Noise Estimate of Pendular Fabry-Perot through Reflectivity Change

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A key issue in developing pendular Fabry-Perot interferometers as very accurate displacement measurement devices, is the noise level. The Fabry-Perot pendulums are the most promising device to detect gravitational waves, and therefore the background and the internal noise should be accurately measured and reduced. In fact terminal masses generates additional internal noise mainly due to thermal fluctuations and vibrations. We propose to exploit the reflectivity change, that occurs in some special points, to monitor the pendulums free oscillations and possibly estimate the noise level. We find that in spite of long transients, it is an effective method for noise estimate. We also prove that to only retain the sequence of escapes, rather than the whole time dependent dynamics, entails the main characteristics of the phenomenon. Escape times could also be relevant for future gravitational wave detector developments.

I. INTRODUCTION

Noise at thermal equilibrium is given by the temperature, i.e. it is connected to the spontaneous flow of energy. In a macroscopic device, where the spontaneous flow of energy often cannot be even defined [1], random perturbations of the dynamical variables can be characterized by some equivalent noise level, by measuring, say, the standard deviation of a representative variable. For instance the noise level in an electronic device can be measured detecting current (or voltage) fluctuations and measuring the background by the power spectral density.

The problem is very complicated for low noise devices, where the measurement itself changes the properties of the devices [2] [3] (see the case of quantum noise characterization [1][4]), and could introduce a substantial amount of extra disturbances.

A preeminent example is the measurement of noise levels in pendular Fabry-Perot (henceforth FP) interferometers, that is the most promising (and intriguing [6][7]) candidate detection system for gravitational waves [8]. This devices consist (see Ref. [9] for details) of multiple suspended pendulums to reduce the external noise, and terminal masses far apart to make the optical distance of several kilometers (in order to enhance sensitivity and enlarges the useful frequency band [10]). Still, it is required that the macroscopic object are subject to background noise as low as to achieve (in the next future) quantum fluctuations of the macroscopic mass. A possibility to measure the noise level is to indirectly retrieve the fluctuations. An indirect effect of fluctuations is given by the Kramers escape [11]. It is well known that a particle (or, in general, a degree of freedom) subject to white noise of intensity (correlator, in statistical terms) $D$ and trapped in a potential of height $\Delta U$ and subject to a friction $\gamma$ overcomes the barrier at a rate:

$$r = r_0 \exp \left( -\frac{\gamma \Delta U}{D} \right)$$

(1)

It is therefore tempting to only measure the rate $r$ detecting the passages through special points related to the barrier $\Delta U$. From the point of view of accuracy, the exponential dependence in Eq.(1) is promising, in that small variations of the noise intensity $D$ induce large changes in the escape rate $r$. We show that this is in fact the case: despite we reduce the sampled dynamics to relatively few points, the noise intensity estimate is rather effective, and the reliability can be analytically estimate as a function of the system parameters.

II. MATHEMATICAL MODEL

The pendulums for gravitational waves detection placed in a FP cavity, in absence of external deterministic signal is described by the following dimensionless equation:

$$\frac{d^2z}{dt^2} + \gamma \frac{dz}{dt} = -z + \Pi_M A(2\pi z) + \xi(t), \quad (2)$$

$$<\xi(t),\xi(t')> = 2D \delta(t-t'). \quad (3)$$

We denote with $z$ the displacement of terminal mass normalized respect to $\lambda$, the wave length of incident laser.
radiation. The symbol \( \xi(t) \) denotes the random process incorporating the whole noise budget of the system applied on the end mass. Furthermore \( r_i \) and \( t_i \) are the reflection and transmission coefficients, \( \Pi_M = R_M/\mu \omega_0^2 \) is the normalized radiation pressure coefficient, \( \mu \) and \( \omega_0 \) are the reduced mass of the pendulum and the mechanical frequency of the end mass, respectively. Time is normalized respect to the inverse of the natural frequency, \( \omega_0 \). The normalized input power is \( R_M = 2P_M/c \) where \( P_M \) is the laser input power. The friction constant \( \gamma = \dot{\gamma}/\omega_0 \) is given by the pendulum dissipation constant \( \dot{\gamma} \) divided by \( \omega_0 \). The function \( A \) is the Airy function \( A = 1/(1 + F \sin^2 \phi) \), where \( F = \pi \sqrt{r_1 r_2}/(1 - r_1 r_2) \) is the Finesse of the cavity \( r_i \) and \( t_i \) are the reflection and transmission coefficients, respectively; \( i = 1, 2 \) refers to the mirrors, see Fig. 1, and \( \phi = 2\pi z \) is the phase of the circulating light. The phases \( \phi \) also determine the half-width \( \Delta \phi \) of the resonance, \( \Delta \phi = \pi/2F \). The maximum stored power corresponds to the peaks of the Airy function \( (\phi = n\pi) \). In Eq. \( 3 \), the noise intensity \( D \) is assumed dominated by the external sources, and therefore does not obey the dissipation fluctuation theorem. This assumptions was also implicitly employed in Eq. \( 1 \). We underline that the dissipation parameter is very important. From the point of view of the detection, damping should be low enough to allow the cavity to reach high quality factors, and in fact actual systems achieve such small damping as \( \gamma = 10^{-6} [12] \).

Equation \( 2 \) describes the motion of a particle in the potential of Fig. 1b.

\[
V(z) = \frac{1}{2} \xi^2 - \frac{\Pi_M}{1 + F} \arctan\left(\sqrt{F + 1} \tan(2\pi z) + \pi \left(2z + \frac{1}{2}\right)\right)
\]

where \( [\cdot] \) denote the floor truncation function. The motion of this particle (without friction and external noise) can be also described by the hamiltonian

\[
H(z, p) = \frac{1}{2}p^2 + V(z)
\]

where \( p \) is the particle momentum.

Placed in a minimum, the particle can escape subject to the random perturbation described by Eq. \( 3 \). In escaping, it crosses the points of sharp change of reflectivity, \( V_+ \) and \( V'_+ \). Thus the time elapsed to overcome the barrier, the escape time (or the first passage time \( \tau_1 \)) given by Eq. \( 1 \) can be detected without mechanical contacts.

To evaluate the performances we have numerically retrieved the ETs distributions employing a quasi-symplectic modified velocity Verlet algorithm with velocity randomization for the integration of stochastic differential equations [13]. Moreover, we have found consistent results with another leapfrog algorithm [14] that has proved to be very efficient even at extremely low dissipation [15]. Finally, the algorithms have been tested against the known estimates for the washboard potential at very low dissipation \( (\gamma << 1) \) and noise \( (D << 1) \) [16]. We have found, for the whole dissipation range \( 10^{-2} > \gamma > 10^{-6} \), that the theory lies within the 95% confidence limit of the numerical simulations. Apart from the difficulty of the numerical simulations, low dissipation has the physical consequence of very long transients, on the scale \( 1/\gamma \). An example is shown in Fig. 2 for the real system (a) (see also [17] and simulations of Eqs. 2). The behavior is quite particular, in that the multiple metastable points of the potential (Fig. 1) can temporarily trap the pendulums. There are therefore several oscillations, with different amplitude, that can be activated by noise. It is particularly interesting that the oscillations at such different scales are obtained by noise alone.

III. NOISE ESTIMATE

The pendular interferometer, being characterized by a very low damping term \( \gamma \approx 10^{-6} [12] \), should be handled
I(6) is that in Fig. 3a. A simple geometrical interpretation of eq. (6) is that one can exploit the following parabolic approximation (by comparison with Fig. 3a) it can be found in acceptable agreement

\[ I(E) \simeq \frac{(V - V_-)2\pi}{\omega_r} + \frac{1}{2}\frac{(V - V_-)^2}{\langle d^2I/dE^2 \rangle_{E=V_-}} \]  

(8)

to compute the average ET, here \( \omega_r \) is the well resonance and \( Ei(\cdot) \) is the exponential integral function, see [21].

The eq. (7) can thus be analytically evaluated:

\[
    r^{-1} = \left(\frac{1 + 2\rho}{r_H}\right) - \frac{1}{\gamma} (1 - 2\rho) \log(1 + \rho \gamma \Delta V/D) + \\
    \frac{1}{\gamma} e^{-1/\rho} (1 + 2\rho) \left[ Ei\left(\frac{1}{\rho}\right) - Ei\left(\gamma \Delta V/D + \frac{1}{\rho}\right) \right] .
\]  

(9)

For

\[ \rho = \frac{D \omega_r}{2\pi \gamma} \left. \frac{d^2I}{dE^2} \right|_{E = V_-} = 0, \]  

(10)

Eq. (9) reduces to the ET of the harmonic oscillator (here \( r_H \) is the rate for the harmonic oscillator and \( \gamma_E \) is the Euler-Mascheroni constant):

\[
    r_H^{-1} = \frac{1}{\gamma} \left[ Ei\left(\frac{\gamma \Delta V}{D}\right) - \log\left(\frac{\gamma \Delta V}{D}\right) - \gamma_E \right] .
\]  

(11)

More sophisticated numerical evaluation of eq. 7 based on the action-angle variable depicted in Fig. 3 introduce a relative error less than 1% in any physical relevant situation.

The distributions of the ETs shown in Fig. 4 is exponential (apart a cutoff) and it is evident that can be exploited to determine the intensity of the noise. This amounts to inferring the noise level from the distributions of Fig. 4 with statistical estimation of the parameter \( D \) in Eq. (9).

The efficacy of the method for the estimate of the noise level has been evaluated computing the variance \( \sigma_N^2 \) of the estimated temperature as a function of the sample size \( N \) and of the noise intensity \( D \), under the hypothesis that ETs are described by an exponential distribution. One can derive the large \( N \) behavior of the Maximum Likelihood estimator for an exponential distribution; in fact the Fisher information \( J \) of the distribution given by Eq. (9) reads: \( \sigma_N^2 \simeq J^{-1}/N \) [22]. Thus, the theoretical estimate shows that the relevant parameter is the noise level \( D \) normalized to the product of damping and the energy barrier, \( \gamma \Delta U \). The dependance is displayed in Fig. 5.

The behavior is therefore shown in Fig. 5 as a function of the noise temperature. The variance of the estimate relative error increases with noise level up to a saturation point (around \( D \simeq \gamma \Delta V \) that depends, as expected, upon \( N^{-1/2} \) and exhibits a weak dependence upon \( \rho \) (see in the extremely low dissipation limit. Some analytical approaches to handle activation processes as those entailed in Eq. 3 have been proposed. These approaches extend the treatment of stochastic differential equations with vanishingly damping up to finite values [19] [20]. To apply the method described by Ref. [16], we insert the total (electromagnetic and mechanical) potential in the energy diffusion limit of the escape rate [20]. In this approach the action angle variable is used

\[ I(E) = \oint pdz \]  

(6)

to describe the diffusive equation, the contour integral is defined by the isoenegy surface \( H(z, p) = E \) (see inset in Fig. 3a). A simple geometrical interpretation of eq. (6) is that \( I(E) \) is the area (in the phase space) of the closed surface determined by the inequality \( H(z, p) \leq E \) (as depicted in the inset of Fig. 3a).

Using the derivative of the action \( I \) in the minimum of the potential \( V_- \) and integrating by parts we obtain:

\[
    r^{-1} = \frac{\gamma}{D^2} \int_{V_-}^{V_+} \frac{I(E)e^{-E\gamma/D}dE}{I(E)} + \frac{\gamma}{D} \int_{V_-}^{V_+} e^{E\gamma/D} \frac{dE'}{I(E')}E' + \frac{\Delta V}{D}.
\]  

(7)
The estimate of the noise level improves by increasing $\gamma \Delta V$ i.e. the dissipation and/or the potential barrier. From the physical consideration that a minimum time occurs before escape [23] (see also Fig. 4), it is evident that the ET density departs from the exponential model for short escapes. Such deviations from the exponential distribution lead to an overestimate of the relative error (as evident from the fitted curve in Fig. 5) numerically evaluated to be about 30% smaller.

Apart the finite size error, the method could also be biased. This has been checked by numerical simulations, and we have found that the distortion of the temperature estimate is below 5%.

Finally, we will hint to the effect of a periodic drive, to mimic the perturbation of a gravitational wave. If such a sinusoidal term is added in Eq.(2), the distribution of the ETs is modified [23] and the ETs can be used for the signal detection [24].

IV. CONCLUSION

We propose to employ the abrupt reflectivity changes to estimate the noise level of pendular FP. We have shown that an effective estimate of the noise can be reached. Realistic models include very low dissipation $\gamma \approx 10^{-6}$. Such low damping, combined with multistable potential determines a peculiar behavior of the transient, that exhibit oscillations at different amplitudes. This could resemble glitches, as shown in Fig. 2. We have also estimated, in the very low damping limit, the average escape rate. Assuming an exponential distribution, the approximation allows for an analytical evaluation of the reliability of the noise estimate. In particular, as also demonstrated by the simulations, we can ascertain that the estimate improves as the square root of the number of detected escapes, as in optimal estimates. The analysis goes further, and gives also the behavior as a function of the system parameters. While noise estimate is a preliminary measurement, the natural question to follow is about signal detection. It has been proved that resonant activation is connected to some sub-optimal strategies, while for optimal strategies it disappears [24]. In perspective we propose that it could be interesting to estimate the noise level when the disturbance is not Gaussian, as
FIG. 5: (a) Relative accuracy $\sqrt{N} \sigma_N / D$ of the detected noise level as a function of the normalized external noise intensity $D/(\gamma \Delta V)$ for different values of the sample numerosity. Symbols refer to simulations of the Langevin Eq. (2) for $N = 1000$, $F = 20$, $\Pi_M = 2$, $\Delta V = 0.042$, $D = 0.00042$ (filled points) and $D = 0.00084$ (empty circles). The dashed line denotes the fitting. (b) The asymptotic behaviors as a function of the parameter $\rho$ that represents the deviations of the model Eq. (2) from the harmonic oscillator ($\rho = 0$).

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