Dynamic and Static Properties of
the Randomly Pinned Planar Flux Array

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Abstract

We report the results of large scale numerical studies of the dynamic and static properties of the random phase model of the randomly pinned flux lines confined in a plane. From the onset of nonlinear IV characteristics, we identify a vortex glass transition at the theoretically anticipated temperature. However, the vortex glass phase itself is much less glassy than expected. No signature of the glass transition has been detected in the static quantities measured.
Flux pinning is crucial to the performance of high-$T_c$ superconductors in a strong magnetic field [1]. It has been conjectured [2,3] that flux lines in a superconductor with random, point-like impurities may form a glass, the vortex glass, in which the flux line configurations are frozen by the defects at low temperatures. It is argued [3,4] that if a glass phase exists, then the sluggish glassy dynamics of the flux lines suppresses dissipation, making the system a true superconductor. Unfortunately, evidence supporting the vortex glass hypothesis is still inconclusive: The phase transition seen experimentally by Koch et al. [5] and by Gammel et al. [6] may be due to pinning by correlated defects [7] such as twin planes and/or screw dislocations. Existing numerical simulations which support the existence of the vortex glass are thus far restricted to models with zero external magnetic field and/or an infinite London penetration length [8]. However, the glass phase seems to be unstable to thermal fluctuations in more realistic models with a finite London penetration length [9].

Without solid evidence from experimental and numerical studies, major support for the vortex glass hypothesis comes from a number of analytic studies [2,4,10,11], which are however restricted to “elastic systems” where topological defects are forbidden to nucleate. While the validity of the elastic approximation for the bulk system in $2+1$ dimensions is a subject of intense current research, the elastic model is well justified to describe flux lines confined in a plane ($1+1$ dimensions) [2,12,13] or vortex lines in planar Josephson junctions [14]. Thus far, the $1+1$ dimensional flux array is the only system for which a vortex glass phase has been demonstrated analytically; as such, it is one of the very few pieces of “solid” support on which the vortex glass hypothesis for bulk superconductors is based. Because of this significance, the static and dynamic properties of the $1+1$ dimensional system have been investigated analytically by many groups in the past several years [4,10,17,19,22]. These studies led to a number of rather different predictions, none of which has been tested experimentally or numerically.

In this paper, we report the results of detailed numerical studies of the random phase model [2,12,14,23] of the planar flux array using the Connection Machine CM5. We observe an apparent vortex glass transition at a finite temperature, $T_g$, below which the IV
characteristics are nonlinear and described by power laws, as anticipated by Refs. [12] and [16]. However, the universal constant found is significantly smaller than that predicted in Ref. [16]. Static quantities such as the equal-time correlation function are also measured. They exhibit no detectable signs of the glass transition.

We consider an array of flux lines confined to the \((x, z)\)-plane, and directed in the \(z\)-direction by an applied magnetic field \(H = H_0 \hat{z}\). Line-line repulsion is modeled by linear elasticity, with an elastic constant \(\kappa = (d\rho/dH_0)^{-1}\), where \(\rho\) is the average line density. Random point-like pinning centers in the plane are described by an uncorrelated Gaussian random potential with a variance \(\Delta_0\). A direct simulation of the flux array is deemed difficult because an interline spacing of \(\rho^{-1} \sim 10\) grid points and a system size of \(\gg 10\rho^{-1}\) is needed to probe the asymptotic behavior. Instead we will study the random phase model,

\[
\mathcal{H} = \int d^2r \left\{ \frac{\kappa}{2} (\nabla u)^2 - \lambda \cos[2\pi u(r) - \beta(r)] \right\},
\]

which is generally believed [2,12,23] to describe the statistical mechanics of the random flux array at length scales large compared to the in-line spacing \(\rho^{-1}\). In Eq. (1), \(r = \{x, z\}\), \(u(r)\) is a displacement-like field, the cosine term captures the discreteness of the flux lines crucial to pinning, \(\beta(r)\) is a random phase uniformly distributed in the interval \([0, 2\pi]\), and \(\lambda = 2\sqrt{\Delta_0 \rho}\) characterizes the strength of the pinning potential. A derivation of this model starting from a system of directed flux lines can be found in Ref. [23].

Renormalization group (RG) studies of Eq. (1) [24,25] found a glass transition at \(T_g = \kappa/\pi\) for small \(\lambda\). For \(T > T_g\), randomness (the cosine term) is irrelevant, and the system is asymptotically described by Eq. (1) with \(\lambda = 0\). But for \(T < T_g\), the system is described by a line of fixed points. A number of properties of the glass phase close to \(T_g\) have been obtained from the RG studies. For instance, the static correlation function is predicted to diverge with a logarithmic anomaly [25],

\[
C(r) \equiv \langle (u(r) - u(0))^2 \rangle = AC_0(r) + B\tau^2[\ln|r|]^2,
\]

where \(\tau \equiv 1 - T/T_g\), \(C_0(r) = T/(\pi\kappa)\ln|r|\) is the correlation function with \(\lambda = 0\), \(A\) is a
nonuniversal constant, and $B = 2/\pi^2 + O(\tau)$ is universal [22]. Overbar and $\langle \ldots \rangle$ denote disorder and thermal averages respectively.

Dynamic RG methods were used to extract the glassy dynamics of the model (1) close to $T_g$ [16,26]. In particular, under a small driving force $F$, the Langevin dynamics

$$\partial_t u = -\frac{\delta H}{\delta u} + F + \eta, \quad (3)$$

where $\eta(r,t)$ is a white noise with $\langle \eta(r,t)\eta(0,0) \rangle = 2T \delta^2(r)\delta(t)$, is expected to yield a nontrivial response, $v(F) \equiv \langle \partial_t u \rangle$, with

$$v(F) \propto |\tau|^\zeta F, \quad \text{for} \quad T > T_g, \quad (4)$$

$$v(F) \propto |\tau|^\zeta F^\alpha, \quad \alpha = 1 + \zeta \tau \quad \text{for} \quad T < T_g, \quad (5)$$

with the universal constant $\zeta \approx 1.78 + O(\tau)$. $F$ describes the effect of a Lorentz force resulting from an electric current perpendicular to the $(x,z)$-plane. $v(F)$, the “IV characteristics”, is proportional to the EMF generated from the in-plane motion of the flux lines when pushed by the Lorentz force [13]. A zero DC resistivity ($\lim_{F \to 0} v(F)/F \to 0$) and hence true superconductivity is expected in the vortex glass phase according to Eq. (5).

Within the RG framework, these predictions are valid in the asymptotic limits of the system size $L \to \infty$ and $F \to 0$. Finite size effects are derived using the length dependence of the dimensionless coupling constant $g(L)$,

$$g(L) = \frac{g(1)L^{2\tau}}{1 + Cg(1)[L^{2\tau} - 1]/(2\tau)}, \quad (6)$$

which characterizes the effective strength of disorder at scale $L$, and is obtained from the RG recursion relation [22,25]. In Eq. (6), $g(1) = \pi \lambda^2/(2T^2)$ is the bare coupling constant, and the nonuniversal coefficient $C \approx 1$ controls the crossover length for the model (1). Finite size effects in the correlation function shall be estimated by replacing $\tau$ in Eq. (2) by $(2/C)g(L)$ [27].

When $F \neq 0$, the equilibration time is $t_{eq} \approx 1/v(F)$, since for $v(F)t \gg 1$, $\langle u \rangle \approx v(F)t \gg 1$, which averages out the pinning effect of the cosine [16]. But during the time $t_{eq}$, only
regions of size $L_{\text{eff}} \approx t_{\text{eq}}^{1/z} (z = 2\alpha \ [16])$ can be equilibrated. Hence the effective system size at a finite drive is $L_{\text{eff}} \approx F^{-1/2}$. While the effect of a large $F$ involves nonequilibrium dynamics [16] and is complicated, at the level of linear response, we shall again estimate the effect of a finite $F$ by replacing the coefficient $|\tau|^{\varsigma}$ in Eq. (5) by $[(2/C)g(F^{-1/2})]^{\varsigma}$.

Although all existing analytic studies agree that $T_g = \kappa/\pi$, some results of the RG analysis disagree with results by other methods. For instance, Toner [13] argued that the form of the IV, for $T < T_g$, should be

$$v(F) \sim F e^{-c\sqrt{\log(F_0/F)}}. \tag{7}$$

Also, Balents and Kardar [19] obtained, at $T = T_g$, a much more strongly divergent correlation function

$$C(r) \sim |r|, \tag{8}$$

using a Bethe ansatz solution of the random flux array. On the other hand, a similar study by Tsvelik [20] yielded $C(r) \propto \ln |r|$ when a different order of thermodynamic limits was taken. One criticism of the RG analysis is that the possibility of “replica symmetry breaking”, often associated with the existence of meta-stable states, has not been considered. A recent variational ansatz with hierarchical replica symmetry breaking scheme [11,21] yields a correlation function even less divergent than Eq. (2),

$$C(r) = \frac{T_g}{\pi \kappa} \ln |r| \tag{9}$$

for all $T < T_g$ and $|r| \gg 1$. However, an earlier attempt [10] yielded Eq. (8).

In view of the myriads of predictions for this system, we perform a large scale numerical study. As we shall be interested in the dynamics, we directly simulate the Langevin equation (3) of the model (1) on an $L \times L$ square lattice with periodic boundary conditions in both the $x$ and $z$ directions. To do this numerical integration, we approximate the time derivative in Eq. (3) by a finite difference, and use a two step, single noise Runge-Kutta integration algorithm [28–30]. This algorithm yields results accurate to $O(dt^2)$, where $dt$ is the the discrete
time step. We ensure that the systematic $dt^2$ errors are as predicted and under control by doing runs at various $dt$’s (with 1024 realizations each) and various temperatures [30]. The simulations reported below are for runs with an optimized $dt = 0.2$, and sizes $L = 64, 128$. The parameters are fixed at $\kappa = 1.0$ and $\lambda = 0.15$. The IV curves are obtained by varying $F$ in the range 0.005 to 0.1, and the glass transition probed by varying $T$ from 0.5 down to 0.25. The number of realizations over which we averaged ranged from 32 ($L = 128$) to 1024 ($L = 64$) and is specified where relevant. Note that the range of parameters are chosen such that $g(1) > g(L) > g(\infty)$ according to Eq. (6). Thus the effective strength of disorder should appear stronger than its asymptotic value due to the finite size effect. However, this effect is not expected to be large, since the crossover length, $L_c$, defined by equating the two terms of the denominator of Eq. (6), is only $L_c = e^{1/\kappa} \approx 15$ at its largest (when $\tau = 0$) [30].

We start from random initial conditions in $u$. Depending on the sizes and temperatures of the systems, we typically use 80,000 to 200,000 time steps to equilibrate, and then 60,000 to 150,000 steps to perform measurements. “Thermal averages” are carried out by time-averaging over the measurement period for each realization. In all cases, equilibration is assured by monitoring the susceptibility, $\chi = \langle (\int d^2r \ \partial_x u)^2 \rangle / (TL_xL_z)$, whose mean is $1/\kappa$, at all $T$, within statistical and $dt^2$ errors [22,30]. As extra precaution, we did some very long runs ($10^6$ steps, 160 realizations) for the $L = 64$ system at $T = 0.285$. All measurements there agree with the shorter runs.

We begin by examining the IV curves $v(F)$. Figure 1 shows the IV curves for a $64 \times 64$ lattice for various temperatures on a log-log plot. (Note that the vertical axis is divided by $F$ to emphasize the nonlinear part of $v(F)$.) According to Eqs. (4) and (5), we expect linear IV curves only above $T_g$. From Fig. 1, we see that the data for $T = 0.4$ satisfy a linear relation [31], while for $T = 0.3$ and below, they are nonlinear and well described by a power law as in Eq. (7). The straightness of the data strongly biases against the prediction (7). The $L = 128$ lattice yields similar results. Fitting Eq. (5) to the IV curves, we can obtain the critical temperature, $T_g$, and the coefficient $\zeta$. From Fig. 2 we see that the exponent $\alpha$ indeed varies linearly with the reduced temperature $\tau$ up to at least $\tau \approx 0.21$ (or $T = 0.25$).
Chi-squared fits to our IV data for $T = 0.25, 0.27, 0.285$ and $0.3$ ($L = 64$) and $T = 0.25, 0.275$ and $0.3$ ($L = 128$), give $T_g = 0.32 \pm 0.01$ and $\zeta = 0.15 \pm 0.02$ for the $L = 64$ system, and $T_g = 0.34 \pm 0.01$ with $\zeta = 0.15 \pm 0.02$ for the $L = 128$ system. The close agreement between the two systems indicates that our result is not limited by the system size $L$. Our value for $T_g$ agrees with the predicted value of $1/\pi \approx 0.318$, but the predicted value for the universal coefficient $\zeta$ is an order of magnitude too large. For comparison, we indicate the predicted slope of the IV curve at $T = 0.25$ as the dashed line in Fig. 1, and the predicted $\alpha(\tau)$ as the dashed line in Fig. 2. Clearly, neither can be accommodated by our numerical results.

One may attempt to attribute the discrepancy between the numerics and the RG prediction to crossover effects, since even the smallest value of $F$ used corresponds to only $L_{\text{eff}} \approx 15$. However, since $\nu(F) \propto [g(F^{-1/2})]^c F^\alpha$ and $g(F^{-1/2})$ increases with $F$ for the range of parameters used, one expects the effective values of $\alpha$ observed to be somewhat larger than the expected asymptotic values [10]. It is therefore not possible to explain the observed discrepancy by such crossover effects. In fact, it is possible (although unlikely) that the observed nonlinear IV’s are the pre-asymptotics of some linear IV. A study at smaller $F$’s is desirable but difficult numerically, since a much longer measurement period will be needed to obtain $\nu(F)$ accurately for each $F$.

Next we consider the static correlation function, $C(r)$, for which we have much better statistics and smaller finite size effects since $F = 0$. In comparing our data for $C(r)$ with the predicted forms, Eqs. (2), (8), and (9), we do not compare directly with $\ln |r|$ since that is the form for the quadratic theory in the continuum. Instead we compare with the correlation function $C_0(r)$ of the quadratic lattice model calculated exactly for $dt = 0.2$. Figure 3 shows the measured values of $C(r)$ at $T = 0.3, 0.285$ and 0.25 for the $L = 64$ system each averaged over 1024 realizations. (The error bars are much smaller than the points and are not shown.) The data are well described by $C(r) = AC_0(r)$ with $A \approx 1.09$ for all $T$, and we find $A \to 1$ as $dt \to 0$ for all $r$ [10]. Our results are significantly different from Eq. (2), whose prediction for $C(r)$ at $T = 0.25$ is plotted as the dashed line in Fig. 3 and should be easily detectable, unless the universal coefficient $B$ is much (two orders of magnitude) smaller than predicted.
Again, it will be difficult to attribute the discrepancies to finite size effects which would have made $C(r)$ appear slightly larger since $g(1) > g(\infty)$ [30]. Our data are also in disagreement with Eqs. (8) and (9). However, the crossover length to Eq. (9) is expected to be long [O($e^{1/\tau}$)] for small $\tau$ [11]. Here, simulations at even lower temperatures are needed to make an unequivocal test.

Finally we examined the variance of the susceptibility $\chi$, which is expected to be nonzero and universal in the glass phase [22]. However, our numerical results yield a variance that is zero within statistical error above and below $T_g$ [11]. It is again in clear disagreement with the RG prediction and in line with the very small effect of disorder observed in all of the other measurements.

To summarize, our numerical simulation of the random phase model of the randomly pinned planar flux array leads to some rather surprising results: Although the onset of nonlinear IV characteristics is observed at the expected glass transition temperature, the low temperature phase is found to be much less glassy than that predicted by the RG analysis. Both the disorder-averaged correlation function and the normalized variance of susceptibility variations are, within statistical errors, indistinguishable from those of the pure system. Our numerical results clearly rule out the RG predictions — either the universal constants $\zeta$ and $B$ are respectively at least one and two orders of magnitude smaller, or the crossover scale is at least two orders of magnitude longer. Other possible scenarios include: (1) We are prevented from reaching the asymptotics of some unknown fixed point (such as those proposed in Refs. [11,19–21]) by a very strong crossover effect. (2) The phase transition observed is purely dynamic and does not show up in the statics. The source of discrepancies between theory and simulation is very puzzling and warrants more detailed theoretical and numerical investigation.

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FIG. 1. The nonlinear part of the IV curves for $L = 64$ at $T = 0.4, 0.3, 0.25$. The solid lines are best straight line fits. The dashed line is Eq. (5) at $T = 0.25$.

FIG. 2. The exponent $\alpha$ for $L = 64$ as a function of the reduced temperature. The dashed line is $\alpha = 1 + 1.78\tau$.

FIG. 3. The correlation function for $L = 64$ at $T = 0.3, 0.285, 0.25$ with 1024 realizations each. The solid lines are $C(r) = 1.09C_0(r)$. The dashed line is Eq. (2) at $T = 0.25$ and $A = 1$. 