Steiner symmetrization and the initial coefficients of univalent functions

V. N. Dubinin

Abstract. We establish the inequality $|a_1|^2 - Re a_1 a_{-1} \geq |a_1^*|^2 - Re a_1^* a_{-1}^*$ for the initial coefficients of any function $f(z) = a_1 z + a_0 + a_{-1}/z + ...$ meromorphic and univalent in the domain $D = \{z : |z| > 1\}$, where $a_1^*$ and $a_{-1}^*$ are the corresponding coefficients in the expansion of the function $f^*(z)$ that maps the domain $D$ conformally and univalently onto the exterior of the result of the Steiner symmetrization with respect to the real axis of the complement of the set $f(D)$. The Pólya–Szegő inequality $|a_1| \geq |a_1^*|$ is already known. We describe some applications of our inequality to functions of class $\Sigma$.

Keywords: Steiner symmetrization, capacity of a set, univalent function, covering theorem.

§ 1. Introduction and statement of results

For arbitrary real $u$ we denote by $l(u)$ the vertical line $Re w = u$. By the Steiner symmetrization of a closed bounded set $E \subset \mathbb{C}_w$ with respect to the real axis we mean the result of transforming this set into a symmetric set

$E^* = \{w = u + iv : E \cap l(u) \neq \emptyset, 2|v| \leq \mu(E \cap l(u))\},$

where $\mu$ stands for linear Lebesgue measure. Let $E$ be a non-degenerate continuum and let a function

$f(z) = a_1 z + a_0 + a_{-1}/z + ...$

map the exterior $D := \{z : |z| > 1\}$ of the unit disc conformally and univalently onto the connected component of $\mathbb{C}_w \setminus E$ containing the point at infinity. We denote by

$f^*(z) = a_1^* z + a_0^* + a_{-1}^*/z + ...$

a function taking the domain $D$ conformally and univalently onto $\mathbb{C}_w \setminus E^*$. The following inequality of Pólya–Szegő [1] is well known:

$|a_1| \geq |a_1^*|,$

(1)
and is equivalent to the following inequality for the logarithmic capacities:

\[ \text{cap} \, E \geq \text{cap} \, E^*. \]

There naturally arises the question of the behaviour of the subsequent coefficients in the decomposition of \( f \) under Steiner symmetrization. There is no meaningful inequality between \( |a_0| \) and \( |a_0^*| \). Indeed, translation of the set \( E \) along the imaginary axis preserves the values \( \text{Re} \, a_0, \text{Re} \, a_0^* \), and \( \text{Im} \, a_0 = 0 \) and makes the value \( \text{Im} \, a_0 \) quite arbitrary. On the other hand, there are examples for which \( \text{Re} \, a_0 \neq \text{Re} \, a_0^* \), and then translation of \( E \) along the real axis leads to the inequality \( |\text{Re} \, a_0| < |\text{Re} \, a_0^*| \). Further, if \( E \) is a line segment forming an acute angle with the real axis, then \( |a_{-1}| > |a_{-1}^*| \). In the case when \( E = \{ w : |w| = 1 \} \), we have the reverse inequality, \( |a_{-1}| = 0 < 1/2 = |a_{-1}^*| \). We shall prove the following result.

**Theorem 1.** The functions \( f \) and \( f^* \) defined above satisfy the inequality

\[ |a_1|^2 - \text{Re} \, a_1 a_{-1} \geq |a_1^*|^2 - \text{Re} \, a_1^* a_{-1}. \]  \hspace{1cm} (2)

This inequality has the following interpretation in terms of capacity. Let a set \( E \) be symmetric with respect to the real axis and let \( H \setminus E \) be a simply connected domain, where \( H := \{ w : \text{Im} \, w > 0 \} \). We denote by \( g \) the function which takes the domain \( H \setminus E \) conformally and univalently onto the half-plane \( H \) in such a way that

\[ \lim_{w \to \infty} [g(w) - w] = 0. \]

The limit

\[ \text{hcap}(E \cap H) = \lim_{w \to \infty} w[g(w) - w] \]

is referred to as the *half-plane capacity (from infinity)* of the set \( E \cap H \) ([2], p. 69). In view of the expansion of \( f \), we conclude that \( g \) has the following form in a neighbourhood of the point at infinity:

\[ g(w) = w + \frac{a_1^2 - a_1 a_{-1}}{w} + \ldots, \]

where \( a_1 \) and \( a_{-1} \) are real numbers. Thus,

\[ \text{hcap}(E \cap H) = |a_1|^2 - \text{Re} \, a_1 a_{-1} \]

and the inequality (2) can be written in the form

\[ \text{hcap}(E \cap H) \geq \text{hcap}(E^* \cap H). \]
Theorem 1 is supplemented by the following assertion.

**Theorem 2.** Let a function \( \tilde{f}(z) = \tilde{a}_1 z + \tilde{a}_0 + \tilde{a}_{-1}/z + \ldots \) map \( D \) conformally and univalently onto the exterior of a continuum \( \tilde{E} \subset E^* \). Then

\[
|a_1^*|^2 - \text{Re } a_1^* a_{-1}^* \geq |\tilde{a}_1|^2 - \text{Re } \tilde{a}_1 \tilde{a}_{-1}.
\]  

(3)

Proofs of Theorems 1 and 2 are given in the concluding part of the paper. To obtain the inequality (2), we need the symmetrization with respect to a circle [3], as described in § 2. The inequality (3) follows from a result of Schiffer which was established using Hadamard’s formula for the variation of the Green function ([4], § 3). As applications of Theorems 1 and 2, we prove covering results for the well-known class \( \Sigma \) of functions \( f(z) = z + a_0 + a_{-1}/z + \ldots \) that are meromorphic and univalent in \( D \) [5].

**Corollary 1.** Let \( f \) be a function belonging to the class \( \Sigma \) and let \( w_0 \) be an arbitrary point of the complement \( E = \mathbb{C}_w \setminus f(D) \). Then the inequality

\[
\frac{m_f^4(w_0, \varphi) + 16R_f^4(w_0)}{8m_f^2(w_0, \varphi)} \leq 1 + \text{Re } e^{-2i\varphi} a_{-1},
\]

holds for any real number \( \varphi \), where \( R_f(w_0) \geq 0 \) stands for the radius of the largest disc centred at the point \( w_0 \) and belonging to the set \( E \), and \( m_f(w_0, \varphi) \) is the linear Lebesgue measure of the intersection of \( E \) with the line \( \{ w = w_0 + te^{i\varphi} : t \in \mathbb{R} \} \). This inequality becomes equation for the functions \( f(z) = w_0 + e^{i\varphi} \lambda^{-1} h^{-1}(\lambda h(e^{-i\varphi}z)) \) with \( h(\zeta) = \zeta + 1/\zeta \) and any \( \lambda > 1 \).

In particular, the following inequalities hold:

\[
\frac{1}{8} m_f^2 \left( w_0, \frac{\arg a_{-1}}{2} \right) - 1 \leq |a_{-1}| \leq 1 - \frac{1}{8} m_f^2 \left( w_0, \frac{\arg a_{-1} + \pi}{2} \right).
\]

The right-hand inequality refines a well-known corollary to the area theorem: \( |a_{-1}| \leq 1 \) ([5], Ch. II, §4). Both inequalities supplement the classical bound

\[
m_f(w_0, \varphi) \leq 4 \quad \forall \varphi,
\]

which follows from (1). Namely,

\[
m_f \left( w_0, \frac{\arg a_{-1}}{2} \right) \leq \sqrt{8(1 + |a_{-1}|)} \leq 4,
\]
These inequalities become equalities when \(|a_{-1}| = 1\) and \(f(z) = z + w_0 + e^{2i\varphi}/z\). It would be of interest to obtain sharp estimates for a fixed \(|a_{-1}| \neq 1\).

**Corollary 2.** Suppose that a function \(f(z) = z + a_0 + a_{-1}/z + \ldots\) of class \(\Sigma\) satisfies the inequality

\[
\mu((\mathbb{C}_w \setminus f(D)) \cap l(u)) \geq \alpha \quad \forall u, \quad \beta \leq u \leq \gamma.
\]

for some \(\alpha, \beta\) and \(\gamma\). Then

\[
\text{Re } a_{-1} \leq 1 - \frac{c^2}{2}(1 - k^2),
\]

where the real constants \(c\) and \(k\) can be found from the condition

\[
c \int_0^1 \sqrt{\frac{\zeta^2 - k^2}{\zeta^2 - 1}} \, d\zeta = \frac{\gamma - \beta}{2} - \frac{i\alpha}{2}, \quad c > 0, \quad 0 < k < 1.
\]

This inequality becomes an equality for a function \(f\) of class \(\Sigma\) mapping \(D\) conformally and univalently onto the exterior of a rectangle with sides lying on the lines \(u = \beta, u = \gamma, \gamma - \beta < 4\), and of an appropriate height \(\alpha\).

Corollaries 1 and 2 are obtained by successively applying the inequalities (2) and (3). The list of assertions of this kind can readily be extended in the same way as the well-known applications of Steiner symmetrization to function theory ([3],[6]).

**§2. Symmetrization with respect to a circle**

Following [3], § 3, we denote by \(r(w)\) the regular branch of the function \(\zeta = i \log w\) mapping the plane \(\mathbb{C}_w\) with a cut along the real negative semi-axis onto the strip \(-\pi < \text{Re } \zeta < \pi\). We define the values of the function \(r(w)\) on the cut in the sense of the boundary correspondence. Let \(E\) be an arbitrary closed set in the plane \(\mathbb{C}_w\) that does not contain both the origin and the point at infinity. By the symmetrization of \(E\) with respect to the circle \(|w| = 1\) we mean the passage from \(E\) to the symmetric set with respect to \(|w| = 1\),

\[
\text{RE} = r^{-1}((r(E))^*),
\]
where, as above, the symbol * stands for the Steiner symmetrization with respect to the real axis, carried out in the strip $-\pi \leq \operatorname{Re} \zeta \leq \pi$. We now give a direct definition of this transformation. For a closed set $E$ lying in $\mathbb{C}_w \setminus \{0\}$, we write

$$E(\theta) = E \cap \{ w : \arg w = \theta \}, \quad R(\theta) = \exp \left( \frac{1}{2} \int_{E(\theta)} \frac{d\rho}{\rho} \right),$$

$$\tilde{E}(\theta) = \begin{cases} \{ w = \rho e^{i\theta} : R^{-1}(\theta) \leq \rho \leq R(\theta) \}, & \text{if } E(\theta) \neq \emptyset, \\ \emptyset, & \text{if } E(\theta) = \emptyset. \end{cases}$$

It can readily be seen that

$$RE = \bigcup_{0 \leq \theta \leq 2\pi} \tilde{E}(\theta).$$

For a fixed $v > 0$ we denote by $L_v(w) := w/v - i$ the linear transformation taking the circle $|w - iv| = v$ to the unit circle $|w| = 1$ and $L_v^{-1}$ the inverse map. By the result of the symmetrization of a closed bounded set $E$ with respect to the circle $|w - iv| = v$ we mean the set

$$R_vE = L_v^{-1}(RL_v(E)).$$

For an open set $B$ containing the points $iv$ and $\infty$, we write

$$S_vB = \overline{\mathbb{C}}_w \setminus R_v(\overline{\mathbb{C}}_w \setminus B).$$

**Lemma 1.** If open sets $B_1$ and $B_2$ satisfy the conditions $\infty \in B_1$ and $\overline{B}_1 \subset B_2$, then the inclusion relation

$$S_vB_1 \subset \overline{\mathbb{C}}_w \setminus (\overline{\mathbb{C}}_w \setminus B_2)^* \quad \quad (R_v(\overline{\mathbb{C}}_w \setminus B_1) \supset (\overline{\mathbb{C}}_w \setminus B_2)^*).$$

holds for all sufficiently large $v > 0$.

The proof of Lemma 1 is clearly of a technical nature, and therefore we omit it. We only note the importance of the condition that $B_1$ is contained in a compact subset of $B_2$. Then the closed set $\overline{\mathbb{C}}_w \setminus B_2$ is contained in $\overline{\mathbb{C}}_w \setminus B_1$ together with some neighbourhood $U$. Near the real axis, the rays passing through the point $iv$ and intersecting the neighbourhood $U$ tend to lines parallel to the imaginary axis as $v \to \infty$. Here the ‘logarithmic measure’ in a neighbourhood of the circle $|w - iv| = v$ tends to the Euclidean measure.
Let \( g_B(z, z_0) \) denote the Green function of the connected component of \( B \) which contains the point \( z_0 \) (with a pole at this point), where \( g_B(z, z_0) \) is defined to be zero outside this connected component. Let \( r(B, z_0) \) be the inner radius of the above component with respect to the point \( z_0 \) [3].

**Lemma 2.** If the connected components of the open set \( B \) have Green’s functions and the points \( iv \) and \( \infty \) belong to \( B \), then

\[
\log[r(B, iv)r(B, \infty)] + 2g_B(iv, \infty) \leq \log[r(S_v B, iv)r(S_v B, \infty)] + 2g_{S_v B}(iv, \infty)
\]

for any \( v > 0 \). Proof. This follows from [3], Theorem 1.7, Proposition 1.11 (see also [7], Theorem 1).

§3. Proofs

**Proof of Theorem 1.** Let \( \{B_n\}_{n=1}^\infty \) be an exhaustion of the domain \( f(D) \) by simply connected domains \( B_n \), where \( \infty \in B_n, \overline{B_n} \subset B_{n+1}, n = 1, 2, \ldots, \) and \( \bigcup_{n=1}^\infty B_n = f(D) \). For any \( n \) and sufficiently large \( v > 0 \), the function \( u_n(w) := g_{B_n}(w, iv) - g_{B_n}(w, \infty) \) is harmonic in the domain \( B_n \setminus \{iv, \infty\} \), and we have \( u_n(w) \to +\infty \) as \( w \to iv \) and \( u_n(w) \to -\infty \) as \( w \to \infty \). In particular, this implies that the sets \( B_1^n := \{w : u_n(w) > 0\} \) and \( B_2^n := \{w : u_n(w) < 0\} \) are disjoint domains. The Green function of the domain \( B_1^n \) with a pole at the point \( iv \) coincides with the function \( u_n(w) \) on \( B_1^n \). Hence,

\[
\log r(B_1^n, iv) = \lim_{w \to iv} (u_n(w) + \log|w - iv|) = \log r(B_n, iv) - g_{B_n}(iv, \infty).
\]

Similarly, the function \(-u_n(w)\) coincides on \( B_2^n \) with the Green function of this domain with a pole at the point \( w = \infty \). Therefore,

\[
\log r(B_2^n, \infty) = \lim_{w \to \infty} (-u_n(w) - \log|w|) = \log r(B_n, \infty) - g_{B_n}(\infty, iv).
\]

Adding these relations, we obtain

\[
\log[r(B_n, iv)r(B_n, \infty)] - 2g_{B_n}(iv, \infty) = \log[r(B_1^n, iv)r(B_2^n, \infty)] =
\]

\[
= \log[r(B_1^n \cup B_2^n, iv)r(B_1^n \cup B_2^n, \infty)] + 2g_{B_1^n \cup B_2^n}(iv, \infty).
\]

By Lemma 2, the last expression does not exceed the sum

\[
\log[r(S_v(B_1^n \cup B_2^n), iv)r(S_v(B_1^n \cup B_2^n), \infty)] + 2g_{S_v(B_1^n \cup B_2^n)}(iv, \infty).
\]
Since $B_n^1 \cap B_n^2 = \emptyset$, it follows that for any $\theta$ the ray $w = iv + \rho e^{i\theta}$, $0 \leq \rho \leq \infty$, meets the set $\overline{C_w \setminus (B_n^1 \cup B_n^2)}$. Hence, by the definition of symmetrization with respect to a circle, the set $R_v(\overline{C_w \setminus (B_n^1 \cup B_n^2)})$ contains the circle $|w - iv| = v$. Therefore, the connected components $\tilde{B}_n^1$ and $\tilde{B}_n^2$ of the set $S_v(B_n^1 \cup B_n^2)$ that contain the points $iv$ and $w = \infty$, respectively, are disjoint, and we have

$$r(S_v(B_n^1 \cup B_n^2), iv) = r(\tilde{B}_n^1, iv), \quad r(S_v(B_n^1 \cup B_n^2), \infty) = r(\tilde{B}_n^2, \infty),$$

$$g_{S_v(B_n^1 \cup B_n^2)}(iv, \infty) = g_{\tilde{B}_n^1 \cup \tilde{B}_n^2}(iv, \infty) = 0.$$ 

We finally obtain

$$\log[r(B_n, iv)r(B_n, \infty)] - 2g_{B_n}(iv, \infty) \leq \log[r(\tilde{B}_n^1, iv)r(\tilde{B}_n^2, \infty)] - 2g_{\tilde{B}_n^1 \cup \tilde{B}_n^2}(iv, \infty) = \log[r(S_v B_n, iv)r(S_v B_n, \infty)] - 2g_{S_v B_n}(iv, \infty). \quad (5)$$

The last equation is established in the same way as (4) in view of the symmetry of the domain $S_v B_n$ with respect to the circle $|w - iv| = v$. Further, we shall use the following fact, which was proved for the first time by Schiffer [4, §3]: if domains $G_1$ and $G_2$ admit Green functions, $G_1 \subset G_2$, and $\zeta$ and $w$ are distinct points of $G_1$, then

$$\log[r(G_1, \zeta)r(G_1, w)] - 2g_{G_1}(\zeta, w) \leq \log[r(G_2, \zeta)r(G_2, w)] - 2g_{G_2}(\zeta, w). \quad (6)$$

By Lemma 1,

$$S_v B_n \subset \overline{C_w \setminus (\overline{C_w \setminus f(D)})^*} \subset f^*(D).$$

Thus, by (5) and (6), the following inequality holds:

$$\log[r(B_n, iv)r(B_n, \infty)] - 2g_{B_n}(iv, \infty) \leq \log[r(f^*(D), iv)r(f^*(D), \infty)] - 2g_{f^*(D)}(iv, \infty). \quad (7)$$

We denote by

$$f_n(z) = a_1^n z + \frac{a_0^n}{z} + \ldots$$

a function which maps $D$ conformally and univalently onto the domain $B_n$. Let

$$h_n(w) = \frac{w}{a_1^n} - \frac{a_0^n}{a_1^n} - \frac{a_{-1}^n}{w} + \ldots$$

be the expansion of the inverse map in a neighbourhood of the point at infinity. Then the following equations hold for any sufficiently large $v$:

$$r(B_n, \infty)|a_1^n| = r(D, \infty) = 1, \quad r(B_n, iv)|h_n(iv)| = r(D, h_n(iv)) = |h_n(iv)|^2 - 1,$$
\[g_{B_n}(iv, \infty) = g_D(h_n(iv), \infty) = \log |h_n(iv)|.\]

This gives
\[
r(B_n, iv)r(B_n, \infty)e^{-2g_{B_n}(iv, \infty)} = \frac{|h_n(iv)|^2 - 1}{|a_1^n h'_n(iv)||h_n(iv)|^2} =
\]
\[
= \frac{1 - \left|\frac{iv}{a_1^n} + O(1)\right|^{-2}}{1 - \frac{a_1^n a_{-1}^n}{v^2} + o\left(\frac{1}{v^2}\right)} = 1 - (|a_1^n|^2 - \Re a_1^n a_{-1}^n)\frac{1}{v^2} + o\left(\frac{1}{v^2}\right), \quad v \to +\infty.
\]

Repeating the above manipulations with the function \(f^*\) instead of \(f_n\), we see from the inequality (7) that
\[|a_1^n|^2 - \Re a_1^n a_{-1}^n \geq |a_1^*|^2 - \Re a_1^* a_{-1}^*.
\]

Passing to the limit as \(n \to \infty\), we obtain the inequality (2). This completes the proof of the theorem.

Proof of Theorem 2. This follows from inequality (6), where one must set \(G_1 = f^*(D)\), \(G_2 = \tilde{f}(D)\), \(\zeta = iv\) and \(w = \infty\), and where \(v\) is sufficiently large. By repeating for \(f^*\) and \(\tilde{f}\) the calculations for \(f_n\) in the last part of the proof of Theorem 1, we finally obtain the inequality (3).

Proof of Corollary 1. We first consider the case in which \(w_0 = 0\) and \(\varphi = \pi/2\). The function
\[
\tilde{f}(z) := iR_f(0)h^{-1}(\lambda h(z)) = i\lambda R_f(0)z + \frac{iR_f(0)}{z}\left(\lambda - \frac{1}{\lambda}\right) + \ldots
\]
maps \(D\) conformally and univalently onto the complement of the set \(\tilde{E} := \{w : |w| \leq R_f(0)\} \cup \{w : \Re w = 0, \ |\Im w| \leq m_f(0, \pi/2)/2\}\) when \(\lambda = \left[m_f^2(0, \pi/2)/(4R_f^2(0)) + 1\right]/[m_f(0, \pi/2)/R_f(0)]^{-1}\). Under the hypotheses of the corollary, we have \(\tilde{E} \subset E^* (E = \overline{\mathbb{C}}_w \setminus f(D))\). Therefore, it follows from the inequalities (2) and (3) that
\[
1 - \Re a_{-1} \geq R_f^2(0)(2\lambda^2 - 1) = \frac{m_f^4(0, \pi/2) + 16R_f^4(0)}{8m_f^2(0, \pi/2)}.
\]

If \(w_0\) and \(\varphi\) are arbitrary, one must apply the above conclusion to the function
\[
e^{i(\pi/2 - \varphi)}[f(e^{i(\varphi - \pi/2)}z) - w_0] = z + \tilde{a}_0 - e^{-2i\varphi}a_{-1}/z + \ldots.
\]
The condition for the equation can be verified immediately.

Proof of Corollary 2. We may assume that $\beta = -\gamma$. The function

$$F(\zeta) = c \int_0^\zeta \sqrt{\frac{\zeta^2 - k^2}{\zeta^2 - 1}} d\zeta + \frac{i\alpha}{2} = c\zeta + c_0 - \frac{c(1-k^2)}{2\zeta} + \ldots$$

maps the upper half-plane $\text{Im}\, \zeta > 0$ conformally and univalently onto the quadrangle $\{w : \text{Im} w > 0\} \setminus \tilde{E}$, where we now have $\tilde{E} = \{w : |\text{Re} w| \leq \gamma, |\text{Im} w| \leq \alpha/2\}$. By the Riemann–Schwarz symmetry principle, the function

$$\tilde{f}(z) := F\left(\frac{1}{2}(z + \frac{1}{z})\right) = \frac{c}{2}z + c_0 - c\left(\frac{1}{2} - k^2\right)\frac{1}{z} + \ldots$$

maps $D$ conformally and univalently onto the exterior of the rectangle $\tilde{E}$. Under the hypotheses of the corollary, we have $\tilde{E} \subset (\overline{\mathbb{C}w \setminus f(D)})^*$. It remains to use the inequalities (2) and (3).

Bibliography

1. G. Pólya and G. Szegő, *Isoperimetric inequalities in mathematical physics*, Ann. of Math. Stud., vol. 27, Princeton Univ. Press, Princeton, NJ 1951; Russian transl., Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow 1962.

2. G. F. Lawler, *Conformally invariant processes in the plane*, Math. Surveys Monogr., vol. 114, Amer. Math. Soc., Providence, RI 2005.

3. V.N. Dubinin, “Symmetrization in the geometric theory of functions of a complex variable”, *Uspekhi Mat. Nauk* 49:1 (1994), 3–76; English transl., *Russian Math. Surveys* 49:1 (1994), 1–79.

4. M. Schiffer, “Some new results in the theory of conformal mappings”, Appendix to the book: R. Courant, *Dirichlet’s Principle, conformal mappings, and minimal surfaces*, Interscience, New York 1950; Russian transl., Inostr. Lit., Moscow 1953.

5. G. M. Goluzin, *Geometric theory of functions of a complex variable*, Nauka, Moscow 1966; English transl., Transl. Math. Monogr., vol. 26, Amer. Math. Soc., Providence, RI 1969.
6. W. K. Hayman, *Multivalent functions*, Cambridge Univ. Press, Cambridge 1958; Russian transl., Inostr. Lit., Moscow 1960.

7. V. N. Dubinin, “Some properties of the reduced inner modulus”, *Sibirsk. Mat. Zh.* 35:4 (1994), 774–792; English transl., *Siberian Math. J.* 35:4 (1994), 689–705.