A SPECIAL CONIC ASSOCIATED WITH THE REULEAUX NEGATIVE PEDAL CURVE

LILIANA GABRIELA GHEORGHE AND DAN REZNIK

Abstract. The Negative Pedal Curve of the Reuleaux Triangle w.r. to a point $M$ on its boundary consists of two elliptic arcs and a point $P_0$. Interestingly, the conic passing through the four arc endpoints and by $P_0$ has a remarkable property: one of its foci is $M$. We provide a synthetic proof based on Poncelet’s polar duality and inversive techniques. Additional intriguing properties of Reuleaux negative pedal are proved using straightforward techniques.

1. Introduction

The Reuleaux triangle $R$, is the (convex) curve formed by the arcs of three circles of equal radius $r$, centered on the vertices $V_1, V_2, V_3$ of an equilateral triangle and that mutually intercepts in these vertices. This triangle is mostly known due to its constant width propriety [3, Reuleaux Triangle].

Here, we study some proprieties of the negative pedal curve $\mathcal{N}$ of $R$ w.r. to a pedal point $M$ lying on one of its sides. This curve is the envelope of lines passing through points $P \in R$ and perpendicular to $PM$, [3, Negative Pedal Curve].

Let $V_3$ denote the center of the circular arc where $M$ lies, and let $V_1, V_2$ the endpoints of said side. Let the arc $A_1A_2$ (resp. $B_1B_2$) be the negative pedal image of the Reuleaux side $V_1V_3$ (resp. $V_2V_3$) where $A_1$ is the image of $V_1$, and $B_1$ the image of $V_2$. The endpoints of $\mathcal{N}$, whose preimage is $V_3$ are respectively $A_2$, when $V_3$ is regarded as a point of the side $V_1V_3$ and $B_2$, when $V_3$ is regarded as a point of the side $V_2V_3$ of the Reuleaux triangle. The negative pedal of arc $V_1V_2$ reduces to a point, $P_0$.

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Main Result. Our main result (Theorem 1, Section 2) is a synthetic proof, based on polar duality and inversive techniques, for a instigating property of the conic $C^*$ that passes through the endpoints $A_1, A_2, B_1, B_2$ of the negative pedal curve, $\mathcal{N}$, and through $P_0$ : that one of its foci is precisely the pedal point, $M$; (see Figure 1).

We also give full description of its other geometric elements (axes, directrix and vertices) and criterion identify its type, according to the location of the pedal point $M$.

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A review of polar reciprocity and other concepts is postponed in Appendix including a definition of the negative pedal curve as a loci, as well as an alternative description as an envelope of lines (Proposition 12).

**Further Results.** In Section 3 we prove other properties of the Reuleaux and its negative pedal curve, involving tangencies, collinearities and homotheties.

2. **Main Result: The Endpoint Conic**

In this section we prove our main result (Theorem 1), that the conic which passes through the five endpoints $A_1, A_2, B_1, B_2$ and $P_0$ of the negative pedal curve of the Reuleaux w.r. to a point $M$ on its boundary has a focus which coincides with $M$.

The proof will require some additional steps which repeatedly use an inversive approach, based on polar duality. Therefore, the reader not familiar with this topic may find useful the details in Appendix, and the references therein.
Perform a polar transform (or a polar duality) w.r. to an inversion circle centered in \( M \) (see Figure 2). Thus, points on the end conic transform into their polars w.r. to the inversion circle, tangent to the dual curve.

By polar duality, the dual of a conic is a circle iff the center of the inversion circle (w.r. to whom the duality is performed) is at the focus of the conic.

Thus, instead of attempting to prove directly that the focus of the end conic is \( M \), we perform a dual transform w.r. to an inversion circle centered on \( M \) and show that there exists a circle to whom the five polars of \( A_1, A_2, B_1, B_2 \) and \( P_0 \) are tangent. Therefore, we need to prove the following.

**Proposition 1.** Let \( R \) be a Reuleaux triangle with vertices \( V_1, V_2, V_3 \). Let \( M \) be a point on the arc \( V_1V_2 \) and fix an inversion circle centered on \( M \). Let \( V_1', V_2', V_3' \) be the inverses of \( V_1, V_2, V_3 \) w.r. to the inversion circle; finally, let \( C \) be the exinscribed circle in \( \triangle V_1'V_2'V_3' \), externally-tangent to side \( V_1'V_2' \). Then the polars of \( A_1, A_2, B_1, B_2 \), and \( P_0 \) are tangent to \( C \).

This indirect approach is feasible since the polars of \( A_1, A_2, B_1, B_2 \), and \( P_0 \) are traceable. We find convenient to state it explicitly.

Referring to Figure 2.

**Lemma 1.** Using the above notation:
i) The polars of \( A_1 \) and \( B_1 \) are the tangents at \( V_1' \) and \( V_2' \) to circular arcs \( V_1'V_3' \) and \( V_2'V_3' \), the inverses of arcs \( V_1V_3 \) and \( V_2V_3 \) of the Reuleaux, respectively.

The poles of the tangents at \( V_1' \) and \( V_2' \) to the arcs \( V_1'V_3' \) and \( V_2'V_3' \), respectively, are the points \( A_1, B_1 \).

ii) The polars of the points \( A_2, B_2 \) are the tangents in \( V_3' \) to arcs \( V_1'V_3' \) and \( V_2'V_3' \), respectively.

The poles of the tangents at \( V_3' \) to arcs \( V_1'V_3' \) and \( V_2'V_3' \), respectively, are the points \( A_2, B_2 \).

iii) The polar of \( P_0 \) is the line \( V_1'V_2' \).

**Proof.** (see Figure 2) Inversion w.r. to a circle maps circles to either circles or lines: thus, the inverse of arcs \( V_1V_3 \) and \( V_2V_3 \) are the two circular arcs \( V_1'V_3' \) and \( V_2'V_3' \), respectively. On the other hand, since the arc \( V_1V_2 \) passes through the inversion center, its image is a union of two half-lines: the line \( V_1'V_2' \) excluding segment \([V_1'V_2']\).

All the other statements are straightforward consequences of the description of the negative pedal curve as a locus, shown in Proposition 13.

Finally, the fact that the negative pedal is the dual of its inverse, (see Appendix, Proposition 12) completes the proof.

Using the same notation as in the previous Lemma:

**Lemma 2.** i) the angles at \( V_1', V_2', \) and \( V_3' \) between arcs \( V_1'V_2', V_1'V_3', \) and \( V_2'V_3' \), respectively (these are the inverses of the sides of the Reuleaux \( R \)), are \( 120^\circ \).

ii) \( \triangle V_1'O'V_2' \) determined by the tangents at \( V_1', V_2' \) to said arcs and the line \( V_1'V_2' \), is equilateral.

**Proof.** This is a direct consequence of the fact that inversion preserves angles between curves. 

Figure 2. The Reuleaux $\mathcal{R}$ triangle and its inverse: (i) the arcs $V'_1V'_3$ (green) and $V'_2V'_3$ (blue) are the inverses of sides $V_1V_3$ and $V_2V_3$ of $\mathcal{R}$ while line $V'_1V'_2$ except for the segment $[V'_1V'_2]$ itself (violet) is the image of arc $V_1V_2$; (ii) the polars of $A_1$ and $B_1$ are the tangents in $V'_1$ and $V'_2$ to arcs $V'_1V'_3$ and $V'_2V'_3$; (iii) the polars of $A_2$ and $B_2$ are the tangents in $V'_3$ to arcs $V'_1V'_3$ and $V'_2V'_3$; (iv) line $V'_1V'_2$ is the polar of $P_0$. (v) all five polars are tangent to the exinscribed circle of $\triangle V'_1O'V'_2$ (purple). The angle between circles $c_0$ (green) and $C_0$ (blue) in $V'_3$ is 120° iff $V'_3$ is on the circle centered in $O$ passing through $V'_1$ and $V'_2$; in this case the tangents at $V'_3$ (dashed green and blue) to circles $c_0$ and $C_0$, are also tangent to the exinscribed circle $c$ (purple) of $\triangle V'_1O'V'_2$.

Proposition 1, hence the assertion that the focus of $C^*$ coincides with the pedal point $M$ is therefore proved, once it is shown that the two tangents at $V'_3$ to arcs $V'_1V'_3$ and $V'_2V'_3$ respectively, also tangent the excircle of triangle $V'_1O'V'_2$. This can be restated as follow.

**Lemma 3** (a scalene lemma). Let $\triangle V'_1O'V'_2$ be an equilateral triangle and let $O$ be the center of its exinscribed triangle, $c$, externally-tangent to side $[V'_1V'_2]$. Let $C$ be a circle (also) centered on $O$ that passes through $V'_1$ and $V'_2$.

Finally, let $c_0$ and $C_0$, be the two circles tangent to the sides $[OV'_1]$ and $[OV'_2]$ at $V'_1$ and $V'_2$, respectively. Then

i) $c_0$ and $C_0$ intersect at an angle of 120° iff the three circles $c_0$, $C_0$, and $C$ pass through one common point.

ii) if the condition above is fulfilled, then the two tangents at the common point $V'_3$ to circles $c_0$ and $C_0$ are also tangent to the exinscribed circle $c$. 
A SPECIAL CONIC ASSOCIATED WITH THE REULEAUX NEGATIVE PEDAL CURVES

See Figure 2

Assertion (i) is fulfilled if we identify circles $c_0$ and $C_0$ with inverses of the sides of the Reuleaux triangle. Assertion (ii) is not obvious and require some additional steps. We shall give a symmetric proof to this apparently scalene result, based on a rudimentary (yet useful) form of Poncelet’s porism for regular hexagons.

**Lemma 4** (a poristic fact). Let $[A_0A_1\ldots A_5]$ a regular hexagon with inscribed circle $c$ and circumcircle $C$. Let $P_0$ be a point on arc $A_0A_1$ of $C$ and let $P_0P_1$, $P_1P_2$, $\ldots$, $P_5P_0$ be the tangents from $P_0$, $P_1$, $\ldots$, $P_5$ to $c$.

Let $c_0$ be the circle tangent to the side $[A_0A_5]$ of the hexagon at $A_0$ and intersecting $C$ at $P_0$ (and $A_0$). Then:

i) Points $P_6$ and $P_0$ coincide and hexagon $[P_0P_1\ldots P_5]$ is regular and congruent with hexagon $[A_0A_1\ldots A_5]$. Both hexagons share the same incircle and circumcircle.

ii) The line $P_0P_1$, from $P_0$ to $c$, also tangents circle $c_0$.

**Proof.** i) When we perform the construction of the tangent lines $P_0P_1$, $\ldots$, $P_5P_6$ the process will end in five steps thanks to Poncelet’s porism, since $c$ and

![Figure 3. The angle between circles $c_0$ (green) and $C_0$ (blue) in $P_0$ is 120° iff $P_0$ is on the circumcircle of the hexagon $[A_0A_1\ldots A_5]$.](image)
\textbf{C} are the incircle and the circumcircle of a hexagon. The regularity of this hexagon is due to the fact that its inscribed and circumscribed circles are concentric.

In fact, a straightforward proof, whose details we omit, can also be used to show that segment \( P_0P_1 \) has the same length as \( A_0A_1 \), hence it is the side of another regular hexagon inscribed in \( \textbf{C} \) and circumscribed to \( \textbf{c} \).

ii) Let \( T_0 \) be the intersection of the perpendicular bisector of segment \([A_0P_0]\) and line \( A_0A_5 \). We shall prove that \( T_0P_0 \) is tangent at \( P_0 \) to circle \( \textbf{c}_0 \) and that points \( T_0, P_0, P_1 \) are collinear.

Let \( \textbf{c}_0 \) be the center of circle \( \textbf{c}_0 \); since \( T_0 \) is a point on the perpendicular bisector of \([A_0P_0]\), and since \( A_0 \) and \( P_0 \) are the two intersections of circles \( \textbf{c}_0 \) and \( \textbf{c} \), then points \( O, \textbf{c}_0 \) and \( T_0 \) are collinear.

Next, \( \triangle T_0A_0c_0 = \triangle T_0P_0c_0 \) as they have respectively-congruent sides, hence

\[
\angle T_0P_0c_0 = \angle T_0A_0c_0 = 90^\circ
\]

which proves that line \( T_0P_0 \) is tangent at \( P_0 \) to circle \( \textbf{c}_0 \).

Furthermore, \( \triangle T_0A_0O = \triangle T_0P_0O \) as they have respectively-congruent sides, hence

\[
\angle T_0P_0O = \angle T_0A_0O
\]

By hypothesis, \( T_0, A_0 \), and \( A_5 \) are collinear, and \( \triangle A_0A_5O \) is equilateral, hence the external angle \( \angle T_0A_0O = 120^\circ \); hence \( \angle T_0P_0O = 120^\circ \) as well.

Since \( \triangle P_0P_1O \) is equilateral then \( \angle OP_0P_1 = 60^\circ \) and \( \angle T_0P_0P_1 = 180^\circ \), proving that points \( T_0, P_0, P_1 \) are collinear.

\[\Box\]

The following fact is a convenient reformulation of Lemma 3. Thus its proof ends the proof of the referred lemma, hence the proof of Proposition 1.

\textbf{Lemma 5} (The key Lemma). Let \([A_0A_1 \ldots A_5]\) be a regular hexagon whose incircle is \( \textbf{c} \) and circumscribed \( \textbf{C} \).

Let \( \textbf{c}_0 \) be the circle tangent to the side \([A_0A_5]\) of the hexagon at \( A_0 \) and let \( \textbf{C}_0 \), be the circle tangents to side \([A_1A_2]\) at \( A_1 \). Then the angle between circles \( \textbf{c}_0 \) and \( \textbf{C}_0 \) is \( 120^\circ \) iff the three circles: \( \textbf{c}_0, \textbf{C}_0 \) and \( \textbf{C} \) have one common point.

\textit{Proof.} \( \iff \) First assume circles \( \textbf{c}_0 \) and \( \textbf{C}_0 \) intersect at a point \( P_0 \) on circumcircle \( \textbf{C} \). Referring to Figure 3, if \( P_0 \) is on arc \( A_0A_1 \) of \( \textbf{C} \) then by Lemma 4, \( P_0P_1 \) is a common tangent to circles \( \textbf{c}_0 \) and \( \textbf{c} \). In particular, \( P_0P_1 \) is the tangent at \( P_0 \) to circle \( \textbf{c}_0 \).

Next, let \( P_0P'_0 \) be the tangent from \( P_0 \) to the incircle \( \textbf{c} \) (distinct from \( P_0P_1 \)). Similarly, let \( P'_0P'_1, P'_1P'_2, \ldots, P'_1P'_t \) be the tangents from points \( P'_0, P'_1, \ldots, P'_t \in \textbf{C} \) to the incircle \( \textbf{c} \).

Then, as above, points \( P'_0 \) and \( P_0 \) coincide and hexagon \([P'_0P'_1 \ldots P'_t]\) is regular; but since the two hexagons \([P_0P_1 \ldots P_5]\) and \([P'_0P'_1 \ldots P'_t]\) have one common point and both are regular, and inscribed in \( \textbf{C} \) they must coincide.

Once again, Lemma 4 guarantees that \( P_0P_3 \) is a common tangent to circles \( \textbf{C}_0 \) and \( \textbf{c} \). In particular, line \( P_0P_3 \) is the tangent at \( P_0 \), to circle \( \textbf{C}_0 \).
Since hexagon \([P_0P_1 \ldots P_5]\) is regular, \(\angle P_1P_0P_5 = 120^\circ\). This guarantees that the angle between circles \(c_0\) and \(C_0\), which is the angle between their tangents at \(P_0\), is also \(120^\circ\).

\(\Rightarrow\) By hypothesis, circles \(c_0\) and \(C_0\) intersect at an angle of \(120^\circ\). We shall prove that, necessarily, point \(P_0\) must be on the circumcircle \(C\).

Let \(P_0\) be the "real" intersection point between circle \(c_0\) and arc \(A_0A_1\) of circle \(C\). We shall prove that \(P_0\) and \(P'_0\) coincide.

Now let \(C'_0\) be the circle tangent at \(A_1\) to line \(A_1A_2\) that passes through \(P'_0\). Then, by the first part of the proof, circles \(c_0\) and \(C'_0\) intersect at an angle of \(120^\circ\). So circles \(C_0\) and \(C'_0\) are both tangent at \(A_1\) to line \(A_1A_2\) and intersect circle \(c_0\) at the same angle. Hence circles \(C_0\) and \(C'_0\) coincide, as do points \(P_0\) and \(P'_0\).

Finally, we state our main result. Refering to Figure 4.

**Theorem 1** (The endpoint conic). The conic \(C^*\) that passes through the endpoints \(A_1, A_2, B_1, B_2\) and \(P_0\) of the negative pedal curve of \(R\) has one focus on \(M\); furthermore its main axis is the line joining the pedal point \(M\) with the Reuleaux center \(G\).

**Proof.** The above lemmas and propositions prove that the focus of the end-point conic coincides with \(M\). We end the proof by showing that the axis of the endpoint conic passes through \(G\), the center-of-mass of the Reuleaux (or equivalently, it passes through the circumcenter of \(\triangle V_1V_2V_3\)).

Once again to prove that the directrix of conic \(C^*\) is perpendicular to the line that joins points \(M\) and \(G\), we employ an inversive argument.

As reminded in Prop 10 the directrix of a conic whose polar-dual is some circle, is precisely the polar of the center of that circle (w.r. to the inversion circle). Therefore, the directrix of the endpoint conic is the polar of point \(O\).

Since points \(V_1', V_2',\) and \(V_3'\) are, respectively, the inverses of \(V_1, V_2, V_3\) w.r. to a circle centered on \(M\), this guarantees that points (i) \(M\), (ii) the circumcenter of \(\triangle V_1'V_2'V_3'\), and (iii) the circumcenter of \(\triangle V_1V_2V_3\) are collinear, i.e., \(G, M, O\) are collinear. In turn, this implies that the polar of \(O\) (that, by definition, is perpendicular to \(OM\)), will be also perpendicular to \(GM\).

Thus, the axis of the endpoint conic and line \(MO\) are parallel. Since \(M\) is the focus of the endpoint conic, this means that the axis is \(MO\) and passes through \(G\).

The above results reveal another interesting fact: the endpoint conic of a Reuleaux is the polar-dual of a special circle which depends on the vertices of the Reuleaux and on the location of \(M\).

**Corollary 1.** The endpoint conic of a Reuleaux is the polar-dual of the circle centered on the circumcenter \(O\) of triangle \(\triangle V_1'V_2'V_3'\) whose radius is the distance from point \(O\) to line \(V_1'V_2'\).

A closer look at the dual circle of the endpoint conic allows one to diagnose its type: (i) ellipse if \(M\) is inside the dual circle of the endpoint conic; (ii) parabola if it is on said circle; (iii) hyperbola when outside.

Therefore one can construct a Reuleaux triangle when a specific endpoint conic is prescribed.
Corollary 2. Let $V_1V_2'$ be a side of a regular hexagon and $C$ and $c$ be its circumscribed and inscribed circles; let $C'$ and $c'$ be the inverses of $C$ and $c$ w.r. to the line $V_1'V_2'$; let $D_1, D_2$ (resp $D_1', D_2'$) be the intersection of circles $C'$ and $c'$ ($C$ and $c'$, respectively). Let $V_3'$ be a point on the arc $V_1'V_2'$ of $C$ and $c_0$ and $C_0$ be the circles tangent at $V_1'$ and $V_2'$ to sides $OV_1'$ and $OV_2'$ of $\triangle V_1'OV_2'$.

Let $M$ be the inverse of $V_3'$ w.r. to $V_1'V_2'$. Choose an inversion circle $I$ centered on $M$.

Then the inverses of arcs $V_1'V_3'$, $V_2'V_3'$, and of line $V_1'V_2'$ w.r. to $I$ determine a Reuleaux triangle, whose endpoint conic will be an ellipse, hyperbola or parabola iff $M$ is on arc $D_1D_2$, outside this arc, or coincides with either $D_1$ or $D_2$, respectively.

Figure 4. The endpoint conic $C^*$ (purple) is the dual of circle $c$ (dashed purple) w.r. to the inversion circle centered on $M$ (dashed black); one of its focus is on $M$, its directrix is the polar of $O$, its focal axis passes through $G$, and its vertices $L_1$ and $L_2$ are the inversions of the antipodal points $N_1$ and $N_2$, the intersection of the diameter of $c$ passing through $M$. ($N_1$ lies outside the figure.
Figure 5. When the point $M$ is outside the arc $D_1 D_2$, the endpoint conic $C^*$ is a hyperbola (purple): shown its two branches and directrices.

Figure 6. The endpoint conic $C^*$ (dark green) is a parabola iff the point $M$ coincides with either $D_1$ or $D_2$; its directrix (dark green) is the polar of the center $O$ of the Reuleaux.

3. Some Elementary Properties

3.1. Collinearity and Tangencies. Referring to Figure 7:
Proposition 2. The negative pedal $\mathcal{N}$ of a Reuleaux consists of two elliptic arcs $\mathcal{E}_A$ and $\mathcal{E}_B$ and a point $P_0$, which is the antipode of $M$ w.r. to the center of the circle where $M$ is located.

$\mathcal{E}_A$, $\mathcal{E}_B$ are centered on $V_1$ and $V_2$, respectively, have one common focus at $M$, and their semi-axes are of length equal to $r$.

Proof. By hypothesis $M$ belongs to the arc $V_1V_2$ of the circle centered in $V_3$ that passes through $V_2$ and $V_3$. Hence, if $P$ is any point on this arc and we draw the perpendicular $p$, through $P$, on $PM$, all these lines will pass through a fixed point, $P_0$, which is the antipode of $M$ w.r. to the center $V_3$.

The second part is a direct consequence of the general construction of a negative pedal of a circle. See Proposition 12 of Appendix.

Proposition 3. The minor axis of either $\mathcal{E}_A$ and $\mathcal{E}_B$ passes through $P_0$.

Proof. By the definition of an antipedal curve, if we regard $V_1$ as a point on the arc $V_1V_2$ of the circle centered on $V_3$ on which $M$ lies, then $P_0V_1$ will be perpendicular to $MV_1$. Since $V_1$ is the center of $\mathcal{E}_A$, and line $MV_1$ is its focal axis, its minor axis will be along $P_0V_1$. Similarly, the minor axis of $\mathcal{E}_B$ will be along $P_0V_2$.

Proposition 4. The points $A_2$ and $B_2$ and $V_3$ are collinear. The line $A_2B_2$ is tangent to both $\mathcal{E}_A$ and $\mathcal{E}_B$.

Proof. By construction, the negative pedal of the arc $V_2V_3$ is the elliptic arc $\mathcal{E}_{A_f}$ delimited by $A_1$ and $A_2$. This implies that lines $MV_3$ and $A_2V_3$ are perpendicular, as are $MV_3$ and $B_2V_3$. Thus $A_2$, $V_3$, and $B_2$ must be collinear.

Also by construction, the perpendicular to $MV_3$ at $V_3$ is tangent to $\mathcal{N}$ at $A_2$ (resp. $B_2$) when $V_3$ is regarded as a point in the $V_2V_3$ (resp. $V_1V_3$) arc. Hence the points $A_2$, $V_3$, and $B_2$ are collinear ($\angle A_2V_3B_2 = 180^\circ$) and $A_2B_2$ is the common tangent to $\mathcal{E}_A$ and $\mathcal{E}_B$, in $A_2$ and $B_2$, respectively.

Proposition 5. The point $A_1$ is on $P_0V_2$ and $B_1$ is on $P_0V_1$.

Proof. If we regard $V_1$ as a point on the arc $V_1V_2$ of the circle centered on $V_3$ whose negative pedal reduces to $P_0$, then, necessarily, $V_1P_0 \perp MV_1$.

Similarly, if we regard $V_1$ as a point on the arc $V_1V_3$ of the circle centered on $V_2$ whose negative pedal is $\mathcal{E}_B$, then by $\mathcal{N}$’s construction $B_1V_1 \perp MV_1$.

Since this perpendicular must be unique, $P_0$, $B_1$, and $V_1$ are collinear as will be $P_0$, $A_1$, and $V_2$.

Proposition 6. The line joining the intersection points of $\mathcal{E}_A$ and $\mathcal{E}_B$ is the perpendicular bisector of the segment $[f_Af_B]$ and also passes through $P_0$.

Proof. Let $U_1$, $U_2$ denote the points where $\mathcal{E}_A$ and $\mathcal{E}_B$ intersect. In order to prove that $P_0$, $U_1$, and $U_2$ are collinear, we show each lies on the perpendicular bisector of $[f_Af_B]$. Since $U_1$ (resp. $U_2$) is on $\mathcal{E}_A$ (resp. $\mathcal{E}_B$), whose foci are $M$ and $f_A$ (resp. $M$ and $f_B$), with main axis of length $2r$, then

$$U_1f_A + U_1M = 2r; \quad U_2f_B + U_1M = 2r.$$

This implies that $U_1f_A = U_1f_B$ and $U_2f_A = U_2f_B$, hence both $U_1$ and $U_2$ belong to the perpendicular bisector of $[f_Af_B]$. Since we already showed that $P_0V_1 \perp MV_1$, and since $V_1$ is the center of $Mf_A$, this means that $P_0V_1$
A SPECIAL CONIC ASSOCIATED WITH THE REULEAUX NEGATIVE PEDAL CURVE

Figure 7. The two branches of the N are arcs of ellipses $E_A$ and $E_B$ (green and blue), centered on Reuleaux vertices $V_1, V_2$, respectively. They have a common focus in $M$, and the other foci are $f_A, f_B$. The lengths of their main axes is $2r$, the same as the diameters of the three Reuleaux circles (dashed). Points $P_0, A_1, V_2$ are collinear and along their minor axis. $P_0, B_1, V_1$ are collinear and along their minor axis. $P_0 A_1$ and $P_0 B_1$ are tangent to $E_A$ and $E_B$, respectively. $A_2 B_2$ is tangent to both ellipses and $A_2, B_2, V_3$ are collinear. The circle (black) passing through $M$ and the other foci $f_A$ and $f_B$ of the ellipses $E_A, E_B$ (green and blue) is centered on $P_0$ (antipodal of $M$ w.r. to $V_3$). Distance between the foci $f_A$ and $f_B$ is constant. Triangle $T = \triangle f_A f_B P_0$ is equilateral and its sides pass through (i) $A_2$, (ii) $B_2$, (iii) $A_1, B_1$, respectively. Both intersections $U_1, U_2$ of $E_A$ with $E_B$ lie on the perpendicular bisector of $f_A f_B$, i.e., they are collinear with $P_0$. $T$ and $V_1 V_2 V_3$ are homothetic ($M$ is their homothety center) and the ratio of their sides is 2.

is the perpendicular bisector of $[Mf_A]$ and this implies that $P_0 f_A = P_0 M$. Similarly, $P_0 f_B = P_0 M$, hence $P_0 f_A = P_0 f_B$. Therefore $P_0$ is also on the perpendicular bisector of $[f_A f_B]$, ending the proof. □

3.2. Triangles and Homotheties. Referring to Figure 7:

**Proposition 7.** The two sides of the triangle $\triangle f_A P_0 f_B$, incident on $P_0$, contain points $A_2$ and $B_2$. The other side contains points $A_1$ and $B_1$.

**Proof.** The construction the negative pedal of arc $V_2 V_3$ implies $A_1 V_2 \perp M V_2$. Since $V_2$ is the center of the $E_A$, $A_1 V_2$ is the perpendicular bisector of $[M f_B]$ hence $A_1 f_B = A_1 M$. Since $A_1$ lies on $E_A$, $M A_1 + f_A A_1 = 2r$, hence $f_B A_1 + f_A A_1 = f_A f_B$. Therefore, by the triangle inequality, $f_B, A_1, f_A$ must be collinear. A similar proof applies to $B_1$. In order to prove that $P_0, B_2,$
and $f_A$ are collinear, we simply show that $P_0f_A = P_0A_2 + A_2f_A$. As noted above, $A_2V_3$ is perpendicular to $P_0M$ and $V_3$ is its midpoint. Hence $A_2V_3$ is the perpendicular bisector of $[P_0M]$; so $P_0A_2 = MA_2$. Since $A_2$ lies on $\mathcal{E}_A$ we have

$$P_0A_2 + A_2f_A' = MA_2 + A_2f_A = 2r$$

The proof for $B_2$ is similar. The triangles $\triangle f_AF_BP_0$ and $\triangle V_1V_2V_3$ are homothetic with ratio 2, and homothety center $M$. Hence, $\triangle f_AF_BP_0$ is equilateral and the distance between the $f_A$ and $f_B$ is the same as the diameter $2r$ of the circles that form the Reuleaux.

**Proposition 8.** Triangles $\triangle f_AF_BP_0$ and $\triangle V_1V_2V_3$ are homothetic at ratio 2, and with $M$ the homothety center. Hence, $\triangle f_AF_BP_0$ is equilateral and the distance between $f_A$ and $f_B$ is the same as $2r$. Furthermore, their barycenters $X_2$ and $X'_2$ are collinear with $M$.

**Proof.** The points $V_1, V_2, V_3$ are the midpoints of $MF_A, MF_B$ and $P_0M$, respectively. Thus, $V_1V_2$ is a mid-base$^1$ of $\triangle f_AMF_B$, $V_2V_3$ is a mid-base of $\triangle f_BP_0f_A$ and $V_3V_1$ is a mid-base of $\triangle P_0Mf_A$. Thus, $\triangle f_AF_BP_0$ is equilateral with sides twice that of the original triangle: $f_AF_B = 2V_1V_2$. This shows that the distance between the pair of foci of $\mathcal{E}_A$ and $\mathcal{E}_B$ is constant and equal to the length of their major axes. Note that the lines $f_AF_A, f_VF_B, P_0V_3$ intercept in $M$, hence the two triangles are perspective at $M$. Due to the parallelism of their sides their medians will be (respectively) parallel; let $X_2$ and $X'_2$ denote the barycenters of triangles $\triangle V_1V_2V_3$ and $\triangle f_AF_BP_0$, respectively. The barycenter divides the medians in equal proportions, which guarantees $\triangle MX'_2V_2 \sim \triangle MX_2f_B$. Since $M, V_2, f_B$ were collinear, $M, X'_2, X_2$ are collinear, as well. □

### 4. Asymmetric Reuleaux

In this section we consider a generalized Reuleaux Triangle where the radii $r_1, r_2, r_3$ of the three circles may be different, and their centers $O_i$ don’t necessarily lie on the vertices of an equilateral.

We also consider a generalized Reuleaux triangle whose sides are three circular arcs of three circles which have three arbitrary radii $r_1, r_2, r_3$ of the three circles may be different, and whose centers $O_1, O_2, O_3$ don’t necessarily form an equilateral triangle.

Without loss of generality, assume $M$ is always on the Reuleaux side contained on the circle centered on $O_3$, with endpoints $V_1, V_2$. The following facts still hold:

(i) The conic $\mathcal{E}^*$ can only have a focus on $M$ if one is at the symmetric Reuleaux configuration.

(ii) For a choice of $O_1, O_2, O_3, r_1, r_2$, the distance $|f'_Af'_B|$ is invariant over $M, r_3$.

(iii) $V_2, A_1, P_0$ are collinear and $V_1, B_1, P_0$ are collinear.

(iv) The line through the two intersections $U_1, U_2$ of $\mathcal{E}_A$ with $\mathcal{E}_B$ is perpendicular to $f'_Af'_B$.

(v) $A_2B_2$ is tangent to both $\mathcal{E}_A$ and $\mathcal{E}_B$.

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$^1$This is a line parallel to a side at half the altitude.
5. Conclusion

We studied properties of a negative pedal of a curve which have a remarkable symmetry: the equilateral Reuleaux triangle. Our methods are entirely synthetic, the results are new and surprising and all the proofs are readable, which is highly gratifying. The only non-elementary (yet synthetic) tool was the use of polar reciprocity, due to Poncelet, invoked in order to prove the result about the focus of the external conic.

Some of the above results still hold if the Reuleaux is asymmetric, i.e., defined by three arcs of arbitrary circles. Other, not. For instance, in general setting, the focus of the endpoint conic will no longer coincide with the pedal point $M$. Below, a few additional questions we couldn’t manage with a synthetic apparatus.

- What is the location of the focus of the endpoint conic if the Reuleaux is asymmetric? When does this coincides with the pedal point?
- What are the bounds on the eccentricity of the endpoint conic associated with a Reuleaux?
- What are the locations of the limiting points $D_1$ and $D_2$ that defines the type of the endpoint conic. Can these points be obtained (construct) geometrically?

We presume one must abdicate the idea of a synthetic proof, in order to provide an answer to all these questions.

Appendix A. Duality and the Negative Pedal Curve

Here we review some inversive concepts and results on polar duals.

Fix a circle $I$ centered at point $M$ and of arbitrary radius, called the inversion circle.

Let $\Gamma'$ denote the inverse of a curve $\Gamma$ and $A'$ the inverse of a point $A$, w.r. to $I$. Assume all inversions below are performed w.r. to $I$. The polars and poles are implicitly performed w.r. to $I$ and $\Gamma^*$ denote the dual (reciprocal curve) of $\Gamma$.

The following result explicitly states the ambivalent definition of a dual curve.

**Proposition 9** (Fundamental Theorem on Dual Curves). Let $\Gamma$ a regular curve. Let $\Gamma^*_1$ be the locus of the poles of its tangents and $\Gamma^*_2$ the envelope of the polars its points. Then $\Gamma^*_1 = \Gamma^*_2$.

Furthermore, $\Gamma^*$ is regular and the polars of the points to the dual curve $\Gamma^*$ are the tangents to $\Gamma$, while the poles of the tangents of $\Gamma^*$ are the points of $\Gamma$.

Further, if we perform the dual of a dual, we obtain the original curve: $[\Gamma^*]^* = \Gamma$.

This result, whose proof is based on the fundamental pole-polar theorem justifies the dual definition of the curve $\Gamma^*$ either as a loci of points, or as an envelope of lines and specifies who the points and the tangents at a dual curve are.

For more details on poles, polars and polar reciprocity, see e.g. [2].

Since the Reuleaux circle is the delimited by circular arcs, below we clarify what a dual of a circle is and how it can be construct.
Figure 8. The dual of a curve $\gamma$ (violet) w.r. to $\mathcal{I}$ (dashed black), is the curve $\Gamma$ (dark green). It is the envelope of polars $E'D'$ of points $D$ on $\gamma$, as well as loci of $E'$, the poles of the tangents $ED$ to $\gamma$, as $D$ sweeps $\gamma$. The dual of a circle is a conic. $\Gamma$ is an ellipse iff $M$ is inside $\gamma$, a hyperbola iff $M$ is outside $\gamma$ and a parabola, iff $M$ is on $\gamma$. Its focus is $M$, and the directrix is the polar of $O$. Its vertices are the inverses of $A'_1$ and $A'_2$, the intersection of $MO$ with $\gamma$.

Referring to Figures 8.

**Proposition 10.** The dual (or the polar dual, or the reciprocal) of a circle $\Gamma$ is a conic $\Gamma^*$ whose:

- focus coincides with the center of the inversion circle $\mathcal{I}$;
- vertices are the inversions of the intersection points of the line joining the centers of the two circles $\Gamma$ and $\mathcal{I}$ and circle $\Gamma$;
- directrix is the polar of the center of $\Gamma$.

The conic $\Gamma^*$ will be

- an ellipse if the center of $\mathcal{I}$ is inside $\Gamma$;
- a parabola, if the center of $\mathcal{I}$ is on $\Gamma$;
- a hyperbola, if the center of $\mathcal{I}$ is outside $\Gamma$.

**Proposition 11.** The dual of a conic $\Gamma$ is a circle $\Gamma^*$ if and only if the center of $\mathcal{I}$ is a focus of the conic $\Gamma$. If this is the case:

- the inverses of the vertices of $\Gamma$ are a pair of antipodal points on the dual circle $\Gamma^*$;
- the directrix of $\Gamma$ is the polar of the center of $\Gamma^*$.

These remarkable results are classic; see [1] or [2, Art.309], for a proof and other details.

There is a natural intertwining between negative pedal curve, inversion, and polar reciprocity.
A SPECIAL CONIC ASSOCIATED WITH THE REULEAUX NEGATIVE PEDAL CURVE

Figure 9. The negative pedal curve $N(\Gamma)$ (dark green) of a circle $\Gamma$ (orange) w.r. to $I$ (dashed black) defines as the envelope of lines $DE'$ (green) where $D$ is a current point of $\Gamma$ and $DE' \perp MD$. Since $DE'$ is (also) the polar of $D'$, (green) the inverse of $D$, situated on circle $\gamma$ (violet) then $N(\Gamma)$ is the envelope of the polars of its inverted circle $\gamma$. Therefore $N(\Gamma)$ is the dual of the its inverse circle $\gamma$, hence a conic with a focus in $M$. Its vertices coincides with points $A_1, A_2$ the diameter of $\Gamma$ that contains $M$. Here, $\Gamma$ is an the ellipses, since $M$ is inside $\gamma$.

**Proposition 12.** The negative pedal curve of $\Gamma$ w.r. to a pedal point is the reciprocal of its inverse, $\Gamma'$ w.r. to a circle centered on that pedal point: $N(\Gamma) = \Gamma'^*$. Therefore:

- the negative pedal curve is the locus of the poles of the tangents to its inverted curve;
- the polars of the points of a negative pedal curve $N(\Gamma)$ are the tangents to its inverted curve $\Gamma'$.

Though this is a known result, for convenience of the reader we include its straightforward proof:

**Proof.** First we prove that the dual of $\Gamma'$ is included into the negative pedal of $\Gamma$. Let $S$ a point in $\Gamma'$; then $S$ is an inverse of some point $L$ in $\Gamma$; $S = L'$; hence, the polar of $S$ is the perpendicular in point $S'$ to the line joining $M$ and $S'$; since $S' = (L')' = L$, the polar of $S = L'$ is the perpendicular in $L$ to line $ML$. Since inversion is bijective (in fact, it is an involution), if $S$ sweeps $\Gamma'$, $L$ sweeps $\Gamma$, hence lines $ML$ corresponds to the set of all the tangents to the negative pedal of $\Gamma$.

The other inclusion is similar, if we refer to negative pedal curves as envelope of lines. \qed
Figure 10. $N(\Gamma)$ (orange) being the dual of its inverse, $\gamma$ (violet), is (also) the locus of the poles $E$ of tangents in $D'$, to $\gamma$, as $D$ sweeps $\Gamma$. Note that $DE$ is the tangent to $N(\Gamma)$ passing through $D$. $N(\Gamma)$ is an ellipse (see Fig 9) iff $M \in [A_1 A_2]$ and $N(\Gamma)$ is a hyperbola iff $M$ is not on segment $[A_1 A_2]$. The negative pedal of a circle is never a parabola.

Thus, the negative pedal curve initially defined as an envelope of lines can also be constructed as a "point curve", i.e., as the locus of the poles of the tangents to its inverse $\Gamma'$.

In order to construct the negative pedal of a circle w.r. to a pedal point that does not lie on $\Gamma$, we first build its inverse, $\Gamma'$ then obtain the latter’s dual. Note that the inverse of can be a circle or a line. The latter case occurs if the center of inversion is on the inverted circle.

Below we describe the negative pedal curve of a circle.

**Proposition 13.** The negative pedal $N(\Gamma)$ of a circle $\Gamma$, w.r. to a pedal point $M$ is a conic, whose:

- focus coincides with $M$;
- center coincides with the center of circle $\Gamma$;
- vertices are the intersection points of the line that joins the pedal point $M$ and the center of $\Gamma$, with the circle $\Gamma$;
- focal axis is the diameter of the circle $\Gamma$.

Then $N(\Gamma)$ will be an ellipse (resp. hyperbola), if the pedal point is interior (resp. exterior) to the circle $\Gamma$.

The negative pedal curve of a circle centered on $O$ w.r. to a point on the circumference reduces to a point, namely the antipodal of $M$ w.r. to $O$.

**Proof.** First assume that $M$ is not on the circle. Build the negative pedal curve of the circle as follows:
A special conic associated with the Reuleaux negative pedal curve

(i) construct $\Gamma'$, the inverse of the circle $\Gamma$. Its diameter will be $[A'_1A'_2]$ where $A'_1$ and $A'_2$ are the inverses of vertices $A_1$ and $A_2$ of the conic $\Gamma$.

(ii) perform the dual of circle $\Gamma'$ to obtain a conic whose focus lies on $M$ (since this is the inversion center) and whose vertices are precisely the inverses of $A'_1$ and $A'_2$, respectively, i.e., $A_1$ and $A_2$.

By the result above, the conic will be an ellipse (resp. hyperbola), if $M$ is inside (resp. outside) $\Gamma$.

If the pedal point $M$ is on the circle, then the inverse is a line, whose reciprocal reduces to a point, its pole.

We emphasize that the negative polar of a circle can never be a parabola!

Parabolas are negative pedals of lines, only.

References

[1] Akopyan, A., Zaslavsky, A.: Geometry of conics, Mathematical World, vol. 26. American Mathematical Society, Providence, RI (2007). DOI 10.1090/mawrdl/026. URL doi.org/10.1090/mawrdl/026. Translated from the 2007 Russian original by Alex Martsinkovsky

[2] Salmon, G.: A treatise on conic sections. Longmans, Green, Reader and Dyer (1869)

[3] Weisstein, E.: Mathworld. MathWorld–A Wolfram Web Resource (2019). URL mathworld.wolfram.com

Liliana Gabriela Gheorghe, Universidade Federal de Pernambuco, Departamento de Matemática, Recife, PE, Brazil

E-mail address: liliana@dmat.ufpe.br

Dan Reznik, Data Science Consulting, Rio de Janeiro, RJ, Brazil

E-mail address: dan@dat-sci.com