We analyze some of the main approaches in the literature to the method of ‘adequality’ with which Fermat approached the problems of the calculus, as well as its source in the παρισότης of Diophantus, and propose a novel reading thereof. Adequality is a crucial step in Fermat’s method of finding maxima, minima, tangents, and solving other problems that a modern mathematician would solve using infinitesimal calculus. The method is presented in a series of short articles in Fermat’s collected works (Tannery and Henry 1981, pp. 133–172). We show that at least some of the manifestations of adequality amount to variational techniques exploiting a small, or infinitesimal, variation ε. Fermat’s treatment of geometric and physical applications suggests that an aspect of approximation is inherent in adequality, as well as an aspect of smallness on the part of ε. We question the relevance to understanding Fermat of 19th century dictionary definitions of παρισότης and adaequare, cited by Breger, and take issue with his interpretation of adequality, including his novel reading of Diophantus, and his hypothesis concerning alleged tampering with Fermat’s texts by Carcavy. We argue that Fermat relied on Bachet’s reading of Diophantus. Diophantus coined the term παρισότης for mathematical purposes and used it to refer to the way in which 1321/711 is approximately equal to 11/6. Bachet performed a semantic calque in passing from παρισότης to adaequare. We note the similar role of, respectively, adequality and the Transcendental Law of Homogeneity in the work of, respectively, Fermat (1896) and Leibniz (1858) on the problem of maxima and minima.
1. The Debate over Adequality

Adequality, or παρισότης (parisotēs) in the original Greek of Diophantus¹, is a crucial step in Fermat’s method of finding maxima, minima, tangents, and solving other problems that a modern mathematician would solve using infinitesimal calculus. The method is presented in a series of short articles in Fermat’s collected works (1891, pp. 133–172). The first article, Methodus ad Disquirendam Maximam et Minimam², opens with a summary of an algorithm for finding the maximum or minimum value of an algebraic expression in a variable \( A \). For convenience, we will write such an expression in modern functional notation as \( f(a) \).³

1.1. Summary of Fermat’s Algorithm

The algorithm can be broken up into six steps in the following way:

1. Introduce an auxiliary symbol \( e \), and form \( f(a + e) \);
2. Set adequal the two expressions \( f(a + e) = AD f(a) \);⁴
3. Cancel the common terms on the two sides of the adequality. The remaining terms all contain a factor of \( e \);
4. Divide by \( e \) (see also next step);
5. In a parenthetical comment, Fermat adds: “or by the highest common factor of \( e \)”;
6. Among the remaining terms, suppress⁵ all terms which still contain a factor of \( e \).⁶

Solving the resulting equation for \( a \) yields the extremum of \( f \).

In modern mathematical language, the algorithm entails expanding the difference quotient

\[
\frac{f(a + e) - f(a)}{e}
\]

1. adaequalitas or adaequare in Latin. See Section 1.2 for a more detailed etymological discussion.
2. In French translation, Méthode pour la Recherche du Maximum et du Minimum. Further quotations will be taken from the French text (1896).
3. Following the conventions established by Viète, Fermat uses capital letters of vowels \( A, E, I, O, U \) for variables, and capital letters of consonants for constants.
4. The notation “\( = AD \)” for adequality is ours, not Fermat’s. There is a variety of differing notations in the literature; see Barner 2011 for a summary.
5. The use of the term “suppress” in reference to the remaining terms, rather than “setting them equal to zero,” is crucial; see n.33.
6. Note that division by \( e \) in Step 4 necessarily precedes the suppression of remaining terms that contain \( e \) in Step 6. Suppressing all terms containing \( e \) or setting \( e \) equal to zero, at stage 3 would be meaningless; see n.22.
in powers of $e$ and taking the constant term.\footnote{Fermat also envisions a division by a higher power of $e$ as in step (5) (see Section 3).} The method (leaving aside step (5) for the moment) is immediately understandable to a modern reader as the elementary calculus exercise of finding the extremum by solving the equation $f'(a) = 0$. But the real question is how Fermat understood this algorithm in his own terms, in the mathematical language of his time, prior to the invention of calculus by Barrow, Leibniz, Newton, et al. There are two crucial points in trying to understand Fermat's reasoning: first, the meaning of “adequality” in step (2), and second, the justification for suppressing the terms involving positive powers of $e$ in step (6). The two issues are closely related because interpretation of adequality depends on the conditions on $e$. One condition which Fermat always assumes is that $e$ is positive. He did not use negative numbers in his calculations. Fermat introduces the term adequality in Methodus with a reference to Diophantus of Alexandria. In the third article of the series, Ad Eamdem Methodum (Sur la Même Méthode), he quotes Diophantus’ Greek term παρισότης, which he translates following Xylander and Bachet, as adaequatio or adaequalitas (1896, p. 126; see A. Weil 1984, p. 28).

1.2. Etymology of παρισότης.

The Greek word παρισότης consists of the prepositional prefix para and the root isos, “equality.” The prefix para, like all “regular” prefixes in Greek, also functions as a preposition indicating position; its basic meaning is that of proximity, but depending upon the construction in which it appears, it can indicate location (“beside”), direction (“to”), or source (“from”) (see Luraghi 2003, pp. 20–22, 131–145).

Compounds with all three meanings are found. Most familiar to mathematicians will be paralléllos (παράλληλος), used of lines that are “next to” one another; the Greek for “nearly resembling” is paraplélēsios (παραπλήσιος); but we also find direction in words like paradosis (παράδοσις), “transmitting, handing over” (from para and dosis, “giving”), and source in words like paralēpis (παράληψις), “receiving” (from para and lépsis, “taking”); there are other meanings for this prefix that do not concern us.

The combination of para and and isos (“equal”) can refer either to a simple equality (Aristotle, Rhetoric 1410a 23: παρίσοσις δ’ ἐὰν ἔσσα τὰ κύκλα, “It is parisósis if the members of the sentence are equal”) or an approximate equality (Strabo 11.7.1, ὡς φησι Πατροκλῆς, ὡς καὶ πάρισον ἴναι τὸ πέλαγος τούτο τῷ Ποντικῷ, “As Patrocles says, who also considers this sea [the Caspian] to be approximately equal [parison] to the Black Sea”). We know of no pas-
sage other than those of Diophantus in which a term involving \( \text{para} \) and \( \text{isos} \) refers to mathematical equality, whether approximate or otherwise. Diophantus himself used the term \( \text{parisos} \) to describe terms that are approximately equal, as we shall demonstrate below (Subsection 8.1).

The term \( \text{isotés} \) denotes a relationship (“equality”), not an action (“setting equal”); the normal term for the action of equalizing would be a form in -\( \text{ōsis} \), and in fact the words \( \text{isós} \) (\( \varepsilon\iota\omicron\sigma\sigma\)σ\) and \( \text{parisós} \) (\( \pi\alpha\rho\iota\iota\sigma\sigma\omega\sigma\)σ\)σ\) are attested with the meaning “making equal” or “equalization”; the latter is a common term in rhetoric for using the same (or nearly the same) number of words or syllables in parallel clauses. The word \( \pi\alpha\rho\iota\sigma\iota\tau\iota\sigma\)s, which occurs only in the two Diophantus passages, is on the face of it more appropriate to the meaning “near equality.”

Fermat himself may not have gotten this far into Greek etymology. On the other hand, Fermat viewed Diophantus through the lens of Bachet’s analysis. Bachet does interpret it as approximate equality. If Fermat follows Bachet, seeking to interpret Fermat’s method of adequality/\( \pi\alpha\rho\iota\sigma\iota\tau\iota\sigma\)s on the basis of the Latin term \( \text{adaequare} \) misses the point (see Subsection 7.1).

1.3. Modern Interpretations

There are differing interpretations of Fermat’s method in the literature. A. Weil notes that Diophantus uses the Greek term

to designate his way of approximating a given number by a rational solution to a given problem. (cf. e.g. \( \text{Dioph. V.11 and 14} \); Weil 1984, p. 28)

According to Weil’s interpretation, approximation is implicit in the meaning of the original Greek term. H. Breger rejects Weil’s interpretation of Diophantus, and proposes his own interpretation of the mathematics of Diophantus’ \( \pi\alpha\rho\iota\sigma\iota\tau\iota\sigma\)s. He argues that \( \pi\alpha\rho\iota\sigma\iota\tau\iota\sigma\)s means equality to Diophantus (Breger 1994, p. 201). Thus, the question of whether there is an element of approximation in the Greek source of the term is itself subject to dispute. There is also a purely algebraic aspect to adequality, based on the ideas of Pappus of Alexandria, and Fermat’s predecessor Viète. In \( \text{Sur la Même Méthode} \) following the comment quoted above, Fermat writes as follows:

En cet endroit, Pappus appelle un rapport minimum \( \mu\omicron\nu\omicron\alpha\chi\omicron\nu \omicron \kappa\omicron \ell\alpha\chi\iota\sigma\omicron\nu \) (singulier et minimum), parce que, si l’on propose une question sur les grandeurs données, et qu’elle soit en général satisfaite par deux points, pour les valeurs maximum et minimum, il n’y aura qu’un point qui satisfasse. C’est pour cela que Pappus
The point is that the extremum of a quadratic expression at the point $a$
corresponds to a double root in $e$ for what would be in modern terms the
equation $f(a + e) - f(a) = 0$. From this point of view, Fermat explains his
method in terms of roots of algebraic equations. In the first paragraph of
the fourth article in the series, *Methodus de Maxima et Minima (Méthode du
Maximum et Minimum)*, Fermat reveals the source of this procedure:

> En étudiant la méthode de la *syncrise* et de la *anastrophe* de Viète, et
en poursuivant soigneusement son application à la recherche de la
constitution des équations corrélatives, il m’est venu à l’esprit d’en
dériver un procédé pour trouver le maximum et le minimum et
pour résoudre ainsi aisément toutes les difficultés relatives aux
conditions limites, qui ont causé tant d’embarras aux géomètres
anciens et modernes. (Fermat 1896, p. 131)

From this point of view, adequality is based on replacing the variable $a$
by the variable $a + e$ in the original algebraic expression and thus creating an
equation in $a$ and $e$ which is required to have a double root at $e = 0$ for an
extremal point $a$. This interpretation is considered by Breger (1994) and
K. Barner (2011) to cover all the examples. They deny that any kind of
"approximation" is involved, and hold adequality to be a formal or alge-
braic procedure of "setting equal."

1.4. Wieleitner, Strømholm, and Giusti
The authors H. Wieleitner (1929), P. Strømholm (1968), and E. Giusti
(2009) argue that both interpretations, algebraic and approximation, are
valid, representing different stages in the development of Fermat’s
method. The algebraic approach, following Pappus and Viète, involves
equating two values $f(a)$ and $f(a + e)$, below the maximum or above the
minimum. However, there is another point of view in which $f(a)$ and
$f(a + e)$ are definitely not assumed by Fermat to be equal, as he writes in
*Sur la Même Méthode* that he compares the two expressions “comme s’ils
étaient égaux, quoiqu’en fait ils ne le soient point” (Fermat 1896, p. 126),
and a little later “une comparaison feinte ou une *adégalité*” (p. 127). Giusti
(2009) and Strømholm (1968) consider this to represent a second stage in
the development of Fermat’s method. Similarly, in the fourth article,
which begins with a reference to Viète and emphasizes the algebraic ap-

8. Sometimes the second point is denoted $e$ rather than $a + e$, and the two expressions
that are equated are $f(a)$ and $f(e)$. 
proach, Fermat introduces an element of approximation, remarking that the difference between the two points $a$ and $e$ goes to zero:

Plus le produit des segments augmentera, plus au contraire diminuera la différence entre $a$ et $e$, jusqu’à ce qu’elle s’évanouisse tout à fait. (1896, p. 132)

What complicates the task is that Fermat does not separate clearly between these two methods. They appear in successive paragraphs. Mahoney understands one of the meanings of adequality as “approximate equality” or “equality in the limiting case” (Mahoney 1973, p. 164 n.46), while emphasizing that the term has multiple meanings. Fermat never gave a full explanation of his method, but he derived it from three sources: Diophantus, Pappus, and Fermat’s predecessor Viète (Vieta). If we consider the source in Diophantus and interpret adequality as approximate equality as did Weil, Strømholm, and Giusti, it is natural to ask whether Fermat considered $e$ to be arbitrarily small and eventually negligible, although he never explicitly stated such an assumption. On the other hand, the algebraic point of view, finding a condition for a unique root of multiplicity 2, following Pappus and Viète, is clearly the point of view in a number of examples mentioned above. These considerations do not resolve the issue of what Fermat thought about the actual magnitude of $e$. Strømholm (1968) and Wieleitner (1929) deal with this question. They distinguish two methods in Fermat. One method is algebraic, following Pappus and Viète, which Strømholm following Wieleitner calls M2. The other method, M1, is interpreted as expanding $f(a + e) - f(a)$ in powers of $e$. The latter approach is most fully expounded in Fermat’s letter to Brûlart (Fermat 1643). The letter attempts to explain why the method guarantees a maximum or minimum without assuming a condition on the size of $e$. The “approximation” interpretation actually branches out into two distinct approaches. The difference between them concerns the interpretation of the symbol $e$ that Fermat uses, to form expressions that in modern notation would be written as “$f(x + e) - f(x)$.” Namely, one can think of $e$ as representing a kinetic process such as “tending to 0,” as in the fourth article, as cited above, or one can think of $e$ as “infinitesimal.” G. Cifoletti (1990) and J. Stillwell (2006) interpret it in accordance with the latter approach.

1.5. Three Approaches to the Nature of $E$
There are therefore at least three different approaches to Fermat’s symbol $E$ as it appears in adequities: algorithmic or formal/algebraic; kinetic $E \to 0$; and infinitesimal. We will argue that the last one is closest to

9. See note 8.
Fermat’s thinking. We note that two distinct issues are sometimes conflated in the literature on Fermat’s method. The first issue is whether adequality means (α) “setting equal,” or whether it is (β) an “approximate equality.” A second, separate issue concerns the question of what the famous symbol $E$ stands for: is it (A) an arbitrary variable, or does it imply (B) some notion of size: small, infinitesimal, tending to zero, etc. The trend in the literature is that scholars following the interpretation (α), also adopt (A), and similarly for the other pair. For instance, Breger (1994, p. 206) rejects the small/infinitesimal idea and supports (A). But the thrust of his argument is to support the (α)-interpretation rather than the (A)-interpretation. These are, in fact, separate issues, as can be seen most readily in the context of an infinitesimal-enriched ring such as the dual numbers $D$ of the form $a + b\varepsilon$ where $\varepsilon^2 = 0$. To differentiate a polynomial $p(x)$, we apply the following purely algebraic procedure: expand in powers of $\varepsilon$; write $p(x + \varepsilon) = p(x) + \varepsilon q(x)$, where $q(x)$ has no $\varepsilon$ terms, then one has $p(x + \varepsilon) - p(x) = \varepsilon q(x)$, and $q(x)$ is the derivative. This produces the derivative of $p$ over $D$. A similar procedure works over any other reasonable infinitesimal-enriched extension of $\mathbb{R}$ such as the hyperreals (see footnote 33). The algebraic nature of this procedure in no way contradicts the infinitesimal nature of $\varepsilon$. In addition to arguing that “adequality” is a purely algebraic procedure of “setting equal,” Breger believes that $E$ is a formal variable with no assumption on size. However, one does not necessarily imply the other. The procedures in non-standard analysis are purely algebraic and can be programmed by a finite algorithm (no need for an infinite limiting process), yet here $E$ is definitely infinitesimal. No analysis of Fermat’s method can be considered complete that does not include a discussion of the application to transcendental curves. Such an analysis appears in Section 5. Concerning the question as to whether Fermat’s method is a purely algorithmic/algebraic one, with $E$ being a formal variable, or whether it involves some notion of “smallness” on the part of $E$, we argue that the answer depends on which stage of Fermat’s method one is dealing with. He certainly did present an algorithmic outline of his method in a way that suggests that $E$ is a formal variable. However, when one examines other applications of the method, one notices additional aspects of Fermat’s method which cannot be accounted for by means of a “formal” story. Thus, Fermat exploits his adequality to solve a least time variational problem for the refraction of light (see Section 9). Here $E$ corresponds to a variation of a physical quantity, and it would seem paradoxical to describe it as a formal variable in this context. Furthermore, in the case of the cycloid, the transcendental nature of the problem creates a situation that cannot be treated algebraically at all (see Section 5). Once it is accepted that at least in some applications, the aspect of “smallness”
on the part of \( E \) is indispensable, one can ask in what sense precisely is \( E \) “small.” Today we know of two main approaches to “smallness,” namely, (1) by means of kinetic ideas related to limits, or (2) by means of infinitesimals. The former ideas were as yet undeveloped in Fermat’s time (though they are already present in Newton only a few decades later), and in fact one finds very little “tends to . . .” material in Fermat. Meanwhile, infinitesimals were already widely used by Kepler, Wallis, and others. Fermat’s 1657 letter to Digby on Wallis’s method (see Section 2) shows that he was intimately familiar with the method of indivisibles. What we argue therefore is that it is more reasonable to assume (2). The question why Fermat wasn’t more explicit about the nature of his \( E \) is an interesting one. Note that Fermat was involved in an acrimonious rivalry with Descartes. Descartes thought that one of the strengths of his own method was that it was purely algebraic. It is possible that Fermat did not wish to elaborate on the meaning of \( E \) because he wished to avoid criticism by Descartes.

2. Methodological Issues in 17th Century Historiography

On 15 August 1657, Fermat sent a letter to Kenelm Digby (1603–1665). The letter was titled “Remarques sur l’arithmétique des infinis de S. J. Wallis.” The letter contains a critique of Wallis’s infinitesimal method that reveals as much about Fermat’s own position as about Wallis’s method. As we will see, certain aspects of Wallis’s method not criticized by Fermat are as interesting as the actual criticisms.

2.1. Fermat’s Letter to Digby

The letter is cited by A. Malet (1996, p. 37, n. 48). It is also mentioned by J. Stedall, who goes on to say that

The [mathematical] details of the subsequent argument need not concern us here. (Stedall 2001, p. 12)

We will be precisely interested in the mathematical, as well as the “metamathematical” issues involved. Fermat summarizes his objection to Wallis’s method in the following terms:

Mais, de même qu’on ne pourroit pas avoir la raison de tous les diamètres pris ensemble des cercles qui composent le cône à ceux du cylindre circonscrit, si on n’avoit la quadrature du triangle; non plus que la raison des diamètres des cercles qui composent le conoïde parabolique à ceux qui font le cylindre circonscrit, si on n’avoit la quadrature de la parabole ; ainsi on ne pourra pas connaître la raison des diamètres de tous les cercles qui composent la
Fermat is making a remarkable claim to the effect that in order to find the quadrature of the circle, Wallis is exploiting the quadrature of the circle itself. Fermat appears to be criticizing an alleged circularity in Wallis’s reasoning. Apart from the issue of the potency of his critique, what is striking about it is the aspect of Wallis’s method that Fermat is not criticizing. Namely, what emerges from Fermat’s presentation is that Fermat is taking the infinitesimal technique itself for granted. In the paragraph preceding the one cited above, Fermat talks about spheres and cylinders being composed of infinite families of parallel circles as a routine matter:

D'où il conclut que, puisqu'on a trouvé aussi la raison de la sphère au cylindre circonscrit, ou celle de l'infini des cercles parallèles, dont on peut concevoir que la sphère est composée, à pareille multitude de ceux qui se peuvent feindre au cylindre, on pourra aussi espérer de pouvoir découvrir la raison des ordonnées en la sphère ou au cercle à celles du cylindre ou quarré, savoir la raison des diamètres des cercles infinis qui composent la sphère aux diamètres des cercles du cylindre. Ce qui seroit avoir la quadrature du cercle. (Fermat 1657, pp. 347–348)

Thus, it is not the infinitesimal method itself that Fermat is criticizing, but rather the logic of Wallis’s reasoning.

2.2. Huygens and Rolle

C. Huygens (1940) declared in 1667 at the French Academy of Sciences that Fermat’s “e” was an infinitely small quantity:

Or, en prenant $e$ infiniment petite, la même équation donnera la valeur de $EG$ lorsqu'elle est égale à $EF$ . . . Ensuite on divise tous les termes par $e$ et on détruit ceux qui, après cette division, contiennent encore cette lettre, puisqu’ils représentent des quantités infinites plus petites par rapport à ceux qui ne renferment plus $e$. (Huygens 1940; Trompler & Noël 2003, p. 110)

Huygens’s interpretation is testimony to the enduring influence of Fermat’s method of adequality already in the 17th century. Yet, Huygens may have been putting in Fermat’s mouth words that did not emanate therefrom.10

10. See also Section 10 for Leibniz’s view on Fermat’s method.
Similarly, Michel Rolle in 1703 claimed a connection between Fermat’s \(a\) and \(e\) and Leibniz’s \(dx\) and \(dy\):

En 1684, Mr de Leibniz donna dans des journaux de Leipzig des exemples de la formule ordinaire des tangentes, et il imposa le nom d’égalité différentielle à cette formule [. . .] Mr de Leibniz n’entreprend point d’expliquer l’origine de ces formules dans ce projet, ni d’en donner la démonstration [. . .] Au lieu de l’\(a\) & de l’\(e\), il prend \(dx \& dy\). (Rolle 1703, p. 6)

Yet Rolle was an enemy of the calculus, and his identification of \(e\) and \(dx\) may have been due to his eagerness to denigrate Leibniz. How are we to avoid this type of pitfall in analyzing Fermat’s oeuvre? In discussing Fermat’s mathematics, two traps are to be avoided:

1. **Whiggish history**, that is, “the study of the past with direct and perpetual reference to the present” (H. Butterfield 1931, p. 11). A convincing reading of Fermat must be solidly grounded in the 17th century and its ideas, rather than 19th or 20th centuries and their ideas.

2. One needs to consider the possibility that Fermat wasn’t working with clear concepts that were destined to become those of the calculus.

We will discuss each of them separately in Subsections 2.3 and 2.4.

### 2.3. Whig History

As far as trap (1) is concerned, it was precisely the risk of tendentious rewriting of mathematical history that prompted Mancosu to observe that the literature on infinity is replete with such ‘Whig’ history. Praise and blame are passed depending on whether or not an author might have anticipated Cantor and naturally this leads to a completely anachronistic reading of many of the medieval and later contributions. (Mancosu 2009, p. 626 [emphasis added])

Thus, Cauchy has been often presented anachronistically as a sort of proto-Weierstrass. Such a Cauchy–Weierstrass tale has been critically analyzed by Blaszczyk et al. (2013), Borovik et al. (2010), Bråting (2007), and Katz & Katz (2011; 2012). Whiggish tendencies in Leibniz scholarship were analyzed by Katz & Sherry (2012). To guard against this trap, we will eschew potential 19th and 20th century ramifications of Fermat’s work, and focus entirely on its 17th century context. More specifically, we will examine a possible connection between Fermat’s adequality and Leibniz’s Tran-
scental Law of Homogeneity (TLH), which play parallel roles in Fermat’s and Leibniz’s approaches to the problem of maxima and minima. Note the similarity in titles of their seminal texts: *Methodus ad Dicquirendam Maximam et Minimam* (Fermat 1891, p. 133) and *Nova methodus pro maximis et minimis . . .* (Leibniz 1858). Leibniz developed the TLH in order to mediate between assignable and inassignable quantities. The TLH governs equations involving differentials. H. Bos interprets it as follows:

A quantity which is infinitely small with respect to another quantity can be neglected if compared with that quantity. Thus all terms in an equation except those of the highest order of infinity, or the lowest order of infinite smallness, can be discarded. For instance,

\[ a + dx = a \]
\[ dx + ddy = dx \]

etc. The resulting equations satisfy this [. . .] requirement of homogeneity. (Bos 1974, p. 33 paraphrasing Leibniz (1858) [1710], pp. 381–382)

The title of Leibniz’s 1710 text is *Symbolismus memorabilis calculi algebraici et infinitesimalis in comparatione potentiarum et differentiarum, et de lege homogeneorum transcendentali.* The inclusion of the transcendental law of homogeneity (*lex homogeneorum transcendentalis*) in the title of the text attests to the importance Leibniz attached to this law. The “equality up to an infinitesimal” implied in TLH was explicitly discussed by Leibniz in a 1695 response to Nieuwentijt, in the following terms:

Caeterum *aequalia* esse puto, non tantum quorum differentia est omnino nulla, sed et quorum differentia est incomparabiler parva; et licet ea Nihil omnino dici non debeat, non tamen est quantitas comparabilis cum ipsis, quorum est differentia. (Leibniz 1846 [1695], p. 322 [emphasis added])

Translation:

Besides, I consider to be equal not only those things whose difference is entirely nothing, but also those whose difference is incomparably small: and granted that it [i.e., the difference] should not be called entirely Nothing, nevertheless it is not a quantity comparable to those whose difference it is.
How did Leibniz use the TLH in developing the calculus? The issue can be illustrated by Leibniz’s justification of the last step in the following calculation:

\[
d \left( uv \right) = (u + du)(v + dv) - uv = \\
udv + vdu + du \, dv = udv + vdu.
\]  

(2.2)

The last step in the calculation (2.2), namely

\[
udv + vdu + du \, dv = udv + vdu
\]

is an application of Leibniz’s TLH. In his 1701 text *Cum Prodiisset* (1846, pp. 46–47), Leibniz presents an alternative justification of the product rule (see Bos 1974, p. 58). Here he divides by \(dx\) and argues with differential quotients rather than differentials. The role played by the TLH in this calculation is similar to that played by adequality in Fermat’s work on maxima and minima.

### 2.4. Clear Concepts?

Was Fermat working with clear concepts that were destined to become those of the calculus? This is a complex question that conflates two separate issues: (a) were Fermat’s ideas clear? and (b) were Fermat’s ideas destined to become those of the calculus? Even the latter formulation is questionable, as the definite article in front of “calculus” disregards that fact, emphasized by H. Bos (1974), that the principles of Leibnizian calculus based on differentials differ from those of modern calculus based on functions. Thus the answer to (b) is certainly “we don’t know,” though there is a parallelism between adequality and TLH as we argued in Subsection 2.3.

As far as question (a) is concerned, it needs to be pointed out that such concerns are as old as the critique of Fermat’s method by Descartes, who precisely thought that Fermat was confused and his method in the category of a lucky guess.\(^{11}\) However, most modern scholars don’t share Descartes’ view. Thus, H. Breger wrote:

> brilliant mathematicians usually are not so very confused when talking about their own central mathematical ideas [. . .] I would like to stress that my hypothesis renders Fermat’s mathematics clear and intelligible, that the hypothesis is supported by several philo-

\(^{11}\) See Mahoney 1994, pp. 180–181.
logical arguments, and that it does not need the assumption that Fermat was confused. (Breger 1994, pp. 193–194)

For all our disagreements with Breger, this is one point we can agree upon.

3. Did Fermat Make a Mistake?
In interpreting Fermat’s adequality, one has to keep in mind that certain crucial components of the conceptual structure of the calculus were as yet unknown to Fermat. A striking illustration of this is what Strømholm refers to as Fermat’s “mistake” (Strømholm 1968, p. 51). Is it really a mistake, a redundancy, or neither? In this section, we will write $A$ and $E$ in place of $a$ and $e$, following Strømholm. After forming the difference $f(A) - f(A + E)$ and cancelling out terms not containing $E$, Fermat writes (in Tannery’s French translation):

On divisera tous les termes par $E$, ou par une puissance de $E$.\(^{12}\) (1896, p. 121)

Strømholm comments as follows concerning this division:

Fermat told his readers that one was to divide by some power of $E$. This, of course, was wrong as can be seen from $f(A + E) - f(A) = \sum_{n=1}^{\infty} \frac{E^n}{n!} f^{(n)}(A)$.

Still, his mistake was understandable (Strømholm 1968, p. 51). Strømholm does not explain how exactly it can be seen from the Taylor series expansion that Fermat was wrong. Strømholm continues as follows:

As he could not possibly foresee the peculiarities and future significance of the $f^{(n)}(A)$, he guarded himself against the possibility that $f'(A)$ be zero, a case which might conceivably (to him) turn up in some future problem.” (1968, p. 51)

In the framework of Strømholm’s narrow interpretation of the method, no such case could conceivably turn up (but see below), and for a reason unrelated to any “future significance of $f^{(n)}(A)$.” Namely, if the derivative vanishes identically (note that Fermat’s calculation is a symbolic manipulation without assigning a particular value to the variable $A$), then the original function itself is identically constant by the fundamental theorem

\(^{12}\) “We divide all the terms by $E$, or by some power of $E$” (the latter clause corresponds to Step (5) of Fermat’s procedure as outlined in Subsection 1.1).
of calculus (FTC). The latter wasn’t proved until the 1670s, by Isaac Barrow. Not being aware of the FTC, Fermat was apparently also unaware of the fact that no such future problem could turn up, and therefore left in the phrase “some power of $E$”, even though only the first power is relevant. Have we shown then that Fermat’s description of his method contains a mistake, or at least a redundancy? This is in fact not the case. Giusti (2009) notes that in the fifth article Appendice à la Méthode du Maximum et Minimum there is an example involving radicals. In this case, the method of adequality as applied by Fermat leads to an expression in which (before division) the least power of $E$ is 2 rather than 1, and one does have to divide by $E^2$ instead of by $E$.\footnote{Strømholm’s mistake is repeated by Mahoney in both editions of his book: “in the problems Fermat worked out, the proviso of repeated division by $[E]$ was unnecessary” (Mahoney 1994, p. 165).} What about the above argument based on the FTC then? What happened is that Fermat performs a series of algebraic simplifications so as to eliminate the radicals from the equation, a point also noted by Andersen (1983, p. 59). The result in this case is an expression where the least power of $E$ is 2. Such manipulations are not mentioned in the algorithmic/formal description of the method of adequacy, and were not taken into account in Strømholm’s description of Fermat’s “mistake.” This example is a striking illustration of the fact that Fermat’s method of adequacy is not a single method but rather a cluster of methods. The algorithmic procedure described by Fermat at the outset is merely a kernel common to all applications, but in each application the kernel is applied somewhat differently. We will analyze one such difference in the next two sections.

### 4. Comparing the First and Second Problems in Méthode

In this section, we will analyze some apparent dissimilarities between Fermat’s approaches to the first two problems in his Méthode pour la recherche du maximum et du minimum (1896, p. 122–123). The first problem involves splitting a segment into two subsegments so as to maximize the area of the rectangle formed by the subsegments as sides. The solution is a straightforward application of the formal algorithmic technique he outlined on the previous page (1896, p. 121). Fermat’s second problem involves finding the equation of a tangent line to a parabola. Mathematically speaking, the second problem is equivalent to the first. Namely, given a point, say $P$, on the parabola, we would write down the point-slope formula for a line through $P$ (with the point fixed and slope, variable), form the difference between the point-slope and the formula for the parabola,
and look for an extremum of the resulting expression. But did Fermat view it that way? He does not appear to have described it that way. Forming the difference of the two formulas is an algebraic procedure. Did Fermat have such an algebraic procedure in mind, or would such an approach go beyond the geometric framework as it actually appears in Fermat? What Fermat did write is that the point on the tangent line lies outside the parabola. Having stated this geometric fact, Fermat proceeds to write down an inequality expressing it. Is the resulting inequality an inessential embellishment of this particular application of the method of maxima and minima, or is it an essential part of the argument? At least on the surface of it, Fermat’s formulation is unlike the earlier case where one obtains an adequality immediately, due to the nature of the problem, without using an intermediate inequality. The passage from the inequality to adequality, depending on whether it is seen as an essential ingredient in the argument, may or may not make the second example different from the first one, as we discuss in Subsection 4.1.

4.1. Tangent Line and Convexity of Parabola
Consider Fermat’s calculation of the tangent line to the parabola (see 1896, pp. 122–123). To simplify Fermat’s notation, we will work with the parabola $y = x^2$, or

$$\frac{x^2}{y} = 1.$$  

To understand what Fermat is doing, it is helpful to think of the parabola as a level curve of the two-variable function $\frac{x^2}{y}$. Given a point $(x, y)$ on the parabola, Fermat wishes to find the tangent line through the point. Fermat exploits the geometric fact that by convexity, a point

$$(p, q)$$  

on the tangent line lies outside the parabola. He therefore obtains an inequality equivalent in our notation to $\frac{p^2}{q} > 1$, or $p^2 > q$. Here $q = y - e$, and $e$ is Fermat’s magic symbol we wish to understand. Thus, we obtain

$$\frac{p^2}{y - e} > 1$$  

(4.1)

At this point Fermat proceeds as follows:

(i) he writes down the inequality $\frac{p^2}{y - e} > 1$, or $p^2 > y - e$;
(ii) he invites the reader to *adégaler* (to “adequate”);
(iii) he writes down the adequality \( \frac{x^2}{p^2} = \alpha \frac{y}{y-e} \);

(iv) he uses an identity involving similar triangles to substitute

\[
\frac{x}{p} = \frac{y + r}{y + r - e}
\]

where \( r \) is the distance from the vertex of the parabola to the point of intersection of the tangent to the parabola at \( y \) with the axis of symmetry,

(v) he cross multiplies and cancels identical terms on right and left, then divides out by \( e \), discards the remaining terms containing \( e \), and obtains the solution \( y = r \).\(^{14}\)

What interests us here are the steps (i) and (ii). How does Fermat pass from an inequality to an adequality? Giusti already noted that

Comme d’habitude, Fermat est autant détaillé dans les exemples qu’il est réticent dans les explications. On ne trouvera donc presque jamais des justifications de sa règle des tangentes. (Giusti 2009)

In fact, Fermat provides no explicit explanation for this step. However, he uses the same principle of applying the defining relation for a curve to points on the tangent line to the curve. Note that here the quantity \( e \), as in \( q = y - e \), is positive: Fermat did not have the facility we do of assigning negative values to variables. Thus, Strømholm notes that Fermat never considered negative roots, and if \( A = 0 \) was a solution of an equation, he did not mention it as it was nearly always geometrically uninteresting. (Strømholm 1968, p. 49)

Fermat says nothing about considering points \( y + e \) “on the other side”, i.e. further away from the vertex of the parabola, as he does in the context of applying a related but different method, for instance in his two letters to Mersenne (see Strømholm 1968, p. 51), and in his letter to Brühlart (Fermat 1643).\(^{15}\) Now for positive values of \( e \), Fermat’s inequality (4.1) would be satisfied by a transverse ray (i.e., secant ray) starting at \((x,y)\) and lying outside the parabola, just as much as it is satisfied by a tangent ray starting at \((x,y)\). Fermat’s method therefore presupposes an additional piece of information, privileging the tangent ray over transverse rays.

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14. In Fermat’s notation \( y = d, y + r = a \). Step (v) can be understood as requiring the expression \( \frac{y}{y-e} - \frac{(y+e)^2}{(y+r-e)^2} \) to have a double root at \( e = 0 \), leading to the solution \( y = r \) or in Fermat’s notation \( a = 2r \).

15. This was noted by Giusti 2009.
4.2. Two Interpretations of the Geometric Ingredient

What is the nature of the additional piece of information that would privilege the tangent ray? There are two possible approaches here:

- one can argue that the additional piece of information is derived from the geometric context: namely, the tangent line provides a better approximation than a transverse line, motivating the passage to an adequality.
- one can argue that both Fermat's geometric context (tangent line being outside the parabola) and his inequality \((p^2 > q)\) are merely incidental, and that Fermat’s procedure here is purely algebraic, namely, equivalent to forming the difference between the formula for a line through \((x,y)\) and the formula for the parabola, and seeking an extremum as before.

In support of the geometric interpretation, it could be said that, in order to understand Fermat’s procedure, other than treating it as a magician’s rabbit out of a hat, we need to relate to the geometric context. To passage from inequality to adequality therefore only becomes intelligible as something less mysterious, if one assumes that \(e\) is small and exploits the geometric background with a better rate of approximation provided by the tangent line as compared to a transverse ray. Strømholm (1968) similarly emphasizes the role of the smallness of \(e\) in Fermat’s thinking. To assert that Fermat’s procedure using the symbol \(e\) is purely formal/algebraic in the context of this particular example, is to assert that Fermat is a magician, not a mathematician. In support of the algebraic interpretation, it could be said that Fermat writes “adégalons donc, d’après la méthode précédente” with reference to the second example, apparently implying that the methods of the first and second examples are comparable. Since both methods contain a common kernel as we discussed in Section 3, the reference to the previous example is not conclusive. Treating the geometric ingredient as an essential part of the proof in this case is the more appealing option, suggesting that the method of tangents is not a direct application of the kernel of the method of maxima and minima, but rather exploits additional geometric information in a crucial way. Similarly, K. Pedersen comments:

The inequality \(IO > IP\) holds for all curves concave with respect to the axis, and the inequality \(IO < IP\) for convex curves. For curves without points of inflection it is possible from these inequalities to find a magnitude depending on \(a - e\) and \(x - e\) which has an extreme value for \(x - e = x\). (Pedersen16 1980, p. 28)

16. This is the same author as Andersen 1983.
She continues:

Neither in his *Methodus* nor in Fermat’s later writings, however, is there any indication that this was the way he related his method of tangents to his method of maxima and minima (idem)

and concludes that

Descartes was right after all in raising the objection that the method of tangents was not a direct application of the method of maxima and minima. (idem)

We saw that the geometric content of the argument dealing with tangents to parabolas tends to go counter to the formal/algebraic interpretation. K. Barner (2011, p. 34) attempts to save the day by declaring that Fermat made a mistake. According to Barner, the point we denoted \((p,q)\) in Section 4.1 should not be outside the parabola at all, but rather should be on the parabola. The line passing through \((p,q)\) should not be the tangent line, but rather a transverse line, whereas \(e\) is the difference in the ordinates of the two points on the parabola. In this way, the *inequality* should not be there in the first place, but should rather be an *equality*.

In fact, what Barner describes with uncanny accuracy is the method of Beaugrand, probably dating from 1638, as analyzed by Strømholm (1968, pp. 64–65). We therefore find Barner’s explanation, as applied to Fermat, to be forced (see Section 5.3 for more details). The formal/algebraic interpretation of adequality is unconvincing in this particular case.

5. Fermat’s Treatment of the Cycloid

The cycloid is generated by marking a point on a circle and tracing the path of the point as the circle rolls along a horizontal straight line. If the marked point is the initial point of contact of the circle with the line, and the circle rolls to the right, then the ordinate\(^{17}\) of the marked point is given by the difference of the length of arc traversed (the distance the center of the circle has moved) and the distance of the point from the vertical line through the center of the circle\(^{18}\).

5.1. Fermat’s Description

Fermat’s description of the cycloid is based on a diagram (1891, Figure 103, p. 163) reproduced in Figure 1. Let \(R\) be a point on the cycloid and \(D\) the point of intersection of the horizontal line \(ℓ\) through \(R\) with the

\[ x = \theta - \sin \theta, \quad y = 1 - \cos \theta. \]

17. In this case the *ordinate* refers to the horizontal coordinate.

18. Assuming the circle to have radius 1, the equation of the cycloid as described above is \(x = \theta - \sin \theta, y = 1 - \cos \theta\).
axis of symmetry of the cycloid generated by one full revolution of the circle. If $M$ is the point of intersection of \ell with the generating circle when centered on the axis of symmetry, and $C$ is the apex of that circle then in the words of Fermat:

La propriété spécifique de la courbe est que la droite $RD$ est égale à la somme de l’arc de cercle $CM$ et de l’ordonnée $DM$.\(^\text{19}\) (1896, 144)

Let $r$ be the tangent line to the cycloid at $R$, and $m$ the tangent line to the circle at $M$. To determine the defining relation of $r$, Fermat considers the horizontal line $NIVOE$ passing through a point $N \in r$. Here $I$ is the first point of intersection with the cycloid, while $V$ is the point of intersection with $m$, and $O$ is the point of intersection with the generating circle, and $E$ is the point of intersection with the axis of symmetry.

The defining relation for $r$ is derived from the defining relation for

\(^{19}\) To compare Fermat’s description with the parametric description given in the previous footnote, we note that length of the segment $RD$ is $\pi - x = \pi - \theta + \sin \theta$, while $\pi - \theta$ is the length of the arc $CM$, and the length $DM$ equals $\sin \theta$.  

the cycloid using adequality. The defining relation for the point $I$ on the cycloid is

$$IE = EO + \text{arc}CO$$

By adequality, Fermat first replaces $I$ by the point $N \in r$:

$$NE = _{AD}OE + \text{arc}CO = OE + \text{arc}CM - \text{arc}MO$$

(5.1)

Then Fermat replaces $O$ by the point $V \in m$, and the arc $MO$, by the length of the segment $MV \subset m$. This produces the linear relation

$$NE = _{AD}VE + \text{arc}CM - MV$$

yielding the equation of the tangent line $r$ to the cycloid at $R$ as a graph over the axis of symmetry. The distance $NE$ is expressed in terms of the distance $VE$, where $V \in m$, and the distance $MV$ along that tangent line. Thinking in terms of slope relative to the variable distance $DE$ (which corresponds to the parameter $e$ in the example of the parabola), Fermat’s equation says that the slope of $r$ relative to $DE$ is the slope of $m$ minus the proportionality factor of $MV$ relative to $DE^{20}$. To summarize, Fermat exploited two adequations in his calculation:

1. the length of a segment along $m$ adequals the length of a segment of a circular arc, and
2. the distance from the axis of symmetry to a point on $r$ (or $m$) adequals the distance from the axis to a corresponding point on the cycloid (or circle).

As Fermat explains,

Il faut donc adégaler (à cause de la proprété spécifique de la courbe qui est à considérer sur la tangente) cette droite $\frac{2\text{arc} - \text{arc}e}{a}$ [i.e., $NE$] à la somme $OE + \text{arc}CO$ [. . .][et] d’après la remarque précédente, substituer, à $OE$, l’ordonnée $EV$ de la tangente, et à l’arc $MO$, la portion de la tangente $MV$ qui lui est adjacente. (Fermat 1896, p. 144; 1891, p. 228)

20. The slope of the tangent line relative to the axis of symmetry, or equivalently, relative to the $y$ axis, given by elementary calculus is $\frac{d}{dx} \left( \frac{\sin \theta}{\sin \theta} \right) = \frac{\cos \theta}{\sin \theta}$. The length $MV$ equals $e(\sin \theta)$ and the slope of the tangent line to the circle relative to the $y$ axis is $\frac{\cos \theta}{\sin \theta}$, in agreement with Fermat’s equation.
The procedure can be summarized in modern terms by the following principle:

**Principle 5.1.** *The tangent line to the curve is defined by using adequality to linearize the defining relation of the curve, or “adégaler (à cause de la propriété spécifique de la courbe qui est à considérer sur la tangente).”*

Fermat uses the same argument in his calculation of the tangents to other transcendental curves whose defining property is similar to the cycloids and involves arc length along a generating curve. For a discussion of some of these examples, see Giusti 2009 and Itard 1949.

5.2. Breger’s Interpretation

Breger claims that Fermat does not make use of an approximate equality between arc length of the circle and a segment of the tangent. The latter interpretation usually is made, but it plainly contradicts the text: Fermat explicitly calls the straight line $DE$ “recta utcumque assumpta” (Fermat 1891, p. 163) [droite arbitraire], that is, $DE$ is not infinitely small or “very small.” (1994, p. 206)

Is it convincing to argue, as Breger does, that the hypothesis of smallness plays no role here? It is certainly true that the axis of symmetry contains points arbitrarily far from the fixed point $D$. However, being able meaningfully to apply the defining relation for the curve to the tangent line, is contingent upon the fact that for points near the point of tangency the tangent line gives a good (second order) approximation to the curve. Why does Fermat’s calculation give the tangent to the cycloid? In modern terms, this is because of the quadratic order in $e$ for the error term which results from making each of the following two substitutions: (1) substituting the point $N$ on the tangent line for the point $I$ on the cycloid and (2) substituting the length of a segment of the tangent to the circle for arc $CO$ on the circle. Breger’s analysis of the example proceeds as follows:

Fermat (1891, p. 163) looks at the expression $NE – OE – CM + MO$ ($CM$ and $MO$ being arcs of the circle) [eq. (5.1)]. This expression takes a minimum, namely zero, if the point $E$ coincides with the point $D$. Then Fermat replaces $MO – OE$ by $MU – UE$ [$U = V$ above]. Now if the point $E$ is not too far away of [sic] the point $D$, then

$$MO - OE \geq -MD$$
as well as

\[ MU - UE \geq -MD \]

and equality holds if and only if the points \( E \) and \( D \) coincide.

Therefore \( NE - UE - CM + MU \) has a minimum, namely zero, if the points \( E \) and \( D \) coincide. \(^{21}\) (Breger 1994, p. 206)

Note that Breger suppresses the term \( e = DE \) by setting it equal to zero (in discussing the expression \( NE - UE - CM + MU \)) at stage 3, namely prior to division by \( e \),\(^{22}\) and with this concludes his argument in favor of a “minimum” interpretation. Such a procedure is however meaningless and certainly cannot be attributed to Fermat. Fermat clearly states:

\[ \text{Divisons par } e; \text{ comme il ne reste ici aucun terme superflu, il n'y a pas d'autre suppression à faire.} \] (Fermat 1896, p. 145)

Fermat thus asserts that there are no terms to be suppressed after division by \( e \) in the example of the cycloid. Breger’s erroneous suppression of \( DE \) prior to division by \( e \) is not accidental, but rather stems from a desire to force a “minimum” interpretation on Fermat. Breger concludes his discussion of the cycloid by pointing out that

\[ \text{It is hard for the modern reader to get rid of the limit ideas in his mind, and so he considers the replacement of a very small arc length of the circle by a very small segment of the tangent to be quite natural. But this is not the way of Fermat.} \] (Breger 1994, p. 206)

Granted, modern “limit ideas” as applied to Fermat are a tell-tale sign of Whiggish history discussed in Section 2. However, setting \( e \) equal to zero prior to division by \( e \) is hardly “the way of Fermat”, either. To conclude, Fermat’s treatment of the cycloid and other transcendental curves cannot be accounted for by means of a formal/algebraic reading, and requires an element of approximation for a convincing interpretation.

5.3. Who Erred: Fermat or Barner?
In a recent article, K. Barner (2011) expresses agreement with Breger, while noting an additional fine point. He claims that, while \textit{aequare} and \textit{adaequare} are semantically equivalent, Fermat uses the term \textit{adaequare}

\(^{21}\) Breger’s inequality involving the segment \( MD \) does not appear in Fermat’s text.
\(^{22}\) See n. 6.
when defining a dependence relation between two variables. Thus, the adequation \( f(x + b) =_{AD} f(x) \) defines the dependence of \( x \) on \( b \). Barner goes on to interpret Fermat’s method in terms of (Dini’s) implicit function theorem. On page 23, Barner writes: “Und genau dies ist die Bedeutung des Wortes adequare: es bezeichnet die Gleichheit zwischen zwei Termen, die keine Identität sondern eine von der Identität verschiedene Relation definiert” [And this is exactly the meaning of the word adequare: it indicates the equality between two terms, an equality which defines not an identity, but rather a relationship distinct from identity]. Barner further claims that there is not really one uniform method, and to assume that Fermat has a clearly formulated method is an anachronism: “Zu unterstellen, da dies für Fermat ebenfalls eine selbstverständliche Routine gewesen sei, das ist ein Anachronismus” [To assume that for Fermat this [method], too, was a self-evident/obvious procedure is an anachronism]. On page 27, Barner goes on to claim that Fermat had to use a bit of ‘trickery’ because his method is not completely correct: “Ich möchte einfach nur verstehen, warum Fermat mit seiner Trickerei Erfolg hat, obwohl der Ansatz, den er dabei macht, die Idee seiner ‘Methode’ nicht ganz korrekt wiedergibt” [I would simply like to understand why Fermat was successful with his little trickery, even though the assumption that he makes does not represent the basic idea of his ‘method’ with full accuracy]. Barner’s approach ultimately leads him astray when dealing with tangents. Barner seeks to replace the tangent line by a secant line (page 34), since his interpretation forces him to assume that the second point near the point of tangency is on the parabola itself (point \( P \) in his diagram on page 34) rather than on the tangent line (point \( O \) in his diagram), so as to get an exact equality: “Aber der Punkt \( P \) (und damit bei Fermat auch \( O \)) liegt damit auf der Kurve und erlaubt es ihm, suivant la propriété spécifique de la ligne courbe, die algebraischen Eigenschaften der vorgelegten Kurve auch für den Punkt \( O \) in Anspruch zu nehmen” [But the point \( P \)—and thus, according to Fermat, the point \( O \) as well!—lies thereby on the curve and allows him, suivant la propriété spécifique de la ligne courbe, to claim the algebraic properties of the curve under consideration for the point \( O \) as well]. How plausible is Barner’s claim? Barner’s claim contradicts Fermat’s explicit statement, which Barner had just quoted:

Après avoir donné le nom, tant à notre parallèle qu’à tous les autres termes de la question, tout le même qu’en la parabole, je considère derechef cette parallèle, comme si le point qu’elle a dans la tangente étoit en effet en la ligne courbe, et suivant la propriété spécifique de la ligne courbe, je compare cette parallèle par adégalité avec l’autre parallèle tirée du point donné à l’axe ou diamètre de la ligne
courbe. Cette comparaison par adégalité produit deux termes inégaux qui enfin produisent l’égalité (selon ma méthode), qui nous donne la solution de la question. (Barner 2011, pp. 35–36)

Thus, Barner has pursued his interpretation to the point of contradicting Fermat’s own comments. Fermat writes explicitly that he is applying the defining property of the curve to points on the tangent line. He says it again in his discussion of the cycloid as we have shown (see Subsection 5.2). On page 36, Barner goes on to say that he can’t understand why Fermat kept making the same mistake: “Warum hat Fermat sein widersprüchliches Vorgehen bei allen seinen Beispielen zur Tangentenmethode immer wieder erneut verwendet? Warum hat er die Sekante, mit der er de facto operiert, nie erwähnt? Ich weiß es nicht” [Why did Fermat continually repeat his inconsistent procedure for all his examples for the method of tangents? Why did he never mention the secant, with which he in fact operated? I do not know]. Barner’s dilemma does not arise in our interpretation, which avoids attributing errors to Fermat.

6. Bachet’s Semantic Calque
The choice of the Latin verb *adaequo* in Bachet’s translation of Diophantus is explained in Bachet’s notes as follows:

> Since in questions of this kind, Diophantus nearly equates the sides of the squares that are being sought, to some side, but he does not properly equate them, he calls this comparison *παρισότης* and not *ίσοτης*. We too call it not equality but adequality, just as we also translate *πάρισον* as adequal.

Here Bachet clearly differentiates the meanings of *aequo* and *adaequo*: the former is “equal,” the latter “nearly equal.” Notice that what Bachet performs here is a semantic calque: the Greek *para* and *iso* are individually translated to *ad* and *aequo*, and recombined to produce *adaequo*. Bachet does not have the Latin meaning of *adaequo* in mind, but rather a new meaning derived from his understanding of the term coined by Diophantus.

23. *proxime* in the Latin.
24. Mathematically speaking, it would apparently have been more correct to write “squares of the sides.”
25. “Quia enim in huiusmodi questionibus Diophantus, cuidam lateri adaequat proxime latera quadratorum quaesitorum, non autem aequat proprie, vocat ille hanc comparitionem *παρισότητα* non autem *ίσοτητα*. Nos etiam non aequalitatem sed adaequalitatem appellamus, sicut etiam *πάρισον* vertimus adaequale.”
26. See a related discussion in Itard 1974, p. 338.
7. Breger On Mysteries of *Adaequare*

In 1994, Breger sought to challenge what he called the “common dogma” to the effect that

*Fermat uses “adaequare” in the sense of “to be approximately equal” or “to be pseudo-equal” or “to be counterfactually equal.”* (1994, p. 194 [emphasis in the original])

After some introductory remarks related to the dating and editing of various manuscripts, he continues

Having made these introductory remarks I want to put forward my hypothesis: Fermat used the word “adaequare” in the sense of “to put equal.” (1994, p. 197)

In this section, we compare Breger’s interpretation and that of the viewpoint of Strømholm and Giusti that there are (at least) two different approaches in Fermat. One is based on the insight from Pappus, and involves a symmetric relation between two equal values near an extremum. The other is based on the insight from Diophantus and the method of *παρισότης*, which exploits an approximation. In the first approach, “adequality” has an operational meaning of “setting equal” as Breger contends. However, in the second approach (see Section 5), as well as in the application of the method to tangents (see Section 4), the element of approximation is essential.

7.1. The Philology of *παρισότης*

Breger presents several arguments in defense of his hypothesis. The first two arguments are based on dictionary definitions, first of the Latin *adaequare* and second of the original Greek term *παρισότης*. Both of these arguments are flawed. Breger questions why Fermat would choose to employ the term *adaequare* in a sense different from that given in standard Latin dictionaries:

There are well established Latin words to indicate an approximate equality, namely “approximare” or, more frequently used in classical Latin as well as in the 17th century, “appropinquare”. It is well known that Fermat’s knowledge of Latin and Greek was excellent, and so if he wanted to tell us that there was an approximate equality, why should he use a word indicating an equality? As far as I know, this question has not been answered nor even discussed by any adherent of the dogma of Fermat interpretation. (Breger 1994, p. 198)
The first point is that there was no pre-existing word meaning “to set two things approximately equal.” Non-mathematical texts do not talk that way; when Polybius says, for example, that Rome and Carthage were πάρισις in their power (Polyb. 1.13.8), nobody takes him to be stating a mathematical equivalence. And the words that Breger suggests, approximate or appropinquare, do not mean what Fermat wants to do: he does not want to “approximate” \( f(a) \), but rather to compare two expressions, \( f(a) \) and \( f(a + e) \). If what he wanted to do was to treat them as being approximately equal, neither the Romans nor the mathematicians of Fermat’s time had a term available for that; and since the Latin prefix ad- commonly translated the Greek prefix par(a)-, the term adaequalitas would be the obvious equivalent for \( \pi\alpha\rho\iota\sigma\sigma\omicron\tau\varsigma \). To give an example of a Greek par transformed into a Latin ad, note that the rhetorical figure that the Greeks called paronomasia (παρονομασία), whereby a speaker makes a pun on two similar but not identical words, was called in Latin adnominatio. But as far as Fermat is concerned, the obvious reason that he used adaequalitas is that that was the term that Bachet used to translate the Greek \( \pi\alpha\rho\iota\sigma\sigma\omicron\tau\varsigma \). Moreover, Fermat himself explicitly refers to the Greek term. The fact that Breger is aware of this problem is evident when he writes:

Therefore Xylander and Bachet were right in using the word “adaequalitas” in their translations of Diophantus [. . .] although that does not imply that they had understood the mathematics of the passage. (Breger 1994, p. 200)

To argue his hypothesis, Breger is led to postulate that Xylander and Bachet misunderstood Diophantus! Breger appears to acknowledge implicitly that Bachet’s intention was to engineer a semantic calque (see Section 6), but argues that Bachet’s calque was a vain exercise, to the extent that adaequo already has the meaning of parisoö, albeit not the meaning Bachet had in mind.

7.2. Greek Dictionary

Breger’s “argument [that] is just based on the Greek dictionary” (1994, p. 199) is misconceived. The Parisian Thesaurus Graecae Linguae of 1831–75 on which Breger bases himself, was a reissue of Henricus Stephanus’ dictionary of the same name (1572). The latter, a work of stupendous scholarship, was published in 1572, three years before Xylander’s original edition of Diophantus. Neither Stephanus’ Thesaurus, nor the cheaper and hence much more widely available pirated abridgement of it by

27. Breger gives the dates 1842–7, understating the magnitude of the task.
Johannes Scapula (1572) (originally printed in 1580 and reprinted innumerable times afterward), nor Stephanus’ 1582 reissue, included the word \( \text{parisostis} \), though both Stephanus and Scapula did have \( \pi\alpha\rho\iota\sigma\omega\sigma\alpha\varsigma \), \( \text{aequalis} \), or \( \text{vel compar} \): “equal or similar” (with the note that the writers on rhetoric used the expression to mean \( \text{prope aequatum} \), “made nearly equal”) and other words formed from \( \pi\alpha\rho\iota\varsigma \) and \( \iota\sigma\omicron\varsigma \).

In the nineteenth-century Paris reissue, the term \( \text{parisostis} \) was added by Karl Wilhelm Dinsdorf, one of the editors, who cited Diophantus and translated “\( \text{Aequalitas} \).” The Paris Stephanus, however, is not today “the best Greek dictionary,” contrary to Breger’s claim. The most (and in fact, for the time being, the only) authoritative dictionary today is Liddell-Scott-Jones, which defines \( \text{parisostis} \) as “approximation to a limit, Dioph. 5.17.” But in fact all Greek and Latin dictionaries (with the exception of the original Stephanus, which was chiefly the result of Stephanus’ own scholarship and that of Guillaume Budé) are secondary sources, recording the meanings that others have given to the words; so both Dindorf and Liddell-Scott-Jones were simply recording the translations then current for Diophantus’ use of \( \text{parisostis} \). Both “equality” and “near equality” are possible meanings for \( \text{parisostis} \); and it is up to the editors and critics of Diophantus to tell the lexicographers which meaning he intended, not the other way around.

Fermat is very unlikely to have used either the hugely expensive and rare Stephanus or the easily available Scapula, when Bachet had printed Diophantus with a facing Latin translation; and his care in saying that he is using the term \( \text{adaequentur ut loquitur Diophantus} \), “as Diophantus says,” seems to make it clear that he is specifically not using the Latin term in its usual meaning of “be set equal,” but, rather, in a meaning peculiar to Diophantus. That meaning, as we argue in Section 8, can only be “be set approximately equal,” as it was correctly understood by Bachet.

8. The Mathematics of \( \text{parisostis} \)

In this section we will analyze the problems in which Diophantus introduces the term \( \text{parisostis} \).

8.1. The \( \text{parisostis} \) of Diophantus.

The relevant problems are in Book Five, problems 12, 14, and 17 in Bachet’s Latin translation. Problem 14 uses the term \( \pi\alpha\rho\iota\sigma\omega\sigma\alpha\varsigma \) (see 1621).

The term \( \pi\alpha\rho\iota\sigma\omega\sigma\alpha\varsigma \) occurs on page 310 of Bachet’s Latin translation.

28. The numbering is the same in Wertheim’s German translation. However, in Ver Eecke’s French translation, the corresponding problems are 9, 11, and 14, which is the numbering referred to by Weil.
Diophanti Alexandrini, Arithmeticorum Liber V. Here the left column is the Greek original, while the right column is the Latin translation. The term \( \pi \acute{\rho} \acute{\iota} \acute{\rho} \acute{i} \acute{o} \acute{s} \) is the first word on line 8 in the left column. In the right column, line 13, we find its Latin translation \( \textit{adaequalem} \).

8.2. Lines 5–8 in Diophantus

In more detail, lines 5–8 in the Greek original contain the following phrase containing \( \pi \acute{\rho} \acute{\iota} \acute{\rho} \acute{i} \acute{o} \acute{s} \):

\[
\text{Latin translation: } 29
\]

\[
\text{Oportet igitur dividere 10 in tres quadratos, ut uniuscuiusque quadrati latus sit adaequale unitatibus } \frac{11}{6}.
\]

\[
\text{English translation:}
\]

So we have to divide ten into three squares, so that the side of each square is adequal to \( \frac{11}{6} \) units.

8.3. Lines 14–16 in Diophantus

Lines 14–16 in the Greek original contain the following phrase containing \( \pi \acute{\rho} \acute{\iota} \acute{\rho} \acute{i} \acute{o} \acute{s} \):

\[
\text{Latin translation: } 29
\]

\[
\text{Oportet igitur horum cuiusvis lateri adaequalem facere } \frac{11}{6}.
\]

\[
\text{English translation:}
\]

So the side of each of these we have to make adequal to \( \frac{11}{6} \).

8.4. Lines 20–21 in Diophantus

Lines 20–21 in the original Greek:

\[
\text{Latin:}
\]

\[
\text{Oportet itaque unumquodque latus adaequare ipsi } 55.
\]

29. The Latin is given according to modern printing conventions.
8.5. Line 25 in Diophantus.
Line 25: Greek:
ταῦτα ἵσα μονάσι ἰ.
Latin:
haec aequantur 10.
English:
These equal 10 units.

8.6. πάρισος As Approximate Equality
Diophantus wishes to represent the number 10 as a sum of three squares. The solution he eventually finds is denoted \((\alpha, \beta, \gamma)\) by Ver Eecke (1926, p. 205). In this notation, Diophantus seeks \(\alpha, \beta, \gamma\) each of which is as close as possible (πάρισος) to the fraction \(\frac{11}{6}\). The solution eventually found is 
\[
\alpha = \frac{1321}{711}, \quad \beta = \frac{1288}{711}, \quad \gamma = \frac{1285}{711}.
\]

Thus, Diophantus specifically uses the term πάρισος to describe the way in which the numbers \(\frac{1321}{711}, \frac{1288}{711}\), and \(\frac{1285}{711}\) approximate the fraction \(\frac{11}{6}\). Interpreting Diophantus’ παρισότης as anything other than “approximate equality” is therefore purely whimsical. In the next subsection we describe the method of Diophantus in more detail.

8.7. The Method of Diophantus
The following description of the method which Diophantus called παρισότης is based on the notes of Ver Eecke (1926), Wertheim (1890), and the paper of Bachmakova (1966). Diophantus does not have a theory of equations, but gives an algorithm for solving a class of problems by solving a particular example. We will explain his method in terms of equations for the reader’s convenience. In all three examples involving παρισότης, the problem is to express a certain number as the sum of two or three rational squares, with a further inequality constraining the individual values.

In problem 12, one seeks two rational squares whose sum is 13 and each one is greater than 6. In problem 14, one seeks three rational squares whose sum is 10 and such that each one is greater than 3. In problem 17,
one seeks two rational squares whose sum is 17 and such that each one is less than 10.

In all three cases, the first step is to find a rational square approximately equal to either one half or one third of the desired sum, depending on whether two or three numbers are sought. In addition, Diophantus uses the existence of a preliminary partial solution involving two or three rational squares giving the desired sum, \( N \), but failing to satisfy the required inequalities. Call these numbers \( a_i \) and denote the rational square close to one half or one third of the desired sum by \( b \). He then expresses \( b \) as a sum \( b = a_i + c_i \) and considers an equation of the form

\[
\sum (a_i + t c_i)^2 = N = \sum a_i^2
\]

Canceling equal terms on the two sides and dividing by \( t \) gives a linear equation for \( t \) with a rational solution \( t_0 \). Since \( t = 1 \) is nearly a solution, the exact solution \( t_0 \) is close to 1 and defines the numbers \( a_i + t_0 c_i \) satisfying the required inequality. As mentioned in Subsection 8.1, Diophantus’ \( \nu \) refers to the approximate equality of each of the \( a_i + t_0 c_i \) to the original fraction \( b \). Ver Eecke explains the matter as follows in a footnote:

\( \nu \) c'est-à-dire "la voie de la quasi-égalité" ou d'approximation vers une limite; méthode dont l'objet est de résoudre des problèmes tels que celui de trouver 2 ou 3 nombres carrés, dont la somme est un nombre donné, et qui sont respectivement au plus rapprochés d'un même nombre (Ver Eecke 1926, p. 203, n. 2).

The method, as explained here, is remarkably similar to Fermat’s. It starts with a near equality and defines a quadratic equation which is solved as an exact equality. The method of \( \nu \) might be called “the method of nearby values” (see Subsection 1.2). G. Lachaud (1988) similarly speaks of \( \nu \) as involving approximation.

8.8. Breger on Diophantus
Breger acknowledges that

At the first step—and this is the relevant step characterizing the method of \( \nu \)—he finds a positive rational \( z \) with the property \( 2z^2 = 2a + 1 \). (1994, p. 200)
After discussing the Diophantine problems, Breger claims that

the approximate equality which in fact occurs in Diophantus’s
Arithmetic is only due to the fact that Diophantus seeks rational
solutions. (1994, p. 202)

Breger’s claim that Diophantus is seeking an approximate rational solu-
tion is in error. In fact, Diophantus starts from an approximate rational so-
lution, to derive an equation for an exact rational solution. The issue of
approximation is not related to the distinction rational/irrational since
irrational solutions were outside the scope of Diophantus’s conceptual
framework. Breger further remarks that

[i]t is strongly misleading to mix Fermat’s notation with our own
and to describe his method in these cases by something like “f(A)
adaequatur f(A − E)”, as is often done. (1994, p. 204)

How is one to interpret Breger’s comment? The fact is that setting the for-
ermer expression adequal to the latter is precisely what Fermat does in the
very first example, using $a + e$ rather than $a − e$. In the French translation
which uses modern notation (in place of Fermat’s original notation à la
Viète for the mathematical expressions), the phrase appears in the follow-
ing form:

Soit maintenant $a + e$ le premier segment de $b$, le second sera $b − a$
$− e$, et le produit des segments: $ba − a^2 + be − 2ae − e^2$; Il doit
être adégalé au précédent: $ba − a^2$. (Fermat 1896, p. 122)

In the original Latin version, the last two lines above read as follows:

$B$ in $A − Aq.$ $+ B$ in $E − A$ in $E$ bis $− Eq.$,
quod debet adaequari superiori rectangulo
$B$ in $A − Aq.$
(see Fermat 1891, p. 134)

Breger continues, “The method consists in finding a polynomial in $E$
which takes a minimum if $E = 0$” (1994, p. 204). However, Fermat does
not merely apply his method to polynomials. He ultimately applied the
method in far greater generality:

At first Fermat applies the method only to polynomials, in
which case it is of course purely algebraic; later he extends it to
increasingly general problems, including the cycloid. (Weil 1973,
p. 1146)

In Section 5 we argued that transcendental curves such as the cycloid nec-
essarily require an element of approximation.
8.9. The Diophantus–Fermat Connection

The mathematical areas Diophantus and Fermat were working in were completely different. It was arithmetic and number theory in the case of Diophantus, and geometry and calculus in the case of Fermat. Such a situation creates a fundamental problem: if one rejects the “approximation” thread connecting Diophantus to Fermat, why exactly did Fermat bring Diophantus and his terminology into the picture when working on problems of maxima and minima?

Breger’s solution to the problem is to declare that Diophantus was talking about minima and Fermat was also talking about minima. What kind of minima was Diophantus talking about? Breger’s answer is that Diophantus’ minimum is $0$. Namely, $|x^2 - y^2|$ is always bigger than zero, but gets arbitrarily close to it:

The minimum evidently is achieved by putting $x$ equal to $y$, and that is why the method received its name “method of $\pi\alpha\rhoισ\sigmaοτης$ or putting equal”. As there is a minimum idea in the Diophantus passage, Fermat’s reference to Diophantus becomes intelligible. (Breger 1994, p. 201)

Does Fermat’s reference to Diophantus become intelligible by means of the observation that $|x^2 - y^2|$ gets arbitrarily close to 0? According to Breger’s hypothesis, Diophantus was apparently led to introduce a new term, $\pi\alpha\rhoισ\sigmaοτης$, to convey the fact that every positive number is greater than zero. To elaborate on Breger’s hypothesis, zero is the infimum of all positive numbers, which is presumably close enough to the idea of a minimum. All this is supposed to explain the connection to Fermat’s method of minima. There are at least two problems with such a reading of Diophantus. First, did Diophantus have the number zero? Second, the condition that Diophantus imposes is merely the bound

$$|x^2 - y^2| < 1$$

rather than any stronger condition requiring the expression $|x^2 - y^2|$ to be arbitrarily close to zero. Thus, Diophantus was not concerned with the infimum of $|x^2 - y^2|$. If one drops the approximation issue following Breger, the entire Diophantus–Fermat connection collapses.

9. Refraction, Adequality, and Snell’s Law

In addition to purely mathematical applications, Fermat applied his adequality in the context of the study of refraction of light, so as to obtain Snell’s law. Thus, in his Analyse pour les réfractions, Fermat sets up the formulas for the length of two segments
representing the two parts of the trajectory of light across a boundary be-
tween two regions of different density, and then writes:

La somme de ces deux radicaux doit être adégalée, d’après les règles
de l’art, a la somme \( mn + bn \). (Fermat 1896, p. 150 [emphasis
added])

Fermat explains the physical underpinnings of this application of ade-
equality in his Synthèse pour les réfractions, where he writes that light travels
slower in a denser medium (Fermat [18, p. 151]). Fermat states the physi-
cal principle underpinning his mathematical technique in the following
terms,

Notre démonstration s’appuie sur ce seul postulat que la nature
opère par les moyens et les voies les plus faciles et les plus aises.
(1896, p. 152)

and goes on to emphasize that this is contrary to the traditional assump-
tion that “la nature opère toujours par les lignes les plus courtes.” Rather,
the path chosen is the one traversed “dans le temps le plus court” (p. 152).
This is Fermat’s principle of least time in optics. Its implicit use by
Fermat in his Analyse pour les réfractions in conjunction with adequality, is
significant. Namely, this physical application of adequality goes against
the grain of the formal/algebraic approach. The latter focuses on the
higher multiplicity of the root of the polynomial
\[
\frac{f(a + e) - f(a)}{e},
\]
where the extremum can be determined by an algebraic pro-
cedure without ever assigning any specific value to \( e \), and obviating
the need to speak of the nature of \( e \). In this method, denoted M2 by Strøm-
holm, the symbol \( e \) could be a formal variable, neither small or large, in
fact without any relation to a specific number system. However, when
we apply a mathematical method in physics, as in the case of the refrac-
tion principle provided by Snell’s law, mathematical idealisations of physi-
cal magnitudes are necessarily numbers. The principle is that the light
chooses a trajectory \( \tau_0 \) which minimizes travel time from point A to point
B. To study the principle mathematically is to commit oneself to compar-
ing such a trajectory to other trajectories \( \tau \) in a family parametrized by a
numerical parameter \( s \). Here we need not assume an identity of the line in
physical space with a number line (the hypothesis of such an identification
is called Cantor’s axiom in the literature); rather, we merely point out that
a number line is invariably what is used in mathematical idealizations of
physical processes. In studying such a physical phenomenon, even before
discussing the size of $e$ (small, infinitesimal, or otherwise), one necessarily commits oneself to a number system rather than treating $e$ as a formal variable. Fermat’s application of adequality to derive Snell’s law provides evidence against a strict formal/algebraic interpretation of adequality.

Applications to physical problems necessarily involve a mathematical implementation based on numbers. Classical physics was done with numbers, not algebraic manipulations. One can’t model phenomena in classical physics by means of formal variables.

Certainly when a physicist performs mathematical operations, he does exactly the same thing as a mathematician does. However, modeling physical phenomena by using mathematical idealization is a stage that precedes mathematical manipulation itself. In such modeling, physical phenomena get numerical counterparts, and therefore necessarily refer to a number system rather than formal variables. Pedersen & Pedersen interpret Fermat’s deduction of the sine law of refraction

as an early example of the calculus of variations rather than as an ordinary application of Fermat’s method of maxima and minima.
(Pedersen 1971)

and speculate that Fermat thus anticipated Jacob Bernoulli who is generally credited with inventing the method of the calculus of variations in 1696. Arguably, at least some of the manifestations of the method of adequality amount to variational techniques exploiting a small or infinitesimal variation $e$.

10. Conclusion
Should Fermat’s $e$ be interpreted as a formal variable, or should it be interpreted as a member of an Archimedean continuum “tending” to zero? Or perhaps should adequality be interpreted in terms of a Bernoullian continuum\textsuperscript{30}, with $e$ infinitesimal? Note that the term “infinitesimal” was not introduced until around 1670\textsuperscript{31}, so Fermat could not have used it. Yet

\textsuperscript{30} G. Schubring attributes the first systematic use of infinitesimals as a foundational concept, to Johann Bernoulli, see 2005, p. 170, 173, 187. To note the fact of such systematic use by Bernoulli is not to say that Bernoulli’s foundation is adequate, or that that it could distinguish between manipulations with infinitesimals that produce only true results and those manipulations that can yield false results. One such infinitesimal distinction was provided by Cauchy (1900) in 1853, thereby resolving an ambiguity inherent in his 1821 “sum theorem” (see Katz & Katz 2011 for details).

\textsuperscript{31} Some contemporary scholars hold that Leibniz coined the term \textit{infinitesimal} in 1673 (see Probst 2008 and 2010). Meanwhile, Leibniz himself attributed the term to Nicolaus Mercator (see Leibniz 1859).
infinitely small quantities were in routine use at the time, by scholars like John Wallis who was in close contact with Fermat.

While discussions of “process” are rare in Fermat when he deals with his ε, we mentioned an instance of such use in Subsection 1.4.\(^\text{32}\) Writes Strømholm:

> It will not do here to drag forth the time-honoured “limiting process” of historians of mathematics [. . .] Fermat was still thinking in terms of equations; I agree that he stood on the verge of a period where mathematicians came to accept that sort of process, but he himself was in this particular case rather the last of the ancients than the first of the moderns. (Strømholm 1968, p. 67)

In the absence of infinitesimals, there is no possibility of interpreting smallness other than by means of a process of tending to zero. But, as Strømholm confirms, such a discussion is uncharacteristic of Fermat, even at the application stage, when he applies his method in concrete instances. Therefore the infinitesimal interpretation (3) is more plausible than the kinetic interpretation (2) (see Section 1).

To return to the question posed in Section 1, as to which of the three approaches is closest to Fermat’s thinking, it could be that the answer to the riddle is . . . it depends. When Fermat presents his definitional characterisation of adequality, as on the first page of his *Méthode pour la recherche du maximum et du minimum*, his algorithmic presentation has a strong formal/algebraic flavor. However, at the application stage, both in geometry and physics, ideas of approximation or smallness become indispensable.

Breger (1994, pp. 205–206) claims that adequality cannot be interpreted as approximate equality. Breger’s argument is based on his contention that the Latin term *adaequare* was not used in the sense of approximate equality by Fermat’s contemporaries. However, the source of adequality is in the Greek *παρισοτέτης* (*parisoten*), rather than the Latin *adaequare*, a fact that undermines Breger’s argument. The question that should be asked is not whether Fermat’s contemporaries used the term *adaequare*, but rather whether they used the infinitely small. The latter were certainly in routine use at the time, by some of the greatest of Fermat’s contemporaries such as Kepler and Wallis.

In addition to the 3-way division: formal, kinetic, and infinitesimal, there is a distinction between (A) Fermat’s definition, i.e., synopsis of the method as it appears in Fermat (1896, p. 121); and (B) what he actually does when he applies his method.

Fermat’s definition (A) does have the air of a kind of a formal algebraic

\(^{32}\) See quotation following n. 9.
manipulation. A formal interpretation of adequality is certainly *mathematically* coherent, regardless of what Fermat meant by it, since one can define differentiation even over a *finite field*. The fact itself of being able to give a purely algebraic account of this mathematical technique is not surprising. What is dubious is the claim that at the application stage (B), he is similarly applying an algebraic procedure, rather than thinking of $e$ geometrically as vanishing, tending to zero, infinitesimal, etc.

In light of the positivity of Fermat’s $e$ in the calculation of the tangent line, the formal story would have difficulty accounting for the passage from inequality to adequality, since the inequality is satisfied for transverse rays as well as the tangent ray. To make sense of what is going on at stage (B), we have to appeal to geometry, to negligible, vanishing, or infinitesimal quantities, or their rate or order.

Breger’s insistence on the formal interpretation (1), when applied to the application stage (B), is therefore not convincing. Fermat may have presented a polished-up algebraic presentation of his method at stage (A) that not even Descartes can find holes in, but he gave it away at stage (B).

Kleiner and Movshovitz-Hadar note that

Fermat’s method was severely criticized by some of his contemporaries. They objected to his introduction and subsequent suppression of the mysterious $e$. Dividing by $e$ meant regarding it as not zero. Discarding $e$ implied treating it as zero. This is inadmissible, they rightly claimed. In a somewhat different context, but with equal justification, Bishop Berkeley in the 18th century would refer to such $e$’s as “the ghosts of departed quantities.” (1994, p. 970 [emphasis added])

Kleiner and Movshovitz-Hadar feel that Fermat’s suppression of $e$ implies treating $e$ as zero, and that the criticisms by his contemporaries and by Berkeley were justified. However, P. Strømholm already pointed out in 1968 that in Fermat’s main method of adequality (M1):

there was never [. . .] any question of the variation $E$ being put equal to zero. The words Fermat used to express the process of suppressing terms containing $E$ was “elido,” “deleo,” and “expunge,” and in French “i’efface” and “i’ôte.” We can hardly believe that a sane man wishing to express his meaning and searching for words, would constantly hit upon such tortuous ways of imparting the simple fact that the terms vanished because $E$ was zero. (Strømholm 1968, p. 51)

Fermat did not have the notion of the derivative. Yet, by insisting that $e$ is being discarded rather than set equal to zero, he planted the seeds of the
solution of the paradox of the infinitesimal quotient and its disappearing \(dx\), a century before George Berkeley ever lifted up his pen to write *The Analyst*.

After summarizing Nieuwentijt’s position on infinitesimals, Leibniz wrote in 1695:

It follows that since in the equations for investigating tangents, maxima and minima (which the esteemed author [i.e., Nieuwentijt] attributes to Barrow, although if I am not mistaken Fermat used them first) there remain infinitely small quantities, their squares or rectangles are eliminated. (Leibniz [1695] (1858), p. 321)

Leibniz held that methods of investigating tangents, minima, and maxima involve infinitesimals. Furthermore, he disagreed with Nieuwentijt as to the priority of developing these methods, specifically attributing them to Fermat. Thus, Leibniz appears to have felt that Fermat’s methods of investigating tangents, minima and maxima did rely on infinitesimals. In the absence of explicit commentary by Fermat concerning the nature of \(E\), Leibniz’s view may be the best 17th century expert view on the matter.

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33. The heuristic procedure of discarding the remaining terms was codified by Leibniz by means of his Transcendental Law of Homogeneity (see Section 2.3). Centuries later, it was implemented mathematically in terms of the standard part function, which associates to each finite hyperreal number, the unique real number infinitely close to it. In 1961, Robinson (1979) constructed an infinitesimal-enriched continuum, suitable for use in calculus, analysis, and elsewhere, based on earlier work by E. Hewitt (1948), J. Łoś (1995), and others. Applications of infinitesimal-enriched continua range from aid in teaching calculus (1990; 2010) to the Boltzmann equation (see L. Arkeryd 1981; 2005) and mathematical physics (see Albeverio et al. 1986). Edward Nelson in 1977 proposed an alternative to ZFC which is a richer (more stratified) axiomatisation for set theory, called Internal Set Theory (IST), more congenial to infinitesimals than ZFC. The hyperreals can be constructed out of integers (see Borovik et al. 2012).
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