Non-spurious solutions to second order BVP by monotonicity methods

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Abstract. We consider the following BVP \( \ddot{x}(t) = f(t, \dot{x}(t), x(t)) - h(t), x(0) = x(1) = 0, \) where \( f \) is continuous and satisfies some other conditions, \( h \in H^1_0(0, 1) \) together with its discretization

\[
-\Delta^2 x(k-1) + \frac{1}{n^2} f\left(\frac{k}{n}, n\Delta x(k-1), x(k)\right) = \frac{1}{n^2} h\left(\frac{k}{n}\right) \text{ for } k \in \{1, 2, \ldots, n\}.
\]

Using monotonicity methods we obtain the convergence of a solutions to a family of discrete problems to the solution of a continuous one, i.e. the existence of non-spurious solutions to the above problems is considered. Continuous dependence on parameters for the continuous problem is also investigated.

Keywords: non-spurious solutions, monotonicity methods, continuous dependence on parameters, boundary value problems.

Mathematics Subject Classification: 39A12, 39A10, 30E25, 34B15.
1 Introduction

In this note we consider non-spurious solutions by using a monotonicity methods to the following second order BVP

\[
\begin{align*}
\begin{cases}
\ddot{x}(t) &= f(t, \dot{x}(t), x(t)) - h(t), \\
x(0) &= x(1) = 0,
\end{cases}
\end{align*}
\]

(1)

where \( f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) is a continuous function such that \( f(t, 0, 0) = 0 \) and \( h : [0, 1] \to \mathbb{R} \) is a continuous function such that \( h(0) = h(1) = 0 \). We will make precise the background further on.

The following assumptions will be used in this work

- **P1** \( \forall r > 0 \exists f_r \in L^1(0, 1) \forall x \in H^1_0(0, 1) \|x\| \leq r \Rightarrow |f(t, \dot{x}(t), x(t))| \leq f_r(t) \text{ a.e. in } (0, 1) \),

- **P2** \( \forall s, t, w, z \in \mathbb{R}, k, l \in [0, 1] (s - t)(f(k, w, s) - f(l, z, t)) \geq 0 \).

Condition \( \textbf{P1} \) is assumed in order to make sure that suitable operator, which we will use, is well defined, while \( \textbf{P2} \) is assumed in order to apply monotonicity methods.

Together with problem (1) we consider its discretization defined as follows. For fixed \( n \in \mathbb{N} \) we consider the following discretization from [3] p. 411

\[-\Delta^2 x(k - 1) + \frac{1}{n^2} f\left(\frac{k}{n}, n \Delta x(k - 1), x(k)\right) = \frac{1}{n^2} h\left(\frac{k}{n}\right) \text{ for } k \in \{1, 2, \ldots, n\},\]

(2)

where \( x : [0, n] \cap \mathbb{N}_0 \to \mathbb{R} \), \( f \) and \( h \) have the same properties as above and \( x(0) = x(n) = 0 \). Again solutions are understood in the weak sense which will be made precise further; \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).
Assume that both continuous boundary value problem (1) and for each fixed \( n \in \mathbb{N} \) the discrete boundary value problem (2) are solvable by \( x \) and \( x^n = (x^n(k)) \), respectively. Moreover, let there exist two positive constants \( Q, N \) such that

\[
\lvert \Delta x^n(k-1) \rvert \leq Q \quad \text{and} \quad \lvert x^n(k) \rvert \leq N
\]

for all \( k = 1, 2, \ldots, n \) and all \( n \geq n_0 \), where \( n_0 \) is fixed (and possibly arbitrarily large). Lemma 2.4 from [3] p. 414 says that for some subsequence \((x^{n_m})_{m \in \mathbb{N}} \) of \((x^n)_{n \in \mathbb{N}} \) it holds

\[
\lim_{m \to \infty} \max_{0 \leq k \leq n_m} \left| \bar{x}^{n_m}(k) - x \left( \frac{k}{n_m} \right) \right| = 0, \quad \lim_{m \to \infty} \max_{0 \leq k \leq n_m} \left| \bar{v}^{n_m}(k) - \dot{x} \left( \frac{k}{n_m} \right) \right| = 0,
\]

where

\[
\bar{x}^n(t) := x^n(k) + n\Delta x^n(k) \left( t - \frac{k}{n} \right), \quad \text{for} \quad \frac{k}{n} \leq t < \frac{k+1}{n},
\]

\[
\bar{v}^n(t) := \begin{cases} 
  n\Delta x^n(k-1) + n^2\Delta^2 x^n(k-1) \left( t - \frac{k}{n} \right), & \frac{k}{n} \leq t < \frac{k+1}{n}, \\ 
  n\Delta x^n(0), & 0 \leq t < \frac{1}{n}.
\end{cases}
\]

Note that if both continuous and discrete problem have unique solution then the convergence holds for the whole sequence, see the last comments in paper [11]. Now following comments in [4], we introduce the idea of a non-spurious solution. The solutions of a family of problems (2) which converge to some solution of problem (1) in the sense described by relation (4) are addressed as non-spurious solutions.

There have been some research in the area of non-spurious solutions addressing mainly problems whose solutions where obtained by the fixed point theorems and the method of lower and upper solutions, [8], [9], [11]. In [4] the variational method is applied, namely the direct method of the calculus of variations. In this note we are aiming at using monotonicity method in order to show that in this setting one can also obtain suitable convergence results. While the approach is somewhat
similar to this of [4], we see that $f$ in contrast to [4] can be dependent on the derivative and as an additional result we get continuous dependence on parameters which seems to be of some novelty by monotonicity approach. As expected we will have to get the uniqueness of solutions for the associated discrete problem, which is not always easy to be obtained, see [10]. Moreover, in our case the first estimation in (3) does not follow from the second one, as the case with $f$ not depending on $\Delta x$, so that it must be derived from the conditions imposed on our problem. As appears with other methods, monotonicity and boundedness of solutions are inherited in the discrete problem from the continuous one. To prove the existence and uniqueness of solution in (1) and for fixed $n \in \mathbb{N}$ in (2), we need following Corollary 6.1.9 from [2] p. 370.

**Corollary 1.1.** Let $H$ be a real Hilbert space and $T : H \to H$ be a continuous and strongly monotone operator, i.e. there exists some $c > 0$ that

$$\langle Tx - Ty, x - y \rangle \geq c \|x - y\|^2$$

for all $x, y \in H$. Then for any $h \in H$ the equation

$$T(u) = h$$

has a unique solution. Let $T(u_1) = h_1$ and $T(u_2) = h_2$. Then

$$\|u_1 - u_2\| \leq \frac{1}{c} \|h_1 - h_2\|$$

with $c > 0$ defined in (5), i.e. $T^{-1}$ is Lipschitz continuous.
2 Non-spurious solutions for (1)

2.1 The continuous problem

Solutions of (1) will be investigated in the real Hilbert space $H_0^1 (0, 1)$ consisting of absolutely continuous functions satisfying the boundary conditions, which have an a.e. derivative being integrable with square. In the space $H_0^1 (0, 1)$ we introduce following norm

$$\| x \| := \left( \int_0^1 (\dot{x} (t))^2 \, dt \right)^{\frac{1}{2}}$$

and with a natural scalar product given by

$$\langle x, y \rangle := \int_0^1 \dot{x} (t) \dot{y} (t) \, dt.$$

Symbol $\| \cdot \|$ will always denote the norm in $H_0^1 (0, 1)$, while for other norms we shall write explicitly.

Since we apply monotonicity methods, we look only for $H_0^1 (0, 1)$ solutions which are called the weak solutions. A function $x \in H_0^1 (0, 1)$ is a weak $H_0^1 (0, 1)$ solution to (1), if the following equality

$$\int_0^1 \dot{x} (t) \dot{y} (t) \, dt + \int_0^1 y (t) f (t, \dot{x} (t), x (t)) \, dt = \int_0^1 y (t) h (t) \, dt \quad (6)$$

holds for all $y \in H_0^1 (0, 1)$, see [1] p. 201. In order to obtain (6) one multiplies the given equation (1) by a test function from $H_0^1 (0, 1)$ and takes integrals. Next we use integration by parts.

Now we must prove that integrals which arise in our problem are finite for any fixed $x, y \in H_0^1 (0, 1)$. Firstly $\int_0^1 \dot{x} (t) \dot{y} (t) \, dt$ is finite since $x, y \in H_0^1 (0, 1)$. Secondly $\int_0^1 y (t) h (t) \, dt$ is finite, because of continuity of both $h$ and $y$. The most demanding is $\int_0^1 y (t) f (t, \dot{x} (t), x (t)) \, dt$. By P1 we see that

$$\left| \int_0^1 y (t) f (t, \dot{x} (t), x (t)) \, dt \right| \leq \int_0^1 |y (t)| |f (t, \dot{x} (t), x (t))| \, dt \leq \int_0^1 |y (t)| f_r (t) \, dt \leq c,$$
where \( c > 0 \) is some constant. We note that any solution of our problem is in fact classical one. Indeed, let us recall the following well know regularity tool, i.e. the du Bois-Reymond Lemma from [7].

**Lemma 2.1.** If \( g \in L^1(0,1) \), \( h \in L^2(0,1) \) and

\[
\int_0^1 (g(t) y(t) + h(t) \dot{y}(t)) \, dt = 0
\]

for all \( y \in H^1_0(0,1) \), then \( \dot{h} = g \) a.e. on \([0,1]\) and \( h \in L^1(0,1) \).

We note that a function \( g \) satisfying (7) is a weak derivative of a function \( h \), see [1] p. 202, for the definition. From Lemma 2.1 it follows that \( h(t) = \int_0^t g(s) \, ds + c \) for some constant \( c \) and for a.e. \( t \in [0,1] \). By standard arguments (see [6]), we see that \( g \) is in fact a classical almost everywhere derivative of \( h \). This observation leads to the conclusion that function any weak solution is such a function from \( H^1_0(0,1) \) that the second derivative exists and it is an \( L^1(0,1) \) function. Such solutions are called classical solution of problem (1).

To prove the existence and the uniqueness of solution we use monotonicity methods. This means that we must find monotone operator which is associated to (1) and use Corollary 1.1. We introduce operator \( K : H^1_0(0,1) \to H^1_0(0,1) \) such that

\[
\langle Kx, y \rangle := \int_0^1 \dot{x}(t) \dot{y}(t) \, dt + \int_0^1 y(t) f(t, \dot{x}(t), x(t)) \, dt, \text{ where } x, y \in H^1_0(0,1) .
\]

We shall see that \( P2 \) implies strong monotonicity of operator \( K \), while continuity of \( K \) follows by continuity of \( f \). Indeed, we have following theorem.

**Theorem 2.2.** Assume that \( f : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \) is continuous, and that conditions \( P1, P2 \) are satisfied. Then problem (1) has exactly one solution.
Proof. Let $f$ satisfies $P1$ and $P2$. We divide our reasoning into two parts.

Monotonicity part

For all $x, y \in H^1(0,1)$ we have

$$
\langle Kx - Ky, x - y \rangle = \langle Kx, x \rangle - \langle Ky, x \rangle + \langle Ky, y \rangle
$$

$$
= \int_0^1 (\dot{x}(t) - \dot{y}(t))^2 \, dt + \int_0^1 (x(t) - y(t)) (f(t, \dot{x}(t), x(t)) - f(t, \dot{y}(t), y(t))) \, dt
$$

$$
= \|x - y\|^2 + \int_0^1 (x(t) - y(t)) (f(t, \dot{x}(t), x(t)) - f(t, \dot{y}(t), y(t))) \, dt
$$

and from $P2$ the second summand is non-negative. Hence

$$
\langle Kx - Ky, x - y \rangle \geq \|x - y\|^2.
$$

Therefore $K$ is strongly monotone with constant $c = 1$.

Continuity part

Let us take a sequence $(x_n)_{n \in \mathbb{N}} \subset H^1_0(0,1)$ convergent (strongly) in $H^1_0(0,1)$ to some $x_0$. This means that both $(x_n)_{n \in \mathbb{N}}$ and $(\dot{x}_n)_{n \in \mathbb{N}}$ are convergent in $L^2(0,1)$. Of course $(x_n)_{n \in \mathbb{N}}$ is bounded, i.e. exists constant $r > 0$ such that $\|x_n\| < r$ for $n \in \mathbb{N}$. Let us fix $y \in H^1(0,1)$. The term

$$
x \to \int_0^1 \dot{x}(t) \dot{y}(t) \, dt
$$

is obviously continuous. We see from $P1$ that

$$
\int_0^1 \|y(t)||f(t, \dot{x}_n(t), x_n(t))| \, dt \leq \int_0^1 \|y(t)||f_r(t) \, dt.
$$

Then by the Lebesgue Dominated Theorem operator $K$ is continuous. Therefore, we can use a Corollary 1.1 and we get existence and uniqueness of solution to (1).
2.2 The discrete problem

Now we consider discretization of problem (1), i.e. problem (2) with fixed $n \in \mathbb{N}$. The space in which the solutions are considered is as follows

$$E := \{ x : [0, n] \cap \mathbb{N}_0 \to \mathbb{R} | \ x(0) = x(n) = 0 \}.$$ 

Clearly, $\dim(E) = n$ and $E$ is a Hilbert space. In the $E$ we choose a norm given by

$$\|x\|_E := \left( \sum_{k=1}^{n} (\Delta x(k-1))^2 \right)^{\frac{1}{2}},$$

with the following scalar product

$$\langle x, y \rangle := \sum_{k=1}^{n} \Delta x(k-1) \Delta y(k-1), \quad x, y \in E.$$ 

Since the space $E$ is finite dimensional the norm $\|\cdot\|_E$ in $E$ is equivalent to the usual norm $\|\cdot\|_0$

$$\|x\|_0 := \left( \sum_{k=1}^{n} |x(k)|^2 \right)^{\frac{1}{2}}.$$ 

We have also the following inequality

$$\|x\|_E \leq 2 \|x\|_0. \quad (8)$$

Function $x \in E$ is a solution to (2) provided that

$$\sum_{k=1}^{n} \Delta x(k-1) \Delta y(k-1) + \frac{1}{n^2} \sum_{k=1}^{n} y(k) f \left( \frac{k}{n}, n\Delta x(k-1), x(k) \right) = \frac{1}{n^2} \sum_{k=1}^{n} y(k) h(k)$$

for all $y \in E$. Let $n$ be a natural number, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. We define operator $K : E \to E$ such that

$$\langle Kx, y \rangle := \sum_{k=1}^{n} \Delta x(k-1) \Delta y(k-1) + \frac{1}{n^2} \sum_{k=1}^{n} y(k) f \left( \frac{k}{n}, n\Delta x(k-1), x(k) \right), \quad \text{where } x, y \in E.$$
Reasoning exactly as in the proof of Theorem 2.2 (the monotonicity part) given the continuity of $K$ (which is obvious by the continuity of $f$ and since we are now in the finite dimensional setting), we have the following theorem.

**Theorem 2.3.** Assume that $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and that condition $P2$ is satisfied. Then problem (2) has exactly one solution.

### 2.3 Main result

In this section we prove the existence of non-spurious solutions to our problem. In order to do this we must first obtain some inequalities concerning the solutions to the discrete problem (2) which would lead to estimations (3) and further to conclusion (4).

**Lemma 2.4.** Assume that $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and that $P2$ is satisfied. By $x^n$ we denote solution of discrete problem (2) with fixed $n \in \mathbb{N}$ and fixed $h$. We have the following inequality

$$\|x^n\|_E \leq \frac{2}{n^2} \|h\|_{C([0,1])}. \tag{9}$$

**Proof.** Let $n \in \mathbb{N}$ and $h \in E$ be fixed. Then with Corollary 1.1, in which we take $h_1 = \frac{1}{n^2}h$, $h_2 = \theta$ and $c = 1$. We have

$$\|x^n - x^n_\theta\|_E \leq \frac{1}{n^2} \|h\|_E.$$

Additionally

$$\|x^n\|_E - \|x^n_\theta\|_E \leq \|x^n - x^n_\theta\|_E.$$

Therefore, for each $h \in E$, we have

$$\|x^n\|_E \leq \frac{1}{n^2} \|h\|_E + \|x^n_\theta\|_E. \tag{10}$$
Moreover, for $y \in \mathbb{E}$ and $h = \theta$, we get
\[
\sum_{k=1}^{n} \Delta x(k-1) \Delta y(k-1) + \frac{1}{n^2} \sum_{k=1}^{n} y(k) f \left( \frac{k}{n}, n \Delta x(k-1), x(k) \right) = \frac{1}{n^2} \sum_{k=1}^{n} y(k) \theta(k),
\]
\[
\sum_{k=1}^{n} \Delta x(k-1) \Delta y(k-1) + \frac{1}{n^2} \sum_{k=1}^{n} y(k) f \left( \frac{k}{n}, n \Delta x(k-1), x(k) \right) = 0.
\]
For $x = \theta$. We have
\[
\sum_{k=1}^{n} \Delta \theta(k-1) \Delta y(k-1) + \frac{1}{n^2} \sum_{k=1}^{n} y(k) f \left( \frac{k}{n}, n \Delta \theta(k-1), \theta(k) \right) = 0,
\]
\[
\sum_{k=1}^{n} 0 \cdot \Delta y(k-1) + \frac{1}{n^2} \sum_{k=1}^{n} y(k) f \left( \frac{k}{n}, 0, 0 \right) = 0.
\]
From $f(t,0,0) = 0$ we have that $x = \theta$ is a solution for $h = \theta$. Additionally we have uniqueness of solution, so
\[
x^n_\theta = \theta.
\]
Hence, from (8) and (10) we have
\[
\|x^n\|_\mathbb{E} \leq \frac{1}{n^2} \|h\|_\mathbb{E} \leq \frac{2}{n^2} \|h\|_0 = \frac{2}{n^2} \left( \sum_{k=1}^{n} |h(k)|^2 \right)^{\frac{1}{2}} \leq \frac{2}{n^2} \left( \sum_{k=1}^{n} \|h\|_{C([0,1])}^2 \right)^{\frac{1}{2}} = \frac{2}{n^2} \|h\|_{C([0,1])}. \]

\[\square\]

**Theorem 2.5.** Assume that conditions \textit{P1, P2} are satisfied and that $f : [0,1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous. Then there exists $x \in H^1_0(0,1)$ which solves uniquely problem (1) and for each $n \in \mathbb{N}$ there exists $x^n$ which solves uniquely problem (2). Moreover relations (4) are satisfied for the sequence $(x^n)_{n \in \mathbb{N}}$.

**Proof.** From Theorem 2.2 we have existence of a solution to problem (1), denoted by $x$. Existence of a solution to problem (2) for each $n \in \mathbb{N}$ is an assertion of Theorem 2.3. We denote this solution
by \( x^n \). Let \( n \in \mathbb{N} \) be fixed and let us take any \( k \in \{1, 2, \ldots, n\} \). Then we have

\[
|\Delta x^n (k - 1)| = \left( (\Delta x^n (k - 1))^2 \right)^{\frac{1}{2}} \leq \left( \sum_{k=1}^{n} (\Delta x^n (k - 1))^2 \right)^{\frac{1}{2}} = \|x^n\|_E. \tag{11}
\]

Multiplying this inequality by \( n \) and from (9), we get

\[
n |\Delta x^n (k - 1)| \leq n \|x^n\|_E \leq \frac{2}{n^{\frac{1}{2}}} \|h\|_{C([0,1])} = \frac{2}{\sqrt{n}} \|h\|_{C([0,1])} \leq 2 \|h\|_{C([0,1])} =: Q \tag{12}
\]

Additionally,

\[
|x^n (k)| = \left| \sum_{j=1}^{k} \Delta x^n (j - 1) \right| \leq \sum_{j=1}^{k} |\Delta x^n (j - 1)| \tag{13}
\]

Moreover from the Cauchy-Schwartz inequality and (11) we get

\[
\left( \sum_{j=1}^{k} |\Delta x^n (j - 1)| \right)^{2} = \left( \sum_{j=1}^{k} 1 \cdot |\Delta x^n (j - 1)| \right)^{2} \leq \sum_{i=1}^{k} |1|^2 \sum_{l=1}^{k} |\Delta x^n (l - 1)|^2
\]

\[
= k \sum_{i=1}^{k} |\Delta x^n (l - 1)|^2 \leq k \sum_{l=1}^{n} |\Delta x^n (l - 1)|^2 = k \|x^n\|_E^2. \tag{14}
\]

Hence

\[
\sum_{j=1}^{k} |\Delta x^n (j - 1)| \leq \sqrt{k} \|x^n\|_E. \tag{14}
\]

Finally from (13) and (14), we obtain

\[
\max_{k \in \{1, 2, \ldots, n\}} |x^n (k)| \leq \sqrt{n} \|x^n\|_E \leq \sqrt{n} \frac{2}{n^{\frac{1}{2}}} \|h\|_{C([0,1])} = \frac{2}{\sqrt{n}} \|h\|_{C([0,1])} \leq 2 \|h\|_{C([0,1])} =: N \tag{15}
\]

Given inequalities (12) and (15) the result now follows from Lemma 2.4 from [3] p. 414, which reads as follows. If we have solutions to problem (1) and problem (2) for each \( n \in \mathbb{N} \) and inequalities

\[
n |\Delta x^n (k - 1)| \leq Q, |x^n (k)| \leq N,
\]

then exist subsequence \((x^n)_{n \in \mathbb{N}}\), denoted still as \((x^n)_{n \in \mathbb{N}}\), such that

\[
\lim_{n \to \infty} \max_{0 \leq k \leq n} |\bar{x}^n (k) - x \left( \frac{k}{n} \right)| = 0, \lim_{n \to \infty} \max_{0 \leq k \leq n} |\bar{v}^n (k) - \dot{x} \left( \frac{k}{n} \right)| = 0,
\]

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where
\[
\bar{x}^n(t) := x^n(k) + n \Delta x^n(k) \left( t - \frac{k}{n} \right), \quad \text{for } \frac{k}{n} \leq t < \frac{k+1}{n},
\]
\[
\bar{v}^n(t) := \begin{cases} 
  n \Delta x^n(k-1) + n^2 \Delta^2 x^n(k-1) \left( t - \frac{k}{n} \right), & \frac{k}{n} \leq t < \frac{k+1}{n}, \\
  n \Delta x^n(0), & 0 \leq t < \frac{1}{n}.
\end{cases}
\]

The mentioned comment from [11] p. 84 reads that we have this relation for whole sequence, not only for subsequence.

**Remark 2.6.** Note that when \( f \) does not depend on the derivative it suffice to obtain inequality (15), compare with Lemma 9.3 from [5] p. 342, which reads as follows. If \( f := f(t, x(t)) \) and if there exists a positive constant \( N \), such that
\[
|x^n(k)| \leq N,
\]
then there exists a positive constant \( Q \), such that
\[
n|\Delta x^n(k-1)| \leq Q.
\]

Hence we can repeat reasoning from proof of Theorem 2.5.

**Remark 2.7.** We observe that in case when \( f := f(t, x(t)) \) our main result provides some improvement over its counterpart from [4] since we also obtain some information on the convergence of derivatives.

Concerning the examples of nonlinear terms any function \( f \) nondecreasing in \( x \) and bounded in \( \dot{x} \) is of order. See for example

a) \( f(t, \dot{x}, x) = g(t) \exp(x - t^2) |\arctan(\dot{x})|, \)

b) \( f(t, \dot{x}, x) = g(t) \arctan(x) |\arctan(\dot{x})|, \)
c) \( f(t, \dot{x}, x) = g(t) x^3 + \exp(x - t^2) \arctan(\dot{x}) \),

where \( g : [0, 1] \to \mathbb{R} \) is a continuous and non-negative function. Recall the Sobolev’s inequality
\[
\max_{t \in [0,1]} |x(t)| \leq \left( \int_0^1 \dot{x}^2(t) \, dt \right)^{1/2}.
\]
All above functions satisfy P2 due to standard monotonicity. Assumption P1 is satisfied since in each case we can calculate for a fixed \( r > 0 \) a function \( f_r \).

Indeed, by Sobolev’s inequality when \( \|x\| \leq r \), we see that \( |x(t)| \leq r \) for all \( t \in [0,1] \). Then we calculate as follows. Let \( g_1(t) = g(t) \exp(-t^2) \) and \( c_1 = \max_{\alpha \in [-r,r]} \exp(\alpha) \). Then, for \( \|x\| \leq r \), we have
\[
 g(t) \exp(x - t^2) |\arctan(\dot{x})| \leq \frac{c_1 \pi}{2} g_1(t)
\]
for all \( t \in [0,1] \). The other examples can be demonstrated likewise.

### 2.4 Additional result

In this section we use the already developed technique to consider the so called continuous dependence on functional parameters. The idea of the continuous dependence on parameters is as follows. Let us consider together with (1) a family of boundary value problems

\[
\begin{align*}
\ddot{x}(t) &= f(t, \dot{x}(t), x(t)) - h_n(t), \\
x(0) &= x(1) = 0,
\end{align*}
\]

where \( f : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \) and \( h_n : [0,1] \to \mathbb{R} \) are continuous functions. Assume that for any \( n \) there exists at least one solution \( x_n \) to (16) and that the sequence \( (h_n)_{n \in \mathbb{N}} \) is convergent to \( h_0 \) in some sense. The problem (1) depends continuously on a functional parameter \( h_n \) if sequence \( (x_n)_{n \in \mathbb{N}} \) is convergent, possibly up to a subsequence, to a solution \( x_0 \) of (1) corresponding to \( h = h_0 \). To prove continuous dependence on parameter in our problem (1) we need the following lemma.
Lemma 2.8. Let \( f : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \) be continuous. By \( x \) we denote solution of problem (1) for functional parameter \( h \in H^1_0(0,1) \). Then we have the inequality
\[
\|x\| \leq \|h\|.
\]

**Proof.** Fix \( h \in H^1_0(0,1) \). Corollary 1.1, in which we take \( h_1 = h, h_2 = \theta \) and \( c = 1 \). We have
\[
\|x\| - \|x_\theta\| \leq \|x - x_\theta\| \leq \|h\|.
\]
So
\[
\|x\| \leq \|h\| + \|x_\theta\|.
\]
Moreover, for \( y \in H^1_0(0,1) \) and \( h = \theta \), we get from (16)
\[
\int_0^1 \dot{x}(t) \dot{y}(t) \, dt + \int_0^1 y(t) f(t, \dot{x}(t), x(t)) \, dt = 0.
\]
Using \( f(t,0,0) = 0 \) we see that \( x = \theta \) is a solution for \( h = \theta \). Additionally we have uniqueness of solution, so \( x_\theta = \theta \). Hence, we have
\[
\|x\| \leq \|h\|.
\]
\[\square\]

**Theorem 2.9.** Assume condition \( P2 \) and that it holds for \( t \in [0,1] \)
\[
f(t, \dot{x}(t), x(t)) = f_1(t, x(t)) + \dot{x}(t) g(t),
\]
where \( f_1 : [0,1] \times \mathbb{R} \to \mathbb{R} \) is continuous and \( g \in L^2(0,1) \). Let \( (h_n)_{n \in \mathbb{N}} \subset H^1_0(0,1) \) be weakly convergent to \( h_0 \). Then for every \( h_n \), where \( n = 0, 1, 2, \ldots \), there exists exactly one solution \( x_n \).
Moreover there exists subsequence \((x_{n_k})_{k \in \mathbb{N}}\), which is weakly convergent to \( x_0 \) and \( x_0 \) is solution for \( h_0 \).
Proof. Let \((x_n)_{n \in \mathbb{N}_0}\) be a sequence of solutions to (16) where each \(x_n\) corresponds to \(h_n\). Such a sequence exists by Theorem 2.2. Note that we do not require condition **P1** to be satisfied but like in (6), we have that \(\int_0^1 \dot{x}_n (t) \dot{y}(t) \, dt\) and \(\int_0^1 y(t) h_n(t) \, dt\) are finite for any \(y \in H^1_0(0,1)\). Additionally \(y\) is continuous, \(x_n, g(t) \in L^2(0,1)\), so \(\int_0^1 y(t) \dot{x}_n(t) g(t) \, dt\) is also finite. Finally \(x_n\) is continuous and \(f_1\) is continuous, hence \(\int_0^1 y(t) f_1(t, x_n(t)) \, dt\) is finite too. Now we need weak convergence of a subsequence of \((x_n)_{n \in \mathbb{N}}\), so we must show that \((x_n)_{n \in \mathbb{N}}\) is bounded. From Lemma 2.8 for each \(n\), we have
\[
\|x_n\| \leq \|h_n\|.
\]
Since \((h_n)_{n \in \mathbb{N}}\) is weakly convergent, there exists positive constant \(c\) such that \(\|h_n\| < c\) for all \(n \in \mathbb{N}\). We finally have that
\[
\forall n \in \mathbb{N} \|x_n\| \leq c.
\]
Therefore, we get existence of a weakly convergent subsequence \((x_{n_k})_{k \in \mathbb{N}}\). We denote the weak limit of \((x_{n_k})_{k \in \mathbb{N}}\) by \(x_0\). Hence for any \(y \in H^1_0(0,1)\) we have
\[
\int_0^1 \dot{x}_{n_k} (t) \dot{y}(t) \, dt + \int_0^1 y(t) f_1(t, x_{n_k}(t)) \, dt + \int_0^1 y(t) \dot{x}_{n_k}(t) g(t) = \int_0^1 y(t) h_{n_k}(t) \, dt.
\]
Applying the definition of weak convergence and the Lebesgue’s Dominated Convergence Theorem to \(\int_0^1 y(t) f_1(t, x_{n_k}(t)) \, dt\) and letting \(n_k \to +\infty\) we get
\[
\int_0^1 \dot{x}_0 (t) \dot{y}(t) \, dt + \int_0^1 y(t) f_1(t, x_0(t)) \, dt + \int_0^1 y(t) \dot{x}_0(t) g(t) = \int_0^1 y(t) h_0(t) \, dt.
\]
Therefore \(x_0\) is a solution of (1) for \(h_0\). \(\square\)

We present examples of functions satisfying the above,

a) \(f(t, \dot{x}, x) = g(t) \exp(x - t^2) + g_1(t) \dot{x},\)
b) \[ f(t, \dot{x}, x) = g(t) \arctan(x) + g_1(t) \dot{x}, \]

c) \[ f(t, \dot{x}, x) = g(t) x^3 + \exp(x - t^2) + g_1(t) \dot{x}, \]

where \( g : [0, 1] \to \mathbb{R} \) is a continuous and non-negative function and \( g_1 \in L^2(0, 1) \).

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