ESTIMATING UNOBSERVED INDIVIDUAL HETEROGENEITY USING PAIRWISE COMPARISONS

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**Abstract.** We propose a new method for studying environments with unobserved individual heterogeneity. Based on model-implied pairwise inequalities, the method classifies individuals in the sample into groups defined by discrete unobserved heterogeneity with unknown support. We establish conditions under which the groups are identified and consistently estimated through our method. We show that the method performs well in finite samples through Monte Carlo simulation. We then apply the method to estimate a model of low-price procurement auctions with unobserved bidder heterogeneity, using data from the California highway procurement market.

**Keywords:** Unobserved Individual Heterogeneity, Discrete Unobserved Heterogeneity, Pairwise Comparisons, Nonparametric Classification, Consistency

**JEL Classification:** C12, C21, C31

1. Introduction

The empirical analysis of many economic settings requires accounting for unobserved individual heterogeneity (UIH) which reflects agent-specific factors that influence agents' decisions but are not recorded in the data. Failing to account for UIH generally leads to biased estimates and affects the validity of counterfactual prediction.

In this paper, we consider a generic economic model where UIH induces a group structure among agents according to their types. We provide conditions for identification of the group structure, and propose a method to recover the group structure from data.

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Our main idea is based on the insight that UIH often implies *pairwise* inequality restrictions on endogenous observable quantities. For instance, in multi-attribute auctions, bidders with higher (unobserved) quality levels have a greater chance of winning the auction, controlling for the bid and the set of competitors. In a labor market setting, agents with higher (unobserved) productivity receive higher wages than less productive ones. In Section 2.2 we provide further examples of economic applications in which pairwise inequality restrictions arise naturally from the behavior of agents in equilibrium.

We develop a statistical method to recover the group structure (that is, to classify individuals into groups defined by UIH) using a pairwise comparison approach. Our method treats UIH as individual-specific discrete parameters which may affect the distribution of other observed or unobserved variables. Such flexibility is important in structural models where individuals interact strategically and the UIH of all agents jointly affects the equilibrium outcome. Our method is nonparametric, i.e., does not assume that UIH belongs to any parametric family of distributions, or that the support of discrete UIH is known.

A naive classification of individuals into groups based on pairwise inequality tests does not necessarily yield a coherent group structure in finite samples in general. Our method recovers the whole group structure by sequentially sub-dividing the set of agents on the basis of the *p*-values of tests of pairwise inequality restrictions. The method recovers the group structure for each assumed number of groups, and then selects the number of groups (and the associated group structure) using a penalization scheme. We show that our estimator of the group structure is consistent under mild regularity conditions and performs well in small samples.

In many settings, classifying individuals into groups defined by UIH offers key economic insights. For example, our method can be used to identify colluding bidders in auctions, and firms’ cost asymmetries or product quality differences. In addition, recovering the group structure also often serves as the first step for estimating structural models with strategic interactions, such as dynamic industry models or auctions with asymmetric bidders.  

This approach offers a feasible way to identify and estimate games with UIH. Specifically, a traditional approach in a setting with UIH would be to treat UIH as “fixed effects”

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1Estimation of *discrete* unobserved individual heterogeneity does not affect subsequent estimation of other structural parameters in terms of pointwise asymptotics. However, establishing uniform asymptotics remains an open question. This problem is analogous to that of post-model-selection inference. For discussion on the issues, see Pötscher (1991), Leeb and Pötscher (2005), and Andrews and Guggenberger (2009) and the references therein. Uniform asymptotics in our setup is complex because of the need to consider every possible direction of local perturbation from the actual group structure in data-generating process. A full theoretical investigation of the issue in our context merits a separate paper.
and estimate them jointly with other structural parameters. This approach poses practical challenges in a setting with agent interdependence, especially when equilibrium outcomes admits no closed-form expressions. First, it is generally not obvious what variation in the data may identify model components including the fixed effects in such settings. One of the contributions of our paper is to point out the variation which identifies the group structure associated with “fixed effects.”

Further, we propose to separate the recovery of the group structure from the estimation of other structural parameters. Recovering the group membership of every agent facilitates, and is often needed for, identifying the remaining structural elements in models with strategic interactions. A classical example is English auctions among bidders with unobserved types (in the sense that bidders’ private values are drawn independently from the distributions “labeled” by bidder type) and where the data only report the transaction price and the identity of the winner. Athey and Haile (2007) show that the distributions of private values cannot be identified in this model if the type of the auction winner is unknown. We provide details and additional examples in the supplemental note to this paper.

Finally, our approach offers a way to estimate games with UIH with substantially low computational cost, compared with the alternative approach of estimating the fixed effects jointly with other structural parameters. For example, consider an environment with a large number of players where many independent games (each containing only a small subset of players) are observed in the data. The numerical optimization (such as simulated GMM or MLE) requires evaluating the objective function which involves computing equilibrium for each value of the fixed effects and other structural parameters. This can be computationally prohibitive in practice. In contrast, our method allows the game to be solved only for the estimated configuration of the agents’ group memberships rather than for every possible configuration as is required under the joint estimation.2

Our method is advantageous especially in settings where the number of agents is moderately large but each market (observation) in the data involves only a small subset of participants. For example, the total number of participants may be several hundreds but each market may contain only several participants. In this case, despite the large number of markets observed, the researcher may have only a small number of markets which contain the same set of participants. We call this issue the problem of the sparsely common set of agents. In such settings, the researcher cannot build inference on the conditional moments given the full set of participating agents in a market, as typically done in the structural empirical literature, because we do not have many such observations. Hence

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2Krasnokutskaya, Song, and Tang (2018) used this classification method as a first step in the structural analysis of an online service market for computer coding.
the researcher needs to “aggregate” the markets or the agents in order to conduct reliable inference with sufficiently many observations. Pairwise restrictions are testable with accuracy even when the data exhibit sparsely common sets of agents since the number of markets where a given pair of agents is present tends to be large even if the number of markets with the same set of participants is small. Thus, pairwise restrictions and the classification procedure offer a natural way to aggregate agents into groups which permits estimation of other primitives.

We investigate the finite sample performance of our classification method in Monte Carlo simulation. The data-generating process (DGP) is a lowest-price procurement auction among asymmetric bidders whose independent private values are drawn from distributions with different means. We report the outcome of classification for DGPs with various numbers of bidders and group structures. Our classification method works well. Its performance is better when the number of bidders and groups are smaller relative to the number of observed markets/games, and when the differences between groups are larger. We also find that the impact of classification errors on subsequent estimation of other structural parameters in the game is non-substantial.

We analyze the California highway procurement market by applying our classification method to a model of asymmetric lowest-price procurement auction. Existing empirical studies of auction markets typically emphasized asymmetry in bidders’ private values associated with their observable characteristics. In comparison, we allow the bidders’ private values to be drawn from heterogeneous distributions with different means. To account for other sources of cost heterogeneity, we control for bidders’ distance to the project site. We also accommodate possible endogeneity in the competitive structure. We use the classification method to recover the unknown group structure (i.e., partition bidders into groups with different mean costs). Then, using this estimated group structure, we estimate group-specific cost distributions using GMM.

Our estimates indicate that the bidders in the data come from several unobserved groups with substantial differences in mean costs. We also find that ignoring such unobserved bidder heterogeneity would lead to biased estimates of how bidders’ costs depend on various factors.

Related Literature. One of the popular methods of accounting for UIH in structural modeling is to adopt finite mixtures. (See Hu (2008), Hu and Schennach (2008), Kashihara and Shimotsu (2009), Hu and Shum (2012), Hu and Shiu (2013), Hu, McAdams, 3

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3For example, Athey, Levin, and Seira (2011), Roberts and Sweeting (2013) and Aradillas-Lopez, Gandhi, and Quint (2013) accounted for the bidder heterogeneity associated with size in the timber market (‘mills’ vs ‘loggers’); Krasnokutskaya and Seim (2011), Jofre-Bonet and Pesendorfer (2003) and Gentry, Komarova, and Shiraldi (2016) incorporated bidder participation differences in highway procurement market (regular’ vs ‘fringe’ bidders); Conley and Decarolis (2016), Asker (2010) and Pesendorfer (2000) allowed for bidder heterogeneity in collusive behaviors.
and Shum (2013), and Henry, Kitamura, and Salanie (2014). See also See Kasahara and Shimotsu (2014) and Kasahara and Shimotsu (2015) for estimating and testing for the number of mixture components in finite mixture models.) The finite mixture modeling assumes that the UIH is a random variable drawn from some unknown distribution. The goal is to identify this distribution and estimate it from data. It does not require each individual to appear in many independent games. In contrast, our approach aims to classify individual agents in the sample into disjoint groups defined by their realized unobserved types, using their participation outcomes in many games. Thus the two approaches are fundamentally different both in their aims and their data requirements. While general identification results have been developed in this literature of finite mixture models (see, e.g., Bonhomme, Jochmans, and Robin (2016)), implementation of the finite-mixture method is impractical in our set-up due to the issue of sparse commonality, and technical issues associated with high dimensionality.\footnote{When each market is drawn from a finite mixture distribution, and there are $I$ agents with each having a type from $S$ values, the number of the mixture components becomes $S^I$ which can be very large in practice, even when $I$ is a moderate number such as five or ten.}

The classification algorithm we propose is related to the clustering method in statistics. (See, e.g., Chapter 14 of Hastie, Tibshirani, and Friedman (2009).) The main difference is that the clustering method groups individuals based on similarity of their observed attributes, whereas our approach groups individuals based on their unobserved attributes. To do so, we exploit the relationship between endogenous outcome and the unobserved types of individuals implied by an economic model. Our method also requires a data structure different from clustering methods. The literature of clustering methods mostly considers a set-up in which each cross-sectional unit is observed once, whereas our method uses many observations per individual in the sample.

Also related to our approach is the literature of panel models with group level heterogeneity. For example, Sun (2005) introduced a linear panel model where parameters take values in a finite set according to a logistic probability, and offered methods of estimating the group structure. Song (2005) considered a panel model with finite-valued nonstochastic parameters and produced an algorithm to recover the unobserved group structure in large panel models. Lin and Ng (2012) provided a method of estimating a panel model using threshold variables when the group membership is unknown. Su, Shi, and Phillips (2016) developed a new Lasso method to recover the unknown group-specific parameters. Bonhomme and Manresa (2014) proposed a k-means clustering algorithm to recover the group structure in a linear panel model. These papers often focus on models which admit a reduced form for the dependent variable in which its functional relation to UIH is made explicit. In contrast, our method targets a set-up where the dependence of the outcome variables on the UIH arises implicitly through equilibrium
contraints in games, and the group structure of UIH is identified only through pairwise
inequality restrictions. Thus, the approaches developed in the panel literature are not
applicable in settings our proposal focuses on.

Roadmap. This paper is organized as follows. Section 2 introduces the basic environment
and defines pairwise inequality restrictions. This section also provides several examples
from various contexts to motivate our classification method.

Section 3 establishes identification of the unobserved group structure using pairwise
inequality restrictions. Section 4 proposes a consistent estimator of the group structure.
Section 5 provides results from Monte Carlo simulation. Section 6 presents the empirical
application. Section 7 concludes. Further examples and mathematical proofs are pro-
vided in the supplemental note of this paper. The note also contains further simulation
results and details in our empirical application.

2. The Model and Examples

2.1. Pairwise Inequalities in Game Models

We consider a setting where the econometrician observes \( L \) games, and in each game,
a set of agents interact with each other. Each agent \( i \) is associated with a non-stochastic
type, \( q_i \), which is not observed by the researcher. We assume that the type is finite-valued
so that \( q_i \in Q_0 = \{ \bar{q}_1, \ldots, \bar{q}_{K_0} \} \), with \( \bar{q}_1 < \cdots < \bar{q}_{K_0} \). This induces an (ordered) partition
\((N_1, N_2, \ldots, N_{K_0})\) of the set \( N \) of agents such that for each \( k = 2, \ldots, K_0 \), \( N_k \) consists of
agents with higher type than those in \( N_{k-1} \). The group structure is characterized by a
function \( \tau : N \to \{1, \ldots, K_0\} \) that links the identity of a player to his unobserved type so
that \( q_i = \bar{q}_{\tau(i)} \) and for each \( k = 1, \ldots, K_0 \),

\[ N_k = \{ i \in N : \tau(i) = k \} . \]

The data available to the researcher contain for every observation \( l = 1, \ldots, L \): a vector
of observable characteristics for all the agents involved, \( \{X_{j,l}\}_{j \in S_l} \), as well as at least one
but possibly multiple vectors of outcome variables, \( Y_l = \{Y_{j,l}\}_{j \in S_l} \), where \( S_l \) is the set of
players involved in each observation (e.g., a game or a market). Our main focus is on
recovering the ordered partition \((N_1, N_2, \ldots, N_{K_0})\) of agents from data.

The main insight of our paper begins with the observation that in many structural mod-
els, the ordering among \( q_i \)'s (or equivalently \( \tau(i) \)'s) coincides with the ordering between
indexes that can be estimated consistently. Specifically, we use pairwise indexes \( \delta_{ij} \) and
that satisfy the following relations:\(^5\)

\[(2.1) \begin{align*}
\delta_{ij} &> 0 \text{ if and only if } \tau(i) > \tau(j); \\
\delta_{ij}^0 & = 0 \text{ if and only if } \tau(i) = \tau(j),
\end{align*}\]

where the indexes \(\delta_{ij}\) and \(\delta_{ij}^0\) can be consistently estimated using the sample. In many applications, we can take the indexes as (a variant of) the following form:

\[(2.2) \begin{align*}
\delta_{ij} &= \int \max\{\mathbb{E}[Y_{i,l}|X_{i,l} = x] - \mathbb{E}[Y_{j,l}|X_{j,l} = x], 0\} dF(x), \text{ and} \\
\delta_{ij}^0 &= \int |\mathbb{E}[Y_{i,l}|X_{i,l} = x] - \mathbb{E}[Y_{j,l}|X_{j,l} = x]| dF(x),
\end{align*}\]

where \(F\) is a known distribution or the distribution of an observable random vector.

For example, suppose that the outcome \(Y_{i,l}\) admits the following reduced form:

\[Y_{i,l} = g(\tau(i), X_{i,l}, \eta_{i,l}),\]

where \(g\) is a function that is strictly increasing in \(\tau(i)\) and \(\eta_{i,l}\) is an unobserved component that is independent of \(X_{i,l}\). Then under regularity conditions, we obtain the pairwise relations \((2.1)\) with \((2.2)\). The main advantage of our approach is that we do not require an explicit characterization of the reduced form \(g\). Due to this flexibility, our approach is most useful for analyzing UIH in structural models where the reduced form for outcomes arises only implicitly through equilibrium constraints. In such a setting, the sign of the indexes \(\delta_{ij}\) represents the pairwise relation which says that between any two agents, one agent’s type is higher than the other if and only if his outcome tends to be higher than that of the other. As we demonstrate through examples below, many structural models imply these pairwise relations through indexes \(\delta_{ij}\) and \(\delta_{ij}^0\).

The main goal of this paper is to develop a statistical procedure to recover the group structure \(\tau\) from data. Our method relies only on the pairwise inequality restrictions in \((2.1)\). Thus so far as the group structure is concerned, the pairwise comparison indexes \(\delta_{ij}\) and \(\delta_{ij}^0\) play the role of a sufficient statistic; the recovery of the group structure does not rely on other details of the structural model.

### 2.2. Examples

We now provide examples of how pairwise inequality restrictions arise as equilibrium implications in a variety of commonly studied empirical contexts.

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\(^5\)For the sake of concreteness, our exposition in the paper focuses on this form of indexes. Our procedures rely on the indexes only through the availability of consistent tests of pairwise inequality restrictions: \(\delta_{ij} > 0\) and \(\delta_{ij}^0 = 0\). As long as such consistent tests are available, one can use our method for other forms of pairwise indexes.
2.2.1. Unobserved Quality in Multi-attribute Auctions. Consider a simplified version of multi-attribute auctions in Krasnokutskaya, Song, and Tang (2018) that abstracts away from observed auction and seller heterogeneity. Let $N$ denote the population of sellers and $S_l$ the set of sellers who submitted bids for a project $l$. Each seller has a discrete unobservable quality: $q_i \in \{\bar{q}_1, ..., \bar{q}_K\}$, with $\bar{q}_k < \bar{q}_{k'}$ whenever $k < k'$. Such a quality is known to buyers but not reported in data. The buyer for project $l$ selects a seller among those who submitted bids or chooses an outside option to maximize his payoff. The payoff to the buyer from engaging services of seller $i \in S_l$ is given by $U_{i,l} = \alpha_l q_i + \epsilon_{i,l} - B_{i,l}$ whereas the payoff from an outside option is $U_{0,l}$. Here $\alpha_l$ is a non-negative weight the buyer gives to the seller’s quality relative to the seller’s bid, whereas $\epsilon_{i,l}$ reflects a buyer-seller match-specific stochastic component.

Let us suppress the auction subscript $l$ and define for any two sellers $i, j$,

$$\rho_{ij}(b) = P\{i \text{ wins} | B_i = b, i \in S, j \notin S\},$$

for all $b$ on the intersection of the supports of $B_i$ and $B_j$. Suppose $\alpha, S, \{C_i, \epsilon_i\}_{i \in S}$ are mutually independent. Proposition 1 of Krasnokutskaya, Song, and Tang (2018) showed that

$$\text{sign}(\rho_{ij}(b) - \rho_{ji}(b)) = \text{sign}(q_i - q_j),$$

for any $b$ on the intersection of bid supports.

On the basis of this property the comparison indexes can be constructed as follows: $\delta_{ij} \equiv \int \max\{\rho_{ij}(b) - \rho_{ji}(b), 0\} db$ and $\delta_{ij}^0 \equiv \int |\rho_{ij}(b) - \rho_{ji}(b)| db$. Note that the comparison indexes do not depend on other details of the structural model such as specific parametric assumptions for the distribution of buyers’ tastes.

2.2.2. Firms’ Cost Efficiency and Pricing Decisions. Consider a population of $n$ firms or brands, each of which produces a single brand of product. The data consists of independent markets indexed by $l = 1, ..., L$. The marginal cost for firm $i$ on market $l$ is $c_{i,l} = \varphi(w_{i,l}, q_i, \eta_{i,l})$, where $w_{i,l}$ are observable cost shifters, $q_i$ a brand-specific unobserved heterogeneity that is fixed across markets, and $\eta_{i,l}$’s are i.i.d. idiosyncratic noises independent from $w_{i,l}$ and $q_i$. We may interpret $q_i$ as a measure of firm $i$’s cost efficiency. Firms have complete information about each others’ cost efficiencies. Firms in the population are partitioned into groups with different levels of $q_i$: $N = \cup_k N_k$ where $i \in N_k$ if $\tau(i) = k$.

Let $\sigma_{i,l}(x_i, p_i, \Omega_l)$ denote firm $i$’s market shares, which is a function of product attributes ($x_i = \{x_{i,l}\}_{l \in S_l}$, where $S_l$ denotes the set of brands in market $l$) and prices ($p_i = \{p_{i,l}\}_{l \in S_l}$)

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6This holds, for example, if sellers are not informed of the weights or outside option of the buyer, or the identities of other sellers in $S_l$.

7This assumption is plausible in certain industries where production efficiency is mostly determined by firms’ technology or equipment that is publicly observable.
conditional on the set of products available in market \( l \) and other market factors denoted by \( \Omega_l \). The profit for firm \( i \) in market \( l \) is:

\[
\pi_{i,l} = (p_{i,l} - c_{i,l})\sigma_{i,l}(x_l, p_l)M_l,
\]

where \( M_l \) is a measure of potential consumers in market \( l \). In any pricing equilibrium with an interior solution, the first-order condition implies

\[
(2.4) \quad c_{i,l} = p_{i,l} + \sigma_{i,l}\frac{\partial \sigma_{i,l}}{\partial p_{i,l}}.
\]

Notice that if \( \eta_{i,l} \) is independent from \( q_i \) and \( w_{i,l} \), and \( \varphi(w_{i,l}, q_i, \eta_{i,l}) \) is strictly monotone in \( q_i \) then so is the right-hand side of (2.4), which can be constructed from estimates of the demand system. Hence, for any pair \( i, j \in N \), \( q_i \geq q_j \) if and only if \( \mathbb{E}[z_{i,l}|w_{i,l} = w_0] \geq \mathbb{E}[z_{j,l}|w_{j,l} = w_0] \), for all \( w_0 \), where \( z_{i,l} \) is defined as the quantity on the right hand side of (2.4). The statement is also true when both inequalities are strict. Thus we can define a pairwise comparison index

\[
(2.5) \quad \delta_{ij} = \int \max\{\mathbb{E}[z_{i,l}|w_{i,l} = w_0] - \mathbb{E}[z_{j,l}|w_{j,l} = w_0], 0\}dF(w_0),
\]

where \( F \) is the distribution of \( w_{i,l} \). In equilibrium \( \delta_{ij} > 0 \) if and only if \( q_i > q_j \). Likewise, define \( \delta_{0ij} \) by replacing \( \max\{\cdot, 0\} \) in the integral in \( \delta_{ij} \) with the absolute value. These pairwise comparison indexes do not condition on specific identities of firms in a market.

2.2.3. Assortative Matching in Labor Market. Sorting of heterogeneous employees across heterogeneous firms has been studied in Lentz and Mortensen (2010), Abowd, Kramarz, and Margolis (1999), and Lise, Meghir, and Robin (2011). In a typical setting, firms are heterogeneous in the productivity from a given worker ceteris paribus. Workers differ in their unobservable ability \( q_i \). Under further restrictions (see Eeckhout and Kircher (2011) and Hagedorn, Law, and Manovskii (2016)), workers with higher ability would in equilibrium earn higher wages than co-workers at the same firm, holding other things equal.

This forms a basis for pairwise comparisons. Specifically, let \( w_{i,f,t} = W(q_i, X_{i,t}, \Omega_{f,t}) \) denote the wage worker \( i \) earns at time \( t \) while employed by firm \( f \), where \( W \) is a non-stochastic function. Here \( \Omega_{f,t} \) captures all the relevant firm-specific unobservable factors while \( X_{i,t} \) reflects worker \( i \)'s characteristics other than \( q_i \). Using \( N_{f,t} \) to denote the set of workers employed by firm \( f \) at time \( t \), we define the comparison index as

\[
\delta_{ij} = \int \max\{\mathbb{E}[w_{i,f,t}|X_{i,t} = x] - \mathbb{E}[w_{j,f,t}|X_{i,t} = x], 0\}dF(x), \ i, j \in N_{f,t},
\]

where \( F \) is the distribution of \( X_{i,t} \). Then \( \delta_{ij} > 0 \) if and only if \( q_i > q_j \), under regularity conditions such as strict mononicity of \( W \) in \( q_i \). Likewise before, define \( \delta_{0ij} \) by replacing the max operator in \( \delta_{ij} \) with its absolute value. In this setting comparison of workers is complicated by the (unobserved) firm heterogeneity and sorting of workers across
firms. Pairwise comparisons allow researchers to circumvent these issues by focusing on workers’ wages earned while they are employed by the same firm.

2.3. Sparsely Common Set of Agents and Pairwise Inequalities

Our pairwise comparison method is most useful in a setting where players appear in a market only sparsely. To express this data feature, define for any $S \subseteq N$,

$$\mathcal{L}(S) = \{1 \leq l \leq L : S_l = S\}.$$ 

Thus $\mathcal{L}(S)$ represents the set of markets where the set of participants in a market $S_l$ is precisely $S$. In this paper, we refer to the setting as that of a sparsely common set of agents, if the proportion $\max_{S \subseteq N} |\mathcal{L}(S)|/L$ is negligible in finite sample. In other words, only a small fraction of the markets in the sample share exactly the same set of participants.

The setting with a sparsely common set of agents is illustrated in Figure 1, where each column symbolizes a “market” and each row an individual agent. The ellipses in each
column represent agents participating in a market. The first panel shows a standard set-up where all the agents appear in all the markets. The second panel shows an example of a data set where only very few markets share exactly the same set of participants \( \{1, 3, 4\} \). Therefore, the conditional choice probability given the same set of agents simultaneously participating in the market cannot be accurately estimated. However, if we focus on only subsets with two agents \( \{1, 3\} \), there are many more markets in which the two agents participate. Aggregating over these markets, one may infer accurately the ordering between the two agents using an inequality test. Given the p-values from inequality tests across pairs of agents, it remains to recover the whole group structure of the agents from these pairwise p-values. We develop an algorithm that recover the group structure from the pairwise p-values consistently.

3. Identification of the Ordered Group Structure

We say that agents \((i, j)\) are comparable if there exist consistently estimable pairwise indexes \(\delta_{ij}\) and \(\delta_{ij}^0\) such that (2.1) holds. In this identification analysis, we assume that a researcher knows whether each pair of agents is comparable through some pairwise comparison index or not. The determination of such comparability can be done in practice by checking whether the data contains sufficiently many markets which allow for reliable estimation of the pairwise indexes.

Let \(\mathcal{E}\) be the collection of pairs \((i, j)\) that are comparable. We refer to comparable agents as adjacent (or linked), so that the set \(\mathcal{E}\) forms the set of edges in a graph on the set of agents \(N\). We call this graph (denoted by \(G = (N, \mathcal{E})\)) the comparability graph.\(^8\) We say a group structure \(\tau\) is identified if it is uniquely determined once the comparability graph \(G\) and the vectors of pairwise indexes \((\delta_{ij}, \delta_{ij}^0)\) \(i, j \in \mathcal{E}\) are known.

Let us explore the identification of \(\tau\) given the comparability graph \(G\) and the vector of pairwise indexes. It is easy to see that if \(\mathcal{E}\) contains only a small subset of possible pairs, we may not be able to identify the group structure. The identification of the ordered group structure \(\tau\) is not guaranteed even when many pairs of agents are comparable. For example, even if \(G\) is a connected graph (where any two agents are connected at least indirectly), the ordered group structure \(\tau\) may not be identified. This is illustrated in a counterexample in Figure 2. Certainly, when every pair of agents are adjacent in the graph \(G\), i.e., \(G\) is a complete graph, the ordered group structure \(\tau\) is identified.\(^9\)

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8In a graph (or network) \(G = (N, \mathcal{E})\) the set \(N\) represents the set of vertices (or nodes) and \(\mathcal{E}\) consists of some pairs \(ij\), with \(i, j \in N\), where each pair \(ij\) is called an edge (or link). If \((i, j) \in \mathcal{E}\), we say that \(i\) and \(j\) are adjacent. A path is a set of vertices \(\{i_1, i_2, ..., i_M\}\) such that \(i_1i_2i_3i_{M-1}i_M \in \mathcal{E}\). Two vertices are called connected if there is a path having \(i\) and \(j\) as end vertices. A graph is called connected if all pairs of vertices are connected in the graph.

9 If all pairs of agents are comparable, we can split the set of agents into one group with the lowest type and the other group with the remaining agents. Then we split these remaining agents into one group with
Figure 2. This figure illustrates an example where the group structure is not identified even when all nodes are connected. The comparability graph is \( G = (N, \mathcal{E}) \), where \( N = \{1, 2, \ldots, 5\} \) and \( \mathcal{E} = \{12, 23, 34, 45\} \). Pairwise comparison is feasible only between nodes linked by solid black lines (a.k.a. links). The two different group structures in this figure are compatible with the same pairwise ordering. Therefore we cannot identify the group structure from pairwise orderings.

Figure 3. This figure shows an example where the condition \( N = N^* \) in Theorem 3.1 is violated. The first panel depicts the comparability graph as one connecting 6 vertices (or nodes). The second panel shows the \( \tau \)-collapsed graph where the two comparable nodes 2 and 3 that have the same type are collapsed into one node named 23. The last panel shows that Nodes 23, 4, and 5 (expressed as solid black nodes) are identified, because they are on a monotone path of length \( K_0 - 1 = 2 \). In this example, Nodes 1 and 6 are not identified and thus the comparable graph does not lead to the identification of the group structure.

Below we establish a necessary and sufficient condition for the group structure to be identified from an incomplete graph \( G \) and the pairwise comparison indexes. Let us introduce some definitions.

the lowest type within these agents and the remaining agents. By continuing this process, we can identify the whole group structure.
Definition 3.1. (i) A graph $G_{\tau}$ is the $\tau$-collapsed graph of $G$ if (a) any two adjacent vertices $i$ and $j$ in $G$ with $\tau(i) = \tau(j)$ collapse to a single vertex (denoted by $(ij)$) in $G_{\tau}$, (b) any edge in $G$ joining a vertex $k$ to either $i$ or $j$ joins vertex $k$ to $(ij)$ in $G_{\tau}$ and (c) all the remaining vertices and edges in $G_{\tau}$ consist of the remaining vertices and edges in $G$. (ii) A path in $G_{\tau}$ is monotone if $\tau(i)$ is monotone as $i$ runs along the path. (iii) A vertex $i$ is said to be identified if its type $\tau(i)$ is identified.

The $\tau$-collapsed graph of $G$ is constructed by reducing any comparable pair of agents in $G$ who have the same type to a single “agent”, and retaining edges as in the original graph of $G$. Certainly, a $\tau$-collapsed graph $G_{\tau}$ is uniquely determined by $\delta_{ij}$’s and $G$. Any pair of adjacent agents in the $\tau$-collapsed graph must have different types, and hence the types of agents on a monotone path are strictly monotone. This means that every vertex on a monotone path in $G_{\tau}$ of length $K_0 - 1$ is identified. Also by similar logic, every vertex on a monotone path with end vertices $i_H$ and $i_L$ is identified if the path has length $\tau(i_H) - \tau(i_L)$ and the end vertices $i_H$ and $i_L$ are identified. Using these two facts, we can recover the set of vertices that are identified as follows.

First, let $N_{[1]} \subset N$ denote the set of vertices such that each vertex in the $\tau$-collapsed graph $G_{\tau}$ is on a monotone path in $G_{\tau}$ of length $K_0 - 1$. For $j \geq 1$ generally, let $N_{[j+1]}$ be the set of vertices each of which belongs to a monotone path, say, $P$, such that its end vertices $i_H$ and $i_L$ are from $N_{[j]}$ and $\tau(i_H) - \tau(i_L)$ is equal to the length of the monotone path $P$. Then define

$$N^* \equiv \bigcup_{j \geq 1} N_{[j]}.$$ 

Given $G_{\tau}$, $N^*$ is uniquely determined as a subset of $N$. It is not hard to see that if $N = N^*$ and $K_0$ is identified, the type structure $\tau$ is identified. The following theorem shows that this condition is in fact necessary for the identification of $\tau$ as well.

Theorem 3.1. Let $G$ be a given comparability graph and $G_{\tau}$ be its $\tau$-collapsed graph. The group structure $\tau$ is identified if and only if there exists a monotone path in $G_{\tau}$ whose length is equal to $K_0 - 1$ and $N = N^*$.

No monotone path in $G_{\tau}$ can have length greater than $K_0 - 1$. Note that there exists a monotone path in $G_{\tau}$ whose length is equal to $K_0 - 1$ if and only if $K_0$ is identified. The conditions in the theorem are obviously satisfied if $G$ contains a monotone path that is monotone and covers all the vertices. The latter condition is trivially satisfied when $G$ is a complete graph. Figure 3 gives a counterexample where the condition that there exists a monotone path in $G_{\tau}$ whose length is equal to $K_0 - 1$ is satisfied, but $N \neq N^*$ so that the comparability graph does not lead to the identification of the group structure.

\(^{10}\)The length of a path is defined as the number of the edges in the path.
4. Consistent Estimation of the Ordered Group Structure

4.1. Pairwise Hypothesis Testing Problems

In this section, we develop a method to estimate the group structure consistently for the case where the comparability graph is complete, so that we take $E$ to be all $ij$ with $i, j \in N, i \neq j$. We first formulate three pairwise hypothesis testing problems for each comparable pair $ij$:

$$H_{0,ij}^+ : \delta_{ij} \leq 0 \text{ against } H_{1,ij}^+ : \delta_{ij} > 0,$$
$$H_{0,ij}^0 : \delta_{ij}^0 = 0 \text{ against } H_{1,ij}^0 : \delta_{ij}^0 \neq 0 \text{ and }$$
$$H_{0,ij}^- : \delta_{ji} \leq 0 \text{ against } H_{1,ij}^- : \delta_{ji} > 0. \tag{4.1}$$

In most examples, we have various tests available. Instead of committing ourselves to a particular method of hypothesis testing, let us assume generally that we are given $p$-values $\hat{p}_{ij}^s$, $s \in \{+, 0, -\}$ from the testing of $H_{0,ij}^+, H_{0,ij}^0$ and $H_{0,ij}^-$, against $H_{1,ij}^+, H_{1,ij}^0$ and $H_{1,ij}^-$ respectively. Let $L$ be the size of the sample (i.e., the number of the markets or games) that is used to construct these $p$-values. We will explain conditions for the $p$-values later and explain details for construction of $p$-values using bootstrap in Section 4.3.

4.2. The Classification Method

Our classification method consists of two generic algorithmic components: the Split Algorithm and the Selection-Split Algorithm which contains the Split Algorithm as a component.

4.2.1. The Selection-Split Algorithm. Let us introduce a method of obtaining an ordered partition $(\hat{N}_1', \hat{N}_2')$ of a given set $N'$ using $p$-values $\hat{p}_{ij}^s$, $s \in \{+, 0, -\}$.

**Definition 4.1.** For a subset $N' \subset N$, we say that the ordered partition of $N'$ into $(\hat{N}_1', \hat{N}_2')$ is obtained by the Split Algorithm if it is obtained as follows. For each $i \in N'$, we let

$$\hat{N}_1'(i) = \{j \in N' \backslash \{i\} : \log \hat{p}_{ij}^+ \leq \log \hat{p}_{ij}^- - r_L\} \text{ and }$$
$$\hat{N}_2'(i) = \{j \in N' \backslash \{i\} : \log \hat{p}_{ij}^- \leq \log \hat{p}_{ij}^+ - r_L\},$$

where $r_L \to \infty$ satisfies Assumption 4.1 below.\(^{11}\) Set $i^r = \arg\min_{i \in N'} \min\{s_1(i), s_2(i)\}$, where

$$s_1(i) = \frac{1}{|\hat{N}_1'(i)|} \sum_{j \in \hat{N}_1'(i)} \log \hat{p}_{ij}^+, \text{ and } s_2(i) = \frac{1}{|\hat{N}_2'(i)|} \sum_{j \in \hat{N}_2'(i)} \log \hat{p}_{ij}^-.$$

\(^{11}\)In many cases, it suffices to consider a sequence such that $r_L / \log L \to 0$. In practice, we propose $r_L = (\log L)^{1/3}$ which satisfies Assumption 4.1 below under lower level regularity conditions. See Section C.3 in the supplemental note for details. From our simulations, we find that even choosing $r_L = 0$ works well in finite samples.
(We set \(s_1(i) = 0\) if \(\hat{N}_i(i)\) is empty, and similarly with \(s(i)\).) Then we take
\[
(\hat{N}_1', \hat{N}_2') = (\hat{N}_i(i^*), N' \setminus \hat{N}_i(i^*)), \text{ if } s_1(i^*) \leq s_2(i^*);
\]
\[
(\hat{N}_1', \hat{N}_2') = (N' \setminus \hat{N}_2(i^*), \hat{N}_2(i^*)), \text{ if } s_1(i^*) > s_2(i^*).
\]

The set \(\hat{N}_i(i)\) estimates the set of agents of lower type than \(i\), and the set \(\hat{N}_2(i)\) estimates the set of agents of higher type than \(i\). Let
\[
N'_i(i) = \{j \in N' \setminus \{i\} : \tau(i) > \tau(j)\}, \text{ and } N_2'(i) = \{j \in N' \setminus \{i\} : \tau(i) < \tau(j)\}.
\]
A necessary condition for \(\hat{N}_i(i)\) to coincide with \(N'_i(i)\) is that \(i\) has higher type than those in \(\hat{N}_i(i)\). The more negative the quantity \(s_1(i)\) is, the more likely that this necessary condition is met. A similar observation applies to \(s_2(i)\) as well. Thus we choose a partition that minimizes \(\min\{s_1(i), s_2(i)\}\) over \(i\).

Suppose that we are given an ordered partition \((\hat{N}_1', ..., \hat{N}_s')\) of \(N\). The Selection-Split Algorithm that we propose produces an ordered partition \((\hat{N}_1'', ..., \hat{N}_s'')\) of \(N\) from \((\hat{N}_1', ..., \hat{N}_s')\) using two steps, the Selection Step and the Split Step, as follows.

1. **The Selection Step:** Let \(\hat{p}_k = \min_{i,j \in \hat{N}_k, i \neq j} \hat{p}_{ij}^0, k = 1, ..., s\), and select \(\hat{N}_k\) such that
\[
\hat{p}_{k*} = \min_{1 \leq k \leq s} \hat{p}_k.
\]

2. **The Split Step:** We split \(\hat{N}_k\) into \((\hat{N}_{k,1}', \hat{N}_{k,2}')\) using the Split Algorithm, and relabel the partition: \((\hat{N}_{1}', ..., \hat{N}_{k-1}', \hat{N}_{k,1}', \hat{N}_{k,2}', \hat{N}_{k+1}', ..., \hat{N}_s') = (\hat{N}_{1}'', ..., \hat{N}_s'')\).

The Selection Step chooses a group \(\hat{N}_k\) that is most likely to contain agents with heterogeneous types and the Split Step splits this group into two sets using the Split Algorithm. The Selection-Split algorithm depends on the data only through the p-values \(\hat{p}_{ij}^0, s \in \{+, 0, -\}\).

4.2.2. **The Classification Method.** For a given positive integer \(K\), partition \(N\) into \(K\) groups as follows. First, split \(N\) into \((\hat{N}_{1}^2, \hat{N}_{2}^2)\) using the Split Algorithm to \(N\), and apply the Selection-Split Algorithm sequentially to obtain \((\hat{N}_{1}^3, \hat{N}_{2}^2, \hat{N}_{3}^3)\), \((\hat{N}_{1}^4, ..., \hat{N}_{4}^4)\), and so on, until we have \((\hat{N}_{1}^K, ..., \hat{N}_{K}^K)\) for a given number \(K\). For each \(K\), we define
\[
\hat{V}(K) = \frac{1}{K} \sum_{k=1}^{K} \min_{i,j \in N_k^K} \log \hat{p}_{ij}^0,
\]
and then select
\[
\hat{K} = \arg\min_{1 \leq K \leq n} \hat{V}(K) + Kg(L),
\]
where \( g(L) \) is slowly increasing in \( L \). We take \( \hat{T}_K = (\hat{N}_1^K, ..., \hat{N}_L^K) \) to be our estimated group structure. The component \( \hat{V}(K) \) measures the goodness-of-fit of the classification, and the second component \( K g(L) \) represents a penalty term that prevents overfitting. We show that \( \hat{T}_K \) is consistent for the underlying group structure \( T \) under regularity conditions.

4.3. Constructing \( p \)-Values Using Bootstrap

In most applications, we can use bootstrap to construct \( p \)-values for testing the inequality restrictions of (4.1). For the sake of concreteness, we explain the bootstrap procedure along the proposal made by Lee, Song, and Whang (2018). Suppose that we are given observations \( \{Z_l\}_{l=1}^L \), where \( Z_l = (Z_{i,l})_{i=1}^n \) denotes the observations pertaining to market \( l \) and \( Z_{i,l} \) denotes the vector of observations specific to agent \( i \). Suppose that for each pair of agents \( i \) and \( j \), there exists a nonparametric function, say, \( r_{ij}(x) \) such that \( \tau(i) \geq \tau(j) \) if and only if \( r_{ij}(x) \geq 0 \) for all \( x \in \mathcal{X} \), where \( \mathcal{X} \) is the common domain of the function \( r_{ij}(\cdot) \), \( i, j \in N \).

To construct a test statistic, we first estimate \( r_{ij}(x) \) using the sample \( \{Z_l\}_{l=1}^L \) to obtain \( \hat{r}_{ij}(x) \) (e.g., using a kernel regression estimator). Then we construct the following indexes:

\[
\hat{\delta}_{ij} = \int \max \{ \hat{r}_{ij}(x), 0 \} \, dx \quad \text{and} \quad \hat{\delta}_{ij}^0 = \int |\hat{r}_{ij}(x)| \, dx.
\]

For \( p \)-values, we re-sample \( \{Z^*_l\}_{l=1}^L \) (with replacement) from the empirical distribution of \( \{Z_l\}_{l=1}^L \) and construct a nonparametric estimator \( \hat{r}_{ij}^*(x) \) for each pair \((i, j)\) in the same way as we did using the original sample. Using these bootstrap estimators, we construct the following bootstrap test statistics:

\[
\hat{\delta}_{ij}^* = \int \max \{ \hat{r}_{ij}^*(x) - \hat{r}_{ij}(x), 0 \} \, dx \quad \text{and} \quad \hat{\delta}_{ij}^{0*} = \int |\hat{r}_{ij}^*(x) - \hat{r}_{ij}(x)| \, dx.
\]

Note that the bootstrap test statistic involves recentering to impose the null hypothesis. Now, the \( p \)-values, \( \hat{p}_{ij}^+, \hat{p}_{ij}^-, \) and \( \hat{p}_{ij}^0 \) can be constructed from the bootstrap distributions of \( \hat{\delta}_{ij}^*, \hat{\delta}_{ji}^*, \) and \( \hat{\delta}_{ij}^{0*} \) respectively, using \( \hat{\delta}_{ij}, \hat{\delta}_{ji} \) and \( \hat{\delta}_{ij}^0 \) as test statistics.

4.4. Consistency of Classification

We prove consistency of the estimated classification \( \hat{T}_K \) as \( L \to \infty \) while \( n \) fixed. (The consistency results and the proof (with high level conditions) for the case of both \( n \) and \( L \) increase to infinity can be found in the supplemental note.) Let \( \mathcal{P} \) be the collection of

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\(^{12}\) The choice of \( g(L) = \log \log L \) appears to work very well from our numerous Monte Carlo simulation experiments.

\(^{13}\) See Bugni (2010), Andrews and Shi (2013), Chernozhukov, Lee, and Rosen (2013), Lee, Song, and Whang (2013), and Lee, Song, and Whang (2018), among many others, and references therein.
the distributions $P$ of the whole vector of the observations in each market $l$. For each $\varepsilon > 0$, $ij \in E$ and $s \in \{+, 0, -\}$, we define

$$P_{0,ij}^s = \{ P \in \mathcal{P} : \delta_{ij}^s(P) \leq \varepsilon \}, \quad \text{and} \quad P_{\varepsilon,ij}^s = \{ P \in \mathcal{P} : \delta_{ij}^s(P) \geq \varepsilon \},$$

where we write the pairwise indexes $\delta_{ij}^s$ as $\delta_{ij}^s(P)$ to reflect that the pairwise indexes depend on $P$. Thus $P_{0,ij}^s$ is the collection of probabilities under the pairwise null hypothesis $H_{0,ij}$, and $P_{\varepsilon,ij}^s$ is the collection of probabilities under the pairwise alternative hypotheses $H_{1,ij}$ such that $\delta_{ij}^s(P)$ is away from zero at least by $\varepsilon$. Then we define

$$P_{0,\varepsilon} = \bigcup_{s \in \{+, 0, -\}} \bigcup_{ij \in E} (P_{0,ij}^s \cup P_{\varepsilon,ij}^s).$$

We assume that the p-value takes the following form:

$$\hat{p}_{ij}^s = 1 - F_{ij}^s(\tilde{T}_{ij}^s), \quad s \in \{+, 0, -\},$$

where $F_{ij}^s$ is a CDF and $\tilde{T}_{ij}^s$ is a random variable both of which depend on the data. Typically, $\tilde{T}_{ij}^s$ represents an appropriately normalized test statistic and $F_{ij}^s$ represents the CDF of the bootstrap distribution of the test statistic after recentering. We make the following assumption.

**Assumption 4.1.** There exist sequences $\lambda_L \to \infty$ and $\rho_L \to 0$ and constants $c_{ij}^s$, $s \in \{+, 0, -\}$ such that along each sequence of probabilities $P_L \in P_{0,\varepsilon}$ and for each pair $i, j \in N$, the following holds for all $s \in \{+, 0, -\}$, as $L \to \infty$.

(i) If $\tau(i) = \tau(j)$, $\tilde{T}_{ij}^s \to_d W_{ij}^s$, for some random variable $W_{ij}^s$.

(ii) If $\tau(i) > \tau(j)$, $\tilde{T}_{ij}^+ / \lambda_L \to_P c_{ij}^+$ and $\tilde{T}_{ij}^- = O_P(1)$.

(iii) If $\tau(i) < \tau(j)$, $\tilde{T}_{ij}^- / \lambda_L \to_P c_{ij}^-$ and $\tilde{T}_{ij}^+ = O_P(1)$.

(iv) If $\tau(i) \neq \tau(j)$, $\tilde{T}_{ij}^0 / \lambda_L \to_P c_{ij}^0$.

(v) $\sup_{t \in \mathbb{R}} |\tilde{F}_{ij}^s(t) - F_{ij,\infty}^s(t)| = O_P(\rho_L)$, where $F_{ij,\infty}^s$ is the CDF of $W_{ij}^s$.

(vi) $r_L^{-1} \log(1 - F_{ij,\infty}^s(c_1 \lambda_L + c_2 \rho_L)) \to -\infty$, for all constants $c_1, c_2 > 0$.

Assumption 4.1 is typically satisfied for various choices of test statistics that arise in the literature of moment inequality testing. See Section C.2 of the supplemental note for some lower level conditions for the case of testing procedures based on Lee, Song, and Whang (2018). In this case, $F_{ij,\infty}^s$ is a standard normal CDF.

**Theorem 4.1.** Suppose that Assumption 4.1 holds, and that $g(L) \to \infty$ and $g(L)/r_L \to 0$ as $L \to \infty$. Then, for any $\varepsilon > 0$, along a sequence of probabilities $P_L$ from $P_{0,\varepsilon}$,

$$P_L\{\hat{K} = K_0\} \to 1, \quad \text{as} \quad L \to \infty,$$

and the estimated group structure $\hat{K}$ satisfies that as $L \to \infty$,

$$P_L\{\hat{K} = T\} \to 1.$$
Table 1: Group Structure in Experiments

| Structure | n  | $K_0$ | $n_k$ |
|-----------|----|-------|-------|
| S1        | 12 | 2     | 6     |
| S2        | 12 | 4     | 3     |
| S3        | 40 | 2     | 20    |
| S4        | 40 | 4     | 10    |

Note: $n$ denotes the total number of the bidders; $K_0$ denotes the number of the groups; $n_k$ denotes the number of actual bidders from group $k$. For each structure in the simulation design, groups all have the same number of bidders.

The proof of Theorem 4.1 proceeds in two steps. First, we show that $\hat{T}_{K_0}$ is consistent for $T$. Second, we show that $\hat{K}$ is consistent for $K_0$. To see the intuition for this second step, note that when $K \geq K_0$, the component $\hat{V}(K)$ is $O_P(1)$, and when $K < K_0$, the component $\hat{V}(K)$ diverges at a rate faster than $g(L)$. From this, we obtain that $\hat{K}$ is consistent for $K_0$.

5. Monte Carlo Simulations

5.1. Finite Sample Performance of the Classification

We use a model of a first-price procurement auction with asymmetric independent private costs to study performance of our classification procedure. (See Appendix B.1 of the supplemental note.) Bidders are classified into $K_0$ groups. We abstract away from the formation of equilibrium strategies, and draw bids from a normal distribution $N(\mu_k, \sigma^2)$. Let $L$ denote the number of auctions in which any given pair of bidders participates. We consider two specifications of $\mu_k$’s. In one specification, $\mu_1 = 2.0, \mu_2 = 2.6, \mu_3 = 3.2$, and $\mu_4 = 3.8$ with increment $D_\mu = 0.6$, and in the other specification, $\mu_1 = 2.0, \mu_2 = 2.2, \mu_3 = 2.4$, and $\mu_4 = 2.6$ with increment $D_\mu = 0.2$. The variance $\sigma^2$ is taken to be 0.25.

Table 1 summarizes the designs of group structures in our simulation. The first two structures involve a total of 12 bidders and the last two 40 bidders. The first and third are designed to be coarser group structures than the second and fourth respectively. We construct $p$-values using the procedure in Section 4.3 and obtain group classification from 500 simulated samples. For each estimate, we used 200 bootstrap iterations to calculate $p$-values.

To evaluate the performance of our classification method, we define a measure of discrepancy between two ordered partitions $T_1$ and $T_2$:

\[
(5.1) \quad \delta (T_1, T_2) = \frac{1}{K_1} \sum_{k=1}^{K_1} \min_{1 \leq j \leq K_2} |N_k^1 \Delta N_j^2|,
\]
Table 2: Performance of the Classification with One Group ($K_0 = 1$ and unknown)

| $n$ | $L$ | $\hat{K}_0$ | EAD | HAD(.10) | HAD(.25) | HAD(.50) |
|-----|-----|-------------|-----|----------|----------|----------|
| 12  | 400 | 1.002       | 0.012 | 0.001 | 0.000 | 0.000 |
| 12  | 200 | 1.003       | 0.014 | 0.002 | 0.000 | 0.000 |
| 12  | 100 | 1.003       | 0.018 | 0.002 | 0.001 | 0.000 |
| 40  | 400 | 1.003       | 0.082 | 0.005 | 0.003 | 0.000 |
| 40  | 200 | 1.006       | 0.084 | 0.008 | 0.002 | 0.000 |
| 40  | 100 | 1.008       | 0.096 | 0.010 | 0.004 | 0.000 |

Note: $n$ is the number of bidders in data; and $L$ the number of markets. $\hat{K}_0$ is the average number of estimated groups in 500 simulation samples. EAD is the average number of mismatched bidders across true groups and simulated samples. HAD($\lambda$) is the hazard rate of average discrepancy. For example, HAD(.10) = 0.002 means that in 499 simulated samples (out of a total of 500) the average number of mismatched bidders is less than 10 percent of the total number of bidders.

Table 3: Performance of the Classification with Multiple Groups ($K_0 \geq 2$ and unknown)

| $n$ | $L$ | $D_\mu$ | $\hat{K}_0$ | EAD | HAD(.25) | HAD(.75) | $\hat{K}_0$ | EAD | HAD(.25) | HAD(.75) |
|-----|-----|--------|-------------|-----|----------|----------|-------------|-----|----------|----------|
| 12  | 400 | 0.6    | 2.00        | 0.00 | 0.00     | 0.00     | 3.96        | 0.03 | 0.01     | 0.00     |
| 12  | 400 | 0.2    | 2.00        | 0.01 | 0.00     | 0.00     | 3.94        | 0.04 | 0.03     | 0.00     |
| 12  | 100 | 0.6    | 2.00        | 0.00 | 0.00     | 0.00     | 3.98        | 0.01 | 0.01     | 0.00     |
| 12  | 100 | 0.2    | 2.03        | 0.52 | 0.07     | 0.004    | 3.24        | 1.53 | 0.24     | 0.00     |
| 40  | 400 | 0.6    | 2.00        | 0.01 | 0.00     | 0.00     | 3.97        | 0.08 | 0.02     | 0.00     |
| 40  | 400 | 0.2    | 2.01        | 0.01 | 0.00     | 0.00     | 3.83        | 0.43 | 0.09     | 0.00     |
| 40  | 100 | 0.6    | 2.01        | 0.01 | 0.00     | 0.00     | 3.95        | 0.13 | 0.03     | 0.00     |
| 40  | 100 | 0.2    | 2.18        | 1.91 | 0.02     | 0.00     | 3.06        | 1.93 | 0.49     | 0.11     |

Note: $\hat{K}_0$, EAD and HAD($\lambda$) are defined as in Table 2. $D_\mu$ is the difference between group means $\mu_1$ and $\mu_2$. Conditional on the number of markets ($L$) and the number of bidders in population ($n$), the classification task is harder when the difference between group means $D_\mu$ is smaller.

where $T_1 = (N^1_1,...,N^1_{K_1})$ and $T_2 = (N^2_1,...,N^2_{K_2})$ are ordered partitions of $N$ and $\Delta$ denotes set-difference. We evaluate our classification method using two criterion: (1) Expected Average Discrepancy (EAD) defined as $E(\delta(T,\hat{T}_{\hat{K}}))$ and (2) the hazard rate of EAD, i.e., $\text{HAD}(\lambda) = P\{\delta(T,\hat{T}_{\hat{K}}) > \lambda n\}$ for $0 < \lambda < 1$.

Table 2 reports estimates when there is no unobserved heterogeneity among bidders ($K_0 = 1$). In this case, our procedure detects the absence of unobserved heterogeneity effectively. For a given $n$, there is a moderate increase in the accuracy of classification as $L$ increases, both in terms of EAD and HAD($\lambda$).

Table 3 reports results for $K_0 = 2$ and $K_0 = 4$. In both cases, the estimates for $K_0$ are mostly correct. Estimation accuracy increases with the difference between group means. For a given number of groups, the performance in terms of EAD and HAD are both better with greater group differences and larger sample sizes.
5.2. Two-Step Estimation in a Structural Model

In this section we use a simple structural model of procurement auctions to investigate the impact of classification errors on subsequent estimation of structural parameters. A set of \( N \) providers (bidders) is partitioned into \( K_0 \) groups, each with a distinct distribution of private costs. Let \( N_k \) denote the set of providers in \( N \) with type \( k \in \{1, 2, ..., K_0\} \), and let \( |N_k| \) denote its cardinality. The cost for a provider \( i \) with type \( \tau(i) \in \{1, 2, ..., K_0\} \) is given by \( c_{i,t} = \mu_{\tau(i)} + \epsilon_{i,t} \), where \( \epsilon_{i,t} \) follows \( N(0, \sigma) \) with the support \([\underline{c}, \bar{c}]\).\(^{14}\)

Auction participants are determined in two steps. First, two out of \( K_0 \) groups, \( \tau_{l,1} \) and \( \tau_{l,2} \), are chosen at random. Next, \( n_1 \) and \( n_2 \) providers are randomly drawn from the corresponding groups \( N_{\tau_{l,1}} \) and \( N_{\tau_{l,2}} \) and their costs were constructed as above. Here \( n_1 \) and \( n_2 \) denote the numbers of actual participants (those who submitted bids in the auction). Then participants bid based on their realized costs. The participant with the lowest bid wins. The identity of each participant and its bid are both reported in data. We use a modified version of the numerical method in Marshall, Meurer, Richard, and Stromquist (1994) to compute \( \mu_{\tau_{l,1}} \) and \( \mu_{\tau_{l,2}} \) for a given \( \theta_0 \) profile of participant types. Standard errors are computed from the analytic expression for the covariance matrix in asymptotic distribution.

To compute \( \mu_{B,k}(\theta_0; I) \) and \( \sigma_{B,k}(\theta_0; I) \) for a given \( \theta_0 \) profile of participant types \( I = (\tau_{l,1}, \tau_{l,2}, n_1, n_2) \) we simulate the equilibrium bidding functions.\(^{15}\) The bidding functions

\(^{14}\)We set the upper and lower bounds of costs to \( \underline{c} = \frac{1}{K_0} \sum_k (\mu_k - 1.96 \times \sigma) \) and \( \bar{c} = \frac{1}{K_0} \sum_k (\mu_k + 1.96 \times \sigma) \). True parameters are chosen so that \( \underline{c} \) is strictly positive.

\(^{15}\)Specifically, we start from the analytical bidding function when all participants belong to the same group, and use a modified version of the numerical method in Marshall, Meurer, Richard, and Stromquist (1994)
Table 4: Simulation Results from Specifications A and B

| Spec A Using True Groups | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\sigma$ |
|--------------------------|---------|---------|---------|---------|---------|
| Rej. Prob.               | 0.0150  | 0.0515  | 0.0523  | 0.0546  | 0.0149  |
| Bias                     | -0.0189 | -0.0252 | -0.0610 | -0.0511 | 0.0242  |
| MSE                      | 0.0005  | 0.0008  | 0.0035  | 0.0039  | 0.0039  |

| Using Est’d Groups       |         |         |         |         |         |
|--------------------------|---------|---------|---------|---------|---------|
| Rej. Prob.               | 0.0148  | 0.0542  | 0.0510  | 0.0485  | 0.0151  |
| Bias                     | 0.0059  | 0.0329  | -0.0241 | -0.0225 | -0.0549 |
| MSE                      | 0.0041  | 0.0083  | 0.0035  | 0.0027  | 0.0383  |

| Spec B Using True Groups | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\sigma$ |
|--------------------------|---------|---------|---------|---------|---------|
| Rej. Prob.               | 0.0120  | 0.0515  | 0.0512  | 0.0514  | 0.0111  |
| Bias                     | -0.0211 | -0.0233 | -0.0621 | -0.0622 | 0.0236  |
| MSE                      | 0.0005  | 0.0007  | 0.0039  | 0.0039  | 0.0034  |

| Using Est’d Groups       |         |         |         |         |         |
|--------------------------|---------|---------|---------|---------|---------|
| Rej. Prob.               | 0.0131  | 0.0550  | 0.0540  | 0.0530  | 0.0160  |
| Bias                     | -0.0213 | -0.0218 | -0.0763 | -0.0765 | 0.0211  |
| MSE                      | 0.0004  | 0.0015  | 0.0411  | 0.0441  | 0.0023  |

Note: Specification A uses $K_0 = 4$, $n_k = 4$, and $L = 200$ and Specification B uses $K_0 = 4$, $n_k = 10$, and $L = 200$. Here $n_k$ is the number of bidders in group $k$, and $L$ the number of markets. The rejection probabilities are from $t$-tests for the individual parameters. The nominal rejection probability is set to 0.05.

are then combined with the cost distributions implied by a vector of trial parameters, $\theta_0$, to obtain the distribution of bids: $F_{B,k}(b|\theta_0, I) = F_{C,k}(\beta_k^{-1}(b)|\theta_0)$. Here $F_{C,k}(.|\theta_0)$ denotes the distribution of project’s cost for a bidder belonging to group $k$ which is correspond to a parameter vector $\theta_0$ and $\beta_k(c)$, $\beta_k^{-1}(b)$ are the bid and the inverse bid functions used by such bidder. We then compute the mean and the standard deviation of this bid distribution.

Tables 4 and 5 report the bias and mean squared errors (MSEs) of two estimators for $(\mu_k)_{k=1}^{K_0}$ and $\sigma$. The first is an “infeasible” estimator that uses the knowledge of the true group structure. The second is the two-step estimator we propose, which requires bidder classification in the first step. These two tables also report the rejection probabilities from t-tests of individual parameters.

Table 4 reports results for a smaller sample with $L = 200$. The rejection probabilities are close to the nominal rejection rate 0.05, except for parameters $\mu_1$ and $\sigma$. The MSE and bias for all parameters are reasonably small. Table 4 also shows that the rejection probabilities for the infeasible estimator using the true groups and those for the actual estimator using the estimated groups are very similar. There is some minor difference to solve for bidding strategies in the presence of multiple groups. We impose a sample version of $c$ and $\bar{c}$ by replacing $\sigma$ with its sample analog.
Table 5: Simulation Results from Specifications C and D

| Spec C Using True Groups | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\sigma$ |
|--------------------------|---------|---------|---------|---------|---------|
| Rej. Prob.               | 0.0149  | 0.0500  | 0.0513  | 0.0520  | 0.0149  |
| Bias                     | -0.0214 | -0.0222 | -0.0615 | -0.0713 | 0.0237  |
| MSE                      | 0.0005  | 0.0007  | 0.0039  | 0.0039  | 0.0006  |

| Spec C Using Est’d Groups |
|---------------------------|
| Rej. Prob.               | 0.0151  | 0.0485  | 0.0526  | 0.0545  | 0.0151  |
| Bias                     | 0.0131  | 0.0412  | -0.0625 | -0.0656 | -0.0241 |
| MSE                      | 0.0060  | 0.0008  | 0.0053  | 0.0031  | 0.0054  |

| Spec D Using True Groups | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\sigma$ |
|--------------------------|---------|---------|---------|---------|---------|
| Rej. Prob.               | 0.0133  | 0.0480  | 0.0510  | 0.0520  | 0.0108  |
| Bias                     | -0.0227 | -0.0219 | -0.0611 | -0.0231 | 0.0236  |
| MSE                      | 0.0005  | 0.0007  | 0.0039  | 0.0039  | 0.0006  |

| Spec D Using Est’d Groups |
|---------------------------|
| Rej. Prob.               | 0.0126  | 0.0498  | 0.0520  | 0.0520  | 0.0128  |
| Bias                     | -0.0229 | -0.0123 | -0.0761 | -0.0361 | 0.0098  |
| MSE                      | 0.0093  | 0.0056  | 0.0068  | 0.0061  | 0.0007  |

Note: Specification C uses $K_0 = 4$, $n_k = 4$, and $L = 400$, and Specification D, $K_0 = 4$, $n_k = 10$, and $L = 400$. Here $n_k$ is the number of bidders in group $k$ and $L$ the number of markets. The nominal rejection probability is set to 0.05.

between these two estimators in the MSE of some group means. The discrepancy seems more pronounced when the size of each group is increased from $|N_k| = 4$ to $|N_k| = 10$.

Table 5 reports the same results for larger samples with $L = 400$. The performance of the estimators improves slightly relative to Table 4. Again, the rejection probabilities for the two estimators are similar. There is also evidence that with a larger number of the markets, our classification method performs better given the same number of within-group bidders. Overall, Table 4 and 5 provide simulation evidence that the classification errors in the first-step do not have any major impact on the finite sample performance of the two-step estimators for structural parameters.

6. Empirical Application: California Market for Highway Procurement

We apply our methodology to analyze procurement auctions conducted by the California Department of Transportation (CalTrans) to allocate projects for highway repair work. Our goal is to demonstrate the performance of our method in the empirical setting, and to highlight the consequences of ignoring agent unobserved heterogeneity in estimation.
Model. We follow the literature in modeling the auction market so that each project attracts a set of potential bidders who decide whether to participate in the auction and, if deciding to participate, choose a bid to submit. Our main innovation is to allow for contractors participating in this market to differ in a way that is not observed by the researcher. Specifically, each contractor is characterized by a contractor-specific cost factor (invariant across projects) $q_i$ which takes discrete values in $\{\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_{K_0}\}$. This unobserved cost factor captures the difference in cost efficiencies across firms generated perhaps by the differences in managerial ability or other factors associated with the firm organization. As in our basic set up this cost factor induces partitioning of the population of firms participating in this market into the groups: $N = \bigcup_{k=1}^{K_0} N_k$ with $N_k = \{i : q_i = \bar{q}_k\}$ and so that $\tau(i) = k$ if and only if $i \in N_k$.

Following the convention in the empirical auction literature, we assume that each project $l$ auctioned in this market is summarized by a set of observable characteristics $X_l$ and an unobservable factor $U_l$. The latter is distributed according to normal distribution with mean zero and standard deviation $\sigma_U$. The set of firms which are potentially interested in project $l$ (potential bidders), denoted here by $S_l$, is exogenously drawn from $N$. A contractor $i$ that is a potential bidder for project $l$ is characterized by private entry costs, $E_{i,l}$, and the private cost of completing the project, $C_{i,l}$. We assume that private costs vary independently across bidders and auctions. The entry costs additionally are independent of $U_l$, and are distributed according to the exponential distribution with a rate parameter $\lambda_{i,l}$. The costs of completing the work are drawn from a log normal distribution with mean $\mu_{i,l}$ and standard deviation $\sigma_C$. The mean of the cost distribution depends on project characteristics including the distance between the project and the bidder’s locations, $D_{i,l}$, as well as an unobserved cost factor $q_i$.\footnote{The groups reflect differences in the contractors’ cost efficiencies related to the project work. While entry costs may also vary across groups, there is no reason for the group differences in project costs to coincide with the group differences in entry costs. For this reason, we explicitly distinguish between the parameters capturing the former ($\bar{q}_k$) and the latter ($\tilde{q}_k$) effects.} \footnote{Following the literature, we distinguish between the bidders who regularly participate in the procurement market (regular bidders) and those who only appear in a very small number of auctions (fringe bidders). We assume that all fringe bidders are associated with the same fixed level of the unobserved cost factor $\bar{q}_0$.} Reflecting these features, we set

$$\mu_{i,l} = X_l\alpha_1 + D_{i,l}\alpha_2 + \sum_{k=1}^{K_0} \bar{q}_k 1\{\tau(i) = k\} + U_l, \quad \text{and} \quad \lambda_{i,l} = X_l\gamma_1 + \sum_{k=1}^{K} \tilde{q}_k 1\{\tau(i) = k\}. $$
A potential bidder decides to participate in the auction for project $l$ if his ex-ante expected profit conditional on participation exceeds entry costs.\footnote{The expected profit reflects his expectation over the participation decisions of other potential bidders, the expectation over his costs of completing the project, and reflects expected probability of winning the project which depends on the costs draws of his competitors.} A bidder who decides to participate observes realization of his costs and the identities of other contractors who decided to participate. He chooses a bid to maximize his interim profit which reflects probability of winning the project conditional on his costs draw and the set of competitors. We assume that the observed outcomes reflect a type-symmetric pure-strategy Bayesian Nash equilibrium (psBNE).\footnote{In such an equilibrium, participants who are \textit{ex ante} identical in an auction $l$ (i.e. $i,j \in S_l$ such that $q_i = q_j$, and $D_{i,l} = D_{j,l}$) adopt the same strategies.}

**Estimation Details.** The estimation methodology consists of two steps. In the first step we use the pairwise comparison indexes to recover the unobserved group structure. In the second step the parameters of the model are estimated through a GMM procedure while imposing the group structure recovered in the first step. We assume bids are rationalized by a single equilibrium.

In the first step we use the pairwise comparison indexes derived in Appendix B.1 in the supplemental note to recover the unobserved group structure.\footnote{The pairwise comparison indexes are derived using Corollary 3 of Lebrun (1999) which for $G_{ij}(b)$ defined as $P\{B_{i,l} \geq b | D_{i,l} = d, D_{j,l} = d, i \in A_l, j \in A_l\}$ establishes that $G_{ij}(b) \leq G_{ji}(b)$ for all $b$ in the common support of $B_{i,l}$ and $B_{j,l}$ whenever $\tau(i) \leq \tau(j)$. The inequality holds strictly at least over some interval with positive Lebesgue measure and holds unconditionally when aggregated over bidder identities and auction characteristics.} Specifically, in accordance with the notation used in the paper, we define $\delta_{ij} \equiv \int \max\{r_{ij}(b), 0\} \, db$ and $\delta^0_{ij} \equiv \int |r_{ij}(b)| \, db$ with $r_{ij}(b) = G_{ji}(b|d) - G_{ij}(b|d)$ and $G_{ij}(b|d) = P\{B_{i,l} \geq b | D_{i,l} = d, D_{j,l} = d, i \in A_l, j \in A_l\}$.\footnote{We recover group structure on the basis of the indexes which aggregate over the values of the distance $d$. As a robustness check we also compute groupings on the basis of subsets of distances. We find that the results of classification are very similar across these approaches.}

In the second stage, we consider the following moments: (a) the first and the second moment of bid distribution for a given level of $d$ and for a given group of bidders; (b) the covariance between bids and the observable project characteristics; (c) the covariance between any two bids submitted in the same auction; (d) the expected number of participants in any given auction for every $(d, q)$-group; (e) the covariance between the number
of participants and the observable project characteristics. We search for the set of parameters which minimizes the distance between the empirical and theoretical counterparts of these moments subject to participation constraints.\footnote{We do not explicitly solve for participation strategies. Instead, we discretize the support of auction characteristics \((X_l, U_l)\) and treat the probabilities of participation for bidders of various \((q, d)\)–types corresponding to these grid values as auxiliary parameters. We follow the spirit of Dube, Fox, and Su (2012) by maximizing a moment-based objective function subject to the constraints that the optimality of the participation strategies is satisfied on the grid of the project characteristics’ values.}

**Estimation Results.** We implement the analysis using the data for California Highway Procurement projects auctioned between 2002 and 2012. The projects in our sample are worth $523,000 and last for around three months on average; 38\% of these projects are partially supported through federal funds. There are 25 firms that participate regularly in this market. The other firms are referred to as “fringes”. An average auction attracts six regular potential bidders and eight fringe bidders. Since only a fraction of potential bidders submits bids, an entry decision plays an important role in this market. Finally, the distance to the company location varies quite a bit and is around 28 miles on average for regular potential bidders.

In the first step, we obtain through our classification method the grouping of the bidders into eight groups that consist of 2, 3, 8, 3, 2, 3, 2 and 2 bidders respectively. The parameter estimates obtained in the second stage of our estimation procedure and their standard errors are summarized in Table 6. We normalize bids by the engineer’s estimate in the estimation. Therefore all the parameters measure the effects relative to the project size. The first two columns present the estimates which are obtained when the unobserved group structure is taken into account in the estimation. The results indicate significant differences in bidders’ costs across the groups. Specifically, fringe bidders (the reference group) tend to have the highest costs whereas the difference in costs between the group of fringe bidders and the groups of regular bidders is comparable in impact to the shortening of the distance to the project site by 42.5 (i.e., by 0.051/0.0012), 10.1, 26.67, 48.33, 11.67, 6.67, 7.5, and 41.67 miles respectively.\footnote{The estimates of the group-specific fixed effects are omitted for brevity. The full table that contains these estimates is found in the supplemental note to the paper.} The distance increases project costs (additional 8.33 miles result in costs which are 1\% higher on average).\footnote{Recall that the coefficients reflect the impact on costs in terms of the fraction of the engineer’s estimate. The distance resulting in 0.01 increase of average costs can thus be computed as 0.01/0.012.}

The entry costs of regular bidders are significantly lower than entry costs of fringe bidders. However, they appear to be quite similar across the groups of regular bidders.

The last two columns of Table 6 show the parameter estimates under the specification when the unobserved group structure of the regular bidders is ignored in the estimation. The parameter estimates are obtained by the GMM estimation procedure using the same
Table 6. Parameter Estimates

|                                | Estimate | Std. Error | Estimate | Std. Error |
|--------------------------------|----------|------------|----------|------------|
| **The Distribution of Project Costs** |          |            |          |            |
| Constant ($\bar{q}_0$)         | 0.127*** | (0.0129)   | 0.113*** | (0.0119)   |
| Eng. Estimate                  | -0.0004*** | (0.0002)   | -0.0005*** | (0.0002)   |
| Duration                       | 0.00026*  | (0.00036)  | 0.00022*  | (0.00027)  |
| Distance                       | 0.0012*** | (0.00022)  | 0.00086*** | (0.00019)  |
| Bridge                         | -0.0092*** | (0.0018)   | -0.012*** | (0.0011)   |
| Federal Aid                    | -0.043*** | (0.0103)   | -0.075*** | (0.009)    |
| Regular Bidder                 |          |            |          |            |
| $\sigma_C$                     | 0.087***  | (0.032)    | 0.112***  | (0.022)    |
| $\sigma_U$                     | 0.021***  | (0.009)    | 0.0207*** | (0.008)    |
| **The Distribution of Entry Costs** |          |            |          |            |
| Constant ($\tilde{q}_0$)       | -0.0114*  | (0.0078)   | -0.0161*  | 0.0091     |
| Eng. Estimate                  | 0.0055*** | (0.0016)   | 0.0051*** | (0.0012)   |
| Number of Items                | 0.0018*   | (0.0011)   | 0.0011*** | (0.0005)   |
| Regular Bidder                 |          |            |          |            |
|                                | -0.022*** | (0.004)    |          |            |

Note: In the results above the distance is measured in miles. The fringe bidders are the reference group. The first two columns correspond to the specification which allows for the unobserved bidder heterogeneity; the last two columns correspond to the specification without unobserved bidder heterogeneity. The results are based on the data for 1,054 medium-sized projects that involve paving and bridge work. Standard errors are computed using bootstrap.

set of moments by imposing that only two groups of sellers are present in the data: fringe and regular bidders. Under this specification, the cost reduction due to the federal aid is estimated to be much higher (7.8% rather than 4.3%), the impact of the distance is estimated to be lower (the distance to the project has to be 11.67 miles higher in order to increase the average cost by 1%). Additionally, the entry costs are estimated to be lower relative to the baseline specification. Our results thus confirm that regular participants in the highway procurement market are characterized by important unobserved cost differences that persist in the data.

7. Conclusion

This paper makes a number of contributions to the literature. First, for models with strategic interdependence between multiple agents, we develop a method to classify these agents based on their discrete unobserved individual heterogeneity, using pairwise inequalities implied by an economic model. Second, we show such pairwise inequalities arise in a number of game-theoretical settings where identification of model primitives is challenging. Third, we propose a computationally feasible method which consistently estimates the group structure defined by unobserved heterogeneity. We apply this method
to California highway procurement data to show that unobserved bidder heterogeneity plays an important role in this procurement market.

The classification method proposed in this paper is especially useful in settings where the analysis of unobserved individual heterogeneity is complicated by the presence of strategic interdependence in the model. We offer new insights into the identification and estimation of such models. Specifically, classification could be used as a first step in the structural studies of many environments where analyses would otherwise be infeasible due to the identification or computational challenges.

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Supplemental Note for “Estimating Unobserved Agent Heterogeneity Using Pairwise Comparisons”

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Appendix A. Introduction

This note is a supplemental note to Krasnokutskaya, Song, and Tang (2019). It consists of five parts. Appendix B gives details about how to derive pairwise comparison index in examples of first-price and English auctions where bidders have asymmetric private values, or collusive behavior. Appendix C gives mathematical proofs of the results in the paper, and provide lower level conditions. Appendix D discusses a bootstrap method to construct a confidence set for the group structure. Appendix E presents further simulation results regarding the performance of our classification algorithm proposed in Krasnokutskaya, Song, and Tang (2019), and Appendix F reports summary statistics for the data used in the empirical application in the paper and some further results.

Appendix B. Bidders with Asymmetric Values or Collusive Behavior

B.1. First-Price Auctions with Asymmetric Bidders

Let the population of bidders, \( N \), be partitioned into \( K_0 \) groups, each of which is characterized by a distinct distribution of private values \( F_k(\cdot) \). To fix ideas, assume that \( F_k(\cdot) \) has the same shape of distribution, but differs only in their location (means) \( \bar{q}_1 < \bar{q}_2 < \ldots < \bar{q}_{K_0} \), with \( \bar{q}_k \) being the mean of \( F_k \). (Our method applies in a more general setting when \( F_k \)'s are stochastically ordered.) In this case, for a bidder \( i \) with \( \tau(i) = k \) the mean of the private value distribution is given by \( q_i = \bar{q}_k \). Let \( S_l \) denote the set of participants in auction \( l \) and \( B_l = \{ B_{i,l} \} \) a vector of bids. In a type-symmetric equilibrium \( B_{i,l} = \beta_k(V_{i,l}) \) if \( \tau(i) = k \), with \( \beta_k(\cdot) \) being a bidding strategy and \( V_{i,l} \) the private value for \( i \) in auction \( l \). Define \( G_{ij}(b) = P\{ B_{i,l} \leq b | i, j \in S_l \} \), where \( S_l \) denotes the set of bidders in auction \( l \). We assume bids are rationalized by a single equilibrium.

Corollary 3 of Lebrun (1999) showed that \( G_{ij}(b) \geq G_{ji}(b) \) for all \( b \) in the common support of \( B_{i,l} \) and \( B_{j,l} \) whenever \( \tau(i) \leq \tau(j) \). The inequality holds strictly at least over some interval with positive Lebesgue measure. This inequality holds unconditionally when aggregated over bidder identities and auction characteristics.\(^{25}\) Thus pairwise comparison

\(^{25}\)A similar property holds in the settings where allocations are implemented through first-price procurement auctions. The only difference is that in these settings \( G_{ij}(b) \) should be defined as \( G_{ij}(b) = P\{ B_{i,l} \geq b | i, j \in S_l \} \).
indexes can be constructed as follows:

(B.1) \[ \delta_{ij} \equiv \int \max \{ G_{ji}(b) - G_{ij}(b), 0 \} \, db. \]

Likewise, define \( \delta^0_{ij} \) by replacing the integrand in \( \delta_{ij} \) by the absolute value of \( G_{ij}(b) - G_{ji}(b) \). These indexes do not condition on the specific identities of bidders participating in each auction \( l \). Thus it allows us to utilize observations from a large number of auctions when constructing a comparison index for any generic pair \( i \) and \( j \).

For the rest of this subsection, we derive the pairwise comparison inequalities in a general model of asymmetric first-price auctions where independent private values are drawn from distributions that are stochastically ordered. That is, \( F_k \) first-order stochastically dominates \( F_{k'} \) whenever \( k' > k \). Also assume that the ordering of the distributions is strict (\( F_1(v) > F_2(v) > \ldots > F_{K_0}(v) \)) at least for \( v \) within some non-degenerate interval on the support. Let \( N(k) \) denote the set of all agents in group \( k \).

For simplicity, suppose that a bidder from group \( k \) becomes active with a fixed probability that is exogenously given. Let \( A \) denote the set of entrants in a given auction and \( \lambda(A) \) denote the structure, or the profile, of entrants. That is, \( \lambda(A) \) is a \( K_0 \)-vector of integers \( (|A(1)|, \ldots, |A(K_0)|) \), with \( A(k) \) being the set of entrants from group \( k \). An entrant \( i \) submits bid \( B_i \) according to his private value \( v_i \), taking into account the competitive structure of an auction \( \lambda(A) \) which he observes at the time of bidding. Across auctions in the data, \( A \) and \( \{v_i\}_{i \in A} \) are independent draws from the same population distribution.

Let \( G_k(\cdot; \lambda) \) be the distribution of \( B_i \) when \( i \in N(k) \). The private values are independent of \( \lambda(A) \) under exogenous entry. Part (i) of Corollary 3 in Lebrun (1999) showed that, given any realization of \( \lambda(A) \), the supremum of the support of bids is the same for all bidder types. That is, for any \( \lambda \), \( \beta_1(\overline{\lambda}|\lambda) = \beta_2(\overline{\lambda}|\lambda) = \ldots = \beta_K(\overline{\lambda}|\lambda) \equiv \eta(\lambda) < \infty \) for some \( \eta(\lambda) \in (\underline{v}, \overline{v}) \), where \( \beta_k \) denotes the equilibrium bidding strategy for a bidder from group \( k \). Furthermore, the corollary also showed that for any \( \lambda(A) \),

\[ F_{k'}(\beta_k^{-1}(b|\lambda(A))) \leq F_k(\beta_k^{-1}(b|\lambda(A))), \]

for all \( b \in [\underline{v}, \eta(\lambda(A))] \) and \( k < k' \), and the inequality holds strictly at least over some interval on \( [\underline{v}, \eta(\lambda(A))] \). Consider \( i \in N(k') \) and \( j \in N(k) \) with \( k' > k \). It then follows that

\[ P \{ B_i \leq b | i, j \in A \} = \sum_{\lambda(A)} F_{k'}(\beta_k^{-1}(b|\lambda(A)))P\{\lambda(A)|i, j \in A\} \]
\[ \leq \sum_{\lambda(A)} F_k(\beta_k^{-1}(b|\lambda(A)))P\{\lambda(A)|i, j \in A\} = P \{ B_j \leq b | i, j \in A \}, \]

with the inequality holding strictly over some non-degenerate interval in the shared bid support. The inequality does not condition on the identities of the entrants other than \( i \) and \( j \).

Finally, note that by a symmetric argument, a similar inequality holds in first-price procurement auctions with \( P \{ B_i \geq b | i, j \in A \} \leq P \{ B_j \geq b | i, j \in A \} \) (with inequality being strict over some non-degenerate interval in the shared bid support), whenever the private cost distribution for \( i \) is stochastically lower than that of \( j \).
B.2. English Auctions with Asymmetric Bidders

Consider the setting in Section B.1, except that the auction format is English (ascending). The data report the identities of entrants in $A$ and the transaction price $W$ in each auction. In a dominant strategy equilibrium, the price in an auction equals the second highest private value among all entrants.

With independent private values, we show below that

\[ P\{W \leq w | i \in A, j \not\in A\} \leq P\{W \leq w | j \in A, i \not\in A\}, \]

for all $w$ over the intersection of support, whenever $\tau(i) > \tau(j)$. Furthermore, the inequality holds strictly for some $w$ over a set of positive measure in common support. This implies

\[ E[W | i \in A, j \not\in A] > E[W | j \in A, i \not\in A]. \]

The intuition behind (B.2) is as follows. Given any structure of entrants who compete with $i$ or $j$ (but not both), the distribution of the transaction price is stochastically higher when $i$ is present but $j$ is not than when $j$ is present but $i$ is not. Loosely speaking, when $j$ is replaced by the stronger type $i$ in the set of entrants, the overall profile of value distributions becomes “stochastically higher”. Then the law of iterated expectations implies (B.2) and (B.3).

To infer the group structure, define the following indexes:

\[ \delta_{ij} = \max\{E[W | i \in A, j \not\in A] - E[W | j \in A, i \not\in A], 0\}, \]
\[ \delta^0_{ij} = |E[W | i \in A, j \not\in A] - E[W | j \in A, i \not\in A]|. \]

One can then use our procedure proposed in the main text to classify the bidders based on pairwise comparison.

We now derive (B.2) formally. Let $V_i$ denote the private value for bidder $i$. Consider the case where $i \in N_{(k^\prime)}$ and $j \in N_{(k)}$ with $k^\prime > k$. Let $\lambda(A)$ denote the $K_0$-vector of integers that summarizes the group structure of the set of entrants $A$. Let $1_k$ denote the unit vector with the $k$-th component being 1. Then define

\[ H_{j,i}(w; \lambda^*) \equiv P\{W \leq w | j \in A, i \not\in A, \lambda(A \backslash \{j\}) = \lambda^*\} \]
\[ = P\left\{ \max_{s \in A} V_s \leq w \mid \lambda(A) = \lambda^* + 1_k \right\} \]
\[ + P\left\{ \max_{s \in A} V_s > w, W \leq w \mid \lambda(A) = \lambda^* + 1_k \right\}, \]
where the first term on the right-hand side equals \( F_k(w) \left( \prod_{l=1}^{K_0} F_l(w)^{s_l} \right) \), and the second on the right-hand side is

\[
P \left\{ \max_{s \in A \setminus \{j\}} V_s \leq w, V_j > w \mid \lambda(A \setminus \{j\}) = \lambda^* \right\}
\]

\[
+ P \left\{ V_j \leq w, \max_{s \in A \setminus \{j\}} V_s > w \mid \lambda(A \setminus \{j\}) = \lambda^* \right\}
\]

\[
= [1 - F_k(w)] \left( \prod_{l=1}^{K_0} F_l(w)^{s_l} \right) + F_k(w) \varphi(w; \lambda^*),
\]

where \( \varphi(w; \lambda^*) \) denotes the probability that the maximum value in \( A \setminus \{j\} \) is strictly greater than \( w \) while the second highest value in \( A \setminus \{j\} \) is less than or equal to \( w \) conditional on the classification \( \lambda(A \setminus \{j\}) = \lambda^* \). Therefore

\[
H_{j,i}(w; \lambda^*) = \left( \prod_{l=1}^{K_0} F_l(w)^{s_l} \right) + F_k(w) \varphi(w; \lambda^*).
\]

By the same argument,

\[
H_{i,j}(w; \lambda^*) = P\{W \leq w|i \in A, j \not\in A, \lambda(A \setminus \{i\}) = \lambda^*\}
\]

\[
= \left( \prod_{l=1}^{K_0} F_l(w)^{s_l} \right) + F_k(w) \varphi(w; \lambda^*).
\]

It is then straightforward to show that for any \( \lambda^* \), that \( F_{k'} \succeq \text{F.S.D.} \) \( F_k \) implies \( H_{i,j}(w; \lambda^*) \leq H_{j,i}(w; \lambda^*) \) over the union of the \( K_0 \) supports of \( \{F_l : 1 \leq l \leq K_0\} \), and the inequality holds strictly at least for some \( w \) in an interval on the intersection of the \( K_0 \) supports of \( \{F_l : 1 \leq l \leq K_0\} \). Under exogenous entry, we get

\[
P\{W \leq w|i \in A, j \not\in A\} \leq P\{W \leq w|j \in A, i \not\in A\},
\]

after integrating out \( \lambda^* \). The inequality holds strictly for some \( w \) over common support.

One may wonder whether we can recover the classification of bidders in the English auction example through a “global” approach when the identity of the winner is reported in the data. That is, by comparing the distribution of transaction prices when \( i \) is the winner versus that when \( j \) is the winner, as opposed to the pairwise comparison approach proposed above. Let us explain why this is not feasible.

For any \( i \in N_{(k')} \) and \( j \in N_{(k)} \) and \( F_{k'} \succeq \text{F.S.D.} \) \( F_k \), let \( A \setminus \{i, j\} \) denote the set of entrants out of \( N \setminus \{i, j\} \) and let \( M(A \setminus \{i, j\}) \equiv \max\{V_s : s \in A \setminus \{i, j\}\} \). Let \( \phi(w; \lambda^*) \) denote the distribution of \( M(A \setminus \{i, j\}) \) conditional on \( \lambda(A \setminus \{i, j\}) = \lambda^* \). Let \( D \) denote the identity of the winner in the auction; and \( S_k \) denote the survival function for the private value of a type-\( k \) bidder. Then,

\[
P\{W \leq w, D = i|i \in A\}
\]

\[
= p_j P\{W \leq w, D = i|i, j \in A\} + (1 - p_j) P\{W \leq w, D = i|i \in A, j \not\in A\},
\]
where \( p_j \) is shorthand for \( j \)'s entry probability. Also note that, by construction, once conditioned on the realized set of entrants from \( A \setminus \{i, j\} \), we have
\[
P \{ W \leq w, D = i | i, j \in A, \lambda(A \setminus \{i, j\}) = \lambda^* \} = \int_{-\infty}^{w} F_k(t)\phi(t; \lambda^*)dF_{k'}(t) + S_{k'}(w)F_k(w)\phi(w; \lambda^*),
\]
and
\[
P \{ W \leq w, D = i | i \in A, j \notin A, \lambda(A \setminus \{i, j\}) = \lambda^* \} = \int_{-\infty}^{w} \phi(t; \lambda^*)dF_{k'}(t) + S_{k'}(w)\phi(w; \lambda^*).
\]
Likewise \( P \{ W \leq w, D = j | j \in A \} \) can be written by swapping the roles of \( i \) and \( j \) and swapping the roles of \( k \) and \( k' \) respectively. Then it can be shown that
\[
B.4 \quad P \{ W \leq w, D = i | i \in A, j \in A \} > P \{ W \leq w, D = j | i \in A, j \in A \}.
\]
To see why the inequality in (B.4) holds, note for any \( \lambda^* \),
\[
P \{ W \leq w, D = i | i \in A, j \in A, \lambda(A \setminus \{i, j\}) = \lambda^* \} > P \{ W \leq w, D = j | i \in A, j \in A, \lambda(A \setminus \{i, j\}) = \lambda^* \},
\]
where the difference is written as
\[
\left[ \int_{-\infty}^{w} F_k(t)\phi(t; \lambda^*)dF_{k'}(t) - \int_{-\infty}^{w} F_{k'}(t)\phi(t; \lambda^*)dF_k(t) \right] + \phi(w; \lambda^*)[S_{k'}(w)F_k(w) - S_k(w)F_{k'}(w)].
\]
The first square bracket in the display above is positive because
\[
\int_{-\infty}^{w} F_k(t)\phi(t; \lambda^*)dF_{k'}(t) > \int_{-\infty}^{w} F_{k'}(t)\phi(t; \lambda^*)dF_k(t) > \int_{-\infty}^{w} F_k(t)\phi(t; \lambda^*)dF_k(t).
\]
Furthermore, the second square bracket in the display is also positive because “\( F_{k'} \succeq_{F,S.D.} F_k \)” implies
\[
S_{k'}(w) \geq S_k(w) \text{ and } F_k(w) \geq F_{k'}(w) \text{ for all } w
\]
and these inequalities hold strictly for some set of \( w \) with positive measure. Integrating out \( \lambda^* \) on both sides of the inequality
\[
P \{ W \leq w, D = i | i, j \in A, \lambda(A \setminus \{i, j\}) = \lambda^* \} > P \{ W \leq w, D = j | i, j \in A, \lambda(A \setminus \{i, j\}) = \lambda^* \},
\]
yields the first inequality in (B.4).

Similarly, the difference between \( P \{ W \leq w, D = i | i \in A, j \notin A, \lambda(A \setminus \{i, j\}) = \lambda^* \} \) and \( P \{ W \leq w, D = j | j \in A, i \notin A, \lambda(A \setminus \{i, j\}) = \lambda^* \} \) equals
\[
\left[ \int_{-\infty}^{w} \phi(t; \lambda^*)dF_{k'}(t) - \int_{-\infty}^{w} \phi(t; \lambda^*)dF_k(t) \right] + \phi(w; \lambda^*)[S_{k'}(w) - S_k(w)]
\]
which must be positive because the two terms in the square brackets are positive.
Now we write
\begin{equation}
P\{W \leq w, D = i| i \in A\} = p_j P\{W \leq w, D = i| i, j \in A\} + (1 - p_j) P\{W \leq w, D = i| i \in A, j \notin A\}
\end{equation}
where $p_j \equiv P(j \in A)$. A similar expression exists for $P\{W \leq w, D = j| j \in A\}$ by swapping the roles of $i$ and $j$ in (B.5). Therefore the difference between $P\{W \leq w, D = i| i \in A\}$ and $P\{W \leq w, D = j| j \in A\}$ is also indeterminate in the absence of knowledge about $p_i, p_j$.

B.3. Bidding Cartel in First-Price Procurement Auctions

Our method can be used to detect the identities of cartel members in a model of first-price procurement auctions in which a bidding cartel competes with competitive non-colluding bidders (Pesendorfer (2000)). Let the population of bidders/firms $N$ be partitioned into a set of colluding firms $N(c)$ and non-colluding firms $N(nc)$. In each auction, the set of potential bidders (who are interested in bidding for the contract) $A$ is partitioned into $A(c)$ and $A(nc)$. The cardinality of $A(c)$ is common knowledge among the bidders. The potential bidders in $A(c)$ collude by refraining from participation except for one bidder $i^*$ who is chosen among them to submit a bid.\(^{27}\)

In an efficient truth-revealing mechanism considered in Pesendorfer (2000), the cartel member that has the lowest cost is selected to be the sole bidder from the cartel. That is, $i^*(A(c)) = \arg \min_{j \in A(c)} C_j$ where $C_j$ is the private cost of bidder $j$. Thus, the set of final entrants who are observed to submit bids in the data is $A^* \equiv \{i^*(A(c))\} \cup A(nc)$. (The set of colluding potential bidders is not reported in the data available to the researcher.)

We maintain that across the auctions in the data bidders’ private costs are independent draws from the same distribution. Bidders are ex ante symmetric in that each bidder’s private cost is drawn independently from the same distribution. Entrants know that a representative of the cartel is participating in bidding, and all follow Bayesian Nash equilibrium bidding strategies.

We are interested in detecting the identities of the set of colluding firms in $N(c)$ from the reported bidding and participation decisions. Let $N'(c) \subset N$ denote the set of bidders such that no two bidders in $N'(c)$ are ever observed to compete with each other in the bidding stage. By construction, $N(c) \subseteq N'(c)$ so the latter should be interpreted as a set of suspects for collusion. However, the set $N'(c)$ could also contain innocent non-colluding bidders who are never observed to compete with each other in the data because of random entry in finite sample. Our goal is to use bidding data to separate $N(c)$ from $N'(c) \setminus N(c) \equiv N(nc) \cap N'(c)$.

Pesendorfer (2000) (Remark 3) shows that in any given auction with participants $A(c) \cup A(nc)$, the distribution of bids from a non-colluding bidder $j$ first-order stochastically dominates the distribution of the bids from the sole bidder representing the cartel.

\(^{27}\)The cartel is sustained through side payments among its members.
Specifically, for any such $i^*$ and $j$,

$$P\{B_{i^*} \leq b | i^* \in A^*, |A^*|\} > P\{B_j \leq b | j \in A^*, |A^*|\}$$

for all $b$ on the common support of the two distributions.

The intuition, as is noted in Pesendorfer (2000), is that the sole bidder representing a cartel has a higher hazard rate than a non-colluding bidder. That is, relative to a competitive bidder, the cartel representative has a higher probability of having a low cost conditional on the costs being above any fixed threshold. Besides, ex ante symmetry among bidders implies that

$$P\{B_i \leq b | i \in A^*, |A^*|\} = P\{B_j \leq b | j \in A^*, |A^*|\}$$

whenever $i, j \in N(c)$ or $i, j \in N(nc) \cap N'(c)$.

We can then construct pairwise comparison indexes $\delta_{ij}, \delta'^{0}_{ij}$ by replacing $G_{ij}$ and $G_{ji}$ in (B.1) with the left- and right-hand side of (B.6).

Appendix C. Mathematical Proofs

C.1. Proof of Identification Results

Proof of Theorem 3.1: Sufficiency is obvious. We focus on necessity. First consider the two facts:

Fact 1: If $G_\tau$ does not contain a monotone path of length $K_0 - 1$, $\tau$ is not identified.

Fact 2: A vertex $i$ is identified if and only if there is a monotone path $P$ containing $i$ such that its end vertices $i_H$ and $i_L$ are identified and

$$\tau(i_H) - \tau(i_L) = \ell(P),$$

where $\ell(P)$ denotes the length of $P$.

By Fact 1, the necessity of $G_\tau$ containing a monotone path of length $K_0 - 1$ follows, and Fact 2 completes the proof of the necessity part of the theorem.

Now let us prove Fact 1. Suppose that $G_\tau$ does not contain a monotone path of length $K_0 - 1$. Let $N_{\max}$ be the set of vertices such that for each vertex $i$ in $N_{\max}$, all his $G_\tau$-neighbors have lower type than the vertex $i$. Then there is no edge in $G_\tau$ which joins any two vertices from the set $N_{\max}$. Choose a vertex $i^*$ from $N_{\max}$ which is an end vertex of a longest monotone path, say, with length $K - 1 < K_0 - 1$. This identifies a lower bound for $K_0$ but there is no upper bound for $K_0$ that we can obtain from $G_\tau$. Take any $\tau'$ such that $\tau'(i^*) > \tau(i^*)$ and $\tau'(i) = \tau(i)$ for all $i \in N \setminus \{i^*\}$. Then $\tau'$ is compatible with $G_\tau$ and the given comparison indexes, proving that $\tau$ is not identified from $G_\tau$.

28. Pesendorfer (2000) proved this result using the implicit assumption that the distribution of costs for non-colluding bidders and that for the sole cartel is common knowledge among all participants in an auction. (See proof of Remark 3 in Pesendorfer (2000).) This assumption is consistent with the informational environment that the partition of $N$ into $N(c)$ and $N(nc)$ is common knowledge among all bidders.

29. Note that the statement is conditional since the bidding strategies depend on the cardinality of the final set of bidders $|A^*|$. 
Let us prove Fact 2. Sufficiency is trivial. Let us focus on necessity. First, suppose to the contrary that there is no monotone path with identified end vertices which contains $i$. Then obviously $i$ is not identified. Therefore, if $i$ is identified, there exists a monotone path with identified end vertices which contains $i$. So it suffices to show that if $i$ is identified, it is necessary that such a monotone path has to have length equal to $\tau(i_H) - \tau(i_L)$.

Suppose to the contrary that every monotone path $P$ that contains $i$ and has identified end vertices $i_H$ and $i_L$ also satisfies $\tau(i_H) - \tau(i_L) > \ell(P)$. Then we will show that $i$ is not identified.

First, assume that there exists a monotone path which contains $i$ but not as one of its end vertices. Let $i^*_H$ be a lowest type vertex among all the identified vertices each of which is on a monotone path that contains $i$ and is of higher type than $i$. Also, let $i^*_L$ be a highest type vertex among all the identified vertices each of which is on a monotone path that contains $i$ and is of lower type than $i$. Let $P$ be a monotone path between $i^*_H$ and $i^*_L$ that passes through $i$. Then by construction, the type difference $\tau(i^*_H) - \tau(i^*_L)$ between the two end vertices is smallest among all the monotone paths that go through $i$. By the condition, we have $\tau(i^*_H) - \tau(i^*_L) > 2$. Therefore, we have multiple different ways to assign $\tau(i^*_H) - 1, \tau(i^*_H) - 2, \ldots, \tau(i^*_L) + 1$ to the vertex $i$ on the path $P$. Hence $i$ is not identified.

Second, assume that all the monotone paths that contain $i$ have $i$ as one of their end vertices. Then either all neighbors of $i$ are of higher type than $i$ or all neighbors of $i$ are of lower type than $i$. Suppose that we are in the former case. (The latter case can be dealt with similarly.) Let $i^*_H$ be a lowest type vertex among all the vertices each of which is on a monotone path that contains $i$ and is of higher type than $i$. Then by the condition, $\tau(i^*_H) - \tau(i) > 1$. Thus we have multiple different ways to assign $\tau(i^*_H) - 1, \tau(i^*_H) - 2, \ldots, \tau(i) + 1, \tau(i)$ to the vertex $i$. Hence $i$ is not identified. ■

C.2. Lower Level Conditions for Assumption 4.1

Lower level conditions for Assumption 4.1 can be found from the literature of testing for moment inequality restrictions. For the sake of concreteness, we focus on the situation where $r_{ij}(x)$ arises from difference between two nonparametric regression functions and the testing procedure is done by the method proposed in Lee, Song, and Whang (2018). Suppose that we have

$$g_i(x) > g_j(x), \forall x \in \mathcal{X} \text{ if and only if } \tau(i) > \tau(j);$$

$$g_i(x) = g_j(x), \forall x \in \mathcal{X} \text{ if and only if } \tau(i) = \tau(j);$$

$$g_i(x) < g_j(x), \forall x \in \mathcal{X} \text{ if and only if } \tau(i) < \tau(j),$$

(C.1)
where \( g_i(x) = \mathbb{E}[Y_{i\ell}|X_{i\ell} = x], i \in N, \) and \( \ell = 1, \ldots, L, \) is the sample unit index. Let us define a kernel estimator of \( g_i(x) \) as follows:
\[
\hat{g}_i(x) = \frac{\sum_{\ell=1}^L Y_{i\ell} K_h(X_{i\ell} - x)}{\sum_{\ell=1}^L K_h(X_{i\ell} - x)},
\]
where \( K_h(x) = K(x/h)/h \), and \( K(\cdot) \) is a multivariate kernel and \( h \) is a bandwidth. We let
\[
\hat{r}_{ij}(x) = \hat{g}_i(x) - \hat{g}_j(x).
\]
Then the test statistics we use are defined as
\[
\begin{align*}
T_{ij}^+ &= \int_X \max\{\hat{r}_{ij}(x), 0\} \, dx, \quad T_{ij}^- = \int_X \max\{\hat{r}_{ji}(x), 0\} \, dx, \quad \text{and} \\
T_{ij}^0 &= \int_X |\hat{r}_{ij}(x)| \, dx.
\end{align*}
\]
As for the bootstrap test statistics, we first obtain bootstrap samples \( \{(Y_{i\ell}^*, X_{i\ell}^*)_{i \in N}\}_{\ell=1}^L \) by resampling the vector \( (Y_{i\ell}^*, X_{i\ell}^*)_{i \in N} \) from the empirical distribution of \( \{(Y_{i\ell}, X_{i\ell})_{i \in N}\}_{\ell=1}^L \) with replacement. Using the bootstrap sample, we construct
\[
\hat{r}_{ij}^*(x) = \hat{g}_i^*(x) - \hat{g}_j^*(x),
\]
where
\[
\hat{g}_i^*(x) = \frac{\sum_{\ell=1}^L Y_{i\ell}^* K_h(X_{i\ell}^* - x)}{\sum_{\ell=1}^L K_h(X_{i\ell}^* - x)}.
\]
Then the bootstrap test statistic we use is defined as
\[
\begin{align*}
T_{ij}^{+*} &= \int_X \max\{\hat{r}_{ij}^*(x) - \hat{r}_{ij}(x), 0\} \, dx, \quad T_{ij}^{-*} = \int_X \max\{\hat{r}_{ji}^*(x) - \hat{r}_{ji}(x), 0\} \, dx, \quad \text{and} \\
T_{ij}^{0*} &= \int_X |\hat{r}_{ij}^*(x) - \hat{r}_{ij}(x)| \, dx.
\end{align*}
\]
Let the CDF of the bootstrap distribution of \( T_{ij}^{*} \) be denoted by \( F_{ij}^* \). Then we set the pairwise \( p \)-value to be \( \hat{p}_{ij}^* = 1 - F_{ij}^*(T_{ij}^{*}) \).

Let \( a_{ij,L}^s \) and \( \sigma_{ij,L}^s \) be sequences of constants such that
\[
\begin{align*}
a_{ij,L}^s &= O(1), \quad \text{and} \quad \sigma_{ij,L}^s \to \sigma_{ij}^s > 0,
\end{align*}
\]
\footnote{One could also use alternative bootstrap statistics using estimated contact sets as in Lee, Song, and Whang (2018) to enhance the power. For simplicity of exposition, here we present the case where we use the least favorable configurations.}
39

as \( n, L \to \infty \). We take

\[
\tilde{T}_{ij}^s = (\sqrt{LT_{ij}} - h^{-d/2}a_{ij,L}^s) / \sigma_{ij,L}^s, \quad \text{and} \\
\tilde{T}_{ij}^* = (\sqrt{LT_{ij}^*} - h^{-d/2}a_{ij,L}^s) / \sigma_{ij,L}^s.
\]

(The researcher does not need to know, estimate, or use the constants \( a_{ij,L}^s \) and \( \sigma_{ij,L}^s \) for construction of the pairwise \( p \)-values and for the implementation of the classification algorithm of this paper.) Then we can rewrite

\[
\hat{p}_{ij}^s = 1 - \tilde{F}_{ij}^s(\tilde{T}_{ij}^s),
\]

where \( \tilde{F}_{ij}^s \) is the CDF of the bootstrap distribution of \( \tilde{T}_{ij}^* \). Let \( \| \cdot \|_{\infty} \) denote the sup norm, i.e., \( \| f \|_{\infty} = \sup_{x} |f(x)| \) for any real function \( f \). Let us consider the following set of assumptions.

**Assumption C.1.** (i) For each \( s \in \{+, 0, -\} \) and each \( i,j \in N \), there exist sequences of constants \( a_{ij,L}^s \) and \( \sigma_{ij,L}^s \) such that the conditions in (C.4) hold, and

\[
\tilde{T}_{ij}^s \to_d N(0, 1),
\]

and \( \| \tilde{F}_{ij}^s - \Phi \|_{\infty} = O_P(L^{-\alpha}) \) for some \( \alpha > 0 \), and \( \Phi \) is the CDF of \( N(0, 1) \).

(ii) \( \sup_{x \in X} |\tilde{r}_{ij}(x) - r_{ij}(x)| = o_P(1) \), as \( L \to \infty \).

(iii) Suppose that \( Lh^d \to \infty \) while \( h \to 0 \) as \( L \to \infty \).

The lower level conditions for Condition (i) can be found in Lee, Song, and Whang (2018). Condition (ii) follows if the kernel regression estimators \( \hat{g}_i(x) \) are uniformly consistent. (See, e.g., Hansen (2008).) Condition (iii) is a standard bandwidth condition in the literature of kernel estimators. Then, we obtain the following lemma.

**Lemma C.1.** Suppose that Assumption C.1 holds, and that the sequence \( r_L \to \infty \) is such that \( r_L / \log L \to 0 \) as \( L \to \infty \). Then Assumption 4.1 holds.

**Proof:** Condition (i) of Assumption 4.1 follows from Condition (i) of Assumption C.1 with \( W_{ij}^s \) being a standard normal random variable. As for Conditions (ii)-(iv) of Assumption 4.1, we focus on only (ii), because the proof for the other two statements is similar. Observe that when \( \tau(i) > \tau(j) \), so that

\[
\tilde{T}_{ij}^+ / \sqrt{L} = (\sigma_{ij,L}^+)^{-1} \left( \int \max\{\tilde{r}_{ij}(x) - r_{ij}(x) + r_{ij}(x), 0\}dx - L^{-1/2}h^{-d/2}a_{ij,L}^+ \right)
\]

\[
= (\sigma_{ij}^+)^{-1} \int \max\{r_{ij}(x), 0\}dx + o_P(1),
\]

by Assumption C.1 (ii). Hence Condition (ii) of Assumption 4.1(ii) follows with

\[
c_{ij}^+ = (\sigma_{ij}^+)^{-1} \int \max\{r_{ij}(x), 0\}dx,
\]

and \( \lambda_L = \sqrt{L} \).
As for Condition (vi) of Assumption 4.1, note that $F_{ij,\infty}^s = \Phi$, the standard normal CDF. Hence there exists $C > 0$ such that for all $t > C$,

$$1 - \Phi(t) \leq C \exp\left(-\frac{Ct^2}{2}\right).$$

Therefore, for any constants $c_1, c_2 > 0$, taking $\lambda_L = \sqrt{L}$ and $\rho_L = L^{-\alpha}$, (from some large $L$ on)

$$r_{L}^{-1} \log\left(1 - \Phi(c_1 \sqrt{L}) + c_2 L^{-\alpha}\right) \leq r_{L}^{-1} \log \left(C \exp\left(-Cc_1^2 L/2\right) + c_2 L^{-\alpha}\right) \rightarrow -\infty,$$

as $L \rightarrow \infty$, by the condition that $r_{L}/ \log L \rightarrow 0$. ■

C.3. Proof of Consistency of Classification

We provide a proof of consistency of the classification allowing both $n$ and $L$ to diverge to infinity jointly. This requires a more elaborated set of assumptions than Assumption 4.1. It is not hard to use the same proof to prove Theorem 4.1.

Assumption C.2. There exist sequences $\rho_L, \kappa_L \rightarrow 0$ and $\lambda_L \rightarrow \infty$ and constants $c_{ij}^s > 0$, $s \in \{+, 0, -\}$ such that along each sequence of probabilities $P_L \in \mathcal{P}_{0, \varepsilon}$, the following holds for all $s \in \{+, 0, -\}$, $s' \in \{+, -\}$, as $n, L \rightarrow \infty$.

(i) If $\tau(i) = \tau(j)$, $\tilde{T}_{ij}^s \to_d W_{ij}^s$ for some random variable $W_{ij}^s$ with continuous CDF $F_{ij, \infty}^s$.

(ii)(a) If $\tau(i) > \tau(j)$, $P\{|\tilde{T}_{ij}^+/\lambda_L - c_{ij}^+/\lambda_L| + |\tilde{T}_{ij}^-/\lambda_L - c_{ij}^-/\lambda_L| \geq \kappa_L\} = O(\rho_L)$.

(ii)(b) If $\tau(i) \geq \tau(j)$, there exists a CDF $F_{ij}^s$ such that for all $t \in \mathbb{R}$,

$$P\{\tilde{T}_{ij}^- \leq t\} \geq F_{ij}^- (t).$$

(iii)(a) If $\tau(i) < \tau(j)$, $P\{|\tilde{T}_{ij}^-/\lambda_L - c_{ij}^-/\lambda_L| + |\tilde{T}_{ij}^+/\lambda_L - c_{ij}^+/\lambda_L| \geq \kappa_L\} = O(\rho_L)$.

(iii)(b) If $\tau(i) \leq \tau(j)$, there exists a CDF $F_{ij}^s$ such that for all $t \in \mathbb{R}$,

$$P\{\tilde{T}_{ij}^+ \leq t\} \geq F_{ij}^+(t).$$

Assumption C.3. Suppose that $F_{ij}^s$ and $F_{ij, \infty}^s$, $s, s' \in \{+, 0, -\}$, are CDFs and $\rho_L, \kappa_L$, and $\lambda_L$ are sequences in Assumption C.2. Then the following holds.

(i) $P\{\|\tilde{F}_{ij}^s - F_{ij, \infty}^s\|_{\infty} > \varepsilon_L\} = O(\rho_L)$ and $\|F_{ij}^s - F_{ij, \infty}^s\|_{\infty} = O(\varepsilon_L)$, where $\varepsilon_L \rightarrow 0$.

(ii) $r_{L}^{-1} \log(1 - F_{ij, \infty}^s((c - \kappa_L)\lambda_L) + \varepsilon_L) \rightarrow -\infty$, for any constant $c > 0$.

(iii) $n^2(\exp(\frac{-r_{L}/2}{2} + \varepsilon_L + \rho_L)) \rightarrow 0$, as $n, L \rightarrow \infty$.

Assumptions C.2 and C.3 are variants of Assumption 4.1 (except for Assumption C.3(iii)) which require the rate of convergences explicitly. Assumption C.3(iii) is new here because we now allow both $n$ and $L$ to increase to infinity. This assumption says that for the consistency of the estimated group structure, we need to have $n$ increase sufficiently faster than $L$. This condition is trivially satisfied when $n$ is fixed and $L$ increases to infinity. Note that the conditions are high level conditions suited to the generic set-up here. With a more detailed specification of the pairwise comparison indexes $\delta_{ij}$ and $\delta_{ij}^0$, test statistics and $p$-values, one may improve the conditions. Using high level conditions also enables us to focus only on more novel aspects of the mathematical proofs.
While existence of such sequences as $\kappa_L$, $\varepsilon_L$, and $\rho_L$ is fairly expected, obtaining their precise forms using lower level conditions require substantial yet tedious arguments depending on the way the test statistics and the $p$-values are constructed. Essentially what one needs to obtain for lower level conditions is the rate in the convergence of the test statistics to the limiting distribution both under the null hypothesis and under the local alternatives. For example, if one follows the approach of Lee, Song, and Whang (2018), the rate of convergence is ultimately delivered by a Berry-Esseen bound (for a sum of independent random variables) used to obtain the asymptotic normality of test statistics (under appropriate mean shifts). However, the final result comes, in combination with this, only with carefully derived rate results on asymptotically negligible terms that arise, among other things, due to the Poissonization technique used in the paper. While these additional developments are feasible, they do not add insights to the main idea of the paper. Hence we do not pursue details in this direction here.

Let us first state the theorem.

**Theorem C.1.** Suppose that Assumptions C.2 - C.3 hold, and that $g(L) \to \infty$ and $g(L)/r_L \to 0$ as $L \to \infty$. Then, for any $\varepsilon > 0$, along a sequence of probabilities $P_{n,L}$ from $P_{0,\varepsilon}$,

$$P_{n,L}\{\hat{K} = K_0\} \to 1,$$

as $n, L \to \infty$, and the estimated group structure $\hat{T}_K$ satisfies that as $n, L \to \infty$,

$$P_{n,L}\{\hat{T}_K = T\} \to 1.$$

The proof of this theorem is long. We first prepare some auxiliary lemmas. Throughout this section, we assume that Assumptions C.2 and C.3 hold. First, for any subset $N' \subset N$, we define $N'(i) = N' \setminus \{i\}$, and let

\begin{align}
N'_1(i) &= \{ j \in N'(i) : \tau(i) > \tau(j) \}, \\
N'_2(i) &= \{ j \in N'(i) : \tau(i) < \tau(j) \},
\end{align}

(C.7)

\begin{align}
\bar{N}'_1(i) &= \{ j \in N'(i) : \tau(i) \geq \tau(j) \}, \text{ and} \\
\bar{N}'_2(i) &= \{ j \in N'(i) : \tau(i) \leq \tau(j) \}.
\end{align}

We also define

\begin{align}
\hat{N}'_1(i) &= \{ j \in N'(i) : \log \hat{p}_{ij}^+ \leq \log \hat{p}_{ij}^- - r_L \} \text{ and} \\
\hat{N}'_2(i) &= \{ j \in N'(i) : \log \hat{p}_{ij}^- \leq \log \hat{p}_{ij}^+ - r_L \}.
\end{align}

Following the convention, given a CDF $G$, we define its generalized inverse $G^{-1}$ as

$$G^{-1}(t) = \inf\{ s \in \mathbb{R} : G(s) \geq t \}, t \in (0, 1).$$
Lemma C.2. (i) For all \( i \in N \),
\[
\begin{align*}
P \left\{ \min_{j \in \mathcal{N}_1(i)} \log \hat{p}_{ij}^- < -\frac{r_L}{2} \right\} & = O(n \omega_{n,L}) \\
P \left\{ \min_{j \in \mathcal{N}_1'(i)} \log \hat{p}_{ij}^+ \geq -3\frac{r_L}{2} \right\} & = O(n \rho_L),
\end{align*}
\]
where \( \omega_{n,L} = \exp(-\frac{r_L}{2}) + \varepsilon_L + \rho_L \).

(ii) For all \( i, j \in N \) such that \( \tau(i) \neq \tau(j) \),
\[
P \left\{ \log \hat{p}_{ij}^0 \geq -\frac{r_L}{2} \right\} = O(\rho_L),
\]
and for all \( i, j \in N \) such that \( \tau(i) = \tau(j) \),
\[
P \left\{ \log \hat{p}_{ij}^0 \leq -\frac{r_L}{2} \right\} = O(\omega_{n,L}).
\]

Proof: (i) We will show the first and the second statements only. The third and the fourth statements can be proved similarly. Let us prove the first statement first. For all \( i \in N \),
\[
P \left\{ \min_{j \in \mathcal{N}_1(i)} \log \hat{p}_{ij}^- < -\frac{r_L}{2} \right\} \leq \sum_{j \in \mathcal{N}_1(i)} P \left\{ \log \hat{p}_{ij}^- \leq -\frac{r_L}{2} \right\}.
\]
Note that
\[
P \left\{ \log \hat{p}_{ij}^- < -\frac{r_L}{2} \right\} = P \left\{ 1 - \left( -\frac{r_L}{2} \right) < F_{ij,\infty}(\hat{T}_{ij}^-) \right\} \leq P \left\{ 1 - \exp \left( -\frac{r_L}{2} \right) < F_{ij,\infty}(\hat{T}_{ij}^-) + \varepsilon_L \right\} + P \left\{ \| \hat{F}_{ij}^- - F_{ij,\infty} \|_{\infty} \geq \varepsilon_L \right\} \leq P \left\{ 1 - \exp \left( -\frac{r_L}{2} \right) < F_{ij,\infty}(\hat{T}_{ij}^-) + \varepsilon_L \right\} + O(\rho_L),
\]
where the last \( O(\rho_L) \) term is due to Assumption C.3(i) and Markov’s inequality. As for the leading probability on the right end side, we use the fact that for any CDF \( G \) and any \( t \in (0, 1) \) and \( x \in \mathbb{R} \), \( G^{-1}(t) \geq x \) if and only if \( t \geq G(x) \), (e.g. Lemma A.1.1. of Reiss (1989)), and bound
\[
P \left\{ 1 - \exp \left( -\frac{r_L}{2} \right) < F_{ij,\infty}(\hat{T}_{ij}^-) + \varepsilon_L \right\} = 1 - P \left\{ \hat{T}_{ij}^- \leq (F_{ij,\infty})^{-1} \left( 1 - \exp \left( -\frac{r_L}{2} \right) - \varepsilon_L \right) \right\} \leq 1 - F_{ij}^- \left( (F_{ij,\infty})^{-1} \left( 1 - \exp \left( -\frac{r_L}{2} \right) - \varepsilon_L \right) \right),
\]
by Assumption C.2(ii)(b) and because \( j \in \overline{N}_1(i) \). By Assumption C.3(i), the last term is equal to

\[
1 - F_{ij,\infty}^- \left( (F_{ij,\infty}^-)^{-1} \left( 1 - \exp\left( -\frac{r_L}{2} \right) - \varepsilon_L \right) \right) + O(\varepsilon_L)
\]

\[
\leq \exp\left( -\frac{r_L}{2} \right) + \varepsilon_L + O(\varepsilon_L) = O(\omega_{n,L}).
\]

Thus we obtain the first statement.

Let us consider the second statement. Suppose that \( \tau(i) > \tau(j) \). We let

\[
A_{1,L} = \left\{ \| \tilde{F}^+_{ij} - F^+_{ij,\infty} \|_{\infty} \leq \varepsilon_L \right\} \quad \text{and} \quad A_{2,L} = \left\{ |\tilde{T}^+_{ij} / \lambda_L - \tilde{c}^+_{ij}| \leq \kappa_L \right\},
\]

and let \( A_L = A_{1,L} \cap A_{2,L} \). Note that

\[
P\{ \log \hat{p}^+_{ij} > -r_L \} \leq P\{ \log \hat{p}^+_{ij} > -r_L \} \cap A_L + PA_L^c
\]

\[
\leq P\{ \log \hat{p}^+_{ij} > -r_L \} \cap A_L + O(\rho_L),
\]

where the last \( O(\rho_L) \) term is due to Assumption C.2(ii)(a) and Assumption C.3(i). For the leading probability on the right hand side, note that

\[
P\{ \log \hat{p}^+_{ij} > -r_L \} \cap A_L = P\{ \log (1 - \tilde{F}^+_{ij,\infty}(\tilde{T}^+_{ij})) > -r_L \} \cap A_L
\]

\[
\leq P\{ \log (1 - \tilde{F}^+_{ij,\infty}(\tilde{T}^+_{ij})) + \varepsilon_L > -r_L \} \cap A_L
\]

\[
\leq PA_L \cdot 1\{ 1 - \tilde{F}^+_{ij,\infty}(c_{ij} + \varepsilon_L) > \exp(-r_L) \}.
\]

The last indicator becomes zero from some large \( L \) on by Condition (ii) of Assumption C.3. Thus, we obtain the second statement.

(ii) The proof of the first statement is the same as that of the second statement of (i), and the proof of the second statement is the same as that of the first statement of (i). Details are omitted. ■

Lemma C.3. Suppose that \( N' \subset N \) contains agents with heterogeneous types. Then, for each \( i \in N \),

\[
P\{ N'_1(i) = \hat{N}'_1(i) \} = 1 + O(n\omega_{n,L});
\]

\[
P\{ N'_2(i) = \hat{N}'_2(i) \} = 1 + O(n\omega_{n,L});
\]

\[
P\{ \overline{N}_2(i) = N'(i) \setminus \hat{N}'_1(i) \} = 1 + O(n\omega_{n,L});
\]

\[
P\{ \overline{N}_1(i) = N'(i) \setminus \hat{N}'_2(i) \} = 1 + O(n\omega_{n,L}).
\]

Proof: We show only the first and the third statements. The remaining statements can be proved similarly. Define

\[
A_L = \left\{ \min_{j \in \overline{N}_1(i)} \log \hat{p}^-_{ij} \geq -\frac{r_L}{2} \right\}.
\]
Note that
\[
P\left\{ N_1'(i) \subseteq \hat{N}_1'(i) \right\} = P\left\{ \max_{j \in N_1'(i)} \log \hat{p}_{ij}^+ - \log \hat{p}_{ij}^- \leq -r_L \right\}
\geq P\left\{ \max_{j \in N_1'(i)} \log \hat{p}_{ij}^+ - \min_{j \in \bar{N}_1'(i)} \log \hat{p}_{ij}^- \leq -r_L \right\}
\geq P\left\{ \max_{j \in N_1'(i)} \log \hat{p}_{ij}^+ \leq -\frac{3r_L}{2} \right\} \cap A_L
\geq P\left\{ \max_{j \in N_1'(i)} \log \hat{p}_{ij}^+ \leq -\frac{3r_L}{2} \right\} - PA_L^c
= 1 - P\left\{ \max_{j \in N_1'(i)} \log \hat{p}_{ij}^+ > -\frac{3r_L}{2} \right\} - PA_L^c = 1 + O(n\omega_{n,L}),
\]
where the last inequality follows by the first and the second statements of Lemma C.2(i). Thus we have
(C.8) \[ P\left\{ N_1'(i) \subseteq \hat{N}_1'(i) \right\} = 1 + O(n\omega_{n,L}), \]
as \( n, L \to \infty \). On the other hand,
\[
P\left\{ \bar{N}_2'(i) \subseteq N'(i) \setminus \hat{N}_1'(i) \right\} \geq P\left\{ \min_{j \in \bar{N}_2'(i)} \log \hat{p}_{ij}^+ - \log \hat{p}_{ij}^- > -r_L \right\}
\geq P\left\{ \min_{j \in \bar{N}_2'(i)} \log \hat{p}_{ij}^+ > -r_L \right\} = 1 + O(n\omega_{n,L}).
\]
The second inequality follows because \( \log \hat{p}_{ij}^- \leq 0 \) and the last equality follows by the third statement of Lemma C.2(i). Since \( N_1'(i) \) and \( \bar{N}_2'(i) \) partition \( N'(i) \), and \( \hat{N}_1'(i) \) and \( N'(i) \setminus \hat{N}_1'(i) \) also partition \( N'(i) \), it follows that
\[
P\{N_1'(i) = \hat{N}_1'(i)\} = 1 + O(n\omega_{n,L}), \text{ and } P\{\bar{N}_2'(i) = N'(i) \setminus \hat{N}_1'(i)\} = 1 + O(n\omega_{n,L}),
\]
as \( n, L \to \infty \). The remaining statements can be shown similarly. ■

**Definition C.1.** (i) An ordered partition \( (N_1', ..., N_s') \) of a subset \( N' \subseteq N \) is said to be a \( \tau \)-**ordered partition**, if for any \( r_1 < r_2, r_1, r_2 = 1, 2, ..., s \), we have \( \tau(i) < \tau(j) \) whenever \( i \in N'_1 \) and \( j \in N'_{r_2} \).

(ii) Let \( \mathcal{N}_\tau(N') \) denote the set of \( \tau \)-ordered partitions of \( N' \).

When an ordered partition \( (N_1', ..., N_s') \) is a \( \tau \)-ordered partition, and \( \tau \) partitions \( N' \) into \( K \) groups (i.e., any two agents, say \( i, j \), from two different groups from the \( K \) groups have \( \tau(i) \neq \tau(j) \)), we must have \( s \leq K \) by the definition of \( \tau \)-ordered partition. Hence some group in the ordered partition \( (N_1', ..., N_s') \) can have agents with different types.

**Definition C.2.** An estimated ordered partition \( (\hat{N}_1', ..., \hat{N}_s') \) of a subset \( N' \subseteq N \) is said to be **asymptotically \( \tau \)-compatible at rate** \( u_{n,L} \), if
\[
P\{(\hat{N}_1', ..., \hat{N}_s') \in \mathcal{N}_\tau(N')\} = 1 + O(u_{n,L}), \text{ as } n, L \to \infty.
\]
Lemma C.4. For each \( \pi = (N'_1, ..., N'_s) \in \mathcal{N}_r(N) \), let \( R_1(\pi) \subset \{1, 2, ..., s\} \) be such that for all \( r \in R_1(\pi) \), \( N'_i \) has some \( i, j \in N'_i \) satisfying \( \tau(i) \neq \tau(j) \), and let \( R_2(\pi) = \{1, 2, ..., s\} \setminus R_1(\pi) \) so that for all \( r \in R_2(\pi) \), and for all \( i, j \in N'_i \), we have \( \tau(i) = \tau(j) \). Let \( B(\pi) \), \( \pi \in \mathcal{N}_r(N') \), be disjoint events. Then,

\[
\sum_{\pi \in \mathcal{N}_r(N')} P \left\{ \exists r_1 \in R_1(\pi), \min_{i,j \in N'_1(\pi)} \log \tilde{p}_{ij}^0 > -r_L \right\} \cap B(\pi) = O(n^2 \rho_L), \text{ and}
\]

\[
\sum_{\pi \in \mathcal{N}_r(N')} P \left\{ \exists r_2 \in R_2(\pi), \min_{i,j \in N'_2(\pi)} \log \tilde{p}_{ij}^0 \leq -r_L \right\} \cap B(\pi) = O(n^2 \omega_{n,L}).
\]

Proof: For each \( \pi \in \mathcal{N}_r(N') \), we write \( K(\pi) = |R_1(\pi)| + |R_2(\pi)| \), i.e., the total number of the groups in \( \pi \), and if \( \pi = (N'_1, ..., N'_{K(\pi)}') \in \mathcal{N}_r(N') \), we write \( (N'_1, ..., N'_{K(\pi)}') = (N'_1(\pi), ..., N'_{K(\pi)}(\pi)) \) to make explicit the dependence of each group on \( \pi \in \mathcal{N}_r(N') \). Let

\[
B_\tau(N') = \bigcup_{\pi \in \mathcal{N}_r(N')} B(\pi).
\]

The first statement of the lemma follows because

\[
\sum_{\pi \in \mathcal{N}_r(N')} P \left\{ \exists r_1 \in R_1(\pi), \min_{i,j \in N'_1(\pi)} \log \tilde{p}_{ij}^0 > -r_L \right\} \cap B(\pi)
\]

\[
\leq \sum_{\pi \in \mathcal{N}_r(N')} \sum_{i,j \in N: \tau(i) \neq \tau(j)} P \left\{ \log \tilde{p}_{ij}^0 > -r_L \right\} \cap B(\pi)
\]

\[
= \sum_{i,j \in N: \tau(i) \neq \tau(j)} P \left\{ \log \tilde{p}_{ij}^0 > -r_L \right\} \cap B_\tau(N')
\]

\[
\leq \sum_{i,j \in N: \tau(i) \neq \tau(j)} P \left\{ \log \tilde{p}_{ij}^0 > -r_L \right\} = O(n^2 \rho_L),
\]

by the assumption that \( B(\pi) \)'s are disjoint. The last equality follows by Lemma C.2(ii). As for the second statement of the lemma, similarly,

\[
\sum_{\pi \in \mathcal{N}_r(N')} P \left\{ \exists r_2 \in R_2(\pi), \min_{i,j \in N'_2(\pi)} \log \tilde{p}_{ij}^0 \leq -r_L \right\} \cap B(\pi) \leq \sum_{i,j \in N: \tau(i) = \tau(j)} P \left\{ \log \tilde{p}_{ij}^0 \leq -r_L \right\}.
\]

Again, the last sum is \( O(n^2 \omega_{n,L}) \) by Lemma C.2(ii). \( \blacksquare \)

Lemma C.5. Suppose that an estimated ordered partition \((\hat{N}_1, ..., \hat{N}_s)\) of \( N \) is asymptotically \( \tau \)-compatible at rate \( u_{n,L} \). Then the Selection Step in the Selection-Split Algorithm applied to this ordered partition selects with a set, say, \( \hat{N}_\tau \), such that

\[
P\{ \exists i, j \in \hat{N}_\tau, \tau(i) \neq \tau(j) \} = 1 + O(n^2 \omega_{n,L} + u_{n,L}),
\]

as \( n, L \to \infty \).

Proof: Let us consider the event that \((\hat{N}_1, ..., \hat{N}_s)\) coincides with a \( \tau \)-ordered partition, say, \( \pi = (N_1(\pi), ..., N_s(\pi)) \in \mathcal{N}_r(N) \), and denote the event by \( A(\pi) \). Note that \( A(\pi) \)'s are
disjoint across $\pi \in \mathcal{M}_r(N)$, and

\[(C.9) \quad \sum_{\pi \in \mathcal{M}_r(N)} PA(\pi) = 1 + O(u_{n,L}),\]

by the assumption that $(\hat{N}_1, \ldots, \hat{N}_s)$ of $N$ is asymptotically $\tau$-compatible with $u_{n,L}$. Corresponding to this $\pi$, let $R_1(\pi)$ and $R_2(\pi)$ be as defined in Lemma C.4. Then,

\[(C.10) \quad \sum_{\pi \in \mathcal{M}_r(N)} P \{ \hat{r} \in R_1(\pi) \} \cap A(\pi) \]

\[\geq \sum_{\pi \in \mathcal{M}_r(N)} P \left\{ \forall r_1 \in R_1(\pi), \forall r_2 \in R_2(\pi), \min_{i,j \in \hat{N}_{r_1}} \log \hat{p}_{ij}^0 < \min_{i,j \in \hat{N}_{r_2}} \log \hat{p}_{ij}^0 \right\} \cap A(\pi).\]

Note that

\[(C.11) \quad \sum_{\pi \in \mathcal{M}_r(N)} P \left\{ \forall r_1 \in R_1(\pi), \forall r_2 \in R_2(\pi), \min_{i,j \in \hat{N}_{r_1}} \log \hat{p}_{ij}^0 < \min_{i,j \in \hat{N}_{r_2}} \log \hat{p}_{ij}^0 \right\} \cap A(\pi) \]

\[\geq \sum_{\pi \in \mathcal{M}_r(N)} P \left\{ \forall r_1 \in R_1(\pi), \forall r_2 \in R_2(\pi), \min_{i,j \in \hat{N}_{r_1}} \log \hat{p}_{ij}^0 < r_L, \min_{i,j \in \hat{N}_{r_2}} \log \hat{p}_{ij}^0 > -r_L \right\} \cap A(\pi) \]

\[\geq \sum_{\pi \in \mathcal{M}_r(N)} P \left\{ \exists r_2 \in R_2(\pi), \min_{i,j \in \hat{N}_{r_2}} \log \hat{p}_{ij}^0 > -r_L \right\} \cap A(\pi) \]

\[= \sum_{\pi \in \mathcal{M}_r(N)} P \left\{ \exists r_2 \in R_2(\pi), \min_{i,j \in \hat{N}_{r_2}(\pi)} \log \hat{p}_{ij}^0 > -r_L \right\} \cap A(\pi).\]

The difference on the right hand side is $1 + O(n^2 \omega_{n,L} + u_{n,L})$ by the second statement of Lemma C.4 and (C.9). On the other hand, the last sum in (C.11) is equal to

\[\sum_{\pi \in \mathcal{M}_r(N)} P \left\{ \exists r_1 \in R_1(\pi), \min_{i,j \in \hat{N}_{r_1}(\pi)} \log \hat{p}_{ij}^0 > -r_L \right\} \cap A(\pi) = O(n^2 \rho_L),\]

by the first statement of Lemma C.4. Since $\rho_L = O(\omega_{n,L})$, we find that

\[(C.12) \quad \sum_{\pi \in \mathcal{M}_r(N)} P \{ \hat{r} \in R_1(\pi) \} \cap A(\pi) = 1 + O(n^2 \omega_{n,L} + u_{n,L}),\]
as \( n, L \to \infty \).

Thus, we have

\[
\sum_{\pi \in \mathcal{M}(N)} P \left\{ \exists i, j \in \hat{N}_p, \tau(i) \neq \tau(j) \right\} 
\geq \sum_{\pi \in \mathcal{M}(N)} P \left\{ \exists i, j \in \hat{N}_p, \tau(i) \neq \tau(j) \right\} \cap A(\pi)
\geq \sum_{\pi \in \mathcal{M}(N)} P \left\{ \exists i, j \in N'_\pi, \tau(i) \neq \tau(j) \right\} \cap A(\pi)
= \sum_{\pi \in \mathcal{M}(N)} P \{ \hat{r} \in R_1(\pi) \} \cap A(\pi) = 1 + O(n^2 \omega_{n,L} + u_{n,L}),
\]

by (C.12). Thus we obtain the desired result. ■

**Lemma C.6.** For any set \( N' \subset N \) with agents of heterogeneous types, the ordered partition \( (\hat{N}'_1, \hat{N}'_2) \) of \( N' \) obtained by the Split Algorithm is asymptotically \( \tau \)-compatible at rate \( n^2 \omega_{n,L} \).

**Proof:** We use the definitions of \( N'_1(i), N'_2(i), \) and \( N'_2(i) \) as in (C.7). By Lemma C.3, for each \( i \in N' \), the ordered partitions \( \hat{T}_1(i) = (\hat{N}'_1(i), N'(i) \setminus \hat{N}'_1(i)) \) and \( \hat{T}_2(i) = (N'(i) \setminus \hat{N}'_1(i), \hat{N}'_2(i)) \) are such that for \( T_1(i) = (N'_1(i), N'_2(i)) \) and \( T_2(i) = (\hat{N}'_1(i), \hat{N}'_2(i)) \), we have

\[
P\{ \hat{T}_1(i) = T_1(i) \} = 1 + O(n \omega_{n,L}), \text{ and } P\{ \hat{T}_2(i) = T_2(i) \} = 1 + O(n \omega_{n,L}).
\]

Therefore,

\[
P\{ \hat{T}_1(i) \neq T_1(i) \} + P\{ \hat{T}_2(i) \neq T_2(i) \} = O(n \omega_{n,L}),
\]

as \( n, L \to \infty \). Define \( \hat{T} = (\hat{N}'_1, \hat{N}'_2) \) and note that

\[
\hat{T} = (N'_1(i^*), (N'(i^*) \setminus \hat{N}'_1(i^*)) \cup \{i^*\}), \text{ or }
\hat{T} = ((N'(i^*) \setminus \hat{N}'_1(i^*)) \cup \{i^*\}, \hat{N}'_2(i^*)),
\]

depending on whether \( s_1(i^*) \leq s_2(i^*) \) or \( s_1(i^*) > s_2(i^*) \). Hence

\[
P\{ \hat{T} \in \mathcal{N}_{\hat{\tau}}(N') \} = \sum_{i \in N'} P\{ \hat{T}_1(i) = T_1(i), i = i^*, s_1(i) \leq s_2(i) \}
+ \sum_{i \in N'} P\{ \hat{T}_2(i) = T_2(i), i = i^*, s_1(i) > s_2(i) \} + O(n^2 \omega_{n,L}),
\]

by (C.14). Now, the leading sum on the right hand side is bounded from below by

\[
\sum_{i \in N'} (P\{ i^* = i, s_1(i) \leq s_2(i) \} - P\{ \hat{T}_1(i) \neq T(i) \}).
\]

Applying the same argument to the last sum in (C.15), we obtain that

\[
P\{ \hat{T} \in \mathcal{N}_{\hat{\tau}}(N') \} \geq 1 - \sum_{i \in N'} \left( P\{ \hat{T}_1(i) \neq T_1(i) \} + P\{ \hat{T}_2(i) \neq T_2(i) \} \right) + O(n^2 \omega_{n,L}),
\]
as $n, L \to \infty$. Hence by (C.14), we conclude that
\[
P\{\hat{T} \in \mathcal{N}_\tau(N')\} = 1 + O(n^2\omega_{n,L}),
\]
as $n, L \to \infty$. ■

**Lemma C.7.** Suppose that for some $s < K_0$, an estimated ordered partition $(\hat{N}_1, ..., \hat{N}_s)$ of $N$ is asymptotically $\tau$-compatible at rate $u_{n,L}$.

Then a new ordered partition $(\hat{N}'_1, ..., \hat{N}'_{s+1})$ of $N$ obtained by applying the Selection-Split Algorithm to $(\hat{N}_1, ..., \hat{N}_s)$ is asymptotically $\tau$-compatible at rate $n^2\omega_{n,L} + u_{n,L}$.

**Proof:** For each $\pi \in \mathcal{N}_\tau(N)$, let us consider the event that $(\hat{N}_1, ..., \hat{N}_s)$ coincides with the $\tau$-ordered partition, say, $\pi = (N'_1(\pi), ..., N'_s(\pi))$, and denote the event by $A(\pi)$. Let $R_1(\pi)$ be as in the proof of Lemma C.5. For each $r \in R_1(\pi)$, let $B_s(r; \pi)$ be the event that the split of $N'_r$ into $\hat{N}'_{r,1} \cup \hat{N}'_{r,2}$ (according to the Split Algorithm) coincides with $N'_{r,1} \cup N'_{r,2}$ such that
\[
(N'_1(\pi), ..., N'_{r-1}(\pi), N'_{r,1}, N'_{r,2}, N'_{r+1}(\pi), ..., N'_s(\pi)) \in \mathcal{N}_\tau(N).
\]
Let $\hat{r}$ be the chosen group index among $1, ..., s$ by the Selection Step. From the proof of Lemma C.5 (see (C.13)), we have
\[
(C.16) \quad \sum_{\pi \in \mathcal{N}_\tau(N)} P\{\hat{r} \in R_1(\pi)\} \cap A(\pi) = 1 + O(n^2\omega_{n,L} + u_{n,L}).
\]
The probability that the new ordered partition $(\hat{N}'_1, ..., \hat{N}'_{s+1})$ of $N$ belongs to $\mathcal{N}_\tau(N)$ is bounded from below by
\[
\sum_{\pi \in \mathcal{N}_\tau(N)} P(B_s(\hat{r}; \pi) \cap A(\pi)) \geq \sum_{\pi \in \mathcal{N}_\tau(N)} P\{\hat{r} \in R_1(\pi)\} \cap A(\pi) \cap B_s(\hat{r}; \pi)
\]
\[
= \sum_{\pi \in \mathcal{N}_\tau(N)} \sum_{r \in R_1(\pi)} P\{\hat{r} = r\} \cap A(\pi) \cap B_s(r; \pi)
\]
\[
= \sum_{\pi \in \mathcal{N}_\tau(N)} \sum_{r \in R_1(\pi)} P\{\hat{r} = r\} \cap A(\pi) + O(n^2\omega_{n,L}),
\]
by Lemma C.6. The last double sum is $1 + O(n^2\omega_{n,L} + u_{n,L})$ by (C.16). Thus, we conclude that
\[
P\{(\hat{N}'_1, ..., \hat{N}'_{s+1}) \in \mathcal{N}_\tau(N)\} = 1 + O(n^2\omega_{n,L} + u_{n,L}),
\]
as $n, L \to \infty$. ■

For each $K \geq 1$, let $\hat{T}_K = (\hat{N}_1, ..., \hat{N}_K)$ be the group structure obtained through the Selection-Split algorithm (until the number of the groups reach $K$) and let $T = (N_1, ..., N_{K_0})$ be the true ordered group structure. For the remainder of the proof, we assume that the conditions of Theorem C.1 holds.

**Lemma C.8.** Along any sequence of probabilities $P_{n,L} \in \mathcal{P}_{0,\varepsilon}$,
\[
P_{n,L}\{\hat{T}_{K_0} = T\} = 1 + O(n^2\omega_{n,L}),
\]
Therefore, the event that \( \tau \) by the Selection-Split Algorithm with Lemma C.9.

(i) If \( K \geq K_0 \), then \( \hat{V}(K) = O_P(1) \), as \( n, L \to \infty \).

(ii) If \( K < K_0 \), then for any \( M > 0 \), as \( n, L \to \infty \),

\[
P\{\hat{V}(K) > g(L)M\} \to 1.
\]

Proof: (i) Let \((\hat{N}_1, \ldots, \hat{N}_{K_0})\) be the ordered partition obtained by the Selection-Split Algorithm. By Lemma C.8, the event that \( \tau(i) = \tau(j) \) for all \( i, j \in \hat{N}_k \) has probability approaching one for all \( k = 1, \ldots, K_0 \). Let \((\hat{N}_1', \ldots, \hat{N}_K')\) be the ordered partition obtained by the Selection-Split Algorithm with \( K \geq K_0 \).

Since \( K \geq K_0 \), due to the sequential split nature of the algorithm, each of the resulting groups, say, \( \hat{N}_k' \), \( k = 1, \ldots, K \), is a subset of a group, say, \( \hat{N}_k \), obtained at step \( K = K_0 \). Therefore, the event that \( \tau(i) = \tau(j) \) for all \( i, j \in \hat{N}_k' \) has probability approaching one for each \( k = 1, \ldots, K \). By Assumption C.2(i), we have

\[
\hat{V}(K) = \frac{1}{K} \sum_{k=1}^{K} \min_{i,j \in \hat{N}_k'} \log \hat{p}_{ij}^0 = O_P(1),
\]
as \( n, L \to \infty \). Thus (i) follows.

(ii) Suppose that \( K < K_0 \), and fix any \( M > 0 \). Let \((\hat{N}_1, \ldots, \hat{N}_K)\) be the ordered partition obtained by the Selection-Split Algorithm. Let the event that \( \pi \in \mathcal{M}_r(N) \) coincides with \((\hat{N}_1, \ldots, \hat{N}_K)\) be denoted by \( A_K(\pi) \). The event is disjoint across \( \pi \in \mathcal{M}_r(N) \). By Lemmas C.6 and C.7, this ordered partition is asymptotically \( \tau \)-compatible at rate \( n^2 \omega_{n,L} \), that is,

\[
\sum_{\pi \in \mathcal{M}_r(N)} PA_K(\pi) = P \left( \bigcup_{\pi \in \mathcal{M}_r(N)} PA_K(\pi) \right) = 1 + O(n^2 \omega_{n,L}).
\]

Then we have

\[
P\{\hat{V}(K) > g(L)M\} = P \left\{ \frac{1}{K} \sum_{k=1}^{K} \min_{i,j \in \hat{N}_k} \log \hat{p}_{ij}^0 < -g(L)M \right\}
\]

\[
\geq \sum_{\pi \in \mathcal{M}_r(N)} P \left\{ \frac{1}{K} \sum_{k=1}^{K} \min_{i,j \in \hat{N}_k} \log \hat{p}_{ij}^0 < -g(L)M \right\} \cap A_K(\pi)
\]

\[
\geq \sum_{\pi \in \mathcal{M}_r(N)} P \left\{ \min_{i,j \in \hat{N}_k(\pi)} \log \hat{p}_{ij}^0 < -g(L)KM \right\} \cap A_K(\pi),
\]
because the log of \( p \)-values are non-positive. Since \( K < K_0 \), for any \( \pi \in \mathcal{M}_r(N) \) such that \( \pi = (\hat{N}_1, \ldots, \hat{N}_K) \), there exists \( \hat{k}(\pi) \in \{1, \ldots, K\} \) such that for some \( i, j \in \hat{N}_{\hat{k}(\pi)} \).
\( \tau(i) \neq \tau(j) \). Hence the last sum in (C.18) is bounded from below by

\[
\sum_{\pi \in \mathcal{M}_c(N)} P \left\{ \forall i, j \in N, \text{ s.t. } \tau(i) \neq \tau(j), \log \hat{p}_{ij}^0 < -g(L)KM \right\} \cap A_K(\pi)
\]

\[
\geq P \left\{ \forall i, j \in N, \text{ s.t. } \tau(i) \neq \tau(j), \log \hat{p}_{ij}^0 < -g(L)KM \right\} + O(n^2\omega_{n,L}),
\]

by (C.17). By the condition that \( g(L)/r_L \to 0 \) as \( n, L \to \infty \), the last probability is bounded from below by (from some large \( L \) on)

\[
P \left\{ \forall i, j \in N, \text{ s.t. } \tau(i) \neq \tau(j), \log \hat{p}_{ij}^0 < -r_L \right\} \leq 1 - P \left\{ \exists i, j \in N, \text{ s.t. } \tau(i) \neq \tau(j), \log \hat{p}_{ij}^0 \geq -r_L \right\} = 1 + O(n^2\rho_L),
\]

by Lemma C.2(ii). ■

\textbf{Lemma C.10.} \( P\{\hat{K} = K_0\} \to 1 \) as \( n, L \to \infty \).

\textbf{Proof:} Choose \( K \) such that \( K_0 < K \) and write

\[
\tilde{Q}(K_0) - \tilde{Q}(K) = \tilde{V}(K_0) - \tilde{V}(K) + (K_0 - K)g(L).
\]

As for the leading term on the left hand side, we have

\[
\tilde{V}(K_0) - \tilde{V}(K) = O_P(1),
\]

by Lemma C.9(i). Since \( g(L) \to \infty \), we find that whenever \( K > K_0 \), we have

\[
P \left\{ \tilde{Q}(K_0) < \tilde{Q}(K) \right\} \to 1.
\]

And for all \( K < K_0 \), we have by Lemma C.9(ii), for any \( M > 0 \),

(C.19) \[
P \left\{ \tilde{V}(K) > g(L)M \right\} \to 1,
\]

whereas \( \tilde{V}(K_0) = O_P(1) \). Therefore, choose \( \varepsilon > 0 \) and take any \( M' > 0 \) such that

\[
P\{\tilde{V}(K_0) \geq M'\} \leq \varepsilon.
\]

We take large \( M > K_0 - K \) and \( 0 < M' \leq (K - K_0 + M)g(L) \). We find that

\[
P \left\{ \hat{Q}(K_0) < \hat{Q}(K) \right\} = P \left\{ \tilde{V}(K_0) < \tilde{V}(K) + (K - K_0)g(L) \right\}
\]

\[
\geq P \left\{ \tilde{V}(K_0) < (K - K_0 + M)g(L) \right\} + o(1), \text{ (by (C.19))}
\]

\[
\geq P \left\{ \tilde{V}(K_0) \leq M' \right\} + o(1) \geq 1 - \varepsilon + o(1),
\]

as \( L \to \infty \). By sending \( \varepsilon \) down to zero, we conclude that \( P\{\hat{K} = K_0\} \to 1 \), as \( n, L \to \infty \). ■

\textbf{Proof of Theorem C.1:} The desired result follows from Lemmas C.8 and C.10. ■
Appendix D. Confidence Sets for the Group Structure

The web appendix of Krasnokutskaya, Song, and Tang (2018) proposes a method to construct a confidence set for each group of agents having the same type. Here for the sake of readers’ convenience, we reproduce the procedure here using the notation of this paper. Let us consider a set-up where we have $K_0$ groups and the set $N$ of agents. Let $\hat{K}$ be the consistent estimator of $K_0$ as proposed in Krasnokutskaya, Song, and Tang (2019). As for confidence sets, we construct a confidence set for each group of agents who have the same type. First, we fix $k = 1, \ldots, \hat{K}$ and construct a confidence set for the $k$-th type group $N_k$. In other words, we construct a random set $\hat{C}_k \subset N$ such that

$$\liminf_{L \to \infty} P\{N_k \subset \hat{C}_k\} \geq 1 - \alpha,$$

For this, we need to devise a way to approximate the finite sample probabilities like $P\{N_k \subset \hat{C}_k\}$. Since we do not know the cross-sectional dependence structure among the agents, we use a bootstrap procedure that preserves the dependence structure from the original sample. The remaining issue is to determine the space in which the random set $\hat{C}_k \subset N$ can take values in. It is computationally infeasible to consider all possible such sets. Instead, we proceed as follows. First we estimate $\hat{N}_k$ as prescribed in the paper and also obtain $\hat{\delta}_{ij}$, the test statistic defined in the main text. Given the estimate $\hat{N}_k$, we construct a sequence of sets as follows:

**Step 1**: Find $i_1 \in N \setminus \hat{N}_k$ that minimizes $\min_{j \in \hat{N}_k} \hat{d}_{i_1,j}$, and construct $\hat{C}_k(1) = \hat{N}_k \cup \{i_1\}$.

**Step 2**: Find $i_2 \in N \setminus \hat{C}_k(1)$ that minimizes $\min_{j \in \hat{C}_k(1)} \hat{d}_{i_2,j}$, and construct $\hat{C}_k(2) = \hat{C}_k(1) \cup \{i_2\}$.

**Step $m$**: Find $i_m \in N \setminus \hat{C}_k(m - 1)$ that minimizes $\min_{j \in \hat{C}_k(m - 1)} \hat{d}_{i_m,j}$ and construct $\hat{C}_k(m) = \hat{C}_k(m - 1) \cup \{i_m\}$.

Repeat Step $m$ up to $n = |N|$.

Now, for each bootstrap iteration $s = 1, \ldots, B$, we construct the sets $\hat{N}_{k,s}$ and $\{\hat{C}_{k,s}(m)\}$ following the steps described above but using the bootstrap sample. (Note that this bootstrap sample should be drawn independently of the bootstrap sample used to construct bootstrap $p$-values $\hat{p}_{ij}$ in the classification.)

Then, we compute the following:

$$\hat{\pi}_k(m) \equiv \frac{1}{B} \sum_{s=1}^{B} 1\{\hat{N}_k \subset \hat{C}_{k,s}(m)\}.$$

Note that the sequence of sets $\hat{C}_{k,s}(m)$ increases in $m$. Hence the number $\hat{\pi}_k(m)$ should also increase in $m$. An $(1 - \alpha)$%-level confidence set is given by $\hat{C}_k(m)$ with $1 \leq m \leq n$ such that

$$\hat{\pi}_k(m - 1) < 1 - \alpha \leq \hat{\pi}_k(m).$$

Note that such $m$ always exists, because $\hat{C}_{k,s}(n) = N$. 

Appendix E. Further Simulation Results

Tables E.1 and E.2 summarize the distribution of estimation errors in our group classification algorithm from 500 simulated data sets, when the number of groups is $K_0 = 2$ and assumed known to the econometrician. The column $D_\mu$ shows the difference between the group means chosen in the simulation.

When $K_0 = 2$, the results show that the estimation error, as measured by the expected average discrepancy (EAD), decreases with the distance between group means. Such a reduction in EAD is most substantial when the number of agents is larger ($n = 40$) and the size of the data is small ($L = 100$). Given group difference, EAD decreases as sample size increases moderately from $L = 100$ to 400. This pattern is most obvious when $D_\mu = 0.2$.

The other measure of estimation errors, HAD(p), also shows encouraging results. HAD(p) is zero for most of the cells in both panels (a) and (b), which shows that the empirical distribution of proportion of mis-classified bidders is reasonably skewed to the right. Besides, the reduction in HAD(p) as the sample size increases is most pronounced with closer group means, regardless of the number of bidders in the population.

When $K_0 = 4$, the results demonstrate very similar patterns. Most remarkably, both measures of mis-classification errors only increase very marginally relative to the case with $K_0 = 2$.

Tables E.3 and E.4 report results from the full, feasible classification procedure when the number of groups is estimated through the penalization scheme proposed in the text. For most of the specifications used in these two tables, the estimates for the number of groups $\hat{K}_0$ are tightly clustered around the correct $K_0$. Compared with the results for infeasible classification under known $K_0$, EAD and HAD(p) increase in most cases. Nonetheless such an increase is quite moderate, suggesting that our feasible classification algorithm performs reasonably well relative to its infeasible counterpart.

In Tables E.3 - E.4, we report the analysis of computation time for the classification procedure. In Table 5.3, we give a decomposition of the time that it took for the classification procedure. The table clearly shows that the major computation time spent is when we construct bootstrap p-values. Once the p-values are constructed, the classification algorithm itself runs fairly fast.

In Table 5.4, the computation time is shown to vary depending on the number of the agents ($n$), the number of the true groups ($K_0$), and the number of the markets ($L$). The results show that the most computation time increase arises when the number of the bidders increases rather than when the number of the markets or the number of the groups increases. Our simulation studies are based on our MatLab code. The program was executed using a computer with the following specifications: Intel(R) Xeon (R) CPU X5690 @3.47 GHz 3.46 GHz.
Table E.1: Performance of the Classification Estimator with Two Groups:
\( (K_0 = 2 \text{ and known}) \)

| \( n \) | \( L \) | \( D_\mu \) | \( \text{EAD} \) | \( \text{HAD(0.10)} \) | \( \text{HAD(0.25)} \) | \( \text{HAD(0.50)} \) | \( \text{HAD(0.75)} \) | \( \text{HAD(0.90)} \) |
|--------|--------|------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 12     | 400    | 0.6        | 0.012           | 0.012           | 0.014           | 0.014           | 0.004           | 0.004           |
| 12     | 400    | 0.4        | 0.014           | 0.014           | 0.014           | 0.014           | 0.004           | 0.004           |
| 12     | 400    | 0.2        | 0.004           | 0.004           | 0.004           | 0.004           | 0.004           | 0.004           |
| 12     | 200    | 0.6        | 0.004           | 0.004           | 0.004           | 0.004           | 0.004           | 0.004           |
| 12     | 200    | 0.4        | 0.006           | 0.006           | 0.006           | 0.006           | 0.006           | 0.006           |
| 12     | 200    | 0.2        | 1.118           | 0.560           | 0.252           | 0.158           | 0.084           | 0.084           |
| 12     | 100    | 0.6        | 0.006           | 0.006           | 0.006           | 0.006           | 0.006           | 0.006           |
| 12     | 100    | 0.4        | 0.084           | 0.078           | 0.006           | 0.006           | 0.006           | 0.006           |
| 12     | 100    | 0.2        | 1.794           | 0.682           | 0.478           | 0.284           | 0.284           | 0.284           |
| 40     | 400    | 0.6        | 0.018           | 0.004           | 0.004           | 0.004           | 0.004           | 0.004           |
| 40     | 400    | 0.4        | 0.022           | 0.004           | 0.004           | 0.004           | 0.004           | 0.004           |
| 40     | 400    | 0.2        | 1.170           | 0.178           | 0.014           | 0.014           | 0.014           | 0.014           |
| 40     | 200    | 0.6        | 0.018           | 0.004           | 0.004           | 0.004           | 0.004           | 0.004           |
| 40     | 200    | 0.4        | 0.020           | 0.004           | 0.004           | 0.004           | 0.004           | 0.004           |
| 40     | 200    | 0.2        | 2.726           | 0.404           | 0.210           | 0.122           | 0.021           | 0.001           |
| 40     | 100    | 0.6        | 0.020           | 0.004           | 0.004           | 0.004           | 0.004           | 0.004           |
| 40     | 100    | 0.4        | 0.452           | 0.010           | 0.004           | 0.004           | 0.004           | 0.004           |
| 40     | 100    | 0.2        | 3.720           | 0.902           | 0.578           | 0.234           | 0.132           | 0.043           |

Note: This table summarizes the distribution of estimation errors in our classification algorithm from 500 Monte Carlo replications when \( K_0 = 4 \) and known. Here \( n \) represents the number of the individual agents, \( L \) the number of the observed games in the data, \( D_\mu \) the distance between population means, \( \text{EAD} \) the expected average discrepancy, and \( \text{HAD(p)} \) the hazard rate of average discrepancies at \( p \).
Table E.2 : Performance of the Classification Estimator with Four Groups:  
\((K_0 = 4 \text{ and known})\)

| \(n\) | \(L\) | \(D_\mu\) | EAD | HAD(.10) | HAD(.25) | HAD(.50) | HAD(.75) | HAD(.90) |
|------|------|--------|-----|---------|---------|---------|---------|---------|
| 12   | 400  | 0.6    | 0.011 | 0.014  | 0.004  | 0       | 0       | 0       |
| 12   | 400  | 0.4    | 0.018 | 0.016  | 0.010  | 0       | 0       | 0       |
| 12   | 400  | 0.2    | 0.017 | 0.022  | 0.006  | 0       | 0       | 0       |
| 12   | 200  | 0.6    | 0.013 | 0.018  | 0.004  | 0       | 0       | 0       |
| 12   | 200  | 0.4    | 0.004 | 0.008  | 0       | 0       | 0       | 0       |
| 12   | 200  | 0.2    | 1.112 | 0.764  | 0.188  | 0.024   | 0.008   | 0       |
| 12   | 100  | 0.6    | 0.003 | 0.006  | 0       | 0       | 0       | 0       |
| 12   | 100  | 0.4    | 0.044 | 0.040  | 0.024  | 0.002   | 0       | 0       |
| 12   | 100  | 0.2    | 1.504 | 0.868  | 0.342  | 0.106   | 0.04    | 0       |
| 40   | 400  | 0.6    | 0.115 | 0.020  | 0.020  | 0       | 0       | 0       |
| 40   | 400  | 0.4    | 0.121 | 0.020  | 0.020  | 0       | 0       | 0       |
| 40   | 400  | 0.2    | 2.450 | 0.680  | 0.368  | 0.018   | 0.018   | 0       |
| 40   | 200  | 0.6    | 0.109 | 0.018  | 0.018  | 0       | 0       | 0       |
| 40   | 200  | 0.4    | 0.140 | 0.026  | 0.026  | 0       | 0       | 0       |
| 40   | 200  | 0.2    | 3.172 | 0.810  | 0.366  | 0.246   | 0.026   | 0       |
| 40   | 100  | 0.6    | 0.141 | 0.024  | 0.024  | 0       | 0       | 0       |
| 40   | 100  | 0.4    | 1.003 | 0.176  | 0.176  | 0.006   | 0       | 0       |
| 40   | 100  | 0.2    | 4.557 | 0.904  | 0.652  | 0.526   | 0.202   | 0.053   |

Note: This table summarizes the distribution of estimation errors in our classification algorithm from 500 Monte Carlo replications when \(K_0 = 4\) and known. Here \(n\) represents the number of the individual agents, \(L\) the number of the observed games in the data, \(D_\mu\) the distance between population means, EAD the expected average discrepancy, and HAD(p) the hazard rate of average discrepancies at \(p\).
Table E.3: Computational Time for Various Steps of the Procedure
\((n = 60, K_0 = 2, \text{ unknown}, \text{ time measured in seconds})\)

| Step | Description | \(L=100\) | \(L=200\) | \(L=400\) |
|------|-------------|------------|------------|------------|
| 1    | generating pairwise indexes from the data | 0.2987 | 0.3543 | 0.4607 |
| 2    | constructing bootstrap pairwise indexes | 81.2178 | 81.4871 | 82.0807 |
| 3    | computing bootstrap p-values | 0.0012 | 0.0014 | 0.0014 |
| 4    | division of a group into two | 0.0008 | 0.0008 | 0.0008 |
| n+4  | number of groups selection | 0.0002 | 0.0002 | 0.0002 |
|      | Total Time | 81.528 | 81.852 | 82.552 |

Note: The table shows a decomposition of a total time it has taken for the classification procedure. The table shows that the major portion of the time comes from constructing the bootstrap pairwise indexes. Once the bootstrap p-values are constructed, the classification algorithm runs quite fast.

Table E.4: Total Computational Time: across \(n, L,\) and \(K_0\)
\((K_0 \text{ unknown}, \text{ time measured in seconds})\)

|      | \(L=100\) | \(L=200\) | \(L=400\) | \(L=200\) | \(L=200\) |
|------|------------|------------|------------|------------|------------|
|      | \(K_0=2\) | \(K_0=2\) | \(K_0=2\) | \(K_0=4\) | \(K_0=6\) |
| \(n = 12\) | 3.246 | 3.224 | 3.239 | 3.216 | 3.219 |
| \(n = 24\) | 13.057 | 13.177 | 13.259 | 13.185 | 13.189 |
| \(n = 48\) | 51.987 | 52.272 | 52.700 | 52.281 | 52.291 |
| \(n = 60\) | 81.528 | 81.852 | 82.552 | 81.862 | 82.874 |
| \(n = 72\) | 116.949 | 117.213 | 117.577 | 116.912 | 117.328 |
| \(n = 96\) | 209.426 | 209.971 | 209.834 | 209.884 | 210.058 |

Note: The table shows the change in the computation time as one changes the number of the groups \((K)\), the number of the markets and the number of the agents (i.e., bidders) \((n)\). The major increase in the computation time arises when the number of the bidders increases rather than when the number of the markets or the groups increases.
Appendix F. Additional Materials for the Empirical Application

Table F.1 reports summary statistics for this set of projects. The table indicates that the projects are worth $523,000 and last for around three months on average; 38% of these projects are partially supported through federal funds. There are 25 firms that participate regularly in this market. All other firms in the data are treated as fringe bidders. An average auction attracts six regular potential bidders and eight fringe bidders. Since only a fraction of potential bidders submits bids, an entry decision plays an important role in this market. Finally, the distance to the company location varies quite a bit and is around 28 miles on average for regular potential bidders.

| Variable                          | Mean  | Std. Dev |
|-----------------------------------|-------|----------|
| Engineer's estimate (mln)         | 0.523 | 0.261    |
| Duration, large projects (months) | 3.01  | 1.56     |
| Federal Aid                       | 0.384 |          |
| Number of Potential Bidders:      |       |          |
| Fringe Bidders                    | 8.2   | 4.8      |
| Regular Bidders                   | 5.5   | 3.3      |
| Number of Entrants:               |       |          |
| Fringe Bidders                    | 3.5   | 2.7      |
| Regular Bidders                   | 1.9   | 1.8      |
| Distance (miles):                 |       |          |
| Fringe Bidders                    | 11.21 | 5.42     |
| Regular Bidders                   | 28.34 | 11.73    |

Note: This table reports summary statistics for the set of medium size bridge work and paving projects auctioned in the California highway procurement market between years of 2002 and 2012. It consists of 1,054 projects. The distance variable is measured in miles. It reflects the driving time between the project site and the nearest company plant. The “Federal Aid” variable is equal to one if the project receives federal aid and zero otherwise.

In Table F.2, we present an extended version of Table 6 in Section 6 of the main paper. This table includes the estimates of the group-specific fixed effects.
Table F.2: Parameter Estimates (Extended Version of Table 6.)

|                                   | Estimate | Std. Error | Estimate | Std. Error |
|-----------------------------------|----------|------------|----------|------------|
| **The Distribution of Project Costs** |          |            |          |            |
| Constant ($\bar{q}_0$)            | 0.127*** | (0.0129)   | 0.113*** | (0.0119)   |
| Eng. Estimate                     | -0.0004*** | (0.0002)  | -0.0005*** | (0.0002)  |
| Duration                          | 0.00026* | (0.00036) | 0.00022* | (0.00027) |
| Distance                          | 0.0012*** | (0.00022) | 0.00086*** | (0.00019) |
| Bridge                            | -0.0092*** | (0.0018)  | -0.012*** | (0.0011)  |
| Federal Aid                       | -0.043*** | (0.0103)  | -0.078*** | (0.009)   |
| Regular Bidder                    |          |            | -0.035*** | (0.003)   |
| \(\bar{q}_1 - \bar{q}_0\)        | -0.051*** | (0.008)   |          |            |
| \(\bar{q}_2 - \bar{q}_0\)        | -0.012*** | (0.005)   |          |            |
| \(\bar{q}_3 - \bar{q}_0\)        | -0.032*** | (0.009)   |          |            |
| \(\bar{q}_4 - \bar{q}_0\)        | -0.058*** | (0.008)   |          |            |
| \(\bar{q}_5 - \bar{q}_0\)        | -0.014*** | (0.007)   |          |            |
| \(\bar{q}_6 - \bar{q}_0\)        | -0.008*** | (0.006)   |          |            |
| \(\bar{q}_7 - \bar{q}_0\)        | -0.009*** | (0.007)   |          |            |
| \(\bar{q}_8 - \bar{q}_0\)        | -0.050*** | (0.006)   |          |            |
| \(\sigma_C\)                      | 0.087*** | (0.032)   | 0.112*** | (0.022)   |
| \(\sigma_U\)                      | 0.021*** | (0.009)   | 0.0207***| (0.008)   |

| **The Distribution of Entry Costs** |          |            |          |            |
| Constant ($\tilde{q}_0$)           | -0.0114* | (0.0078)  | -0.0161* | 0.0091    |
| Eng. Estimate                      | 0.0055*** | (0.0016) | 0.0051*** | (0.0012)  |
| Number of Items                    | 0.0018*  | (0.0011) | 0.0011*** | (0.0005)  |
| Regular Bidder                     |          |            | -0.022***| (0.004)   |
| \(\bar{q}_1 - \tilde{q}_0\)      | -0.019*** | (0.005)  |          |            |
| \(\bar{q}_2 - \tilde{q}_0\)      | -0.018*** | (0.007)  |          |            |
| \(\bar{q}_3 - \tilde{q}_0\)      | -0.016*** | (0.007)  |          |            |
| \(\bar{q}_4 - \tilde{q}_0\)      | -0.024*** | (0.006)  |          |            |
| \(\bar{q}_5 - \tilde{q}_0\)      | -0.022*** | (0.008)  |          |            |
| \(\bar{q}_6 - \tilde{q}_0\)      | -0.018*** | (0.006)  |          |            |
| \(\bar{q}_7 - \tilde{q}_0\)      | -0.017*** | (0.008)  |          |            |
| \(\bar{q}_8 - \tilde{q}_0\)      | -0.019*** | (0.008)  |          |            |

Note: In the results above the distance is measured in miles. The fringe bidders are the reference group. The first two columns correspond to the specification which allows for the unobserved bidder heterogeneity; the last two columns correspond to the specification without unobserved bidder heterogeneity. The results are based on the data for 1,054 medium-sized projects that involve paving and bridge work.

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