The equation of state of two dimensional Yang-Mills theory

Nikhil Karthik and Rajamani Narayanan

Department of Physics, Florida International University, Miami, FL 33199.

Abstract

We study the pressure, $P$, of SU($N$) gauge theory on a two-dimensional torus as a function of area, $A = l/t$. We find a cross-over scale that separates the system on a large circle from a system on a small circle at any finite temperature. The cross-over scale approaches zero with increasing $N$ and the cross-over becomes a first order transition as $N \to \infty$ and $l \to 0$ with the limiting value of $\frac{2Pl}{(N-1)t}$ depending on the fixed value of $Nl$.

*Electronic address: nkarthik@fiu.edu
†Electronic address: rajamani.narayanan@fiu.edu
I. INTRODUCTION

The partition function for SU($N$) gauge theory on a 2d torus with spatial extent $l$ and temperature $t$ is only a function of the area, $A = l/t$, and is given by [1]

$$Z_N(A) = \sum_r \exp \left( -\frac{C_r^{(2)} l}{N t} \right),$$  \hspace{1cm} \text{(1)}

where $C_r^{(2)}$ is the value of Casimir in the representation $r$. One can arrive at (1) by taking the continuum limit of a lattice formalism on a finite lattice [2]. The asymptotic behavior at large $N$ was studied in [3] where only representations with $C_r^{(2)}$ of $O(N)$ dominate. Since the partition function is a sum over string like states with energies proportional to the spatial extent, $l$, the pressure given by

$$P \equiv t \frac{\partial}{\partial l} \ln Z = \frac{\partial}{\partial A} \ln Z = -\frac{1}{N} \langle C_r^{(2)} \rangle,$$  \hspace{1cm} \text{(2)}

is negative.

The partition function for SU(2) is simple and given by

$$Z = \sum_{\lambda=0}^{\infty} e^{-\frac{(\lambda^2 + 2\lambda) A}{4}} = \frac{1}{2} e^{\frac{A}{4}} \left[ \sum_{\lambda=-\infty}^{\infty} e^{-\frac{\lambda^2 A}{4}} - 1 \right] = \frac{1}{2} e^{\frac{A}{4}} \left[ \sqrt{\frac{4\pi}{A}} \sum_{\lambda=-\infty}^{\infty} e^{-\frac{4\pi^2 \lambda^2}{A}} - 1 \right].$$  \hspace{1cm} \text{(3)}

The asymptotic behavior of the equation of state is

$$\frac{Pl}{t} = -\frac{3l}{4t} e^{-\frac{a}{t}} \quad \text{as} \quad l \to \infty,$$  \hspace{1cm} \text{(4)}

and

$$\frac{Pl}{t} = -\frac{1}{2} \quad \text{as} \quad l \to 0.$$  \hspace{1cm} \text{(5)}

The behavior at large $l$ is dominated by a few low lying energy states where as the behavior at small $l$ comes from a sum over all states and could be interpreted as the equipartition limit with the number of degrees of freedom being 1 for SU(2). The cross-over from the behavior on a large circle to a small circle is shown in Figure [\text{Figure}].

Expecting that the equipartition limit is given by

$$\frac{Pl}{t} = -\frac{N-1}{2} \quad \text{as} \quad l \to 0,$$  \hspace{1cm} \text{(6)}

for all $N$, we define

$$Q(\alpha) \equiv -\frac{2Pl}{(N-1)t}; \quad \text{with} \quad \alpha = \frac{Nl}{t},$$  \hspace{1cm} \text{(7)}

and study this quantity in this paper.
FIG. 1: The equation of state for SU(2) gauge theory on a two-dimensional torus is shown as the solid curve. The asymptotic values of $P l/t$ at small area is $-0.5$. At very large area, $P l/t$ behaves as $0.75 \exp(-0.75 l/t)l/t$, which is shown as the dotted curve. There is a cross-over between the two limits.

II. SUMMARY OF RESULTS

We will show the following results in this paper using a numerical simulation of the partition function in Eq. (1):

1. $Q(\alpha)$ falls on a universal curve as $N \to \infty$.

2. $Q(\alpha)$ goes to zero as $\alpha$ goes to infinity. This result implies that the pressure at infinite $N$ is zero for all $l$ at any $t$ as long as one takes $N \to \infty$ keeping $l$ and $t$ finite and is consistent with physics being independent of temperature and spatial extent in the infinite $N$ limit [4, 5].

3. $Q(\alpha)$ goes to unity as $\alpha$ goes to zero. This limit is reached from a finite $l$ and $t$ only at finite $N$.

4. There is a cross-over point defined as a peak in the susceptibility,

$$\chi = A \frac{\partial}{\partial A} Q = \alpha \frac{\partial}{\partial \alpha} Q.$$  (8)
(a) The large side of the cross-over is dominated by representations where $C_r^{(2)}$ are of $\mathcal{O}(N)$. This is the case of interest for all non-zero $l$ at infinite $N$ and studied in [3].

(b) The small side of the cross-over is dominated by representations where $C_r^{(2)}$ are of $\mathcal{O}(N^2)$.

5. Since the value of $Q$ at infinite $N$ and $l = 0$ (or equivalently $t = \infty$) depends on the approach to the limit, $N \to \infty$ and $l \to 0$, there is a first order transition confirming the argument in [6].

III. PROPERTIES OF CASIMIR FOR SU(N)

The representations of SU($N$) are specified by the sequence of integers $\Lambda_r = (\lambda_1, \lambda_2, \ldots, \lambda_{N-1})$, subjected to the ordering $\lambda_i \geq \lambda_{i+1}$ and the value of $C_r^{(2)}$ is

$$C_r^{(2)} = \sum_{i=1}^{N-1} \lambda_i^2 - \sum_{i=1}^{N-1} i\lambda_i - \frac{\lambda^2}{N} + (N + 1)\lambda \quad \text{where} \quad \lambda = \sum_{i=1}^{N-1} \lambda_i. \quad \text{(9)}$$

The maximum and the minimum value of Casimir, given the constraint that $\lambda$ has to be kept fixed, would be used in the subsequent sections. The representation with the maximum value of $C_r^{(2)}$ for a given $\lambda$ is given by

$$\Lambda_{\text{max}} = (\lambda, 0, \ldots, 0). \quad \text{(10)}$$

The minimum value of $C_r^{(2)}$ is given by the sequence $\Lambda_{\text{min}}$:

$$\lambda_i = \begin{cases} \left\lfloor \frac{\lambda}{N-1} \right\rfloor + 1 & \text{if } i \leq k \equiv \lambda - (N - 1)\left\lfloor \frac{\lambda}{N-1} \right\rfloor \\ \left\lfloor \frac{\lambda}{N-1} \right\rfloor & \text{if } i > k. \end{cases} \quad \text{(11)}$$

To prove that the two sequences extremize the Casimir, note that the Casimir decreases under the transformation $(\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_{N-1})$ to $(\lambda_1, \lambda_2, \ldots, \lambda_i - 1, \ldots, \lambda_j + 1, \ldots, \lambda_{N-1})$ for $j > i$, provided this transformation is allowed. Such a transformation is not possible for $\Lambda_{\text{min}}$. Similarly, the reverse of that transformation is not possible on $\Lambda_{\text{max}}$. One can prove by contradiction that $\Lambda_{\text{min}}$ and $\Lambda_{\text{max}}$ are unique to satisfy these properties.

We have shown the behaviour of the maximum and the minimum value of $C_r^{(2)}$ as a function of $\lambda$ in Figure 2. The minimum of $C_r^{(2)}$ shows a quasi-periodic behaviour, with
FIG. 2: Behaviour of Casimir as a function of $\lambda$. The upper solid curve is the maximum value of Casimir given a value of $\lambda$, as a function of $\lambda$. Similarly, the lower solid curve is the minimum value of Casimir given a value of $\lambda$, as a function of $\lambda$. The dotted line is where $C_r^{(2)} = N^2$.

troughs at $\lambda = qN$ for integer $q$. The values of Casimir at these troughs are

$$C_{\text{min}} = N \left( 1 + \left\lfloor \frac{q}{N-1} \right\rfloor \right) \left( 2q - \left\lfloor \frac{q}{N-1} \right\rfloor (N-1) \right), \quad (12)$$

whose dependence on $N$ is linear for $q$ between two multiples of $(N-1)$ and is quadratic for $q$ that are multiples of $(N-1)$. On very large circles (or at very low temperatures), one would expect that only the excitations around these troughs at small $q$ would be important. On very small circles (or at very high temperatures), large values of $q$ would become accessible, where all possible Casimir are $\mathcal{O}(N^2)$. This is the region above the red dotted line in Figure 2 where $C_r^{(2)}$ is larger than $N^2$. Qualitatively, this is the difference one might expect between the low and high temperature phases.

IV. HEAT-BATH ALGORITHM

We simulated the partition function in Eq. (1) by updating $\Lambda_r$ by the heat-bath algorithm. Each heat-bath update is a sequence of local updates from $\lambda_1$ to $\lambda_{N-1}$, in that order, such that the ordering of $\lambda_i$ is preserved. For the local update of $\lambda_i$, the probability distribution
of \(\lambda_i\) is given by a discrete version of the Gaussian distribution

\[
T(\lambda_i) \propto e^{-(\lambda_i - \mu_i)^2/2\sigma_i^2},
\]

subject to the condition \(\lambda_{i+1} \leq \lambda_i \leq \lambda_{i-1}\) for \(i > 1\) and \(\lambda_2 \leq \lambda_1\). The \(\mu_i\) and \(\sigma_i\) for the above discrete Gaussian distribution are functions of the rest of the \(\lambda_i\)'s forming the heat-bath:

\[
\mu_i = \overline{\lambda} + \frac{N(2i - N - 1)}{2(N - 1)} \quad \text{and} \quad \sigma_i^2 = \frac{N^2}{2A(N - 1)},
\]

where \(\overline{\lambda} = \sum_{j \neq i} \lambda_j\). For \(i > 1\), the set of allowed values for \(\lambda_i\) is bounded from above and below. Hence, we included all the allowed possibilities weighted by Eq. (13) as candidates for the update. Since Eq. (14), along with the inequality \(\lambda - \lambda_1 < (N - 2)\lambda_2\), implies that \(\mu_1 < \lambda_2\), the probability for \(\lambda_1\) is a monotonically decreasing function. This enables one to put an upper cut-off on \(\lambda_1\). In our calculation, we used an upper cut-off of \(\lambda_2 + 3\sigma\). We also checked that changing this value to \(\lambda_2 + 10\sigma\) does not cause any statistically significant changes. Since a representation \(r\) and its conjugate representation \(\overline{r}\) have the same Casimir, one can do an over-relaxation step by a global update \(\lambda'_i = \lambda_1 - \lambda_{N-i+1}\).

In our simulations, the successive measurements were separated by 100 iterations of 2 heat-bath and 1 over-relaxation steps. The first 2000 measurements were discarded for thermalization. In this way, we collected \(10^4\) configurations of \(\Lambda_r\) at all area and \(N\).

V. RESULTS

In the top panel of Figure 3, we show the behaviour of \(Q\) as a function of the scaled area \(\alpha\) for various values of \(N\). The important thing to notice is that \(Q\) has a large-\(N\) limit when plotted as a function of \(\alpha\). For \(\alpha \ll 1\), \(Q\) seems to approach 1 for all \(N\). This is in agreement with our intuition based on the equipartition theorem. The non-trivial observation is that this cross-over to the equipartition limit happens at a finite value of \(\alpha\) in the large-\(N\) limit. For \(\alpha \gg 1\), \(Q\) seems to behave as \(N^{-1}\exp(-\sigma\alpha/N)\) for a constant \(\sigma \approx 0.81\) in the large-\(N\) limit. This is shown in the bottom panel of Figure 3. Thus, it can also be seen as a cross-over from strong-coupling regime, which has a scale \(\sigma\), to the weak-coupling regime with no underlying scale.

We determined the cross-over point \(\alpha_c\) using the peak-position of the susceptibility \(\chi\), after interpolating using multi-histogram reweighting. We show \(\chi\) as a function of \(\alpha\) in...
FIG. 3: $Q$ as a function of the scaled area $\alpha = N A$ is shown in the top panel. It is seen that $Q$ as a function of $\alpha$ has a large-$N$ limit. For very small values of $\alpha$, $Q$ approaches 1. In the bottom panel, the large area behaviour of $Q$ in the large-$N$ limit is shown. In this case, $QN$ behaves as $\exp (-\sigma \alpha / N)$.

Figure 4 for various $N$. The susceptibility also has a large-$N$ limit when plotted as a function of $\alpha$. The peak positions of susceptibility for $N > 19$ agree within errors, giving us an estimate $\alpha_c = 12.1(2)$. This implies that the cross-over area $A_c = \alpha_c / N$ shifts to smaller values at larger $N$. The width of the susceptibility when expressed in terms of the area $A$ decreases inversely as $N$. This is characteristic of finite volume scaling near a first order phase transition, with the large-$N$ limit replacing the thermodynamic limit in this case.
FIG. 4: Susceptibility $\chi$ as a function of the scaled area $\alpha = NA$. The cross-over coupling $\alpha_c$ is shown by the vertical line.

The reason for this cross-over can be understood from the scatter plot of $C_r^{(2)}$ versus $\lambda$ measured during the course of the Monte Carlo run using a value of $\alpha$. Such scatter plots at various $\alpha$ are shown in Figure 5 for two different $N$. We also show the maximum and minimum value of Casimir at a fixed $\lambda$, as a function of $\lambda$. As discussed earlier, the minimum Casimir shows a quasi-periodic behaviour forming wells with a periodicity $N$. At large values of $\alpha$, the representations near the troughs of these wells at small values of $\lambda$ get populated. The representations within these wells are sparse, and this discreteness govern the large area behaviour. At very small area, the most probable $C_r^{(2)}$ moves away from the line of minimum $C_r^{(2)}$ and remains in a region where one can approximate the distribution of Casimir by a continuum. The cross-over between the two behaviours is what shows up as a peak in $\chi$. As discussed in Section III, the Casimir near the troughs at small $\lambda$ is of $\mathcal{O}(N)$, while the Casimir at very large $\lambda$ is of $\mathcal{O}(N^2)$. As shown by the dotted line in Figure 5, this cross-over at $\alpha \approx 12.1$ roughly occurs when the dominant behaviour $C_r^{(2)}$ changes from $\mathcal{O}(N)$ to $\mathcal{O}(N^2)$. 
VI. CONCLUSIONS

Yang-Mills theory in two dimensions is always in the confined phase. We focused on the quantity, \( Q = -\frac{2PL}{(N-1)t} \), to study the equation of state. We showed that the equation of state shows a cross-over from strong coupling (large spatial extent) to weak coupling (small spatial extent) within the confined phase. Viewed as a function of \( \alpha = \frac{ln t}{t} \), \( Q(\alpha) \) approaches a universal curve as \( N \to \infty \) as shown in Figure 6. This behavior is similar to the Durhuus-Olesen transition [7, 8] with the double scaling limit for the equation of state being \( N \to \infty \) and \( l \to 0 \) (or \( t \to \infty \)) keeping \( \alpha = \frac{ln l}{t} \) fixed. There is a line of cross-over, \( \frac{ln l}{t} = \alpha_c \), extending
from the origin in the $\frac{l}{t} - \frac{1}{N}$ diagram as shown in Figure 7. Well above this line, $Q \ll 1$ and it behaves as $\exp(-\sigma A)/N$. Well below this line, $Q$ is approximately 1. Depending on the slope, $\alpha$, of the line along which the $N \to \infty$ and $\frac{l}{t} \to 0$ limit is taken, the limiting value of $Q$ differs. Specifically, if $N \to \infty$ limit is taken after the $A \to 0$ limit is taken, then $Q$ is 1. When the two limits are reversed, $Q$ becomes 0. Therefore, the cross-over along $AN = \alpha_c$ becomes a first order transition at vanishing area in the large-$N$ limit.

The equation of state in four dimensional Yang-Mills theories for several different values of $N$ has been recently studied [9]. The pressure is found to be close to zero in the confined phase. In light of this paper, it would be interesting to perform a careful study of the equation of state in the confined phase in three and four dimensions and see if one can see a cross-over similar to the one seen here in two dimensions.

Acknowledgments

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[1] A. A. Migdal, Sov.Phys.JETP 42, 413 (1975).

[2] J. Kiskis, R. Narayanan, and D. Sigdel, Phys.Rev. D89, 085031 (2014), 1403.1770.
FIG. 7: Phase diagram. Various approaches to vanishing area at large-$N$ are indicated by lines with arrows. The critical value of the slope $AN = \alpha_c$ is shown as the dot-dashed line. For values of $AN \gg \alpha_c$ (the dotted line), $Q$ decays exponentially with area. For values of $AN \ll \alpha_c$ (the dashed line), $Q \approx 1$. In particular, when $A$ is reduced to 0 after taking the large-$N$ limit (i.e., along $y$-axis), $Q$ vanishes. When the two limits are interchanged (i.e., along $x$-axis), $Q$ becomes 1.

[3] D. J. Gross and W. Taylor, Nucl.Phys. B400, 181 (1993), hep-th/9301068.
[4] D. Gross and E. Witten, Phys.Rev. D21, 446 (1980).
[5] T. Eguchi and H. Kawai, Phys.Rev.Lett. 48, 1063 (1982).
[6] L. D. McLerran and A. Sen, Phys.Rev. D32, 2794 (1985).
[7] B. Durhuus and P. Olesen, Nucl.Phys. B184, 461 (1981).
[8] R. Narayanan and H. Neuberger, JHEP 0712, 066 (2007), 0711.4551.
[9] S. Datta and S. Gupta, Phys.Rev. D82, 114505 (2010), 1006.0938.