The Volume Entropy of a Riemannian Metric Evolving by the Ricci Flow on a Manifold of Dimension 3 or Above

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Abstract

In this paper it is proven that the volume entropy of a riemannian metric evolving by the Ricci flow, if it does not collapse, nondecreases. Therefore it provides a sufficient condition for a solution to collapse. Then, for the limit solutions of type I or III, the limit entropy is the limit of the entropy as $t$ approaches the singular (finite or not) time.

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1 Introduction

Consider on a compact manifold $M$ of dimension greater the 3 a riemannian metric evolving by the Ricci Flow: $\frac{\partial}{\partial t} g(t) = -2Rc_g(t)$. In [MANN] A. Manning studies the volume entropy of a surface evolving under the normalized Ricci Flow. Here, the author studies the volume entropy of a manifold of dimension 3 or above, evolving by the unnormalized flow, in connection with collapsing riemannian manifolds. In [PER] G. Perelman proves a non-collapsing theorem for a solution to the Ricci flow that develops finite time singularities. Despite this, a solution $g(t)$ that exists $\forall t$, could collapse. The present paper deals with this case.

Let $\pi: \tilde{M} \to M$ be the universal cover of $M$. Geometric data on $\tilde{M}$ will be denoted with tilde (such as $\tilde{R}$ for the scalar curvature) to be distinguished from the data on $M$. Consider The Ricci Flow on $\tilde{M}$: $\frac{\partial}{\partial t} \tilde{g}(t) = -2Rc_{\tilde{g}(t)}$ and the one on $M$: $\frac{\partial}{\partial t} g(t) = -2Rc_{g(t)}$. If $\tilde{g}(0)$ is the covering riemannian metric of $g(0)$ then, as symmetries are preserved by the Ricci flow, $\tilde{g}(t)$ will be the covering riemannian metric of $g(t)$, for all $t$ for which the two solution exist.
By definition, the volume growth function of a compact riemannian manifold \((M, g)\) is 
\[
\omega(M, g, r) = \frac{1}{r} \log Vol B(r),
\]
where \(B(r)\) is the radius \(r\) ball in the universal cover \((\tilde{M}, \tilde{g})\) (the center of the ball doesn’t matter). The volume entropy is the quantity 
\[
h(M, g) = \lim_{r \to \infty} \omega(M, g, r).\]

## 2 Evolution of The Volume Entropy

**Theorem 2.1.** If the injectivity radius \(i(M, g(t)) \geq i \in \mathbb{R}\) uniformly in time then the volume entropy \(h(M, g(t))\) is nondecreasing. If 
\[
\int_{B(\tau(t))} \tilde{R}d\tilde{\mu}_{\tilde{g}(t)} - \int_{\tilde{B}(\tau(t))} \tilde{H}d\tilde{\mu}_{\tilde{g}(t)} \geq 0
\]
for some positive \(C\), where \(\tilde{H}\) is the injectivity radius of the universal cover \((\tilde{M}, \tilde{g})\) (the center of the ball doesn’t matter). The volume entropy is the quantity 
\[
h(M, g) = \lim_{r \to \infty} \omega(M, g, r).\]

**Proof.** Consider the volume growth function of \((M, g(t))\), \(\omega(M, g(t), r) = \frac{1}{r} \log Vol \tilde{B}(r(t))\). We have that 
\[
\begin{align*}
\frac{\partial}{\partial t} Vol \tilde{B}(r(t)) &= \lim_{h \to 0} \frac{1}{h} \left[ \int_{B(r(t+h))} d\tilde{\mu}_{\tilde{g}(t+h)} - \int_{B(r(t))} d\tilde{\mu}_{\tilde{g}(t)} \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[ \int_{B(r(t+h))} d\tilde{\mu}_{\tilde{g}(t+h)} - \int_{B(r(t))} d\tilde{\mu}_{\tilde{g}(t)} \right]
\end{align*}
\]
we shall have 
\[
\frac{\partial}{\partial t} Vol \tilde{B}(r(t)) = - \int_{B(r(t))} \tilde{R}d\tilde{\mu}_{\tilde{g}(t)} + \lim_{h \to 0} \frac{1}{h} \int_{B(r(t+h))} d\tilde{\mu}_{\tilde{g}(t)}.
\]
Thus, since the second term vanishes, we obtain the evolution 
\[
\frac{\partial}{\partial t} \omega(M, g(t), r) = - \frac{1}{r} \frac{1}{Vol(B(r(t)))} \int_{B(r(t))} \tilde{R}d\tilde{\mu}_{\tilde{g}(t)}
\]
for the volume growth function. Now, the injectivity radius of the universal riemannian cover satisfies 
\[
i(M, \tilde{g}(t)) \geq i(M, g(t)) \geq i\]
so, by a theorem from [SHEN] that generalizes [GREEN] we have that 
\[
\lim_{r \to \infty} \frac{1}{Vol(B(r(t)))} \int_{B(r(t))} \tilde{R}d\tilde{\mu}_{\tilde{g}(t)} \leq n(n - 1)C
\]
for some positive \(C \in \mathbb{R}\), so 
\[
- \lim_{r \to \infty} \frac{1}{Vol(B(r(t)))} \int_{B(r(t))} \tilde{R}d\tilde{\mu}_{\tilde{g}(t)} \geq -n(n - 1)C.
\]
Therefore, from equation \([2]\) for \(r \to \infty\), we get 
\[
\frac{\partial}{\partial t} h(M, g(t)) \geq 0.
\]
For the second assertion, we have that \( \frac{\partial^2}{\partial r^2} \omega(M, g(t), r) = \frac{2}{r^2} \omega(M, g(t), r) - \frac{1}{r^2 V ol B(r(t))} \int V ol^2 \tilde{S}(r(t)) \frac{1}{r^2 V ol B(r(t))} \int \hat{H}d\sigma_{\tilde{g}(t)} \), where \( d\sigma_{\tilde{g}(t)} \) is the restriction of the riemannian measure of the universal cover to the sphere \( \tilde{S}(r(t)) \), and \( \hat{H} \) denotes the mean curvature of the sphere. Thus, via equation (2), we get

\[
\frac{\partial}{\partial t} \omega(M, g(t), r) = \frac{\partial}{\partial r^2} \omega(M, g(t), r) - \frac{2}{r^2} \omega(M, g(t), r) + \frac{Vol \tilde{S}(r(t))}{r^2 V ol B(r(t))} + \frac{Vol^2 \tilde{S}(r(t))}{r V ol B(r(t))} + \frac{1}{r V ol B(r(t))} \left( \int_{\tilde{B}(r(t))} \hat{R}d\mu_{\tilde{g}(t)} - \int_{\tilde{S}(r(t))} \tilde{H}d\mu_{\tilde{g}(t)} \right) \tag{3}
\]

and this is a heat equation! Now, assuming by hypothesis that \( \int_{\tilde{B}(r(t))} \hat{R}d\mu_{\tilde{g}(t)} - \int_{\tilde{S}(r(t))} \tilde{H}d\mu_{\tilde{g}(t)} \geq 0 \) \forall r, in equation (3) above the last three terms are positive, which implies that

\[
\frac{\partial}{\partial t} \omega(M, g(t), r) \geq \frac{\partial}{\partial r^2} \omega(M, g(t), r) - \frac{2}{r^2} \omega(M, g(t), r). \tag{4}
\]

In this point of the proof, we can apply the maximum principle for a supersolution to the equation (4). In order to do this, consider the ODE \( \frac{dx}{dt} = -\frac{2}{r^2} x \) which has the solution \( x = Ce^{(-2/r^2)t} \). By the maximum principle, the ODE gives pointwise bounds to the PDE, so

\[
\omega(M, g(t), r) \geq \omega(M, g(0), r)e^{(-2/r^2)t}
\]

and this happens for all \( r \). Taking limit as \( r \to \infty \), we get

\[
h(M, g(t)) \geq h(M, g(0)). \tag{5}
\]

Let \( t_k \not\to \infty \). By repeating the argument above with \( t_k \) as origin, we obtain that \( h(M, g(t)) \) is nondecreasing. \( \square \)

**Remark 2.2.** On a gradient Ricci soliton, the volume entropy is constant. Indeed, a gradient Ricci soliton satisfies

\[
-R = \Delta f + \frac{n\epsilon}{2t}
\]

Where \( \epsilon = -1, 0, 1 \) according to the gradient Ricci soliton is shrinking, steady or expanding. Then, by equation (2), since by the divergence theorem we have \( \int_{\tilde{B}(r(t))} \Delta f d\mu_{\tilde{g}(t)} = 0 \), follows that

\[
\frac{\partial}{\partial t} \omega(M, g(t), r) = \frac{1}{r V ol(B(r(t)))} \int_{B(r(t))} \Delta f \, d\mu_{\tilde{g}(t)} = \frac{n\epsilon}{2} \frac{d}{dt} h(M, g(t)) = 0
\]

thus, by taking limit as \( r \to \infty \), follows that \( \frac{d}{dt} h(M, g(t)) = 0. \)
3 The Volume Entropy and The Limit Solutions

Lemma 3.1. Under a parabolic rescaling of the solution to the Ricci flow the volume growth function changes by

\[ \omega(M, g_i(t), r) = \frac{n}{2} \log \frac{|Rm(x_i, t_i)|}{r} + \omega(M, g(t_i + \frac{t}{|Rm(x_i, t_i)|}), r). \]  

(6)

Proof. Since a riemannian covering \( \tilde{M} \to M \), being a local isometry, has \(|\tilde{Rm}(y)| = |Rm(\pi(y))|\), we shall omit the tilde in the following computation, to simplify notations.

Let \((x_i, t_i)\) a sequence of points \(x_i \in M\) and times \(t_i \to T \in (0, \infty]\) which converges to the singular time \(T\). Then, a parabolic rescaling of the metric is given by the formula

\[ g_i(t) = \frac{|Rm(x_i, t_i)|}{g(t_i + \frac{t}{|Rm(x_i, t_i)|})}. \]

(7)

Hence, the volume of balls changes according to

\[ \int_{B(x_i, r)} d\mu_{g_i(t)} = \int_{B(x_i, r)} |Rm(x_i, t_i)|^{n/2} d\mu_{g(t_i + \frac{t}{|Rm(x_i, t_i)|})}(x, r) \]

so

\[ \omega(M, g_i(t), r) = \frac{1}{r} \log VolB_{g_i(t)}(x_i, r) = \frac{1}{r} \log(|Rm(x_i, t_i)|^{n/2} VolB_{g(t_i + \frac{t}{|Rm(x_i, t_i)|})}(x_i, r)) \]

(8)

and equation (6) follows. \(\square\)

Let \((M_\infty, g_\infty(t))\) be the limit solution.

Proposition 3.2. If \((M, g(t))\) is a type I or III solution to the Ricci flow, then \(h(M, g_\infty) = \lim_{t \to T} h(M, g(t))\).

Proof. Consider a type I solution, i.e. a solution that exhibits a singularity in finite time \(T\) and \((T-t)|Rm(x, t)| \leq \infty\). Assume \(|Rm(x_i, t_i)|\) has not polynomial growth, i.e. \(|Rm(x_i, t_i)| \leq \frac{1}{(T-t_i)^p}\) for some positive \(p\), hence \((T-t_i)^p|Rm(x_i, t_i)| = (T-t_i)^{p-1}(T-t_i)|Rm(x_i, t_i)| \geq 1\), so \((T-t_i)^{p-1}\) finite \(\geq 1\) and this contradicts the fact that \(t_i \to T\). Therefore \(|Rm(x_i, t_i)|\) has polynomial growth, and by equation (6)

\[ \lim_{i \to \infty} \omega(M, g_i(t), r) = \lim_{i \to \infty} \omega(M, g(t_i + \frac{t}{|Rm(x_i, t_i)|}), r) \]

(9)
so $h(M_\infty, g_\infty(t)) = \lim_{t \to T} h(M, g(t))$.

Consider now a type III solution, i.e. a solution that does not exhibit a singularity in finite time, and $t |Rm(x, t)| \leq \infty$. Assume again $|Rm(x_i, t_i)|$ has not polynomial growth, i.e. $|Rm(x_i, t_i)| \geq t_i^p$ for some positive $p$ and repeat the above reasoning. Follows that $h(M_\infty, g_\infty(t)) = \lim_{t \to \infty} h(M, g(t))$ which completes the proof. □

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