1997

Factors from trees

Jacqueline Ramagge
University of Wollongong, ramagge@uow.edu.au

A G. Robertson
Newcastle University (UK)

Publication Details
Ramagge, J. & Robertson, A. G. (1997). Factors from trees. Proceedings of the American Mathematical Society, 125 2051-2055.
Factors from trees

Abstract
We construct factors of type III,\(n\) for \(n \in \mathbb{N}, n > 2\), from group actions on homogeneous trees and their boundaries. Our result is a discrete analogue of a result of R.J Spatzier, where the hyperfinite factor of type III1 is constructed from a group action on the boundary of the universal cover of a manifold.

Keywords
factors, trees

Disciplines
Physical Sciences and Mathematics

Publication Details
Ramagge, J. & Robertson, A. G. (1997). Factors from trees. Proceedings of the American Mathematical Society, 125 2051-2055.

This journal article is available at Research Online: http://ro.uow.edu.au/infopapers/1237
FACTORS FROM TREES

JACQUI RAMAGGE AND GUYAN ROBERTSON

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We construct factors of type $\text{III}_{1/n}$ for $n \in \mathbb{N}, n \geq 2$, from group actions on homogeneous trees and their boundaries. Our result is a discrete analogue of a result of R.J. Spatzier, where the hyperfinite factor of type $\text{III}_1$ is constructed from a group action on the boundary of the universal cover of a manifold.

1. INTRODUCTION

Let $\Gamma$ be a group acting simply transitively on the vertices of a homogeneous tree $T$ of degree $n + 1 < \infty$. Then, by [FTN, Ch. I, Theorem 6.3],

$$\Gamma \cong \mathbb{Z}_2 * \cdots * \mathbb{Z}_2 * \mathbb{Z} * \cdots * \mathbb{Z}$$

where there are $s$ factors of $\mathbb{Z}_2$, $t$ factors of $\mathbb{Z}$, and $s + 2t = n + 1$. Thus $\Gamma$ has a presentation

$$\Gamma = \langle a_1, \ldots, a_{s+t} : a_i^2 = 1 \text{ for } i \in \{1, \ldots, s\} \rangle,$$

we can identify the Cayley graph of $\Gamma$ constructed via right multiplication with $T$ and the action of $\Gamma$ on $T$ is equivalent to the natural action of $\Gamma$ on its Cayley graph via left multiplication.

We can associate a natural boundary to $T$, namely the set $\Omega$ of semi-infinite reduced words in the generators of $\Gamma$. The action of $\Gamma$ on $\Omega$ induces an action of $\Gamma$ on $\Omega$.

For each $x \in \Gamma$, let

$$\Omega^x = \{ \omega \in \Omega : \omega = x \cdots \}$$

be the set of semi-infinite reduced words beginning with $x$. The set $\{\Omega^x\}_{x \in \Gamma}$ is a set of basic open sets for a compact Hausdorff topology on $\Omega$. Denote by $|x|$ the length of a reduced expression for $x$. Let $V^m = \{x \in \Gamma : |x| = m\}$ and define $N_m = |V^m|$. Then $\Omega$ is the disjoint union of the $N_m$ sets $\Omega^x$ for $x \in V^m$.

We can also endow $\Omega$ with the structure of a measure space. $\Omega$ has a unique distinguished Borel probability measure $\nu$ such that

$$\nu(\Omega^x) = \frac{1}{n+1} \left( \frac{1}{n} \right)^{|x|-1}$$

for every nontrivial $x \in \Gamma$. The sets $\Omega^x, x \in \Gamma$, generate the Borel $\sigma$-algebra.
This measure $\nu$ on $\Omega$ is quasi-invariant under the action of $\Gamma$, so that $\Gamma$ acts on the measure space $(\Omega, \nu)$ and enables us to extend the action of $\Gamma$ to an action on $L^\infty(\Omega, \nu)$ via
\[ g \cdot f(\omega) = f(g^{-1} \cdot \omega) \]
for all $g \in \Gamma$, $f \in L^\infty(\Omega, \nu)$, and $\omega \in \Omega$. We may therefore consider the von Neumann algebra $L^\infty(\Omega, \nu) \rtimes \Gamma$ which we shall write as $L^\infty(\Omega) \rtimes \Gamma$ for brevity.

2. THE FACTORS

We note that the action of $\Gamma$ on $\Omega$ is free since if $g\omega = \omega$ for some $g \in \Gamma$ and $\omega \in \Omega$ then we must have either $\omega = ggg \cdots$ or $\omega = g^{-1}g^{-1}g^{-1} \cdots$ and $\nu \{ggg \cdots, g^{-1}g^{-1}g^{-1} \cdots\} = 0$.

The action of $\Gamma$ on $\Omega$ is also ergodic by the proof of [PS, Proposition 3.9], so that $L^\infty(\Omega) \rtimes \Gamma$ is a factor. Establishing the type of the factor is not quite as straightforward. We begin by recalling some classical definitions.

**Definition 2.1.** Given a group $\Gamma$ acting on a measure space $\Omega$, we define the full group, $[\Gamma]$, of $\Gamma$ by
\[ [\Gamma] = \{T \in \text{Aut}(\Omega) : T\omega \in \Gamma\omega \text{ for almost every } \omega \in \Omega\}. \]
The set $[\Gamma]_0$ of measure preserving maps in $[\Gamma]$ is then given by
\[ [\Gamma]_0 = \{T \in [\Gamma] : T\nu = \nu\}. \]

**Definition 2.2.** Let $G$ be a countable group of automorphisms of the measure space $(\Omega, \nu)$. Following W. Krieger, define the ratio set $r(G)$ to be the subset of $[0, \infty)$ such that if $\lambda \geq 0$ then $\lambda \in r(G)$ if and only if for every $\epsilon > 0$ and Borel set $E$ with $\nu(E) > 0$, there exist a $g \in G$ and a Borel set $F$ such that $\nu(F) > 0$, $F \cup gF \subseteq E$ and
\[ \left| \frac{d\nu \circ g}{d\nu} (\omega) - \lambda \right| < \epsilon \]
for all $\omega \in F$.

**Remark 2.3.** The ratio set $r(G)$ depends only on the quasi-equivalence class of the measure $\nu$; see [HO, §1-3, Lemma 14]. It also depends only on the full group in the sense that $[H] = [G] \Rightarrow r(H) = r(G)$.

The following result will be applied in the special case where $G = \Gamma$. However, since the simple transitivity of the action doesn't play a role in the proof, we can state it in greater generality.

**Proposition 2.4.** Let $G$ be a countable subgroup of $\text{Aut}(T) \leq \text{Aut}(\Omega)$. Suppose there exist an element $g \in G$ such that $d(ge, e) = 1$ and a subgroup $K$ of $[G]_0$ whose action on $\Omega$ is ergodic. Then
\[ r(G) = \{ nk : k \in \mathbb{Z}\} \cup \{0\}. \]

**Proof.** By Remark 2.3, it is sufficient to prove the statement for some group $H$ such that $[H] = [G]$. In particular, since $[G] = [(G, K)]$ for any subgroup $K$ of $[G]_0$, we may assume without loss of generality that $K \leq G$.

By [FTN, Chapter II, part 1]), for each $g \in G$ and $\omega \in \Omega$ we have
\[ \frac{d\nu \circ g}{d\nu} (\omega) \in \{ nk : k \in \mathbb{Z}\} \cup \{0\}. \]
Since $G$ acts ergodically on $\Omega$, $r(G) \setminus \{0\}$ is a group. It is therefore enough to show that $n \in r(G)$. Write $x = ge$ and note that $\nu_x = \nu \circ g^{-1}$. By [FTN, Chapter II, part 1] we have

$$\frac{d\nu_x}{d\nu}(\omega) = n, \text{ for all } \omega \in \Omega^x_e.$$  

Let $E \subseteq \Omega$ be a Borel set with $\nu(E) > 0$. By the ergodicity of $K$, there exist $k_1, k_2 \in K$ such that the set

$$\mathcal{F} = \{\omega \in E : k_1 \omega \in \Omega^x_e \text{ and } k_2 g^{-1} k_1 \omega \in E\}$$

has positive measure.

Finally, let $t = k_2 g^{-1} k_1 \in G$. By construction, $\mathcal{F} \cup t \mathcal{F} \subseteq E$. Moreover, since $K$ is measure-preserving,

$$\frac{d\nu tr}{d\nu}(\omega) = \frac{d\nu g^{-1}}{d\nu}(k_1 \omega) = \frac{d\nu_x}{d\nu}(k_1 \omega) = n \text{ for all } \omega \in \mathcal{F}$$

by (1), since $k_1 \in \Omega^x_e$. This proves $n \in r(G)$, as required. \qed

**Corollary 2.5.** If, in addition to the hypotheses for Proposition 2.4, the action of $G$ is free, then $L^\infty(\Omega) \rtimes G$ is a factor of type $III_{1/n}$.

*Proof.* Having determined the ratio set, this is immediate from [C1, Corollaire 3.3.4]. \qed

Thus, if we can find a countable subgroup $K \leq [\Gamma]_0$ whose action on $\Omega$ is ergodic we will have shown that $L^\infty(\Omega) \rtimes G$ is a factor of type $III_{1/n}$. To this end, we prove the following sufficiency condition for ergodicity.

**Lemma 2.6.** Let $K$ be group which acts on $\Omega$. If $K$ acts transitively on the collection of sets $\{\Omega^x : x \in \Gamma, |x| = m\}$ for each natural number $m$, then $K$ acts ergodically on $\Omega$.

*Proof.* Suppose that $X_0 \subseteq \Omega$ is a Borel set which is invariant under $K$ and such that $\nu(X_0) > 0$. We show that this necessarily implies $\nu(\Omega \setminus X_0) = 0$, thus establishing the ergodicity of the action.

Define a new measure $\mu$ on $\Omega$ by $\mu(X) = \nu(X \cap X_0)$ for each Borel set $X \subseteq \Omega$. Now, for each $g \in K$,

$$\mu(gX) = \nu(gX \cap X_0) = \nu(X \cap g^{-1}X_0) = \nu(X \cap X_0) = \mu(X),$$

and therefore $\mu$ is $K$-invariant. Since $K$ acts transitively on the basic open sets $\Omega^x$ associated to words $x$ of length $m$ this implies that

$$\mu(\Omega^x) = \mu(\Omega^y)$$

whenever $|x| = |y|$. Since $\Omega$ is the union of $N_m$ disjoint sets $\Omega^x, x \in V^m$, each of which has equal measure with respect to $\mu$, we deduce that

$$\mu(\Omega^x) = \frac{c}{N_m}$$

for each $x \in V^m$, where $c = \mu(X_0) = \nu(X_0) > 0$. Thus $\mu(\Omega^x) = cv(\Omega^x)$ for every $x \in \Gamma$. 

Thus, if we can find a countable subgroup $K \leq [\Gamma]_0$ whose action on $\Omega$ is ergodic we will have shown that $L^\infty(\Omega) \rtimes G$ is a factor of type $III_{1/n}$. To this end, we prove the following sufficiency condition for ergodicity.
Since the sets $\Omega^x$, $x \in \Gamma$ generate the Borel $\sigma$-algebra, we deduce that $\mu(X) = c\nu(X)$ for each Borel set $X$. Therefore
\[
\nu(\Omega \setminus X_0) = c^{-1}\mu(\Omega \setminus X_0) = c^{-1}\nu((\Omega \setminus X_0) \cap X_0) = 0,
\]
thus proving ergodicity.

In the last of our technical results, we give a constructive proof of the existence of a countable ergodic subgroup of $[\Gamma]_0$.

**Lemma 2.7.** There is a countable ergodic group $K \leq \text{Aut}(\Omega)$ such that $K \leq [\Gamma]_0$.

**Proof.** Let $x, y \in V^m$. We construct a measure preserving automorphism $k_{x,y}$ of $\Omega$ such that

1. $k_{x,y}$ is almost everywhere a bijection from $\Omega^x$ onto $\Omega^y$,
2. $k_{x,y}$ is the identity on $\Omega \setminus (\Omega^x \cup \Omega^y)$.

It then follows from Lemma 2.6 that the group $K = \langle k_{x,y} : \{x, y\} \subseteq V^m, m \in \mathbb{N} \rangle$ acts ergodically on $\Omega$ and the construction will show explicitly that $K \leq [\Gamma]_0$.

Fix $x, y \in V^m$ and suppose that we have reduced expressions $x = x_1 \ldots x_m$, and $y = y_1 \ldots y_m$.

Define $k_{x,y}$ to be left multiplication by $yx^{-1}$ on each of the sets $\Omega^{xz}$ where $|z| = 1$ and $z \notin \{x_m^{-1}, y_m^{-1}\}$. Then $k_{x,y}$ is a measure preserving bijection from each such set onto $\Omega^{yz}$. If $y_m = x_m$ then $k_{x,y}$ is now well defined everywhere on $\Omega^x$.

Suppose now that $y_m \neq x_m$. Then $k_{x,y}$ is defined on the set $\Omega^{x \setminus \Omega^{x y_m^{-1}}}$, which it maps bijectively onto $\Omega^{u \setminus \Omega^{u y_m^{-1}}}$.

Next define $k_{x,y}$ to be left multiplication by $yx_m^{-1}y_m x^{-1}$ on the sets $\Omega^{x y_m^{-1} z}$ where $|z| = 1$ and $z \notin \{x_m, y_m\}$. Then $k_{x,y}$ is a measure preserving bijection of each such $\Omega^{x y_m^{-1} z}$ onto $\Omega^{y z_m^{-1} z}$.

Thus we have extended the domain of $k_{x,y}$ so that it is now defined on the set $\Omega^x \setminus \Omega^{x y_m^{-1} z}$, which it maps bijectively onto $\Omega^x \setminus \Omega^{y z_m^{-1} y_m}$.

Um continues in this way. At the jth step $k_{x,y}$ is a measure preserving bijection from $\Omega^x \setminus X_j$ onto $\Omega^x \setminus Y_j$ where $\nu(X_j) \to 0$ as $j \to \infty$ so that eventually $k_{x,y}$ is defined almost everywhere on $\Omega$. Finally, define
\[
k_{x,y}(x y_m^{-1} x_m y_m^{-1} x_m x_m^{-1} x_m \ldots) = y x_m^{-1} y_m x_m^{-1} y_m x_m^{-1} y_m \ldots
\]
thus defining $k_{x,y}$ everywhere on $\Omega$ in such a way that its action is pointwise approximable by $\Gamma$ almost everywhere. Hence
\[
K = \langle k_{x,y} : \{x, y\} \subseteq V^m, m \in \mathbb{N} \rangle
\]
is a countable group with an ergodic measure-preserving action on $\Omega$ and $K \leq [\Gamma]_0$.

We are now in a position to prove our main result.

**Theorem 2.8.** The von Neumann algebra $L^\infty(\Omega) \rtimes \Gamma$ is the hyperfinite factor of type $\text{III}_{1/\gamma}$.\[\Box\]
**Proof.** By applying Corollary 2.5 with \( G = \Gamma, \ g \in \Gamma \) any generator of \( \Gamma \), and \( K \) as in Lemma 2.7 we conclude that \( L^\infty(\Omega) \rtimes \Gamma \) is a factor of type III\(_{1/n}\).

To see that the factor is hyperfinite simply note that the action of \( \Gamma \) is amenable as a result of [A, Theorem 5.1]. We refer to [C2, Theorem 4.4.1] for the uniqueness of the hyperfinite factor of type III\(_{1/n}\). \( \square \)

**Remark 2.9.** In [Sp1], Spielberg constructs III\(_A\) factor states on the algebra \( \mathcal{O}_2 \). The reduced C*-algebra \( C(\Omega) \rtimes \Gamma \) is a Cuntz-Krieger algebra \( \mathcal{O}_A \) by [Sp2]. What we have done is construct a type III\(_{1/n}\) factor state on some of these algebras \( \mathcal{O}_A \).

**Remark 2.10.** From [C2, p. 476], we know that if \( \Gamma = \mathbb{Q} \times \mathbb{Q}^* \) acts naturally on \( \mathbb{Q}_p \), then the crossed product \( L^\infty(\mathbb{Q}_p) \rtimes \Gamma \) is the hyperfinite factor of type III\(_{1/p}\). This may be proved geometrically as above by regarding the the boundary of the homogeneous tree of degree \( p+1 \) as the one point compactification of \( \mathbb{Q}_p \) as in [CKW].

**References**

[A] S. Adams. *Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups*. Topology 33 (1994), 765–783.

[CKW] D. I. Cartwright, V. Kaimanovich, and W. Woess. *Random walks on the affine group of local fields and homogeneous trees*. M.S.R.I. Preprint No. 022-94, 1993. MR 96f:60121

[C1] A. Connes. *Une classification des facteurs de type III*. Ann. Scient. Ec. Norm. Sup., 6 (1973), pp. 133–252. MR 49:5865

[C2] A. Connes. *On the classification of von Neumann algebras and their automorphisms*. Symposia Mathematica XX, pp. 435–478, Academic Press 1978. MR 56:9278

[FTN] A. Figa-Talamanca and C. Nebbia, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees*, Cambridge University Press, London 1991. MR 93f:22004

[HO] T. Hamachi and M. Osikawa, *Ergodic Groups Acting of Automorphisms and Krieger’s Theorems*, Seminar on Mathematical Sciences No. 3, Keio University, Japan, 1981.

[PS] C. Pensavalle and T. Steger, *Tensor products and anisotropic principal series representations for free groups*, Pac. J. Math. 173 (1996), 181–202.

[S] R. J. Spatzier, *An example of an amenable action from geometry*, Ergod. Th. & Dynam. Sys. (1987), 7, 289–293. MR 88j:58100

[Sp1] J. Spielberg, *Diagonal states on \( \mathcal{O}_2 \)*, Pac. J. Math. 144 (1990), 351–382. MR 91k:46065

[Sp2] J. Spielberg, *Free product groups, Cuntz-Krieger algebras and covariant maps*, Int. J. Math. (1991), 2, 457–476. MR 92j:46120

**Department of Mathematics, University of Newcastle, Callaghan, New South Wales 2308, Australia**

E-mail address: jacqui@maths.newcastle.edu.au

E-mail address: guyan@maths.newcastle.edu.au