I. INTRODUCTION

Supersymmetry is a symmetry between bosons and fermions. In nature, the symmetry is broken, i.e. in hadron spectroscopy, spectra of baryons which are fermions do not always have corresponding spectra of mesons which are bosons. Similarly the ground state of the standard model consists of a boson and there does not exist a fermion partner.

Spectroscopies are defined in Hilbert space. In the formulation of Wess-Zumino (WZ) model\cite{1}, a conformal structure of $\alpha(x)$ at position $x$ and metrics $\eta_{\mu\nu}$ of Lagrangian for Majorana spinors\cite{2,5} are defined. The coupling constant $g$ in Lagrangian was chosen to be positive, or interaction between Majorana spinors are repulsive and two spinors cannot have a zero distance, and the fields are conformal.

Magnetic monopoles are solitons in 3D space, which are confined by superconductor produced by Higgs field\cite{4,5,10}. ’t Hooft discussed open string theory by transforming magnetic monopoles which can be regarded as 3 dimensional solitons to quarks, and possibility of describing hadrons. We do not consider solitons in 3D space in this review, but explain the string theory and study propagation of solitons in 2D space that appear by nonlinear interactions\cite{5,6}.

Procedures to keep conformality by adding extra Lagrangians are discussed in the paper\cite{5}. Magnetic monopoles of $SU(2) \times U(1)$ gauge group of George-Glashaw\cite{11} were derived by ’t Hooft\cite{6}. Instantons in 3D non-abelian gauge theory was found to contain a monopole solution by Polyakov\cite{12}. Reviews of monopole solutions are given in\cite{13,14} as examples.

In an application of gauge theory to four dimensional topology, Donaldson\cite{16} proved an existence of a smooth ball of 5D with only finite number of singularities. The $r$'th homotopy group of n-sphere $\pi_r(S^n)$ was systematically studied by Rohlin, whose theory is summarized in\cite{67}, and Donaldson extended Rohlin’s theory. For homotopy groups on Riemann surface, please refer to\cite{17}.

The super gauge transformation that links fermions and bosons defined by WZ in space 3-dimension, 1 time dimensional system was applied to conformal quantum mechanics of space 1-dimension, 1 time dimensional system by de Alfaro, Fubini and Furlan (dAFF)\cite{20}. They defined the Hailtonian operator $H$, dilation generator $D$ and conformal generator $K$, that satisfy

$$[H, K] = -\sqrt{D} H, [K, D] = -\sqrt{D} K, [H, K] = 2\sqrt{D}.$$ 

In dAFF, 4 components of gauge vector $A_m$ is gauge fixed to the light-cone $A_+ = 0$, and anti-selfdual field $F_{\mu\nu}^\ast$ satisfies $F_{\mu\nu}^\ast = - F_{\mu\nu}^\ast$, where * means Hodge dual\cite{11,15,19}.

Superconformal quantum mechanics in de Sitter space was formulated by Fubini and Rabinovici\cite{21}. de Sitter space is the solution of Einstein field equation, whose
coordinates satisfy
\[ \sum_{i=1}^{4} (x^i)^2 - (x^0)^2 = R^2 = \frac{3}{G \lambda} \]
where \( \lambda \) is a cosmological constant.

Maldacena conjectured that our 4 dimensional world can be represented by the conformal field theory (CFT) projected on the boundary of \( 4 + 1 \) dimensional anti de Sitter (AdS) space, and called the principle as AdS/CFT correspondence. Superconformal quantum mechanics in AdS space was formulated by de Téramond, Brodsky and their collaborators using 5D holographic mapping of physical space to AdS space. For the review [24] is helpful. They considered light cone gauge, and the light front holographic QCD, the Dirac’s relativistic treatment [34].

When one extends the metrics of the Hilbert space from 4D Minkowsky space to 5D ADS space, and assumes the holographic principle, one can show relations between baryon spectroscopies and meson spectroscopies [24], although the mass of lightest meson pion remains massless.

A Higgs boson in the standard model, whose supersymmetric partner which is called higgsino is not observed experimentally. However, Higgs boson interacts with fermions and cause them to be massive, and it decays into two photons, vector boson pairs and \( b\bar{b} \) pairs [25, 26]. Decay modes of a heavior Higgs boson is also proposed [27].

Dirac stated that in use of mathematics of transformations of relativity and quantum mechanics, there are two ways, i) symbolic methods that deals with invariants and ii) the methods of coordinates or representations that deals with numbers corresponding to these quantities, expressed by real \( \mathbb{R} \) numbers. He has established quantum electrodynamics (QED) based on the first method using a complex number \( \mathbb{C} \) defining the \( U(1) \) gauge transformation as an aid to practical calculations.

Extension of standard model in \( \mathbb{R}^{1,1} \times \mathbb{R}^3 \) space-time in the framework of noncommutative geometry was proposed by Connes and his collaborators. Noncommutative geometry is based on K-theory in which real numbers, complex numbers and quaternions play their roles.

Supersymmetry plays a role in fluctuation of fermionic systems (instantons) near the ground state via occurrence of bosonic soliton waves. When the algebra \( \wedge(\mathbb{R}) \) is one dimensional vector space normal to a 2-dimensional plane in hysteretic solids, a similar analysis of spectroscopy of sound wave (phonon) is possible by projecting dynamics to 2 dimensional projected quaternion space, and by taking the time parameter \( \tau = t \pm z/v \) where \( v \) is the sound velocity in a material.

Hysteretic effects in phonon systems and instanton effects are compared via the holonomy theory.

A Phonon can be regarded as a soliton which is a boson that propagates in fermionic media. It does not have supersymmetric partner, but its propagation depends on even or odd numbers of fermions that it interacts.

In the following subsections, WZ model, dAFF model, Witten’s model, anti self-dual Yang Mills model, Penrose’s model, dynamical models using quaternions are reviewed as a preparation of hadron spectroscopy in 5D projective spaces and soliton dynamics in 2D projective quaternion space, both related to the supersymmetry.

### A. Wess-Zumino Model and supergauge transformation

Wess and Zumino [1] defined supergauge transformation in four dimension as gauge transformation whose commutators turns out to be a combination of a conformal transformation and a \( \gamma_5 \) transformation.

Consider Majorana spinors \( \alpha(x) \) that satisfy
\[ (\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu - \frac{1}{2} \eta_{\mu\nu} \gamma^5 \partial_\lambda) \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right) = 0, \]

where
\[ \gamma^0 = \gamma_0 = \left( \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right), \quad \gamma^k = -\gamma_k = \left( \begin{array}{cc} 0 & -\sigma_k \\ \sigma_k & 0 \end{array} \right), \]

where \( k = 1, 2, 3 \) and
\[ \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I, \]
\[ \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma_5 = \left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right), \]
\[ \sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{array} \right), \]
\[ \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). \]

The infinitesimal parameters \( \xi_\mu \) satisfy
\[ \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{1}{2} \eta_{\mu\nu} \gamma^5 \partial_\lambda. \]

Then it follows that
\[ \xi_\mu = 2\sqrt{-1} \alpha_1 \gamma_\mu \alpha_2, \]
and by multiplying \( \gamma^\mu \) to (2), one obtains
\[ \partial_\mu \alpha = \frac{1}{4} \gamma_\mu \gamma^5 \partial_\lambda \alpha. \]

It was shown that \( \alpha \) can be expressed in the form
\[ \alpha = \left( \begin{array}{c} \alpha_0^{(0)} \\ \alpha_0^{(1)} \\ \alpha_2^{(0)} \\ \alpha_2^{(1)} \end{array} \right), \]
and
\[ \xi_{\mu} = c_{\mu} + \omega_{\mu\nu} x^\nu + \epsilon x_{\mu} + a_{\mu} x^2 - 2x_{\mu} \alpha \cdot x. \] (8)

The metric tensor becomes
\[ \eta = 4\sqrt{-1}(\alpha_1^{(1)} \gamma\alpha_2^{(0)} - \alpha_2^{(1)} \gamma\alpha_1^{(0)}), \] (9)
which mixes \( \bar{\alpha} \psi \) and \( \alpha \gamma\psi \).

The supergauge transformation was applied to spin systems with superfield \( \phi(t, \theta, \bar{\theta}) \) where \( \theta, \bar{\theta} \) are totally anti-commuting quantities and \( t \in \mathbb{R} \) by Nicolai [32].
\[ \phi(t, \theta, \bar{\theta}) = A(t) + \theta \psi(t) + \bar{\psi}(t) \theta + \bar{\theta} \bar{\psi}(t), \] (10)
where \( A \) is a fermionic function that satisfies \( A^* = A \), \( \psi_j \psi_k, \psi_i \psi_j \psi_k \) etc. are fermi fields, \( F(t) \) is a bosonic function that satisfies \( F^* = F \). Similar model was constructed also by Polyakov [14].

The Hamiltonian \( H = \{Q, Q^\dagger\} \) where \( Q = \sum_{j \in G} \sum_{n} a_j^{(1)} Q_j + i n \cdot a_j \) is a constant with dimension of length. Al-

A review of super gauge transformation is also given by Green et al [18, 19].

B. de Alfaro-Fubini-Furlan model

de Alfaro, Fubini and Furlan (dAFF) [20] studied a field theory in one over-all time dimension, invariant under the full conformal group, whose Lagrangian of a physical operator \( Q(t) \) with a coupling constant \( g > 0 \) is given by
\[ L = \frac{1}{2}(\dot{Q}^2 - \frac{g}{Q^2}). \] (12)

They considered fundamental operators: 1) \( H = \frac{1}{2} x^2 + \frac{g}{Q^2} \), corresponding to hamiltonian [34, 53], 2) \( D \) corre-

sponding to dilatation operation, and 3) \( K \) correspon-

ding to conformal transformation. They constructed \( O(2,1) \) conformal group of non-compact rotations
\[ R = \frac{1}{2}(1 - aK + aH), \quad S = \frac{1}{2}(1 - aK - aH), \] (13)
where \( a \) is a constant with dimension of length. Al-

though the lowest eigenstate \( H \) is not normalizable due to infrared divergence, eigenstates of \( R \) are meaningful. Combinations
\[ G = uH + vD + wK \] (14)
satisfies the evolution equation
\[ \frac{\partial G}{\partial t} = \sqrt{-1}[H, G] = 0, \] (15)
and rotations become compact when \( \Delta = v^2 - 4u w > 0 \).

C. Witten’s model and Fubini-Rabinovici model in de Sitter space

In nature supersymmetry is broken, and constraints on supersymmetry breaking in the Yang-Mills field [33] was discussed by Witten [30, 31]. In zero momentum space, supersymmetry charge \( Q_1, \cdots, Q_k \) are chosen to satisfy
\[ Q_1^2 = Q_2^2 = \cdots = Q_k^2 = H, \]
\[ Q_i Q_j + Q_j Q_i = 0, \text{ for } i \neq j. \] (16)

Bosonic states are defined by
\[ exp(2\pi\sqrt{-1}J_z)|b\rangle = |b\rangle \] (17)
and fermionic states are defined by
\[ exp(2\pi\sqrt{-1}J_z)|f\rangle = |f\rangle, \] (18)
where \( J_z \) is a rotation operator, and the operator
\[ (-1)^F = exp(2\pi\sqrt{-1}J_z) \] (19)
distinguishes fermions with half integer \( J_z \) and bosons with integer \( J_z \). For a particle moving in one dimension, he considered supersymmetric charge
\[ Q_1 = \frac{1}{2}(\sigma_1 p + \sigma_2 W(x)), \]
\[ Q_2 = \frac{1}{2}(\sigma_1 p - \sigma_2 W(x)), \] (20)
where \( W(x) \) is an arbitraly function.

The number of zero energy states of bosons \( n_{B}^{E=0} \) and that of fermions \( n_{F}^{E=0} \) have the relation
\[ Tr(-1)^F = n_{B}^{E=0} - n_{F}^{E=0}, \]
and when \( n_{B}^{E=0} = n_{F}^{E=0} = 0 \), there are two possibilities, 1) \( n_{B}^{E=0} + n_{F}^{E=0} = 0 \), supersymmetry is broken, 2) \( n_{B}^{E=0} = n_{F}^{E=0} \) are equal but non zero, supersymmetry is unbroken.

When \( n_{B}^{E=0} - n_{F}^{E=0} \neq 0 \), supersymmetry is not spontaneously broken. Spontaneous symmetry breaking occurs in the Higgs mechanism.

Fubini and Rabinovici [21] extended the operator \( Q(t) \) as
\[ Q = \sum_{\alpha=1}^{D} \psi_\alpha^\dagger (-\sqrt{-1}p_\alpha + \frac{dW}{dx_\alpha}), \]
\[ Q^\dagger = \sum_{\alpha=1}^{D} \psi_\alpha (\sqrt{-1}p_\alpha + \frac{dW}{dx_\alpha}), \] (21)
where \( W(x) \) is the superpotential and \( p_\alpha = -\sqrt{-1}(\partial/\partial x_\alpha) \). The \( N = 1 \) supersymmetry was extended to \( N = 2 \) de Sitter supersymmetry.
The algebra of dAFF was extended by defining
\[ G = uH + vD + wK, \quad \Delta = v^2 - 4uw < 0, \]
and incorporating Witten’s algebra
\[ \frac{1}{2}(Q, Q^\dagger) = H, \quad \{Q, Q\} = \{Q^+, Q^\dagger\} = 0. \]
Here
\[ Q = \psi^\dagger(-\sqrt{-1}p + \frac{dW}{dx}), \quad Q^+ = \psi(\sqrt{-1}p + \frac{dW}{dx}) \]
and super potential \( W(x) = \frac{1}{2} f \log x^2 \). An extension to spacially many dimensions and system with number of fermions and bosons doubled \( \mathcal{N} = 2 \) and transitions from strongly coupled vacuum to weakly coupled monopole condensates follows the works of Bogomol’nyi, Prasad and Sommerfield were presented by Seiberg and Witten.

D. Anti self-dual Yang-Mills (ASDYM) field model

In the anti self-dual Yang-Mills (ASDYM) equation framework, one can construct integrable \( SU(2) \) chiral model, including the Higgs field \( \lambda \) and a vector field \( A_\mu \)
\[
\hat{\Phi} = \frac{\Phi}{|\Phi|}, \\
A_\mu^a = -\epsilon^{abc} \partial_\mu \hat{\Phi} b c + k_i \Phi^a, \tag{22}
\]
for some vector \( k_i \). The magnetic field is \( B_k = i \epsilon_{ijk} F_{ij}^a \Phi^a \).

In Minkowski space defined by \( ds^2 = -dt^2 + dx^2 + dy^2 \), the Lagrangian density
\[
\mathcal{L} = \frac{1}{2} Tr(F_{\mu
u} F^{\mu
u}) - Tr(\partial_\mu \Phi D^\mu \Phi), \\
F_\mu^a = \partial_\mu A^a - \partial_\mu A^a - \epsilon_{abc} A^b A^c_i, \\
\epsilon^{abc} \partial_\mu \Phi b c - (\epsilon^{apr} \partial_\mu \Phi a d \Phi^d) \Phi^a, \\
D_\mu \Phi = F_{\mu t}, \quad D_Y \Phi = F_{\mu x}, \quad D_t \Phi = F_{\mu y}. \tag{23}
\]
Solutions of the equation
\[ \partial_\mu K_\nu - \partial_\nu K_\mu = \frac{1}{4} \eta_{\mu \nu} \partial_\rho K^\rho \tag{24} \]
is called Killing vectors and one parametrizes \( K = \partial / \partial \tau \) and transform real coordinates \((t, x, y, \tau)\) to
\[ z = \frac{x - \tau}{\sqrt{2}}, \quad \bar{z} = \frac{x + \tau}{\sqrt{2}}, \quad w = \frac{t + y}{\sqrt{2}}, \quad \bar{w} = \frac{t - y}{\sqrt{2}}. \]
Using the Hodge operator \( * \) on \( \mathbb{R}^{2,1} \), the Lax pair equation can be written as
\[ D\Phi = sF. \tag{26} \]

For \( GL(2, \mathbb{R}) \) valued functions \( A_x, A_y \) which depend on \((x, y)\), one defines a two component vector \( v \), which satisfies
\[ D_x v = \partial_x v + A_x v = 0, \quad D_y v = \partial_y v + A_y v = 0, \tag{27} \]
and
\[ \partial_y \partial_x v - \partial_x \partial_y v = -\partial_y (A_x v) + \partial_x (A_y v) = (\partial_x A_y - \partial_y A_x + [A_x, A_y]) v = 0. \]
The above equation is equivalent to
\[ F_{xy} = [D_x, D_y] = 0 \tag{28} \]
When one considers with a complex projective parameter \( \lambda \in \mathbb{CP}^1 \),
\[ L = D_{\bar{z}} - \lambda D_w, \quad M = D_{\bar{w}} - \lambda D_z, \tag{29} \]
ASDYM becomes
\[ [L, M] = F_{\bar{w}z} - \lambda (F_{w\bar{z}} - F_{\bar{z}z}) + \lambda^2 F_{wz} = 0. \tag{30} \]
The Lax pair equations for \( GL(n, \mathbb{C}) \) valued functions \( \Psi(w, z, \bar{w}, \bar{z}, \lambda) \), where \( w, z, \bar{w}, \bar{z} \in \mathbb{R}, \lambda \in \mathbb{C} \), satisfy
\[ L_0 \Psi = (D_y + D_t - \lambda (D_{\bar{z}} + \Phi)) \Psi = 0, \]
\[ L_1 \Psi = (D_x - \Phi - \lambda (D_t - D_y)) \Psi = 0, \tag{31} \]
and
\[ \Psi(x^\mu, \bar{\lambda})^* \Psi(x^\mu, \lambda) = 1. \tag{32} \]
The function \( \Psi \) can be chosen to satisfy gauge transformations \( J : \mathbb{R}^3 \rightarrow U(n) \) such that
\[ A_t = A_y = \frac{1}{2} J^{-1} (J_t + J_y), \quad A_x = -\Phi = \frac{1}{2} J^{-1} J_x, \tag{33} \]
and the equation \( (35) \) becomes
\[ (J^{-1} J_t)_x - (J^{-1} J_x)_t - (J^{-1} J_y)_y - [J^{-1} J_t, J^{-1} J_y] = 0. \tag{34} \]
The last term is called the Wess-Zumino-Witten (WZW) term. The Lax pair have soliton solutions[61].

E. Twisters and conformal model

In order to combine the electromagnetic force described by Maxwell equation and the gravity described by Einstein equation, Penrose proposed a twistor method using a function \( f \) of \( \mathbb{R}^3 \rightarrow \mathbb{R} \)
\[
v(x, y, \zeta, t) = \frac{1}{2\pi \sqrt{1}} \int_{\Gamma \subset \mathbb{CP}^1} f(-(x + \sqrt{-1}y) + \lambda(t - \zeta), \quad (t + \zeta) + \lambda(-x + \sqrt{-1}y), \lambda) d\lambda.
\]
The integral is along a curve on a manifold \( \mathbb{CP}^1 \), which is the quotient of \( \mathbb{C}^2 \times \mathbb{C}^2 \) by the equivalence relation
\[
(Z^0, Z^1) \sim (cZ^0, cZ^1)
\]
for \( c \in \mathbb{C} \), and \( \lambda = Z^1/Z^0 \), \( \bar{\lambda} = 1/\lambda \).

Mapping between local coordinate between \( S^2 \) and \( \mathbb{CP}^1 \) is given by
\[
(u_1, u_2, u_3) \rightarrow \{1 - u_3, u_1 + \sqrt{-1}u_2\},
\]
and by patching two \( S^2 \), one obtains the Riemann sphere \( S^3 \).

On the manifold \( S^3 \times S^1 \), a Lie group \( GL(2, \mathbb{C}) \) operates.

F. Dynamics on \( S^3 \) manifold and quaternions \( \mathbf{H} \)

In quantum chromo dynamics (QCD), there is axial gauge transformation \( U(1)_A \), whose symmetry is spontaneously broken. For practical calculations, numbers other than complex numbers could be useful. In 1877, Frobenius showed that there are only three isomorphically distinct real finite dimensional associative division algebra \( \mathbb{R}, \mathbb{C} \) and quaternion \( \mathbf{H} \). Quaternions have bases
\[
\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}
\]
\[
\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.
\]

Real quaternion fields, which preserve time-reversal symmetry was extensively studied in \([102]\). The authors defined time in real algebra \( (\mathcal{R}) \) and quaternion algebra \( (\mathcal{Q}) \). In \( \mathcal{R} \), time reversal corresponds to \( u \rightarrow \bar{u} \), while in \( \mathcal{Q} \), \( \sigma(u) = Q \cdot \bar{u} \cdot Q^{-1} \), where
\[
Q = \begin{pmatrix}
0 & -1 & & & \\
1 & 0 & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & 0 & -1 \\
& & & & & 1 & 0
\end{pmatrix}
\]
is the symplectic matrix \([203, 204]\). The symplectic group \( Sp(n, \mathbb{R}) \) is isomorphic to \( GL(2n, \mathbb{C}) \).

For \( \epsilon_n \), the unit matrix of degree \( n \), let matrix \( J \) be
\[
J = \begin{pmatrix} 0 & \epsilon_n \\ -\epsilon_n & 0 \end{pmatrix},
\]
matrices \( \sigma \) that satisfy \( \sigma J \sigma = J \) are \( Sp(n, \mathbb{C}) \) matrices.

The quaternions \( \mathbf{H} \) has orthogonal bases of the Hilbert space consisting of eigens elements \( \mathbf{i}, \mathbf{j}, \mathbf{j}, \mathbf{k} \) which satisfy
\[
ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j
\]
In the \( SL(2, \mathbb{C}) \) representation
\[
\mathbf{H} = a\mathbf{I} + bi + cj + dk
\]
\[
= \begin{pmatrix} a + \sqrt{-1}d & -c + \sqrt{-1}b \\ c + \sqrt{-1}b & a - \sqrt{-1}d \end{pmatrix},
\]
where \( a, b, c, d \in \mathbb{R} \) and \( \mathbf{H}^* = a\mathbf{I} - bi - cj - dk \).

Topology of manifolds on which functions of these numbers are defined turned out to be important to understanding physical phenomena \([29, 30]\).

G. Supersymmetry in soliton-fermion interactions

Witten’s formulation of supersymmetry was applied to non-linear \( \sigma \) model and the effect of instantons in ferromagnetic and anti-ferromagnetic materials were studied. A good review on supersymmetry in particle physics and introduction to the super gauge transformation of Wess and Zumino \([1]\) are written in the book of Labelle \([3]\).

Electrons in 2D strong magnetic field show Quantum Hall Effects (QHE). In calculation of conductivity tensors \( \sigma_{xy} \) and density of states, supersymmetric analyses was performed. Supersymmetry in disorder and chaos is discussed in \([94]\).

In fermionic materials, non-linear interactions sometimes induce solitons which are observed as phonons. Phonons are bosons but do not have supersymmetric partners. In 1 dimensional antiferromagnetic Heisenberg chains, Haldane \([91]\) showed that spectroscopy of solitons in integer spin system and in half-integer spin system are distinguished.

In order to study these physical processes, usual methods is to define state space of fermions and bosons using 2nd quantization \([31]\). State space of bosons is described by
\[
\mathcal{H}_B = \sum_{n=0}^{\infty} \oplus \mathcal{H}^n_B
\]
and vectors in the space is described by creation operators \( a^*(x_i) \) as
\[
\Phi(a^*) = \sum_{n} \frac{1}{(n!)^{1/2}} \int K_n(x_1, \cdots, x_n) a^*(x_1) \cdots a^*(x_n) d^n x,
\]
where \( K_n(x_1, \cdots, x_n) \) are symmetric functions. The annihilation operator \( a(x_j) \) is complex conjugate of \( a^*(x_i) \).
State space of fermions is described by $\mathcal{H}_F$ is similar to $\Phi(a^*)$, but creation operator and annihilation operator are generators of Grassmann algebra
\[ \{a(x), a(x')\} = \{a^*(x), a^*(x')\} = \{a^*(x), a(x')\} = 0. \]
Vectors in the space is expressed as
\[ \Phi_{n_1, n_2, \ldots, n_k, \ldots} = \frac{1}{(n_1! n_2! \cdots)^{1/2}} (a_1^*)^{n_1} (a_2^*)^{n_2} \cdots. \]
The symmetry between fermions which have a half-integer spin quantum number $s$ and an integer angular momentum number $L$ forming a half-integer total spin quantum number $J$, and bosons which have an integer $s$ and an integer $L$ forming an integer $J$ quantum numbers is the supersymmetry. Berezin treated bosons and fermions in Hilbert space not using complex numbers is the supersymmetry. Berezin treated bosons and fermions in Hilbert space not using complex numbers, but generators of Grassmann algebra $J$ and an integer angular momentum number $L$ are generators of Grassmann algebra $J$ and an integer $L$ forming a half-integer total spin quantum number $J$. With each Grassmann algebra, there is closely connected $2n$ dimensional Clifford algebra, whose operators $P_i = \frac{1}{\sqrt{-1}}(\tilde{x}_k - (\partial/\partial x_k))$, $Q_i = \tilde{x}_k + (\partial/\partial x_k)$ satisfy
\[ \{P_i, Q_j\} = 0, \quad \{P_i, P_j\} = \{Q_i, Q_j\} = 2\delta_{ij}. \]
The mapping from $\mathbb{R}^{1,1} \rightarrow \mathcal{H}$ given by
\[ [(x_1 e_1 + x_2 e_2)] \rightarrow [x_1 i + x_2 j] = \begin{pmatrix} x_1 & 0 \\ 0 & -x_1 + x_2 \end{pmatrix} \]
is a Clifford algebra.

An aim of this presentation is to show that the supergroup $\mathbb{R}^{1,1}$ plays the common role in supersymmetric studies of elementary particles and in solitons.

The structure of this presentation is as follows. In section 2, we discuss supersymmetry in hadron dynamics. In section 3, supersymmetry in low dimensional spin systems is discussed. A bosonic excitation in fermionic systems sometimes appear as solitons. Symmetry in propagation of solitary waves in 2D media is discussed in section 4. In section 5, we present discussion on future research and conclusion.

II. SUPERSYMMETRY IN HADRON DYNAMICS

A. Light front holographic QCD in anti de Sitter space

To construct renormalizable standard model of Quantum Chromo Dynamics (QCD) [66], Srivastava and Brodsky [68, 69] proposed light-front (LF) quantization in the light-cone (LC) gauge, using the Dirac method [34]. In this framework and ‘t Hooft renormalization gauge [1, 3], they extended the Glashow-Weinberg-Salam model of electroweak interaction without including ghosts that appear in loop calculations. In LF-form, the vacuum state is trivial up to possible $k^+ = k^0 + k^z = 0$ zero mode, in contrast to the Higgs zero-mode in instant-form.

To include gravitation field in the standard model, Boussou conjectured that local field theory fails and holographic principle, i.e. restricting number of fundamental degrees of freedom in 4D space-time is related to the area of surface in 5D anti de Sitter space (AdS) should be utilized [104]. The gauge/string duality in low energy hadron study is discussed in [72, 105].

AdS space is the maximally symmetric space-time with negative scalar curvature. $AdS_{d+1}$ space is defined by a surface
\[ X_2^2 + X_0^2 - X_1^2 - \cdots - X_d^2 = R^2 \]
and the metric of distance
\[ ds^2 = dX^2 + dX_0^2 - dX_1^2 - \cdots - dX_d^2. \]
The plane $X_{-1} = X_1$ splits the $AdS_{d+1}$ into two regions defined by light cone coordinates
\[ u = \frac{1}{R^2} (X_{-1} - X_d), \quad v = \frac{1}{R^2} (X_{-1} + X_d). \]
The correspondence between 10-dimensional supergravity and super Yang-Mills equations on the boundary, which is conformal field theory (CFT) was called $AdS/CFT$ correspondence [22, 23].

de Téramond et al. [24, 74] extended the LF quantization in the LC gauge based on the method of [20] to the light front space-time with holographic embedding, by using holographic coordinates
\[ z = R^2/(X_{-1} - X_d). \]

In their formulation, the spectrum of squared mass $M^2$ of mesons and baryons show a similarity. In [72], superconformal quantum mechanics of [21] applied to the fermionic light-front bound state equations was shown to be dual to the bosonic $AdS_5$. The LF wave function is $\Psi(x_i, k_{\perp i}, \lambda_i)$ where
\[ x_i = k^+/P^+|_i = (k^0 + k^z)/(P^0 + P^z)|_i \]
is the LC fraction of a quark, $k_{\perp i}$ are the transverse momenta, and $\lambda_i$ are spin projections. Metric of $AdS$ is
\[ ds^2 = \frac{R^2}{z^2} (dx^+ dx^- - dx_{\perp}^2 - dz^2), \]
where $x^\pm = x^0 \pm x^3$ and $z$ is the holographic variable. Boussou [104] and Susskind and Lindesay [23] explain utility of holographic principles in AdS space.
The correspondence between the LF holographic QCD (LFHQCD) and supersymmetric quantum mechanism is called $AdS/QCD$ correspondence. Since the longitudinal mode is separated from the transverse modes and the parameter of evolution is taken to be $\tau = x^+ = t + z/c^2$, the LF time evolution operator $P^- = P^0 - P^z = d/d\tau$, and $P^+ = P^0 + P^z$, the LF hamiltonian that operates on $\Psi(x_i, k_{\perp}, \lambda_i)$ is given by

$$H_{LF} = P^+ P^- - \vec{P}_\perp^2 = \frac{k_{\perp}^2 + m^2}{x(1 - x)} + U_{eff}(x, k_{\perp}).$$

As mentioned by Srivastava and Brodsky, Faddeev-Popov or Gupta-Bleuler ghost terms are absent in the LCHQCD, and the mass scale of hadrons given by

$$H_{LF} |\Psi(x,k_{\perp},\lambda)\rangle = \mathcal{M}^2 |\Psi(x,k_{\perp},\lambda)\rangle$$

are defined by a parameter in $U_{eff}(x, k_{\perp})$.

In the LFHQCD approach, Brodsky and de Téramond found low energy hadron spectrum of the Regge behavior $\mathcal{M}^2 \sim n + L$, but the behavior is not identical to the Regge behavior of open strings $\mathcal{M}^2 \sim \frac{n - \alpha}{\alpha}$. Needs of better understandings of chirality breaking and confinement as pointed out by 't Hooft were discussed. Waves of relativistic dynamics can be expressed in the instant form, the front form and the point form. The front form is the wave surface expressed by $\sqrt{g}$ and specify spaces with $u, v > 0$ as $AdS_{d+1}^+$, and define a group $Z$ generated by

$$u \rightarrow \lambda^{-1} u, \quad v \rightarrow \lambda v, \quad x_i \rightarrow x_i,$$

with $\lambda$ a fixed real number. The quotient $AdS_{d+1}^+/Z$ is defined as $X_1$. Choosing $1 \leq v \leq \lambda$, with $v = 1$ and $v = \lambda$ identified. $X_1$ is topologically $R^4 \times S^1$.

At infinity of $X_1$

$$u \sum_{i=1}^{d} x_i^2 = 0$$

and $u, v$ and $x_i$ can be regarded as homogeneous coordinates, and set

$$\sum_{i=1}^{d} x_i^2 = 1.$$

The boundary of $X_1$ is a copy of $M = S^1 \times S^{d-1}$. Boundary conformal field theory on $S^1 \times S^{d-1}$ is to be compared with that of $X_1 = S^1 \times R^d$.

In the $AdS/CFT$ correspondence of Maldacena, Euclidean metric

$$-\tilde{X}_0^2 = \tilde{X}_0^2 + \tilde{X}_1^2 + \cdots + \tilde{X}_d^2 = -R^2$$

and

$$U = \tilde{X}_0 + \tilde{X}_d, \quad V = \tilde{X}_0 - \tilde{X}_d = \frac{x^2 u - R^2}{u}, \quad x_\alpha = \frac{\tilde{X}_\alpha R}{U}, \quad \alpha = 0, 1, \cdots, d - 1$$

was defined.

B. Tetraquark states or colour singlet quark pair states

Negative chirality component of quark in a meson which yield negative chirality leptons, could produce negative chirality component of quarks in baryons. In LFHQCD, a fermion operator $\bar{R}_1$, that connects $q\bar{q}$ mesons to $qqq$ baryons can create tetra quark $qq\bar{q}\bar{q}$ states.

When $qq\bar{q}\bar{q}$ states dominate over $qqq\bar{q}$ states is an interesting problem for the study of supersymmetry. Brodsky et al emphasized that the supersymmetric features of hadron physics derived from superconformal quantum mechanics refers to the symmetry properties of bound state wave functions of hadrons and not to quantum fields. Similar argument based on ontological quantum mechanics was proposed also by 't Hooft.

Recently the CMS collaboration at LHC announced detection of $\Upsilon(1S) T(1S) \rightarrow \mu^+\mu^-\mu^+\mu^-$ in proton-proton collisions at $\sqrt{s} = 13$ TeV.
\(\Upsilon(1S), \Upsilon(2S), \Upsilon(3S)\) are \(b\bar{b}\) bosons of \(J^{PC} = 1^{--}\) and have mass 9.460 GeV, 10.023 GeV and 10.355 GeV, respectively. There is also \(\Upsilon(1D)\) of \(J^{PC} = 2^{--}\), but its decay to \(\mu^+\mu^-\) is not detected.

The 4 \(\mu\) decay modes seems to be dominated via \(\Upsilon(1S)\) resonances and sum of single parton scattering (SPS) and double parton scattering (DPS) fits the experimental data quite well, and tetraquark contribution seems to be small, although there are uncertainty in the quark spin polarisation of \(\Upsilon(1S)\).

When the \(b\) quark is replaced by \(c\) quark, \(c\bar{c}\) makes \(J/\psi\) of \(J^{PC} = 1^{--}\), and \(J/\psi(1S), J/\psi(2S)\) have masses 3.097 GeV and 3.686 GeV, respectively. \(c\bar{c}\) contribution to the proton electromagnetic form factor were calculated using lattice data and \(s\bar{s}\) contribution was discussed\[84\]. Similar analysis will become possible if \(b\bar{b}\) amplitudes are similar to those of \(c\bar{c}\).

Concerning \(J/\psi, J/\psi, \Upsilon\) and \(J/\psi\) productions, there is a proposal of single and double quarkonium productions in the colour-evapolation model\[53\], which is an extension of a model for \(J/\psi, J/\psi\) production\[54\]. The colour evapolation model (CSM) assumes direct production of vector quarkonium from gluons.

In the case of production of \(J/\psi, J/\psi, \Upsilon\) and \(J/\psi\) cross section per quarkonium transvers momentum \(d\sigma/dP_T\) in perturbative QCD was found to be larger than that of \(J/\psi + \eta_c + g\) including QCD coupling constants \(\alpha_s(m_q)\)\[84\]. Experiments at LHC indicate that NLO \(\alpha_s(m_q)\) and NNLO \(\alpha_s(m_q)\) were found to be larger than the leading order (LO)\[87\].

The calculated cross section of \(\Upsilon\) production in CSM turned out to be smaller than that obtained by the sum of SPS and DPS evaluated by the CMS collaboration. There remain problems on tetra quark models and quark pair models.

C. Noncommutative geometry approach

Connes and Lott\[118\] proposed a quantum field theory of Yang-Mills type equations\[33\], by adopting a gauge fixing excluding ghost fields, based on noncommutative geometry\[119, 120\].

We define the complex projective space \(P_m\) by taking \(C_{m+1} - 0\), where \(C_{m+1}\) is \((m+1)\)-tuples \((z_0, z_1, \ldots, z_m)\) and 0 is the point \((0, \ldots, 0)\[129\]. A natural projection

\[\psi: C_{m+1} - 0 \to P_m\]  

(57)

and inverse image of each point is \(C^* = C_1 - 0\). In \(\psi^{-1}(U_i)\) where \(U_i\) is an open neighbourhood of \(z^i\) we use coordinates \(z^h = z^h/z^i, (0 \leq h \leq m, h \neq i)\), and \(z^i\). To a point \(p \in P_m, \psi^{-1}(p)\) are called its homogeneous coordinates.

The equation

\[\sum_{k=0}^{m} z^k z^k = 1\]  

(58)

defines a \(S^{2m+1}\) sphere. The mapping \(\psi: S^{2m+1} \to P_m\), whose inverse image of each point is a circle is called Hopf fibering of \(S^{2m+1}\).

Let \(\Gamma\) be the discontinuous group generated by \(2m\) translations of \(C_m\). Then \(C_m/\Gamma\) is called the complex torus.

Let \(\Delta\) be the discontinuous group generated by \(z^k \to 2z^k, (1 \leq k \leq m)\). The quotient manifold \((C_m - 0)/\Delta\) is called the Hopf manifold. It is homeomorphic to \(S^1 \times S^{2m-1}\).

The Dirac matrices, which are familiar to physicists represent Clifford algebra \(A_{3,1}\) which is a subalgebra of \(A_{3,2} \sim M_4(C)\). The algebra \(A_{3,1}\) is isomorphic to \(M_2(H)\), where \(H\) is the quaternion field\[130\]. Spinor structures and Clifford modules were studied in\[108\] and conformal transformations of Dirac spinors were used in\[1\], and relations to complex Grassmannian are explained in\[31\].

In an article of Connes et al\[122\], an extension of noncommutative geometry formalism to unify gravity and the supersymmetric standard model\[121\] in Einstein-Hilbert space was discussed. He defined spectral triple \((A, H, D)\), where \(A\) is the algebra generated by \(R, C\) and \(H, \mathcal{H}\) is the Hilbert space and \(D\) is a self-adjoint operator.

By a proper choice of foliation \(F\) in the space \(V = R^n\) of noncommutative algebra, longitudinal components could be eliminated from effective cohomology, and extract transverse components in the Hilbert space, and ghosts were not required for construction of consistent gauge theory\[120\].

Supersymmetric field theory on noncommutative geometry is based on the supergroup \(R^{1,1}\). The Hopf algebra \(H\) of smooth functions on the \(R^{1,1}\) is given as

\[H = C^\infty(R^{1,1}) = C^\infty(R) \otimes \wedge(R)\]  

(59)

where \(\wedge(R)\) is the exterior algebra of one-dimensional vector space. Every element of \(H\) can be expressed as \(f + g\theta_1\), where \(f, g \in C^\infty(R)\) and \(\theta_1^2 = 0\). Thus supersymmetry becomes manifest in \(H\). In terms of Grassmann variables \(\theta = (\theta_1, \theta_2)\), \(F(\theta_1) = c_0 + c_1\theta_1\) has the similar structure as \(H\).

Jaffe et al\[113, 117\] extended the \(N = 2\) supersymmetric Wess-Zumino quantum fields adopted by Witten\[28\], using an extension of Connes’ triple as spectral quadruple \((A, \mathcal{H}, \Gamma, Q)\) where \(\Gamma\) is \(Z_2\) grading of \(A\) and \(Q\) is the Dirac operator,

\[Q = \left( \begin{array}{cc} 0 & Q_- \\ Q_+ & 0 \end{array} \right) \]  

(60)

where \(Q_-\) operates on \(\mathcal{H}_- \to \mathcal{H}_+\) and \(Q_+\) operates reversely.

The Hamiltonian \(H = Q^2\) is

\[H = \left( \begin{array}{cc} Q_+ & Q_- \\ Q_- & Q_+ \end{array} \right) \]  

(61)
and considered the heat kernel $e^{-\beta H}$ ($\beta > 0$), state vectors $a(t) = e^{-tH}ae^{tH}$ and cochains $C(A)$ consisting of $(f_0, f_1, f_2, \cdots)$. Grading of $\Gamma$ allows decompositions of $C(A)$

$$C(A) = C^e(A) \oplus C^o(A),$$

where $(f_0, f_2, f_4, \cdots) \in C(A)^e, (f_1, f_3, f_5, \cdots) \in C(A)^o$, and

$$C(A)^e = \frac{1 + \Gamma}{2} C(A)^e + \frac{1 - \Gamma}{2} C(A)^o = C(A)^e_+ + C(A)^o_-. \quad (62)$$

They showed that the coboundary operator $\partial$ exists that show the cohomology sequence

$$\cdots \rightarrow C^e_+(A) \xrightarrow{\partial} C^o_+(A) \xrightarrow{\partial} C^e_+(A) \rightarrow \cdots \quad (63)$$

In the case of $N = 2$ Wess-Zumino supersymmetric model, one can choose

$$Q_1 = \sqrt{-1} \delta \phi_1 - \sqrt{-1} \delta \phi_2 (\partial V)^*, \quad Q_2 = \sqrt{-1} \delta \phi_2 \delta + \sqrt{-1} \delta \phi_1 V,$$

and

$$H = Q^2 = -\partial \delta - \delta \bar{\psi}_1 \theta - \bar{\psi}_2 \theta (\partial V)^* + |\partial V|^2, \quad (66)$$

is hermitian due to $\psi \psi = (\bar{\psi}_2 \psi)^*$. One can consider spin $1/2$ fermion $SU(2)$ gauge theory with Yukawa coupling, but the pion remains massless.

### D. Supersymmetry and the BRST charge

In topological quantum field theory, Witten defined the charge of Becchi-Rouet-Stora-Tyutin (BRST) transformations [33, 34] which contain Faddeev-Popov ghosts as follows [39].

One begins with gauge fields $A_\alpha^a(x)$ on a three manifold $Y$. Here $i = 1, 2, 3$ labels the components of a tangent vector to a manifold $M$, $a$ runs over the generators of a gauge group $G$, $x$ labels a point in $Y$, which is endowed with a metric tensor $g_{ij}$. The variation $\delta A_\alpha^a(x)$ can be regarded as a differential operator on differential forms $\omega$ on the space $A$: $\omega \rightarrow \delta A_\alpha^a(x) \wedge \omega$. The exterior derivative and its adjoint on $A$

$$d = \int d^3 x \psi^a(x) \frac{\delta}{\delta A_\alpha^a(x)}, \quad d^* = -\int d^3 x \chi^a_\alpha(x) \frac{\delta}{\delta A_\alpha^a(x)}, \quad (67)$$

where $\chi^a_\alpha(x)$ is a vector dual to $\psi^a(x)$. With a real parameter $t$, and Chern-Simons functional

$$W = \frac{1}{2} \int_Y Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad (68)$$

one defines $d_t = e^{-tW}de^{tW}, d^*_t = e^{tW}d^*e^{-tW}$ that satisfy

$$d^*_td_t = 0, \quad d^*_t d_t = 0, \quad d_t d^*_t + d^*_t d_t = 2H. \quad (69)$$

The group of $Y$ is graded and additive quantam number $U$ is $-1$ for $\chi$ and $+1$ for $\psi$.

In order to obtain a relativistic version, one works on space $Y$ and time $R^1$, and in the space $Y \times R^1$ name the supersymmetric generator $d_t$ as $Q$. One tries the Lagrangian

$$L = \int_M dx Tr\left[\frac{1}{4} F_{\alpha \beta} \bar{F}^{\alpha \beta} - i \eta D_\alpha \psi^\alpha + i (D_\alpha \psi_\beta) \chi^{\alpha \beta}\right]. \quad (70)$$

The supersymmetry transformation law is

$$\delta A_\alpha = i \epsilon \psi^\alpha, \quad \delta \eta = 0, \quad \delta \psi^\alpha = 0, \quad \delta \chi^{\alpha \beta} = \epsilon (F_{\alpha \beta} + \frac{1}{2} \epsilon_{\alpha \beta \gamma} F^{\gamma \delta}). \quad (71)$$

The propagating modes of $A_\alpha$ have helicities $(1, -1)$, and those of $(\eta, \psi, \chi)$ have helicities $(1, -1, 0, 0)$.

The Lagrangian invariant under the fermionic symmetry

$$\delta A_\alpha = i \epsilon \psi^\alpha, \quad \delta \phi = 0, \quad \delta \lambda = 2i \epsilon \eta, \quad \delta \eta = \frac{1}{2} \{\phi, \lambda\},$$

$$\delta \psi^\alpha = -\epsilon D_\alpha \phi, \quad \delta \chi^{\alpha \beta} = \epsilon (F_{\alpha \beta} + \frac{1}{2} \epsilon_{\alpha \beta \gamma} F^{\gamma \delta}). \quad (71)$$

where $\lambda, \phi$ are new scalar fields, becomes

$$L_0 = \int_M d^4 x Tr\left[\frac{1}{4} F_{\alpha \beta} \bar{F}^{\alpha \beta} + \frac{1}{2} \phi D_\alpha D^\alpha \lambda + i D_\alpha \psi_\beta \cdot \chi^{\alpha \beta} - \frac{i}{8} [\chi^{\alpha \beta}, \chi^{\alpha \beta}] - i [\lambda_\alpha, \psi^\alpha] \right]. \quad (72)$$

For any functional $O$, its variation under the fermionic symmetry becomes

$$\delta O = -i \epsilon \cdot \{Q, O\}. \quad (73)$$

For $O = \frac{1}{4} Tr[\phi, \lambda, \eta]$, there is a possibility to add a Lagrangian

$$L_1 = s \int d^4 x \{Q, O\} = s \int d^4 x Tr\left[\frac{i}{2} \phi [\eta, \eta] + \frac{i}{8} [\phi, \lambda]^2\right], \quad (74)$$

with $s$ an arbitrary parameter.

By choosing $s = -1$, the relativistic Lagrangian becomes

$$L = \int_M d^4 x \sqrt{g} Tr\left[\frac{1}{4} F_{\alpha \beta} \bar{F}^{\alpha \beta} + \frac{1}{2} \phi D_\alpha D^\alpha \lambda - i \eta D_\alpha \psi^\alpha + i D_\alpha \psi_\beta \cdot \chi^{\alpha \beta} - \frac{i}{8} [\chi^{\alpha \beta}, \chi^{\alpha \beta}] - \frac{i}{2} [\lambda_\alpha, \psi^\alpha] - \frac{i}{2} \phi [\eta, \eta] - \frac{1}{8} [\phi, \lambda]^2\right]. \quad (75)$$
One defines a tensor $T_{\alpha\beta}$ such that under an infinitesimal change of metric $g^{\alpha\beta} \rightarrow g^{\alpha\beta} + \delta g^{\alpha\beta}$, the change of the action becomes

$$\delta L = \frac{1}{2} \int_M \sqrt{g} \delta g^{\alpha\beta} T_{\alpha\beta}. \quad (76)$$

$T_{\alpha\beta}$ can be expressed in the form of BRST commutator

$$T_{\alpha\beta} = \{ Q, \lambda_{\alpha\beta} \} \quad (77)$$

with

$$\lambda_{\alpha\beta} = \frac{1}{2} Tr(F_{\alpha\sigma} \chi^\sigma + F_{\beta\sigma} \chi^\sigma - \frac{1}{2} g_{\alpha\beta} F_{\alpha\sigma} \chi^\sigma$$
$$+ \frac{1}{2} Tr(\psi_\alpha D_\beta \lambda + \psi_\beta D_\alpha \lambda - g_{\alpha\beta} \psi_\sigma D^\sigma \lambda)$$
$$+ \frac{1}{4} g_{\alpha\beta} Tr(\eta[\phi, \lambda]). \quad (78)$$

One can show that $D_\alpha T^{\alpha\beta} = 0$ \[39\].

In contrast to the $Q_{BRST} \sim c \partial c b + c(\partial X)^2$, where $c, b$ are conformal ghosts and $X$ is the matter field of Kugo and Ojima \[133, 136\], the supercharge $Q$ is defined on twisted manifolds $M = Y \times R^4$, and ghosts are absent in the supersymmetric theory.

We investigated the Kugo-Ojima parameter $u(0) = c$ in lattice Landau gauge OCD, and observed a small deviation from $c = 1$ \[137, 138\]. In lattice simulations, it is necessary to extrapolate finite size simulation data to continuum limit, and it is difficult to check whether nature satisfies the BRST symmetry \[139, 188\].

Schaden and Zwanziger \[141\] studied BRST cohomology by choosing physical subspace $V$ to be the physical sector of the Gribov-Zwanziger (GZ) theory \[140\]. The GZ vacuum breaks BRST symmetry and the Lagrangian is written as

$$L = L^YM + L^{\text{ghosts}} = L^YM + s\Psi \quad (79)$$

The expectation value $\langle s\Psi \rangle$ becomes zero due to cancellation of fermion boson and boson ghosts.

Watson and Alkofer \[142\] verified the Kugo-Ojima confinement criterion with an assumption that the physical subspace is $V = Ker(Q_{BRST})$ and physical states are given by the cohomology $H(Q_{BRST}, V) = Ker(Q_{BRST})/Im(Q_{BRST})$. They showed the leading infrared powers of the gluon $d_0$ and the ghost propagator $e_0$ are related, and the enhancement of the ghost propagator and suppression of the gluon propagator in infrared region as compared to tree level calculations can be understood.

There is an attempt to construct BRST complex that gives Gupta-Bleuler space of physical states using Cohomological reductions \[143\]. Ghosts can be regarded as supersymmetric partners of fermions that were required to keep 4D space inside the Gupta-Bleuler space. Theories based on quantum K-theory \[117, 120\] in which ghosts are absent may be more favorable to understand symmetries of nature.

Experimental check of the standard model, or searches of theories beyond the standard model was performed at LHC, Belle and other laboratories through two photon emission measurements. At LHC gluon fusions $gg \rightarrow X \rightarrow \gamma \gamma$ where $X$ scalar bosons including $125$GeV Higgs boson. The standard model of the Higgs spectrum is based on $SO(5)/SO(4)$ coset.

The group $SO(n)$ is a kernel of determinant map $O(n) \rightarrow \{ \pm 1 \}$. $SO(3)$ is homeomorphic to projection space $RP^3$ \[114\]. $SO(4)$ is homeomorphic to $S^3 \times SO(3)$. Identifying $R^4$ with the quaternions $H$ and $S^3$ as the unit of quaternions, one obtains the homeomorphism $S^3 \times SO(3) \approx SO(4)$ \[114\]. The homomorphism $\psi : S^3 \times S^3 \rightarrow SO(4)$ that sends a pair of unit quaternions

$$(u, v) = (u_0 I + u_1 i + u_2 j + u_3 k, v_0 I + v_1 i + v_2 j + v_3 k) \quad (80)$$
to the isometry $w \rightarrow uvw^{-1}$ of $H$ is surjective with the kernel $Z_2 = \{ \pm (1, 1) \}$.

Using octonions $O$, one can construct a homeomorphism $SO(8) \approx S^7 \times SO(7)$, but in other $n$, $SO(n)$ is only a twisted product of $S(n - 1) \times S^{n-1}$ \[114\]. As an example, $SO(5)$ is homeomorphic to a twisted product of $SO(4)$ and $S^4$, $SO(5)/SO(4)$ is homeomorphic to $S^4 \times Z_2$.

When Kaluza-Klein 5 dimensional vector gauge theory works, an extension of the Higgs model based on $SO(5)/SO(4)$ which includes the WZW term is possible \[144, 145\]. The model is based on spontaneous CP symmetry breaking mechanism of Pececi and Quinn (PQ) \[146\].

The PQ model is a model of fermions which couple to non-abelian gauge fields, whose effective action is given by

$$S_{eff}^q = \int d^4x (L + \sqrt{-1}q) \eta, \quad (81)$$

where $q = (g^2/32\pi^2) \int d^4xF_{\mu\nu}^a F^{a\mu\nu}$. The rotation of the fermion field by $exp(\sqrt{-1}Y\gamma_5\eta)$ induces a change in the effective action

$$\delta S_{eff}^q = -\sqrt{-1}\int d^4x (\partial^\mu j_5^5)\eta = -2\sqrt{-1}q\eta \quad (82)$$

and the net effect is the rotation $\theta \rightarrow \theta - 2q$. It induces spontaneous symmetry breaking which creates Higgs bosons.

In topological classification of $2n$ free fermion systems \[92\], 4 Majorana fermion system $c_1 \bar{c}_2 \bar{c}_3 \bar{c}_4$ with the $so(3)$ symmetry, and 8 Majorana fermion system $\hat{c}_1, \ldots, \hat{c}_8$ with the $so(8)$ symmetry were studied. In the $so(8)$ space, a subspace $S_+$ and a subspace $S_-$ which are invariant under $so(7)$ were chosen and the triality symmetry of octonions were taken into account.
The Hopf theorem says that there exists a continuous mapping of $S^{n-1} \times S^{n-1} \rightarrow S^{n-1}$ where $n$ is a power of 2. When $n = 4$, the mapping $S^3 \times S^3 \rightarrow S^3$ can be expressed by transformations of three unit vectors in $S^3$ represented by quaternion algebra. When $n = 8$, $S^7 \times S^7 \rightarrow S^7$ is represented by octonions, and this $n = 8$ is largest due to the Bott periodicity.

The PQ model predicts Goldstone bosons and axion-like particles. Axion-like particles are (pseudo) scalar particles $\phi$ of mass $m_\phi$ interacting with the standard model electroweak gauge field-tensor $B_{\mu\nu}$ and $G_{\mu\nu}$ whose interaction Lagrangian is

$$\mathcal{L}_\phi \supset - \frac{1}{4} g_{\phi BB} \phi B_{\mu\nu} \tilde{B}_{\mu\nu} - \frac{1}{4} g_{\phi gg} \phi G_{\mu\nu} \tilde{H} gg G_{\mu\nu}, \quad (83)$$

where $B_{\mu\nu}$ is electro weak gauge field-tensor and $G_{\mu\nu}$ is the Goldstone boson field-tensor.

### E. Extensions of Higgs boson model and supersymmetry

In minimal supersymmetric standard model (MSSM) $[2]$, Higgs field consists of $SU(2)_L$ doublet

$$H_u = \left( \begin{array}{c} H_u^+ \\ H_u^0 \end{array} \right) \geq \left( \begin{array}{c} 0 \\ \nu_u \end{array} \right)$$

and

$$H_d = \left( \begin{array}{c} H_d^0 \\ H_d^- \end{array} \right) \geq \left( \begin{array}{c} \nu_d \\ 0 \end{array} \right)$$

where $H^+, H^0$ are complex scalar fields. The Higgs bosons in the MSSM consists of three massless and five massive ($h^0, H^0, H^+, A^0$), and $M_h \sim 125\text{GeV}$.

In $[149]$ the tan $\beta$ parameter in the MSSM is defined as

$$\tan \beta = \frac{\nu_u}{\nu_d}$$

The partner of the standard model fermion are expected to exist beyond TeV scale and $\tan \beta < 5$ is ruled out.

In the framework of search of beyond the standard model, there is the two Higgs doublet model (THDM) $[150, 151]$ as an effective field theory below the SUSY scale. In this model, renormalization on $\tan \beta$ was reconsidered by removing sleptons and squarks.

In the standard model, CP violation occurs through Kobayashi-Maskawa $SU(3)$ matrix phases in quark and s quark dynamics $D^0 \rightarrow K^+ K^- e^+ e^-$ and $D^0 \rightarrow \pi^+ \pi^- e^+ e^-$. Authors of $[152]$ studied CP violation which can be detected in 500 GeV $e^+ e^-$ collision in International Linear Collider (ILC) and 380 GeV $e^+ e^-$ collision in Compact Linear Collider (CLIC).

In the mathematical model of strings with Dirichlet boundary condition (D-Brane), one can assume the flipped $SU(5) \times U(1)$ model for hadron interactions $[153]$ and estimate effects of supersymmetric partner of materials to the mass of Higgs bosons etc.

### F. Dark Matter and supersymmetry

Duan et al. $[156]$ proposed a probe of bino-wino coannihilation dark matter below the neutrino floor at the LHC. Winos are SUSY partner of $W^\pm$ of the MSSM theory, and the proton-proton collision in LHC that creates winolike $\chi_2^0$

$$pp \rightarrow \chi_2^0(\rightarrow l^+ l^- \chi_1^0) \chi_1^\pm + \text{jets}$$

was considered. Here $\chi_1^0$ is the lightest neutralino is a weakly interacting massive particle (WIMP) dark matter (DM) which is a linear combination of gauge eigenstates $(\tilde{B}, \tilde{W}, \tilde{H}_d^0, \tilde{H}_u^0)$, i.e.

$$\chi_1^0 = N_{11} \tilde{B} + N_{12} \tilde{W} + N_{13} \tilde{H}_d^0 + N_{14} \tilde{H}_u^0.$$ 

When $N_{11}^2 > \max \{N_{12}^2, N_{13}^2 + N_{14}^2\}$, $\chi_1^0$ is called bino-like.

Possibility of detecting DM relic density is discussed. Relations between DM and Big Bang Nucleosynthesis (BBN) was studied by Hamdan and Unwin $[157]$. In the BBN, DM decomposition is assumed to occur during radiation domination. In presence of additional dimension to 4 dimensional space-time, DM can decouple when the matter-like universe exists.

### G. Non-local deformation of a supersymmetric field theory

Heisenberg uncertainty problem is related to the length scale of physics and it is an important ingredient for $N = 1$ supersymmetric field theory and conformal field theory $[168]$.

Materials that are considered are Zeeman driven Lifshitz transition, which is a phase transition caused by changes of Fermi surface in a heavy fermion metal $[170]$, and theoretical bases were taken from $[169]$.

### H. New colored scalar bosons

In the search of beyond the standard model, Gabrielli et al. $[158]$ proposed a colored scalar boson of mass 750 GeV, as a cause of bumps of two-photon final state observed at LHC. The LHC Run 1 and Run 2 data at 7 TeV and 8 TeV, the cross section was $4.5 \pm 1.9$ fb, and the data at 13 TeV was $10.6 \pm 2.9$ fb. Their expectations
is $gg \rightarrow S \rightarrow \gamma \gamma$ with triangle diagrams of $S \rightarrow gg$ and $S \rightarrow \gamma \gamma$.

Triangle diagrams appear in the theory of anomaly.

Higgs boson decays to dibosons including $\gamma \gamma$ was measured at LHC and analyzed as an example in

\[ \text{I. Superconformal field theory and M-Theory} \]

In the string theory and M-theory, the flux compactification in background field (Gravitational field) becomes necessary. Lazaroiu et al [123] studied supersymmetric flux compactification of supergravity theories, by defining constrained generalized Killing spinors. A Killing vector in a Riemann space with metrics $g_{\alpha \mu}$ is written as $X = \xi^\lambda \partial / \partial x^\lambda$ where $\xi$ satisfies

\[
\xi^\alpha \partial g_{\alpha \mu} + g_{\alpha \beta} \partial x^\mu + g_{\alpha \mu} \partial \xi^\alpha \\
\nabla_\mu \xi^\alpha + \nabla_\alpha \xi^\mu = 0. \tag{84}
\]

Killing spinor $\psi$ satisfies

\[
\nabla_X \psi = \lambda X \cdot \psi,
\]

where $\cdot$ is the Clifford multiplication and considered Kaehler-Atiyah bundle.

Kaehler metric in complex projective spaces are hermitian metric $\omega = \sqrt{-1} \sum_{\alpha, \beta} g_{\alpha \beta} dz^\alpha \wedge \bar{dz}^\beta$ that satisfies $d \omega = 0$. Lazaroiu et al [123] referred Gauntlett et al [125] for the Kaehler-Atiyah metric. Gauntlett et al discussed superstring with intrinsic torsion. The are two types of string theories: 1) open string theory and 2) closed string theory. Gravitational effects in string theories can be studied in [18, 19]. For an arbitrary manifold, one can not necessarily define a Kaehler metric. Gauntlett et al. [127] considered complex structures on the K3 manifold also. Their manifold allows intrinsic torsion on non-Kaehler manifold, like Hopf manifold.

It is well known that superconductivity is well described by the Landau-Ginzburg description $N = 2$ superconformal model [124]. Witten used the Landau-Ginzburg description in $N = 2$ superconformal model [124, 127], and Davenport and Melnikov studied (2, 2) compactified supersymmetry theory with central charge $c = \frac{N(N+1)}{2} < 6$.

Kiyoshige and Nishinaka derived formula of three-point functions of 4-dimensional (4D) $N = 2$ superconformal field theories using the Calabi-Yau Metric. $N = 2$ super-conformal field theories in 4D was proposed by Seiberg and Witten for introducing electric-magnetic duality, monopole condensate and confinement in Yang-Mills theory, and extended to $SU(3)$ supersymmetric gauge theory by Argyres and Douglas. Topological structures of closed strings were studied.

The conformal field theory (CFT) is a starting point for describing infrared (IR) property of asymptotically free field theory in which the world sheet described by $2 \times 2$ symmetric tensor $h_{\alpha \beta}$ is parametrized as $n_{\alpha \beta} \phi^0$, where $\phi^0$ is an unknown conformal factor. A $N = 2$ superconformal field theory (SCFT) in 4D was proposed in [10].

A 4D SCFT on $S^3 \times \mathbb{R}$ for $N = 1, 2, 4$ was studied by Kinney et al. [17] and in the case of $N = 4$, AdS/CFT correspondence of entropy of Bogomolnyi-Prasad-Sommerfeld (BPS) black hole in strong coupling $AdS_5 \times S^5$ frameworks.

Gadde et al. [18] showed that the superconformal index (the partition function on $S^3 \times S^1$) of a certain class of 4D supersymmetric field theories is exactly equal to a partition function of $q$- deformed nonsupersymmetric 2D Yang-Mills theory.

Beem et al. [19] classified $N = 2$ superconformal field theory using Schur operators [123], that operates on quaternions.

Using BRST cohomology, they conjectured that 4D CFT with extended supersymmetry and 2D chiral algebras have a correspondence.

\[ \text{J. Electron-Boson quantum manybody systems} \]

In the review article of Caliceti et al. [88], problem of getting physical predictions from slowly convergent series was discussed. Mera et al. [89] applied hypergeometric resummation technique developed by themselves and showed that in the calculation of self-consistent Dyson equations of electron-boson quantum many-body system, hypergeometric resummation is better than Padé approximant method.

Authors claim that their method is useful in calculation of physical observables in a non-equilibrium steady state electron-boson quantum many-body systems. Dynamics of electrons interacting with phonons in the presence of applied bias voltage is also discussed.

\[ \text{K. The thermal tensor network} \]

Dong et al [92] analyzed thermal tensor networks of a Ising chain, whose Hamiltonian is

\[ H = - \sum_i^L (J S_i^\alpha S_{i+1}^\alpha + B S_i^z). \]

They used the nonlocal fermionization transformation:

\[ c_i = e^{\pi i} \sum_{j=1}^{i-1} (S_j^z + \frac{1}{2}) S_i^- \]
\[ c_i^\dagger = \exp[-\pi \sum_{j=1}^{i-1} (S^z_j + \frac{1}{2})] S^z_i, \]
\[ c_i^\dagger c_i = S^z_i + \frac{1}{2}, \]
where \( S^z_i = S^x_i + \sqrt{-1} S^y_i \), \( S^z_i = S^x_i - \sqrt{-1} S^y_i \).

The Hamiltonian is transformed to
\[ H = \frac{B L}{2} - B \sum_{i=1}^{L} c_i^\dagger c_i - \frac{J}{4} \sum_{i=1}^{L} (c_i^\dagger - c_i)(c_{i+1}^\dagger + c_{i+1}). \]

Linearized tensor renormalization group (LTRG) for bilayers was applied to a fermionic extended Hubbard model (EHM).

### III. SUPERSYMMETRY IN LOW DIMENSIONAL SPIN SYSTEM

In 1983, Haldane [91] showed in one-dimensional non-linear sigma model, difference of half-integer spin states and integer spin states, which have different instability induces solitons. Couplings of nonlinear zero-point fluctuation (instantons) were discussed. He studied stabilities of half-integer spin fermionic systems and integer spin fermionic systems.

In the same year, Efetov [92] studied non-linear supermatrix sigma model, using supervectors and supermatrices, which are based on Berezinian (Grassmann algebra for fermions) [31].

Al'ttand and Zirnbauer [93] studied mesoscopic junctions of normal-conducting metals and superconducting metals, using the Efetov’s method. They showed that the system is classified by the symmetry operations of time reversal and rotation of electron's spin, and the 4 symmetry classes of Hamiltonian corresponding to time-reversal symmetry (TRS), particle-hole symmetry (PHS) and chiral symmetry (SLS) which is a product space of TRS and PHS, corresponding to Cartan’s symmetric spaces of type \( C, CI, D \) and \( DIII \) which are called Bogolyubov-de Gennes symmetry classes.

Classification of topological insulators and superconductors in three spacial dimension was studied by Schnyder et al. [100].

In Witten’s model, triangle diagrams appear in the argument of symmetry-protected topological (SPT) bosonic phases through Chern-Simons theory [42].

Symmetry protected topological phases in two dimensional fermionic system was studied by Gu and Wen [165] and gapless modes at boundaries were studied by Zhang et al. [164]. They classified phases of 2D fermionic symmetry protected topological phase (FSPT) using the technique of induced representations applied to quantum mechanics [167].

For spinless fermions, the authors consider cyclic groups \( C_{2m-1} \) and \( C_{2m} \) which has \( Z_{2m-1} \) and \( Z_{2m} \) symmetries, respectively. Semidirect products of rotation \( C_n \) and reflection symmetry \( Z_2^M \) which are dihedral group \( D_n = C_n \times Z_2^M \) for even \( m \) and \( Z_{8m} \) for odd \( m \). \( D_{2m-1} = D_{2m} = Z_2 \).

The classification of \( C_{2m-1} \) for spin 1/2 fermions is same as spinless fermions, but \( C_{2m} = Z_2 \times Z_{4m} \) for even \( m \) and \( Z_{8m} \) for odd \( m \). \( D_{2m-1} \) and \( D_{2m} \) for spin 1/2 fermion is \( Z_1 \) and \( Z_2 \times Z_2 \), respectively.

Efetov considered in ref. [94] section 10.3, one dimensional fermionic system using the partition function
\[ Z = \int \exp(-S[\Phi]) D\Phi, \]
where
\[ S[\Phi] = \int_{-\infty}^{\infty} \left( \frac{\text{tr} m}{2} \frac{\partial \Phi(t)}{\partial t} \right)^2 + V(\Phi) \, dt, \]
and \( \Phi \) may be real symmetric, general Hermitian or composed of real quaternions. Correlatation functions of interacting fermions were calculated. The topological insulator predicted in [95] was applied to the integer quantum Hall effect and quantum spin Hall effect discovered by [97] the article of [100, 101].

Kennedy and Zirnbauer [96] studied \( Z_2 \) symmetric ground states of gapped superconductors using classes in the symmetric space. Interactions of electrons are taken into account, the \( Z_2 \) symmetry can be broken to \( Z_8 \) symmetry [97]. The authors adopted the interaction similar to that of [20].

\[ \tilde{H} = t \Phi + w W_{\text{tot}} + v V_{\text{tot}}. \]

White [171] studied \( S = 1/2 \) and \( S = 1 \) antiferromagnetic one dimensional spin chain, using the density matrix formulation, including configurations of fractional \( S = 1/2 \) spins at the ends of open \( S = 1 \) chains.

The supersymmetry method based on using commuting and anticommuting variables for bosons and fermions, respectively, is a useful tool for studying disordered and chaotic [94].

Supersymmetric extensions of Korteweg-de Vries (KdV) soliton equation [172], Kadomtzev-Petviashvili (KP) equation [173] and the non-linear equation of Ablovitz-Kaup-Newell-Segur (AKNS) [174] were studied by several authors [175, 180].

Using the \( D \) operator
\[ D_t f_n f_{n-1} = f_{n+1} f_{n-2} - f_{n} f_{n-1}, \quad D_x f g = \frac{\partial f}{\partial x} g - f \frac{\partial g}{\partial x}, \]
the KdV equation is
\[ \frac{\partial}{\partial t} u + \frac{\partial^3}{\partial x^3} u + 6 u \frac{\partial}{\partial x} u = 0, \quad u = 2 \frac{\partial^2}{\partial x^2} (\log f), \]
\[ (D_x^2 - 4 D_t D_x) f \cdot f = 0, \]
and the KP equation is
\[
\frac{\partial}{\partial x_1} \left( \frac{\partial}{\partial x_3} u + \frac{\partial^3}{\partial x_1^3} u + 6u \frac{\partial}{\partial x_1} u + \frac{\partial^2}{\partial x_2^2} u \right) = 0,
\]
\[
u = 2 \frac{\partial^2}{\partial x_1^2} \log \tau
\]
\[
(D^4_{x_1} - 4D^3_{x_1} D_{x_3} + 3D^2_{x_2}) \tau \cdot \tau = 0 \tag{90}
\]

The solution of AKNS equation in the inverse scattering transform (IST) of Zakharov-Shabat\[176, 177\] is
\[
q_t - \sqrt{-1} q_{xx} - 2\sqrt{-1} q^2 q^* = 0. \tag{91}
\]

In the $\mathcal{N} = 2$ supersymmetric extension of KdV equation\[180\], the resolvent kernel $R = (-L + \zeta)^{-1}$ where $L = \partial^2_x - u(x)$, is expressed as
\[
R(x, \zeta) = \lim_{x' \to x} R(x, x'; \zeta) = \sum_n R_n[u] \zeta^{-n-1/2}, \tag{92}
\]
and the residue or the coefficient of $\partial^{-1}$ becomes two times Gelfand-Dikii polynomials $R_n$,
\[
\text{res} L^{n-1/2} = 2R_n. \tag{93}
\]
The KdV differential equation is expressed by the differential equation of the Lax operator $L$, that satisfy
\[
[L, L_+^{n+1/2}] = \partial \partial t_n L \tag{94}
\]
which is equivalent to
\[
\partial \partial t_n = 4\partial_x R_{n+1}. \tag{95}
\]

Supersymmetric extension of the nonlinear AKNS equation was performed by introducing Lie superalgebra\[179\], in which the linear problem
\[
\Psi_x = (qE_2 + i\lambda E_0 + rE_1 + \epsilon E_3 + \beta E_4)\Psi,
\]
\[
\psi_\lambda = V\psi, \tag{96}
\]
where $\epsilon, \beta$ are anticommuting variable, $r, q$ are commuting fields,
\[
V = \begin{pmatrix} A & C & \alpha \\ B & -A & \rho \\ \rho & -\alpha & 0 \end{pmatrix} \tag{97}
\]
is the temporal evolution operator, $A, C$ are commuting and $\alpha, \rho$ anticommuting values, are solved. $\lambda$ is the eigenvalue of the problem and $\psi$ is the temporal evolution.

The state vectors $\{B_j, C_j, \rho_j, \alpha_j\}$ satisfy with Lie superpercusion operator $\mathcal{L}$ as
\[
2\sqrt{-1} \{B_{j+1}, C_{j+1}, \rho_{j+1}, \alpha_{j+1}\} = \mathcal{L} \{B_j, C_j, \rho_j, \alpha_j\}. \tag{98}
\]

The $E_i$'s satisfy commutation rules
\[
\{E_0, E_1\} = -2E_2, \quad \{E_0, E_2\} = 2E_1, \quad \{E_1, E_2\} = E_0, \\
\{E_0, E_3\} = E_3, \quad \{E_0, E_4\} = -E_4, \\
\{E_1, E_3\} = 0, \quad \{E_1, E_4\} = E_3, \quad \{E_2, E_4\} = 0, \\
\{E_3, E_4\} = E_0, \quad \{E_3, E_3\} = -2E_1, \\
\{E_4, E_4\} = 2E_2. \tag{99}
\]

In the case of supersymmetric KP equation\[178\], pseudo-differential operator $L = \partial + \sum_{i=1}^{\infty} \alpha_i \partial^{-i}$ and the differential equation
\[
\partial_i L = [L^i_+, L], \quad \partial_i = \frac{\partial}{\partial t_i}, \tag{100}
\]
where $L_+$ is the differential part of the operator $L$ was solved. Here even-odd pairs of space variables $(x, \xi)$ and even-odd times $(\tau_1, t_2, \tau_3, t_4, \cdots)$ were adopted.

IV. SYMMETRIES IN PROPAGATION OF SOLITARY WAVES IN MATTERS

Although magnetic monopoles are not detected, they can be regarded as 3D solitons, and supersymmetry have relevance to solitons. Time reversal symmetry based nonlinear elastic wave spectroscopy, in which one optimizes the convolution of the scattered wave from defects in materials and its time reversed wave show peaks was an effective method for non-destructive testing (NDT)\[172\].

Getting physical information of nonlinear acoustic equations was performed by using the Lie symmetry of the Khokhlov-Zabolotskaya(KhZa) equation which uses the front form coordinate $\eta = \omega(t - x/c_0)$ and a special coordinate $\eta(r, z, u)$\[190\][191\]. Details on possibility of detecting gravitational effects using AI technique is discussed in \[192\].

A. Conformality in quaternion projective space

In\[116\], orthogonal projection of the Dirac operator $Q_+ = P_+QP_+$, where $P_+ = \frac{1 + \Gamma}{2}$ and index $i(Q_+)$ for Wess-Zumino model defined by the superpotential $V(\varphi)$ where $\varphi$ is the holomorphic function ($\varphi \in \mathbb{C}$) becomes Atiyah-Singer index
\[
i(Q_+) = \dim \ker Q_+ - \dim \ker Q_- = n_+ - n_- \tag{101}
\]
was verified. The grading $\Gamma$ is defined by fermionic particle number $N_f$ as $\Gamma = (-1)^{N_f}$.

In TR-NEWS experiment, propagation of a soliton is restricted to a 2D plane, and the support of $\varphi$ has rectangular boundary. An index theorem for Dirac operators in a bounded region was formulated by Atiyah, Patodi and Singer\[111\], and discussed by Witten\[12\] and
Yu et al. [196]. $\mathbb{Z}_2$ topological order and quantum spin Hall effect, which are related to boundary conditions of conformal field theory was investigated in 2005 by Kane and Mele [92, 93].

Analytical extensions of differentiable functions defined in closed sets was discussed in 1933 by Whitney [17], refined by Seeley [112], and boundary conditions following the heat equation was studied by Atiyah, Bott and Patodi [110]. A review of differential calculus of functions that map open interval $I$ to $nD$ real functions $f = (f_1, \cdots, f_n) \in C^k$ by Taylor expansions

$$f(x + y) = \sum_{j=0}^{k-1} f^{(j)}(x; y, \cdots, y) / j! + \int_0^1 f^{(k)}(x + ty; y, \cdots, y)(1-t)^{k-1} dt / (k-1)!,$$

(102)

where $[x, x + y] \in I$ and their Fourier-Laplace transform were shown by Hoermander [131]. A good review on Atiyah-Singer’s index theorem [107] and Atiyah-Bott-Shapiro’s Clifford modules [108] are written in the book of Hirzebruch [113]. Stability of index formulas and boundary problems for elliptic differential operators are explained in [132].

In application of mathematical theories to speech recognitions or image recognitions, recurrent neural networks are commonly used [206]. Parcollet et al. [207] applied quaternion neural networks to image recognitions. Although the subject is not directly related to the supersymmetry, in the analysis of propagation of solitons and edge effects of solitoic wave should be taken into account, where Atiyah-Padobi-Singer index was shown by Hoermander [131]. A good review on Atiyah-Singer’s index theorem [107] and Atiyah-Bott-Shapiro’s Clifford modules [108] are written in the book of Hirzebruch [113]. Stability of index formulas and boundary problems for elliptic differential operators are explained in [132].

In application of mathematical theories to speech recognitions or image recognitions, recurrent neural networks are commonly used [206]. Parcollet et al. [207] applied quaternion neural networks to image recognitions. Although the subject is not directly related to the supersymmetry, in the analysis of propagation of solitons and edge effects of solitoic wave should be taken into account, where Atiyah-Padobi-Singer index was shown by Hoermander [131]. A good review on Atiyah-Singer’s index theorem [107] and Atiyah-Bott-Shapiro’s Clifford modules [108] are written in the book of Hirzebruch [113]. Stability of index formulas and boundary problems for elliptic differential operators are explained in [132].

In application of mathematical theories to speech recognitions or image recognitions, recurrent neural networks are commonly used [206]. Parcollet et al. [207] applied quaternion neural networks to image recognitions. Although the subject is not directly related to the supersymmetry, in the analysis of propagation of solitons and edge effects of solitoic wave should be taken into account, where Atiyah-Padobi-Singer index was shown by Hoermander [131]. A good review on Atiyah-Singer’s index theorem [107] and Atiyah-Bott-Shapiro’s Clifford modules [108] are written in the book of Hirzebruch [113]. Stability of index formulas and boundary problems for elliptic differential operators are explained in [132].

In application of mathematical theories to speech recognitions or image recognitions, recurrent neural networks are commonly used [206]. Parcollet et al. [207] applied quaternion neural networks to image recognitions. Although the subject is not directly related to the supersymmetry, in the analysis of propagation of solitons and edge effects of solitoic wave should be taken into account, where Atiyah-Padobi-Singer index was shown by Hoermander [131]. A good review on Atiyah-Singer’s index theorem [107] and Atiyah-Bott-Shapiro’s Clifford modules [108] are written in the book of Hirzebruch [113]. Stability of index formulas and boundary problems for elliptic differential operators are explained in [132].

In application of mathematical theories to speech recognitions or image recognitions, recurrent neural networks are commonly used [206]. Parcollet et al. [207] applied quaternion neural networks to image recognitions. Although the subject is not directly related to the supersymmetry, in the analysis of propagation of solitons and edge effects of solitoic wave should be taken into account, where Atiyah-Padobi-Singer index was shown by Hoermander [131]. A good review on Atiyah-Singer’s index theorem [107] and Atiyah-Bott-Shapiro’s Clifford modules [108] are written in the book of Hirzebruch [113]. Stability of index formulas and boundary problems for elliptic differential operators are explained in [132].

In application of mathematical theories to speech recognitions or image recognitions, recurrent neural networks are commonly used [206]. Parcollet et al. [207] applied quaternion neural networks to image recognitions. Although the subject is not directly related to the supersymmetry, in the analysis of propagation of solitons and edge effects of solitoic wave should be taken into account, where Atiyah-Padobi-Singer index was shown by Hoermander [131]. A good review on Atiyah-Singer’s index theorem [107] and Atiyah-Bott-Shapiro’s Clifford modules [108] are written in the book of Hirzebruch [113]. Stability of index formulas and boundary problems for elliptic differential operators are explained in [132].

In application of mathematical theories to speech recognitions or image recognitions, recurrent neural networks are commonly used [206]. Parcollet et al. [207] applied quaternion neural networks to image recognitions. Although the subject is not directly related to the supersymmetry, in the analysis of propagation of solitons and edge effects of solitoic wave should be taken into account, where Atiyah-Padobi-Singer index was shown by Hoermander [131]. A good review on Atiyah-Singer’s index theorem [107] and Atiyah-Bott-Shapiro’s Clifford modules [108] are written in the book of Hirzebruch [113]. Stability of index formulas and boundary problems for elliptic differential operators are explained in [132].

In application of mathematical theories to speech recognitions or image recognitions, recurrent neural networks are commonly used [206]. Parcollet et al. [207] applied quaternion neural networks to image recognitions. Although the subject is not directly related to the supersymmetry, in the analysis of propagation of solitons and edge effects of solitoic wave should be taken into account, where Atiyah-Padobi-Singer index was shown by Hoermander [131]. A good review on Atiyah-Singer’s index theorem [107] and Atiyah-Bott-Shapiro’s Clifford modules [108] are written in the book of Hirzebruch [113]. Stability of index formulas and boundary problems for elliptic differential operators are explained in [132].

In application of mathematical theories to speech recognitions or image recognitions, recurrent neural networks are commonly used [206]. Parcollet et al. [207] applied quaternion neural networks to image recognitions. Although the subject is not directly related to the supersymmetry, in the analysis of propagation of solitons and edge effects of solitoic wave should be taken into account, where Atiyah-Padobi-Singer index was shown by Hoermander [131]. A good review on Atiyah-Singer’s index theorem [107] and Atiyah-Bott-Shapiro’s Clifford modules [108] are written in the book of Hirzebruch [113]. Stability of index formulas and boundary problems for elliptic differential operators are explained in [132].
expressed by quaternions \[109, 188, 189\].

In the twistor formulation of Dunajski \[53\], projection of \( C^2 \rightarrow \mathbf{CP}^1 \) was considered and stability of exponential of infinitesimal deformation proven by Kodaira \[52\] was applied. On a compact surface \( S \), Kodaira \[50\] defined an automorphism \( g \) of a domain \( W \) given by \((z_1, z_2) = C^2 - (0, 0) \sim \mathbf{CP}^1 \),

\[
g : (z_1, z_2) \rightarrow (z'_1, z'_2) = (sz_1 + \lambda z_2^m, tz_2) \quad (103)
\]

where \( m \) is a positive integer and \( s, t, \lambda \) are constants such that

\[
0 < |s| \leq |t| < 1, \quad (tm - s)\lambda = 0. \quad (104)
\]

and for an infinite cyclic group \( \mathbb{Z} \), the quotient surface \( W/\mathbb{Z} \) was homeomorphic to \( S^3 \times S^3 \). It was shown that the cohomology groups \( H^\ast(S^3 \times S^3, \mathbb{C}(F_{\mu})) \), \( \nu = 0, 1, 2, \ldots \) vanish for \( \mu \neq 1 \), which means that there is a curve on the patched complex areas on the surface \( S \).

In the analysis of solitons propagating on 2D planes, quaternion projective space may be useful, by identifying real quaternions \( p_1 \) and \( p_2 \), if there exists a quaternion \( H_\sigma \) \{\( 0 \{1 \} \) such that \( h_{p1} = p_2 h \) for reducing parameters \[203\]. In literatures \[194\], projective quaternion spaces were defined on tangent bundle of \( S^4 \) manifold. But they can be defined on 2D plane bundle parametrized by \( \tau^\pm = t \pm z/c \[4, 189\].

As a model of Higgs boson based on \( SU(4)/Sp(4) \) was also discussed in \[147, 148\]. Experiments of enhancement of Higgs boson decay into \( \gamma \gamma \) due to WZW term via the technicolor was investigated.

As an extension of the Maxwell-Gravitation model, complex octonion model \[202\] in which a quaternion \( H_\sigma \) is used for the electromagnetic field and the second quaternion \( H_p \) is used for gravitational field and complex octonion field

\[
H(h^\alpha) = \sqrt{-1}h^0i_0 + h^ri_r + \sqrt{-1}h^4i_4 + h^{4+r}i_{4+r}, \quad (r = 1, 2, 3)
\]

was proposed. In the curved space metric was defined as \( g_{\alpha\beta}\sqrt{e}du^\alpha du^\beta \), depending on the coordinates of quaternions. It is not clear whether the affine connection of complex octonions can express correct curvature in Maxwell-Gravitation models. Metric tensors \( g_{\alpha\beta} \) should satisfy

\[
g_{\alpha\beta}(x) = \eta_{ab}e_\alpha^a(x)e_\beta^b(x) = e_\alpha^ae_\beta^b \quad (105)
\]

where \( \eta \) is the Minkowski metric, and \( \eta_{ab}e_\beta^b \). Whether such a metric can be defined by quaternion basis vectors instead of usual spinors is not obvious.

In a \( SU(2) \) sharenol model of Yang-Mills equation \[109\], the Riemannian metric and the curvature of \( S^3 \times \mathbb{R} \) manifold were calculated.

In LHFLQCD \[75\], the metric is contained in the dilaton background field factor and absorbed in the Rarita-Schwinger spinor.

Constraints on supersymmetry breaking was studied by Witten \[57\] and spontaneous symmetry breaking of \( SU(3)_C \times SU(3)_L \) to the diagonal \( SU(3) \) was discussed in \[39\]. Bases of these theories seems to exist in Wess-Zumino’s theory \[1\], as written in my review on supersymmetry. There are no BRST ghosts in the holoraphic lightfront QCD (HLFQCD) approach of Brodsky et al. and K-theory approach of Connes.

From the 3 dimensional linear subspace of \( \mathbf{R}^3 \), one can define the sphere \( S^2 \) and mapping \( S^2 \rightarrow \mathbf{D}^2 \) i.e. projective subspace in manifold \( S^3 \). Quaternion projective space was considered to reduce parameters to fit physical data. I would like to point out that supergroup functions in 2D using projective quaternion space can be used in physics and engineering which was be discussed elsewhere \[5, 14\]. It is my impression that the conformality of holonomic functions on projective space which allow patching local coordinates to global one is crucial, and the Hopf manifold proposed by Chern \[121\] can be used to understand dynamics of nature.

The supersymmetry was applied to hadron spectrum region, solid state physics region and Physics of universe including the dark matter. I presented a biased view of supersymmetry based on the theory of Fubini and his collaborators, and Brodsky and his collaborators related quark model to particle physics and supersymmetry based on the theory of Efetov and his collaborators related to low dimensional spin dynamics and solitons.

There are many good references on supersymmetry, and I apologize to authors whom I did not mention in this review. From mathematiccal point of view of supersymmetry, DeWitt’s book \[208\], Freed’s book \[201\], and a concise encyclopedia of supersymmetry edited by Duplij, Siegel and Bagger \[43\] and references therein are helpful.

Acknowledgment I thank Stanley Brodsky for sending information of works in his group, Guy de Téramond and Serge Dos Santos for helpful discussions, late Professor Hideo Nakajima for collaboration on lattice simulations, physics research organizations in France, Germany for giving me opportunities to study symmetry between 1972 and 1989. Dr. Claude Amsler for attracting my attention to \[83\] and Professor Steven Duplij of Kharkov University in Ukraine for sending me the book of encyclopedia of supersymmetry \[43\] in 2005. Thanks are also due to libraries of Tokyo Institute of Technology and the library of the institute of mathematical science of the University of Tokyo for allowing consulting references.
troscopy, Structure, and Dynamics, rXiv:2004.07756v1 [hep-ph] (2020).

[82] Gerard ’t Hooft, *Deterministic Quantum Mechanic: The Mathematical Equations*, Frontiers in Physics, 8 Article 253 (2020).

[83] The CMS Collaboration, *Measurement of the Upsilon(1S) pair production cross section and search for resonances decaying to $\Upsilon(1S)\mu^+\mu^-$ in proton-proton collisions at $\sqrt{s} = 13\text{TeV}$*, To be published in Phys. Lett B (2020).

[84] R.S. Sufian, T. Liu, A. Alexandru, S.J. Brodsky, G. de Téramond, H.G. Dosch, T. Draper, K-F. Liu and Y-B. Yang, *Constraints on charm-anticharm asymmetry in the nucleon from lattice QCD*, To be published in Phys. Lett. B (2020). [arXiv:2003.01078v2 [hep-ph]].

[85] J-P. Landesberg, H-S. Shao, N. Yamanaka, Y-J. Zhang and C. Nois, *Complete NLO QCD study of single and double-quarkonium hadroproduction in the colour-evaporation model at the Tevatron and the LHC*, To be published in Nucl. Phys. B (2020).

[86] J-P. Landesberg and H-S. Shao, *Production of $J/\psi+\eta_c$ versus $J/\psi+J/\psi$ at the LHC: Importance of Real $\alpha_s$ Corrections*, Phys. Rev. Lett. 111, 122001 (2013).

[87] J-P. Landesberg, *New Observables in Inclusive Production of Quarkonia*, arXiv:1903.09185v1 [hep-ph].

[88] E. Caliceti, M. Meyer-Hermann P. Ribeca, A. Surzhykov and U.D. Jentschura, *From useful algorithm for slowly convergent series to physical predictions based on divergent perturbative expansions*, Phys. Rept. 446 (1-3) 1-96 (2007).

[89] H. Mera, T. G. Pedersen and B.K. Nikolic, *Hypergeometric resummation of self-consistent sunset diagram for steady-state electron-boson quantum many-body systems out of equilibrium*, Phys. Rev. B 94, 165429 (2016).

[90] H. Mera, T. G. Pedersen and B.K. Nikolic, *Nonperturbative Quantum Physics from Low-Order Perturbation Theory*, Phys. Rev. Lett. 115, 143001 (2015).

[91] F.D.M. Haldane, *Nonlinear Field Theory of Large-Spin Heisenberg Antiferromagnets: Semiclassically Quantized Solutions of the One-Dimensional Easy-Axis Néel State*, Phys. Rev. Lett. 50 (15) 1153-1156 (1983).

[92] Y-L. Dong, L. Chen, Y-J. Liu and W. Li, *Bilayer linearized tensor renormalization group approach for thermal tensor networks*, Phys. Rev. B 95, 144428 (2017).

[93] K. B. Efetov, *Supersymmetry and theory of disordered metals*, Advances in Physics, 32(1) 53-127 (1983)

[94] Konstantin Efetov, *Supersymmetry in Disorder and Chaos*, Cambridge University Press (1997).

[95] Alexander Altland and Martin R. Zirnbauer, *Non-standard symmetry classes in mesoscopic normal-superconducting hybrid structures*, Phys. Rev. B 55 (2) 1142-1161, (1997).

[96] R. Kennedy and M.R. Zirnbauer, *Bott Periodicity for $Z_2$ Symmetric Ground States of Gapped Free-Fermion Systems*,Comm. Math. Phys. 342, 909-963 (2016).

[97] C.L. Kane and E. J. Mele, *Z_2 Topological Order and the Quantum Spin Hall Effect*, Phys.Rev. Lett 95, 146802 (2005).

[98] C.L. Kane and E. J. Mele, *Quantum Spin Hall Effects in Graphene*, Phys.Rev. Lett 95, 226801 (2005).

[99] L. Fidkowski and A. Kitaev, *Effects of interactions on the topological classification of free fermion systems*, Phys. Rev. B 81, 134509 (2010).

[100] A.P. Schnyder, S. Ryu, A. Furusaki and A.W. W. Ludwig, *Classification of Topological insulators and superconductors in three spatial dimensions*, Phys. Rev. B 78, 195125 (2008).

[101] G. De Nittis and K. Gomi, *Classification of “Quaternionic” Bloch-Bundles, Topological Quantum System of Type AII*, Commun. Math. Phys. 339, 1-55 (2015).

[102] G. De Nittis and K. Gomi, *Differential Geometric Invariants for Time-Reversal Symmetric Bloch-Bundles: The “Real” Case*, [arXiv:1502.01232v2 [math-ph]] (2016).

[103] Ruben Sandapen, *An overview of light-front holography*, [arXiv:2001.03479v1[hep-th]].

[104] Raphael Bousso, *The holographic principle*, Rev. Mod. Phys. 74 (3), 825-874, (2002).

[105] S. Ryu and T. Takayanagi, *Holographic Derivation of Entanglement Entropy from the anti-de Sitter Space/Conformal Field Theory Correspondence*, Phys. Rev. Lett. 96, 181602 (2006).

[106] T. Liu, R.S. Sufian, Guy F. de Téramond, H. Dosch, S.J. Brodsky and A. Deur (HLFHS Collaboration), *Unified Description of Polarized and Unpolarized Quark Distributions in the Proton*, Phys. Rev. Lett. 124 082003 (2020).

[107] M.F. Atiyah and I.M. Singer, *The Index of Elliptic Operators on Compact Manifolds*, Bull. Am. Math. 87, 422-433 (1963).

[108] M.F. Atiyah, R. Bott and A. Shapiro, *Clifford Modules*, Topology 3, suppl 1 3-38 (1964): The University of Edinburg, www.math.ed.ac.uk/ vranick/papers/abs.pdf.

[109] M.F. Atiyah, H.J. Hiitchin, V.G. Drinfeld and Yu.I. Manin, *Construction of Instantons*, Phys. Lett. B 65A (3) 185-187 (1977).

[110] M. Atiyah, R. Bott and V.K. Patodi, *On the Heat Equation and the Index Theorem*, Inventions Math. 19 279-330 (1973); Errata Inventions Math. 28 277-280 (1975).

[111] M.F. Atiyah, V.K. Patodi and I.M. Singer, *Spectral asymmetry and Riemannian geometry*, J. Math. Proc. Cambridge Philos. Soc. 77 43-69 (1975).

[112] R.T. Seeley, *Extension of $C^\infty$ Functions Defined in a Half Space*, Proc. Amer. Math. Soc. 15 625-626 (1964).

[113] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, Berlin (1978).

[114] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge(UK), (2001).

[115] A. Jaffe, A. Lesniewski and M. Lewenstein, *Ground State Structure in Supersymmetric Quantum Mechanics*,Annals of Physics 178, 313-329 (1987).

[116] A. Jaffe, A. Lesniewski and J. Weitsman, *Index of a family of Dirac operators on loop space*, Comm. Math. Phys. 112 75-88 (1983).

[117] A. Jaffe, A. Lesniewski and K.Osterwalder, *Quantum K-Theory*, Commun. Math. Phys. 118 1-14 (1988).

[118] Alain Connes and John Lott, *Particle Models and Non-commutative Geometry*, Nucl. Phys. B (Proc.Suppl.) 18B 29-47 (1990).

[119] Alain Connes, *Geometric non commutative*, Inter Editions, Paris (1990); Translated to Japanese by F.
Maruyama, Iwanami Shoten Pub. (1999).

[120] Alain Connes, *Noncommutative Geometry*, Academic Press, An Imprint of Elsevier, San Diego, New York, Boston (1994).

[121] A.H. Chamseddine and A. Connes, *The Spectral Action Principle*, Commun. Math. Phys. 186 731-750 (1997).

[122] Alain Connes, *Geometry and the Quantum*, arXiv:1703.02470 v1 [hep-th].

[123] C. I. Lazaroiu, E.M. Babalic and I.A. Coman, *Geometric Algebra techniques in Flux Quantifications*, Adv. High Energy Phys. 2016, 7292534 (2016).

[124] A.A. Rosly , A.S. Schwarz and A.A.Voronov, *Geometry of Superconformal Manifolds*, Commun. Math. Phys. 119, 129-152 (1988).

[125] J.P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with intrinsic torsion*, Phys. Rev. D69 086002 (2004).

[126] Charles Kittel, *Introduction to Solid State Physics*, 3rd Edition, John Wiley and Sons Inc, (1966); Translated to Japanese by Uno et al. Maruzen (1968).

[127] Edward Witten, *On the Landau-Ginzburg Description of N = 2 Minimal Models*, Nucl. Phys. B403 159 (1993).

[128] I.C. Davenport and I.V. Melnikov, *Landau-Ginzburg skeletons*, JHEP05, 050 (2017).

[129] S.S. Chern, *Complex Manifolds Without Potential Theory*, D. Van Nostrand Company, INC, Princeton NewJersey (1967).

[130] D.J.H. Garling, *Clifford Algebras: An Introduction*, London Mathematical Society Student Texts 78, Cambridge University Press, New York (2011).

[131] Lars Hoermander, *Linear Partial Differential Equation I*, Springer- Verlag, Berlin (1983).

[132] Lars Hoermander, *Linear Partial Differential Equation III*, Springer- Verlag, Berlin (1987).

[133] C. Becchi, A. Rouet and R. Stora, *Renormalization of the abelian Higgs-Kibble model*, Commun. Math. Phys. 42 127 (1975).

[134] I.V. Tyutin, *Gauge invariance in field theory and statistical mechanics*, Lebedev preprint FIAN, no. 39 (1975).

[135] Taichiro Kugo and Izumi Ojima, *Manifestly Covariant Canonical Formulation of Yang-Mills Theories, Physical state Subsidiary Conditions and Physical S-Matrix Unitarity*, Phys. Lett. 73B 459-462 (1978).

[136] Taichiro Kugo and Izumi Ojima, *Local Covariant Operator Formalism of Non-Abelian Gauge Theories and Quark Confinement Problems*, Supp. of Prog. Theor. Phys. 661-129 (1979)

[137] Hideo Nakajima and Sadataka Furui, *Test of the Kugo-Ojima Confinement Criterion in the Lattice Landau Gauge*, Lattice ’99 proceedings, University of Pisa, Italy, July 1999, Nucl. Phys. B(Proc. Suppl.) 83-84 521-523 (2000), arXiv:hep-lat/9909008.

[138] Hideo Nakajima and Sadataka Furui, *Infrared Features of the Lattice Landau Gauge QCD*, Nucl. Phys. B(Proc Suppl.) 129 - 130, 730-732 (2004).

[139] Sadataka Furui and Hideo Nakajima, *Effects of the quark field on the ghost propagator of Lattice Landau Gauge QCD*, Phys. Rev. D 73, 094506 (2006), arXiv: 0602027[hep-lat].

[140] Daniel Zwanziger, *Renormalizability of the critical limit of lattice gauge theory by BRST invariance*, Nucl. Phys. B 399 477-513 (1993).

[141] Martin Schaden and Daniel Zwanziger, *BRST cohomology and physical space of GZ model*, arXiv:1412.4823 v3 [hep-ph] (2015).

[142] P. Watson and R. Alkofer, *Verifying the Kugo-Ojima Confinement Criterion in Landau Gauge Yang-Mills theory*, arXiv: hep-ph/0102332v2 (2018).

[143] Zbigniew Hasiewicz and Jan L. Ciesinski, *BRST Cohomologies of Mixed and Second Class Constraints*, Symmetry 2020, 12, 428 (2020).

[144] S. Knapen, T. Melia, M. Papucci and K.M. Zurek, *Rays of light from the LHC*, Phys. Rev. D 93, 075020 (2016).

[145] R.D. Peccei and H.R. Quinn, *CP Conservation in the Presence of Pseudoparticles*, Phys. Rev. Lett. 38 (25) 1440-1443 (1977).

[146] J. Jaeckel, M. Jankowiak and M. Spannowsky, *LHC probes the hidden sector*, Physics of the Dark Universe 2 111-117 (2013).

[147] J. Barnard, T. Ghergetta and T.S. Ray, *UV descriptions of composite Higgs models without elementary scalars*, JHEP 1402 002 (2014), arXiv: 1311.6562 [hep-ph].

[148] A. Arbey, G. Cacciapaglia, H. Cai, A. Deandrea, S. Le Corre and F. Sannino, *Fundamental Composite Electroweak Dynamics: Status at the LHC*, Phys. Rev. D 95, 015028 (2017).

[149] H. Bahl, S. Lieber and T. Stefaniak, *MSSM Higgs benchmark scenarios for Run2 and beyond: the low tanβ region*, Eur. Phys. J. C79 279 (2019).

[150] W. Bernreuther, L. Chen, I. Garcia, M. Perelló, R. Poeschl, F. Richard, E. Ros and M. Vos, *CP-violating top quark couplings at future linear e+e− colliders*, Eur. Phys. J. C78 155 (2018).

[151] H. Bahl and W. Hollik, *Precise prediction of the MSSM Higgs boson masses for low Mχ*, JHEP07 182 (2018).

[152] M. Kobayashi and T. Maskawa, *CP Violation in the Renormalizable Theory of Weak Interaction*, Prog. Theor. Phys. 49 (2), 652-657 (1973).

[153] J. Miller, *Charm-quark decays violate charge-parity symmetry*, Physics Today72 (8) (2019).

[154] R. Aaij et al, (LHCb Collab.), *Evidence for CP Violation in Time-Integrated D^0 → h^+ h^- Decay Rates*, Phys Rev. Lett. 108 11602 (2012).

[155] R. Aaij et al, (LHCb Collab.), *Observation of CP Violation in Charm Decays*, Phys. Rev. Lett. 122, 211803 (2019).

[156] G. H. Duan, K.-i. Hikasa, J. Ren, L. Wu and J.M. Yang, *Probing bino-wino coannihilation dark matter below the neutrino floor at the LHC*, Phys. Rev. Lett. D98, 015010 (2018).

[157] S. Hamdan and J. Unwin, *Dark matter freeze-out during matter domination*, Modern Physics Letters A 33 (29) 1850181 (2018).

[158] E. Gabrielli, K. Kannike, B. Mele, M. Raidal, C. Spethmann and H. Veermoe, *A SUSY inspired simplified model fo the 750GeV diphoton excess*, Physics Letters B756 36-41 (2016).

[159] R. De Benedetti, C. Li, T. Li, A. Lux, J.A. Maxin and D. V. Nanopoulos, *Inspiration from intersecting D-branes: general supersymmetry breaking soft terms
in no-scale \( F - SU(5) \), Eur. Phys. J. C 78 958 (2018).

[160] K. Fujikawa, Path-Integral Measures for Gauge-Invariant Fermion Theories, Phys. Rev. Lett. 42 (18) 1195-1198 (1979).

[161] ATLAS Collaboration, Measurements of Higgs boson production and couplings in diboson final states with the ATLAS detector at the LHC, Phys. Letters B 726 88-119 (2013).

[162] ATLAS Collaboration, Corregium, Phys. Letters B 734 406 (2014)

[163] Sadataka Furui, Cartan’s Supersymmetry and the Decay of a \( H^0(0^+) \), arXiv: 1504.03795v5 [hep-ph] (2016).

[164] Jian-Hao Zhang, Qing-Rui Wang, Shuo Yang, Yang Qi and Zheng-Cheng Gu, Construction and classification of point-group symmetry-protected topological phases in two-dimensional interacting fermionic systems, Phys. Rev. B 101, 100501 (2020).

[165] Zheng-Cheng Gu and Xiao-Gang Wen, Symmetry-protected topological order for interacting fermions: Fermionic topological nonlinear \( \sigma \) models and a spectral group supercohomology theory, Phys. Rev. B 90, 115141 (2014).

[166] Qing-Rui Wang and Zheng-Cheng Gu, Construction and classification of symmetry protected topological phases in interacting fermion systems, arXiv:1811.00530v3 [cond-mat.str-el].

[167] G. W. Mackey, Induced Representations of Groups and Quantum Mechanics, W.A. Benjamin, INC, New York (1968).

[168] Z. Zhao, M. Faizal, M.B. Shah, A. Bhat, P.A. Ganal, Z. Zaz, S. Massoud, J. Raza and R.M. Irfan. Nonlocal deformation of a supersymmetric field theory, Eur. Phys. J. C. 77 612 (2017).

[169] B. Berxc and F.F. Assad, Metamagnetism and Lifshitz transitions in metals for heavy fermions, Phys. Rev. B86, 075108 (2012).

[170] A. Hakie and M. Vojta, Zeeman-Driven Lifshitz Transition: A model for Experimentally observed Fermi-Surface Reconstruction in Yb2Rh2Si2, Phys. Rev. Lett.106, 137002 (2011).

[171] Steven R. White, Density Matrix Formulation for Quantum Renormalization Group, Phys. Rev. Lett. 69 (19) 2863-2866 (1992).

[172] V. Bacot, M. Labousse, A. Eddi, M. Fink and E. Fort, Time reversal and holography with spacetime transformations, Nature Physics 12 972-977 (2016).

[173] L.D. Landau and E.M. Lifshitz, Theory of Elasticity, (1959). Translated from Russian by J.B. Sykes and W.H. Reid , Pergamon (1964); E. Lifshitz and V. Bérestetski, 4th edition (1987), Translated to Japanese by J. Sato and Z. Ishibashi, Tokyo Tosho Pub. (1989).

[174] T. Miwa, M. Jimbo and E. Date, Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras, Cambridge University Press, Cambridge (2000), Translated from Japanese edition published by Iwanami Shoten (1993).

[175] M.J. Ablowitz, D. J. Kaup, A.C. Newell and H. Segur, The Inverse Scattering Transform-Fourier Analysis for Nonlinear Problems, Studies in Applied Mathematics, vol LIII (4) M.I.T. (1974).

[176] T. Miwa, M. Jimbo and E. Date, Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras, Cambridge University Press, Cambridge (2000), Translated from Japanese edition published by Iwanami Shoten (1993).

[177] V.E. Zakharov and A.B. Shabat, Exact Theory of Two-Dimensional Self-focusing and One-Dimensional Self-Modulation of Waves in Nonlinear Media, Sov. Phys. JETP 34 (1) 62-69 (1972).

[178] V.E. Zakharov and A.B. Shabat, Interaction between solitons in a stable medium, Sov. Phys. JETP 37 (5) (1973).

[179] Y. I. Manin and A.O. Radul, A Supersymmetric Extension of the Kadomtsev-Petviashvili Hierarchy, Commun. Mathematical Physics. 98, 65-77 (1985).

[180] A.R. Choudhury and S. Roy, On the Baecklund transformation and Hamiltonian properties of super evaluation equations, J. Math. Phys. 27 (10) (1986).

[181] I.N. McAtourh and C.M. Yung, Gelfand-Dikii Analysis for \( N = 2 \) Supersymmetric Kad Equations, Commun. Math. Phys. 152 1-18 (1993).

[182] L. O. Chua, Memristor - The missing circuit element, IEEE Trans. Circuit Th. CT-18 507-519 (1971).

[183] L. O. Chua, Five non-volatile memristor enigmas solved, Appl. Phys. A 124 563 (2018).

[184] S. Dos Santos, Symmetry of nonlinear acoustic equations using group theoretic methods : a signal processing tool for extracting judicious physical variables, In proceedings of the joint congress CFA/DAGA, Strasbourg pp. 549-550 (2004).

[185] S. Furui and T. Takano, On the amplitude of External Perturbation and Chaos via Devil’s Staircase in Muthuswamy-Chua System, International Journal of Bifurcation and Chaos, 23 1350136 (2014), arXiv: [nlin], CD 1406.4346

[186] I.Y. Solodov and B. A. Korshak, Instability, Chaos, and "Memory" in Acoustic-Wave-Crack Interaction, PRL 88 (1) 014303 (2002).

[187] N.D. Mermin and H. Wagner, Absence of Ferromagnetism and Antiferromagnetism in One or Two-Dimensional Isotropic Heisenberg Models, Phys. Rev. Lett. 17(22)1133-1136 (1966).

[188] P.C. Hohenberg, Existence of Long-Range Order in One and Two Dimensions, Phys. Rev. 158 (2) 383 (1967).

[189] S. Furui, A Closer Look at Gluons, Chapter 6 of a book “Horizon in World Physics vol. 302”, Ed. by Albert Reimer, Nova Scientific Pub. (2020).

[190] S. Furui, Understanding Quaternions, A chapter of a book to be published from Nova Scientific Pub.

[191] S. Dos Santos, Symmetry of nonlinear acoustics equation using group theoretic methods : a signal processing tool for extracting judicious physical variables, Proceedings of the Joint Congress CFA/DAGA’04, Strasbourg 2004, 549-550 (2004).

[192] S. Dos Santos and O. Bou Matar, Symmetry of KZ(khokhlov-Zabolotskaya) equation, preprint (2004).

[193] S. Dos Santos and C. Plag, Excitation Symmetry Analysis Method (ESAM) for Calculation of Higher Order Nonlinearity, Int. J. Nonlinear Mech. 43 p. 114-118 (2008).

[194] S. Dos Santos, Advanced ground truth multimodal imaging using Time Reversal (TR) based Nonlinear Elastic Wave Spectroscopy (NEWS): medical imaging trends versus non-destructive testing applications, Chapter 4 in S. Dos Santos et al.(eds), Recent Advances in Mathematics and Technology, Applied and Numerical Harmonic Analysis, Springer Nature Switzerland AG (2020).
[194] R.G. Swan, *Vector Bundles and Projective Modules*, Trans. American Mathematical Society, **105** (2) 264-277 (1962).

[195] Bochicchio and A. Pilloni, *Gauge theories in anti-selfdual variables*, JHEP **09** 39 (2013).

[196] Y. Yu, Y-S. Wu and X. Xie, *Bulk-edge correspondence, spectral flow and Atiyah-Patodi-Singer theorem for the $\mathbb{Z}_2$-invariant in topological insulators*, Nucl. Phys. B **916** 550-566 (2017).

[197] Peter J. Braam and Pierre van Baal, *Nahm’s Transformation for Instantons*, Commun. Math. Phys. **122**, 267 (1989).

[198] Pierre van Baal and Bas van den Heuvel, *Zooming-in on the SU(2) fundamental domain*, Nucl. Phys. B **417** 215-237 (1994).

[199] Pierre van Baal and N.D. Hari Dass, *The theta Dependence Beyond Steepest Descent*, Nucl. Phys. B **385**, 185-226 (1992).

[200] Sidney Coleman, *Aspects of Symmetry*, Selected Erice Lectures, Cambridge Univ. Press (1985).

[201] J.P. Morais, S. Georgiev and W. Sproessig, *Real Quaternionic Calculus Handbook*, Birkhaeuser, Springer Basel (2014).

[202] Zi-Hua Weng, *Forces in the complex octonion curved space*, International Journal of Geometric Methods in Modern Physics, **13** (6)1650076 (2016).

[203] Claude Chevalley, *Theory of Lie Groups*, Princeton University Press, Princeton (1946), Asian Text; Overseas Publications LTD, Tokyo (1965).

[204] J. -M. Souriau, *Structure des Systèmes Dynamiques*, maitrise de mathématiques, Dunod Université, Paris (1970).

[205] Dietmer Ebert, *Eichtheorien*, Grundlage der Elementarteilchenphysik, VCH Verlagsgesellschaft, Weinheim (1989).

[206] C.C. Aggarwal, *Neural Networks and Deep Learning*, Springer International Publishing A.G. part of Springer Nature (2018).

[207] T. Parcollet, M. Ravanelli, M. Morchild, G. Linar’es, C. Trabelsi, R. De Mori and Y. Bengio, *Quaternion Recurrent Neural Networks*, ICLR 2019 Conference Paper (2019).

[208] Bryce DeWitt, *Supermanifolds*, Cambridge Monographs on mathematical physics, Cambridge University Press, Cambridge (1992).

[209] Daniel S. Freed, *Five lectures on supersymmetry*, American Mathematical Society (1999).