New exact solutions of 2DKK and 2DSK equations

V.G. Dubrovsky, A.V. Topovsky, and M.Yu. Basalaev
Novosibirsk State Technical University, Karl Marx prosp. 20, Novosibirsk 630092, Russia.

(Dated: 16 February 2022)

The 2+1-dimensional integrable generalization of Kaup-Kuperschmidt and Sawada-Kotera equations are studied by \( \bar{\partial} \)-dressing method of Zakharov and Manakov. The solutions with functional parameters and periodic solutions are constructed.

PACS numbers: 02.30.Ik, 02.30.Jr, 02.30.Zz, 05.45.Yv

I. INTRODUCTION

In the last three decades the Inverse Spectral Transform (IST) method has been generalized and successfully applied to various 2+1-dimensional nonlinear evolution equations such as Kadomtsev-Petviashvili, Davey-Stewardson, Veselov-Novikov, Zakharov-Manakov system, Ishimory, two dimensional integrable sine-Gordon and others, see the books 1–4 and references there. The nonlocal Riemann-Hilbert 5, \( \partial \)-problem 6 and more general \( \bar{\partial} \)-dressing method of Zakharov and Manakov 7–9 are now basic tools for solving 2+1-dimensional integrable nonlinear equations, see also the books 1–4 and reviews 10–12.

In the present paper the \( \bar{\partial} \)-dressing method is applied for the construction of new classes of solutions with functional parameters and as their particular cases periodic solutions of 2+1-dimensional integrable generalizations of Kaup-Kupershmidt (2DKK),

\[
\frac{\partial}{\partial t} + u_{xxxxx} + \frac{25}{2} u_x u_{xx} + 5u u_{xxx} + 5u^2 u_x + 5u u_{xxy} - 5 u^{-1} u_y u + 5u_y u + 5u_x u^{-1} u = 0;
\]

and Sawada-Kotera (2DSK),

\[
\frac{\partial}{\partial t} + u_{xxxxx} + 5u_x u_{xx} + 5u u_{xxx} + 5u^2 u_x + 5u u_{xxy} - 5 u^{-1} u_y u + 5u_y u + 5u_x u^{-1} u = 0,
\]
equations. These equations have been discovered in paper 13. Now it is well known that the Sawada-Kotera equation belongs to the BKP hierarchy, and the Kaup-Kupershmidt equation to the CKP hierarchy 14. These equations can be represented as the compatibility conditions in the Lax form \([L_1, L_2] = 0\); for the 2DKK equation of the following two linear auxiliary problems 13,

\[
L_1 \psi = (\partial_x^3 + u \partial_x + \frac{1}{2} u_x + \partial_y) \psi = 0,
\]

\[
L_2 \psi = [\partial_t - 9 \partial_x^5 - 15u \partial_x^3 - \frac{45}{2} u_x \partial_x^2 - (\frac{35}{2} u_{xxx} + 5u^2 - 5 \partial_x^{-1} u_y) \partial_x - (5uu_y - \frac{5}{2} u_y + 5u_x)] \psi = 0;
\]

and for 2DSK equation of another two linear auxiliary problems 13,

\[
L_1 \psi = (\partial_x^3 + u \partial_x + \partial_y) \psi = 0,
\]

\[
L_2 \psi = [\partial_t - 9 \partial_x^5 - 15u \partial_x^3 - 15u_x \partial_x^2 - (10u_{xxx} + 5u^2 - 5 \partial_x^{-1} u_y) \partial_x] \psi = 0.
\]

and bellow \( \partial_x \equiv \partial / \partial x, \ldots \) and \( \partial_x^{-1} \) is an operator inverse to \( \partial_x \).

The first linear auxiliary differential problems in (3) and (5) are of the third order on \( \partial_x \), such problems in general position have several fields as the coefficients at the various degrees of \( \partial_x \). The 2DKK (1) and 2DSK (2) equations arise as special reductions of some integrable nonlinear systems for these fields. It is well known that the study of

\[ \text{a) E-mail: dubrovsky@nngs.ru} \]
special reductions requires more attention and may be more difficult than the consideration of nonlinear equations integrable by auxiliary linear problems in general position.

The scheme for construction of solutions with functional parameters for 2+1-dimensional integrable equations by the example of KP equation was developed earlier in the famous papers\textsuperscript{15,16} of Zakharov and Shabat in the framework of their variant of dressing method, see also in this connection the books.\textsuperscript{17,18}

The present paper is natural continuation of the paper \textsuperscript{17} of the first author. In the paper\textsuperscript{17} \(\bar{\varphi}\)-dressing method of Zakharov and Manakov was at first used for construction of multiline soliton solutions of 2DKK and 2DSK equations; some line solitons of considered equations were constructed earlier by another means, see for example the paper\textsuperscript{18}. The application of \(\bar{\varphi}\)-dressing method in nonstandard situations, in our case some nonlinear constraints on the wave functions of the linear auxiliary problems must be satisfied as special reductions, may be very useful and instructive. Let us underline that all of our constructions of exact solutions of considered equations are based exclusively on \(\bar{\varphi}\)-dressing method and not depend on the relations of 2DKK, 2DSK equations with CKP and BKP hierarchies correspondingly.

Our paper is organized as follows. In the section II the basic ingredients of \(\bar{\varphi}\)-dressing method are shortly reviewed and some useful formulas derived in the paper\textsuperscript{17} for 2DKK and 2DSK equations \textsuperscript{11} and \textsuperscript{12} are presented: reconstruction formulas, nonlinear constraints on the wave functions, the conditions of reality of solutions and so on. The new classes of exact solutions with functional parameters for the 2DKK and 2DSK equations are constructed correspondingly in sections III and IV. In section V as a particular cases of solutions with functional parameters the periodic solutions of 2DKK and 2DSK equations are calculated. Section VI contains some conclusions and acknowledgements.

II. BASIC FORMULAS OF \(\bar{\varphi}\)-DRESSING METHOD FOR 2DKK AND 2DSK EQUATIONS.

In this section for convenience we are going to remind some useful general formulas of \(\bar{\varphi}\)-dressing method for 2DKK and 2DSK equations \textsuperscript{11} and \textsuperscript{12}, see the paper\textsuperscript{17} for more details.

At first one postulates non-local \(\bar{\varphi}\)-problem\textsuperscript{12,17,18},

\[
\frac{\partial \chi}{\partial \lambda} = (\chi \ast R)(\lambda, \lambda) = \int \int_{C} d\mu \wedge d\overline{\mu} \chi(\mu; \overline{\mu}) R(\mu, \overline{\mu}; \lambda, \lambda),
\]

\begin{equation}
(7)
\end{equation}

here \(\chi\) and \(R\) in considered case are scalar complex valued functions. For wave function \(\chi\) we choose canonical normalization: \(\chi \rightarrow 1\) as \(\lambda \rightarrow \infty\). We assume also that problem \textsuperscript{(7)} is uniquely solvable. The solution of \(\bar{\varphi}\)-problem \textsuperscript{(7)} with constant normalization is equivalent to solution of the following singular integral equation:

\[
\chi(\lambda) = 1 + \int \int \frac{d\lambda' \wedge d\lambda}{2\pi i (\lambda' - \lambda)} \int \frac{d\lambda}{\lambda} \chi(\mu, \overline{\mu}; \lambda') R(\mu, \overline{\mu}; \lambda', \lambda) R(\mu, \overline{\mu}; \lambda, \lambda),
\]

\begin{equation}
(8)
\end{equation}

Then one introduces the dependence of a kernel \(R\) of \(\bar{\varphi}\)-problem \textsuperscript{(7)} on space and time variables \(x, y, t\),

\[
R(\mu, \overline{\mu}; \lambda, x, y, t) = R_{0}(\mu, \overline{\mu}; \lambda, x, y, t) e^{-F(\mu; x, y, t) - F(\lambda, x, y, t)}; \quad F(\lambda; x, y, t) := i(\lambda x + \lambda^{3} y + 9\lambda^{5} t).
\]

\begin{equation}
(9)
\end{equation}

At the next stage of \(\bar{\varphi}\)-dressing method\textsuperscript{2,8} one construct auxiliary linear problems for 2DSK and 2DKK equations, which in general form are given by expressions\textsuperscript{12,17,18}.

\[
L_{1}\psi = (\partial_{y} + \partial_{x}^{2} + u\partial_{x} + v)\psi = 0,
\]

\[
L_{2}\psi = (\partial_{t} - 9\partial_{x}^{2} + w_{3}\partial_{x}^{3} + w_{2}\partial_{x}^{2} + w_{1}\partial_{x} + w_{0})\psi = 0.
\]

\begin{equation}
(10)
\end{equation}

The wave function \(\psi\) in \textsuperscript{11} is connected with wave function \(\chi\) by the relation \(\psi := \chi e^{F(\lambda; x, y, t)}\).

Reconstruction formulas for the potentials of problems \textsuperscript{11} express these potentials through some coefficients of series expansions of wave function \(\chi\) in terms of powers of spectral variable \(\lambda\) near the points \(\lambda = 0\) and \(\lambda = \infty\),

\[
\chi = \chi_{0} + \chi_{1}\lambda + \chi_{2}\lambda^{2} + \ldots, \quad \chi = \chi_{\infty} + \frac{\chi_{-1}}{\lambda} + \frac{\chi_{-2}}{\lambda^{2}} + \ldots;
\]

\begin{equation}
(11)
\end{equation}

these formulas for the potentials of the first linear problem \textsuperscript{11} have the form\textsuperscript{17}.

\[
v = -3i\chi_{-1}x_{1} + 3\chi_{-2}x_{2} - 3\chi_{-1}x_{1}x_{2}, \quad u = -3i\chi_{-1}x_{2}.
\]

\begin{equation}
(12)
\end{equation}

The coefficients \(\chi_{-1}\) and \(\chi_{-2}\) due to \textsuperscript{8} are given by expressions,

\[
\chi_{-1} = -\int \int \frac{d\lambda \wedge d\overline{\lambda}}{2\pi i} \int \frac{d\lambda}{\lambda} \chi(\mu, \overline{\mu}) R_{0}(\mu, \overline{\mu}; \lambda, \lambda) e^{-F(\mu) - F(\lambda)} d\mu \wedge d\overline{\mu},
\]

\begin{equation}
(13)
\end{equation}
New exact solutions of 2DKK and 2DSK equations

\[ \chi_{-2} = - \int \int \frac{\lambda d\lambda \wedge d\bar{\lambda}}{2\pi i} \int \int \chi(\mu, \bar{\mu}) R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) e^{F(\mu)-F(\bar{\lambda})} d\mu \wedge d\bar{\mu}, \]  

(14)

\[
\text{here and below, for abbreviation, short notations } F(\lambda) \text{ for } F(\lambda; x, y, t) \text{ will be used.}
\]

It was shown in the paper\textsuperscript{11} that to 2+1-dimensional integrable generalizations of nonlinear Kaup-Kuperschmidt \textsuperscript{11} and Sawada-Kotera \textsuperscript{2} equations correspond the reductions,

\[
(2DKK) : \quad v = \frac{1}{2} u_x; \quad (2DSK) : \quad v = 0.
\]

(15)

In terms of the wave function \( \chi \) the reductions \textsuperscript{15} can be expressed as following nonlinear constraints on the coefficients \( \chi_{-1} \) and \( \chi_{-2}\).

\[
(2DKK) : \quad \chi_{-2x} - i \frac{1}{2} \chi_{-1xx} - \chi_{-1x} = 0;
\]

(16)

\[
(2DSK) : \quad \chi_{-2x} - i \chi_{-1xx} - \chi_{-1x} = 0.
\]

(17)

Reconstruction formulas for the potentials of the second auxiliary problem \textsuperscript{10} due to \textsuperscript{12} and reductions \textsuperscript{15} have the forms\textsuperscript{17},

\[
(2DKK) : \quad w_1 = -\frac{35}{2} u_{xx} + 5 \partial_x^{-1} u_y - 5 u_x^2,
\]

\[
w_2 = -\frac{45}{2} u_x, \quad w_3 = -15 u, \quad w_0 = \frac{5}{2} u_y - 5 u_{xxx} - 5 u u_x,
\]

(18)

in the case of 2DKK equation \textsuperscript{11} and,

\[
(2DSK) : \quad w_1 = -10 u_{xx} + 5 \partial_x^{-1} u_y - 5 u_x^2,
\]

\[
w_2 = -15 u_x, \quad w_3 = -15 u, \quad w_0 = 0,
\]

(19)

One can easily obtain from \textsuperscript{12} and \textsuperscript{13} the restrictions on the kernel \( R_0 \) of \( \bar{\partial} \)-problem \textsuperscript{7} following from reality of \( u \); one derives in the limit of weak fields\textsuperscript{17},

\[
R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \overline{R_0(-\mu, -\bar{\mu}; -\lambda, -\bar{\lambda})}; \quad R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = R_0(\lambda, \bar{\lambda}; \mu, \bar{\mu}).
\]

(20)

The conditions of reductions \textsuperscript{15} or \textsuperscript{16}, \textsuperscript{17} and reality \textsuperscript{20} for \( u \) lead to some restrictions on the kernel \( R_0 \) of \( \bar{\partial} \)-problem \textsuperscript{7} in the cases of the 2DSK and 2DKK equations. It is evident that conditions \textsuperscript{20} are the same for both 2DKK and 2DSK equations \textsuperscript{11} and \textsuperscript{2} but nonlinear constraints \textsuperscript{16} and \textsuperscript{17} for these equations have different forms. So in order to calculate exact solutions of 2DKK \textsuperscript{11} and 2DSK \textsuperscript{2} equations via \( \bar{\partial} \)-dressing method one must satisfy to conditions of nonlinear constraints \textsuperscript{16}, \textsuperscript{17} and reality \textsuperscript{20}; this is main and difficult part of all constructions.

For degenerate kernel \( R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) \) of \( \bar{\partial} \)-problem \textsuperscript{7},

\[
R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \pi \sum_{k=1}^{N} f_k(\mu, \bar{\mu}) g_k(\lambda, \bar{\lambda}),
\]

(21)

one can easily derive general determinant formula for the class of exact solutions \( u(x, y, t) \) with functional parameters of 2DKK and 2DSK equations \textsuperscript{11}, \textsuperscript{2}. Indeed, from \textsuperscript{13}, \textsuperscript{14} and \textsuperscript{21} follow compact formulas for the coefficients \( \chi_{-1}, \chi_{-2} \) of the expansion \textsuperscript{11} of function \( \chi \),

\[
\chi_{-1} = -\frac{1}{2i} \sum_{l,k=1}^{N} A^{-1}_{lk} a_l \beta_k, \quad \chi_{-2} = -\frac{1}{2} \sum_{l,k=1}^{N} A^{-1}_{lk} a_l \beta_k x,
\]

(22)

where the matrix \( A \) has the form:

\[
A_{lk} = \delta_{lk} + \frac{1}{2} \partial_x^{-1} a_l \beta_k.
\]

(23)
The functions $\alpha_k(x, y, t), \beta_k(x, y, t)$ in (22) and (23), which given by formulas,

$$
\alpha_l(x, y, t) := \int_C f_l(\mu, \bar{\mu}) e^{F(\mu)} d\mu \wedge d\bar{\mu}, \quad \beta_l(x, y, t) := \int_C g_l(\lambda, \bar{\lambda}) e^{-F(\lambda)} d\lambda \wedge d\bar{\lambda},
$$

(24)

are known as functional parameters (in coordinate representation). The functions $f_k(\mu, \bar{\mu}), g_k(\lambda, \bar{\lambda})$ can be named as functional parameters in spectral representation. By definitions (24) and (25) the functional parameters $\alpha_n$ and $\beta_n$ satisfy to following linear equations,

$$
\alpha_{n,y} + \alpha_{n,xx} = 0, \quad \alpha_{n,t} + \alpha_{n,xxx} + 5\alpha_{n,xy} - 5\partial_x^{-1} \alpha_{n,yy} = 0;
$$

$$
\beta_{n,y} + \beta_{n,xx} = 0, \quad \beta_{n,t} + \beta_{n,xxx} + 5\beta_{n,xy} - 5\partial_x^{-1} \beta_{n,yy} = 0.
$$

(25)

Here and below useful determinant identities,

$$
\text{Tr}(\frac{\partial A}{\partial x} A^{-1}) = \frac{\partial}{\partial x} \ln(\det A), \quad \text{Tr}(BA^{-1}) = \frac{\det(A+B)}{\det A} - 1, \quad 1 + \text{Tr}B = \det(1+B),
$$

(27)

will be used. The matrices $B$ and $BA^{-1}$ in (27) are degenerate with rank 1. With help of the first identity in (27) expression for $\chi_{-1}$ (22) takes the form,

$$
\chi_{-1} = i \sum_{k,l=1}^N A^{-1}_{kl} \frac{\partial A_{lk}}{\partial x} = i \text{Tr}(A^{-1} \frac{\partial A}{\partial x}) = i \frac{\partial}{\partial x} (\ln \det A).
$$

(28)

From (28) by use of reconstruction formula (12) one obtains general determinant formula for the solution $u$ with functional parameters $\alpha_k(x, y, t), \beta_k(x, y, t)$ of 2DKK (11) and 2DSK (2) equations,

$$
u(x, y, t) = -3i \chi_{-1} x = 3 \frac{\partial^2}{\partial x^2} \ln \det A,
$$

(29)

here the elements of matrix $A$ are given by (28).

In following sections III and IV calculations of exact solutions $u(x, y, t)$ via simultaneous satisfaction to conditions of nonlinear reductions (16), (17) and reality (20) are performed for convenience in cases of 2DKK (11) and 2DSK (2) equations separately; analogous problem of calculations of multiline soliton solutions for these equations was solved in the previous paper [4].

### III. SOLUTIONS OF 2DKK EQUATION.

In this section we calculate new classes of solutions with functional parameters and as their particular cases reproduce multiline solitons [5].

The reduction condition (16) imposes some restrictions on functional parameters $\alpha_l(x, y, t), \beta_l(x, y, t)$ (24). Substituting coefficients $\chi_{-1}$ and $\chi_{-2}$ from (22) into (16) the reduction condition for 2DKK equation can be rewritten in the form,

$$
\frac{\partial}{\partial x} \left[ \chi_{-2} - \frac{i}{2} \chi_{-1} x - \frac{1}{2} \chi_x^2 \right] = \sum_{k,l=1}^N \frac{\partial}{\partial x} \left[ \alpha_l \beta_k - \alpha_k \beta_l \right] A_{kl}^{-1} = \frac{\partial}{\partial x} \text{Tr}(BA^{-1}) = 0,
$$

(30)

where matrix $B$ has elements,

$$
B_{lk} = \alpha_l \beta_k - \alpha_k \beta_l.
$$

(31)

It will be shown below that reduction condition (16), or equivalently (30), is satisfied by several choices of the kernel $R_{ij}$ (21).

The condition I of reduction (16). It is clear from (30) that condition (16) of reduction for the simplest case $R = \pi f_1 g_1$ of $N = 1$ of one term in the sum (21) is satisfied if $\alpha_1$ and $\beta_1$ proportional to each other: $\alpha_1(x, y, t) =$
New exact solutions of 2DKK and 2DSK equations

\[ c_1 \beta_1(x, y, t). \] Now let us prove that under the same interrelations between \( \alpha_k \) and \( \beta_k \), with some constants \( c_k \), for all \( k \),

\[ \alpha_k(x, y, t) = c_k \beta_k(x, y, t), \quad (k = 1, \ldots, N), \tag{32} \]

the condition \( \text{(30)} \) is satisfied also for more general case of \( N \neq 1 \) terms in the kernel \( \text{(24)} \). At first due to \( \text{(30)} \) one transforms matrix \( A \) given by \( \text{(31)} \),

\[ A_{lk} = \delta_{lk} + \frac{1}{2c_k} \delta_x^{-1} \alpha_l \alpha_k = \sqrt{c_l}(\delta_{lk} + \frac{1}{2c_k c_l} \delta_x^{-1} \alpha_l \alpha_k) \frac{1}{\sqrt{c_k}} = \sqrt{c_l} A_{lk} \frac{1}{\sqrt{c_k}}, \tag{33} \]

where \( \hat{A} = \delta_{lk} + \frac{1}{2c_k} \delta_x^{-1} \alpha_l \alpha_k \) is symmetrical matrix. One continues in this fashion transforming matrix \( B \) \( \text{(31)} \),

\[ B_{lk} = \frac{1}{c_k} \alpha_l \alpha_k x_k - \frac{1}{c_k} \alpha_l \alpha_k = \sqrt{c_l}(\frac{1}{\sqrt{c_k c_l}} \alpha_l \alpha_k x_k - \frac{1}{\sqrt{c_k c_l}} \alpha_l \alpha_k) \frac{1}{\sqrt{c_k}} = \sqrt{c_l} B_{lk} \frac{1}{\sqrt{c_k}}, \tag{34} \]

where \( \hat{B} = \frac{1}{\sqrt{c_k c_l}} (\alpha_l \alpha_k x_k - \alpha_l \alpha_k) \) is antisymmetrical matrix. Consequently under conditions \( \text{(32)} \) imposed on functional parameters the reduction condition \( \text{(10)} \) or \( \text{(30)} \) for the 2DKK equation due to \( \text{(33)} \) and \( \text{(34)} \) transforms to the form,

\[ \frac{\partial}{\partial x} Tr(\hat{B} \hat{A}^{-1}) = 0, \tag{35} \]

which satisfies for every \( x, y, t \) due to the facts that \( \hat{A}^{-1} \) – symmetrical as matrix inverse to symmetrical matrix \( \hat{A} \) and \( \hat{B} \) – antisymmetrical matrix.

As a result of performed consideration it is shown that the condition of reduction \( \text{(10)} \) or \( \text{(30)} \) for the 2DKK equation is fulfilled for the interrelations between of functional parameters \( \alpha_k \) and \( \beta_k \) of the type \( \text{(32)} \). From definitions of the \( \alpha_k, \beta_k \) \( \text{(24)} \) follow due to \( \text{(32)} \) the interrelations between functional parameters \( f_k \) and \( g_k \) in spectral representation,

\[ f_k(\mu, \nu) = c_k g_k(-\mu, -\nu), \quad (k = 1, \ldots, N). \tag{36} \]

So in considered relatively simple but important and interesting case, which will be named as condition I of reduction \( \text{(10)} \), the kernel \( R_0 \) satisfying to the condition of reduction \( \text{(10)} \) or \( \text{(30)} \) has due to \( \text{(24)} \) and \( \text{(36)} \) the form,

\[ R_0(\mu, \nu, \lambda, \bar{\lambda}) = \sqrt{\pi} \sum_{k=1}^{N} c_k^{-1} f_k(\mu, \nu) f_k(-\lambda, -\bar{\lambda}). \tag{37} \]

To the first condition of reality from \( \text{(20)} \) one can satisfy by imposing on each term of the sum \( \text{(37)} \) the following restrictions,

\[ c_k^{-1} f_k(\mu, \nu) f_k(-\lambda, -\bar{\lambda}) = \overline{c_k^{-1} f_k(-\mu, -\nu) f_k(\lambda, \bar{\lambda})}. \tag{38} \]

The meaning of such kind of condition is very simple: each term of the sum \( \text{(37)} \) after applying reality condition does not change. By separating variables,

\[ \frac{f_k(\mu, \nu)}{f_k(-\mu, -\nu)} = \frac{c_k^{-1} f_k(\lambda, \bar{\lambda})}{c_k^{-1} f_k(-\lambda, -\bar{\lambda})} = v_k, \tag{39} \]

with some complex constants \( v_k \), one obtain the following restrictions on the functions \( f_k(\mu, \nu) \),

\[ f_k(\mu, \nu) = v_k f_k(-\mu, -\nu), \quad \frac{v_k}{c_k} = \sqrt{\frac{v_k}{c_k}}, \quad v_k^2 = \frac{c_k}{c_k}, \quad |v_k| = 1. \tag{40} \]

Second condition of reality from \( \text{(20)} \) leads effectively to exactly the same relations as in \( \text{(10)} \) and will not be considered here.

Due to definitions \( \text{(24)} \) and \( \text{(30)} \) one derive the following interrelations between different functional parameters,

\[ \alpha_k := v_k \int_C f_k(-\mu, -\nu) e^{F(\mu)} d\mu \wedge d\nu = -v_k \alpha_k, \quad \beta_k := v_k^{-1} \int_C f_k(-\lambda, -\bar{\lambda}) e^{-F(\lambda)} d\lambda \wedge d\bar{\lambda} = v_k^{-1} \alpha_k. \tag{41} \]
So due to (41) the sets of functional parameters are characterized by the following properties,

\((\alpha_1, \ldots, \alpha_N) := (-v_1 \alpha_1, \ldots, -v_N \alpha_N), \quad (\beta_1, \ldots, \beta_N) := (c_1^{-1} \alpha_1, \ldots, c_N^{-1} \alpha_N)\) (42)

i.e. both sets \(\alpha_n\) and \(\beta_n\) express through \(N\) independent complex functional parameters \((\alpha_1(x, y, t), \ldots, \alpha_N(x, y, t))\) satisfying to the first relation from (41).

General determinant formula (29) with matrix \(A\) (23) corresponding to kernel \(R_0\) (37) of \(\overline{\partial}\)-problem (7) gives the class of exact solutions \(u\) with functional parameters for 2DKK equation (11). By construction due to (42) these solutions depend on \(N\) functional parameters \((\alpha_1, \ldots, \alpha_N)\). In the simplest case \(N = 1\), \(\beta_1 := c_1^{-1} \alpha_1\) and due to (41) determinant of matrix \(A\) (23) has the form,

\[
det A = \left(1 + \frac{1}{2c_1} \partial_x^{-1} \alpha_1^2\right) \left(1 - \frac{v_1}{2c_1} \partial_x^{-1} |\alpha_1|^2\right).
\]

The corresponding solution \(u\) with functional parameters of 2DKK equation due to (29) and (43) is given by expression,

\[
u(x, y, t) = -\frac{3v_1}{2c_1 \det A^2} \left(\det A \cdot |\alpha_1|^2 + \frac{v_1}{2c_1} |\alpha_1|^4\right),
\]

where \(v_1/c_1\) due to (40) is some real parameter. In the case \(N = 2\) \((\beta_1, \beta_2) := (c_1^{-1} \alpha_1, c_2^{-1} \alpha_2)\) and due to (41) determinant of matrix \(A\) (23) has the form,

\[
det A = \left(1 + \frac{1}{2c_1} \partial_x^{-1} \alpha_1^2\right) \left(1 + \frac{1}{2c_2} \partial_x^{-1} \alpha_2^2\right) - \frac{1}{4c_1c_2} \left(\partial_x^{-1} \alpha_1 \alpha_2\right)^2 = \left(1 - \frac{v_1}{2c_1} \partial_x^{-1} |\alpha_1|^2\right) \left(1 - \frac{v_2}{2c_2} \partial_x^{-1} |\alpha_2|^2\right) - \frac{v_1v_2}{4c_1c_2} \partial_x^{-1} \alpha_1 \alpha_2^2,
\]

where \(v_k/c_k\), \((k = 1, 2)\) due to (40) are some real parameters. The corresponding solution \(u\) is calculated with help of reconstruction formula (29). It is interesting to note that considered simplest solutions \(u(x, y, t)\) of 2DKK equation with functional parameters for cases \(N = 1, 2\) are nonsingular for negative values of constants \(\frac{v_k}{c_k} < 0\), \((k = 1, 2)\); for \(N = 2\) it follows from (43) particularly due to Cauchy-Buniakovskii inequality \((\partial_x^{-1} |\alpha_1|^2)(\partial_x^{-1} |\alpha_2|^2) \geq |\partial_x^{-1} \alpha_1 \alpha_2|^2\).

In particular case of kernel \(R_0\) (37) of delta-functional type with \(f_k(\mu, \overline{\nu}) = A_k \delta(\mu - i\mu_0), \quad (k = 1, \ldots, N)\), which satisfy to conditions (40) and (41), due to definitions (23) functional parameters \(\alpha_k\) have the following forms,

\[
\alpha_k = -2i A_k e^{F(i\mu_0)},
\]

where in accordance with (40) \(A_k = v_k A_k\) and \(\mu_0\) are some real parameters. Such kernel leads to corresponding exact multiline soliton solutions.

In the simplest case of \(N = 1\) from (41), (40), under the condition \(\frac{v|A|^2}{c_1 \mu_0} = e^{2\varphi_0} > 0\), one obtains the exact nonsingular one line soliton solution of the 2DKK equation,

\[
u(x, y, t) = \frac{3\mu_0^2}{\cosh^2(\mu_0 x - \mu_0 y + 9 \mu_0^2 t - \varphi_0)}.
\]

This one line soliton solution was derived earlier in paper of first author. In the case of \(N = 2\) from (29), (43) and (40) can be easily calculated exact two line soliton solution of the 2DKK equation.

To conditions of reality (10) of the solution of 2DKK equation (11) one can satisfy more nontrivially by imposing another restrictions on functional parameters \(f_k(\mu, \overline{\nu}), g_k(\lambda, \overline{\lambda})\) or \(\alpha_k(x, y, t), \beta_k(x, y, t)\). For this purpose the terms in the sum (24) or (37) for the kernel \(R_0\) can be grouped by pairs. The kernel of the \(\overline{\partial}\)-problem for which condition of reduction (10) fulfills, has to due (37) the form,

\[
R_0(\mu, \overline{\nu}, \lambda, \overline{\lambda}) = \pi \sum_{k=1}^{2N} c_k^{-1} f_k(\mu, \overline{\nu}) f_k(-\lambda, -\overline{\lambda}) = \pi \sum_{k=1}^{N} \left[c_k^{-1} p_k(\mu, \overline{\nu}) p_k(-\lambda, -\overline{\lambda}) + \overline{c_k^{-1} p_k(\mu, \overline{\nu})} \overline{p_k(-\lambda, -\overline{\lambda})}\right],
\]

where \((f_1, \ldots, f_{2N}) = (p_1(\mu, \overline{\nu}), \ldots, p_N(\mu, \overline{\nu}); \overline{p_1}(\mu, \overline{\nu}), \ldots, \overline{p_N}(\mu, \overline{\nu}))\). To the first and the second conditions of reality from (29) one can satisfy by imposing on each pair of terms in the sum (48) the following restrictions,

\[
c_k^{-1} p_k(\mu, \overline{\nu}) p_k(-\lambda, -\overline{\lambda}) = \overline{c_k^{-1} p_k(-\mu, -\overline{\nu}) p_k(-\lambda, -\overline{\lambda})},
\]

i.e. the first term in square bracket of (48) goes under considered reality conditions to second one. By separating variables in last expression (49),

\[
p_k(\mu, \overline{\nu}) = \frac{\overline{c_k^{-1} p_k(-\mu, -\overline{\nu})}}{c_k^{-1} p_k(-\lambda, -\overline{\lambda})} = \overline{v_k^{-1}},
\]
New exact solutions of 2DKK and 2DSK equations

with some complex constants $v_k$, one obtain the interrelations between the functional parameters $\tilde{p}_k(\mu, \overline{\mu})$ and $p_k(\mu, \overline{\mu})$,

$$\tilde{p}_k(\mu, \overline{\mu}) = v_k p_k(-\overline{\mu}, -\mu), \quad c_k = v_k^2, \quad (k = 1, ..., N).$$

(51)

So the kernel which satisfies to the first and the second conditions of reality from (20) and condition of reduction (16) or (30) due to (36), (48), (51) has the form,

$$R_0(\mu, \overline{\mu}, \lambda, \overline{\lambda}) = \pi \sum_{k=1}^{N} \left[ c_k^{-1} p_k(\mu, \overline{p}_k(-\lambda, -\overline{\lambda}) + \tau_k^{-1} p_k(-\overline{\mu}, -\mu) \overline{p}_k(\lambda, \overline{\lambda}) \right],$$

(52)

and due to (48), (52) one can choose the following convenient sets $f$ and $g$ of functions $f_n, g_n, n = 1 \ldots 2N$,

$$f := (f_1, \ldots, f_{2N}) = (p_1(\mu, \overline{\mu}), \ldots, p_N(\mu, \overline{\mu}), \overline{p}_1(-\overline{\mu}, -\mu), \ldots, \overline{p}_N(-\overline{\mu}, -\mu)),$$

(53)

$$g := (g_1, \ldots, g_{2N}) = (c_1^{-1} p_1(-\lambda, -\overline{\lambda}), \ldots, c_N^{-1} p_N(-\lambda, -\overline{\lambda}), \tau_1^{-1} \overline{p}_1(\overline{\lambda}, \lambda), \ldots, \tau_N^{-1} \overline{p}_N(\overline{\lambda}, \lambda)).$$

(54)

Due to definitions (24) and (53), (54) one derive the following interrelations between different functional parameters in coordinate representation,

$$\beta_k := c_k^{-1} \int_C p_k(-\lambda, -\overline{\lambda}) e^{-F(\lambda)} d\lambda \wedge d\overline{\lambda} = c_k^{-1} \alpha_k, \quad \beta_{k+N} := c_k^{-1} \int_C \overline{p}_k(\lambda, \overline{\lambda}) e^{-F(\lambda)} d\lambda \wedge d\overline{\lambda} = -\tau_k^{-1} \alpha_k,$$

(55)

$$\alpha_{k+N} := \int_C p_k(-\overline{\mu}, -\mu) e^{F(\mu)} d\mu \wedge d\overline{\mu} = -\overline{\alpha_k},$$

(56)

in formulas (55) and (56) index $k$ takes the values: $k = 1, ..., N$. So due to (40), (49) the sets of functional parameters $\alpha_n$ and $\beta_n$ have the following properties,

$$(\alpha_1, \ldots, \alpha_{2N}) := (\alpha_1, \ldots, \alpha_N; -\overline{\alpha_1}, \ldots, -\overline{\alpha_N})$$

(57)

$$(\beta_1, \ldots, \beta_{2N}) := (c_1^{-1} \alpha_1, \ldots, c_N^{-1} \alpha_N; -\tau_1^{-1} \overline{\alpha_1}, \ldots, -\tau_N^{-1} \overline{\alpha_N}),$$

(58)

i.e. both sets express through $N$ independent complex functional parameters $(\alpha_1, \ldots, \alpha_N)$.

General determinant formula (29) with matrix $A_{23}$ corresponding to kernel $R_0$ (52) of $\overline{\partial}$-problem (7) gives another class of exact solutions $u$ with functional parameters of 2DKK equation (11). By construction due to (47), (58) these solutions depend on $N$ functional parameters $(\alpha_1, \ldots, \alpha_N)$ given by expressions (24). In the simplest case $N = 1$ $(\alpha_1, \alpha_2) := (\alpha_1, -\overline{\alpha_1})$, $(\beta_1, \beta_2) := (c_1^{-1} \alpha_1, -\overline{\alpha_1})$ and due to (55), (56) the determinant of matrix $A_{23}$ is given by expression:

$$\det A = \left(1 + \frac{1}{2c_1} \partial_x^{-2} \alpha_1^2\right) \left(1 + \frac{1}{2c_1} \partial_x^{-1} \alpha_1^2\right) - \frac{1}{4c_1^2} (\partial_x^{-1} \alpha_1 \partial_x^{-1} \overline{\alpha_1})^2 = \left|1 + \frac{1}{2c_1} \partial_x^{-1} \alpha_1^2\right|^2 - \frac{1}{4c_1^2} (\partial_x^{-1} |\alpha_1|^2)^2.$$  

(59)

Corresponding solution $u$ is calculated with help of reconstruction formula (29) and has the form,

$$u = \frac{3}{\det A} \left( \det A \left( \frac{\alpha_1 \alpha_1 x}{c} + \overline{\alpha_1} \alpha_1 x \overline{c} \right) + \frac{1}{2c_1^2} \left( \alpha_1 \alpha_1 x \partial_x^{-1} \alpha_1^2 + \overline{\alpha_1} \overline{\alpha_1} \partial_x^{-1} \overline{\alpha_1}^2 - |\alpha_1|^2 \partial_x^{-1} |\alpha_1|^2 \right) \right) -$$

$$- \frac{\alpha_1^2}{2c_1} + \frac{\overline{\alpha_1}^2}{2c_1} + \frac{1}{4c_1^2} \left( \alpha_1^2 \partial_x^{-1} \alpha_1^2 + \overline{\alpha_1}^2 \partial_x^{-1} \overline{\alpha_1}^2 - 2|\alpha_1|^2 \partial_x^{-1} |\alpha_1|^2 \right)^2,$$

(60)

and due to (59) evidently is singular.

In the case of kernel $R_0$ (52) of delta-functional type with $p_k(\mu, \overline{\mu}) = A_k(\mu - \mu_k), (k = 1, \ldots, N)$, which satisfies to the conditions (30) and (51), due to the definitions (24) functional parameters $\alpha_k$ have the following form,

$$\alpha_k = -2i A_k \varepsilon^{F(\mu_k)}, \quad (k = 1, \ldots, N).$$

(61)
Such kernel leads to corresponding exact multinline soliton solutions. In the simplest case of \( N = 1 \) from \((59)\) and \((60)\) due to \((61)\) follows the exact one line soliton solution of 2DKK equation,

\[
(62)
\]

where determinant of the matrix \( A \) has form,

\[
(63)
\]

Due to the expression \((63)\) for \( \det A \) last calculated solution evidently is singular.

**The condition II of reduction \((16)\).** The condition of reduction \((16)\) or \((30)\) can be satisfied by another restriction on functional parameters. One groups for this the terms in the kernel \( R_0 \) by pairs,

\[
(64)
\]

One define the following sets \( f \) and \( g \) of functions \( f_n, g_n, n = 1 \ldots 2N \),

\[
(65)
\]

For simplicity let us rewrite the last expression in \((30)\) for case \( N = 1 \) of one pair of terms in kernel \((64)\),

\[
(66)
\]

where \( A_{ij}, B_{ij} \) are elements of matrices \( A \) \((23)\) and \( B \) \((31)\). Substituting \( A_{ij}, B_{ij} \) from \((23)\) and \((31)\) into \((66)\) one obtains, for one pair of terms in kernel \((64)\), from the equality \( B_{11}A_{22} - B_{12}A_{21} - B_{21}A_{12} + B_{22}A_{11} = 0 \) new condition of reduction in terms of functional parameters,

\[
(67)
\]

It is easy to check that choice \( \alpha_2 = c\beta_1 \) and \( \beta_2 = c^{-1}\alpha_1 \), or due to \((24)\) corresponding choice in terms of functional parameters in spectral representation, \( f_2(\mu, \overline{\nu}) = cg_1(-\mu, -\overline{\nu}) \) and \( g_2(\mu, \overline{\nu}) = c^{-1}f_1(-\mu, -\overline{\nu}) \), satisfies to condition \((67)\). These conditions can be generalized for case of kernel consisting of \( N \) pairs in \((64)\) with following identification of multipliers,

\[
(68)
\]

So to the condition of reduction \((16)\) or \((30)\) is satisfied due to \((68)\) following kernel \( R_0 \) \((31)\) of the \( \overline{\nu} \)-problem \( \overline{7} \),

\[
(69)
\]

Formulated conditions \((68)\) will be named as **condition II of reduction \((16)\)**.

The first condition of reality from \((20)\) and condition of reduction \((16)\) or \((30)\) are satisfied simultaneously by imposing on each pair of terms in the sum \((69)\) the following restriction,

\[
(70)
\]

Due to \((70)\) two cases are possible,

A. \( p_k(\mu, \overline{\nu}) q_k(\lambda, \overline{\lambda}) = \frac{p_k(-\overline{\mu}, -\overline{\nu})}{p_k(\lambda, \overline{\lambda})} q_k(-\overline{\lambda}, -\overline{\lambda}), \)
New exact solutions of 2DKK and 2DSK equations

\[ B. \quad p_k(\mu, \overline{\mu})q_k(\lambda, \overline{\lambda}) = \overline{q_k(\overline{\mu}, \mu) p_k(\lambda, \overline{\lambda})}. \]  

(72)

In the case A, by separating variables,

\[ \frac{p_k(\mu, \overline{\mu})}{p_k(-\overline{\mu}, -\mu)} = \frac{q_k(-\overline{\lambda}, -\lambda)}{q_k(\lambda, \overline{\lambda})} = v_k, \]  

(73)

with some complex constants \( v_k, (k = 1, \ldots, N), \) one obtain the following restrictions on the functions \( p_k(\mu, \overline{\mu}) \) and \( q_k(\lambda, \overline{\lambda}), \)

\[ p_k(\mu, \overline{\mu}) = v_k p_k(-\overline{\mu}, -\mu), \quad q_k(\lambda, \overline{\lambda}) = v_k^{-1} q_k(-\overline{\lambda}, -\lambda). \]  

(74)

In the case B, by separating variables,

\[ \frac{p_k(\mu, \overline{\mu})}{q_k(\overline{\mu}, \mu)} = \frac{p_k(\lambda, \overline{\lambda})}{q_k(\lambda, \overline{\lambda})} = v_k, \]  

(75)

with some another constants \( v_k, (k = 1, \ldots, N), \) one obtain another restrictions on the functions \( p_k(\mu, \overline{\mu}) \) and \( q_k(\lambda, \overline{\lambda}), \)

\[ q_k(\mu, \overline{\mu}) = v_k^{-1} p_k(\overline{\mu}, \mu), \quad v_k = v_k. \]  

(76)

One can show that second condition of reality from (20) leads to the same restrictions on the kernel \( R_0 \) as obtained above; i.e. only the cases A. (71) and B. (72) will be discussed further.

So for the case A. (71) the kernel \( R_0 \) which satisfies simultaneously to conditions of reduction (16) or (30) and reality (20) due to (68) and (69) takes the form,

\[ R_0(\mu, \overline{\mu}, \lambda, \overline{\lambda}) = \pi \sum_{k=1}^{2N} f_k(\mu, \overline{\mu}) g_k(\lambda, \overline{\lambda}) = \pi \sum_{k=1}^{N} \left[ p_k(\mu, \overline{\mu}) q_k(\lambda, \overline{\lambda}) + q_k(-\mu, -\overline{\mu}) p_k(-\lambda, -\overline{\lambda}) \right]. \]  

(77)

where functions \( p_k \) and \( q_k \) are satisfy to conditions (74).

For the kernel (77) one choose the following convenient sets \( f \) and \( g \) of functions \( f_n, g_n, n = 1 \ldots 2N, \)

\[ f := (f_1, \ldots, f_{2N}) = (p_1(\mu, \overline{\mu}), \ldots, p_N(\mu, \overline{\mu}); q_1(-\mu, -\overline{\mu}), \ldots, q_N(-\mu, -\overline{\mu})), \]  

(78)

\[ g := (g_1, \ldots, g_{2N}) = (g_1(\lambda, \overline{\lambda}), \ldots, g_N(\lambda, \overline{\lambda}); p_1(-\lambda, -\overline{\lambda}), \ldots, p_N(-\lambda, -\overline{\lambda})). \]  

(79)

Due to definitions (24) and (74), (76), (79) one derive the following interrelations between different functional parameters,

\[ \alpha_k := v_k \int_C \frac{p_k(-\overline{\mu}, -\mu)e^{F(\mu)}}{\overline{\mu} - \overline{\mu}} d\mu \wedge d\overline{\mu} = -v_k \overline{\alpha_k}, \quad \beta_k := v_k^{-1} \int_C q_k(\overline{\lambda}, -\lambda) e^{-F(\lambda)} d\lambda \wedge d\overline{\lambda} = -v_k^{-1} \overline{\beta_k}, \]  

(80)

\[ \alpha_{k+N} := \int_C q_k(-\mu, -\overline{\mu}) e^{F(\mu)} d\mu \wedge d\overline{\mu} = \beta_k, \quad \beta_{k+N} := \int_C p_k(-\lambda, -\overline{\lambda}) e^{-F(\lambda)} d\lambda \wedge d\overline{\lambda} = \alpha_k, \]  

(81)

where in formulas (80) and (81) index \( k \) takes the values: \( k = 1, \ldots, N. \) So by the use (80), (81) one concludes that the sets of functional parameters have the following structure,

\[ (\alpha_1, \ldots, \alpha_{2N}) := (\alpha_1, \ldots, \alpha_N; \beta_1, \ldots, \beta_N) \]  

(82)

\[ (\beta_1, \ldots, \beta_{2N}) := (\beta_1, \ldots, \beta_N; \alpha_1, \ldots, \alpha_N) \]  

(83)

i.e. both sets are expressed through \( 2N \) independent functional parameters \( (\alpha_1, \ldots, \alpha_N) \) and \( (\beta_1, \ldots, \beta_N) \) with properties (80).

General determinant formula (29) with matrix \( A \) (23) corresponding to kernel \( R_0 \) (77) of \( \overline{\overline{J}} \)-problem (7) gives the class of exact solutions \( u \) with functional parameters of 2DKK equation (1). By construction due to (82), (83) these solutions depend on \( 2N \) functional parameters \( (\alpha_1, \ldots, \alpha_N) \) and \( (\beta_1, \ldots, \beta_N) \) with properties (80).
In the simplest case \( N = 1 \) \((\alpha_1, \alpha_2) := (\alpha_1, \beta_1), \ (\beta_1, \beta_2) := (\beta_1, \alpha_1)\) the determinant of matrix \( A \) due to (23) is given by expression,

\[
\det A = \left(1 + \frac{1}{2} \partial_x^{-1} \alpha_1 \beta_1 \right)^2 - \frac{1}{4} \partial_x^{-1} | \alpha_1 |^2 \partial_x^{-1} | \beta_1 |^2
\]

(84)

where due to (50) \( \alpha_1 \beta_1 = \overline{\alpha_1} \overline{\beta_1} \). The corresponding solution \( u \) is calculated with help of reconstruction formula (29) and has the form,

\[
u = \frac{3}{(\det A)^2} \left( \det A \left( \frac{1}{2} \alpha_1^2 \beta_1^2 - \frac{1}{2} | \alpha_1 |^2 | \beta_1 |^2 + (\alpha_1 \beta_1)_x (1 + \frac{1}{2} \partial_x^{-1} \alpha_1 \beta_1) - \frac{1}{4} | \alpha_1 |^2 \partial_x^{-1} | \beta_1 |^2 - \frac{1}{4} | \beta_1 |^2 \partial_x^{-1} | \alpha_1 |^2 \right) - \\
- \left( \alpha_1 \beta_1 (1 + \frac{1}{2} \partial_x^{-1} \alpha_1 \beta_1) - \frac{1}{4} | \alpha_1 |^2 \partial_x^{-1} | \beta_1 |^2 - \frac{1}{4} | \beta_1 |^2 \partial_x^{-1} | \alpha_1 |^2 \right)^2 \right).
\] (85)

The calculated simplest solutions \( u(x, y, t) \) with functional parameters of 2DKK equation for case \( N = 1 \) may be due singular or nonsingular, it depends on concrete choice of functional parameters.

For the case of kernel \( R_0 \) of delta-functional type, with \( p_k(\mu, \overline{\mu}) = A_k \delta(\mu - i \mu_0), \ q_k(\lambda, \overline{\lambda}) = B_k \delta(\lambda - i \lambda_0) \), \((k = 1, \ldots, N)\), satisfies the conditions (68) and (72), the functional parameters have the following form,

\[
\alpha_k = -2i A_k e^F(\mu_{10}), \quad \beta_k = -2i B_k e^{-F(\lambda_{10})}, \quad (k = 1, \ldots, N),
\]

(86)

where in accordance with (74) \( A_k = v_k \overline{\alpha_k}, \ B_k = v_k^{-1} \overline{\beta_k} \) and \( \mu_{10}, \lambda_{10} \) - some real parameters. Such kernel leads to corresponding exact multiline soliton solutions. In the simplest case of \( N = 1 \) one obtains from (84), (85) via (80) the exact one line soliton solution of the 2DKK equation,

\[
\det A = 1 + \frac{4a}{\mu_{10} - \lambda_{10}} e^{\varphi(x, y, t)} + \frac{a^2}{\mu_{10} \lambda_{10}} \left( \mu_{10} + \lambda_{10} \right)^2 e^{2 \varphi(x, y, t)},
\]

\[
u(x, y, t) = \frac{12a(\mu_{10} - \lambda_{10}) e^{\varphi}}{\det A^2} \left[ \det A + \frac{a(\mu_{10} - \lambda_{10}) e^{\varphi}}{\mu_{10} \lambda_{10}} \right],
\]

(87)

where \( a = A_1 B_1 = \overline{a} \) - some real parameter, \( \varphi(x, y, t) := F(i \mu_{10}) - F(i \lambda_{10}) \). This one line soliton solution for the values of parameters, \( \frac{\mu_{10} - \lambda_{10}}{a} > 0, \mu_{10} \lambda_{10} > 0 \), is nonsingular and was derived earlier in paper of first author.

For the case \( B \) the kernel \( R_0 \) which satisfies conditions of reduction (110) or (111) and reality (20) due to (68), (69) and (76) has the form,

\[
R_0(\mu, \overline{\mu}, \lambda, \overline{\lambda}) = \pi \sum_{k=1}^{2N} f_k(\mu, \overline{\mu}) g_k(\lambda, \overline{\lambda}) = \pi \sum_{k=1}^{N} \left[v_k^{-1} p_k(\mu, \overline{\mu}) p_k(\overline{\lambda}, \lambda) + v_k^{-1} p_k(\overline{\mu}, \mu) p_k(-\lambda, -\overline{\lambda}) \right].
\]

(88)

From (88) one choose the following convenient sets \( f, g \) of functions \( f_n, g_n, n = 1 \ldots 2N \),

\[
f := (f_1, \ldots, f_{2N}) = (p_1(\mu, \overline{\mu}), \ldots, p_N(\mu, \overline{\mu}); v_1^{-1} p_1(\overline{\mu}, -\mu), \ldots, v_N^{-1} p_N(\overline{\mu}, -\mu)),
\]

(89)

\[
g := (g_1, \ldots, g_{2N}) = (v_1^{-1} p_1(\overline{\lambda}, \lambda), \ldots, v_N^{-1} p_N(\overline{\lambda}, \lambda); p_1(-\lambda, -\overline{\lambda}), \ldots, p_N(-\lambda, -\overline{\lambda})).
\]

(90)

Due to definitions (24) and (89), (91) one derive the following interrelations between different functional parameters,

\[
\beta_k := v_k^{-1} \int_C \overline{p_k(\lambda, \overline{\lambda}) e^{-F(\lambda)}} d\lambda \wedge d\overline{\lambda} = -v_k^{-1} \overline{\alpha_k}, \quad k = 1, \ldots, N;
\]

\[
\alpha_{k+N} := v_k^{-1} \int_C \overline{p_k(\overline{\mu}, \mu) e^{-F(\mu)}} d\mu \wedge d\overline{\mu} = -v_k^{-1} \overline{\alpha_k}, \quad \beta_{k+N} := \int_C p_k(-\lambda, -\overline{\lambda}) e^{-F(\lambda)} d\lambda \wedge d\overline{\lambda} = \alpha_k.
\]

(91)

(92)

So using (91), (92) one concludes that the sets of functional parameters have the following structure,

\[
(\alpha_1, \ldots, \alpha_{2N}) := (\alpha_1, \ldots, \alpha_N; -v_1^{-1} \overline{\alpha_1}, \ldots, -v_N^{-1} \overline{\alpha_N}),
\]

(93)
New exact solutions of 2DKK and 2DSK equations

\[(\beta_1, \ldots, \beta_{2N}) := (-v_1^{-1}\alpha_1, \ldots, -v_N^{-1}\alpha_N, \alpha_1, \ldots, \alpha_N), \] (94)
i.e. both sets express through \(N\) independent complex functional parameters \((\alpha_1, \ldots, \alpha_N)\).

General determinant formula (29) with matrix \(A\) (23) corresponding to kernel \(R_0\) (88) of \(\bar{\partial}\)-problem (7) gives the class of exact solutions \(u\) with functional parameters of the 2DKK equation (11). By construction due to (93), (94) these solutions depend on \(N\) functional parameters \((\alpha_1, \ldots, \alpha_N)\) given by (24).

In the simplest case \(N = 1\) \((\alpha_1, \alpha_2) := (\alpha_1, -v_1^{-1}\alpha_1)\), \((\beta_1, \beta_2) := (-v_1^{-1}\alpha_1, \alpha_1)\) the determinant of \(A\) due to (29) is given by expression,

\[
det A = (1 - \frac{1}{2v_1^2}\partial_x^{-1}|\alpha_1|^2)^2 - \frac{1}{4v_1^2}|\partial_x^{-1}\alpha_1|^2, \] (95)

where due to (76) \(v_1 = \bar{v}_1\). The corresponding solution \(u\) is calculated with help of reconstruction formula (29); for \(v_1 < 0\), due to (95) and to Cauchy-Bunyakovskii inequality \((\partial_x^{-1}|\alpha_1|^2)(\partial_x^{-1}|\alpha_1|^2) \geq |\partial_x^{-1}\alpha_1\alpha_1|^2\), this solution is nonsingular and has the form,

\[
u = \frac{3}{(\det A)^2} \left( \frac{\det A}{2v_1^2} \left( -2v_1|\alpha_1|^2(1 - \frac{1}{2v_1^2}\partial_x^{-1}|\alpha_1|^2) - \alpha_1\alpha_1\partial_x^{-1}\alpha_1^2 - \bar{\alpha}_1\bar{\alpha}_1\partial_x^{-1}\alpha_1^2 \right) - \frac{1}{16v_1^2} \left( \alpha_1^2\partial_x^{-1}\alpha_1^2 + \bar{\alpha}_1^2\partial_x^{-1}\bar{\alpha}_1^2 + 4v_1|\alpha_1|^2(1 - \frac{1}{2v_1^2}\partial_x^{-1}|\alpha_1|^2) \right)^2 \right). \] (96)

In the case of kernel \(R_0\) (88) of delta-functional type, with \(p_k(\mu, \bar{\mu}) = A_k\delta(\mu - \mu_k)\), \((k = 1, \ldots, N)\), which satisfies the conditions (68) and (76), due to definitions (24), functional parameters \(\alpha_k\) have the following form:

\[
\alpha_k = -2iA_ke^{\varphi(\mu_k)}, \quad (k = 1, \ldots, N). \] (97)

Such kernel leads to corresponding multinline soliton solutions. In the simplest case of \(N = 1\) one obtains from (96), (97) via (29) the exact one line soliton solution of the 2DKK equation,

\[
det A = 1 + \frac{2a}{\mu_{1f}}e^{\varphi(x,y,t)} + \frac{a^2}{|\mu_{1f}|^2}\frac{\mu_{1f}^2}{\mu_{1f}^2}e^{2\varphi(x,y,t)} = \left( 1 + \frac{a}{|\mu_{1f}|^2}e^{\varphi(x,y,t)} \right)^2 - \frac{a^2}{|\mu_{1f}|^2}e^{2\varphi(x,y,t)},
\]

\[
u(x,y,t) = \frac{2a\mu_{1f}e^{\varphi}}{(\det A)^2} \left[ \det A - \frac{2a\mu_{1f}e^{\varphi}}{|\mu_{1f}|^2} \right], \] (98)

where \(a = v_1^{-1}|\alpha_1|^2 = \bar{a}\) is some real parameter, \(\varphi(x,y,t) := F(\mu_1) - F(\bar{\mu}_1)\); for \(\frac{a}{\mu_{1f}} = \frac{|A_1|^2}{v_1\mu_{1f}} > 0\), due to expression for \(\det A\) in (98), this solution is nonsingular.

IV. SOLUTIONS OF 2DSK EQUATION.

In present section the classes of solutions with functional parameters are calculated for 2DSK equation (2). This equation has condition of reduction (17) which is different from that one (16) for 2DKK equation. It is convenient to transform this condition of reduction to determinant form, more appropriate for calculations with 2DSK equation (2). Substituting coefficients \(\chi_{-1}\) and \(\chi_{-2}\) from (22) into the condition of reduction (17) one obtains,

\[
\sum_{k,l=1}^{N} (\alpha_k\beta_l)A_{kl}^{-1} - \left( \sum_{k,l=1}^{N} \frac{\partial A_{lk}}{\partial x} A_{kl}^{-1} \right)^2 = 0. \] (99)

Defining degenerate matrix \(V\) with elements \(V_{lk} = \alpha_k\beta_l\) and rank equal to unity, rank\(V\) = 1, one rewrites condition (99) in the form,

\[
Tr(VA^{-1}) - \left[ Tr\left( \frac{\partial A}{\partial x} A^{-1} \right) \right]^2 = 0. \] (100)

By the use of identities (27) one obtains from (100) condition of reduction (17) in determinant form,

\[
T(S - T) - T^2_x = 0, \] (101)
where $T = \det A$, $S = \det (A + V)$.

The conditions of reality (20) and reduction (17) or (101) impose some restrictions on functional parameters. In order to satisfy these conditions the terms in kernel $R_0$ (21) will be grouped by pairs,

$$R_0(\mu, \overline{\nu}, \lambda, \overline{\lambda}) = \pi \sum_{k=1}^{2N} f_k(\mu, \overline{\nu}) g_k(\lambda, \overline{\lambda}) = \pi \sum_{k=1}^{N} [p_k(\mu, \overline{\nu}) q_k(\lambda, \overline{\lambda}) + \tilde{p}_k(\mu, \overline{\nu}) \tilde{q}_k(\lambda, \overline{\lambda})].$$

(102)

It is convenient to define via (102) the following sets $f$ and $g$ of functions $f_n$, $g_n$, $n = 1 \ldots 2N$,

$$f := (f_1, \ldots, f_{2N}) = (p_1(\mu, \overline{\nu}), \ldots, p_N(\mu, \overline{\nu}); \tilde{p}_1(\mu, \overline{\nu}), \ldots, \tilde{p}_N(\mu, \overline{\nu})),

\quad g := (g_1, \ldots, g_{2N}) = (q_1(\lambda, \overline{\lambda}), \ldots, q_N(\lambda, \overline{\lambda}); \tilde{q}_1(\lambda, \overline{\lambda}), \ldots, \tilde{q}_N(\lambda, \overline{\lambda})).$$

(103)

One shows easily that for case of $N = 1$, i.e. for kernel (102) of the $\overline{\partial}$-problem with one pair of terms, the condition of reduction (17) or (101) is fulfilled for the choice of interrelations between functional parameters.

In terms functional parameters in spectral representation, due to definitions (21), last relation is rewritten equivalently for the choice, $\tilde{p}_1(\mu, \overline{\nu}) = c_1^{-1} \mu^{-1} q_1(-\mu, -\overline{\nu})$, $\tilde{q}_1(\lambda, \overline{\lambda}) = c_1 \lambda p_1(-\lambda, -\overline{\lambda})$. By the use of symbolic calculations it was verified that the condition of the reduction (17) or (101) is satisfied for two such pairs of terms (i.e. $N = 2$) in the kernel $R_0$ (102). Generalizing last observation to the case of $N > 2$ pairs of terms in (102) one choose the following sets of functional parameters in coordinate and in spectral representations relating to each other by expressions,

$$\alpha_k = ic_1^{-1} \partial_x^{-1} \beta_k, \quad \beta_{k+N} = ic_k \alpha_k, \quad k = 1, \ldots, N,

\tilde{p}_k(\mu, \overline{\nu}) = c_k^{-1} \mu^{-1} q_k(-\mu, -\overline{\nu}), \quad \tilde{q}_k(\lambda, \overline{\lambda}) = c_k \lambda p_k(-\lambda, -\overline{\lambda}),$$

(104)

where $c_k, (k = 1, \ldots, N)$ are some complex constants and index $k$ numerates the pair of terms in kernel $R_0$; for such choice the condition of reduction (17) or (101) is fulfilled for the choice of interrelations between functional parameters of each pair in (102) given by expressions (104).

So due to (104) the condition of reduction (17) or (101) is satisfied by choosing kernel $R_0$ (102) of $\overline{\partial}$-problem (7) in the following form,

$$R_0(\mu, \overline{\nu}, \lambda, \overline{\lambda}) = \pi \sum_{k=1}^{2N} f_k(\mu, \overline{\nu}) g_k(\lambda, \overline{\lambda}) = \pi \sum_{k=1}^{N} [p_k(\mu, \overline{\nu}) q_k(\lambda, \overline{\lambda}) + \lambda \mu q_k(-\mu, -\overline{\nu}) p_k(-\lambda, -\overline{\lambda})].$$

(105)

In accordance with (105) the sets $f$ and $g$ of functions $f_n$, $g_n$, $n = 1, \ldots, 2N$ in (103) are taken the forms,

$$f := (f_1, \ldots, f_{2N}) = (p_1(\mu, \overline{\nu}), \ldots, p_N(\mu, \overline{\nu}); \frac{1}{\mu} q_1(-\mu, -\overline{\nu}), \ldots, \frac{1}{\mu} q_N(-\mu, -\overline{\nu})),

\quad g := (g_1, \ldots, g_{2N}) = (q_1(\lambda, \overline{\lambda}), \ldots, q_N(\lambda, \overline{\lambda}); \lambda p_1(-\lambda, -\overline{\lambda}), \ldots, \lambda p_N(-\lambda, -\overline{\lambda})).$$

(106)

The first condition of reality from (20) is satisfied by imposing on each pair of terms in sum (105) the following restrictions,

$$p_k(\mu, \overline{\nu}) q_k(\lambda, \overline{\lambda}) + \frac{\lambda}{\mu} q_k(-\mu, -\overline{\nu}) p_k(-\lambda, -\overline{\lambda}) =

= p_k(-\overline{\nu}, -\mu) q_k(-\overline{\lambda}, -\lambda) + \frac{\lambda}{\mu} q_k(\overline{\nu}, \mu) p_k(\overline{\lambda}, \lambda).$$

(107)

Due to (107) two cases are possible,

A. $p_k(\mu, \overline{\nu}) q_k(\lambda, \overline{\lambda}) = \frac{p_k(-\overline{\nu}, -\mu) q_k(-\overline{\lambda}, -\lambda)}{p_k(-\overline{\nu}, -\mu) q_k(-\overline{\lambda}, -\lambda)}$, 

(108)

B. $p_k(\mu, \overline{\nu}) q_k(\lambda, \overline{\lambda}) = \frac{\lambda q_k(\overline{\nu}, \mu)}{p_k(\overline{\lambda}, \lambda)}$.

(109)

In the case $A$ by separating variables,

$$\frac{p_k(\mu, \overline{\nu})}{p_k(-\overline{\nu}, -\mu)} = \frac{q_k(-\overline{\lambda}, -\lambda)}{q_k(\lambda, \overline{\lambda})} = v_k,$$

(110)
with some complex constants \( v_k, (k = 1, ..., N), \) one obtain following restrictions on the functions \( p_k(\mu, \overline{\mu}) \) and \( q_k(\lambda, \overline{\lambda}), \)

\[
p_k(\mu, \overline{\mu}) = v_k \overline{p_k(-\overline{\mu}, -\mu)}, \quad q_k(\lambda, \overline{\lambda}) = v_k^{-1} \overline{q_k(-\overline{\lambda}, -\lambda)}, \quad |v_k| = 1.
\]  

(111)

In the case \( B \) by separating variables,

\[
\frac{\mu p_k(\mu, \overline{\mu})}{q_k(\overline{\mu}, \mu)} = \frac{\lambda p_k(\overline{\lambda}, \lambda)}{q_k(\lambda, \overline{\lambda})} = v_k,
\]  

(112)

with some real constants \( v_k, (k = 1, ..., N), \) one obtain another restrictions on the functions \( p_k(\mu, \overline{\mu}), \)

\[
q_k(\mu, \overline{\mu}) = \frac{\mu}{v_k} \overline{p_k(\overline{\mu}, \mu)}, \quad v_k = \overline{v_k} = v_{k0}.
\]  

(113)

Second condition of reality from (21) for 2DSK equation (2) will be satisfied by imposing on each terms in sum (105) the following restriction,

\[
p_k(\mu, \overline{\mu}) q_k(\lambda, \overline{\lambda}) + \frac{\lambda}{\mu} q_k(-\mu, -\overline{\mu}) p_k(-\lambda, -\overline{\lambda}) = \frac{\lambda}{\mu} q_k(-\overline{\lambda}, -\lambda) p_k(-\overline{\mu}, -\mu).
\]  

(114)

Due to (114), also as for (107), two cases are possible,

\[
A'. \quad p_k(\mu, \overline{\mu}) q_k(\lambda, \overline{\lambda}) = \frac{\lambda}{\mu} q_k(-\mu, -\overline{\mu}) p_k(-\lambda, -\overline{\lambda}) = \frac{\lambda}{\mu} q_k(-\overline{\lambda}, -\lambda) p_k(-\overline{\mu}, -\mu).
\]  

(115)

\[
B'. \quad p_k(\mu, \overline{\mu}) q_k(\lambda, \overline{\lambda}) = \frac{\lambda}{\mu} q_k(-\mu, -\overline{\mu}) p_k(-\lambda, -\overline{\lambda}) = \frac{\lambda}{\mu} q_k(-\overline{\lambda}, -\lambda) p_k(-\overline{\mu}, -\mu).
\]  

(116)

From first case \( A' \) (115) by separating variables one obtain the following expressions,

\[
\frac{p_k(\mu, \overline{\mu})}{q_k(\mu, \overline{\mu})} = \frac{\lambda}{\mu} \frac{q_k(-\mu, -\overline{\mu})}{q_k(-\lambda, -\lambda)} = v_k,
\]  

\[
\frac{\lambda^2 p_k(-\lambda, -\overline{\lambda})}{q_k(-\overline{\lambda}, -\lambda)} = \frac{\mu^2 p_k(-\overline{\mu}, -\mu)}{q_k(-\mu, -\overline{\mu})} = \tilde{v}_k,
\]  

(117)

with some complex constants \( v_k \) and \( \tilde{v}_k \) \( (k = 1, ..., N). \) It follows from the last equation in (117), due to the first equation from (117), the relation \( \lambda^2 v_k = \mu^2 v_k = \tilde{v}_k \) which is impossible for arbitrary \( \lambda, \mu. \)

From second case \( B' \) (116) by separating variables one obtain the following expressions,

\[
\frac{p_k(\mu, \overline{\mu})}{\mu p_k(-\overline{\mu}, -\mu)} = \frac{\lambda}{\mu} \frac{q_k(-\mu, -\overline{\mu})}{q_k(-\lambda, -\lambda)} = v_k,
\]  

\[
\frac{\lambda^2 p_k(-\mu, -\overline{\mu})}{q_k(-\mu, -\overline{\mu})} = \frac{\mu^2 p_k(-\overline{\lambda}, -\lambda)}{p_k(-\overline{\mu}, -\mu)} = \tilde{v}_k,
\]  

(118)

with some complex constants \( v_k \) and \( \tilde{v}_k \) \( (k = 1, ..., N). \) It follows from the last equation in (118), due to the first equation from (118), the relation \( \mu^2 v_k = -\tilde{v}_k \) which is impossible for arbitrary \( \mu. \) So bellow will be considered only the cases \( A \) and \( B \) defined by relations (108) and (109).

For case \( A \) (108) the kernel \( R_0 \), which satisfies to conditions of reality (20) and reduction (17) or (101), has the form (105), where functions \( p_k(\mu, \overline{\mu}) \) and \( q_k(\lambda, \overline{\lambda}) \) are characterized by properties (111). Due to definitions (24) and (104), (111) one derive the following interrelations between different functional parameters,

\[
\alpha_k := v_k \int_C \frac{p_k(-\overline{\mu}, -\mu)}{\overline{p_k(-\mu, \mu)}} e^{F(\mu)} d\mu d\overline{\mu} = -v_k^{-1} \alpha_k, \quad \beta_k := \overline{v_k} \int_C \frac{q_k(-\overline{\lambda}, -\lambda)}{\overline{q_k(-\lambda, \lambda)}} e^{-F(\lambda)} d\lambda d\overline{\lambda} = -\overline{v_k} \beta_k,
\]  

(119)
New exact solutions of 2DKK and 2DSK equations

\[ \alpha_{k+N} := c_k^{-1} \int_\mathcal{C} \frac{1}{\mu} q_k(-\mu, -\nabla_x) e^{F(\mu)} d\mu \wedge d\mu = i e^{-1} \partial_x^{-1} \beta_k, \quad \beta_{k+N} := c_k \int_\mathcal{C} \lambda q_k(-\lambda, -\nabla_x) e^{-F(\lambda)} d\lambda \wedge d\lambda = i e^{-1} \alpha_{k+N}, \]

where \(|v_k|^2 = 1\) and \((k = 1, \ldots, N)\). So due to (119), (120) the sets of functional parameters have the following structure,

\[ (\alpha_1, \ldots, \alpha_{2N}) := (\alpha_1, \ldots, \alpha_N; i e^{-1} \partial_x^{-1} \beta_1, \ldots, i e^{-1} \partial_x^{-1} \beta_N), \]

\[ (\beta_1, \ldots, \beta_{2N}) := (\beta_1, \ldots, \beta_N; ic_1 \alpha_1 x, \ldots, ic_N \alpha_N x), \]

i.e. both sets are expressed through 2 independent complex functional parameters \((\alpha_1, \ldots, \alpha_N)\) and \((\beta_1, \ldots, \beta_N)\) with properties (119).

General determinant formula (29) with matrix \(A\) (23) corresponding to the kernel \(R_0\) (105) of the \(\overline{\partial}\)-problem (7) due to (114) gives the class of exact solutions \(u\) with functional parameters of the 2DSK equation (4). By construction due to (123), (122) these solutions depend on 2N functional parameters \((\alpha_1, \ldots, \alpha_N)\) and \((\beta_1, \ldots, \beta_N)\).

In the simplest case \(N = 1\) \((\alpha_1, \alpha_2) := (\alpha_1, i e^{-1} \partial_x^{-1} \beta_1), \) \((\beta_1, \beta_2) := (\beta_1, ic_1 \alpha_1 x)\) due to (119), (120) the determinant of the matrix \(A\) (23) is given by expression

\[ \det A = \left(1 - \frac{1}{4} \alpha_1 \partial_x^{-1} \beta_1 + \frac{1}{2} \partial_x^{-1} \alpha_1 \beta_1\right)^2 = \Delta^2. \]

The corresponding solution \(u\) is calculated with help of reconstruction formula (29) and has the form,

\[ u = \frac{3}{2 \Delta^2} \left[ \Delta \left( \alpha_1 \beta_1 - \alpha_1 \beta_1 \partial_x^{-1} \beta_1 \right) - \frac{1}{2} \Delta \left( \alpha_1 \beta_1 - \alpha_1 \beta_1 \partial_x^{-1} \beta_1 \right) \right] \]

Due to expression (123) for \(\det A\) this solution is nonsingular for choices of functional parameters \(\alpha_1, \beta_1\) satisfying to inequality, \(-\frac{1}{4} \alpha_1 \partial_x^{-1} \beta_1 + \frac{1}{2} \partial_x^{-1} \alpha_1 \beta_1 > 0\).

In the case of kernel \(R_0\) (105) of delta-functional type with \(p_k(\mu, \nabla_x) = A_k \delta(\mu - i \lambda_k), \) \(q_k(\lambda, \nabla_x) = B_k \delta(\lambda - i \lambda_k), \)

\(k = 1, \ldots, N,\) which satisfy to conditions (104) and (111), due to definitions (24), the functional parameters have the following form,

\[ \alpha_k = -2i A_k e^{F(\mu_k)}, \quad \beta_k = -2i B_k e^{-F(\mu_k)}, \]

where in accordance with (111) \(A_k = v_k \overline{A_k}, \) \(B_k = v_k \overline{B_k}\) and \(\mu_k, \lambda_k - \) some real parameters. Such kernel leads to corresponding multilinear soliton solutions.

In the simplest case of \(N = 1\) from (123), (124) due to (125) one obtains, under the condition \(A_1 B_1 \frac{\mu_{10} + \lambda_{10}}{\lambda_{10} (\mu_{10} - \lambda_{10})} = e^{\varphi_0} > 0,\) the exact nonsingular one line soliton solution of the 2DSK equation,

\[ u(x, y, t) = \frac{3(\mu_{10} - \lambda_{10})^2}{2 \cosh^2(\sqrt{2+\varphi_0})}, \]

where \(\varphi = F(\mu_{10}) - F(\lambda_{10}).\) This one line soliton solution was derived earlier in the paper of first author. For the case of \(N = 2\) one obtains the exact two line soliton solution of the 2DSK equation.

For case \(B\) (109) the kernel \(R_0,\) which satisfies conditions of reality (20) and reduction (17) or (111), due to (104), (106), (113) has the form,

\[ R_0(\mu, \nabla_x, \lambda, \nabla_x) = \sum_{k=1}^{2N} \int_\mathcal{C} f_k(\mu, \nabla_x) g_k(\lambda, \nabla_x) = \sum_{k=1}^{2N} \left[ v_k^{-1} \lambda q_k(\mu, \nabla_x) p_k(\lambda, \nabla_x) - v_k^{-1} \lambda q_k(-\nabla_x, -\mu) p_k(-\lambda, -\nabla_x) \right], \]

where \(v_k = v_{k0}\) - some real parameters. From (127) one choose convenient sets \(f\) and \(g\) of functions \(f_n, g_n, n = 1, \ldots, 2N,\)

\[ f := (f_1, \ldots, f_{2N}) = (p_1(\mu, \nabla_x), \ldots, p_N(\mu, \nabla_x); -v_1^{-1} p_1(-\nabla_x, -\mu), \ldots, -v_N^{-1} p_{N}(-\nabla_x, -\mu)), \]

\[ g := (g_1, \ldots, g_{2N}) = (v_1^{-1} \lambda p_1(\nabla_x, \lambda), \ldots, v_N^{-1} \lambda p_{N}(\nabla_x, \lambda); \lambda p_1(-\lambda, -\nabla_x), \ldots, \lambda p_{N}(-\lambda, -\nabla_x)). \]

Due to definitions (24) from (128), (129) one derive the following interrelations between different functional parameters,

\[ \beta_k := v_k^{-1} \int_\mathcal{C} \lambda q_k(\lambda, \nabla_x) e^{-F(\lambda)} d\lambda \wedge d\lambda = -i v_k^{-1} \alpha_k x, \quad k = 1, \ldots, N \]
New exact solutions of 2DKK and 2DSK equations

\[ \alpha_{k+N} := -v_k^{-1} \int_\mathcal{C} p_k(-\lambda, -\mu)e^{F(\mu)}d\mu \wedge d\overline{\mu} = v_k^{-1} \overline{\alpha}, \quad \beta_{k+N} := \int_\mathcal{C} \lambda p_k(-\lambda, -\overline{\lambda})e^{-F(\lambda)}d\lambda \wedge d\overline{\lambda} = i\alpha_{kz}, \quad (131) \]

where \( v_k = v_{k0} \) - some real parameters. So due to (130), (131) the sets of functional parameters have the following structure,

\[ (\alpha_1, \ldots, \alpha_{2N}) := (\alpha_1, \ldots, \alpha_N; v_1^{-1} \overline{\alpha}_1, \ldots, v_N^{-1} \overline{\alpha}_N), \quad (132) \]

\[ (\beta_1, \ldots, \beta_{2N}) := (-iv_1^{-1} \overline{\alpha}_{1z}, \ldots, -iv_N^{-1} \overline{\alpha}_{Nz}; i\alpha_{1z}, \ldots, i\alpha_{Nz}), \quad (133) \]

i.e. both sets express through \( N \) independent complex functional parameters \((\alpha_1, \ldots, \alpha_N)\).

General determinant formula (29) with matrix \( A \) (23) corresponding to kernel \( R_0 \) (127) of \( \overline{\partial} \)-problem (1) gives the class of exact solutions \( u \) with functional parameters of 2DSK equation (2). By construction, due to (152) and (133), these solutions depend on \( N \) complex functional parameters \((\alpha_1, \ldots, \alpha_N)\) given by (24).

In the simplest case \( N = 1 \), \((\alpha_1, \alpha_2) := (\alpha_1, v_1^{-1} \overline{\alpha}_1)\) and \((\beta_1, \beta_2) := (-iv_1^{-1} \overline{\alpha}_{1z}, i\alpha_{1z})\), the determinant of the matrix \( A \) due to (23) is given by expression (134).

\[ \det A = (1 + \frac{i}{4v_1} \partial_x^{-1}(\alpha_1 \overline{\alpha}_1 - \alpha_1 \overline{\alpha}_{1z}))^2 = \Delta^2. \quad (134) \]

The corresponding solution \( u \) is calculated with help of reconstruction formula (29) and due to (134) has the form:

\[ u = \frac{3}{2v_1 \Delta^2} \left[ i \Delta (\alpha_1 \overline{\alpha}_1 - \alpha_1 \overline{\alpha}_{1z}) + \frac{1}{4v_1} (\alpha_1 \overline{\alpha}_1 - \alpha_1 \overline{\alpha}_{1z})^2 \right]. \quad (135) \]

Due to expression (134) for \( \det A \) this solution is nonsingular for choices of functional parameter \( \alpha_1 \) satisfying to inequality, \( \frac{1}{4v_1} \partial_x^{-1}(\alpha_1 \overline{\alpha}_1 - \alpha_1 \overline{\alpha}_{1z}) > 0 \).

In the case of kernel \( R_0 \) (105) of delta-functional type with \( p_k(\mu, \overline{\mu}) = A_k \delta(\mu + \mu_k), k = 1, \ldots, N \), which satisfy to conditions (104) and (113), due to definitions from (24) functional parameters \( \alpha_k \) have the following form,

\[ \alpha_k = -2iA_k e^{F(\mu_k)}. \quad (136) \]

The kernels \( R_0 \) with such kind of functional parameters (136) lead to corresponding exact multiline soliton solutions.

In the simplest case of \( N = 1 \) from (124), (135) and due to (136), under condition \( \frac{A_12}{v_1} \frac{\mu_1}{\mu_1} = e^{\varphi_0} > 0 \), one obtains the exact nonsingular one line soliton solution of the 2DSK equation,

\[ u(x, y, t) = \frac{6\mu_1^2}{\cosh^2\left(\frac{\varphi}{\sqrt{2}}\right)}, \quad (137) \]

where \( \varphi = F(\mu_1) - F(\overline{\mu}_1) \). This one line soliton solution was derived earlier in paper of the first author. In the case of \( N = 2 \) one obtains the exact two line soliton solution of the 2DSK equation.

V. PERIODIC SOLUTIONS OF 2DKK AND 2DSK EQUATIONS

In this section there will be calculated also the periodic solutions of 2DKK (11) and 2DSK (2) equations as particular cases from corresponding solutions with functional parameters obtained in sections III and IV.

Periodic solutions of 2DKK equation. At first using only reduction condition (11) or (30) one calculates via general formulas (29), (23) (without using reality conditions), taking in to account (32) or (36) (condition I of reduction (16)), complex solution of 2DKK equation. This solution for the simplest case \( N = 1 \) in (57) has the form,

\[ u = \frac{3}{c_1} \frac{\alpha_1 \alpha_{1z}}{1 + \frac{1}{2c_1} \partial_x^{-1} \alpha_1^2} - \frac{1}{c_1} \alpha_1^4 \left(1 + \frac{1}{2c_1} \partial_x^{-1} \alpha_1^2\right)^2. \quad (138) \]

For delta-functional kernel \( R_0 \), with \( f_1 := A_1 \delta(\mu - \mu_1) \) one obtains \( \alpha_1 = -2iA_1 e^{F(\mu_1)} \), and from (138) it follows for \( u \) at this stage complex expression,

\[ u = \frac{-12A_1^2 \mu_1 e^{\varphi}}{(1 + \frac{4A_1^2}{\mu_1} e^{\varphi})^2}. \quad (139) \]
where \( 2F(\mu_1) = \phi, \ iA_{1I} = A^2 \). Using (139) one formulates the reality condition \((u = \bar{u})\),

\[
\frac{-12A^2\mu_1 e^{\phi}}{(1 + \frac{A^2}{\mu_1} e^{\phi})^2} = \frac{-12A^2\mu_1 e^{-\phi}}{(1 + \frac{A^2}{\mu_1} e^{-\phi})^2}.
\]

(140)

Under assumptions \( \mu_1 = \bar{\mu}_1 = \mu_{10} \) and \( A^2 = |\mu_1| e^{i\phi_0} \) (where \( \phi_0 \) - is arbitrary real constant) condition (140) fulfills, consequently the phase \( \phi = -\bar{\phi} = i\phi = 2i(\mu_1 x + \mu_1^* y + 9\mu_1^5 t) \) is pure imaginary. Doing by this way and imposing on (139) formulated restrictions one obtain singular periodic solutions of 2DKK equation,

\[
\mu_{10} > 0: \ u(x, y, t) = \frac{-3\mu_{10}^2}{\cos^2(\frac{\phi_0}{2})}; \quad \mu_{10} < 0: \ u(x, y, t) = \frac{-3\mu_{10}^2}{\sin^2(\frac{\phi_0}{2})}.
\]

(141)

Another periodic solutions of 2DKK equation will be derived by use condition II of reduction (68). One obtains directly from (29), due to (23), the non real solution \( u \neq \bar{u} \), which satisfies only to condition (68) but doesn’t satisfies to condition of reality. This solution for the simplest case one pair of terms \((N = 1)\) in the kernel (69) is derived by reconstruction formula (29) and due to (68), where determinant of matrix \( A \) has the form,

\[
det A = (1 + \frac{1}{2}\beta x^1 \alpha y^1) - \frac{1}{4}\beta^{-1} x^1 \beta^{-1} \beta^1.
\]

(142)

For the choice \( f_1 := A_1 \delta(\mu - \mu_1) \) and \( g_1 := B_1 \delta(\lambda - \lambda_1) \), due to definitions (24) \( \alpha_1 = -2iA_1 e^{F(\mu_1)} \) and \( \beta_1 = -2iB_1 e^{-F(\lambda_1)} \), for \( u \) one obtains \((a := iA_1 B_1)\) the expression,

\[
det A = 1 + \frac{4a}{\mu_1 - \lambda_1} \cos(\phi(x, y, t)) + \frac{a^2}{\mu_1 \lambda_1} (\mu_1 + \lambda_1)^2 e^{2\phi(x, y, t)},
\]

\[
u(x, y, t) = \frac{-12\alpha(\mu_1 - \lambda_1) e^{\phi}}{\det A} \left[ \det A + \frac{a(\mu_1 - \lambda_1) e^{\phi}}{\mu_1 \lambda_1} \right],
\]

(143)

where \( \phi = F(\mu_1) - F(\lambda_1) \).

The reality condition \((u = \bar{u})\) with requirement of imaginary phase \( \varphi = -\bar{\varphi} = i\phi \) leads to following conditions on the parameters:

\[
\mu_1 = \bar{\mu}_1 = \mu_{10}, \quad \lambda_1 = \bar{\lambda}_1 = \lambda_{10} = c\mu_{10}, \quad a = \pm\sqrt{c}\mu_{10} \left( \frac{1-c}{1+c} \right),
\]

(144)

where \( c \) is real parameter. When one imposes on (143) these restrictions one obtain the singular periodic solutions, for \( c > 0 \),

\[
u(x, y, t) = \pm 12\sqrt{c}\mu_{10} (1-c)^2 \frac{(1-c)^2}{(1+c)} \left( 2 \cos \phi \pm \frac{1+c}{\sqrt{|c|}} \right); \]

(145)

and for \( c < 0 \),

\[
u(x, y, t) = \pm 12\sqrt{|c|}\mu_{10} (1-|c|) \frac{(1-|c|)^2}{(1+|c|)} \cos \phi \pm \frac{1-|c|}{\sqrt{|c|}} \frac{\sqrt{|c|}}{1+|c|} \]

(146)

where \( \phi = \mu_{10}(1-c)x + \mu_{10}^3(1-c^3)y + 9\mu_{10}^5(1-c^5)t \).

Another possibility to satisfy the reality condition \((u = \bar{u})\) for \( u \) given by (143) with requirement of imaginary phase \( \varphi = -\bar{\varphi} = i\phi \) leads to another conditions on the parameters,

\[
\lambda_1 = \bar{\mu}_1, \quad a^2 = \pm|\mu_1|^2 \left( \frac{\mu_{1R}}{\mu_{1I}} \right)^2.
\]

(147)

When one imposes these restrictions on (143) one obtains the nonsingular (for arbitrary complex constant \( \mu_1 \)) periodic solutions; for the choice \( a = \pm|\mu_1| \left( \frac{\mu_{1R}}{\mu_{1I}} \right) \),

\[
u(x, y, t) = \pm 12|\mu_1| \left( \frac{\mu_{1R}}{\mu_{1I}} \right)^2 \cos \phi \pm \frac{\mu_{1R}}{\mu_{1I}} \frac{\mu_{1I}}{(\cos \phi \pm \frac{\mu_{1R}}{\mu_{1I}})^2},
\]

(148)
and for the choice \( a = \pm i|\mu_1| (\mu_{1R} / \mu_{1I}) \),
\[
 u(x, y, t) = \pm 12|\mu_1| \frac{\mu_{1R}^2}{\mu_{1I}} \sin \phi \mp \frac{\mu_{1I}}{\mu_{1R}} \phi_{x} \pm \frac{\mu_{1I}}{\mu_{1R}} \phi_{y}^2, \tag{149}
\]
where \( \phi = (\mu_{1R} + \overline{\mu_{1I}})x + (\mu_{1I}^2 + \overline{\mu_{1I}})y + (\mu_{1R}^2 + \overline{\mu_{1R}})t \).

**Periodic solutions of 2DSK equation.** At this point is shown how one calculates periodic solution \( u \) for 2DSK equation. The solution \( u \) which satisfies only condition of reduction \( (17) \) or \( (101) \) for case of \( N = 1 \) in \( (105) \) has the form \( (20) \) with \( \det A \) due to \( (29) \) and \( (101) \) given by expression,
\[
 \det A = \left( 1 - \frac{1}{4} \alpha_1 \partial_x^{-1} \beta_1 + \frac{1}{2} \partial_x^{-1} \alpha_1 \beta_1 \right)^2 = \Delta^2. \tag{150}
\]
For delta-functional \( f_1 := A_1 \delta(\mu - \mu_1) \) and \( g_1 := B_1 \delta(\lambda - \lambda_1) \) due to definitions \( (24) \) follow expressions for functional parameters, \( \alpha_1 = -2iA_1 e^{F(\mu_1)} \) and \( \beta_1 = -2iB_1 e^{-F(\lambda_1)} \), and for \( u \) one obtains \( (a := iA_1 B_1) \),
\[
 \det A = \left[ 1 + a \frac{\mu_{1R}^2 + \lambda_1}{\mu_{1I}} e^{\varphi(x, y, t)} \right]^2, \tag{151}
\]
where \( \varphi = F(\mu_1) - F(\lambda_1) \). The reality condition \( (u = \bar{u}) \) with requirement of imaginary phase \( \varphi = -\varphi = i\phi \) leads to following conditions on the parameters,
\[
 \mu_1 = \overline{\mu_{1I}} = \mu_{10}, \quad \lambda_1 = \lambda_{10}, \quad a = |a| e^{i\phi_a} = \left| \frac{\lambda_{10}(\mu_{10} - \lambda_{10})}{\mu_{10} + \lambda_{10}} \right| e^{i\phi_a}. \tag{152}
\]
When one imposes on \( (151) \) these restrictions one obtain the singular periodic solutions of 2DSK equation \( (2) \),
\[
 \frac{\lambda_{10}(\mu_{10} - \lambda_{10})}{\mu_{10} + \lambda_{10}} > 0: \quad u(x, y, t) = \frac{-3(\mu_{10} - \lambda_{10})^2}{2 \cos^2(\frac{\mu_{10} + \lambda_{10}}{2})}; \quad \frac{\lambda_{10}(\mu_{10} - \lambda_{10})}{\mu_{10} + \lambda_{10}} < 0: \quad u(x, y, t) = \frac{-3(\mu_{10} - \lambda_{10})^2}{2 \sin^2(\frac{\mu_{10} + \lambda_{10}}{2})}, \tag{153}
\]
where \( \phi = (\mu_{10} - \lambda_{10})x + (\mu_{10}^2 - \lambda_{10}^2)y + 9(\mu_{10}^5 - \lambda_{10}^5)t \) and \( \phi_a \) - is arbitrary real constant.

Another possibility to satisfy the reality condition \( (u = \bar{u}) \) with requirement of imaginary phase \( \varphi = -\varphi = i\phi \) leads to another conditions on the parameters,
\[
 \lambda_1 = -\overline{\lambda_{1I}}, \quad a = |a| e^{i\phi_a} = \left| \frac{\mu_{1R}}{\mu_{1I}} \right| |\mu_1| e^{i\phi_a}. \tag{154}
\]
By imposing on \( (151) \) these restrictions one obtain the another singular periodic solutions of 2DSK equation \( (2) \),
\[
 \frac{\mu_{1R}}{\mu_{1I}} > 0: \quad u(x, y, t) = \frac{-6\mu_{1R}^2}{\cos^2(\frac{\mu_{1R} + \phi_a}{2})}; \quad \frac{\mu_{1R}}{\mu_{1I}} < 0: \quad u(x, y, t) = \frac{-6\mu_{1R}^2}{\sin^2(\frac{\mu_{1R} + \phi_a}{2})}, \tag{155}
\]
where \( \phi = (\mu_{1R} + \overline{\mu_{1I}})x + (\mu_{1I}^2 + \overline{\mu_{1I}})y + (\mu_{1R}^2 + \overline{\mu_{1R}})t \) and \( \phi_a \) - is arbitrary real constant.

**VI. CONCLUSIONS AND ACKNOWLEDGEMENTS**

The integrable 2DKK \((11)\) and 2DSK \((12)\) equations differ only by coefficients \((25/2 \quad \text{for 2DKK, and 5 \quad \text{for 2DSK})}\) at nonlinear terms \( u_x u_{xx} \). These equations arise as special reductions \((10)\), \( v = \frac{5}{4} u_x \) for 2DKK equation and \( v = 0 \) - for 2DSK equation, of more general integrable nonlinear system of equations for the corresponding fields in general position. Such reductions lead to some nonlinear constraints \((16), (17)\) on the wave functions of corresponding linear auxiliary problems.

In the present paper by the use of \( \overline{\sigma} \)-dressing method of Zakharov and Manakov it is shown how nonlinear constraints on wave functions are satisfied. By this way new classes of solutions with functional parameters of considered equations were constructed, as particular cases some periodic solutions also were obtained.

It is interesting to note that in the paper the gauge-invariant formulation of integrable 2DKK-2DSK system of equations was given. It was shown that 2DKK and 2DSK equations admit formulation in terms of corresponding
gauge invariants and these equations are gauge nonequivalent to each other. Both of considered in the present paper
equations have different dispersionless partners, the investigation of these partners and calculation of theirs exact
solutions is also interesting task and will be considered elsewhere.

It was shown also that Nizhnik-Veselov-Novikov (NVN) and modified Nizhnik-Veselov-Novikov (mNVN) equations
also admit gauge-invariant formulation\(^1\), but in contrast to the case of 2DKK and 2DSK equations, NVN and mNVN
equations belong to gauge-equivalent classes of integrable nonlinear equations, their solutions can be connected by
Miura-type transformations.

This research work is supported: 1. by scientific Grant of fundamental researches of Novosibirsk State Technical
University (2010); 2. by the Grant (registration number 2.1.1/1958) of Ministry of Science and Education of Russia
Federation via analytical departmental special program "Development of potential of High School" (2009-2011); 3.
international research Grant RFFR # 09-01-92442-Kea (2009-2010).

1. Novikov, S.P., Zakharov, V.E., Manakov, S.V., and Pitaevskii, L.V., Soliton Theory: The inverse Scattering Method (Plenum, New
York, 1984).
2. Ablowitz, M.J., Clarkson, P.A., Solitons, Nonlinear Evolution Equations and Inverse Scattering, London Mathematical Society Lecture
Notes Series (Cambridge University Press, Cambridge 1991), Vol. 149.
3. Konopelchenko, B.G., Introduction to the multidimensional Inegible equations: The inverse Spectral Transform in 2+1 Dimensions
(Plenum, New York, 1992).
4. Konopelchenko, B.G., Solitons in Multidimensions: Inverse Spectral Transform Method (World Scientific, Singapore, 1993).
5. Manakov, S.V., "The inverse scattering transform for the time-dependent Schrödinger equation and Kadomtsev-Petviashvili equation," Physica D 3, 420 (1981).
6. Beals, R. and Coifman, R.R., "The \(\mathcal{F}\) approach to inverse scattering and nonlinear equations," Physica D 18, 242 (1986).
7. Zakharov, V.E. and Manakov, S.V., "Construction of higher-dimensional systems and of their solutions," (English. Russian original)
Func. Anal. and Appl. 19(2), 89-101 (1985); translation from Funk. Anal. Priloz. 19(2), 11-25 (1985).
8. Zakharov, V.E., Proceedings of the International Workshop "Plasma Theory and Nonlinear and Turbulent Processes in Physics"
(Naukova Dumka, Kiev, 1988), Vol.I, p.152.
9. Bogdanov, L.V. and Manakov, S.V., "The non-local \(\mathcal{F}\)-problem and (2+1)-dimensional soliton equations," J. Phys., A 21, L537 (1988).
10. Fokas, A.S. and Ablowitz, M.J., "The inverse scattering transform for multidimensional (2+1) problems in "Nonlinear Phenomena",
Proc. Oaxtepec, Mexico 1982, ed. K.B. Wolf, Lecture Notes in Physics, 189, 137-183 (1983).
11. Beals, R. and Coifman, R.R. "Linear Spectral problems, nonlinear equations, and \(\mathcal{F}\)-method," Inverse Problems 5, 87-130 (1989).
12. Zakharov, V.E., On the dressing method in "Inverse Methods in Action" ed. Sabatier, P.C. (Springer, Berlin, 1990), p. 602.
13. Konopelchenko, B.G. and Dubrovin, V.G., "Some new integrable nonlinear evolution equations in 2+1-dimensions," Phys. Lett. A 102, 15 (1984).
14. Date, E., Jimbo, M., Kashiwara, M. and Miwa, T., "Operator approach to the Kadomtsev-Petviashvili equation: Transformation groups
for soliton equation III," J.Phys. Soc. Japan, 50, 3806 (1981).
15. Zakharov, V.E. and Shabat, A.B., "A scheme for integrating the nonlinear equations of mathematical physics by the method of the
inverse scattering problem. I," (English. Russian original) Funkt. Anal. and Appl. 8, 226-235 (1974); translation from Funk. Anal.
Priloz. 8(3), 43-53 (1974).
16. Zakharov, V.E. and Shabat, A.B., "Integration of the nonlinear equations of mathematical physics by the method of the inverse scattering
transform," (English. Russian original) Funkt. Anal. and Appl. 13, 166 (1979); translation from Funkc. Anal. Priloz. 13(3), 13 (1979).
17. Dubrovin, V.G. and Liisitsyn, Ya.V., "The construction of exact solutions of two-dimensional generalizations of Sawada-Kotera and
Kaup-Kupershmidt integrable nonlinear equations via \(\mathcal{F}\)-dressing method", Phys. Lett., A 295, 198-207 (2002).
18. Xing-biao Hu, Dao-liu Wang, Xian-min Qian, "Soliton solutions and symmetries of the 2+1 dimensional Kaup-Kupershmidt equation
Phys. Lett A 262, 409 (1999).
19. Dubrovin, V.G. and Gramolin, A.V., "Gauge-invariant description of several (2 + 1)-dimensional integrable nonlinear evolution equations,") (English. Russian original) Teor. Math. Phys. 160(1), 905-916 (2009); translation from Teor. Mat. Fiz. 160(1), 35-48 (2009).