MARKOVIAN VERSUS NON-MARKOVIAN STOCHASTIC QUANTIZATION OF A COMPLEX-ACTION MODEL

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We analyze the Markovian and non-Markovian stochastic quantization methods for a complex action quantum mechanical model analog to a Maxwell-Chern-Simons electrodynamics in Weyl gauge. We show through analytical methods convergence to the correct equilibrium state for both methods. Introduction of a memory kernel generates a non-Markovian process which has the effect of slowing down oscillations that arise in the Langevin-time evolution toward equilibrium of complex action problems. This feature of non-Markovian stochastic quantization might be beneficial in large scale numerical simulations of complex action field theories on a lattice.

Keywords: Stochastic Quantization; Complex Actions; Topological Quantum Mechanics

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1. Introduction

Recent years have witnessed a vigorous revival of the method of stochastic quantization of systems with a complex action. The revival comes after drawbacks of the method, pointed out long ago related to lack of convergence or convergence to a wrong limit of solutions of the associated Langevin equations. The interest in stochastic quantization of complex actions is driven mostly by the pressing need of simulation techniques to study the phase diagram of quantum chromodynamics (QCD) at finite temperature and baryon chemical potential. The traditional Monte Carlo methods widely used in studies of the hadron spectrum, based on importance
sampling with a Boltzmann weight given in terms of the real, positive Euclidean action of QCD is inapplicable to problems with a baryon chemical potential because the action is complex. This difficulty is not exclusive to QCD, it also occurs in several other physics problems; notorious examples are problems of cold atoms and strongly-correlated electrons in condensed matter physics. The main difficulty with a complex-action Langevin simulation can be better posed in terms of the stationary solutions of the associated Fokker-Planck equation whereas in the case of a real action the Boltzmann weight of the Euclidean path integral can be shown to be given by the stationary solution of the Fokker-Planck equation, such a proof is still lacking for a complex action. Pragmatically, however, the correctness of a given complex-action Langevin simulation can be assessed to some extent with the use of a set of rather simple and general criteria that calculated observables must satisfy.

The recent renewed optimism with stochastic quantization of complex actions has grown from robust evidence that problems with instabilities and incorrect convergence of solutions of the Langevin field equations can be controlled by choosing a small enough Langevin step-size and also with the use of more elaborate algorithms, like of adaptive step-size and of higher order. A complex action in general introduces oscillatory behavior in the time evolution toward equilibrium of the solutions of the Langevin equation; depending on the problem, the oscillations become irregular and of high frequency. High-frequency oscillations are the main reason for the need of smaller step-sizes. With this in mind, in the present paper we advocate the use of non-Markovian stochastic quantization for complex actions. Non-Markovian stochastic quantization amounts to a modification of the Langevin equation by introducing a memory kernel and use of colored noise according to the fluctuation and dissipation theorem. A judiciously chosen memory kernel can soften considerably the oscillatory behavior induced by a complex action and hence larger step-sizes can be used to sample the time evolution. We illustrate this by making use of a simple quantum mechanical model with a complex action that is solvable analytically. Specifically, we employ a topological quantum mechanical model which is analogous to the three-dimensional topologically massive, Chern-Simons electrodynamics in the Weyl gauge.

Although our primary interest in the quantum mechanical model is non-Markovian stochastic quantization, we emphasize that topological actions find interesting applications in several situations of physical interest. For instance, coupling of Maxwell and Chern-Simons Lagrangians in $2 + 1$ dimensions yields a different form of gauge field mass generation, known in the literature as a topologically massive gauge theory. In addition, it has been argued that topological Chern-Simons fields may play an important role in the three-dimensional dynamics in planar condensed-matter settings. As discussed in Refs. such topological field configurations are present in models for the fractional quantum Hall effect which encompasses quasiparticles with magnetic fluxes attached to charged particles. More recently, models with Chern-Simons terms were also employed in the
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study of the pseudogap phase in High-$T_c$ superconductors.\textsuperscript{31} Further discussions on peculiar features on the quantization of topological field theories can be found in Ref.\textsuperscript{32}.

The organization of the paper is as follows. In section 2, we discuss the topological model we study in the present paper. In section 3, we present the Markovian stochastic quantization of the model and show that the associated Langevin equation converges to the correct limit. The non-Markovian stochastic quantization of the same model is discussed in section 4. Explicit numerical solutions are presented in section 5 for both Markovian and non-Markovian processes. Conclusions and perspectives are presented section 6. The paper also includes an Appendix, where we present details of some lengthy derivations.

2. The model

The topological model we consider describes the motion of a particle of mass $m$ and electrical charge $e$ in external electromagnetic fields. The external fields give rise to Lorentz forces and the motion of the particle is governed by the standard Lagrangian:\textsuperscript{25}

$$L = \frac{m}{2} \dot{q}^2 + \frac{e}{c} \dot{q} \cdot A(q) - eV(q), \quad (1)$$

where $q = (q^1, q^2)$ is the only dynamical variable of the system, and $A(q) = (A^1(q), A^2(q))$ and $V(q)$ are the external vector and scalar electromagnetic potentials; the magnetic and electric fields are given by $B = \nabla \times A$ and $E = -\nabla V$.

The model is exactly solvable, classically and quantum-mechanically, for a constant magnetic field $B$, $A^i = -\zeta^{ij} q^j B / 2$, and a quadratic scalar potential $V(q) = kq^2 / 2$ – summation over repeated indices is implied. In this case, Eq. (1) becomes

$$L = \frac{m}{2} \dot{q}^2 + \frac{eB}{2c} \dot{q} \times \dot{q} - \frac{ek}{2} q^2. \quad (2)$$

As mentioned in the Introduction, this Lagrangian is analogous to the Lagrangian density of three-dimensional, topologically massive electrodynamics in the $A^0 = 0$ gauge:

$$\mathcal{L} = \frac{1}{2} \dot{A}^2 + \frac{\mu}{2} \dot{A} \times A - \frac{1}{2} (\nabla \times A)^2. \quad (3)$$

Here, $A$ is the only dynamical variable of the problem. The formal correspondence between $L$ and $\mathcal{L}$ is such that the kinetic and potential terms $m\dot{q}^2 / 2$ and $-ekq^2 / 2$ are analogous to $\dot{A}^2 / 2$ and $-(\nabla \times A)^2 / 2$ respectively, and the term corresponding to the Lorentz force $eBq \times \dot{q} / 2c$ is analogous to the Chern-Simmons term $\mu \dot{A} \times A / 2$.

Henceforth, we set $c$ and $e$ to unity.

The Hamiltonian corresponding to the Lagrangian of Eq. (2) is given by

$$H = \frac{1}{2m} \left( p^i + \frac{B}{2} \epsilon^{ijk} q^j q^k \right) \left( p^i + \frac{B}{2} \epsilon^{ijk} q^j q^k \right) + \frac{k}{2} q^2 q^2 i, \quad (4)$$
with
\[ p^i(t) = \frac{\partial L}{\partial \dot{q}^i(t)} = m\dot{q}^i(t) - \frac{B}{2} \epsilon^{ij} q^j(t). \]  

The Hamiltonian can be brought to diagonal form:
\[ H = \frac{1}{2} \left( p_+^2 + \omega_+^2 q_+^2 \right) + \frac{1}{2} \left( p_-^2 + \omega_-^2 q_-^2 \right), \]

where \((p_\pm, q_\pm)\) are canonical variables given in terms of the original \((p^i, q^i)\) as
\[ p_\pm = \left( \frac{\omega_\pm}{2m \Omega} \right)^{1/2} p^1 \mp \left( \frac{2m \Omega \omega_\pm}{4m \Omega - \omega_\pm^2} \right)^{1/2} q^1, \]
\[ q_\pm = \left( \frac{2m \Omega}{\omega_\pm} \right)^{1/2} q^1 \mp \left( \frac{4m \Omega^2}{2m \Omega \omega_\pm} \right)^{1/2} p^2, \]

with the frequencies \(\omega_\pm\) given by
\[ \omega_\pm = \frac{\Omega \pm B \sqrt{m^2 + k m}}{2m}. \]

Path integral quantization of the model proceeds via the probability amplitude \(Z\) of finding the particle at position \(q\) at time \(t\), when one knows that it was located at point \(q_0\) at time \(t_0\) – for simplicity of presentation, we set \(q_0 = 0\) and \(t_0 = 0\). An analytical continuation of \(Z\) to imaginary time \(t \to -i \tau\) leads to the following path integral representation of \(Z\):
\[ Z = \int Dq(t) e^{-S[q]}, \]

where the action \(S[q]\) corresponding to the Lagrangian in Eq. (2) is complex and given by
\[ S[q] = \int dt \left[ \frac{m}{2} \dot{q}^i(t)\dot{q}^i(t) - B \epsilon^{ij} q^i(t)\dot{q}^j(t) + k \dot{q}^i(t)q^i(t) \right]. \]

Equivalently, using the the coordinates and velocities \((q_\pm, \dot{q}_\pm)\) corresponding to the canonical variables \((q_\pm, p_\pm)\) given in Eq. (7), the action is given by
\[ S[q_\pm] = \int dt \left[ \frac{1}{2} (\dot{q}_+^2(t) + \dot{q}_-^2(t)) + \frac{1}{2} (\omega_+^2 q_+^2(t) + \omega_-^2 q_-^2(t)) \right]. \]

From the correlation function
\[ \Delta^{ij}(t, t') = \frac{1}{Z[q]} \int Dq(t) q^i(t)q^j(t') e^{-S[q]}, \]

one can obtain the energy gap between the ground- and the first excited-state from the large-time \(|t - t'| \to \infty\) falloff of \(\Delta^{ij}(t, t')\). Specifically, for the present model:
\[ \lim_{|t-t'| \to \infty} \Delta^{\pm \pm}(t, t') \sim e^{-\Delta E_{\pm}|t-t'|}, \]
where $\Delta E_\pm$ is given by
\[
\Delta E_\pm = \omega_\pm = \sqrt{\frac{B^2}{4m^2} + \frac{k}{m}} \pm \frac{B}{2m}.
\] (15)

For the so-called reduced theory considered in Ref.\cite{25} for which $m = 0$, one can show that
\[
\Delta E_\pm = \pm \frac{k}{B}.
\] (16)

3. Markovian stochastic quantization

Markovian stochastic quantization (MSQ) is based on a Langevin equation of the form
\[
\frac{\partial}{\partial \tau} q^i(\tau, t) = -\frac{\delta S}{\delta q^i(\tau, t)} + \eta^i(\tau, t),
\] (17)

where time $\tau$ is a fictitious time variable, $S$ is the action given by Eq. (11), and $\eta^i(\tau, t)$ is postulated to satisfy
\[
\langle \eta^i(\tau, t) \rangle_\eta = 0, \quad \langle \eta^i(\tau, t) \eta^j(\tau', t') \rangle_\eta = 2\delta^{ij}\delta(\tau - \tau')\delta(t - t').
\] (18) (19)

Here, $\langle \cdots \rangle_\eta$ means ensemble average over noise realizations. Expectation values $\langle O[q(t)] \rangle$ of quantum mechanical operators $O[q(t)]$ are obtained as ensemble averages of the functions $O[q(\tau, t)]$ in the $\tau \to \infty$ limit. In particular, the correlation function $\Delta^{ij}(t, t')$ defined in Eq. (13) is obtained as
\[
\Delta^{ij}(t, t') = \lim_{\tau' \to \tau \to \infty} \langle q^i(\tau, t) q^j(\tau', t') \rangle_\eta,
\] (20)

where the $q(\tau, t)$ are solutions of the Langevin equation in Eq. (17).

Solutions of the Langevin equation in Eq. (17) can be obtained as follows. Since the equation is linear, it can be solved using Fourier transforms in $t$ for $q^i(\tau, t)$ and $\eta^i(\tau, t)$:
\[
\begin{pmatrix} q^i(\tau, t) \\ \eta^i(\tau, t) \end{pmatrix} = \int_{-\infty}^{+\infty} d\omega e^{i\omega t} \begin{pmatrix} q^i(\tau, \omega) \\ \eta^i(\tau, \omega) \end{pmatrix}.
\] (21)

Using these in Eqs. (17) and (19), one obtains
\[
\frac{\partial}{\partial \tau} q^i(\tau, t) = -\left[(m\omega^2 + k)\delta^{ij} + B\varepsilon^{ij}\omega\right] q^i(\tau, \omega) + \eta^i(\tau, \omega),
\] (22)

and
\[
\langle \eta^i(\tau, \omega) \eta^j(\tau', \omega') \rangle_\eta = 4\pi\delta^{ij}\delta(\tau - \tau')\delta(\omega + \omega').
\] (23)

It is instructive to represent the solution of Eq. (22) in terms of a retarded matrix-valued Green’s function with elements $g^{ij}(\tau, \omega)$:
\[
q^i(\tau, \omega) = \int_0^\tau d\tau' g^{ij}(\tau - \tau', \omega) \eta^j(\tau', \omega),
\] (24)
where we assumed \( q^i(t, \omega) = 0 \), with \( g^{ij}(\tau, \omega) \) obeying the differential equation:

\[
\frac{\partial}{\partial \tau} g^{ij}(\tau, \omega) = - \left[ (m\omega^2 + k) \delta^{ik} + B \varepsilon^{ik} \omega \right] g^{kj}(\tau, \omega) + \delta^{ij} \delta(\tau).
\]

The solution of this equation is:

\[
g^{ij}(\tau, \omega) = \theta(\tau) \left[ \delta^{ij} \cos(B\omega \tau) - \varepsilon^{ij} \sin(B\omega \tau) \right] e^{-\left( m\omega^2 + k \right) \tau}, \tag{25}
\]

where \( \theta(\tau) \) is the usual step function. Plainly, \( g^{ij}(\tau, \omega) \) is oscillatory in \( \tau \) because \( B \neq 0 \) - recall that \( B \neq 0 \) implies a complex Euclidean action, Eq. (11). The oscillatory behavior of the Langevin evolution is a generic feature of a complex action problem, and is the main cause of instabilities or of convergence to wrong limits in numerical integration procedures.

For \( B \) purely imaginary, the trigonometric functions in Eq. (25) lead to an exponentially growing factor \( e^{\left| B \right| \omega \tau} \). Also, for the purely topological theory, i.e. \( m = k = 0 \), one immediately sees that there will be no large-\( \tau \) limit for the solution \( q_i(\tau, \omega) \) and their correlation functions. In other words, the stochastic process described by the Langevin equation in Eq. (17) never approaches an equilibrium solution for the purely topological theory. These quantities can only be set to zero in the equilibrium results. Still, we are free to take one of the parameters \( m, k \) as zero while maintaining the other finite.

Using standard methods, one can calculate easily the two-point correlation function \( \langle q^i(\tau, \omega) q^j(\tau, \omega') \rangle_\eta \) in the large-\( \tau \) limit. Using Eqs. (23), (24) and (25), we obtain for the two-point correlation function

\[
\Delta^{ij}(\omega, \omega') = \lim_{\tau \to \infty} \langle q^i(\tau, \omega) q^j(\tau, \omega') \rangle_\eta = \frac{2\pi \delta(\omega + \omega')}{p^4 + \omega^2 B^2} \left( \delta^{ij} p^2 - \varepsilon^{ij} \omega B \right), \tag{26}
\]

where \( p^2 = m\omega^2 + k \). From this, for the purely topological theory one obtains

\[
\Delta^{ij}(\omega, \omega') = -\frac{2\pi \delta(\omega + \omega')}{\omega B} \varepsilon^{ij}. \tag{27}
\]

As it stands, the result in Eq. (26) indicates that the considered stochastic process converges and the natural question is that if the converged result is the correct one. The question can be answered by checking the asymptotic behavior for the inverse Fourier transform of the two-point correlation function. As remarked at the end of the previous section, the first energy gap can be extracted from the large relative time behavior of the two-point correlation function. It is easy to prove that Markovian stochastic quantization leads to the correct limit given in Eq. (14).

Since we are working in Euclidean space, Eq. (5) must be analytically continued to imaginary time. In Fourier space:

\[
p^i(\omega) = -m\omega q^i(\omega) - \frac{B}{2} \varepsilon^{ij} q^j(\omega), \tag{28}
\]

and hence

\[
q_{\pm}(\omega) = m \left( \frac{1}{2m\Omega \omega_{\pm}} \right)^{1/2} \left[ \omega_{\mp} q^1(\omega) \pm \omega q^2(\omega) \right]. \tag{29}
\]
With the help of Eq. (26), one obtains
\[
\Delta \pm \pm (\omega, \omega') = \lim_{\tau \to \infty} \langle q_\pm(\tau, \omega)q_\pm(\tau, \omega') \rangle_{\eta}
\]
\[
= \frac{2\pi \delta(\omega + \omega')}{2m\Omega\omega_\pm(\omega^2 + \omega_0^2)(\omega^2 + \omega_\pm^2)} \left[ (\omega_\pm^2 - \omega^2) p^2 + 2B\omega^2\omega_\mp \right],
\]
where we used
\[
p^4 + \omega^2 B^2 = m^2 \left[ \omega^4 + \omega^2 \left( \frac{B^2}{m^2} + \frac{2k}{m} \right) + \frac{k^2}{m^2} \right] = m^2 (\omega^2 + \omega_0^2) (\omega^2 + \omega_\pm^2).
\]
Performing the inverse Fourier transforms of \(q_\pm(\tau, \omega)\), one obtains
\[
\Delta \pm \pm (t, t') = \frac{1}{2m\Omega\omega_\pm(\omega^2 + \omega_0^2)} \left\{ \frac{e^{-\omega_\pm|t-t'|}}{2\omega_\pm} \left[ (\omega_0^2 + \omega_\pm^2)(-m\omega_\pm^2 + k) + 2B\omega_\pm^2 \omega_\mp \right] \right\}.
\]
Using now the results:
\[
2\omega_\pm^2 (-m\omega_\pm^2 + k) \pm 2B\omega_\pm^2 = \mp2B\omega_\mp^2 (\pm B/2m + \Omega) \pm 2B\omega_\mp^2 = 0,
\]
\[
\frac{1}{\omega_\pm} \left[ (\omega_0^2 + \omega_\pm^2)(-m\omega_\pm^2 + k) \mp 2B\omega_\pm^2 \omega_\mp \mp 2B\omega_\mp^2 \right] = \mp4B\Omega^2,
\]
\[
\omega_\pm^2 - \omega_0^2 = \frac{-2\Omega B}{m},
\]
Eq. (32) can be cast in a simpler form as
\[
\Delta \pm \pm (t, t') = \frac{e^{-\omega_\pm|t-t'|}}{2\omega_\pm},
\]
which agrees with the result from the path integral calculation, Eq. (14). We also note that for \(m = 0\), one obtains \(\omega_\pm \equiv \pm k/B\), which yields the first energy gaps in the reduced theory.\(^{25}\)

To conclude this Section, we mention that Eq. (17) can be considered as the high-friction (or overdamped) limit of appropriate phase-space equations.\(^{33,34}\) Ref. \(^{35}\) presents a recent discussion of problems with the continuum limit of second-order Langevin equations in numerical simulations of lattice field theories.

4. Non-Markovian stochastic quantization

In the present section we consider the non-Markovian stochastic quantization (NMSQ) of the model. NMSQ amounts to introduce a memory kernel \(M_\Lambda(\tau - \tau')\) in the Langevin equation as
\[
\frac{\partial}{\partial \tau} q^i(\tau, t) = - \int_0^\tau d\tau' M_\Lambda(\tau - \tau') \frac{\delta S}{\delta q^i(\tau', t)} + \eta^i(\tau, t),
\]
where \(\Lambda\) is a parameter that controls the memory decay, such that \(M_\Lambda(\tau - \tau') \to \delta(\tau - \tau')\) as \(\Lambda \to \infty\), recovering the Markovian Langevin equation of Eq. (17) in
this limit. In order to obtain the correct equilibrium distribution \( \exp(-S[q]) \), with the quadratic action \( S[q] \) given by Eq. (11), one must impose the colored-noise correlation:

\[
\langle \eta^i(\tau, t) \eta^j(\tau', t') \rangle_\eta = 2 \delta^{ij} M_A(\tau - \tau') \delta(t - t'),
\]

(38)

instead of the white-noise form of Eq. (19). An interesting situation with a non-Markovian approach was considered in Ref. [36]. However in such a reference the author is concerned with the physical consequences associated with a non-Markovian friction, whereas here the generalized stochastic process is described by regarding a non-Markovian driving term in the Langevin equation.

As previously, performing a Fourier transform in \( t \) for \( q_i \) and \( \eta_i \) one obtains:

\[
\frac{\partial}{\partial \tau} q_i(\tau, \omega) = -(p^2 \delta_{ij} + B \varepsilon_{ij} \omega) \int_0^\tau d\tau' M_A(\tau - \tau') q_j(\tau', \omega) + \eta_i(\tau, \omega),
\]

(39)

with

\[
\langle \eta_i(\tau, \omega) \eta_j(\tau', \omega') \rangle_\eta = 4 \pi \delta_{ij} M_A(|\tau - \tau'|) \delta(\omega + \omega'),
\]

(40)

and, of course, \( \langle \eta_i(\tau, \omega) \rangle = 0 \). The solution of this Langevin can be written as

\[
q_i(\tau, \omega) = \int d\tau' G_{ij}(\tau - \tau', \omega) \eta_j(\tau', \omega),
\]

(41)

where we assumed \( q_i(0, \omega) = 0 \) and \( G_{ij} \) is the retarded Green’s function obeying the differential equation

\[
\frac{\partial}{\partial \tau} G_{ij}(\tau, \omega) = -(p^2 \delta_{ik} + B \varepsilon_{ik} \omega) \int_0^\tau d\tau' M_A(\tau - \tau') G_{kj}(\tau', \omega) + \delta_{ij} \delta(\tau).
\]

(42)

This equation can be solved via the use of Laplace transformation. Formally, one can write the solution as \( G(\tau, \omega) = \Gamma(\tau, \omega) \theta(\tau) \), with the \( \Gamma \) matrix defined through the Laplace transform of its inverse:

\[
\Gamma_{ij}^{-1}(z, \omega) = z \delta_{ij} + D_{ij}^{-1}(\omega) M_A(z),
\]

(43)

with

\[
D_{ij}^{-1}(\omega) = \delta_{ij} p^2 + \varepsilon_{ij} \omega B,
\]

(44)

and \( M_A(z) \) is the Laplace transform of \( M_A(\tau) \). For an exponential kernel, \( M_A(z) \) is given explicitly in Eq. (A.2); for such a kernel, the \( \Gamma \) matrix can be inverted analytically. As outlined in the Appendix, after a rather lengthy calculation one can write \( \Gamma_{ij}(\tau, \omega) \) as

\[
\Gamma_{ij}(\tau, \omega) = \delta_{ij} I_+(\tau, \omega) - i \varepsilon_{ij} I_-(\tau, \omega),
\]

(45)

where

\[
I_{\pm}(\tau, \omega) = \frac{1}{2} [G_+(\tau, \omega) \pm G_-(\tau, \omega)],
\]

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with

\[ G_\pm (\tau, \omega) = \left[ \frac{1}{\beta_\pm} \sinh \left( \frac{\beta_\pm \Lambda \tau}{2} \right) + \cosh \left( \frac{\beta_\pm \Lambda \tau}{2} \right) \right] e^{-\Lambda \tau^2}, \]  

(47)

and

\[ \beta_\pm = a_+ \pm a_- \quad a_\pm = \pm \frac{1}{\sqrt{2}} \left[ \rho \pm \left( 1 - \frac{2p^2}{\Lambda} \right) \right]^{1/2}, \]

(48)

Close inspection of Eq. (47) reveals that convergence in the \( \tau \to \infty \) demands \( a_+ < 1 \), which implies the following constraint on the value of the memory parameter \( \Lambda \):

\[ \Lambda > \frac{B^2 \omega^2}{2(m \omega^2 + k) \to B^2}{2m}. \]  

(49)

This convergence criterium may be compared to the one found in Ref.[23] Note that, because \( \beta_\pm \) is complex, the non-Markovian evolution is also oscillatory. However, as will be discussed in the next section, for \( \Lambda \neq \infty \), the effective non-Markovian oscillation frequency can be significantly smaller that the corresponding Markovian frequency.

Next, we consider the two-point correlation function - we follow the derivation strategy developed in Ref. [22]. From Eqs. (40) and (41), we have

\[ \langle q_i(\tau, \omega) q_j(\tau', \omega') \rangle_\eta = 4\pi \delta(\omega + \omega') \Delta_{ij}(\tau, \omega; \tau' \omega'), \]  

(50)

where

\[ \Delta_{ij}(\tau, \omega; \tau' \omega') = \int_0^\tau d\tau_1 \int_0^{\tau'} d\tau_2 \Gamma_{im}(\tau - \tau_1, \omega) \Gamma_{mj}(\tau' - \tau_2, \omega) M(\tau_1 - \tau_2). \]  

(51)

Using double Laplace transformations of \( \Delta_{ij}(\tau, \omega; \tau' \omega') \), one obtains

\[ \Delta_{ij}(z, \omega; z', \omega') = \int_0^\infty d\tau \int_0^\infty d\tau' e^{-z\tau} e^{-z'\tau'} \Delta_{ij}(\tau, \omega; \tau' \omega') \]

\[ = \Gamma_{im}(z, \omega) \Gamma_{mj}(z', \omega) \left[ \frac{M(z) + M(z')}{z + z'} \right]. \]  

(52)

Now, using Eq. (48) to eliminate \( M(z) \) and \( M(z') \), one can write

\[ \Delta_{ij}(z, \omega; z', \omega') = \left[ \frac{\Gamma_{il}(z, \omega) + \Gamma_{il}(z', \omega)}{z + z'} - \Gamma_{im}(z, \omega) \Gamma_{mj}(z', \omega) \right] D_{ij}(\omega), \]  

(53)

Using double inverse Laplace transformations in this equation, leads to the non-Markovian two-point correlation function

\[ \langle q_i(\tau, \omega) q_j(\tau', \omega') \rangle_\eta = 4\pi \delta(\omega + \omega') [\Gamma(|\tau' - \tau|, \omega) - \Gamma(\tau, \omega) \Gamma(\tau', \omega)] \Delta_{ij}(\omega). \]  

(54)

Employing Eq. (45) for \( \Gamma \) in this expression, gives the complete and explicit solution for the non-Markovian two-point correlation function in Fourier space.
It is not difficult to show that the large-$\tau$ limit of the non-Markovian two-point correlation function is given by

$$\lim_{\tau \to \infty} \langle q_i(\tau, \omega) q_j(\tau, \omega') \rangle = \frac{2\pi \delta(\omega + \omega')}{p^4 + \omega^2 B^2} (\delta_{ij} p^2 - \omega B \epsilon_{ij}). \quad (55)$$

As expected physically, the asymptotic limit is the same as in the Markovian case. Differences arise at finite $\tau$. In particular, the memory kernel implies a slower convergence toward equilibrium, but with a less oscillatory Green’s function then the corresponding Markovian one, as will be shown by an explicit numerical calculation in the next section.

5. Evolution towards equilibrium - numerical results

In the present section we explore the qualitative differences between the evolution toward equilibrium of the Markovian and non-Markovian processes for the present complex-action problem. As argued previously, the introduction of a memory kernel can be helpful with the requirements of small-step Langevin times in a numerical simulation of complex-action Langevin equations. Specifically, we will show that the Green’s function of the non-Markovian process is less oscillatory in $\tau$ than the one corresponding to the Markovian process with the same model parameters.

We consider first Markovian evolution. The matrix-valued retarded Green’s function has elements $g_{ij}(\tau, \omega)$ given in Eq. (25). For our purposes, it is sufficient to consider just one of its entries, since $g_{11}(\tau, \omega) = g_{22}(\tau, \omega)$, $g_{12}(\tau, \omega)$ has the same qualitative behavior as $g_{11}(\tau, \omega)$, and $g_{12}(\tau, \omega) = -g_{21}(\tau, \omega)$. Therefore, let us focus on $g_{11}(\tau, \omega)$. In Fig. 1 we plot $g_{11}(\tau, \omega)$ for a specific value of $\omega$ and arbitrarily chosen values of $k$, $m$ and $B$ - the value of $B$ is intentionally chosen somewhat larger than other parameters in Planck units to highlight more clearly the oscillatory character of the Green’s function. The convergence of the Markovian process is clearly seen in the Fig. 1, as well as its oscillatory behavior. From Eq. (25), it should be clear that for non-zero values of $k$ the convergence to equilibrium is faster than for $k = 0$ case.

Next, using the same values of $k$, $m$ and $B$, we examine the non-Markovian retarded Green’s function, $G(\tau, \omega) = \Gamma(\tau, \omega) \theta(\tau)$, with the matrix $\Gamma$ given by Eq. (45). Similarly to the Markovian case, we have $G_{11}(\tau, \omega) = G_{22}(\tau, \omega)$, $G_{12}(\tau, \omega) = -G_{21}(\tau, \omega)$ and $G_{21}$ has the same qualitative behavior as $G_{11}$. Therefore let us consider $G_{11}(\tau, \omega)$; its $\tau$ dependence is illustrated in Figure 2. The figure clearly shows that the pattern of oscillations is much broader than the Markovian counterpart in Fig. 1. That is, the oscillations are of lower frequency. We mention also that similarly to the Markovian case, for non-zero values of $k$ we notice that the convergence to equilibrium is faster than the $k = 0$ case, even though this conclusion is not obvious for NMSQ.

An important feature of the non-Markovian process, at least in the context of the present model, is that the lowering of the oscillation frequency saturates for some value of the memory parameter $\Lambda$ - smaller values of $\Lambda$ do not decrease the frequency
of oscillations, they only retard more the evolution toward equilibrium. Therefore, there is a compromise between lowering of the oscillation frequency and time of equilibration that has to be verified case by case to benefit from a non-Markovian stochastic quantization in a real, large scale numerical simulation.

Although our results are for a specific example, it should be clear that the introduction of a memory kernel will have a similar effect in other situations. This is so because on general physical grounds, a memory kernel as introduced here has the effect of delaying equilibration and hence slowing down eventual oscillations in the time evolution - examples in other contexts can be found in Refs. 37, 38. The obvious consequence for large-scale numerical simulations is that the Langevin time evolution can be sampled with with larger step-sizes. Evidently, delayed equilibration has computational costs. However, such costs might be a price to be payed for smoother time evolution.

6. Conclusions and Perspectives

In the present paper we studied qualitative differences between Markovian and non-Markovian stochastic quantization in a simple model with a complex action - a topological quantum mechanical action which is analog to a Maxwell-Chern-Simons action in the Weyl gauge. Complex actions introduce oscillations in the Langevin
Fig. 2. Non-Markovian retarded Green’s function $G_{11}$ for the diffusion problem. The values used were the same as the previous figure with a memory parameter $\Lambda = 10000$ in Planck units.

time evolution toward equilibrium. Such oscillations can introduce difficulties in numerical simulations, as the requirement of short Langevin time-steps for achieving convergence to the correct equilibrium state. The introduction of a memory kernel in the Langevin equation has the effect of delaying equilibration which in turn slows down oscillations in the time evolution. The practical consequence for large-scale numerical simulations is that the Langevin time evolution can be sampled with larger step-sizes.

As we remarked in the previous section, although our results are for a specific example, the effect of softening oscillations is a generic physical feature of memory kernels and because of this one expects that non-Markovian stochastic quantization might be helpful for other, more complicated problems with complex actions.

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Appendix A. Calculation of the non-Markovian Green’s function

In this Appendix we outline the derivation of the inverse Laplace transform for the $\Gamma$ matrix. Our derivation is for a memory kernel $M(\tau)$ of exponential form:

$$M(\tau) = \frac{1}{2} \Lambda e^{-\Lambda |\tau|}, \quad (A.1)$$

whose Laplace transform is

$$M(z) = \int_0^\infty d\tau M(\tau) e^{-z\tau} = \frac{\Lambda}{2} \frac{1}{z + \Lambda}. \quad (A.2)$$

From Eq. (A.2), one has that $\Gamma_{ij}(z, \omega)$ can be written as

$$\Gamma_{ij}(z, \omega) = \delta_{ij} \left[ \left( \frac{z + p^2 M(z)}{z + p^2 M(z)^2 + \omega^2 B M^2(z)} \right) B M^2(\Lambda) \right]. \quad (A.3)$$

In order to obtain the inverse Laplace transform of this equation, one needs to find the zeros of the denominator. Expanding the denominator using the explicit form of the memory kernel of Eq. (A.2), one has to find the zeros of the quartic equation

$$z^4 + 2\Lambda z^3 + (\Lambda^2 + p^2 \Lambda) z^2 + p^2 \Lambda^2 z + (p^2 + B^2 \omega^2) \Lambda^2 / 4 = 0. \quad (A.4)$$

The four roots are given by

$$z_1 = -\frac{1}{2} \left[ \Lambda + (a_+ + ia_-) \right], \quad z_2 = -\frac{1}{2} \left[ \Lambda - (a_+ + ia_-) \right],$$

$$z_3 = -\frac{1}{2} \left[ \Lambda + (a_+ - ia_-) \right], \quad z_4 = -\frac{1}{2} \left[ \Lambda - (a_+ - ia_-) \right] \quad (A.5)$$

where the $a_\pm$ are given in Eq. (A.2). Obtaining the inverse Laplace transformation with a denominator as $(z - z_1)(z - z_2)(z - z_3)(z - z_4)$ is a straightforward, albeit tedious procedure. The final result for $\Gamma_{ij}(\tau, \omega)$ can be written in the form presented in Eqs. (45)-(48).

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