IS QUANTUM FIELD THEORY A GENERALIZATION OF QUANTUM MECHANICS?

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ABSTRACT. We construct a mathematical model analogous to quantum field theory, but without the notion of vacuum and without measurable physical quantities. This model is a direct mathematical generalization of scattering theory in quantum mechanics to path integrals with multidimensional trajectories (whose mathematical interpretation has been given in a previous paper). In this model the normal ordering of operators in the Fock space is replaced by the Weyl-Moyal algebra. This model shows to be useful in proof of various results in quantum field theory: one first proves these results in the mathematical model and then “translates” them into the usual language of quantum field theory by more or less “ugly” procedures.

INTRODUCTION

The purpose of this paper is to answer the question stated in the title; let us state it in more detail. Mathematically, it is natural to ask whether one can generalize the rich and beautiful apparatus of the theory of linear partial differential equations to multidimensional variational problems, so that the bicharacteristics be replaced by multidimensional surfaces. This question is closely related with presentation of solutions of partial differential equations in the form of a path integral. A mathematical interpretation of the notion of path integral has been given in [1]. One can ask whether the theory of linear PDE’s, in particular the scattering theory in quantum mechanics, can be generalized to path integrals over multidimensional trajectories. Since the works of Feynman and others, it is conventional to think that such a generalization is given by quantum field theory. However, when trying to give a mathematical sense to this statement, one meets difficulties. The main principal problem is that quantum field theory has no rigorous mathematical sense and, moreover, is self-contradictory. The logical contradiction is, according to [2], in that particles are considered separately from the vacuum in which they move when they are sufficiently far from one another. However, in fact “particles continuously interact with vacuum as with the physical medium in which they
move” ([2], p. 139). The logically correct description of quantum fields should be a synthesis of the notions of particle and field and, therefore, should exclude the very notion of vacuum, replacing it with quantum field, the universal form of matter.

Mathematically, this means that one should not use the Fock space. But then one looses the opportunity to measure physical quantities such as scattering amplitudes. So we are left with a non-contradictory but non-physical theory. The description of this theory is the subject of the present paper. The key role in it is played by the infinite dimensional Weyl–Moyal algebra \( W_0 \), see its definition in §2. See the book [3] or the paper [4] for the detailed motivation of introducing this algebra instead of the algebra of operators in the Fock space. In [3,4] it is said that there is a (not everywhere defined) homomorphism from the algebra \( W_0 \) to the algebra of operators in the Fock space. However, when computing matrix elements in the Fock space of operators like \( \varphi(x)^4 \) from \( W_0 \), we see that these matrix elements are infinite. To make them finite, one should replace \( \varphi(x)^4 \) by the normally ordered operator in the Fock space, a procedure which has no mathematical and logical meaning.

Similarly, when proving many results in quantum field theory, one first states them in our mathematical model and, second, one translates them into the language of quantum field theory, introducing the Fock space, normal orderings etc.; this translation is an “ugly” procedure, but necessary from the physical point of view. This holds for the construction of S-matrix and Green functions in renormalized perturbation theory (this translation procedure was erroneously omitted in [3,4]), for the Maslov quantum field theory complex germ (compare its “mathematical” counterpart in [3,4] with the “physical” exposition in [5] or [6]), for two-dimensional conformal field theory (to appear elsewhere), hopefully for ultrasoldy quantization [7], for pseudodynamical evolution [8] (whose meaning is to be clarified), etc. Thus, our mathematical model proves to be useful to physics as well as to mathematics.

Let us describe the contents of the present paper. In §1 a one-dimensional model of scattering in quantum mechanics is given in a form suitable for generalization to many dimensions. In §2 the mathematical model of renormalized perturbation theory for the Weyl–Moyal algebra is stated on the example of the \( \varphi^4 \) model in \( \mathbb{R}^{3+1} \).

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1. ONE-DIMENSIONAL MODEL: SCATTERING THEORY

Consider the Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi, \]

where \( \psi = \psi(t, q)(dq_1 \ldots dq_n)^{1/2} \) is a half-form (the wave function) on configuration space, \( q = (q_1, \ldots, q_n) \), \( H(t, p, q) = H_0(p, q) + V(t, p, q) \) is the classical Hamiltonian, where \( H_0(p, q) \) is quadratic in \( p, q \) and independent of \( t \), and \( V(t, p, q) \) has compact support in \( t \); \( \hat{H} = \hat{H}(t) \) is the quantum Hamiltonian operator in the Weyl calculus \([9, 3, 4]\) with the Weyl symbol \( H(t, p, q) \). Let the interval \([T_1, T_2]\) contain the support of \( V \). Denote by \( U \) (resp. \( U_0 \)) the evolution operator of equation (1) (resp. of equation (1) with \( H_0 \) instead of \( H \)) from \( T_1 \) to \( T_2 \). Denote by \( W \) the algebra of Weyl quantum observables \( \hat{\Phi}(t) \), \( \Phi = \Phi(t, p, q) \), satisfying the Heisenberg equation

\[ \frac{d\hat{\Phi}}{dt} = \frac{1}{i\hbar} [\hat{H}_0, \hat{\Phi}] = \{\hat{\Phi}, H_0\}, \]

where \( \{\Phi, H_0\} \) is the Poisson bracket (the latter equality in (2) holds, because \( H_0 \) is quadratic). Then the operator \( S = S(T_1) = U_0^{-1} U \) naturally belongs to the algebra \( W \). We call \( S \) the scattering operator.

If \( V(t, p, q) = V_0(t, p, q) + j(t)q \) for a vector function \( j(t) = (j_k(t)) \) with compact support, then the operator \( S = S(j) \) becomes dependent on \( j \). We call

\[ Tq_{k_1}(t_1)*_{V_0} \ldots *_{V_0} q_{k_N}(t_N) \overset{\text{def}}{=} \frac{\delta^N S(j)}{\delta j_{k_1}(t_1) \ldots \delta j_{k_N}(t_N)} \bigg|_{j=0} \]

the operator Green function of equation (1); it is a \( W \)-valued distribution of \( t_1, \ldots, t_N \). According to [1], there exists a \( W \)-valued distribution \( \mu \) on the space \( R \) of trajectories \( q(t) \), \(-\infty < t < \infty\), such that

\[ Tq_{k_1}(t_1)*_{V_0} \ldots *_{V_0} q_{k_N}(t_N) = \int_R q_{k_1}(t_1) \ldots q_{k_N}(t_N) D\mu(q(\cdot)). \]

We call \( \mu \) the operator Feynman measure on the space of trajectories.

The operator \( S = S(V) \) satisfies the following conditions:

1) unitarity: \( S*\bar{S} = 1 \), where \(*\) is the (Moyal) product in the Weyl algebra, \( \bar{S} \) has the Weyl symbol complex conjugate to \( S \);

2) causality: if \( V_1(t) = V_2(t) \) for \( t < t_0 \), then the operator \( S(V_1)*S(V_2)^{-1} \) does not depend on the behavior of the functions \( V_1(t), V_2(t) \) at \( t < t_0 \).
2. Generalization to higher dimensions

We are going to generalize the constructions of §1 to multidimensional space-time $\mathbb{R}^{n+1}$ with coordinates $x = (x^0, \ldots, x^n)$. In this case, instead of configuration space we have the infinite dimensional Schwartz space of real functions $\varphi(s)$ on an $n$-dimensional space-like surface $C$ given by smooth parameterization $x = x(s), s = (s^1, \ldots, s^n)$. Since there are no half-forms and the theory of differential equations is ill-defined [3,4], it remains to generalize the scattering theory.

We shall restrict ourselves with theory of real scalar field, the generalizations to the vector case and to fermionic case being straightforward.

2.1. Hamiltonian formulation of classical field theory. Let us first recall the covariant Hamiltonian formulation of classical field theory [3,10].

Consider the action functional of the form

$$J = \int L(x, \varphi(x), \varphi_{x^j}(x))dx.$$ (5)

Then the Euler–Lagrange equations can be written in the following Hamiltonian form. Introduce the conjugate variables $\pi(s)$ to $\varphi(s)$ by the formula

$$\pi(s) = \sum_l (-1)^l L_{\varphi_{x^l}} \frac{\partial(x^0, \ldots, \hat{x}^l, \ldots, x^n)}{\partial(s^1, \ldots, s^n)},$$ (6)

where the fraction means Jacobian. Introduce also the covariant Hamiltonian densities $H^j(s) = H^j(x^l(s), x^l_{\varphi_k}, \varphi(s), \varphi_{\pi_k}, \pi(s)), j = 0, \ldots, n$, by the formulas

$$H^j = \sum_{l \neq j} (-1)^l L_{\varphi_{x^l}} \frac{\partial(x^0, \ldots, \hat{x}^l, \ldots, x^n)}{\partial(s^1, \ldots, s^n)}$$ (7)

$$+ (-1)^l (L_{\varphi_{x^j}} \varphi_{x^j} - L) \frac{\partial(x^0, \ldots, \hat{x}^j, \ldots, x^n)}{\partial(s^1, \ldots, s^n)}.$$

Then the equations of motion can be written in the form

$$\frac{\delta \Phi(\varphi(\cdot), \pi(\cdot); x^j(\cdot))}{\delta x^j(s)} = \{\Phi, H^j(s)\},$$ (8)

where $\Phi(\varphi(\cdot), \pi(\cdot); x^j(\cdot))$ is an arbitrary functional of functions $\varphi(s)$, $\pi(s)$ changing together with the surface $x^j = x^j(s)$, and

$$\{\Phi_1, \Phi_2\} = \int \left( \frac{\delta \Phi_1}{\delta \varphi(s)} \frac{\delta \Phi_2}{\delta \pi(s)} - \frac{\delta \Phi_1}{\delta \pi(s)} \frac{\delta \Phi_2}{\delta \varphi(s)} \right) ds$$ (9)

is the Poisson bracket of two functionals $\Phi_l(\varphi(\cdot), \pi(\cdot)), l = 1, 2$. 

2.2. Definition of the Weyl–Moyal algebra $W_0$. The Weyl–Moyal algebra of the surface $\mathcal{C}$ is defined as the algebra of infinitely differentiable functionals $\Phi(\varphi(\cdot), \pi(\cdot))$ of functions $\varphi(s)$, $\pi(s)$ from the Schwartz space with respect to the following (not everywhere defined) Moyal product:

\[
\begin{align*}
(\Phi_1 \ast \Phi_2)(\varphi(\cdot), \pi(\cdot)) &= \exp \left( -\frac{i\hbar}{2} \int \left( \frac{\delta}{\delta \varphi_1(s)} \frac{\delta}{\delta \pi_2(s)} - \frac{\delta}{\delta \pi_1(s)} \frac{\delta}{\delta \varphi_2(s)} \right) ds \right) \\
\Phi_1(\varphi_1(\cdot), \pi_1(\cdot))\Phi_2(\varphi_2(\cdot), \pi_2(\cdot))|_{\varphi_1=\varphi_2=\varphi, \pi_1=\pi_2=\pi}.
\end{align*}
\]

Consider the case

\[
L(x, \varphi(x), \varphi_{x^j}(x)) = L_0(\varphi(x), \varphi_{x^j}(x)) + V(x, \varphi(x), \varphi_{x^j}(x)),
\]

where $L_0$ is quadratic in $\varphi$ and $\varphi_{x^j}$ and independent of $x$, more concretely,

\[
L_0 = \frac{1}{2} \left( \varphi_{x^0}^2 - \sum_{j=1}^{n} \varphi_{x^j}^2 - m^2 \varphi^2 \right),
\]

and $V$ has compact support in $x$. Then the Weyl–Moyal algebra $W_0$ is defined as the algebra of functionals $\Phi(\varphi(\cdot), \pi(\cdot); x(\cdot))$ with the Moyal product (10) subject to the following Heisenberg equation:

\[
\begin{align*}
\frac{\delta \Phi}{\delta x^j(s)} &= \frac{1}{i\hbar} [H_0^j(s), \Phi] = \{\Phi, H_0^j(s)\},
\end{align*}
\]

where $H_0^j(s)$ is the covariant Hamiltonian density corresponding to the Lagrangian $L_0$. The latter equality in (11) holds because $H_0^j(s)$ is quadratic, and solutions of equation (11) are well defined just because of this equality.

In fact, the algebra $W_0$ is identified with the Weyl–Moyal algebra of the symplectic vector space of solutions $\varphi(x)$ of the Klein–Gordon equation

\[
\Box \varphi - m^2 \varphi \equiv -\frac{\partial^2 \varphi}{(\partial x^0)^2} + \sum_{j=1}^{n} \frac{\partial^2 \varphi}{(\partial x^j)^2} - m^2 \varphi = 0
\]

on the whole space-time, i.e., the algebra of functionals $\Phi(\varphi(\cdot))$ of a solution $\varphi(x)$ with the Moyal product corresponding to the Poisson bracket on the space of solutions. Below we shall use this realization of the algebra $W_0$. 

2.3. Perturbation theory: the $\varphi^4$ model. Further on we restrict ourselves by the typical example of the $\varphi^4$ model in $\mathbb{R}^{3+1}$, i.e.

\[ V(x, \varphi(x), \varphi_x(x)) = V_0 + j(x)\varphi(x), \quad V_0 = \frac{g(x)}{4!} \varphi^4(x), \]

where $g(x)$ and $j(x)$ are real smooth functions with compact support.

**Theorem.** There exists a map from the set of smooth functions $g = g(x), j = j(x)$ with compact support to the set of functionals $S(g, j) \in W_0$ with the following properties.

1) $S(g, j)$ is a formal series in $g, j$ with the first three terms

\[ S(g, j) = 1 + \frac{1}{\hbar} \int \left( \frac{g(x)}{4!} \varphi(x)^4 + j(x)\varphi(x) \right) dx + \ldots. \]

2) Classical limit: $S(g, j) = a(g, j; h) \exp(iR(g, j)/\hbar)$, where $a(g, j; h)$ is a formal series in $h$, and conjugation by $\exp(iR(g, j)/\hbar)$ in the Weyl algebra $W_0$ up to $O(h)$ yields the perturbation series for the evolution operator of the nonlinear classical field equation

\[ \Box \varphi(x) - m^2 \varphi(x) - g(x)\varphi^3(x)/3! = j(x) \]

in the space of functionals $\Phi(\varphi(\cdot))$ from $t = x_0 = -\infty$ to $t = \infty$.

3) The Lorentz invariance condition:

\[ \Lambda S(\Lambda^{-1}g, \Lambda^{-1}j) = S(g, j) \]

for a Lorentz transformation $\Lambda$.

4) The unitarity condition:

\[ S(g, j) \ast \overline{S(g, j)} = 1, \]

where $\overline{S(g, j)}$ is complex conjugate to $S(g, j)$.

5) The causality condition: for two sets of functions $(g_1, j_1); (g_2, j_2)$ equal for $t \leq t_0$, the product $S(g_1, j_1) \ast S(g_2, j_2)^{-1}$ does not depend on the behavior of the functions $g_1, j_1, g_2, j_2$ for $t < t_0$. The same holds for any space-like surface $C$ instead of the surface $t = t_0$.

6) The quasiclassical dynamical evolution (cf. with the Maslov-Shvedov quantum field theory complex germ [5,6]): for any space-like surfaces $C_1, C_2$ there exists a limit $S_{C_1,C_2}$ of $S(g, j)$ modulo $o(h)$ as the function $g(x)$ tends to $g = \text{const}$ if $x$ belongs to the strip between the space-like surfaces and to 0 otherwise. This limit possesses the property

\[ S_{C_2,C_3}(g) \ast S_{C_1,C_2}(g) = S_{C_1,C_3}(g_1) + o(h), \]

where $g_1 = g + O(h)$ is a formal series in $g$.

7) The adiabatical interaction switch off: there exists a limit $S$ of $S(g, j)$ as $g(x)$ tends to the function $g = \text{const}$. This $S$ is a formal power series in $g$. 

Any other choice of \( S(g, j) \) with the properties 1–7 above is equivalent to some change of parameters \( g(x) \).

**Sketch of the proof.** The proof is based on the same ideas as the construction of Bogolubov \( S \)-matrix in [2] using the renormalization and the Bogolubov–Parasyuk theorem. The main difference with [2] is that instead of algebra of operators in the Fock space one uses the Weyl algebra of functionals with the Moyal product, instead of the normal ordering of operators in the Fock space one uses the usual (commutative) product of functionals, and instead of the Feynman propagator \( 1/(p^2 - m^2 + i\varepsilon) \) one uses the function \( \mathcal{P} \mathcal{V} 1/(p^2 - m^2) \), where \( \mathcal{P} \mathcal{V} \) denotes the Cauchy principal value.

Similarly to §1 one introduces the operator Green functions and the operator Feynman measure \( \mu \).

One can conjecture that the scattering operator \( S(g, j) \in W_0 \) exists outside perturbation theory as a formal series in \( j \), subject to conditions 2–7 of the Theorem.

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