Saturation region of Freeway Networks under Safe Microscopic Ramp Metering

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Abstract

We consider ramp metering at the microscopic level subject to vehicle safety constraint. The traffic network is abstracted by a ring road with multiple on- and off-ramps. The arrival times of vehicles to the on-ramps, as well as their destination off-ramps are modeled by exogenous stochastic processes. Once a vehicle is released from an on-ramp, it accelerates towards the free flow speed if it is not obstructed by another vehicle; once it gets close to another vehicle, it adopts a safe behavior. The vehicle exits the traffic network once it reaches its destination off-ramp. We design traffic-responsive ramp metering policies which maximize the saturation region of the network. The saturation region of a policy is defined as the set of demands, i.e., arrival rates and the routing matrix, for which the queue lengths at all the on-ramps remain bounded in expectation. The proposed ramp metering policies operate under synchronous cycles during which an on-ramp does not release more vehicles than its queue length at the beginning of the cycle. We provide three policies under which, respectively, each on-ramp (i) pauses release for a time-interval at the end of the cycle, or (ii) modulates the release rate during the cycle, or (iii) adopts a conservative safety criterion for release during the cycle. None of the policies, however, require information about the demand. The saturation region of these policies is characterized by studying stochastic stability of the induced Markov chains, and is proven to be maximal when the merging speed of all on-ramps equals the free flow speed. Simulations are provided to illustrate the performance of the policies.

I. INTRODUCTION

Traffic congestion is caused by high demand competing to use the limited supply of the road systems. One of the most efficient tools to combat this congestion is Ramp Metering (RM) in which the inflows to a freeway are regulated in order to balance the supply-demand chain and ultimately improve some measure of performance in the freeway network [1], [2]. The problem of regulating the inflow rates is often studied by adopting macroscopic models. In spite of their generality, these models do not have the resolution to distinguish between safety protocols under different connectivity and automation scenarios. On the other hand, the usage of microscopic models in studying the interplay between safety, connectivity and traffic network performance in the context of RM is limited to heuristics or simulations. The objective of this paper is to address this gap.

There is an overwhelming body of literature on designing RM policies using macroscopic models. We review them here only briefly. Interested readers are referred to [1] and [2] for comprehensive reviews of the macroscopic and microscopic approaches, respectively. RM can be generally classified into fixed-time and traffic-responsive strategies [1]. Fixed-time strategies such as [3] are fine-tuned offline and operate based on historical traffic data. Due to the uncertainty in the traffic demand and the absence of real-time measurements, these strategies would either lead to congestion or under-utilization of the capacity of the freeway [4]. Traffic-responsive policies, on the other hand, use real-time measurements. These policies can be further sub-classified into local and coordinated policies depending on whether the on-ramps make use of the measurements obtained from their vicinity (local) or other regions of the network (coordinated) [1], [5]. A well-known example of a local strategy is ALINEA [6] which have been shown, both analytically and in practice, to yield a good performance. A caveat in employing local policies is that there is no guarantee that they can improve the overall performance of a freeway network while providing a fair access to vehicles on different on-ramps [4]. This motivates the study of coordinated policies such as the ones considered in [4], [7], [8].

The aforementioned research studies adopt macroscopic models for design purposes. The introduction of Connected and Automated Vehicles (CAVs) to the existing freeway systems creates new challenges and opportunities in the design of RM policies. On one hand, CAVs promise to compensate for human errors and provide more accurate traffic measurements by means of Vehicle-to-Vehicle (V2V), or Vehicle-to-Infrastructure (V2I) communications which can be incorporated in the RM design [2], [9]. On the other hand, they are required to obey certain safety rules during and after merging, which affects the overall performance of the freeway network. In order to adequately address the RM design problem in the presence of CAVs, microscopic models are an appropriate choice [2]. The majority of the works in this direction consider an isolated on-ramp and assume some levels of autonomy and communication among the vehicles [2]. The problem is then to determine safe merging maneuvers that respect vehicle limitations such as speed, safety, and comfort, while improving the mainstream flow by avoiding stop-and-go driving [2]. Relatively little attention has been given to analyzing the implications of a merging policy on the overall performance of the freeway network. For example, do such local merging policies optimize certain system
level performance? Indeed it is possible that a greedy merging strategy could lead to spillback upstream and thereby affecting merging at the upstream on-ramps. Motivated by increasing connectivity among vehicles and ramp meters, the primary objective of this paper is to systematically design and analyze performance of traffic-responsive ramp metering policies that optimize the system level performance.

We consider a ring road with multiple on/off-ramps as as an abstraction of a traffic network with space constraint. Previous studies on a “closed” ring road, i.e., with no on/off-ramps, have suggested that this modeling framework has some theoretical and practical advantages over its straight-line analogue [10]–[13]. However, the results of this paper are transferable to a fixed-length straight-line model with suitable boundary conditions. Vehicles in this network are assumed to have the same length, same acceleration and brake capabilities, and follow the standard rules for the safety and speed: a vehicle accelerates towards and maintains the free flow speed when it is sufficiently far away from the preceding vehicle, or adopts a safe behavior if it gets close. We do not specify the exact safe behavior, nor require vehicles to adopt the same safe behavior, except when they are moving at the constant free flow speed. In the free flow regime, a vehicle is assumed to keep a safe constant time headway plus an additional constant gap from its preceding vehicle [14]. Vehicles are equipped with V2V and V2I communication systems, but are not required to be autonomous or to coordinate with one another. Vehicle arrivals at each on-ramp is modeled by a Bernoulli process that is independent across different on-ramps. The destination off-ramp for every vehicle is sampled independently from a routing matrix. It should be emphasized that the main results of this paper do not depend on this specific demand model. Once released into the mainline, every vehicle follows the aforementioned safety and speed rules on the mainline until it reaches its destination off-ramp, at which point it exits the network. We design traffic-responsive ramp metering policies which give maximal saturation region for the traffic network. Saturation region is defined as the set of combinations of arrival demands under which the network is under-saturated.

The proposed ramp metering policies operate under synchronous cycles during which an on-ramp does not release more vehicles than its queue length at the beginning of the cycle. We provide three policies under which, respectively, each on-ramp (i) pauses release for a time-interval at the end of the cycle, or (ii) modulates the time between successive releases during the cycle, or (iii) adopts a conservative safety criterion for release during the cycle, all based on the traffic state. None of the policies however require information about the arrival rates or the routing matrix. The saturation region under these policies is characterized by studying stochastic stability of the induced Markov chains, and is proven to be maximal when the merging speed at all on-ramps is the free flow speed. In summary, the three main contributions of the paper are as follows:

- Formalism of the saturation region of a traffic network. The saturation region is a generalization of the saturation flow of a single-on-ramp, and hence, is a compelling performance index for a traffic network.
- Design of traffic-responsive ramp metering policies at the microscopic level. This allows to understand the impact of different safety and connectivity protocols on the saturation region.
- Design of policies that do not require vehicle autonomy or coordination, yet give maximal saturation region. Hence, they are suitable to implement in scenarios where there is a mixture of autonomous and human-driven vehicles.

The rest of the paper is organized as follows: in Section II we state the problem setup, vehicle-level objectives, demand model, and a summary of ramp metering policies studied in this paper. The design of RM policies and their performance analysis takes place in Section III. We verify the performance of the proposed policies in Section IV and conclude the paper in Section V.

The following standard notations are used throughout the paper. Let \( \mathbb{N} \), \( \mathbb{N}_0 \), and \( \mathbb{R} \) respectively denote the set of positive integers, non-negative integers and real numbers. For \( m \in \mathbb{N} \), \([m]\) denotes the set \( \{1, \ldots, m\} \).

## II. Problem Formulation

Consider a simple model for a freeway network with \( m \) on- and off- ramps, where the mainline is abstracted as a circular road of perimeter \( P \), as illustrated in Figure 1. The direction of travel on the mainline is assumed to be clockwise. The on- and off-ramps are placed alternately, and they are numbered in an increasing order along the clockwise direction, such that, for all \( i \in [m] \), off-ramp \( i \) comes after on-ramp \( i \). The section of the mainline between the \( i \)-th on- and off- ramps is referred to as link \( i \). Vehicle arrivals to the on-ramps and their choice of destination off-ramp is determined exogenously; see Section II-B.

We assume a point queue model for vehicles waiting at the on-ramps, with the queue on an on-ramp co-located with its ramp meter.

### A. Vehicle-Level Objectives

We consider vehicles of length \( L \) with the same acceleration and brake capabilities. Vehicles are equipped with V2V and V2I communication systems but are not required to be autonomous or to coordinate with each other. We use the term ego vehicle to refer to a specific vehicle under consideration, and denote it by \( e \). We define the safety distance as the minimum distance required in an emergency braking scenario in order to avoid collision. Let \( v_e \) (resp. \( v_p \)) be the speed of the ego vehicle (resp.
its preceding vehicle), and $S_e$ be the safety distance between the two vehicles. We consider the emergency braking scenario of [14] under which

$$S_e = h v_e + S_0 + \frac{v_e^2 - v_p^2}{2 |a_{\text{min}}|},$$

where, $h > 0$ is a safe time headway constant, $S_0 > 0$ is an additional constant gap, and $a_{\text{min}} < 0$ is the minimum possible deceleration of the preceding vehicle. We consider two general modes of operation for each vehicle: the speed tracking mode and the safety mode. The main objective in the speed tracking mode is to adjust the speed to the free flow speed $V_f$ as explained in Appendix A-A where we have assumed a third-order vehicle dynamics for simplicity; the main objective in the safety mode is to avoid collision. We define the acceleration lane of an on-ramp as the section of the network starting immediately downstream of the ramp meter and ending on the mainline, such that if the ego vehicle is in the speed tracking mode throughout the entire section, it achieves the speed $V_f$ at the end of it. With this definition, the acceleration lane may or may not overlap with the mainline, depending on the distance from the ramp meter to the merging point with the mainline; see Figure 2. If the ego vehicle is in the speed tracking mode throughout the acceleration lane, its speed at the merging point will be referred to as the merging speed of the on-ramp.

In what follows place some assumptions on the vehicle-level controller. Consider a merging scenario as shown in Figure 2. Without loss of generality, let the ego vehicle be at the on-ramp. At each time $t_0 \geq 0$ before merging, the ego vehicle predicts its speed and merging time $t_m$, where we have dropped the dependence on $t_0$ for brevity. By using V2V communication, it broadcasts this information to the surrounding vehicles, and receives their projected position and speed at time $t_m$. Let $\hat{v}_e(t)$ be the projected speed of the ego vehicle at time $t \geq t_0$. We assume that the ego vehicle uses the following speed prediction rule: if in the speed tracking mode at time $t_0$, then $\hat{v}_e(t)$ is calculated assuming that it stays in this mode in the future; if in the safety mode, then $\hat{v}_e(t) = v_e(t_0)$ for all $t \geq t_0$. We define the projected preceding vehicle of the ego vehicle as the vehicle that is projected to precede the ego vehicle on the mainline at time $t_m$. We assume that vehicles obey the following rules:
(VC1) if the ego vehicle is moving at the constant free flow speed $V_f$, then it maintains its speed if and only if:

(a) its distance $y_e$ to the preceding vehicle is no less than the safety distance $S_e$. If the preceding vehicle is also moving at the constant speed $V_f$, we must have $y_e \geq S_e = hV_f + S_0$. This distance is equivalent to a time headway of at least $\tau := h + (S_0 + L)/V_f$. This rule is believed to be widely adopted by human drivers as well as standard adaptive cruise control systems \cite{14}.

(b) its projected distance $\hat{y}_e$ to the projected preceding vehicle at the moment of merging is not less than the projected safety distance. In other words, if and only if $\hat{y}_e(t_m) \geq \hat{S}_e(t_m) := h\hat{v}_e(t_m) + S_0 + \frac{\hat{v}_p^2(t_m) - \hat{v}_e^2(t_m)}{2\|\dot{v}_\text{min}\|}$, where $\hat{v}_e$ is the projected speed of the projected preceding vehicle.

(VC2) upon being released from the on-ramp, the ego vehicle is initialized to be in the speed tracking mode. It changes mode if the ego vehicle is moving at the constant free flow speed $V_f$. Demand Model and Saturation Region

only if and when needed for performance analysis of ramp metering policies in the paper. Also, the control logic in the safety Section II-A. Let vehicles arrive to on-ramp $V_i$ the minimum safe time headway between two consecutive vehicles that are moving at the constant speed $V_i$.

Remark

Note that we intentionally do not specify the total number of submodes within the safety mode, the exact control logic within each submode, or the exact logic for switching back to the speed tracking mode. Such details will be introduced only if and when needed for performance analysis of ramp metering policies in the paper. Also, the control logic in the safety mode is allowed to be different for different vehicles.

B. Demand Model and Saturation Region

It will be convenient for performance analysis later on to adopt a discrete time setting. Let the time step be $\tau$, representing the minimum safe time headway between two consecutive vehicles that are moving at the constant speed $V_f$; see (VC1) in Section II-A. Let vehicles arrive to on-ramp $i \in [m]$ according to an i.i.d Bernoulli process with parameter $\lambda_i \in [0, 1]$. These processes are independent across the on-ramps. That is, in any given time step, the probability that a vehicle arrives at the $i$-th on-ramp is $\lambda_i$ independent of everything else. We refer to the parameter $\lambda_i$ as the arrival rate to on-ramp $i$ and we let $\lambda := \lambda_i$ be the vector of arrival rates to the network. The destination-off-ramp for individual arriving vehicles is i.i.d. and is given by a routing matrix $R = [R_{ij}]$, where $0 \leq R_{ij} \leq 1$ is the probability that an arrival to on-ramp $i$ wants to exit from off-ramp $j$. Naturally, for every on-ramp $i$ we have $\sum_j R_{ij} = 1$. Finally, we let $\tilde{R} = [\tilde{R}_{ij}]$ be the cumulative routing matrix, where $\tilde{R}_{ij}$ is the fraction of arrivals $n_i$ that need to cross link $j$ in order to reach their destination. Note that $\sum_j \tilde{R}_{ij}$ indicates the fraction of arrivals at all on-ramps that need to cross link $j$ in order to reach their destination. Let $\rho_j := \sum_i \lambda_i \tilde{R}_{ij}$ be the average rate of arrivals that need to cross link $j$ in order to reach their destination off-ramp, and let $\rho := \max_{j \in [m]} \rho_j$ be the average load in the network.

Example 1. Let the routing matrix for a 3-ramp network (see, for example, Figure II) be given by

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}.$$ 

Then, the cumulative routing matrix is calculated as follows:

$$\tilde{R} = \begin{pmatrix} 1 & 1 - R_{11} & 1 - (R_{11} + R_{12}) \\ 1 - R_{21} & 1 - R_{22} & 1 - R_{12} \\ 1 - R_{31} & 1 - R_{32} & 1 - R_{33} \end{pmatrix} = \begin{pmatrix} 1 & R_{12} & R_{13} \\ R_{21} & 1 & R_{21} + R_{23} \\ R_{31} + R_{32} & R_{32} & 1 \end{pmatrix}.$$ 

The key performance metric in this paper is the saturation region of the network. For $i \in [m]$, let $Q_i(t)$ be the vector of destination off-ramps of the vehicles waiting at on-ramp $i$, arranged in the order of their arrival, at $t$. Therefore, $|Q_i(t)|$ denotes the queue length at on-ramp $i$ at time $t$. Let $|Q(t)| = [Q_i(t)]$ be the vector of queue lengths at all on-ramps at time $t$. The network is said to be under-saturated under a given demand $(\lambda, R)$ and a ramp metering policy if $\lim_{t \to \infty} \mathbb{E}[|Q_i(t)|] < \infty$ for all $i \in [m]$; otherwise, it is called saturated. We are interested in finding ramp metering policies which keep the network under-saturated for maximal combinations of $(\lambda, R)$.

C. Ramp Metering

To conveniently track vehicle locations in discrete time, we introduce the notion of slot. A slot is associated with a particular point on the mainline or acceleration lanes at a particular time. We first define the mainline slots. Let $n_c$ be the maximum number of distinct points that can be placed on the mainline, such that the distance between adjacent points is $hV_f + S_0 + L$. This distance is governed by safety considerations as explained in Section II-A. In particular, $n_c$ is the maximum number of vehicles that can safely travel at the constant free flow speed on the mainline. Consider a configuration of these $n_c$ points at $t = 0$. Each point represents a slot on the mainline that moves at the free flow speed; without loss of generality, we let $n_c$
be an integer so that each slot replaces the next slot at the end of each time step $\tau$. The mainline slots are numbered in an increasing order in the clockwise direction, with the first slot after on-ramp 1 always assigned number 1.

We next define the acceleration lane slots. Suppose that the ego vehicle is released from the $i$-th on-ramp at $t = 0$ such that it remains in the speed tracking mode in the future. Consider the ego vehicle’s location at the end of each time step $\tau$ until after it exits the acceleration lane. Each of these location points represents a slot for the $i$-th acceleration lane at $t = 0$. For example, if the ego vehicle exits the acceleration lane after $2.5\tau$ units of time of being released, there are three slots corresponding to its location at times $\tau$, $2\tau$, and $3\tau$. Let $n_i$ be the number of $i$-th acceleration lane slots, and $n_a = \sum_i n_i$. The acceleration lane slots are numbered from $n_c + 1$ to $n_c + n_a$, with the first slot of on-ramp 1 numbered $n_c + 1$, and the last slot of on-ramp $m$ numbered $n_c + n_a$. Note that by definition, the last acceleration lane slot of every on-ramp is on the mainline.

Therefore, for a given configuration of mainline slots at $t = 0$, the last acceleration lane slot of on-ramp $i$ at $t = t_i$ coincides with a mainline slot for some $t_i \in [0, \tau)$. Thereafter, the last acceleration lane slot coincides with a mainline slot at the end of every time step $\tau$, i.e., at $t = k\tau + t_i$ for all $k \in \mathbb{N}$. Without loss of generality, consider a configuration of slots at $t = 0$, such that all the last acceleration lane slots coincide with a mainline slot, i.e., $t_i = 0$ for all $i \in [m]$.

We consider ramp metering policies of the form $\pi(t) \in \{0, 1\}^m$, where $\pi_i(t) = 1$ means that a vehicle is released from on-ramp $i$ at time $t$. The information available to a policy will be a combination of $|Q(t)|$ and the state $X(t)$ defined as follows: for the ego vehicle, let $x_t^e = (p_e, v_e, a_e, I_e)$, where $p_e$ is the location, $a_e$ is the acceleration, and $I_e(t)$ is a binary variable which is equal to one if the ego vehicle has been in speed tracking mode at all times since being released, and zero otherwise. The state of all the vehicles is collectively denoted as $X^T = \{x_t^e \mid e \in [n]\}$, where $n$ is the number of vehicles on the mainline and acceleration lanes. Table I provides a summary of the ramp metering policies considered in this paper, and their communication costs which is explained in Appendix A-B. In Table I, $d$ is a constant, $dn_a$ specifies the required communication range near an on-ramp in order to detect safe gaps, and $T_{per}$ is a design update period.

### III. Ramp Metering Policies and Performance Analysis

All the policies presented in Sections III-A III-D are traffic-responsive. The policies in Sections III-A III-B and III-D are centralized, and the one in Section III-C is a distributed version of the centralized policy in Section III-B. All policies operate under synchronous cycles during which an on-ramp does not release more vehicles than its queue length at the beginning of the cycle. The synchronization of cycles is done in real time in Section III-A whereas in Sections III-B III-D it is done once offline. The policies differ in using the information to either pause release for a time-interval at the end of the cycle (Section III-A), to modulate the time between successive releases during the cycle (Section III-B), or to adopt a conservative dynamic safety criterion for release during the cycle (Section III-D); see Table II for a summary of the information used by each policy. An inner approximation to the saturation region is provided for each policy, and is then compared to an outer approximation in Section III-E.

#### A. Renewal Policy

The first policy is inspired by the queuing theory literature in the context of communication networks, e.g., see [15], [16]. Once an on-ramp releases all the vehicles waiting at the beginning of the cycle, it pauses release until all other on-ramps have done so, and these vehicles exit the network, i.e., until the mainline and acceleration lanes are empty – hence we refer to it as the renewal policy.

**Definition 1. (Renewal Ramp Metering (R-RM) policy)** No vehicle is released until all the initial vehicles exit the network, i.e., until the mainline and acceleration lanes are empty, say at time $t_1$. Thereafter, the policy works in cycles of variable length. At the beginning of the $k$-th cycle at time $t_k$, each on-ramp allocates itself a “quota” equal to the queue length at that on-ramp at $t_k$. At time $t$ during the cycle, an on-ramp releases the ego vehicle if and only if:

\[ (M1) \quad t = k\tau \text{ for some } k \in \mathbb{N}_0. \]
\[ (M2) \quad y_e(t) \geq S_e(t), \text{ i.e., it is safe to release at time } t \text{ (cf. VC2)}. \]
\[ (M3) \quad \text{the on-ramp has not reached its quota.} \]
\[ (M4) \quad \text{the ego vehicle is projected to remain in the speed tracking mode until it exits the acceleration lane; its projected following vehicle is projected not to violate the safety distance until the ego vehicle exits the acceleration lane.} \]
Once an on-ramp reaches its quota, it does not release a vehicle during the rest of the cycle. The next cycle begins when the mainline and acceleration lanes are empty.

**Remark 2.** A simpler form of this policy, called the *quota policy*, is analyzed in [15]. Direct adaptation of the quota policy to the current transportation network requires additional analysis, mainly because of the vehicle dynamics.

We need an additional notation for future results. Consider a situation where vehicles upstream of on-ramp $i$ are moving at the constant speed $V_f$. Suppose that on-ramp $i$ releases the ego vehicle under (M4) such that the ego vehicle remains in the speed tracking mode in the future, and its projected following vehicle maintains the speed $V_f$. We let $\tau_i$ be the minimum time headway between the two vehicles at the moment of merging. Note that $\tau_i \geq \tau$, where the equality holds if and only if the merging speed of on-ramp $i$ is $V_f$.

**Theorem 1.** For any safe initial condition, the R-RM policy keeps the network under-saturated if $\left[ \frac{\sigma}{\tau} \right] \rho_i - (\left[ \frac{\rho_i}{\tau} \right] - 1) \lambda_i - 1 < 1$ for all $i \in [m]$.

**Proof.** See Appendix C-A.

**Communication requirements:** the R-RM policy uses information about $|Q|$ and $X$. It worst-case communication cost is calculated as follows: at each time step during a cycle, any vehicle that is on the mainline or an acceleration lane must communicate its state to all on-ramps. After a finite time, the number of these vehicles is no more than $n_c + n_a$. Hence, the communication cost $C$ is upper-bounded by $(n_c + n_a)m$.

### B. Dynamic Release Rate Policy

This policy imposes dynamic minimum time gap criterion, in addition to (M1), between release of successive vehicles by an on-ramp is akin to changing its release rate, and hence the name of the policy.

**Definition 2. (Dynamic Release Rate Ramp Metering (DRR-RM) policy)** The policy works in cycles of fixed length $T_{cyc}\tau$, where $T_{cyc} \in \mathbb{N}$. At the beginning of the $k$-th cycle at $t_k = (k - 1)T_{cyc}\tau$, each on-ramp allocates itself a “quota” equal to the queue length at that on-ramp at $t_k$. At time $t \in [t_k, t_{k+1}]$ during the $k$-th cycle, on-ramp $i$ releases the ego vehicle if and only if (M1)-(M4), and the following condition are satisfied:

- (M5) at least $g(t)$ time has passed since the release of the last vehicle from on-ramp $i$.

Once an on-ramp reaches its quota, it does not release a vehicle during the rest of the cycle. The minimum time gap $g(t)$ is piecewise constant, updated periodically at $t = T_{per}, 2T_{per}, \ldots$, as described in Algorithm 1. In Algorithm 1, $X^G$ is defined as follows: consider a merging scenario as described in Section 11-A at time $t$, and recall that $t_m$ is the time at which a merging is projected to occur. Let $t_f$ be the time at which the merging vehicle is projected to leave the acceleration lane. For the ego vehicle, let

$$x_c^G(t) := \begin{cases} 
\min\{0, y_e(t) - S_c(t), \inf_{s \in [t_m, t_f]} \hat{y}_e(s) - \hat{S}_e(s), v_e(t) - V_f, a_e(t)\} & \text{if } I_e(t) = 0 \\
\min\{0, y_e(t) - S_c(t), \inf_{s \in [t_m, t_f]} \hat{y}_e(s) - \hat{S}_e(s), 0, 0\} & \text{otherwise}
\end{cases}$$

where the term $\inf_{s \in [t_m, t_f]} \hat{y}_e(s) - \hat{S}_e(s)$ is set to zero whenever the ego vehicle is not in a merging scenario. Note that $\min\{0, y_e(t) - S_c(t), \inf_{s \in [t_m, t_f]} \hat{y}_e(s) - \hat{S}_e(s)\} = 0$ implies: (i) $y_e(t) \geq S_c(t)$; so the ego vehicle is at a safe distance with respect to its preceding vehicle at time $t$, and (ii) $\hat{y}_e(s) \geq \hat{S}_e(s)$ for all $s \in [t_m, t_f]$; so the ego vehicle is projected to be at a safe distance with respect to its projected preceding vehicle in $[t_m, t_f]$. The state $X^G$ is the collection of $x_c^G$ for all the vehicles on the mainline and acceleration lanes.

**Algorithm 1** Update rule for the minimum time gap between release of vehicles under the DRR-RM policy

**Input:** $T_{per} > 0, \alpha_1 > 0, \alpha_2 > 0, \theta^2 > 0, \beta > 1$

**g(0) = 0, \theta(0) = \theta^2**

**for** $t = T_{per}, 2T_{per}, \ldots$ **do**

**if** $\|X^G(t)\| \leq \max\{\|X^G(t - T_{per})\| - \alpha_1, 0\}$ **then**

$\theta(t) \leftarrow \theta(t - T_{per})$

$g(t) \leftarrow \max\{g(t - T_{per}) - \alpha_2, 0\}$

**else**

$\theta(t) \leftarrow \beta\theta(t - T_{per})$

$g(t) \leftarrow g(t - T_{per}) + \theta(t)$

**end if**

**end for**
Theorem 2. For any initial condition, $T_{\text{cyc}} \in \mathbb{N}$, positive design constants $T_{\text{per}}, \alpha_1, \alpha_2, \theta^o$, and $\beta > 1$, the DRR-RM policy keeps the network under-saturated if $\lceil \frac{T_{\text{cyc}}}{T_{\text{per}}} \rceil \beta_i < 1$ for all $i \in [m]$.

**Proof.** See Appendix C-B \qed

Communication requirements: This policy uses information about $|Q|$ and $X$. Its worst-case communication cost is calculated as follows: at the end of each update period $T_{\text{per}}$, $X$ is communicated to all on-ramps. After a finite time, the number of vehicles that constitute $X$ is no more than $n_c + n_a$. Furthermore, at each time step during a cycle, the neighboring vehicles of every on-ramp communicate their state to that on-ramp. For every on-ramp $i \in [m]$ and after a finite time, the number of such neighboring vehicles is no more than $dn_i$, for some positive constant $d$. Hence, $C$ is upper bounded by $m(n_c + n_a)/T_{\text{per}} + dn_a$.

C. Distributed Dynamic Release Rate Policy

This policy imposes dynamic minimum time gap criterion, just like its coordinated counterpart. However, the minimum time gap requirements for an on-ramp is updated based on the state of vehicles in its vicinity, and the minimum time gap at its immediately downstream on-ramp.

Definition 3. (Distributed Dynamic Release Rate Ramp Metering (DisDRR-RM) policy) The policy works in cycles of fixed length $T_{\text{cyc}}$, where $T_{\text{cyc}} \in \mathbb{N}$. At the beginning of the $k$-th cycle at $t_k = (k-1)T_{\text{cyc}}$, each on-ramp allocates itself a “quota” equal to the queue length at that on-ramp at $t_k$. At time $t \in [t_k, t_{k+1}]$ during the $k$-th cycle, on-ramp $i$ releases the ego vehicle only if (M1)-(M4), and the following condition are satisfied:

(M5) at least $g_i(t)$ time has passed since the release of the last vehicle from on-ramp $i$.

Once an on-ramp reaches its quota, it does not release a vehicle during the rest of the cycle. The minimum time gap $g_i(t)$ for on-ramp $i$ is piecewise constant, updated periodically at $t = T_{\text{per}}, 2T_{\text{per}}, \cdots$ according to Algorithm 2. For $i \in [m]$, let $X^G_i(t)$ be the part of $X^G(t)$ associated with all the vehicles located on the mainline between the $i$-th and $(i+1)$-th on-ramps or on the $i$-th acceleration lane at time $t$.

**Algorithm 2** Update rule for the minimum time gap between release of vehicles under the distributed DisDRR-RM policy

**Input:** $T_{\text{per}} > 0, T_{\text{max}} > 0, \alpha_1 > 0, \alpha_2 > 0, \theta^o > 0, \beta > 1$, $(g_i(0), g_i(T_{\text{per}})) = (0, 0), (\theta_i(0), \theta_i(T_{\text{per}})) = (\theta_i, \theta_1^o)$, $i \in [m]$

**for** $t = 2T_{\text{per}}, 3T_{\text{per}}, \cdots$ **do** the following for each on-ramp $i \in [m]$

**if** $g_{i+1}(t - T_{\text{per}}) \leq T_{\text{max}}$ **then**

**if** $||X^G_i(t)|| \leq \max(||X^G_i(t - T_{\text{per}})|| - \alpha_1, 0)$ **then**

$\theta_i(t) \leftarrow \theta_i(t - T_{\text{per}})$

$g_i(t) \leftarrow \max(g_i(t - T_{\text{per}}) - \alpha_2, 0)$

**else**

$g_i(t) \leftarrow g_i(t - T_{\text{per}}) + \theta_i(t)$

**if** $||X^G_i(t - T_{\text{per}})|| \leq \max(||X^G_i(t - 2T_{\text{per}})|| - \alpha_1, 0)$ **then**

$\theta_i(t) \leftarrow \beta \theta_i(t - T_{\text{per}})$

**else**

$\theta_i(t) \leftarrow \theta_i(t - T_{\text{per}})$

**end if**

**else**

$\theta_i(t) \leftarrow \theta_i(t - T_{\text{per}})$

$g_i(t) \leftarrow g_i(t - T_{\text{per}}) + \theta_i(t)$

**end if**

**end for**

**Proposition 1.** For any initial condition, $T_{\text{cyc}} \in \mathbb{N}$, positive design constants $T_{\text{per}}, T_{\text{max}}, \alpha_1, \alpha_2, \theta^o$, $i \in [m]$, and $\beta > 1$, the DisDRR-RM policy keeps the network under-saturated if $\lceil \frac{T_{\text{cyc}}}{T_{\text{per}}} \rceil \beta_i < 1$ for all $i \in [m]$.

**Proof.** See Appendix C-C \qed

Communication Requirement: This policy uses information about $|Q|$ and $X$. Its worst-case communication cost is calculated similar to the DRR-RM policy, except that at the end of each update period $T_{\text{per}}$, $X$ is not communicated to all on-ramps. Instead, for all $i \in [m]$, only the part $X$ associated with the vehicles located on the mainline between the $i$-th and $i + 1$-th on-ramps or on the $i$-th acceleration lane is communicated to on-ramp $i$. Thus, the worst-case communication cost is reduced to $(n_c + n_a)/T_{\text{per}} + dn_a$. 

D. Dynamic Space Gap Policy

In this policy, on-ramps require an additional space gap on top of the safety distance before releasing a vehicle. This additional space gap is updated periodically based on the state of all vehicles. Recall that the DRR-RM policy enforces an additional time gap between release of successive vehicles, which is updated based on the current state of vehicles as well as their state in the past. The dynamic space gap policy only requires the current state of vehicles. However, it requires the following assumptions on the vehicle-level controller: consider $n$ vehicles over a time interval during which at least one vehicle is in the speed tracking mode, and no vehicle leaves through an off-ramp. We assume that:

- (VC4) each vehicle switches to the safety mode because of at most one vehicle.
- (VC5) if no vehicle changes mode, then $X^G$ converges to zero globally exponentially.

**Definition 4. (Dynamic Space Gap Ramp Metering (DSG-RM) policy)** The policy works in cycles of fixed length $T_{cyc}\tau$, where $T_{cyc} \in \mathbb{N}$. At the beginning of the $k$-th cycle at $t_k = (k-1)T_{cyc}\tau$, each on-ramp allocates itself a “quota” equal to the queue length at that on-ramp at $t_k$. At time $t \in [t_k, t_{k+1}]$ during the $k$-th cycle, on-ramp $i$ releases the ego vehicle if and only if (M1)-(M3), and the following condition is satisfied:

- (M6) its projected following vehicle is in the speed tracking mode at time $t$. Moreover, $y_k(t) \geq S_i(t) + K(X(t))$, $\tilde{y}_k(t_m) \geq \hat{S}_i(t_m) + K(X(t))$, and $\tilde{y}_f(t_m) \geq \hat{S}_f(t_m) + K(X(t))$, where $K(\cdot)$ is the additional space gap, $\tilde{y}_f$ is the projected distance between the ego vehicle and its projected following vehicle, and $\hat{S}_f$ is the projected safety distance between the two vehicles.

Once an on-ramp reaches its quota, it does not release a vehicle during the rest of the cycle. The additional space gap is piecewise constant and updated periodically at each time step according to the rule $K(X(t)) = a \tilde{y}(t) + b \cdot f(X(t))$, where $n(t)$ is the number of vehicles on the mainline and acceleration lanes at time $t$, $a$ and $b$ are constants that depend on the vehicle-level controller, $f(\cdot)$ depends on the state of vehicles near the on-ramps and satisfies $f(X) = 0$ if $||X^G|| = 0$. These parameters will be determined in the proof of Theorem 3.

**Theorem 3.** Let the vehicle dynamics also satisfy (VC4)-(VC5). There exists $a$, $b$, and $f(\cdot)$ such that for all initial condition, and $T_{cyc} \in \mathbb{N}$, the DSG-RM policy keeps the network under-saturated if $\lfloor \frac{\tau}{\tau} \rfloor \rho_i < 1$ for all $i \in [m]$.

**Proof.** See Appendix C-D.

**Communication Requirement:** This policy uses information about $|Q|$ and $X$. Except for an initial finite time, $X$ is communicated to every on-ramp at each time step. Thus, the communication cost is upper bounded by $(n_c + n_a)m$.

E. A Necessary Condition

We now provide a necessary condition for under-saturation, against which we benchmark the sufficient condition for the ramp metering policies from the previous sections. To that purpose, let $D_{\pi,p}(k\tau)$ be the cumulative number of vehicles that has crossed point $p$ on the mainline up to time $k\tau$, $k \in \mathbb{N}_0$, under the ramp-metering policy $\pi$. Then, the crossing rate at point $p$ is defined as $D_{\pi,p}(k\tau)/k$ and the “long-run” crossing rate is $\limsup_{k \to \infty} D_{\pi,p}(k\tau)/k$.

**Theorem 4.** Suppose that the long-run crossing rate is no more than one for at least one point $p_i$ on the $i$-th link, $i \in [m]$. If the network is under-saturated, then the demand must satisfy $\rho_i \leq 1$.

**Proof.** See Appendix C-E.

**Remark 3.** In all of the policies studied in previous sections, vehicles move at the constant speed $V_f$ at some point $p_i$ on the $i$-th link, $i \in [m]$, after a finite time. (VC1) then implies that the number of vehicles that cross $p_i$ at each time step is no more than one. Therefore, $\limsup_{k \to \infty} D_{\pi,p}(k\tau)/k \leq 1$ for all $i \in [m]$, i.e., the assumption of Theorem 4 holds.

More generally, $D_{\pi,p}(k+1\tau) - D_{\pi,p}(k\tau)$ represents the traffic flow at point $p$ in number of vehicles per time step. Macroscopic traffic models suggest that the flow is no more than the road capacity. The capacity of the mainline under the constant time headway safety rule in Section II-A is 1 vehicle per time step. Hence, $D_{\pi,p}(k+1\tau) - D_{\pi,p}(k\tau) \leq 1$ for all $k \in \mathbb{N}_0$ and any ramp metering policy $\pi$. This implies that $\limsup_{k \to \infty} D_{\pi,p}(k\tau)/k \leq 1$ for all $i \in [m]$.

**Remark 4.** If the merging speed at all the on-ramps is $V_f$, then the minimum time headways $\tau_i$, $i \in [m]$, are all equal to $\tau$. In this case, one can check that the sufficient conditions on $(\lambda, R)$ in the previous sections become $\rho < 1$. Comparing with Theorem 4, this implies that all the four policies R-RM, DSG-RM, DRR-RM, and DisDRR-RM, give the maximum possible saturation region when the merging speed of all on-ramps is $V_f$.

F. Decentralized and Greedy Policies

The analysis of the performance of the three policies in Sections III-B-III-D can be divided into two phases. The first phase concerns the transient from the initial condition to the state where each vehicle occupies some slot, and the second phase is from this state onwards. Since the performance criterion of saturation region is an asymptotic notion, it is compelling to
examine the policies specifically in the second phase. Indeed, in the second phase, the actions of all the three policies can be shown to be equivalent to the following policy.

**Definition 5. (Fixed-Cycle Quota Ramp Metering (FCQ-RM) policy)** The policy works in cycles of fixed length $T_{\text{cyc}}$, where $T_{\text{cyc}} \in \mathbb{N}$. At the beginning of the $k$-th cycle at $t_k = (k - 1)T_{\text{cyc}}$, each on-ramp allocates itself a “quota” equal to the queue length at that on-ramp. During a cycle, the $i$-th on-ramp releases the ego vehicle if and only if (M1)-(M4) are satisfied. Once an on-ramp reaches its quota, it does not release a vehicle during the rest of the cycle.

Note that the FCQ-RM policy is decentralized, except for synchronizing the beginning of its cycles with other on-ramps, which can be done once offline. In the special case of $T_{\text{cyc}} = 1$, the FCQ-RM becomes the simple greedy (and decentralized) policy under which the on-ramps do not need to synchronize cycles with other on-ramps or keep track of their quota. One can see from the proofs of the DRR-RM, DSG-RM, and DisDRR-RM policies that, for the aforementioned second phase initial condition, the traffic network is under-saturated under the FCQ-RM policy if $\lfloor \tau_i / \rho_i \rfloor < 1$ for all $i \in [m]$. It is natural to wonder if such a result holds true for arbitrary initial conditions, or under *any* other greedy policy (not just the slot-based).

### IV. Simulations

The following setup is common to all the simulations in this section. Let $P = 620$ [m] with $m = 2$ on-/off-ramps. Let $h = 1.5$ [s], $S_0 = 4$ [m], $L = 4.5$ [m], and $V_f = 15$ ($\frac{m}{s}$). For these parameters, we obtain $n_c = 20$. The two on-ramps are located at 0 and $P/2$; the corresponding off-ramps are located at $P - 3(hV_f + S_0 + L)$ and $P/2 - 3(hV_f + S_0 + L)$. The initial queue length on both the on-ramps is assumed to be zero. Vehicles arrive at both the on-ramps according to i.i.d Bernoulli processes with the same rate $\lambda$; their destinations are determined by

$$R = \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}.$$  

Note that on average, most of the vehicles exit from off-ramp 1. Thus, under the policies introduced in this paper, one should expect that, on average, on-ramp 1 finds more empty virtual slots than on-ramp 2. Equivalently, on-ramp 2’s queue should be longer than that of on-ramp 1 on average.

**A. Greedy Policy for Low Merging Speed**

Recall the greedy policy, i.e., the FCQ-RM policy with $T_{\text{cyc}} = 1$, from Section III-F. Let the mainline and acceleration lanes be empty at $t = 0$. Let $\lambda_1 = \lambda_2$. When the merging speed at both of the on-ramps is $V_f$, then for the given $R$, the saturation region is given by $\lambda_1 = \lambda_2 < 5/9$. The queue length dynamics for $\lambda_1 = \lambda_2 = 0.5$, which corresponds to $\rho = 0.9$, is shown in Figure 3. As expected, $|Q_2(t)|$ is generally larger than $|Q_1(t)|$.

Next consider the case when the merging speed at on-ramp 1 is $V_f$, i.e., $\tau_1 = \tau$, but is smaller at on-ramp 2 such that $\tau_2 = 3\tau$. For $\lambda_1 = \lambda_2 = 0.5$, which again corresponds to $\rho = 0.9$, $|Q_2(t)|$ increases steadily which suggests that the network is over-saturated even if $\rho < 1$. A boundary point of the capacity region in this case is estimated from simulations to be $\lambda_1 = \lambda_2 < 0.365$. The estimates of the capacity region under the R-RM and DRR-RM policies, given by Theorems 1 and 2 respectively, are $\lambda_1 = \lambda_2 < 0.4$ and $\lambda_1 = \lambda_2 < 0.2$. Combining this with the simulation results suggest that the R-RM policy performs strictly better than the greedy policy in terms of the saturation region.
Figure 4: Effect of cycle length $T_{cyc}$ on average queue length (a) for different $\rho$ when both on-ramps are long, (b) for a fixed $\rho$ when on-ramp 2 is short. Both plots are under the FCQ-RM policy.

B. Effect of Cycle Length in the FCQ-RM Policy

The mainline and acceleration lanes are assumed to be initially empty for the simulations in this subsection. The average queue lengths are computed using the \textit{batch means} approach, with warm-up period $10^5$, i.e., the first $10^5$ observations are not used, and batch size $10^5$. In each case, the simulations are run until the margin of error of the 95\% confidence intervals are 1\%.

Figure 4a shows average queue length, i.e., $\limsup_{t \to \infty} \mathbb{E}[|Q_1(t)| + |Q_2(t)|]$, vs. $\rho$ under the FCQ-RM policy for different $T_{cyc}$, when the merging speed on both of the on-ramps is $V_f$. The plot suggests that for all $\rho$, the average queue length increases monotonically with $T_{cyc}$. However, this does not hold true when the merging speeds are low. For example, Figure 4b shows average queue length vs. $T_{cyc}$ under the FCQ-RM policy for $\lambda_1 = \lambda_2 = 0.4$, which corresponds to $\rho = 0.72$, when the merging speed from on-ramp 1 is $V_f$, and the merging speed from on-ramp 2 is such that $T_2 = 3T_1$. The plot shows that the network is in fact over-saturated for small $T_{cyc}$, and that when $T_{cyc}$ is greater than a certain threshold ($T_{cyc} = 9$ in the plot), the dependence of the average queue length on $T_{cyc}$ is not monotonic.

V. Conclusion and Future Work

We provided a ramp metering framework which regulates entry of vehicles into the mainline at the microscopic level. This allows to explicitly take into account the V2X communication scenarios into the ramp metering design, and study the impact on traffic network performance. We specifically provided ramp metering policies for a few such scenarios, and analyzed performance in terms of saturation region of the network abstracted by a ring road. There are several avenues for generalizing the setup and methodologies initiated in this paper. Of immediate interest would be to consider a general network structure, and to derive sharper lower bounds on the saturation region for the low merging speed case. We are also interested in expanding performance analysis to include travel time, possibly by leveraging waiting time analysis from queuing theory.

\section*{References}

[1] M. Papageorgiou and A. Kotsialos, “Freeway ramp metering: An overview,” \textit{IEEE transactions on intelligent transportation systems}, vol. 3, no. 4, pp. 271–281, 2002.

[2] J. Rios-Torres and A. A. Malikopoulos, “Automated and cooperative vehicle merging at highway on-ramps,” \textit{IEEE Transactions on Intelligent Transportation Systems}, vol. 18, no. 4, pp. 780–789, 2016.

[3] J. A. Wattleworth, “Peak-period analysis and control of a freeway system,” tech. rep., Texas Transportation Institute, 1965.

[4] I. Papamichail, A. Kotsialos, I. Margonis, and M. Papageorgiou, “Coordinated ramp metering for freeway networks—a model-predictive hierarchical control approach,” \textit{Transportation Research Part C: Emerging Technologies}, vol. 18, no. 3, pp. 311–331, 2010.

[5] M. Papageorgiou, C. Diakaki, V. Dinopoulou, A. Kotsialos, and Y. Wang, “Review of road traffic control strategies,” \textit{Proceedings of the IEEE}, vol. 91, no. 12, pp. 2043–2067, 2003.

[6] M. Papageorgiou, H. Hadj-Salem, J.-M. Blosseville, \textit{et al.}, “Alinea: A local feedback control law for on-ramp metering,”

[7] M. Papageorgiou, J.-M. Blosseville, and H. Haj-Salem, “Modelling and real-time control of traffic flow on the southern part of boulevard périphérique in paris: Part ii: Coordinated on-ramp metering,” \textit{Transportation Research Part A: General}, vol. 24, no. 5, pp. 361–370, 1990.

[8] G. Gomes and R. Horowitz, “Optimal freeway ramp metering using the asymmetric cell transmission model,” \textit{Transportation Research Part C: Emerging Technologies}, vol. 14, no. 4, pp. 244–262, 2006.

[9] L. Li, D. Wen, and D. Yao, “A survey of traffic control with vehicular communications,” \textit{IEEE Transactions on Intelligent Transportation Systems}, vol. 15, no. 1, pp. 425–432, 2013.

[10] Y. Sugiya, M. Fukui, M. Kiikuchi, K. Hasebe, A. Nakayama, K. Nishinari, S. Ichii Tadaki, and S. Yukawa, “Traffic jams without bottle-necks—experimental evidence for the physical mechanism of the formation of a jam,” 2008.
A. Dynamics In The Speed Tracking Mode

Suppose that the ego vehicle is in the speed tracking mode for all \( t \geq 0 \), and \( v_c(t) = v_0 \in [0, V_f] \), \( a_c(t) = a_0 \in [a_{\min}, a_{\max}] \), where \( a_c \) is the acceleration and \( a_{\max} \) is the maximum possible acceleration. Then, its dynamics for all \( t \geq 0 \) is as follows:

\[
v_c(t) = v_0 + \int_0^t a_c(\xi)d\xi
\]

\[
a_c(t) = \begin{cases} 
J_{\max}t + a_0 & \text{if } 0 \leq t < t_1 \\
[a_{\max} & \text{if } t_1 \leq t < t_1 + t_2 \\
[a_{\max} - J_{\max}t & \text{if } t_1 + t_2 \leq t < t_1 + t_2 + t_3 \\
0 & \text{if } t \geq t_1 + t_2 + t_3 
\end{cases}
\]

where \( J_{\max} \) is the maximum possible jerk, \( t_1 \) is the time at which the ego vehicle reaches the maximum acceleration, \( t_2 \) is the additional time required to reach a desired speed before it decelerates, and \( t_3 \) is the time required to reach the zero acceleration in order to avoid exceeding the speed limit \( V_f \). Hence, \( v_c(t_1 + t_2 + t_3) = V_f \), and \( a_c(t_1 + t_2 + t_3) = 0 \). The dynamics for the \( v_0 > V_f \) case is similar.

B. Communication Cost of a Ramp Metering Policy

The calculation of the communication cost for a policy is inspired by the robotics literature [17, Remark 3.27]. Let \( c_{ij}(t) \) be the communication cost of vehicle \( i \) that is on the mainline or an acceleration lane to on-ramp \( j \) at time \( t \). We let \( c_{ij}(t) = 1 \) if vehicle \( i \) communicates with on-ramp \( j \), and \( c_{ij}(t) = 0 \) otherwise, where we have normalized the cost for simplicity independent of the type of information communicated. Then, the communication cost at time \( t \) is \( C(t) = \sum_{i\in[n],j\in[m]} c_{ij}(t) \), where \( n \) is the number of vehicles on the mainline. The communication cost of the policy is obtained as follows:

\[
C = \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} C(t).
\]
occupying slot $\ell$ at time $t$ is off-ramp $j$, and $M_k(t) = 0$ if slot $\ell$ is empty at time $t$. $|M(t)|$ is therefore the number of vehicles on the mainline and acceleration lanes at time $t$. Consider the following discrete-time Markov chain with the state
\[ Z_{\Delta}(t) := (Y(t), Y(t − 1), \ldots, Y(t − \Delta + 1)), \quad t \geq \Delta − 1, \]
where $Y(t) := (Q(t), M(t))$, and $\Delta \in \mathbb{N}$ will be specified for the policy being analyzed. The transition probabilities are determined by the ramp metering policy being analyzed, but will not be specified explicitly for brevity. For all the ramp metering policies considered in this paper, the state $Y(t) = (0, 0)$ is reachable from all other states, and $\mathbb{P}(Y(t + 1) = (0, 0) \mid Y(t) = (0, 0)) > 0$. Hence, the Markov chain $Z_\Delta$ is irreducible and aperiodic.

The following is an adaptation of a well-known result, e.g., see [18] Theorem 14.0.1, for the setting of our paper.

**Theorem 5. (Foster-Lyapunov drift criterion)** Let $\{Z(t)\}_{t=1}^\infty$ be an irreducible and aperiodic discrete time Markov chain evolving on a countable state space $\mathcal{Z}$. Suppose that there exist $V : \mathcal{Z} \to [0, \infty), f : \mathcal{Z} \to [1, \infty)$, a finite constant $b$, and a finite set $B \subseteq \mathcal{Z}$ such that, for all $z \in \mathcal{Z}$,
\[ \mathbb{E}[V(Z(t + 1)) − V(Z(t)) \mid Z(t) = z] \leq −f(z) + b\mathbb{I}_B(z), \]
where $\mathbb{I}_B(z)$ is the indicator function of the set $B$. Then, $\lim_{t \to \infty} \mathbb{E}[f(Z(t))]$ exists and is finite.

**Remark 5.** If the conditions of Theorem 5 hold true, then $V$ is referred to as a Lyapunov function. Additionally, if $f(Z(t)) = \|Q(t)\|_\infty$, where $\|Q(t)\|_\infty$ is the $\infty$-norm of the vector of queue lengths $|Q(t)|$, then $\lim_{t \to \infty} \mathbb{E}[\|Q(t)\|_\infty] < \infty$, and hence $\lim_{t \to \infty} \mathbb{E}[\|Q(t)\|_\infty] < \infty$ for all $i \in [m]$.

**APPENDIX C**

**PROOF OF THE MAIN RESULTS**

A. Proof of Theorem 7

For the sake of readability, we present proofs of intermediate claims at the end. Since the mainline and acceleration lanes become empty after at most $T_{\text{empty}}$ time (see (VC3)), we have $t_1 \leq T_{\text{empty}}$. We may assume without loss of generality that $t_1 = 0$. Thereafter, we can adopt the Markov chain setting from Section B with $\{Y(t_k)\}_{k \geq 1}$ as the Markov chain, where $t_k$ is the beginning of the $k$-th cycle. Consider the function $V : \mathcal{Y} \to [0, \infty)$,
\[ V(Y(t_k)) = T_{\text{cyc}}(k)^2, \]
where $\mathcal{Y}$ is the range of values of $Y$, and $T_{\text{cyc}}(k) = t_{k+1} − t_k$ is the length of the $k$-th cycle. We let $V(Y(t)) \equiv V(t)$ for brevity. It is shown at the end of the proof that
\[ T_{\text{cyc}}(k + 1) \leq \tilde{A}(T_{\text{cyc}}(k)) + 2n_c + n_a, \]
(2)
where $\tilde{A}(T_{\text{cyc}}(k))$ satisfies the following:
\[ \lim_{t \to \infty} \mathbb{E}\left[ \left( \frac{\tilde{A}(T_{\text{cyc}}(k))}{T_{\text{cyc}}(k)} \right)^2 T_{\text{cyc}}(k) = t \right] = \left( \max_{i \in [m]} \frac{\tau_i}{\tau} |\rho_i - \left( \frac{\tau_i}{\tau} - 1 \right)\lambda_i \right)^2. \]
(3)
By assumption, $\lfloor \tau_i / \tau \rfloor |\rho_i - \lfloor \tau_i / \tau \rfloor |\lambda_i < 1$ for all $i \in [m]$. Combining this with (2) and (3) then imply that
\[ \lim_{t \to \infty} \mathbb{E}\left[ \left( \frac{T_{\text{cyc}}(k + 1)}{T_{\text{cyc}}(k)} \right)^2 T_{\text{cyc}}(k) = t \right] \leq \lim_{t \to \infty} \mathbb{E}\left[ \left( \frac{\tilde{A}(T_{\text{cyc}}(k))}{T_{\text{cyc}}(k)} \right)^2 T_{\text{cyc}}(k) = t \right] = \left( \max_{i \in [m]} \frac{\tau_i}{\tau} |\rho_i - \left( \frac{\tau_i}{\tau} - 1 \right)\lambda_i \right)^2 < 1. \]
Therefore, there exists $\delta, T > 0$ such that for all $t > T$ we have
\[ \mathbb{E}\left[ \left( \frac{T_{\text{cyc}}(k + 1)}{T_{\text{cyc}}(k)} \right)^2 T_{\text{cyc}}(k) = t \right] < 1 - \delta, \]
which in turn implies that
\[ \mathbb{E}\left[ T_{\text{cyc}}(k + 1)^2 - T_{\text{cyc}}(k)^2 \mid T_{\text{cyc}}(k) = t \right] < -\delta T_{\text{cyc}}(k)^2. \]
Since $\|Q(t_k)\|_\infty \leq T_{\text{cyc}}(k) \leq T_{\text{cyc}}(k)^2$, it follows that $\mathbb{E}\left[ T_{\text{cyc}}(k + 1)^2 - T_{\text{cyc}}(k)^2 \mid T_{\text{cyc}}(k) > T \right] < -\delta \|Q(t_k)\|_\infty$. Finally, if $T_{\text{cyc}}(k) \leq T$, then $\tilde{A}(T_{\text{cyc}}(k)) \leq mT$ because the number of arrivals at each time step does not exceed $m$. Combining this with (2) gives $V(t_{k+1}) = T_{\text{cyc}}(k + 1) \leq mT \max_{i \in [m]} \tau_i / \tau + 2n_c + n_a$. Therefore,
\[ \mathbb{E}\left[ V(t_{k+1}) - V(t_k) \mid V(t_k) \right] \leq -\delta \|Q(t_k)\|_\infty + \left( mT \max_{i \in [m]} \frac{\tau_i}{\tau} + 2n_c + n_a \right)^2 \mathbb{I}_B, \]
where $B = \{Y(t_k) : V(t_k) \leq T\}$ (a finite set). The result then follows from Theorem 5.
Proof of (2)
Consider the \((k+1)\)-th cycle. Let \(s_i \in [t_{k+1}, t_{k+2})\) be the time at which on-ramp \(i \in [m]\) empties its quota during the cycle. We claim that
\[
s_i - t_{k+1} \leq \left\lceil \frac{\tau_i}{\tau} \right\rceil N_i(t_{k+1}) - \left(\left\lfloor \frac{\tau_i}{\tau} \right\rfloor - 1\right)Q_i(t_{k+1}),
\]
where \(N_i(t_{k+1})\) is the degree of on-ramp \(i\) representing the number of vehicles in the network at time \(t_{k+1}\) that need to cross the merging point of on-ramp \(i\) in order to reach their destination. Suppose not; consider all the mainline slots upstream of the merging point that are at most \(\lfloor \tau_i/\tau \rfloor\) time steps away. Whenever at least one of these slots is occupied during the time interval \([t_{k+1}, s_i]\), the last acceleration lane slot of on-ramp \(i\) that is not on the mainline must be empty (see (M4)). Since the number of vehicles from other on-ramps that need to cross the merging point of on-ramp \(i\) is \(N_i(t_{k+1}) - Q_i(t_{k+1})\), the aforementioned acceleration lane slot is empty for at most \(\lfloor \tau_i/\tau \rfloor (N_i(t_{k+1}) - Q_i(t_{k+1}))\) time steps. Therefore, it must be occupied for \(s_i - t_{k+1} - (\lfloor \tau_i/\tau \rfloor (N_i(t_{k+1}) - Q_i(t_{k+1})))\) time steps, which by assumption is greater than \(Q_i(t_{k+1})\). This however contradicts the feature of the R-RM policy under which the number of vehicles released by an on-ramp during a cycle does not exceed the queue length at the beginning of the cycle.

Since \(T_{\text{cyc}}(k+1) \leq \max_{i \in [m]} s_i - t_{k+1} + 2n_c + n_a\), it follows that
\[
T_{\text{cyc}}(k+1) \leq \max_{i \in [m]} \left\{ \left\lceil \frac{\tau_i}{\tau} \right\rceil N_i(t_{k+1}) - \left(\left\lfloor \frac{\tau_i}{\tau} \right\rfloor - 1\right)Q_i(t_{k+1}) \right\} + 2n_c + n_a. \tag{4}
\]

Let \(A_i(s)\) be the number of arrivals to on-ramp \(i\), and \(A_{t,i}(s)\) be the number of arrivals to all on-ramps that need to cross link \(i\) at time \(s\). We let \(\tilde{A}_i(t_{k+1} - t_k)\) denote the cumulative number of arrivals to on-ramp \(i\) during the interval \([t_{k+1}, t_{k+1}]\) i.e.,
\[
\tilde{A}_i(t_{k+1} - t_k) = \sum_{s=t_{k+1}}^{t_{k+1}} A_i(s).
\]

We define \(\tilde{A}_{t,i}(t_{k+1} - t_k)\) similarly. Note that \(Q_i(t_{k+1})\) is precisely the cumulative number of arrivals to on-ramp \(i\) in \([t_{k+1}, t_{k+1}]\), i.e., \(Q_i(t_{k+1}) = \tilde{A}_i(t_{k+1} - t_k)\). Similarly, \(N_i(t_{k+1}) = \tilde{A}_{t,i}(t_{k+1} - t_k)\). This and (4) imply (2) with
\[
\hat{A}(T_{\text{cyc}}(k)) = \max_{i \in [m]} \left\{ \left\lceil \frac{\tau_i}{\tau} \right\rceil \tilde{A}_{t,i}(t_{k+1} - t_k) - \left(\left\lfloor \frac{\tau_i}{\tau} \right\rfloor - 1\right)\tilde{A}_i(t_{k+1} - t_k) \right\}.
\]

Proof of (3)
Consider the sequences \(\{A_i(s)\}_{s=t_{k+1}}^{\infty}\) and \(\{A_{t,i}(s)\}_{s=t_{k+1}}^{\infty}\). Each sequence is i.i.d and \(\mathbb{E}[A_i(s)] = \lambda_i, \mathbb{E}[A_{t,i}(s)] = \rho_i\). By the strong law of large numbers, with probability one,
\[
\lim_{T_{\text{cyc}}(k) \to \infty} \frac{[\tau_i/\tau] \tilde{A}_{t,i}(t_{k+1} - t_k) - \left(\left\lfloor \tau_i/\tau \right\rfloor - 1\right)\tilde{A}_i(t_{k+1} - t_k)}{T_{\text{cyc}}(k)} = \left[\frac{\tau_i}{\tau}\right] \rho_i - \left(\left\lfloor \frac{\tau_i}{\tau} \right\rfloor - 1\right)\lambda_i.
\]

Therefore, with probability one,
\[
\lim_{T_{\text{cyc}}(k) \to \infty} \frac{\hat{A}(T_{\text{cyc}}(k))}{T_{\text{cyc}}(k)} = \lim_{T_{\text{cyc}}(k) \to \infty} \max_{i \in [m]} \frac{1}{T_{\text{cyc}}(k)} \left(\left\lceil \frac{\tau_i}{\tau} \right\rceil \tilde{A}_{t,i}(t_{k+1} - t_k) - \left(\left\lfloor \frac{\tau_i}{\tau} \right\rfloor - 1\right)\tilde{A}_i(t_{k+1} - t_k)\right) = \max_{i \in [m]} \frac{\left\lceil \frac{\tau_i}{\tau} \right\rceil}{\tau} \rho_i - \left(\left\lfloor \frac{\tau_i}{\tau} \right\rfloor - 1\right)\lambda_i.
\]

Moreover, since the real function \(x^n\) is continuous for all \(n \in \mathbb{N}\), we obtain, with probability one,
\[
\lim_{T_{\text{cyc}}(k) \to \infty} \left(\frac{\hat{A}(T_{\text{cyc}}(k))}{T_{\text{cyc}}(k)}\right)^n = \left(\lim_{T_{\text{cyc}}(k) \to \infty} \frac{\hat{A}(T_{\text{cyc}}(k))}{T_{\text{cyc}}(k)}\right)^n = \left(\max_{i \in [m]} \frac{\left\lceil \frac{\tau_i}{\tau} \right\rceil}{\tau} \rho_i - \left(\left\lfloor \frac{\tau_i}{\tau} \right\rfloor - 1\right)\lambda_i\right)^n.
\]

Finally, if the sequence \(\left\{\left(\frac{\hat{A}(T_{\text{cyc}}(k))}{T_{\text{cyc}}(k)}\right)^n\right\}_{T_{\text{cyc}}(k)=1}^{\infty}\) is upper bounded by an integrable function, then the dominated convergence theorem implies
\[
\lim_{T_{\text{cyc}}(k) \to \infty} \mathbb{E} \left[\left(\frac{\hat{A}(T_{\text{cyc}}(k))}{T_{\text{cyc}}(k)}\right)^n\right] = \mathbb{E} \left[\lim_{T_{\text{cyc}}(k) \to \infty} \left(\frac{\hat{A}(T_{\text{cyc}}(k))}{T_{\text{cyc}}(k)}\right)^n\right] = \left(\max_{i \in [m]} \frac{\left\lceil \frac{\tau_i}{\tau} \right\rceil}{\tau} \rho_i - \left(\left\lfloor \frac{\tau_i}{\tau} \right\rfloor - 1\right)\lambda_i\right)^n.
\]

1In this context, the beginning and end of each interval is specified whenever needed. Thus, the notation \(\tilde{A}_i(t_{k+1} - t_k)\) should not create any confusion.
which in turn gives \(3\). The upper bound follows from the following fact: the total number of arrivals to the network is bounded by \(m\) at each time step. Thus,
\[
\frac{\hat{A}(T_{\text{cyc}}(k))}{T_{\text{cyc}}(k)} = \frac{1}{T_{\text{cyc}}(k)} \max_{i \in [m]} \left( \frac{\tau_1}{\tau} \hat{A}_{\text{cyc}}(t_{k+1} - t_k) - \frac{\tau_1}{\tau} \hat{A}_i(t_{k+1} - t_k) \right)
\]
\[
\leq \frac{1}{T_{\text{cyc}}(k)} \max_{i \in [m]} \left( \frac{\tau_1}{\tau} \hat{A}_{\text{cyc}}(t_{k+1} - t_k) \right) \leq \max_{i \in [m]} \frac{\tau_1}{\tau} m,
\]

as desired.

### B. Proof of Theorem 2

For any initial condition, we first show that \(\|X^G(t_0)\| = 0\) for some finite time \(t_0 \geq 0\). That is, at time \(t_0\), \(\min \{0, y_e(t_0) - S_e(t_0)\}\) is to be determined. Consider the function \(V(t)\). This implies that \(\theta(t) = \infty\), which in turn implies that \(\lim \sup_{t \to \infty} \theta(t) = \infty\), where \(k\) is such that \(g(kT_{\text{per}}) > mT_{\text{empty}}(1 + \alpha_T/T_{\text{per}})\). Note that for all \(t \in [t_f, t_f + mT_{\text{empty}}]\), \(g(t) > mT_{\text{empty}}\). Thus, each on-ramp releases at most one vehicle during the interval \([t_f, t_f + mT_{\text{empty}}]\). Hence, there exists a time interval of length at least \(T_{\text{empty}}[t_f, t_f + mT_{\text{empty}}]\) during which no on-ramp releases a vehicle. Condition (VC4) then implies that the mainline and acceleration lanes become empty after such \(T_{\text{empty}}\) time units, at the end of which \(\|X^G(t)\| = 0\); a contradiction to the assumption that \(\|X^G(kT_{\text{per}})\| \neq 0\) for all \(k\).

Without loss of generality, we let \(\|X^G(0)\| = 0\), \(g(0) = 0\), and we assume that the location of all the existing vehicles coincide with a slot at \(t = 0\) as in Section B. For the sake of readability, we present proofs of intermediate claims at the end. We adopt the Markov chain setting from Section B with \(\{Z_{\Delta}(t)\}_{t \geq \Delta - 1}\) as the Markov chain, where \(\Delta = kT_{\text{cyc}}\) and \(k \in \mathbb{N}\) is to be determined. Consider the function \(V : \mathcal{X}^{kT_{\text{cyc}}} \to [0, \infty)\):
\[
V(t) = V \left( Z_{kT_{\text{cyc}}}(t) \right) := \sum_{s = t - kT_{\text{cyc}} + 1}^{t} N^2(s),
\]

where \(\mathcal{X}\) is the range of values of \(X\) in Section B, and \(N^2(s) = \max_{i \in [m]} N_i(s)\), where \(N_i(t)\) is the degree of on-ramp \(i\) representing the number of vehicles in the network at time \(t\) that need to cross the merging point of on-ramp \(i\) in order to reach their destination. Note that \(3\) implies \(V(t + 1) - V(t) = N^2(t + 1) - N^2(t - kT_{\text{cyc}} + 1)\). We claim that if \(V(t)\) is large enough, specifically if \(V(t) > L := \sum_{s = t - kT_{\text{cyc}} + 1}^{t} (m kT_{\text{cyc}} + n_c + n_a + m(s - t + kT_{\text{cyc}} - 1))\), then there exists at least one on-ramp, say \(q \in [m]\), whose degree decreases by at least one at every \(\lceil \tau_q/\tau \rceil\) time steps in the interval \([t - kT_{\text{cyc}} + 1, t]\), without considering new arrivals. We claim that \(V(t) > L\) implies \(N(t - kT_{\text{cyc}} + 1) > m kT_{\text{cyc}} + n_c + n_a\). If not, then since the total number of arrivals to the network is bounded by \(m\) at each time step, it follows for all \(s \in [t - kT_{\text{cyc}} + 1, t]\) that
\[
N(s) \leq N(t - kT_{\text{cyc}} + 1) + m(s - t + kT_{\text{cyc}} - 1) \leq m kT_{\text{cyc}} + n_c + n_a + m(s - t + kT_{\text{cyc}} - 1).
\]

This would lead to the contradiction that
\[
V(t) = \sum_{s = t - kT_{\text{cyc}} + 1}^{t} N^2(s) \leq \sum_{s = t - kT_{\text{cyc}} + 1}^{t} (m kT_{\text{cyc}} + n_c + n_a + m(s - t + kT_{\text{cyc}} - 1))^2 = L.
\]

The inequality \(N(t - kT_{\text{cyc}} + 1) > m kT_{\text{cyc}} + n_c + n_a\) in turn implies that there exists at least one on-ramp, say \(q \in [m]\), such that \(|Q_q(t - kT_{\text{cyc}} + 1)| > kT_{\text{cyc}}\). If not, then \(N(t - kT_{\text{cyc}} + 1) \leq m \sum_{j} |Q_j(t - kT_{\text{cyc}} + 1)| + n_c + n_a \leq m kT_{\text{cyc}} + n_c + n_a\),
a contradiction. Let \( k > K_1 := \max_{i \in [m]} \{ \lceil \tau_i / \tau \rceil / T_{\text{cyc}} \} \). If, in addition, \( k \geq 1 + \sqrt{3(1/m + 1)} \), then we show at the end of the proof that for all \( w \in [t - kT_{\text{cyc}} + 1, t - \lceil \tau_q / \tau \rceil + 1] \) we have

\[
N_q(w + \lceil \tau_q / \tau \rceil) \leq N_q(w) - 1 + \sum_{s = w + 1}^{w + \lceil \tau_q / \tau \rceil} A_{f_q}(s),
\]

where \( A_{f_q}(s) \) is the total number of arrivals at time \( s \) that need to cross on-ramp \( q \). By summing up (6) over disjoint sub-intervals of length \( \tau_q / \tau \), we obtain

\[
N_q(t + 1) \leq N_q(t - kT_{\text{cyc}} + 1) - kT_{\text{cyc}} \left( \frac{\tau_q}{\tau} \right) + 1 + \sum_{s = t - kT_{\text{cyc}} + 2}^{t + 1} A_{f_q}(s)
\]

\[
\leq N(t - kT_{\text{cyc}} + 1) - kT_{\text{cyc}} \left( \frac{\tau_q}{\tau} \right) + 1 + \sum_{s = t - kT_{\text{cyc}} + 2}^{t + 1} A_{f_q}(s).
\]

We show at the end of the proof that a certain form of (7) holds true for all off-ramps. Letting \( K_2 := \max_{i \in [m]} \{ 1 + \sqrt{3(1/m + 1)} \} \), it then follows for all \( k \geq \max\{ K_1, K_2 \} \) that

\[
N(t + 1) \leq N(t - kT_{\text{cyc}} + 1) - \delta kT_{\text{cyc}} + C + \tilde{A}(kT_{\text{cyc}}),
\]

where \( \tilde{A}(kT_{\text{cyc}}) \) satisfies the following: there exist \( \varepsilon, K_3 > 0 \) such that \( \varepsilon < \delta \) and for all \( k \geq K_3 \) we have

\[
\mathbb{E} \left[ \tilde{A}^2(kT_{\text{cyc}}) \right] < 2\varepsilon(kT_{\text{cyc}})^2, \quad \mathbb{E} \left[ \tilde{A}(kT_{\text{cyc}}) \right] < \varepsilon kT_{\text{cyc}}
\]

The inequality (8) implies that

\[
N^2(t + 1) \leq (N(t - kT_{\text{cyc}} + 1) - \delta kT_{\text{cyc}} + C + \tilde{A}(kT_{\text{cyc}}) + 2(N(t - kT_{\text{cyc}} + 1) - \delta kT_{\text{cyc}} + C) + \tilde{A}(kT_{\text{cyc}}) + 2(\delta kT_{\text{cyc}} - C)^2
\]

\[
+ 2\varepsilon kT_{\text{cyc}}(kT_{\text{cyc}} - \delta kT_{\text{cyc}} + C).
\]

By choosing \( k \geq K_3 \) and taking conditional expectation from both sides of (10) we obtain

\[
\mathbb{E} \left[ N^2(t + 1) - N^2(t - kT_{\text{cyc}} + 1) \mid V(t) > L \right] \leq -2N(t - kT_{\text{cyc}} + 1)((\delta - \varepsilon)kT_{\text{cyc}} - C - \varepsilon kT_{\text{cyc}}(kT_{\text{cyc}} - 2\delta kT_{\text{cyc}} + 2C)
\]

\[
= -2 \max_{s \in [t - kT_{\text{cyc}} + 1, t]} \| Q(s) \|_{\infty} - 2 \max_{s \in [t - kT_{\text{cyc}} + 1, t]} \| Q(s) \|_{\infty} \left( \frac{(\delta - \varepsilon)kT_{\text{cyc}} - C}{2} \right)
\]

\[
+ (\delta kT_{\text{cyc}} - 2kT_{\text{cyc}} - \delta kT_{\text{cyc}} - C).
\]

Since \( \max_{s \in [t - kT_{\text{cyc}} + 1, t]} \| Q(s) \|_{\infty} \geq Q_q(t - kT_{\text{cyc}} + 1) \geq k^2 T_{\text{cyc}} \), for all \( k \geq \max\{ K_1, K_2, K_3, K_4 := \frac{C + 1/2}{\sigma T_{\text{cyc}}} \} \) we have

\[
\mathbb{E} \left[ N^2(t + 1) - N^2(t - kT_{\text{cyc}} + 1) \mid V(t) > L \right] \leq 0 - 2 \max_{s \in [t - kT_{\text{cyc}} + 1, t]} \| Q(s) \|_{\infty} - 2(\delta - \varepsilon)T_{\text{cyc}}^2 k^3
\]

\[
+ (\delta^2 T_{\text{cyc}} + 2(1 - \varepsilon)\delta T_{\text{cyc}} + 2C + 1) T_{\text{cyc}} k^2 - 2(\varepsilon + 1)CT_{\text{cyc}} k + C^2.
\]

In addition to the previously stated lower bound \( k \geq \max\{ K_1, K_2, K_3, K_4 \} \), if we also choose \( k \) such that

\[
-2(\delta - \varepsilon)T_{\text{cyc}}^2 k^3 + (\delta^2 T_{\text{cyc}} + 2(1 - \varepsilon)\delta T_{\text{cyc}} + 2C + 1) T_{\text{cyc}} k^2 - 2(\varepsilon + 1)CT_{\text{cyc}} k + C^2 < 0,
\]

then \( \mathbb{E} \left[ V(t + 1) - V(t) \mid V(t) > L \right] \leq -\max_{s \in [t - kT_{\text{cyc}} + 1, t]} \| Q(s) \|_{\infty} \). Such a \( k \) always exists because the \( -2(\delta - \varepsilon)T_{\text{cyc}}^2 k^3 \) term in (12) dominates for sufficiently large \( k \).

Finally, if \( V(t) = \sum_{s = t - kT_{\text{cyc}} + 1}^{t} N^2(s) \leq L \), then \( N^2(s) \leq L \) for every \( s \in [t - kT_{\text{cyc}} + 1, t] \). Therefore,

\[
\max_{s \in [t - kT_{\text{cyc}} + 1, t]} \| Q(s) \|_{\infty} \leq \sqrt{L}
\]
because $\|Q(s)\|_\infty \leq N(s)$ for all $s \in [t - kT_{cy}, t]$. Also, $V(t + 1) \leq V(t) + N^2(t + 1) \leq V(t) + (\sqrt{T} + m)^2$ because the total number of arrivals at each time step does not exceed $m$. Combining this with the previously considered case of $V(t) > L$, we get

$$E[V(t + 1) - V(t) | Z(t)] \leq -\max_{s \in [t - kT_{cy}, t]} \|Q(s)\|_\infty + (\sqrt{T} + m)^2 + (1 + \sqrt{T}) \mathbb{I}_B,$$

where $B = \{Z_{kT_{cy}}(t) \in \mathcal{Y}^{kT_{cy}} : V(t) \leq L\}$ (a finite set). The result then follows from Theorem 5.

**Proof of (5)**

Let $w \in [t - kT_{cy}, t - [\tau_q/\tau] + 1]$ and consider all the mainline slots upstream of the merging point of on-ramp $q$ that are at most $[\tau_q/\tau]$ time steps away at time $w$. If at least one of these slots is occupied, then (5) obviously follows. If not, consider the last acceleration lane slot of on-ramp $q$ that is not on the mainline at time $w$. This slot satisfies (M1)-(M5), and so it must be occupied at time $w$ if on-ramp $q$ had non-zero quotas at the time this slot was the first acceleration lane slot, which is at most $n_q$ time steps before $w$. This would again give (5).

Let $t'$ be the start of the most recent cycle at least $n_q$ time steps before $t - kT_{cy} + 1$. Considering at most one arrival per time step gives $|Q_q(t')| \geq |Q_q(t - kT_{cy} + 1)| - (t - kT_{cy} + 1 - t') \geq k^2T_{cy} - (n_q + T_{cy})$, which is greater than $(k + [n_q/T_{cy}] + 1)T_{cy} + n_q + T_{cy}$ for

$$k \geq 1 + \sqrt{3[\frac{n_q}{T_{cy}}] + 1}.$$

Now, let $t'' \geq t'$ be the the most recent cycle at least $n_q$ time steps before $w$. Since at most one vehicle is released from on-ramp $q$ per time step, it follows that $|Q_q(t'')| \geq |Q_q(t')| - (t'' - t') > n_q + T_{cy}$, where the last inequality follows from $t'' - t' \leq (k + [n_q/T_{cy}] + 1)T_{cy}$. Therefore, on-ramp $q$ has non-zero quotas at the time the aforementioned slot was the first acceleration lane slot.

**Proof of (6)**

Consider on-ramp $i \in [m]$. If at every $[\tau_i/\tau]$ time steps in the interval $[t - kT_{cy}, t]$ a vehicle crosses on-ramp $i$, then for all $w \in [t - kT_{cy} + 1, t - [\tau_i/\tau] + 1]$ we have

$$N_i(w + [\tau_i/\tau]) \leq N_i(w) - 1 + \sum_{s = w + 1}^{w + [\tau_i/\tau]} A_{f,i}(s).$$

Hence,

$$N_i(t + 1) \leq N_i(t - kT_{cy} + 1) - \frac{kT_{cy}}{[\tau_i/\tau]} + 1 + \sum_{s = t - kT_{cy} + 2}^{t + 1} A_{f,i}(s)$$

$$\leq N(t - kT_{cy} + 1) - \frac{kT_{cy}}{[\tau_i/\tau]} + 1 + \sum_{s = t - kT_{cy} + 2}^{t + 1} A_{f,i}(s),$$

as desired. If not, let $s_i \in [t - kT_{cy} + 1, t - [\tau_i/\tau] + 1]$ be the last time at which a vehicle does not cross on-ramp $i$ in $[s_i, s_i + [\tau_i/\tau])$, i.e., $N_i(s_i + [\tau_i/\tau]) = N_i(s_i) + \sum_{s = s_i + 1}^{s_i + [\tau_i/\tau]} A_{f,i}(s)$. Hence, for all $w \in [s_i + 1, t - [\tau_i/\tau] + 1], N_i(w + [\tau_i/\tau]) \leq N_i(w) - 1 + \sum_{s = w + 1}^{w + [\tau_i/\tau]} A_{f,i}(s)$. This further gives

$$N_i(t + 1) \leq N_i(s_i + 1) - \frac{t - s_i}{[\tau_i/\tau]} + 1 + \sum_{s = s_i + 1}^{t + 1} A_{f,i}(s).$$ (13)

Furthermore, we claim that the queue length of on-ramp $i$ at time $s_i + 1$ cannot exceed $T_{cy} + n_i$, i.e., $|Q_i(s_i + 1)| \leq T_{cy} + n_i$. Note that since no upstream vehicles crosses on-ramp $i$ in $[s_i, s_i + [\tau_i/\tau])$, it must be that: (i) all the mainline slots upstream of the merging point that are at most $[\tau_i/\tau]$ time steps away are empty at time $s_i$, (ii) the last acceleration lane slot of on-ramp $i$ that is not on the mainline at time $s_i$ is empty. Moreover, (i) implies that the aforementioned acceleration lane slot satisfies (M1)-(M4) at time $s_i$. Therefore, it must be that on-ramp $i$ had zero quotas at the time this slot was the first acceleration lane slot, which is at most $n_i$ time steps before $s_i$. Since the number of on-ramp arrivals is no more than one per time step, it follows that $|Q_i(s_i + 1)| \leq T_{cy} + n_i$. Hence,

$$N_i(s_i + 1) \leq N_i(s_i + 1) - |Q_i(s_i + 1)| + n_c + n_i$$

$$\leq N_i(s_i + 1) + T_{cy} + n_c + 2n_i.$$
Combining this with (13) gives
\[ N_i(t+1) \leq N_{i-1}(s_i + 1) - \frac{t-s_i}{\tau_i} + 1 + \sum_{s=s_i+2}^{s_i+1} A_{f,i}(s) + T_{\text{cyc}} + n_c + 2n_i. \] (14)

Repeating the above steps for on-ramp \( i - 1 \) gives
\[ N_{i-1}(s_i + 1) \leq N_{i-2}(s_i - 1) + \frac{s_i - s_i}{\tau_i} + 1 + \sum_{s=s_i+1+1}^{s_i+1} A_{f,i-1}(s) + T_{\text{cyc}} + n_c + 2n_{i-1}, \]
where \( s_{i-1} \in [t - kT_{\text{cyc}} + 1, s_i - \lceil \tau_{i-1}/\tau \rceil + 1] \). This process can be repeated until we find an on-ramp, indexed by \( i - m_i \) for some \( m_i \in \{0\} \cup \{m-1\} \), such that for all \( w \in [t - kT_{\text{cyc}} + 1, s_{i-m_i} - \lceil \tau_{i-m_i}/\tau \rceil + 1] \), \( N_{i-m_i}(w + \lceil \tau_{i-m_i}/\tau \rceil) \leq N_{i-m_i}(w) - 1 + \sum_{s=w+\lceil \tau_{i-m_i}/\tau \rceil}^{s_{i-m_i}+1} A_{f,i-m_i}(s) \). Indeed, one such on-ramp is always \( q \); see the argument around (6). Therefore,
\[ N_{i-m_i}(s_{i-m_i} + 1) \leq N(t - kT_{\text{cyc}} + 1) - \frac{s_i - m_i + 1 - t + kT_{\text{cyc}}}{\lceil \tau_{i-m_i}/\tau \rceil} + 1 + \sum_{s=t-kT_{\text{cyc}}+2}^{s_i-m_i+1+1} A_{f,i-m_i}(s). \]

By combining all the inequalities for on-ramps \( i - p \), \( p \in \{0\} \cup \{m_i\} \), we have
\[ N_i(t+1) \leq N(t - kT_{\text{cyc}} + 1) + \sum_{p=0}^{m_i} A_{f,i-p}(s_{i-p+1} - s_{i-p}) - \frac{s_i - p - s_i - p}{\lceil \tau_{i-p}/\tau \rceil} + (T_{\text{cyc}} + n_c + 1)m + 2n_i, \]
where \( s_{i+1} = t, s_{i-1} = t - kT_{\text{cyc}}, \) and \( A_{f,i}(s_{i-p+1} - s_{i-p}) \) is the cumulative number of arrivals in the interval \([s_{i-p} + 2, s_{i-p+1} + 1]\) as defined in the proof of Theorem 1. By the assumption \( \lceil \tau_{i}/\tau \rceil \rho_i < 1 \), there exists \( \delta > 0 \) such that for all \( i \in [m] \),
\[ \rho_i + \delta < \frac{1}{\lceil \tau_{i}/\tau \rceil}. \]

Thus, for all \( i \in [m] \) we have
\[ \sum_{p=0}^{m_i} A_{f,i-p}(s_{i-p+1} - s_{i-p}) - \frac{s_i - p - s_i - p}{\lceil \tau_{i-p}/\tau \rceil} < -\delta kT_{\text{cyc}} + \sum_{p=0}^{m_i} A_{f,i-p}(s_{i-p+1} - s_{i-p}) - \rho_i(s_{i-p+1} - s_{i-p}). \]

Hence, (8) follows with
\[ \hat{A}(kT_{\text{cyc}}) = \max_{i \in [m]} \sum_{p=0}^{m_i} A_{f,i-p}(s_{i-p+1} - s_{i-p}) - \rho_i(s_{i-p+1} - s_{i-p}). \] (15)

Proof of (9)
Consider the sequence \( \{A_{f,i_s}(s)\}_{s=t-kT_{\text{cyc}}+2}^{s=\infty} \), where the indices \( i_s \in [m] \) are allowed to depend on time \( s \). For a given \( s \in [t - kT_{\text{cyc}} + 2, \infty) \), the term \( A_{f,i}(s) \) is independent of the other terms in the sequence, \( \rho_i = \mathbb{E}[A_{f,i}(s)] \) is bounded, and \( \sigma_i^2 := \mathbb{E}[A_{f,i}^2(s) - \rho_i^2] \) is (uniformly) bounded for all \( i_s \in [m] \). As a result, \( \lim_{k \to \infty} \sum_{s=t-kT_{\text{cyc}}+1}^{s} (s - t + kT_{\text{cyc}})^{-2} \sigma_i^2 \) is also bounded. From Kolmogorov’s strong law of large numbers [19, Theorem 10.12], we have, with probability 1,
\[ \lim_{k \to \infty} \frac{1}{kT_{\text{cyc}}} \left( \sum_{s=t-kT_{\text{cyc}}+2}^{s} A_{f,i_s}(s) - \sum_{s=t-kT_{\text{cyc}}+2}^{s} \rho_i \right) = 0. \]

By following similar steps to the proof of (3) in Theorem 1, it follows for all \( n \in \mathbb{N} \) that
\[ \lim_{k \to \infty} \mathbb{E} \left[ \left( \frac{\hat{A}(kT_{\text{cyc}})}{kT_{\text{cyc}}} \right)^n \right] = 0, \]
which in turn gives (9).
C. Proof of Proposition 1

It is sufficient to show that \( |X^G(kT_{\text{per}})| = 0 \) for some \( k \in \mathbb{N} \). This would then imply that \( g(\cdot) = 0 \) after a finite time. Therefore, the rest of the proof follows along the lines of the proof of Theorem 2.

Suppose that \( |X^G(kT_{\text{per}})| \neq 0 \) for all \( k \in \mathbb{N} \). Hence, there exists \( q \in [m] \) and an infinite sequence \( \{k_n\}_{n \geq 1} \) such that \( |X^G_q(k_nT_{\text{per}})| \neq 0 \) for all \( n \geq 1 \). We prove \( \limsup_{k \to \infty} g_q(kT_{\text{per}}) = \infty \) by considering the following two cases:

(i) if
\[
|X^G_q(kT_{\text{per}})| \leq \max\{ |X^G_q((k - 1)T_{\text{per}})| - \alpha, 0 \}
\]
holds for finitely many \( k \)'s, then Algorithm 2 implies \( \limsup_{k \to \infty} g_q(kT_{\text{per}}) = \infty \).

(ii) if (16) holds for an infinite sequence \( \{k'_n\}_{n \geq 1} \), then there exists an infinite subsequence \( \{k''_n\}_{n \geq 1} \) for which (16) does not hold at \( k = k''_{n} + 1 \) for all \( n \geq 1 \). If not, then there exists \( M \in \mathbb{N} \) such that (16) holds for all \( k \geq k_M \).

Since \( |X^G_q((k - 1)T_{\text{per}})| \) is bounded, this implies that \( |X^G_q(kT_{\text{per}})| = 0 \) for all \( k \) sufficiently greater than \( k_M \) — a contradiction. With respect to the subsequence \( \{k''_n\}_{n \geq 1} \), Algorithm 2 implies that \( \theta_q(k''_{n} + 1T_{\text{per}}) = \beta_0 \theta_q(k''_{n}T_{\text{per}}) \) for all \( n \geq 1 \). Since \( \theta_q(\cdot) \) is non-decreasing and \( \beta > 1 \), this implies \( \lim_{n \to \infty} g_q(k''_{n} + 1T_{\text{per}}) \geq \lim_{n \to \infty} \theta_q(k''_{n} + 1T_{\text{per}}) = \infty \).

That is, \( \limsup_{k \to \infty} g_q(kT_{\text{per}}) = \infty \).

Let \( \theta := \max_{i \in [m]} \theta_q \). Algorithm 2 implies that, while \( g_q(\cdot) > 2 \max\{T_{\text{max}}, mT_{\text{empty}}(1 + \alpha/2T_{\text{per}})\} \), it takes at most \( \max\{T_{\text{max}}, mT_{\text{empty}}(1 + \alpha/2T_{\text{per}})\}/\theta \) periods for \( g_q(\cdot) \) to exceed \( \max\{T_{\text{max}}, mT_{\text{empty}}(1 + \alpha/2T_{\text{per}})\} \). When \( g_{q-1}(\cdot) \) exceeds \( T_{\text{max}}, g_{q-2}(\cdot) \) starts to increase, and so on. Therefore, while \( g_q(\cdot) > 2 \max\{T_{\text{max}}, mT_{\text{empty}}(1 + \alpha/2T_{\text{per}})\} \), it takes at most \( k_f = 1 + (m - 2)\max\{T_{\text{max}}, mT_{\text{empty}}(1 + \alpha/2T_{\text{per}})\}/\theta \) periods such that, for all \( i \in [m] \), \( g_{q}(\cdot) \) exceeds \( \max\{T_{\text{max}}, mT_{\text{empty}}(1 + \alpha/2T_{\text{per}})\} \). Let \( \hat{k} \) be such that \( g_q(kT_{\text{per}}) > \max\{T_{\text{max}}, mT_{\text{empty}}(1 + \alpha/2T_{\text{per}})\} + 2k_f \). This implies that \( g_q(kT_{\text{per}}) > \max\{T_{\text{max}}, mT_{\text{empty}}(1 + \alpha/2T_{\text{per}})\} \) for \( k = \hat{k} + 1, \ldots, \hat{k} + k_f \). Hence, \( g_i((\hat{k} + k_f)T_{\text{per}}) \geq \max\{T_{\text{max}}, mT_{\text{empty}}(1 + \alpha/2T_{\text{per}})\} \) for all \( i \in [m] \).

Note that, for all \( i \in [m] \), \( g_i(t) \geq mT_{\text{empty}} \) for all \( t \in [t_f, t_f + mT_{\text{empty}}] \), where \( t_f := kT_{\text{per}} \). Thus, each on-ramp releases at most one vehicle during the interval \( [t_f, t_f + mT_{\text{empty}}] \). Hence, there exists a time interval of length at least \( T_{\text{empty}} \) in \( [t_f, t_f + mT_{\text{empty}}] \) during which no on-ramp releases a vehicle. Condition (VC3) then implies that the mainline and acceleration lanes become empty after such \( T_{\text{empty}} \) time units, at the end of which \( |X^G(\cdot)| = 0 \); a contradiction to the assumption that \( |X^G(kT_{\text{per}})| \neq 0 \) for all \( k \).

D. Proof of Theorem 2

We show that, for sufficiently large \( K(\cdot) \): (i) no vehicle initially present on the mainline or acceleration lanes switches to the safety mode because of a vehicle that is released thereafter; and (ii) no released vehicle ever switches to the safety mode. (i) ensures that all initial vehicles on the mainline reach their destination by time \( T_{\text{empty}} \). Combined with (ii), it also ensures \( |X^G(T_{\text{empty}})| = 0 \). The rest of the proof then follows along the lines of the proof of Theorem 2. We provide proof for (ii); the proof for (i) is similar. To avoid tedious algebra and without loss of generality, we assume that merging occurs instantaneously and the merging speed at all on-ramps is \( V_f \), i.e., the ego vehicle enters the mainline immediately after being released at the free flow speed. Pointers for the proof of the general case is presented at the end.

Suppose that (ii) does not hold true: let \( t \in [0, T_{\text{empty}}] \) be the first time at which a vehicle that is released at \( t_0 \in [0, t] \), indexed by \( e \), switches to the safety mode, because of vehicle \( p \) that had been present at or before \( t_0 \). Without loss of generality, let \( t_0 = 0 \). We have
\[
y_e(t) = y_e(0) + \int_0^t v_p(\eta) - v_e(\eta)d\eta = y_e(0) - \int_0^t |v_p(\eta) - V_f|d\eta \geq S_e(0) + K(X(0)) - \int_0^t |v_p(\eta) - V_f|d\eta.
\]
We show that, for sufficiently large \( K(\cdot) \),
\[
\int_0^t |v_p(\eta) - V_f|d\eta + S_e(t) \leq K(X(0)) + S_e(0),
\]
which combined with (17) implies that vehicle \( e \) does not satisfy the switching criterion to the safety mode (see (VC2)) — a contradiction.

Let \( 0 \leq \xi_1 \leq \ldots \leq \xi_\ell \leq t \) be the “jump” time instants, i.e., when a vehicle that is present at time \( 0 \) (I) leaves through an off-ramp; or (II) changes mode. In between these jump events, e.g., \( \eta \in [\xi_j, \xi_{j+1}], j \in [\ell - 1] \), (VC5) implies that \( |X^G(\eta)| \leq ce^{-c(\eta - \xi)} ||X^G(\xi_j)|| \) for some \( c, r > 0 \). Moreover, for all \( j \in [\ell] \), in jump event (I) we have \( |X^G(\xi_j)| \leq |X^G(\xi_j)| \); and in (II) if vehicle \( i \) changes mode and \( I_i(\xi_j) = 0 \), then \( |X^G(\xi_j)| = |X^G(\xi_j)| \). Otherwise,
\[
|X^G(\xi_j)| \leq ||X^G(\xi_j)|| + |v_i(\xi_j) - V_f| + |a_i(\xi_j)|,
\]
where the equality holds if vehicle \( i \) is outside an acceleration lane at time \( \xi_j \), which in turn implies that \( |v_i(\xi_j) - V_f| + |a_i(\xi_j)| = 0 \). Since the merging occurs instantaneously and the merging speed is \( V_f \), we have \( |X^G(\xi_j)| = |X^G(\xi_j)| \) in jump event (II). Moreover, for all \( \xi \in [0, t] \), \( |v_p(\eta) - V_f| \leq |X^G(\xi)| \). We can now bound \( |X^G(\xi)| \) in terms of \( |X^G(0)| \) as follows:
for all \( j \in [\ell - 1] \) and \( \eta \in [\xi_j, \xi_{j+1}] \), we have \( |v_p(\eta) - V_f| \leq \|X^G(\eta)\| \leq c e^{-r(\eta - \xi_j)} \|X^G(\xi_j)\| \leq \cdots \leq c^{\ell+1} e^{-r\eta} \|X^G(0)\| \). Hence,

\[
\int_0^t |v_p(\eta) - V_f| d\eta + \frac{V_f^2 - v_p(t)^2}{2|a_{\min}|} \leq \left( \frac{1}{r} (1 - e^{-rt}) + \frac{V_f + V}{2|a_{\min}|} e^{-rt} \right) c^{\ell+1} \|X^G(0)\| \\
\leq \left( \frac{1 - e^{-rT_{\text{merge}}}}{r} + \frac{V_f + V}{2|a_{\min}|} \right) c^{3n(0)} \|X^G(0)\|,
\]

where in the second inequality, we have used \( \ell + 1 \leq 3n(0) \) by (VC4). Therefore, picking \( a = \left( \frac{1 - e^{-rT_{\text{merge}}}}{r} + \frac{V_f + V}{2|a_{\min}|} \right) \), \( b = c^3 \), \( f(X) = \frac{v^2 - V^2}{2|a_{\min}|} \), and setting \( K(X) = ab^n \|X^G\| + f(X) \) gives (17).

When the merging is not instantaneous, additional arguments are needed near (17) to show that between the release and merge of vehicle \( e \), its projected distance with respect to vehicle \( p \) is no less than the projected safety distance. Moreover, the speed of vehicle \( e \) is less than \( V_f \) before leaving the acceleration lane. All this would only result in different \( a, b, \) and \( f \).

E. Proof of Theorem 4

Let \( t = 0, 1, \ldots \) with time steps of length \( \tau \). Without loss of generality, let the point \( p_i \) be the merging point of on-ramp \( i \) for all \( i \in [m] \). We have \( N_i(t) = N_i(0) + \sum_{s=1}^t A_{f,i}(s) - D_i(t) \), where we have dropped the dependence on the ramp metering policy for brevity. Note that the previous equation holds for any point on the \( i \)-th link, which justifies the no loss in generality.

Since the arrival processes are assumed to be i.i.d. across on-ramps, the strong law of large numbers implies that for all \( i \in [m] \), with probability one,

\[
\liminf_{t \to \infty} \frac{\sum_{s=1}^t A_{f,i}(s)}{t} = \rho_i,
\]

and hence

\[
\liminf_{t \to \infty} \frac{N_i(t)}{t} = \rho_i - \limsup_{t \to \infty} \frac{D_i(t)}{t}.
\]

If \( \rho_i > 1 \) for some \( i \in [m] \), then, with probability one, \( \liminf_{t \to \infty} N_i(t)/t \) is bounded away from zero, and hence \( \liminf_{t \to \infty} N_i(t) = \infty \). Thus, \( \liminf_{t \to \infty} |Q_i(t)| = \infty \) for some on-ramp \( j \in [m] \). Combining this with Fatou’s lemma imply that the average queue length grows unbounded at on-ramp \( j \). This contradicts the network being under-saturated.