BOUNDARY ACTIONS OF CAT(0) SPACES AND THEIR C*-ALGEBRAS

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Abstract. In this paper, we study boundary actions of CAT(0) spaces from a point of view of topological dynamics and C*-algebras. First, we investigate the actions of right-angled Coxeter groups and right-angled Artin groups with finite defining graphs on the visual boundaries and the Nevo-Sageev boundaries of their natural assigned CAT(0) cube complexes. In particular, we establish (strongly) pure infiniteness results for reduced crossed product C*-algebras of these actions through investigating the corresponding CAT(0) cube complexes and establishing necessary dynamical properties such as minimality, topological freeness and pure infiniteness of the actions. In addition, we study actions of fundamental groups of graphs of groups on the visual boundaries of their Bass-Serre trees. We show that the existence of repeatable paths essentially implies that the action is 2-filling, from which, we also obtain a large class of unital Kirchberg algebras. Furthermore, our result also provides a new method in identifying C*-simple generalized Baumslag-Solitar groups. The examples of groups obtained from our method have n-paradoxical towers in the sense of [26]. This class particularly contains non-degenerated free products, Baumslag-Solitar groups and fundamental groups of n-circles or wedge sums of n-circles.

1. Introduction

Boundaries of certain CAT(0) spaces and group actions on them play important roles in the study of groups, geometry and topology. Motivating examples include the Gromov boundaries of hyperbolic spaces as well as hyperbolic groups acting on their Gromov boundaries. For CAT(0) spaces beyond the hyperbolic world, there are many boundaries with similar flavor that one may consider, as we will list below. Suppose a group $G$ acts on a CAT(0) space $X$ by isometry. Then under some natural assumptions, there is an induced topological action of $G$ on these boundaries. Dynamical properties on the boundaries have been proved to play a significant role in investigating many useful properties of the acting groups or the spaces themselves such as the Tits alternative, Yu’s Property A, a-T-amenability, and thus leads to many other applications in topology.

On the other hand, as one of our main motivation in the current paper, reduced crossed products of the form $C(X) \rtimes_r G$ arising from topological dynamical systems, say from $(X, G, \alpha)$ for a countable discrete group $G$, a locally compact Hausdorff space $X$ and a continuous action $\alpha$, have long been an important source of examples and motivation for the study of C*-algebras. Our one goal in this paper is to continue the study of the first author in [42] and [43] to investigate the pure infiniteness of reduced crossed product C*-algebras. See also [1], [41], [36], [48] and a very recent progress [26] for more information in this direction.

Pure infiniteness of a C*-algebra, reflecting a kind of paradoxical nature, is an important regularity property of C*-algebras. It has many characterizations (see [39], [40] and [45]) and plays an essential role in the celebrated classification theorem by Kirchberg and Phillips (see, e.g., [46]). On the other hand, beyond the

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classification theorem, the property of pure infiniteness, and its variant strong pure infiniteness have their own interest to be studied as well.

Therefore, in this paper, we study boundary actions of certain CAT(0) spaces from a topological dynamical and operator algebraic viewpoints to determine when the reduced crossed product C*-algebras of the boundary actions are purely infinite. This study will yield new and interesting examples belonging to the class of strongly purely infinite C*-algebras.

Our first motivating examples are actions of certain non-amenable groups on the visual boundaries that have a strong paradoxical flavor. For example, as a generalization of the hyperbolic case, if a group $G$ acts on a proper CAT(0) space $X$ by isometry in a non-elementary way (see [30]), then any rank-one element $g$ in $G$ performs the classical North-South dynamics on the visual boundary $\partial X$ (see [30]), i.e., there exist attracting and repelling fixed points of $g$ such that the positive powers of $g$ contract the whole boundary except the repelling fixed point into the attracting points. This strong contracting dynamics implies that the action of $G$ on $\partial X$ has dynamical comparison in the sense of [35, Definition 3.2] and has no $G$-invariant measures. This condition of comparison is also equivalent to the so-called pure infiniteness of the action in the sense of [43, Definition 3.5]. Moreover, it was shown in [30] that such an action is also minimal. Then under the assumption that the action is topologically free, its reduced crossed product is simple and purely infinite. See e.g. [42].

Enlarging our scope, observe that many examples of purely infinite reduced crossed product appeared in the literature arise from boundary actions that have similar strong contracting dynamics. This implies that it is worth investigating boundary actions of CAT(0) spaces in a more systematical way. The first step is to determine which boundary one should look at because there are a lot of candidates beyond the hyperbolic world. We enumerate several here and warn that this is not a complete list at all. Let $X$ be a CAT(0) space, one may consider

1. the visual boundary $\partial X$,
2. the horofunction boundary,
3. Contracting boundary or equivalently in the CAT(0) case, the Morse boundary (see [15], [24] and [16]),
4. $\kappa$-Morse boundary (see [19]).

In this list, the visual boundary might be the most “transparent” compact Hausdorff boundary associated to a CAT(0) space. Similarly to the Gromov boundary, it contains the equivalence class of geodesics that are almost in the same direction. The horofunction boundary are equivalent to the visual boundary. See [9] for more information. However, the visual boundary is not a quasi-isometric invariant and that is one of the motivation why the boundaries in (3) and (4) above are invented. However, the topologies on contracting and ($\kappa$-)Morse boundaries are no longer compact if the space is not hyperbolic. See [16, Theorem 10.1], [47, Proposition 6.6] and [19, Theorem 1.1]. It also seems unknown whether there exists a non-hyperbolic CAT(0) space with locally compact contracting or $\kappa$-Morse boundary. Therefore, boundaries in (3) and (4) are out of our scope at this moment because topological spaces considered for C*-algebras are usually assumed to be locally compact Hausdorff.

Nevertheless, If we consider additional structures on the CAT(0) spaces, e.g. cube complexes, we have more boundaries at hands which are of combinatorial flavors,

5. Roller boundary $R(X)$ (see [25]) and
6. Nevo-Sageev boundary $B(X)$ as a subset of $R(X)$ (see [44]).

We remark that the Roller boundary can be identified with the horofunction boundary of the 1-skeleton $X^1$ of the complex $X$ with $\ell_1$-metric. Based on these discussions, we mainly consider the visual boundary, the Roller boundary and the
Nevo-Sageev boundary in this paper and consider the actions of right-angled Coxeter groups (RACGs) and right-angled Artin groups (RAAGs) as well as the actions of fundamental groups of graph of groups on the visual boundaries of their Bass-Serre trees in this paper.

Our first contribution in this paper is to detect the structure of the reduced crossed products of actions of RACGs and RAAGs on the boundaries above based only on the information of their defining graphs. The main outcome are (strongly) pure infiniteness results for the reduced crossed products. To establish this, our main method is to verify necessary dynamical properties such as minimality, topological freeness and pure infiniteness of the actions (see Theorem 3.6 below) as well as using the algebraic structure of tensor products. Note that the above dynamical conditions are in general not easy to establish. Nevertheless, actually as our another motivation to consider boundary actions, in some cases, our actions are boundary actions in the sense of Furstenberg and thus are topologically free provided that the acting groups are $C^*$-simple and there exists points with amenable stabilizers (see [3]).

As a result, we have the following main theorems on CAT(0) cube complexes. The involved notions will be introduced in next sections. Let $\Gamma = (V, E)$ be a finite simple graph. For simplicity, we denote by $G_\Gamma$ the RACG $W_\Gamma$ or the RAAG $A_\Gamma$ and $X_\Gamma$ the corresponding Davis complex $\Sigma_\Gamma$ or the universal cover of the Salvetti complex $\tilde{\Sigma}_\Gamma$, respectively. There is a canonical way to write all $G_\Gamma$ into direct products of special subgroups by using joins of the defining graph $\Gamma$ with a form $G_\Gamma = G_{\Gamma, 1} \times \ldots G_{\Gamma, m}$. Similarly, using the joins of $\Gamma$, one can also decompose $X_\Gamma$ to be $X_\Gamma = X_{\Gamma, 1} \times \cdots \times X_{\Gamma, m}$ in a corresponding way.

**Theorem A.** (Theorem 4.11) Let $G_\Gamma \curvearrowright X_\Gamma$ where $X_\Gamma$ is essential and has at least one non-Euclidean irreducible factor $X_{\Gamma, i}$ in the decomposition above. Suppose

1. $G_\Gamma = W_\Gamma$ has no special subgroup $D_\infty$ in the RACG case;
2. $G_\Gamma = A_\Gamma$ has no special subgroup $\mathbb{Z}$ in the RAAG case.

Then the reduced crossed product $A = C(B(X_\Gamma)) \rtimes_r G_\Gamma$ of the induced action on the Nevo-Sageev boundary $\beta : G_\Gamma \curvearrowright B(X_\Gamma)$ is unital simple separable and purely infinite. In addition, in the RACG case, $A$ is nuclear as well and thus a Kirchberg algebra satisfying the UCT.

We remark that the absence of a special subgroup $D_\infty$ or $\mathbb{Z}$ for $G_\Gamma$ is equivalent to that there is no factor of $D_\infty$ or $\mathbb{Z}$, which is called Euclidean factors, in the direct product decomposition $G_\Gamma = G_{\Gamma, 1} \times \ldots G_{\Gamma, m}$ above. On the other hand, such a Euclidean factor $D_\infty$ or $\mathbb{Z}$, has its own action on its own Nevo-Sageev boundary, which is exactly a set consisting two points, denoted by $\{0, 1\}$ in this paper. Moreover, in the RACG case, $D_\infty$ acts on the set by alternating while in the RAAG case, $\mathbb{Z}$ acts on it trivially. See more in Section 4.

Now suppose $G_\Gamma$ has such Euclidean factors. One can still verify pure infiniteness in the sense of Definition 3.3 of the action $\beta$ on $B(X_\Gamma)$ by using Proposition 3.9. To observe more, in the RACG case, $\beta$ is still minimal and in the RAAG case, $\beta$ has finitely many closed invariant sets (see Section 4). However, it follows from Proposition 3.9 that $\beta$ is no longer topologically free and in particular is not essentially free so that Theorem 3.6 cannot be applied here to establish the (strongly) pure infiniteness of the crossed product. From the viewpoint of $C^*$-algebras, after we add the Euclidean factors $D_\infty$ or $\mathbb{Z}$ in the group, the Nevo-Sageev boundary increases in a way that the crossed product do not have a good ideal structure any more. Nevertheless, because the Nevo-Sageev boundary preserves the product structure, we still know what does $C(B(X_\Gamma)) \rtimes_r G_\Gamma$ look like and are able to show they are strongly purely infinite. First it follows from Theorem 4.11 that any $G_\Gamma$ can be decomposed into a direct product $G_\Gamma = G_{\Gamma'} \times H^n$, for some $n \in \mathbb{N}$, where $\Gamma'$ is a subgraph of $\Gamma$ and $H$ is the Euclidean factor, i.e., $H = D_\infty$ if $G = W_\Gamma$ and $H = \mathbb{Z}$ if $G = A_\Gamma$. 
Corollary 1.1. (Corollary 4.12) Let $G_\Gamma = G_\Gamma \times H^n$ be the decomposition mentioned above. Write $A = C(\partial X_\Gamma) \rtimes_r G_\Gamma$. Then one has

1. In the RACG case, one has $A = (C(\partial X_{\Gamma'}) \rtimes_r G_{\Gamma'}) \otimes \bigotimes_{i=1}^n (C(\{0, 1\}) \rtimes_r D_{\infty})$, where the action of $D_{\infty}$ on $\{0, 1\}$ is by alternating and $C(X_{\Gamma'}) \rtimes_r G_{\Gamma'}$ is unital simple separable purely infinite.

2. In the RAAG case, one has $A = (C(\partial X_{\Gamma'}) \rtimes_r G_{\Gamma'}) \otimes C(\{0, 1\}^n) \otimes C(\mathbb{T}^n)$ in which $\mathbb{T}$ is the unit circle and $C(X_{\Gamma'}) \rtimes_r G_{\Gamma'}$ is unital simple separable purely infinite.

In the RACG case, $A$ is $O_\infty$-stable and actually, in either case, $A$ is strongly purely infinite.

For the visual boundaries of the cube complexes $X_\Gamma = \Sigma_\Gamma$ or $S_\Gamma$, in the irreducible case, by establishing the necessary topological dynamical properties mentioned above, we have the following result.

Theorem B. (Theorem 5.7) Let $\Gamma = (V, E)$ be a finite simple graph without joins.

1. If $|V| \geq 3$ then $C(\partial \Sigma_\Gamma) \rtimes_r W_\Gamma$ is simple and purely infinite.
2. If $|V| \geq 2$ then $C(\partial S_\Gamma) \rtimes_r A_\Gamma$ is simple and purely infinite.

Another type of actions of groups on the visual boundary considered in this paper is the fundamental group of a graph of groups acting on its Bass-Serre tree. Follow the notations in [12] for graph of groups $\mathcal{G} = (\Gamma, G)$ and denote by $\pi_1(\mathcal{G}, v)$ the fundamental group of $\mathcal{G}$ at a base vertex $v$ and $v \partial X_\mathcal{G}$ the boundary of the Bass-Serre tree, we have the following other main theorems. We remark that the following assumptions in Theorem C and D on graphs of groups $\mathcal{G}$ are very mild so that one can easily construct examples satisfying the theorems, e.g., examples in Theorem E below. One may also want to compare Theorems C and D to the pure infiniteness result of $C^*$-algebras obtained in [12] originally. The main tool there is the local contraction of the action introduced in [11] and [11]. The relation between the local contraction and our pure infiniteness of actions were discussed in [13] Theorem 5.8. On the other hand, it follows from [13] Theorem 6.15 that there exists purely infinite actions of $\mathbb{Z}_2 * \mathbb{Z}_3$ on $\{0, 1\}^N \times \mathbb{R}$, which is not locally contracting. For the other notions appeared in the following theorems, we also refer to Section 5 for the definitions.

Theorem C. (Theorem 5.17) Let $\Gamma = (V, E)$ be a locally finite non-singular graph and $\mathcal{G} = (\Gamma, G)$ a graph of groups. Suppose

1. $v \partial X_\mathcal{G}$ is infinite;
2. $\xi$ can flow to $e$ for any $\xi \in \partial X_\mathcal{G}$ and $e \in E$; and
3. there is a repeatable path $\mu = g_1 e_1 \ldots g_n e_n$ with $|\Sigma e_n| \geq 2$.

Then the natural action $\beta : \pi(\mathcal{G}, v) \lhd v \partial X_\mathcal{G}$ is a strong boundary action. In particular, $\beta$ is a $\pi_1(\mathcal{G}, v)$-boundary action. If, in addition, each $G_e$ is amenable and $\pi_1(\mathcal{G}, v)$ is $C^*$-simple, then the action $\beta$ is topologically free and thus the crossed product $C(v \partial X_\mathcal{G}) \rtimes_r \pi_1(\mathcal{G}, v)$ is a unital Kirchberg algebra satisfying the UCT.

In particular, if we restrict to Generalized Baumslag-Solitar (GBS) groups/graphs, we have the following theorem. Note that this result also provides a new method in identifying $C^*$-simple GBS groups.

Theorem D. (Theorem 5.20) Let $\mathcal{G} = (\Gamma, G)$ be a locally finite non-singular GBS graph of groups in which $\Gamma = (V, E)$ is a finite graph. Suppose

1. $v \partial X_\mathcal{G}$ is infinite;
2. $\xi$ can flow to $e$ for any $\xi \in \partial X_\mathcal{G}$ and $e \in E$;
3. there is a repeatable path $\mu = g_1 e_1 \ldots g_n e_n$ with $|\Sigma e_n| \geq 2$; and
4. $\mathcal{G}$ is not unimodular.
Then the natural action $\beta : \pi_1(G, v) \curvearrowright v\partial X_G$ is an amenable topologically free strong boundary action and the crossed product $C(v\partial X_G) \rtimes_r \pi_1(G, v)$ is a unital Kirchberg algebra satisfying the UCT. Furthermore, $\pi_1(G, v)$ is $C^*$-simple.

Denote by $\mathcal{C}$ the class of all fundamental groups of graph of groups satisfying Theorems C and D, which includes the following specific examples. We remark all groups in $\mathcal{C}$ have 2-paradoxical towers in the sense of [20]. Then any minimal topologically free amenable actions of these groups on a compact metrizable space yield a unital Kirchberg algebra satisfying the UCT. See more in Theorem 5.26. To detect members in $\mathcal{C}$, we have the following examples.

**Theorem E.** (Theorem 5.25) Still write $\mathcal{G} = (\Gamma, G)$, the class $\mathcal{C}$ particularly contains the following groups.

1. $C^*$-simple $\pi_1(G, v)$ in which $G$ satisfies assumptions (1)-(3) of Theorem C and each $G_e$ is amenable. This includes Example 5.18. In particular, this includes $G \ast F$ such that $(|G| - 1)(|F| - 1) \geq 2$.
2. $C^*$-simple GBS groups $\pi_1(G, v)$ appeared in Theorem D. This includes non-degenerated BS($k, l$) where $(|k| - 1)(|l| - 1) \geq 2$ in Example 5.21 and certain GBS groups of $n$-circles in Example 5.22 as well as some GBS groups of wedge sum of $n$-circles for $m$ times in Example 5.23. In addition, if $n \geq 2$ or $m \geq 2$, these are not non-degenerated BS groups by Remark 5.24.

**Remark 1.2.**

1. During the preparation of the current paper, Gardella, Guffen, Kranz and Naryshkin posted on arXiv a paper [20] on similar topics. We remark that our Theorem A(1) has generalized their result in [20, Example 4.8]. On the other hand, once the minimality and the topological freeness has been proved through our method (see Section 4), by combining the amenability of action on $\mathcal{R}(X)$ obtained in [34], one can also apply their results to obtain a different proof of Theorem A(1). However, the other theorems in this paper cannot be established in this way, because to the best knowledge of the authors, it is unknown whether those actions are amenable. See more in Remark 4.13.
2. In the same day that the authors of this paper submit the first version of the current paper to arXiv, there is a new version of [20] appeared on arXiv in which a different approach is used to show non-degenerated free products and Baumslag-Solitar groups as new examples of groups with $n$-paradoxical towers. These examples are covered by our Theorem E as well. Furthermore, in the current second version, we added more examples other than BS groups obtained in the first version.

Our paper is organized in the following way. In Section 2, we review some necessary concepts, definitions and preliminary results. In Section 3, we establish all topological dynamical theorems and link them to pure infiniteness of reduced crossed products for use later. In Section 4, we focus on the action of RACGs and RAAGs on the Roller boundaries and the Nevo-Sageev boundaries to establish all necessary dynamical properties described in Section 3 and show the reduced crossed product is (strongly) purely infinite. In Section 5, we investigate actions of irreducible RACGs and RAAGs on visual boundaries of their complexes as well as the fundamental groups of graphs of groups on the visual boundaries of their Bass-Serre trees to study pure infiniteness of related reduced crossed products, $C^*$-simplicity of certain GBS groups and the groups with $n$-paradoxical towers.

2. Preliminaries

In this section, we recall some terminologies and definitions used in the paper.
2.1. Groups, topological dynamical systems and their C*-algebras. Let $G$ be a countable discrete group, $X$ a locally compact Hausdorff space and $\alpha : G \curvearrowright X$ denotes a continuous action of $G$ on $X$. We write $M_G(X)$ for the set of all $G$-invariant regular Borel probability measures on $X$.

We say an action $\alpha : G \curvearrowright X$ is minimal if all orbits are dense in $X$. Recall that an action $\alpha : G \curvearrowright X$ is said to be essentially free provided that, for every closed $G$-invariant subset $Y \subset X$, the subset of points in $Y$ with the trivial stabilizer, say $\{x \in Y : \text{Stab}_G(x) = \{e\}\}$, is dense in $Y$, where $\text{Stab}_G(x) = \{t \in G : tx = x\}$. An action is said to be topologically free provided that the set $\{x \in X : \text{Stab}_G(x) = \{e\}\}$, is dense in $X$ and this is equivalent to that the fixed point set $\{x \in X : tx = x\}$ of each nontrivial element $t$ of $G$ is nowhere dense. It is not hard to see that essentially freeness means that the restricted action to each $G$-invariant closed subspace is topologically free with respect to the relative topology and thus these two concepts are equivalent when the action is minimal. We refer to [11] for standard construction of (reduced) crossed product $C^*$-algebras $C_0(X) \rtimes_r G$ for topological dynamical systems.

In the case that $X$ is compact, it is well known that if the action $G \curvearrowright X$ is topologically free and minimal then the reduced crossed product $C(X) \rtimes_r G$ is simple (see [3]) and it is also known that the crossed product $C(X) \rtimes_r G$ is nuclear if and only if the action $G \curvearrowright X$ is amenable (see [11]). Archbold and Spielberg [3] showed that $C(X) \rtimes G$ is simple if and only if the action is minimal, topologically free and regular (meaning that the reduced crossed product coincides with the full crossed product). These imply that $C(X) \rtimes_r G$ is simple and nuclear if and only if the action is minimal, topologically free and amenable.

A type of topological dynamical systems of the particular interest are $G$-boundary actions. Now, let $X$ be compact and denote by $P(X)$ the set of all probability measures on $X$. Furstenberg provided the following definition in [25].

**Definition 2.1.** (1) A $G$-action $\alpha$ on $X$ is called strongly proximal if for any probability measure $\eta \in P(X)$, the closure of the orbit $\{g\eta : g \in G\}$ contains a Dirac mass $\delta_x$ for some $x \in X$.

(2) A $G$-action $\alpha$ on a compact Hausdorff space $X$ is called a $G$-boundary action if $\alpha$ is minimal and strongly proximal.

Topological freeness of a $G$-boundary action is linked to $C^*$-simplicity of $G$, i.e., $C^*_r(G)$ is simple. We refer to [31], [8] and [32] for this topic.

Let $A$ be a unital $C^*$-algebra. A non-zero positive element $a$ in $A$ is said to be properly infinite if $a \preceq a \preceq a$, where $\preceq$ is the Cuntz subequivalence relation, for which we refer to [2] as a standard reference. A $C^*$-algebra $A$ is said to be purely infinite if there are no characters on $A$ and if, for every pair of positive elements $a, b \in A$ such that $b$ belongs to the closed ideal in $A$ generated by $a$, one has $b \preceq a$. It was proved in [39] that a $C^*$-algebra $A$ is purely infinite if and only if every non-zero positive element $a$ in $A$ is properly infinite. In addition, in [40] Theorem 5.1, Kirchberg and Rørdam also introduced a stronger version of pure infiniteness for $C^*$-algebras called strongly pure infiniteness. Denote by $\mathcal{O}_\infty$ the Cuntz algebra of infinite generators. The condition of $\mathcal{O}_\infty$-stability for $A$, i.e., $A \otimes \mathcal{O}_\infty \simeq A$, implies that $A$ is strongly purely infinite, and if $A$ is separable and nuclear, these two conditions are equivalent. See by [40] Theorem 8.6.

In this paper, we will address on right-angled Coexter groups and right-angled Artin groups, abbreviated by RACGs and RAAGs, respectively. We recall their definitions by using defining graphs, which are finite simple graph $\Gamma = (V, E)$, i.e., the vertex set $V$ is finite.

**Definition 2.2.** For a finite simple graph $\Gamma = (V, E)$, the corresponding RACG $W_\Gamma$ is defined to be

$$W_\Gamma = \langle V : v_i^2 = e \text{ for any } 1 \leq i \leq n \text{ and } v_iv_j = v_jv_i \text{ for any } (v_i, v_j) \in E \rangle.$$
The corresponding RAAG $A_{\Gamma}$ is defined to be

$$A_{\Gamma} = \langle V : v_i v_j = v_j v_i \text{ for any } (v_i, v_j) \in E \rangle.$$ 

Let $\Gamma = (V, E)$ be a finite simple graph. Let $\Lambda$ be a subgraph of $\Gamma$. Then the corresponding subgroup $W_{\Lambda}$ (resp. $A_{\Lambda}$) of $W_{\Gamma}$ (resp. $A_{\Gamma}$) is called a special subgroup. We say $\Gamma$ is a join if there are two subgraphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ of $\Gamma$ such that $V_1$ and $V_2$ form a partition of $V$ and every vertex in $V_1$ is adjacent to every vertex in $V_2$. In this situation, we write $\Gamma = \Gamma_1 \ast \Gamma_2$. If there is no join for $\Gamma$, we call the corresponding RACG $W_{\Gamma}$ and RAAG $A_{\Gamma}$ irreducible. In addition, for each RACG and RAAG, one naturally assign it with a CAT(0) cube complex constructed from its Cayley graph, which is called Davis complex and the universal cover of the Salvetti complex, respectively so that the RACG and the RAAG act on them by isometry cocompactly, respectively. We will leave the definitions and discussions of these two specific complexes until Section 4. Instead, we recall some general facts on CAT(0) cube complexes here.

2.2. CAT(0) cube complexes and their boundaries. We refer to [14, 15, 35] and [17] for general information of CAT(0) cube complexes.

Definition 2.3. A CAT(0) cube complex is a simply connected cell complex whose cells are Euclidean cubes $[0, 1]^d$ of various dimensions. In addition, the link of each 0-cell, i.e., vertex, is a flag complex, which is a simplicial complex such that any $n + 1$ adjacent vertices belong to an $n$-simplex.

We say a CAT(0) cube complex $X$ finite dimensional if there is a uniform upper bound on the dimension of cubes in $X$. Let $X$ be a CAT(0) cube complex. A midcube of a cube $[0, 1]^d$, is the restriction of a coordinate of the cube to be 1/2. A hyperplane $h$ is a connected subspace of $X$ with the property that for each cube $C$ in $X$, the intersection $h \cap C$ is either a midcube of $C$ or empty. Let $e$ be an edge in $X^1$, we say a hyperplane $h$ is dual to $e$ if $h \cap e \neq \emptyset$. In general, $h$ separates $X$ into precisely two components, called halfspaces, denoted by $h$ and $h^\ast$. $X$ is said to be essential if given any half space $h$ in $X$, there is a vertex in $h$ arbitrary far from $h$. Similarly, we say a group $G \leq \text{Aut}(X)$ acts essentially on $X$ if no $G$-orbit remains in a bounded neighborhood of a halfspace of $X$. Here $\text{Aut}(X)$ is the automorphism group of $X$ consisting all isometries that preserve the cubical structures. A CAT(0) cube complex $X$ is said to be cocompact if the action on $X$ of the group $\text{Isom}(X)$ consisting all isometries of $X$ is cocompact.

A CAT(0) cube complex $X$ is said to be irreducible if it cannot be written as a product of two CAT(0) cube complexes. Otherwise, we say $X$ is reducible. Let $n \in \mathbb{N}$. An $n$-dimensional flat is an isometrically embedded copy of $n$-dimensional Euclidean space $E^n$ (in the usual CAT(0) metric). A unbounded cocompact CAT(0) cube complex $X$ is said to be Euclidean if $X$ contains a $\text{Aut}(X)$-invariant flat. Otherwise, we say $X$ is non-Euclidean. An unbounded essential CAT(0) cube complex whose irreducible factors are all non-Euclidean is called a strictly non-Euclidean complex.

We also consider the 1-skeleton of a CAT(0) cube complex, which usually equipped with the usual $\ell_1$-metric (called the path metric or the combinatorial metric as well). For the finite-dimensional case, the $\ell_1$-metric and the usual CAT(0) metric on $X$ are quasi-isometric to each other.

Lemma 2.4. [15] Lemma 2.2] Let $(X, d)$ be a finite-dimensional CAT(0) cube complex, where $d$ is the usual CAT(0) metric. Then $(X, d)$ is quasi-isometric to its 1-skeleton endowed with the combinatorial metric.

Finally, we recall that one may assign several compact Hausdorff boundaries to a CAT(0) cube complex $X$. We refer to [14] Section 1.3 and [9] for more detailed
information. If a group $G$ acting on $X$ by isometry, sometimes, the action can be naturally extended to the boundary as a topological action, which will yield interesting topological dynamical systems and $C^*$-algebras. In this paper, we mainly care about the visual boundary (see, e.g. [9]) and the Nevo-Sageev boundary introduced in [44]. We still leave the definitions to Section 4 and 5.

3. Comparison properties and pure infiniteness of dynamical systems

In this section, we recall several useful dynamical notions appeared in the literature that have a paradoxical flavor and in fact imply the reduced crossed products are purely infinite. We also provide some new criteria for these notions to hold, which will be applied in the following sections. Let $G$ be a countable discrete group and $X$ a locally compact Hausdorff space. Let $G \curvearrowright X$ be a continuous action. The following definition appeared in [41]. See also [28].

**Definition 3.1.** [41, Definition 1] Let $X$ be a compact Hausdorff space. We say an action $G \curvearrowright X$ is a strong boundary action (or extreme proximal) if for any compact set $F$ and non-empty open set $O$ there is a $g \in G$ such that $gF \subset O$.

We remark that $G \curvearrowright X$ is a strong boundary action, then in [27], Glasner showed that $X$ is a $G$-boundary in the sense of Definition 2.1. On the other hand, it was proved in [41] that the reduced crossed product of a topological free strong boundary action is simple and purely infinite. Then the notion of strong boundary action has been generalized in [36] to $n$-filling actions.

**Definition 3.2.** [36] An action $\alpha : G \curvearrowright X$ on a compact Hausdorff space $X$ is said to be $n$-filling if for any non-empty open sets $O_1, \ldots, O_n$ there are $n$ group elements $g_1, \ldots, g_n \in G$ such that $\bigcup_{i=1}^{n} g_i O_i = X$.

It is not hard to see strong boundary actions are exactly the 2-filling actions and it was proved in [36] that reduced crossed products of topologically free $n$-filling actions are also simple and purely infinite. Note that all $n$-filling actions are necessarily minimal. Then, in [42], the first author observed that the dynamical comparison, first introduced by Winter and then refined by Kerr in [38], also serves as a generalization of the $n$-filling property and still implies the pure infiniteness of the reduced crossed products in the case that $\alpha$ is minimal and there is no $G$-invariant probability measure on $X$. To move furthermore, in [42] and [19], under the assumption that there is no invariant measures, which is usually necessary for a reduced crossed product to be purely infinite, the theory surrounding dynamical comparison property actually has been established in a more general setting of non-minimal actions and even for locally compact Hausdorff étale groupoids. In the current paper, we only deal with the transformation groupoids case, i.e., countable discrete group $G$ acting on locally compact Hausdorff spaces $X$. In addition, throughout the paper, we write $A \sqcup B$ to indicate that the union of sets $A$ and $B$ is a disjoint union and denote by $\bigsqcup_{i \in I} A_i$ the disjoint union of the family $\{A_i : i \in I\}$.

**Definition 3.3.** ([38], [42]) Let $G \curvearrowright X$ be a continuous action on a locally compact Hausdorff space $X$. Let $O, V$ be non-empty open sets in $X$ and $F$ a compact set in $X$.

(i) We write $F \prec O$ if there is an open cover $\{U_1, \ldots, U_n\}$ of $F$ and group elements $g_1, \ldots, g_n \in G$ such that $\{g_1 U_1, \ldots, g_n U_n\}$ is a disjoint family of open sets contained in $O$, i.e., $\bigsqcup_{i=1}^{n} g_i U_i \subset O$.

(ii) We say $V$ is dynamical subequivalent to $O$, denoted by $V \prec O$, if $F \prec O$ for any compact set $F \subset V$.

(iii) We say $V$ is paradoxical subequivalent to $O$, denoted by $V \prec_2 O$, if $F \prec_2 O$ for any compact set $F \subset V$ in the sense that there are disjoint non-empty open sets $O_1, O_2 \subset O$ such that $F \prec O_1$ and $F \prec O_2$. 
The following concepts were introduced in [38], [42] and [43].

**Definition 3.4.** Let \( \alpha : G \rtimes X \) be a continuous action on a locally compact Hausdorff space \( X \).

(i) \( \alpha \) is said to have dynamical comparison if \( U \prec V \) whenever \( \mu(U) < \mu(V) \) for any \( \mu \in M_G(X) \).

(ii) \( \alpha \) is said to have paradoxical comparison if \( O \prec 2O \) for any non-empty open set \( O \) in \( X \).

(iii) \( \alpha \) is said to be purely infinite if \( U \prec 2V \) for any non-empty open sets \( U, V \) satisfying \( U \subset G \cdot V \).

(iv) \( \alpha \) is said to be weakly purely infinite if \( U \prec V \) for any non-empty open sets \( U, V \) satisfying \( U \subset G \cdot V \).

It has been observed in [42] that all \( n \)-filling actions, thus including all strong boundary actions, satisfy dynamical comparison and have no invariant probability measures. For the relation among notions above, the first author proved the following theorem in [43], which was written in the language of groupoids.

**Theorem 3.5.** [43, Theorem 5.1] Let \( \alpha : G \rtimes X \). Consider the following conditions.

(i) \( \alpha \) has dynamical comparison and \( M_G(X) = \emptyset \).

(ii) \( \alpha \) is purely infinite.

(iii) \( \alpha \) has paradoxical comparison.

(iv) \( \alpha \) is weakly purely infinite.

Then (i) \( \Rightarrow \) (ii) \( \Leftrightarrow \) (iii) \( \Rightarrow \) (iv). If \( \alpha \) is minimal then they are equivalent.

The following result was essentially established in [42] as our main tool for application in the next sections. We remark that a version of locally compact Hausdorff \( \breve{\text{e}} \)tale groupoids of the following results have been established in [43].

**Theorem 3.6.** [42, Theorem 1.1 Corollary 1.4] Let \( G \) be a countable discrete infinite group, \( X \) a compact Hausdorff space and \( \alpha : G \rtimes X \) an action of \( G \) on \( X \). Suppose \( \alpha \) is purely infinite. If either

1. \( \alpha \) is minimal and topologically free, or
2. \( G \) is exact and \( \alpha \) is essentially free as well as there are only finitely many \( G \)-invariant closed sets in \( X \),

then the reduced crossed product \( C(X) \rtimes_r G \) is strongly purely infinite. In the first case \( C(X) \rtimes_r G \) is simple. In the second case \( C(X) \rtimes_r G \) has finitely many ideals.

**Proof.** Under the assumptions, it has been proved that \( C(X) \rtimes_r G \) is purely infinite in [42]. If \( \alpha \) is minimal and topologically free then \( C(X) \rtimes_r G \) is simple and thus strongly purely infinite. In the second case, first it was proved in [51, Theorem 1.20] that \( C(X) \) separates the ideals of \( A = C(X) \rtimes_r G \). Then since there are only finitely many \( G \)-invariant closed ideals in \( C(X) \) by assumption, the primitive ideal space \( \text{Prim}(A) \) is finite (exactly happens when \( A \) has finitely many ideals) and thus has a basis of compact-open sets. Therefore \( A \) also has the ideal property (IP), whence \( A \) is strongly purely infinite by [45, Proposition 2.11, 2.14] ∎

In the rest of this section, we mainly show that existence of contractible sets for an action, which is common in certain boundary actions, usually implies pure infiniteness of the action.

**Definition 3.7.** Let \( G \rtimes X \) be a continuous action on a compact Hausdorff space \( X \). An open set \( V \) in \( X \) is called contractible if there exists an \( x \in X \) such that for any neighborhood \( U \) of \( x \), there is a \( g \in G \) such that \( gV \subset U \).

Let \( G \rtimes X \) be a continuous action on a compact space. If the action is minimal and \( X \) is not finite, then it has to be perfect in the sense that there is no isolated points. Suppose the contrary, there exists an \( x \in X \) such that \( \{x\} \) is open. Then
the minimality of the action implies that \( X = G \cdot x \), which has to be finite. This is a contradiction.

**Proposition 3.8.** Let \( X \) be an infinite compact Hausdorff space. If the action \( \alpha : G \curvearrowright X \) is minimal and there is a contractible open set \( V \), then \( \alpha \) is purely infinite.

*Proof.* By Theorem 3.3, it suffices to show \( F \prec O \) for any compact set \( F \) and non-empty open set \( O \) in \( X \). Indeed, let \( F, O \) be such sets. First, since \( \alpha \) is minimal, there are \( g_1, \ldots, g_n \in G \) such that \( F \subset \bigcup_{i=1}^{n} g_i V \). In addition, because \( X \) is perfect, one can choose \( n \) disjoint non-empty open subsets \( O_1, \ldots, O_n \) of \( O \).

On the other hand, let \( x_0 \in X \) be an element that makes \( V \) contractible. Since \( \alpha \) is minimal, for each \( O_i \) where \( 1 \leq i \leq n \), there is an \( h_i \in G \) and a neighborhood \( U_i \) of \( x_0 \) such that \( h_i U_i \subset O_i \). Then the contractibility of \( V \) implies that there is a \( t_i \in G \) such that \( t_i V \subset U_i \). This implies that \( t_i h_i V \subset O_i \). Therefore, one has \( \bigcup_{i=1}^{n} t_i h_i^{-1}(g_i V) \subset O \), which establishes \( F \prec O \). \( \Box \)

We also need the following permanence result later.

**Proposition 3.9.** Let \( \alpha : G \curvearrowright X \) be a purely infinite action on an infinite locally compact Hausdorff space \( X \) and \( \beta : H \curvearrowright Y \) an action of a discrete group \( H \) on a finite set \( Y \), equipped with the discrete topology. Then the product action \( \alpha \times \beta \) is still purely infinite. If there is an \( h \in H \) such that \( h \neq e_H \) but \( hy = y \) for some \( y \in Y \). Then the product action \( \alpha \times \beta \) is not topologically free. In particular, it is not essentially free.

*Proof.* Write \( Y = \{ y_1, \ldots, y_m \} \) and it suffices to verify paradoxical comparison by Theorem 3.3. For any non-empty open set \( O \subset X \times Y \), there are open sets \( O_i \subset X \) for each \( 1 \leq i \leq m \) such that \( O = \bigcup_{i=1}^{m} O_i \times \{ y_i \} \) (some \( O_i \) may be empty). One can do this by observing that \( O_i \times \{ y_i \} = O \cap \pi_{Y}^{-1}(\{ y_i \}) \). For any compact set \( F \subset O \), using the same trick and the fact \( \pi_{Y}^{-1}(\{ y_i \}) \) is clopen in \( X \times Y \), one can find compact sets \( F_i \subset O_i \) for \( i = 1, \ldots, m \) such that \( F = \bigcup_{i=1}^{m} F_i \times \{ y_i \} \), where \( \pi_Y \) is the natural projection from \( X \times Y \) to \( Y \). Then since \( \alpha \) is purely infinite, for each \( i \) with \( O_i \neq \emptyset \), there are disjoint non-empty open sets \( U_{i,1}, U_{i,2} \subset O_i \) such that \( F_i \prec U_{i,j} \) for both \( j = 1, 2 \), which implies that there are collections of open sets \( \{ V_{i,j}^k \subset X \} = k = 1, \ldots, n_{i,j} \} \) and group elements \( \{ g_{i,j}^k \in G : k = 1, \ldots, n_{i,j} \} \) such that \( F_i \subset \bigcup_{k=1}^{n_{i,j}} V_{i,j}^k \) and \( \bigcup_{k=1}^{n_{i,j}} g_{i,j}^k V_{i,j}^k \subset U_{i,j} \) for \( j = 1, 2 \).

We denote by \( I = \{ 1 \leq m : O_i \neq \emptyset \} \). Observe that \( U_j = \bigcup_{i \in I} U_{i,j} \times \{ y_i \} \subset O \) for \( j = 1, 2 \) and \( U_1, U_2 \) are disjoint non-empty open sets. In addition, one has

\[
F = \bigcup_{i=1}^{m} F_i \times \{ y_i \} \subset \bigcup_{i \in I} \bigcup_{k=1}^{n_{i,j}} (V_{i,j}^k \times \{ y_i \})
\]

and

\[
\bigcup_{i \in I} \bigcup_{k=1}^{n_{i,j}} (g_{i,j}^k, e_H)(V_{i,j}^k \times \{ y_i \}) \subset \bigcup_{i \in I} U_{i,j} \times \{ y_i \} = U_j
\]

for \( j = 1, 2 \). These establish that \( \alpha \times \beta \) is purely infinite.

Now, choose an \( h \in H \) with \( h \neq e_H \) and \( y \in Y \) such that \( hy = y \). Then \( (e_G, h) \) fixes \( X \times \{ y \} \), which is open in \( X \times Y \). This entails that \( \alpha \times \beta \) is not topologically free and thus not essentially free. \( \Box \)

4. Roller Boundary \( \mathcal{R}(X) \) and Nevo-Sageev Boundary \( B(X) \)

In this section, we mainly study group actions on CAT(0) cube complexes \( X \) and their Roller boundary \( \mathcal{R}(X) \) as well as a particular subset \( B(X) \) of \( \mathcal{R}(X) \), which was introduced by Nevo and Sageev in [41]. These two boundaries are of combinatorial
nature. We begin with recalling the necessary concepts. We refer to [44] and [35] for more information.

We denote by $\mathcal{H}$ the collection of all hyperplanes of $X$ and $\mathcal{H}$ the set of all halfspaces. Similar to Stone-Čech compactification, one can use ultrafilters consisting of certain halfspaces to define the Roller compactification. See [29]. Recall an ultrafilter $\alpha$ on $\mathcal{H}$ is a subset of $\mathcal{H}$ satisfying

(1) For any hyperplane $\hat{h}$, either $h \in \alpha$ or $h^* \in \alpha$ and
(2) If $h \in \alpha$ and $h \subset h'$ then $h' \in \alpha$.

We denote by $\mathcal{U}(X)$ the collection of all ultrafilters on $X$, which can be viewed as a closed subset of $\prod_{h \in \mathcal{H}} \{ h, h^* \}$ and thus a compact metrizable space if $X$ is locally finite. In addition, by identifying each vertex $x \in X^0$ by the principal ultrafilter $\alpha_x = \{ h \in \mathcal{H} : x \in h \}$, the Roller boundary is defined to be $\mathcal{R}(X) = \mathcal{U}(X) \setminus X^0$, which is also a compact metrizable space if $X$ is locally finite. Nevo and Sageev in [44] also consider the following special subset $B(X)$ of $\mathcal{R}(X)$, which is referred as the Nevo-Sageev boundary in this paper. Consider the following set $\mathcal{U}_{NT}(X)$ consisting all non-terminating ultrafilters:

$$\mathcal{U}_{NT}(X) = \{ \alpha \in \mathcal{U}(X) : h \in \alpha \Rightarrow \text{there exists } h' \in \alpha \text{ with } h' \subset h \}$$

and define $B(X) = \mathcal{U}_{NT}(X)$ in $\mathcal{U}(X)$. Such a boundary is always non-empty if $X$ is essential and cocompact by [44] Theorem 3.1. Unlike the visual boundary that will be addressed in the next section, $B(X)$ has a very nice property that if $X$ is not irreducible and decomposes as $X = \prod_{i=1}^n X_i$, then $B(X) = \prod_{i=1}^n B(X_i)$ so that the dynamics on the $B(X)$ is more convenient to deal with.

Let $\Gamma$ be a finite simple graph and $W_\Gamma$ and $A_\Gamma$ be its RACG and RAAG, respectively. Following the notation in [17], we denote by $\Sigma_\Gamma$ the Davis complex for $W_\Gamma$ and $\tilde{S}_\Gamma$ the universal cover of the Salvetti complex $S_\Gamma$ for $A_\Gamma$. Note that both of $\Sigma_\Gamma$ and $\tilde{S}_\Gamma$ are finite dimensional CAT(0) cube complexes and the 1-skeleton of them are exactly the Cayley graph of $W_\Gamma$ and $A_\Gamma$, respectively. From this view, for Davis complex $\Sigma_\Gamma$, the set of vertices $C \subset W_\Gamma$ spans a cube if and only if it forms a coset $uW_\Lambda$ for some finite special subgroup $W_\Lambda$. Here $\Lambda$ is allowed to be empty in which case $uW_\Lambda = \{ w \}$.

Such a graph theoretical description of $\tilde{S}_\Gamma$ is analogous to that of $\Sigma_\Gamma$. In fact, for any special subgroup $W_\Lambda$ of the Coxeter group $W_\Gamma$, with generating vertices $\{ v_1, \ldots, v_k \} \subset V$, one can lift $W_\Lambda$ to the subset $\tilde{W}_\Lambda \subset A_\Lambda$ (not subgroup) consisting of elements of the form $v_1^{\epsilon_1} \cdots v_k^{\epsilon_k}$ where $\epsilon_i = 0, 1$. Then $\tilde{S}_\Gamma$ is the cube complex whose 1-skeleton is the Cayley graph of $A_\Gamma$ and whose cubes are set of vertices of the form $a\tilde{W}_\Lambda$ for some $a \in A_\Gamma$ and some finite special subgroup $W_\Lambda$.

Both $\Sigma_\Gamma$ and $\tilde{S}_\Gamma$ have very nice properties. Write $X_\Gamma = \Sigma_\Gamma$ or $\tilde{S}_\Gamma$ for simplicity. First, $X_\Gamma$ is always cocompact since $G_\Gamma$ acting on it cocompactly. Using the fact that 1-skeleton of $\tilde{S}_\Gamma$ is exactly the Cayley graph of the corresponding group $W_\Gamma$ or $A_\Gamma$, the edges $X_1$ of $X_\Gamma$ naturally are labeled by the vertex set $V$ of the defining graph $\Gamma$. To observe more, for any vertex $x \in X_1^0$ ($x$ thus actually belongs to $W_\Gamma$ or $A_\Gamma$), the 1-skeleton of the link of $x$ is isomorphic to the defining graph $\Gamma$. In addition, the edges in the 1-skeleton of the link of $x$ labeled by a subset $\{ v_{k_1}, \ldots, v_{k_m} \}$ of $V$ belong to the same cube if and only if $(v_{k_i}, v_{k_j}) \in E$ for any $1 \leq i, j \leq m$. Moreover, the labels of edges dual to one hyperplane $\hat{h}$ are all the same $v \in V$, which is called the type of $\hat{h}$. Finally, If in $X_\Gamma$, two hyperplanes $\hat{h}_1 \cap \hat{h}_2 \neq \emptyset$, then their types $v_1$ and $v_2$ satisfy $(v_1, v_2) \in E$. See more in [35] Section 8]. For essentialness of $X_\Gamma$, the following theorem was also proved in [35]. For a simple graph $\Gamma = (V, E)$, the graph $\Gamma^c = (V, E^c)$, in which $E^c = \{ (u, v) : u, v \in V \text{ and } (u, v) \notin E \}$ is called its complemented graph.
Proposition 4.1. [35] Proposition 8.1, Lemma 8.3] Let $\Gamma$ be a finite simple graph. The Davis simplex $\Sigma_\Gamma$ of the corresponding RACG $W_\Gamma$ is essential if and only if the complement graph $\Gamma^c$ does not have an isolated vertex. In addition, the universal cover of the Salvetti complex $\hat{S}_\Gamma$ of the corresponding RAAG $A_\Gamma$ is always essential.

Therefore $B(\hat{S}_\Gamma)$ is always never empty and so are $B(\Sigma_\Gamma)$ whenever $\Gamma^c$ has no isolated vertex. We now study the reducibility of $\hat{S}_\Gamma$ and $\Sigma_\Gamma$. Let $\Gamma = (V, E)$. Simply observe that $\Gamma^c$ has an isolated vertex $v$ if and only if $(v, w) \in E$ for any $w \in V \setminus \{v\}$ if and only if $W_\Gamma$ has a factor of $\mathbb{Z}_2$ as a special subgroup, i.e., $W_\Gamma = W_\Gamma \times \mathbb{Z}_2$ for some subgraph $\Gamma'$. Therefore, the Davis complex $\Sigma_\Gamma$ is essential if and only if $W_\Gamma$ has no factor $\mathbb{Z}_2$ as a special subgroup.

Lemma 4.2. [35] Lemma 2.5] A decomposition of a CAT(0) cube complex $X$ as a product of cube complexes corresponds to a partition of the collection of hyperplanes of $X$, $\bar{H} = \bar{H}_1 \sqcup \bar{H}_2$ such that every hyperplane in $\bar{H}_1$ meets every hyperplane in $\bar{H}_2$.

Now we have the following result, which seems well-known to experts. However, to be self-contained, we include the proof here.

Proposition 4.3. Let $\Gamma = (V, E)$ be a simple finite graph with $|V| \geq 2$ and $X_\Gamma = \Sigma_\Gamma$ or $\hat{S}_\Gamma$. Then $X_\Gamma$ can be written as a direct product of CAT(0) cube complexes if and only if $\Gamma$ is a join.

Proof. Suppose $X$ is a direct product. Then Lemma 4.2 implies that one can partition the whole $\bar{H} = \bar{H}_1 \sqcup \bar{H}_2$ non-trivially such that $\bar{h}_1 \cap \bar{h}_2 \neq \emptyset$ for any $\bar{h}_1 \in \bar{H}_1$ and $\bar{h}_2 \in \bar{H}_2$. Now, observe that there are $\hat{h}_1 \in \bar{H}_1$ and $\hat{h}_2 \in \bar{H}_2$ with different types $v$ and $w$, respectively. Otherwise, there is only one type for all hyperplanes in $\bar{H}$, which means $|V| = 1$. This is a contradiction.

We then define $J_{1,0} = \{h_1\}$ and $J_{2,0} = \{h_2\}$. Now we enumerate $V \setminus \{v, w\}$ by $\{v_1, v_2, \ldots, v_n\}$. Suppose we have defined $J_{1,m}$ and $J_{2,m}$ for $0 \leq m < n$. Then for $v_{m+1}$, choose $\hat{h} \in \bar{H}$ with type $v_{m+1}$. Observe that either $\hat{h} \in \bar{H}_1$ or $\bar{H}_2$. Then for $i, j = 1, 2$ and $i \neq j$, define $J_{i,m+1} = J_{i,m} \cup \{\hat{h}\}$ if $\hat{h} \in \bar{H}_1$ and $J_{j,m+1} = J_{j,m}$. Finally one defines $V_i = \{v \in V : \text{there is a } \hat{h} \in J_{i,n} \text{ of type } v\}$ for $i = 1, 2$. Our construction implies $V_1 \sqcup V_2 = V$. In addition, because $J_{i,n} \subset \bar{H}_i$ for $i = 1, 2$, one has $h_1 \cap h_2 \neq \emptyset$ for any $\bar{h}_1 \in J_{i,n}$ for $i = 1, 2$, which implies that $(v_1, v_2) \in E$ for any $v_1 \in V_1$ and $v_2 \in V_2$. This means that $\Gamma$ is a join. The converse direction is trivial by observing $X_\Gamma = X_{\Gamma_1} \times X_{\Gamma_2}$ whenever $\Gamma$ has a non-trivial join $\Gamma = \Gamma_1 \star \Gamma_2$ for some subgraphs $\Gamma_1$ and $\Gamma_2$.

Now if $X_\Gamma$ can be written as a non-trivial direct product, then $\Gamma$ has a non-trivial join, e.g., $\Gamma = \Gamma_1 \star \Gamma_2$, which implies $X_\Gamma = X_{\Gamma_1} \times X_{\Gamma_2}$. If one of these $\Gamma_i$, $i = 1, 2$, is still a join, we can decompose $X_{\Gamma_i}$ further in the same manner. Following this strategy, since $\Gamma$ is finite, one can decompose $X_\Gamma = X_{\Gamma_1} \times \cdots \times X_{\Gamma_m}$, in which each factor is irreducible. We remark that such a factorization is unique up to a permutation of factors. See the proof of [35] Proposition 2.6]. Then the natural action of $W_\Gamma \curvearrowright \Sigma_\Gamma$ is exactly the product of all actions $W_{\Gamma_i} \curvearrowright \Sigma_{\Gamma_i}$, i.e., $W_\Gamma = W_{\Gamma_1} \times \cdots \times W_{\Gamma_m} \curvearrowright \Sigma_{\Gamma_1} \times \cdots \times \Sigma_{\Gamma_m}$ coordinatewise. The same also holds for $A_\Gamma$. We now establish the following result on the structure of $X_\Gamma$. We first recall concepts in [35] Section 4.4] that a CAT(0) cube complex $X$ is called $\mathbb{R}$-like if there is an $\text{Aut}(X)$-invariant geodesic line $\ell \subset X$. In addition, we remark that if $\text{Aut}(X)$ acts cocompactly, then $X$ is quasi-isometric to the real line $\mathbb{R}$.

Lemma 4.4. Let $X_\Gamma = \Sigma_\Gamma$ or $\hat{S}_\Gamma$ such that $X_\Gamma$ is essential. Let $X_\Gamma = X_{\Gamma_1} \times \cdots \times X_{\Gamma_m}$ be a decomposition of $X_\Gamma$ into irreducible factors described above. Suppose one $X_{\Gamma_i}$ is Euclidean. Then

(1) in the case $X_{\Gamma_i} = \Sigma_{\Gamma_i}$ one has $W_{\Gamma_i} \simeq D_\infty$; and
(2) in the case $X_{\Gamma_i} = \tilde{S}_{\Gamma_i}$ one has $A_{\Gamma_i} \cong \mathbb{Z}$.

Proof. Let $X_{\Gamma} = \Sigma_{\Gamma}$ or $\tilde{S}_{\Gamma}$ with a decomposition $X_{\Gamma} = X_{\Gamma_1} \times \cdots \times X_{\Gamma_n}$, where each $X_{\Gamma_i} = \Sigma_{\Gamma_i}$ or $\tilde{S}_{\Gamma_i}$ is irreducible. Since $X_{\Gamma}$ is essential, Proposition 4.3 implies that each factor $X_{\Gamma_i}$ is essential and thus unbounded.

Write $G_i = W_{\Gamma_i}$ or $A_{\Gamma_i}$, respectively for simplicity. Suppose a factor $X_{\Gamma_i}$ is Euclidean. Then there is a $\text{Aut}(X_{\Gamma_i})$-invariant flat in $X_{\Gamma_i}$. Because $X_{\Gamma_i}$ is essential and the 1-skeleton of $X_{\Gamma_i}$ is exactly the Cayley graph of $G_i$, on which the action of $G_i$ is transitive, the $G_i$-action on $X_{\Gamma_i}$ is essential and cocompact. Therefore, $\text{Aut}(X_{\Gamma_i})$ acts on $X_{\Gamma_i}$ essentially and cocompactly since $G_i \leq \text{Aut}(X_{\Gamma_i})$. Then Lemma 7.1 implies that $X_{\Gamma_i}$ is $\mathbb{R}$-like and thus $X_{\Gamma_i}$ is quasi-isometric to the real line $\mathbb{R}$. This implies that $G_i$ is quasi-isometric to $\mathbb{Z}$ by Lemma 7.2 and thus $G_i$ is virtually $\mathbb{Z}$ and has exactly two ends.

In the case $G_i = W_{\Gamma_i}$, applying [23] Theorem 8.7.3], one has that $G_i$ is the product of a finite special subgroup and a special subgroup that is an infinite dihedral group $D_{\infty}$. Now if the finite special group factor of $G_i$ is nontrivial, then $X_{\Gamma_i}$ is reducible by Proposition 4.3, which is a contradiction to the fact that $X_{\Gamma_i}$ is irreducible. This implies $G_i = D_{\infty}$. In the case $G_i = A_{\Gamma_i}$, for any vertices $v, w$ of the graph $\Gamma_i$, the two-generator subgroup $\langle v, w \rangle \leq A_{\Gamma_i}$ has to be either free or abelian by a classical result of Baudisch in [4], which implies that $v, w$ have to commute because $A_{\Gamma_i}$ is virtually $\mathbb{Z}$, which is amenable. This implies that $A_{\Gamma_i}$ has to be isomorphic to $\mathbb{Z}$. □

These lead to the following result.

**Theorem 4.5.** Let $G_{\Gamma} \subset X_{\Gamma}$ where $G_{\Gamma} = W_{\Gamma}$ or $A_{\Gamma}$ and $X_{\Gamma} = \Sigma_{\Gamma}$ or $\tilde{S}_{\Gamma}$, respectively. Then $G_{\Gamma} = G_{\Gamma'} \times H^n$ for some subgraph $\Gamma'$ of $\Gamma$ and a group $H$ and an $n \in \mathbb{N}$, $m$ which $H = D_{\infty}$ if $G_{\Gamma} = W_{\Gamma}$ and $H = \mathbb{Z}$ if $G_{\Gamma} = A_{\Gamma}$. In addition, the corresponding complex $X_{\Gamma'}$ of $G_{\Gamma'}$ is strictly non-Euclidean.

Proof. For the action $G_{\Gamma} \subset X_{\Gamma}$, by the reduction, it can be written as

$$G_{\Gamma_1} \times \cdots \times G_{\Gamma_m} \subset X_{\Gamma_1} \times \cdots \times X_{\Gamma_m},$$

in which each $X_{\Gamma_i}$ is irreducible. Collecting all Euclidean factors $X_{\Gamma_i}$ together. Without loss of generality, one may assume they are exactly the final $n$ factors. Then Lemma 4.4 implies that for $m - n + 1 \leq i \leq m$, the group $H = G_{\Gamma_i} \cong D_{\infty}$ or $\mathbb{Z}$ depending on which case is under consideration. Now $\Gamma'$ is defined to be the join of all graphs $\Gamma_1, \ldots, \Gamma_{m-n}$ such that $X_{\Gamma'} = X_{\Gamma_1} \times \cdots \times X_{\Gamma_{m-n}}$, which is strictly non-Euclidean by definition. □

On the other hand, ultrafilters and the Nevo-Sageev boundary work compatible with the product of $\text{CAT}(0)$ cube complexes. See [44]. Given a decomposition $X \cong \prod_{i=1}^{m} X_i$, one actually has $U(X) \cong \prod_{i=1}^{m} U(X_i)$ and $B(X) \cong \prod_{i=1}^{m} B(X_i)$. In our case, since the action $G_{\Gamma} \subset X_{\Gamma}$ action can be decomposed to be

$$G_{\Gamma_1} \times \cdots \times G_{\Gamma_m} \subset X_{\Gamma_1} \times \cdots \times X_{\Gamma_m},$$

Then the action $G_{\Gamma} \subset U(X_{\Gamma})$ is exactly the product action

$$G_{\Gamma} = \prod_{i=1}^{m} G_{\Gamma_i} \subset \prod_{i=1}^{m} U(X_{\Gamma_i})$$

and therefore the action on the Nevo-Sageev boundary $G_{\Gamma} \subset B(X_{\Gamma})$ can be written as

$$G_{\Gamma} = \prod_{i=1}^{m} G_{\Gamma_i} \subset \prod_{i=1}^{m} B(X_{\Gamma_i}).$$

Now we study the dynamics of $G_{\Gamma}$ on the Nevo-Sageev boundary $B(X_{\Gamma})$. In the irreducible Euclidean case, let $G_{\Gamma} = D_{\infty} \text{ or } \mathbb{Z}$. Then by a simple observation, the corresponding Roller boundary and the Nevo-Sageev boundary $\mathcal{R}(X_{\Gamma}) = B(X_{\Gamma})$ is
a set consisting exactly two points, which can be identified by the only two infinite geodesics in this case. However, the action on them are different. In the RACG case, \( W_G = D_\infty \) when the defining graph \( \Gamma = (V = \{u,v\}, E = \emptyset) \). If we denote by \( \hat{0} \) the infinite geodesic \( uuu \ldots \) and \( \hat{1} \) the geodesic \( vvv \ldots \) for simplicity, then \( \mathcal{R}(\Sigma_\Gamma) = B(\Sigma_\Gamma) \) can be identified by \( \{\hat{0}, \hat{1}\} \) and therefore the action of \( W_\Gamma \) on the boundary is generated by the permutations \( u \cdot \hat{0} = \hat{1} \) and \( v \cdot \hat{1} = \hat{0} \) as well as \( v \cdot \hat{0} = \hat{1} \) and \( v \cdot \hat{1} = \hat{0} \). In the RAAG case that \( A_\Gamma = \mathbb{Z} \), it is easy to see the the boundary \( \mathcal{R}(\hat{S}_\Gamma) = B(\hat{S}_\Gamma) \) are exactly the two ends of it, on which the action of \( A_\Gamma = \mathbb{Z} \) is trivial. Now, we write the boundary \( B(X_\Gamma) = \{\hat{0}, \hat{1}\} \) for simplicity in both cases.

In the strictly non-Euclidean case, the following theorem was established in [44]. See [44] Theorem 5.1 and the proof of [44] Theorem 5.8.

**Theorem 4.6.** [44] Theorem 5.1, Theorem 5.8] Let \( X \) be an essential strictly non-Euclidean CAT(0) cube complex admitting a proper cocompact action of \( G \leq \text{Aut}(X) \). Then \( B(X) \) is a \( G \)-boundary and there is a contractible open set for the action in \( B(X) \).

The following theorem was proved in [10]. See also [44] Theorem 7.4.

**Theorem 4.7.** [10] Theorem 4.2] Let countable discrete group \( G \) acts properly on a finite-dimensional CAT(0) cube complex \( X \). Then the stabilizer group \( \text{Stab}_G(x) \) is amenable for any \( x \in \mathcal{R}(X) \). In particular, \( \text{Stab}_G(x) \) is amenable for any \( x \in B(X) \).

Recall a classical result that any infinite irreducible RACG \( W_\Gamma \) (\( \Gamma \) is finite) is \( C^* \)-simple whenever \( |V(\Gamma)| \geq 3 \) (see e.g., [31]). We then establish the following result for irreducible RAAGs, which might be known to experts.

**Lemma 4.8.** Let \( A_\Gamma \neq \mathbb{Z} \) be an irreducible RAAG in which the defining graph \( \Gamma = (V,E) \) is finite. Then \( A_\Gamma \) is \( C^* \)-simple.

**Proof.** We follow the embedding arguments in [22] to show that such a \( A_\Gamma \) can be embedded into an irreducible non-amenable RACG. Indeed, since \( A_\Gamma \) is irreducible, the defining graph \( \Gamma \) has no joins. Define a new graph \( \Gamma' = (V', E') \) in the way that the vertex set \( V' = V \times \{0,1\} \) and

1. \( ((v,1), (w,1)) \in E' \) if and only if \( (v,w) \in E \);
2. \( ((v,0), (w,0)) \in E' \) for any \( v, w \in V \); and
3. \( ((v,0), (w,1)) \in E' \) if and only if \( v \neq w \).

It was proved in [22] that \( A_\Gamma \) can be embedded in \( W_{\Gamma'} \) as a subgroup with a finite index. We claim that \( W_{\Gamma'} \) is irreducible by showing that \( \Gamma' \) has no joins. Suppose the contrary that \( \Gamma' = \Gamma'_1 \ast \Gamma'_2 \) for two non-trivial subgraph \( \Gamma'_1 = (V_1', E_1') \) and \( \Gamma'_2 = (V_2', E_2') \). Then for any pair of vertices \( \{v\} \times \{0,1\} \), one has either \( \{v\} \times \{0,1\} \subset V'_1 \) or \( \{v\} \times \{0,1\} \subset V'_2 \) because there is no edge in \( E' \) between \( (v,0) \) and \( (v,1) \) in \( \Gamma' \).

This implies that \( V'_i = \{v \in V : \{v\} \times \{0,1\} \subset V'_i\} \) for \( i = 1, 2 \) form a non-trivial partition of \( V \). Now for any \( v \in V_1 \) and \( w \in V_2 \), since \( \Gamma'_1 \) and \( \Gamma'_2 \) form a join, one has \( ((v,1), (w,1)) \in E' \), which implies that \( (v,w) \in E \). Therefore \( \Gamma \) itself has a join, which is a contradiction. Now since \( A_\Gamma \neq \mathbb{Z} \), then \( |V| \geq 2 \) and thus \( |V'| \geq 4 \) by our construction. Now because \( W_{\Gamma'} \) is also irreducible, one has \( W_{\Gamma'} \) is \( C^* \)-simple, whence \( A_\Gamma \) is also \( C^* \)-simple because \( A_\Gamma \) is a subgroup of \( W_{\Gamma'} \) with a finite index by [31] Proposition 19. \( \square \)

**Remark 4.9.** Let \( \Gamma = (V,E) \) be a finite simple graph. In the infinite irreducible RACG case, if \( |V| = 2 \) then \( W_\Gamma = D_\infty \) and then \( \Sigma_\Gamma \) is Euclidean. In the irreducible RAAG case, if \( |V| = 1 \) then \( A_\Gamma = \mathbb{Z} \) and the complex \( \hat{S}_\Gamma \) is Euclidean. Now, we still write \( G_\Gamma \) as \( X_\Gamma \) for simplicity as usual, where \( G_\Gamma = W_\Gamma \) or \( A_\Gamma \) and \( X_\Gamma = \Sigma_\Gamma \) or \( \hat{S}_\Gamma \), respectively. Therefore, If \( X_\Gamma \) is irreducible and non-Euclidean then \( G_\Gamma \) is \( C^* \)-simple.
Now we are ready to prove the following main theorem in this section. We remark that in the following theorem it is necessary to assume there is one non-Euclidean factor $X_Γ$, in the canonical decomposition $X_Γ = X_Γ_1 \times \cdots \times X_Γ_m$ discussed above because if not, as we have shown above, the boundary $B(X_Γ)$ will be finite and the group $G_Γ$ is amenable, in which case the action $β$ below cannot be purely infinite.

**Theorem 4.10.** Let $G_Γ \curvearrowright X_Γ$ where $X_Γ$ is essential and has at least one non-Euclidean irreducible factor $X_Γ_i$ in the canonical decomposition above. Then the induced action $β : G_Γ \curvearrowright B(X_Γ)$ is purely infinite. In addition, in the RAAG case, the action $β$ has finitely many $G_Γ_i$-invariant closed sets. In the RACG case, $β$ is minimal. However, if $G_Γ$ has special subgroups $D_∞$ or $Z$, depending on the RACG case or the RAAG case, then the action is not topologically free.

**Proof.** Theorem 4.5 implies that $G_Γ = G_Γ × H^n$ where $H = D_∞$ or $Z$ and the complex $X_Γ$ is strictly non-Euclidean. Observe that $β$ is exactly the product action of $β_1 : G_Γ \curvearrowright B(X_Γ)$ and the action $β_2 : H^n \curvearrowright \{0, 1\}^n$. Theorem 4.6 implies that there is a contractible set $V$ in $B(X_Γ)$ for $β_1$ and also $β_1$ is minimal. Then Proposition 3.9 entails that $β_1$ is purely infinite. In addition, because $β_2$ is an action on a finite set, Proposition 3.9 implies that $β = β_1 × β_2$ is purely infinite.

Now observe that $G_Γ$ is a finite direct product of $C^*$-simple groups by Remark 4.9 and thus $G_Γ$ itself is $C^*$-simple by [31 Proposition 19]. Now since $β_1$ is a $G_Γ$-boundary action by Theorem 4.6 and the stabilizer group $Stab_G(x)$ of each $x ∈ B(X)$ is amenable by Theorem 4.7 one has $β_1$ is topologically free by [8 Proposition 1.9].

Now if $n > 0$, in the RAAG case, note that $H = Z$ and the action $β_2$ is the trivial action. Thus $β$ has in total $2^n G_Γ$-invariant closed sets. In the RACG case, note that the corresponding action $β_2$ of $H = D_∞$ on $\{0, 1\}^n$ is minimal. Then it is easy to verify that $β$ is minimal.

However, because there is an non-trivial element in $H$ fixing a point in $\{0, 1\}^n$, it follows from Proposition 3.9 that $β$ is not topologically free. In particular, $β$ is not essentially free.

Applying Theorem 3.6 one immediately has the following theorem.

**Theorem 4.11.** Let $G_Γ \curvearrowright X_Γ$ where $X_Γ$ is essential and has at least one non-Euclidean irreducible factor $X_Γ_i$ in the decomposition above. Suppose

1. $G_Γ = G_Γ × H^n$ where $H = D_∞$ or $Z$ and the complex $X_Γ$ is strictly non-Euclidean. Now if $G_Γ$ has no special subgroups of $D_∞$ or $Z$, then $n = 0$ and thus $X_Γ$ is strictly non-Euclidean. Then Theorem 4.10 implies that $β$ is minimal topologically free and purely infinite and thus $A = C(B(X_Γ)) ×_r G_Γ$ is unital simple separable and purely infinite. In addition, in the RACG case, $A$ is nuclear as well and thus a Kirchberg algebra satisfying the UCT.

**Proof.** Theorem 4.5 implies that $G_Γ = G_Γ × H^n$ where $H = D_∞$ or $Z$ and the complex $X_Γ$ is strictly non-Euclidean. Now if $G_Γ$ has no special subgroups of $D_∞$ or $Z$, then $n = 0$ and thus $X_Γ$ is strictly non-Euclidean. Then Theorem 4.10 implies that $β$ is minimal topologically free and purely infinite and thus $A = C(B(X_Γ)) ×_r G_Γ$ is unital simple separable and purely infinite by Theorem 3.6.

It is left to show $A$ is nuclear in the RACG case. First, for the Davis complex $X_Γ = Σ_Γ$, in the irreducible case, it was proved by [34 Theorem D] that $R(Σ_Γ) = B(Σ_Γ)$. In addition, $R(Σ_Γ)$ can be identified with the horofunction boundary of the Cayley graph of $W_Γ$ with the usual $ℓ_1$-metric by a unpublished work of U. Bader and D. Guralnik (see e.g. [44 Section 1.3]). On the other hand, the horofunction boundary of the Cayley graph of $W_Γ$ with the usual $ℓ_1$-metric also coincides with the minimal combinatorial boundary $C_1(Σ_Γ) \setminus Σ_Γ$ introduced in [14]. See [14] Theorem 3.1. Then finally it was proved in [34] that the action of $W_Γ$ on the minimal combinatorial boundary $C_1(Σ_Γ) \setminus Σ_Γ$ is amenable. Therefore, in the irreducible
case, the action $\beta : \Gamma \curvearrowright B(\Sigma \Gamma)$ is amenable using this chain of identifications. However, in our more general case that $X_\Gamma$ is strictly non-Euclidean, the action $\beta$ is a product of amenable actions and thus amenable. Therefore, $A$ is nuclear and thus a Kirchberg algebra. Also, $A$ satisfies the UCT by the result of Tu in [52].

If $G_\Gamma$ have sepicial subgroups of $D_\infty$ or $Z$, we then have the following result on the structure of the reduced crossed products.

**Corollary 4.12.** Let $G_\Gamma = G_{\Gamma'} \times H^n$ be the decomposition in Theorem 4.11. Write $A = C(\partial X_\Gamma) \rtimes_r G_\Gamma$. Then one has

1. In the RACG case, one has $A = (C(\partial X_{\Gamma'}) \rtimes_r G_{\Gamma'}) \otimes \bigotimes_{n=1}^\infty (C(\{0, 1\}) \rtimes_r D_\infty)$, where $C(X_{\Gamma'}) \rtimes_r G_{\Gamma'}$ is unital simple separable purely infinite.

2. In the RAAG case, one has $A = (C(\partial X_{\Gamma'}) \rtimes_r G_{\Gamma'}) \otimes (C(\{0, 1\})^n \otimes C(\mathbb{T}^n))$ in which $T$ is the unit circle and $C(X_{\Gamma'}) \rtimes_r G_{\Gamma'}$ is unital simple separable purely infinite.

In the RACG case, $A$ is $\mathcal{O}_\infty$-stable and actually, in either case, $A$ is strongly purely infinite.

**Proof.** In the context of the proof of Theorem 4.11 one can decompose $\beta : \Gamma \curvearrowright B(X_\Gamma)$ into the product action of $\beta_1 : G_{\Gamma'} \curvearrowright B(X_{\Gamma'})$ and $\beta_2 : H^n \curvearrowright \{0, 1\}^n$, which implies that $C(\partial X\rtimes_r G_\Gamma$ is the tensor product of the reduced product $C(\partial X_{\Gamma'}) \rtimes_r G_{\Gamma'}$ with the reduced crossed product of $\beta_2$. Note that $C(\partial X_{\Gamma'}) \rtimes_r G_{\Gamma'}$ is unital simple separable purely infinite by Theorem 4.11.

On the other hand, in the RACG case, the reduced crossed products of $\beta_2$ is exactly $\bigotimes_{n=1}^\infty (C(\{0, 1\}) \rtimes_r D_\infty)$. In the RAAG case, because the action $\beta_2$ is trivial, the corresponding crossed product is $C(\{0, 1\})^n \otimes C_\alpha(\mathbb{Z}^n) = C(\{0, 1\})^n \otimes C(\mathbb{T}^n)$.

In the RACG case, because $C(\partial X_{\Gamma'}) \rtimes_r G_{\Gamma'}$ is a Kirchberg algebra and thus $\mathcal{O}_\infty$-stable. Therefore, $A$ is also $\mathcal{O}_\infty$-stable and thus strongly purely infinite. In the RAAG case, $C(\partial X_{\Gamma'}) \rtimes_r G_{\Gamma'}$ is simple and purely infinite, and thus strongly purely infinite by [40, Theorem 9.1]. Then it follows from [33, Theorem 1.3] that $A$ is strongly purely infinite as well.

**Remark 4.13.** We remark that Theorem 4.11(1) has generalized the result in [26, Example 4.8] because the boundary considered there can also be identified with the horofunction boundary of the Cayley graph of $W_\Gamma$ with the usual $\ell_1$-metric. which is exactly the Nevo-Sageev boundary as we have shown in the proof of Theorem 4.11.

On the other hand, after the minimality and topologically freeness as well as the amenability of the action $\beta$ have been established by using the method in this section, one can apply the result [26, Theorem B] to obtain a different proof of this result. However, at this moment, the results in [26] do not apply to 4.11(2). The main obstruction is that, to the best knowledge of the authors, it is unknown whether the action of an irreducible non-amenability RAAG on the Nevo-Sageev boundary $B(X)$ is amenable.

5. Visual Boundary $\partial X$

5.1. Actions on irreducible CAT(0) cube complexes. In this section, we focus on actions of certain groups on the visual boundaries of some CAT(0) spaces, especially on the visual boundaries of CAT(0) cube complexes in this subsection. We will deal with actions on the visual boundaries of trees in the next subsection.

We begin with the definition of the visual boundary. Let $(X, d)$ be a CAT(0) space, we say two geodesic rays $c_1, c_2 : [0, \infty) \to X$ are asymptotic if there is a $C > 0$ such that $d(c_1(t), c_2(t)) < C$ for any $t \in [0, \infty)$. We remark that being asymptotic is an equivalence relation for geodesic rays. Denote by $\partial X$ the set of equivalence classes, which is called the boundary set of $X$. In addition, for any geodesic $c : [0, \infty) \to X$, we denote by $c(\infty)$ the equivalence class containing $c$. We further remark that
isometry. Then \( \Lambda \) the action is called elementary [30, Theorem 1.1] \( G \) or if \( X \) space see more.

Then one observes that \( G \), \( X \) acts on a proper CAT(0) space

repelling fixed points of \( g \)

Proof. \( \alpha \) thus purely infinite. Then \( \gamma \) are

find two open neighborhoods \( U \) of \( \gamma \) are 2-filling and thus is purely infinite. Indeed, we have the following proposition. We remark that similar arguments also appeared in \([1]\) and \([11]\).

**Proposition 5.3.** Let \( \alpha : G \curvearrowright Y \) be a continuous minimal action of a discrete group \( G \) on a infinite compact Hausdorff space \( Y \). Suppose there is a \( g \in G \) performing the north-south dynamics. Then \( \alpha \) is 2-filling and thus is purely infinite. In addition, \( \alpha \) is a \( G \)-boundary action.

**Proof.** Let \( O_1, O_2 \) be non-empty open sets in \( Y \). Suppose \( x, y \) are attracting and repelling fixed points of \( g \), respectively. First, by minimality of the action, one can find two open neighborhoods \( U, V \) of \( x, y \), respectively, small enough such that there are \( \gamma_1, \gamma_2 \in G \) such that \( \gamma_1V \subset O_1 \) and \( \gamma_2U \subset O_2 \). Now our assumption on \( g \) implies that there is an \( m \in \mathbb{N} \) such that \( g^m(Y \setminus V) \subset U \), which implies \( \gamma_2g^m(Y \setminus V) \subset O_2 \). Then one observes that \( Y = (\gamma_2g^m)^{-1}O_2 \cup \gamma_1^{-1}O_1 \). This shows that \( \alpha \) is 2-filling and thus purely infinite. Then \( \alpha \) is a strong boundary action and thus is a \( G \)-boundary action.

We then mainly focus on the case that a countable discrete group \( G \) acting on a proper CAT(0) cube complex \( X \). We first recall the following result in \([30]\). If \( G \) acts on a proper CAT(0) space \( X \) by isometry. The limit set \( \Lambda \) of \( G \) is the set of accumulation points in \( \partial X \) of an orbit of the action, which is closed and \( G \)-invariant. The action is called elementary if either its limit set \( \Lambda \) consists of at most two points or if \( G \) fixes a point in \( \partial X \).

**Proposition 5.4.** [30] Theorem 1.1] Let \( G \) acts by isometry on a proper CAT(0) space \( X \) non-elementarily with the limit set \( \Lambda \). Suppose \( G \) contains a rank-one isometry. Then \( \Lambda \) is perfect and the induced action \( \alpha : G \curvearrowright \Lambda \) is minimal.

In fact, if the action is cocompact and the visual boundary is non-trivial, we may see more.
Remark 5.5. It was proved in [5] that if $|\partial X| > 2$ and the action of $G$ on $X$ cocompactly by isometry then the action is necessarily non-elementary and the limit set $\Lambda = \partial X$. Note that it follows that there is no global fixed point in $\partial X$ for $G$ in this case. See more in [6] and [30].

We denote by $P(G)$ the set of all probability measures on $G$. Let $G$ acts on a metric space $(X, d)$ by isometry. A probability measure $\mu \in P(G)$ is said to have finite first moment if $\sum_{g \in G} d(gy, gy) \mu(g) < \infty$. A measure $\mu \in P(G)$ is said to be generating if the support of $\mu$ generates $\Gamma$ as a semigroup, i.e., for any $g \in G$ there are several $h_1, \ldots, h_n \in \text{supp}(\mu)$ such that $g = h_1 \ldots h_n$. For a $\mu \in P(G)$, we denote by $\mu$ the reflected measure of $\mu$ on $G$ defined by $\hat{\mu}(g) = \mu(g^{-1})$ for $g \in G$. Note that it is very easy to find a generating $\mu \in P(G)$ with finite first moment for a finitely generated group $G$. Now we have the following result.

Lemma 5.6. Let $G$ be a $C^*$-simple finitely generated group and acts properly, freely, essentially and cocompactly by isometry on a proper irreducible finite dimensional CAT(0) cube complex $X$ such that

1. $|\partial X| > 2$ and
2. there is a vertex $x \in X^0$ such that the map $g \mapsto gx$ from $G$ to $X$ is a $(c, b)$-quasi-isometric map for some $c, b > 0$.

Then the reduced crossed product $C(\partial X) \rtimes_r G$ of the induced action $\alpha: G \curvearrowright \partial X$ is simple and purely infinite.

Proof. First, Remark 5.5 implies that there is no global fixed point of $G$ on $\partial X$. Then because $X$ is irreducible and the action of $G$ on $X$ by isometry is essential and proper, it follows from [15] Theorem A) that $G$ has a rank-one isometry $g$. Now, Proposition 5.4 implies that $\partial X$ is perfect and the induced action of $G$ on $\partial X$ is minimal. Note that the rank-one isometry $g$ performs the north-south dynamics on $\partial X$. Therefore, Proposition 5.3 implies that $\alpha$ is a $G$-boundary action and thus $2$-filling.

Choose a generating probability measure $\nu$ on $G$ with finite first moment. Denote by $\rho$ the usual word metric on $G$ with respect to a finite generating set. The assumption (2) implies that

$$(1/c) \rho(1_G, g) - b \leq d(x, gx) \leq c \rho(1_G, g) + b,$$

whence for any $n \in \mathbb{N}$, one has $\{g \in G : d(x, gx) \leq n\} \subset B_\rho(1_G, c(b + n))$, where $B_\rho(1_G, c(b + n))$ is the ball in $G$ centered at $1_G$ with radius $c(b + n)$. Since $G$ is finitely generated, it has to be at most exponential growth. Therefore, there is a $C > 0$ such that

$$\{|g \in G : d(x, gx) \leq n\} \leq |B_\rho(1_G, c(b + n))| \leq e^{Cn}.$$  

Then, since the action is additionally minimal, it follows from [37] Corollary 6.2 that there are two Probability measure $\mu_+$ and $\mu_-$ on the visual boundary $\partial X$ such that $(\partial X, \mu_+)$ and $(\partial X, \mu_-)$ are Furstenberg-Poisson boundaries of $(G, \nu)$ and $(G, \hat{\nu})$, which forms a boundary pair in the sense of [4] Definition 2.3 by [24] Theorem 5.5. Then since the action of $G$ is non-elementary by Remark 5.5, it follows from [24] Theorem 7.1 that there is a measurable $G$-equivariant map $\varphi: \partial X \rightarrow \mathcal{R}(X)$, which implies that for any $y \in \partial X$ and $g \in G$, if $gy = y$ then $g \varphi(y) = \varphi(y)$. Therefore, the stabilizer group $\text{Stab}_G(y)$ is a subgroup of $\text{Stab}_G(\varphi(y))$, which is amenable by Theorem 4.7. It follows that $\text{Stab}_G(y)$ is amenable as well. Now since the $\partial X$ is a $G$-boundary and $G$ is $C^*$-simple, the action $\alpha$ is topologically free by [8] Proposition 1.9. Therefore the reduced crossed product $C(\partial X) \rtimes_r G$ is simple and purely infinite by Theorem 3.6. □

Our main application is still for actions of $G_1$ on $X_1$, where $G_1$ is a RACG or a RAAG while $X_1$ is the corresponding Davis complex or the universal cover of the Salvetti complex. We have the following main theorem in this subsection.
Theorem 5.7. Let $\Gamma = (V, E)$ be a finite simple graph without joins.

1. if $|V| \geq 3$ then $C(\partial\Sigma) \times_r W_\Gamma$ is simple and purely infinite.
2. If $|V| \geq 2$ then $C(\partial\mathcal{H}_\Gamma) \times_r A_\Gamma$ is simple and purely infinite.

Proof. We still write $G_\Gamma$ for $W_\Gamma$ or $A_\Gamma$ for the corresponding complexes for simplicity. For the cases mentioned above, $G_\Gamma$ is finitely generated and $C^*$-simple. In addition, it is known in this irreducible case, the boundary $\partial\mathcal{X}_\Gamma$ contains more than 2 elements. Finally, since the 1-skeletons of $\mathcal{X}_\Gamma$ is the Cayley graph of $G_\Gamma$, the other conditions of Lemma 5.6 are easy to be verified. □

5.2. Actions on Bass-Serre trees. Another important case involving the visual boundary is that groups act on the visual boundary of trees, especially actions of the fundamental group of a graph of groups on the boundary of its Bass-Serre tree. We refer to [50] and [12] for notations and backgrounds. However, we follow the notations in [12] and still recall necessary concepts here. Given a graph $\Gamma = (V, E)$, from the viewpoint of groupoids, one may identify the vertex set $V$ with the unit space of $\Gamma$ and the edge set $E$ with “arrows” in the groupoids. Then one may define the source and range maps of an edge, which provides a direction of each edge. We abuse the notation by still denoting $E$ for all directed edges. This also allows to define the “edge-reversing” map from $E$ to $E$ by $e \mapsto \bar{e} = e^{-1}$. It is not hard to see $\bar{e} \neq e$, $\bar{e} = e$, $s(e) = r(\bar{e})$ and $r(e) = s(\bar{e})$. If we assign indexes for $e$, e.g., $e_i$, then we write $\bar{e}_i$ for its reverse.

Definition 5.8. A graph of groups $\mathcal{G} = (\Gamma, G)$ consists a connected graph $\Gamma = (V, E)$ and a system of groups:

1. a vertex group $G_v$ for each $v \in V$;
2. an edge group $G_e$ for each $e \in E$ such that $G_e = G_\Gamma$; and
3. a monomorphism $\alpha_e : G_e \to G_{r(e)}$ for each $e \in E$.

For simplicity, we denote by $1_v$ and $1_e$ the identity elements in $G_v$ and $G_e$, respectively. We also write 1 if the context is clear. A graph $\Gamma = (V, E)$ is said to be locally finite if $|r^{-1}(v)| < \infty$ for any $v \in V$. We also say a graph of groups $\mathcal{G} = (\Gamma, G)$ is locally finite if

1. the underlying graph $\Gamma$ is locally finite; and
2. $[G_{r(e)} : \alpha_e(G_e)] < \infty$ for any $e \in E$.

The graph of groups $\mathcal{G} = (\Gamma, G)$ is also called non-singular if $[G_{r(e)} : \alpha_e(G_e)] > 1$ whenever $r^{-1}(r(e)) = \{e\}$. We remark that all graphs $\mathcal{G} = (\Gamma, G)$ in this subsection are locally finite and non-singular.

Definition 5.9. [12] Definition 2.5] Let $\mathcal{G} = (\Gamma, G)$ be a graph of groups. The path group, denoted by $\pi(\mathcal{G})$ is the group generated by the set $E \cup \bigcup_{v \in V} G_v$ modulo the relations

(R1) $\bar{e} e = 1$ for all $e \in E$
(R2) $e \alpha_e(g) \bar{e} = \alpha_e(g)$ for all $e \in E$ and $g \in G_e = G_\Gamma$.

Definition 5.10. [12] Definition 2.4] Let $\mathcal{G} = (\Gamma, G)$ be a graph of groups. For each $e \in E$, we fix a transversal $\Sigma_e$ for $G_{r(e)}/\alpha_e(G_e)$ with $1_{r(e)} \in \Sigma_e$.

1. A $\mathcal{G}$-word (of length $n$) is a sequence of the form $g_1, \ldots, g_{n} e_n$, or $g_1 e_1 g_2 e_2 \ldots g_n e_n$ such that $s(e_i) = r(e_{i+1})$ for $1 \leq i \leq n - 1$, $g_j \in G_{r(e_j)}$ for $1 \leq j \leq n$ and $g_{n+1} \in G_{s(e_n)}$. In the case $n = 0$, the element $g_1 \in G_v$ for some $v \in V$.
2. A reduced $\mathcal{G}$-word is a $\mathcal{G}$-word in which if $n > 0$ then $g_j \in \Sigma_{e_j}$ for $1 \leq j \leq n$ and $g_{n+1} \neq 1_{r(e_{n+1})}$ whenever $e_i = \bar{e}_{i+1}$. Note that there is no restriction for $g_{n+1} \in G_{s(e_n)}$.
3. A $\mathcal{G}$-path is a reduced $\mathcal{G}$-word of the form $g_1 = 1$ or $g_1 e_1 g_2 e_2 \ldots g_n e_n$. 
**Definition 5.11.** [12] Definition 2.6, 2.7 Let $\mathcal{G} = (\Gamma, G)$ be a graph of groups. For $v, w \in V$, define $\pi[v, w] \subset \pi(\mathcal{G})$ to be the set of images in $\pi(\mathcal{G})$ of the $\mathcal{G}$-words with the range $v$ and the source $w$. In the case that $v = w$, we write $\pi_1(\mathcal{G}, v)$ for $\pi[v, v]$, which is a subgroup of $\pi(\mathcal{G})$. We call $\pi_1(\mathcal{G}, v)$ the fundamental group of $\mathcal{G}$ based at $v$.

Note that by using relations (R1) and (R2), the image of any $\mathcal{G}$-word in $\pi[v, w]$ can be represented by a reduced $\mathcal{G}$-word. In particular, a typical element in the fundamental group $\pi_1(\mathcal{G}, v)$ is represented by a reduced $\mathcal{G}$-word with the source and range $v$.

**Definition 5.12.** [12] Definition 2.13 Let $\Gamma = (V, E)$ be a graph and $\mathcal{G} = (\Gamma, G)$ a graph of groups with a base vertex $v \in V$. The Bass-Serre tree $X_\mathcal{G}, v$ of $\mathcal{G}$ has vertex set

$$X_{\mathcal{G}, v}^0 = \bigcup_{w \in V} \pi[v, w]/G_w = \{ \gamma G_w : \gamma \in \pi[v, w], w \in V \}.$$ 

Then there is an edge between vertexes $\gamma G_w$ and $\gamma' G_w$ if $\gamma^{-1} \gamma' \in G_w e G_w$, for some $e \in E$ with $r(e) = w$ and $s(e) = w'$.

It was proved by [6] Theorem 1.17 that $X_{\mathcal{G}, v}$ is indeed a tree. The natural action of $\pi(\mathcal{G}, v)$ on $X_{\mathcal{G}, v}$ is given as follows. Let $\gamma \in \pi_1(\mathcal{G}, v) = \pi[v, v]$ and $\gamma' G_w \in X_{\mathcal{G}, v}^0$. One defines $\gamma \cdot \gamma' G_w = \gamma \gamma' G_w$ and this action extends to an action on the edges of $X_{\mathcal{G}, v}$. We also remark that the Fundamental Theorem of Bass-Serre Theory (see [9] and [12] Theorem 2.16)) implies that the whole process above is independent of the choice of the base vertex. Let $v$ be a base vertex in $V$ and then we choose $1_v G_v \in X_{\mathcal{G}, v}^0$ as our base vertex of $X_{\mathcal{G}, v}$.

In general, for a tree $T$ with a base vertex $x_0$. By definition, the visual boundary $\partial T$ is exactly the set of infinite branches starting from $x_0$, equipped with cone topology generated by cylinder sets of the form $Z(\mu)$. Here, $Z(\mu)$ is the set that consists all infinite branches with a common initial segment $\mu$, where $\mu$ is a finite path starting from $x_0$. One can easily verify that under the cone topology, $\partial T$ is a compact Hausdorff totally disconnected space.

In particular, we denoted by $v \partial X_{\mathcal{G}}$ the visual boundary of $X_{\mathcal{G}, v}$ with respect to the base vertex $1_v G_v$. We also write $\partial X_{\mathcal{G}} = \bigcup_{v \in V} v \partial X_{\mathcal{G}}$ the union of all boundaries defined from all base vertices of $V$. As one may observe, each vertex of $X_{\mathcal{G}, v}$ has a unique representative of a $\mathcal{G}$-path $g_1 e_1 \ldots g_n e_n$ where $r(e_1) = v$. There is an edge between two vertexes $\gamma G_v$ and $\gamma' G_v'$ if and only if the representative of one of these two vertices extends another with length plus one. See more in [12] Definition 2.13 and [6] Remark 1.18. From this identification, one may view the visual boundary of $X_{\mathcal{G}, v}$ by all infinite reduced $\mathcal{G}$-words with range $v$, i.e., the infinite sequences $g_1 e_1 g_2 e_2 \ldots$ such that each initial finite subsequence $g_1 e_1 \ldots g_n e_n$ is a reduced $\mathcal{G}$-word. In addition, the natural induced action of $\pi_1(\mathcal{G}, v)$ on $v \partial X_{\mathcal{G}}$ by homeomorphism can be described in a symbolical way. Let $\gamma = [g_1 e_1 \ldots g_n e_n g_{n+1}] \in \pi_1(\mathcal{G}, v)$ in which $g_1 e_1 \ldots g_n e_n g_{n+1}$ is a reduced $\mathcal{G}$-word and $r(e_1) = s(e_{n+1}) = v$, and an infinite reduced words $\eta = h_1 f_1 h_2 f_2 \ldots \in v \partial X_{\mathcal{G}}$, then one has that $\gamma \cdot \eta$ is exactly the infinite reduced words uniquely determined by $g_1 e_1 \ldots g_n e_n g_{n+1} f_1 h_1 h_2 f_2 \ldots$ by doing reduction by using relation (R1) and (R2) even possibly infinite times. See more in [12] Section 2.3.1. Now we need the following key concept.

**Definition 5.13.** [12] Definition 5.14 We say a $\mathcal{G}$-path $g_1 e_1 \ldots g_n e_n$ is repeatable if $r(e_1) = s(e_n)$ and $g_1 e_1 \neq 1_{\pi(e_n)} e_n$.

Let $\mu = g_1 e_1 \ldots g_n e_n$ be a repeatable path. Then denote by $\mu^m$ the concatenation of $\mu$ by itself for $m$ times. Note that the repeatability of $\mu$ implies that $\mu^m$ is a reduced word. We also allow $m = \infty$, in which case, $\mu^\infty$ is an infinite reduced words located in the boundary $v \partial X_{\mathcal{G}}$. 

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**References**

[6], [9], [12]
Proposition 5.14. Let $\Gamma = (V, E)$ be a graph and $G = (\Gamma, G)$ a graph of groups. Suppose $\nu \partial X_G$ is an infinite set and there is a repeatable word $\mu = g_1 e_1 \ldots g_n e_n$ with $|\Sigma_{e_n}| \geq 2$ and the natural action $\beta : \pi(G, v) \curvearrowright \nu \partial X_G$ is minimal. Then $\beta$ is 2-filling and thus a strong boundary action.

Proof. Let $\mu = g_1 e_1 \ldots g_n e_n$ be the repeatable path. Then let $v = r(e_1) = s(e_n)$ be our base vertex and consider the corresponding fundamental group $\pi_1(G, v)$ and the Bass-Serre tree $X_G, v$, together with the visual boundary $\nu \partial X_G$. Note that by definition $\mu^m \in \pi_1(G, v)$ for any $m \in \mathbb{N}$.

Let $m \in \mathbb{N}$. For any $G$-word $sf$ with length 1 such that $f \in E$ with $r(f) = v$ and $s \in \Sigma_f$. If $f \neq e_n$, then for any $\eta \in Z(sf)$, one has that the concatenation $\mu^m \cdot \eta$ is still an infinite reduced word, whence $\mu^m \cdot Z(sf) \subset Z(\mu^m)$. Define

$A_1 = \bigcup_{f \in E, r(f) = v, f \neq e_n, s \in \Sigma_f} Z(sf)$

and one actually has $\mu^m \cdot A_1 \subset Z(\mu^m)$. Now suppose $f = e_n$. Then for any $s \neq 1_{r(e_n)}$ and any $\xi \in Z(s e_n)$, the concatenation $\mu^m \cdot \xi$ is still an infinite reduced word. Define

$A_2 = \bigcup_{s \in \Sigma_f \setminus \{1_{r(e_n)}\}} Z(s e_n)$

and one has $\mu^m \cdot A_2 \subset Z(\mu^m)$. Define $A = A_1 \cup A_2$. Finally, for the case $B = Z(1_{r(e_n)} e_n)$, since $|\Sigma_{e_n}| \geq 2$, one can choose a $t \in \Sigma_{e_n} \setminus \{1_{r(e_n)}\} \subset G_{r(e_n)} = G_{s(e_n)}$. This shows that $\mu^m t$ is still an group element in $\pi_1(G, v)$. Now for any $\xi \in Z(1_{r(e_n)} e_n)$, which is of the form $\xi = 1_{r(e_n)} e_n^\rho$ and thus $\mu^m t \cdot \xi = \mu^m t e_n^\rho$, which is an infinite reduced word in $Z(\mu^m)$. This implies that $\mu^m t \cdot B \subset Z(\mu^m)$.

Now observe that $\{Z(\mu^m) : m \in \mathbb{N}\}$ forms a neighborhood basis of $\mu^\infty$. For any non-empty open sets $O_1, O_2$ in $\nu \partial X_G$, since the action $\beta$ is minimal, there are $\gamma_1, \gamma_2 \in \pi_1(G, v)$ and an $m \in \mathbb{N}$ such that $\gamma_i Z(\mu^m) \subset O_i$ for $i = 1, 2$. Now define group elements $h_1 = \gamma_1 \mu^m$ and $h_2 = \gamma_2 \mu^m t$ and observe that $h_1 A \subset O_1$ and $h_2 B \subset O_2$. This implies that the whole boundary $\nu \partial X_G = A \cup B \subset h_1^{-1} O_1 \cup h_2^{-1} O_2$ and thus $\beta$ is 2-filling.

On the other hand, a characterization of minimality of the action $\pi(G, v) \curvearrowright \nu \partial X_G$ was proved in [12].

Definition 5.15. [12] Definition 5.3 Let $\Gamma = (V, E)$ be a graph and $G = (\Gamma, G)$ a graph of groups. Let $e, f \in E$. We say $f$ can flow to $e$ if $f$ occurs in an infinite reduced words $\xi \in Z(1_{r(e)} e)$ and $f$ is not the range-most edge of $\xi$. We say a boundary point $\xi \in \partial X_G$ can flow to $e$ if there is an $f$ occurs in $\xi$ can flow to $e$.

See more in [12] Lemma 5.4 for an elementary and more explicit description of Definition 5.15. Then we record the following theorem on minimality.

Theorem 5.16. [12] Theorem 5.5] The action $\beta : \pi(G, v) \curvearrowright \nu \partial X_G$ is minimal if and only if $\xi$ can flow to $e$ for any $\xi \in \partial X_G$ and $e \in E$.

Combining these result, we have the following result immediately.

Theorem 5.17. Let $\Gamma = (V, E)$ be a graph and $G = (\Gamma, G)$ a graph of groups. Suppose

1. $\nu \partial X_G$ is infinite;
2. $\xi$ can flow to $e$ for any $\xi \in \partial X_G$ and $e \in E$; and
3. there is a repeatable path $\mu = g_1 e_1 \ldots g_n e_n$ with $|\Sigma_{e_n}| \geq 2$.

Then the natural action $\beta : \pi(G, v) \curvearrowright \nu \partial X_G$ is a strong boundary action. In particular, $\beta$ is a $\pi_1(G, v)$-boundary action. If, in addition, each $G_e$ is amenable and $\pi_1(G, v)$ is $C^*$-simple, then the action $\beta$ is topologically free and thus the crossed product $C(\nu \partial X_G) \rtimes_\beta \pi_1(G, v)$ is a unital Kirchberg algebra satisfying the UCT.
Proof. By assumptions, observe that $\beta$ is a strong boundary action by Proposition 5.14 and thus a $\pi_1(G, v)$-boundary action. Now if all $G_e$ are amenable groups, then so are all $G_v$ because $[G_{r(v)}, \alpha_v(G_v)]$ is finite for all $e \in E$. Therefore, the action $\beta$ is amenable by [11 Proposition 5.2.1]. Now in the case that $\pi_1(G, v)$ is $C^*$-simple, the crossed product $C(\partial X_G) \rtimes_\pi \pi_1(G, v)$ is simple and nuclear. Therefore, $\beta$ is topologically free by [3]. Then Theorem 5.16 implies that $C(\partial X_G) \rtimes_\pi \pi_1(G, v)$ is purely infinite and thus a unital Kirchberg algebra satisfying the UCT by the classical result of Tu in [52].

Example 5.18. Consider the classical case that $\Gamma = (V, E)$ such that $V = \{u, v\}$ and $E = \{e\}$ with $e = (u, v)$. Now define $n_f = |\Sigma_f| = [G_{r(f)} : \alpha_f(G_f)]$ for $f = e$ or $\bar{e}$. In the non-degenerated case, i.e., $(n_e - 1)(n_e - 1) \geq 2$, observe that the graph of groups $\mathcal{G}$ satisfies Theorem 5.17.

Indeed, first, since $n_e, n_{\bar{e}} \geq 2$, one can choose $g, h \not= 1$ such that the path $geh\bar{e}$ is repeatable. For the minimality of the action, since there are only two edges, i.e., $e, \bar{e}$ in $E$, simply observe the infinite reduced words $1e1e\ldots$ and $1eh\bar{e}\ldots$, where $h \not= 1$ in $Z(1e)$, witness that all $\xi \in \partial X_G$ flow to $e$ because the only possible edges appeared in $\xi$ are $e$ and $\bar{e}$. The same argument shows all $\xi \in \partial X_G$ flow to $\bar{e}$ as well. Then it follows from Theorem 5.16 that the action is minimal. On the other hand, when $n_e$ or $n_{\bar{e}} \geq 3$, the boundary $v\partial X_G$ is infinite (see e.g., [12 Example 2.14(E1)]). Therefore, the action in Theorem 5.17 for non-degenerated free product with amalgamation $\pi_1(G, v)$ is a strong boundary action. To observe more, if $G_e$ is amenable, then so are $G_e$ and $G_u$ because $n_e, n_{\bar{e}}$ are assumed to be finite in our locally finite setting (Definition 5.8) and the action $\beta$ is amenable by [11 Proposition 5.2.1]. Now if $\pi_1(G, v)$ is $C^*$-simple, (e.g., when $G_e$ is trivial by [31 Corollary 12]) then $\beta$ is topologically free. In this case $\pi_1(G, v)$, the reduced crossed product $C(\partial X_G) \rtimes_\pi \pi_1(G, v)$ is a unital Kirchberg algebra satisfying the UCT.

From now on, we focus on the generalized Baumslag-Solitar groups (GBS groups for simplicity), which can be regarded as a fundamental group $\pi_1(G, v)$ for a graph of groups $\mathcal{G} = (\Gamma, G)$ in which vertex groups $G_v$ and edge groups $G_e$ are isomorphic to $\mathbb{Z}$. In this case, we also call the graph of groups $\mathcal{G} = (\Gamma, G)$ a GBS graph of groups.

Let $\mathcal{G} = (\Gamma, G)$ be a locally finite non-singular GBS graph of groups in which $\Gamma = (V, E)$. Then in Definition 5.10 one can choose the transversal $\Sigma_e = \{0, 1, \ldots, |k_e| - 1\}$ for some $k_e \in \mathbb{Z}$ and actually $|k_e| = [G_{r(e)} : \alpha_e(G_e)]$. For each $G$-word $\mu = g_1e_1 \ldots g_ne_n g_{n+1}$, one may assign a rational number $\nu(\gamma) = \prod_{i=1}^{n} (k_{e_i}/|k_{e_i}|)$ and verify that the restriction of $\nu$ on $\pi_1(G, v)$ to $\mathbb{Q}^\times$ is a group homomorphism. The graph of groups $\mathcal{G}$ is called unimodular if $|\nu(\gamma)| = 1$ for any $G$-word $\gamma$ with $s(\gamma) = r(\gamma)$.

It was also provided in [12 Theorem 7.5] a characterization of when the natural action $\beta : \pi_1(G, v) \subseteq v\partial X_G$ is topologically free. However, if we restrict to finite graphs, one actually has a very nice characterization.

Proposition 5.19. [12 Corollary 7.11] Let $\mathcal{G} = (\Gamma, G)$ be a GBS graph of groups in which $\Gamma = (V, E)$ is a finite graph. Then the natural action $\beta : \pi_1(G, v) \subseteq v\partial X_G$ is topologically free if and only if $\mathcal{G}$ is not unimodular.

Combining Theorem 5.17 and Proposition 5.19, one has the following pure finiteness result, which also yields a new method to find $C^*$-simple GBS groups.

Theorem 5.20. Let $\mathcal{G} = (\Gamma, G)$ be a GBS graph of groups in which $\Gamma = (V, E)$ is a finite graph. Suppose

1. $v\partial X_G$ is infinite;
2. $\xi$ can flow to $e$ for any $\xi \in \partial X_G$ and $e \in E$;
3. there is a repeatable path $\mu = g_1e_1 \ldots g_ne_n$ with $|\Sigma_{\bar{e}_n}| \geq 2$; and
4. $\mathcal{G}$ is not unimodular.
Then the natural action $\beta : \pi_1(\mathcal{G}, v) \curvearrowright v\partial X_{\mathcal{G}}$ is an topological amenable topologically free strong boundary action and the crossed product $C(v\partial X_{\mathcal{G}}) \rtimes_{\pi_1(\mathcal{G}, v)}$ is a unital Kirchberg algebra satisfying the UCT. Furthermore, $\pi_1(\mathcal{G}, v)$ is $C^*$-simple.

Proof. Amenability of the action $\beta$ follows from the fact that each vertex group $G_v \simeq \mathbb{Z}$ and [11 Proposition 5.2.1]. Therefore the reduced crossed product is nuclear and thus satisfies the UCT by [32]. it follows from Theorems 5.16 and 3.6 as well as Proposition 5.19 that $C(v\partial X_{\mathcal{G}}) \rtimes_{\pi_1(\mathcal{G}, v)}$ is a unital Kirchberg algebra satisfying the UCT. For $C^*$-simplicity part, apply [32 Theorem 1.5].

One may also want to compare our Theorems 5.20 and 5.17 with results obtained in [13]. In particular, we also have the following examples satisfying Theorem 5.20.

Example 5.21. First, we claim that Baumslag-Solitar groups (BS groups) $BS(k, l)$ where $\{k \mid 1 \leq k \leq \{l \mid 1 \geq 2, satisfies Theorem 5.20. In fact, all $B(k, l)$ can be written as $\pi_1(\mathcal{G}, v)$, in which $\mathcal{G} = (\Gamma = (V, E), G)$ such that $V = \{v\}$ and $E = \{e\}$ with $s(e) = r(e) = v$ and $|k| = |\Sigma_e| = |G_v : \alpha_e(G_e)|$ as well as $|l| = |\Sigma_e| = |G_v : \alpha_e(G_e)|$. First, 1e is a desirable repeatable element. Then using the same argument in Example 5.18, if $|k|, |l| \geq 2$ then any $\xi$ flows to any $e$ or $\bar{e}$ and thus the action $\beta$ is minimal. In addition, in this case $v\partial X_{\mathcal{G}}$ is infinite (see e.g., [12 Example 2.14(E2)]. Finally, $\mathcal{G}$ is unimodular if and only if $|k| \neq |l|$.

Example 5.22. (n-circle) One may generalize the construction in Example 5.21 by considering the GBS graph $\Gamma = (V, E)$ in which $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_n\}$ such that $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq n - 1$ and $e_n = (v_n, v_1)$. Actually, $\Gamma$ is exactly a circle with $n$ vertices and $n$ edges. From this viewpoint, the graph for Example 5.21 is exactly a 1-circle. We now assume each $|k_{e_i}|$ and $|k_{e_i}| \geq 2$ and it is not hard to see that in this case the boundary is infinite.

First, a path of the form $g_1e_1 \ldots g_ne_n$ is repeatable. Then, to verify minimality, similar to the argument in Example 5.18 since only $e_i, \bar{e}_i$ for $i = 1, \ldots, n$ may occur in any infinite path, for $e_i$, the branch

$1e_11e_2 \ldots 1e_ng_n1e_{n-1} \ldots 1e_1 \ldots ,$

where $g \neq 1_{e_i}$, witnesses that any infinite path $\xi \in \partial X_{\mathcal{G}}$ flows to $e_1$. By symmetry and the fact that all $|k_{e_i}|$ and $|k_{e_i}| \geq 2$, we have $\xi$ flows to $e_i$ for any $\xi \in \partial X_{\mathcal{G}}$ and any $e_i$ and $\bar{e}_i$ for $i = 1, \ldots, n$. This establishes the minimality of the action. Now since there is only one circle in the graph $\mathcal{G}$, the $\Gamma$ is not unimodular if and only if $q = \prod_{1 \leq i \leq n} k_{e_i}/k_{\bar{e}_i} \neq \pm 1$. Therefore, with a proper assignment of all $k_{e_i}$ and $k_{\bar{e}_i}$ for $1 \leq i \leq n$ such that

1. all $|k_{e_i}|$ and $|k_{\bar{e}_i}| \geq 2$ and
2. $q = \prod_{1 \leq i \leq n} k_{e_i}/k_{\bar{e}_i} \neq \pm 1$,

the group $\pi_1(\mathcal{G}, v)$ satisfy Theorem 5.20.

Example 5.23. (wedge sums of n-circles) To be even more general, we consider the graph $\Gamma = (V, E)$, which is a wedge sum of two $n$-circles, which means

1. $V = V_1 \cup V_2$ such that there is a $v \in V_1 \cap V_2$ and $(V_1 \setminus \{v\}) \cap (V_2 \setminus \{v\}) = \emptyset$;
2. $E = E_1 \sqcup E_2$ and
3. the subgraph $\Gamma_i = (V_i, E_i)$ is a $n$-circle for $i = 1, 2$.

Similar work would show that $\pi_1(\mathcal{G}, v)$ still satisfies Theorem 5.20 for proper choices of $k_e$ and $k_{\bar{e}}$ for $e \in E$. We also assume all $|k_e|$ and $|k_{\bar{e}}|$ $\geq 2$ for $e \in E$. Then, first, each $n$-circle yields a repeatable path and the boundary $v\partial X_{\mathcal{G}}$ is infinite.

Now we write $E_1 = \{e_1, \ldots, e_n\}$ and $E_2 = \{f_1, \ldots, f_n\}$. Considering the directions, without loss of any generality, one may assume

1. $r(e_1) = r(f_1) = v$;
2. $s(e_i) = r(e_{i+1})$ and $s(f_i) = r(f_{i+1})$ for $1 \leq i \leq n - 1$;
3. $s(e_n) = r(e_1)$ and $s(f_n) = r(f_1)$. 

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Then since edges may appear in any infinite path are just all $e_k, \bar{e}_k$ and $f_l, \bar{f}_l$, for any $e_i$, the following infinite reduced word
\[1e_1e_{l+1} \ldots 1e_n1f_1 \ldots 1f_k1e_{l}1e_2 \ldots 1e_{l-1}g\bar{e}_{l-1}1e_{l-2} \ldots 1e_11\bar{f}_n \ldots 1\bar{f}_l1e_n \ldots 1e_i \ldots ,\]
where $g \neq 1$, witnesses that any infinite branch $\xi \in \partial X_G$ flows to $e_i$. Then the same argument shows that any infinite branch $\xi$ flows to any $e$ or $\bar{e} \in E$. Therefore the action of $\pi_1(G, v)$ on $\partial X_G$ is minimal. Finally, for $n$-circles $\Gamma_1$ and $\Gamma_2$, define $q_1 = \prod_{i=1}^k e_i/k\bar{e}_i$ and $q_2 = \prod_{i=1}^k f_i/k\bar{f}_i$. Then $G$ is unimodular if and only if $q_1$ or $q_2 \neq \pm 1$ or $q_1^{\pm 1} \cdot q_2^{\pm 1} \neq \pm 1$. Combining this, for a wedge sum of $n$-circles $\Gamma = (V, E)$, we have

1. $|k_e|, |k_{\bar{e}}| \geq 2$ for any $e \in E$ and
2. $q_1$ or $q_2$ or $q_1^{\pm 1} \cdot q_2^{\pm 1} \neq \pm 1$;

then $\pi_1(G, v)$ satisfies Theorem 5.20. To deal with wedge sums of $n$-circles for $m$-times, i.e., the underlying graph $\Gamma = (V, E)$ contains $m$ subgraph $\Gamma_i = (V_i, E_i)$, for $1 \leq i \leq m$, which are all $n$-circles and there is exactly one common vertex $v$ in all $V_i$ as our base vertex. For each such $\Gamma_i$, we can define corresponding $q_i \in \mathbb{Q}$. Using the same argument, we have if

1. $|k_e|, |k_{\bar{e}}| \geq 2$ for all $e \in E$ and
2. there is a finite set $F \subset \{1, \ldots, m\}$ such that $\prod_{i \in F} q_i^{\pm 1} \neq \pm 1$,

then $\pi_1(G, v)$ satisfies Theorem 5.20.

**Remark 5.24.** We finally remark that GBS group $\pi_1(G, v)$ in Example 5.22 and 5.23 are not isomorphic to a BS group $BS(k, l)$ in Example 5.21 whenever $n \geq 2$ or $m \geq 2$. Even the isomorphism problem of GBS groups have not been fully solved, in our specific cases, there is a way to tell this. We refer readers to 20. First, in general, if two non-elementary (see 20 Section 2.2 for its definition) GBS groups $G$ and $H$ are isomorphic, then their corresponding Bass-Serre trees are in the same deformation space in the sense of 20. Definition 2.1], which implies that there is a path from the underlying graph $\Gamma_G$ to $\Gamma_H$ by using moves of three types defined in 20. Definition 2.2. Moreover, the first Betti numbers of $\Gamma_G$ and $\Gamma_H$ are same.

In our case, first, we consider wedge sums of $n$-circles. Let $G_1 = (\Gamma_1, G)$ such that $\Gamma_1 = (V_1, E_1)$ is a wedge sum of $m$ summands in Example 5.23. Also let $G_2 = (\Gamma_2, G)$ such that $\Gamma_2$ be the graph of a non-degenerated BS group in Example 5.21. First, the corresponding Bass-Serre trees are non-elementary because $\pi_1(G_1, v_1)$ are $\pi_1(G_2, v_2)$ are not ameniable. Then observe their first Betti numbers satisfying $b(\Gamma_1) = m \neq 1 = b(\Gamma_2)$, which implies that $\pi_1(G_1, v_1)$ is not a non-degenerated BS group.

In the case of $n$-circle, where $n \geq 2$, let $G_3 = (\Gamma_3, G)$ in which $\Gamma_3$ is the graph of a $n$-circle in Example 5.22. Also the Bass-Serre trees of $\Gamma_3$ and $\Gamma_2$ are non-elementary and reduced in the sense of 20. Definition 2.1]. Then if $\pi_1(G_3, v_3)$ is a non-degenerated BS group, then, as we mentioned above, there is a path of moves of the three types defined in 20. Definition 2.2] from the $\Gamma_2$, which is a 1-circle to the graph $\Gamma_3$, which is a $n$-circle such that any intermediate graph is still reduced by 20. Theorem 2.5]. However, the only possible first move is a slide-move, which happens exactly when the 1-circle $\Gamma_2$ is a strict virtual ascending loop. Otherwise, there is no transformation of $\Gamma_2$ by using these types of moves. After the first possible slide-move, the next moves have to be the coninations of slide-moves and induction moves, which however, either keep the graph structure, which is a “lollipop” or yield a non-reduced graph. This is a contradiction and thus establish that no groups $\pi_1(G_2, v_2)$ of $n$-circles in Example 5.22 are BS groups in Example 5.21 whenever $n \geq 2$.

We now denote by $C$ the class of all groups of graph of groups satisfying Theorems 5.17 and 5.20. As a summary, we have the following result.

**Theorem 5.25.** Still write $G = (\Gamma, G)$, the class $C$ includes the following groups.
(1) $C^*$-simple $\pi_1(\mathcal{G}, v)$ in which $\mathcal{G}$ satisfies assumptions (1)-(3) of Theorem 5.17 and each $G_e$ is amenable. This includes Example 5.18. In particular, this includes $G \ast F$ such that $\left| \left| (G - 1)(F - 1) \right| \right| \geq 2$.

(2) $C^*$-simple GBS groups $\pi_1(\mathcal{G}, v)$ appeared in Theorem 5.20. This includes non-degenerated BS$(k,l)$ where $\left| (k - 1)(l - 1) \right| \geq 2$ in Example 5.21 and certain GBS groups of $n$-circles in Example 5.22 as well as some GBS groups of wedge sum of $n$-circles for $m$ times in Example 5.23. In addition, if $n \geq 2$ or $m \geq 2$, these are not non-degenerated BS groups.

We finally remark that all groups in $\mathcal{C}$ owns a 2-paradoxical towers in the sense of [26], from which, we have the following application.

**Theorem 5.26.** Let $\pi_1(\mathcal{G}, v)$ be the fundamental group appeared in $\mathcal{C}$, which in particular, contains all examples recorded in Theorem 5.20. Let $H$ be another countable discrete group. Suppose $\pi_1(\mathcal{G}, v) \rtimes H \simeq X$ is a purely infinite topological amenable minimal topologically free action on a compact metric space $X$. Then its reduced crossed product is a UCT unital Kirchberg algebra and thus Classifiable by its Elliott invariant.

**Proof.** Theorem 5.20 actually implies that $\pi_1(\mathcal{G}, v)$ admits $n$-paradoxical towers in the sense of [26], Definition A. Then simply apply [26, Theorem B]. \qed

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