Unconditional Convergence of Conservative Spectral Galerkin Methods for the Coupled Fractional Nonlinear Klein–Gordon–Schrödinger Equations

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Abstract
In this work, two novel classes of structure-preserving spectral Galerkin methods are proposed which based on the Crank–Nicolson scheme and the exponential scalar auxiliary variable method respectively, for solving the coupled fractional nonlinear Klein–Gordon–Schrödinger equation. The paper focuses on the theoretical analyses and computational efficiency of the proposed schemes, the Crank–Nicolson scheme is proved to be unconditionally convergent and has maximum-norm boundness of numerical solutions. The exponential scalar auxiliary variable scheme is linearly implicit and decoupled, but lack of the maximum-norm boundness, also, the energy structure is modified. Subsequently, the efficient implementations of the proposed schemes are introduced in detail. Both the theoretical analyses and the numerical comparisons show that the proposed spectral Galerkin methods have high efficiency in long-time computations.

Keywords Riesz fractional derivative · Spectral Galerkin method · Structure-preserving algorithm · Unique solvability · Convergence

Mathematics Subject Classification 26A33 · 70H0 · 65M125 · 65N35

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1 Introduction

Fractional differential equations have been widely applied in many fields due to the non-local properties of fractional calculus, they can be employed to simulate some complex anomalous (super) diffusion phenomena which cannot be simulated by classical differential equations, and gained many researches and considerations by physicists and mathematicians, such that there has created extensive excellent theoretical achievements and realistic value. For instance, Laskin [20] proposed a class of space-fractional nonlinear Schrödinger equation by adopting $\alpha$-stable Lévy process instead of Brownian process, which plays an essential role in quantum mechanics, water wave dynamics and optics [18, 21] etc. Almost simultaneously, Alfmov and Vázquez et al. [1] established a kind of space-fractional nonlinear wave equation, which is utilized to describe the propagation of fluxons in Josephson junctions. Recently, the study on the well-posedness of solutions of nonlinear partial differential equations [9, 25]. Therefore, it is essential to develop some efficient and stable numerical methods for fractional differential equations [2, 10, 26, 36, 40, 41]. Interestingly, a special dynamical system, called as the Klein–Gordon–Schrödinger equation which consists of the Schrödinger equation and the wave equation in the normal region [22, 34, 42] or the nonrelativistic limit regime [7], has attracted considerable attention in recent years. In this paper, we focus on effective spectral Galerkin methods for the following strong coupled fractional nonlinear Klein–Gordon–Schrödinger (FNKGS) equation:

$$
\begin{align*}
    iu_t - \frac{\lambda}{2} (-\Delta)^{\alpha} u + \kappa_1 u \phi + 2\kappa_2 |u|^2 u \phi &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
    \phi_{tt} + \gamma (-\Delta)^{\beta} \phi + \eta^2 \phi - \kappa_1 |u|^2 - \kappa_2 |u|^4 &= 0 \quad \text{in} \quad \Omega \times (0, T),
\end{align*}
$$

subject to the initial-boundary conditions

$$
\begin{align*}
    u(x, 0) &= u_0(x), \quad \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x) \quad \text{in} \quad \Omega, \\
    u(x, t) &= 0, \quad \phi(x, t) = 0 \quad \text{on} \quad \partial\Omega \times (0, T).
\end{align*}
$$

The system (1.1)–(1.4) is the fractional model describing the dynamics of conserved complex nucleon fields $u$ interacting with neutral real scalar meson fields $\phi$ in the bounded domain $\Omega = (a, b)$, where $1 < \alpha < 2$, $i^2 = -1$, $\lambda, \kappa_1, \kappa_2, \gamma \geq 0$ and $\eta \in \mathbb{R}$ are any given constants, $u_0(x), \phi_1(x)$ and $\phi_1(x)$ are given smooth functions. The Riesz fractional derivative is represented by

$$
(-\Delta)^{\alpha} u = \frac{1}{2 \cos \left( \frac{a \pi}{2} \right)} \left( RL D_x^\alpha + RL D_y^\beta \right) u,
$$

where the left and right Riemann–Liouville fractional derivatives are defined as

$$
\begin{align*}
    RL D_x^\alpha u &= \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_a^x u(\xi, t) d\xi, \\
    RL D_y^\beta u &= \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_x^b u(\xi, t) d\xi,
\end{align*}
$$

in which $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$.

It is worth noticing that the solutions of FNKGS equation conserve the mass of the nucleon field as follows:

$$
\mathcal{M}(t) = \int_\Omega |u|^2 dx = \mathcal{M}(0), \quad t \in (0, T],
$$

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and the total energy as follows:

\[
E(t) = \int_{\Omega} (\phi_t)^2 + \gamma \phi (-\Delta)^{\frac{\alpha}{2}} \phi + \eta^2 \phi^2 + \lambda \tilde{u} (-\Delta)^{\frac{\alpha}{2}} u (2\kappa_1 |u|^2 + 2\kappa_2 |u|^4) \phi dx \\
= E(0), \quad t \in (0, T],
\]

where \( \tilde{u} \) is the complex conjugate of \( u \).

Our object is to investigate efficient numerical schemes and the associated convergence analyses of the FNKGS equation. The main difficulties of the studies on numerical schemes for the FNKGS equation can be summarized as follows: (i) The proposed numerical algorithms should preserve the inherent physical invariants as much as possible. (ii) The unconditional convergence of numerical methods should be discussed without any Lipschitz continuity restrictions on nonlinear terms of original system. (iii) The numerical scheme should be linearly implicit and decoupled, also, the convergence can be obtained. Based on the aforementioned issues, we try to construct a numerical scheme which enjoys high-order accuracy in space and high efficiency in time, for solving the FNKGS equation to overcome some of the above challenges.

As is known to us, structure-preserving algorithm is famous for preserving the inherent invariants of nonlinear conservative systems. In generally, traditional numerical algorithms are proposed to solve the conservative system, which often ignore the intrinsic conservative structure of the system. Although they have the same stable computational ability as the structure-preserving algorithms in a short time, the latter are superior in long-time simulations because they can inherit the intrinsic physical characteristics of a given dynamical system, please refer to [8, 35] and references therein. Undoubtedly, time-stepping method is the key of the construction of structure-preserving algorithms. For classical examples, the averaged vector field method [28], the symplectic Runge–Kutta method [13], the Hamiltonian boundary value method [6] and so on have caught much attention in recent years. However, the above methods share a common drawback that the numerical implementations are usually fully implicit and the convergence analyses are hard to obtain. It is worth noticing that some novel tools are developed to provide the possibilities for constructing the high-order linearly implicit structure-preserving methods. For instances, readers can refer to the invariant energy quadratization (IEQ) method [38], the scalar auxiliary variable (SAV) method [31], the exponential scalar auxiliary variable (ESAV) method [24] and so forth, which possess a modified energy function. In recent years, there exist some numerical researches on the FNKGS equation. For examples, Fu and Cai et al. [12] proposed a conservative scheme by combining the partitioned averaged vector field method and pseudo spectral method. The numerical scheme is proved to be linearly implicit and decoupled, but the convergence analysis is unacquirable. Li and Huang et al. [23] derived a class of structure-preserving scheme with the leapfrog method and standard finite element method. Although the convergence is obtained, the energy structure of the proposed method is modified and it suffers a low-order accuracy in space. Simultaneously, Shi and Ma et al. [32] constructed a kind of numerical scheme by using the leapfrog method and center difference method, which is linearly implicit and unconditionally convergent, but the energy structure is still modified and the long-time conservation of the proposed scheme is less efficient.

The purpose of this paper is to construct a conservative Crank–Nicolson spectral Galerkin method (CN-SGM) which enjoys the original mass and energy structures as well as spectral accuracy in space for the FNKGS equation. Meanwhile, we discuss the unconditional convergence analysis and unique solvability. It is worth noticing that the existing theoretical result [5, 33] shows that the solution of the FNKGS equation blows up in finite time if the initial
energy $\mathcal{E}(0) < 0$. For numerical comparisons, we propose the other conservative numerical scheme by combining the exponential scalar auxiliary variable method and spectral Galerkin method (ESAV-SGM). Extensive numerical results illustrate two numerical phenomena: (1) The ESAV-SGM is linearly implicit and decoupled, it possesses the higher computational efficiency than CN-SGM. (2) Although the CN-SGM is fully implicit and coupled, it is more accurate and enjoys the superior capability of capturing blow-up than ESAV-SGM when $\mathcal{E}(0) < 0$.

The rest of this paper are arranged as follows: In Sect. 2, some essential definitions and lemmas of the Galerkin method are introduced. In Sect. 3, a class of structure-preserving numerical method is constructed by combining the spectral Galerkin method and the Crank–Nicolson method. Then the invariant conservations and unconditional convergence are proved. In Sect. 4, a linearly implicit and decoupled ESAV-SGM is derived for numerical comparison. Subsequently, the detailed numerical implementations of the proposed schemes are offered in Sect. 5. Next, numerical results are reported to illustrate the high efficiency of the proposed schemes in Sect. 6. Finally, some conclusions are drawed in Sect. 7.

2 Preliminaries

In this section, we introduce some essential definitions and lemmas of Galerkin methods which can be found in Refs. [27, 29].

Definition 2.1 Define the inner product, $L^p$ norm ($1 \leq p < \infty$) and $L^\infty$ norm as

$$(u, v) := \int_\Omega \tilde{v} u \, dx, \quad \|u\| := \sqrt{(u, u)}, \quad \|u\|_p := \sqrt[p]{\int_\Omega |u|^p \, dx}, \quad \|u\|_\infty := \text{ess sup}_{x \in \Omega} \{ |u(x)| \}.$$  

Definition 2.2 (Symmetric fractional derivative space) For $\mu > 0$, where $\mu \neq n - 1/2$, $n \in \mathbb{N}$, define the semi-norm and the norm as

$$\|u\|_{J^\mu_S(\Omega)} := \left( \frac{a D^\mu_x u, b D^\mu_x u}{\|a D^\mu_x u\|^2 + \|b D^\mu_x u\|^2} \right)^{1/2}, \quad \|u\|_{J^\mu_S(\Omega)} := \left( \|u\|^2 + \|u\|^2_{J^\mu_S(\Omega)} \right)^{1/2}. \quad J^\mu_S(\Omega) \quad (or \quad J^\mu_{S,0}(\Omega)) \quad \text{denotes the closure of} \quad C^\infty(\Omega) \quad \text{or} \quad C_0^\infty(\Omega) \quad \text{with respect to} \quad \| \cdot \|_{J^\mu_S(\Omega)}.$$  

Definition 2.3 (Fractional Sobolev space) For $\mu > 0$, define the semi-norm and the norm as

$$\|u\|_{H^\mu(\Omega)} := \|\xi|^\mu \mathcal{F} [u, \xi] \|, \quad \|u\|_{H^\mu(\Omega)} := \left( \|u\|^2 + \|u\|^2_{H^\mu(\Omega)} \right)^{1/2}. \quad H^\mu(\Omega) \quad (or \quad H^\mu_0(\Omega)) \quad \text{denotes the closure of} \quad C^\infty_0(\Omega) \quad \text{with respect to} \quad \| \cdot \|_{H^\mu(\Omega)}, \quad \text{and} \quad \mathcal{F} \quad \text{represents the Fourier transform}.$$  

Lemma 2.1 Suppose $\mu > 0$ and $\mu \neq n - 1/2$, $n \in \mathbb{N}$. Then the spaces $J^\mu_S(\Omega)$ and $H^\mu(\Omega)$ are equal with equivalent semi-norms and norms, and the spaces $J^\mu_{S,0}(\Omega)$ and $H^\mu_0(\Omega)$ are equal with equivalent semi-norms and norms.

Lemma 2.2 Suppose $1 < \mu < 2$. For any $u \in H^{\mu}_0(\Omega)$ and $v \in H^{\mu/2}_0(\Omega)$, we have

$$\left( a D^\mu_x u, v \right) = \left( a D^{\mu/2}_x u, a D^{\mu/2}_x v \right), \quad \left( x D^\mu_b u, v \right) = \left( x D^{\mu/2}_b u, x D^{\mu/2}_b v \right).$$  

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3 Crank–Nicolson Spectral Galerkin Method

Define the following bilinear forms
\[ B(u, w) = \frac{1}{2 \cos(\frac{\alpha \pi}{2})} \left( (RLD_{b}^{\alpha/2} u, RL D_{b}^{\alpha/2} w) + (RLD_{b}^{\alpha/2} u, RL D_{b}^{\alpha/2} w) \right), \quad \forall u, w \in H_{0}^{\alpha/2}(\Omega). \]

For simplicity, introduce the semi-norms and the norms as
\[ \| u \|_{\alpha/2} := \sqrt{B(u, u)}, \quad \| u \|_{\alpha/2} := \sqrt{\| u \|^{2} + \| u \|^{2}_{\alpha/2}}. \]

We observe from Lemma 2.1 that \( | \cdot |_{\alpha/2} \) and \( \| \cdot \|_{\alpha/2} \) are equivalent with the semi-norms and norms of \( J_{S}^{\mu}(\Omega) \) and \( H^{H}(\Omega) \). From Ref. [29], the bilinear form \( B(u, w) \) has the continuous and coercive properties, i.e., there exist positive constants \( C_{1} \) and \( C_{2} \), for any \( u, w \in H_{0}^{\alpha/2}(\Omega) \), such that
\[ |B(u, w)| \leq C_{1} \| u \|_{\alpha/2} \| w \|_{\alpha/2}, \quad |B(u, u)| \geq C_{2} \| u \|_{\alpha/2}^{2}. \]

Based on Lemma 2.2 and the definition of Riesz fractional derivative, the weak form of (1.1) reads: finding \( u, \phi \in H_{0}^{\alpha/2}(\Omega) \), such that
\begin{align*}
\frac{d}{dt} u(t) &- \frac{\lambda}{2} B(u, w) + \kappa_{1}(u\phi, w) + 2\kappa_{2}(|u|^{2}u\phi, w) = 0, \quad \forall w \in H_{0}^{\alpha/2}(\Omega), \\
(\phi_{t}, w) + \gamma B(\phi, w) + \eta^{2}(\phi, w) - \kappa_{1}(|u|^{2}, w) - \kappa_{2}(|u|^{4}, w) &= 0, \quad \forall w \in H_{0}^{\alpha/2}(\Omega),
\end{align*}
with the initial conditions given by
\[ u(x, 0) = u_{0}(x), \quad \phi(x, 0) = \phi_{0}(x), \quad \phi_{t}(x, 0) = \phi_{1}(x). \]

3.1 Fully Discrete Scheme

Introduce \( \psi = \phi \) to reformulate (3.1)–(3.2) into an equivalent system as follows, i.e., finding \( u, \phi, \psi \in H_{0}^{\alpha/2}(\Omega) \), for any \( w \in H_{0}^{\alpha/2}(\Omega) \), such that
\begin{align*}
\frac{d}{dt} u(t) &- \frac{\lambda}{2} B(u, w) + \kappa_{1}(u\phi, w) + 2\kappa_{2}(|u|^{2}u\phi, w) = 0, \\
(\phi_{t}, w) &= (\psi, w), \\
(\psi_{t}, w) + \gamma B(\phi, w) + \eta^{2}(\phi, w) - \kappa_{1}(|u|^{2}, w) - \kappa_{2}(|u|^{4}, w) &= 0.
\end{align*}

For simplicity, we introduce the following notations for \( n = 0, 1, \ldots, N_{t} \),
\[ u^{n} = u(x, t_{n}), \quad \delta_{t} u^{n+\frac{1}{2}} = \frac{u^{n+1} - u^{n}}{\tau}, \quad u^{n+\frac{3}{2}} = \frac{u^{n+1} + u^{n}}{2}, \quad u^{n+\frac{1}{2}} = \frac{3u^{n} - u^{n-1}}{2}, \]
where \( t_{n} = n \tau, \) and \( \tau = T / N_{t} \) is the time step size. We denote \( u^{n} \) and \( U_{N}^{n} \) as the exact solution and numerical approximation at \( t = t_{n} \), respectively. Without loss of generality, let \( C \) be a general positive constant which is independent of \( \tau \) and \( N \). For a fixed positive integer \( N \), denote \( P_{N}(\Omega) \) to be the polynomial space with the degree no more than \( N \) in interval \( \Omega \). The approximation space \( X_{N}^{0}(\Omega) \) is defined as
\[ X_{N}^{0}(\Omega) = P_{N}(\Omega) \cap H_{0}^{\alpha/2}(\Omega). \]
It is clear that $X^0_N(\Omega)$ is a subspace of $H^{1/2}_0(\Omega)$. By using the modified Crank–Nicolson scheme for temporal derivative in (3.4)–(3.6), the Crank–Nicolson scheme with truncation errors reads: finding $u^n, \psi^n, \phi^n, \in H^{1/2}_0(\Omega)$, for any $w \in H^{1/2}_0(\Omega)$, such that

$$i \left( \partial_t u^{n+1/2} + w \right) - \frac{\lambda}{2} B (u^{n+1/2}, w) + \kappa_1 (u^{n+1/2} \phi^{n+1/2}, w) + \kappa_2 \left( (|u^{n+1}|^2 + |u^n|^2) u^{n+1/2} \phi^{n+1/2}, w \right) = \left( \hat{\mathcal{R}}_1^n, w \right),$$  \hspace{1cm} (3.7)

$$\left( \partial_t \phi^{n+1/2}, w \right) = \left( \psi^{n+1/2}, w \right) + \left( \hat{\mathcal{R}}_2^n, w \right),$$ \hspace{1cm} (3.8)

$$\left( \partial_t \psi^{n+1/2}, w \right) + \gamma B (\phi^{n+1/2}, w) + \eta^2 (\phi^{n+1/2}, w) - \frac{\kappa_1}{2} \left( (|u^{n+1}|^2 + |u^n|^2) \phi^{n+1/2}, w \right) - \frac{\kappa_2}{2} \left( (|u^{n+1}|^4 + |u^n|^4) \phi^{n+1/2}, w \right) = \left( \hat{\mathcal{R}}_3^n, w \right).$$ \hspace{1cm} (3.9)

Using the Taylor’s expansion, suppose $u(\cdot, t), \phi(\cdot, t) \in C^3([0, T])$. One easily arrives at the following truncation errors

$$\left| \left( \hat{\mathcal{R}}_1^n \right) \right| \leq C \tau^2, \quad \left| \left( \hat{\mathcal{R}}_2^n \right) \right| \leq C \tau^2, \quad \left| \left( \hat{\mathcal{R}}_3^n \right) \right| \leq C \tau^2.$$ \hspace{1cm} (3.10)

Similar to the proof of [17], assume that $\phi(\cdot, t) \in C^4([0, T])$, one has

$$\left| \delta_t \left( \hat{\mathcal{R}}^{n+1/2}_1 \right) \right| \leq C \tau^2,$$ \hspace{1cm} (3.11)

whose proof is collected in “Appendix A”. Omitting the residual terms in (3.7)–(3.9) and replacing the exact solutions $u^n, \psi^n$ and $\phi^n$ with the numerical solutions $U^n_N, \Psi^n_N$ and $\Phi^n_N$, we propose the fully discrete CN-SGM scheme as follows: finding $U^n_N, \Psi^n_N, \Phi^n_N \in X^n_N(\Omega)$, for any $w_N \in X^0_N(\Omega)$, such that

$$i \left( \partial_t U^{n+1/2}_N + w_N \right) - \frac{\lambda}{2} B (U^{n+1/2}_N, w_N) + \kappa_1 \left( U^{n+1/2}_N \Phi^{n+1/2}_N, w_N \right) + \kappa_2 \left( (|U^{n+1}_N|^2 + |U^n_N|^2) U^{n+1/2}_N \Phi^{n+1/2}_N, w_N \right) = 0,$$ \hspace{1cm} (3.12)

$$\left( \partial_t \Phi^{n+1/2}_N, w_N \right) = \left( \Psi^{n+1/2}_N, w_N \right),$$ \hspace{1cm} (3.13)

$$\left( \partial_t \Psi^{n+1/2}_N, w_N \right) + \gamma B \left( \Phi^{n+1/2}_N, w_N \right) + \eta^2 \left( \Phi^{n+1/2}_N, w_N \right) - \frac{\kappa_1}{2} \left( (|U^{n+1}_N|^2 + |U^n_N|^2) \Phi^{n+1/2}_N, w_N \right) - \frac{\kappa_2}{2} \left( (|U^{n+1}_N|^4 + |U^n_N|^4) \Phi^{n+1/2}_N, w_N \right) = 0,$$ \hspace{1cm} (3.14)

with the initial conditions

$$U^0_N = \Pi^{0,\alpha/2}_N u_0(x), \quad \Phi^0_N = \Pi^{0,\alpha/2}_N \phi_0(x), \quad \Psi^0_N = \Pi^{0,\alpha/2}_N \phi_1(x),$$ \hspace{1cm} (3.15)

where the orthogonal projection operator $\Pi^{0,\alpha/2}_N: H^{\alpha/2}_0(\Omega) \rightarrow X^0_N(\Omega)$ is defined as

$$B (u - \Pi^{0,\alpha/2}_N u, w_N) = 0, \quad \forall w_N \in X^0_N(\Omega).$$

### 3.2 Conservations and Boundness

**Theorem 3.1** (Mass and energy conservative laws) The spectral Galerkin scheme (3.12)–(3.14) possesses the mass and energy conservative laws in the following discrete sense that

$$M^n = \cdots = M^0, \quad E^n = \cdots = E^0, \quad n = 1, 2, \ldots, N_t,$$ \hspace{1cm} (3.16)
in which
\[ \mathcal{M}^n = \| U_N^n \|^2 \]
and
\[ \mathcal{E}^n = \| \Psi_N^n \|^2 + \gamma | \Phi_N^n |^2 + \eta^2 \| \Phi_N^n \|^2 + \lambda \left( U_N^n \right)^2 - 2 \int_\Omega \left( \kappa_1 \left( U_N^n \right)^2 + \kappa_2 \left( U_N^n \right)^4 \right) \Phi_N^n dx. \]

**Proof** Choosing \( w_N = U_N^{n+1/2} \) in (3.12) and taking the imaginary part, we obtain
\[ \| U_N^{n+1} \|^2 = \| U_N^n \|^2 \]
which implies the mass conservation.

Selecting \( w_N = \delta_i U_N^{n+1/2} \) in (3.12) and taking the real part, we have
\[
\begin{align*}
\Re \left( i \delta_i U_N^{n+1/2}, \delta_i U_N^{n+1/2} \right) - \frac{\lambda}{2} \Re \left[ B \left( U_N^{n+1/2}, \delta_i U_N^{n+1/2} \right) \right] + \kappa_1 \Re \left( U_N^{n+1/2} \Phi_N^{n+1/2}, \delta_i U_N^{n+1/2} \right) \\
+ \kappa_2 \Re \left( \left( U_N^{n+1/2} + U_N^n \right) U_N^{n+1/2} \Phi_N^{n+1/2}, \delta_i U_N^{n+1/2} \right) = 0, \tag{3.17}
\end{align*}
\]
where “\( \Re \)” represents the real part of a complex number.

By some calculations, we have
\[
\begin{align*}
\Re \left( i \delta_i U_N^{n+1/2}, \delta_i U_N^{n+1/2} \right) &= 0, \quad \Re \left[ B \left( U_N^{n+1/2}, \delta_i U_N^{n+1/2} \right) \right] = \frac{\left| U_N^{n+1} \right|^2}{a/2} - \frac{\left| U_N^n \right|^2}{a/2}, \tag{3.18} \\
\Re \left( U_N^{n+1/2} \Phi_N^{n+1/2}, \delta_i U_N^{n+1/2} \right) &= \frac{1}{2 \tau} \int_\Omega \left( \left| U_N^{n+1} \right|^2 - \left| U_N^n \right|^2 \right) \Phi_N^{n+1/2} dx \\
&= \frac{1}{4 \tau} \int_\Omega \left( \left| U_N^{n+1} \right|^2 - \left| U_N^n \right|^2 \right) \left( \Phi_N^{n+1} + \Phi_N^n \right) dx \tag{3.19}
\end{align*}
\]
and
\[
\begin{align*}
\Re \left( \left( U_N^{n+1/2} + U_N^n \right) U_N^{n+1/2} \Phi_N^{n+1/2}, \delta_i U_N^{n+1/2} \right) \\
&= \frac{1}{2 \tau} \int_\Omega \left( \left| U_N^{n+1} \right|^4 - \left| U_N^n \right|^4 \right) \Phi_N^{n+1/2} dx \\
&= \frac{1}{4 \tau} \int_\Omega \left( \left| U_N^{n+1} \right|^4 - \left| U_N^n \right|^4 \right) \left( \Phi_N^{n+1} + \Phi_N^n \right) dx. \tag{3.20}
\end{align*}
\]
Substituting (3.18)–(3.20) into (3.17), we conclude
\[
\begin{align*}
\lambda \left( \frac{\left| U_N^{n+1} \right|^2}{a/2} - \frac{\left| U_N^n \right|^2}{a/2} \right) \\
- 2 \kappa_1 \int_\Omega \left( \left| U_N^{n+1} \right|^2 \Phi_N^{n+1} - \left| U_N^n \right|^2 \Phi_N^n \right) + \left| U_N^{n+1} \right|^2 \Phi_N^n - \left| U_N^n \right|^2 \Phi_N^{n+1} \right) dx \\
- 2 \kappa_2 \int_\Omega \left( \left| U_N^{n+1} \right|^4 \Phi_N^{n+1} - \left| U_N^n \right|^4 \Phi_N^n \right) + \left| U_N^{n+1} \right|^4 \Phi_N^n - \left| U_N^n \right|^4 \Phi_N^{n+1} \right) dx = 0. \tag{3.21}
\end{align*}
\]
Setting \( w_N = \delta \Phi_N^{n+1/2} \) in (3.14), we get

\[
\left( \delta t \Psi_N^{n+1/2}, \delta t \Phi_N^{n+1/2} \right) + \gamma B \left( \Phi_N^{n+1/2}, \delta_t \Phi_N^{n+1/2} \right) + \eta^2 \left( \Phi_N^{n+1/2}, \delta_t \Phi_N^{n+1/2} \right) - \frac{\kappa_2}{2} \left( \left( |U_N^{n+1}|^2 + |U_N^n|^2 \right), \delta_t \Phi_N^{n+1/2} \right) = 0,
\]

in which

\[
\left( \delta_t \Psi_N^{n+1/2}, \delta_t \Phi_N^{n+1/2} \right) = \left( \Psi_N^{n+1/2}, \delta_t \Phi_N^{n+1/2} \right) = \frac{\| \Psi_N^{n+1} \|^2 - \| \Psi_N^n \|^2}{2 \tau},
\]

\[
B \left( \Phi_N^{n+1/2}, \delta_t \Phi_N^{n+1/2} \right) = \frac{\| \Phi_N^{n+1} \|^2_{\alpha/2} - \| \Phi_N^n \|^2_{\alpha/2}}{2 \tau},
\]

\[
\left( \Phi_N^{n+1/2}, \delta_t \Phi_N^{n+1/2} \right) = \frac{\| \Phi_N^{n+1} \|^2 - \| \Phi_N^n \|^2}{2 \tau},
\]

\[
\left( \left( |U_N^{n+1}|^2 + |U_N^n|^2 \right), \delta_t \Phi_N^{n+1/2} \right) = \frac{1}{\tau} \int \left( |U_N^{n+1}|^2 + |U_N^n|^2 \right) (\Phi_N^{n+1} - \Phi_N^n) dx. \quad (3.26)
\]

Substituting (3.23)–(3.26) into (3.22), we obtain

\[
\left( \| \Psi_N^{n+1} \|^2 - \| \Psi_N^n \|^2 \right) + \gamma \left( \| \Phi_N^{n+1} \|^2 - \| \Phi_N^n \|^2 \right) + \eta^2 \left( |\Phi_N^{n+1}|^2_{\alpha/2} - |\Phi_N^n|^2_{\alpha/2} \right) - 2\kappa_1 \int \left( |U_N^{n+1}|^2 \Phi_N^{n+1} - |U_N^n|^2 \Phi_N^n + |U_N^{n+1}|^2 \Phi_N^{n+1} - |U_N^n|^2 \Phi_N^n \right) dx
\]

\[
- 2\kappa_2 \int \left( |U_N^{n+1}|^4 \Phi_N^{n+1} - |U_N^n|^4 \Phi_N^n + |U_N^{n+1}|^4 \Phi_N^{n+1} - |U_N^n|^4 \Phi_N^n \right) dx = 0. \quad (3.28)
\]

Summing up (3.21) and (3.28), the energy conservation (3.16) is immediate. This ends the proof.

\[ \square \]

**Assumption 3.1** For the purpose of numerical analyses, two classes of assumptions of exact solutions \( u, \phi \) and \( \psi \) are imposed

(A) \( \max \left\{ \| u_0 \|, \| u_0 \|_{\alpha/2}, \| \phi_0 \|, \| \phi_0 \|_{\alpha/2}, \| \phi_1 \| \right\} \leq C_E \),

(B) \( \max_{1 \leq n \leq N_1} \left\{ \| u^n \|_{\infty}, \| \phi^n \|_{\infty} \right\} \leq \tilde{C}_E \),

where \( C_E \) and \( \tilde{C}_E \) are two positive constants only dependent of exact solutions \( u, \phi \) and \( \psi \).

**Lemma 3.1** ([15, 41]) Let \( \mu \) and \( r \) be arbitrary real numbers satisfying \( 1 < \alpha < 2, r \geq 1 \). Then there exists a positive constant \( C \) independent of \( N \), such that, for any function \( u \in \)

\[ \square \] Springer
Thus, we have
\[ M = \text{Proof} \]
From the triangle inequality, condition (A) of Assumption 3.1, and Lemma 3.1, for
\[ \text{It follows from energy conservation (3.16) and the Young’s inequality that} \]
\[ \text{Lemma 3.2 (Fractional Gagliardo–Nirenberg inequality [14, 37]) For } 1 < \alpha < 2, \text{ there exist positive constants } C_4 \text{ and } C_6, \text{ such that} \]
\[ \|u \|_4^4 \leq C_4 |u|_{\alpha/2}^{2/\alpha} |u|^{4-2/\alpha}, \quad \|u \|_8^8 \leq C_6 |u|_{\alpha/2}^{2/\alpha} |u|^{8-2/\alpha}. \]
\[ \text{Lemma 3.3 (Sobolev inequality [19]) For } \frac{1}{2} < \mu < 1, \text{ then there exists a positive constant } C, \text{ such that} \]
\[ \|u\|_{\infty} \leq C \|u\|_{\mu}, \quad \forall u \in H_0^0(\Omega). \]
In theoretical analyses, we will frequently use the following Young’s inequality
\[ ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \text{for } a, b \geq 0, \quad \varepsilon > 0. \]
\[ \textbf{Theorem 3.2} \quad \text{(Boundness of numerical solution)} \quad \text{Under Assumption (A), the numerical solution of spectral Galerkin scheme (3.12)–(3.14) is long-time bounded in the following sense} \]
\[ \|U^n_N\| \leq C, \quad \|U^n_n\|_{\alpha/2} \leq C, \quad \|U^n_n\|_{\infty} \leq C, \quad n = 0, 1, \ldots, N_t \]
and
\[ \Psi^n_N^2 \leq C, \quad \Phi^n_N \leq C, \quad \Phi^n_n_{\alpha/2} \leq C, \quad \Phi^n_n_{\infty} \leq C, \quad n = 0, 1, \ldots, N_t. \]
\[ \textbf{Proof} \quad \text{From the triangle inequality, condition (A) of Assumption 3.1, and Lemma 3.1, for sufficient small } N^{-1}, \text{ we derive} \]
\[ \|U^0_n\| = \|\Pi^2_{N/2} u_0\| \leq \|u_0\| + \|u_0 - \Pi^2_{N/2} u_0\| \leq C. \]
\[ \text{Thus, we have } M^0 \leq C. \text{ By similar procedure, we deduce } E^0 \leq C. \]
\[ \text{According to the mass conservation (3.16), we have} \]
\[ \|U^n_n\|^2 = M^n = M^0 = \|U^0_n\|^2 \leq C, \quad n = 0, 1, \ldots, N_t. \]
\[ \text{It follows from energy conservation (3.16) and the Young’s inequality that} \]
\[ \|\Psi^n_N\|^2 + \gamma \|\Phi^n_N_{\alpha/2}\|^2 + \eta^2 \|\Phi^n_N\|^2 + \lambda \|U^n_N\|^2 = E^0 + 2 \int_{\Omega} (\kappa_1 |U^n_N|^2 + \kappa_2 |U^n_n|^4) \Phi^n_N dx \]
\[ \leq E^0 + \eta^2 \|\Phi^n_N\|^2 + \frac{2}{\eta^2} \int_{\Omega} (\kappa_1 |U^n_N|^2 + \kappa_2 |U^n_n|^4) dx \]
\[ \leq E^0 + \frac{\eta^2}{2} \|\Phi^n_N\|^2 + \frac{4\kappa_1^2}{\eta^2} \|U^n_N\|^4_4 + \frac{4\kappa_2^2}{\eta^2} \|U^n_n\|^8_8. \quad \text{(3.29)} \]
By using Lemma 3.2 and the Holder inequality, we derive

\[
\left\| U_N^n \right\|_4^4 \leq C_4 \left\| \frac{U_N^n}{\alpha/2} \right\|_4^{4-2/\alpha} \leq C_4 \epsilon_1 \left\| U_N^n \right\|_{\alpha/2}^2 + C_4 C(\epsilon_1) \left\| U_N^n \right\|_{\alpha/2}^{\frac{\alpha-1}{\alpha+1}} \leq C_4 \epsilon_1 \left\| U_N^n \right\|_{\alpha/2}^2 + C,
\]

(3.30)

where \( \epsilon_1 \) is a positive constant, and \( C(\epsilon_1) > 0 \) is a constant dependent on the positive constant \( \epsilon_1 \). Similarly, we obtain

\[
\left\| U_N^n \right\|_8^8 \leq C_8 \epsilon_2 \left\| U_N^n \right\|_{\alpha/2}^2 + C. \quad (3.31)
\]

Substituting (3.30) and (3.31) into (3.29), we conclude

\[
\left\| \Psi_N^n \right\|^2 + \gamma \left\| \Phi_N^n \right\|^2_{\alpha/2} + \frac{\eta^2}{2} \left\| \Phi_N^n \right\|^2 + \lambda \left\| U_N^n \right\|^2_{\alpha/2} \leq \mathcal{E} + \left( \frac{4 \kappa_1^2 C_4 \epsilon_1}{\eta^2} + \frac{4 \kappa_2^2 C_8 \epsilon_2}{\eta^2} \right) \left\| U_N^n \right\|^2_{\alpha/2}.
\]

(3.32)

Taking appropriate \( \epsilon_1 \) and \( \epsilon_2 \), such that

\[
\frac{4 \kappa_1^2 C_4 \epsilon_1}{\eta^2} + \frac{4 \kappa_2^2 C_8 \epsilon_2}{\eta^2} = \frac{\lambda}{2}.
\]

(3.33)

Substituting (3.33) into (3.32), under condition (A) of Assumption 3.1, we derive

\[
\left\| \Psi_N^n \right\|^2 + \gamma \left\| \Phi_N^n \right\|^2_{\alpha/2} + \frac{\eta^2}{2} \left\| \Phi_N^n \right\|^2 + \frac{\lambda}{2} \left\| U_N^n \right\|^2_{\alpha/2} \leq \mathcal{C}.
\]

Therefore, we have

\[
\left\| \Psi_N^n \right\|^2 \leq \mathcal{C}, \quad \left\| \Phi_N^n \right\|^2_{\alpha/2} \leq \mathcal{C}, \quad \left\| \Phi_N^n \right\|^2 \leq \mathcal{C}, \quad \left\| U_N^n \right\|^2_{\alpha/2} \leq \mathcal{C}, \quad n = 0, 1, \ldots, N_t.
\]

By admitting Lemma 3.3, we obtain

\[
\left\| U_N^n \right\|_\infty \leq \mathcal{C} \sqrt{\left\| U_N^n \right\|^2_{\alpha/2} + \left\| U_N^n \right\|^2} \leq \mathcal{C}, \quad \left\| \Phi_N^n \right\|_\infty \leq \mathcal{C} \sqrt{\left\| \Phi_N^n \right\|^2_{\alpha/2} + \left\| \Phi_N^n \right\|^2} \leq \mathcal{C}.
\]

This completes the proof.
3.3 Unique Solvability

**Lemma 3.4** Suppose that \( X_i, Y_i \) (\( i = 1, 2, 3, 4 \)) are complex numbers. Then the following inequalities hold

\[
\begin{align*}
(1) & \quad |X_1X_2 - Y_1Y_2| \leq \max \left\{ |X_1|, |Y_2| \right\} \left( |X_1 - Y_1| + |X_2 - Y_2| \right), \\
(2) & \quad |X_1|^2 - |Y_1|^2 \leq 2 \max \left\{ |X_1|, |Y_1| \right\} |X_1 - Y_1|, \\
(3) & \quad |X_1|^4 - |Y_1|^4 \leq 2( |X_1|^2 + |Y_1|^2 ) \max \left\{ |X_1|, |Y_1| \right\} |X_1 - Y_1|, \\
(4) & \quad (|X_3|^2 + |X_4|^2)X_1X_2 - (|Y_3|^2 + |Y_4|^2)Y_1Y_2 \\
& \quad \leq 2 \max \left\{ |X_3|^2, |Y_1||Y_2| \right\} \max \left\{ |X_3|, |Y_3| \right\} |X_3 - Y_3| \\
& \quad + 2 \max \left\{ |X_4|^2, |Y_1||Y_2| \right\} \max \left\{ |X_4|, |Y_4| \right\} |X_4 - Y_4| \\
& \quad + \max \left\{ |X_3|^2, |Y_1||Y_2| \right\} \max \left\{ |X_1|, |Y_2| \right\} \left( |X_1 - Y_1| + |X_2 - Y_2| \right) \\
& \quad + \max \left\{ |X_4|^2, |Y_1||Y_2| \right\} \max \left\{ |X_1|, |Y_2| \right\} \left( |X_1 - Y_1| + |X_2 - Y_2| \right). \quad (3.37)
\end{align*}
\]

**Proof** From the triangle inequality and some careful calculations, one derives

\[
\begin{align*}
|X_1X_2 - Y_1Y_2| &= |X_1X_2 + (X_1 - Y_1)Y_2 - X_1Y_2| \\
&= |(X_1 - Y_1)Y_2 + X_1(X_2 - Y_2)| \\
&\leq |X_1 - Y_1||Y_2| + |X_1||X_2 - Y_2| \\
&\leq \max \left\{ |X_1|, |Y_2| \right\} \left( |X_1 - Y_1| + |X_2 - Y_2| \right).
\end{align*}
\]

Following the inequality (3.34), one arrives at

\[
\begin{align*}
|X_1|^2 - |Y_1|^2 &= |X_1 \overline{X_1} - Y_1 \overline{Y_1}| \\
&\leq \max \left\{ |X_1|, |Y_1| \right\} \left( |X_1 - Y_1| + |\overline{X_1} - \overline{Y_1}| \right) \\
&= 2 \max \left\{ |X_1|, |Y_1| \right\} |X_1 - Y_1|.
\end{align*}
\]

Similarly, we conclude

\[
\begin{align*}
|X_1|^4 - |Y_1|^4 &= \left| (|X_1|^2 + |Y_1|^2)(|X_1|^2 - |Y_1|^2) \right| \\
&\leq ( |X_1|^2 + |Y_1|^2 ) |X_1|^2 - |Y_1|^2 \\
&\leq 2( |X_1|^2 + |Y_1|^2 ) \max \left\{ |X_1|, |Y_1| \right\} |X_1 - Y_1|.
\end{align*}
\]
It follows from the relation (3.34) and (3.35) that
\[
\left| (|X_3|^2 + |X_4|^2)X_1X_2 - (|Y_3|^2 + |Y_4|^2)Y_1Y_2 \right|
\leq \max \left\{ |X_3|^2, |Y_1||Y_2| \right\} \left( |X_3|^2 - |Y_3|^2 + X_1X_2 - Y_1Y_2 \right)
+ \max \left\{ |X_4|^2, |Y_1||Y_2| \right\} \left( |X_4|^2 - |Y_4|^2 + X_1X_2 - Y_1Y_2 \right)
\leq 2 \max \left\{ |X_3|^2, |Y_1||Y_2| \right\} \max \left\{ |X_3|, |Y_3| \right\} |X_3 - Y_3|
+ 2 \max \left\{ |X_4|^2, |Y_1||Y_2| \right\} \max \left\{ |X_4|, |Y_4| \right\} |X_4 - Y_4|
+ \max \left\{ |X_3|^2, |Y_1||Y_2| \right\} \max \left\{ |X_1|, |Y_1| \right\} \left( |X_1 - Y_1| + |X_2 - Y_2| \right)
+ \max \left\{ |X_4|^2, |Y_1||Y_2| \right\} \max \left\{ |X_1|, |Y_2| \right\} \left( |X_1 - Y_1| + |X_2 - Y_2| \right).
\]

This completes the proof.

**Lemma 3.5** (Browder fixed point theorem [4]) Let \((\mathcal{H}, (\cdot, \cdot))\) be a finite dimensional inner product space, \(|| \cdot ||\) the associated norm, and \(\mathcal{F} : \mathcal{H} \to \mathcal{H}\) be continuous, such that
\[
\exists \delta > 0, \forall \ z \in \mathcal{H}, ||z|| = \delta, \text{ s.t. } Re(\mathcal{F}(z), z) \geq 0,
\]
there exists \(z^* \in \mathcal{H}, ||z^*|| \leq \delta\) such that \(\mathcal{F}(z^*) = 0\).

**Theorem 3.3** Under Assumption (A), for the given initial values and sufficient small \(\tau\), the numerical solutions of spectral Galerkin scheme (3.12)–(3.14) are uniquely solvable.

**Proof** For the sake of readability, we leave the proof of this theorem to “Appendix B”.

### 3.4 Convergence Analysis

**Lemma 3.6** ([16]) For time sequences \(w = \{w^0, w^1, \ldots, w^n, w^{n+1}\}\) and \(g = \{g^0, g^1, \ldots, g^n, g^{n+1}\}\), there is
\[
|2\tau \sum_{k=0}^{n} g^k \delta^k w^{k+1/2}| \leq \tau \sum_{k=0}^{n} |w^{k}|^2 + \tau \sum_{k=0}^{n-1} |\delta^k g^{k+1/2}|^2 + \frac{1}{2} |w^{n+1}|^2
+ 2 |g^n|^2 + |w^0|^2 + |g^0|^2.
\]

**Lemma 3.7** (Gronwall inequality I [39]) Suppose that the discrete grid function \(\{w^n | n = 0, 1, 2, \ldots, N\}; N; \tau = T\) satisfies the following inequality
\[
w^n - w^{n-1} \leq A \tau w^n + B \tau w^{n-1} + C_n \tau,
\]
where \(A, B\) and \(C_n\) are non-negative constants, then
\[
\max_{1 \leq n \leq N} |w^n| \leq \left( w^0 + \tau \sum_{k=1}^{N} C_k \right) e^{2(A+B)\tau},
\]
where \(\tau\) is sufficiently small, such that \((A + B)\tau < \frac{1}{2}, (N > 1)\).
Lemma 3.8 (Gronwall inequality II [39]) Suppose that the discrete grid function \( \{w^n \mid n = 0, 1, 2, \ldots, N_t; N_t \tau = T \} \) satisfies the following inequality

\[
w^n \leq A + \tau \sum_{k=1}^{n} B_k w^k,
\]

where \( A \) and \( B_k \) (\( k = 0, 1, 2, \ldots, N_t \)) are non-negative constants, then

\[
\max_{1 \leq n \leq N_t} |w^n| \leq A \exp \left( 2\tau \sum_{k=1}^{N_t} B_k \right),
\]

where \( \tau \) is sufficiently small, such that \( \tau \max_{1 \leq k \leq N_t} B_k \leq 1/2 \).

Theorem 3.4 Suppose that \( u \in C^3(H^2(\Omega) \cap H^r(\Omega), [0, T]) \) and \( \phi \in C^4(H^2_0(\Omega) \cap H^r(\Omega), [0, T]) \) \((r > 1)\) are solutions of (1.1)--(1.2), \( U^n_N \), \( \Psi^n_N \) and \( \Phi^n_N \) are solutions of (3.12)--(3.14). Under Assumptions (A) and (B), for sufficient small \( \tau \) and \( N^{-1} \), the CN-SGM scheme is unconditionally convergent in the sense that

\[
\begin{align*}
\left\| u^n - U^n_N \right\| &\leq \mathcal{C}(\tau^2 + N^{-r}), \\
\left\| \phi^n - \Phi^n_N \right\| &\leq \mathcal{C}(\tau^2 + N^{-r}), \quad \alpha \neq \frac{3}{2}, \\
\left\| \psi^n - \Psi^n_N \right\| &\leq \mathcal{C}(\tau^2 + N^{-r}), \quad \alpha = \frac{3}{2}, \quad 0 < \epsilon < \frac{1}{2}
\end{align*}
\]

and

\[
\left\| \phi^n - \Phi^n_N \right\|_\infty \leq \mathcal{C}(\tau^2 + N^{-2-r}).
\]

Proof Denote the error functions

\[
\begin{align*}
\xi^n_u &= u^n - U^n_N = (u^n - \Pi^{\alpha/2,0}_N u^n) + (\Pi^{\alpha/2,0}_N u^n - U^n_N) = \eta^n_u + \xi^n_u, \\
\xi^n_\phi &= \phi^n - \Phi^n_N = (\phi^n - \Pi^{\alpha/2,0}_N \phi^n) + (\Pi^{\alpha/2,0}_N \phi^n - \Phi^n_N) = \eta^n_\phi + \xi^n_\phi, \\
\xi^n_\psi &= \psi^n - \Psi^n_N = (\psi^n - \Pi^{\alpha/2,0}_N \psi^n) + (\Pi^{\alpha/2,0}_N \psi^n - \Phi^n_N) = \eta^n_\psi + \xi^n_\psi, \\
\xi^0_u &= u_0 - \Pi^{\alpha/2,0}_N u_0, \quad \xi^0_\phi = \phi_0 - \Pi^{\alpha/2,0}_N \phi_0, \quad \xi^0_\psi = \psi_1 - \Pi^{\alpha/2,0}_N \phi_1.
\end{align*}
\]

Noticing that

\[
\begin{align*}
\xi^0_u &= \Pi^{\alpha/2,0}_N u_0 - U^n_N = \Pi^{\alpha/2,0}_N u_0 - \Pi^{\alpha/2,0}_N u_0 = 0, \\
\xi^0_\phi &= \Pi^{\alpha/2,0}_N \phi_0 - \Phi^n_N = \Pi^{\alpha/2,0}_N \phi_0 - \Pi^{\alpha/2,0}_N \phi_0 = 0, \\
\xi^0_\psi &= \Pi^{\alpha/2,0}_N \phi_1 - \Psi^n_N = \Pi^{\alpha/2,0}_N \phi_1 - \Pi^{\alpha/2,0}_N \phi_1 = 0.
\end{align*}
\]
Subtracting (3.12)–(3.14) from (3.7)–(3.9), in view of the definition of orthogonal projection operator $\Pi_N^{n+2,0}$, we have

\[
i(\delta_t \xi^{n+1/2}_u, w_N) - \frac{\lambda}{2} B(\xi^{n+1/2}_u, w_N) + \kappa_1 (u^{n+1/2} \phi^{n+1/2} - U_N^{n+1/2} \Phi_N^{n+1/2}, w_N)
+ \kappa_2 \left( |u^{n+1/2}|^2 + |u^n|^2 \right) u^{n+1/2} \phi^{n+1/2} - \left( |U_N^{n+1/2}|^2 + |U_N^n|^2 \right) U_N^{n+1/2} \Phi_N^{n+1/2}, w_N)
\]

\[
= (\mathcal{R}_1^n, w_N), \quad \forall w_N \in X_N^N(\Omega),
\]

(3.38)

\[
(\delta_t \xi^{n+1/2}_\psi, w_N) = (\epsilon^{n+1/2}_\psi, w_N) + (\mathcal{R}_2^n, w_N), \quad \forall w_N \in X_N^0(\Omega),
\]

(3.39)

\[
(\delta_t \xi^{n+1/2}_\phi, w_N) + \gamma B(\xi^{n+1/2}_\phi, w_N) + \eta^2 (\xi^{n+1/2}_\phi, w_N)
- \frac{\kappa_1}{2} \left( |u^{n+1/2}|^2 + |u^n|^2 \right) - \left( |U_N^{n+1/2}|^2 + |U_N^n|^2 \right), w_N)
- \frac{\kappa_2}{2} \left( |u^{n+1/2}|^2 + |u^n|^2 \right) - \left( |U_N^{n+1/2}|^2 + |U_N^n|^2 \right), w_N)
= (\mathcal{R}_3^n, w_N), \quad \forall w_N \in X_N^0(\Omega),
\]

(3.40)

where

\[
\mathcal{R}_1^n = \mathcal{R}_1^n - i \delta_t \eta^{n+1/2}_u,
\]

\[
\mathcal{R}_2^n = \mathcal{R}_2^n + \eta^{n+1/2}_\psi - \delta_t \eta^{n+1/2}_\psi,
\]

\[
\mathcal{R}_3^n = \mathcal{R}_3^n - \delta_t \eta^{n+1/2}_\phi - \eta^2 \eta^{n+1/2}_\phi.
\]

For the case $\alpha \neq \frac{3}{2}$, according to Taylor’s expansion and Lemma 3.1, we have

\[
\left\| \eta^{n+1/2}_\psi \right\| \leq \left\| \eta^{n+1/2}_\psi - \eta_\psi(\cdot, t_{n+1/2}) \right\| + \left\| \eta_\psi(\cdot, t_{n+1/2}) \right\| \leq C(\tau^2 + N^{-r}),
\]

\[
\left\| \delta_t \eta^{n+1/2}_\phi \right\| \leq \left\| \delta_t \eta^{n+1/2}_\phi - \delta_t \eta_\phi(\cdot, t_{n+1/2}) \right\| + \left\| \delta_t \eta_\phi(\cdot, t_{n+1/2}) \right\| \leq C(\tau^2 + N^{-r}).
\]

Similarly, we have

\[
\left\| \eta^{n+1/2}_\phi \right\| \leq C(\tau^2 + N^{-r}), \quad \left\| \delta_t \eta^{n+1/2}_\psi \right\| \leq C(\tau^2 + N^{-r}), \quad \left\| \delta_t \eta^{n+1/2}_u \right\| \leq C(\tau^2 + N^{-r}).
\]

Thus, we obtain

\[
\left\| \mathcal{R}_1^n \right\| \leq C(\tau^2 + N^{-r}), \quad \left\| \mathcal{R}_2^n \right\| \leq C(\tau^2 + N^{-r}), \quad \left\| \mathcal{R}_3^n \right\| \leq C(\tau^2 + N^{-r}).
\]

By admitting the above relations and (3.11), we deduce

\[
\left\| \delta_t \mathcal{R}_2^{n+1/2} \right\| \leq C(\tau^2 + N^{-r}).
\]

Selecting $w_N = \xi^{n+1/2}_u$ in (3.38) and taking the imaginary part, we deduce

\[
\text{Im}\left\{ i(\delta_t \xi^{n+1/2}_u, \xi^{n+1/2}_u) \right\} = -\kappa_1 \text{Im}\left( u^{n+1/2} \phi^{n+1/2} - U_N^{n+1/2} \Phi_N^{n+1/2}, \xi^{n+1/2}_u \right)
- \kappa_2 \text{Im} \left( |u^{n+1/2}|^2 + |u^n|^2 \right) u^{n+1/2} \phi^{n+1/2} - \left( |U_N^{n+1/2}|^2 + |U_N^n|^2 \right) U_N^{n+1/2} \Phi_N^{n+1/2}, \xi^{n+1/2}_u \right)
+ \text{Im}(\mathcal{R}_1^n, \xi^{n+1/2}_u).
\]

(3.41)

By some careful calculations, we obtain

\[
\text{Im} \left\{ i(\delta_t \xi^{n+1/2}_u, \xi^{n+1/2}_u) \right\} = \frac{1}{2\tau} \left( \left\| \xi^{n+1/2}_u \right\|^2 - \left\| \xi^{n}_u \right\|^2 \right).
\]

(3.42)
Employing Lemma 3.1, Lemma 3.4 and the Cauchy–Schwarz inequality, we conclude

\[ -\kappa_1 \text{Im} \left( u^{n+1/2} \phi^{n+1/2} - U_N^{n+1/2} \Phi_N^{n+1/2}, \xi_u^{n+1/2} \right) \]

\[ \leq \kappa_1 \left\| u^{n+1/2} \phi^{n+1/2} - U_N^{n+1/2} \Phi_N^{n+1/2} \right\| \left\| \xi_u^{n+1/2} \right\| \]

\[ \leq C \left( \left\| u^{n+1/2} \phi^{n+1/2} - U_N^{n+1/2} \Phi_N^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 \right) \]

\[ \leq C \left( \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 \right) \]

\[ \leq C \left( \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 \right) \]

\[ \leq C \left( \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 \right) \]

(3.43)

\[ -\kappa_2 \text{Im} \left( \left( u^{n+1/2} \phi^{n+1/2} - U_N^{n+1/2} \Phi_N^{n+1/2} \right) \xi_u^{n+1/2} \right) \]

\[ \leq \kappa_2 \left( \left\| u^{n+1/2} \phi^{n+1/2} - U_N^{n+1/2} \Phi_N^{n+1/2} \right\| \left\| \xi_u^{n+1/2} \right\| \right) \]

\[ \leq C \left( \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 \right) \]

(3.44)

and

\[ \text{Im} \left( \mathscr{R}_1^{n+1/2}, \xi_u^{n+1/2} \right) \leq C \left( \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \tau^4 \right). \]

(3.45)

Substituting (3.42)–(3.45) into (3.41), one has

\[ \left\| \xi_u^{n+1/2} \right\|^2 \leq \left\| \xi_u^n \right\|^2 + C \tau \left( \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^n \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^n \right\|^2 + \tau^4 + N^{-2r} \right). \]

(3.46)

Choosing \( w_N = \xi_u^{n+1/2} \) in (3.39), we have

\[ \left\| \xi_u^{n+1/2} \right\|^2 - \left\| \xi_u^n \right\|^2 = 2 \tau \left( \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^n \right\|^2 \right) \]

\[ \leq C \tau \left( \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^n \right\|^2 + \left\| \xi_u^{n+1/2} \right\|^2 + \left\| \xi_u^n \right\|^2 + \tau^4 + N^{-2r} \right). \]

(3.47)

Employing the Gronwall inequality I (Lemma 3.7), one obtains

\[ \left\| \xi_u^{n+1/2} \right\|^2 \leq C \tau \sum_{k=0}^{n} \left\| \xi_u^{k+1/2} \right\|^2 + C(\tau^4 + N^{-2r}), \]

(3.48)

which implies

\[ \tau \sum_{k=0}^{n} \left\| \xi_u^{k+1} \right\|^2 \leq C \tau^2 \sum_{k=0}^{n} \sum_{j=0}^{k} \left\| \xi_u^{j+1} \right\|^2 + C(\tau^4 + N^{-2r}) \]

\[ \leq C \tau^2 \sum_{k=0}^{n} \sum_{j=0}^{n} \left\| \xi_u^{j+1} \right\|^2 + C(\tau^4 + N^{-2r}) \]

\[ \leq C \tau \sum_{k=0}^{n} \left\| \xi_u^{k+1} \right\|^2 + C(\tau^4 + N^{-2r}). \]

(3.49)
Taking \( w_N = \delta_t \xi_{n+1/2}^\phi \) in (3.40), we have

\[
\left( \delta_t \xi_{n+1/2}^\psi, \delta_t \xi_{n+1/2}^\phi \right) + \gamma \frac{\left| \xi_{n+1/2}^\phi \right|^2_{\alpha/2} - \left| \xi_{n+1/2}^\psi \right|^2_{\alpha/2} + \eta^2 \left( \left\| \xi_{n+1/2}^\phi \right\| - \left\| \xi_{n+1/2}^\psi \right\| \right)}{2\tau} = - \frac{\kappa_1}{2} \left( \left( |u_n| + 1 \right)^2 + |u_n|^2 \right) - \frac{\kappa_2}{2} \left( \left( |U_{n+1}^N| + 1 \right)^2 + |U_{n}^N|^2 \right) \cdot \delta_t \xi_{n+1/2}^\phi.
\]

It follows from (3.39) that

\[
\left( \delta_t \xi_{n+1/2}^\phi, \delta_t \xi_{n+1/2}^\psi \right) = \left( \xi_{n+1/2}^\phi, \delta_t \xi_{n+1/2}^\psi \right) + \left( \mathcal{R}_N^n, \delta_t \xi_{n+1/2}^\phi \right).
\]

Setting \( w_N = \delta_t \Phi_{n+1/2}^\psi - \Psi_{n+1/2}^\phi \) in (3.13), we deduce

\[
\delta_t \Phi_{n+1/2}^\psi = \Psi_{n+1/2}^\phi.
\]

It is trivial to check that

\[
\left( \xi_{n+1/2}^\phi, w_N \right) = \left( \delta_t \xi_{n+1/2}^\phi + \Pi_{n+1/2}^\alpha, \psi_{n+1/2} - \Pi_{n+1/2}^\alpha, \delta_t \phi_{n+1/2}, w_N \right), \quad \forall w_N \in X_N^0(\Omega),
\]

which implies

\[
\delta_t \xi_{n+1/2}^\phi = \xi_{n+1/2}^\phi + \Pi_{n+1/2}^\alpha, \delta_t \phi_{n+1/2} - \Pi_{n+1/2}^\alpha, \psi_{n+1/2}.
\]

By employing Lemmas 3.4, 3.1, Assumption 3.1 and the Cauchy–Schwarz inequality that

\[
\begin{align*}
&- \frac{\kappa_1}{2} \left( \left( |u_n| + 1 \right)^2 + |u_n|^2 \right) - \left( \left| U_{n+1}^N \right|^2 + \left| U_{n}^N \right|^2 \right) \cdot \delta_t \xi_{n+1/2}^\phi \\
&\leq \frac{\kappa_1}{2} \left( \left( |u_n| + 1 \right)^2 + |u_n|^2 \right) - \left( \left| U_{n+1}^N \right|^2 + \left| U_{n}^N \right|^2 \right) \cdot \delta_t \xi_{n+1/2}^\phi \\
&\leq C \left( \left( |u_n| + 1 \right)^2 + |u_n|^2 \right) - \left( \left| U_{n+1}^N \right|^2 + \left| U_{n}^N \right|^2 \right) \cdot \delta_t \xi_{n+1/2}^\phi \\
&\leq C \left( \left( |u_n| + 1 \right)^2 + |u_n|^2 \right) - \left( \left| U_{n+1}^N \right|^2 + \left| U_{n}^N \right|^2 \right) \cdot \delta_t \xi_{n+1/2}^\phi \\
&\leq C \left( \left( |u_n| + 1 \right)^2 + |u_n|^2 \right) - \left( \left| U_{n+1}^N \right|^2 + \left| U_{n}^N \right|^2 \right) \cdot \delta_t \xi_{n+1/2}^\phi \\
&\leq \left( \left( \xi_{n+1/2}^\phi \right)^2 + \left( \xi_{n+1/2}^\psi \right)^2 \right) + \left( \left| \xi_{n+1/2}^\phi \right| \right)^2 - \left( \left| \xi_{n+1/2}^\psi \right| \right)^2 + \left( \left| \xi_{n+1/2}^\phi \right| \right)^2 - \left( \left| \xi_{n+1/2}^\psi \right| \right)^2 + N^{-2r} \\
&\leq \left( \left( \xi_{n+1/2}^\phi \right)^2 + \left( \xi_{n+1/2}^\psi \right)^2 \right) + \left( \left| \xi_{n+1/2}^\phi \right| \right)^2 - \left( \left| \xi_{n+1/2}^\psi \right| \right)^2 + \left( \left| \xi_{n+1/2}^\phi \right| \right)^2 - \left( \left| \xi_{n+1/2}^\psi \right| \right)^2 + N^{-2r} \\
&\leq \left( \left( \xi_{n+1/2}^\phi \right)^2 + \left( \xi_{n+1/2}^\psi \right)^2 \right) + \left( \left| \xi_{n+1/2}^\phi \right| \right)^2 - \left( \left| \xi_{n+1/2}^\psi \right| \right)^2 + \left( \left| \xi_{n+1/2}^\phi \right| \right)^2 - \left( \left| \xi_{n+1/2}^\psi \right| \right)^2 + N^{-2r} \\
&\leq \left( \left( \xi_{n+1/2}^\phi \right)^2 + \left( \xi_{n+1/2}^\psi \right)^2 \right) + \left( \left| \xi_{n+1/2}^\phi \right| \right)^2 - \left( \left| \xi_{n+1/2}^\psi \right| \right)^2 + \left( \left| \xi_{n+1/2}^\phi \right| \right)^2 - \left( \left| \xi_{n+1/2}^\psi \right| \right)^2 + N^{-2r} \\
&\leq C \left( \left( \xi_{n+1/2}^\phi \right)^2 + \left( \xi_{n+1/2}^\psi \right)^2 \right) + \left( \left| \xi_{n+1/2}^\phi \right| \right)^2 - \left( \left| \xi_{n+1/2}^\psi \right| \right)^2 + \left( \left| \xi_{n+1/2}^\phi \right| \right)^2 - \left( \left| \xi_{n+1/2}^\psi \right| \right)^2 + N^{-2r}.
\end{align*}
\]

(3.52)
where the last inequality has used the following estimate: by taking $w_N = \delta_t \phi^{n+1/2} - \psi^{n+1/2}$ in (3.8), one has
\[
\left\| \delta_t \phi^{n+1/2} - \psi^{n+1/2} \right\| \leq \left\| \mathcal{R}_2 \right\| \leq C(\tau^4 + N^{-2r}).
\]

Similar estimates, we obtain
\[
- \frac{\kappa_2}{2} \left( (|u^{n+1}|^4 + |u^n|^4) - (|U_N^{n+1}|^4 + |U_N^n|^4) \right) \delta_t \xi_{\phi}^{n+1/2} \\
\leq C \left( \left\| \xi_{\phi}^{n+1} \right\|^2 + \left\| \xi_{\psi}^n \right\|^2 + \left\| \xi_{\phi}^{n+1/2} \right\|^2 + \left\| \xi_{\psi}^{n+1/2} \right\|^2 + \tau^4 + N^{-2r} \right) \tag{3.53}
\]
and
\[
\left( \mathcal{R}_3, \delta_t \xi_{\phi}^{n+1/2} \right) \leq C \left( \left\| \xi_{\phi}^{n+1} \right\|^2 + \left\| \xi_{\psi}^n \right\|^2 + \tau^4 + N^{-2r} \right). \tag{3.54}
\]

Substituting (3.51)–(3.54) into (3.50), we arrive at
\[
\left\| \xi_{\phi}^{n+1} \right\|^2 + \gamma \left\| \xi_{\phi}^{n+1} \right\|^2_{\frac{\alpha}{2}} + \eta^2 \left\| \xi_{\phi}^{n+1} \right\|^2 \\
\leq \left\| \xi_{\phi}^n \right\|^2 + \gamma \left\| \xi_{\phi}^n \right\|^2_{\frac{\alpha}{2}} + \eta^2 \left\| \xi_{\phi}^n \right\|^2 + C \tau \left( \left\| \xi_{\phi}^{n+1} \right\|^2 + \left\| \xi_{\phi}^n \right\|^2 + \left\| \xi_{\phi}^{n+1/2} \right\|^2 + \left\| \xi_{\phi}^n \right\|^2 + \tau^4 + N^{-2r} \right) \\
- 2\tau \left( \mathcal{R}_2, \delta_t \xi_{\phi}^{n+1/2} \right). \tag{3.55}
\]

Summing up (3.46) and (3.55), we have
\[
\left\| \xi_{\phi}^{n+1} \right\|^2 + \left\| \xi_{\phi}^n \right\|^2 + \left\| \xi_{\phi}^{n+1/2} \right\|^2 + \gamma \left\| \xi_{\phi}^{n+1} \right\|^2_{\frac{\alpha}{2}} + \eta^2 \left\| \xi_{\phi}^{n+1} \right\|^2 \\
\leq \left\| \xi_{\phi}^n \right\|^2 + \left\| \xi_{\phi}^n \right\|^2 + \gamma \left\| \xi_{\phi}^n \right\|^2_{\frac{\alpha}{2}} + \eta^2 \left\| \xi_{\phi}^n \right\|^2 + C \tau \left( \left\| \xi_{\phi}^{n+1} \right\|^2 + \left\| \xi_{\phi}^n \right\|^2 + \left\| \xi_{\phi}^{n+1/2} \right\|^2 + \left\| \xi_{\phi}^n \right\|^2 + \tau^4 + N^{-2r} \right) \\
+ \left\| \xi_{\phi}^{n+1} \right\|^2 + \left\| \xi_{\phi}^n \right\|^2 + \tau^4 + N^{-2r} - 2\tau \left( \mathcal{R}_2, \delta_t \xi_{\phi}^{n+1/2} \right) \\
\leq C \tau \sum_{k=0}^n \left( \left\| \xi_{\phi}^{k+1} \right\|^2 + \left\| \xi_{\phi}^k \right\|^2 + \left\| \xi_{\phi}^{k+1/2} \right\|^2 \right) + C \left( \tau^4 + N^{-2r} \right) - 2\tau \sum_{k=0}^n \left( \mathcal{R}_2, \delta_t \xi_{\phi}^{k+1/2} \right). \tag{3.56}
\]

It follows from (3.49) and Lemma 3.6 that
\[
\left\| \xi_{\phi}^{n+1} \right\|^2 + \left\| \xi_{\phi}^{n+1/2} \right\|^2 + \gamma \left\| \xi_{\phi}^{n+1} \right\|^2_{\frac{\alpha}{2}} + \eta^2 \left\| \xi_{\phi}^{n+1} \right\|^2 \\
\leq C \tau \sum_{k=0}^n \left( \left\| \xi_{\phi}^{k+1} \right\|^2 + \left\| \xi_{\phi}^k \right\|^2 \right) + C \left( \tau^4 + N^{-2r} \right) - 2\tau \sum_{k=0}^n \left( \mathcal{R}_2, \delta_t \xi_{\phi}^{k+1/2} \right) \\
\leq C \tau \sum_{k=0}^n \left( \left\| \xi_{\phi}^{k+1} \right\|^2 + \left\| \xi_{\phi}^k \right\|^2 \right) + \frac{1}{2} \left\| \xi_{\phi}^{n+1} \right\|^2 + C \left( \tau^4 + N^{-2r} \right). \tag{3.57}
\]
Then, we reformulate (3.57) as
\[
\left\| \xi_{n+1}^u \right\|^2 + \frac{1}{2} \left\| \xi_{n+1}^\psi \right\|^2 + \gamma \left\| \xi_{n+1}^\phi \right\|_{\alpha/2}^2 + \eta^2 \left\| \xi_{n+1}^\phi \right\|^2 \\
\leq C \tau \sum_{k=0}^{n} \left( \left\| \xi_{k+1}^u \right\|^2 + \frac{1}{2} \left\| \xi_{k+1}^\psi \right\|^2 \right) + C \left( \tau^4 + N^{-2r} \right)
\]
\[
\leq C \tau \sum_{k=0}^{n} \left( \left\| \xi_{k+1}^u \right\|^2 + \frac{1}{2} \left\| \xi_{k+1}^\psi \right\|^2 + \gamma \left\| \xi_{k+1}^\phi \right\|_{\alpha/2}^2 + \eta^2 \left\| \xi_{k+1}^\phi \right\|^2 \right) + C \left( \tau^4 + N^{-2r} \right).
\]
(3.58)

Utilizing the Gronwall inequality II (Lemma 3.8) for (3.58), we conclude
\[
\left\| \xi_{n+1}^u \right\| \leq C(\tau^2 + N^{-r}), \quad \left\| \xi_{n+1}^\psi \right\| \leq C(\tau^2 + N^{-r}), \quad \left\| \xi_{n+1}^\phi \right\|_{\alpha/2} \leq C(\tau^2 + N^{-r}).
\]

By applying Lemma 3.1, we derive
\[
\left\| \phi_{n+1} - \Phi_{N}^{n+1} \right\|_{\alpha/2} \leq \left\| \eta_{n+1}^\phi \right\|_{\alpha/2} + \left\| \xi_{n+1}^\phi \right\|_{\alpha/2} \leq C(\tau^2 + N^{\alpha/2-r}),
\]
\[
\left\| \eta_{n+1}^u \right\| + \left\| \xi_{n+1}^u \right\| \leq C(\tau^2 + N^{-r})
\]
and
\[
\left\| \phi_{n+1} - \Phi_{N}^{n+1} \right\| \leq \left\| \eta_{n+1}^\phi \right\| + \left\| \xi_{n+1}^\phi \right\| \leq C(\tau^2 + N^{-r}).
\]

Therefore, combining the above estimates and Lemma 3.3, we get
\[
\left\| \phi_{n+1} - \Phi_{N}^{n+1} \right\|_{\infty} \leq C \sqrt{\left\| \phi_{n+1} - \Phi_{N}^{n+1} \right\|^2 + \left\| \phi_{n+1} - \Phi_{N}^{n+1} \right\|_{\alpha/2}^2} \leq C(\tau^2 + N^{\alpha/2-r}).
\]

When $\alpha = \frac{3}{2}$, the inference procedures are similar to the case $\alpha \neq \frac{3}{2}$, here we omit it. We obtain the following consequences
\[
\left\| u_{n+1} - U_{N}^n \right\| \leq C(\tau^2 + N^{\epsilon-r}), \quad \left\| \phi_{n} - \Phi_{N}^{n} \right\| \leq C(\tau^2 + N^{\epsilon-r}),
\]
\[
\left\| \phi_{n} - \Phi_{N}^{n} \right\|_{\infty} \leq C(\tau^2 + N^{\frac{\alpha}{2}-r}), \quad 0 < \epsilon < \frac{1}{2}.
\]

Thus, the proof is completed. \(\square\)

### 4 Linearly Implicit and Decoupled Conservative Scheme

The ESAV formulation of the FNKGS equation (1.1)–(1.2) introduces the exponential scalar auxiliary variables

\[
p(t) = \exp \left( 2 \int_{\Omega} (\kappa_1 |u|^2 + \kappa_2 |u|^4) \phi dx \right).
\]
Taking the derivative of $\ln(p)$, we have
\[
\frac{d}{dt} \ln(p) = 2 \int_{\Omega} \left( 2\kappa_1 \text{Re}(\bar{u}u_t) + 4\kappa_2 |u|^2 \text{Re}(\bar{u}u_t) \right) \phi + \left( \kappa_1 |u|^2 + \kappa_2 |u|^4 \right) \phi_t \, dx
\]
\[
= 4\Re \left( ( \kappa_1 + 2\kappa_2 |u|^2 ) u\phi, u_t \right) + 2 \left( \kappa_1 |u|^2 + \kappa_2 |u|^4 \right) \phi_t
\]
\[
= 4\Re \left( p F(u, \phi) u\phi, u_t \right) + 2 \left( p G(u, \phi), \phi_t \right),
\]
where
\[
F(u, \phi) = \frac{\kappa_1 + 2\kappa_2 |u|^2}{\exp \left( 2 \int_{\Omega} (\kappa_1 |u|^2 + \kappa_2 |u|^4) \phi \, dx \right)}, \quad
G(u, \phi) = \frac{\kappa_1 |u|^2 + \kappa_2 |u|^4}{\exp \left( 2 \int_{\Omega} (\kappa_1 |u|^2 + \kappa_2 |u|^4) \phi \, dx \right)}.
\]

Then, the system (3.4)–(3.6) can be reformulated into an equivalent form
\[
i (u_t, w) - \frac{\lambda}{2} B(u, w) + (p F(u, \phi) u\phi, w) = 0,
\]
\[
(\phi_t, w) = (\psi, w),
\]
\[
(\psi_t, w) + \gamma B(\phi, w) + \eta^2 (\phi, w) - (p G(u, \phi), w) = 0,
\]
\[
\frac{d}{dt} \ln(p) = 4\Re \left( p F(u, \phi) u\phi, u_t \right) + 2 \left( p G(u, \phi), \phi_t \right),
\]
with
\[
u(x, 0) = u_0(x), \quad \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x) \quad p_0
\]
\[
= \exp \left( 2 \int_{\Omega} (\kappa_1 |u_0|^2 + \kappa_2 |u_0|^4) \phi \, dx \right).
\]

The energy conservation in the ESAV formulation is
\[
\frac{d}{dt} \left( \|\psi\|^2 + \gamma \|\phi\|_{a/2}^2 + \eta^2 \phi^2 + \kappa_1 |u|^2 - \ln(p) \right) = 0.
\]

Combining the implicit midpoint method and spectral Galerkin method, the ESAV-SGM for (1.1)–(1.2) is constructed as: finding $U_n^p, \Phi_n^p, \Psi_n^p \in X_N^0(\Omega)$, for $w_N \in X_N^0(\Omega)$, such that
\[
\frac{\ln(p^{n+1}) - \ln(p^n)}{\tau} = 4 \tilde{P}^{n+1/2} \text{Re} \left( F \left( \bar{U}_N^{n+1/2}, \tilde{\Phi}_N^{n+1/2} \right), \bar{U}_N^{n+1/2}, \Phi_N^{n+1/2} \right) + 2 \tilde{P}^{n+1/2} \left( G \left( \tilde{U}_N^{n+1/2}, \Phi_N^{n+1/2} \right), \delta_t U_N^{n+1/2} \right),
\]
with the startup scheme
\[
i\left(\delta_t U_n^{1/2}, w_N\right) - \frac{\lambda}{2} \mathcal{B} \left(U_n^{1/2}, w_N\right) + P^{1/2} \left(F \left(U_n^{1/2}, \Phi_N^{1/2}\right) U_n^{1/2} \Phi_N^{1/2}, w_N\right) = 0, \quad (4.9)
\]
\[
\left(\delta_t, \Phi_N^{1/2}, w_N\right) = \left(\Psi_N^{1/2}, w_N\right), \quad (4.10)
\]
\[
\left(\delta_t, \Psi_N^{1/2}, w_N\right) + \gamma \mathcal{B}(\Phi_N^{1/2}, w_N) + \eta^2(\Phi_N^{1/2}, w_N) - P^{1/2} \left(G \left(U_n^{1/2}, \Phi_N^{1/2}\right), w_N\right) = 0, \quad (4.11)
\]
\[
\frac{\ln(P^1) - \ln(P^0)}{\tau} = 4P^{1/2} \text{Re} \left( F \left(U_n^{1/2}, \Phi_N^{1/2}\right) U_n^{1/2} \Phi_N^{1/2}, \delta_t U_n^{1/2}\right) + 2P^{1/2} \left(G \left(U_n^{1/2}, \Phi_N^{1/2}\right), \delta_t \Phi_N^{1/2}\right). \quad (4.12)
\]

**Theorem 4.1** The numerical solution of the ESAV-SGM (4.5)–(4.8) enjoys an energy preservation in the sense that
\[
\mathcal{E}^n = \cdots = \mathcal{E}^0, \quad n = 0, 1, \ldots, N_t,
\]
where
\[
\mathcal{E}^n = \left\| \Psi_N^n \right\|^2 + \gamma \left\| \Phi_N^n \right\|^2 + \eta^2 \left\| \Phi_N^n \right\|^2 + \lambda \left| U_n^n \right|^2_{\alpha/2} - \ln(P^0).
\]

**Proof** Setting \( w_N = \delta_t U_n^{n+1/2} \) in (4.5) and taking the real part, then choosing \( w_N = \delta_t \Phi_n^{n+1/2} \) in (4.7), summing up the above resulting formulae similar to Theorem 3.1, the energy preservation is readily and immediately available. \( \square \)

## 5 Decoupled Iterative Implementation

In this section, we focus on the implementation of CN-SGM (3.12)–(3.14), the implementation of ESAV-SGM can be similarly modified. The associated numerical solutions \( U_n^0, \Phi_N^n \) and \( \Psi_N^n \) are of the form
\[
U_N^n = \sum_{k=0}^{N-2} \tilde{U}_k^n \xi_k(x), \quad \Phi_N^n = \sum_{k=0}^{N-2} \tilde{\Phi}_k^n \xi_k(x), \quad \Psi_N^n = \sum_{k=0}^{N-2} \tilde{\Psi}_k^n \xi_k(x)
\]
belong to the \( N \)-dimensional polynomial space \( X_N^0(\Omega) \) which is given by
\[
X_N^0(\Omega) = \text{span}\{\xi_k(x) : k = 0, 1, \ldots, N - 2\},
\]
where \( \xi_k(x) \) is determined by the following recurrence relation
\[
\xi_k(x) = L_k(\hat{x}) - L_{k+2}(\hat{x}), \quad \hat{x} \in [-1, 1], \quad x = \frac{(b - a)\hat{x} + (a + b)}{2} \in [a, b],
\]
in which \( L_k(\hat{x}) \) represents Legendre orthogonal polynomial [30] which satisfies the following three-term recurrence relation
\[
\begin{cases}
L_0(\hat{x}) = 1, & L_1(\hat{x}) = \hat{x}, \\
(k + 1)L_{k+1}(\hat{x}) = (2k + 1)\hat{x}L_k(\hat{x}) - kL_{k-1}(\hat{x}), & k \geq 1.
\end{cases}
\]
The associated mass and stiff matrices of CN-SGM are all symmetric and computed by
\[ M_{i,k} = (\zeta_k(x), \zeta_l(x)), \quad S_{i,k} = \frac{1}{2 \cos(\frac{\pi x}{2})} \left( RL D_x^{\alpha/2} \zeta_k(x), RL D_x^{\alpha/2} \zeta_l(x) \right) + \left( RL D_b^{\alpha/2} \zeta_k(x), RL D_b^{\alpha/2} \zeta_l(x) \right). \]

The linearized and decoupled fixed-point iterative algorithm is presented for the CN-SGM in the following form
\[
\left\{
\begin{aligned}
(iM - \frac{\pi x}{4} S) \hat{U}^{n+1} & = (iM + \frac{\pi x}{4} S) \hat{U}^n - \frac{\pi}{4} (\kappa_1 \Lambda_1^{n+1,s} + \kappa_2 \Lambda_2^{n+1,s}), \quad s = 0, 1, \ldots, \\
(M + \frac{\pi x}{4} S + \frac{\pi x^2}{4} M) \Phi^{n+1,s+1} & = (M - \frac{\pi x}{4} S - \frac{\pi x^2}{4} M) \Phi^n + \frac{\pi}{4} (\kappa_1 \Lambda_3^{n+1,s} + \kappa_2 \Lambda_4^{n+1,s}),
\end{aligned}
\right.
\]
where
\[
\hat{U}^{n+1,0} = \begin{cases} \hat{U}^0, & n = 0, \\ 2\hat{U}^n - \hat{U}^{n-1}, & n \geq 1 \end{cases}, \quad \Phi^{n+1,0} = \begin{cases} \Phi^0, & n = 0, \\ 2\Phi^n - \Phi^{n-1}, & n \geq 1 \end{cases}
\]
and
\[
\left\{
\begin{aligned}
(\lambda_1^{n+1,s})_l & = (U_N^{n+1} + U_N^{n+1,s}) (\Phi_N^{n} + \Phi_N^{n+1,s}), \quad s = 1, 2, \\
(\lambda_2^{n+1,s})_l & = (|U_N^{n+1}|^2 + |U_N^{n+1,s}|^2) (U_N^{n+1} + U_N^{n+1,s}) (\Phi_N^{n} + \Phi_N^{n+1,s}), \\
(\lambda_3^{n+1,s})_l & = |U_N^{n+1}|^2 + |U_N^{n+1,s}|^2, \\
(\lambda_4^{n+1,s})_l & = |U_N^{n+1}|^4 + |U_N^{n+1,s}|^4,
\end{aligned}
\right.
\]
Then, \( U_N^{n+1,s+1} \) and \( \Phi_N^{n+1,s+1} \) numerically converge to the numerical solutions \( U_N^{n+1} \) and \( \Phi_N^{n+1} \), respectively, if there satisfies
\[
\| U_N^{n+1,s+1} - U_N^{n+1} \| + \| \Phi_N^{n+1,s+1} - \Phi_N^{n+1} \| \leq tol
\]
for the given stopping criterion \( tol = 10^{-14} \).

### 6 Numerical Experiments

In this section, some numerical results are reported to verify the proposed spectral Galerkin method. All the simulations are implemented by using Matlab R2018a software on a computer with Intel Core i7 and 16 GB RAM. Without special instructions, we always take the stopping criterion \( tol = 10^{-14} \).

In numerical tests, we compute the \( L^2 \)- and \( L^\infty \)-norm errors at \( t = N_t \tau \) by
\[
\| u^{N_t} - U_N^{N_t} \| = \left( \int_{\Omega} \left| u^{N_t}(x) - U_N^{N_t}(x) \right|^2 \, dx \right)^{1/2} \approx \sqrt{\frac{b-a}{2} \sum_{j=0}^{M} |u_N^{N_t}(x_j) - U_N^{N_t}(x_j)|^2} \sigma_j
\]
and
\[
\| u^{N_t} - U_N^{N_t} \|_\infty := \max_{0 \leq j \leq M} |u^{N_t}(x_j) - U_N^{N_t}(x_j)| \quad \text{with} \quad x_j = \frac{(b-a)\tilde{x}_j + (a+b)}{2},
\]
where \( \{\tilde{x}_j\} \) and \( \{\sigma_j\} \) are points and weights of the Legendre–Gauss–Lobatto quadrature, respectively, and \( M = \mu N \) \( (\mu > 1, \mu \in \mathbb{N}_+) \). In convergence test, for the case that the
solution of equation is unknown, we intend to regard the more accurate numerical solutions as the reference solutions. To illustrate the energy preservation of the proposed numerical schemes, define the following relative mass and energy deviations

\[ \text{RM}^n = \frac{|M^n - M^0|}{M^0}, \quad \text{RE}^n = \frac{|E^n - E^0|}{E^0}. \]

Example 6.1 Consider the FNKGS system (1.1)–(1.2) with \( \lambda = 1, \kappa_1 = 1, \kappa_2 = 0, \gamma = 1 \) and \( \eta = 1 \) in domain \((-20, 20) \times (0, T]\). When \( \alpha = 2 \), the system has the exact solitary wave solutions as follows:

\[ u(x, t, \nu) = \frac{3\sqrt{2}}{4\sqrt{1 - \nu^2}} \text{sech}^2 \frac{x - \nu t - \chi_0}{2\sqrt{1 - \nu^2}} \exp \left( i \left( \nu x + \frac{1 - \nu^2 + \nu^4}{2(1 - \nu^2)} \right) t \right), \quad (6.1) \]

\[ \phi(x, t, \nu) = \frac{3}{4(1 - \nu^2)} \text{sech}^2 \frac{x - \nu t - \chi_0}{2\sqrt{1 - \nu^2}}, \quad (6.2) \]

\[ \phi_t(x, t, \nu) = \frac{3\nu}{4(1 - \nu^2)^{3/2}} \text{sech}^2 \frac{x - \nu t - \chi_0}{2\sqrt{1 - \nu^2}} \tanh \frac{x - \nu t - \chi_0}{2\sqrt{1 - \nu^2}}. \quad (6.3) \]

Here, the initial datum \( u_0(x), \phi_0(x) \) and \( \phi_1(x) \) are determined by the exact solutions (6.1)–(6.3), where \( \nu = 0.8 \) and \( \chi_0 = -10 \).

Firstly, we carry out testing the convergence accuracy of CN-SGM in time and space. For the fixed temporal step, we find from Fig. 1 that the errors are exponentially decaying in \( L^2 \) and \( L^\infty \) norm. For the fixed polynomial degree \( N = 150 \), the temporal numerical results are listed in Table 1. The numerical results show that the proposed CN-SGM possesses the second-order accuracy in \( L^2 \) and \( L^\infty \) norm, which are in great accordance with the theoretical results. Secondly, we show the effect of fractional order \( \alpha \) on numerical profiles. It is obvious to observe from Fig. 2 that the fractional order \( \alpha \) will dramatically affect the shapes of the solitons, we refer readers to [23] for more details. Finally, for the purpose of numerical comparisons, we utilize the CN-SGM, the ESAV-SGM and the finite difference scheme (LF-FDM [32]) to solve Example 6.1 until \( t = 100 \). Figures 3 and 4 illustrates that the CN-SGM uniformly preserves the mass and energy to machine accuracy. Figure 4 verifies that the ESAV-SGM preserves the energy well, but the mass preservation cannot be done from Fig. 3. Actually, the mass conservation is also an important structure for the convergence and stability analyses of a conservative algorithm. Figures 3 and 4 shows that the LF-FDM is not up to the task of long-time mass and energy preservations. These numerical pictures illustrate that the CN-SGM enjoys a great superior than the ESAV-SGM and the LF-FDM for energy preservation. Subsequently, we plot the numerical errors and associated CPU time of the CN-SGM and the ESAV-SGM in Fig. 5 by setting the various temporal step \( \tau \) and the
| $\alpha$ | $\tau$ | $\|u^{Nt}_N - U^{Nt}_N\|$ | Conv. rate | $\|\phi^{Nt} - \Phi^{Nt}_N\|$ | Conv. rate | $\|\phi^{Nt} - \Phi^{Nt}_N\|_\infty$ | Conv. rate | CPU time (s) |
|---|---|---|---|---|---|---|---|---|
| 1.2 | 0.1 | $4.5813e-03$ | – | $1.1683e-03$ | – | $12134e-03$ | – | 1.03 |
|  | 0.05 | $1.1492e-03$ | 1.9951 | $2.9386e-04$ | 1.9912 | $30651e-04$ | 1.9851 | 1.54 |
|  | 0.025 | $2.8769e-04$ | 1.9981 | $7.3579e-05$ | 1.9978 | $76822e-05$ | 1.9963 | 2.43 |
|  | 0.0125 | $7.2149e-05$ | 1.9955 | $1.8402e-05$ | 1.9994 | $19218e-05$ | 1.9991 | 4.00 |
| 1.5 | 0.1 | $4.8761e-03$ | – | $1.0746e-03$ | – | $1.0233e-03$ | – | 1.05 |
|  | 0.05 | $1.2232e-03$ | 1.9951 | $2.7114e-04$ | 1.9866 | $2.6063e-04$ | 1.9731 | 1.69 |
|  | 0.025 | $3.0612e-04$ | 1.9985 | $6.7948e-04$ | 1.9965 | $6.5450e-05$ | 1.9935 | 2.69 |
|  | 0.0125 | $7.6661e-05$ | 1.9975 | $1.6996e-04$ | 1.9992 | $1.6384e-05$ | 1.9981 | 4.30 |
| 1.8 | 0.1 | $5.4219e-03$ | – | $9.0221e-04$ | – | $8.5874e-04$ | – | 1.04 |
|  | 0.05 | $1.3661e-03$ | 1.9888 | $2.2578e-04$ | 1.9985 | $2.1491e-04$ | 1.9989 | 1.69 |
|  | 0.025 | $3.4227e-04$ | 1.9968 | $5.6461e-05$ | 1.9996 | $5.3741e-05$ | 1.9997 | 2.43 |
|  | 0.0125 | $8.5705e-05$ | 1.9977 | $1.4124e-05$ | 1.9991 | $1.3429e-05$ | 2.0012 | 3.99 |
Fig. 2 Time evolutions of the relative energy and mass of CN-SGM for Example 6.1 with $\tau = 1/10$ and $N = 100$

Fig. 3 Time evolutions of the relative mass deviations of CN-SGM (Left), ESAV-SGM (Middle) and LF–FDM (Right) for Example 6.1 with $\tau = 1/10$ and $N = 100$

same polynomial degree $N$. From the numerical results, we observe that the ESAV-SGM performs more effective than the CN-SGM, in the other word, the ESAV-SGM needs less time to obtain the same errors. It is worth noticing that the CN-SGM can obtain the more accurate numerical results than the ESAV-SGM for the same temporal step $\tau$ and polynomial degree $N$. Therefore, the CN-SGM and the ESAV-SGM each have their own advantages in long-time computations.

Example 6.2 Consider the FNKGS system (1.1)–(1.2) with $\lambda = 1, \kappa_1 = 1, \kappa_2 \geq 0, \gamma = 1$ and $\eta = 1$ in domain $(-20, 20) \times (0, T]$. We choose the following initial datum with two solitons:

$$ u_0(x) = u(x - p_1, 0, v_1) + u(x - p_2, 0, v_2), $$

$$ \phi_0(x) = \phi(x - p_1, 0, v_1) + \phi(x - p_2, 0, v_2), $$

$$ \phi_1(x) = \phi_t(x - p_1, 0, v_1) + \phi_t(x - p_2, 0, v_2), $$

where $p_1 = -p_2 = -10, v_1 = -v_2 = 0.8$ and $\chi_0 = 0$.

For the FNKGS system with Yukawa interaction, we intend to perform the effect of fractional orders on the numerical accuracy and discrete conservation laws. We apply the CN-SGM to solve Example 6.2, the convergence rate and the associated CPU time for various $\kappa_2$ are shown in Tables 2, 3 and 4, the numerical results verify the second-order accuracy
in time, which are in good agreement with theoretical results. For the fixed temporal step $\tau = 1/50$ and polynomial degree $N = 200$, the numerical solutions $|U_N^n(x)|$ and $\phi_N^n(x)$ are plotted in Fig. 6, we observe that the collisions will occur earlier as $\alpha$ increases, and the effect of fractional order $\alpha$ on two solitons collide is distinct. By setting $\tau = 1/50$, $N = 150$ and $\kappa_2 = 0.01$, the time evolution of relative mass and relative energy errors are presented in Fig. 7, the numerical results exactly confirm that the proposed spectral Galerkin scheme preserves the mass and energy well in long-time computations.

In the end, for the special case that the initial energy functional $E(t_0)$ is negative for $\kappa_2 = 1$ in system (1.1)–(1.2). In such case, we apply the proposed spectral Galerkin schemes CN-SGM and ESA-V-SGM for solving it, it is clear from Fig. 8 that the discrete energy exactly occurs negative. which demonstrates that the analytical solutions of the FNKGS system will be blow-up in finite time [5]. From the perspective of numerical solution, the existing work [11] claimed that the system has a singularity as soon as its solution becomes three (or more) times bigger than the initial value in $L^\infty$ norm. As a result, for the mesh sizes are small enough, we can conclude that the ESA-V-SGM scheme is not, but the CN-SGM is, capable of capturing the blow-up phenomenon of the FNKGS equation.

7 Conclusions

In this paper, we focus on the spectral Galerkin methods for the coupled fractional nonlinear Klein–Gordon–Schrödinger equation. We have proposed two structure-preserving schemes, also analyzed the advantages and disadvantages of each other by numerical comparisons. In theoretical aspects, the maximum-norm boundness of numerical solutions of the CN-SGM are proved, and the unique solvability is obtained with the help of Browder fixed point theorem and maximum-norm boundness of the numerical solutions. Moreover, the unconditional convergence are analyzed without any restriction on grid ratio. Numerical
Table 2 Temporal accuracy of CN-SGM for Example 6.2 with $\kappa_2 = 0$ and $N = 150$ at $t = 1$

| $\alpha$ | $\tau$ | $\|u_{Nt} - U_{Nt}^{N}\|$ | Conv. rate | $\|\phi_{Nt} - \Phi_{Nt}^{N}\|$ | Conv. rate | $\|\phi_{Nt} - \Phi_{Nt}^{N}\|_\infty$ | Conv. rate | CPU time (s) |
|----------|--------|-----------------|-----------|-----------------|-----------|-----------------|-----------|-------------|
| 1.2      | 0.1    | 6.4901e-03      | –         | 1.6517e-03      | –         | 1.2104e-03      | –         | 1.15        |
| 0.05     |        |                 |           |                 |           |                 |           |             |
| 0.025    |        |                 |           |                 |           |                 |           |             |
| 0.0125   |        |                 |           |                 |           |                 |           |             |
| 1.5      | 0.1    | 6.8998e-03      | –         | 1.5194e-03      | –         | 1.0228e-03      | –         | 1.03        |
| 0.05     |        |                 |           |                 |           |                 |           |             |
| 0.025    |        |                 |           |                 |           |                 |           |             |
| 0.0125   |        |                 |           |                 |           |                 |           |             |
| 1.8      | 0.1    | 7.6685e-03      | –         | 1.2759e-03      | –         | 8.5880e-04      | –         | 1.01        |
| 0.05     |        |                 |           |                 |           |                 |           |             |
| 0.025    |        |                 |           |                 |           |                 |           |             |
| 0.0125   |        |                 |           |                 |           |                 |           |             |
| $\alpha$ | $\tau$ | $\|u^N_{Nt} - U^N_{Nt}\|$ | Conv. rate | $\|\phi^N_{Nt} - \Phi^N_{Nt}\|$ | Conv. rate | $\|\phi^N_{Nt} - \Phi^N_{Nt}\|_\infty$ | Conv. rate | CPU time (s) |
|----------|-------|----------------|-----------|----------------|-----------|----------------|-----------|-----------|
| 1.2      | 0.1   | 7.6888e−03    | –         | 1.6980e−03    | –         | 1.2975e−03    | –         | 1.09      |
| 0.05     |       | 1.9294e−03    | 1.9946    | 4.2726e−04    | 1.9906    | 3.2780e−04    | 1.9848    | 1.62      |
| 0.025    |       | 4.8297e−04    | 1.9981    | 1.0699e−04    | 1.9976    | 8.2164e−05    | 1.9962    | 2.52      |
| 0.0125   |       | 1.2105e−04    | 1.9963    | 2.6759e−05    | 1.9994    | 2.0555e−05    | 1.9990    | 4.09      |
| 1.5      | 0.1   | 8.1416e−04    | –         | 1.6199e−03    | –         | 1.2303e−03    | –         | 1.05      |
| 0.05     |       | 2.0423e−04    | 1.9951    | 4.0976e−04    | 1.9830    | 3.1367e−04    | 1.9716    | 1.64      |
| 0.025    |       | 5.1110e−04    | 1.9951    | 1.0275e−04    | 1.9956    | 7.8783e−05    | 1.9933    | 2.62      |
| 0.0125   |       | 1.2794e−04    | 1.9981    | 2.5706e−05    | 1.9990    | 1.9725e−05    | 1.9978    | 4.18      |
| 1.8      | 0.1   | 9.2848e−03    | –         | 1.0966e−03    | –         | 6.9719e−04    | –         | 1.17      |
| 0.05     |       | 2.3492e−03    | 1.9827    | 2.7501e−04    | 1.9955    | 1.7479e−04    | 1.9959    | 1.55      |
| 0.025    |       | 5.8940e−04    | 1.9948    | 6.8813e−05    | 1.9987    | 4.3738e−05    | 1.9987    | 2.56      |
| 0.0125   |       | 1.4756e−04    | 1.9979    | 1.7233e−05    | 1.9975    | 1.0917e−05    | 2.0024    | 4.04      |
Table 4 Temporal accuracy of CN-SGM for Example 6.2 with $\kappa_2 = 0.1$ and $N = 150$ at $t = 1$

| $\alpha$ | $\tau$ | $\|u_{Nt}^{Nt} - U_{Nt}^{Nt}\|$ | Conv. rate | $\|\phi_{Nt}^{Nt} - \Phi_{Nt}^{Nt}\|$ | Conv. rate | $\|\phi_{Nt}^{Nt} - \Phi_{Nt}^{Nt}\|_\infty$ | Conv. rate | CPU time (s) |
|----------|--------|-------------------------------|------------|--------------------------------|------------|--------------------------------|------------|------------|
| 1.2      | 0.1    | 5.0647e-02                    | –          | 3.9455e-03                     | –          | 3.7289e-03                     | –          | 1.78       |
|          | 0.05   | 1.2893e-02                    | 1.9739     | 1.0040e-03                     | 1.9745     | 9.5011e-04                     | 1.9726     | 2.19       |
|          | 0.025  | 3.2382e-03                    | 1.9933     | 2.5212e-04                     | 1.9935     | 2.3866e-04                     | 1.9931     | 3.34       |
|          | 0.0125 | 8.1054e-04                    | 1.9982     | 6.3100e-05                     | 1.9984     | 5.9737e-05                     | 1.9983     | 5.28       |
| 1.5      | 0.1    | 5.4014e-02                    | –          | 6.6000e-03                     | –          | 5.9641e-03                     | –          | 1.76       |
|          | 0.05   | 1.3701e-02                    | 1.9790     | 1.6773e-03                     | 1.9763     | 1.5190e-03                     | 1.9731     | 2.22       |
|          | 0.025  | 3.4379e-03                    | 1.9947     | 4.2105e-04                     | 1.9941     | 3.8152e-04                     | 1.9933     | 3.41       |
|          | 0.0125 | 8.6037e-04                    | 1.9985     | 1.0537e-04                     | 1.9985     | 9.5486e-05                     | 1.9984     | 5.46       |
| 1.8      | 0.1    | 5.1174e-02                    | –          | 6.3376e-03                     | –          | 5.2555e-03                     | –          | 1.67       |
|          | 0.05   | 1.2977e-02                    | 1.9795     | 1.6217e-03                     | 1.9664     | 1.3529e-03                     | 1.9578     | 2.12       |
|          | 0.025  | 3.2563e-03                    | 1.9946     | 4.0782e-04                     | 1.9915     | 3.4072e-04                     | 1.9894     | 3.22       |
|          | 0.0125 | 8.1506e-04                    | 1.9982     | 1.0211e-04                     | 1.9978     | 8.5323e-05                     | 1.9976     | 5.18       |
Results are reported to verify that the proposed numerical schemes have excellent capabilities in numerical accuracy and long-time conservations. In future work, the convergence proof of the ESAV-SGM will be carried out, also, our schemes and theoretical results will be extended to the high-dimensional FNKGS equation.
Employing the Lagrange’s mean value theorem, we obtain

\[ \varphi(t, t_{n+1/2}) = \frac{\varphi(t, t_{n+1/2}) - \varphi(t, t_{n+1/2})}{t - t_{n+1/2}} = \frac{1}{2} \varphi'(t_{n+1/2}) - \frac{1}{2} \varphi'(t_{n+1/2}). \]

Assume that \( T \)

Proof Utilizing the Taylor’s expansion for \( \varphi \) and \( \psi \) of (3.5) at \( t = t_{n+1/2} \), we have

\[ \mathcal{R}_2 = \varphi(t, t_{n+1/2}) - \varphi(t, t_{n+1/2}) + \psi_{n+1/2} - \psi(t, t_{n+1/2}) = T_1 + T_2, \]

in which

\[ T_1 = \varphi(t, t_{n+1/2}) - \varphi(t, t_{n+1/2}) = -\frac{\tau^2}{16} \int_0^1 \left[ \varphi_{ttt}(t_{n+1/2} + \frac{s}{2}) + \varphi_{ttt}(t_{n+1/2} - \frac{s}{2}) \right] (1 - s)^2 ds \]

and

\[ T_2 = \varphi_{n+1/2} - \varphi(t, t_{n+1/2}) = \frac{\tau^2}{8} \int_0^1 \left[ \psi_{ttt}(t_{n+1/2} + \frac{s}{2}) + \psi_{ttt}(t_{n+1/2} - \frac{s}{2}) \right] (1 - s) ds \]

Assume that \( \varphi(t, t) \in C^3([0, T]) \). We deduce

\[ \left| \mathcal{R}_2 \right| \leq C \tau^2, \quad n = 0, 1, \ldots, N_t - 1. \]

Simultaneously,

\[ \delta_t \mathcal{R}_{2}^{n+1/2} = -\frac{\tau^2}{16} \int_0^1 \left[ \delta_t \varphi_{ttt}(t_{n+1/2} + \frac{s}{2}) + \delta_t \varphi_{ttt}(t_{n+1/2} - \frac{s}{2}) \right] (1 - s)^2 ds + \frac{\tau^2}{8} \int_0^1 \left[ \delta_t \psi_{ttt}(t_{n+1/2} + \frac{s}{2}) + \delta_t \psi_{ttt}(t_{n+1/2} - \frac{s}{2}) \right] (1 - s) ds. \]

Employing the Lagrange’s mean value theorem, we obtain

\[ \varphi_{ttt}(t_{n+1/2} + \frac{s}{2}) - \varphi_{ttt}(t_{n+1/2} + \frac{s}{2}) = \tau \varphi^{(4)}(\cdot, \xi), \]

\[ t_{n+1/2} + \frac{s}{2} < \xi < t_{n+1/2} + \frac{s}{2}, \]

and

\[ \varphi_{ttt}(t_{n+1/2} - \frac{s}{2}) - \varphi_{ttt}(t_{n+1/2} - \frac{s}{2}) = \tau \varphi^{(4)}(\cdot, \xi'), \]

\[ t_{n+1/2} - \frac{s}{2} < \xi' < t_{n+1/2} - \frac{s}{2}. \]
Suppose that $\phi(\cdot, t) \in C^4([0, T])$. We get

$$\left| \delta_t \tilde{\mathcal{F}}_2^{n+1/2} \right| \leq C \tau^2, \quad n = 0, 1, \ldots, N_t - 1.$$ 

This completes the proof of (3.11). 

\[\square\]

### B Appendix: Proof of Theorem 3.3

The proof of Theorem 3.3 is divided into two parts, including the existence and uniqueness.

#### (1) Existence:

**Proof** It is worth noting that $U_{N}^{n+1} = 2U_{N}^{n+1/2} - U_{N}^{n}$ and $\delta_t \Phi_N^{n+1/2} = \Psi_N^{n+1/2}$. We reformulate the spectral Galerkin scheme (3.12)–(3.14) into the following form

\[
\begin{align*}
(U_{N}^{n+1/2} - U_{N}^{n}, w_N) + & \frac{\lambda \tau}{4} i B (U_{N}^{n+1/2}, w_N) - \frac{\kappa_1 \tau}{2} i (U_{N}^{n+1/2} \Phi_N^{n+1/2}, w_N) \\
- & \frac{\kappa_2 \tau}{2} i ([2U_{N}^{n+1/2} - U_{N}^{n}]^2 + |U_{N}^{n}|^2) U_{N}^{n+1/2} \Phi_N^{n+1/2}, w_N) = 0, \quad (B.1) \\
(2\Phi_{N}^{n+1/2} - 2\Phi_N^{n} - \tau \Psi_N^{n}, w_N) + & \frac{\eta \tau}{2} B (\Phi_N^{n+1/2}, w_N) + \frac{\eta^2 \tau^2}{2} (\Phi_N^{n+1/2}, w_N) \\
- & \frac{\kappa_1 \tau^2}{4} ([2U_{N}^{n+1/2} - U_{N}^{n}]^2 + |U_{N}^{n}|^2), w_N) - \frac{\kappa_2 \tau^2}{4} ([2U_{N}^{n+1/2} - U_{N}^{n}]^4 + |U_{N}^{n}|^4), w_N) = 0. \quad (B.2)
\end{align*}
\]

Now, we carry out proving the existence of $U_{N}^{n+1/2}$ and $\Phi_N^{n+1/2}$. For convenience, we define the map $s = (s_1, s_2), \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : (X_0^0(\Omega), X_0^0(\Omega)) \rightarrow (X_0^0(\Omega), X_0^0(\Omega))$, such that

\[
(\mathcal{F}_1(s), w_N) = (s_1 - U_{N}^{n}, w_N) + \frac{\lambda \tau}{4} i B(s_1, w_N) - \frac{\kappa_1 \tau}{2} i (s_1 s_2, w_N) \\
- \frac{\kappa_2 \tau}{2} i ([2s_1 - U_{N}^{n}]^2 + |U_{N}^{n}|^2), s_1 s_2, w_N), \quad \forall w_N \in X_0^0(\Omega) \quad (B.3)
\]

and

\[
(\mathcal{F}_2(s), w_N) = (2s_2 - 2\Phi_N^{n} - \tau \Psi_N^{n}, w_N) + \frac{\eta \tau}{2} B(s_2, w_N) + \frac{\eta^2 \tau^2}{2} (s_2, w_N) \\
- \frac{\kappa_1 \tau^2}{4} ([2s_1 - U_{N}^{n}]^2 + |U_{N}^{n}|^2, w_N) \\
- \frac{\kappa_2 \tau^2}{4} ([2s_1 - U_{N}^{n}]^4 + |U_{N}^{n}|^4, w_N), \quad \forall w_N \in X_0^0(\Omega). \quad (B.4)
\]

Choosing $w_N = s_1$ in (B.3) and taking the real part, from the Young’s inequality, we derive

\[
\text{Re}(\mathcal{F}_1(s), s_1) = \text{Re}(s_1 - U_{N}^{n}, s_1) = \|s_1\|^2 - \text{Re}(U_{N}^{n}, s_1) \geq \frac{1}{2} \|s_1\|^2 - \frac{1}{2} \|U_{N}^{n}\|^2 \\
\geq \frac{1}{2} \|s_1\|^2 - C.
\]
Setting $w_N = s_2$ in (B.4), by utilizing the Cauchy–Schwarz inequality, we obtain

\[
(\mathcal{F}_2(s), s_2) \geq (2s_2 - 2\Phi^N_n - \tau \Psi^N_n, s_2) - \frac{\kappa_1 \tau^2}{4} \left( |2s_1 - U^N_n|^2 + |U^N_n|^2, s_2 \right) - \frac{\kappa_2 \tau^2}{4} \left( |2s_1 - U^N_n|^4 + |U^N_n|^4, s_2 \right)
\]

\[
= 2 \|s_2\|^2 - (2\Phi^N_n + \tau \Psi^N_n, s_2) - \frac{\kappa_1 \tau^2}{4} \left( |2s_1 - U^N_n|^2 + |U^N_n|^2, s_2 \right) - \frac{\kappa_2 \tau^2}{4} \left( |2s_1 - U^N_n|^4 + |U^N_n|^4, s_2 \right).
\]

(B.5)

By employing the boundness of the numerical solutions (see Theorem 3.2) and the Young’s inequality, for sufficient small $\tau$, we conclude

\[
(2\Phi^N_n + \tau \Psi^N_n, s_2) \leq \left\| 2\Phi^N_n + \tau \Psi^N_n \right\| \|s_2\| \leq \frac{1}{2} \left\| 2\Phi^N_n + \tau \Psi^N_n \right\|^2 + \frac{1}{2} \|s_2\|^2,
\]

(B.6)

\[
\frac{\kappa_1 \tau^2}{4} \left( |2s_1 - U^N_n|^2 + |U^N_n|^2, s_2 \right) \leq \frac{\kappa_1^2 \tau^4}{32} \left\| 2s_1 - U^N_n \right\|^2 + \frac{1}{2} \|s_2\|^2 \leq C + \frac{1}{2} \|s_2\|^2
\]

and

\[
\frac{\kappa_2 \tau^2}{2} \left( |2s_1 - U^N_n|^4 + |U^N_n|^4, s_2 \right) \leq C + \frac{1}{2} \|s_2\|^2.
\]

(B.7)

Substituting (B.6)–(B.8) into (B.5), we arrive at

\[
(\mathcal{F}_2(s), s_2) \geq \frac{1}{2} \|s_2\|^2 - \frac{1}{2} \left\| 2\Phi^N_n + \tau \Psi^N_n \right\|^2 - C.
\]

Therefore, we have

\[
\text{Re}(\mathcal{F}(s), s_1) = \text{Re}(\mathcal{F}_1(s), s_2) + \text{Re}(\mathcal{F}_2(s), s)
\]

\[
\geq \frac{1}{2} \|s\|^2 - \frac{1}{2} \left\| 2\Phi^N_n + \tau \Psi^N_n \right\|^2 - C = \frac{1}{2} \left( \|s\|^2 - \left\| \left( 2\Phi^N_n + \tau \Psi^N_n, \sqrt{2C} \right) \right\|^2 \right).
\]

Taking $\delta = \left\| \left( 2\Phi^N_n + \tau \Psi^N_n, \sqrt{2C} \right) \right\|$, which satisfies the condition of Lemma 3.5, we derive

\[
\text{Re}(\mathcal{F}(s), s) \geq 0, \quad \forall s : \|s\| = \delta.
\]

This proves the existence of the numerical solution of (3.12)–(3.14).

Next, we prove the uniqueness of the numerical solution.

II Uniqueness:

Proof We prove the theorem by introduction. It is obvious to find from (3.15) that the numerical solution $(U^N_n, \Phi^N_n, \Psi^N_n) \in (X^0_n(\Omega), X^1_n(\Omega), X^0_n(\Omega))$ exists and is unique. Assume that $(U^N_n, \Phi^N_n, \Psi^N_n)$ is the unique solution of (3.12)–(3.14) for $n = 0, 1, \ldots, N_t - 1$. Next, we prove the uniqueness of the solution $(U^{n+1/2}_N, \Phi^{n+1/2}_N)$. Assume there are two solutions $X^{n+1/2} = (X^{n+1/2}_1, X^{n+1/2}_2)$ and $Y^{n+1/2} = (Y^{n+1/2}_1, Y^{n+1/2}_2)$ for scheme (3.12)–(3.14). Then $X^{n+1/2}_1 - Y^{n+1/2}_1$ and $X^{n+1/2}_2 - Y^{n+1/2}_2$ satisfy (B.3) and (B.4) as follows

\[
(\mathcal{F}_1(X^{n+1/2}) - \mathcal{F}_1(Y^{n+1/2}), X^{n+1/2}_1 - Y^{n+1/2}_1) = 0,
\]

\[
(\mathcal{F}_2(X^{n+1/2}) - \mathcal{F}_2(Y^{n+1/2}), X^{n+1/2}_2 - Y^{n+1/2}_2) = 0.
\]
By the definition of $\mathcal{F}_1$, we have

\[
\begin{aligned}
\left\| X_1^{n+1/2} - Y_1^{n+1/2} \right\|^2 &= + \frac{\lambda \tau}{4} i B \left( X_1^{n+1/2} - Y_1^{n+1/2} , X_1^{n+1/2} - Y_1^{n+1/2} \right) \\
&- \frac{\kappa_1 \tau}{2} i ( X_1^{n+1/2} X_2^{n+1/2} - Y_1^{n+1/2} Y_2^{n+1/2} , X_1^{n+1/2} - Y_1^{n+1/2} ) \\
&- \frac{\kappa_2 \tau}{2} i \left( \left( 2X_1^{n+1/2} - U_n^1 \right)^2 + \left| U_n^1 \right|^2 \right) X_1^{n+1/2} X_2^{n+1/2} \\
&- \left( 2Y_1^{n+1/2} - U_n^1 \right)^2 + \left| U_n^1 \right|^2 Y_1^{n+1/2} Y_2^{n+1/2} , X_1^{n+1/2} - Y_1^{n+1/2} \right) = 0.
\end{aligned}
\]  

(B.9)

Taking the real part of (B.9), by admitting the boundness of the numerical solutions (see Theorem 3.2), Lemma 3.4 and the Cauchy–Schwarz inequality, we obtain

\[
\begin{aligned}
\left\| X_1^{n+1/2} - Y_1^{n+1/2} \right\|^2 &= - \frac{\kappa_1 \tau}{2} \text{Im} \left( X_1^{n+1/2} X_2^{n+1/2} - Y_1^{n+1/2} Y_2^{n+1/2} , X_1^{n+1/2} - Y_1^{n+1/2} \right) \\
&+ \frac{\kappa_2 \tau}{2} \text{Im} \left( \left( 2X_1^{n+1/2} - U_n^1 \right)^2 + \left| U_n^1 \right|^2 \right) X_1^{n+1/2} X_2^{n+1/2} \\
&- \left( 2Y_1^{n+1/2} - U_n^1 \right)^2 + \left| U_n^1 \right|^2 Y_1^{n+1/2} Y_2^{n+1/2} , X_1^{n+1/2} - Y_1^{n+1/2} \right) \\
&\leq C \tau \left( \left\| X_1^{n+1/2} - Y_1^{n+1/2} \right\|^2 + \left\| X_1^{n+1/2} - Y_1^{n+1/2} \right\|^2 \right).
\end{aligned}
\]

(B.10)

Taking into account of the definition of $\mathcal{F}_2$, we get

\[
\begin{aligned}
\left\| X_2^{n+1/2} - Y_2^{n+1/2} \right\|^2 &= + \frac{\eta \tau^2}{4} B \left( X_2^{n+1/2} - Y_2^{n+1/2} , X_2^{n+1/2} - Y_2^{n+1/2} \right) \\
&+ \frac{\eta \tau^2}{4} \left( X_2^{n+1/2} - Y_2^{n+1/2} , X_2^{n+1/2} - Y_2^{n+1/2} \right) \\
&- \frac{\kappa_1 \tau^2}{8} \left( \left( 2X_2^{n+1/2} - U_n^2 \right)^2 - \left( 2Y_2^{n+1/2} - U_n^2 \right)^2 , X_2^{n+1/2} - Y_2^{n+1/2} \right) \\
&- \frac{\kappa_2 \tau^2}{8} \left( \left( 2X_2^{n+1/2} - U_n^2 \right)^2 - \left( 2Y_2^{n+1/2} - U_n^2 \right)^2 , X_2^{n+1/2} - Y_2^{n+1/2} \right) = 0.
\end{aligned}
\]

(B.11)
Summing up \((B.10)\) and \((B.11)\), for a sufficient small \(\tau\), we arrive at
\[
\left\| X^{n+1/2} - Y^{n+1/2} \right\|^2 = \left\| X_1^{n+1/2} - Y_1^{n+1/2} \right\|^2 + \left\| X_2^{n+1/2} - Y_2^{n+1/2} \right\|^2 \\
\leq C\tau \left\| X_1^{n+1/2} - Y_1^{n+1/2} \right\|^2 + C\tau \left\| X_2^{n+1/2} - Y_2^{n+1/2} \right\|^2 \\
+ C\tau^2 \left\| X_2^{n+1/2} - Y_2^{n+1/2} \right\|^2 \\
\leq C\tau \left\| X_1^{n+1/2} - Y_1^{n+1/2} \right\|^2 + C\tau \left\| X_2^{n+1/2} - Y_2^{n+1/2} \right\|^2 \\
= C\tau \left\| X^{n+1/2} - Y^{n+1/2} \right\|^2.
\]

By assuming \(C\tau < 1\), it leads to
\[
\left\| X^{n+1/2} - Y^{n+1/2} \right\| = 0,
\]
which implies \(X_1^{n+1/2} = Y_1^{n+1/2}\) and \(X_2^{n+1/2} = Y_2^{n+1/2}\). This ends the proof of the uniqueness. \(\square\)

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