is the same level of ambiguity present in the definition of cork twist.

**Remark 3.** The manifolds (up to diffeomorphism) that can be produced from \(X\) by twisting \((P_0(\Gamma, \mathcal{R}), P_1(\Gamma, \mathcal{R}), id)\) do not depend on the specific realization chosen, what matters is that we use the same realization for the protocork
and its reflection. Indeed if \( R' \) is a different realization, there exist diffeomorphisms \( \varphi_i : P_i(\Gamma, R) \to P_i(\Gamma, R') \), \( i = 0, 1 \) that coincide over the boundary, i.e. \( \varphi_0|_{\partial P_0(\Gamma, R)} = \varphi_1|_{\partial P_1(\Gamma, R)} \); these can be constructed by isotopying the surgery data.

Similarly, the manifolds that can be produced from \( X \) by twisting an abstract symmetric protocork \( (P_0, \iota) \), where \( P_0 \simeq P_0(\Gamma) \) depend only on the isomorphism class of \( \Gamma \). It was not immediate from the definition because the involution \( \tau \) depends on the realization datum.

**Proposition 2.13.** For any exotic pair \((X_0, X_1)\) of simply connected, oriented, closed 4-manifolds there exists an abstract symmetric protocork \((P_0, \iota)\) with connected protocork plumbing graph and an embedding \( e : P_0 \to X_0 \) such that \( X_1 \) is diffeomorphic to \( X_0 \cup P_0 \cup e \).

The proof is along the lines of the first part of the argument of [CFHS96, Mat95], the difference is that we do not need to complete the subcobordism to obtain contractible ends.

**Proof.** The first part of the proof of the h-cobordism theorem ([Sma62], see also [Mil65]) ensures the existence of an h-cobordism \( W^5 : X_0 \to X_1 \) with the following properties:

- \( W^5 \) is obtained by gluing simultaneously \( n \geq 0 \) 2-handles \( \{h^2_j\}_{j=1}^n \) to \( X_0 \times \{1\} \subset X_0 \times [0, 1] \) obtaining \( W' \) and then by gluing simultaneously \( n \) 3-handles \( \{h^3_i\}_{j=1}^n \) to \( \partial_+ W' \). In particular there are no handles of index \( \lambda \neq 2, 3 \).

- Let the 4-manifold \( W_{1/2} \subset W \) be the middle level of the cobordism obtained after attaching all the 2-handles. Let \( B_j \subset W_{1/2} \) be the belt sphere of \( h^2_j \) and let \( A_i \subset W_{1/2} \) be the attaching sphere of \( h^3_i \). Then their algebraic intersection number in \( W_{1/2} \) is equal to the Kronecker delta: \( B_j : A_i = \delta_{ij} \) and the normal bundle of \( B_j, A_i \) is trivial for any \( i, j \).

It follows that the intersection graph of the spheres \( \{B_j, A_i\}_{i,j} \) induces a protocork plumbing graph \( \Gamma \). If \( \Gamma \) is not connected, then we can add extra pairs +, - of geometrically cancelling intersections by pushing one of the spheres into another. In a similar manner if \( \Gamma \) is not symmetric we can symmetrize it by adding some *geometrically* cancelling intersections between the spheres \( \{B_j, A_i\}_{i,j} \).

Now let \( P_{1/2} \subset W_{1/2} \) be a tubular neighbourhood of the intersection of the spheres. \( P_{1/2} \) is clearly diffeomorphic to \( P_{1/2}(\Gamma, R) \) for some realization datum \( R \) by a diffeomorphism that sends \( A_i \subset P_{1/2} \) to the sphere associated to the
vertex \( v_i^A \) in \( P_{1/2}(\Gamma, \mathcal{R}) \) and \( B_j \subset P_{1/2} \) to the sphere associated to the vertex \( v_j^B \) in \( P_{1/2}(\Gamma, \mathcal{R}) \).

Notice that \( P_{1/2} \) is the middle level of a subcobordism \( A^5 \subset W^5 \) obtained by flowing \( P_{1/2} \) with the gradient of the Morse function of \( W \) forward and backward in time (and adding the limiting critical points). Since the belt spheres of the \( h^2 \)s and the attaching spheres of the \( h^3 \)s lie in \( P_{1/2} \), \( A^5 \) will contain all critical points of the Morse function and therefore \( W \setminus \text{int}(A) \) will be a trivial cobordism, i.e. diffeomorphic to \( M \times [0, 1] \) where \( M \) is a compact 4-manifold with boundary.

It follows from Morse theory that:

- the incoming boundary of \( A, A_0 \subset X_0 \), is diffeomorphic to the manifold \( P_0 \) obtained from \( P_{1/2} \) by surgering the belt spheres \( B_j \) of the 2-handles.

- Similarly, the outcoming boundary of \( A, A_1 \subset X_1 \), is diffeomorphic to \( P_1 \), the manifold obtained from \( P_{1/2} \) by surgering the attaching spheres \( A_i \) of the 3-handles.

- \( X_0 \simeq M \cup P_0 \) and \( X_1 \simeq M \cup P_1 \), where we can assume that \( P_0 \) and \( P_1 \) have the same boundary and the gluing maps are the same because surgery interests only the interior of \( P_{1/2} \).

Clearly there are diffeomorphisms \( f_t : P_t \to P_t(\Gamma, \mathcal{R}) \) for \( t \in \{0, 1/2, 1\} \) such that \( f_t|_{\partial P_t(\Gamma, \mathcal{R})} \) is independent of \( t \), these can be constructed with an appropriate choice of \( \mathcal{R} \) or by isotopying the surgery data. Therefore we obtain diffeomorphisms \( X_0 \simeq M \cup P_0(\Gamma, \mathcal{R}) \) and \( X_1 \simeq M \cup P_1(\Gamma, \mathcal{R}) \). Since \( \Gamma \) is symmetric, we obtain the conclusion using the diffeomorphism \( \hat{\tau} \) of Subsection 2.5. \( \Box \)

In Figure 5 we have seen how the Akbulut cork is, informally speaking, supported by a protocork. This happens in general to all corks, more precisely we have the following.

**Proposition 2.14.** Given a twisting pair \( (C, f) \) with \( C \) contractible and \( f^2 = \text{id} \), there exists a symmetric, connected protocork plumbing graph \( \Gamma \) with symmetric realization datum \( \mathcal{R} \) such that \( P_0(\Gamma, \mathcal{R}) \) embeds in \( \text{int}(C) \) and there is a diffeomorphism

\[
F : C \to (C \setminus \text{int}(P_0(\Gamma, \mathcal{R}))) \bigcup_{\tau} P_0(\Gamma, \mathcal{R})
\]

restricting to \( f \) on \( \partial C \).
When \((C, f)\) and \(P_0(\Gamma, \mathcal{R})\) are as in the above proposition, we will say that \((C, f)\) is supported by the protocork \(P_0(\Gamma)\); the redundancy of the realization datum follows from Remark 3. Notice that in this case, if \(C\) embeds in a 4-manifold \(X_0\), then the result of the cork twist, \(X_1\) is diffeomorphic to

\[
(X_0 \setminus \text{int}(C)) \bigcup ((C \setminus \text{int}(P_0(\Gamma, \mathcal{R}))) \bigcup P_0(\Gamma, \mathcal{R}),
\]

i.e. is obtained by twisting the supporting protocork \(P_0(\Gamma, \mathcal{R})\).

**Proof of Proposition 2.14.** Let \((C, f)\) be as in the statement. Since \(C\) is contractible \(\bar{C} \cong f\) is a smooth simply connected homology 4-sphere and therefore by Freedman is homeomorphic to \(S^4\).

We claim that there exists a contractible smooth 5-manifold \(W^5\), with boundary \(\partial W \cong \bar{C} \cup_f C\). Indeed since \(\bar{C} \cup_f C \cong_{\text{TOP}} S^4\) there is, by Wall’s theorem [Wal64] an h-cobordism \(\tilde{W} : \bar{C} \cup_f C \to S^4\), this is constructed just using handles of index 2 and 3 so it is also simply connected. We construct \(W\) by capping \(S^4\) with \(D^5\).

Being contractible, \(W^5\) gives an h-cobordism \(W : C \to C\) with a diffeomorphism of the boundary

\[
\partial W \cong \bar{C} \bigcup \limits_{\text{id}} (\partial \bar{C} \times I) \bigcup \limits_f C
\]

in particular, it restricts to the mapping cylinder of the involution on \(\partial \bar{C} \times I\). We consider a Morse function \(\bar{F} : W \to \mathbb{R}\) such that \(\bar{F}(\partial_- W) = 0\), \(\bar{F}(\partial_+ W) = 1\) and \(\bar{F}|_{\partial \bar{C} \times I}\) is the projection onto the second factor. Now we can repeat the same argument of the proof of Proposition (2.13) obtaining that there is a subcobordism \(A \subset W\), \(A : A_0 \to A_1\), such that \(W \setminus A\) is a trivial cobordism and \(A_0 \cong P_0(\Gamma, \mathcal{R})\), \(A_1 \cong P_1(\Gamma, \mathcal{R})\) with \(\Gamma\) connected and symmetric. Moreover, setting \(M_0 := C \setminus A_0\) and \(M_1 := C \setminus A_1\), we have that

\[
\partial_- \bar{W} = M_0 \bigcup A_0 \cong M_0 \bigcup P_0(\Gamma, \mathcal{R})
\]

and

\[
\partial_+ \bar{W} = M_1 \bigcup A_1 \cong M_0 \bigcup P_1(\Gamma, \mathcal{R})
\]

where the diffeomorphism \(\partial_+ \bar{W} \cong M_0 \bigcup P_1(\Gamma, \mathcal{R})\) restricts to \(f^{-1}\) on \(\partial C\) because of our choice of \(\bar{F}|_{\partial \bar{C} \times I}\). On the other hand, \(\partial_+ W = C\), therefore we obtain a diffeomorphism \(C \to M_0 \bigcup P_1(\Gamma, \mathcal{R})\) restricting to \(f\) on the boundary. Now,
since \( \Gamma \) is symmetric we can construct the diffeomorphism \( C \to M_0 \cup \tau P_0(\Gamma, \mathcal{R}) \) of the thesis using the diffeomorphism \( \tilde{\tau} \) of Subsection 2.5.

\[ \square \]

**Remark 4 (Protocorks as obstruction.).** We observe from the proof, that an involution \( f : \partial C \to \partial C \) extends to a self-diffeomorphism of \( C \), if and only if \( C \) has a supporting symmetric protocork \( P_0(\Gamma) \) which is trivial i.e. the involution \( \tau \) of the protocork boundary extends to the full protocork. Equivalently we can say that a twisting pair \((C, f)\) with \( C \) contractible and \( f^2 = id \) is a cork if and only if any protocork supporting it is non-trivial.

Indeed if \( f \) extends to \( C \), then we can take \( W^5 \) in the proof to be \( C \times [0, 1] \) and thus \( C \) admits a supporting protocork with \( \Gamma \) made of two vertices and one edge. Vice versa, if \( \tau \) extends to \( \tilde{\tau} : P_0(\Gamma) \to P_0(\Gamma) \), then we can construct an extension of \( f \):

\[
\begin{array}{c}
\begin{array}{c}
C \\
\partial C
\end{array}
\end{array}
\xrightarrow{f}
\begin{array}{c}
\begin{array}{c}
M_0 \cup \tau P_0(\Gamma) \\
\partial C
\end{array}
\end{array}
\xrightarrow{id \cup \tilde{\tau}}
\begin{array}{c}
\begin{array}{c}
M_0 \cup P_0(\Gamma) = C \\
\partial C
\end{array}
\end{array}
\]

where \( M_0 \) is the complement of the protocork and the horizontal maps are diffeomorphisms. Notice however that the existence of one non-trivial supporting protocork is not sufficient to conclude that \( f \) does not extend to \( C \). For example, the protocork of Figure 5a, embeds in \( \mathbb{D}^4 \) and the involution \( \tau \) extends to an involution of \( S^3 \), this can be seen by gluing two 0-framed 2-handles along the blue curves of Figure 5b. However, any automorphism of \( S^3 \) extends to \( \mathbb{D}^4 \) [Cer68].

### 3 On the monopole Floer homology of protocork boundaries.

**Outline.** The section is divided in four parts. The first subsection, Subsection 3.1 recaps the relevant background and notation from Kronheimer and Mrowka’s book [KM07]. The reader accustomed with [KM07] may well skip it.

The second subsection, Subsection 3.2, describes in a general setting what we call Morselike perturbations, these are are perturbations of the Chern-Simons-Dirac functional obtained by pulling back a Morse-Smale function defined on the torus of flat connections. We will use them to gain control over the reducible
critical points. This idea is not new, in fact ad hoc applications of Morselike
perturbations appear in several places in the literature as a device to pass from
a Morse-Bott critical submanifold to a Morse one, e.g. [BD95]. In particular, we
borrowed the idea from Section 35.1 and Chapter 36 of [KM07]. We need however
some stronger results about them with respect to those used in the book; we
develop them in full generality in Proposition 3.3 and report the technical proof
in Appendix A.

The third and fourth subsection are specific to the case of \( Y^3 \) being a protocork
boundary, and constitute the core of the section. In Subsection 3.3 we describe
the relevant geometric setting over a protocork boundary \( Y \) in order to define its
Floer homology and point out some key properties of the chain complex. We will
need them in Section 4 and in the proof of Theorem 1.1. In the last subsection
Subsection 3.4, we will prove Theorem 1.4 which provides a splitting of the Floer
homology of \( Y \). The proof will rely only on the functorial properties of Floer
homology so it will be independent of the previous subsections.

3.1 Background on generators of the chain complex.

Outline. In this section we will use the monopole Floer machinery developed by
Kronheimer and Mrowka in their book [KM07]. The aim of this subsection is to
recap definitions and notation from [KM07] that we will need in the rest of the
section. We make no claim of originality.

We recall that the abstract recipe for Floer homology is to construct a chain
complex where the generators are the restpoints of a vector field (usually gra-
dient of a functional) and the differential counts flowlines between them. In the
case of monopole Floer homology, the vector field we are interested in is the for-
mal gradient of the Chern-Simons-Dirac functional \( \mathcal{L} \) defined over the quotient
configuration space \( \mathcal{B}(Y, s) \) in the following). The quotient configuration space
however is not a manifold in general due to the presence of reducible config-
urations. This issue is solved by Kronheimer and Mrowka by working on the
blown-up quotient configuration space \( \mathcal{B}_k^\sigma(Y, s) \) which is an Hilbert manifold
with boundary. The gradient of \( \mathcal{L} \) admits a lift to the blown-up, \( (\text{grad}(\mathcal{L}))^\sigma \),
which is used to define the chain complex. We briefly review how these objects
are defined.

BG. 1 (Classical 3D configuration spaces). Let \( Y \) be a closed, oriented Rie-
mannian 3-manifold, and let \( s \) be a spin\(^c\) structure on \( Y \), with associated spinor
bundle \( S \to Y \) and Clifford multiplication \( \rho : TY \to \text{End}(S) \). We suppose that \( \rho \)
is compatible with the orientation of \( Y \), i.e. if \( e_1, e_2, e_3 \) is an oriented orthonormal frame then \( \rho(e_1)\rho(e_2)\rho(e_3) = 1 \). Given a spin\(^c\) connection \( B \), we denote by \( B^\flat \) the induced connection on the determinant bundle \( \Lambda^2 S \) and by \( F_{B^\flat} \in \Lambda^2(Y, i\mathbb{R}) \) its curvature. \( D_B : C^\infty(Y; S) \to C^\infty(Y; S) \) will denote the spin\(^c\) Dirac operator induced by \( B \). Let \( k \geq 2 \) be a natural number, \( k \) will be the regularity parameter of our Sobolev spaces of sections. In particular \( k > 2 \) will guarantee the embedding into the space of continuous functions. By \( \mathcal{A}_k(Y, s) \) we denote the space of \( L^2 \) Sobolev spin\(^c\) connections on \( S \). The configuration space is defined as

\[
\mathcal{C}_k(Y, s) = \mathcal{A}_k(Y, s) \times L^2_k(Y; S)
\]  

(39)

where \( L^2_k(Y; S) \) denotes the Sobolev space of \( L^2 \)-sections of the spinor bundle \( S \to Y \). As a general rule, omission of the subscript \( k \) means that smooth sections (or connections) are considered. We will denote by \( \mathcal{T}_j = L^2_j(Y; iT^*Y \otimes S) \) the tangent space of \( \mathcal{C}_k(Y, s) = \mathcal{A}_k(Y, s) \times L^2_j(Y; S) \) consisting of \( L^2 \)-sections \( j \leq k \) (see [KM07, Section 9]). The gauge group is defined as

\[
\mathcal{G}_{k+1}(Y) = \{ u \in L^2_{k+1}(Y; C) \mid \| u(y) \| = 1 \ \forall y \in Y \}
\]  

(40)

with the subspace topology. Notice that the evaluation at a point makes sense because \( k > 2 \) guarantees that \( u \in L^2_{k+1}(Y) \subset C^0(Y) \) by Sobolev embedding theorem. \( \mathcal{G}_{k+1}(Y) \) acts on \( \mathcal{C}_k(Y, s) \) by

\[
u \cdot (B, \psi) = (B - \frac{du}{u} \otimes 1_s, u \psi)
\]  

(41)

for \( u \in \mathcal{G}_{k+1}(Y) \) and \( (B, \psi) \in \mathcal{C}_k(Y, s) \), here \( 1_s \in \text{End}(S) \) is the identity map, so that \( \frac{du}{u} \otimes 1_s \in L^2_{k+1}(Y; \Lambda^1 Y \otimes \text{End}(S)) \). Configurations that have trivial (non-trivial) stabilizer under the gauge group action are said to be irreducible (reducible). The reducible configurations are precisely those with identically zero spinor component. The quotient configuration space \( \mathcal{R}_k(Y, s) := \mathcal{C}_k(Y, s)/\mathcal{G}_{k+1}(Y, s) \) is the quotient of the configuration space by the action of the gauge group. We will denote by \( \mathbb{T} \subset \mathcal{R}_k(Y, s) \) the torus of flat connections, i.e.

\[
\mathbb{T} := \{ [(B, 0)] \in \mathcal{R}_k(Y, s) \mid F_B = 0 \},
\]  

(42)

clearly \( \mathbb{T} \neq \emptyset \) only when \( s \) is torsion and in this case \( \mathbb{T} \cong H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z}) \) is a torus of dimension \( b_1(Y) \).

**BG. 2 (Blow-up of configuration spaces).** The configuration spaces \( \mathcal{C}_k(Y, s) \) have the inconvenience that the action of the gauge group is not free and therefore \( \mathcal{R}_k(Y, s) \) is not a manifold in general. This issue is solved by Kronheimer
and Mrowka by introducing blown-up configuration spaces (see Section 6 and 9 of [KM07]) defined as follows.

$$C^\sigma_k Y \cdot \mathfrak{g} := \mathfrak{g}_k Y \cdot \mathfrak{g} \times \mathbb{R}_{\geq 0} \times S(L^2_k Y; S),$$

where $S(L^2_k Y; S)$ is the unit sphere with respect to the $L^2$-norm (notice it is not the $L^\infty_k$-norm). We call elements with zero $\mathbb{R}_{\geq 0}$-component reducibles and their complement irreducibles. The tangent bundle of $C^\sigma_k Y \cdot \mathfrak{g}$ is denoted by $T^\sigma_k$, see [KM07, Section 9] for an explicit definition.

$G_k Y \cdot \mathfrak{g}$ acts freely on $C^\sigma_k Y \cdot \mathfrak{g}$ by

$$u \cdot (B, r, \psi) = (B - \frac{du}{u} \otimes 1, r, u\psi)$$

for $u \in C_{k+1}(Y), (B, r, \psi) \in C^\sigma_k(Y, \mathfrak{g})$. The quotient space is denoted by $\mathcal{B}^\sigma_k(Y, \mathfrak{g})$ and is an Hilbert manifold with boundary [KM07, Corollary 9.3.8]. The boundary consists of equivalence classes of reducible configurations. There is a projection map $\pi : C^\sigma_k(Y, \mathfrak{g}) \to C_k(Y, \mathfrak{g})$ given by $(B, r, \psi) \mapsto (B, r\psi)$. $\pi$ is $G_k(Y, \mathfrak{g})$-equivariant, hence defines a map between the quotient configuration spaces $\pi : \mathcal{B}^\sigma_k(Y, \mathfrak{g}) \to \mathcal{B}_k(Y, \mathfrak{g})$ which is a diffeomorphism over the irreducible locus.

**BG. 3 (Chern-Simons-Dirac functional and the lifted gradient).** Fix a reference connection $B_0 \in \mathfrak{g}_k(Y, \mathfrak{g})$ then the Chern-Simons-Dirac functional $\mathcal{L} : C_k(Y, \mathfrak{g}) \to \mathbb{R}$ is defined as

$$\mathcal{L}(B, \psi) = \frac{1}{8} \int_Y (B' - B_0') \wedge (F_{B'} - F_{B_0}) + \frac{1}{2} \int_Y (D_B \Psi, \Psi) d \text{vol}.$$  

(45)

It can be checked that $\mathcal{L}$ does not depend on the choice of the connection $B_0$. $\mathcal{L}$ is not $G_{k+1}(Y, \mathfrak{g})$-invariant in general, however we will be concerned only with torsion spin$^c$ structures and in this case $\mathcal{L}$ will be gauge invariant and thus will descend to a functional $\mathcal{L} : \mathcal{B}_k(Y, \mathfrak{g}) \to \mathbb{R}$. The formal gradient of $\mathcal{L}$, is a map $\text{grad} \mathcal{L} : C_k(Y, \mathfrak{g}) \to \mathcal{T}_{k-1}$ explicitly defined in [KM07, Eq. (4.3)]. In principle, this is the vector field that we would like to use to define monopole Floer homology, however as said above, due to technical issues, the actual definition uses another vector field, $(\text{grad} \mathcal{L})^\sigma : C_k(Y, \mathfrak{g}) \to \mathcal{T}_{k-1}^\sigma$, which is a lift of grad $\mathcal{L}$ to the blow-up. $(\text{grad} \mathcal{L})^\sigma$ is defined as follows [KM07, pg. 117]:

$$(\text{grad} \mathcal{L})^\sigma(B, r, \psi) = \begin{pmatrix} \frac{1}{2} * F_{B'} + r^2 \rho^{-1}(\psi\psi^*)_{0} \\ \Lambda(B, r, \psi) r \\ D_B \psi - \Lambda(B, r, \psi) \psi \end{pmatrix} \in \mathcal{T}_{k-1}^\sigma$$

(46)
where $\Lambda(B, r, \psi) = \langle \psi, D_B \psi \rangle_{L^2(Y)}$ and $(\psi \psi^*)_0$ denotes the traceless part of the hermitian endomorphism $\psi \psi^*$. Contrarily to $\text{grad} \mathcal{L}$, the lifted gradient is not, in general, the gradient of a functional defined on the blow-up.

**BG. 4 (Taxonomy of critical points).** Although $(\text{grad} \mathcal{L})^{\sigma}$ is not the formal gradient of a functional over the blow-up, its restpoints are called critical points since they generate the chain complexes, in analogy with Morse homology. According to [KM07, Proposition 6.2.3], if $(B, r, \psi) \in \mathcal{C}^\sigma_k(Y, s)$ is a critical point of $(\text{grad} \mathcal{L})^{\sigma}$ then either

- $(B, r, \psi)$ is irreducible (i.e. $r \neq 0$), and $(B, r \psi) = \pi(B, r, \psi)$ is a critical point of $\text{grad} \mathcal{L}$ or
- $(B, r, \psi)$ is reducible (i.e. $r = 0$) and $(B, 0) = \pi(B, r, \psi)$ is a critical point of $\text{grad} \mathcal{L}$ and $\psi$ is an eigenvector of $D_B$.

Thus over the irreducibles the critical points of $(\text{grad} \mathcal{L})^{\sigma}$ and $\text{grad} \mathcal{L}$ coincide under $\pi$ while each reducible critical point of $\text{grad} \mathcal{L}$ introduces countably many (one for each eigenvector) critical points in the blowup. Reducible critical points are those whose equivalence class lie in the boundary of $\mathcal{B}^\sigma_k(Y, s)$, we can thus divide them in two classes:

(a) critical points relative to a **positive** eigenvalue, called boundary-stable and

(b) critical points relative to a **negative** eigenvalue, called boundary-unstable.

The name comes from the fact that flowlines of $-(\text{grad} \mathcal{L})^{\sigma}$ which are not contained entirely in $\partial \mathcal{B}^\sigma_k(Y, s)$ either start from boundary-unstable to arrive at an irreducible critical point or start from an irreducible to arrive at a boundary-stable critical point.

**BG. 5 (Perturbations).** The Chern-Simons-Dirac functional may have degenerate critical points [KM07, Definition 12.1.1]) and furthermore the trajectories of $-(\text{grad} \mathcal{L})^{\sigma}$ may be singular moduli spaces. This problem is soved in [KM07] by perturbing $\text{grad} \mathcal{L}$ with a generic **tame perturbation**. A tame perturbation [KM07, Definition 10.5.1]) is a continuous map

$$ q : \mathcal{C}(Y, s) \to \mathcal{T}_0 $$

satisfying some conditions ensuring nice properties of the perturbed moduli spaces (e.g. compactness), we will denote by $q^0$ and $q^1$ the connection and
spinor component of \( q \) respectively. We can use \( q \) to perturb \((\text{grad } \mathcal{L})^\sigma\) obtaining \((\text{grad } \mathcal{L})^\sigma(B, r, \psi)\) as follows:

\[
(\text{grad } \mathcal{L})^\sigma(B, r, \psi) := \begin{pmatrix}
\frac{1}{2} * F_B r + r^2 \rho^{-1}(\psi \psi^*)_0 + q^0(B, r \psi) \\
\Lambda_q(B, r, \psi)r \\
D_B \psi + \tilde{\gamma}^1(B, r, \psi) - \Lambda_q(B, r, \psi) \psi
\end{pmatrix} \in \mathcal{G}_{k-1}^\sigma
\] (48)

where denoting by \( D_x \) the Fréchet derivative at \( x \),

\[
\tilde{\gamma}^1(B, r, \psi) = \begin{cases}
\frac{1}{2} q^1(B, r \psi) & \text{if } r \neq 0 \\
D_{(B,0)} q^1(0, \psi) & \text{if } r = 0,
\end{cases}
\] (49)

and \( \Lambda_q(B, r, \psi) = \text{Re} \left( \psi, D_B \psi + \tilde{\gamma}^1(B, r, \psi) \right)_{L^2} \). The classification of critical points explained in the previous paragraph makes sense even in the perturbed setting. In fact, \((\text{grad } \mathcal{L})^\sigma\) is the lift of \( \text{grad } \mathcal{L} = \text{grad } \mathcal{L} + q \) and, similarly to \((\text{grad } \mathcal{L})^\sigma\), the critical points of \((\text{grad } \mathcal{L})^\sigma\) are either the lift of an irreducible critical point of \( \text{grad } \mathcal{L} \) or triples \((B, 0, \psi)\) with \( \psi \) an eigenvector of

\[
D_{Bq} := D_B + D_{(B,0)} q^1(0, \cdot).
\] (50)

Thus it still makes sense to speak of boundary-stable and unstable critical points.

The authors of [KM07] construct a large Banach space of perturbations \( \mathcal{P} \) in [KM07, Section 11.6], consisting of tame perturbations, which is used in several constructions as an input for the Sard-Smale theorem to obtain a non-degenerate functional and regular moduli spaces (Theorem 12.1.2 and Theorem 5.1.1 in [KM07]). Despite the notation, a tame perturbation does not have to be the formal gradient of a function, however this is true for \( q \in \mathcal{P} \), thus we have also a perturbed functional \( \mathcal{L} = \mathcal{L} + f_q \) where \( f_q : \mathcal{C}(Y, s) \rightarrow \mathbb{R} \) is a primitive of \( q \).

**BG. 6 (Chain complexes).** Section 22 of [KM07] defines the chain complexes \( \hat{\mathcal{C}}, \hat{\mathcal{C}}, \hat{\mathcal{C}} \) giving rise to Floer homology groups. We briefly review how these are generated by the critical points of \((\text{grad } \mathcal{L})^\sigma\). First of all, we choose an admissible perturbation \( q \in \mathcal{P} \) [KM07, Definition 22.1.1]], in particular the critical points of \((\text{grad } \mathcal{L})^\sigma\) are non-degenerate and the moduli spaces of trajectories are regular. For any \( \Lambda = \{x, y\} \) 2-element set, \( \mathbb{Z} \Lambda \) will denote the coefficient group

\[
\mathbb{Z} \Lambda = \langle x, y \mid x = -y \rangle.
\] (51)

Notice that a choice of a generator \( x \) or \( y \) establishes an isomorphism \( \mathbb{Z} \Lambda \cong \mathbb{Z} \), indeed we could have worked with \( \mathbb{Z} \)-coefficients instead of using \( \mathbb{Z} \Lambda \) but then...
we would have to choose an orientation for each generator of the complex. Now let \( C^o, C^s, C^u \) be the set of gauge-equivalence classes of irreducible, boundary-stable and boundary unstable critical points of \((\mathcal{L})^\sigma\) respectively and set

\[
\begin{align*}
C^o &= \bigoplus_{[\alpha] \in C^o} \mathbb{Z} \Lambda([\alpha]), \\
C^s &= \bigoplus_{[\alpha] \in C^s} \mathbb{Z} \Lambda([\alpha]), \\
C^u &= \bigoplus_{[\alpha] \in C^u} \mathbb{Z} \Lambda([\alpha]).
\end{align*}
\]

(52)

where \( \Lambda([\alpha]) \) is a 2-element set of orientations of a moduli space associated to \([\alpha]\) [KM07, Section 20.3]. In Section 22 of [KM07] are defined homomorphisms \( \partial^x : C^x \to C^y \) and \( \partial^y : C^x \to C^y \) for \( x, y \in \{s, u, o\} \) between these complexes obtained by counting 1-dimensional moduli spaces of trajectories (only reducible ones in the latter case) of \((\mathcal{L})^\sigma\). Now define

\[
\begin{align*}
\check{C} &= C^o \oplus C^s \\
\hat{C} &= C^o \oplus C^u \\
\bar{C} &= C^s \oplus C^u.
\end{align*}
\]

(53)

The Floer homology groups \( \check{HM}_*(Y, s), \hat{HM}_*(Y, s), \bar{HM}_*(Y, s) \) are the homologies of the chain complexes \((\check{C}, \partial), (\hat{C}, \partial), (\bar{C}, \partial)\), where \( \partial, \hat{\partial}, \bar{\partial} \) are constructed from the abovementioned homomorphisms \( \partial^x, \) and \( \partial^y \). For a precise definition we refer the reader to [KM07, Section 22]. Although as defined the Floer homology groups depend on the perturbation \( q \) and on the metric there is a canonical isomorphism between the Floer homologies arising from a different choice of data (Riemannian metric and perturbation) [KM07, Corollary 23.1.6]. Thus the Floer homology groups are topological invariants of \( Y \).

### 3.2 Morselike perturbations.

**Outline.** A Morselike perturbation is a perturbation of \( \mathcal{L} \) that is the pullback of a Morse function defined on the torus of flat connections \( \mathbb{T} \); we give a formal definition below. This has the advantage of giving us a complete understanding of the reducible critical points and of the flow between them in the blow-down. In general such a perturbation is not admissible: critical points may be degenerate (even the reducible ones in the blow-down if the Dirac operator is not invertible) and the moduli spaces of trajectories may be singular. We can use the theorems of Chapter 12 and Theorem 15.1.1 in [KM07] to obtain an admissible perturbation \( \mathcal{L} = \mathcal{L} + f + f' \) where \( \text{grad } f' \in \mathcal{P} \), however, this would defy the purpose of the Morselike perturbation \( f \) because the perturbation \( f' \) may alter reducible critical points and the flow between them. This issue is only apparent, indeed a slight modification of the proofs of [KM07] shows that in the case of a Morselike perturbation, we can sharpen the result of [KM07] and assume that the connection component of \( \text{grad } f' \) vanishes on the reducible locus at the cost of possibly
perturbing slightly the Morse function. We state this as Proposition 3.3 here and give a proof in Appendix A.

**Definition of Morselike perturbations.** We continue with the notation of Subsection 3.1, thus \((Y, g)\) will denote a closed oriented Riemannian 3-manifold with torsion spin^c structure \(s\). Fix a reference flat spin^c connection \(B_0 \in \mathcal{A}_k(Y, s)\) so that \(\mathcal{A}_k(Y, s) = B_0 + L_k^2(iT^*Y)\). Define \(P : \mathcal{C}_k(Y, s) \to \mathcal{C}_k(Y, s)\) by

\[
P(B_0 + b \otimes 1_S, \psi) = (B_0 + (P_{\ker \Delta} b) \otimes 1_S, 0),
\]

(54)

where \(P_{\ker \Delta} : L_k^2(Y, i\mathbb{R}) \to L_k^2(Y, i\mathbb{R})\) is the \(L^2\)-projector on the space of harmonic 1-forms i.e. the kernel of the Hodge Laplacian \(\Delta : L_k^2(iT^*Y) \to L_k^2(iT^*Y)\). The existence of \(P_{\ker \Delta}\) is ensured by Hodge theory, notice that the image of \(P\) is the set of flat connections. The map \(P\) passes to the quotient defining a smooth retraction

\[
p_T : \mathcal{B}_k(Y, s) \to \mathbb{T}.
\]

(55)

**Definition 3.1.** A functional \(f : \mathcal{C}(Y, s) \to \mathbb{R}\) of the form

\[
f(x) := f_T([P(x)]) \quad \text{for all } x \in \mathcal{C}(Y, s)
\]

(56)

where \(f_T : \mathbb{T} \to \mathbb{R}\) is a Morse-Smale function, is called **Morselike perturbations**.

Notice that the *reducible* critical points of \(\mathcal{L} + f\) are the critical points of \(f\) and \(\mathbb{T}\) is invariant under the gradient flow of \(\mathcal{L} + f\), the reducible trajectories (in the blow-down) are precisely the gradient trajectories of the Morse function \(f\). The gradient of \(f\) is a tame perturbation.

**Definition 3.2 \((\mathcal{P}^\bot)\).** We denote by \(\mathcal{P}^\bot < \mathcal{P}\) the Banach subspace of perturbations that vanish on the reducible locus. In formulae, \(q \in \mathcal{P}^\bot\) if \(q(B, 0) = 0\) for all \(B \in \mathcal{A}(Y, s)\).

**Remark 5.** We could have defined \(\mathcal{P}^\bot\) by requiring that only the connection component vanishes on the reducible locus, i.e. \(q^0(B, 0) = 0\) for all \(B \in \mathcal{A}(Y, s)\). Indeed equivariance of tame perturbations implies that \(q^1(u \cdot (B, \psi)) = u q^1(B, \psi)\) for any \(u \in \mathcal{C}(Y), (B, \psi) \in \mathcal{C}(Y)\), thus the spinor component always vanishes over the reducible locus; its derivative instead may be non-zero and perturb the equations in the blown-up model. In light of this, a perturbation \(q \in \mathcal{P}\) with primitive \(f_q : \mathcal{C}(Y, s) \to \mathbb{R}\) belongs to \(\mathcal{P}^\bot\) if and only if \(f_q\) is constant on the reducible locus.
The subspace $\mathcal{P}^\perp$ of perturbations vanishing on the reducible locus is clearly a proper closed subspace of $\mathcal{P}^\perp$. These perturbations are important to us, because they do not alter the reducible trajectories in the (classical) configuration space.

**Proposition 3.3.** Let $f$ be a Morselike perturbation induced by $f_T : \mathbb{T} \to \mathbb{R}$. Then there is a residual subset $U_0 \subset \mathcal{P}^\perp$ such that for any $u_0 \in U_0$, after possibly enlarging $\mathcal{P}$, there are a closed subspace $Z < \mathcal{P}$ depending on $u_0$ and a neighbourhood of zero $U \subset Z$ such that $u \in U$ implies that

(a) the critical points of $(\nabla(\mathcal{L} + f) + u_0)^\sigma$ are non-degenerate,

(b) $u$ vanishes in a neighbourhood of the critical points of $\nabla(\mathcal{L} + f) + u_0$,

(c) $(\nabla(\mathcal{L} + f) + u_0 + u)^\sigma$ has the same critical points of $(\nabla(\mathcal{L} + f) + u_0)^\sigma$,

(d) $u = \nabla(h_T([P(\cdot)]) + q^\perp$ where $q^\perp \in \mathcal{P}^\perp$ and $f_T + h_T$ is Morse-Smale.

(e) In addition, the set of $u \in U$ for which $\nabla f + u_0 + u$ is admissible is residual in $U$.

The proof is given in Appendix A. Notice that since $u_0 \in U_0 \subset \mathcal{P}^\perp$, the reducible critical points of $\nabla(\mathcal{L} + f) + u_0 + u$ are the critical points of $\nabla(\mathcal{L} + f)$ in the blow-down.

### 3.3 Critical points for $Y$ a protocork boundary.

**Outline.** In this subsection we will describe our setting, in particular the data (metric and perturbation) that we will use and the critical points. We will finally draw some conclusions on the boundary maps and relative degree (Proposition 3.5) and define an homomorphism that will come in handy in the proof of Theorem 1.1. The main ingredients will be a metric constructed by Morgan and Szabó and Morselike perturbations as described in Subsection 3.2. The last part of the section deals with Morgan-Szabó’s number, introduced in Theorem 1.1 item (b), we show that the original definition of [MS99] coincides with the one that we use here.

**Setting for Floer homology.** The following result due to Morgan and Szabó will be fundamental.
Lemma 3.4 (Lemma 2.1 in [MS99]). Let $Y$ be the boundary of a protocork and let $s$ be the trivial spin$^c$ structure. Then there exists a Riemannian metric on $Y$ such that for any flat spin$^c$ connection $B$ the kernel of the Dirac operator $D_B : \Gamma(S) \to \Gamma(S)$ is trivial.

Notice that the conclusion of the above lemma is true in general for manifolds admitting a metric with positive scalar curvature. The proof of Theorem 3.4 does not show that $Y$ has a metric with positive scalar curvature (which is false unless the protocork is trivial) but is based instead on a neck stretching argument.

The metric of Lemma 3.4 can be described as follows. Recall that $Y$ is obtained by gluing two trivial $S^1$-bundles over $\Sigma_A, \Sigma_B$, surfaces diffeomorphic to $S^2$ minus some open disks. The bundles $\Sigma_A \times S^1, \Sigma_B \times S^1$ are endowed with the product metric where the metric over $\Sigma_A$ and $\Sigma_B$ is such that the boundary circles have unit length and restricts to a product metric in a collar of the boundary. In this way, there are necks isometric to $[-T, T] \times \mathbb{T}^2$ embedded in $Y$ for some $T > 0$. The authors of [MS99] show that if $T$ is large enough then the thesis of Lemma 3.4 follows.

The form of the metric allows us to assume that the involutions $\rho_A, \rho_B$ and $\tau$ (when defined) are isometries, indeed it is not difficult, using the realization datum described in Subsection 2.5, to construct metrics that satisfy the above hypothesis and for which the involutions are isometries.

For the rest of this section, $(Y, g)$ will be the boundary of a protocork endowed with a metric $g$ given by 3.4. Let $s$ be a trivial spin$^c$ structure on $Y$, this is unique up to isomorphism since $H^*(Y; \mathbb{Z})$ has no torsion. We will use a perturbation $q \in \mathcal{P}$ of the form

$$q := \text{grad} f + q'$$

where $f$ is Morselike perturbation $f$ and $q' \in \mathcal{P}^\perp$. We will also assume without loss of generality that

Assumption 1. (a) $q$ is admissible,

(b) $f_T$ has a unique critical point of Morse index $b_1(Y) = \dim \mathbb{T}$ on the torus of flat connections $\mathbb{T}$, and

(c) $q'$ is so small in $\mathcal{P}$-norm that for any $[(B, 0)] \in \mathbb{T}$, the perturbed Dirac operator $D_{B,q}$ (c.f. (50)) is invertible.

The existence of such $q$ is ensured by Proposition 3.3 observing that, thanks to Lemma 3.4, the unperturbed Dirac operator $D_B$ is invertible for all $[(B, 0)] \in \mathbb{T}$.
and invertibility is an open condition. We denote the primitive of \( q' \) by \( f' \) and set
\[
\mathcal{L} = \mathcal{L} + f + f'.
\] (58)

**Critical points.** The critical point set of \( \mathcal{L} \) in \( \mathcal{B}(Y, s) \) is finite and consisting of \( N_{irr} \) irreducibles and \( N_{red} \) reducibles \([a_1], \ldots, [a_{N_{red}}]\), where \( a_i \in \mathbb{T} \) is a flat connection and a critical point of \( f_T \). We will assume \([a_{N_{red}}]\) to be the unique critical point of \( f_T \) of maximal Morse index.

On the blow-up \( \mathcal{B}^\sigma(Y, s) \), the zero locus of \( (\text{grad}\mathcal{L})^\sigma \) consists of

- \([b_1], \ldots, [b_{N_{irr}}]\) \( \in \mathcal{B}^\sigma_k(Y, s) \), the lift of the irreducibles
- and a tower of reducibles for each \([a_i]\), \( i = 1, \ldots, N_{red} \).

Recall that the elements of the tower correspond to eigenvalues of the Dirac operator \( D_{B_i,q} \) (which has simple spectrum) where \( B_i \) is the connection associated to \( a_i \). These eigenvalues are real and if we order them in increasing order:
\[
\cdots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_0 < \lambda_1 < \ldots
\] (59)

We can denote the elements of the tower over \([a_i]\) by \([a_{i,j}]\) where \( j \in \mathbb{Z} \) indicates that \([a_{i,j}]\) corresponds to the eigenvalue \( \lambda_j \). So in particular \([a_{i,0}]\) corresponds to the **smallest positive** eigenvalue, \([a_{i,-1}]\) to the larger negative one etc. Therefore \([a_{n,i}]\) will be \textit{boundary-unstable} for \( i < 0 \) and \textit{boundary-stable} for \( i \geq 0 \).

**Remark 6.** Thanks to (c) of Assumption 1, the spectral flow of any path of perturbed Dirac operators \( D_{B_i,q}, ([B_t, 0]) \in \mathbb{T} \) is zero. Therefore we will enumerate the eigenvalues so that the numbering is consistent with the spectral flow, i.e. the 0-th eigenvalue of \( D_{B_0,q} \) and the 0-th eigenvalue of \( D_{B_1,q} \) are the endpoints of a continuous curve \( \{ (\lambda(t), D_{B_t,q}) \mid \lambda(t) \in \sigma(D_{B_t,q}) \} \) where \( B_t \) is flat for each \( t \).

**On the moduli spaces between reducible critical points.** Consider two reducible critical points in the blow-up \([a_{n,i}], [a_{m,j}]\), then we have \( M^\text{red}_z([a_{n,i}], [a_{m,j}]) \), the moduli space of reducible trajectories on \( \mathbb{R} \times Y \) limiting to \([a_{n,i}]\) for \( t \to -\infty \) and to \([a_{m,j}]\) for \( t \to +\infty \) of homotopy class \( z \) [KM07, Definition 13.1.1]. Adopting the convention of [KM07, Chapter 16.6], we denote by
\[
\text{gr}([a_{n,i}], [a_{m,j}]) = \dim M^\text{red}_z([a_{n,i}], [a_{m,j}])
\] (60)
the virtual dimension of the moduli space. Notice that since the spin$^c$ structure is torsion $\text{gr}([a_{n_1}], [a_{m_j}])$ does not depend from the path $z$, so we have omitted it from the notation. It can be shown that $\text{gr}$ is additive, meaning that $\text{gr}([a_{n_1}], [a_{m_j}]) + \text{gr}([a_{m_j}, [a_{r_k}]) = \text{gr}([a_{n_1}], [a_{r_k}])$.

For a reducible critical point $[a_{n_1}]$, denote by

$$\text{index}_f[a_{n_1}] \in \mathbb{N}$$ (61)

the index of the Morse function $f_T$ at $[a_i] \in \mathcal{T}$. Remark 6, and the Morselike perturbation $f$ imply the following properties analogous to the case of $\mathbb{S}^1 \times \mathbb{S}^2$ (cf. [KM07, Section 36.1] and the related [KM07, Lemma 33.3.3]).

**Proposition 3.5.**

(a) $\text{gr}([a_{n_1}], [a_{m_j}]) = \text{index}_f[a_{n_1}] - \text{index}_f[a_{m_j}] + 2(i - j)$.

(b) $\partial_u = 0$.

Notice also that despite this, the proof given for $\mathbb{S}^1 \times \mathbb{S}^2$ that $\partial_u = 0$ does not go through because the metric of $Y$ does not have non-negative scalar curvature.

**The homomorphism** proj. Recall from Background 6 that the group of chains defining $\hat{HM}_*(Y)$ and $\hat{HM}_*(Y)$ are respectively $\hat{C} = C^o \oplus C^u$ and $\hat{C} = C^o \oplus C^u$. Suppose for simplicity to have chosen an orientation for all the generators of all the chain complexes involved, so that the group of chains will have coefficients in $\mathbb{Z}$ instead of $\mathbb{Z}\Lambda([c])$. Then we define $\mathbb{Z}$-homomorphisms

$$\text{proj}: \hat{HM}_*(Y) \to \mathbb{Z} \quad \text{proj}: \hat{HM}_*(Y) \to \mathbb{Z}$$ (62)

by projecting along the component $[a_{N^{\text{red}},-1}] \in C^u$. The next lemma guarantees that proj is well defined in homology.

**Lemma 3.6.** Consider a generator $[c] \in \hat{C} = C^o \oplus C^u$, then $\langle \hat{\partial}[c], [a_{N^{\text{red}},-1}] \rangle = 0$ i.e. $\hat{\partial}[c]$ has no component along $[a_{N^{\text{red}},-1}]$. The same holds in $\hat{C}$, i.e. if $[c] \in \hat{C}$ is a generator then $\langle \hat{\partial}[c], [a_{N^{\text{red}},-1}] \rangle = 0$. Consequently reading the component along $[a_{N^{\text{red}},-1}]$ gives well defined $\mathbb{Z}$-homomorphisms proj : $\hat{HM}_*(Y) \to \mathbb{Z}$, proj : $\hat{HM}_*(Y) \to \mathbb{Z}$

**Proof of Lemma 3.6.** Recall that the boundary map of $\hat{C}$ is given by

$$\hat{\partial} = \begin{bmatrix} \partial^o \\ -\partial^u \partial^o \\ -\partial^u \partial^o \\ -\partial^u \partial^u \end{bmatrix} : C^o \oplus C^u \to C^o \oplus C^u.$$
By Proposition 3.5, \( \partial_i \tilde{\eta} = 0 \), therefore the only trajectories to \([a_{N_{\text{red}}, -1}]\) come from elements of \( C^u \). On the other hand if \([c] = [a_{k, -1}] \in C^u\) then an application of Proposition 3.5 shows that \( \overline{\text{gr}}([a_{k, -1}], [a_{N_{\text{red}}, -1}]) = \text{index}_f([a_k]) - b_1 + 2(1 - i) \). On the other hand, \( \text{index}_f([a_k]) - b_1 = 0 \) because otherwise we would not have a flow of the connection component. Consequently \( \overline{\text{gr}}([a_{k, -1}], [a_{N_{\text{red}}, -1}]) = 2(1 - i) \neq 1 \) for \( i \neq 1 \). Therefore there are no 1-dimensional moduli spaces of trajectories from \([a_{k, -1}]\) to \([a_{N_{\text{red}}, -1}]\). \( \square \)

Another useful corollary of the above lemma is the following sufficient condition for belonging to the reduced homology.

**Lemma 3.7.** If \( \sigma = (\sigma^o, k[a_{N_{\text{red}}, -1}]) \in \hat{C} = C^o \oplus C^u, k \in \mathbb{Z} \) is a cycle and \([\sigma] \in H^{\text{red}}_M(Y), then k = 0, therefore \( \text{proj}(H^{\text{red}}_M(Y)) = 0 \).**

**Proof of Lemma 3.7.** By definition, \( H^{\text{red}}_M(Y) = \ker p^* : \overline{H}_*(Y) \to \overline{H}_*(Y) \), where \( p \) is the anti-chain map

\[
p = \begin{bmatrix}
\partial^o_s & \partial^u_s \\
0 & 1
\end{bmatrix} : C^o \oplus C^u \to C^s \oplus C^u.
\]

Therefore \( p(\sigma) = (\partial^o_s \sigma^o + \partial^u_s [a_{N_{\text{red}}, -1}], k[a_{N_{\text{red}}, -1}]) \) is equal to \( \partial \tilde{\eta} \) for some \( \tilde{\eta} \in \hat{C} \). Now Theorem 3.6 implies that \( k = 0 \). \( \square \)

**The Morgan-Szabó number.** In view of the notation just introduced, the Morgan-Szabó number defined in Theorem 1.1 (b) is equal to

\[
n_{\text{MS}} = 2 + \max_{i, j \in \{1, \ldots, N_{\text{irr}}\}} \text{gr}([b_i], [b_j]). \tag{63}
\]

Notice that this number is dependent on the metric \( g \) and the perturbation \( q \).

Morgan and Szabó’s paper [MS99] came out before that monopole Floer homology was developed by Kronheimer and Mrowka [KM07], therefore the original definition of \( n_{\text{MS}} \), which is given by the definition of \( k \) in [MS99, Proposition 2.3], is slightly different. Now we want to explain why the two definitions are equivalent.

In [MS99], the Chern-Simons-Dirac functional is perturbed using an exact 2-form \( \mu = d\eta \in \Omega^2(Y, i\mathbb{R}) \), i.e.

\[
\mathcal{L}_{\text{MS}}(B, \psi) := \mathcal{L}(B, \psi) + \frac{1}{4} \int_Y (B - B_0)^t \wedge d\eta \tag{64}
\]
The resulting 3-monopole equations are:

\[
\frac{1}{2} \ast (F_{B'} - d\eta) + \rho^{-1}(\psi\psi^*) = 0
\]

\[D_B\psi = 0\]  \hspace{1cm} (65)

for \((B, \psi) \in \mathbb{C}(Y, s)\). The 1-form \(\eta\) is chosen in such a way that the irreducible solutions are non-degenerate and the moduli spaces of trajectories are regular. We will also suppose without any loss of generality that \(\eta\) has \(L^2_k\)-norm so small that the perturbed Dirac operator \(\psi \mapsto D_B\psi + \frac{1}{2}\rho(\eta)\psi\) is still injective for any flat \(B\). Then the authors of [MS99] define \(k\) as 2 plus the maximum relative degree (i.e. the dimension of the space of trajectories of \(-\text{grad} \mathcal{L}_{MS}\)) between two irreducible solutions. Notice that \(\mathcal{L}_{MS} - \mathcal{L}\) is not a perturbation of the type considered in the beginning of Subsection 3.3, as can be easily seen from the fact that the reducible solutions are precisely a translation of the torus of flat connections.

**Proposition 3.8.** There exists a perturbation \(\eta\) of the form (57) satisfying Assumption 1 such that the number \(n_{MS}\) as defined in (63) is equal to \(k\).

**Proof.** Define the diffeomorphism \(P_\eta : \mathbb{C}_k(Y, s) \to \mathbb{C}_k(Y, s)\) as

\[
P_\eta(B, \psi) = (B - \frac{1}{2}\eta \otimes 1_s, \psi). \hspace{1cm} (66)
\]

Then \(P_\eta\) establishes a diffeomorphism from the moduli space of solutions of (65) to the moduli space of solutions of

\[
\frac{1}{2} \ast F_{B'} + \rho^{-1}(\psi\psi^*) = 0
\]

\[D_B\psi + \frac{1}{2}\rho(\eta)\psi = 0, \hspace{1cm} (67)
\]

\((B, \psi) \in \mathbb{C}(Y, s)\), i.e. the critical points of

\[
\mathcal{L}_1(B, \psi) := \mathcal{L}(B, \psi) + \frac{1}{4}\text{Re} \langle \rho(\eta)\psi, \psi \rangle_{L^2(Y)}. \hspace{1cm} (68)
\]

In addition \(P_\eta\) preserves the relative degree between two solutions. Indeed the relevant Fredholm operators differ by post-composition with a translation and hence have the same index.
Now we consider a cylinder function of the form $f \sum [\alpha] \beta_{[\alpha]}$ where $f$ is a chosen Morselike perturbation, $[\alpha]$ ranges over irreducible critical points and $\beta_{[\alpha]}$ is a cylinder function that vanishes in a neighbourhood of $[\alpha]$ and is equal to 1 outside of a slightly larger neighbourhood disjoint from the reducible locus. These functions are constructed as in the paragraph preceding the definition of $P_f$ in the proof of Proposition 3.3.

We claim that exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the perturbed functional

$$L_{2,\epsilon} = L_1 + \epsilon f \sum_{[\alpha]} \beta_{[\alpha]}$$

(69)

has as reducible critical points the critical points those $f$ and as irreducible critical points those of $L_1$. By contradiction suppose $\epsilon_n \to 0$ and let $(B_n, \psi_n) \in C(Y)$ be irreducible critical points of $L_{2,\epsilon_n}$ that are not critical points of $L_1$. Then, by the properness property of tame perturbations [KM07, Proposition 11.6.4], up to applying a gauge transformation or passing to a subsequence we can assume that $(B_n, \psi_n) \to (B_\infty, \psi_\infty)$ critical point of $L_1$. The configuration $(B_\infty, \psi_\infty)$ cannot be irreducible because, thanks to the bump function $\beta_{[\alpha]}$, each irreducible critical point of $L_1$ has a neighbourhood where it is the only critical point of $L_{2,\epsilon_n}$. Thus $(B_\infty, \psi_\infty)$ is reducible, $\psi_\infty = 0$ and $B_\infty$ is flat. Consider the normalized sequence $\Phi_n = \psi_n/\|\psi_n\|_{L^2(Y)}$, these unit spinors satisfy the elliptic equation $D_{B_n} \Phi_n + \frac{1}{2} \rho(\eta) \Phi_n = 0$. Thanks to elliptic regularity we can extract a converging subsequence $\Phi_n \to \Phi_\infty$ to a unit spinor in the kernel of $D_{B_\infty} + \frac{1}{2} \rho(\eta)$, but the latter operator is injective by assumption. This concludes the proof of the claim.

We choose $\epsilon \in (0, \epsilon_0)$ and put $L_2 := L_{2,\epsilon}$. The relative degree between the irreducible critical points of $L_2$ is the same of $L_1$ because the relative Fredholm operator varies by a compact perturbation.

It is possible that the reducible critical points of $L_2$ are degenerate in the blow-up. We can remedy this by adding a small perturbation $q_3 \in P^\perp$ vanishing in a neighbourhood $\mathcal{O}$ of the irreducibles critical points (cf. the definition of $P^\perp_{\mathcal{O}}$ in Appendix A with the caveat that therein $\mathcal{O}$ is also a neighbourhood of reducible critical points). If the perturbation is small enough no irreducible critical point are introduced in the blow-down (the reducible critical points are not affected by a perturbation in $P^\perp$). This follows from [KM07, Proposition 11.6.4] and an application of the implicit function theorem exploiting that if $\text{grad} L_2(B, 0) = 0$, $q_3 \in P^\perp$ then $\text{grad} L_2(B, 0) + q_3(B, 0) = 0$ and that reducible critical points are non-degenerate in the blow-down, i.e. the relevant operator is an isomorphism when acting over the Coulomb slice at the critical point.
We denote the perturbed functional as $\mathcal{L}_3 = \mathcal{L}_2 + f_3$, where $f_3$ is a primitive of $q_3$.

After possibly enlarging the large Banach space of perturbations $\mathcal{P}$, we see that the tame perturbation associated to $\mathcal{L}_3$ takes the form: $\text{grad}(\epsilon f) + u_0$, where $u_0 \in \mathcal{P}^\perp$ and $\epsilon f$ is Morselike. This is almost of the kind of perturbations required, except for the fact that the moduli spaces of trajectories might not be regular. This issue can be solved by adding another small perturbation $u = q_{\epsilon f} + q''$ as in Lemma A.2 in the proof of Proposition 3.3.

This shows that $k$ as defined [MS99, Proposition 2.3] is equal to $n_{MS}$ as defined in (63) using the perturbation $q := \text{grad}(\epsilon f) + u_0 + u$. \hfill $\Box$

### 3.4 A splitting theorem for Floer homology.

**Outline.** The aim of this subsection is to prove Theorem 1.4. We will construct two cobordism $W$ and $Q$ such that $Q \circ W$ is trivial, this will provide us with a first splitting, then the vanishing of the triple cap product (Proposition 2.4) will allow us to characterize the two factors. The proof will not rely on the other parts of this section.

**Notation.** Throughout this subsection, $\Gamma$ is a given protocork plumbing graph with sphere-number $n$, $R$ a given realization datum for $\Gamma$ and we denote by $N_0, N_1$ and $Y$ respectively the realization $P_0(\Gamma, R)$, its reflection $P_1(\Gamma, R)$ and their common boundary $\partial P_0(\Gamma, R)$. We will also adopt the shorthand notation

$$b_1 := b_1(Y),$$

and $nM := M \# \ldots \# M$ to denote the $n$-fold connected sum of a 3-manifold $M$ with itself, thus $b_1 S^1 \times S^2 = \#_{i=1}^{b_1(Y)} (S^1 \times S^2)$. Also for a cobordism $W : M_0 \to M_1$ we will denote by $\partial_+ W = M_0$ the incoming boundary (with opposite orientation) and by $\partial_- W = M_1$ the outcoming boundary.

**The cobordism $W$.** We construct a cobordism $W : b_1 S^1 \times S^2 \to Y$. Start with a Kirby diagram $\mathcal{D}$ for $N_0$ of the type described in Subsection 2.3. The diagram $\mathcal{D}$ will consists of a link with components of three types:

- the 0-framed knots $\alpha_1, \ldots, \alpha_n$, relative to the vertices $v_i^A, i = 1, \ldots, n$,
- the dotted circles $\beta_1, \ldots, \beta_n$ relative to the vertices $v_i^B, i = 1, \ldots, n$, and

the dotted circles $\mu_1, \ldots, \mu_b$, associated to edges in the complement of a spanning tree of $\Gamma$.

Considering the collection of the $\mu_i$s alone induces a surgery presentation for $b_1 S^1 \times S^2$, indeed when looking at the boundary of a handlebody induced by a Kirby-diagram the dotted circles contribute as 0-framed knots. Then we construct $W$ in two steps: firstly we add $n$ 1-handles to $(b_1 S^1 \times S^2) \times \{1\} \subset (b_1 S^1 \times S^2) \times [0, 1]$ obtaining $W'$. The outcome boundary $\partial W'$ has a surgery presentation given by considering only the $\mu_i$s and the $\beta_i$s in $D$ and regarding them as 0-framed. The second step consist in gluing $n$ 2-handles to $\partial W'$, with reference to the surgery presentation just described, the attaching curves of the 2-handles will be given by the $\alpha_i$s and will be 0-framed. We notice also that $W$ is an homology cobordism because the 1-handles are algebraically cancelled by the 2-handles.

The cobordism $Q$. We will construct a cobordism $Q : Y \to b_1 S^1 \times S^2$ such that the composition $Q \circ W : b_1 S^1 \times S^2 \to b_1 S^1 \times S^2$ is a trivial cobordism, i.e. $Q \circ W \simeq (b_1 S^1 \times S^2) \times I$. Consider the surgery presentation for $Y$ induced by the diagram $D$. We construct $Q$ in two steps. Firstly for each $i = 1, \ldots, n$, we add a 0-framed 2-handle meridional to $\beta_i$ to $(b_1 S^1 \times S^2) \times \{1\} \subset (b_1 S^1 \times S^2) \times [0, 1]$ obtaining a cobordism $Q'$. A surgery presentation for $\partial Q'$ is given by the surgery presentation for $Y$ induced by the diagram $D$, together with the attaching circles (meridional to the $\beta_i$s) of the 0-framed 2-handles we just add. Denote by $\gamma_i$, $i = 1, \ldots, n$ a loop in $\partial Q'$ meridional to $\alpha_i$. As a second step, for each $i = 1, \ldots, n$ we glue a 3-handle to $\partial Q'$ in such a way that, its attaching sphere meets $\gamma_i$ exactly in one point. Such a sphere exists in $\partial Q'$, because one can slide $\alpha_i$ over the newly added 0-framed 2-handles meridional to the $\beta_i$s so that $\alpha_i$ becomes a 0-framed unknot unlinked from the rest of the diagram having $\gamma_i$ as one of its meridians.

Lemma 3.9. $Q$ is a left inverse of $W$, i.e. $Q \circ W : b_1 S^1 \times S^2 \to b_1 S^1 \times S^2$ is a trivial cobordism.

Proof. The cobordism $Q \circ W$ is constructed from $b_1 S^1 \times S^2 \times [0, 1]$ by attaching handles to $b_1 S^1 \times S^2 \times \{1\}$ in four steps by repeating the process described to construct $W$ and then $Q$. Now it is sufficient to note that by construction the addition of the 2-handles of $Q$ will cancel the 1-handles of $W$ and the 3-handles will cancel the 2-handles of $W$. We are thus left with a cobordism with no handles hence trivial. \hfill \Box
Recall that, as stated in [KM07, Theorem 3.4.3], Floer homology defines a covariant functor from the cobordism category to the category of \( \mathbb{Z}[[U]] \)-modules. In particular, we have \( \mathbb{Z}[[U]] \)-homomorphisms

\[
\begin{align*}
\overline{HM}_* (W) : & \overline{HM}_* (b_1 \mathbb{S}^1 \times \mathbb{S}^2) \to \overline{HM}_* (Y) \\
\overline{HM}_* (Q) : & \overline{HM}_* (Y) \to \overline{HM}_* (b_1 \mathbb{S}^1 \times \mathbb{S}^2)
\end{align*}
\]

(71)

such that, thanks to Lemma 3.9,

\[
\overline{HM}_* (Q) \circ \overline{HM}_* (W) = \overline{HM}_* ((b_1 \mathbb{S}^1 \times \mathbb{S}^2) \times I) = \text{id},
\]

(72)

the identity \( \overline{HM}_* (b_1 \mathbb{S}^1 \times \mathbb{S}^2) \to \overline{HM}_* (b_1 \mathbb{S}^1 \times \mathbb{S}^2) \). With analogous maps \( \overline{HM}_* (W), \overline{HM}_* (Q) \) and \( \overline{HM}_* (Q) \) composing to \( \text{id} \) and \( \text{id} \) respectively.

Since \( b_1 \mathbb{S}^1 \times \mathbb{S}^2 \) has a metric with positive scalar curvature by [KM07, Proposition 36.1.3] the Floer homology groups of \( b_1 \mathbb{S}^1 \times \mathbb{S}^2 \) are isomorphic as \( \mathbb{Z}[[U]] \)-modules to

\[
\begin{align*}
\overline{HM}_* (b_1 \mathbb{S}^1 \times \mathbb{S}^2) & \cong \Lambda^* (Z^b) \otimes \mathbb{Z}[[U]] \\
\overline{HM}_* (b_1 \mathbb{S}^1 \times \mathbb{S}^2) & \cong \Lambda^* (Z^b) \otimes \mathbb{Z}[[U^{-1},U]/\mathbb{Z}[U]] \\
\overline{HM}_* (b_1 \mathbb{S}^1 \times \mathbb{S}^2) & \cong \Lambda^* (Z^b) \otimes \mathbb{Z}[[U^{-1},U]].
\end{align*}
\]

(73)

We will denote by

\[
\begin{align*}
\hat{G} & < \overline{HM}_* (Y) \\
\check{G} & < \overline{HM}_* (Y) \\
\breve{G} & < \overline{HM}_* (Y)
\end{align*}
\]

(74)

The image of \( \overline{HM}_* (W), \overline{H}_{\mathfrak{m}}(W) \) and \( \overline{HM}_* (W) \) respectively.

Before proceeding to the next proposition, we recall that for any 3-manifold \( M \), there is a long exact sequence [KM07, pg. 52]

\[
\cdots \to \overline{HM}_* (M) \to \overline{HM}_* (M) \to \overline{HM}_* (M) \to \cdots,
\]

(75)

and the reduced homology \( HM^{\text{red}} (M) < \overline{HM}_* (M) \) is defined as the kernel of the map \( p_* : \overline{HM}_* (M) \to \overline{HM}_* (M) \) [KM07, Definition 3.6.3]. For any 3-manifold, the reduced homology has finite rank.

The next proposition clearly implies Theorem 1.4.

**Proposition 3.10.** Let \( \mathfrak{s} \) be the the unique (up to isomorphism) torsion spin\(^c \) structure of \( Y \). The cobordism map \( \overline{HM}_* (W) \) has image contained in \( \overline{HM}_* (Y, \mathfrak{s}) \), preserves the absolute \( \mathbb{Q} \)-grading and has a left inverse. The same applies to \( HM_* (W) \)
and $\overline{H \! M}_{\bullet}(W)$, hence $\hat{G}, \tilde{G}, \tilde{G}$ are isomorphic to the $\mathbb{Z}[[U]]$-modules on the right of (73). In addition, we have that
\[
\overline{H \! M}_{\bullet}(Y) \simeq \hat{G} \oplus H \! M_{\bullet}^{\text{red}}(Y) \\
\overline{H \! M}_{\bullet}(Y) \simeq \tilde{G} \oplus H \! M_{\bullet}^{\text{red}}(Y) \\
\overline{H \! M}_{\bullet}(Y) \simeq \tilde{G}
\] (76)

With respect to this splitting, the map $j_{\ast} : \hat{G} \oplus H \! M_{\bullet}^{\text{red}}(Y) \to \hat{G} \oplus H \! M_{\bullet}^{\text{red}}(Y)$ is given by $j_{\ast}(g,q) = (0,q)$ and the map $p_{\ast} : \hat{G} \oplus H \! M_{\bullet}^{\text{red}}(Y) \to \hat{G}$ is given by $p_{\ast}(g,q) = p_{\ast}(b_1 S^1 \times S^2)(q)$ where $p_{\ast}(b_1 S^1 \times S^2)$ is conjugated to $p_{\ast} : \overline{H \! M}_{\bullet}(b_1 S^1 \times S^2) \to H \! M_{\bullet}(b_1 S^1 \times S^2)$ via $H \! M_{\bullet}(W)$ and $H \! M_{\bullet}(W)$.

**Proof.** The Floer homology of $b_1 S^1 \times S^2$ is supported in its torsion spin$^c$ structure, $s_{b_1 S^1 \times S^2}$ [KM07, Proposition 36.1.3] and since $W$ is an homology cobordism there is a unique spin$^c$ structure over $W$ extending $s_{b_1 S^1 \times S^2}$ (up to isomorphism). By considering a trivial spin$^c$ structure we see that it must restrict to $s_{0}$ over $Y$. To see that $\text{gr} \overline{Q}$ is preserved it is enough to compute the degree of the map [KM07, Equation (28.3)]
\[
\frac{1}{4}(c_1^2 - \sigma(W)) - \frac{1}{2}(\chi(W) + \sigma(W) + b_1(b_1 S^1 \times S^2) - b_1(Y)) = 0.
\] (77)

We proceed to prove the splitting part of the proposition. $Q$ provides the left inverse for the cobordism maps induced by $W$ (see Lemma 3.9 and (72)). Consequently there are submodules $\overline{Q}, \overline{Q}, \overline{Q}$ given by the kernel of $H \! M_{\bullet}(Q), H \! M_{\bullet}(Q), H \! M_{\bullet}(Q)$ such that
\[
\overline{H \! M}_{\bullet}(Y) = \hat{G} \oplus \overline{Q} \\
\overline{H \! M}_{\bullet}(Y) = \tilde{G} \oplus \overline{Q} \\
\overline{H \! M}_{\bullet}(Y) = \tilde{G} \oplus \overline{Q}.
\] (78)

Firstly we will show that $\overline{Q} = \{0\}$. The diagram
\[
\overline{H \! M}_{\bullet}(b_1 S^1 \times S^2) \xrightarrow{\overline{H \! M}_{\bullet}(W)} \overline{H \! M}_{\bullet}(Y) \xrightarrow{\overline{H \! M}_{\bullet}(Q)} \overline{H \! M}_{\bullet}(b_1 S^1 \times S^2)
\] (79)
is, up to isomorphism,
\[
\hat{G} \hookrightarrow \hat{G} \oplus \overline{Q} \twoheadrightarrow \hat{G}
\] (80)
where the first map is the inclusion into the first summand and the second map the projection to the first summand. Now we use that, as showed in Proposition 2.4, the triple cup product of $Y$ vanishes, therefore [KM07, pg. 687-688] implies that $\overline{H \! M}_{\bullet}(Y) \simeq \Lambda^\ast(\mathbb{Z}^{b_1}) \otimes_\mathbb{Z} \mathbb{Z}[U^{-1}, U]]$. 

The maps of (80) can be promoted to \( \mathbb{Z}[U^{-1}, U] \)-module homomorphisms. Tensoring by \( \mathbb{R} \) so that \( \mathbb{R}[U^{-1}, U] \) is a PID, and \( \overline{HM}_\bullet(Y) \otimes \mathbb{R} \) is a free \( \mathbb{R}[U^{-1}, U] \)-module of rank \( 2^h_1 \), we see that \( \overline{Q} \otimes \mathbb{R} = 0 \), because the maps of (80) tensored with \( \mathbb{R} \) have to be isomorphisms for dimensional reasons. This also forces the maps of (80) to be isomorphisms over \( \mathbb{Z} \) because if \( \overline{Q} \neq 0 \) also \( \overline{Q} \otimes \mathbb{R} \neq 0 \) as \( \overline{HM}_\bullet(Y) \) is torsion free.

This shows that \( \overline{Q} = \{0\} \), now we will show that \( \hat{Q} \simeq \hat{Q} = HM^{\text{red}}(Y) \). The commutative diagram

\[
\begin{array}{cccccc}
\overline{HM}_\bullet(b_1 \mathbb{S}^1 \times \mathbb{S}^2) & \xrightarrow{\overline{HM}_\bullet(W)} & \overline{HM}_\bullet(Y) & \xrightarrow{\overline{HM}_\bullet(Q)} & \overline{HM}_\bullet(b_1 \mathbb{S}^1 \times \mathbb{S}^2) \\
\downarrow{j_*} & & \downarrow{j_*} & & \downarrow{j_*} \\
\overline{HM}_\bullet(b_1 \mathbb{S}^1 \times \mathbb{S}^2) & \xrightarrow{\overline{HM}_\bullet(W)} & \overline{HM}_\bullet(Y) & \xrightarrow{\overline{HM}_\bullet(Q)} & \overline{HM}_\bullet(b_1 \mathbb{S}^1 \times \mathbb{S}^2) \\
\downarrow{p_*} & & \downarrow{p_*} & & \downarrow{p_*} \\
\overline{HM}_\bullet(b_1 \mathbb{S}^1 \times \mathbb{S}^2) & \xrightarrow{\overline{HM}_\bullet(W)} & \overline{HM}_\bullet(Y) & \xrightarrow{\overline{HM}_\bullet(Q)} & \overline{HM}_\bullet(b_1 \mathbb{S}^1 \times \mathbb{S}^2)
\end{array}
\]  

(81)

is isomorphic to

\[
\begin{array}{cccccc}
\hat{G} & \xrightarrow{\hat{G} \oplus \hat{Q} \quad pr_1} & \hat{G} \\
\downarrow{0} & & \downarrow{0} \\
\hat{G} & \xrightarrow{\hat{G} \oplus \hat{Q} \quad pr_1} & \hat{G} \\
\downarrow{\cong} & & \downarrow{\cong} \\
\hat{G} & \xrightarrow{\cong} \hat{G} & \xrightarrow{\cong} \hat{G}
\end{array}
\]  

(82)

where the first and last column are the exact sequence (75) for \( b_1 \mathbb{S}^1 \times \mathbb{S}^2 \) (\( j_* \) is zero by [KM07, Proposition 36.1.3]). The middle column is the exact sequence (75) for \( Y \), and since the diagram is commutative and horizontally the first maps are inclusions, we see that when restricted to the \( G \)-factors the middle column coincides with exact sequence for \( b_1 \mathbb{S}^1 \times \mathbb{S}^2 \). The claim \( \hat{Q} = HM^{\text{red}}(Y) \) is equivalent to \( \hat{Q} = \ker(\hat{G} \oplus \hat{Q} \to \hat{G}) \) which is now clear from the last two rows of the diagram. To see that \( j_* : \hat{Q} \xrightarrow{\cong} \hat{Q} \) is an isomorphism we consider again the diagram above for surjectivity and for injectivity the exact sequence \( \hat{G} \to \hat{G} \oplus \hat{Q} \to \hat{G} \oplus \hat{Q} \) restricting over the \( G \)s to the exact sequence of \( b_1 \mathbb{S}^1 \times \mathbb{S}^2 \). \qed
4 Moduli spaces of reducible monopoles over a protocork.

Outline. In this section we study the moduli spaces of perturbed reducible monopoles over a protocork $P_0(\Gamma)$. In Subsection 4.1 we recap the relevant background and notation from the book [KM07] by Kronheimer and Mrowka that we will need. In Subsection 3.3 we compute the formal dimension of the reducible moduli spaces of monopoles asymptotic to the critical points described in Subsection 3.3. This computation will be crucial in the proof of item (b) of Theorem 1.1. In the last subsection we show that the moduli space of reducible monopoles limiting to $\mathfrak{a} N_{\text{red}}$ consist of a single point when appropriate perturbations are considered. This is a fact of its own interest, because this moduli space contributes to the Seiberg-Witten invariants. Indeed it can be used to prove item (a) of the main theorem, even though in our final manuscript we decided to prove (a) using only the functorial properties of $HM_\bullet$ to make the proof accessible to a larger audience. Our propositions will descend from some nice properties of protocorks, in particular:

(a) $H^1(P_0(\Gamma)) \to H^1(\partial P_0(\Gamma))$ is surjective,

(b) the diffeomorphism $\rho_A$ (see Subsection 2.5) is orientation reversing and preserve $H^1(\partial P_0(\Gamma))$,

(c) in the classical moduli space of unperturbed 3-monopoles over $\partial P_0(\Gamma)$, the torus of reducibles is a Morse-Bott singularity and

(d) the Dirac operator on $\partial P_0(\Gamma)$ associated to flat connections is invertible.

The main difficulty is to deal with the and perturbations and the blown-up moduli spaces where all the spectrum of the Dirac operator matters.

4.1 Background on 4D moduli spaces.

Outline. In this subsection we review the relevant background from [KM07] and establish our notation. The definition of the perturbed moduli space asymptotic to a 3-monopole, together with the functional spaces used therein will be particularly important to us. We make no claim of originality.

BG. 7 (Classical and blown-up configuration spaces on a compact 4-manifold with boundary). Let $X$ be a compact, smooth, oriented 4-manifold with boundary $\partial X = Y$ endowed with a Riemannian metric and spin$^c$ structure $s_X$
consisting of spinor bundle $S_X \to X$ and Clifford multiplication $\rho_X : TX \to \text{Hom}(S_X, S_X)$ extended as usual to $\Lambda^*X$. The bundle $S_X$ splits as $S^+ \oplus S^-$, which are respectively the $-1$ and $+1$ eigenbundles of $\rho_X(\text{vol}_X)$, where $\text{vol}_X$ is the Riemannian volume form. We will denote the Dirac operator induced by $A \in \mathcal{A}(X, s_X)$ acting on positive spinors as $D^+_A : \Gamma(S^+) \to \Gamma(S^-)$. As in the 3D case, given a spin$^c$ connection $A$, we will denote by $A'$ the induced connection on the determinant bundle. Let $k \geq 0$, then similarly to the 3-dimensional case, we can define
\begin{align*}
\mathcal{C}_k(X, s_X) &= \mathcal{A}_k(X, s_X) \times \mathbb{R}_{\geq 0} \times \{ \varphi \in L^2_k(X; S^+) \mid \|\varphi\|_{L^2(X)} = 1 \} \\
\mathcal{G}_{k+1}(X) &= \{ u \in L^2_{k+1}(X; \mathbb{C}) \mid |u(x)| = 1 \} \\
\mathcal{B}_k(X, s_X) &= \mathcal{C}_k(X, s_X)/\mathcal{G}_{k+1}(X),
\end{align*}
which are respectively the configuration space, gauge group and quotient configuration space. The action of $u \in \mathcal{G}_{k+1}(X)$ is the same as in (41). In the following we will always assume that $k > 2$ and if $k$ is omitted, smooth objects are considered. We also have the blown-up spaces:
\begin{align*}
\mathcal{C}_k^\sigma(X, s_X) &= \mathcal{A}_k(X, s_X) \times \mathbb{R}_{\geq 0} \times \{ \varphi \in L^2_k(X; S^+) \mid \|\varphi\|_{L^2(X)} = 1 \} \\
\mathcal{B}_k^\sigma(X, s_X) &= \mathcal{C}_k^\sigma(X, s_X)/\mathcal{G}_{k+1}(X),
\end{align*}
where $u \in \mathcal{G}_{k+1}(X)$ acts as in (44). The completion of the tangent spaces of $\mathcal{C}_k(X, s_X)$ and $\mathcal{C}_k^\sigma(X, s_X)$ in the $L^2_j$-norm, $j \leq k$ are denoted by $\mathcal{T}_j$ and $\mathcal{T}_j^\sigma$ respectively. As in the 3-dimensional case there is a blow-down map $\pi : \mathcal{C}_k(X, s_X) \to \mathcal{C}_k^\sigma(X, s_X)$ given by $(A, s, \varphi) \mapsto (A, s\varphi)$ that is a diffeomorphism over the irreducibles. We also introduce the vector bundle $\mathcal{V}_j \to \mathcal{C}_k(X, s_X)$, $j \leq k$ defined by
\begin{align*}
\mathcal{V}_j &= L^2_j(X; isu(S^+) \oplus S^-) \quad (85)
\end{align*}
where $su(S^+)$ is the bundle of traceless skew-hermitian endomorphisms of $S^+$. The blow-up $\mathcal{V}_j^\sigma \to \mathcal{C}_k^\sigma(X, s_X)$ is defined as the pull-back along $\pi$ of $\mathcal{V}_j^\sigma$. This will be the codomain for the Seiberg-Witten map.

**BG. 8 (Special notation in the case of the cylinder).** Let $Z = I \times Y$, where $I \subset \mathbb{R}$ is an interval, and let $s$ be a spin$^c$ structure on $Y$ with spinor bundle $S$ and Clifford multiplication $\rho$ as in Subsection 3.1. We can endow $Z$ with a spin$^c$ structure $s_Z$ where, denoting by $\pi : Z \to Y$ the projection onto the second factor, the spinor bundle is $S_Z := \pi^*S \oplus \pi^*S$, and Clifford multiplication $\rho_Z : TZ \to \text{End}(S_Z)$ is given by
\[
\rho_Z(\partial/\partial t) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \rho_Z(v) := \begin{bmatrix} 0 & -\rho(v)^* \\ \rho(v) & 0 \end{bmatrix}
\]
for $v \in TY \hookrightarrow TZ$. In particular the bundles of positive and negative spinors are $S^+ = S^- = \pi^*S$. A configuration $(A, \phi) \in \mathcal{A}_k(Z, s_Z) \times L^2_k(S^+)$ induces a time-dependent configuration $(\tilde{A}(t), \tilde{\phi}(t)) \in \mathcal{C}_k(Y)$ defined implicitly by putting
\[
(A, \phi)|_{\{t\} \times Y} = (\tilde{A}(t) + c(t)dt \otimes 1_s, \tilde{\phi}(t))
\] (86)
for some $c : Z \to \mathbb{R}$, so that $\tilde{A}(t)$ is in temporal-gauge. Moreover, we have an isomorphism of vector bundles
\[
\pi^*\Lambda^1(Y) \sim \text{i}su(S^+)
\]
\[
\pi^*\alpha \mapsto \frac{1}{2} \rho_Z (\pi^*(\ast \alpha) + \pi^*(\alpha) \wedge dt).
\] (87)
In particular $\mathcal{V}_0(Z) \simeq L^2(Z; \pi^*\Lambda^1(Y) \oplus \pi^*S)$. Now, consider a perturbation $q : \mathcal{C}(Y, s) \to \mathcal{F}_0(Y)$. This determines a map
\[
\hat{q} : \mathcal{C}(Z, s_Z) \to \mathcal{V}_0(Z) \simeq L^2(Z; \pi^*\Lambda^1(Y) \oplus \pi^*S),
\] (88)
defined by
\[
\hat{q}(A, \varphi)|_{\{t\} \times Y} = q(\tilde{A}(t), \tilde{\phi}(t)) \in \mathcal{F}_0(Y) = L^2(iT^*Y \oplus S).
\] (89)
More information may be found in [KM07, Chapter 4].

**BG. 9 (Perturbations).** We will briefly describe the structure of the perturbations used over 4-manifolds, more details may be found in [KM07, Section 24]. Suppose that $(X, s_X)$ and $(Y, s)$, $Y = \partial X$ are as above. In particular, $X$ is compact, and we will suppose that there is a collar of $\partial X$ isometric to $(-C, 0] \times Y$ for some $C > 0$, where $s_X$ is constructed with the spin$^c$ structure of $Y$, $s$ as in the previous paragraph. The perturbation will be supported in this collar. Fix $q$, an admissible perturbation for $(Y, s)$, and let $\beta \in C^\infty([-C, +\infty))$ be a cut-off function, such that $\beta(t) = 1$ for $t \geq 0$, $\beta(t) = 0$ for $t \leq -C/2$. Let $\beta_0 \in C^\infty([-C, +\infty))$ be a bump-function with $\text{supp}(\beta_0) \subset (-C, -C/2)$. For any perturbation $q_0 \in \mathcal{P}(Y, s)$, we define a perturbation over $(-C, 0] \times Y$ by setting
\[
\hat{p} = \beta_0 \hat{q}_0 + \beta \hat{q} : \mathcal{C}((-C, 0] \times Y, s_X|_{(-C, 0] \times Y}) \to \mathcal{V}_0((-C, 0] \times Y)
\] (90)
where we are using (88). This induces a perturbation
\[
\hat{p} : \mathcal{C}_k(X, s_X) \to \mathcal{V}_k(X)
\] (91)
that depends only on the behaviour of the configuration on the collar. Notice that \( \hat{\mathfrak{p}} \) will be slice-wise equal to \( \mathfrak{q} \) in a neighbourhood of \( \{0\} \times Y \). \( \hat{\mathfrak{p}} \) induces a perturbation on the blow-up as follows. Writing \( \mathcal{V}_k = L^2_k(X, i\mathfrak{su}(S^+) \oplus S^-) \), we have the decomposition \( \hat{\mathfrak{p}} = (\hat{\mathfrak{p}}^0, \hat{\mathfrak{p}}^1) \). We define \( \hat{\mathfrak{p}}^\sigma \) on the blow-up to be the section

\[
\hat{\mathfrak{p}}^\sigma : \mathcal{C}^\sigma_k(X, s_X) \to \mathcal{V}_k^\sigma
\]

\[
\hat{\mathfrak{p}}^{0,\sigma}(A, s, \phi) = \mathfrak{p}^0(A, s\phi)
\]

\[
\hat{\mathfrak{p}}^{1,\sigma}(A, s, \phi) = \begin{cases} 
\frac{1}{s} \hat{\mathfrak{p}}^1(A, s\phi) & \text{if } s \neq 0 \\
D_{(A,0)} \hat{\mathfrak{p}}^1(\phi) & \text{if } s = 0
\end{cases}
\]

\[\text{(92)}\]

**BG. 10 (Blown-up moduli spaces over compact } X)\). The Seiberg-Witten map (in the blown-up setting) is defined as

\[
\mathcal{F}^\sigma : \mathcal{C}^\sigma_k(X, s_X) \to \mathcal{V}_k^\sigma
\]

\[
(A, s, \varphi) \mapsto (\frac{1}{2} \rho_X(F^+_A - s^2(\varphi\varphi^*)_0, D^+_A\varphi)
\]

\[\text{(93)}\]

which is a lift of the classical Seiberg-Witten map under the blow-down map \( \pi : \mathcal{C}^\sigma(X, s_X) \to \mathcal{C}(X, s_X) \). Given a perturbation \( \hat{\mathfrak{p}}^\sigma \) as in the above paragraph, we can define the perturbed Seiberg-Witten map

\[
\mathcal{F}^\sigma_p = \mathcal{F}^\sigma + \hat{\mathfrak{p}}^\sigma : \mathcal{C}^\sigma_k(X, s_X) \to \mathcal{V}_k^\sigma.
\]

\[\text{(94)}\]

The zero locus of \( \mathcal{F}^\sigma_p \) is invariant under \( \mathcal{G}_{k+1}(X) \) and we have the moduli spaces of solutions

\[
M(X, s_X) = \{ x \in \mathcal{C}_k(X, s_X) \mid \mathcal{F}^\sigma_p(x) = 0 \} / \mathcal{G}_{k+1}(X) \subset \mathcal{B}_k(X, s_X)
\]

\[\text{(95)}\]

which we will refer to as the moduli space of perturbed monopoles.

**BG. 11 (Moduli spaces for manifolds with cylindrical ends)\). We will denote by \( X^* \), the manifold \( X \) with cylindrical ends attached to the boundary using the diffeomorphism \( \partial X \approx Y \),

\[
X^* = X \bigcup ([0, +\infty) \times Y).
\]

\[\text{(96)}\]

On \( X^* \) we extend the Riemannian metric of \( X \) so that the infinite cylinder is isometric to \( (-C, +\infty) \times Y \). The relevant spaces in this context are the \( L^2_{k,loc} \)-configuration spaces:

\[
\mathcal{C}_{k,loc}(X^*, s_X) = \mathcal{A}_{k,loc}(X^*, s_X) \times L^2_{k,loc}(X^*; S^+).
\]

\[\text{(97)}\]
Note however that $L^2_{k,\text{loc}}(X^*)$ is not a Banach space as it is not normable. Also notice that the $L^2_k$-norm of an element of $L^2_{k,\text{loc}}(X^*)$ can be infinite in general. The $L^2_{k,\text{loc}}$-blown-up configuration spaces are defined as follows. Let $S := (L^2_{k,\text{loc}}(X^*, S^*) \setminus \{0\}) / \mathbb{R}_+$, then put

$$G_{k,\text{loc}}(X^*, s) := \{(A, \phi, \mathbb{R}_+ \phi) \mid (A, \phi) \in G_{k,\text{loc}}(X^*, s), \phi \neq 0\} \subset G_{k,\text{loc}}(X^*, s) \times S.$$ 

The relevant group of tranformations is the gauge group $G_{k+1,\text{loc}}(X^*)$ defined as in (83) but using $L^2_{k,\text{loc}}$ Sobolev spaces. We can define as usual $G_{k,\text{loc}}(X^*, s_X)$ and $R_{k,\text{loc}}(X^*, s_X)$ by quotienting out the above defined configuration spaces by the action of $G_{k,\text{loc}}(X^*)$. Over $G_{k,\text{loc}}(X^*, s_X)$ we have a bundle $\mathcal{V}_{j,\text{loc}}(X^*)$ defined as in (85) but using $L^2_{k,\text{loc}}$-coefficients. In the blow-up setting, the bundle $\mathcal{V}_{\tau,\text{loc}}(X^*)$, is defined as follows. First of all, we define the tautological line bundle

$$\mathcal{O}(-1) := \{(A, \phi, \mathbb{R}_+ \phi, z \phi) \mid (A, \phi, \mathbb{R}_+ \phi) \in G_{k,\text{loc}}(X^*), z \in \mathbb{C}\} \to G_{k,\text{loc}}(X^*)$$

then we put

$$\mathcal{V}_{\tau,\text{loc}}(X^*) := \mathcal{O}(-1)^* \otimes \pi^* \mathcal{V}_{j,\text{loc}}(X^*) \to G_{k,\text{loc}}(X^*).$$

This definition generalizes the one given for $\mathcal{V}_j^\sigma(X)$, indeed in the $L^2_j$-case, the norm allows us to trivialize the bundle $\mathcal{O}(-1)$, hence $\mathcal{V}_j^\sigma(X)$ is just the pullback via the blow-down map $\pi$. The operator $F_p^\sigma$ defined in (94) induces an a continuous section of $\mathcal{V}_{\tau,\text{loc}}^\sigma$ invariant under the action of $G_{k+1,\text{loc}}(X^*)$ that we will denoted again by $F_p^\sigma$. We can define then $M(X^*, s_X)$ as the zero locus of $F_p^\sigma$ quotiented out by $G_{k,\text{loc}}(X^*)$.

BG. 12 ($\tau$-model). In the case of a cylinder $Z = I \times Y$ ($I \subset \mathbb{R}$ possibly unbounded), there is another model, called $\tau$-model [KM07, Section 13.1]. The $L^2_{k,\text{loc}}$-version of it is the set

$$G_{k,\text{loc}}^\tau(Z, s_Z) \subset s^A_{k,\text{loc}}(Z) \times L^2_{k,\text{loc}}(I) \times L^2_{k,\text{loc}}(Z, S^+)$$

consisting of triples $(A, s, \varphi)$ such that $s(t) \geq 0$ and $\|\varphi(t)\|_{L^2(Y)} = 1$ for all $t \in I$. The spin$^c$ structure over $Z$ is always the one induced by $Y$ and defined in Background 8. The gauge group $G_{k+1,\text{loc}}(Z)$ acts on $G_{k,\text{loc}}^\tau(Z, s_Z)$ with quotient $R_{k,\text{loc}}^\tau(Z, s_Z)$. Elements of $G_{k,\text{loc}}^\tau(Z, s_Z)$ with connection component in temporal gauge naturally represent a path of configurations in $G_{k}^\sigma(Y, s)$. Similarly to the
case of the blown-up model, there is a bundle $\mathcal{V}_{k,\text{loc}}^\tau \to \mathcal{C}_{k,\text{loc}}^\tau (Z, s_Z)$ and a perturbed Seiberg-Witten map $\mathcal{F}^\tau_p$ which is a section of it. Suppose that $I = \mathbb{R}_{\geq 0}$ and we are given an element $[y] \in \mathcal{B}_{k,\text{loc}}^\tau (Z, s_Z)$ and a critical point $[b] \in \mathcal{B}_{k}^\tau (Y, s_Y)$. Then we write that
\[
\lim_{t \to -\infty} [y] = [b]
\]
if $[y(t + \cdot, \cdot)]$ tends in $L^2_{k,\text{loc}}(Z)$ to the translation invariant solution defined by $[b]$ as $t \to +\infty$. In an analogous way one can define $\lim_{t \to -\infty} [y]$, see [KM07, Section 13.1].

**BG. 13 (Moduli spaces of monopoles asymptotic to a 3-monopole).** Now we want to define the moduli space of solutions limiting to a critical point $[b] \in \mathcal{B}_{k}^\tau (Y, s_Y)$. First of all, notice that there is a restriction map defined slicewise by renormalizing the spinor component
\[
R_Z : \{ [y] \in \mathcal{B}_{k,\text{loc}}^\sigma (X^*, s_X) \mid \mathcal{F}^\sigma_p (y) = 0 \} \to \mathcal{B}_{k,\text{loc}}^\tau (Z, s_Z),
\]
where $Z = [0, +\infty) \times Y$, and we are implicitly using the diffeomorphism $\partial X \cong Y$.

Notice that (101) can be defined only on the solution space because we need to appeal to the unique continuation property for the solutions of the equations. We define
\[
M(X^*, s_X, [b]) \subset \mathcal{B}_{k,\text{loc}}^\sigma (X^*, s_X)
\]
as the set of $[y]$ such that $\mathcal{F}^\sigma_p (y) = 0$ and the restriction $\lim_{t \to -\infty} R_Z [y] = [b]$. In [KM07], the notation $M_z(X^*, s_X, [b])$ is used to denote the component of the moduli space consisting of monopoles of homotopy class $z$ [KM07, pg. 474] we will not need it because in our specific case there will be only one class $z$.

**BG. 14 (Index function).** Suppose now that our 4-manifold is a cobordism $W^4 : Y_0 \to Y_1$. We will assume to have fixed spin$^c$ structure $s_W$ over $W$ and that $[a]$ and $[b]$ are non-degenerate critical points respectively over $Y_0$ and $Y_1$ relative to the spin$^c$ structure obtained restricting $s_W$.

To this data, Kronheimer and Mrowka associate an integer $\text{gr}([a], W^*, [b])$ [KM07, pg. 475], the index of a Fredholm operator. In general $\text{gr}$ depends also on the choice of a $W$-path $z$ from $[a]$ to $[b]$, [KM07, Def. 23.3.2] modulo homotopy fixing the endpoints, however in all our applications the restrictions of $s_W$ will be torsion and in this case $\text{gr}$ becomes independent of $z$, thus we drop it from the notation.
The function $\text{gr}$ relates to the formal dimension of the moduli spaces in the following way:

$$\dim M([a], W^*, [b]) = \text{gr}([a], W^*, [b]) + \varepsilon,$$  \hspace{1cm} (103)

where $\varepsilon$ is computed in the following way. Let $n_+$ be the number of *outcoming* connected components of $\partial W$ with a boundary-unstable critical point associated similarly let $n_-$ be the number of *incoming* components of $\partial W$ with a boundary-stable critical point associated. Now set $c := n_+ + n_- - 1$, if $c > 0$, we say that the moduli space is boundary-obstructed with corank $c$ and $\varepsilon := c$, otherwise $\varepsilon := 0$.

The function $\text{gr}$ enjoys the following *additivity property*:

$$\text{gr}([A], W_1^*, [B]) + \text{gr}([B], W_2^*, [C]) = \text{gr}([A], (W_1 \circ W_2)^*, [C]),$$  \hspace{1cm} (104)

which instead does not hold for $\dim$.

### 4.2 Formal dimension of moduli spaces asymptotic to a reducible critical point.

*Outline.* The aim of this subsection is to prove Proposition 4.2 below which computes the formal dimension of reducible moduli spaces $M^\text{red}(N^*, [a_{k,j}])$ where $N$ is a protocork. The idea of the proof is to reduce to the case of the classical (unperturbed) moduli space limiting to the torus of reducibles, which is a Morse-Bott singularity of $\mathcal{L}$, and apply the Atiyah-Patodi-Singer (APS) index theorem. This is done in a several steps. We use the characterization given by Lemma 4.1 below to pass from $L^2_{k,\text{loc}}$-monopoles to monopoles in a weighted Sobolev space, this is probably known to experts, but we decided to give an explicit proof in Appendix B for future reference because, to the authors’ knowledge, it is not present in the literature. Then we reduce to the $k = N^\text{red}, j = 0$ case, i.e. $M^\text{red}(N^*, [a_{N^\text{red},0}])$, for two reasons: $j = 0$ allows us to pass from weighted Sobolev spaces to $L^2$-spaces (Lemma 4.3) and $k = N^\text{red}$ allows us to kill the perturbative term (Lemma 4.4). At this point the computation of the APS index is carried out with the help of the nice properties of $Y$ showed in Subsection 3.3.

*Setting.* For the rest of this section $N$ will be an oriented, compact 4-manifold diffeomorphic to a protocork $P_0(\Gamma)$ for some protocork plumbing graph $\Gamma$. The boundary of $N$ will be denoted by $Y$. We suppose also that $N$ is endowed with a Riemannian metric so that on a collar neighbourhood of the boundary is isometric to $(-C, 0] \times Y$, for some $C > 0$ where the Riemannian metric on $Y$ satisfies
the hypothesis of Subsection 3.3. We denote by \( s \) a \textit{trivial} spin\(^c\) structure over \( Y \) and we consider a \textit{trivial} spin\(^c\) structure \( s_N \) over \( N \) extending \( s \), such that on the collar \( (-C, 0] \times Y \) is of the form described in Background 8.

Since these are the only spin\(^c\) structures we are going to deal with, we will omit them in our notation. The setting over \( Y \) is that described in Subsection 3.3, in particular the perturbation \( q \) is defined in (57) and in \( \mathcal{B}_k^q(Y) \), we have irreducible critical points \([b_i], i = 1, \ldots, N^{irr}\) and reducible critical points \([a_{k,i}], k = 1, \ldots, N^{red}, i \in \mathbb{Z}\) where \( i \geq 0 \) \((i < 0)\) corresponds to boundary-stable (unstable) critical points.

**Premise on moduli spaces of reducible monopoles.** For any critical point \([c] \in \mathcal{B}_k^q(Y)\), we denote by \( M^\text{red}(N^*; [c]) \subset M(N^*; [c]) \) the moduli space of (perturbed) \textit{reducible} monopoles. If \([c] \) is irreducible, this space is empty, if \([c] \) is reducible and boundary-unstable then it is the whole of \( M(N^*; [c]) \) while if \([c] \) is reducible and boundary-stable it consists of the boundary of \( M(N^*; [c]) \) (assuming regularity of the moduli spaces).

We remark that in general the formal dimension depends on the homotopy class of the monopoles [KM07, pg. 474]. However for any \([c] \) critical points, \( \pi_0(\mathcal{B}_k^q(N; [c])) = \{ * \} \) consists of a point; indeed it is a principal homogeneous space over \( H^1(Y; \mathbb{Z})/H^1(N; \mathbb{Z}) \) which is trivial by Proposition 2.4. Consequently, we can omit the subscript \( z \in \pi_0(\mathcal{B}_k^q(N; [c])) \) in \( M_z(N^*; [c]) \).

In the next lemma we will denote by \( L^2_{k, \delta}(N^*) \), \( \delta \in \mathbb{R} \), the weighted Sobolev space defined using as weight function \( t \mapsto e^{2 \delta t \beta(t)} \), in particular, \( L^2_{k, \delta}(N^*) \rightarrow L^2_{k, \delta}(N^*) = f \mapsto e^{-2 \delta t \beta(t)} f \) is an isometry. Here \( \beta \in C^\infty(N^*, [0, 1]) \) is the cut-off function defined in Background 9 extended to zero over the complement of the tube.

It is convenient to define the following Hilbert manifolds, associated to a spinor \( \psi_0 \in L^2_k(Y) \) and \( \lambda_0 \in \mathbb{R}, \delta > 0 \),

\[
X^\pm_{k, \lambda_0, \delta}(\psi_0) := \left\{ \Phi \in L^2_k(N^*; S^\pm) \mid \Phi_{|_{\partial_+ \times Y}}(t, y) = e^{-\lambda_0 t} \psi_0(y) + \varphi(t, y), \ \varphi \in L^2_{k, \lambda_0 + \delta}(N^*; S^\pm) \right\}
\]

endowed with the smooth structure defined by requiring that

\[
L^2_{k, \lambda_0 + \delta}(N^*; S^\pm) \rightarrow X^\pm_{k, \lambda_0, \delta}(\psi_0) = \varphi \mapsto e^{-\lambda_0 (-) \beta(\cdot)} \beta(\cdot) \psi_0 + \varphi \quad \text{(105)}
\]

is a diffeomorphism.
Remark 7. Let $A \in \mathcal{A}_{k,\text{loc}}(N^*)$ be translation invariant and equal to $B$ over the tube. Then the Dirac operator defines a smooth map

$$D_A^+: X^+_{k,\lambda,\delta}(\psi_0) \to X^-_{k-1,\lambda,\delta}(\lambda_0\psi_0 + D_B\psi_0),$$

which is conjugated via the diffeomorphism (105), to

$$D_A^+ + K : L^2_{k,\lambda,\delta}(N^*, S^+) \to L^2_{k-1,\lambda,\delta}(N^*, S^-)$$

where $K : L^2_{k,\lambda,\delta}(N^*, S^+) \to L^2_{k-1,\lambda,\delta}(N^*, S^-)$ is a compact operator.

Lemma 4.1 (Characterization of reducible moduli spaces with cylindrical ends.). Suppose that $[c] = [(B_\epsilon, 0, \psi_\epsilon)] \in \mathcal{B}_k^e(Y)$ is a non-degenerate, reducible critical point with associated eigenvalue $\lambda_\epsilon$. Then there exists $\bar{\delta} > 0$, such that for any $\delta \in (0, \bar{\delta}]$, $M^\text{red}(N^*; [c])$ is naturally identified with the set of pairs $(A, \Phi) \in A_\epsilon + L^2_k(N^*, i\Lambda^1 N^*) \times X^+_{k,\lambda,\delta}(\psi_\epsilon)$ such that

$$F^+_{A_\epsilon} + \hat{\nabla}^0(A_\epsilon, 0) = 0$$

$$D^+_A\Phi + D_{(A, 0)}\hat{\nabla}^1(\Phi) = 0$$

quotiented out by the action of the gauge group

$$\mathcal{G}_{k+1,\delta}(N^*; 1) := \{u \in L^2_{k+1,\text{loc}}(N^*) \mid 1 - u \in L^2_{k+1,\delta}(N^*), |u| = 1\}$$

Here $A_\epsilon$ denotes a connection that over the cylindrical end is translation invariant and equal to $B_\epsilon$ and the Sobolev norm on $L^2_k(N^*, S^*)$ is computed using the connection $A_\epsilon$.

The proof of this Lemma is in Appendix B. This characterization relies on the fact that $\pi_0(\mathcal{B}_k^e(N; [c])) = \{\ast\}$, but can be easily modified to cover the general case by defining spaces $X^+_{z,k,\lambda,\delta}(\psi_\epsilon)$ dependent on the path $z$.

Proposition 4.2. The formal dimension of $M(N^*; [a_{k,j}])$ is equal to

$$d(M(N^*; [a_{k,j}]))) = \begin{cases} 
  b_1(Y) - \text{index}_f[a_k] - 2i - 1 & \text{for } i \geq 0 \\
  b_1(Y) - \text{index}_f[a_k] - 2i - 2 & \text{for } i < 0 
\end{cases}$$

where $\text{index}_f[a_k]$ is defined in (61).
Proof of Proposition 4.2. Consider \([a_{\text{red},0}] \in \mathcal{B}_k(Y)\), the only critical point such that \(\text{index}_f[a_{\text{red}}] = b_1(Y)\) (see Subsection 3.3). We will show that \(d(M(N^*; [a_{\text{red},0}]) = -1\). The general case will then follow from the additivity of the index:

\[
d(M(N^*; [a_{k,j}])) = d(M(N^*; [a_{\text{red},0}]) + \text{gr}([a_{\text{red},0}], [a_{k,j}])
\]

\[
= d(M(N^*; [a_{\text{red},0}]) + \text{gr}([a_{\text{red},0}], [a_{k,j}]) - \epsilon(j),
\]

where \(\epsilon = 1\) if \(j < 0\) and 0 otherwise and \(\text{gr}\) has been defined in Proposition 3.5. Since reducible solutions constitute the boundary of the moduli space: \(M^\text{red}(N^*; [a_{\text{red},0}]) = \partial M(N^*; [a_{\text{red},0}])\), it is enough to show that the virtual dimension of \(M^\text{red}(N^*; [a_{\text{red},0}])\) is equal to -2.

Denote the components of \(a_{\text{red},0}\) as \(a_{\text{red},0} = (B_\infty, 0, \psi_\infty) \in \mathcal{C}_k^0(Y)\). Since \(a_{\text{red},0}\) is non-degenerate we can invoke Lemma 4.1, so that \(M^\text{red}(N^*; [a_{\text{red},0}])\) consists of equivalence classes of perturbed monopoles \((B_0 + a \otimes 1_S, \mathbb{R}, \Phi)\) where \(B_0\) is a smooth connection equal to \(B_\infty\) over the end and \((a, \Phi) \in L^2_k(N^*; i\Lambda^1(N^*)) \times X_{+\lambda_0,\delta}^+(\psi_\infty)\). Here \(\lambda_0 > 0\) is the eigenvalue associated to \(\psi_\infty\) and \(\delta > 0\) is arbitrarily small. Lemma 4.1, will allow us to pass from \(L^2_{k,\text{loc}}\) to \(L^2_k\) sections, this is necessary step since we want to invoke the Atiyah-Patodi-Singer’s index theorem [APS75] later.

**Lemma 4.3.** The formal dimension of \(M^\text{red}(N^*; [a_{\text{red},0}])\) is equal to \(\text{ind}_{L^2} Q - 1\) where \(Q\) is the operator

\[
Q : L^2_k(N^*; i\Lambda^1(N^*) \oplus S^+) \to L^2_{k-1}(N^*; i\Lambda^2_k \oplus S^- \oplus \mathbb{R})
\]

\[
(a, \Phi) \mapsto (d^+ a + D_Y \hat{\phi}^0(a), D_{B_0}^+ \Phi + D_Y \hat{\phi}^1(\Phi), -d^+ a),
\]

and \(\gamma = (B_0, 0) \in \mathcal{C}_{k,\text{loc}}(N^*)\).

**Proof.** The formal dimension is given by the index of the Fredholm operator obtained by linearizing the equations (108) and adding a gauge fixing condition.

Pick \(\Phi_0 \in X_{+\lambda_0,\delta}^+(\psi_\infty)\), then the linearization of (108) at \((B_0, \Phi_0)\) is conjugated via (105) to

\[
\hat{Q} : L^2_{k,\delta}(N^*; i\Lambda^1 N^*) \oplus L^2_{k,\lambda_0+\delta}(N^*; S^+) \to L^2_{k-1,\delta}(N^*; i\Lambda^1 N^+) \oplus L^2_{k-1,\lambda_0+\delta}(N^*; S^-)
\]

\[
(a, \Phi) \mapsto (d^+ a + D_Y \hat{\phi}^0(a), D_{B_0}^+ \Phi + D_Y \hat{\phi}^1(\Phi)) + K(a, \Phi)
\]

where \(K\) is a compact operator (see also Remark 7). Choosing a different \((B_0, \Phi_0)\) will perturb \(\hat{Q}\) by a compact operator (hence irrelevant for the computation of the index) due to compact embedding theorems for weighted Sobolev spaces.
To take care of the action of the gauge group, we add a gauge-fixing condition, i.e. we consider the operator \( \tilde{Q} \) where \( \tilde{d} = \gamma \) where \( \gamma \): \( L^2_k(i\Lambda^1(N^*) \oplus S^+) \rightarrow L^2_{k-1,\delta}(i\mathbb{R}) \) is the operator defined at [KM07, pg. 143]; the \( L^2 \)-kernel of \( \tilde{d} \) is the tangent space to the Coulomb slice at \( \gamma \). \( \tilde{Q} \) is equal, modulo compact operators, to the operator

\[
Q : L^2_{k,\delta}(N^*; i\Lambda^1 N^*) \oplus L^2_{k,\lambda_0+\delta}(N^*; S^+) \rightarrow L^2_{k-1,\delta}(N^*; i\Lambda^1 N^*) \oplus L^2_{k-1,\lambda_0+\delta}(N^*; S^-) \oplus L^2_{k-1,\delta}(N^*; i\mathbb{R})
\]

\[
(a, \Phi) \mapsto (d^* a + D_\gamma p^0(a), D^+_{B_0} \Phi + D_\gamma p^1(\Phi), -d^* a).
\]

\( Q \) is not quite the operator appearing in the thesis because of the weighted Sobolev spaces. Since \( \lambda_0 \) is the first positive eigenvalue of the limiting perturbed Dirac operator and the latter has simple spectrum, by choosing \( \delta > 0 \) small, we have that \( \text{ind}_{L^2_{k,\lambda_0+\delta}}(D^+_{B_0} + D_\gamma p^1) = \text{ind}_{L^2}(D^+_{B_0} + D_\gamma p^1) - 2 \) (see also the proof of Lemma 4.9). Furthermore, \( \text{ind}_{L^2}(d^* + D_\gamma p^0, -d^*) = \text{ind}_{L^2}(d^* + D_\gamma p^0, -d^*) + 1 \) because the exponential decay cuts off an \( \mathbb{R} \)-summand in the cokernel due to constant functions. In conclusion the difference between the above operator \( Q \) and the operator \( \tilde{Q} \) in the thesis is \( -1 \).

In view of Lemma 4.3, in order to conclude the proof we will show that the \( L^2 \)-index of \( Q \) is equal to \( -1 \).

**Lemma 4.4.**

\[
\text{ind } Q = \text{ind } ASD \oplus D^+_{B_0}
\]

where \( ASD = d^* - d^* \) is the anti-self duality operator.

**Proof.** Recall that \( D_{B_0} : L^2_k(Y; S) \rightarrow L^2_{k-1}(Y; S) \) is invertible by our choice of metric and we choose our perturbation \( q \) so small in the Banach space of perturbations that \( D_{B_0,q} \) remains injective, consequently, the spectral flow will be zero and the index of \( D^+_{B_0} + D_\gamma p^1 \) will be equal to the index of \( D^+_{B_0} \) alone. Therefore

\[
\text{ind } Q = \text{ind } (ASD + D_\gamma p^0) \oplus D^+_{B_0}
\]

where \( ASD = d^* - d^* \) is the anti-self duality operator.

The rest of the proof deals with \( ASD + D_\gamma p^0 \). Let \( \mathcal{H}^0 \) denote the imaginary valued harmonic \( p \)-forms over \( Y \) and let \( \text{Hess}_{B_0}^f : \mathcal{H}^1 \rightarrow \mathcal{H}^1 \) be the Hessian at zero of the Morse function

\[
\alpha \in \mathcal{H}^1 \mapsto f(B_0 + \alpha) \in \mathbb{R}.
\]
The operator \( D_\gamma p^0 : L^2_k(N^*;i\Lambda^1) \to L^2_{k-1}(N^*;i\Lambda^1) \) is supported over the cylindrical end and acts on 1-forms as

\[
D_\gamma p^0(a)(t) = dt \wedge \beta(t)(\text{Hess}_{B_\infty}^f(\bar{a}_{\text{harm}}(t))) + dt \wedge \beta_0(t)D_{\gamma(t)}q^0_0(\bar{a}(t)) \tag{116}
\]

(see the discussion of perturbations in Background 9). Since \( \beta_0 \) is compactly supported and \( D_{\gamma(t)}q^0_0 \) is compact (\( q_0 \) is a tame perturbation), \( D_\gamma(\beta_0q^0_0) \) is a compact operator over the cylinder hence is not relevant for index computations. We will thus assume from now on that \( q^0_0 = 0 \).

Equation (116) tells us that the operator \( ASD + D_\gamma p^0 \) is an Atiyah-Patodi-Singer operator [APS75, Nic00], and after the usual identifications, the operator over the slice \( \{t\} \times Y \) takes the form (see for example [Nic00, pg. 312])

\[
\partial_t + \text{SIGN} + \beta(t) \text{Hess} f \circ \mathcal{P}_{\ker \Delta} \tag{117}
\]

where

\[
\text{SIGN} = \begin{bmatrix}
*d & -d \\
-d^* & 0
\end{bmatrix} : L^2_k(Y;\Lambda^1 \oplus \Lambda^0) \to L^2_{k-1}(Y;\Lambda^1 \oplus \Lambda^0) \tag{118}
\]

is the odd signature operator. The gluing formulas for the index [APS75] give:

\[
\text{ind} (ASD + D_\gamma p^0) = \text{ind} (ASD) + \dim \ker (\text{SIGN}) + \dim \left( \partial_t + \text{SIGN} + \beta(t) \text{Hess}_{B_\infty}^f \circ \mathcal{P}_{\ker \Delta} \right) \tag{119}
\]

where the last term is the index over the infinite cylinder \( \mathbb{R} \times Y \).

Clearly \( \ker (\text{SIGN}) \simeq H^0(Y) + H^1(Y) \), we claim that the last summand in (119) is equal to \( -1 - b_1(Y) \). This will imply that \( \text{ind} (ASD + D_\gamma p^0) = \text{ind} (ASD) \) and conclude the proof of the lemma. The operator \( \text{SIGN} + \beta(t) \text{Hess}_{B_\infty}^f \circ \mathcal{P}_{\ker \Delta} \) splits as

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & \beta(t) \text{Hess}_{B_\infty}^f & 0 \\
0 & 0 & \text{SIGN}'
\end{bmatrix} : \begin{array}{c}
\mathcal{H}^0 \\
\mathcal{H}^1 \\
(\ker \text{SIGN})^\perp
\end{array} \rightarrow \begin{array}{c}
\mathcal{H}^0 \\
\mathcal{H}^1 \\
(\ker \text{SIGN})^\perp
\end{array}
\]

with \( \text{SIGN}' \) being the restriction of \( \text{SIGN} \). It follows that

\[
\text{ind} \left( \partial_t + \text{SIGN} + \beta(t) \text{Hess}_{B_\infty}^f \circ \mathcal{P}_{\ker \Delta} \right) = \text{ind} \left( \partial_t + \begin{bmatrix}
0 & 0 \\
0 & \text{SIGN}'
\end{bmatrix} \right) + \text{ind} \left( \partial_t + \beta(t) \text{Hess}_{B_\infty}^f \right).
\]
where all the operators are over the infinite cylinder $\mathbb{R} \times Y$. We recall the the Atiyah-Patodi-Singer index is equal to the dimension the $L^2$-kernel minus the dimension of the $L^2_{ex}$ kernel of the adjoint, where $L^2_{ex}$ denotes the extended $L^2$ sections [APS75, Corollary 3.14]. We can easily show by separation of variables that $\ker_{L^2}(\partial_t + \begin{bmatrix} 0 & 0 \\ 0 & \text{SIGN}' \end{bmatrix}) = \{0\}$ whilst $\ker_{L^2_{ex}}(\partial_t - \begin{bmatrix} 0 & 0 \\ 0 & \text{SIGN}' \end{bmatrix})$ has dimension one, given by the constant solutions in $H^0$. Similarly $\ker_{L^2}(\partial_t + \beta(t) \text{Hess}^f_{B_{\infty}}) = \{0\}$ because $\beta(t) = 0$ for $t < 0$ forces a solution to be constant, hence it cannot be in $L^2$ unless it is trivial. On the other hand an element in $\ker_{L^2_{ex}}(\partial_t - \beta(t) \text{Hess}^f_{B_{\infty}})$ will be constant as $t \to -\infty$ and $O(e^{\lambda t})$ for $t \to +\infty$ where $\lambda$ is an eigenvalue of $\text{Hess}_f^f_{B_{\infty}}$, therefore the $L^2_{ex}$-kernel has dimension equal to the number of negative eigenvalues of the Hessian, which is $b_1(Y)$ by construction. This concludes the proof of the claim and hence the proof of Lemma 4.4.

We have shown that $\text{ind } Q = \text{ind}(APS @ D^+_{B_{\infty}})$. The APS index of such operator is (see for example [Nic00, pg. 312]) equal to

$$\frac{1}{4} (c_1^2(s) - 2\chi(N_0) - 3\sigma(N_0)) - \frac{b_1(Y) + 1}{2} - \dim(\ker_C D_{B_{\infty}}) - \eta(D_{B_{\infty}}) - \frac{1}{4} \eta(\text{SIGN})$$

The $\eta$-invariant of the Dirac operator $D_{B_{\infty}}$ and that of the signature operator vanish because we have an orientation reversing isometry of $Y$ that fixes $B_{\infty}$ given by $\rho_B$ defined in Subsection 2.5. In addition, $\ker_C D_{B_{\infty}} = 0$ because we used a metric satisfying Theorem 3.4 and $\chi(N_0) = 1 - b_1(Y), c_1^2(s) = \sigma(N_0) = 0$. Hence $\text{ind } Q = -1$. This concludes the proof of Proposition 4.2.

4.3 Pointlike moduli spaces.

Outline. We recall that we defined $[a_{\text{red}}] \in \mathbb{T}$ to be a critical point of $f_T$ of maximal index on the torus, i.e. $\text{index}_f[a_{\text{red}}] = \dim \mathbb{T} = b_1(Y).$ The aim of this subsection is to prove Proposition 4.2 below which informally says that, when suitable perturbations are used, the moduli space of monopoles over a protocork limiting to $[a_{\text{red}}]$ is pointlike (in particular is not empty) and the same applies to its reflection. We know that the moduli space, if not-empty, has dimension zero from Proposition 4.2. Lemma 4.8 will provide us with perturbations making the moduli spaces regular without perturbing the connection equation. Then we will solve the connection equation (Lemma 4.6, Lemma 4.7) showing that it has a unique solution modulo gauge equivalence. In the second part of the proof
we will deal with the spinor equation. The main difficulty here is to show that the solution space is not empty. In fact by Lemma 4.9 it is easy to show that the equation will have a solution but this is not enough as we need solutions with the correct asymptotics. Here the perturbation chosen in the previous step will play a crucial role, since will allow us to pass from a general asymptotically cylindrical operator to a cylindrical one. This will improve greatly our understanding of the behaviour of the solutions at infinity.

We recall that the subspace of perturbations $\mathcal{P}^\perp$ has been defined in Definition 3.2 and the structure of perturbations of 4D moduli spaces has been reviewed in Background 9.

**Proposition 4.5.** There is a subset of perturbations $S \subset \mathcal{P}^\perp$, residual in $\mathcal{P}^\perp$, such that for any $\mathbf{q}_0 \in S$, the perturbation $\mathbf{p} = \beta_0(t)\mathbf{q}_0 + \beta(t)\mathbf{q}$ is such that the moduli space $M(N^*; [\mathbf{a}_{\text{red},-1}^{\text{red}}])$ is regular and consists of a single point, in particular is not empty. Moreover, if $\Phi : N \to P_0(\Gamma, \mathcal{R})$ is a diffeomorphism of $N$ with a realization of a protocork, then the same statement above holds for the moduli space over the reflection of the protocork $M(P_1(\Gamma, \mathcal{R})^*; [\mathbf{a}_{\text{red},-1}^{\text{red}}])$ where the identification $Y \simeq \partial P_1(\Gamma, \mathcal{R})$ is given by $\Phi|_Y$.

**Proof.** Let $(B_{\infty}, 0, \psi_{-1}) \in \mathcal{G}_k^\sigma(Y)$ be a smooth representative of $[\mathbf{a}_{\text{red},-1}^{\text{red}}]$. Since by Proposition 2.4 $H^1(N) \to H^1(Y)$ is an isomorphism we can also assume without loss of generality that exists a flat connection $A_0 \in \mathcal{A}(N^*)$ that restricts over the end to a translation invariant connection equal to $B_{\infty}$ slicewise. $A_0$ is unique up to gauge-transformation. Since $[\mathbf{a}_{\text{red},-1}^{\text{red}}]$ is a boundary unstable critical point $M(N_0^*; [\mathbf{a}_{\text{red},-1}^{\text{red}}])$ consists entirely of reducible monopoles. We can therefore apply Lemma 4.1, which tells us that the elements of the moduli space can be represented by pairs $(A, \Phi) \in \left(\tilde{A}_0 + L_{k,\delta}^2(N^*; i\Lambda^1 N^*)\right) \times X_{k,\lambda_{-1,0}}^+ (\psi_{-1})$ for some $\delta > 0$. Firstly we focus on the first equation of (108). Making apparent the perturbative terms this equation is:

$$
(F_{A_0^+}^+ + d^+a) + \beta_0 \hat{q}_0^0(A_0 + a, 0) + \beta \hat{q}_0^0(A_0 + a, 0) = 0 \tag{120}
$$

where $a := A - A_0 \in L_{k,\delta}^2(N^*, i\Lambda^1 N^*)$.

Now we invoke Lemma 4.6 below, that tells us that we can ignore the perturbation $\beta \mathbf{q}_0^0$ so that the equation becomes

$$
(F_{A_0^+}^+ + d^+a) + \beta_0 \hat{q}_0^0(A_0 + a) = 0. \tag{121}
$$
This is false in general and relies on our assumption that \([a_{N\text{rea}}] \in \mathbb{T}\) is a critical point of \(f_\|\) of maximal index.

**Lemma 4.6 (The Morselike perturbation can be omitted).** Regardless of \(q_0\), the moduli space of solutions to (120) is equal to the moduli space of \(L^2_{k,\delta}\)-solutions to

\[
(F^{+}_{A_0} + d^*a) + \beta_0 \, \hat{q}_0 (A_0 + a) = 0
\]

(122)

**Proof.** Recall that \(q^0\) is the formal gradient of the Morselike perturbation \(f : C(Y) \to \mathbb{R}\). Denote by \(\mathcal{H}^1 \subset L^2(Y; i\Lambda^1 Y)\) the space of imaginary valued 1-forms. Define \(V : \mathcal{H}^1 \to \mathcal{H}^1\) to be the \(L^2\)-gradient of the function

\[
b \mapsto f(B_\infty + b, 0) : \mathcal{H}^1 \to \mathbb{R}.
\]

(123)

Then since Morselike perturbations are pulled back from the torus of flat connections we have that

\[
q^0(B_\infty + b) = V(b_{\text{harm}}) \in \mathcal{H}^1
\]

(124)

for any \(b \in L^2(Y; i\Lambda^1 Y)\), here \(b_{\text{harm}} = \mathbb{P}_{\ker \Delta}(b)\) is the \(L^2\)-projection to \(\mathcal{H}^1\). Then over the end \(Z := [-C/2, +\infty) \times Y\) with the usual identifications in place [KM07, Section 4.3], (120) takes the form

\[
\partial_t \hat{a}(t) = - * d(\hat{a}(t)) + dc(t) - \beta(t) V(\hat{a}_{\text{harm}}(t)).
\]

(125)

where \(a(t) = \hat{a}(t) + c(t) dt\), \(\hat{a}(t) \in L^2_{k,\delta}(Y; i\Lambda Y)\); here we exploited that \(A_0\) is flat and \(\beta_0([-C/2, +\infty)) = \{0\}\). With respect to the Hodge decomposition \(L^2(Y; i\mathbb{R}) \cong Im \, d \oplus Im \, d^* \oplus \mathcal{H}^1\), the equation splits as :

\[
\partial_t \hat{a}_{\text{harm}}(t) = -\beta(t) V(\hat{a}_{\text{harm}}(t))
\]

\[
\partial_t \hat{a}^+(t) = - * d(\hat{a}^+(t)) + dc(t)
\]

(126)

where \(a^+(t)\) is the component of \(\hat{a}(t)\) in \(Im \, d \oplus Im \, d^*\). This implies that \(\hat{a}_{\text{harm}}(t) \equiv 0\) for any \(t \geq -C/2\), because otherwise \(t \mapsto [B_\infty + \hat{a}_{\text{harm}}(t)] \in \mathbb{T}\) would be a non trivial trajectory of \(- \text{grad} \, f_\|\) converging to \([B_\infty]\) which cannot be because \([B_\infty]\) is a maximum of the Morse function \(f_\|\) by assumption. It follows that solutions to (120) satisfy (122). Vice versa, the harmonic part \(\hat{a}_{\text{harm}}(t)\) of a solution to (122) must be constant since it satisfies (126) with \(\beta(t) = 0\) for any \(t\), however such a connection cannot be in \(L^2_k\) unless \(\hat{a}_{\text{harm}}(t) = 0\) for all \(t \geq -C/2\). The proof that the Morselike perturbation can be omitted is concluded. \(\square\)
The next lemma, Lemma 4.7 shows that if \( q_0 = 0 \) then there is a unique solution to (121) modulo gauge, given by \( a = 0 \). This is enough, in fact Lemma 4.8 below ensures the existence of a residual set of perturbations \( q_0 \in \mathcal{P}^\perp \) (so that \( q_0 = 0 \)) that makes the moduli spaces regular.

**Lemma 4.7 (Unique solution for connection eq.)**. If \( q_0 = 0 \) there is a unique solution to (121) modulo gauge.

**Proof.** When \( q_0 = 0 \), then (122) is the ASD equation over \( N^* \). First of all we show that \( A = A_0 + a \) must be flat. Indeed \( A' \in L^2 \) limits exponentially to flat connections over the end therefore its curvature \( F_{A'} \in L^2_k \). Since \( F_{A'} \) is anti-self-dual and closed it is also harmonic; the space of harmonic self dual forms has dimension \( b^-(N) = 0 \) [APS75], hence \( F_{A'} = 0 \) as we wanted. Since \( H^1(N) \rightarrow H^1(Y) \) is an isomorphism, we have a unique gauge-equivalence class of flat connections extending \( B_\infty \) (given by \( A_0 \)). \( \square \)

**Lemma 4.8 (Perturbations).** The set of perturbations \( q_0 \in \mathcal{P}^\perp \) such that the moduli spaces \( \mathcal{M}^{\text{red}}(N^*; [a_{N\text{red},j}]) \) are regular \( \forall j \in \mathbb{Z} \) is residual in \( \mathcal{P}^\perp \).

Essentially this lemma tells us that in our specific case, in particular the moduli space in the blow-down is regular, we can improve the transversality result [KM07, Section 24.4.7] from \( q_0 \in \mathcal{P} \) to \( q_0 \in \mathcal{P}^\perp \). Notice that the lemma is only about reducible monopoles. The proof will be a modification of [KM07, Proposition 24.4.7], notice however that our hypothesis are not strong enough to prove a generalization of the density result [KM07, Lemma 24.4.8], we can though use some ideas used to prove [KM07, Lemma 24.4.8] to show a partial result on the spinor component and then use our specific hypothesis to deal with the connection component.

**Proof.** Let \( \mathfrak{M}^{\text{red}}(N) \subset \partial \mathcal{B}^{\sigma}_{k-\frac{1}{2}}(N) \times \mathcal{P}^\perp \) be the parametrized moduli space, i.e. the quotient of the zero locus of \( \mathfrak{M} = (y, q_0) \mapsto \mathcal{F}_{q_0}(y) \) by the action of the gauge group where with an abuse of language we wrote \( \mathcal{F}_{q_0} \) to denote the Seiberg-Witten map with perturbation \( \beta q_0 + \beta q \). Notice that this is a moduli space over the compact \( N \) not \( N^* \). Put \( Z = [0, +\infty) \times Y \), and let \( R_+: \mathfrak{M}^{\text{red}}(N) \rightarrow \partial \mathcal{B}^{\sigma}_{k-\frac{1}{2}}(Y) \) and \( R_-: M^{\text{red}}(Z; [a_{N\text{red},j}]) \rightarrow \partial \mathcal{B}^{\sigma}_{k-\frac{1}{2}}(Y) \) denote the restriction to the boundary \( \{0\} \times Y \). We have to show that the fiber product is regular, this is equivalent to showing that for each element \( ([\alpha], q_0), [\beta]) \) in the fiber product, the map

\[
D_{([\alpha], q_0)}R_+ - D_{[\beta]}R_-, \quad T_{([\alpha], q_0)} \mathfrak{M}^{\text{red}}(N) \oplus T_{[\beta]} M^{\text{red}}(Z; [a_{N\text{red},j}]) \rightarrow T_{([\alpha], q_0)} \mathcal{B}^{\sigma}_{k-\frac{1}{2}}(Y)
\]
is surjective.

Note that we already know that this map is Fredholm by [KM07, Lemma 24.4.1], moreover using the regularity result [KM07, Lemma 17.2.9] we reduce to showing surjectivity in the case when \( k = 1 \). Notice also that we can assume that, possibly after gauge transformation, \( \alpha|_{\partial N} = \beta|_{\partial Z} \) so that they glue to a configuration over \( N^* \), this follows from [KM07, Lemma 24.2.2].

Let \( \mathcal{Q} = D_{(\alpha,a_0)} \mathcal{W}^d \oplus d_{\alpha}^\dagger : (\mathcal{T}_1^\alpha(N))^d \oplus T_{a_0} \mathcal{R}^\perp \to L^d_0 \) be the linearization of the Seiberg-Witten map with an added gauge-fixing condition, and similarly let \( \mathcal{Q}^\tau = D_{\beta} \mathcal{T}_2^\tau \oplus d_{\beta}^\dagger : (\mathcal{T}_1^\tau(Z))^d \to L^d_0 \). The kernel of these maps is isomorphic to \( T^\perp_1 \mathcal{W}^e(N) \) and \( T^\perp_1 M^e(Z; [a_{N^*}, j]) \) respectively, moreover both are surjective operators because \( N \) and \( Z \) have non-empty boundary and their formal adjoint enjoys the unique continuation property [KM07, Proposition 14.1.5]. Under the usual identifications, we see that the surjectivity of (127) is equivalent to showing that the range of

\[
(\mathcal{Q}, r_+) - (\mathcal{Q}^\tau, r_-) : \\
\mathcal{R}^\perp \oplus L^2_1(Z; i T^\tau Z) \oplus \mathcal{R}^\perp \oplus L^2_1(Y; i T^\tau Y) \to L^2(N; i T^* N \oplus S^+) \oplus L^2(N; i T^* Y \oplus S^+ \oplus i \mathcal{R})
\]

contains all elements of the form

\[
W = (\omega_N, \psi_N, 0) \oplus (\omega_Z, \psi_Z, 0) \oplus (v, \psi, 0).
\]

In the above definition \( r_+, r_- \) are the differentials of the restriction map.

Suppose by contradiction that such a \( W \) is \( L^2 \)-orthogonal to the range of \( (\mathcal{Q}, r_+) - (\mathcal{Q}^\tau, r_-) \). By considering variations supported away from the boundary that leave the perturbation component untouched it can be shown, as in the proof of [KM07, Lemma 24.4.8], that \( W_N|_{\partial N} = -w = W_Z|_{\partial Z} \). Since \( \mathcal{Q}^* W_N = 0 \), \( (\mathcal{Q}^\tau)^* W_Z = 0 \), we have that \( W_N, W_Z \in L^2_1 \). Hence \( W_N \) and \( W_Z \) can be glued to obtain an \( L^2 \) configuration \( W_N \# W_Q \) over \( N^* \) because their boundary values coincide.

By considering gauge-orbit directions, we see that \( w \) is orthogonal to the gauge orbits and, proceeding as in the proof of [KM07, Lemma 15.1.4], we obtain that \( W_N|_{\partial N} = -w \) implies that also \( \dot{W}_N(t) \) is orthogonal to the orbit for each \( t \) in the collar and never vanishing unless \( W = 0 \).
We are now in the position to run the same argument in [KM07, Section 15.1.4] and showing that, if \( \psi_N \neq 0 \) then we can find an element \( q_0 \in \mathcal{P}^\perp \) (the dot here is just to remind us that is a variation) such that

\[
\langle \beta_0 q_0(\alpha), \psi_N \rangle_{L^2(N)} > 0.
\]  

(130)

Indeed, the cylinder function constructed at [KM07, pg. 271] is constant on the reducibles hence its gradient is in \( \mathcal{P}^\perp \). This however contradicts the assumption that \( W \) is orthogonal to the range of \( (\Xi, r_+) \) as can be seen by using directions tangent to \( \mathcal{P}^\perp \). Therefore \( \psi_N = 0 \). And consequently, by the unique continuation property, \( \psi_N \# \psi_Z = 0 \) hence \( \psi_Z = 0 \).

Notice that up to this point we did not use our specific hypothesis. We will use it to show that \( \omega_N \# \omega_Z = 0 \). Consider variations \( (\dot{a}_N, \dot{a}_Z) \in L^2_1(N; IT^*N) \oplus L^2_1(Z; IT^*Z) \) such that \( \dot{a}_N \mid \omega_N = \dot{a}_Z \mid \omega_Z \) (the dot here is just our notation to keep track of the variation). In this case we can glue the forms to get \( \dot{a}_N \# \dot{a}_Z \in L^2_1(N^*; T^*N^*) \) and \( r_+ \dot{a}_N = r_- \dot{a}_Z \). Now the orthogonality relation can be rewritten as

\[
0 = \langle \Xi(\dot{a}_N), \omega_N \rangle_{L^2(N)} + \langle \Omega^\mathcal{F}(\dot{a}_Z), \omega_Z \rangle_{L^2(Z)} = \langle (d^* + \beta D_{A_0} q^0)(\dot{a}_N \# \dot{a}_Z), \omega_N \# \omega_Z \rangle_{L^2(N^*)},
\]

(131)

in fact \( p^0 = \beta_0 q_0^0 + \beta q^0 = \beta q^0 \) over the reducibles because \( q_0 \in \mathcal{P}^\perp \).

We claim that (131) cannot hold for all \( \dot{a}_N \# \dot{a}_Z \) unless \( \omega_N \# \omega_Z \) is zero. First of all, notice that the blown-down moduli space of solutions to

\[
F_{A_0 + a}^+ + \beta \dot{q}^0(a) = 0
\]

(132)

limiting to \( [a_{\text{red}}] = \pi[a_{\text{red}, j}] \) modulo gauge equivalence is regular. Indeed the deformation complex

\[
L^2_{2,ex}(N^*, i\mathbb{R}) \xrightarrow{d} L^2_1(N^*; iA^1) \xrightarrow{d^* + \beta D_{A_0} q^0} L^2(N^*; iA^+)
\]

(133)

has index equal to \(- \text{ind}(\text{ASD} \oplus \beta D_{A_0} q^0) = - \text{ind}(\text{ASD})\) by the same argument used in the proof of Lemma 4.4 and the latter is equal to

\[
- (b^3(N_0) + b^+(N_0) + \dim(\text{Im}(H^2(N_0) \to H^2(Y))) + 1 = 1.
\]

(134)

On the other hand, denoting by \( H^0_{\text{ex}}, H^1, H^2 \) the cohomology groups of the deformation complex, we have that \( H^0_{\text{ex}} \cong \mathbb{R} \) due to constant functions and \( H^1 \cong (0) \).
because by Lemma 4.7 the moduli space is a point when $q_0^0 = 0$. Consequently $H^2 = (0)$ as well. This shows that the second map in the deformation complex is surjective, but this is precisely the map in the second term in (131), thus $\omega_N\#\omega_Z = 0$ and the proof of Lemma 4.8 is concluded.

Thanks to Lemma 4.8, we can assume for the rest of the proof that we are working with a perturbation of the form $p = \beta_0 q_0 + \beta q$ where $q_0^0 = 0$ such that the moduli spaces $M(N^*; [a_{i\text{red}}, j])$ are regular $\forall j \in \mathbb{Z}$. Let $A_0$ denote the unique solution (modulo gauge equivalence) to (120), we can assume that $A_0$ is constant over the end, and equal to $B_1$. In order to conclude the proof of Theorem 4.5, we need to show that the second equation of (108), i.e. the one involving the spinor, has a unique solution. This is not obvious mainly because the moduli space can be both regular and empty.

The second equation of (108) reads

$$D_{A_0}^+ \Phi + \beta_0 D_{(A_0,0)}^1 q_0^1(\Phi) + \beta D_{(A_0,0)} q_1(\Phi) = 0 \quad (135)$$

where $\Phi \in X_{k,\lambda-1,\delta}^+ (\psi_{-1})$. Set $\mu := \lambda_{-1} - \epsilon$, where $\epsilon > 0$ is so small that $\lambda_{-1} - \epsilon > \lambda_{-2}$. The space of $L^2_{k,\mu}$-solutions to (135) contains the $X_{k,\lambda-1,\delta}^+ (\psi_{-1})$-solutions because $\psi_{-1}$ is smooth, hence $X_{k,\lambda-1,\delta}^+ (\psi_{-1}) \subset e^{-(\lambda_{-1} - \epsilon)} L^2_{k,\delta} \subset L^2_{k,\mu}$. As a first step, we will show that the space of $L^2_{k,\mu}$-solutions to (135) is not empty. This follows from the fact that the $L^2_{k,\mu}$-solutions to the operator in (135) is positive, hence the kernel must be nontrivial.

**Lemma 4.9.** The (real) index of the operator in (135) as an operator $L^2_{k,\mu} (N^*, S^+) \to L^2_{k-1,\mu} (N^*, S^-)$ is equal to 2.

**Proof.** Let $W_{\mu} : N^* \to \mathbb{R}$ be a smooth weight function equal to 1 on $N$ and equal to $e^{-2\mu(\cdot)}$ on the infinite cylinder. First of all, notice that modulo compact terms and continuous perturbations in the space of Fredholm operators $L^2_{k,\delta} \to L^2_{k-1,\delta}$, the operator of (135) becomes $D_{A_0}^+ + \beta D_{(B_0,0)} q_1^1$. Multiplication by $W_{\mu}^{-1}$ gives an isometry $L^2_k \to L^2_{k,\mu}$ under which the operator $D_{A_0}^+ + \beta D_{(B_0,0)} q_1^1 : L^2_{k,\mu} \to L^2_{k-1,\mu}$ is conjugated to $D_{A_0}^+ + \beta D_{(B_0,0)} q_1^1 + \beta(t-2)(-\lambda_{-1} + \epsilon) : L^2_k \to L^2_{k-1}$ modulo a compact operator due to the behaviour of $W_{\mu}^{-1}$ on the complement of the cylinder.
where we have used the usual gluing formulas for the index [APS75], and
\( sf\{A_t\}_t \) denotes the spectral flow of the family of operators \( \{A_t\}_t \) (for a definition see [KM07, pg. 244] or [Nic00, Section 4.1.3]). Now, \( \text{ind}_{L^2} (D^+_{A_0}) = 0 \) because \( D_{B_\infty} \) is injective and \( \eta(D_{B_\infty}) = 0 \) due to the orientation reversing isometry \( \rho_B \) of \( Y \) (see Subsection 2.5). \( sf\{D_{B_\infty} + \beta(t)D_{(B_\infty,0)}q^1\}_t = 0 \) because we can choose a priori a perturbation so small that the operator is injective for any \( t \), therefore we have no crossings. We claim that \( sf\{D_{B_\infty} + D_{(B_\infty,0)}q^1 + \beta(t)(-\lambda_1 + \epsilon)\}_t = 2 \).
Indeed we only have one crossing due to the flow of the eigenvalue \( \lambda_1 \), which goes from negative \( (\lambda_1 < 0) \) to positive \( (\epsilon > 0) \), thus the crossing has positive sign and the spectral flow equals the real dimension of the \( \lambda_1 \)-eigenspace which is 2 since the spectrum is simple over \( \mathbb{C} \).

This shows that the space of \( L^2_{k,\mu} \)-solutions to (4.2) is not empty. The second step consists in showing that up to multiplication by an element in \( \mathbb{C}^\times \), these solutions lie in \( X^+_{k,\lambda_1,\delta}(\psi_1) \). Observe that over \([0,+\infty) \times Y\), (135) takes the form

\[
\partial_t \Phi(t) = -D_{B_\infty,q} \Phi(t)
\]

where the right hand side is defined by (50). Consequently, by separation of variables, a general \( L^2_{k,\mu,\text{loc}} \)-solution of (136) takes the form of a linear combination

\[
\Phi(t) = \sum_{i \in \mathbb{Z}} c_i e^{-\lambda_i t} \psi_i
\]

where \( \psi_i \) is a \( \lambda_i \)-eigenvector of \( D_{B_\infty,q} \), \( c_i \in \mathbb{C} \). Notice that here we used that, thanks to our earlier discussion, the connection is translation invariant over the tube. Let \( \Phi \neq 0 \) be a solution given by step one above. Since \( \Phi \in L^2_{k,\mu} \) only those \( i \) with \( \lambda_i > \mu \) can appear in the expansion (137), thus since \( \mu = \lambda_{-1} - \epsilon \) and \( \mu > \lambda_{-2} \) we obtain that

\[
\Phi(t) = \sum_{i \geq -1} c_i e^{-\lambda_i t} \psi_i.
\]

Notice that if \( c_{-1} \neq 0 \) then \( \frac{1}{c_{-1}} \Phi \in X^+_{k,\lambda_{-1},\delta}(\psi_{-1}) \). We will show that \( c_{-1} = 0 \) is impossible. Indeed, suppose by contradiction that \( c_{-1} = 0 \), then \( \Phi(t) \in L^2_{k,\mu} \) and
thus by Lemma 4.1 we obtain an element of $M^\text{red}(N^*, [a_{N^*}])$, $j \geq 0$. This would show that $M^\text{red}(N^*; [a_{N^*}]) \neq \emptyset$ for some $j \geq 0$, but this is not possible because $M^\text{red}(N^*; [a_{N^*}])$ is a regular moduli space, thanks to our assumption on $p$, with a negative formal dimension thanks to Proposition 4.2.

Consequently, rescaling by $\mathbb{C}$ the vector space of solutions, isomorphic to $\mathbb{C}$, guaranteed by the index theorem, we obtain one solution in $X^+_{k, \lambda-1, \delta}(\psi_{-1})$. Notice that we cannot have more solutions because solutions to the spinor equation form a vector space, therefore they would make the dimension of the moduli space higher than the formal dimension which is zero. This shows that the moduli space is not empty and is a point. The part of the thesis about the reflection $P_1(\Gamma, \mathcal{R})$ follows by the same argument used for $N$ as all the key properties of $P_0(\Gamma, \mathcal{R})$ are satisfied also by its reflection. This concludes the proof of Proposition 4.5.

\[ \square \]

5 On the variation of the Seiberg-Witten invariants.

Outline. The aim of this section is to give a proof of the main theorem, Theorem 1.1. The proof of items (a) and (c) rely on the study of some cobordisms maps and, in the case of (a), on Theorem 1.4, used in the form of Proposition 3.10. In particular the proofs of part (a) and (c) do not require the results of Section 4 which we use only to prove (b). More precisely we will need Proposition 4.2 to compute the dimension of some moduli spaces. Although proposition 4.5, about pointlike moduli spaces will not be used in the proof we mention that it can be used to given an alternative proof of part (a) which does not use the cobordism $C$ (see below). We point out that (c) is a non-trivial statement due to the fact that even though we have a well defined notion of composition of homology orientations, in practice is hard to compute the composite orientation from the definition. The idea of our proof is to reduce to the case of an isomorphism between cobordisms, where the variation of the homology orientation is easily computed from the action on homology.

Setting. In this section we adopt the same notation of Subsection 3.4, in particular $N_0 = P_0(\Gamma, \mathcal{R})$ is a protocork, $N_1 = P_1(\Gamma, \mathcal{R})$ its reflection, $Y = \partial N_0 = \partial N_1$ is their common boundary and $b_1 = b_1(Y)$. For each $i = 0, 1$, we write $N_i \setminus \mathbb{B}^4$ to denote the manifold $N_i$ minus a ball removed from its interior, and we regard it as a cobordism $N_i \setminus \mathbb{B}^4 : S^3 \to Y$. We recall that an homology orientation [KM07,
Definition 3.4.1] for a cobordism $C : \partial_- C \to \partial_+ C$ is an orientation of
\[
\Lambda^\text{max} H^1(C; \mathbb{R}) \otimes \Lambda^\text{max} I^+(C) \otimes \Lambda^\text{max} H^1(\partial_+ C; \mathbb{R})
\]
where $I^+(C)$ is a chosen maximal non-negative subspace for the intersection form on $I^2(C) := \ker \{ H^2(C; \mathbb{R}) \to H^2(\partial C; \mathbb{R}) \}$. We recall that the homology orientation is part of the data necessary to define the cobordism maps induced by a cobordism. Choose $\mu_0$, an homology orientation for $N_0 \setminus \mathbb{B}^4$.

**Proof of Theorem 1.1 (a).** By Proposition 3.10, we have a splitting $\widehat{HM}_\bullet(Y) \simeq \hat{G} \oplus H M^\text{red} (Y)$ where $\hat{G} = \text{Im} (\widehat{HM}_\bullet(W))$, and $W : b_1S^1 \times S^2 \to Y$ has been defined in Subsection 3.4. From the construction of $W$, it is clear that $N_0 \setminus \mathbb{B}^4 = W \circ F$ where $F : S^3 \to b_1S^1 \times S^2$ is the cobordism obtained by removing a ball from $b_{i=1}^{b_1} S^1 \times D^3$

Choosing homology orientations on $W$ and $F$ so that their composition agrees with $\mu_0$, we see that
\[
x_0 = \widehat{HM}_\bullet(N_0 \setminus \mathbb{B}^4)(\hat{1}) = \widehat{HM}_\bullet(W) \circ \widehat{HM}_\bullet(F)(\hat{1}),
\]
therefore $x_0 \in \hat{G}$.

We recall that since we are dealing with torsion spin$^c$ structures, the Floer homologies of $S^3, b_1S^1 \times S^2$ and $Y$ are endowed with an absolute $\mathbb{Q}$-grading [KM07, Section 28.3] such that multiplication by $U$ is an endomorphism of degree $-2$. The generator $\hat{1} \in \widehat{HM}_\bullet(S^3) \simeq \mathbb{Z}[[U]]$ has degree $\text{gr}^\mathbb{Q}(\hat{1}) = -1$. And, as $\mathbb{Q}$-graded $\mathbb{Z}[[U]]$-modules,
\[
\widehat{HM}_\bullet(b_1S^1 \times S^2) \simeq \Lambda^* (\mathbb{Z}^{b_1}) \otimes_{\mathbb{Z}} \mathbb{Z}[[U]],
\]
\[
\text{gr}^\mathbb{Q} \left((e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes 1\right) = (k - 1) - b_1,
\]
where $\{e_i\}_{i=1}^{b_1}$ denotes the standard basis of $\mathbb{Z}^{b_1}$. This follows from the discussion of [KM07, Chapter 36], alternatively we can use computations analogous to those of Proposition 4.2.

Now, $\widehat{HM}_\bullet(F)$ is a homomorphism of degree [KM07, Eq. (28.3)]
\[
\frac{1}{4} \left( \begin{array}{cc}
-2 & \chi(F) - 3 \sigma(F)
\end{array} \right) - \frac{1}{2} \left( \begin{array}{cc}
b_1(b_1S^1 \times S^2) - b_1(S^3)
\end{array} \right) = 0.
\]

Thus, $\widehat{HM}_\bullet(F)(\hat{1})$ belongs to the $\mathbb{Z}$-submodule generated by $(e_1 \wedge \cdots \wedge e_{b_1}) \otimes 1$. Moreover, since also $\widehat{HM}_\bullet(W)$ has degree zero (see Subsection 3.4), the elements of $\hat{G}$ of degree $\text{gr}^\mathbb{Q}(x_0) = -1$ constitute a free $\mathbb{Z}$-module of rank 1.
We claim that this $\mathbb{Z}$-module is generated by $x_0$. To see this, we consider
the cobordism $C: Y \to \mathbb{S}^3$ that, with respect to the surgery presentation of $Y$
described in Subsection 2.3, is obtained by adding $b_1(Y)$ 0-framed 2-handles to
$Y \times \{1\} \subset Y \times [0,1]$ meridional to the dotted circles generating $H_1(Y)$. Then, from
the Kirby diagram described in Subsection 2.3 it is clear that in the composition
$C \circ (N_0 \setminus \mathbb{B}^4)$ is a trivial cobordism because the 2-handles coming from $C$
cancel $b_1(Y)$ 1-handles of $N_0 \setminus \mathbb{B}^4$ and, at this point, the remaining 1-handles
can be cancelled in pairs with the 2-handles. Consequently, for some homology
orientation of $C, \mu_C,$
\[
\overline{HM}_\bullet(C, \mu_C)(x_0) = \overline{HM}_\bullet(C \circ (N_0 \setminus \mathbb{B}^4))(\hat{1}) = \hat{1} \in H_{\overline{HM}}(\mathbb{S}^3) \tag{143}
\]
which proves the claim because $\hat{1}$ is a generator of the $\mathbb{Z}$-submodule of elements
of its degree.

By looking at the Kirby diagram for $N_1$ given in Subsection 2.3, we see that
also the composition $C \circ (N_1 \setminus \mathbb{B}^4)$ is a trivial cobordism hence $\overline{HM}_\bullet(C, \mu_C)(x_1) = ± \hat{1}$. Let $\mu_1$ be the homology orientation for $N_1 \setminus \mathbb{B}^4$ such that $\overline{HM}_\bullet(C, \mu_C)(x_1) = \hat{1}$. Now, with respect to the splitting $\overline{HM}(Y) \simeq \hat{G} \oplus H_{\text{red}}(Y), x_1$ splits as
$x_1 =: (g, q) \in \hat{G} \oplus H_{\text{red}}(Y)$. Since $\overline{HM}_\bullet(C, \mu_0)$ maps $x_1$ and $x_0$ to the same
element, $\text{gr}^\mathbb{Q}(x_1) = \text{gr}^\mathbb{Q}(x_0)$; consequently $g = kx_0$ for some $k \in \mathbb{Z}$. Moreover $H_{\text{red}}(\mathbb{S}^3) = (0)$ implies that $\overline{HM}_\bullet(C, \mu_C)(q) = 0$, hence we obtain that, with the homology orientation $\mu_C,$
\[
\overline{HM}_\bullet(C)(x_1) = \overline{HM}_\bullet(C)(g, q) = \overline{HM}_\bullet(C)(g, 0) = \overline{HM}_\bullet(C)(kx_0) = k\overline{HM}_\bullet(x_0) = k\hat{1},
\]
and consequently $k = 1$. Hence $\Delta = x_0 - x_1 = q \in H_{\text{red}}(Y)$ and item (a) of
Theorem 1.1 is proved. \hfill \Box

Proof of Theorem 1.1 (b). In this proof the geometric setting over $Y$, and the $N_i$s
is that used in Subsection 4.3 and Subsection 4.2. In particular the generators
of the chain complex $\hat{C}$ (see Background 6) are those described in Subsection 3.3
and we can express the Morgan-Szabó number as
\[
n_{MS} = 2 + \max_{[b_i], [b_j]} \text{gr}([b_i], [b_j]) \tag{144}
\]
where $[b_i], [b_j] \in \mathbb{B}^\mathbb{C}_k(Y, s)$ vary on irreducible critical points of the perturbed
Chern-Simons-Dirac functional. Beware that the parenthesis of $[b_i]$ are due to
gauge equivalence class and not to the Floer homology class (which may be even
undefined since we do not know whether $[b_i]$ is a cycle).
We want to show that $U^{n_{MS}/2} \Delta = 0$. Proposition 4.2, implies that $\text{gr}^Q(\Delta) = \text{gr}^Q([a_{N_{\text{red},-1}}])$ because $HM_* (N_i \setminus \mathbb{B}^4)(\hat{1})$ considers zero-dimensional moduli spaces. Furthermore, $[a_{N_{\text{red},-1}}]$ is the only boundary-unstable critical point of that degree. On the other hand, $\Delta \in HM^{\text{red}}(Y)$ and Lemma 3.7 imply that $\text{proj} (\Delta) = 0$, hence

$$\Delta = \left[ \sum_{i=1}^{N_{\text{irr}}} c_i [b_i] \right] \in \widetilde{HM}_*(Y)$$  \hspace{1cm} (145)

is represented by an irreducible cycle for some coefficients $c_i \in \mathbb{Z}$, $i = 1, \ldots, N_{\text{irr}}$. Most importantly, we can assume that not all of the $c_i$s are zero for otherwise $\Delta = 0$ and the statement would be trivial.

Now, the definition of $n_{MS}$ implies that

$$U^{n_{MS}/2} \left[ \sum_{i=1}^{N_{\text{irr}}} c_i [b_i] \right] = [r] \in \widetilde{HM}_*(Y)$$  \hspace{1cm} (146)

for some $r \in C^u$ cycle of reducibles. Indeed, let $i_0$ be an index such that $c_{i_0} \neq 0$, and suppose that $U^{n_{MS}/2} \left[ \sum_{i=1}^{N_{\text{irr}}} c_i [b_i] \right]$ has a representative with a non zero component along an irreducibles $[b_j]$, then $\text{gr}([b_{i_0}], [b_j]) = n_{MS}$ which is impossible.

Since $\Delta \in HM^{\text{red}}(Y)$, also $U^{n_{MS}/2} \Delta \in HM^{\text{red}}(Y)$, therefore $p_* [r] = 0 \in \widetilde{HM}_*(Y)$. This tells us that there exists $(\eta^s, \eta^u) \in \hat{C}$ such that $\partial (\eta^s, \eta^u) = p(r) = (\partial^u_0 r, r)$. From which we obtain, thanks to $\partial^u_0 = 0$ (see Proposition 3.5), that $\partial^u_0 \eta^u = r$. This implies that $[r] = 0$ in $\widetilde{HM}_*(Y)$ because

$$\partial^u \eta^u = \partial^u_0 \eta^u - \partial^u_0 \eta^u = -r,$$  \hspace{1cm} (147)

where $\partial^u_0 \eta^u = 0$ for the definition of $n_{MS}$ as above. Consequently $[r] = 0$. \hfill \Box

**Proof of Theorem 1.1 (c).** For the rest of this section we suppose that $N_0$ is symmetric. The map $r$ has been defined in Subsection 2.5. The proof of the statement (c) is based on the following lemma:

**Lemma 5.1.** There exists a cobordism $T : Y \to \mathbb{S}^3$, such that

(a) $\tau : Y \to Y$ extends to an orientation preserving self-diffeomorphism of $T$, $f : T \to T$ which restricts to the identity on $\mathbb{S}^3$,

(b) $T \circ (N_0 \setminus \mathbb{B}^4) : \mathbb{S}^3 \to \mathbb{S}^3$ is the trivial cobordism,

(c) $H^1(T) = 0$, $\text{ker}(H^2(T, \mathbb{R}) \to H^2(\partial T, \mathbb{R})) = 0$. 

We postpone the proof of this lemma to the next paragraph and we continue the proof. Since \( x_0 = \overline{HM} \cdot (N_0 \setminus \mathbb{B}^4, \mu_0)(\hat{1}) \), the second item of Lemma 5.1 implies that \( \overline{HM} \cdot (T, \mu_T)(x_0) = \hat{1} \) for some homology orientation \( \mu_T \). Denote by \((N_0 \setminus \mathbb{B}^4)_\tau : \mathbb{S}^3 \to Y \) the cobordism obtained from \( N_0 \setminus \mathbb{B}^4 \) by replacing the identification map of the outcoming boundary with \( \tau \). Clearly, from the definition of \( \tau \) and \( \hat{\tau} \) in Subsection 2.5 it follows that

\[
x_1 = \overline{HM} \cdot (N_1, \hat{\tau}_* \mu_0)(\hat{1}) = \overline{HM} \cdot ((N_0 \setminus \mathbb{B}^4)_\tau, \mu_0)(\hat{1}).
\]

(148)

On the other hand, the first item of Lemma 5.1 tells us that

\[
(T, \mu_T) \circ ((N_0 \setminus \mathbb{B}^4)_\tau, \mu_0) = (T, f_* \mu_T) \circ (N_0 \setminus \mathbb{B}^4, \mu_0),
\]

(149)

which implies that \( \overline{HM} \cdot (T, \mu_T)(x_1) = \pm \hat{1} \); to conclude the proof we have to show that this is equal to \( +\hat{1} \).

The third item of Lemma 5.1 implies that for \( T \) the vector space (139) is zero dimensional hence \( f \) preserves the homology orientation, i.e. \( f_*(\mu_T) = \mu_T \). Consequently

\[
\hat{1} = \overline{HM} \cdot (T, \mu_T)(x_i)
\]

(150)

for any \( i \in \{0, 1\} \) which concludes the proof.

We are left to prove Lemma 5.1.

**Proof of Lemma 5.1.** Let \( \Gamma \) be a symmetric protocork plumbing graph. We will construct a Kirby diagram for \( P_0(\Gamma) \) and closed curves \( \gamma_1, \ldots, \gamma_{b_1(Y)} \) in its boundary such that \( \gamma_i \) intersects the belt sphere of the \( i \)-th 1-handle of \( P_0(\Gamma) \) exactly once and \( \tau \) (for some realization \( \hat{R} \)) preserves the support of \( \{\gamma_i\}_i \). At this point \( T \) will be constructed by gluing 2-handles to \( Y \times \{1\} \subset Y \times I \) along the curves \( \{\gamma_i\}_{i=1}^{b_1} \), this will allow us to extend \( \tau \) to \( T \) and the other property will cause the cancellation of the 1-handles of \( N_0 \setminus \mathbb{B}^4 \) when composed with \( T \).

**Embedding \( \Gamma \) and the curves \( \gamma_i \).** Let \( n \) be the sphere number of \( \Gamma \), since \( \Gamma \) is symmetric, it is obtained by adding some edges to the graph \( \Gamma' \) embedded in \( \mathbb{R}^3 \) showed in Figure 7a for the case \( n = 3 \). We choose as spanning tree for \( \Gamma \), the gray edges in Figure 7a, i.e. all the horizontal edges and the lower diagonal edges. All the other edges \( e_1, \ldots, e_k \) are in excess and will contribute to generate \( b_1(Y) \). To each of them we associate an 8-shaped loop \( \gamma_i \) on the embedding of \( \Gamma \) as showed in Figure 7b, if the edge is \( e_k \), the path \( \gamma_k \) will be given by \( e_k \), the two horizontal edges meeting \( e_k \) and the diagonal edge below \( e_k \).
Now pick an edge of $\Gamma$ not in $\Gamma'$, and call it $e$. Suppose $e$ joins $v^A_i$ to $v^B_j$ and $i < j$. Then we embed it as in 7c, in such a way that is above all the other edges. If instead $i > j$ then we embed it as in Figure 7d, so that it is below all the other edges. If instead $i = j$ then we embed it in the plane of the vertices as in Figure 7e. Then we associate to $e$ loop $\gamma_e$, given by a zig-zag (diagonal edge, horizontal edge) as showed in Figure 7c, Figure 7d, Figure 7e depending on $i, j$. Since $\Gamma$ is symmetric, there is an edge $e'$ associated to $e$, such with $e' \approx (v^A_i, v^B_j, \pm)$. We apply the same procedure to embed $e'$ obtaining a new loop $\gamma_{e'}$. Iterating the previous steps we embed all of $\Gamma$.

We use this data to construct a Kirby diagram for $P_0(\Gamma)$ as in Subsection 2.3. The loops $\gamma_i$ induce up to isotopy some loops in the diagram. Notice that we will have also an associated symmetric realization datum given by the pairs $e, e'$ and thus a diffeomorphism $\tau$ that exchanges $\gamma_e$ and $\gamma_{e'}$ (we parametrize the curves so that $\tau$ preserves the orientation of $\gamma_e$ and $\gamma_{e'}$).

Properties of $\gamma_i$ The set of curves $\gamma_i$ has some properties. First of all, with reference to the Kirby diagram, each curve will pass exactly once through the dotted circle of its associated edge and through of a certain number of dotted-circles associated to the diagonal edges of $\Gamma'$ in the case that $\gamma$ passes through the upper diagonal edges, i.e. if $e$ joins $v^A_i$ to $v^B_j$ and $i < j$.

Secondly, if we add to $P_0(\Gamma)$ $b_1(Y)$ 0-framed 2-handles along the curves $\gamma_i$ then these will cancel out the 1-handles of $P_0(\Gamma)$. Indeed, the $\gamma_i$ relative to the 8-figure shaped paths will appear as loops meridional to the dotted circles of the clasps associated to the diagonal edges. Hence we can cancel these dotted circles. After this cancellation, the remaining loops $\gamma_i$ can be isotoped in the diagram to curves that are meridional to the remaining dotted circles. This is because they do not interact between each other, as can be seen from the fact that the $\gamma_i$'s can be pushed to the boundary of a tubular neighbourhood $v\Gamma \subset \mathbb{R}^3$ of $\Gamma$ in in such a way that their image bound disks in the complement of $v\Gamma$. Figure 7c, Figure 7d, Figure 7e provide some intuition for these disks. As a result, we can cancel also the remaining dotted circles. Notice that the result of the 2-handle addition is $\mathbb{B}^4$ since we will have cancelled all the clasps from the diagram except one for each pair $v^A_i, v^B_i$.

Let $T$ be the cobordism from $Y$ to $S^3$ obtained by gluing the 2-handles as above. Since the set of attaching curves $\gamma_i$ is preserved by $\tau$, we can extend $\tau$ to a diffeomorphism $f : T \to T$, which exchanges the 2-handles relative to each pair of $\gamma_e$ and $\gamma_{e'}$ exchanged by $\tau$ and if $\tau$ sends $\gamma_i \to \pm \gamma_i$ then we extend it to a diffeo-
Figure 7: (7a) The graph $\Gamma'$ for $n = 3$. The spanning tree is in gray while the black edges correspond to the excess arcs $e_1, e_2$. (7b) In red, the 8-shaped loop $\gamma_1$ corresponding to $e_1$. (7c) Case $i < j$. Notice the arc is above the rest of the graph. In red, the loop $\gamma_e$ with the disk it bounds in light red. (7d) Case $i > j$. Notice the arc is below the rest of the graph. In red, the loop $\gamma_e$ with the disk it bounds in light red. (7e) Case $i = j$. In red, the loop $\gamma_e$ with the disk it bounds in light red.
morphism of the handle $\mathbb{D}^2 \times \mathbb{D}^2$ accordingly. The requirement that $f$ restricts to the identity on $\mathbb{S}^3$ is simple to fulfill because any orientation preserving diffeomorphism of $\mathbb{S}^3$ is isotopic to the identity, therefore we can just modify $f$ in a collar neighbourhood of the boundary to satisfy this requirement.

**Homological properties.** To prove the third item of Lemma 5.1, we notice that $H^1(T) = 0$ because $T$ is obtained by gluing 2-handles along the curves generating $H^1(Y)$ hence these generators are killed. Moreover $H^2(T, Y) \cong \mathbb{Z}^{b_1}$ and $H^3(T, Y) = 0$, because $H^*(T, Y)$ is generated by the handles used in the construction and these account only to $b_1$ 2-handles. Now the long exact sequence of $(T, Y)$ gives

$$0 \rightarrow H^1(Y) \rightarrow H^2(T, Y) \rightarrow H^2(T) \rightarrow H^2(Y) \rightarrow 0$$

from which is clear that

$$\ker (H^2(T, \mathbb{R}) \rightarrow H^2(\partial T, \mathbb{R}) \cong H^2(Y, \mathbb{R}) \oplus H^2(\mathbb{S}^3, \mathbb{R})) = 0.$$  

This concludes the proof of Lemma 5.1. \qed

### A Proof of Proposition 3.3.

For the sake of this appendix, $f$ will denote a Morselike perturbation.

**Proof sketch.** The simpler part of the proof is to achieve non-degeneracy of the critical points. In fact the proof of the book without modification gives perturbations in $\mathcal{D}^\perp$ in our setting (see Lemma A.1 below). The more delicate part is the regularity of the moduli spaces. The perturbations used in the book to achieve such result are perturbations vanishing in a fixed neighbourhood of the critical points. The natural thing to do would be to use the subset of such perturbations with connection-component vanishing on the reducible locus (i.e. in $\mathcal{D}^\perp$). The problem is that this is not sufficient to prove regularity in the case of irreducible trajectories between two reducible critical points. In this case we have to allow our perturbations to be non-constant on the reducible locus. At the same time it is important to us to obtain in the end a perturbation in $h + \mathcal{D}^\perp$ where $h$ is Morselike. To this end we define a second space of perturbations $\mathcal{D}_f$. This consists of perturbations which are pull back of functions on the torus of reducibles (not necessarily Morse) vanishing in a neighbourhood of the critical points of $f$. 

and are made constant in a neighbourhood of the critical points of $\mathcal{L} + f$ with the help of a bump function. Using the space of perturbations $\mathcal{P}_f \times \mathcal{P}_0^\perp$ we manage to achieve generic regularity (Lemma A.3).

**Non-degeneracy** From [KM07, pg. 212] it follows that a *reducible* critical point $[(B, 0, \psi)] \in \mathcal{B}^c_k(Y, s)$ in the blow-up is non-degenerate iff and only if the perturbed Dirac operator $D_B\alpha$ is invertible and has simple spectrum over $\mathbb{C}$ and the operator $*d + D_{(B,0)} q$ is invertible. In particular, for a Morselike perturbation $q = \text{grad} f$ the second condition is automatically satisfied thanks to $f_\Gamma$ being a Morse function.

**Lemma A.1.** The set of perturbations $q' \in \mathcal{P}^\perp$ such that $(\text{grad}(\mathcal{L} + f) + q')^\sigma$ has non-degenerate critical points (in the blow-up) is residual in $\mathcal{P}^\perp$.

**Proof.** Firstly we take care of the reducibles critical points. Let $Op^\text{sa}$ be the set of self-adjoint Fredholm maps defined in [KM07, pg. 215]. Let $\Sigma \subset Op^\text{sa}$ be the subset of operators with non-simple spectrum union the set of non-injective operators. $\Sigma$, is stratified by manifolds and the map

$$q \in \mathcal{P}^\perp \mapsto D_B + D_{(B,0)} q^\dagger(0, \cdot) \in Op^\text{sa}$$

is transverse to the stratification by [KM07, Lemma 12.6.2]. Since each stratum has positive codimension and the set of reducibles critical points in the blow-down, $\{(B,0)\}$, is 0-dimensional it follows that for generic $q' \in \mathcal{P}^\perp$ the perturbed Dirac operator will have simple spectrum and be invertible.

Secondly, the set of such $q'$ such that all the *irreducible* critical points are non-degenerate is residual. The proof of this fact follows the same course of the proof of [KM07, Lemma 12.5.2]. Indeed $D_\alpha \text{grad}(\mathcal{L} + f)$ has the same key properties of $D_\alpha \text{grad}(\mathcal{L})$ and we may assume that the perturbation in Corollary 11.2.2 of [KM07] vanishes on the reducibles (hence can be approximated with elements of $\mathcal{P}^\perp$) because the critical point we are considering is irreducible. \hfill \Box

Now we want to prove the generic regularity of the moduli spaces of trajectories. We will introduce the first class of perturbations that we will use. These are similar to those used in the book (i.e. vanishing in a neighbourhood of the critical points) with the additional constraint that their connection component vanishes over the reducibles.
Perturbations $\mathcal{P}_{\mathcal{O}}$. Now let $q'$ be a perturbation given by Lemma A.1 and let $f'$ a primitive for it. Set

$$L' = L + f + f'.$$

For each critical point $[\alpha]$ of $L'$ in the blow-down, we may find a gauge-invariant open neighbourhood $\mathcal{O}_{[\alpha]} \subset \mathcal{B}(Y, s)$ as defined at page 265 of [KM07]. Here $\mathcal{B}(Y, s)$ is the based quotient configuration space (see pg 173 of [KM07]). We then set $\mathcal{O} := \bigsqcup_{[\alpha]} \mathcal{O}_{[\alpha]} \subset \mathcal{B}(Y, s)$ where $[\alpha]$ varies on the set of critical points of $L'$. Define $\mathcal{P}_{\mathcal{O}}$ as the closed subspace of perturbations $q \in \mathcal{P}$ vanishing over $\mathcal{O}$.

Perturbations $\mathcal{P}_f$. The perturbations $\mathcal{P}_{\mathcal{O}}$ are not enough to prove regularity because in the case of irreducible trajectories between reducible critical points, we may need to use cylinder functions that are not constant on the reducibles. To remedy this, we introduce another space of perturbations which will ensure us to get, in the end, a perturbation which is still Morselike over the reducibles.

Let $\mathcal{O}_\mathbb{T} \subset \mathbb{T}$ be a union of disjoint contractible open neighbourhoods of the reducible critical points of $f|_\mathbb{T}$. Withou loss of generality we may assume that $\mathcal{O}$ and $\mathcal{O}_\mathbb{T}$ are chosen in such a way that for each $[\alpha]$ reducible critical point of $L'$, $\mathcal{O}_{[\alpha]} \subset p^{-1}_\mathbb{T}(\mathcal{O}_\mathbb{T})$ where $p_\mathbb{T} : \mathcal{B}(Y, s) \to \mathbb{T}$ is a retraction to the torus of flat connections analogous to (55). Up to shrinking $\mathcal{O}_{[\alpha]}$, we may also assume that for any $[\alpha]$ irreducible, $\mathcal{O}_{[\alpha]} \cap \mathcal{O}_0 = \emptyset$, some slightly larger neighbourhoods of $[\alpha]$ with the same properties.

It follows that $\mathcal{O}_{[\alpha]}$ is $\mathcal{G}(Y)$-invariant cylinder function [KM07, Definition 11.1.1]. The existence of such functions relies on the assumption that the sets $\mathcal{O}_{[\alpha]}$ do not intersect the reducible locus.

Next we choose a countable family $\{\tilde{h}_n\}_{n \in \mathbb{N}}$ of smooth functions $\tilde{h}_n : \mathbb{T} \to \mathbb{R}$ such that

(a) $\rho$ is of the kind of functions appearing in the first item of [KM07, Definition 11.1.1],

(b) $b_{[\alpha]} : \mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m \to \mathbb{R}$ is of the kind of functions appearing in the second item of [KM07, Definition 11.1.1],

(c) setting $\beta_{[\alpha]} := b_{[\alpha]} \circ \rho$, it holds that $\beta_{[\alpha]} \equiv 0$ on $\mathcal{O}_{[\alpha]}$ and $\beta_{[\alpha]} \equiv 1$ outside of $\mathcal{O}'_{[\alpha]}$, some slightly larger neighbourhoods of $[\alpha]$ with the same properties.
(a) $\tilde{h}_n$ is constant over $\Theta_T$.

(b) $\{h_n\}_{n \in \mathbb{N}}$ is dense in the set of smooth functions $T \to \mathbb{R}$ constant over $\Theta_T$ endowed with the $C^\infty$ topology.

Define $\mathcal{P}_f$ to be the Banach space of tame perturbations [KM07, Theorem 11.6.1] generated by the countable family of cylinder functions

$$h_n(x) := \tilde{h}(p_T([x])) \sum_{[\alpha]} \beta_{[\alpha]}(x) = \tilde{h}_n(pr_2(p([x]))) \sum_{[\alpha]} b_{[\alpha]}(p([x]))$$  \hspace{1cm} (153)

$n \in \mathbb{N}$, where $pr_2 : \mathbb{R}^n \times T \times \mathbb{C}^m \to T$ is the projection onto the second factor.

Notice that this implies that if $q \in \mathcal{P}_f$, then $q|_{\Theta} \equiv 0$. Indeed this is ensured by the functions $\beta_{[\alpha]}$ for $[\alpha]$ irreducible and by the assumption $\Theta_{[\alpha]} \subset p_T^{-1}(\Theta_T)$ together with the first condition on $\tilde{h}_n$ for $[\alpha]$ reducible.

Remark 8. As defined, $\mathcal{P}_f$ is not a subspace of $\mathcal{P}$ in general, however, once $f$ and $f'$ are given we can form the large Banach space of tame perturbations $\mathcal{P}$ using a countable family of cylinder functions that includes $\{h_n\}_{n \in \mathbb{N}}$, in this way we have an inclusion $\mathcal{P}_f \subset \mathcal{P}$ as a closed subspace. This is the possible enlarging of $\mathcal{P}$ to which the statement of Proposition 3.3 refers to.

Lemma A.2. There exists a neighbourhood of zero $\mathcal{N} \subset \mathcal{P}_f \times \mathcal{P}_0^+$ such that if $(q_f, q'') \in \mathcal{N}$ and we denote by $h$ and $f''$ a primitive of $q_f$ and $q''$ respectively, then the critical points of $\mathcal{L} := \mathcal{L}^r + f'' + h$ are the same as the critical points of $\mathcal{L}'$, are non-degenerate and $(f + h)$ is Morselike in a neighbourhood of the reducible locus.

Proof. Since $(q_f + q'')|_{\Theta} = 0$ all the critical points of $\mathcal{L}'$ will persist the perturbation, moreover they will still be non-degenerate since $\Theta$ is open. Finally, the properness property of tame perturbations [KM07, Proposition 11.6.4] ensures that if the perturbation $q_f + q''$ is small enough, it does not introduce any new critical point in the complement of $\Theta$.

It remains to show that for such small perturbations $(f + h)$ is Morselike in a neighbourhood of the reducible locus. Notice that by construction (see (153)), $f + h$ restricted to a neighbourhood of the reducible locus is the pullback of a function defined over the torus of reducibles, thus we only need to check that $f + h$ is Morse-Smale on $T$. We already know that it is Morse, because $h$ does not introduce new critical points and vanishes in a neighbourhood of the critical points of $f$. To conclude, we observe that the property regarding the intersection of stable and unstable manifolds of $f + h$ is preserved if the perturbations is small enough. \hfill $\square$
In the following we let $N \subset P_f \times P_0^+$ be given by Lemma A.2.

**Lemma A.3.** The subset of $N$ consisting of $(q_f, q'')$ such that all the moduli spaces of trajectories of $(\text{grad}(L') + q_f + q'')^\sigma$ between any two critical points are regular is residual in $N$.

**Proof.** Let $[a], [b] \in R^*_k(Y, s)$ be critical points of $(\text{grad}(L'))^\sigma$. Our proof is a modification of the proof of [KM07, Theorem 15.1.1.] which follows the standard approach of showing that the parametrized moduli spaces $\mathcal{M}_z([a], [b]) \subset R^*_k([a], [b]) \times P_0$ are regular (see pg. 267 of the book). There are three cases to distinguish depending on $[a], [b]$:

(a) If $[a], [b]$ are both reducibles and project to the same reducible in the blow-down moduli space, then the original proof goes through as the regularity of $M([a], [b])$ follows from the non-degeneracy of the critical points.

(b) If $[a], [b]$ are reducibles but do not project to the same reducible downstairs and the trajectory is reducible, then the proof of Theorem 15.1.1. in [KM07] boils down to ensuring that an element $\tilde{V} = (\omega, \psi) \in L^2_0(\mathbb{R} \times Y; iT^* Y \oplus S)$, orthogonal to the range of the map defining the parametrized moduli space, is zero (see pg 270-271 in [KM07]). The proof that $\psi = 0$ actually uses only perturbations that are gradient of functions which are constant on the reducible locus, thus belong to $P^+$ so we are left to showing that $\omega = 0$. Here our proof diverges from the book. In our case, $\omega = 0$ is ensured by the fact that over the reducibles,

\[(\text{grad}(L') + q_f + q'')^0 = (\text{grad}(L') + q_f)^0 = (\text{grad} L + \text{grad} f + \text{grad} h)^0,\]

and $L + f + h$ is a Morse-like perturbation. Hence the operator $d^* + D_{\gamma(t)}(\text{grad}(f + h))^0$ acting on $Y \times \mathbb{R}$ is surjective, and this is the same operator shown in pg. 272 of [KM07].

(c) Suppose now that the trajectory considered is irreducible. Then reasoning as in the proof of Theorem 15.1.1 in [KM07], we arrive at a situation where we have a curve $\gamma$ downstairs, image of the trajectory $\gamma$, limiting to the critical points. Moreover, we have a never-vanishing vector field $\vec{V} = q_f + q''$ such that $\int_{\mathbb{R}} \langle \vec{V}(t), \partial_0 \gamma(t) \rangle dt > 0$. Here is where we need to use $P_f$. Indeed if $\vec{V}$ is not tangent to $\gamma$ for some $t_0$ and $\gamma(t_0) \notin \emptyset$, then we can
find a cylinder function with gradient vanishing outside a neighbourhood of \( y(t_0) \) consisting of irreducibles that satisfies the inequality above. In particular, this perturbation will be in \( \mathcal{P}_0^\perp \). However if \( V(t) \) is tangent to \( y \) for all \( t \) such that \( [y(t)] \notin \mathcal{O} \), and in addition \([a], [b] \) are irreducibles, then we cannot find such a function. Indeed such a function, say \( g \), must be constant on the irreducibles, but along \( y \), \( g \) has to be increasing, forcing the values of \( g \) at the endpoints to be different. This is solved using a cylinder function with gradient \( \mathcal{P}_f \) which can be made increasing along \( y \) because it is allowed to take different values at the two limiting irreducibles.

\[ \square \]

### B Proof of Lemma 4.1.

Call \( S \) the set of pairs satisfying (108). We will construct a map \( M^{\text{red}}(N^*; [c]) \rightarrow S/\mathcal{G}_{k+1,\delta}(N^*; 1) \). Suppose \([A, 0, \Phi] \in M^{\text{red}}(N^*; [c]) \), so in particular \((A, \Phi) \in L_{k,\text{loc}}^2(N^*) \). Denote by \( Z = [0, +\infty) \times Y \subset N^* \) the tube and define \( \psi \in L_{k,\text{loc}}^2(Z, S^*) \) by \( \psi(t, y) = \Phi(t, y) ||\Phi(t, \cdot)||_{L^2(Y)} \).

\((A, 0, \psi) \in \mathcal{G}_{k,\text{loc}}^\perp(Z) \) solves the perturbed flow equations in [KM07, Eq. (6.11)]. Since \([c] \) is non-degenerate, [KM07, Proposition 13.6.1] ensures the existence of \( \delta > 0 \) (depending on \( c \)) and a gauge transformation \( u \in \mathcal{G}_{k+1,\text{loc}}(Z) \) such that

\[ (u \cdot A, 0, u\psi) = \gamma_c \in e^{-\delta t}L_{k, B_k}^2(Z), \]  

where \( \gamma_c \in \mathcal{G}_{k}^\perp(Z) \) represents the translation invariant solution induced by \( c \). Since \( H^1(N) \rightarrow H^1(Y) \) is an isomorphism, \( u \) extends to \( N^* \), so we can suppose that \( u \in \mathcal{G}_{k+1,\text{loc}}(N^*) \).

Now we will show that up to rescaling \( \Phi \), \((u \cdot A, u\Phi) \in S \). Equation (108) is clearly satisfied, what is not obvious is that up to rescaling \( u\Phi \in X_{k,\lambda,\delta}^+(\psi_c) \).

Consider the function \( s(t) = ||u\Phi(t)||_{L^2(Y)} \). Since \((uA, 0, u\psi)\) satisfies the flow equations we have that

\[ \frac{d}{dt}s(t) = -\Lambda(t)s(t), \]

where \( \Lambda(t) = \text{Re}(D_{uA}^* u\psi, \rho Z(dt)^{-1}u\psi)_{L^2([t] \times Y)} \). By Lemma B.1 proved below, \( \Lambda(\cdot) - \lambda_c \in L_{k-1,\delta}^2(\mathbb{R}_+^*) \) and by Lemma B.2, \( s(t) = e^{-\lambda_c t}c + g(t) \) for some \( c \in \).
\( \mathbb{R}^X, g \in L^2_{k,\lambda+\delta}(\mathbb{R}_+). \) It follows that, over the end

\[
u(t, y) = s(t)u(t, y) = s(t)\psi(t) + s(t)(u\psi(t, y) - \psi(t))
\]
\[
= e^{-\lambda t}c(t)\psi(t) + g(t)\psi(t) + s(t)(u\psi(t, y) - \psi(t)).
\]

(157)

Notice that since \((u\psi(t, \cdot) - \psi(t)) \in L^2_{k,\delta}\) and \(s \in L^2_{k,\lambda+\delta}\), their product lies in \(L^2_{k,\lambda+\delta}\). Consequently, we obtain that \((u \cdot A, u/c\Phi) \in A \times L^2_{k,\delta}(N^*, i\Lambda^1 N^*) \times X^{-}\).

It can be showed as in the proof of [KM07, Lemma 13.3.1] that any other gauge transformation \(\tilde{u} \in L^2_{k+1,\lambda+\delta}(N^*)\) such that (155) holds satisfies \(\tilde{u}^{-1} u \in \mathbb{G}_{k+1,\lambda}(N^*; 1)\), therefore we established an injection \(M^\text{red}(N^*; [\xi]) \to S/\mathbb{G}_{k+1,\lambda}(N^*; 1).\)

Now suppose that \([A, \Phi] \in S/\mathbb{G}_{k+1,\lambda}(N^*; 1);\) we want to prove that \([A, 0, \Phi] \in M^\text{red}(N^*; [\xi])\). Equation (108) implies that \((A, \Phi)\) satisfies the Seiberg-Witten equations, and clearly \(L^2_k \subset L^2_{k,\text{loc}}\) therefore the only thing to check is that \(\Phi\) has the correct asymptotics. This follows from special properties of \(X^+_{k,\lambda,\delta}(\psi_t)\) as can be seen by a direct computation.

**Lemma B.1.** Let \(Z = \mathbb{R}^X \times Y\). Let \((A, 0, \psi) \in \mathbb{G}^\tau_{k,\text{loc}}(Z)\) and define \(c \in L^2_k(\mathbb{R}_+)\) by \(A(t) = \hat{A}(t) + c(t)dt \otimes 1_S\). Let \((A_b, 0, \psi_b) \in \mathbb{G}^\tau_k(Z, Y)\) and denote by the same symbol \((A_b, 0, \psi_b) \in \mathbb{G}^\tau_{k,\text{loc}}(Z)\) the translation invariant configuration induced by it. Suppose that \((A - A_b, \psi - \psi_b) \in L^2_{k}(Z, iT^*Z) \times L^2_{k}(Z, S^*).\) Then the map defined by

\[
\Lambda(t) = \left\{D_{\hat{A}(t),a}\psi(t) + c(t)dt \cdot \psi(t), \psi(t)\right\}_{L^2(Y)} - \left\{D_{\hat{A}_b,a}\psi_b, \psi_b\right\}_{L^2(Y)}
\]

(158)
is in \(L^2_{k-1}(\mathbb{R}_+).\) If in addition the data decays exponentially, i.e. \((A - A_b, \psi - \psi_b) \in L^2_{k,\delta}\) for some, then also (158) is in \(L^2_{k-1,\delta}\).

**Proof.** The hypothesis \((A - A_b, \psi - \psi_b) \in L^2_{k}(Z, iT^*Z) \times L^2_{k}(Z, S^*).\) implies that

\[
t \mapsto \psi(t) - \psi_b
\]

(159)

belongs to \(L^2_{k-1}(\mathbb{R}_+, L^2(Y, S)).\) Indeed the Sobolev spaces for Hilbert valued functions are defined by requiring that the weak derivatives (defined by integrating against \(C^\infty_0(\mathbb{R}, \mathbb{R})\)) lie in \(L^2\) and this is clearly seen to be implied by the hypothesis using the Pettis integral. Similarly, the map \(t \mapsto A(t) - A_b\) belongs
to $L^2_{k-1}(\mathbb{R}_+, L^2(Y; iT^*Y \oplus \mathbb{R}))$. Now

$$
\Lambda(t) = \left\langle D_{\dot{A}(t)} \psi(t) + c(t) dt \cdot \psi(t), \psi(t) \right\rangle_{L^2(Y)} - \left\langle D_{A_b} \psi_b, \psi_b \right\rangle_{L^2(Y)}
= \left\langle D_{\dot{A}(t)} \psi(t) + c(t) dt \cdot \psi(t), \psi(t) \right\rangle_{L^2(Y)} - \left\langle D_{A_b} \psi_b, \psi(t) \right\rangle_{L^2(Y)}
+ \left\langle D_{\dot{A}_b} \psi_b, \psi(t) \right\rangle_{L^2(Y)} - \left\langle D_{A_b} \psi_b, \psi_b \right\rangle_{L^2(Y)}
= \left\langle D_{\dot{A}(t)} \psi(t) + c(t) dt \cdot \psi(t) - D_{\dot{A}_b} \psi_b, \psi(t) \right\rangle_{L^2(Y)} + \left\langle D_{\dot{A}_b} \psi_b, \psi(t) - \psi_b \right\rangle_{L^2(Y)}.
$$

(160)

We will use the following lemma: if $X$ is a Banach space, $g \in L^2_r(\mathbb{R}, X)$, $r > 1$ and $F : X \rightarrow \mathbb{R}$ is a smooth function such that $F(0) = 0$ then $F \circ g \in L^2_r(\mathbb{R})$. The proof of this fact for vector valued functions is analogous to the case when $X = \mathbb{R}$. The map $L^2(Y, S) \rightarrow \mathbb{R}$ defined by $\varphi \mapsto \left\langle D_{\dot{A}_b} \psi_b, \varphi \right\rangle_{L^2(Y)}$ clearly satisfies these hypotheses, therefore $t \mapsto \left\langle D_{\dot{A}_b} \psi_b, \psi(t) - \psi_b \right\rangle_{L^2(Y)}$ belongs to $L^2_{k-1}(\mathbb{R}_+, \mathbb{R})$. Now define

$$
Q(t) := D_{\dot{A}(t)} \psi(t) + c(t) dt \cdot \psi(t) - D_{\dot{A}_b} \psi_b.
$$

(161)

More precisely:

$$
Q(t) = D_{A_b} (\psi(t) - \psi_b) + \rho_Y (A(t) - A_b) (\psi(t)) + D_{(\dot{A}(t), 0)} q^1(\psi(t)) - D_{(A_b, 0)} q^1(\psi_b).
$$

(162)

We can write

$$
\left\langle Q(t), \psi(t) \right\rangle_{L^2(Y)} = \langle G(\dot{A}(t) - A_b, c(t), \psi(t) - \psi_b, \psi(t) \rangle_{L^2(Y)}
$$

(163)

where $G : L^2(Y, iT^*Y \oplus \mathbb{R}) \oplus L^2(Y, S) \rightarrow L^2(Y, S)$ is defined as

$$
G(a, c, \phi) = D_{A_b} \phi + \rho_Y (a + c) \cdot (\phi + \psi_b) + D_{(a + A_b, 0)} q^1(\phi + \psi_b) - D_{(A_b, 0)} q^1(\psi_b).
$$

(164)

$G$ is smooth and $G(0) = 0$ as is the map $(a, c, \phi) \mapsto \langle G(a, c, \phi), \phi + \psi_b \rangle_{L^2(Y)}$. Consequently $t \mapsto \langle Q(t), \psi(t) \rangle_{L^2(Y)}$ is in $L^2_{k-1}(\mathbb{R}_+)$. To prove the statement about exponential decay we apply the same argument using Sobolev spaces with weights.

\[ \square \]

**Lemma B.2.** Let $\lambda \in \mathbb{R}, \delta > 0$ and $\Lambda \in \lambda + L^2_{k-1}(\mathbb{R}_+)$ with $k > 2$. Set $s(t) = \exp(-\int_0^t \Lambda(t) d\tau) : \mathbb{R}_+ \rightarrow \mathbb{R}$. Then $s(t) = e^{-\lambda t} c + g(t)$ where $c = e^{-\int_0^t (\Lambda(t) - \lambda) d\tau}$ and $g \in L^2_{k, \lambda + \delta}(\mathbb{R}_+)$. In particular $s \in L^2_{k, \lambda}(\mathbb{R}_+)$. 
Proof. Since $L^2_{k,\delta}(\mathbb{R}_+^\times) \subset L^1(\mathbb{R}_+)$, the integral $\int_0^{\Lambda(\tau) - \lambda} d\tau$ is finite and $c \in \mathbb{R}^\times$ is well defined. Now we have that
\[
e^{-\int_0^t (\Lambda(\tau) - \lambda)\,d\tau} = e^{-\int_0^t (\Lambda(\tau) - \lambda)\,d\tau} (1 - e^{-\int_t^\infty (\Lambda(\tau) - \lambda)\,d\tau})
\]
(165)

Using Sobolev multiplication theorems [KM07, Section 13.2] and that $|\int_t^\infty (\Lambda(\tau) - \lambda)\,d\tau| \leq \text{const} e^{-\delta t}$, we find that
\[
(1 - e^{-\int_t^\infty (\Lambda(\tau) - \lambda)\,d\tau}) \in L^2_{k,\delta}
\]
\[
e^{-\int_0^t (\Lambda(\tau) - \lambda)\,d\tau} \in L^\infty_2(\mathbb{R}_+)
\]
\[
\partial_t (e^{-\int_0^t (\Lambda(\tau) - \lambda)\,d\tau}) \in L^2_{k-1,\delta}(\mathbb{R}_+).
\]
(166)

Consequently, $e^{\lambda t} s(t) - c \in L^2_{k,\delta}(\mathbb{R}_+)$ by applying Sobolev multiplication theorems to (165). Therefore there exists $\tilde{g} \in L^2_{k,\delta}(\mathbb{R}_+)$, such that $e^{\lambda t} s(t) = c + \tilde{g}(t)$. \qed

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