FI-modules and stability for representations of symmetric groups

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September 10, 2014

Abstract

In this paper we introduce and develop the theory of FI-modules. We apply this theory to obtain new theorems about:

• the cohomology of the configuration space of $n$ distinct ordered points on an arbitrary (connected, oriented) manifold;
• the diagonal coinvariant algebra on $r$ sets of $n$ variables;
• the cohomology and tautological ring of the moduli space of $n$-pointed curves;
• the space of polynomials on rank varieties of $n \times n$ matrices;
• the subalgebra of the cohomology of the genus $n$ Torelli group generated by $H^1$;

and more. The symmetric group $S_n$ acts on each of these vector spaces. In most cases almost nothing is known about the characters of these representations, or even their dimensions. We prove that in each fixed degree the character is given, for $n$ large enough, by a polynomial in the cycle-counting functions that is independent of $n$. In particular, the dimension is eventually a polynomial in $n$. In this framework, representation stability (in the sense of Church–Farb) for a sequence of $S_n$-representations is converted to a finite generation property for a single FI-module.

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*The authors gratefully acknowledge support from the National Science Foundation. The second author’s work was partially supported by a Romnes Faculty Fellowship.
1 Introduction

In this paper we develop a framework in which we can deduce strong constraints on naturally occurring sequences of $S_n$-representations using only elementary structural symmetries. These structural properties are encoded by objects we call FI-modules.

Let $\text{FI}$ be the category whose objects are finite sets and whose morphisms are injections. The category $\text{FI}$ is equivalent to the category with objects $\{1, \ldots, n\}^+$ for $n \in \mathbb{N}$ and morphisms from $m$ to $n$ being the injections $m \hookrightarrow n$.

**Definition 1.1 (FI-module).** An $\text{FI}$-module over a commutative ring $k$ is a functor $V$ from $\text{FI}$ to the category of $k$-modules. We denote the $k$-module $V(n)$ by $V_n$.

Since $\text{End}_{\text{FI}}(n) = S_n$, any FI-module $V$ determines a sequence of $S_n$-representations $V_n$ with linear maps between them respecting the group actions. One theme of this paper is the conceptual power of encoding this large amount of (potentially complicated) data into a single object $V$.

Many of the familiar notions from the theory of modules, such as submodule and quotient module, carry over to FI-modules. In particular, there is a natural notion of finite generation for FI-modules.

**Definition 1.2 (Finite generation).** An FI-module $V$ is finitely generated if there is a finite set $S$ of elements in $\bigsqcup_i V_i$ so that no proper sub-FI-module of $V$ contains $S$; see Definition 2.3.4.

It is straightforward to show that finite generation is preserved by quotients, extensions, tensor products, etc. Moreover, finite generation also passes to submodules when $k$ contains $\mathbb{Q}$, so this is a robust property.

**Theorem 1.3 (Noetherian property).** Let $k$ be a Noetherian ring containing $\mathbb{Q}$. The category of FI-modules over $k$ is Noetherian: that is, any sub-FI-module of a finitely generated FI-module is finitely generated.

A theorem of Snowden [Sn, Theorem 2.3] can be used to give another proof of Theorem 1.3. In a sequel to the present paper, the authors, in joint work with Rohit Nagpal, removed the restriction in Theorem 1.3 that $\mathbb{Q} \subset k$, proving that the category of FI-modules over any Noetherian ring is Noetherian [CEFN, Theorem A].
Examples of FI-modules. Finitely generated FI-modules are ubiquitous. To illustrate this we present in Table 1 a variety of examples of FI-modules that arise in topology, algebra, combinatorics and algebraic geometry. In the course of this paper we will prove that each entry in this list is a finitely generated FI-module. The exact definitions of each of these objects will be given later in the paper. Any parameter here not equal to \( n \) should be considered fixed and nonnegative.

| FI-module \( V = \{ V_n \} \) | Description |
|---------------------------------|-------------|
| 1. \( H^i(\text{Conf}_n(M); \mathbb{Q}) \) | Configuration space of \( n \) distinct ordered points on a connected, oriented manifold \( M \) (§6) |
| 2. \( R^{(r)}_J(n) \) | \( J = (j_1, \ldots, j_r) \), \( R^{(r)}(n) = \bigoplus_J R^{(r)}_J(n) \) = \( r \)-diagonal coinvariant algebra on \( r \) sets of \( n \) variables (§5.1) |
| 3. \( H^i(\mathcal{M}_{g,n}; \mathbb{Q}) \) | \( \mathcal{M}_{g,n} \) = moduli space of \( n \)-pointed genus \( g \geq 2 \) curves (§7.1) |
| 4. \( \mathcal{R}^i(\mathcal{M}_{g,n}) \) | \( i \)-th graded piece of tautological ring of \( \mathcal{M}_{g,n} \) (§7.1) |
| 5. \( \mathcal{O}(X_{P,r}(n))_i \) | space of degree \( i \) polynomials on \( X_{P,r}(n) \), the rank variety of \( n \times n \) matrices of \( P \)-rank \( \leq r \) (§5.3) |
| 6. \( G(A_n/\mathbb{Q})_i \) | degree \( i \) part of the Bhargava–Satriano Galois closure of \( A_n = \mathbb{Q}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^2 \) (§5.2) |
| 7. \( H^i(\mathcal{I}_n; \mathbb{Q})_{\text{alb}} \) | degree \( i \) part of the subalgebra of \( H^*(\mathcal{I}_n; \mathbb{Q}) \) generated by \( H^1(\mathcal{I}_n; \mathbb{Q}) \), where \( \mathcal{I}_n \) = genus \( n \) Torelli group (§7.2) |
| 8. \( H^i(\text{IA}_n; \mathbb{Q})_{\text{alb}} \) | degree \( i \) part of the subalgebra of \( H^*(\text{IA}_n; \mathbb{Q}) \) generated by \( H^1(\text{IA}_n; \mathbb{Q}) \), where \( \text{IA}_n \) = Torelli subgroup of \( \text{Aut}(F_n) \) (§7.2) |
| 9. \( \text{gr}(\Gamma_n)_i \) | \( i \)-th graded piece of associated graded Lie algebra of many groups \( \Gamma_n \), including \( \mathcal{I}_n, \text{IA}_n \) and pure braid group \( P_n \) (§7.3) |

Table 1: Examples of finitely generated FI-modules

Character polynomials. Each of the vector spaces in Table 1 admits a natural action of the symmetric group \( S_n \). As noted above, any FI-module \( V \) provides a linear action of \( \text{End}_{\text{FI}}(n) = S_n \) on \( V_n \). If \( V \) is finitely generated, the representations \( V_n \) satisfy strong constraints, as we now explain.

For each \( i \geq 1 \) and any \( n \geq 0 \), let \( X_i: S_n \to \mathbb{N} \) be the class function defined by

\[
X_i(\sigma) = \text{number of } i\text{-cycles in the cycle decomposition of } \sigma.
\]

Polynomials in the variables \( X_i \) are called character polynomials. Though perhaps not widely known, the study of character polynomials goes back to work of Frobenius, Murnaghan, Specht, and Macdonald (see e.g. [Mac, Example I.7.14]).

It is easy to see that for any \( n \geq 1 \) the vector space of class functions on \( S_n \) is spanned by character polynomials, so the character of any representation can be described by such a polynomial. For example, if \( V \simeq \mathbb{Q}^n \) is the standard permutation representation of \( S_n \), the character \( \chi_V(\sigma) \) is the
number of fixed points of $\sigma$, so $\chi_V = X_1$. If $W = \wedge^2 V$ then $\chi_W = \left(\frac{X_1}{2}\right) - X_2$, since $\sigma \in S_n$ fixes those basis elements $x_i \wedge x_j$ for which the cycle decomposition of $\sigma$ contains the pair of fixed points $(i)(j)$, and negates those for which it contains the 2-cycle $(i \, j)$.

One notable feature of these two examples is that the same polynomial describes the characters of an entire family of similarly-defined $S_n$-representations, one for each $n \geq 0$. Of course, the expression of a class function on $S_n$ as a character polynomial is not unique; for example $X_N$ vanishes identically on $S_n$ for all $N > n$. However, two character polynomials that agree on $S_n$ for infinitely many $n$ must be equal, so for a sequence of class functions $\chi$ on $S_n$ it makes sense to ask about “the” character polynomial, if any, that realizes $\chi$.

**Definition 1.4 (Eventually polynomial characters).** A sequence $\chi_n$ of characters of $S_n$ is eventually polynomial if there exist integers $r$, $N$ and a character polynomial $P(X_1, \ldots, X_r)$ such that

$$\chi_n(\sigma) = P(X_1, \ldots, X_r)(\sigma) \quad \text{for all } n \geq N \text{ and all } \sigma \in S_n.$$ 

The degree of the character polynomial $P(X_1, \ldots, X_r)$ is defined by setting $\deg(X_i) = i$.

One of the most striking properties of FI-modules in characteristic 0 is that the characters of any finitely generated FI-module have such a uniform description.

**Theorem 1.5 (Polynomiality of characters).** Let $V$ be an FI-module over a field of characteristic 0. If $V$ is finitely generated then the sequence of characters $\chi_{V_n}$ of the $S_n$-representations $V_n$ is eventually polynomial. In particular, $\dim V_n$ is eventually polynomial.

The character polynomial of a finitely generated FI-module is the polynomial $P(X_1, \ldots, X_r)$ which gives the characters $\chi_{V_n}$. It will always be integer-valued, meaning that $P(X_1, \ldots, X_r) \in \mathbb{Z}$ whenever $X_1, \ldots, X_r \in \mathbb{Z}$. In situations of interest one can typically produce an explicit upper bound on the degree of the character polynomial by computing the weight of $V$ as in Definition 3.2. In particular this gives an upper bound for the number of variables $r$. Moreover, computing the stability degree of $V$ (§3.1) gives explicit bounds on the range $n \geq N$ where $\chi_{V_n}$ is given by the character polynomial. This converts the problem of finding all the characters $\chi_{V_n}$ into a concrete finite computation. Note that the second claim in Theorem 1.5 follow from the first since

$$\dim V_n = \chi_{V_n}(\text{id}) = P(n, 0, \ldots, 0).$$

One consequence of Theorem 1.5 is that $\chi_{V_n}$ only depends on “short cycles”, i.e. on cycles of length $\leq r$. This is a highly restrictive condition when $n$ is much larger than $r$. For example, the proportion of permutations in $S_n$ that have no cycles of length $r$ or less is bounded away from 0 as $n$ grows, and $\chi_{V_n}$ is constrained to be constant on this positive-density subset of $S_n$.

Except for a few special (e.g. $M = \mathbb{R}^d$) and low-complexity (i.e. small $i$, $d$, $g$, $J$, etc.) cases, little is known about the characters of the $S_n$-representations in Table 1 or even their dimension. In many cases, closed form computations seem out of reach. By contrast, the following result gives an answer, albeit a non-explicit one, in every case. Since each sequence (1)–(9) comes from a finitely generated FI-module, Theorem 1.5 applies to each sequence, giving the following.

**Corollary 1.6.** The characters $\chi_{V_n}$ of each of the sequences (1)–(9) of $S_n$-representations in Table 1 are eventually polynomial. In particular $\dim(V_n)$ is eventually polynomial.

Apart from a few special cases, we do not know how to specify the polynomials produced by Corollary 1.6 although we can give explicit upper bounds for their degree. A primary obstacle is
that our theorems on finite generation depend on the Noetherian property of FI-modules proved in Theorem 1.3 and such properties cannot in general be made effective.

As a contrasting example, the dimension of $H^2(\overline{M}_{g,n};\mathbb{Q})$ grows exponentially with $n$, where $\overline{M}_{g,n}$ is the Deligne-Mumford compactification of the moduli space of $n$-pointed genus $g$ curves. Although the cohomology groups $H^2(\overline{M}_{g,n};\mathbb{Q})$ do form an FI-module, this FI-module is not finitely generated.

**FI$\mathbb{Z}$-modules.** It is often the case that FI-modules arising in nature carry an even more rigid structure. An FI$\mathbb{Z}$-module is a functor from the category of partial injections of finite sets to the category of $k$-modules (see [4.1.1]). In contrast with the category of FI-modules, the category of FI$\mathbb{Z}$-modules is close to being semisimple (see Theorem 4.1.5 for a precise statement).

**Theorem 1.7.** Let $V$ be an FI$\mathbb{Z}$-module over any field $k$. The following are equivalent:

1. $\dim(V_n)$ is bounded above by a polynomial in $n$.
2. $\dim(V_n)$ is exactly equal to a polynomial in $n$ for all $n \geq 0$.

The power of Theorem 1.7 comes from the fact that in practice it is quite easy to prove that $\dim(V_n)$ is bounded above by a polynomial. The rigidity behind this theorem holds even for FI$\mathbb{Z}$-modules over $\mathbb{Z}$ or other rings. When $k$ is a field of characteristic 0, we strengthen Theorem 1.5 to show that the character of $V_n$ is given by a single character polynomial for all $n \geq 0$.

We now focus in greater detail on two of the most striking applications of our results. Many other applications are given in Sections 5, 6 and 7.

**Cohomology of configuration spaces.** In [6] we prove a number of new theorems about configuration spaces on manifolds. Let $\text{Conf}_n(M)$ denote the configuration space of ordered $n$-tuples of distinct points in a space $M$:

$$\text{Conf}_n(M) := \{(p_1, \ldots, p_n) \in M^n \mid p_i \neq p_j\}$$

Configuration spaces and their cohomology are of wide interest in topology and in algebraic geometry; for a sampling, see Fulton-MacPherson [FMac], McDuff [McD], or Segal [Se].

An injection $f: m \hookrightarrow n$ induces a map $\text{Conf}_n(M) \to \text{Conf}_m(M)$ sending $(p_1, \ldots, p_n)$ to $(p_{f(1)}, \ldots, p_{f(m)})$. This defines a contravariant functor $\text{Conf}(M)$ from FI to the category of topological spaces. Thus for any fixed $i \geq 0$ and any ring $k$, we obtain an FI-module $H^i(\text{Conf}(M); k)$. Using work of Totaro [10] we prove that when $M$ is a connected, oriented compact manifold of dimension $\geq 2$, the FI-module $H^i(\text{Conf}(M); \mathbb{Q})$ is finitely generated for each $i \geq 0$. Moreover, when $\dim M \geq 3$ we bound the weight and stability degree of this FI-module to prove the following.

**Theorem 1.8.** If $\dim M \geq 3$, there is a character polynomial $P_{M,i}$ of degree $\leq i$ so that

$$\chi_{H^i(\text{Conf}_n(M);\mathbb{Q})}(\sigma) = P_{M,i}(\sigma) \quad \text{for all } n \geq 2i \text{ and all } \sigma \in S_n.$$  

In particular, the Betti number $b_i(\text{Conf}_n(M))$ agrees with a polynomial of degree $i$ for all $n \geq 2i$.

If $M$ is the interior of a compact manifold with nonempty boundary, we prove that $H^i(\text{Conf}(M); k)$ is in fact an FI$\mathbb{Z}$-module for any ring $k$. This implies that the character polynomial $P_{M,i}$ from Theorem 1.8 agrees with the character of $H^i(\text{Conf}_n(M);\mathbb{Q})$ for all $n \geq 0$. It also implies sharp constraints on the integral and mod-$p$ cohomology of $\text{Conf}_n(M)$.

**Theorem 1.9.** If $M$ is the interior of a compact manifold with nonempty boundary, each of the following invariants of $\text{Conf}_n(M)$ is given by a polynomial in $n$ for all $n \geq 0$ (of degree $i$ if $\dim M \geq 3$, and of degree $2i$ if $\dim M = 2$):
1. the $i$-th rational Betti number $b_i(\text{Conf}_n(M))$, i.e. $\dim_\mathbb{Q} \text{Conf}_n(M; \mathbb{Q})$;
2. the $i$-th mod-$p$ Betti number of $\text{Conf}_n(M)$, i.e. $\dim_{\mathbb{F}_p} \text{Conf}_n(M; \mathbb{F}_p)$;
3. the minimum number of generators of $H^i(\text{Conf}_n(M); \mathbb{Z})$;
4. the minimum number of generators of the $p$-torsion part of $H^i(\text{Conf}_n(M); \mathbb{Z})$.

We believe that each of these results is new. Our theory also yields a new proof of [Ch Theorem 1], which was used by Church [Ch] to give the first proof of rational homological stability for unordered configuration spaces of arbitrary manifolds. Our proof here is in a sense parallel to that of [Ch], but the new framework simplifies the mechanics of the proof considerably, and allows us to sharpen the bounds on the stable range. When $\dim(M) > 2$ we improve Church’s stable range from [Ch Theorem 5] on homological stability for configurations of “colored points” in $M$.

As a simple illustration of the above results, the character of the $S_n$-representation $H^2(\text{Conf}_n(\mathbb{R}^2); \mathbb{Q})$ is given for all $n \geq 0$ by the single character polynomial

$$
\chi_{H^2(\text{Conf}_n(\mathbb{R}^2); \mathbb{Q})} = 2\left(\frac{X_1}{3}\right) + 3\left(\frac{X_2}{4}\right) X_2 - \left(\frac{X_2}{2}\right) - X_3 - X_4.
$$

**Diagonal coinvariant algebras.** In §5.1 we obtain new results about a well-studied object in algebraic combinatorics: the multivariate diagonal coinvariant algebra. The story begins in classical invariant theory. Let $k$ be a field of characteristic 0, and fix $r \geq 1$. For each $n \geq 0$ we consider the algebra of polynomials

$$
k[\mathcal{X}^{(r)}(n)] := k[x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}]
$$
in $r$ collections of $n$ variables. The symmetric group $S_n$ acts on this algebra diagonally:

$$
\sigma \cdot x_j^{(i)} := x_{\sigma(j)}^{(i)}
$$

Chevalley and Weyl computed the $S_n$-invariants under this action, the so-called *multisymmetric polynomials* (see [Wey], II.A.3). Let $I_n$ be the ideal in $k[\mathcal{X}^{(r)}(n)]$ generated by the multisymmetric polynomials with vanishing constant term. The $r$-diagonal coinvariant algebra is the $k$-algebra

$$
R^{(r)}(n) := k[\mathcal{X}^{(r)}(n)]/I_n.
$$

Each coinvariant algebra $R^{(r)}(n)$ is known to be a finite-dimensional $S_n$-representation. These representations have been objects of intense study in algebraic combinatorics. Borel proved that $R^{(1)}(n)$ is isomorphic as an $S_n$-representation to the cohomology $H^*(\text{GL}_n \mathbb{C}/B; k)$ of the complete flag variety $\text{GL}_n \mathbb{C}/B$. Furthermore Chevalley [Che Theorem B] proved that $R^{(1)}(n)$ is isomorphic to the regular representation of $S_n$, so $\dim R^{(1)}(n) = n!$. Haiman [Ha] gave a geometric interpretation for $R^{(2)}(n)$ and used it to prove the “$(n+1)^{n-1}$ Conjecture”, which stated that

$$
\dim(R^{(2)}(n)) = (n + 1)^{n-1}.
$$

For $r > 2$ the dimension of $R^{(r)}(n)$ is not known.

The polynomial algebra $k[\mathcal{X}^{(r)}(n)]$ naturally has an $r$-fold multi-grading, where a monomial has multi-grading $J = (j_1, \ldots, j_r)$ if its total degree in the variables $x_1^{(k)}, \ldots, x_n^{(k)}$ is $j_k$. This multi-grading is $S_n$-invariant, and descends to an $S_n$-invariant multi-grading

$$
R^{(r)}(n) = \bigoplus_J R^{(r)}_J(n)
$$
on the $r$-diagonal coinvariant algebra $R^{(r)}(n)$. It is a well-known problem to describe these graded pieces as $S_n$-representations.
Particular there exists a polynomial $P$ a priori. Theorem 1.5. It is clear that the tensor product $V_P^n$ of the character of this dimension eventually coincides exactly with a polynomial is new, as is the polynomial behavior of no information on the polynomials $P$. Given a field of characteristic 0 and a partition $Murnaghan’s theorem. (Problem 1.12 for all sufficiently large $g$ but the coefficients are generated. From this point of view, Murnaghan’s theorem is not merely an assertion about a list of finitely generated FI-modules is finitely generated. For example, it is easy to check that:

$$\begin{align*}
\dim R_1^{(1)}(n) & = n - 1 & \chi_{R_1^{(1)}}(n) & = X_1 - 1 & \text{for } n \geq 1 \\
\dim R_2^{(1)}(n) & = \binom{n}{2} - 1 & \chi_{R_2^{(1)}}(n) & = \left(\frac{X_1}{2}\right) + X_2 - 1 & \text{for } n \geq 2 \\
\dim R_{11}^{(2)}(n) & = 2 \binom{n}{2} - n & \chi_{R_{11}^{(2)}}(n) & = 2 \left(\frac{X_1}{2}\right) - X_1 & \text{for } n \geq 2
\end{align*}$$

Note that in these low-degree cases, once $n$ is sufficiently large the dimension of $R_J^{(r)}(n)$ is polynomial in $n$, as is its character. For $r > 2$ it seems that almost nothing is known about Problem 1.10 except in low-degree cases (see [Be, §4] for a discussion of these cases). However, the descriptions in [1] are just the simplest examples of a completely general phenomenon.

Theorem 1.11 (Character polynomials for diagonal coinvariant algebras). Describe the polynomial $P_J^{(r)}(n)$ whose existence is guaranteed by Theorem 1.11.

Murnaghan’s theorem. Given a field of characteristic 0 and a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, for any $n \geq |\lambda| + \lambda_1$ we define $V(\lambda)_n$ to be the irreducible representation of $S_n$ corresponding to the partition $\lambda[n] := (n - |\lambda|, \lambda_1, \ldots, \lambda_\ell)$.

Murnaghan’s theorem states that, for any partitions $\lambda$ and $\mu$, there are coefficients $g^\nu_{\mu, \lambda}$ such that the tensor product $V(\lambda)_n \otimes V(\mu)_n$ decomposes into $S_n$-irreducibles as

$$V(\lambda)_n \otimes V(\mu)_n = \bigoplus_\nu g^\nu_{\lambda, \mu} V(\nu)_n$$

for all sufficiently large $n$. The first complete proof of this theorem was given by Littlewood [1] in 1957, but the coefficients $g_{\lambda, \mu}$ remain unknown in general. We will show in [3.4] that Murnaghan’s theorem is an easy consequence of the fact that the tensor product of finitely generated FI-modules is finitely generated. From this point of view, Murnaghan’s theorem is not merely an assertion about a list
of numbers, but becomes a structural statement about a single mathematical object: the FI-module $V(\lambda) \otimes V(\mu)$.

**Connection with representation stability.** In [CF], Church and Farb introduced the theory of representation stability. Stability theorems in topology and algebra typically assert that in a given sequence of vector spaces with linear maps

$$\cdots \to V_n \to V_{n+1} \to V_{n+2} \to \cdots$$

the maps $V_n \to V_{n+1}$ are isomorphisms for $n$ large enough. The goal of representation stability is to provide a framework for generalizing these results to situations when each vector space $V_n$ has an action of the symmetric group $S_n$ (or other natural families of groups). Representation stability provides a formal way of saying that the “names” of the $S_n$-representations $V_n$ stabilize, and a language to describe this stabilization rigorously; see §3.3 below for the precise definition. Representation stability has been proved in many cases, and gives new conjectures in others; see [CF] and [Ch], as well as [JR1, Wi1]. The theory of FI-modules converts representation stability into a finite generation property.

**Theorem 1.13 (Finite generation vs. representation stability).** An FI-module $V$ over a field of characteristic 0 is finitely generated if and only if the sequence $\{V_n\}$ of $S_n$-representations is uniformly representation stable in the sense of [CF] and each $V_n$ is finite-dimensional. In particular, for any finitely generated FI-module $V$, we have for sufficiently large $n$ a decomposition:

$$V_n \simeq \bigoplus c_\lambda V(\lambda)_n$$

where the coefficients $c_\lambda$ do not depend on $n$.

In the language of [CF], the new result here is that “surjectivity” implies “uniform multiplicity stability” for FI-modules. This turns out to be very useful, because in practice finite generation is much easier to prove than representation stability. A key ingredient in this theorem is the “monotonicity” proved by the first author in [Ch, Theorem 2.8], and as a byproduct of the proof we obtain that finitely generated FI-modules are monotone in this sense. Once again, the stability degree defined in Section 3.1 allows us, in many cases of interest, to replace “sufficiently large $n$” with an explicit range.

Our new point of view also simplifies the description of many representation-stable sequences. As a simple example, in [CF] we showed that

$$H^2(\text{Conf}_n(\mathbb{R}^2); \mathbb{Q}) = V(1)_n^{ \oplus 2} \oplus V(1,1)_n^{ \oplus 2} \oplus V(2)_n^{ \oplus 2} \oplus V(2,1)_n^{ \oplus 2} \oplus V(3)_n \oplus V(3,1)_n$$

for all $n \geq 7$, and separate descriptions were necessary for smaller $n$. In the language of the present paper we can simply write

$$H^2(\text{Conf}(\mathbb{R}^2); \mathbb{Q}) = M(\square) \oplus M(\square\square)$$

(2)

to simultaneously describe $H^2(\text{Conf}_n(\mathbb{R}^2); \mathbb{Q})$ for all $n \geq 0$ (see §2.2 for notation, and §6 for the proof). One appealing feature here is that even the “unstable” portions of the sequence $H^2(\text{Conf}_n(\mathbb{R}^2); \mathbb{Q})$, meaning the qualitatively different descriptions that are necessary for $n < 7$, are already encoded in the right-hand side of (2). Based on computer calculations, John Wiltshire-Gordon (personal communication) has formulated a precise conjecture for the decomposition of $H^1(\text{Conf}(\mathbb{R}^2); \mathbb{Q})$ as in (2) for all $i \geq 0$. Our results give a decomposition as in (2) with $\mathbb{R}^2$ replaced by any open manifold, but we do not in general know the right-hand side explicitly.

**Relation with other work.** We record here some relationships between the material in this paper and the work of others, both before and after this paper was first posted.
• Modules over EI-categories, namely those where all endomorphisms are isomorphisms, have been previously studied in the context of transformation groups (see e.g. [L]). In particular, the analogue of Theorem [1.3] with FI replaced by a finite EI-category was proved by Lück in [L, Lemma 16.10b]; however, he has explained to us that his methods cannot be extended to infinite categories such as FI. Building on the present paper, Gan–Li [GL] have generalized Theorem [1.3] to a wide class of infinite EI-categories, and Snowden–Sam [SSS] have extended the generalization [CEFN, Theorem A] to an even wider class of combinatorial categories.

• The classification of FI♯-modules in Theorem [4.1.5] below parallels the main theorem of Pirashvili in [P], which establishes a similar classification for modules over a larger category (that of finite pointed sets). Although Theorem [4.1.5] is not precisely a special case of Pirashvili’s theorem, the ideas involved are closely related in spirit.

• The category of FI-modules, along with related abelian categories, has been considered by researchers in the field of polynomial functors. See for example the recent work of Djament and Vespa [DV] which studies the stable behavior of the cohomology groups $H^i(S_n, V_n)$ for FI-modules $V$, or the earlier results of Pirashvili [Pi].

• The recent work of Snowden and Sam on “twisted commutative algebras” has some overlap with our own, because FI-modules can be viewed as modules for the “exponential” twisted commutative algebra (see [AM]; for an introduction to TCAs, see [SS2]). In particular, as mentioned above, Snowden proves in [Sn, Theorem 2.3] a Noetherian property for modules over a broad class of TCAs in characteristic 0, which could be used to give a different proof of Theorem [1.3]. Applying this perspective to FI♯-modules yields examples of (divided power) D-modules both in characteristic 0 and in positive characteristic, as we hope to explain in a future paper. More recently, Sam–Snowden have given in [SS] a more detailed analysis of the algebraic structure of the category of FI-modules in characteristic 0.

• Objects in Deligne’s category $\text{Rep}(S_t)$, where $t$ is a complex number, are closely related to sequences of $S_n$-representations whose characters are polynomial (see Deligne [De], Knop [Kn], or Etingof’s lecture [Et]). Is there a sense in which a finite-type object of $\text{Rep}(S_t)$ can be specialized to a finitely generated FI-module? An interesting example is provided by the recent work of Ren and Schedler [RS] on spaces of invariant differential operators on symplectic manifolds. Their results are consistent with the proposition that their sequence $\text{Inv}_n(V)$ carries the structure of a finitely generated FI-module. Does it?

• A recent paper of Putman [Pu] proposes an alternative representation stability condition called “central stability” and proves that it holds for the mod-$p$ cohomology of congruence groups. We prove with Rohit Nagpal in [CEFN] that in the language of FI-modules, central stability is equivalent to finite presentation. We also extend Theorem [1.3] from characteristic 0 to any Noetherian ring in [CEFN]. Therefore any finitely generated FI-module is finitely presented, so finite generation for FI-modules, representation stability for $S_n$-representations (in characteristic 0), and central stability are all equivalent.

• In [CEP], we apply Theorem [1.3] to the étale cohomology of hyperplane arrangements to prove asymptotic stabilization for certain statistics for polynomials over finite fields. In [CP], Church–Putman extend the notion of finite generation from FI-modules to nonabelian FI-groups (see [7.2]), and use this to construct generating sets for the Johnson filtration of the Torelli group with bounded complexity. In [CE], Church–Ellenberg prove a quantitative bound on FI-homology which says that in an approximate sense, FI-modules behave as if the homological dimension of
The results of this paper on $S_n$-representations have been extended to other Weyl groups by Wilson by constructing and analyzing the category of $\text{FI}_W$-modules $[\text{Wi}2, \text{Wi}3]$. The analogous categories where $S_n$ is replaced by $\text{GL}_n(R)$ for a finite ring $R$ have been studied by Putman–Sam $[\text{PuS}]$; modules over these categories turn out to have many features in common with $\text{FI}$-modules, including the Noetherian property of Theorem 1.3 and $[\text{CEFN}$, Theorem A$]$.

Acknowledgements. We are grateful to Wolfgang Lück for conversations regarding his work in $[\text{L}]$ and its relation with the present paper. We thank Persi Diaconis for many helpful conversations regarding the polynomials considered in Theorem 3.3.4. We also thank Marcelo Aguiar, Ralph Cohen, Pierre Deligne, Aurélien DJament, Pavel Etingof, James Griffin, Jim Haglund, Rita Jimenez Rolland, Aaron Lauve, Daniel Litt, Jacob Lurie, Andrew Putman, Claudiu Raicu, Steven Sam, Andrew Snowden, Bernd Sturmfels, David Treumann, Julianna Tymoczko, Christine Vespa, Jennifer Wilson, and John Wiltshire-Gordon for helpful discussions. We are especially grateful to Rosona Eldred and the Hamburg/Bremen reading group for numerous helpful comments on an earlier version of this paper. Finally, we would like to thank the referees, whose comments and suggestions have greatly improved the exposition in this paper.

2 FI-modules: basic properties

In this section we develop the basic theory of FI-modules, including definitions and foundational properties.

2.1 FI-objects

Fix a commutative ring $k$. The case when $k$ is a field will be foremost in our minds, and our notation is chosen accordingly. For example, by a representation over $k$ of a group $G$ we mean a $k$-module $V$ together with an action of $G$ on $V$ by $k$-module automorphisms, i.e. a $kG$-module. The dual representation $V^*$ is the $k$-module $\text{Hom}_k(V, k)$ together with its induced $G$-action.

Given two groups $G$ and $H$, if $V$ is a representation of $G$, then we denote by $V \boxtimes k$ the same $k$-module $V$ interpreted as a representation of $G \times H$ on which $H$ acts trivially. All tensor products are taken over $k$ unless otherwise specified.

**FI-modules.** Recall from Definition 1.1 that $\text{FI}$ denotes the category whose objects are finite sets and whose morphisms from $S$ to $T$ are injections $S \hookrightarrow T$. We will later consider three other categories with the same objects: $\text{FB}$ will denote the category whose morphisms from $S$ to $T$ are the bijections $S \cong T$; $\text{co-FI}$ will denote the opposite category $\text{FI}^{\text{op}}$, whose morphisms from $S$ to $T$ are the injections $T \hookrightarrow S$; and $\text{FI}^\#$ will denote the category whose morphisms from $S$ to $T$ are the partial injections from $S$ to $T$ (see §4.1 for details).

**Definition 2.1.1.** The category of FI-modules is the category $\text{FI-Mod} := [\text{FI}, k\text{-Mod}]$ of functors from $\text{FI}$ to the category of $k$-modules, with natural transformations of functors as morphisms.

If $V$ is an FI-module and $S$ is a finite set, we write $V_S$ for $V(S)$, and write $V_n$ for $V(n)$. Given a morphism $f : S \rightarrow T$ or $f : m \rightarrow n$, we often write $f_* : V_S \rightarrow V_T$ or $f_* : V_m \rightarrow V_n$ for the map $V(f)$. For any FI-module $V$, the endomorphisms $\text{End}_{\text{FI}}(V)$ act on $V$, so we can consider $V$ as an $S_n$-representation. Given a map $V \rightarrow W$ of FI-modules, the induced map $V_n \rightarrow W_n$ is $S_n$-equivariant.

**Remark 2.1.2.** $\text{FI-Mod}$ inherits the structure of an abelian category from $k\text{-Mod}$, with all notions such as kernel, cokernel, subobject, quotient object, injection, and surjection being defined “pointwise”
from the corresponding notions in $k$-$\text{Mod}$. This is true for any category of functors from any small category to an abelian category $\text{A.3.3}$. In particular, the same holds for FB-modules, FI-modules, etc.; see Remark 2.1.4 below.

For example, a map $F: V \to W$ of FI-modules is a surjection if and only if the maps $F_S: V_S \to W_S$ are surjections for all finite sets $S$. Since every finite set is isomorphic to some $n$, an equivalent condition is that $V_n \to W_n$ is surjective for all $n \in \mathbb{N}$. We will frequently use this equivalence to verify some property of an FI-module $V$ by verifying it for $V_n$ for all $n \in \mathbb{N}$. For another example, the kernel of this map $F$ is an FI-module $\ker[F]$, which to $S$ assigns the $k$-module $\ker[F]_S := \ker(F_S: V_S \to W_S)$. To a morphism $f: S \hookrightarrow T$, it assigns the restriction of $V(f): V_S \to V_T$ to $\ker(F_S)$ (which must have image in $\ker(F_T)$ if $F$ is a map of FI-modules).

Remark 2.1.3. In some sense the category of FI-modules might be thought of as a “limit” of the category of representations of $S_n$ as $n \to \infty$. But care is necessary. For example, the category FI-$\text{Mod}$ is not semisimple, even when $k$ is a field of characteristic 0.

Consider the FI-module $V$ with $V_S = k$ for all finite sets $S$ and $V_f = \text{id}$ for all injections $f$. Let $W$ be the FI-module with $W_0 = k$ and $W_S = 0$ if $S \neq \emptyset$, with $W_f = 0$ except for $f: \emptyset \to \emptyset$. There is an obvious surjection of FI-modules $F: V \to W$ defined by $F_\emptyset = \text{id}: k \to k$ and $F_S = 0: k \to 0$ for $S \neq \emptyset$. However, this surjection is not split: for $f: \emptyset \hookrightarrow 1$ the map $f_*: W_0 \to W_1$ is zero while $f_*: V_0 \to V_1$ is an isomorphism, so no nonzero map $s: W \to V$ can exist.

The category of FI-modules is closed under any covariant functorial construction on $k$-modules, by applying functors pointwise. For example, if $V$ and $W$ are FI-modules, then $V \oplus W$ and $V \otimes W$ are FI-modules. Concretely, the FI-module $V \otimes W$ assigns to a finite set $S$ the $k$-module $V_S \otimes W_S$, and to an inclusion $f: S \hookrightarrow T$ the homomorphism $V(f)_* \otimes W(f)_*: V_S \otimes W_S \to V_T \otimes W_T$. Symmetric products and exterior products of FI-modules are also FI-modules, once we fix a functorial definition of these constructions. The dual $V^*$ of an FI-module $V$, on the other hand, is a co-FI-module; to a finite set $S$ the co-FI-module assigns the $k$-module $V(S)^*$, and to an inclusion $f: S \hookrightarrow T$ the homomorphism $V(f)^*: V(T)^* \to V(S)^*$.

Remark 2.1.4. We use the same notation for other categories: for example, an $\text{FB-module}$ is a functor from FB to the category of $k$-modules; an $\text{FI-group}$ is a functor from FI to the category of groups; an $\text{FI}_k$-$\text{space}$ is a functor from $\text{FI}_k$ to the category of topological spaces, and so on. In all these cases, we take natural transformations as our morphisms, so $\text{FB-Mod} := [\text{FB}, k$-$\text{Mod}]$ is the category of FB-modules, $\text{FI-Groups} := [\text{FI, Groups}]$ is the category of FI-groups, and so on.

Remark 2.1.5. By a graded $k$-module we always mean an $\mathbb{N}$-graded $k$-module. In keeping with the approach of this paper, we can think of a graded $k$-module as a functor $A: \mathbb{N} \to k$-$\text{Mod}$, where $\mathbb{N}$ is the discrete category with objects $\mathbb{N}$, so that $A_i = A(i)$ is the part of $A$ in grading $i$. A morphism $A \to B$ of graded $k$-modules is a natural transformation, i.e. a collection of homomorphisms $A_i \to B_i$ (so we only consider morphisms of “degree 0”). To avoid phrases such as “FI-(graded $k$-module)”, we will say instead “graded FI-module”, and so on. That is, a graded FI-module $W$ is a functor $W: \text{FI} \to \text{gr-k-Mod}$, and $\text{gr-FI-Mod} = [\text{FI, gr-k-Mod}]$ is the category of graded FI-modules. If $W$ is a graded FI-module, restricting to a fixed $i \in \mathbb{N}$ yields an FI-module $W^i$.

Remark 2.1.6. A graded $k$-algebra is a graded $k$-module $A$ endowed with a unital multiplication $A \otimes A \to A$. We require that this multiplication be unital, but it need not be associative. An ideal $I \subset A$ is a graded submodule $I \subset A$ such that under multiplication $A \otimes I$ and $I \otimes A$ map into $I$; in this case, the quotient $A/I$ is a graded $k$-algebra. In particular, a graded FI-algebra $V$ is a functor from FI to the category of graded $k$-algebras, and an FI-ideal $I \subset V$ yields for each finite set $S$ an ideal $I_S \subset V_S$.
2.2  Free FI-modules

In this section we define certain families of FI-modules that can be thought of as the FI-modules “freely generated” by an $S_a$-representation.

**Definition 2.2.1 (FB-modules).** Let FB denote the category of finite sets and bijections. An FB-module $W$ is a functor $W : FB \to k$-Mod.

The category $FB$-Mod = $[FB$-$k$-Mod] of FB-modules is also known as the category of vector species (see e.g. [AM]). Since $End_{FB}(n) \simeq S_n$, an FB-module $W$ determines a collection of $S_n$-representations $W_n$ for all $n \in \mathbb{N}$, just as an FI-module did. But in contrast with FI-modules, an FB-module $W$ is determined by the $S_n$-representations $W_n$, with no additional data such as maps between them. Indeed, we can consider any $S_a$-representation $W_n$ as an FB-module by setting $W_n = 0$ for $n \neq a$, and extending to all finite sets $T$ by choosing isomorphisms $T \simeq n$. With this convention we have an isomorphism of FB-modules $W \simeq \bigoplus_{n=0}^{\infty} W_n$.

The obvious inclusion $FB \hookrightarrow FI$ induces a forgetful functor $\pi : FI$-Mod $\to$ FB-Mod, which remembers the $S_n$-actions on $V_n$ but forgets the maps $V_m \to V_n$ for $m < n$.

**Definition 2.2.2 (The FI-module $M(W)$).** We define the functor $M(-) : FB$-Mod $\to$ FI-Mod as the left adjoint of $\pi : FI$-Mod $\to$ FB-Mod. Explicitly, if $W$ is an FB-module, then by [MacL, (Eq. X.3.10)] the FI-module $M(W)$ satisfies

$$M(W)_S = \text{colim}_{f : T \to S} W_T = \bigoplus_{T \subseteq S} W_T, \quad (3)$$

with the map $f_* : M(W)_S \to M(W)_{S'}$ induced by $f : S \hookrightarrow S'$ being the sum of $(f|_{T})_* : W_T \to W_{f(T)}$.

The unit of this adjunction is the inclusion of FB-modules $W \hookrightarrow \pi(M(W))$ which sends $W_S \hookrightarrow \bigoplus_{T \subseteq S} W_T = M(W)_S$. The co-unit provides a surjection of FI-modules $\bigoplus_{n \geq 0} M(V_n) \to V$.

From (3) we see that $M(-)$ is an exact functor. Also, as an $S_n$-representation, we can identify

$$M(W)_n \simeq \bigoplus_{a \leq n} \text{Ind}_{S_a \times S_{n-a}}^{S_n} W_a \boxtimes k. \quad (4)$$

In particular, when $k$ is a field, the dimension of the vector space $M(W)_n$ is:

$$\dim M(W)_n = \sum_{a \geq 0} \dim(W_a) \cdot \binom{n}{a}$$

This shows that for any $W$, the dimension of $M(W)_n$ is given for all $n \geq 0$ by a single polynomial in $n$; we will see in Theorem 4.1.7 that a similar statement holds for the character of $M(W)_n$.

Consider the regular $S_m$-representation $k[S_m]$ as an FB-module. The resulting functor assigns to a finite set $T$ the $k$-module with basis indexed by the bijections $m \to T$. Applying the functor $M(-)$, we obtain the following important family of FI-modules.

**Definition 2.2.3 (The FI-module $M(m)$).** For any $m \geq 0$, we denote by $M(m)$ the “representable” FI-module $k[\text{Hom}_{FI}(m, -)] \simeq M(k[S_m])$. For any finite set $S$, the $k$-module $M(m)_S$ has basis indexed by the injections $m \hookrightarrow S$; in other words, by ordered sequences of $m$ distinct elements of $S$. A map of FI-modules $F : M(m) \to V$ is determined by the choice of an element $v = F(id_m) \in V_m$, via the adjunction defining $M(-)$. 

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Example 2.2.4. Since there is precisely one injection \( \emptyset = 0 \hookrightarrow S \) for every finite set \( S \), the FI-module \( M(0) \) is the constant functor with value \( k \); as an \( S_n \)-representation \( M(0)_n \simeq k \) is the 1-dimensional trivial representation. Since injections \( 1 \hookrightarrow S \) correspond simply to elements of \( S \), the FI-module \( M(1) \) assigns to a finite set \( S \) the free \( k \)-module with basis \( S \); as an \( S_n \)-representation \( M(1)_n \simeq k^n \) is the usual permutation representation.

The permutation group \( S_n \simeq \text{End}_{FI}(m) \) acts on \( M(m) \) by FI-module automorphisms (as is true of any representable functor). This action is quite concrete: it is simply the standard action of \( S_m \) on ordered sequences \((s_1, \ldots, s_m)\) by reordering the elements of a sequence.

Irreducible representations of \( S_n \) in characteristic 0. When \( k \) is a field of characteristic 0, every \( S_n \)-representation decomposes as a direct sum of irreducible representations. In characteristic 0 every irreducible representation of \( S_n \) is defined over \( \mathbb{Q} \). As a result this decomposition does not depend on the field \( k \); moreover, every representation of \( S_n \) is self-dual.

Recall that the irreducible representations of \( S_n \) over any field \( k \) of characteristic 0 are classified by the partitions \( \lambda \) of \( n \). A partition of \( n \) is a sequence \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell > 0) \) with \( \lambda_1 + \cdots + \lambda_\ell = n \); we write \( |\lambda| = n \) or \( \lambda \vdash n \). We denote by \( V_\lambda \) the irreducible \( S_n \)-representation corresponding to the partition \( \lambda \vdash n \). The representation \( V_\lambda \) can be obtained as the image \( k[S_n] \cdot c_\lambda \) of a certain idempotent \( c_\lambda \) (the “Young symmetrizer”) in the group algebra \( k[S_n] \).

Definition 2.2.5 (The irreducible representation \( V(\lambda)_n \)). Given a partition \( \lambda \), for any \( n \geq |\lambda| + \lambda_1 \) we define the padded partition

\[
\lambda[n] := (n - |\lambda|, \lambda_1, \ldots, \lambda_\ell).
\]

For \( n \geq |\lambda| + \lambda_1 \), we define \( V(\lambda)_n \) to be the irreducible \( S_n \)-representation

\[
V(\lambda)_n := V_\lambda[n].
\]

Since every partition of \( n \) can be written as \( \lambda[n] \) for a unique partition \( \lambda \), every irreducible \( S_n \)-representation is isomorphic to \( V(\lambda)_n \) for a unique partition \( \lambda \). We sometimes replace \( \lambda \) by its corresponding Young diagram.

In this notation the trivial representation of \( S_n \) is \( V(0)_n \), and the standard \((n-1)\)-dimensional irreducible representation is \( V(1)_n = V([\square])_n \) for all \( n \geq 2 \). Many of the usual linear algebra operations behave well with respect to this notation. For example, the identities

\[
\wedge^3 V([\square])_n \simeq V([\bar{\square}])_n \quad \text{and} \quad V([\square])_n \otimes V([\square])_n \simeq V([\bar{\square}])_n \oplus V([\square])_n \oplus V([\square])_n \oplus V(0)_n
\]

hold whenever both sides are defined, namely whenever \( n \geq 4 \). For a similar decomposition of any tensor product in general, see Theorem \((3.4.2)\) in \((3.4)\) below.

Definition 2.2.6 (The FI-module \( M(\lambda) \)). When \( k \) is a field of characteristic 0, given a partition \( \lambda \) we write \( M(\lambda) \) for the FI-module \( M(\lambda) := M(V_\lambda) \).

Many natural combinatorial constructions correspond to FI-modules of this form. For example, since \([\square\square\square]\) is the trivial representation of \( S_3 \), a basis for \( M([\square\square\square])_S \) is given by the 3-element subsets of \( S \); in other words, the FI-module \( M([\square\square\square]) \) is the linearization of the functor \( S \mapsto \binom{S}{3} \). Similarly, the FI-module \( M([\square\square]) \) is the linearization of the functor sending \( S \) to the collection of oriented edges \( x \to y \) between distinct elements of \( S \) (oriented in the sense that \( y \to x \) is the negative of \( x \to y \)).
Remark 2.2.7 (Projective FI-modules). A projective FI-module is a projective object in the abelian category FI-Mod. It follows from general considerations (see [Wei, Exercise 2.3.8]) that the category FI-Mod has enough projectives. Indeed if $P_n$ is a projective $k[S_n]$-module, then $M(P_n)$ is projective, since $M(-)$ is the left adjoint to an exact functor and thus preserves projectives [Wei, Proposition 2.3.10]. In fact, it is not hard to show that every projective FI-module is a direct sum $\bigoplus_{n \geq 0} M(P_n)$ where $P_n$ is a projective $k[S_n]$-module; see e.g. [L] Corollary 9.40.

2.3 Generators for an FI-module

Definition 2.3.1 (Span). If $V$ is an FI-module and $\Sigma$ is a subset of the disjoint union $\bigsqcup V_n$, the span $\text{span}_V(\Sigma)$ is the minimal sub-FI-module of $V$ containing each element of $\Sigma$. We say that $\text{span}_V(\Sigma)$ is the sub-FI-module of $V$ generated by $\Sigma$. We sometimes write $\text{span}(\Sigma)$ when there is no ambiguity.

Lemma 2.3.2. Given an FI-module $V$, a choice of element $v \in V_m$ determines a map $M(m) \to V$. The image of this map is $\text{span}_V(v)$. More generally, if $\Sigma$ is the disjoint union of $\Sigma_n \subset V_n$, the image of the natural map $\bigoplus_{n \geq 0} M(n)^{\oplus \Sigma_n} \to V$ is $\text{span}_V(\Sigma)$.

Proof. Let $W$ be the image of the map $M(m) \to V$ sending $id_m \in M(m)_m$ to $v \in V_m$. Certainly $v \in W_m$, so $span_V(v) \subset W$. A basis for $M(m)_S$ consists of the morphisms $f : m \mapsto S$, and $f \in M(m)_S$ is sent to $f_*(v) \in W_S$ (as it must be for $M(m) \to V$ to be a map of FI-modules). Thus $W_S$ is the submodule of $V_S$ spanned by the elements $f_*(v)$, where $f$ ranges over injections $f : m \mapsto S$. Since any sub-FI-module of $V$ containing $v$ must contain all these elements $f_*(v)$, we have $W \subset \text{span}_V(v)$ as well. This completes the proof for $\text{span}_V(v)$; the argument for general $\Sigma$ is identical. $\square$

Taking $\Sigma_n = V_n$ in Lemma 2.3.2 gives a surjection $\bigoplus_{n \geq 0} M(n)^{\oplus V_n} \to V$, which is a slightly less efficient version of the surjection $\bigoplus_{n \geq 0} M(V_n) \to V$ from Definition 2.2.2; this amounts to including every element of $V$ in our generating set.

Definition 2.3.3 (Generation in degree $\leq m$). We say that an FI-module $V$ is generated in degree $\leq m$ if $V$ is generated by elements of $V_k$ for $k \leq m$, i.e. if $\text{span}(V_{\leq m}) = V$.

Definition 2.3.4 (Finite generation). We say that an FI-module $V$ is finitely generated if there is a finite set of elements $v_1, \ldots, v_k$ with $v_i \in V_{m_i}$ which generates $V$, meaning that $\text{span}(v_1, \ldots, v_k) = V$.

We will say that $V$ is finitely generated in degree $\leq m$ if $V$ is generated in degree $\leq m$ and $V$ is finitely generated; in other words, there exists a finite generating set $v_1, \ldots, v_k$ with $v_i \in V_{m_i}$ for which $m_i \leq m$ for all $i$. Lemma 2.3.2 implies the following characterization of finitely generated FI-modules in terms of the free FI-modules $M(m)$.

Proposition 2.3.5 (Finite generation in terms of $M(m)$). An FI-module $V$ is finitely generated if and only if it admits a surjection $\bigoplus_i M(m_i) \to V$ for some finite sequence of integers $\{m_i\}$.

One consequence of Proposition 2.3.5 is that if $V$ is a finitely generated FI-module then $V_S$ is a finitely generated $k$-module for any finite set $S$. Proposition 2.3.5 will allow us to reduce questions of finite generation to the corresponding question for the particular FI-modules $M(m)$, as in the proof of the following proposition.

Proposition 2.3.6 (Tensor products of f.g. FI-modules). If $V$ and $W$ are finitely generated FI-modules, so is $V \otimes W$. If $V$ is generated in degree $\leq m_1$ and $W$ is generated in degree $\leq m_2$, then $V \otimes W$ is generated in degree $\leq m_1 + m_2$.
Proof. By Proposition 2.3.5, it suffices to show that $U := M(m_1) \otimes M(m_2)$ is finitely generated in degree $\leq m_1 + m_2$. Each $k$-module $U_n$ is certainly finitely generated (of rank $\binom{n}{m_1}\cdot \binom{n}{m_2}$), so it suffices to prove that $U = \text{span}(U_{\leq m_1 + m_2})$. A basis for $U_S$ is given by pairs $(f_1: m_1 \hookrightarrow S, f_2: m_2 \hookrightarrow S)$. The basis element $(f_1, f_2)$ lies in the image of $U_T$, where $T = \text{im}(f_1) \cup \text{im}(f_2)$. Since $|T|$ is at most $m_1 + m_2$, this shows that $\text{span}(U_{\leq m_1 + m_2})$ is all of $U$, as desired. \hfill $\square$

$H_0(V)$ and generators for $V$. We conclude this section with another perspective on generators for an FI-module $V$ that will be used in later sections. Given an FB-module $W$, we can consider $W$ as an FI-module by declaring $f_*: W_S \to W_T$ to be 0 whenever $f: S \hookrightarrow T$ is not bijective. This defines an inclusion of categories $\text{FB-Mod} \hookrightarrow \text{FI-Mod}$.

Definition 2.3.7 (The functor $H_0$). We define the functor $H_0: \text{FI-Mod} \to \text{FB-Mod}$ as the left adjoint of this inclusion $\text{FB-Mod} \hookrightarrow \text{FI-Mod}$. Explicitly, given an FI-module $V$, the FB-module $H_0(V)$ satisfies $$H_0(V)_S := V_S / \text{span}(V_{<|S|})_S;$$ given a bijection $f: S \to S'$, the map $f_*: H_0(V)_S \to H_0(V)_{S'}$ is that induced by $f_*: V_S \to V_{S'}$.

Remark 2.3.8. An FI-module $V$ is generated in degree $\leq m$ if and only if $H_0(V)_n$ vanishes for $n > m$. Similarly, $V$ is finitely generated if and only if the $k$-module $\bigoplus_{n \geq 0} H_0(V)_n$ is finitely generated.

If $W_a$ is a $k[S_a]$-representation, the FI-module $M(W_a)$ is generated by $M(W_a)_a \simeq W_a$, so $H_0(M(W_a))_n$ vanishes for all $n \neq a$, and we also have $H_0(M(W_a))_a \simeq W_a$. More generally, for any FB-module $W$ we have a natural isomorphism $H_0(M(W)) \simeq W$.

Remark 2.3.9. The functor $H_0: \text{FI-Mod} \to \text{FB-Mod}$ is right-exact but not exact. In the paper [CE] we investigate the higher FI-homology $H_i(V)$ of an FI-module $V$, namely the left-derived functors of $H_0$. In particular, we show that the vanishing of higher FI-homology is connected with the existence of inductive presentations for $V$, and apply this to homological stability for FI-groups.

3 FI-modules: representation stability and character polynomials

In this section we introduce various quantitative measures of how quickly an FI-module $V$ stabilizes. We use these to give, in characteristic 0, an equivalence between finite generation for the FI-module $V$ and representation stability (in the sense of Church–Farb) for the sequence of $S_n$-representations $V_n$.

We give a number of strong constraints on $S_n$-representations arising in FI-modules, and prove that the characters $\chi_{S_n}$ are eventually given by a character polynomial. Finally, we prove stability for Schur functors, and we demonstrate how a classical theorem of Murnaghan follows easily from the general theory of FI-modules.

3.1 Stability degree

In this subsection we introduce the notion of a stability degree for an FI-module over an arbitrary ring $k$. In characteristic 0, the stability degree provides a counterpart to the “stable range” for representation stability in [CE], as we prove later in Proposition 3.3.3.

Coinvariants of FI-modules. If $V_n$ is a representation of $S_n$, recall that the coinvariant quotient $(V_n)_{S_n}$ is the $k$-module $V_n \otimes_{k[S_n]} k$; this is the largest $S_n$-equivariant quotient of $V_n$ on which $S_n$ acts trivially. The functor taking an $S_n$-representation $V_n$ to the coinvariants $(V_n)_{S_n}$ is automatically right exact, and is also left exact when $|S_n| = n!$ is invertible in $k$ (by averaging over $S_n$).
Given an FI-module $V$, we can apply this for all $n \geq 0$ simultaneously, yielding $k$-modules $(V_n)_{S_n}$ for each $n \geq 0$. Moreover, all the various maps $f_s: V_n \to V_m$ induce just a single map $(V_n)_{S_n} \to (V_m)_{S_m}$ for each $n \leq m$. This data basically amounts to a graded $k[T]$-module, which we define below as $\Phi_0(V)$.

**Definition 3.1.1.** A graded $k[T]$-module $U$ consists of a collection of $k$-modules $U_i$ for each $i \in \mathbb{N}$, endowed with a map $T: U_i \to U_{i+1}$ for each $i \in \mathbb{N}$ (i.e. $T$ acts by a map of grading 1).

For each $a \geq 0$, fix once and for all some set $\overline{a}$ with cardinality $a$. The specific set $\overline{a}$ is irrelevant; the authors suggest the choice $\overline{a} := \{-1, \ldots, -a\}$, to minimize mental collisions with the finite sets the reader is likely to have in mind. Fix also a functorial definition of the disjoint union $S \sqcup T$ of sets.

**Definition 3.1.2 (The graded $k[T]$-module $\Phi_a(V)$).** Given $a \geq 0$ and an FI-module $V$, we define the graded $k[T]$-module $\Phi_a(V)$ as follows. For $n \in \mathbb{N}$, let $\Phi_a(V)_n$ be the coinviant quotient

$$\Phi_a(V)_n := (V_{\overline{n}})_{S_n}.$$  

To define $T: \Phi_a(V)_n \to \Phi_a(V)_{n+1}$, choose any injection $f: n \hookrightarrow n + 1$. This determines an injection $\text{id} \sqcup f: \overline{n} \sqcup \mathbb{N} \hookrightarrow \overline{n+1} \sqcup \mathbb{N} + 1$, and thus a map $(\text{id} \sqcup f)_*: V_{\overline{n}} \to V_{\overline{n+1}}$. The map $T: \Phi_a(V)_n \to \Phi_a(V)_{n+1}$ is defined to be the induced map $(V_{\overline{n}})_{S_n} \to (V_{\overline{n+1}})_{S_{n+1}}$. All such injections $\text{id} \sqcup f$ are equivalent under post-composition by $S_n$, so the map $T$ is well-defined, independent of the choice of $f$.

A morphism $V \to W$ of FI-modules induces a homomorphism $\Phi_a(V) \to \Phi_a(W)$ of graded $k[T]$-modules, and $\Phi_a$ is a functor from FI-modules to graded $k[T]$-modules. The functor $\Phi_a$ is always right exact. When $k$ contains $\mathbb{Q}$ the functor $\Phi_a$ is exact, for the same reason that $W \mapsto (W)_{S_n}$ is exact.

**Definition 3.1.3 (Stability degree).** The stability degree $\text{stab-deg}(V)$ of an FI-module $V$ is the smallest $s \geq 0$ such that for all $a \geq 0$, the map $T: \Phi_a(V)_n \to \Phi_a(V)_{n+1}$ is an isomorphism for all $n \geq s$. (If no such $s$ exists, we take $\text{stab-deg}(V) = \infty$.)

We can rewrite this definition to fit our usual indexing: if $f_n: n \hookrightarrow n + 1$ is the inclusion, for all $a \geq 0$ the maps $(f_n)_*: (V_n)_{S_{n-a}} \to (V_{n+1})_{S_{n+1-a}}$ are isomorphisms for all $n \geq \text{stab-deg}(V) + a$. When FI-modules are thought of as modules for the exponential twisted commutative algebra as in [AM], the functor $\Phi_0$ is the bosonic Fock functor that plays a crucial role in that theory.

**Remark 3.1.4.** The permutations of $\overline{a}$ act on the functor $\Phi_a$, so $\Phi_a(V)_n$ is an $S_a$-representation; explicitly, $\Phi_a(V)_n \simeq (\text{Res}_{S_a \times S_{n-a}}^S V_{a+n})_{S_a}$ as an $S_a$-representation. The map $T: \Phi_a(V)_n \to \Phi_a(V)_{n+1}$ is $S_a$-equivariant, so it is equivalent to ask in Definition 3.1.3 that $T$ be an isomorphism of $k$-modules or of $S_a$-representations. We make use of this equivalence in Proposition 3.3.3 and Corollary 6.3.4 below.

It turns out to be useful to refine the notion of stability degree slightly. This will be especially important in Section 6 when we study the behavior of FI-modules in spectral sequences.

**Definition 3.1.5.** We say that the injectivity degree $\text{inj-deg}(V)$ (resp. surjectivity degree $\text{surj-deg}(V)$) of an FI-module $V$ is the smallest $s \geq 0$ such that for all $a \geq 0$, the map $\Phi_a(V)_n \to \Phi_a(V)_{n+1}$ induced by multiplication by $T$ is injective (resp. surjective) for all $n \geq s$. By definition, $\text{stab-deg}(V) = \max(\text{inj-deg}(V), \text{surj-deg}(V))$.

**Lemma 3.1.6.** Let $V$ be an FI-module. Any quotient $V \to W$ satisfies $\text{surj-deg}(W) \leq \text{surj-deg}(V)$. When $k$ contains $\mathbb{Q}$, any submodule $W \subset V$ satisfies $\text{inj-deg}(W) \leq \text{inj-deg}(V)$.

**Proof.** Given $V \to W$, the maps $\Phi_a(V)_n \to \Phi_a(W)_n$ are surjective since $\Phi_a$ is right-exact. Thus if $T: \Phi_a(V)_n \to \Phi_a(V)_{n+1}$ is surjective, the same is true of $T: \Phi_a(W)_n \to \Phi_a(W)_{n+1}$. The proof for $W \subset V$ is identical, using that $\Phi_a$ is left-exact when $k$ contains $\mathbb{Q}$. 

\[\square\]
The following computation will be needed in Section 6; it also provides an example where the injectivity degree and surjectivity degree differ quite drastically.

**Proposition 3.1.7.** For any FB-module $W$, the FI-module $M(W)$ has inj-deg$(M(W)) = 0$; if $W_i = 0$ for $i > m$, then surj-deg$(M(W)) \leq m$. The FI-module $M(m)$ has inj-deg$(M(m)) = 0$ and surj-deg$(M(m)) = \text{stab-deg}(M(m)) = m$.

**Proof.** We will directly compute the graded $k[T]$-module $\Phi_a M(W)$. For any FB-module $W$, by [3] we have

$$M(W)_{\pi \cdot n} = \bigoplus_{T \subset \pi \cdot n} W_T.$$  

Since $\Phi_a M(W)_n$ is by definition the $S_n$-coinvariants of $M(W)_{\pi \cdot n}$, it splits as a direct sum over the orbits of $S_n$ acting on $\{T \subset \pi \cdot n\}$. Such an orbit is determined by the intersection $T \cap \pi$ (which can be any subset $U \subset \pi$) together with the cardinality $|T \cap n|$ (which can be any integer $0 \leq k \leq n$).

As a representative for the orbit corresponding to $(U, k)$ we can take $T = U \cup k$. The stabilizer in $S_n$ of this set consists of permutations of $n$ preserving the subset $k$, which we can identify with $S_k \times S_{n-k}$. This summand thus contributes $(W_{U \cup k})_{S_k \times S_{n-k}}$ to the coinvariants. Since the subgroup $S_{n-k} \leq S_k \times S_{n-k}$ acts trivially on $U \cup k$ and thus on $W_{U \cup k}$, we have $(W_{U \cup k})_{S_k \times S_{n-k}} = (W_{U \cup k})_{S_k}$. This last can be identified as a $k$-module with $\Phi_a |_{[U]} (W_k)$. (Recall from Definition 2.3.7 that every FB-module can be regarded as an FI-module, so it is meaningful to talk about $\Phi_a (W)$.) In conclusion, we have an isomorphism of $k$-modules

$$\Phi_a M(W)_n \cong \bigoplus_{U \subset \pi} \Phi_a |_{[U]} (W)_k \tag{6}$$

The map $T: \Phi_a M(W)_n \to \Phi_a M(W)_{n+1}$ is induced by $\text{id} \cup f: \pi \cup n \to \pi \cup n + 1$ for any $f: n \to n + 1$. Such an injection preserves the subset $T \cap \pi$ and the cardinality $|T \cap n| = |f(T) \cap n + 1|$. In other words, $T$ preserves the decomposition [6], and takes the factor $\Phi_a |_{[U]} (W)_k$ of $\Phi_a M(W)_n$ to the factor $\Phi_a |_{[U]} (W)_k$ of $\Phi_a M(W)_{n+1}$ via the identity. This demonstrates that inj-deg$(M(W))$ has injectivity degree 0, since $T: \Phi_a M(W)_n \to \Phi_a M(W)_{n+1}$ is always injective.

If $W_i = 0$ for $i > m$, certainly $\Phi_a |_{[U]} (W)_i = 0$ for $i > m$ (indeed for $i > m - |U|$). When $n \geq m$, the condition $0 \leq k \leq n$ in [6] is therefore vacuous, so this description of $\Phi_a M(W)_n$ is independent of $n$ for $n \geq m$. Since on each factor the map $T$ is an isomorphism, we conclude that $T: \Phi_a M(W)_n \to \Phi_a M(W)_{n+1}$ is an isomorphism for $n \geq m$, so $M(W)$ has surjectivity degree $\leq m$.

Finally, for $M(m) = M(k[S_m])$, we can compute that $\Phi_a (k[S_m])_m \cong (k[S_m])_m$ $\cong k \neq 0$. This summand contributes to $\Phi_a M(m)_n$ only when $n \geq m$, so surj-deg$(M(m)) = m$. \[\square\]

**Proposition 3.1.8.** If $V$ is generated in degree $\leq d$ then surj-deg$(V) \leq d$.

**Proof.** If $V$ is generated in degree $\leq d$, Lemma 2.3.2 shows that $V$ is a quotient of $\bigoplus_{m \leq d} M(V_m)$. The latter has surjectivity degree $\leq d$ by Proposition 3.1.7 and surj-deg$(V) \leq d$ by Lemma 3.1.6. \[\square\]

The converse of Proposition 3.1.8 is not true. For fixed $d \geq 0$, any FI-module $V$ with $(V_{d+1})_{S_{d+1}} = 0$ and $V_n = 0$ for $n > d + 1$ will have surj-deg$(V) \leq d$. Indeed, these assumptions guarantee that $\Phi_a (V)_{n+1} = 0$ for all $n \geq d$ and all $a \geq 0$, so surjectivity of $\Phi_a (V) \to \Phi_a (V)_{n+1}$ is automatic. But if $V_{d+1} \neq 0$, this FI-module is not generated in degree $\leq d$.
3.2 Weight and stability degree in characteristic 0

In this section we define and characterize certain structural properties of FI-modules over a field of characteristic 0 that will be used in §3.3 below.

**Definition 3.2.1 (Weight of an FI-module).** Let $V$ be an FI-module over a field of characteristic 0. The **weight** $\text{weight}(V)$ of $V$ is the maximum of $|\lambda|$ over all irreducible constituents $V(\lambda)_n$ occurring in the $S_n$-representations $V_n$. If $V = 0$ we set $\text{weight}(V) = 0$; if $|\lambda|$ is unbounded then $\text{weight}(V) = \infty$.

If $W$ is a subquotient of $V$, then $\text{weight}(W) \leq \text{weight}(V)$. We also have the following proposition, which follows from a well-known property of Kronecker coefficients: if $V(\nu)_n$ occurs in the tensor product $V(\lambda)_n \otimes V(\mu)_n$, then $|\nu| \leq |\lambda| + |\mu|$.

**Proposition 3.2.2.** If $V$ and $W$ are FI-modules over a field of characteristic 0, the tensor product $V \otimes W$ satisfies $\text{weight}(V \otimes W) \leq \text{weight}(V) + \text{weight}(W)$.

In our analysis of the notion of weight, we will make frequent use of the classical branching rule for $S_n$-representations (see e.g. [FH]). The same result holds if $\mathbb{Q}$ is replaced by any field of characteristic 0.

**Lemma 3.2.3 (Branching rule for $S_n$-representations).** Let $\lambda \vdash n$ be a partition, and $V_{\lambda}$ the corresponding irreducible $S_n$-representation over $\mathbb{Q}$.

(i) As $S_{n+k}$-representations, we have the decomposition

$$\text{Ind}_{S_n \times S_k}^{S_{n+k}} V_{\lambda} \otimes \mathbb{Q} \simeq \bigoplus_{\mu} V_{\mu}$$

over those partitions $\mu \vdash n + k$ obtained from $\lambda$ by adding one box to $k$ different columns.

(ii) As $S_{n-k}$-representations, we have the decomposition

$$\text{Res}_{S_n \times S_k}^{S_{n-k}} V_{\lambda} \otimes \mathbb{Q} \simeq \bigoplus_{\nu} V_{\nu}$$

over those partitions $\nu \vdash n - k$ obtained from $\lambda$ by removing one box from $k$ different columns.

**Proposition 3.2.4.** For any partition $\mu \vdash m$, the FI-module $M(\mu)$ over a field of characteristic 0 has weight $M(\mu) = m$.

*Proof.* The identification (4) shows that $M(\mu)_n = M(V_{\mu})_n$ is isomorphic as an $S_n$-representation to $\text{Ind}_{S_m \times S_{n-m}}^{S_n} V_{\mu} \otimes k$. Thus by Lemma 3.2.3(i), those $\nu \vdash n$ for which $V_{\nu}$ occurs in $M(\mu)_n$ are those obtained from $\mu$ by adding one box to $n - m$ different columns. Since $\nu_1$ is the number of columns of $\nu$, we must have $\nu_1 \geq n - m$. When writing $\nu = \lambda[n]$ we have $|\lambda| = n - \nu_1$, so we can rephrase this as saying that $|\lambda| \leq m$ for any constituent $V(\lambda)_n$ of $M(\mu)_n$. This shows that $\text{weight}(M(\mu)) \leq m$. For any $n \geq m + \nu_1$, adding one box to the first $n - m$ columns yields the partition $\mu[n]$, so $V(\mu)_n$ itself occurs in $M(\mu)_n$; we conclude that $\text{weight}(M(\mu)) = m$. \(\square\)

Proposition 3.2.4 shows that weight($M(V_m)$) $\leq m$ for any $S_m$-representation $V_m$. Since any FI-module $V$ generated in degree $\leq d$ is a quotient of $\bigoplus_{m \leq d} M(V_m)$ by Lemma 2.3.2 we obtain the following proposition as a corollary.

**Proposition 3.2.5.** Let $V$ be an FI-module over a field of characteristic 0. If $V$ is generated in degree $\leq d$, then $\text{weight}(V) \leq d$.
Stability degree in characteristic 0. The notion of stability degree over a field $k$ of characteristic 0 is richer than in the general case. For example, Proposition 3.1.7 states that the FI-module $M(\lambda) = M(V_\lambda)$ has stab-deg($M(\lambda)$) $\leq |\lambda|$, but in fact this can be improved. Recall that $\lambda_1$ is the length of the first row of the partition $\lambda$.

**Proposition 3.2.6.** For any partition $\lambda$ the FI-module $M(\lambda)$ over a field of characteristic 0 has stab-deg($M(\lambda)$) $= \lambda_1$.

To prove Proposition 3.2.6 we will need the following elementary consequences of the branching rule. If $V_n$ is an $S_n$-representation, we will write $(V_n)_S$ for the $S_{n-k}$-representation $(\Res_{S_{n-k}}^{S_n}) V_n$.

**Lemma 3.2.7.** Let $\mu$ be a partition of $n$, and write $\mu = \lambda|n|$ as in Definition 2.2.5. Given $a \leq n$, set $k := n - a$, and consider the $S_a$-representation $(V_\mu)_S = (V(\lambda)_n)_{S_{n-a}}$ over $\mathbb{Q}$.

(i) $(V_\mu)_S = 0 \iff k > \mu_1$; equivalently, $(V(\lambda)_n)_{S_{n-a}} = 0 \iff a < |\lambda|$.

(ii) If $k = \mu_1$ (i.e. if $a = |\lambda|$), we have $(V_\mu)_S = (V(\lambda)_n)_{S_{n-a}} \simeq V_\lambda$.

(iii) For fixed $\lambda$ and $a \geq |\lambda|$, the $S_a$-representation $(V(\lambda)_n)_{S_{n-a}}$ is independent of $n$ once $n \geq a + \lambda_1$; in fact

$$(V(\lambda)_n)_{S_{n-a}} \simeq \Ind_{S_{n-a}}^{S_n} S_{|\lambda| -a} \otimes_{\mathbb{Q}} V_\lambda \quad \text{for all } n \geq a + \lambda_1.$$ (7)

(iv) If $V$ is an FI-module with weight $(V) \leq a$, any subquotient $W_n$ of the $S_n$-representation $V_n$ satisfies $W_n = 0 \iff (W_n)_{S_{n-a}} = 0$.

**Proof.** Lemma 3.2.3(ii) states that $(V_\mu)_S$ is the sum of $V_\nu$ over partitions $\nu \vdash a$ obtained from $\mu$ by removing boxes from $k$ different columns. Since $\mu$ has only $\mu_1$ columns, this is impossible when $k > \mu_1$. This demonstrates (i), from which (iv) follows immediately. When $k = \mu_1$, this can be done only by removing one box from each of the $\mu_1$ columns of $\mu$; by definition, to say that $\mu = \lambda|n|$ means that the resulting partition is $\lambda$, demonstrating (ii).

To prove (iii), let $r = n - a$ and $c = a - |\lambda|$; our assumption is that $r \geq \lambda_1$. By Lemma 3.2.3(i), the right side of (7) consists of those $V_\nu$ for which $\nu \vdash a$ is obtained from $\lambda$ by adding one box to $c$ different columns. Observe that these columns must lie within the first $\lambda_1 + c$ columns, or the result would not be a partition.

By definition, $\lambda|n|$ is obtained from $\lambda$ by adding one box to each of the first $r + c$ columns. By Lemma 3.2.3(ii) the left side of (7) consists of those $V_\nu$ for which $\nu$ is obtained from $\lambda|n|$ by removing boxes from $r$ different columns; in other words, those $\nu$ obtained from $\lambda$ by adding one box to $c$ different columns within the first $r + c$ columns. When $r \geq \lambda_1$, the observation above shows this last condition is vacuous, and the collection of $V_\nu$ occurring in the left and right side of (7) coincide.

**Proof of Proposition 3.2.6.** From our earlier computation of $\Phi_\lambda M(V_\lambda)$ in (6) we have:

$$\Phi_\lambda M(V_\lambda)_n \simeq \bigoplus_{k \leq n} \Phi_{U \subseteq \Pi} (V_\lambda)_k = \bigoplus_{k \leq n \atop |U| + k = |\lambda|} (V_\lambda)_k$$ (8)

We saw in the proof of Proposition 3.1.7 that $T: \Phi_\lambda M(V_\lambda)_n \to \Phi_\lambda M(V_\lambda)_{n+1}$ is the identity on each factor of this decomposition, so our goal is to show that no new factors occur in $\Phi_\lambda M(V_\lambda)_n$ for $n > \lambda_1$.

By Lemma 3.2.7(i), $(V_\lambda)_S = 0$ when $k > \lambda_1$. Thus we can add the condition $k \leq \lambda_1$ to (8); the condition $k \leq n$ is then vacuous for $n \geq \lambda_1$, showing that no new factors occur after $n = \lambda_1$. By Lemma 3.2.7(ii), $(V_\lambda)_S \neq 0$ when $k = \lambda_1$. Therefore the factors with $k = \lambda_1$, which occur only for $n \geq \lambda_1$, are nonzero. We conclude that $T: \Phi_\lambda M(V_\lambda)_n \to \Phi_\lambda M(V_\lambda)_{n+1}$ is an isomorphism for $n \geq \lambda_1$ but is not surjective for $n = \lambda_1 - 1$, so $M(V_\lambda)$ has stability degree $\lambda_1$. 

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The stability degree bounds the width of the irreducible constituents of the \( S_n \)-representations \( V_n \).

**Proposition 3.2.8.** Let \( V \) be an FI-module over a field of characteristic 0. For all \( n \geq 0 \), every irreducible constituent \( V(\lambda)_n \) of the \( S_n \)-representation \( V_n \) satisfies \( \lambda_1 \leq \text{stab-deg}(V) \).

**Proof.** Let \( s := \text{stab-deg}(V) \). Consider the filtration \( F^i V := \text{span}(V_{<i}) \), which satisfies \( V = \bigcup F^i V \). We will prove the inequality \( \lambda_1 \leq s \) for the irreducible constituents of each quotient \( F^m V/F^{m-1} V \), and this implies the same inequality for \( V \).

We first prove that for all \( m \geq 0 \), every irreducible constituent \( V_\mu \) of \( H_0(V)_m \) satisfies \( \mu_1 \leq s \). Fix \( m \geq 0 \), and let \( \nabla := V/F^{m-1} V \). We have \( \nabla_m = V_m/\text{span}(V_{<m})_m = H_0(V)_m \) as \( S_m \)-representations, and of course \( \nabla_n = 0 \) for \( n < m \). Being a quotient of \( V \), we have \( \text{surj-deg}(\nabla) \leq s \) by Lemma 3.1.6, so the map \( T: \Phi_{m-s-1}(\nabla)_s \to \Phi_{m-s-1}(\nabla)_{s+1} \) is surjective. The domain \( \Phi_{m-s-1}(\nabla)_s \) is a quotient of \( \nabla_{m-1} = 0 \) and thus vanishes, so \( \Phi_{m-s-1}(\nabla)_{s+1} = 0 \) as well. The latter is isomorphic to \( (\nabla_m)_{s+1} \), so Lemma 3.2.7(i) states that \( s + 1 > \mu_1 \) for every irreducible constituent \( V_\mu \) of \( \nabla_m \cong H_0(V)_m \).

We next show that every irreducible constituent \( V(\lambda)_n \) of \( M(\lambda)_n \) satisfies \( \lambda_1 \leq \mu_1 \). As we saw in the proof of Proposition 3.2.4, \( M(\lambda)_n \) consists of those \( V_\nu \) for which \( \nu \vdash n \) obtained from \( \mu \) by adding boxes to \( n - m \) different columns. The length of the second row of \( \nu \) is thus bounded by the length of the first row of \( \mu \). When we write \( V_\nu = V(\lambda)_n \) we have \( \lambda_1 = \nu_2 \), which verifies the claim.

Consider the quotient \( W := F^m V/F^{m-1} V \); to complete the proof we must show that every irreducible constituent \( V(\lambda)_n \) of \( W_n \) satisfies \( \lambda_1 \leq s \). By definition \( F_m V \) is generated in degree \( \leq m \), so the same is true of \( W \). By Lemma 2.3.2 there exists a surjection \( \bigoplus_{n \leq m} M(W_n) \to W \). As before we have \( W_n = 0 \) for \( n < m \) and \( W_m \cong \nabla_m \cong H_0(V)_m \), so this simplifies to \( M(H_0(V)_m) \to W \). As an \( S_n \)-representation \( H_0(V)_m \) is a sum of irreducibles \( V_\mu \), which according to the second paragraph all satisfy \( \mu_1 \leq s \), so the third paragraph implies that every irreducible constituent \( V(\lambda)_n \) of \( M(H_0(V)_m)_n \) satisfies \( \lambda_1 \leq s \). As an \( S_n \)-representation \( W_n \) is a quotient of \( M(H_0(V)_m)_n \), so the desired property holds for \( W_n \) as well. This verifies the claim for \( F^m V/F^{m-1} V \), and completes the proof.

Finally, we prove the Noetherian property for FI-modules over rings containing \( \mathbb{Q} \).

**Proof of Theorem 1.3.** Let \( k \) be a Noetherian ring, let \( V \) be a finitely generated FI-module over \( k \) generated in degree \( \leq a \), and let \( W \) be a sub-FI-module of \( V \); our goal is to prove that \( W \) is finitely generated. Proposition 2.3.5 implies that \( V_n \) is a finitely generated \( k \)-module for each \( n \). Since \( k \) is Noetherian, its submodule \( W_n \) is finitely generated as well.

The functor \( \Phi_a \) of Definition 3.1.2 is exact over rings containing \( \mathbb{Q} \), so \( \Phi_a(W) \subset \Phi_a(V) \). The graded \( k[T] \)-module \( \Phi_a(V) \) is finitely generated by Proposition 3.1.8. Since \( k[T] \) is a Noetherian ring, its submodule \( \Phi_a(W) \) is finitely generated as a \( k[T] \)-module (equivalently, as a graded \( k[T] \)-module).

**The sub-FI-module \( \tilde{W} \).** Choose a finite set of generators \( x_1, \ldots, x_r \) of \( \Phi_a(W) \) as a graded \( k[T] \)-module, with \( x_i \in \Phi_a(W)_{n_i} \). Since \( \Phi_a(W)_{n} \cong (W_{a+n})_{S_n} \) is a quotient of \( W_{a+n} \), we can choose lifts \( w_i \in W_{a+n} \), projecting to \( x_i \). Let \( \tilde{W} \) be the finitely generated sub-FI-module of \( W \) generated by the lifts \( w_1, \ldots, w_r \), together with finite generating sets for \( W_0, \ldots, W_1 \).

Since \( \tilde{W} \subset W \) we have \( \Phi_a(\tilde{W}) \subset \Phi_a(W) \). Since \( \Phi_a(W) \) contains the generating set \( \{ x_i \} \) we have \( \Phi_a(\tilde{W}) \supset \Phi_a(W) \). Thus \( \Phi_a(\tilde{W}) = \Phi_a(W) \). Since \( \Phi_a \) is exact, this implies that \( \Phi_a(W/\tilde{W}) = 0 \).

**The difference \( W/\tilde{W} \).** For any \( n \) we can decompose \( (W/\tilde{W})_n \) as a \( k[S_n] \)-module into isotypic components \( (W/\tilde{W})_n \cong \bigoplus N_\lambda \otimes \mathbb{Q} V(\lambda)_n \), where \( V(\lambda)_n \) is the irreducible \( \mathbb{Q}[S_n] \)-module and \( N_\lambda \) is a \( k \)-module. We claim that only those \( \lambda \) with \( |\lambda| \leq a \) appear in this decomposition. This is essentially nothing more than the claim weight(\( W/\tilde{W} \)) \( \leq a \), except that weight is only defined for FI-modules over a field of characteristic 0.

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However, by restricting to $Q \subset k$, we can consider any FI-module $U$ over $k$ as an FI-module $U^Q$ over $Q$. To say that $V$ is generated in degree $\leq a$ means the natural map $\bigoplus_{m \leq a} M(V_m) \to V$ is surjective. It certainly remains surjective when considered as a map of $Q$-vector spaces, so since $M(V_m)^Q = M(V_m^Q)$ this provides a surjection $\bigoplus_{m \leq a} M(V_m^Q) \to V^Q$. Therefore $V^Q$ is generated in degree $\leq a$ as well, weight($V^Q$) $\leq a$ by Proposition 3.2.5 states that weight($V^Q$) $\leq a$. Since $(W/\tilde{W})^Q$ is a subquotient of $V^Q$ we have weight($(W/\tilde{W})^Q) \leq a$, as claimed.

For any $n \geq a$, we have

$$\Phi_a(W/\tilde{W})_{n-a} = (W/\tilde{W})_n \otimes_{k[S_{n-a}]} k \cong \bigoplus N_\lambda \otimes_Q (V(\lambda)_n \otimes_{Q[S_{n-a}]} Q).$$

Lemma 3.2.7(i) states that $V(\lambda)_n \otimes_{Q[S_{n-a}]} Q = (V(\lambda)_n)_{S_{n-a}}$ is nonzero when $|\lambda| \leq a$. But we proved above that $\Phi_a(W/\tilde{W}) = 0$. This is only possible if $N_\lambda = 0$ for all $\lambda$, or in other words if $(W/\tilde{W})_n = 0$.

This shows that $(W/\tilde{W})_n = 0$ for all $n \geq a$, and $\tilde{W}_n = W_n$ by definition for $n < a$, so $W = \tilde{W}$. This demonstrates that $W$ itself is finitely generated, and thus completes the proof of the theorem.

### 3.3 Representation stability and character polynomials

Our main goal in this subsection is to prove Proposition 3.3.3 relating the weight and stability degree of FI-modules with uniform representation stability in the sense of Church–Farb [CF]. We then use this to prove Theorem 1.13 which states the equivalence of finite generation and representation stability.

We first quickly recall the definition of representation stability; see [CF] for a detailed treatment.

As mentioned in the introduction, an FI-module $V$ provides a sequence $\{V_n\}$ of $S_n$-representations for all $n \in \mathbb{N}$. Moreover, if $f_{n,n+k} : n \mapsto n + k$ denotes the standard inclusion, the injections $f_{n,n+1} : n \mapsto n + 1$ induce $S_n$-equivariant maps $\phi_n : V_n \to V_{n+1}$. Such a sequence $\{V_n, \phi_n\}$ of $S_n$-representations and $S_n$-equivariant maps was called a consistent sequence in [CF].

**Remark 3.3.1.** Not every consistent sequence can arise from an FI-module. For any $\sigma \in S_{n+k}$ with $\sigma|_{n} = \text{id}$, we have the identity $f_{n,n+k} = \sigma \circ f_{n,n+k}$ in $\text{Hom}_{FI}(n, n+k)$. Therefore on any consistent sequence arising from an FI-module,

$$\text{all } \sigma \in S_{n+k} \text{ with } \sigma|_{n} = \text{id} \text{ must act trivially on } \text{im}(V_n \to \cdots \to V_{n+k}) \subset V_{n+k}.$$  

(9)

For example, the consistent sequence of regular representations $k \to k \to k[S_2] \to k[S_3] \to \cdots$ induced by the standard inclusions $S_m \hookrightarrow S_{m+1}$ does not satisfy (9), and thus cannot arise from an FI-module. Conversely, it is not difficult to check that this condition is also sufficient: any consistent sequence satisfying (9) can be “promoted” to an FI-module. (In fact, it suffices that (9) holds when $k = 2$.)

**Definition 3.3.2 (Representation stability [CF]).** Let $\{V_n, \phi_n\}$ be a consistent sequence of $S_n$-representations over a field of characteristic 0. The sequence $\{V_n, \phi_n\}$ is uniformly representation stable with stable range $n \geq N$ if each of the following conditions holds.

**I. Injectivity:** The map $\phi_n : V_n \to V_{n+1}$ is injective for all $n \geq N$.

**II. Surjectivity:** The span of the $S_{n+1}$-orbit of $\phi_n(V_n)$ equals all of $V_{n+1}$ for all $n \geq N$.

**III. Multiplicities:** Decompose $V_n$ into irreducible representations as

$$V_n = \bigoplus_{\lambda} c_{\lambda,n} V(\lambda)_n$$

with multiplicities $0 \leq c_{\lambda,n} \leq \infty$. For each $\lambda$, the multiplicities $c_{\lambda,n}$ are independent of $n$ for $n \geq N$. 

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We say that $V$ is uniformly representation stable if Definition 3.3.2 holds for some $N$. The following proposition shows that a stability degree for an FI-module $V$ guarantees that it is uniformly representation stable with a stable range that can be specified quite precisely.

**Proposition 3.3.3 (Stability degree and representation stability).** Let $V$ be an FI-module over a field of characteristic 0. The sequence of $S_n$-representations $\{V_n, \phi_n\}$ is uniformly representation stable with stable range $n \geq \text{weight}(V) + \text{stab-deg}(V)$.

**Proof.** Set $s := \text{stab-deg}(V)$ and $d := \text{weight}(V)$. Let $K_n$ be the kernel of the $S_n$-equivariant map $\phi_n : V_n \to V_{n+1}$, and let $C_{n+1}$ be the cokernel of the $S_{n+1}$-equivariant map $\phi'_n := \text{Ind}_{S_n}^{S_{n+1}} \phi_n : \text{Ind}_{S_n}^{S_{n+1}} V_n \to V_{n+1}$ induced by $\phi_n$.

To prove Conditions 1 and 2 of Definition 3.3.2, we must show that $K_n = 0$ and $C_{n+1} = 0$ for all $n \geq s + d$. Since $K_n$ is a subrepresentation of $V_n$ and $C_{n+1}$ is a quotient of $V_{n+1}$, Lemma 3.2.7(iv) implies for any $n \geq d$ that $K_n = 0 \iff (K_n)_{S_{n-d}} = 0$ and $C_{n+1} = 0 \iff (C_{n+1})_{S_{n+1-d}} = 0$. Since taking $S_n$-coinvariants in characteristic 0 is exact, $(K_n)_{S_{n-d}}$ is the kernel of $(\phi_n)_{S_{n-d}} : (V_n)_{S_{n-d}} \to (V_{n+1})_{S_{n-d}}$, and $(C_{n+1})_{S_{n+1-d}}$ is the cokernel of $(\phi'_n)_{S_{n+1-d}} : (\text{Ind}_{S_n}^{S_{n+1}} V_n)_{S_{n+1-d}} \to (V_{n+1})_{S_{n+1-d}}$.

The map $T : \Phi_d(V)_{n-d} \to \Phi_d(V)_{n+1-d}$ of Definition 3.1.3 can be factored in two ways:

$$T : (V_n)_{S_{n-d}} \to (\text{Ind}_{S_n}^{S_{n+1}} V_n)_{S_{n+1-d}} \to (V_{n+1})_{S_{n+1-d}}.$$  

By definition of stability degree, this composition $T$ is an isomorphism when $n - d \geq s$. It follows that $(\phi_n)_{S_{n-d}}$ is injective and $(\phi'_n)_{S_{n+1-d}}$ is surjective when $n - d \geq s$. We conclude for $n \geq s + d$ that $(K_n)_{S_{n-d}} = 0$ and $(C_{n+1})_{S_{n+1-d}} = 0$; therefore $K_n = 0$ and $C_{n+1} = 0$, as desired.

It remains to prove that the multiplicity $c_{\lambda,n}$ of $V(\lambda)_n$ in $V_n$ is constant when $n \geq s + d$. We prove this by induction on $|\lambda|$; no base case will be necessary. If $|\lambda| > d$, by definition of weight $c_{\lambda,n} = 0$ for all $n$, so it suffices to prove this for $|\lambda| \leq d$. For fixed $m \leq d$, assume by induction that for all $\mu$ with $|\mu| < m$ the multiplicity $c_{\mu,n}$ of $V(\mu)_n$ in $V_n$ is constant for $n \geq s + d$.

By definition of stability degree, that the isomorphism class of the $S_m$-representation $(V_n)_{S_{n-m}}$ is independent of $n$ for $n \geq s + m$. By Proposition 3.2.8, each $V(\lambda)_n$ which occurs in $V_n$ satisfies $\lambda_1 \leq s$. The results of Lemma 3.2.7 apply once $n \geq \lambda_1 + m$, so they hold in our range $n \geq s + d$.

By Lemma 3.2.7(ii), only those $V(\lambda)_n$ with $|\lambda| \leq m$ contribute to $(V_n)_{S_{n-m}}$, so we can write

$$V_n S_{n-m} = \bigoplus_{|\mu| < m} c_{\mu,n} V(\mu)_n S_{n-m} \oplus \bigoplus_{|\lambda| = m} c_{\lambda,n} V(\lambda)_n S_{n-m} \quad (10)$$

For each $\mu$ appearing in the first summand, Lemma 3.2.7(iii) states that the $S_m$-representation $(V(\mu)_n)_{S_{n-m}}$ is independent of $n$ in our range. By induction the multiplicities $c_{\mu,n}$ are also constant for $n \geq s + d$, so the isomorphism class of the first summand of (10) is constant for $n \geq s + d$. Since the same is true of $(V_n)_{S_{n-m}}$ itself, the isomorphism class of the second summand of (10) must also be constant for $n \geq s + d$. By Lemma 3.2.7(ii), the second summand of (10) is simply $\bigoplus_{|\lambda| = m} c_{\lambda,n} V\lambda$. Therefore for each $\lambda$ with $|\lambda| = m$, the multiplicity $c_{\lambda,n}$ is constant for $n \geq s + d$, as desired.

We can now complete the proof of Theorem 1.13 which states that representation stability is equivalent to finite generation for FI-modules over a field of characteristic 0. We also observe along the way that finitely generated FI-modules are monotone in the sense of [Ch, Definition 1.2].
Proof of Theorem 1.13. Assume that $V$ is finitely generated, say in degree $\leq g$. We have surj-deg$(V) \leq g$ by Proposition 3.1.8 and we would like to bound inj-deg$(V)$ as well. By Lemma 2.3.2 there is a free FI-module $M := \bigoplus_{i=0}^{g} M(i)\otimes b$ with a surjection $M \to V$; let $K$ be the kernel of this map. By Theorem 1.3, the submodule $K$ of $M$ is finitely generated, say in degree $\leq r$.

Since $\Phi_n$ is exact over rings containing $\mathbb{Q}$, we have $\Phi_n(V)_n \cong \Phi_n(M)_n/\Phi_n(K)_n$. Proposition 3.1.7 states that inj-deg$(M) = 0$, so the maps $T : \Phi_n(M)_n \to \Phi_n(M)_{n+1}$ are injective for all $n \geq 0$. Proposition 3.1.8 implies surj-deg$(K) \leq r$, so for $n \geq r$ the maps $T : \Phi_n(K)_n \to \Phi_n(K)_{n+1}$ are surjective. We conclude that for $n \geq r$ the maps $T : \Phi_n(V)_n \to \Phi_n(V)_{n+1}$ are injective, or in other words that inj-deg$(V) \leq r$. Therefore stab-deg$(V) \leq N := \max(g, r)$, so Proposition 3.3.3 implies that $\{V_n, \phi_n\}$ is uniformly representation stable in degrees $\geq N + g$.

Another application of Proposition 3.3.3 shows that $\{K_n\}$ is uniformly representation stable in some appropriate range, and Church [Ch] Theorem 2.8 states that $\{\bigoplus_{i=0}^{g} M(i)\otimes b\}$ is monotone. It then follows from [Ch] Proposition 2.3 that $\{V_n\}$ is monotone in some stable range.

It remains to show that uniform representation stability implies finite generation. Assume that $\{V_n, \phi_n\}$ is uniformly representation stable for $n \geq N$. Definition 3.3.2 states that if $|\text{for } n \geq N$ the $S_{n+1}$-span of $\text{im}(\phi_n, V_n \to V_{n+1})$ is all of $V_{n+1}$. Equivalently, $V_{n+1}$ is spanned by the images of all FI-maps $f : V_n \to V_{n+1}$ for $f : n \in [n+1]$. By induction, this implies that $V = \text{span}(V_{\leq N})$. Since each $V_n$ is finite-dimensional by assumption, this shows that $V$ is finitely generated.

Character polynomials. We finish this section by proving a refined version of Theorem 1.5. Recall that for each $i \geq 1$ and any $n \geq 0$, the class function $X_i : S_n \to \mathbb{N}$ is defined by

$$X_i(\sigma) := \text{number of } i\text{-cycles in } \sigma.$$  

For example, $X_1(\sigma)$ is the number of fixed points of the permutation $\sigma$. Polynomials in the variables $X_i$ are called character polynomials. Class functions form a ring under pointwise product, so any character polynomial $P \in \mathbb{Q}[X_1, X_2, \ldots]$ also defines a class function $P : S_n \to \mathbb{Q}$ for all $n \geq 0$. The degree of a character polynomial is defined by setting $\deg(X_i) = i$.

Theorem 3.3.4 (Polynomiality of characters). Let $V$ be a finitely generated FI-module over a field of characteristic 0. There is a unique polynomial $P_V \in \mathbb{Q}[X_1, X_2, \ldots]$ with $\deg P_V \leq \text{weight}(V)$ such that for all $n \geq \text{stab-deg}(V) + \text{weight}(V)$ and all $\sigma \in S_n$,

$$\chi_{V_n}(\sigma) = P_V(\sigma).$$

Proof. Classically, the interest in character polynomials is driven by the following fact: for each partition $\lambda$, there exists a polynomial $P_\lambda \in \mathbb{Q}[X_1, X_2, \ldots]$ of degree $|\lambda|$ such that $\chi_{V_n}(\sigma) = P_\lambda(\sigma)$ for all $n \geq |\lambda| + \lambda_1$ and all $\sigma \in S_n$ [Mac, Example I.7.14]. This seems to have been known, at least implicitly, as far back as Murnaghan; Macdonald traces it back to work of Frobenius in 1904. For a more recent reference, see Garsia–Goupil [GG].

Proposition 3.3.3 implies that for every finitely generated FI-module $V$, there exist constants $c_\lambda$ so that $V_n \cong \bigoplus_\lambda c_\lambda V(\lambda)_n$ for all $n \geq \text{stab-deg}(V) + \text{weight}(V)$. If $\lambda$ is a partition with $c_\lambda \neq 0$, the definition of weight implies that $|\lambda| \leq \text{weight}(V)$ and Proposition 3.2.8 states that $\lambda_1 \leq \text{stab-deg}(V)$. The polynomial $P_V := \sum_\lambda c_\lambda P_\lambda$ thus satisfies the desired claim.

3.4 Murnaghan’s theorem and stability of Schur functors

The FI-module $V(\lambda)$. Given a partition $\lambda$, in Definition 2.2.5 we defined the padded partition $\lambda[n]$ and the irreducible $S_n$-representation $V(\lambda)_n := V_{\lambda[n]}$. The following proposition says that all these $S_n$-representations arise from a single finitely-generated FI-module $V(\lambda)$ with $V(\lambda)_n \cong V(\lambda)_n$. 23
Proposition 3.4.1 (The FI-module \(V(\lambda)\)). Let \(k\) be a field of characteristic 0. For any partition \(\lambda\), there is an FI-module \(V(\lambda)\) satisfying

\[
V(\lambda)_n \simeq \begin{cases} 
V_{\lambda[n]} & \text{if } n \geq |\lambda| + \lambda_1 \\
0 & \text{otherwise}
\end{cases}
\]

that is finitely generated in degree \(|\lambda| + \lambda_1\). It has \(\text{stab-deg}(V(\lambda)) = \lambda_1\) and \(\text{weight}(V(\lambda)) = |\lambda|\).

Proof. We will define \(V(\lambda)\) as a sub-FI-module of \(M(\lambda)\). Let \(d := |\lambda| + \lambda_1\). Since every finite set is isomorphic to \(n\) for some \(n \geq 0\), it suffices to define the subspace \(V(\lambda)_n \subset M(\lambda)_n\) for \(n \geq d\).

Lemma 3.2.3(i) states that \(V_\mu\) occurs in \(M(\lambda)_n\) if \(\mu \vdash n\) is obtained from \(\lambda\) by adding one box to \(n - |\lambda|\) different columns. When \(n - |\lambda| \geq \lambda_1\), adding boxes to the first \(n - |\lambda|\) columns yields the partition \(\mu = \lambda[n]\). (For future reference, we note that \(\lambda[n]\) has \(|\lambda|\) boxes below the first row, while all other partitions \(\mu \neq \lambda[n]\) that occur have \(< |\lambda|\) boxes below the first row.) The condition (11) that \(V(\lambda)_n \simeq V_{\lambda[n]}\) for \(n \geq d\) thus uniquely defines the subspace \(V(\lambda)_n \subset M(\lambda)_n\).

We must now verify that these subspaces define a sub-FI-module \(V(\lambda)\) of \(M(\lambda)\). That is, we must show that for every \(f \in \Hom_{\mathfrak{FI}}(\mathfrak{m}, \mathfrak{n})\) the map \(f_*: M(\lambda)_m \to M(\lambda)_n\) satisfies \(f_*(V(\lambda)_m) \subset V(\lambda)_n\). As we will see, this follows just from the fact that \(f_*\) is \(S_m\)-equivariant.

According to Pieri’s rule [FH, Exercise 4.44], the \(S_m\)-irreducible representations \(V_\nu\) which occur in the restriction \(\text{Res}_{S_m}^{S_n} V_\mu\) are exactly those for which \(\nu \vdash m\) can be obtained from \(\mu \vdash n\) by removing \(n - m\) boxes. We noted above that every \(V_\mu\) with \(\mu \neq \lambda[n]\) occurring in \(M(\lambda)_n\) has \(< |\lambda|\) boxes below the first row. Any partition \(\nu\) obtained by removing boxes from such a \(\mu \neq \lambda[n]\) must also have \(< |\lambda|\) boxes below the first row. In particular, \(V(\lambda)_m\) cannot occur in \(\text{Res}_{S_m}^{S_n} V_\mu\) for any such \(\mu \neq \lambda[n]\).

We conclude that any \(S_m\)-equivariant map from \(V(\lambda)_m\) to \(M(\lambda)_n\) has image contained in \(V(\lambda)_n\). In particular this applies to \(f_*\), which shows that \(V(\lambda)\) is a sub-FI-module of \(M(\lambda)\).

We now show that \(\text{span}(V_d) = V\). For any \(n \geq d\), choose any injection \(f: d \hookrightarrow n\). The map \(f_*: V(\lambda)_d \to V(\lambda)_n\) is injective, since \(f_*: M(\lambda)_d \to M(\lambda)_n\) is. Its image \(f_*(V(\lambda)_d) \subset V(\lambda)_n\) is thus nonzero, showing that \(\text{span}(V_d)_n \neq 0\). Since \(V(\lambda)_n\) is irreducible, it follows that \(\text{span}(V_d)_n = V_n\), as desired. This argument (a special case of the “monotonicity” proved for \(M(\lambda)\) in Church [Ch, Theorem 2.8]) shows that \(V\) is generated in degree \(d = |\lambda| + \lambda_1\).

That \(\text{weight}(V(\lambda)) = |\lambda|\) is immediate from (11). Since \(V(\lambda)\) is a sub-FI-module of \(M(\lambda)\), Lemma 3.1.6 implies that \(V(\lambda)\) has injectivity degree 0. Let \(\mu = (\lambda_1, \lambda_1, \ldots, \lambda_d) = |d|\). Since \(V(\lambda)\) is generated by \(V(\lambda)_d \simeq V_\mu\), the FI-module \(V(\lambda)\) is a quotient of \(M(\mu)\). By Proposition 3.2.6 we know \(\text{surj-deg}(M(\mu)) = \mu_1 = \lambda_1\), so by Lemma 3.1.6 we have \(\text{surj-deg}(V(\lambda)) \leq \lambda_1\). Lemma 3.2.7(ii) implies that \(\Phi|\lambda|V(\lambda)_{\lambda_1} \simeq V_{\lambda_1} \neq 0\), while \(\Phi|\lambda|V(\lambda)_{\lambda_1-1} = 0\), so this bound is sharp. \(\square\)

Murnaghan’s theorem. In 1938 Murnaghan stated the following theorem; the first complete proof of this theorem was given in 1957 by Littlewood [Ll].

Theorem 3.4.2 (Murnaghan’s Theorem). For each pair of partitions \(\lambda\) and \(\mu\) there exists a finite set \(S\) of partitions \(\nu\) and a set of nonnegative integers \(g^\nu_{\lambda,\mu}\) such that for all sufficiently large \(n\):

\[
V(\lambda)_n \otimes V(\mu)_n = \bigoplus_{\nu \in S} g^\nu_{\lambda,\mu} V(\nu)_n.
\]

As we now demonstrate, Murnaghan’s Theorem follows rather easily from the theory that we have already built up as a structural statement about a single object, the FI-module \(V(\lambda) \otimes V(\mu)\).
Proof. Since $V(\lambda)$ and $V(\mu)$ are finitely generated, Proposition 2.3.6 implies that $V(\lambda) \otimes V(\mu)$ is finitely generated. Theorem 1.13 then implies that the sequence of $S_n$-representations

$$(V(\lambda) \otimes V(\mu))_n = V(\lambda)_n \otimes V(\mu)_n$$

is uniformly representation stable, which implies (12).

**Stability for Schur functors.** Let $k$ be a field of characteristic 0. Given a partition $\lambda$, the Schur functor $S_\lambda$ is a functor from $k$-vector spaces to $k$-vector spaces whose properties are the subject of substantial interest in combinatorics and representation theory. See [FH, Lecture 6] for the basic properties of Schur functors. Since $S_\lambda$ is a covariant functor, for any FI-module $V$ we can consider the FI-module $S_\lambda V$, which satisfies $(S_\lambda V)_S := S_\lambda(V_S)$.

**Proposition 3.4.3 (Finite generation of Schur functors).** Let $V$ be an FI-module over a field of characteristic 0. Assume that $V$ is finitely generated (resp. generated in degree $\leq m$, resp. of weight $\leq m$). Then for any partition $\lambda$, the FI-module $S_\lambda(V)$ is finitely generated (resp. generated in degree $\leq m \cdot |\lambda|$, resp. of weight $\leq m \cdot |\lambda|$).

Proof. Let $d = |\lambda|$, and let $c_\lambda \in k[S_d]$ be the Young symmetrizer associated to the partition $\lambda$. The permutation group $S_d$ acts on the FI-module $V^{\otimes d}$ by permuting the factors, so $c_\lambda$ defines an idempotent endomorphism of $V^{\otimes d}$ whose image is isomorphic to $S_\lambda(V)$. Propositions 2.3.6 and 3.2.2 imply that the FI-module $V^{\otimes d}$ has the desired properties, so the same is true of its quotient $S_\lambda(V)$.

Applying Theorem 1.13 and Proposition 3.4.3 to the FI-module $V = M(1)$ over $\mathbb{Q}$ yields the following corollary, which resolves a basic issue left open in [CF] (cf. Theorem 3.1 of [CF]). The relative simplicity of the proof given here illustrates the power of the FI-module point of view.

**Corollary 3.4.4 (Stability of Schur functors).** The sequence of $S_n$–representations $\{S_\lambda(\mathbb{Q}^n)\}$ is monotone in the sense of [Ch] and is uniformly representation stable.

Applying Proposition 3.4.3 to the finitely generated FI-module $V(\mu)$ from Proposition 3.4.1 shows that the FI-module $S_\lambda(V(\mu))$ is finitely generated. Theorem 1.13 then implies the following.

**Proposition 3.4.5 (Schur functors of irreducibles).** Let $\lambda, \mu$ be partitions. There exists a finite set $S$ of partitions $\nu$ and a set of nonnegative integers $\beta^\nu_{\lambda, \mu}$ such that for all sufficiently large $n$:

$$S_\lambda(V(\mu)_n) = \bigoplus_{\nu \in S} \beta^\nu_{\lambda, \mu} V(\nu)_n$$

4 FI$^\#$-modules and graded FI-modules

In this section we describe and analyze two types of additional structure frequently carried by the FI-modules that arise in applications.

4.1 FI$^\#$-modules

Many of the sequences of $S_n$-representations we encounter in applications simultaneously carry both an FI-module and a co-FI-module structure, which are compatible in a certain precise sense. Together these give such sequences an FI$^\#$-module structure, which is extremely rigid.
Definition 4.1.1 (FI\textsuperscript{♯}-modules). Let FI\textsuperscript{♯} be the category whose objects are finite sets, with morphisms being partially-defined injections \( S \supset A \xrightarrow{\phi} B \subset T \). That is, \( \text{Hom}_{\text{FI}\textsuperscript{♯}}(S,T) \) consists of triples \((A,B,\phi)\) with \( A \subseteq S, B \subseteq T \) and \( \phi: A \to B \) an isomorphism, with the composition of

\[
S \supset \phi^{-1}(B) \xrightarrow{\phi} B \subset T \quad \text{and} \quad T \supset C \xrightarrow{\psi} \psi(C) \subset U \quad \text{being} \quad S \supset \phi^{-1}(B \cap C) \xrightarrow{\psi \circ \phi} \psi(B \cap C) \subset U.
\]

An FI\textsuperscript{♯}-module \( V \) is a functor from FI\textsuperscript{♯} to the category of \( k \)-modules, and \( \text{FI}\textsuperscript{♯}-\text{Mod} = [\text{FI}\textsuperscript{♯}, \text{k-Mod}] \) is the category of FI\textsuperscript{♯}-modules.

Example 4.1.2. The most basic example is the FI\textsuperscript{♯}-module \( M(1) \) taking a finite set \( S \) to the free \( k \)-module \( M(1)_S = \langle e_s | s \in S \rangle \) with basis \( S \). A morphism \( f: S \supset A \xrightarrow{\phi} B \subset T \) induces the map \( f_*: M(1)_S \to M(1)_T \) defined by \( f_*(e_s) = e_{\phi(s)} \) if \( s \in A \), and \( f_*(e_s) = 0 \) otherwise.

Remark 4.1.3. The category FI embeds in FI\textsuperscript{♯} by taking only those morphisms of the form \( S = S \xrightarrow{\phi} B \subset T \), so every FI\textsuperscript{♯}-module can be considered as an FI-module. The category co-FI also embeds in FI\textsuperscript{♯} by taking only those morphisms of the form \( S \supset A \xrightarrow{\phi} T \), so every FI\textsuperscript{♯}-module can be considered as a co-FI-module as well. The relations in FI\textsuperscript{♯} impose conditions on how these FI-module and co-FI-module structures interact.

In particular, the endomorphisms \( \text{End}_{\text{FI}\textsuperscript{♯}}(n) \) form the so-called rook algebra of rank \( n \), so for any FI\textsuperscript{♯}-module \( V \) the \( k \)-module \( V_n \) is a representation of the rook algebra. For the FI\textsuperscript{♯}-module \( M(1) \) in Example 4.1.2, \( M(1)_n \) is the standard representation of the rook algebra on \( k^n \). The basic properties of the rook algebra and its representation theory were determined by Munn and Solomon [So].

The category FI\textsuperscript{♯} is naturally isomorphic to FI\textsuperscript{♯}op, by simply taking \( S \supset A \xrightarrow{\phi} B \subset T \in \text{Hom}_{\text{FI}\textsuperscript{♯}}(S,T) \) to \( T \supset B \xrightarrow{\phi^{-1}} A \subset S \in \text{Hom}_{\text{FI}\textsuperscript{♯}op}(S,T) \). Therefore the dual of an FI\textsuperscript{♯}-module is naturally an FI\textsuperscript{♯}-module.

Example 4.1.4 (\( M(W) \) is an FI\textsuperscript{♯}-module). Given an FB-module \( W \), we defined the FI-module \( M(W) \) in Definition 2.2.2. It turns out this extends to a natural FI\textsuperscript{♯}-module structure on \( M(W) \), as follows. To a finite set \( S \), the FI\textsuperscript{♯}-module \( M(W) \) assigns

\[
M(W)_S := \bigoplus_{T \subseteq S} W_T.
\]

To a morphism \( f: S \supset A \xrightarrow{\phi} B \subset S' \), it assigns the map \( f_*: M(W)_S \to M(W)_{S'} \) which on the factor \( W_T \) is \( (\phi|_T)_*: W_T \to W_{\phi(T)} \) if \( T \subseteq A \), and is 0 if \( T \not\subseteq A \).

When restricted to FI-morphisms (those with \( A = S \)), the condition \( T \subseteq A \) is always satisfied, and so this definition agrees with Definition 2.2.2. This means that when considered as FI-modules, the two definitions of \( M(W) \) agree, so there is no conflict of notation. For example, the FI\textsuperscript{♯}-module \( M(1) \) of Example 4.1.2 becomes the FI-module \( M(1) \) defined in Definition 2.2.3.

The classification of FI\textsuperscript{♯}-modules. The following classification theorem shows that every FI\textsuperscript{♯}-module is of the form \( M(W) \) for some FB-module \( W \). Considering an FI\textsuperscript{♯}-module as an FI-module provides a forgetful functor \( \text{FI}\textsuperscript{♯}-\text{Mod} \to \text{FI-Mod} \). By slight abuse of notation, we denote the composition \( \text{FI}\textsuperscript{♯}-\text{Mod} \to \text{FI-Mod} \xrightarrow{H_0} \text{FB-Mod} \) also by \( H_0 \). To reiterate, if \( V \) is an FI\textsuperscript{♯}-module, then \( H_0(V) \) is always defined by considering \( V \) as an FI-module and applying Definition 2.3.7.

Theorem 4.1.5 (Classification of FI\textsuperscript{♯}-modules). The category of FI\textsuperscript{♯}-modules is equivalent to the category of FB-modules, via the equivalence of categories

\[
M(-): \text{FB-Mod} \rightleftarrows \text{FI}\textsuperscript{♯}-\text{Mod} : H_0(-).
\]

Thus every FI\textsuperscript{♯}-module \( V \) is of the form \( V = \bigoplus_{i=0}^\infty M(W_i) \), where \( W_i \) is the \( S_i \)-representation \( H_0(V)_i \).
Remark 4.1.6. Pirashvili established in [Pi] Theorem 3.1 an equivalence between the category of Γ-modules and FS-modules; here Γ is the category of finite based sets and all based maps, and FS is the category of finite sets and surjections. The classification of FI♯-modules in Theorem 4.1.5 is closely related to Pirashvili's result, although technically neither follows from the other.

Since FB is a full subcategory of FS, an FS-module can be thought of as an FB-module with additional morphisms. Moreover, the full subcategory of Γ whose morphisms are based maps that are closely related to Pirashvili's result, although technically neither follows from the other.

♯ is the category of finite sets and surjections. The classification of FI♯-modules and FS-modules; here Γ is the category of finite based sets and all based maps, and FS

Theorem 4.1.5, one could think of Pirashvili's theorem saying that the necessary additional information to promote an FI♯-module to a Γ-module is the same as the information needed to promote an FB-module to an FS-module (and indeed, in spirit this is how Pirashvili proves his result).

Proof of Theorem 4.1.5. We have already seen the canonical isomorphism $H_0(M(W)) \simeq W$ for any FB-module $W$ in Remark 2.3.8. It remains to prove that for any FI♯-module, we have a canonical isomorphism $V \simeq M(H_0(V))$.

Fix $n \geq 0$. We will analyze the category of FI♯-modules $V$ satisfying the condition

$$V_S = 0 \quad \text{for all } S \text{ with } |S| < n. \quad (13)$$

The assumption (13) implies that the quotient defining $H_0(V)_n$ is vacuous, so $H_0(V)_n \simeq V_n$ as $S_n$-representations. Our goal will be to prove that we have a natural isomorphism

$$V \simeq M(V_n) \oplus V' \simeq M(H_0(V)_n) \oplus V', \quad (14)$$

where $V'$ is an FI♯-module for which $V_S = 0$ whenever $|S| \leq n$. The theorem then follows by induction on $n$ (beginning with $n = 0$, when the assumption (13) is vacuous), since $H_0(V) = \bigoplus_n H_0(V)_n$.

Notation. In this proof we use the following notation. Given a subset $C \subseteq S$, we denote by $I_C \in \text{End}_{\text{FI}^{\#}}(S)$ the morphism $I_C : S \supset C \xrightarrow{\text{id}} C \subset S$. Given a morphism $f : S \supset A \xrightarrow{\phi} \phi(A) \subset T$ and subset $C \subset A$, we denote by $f|_C$ the morphism $f|_C := f \circ I_C = S \supset C \xrightarrow{\phi} \phi(C) \subset T$.

The endomorphism $E$. Under the assumption (13), we define an endomorphism $E : V \to V$ of FI♯-modules by taking $E_S : V_S \to V_S$ to be:

$$E_S = \sum_{C \subseteq S, |C| = n} (I_C)_* \quad (15)$$

We first verify that $E : V \to V$ is a map of FI♯-modules: given a morphism $f : S \supset A \xrightarrow{\phi} B \subset T$, we must check that $f_* \circ E_S = E_T \circ f_* : E_S \to E_T$.

Expanding $f_* \circ E_S$ yields a sum of terms $f_* \circ (I_{C})_* = (f \circ I_C)_* = (f|_{C \cap A})_*$, each of which factors as $(f|_{C \cap A})_* : V_S \to V_{C \cap A} \to V_T$. But if $C \not\subseteq A$ we have $|C \cap A| < |C| = n$, in which case our assumption guarantees $V_{C \cap A} = 0$. Thus $(f|_{C \cap A})_* = 0$ when $C \not\subseteq A$, and $f_* \circ E_S$ reduces to the sum over $C \subseteq A$ of $(f|_{C \cap A})_* = (f|_C)_*$. Applying a similar analysis to $I_D \circ f = f|_{\phi^{-1}(D \cap B)}$ shows that

$$f_* \circ E_S = \sum_{C \subseteq A, |C| = n} (f|_C)_* \quad \text{and} \quad E_T \circ f_* = \sum_{D \subseteq B, |D| = n} (f|_{\phi^{-1}(D)})_*.$$

Since $C$ and $\phi^{-1}(D)$ range over the same sets, these sums coincide, and so $f_* \circ E_S = E_T \circ f_*$ as desired.
For future reference, we point out that if \( V_n = 0 \), then \( E: V \to V \) is the zero map. Indeed, this assumption implies that for any \( C \) with \( |C| = n \) we have \( V_C = 0 \). Since the map \((I_C)_*: V_S \to V_C \to V_S\), we have \((I_C)_* = 0\). It follows that \( E_S = \sum_C (I_C)_* = 0 \), as claimed.

If \( U \) is another \( \text{FI}_\mathbb{Z} \)-module satisfying (13), the definition (13) applies equally well to \( U \) and defines an endomorphism \( E^U: U \to U \). Any map \( F: U \to V \) of \( \text{FI}_\mathbb{Z} \)-modules respects the \( \text{FI}_\mathbb{Z} \)-morphisms \((I_C)_*\); since \( E \) is defined in terms of these maps \((I_C)_*\), we have \( E^V \circ F = F \circ E^U \).

**The decomposition** \( V \simeq EV \oplus \ker[E] \). The endomorphism \( E: V \to V \) is idempotent, as we now verify. Using the identity \( I_C \circ I_C = I_{C \cap C'} \), we compute

\[
E_S \circ E_S = \sum_{C, C' \subset S, |C|=|C'|=n} (I_{C \cap C'})_* = \sum_{C \subset S} (I_C)_* = E_S.
\]

In the second equality, we used that \((I_{C \cap C'})_*\) factors through \( V_{C \cap C'} \), and thus vanishes whenever \( |C \cap C'| < n \), i.e. whenever \( C \neq C' \).

Since \( E^2 = E \in \text{End}_{\text{FI}_\mathbb{Z}-\text{Mod}}(V) \), the \( \text{FI}_\mathbb{Z} \)-module \( V \) splits as a direct sum \( V \simeq EV \oplus \ker[E] \). Moreover, since \( E^V \circ F = F \circ E^U \) for any map \( F: U \to V \) of \( \text{FI}_\mathbb{Z} \)-modules satisfying (13), this decomposition \( V \simeq EV \oplus \ker[E] \) is natural for all \( \text{FI}_\mathbb{Z} \)-modules satisfying (13). We will show that this provides the desired decomposition (14).

When \( |S| = n \) the definition of \( E_S \) reduces to \( E_S = (I_S)_* = \text{id}: V_S \to V_S \), so \( \ker[E]_S = 0 \) for such \( S \). Since \( V_S = 0 \) when \( |S| < n \), certainly \( \ker[E]_S = 0 \) in this case as well. Therefore \( \ker[E] \) satisfies the desired properties of \( V' \) from (14), so it remains to exhibit an isomorphism \( M(V_n) \simeq EV \).

**The map** \( M(V_n) \to V \). We now construct a natural map of \( \text{FI}_\mathbb{Z} \)-modules \( F: M(V_n) \to V \). We emphasize that without our assumption (13) that \( V_S = 0 \) when \( |S| < n \), such a map need not exist!

For readability, let \( M := M(V_n) \). From Example 4.1.4 we have that

\[
M_S = M(V_n)_S = \bigoplus_{T \subset S} V_T.
\]

Given \( T \subset S \), set \( g_T: T = T \overset{id}{\to} T \subset S \); then we define the map \( F_S: M_S \to V_T \) on the factor \( V_T \) to be \((g_T)_*: V_T \to V_S \). We must verify that \( F: M \to V \) is a map of \( \text{FI}_\mathbb{Z} \)-modules, i.e. for any morphism \( f: S \supset A \overset{\phi}{\to} \phi(A) \subset S' \) we have \( f_T \circ F_S = F_{S'} \circ f_A : M_S \to V_{S'} \).

On the factor \( V_T \), the map \( F_{S'} \circ f_A \) is 0 if \( T \nsubseteq A \), and is \((g_{\phi(T)})_* \circ (f|_T)_* \) if \( T \subset A \). In contrast, the map \( f_T \circ F_S \) on the factor \( V_T \) is \((f \circ g_T)_*: T = T \overset{\phi_T}{\to} \phi(T) \subset S', \) which is indeed equal to \( g_{\phi(T)} \circ \phi_T \). In the remaining case, we can factor \((f \circ g_T)_*: V_T \to V_{T \cap A} \overset{\phi(T \cap A)}{\to} \phi(T \cap A) \subset S' \). When \( T \subset A \) this is simply \( f \circ g_T: T = T \overset{\phi_T}{\to} \phi(T) \subset S' \), which is indeed equal to \( g_{\phi(T)} \circ \phi_T \). In the remaining case, we can factor \((f \circ g_T)_*: V_T \to V_{T \cap A} \to V_{S'} \). When \( T \nsubseteq A \) we have \(|T \cap A| < |T| = n\), so our assumption implies \( V_{T \cap A} = 0 \). Thus in this case \((f \circ g_T)_* = 0 \), as desired. This verifies that \( F: M(V_n) \to V \) is a map of \( \text{FI}_\mathbb{Z} \)-modules.

**The isomorphism** \( M(V_n) \simeq EV \). Since \( M \) satisfies (13), the map \( F: M \to V \) must satisfy \( E^V \circ F = F \circ E^M \). We can easily analyze \( E^M \): on the factor \( V_T \) of \( M_S \), the map \((I_T)_*\) is 0 unless \( C = T \), when it is \((I_T)_* = \text{id} \). Thus the sum \( E_S = \sum_C (I_C)_* \) acts as the identity on each factor, so \( E^M = \text{id}: M \to M \).

We conclude that \( E^V \circ F = F \circ E^M = F \), so the image of \( F: M \to V \) is contained in \( EV \). We therefore have

\[
0 \to K \xrightarrow{\phi} M \xrightarrow{F} EV \to U \to 0
\]
where \( K := \ker[F] \) and \( U := \coker[F] \). Since \( M_n = V_n = E V_n \) we have \( K_n = 0 \) and \( U_n = 0 \), and we observed above that this implies \( E^K = 0 \) and \( E^U = 0 \). However \( E^M = \operatorname{id} \) implies \( E^K = \operatorname{id} \), and similarly \( E^{EV} = \operatorname{id} \) implies \( E^U = \operatorname{id} \). We conclude that \( K = 0 \) and \( U = 0 \), so the natural map \( F: M(V_n) \to EV \) is an isomorphism, as desired. \( \square \)

From the classification in Theorem 4.1.5 we see that if an \( \text{FI}_U \)-module \( V \) is generated in degree \( \leq d \) as an \( \text{FI} \)-module, the same is true of any sub-\( \text{FI}_U \)-module of \( V \). More generally, this classification has the following consequences, which in particular imply Theorem [L.7] In any abelian category, one says that an object \( V \) is finitely generated if for any directed family \( \{W^i\}_{i \in I} \) of subobjects \( W^i \subset V \) with \( \bigcup W^i = V \), there exists \( N \in I \) such that \( W^N = V \). (For \( \text{FI} \)-modules, it is easy to see that this is equivalent to Definition [2.3.4].)

**Theorem 4.1.7.** Let \( V \) be an \( \text{FI}_U \)-module over \( k \). The following conditions are equivalent:

(i) \( V \) is finitely generated as an \( \text{FI}_U \)-module.

(ii) \( V \) is finitely generated as an \( \text{FI} \)-module.

(iii) \( \bigoplus_{n \geq 0} H_0(V)_n \) is finitely generated as a \( k \)-module.

(iv) There exists \( d \geq 0 \) such that \( V_n \) is generated as a \( k \)-module by \( O(n^d) \) elements.

When \( k \) is a field, these are also equivalent to the following conditions:

(v) There exists \( d \geq 0 \) such that \( \dim_k V_n = O(n^d) \).

(vi) There exists an integer-valued polynomial \( P \in \mathbb{Q}[T] \) such that \( \dim_k V_n = P(n) \) for all \( n \geq 0 \).

When \( k \) is a field of characteristic 0, these are also equivalent to the following condition:

(vii) There exists a polynomial \( P_V \in \mathbb{Q}[X_1, X_2, \ldots] \) s.t. \( \chi_{V_n}(\sigma) = P_V(\sigma) \) for all \( n \geq 0 \) and all \( \sigma \in S_n \).

**Proof.** It is immediate that (ii) \( \implies \) (i), and we have already seen in Remark 2.3.8 that (ii) \( \iff \) (iii). Consider the FB-module \( W = H_0(V) \), so that \( V \cong M(W) \). Theorem 4.1.5 states that (i) holds if and only if \( H_0(V) \) is finitely generated as an FB-module. Since \( k[S_n] \) is finite over \( k \), this implies that \( \bigoplus_{n \geq 0} H_0(V)_n \) is finitely generated as a \( k \)-module, so (i) \( \implies \) (iii).

Let \( c_i \) be the smallest number of generators for the \( k \)-module \( H_0(V)_i = W_i \). Condition (iii) states that there exists \( d \) such that \( c_i < \infty \) for all \( i \geq d \) and \( c_i = 0 \) for all \( i > d \). From (iii) we have \( M(W)_n \cong \bigoplus_{T \subset n} W_T \), so as \( k \)-modules we have \( V_n = M(W)_n \cong \bigoplus_{i \geq 0} W_i^{|\sigma_i|} \). This is generated by \( \sum_{i \geq 0} c_i \cdot \binom{n}{i} \) elements, so (iii) \( \implies \) (iv). Conversely, assume that \( V_n \) is generated by \( O(n^d) \) elements, so \( V_n \) admits a surjection from \( k^{cn^d} \) for some constant \( c \). Suppose furthermore that \( W_i \) is nonzero for some \( i > 0 \), and let \( m \) be a maximal ideal of \( k \) with quotient field \( \mathbb{F} := k/m \) such that \( W_i \otimes_k \mathbb{F} \neq 0 \). The hypothesized surjection shows that the \( \mathbb{F} \)-dimension of \( k^{cn^d} \otimes_k \mathbb{F} \cong \mathbb{F}^{cn^d} \) is at least that of \( W_i^{|\sigma_i|} \otimes_k \mathbb{F} \cong (W_i \otimes_k \mathbb{F})^{\binom{|\sigma_i|}{i}} \) for all \( n \), which implies \( i \leq d \). Therefore \( W_i \) is zero for all \( i > d \). Since \( V_i \) surjects to \( W_i \), the \( k \)-module \( W_i \) must be finitely generated for \( i \leq d \). This shows that (iv) \( \implies \) (iii).

When \( k \) is a field, (iv) \( \iff \) (v) are equivalent by definition, and (vi) \( \implies \) (v) is immediate. Under the assumption of (iii), we can take \( P(n) \) to be the polynomial \( P(n) = \sum_{i=0}^d \dim W_i \cdot \binom{n}{i} \). Then \( \dim_k V_n = P(n) \) for all \( n \geq 0 \), so (iii) \( \implies \) (vi).

When \( k \) is a field of characteristic 0, specializing the polynomial \( P_V \) of (vii) to \( P(n) := P_V(n, 0, 0, \ldots) \) shows that (vii) \( \implies \) (vi). We will prove that (iii) \( \implies \) (vii) by providing an explicit formula for the character polynomial \( P_V \) under the assumption that \( V \cong M(W) \).
Given a partition $\lambda \vdash d$, let $n_i(\lambda)$ be the number of parts of $\lambda$ equal to $i$, so that $\sum i \cdot n_i(\lambda) = |\lambda|$. Define the polynomial $\left( \frac{X_i}{\lambda \vdash d} \right) \in \mathbb{Q}[X_1, X_2, \ldots]$ by:

$$
\left( \frac{X_i}{\lambda \vdash d} \right) := \left( \frac{X_1}{n_1(\lambda)} \right) \left( \frac{X_2}{n_2(\lambda)} \right) \ldots \left( \frac{X_d}{n_d(\lambda)} \right)
$$

For $\lambda \vdash d$, let $\chi_V(\lambda)$ denote the character of $W_\lambda$ on the conjugacy class of $S_d$ whose cycle decomposition is encoded by $\lambda$. We then define the character polynomial $P_V \in \mathbb{Q}[X_1, X_2, \ldots]$ to be:

$$
P_V := \sum_\lambda \chi_W(\lambda) \left( \frac{X_i}{\lambda \vdash d} \right)
$$

The assumption $(iii)$ guarantees that this sum is finite.

To verify that $\chi_{V_n} = P_V$, we directly compute the character of $M(W)_n$. Fix $n \geq 0$ and $\sigma \in S_n$. As above, we have $M(W)_n = \bigoplus_{T \subset \sigma} W_T$. Only those summands with $\sigma(T) = T$ will contribute to the character of $\sigma$. A subset $T$ fixed by $\sigma$ is a union of orbits; if $T$ is the union of $n_i(\lambda) i$-cycles, $\sigma|_T$ has cycle type $\lambda$ and thus contributes $\chi_W(\lambda)$ to $\chi_{M(W)_n}(\sigma)$. The number of such subsets is $\binom{n_1(\lambda)}{\lambda_1} \cdots \binom{n_d(\lambda)}{\lambda_d} = \binom{X_i}{\lambda}(\sigma)$. We conclude that $\chi_{M(W)_n}(\sigma) = \sum \chi_W(\lambda) \binom{X_i}{\lambda}(\sigma) = P_V(\sigma)$ for any $n \geq 0$ and any $\sigma \in S_n$. This shows that $(iii) \implies (vii)$, completing the proof of the theorem.

Finally, we have the following corollary regarding representation stability for $\text{FI}_\mathbb{F}$-modules.

**Corollary 4.1.8.** Let $V$ be an $\text{FI}_\mathbb{F}$-module over a field of characteristic 0. The sequence of $S_n$-representations $\{V_n, \phi_n\}$ is uniformly representation stable with stable range $n \geq 2 \cdot \text{weight}(V)$.

**Proof.** We first show that $\text{stab-deg}(V) \leq \text{weight}(V)$. By Theorem 4.1.5 we have an isomorphism $V \cong \bigoplus_\lambda M(\lambda)^{\text{dim} \lambda}$. Applying Propositions 3.2.6 and 3.2.4 each factor $M(\lambda)$ occurring in this sum satisfies $\text{stab-deg}(M(\lambda)) = \lambda_1 \leq |\lambda| = \text{weight}(M(\lambda))$, so the same inequality holds for $V$. The corollary now follows from Proposition 3.3.3.

**Remark 4.1.9 (Finite projective resolutions).** Over a field of characteristic 0, Remark 2.2.7 implies that every projective FI-module extends to an $\text{FI}_\mathbb{F}$-module. So Theorem 4.1.7 implies that the characters of any finitely generated projective FI-module are polynomial for all $n \geq 0$. Since characters are additive in exact sequences, the same is true for any finitely generated FI-module admitting a finite projective resolution: the character $\chi_{V_n}$ is given by the character polynomial $P_V$ for all $n \geq 0$.

So if this is not the case, then $V$ does not have finite projective dimension. For example, let $W$ be the FI-module from Remark 2.1.3 with $W_0 = k$ and $W_n = 0$ for each $n > 0$. This has character polynomial $P_V = 0$, but when $n = 0$, the character $\chi_{V_0}(\text{id}) = \dim V_0 = 1$ fails to agree with $P_V(0, \ldots, 0) = 0$, so this FI-module has no finite projective resolution.

**Tensor products of $\text{FI}_\mathbb{F}$-modules.** When $k$ is a field of characteristic 0, Remark 2.2.7 and Theorem 4.1.5 together imply that the projective FI-modules are precisely those that can be extended to $\text{FI}_\mathbb{F}$-modules. The tensor product of two $\text{FI}_\mathbb{F}$-modules is an $\text{FI}_\mathbb{F}$-module, so in particular the tensor project of two projective FI-modules over a field of characteristic 0 is projective.

Recall the $\text{FI}_\mathbb{F}$-module $M(\lambda) = M(V_\lambda)$ generated by the single irreducible representation $V_\lambda$. Theorem 4.1.5 states that every $\text{FI}_\mathbb{F}$-module is a direct sum of $\text{FI}_\mathbb{F}$-modules $M(\nu)$. In particular, every tensor product $M(\lambda) \otimes M(\mu)$ decomposes as a direct sum of finitely many $\text{FI}_\mathbb{F}$-modules $M(\nu)$. 

\[ 30 \]
It is not hard to compute small examples by hand; for instance:

\[
M(\square) \otimes M(\square) = M(\square) \oplus M(\square) + M(\square)
\]
\[
M(\square) \otimes M(\square) = M(\square) \oplus M(\square) + M(\square) + M(\square)
\]
\[
M(\square) \otimes M(\square) = M(\square) \oplus M(\square) + M(\square) + M(\square)
\]
\[
\quad + M(\square) \oplus M(\square)^2 + M(\square) + M(\square).
\]  

(16)

These computations have natural combinatorial interpretations. For example, \(M(1) = M(\square)\) takes a finite set \(S\) to the vector space with basis \(S\), so \(M(\square) \otimes M(\square)\) associates to \(S\) the vector space with basis the ordered pairs \((x, y)\) from \(S\). We can partition these into those pairs with \(x = y\), yielding the summand \(M(\square)\), and those with \(x \neq y\), yielding the summand \(M(2) = M(\square) \oplus M(\square)\).

For any partitions \(\lambda\) and \(\mu\) we have the direct sum decomposition

\[
M(\lambda) \otimes M(\mu) = \bigoplus_{\nu} d_{\lambda,\mu}^{\nu} M(\nu)
\]  

(17)

The coefficients \(d_{\lambda,\mu}^{\nu}\) are nonnegative integers, and it can be shown that \(d_{\lambda,\mu}^{\nu}\) is only nonzero when \(\max(|\lambda|, |\mu|) \leq |\nu| \leq |\lambda| + |\mu|\). It is straightforward to check that when \(|\lambda| = |\mu| = |\nu| = n\), the coefficient \(d_{\lambda,\mu}^{\nu}\) is equal to the Kronecker coefficient (the multiplicity of \(V_{\lambda}\) in \(V_{\lambda} \otimes V_{\mu}\)).

Moreover, any Schur functor \(S_{\lambda}\) yields an \(F I_{\ast}\)-module \(S_{\lambda}(M(\square))\) whose “leading term” is \(M(\lambda)\), in the sense that \(S_{\lambda}(M(\square)) = M(\lambda) \oplus V\) where \(V\) is generated in degrees \(< |\lambda|\). It follows that when \(|\nu| = |\lambda| + |\mu|\), the coefficient \(d_{\lambda,\mu}^{\nu}\) is equal to the Littlewood-Richardson coefficient \(c_{\lambda,\mu}^{\nu}\) (the coefficient of \(S_{\nu}W\) in \(S_{\lambda}W \otimes S_{\mu}W\)). The honeycomb model used by Knutson–Tao in their proof of the saturation conjecture [KT] gives a geometric interpretation for the Littlewood–Richardson coefficients \(c_{\lambda,\mu}^{\nu}\) as the number of integer points in a certain Berenstein–Zelevinsky polytope. It would be very interesting to find a similar geometric interpretation for the coefficients \(d_{\lambda,\mu}^{\nu}\).

**Problem 4.1.10 (FI_{\ast}-module tensor coefficients).** Give a geometric interpretation of the structural coefficients \(d_{\lambda,\mu}^{\nu}\) in (17). Give a method for determining which of these coefficients are nonzero.

### 4.2 Graded FI-modules and FI-algebras

The FI-modules that arise in the applications of [3], [6], and [7] all come with a natural grading. When this happens, the FI-module will almost never be finitely generated, even if the graded pieces are. For example, the FI-module that associates to a finite set \(S\) the polynomial algebra \(R_{\ast} := \mathbb{Q}[x_s \mid s \in S]\) is definitely not finitely generated, since \(R_{\ast}\) is not even finite-dimensional for a fixed set \(S\). However, for any fixed \(i \geq 0\), the homogeneous degree-\(i\) polynomials form a sub-FI-module \(R_{\ast}^i\) which is finitely generated in degree \(\leq i\). This example motivates the following definition.

Recall that a graded FI-module \(V\) is a functor from \(\mathcal{FI}\) to \(\mathbb{N}\)-graded \(k\)-modules. For each \(i \geq 0\), restricting \(V_{\ast}\) to the piece \(V_{\ast}^i\) in grading \(i\) yields an FI-module \(V^i\), and \(V\) is equivalent to the collection of FI-modules \(\{V^i\}_{i \in \mathbb{N}}\).

**Definition 4.2.1 (Graded FI-modules of finite type).** Let \(V\) be a graded FI-module. We say that \(V\) is of finite type if each FI-module \(V^i\) is a finitely generated FI-module. If \(k\) is a field of characteristic 0, we say that \(V\) has slope \(\leq m\) if weight(\(V^i\)) \(\leq m \cdot i\) for all \(i \in \mathbb{N}\). (For instance, this occurs when \(V^i\) is generated in degree \(\leq m \cdot i\), via Proposition 3.2.5)

If \(V\) is a graded FI-module of finite type, any quotient of \(V\) is of finite type. When \(k\) contains \(\mathbb{Q}\), Theorem [13] implies that any subquotient of \(V\) is of finite type. If \(V\) has slope \(\leq m\), any subquotient of \(V\) has slope \(\leq m\). (Whenever we speak about slope, we implicitly assume that \(k\) is a field of
characteristic 0, since the weight of an FI-module is only defined in this case.) Although a graded FI-module over $k$ can be of finite type without having finite slope (if $\text{weight}(V^i)$ grows faster than linearly), we do not know of any interesting examples where this is the case.

**Tensor products of FI-modules of finite type.** If $A$ and $B$ are graded $k$-modules, their tensor product $A \otimes B$ is the graded $k$-module defined by

$$(A \otimes B)^k = \bigoplus_{i+j=k} A^i \otimes B^j. \quad (18)$$

Given graded FI-modules $V$ and $W$, their tensor product $V \otimes W$ is the graded FI-module obtained by applying (18) pointwise. The following two propositions show that we can safely take tensor products of graded FI-modules of finite type.

**Proposition 4.2.2.** Let $U$ and $W$ be graded FI-modules. If $U$ and $W$ are of finite type, the tensor product $U \otimes W$ is of finite type. If $U$ and $W$ have slope $\leq m$, then $U \otimes W$ has slope $\leq m$.

**Proof.** By Proposition 2.3.6, $U^i \otimes W^j$ is a finitely generated FI-module for each fixed $i$ and $j$. For fixed $k$ the graded piece $(U \otimes W)^k$ is the direct sum of $k+1$ such summands, and thus is finitely generated. The second claim follows from the inequality $\text{weight}(U^i \otimes W^j) \leq \text{weight}(U^i) + \text{weight}(W^j) \leq mi + mj = mk$ from Proposition 3.2.2. 

**FI-algebras.** Recall from Remark 2.1.6 that a graded FI-algebra $A$ is a functor from $\text{FI}$ to the category of graded $k$-algebras. We say that a graded FI-algebra is of finite type if the underlying graded FI-module is of finite type.

Given a $k$-module $W$, we denote by $k\{W\}$ the free non-associative algebra on $W$; if $W$ is a graded $k$-module, $k\{W\}$ inherits a grading just as in (18). Thus if $V$ is a graded FI-module, the free algebra $k\{V\}$ is a graded FI-algebra.

Given a graded sub-FI-module $V \subset A$, we say that $A$ is generated by $V$ if $V_S$ generates $A_S$ as a $k$-algebra for all finite sets $S$. The inclusion $V \subset A$ induces a natural map $k\{V\} \to A$, and $A$ is generated by $V$ exactly when this map is a surjection $k\{V\} \twoheadrightarrow A$.

Given $\Sigma \subset \coprod A_n$, we denote by $(\Sigma) \subset A$ the FI-ideal generated by $\Sigma$, i.e. the smallest FI-ideal of $A$ containing $\Sigma$. If $V \subset A$ is a graded sub-FI-module, the FI-ideal $(V) \subset A$ is easy to describe: $(V)_S \subset A_S$ is simply the ideal in $A_S$ generated by $V_S$.

**Theorem 4.2.3 (Algebras generated by finite type FI-modules).** Let $A$ be a graded FI-algebra generated by a graded sub-FI-module $V \subset A$ with $V^0 = 0$. If $V$ is of finite type then $A$ is of finite type. If $V$ has slope $\leq m$ then $A$ has slope $\leq m$.

**Proof.** We first consider $k\{V\}$, which decomposes as a graded FI-module as

$$k\{V\} \cong \bigoplus_{j=0}^\infty (V^{\otimes j})^{\otimes P_j}$$

where $P_j$ is the set of parenthesizations of a $j$-fold product (so that $|P_j|$ is the $j$-th Catalan number).

If $V$ is of finite type (resp. has slope $\leq m$), the same is true of each summand $(V^{\otimes j})^{\otimes P_j}$ by Proposition 4.2.2. Moreover, since $V^i \neq 0$ only for $i \geq 1$, we have $(V^{\otimes j})^{\otimes i} \neq 0$ only for $i \geq j$. For fixed $i \in \mathbb{N}$ we thus have a finite sum $k\{V\}^i = \bigoplus_{j=0}^i ((V^{\otimes j})^{\otimes P_j})$. We conclude that the FI-module $k\{V\}^i$ is finitely generated (resp. of weight $\leq m \cdot i$) for each $i$. By assumption we have a surjection $k\{V\} \twoheadrightarrow A$, so the same is true of $A^i$; this completes the proof.
Theorem 4.2.3 applies to each of the FI-algebras
\[ k\{V\} := \text{the free algebra on } V \]
\[ k[V] := \text{the free associative algebra on } V \]
\[ k[V] := \text{the free commutative algebra on } V \]
\[ \mathcal{L}(V) := \text{the free Lie algebra on } V \]
\[ \Gamma[V] := \text{the free graded-commutative } k\text{-algebra on } V \]
as well as any quotient of these algebras by any FI-ideal. Many of the applications in the second half of this paper will follow directly from these results.

**Co-FI-algebras.** Let \( W \) be a graded co-FI-module over a field \( k \). We say that \( W \) is of finite type (resp. has slope \( \leq m \)) if its dual \( W^* \) is of finite type (resp. has slope \( \leq m \)) as a graded FI-module. We say a graded co-FI-submodule \( A \subset W \) if the natural map \( k\{W\} \to A \) is surjective.

If \( W \) is of finite type then any \( U \subset W \) is of finite type since the injection \( U \to W \) induces a surjection \( W^* \to U^* \). Similarly, when \( \text{char}(k) = 0 \), any quotient of \( W \) has finite type by Theorem 1.3. The graded FI-module \( k\{W\}^* \cong k\{W^*\} \) is of finite type by Theorem 4.2.3, so we obtain the following proposition as a corollary.

**Proposition 4.2.4.** Let \( A \) be a graded co-FI-algebra over a field of characteristic 0, generated by a graded co-FI-submodule \( W \subset A \) with \( W^0 = 0 \). If \( W \) is of finite type, \( A \) is of finite type. If \( V \) has slope \( \leq m \), then \( A \) has slope \( \leq m \).

**Corollary 4.2.5.** If a graded FI-algebra (resp. co-FI-algebra) \( A \) is generated by a sub-FI-module (resp. co-FI-module) \( V \) concentrated in grading 1 (i.e. \( V^i = 0 \) for \( i \neq 1 \)), then \( A \) has slope \( \leq \text{weight}(V) \).

**Another tensor product** \( V^\otimes \cdot \). In Sections 5 and 6 we will need to consider a different sort of tensor product. For any set \( S \) and any \( k\)-module \( W \), let \( W^\otimes S := \bigotimes_{s \in S} W \). Given \( F: W \to U \) and a bijection \( f: S \to T \), let \( F^\otimes f: W^\otimes S \to U^\otimes T \) be the map that takes the factor labeled by \( s \in S \) to the factor labeled by \( f(s) \in T \) via \( F \).

**Definition 4.2.6.** Given an FI-\( \mathcal{S} \)-module \( V \) equipped with a splitting \( M(0) \hookrightarrow V \to M(0) \), define the FI-\( \mathcal{S} \)-module \( V^\otimes \cdot \) by \((V^\otimes \cdot)_S := (V^\otimes)_S^\otimes S\). Given \( f: S \supset A \supset B \subset T \), let \( f_*: (V^\otimes \cdot)_S \to (V^\otimes \cdot)_T \) be the composition
\[ \bigotimes_{s \in S} V_S \to \bigotimes_{s \in A} V_S \xrightarrow{(f_*)^\otimes S} \bigotimes_{t \in B} V_T \to \bigotimes_{t \in T} V_T \tag{19} \]
Here the first map applies the surjection \( V_S \to M(0)_S \cong k \) to each factor with \( s \not\in A \), followed by the canonical isomorphism \((V_S)^\otimes A \otimes k^\otimes S^{-A} \cong (V_S)^\otimes A\). Similarly the last map composes the isomorphism \((V_T)^\otimes B \cong (V_T)^\otimes B \otimes k^\otimes T^{-B}\) with the inclusions \( k \cong M(0)_T \to V_T \).

We can apply the construction (19) to FI-modules and co-FI-modules as well. Given an FI-module \( V \) equipped with an injection \( M(0) \hookrightarrow V \), let \( V^\otimes \cdot \) be the FI-module with \( (V^\otimes \cdot)_S := (V^\otimes)_S^\otimes S \), where \( f: S \to T \) acts by
\[ f_*: (V_S)^\otimes S \xrightarrow{(f_*)^\otimes f} (V_T)^\otimes f(S) \xrightarrow{(f_*)^\otimes f^{-1}} (V_S)^\otimes S. \]

Given a co-FI-module \( V \) equipped with a surjection \( V \to M(0) \), let \( V^\otimes \cdot \) be the co-FI-module with \( (V^\otimes \cdot)_S := (V^\otimes)_S^\otimes S \), where \( f: S \to T \) acts by
\[ f_*: (V_T)^\otimes T \xrightarrow{(f_*)^\otimes f^{-1}} (V_T)^\otimes f(S) \xrightarrow{(f_*)^\otimes f} (V_S)^\otimes S. \]
If $V$ is a graded FI$^*$-module (or FI-module, or co-FI-module), the individual pieces $(V_S)^{\otimes S}$ are graded $k$-modules. In this case we consider $M(0)$ as concentrated in grading $0$ (i.e. $M(0)^i = 0$ for $i \neq 0$), and require that the maps $M(0) \hookrightarrow V$ and/or $V \twoheadrightarrow M(0)$ preserve the grading. The grading is then preserved by the morphisms $f_*: (V_S)^{\otimes S} \to (V_T)^{\otimes T}$, and so $V^{\otimes *}$ is a graded FI$^*$-module (or FI-module, or co-FI-module).

If $V$ is an FI$^*$-algebra (or FI-algebra, or co-FI-algebra), the individual pieces $(V_S)^{\otimes S}$ are $k$-algebras. In this case we consider $M(0)$ as an FI$^*$-algebra with multiplication $M(0)_S \otimes M(0)_S = k \otimes k \cong k = M(0)_{(0)}$, and require that the maps $M(0) \hookrightarrow V$ and/or $V \twoheadrightarrow M(0)$ are maps of FI$^*$-algebras (or FI-algebras, or co-FI-algebras). The morphisms $f_*: (V_S)^{\otimes S} \to (V_T)^{\otimes T}$ are then maps of $k$-algebras, and so $V^{\otimes *}$ is an FI$^*$-algebra (or FI-algebra, or co-FI-algebra).

**Remark 4.2.7.** If $U$ is a graded $k$-module with splitting $k \hookrightarrow U^0 \twoheadrightarrow k$, we can consider $U$ as a “constant” graded FI$^*$-module, sending every finite set $S$ to $U$ and every morphism to the identity. Definition 4.2.6 thus defines a graded FI$^*$-module $U^{\otimes *}$, which satisfies $(U^{\otimes *})_n \cong U^{\otimes n}$.

**Proposition 4.2.8.** Let $V$ be a graded FI-module endowed with an isomorphism $V^0 \cong M(0)$. If $V$ is of finite type then the FI-module $V^{\otimes *}$ is of finite type.

**Proof.** Consider a graded FI-module $M$ that is free in each grading, meaning that there exists an index set $L$, and for each $\ell \in L$ numbers $m_\ell \in \mathbb{N}$ and $i_\ell \in \mathbb{N}$, so that $M^\ell \cong \bigoplus_{i=0}^{\infty} M(m_\ell)$. Every graded FI-module $V$ admits a surjection $M \twoheadrightarrow V$ from such a graded FI-module $M$. If $V$ is of finite type, then we can take $M$ to be of finite type, meaning that $\{\ell \in L \mid i_\ell = j\}$ is finite for each $j$. If $V^0 \cong M(0)$, we can assume that there is a unique $\ell_0 \in L$ with $i_{\ell_0} = 0$, and it satisfies $m_{\ell_0} = 0$. Since a surjection $M \twoheadrightarrow V$ induces a surjection $M^{\otimes *} \twoheadrightarrow V^{\otimes *}$, it suffices to prove that $M^{\otimes *}$ is of finite type.

A basis for $M_S^{\otimes *} \cong (M_S)^{\otimes S}$ is given by a choice of indices $\eta: S \to L$, and for each $s \in S$ an injection $g_s: \mathbf{m}_{\eta(s)} \hookrightarrow S$. For such a basis element, the multiset $\eta(S)$ can be written uniquely for some $j \leq |S|$ as $\{\ell_1, \ldots, \ell_j\} \cup \{0, \ldots, 0\}$ with $\ell_1, \ldots, \ell_j \in L - \{0\}$. Under any FI-map $f_*: M_S^{\otimes *} \to M_T^{\otimes *}$, such a basis element is taken to another basis element determining the same multiset $\ell = \{\ell_1, \ldots, \ell_j\}$. Therefore the graded FI-module $M^{\otimes *}$ splits as a direct sum $M^{\otimes *} = \bigoplus_{\ell} M^{\otimes *}_{\ell}$ indexed by finite multisets $\ell = \{\ell_1, \ldots, \ell_j\}$ of elements of $L - \{0\}$.

We now show that each summand $M^{\otimes *}_{\ell}$ is finitely generated. Fix a multiset $\ell = \{\ell_1, \ldots, \ell_j\}$ of elements of $L - \{0\}$. For any finite set $T$, a basis for $(M^{\otimes *}_{\ell})_T$ is determined by

\[
\{ (\eta: T \to L, g_t: \mathbf{m}_{\eta(t)} \to T) \mid \eta(T) = \{\ell_1, \ldots, \ell_j\} \cup \{0, \ldots, 0\} \}
\]

(20)

If $|T| > j + m_{\ell_1} + \cdots + m_{\ell_j}$, there must exist some $t \in T$ with $\eta(t) = 0$ such that $t \not\in \text{im}(g_t)$ for all $t' \in T$. Set $S := T - \{t\}$, so that $\text{im}(g_s) \subset S$ for all $s \in S$, and let $f: S \to T$ be the inclusion. Under the induced map $f_*: (M^{\otimes *}_{\ell})_S \to (M^{\otimes *}_{\ell})_T$, the basis element $(\eta|_S, g_s)$ of $(M^{\otimes *}_{\ell})_S$ is sent to the basis element $(\eta, g_t)$ of $(M^{\otimes *}_{\ell})_T$. This shows that the FI-module $M^{\otimes *}_{\ell}$ is generated in degree $\leq j + m_{\ell_1} + \cdots + m_{\ell_j}$.

Since the basis (20) is finite for each $T$, it follows that $M^{\otimes *}_{\ell}$ is finitely generated.

The summand $M^{\otimes *}_{\ell}$ contributes to $(M^{\otimes *})^i$ only in grading $i = i_{\ell_1} + \cdots + i_{\ell_j}$. Since $\{\ell \in L \mid i_\ell = j\}$ is finite for each $j$, for fixed $i \in \mathbb{N}$ there are only finitely many multisets $\ell$ with $i = i_{\ell_1} + \cdots + i_{\ell_j}$. Therefore for each $i \in \mathbb{N}$ the FI-module $(M^{\otimes *})^i$ is a finite direct sum of finitely generated FI-modules $M^{\otimes *}_{\ell}$, so the graded FI-module $M^{\otimes *}$ is of finite type, as desired. □

## 5 Applications: Coinvariant algebras and rank varieties

In this section we apply the theory of FI-modules to explicit FI-algebras arising in algebraic combinatorics and algebraic geometry, obtaining concrete results about important families of $S_n$-representations.
5.1 Algebras with explicit presentations

One way that FI-algebras and co-FI-algebras naturally arise is from explicit presentations by generators and relations. For example, we can consider the free commutative graded FI$\sharp$-algebra $k[M(1)]$ generated by $M(1)$, so that $k[M(1)]_n$ is the polynomial algebra $k[x_1, \ldots, x_n]$. We will consider $k[M(1)]$ as an FI$\sharp$-algebra, an FI-algebra, or a co-FI-algebra, depending on the application.

**Definition 5.1.1 (Symmetrically presented FI-algebra).** Given a collection $P = \{P_i(x_1, \ldots, x_{m_i})\}$ of homogeneous polynomials, the symmetrically presented FI-algebra $R(P)$ is the quotient $k[M(1)]/((P))$.

By Theorem 4.2.3, $R(P)$ is a graded FI-algebra of finite type. Concretely, $R(P)_S$ is the quotient of the polynomial ring $k[x_S | S \subseteq S]$ by the ideal generated by the polynomials $P_i(x_{f(1)}, \ldots, x_{f(m_i)})$ for all $i$ and all injections $f : m_i \hookrightarrow S$.

**Example 5.1.2 (Nilpotent rings).** For $d \geq 2$, let $P_d = \{x_i^d\}$. The symmetrically presented FI-algebra $R(P_d)$ over $\mathbb{Q}$ corresponds to the sequence of nilpotent rings

$$R(P_d)_n \cong \mathbb{Q}[x_1, \ldots, x_n]/(x_1^d, \ldots, x_n^d)$$

on which $S_n$ acts by permutations. Since $R(P_d)$ is of finite type, for each $i \geq 0$ the sequence of $S_n$-representations $R(P_d)_n^i$ is uniformly representation stable. When $d = 2$, this fact was first proved by Ashraf–Azam–Berceneau in [AAAB, Corollary 5.2].

**Example 5.1.3 (The Arnol'd algebra).** Braid groups were first studied by Artin and Hurwitz over a century ago. In this section we focus on the pure braid group $\Gamma_n$, which is the fundamental group of the configuration space of $n$ distinct complex numbers $(z_1, \ldots, z_n)$ with $z_i \in \mathbb{C}$ and $z_i \neq z_j$. This space is aspherical, so its cohomology coincides with $H^*(P_n; \mathbb{Q})$. These cohomology algebras $H^*(P_n; \mathbb{Q})$ fit together into a graded FI$\sharp$-algebra $H^*(P_n; \mathbb{Q})$; this follows from a general result that we will prove below in Proposition 6.4.2. In this section we combine this FI$\sharp$-algebra structure with a presentation of $H^*(P_n; \mathbb{Q})$ due to Arnol’d [Ar] to describe more precisely the cohomology of the pure braid groups.

From Arnold’s description, a basis for $H^1(P_n; \mathbb{Q})$ is given by the symbols $\{w_{ij} | i, j \in S\}$ modulo the relations $w_{ij} = w_{ji}$ and $w_{ii} = 0$, and a partial injection $f : S \supset A \to B \subset T$ acts by $f_* w_{ij} = w_{\phi(i), \phi(j)}$ when $i, j \in A$ and $f_* w_{ij} = 0$ otherwise. We can immediately identify this as $H^1(P_n; \mathbb{Q}) \cong \text{Sym}^2 M(1)/M(1) \cong M([A])$.

Arnol’d [Ar] proved that the cohomology ring $H^*(P_n; \mathbb{Q})$ is generated by $H^1(P_n; \mathbb{Q})$. By Theorem 4.2.3, the graded FI$\sharp$-algebra $H^*(P_n; \mathbb{Q})$ is of finite type. Since $H^1(P_n; \mathbb{Q})$ is generated in degree 2, Corollary 4.2.3 implies that $H^*(P_n; \mathbb{Q})$ has slope 2 and that $H^*(P_n; \mathbb{Q})$ is generated in degree $\leq 2i$. Corollary 4.1.8 thus implies that for each $i \geq 0$, the sequence $\{H^i(P_n; \mathbb{Q})\}$ of $S_n$-representations is uniformly representation stable with stable range $n \geq 4i$. This yields a new proof of one of the main theorems (Theorem 4.1) in [CF].

**Computing the character of $H^i(P_n; \mathbb{Q})$.** Consider $H^i(P_n; \mathbb{Q})$ as a graded FI-module $W$ with $W^1 = H^1(P_n; \mathbb{Q})$ and $W^i = 0$ for $i \neq 1$. Thus $\Gamma[W]$ is a graded-commutative FI$\sharp$-algebra, which as an FI-algebra is of finite type by Theorem 4.2.3. Let $I$ be the FI$\sharp$-ideal in $\Gamma[W]$ generated by the quadratic relation

$$w_{12}w_{23} + w_{23}w_{31} + w_{31}w_{12} \in \Gamma[W]_3. \tag{21}$$

In this case, $I$ is also the FI-ideal generated by this relation $\{21\}$. It follows from Arnol’d [Ar] that $H^*(P_n; \mathbb{Q})$ is isomorphic to the graded FI$\sharp$-algebra $\Gamma[W]/I$. Thanks to the classification of FI$\sharp$-modules, this reduces the description of $H^i(P_n; \mathbb{Q})$ to a finite computation for each $i \geq 0$. For example, we have

$$H^2(P_n; \mathbb{Q}) \simeq \bigwedge^2 H^1(P_n; \mathbb{Q})/I^2.$$
The quadratic part $I^2$ is just the FI-module generated by the relation (21); since $S_3$ acts on this element of $\Gamma[W]_3$ by the sign representation, this is $I^2 \simeq M(\frac{3}{2})$. We computed in (16) that

$$M(\square) \otimes M(\square) = M(\square \square) + M(\square \square) + M(\square \square) + M(\square \square) + M(\square \square) + M(\square \square).$$

Exchanging the two factors gives an isomorphism from $M(\square) \otimes M(\square)$ to itself. As an isomorphism of FI*-modules, it necessarily preserves each of the summands in the decomposition above. By examining the details of the computation of (16) we can check that this involution acts trivially on $M(\square \square), M(\square), M(\square)$; it acts by negation on $M(\square \square)$ and $M(\square)$; and it exchanges the two $M(\square \square)$ summands. We conclude that $\Lambda^2 M(\square) = M(\square) + M(\square) + M(\square)$. This yields the description of $H^2(P_n; \mathbb{Q})$ mentioned in the introduction:

$$H^2(P_n; \mathbb{Q}) \simeq \Lambda^2 M(\square) / M(\square)$$

$$\simeq M(\square) + M(\square) + M(\square) / M(\square)$$

which specifies $H^2(P_n; \mathbb{Q})$ for all $n$ simultaneously. Based on computer calculations, John Wiltshire-Gordon (personal communication) has formulated a precise conjecture for the decomposition of $H^i(P_n; \mathbb{Q})$ as a sum of FI*-modules $M(\lambda)$ for all $i \geq 0$.

**Coinvariant algebras.** The symmetric polynomials are the $S_n$-invariants in the ring of polynomials $\mathbb{Q}[x_1, \ldots, x_n]$, where $S_n$ acts by permuting the variables. Let $J_n = (\mathbb{Q}[x_1, \ldots, x_n]^{S_n})$ be the ideal generated by symmetric polynomials with vanishing constant term. The classical coinvariant algebra $R(n)$ is the quotient $R(n) := \mathbb{Q}[x_1, \ldots, x_n] / J_n$. Chevalley [Che, Theorem B] proved that $R(n)$ is isomorphic as an $S_n$-representation to the regular representation $\mathbb{Q}[S_n]$.

We cannot analyze $R(n)$ using Theorem 4.2.3 because the ideals $J_n$ do not together form an FI-ideal. For example, the inclusion $n \mapsto n + 1$ takes the symmetric polynomial $x_1 + \cdots + x_n \in \mathbb{Q}[x_1, \ldots, x_n]^{S_n}$ to the non-symmetric polynomial $x_1 + \cdots + x_n \notin \mathbb{Q}[x_1, \ldots, x_n, x_{n+1}]^{S_{n+1}}$. Fortunately, we can understand the coinvariant algebra by instead using the natural co-FI-module structure on the algebra of polynomials.

Given a co-FI-module $W$, let $W^\text{inv} \subset W$ be the submodule consisting of the invariants $W^\text{inv}_T := (W_T)^{S_T}$. Given $f : T \hookrightarrow T'$, for any $\sigma : T \hookrightarrow T$ there exists $\sigma' : T' \hookrightarrow T'$ such that $f \circ \sigma = \sigma' \circ f$. This shows that $f_* : W_T \to W_{T'}$ satisfies $f_*(W^\text{inv}) \subset f_*(W^\text{inv})$, so $W^\text{inv}$ is indeed a co-FI-module. If $W$ is a graded co-FI-module, let $W_{>0} \subset W$ be the part in positive grading, defined by $W^i_{>0} = 0$ and $W^i_{>0} = W^i$ for $i > 0$.

**Definition 5.1.4.** Given a graded co-FI-algebra $A$, the **coinvariant algebra** $\text{coinv}(A)$ is the graded co-FI-algebra $\text{coinv}(A) := A/((A^\text{inv})_T)$, which assigns to $T$ the algebra $\text{coinv}(A)_T = A_T/((A^\text{inv})^T_{>0})$.

**Proposition 5.1.5.** Let $A$ be a graded co-FI-algebra over a field of characteristic 0. If $A$ is of finite type, $\text{coinv}(A)$ is of finite type. If $A$ has slope $\leq m$, then $\text{coinv}(A)$ has slope $\leq m$.

**Example 5.1.6.** Consider the graded co-FI-algebra $R := \text{coinv}(\mathbb{Q}[M(1)])$, which is of finite type with slope $\leq 1$ by Proposition 5.1.5. As $S_n$-representations, $R_n \simeq \mathbb{Q}[x_1, \ldots, x_n]/(\mathbb{Q}[x_1, \ldots, x_n]^{S_n})_{>0} \simeq R(n)$. In other words, we have combined the classical coinvariant algebras $R(n)$ into a single co-FI-algebra $R$. Theorem 1.13 implies that for each $i \geq 0$, the graded pieces $R^i_n \simeq R(n)^i$ of the coinvariant algebra form a uniformly representation stable sequence of $S_n$-representations. A different proof of this result was given in [CF, Theorem 7.4], using work of Stanley, Lusztig, and Kraskiewicz–Weyman.
We can now prove Theorem 1.11 from the introduction, regarding the characters of multivariate diagonal coinvariant algebras, using Proposition 5.1.5.

Proof of Theorem 1.11. For fixed \( r \geq 1 \), let \( R^{(r)} \) be the graded co-FI-algebra \( R^{(r)} := \text{coinv}(\mathbb{Q}[M(1)^{\oplus r}]) \) from Definition 5.1.4 with \( M(1)^{\oplus r} \) concentrated in grading 1. To the set \( n \) this associates \( R_n^{(r)} = \text{coinv}(\mathbb{Q}[M(1)^{\oplus r}_n]) \), which is naturally isomorphic to the diagonal coinvariant algebra

\[
R^{(r)}(n) := \mathbb{Q}[x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}]/(\mathbb{Q}[x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}]_{>0})
\]

defined in the introduction. By Proposition 3.2.4 we have weight(\( M(1)^{\oplus r} \)) = 1, so Proposition 4.2.4 implies that \( \mathbb{Q}[M(1)^{\oplus r}] \) is of finite type with slope 1. By Proposition 5.1.5, \( R^{(r)} \) is also of finite type with slope 1. In other words, for each \( 0 \leq i \leq r \) the FI-module \( (R^{(r)},i)^* \) is finitely generated with weight \( \leq i \). The multi-grading by \( J = (j_1, \ldots, j_r) \in \mathbb{N}^r \) considered there is a refinement of our grading; setting \( |J| := j_1 + \cdots + j_r \) we have a decomposition \( (R^{(r)})^i = \bigoplus_{|J|=i} R_{i,J}^{(r)} \) as co-FI-modules. Therefore for any \( J \in \mathbb{N}^r \), the FI-module \( (R_{J}^{(r)})^* \) is finitely generated, and weight((\( R_{J}^{(r)})^* )) \leq J \). By Theorem 3.3.4 this implies that the characters \( \chi_{R_{J}^{(r)}} = \chi_{(R_{J}^{(r)})^*} \) are eventually given by a polynomial in \( X_1, \ldots, X_{|J|} \) of degree \( \leq J \), as claimed. \( \square \)

5.2 The Bhargava–Satriano algebra

In [BS] Bhargava and Satriano develop a notion of Galois closure for an arbitrary finite-dimensional algebra \( R \) over a field \( k \), which reduces to the usual notion when \( R \) is a field extension (or even an étale algebra). The algebra \( G(R/k) \) is a quotient of \( R^{\otimes d} \), where \( d = \dim_k R \), and the natural action of \( S_d \) on \( R^{\otimes d} \) descends to an action of \( S_d \) on \( G(R/k) \) by “Galois automorphisms”, just as in the classical case.

In many ways, the most interesting case is the most degenerate: consider the ring

\[
R_n := k[x_1, \ldots, x_n]/(x_i x_j)_{i,j \in \{1, \ldots, n\}}
\]

which has dimension \( n + 1 \) over \( k \) and which carries an action of \( S_n \) by permutation of the variables. We call the Galois closure \( G(R_n/k) \) the Bhargava–Satriano algebra. Bhargava–Satriano show [BS, Theorem 7] that this ring is “maximally degenerate” among rings of the same dimension, in the sense that \( \dim G(R_n/k) \geq \dim G(T_n/k) \) for any other algebra \( T_n \) with \( \dim T_n = \dim R_n \).

The ring \( G(R_n/k) \) carries an action of \( S_n \times S_{n+1} \), the first factor coming from the action of \( S_n \) on \( R_n \) by field automorphisms, the second factor coming from the Galois automorphisms. Bhargava–Satriano compute the decomposition of \( G(R_n/k) \) as a representation of the Galois group \( S_{n+1} \), and use this to show that \( \dim G(R_n/k) > (n + 1)! \) for \( n \geq 3 \). The decomposition of \( G(R_n/k) \) as an \( S_n \times S_{n+1} \)-representation is unknown.

Proposition 5.2.1. Assume that \( k \) is a field of characteristic 0. There is a graded co-FI-algebra \( BS \) of finite type satisfying \( BS_n \simeq G(R_n/k) \) as graded \( S_n \)-representations, where the \( S_n \)-action on \( G(R_n/k) \) is via the diagonal subgroup \( S_n \subset S_n \times S_{n+1} \). In particular, the \( S_n \)-representations \( G(R_n/k)^i \) are uniformly representation stable.

Proof. The Galois closure \( G(R_n/k) \) is defined as the quotient of \( R_n^{\otimes n+1} \) by certain relations, and Bhargava–Satriano show [BS, §11.2] in this case that the ideal \( J_n \) of relations in \( R_n^{\otimes n+1} \) is generated by the elements:

\[
\gamma_n(x_i) := x_i \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes x_i \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes x_i
\]
Let $R$ be the graded co-FI-algebra $R$ with $R^0 \simeq M(0)$, $R^1 \simeq M(1)^*$, and $R^i = 0$ for $i > 1$; the multiplication is uniquely determined, since $R^1 \otimes R^1 \rightarrow R^2 = 0$. To the set $\mathbf{n}$ this assigns the ring $R_\mathbf{n} = k[x_1, \ldots, x_\mathbf{n}] / (x_ix_j)$ defined above.

Recall the graded co-FI-algebra $R^\otimes^\bullet$ from Definition 4.2.6, which has $(R^\otimes^\bullet)_n \simeq R^{\otimes^\bullet n}$. The action of $S_n$ on $(R \otimes R^\otimes^\bullet)_n$ realizes the diagonal action of $S_n$ on $R^{\otimes^\bullet n+1}$. We define the co-FI-ideal $J \subset R \otimes R^\otimes^\bullet$ as follows. For any finite set $T$ and any $t \in T$, define $\gamma_T(x_t) \in (R \otimes R^\otimes^\bullet)_T$ by

$$\gamma_T(x_t) := x_t \otimes (1 \otimes \cdots \otimes 1) + 1 \otimes \left( \sum_{\nu \in T} 1 \otimes \cdots \otimes x_\nu \otimes \cdots \otimes 1 \right).$$

Let $J_T \subset (R \otimes R^\otimes^\bullet)_T$ be the ideal generated by $\gamma_T(x_t)$ for all $t \in T$. For any $f: S \rightarrow T$, examination shows that the co-FI-module map $f_*: (R \otimes R^\otimes^\bullet)_T \rightarrow (R \otimes R^\otimes^\bullet)_S$ satisfies $f_*(\gamma_T(x_t)) = \gamma_S(x_{f(t)})$ if $t = f(s)$, and $f_*(\gamma_T(x_t)) = 0$ if $t \not\in f(S)$. Therefore the ideals $J_T$ form a co-FI-ideal $J \subset R \otimes R^\otimes^\bullet$.

We define the graded co-FI-algebra $BS$ to be the quotient $BS := R \otimes R^\otimes^\bullet / J$, so that

$$BS_\mathbf{n} \simeq R^{\otimes^\bullet n+1}_\mathbf{n} / J_\mathbf{n} \simeq G(R_\mathbf{n} / k).$$

Since $R$ is of finite type, Propositions 4.2.8 and 4.2.2 imply that the graded co-FI-algebra $R \otimes R^\otimes^\bullet$ is of finite type. Since $\text{char}(k) = 0$, its quotient $BS$ is a graded co-FI-algebra of finite type. The representation stability of $(BS^i)^* \simeq BS^i \simeq G(R_\mathbf{n} / k)^i$ follows from Theorem 1.13.

The graded pieces $BS^i$ can be computed for small $i$ by hand. For example, the dual of $BS^1$ is isomorphic as an FI-module to

$$(BS^1)^* \simeq M(1) \otimes M(1) = M(\square) \oplus M(\square) \oplus M(\square).$$

This raises the question: does the co-FI-algebra $BS$ in fact have the structure of an FI$^\square$-algebra?

### 5.3 Polynomials on rank varieties

Let $k$ be a field of arbitrary characteristic. For $n \geq 1$ let $k[x_{ij} | i, j \in \mathbf{n}]$ be the coordinates on the space $\text{Mat}_{n \times n}(k)$ of $n \times n$ matrices over $k$. Let $P = (p_1, \ldots, p_m)$ be an $m$-tuple of polynomials $p_\ell \in k[T]$, and let $r = (r_1, \ldots, r_m)$ be an $m$-tuple of positive integers. With this data Eisenbud–Saltman [BS] define the rank variety of matrices:

$$X_{P,r}(n) := \{ A \in \text{Mat}_{n \times n}(k) \mid \text{rank}(p_\ell(A)) \leq r_\ell, 1 \leq \ell \leq m \}$$

The special case when $m = 1$ and $p_1(T) = T$ is the fundamental example of the determinantal variety of matrices of rank $\leq r_1$. We consider $X_{P,r}(n)$ as an affine variety, defined by the polynomial equations

$$\{ \det(B(p_\ell(A))) = 0 \mid B \in B_\ell, 1 \leq \ell \leq m \}$$

where $B_\ell$ denotes the set of $(r_\ell + 1) \times (r_\ell + 1)$ minors of the matrix $(x_{ij})$. Let $J_\mathbf{n}$ be the ideal of $k[\{x_{ij}\}]$ generated by the homogeneous polynomials $\det(B(p_\ell(A)))$. The coordinate ring $O(X_{P,r}(n))$ is isomorphic by definition to $k[\{x_{ij}\}] / J_\mathbf{n}$, and inherits a grading from $k[\{x_{ij}\}]$.

**Question 5.3.1.** What is the dimension of the space of degree $d \geq 1$ polynomials on $X_{P,r}(n)$; that is, what is the dimension of the degree $d \geq 1$ component of $O(X_{P,r}(n)) \simeq k[\{x_{ij}\}] / J_\mathbf{n}$?

In the special case of determinantal varieties in characteristic 0, an answer to Question 5.3.1 can be deduced from work of Lascoux and Weyman (see [Wey], Corollary 6.1.5(d)). While we cannot resolve Question 5.3.1 completely, the theory of FI$^\square$-modules imposes strong constraints on the answer.
Theorem 5.3.2 (Polynomiality for rank varieties). Fix $P$, $r$, and $d$. Then the dimension of the space $\mathcal{O}(X_{P,r}(n))^d$ of degree-$d$ polynomials on the rank variety $X_{P,r}(n)$ is polynomial in $n$ of degree at most $2d$ for all $n \geq 0$. If $\text{char } k = 0$ then the same is true of the characters $\chi_{\mathcal{O}(X_{P,r}(n))^d}$.

Proof. The matrix varieties $\text{Mat}_{n \times n}$ fit together into an FI-$\sharp$-scheme $\text{Mat}$; the affine scheme $\text{Mat}_{S \times S}$ has coordinate ring $\mathcal{O}(\text{Mat}_{S \times S}) = k[\{x_{s,s'} | s, s' \in S\}]$. Given $f: S \subseteq A \xrightarrow{\phi} A \subseteq T$, the map $f_*: \text{Mat}_{S \times S} \to \text{Mat}_{T \times T}$ is given in coordinates by $x_{\phi(s),\phi(s')} \mapsto x_{s,s'}$ and $x_{t,t'} \mapsto 0$ if $t \notin \phi(A)$ or $t' \notin \phi(A)$. The rank varieties $X_{P,r}(S) \hookrightarrow \text{Mat}_{S \times S}$ are preserved by these maps, and thus form an FI-$\sharp$-subscheme $X_{P,r} \hookrightarrow \text{Mat}$.

The coordinate rings $\mathcal{O}(\text{Mat})$ and $\mathcal{O}(X_{P,r})$ are naturally FI-$\sharp$-algebras, but via the isomorphism $\text{FI}_\sharp \simeq \text{FI}_\sharp^{op}$ from Remark 4.1.3 we can consider them as FI-$\sharp$-algebras. For example, from the above formula we see that $\mathcal{O}(\text{Mat}) \simeq k[M(1) \otimes M(1)]$ as a graded FI-$\sharp$-algebra. Its quotient $\mathcal{O}(X_{P,r})$ is thus generated as a graded FI-$\sharp$-algebra by the image of $M(1) \otimes M(1) \subset \mathcal{O}(X_{P,r})^1$.

$M(1) \otimes M(1)$ is finitely generated in degree 2, so in particular $\text{weight}(M(1) \otimes M(1)) = 2$. Thus Theorem 4.2.3 implies that $\mathcal{O}(X_{P,r})$ is a graded FI-$\sharp$-algebra of finite type and slope $\leq 2$. The theorem now follows from Theorems 3.3.4 and 4.1.7.

Remark 5.3.3. The rank variety $X_{P,r}(n)$ of Theorem 5.3.2 may be non-reduced as a scheme. However, the same theorem holds for the underlying reduced variety. The proof amounts to replacing the ideal $J_n$ by its radical and applying the same argument.

Recent work of Draisma and Kuttler [DK] proves a bounded-generation result for “border rank varieties”, which generalize the rank varieties discussed here to the case of tensors of rank higher than 2. The key ingredient of their theorem is a Noetherian-type property. It would be interesting to understand whether their work can be phrased in the language of FI-modules.

6 Applications: cohomology of configuration spaces

In this section we apply the theory of FI-modules to the cohomology of configuration spaces of distinct points in a manifold $M$. We obtain in Theorem 1.8 and Corollary 6.3.4 strengthenings of earlier results of Church [Ch] Theorems 1 and 5]. When $M$ is an open manifold, we apply the theory of FI-$\sharp$-modules to obtain a number of new theorems on the rational, integral, and mod-$p$ cohomology of the configuration spaces of $M$.

6.1 FI-spaces and co-FI-spaces

An FI-space $X$ is a functor from FI to the category Top of topological spaces. These have been considered elsewhere in the topological literature under the names “$T$-spaces” or “$A$-spaces”; see Remark 6.1.3 below. Since homology is functorial, for any FI-space $X$ the homology $H_*(X;k)$ forms a graded FI-module. Similarly, a co-FI-space $X$ is a functor $X: \text{co-FI} \to \text{Top}$. Since cohomology is contravariantly functorial, for any co-FI-space $X$ the cohomology $H^*(X;k)$ forms a graded FI-algebra.

Definition 6.1.1. Given a topological space $M$, we define the co-FI-space $M^*$ as follows. For any finite set $S$, we have the topological space $M^S$ of functions from $S$ to $M$. An injection $f: T \hookrightarrow S$ induces a projection $f_*: M^S \twoheadrightarrow M^T$ by restriction, given explicitly by $f_*(\varphi) := \varphi \circ f: T \to S \to M$.

Proposition 6.1.2. Let $M$ be a connected topological space, and let $k$ be a field. If $H^i(M;k)$ is finite-dimensional for all $i \geq 0$, the graded FI-algebra $H^*(M^*;k)$ is of finite type.

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Proof. The graded FI-algebra $H^\ast(M^\bullet; k)$ associates to a finite set $S$ the cohomology $H^\ast(M^S; k)$. Since $k$ is a field, the Künneth theorem implies that $H^\ast(M^S; k) \simeq H^\ast(M; k)^{\otimes S}$. It thus seems that $H^\ast(M^\bullet; k)$ should be isomorphic to the FI-module $H^\ast(M; k)^{\otimes \cdot}$ defined in Definition 4.2.6. This is almost true; the only difference is that the Künneth isomorphism $H^\ast(M^S; k) \simeq H^\ast(M; k)^{\otimes S}$ depends on an ordering of $S$, and when these factors are permuted, a sign is sometimes introduced depending on the grading.

We can nevertheless deduce the proposition from Proposition 4.2.8 since it is easy to see that the proof of that proposition is not affected by this sign. To apply Proposition 4.2.8, we consider the graded $k$-module $H^\ast(M; k)$ as a “constant” graded FI-module $V$. The assumption that $M$ is connected guarantees that $H^0(M; k) \simeq k$, which gives an isomorphism $V^0 \simeq M(0)$. The remaining hypothesis of Proposition 4.2.8 is that $V$ is of finite type, or in other words that $H^i(M; k)$ is finite-dimensional for each $i \geq 0$. We conclude that $H^\ast(M^\bullet; k)$ is of finite type, as claimed. \qed

Remark 6.1.3. Let $M$ be a connected well-based topological space (i.e. with specified basepoint $\ast \in M$). We can extend the co-FI-space structure of Definition 6.1.1 to consider $M^\bullet$ as an FI$^\ast$-space: given $f: S \subseteq A \overset{\phi}{\to} B \subseteq T$ and $\varphi: S \to M$, we define $f_*: M^S \to M^T$ by

$$f_*(\varphi)(t) = \begin{cases} \varphi(\phi^{-1}(t)) & \text{if } t \in B \\ \ast & \text{if } t \in T - B \end{cases} \quad (22)$$

Considering $M^\bullet$ as an FI-space, given a map defined on $S$, an inclusion $f: S \to T$ simply extends it to $T$ by sending $T - f(S)$ to the basepoint $\ast$. We point out that the direct limit colim$_{FI} M^\bullet$ computes the infinite symmetric product $\text{Sym}_{\infty}^\infty M \simeq \text{colim}_{FI} M^\bullet$, which by the Dold–Thom theorem satisfies $\pi_i(\text{Sym}_{\infty}^\infty M) \simeq \tilde{H}_i(M; \mathbb{Z})$.

In fact, $M^\bullet$ provides an example of an $\mathcal{I}$-monoid in the sense of Sagave–Schlichtkrull [SSc] (a commutative monoid in the category of FI-spaces). Sagave–Schlichtkrull prove that any infinite loop space can be modeled by an $\mathcal{I}$-monoid. For example, the Barratt–Priddy–Quillen theorem implies that $\Omega^\infty \Sigma^\infty M \simeq \text{hocolim}_{FI} M^\bullet$. Similarly, Quillen’s $K$-theory space $K(R)$ (or rather the identity component) is realized by $\text{hocolim}_{FI} BGL_n(R)$ for the natural FI-space $BGL_n(R)$. We will not exploit this connection further in this paper; see [SSc] Examples 1.3 and 1.5 for these examples and other applications of $\mathcal{I}$-monoids.

### 6.2 Configuration spaces as co-FI-spaces

We define the co-FI-space $\text{Conf}(M)$ as a subspace of $M^\bullet$: let $\text{Conf}_S(M) \subset M^S$ be the subspace of embeddings $S \hookrightarrow M$:

$$\text{Conf}_S(M) := \text{Emb}(S, M) \subset M^S := \text{Map}(S, M)$$

If $\varphi: T \hookrightarrow M$ and $f: S \hookrightarrow T$ are injective, then $f_* (\varphi) = \varphi \circ f: S \hookrightarrow T$ is injective. Thus $\text{Conf}(M)$ is indeed a co-FI-space, and the inclusion $\text{Conf}(M) \hookrightarrow M^\bullet$ is a map of co-FI-spaces. Of course, the space $\text{Conf}_n(M) = \text{Emb}(n, M)$ can be identified with the configuration space of ordered $n$-tuples of distinct points on $M$:

$$\text{Conf}_n(M) = \{(x_1, \ldots, x_n) \in M^n \mid x_i \neq x_j\}$$

The following theorem is proved (in different language) in [Ch]. The assumption $\dim M \geq 2$ guarantees that $\text{Conf}_S(M)$ is connected; the condition $\dim_{\mathbb{Q}}(H^\ast(M; \mathbb{Q})) < \infty$ is satisfied whenever $M$ is compact.

**Theorem 6.2.1 (Rational cohomology of configuration spaces).** Let $M$ be a connected, oriented manifold of dimension at least 2 with $\dim_{\mathbb{Q}}(H^\ast(M; \mathbb{Q})) < \infty$. Then the graded FI-algebra $H^\ast(\text{Conf}(M); \mathbb{Q})$ is of finite type.
Proof. We will consider the Leray spectral sequence of the inclusion of co-FI-spaces $\text{Conf}(M) \hookrightarrow M^\bullet$. Recall that for any continuous map of topological spaces $f: X \to Y$, the Leray spectral sequence with $\mathbb{Q}$ coefficients is a cohomological spectral sequence converging to $H^*(X; \mathbb{Q})$ with $E_2$ page $E_2^{p,q} \simeq H^p(Y; R^f_* \mathbb{Q})$. These entries are in general difficult to compute, with the exception of the edge terms $E_2^{p,0} \simeq H^p(Y; \mathbb{Q})$. Recall further that the Leray spectral sequence is functorial: if $f'': Y' \to X'$ is another continuous map, and $g_X, g_Y$ are maps $g_X: X' \to X$ and $g_Y: Y' \to Y$ such that $f'' \circ g_X = g_Y \circ f'$, then $g$ induces a map of spectral sequences from the Leray spectral sequence for $f: X \to Y$ to the Leray spectral sequence for $f': X' \to Y'$.

For any finite set $S$ we have a continuous map $i_S: \text{Conf}_S(M) \hookrightarrow M^S$, so we have a Leray spectral sequence with $E_2^{p,0} \simeq H^p(M^S; \mathbb{Q})$ converging to $H^*(\text{Conf}_S(M); \mathbb{Q})$. Moreover for any $f: T \hookrightarrow S$ we have $i_{S} \circ f_* = f_* \circ i_T$ (this is what it means for $i$: $\text{Conf}(M) \hookrightarrow M^\bullet$ to be a map of co-FI-spaces), so we obtain maps $f_*$ between these spectral sequences by functoriality. In other words, all the Leray spectral sequences for $\text{Conf}_S(M) \hookrightarrow M^S$ together form a cohomological spectral sequence of FI-modules with edge terms $E_2^{p,0} \simeq H^p(M^\bullet; \mathbb{Q})$ converging to the graded FI-algebra $H^*(\text{Conf}(M); \mathbb{Q})$.

Each page $E_r$ forms a bigraded FI-algebra, which we can think of as a graded FI-algebra by $E_r^{p,q} = \bigoplus_{p+i \leq q} E_{r,i}^{p,i-p}$. Since this sum is finite, there is no ambiguity in asking whether $E_r$ is of finite type: this means that $E_r^{p,q}$ is a finitely generated FI-module for all $p \geq 0$ and all $q \geq 0$.

Totaro [13] provides an explicit description of the $E_2$ page, which in particular gives an isomorphism $E_2^{0,d-1} \simeq H^{d-1}(\text{Conf}([d]; \mathbb{Q}))$. This FI-module can be identified with $M([d])$ for $d = 1$, and with $M([d])$ when $d$ is odd; in particular, the FI-module $E_2^{0,d-1}$ is finitely generated. Moreover, Totaro shows that the graded FI-algebra $E_2$ is generated by $E_2^{0,d-1}$ together with $E_2^{p,0}$ for all $p \geq 0$. (For future reference, this implies that $E_2^{p,q} = 0$ unless $p = q(d-1)$.)

$E_2^{*,0} \simeq H^*(M^\bullet; \mathbb{Q})$ is of finite type by Proposition 6.1.2 and $E_2^{0,d-1}$ is finitely generated, so Totaro’s generating set is a graded FI-module of finite type. Theorem 4.2.3 then implies that the graded FI-algebra $E_2$ is of finite type. Since $E_\infty$ is a subquotient of $E_2$, Theorem 1.3 implies that $E_\infty$ is of finite type. The FI-module $H^i(\text{Conf}(M); \mathbb{Q})$ has a finite filtration with graded quotients $E_\infty^{p,i-p}$, so $H^*(\text{Conf}(M); \mathbb{Q})$ is of finite type if and only if $E_\infty$ is; this concludes the proof.

Combined with Theorem 1.13, Theorem 6.2.1 implies the theorem of Church [Ch, Theorem 1] that the sequence of $S_n$-representations $\{H^i(\text{Conf}_n(M); \mathbb{Q})\}$ is uniformly representation stable. By applying this result to the trivial representation, Church extended the classical theorem on rational homological stability for the configuration spaces of unordered points in an open manifold to configuration spaces of unordered points in an arbitrary manifold.

Remark 6.2.2. The key technical innovation in [Ch] was the notion of monotonicity, which allows representation stability to be transmitted from the initial term of a spectral sequence to its $E_\infty$ term. We showed in the proof of Theorem 1.13 that, for FI-modules, finite generation implies monotonicity. Thus it is not surprising that we are able to imitate the argument of [Ch] in the present paper.

6.3 Bounding the stability degree for cohomology of configuration spaces

The results of Church [Ch] not only show that $H^i(\text{Conf}_n(M); \mathbb{Q})$ is representation stable, but they provide an explicit stable range. In this section we strengthen those results by computing explicit bounds on the stability degree for the FI-module $H^i(\text{Conf}(M); \mathbb{Q})$, which let us improve Church’s stable range.

Recall from Definition 3.1.5 that we divided the notion of stability degree into injectivity degree and surjectivity degree. For succinctness, for an FI-module $V$ we write $V \preceq (A,B)$ if inj-deg($V$) $\leq A$ and surj-deg($V$) $\leq B$; this implies stab-deg($V$) $\leq \max(A,B)$.
Theorem 6.3.1. Let $M$ be a connected, oriented manifold of dimension $d \geq 3$. For any $i \geq 0$, the FI-module $H^i(\text{Conf}(M); \mathbb{Q})$ has weight $\leq i$ and $H^i(\text{Conf}(M); \mathbb{Q}) \simeq (i + 2 - d, i)$; in particular stab-deg($H^i(\text{Conf}(M); \mathbb{Q})$) $\leq i$.

We first record two elementary homological facts that we will use in the proof, which follow from the fact that $\Phi_\alpha$ is exact over rings containing $\mathbb{Q}$. Both can be verified by a simple diagram chase.

Lemma 6.3.2. Assume that $k$ contains $\mathbb{Q}$.

(i) Consider a complex of FI-modules $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $g \circ f = 0$. If $X \leq (A, B)$, $Y \leq (C, D)$, and $Z \leq (E, F)$, then the homology $\ker g/\im f$ satisfies $\ker g/\im f \leq (\max(B, C), \max(D, E))$.

(ii) If $V$ is an FL-module with a finite filtration by sub-FL-modules $F^iV \subset V$, then $V \leq (A, B)$ if and only if $F^iV/F^{i-1}V \leq (A, B)$ for all $i$.

Proof of Theorem 6.3.1. We continue with the notation from the proof of Theorem 6.2.1. Let $D = d - 1$. We saw in that proof that $E_2^{*,*} = 0$ unless $* = qD$. Moreover, Church [Ch §3.3] gives an explicit description of $E_2^{p,qD}$ as a direct sum $E_2^{p,qD} \simeq \bigoplus_{k=0}^{p+2q} M(W_k)$, where $W_k$ is a certain representation of $S_k$. By Proposition 3.1.7, this implies that $E_2^{p,qD} \simeq (0, p + 2q)$ and weight($E_2^{p,qD}$) $\leq p + 2q$.

The argument proceeds by induction on the pages of the Leray spectral sequence. To be precise, the fact that $E_2^{p,*} = 0$ unless $* = qD$ implies that the only nontrivial differentials occur on pages $E_k^{p,qD}$, with $E_{(k-1)+2} \simeq \cdots \simeq E_{kD} \simeq E_{kD+1}$. We will prove the following claims $(a_k)$ and $(b_k)$ for all $k \geq 2$:

$(a_k)$ inj-deg($E_k^{p,qD}$) $\leq p + 2q + 1 - D$ for all $p \geq 0$ and all $q \geq 0$

$(b_k)$ surj-deg($E_k^{p,qD}$) $\leq p + 2q + \min(k - 2, q - 1)(D - 2)$ for all $p \geq 0$ and all $q > 0$.

We begin with the base case $k = 2$. We have isomorphisms $E_2 \simeq E_{D+1}$ and $E_{D+2} \simeq E_{2D+1}$. The only nontrivial intervening differential is on the $E_{D+1}$ page, where $E_{2D+1}$ is computed as the cohomology of the complex

$$E_{D+1}^{p-D+1,(q+1)D} \to E_{D+1}^{p,qD} \to E_{D+1}^{p+D+1,(q-1)D}.$$  

From $E_2 \simeq E_{D+1}$ we know that $E_{D+1}^{p,qD} \simeq (0, p + 2q)$, so

$$E_{D+1}^{p-D+1,(q+1)D} \simeq (0, p + 2q + 1 - D), \quad E_{D+1}^{p,qD} \simeq (0, p + 2q), \quad E_{D+1}^{p+D+1,(q-1)D} \simeq (0, p + 2q + D - 1).$$  

Applying Lemma 6.3.2(i) shows that $E_{2D+1}^{pq} \simeq (p + 2q + 1 - D, p + 2q)$, verifying $(a_2)$ and $(b_2)$.

Suppose that $(a_k)$ and $(b_k)$ hold. We can write $E_{(k+1)+1}^{p,qD}$ as the cohomology of a three-term complex:

$$E_{kD+1}^{p-kD-1,(q+k)D} \to E_{kD+1}^{p,qD} \to E_{kD+1}^{p+kD+1,(q-k)D}.$$  

For readability, let $X \to Y \to Z$ denote the FI-modules making up this complex (23). By $(a_k)$ we have inj-deg($Y$) $\leq p + 2q + 1 - D$, and by $(b_k)$ we have

$$\text{surj-deg}(X) \leq p - kD - 1 + 2(q + k) + (k - 2)(D - 2) = p + 2q + 3 - 2D \leq p + 2q + 1 - D,$$

where the last inequality holds because $D \geq 2$. We conclude that inj-deg($E_{(k+1)+1}^{p,qD}$) $\leq p + 2q + 1 - D$ by Lemma 6.3.2(i), verifying $(a_{k+1})$. 

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Assume that \( k > q > 0 \). In this case \( Z = 0 \), since \( E_{kD+1}^{p+kD+1,(q-k)D} \) lies outside the first quadrant, so \( \text{surj-deg}(E_{(k+1)D+1}^{p,qD}) \leq \text{surj-deg}(Y) \). By \((b_k)\) we have \( \text{surj-deg}(Y) \leq p+2q+\min(k-2,q-1)(D-2) \leq p+2q+\min(k+1-2,q-1)(D-2) \), verifying \((b_{k+1})\) when \( k > q > 0 \).

Finally, assume that \( k \leq q \). By \((a_k)\), \( \text{inj-deg}(Z) \leq p+kD+1+2(q-k)+1-D = p+2q+(k-1)(D-2) \). Similarly, \((b_k)\) implies \( \text{surj-deg}(Y) \leq p+2q+\min(k-2,q-1)(D-2) = p+2q+(k-2)(D-2) \). By Lemma 6.3.2(i), \( \text{surj-deg}(E_{(k+1)D+1}^{p,qD}) \leq p+2q+(k-1)(D-2) \), verifying \((b_{k+1})\) when \( k \leq q \).

We now deduce the theorem from the claims \((a_k)\) and \((b_k)\). The FI-module \( H^i(\text{Conf}(M); \mathbb{Q}) \) has a finite filtration with graded quotients those \( E_{\infty}^{p,qD} \) with \( p+qD = i \). Thus by Lemma 6.3.2(ii), it suffices to show that \( E_{\infty}^{p,qD} \cong (i+2-d,i) \) and \( \text{weight}(E_{\infty}^{p,qD}) \leq i \) whenever \( p+qD = i \).

Since \( E_{\infty}^{p,qD} \) is a subquotient of \( E_2^{p,qD} \), we automatically have \( \text{weight}(E_{\infty}^{p,qD}) \leq p+2q \leq i \). As \( k \to \infty \), the claims \((a_k)\) imply \( \text{inj-deg}(E_{\infty}^{p,qD}) \leq p+2q+1-D \leq i+2-d \). Similarly, when \( q > 0 \) the claims \((b_k)\) imply \( \text{surj-deg}(E_{\infty}^{p,qD}) \leq p+qD+2-D \leq i \), so in this case we have \( E_{\infty}^{p,qD} \cong (i+2-d,i) \) as desired. When \( q = 0 \), the FI-module \( E_{\infty}^{\infty,p} \) is a quotient of \( E_2^{\infty,p} \), and we showed at the beginning of the proof that \( \text{surj-deg}(E_2^{p,0}) \leq p \). We therefore have \( \text{surj-deg}(E_{\infty}^{p,0}) \leq p = i \), with no need for induction, verifying that \( E_{\infty}^{\infty} \cong (i+2-d,i) \) as well.

Combining Theorem 6.3.1 with Theorem 3.3.4 proves Theorem 1.8 from the introduction.

Remark 6.3.3. If \( M \) is a connected oriented manifold of dimension 2 with \( \dim_{\mathbb{Q}}(H^*(M; \mathbb{Q})) < \infty \), the weight of the FI-module \( H^i(\text{Conf} M; \mathbb{Q}) \) is bounded by that of \( \bigoplus E_2^{i-\mu} \), just as in the proof of Theorem 6.3.1. This weight is \( p+2(i-p) \), which is maximized when \( p = 0 \). So in this case \( H^i(\text{Conf} M; \mathbb{Q}) \) is a finitely generated FI-module of weight at most \( 2i \), whence the characters and Betti numbers of \( H^i(\text{Conf}_n(M); \mathbb{Q}) \) are eventually polynomial of degree \( \leq 2i \).

A “classical” application. We can apply Theorem 6.3.1 to prove a cohomological stability result, in the usual sense, for unordered configuration spaces with some population of colored points. This result is inspired by Vakil–Wood [VW], who proved a motivic analogue when \( M \) is an algebraic variety.

Given a partition \( \mu = (\mu_1, \ldots, \mu_k) \), we denote by \( B_{n,\mu}(M) \) the configuration space of sets of \( n \) distinct unordered points on \( M \), with \( \mu_i \) of the points labeled with color \( i \) and \( n-|\mu| \) of the points left uncolored. The following corollary is a direct improvement on [Ch. Theorem 5], where the bound \( n \geq \max(2i,2|\mu|) \) was obtained. It is the bound on stability degree in Theorem 6.3.1 that allows us to improve the stable range from \( \max(2i,2|\mu|) \) to \( i+|\mu| \).

Corollary 6.3.4. Let \( M \) be a connected, oriented manifold of dimension \( \geq 3 \). Then for all \( n \geq i+|\mu| \),

\[
H^i(B_{n,\mu}(M); \mathbb{Q}) \simeq H^i(B_{n+1,\mu}(M); \mathbb{Q}).
\]

Proof. The invariant subspace \( H^i(\text{Conf}_n(M); \mathbb{Q})^{S_{n-|\mu|}} \) carries an action of \( S_{|\mu|} \), and the dimension of \( H^i(B_{n,\mu}(M); \mathbb{Q}) \) is the dimension of the space fixed by the action of \( S_{|\mu|} = S_{\mu_1} \times \cdots \times S_{\mu_k} \subset S_{|\mu|} \).

The statement that \( H^i(\text{Conf}_n(M); \mathbb{Q}) \) has stability degree \( \leq i \) means by definition that the isomorphism class of \( H^i(\text{Conf}_n(M); \mathbb{Q})^{S_{n-|\mu|}} \simeq H^i(\text{Conf}_n(M); \mathbb{Q})^{S_{n-|\mu|}} \) as an \( S_{|\mu|} \)-representation is constant for \( n \geq i+|\mu| \). The dimension of the \( S_{|\mu|} \)-invariant subspace is thus constant for \( n \geq i+|\mu| \) as well. \( \square \)
6.4 Configurations on manifolds with boundary as homotopy FI\textsuperscript{♯}-space

We saw in Remark 6.1.3 that \( M^\bullet \) could be extended from a co-FI-space to an FI\textsuperscript{♯}-space. This is not possible for \( \text{Conf}(M) \). However, when \( M \) is the interior of a compact manifold with nonempty boundary, we can in fact realize the configuration space \( \text{Conf}(M) \) as an FI\textsuperscript{♯}-space up to homotopy.

**Definition 6.4.1.** An homotopy FI\textsuperscript{♯}-space is a functor \( X \) from FI\textsuperscript{♯} to hTop, the category of topological spaces and homotopy classes of continuous maps. Concretely, this means that for each \( n \) we have a space \( X_n \), for each partial injection \( f \in \text{Hom}_{\text{FI}\textsuperscript{♯}}(\mathbf{m}, \mathbf{n}) \) we have a map \( f_* : X_m \to X_n \) (or rather a homotopy class of maps), and the corresponding diagrams all commute up to homotopy. Since a homotopy class of maps induces a well-defined map on homology and cohomology, both the homology groups \( H_*(X;k) \) and cohomology groups \( H^*(X;k) \) form an FI\textsuperscript{♯}-module.

**Proposition 6.4.2.** Let \( M \) be the interior of a connected, oriented, compact manifold \( \overline{M} \) of dimension \( d \geq 2 \) with nonempty boundary \( \partial \overline{M} \). We can extend the FI-space \( \text{Conf}(M) : \text{co-FI} \to \text{Top} \) to a homotopy FI\textsuperscript{♯}-space \( \text{Conf}(M) : \text{FI}_\text{♯} \to \text{hTop} \).

**Proof.** We will use a variant of the FI\textsuperscript{♯}-space structure on \( M^\bullet \) defined in (22), modified so as to preserve the injectivity of our configurations \( S \hookrightarrow M \). The fact that \( \partial \overline{M} \neq \emptyset \) lets us define, for any inclusion of finite sets \( B \subseteq T \), a map (defined up to homotopy)

\[
\Psi_B^T : \text{Conf}_B(M) \to \text{Conf}_T(M)
\]

which “adds points at infinity”, as follows. Fix a collar neighborhood \( R \) of one component of \( \partial \overline{M} \) (so that \( R \) is connected), and fix a homeomorphism \( \Phi : M \xrightarrow{\cong} M \setminus R \) isotopic to the identity \( \text{id} : M \to M \).

If \( T = B \) we set \( \Psi_B^T = \text{id} \). Otherwise, fix a configuration \( q_B^T : (T - B) \to R \) in \( \text{Conf}_{T - B}(R) \). Then any embedding \( \varphi : B \hookrightarrow M \) in \( \text{Conf}_B(M) \) can be extended to a function \( \Psi_B^T(\varphi) : T \hookrightarrow M \) by defining:

\[
\Psi_B^T(\varphi)(t) = \begin{cases} 
\Phi(\varphi(t)) & t \in B \\
qu_B^T(t) & t \in T - B
\end{cases}
\]

Since \( \text{Conf}_{T - B}(R) \) is connected, different choices of \( q_B^T \in \text{Conf}_{T - B}(R) \) induce homotopic maps, so the map \( \Psi_B^T \) is well-defined up to homotopy.

For a morphism \( f : S \supset \psi^{-1}(B) \xrightarrow{\psi} B \subseteq T \), let \( f_* : \text{Conf}_S(M) \to \text{Conf}_T(M) \) be the composition

\[
f_* : \text{Conf}_S(M) \xrightarrow{-\circ \psi^{-1}} \text{Conf}_B(M) \xrightarrow{\Psi_B^T} \text{Conf}_T(M).
\]

Let \( g : T \supset C \xrightarrow{\psi} \psi(C) \subset U \) be another morphism, and let us compare the two maps \( g_* \circ f_* \) and \( (g \circ f)_* : \text{Conf}_S(M) \to \text{Conf}_U(M) \). The restriction \( (g \circ f)_*|_{U - \psi(B \cap C)} \) is equal to \( q_B^T \) for all \( \varphi \in \text{Conf}_S(M) \); the restriction \( g_* \circ f_*(\varphi)|_{U - \psi(B \cap C)} \) is similarly constant (though not the same). On the remaining elements \( \psi(B \cap C) \), if \( u = \psi(\varphi^{-1}(s)) \) we have \( (g \circ f)_*(\varphi)(u) = \Phi(\varphi(s)) \) and \( g_* \circ f_*(\varphi)(u) = \Phi(\Phi(\varphi(s))) \). Using the isotopy from \( \Phi \) to the identity, we can thus construct a homotopy from \( g_* \circ f_* \) to \( (g \circ f)_* \). This completes the proof that \( \text{Conf}(M) \) is a homotopy FI\textsuperscript{♯}-space.

It follows from Proposition 6.4.2 that the FI-module \( H^*(\text{Conf}(M);k) \) is in fact an FI\textsuperscript{♯}-module in this case, for any ring \( k \). In particular, all of the maps \( H^*(\text{Conf}_n(M);k) \to H^*(\text{Conf}_{n+m}(M);k) \) are injective since every inclusion \( f : S \hookrightarrow T \) has a left inverse in FI\textsuperscript{♯}; similarly, all of the maps \( H^*(\text{Conf}_{n+m}(M);k) \to H^*(\text{Conf}_n(M);k) \) are surjective.
Theorem 6.4.3. Let $M$ be a connected, oriented manifold of dimension $d \geq 2$ which is the interior of a compact manifold with nonempty boundary. Let $k$ be a Noetherian ring of finite Krull dimension. For each $i \geq 0$, the $\FI_\sharp$-module $H^i(\Conf(M); k)$ is finitely generated in dimension $\leq i$ if $d \geq 3$, and in dimension $\leq 2i$ if $d = 2$.

Proof. We have already proved the theorem in the case $k = \mathbb{Q}$; the content lies in the case when $k$ has positive characteristic, or is not a field. We recall from the proof of Theorem [6.3.1] that each entry $E_2^{pq}$ of the Leray spectral sequence $\Conf(M) \hookrightarrow M^\bullet$ is a finitely generated $\FI_\sharp$-module. However, we cannot immediately conclude that $E_\infty^{pq}$ or $H^i(\Conf(M); k)$ is finitely generated, even though this spectral sequence converges to $H^*(\Conf(M); k)$. The problem is that $\Conf(M)$ is only a homotopy $\FI_\sharp$-space, and the Leray spectral sequence is not homotopy invariant. As a result it is not a spectral sequence of $\FI_\sharp$-modules.

Our hypotheses on $k$ will allow us to circumvent this issue. We first explain the proof in the cases when $k$ is either $\mathbb{Z}$ or a field, which were mentioned in the introduction, and afterwards explain how to get the general case. First assume that $k$ is a field. From the proof of Theorem [6.3.1], each entry $E_2^{pq}$ of the Leray spectral sequence is a finitely generated $\FI_\sharp$-module generated in degree $\leq p + 2q$.

When $d \geq 3$, it follows that $\dim_k(E_2^{pq})_n = O(n^{p+q})$. As a vector space $(E_\infty^{pq})_n$ is a quotient of $(E_2^{pq})_n$, so $\dim_k(E_\infty^{pq})_n = O(n^{p+q})$ as well. Finally, $H^i(\Conf_n(M); k)$ has a finite filtration with quotients $(E_\infty^{pq-i})_n$, so we conclude that $\dim_k H^i(\Conf_n(M); k) = O(n^i)$. By Theorem 4.1.7, this implies that the $\FI_\sharp$-module $H^i(\Conf(M); k)$ is finitely generated in degree $\leq i$, as the theorem claims. When $d = 2$ we have $\dim_k(E_2^{pq})_n = O(n^{p+2q})$, so $\dim_k H^i(\Conf_n(M); k) = O(n^{2i})$.

The same argument works when $k = \mathbb{Z}$, using the fact that the number of generators of a $\mathbb{Z}$-module decreases when passing to submodules. Of course, this fact is not true for a general ring; for example, $\mathbb{C}[x, y]$ has rank 1 as a module over itself, but the ideal $(x, y)$ cannot be generated by fewer than 2 elements. However, Forster [Fo, Satz 1] proved the following theorem for a Noetherian ring $k$ of Krull dimension $d$: if a $k$-module $A$ satisfies

$$\dim_F(A \otimes_k F) \leq N$$

for all quotient fields $F := k/m$,

then $A$ is generated by at most $N + d$ elements.

We want to apply Forster’s theorem to $H^i(\Conf_n(M); k)$. The discussion above shows for $d \geq 3$ that $\dim_F(E_2^{pq})_n \otimes_k F = O(n^{p+q})$. Since $F$ is a field, this bound passes to subquotients, so $\dim_F H^i(\Conf_n(M); k) \otimes F = O(n^i)$ for all quotient fields $F$. (Despite the use of big-$O$ notation, we in fact have an explicit upper bound on this dimension which is independent of $F$, coming from the number of generators for $(E_2^{pq})_n$ in the case $k = \mathbb{Z}$.) Applying Forster’s theorem, we conclude that the $k$-module $H^i(\Conf_n(M); k)$ is generated by $O(n^i + d) = O(n^i)$ elements. The equivalence of (iv) and (i) in Theorem 4.1.7 then implies that the $\FI_\sharp$-module $H^i(\Conf(M); k)$ is finitely generated in degree $\leq i$, as claimed. The proof for $d = 2$ is identical.

Theorem 1.9 follows immediately from Theorem 6.4.3 by applying the classification of $\FI_\sharp$-modules from Theorem 4.1.5 just as in the proof of Theorem 4.1.7.

7 Applications: cohomology of moduli spaces

7.1 Stability for $\mathcal{M}_{g,n}$ and its tautological ring

Let $S_g$ be a closed, oriented surface of genus $g \geq 2$. For each $n \geq 1$ let $(y_1, \ldots, y_n)$ be an ordered $n$-tuple of $n$ distinct points on $S_g$. Let $\mathcal{M}_{g,n}$ denote the moduli space of $n$-pointed Riemann surfaces
(X; y1, ..., yn) homeomorphic to \((S_g; y_1, ..., y_n)\). The space \(\mathcal{M}_{g,n}\) has the structure of a complex orbifold, i.e. the quotient of a complex manifold by a finite group of holomorphic automorphisms.

The map “forget the nth point” yields a fibration \(p_n: \mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}\) whose fiber is an \((n-1)\)-punctured surface of genus \(g\). For each \(k \geq 0\), the map \(p_n\) induces a homomorphism

\[ p_n^*: H^k(\mathcal{M}_{g,n-1}; \mathbb{Q}) \to H^k(\mathcal{M}_{g,n}; \mathbb{Q}). \]

For each \(i = 1, ..., n\), we let \(L_i \to \mathcal{M}_{g,n}\) be the complex line bundle over \(\mathcal{M}_{g,n}\) whose fiber at \((X; y_1, ..., y_n)\) is the cotangent space to \(X\) at \(y_i\). Let \(\psi_i = c_1(L_i) \in H^2(\mathcal{M}_{g,n}; \mathbb{Q})\) be the first Chern class of the line bundle \(L_i\). Integration along the fiber yields a Gysin homomorphism

\[ (p_n!): H^j(\mathcal{M}_{g,n}; \mathbb{Q}) \to H^{j-2}(\mathcal{M}_{g,n-1}; \mathbb{Q}). \]

For each \(j \geq 0\) define \(\kappa_j := (p_{n+1})!(\psi^{j+1}_{n+1}) \in H^{2j}(\mathcal{M}_{g,n}; \mathbb{Q})\).

The tautological ring \(\mathcal{R}(\mathcal{M}_{g,n})\) is defined to be the subring of \(H^*(\mathcal{M}_{g,n}; \mathbb{Q})\) generated by \({\psi_i | 1 \leq i \leq n}\) \(\cup\) \({\kappa_j | j \geq 0}\).

This ring has been intensively studied by algebraic geometers (see e.g. [Ya]). The usual grading on \(\mathcal{R}(\mathcal{M}_{g,n})\) is half the cohomological grading, so that \(\psi_i\) has grading 1 and \(\kappa_j\) has grading \(j\).

\(S_n\) acts on \(\mathcal{M}_{g,n}\) by permuting the marked points. The induced action on \(H^*(\mathcal{M}_{g,n}; \mathbb{Q})\) satisfies

\[ \sigma \cdot \kappa_j = \kappa_j \quad \text{and} \quad \sigma \cdot \psi_i = \psi_{\sigma(i)} \]

for each \(j \geq 1\) and each \(1 \leq i \leq n\), so \(\mathcal{R}(\mathcal{M}_{g,n})\) is a subrepresentation of \(H^*(\mathcal{M}_{g,n}; \mathbb{Q})\). The grading-j components \(\mathcal{R}^j(\mathcal{M}_{g,n})\) can be quite complicated as \(S_n\)-representations, and for \(g \geq 3\) they are poorly understood. However, we have the following strong constraint once \(n\) is sufficiently large.

**Theorem 7.1.1.** For each \(g \geq 2\) the tautological ring \(\mathcal{R}(\mathcal{M}_{g,*})\) is a graded FI-algebra of finite type. Thus for each \(j \geq 1\) the characters \(\chi_{\mathcal{R}^j(\mathcal{M}_{g,n})}\) are eventually polynomial of degree \(\leq j\). In particular, \(\dim \mathcal{R}^j(\mathcal{M}_{g,n})\) is eventually polynomial in \(n\) of degree \(\leq j\).

**Proof.** Let \(\mathcal{M}_{g,S}\) be the moduli space of genus \(g\) Riemann surfaces \(X\) endowed with an injection \(S \hookrightarrow X\). These spaces form a co-FI-space \(\mathcal{M}_{g,*}\) just as in \([6.2]\), so the cohomology \(H^*(\mathcal{M}_{g,*}; \mathbb{Q})\) is a graded FI-algebra.

For each \(s \in S\) we have a line bundle \(L_s \to \mathcal{M}_{g,S}\) with first Chern class \(\psi_s := c_1(L_s) \in H^2(\mathcal{M}_{g,S}; \mathbb{Q})\). Similarly, we consider the map \(p_*: \mathcal{M}_{g,S;\{s\}} \to \mathcal{M}_{g,S}\), and for each \(j \geq 0\) we define \(\kappa_j := (p_*)!(\psi^{j+1}_s) \in H^{2j}(\mathcal{M}_{g,S}; \mathbb{Q})\). These classes define a map of graded FI-modules \(t: V \to H^*(\mathcal{M}_{g,*}; \mathbb{Q})\), where \(V\) is the graded FI-module with \(V^{2i+1} = 0\) and

\[ V^2 \cong M(1) \oplus M(0), \quad V^{2i} = \mathbb{Q} \kappa_i \cong M(0) \text{ for } i > 1. \]

By definition, \(\mathcal{R}(\mathcal{M}_{g,*})\) is the sub-FI-algebra of \(H^*(\mathcal{M}_{g,*}; \mathbb{Q})\) generated by the image \(t(V)\). By construction, \(V\) is a graded FI-module of finite type with slope \(\frac{1}{2}\), so the same is true of \(t(V)\). Theorem [4.2.3] implies that the sub-FI-algebra of \(H^*(\mathcal{M}_{g,*}; \mathbb{Q})\) generated by \(t(V)\) is a graded FI-module of finite type which has slope \(\leq \frac{1}{2}\). Recalling the difference in the grading, we conclude that the FI-module \(\mathcal{R}^j(\mathcal{M}_{g,*}) \subset H^{2j}(\mathcal{M}_{g,*}; \mathbb{Q})\) is finitely generated in degree \(\leq i\). The desired conclusion follows from Theorem [3.3.4].

**Remark 7.1.2.** Jimenez Rolland [JR1] proved the related theorem that for each fixed \(g \geq 2\) and \(i \geq 0\), the sequence \(\{H^i(\mathcal{M}_{g,n}; \mathbb{Q})\}\) is a uniformly representation stable sequence of \(S_n\)-representations. Since \(H^*(\mathcal{M}_{g,*}; \mathbb{Q})\) is an FI-module, and since \(H^*(\mathcal{M}_{g,n}; \mathbb{Q})\) is finite dimensional, Theorem [1.13] together with Jimenez Rolland’s theorem shows that \(\{H^*(\mathcal{M}_{g,*}; \mathbb{Q})\}\) is a graded FI-module of finite type. This result, with bounds on the stability degree, etc., has been worked out by Jimenez Rolland [JR2] using the theorems in the present paper.
7.2 Albanese cohomology of Torelli groups

The Torelli subgroups $T_g^1$ and $I_A_n$ of the mapping class groups and automorphism groups of free groups, respectively, are of interest in low-dimensional topology and combinatorial group theory (see below for definitions). However, very little is known about $H^*(T_g^1; \mathbb{Q})$ or $H^*(I_A_n; \mathbb{Q})$. In this section we apply the theory of FI-modules to the subalgebra generated by first cohomology.

**Definition 7.2.1 (Albanese cohomology).** Let $\Gamma$ be a finitely generated group, let $\Gamma^{ab}$ be its abelianization, and let $\psi: \Gamma \to \Gamma^{ab}$ be the natural quotient map. Since $H^1(\Gamma^{ab}; \mathbb{Q}) \simeq H^1(\Gamma; \mathbb{Q})$ and $H^*(\Gamma^{ab}; \mathbb{Q}) \simeq \bigwedge^* H^1(\Gamma^{ab}; \mathbb{Q})$, the map $\psi$ induces a homomorphism

$$\psi^*: H^*(\Gamma^{ab}; \mathbb{Q}) \simeq \bigwedge^* H^1(\Gamma; \mathbb{Q}) \to H^*(\Gamma; \mathbb{Q}).$$

We define the Albanese cohomology $H^i_{\text{Alb}}(\Gamma; \mathbb{Q})$ of $\Gamma$ to be the image of this map:

$$H^i_{\text{Alb}}(\Gamma; \mathbb{Q}) := \psi^*(H^*(\Gamma^{ab}; \mathbb{Q})) \subset H^*(\Gamma; \mathbb{Q}).$$

We use the name “Albanese cohomology” in order to keep in mind the frequently encountered case where $\Gamma$ is the fundamental group of a compact Kahler manifold, in which case $H^*_{\text{Alb}}$ is the part of the cohomology coming from the associated Albanese variety. Clearly $H^*_{\text{Alb}}(\Gamma; \mathbb{Q})$ can also be described as the subalgebra of $H^*(\Gamma; \mathbb{Q})$ generated by $H^1(\Gamma; \mathbb{Q})$.

**Albanese cohomology of the Torelli group.** Let $S_g^1$ be a compact, oriented genus $g \geq 2$ surface with one boundary component. The mapping class group $\text{Mod}(S_g^1)$ is the group of path components of the group $\text{Homeo}^+(S_g^1, \partial S_g^1)$ of orientation-preserving homeomorphisms of $S_g^1$ fixing the boundary pointwise. The action of $\text{Mod}(S_g^1)$ on $H_1(S_g^1; \mathbb{Z})$ preserves algebraic intersection number, which is a symplectic form on $H_1(S_g^1; \mathbb{Z})$. The Torelli group $T_g^1$ is defined to be the kernel of this action, and we have a well-known (see e.g. [FM]) exact sequence, where $\text{Sp}_{2g}$ is the integral symplectic group:

$$1 \to T_g^1 \to \text{Mod}(S_g^1) \to \text{Sp}_{2g} \mathbb{Z} \to 1$$

Very little is known about the cohomology $H^*(T_g^1; \mathbb{Q})$ of the Torelli group, or even about the subalgebra $H^i_{\text{Alb}}(T_g^1; \mathbb{Q})$. For $g = 2$, it follows from Mess [Me] that $H^1_{\text{Alb}}(T_g^1; \mathbb{Q}) = H^1(T_g^2; \mathbb{Q}) \simeq \mathbb{Q}^8$. For $g \geq 3$, Johnson proved that $T_g^1$ is finitely generated, and gave an isomorphism

$$H^1_{\text{Alb}}(T_g^1; \mathbb{Q}) = H^1(T_g^1; \mathbb{Q}) \simeq \bigwedge^3 \mathbb{Q}^{2g}$$

as $\text{Sp}_{2g}$-modules. Hain [Ha], §10 computed $H^2_{\text{Alb}}(T_g^1; \mathbb{Q})$ as an $\text{Sp}_{2g}$-module, and Sakasai [Sa] did the same for $H^3_{\text{Alb}}(T_g^1; \mathbb{Q})$. We found many examples of nontrivial classes in $H^i_{\text{Alb}}(T_g^1; \mathbb{Q})$ in [CF2]; these give a lower bound of order $g^{i+2}$ for the dimension of $H^i_{\text{Alb}}(T_g^1; \mathbb{Q})$. Beyond these coarse bounds, nothing is known about the precise dimension of $H^i_{\text{Alb}}(T_g^1; \mathbb{Q})$.

**Theorem 7.2.2.** For each $i \geq 0$ there exists a polynomial $P_i(T)$ of degree at most $3i$ such that $\dim H^i_{\text{Alb}}(T_g^1; \mathbb{Q}) = P_i(g)$ for $g \gg i$.

Although it follows from Johnson’s theorem that $\dim H^i_{\text{Alb}}(T_g^1; \mathbb{Q})$ grows no faster than $O(g^{3i})$, to say that this dimension coincides exactly with a polynomial for large $g$ is much stronger. We emphasize that the dimension of $H^i_{\text{Alb}}(T_g^1; \mathbb{Q})$ is unknown when $i \geq 3$; in particular we do not know what the polynomials produced by Theorem 7.2.2 are.

In order to prove Theorem 7.2.2 we will prove that $H^*_{\text{Alb}}(T_g^1; \mathbb{Q})$ is a graded FI-module of finite type. One novelty here is that the FI-module structure on $H^*_{\text{Alb}}(T_g^1; \mathbb{Q})$ is in some sense not natural, but its existence nevertheless allows us to prove Theorem 7.2.2.

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We say an FI-group up to conjugacy is a functor $\Gamma$ from FI to the category of groups and homomorphisms-up-to-conjugacy; in other words, for each finite set $S$ we have a group $\Gamma_S$, and for each $f: S \leftrightarrow T$ we have maps $f_*: \Gamma_S \to \Gamma_T$ so that the relevant diagrams commute up to conjugacy. Since conjugation acts trivially on group homology and cohomology, if $\Gamma$ is an FI-group up to conjugacy, then $H_\ast(\Gamma; k)$ is a graded FI-module, and $H^\ast(\Gamma; k)$ is a graded co-FI-algebra.

Proof of Theorem 7.2.3. We define the FI-group up to conjugacy $T^1_\ast$ as follows. Fix for each $g$ an ordered symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_1(S^1_g; \mathbb{Z})$. Any injection $f: \mathbf{m} \hookrightarrow \mathbf{n}$ determines an injection $f_*: H_1(S^1_m; \mathbb{Z}) \hookrightarrow H_1(S^1_n; \mathbb{Z})$ by $f_*(a_i) = a_{f(i)}$ and $f_*(b_i) = b_{f(i)}$. Any embedding $S^1_m \hookrightarrow S^1_n$ determines an inclusion $H_1(S^1_m; \mathbb{Z}) \hookrightarrow H_1(S^1_n; \mathbb{Z})$ as a symplectic summand. Such an embedding also determines an injection $T^1_m \hookrightarrow T^1_n$, given by extending each map to be the identity in the complement. Moreover, although there are many embeddings $S^1_m \hookrightarrow S^1_n$ inducing a given map on homology, Johnson [Jo2] Theorem 1A] proved that the resulting injections $T^1_m \hookrightarrow T^1_n$ are all conjugate in $T^1_n$. As a consequence, these maps $f_*: T^1_m \hookrightarrow T^1_n$ make $T^1_\ast$ into an FI-group up to conjugacy.

Of course, the co-FI-module structure on $H^\ast(T^1_\ast; Q)$ is not unique or canonical; it depends on the initial choice of bases for $H_1(S^1_g; \mathbb{Z})$. But all that matters for us is the existence of a co-FI-module structure on $H^\ast(T^1_\ast; Q)$ such that $H_1(T^1_\ast; Q) = H^1(T^1_\ast; Q)^\ast$ is a finitely generated FI-module.

Johnson’s identification in (24) comes from a natural isomorphism $H_1(T^1_\ast; Q) \simeq \bigwedge^3(H_1(S^1_3; Q))$. With respect to the FI-module structure above, we have $H_1(S^1_3; Q) \simeq M(1) \oplus M(1)$, so we obtain an isomorphism $H_1(T^1_\ast; Q) \simeq \bigwedge^3(M(1) \oplus M(1))$. In particular, $H_1(T^1_\ast; Q)$ is a finitely generated FI-module of weight 3 by Proposition 3.4.3. Proposition 4.2.4 implies that $H^\ast_{\text{Alb}}(T^1_\ast; Q)$ is a co-FI-algebra of finite type and slope $\leq 3$, and the desired conclusion follows from Theorem 3.3.4.

Albanese cohomology of IA$_n$. Let $F_n$ denote the free group of rank $n \geq 0$. The Torelli subgroup IA$_n$ of Aut($F_n$) is defined to be the subgroup consisting of those automorphisms that act trivially on $H_1(F_n; \mathbb{Z}) \simeq \mathbb{Z}^n$, giving the exact sequence $1 \to$ IA$_n$ $\to$ Aut($F_n$) $\to$ GL$_n \mathbb{Z} \to$ 1.

Magnus proved in 1934 that IA$_n$ is finitely generated for all $n \geq 0$. The conjugation action of Aut($F_n$) on IA$_n$ induces an action of GL$_n \mathbb{Z}$ on the homology groups $H_1(\text{IA}_n; \mathbb{Q})$. Farb, Cohen–Pakianathan and Kawazumi (see e.g. [Ka]) independently proved that for all $n \geq 0$, there exists a natural isomorphism $H_1(\text{IA}_n; \mathbb{Z}) \simeq \bigwedge^2 F_n^{ab} \otimes_{\mathbb{Z}} (F_n^{ab})^\ast$ (25)
as GL$_n \mathbb{Z}$-representations. As with the Torelli group, basically nothing is known about $H^\ast_{\text{Alb}}(\text{IA}_n; \mathbb{Q})$; other than $H^1_{\text{Alb}} = H^1$; the only progress is Pettet’s computation [Pe] of $H^\ast_{\text{Alb}}(\text{IA}_n; \mathbb{Q})$.

Theorem 7.2.3. For each $i \geq 0$ there exists a polynomial $P_i(T)$ of degree at most $3i$ such that $\dim H^i_{\text{Alb}}(\text{IA}_n; \mathbb{Q}) = P_i(n)$ for $n \gg i$.

Proof. The proof proceeds along much the same lines as that of Theorem 7.2.2. One simplifying factor is that in this case we can actually define an FI-group IA$_\ast$ as follows. To $\mathbf{n}$ we associate the group IA$_n$ defined above, and to an injection $f: \mathbf{m} \hookrightarrow \mathbf{n}$ we associate the following homomorphism $f_*: \text{IA}_m \hookrightarrow \text{IA}_n$. Let $f_*: F_m \hookrightarrow F_n$ denote the inclusion induced by $x_i \mapsto x_{f(i)}$. Given $\varphi \in \text{IA}_m$, we define $f_* \varphi \in \text{IA}_n$ to be the automorphism sending $x_{f(i)} \mapsto f_*(\varphi(x_i))$ and $x_j \mapsto x_j$ for those $j \not\in f(\mathbf{m})$. It is easy to check that $(f \circ g)_* = f_* \circ g_*$, so this defines an FI-group IA$_\ast$. In particular, $H^\ast(\text{IA}_\ast; \mathbb{Q})$ is a graded co-FI-algebra.
The isomorphism \((25)\) is provided by the Johnson homomorphism, which is compatible with the maps \(f_\ast\), so we have an isomorphism of FI-modules

\[
H_1(\mathcal{I}A\bullet; \mathbb{Q}) \simeq \bigwedge^2 M(1) \otimes M(1) \simeq M(\bigotimes) \oplus M(\bigoplus) \oplus M(\bigoplus). 
\]

The FI-module \(H_1(\mathcal{I}A\bullet; \mathbb{Q}) \simeq H^1(\mathcal{I}A\bullet; \mathbb{Q})\) is finitely generated of weight 3 (we could also deduce this from Propositions 2.3.6 and 3.4.3). By Proposition 4.2.4 the graded co-FI-module \(H^\ast_{\text{Ab}}(\mathcal{I}A\bullet; \mathbb{Q})\) is of finite type and slope \(\leq 3\). Theorem 3.3.4 now implies the theorem, and moreover that the character of \(H_1^\ast_{\text{Ab}}(\mathcal{I}A_n; \mathbb{Q})\) as an \(S_n\)-representation is eventually given by a character polynomial. \(\square\)

### 7.3 Graded Lie algebras associated to the lower central series

Recall that the lower central series

\[
\Gamma = \Gamma_1 > \Gamma_2 > \cdots
\]

of a group \(\Gamma\) is defined inductively by \(\Gamma_1 := \Gamma\) and \(\Gamma_{j+1} := [\Gamma, \Gamma_j]\). For simplicity, assume that \(k\) is a field. The associated graded Lie algebra of \(\Gamma\), denoted \(\text{gr}(\Gamma)\), is the Lie algebra over \(k\) defined by

\[
\text{gr}(\Gamma) := \bigoplus_{j=1}^{\infty} \text{gr}(\Gamma)_j = \bigoplus_{j=1}^{\infty} (\Gamma_j/\Gamma_{j+1}) \otimes k
\]

where the Lie bracket is induced by the group commutator.

When \(\Gamma\) is finitely generated the graded vector space \(\text{gr}(\Gamma)\) is of finite type. The natural action of the automorphism group \(\text{Aut}(\Gamma)\) on \(\Gamma\) preserves each \(\Gamma_j\), and so acts on \(\text{gr}(\Gamma)\). It is easy to see that this action factors through an action of \(\text{Aut}(\Gamma/\Gamma, \Gamma)\). In particular, the action of inner automorphisms on \(\text{gr}(\Gamma)\) is trivial, so the group \(\text{Out}(\Gamma)\) of outer automorphisms of \(\Gamma\) naturally acts on \(\text{gr}(\Gamma)\). The action on \(\text{gr}(\Gamma)_1\) in grading 1 is nothing more than the representation of \(\text{Aut}(H_1(\Gamma; \mathbb{Z})) = \text{Aut}(H_1(\Gamma; \mathbb{Z}))\) on \(H_1(\Gamma; k) \simeq H_1(\Gamma; \mathbb{Z}) \otimes k\).

The fundamental group of an FI-space. The fundamental group \(\pi_1\) is not a functor from topological spaces to groups, since it depends on a choice of basepoint; a map \(Y \to Z\) only determines a homomorphism \(\pi_1(Y) \to \pi_1(Z)\) up to conjugation. As a result, even though the configuration space \(\text{Conf}\bullet(M)\) from §6.2 is a co-FI-space, there is no co-FI-group \(\pi_1(\text{Conf}\bullet(M))\) (only a co-FI-group up to conjugacy).

However, since conjugation acts trivially on \(\text{gr}(\pi_1(Z))\), there is a functor from \(\text{Top}\) to the category of graded Lie algebras which sends a space \(Z\) to the graded Lie algebra \(\text{gr}(\pi_1(Z))\). Thus if \(X\) is an FI-space \(X\) (or even just a homotopy FI-space), the associated graded Lie algebra \(\text{gr}(\pi_1(X))\) is a graded FI-algebra. Similarly we do have a graded co-FI-algebra \(\text{gr}(\pi_1(\text{Conf}\bullet(M)))\) for any \(M\). When \(M\) is the interior of a manifold with boundary as in §6.4 Proposition 6.4.2 implies that \(\text{gr}(\pi_1(\text{Conf}\bullet(M)))\) is a graded FI-\#-algebra. In general, if \(\Gamma\) is an FI-group up to conjugacy then \(\text{gr}(\Gamma)\) is a graded FI-module.

**Theorem 7.3.1 (Finite generation of \(\text{gr}(\Gamma)\)).** If \(\Gamma\) is an FI-group up to conjugacy (e.g. the fundamental group of a homotopy FI-space) and the FI-module \(H_1(\Gamma; k)\) is finitely generated, the graded FI-module \(\text{gr}(\Gamma)\) is of finite type.

**Proof.** For any group \(\Gamma\), the terms \(\Gamma_i\) of the lower central series are by definition generated by iterated commutators of elements of \(\Gamma\). This shows that \(\text{gr}(\Gamma)\) is always generated as a Lie algebra by \(\text{gr}(\Gamma)_1 \simeq H_1(\Gamma; k)\). Thus if \(H_1(\Gamma; k)\) is finitely generated, Theorem 4.2.3 implies the graded FI-algebra \(\text{gr}(\Gamma)\) is of finite type. \(\square\)
Example 7.3.2 (Free groups). Let \( \Gamma \) be the FI\(_1\) group which assigns to \( S \) the free group \( F_S = \langle x_s \mid s \in S \rangle \), and to a morphism \( f: S \supset A \xrightarrow{\phi} B \subset T \) assigns the map \( F_S \rightarrow F_T \) given by \( \phi(x_s) = x_{\phi(s)} \) if \( i \in A \), and \( \phi(x_s) = 1 \) if \( i \in S - A \). It is easy to compute that \( \text{gr}(\Gamma)_1 \cong M(1) \). Since \( \text{gr}(\Gamma)_1 \) is finitely generated, Theorem 7.3.1 implies that \( \text{gr}(\Gamma) \) is of finite type. It is known that \( \text{gr}(F_n) \) is isomorphic to the free Lie algebra on \( n \) variables. Applying Theorem 1.13, we conclude that the graded pieces of the free Lie algebra on \( n \) variables are representation stable as representations of \( S_n \). A variant of this result, for \( \text{GL}_n \) \( C \)-representations rather than \( S_n \)-representations, was proved in [CF, Corollary 5.7].

Example 7.3.3 (Pure braid groups). We proved in Proposition 6.4.2 that the configuration space \( \text{Conf}_n(\mathbb{R}^2) \) of ordered \( n \)-tuples of distinct points in the plane is a homotopy FI\(_1\)-space. The pure braid group \( P_n \) on \( n \) strands is the fundamental group \( P_n = \pi_1(\text{Conf}_n(\mathbb{R}^2)) \), so this shows that \( P \) is an FI\(_1\)-group up to conjugacy. We proved in Example 5.1.3 that \( H_1(P_n; \mathbb{Q}) \cong \mathbb{M}(\square) \), so Theorem 7.3.1 implies that \( \text{gr}(P_n) \) is isomorphic to the Malcev Lie algebra of \( P_n \). Applying Theorem 1.13 to Example 7.3.3 implies the following theorem, which confirms Conjecture 5.15 of [CF].

Theorem 7.3.4 (Representation stability for \( p_n \)). For each fixed \( i \geq 1 \), the sequence \( \{p_n^i\} \) of grading-\( i \) pieces of \( p_n \) is a uniformly representation stable sequence of \( S_n \)-representations.

Drinfeld–Kohno (see [Ko]) actually found an explicit presentation of \( p_n \); for another approach to Theorem 7.3.4, we could apply Theorem 4.2.3 directly to their presentation.

Remark 7.3.5. The pure braid groups \( P_n \) are examples of pseudo-nilpotent groups, so that \( H^i(p_n; \mathbb{Q}) \cong H^i(P_n; \mathbb{Q}) \) for all \( n \). It was already proved in [CF] that the sequence \( \{H^i(P_n; \mathbb{Q})\} \) is uniformly representation stable for each \( i \geq 0 \), so one is tempted to derive Theorem 7.3.4 directly from [CF, Theorem 5.3], which states the equivalence of uniform representation stability for a Lie algebra and for its (co)homology. However, that theorem was only proved in [CF] in the context of stability for \( \text{SL}_n \) \( C \)-representations and \( \text{GL}_n \) \( C \)-representations. Indeed the “strong stability” hypothesis assumed in that theorem almost never holds for \( S_n \)-representations (and it does not hold here).

Example 7.3.6 (The Torelli group). In [7.2] we encountered the Torelli group \( T_g^1 \). The action of the mapping class group \( \text{Mod}(S_g^1) \) on \( T_g^1 \) by conjugation induces a well-defined action of \( \text{Sp}_{2g} \mathbb{Z} \) on \( \text{gr}(T_g^1) \), taking \( k = \mathbb{Q} \). A finite presentation for \( \text{gr}(T_g^1) \) as a Lie algebra has been given by Habegger–Sorger [HS], extending the fundamental computation of Hain [Ha] in the case of closed surfaces. Hain also worked out the first few graded terms of this Lie algebra explicitly as \( \text{Sp}_{2g} \mathbb{Z} \)-representations. Getzler–Garoufalidis (personal communication) have recently given more detailed computations in this direction. However, exact computations in arbitrary degrees seem out of reach. The situation for \( \text{gr}(IA_n) \) as a \( \text{GL}_n \mathbb{Z} \)-representation is the same. Even so, we have the following theorems.

Theorem 7.3.7. For each \( i \geq 0 \) the dimension \( \dim(\text{gr}(T_g^1)^i) \) is polynomial in \( g \) for \( g \gg i \).

Theorem 7.3.8. For each \( i \geq 0 \) the dimension \( \dim(\text{gr}(IA_n)^i) \) is polynomial in \( n \) for \( n \gg i \).

Proof. We showed in the proof of Theorems 7.2.2 and 7.2.3 that \( T_g^1 \) is an FI-group up to conjugacy and \( IA_\bullet \) is an FI-group. Moreover, we saw that the FI-modules \( H_1(T_g^1; \mathbb{Q}) \cong \bigwedge^3(M(1) \oplus M(1)) \) and \( H_1(IA_n; \mathbb{Q}) \cong \bigwedge^2 M(1) \otimes M(1) \) are finitely generated. Theorem 7.3.1 now implies that the graded FI-algebras \( \text{gr}(T_g^1) \) and \( \text{gr}(IA_\bullet) \) are of finite type, and the claim follows from Theorem 3.3.4. \( \square \)
References

[AM] M. Aguiar and S. Mahajan, *Monoidal Functors, Species, and Hopf Algebras*, CRM Monograph Series, Vol 29 (2010).

[AAB] S. Ashraf, H. Azam and B. Berceanu, Representation stability of power sets and square free polynomials, arXiv:1106.4926v1, preprint 2011.

[Ar] V.I. Arnol’d, The cohomology ring of the colored braid group, *Mathematical Notes* 5, no. 2 (1969), 138–140.

[Be] F. Bergeron, Multivariate diagonal coinvariant spaces for complex reflection groups, *Adv. Math.* 239 (2013), 97–108. arXiv:1105.4358

[BS] M. Bhargava and M. Satriano, On a notion of “Galois closure” for extensions of rings, to appear in *J. Eur. Math. Soc.* arXiv:1006.2562.

[Che] C. Chevalley, Invariants of finite groups generated by reflections, *Amer. J. Math.* 77 (1955), 778–782.

[CE] T. Church and J. S. Ellenberg, Homology of FI-modules and stability, in preparation.

[CEF] T. Church, J. S. Ellenberg, and B. Farb, Representation stability in cohomology and asymptotics for families of varieties over finite fields, *Contemp. Math.* 620 (2014), 1–54. arXiv:1309.6038

[CEFN] T. Church, J. S. Ellenberg, B. Farb, and R. Nagpal, FI-modules over Noetherian rings, to appear in *Geometry & Topology*. arXiv:1210.1854

[CF] T. Church and B. Farb, Representation theory and homological stability, *Adv. Math.* (2013), 250–314. arXiv:1008.1368

[CF2] T. Church and B. Farb, Parameterized Abel–Jacobi maps and abelian cycles in the Torelli group, *J. Topol.* 5 (2012), no. 1, 15–38. arXiv:1001.1114.

[Ch] T. Church, Homological stability for configuration spaces of manifolds, *Invent. Math.* 188 (2012), no. 2, 465–504. arXiv:1103.2441

[CP] T. Church and A. Putman, Generating the Johnson filtration, arXiv:1311.7150, preprint 2013.

[De] P. Deligne, La Catégorie des Représentations du Groupe Symétrique $S_t$, lorsque $t$ n’est pas un Entier Naturel, in *Algebraic groups and homogeneous spaces*, 209–273, Tata Inst. Fund. Res. Stud. Math., Mumbai, 2007.

[DK] J. Draisma and J. Kuttler, Bounded-rank tensors are defined in bounded degree, *Duke Math. J.* 163 (2014), no. 1, 35–63, 2014. arXiv:1103.5336.

[DV] A. Djament and C. Vespa, Sur l’homologie des groupes orthogonaux et symplectiques à coefficients tordus, *Ann. Sci. École Norm. Sup* 43 (2010) 4, 395–459. arXiv:0808.4035

[ES] D. Eisenbud and D. Saltman, Rank varieties of matrices, in *Commutative Algebra*, ed. by M. Hochster, C. Huneke and J.D. Sally, Springer-Verlag, 1989.

[Et] P. Etingof, Representation theory in complex rank, Conference talk at the Isaac Newton Institute for Mathematical Sciences, 2009. Available at: http://www.newton.ac.uk/programmes/ALT/seminars/032716301.html

[FM] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, Princeton Mathematical Series 49, Princeton University Press, 2012.

[Fo] O. Forster, Über die Anzahl der Erzeugenden eines Ideals in einem Noetherschen Ring, *Math. Z.* 84 (1964), 80–87.
[FH] W. Fulton and J. Harris, *Representation theory. A first course*, Graduate Texts in Mathematics 129, Readings in Mathematics, Springer-Verlag, New York, 1991.

[FMac] W. Fulton and R. MacPherson, A compactification of configuration spaces, *Ann. Math* 139 (1994) 1, 183–225.

[GG] A. Garsia and A. Goupil, Character polynomials, their $q$-analogs and the Kronecker product, *Elect. Jour. Combin.* 16(2) 2009, #R19.

[GL] W. L. Gan and L. Li, Noetherian property of infinite EI categories. preprint 2014.

[Ha] R. Hain, Infinitesimal presentations of the Torelli groups, *J. Amer. Math. Soc.* 10 (1997), no. 3, 597–651.

[Hai] M. Haiman, Combinatorics, symmetric functions, and Hilbert schemes, in *Current developments in mathematics, 2002*, 39–111, Int. Press, Somerville, MA, 2003.

[Hai2] M. Haiman, Vanishing theorems and character formulas for the Hilbert scheme of points in the plane, *Invent. Math.* 149 (2002), 371–407. arXiv:math/0201148

[HHLRU] J. Haglund, M. Haiman, N. Loehr, J. Remmel and A. Ulyanov, A combinatorial formula for the character of the diagonal coinvariants, *Duke Math. J.* 126 (2005), pp. 195–232. arXiv:math/0310424

[HS] N. Habegger and C. Sorger, An infinitesimal presentation of the Torelli group of a surface with boundary, preprint 2000.

[JR1] R. Jimenez Rolland, Representation stability for the cohomology of moduli space $M_{g,n}$, *Algebr. Geom. Topol.* 11 (2011), 3011–3041. arXiv:1106.0947

[JR2] R. Jimenez Rolland, On the cohomology of pure mapping class groups as FI-modules, to appear in *J. Homotopy Relat. Struct.* arXiv:1207.6828.

[Jo2] D. Johnson, Conjugacy relations in subgroups of the mapping class group and a group-theoretic description of the Rochlin invariant, *Math. Ann.* 249 (1980), no. 3, 243–263.

[Ka] N. Kawazumi, Cohomological aspects of Magnus expansions. arXiv:math/0505497v3 preprint 2006.

[Ku] F. Knop, Tensor envelopes of regular categories, *Adv. Math.* 214 (2007) 2, 571–617. arXiv:math/0610552

[KT] A. Knutson and T. Tao, The honeycomb model of $GL_n(C)$ tensor products I: Proof of the saturation conjecture, *J. Amer. Math. Soc.* 12 (1999) 4, 1055–1090. arXiv:math/9807160.

[Ko] T. Kohno, Série de Poincaré-Koszul associée aux groupes de tresses pures, *Invent. Math.* 82 (1985), 57–75.

[Li] D.E. Littlewood, Products and plethysms of characters with orthogonal, symplectic and symmetric groups, *Canad. J. Math.* 10 (1958), 17–32.

[L] W. Lück, *Transformation Groups and Algebraic K-Theory*, Lecture Notes in Mathematics 1408, Mathematika Gottingensis, Springer-Verlag, Berlin, 1989.

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Math. Mon., Clarendon Press, Oxford, 1995.

[McD] D. McDuff, Configuration spaces of positive and negative particles, *Topology* 14 (1975), 91–107.

[MacL] S. Mac Lane, *Categories for the working mathematician*, Springer GTM Series Vol. 5, 2nd ed., 1998.
G. Mess, The Torelli groups for genus 2 and 3 surfaces, Topology 31 (1992), 775–790.

A. Pettet, The Johnson homomorphism and the second cohomology of \( \text{IA}_n \), Algebr. Geom. Topol. 5 (2005), 725–740. arXiv:math/0501053

T. Pirashvili, Dold-Kan type theorem for \( \Gamma \)-groups, Math. Ann. 318 (2000) 2, 277–298.

A. Putman, Stability in the homology of congruence subgroups, arXiv:1201.4876v4, preprint 2012.

A. Putman and S. Sam, Representation stability and finite linear groups, arXiv:1408.3694v1 preprint 2014.

Q. Ren and T. Schedler, On the asymptotic \( S_n \)-structure of invariant differential operators on symplectic manifolds, J. Algebra 356 (1) 39–89. arXiv:1006.0268

T. Sakasai, The Johnson homomorphism and the third rational cohomology group of the Torelli group, Topology and its Applications 148 (2005) 83–111.

S. Sagave and C. Schlichtkrull, Diagram spaces and symmetric spectra, Adv. Math. 231 (2012), no. 3-4, 2116–2193. arXiv:1103.2764

S. Sam and A. Snowden, GL-equivariant modules over polynomial rings in infinitely many variables, arXiv:1206.2233v2 preprint 2013.

S. Sam and A. Snowden, Introduction to twisted commutative algebras, arXiv:1209.5122v1 preprint 2012.

S. Sam and A. Snowden, Gröbner methods for representations of combinatorial categories, preprint 2014.

G. Segal, The topology of spaces of rational functions, Acta Mathematica 143 (1979), no. 1, 39–72.

A. Snowden, Syzygies of Segre embeddings and \( \Delta \)-modules, Duke Math J. 162 (2013), no. 2, 225–277. arXiv:1006.5248

L. Solomon, Representations of the rook monoid, J. Algebra 256 (2002), no. 2, 309–342.

B. Totaro, Configuration spaces of algebraic varieties, Topology 35 (1996), no. 4, 1057–1067.

R. Vakil, The moduli space of curves and its tautological ring, Notices Amer. Math. Soc. 50 (2003), no. 6, 647-658. Available at: http://www.ams.org/notices/200306/fea-vakil.pdf

R. Vakil and M.M. Wood, Discriminants in the Grothendieck ring, arXiv:1208.3166v1 preprint 2012.

C. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, Cambridge, 1994.

J. Weyman, Cohomology of Vector Bundles and Syzygies, Cambridge Tracts in Math., Vol. 149, Cambridge Univ. Press, 2003.

H. Weyl, The Classical Groups, their Invariants and Representations, Princeton Math. Series, Vol. 1, Princeton, 1946.

J. Wilson, Representation stability for the cohomology of the pure string motion groups, Alg. Geom. Top. 12 (2012) 909–931. arXiv:1108.1255

J. Wilson, \( \text{FI}_W \)-modules and stability criteria for representations of classical Weyl groups. Available at: http://math.uchicago.edu/~wilsonj/research.html
[Wi3] J. Wilson, $\text{FI}_W$-modules and constraints on classical Weyl group characters. Available at: http://math.uchicago.edu/~wilsonj/research.html

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