ON THE CAUCHY PROBLEM FOR THE WAVE EQUATION WITH DATA ON THE BOUNDARY

M. N. Demchenko

The Cauchy problem for the wave equation in $\Omega \times \mathbb{R}$ with data given on some part of the boundary $\partial \Omega \times \mathbb{R}$ is considered. A reconstruction algorithm for this problem based on analytic expressions is given. This result is applicable to the problem of determining a nonstationary wave field arising in geophysics, photoacoustic tomography, tsunami wave source recovery. Bibliography: 18 titles.

1. INTRODUCTION

Consider the wave equation

$$\partial_t^2 u - \Delta u = 0$$

in a space-time cylinder $\Omega \times \mathbb{R}$ ($\Omega$ is a domain in $\mathbb{R}^n$, $n \geq 2$, $\Delta$ is the Laplace operator in $\mathbb{R}^n$). We study the problem of determining a solution $u$ from Cauchy data $u$, $\partial_n u$ ($\nu$ is the outward unit normal to $\partial \Omega$) given on some part of the boundary $\partial \Omega \times \mathbb{R}$. In contrast to the classical Cauchy problem for the wave equation with the data on a space-like surface, the problem in consideration is ill-posed [1, 2]. However, a solution $u$ is uniquely determined in some part of $\Omega \times \mathbb{R}$ dependent on the set, on which the Cauchy data are given. This can be inferred from the unique continuation property for the wave equation, which is provided by Holmgren’s theorem (or Tataru’s theorem [3] in the case of a hyperbolic equation with variable coefficients).

We will denote points in $\mathbb{R}^n$ by $(x, y)$, where $x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$. We will consider the case where $\Omega$ is a subgraph of a $C^\infty$-smooth function $Y(x)$ that satisfies a certain growth condition:

$$\Omega = \{(x, y)|x \in \mathbb{R}^{n-1}, -\infty < y < Y(x), \quad |Y(x)| \leq C_1 + C_2|x|, \quad C_2 < 1$$

(2)

(see Fig. 1). In the case $Y(x) \equiv \text{const}$, the domain $\Omega$ is a half-space. Fix a point $(x^*, y^*) \in \Omega$ and $t^* \in \mathbb{R}$. We will obtain an algorithm (formula (18), Sec. 4), which allows determining $u(x^*, y^*, t^*)$ from the Cauchy data on the set

$$\{(x, y, t)|(x, y) \in S, T_- (x) \leq t \leq T_+ (x)\}.$$  

(3)

Here $T_\pm (x) = t^* \pm (Y(x) - y^*)$, $S$ is a bounded relatively open subset of $\partial \Omega$ that contains the intersection $K \cap \partial \Omega$, where

$$K = \{(x, y) \in \mathbb{R}^n|y - y^* \geq |x - x^*|\}$$

(4)

(note that the intersection $K \cap \partial \Omega$ is itself bounded by the growth condition in (2)). The cone $K$ and the set $S$ are shown in Fig. 1. Evidently, $Y(x) > y^*$ whenever $(x, y) \in K \cap \partial \Omega$. We assume that $S$ is chosen in such a way that the same inequality holds true whenever $(x, y)$ belongs to the closure $\overline{S}$, which leads to the inequality $T_- (x) < T_+ (x)$. Thus the condition on $t$ in (3) makes sense.

Note that formula (18) allows for determining $u(x^*, y^*, t^*)$ from Cauchy data on various subsets of the boundary. Indeed, one can take a cone $K$ with a different orientation than that in (4) (though its vertex should be at $(x^*, y^*)$), which yields a different set $S \subset \partial \Omega$. Next we may choose a Cartesian coordinate system in $\mathbb{R}^n$, in which $K$ is represented by (4), and then apply formula (18). However, it is required that the domain $\Omega$ can be represented by (2) in...
chosen coordinates. Recall that in Cauchy problems for elliptic equations and in the analytic continuation problem, the solution is also uniquely determined from Cauchy data on various sets.

Fig. 1. The domain $\Omega$ is a subgraph of the function $Y(x)$. The darkened region in $\Omega$ is the set $\omega$ containing scatterers and inhomogeneities. The hatched region is the cone $K$.

The wave equation (1) describes wave processes of various nature in homogeneous media. Thus our result can be used for determination of the wave field in a domain from boundary measurements. It will be shown at the end of Sec. 4 that our algorithm is applicable also in the case where the solution $u(x, y, t)$ is defined for $(x, y)$ belonging to some subset of $\Omega$. This may correspond, for example, to the wave process in the medium that occupies the domain $\Omega$ and contains inhomogeneities and scatterers located in some set $\omega$ (see Fig. 1). Under the condition $\omega \cap K = \emptyset$, formula (18) can be used to find the wave field in homogeneous part of the medium (i.e., in $\Omega \setminus \omega$) without knowledge of the structure of inhomogeneities and scatterers.

The problem of determination of nonstationary wave field arises in various applications, such as geophysics [4], photoacoustic tomography [5,6], tsunami wave source recovery [7], and coefficient inverse problems [8–10].

The Cauchy problem for hyperbolic equations with data on the boundary has been extensively studied. We refer the reader to monographs [1,2] containing an overview of related results. Most of these results are Carleman type estimates, which provide the uniqueness of a solution and conditional stability estimates. In the case of the wave equation with constant coefficients, reconstruction algorithms based on analytic expressions were obtained. However, most of them require the nonlocal Cauchy data, which means that the surface $S$ coincides with the entire boundary $\partial \Omega$ [5,11–14]. As to the case of local data, we mention the famous result of R. Courant on the Cauchy problem for the ultrahyperbolic equation in a half-space (see [15]), and papers [16,17], in which two-dimensional domains were considered.

2. A SPECIAL SOLUTION OF THE LAPLACE EQUATION

The proof of our main result will be based on some special solution of the wave equation, which depends on a small parameter $h$ and enjoys a certain localization property as $h \to 0$. To construct such a solution, we will need a special solution of the Laplace equation, which is the subject of this section.

We will use the notation $\Delta_x = \sum_j \partial^2_{x_j}$ (thus for Laplace operator $\Delta$ in $\mathbb{R}^n$ we have $\Delta = \Delta_x + \partial^2_y$).

Consider the following Cauchy problem for the Laplace equation in $\mathbb{R}^n$:
\[ \Delta \varphi = 0, \]
\[ \varphi|_{y=0} = \frac{e^{-x^2/h}}{(\pi h)^m}, \quad \partial_y \varphi|_{y=0} = 0, \tag{6} \]

where \( h > 0 \), \( m = (n-1)/2 \), and for a real or complex vector \( \zeta = (\zeta_1, \ldots, \zeta_{n-1}) \) we put \( \zeta^2 = \sum_j \zeta_j^2 \). Note that \( \varphi(x,0) \) is the Gaussian distribution in \( \mathbb{R}^{n-1} \).

**Lemma 1.** For any \( h > 0 \) there is a unique \( C^\infty \)-smooth function \( \varphi(x,y) \) in \( \mathbb{R}^n \) that satisfies Eq. (5) and initial conditions (6). Moreover,
\[ \varphi, \partial_{x,y} \varphi, \partial_{x,y}^2 \varphi \to 0, \quad h \to 0, \quad \text{if} \ |x| > |y|; \tag{7} \]
the convergence is uniform on every compact set in \( \{|x| > |y|\} \).

**Proof.** We will study the Cauchy problem (5), (6), using a method, which is similar to that described in [18] (Lemma 9.1.4), where the Laplace equation is reduced to the wave equation. This method provides a representation of the solution \( \varphi(x,y) \) in terms of the fundamental solution \( G(x,y) \) of the wave equation. The latter is defined by the following relations:
\[ \partial^2_y G - \Delta_x G = 0, \tag{8} \]
\[ G|_{y=0} = 0, \quad \partial_y G|_{y=0} = \delta(x) \tag{9} \]
(\( \delta \) is the Dirac delta function). For any \( y \), the distributions \( \partial^\beta_y G(\cdot,y) \), \( \beta \geq 0 \), are supported in the ball \( \{x \mid |x| \leq |y|\} \). The Fourier transform of \( \partial_y G(\cdot,y) \) is equal to \( \cos(|x|) \). Put
\[ \varphi(x,y) = \frac{1}{(\pi h)^m} \left( \partial_y G(x',y), e^{-(x-x')^2/h} \right). \tag{10} \]

Throughout this proof, the angle brackets mean the pairing of a distribution in the variable \( x' \) with a test function. Although the smooth function \( e^{-(x-x')^2/h} \) is not compactly supported with respect to \( x' \), the right-hand side makes sense since the distribution \( \partial_y G(\cdot,y) \) is compactly supported. Clearly the derivatives of \( \varphi \) can be represented in a similar way:
\[ \partial^\alpha_x \partial^\beta_y \varphi(x,y) = \frac{1}{(\pi h)^m} \left( \partial^{\beta+1}_y G(x',y), \partial^\alpha_x e^{-(x-x')^2/h} \right). \tag{11} \]

This representation and the fact that \( \partial^2_y G(x,0) = 0 \) imply the second relation in (6). The first relation in (6) follows from the definition (10) and the second condition in (9).

The function \( \varphi(x,y) \) and its derivatives with respect to \( x, y \) can be considered as analytic functions in complex variables \( x_1, \ldots, x_{n-1} \), which follows from (10) and (11). The function
\[ \psi(x,y) = \varphi(ix,y) = \frac{1}{(\pi h)^m} \left( \partial_y G(x',y), e^{(x-x')^2/h} \right) \]
is a convolution of \( (\pi h)^{-m} e^{x^2/h} \) with \( \partial_y G(x,y) \) with respect to \( x \) and thus by (8) satisfies the wave equation
\[ \partial^2_y \psi - \Delta_x \psi = 0 \]
for all \( (x,y) \in \mathbb{R}^n \). For any fixed real \( y \) the left-hand side here is an analytic function in \( x_1, \ldots, x_{n-1} \). Therefore, this equation is satisfied for all complex \( x_1, \ldots, x_{n-1} \), which yields
\[ (\partial^2_y \varphi + \Delta_x \varphi)(x,y) = (\partial^2_y \psi - \Delta_x \psi)(-ix,y) = 0. \]
Thus Eq. (5) is satisfied. As is well known, the solution \( \varphi \) of problem (5), (6) is unique.
Now turn to assertion (7). Since the distribution \( \partial^\beta_y G(\cdot, y) \) is supported in the ball \( \{ x \mid |x| \leq |y| \} \), its pairing with a test function \( f \) is determined by the restriction of \( f \) to any neighborhood of the specified ball. Applying the Fourier transform, one can easily obtain the estimate

\[
\left| \left\langle \partial^\beta_y G(x', y), f(x') \right\rangle \right| \leq C_{y, \varepsilon} \max_{|x'|^2 \leq |y|^2 + \varepsilon, |a| \leq n + \beta - 1} |\partial^a f(x')|,
\]

where \( \varepsilon > 0 \), and the constant \( C_{y, \varepsilon} \) is bounded whenever \( y \) is bounded. Suppose \( |x| > |y| \). Place \( f(x') = \frac{\pi}{h} e^{-|x|/2h} \), \( \varepsilon = (|x|^2 - |y|^2)/2 \). For \( |x'|^2 \leq |y|^2 + \varepsilon \) we have

\[
|e^{-|x|^2/2h}/h^m| = e^{(|x|^2 - |y|^2)/(2h)} / h^m \rightarrow 0, \quad h \rightarrow 0.
\]

The derivatives \( \partial^a_x f(x') \) are treated in the same way. In view of (10), (12), we obtain (7) for the function \( \phi \). The same assertion for the derivatives of \( \phi \) is proved by a similar argument using (11) instead of (10). □

3. A special solution of the wave equation

The following lemma provides a transformation of a solution of the Laplace equation to a solution of the wave equation (1).

**Lemma 2.** Suppose a \( C^\infty \)-smooth function \( \varphi(x, y) \) in \( \mathbb{R}^n \) satisfies the relations

\[
\Delta \varphi = 0, \quad \partial_y \varphi|_{y=0} = 0.
\]

Then the function

\[
w(x, y, t) = \frac{1}{\pi} \int_0^{\pi/2} \varphi \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) ds
\]

is \( C^\infty \)-smooth in the set \( x \in \mathbb{R}^{n-1}, t \in \mathbb{R}, y > |t| \), and satisfies the wave equation

\[
\partial^2_t w - \Delta w = 0.
\]

**Proof.** We have

\[
\pi \partial_y w = \frac{y}{\sqrt{y^2 - t^2}} \int_0^{\pi/2} (\partial_y \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) \sin s ds,
\]

\[
\pi \partial_t w = \frac{-t}{\sqrt{y^2 - t^2}} \int_0^{\pi/2} (\partial_t \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) \sin s ds.
\]

Next

\[
\pi \partial^2_y w = \frac{-t^2}{(y^2 - t^2)^{3/2}} \int_0^{\pi/2} (\partial^2_y \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) \sin s ds
\]

\[
+ \frac{y^2}{y^2 - t^2} \int_0^{\pi/2} (\partial^2_y \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) (\sin s)^2 ds,
\]

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\[
\pi \partial_t^2 w = \frac{-y^2}{(y^2 - t^2)^{3/2}} \int_0^{\pi/2} (\partial_y \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) \sin s \, ds \\
+ \frac{t^2}{y^2 - t^2} \int_0^{\pi/2} (\partial_y^2 \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) (\sin s)^2 \, ds.
\]

Hence
\[
\pi (\partial_t^2 - \partial_y^2) w = \frac{-1}{\sqrt{y^2 - t^2}} \int_0^{\pi/2} (\partial_y \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) \sin s \, ds \\
- \int_0^{\pi/2} (\partial_y^2 \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) (\sin s)^2 \, ds.
\]

The first term on the right-hand side equals
\[
\frac{1}{\sqrt{y^2 - t^2}} \int_0^{\pi/2} (\partial_y \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) \frac{d}{ds} \cos s \, ds \\
= -\frac{(\partial_y \varphi)(x,0)}{\sqrt{y^2 - t^2}} - \int_0^{\pi/2} (\partial_y^2 \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) (\cos s)^2 \, ds.
\]

Now applying the second relation in (13), we obtain
\[
\pi (\partial_t^2 - \partial_y^2) w = - \int_0^{\pi/2} (\partial_y^2 \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) \, ds.
\]

Adding this to
\[
-\pi \Delta_x w = - \int_0^{\pi/2} (\Delta_x \varphi) \left( x, \sqrt{y^2 - t^2} \cdot \sin s \right) \, ds,
\]
we obtain (15) in view of the first relation in (13).

It can easily be seen from the definition (14) that \( w, \partial_x w \) have continuous extensions to the set \( \{ y \geq |t| \} \). This is also true for \( \partial_y w \) and \( \partial_t w \), which follows from the expressions for these derivatives given at the beginning of the previous proof and from the second condition in (13). Henceforth we denote by \( w \) and \( \partial_{x,y,t} w \) these continuous extensions defined on \( \{ y \geq |t| \} \). We have
\[
(\partial_y w \pm \partial_t w) \big|_{t=\pm y} = 0.
\]\( (\partial_y w \pm \partial_t w) \big|_{t=\pm y} = 0. \) (16)

**Lemma 3.** Suppose \( \varphi \) is the solution of the Cauchy problem (5), (6). Then for \( w(x,y,t) \) defined by (14) we have
\[
w, \partial_{x,y} w \to 0, \quad h \to 0, \quad \text{if } |x| > y \geq |t|; \quad (17)
\]
the convergence is uniform on every compact set in \( \{|x| > y \geq |t|\} \).
Proof. For $w$ and $\partial_x w$ assertion (17) follows from (7) and formula (14). To estimate $\partial_y w$, observe that owing to $\partial_y \varphi|_{y=0} = 0$ we have
\[
\left| (\partial_y \varphi) \left( x, \sqrt{y^2 + t^2} \cdot \sin s \right) \right| \leq \sqrt{y^2 + t^2} \max_{\tau \in [0, \sqrt{y^2 + t^2}]} \left| (\partial_y^2 \varphi)(x, \tau) \right|.
\]
Now applying (7) and the formula for $\partial_y w$ given at the beginning of the proof of Lemma 2, we obtain (17) for $\partial_y w$.

Note that the condition $|x| > y \geq |t|$ in (17) can be weakened to $\sqrt{x^2 + t^2} > y \geq |t|$ (although we will not use this generalization). Note also that if $\sqrt{x^2 + t^2} < y$, then generally $w$ grows as $h \to 0$.

4. Determination of the solution $u$

In this section we will obtain our main result, that is, formula (18), which relates $u(x^*, y^*, t^*)$ to the Cauchy data on the set (3).

**Theorem 1.** Under the conditions indicated in Sec. 1, for any solution $u \in C^2(\overline{\Omega} \times \mathbb{R})$ of the wave equation (1) we have
\[
u(x^*, y^*, t^*) = \frac{1}{2} \sum_{\pm} u(x^*, Y(x^*), T_\pm(x^*)) + \lim_{h \to 0} \int_S d\sigma_{x,y} \int_{T_+(x)}^{T_-(x)} (u_{\partial_x} w^* - \partial_x u \cdot w^*) dt,
\]
where $w^*(x, y, t) = w(x - x^*, y - y^*, t - t^*)$, the function $w$ being defined in Lemma 3, $d\sigma$ is the surface measure on $\partial \Omega$.

From the remarks given at the end of Sec. 3, it follows that $w^*$ grows in the cone $K$ as $h \to 0$. Hence the integrand on the right-hand side of (18) grows, which means that if we plug there arbitrary smooth functions instead of the Cauchy data $u, \partial_x u$, the limit of the integral generally does not exist. So if the Cauchy data are given with some error (which is always the case in practice), one should approximate this limit by the corresponding integral computed for some positive $h$. In fact, the optimal value of $h$ depends on the accuracy of the Cauchy data. This is a specific feature of other problems, which require a regularization, including the analytic continuation problem.

**Proof of Theorem 1.** Assume the coordinates are chosen so that $(x^*, y^*, t^*) = (0, 0, 0)$. In this case we have $T_\pm(x) = \pm Y(x)$. Our condition $Y(x) > y^*, (x, y) \in \overline{S}$ imposed on $S$ in Sec. 1 now reads $Y(x) > 0, (x, y) \in \overline{S}$. First we prove (18), assuming that $Y(x) > 0$ for all $x \in \mathbb{R}^{n-1}$. After that we will eliminate this restriction.

Note that in view of the growth condition in (2), the set $K_\Omega = K \cap \overline{\Omega}$ is compact. Hence there exists a compactly supported $C^\infty$-smooth function $\chi(x, y)$ in $\mathbb{R}^n$ such that $\chi = 1$ in some neighborhood of $K_\Omega$. Pick a number $R$ such that the projection of the support $supp_\chi$ to the hyperplane $(x_1, \ldots, x_{n-1})$ is contained in the ball $\{|x| < R\}$. Put
\[
V = \{(x, y, t) ||x| < R, |t| < y < Y(x)\} \subset \Omega \times \mathbb{R}.
\]
The set $V$ is a bounded Lipschitz domain in $\mathbb{R}^{n+1}$. Indeed, the diffeomorphism
\[
(x, y, t) \mapsto (x, y/Y(x), t/Y(x))
\]
is well defined in a neighborhood of $\overline{V}$, since $Y(x)$ is separated from zero for bounded $x$. It remains to observe that the specified diffeomorphism maps $V$ to the Cartesian product $\{x \ | \ |x| < R\} \times \{(y, t) \ | \ |t| < y < 1\}$ of Lipschitz domains.
Put \( \tilde{u} = \chi u \). We have
\[
\int_{\mathcal{V}} \left[ w \cdot (\partial_t^2 - \Delta) \tilde{u} - \tilde{u} \cdot (\partial_t^2 - \Delta) w \right] \, dx dy dt
= \int_{\partial \mathcal{V}} \left[ (\tilde{u} \partial_x w - w \partial_x \tilde{u}) \nu_x + (\tilde{u} \partial_y w - w \partial_y \tilde{u}) \nu_y + (-\tilde{u} \partial_t w + w \partial_t \tilde{u}) \nu_t \right] \, d\gamma.
\]
(19)

Here \( \nu = (\nu_x, \nu_y, \nu_t) \) is the outward unit normal to \( \partial \mathcal{V} \), \( d\gamma \) is the surface measure on \( \partial \mathcal{V} \) (we use the notation \( \nu_x = (\nu_{x_1}, \ldots, \nu_{x_{n-1}}) \)). This formula is applicable, provided that the following conditions are satisfied: (i) the functions \( \tilde{u} \) and \( w \) are \( C^2 \)-smooth in \( V \); (ii) the integral on the left-hand side is absolutely convergent; (iii) the functions \( \tilde{u} \), \( w \) and their first order derivatives have continuous extensions from \( V \) to the closure \( \overline{V} \) (the right-hand side involves the values of these extensions on \( \partial \mathcal{V} \)). The condition (i) is obviously satisfied. The condition (ii) follows from (15) and the fact that \( w \) is bounded in \( V \), while \( \tilde{u} \) is \( C^2 \)-smooth in \( \overline{V} \). The last condition (iii) is obvious for \( \tilde{u} \); by the remark given right after the proof of Lemma 2, (iii) is satisfied for \( w \).

Define the set \( \Gamma \) as the intersection of \( \partial \Omega \) with \( \partial \mathcal{V} \times \mathbb{R} \), while the sets \( \Gamma^\pm \) are defined as the intersections of \( \partial \mathcal{V} \) with the hyperplanes \( \{ \pm t = y \} \). For any point \((x, y, t)\) from \( \partial \mathcal{V} \setminus (\Gamma \cup \Gamma^+ \cup \Gamma^-) \) we have \( |x| = R \). By our choice of \( R \), at such points we have \( \chi = 0 \), and so \( \tilde{u} = 0 \). Therefore, the integral over \( \partial \mathcal{V} \) in (19) equals the sum of the corresponding integrals over \( \Gamma^+ \) and \( \Gamma^- \). The integral over \( \Gamma^+ \) equals
\[
\int_{\Gamma^+} \left[ (\tilde{u} \partial_y w - w \partial_y \tilde{u}) \nu_y + (-\tilde{u} \partial_t w + w \partial_t \tilde{u}) \nu_t \right] \, d\gamma
= \frac{1}{\sqrt{2}} \int_{\Gamma^+} \left[ w(\partial_y \tilde{u} + \partial_t \tilde{u}) - \tilde{u}(\partial_y w + \partial_t w) \right] \, d\gamma.
\]
(20)

(on \( \Gamma^+ \) we have \( \nu_x = 0 \), \( \nu_t = -\nu_y = 1/\sqrt{2} \)). By (16), we have \( \partial_y w + \partial_t w = 0 \) on \( \Gamma^+ \). Now taking into account the equality \( w(x, y, \pm y) = \varphi(x, 0)/2 \), which follows from the definition (14), the previously obtained integral equals
\[
\frac{1}{2} \int_{|x| < R} \frac{1}{2} \int_{|y| < R} \varphi(x, 0) \left( \partial_y \tilde{u} + \partial_t \tilde{u} \right)(x, y, y) \, dy = \frac{1}{2} \int_{|x| < R} \varphi(x, 0) \left( \partial_y \tilde{u}(x, y, y) \right) \, dy
= \frac{1}{2} \int_{|x| < R} \varphi(x, 0) \left( \tilde{u}(x, Y(x), Y(x)) - \tilde{u}(x, 0, 0) \right) \, dx.
\]

In the same way, we derive that the integral (20), in which \( \Gamma^+ \) is replaced by \( \Gamma^- \), equals
\[
\frac{1}{2} \int_{|x| < R} \varphi(x, 0) \left[ \tilde{u}(x, Y(x), -Y(x)) - \tilde{u}(x, 0, 0) \right] \, dx.
\]

We conclude that the integral on the right-hand side of (19) equals
\[
\int_{|x| < R} \varphi(x, 0) \left[ \frac{1}{2} \tilde{u}(x, Y(x), Y(x)) + \frac{1}{2} \tilde{u}(x, Y(x), -Y(x)) - \tilde{u}(x, 0, 0) \right] \, dx
+ \int_{\partial \mathcal{V}} \left( \tilde{u} \partial_\nu w - \partial_\nu \tilde{u} \cdot w \right) \, d\gamma.
\]
(21)
Now we pass to the limit as \( h \to 0 \) in (19). The left-hand side tends to zero. Indeed, the second term of the integrand vanishes in view of (15). The function \((\partial_{t}^{2} - \Delta)\tilde{u}\) is nonzero only if \( \partial_{x,y}\chi \neq 0 \). However, the corresponding set of points \((x, y)\) is separated from \( K_{\Omega} \) owing to our choice of \( \chi \). In combination with (17), this gives that the first term of the integrand tends to zero as \( h \to 0 \).

As was previously shown, the right-hand side of (19) equals the expression (21). By the first relation in (6), \( \varphi(x, 0) \) is the Gaussian distribution in \( \mathbb{R}^{n-1} \), which tends to \( \delta(x) \) as \( h \to 0 \). Thus the first integral in (21) tends to

\[
\frac{1}{2} \sum_{\pm} \tilde{u}(0, Y(0), \pm Y(0)) - \tilde{u}(0, 0, 0) = \frac{1}{2} \sum_{\pm} u(0, Y(0), \pm Y(0)) - u(0, 0, 0)
\]

as \( h \to 0 \) (we used the fact that \( \chi|_{K_{\Omega}} = 1 \)). From the definition of the hypersurface \( \Gamma \) it follows that the integral over \( \Gamma \) in (21) equals

\[
\int_{S_{R}} d\sigma_{x,y} \int_{-Y(x)}^{Y(x)} (\tilde{u}\partial_{x}w - \partial_{x}\tilde{w} \cdot w) dt,
\]

where \( S_{R} = \{(x, y) \mid |x| < R, y = Y(x)\} \subset \partial \Omega \). Thus passing to the limit \( h \to 0 \) in (19) yields

\[
u(0, 0, 0) = \frac{1}{2} \sum_{\pm} u(0, Y(0), \pm Y(0)) + \lim_{h \to 0} \int_{S_{R}} d\sigma_{x,y} \int_{-Y(x)}^{Y(x)} (\tilde{u}\partial_{x}w - \partial_{x}\tilde{w} \cdot w) dt.
\]

The integral over \( S_{R} \) can be replaced by that over \( S \). This follows from (17) and from the fact that the set \( S_{R} \setminus S \) is separated from \( K_{\Omega} \), which means that the integrand tends to zero as \( h \to 0 \). Analogously, \( \tilde{u} \) can be replaced by \( u \), since the set of points, at which \( \chi \neq 1 \), is separated from \( K_{\Omega} \). Thus (18) is proved in the case \( Y(x) > 0, x \in \mathbb{R}^{n-1} \). Next we turn to the general case where \( Y(x) \) is assumed to be positive only if \((x, y) \in \bar{S} \).

Let \( X \) be the projection of the closure \( \bar{S} \) to the hyperplane \((x_{1}, \ldots, x_{n-1})\). Thus \( Y(x) > 0 \) for \( x \in X \), which implies that \( Y \) is positive in some neighborhood of \( X \). Hence there exists a \( C^{\infty} \)-smooth function \( \tilde{Y} \) in \( \mathbb{R}^{n-1} \) satisfying \( \tilde{Y}|_{X} = Y|_{X} \) and \( 0 < \tilde{Y}(x) \leq C \) for \( x \in \mathbb{R}^{n-1} \). Now we introduce the domain

\[
\bar{\Omega} = \{(x, y) \mid x \in \mathbb{R}^{n-1}, -\infty < y < \tilde{Y}(x)\}.
\]

Despite the fact that \( \tilde{Y} \) is everywhere positive, the preceding proof of (18) cannot be applied directly to the domain \( \bar{\Omega} \), since generally the solution \( u \) is not defined in \( \bar{\Omega} \times \mathbb{R} \). However, this issue is handled by a proper choice of the function \( \chi \) introduced at the beginning of the proof. It is possible to take \( \chi \) such that \( \chi = 1 \) in some neighborhood of \( K_{\Omega} \) and \( \chi(x, y) = 0 \) whenever \( x \notin X \), since the projection of \( K_{\Omega} \) to the hyperplane \((x_{1}, \ldots, x_{n-1})\) is contained in the interior of \( X \). Then the product \( \tilde{u} = \chi u \) is well defined for \( x \in X, y < \tilde{Y}(x) = Y(x), t \in \mathbb{R} \), and can be smoothly continued by zero to the remainder of \( \bar{\Omega} \times \mathbb{R} \). After that the preceding proof of (18) goes through.

The proof of the preceding theorem does not require the solution \( u \) of Eq. (1) to be defined everywhere in \( \Omega \times \mathbb{R} \). Suppose that Eq. (1) is satisfied for \( t \in \mathbb{R} \), and \((x, y) \in \Omega \setminus \omega \), where \( \omega \) is some relatively closed subset of \( \Omega \). Then formula (18) holds true, provided that the cone \( K \) is separated from \( \omega \) (which implies, in particular, that \((x^{*}, y^{*}) \in \Omega \setminus \omega \), see Fig. 1). The function \( \chi \) in the proof should be chosen so that it satisfies \( \chi = 0 \) in a neighborhood of \( \omega \). Then the product \( \chi u \) can be smoothly continued by zero to \( \omega \times \mathbb{R} \) and thus it can be viewed
as a function defined everywhere in $\Omega \times \mathbb{R}$. After that formula (18) is derived by the same argument.

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