Open markets

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Abstract
An open market is a subset of a larger equity market, composed of a certain fixed number of top-capitalization stocks. Though the number of stocks in the open market is fixed, their composition changes over time, as each company’s rank by market capitalization fluctuates. When one is allowed to invest also in a money market, an open market resembles the entire “closed” equity market in the sense that the market viability (lack of arbitrage) is equivalent to the existence of a numéraire portfolio (which cannot be outperformed). When access to the money market is prohibited, the class of portfolios shrinks significantly in open markets; in such a setting, we discuss the Capital Asset Pricing Model, how to construct functionally generated portfolios, and the concept of universal portfolio.

Keywords
functionally generated portfolio, universal portfolio, local martingale deflator, market viability, numéraire portfolio, open market, optional decomposition

1 | INTRODUCTION

Equity markets are conventionally thought of as being closed, in the sense that they are almost universally assumed to consist of a given, fixed number of stocks at all times. However, this assumption fails to represent most real markets, where new stocks enter and some others exit due to privatization, bankruptcy, or simply bad luck.

The number of companies in the U.S. stock market has undergone wide fluctuations. In 1975, there were around 4,800 U.S. domiciled firms listed on the NYSE, Amex, and Nasdaq. This number reached a peak of 7,500 listed firms in 1997, and then decreased by more than half to 3,600 firms 20 years later in 2017.
To mitigate the assumption of a fixed, immutable collection of companies, and to model stock markets more realistically, we study here markets that are open. These are constructed by restricting, at any given time, our “investing space” from the entire market to the subset composed of a certain fixed number $n$ of top-capitalization stocks at that time. More precisely, within the entire stock market, we keep track of the price dynamics of all stocks, rank them by order of market capitalization, consider an open market consisting of the top $n$ stocks, and only invest in stocks that belong to this open market. High-capitalization indices, such as the S&P 500 index, where one invests only in the $n = 500$ highest capitalization companies and any given stock is replaced by another one when its capitalization falls, are of this type.

In this paper, we present some results from closed markets, which remain valid also in open markets. The main result of this type involves the concept of “market viability,” which is understood as “lack of a certain egregious form of arbitrage”; this condition prohibits financing nontrivial liabilities starting from arbitrarily small initial capital. The result shows that in an open equity market, and with access to a money market, viability is equivalent to any one of the following conditions: (i) a portfolio with the local martingale numéraire property exists, (ii) a local martingale deflator exists, (iii) the market has locally finite maximal growth, (iv) a growth-optimal portfolio exists, and (v) a portfolio with the log-optimality property exists. We provide precise definitions for these terms, and show that this equivalence can be formulated in terms of the drifts and covariances of the underlying stock prices, modeled by continuous semimartingales.

When access to a money market is forbidden, and one is only allowed to invest in a fixed number $n$ of top-capitalization stocks, the investing space diminishes significantly, as portfolios must satisfy conditions of “self-financibility.” Under this extra condition, we provide a connection of the above viability theory to the Capital Asset Pricing Model (CAPM), develop a way for constructing functionally generated portfolios, and discuss the concept of universal portfolio in an open market.

**Preview**: Section 2 defines open markets, investment strategies, and portfolios, as well as other notions needed throughout this paper. Section 3 develops arbitrage theory in open markets, along with the concepts of market viability and numéraire. We provide definitions and properties for these concepts, then state and prove the main result. Section 4 explores stock portfolios, the CAPM, functional generation of portfolios, as well as the so-called “universal portfolio,” in the open market context. Section 5 provides some concluding remarks.

## 2 PORTFOLIOS IN OPEN MARKETS

Let us suppose that the “whole equity market universe” is composed of $N$ stocks, and that we are only interested in investing in the top $n$ largest capitalization stocks, for some fixed $1 \leq n < N$. For example, when our investing universe is the entire U.S. stock market, by setting $n = 500$ we are investing in those large companies which form the S&P 500 index. In order to invest in these top $n$ stocks, we must keep track of the rank of each stock’s capitalization at all times, and put together a portfolio composed of the $n$ stocks with the largest capitalizations.

Throughout this paper, we fix two positive integers $n$ and $N$ satisfying $1 \leq n < N$ as above. We suppose that trading is continuous in time, with no transaction costs or taxes, and that shares are infinitely divisible. We assume without loss of generality that each stock has a single share outstanding, and the price of a stock is equal to its capitalization; thus, we use the terms “price of stock” and “capitalization of stock” interchangeably. We also assume that stock prices are
discounted by the money market, and adopt the convention that the money market pays and charges zero interest.

## 2.1 Stock prices and their ranks

We start by presenting the following definition of price process in the market described above.

**Definition 2.1 (Price process).** For an $N$-dimensional vector $S \equiv (S_1, ..., S_N)$ of continuous semimartingales on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(\cdot), \mathbb{P})$, we call $S$ a *vector of price processes* if each component is strictly positive, i.e., the inequalities $S_i(t) > 0$ hold for all $i \in \{1, ..., N\}$ at any time $t \geq 0$. The component processes of $S$ represent the stock prices, or the capitalizations, of $N$ companies.

We need now to clarify some notation regarding ranks. Given the vector $S$ of price processes, we define the $k$-th ranked process $S(k)$ of $S_1, ..., S_N$ by

$$S(k)(t) := \max_{1 \leq i_1 < \cdots < i_k \leq N} \min\{S_{i_1}(t), ..., S_{i_k}(t)\}, \quad t \geq 0. \quad (1)$$

To be more specific, for any $t \geq 0$, we have

$$\max_{i=1,...,N} S_i(t) = S(1)(t) \geq S(2)(t) \geq \cdots \geq S(N-1)(t) \geq S(N)(t) = \min_{i=1,...,N} S_i(t); \quad (2)$$

that is, we rank the components of the vector process $S = (S_1, ..., S_N)$ in descending order, with the lexicographic rule for breaking ties that always assigns a higher rank (i.e., a smaller $(k)$) to the smallest index $i$.

**Definition 2.2 (Price process by rank).** For the vector $S$ of price processes in Definition 2.1, we call the $N$-dimensional vector process

$$S(t) \equiv (S(1)(t), ..., S(N)(t)), \quad t \geq 0, \quad (3)$$

where each component is defined via (1), the vector of price processes by rank. In particular, the $k$-th component $S_k(t) = S(k)(t)$ of the vector $S(t)$ represents the price of the $k$-th ranked stock among all $N$ companies at time $t$.

Each component of the vector process $S(\cdot)$ is also a continuous semimartingale, from the results in Banner and Ghomrasni (2008). Along with the notation (1), we define a process $\{1, ..., N\} \times [0, \infty) \ni (i, t) \mapsto u_i(t) \in \{1, ..., N\}$, such that each $u_i(\cdot)$ is predictable and satisfies

$$S_i(t) = S(u_i(t))(t), \quad \forall t \geq 0, \quad (4)$$

for every $i = 1, ..., N$. In other words, $u_i(t)$ is the rank of the $i$-th stock $S_i(t)$ at time $t$, for any given index $i = 1, ..., N$. Note that for every fixed $t \geq 0$, the function $u_i(t) : \{1, ..., N\} \rightarrow \{1, ..., N\}$ is a bijection, because we break ties using the lexicographic rule when defining (1).
2.2  Cumulative return processes

In this subsection, we present the notion of cumulative returns of the market. We first define the stochastic logarithm \( \mathcal{L}(Y) \) of a positive continuous semimartingale \( Y \) with \( Y(0) = 1 \) by

\[
\mathcal{L}(Y) := \int_0^t \frac{dY(t)}{Y(t)},
\]

and consider the vector \( R \equiv (R_1, ..., R_N) \), whose every component is the stochastic logarithm of the corresponding normalized component of \( S \) in Definition 2.1:

\[
R_i := \mathcal{L}\left(\frac{S_i}{S_i(0)}\right), \quad i = 1, ..., N.
\]

Each component process \( R_i \) is again a semimartingale and represents the cumulative returns of the \( i \)-th stock, since it has the dynamics

\[
dR_i(t) = \frac{dS_i(t)}{S_i(t)}, \quad t \geq 0, \quad \text{and} \quad R_i(0) = 0 \quad \text{for} \quad i = 1, ..., N.
\]

We posit the semimartingale decomposition

\[
R_i = A_i + M_i, \quad i = 1, ..., N,
\]

for each component of the vector \( R = (R_1, ..., R_N) \). Here, the component \( A_i \) of the vector process \( A \equiv (A_1, ..., A_N) \) with \( A_i(0) = 0 \) is adapted, continuous, and of finite variation on compact time intervals; whereas each component \( M_i \) of the vector process \( M \equiv (M_1, ..., M_N) \) is a continuous local martingale with \( M_i(0) = 0 \), for \( i = 1, ..., N \). We think of the finite variation processes \( A_i \) as the “drift components,” and of the local martingales \( M_i \) as the “noise components,” of \( R \).

We define next the continuous, nondecreasing scalar process

\[
O := \sum_{i=1}^N \left( \int_0^T |dA_i(t)| + d[M_i, M_i](t) \right),
\]

where \( \int_0^T |dA_i(t)| \) denotes the total variation of \( A_i \) on the interval \([0, T]\) for \( T \geq 0 \) and \([M_i, M_i]\) represents the covariation process of the continuous semimartingales \( M_i \) and \( M_j \) for \( 1 \leq i, j \leq N \). We note that \([R_i, R_j] = [M_i, M_j]\) holds from (8). This scalar process \( O \) plays the role of an “operational clock” for the vector \( R \). All processes \( A_i \) and \([M_i, M_j]\) for \( 1 \leq i, j \leq N \) are absolutely continuous with respect to this clock, and thus, by the Radon–Nikodým Theorem, there exist two predictable processes

\[
\alpha \equiv (\alpha_i)_{1 \leq i \leq N} \quad \text{and} \quad c \equiv (c_{i,j})_{1 \leq i, j \leq N},
\]

vector-valued and matrix-valued, respectively, such that

\[
A = \int_0^T \alpha(t) dO(t), \quad \text{and} \quad C \equiv [M, M] = \int_0^T c(t) dO(t).
\]
Here and in what follows, we write \( C \equiv (C_{i,j})_{1 \leq i, j \leq N} \) for the nonnegative-definite, matrix-valued process of covariations

\[
C_{i,j} := [M_i, M_j] = [R_i, R_j], \quad \text{for } 1 \leq i, j \leq N. \tag{12}
\]

The component \( \alpha_i \) in (10) represents the local rate of return of the \( i \)-th stock in the market; whereas the entry \( c_{i,j} \) stands for the local covariation rate of the \( i \)-th and \( j \)-th stocks. We call the collection of local rates \( \alpha, c \) in (10) the local characteristics of the market, and these rates are measured with respect to the operational clock \( O \) in (9).

For a continuous vector-valued semimartingale \( Y = (Y_1, \ldots, Y_N) \), we denote by \( I(Y) \) the class of predictable vector processes \( \pi = (\pi_1, \ldots, \pi_N) \), which are integrable with respect to the vector process \( Y \). In particular, for the collection \( I(R) \) of the vector process \( R \) in (6) and (8), we have a very convenient characterization: A predictable vector process \( \pi = (\pi_1, \ldots, \pi_N) \) belongs to \( I(R) \), if and only if

\[
\int_0^T \left( |\pi'(t)\alpha(t)| + \pi'(t)c(t)\pi(t) \right) dO(t) < \infty \quad \text{holds for any } \quad T \geq 0. \tag{13}
\]

We denote then by

\[
\int_0^T \sum_{i=1}^N \pi_i(t)dR_i(t) = \int_0^T \pi'(t)dR(t) = \int_0^T \pi'(t)dA(t) + \int_0^T \pi'(t)dM(t)
\]

the stochastic integral of \( \pi \in I(R) \) with respect to the vector semimartingale \( R \).

### 2.3 Investment strategies and portfolios

Along with the \( N \)-dimensional vector \( S \) of Definition 2.1, representing the stock prices of the market, we introduce the following notions.

**Definition 2.3** (Investment strategy, wealth process, and numéraire). We call an \( N \)-dimensional vector of predictable process \( \theta \equiv (\theta_1, \ldots, \theta_N) \) investment strategy, if it is integrable with respect to the price vector \( S \), i.e., \( \theta \in I(S) \). For any nonnegative real number \( x \), we call

\[
X(\cdot; x, \theta) := x + \int_0^\cdot \theta'(t)dS(t) \equiv x + \int_0^\cdot \sum_{i=1}^N \theta_i(t)dS_i(t) \tag{14}
\]

the wealth process generated by \( \theta \) with initial capital \( x \). We call the wealth process numéraire, if \( X(\cdot; 1, \theta) > 0 \) holds for the normalized initial capital \( x = 1 \). The collection of all numéraires is denoted by \( \mathcal{X} \).

The \( i \)-th component \( \theta_i(t) \) represents the units of investment (or number of shares) held in the \( i \)-th stock at time \( t \), and plays the role of integrand with integrator \( dS_i(t) \) in the stochastic integral of (14). The requirement \( X(0) = x = 1 \) in defining numéraires is a simple normalization, because \( X(\cdot; cx, c\theta) = cX(\cdot; x, \theta) \) holds for any positive real number \( c \).
Since we consider investment only in the top $n$ stocks, we need a similar definition of investment strategy for this particular case.

**Definition 2.4** (Investment strategy among the top $n$ stocks). We call an investment strategy $\vartheta \in I(S)$ an investment strategy among the top $n$ stocks, if the “censoring” equalities

$$\vartheta_i(t)1_{\{u_i(t)>n\}} = 0, \quad \text{for} \quad i = 1, \ldots, N, \quad t \geq 0,$$

(15)

hold with the notation (4).

The wealth process and the numéraire associated with this investment strategy $\vartheta$ among the top $n$ stocks, are defined in the same manner as in Definition 2.3. We denote by $T(n)$ the collection of $N$-dimensional predictable processes $\vartheta$ satisfying the condition (15), and by $I(S) \cap T(n)$ the collection of investment strategies among the top $n$ stocks.

The collection of all numéraires generated by investment strategies $\vartheta \in I(S) \cap T(n)$ among the top $n$ stocks, is denoted by $\mathcal{K}^n$.

The condition (15) prohibits the strategy $\vartheta$ from investing in stock $i$ at time $t \geq 0$, if this stock fails to rank at that time among the top $n$ stocks in terms of capitalization. We present another definition, that of a portfolio rule, which plays the role of integrand with respect to the integrator $dR_i(t)$ of (6).

**Definition 2.5** (Portfolio). We call an $N$-dimensional predictable, vector-valued process $\pi \equiv (\pi_1, \ldots, \pi_N) \in I(R)$ a portfolio, if it is integrable with respect to the cumulative return vector $R$ of (6). We call a portfolio $\pi \in I(R)$ a portfolio among the top $n$ stocks, if the equalities

$$\pi_i(t)1_{\{u_i(t)>n\}} = 0, \quad \text{for} \quad i = 1, \ldots, N, \quad t \geq 0,$$

(16)

hold with the notation (4). We denote the collection of portfolios among the top $n$ stocks by $I(R) \cap T(n)$.

Since the function $u_i(t): \{1, \ldots, N\} \to \{1, \ldots, N\}$ is bijective for every $t \geq 0$, the collection $\{1_{\{u_i(t)=k\}}\}_{k=1,\ldots,N}$ constitutes a partition of unity for any given $i = 1, \ldots, N, t \geq 0$, and the conditions (15), (16) can also be formulated, respectively, as

$$\vartheta_i(t) = \sum_{k=1}^n \vartheta_i(t)1_{\{u_i(t)=k\}} = \vartheta_i(t)1_{\{u_i(t)\leq n\}}, \quad \text{for} \quad i = 1, \ldots, N, \quad t \geq 0,$$

(17)

$$\pi_i(t) = \sum_{k=1}^n \pi_i(t)1_{\{u_i(t)=k\}} = \pi_i(t)1_{\{u_i(t)\leq n\}}, \quad \text{for} \quad i = 1, \ldots, N, \quad t \geq 0.$$

(18)

We present next the connection between investment strategies $\vartheta$ and portfolios $\pi$. For any scalar continuous semimartingale $Z$ with $Z(0) = 0$, we denote the stochastic exponential of $Z$ by

$$\mathcal{E}(Z) := \exp \left( Z - \frac{1}{2} [Z, Z] \right).$$

(19)
It can be shown that this is also the unique process satisfying the linear stochastic integral equation

\[ \mathcal{E}(Z) = 1 + \int_0^\cdot \mathcal{E}(Z)(t) dZ(t). \]  

(20)

It is straightforward to check that the stochastic logarithm operator \( \mathcal{L}(\cdot) \) in (5), is the inverse of the stochastic exponential operator \( \mathcal{E}(\cdot) \) in (19).

We introduce now the \textit{cumulative returns process of a portfolio} \( \pi \) as in Definition 2.5, via the vector stochastic integral

\[ R_\pi := \int_0^\cdot \pi'(t)dR(t) = \int_0^\cdot \sum_{i=1}^N \pi_i(t)dR_i(t), \]  

(21)

and consider its stochastic exponential

\[ X_\pi := \mathcal{E}(R_\pi) = \mathcal{E} \left( \int_0^\cdot \sum_{i=1}^N \pi_i(t)dR_i(t) \right). \]  

(22)

In particular, we note that \( X_\pi \) is positive. Then, from (22), (21), and (7), we obtain the dynamics

\[ \frac{dX_\pi(t)}{X_\pi(t)} = dR_\pi(t) = \sum_{i=1}^N \pi_i(t)dR_i(t) = \sum_{i=1}^N \pi_i(t)\frac{dS_i(t)}{S_i(t)}, \quad X_\pi(0) = 1. \]  

(23)

By setting

\[ \vartheta_i := \frac{X_\pi \pi_i}{S_i} \quad \text{for} \quad i = 1, ..., N, \]  

(24)

we arrive at Equation (14) with \( X(\cdot; 1, \vartheta) \) replaced by \( X_\pi(\cdot) \). Thus, from the portfolio \( \pi \in I(R) \), we can obtain the corresponding investment strategy \( \vartheta \) and its numéraire \( X(\cdot; 1, \vartheta) \), via the recipe (24). Here, we denote the numéraire generated by the portfolio \( \pi \) by \( X(\cdot; 1, \vartheta) := X_\pi \), as in (22).

Conversely, for a given investment strategy \( \vartheta \) generating a positive wealth process, i.e., the numéraire \( X(\cdot; 1, \vartheta) \), we define a predictable, vector-valued process \( \pi \equiv (\pi_1, ..., \pi_N) \) as

\[ \pi_i := \frac{S_i \vartheta_i}{X(\cdot; 1, \vartheta)} \quad \text{for} \quad i = 1, ..., N. \]  

(25)

It can be easily checked that \( \pi \) is indeed a portfolio, i.e., \( R \)-integrable and (14) can be written as

\[ X(\cdot; 1, \vartheta) = 1 + \int_0^\cdot X(t; 1, \vartheta) \sum_{i=1}^N \pi_i(t)dR_i(t), \]  

with the help of (7). This last equation gives the dynamics in (23) with \( X_\pi(\cdot) \equiv X(\cdot; 1, \vartheta) \).

Thus, whether we start from an investment strategy \( \vartheta \) generating a numéraire, or from a portfolio \( \pi \), the counterpart can always be obtained via (25) or (24), respectively. We will denote the corresponding numéraire \( X(\cdot; 1, \vartheta) \) in (14) by \( X_\pi \) as in (22).
In the relationship (25), the product \( S_i(t) \bar{\theta}_i(t) \) represents the amount of wealth invested in the \( i \)-th stock at time \( t \); thus \( \pi_i(t) \) can be interpreted as the proportion of current wealth invested in the \( i \)-th stock at time \( t \). The remaining proportion of wealth

\[
\pi_0 := 1 - \sum_{i=1}^{N} \pi_i
\]  

(26)
is then considered to be placed in the money market.

We present now a few more concepts regarding portfolios. For any two portfolios \( \pi, \rho \) in \( I(R) \), we consider the covariation process between the cumulative returns \( R_\pi, R_\rho \) in (21), namely,

\[
C_{\pi\rho} := [R_\pi, R_\rho] = \int_{0}^{\cdot} c_{\pi\rho}(t) d\mathcal{O}(t), \quad \text{with} \quad c_{\pi\rho} := \pi' c \rho = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i c_{i,j} \rho_j.
\]  

(27)

Here, we recall the definitions of the matrix-valued processes \( c \) and \( C \) in (10), (11) and note the notational consistency with (27). In particular, when the portfolio is given as the unit vector \( e^i \) of \( \mathbb{R}^N \) for some \( i = 1, \ldots, N \), we use the subscript “\( i \)” instead of “\( e^i \)” to write \( C_{i\rho} \equiv C_{e^i \rho} \) and \( c_{i\rho} \equiv c_{e^i \rho} \) in order to ease notation. This convention is consistent with the actual equalities \( C_{i,j} = C_{e^i e^j} \) and \( c_{i,j} = c_{e^i e^j} \) for \( 1 \leq i, j \leq N \).

Recalling the wealth process \( X_\pi \) generated by the portfolio \( \pi \) as in (22), (23), we can express the logarithm of \( X_\pi \) as

\[
\log X_\pi = R_\pi - \frac{1}{2} C_{\pi\pi} = \int_{0}^{\cdot} \pi'(t) dA(t) - \frac{1}{2} C_{\pi\pi} + \int_{0}^{\cdot} \pi'(t) dM(t).
\]  

(28)

We call the finite-variation part of \( \log X_\pi \) the cumulative growth of the portfolio \( \pi \), and denote it by

\[
\Gamma_\pi := A_\pi - \frac{1}{2} C_{\pi\pi}, \quad \text{where} \quad A_\pi := \int_{0}^{\cdot} \pi'(t) dA(t).
\]  

(29)

In a similar manner, the local martingale part of the decomposition in (28) is denoted by

\[
M_\pi := \int_{0}^{\cdot} \pi'(t) dM(t).
\]  

(30)

In particular, the cumulative return \( R_\pi \) in (22) is the stochastic logarithm \( \mathcal{L}(X_\pi) \) of \( X_\pi \), and has “drift” component \( A_\pi \) as in (29), from (21) and (8); whereas the natural logarithm \( \log X_\pi \) in (28) of \( X_\pi \) has “drift” term \( \Gamma_\pi \).

We further define the predictable processes

\[
\alpha_\pi := \pi' \alpha, \quad \gamma_\pi := \pi' \alpha - \frac{1}{2} \pi' c \pi = \alpha_\pi - \frac{1}{2} c_{\pi\pi},
\]  

(31)

and call \( \alpha_\pi \) the rate of return, and \( \gamma_\pi \) the growth rate, of the portfolio \( \pi \). The “drift parts” \( A_\pi \) and \( \Gamma_\pi \), of \( \mathcal{L}(X_\pi) \) and \( \log X_\pi \), respectively, are then represented as the integrals of these rates with
respect to the “operational clock” in (9):

\[ A_\pi = \int_0^t \alpha_\pi(t) d\mathcal{O}(t), \quad \Gamma_\pi = \int_0^t \gamma_\pi(t) d\mathcal{O}(t). \tag{32} \]

### 2.4 Portfolios among the top \( n \) stocks

In this subsection, we provide definitions, similar to those introduced in the previous subsections, for portfolios that invest only among the top \( n \) stocks.

For \( \vartheta \in I(S) \cap T(n) \) and \( \pi \in I(R) \cap T(n) \), representing a strategy that invests only among the top \( n \) stocks and a portfolio among the top \( n \) stocks, respectively, Equations (21)–(26) can be used in the same manner. In particular, the bidirectional connections (24) and (25) between \( \vartheta \in I(S) \cap T(n) \) and \( \pi \in I(R) \cap T(n) \) still hold, because of the similarity in the conditions (15) and (16).

We define next a new \( N \)-dimensional vector \( \tilde{R} \equiv (\tilde{R}_1, \ldots, \tilde{R}_N) \) by

\[ \tilde{R}_i(t) := \int_0^t 1_{[u_i(s) \leq n]} dR_i(s), \quad \text{for} \quad i = 1, \ldots, N, \quad t \geq 0. \tag{33} \]

Each component \( \tilde{R}_i(t) \) represents the cumulative return of the \( i \)-th stock, accumulated over \([0, t]\) but only at times when this stock ranks among the top \( n \) by capitalization. Then, for \( \pi \in I(R) \cap T(n) \), the stochastic integral in (21) can be also cast as

\[ R_\pi = \int_0^t \sum_{i=1}^N \pi_i(t) dR_i(t) = \int_0^t \sum_{i=1}^N \pi_i(t) 1_{[u_i(t) \leq n]} dR_i(t) = \int_0^t \sum_{i=1}^N \pi_i(t) d\tilde{R}_i(t), \tag{34} \]

where the second equality follows from (18). We then have the semimartingale decomposition

\[ \tilde{R}_i = \tilde{A}_i + \tilde{M}_i, \quad i = 1, \ldots, N, \tag{35} \]

where

\[ \tilde{A}_i(t) := \int_0^t 1_{[u_i(s) \leq n]} dA_i(s), \quad \tilde{M}_i(t) := \int_0^t 1_{[u_i(s) \leq n]} dM_i(s), \quad i = 1, \ldots, N, \tag{36} \]

from (8). In the decomposition \( R_\pi = A_\pi + M_\pi \), with \( A_\pi \) as in (29) and \( M_\pi \) as in (30), we note that \( A_\pi \) and \( M_\pi \) can be expressed in terms of the components of \( \tilde{A} \) and \( \tilde{M} \), respectively, as

\[ A_\pi = \int_0^t \sum_{i=1}^N \pi_i(t) dA_i(t) = \int_0^t \sum_{i=1}^N \pi_i(t) d\tilde{A}_i(t), \quad M_\pi = \int_0^t \sum_{i=1}^N \pi_i(t) dM_i(t) = \int_0^t \sum_{i=1}^N \pi_i(t) d\tilde{M}_i(t). \tag{37} \]

by analogy with (34). Also in a manner similar to (12), we define

\[ \tilde{C}_{i,j} := [\tilde{M}_i, \tilde{M}_j] = [\tilde{R}_i, \tilde{R}_j], \quad \text{for} \quad 1 \leq i, j \leq N. \tag{38} \]
Note the relationship

\[ d\tilde{C}_{i,j}(t) = d[\tilde{R}_i, \tilde{R}_j](t) = 1_{[u_i(t) \leq n]}1_{[u_j(t) \leq n]} d[R_i, R_j](t) = 1_{[u_i(t) \leq n]}1_{[u_j(t) \leq n]} dC_{i,j}(t) \quad (39) \]

between \( \tilde{C} \) and \( C \). We further define a vector-valued process \( \tilde{\alpha} \equiv (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N) \) and a matrix-valued process \( \tilde{c} \equiv (\tilde{c}_{i,j})_{1 \leq i, j \leq N} \) as

\[ \tilde{\alpha}_i(t) := 1_{[u_i(t) \leq n]} \alpha_i(t), \quad i = 1, \ldots, N; \quad (40) \]

\[ \tilde{c}_{i,j}(t) := 1_{[u_i(t) \leq n]}1_{[u_j(t) \leq n]} c_{i,j}(t), \quad 1 \leq i, j \leq N; \quad (41) \]

it is then straightforward to obtain the relationships

\[ \tilde{A} = \int_0^t \tilde{\alpha}(t) d\tilde{O}(t) \quad \text{and} \quad \tilde{C} = [\tilde{M}, \tilde{M}] = \int_0^t \tilde{c}(t) d\tilde{O}(t) \quad (42) \]

in accordance with (11), where the vector-valued and matrix-valued processes \( \tilde{A} \equiv (\tilde{A}_1, \ldots, \tilde{A}_N) \) and \( \tilde{C} \equiv (\tilde{c}_{i,j})_{1 \leq i, j \leq N} \), respectively, are as in (36), (38).

The definition of \( C_{\pi, \rho} \) in (27) can be also invoked when \( \pi, \rho \) are in \( I(R) \cap T(n) \), but we have also with the help of (34) and (38) the alternative representation

\[ C_{\pi, \rho} = [R_\pi, R_\rho] = \left[ \int_0^t \sum_{i=1}^N \pi_i(t) d\tilde{R}_i(t), \int_0^t \sum_{j=1}^N \rho_j(t) d\tilde{R}_j(t) \right] = \int_0^t \sum_{i=1}^N \sum_{j=1}^N \pi_i(t) \rho_j(t) d\tilde{C}_{i,j}(t) \quad (43) \]

In particular, consider the portfolio \( \pi \) among the top \( n \) stocks, defined as

\[ \pi(\cdot) := 1_{[u_i(\cdot) \leq n]} e^i \quad (44) \]

for a fixed \( i = 1, \ldots, N \). This portfolio \( \pi \) invests all wealth in the \( i \)-th stock, when this stock ranks among the top \( n \); otherwise, it places all wealth in the money market. From (34), the identity

\[ R_\pi = \tilde{R}_i \quad (45) \]

holds, and we shall use the subscript “\( \tilde{\cdot} \)” instead of “\( \pi \)” to write \( X_\pi \equiv X_{\tilde{\pi}} \) and

\[ C_{\tilde{\pi}, \rho} \equiv C_{\pi, \rho} = \int_0^t \sum_{j=1}^N \rho_j(t) d\tilde{C}_{i,j}(t), \quad \text{as well as} \quad c_{\tilde{\pi}, \rho} \equiv c_{\pi, \rho} = \sum_{j=1}^N \rho_j(t) \tilde{c}_{i,j}(t), \quad (46) \]

in order to ease notation for the specific \( \pi \) in (44). This convention is consistent with the equalities

\[ C_{\tilde{\pi}, \rho} = [\tilde{R}_i, \tilde{R}_j] = \tilde{C}_{i,j} \quad \text{for} \ 1 \leq i, j \leq N. \]

It is useful to write succinctly the above relationships in this subsection, between symbols with tilde and corresponding symbols without tilde, in matrix notation. We do this by introducing the
predictable matrix-valued process \( D \equiv (D_{i,j})_{1 \leq i, j \leq N} \) with entries

\[
D_{i,j}(t) := \begin{cases} 
1_{\{u_i(t) \leq n\}}, & i = j, \\
0, & i \neq j,
\end{cases}
\]

for each \( t \geq 0 \). Here, we note that \( D(t) \) is a diagonal, idempotent matrix, whose \((i, i)\)-th entry is 1 if the \( i \)-th stock belongs to the top \( n \) stocks at time \( t \geq 0 \), otherwise it is zero. Because at least \( N - n \) diagonal entries of \( D(t) \) are zero, \( D(\cdot) \) is always singular. Then, any \( N \)-dimensional predictable process \( \nu \) in \( \mathcal{T}(n) \) as in Definition 2.4, satisfies \( D\nu = \nu \); in particular,

\[
D\vartheta = \vartheta, \quad D\pi = \pi,
\]

hold for all \( \vartheta \in I(S) \cap \mathcal{T}(n) \) and \( \pi \in I(R) \cap \mathcal{T}(n) \) from the conditions (17) and (18). Also, the identities (33), (36), (39), (40), and (41) can be reformulated as

\[
d\bar{R}(t) = D(t)dR(t), \quad d\bar{A}(t) = D(t)dA(t), \quad d\bar{M}(t) = D(t)dM(t),
\]

\[
d\bar{C}(t) = D(t)dC(t)D(t),
\]

as well as

\[
\bar{\alpha} = D\alpha, \quad \bar{c} = DcD.
\]

Moreover, we have another expression of the type (31) for \( \pi \in I(R) \cap \mathcal{T}(n) \): using the property (48), we write

\[
\alpha_\pi = \pi'\alpha = \pi'D\alpha = \pi'\bar{\alpha}, \quad \gamma_\pi = \pi'\alpha - \frac{1}{2}\pi'c\pi = \pi'\bar{\alpha} - \frac{1}{2}\pi'\bar{c}\pi = \pi'\bar{\alpha} - \frac{1}{2}\pi'\bar{c}\pi.
\]

We present now the following results regarding the integrability condition with respect to \( R \) (or \( \bar{R} \)), which will be used in the next section.

**Lemma 2.6** (Null portfolio). For an \( N \)-dimensional predictable process \( \eta \in \mathcal{T}(n) \), suppose that \( \eta'\bar{\alpha} = 0 \) and \( \bar{\eta} = 0 \) hold in the \( (\mathbb{P} \otimes O) \)-a.e. sense.

Then \( \eta \) is a portfolio, i.e., \( \eta \in I(R) \cap \mathcal{T}(n) \), and the identity \( R_\eta = \int_0^T \eta'(t)d\bar{R}(t) \equiv 0 \) holds. In this case, we call \( \eta \) a null portfolio.

**Lemma 2.7** (Integrability condition with respect to \( R \)). An \( N \)-dimensional predictable vector process \( \pi \in \mathcal{T}(n) \) belongs to \( I(R) \), if and only if

\[
\int_0^T (|\pi'(t)\bar{\alpha}(t)| + \pi'(t)\bar{c}(t)\pi(t))dO(t) < \infty, \quad \text{for any} \quad T \geq 0.
\]
This section presents the fundamental result of Karatzas and Kardaras (2007) in the arbitrage theory of an equity market, in open market context. Before we state and prove the result, we explain in succession several necessary concepts.

3.1 Auxiliary market

Consider a portfolio $\rho \in I(R)$, which generates the numéraire $X_\rho$ as in Definition 2.5 and (22), and fix $\rho$ throughout this subsection. We regard this portfolio $\rho$ as a “baseline,” in the sense that we want to compare the relative performance of any other portfolio $\pi \in I(R)$ with respect to $\rho$, by understanding the relative wealth process

$$X^\rho_\pi := \frac{X_\pi}{X_\rho}. \quad (53)$$

As the wealth $X_\pi$ is denominated relative to $X_\rho$ in (53), we consider an auxiliary market, in which all the components of the price vector $S$ in Definition 2.1 are denominated in units of $X_\rho$:

$$S^\rho_i := \frac{S_i}{X_\rho}, \quad i = 1, \ldots, N. \quad (54)$$

Here, we also consider the money market $S_0 \equiv 1$, with $S^\rho_0 := 1/X_\rho$, as we assume that the money market pays and charges zero interest in the introductory part of Section 2. Since $S^\rho_0$ is no longer trivial, we will consider the $(N + 1)$-dimensional vector $S^\rho \equiv (S^\rho_0, S^\rho_1, \ldots S^\rho_N)$ as the price process vector in this auxiliary market.

Recalling the notation (21) and (27), we define two $(N + 1)$-dimensional vectors of semimartingales $R^\rho \equiv (R^\rho_0, \ldots, R^\rho_N)$, and $\bar{R}^\rho \equiv (\bar{R}^\rho_0, \ldots, \bar{R}^\rho_N)$ with components

$$R^\rho_0 := C_{\rho\rho} - R_\rho, \quad \text{and} \quad \bar{R}^\rho_0 := \bar{R}^\rho_0 = C_{\rho\rho} - R_\rho, \quad \text{for} \quad i = 1, \ldots, N. \quad (55)$$

$$R^\rho_i := (R_i - C_{i\rho}), \quad \text{and} \quad \bar{R}^\rho_i := (\bar{R}_i - C_{i\rho}), \quad \text{for} \quad i = 1, \ldots, N. \quad (56)$$

Proposition 1.29 of Karatzas and Kardaras (2021) shows that the vector $R^\rho$ plays the role of cumulative return in the auxiliary market, as the relative wealth process $X^\rho_\pi$ of (53) admits the representation

$$X^\rho_\pi = \mathcal{E}(R^\rho_\pi), \quad \text{where} \quad R^\rho_\pi := R_{\pi - \rho} - C_{\pi - \rho, \rho} = \int_0^t \sum_{i=0}^N \pi_i(t) dR^\rho_i(t). \quad (57)$$

Here, we also recall the “money market proportion” $\pi_0$ of a portfolio $\pi$ in (26). We present the additional representation in particular for portfolios $\rho, \pi$ among the top $n$ stocks.
**Proposition 3.1.** For any two portfolios \( \rho, \pi \in I(R) \cap T(n) \) among the top \( n \) stocks, the process \( R^\rho_\pi \) in (57) admits the additional representation

\[
R^\rho_\pi = \int_0^t \sum_{i=0}^N \pi_i(t) d\tilde{R}_i^\rho(t).
\]

**Proof.** Since \( \tilde{R}^\rho_0 = R^\rho_0 \) in (55) and (56), it is enough to show

\[
\int_0^t \sum_{i=1}^N \pi_i(t) d(R_i - C_{i\rho})(t) = \int_0^t \sum_{i=1}^N \pi_i(t) d\left(\tilde{R}_i - C_{i\rho}\right)(t).
\]

Thanks to the condition (18) and the definition (33), this can be easily checked:

\[
\int_0^t \sum_{i=1}^N \pi_i(t) d(R_i - C_{i\rho})(t) = \int_0^t \sum_{i=1}^N \pi_i(t) d\left(\tilde{R}_i - C_{i\rho}\right)(t) = \int_0^t \sum_{i=1}^n \pi_i(t) d\left(\tilde{R}_i - C_{i\rho}\right)(t),
\]

where, in the last equality, we used the string of identities

\[
1_{[u_i(t) \leq n]} dC_{i\rho}(t) = 1_{[u_i(t) \leq n]} d[R_i, R_\rho](t) = d[\tilde{R}_i, R_\rho](t) = dC_{i\bar{\rho}}.
\]

\[
(59)
\]

In the special case \( \pi \equiv e^i \), that is, when the portfolio \( \pi \) invests all wealth in the \( i \)-th stock at all times, the relative wealth process \( X^\rho_\pi \) and its stochastic logarithm \( R^\rho_\pi \) in (53), (57) become

\[
X^\rho_\pi = \frac{S_i}{X^\rho} = S^\rho_i, \quad R^\rho_\pi = R^\rho_i,
\]

and Proposition 3.1 yields

\[
S^\rho_i = \mathcal{E}(R^\rho_i)
\]

for any given \( i = 1, \ldots, N \). Therefore, the component \( R^\rho_i \) of (55) is the stochastic logarithm of the \( i \)-th component of the price vector \( S^\rho \) in the auxiliary market, and the vector \( R^\rho \) plays the role of cumulative returns in the auxiliary market.

By analogy with (23), we also have

\[
\frac{dX^\rho_\pi(t)}{X^\rho_\pi(t)} = dR^\rho_\pi(t) = \sum_{i=0}^N \pi_i(t) dR^\rho_i(t) = \sum_{i=0}^N \pi_i(t) \frac{dS^\rho_i(t)}{S^\rho_i(t)}, \quad X^\rho_\pi(0) = 1,
\]

for \( \rho, \pi \) in \( I(R) \), from (57), (60). It is very important that the summation in (61) should include the index \( i = 0 \), as indeed it does, in contrast to the summation in (23).
3.2 Supermartingale numéraire and local martingale numéraire

We introduce now the notions of supermartingale numéraire and local martingale numéraire.

**Definition 3.2** (Supermartingale numéraire and local martingale numéraire). A given portfolio $\rho \in I(R)$ is called supermartingale numéraire portfolio (local martingale numéraire portfolio) in the whole market, if the relative wealth process $X_\rho^\pi = X_\pi / X_\rho$ of (53) is a supermartingale (local martingale) for every portfolio $\pi \in I(R)$ in the market. In this case, the wealth process $X_\rho$ is called a supermartingale numéraire (local martingale numéraire, respectively) in the whole market.

Similarly, a given portfolio $\rho \in I(R) \cap T(n)$ among the top $n$ stocks is called supermartingale numéraire portfolio (local martingale numéraire portfolio) among the top $n$ stocks, if the relative wealth process $X_\rho^\pi$ is a supermartingale (local martingale) for every portfolio $\pi \in I(R) \cap T(n)$ among the top $n$ stocks. In this case, the wealth process $X_\rho$ is called supermartingale numéraire (local martingale numéraire, respectively) among the top $n$ stocks.

By Fatou’s lemma, every nonnegative local martingale is a supermartingale; thus, every local martingale numéraire is in particular a supermartingale numéraire. We also have the following uniqueness result for supermartingale (local martingale) numéraires (respectively, among the top $n$ stocks).

**Lemma 3.3.** There is a unique (modulo null portfolios) supermartingale (local martingale) numéraire portfolio in the entire market, and also among the top $n$ stocks.

**Proof.** Suppose that there are two local martingale (or two supermartingale) numéraire portfolios $\rho$ and $\nu$ with the same initial wealth $X_\rho(0) = X_\nu(0)$. Then, the relative wealth process $X_\rho^\pi / X_\nu^\pi$ and its reciprocal $X_\nu^\pi / X_\rho^\pi$ are positive supermartingales. From the Doob–Meyer decomposition of semimartingales, it is easy to show that a continuous, positive supermartingale $Y$ is almost everywhere constant, if its reciprocal is also a supermartingale. Thus, $X_\rho \equiv X_\nu$ almost everywhere, and the two portfolios $\rho$ and $\nu$ generate the same wealth process. □

It can be shown that the supermartingale numéraire is actually the local martingale numéraire, thus the two numéraires are equivalent, in the whole market where no constraint is imposed on portfolios. This is Proposition 2.4 of Karatzas and Kardaras (2021), which we repeat here for the convenience of the reader.

**Proposition 3.4.** For a portfolio $\rho \in I(R)$, the following statements are equivalent:

1. $\rho$ is a supermartingale numéraire portfolio in the whole market.
2. $\rho$ is a local martingale numéraire portfolio in the whole market.
3. The equality $A_i = C_{i\rho}$ holds for all $i = 1, \ldots, N$.

The statement (3) gives a very simple structural condition, derived from the cumulative return process of the market, which characterizes this equivalence. It is no surprise that the result also holds for portfolios among the top $n$ stocks; but in this case, the cumulative return process vector $R$ in (6) should be replaced by $\tilde{R}$ of (33).
**Proposition 3.5.** For a portfolio \( \rho \in \mathcal{I}(R) \cap \mathcal{T}(n) \), the following statements are equivalent:

1. \( \rho \) is a supermartingale numéraire portfolio among the top \( n \) stocks.
2. \( \rho \) is a local martingale numéraire portfolio among the top \( n \) stocks.
3. The equality \( \tilde{A}_i = C_{\tilde{\rho}_i} \) holds for all \( i = 1, \ldots, N \).

**Proof.** We first assume statement (3), which is equivalent to the requirement that \( \tilde{R}_i - C_{\tilde{\rho}_i} = \tilde{M}_i \) is a local martingale for all \( i = 1, \ldots, N \) from (35). Recalling the notation of (56), (27) with the identities (18), (34), and (59), we obtain that the process

\[
\tilde{R}_0^\rho = C_{\rho\rho} - R_\rho = \int_0^t \sum_{i=1}^N \rho_i(t) dC_{\tilde{\rho}_i}(t) - \int_0^t \sum_{i=1}^N \rho_i(t) d\tilde{R}_i(t)
\]

is then also a local martingale. This in turn implies that all the components \( \tilde{R}_i^\rho = \tilde{R}_0^\rho + (\tilde{R}_i - C_{\tilde{\rho}_i}) \) for \( i = 1, \ldots, N \) in (56) are local martingales as well. Moreover, from Proposition 3.1, the processes \( \tilde{R}_\pi^\rho \) and \( X_\pi^\rho \) are also local martingales for every portfolio \( \pi \in \mathcal{I}(R) \cap \mathcal{T}(n) \) among the top \( n \) stocks, so the implication (3) \( \Rightarrow \) (2) has been proved.

Since statement (2) trivially implies statement (1), it remains to establish the implication (1) \( \Rightarrow \) (3). Assuming statement (1), we first fix any \( i \) in \( \{1, \ldots, N\} \), consider a specific portfolio \( \pi \) among the top \( n \) stocks defined as in (44), and recall the notation \( X_\pi \equiv X_i \) as well as \( R_\pi \equiv \tilde{R}_i \). Then, the processes

\[
X_{\rho+i}^\rho = \frac{X_{\rho+i}}{X_\rho}, \quad X_{\rho-i}^\rho = \frac{X_{\rho-i}}{X_\rho},
\]

are supermartingales. In view of Proposition 3.1 along with (45), all processes

\[
\mathcal{L}(X_{\rho+i}^\rho) = R_{\rho+i}^\rho = \tilde{R}_i - C_{\tilde{\rho}_i}, \quad \mathcal{L}(X_{\rho-i}^\rho) = R_{\rho-i}^\rho = - (\tilde{R}_i - C_{\tilde{\rho}_i}),
\]

are local supermartingales, implying that \( \tilde{R}_i - C_{\tilde{\rho}_i} \) is a local martingale. Since \( i \in \{1, \ldots, N\} \) can be chosen arbitrarily, we arrive at statement (3).

**Remark 3.6 (Representation of wealth relative to the supermartingale numéraire).** When \( \rho \in \mathcal{I}(R) \cap \mathcal{T}(n) \) is a supermartingale numéraire portfolio among the top \( n \) stocks, statement (3) of Proposition 3.5 implies that \( \tilde{R}_i - C_{\tilde{\rho}_i} = \tilde{M}_i \) is a local martingale for every \( i = 1, \ldots, N \).
Then, Proposition 3.1 with the notation (56) yields the following representation of the relative wealth process $X^{\rho}_{\pi}$ for any portfolio $\pi \in I(R) \cap I(n)$ among the top $n$ stocks, namely,

$$X^{\rho}_{\pi} = \mathcal{E} \left( \int_0^{N} \sum_{i=0}^{N} \pi_i(t) d\tilde{R}^\rho_i(t) \right) = \mathcal{E} \left( \tilde{R}^\rho_0 + \int_0^{N} \sum_{i=1}^{N} \pi_i(t) d\tilde{M}_i(t) \right)$$

$$= \mathcal{E} \left( \int_0^{N} \sum_{i=1}^{N} (\pi_i(t) - \rho_i(t)) d\tilde{M}_i(t) \right) = 1 + \int_0^{N} X^{\rho}_{\pi}(t) \sum_{i=1}^{N} (\pi_i(t) - \rho_i(t)) d\tilde{M}_i(t),$$

where the second-to-last equality is from (62). Thus, the relative wealth process $X^{\rho}_{\pi}$ is a stochastic integral with respect to the local martingale vector $\tilde{M}$, defined in (36).

**Remark 3.7** (A reformulation of statement (3)). The statement (3) of Proposition 3.5 can be reformulated using the “rate processes” $\tilde{\alpha}, \tilde{c}$ of (42), namely,

$$\int_0^{N} \tilde{\alpha}_i(t) dO(t) = \tilde{A}_i = C_{i\rho} = \int_0^{N} \sum_{j=1}^{N} \rho_j(t) d\tilde{C}_{i,j}(t) = \int_0^{N} \sum_{j=1}^{N} \rho_j(t) \tilde{c}_{i,j}(t) dO(t),$$

with the help of (46). Thus, we have the following statement (3) of Proposition 3.5, namely,

$$(3) \quad \tilde{\alpha} = \tilde{c}\rho, \quad (\mathbb{P} \otimes O) - \text{a.e.} \quad (63)$$

in matrix notation. In the same manner, the statement (3) of Proposition 3.4 also has the equivalent formulation:

$$(3)' \quad \alpha = c\rho, \quad (\mathbb{P} \otimes O) - \text{a.e.} \quad (64)$$

**3.3 Structural conditions**

In this subsection, we present yet another equivalent requirement for statement (3) of Proposition 3.5. This new formulation does not involve $\rho$, the supermartingale numéraire portfolio among the top $n$ stocks, and is in the form of what we call “structural conditions.” First, we note that $\tilde{c}$ of (50) is a singular symmetric matrix, thus not invertible, from the fact that $D$ is singular. Before proceeding to the next result, we need the following definition of “pseudo-inverse” for the matrix-valued process $\tilde{c}$ of (41):

$$\tilde{c}^\dagger := \lim_{m \to \infty} \left( \left( \tilde{c} + \frac{1}{m} I \right)^{-2} \tilde{c} \right),$$

where $I$ is the identity operator on $\mathbb{R}^N$. This process $\tilde{c}^\dagger$ will play the role of “pseudo-inverse” for $\tilde{c}$, because it is easily checked that

(a) $\tilde{c}^\dagger$ is the inverse of $\tilde{c}$ when restricted on $\text{range}(\tilde{c})$,

(b) $\tilde{c}^\dagger \tilde{c}$ coincides with the projection operator of $\mathbb{R}^N$ onto $\text{range}(\tilde{c})$,
(c) $\tilde{c}^\dagger$ is predictable, since matrix inversion is a continuous operation when restricted to strictly positive-definite matrices.

We are now ready to present the structural conditions.

**Proposition 3.8.** The existence of the supermartingale numéraire portfolio among the top $n$ stocks, is equivalent to the conjunction of the conditions:

\begin{align}
(i) & \quad \tilde{\alpha} \in \text{range}(\tilde{c}), \quad (\mathbb{P} \otimes O) - a.e., \quad (66) \\
(ii) & \quad \int_0^T \tilde{\alpha}'(t)\tilde{c}^\dagger(t)\tilde{\alpha}(t)dO(t) < \infty, \quad \text{for any } T \geq 0. \quad (67)
\end{align}

**Proof.** First, we assume that the supermartingale numéraire portfolio $\rho$ among the top $n$ stocks exists; then, from statement (3) of (63), the identity $\tilde{\alpha} = \tilde{c}\rho$ holds. The condition (i) follows immediately, and we obtain $\tilde{c}\tilde{c}^\dagger \tilde{\alpha} = \tilde{\alpha}$ from the property (b) above. This also implies that the set $\{\tilde{\alpha} \in \text{range}(\tilde{c})\}$ is predictable. We set the predictable process

$$
\nu := \tilde{c}^\dagger \tilde{\alpha},
$$

which is $\text{range}(\tilde{c})$-valued in the $(\mathbb{P} \otimes O)$-a.e. sense, and satisfies $\tilde{\alpha} = \tilde{c}\nu$. Then, every supermartingale numéraire portfolio among the top $n$ stocks should be of the form

$$
\rho = \nu + \eta = \tilde{c}^\dagger \tilde{\alpha} + \eta, \quad (69)
$$

for a suitable predictable process $\eta$, which is in $\text{ker}(\tilde{c})$, the kernel of $\tilde{c}$, $(\mathbb{P} \otimes O)$-a.e. We have $\tilde{c}\eta = 0$ and $\eta' \tilde{\alpha} = 0$, thus $\eta$ is a null portfolio in the sense of Lemma 2.6.

On the other hand, the assumption that the supermartingale numéraire portfolio among the top $n$ stocks exists, implies that some $N$-dimensional process of the form $\rho = \tilde{c}^\dagger \tilde{\alpha} + \eta$ of the form (69) should be a portfolio, i.e., $R$-integrable. The integrability condition (52) in Lemma 2.7 with the observation

$$
\rho'\tilde{c}\rho = \rho'\tilde{c}(\tilde{c}^\dagger \tilde{\alpha} + \eta) = \rho'\tilde{\alpha} = \tilde{\alpha}'\rho = \tilde{\alpha}'\tilde{c}^\dagger \tilde{\alpha},
$$

gives the condition (ii).

We next assume the conjunction of conditions (i), (ii), and find the supermartingale numéraire portfolio among the top $n$ stocks. We define the two predictable processes

$$
\nu := \tilde{c}^\dagger \tilde{\alpha}, \quad \text{and} \quad \rho := D\nu = D\tilde{c}^\dagger \tilde{\alpha}, \quad (70)
$$

and claim that $\rho$ is the supermartingale numéraire portfolio among the top $n$ stocks. Thanks to the condition (i), we obtain the identity $\tilde{c}\nu = \tilde{c}\tilde{c}^\dagger \tilde{\alpha} = \tilde{\alpha}$, $(\mathbb{P} \otimes O)$-a.e. Then, we note the series of
identities
\[ v' \tilde{c} v = v' \tilde{\alpha} = \tilde{\alpha}' v = \tilde{\alpha}' \tilde{c}^{\dagger} \tilde{\alpha}, \quad (\mathbb{P} \otimes O) - \text{a.e.}, \]
as well as
\[ \rho' \tilde{c} \rho = v' D \overline{c} v = v' \tilde{\alpha} = \tilde{\alpha}' \tilde{c}^{\dagger} \tilde{\alpha}, \quad \rho' \tilde{\alpha} = v' D \tilde{\alpha} = v' \tilde{\alpha} = \tilde{\alpha}' \tilde{c}^{\dagger} \tilde{\alpha}, \quad (\mathbb{P} \otimes O) - \text{a.e.} \quad (71) \]

Here, we used the identities \( D \tilde{\alpha} = \tilde{\alpha} \), and \( D \overline{c} = \overline{c} \), which can be obtained from (50). Combining equations of (71) with the condition (ii) yields the integrability condition (52) for \( \rho \equiv \pi \) in Lemma 2.7, i.e., \( \rho \in I(R) \). Also, from the construction (70), we have \( D \rho = D \overline{c} v = D v = \rho \), thus \( \rho \in \mathcal{T}(n) \). Therefore, we have shown that \( \rho \) is a portfolio among the top \( n \) stocks, i.e., \( \rho \in I(R) \cap \mathcal{T}(n) \).

Furthermore, we deduce
\[ \tilde{c} \rho = \overline{c} v = \overline{c} \overline{c}^{\dagger} \tilde{\alpha} = \tilde{\alpha}, \quad (\mathbb{P} \otimes O) - \text{a.e.}, \quad (72) \]
where the second equation uses the identity \( \overline{c} D = \overline{c} \), a consequence of (50) and of the fact that \( D \) is idempotent. Thus, we have obtained the condition (63), which is equivalent to statement (3) of Proposition 3.5, and \( \rho \) is indeed the supermartingale numéraire portfolio among the top \( n \) stocks.

The conjunction of the two conditions in Proposition 3.8 can be formulated as one equivalent condition, as follows. We first recall the “growth rate” \( \gamma_{\pi} \) of the portfolio \( \pi \in I(R) \cap \mathcal{T}(n) \) among the top \( n \) stocks in (51). We denote \( \mathbb{R}^N \cap \mathcal{T}(n) \) the collection of elements in \( \mathbb{R}^N \) such that at most \( n \) components are nonzero; then \( \pi(t) \) takes values in \( \mathbb{R}^N \cap \mathcal{T}(n) \) for each \( t \geq 0 \), by the property (16). Let us define the \([0, \infty]\)-valued process

\[ \bar{g} := \sup_{p \in \mathbb{R}^N} \left( p' \tilde{\alpha} - \frac{1}{2} p' \overline{c} p \right) = \sup_{p \in \mathbb{R}^N \cap \mathcal{T}(n)} \left( p' \tilde{\alpha} - \frac{1}{2} p' \overline{c} p \right). \quad (73) \]

The last equality follows because of the identities
\[ p' \tilde{\alpha} - \frac{1}{2} p' \overline{c} p = p' D \tilde{\alpha} - \frac{1}{2} p' D \tilde{c} D p = \tilde{p}' \tilde{\alpha} - \frac{1}{2} \tilde{p}' \overline{c} \tilde{p}, \]
valid for any \( p \in \mathbb{R}^N \), where \( \tilde{p} := D p \in \mathbb{R}^N \cap \mathcal{T}(n) \), by recalling the properties \( \tilde{\alpha} = D \tilde{\alpha} \) and \( \overline{c} = D \overline{c} \) which can be deduced from (50).

This process \( \bar{g} \) in (73) can be interpreted as the maximal growth rate achievable for all portfolios among the top \( n \) stocks. Note that \( \bar{g} \) is predictable, because the supremum can be restricted over a countable, dense subset of \( \mathbb{R}^N \). We then easily rewrite the process \( \bar{g} \) in the form

\[ \bar{g} = \frac{1}{2} (\tilde{\alpha}' \tilde{c}^{\dagger} \tilde{\alpha}) 1\{\tilde{\alpha} \in \text{range}(\overline{c})\} + \infty 1\{\tilde{\alpha} \notin \text{range}(\overline{c})\}, \quad (74) \]
and the supremum of (73) is attained if and only if \( \tilde{g} < \infty \), at \( \rho \equiv \rho := D \tilde{c}^\dagger \tilde{\alpha} \) as in (70) and (71). Then, the conjunction of conditions (i) + (ii) in Proposition 3.8 becomes simply

\[
\tilde{G}(T) < \infty, \quad \text{for all} \quad T \geq 0,
\]

where \( \tilde{G} \) is an adapted nondecreasing process

\[
\tilde{G} := \int_0^\cdot \tilde{g}(t) dO(t).
\]

We call this \( \tilde{G} \) the aggregate maximal growth from portfolios among the top \( n \) stocks; and say that the market consisting of the top \( n \) stocks has locally finite growth, if the process \( \tilde{G} \) satisfies the condition (75). We formalize this argument into the next proposition.

**Proposition 3.9.** The requirement of (75) of locally finite growth among the top \( n \) stocks, is equivalent to the conjunction of the two conditions (i) + (ii) of Proposition 3.8, thus sufficient and necessary for a supermartingale numéraire portfolio among the top \( n \) stocks to exist. In this case, we have

\[
\tilde{G} = \Gamma \rho,
\]

where \( \rho \) is a supermartingale numéraire portfolio among the top \( n \) stocks.

We present the following results, which will be used later.

**Lemma 3.10.** Suppose the market has locally finite growth among the top \( n \) stocks, i.e., that (75) holds, and let \( \rho \) be a supermartingale numéraire portfolio among the top \( n \) stocks. Recalling (51), (32), (27), and (37), we have

\[
\tilde{G} = \Gamma \rho = \frac{1}{2} C_{\rho \rho},
\]

as well as the representation

\[
\frac{1}{X^\rho} = \mathcal{E}(-M_\rho).
\]

**Proof.** As with (70) in the proof of Proposition 3.8, the supermartingale numéraire portfolio \( \rho \) among the top \( n \) stocks is of the form \( D \tilde{c}^\dagger \tilde{\alpha} \). With (71), the claim \( \tilde{G} = \Gamma \rho \) is easily obtained. Furthermore, again by (71) with (51), we have

\[
\gamma_\rho = \rho^\dagger \tilde{\alpha} - \frac{1}{2} \rho^\dagger \tilde{c} \rho = \frac{1}{2} \rho^\dagger \tilde{c} \rho = \frac{1}{2} C_{\rho \rho} = \tilde{g}
\]

thus \( \Gamma \rho = \frac{1}{2} C_{\rho \rho} \), as well as \( A_\rho = C_{\rho \rho} \). We then write (57), (58) with \( \pi \equiv (0,\ldots,0) \in I(R) \cap T(n) \):

\[
\frac{1}{X^\rho_\pi} = X^\rho_{\pi} = \mathcal{E}(\tilde{R}^\rho_0) = \mathcal{E}(C_{\rho \rho} - R_\rho) = \mathcal{E}(C_{\rho \rho} - A_\rho - M_\rho) = \mathcal{E}(-M_\rho).
\]
Lemma 3.11. Let \( \rho \) be a supermartingale numéraire portfolio among the top \( n \) stocks. For any investment strategy \( \vartheta \in I(S) \cap T(n) \) among the top \( n \) stocks, and for any initial capital \( x \geq 0 \), let us recall the wealth process \( X \equiv X(\cdot; x, \vartheta) \) generated by \( \vartheta \) and \( x \) in the manner of (14). Then there exists a process \( \eta = (\eta_1, \ldots, \eta_N) \in I(\tilde{M}) \cap T(n) \), such that

\[
\frac{X}{X_\rho} = x + \int_0^T \sum_{i=1}^N \eta_i(t) \, d\tilde{M}_i(t). \tag{79}
\]

Conversely, for any \( x \geq 0 \) and \( \eta \in I(M) \cap T(n) \), there exists a process \( \vartheta \in I(S) \cap T(n) \) such that (79) holds.

Proof. From (77) and (37), we have \( d(1/X_\rho(t)) = (1/X_\rho(t)) \sum_{i=1}^N (-\rho_i(t)) d\tilde{M}_i(t) \), as well as the dynamics

\[
dx(t) = \sum_{i=1}^N \delta_i(t) dS_i(t) = \sum_{i=1}^N \delta_i(t) S_i(t) d\tilde{R}_i(t) = \sum_{i=1}^N \delta_i(t) S_i(t) (d\tilde{A}_i(t) + d\tilde{M}_i(t)),
\]

from (35). Combining two equations via Itô’s formula, we obtain

\[
d(X(t)/X_\rho(t)) = \sum_{i=1}^N \frac{\delta_i(t) S_i(t)}{X_\rho(t)} (d\tilde{A}_i(t) + d\tilde{M}_i(t)) + \frac{X(t)}{X_\rho(t)} \sum_{i=1}^N (-\rho_i(t)) d\tilde{M}_i(t)
\]

\[
+ \sum_{i=1}^N \sum_{j=1}^N \frac{\delta_i(t) S_i(t)}{X_\rho(t)} (-\rho_j(t)) d[\tilde{M}_i, \tilde{M}_j](t).
\]

Here, the finite variation terms vanish because of the relationship \( d\tilde{A}_i(t) = \sum_{j=1}^N \rho_j(t) d[\tilde{M}_i, \tilde{M}_j](t) \) for \( i = 1, \ldots, N \), which is valid on the strength of condition (3) in Proposition 3.5. Thus, by setting

\[
\eta_i(t) := \frac{\delta_i(t) S_i(t) - X(t) \rho_i(t)}{X_\rho(t)}, \quad i = 1, \ldots, N,
\]

it is straightforward to check \( \eta \in T(n) \), and the result follows. The converse can be easily shown by reversing the above procedure.

3.4 Local martingale deflator and market viability

We present a few more concepts and state the main result.

Definition 3.12 (Local martingale deflator). An adapted, right-continuous and left-limited (RCLL) process \( Y \) is called local martingale deflator among the top \( n \) stocks, if it satisfies \( Y(0) = 1, Y > 0 \), and the process \( YX \) is a local martingale for every \( X \in \mathcal{X}^n \) of Definition 2.4. We denote by \( \mathcal{Y}^n \) the collection of all local martingale deflectors among the top \( n \) stocks.
Since $X \equiv 1 \in \mathcal{X}$, every deflator $Y \in \mathcal{Y}$ is in particular local martingale.

**Definition 3.13** (Cumulative withdrawal stream). We denote by $\mathcal{K}$ the collection of all nondecreasing, adapted, and right-continuous processes $K$ with $K(0) = 0$. Any element $K$ of $\mathcal{K}$ is called *cumulative withdrawal process*, and $K(t)$ represents for the cumulative capital withdrawn up to time $t \geq 0$; actual withdrawals in each infinitesimal interval $(t, t + dt)$ are represented as $dK(t)$. We say that $K \in \mathcal{K}$ is nonzero, if $\mathbb{P}(K(\infty) > 0) > 0$.

For $x \geq 0$, $\vartheta \in I(S)$ or $\vartheta \in I(S) \cap T(n)$, the wealth process $X(\cdot;x,\vartheta)$ defined in (14) is said to finance a given cumulative withdrawal process $K \in \mathcal{K}$, if $X \geq K$ holds. In this case, we say the process $K$ is *financeable from the initial capital $x \geq 0$ with the investment strategy $\vartheta$*. We denote by $\mathcal{K}(x)$, $\mathcal{K}^n(x)$ the subset of $\mathcal{K}$ consisting of cumulative capital withdrawal processes financeable from initial capital $x$; namely,

$$\mathcal{K}(x) := \{K \in \mathcal{K} \mid \exists \vartheta \in I(S) \text{ such that } X(\cdot;x,\vartheta) \geq K\},$$

$$\mathcal{K}^n(x) := \{K \in \mathcal{K} \mid \exists \vartheta \in I(S) \cap T(n) \text{ such that } X(\cdot;x,\vartheta) \geq K\}.$$

We introduce also the collection of cumulative withdrawal processes in $\mathcal{K}$, which can be financed starting from any positive initial capital:

$$\mathcal{K}'(0+) := \bigcap_{x > 0} \mathcal{K}(x) \subset \mathcal{K}, \quad \mathcal{K}'^n(0+) := \bigcap_{x > 0} \mathcal{K}^n(x) \subset \mathcal{K}.$$

**Definition 3.14** (Superhedging capital). For any cumulative withdrawal process $K \in \mathcal{K}$, we call the quantities

$$x(K) := \inf\{x \geq 0 \mid K \in \mathcal{K}(x)\} = \inf\{x \geq 0 \mid \exists \vartheta \in I(S) \text{ such that } X(\cdot;x,\vartheta) \geq K\},$$

$$x^n(K) := \inf\{x \geq 0 \mid K \in \mathcal{K}^n(x)\} = \inf\{x \geq 0 \mid \exists \vartheta \in I(S) \cap T(n) \text{ such that } X(\cdot;x,\vartheta) \geq K\}$$

the *superhedging capital* associated with the withdrawal stream $K$ in the entire market, and in the market consisting of the top $n$ stocks, respectively. We follow here the standard convention that the infimum of an empty set is equal to infinity.

**Lemma 3.15.** Suppose that $\mathcal{Y}^n$ is nonempty. For a fixed cumulative withdrawal process $K \in \mathcal{K}$, we assume that it is financeable from the initial capital $x \geq 0$ with investment strategy $\vartheta \in I(S) \cap T(n)$, i.e.,

$$X \equiv X(\cdot;x,\vartheta) = x + \int_0^t \sum_{i=1}^N \vartheta_i(t)dS_i(t) \geq K.$$
Then, the process
\[ Y(X - K) + \int_0^\cdot Y(t-)dK(t) \]
is a nonnegative local martingale, thus also a supermartingale, for every local martingale deflator \( Y \in \mathcal{Y}^n \) among the top \( n \) stocks. In particular, \( Y(X - K) \) is nonnegative supermartingale, for every \( Y \in \mathcal{Y}^n \). Furthermore, for the quantity \( x^n(K) \) of (80) we have the inequality
\[ x^n(K) \geq \sup_{Y \in \mathcal{Y}^n} \mathbb{E}^P \left[ \int_0^\infty Y(t-)dK(t) \right]. \] (81)

Proof. For every \( Y \in \mathcal{Y}^n \), integration by parts gives
\[ Y(X - K) = YX - \int_0^\cdot Y(t-)dK(t) - \int_0^\cdot K(t-)dY(t), \]
thus
\[ Y(X - K) + \int_0^\cdot Y(t-)dK(t) = YX - \int_0^\cdot K(t-)dY(t). \] (82)
Both terms on the right-hand side of (82) are local martingales, and the terms on the left-hand side of (82) are nonnegative; thus the first claim follows. Also, the process \( \int_0^\cdot Y(t-)dK(t) \) is non-decreasing, therefore \( Y(X - K) \) is a nonnegative supermartingale. We denote the left-hand side of (82) by \( Q := Y(X - K) + \int_0^\cdot Y(t-)dK(t) \), then we obtain
\[ Q(0) = x \geq \mathbb{E}^P[Q(\infty)] \geq \mathbb{E}^P \left[ \int_0^\infty Y(t-)dK(t) \right]. \]
By taking the supremum over \( Y \in \mathcal{Y}^n \) and then the infimum over the initial capital \( x \geq 0 \), the last claim follows. \( \square \)

Definition 3.16 (Viability). We say that the entire market is viable if, whenever \( x(K) = 0 \) holds for some cumulative withdrawal process \( K \in \mathcal{K} \), we have \( K \equiv 0 \).

In the same manner, we say the market consisting of the top \( n \) stocks is viable, if whenever \( x^n(K) = 0 \) holds for some cumulative withdrawal process \( K \in \mathcal{K} \), we have \( K \equiv 0 \).

The viability of the market consisting of the top \( n \) stocks, is actually equivalent to the identity
\[ \mathcal{K}^n(0+) = \{0\}; \]
whereas the failure of such viability implies the strict inclusion \( \mathcal{K}^n(0+) \supset \{0\} \). When the viability of the market consisting of the top \( n \) stocks fails, there exists a nonzero cumulative withdrawal process \( K \in \mathcal{K} \), which is financeable from any initial capital \( x > 0 \), no matter how minuscule; or
equivalently, there exists an investment strategy \( \hat{s}_m \in I(R) \cap T(n) \) for each \( m \in \mathbb{N} \), such that

\[
X(\cdot; \frac{1}{m}, \hat{s}_m) \geq K. 
\]

We further present the following lemma; it can be proven in the same manner as Exercise 2.22 of Karatzas and Kardaras (2021).

**Lemma 3.17.** The market consisting of the top \( n \) stocks fails to be viable if, and only if, there exist a real number \( T \geq 0 \) and a nonnegative \( F(T) \)-measurable random variable \( h \) with \( \mathbb{P}[h > 0] > 0 \) such that for every \( m \in \mathbb{N} \), there exists an \( X^m \in \mathcal{X}^m \) with \( X^m(T) \geq mh \).

The following result presents another equivalent characterization of viability for the market consisting of the top \( n \) stocks.

**Proposition 3.18 (Boundedness in probability).** The market consisting of the top \( n \) stocks is viable if, and only if,

\[
\lim_{m \to \infty} \sup_{X \in \mathcal{X}^n} \mathbb{P}[X(T) > m] = 0, \quad \forall T \geq 0. \tag{83}
\]

**Proof.** We first assume that the market consisting of the top \( n \) stocks is not viable. Then, from Lemma 3.17, there exist a real number \( T \geq 0 \), a nonnegative \( F(T) \)-measurable random variable \( h \) with \( \mathbb{P}[h > 0] > 0 \), and a sequence \( (X^m)_{m \in \mathbb{N}} \) of wealth processes \( X^m \in \mathcal{X}^m \) satisfying \( X^m(T) \geq mh \). Pick \( \varepsilon > 0 \) sufficiently small, so that \( \mathbb{P}[h > \varepsilon] > \varepsilon \) holds. We then have

\[
\liminf_{m \to \infty} \mathbb{P}[X^m(T) > \varepsilon m] \geq \liminf_{m \to \infty} \mathbb{P}[X^m(T) > mh, h > \varepsilon] \geq \varepsilon,
\]

thus the condition (83) is violated.

Conversely, we assume that for some \( T \geq 0 \), there exist \( \varepsilon > 0 \) and a sequence \( (X^m)_{m \in \mathbb{N}} \subset \mathcal{X}^n \) such that \( \mathbb{P}[X^m(T) > m2^m] > \varepsilon \) hold for all \( m \in \mathbb{N} \). Consider the set

\[
H := \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{X^k(T) > k2^k\} \in F(T),
\]

and note that \( \mathbb{P}(H) \geq \varepsilon \). For every \( m \in \mathbb{N} \), the inclusion

\[
H \subseteq \bigcup_{k=m+1}^{\infty} \{X^k(T) > k2^k\}
\]

holds, so there exists a sufficiently large number \( K_m > m \) such that the set

\[
H_m := H \cap \left( \bigcup_{k=m+1}^{K_m} \{X^k(T) > k2^k\} \right) \in F(T)
\]
satisfies \( \mathbb{P}[H \setminus H_m] \leq \frac{\mathbb{P}[H]}{2^{m+1}} \). Then, the countable intersection

\[
E := \bigcap_{m=1}^{\infty} H_m \in \mathcal{F}(T)
\]

is a subset of \( H \), and we have

\[
\mathbb{P}[H \setminus E] = \mathbb{P}\left[ \bigcup_{m=1}^{\infty} (H \setminus H_m) \right] \leq \sum_{m=1}^{\infty} \frac{\mathbb{P}[H]}{2^{m+1}} = \frac{\mathbb{P}[H]}{2},
\]

thus, \( \mathbb{P}[E] \geq \frac{\mathbb{P}[H]}{2} \) and \( \mathbb{P}[E] \geq \frac{\varepsilon}{2} > 0 \). Let us define a sequence of numéraires \((\Xi^m)_{m \in \mathbb{N}}\)

\[
\Xi^m := \sum_{k=m+1}^{K_m} 2^{-(k-m)} X^k,
\]

for each \( m \in \mathbb{N} \), and it is straightforward that \( \Xi^m \in \mathcal{X}^n \), as all \( X^k \in \mathcal{X}^n \) for \( k \in \mathbb{N} \). Furthermore, for every \( m \in \mathbb{N} \), we have \( E \subseteq H_m \subseteq \{\Xi^m(T) > m\} \), from which \( \Xi^m(T) \geq m 1_E \) follows. Set \( h := 1_E \in \mathcal{F}(T) \), then

\[
\mathbb{P}[h > 0] = \mathbb{P}[E] \geq \frac{\varepsilon}{2} > 0.
\]

Lemma 3.17 yields that the market consisting of the top \( n \) stocks is not viable. \( \square \)

We are now ready to state and prove the main result of this section.

**Theorem 3.19.** The following statements are equivalent:

1. The market consisting of the top \( n \) stocks is viable.
2. There exists a local martingale deflator among the top \( n \) stocks, i.e., \( \mathcal{Y}^n \neq \emptyset \).
3. The supermartingale numéraire among the top \( n \) stocks exists.
4. The market consisting of the top \( n \) stocks has locally finite growth; namely, the condition \((75)\) of the aggregate maximal growth process \( \tilde{G} \) among the top \( n \) stocks of \((76)\) holds.

**Proof.** The implication \((4) \Rightarrow (3)\) follows from Proposition 3.9. The implication \((3) \Rightarrow (2)\) also follows easily, because the supermartingale numéraire among the top \( n \) stocks is a local martingale numéraire among the top \( n \) stocks from Proposition 3.5, and the reciprocal of the local martingale numéraire among the top \( n \) stocks is a local martingale deflator among the top \( n \) stocks.

In order to prove \((2) \Rightarrow (1)\), let \( Y \in \mathcal{Y}^n \) be a local martingale deflator and pick a cumulative withdrawal process \( K \in \mathcal{K} \) such that \( x^n(K) = 0 \). From \((81)\) of Lemma 3.15, we have

\[
\mathbb{E}^\mathbb{P} \left[ \int_0^{\infty} Y(t-)dK(t) \right] = 0.
\]

Since \( Y \) is strictly positive and \( K \) is nondecreasing with \( K(0) = 0 \), it follows that \( K(\infty) = 0 \) holds \( \mathbb{P}\)-a.e., which is equivalent to \( K \equiv 0 \). The market consisting of the top \( n \) stocks is then viable.
The remaining part is to show the implication (1) ⇒ (4), which is quite technical. Suppose that the market fails to have locally finite growth among the top \( n \) stocks, i.e., one of the structural conditions (66), (67) is violated. Thus, we need to consider two cases:

(A) the set \( \{ \bar{a} \notin \text{range}(\bar{c}) \} \) fails to be \((\mathbb{P} \otimes O)\)-null,
(B) the set \( \{ \bar{a} \notin \text{range}(\bar{c}) \} \) is \((\mathbb{P} \otimes O)\)-null, but \( \mathbb{P}[\bar{G}(T) = \infty] > 0 \) holds for some \( T > 0 \).

We shall show that the market is not viable in each of the cases (A) and (B) below.

* Case (A). Recalling the notation (65) with its properties (a)–(c), we first note that the predictable process

\[
\varphi := \frac{1}{||\bar{a} - \bar{c}^\top \bar{a}||^2} (\bar{a} - \bar{c}^\top \bar{a}) 1_{\{\bar{a} \notin \text{range}(\bar{c})\}}, \tag{84}
\]

is well defined, because \( \bar{a} \notin \text{range}(\bar{c}) \) holds if and only if \( \bar{c}^\top \bar{a} \neq \bar{a} \). Note that \( D\varphi = \varphi \), thus \( \varphi \in \mathbb{Y}(\mathcal{R}) \), thanks to the properties \( D\bar{a} = \bar{a}, D\bar{c} = \bar{c} \) from (50). Since the process \( \bar{a} - \bar{c}^\top \bar{a} \) is orthogonal to \( \text{range}(\bar{c}) \), we have \( \bar{c}\varphi = 0 \). Furthermore, we have \( \varphi'(\bar{a}) = 1_{\{\bar{a} \notin \text{range}(\bar{c})\}} \), because

\[
(\bar{a} - \bar{c}^\top \bar{a})'(\bar{a}) = ||\bar{a} - \bar{c}^\top \bar{a}||^2 + (\bar{a} - \bar{c}^\top \bar{a})'(\bar{c}^\top \bar{a}) = ||\bar{a} - \bar{c}^\top \bar{a}||^2.
\]

Thus, from Lemma 2.7, \( \varphi \) is a portfolio among the top \( n \) stocks, i.e., \( \varphi \in I(\mathcal{R}) \cap \mathcal{T}(n) \). Also, the local martingale vanishes: \( \int_0^t \varphi'(t)d\bar{M}(t) \equiv 0 \), because its quadratic variation process vanishes

\[
\left[ \int_0^t \varphi'(t)d\bar{M}(t) \right] = \int_0^t \varphi'(t)\bar{c}(t)\varphi(t)dO(t) \equiv 0. \tag{85}
\]

Thus,

\[
\int_0^t \varphi'(t)d\bar{R}(t) = \int_0^t \varphi'(t)d\bar{A}(t) = \int_0^t \varphi'(t)\bar{a}(t)dO(t) = \int_0^t 1_{\{\bar{a} \notin \text{range}(\bar{c})\}}(t)dO(t) = : K.
\]

We define the vector process \( \vartheta \equiv (\vartheta_1, \ldots, \vartheta_N) \) with components given by \( \vartheta_i = \varphi_i/S_i \) for \( i = 1, \ldots, N \). It is then easy to check that \( m\vartheta \) is an investment strategy among the top \( n \) stocks, i.e., \( m\vartheta \in I(S) \cap \mathcal{T}(n) \), for any \( m \in \mathbb{N} \), and

\[
X(\cdot; 0, m\vartheta) = \int_0^t m\vartheta'(t)dS(t) = m \int_0^t \varphi'(t)dR(t) = m \int_0^t \varphi'(t)d\bar{R}(t) = mK.
\]

In other words, for any \( m \in \mathbb{N} \), the wealth process generated by the investment strategy \( m\vartheta \) among the top \( n \) stocks has vanishing local martingale part, and is equal to the nontrivial, nondecreasing part \( mK \) of finite variation. This process \( mK \) can be arbitrarily scaled by the multiplicative constant \( m \in \mathbb{N} \), and thus \( X^a(K) = 0 \), by recalling (80). We conclude that the market consisting of the top \( n \) stocks is not viable.

* Case (B). We assume that the set \( \{ \bar{a} \notin \text{range}(\bar{c}) \} \) is \((\mathbb{P} \otimes O)\)-null, but \( \mathbb{P}[\bar{G}(T) = \infty] > 0 \) holds for some \( T > 0 \). In this case, the aggregate maximal growth process \( \bar{G} \) of (76) becomes

\[
\bar{G} = \frac{1}{2} \int_0^t \bar{a}'(t)\bar{c}(t)\varphi(t)dO(t). \tag{86}
\]
We consider first the portfolio \( \rho := D\tilde{\alpha}^{\dagger} \in \tau(n) \) as in (70), and also set \( \rho^m := \rho 1_{||\rho|| \leq m} \in I(R) \cap \tau(n) \). The log-wealth process of (28) can be represented, with the help of (37) and (71), as

\[
\log X_{\rho m} = \frac{1}{2} \int_0^T 1_{||\rho(t)|| \leq m} \rho'(t)\tilde{\alpha}(t)\rho(t) dO(t) + \int_0^T 1_{||\rho(t)|| \leq m} \rho'(t) d\tilde{M}(t). \tag{87}
\]

Note that the first integral on the right-hand side of (87), namely,

\[
2G^m := \int_0^T 1_{||\rho(t)|| \leq m} \rho'(t)\tilde{\alpha}(t)\rho(t) dO(t),
\]

is the quadratic variation of the local martingale \( \int_0^T 1_{||\rho(t)|| \leq m} \rho'(t) d\tilde{M}(t) \), which is the second integral on the right-hand side of (87). The Dambis–Dubins–Schwarz representation (cf. Theorem 3.4.6 and Problem 3.4.7 of Karatzas & Shreve, 1991), with the scaling property of Brownian motion, implies that there exists a Brownian motion \( W^m \), on a possibly enlarged filtered probability space, such that

\[
\log X_{\rho m} = G^m + \sqrt{2}W^m(G^m), \tag{88}
\]

for every \( m \in \mathbb{N} \). The sequence \( \{G^m(T)\}_{m \in \mathbb{N}} \) is nondecreasing and converges to

\[
\frac{1}{2} \int_0^T \rho'(t)\tilde{\alpha}(t)\rho(t) dO(t) = \frac{1}{2} \int_0^T \tilde{\alpha}'(t)\tilde{\alpha}^{\dagger}(t)\tilde{\alpha}(t) dO(t) = \tilde{G}(T),
\]

as in (86), again with the help of (71). The strong law of large numbers for Brownian motion gives

\[
\lim_{m \to \infty} \mathbb{P} \left[ \frac{W^m(G^m(T))}{G^m(T)} \leq -\frac{1}{2\sqrt{2}}, \quad \tilde{G}(T) = \infty \right] = 0.
\]

From the representation (88), we obtain

\[
\lim_{m \to \infty} \mathbb{P} \left[ \frac{\log X_{\rho m}(T)}{G^m(T)} \leq \frac{1}{2}, \quad \tilde{G}(T) = \infty \right] = 0.
\]

Therefore, in case (B), the collection of random variables \( \{X_{\rho m}(T) \mid m \in \mathbb{N}\} \subseteq \{X(T) \mid X \in \mathcal{X}^n\} \) fails to be bounded in probability, and Proposition 3.18 concludes that the market consisting of the top \( n \) stocks is not viable. \( \square \)

### 3.5 Growth optimality and relative log-optimality

The results in the previous subsection characterize the supermartingale numéraire portfolio among the top \( n \) stocks, in terms of \( \tilde{\alpha} \) and \( \tilde{\epsilon} \), via the “structural condition.” More specifically, in the argument leading to Proposition 3.8 and in the proof of Lemma 3.10, the maximal growth rate among the top \( n \) stocks \( \tilde{g} \) of (73) is attained when the portfolio \( \rho \) has the supermartingale numéraire property among the top \( n \) stocks, as in (78). In this subsection, we reformulate this property and
show that a portfolio with the supermartingale numéraire property is “optimal” in some sense among portfolios of top \( n \) stocks.

**Definition 3.20** (Relative growth and growth optimality). We define the relative growth of a given portfolio \( \pi \in \mathcal{I}(R) \) with respect to another portfolio \( \rho \in \mathcal{I}(R) \) as

\[
\Gamma_{\pi}^{\phi} := \Gamma_{\pi} - \Gamma_{\rho},
\]

namely, the difference between the finite variation process of the log-relative wealth process \( \log X_{\pi}^{\phi} = \log(X_{\pi} / X_{\rho}) \) from (29), (53).

We call a portfolio \( \rho \in \mathcal{I}(R) \cap \mathcal{T}(n) \) growth-optimal among the top \( n \) stocks, if for every portfolio \( \pi \in \mathcal{I}(R) \cap \mathcal{T}(n) \) the process \( \Gamma_{\pi}^{\phi} = \Gamma_{\pi} - \Gamma_{\rho} \) is nonincreasing.

**Proposition 3.21.** A portfolio is growth-optimal among the top \( n \) stocks, if and only if it is a supermartingale numéraire portfolio among the top \( n \) stocks.

**Proof.** (i) Let us first assume that \( \rho \in \mathcal{I}(R) \cap \mathcal{T}(n) \) is a supermartingale numéraire portfolio among the top \( n \) stocks. From Proposition 3.8 and (63), we know that \( \bar{\alpha} \in \text{range}(\tilde{\gamma}) \) and \( \bar{\alpha} = \bar{c}_{\rho} \) hold \((\mathbb{P} \otimes O) - a.e.$$ Recalling (51), (73) and the fact that the supremum of \( g \) is attained by a supermartingale numéraire portfolio among the top \( n \) stocks, the comparison \( \gamma_{\rho} = \tilde{g} \geq \gamma_{\pi} \) holds \((\mathbb{P} \otimes O) - a.e.$$ for every \( \pi \in \mathcal{I}(R) \cap \mathcal{T}(n) \). Thus, \( \rho \) is growth-optimal.

(ii) Next, we assume that \( \nu \in \mathcal{I}(R) \cap \mathcal{T}(n) \) is a growth-optimal portfolio among the top \( n \) stocks. We pick a portfolio \( \varphi \in \mathcal{I}(R) \cap \mathcal{T}(n) \) satisfying \( \bar{c}_{\varphi} = 0 \) and \( \varphi' \bar{\alpha} = 1 \) on the set \( \{ \bar{\alpha} \notin \text{range}(\tilde{\gamma}) \} \) (for example, as in (84) in the proof of Theorem 3.19). We then have \( \gamma_{\nu + \varphi} = \gamma_{\nu} + 1 \) on \( \{ \bar{\alpha} \notin \text{range}(\tilde{\gamma}) \} \) from (51), violating the growth-optimality of \( \nu \). This implies that the latter set is \((\mathbb{P} \otimes O)\)-null. In particular, \( \tilde{g} < \infty \) in the \((\mathbb{P} \otimes O) - a.e.$$ sense, from (74).

On the other hand, we let \( \rho := D \bar{c}_{\nu}^{\dagger} \bar{\alpha} \in \mathcal{T}(n) \) and define \( \rho^{m} := \rho 1_{\|\varphi\| \leq m} \in \mathcal{I}(R) \cap \mathcal{T}(n) \) for \( m \in \mathbb{N} \). Equation (78) yields \( \gamma_{\nu} \geq \gamma_{\rho^{m}} = \tilde{g} 1_{\|\varphi\| \leq m} \), and thus \( \gamma_{\nu} \geq \tilde{g} \) holds \((\mathbb{P} \otimes O) - a.e.$$ by taking the limit \( m \to \infty \). We conclude that \( \nu \) is also a supermartingale numéraire portfolio among the top \( n \) stocks.

A supermartingale numéraire portfolio among the top \( n \) stocks is “optimal” also in another sense, as follows.

**Definition 3.22.** A portfolio \( \rho \in \mathcal{I}(R) \cap \mathcal{T}(n) \) is called relatively log-optimal among the top \( n \) stocks, if for all portfolios \( \pi \in \mathcal{I}(R) \cap \mathcal{T}(n) \) and for all stopping times \( \tau \) of \( F \), we have

\[
\mathbb{E}^{\mathbb{P}}[(\log X_{\pi}^{\rho}(\tau))^{+}] < \infty, \quad \text{and} \quad \mathbb{E}^{\mathbb{P}}[\log X_{\pi}^{\rho}(\tau)] \leq 0.
\]  

(90)

**Proposition 3.23.** A portfolio is relatively log-optimal among the top \( n \) stocks, if and only if it is a supermartingale numéraire portfolio among the top \( n \) stocks.

**Proof.** (i) We first suppose that \( \rho \in \mathcal{I}(R) \cap \mathcal{T}(n) \) is a supermartingale numéraire portfolio among the top \( n \) stocks. Then, we obtain

\[
\mathbb{E}^{\mathbb{P}}[(\log X_{\pi}^{\rho}(\tau))^{+}] = \int_{0}^{\infty} \mathbb{P}(X_{\pi}^{\rho}(\tau) > e^{t}) dt \leq \int_{0}^{\infty} \mathbb{P}(X_{\pi}^{\rho}(\tau) > t) dt \leq \mathbb{E}^{\mathbb{P}}[X_{\pi}^{\rho}(\tau)] \leq 1,
\]
where the last inequality is from the Optional Sampling Theorem. By applying Jensen’s inequality to this last inequality, the second condition of (90) also holds, and we conclude that \( \rho \) is relatively log-optimal among the top \( n \) stocks.

(ii) For the converse implication, we assume that \( \nu \in I(R) \cap T(n) \) is relatively log-optimal among the top \( n \) stocks. As in the proof of Proposition 3.21, we pick a portfolio \( \varphi \in I(R) \cap T(n) \) as in (84), satisfying \( \bar{c}\varphi = 0 \) and \( \varphi'\bar{\alpha} = 1 \) on the set \( \{\bar{\alpha} \notin \text{range}(\Bar{c})\} \). By recalling (28), (29), (32), (37), and (51), straightforward computations show

\[
\log X_{Y+\varphi} = \log X_{Y+\varphi} - \log X_{\nu} = \int_0^T (\gamma_{Y+\varphi}(t) - \gamma_Y(t)) dO(t) + \int_0^T \varphi'(t) d\bar{M}(t) \tag{91}
\]

Here, the last integral on the right-hand side of (91) vanishes, because of Equation (85) above. The relative log-optimality of \( \nu \) implies that the set \( \{\bar{\alpha} \notin \text{range}(\Bar{c})\} \) is \( (\mathbb{P} \otimes O) - \text{null} \). We then consider a process \( \rho := \bar{D}\bar{c}^\top \bar{\alpha} \in T(n) \) of (70), as in the proof of Proposition 3.21. Note that \( \bar{\alpha} = \bar{c}\rho \) holds \( (\mathbb{P} \otimes O) - a.e. \) from (72), or equivalently, \( \bar{A}_i = C_{i\rho} \) hold for \( i = 1, \ldots, N \), from Remark 3.7. This last requirement implies that \( A_\pi = C_{\pi\rho} \), thus \( R_\pi - C_{\pi\rho} \) is a local martingale for every \( \pi \in I(R) \cap T(n) \). We further define

\[
\nu^m := \nu\mathbf{1}_{\{\bar{\alpha} = \bar{c}\nu\}} + \nu\mathbf{1}_{\{\bar{\alpha} \neq \bar{c}\nu\}} \mathbf{1}_{\{||\rho|| > m\}} + \rho\mathbf{1}_{\{\bar{\alpha} \neq \bar{c}\nu\}} \mathbf{1}_{\{||\rho|| \leq m\}}, \quad \text{for } m \in \mathbb{N},
\]

and it is easy to check that \( \nu^m \in I(R) \cap T(n) \) for all \( m \in \mathbb{N} \).

We now claim that the ratio \( X_\nu/X_{\nu^m} \) for every \( m \in \mathbb{N} \) is a local martingale. Proposition 3.1 implies that it is sufficient to show \( R_{\nu^m} = R_\nu - C_{\nu^m} = Q \) is a local martingale, where we set \( \pi := \nu - \nu^m \in I(R) \cap T(n) \). On the set \( \zeta := \{\bar{\alpha} \neq \bar{c}\nu, ||\rho|| \leq m\} \), we have \( \nu^m = \rho \), thus \( Q \) is a local martingale. On the complement set \( \zeta^c \), we have \( \pi = \nu - \nu^m = 0 \), thus \( Q = 0 \). In other words, we showed that

\[
Q = \int_0^T \mathbf{1}_\zeta(t) dQ(t) = \int_0^T \mathbf{1}_\zeta(t) d(R_\pi - C_{\pi\rho})(t)
\]

is a local martingale, verifying our claim that \( X_\nu/X_{\nu^m} \) is a local martingale for every \( m \in \mathbb{N} \). As the ratio is positive, \( X_\nu/X_{\nu^m} \) is also a supermartingale.

If we assume that \( \mathbb{P}[X_\nu(T) \neq X_{\nu^m}(T)] > 0 \) were true for some \( T > 0 \), we obtain

\[
\mathbb{E}^\mathbb{P} \left[ \log \frac{X_\nu(T)}{X_{\nu^m}(T)} \right] < \log \mathbb{E}^\mathbb{P} \left[ \frac{X_\nu(T)}{X_{\nu^m}(T)} \right] \leq 0,
\]

contradicting the relative log-optimality of \( \nu \). Thus, we conclude that \( X_\nu = X_{\nu^m} \), from the continuity of \( X_\nu/X_{\nu^m} \), and \( \nu - \nu^m \) is a null portfolio in the sense of Lemma 2.6. We then have \( \bar{c}\nu^m = \bar{c}\nu = \bar{c}\rho = \bar{\alpha} \), \( (\mathbb{P} \otimes O) - a.e. \) on the set \( \zeta = \{\bar{\alpha} \neq \bar{c}\nu, ||\rho|| \leq m\} \) defined above, which implies that \( \zeta \) is \( (\mathbb{P} \otimes O) \)-null. Since this property is true for every \( m \in \mathbb{N} \), the identity \( \bar{\alpha} = \bar{c}\nu \) is valid \( (\mathbb{P} \otimes O) - a.e. \), thus \( \nu \) is a supermartingale numéraire portfolio among the top \( n \) stocks.

In part (ii) of the proofs of both Proposition 3.21 and Proposition 3.23, we did not assume the existence of a supermartingale numéraire portfolio among the top \( n \) stocks. Thus, the existence of
a growth-optimal or relatively log-optimal portfolio among the top \( n \) stocks, is equivalent to the existence of a supermartingale numéraire portfolio among the top \( n \) stocks, and we can add the following two statements to the list of equivalences in Theorem 3.19:

(5) A growth-optimal portfolio among the top \( n \) stocks exists.
(6) A relatively log-optimal portfolio among the top \( n \) stocks exists.

### 3.6 The optional decomposition

Suppose that we are given a nonnegative, adapted process with RCLL paths and \( X(0) = x \geq 0 \). In this subsection, we characterize the condition when \( X \) belongs to \( \mathcal{X}^n \) of Definition 2.4, i.e., when \( X \) is the wealth process generated by an investment strategy that invests in the top \( n \) stocks of the market, and study how can we construct this strategy from \( X \). The following Theorem 3.25, which we call Optional Decomposition Theorem, gives, along with its Corollary 3.26, the answer to this question.

We first present the following result, originally from Theorem 1 of Schweizer (1995). See also Propositions 2.3 and 3.2 of Larsen and Žitković (2007). We recall for this purpose the semimartingale vector \( \tilde{M} \) defined in (36) and write \( \mathcal{M}^{\perp}_{loc}(\tilde{M}) \) the collection of scalar local martingales \( L \) with RCLL paths, satisfying \( L(0) = 0 \) and the orthogonality \( [L, \tilde{M}_i] = 0 \) for all \( i = 1, \ldots, N \).

#### Lemma 3.24.

If a supermartingale numéraire portfolio \( \rho \) among the top \( n \) stocks exists, then the collections \( \mathcal{Y}^n \) of local martingale deflators among the top \( n \) stocks, defined in Definition 3.12, admits the representation

\[
\mathcal{Y}^n = \left\{ \frac{1}{X_\rho} \mathcal{E}(L) : L \in \mathcal{M}^{\perp}_{loc}(\tilde{M}) \text{ with } \Delta L > -1 \right\}. \tag{92}
\]

In order to simplify the proof of the Optional Decomposition Theorem, we shall work under the following assumption. The general case of the Theorem can be proved as in Subsection 3.1.3 of Karatzas and Kardaras (2021).

**Assumption (A)**: All local martingales on the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}(\cdot), \mathbb{P})\) have continuous paths.

#### Theorem 3.25 (Optional Decomposition).

Suppose that the market consisting of the top \( n \) stocks is viable. For a nonnegative, adapted process \( X \) with RCLL paths satisfying \( X(0) = x \geq 0 \), the following statements are equivalent:

1. The process \( YX \) is a supermartingale, for every \( Y \in \mathcal{Y}^n \).
2. There exist an investment strategy \( \theta \in \mathcal{I}(S) \cap \mathcal{T}(n) \) among the top \( n \) stocks, and a cumulative withdrawal process \( K \in \mathcal{K} \), such that

\[
X = x + \int_0^\cdot \sum_{i=1}^N \theta_i(t) dS_i(t) - K. \tag{93}
\]
Proof. We first show the implication \((2) \implies (1)\). For any \(Y \in \mathcal{Y}^n\), write \(Y = \mathcal{E}(L)/X_\rho\) for some \(L \in \mathcal{M}_{1,loc}(\tilde{M})\) with \(\Delta L > -1\) from Lemma 3.24, where we denote by \(\rho\) a supermartingale numéraire portfolio among the top \(n\) stocks. Then, we have from Lemma 3.11,

\[
YX + YK = \frac{x + \int_0^t \sum_{i=1}^N \tilde{\vartheta}_i(t) dS_i(t)}{X_\rho(t)} \mathcal{E}(L) = \left( x + \int_0^t \sum_{i=1}^N \eta_i(t) d\tilde{M}_i(t) \right) \mathcal{E}(L),
\]

for some process \(\eta \in I(\tilde{M}) \cap \mathcal{T}(n)\). The last expression is a product of two nonnegative, orthogonal local martingales, thus it is a nonnegative local martingale. The claim that \(YX\) is a supermartingale follows.

We now show the implication \((1) \implies (2)\), which is more involved, under the above Assumption (A). We assume that \((1)\) holds and recall the collections \(\mathcal{Y}^n\) and \(\mathcal{M}_{1,loc}(\tilde{M})\) of (92). All processes in \(\mathcal{M}_{1,loc}(\tilde{M})\) have continuous paths under the Assumption (A). From Lemma 3.24, \((X/X_\rho)\mathcal{E}(L)\) is a supermartingale for every \(L \in \mathcal{M}_{1,loc}(\tilde{M})\), and in particular, \(X/X_\rho\) is a supermartingale itself. The Doob–Meyer and Kunita–Watanabe decompositions give

\[
\frac{X}{X_\rho} = x + M_\eta + L - B, \quad \text{where} \quad M_\eta := \int_0^t \sum_{i=1}^N \eta_i(t) d\tilde{M}_i(t).
\]

Here, \(\eta \equiv (\eta_1, \ldots, \eta_N) \in I(\tilde{M})\), \(L \in \mathcal{M}_{1,loc}(\tilde{M})\) and \(B\) is an adapted, nondecreasing and right-continuous process with \(B(0) = 0\), i.e., \(B\) is a cumulative withdrawal process in \(\mathcal{K}\). Recalling the diagonal matrix \(D\) of (47) with its property \(Dd\tilde{M}(t) = d\tilde{M}(t)\), we further define \(\tilde{\eta} := D\eta \in I(\tilde{M}) \cap \mathcal{T}(n)\), and we have

\[
M_\eta = \int_0^t \eta'(t) d\tilde{M}(t) = \int_0^t (D\eta)'(t) d\tilde{M}(t) = M_{\tilde{\eta}}.
\]

Consequently, we obtain

\[
\frac{X}{X_\rho} = x + M_{\tilde{\eta}} + L - B, \quad \text{with} \quad \tilde{\eta} \in I(\tilde{M}) \cap \mathcal{T}(n), \quad L \in \mathcal{M}_{1,loc}(\tilde{M}). \quad \text{(94)}
\]

We next show that \(L \equiv 0\) in (94). Again from Lemma 3.24, \((1/X_\rho)\mathcal{E}(mL)\) is a local martingale, thus \((X/X_\rho)\mathcal{E}(mL)\) is a supermartingale for every \(m \in \mathbb{N}\). Since \([\mathcal{E}(mL), \tilde{M}_i] = 0\) for \(i = 1, \ldots, N\), we have \([\mathcal{E}(mL), M_{\tilde{\eta}}] = 0\) and consequently, \(\mathcal{E}(mL)M_{\tilde{\eta}}\) is a local martingale as a product of two orthogonal local martingales. Thus, from (94), the process

\[
\mathcal{E}(mL)(L - B) = \mathcal{E}(mL) \frac{X}{X_\rho} - \mathcal{E}(mL)(x + M_{\tilde{\eta}})
\]

is a local supermartingale for every \(m \in \mathbb{N}\). On the other hand, the integration by parts gives

\[
\mathcal{E}(mL)(L - B) = \int_0^t (L - B)(t-)d\mathcal{E}(mL)(t) + \int_0^t \mathcal{E}(mL)(t)dL(t) + \int_0^t \mathcal{E}(mL)(t)d([mL, L] - B)(t).
\]
Then, the last integrator $m[L,L] - B$ should be a local supermartingale for every $m \in \mathbb{N}$, which implies $[L,L] \equiv 0$, thus $L \equiv 0$.

As a result, Equation (94) becomes

$$\frac{X}{X_\rho} = x + M_{\tilde{\eta}} - B,$$

and we apply the product rule to obtain the decomposition of $X = X_\rho(X/X_\rho)$:

$$X = x + \int_0^T X(t-)\rho'(t)dR(t) + \int_0^T X_\rho(t)d(M_{\tilde{\eta}} - B)(t) + \int_0^T X_\rho(t)dC_{\tilde{\eta}\rho}(t),$$

in conjunction with (23) and (43). Moreover, the condition (3) of Proposition 3.5 implies $C_{\tilde{\eta}\rho} = A_{\tilde{\eta}} = R_{\tilde{\eta}} - M_{\tilde{\eta}}$, and we deduce

$$X = x + \int_0^T (X(t-)\rho'(t) - X_\rho(t)\tilde{\eta}'(t))dR(t) - \int_0^T X_\rho(t)dB(t).$$

Therefore, if we define

$$\vartheta_i(t) := \frac{X(t-)\rho'(t) - X_\rho(t)\tilde{\eta}'(t)}{S_i(t)}, \quad i = 1,\ldots,N, \quad K := \int_0^T X_\rho(t)dB(t),$$

then it is easy to check that

$$\vartheta \equiv (\vartheta_1,\ldots,\vartheta_N) \in I(S) \cap T(n) \cap K.$$ 

**Corollary 3.26.** Suppose that the market consisting of the top $n$ stocks is viable. For a nonnegative, adapted process $X$ with RCLL paths satisfying $X(0) = x \geq 0$, the following statements are then equivalent:

1. The process $YX$ is a local martingale, for every $Y \in \mathcal{Y}^n$.
2. There exists an investment strategy $\vartheta \in I(S) \cap T(n)$ among the top $n$ stocks, such that

$$X = x + \int_0^T \sum_{i=1}^N \vartheta_i(t)dS_i(t).$$

(95)

**Proof.** We first assume (1); then $YX$ is a supermartingale for every $Y \in \mathcal{Y}^n$. From Theorem 3.25, we have a decomposition (93) for some $\vartheta \in I(S) \cap T(n)$ and $K \in \mathcal{K}$. In particular, if we take $Y = 1/X_\rho$, the reciprocal of the local martingale numéraire, we obtain

$$YK = \frac{X(\cdot;x,\vartheta)}{X_\rho} - YX,$$

with the notation in (14). Since the terms on the right-hand side are local martingales, $YK$ is a local martingale, and so is

$$YK - \int_0^T K(t-)dY(t) = \int_0^T Y(t)dB(t).$$
However, the last integral is nondecreasing and is a supermartingale (as a nonnegative local martingale), and therefore identically equal to zero. Thus, \( K \equiv 0 \) as \( Y \) is positive, and the statement (2) follows.

In order to show the reverse implication, we assume (2), then \( X/X_\rho \) is a local martingale where \( \rho \) is a local martingale numéraire portfolio among the top \( n \) stocks, as before. From Lemma 3.11, \( X/X_\rho \) can be cast as a stochastic integral with respect to the local martingale vector \( \tilde{M} \). Furthermore, from Lemma 3.24, every \( Y \in \mathcal{Y}^n \) is of the form \( Y = (1/X_\rho)\mathcal{E}(L) \) for some local martingale \( L \) satisfying \( [L, \tilde{M}_i] = 0 \) for \( i = 1, \ldots, N \). Therefore, the product \( YX = (X/X_\rho)\mathcal{E}(L) \) of these two orthogonal local martingales is again a local martingale.

\[ \square \]

### 3.7 Entire market versus top \( n \) market

We present first the following result, which can be easily proven from the equivalence between the existence of supermartingale numéraire portfolio and market viability.

**Theorem 3.27.** The existence of a supermartingale numéraire portfolio in the whole market, implies the existence of a supermartingale numéraire portfolio among the top \( n \) stocks.

**Proof.** From Theorem 2.34 of Karatzas and Kardaras (2021), the existence of a supermartingale numéraire portfolio in the whole market, is equivalent to the viability of the whole market. The viability of the whole market implies the viability of the market consisting of the top \( n \) stocks, thanks to the inequality \( 0 \leq x(K) \leq x^n(K) \) in Definition 3.14. We conclude that there exists a supermartingale numéraire portfolio among the top \( n \) stocks, from Theorem 3.19.

\[ \square \]

Theorem 3.27 shows that the viability of the entire market, composed of \( N \) stocks, implies the viability of the “top \( n \) market.” Thus, if the entire market is viable, there exist both a supermartingale numéraire portfolio for the whole market, and a supermartingale numéraire portfolio among the top \( n \) stocks, and the former dominates the latter in the sense of growth-optimality. In the following proposition, we study this dominance by expressing the asymptotic behavior of log-relative wealth process between these two portfolios in terms of the “local characteristics” of the market. We first need the following definitions, which are similar to those in (73)–(76).

We call the \([0, \infty]\)-valued, predictable process

\[
g := \sup_{\rho \in \mathbb{R}^N} \left( p'\alpha - \frac{1}{2} p'cp \right)
\]  

(96)

the **maximal growth rate** achievable in the whole market. This process can be rewritten in the form

\[
g = \frac{1}{2} (\alpha'c^\dagger\alpha) \mathbf{1}_{[\alpha \in \text{range}(c)]} + \infty \mathbf{1}_{[\alpha \not\in \text{range}(c)]},
\]  

(97)

and the supremum of (96) is attained if and only if \( g < \infty \), at \( p \equiv \rho := c^\dagger\alpha \), i.e., when \( \rho \) is a supermartingale numéraire portfolio for the whole market. Here, \( c^\dagger \) is the “pseudo-inverse” of \( c \), defined as in (65). Then, the viability of the whole market can be shown to be equivalent to the condition

\[
G(T) := \int_0^T g(t)dO(t) < \infty, \quad \text{for all} \quad T \geq 0.
\]  

(98)
Here, the adapted nondecreasing process $G$ is called aggregate maximal growth of the whole market.

When the whole market is viable, the growth rates $\bar{g}$ of (73) and $g$ of (96) have simpler forms
\begin{equation}
\bar{g} = \frac{1}{2} \bar{\alpha}' \bar{c}^+ \bar{\alpha} = \gamma_{\bar{\alpha}}, \quad g = \frac{1}{2} \alpha' c^+ \alpha = \gamma_{\alpha},
\end{equation}
respectively, as from (78), with $\bar{\alpha} : = D\bar{c}^+ \bar{\alpha}$ a supermartingale numéraire portfolio among the top $n$ stocks, and $\rho := c^+ \alpha$ a supermartingale numéraire portfolio for the whole market. We denote the difference of aggregate maximal growth between the whole market and the top $n$ market by
\begin{equation}
J := G - \bar{G} = \int_{0}^{T} (g(t) - \bar{g}(t))dO(t) = \int_{0}^{T} (\gamma_{\rho}(t) - \gamma_{\bar{\rho}}(t))dO(t) = \Gamma_{\rho} - \Gamma_{\bar{\rho}}.
\end{equation}

Since $\rho$ is also a growth-optimal portfolio (as a supermartingale numéraire portfolio) in the whole market, the relative growth $\Gamma_{\rho}^\rho = \Gamma_{\bar{\rho}} - \Gamma_{\rho}$ of Definition 3.20 is nonincreasing. We conclude that $J$ is nondecreasing and nonnegative.

**Proposition 3.28.** Suppose that the whole market is viable, and let $\rho$ and $\bar{\rho}$ be a supermartingale numéraire portfolio for the whole market and a supermartingale numéraire portfolio among the top $n$ stocks, respectively. Then, the asymptotic growth rate of the log-relative wealth process $\log(X_{\bar{\rho}}/X_{\rho})$ is the same as $-J$ of (100), namely,
\begin{equation}
\lim_{T \to \infty} \frac{1}{J(T)} \log \left( \frac{X_{\bar{\rho}}(T)}{X_{\rho}(T)} \right) = -1 \quad \text{holds \ P - a.e. on the set} \quad \{ \lim_{T \to \infty} J(T) = \infty \}.
\end{equation}

**Proof.** We recall the notations (28)–(32) and write for $T \geq 0$,
\begin{equation}
\log(X_{\bar{\rho}}/X_{\rho})(T) = \log X_{\bar{\rho}}(T) - \log X_{\rho}(T) = \int_{0}^{T} (\gamma_{\bar{\rho}}(t) - \gamma_{\rho}(t))dO(t) + \int_{0}^{T} (\bar{\rho}(t) - \rho(t))'dM(t).
\end{equation}
The first integral on the right-hand side is just $-J(T)$ of (100) and it can be rewritten as
\begin{equation}
-J(T) = \int_{0}^{T} (\gamma_{\bar{\rho}}(t) - \gamma_{\rho}(t))dO(t) = \frac{1}{2} \int_{0}^{T} (\bar{\alpha}' \bar{c}^+ \bar{\alpha} - \alpha' c^+ \alpha)(t)dO(t)
\end{equation}
from (99). On the other hand, from $\rho = c^+ \alpha$ and $\bar{\rho} = D\bar{c}^+ \bar{\alpha}$, we obtain series of equations as in (71):
\[ (\bar{\rho} - \rho)'c(\bar{\rho} - \rho) = \bar{\rho}'c\bar{\rho} + \rho'c\rho - \bar{\rho}'c\rho - \rho'c\bar{\rho} = \bar{\alpha}' \bar{c}^+ \bar{\alpha} + \alpha' c^+ \alpha - 2\bar{\rho}'c\rho, \]
as well as
\[ \bar{\rho}'c\rho = \bar{\rho}'cc^+ \alpha = \bar{\rho}' \alpha = \bar{\alpha}'(\bar{c}^+)D\alpha = \bar{\alpha}'(\bar{c}^+)\bar{\alpha}. \]
Combining these equations, we have
\[ (\bar{\rho} - \rho)'c(\bar{\rho} - \rho) = \alpha' c^+ \alpha - \bar{\alpha}' \bar{c}^+ \bar{\alpha}. \]
Thus, the quadratic variation of the last integral on the right-hand side of (102) is written as

\[
\left[ \int_0^T (\tilde{\rho}(t) - \rho(t))' dM(t) \right] = \int_0^T (\tilde{\rho} - \rho)' c(\tilde{\rho} - \rho)(t) dO(t) \\
= \int_0^T (\alpha' c \alpha - \tilde{\alpha}' \tilde{c} \tilde{\alpha})(t) dO(t) = 2J(T).
\]

The Dambis–Dubins–Schwarz representation—Theorem 3.4.6, Problem 3.4.7 of Karatzas and Shreve (1991)—along with the scaling property of Brownian motion, implies that there exists a Brownian motion \( W \), on a possibly enlarged filtered probability space, such that

\[
\log X_{\tilde{\rho}}(T) = -J(T) + \sqrt{2W(J(T))}.
\]

The strong law of large numbers for Brownian motion gives the result (101). □

The expression (103) shows that the asymptotic growth rate of the log-relative wealth process \( \log(X_{\tilde{\rho}}/X_{\rho}) \) is expressed in terms of the “local characteristics” of the market: \( \alpha, \tilde{\alpha}, c, \) and \( \tilde{c} \).

4 | STOCK PORTFOLIOS IN OPEN MARKETS

The open market described in the previous section consists of the top \( n \) stocks in terms of capitalization, and of the money market. The existence of this money market gives us flexibility to construct portfolios among the top \( n \) stocks. To be more specific, for any given portfolio \( \pi \in I(R) \), premultiplying it the diagonal matrix \( D \) of (47) transforms it into a new portfolio \( D\pi \) among the top \( n \) stocks. The proportion of assets, which is supposed to be invested in “bottom” \( N - n \) stocks by \( \pi \), is now assigned to the money market by \( D\pi \). In the absence of the money market, building portfolios among the top \( n \) stocks is more subtle, and this section focuses on these subtleties.

4.1 | Stock portfolios and the market portfolio

An important subclass of portfolios in Definition 2.5 is the collection of portfolios \( \pi \) satisfying \( \sum_{i=1}^N \pi_i \equiv 1 \), or \( \pi_0 \equiv 0 \) in (26). Such a portfolio never invests in the money market; and this condition can be formulated as \( \pi \in \Delta^{N-1} \), where we denote

\[
\Delta^{N-1} := \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \sum_{i=1}^N x_i = 1 \right\}.
\]

**Definition 4.1 (Stock Portfolio).** We call a portfolio \( \pi \in I(R) \) stock portfolio, if it takes values in \( \Delta^{N-1} \), i.e., satisfies \( \sum_{i=1}^N \pi_i \equiv 1 \). We denote the collection of stock portfolios by \( I(R) \cap \Delta^{N-1} \).

We call a stock portfolio \( \pi \) stock portfolio among the top \( n \) stocks, if in addition it belongs to \( \mathcal{T}(n) \), i.e., satisfies the condition (16), or equivalently, (18). We denote the collection of stock portfolios among the top \( n \) stocks by \( I(R) \cap \Delta^{N-1} \cap \mathcal{T}(n) \).
Remark 4.2 (Self-financibility of stock portfolios). For any stock portfolio \( \pi \), we sum over \((25)\) for all indices \( i = 1, \ldots, N \) to obtain
\[
1 \equiv \sum_{i=1}^{N} \pi_i(t) = \sum_{i=1}^{N} S_i(t) \theta_i(t) / X(t; 1, \theta), \quad t \geq 0
\]
and from \((14)\),
\[
X(T; 1, \theta) = 1 + \int_{0}^{T} \sum_{i=1}^{N} \theta_i(t) dS_i(t) = \sum_{i=1}^{N} \theta_i(T) S_i(T), \quad T \geq 0.
\]
This last equation shows the “self-financing” property of stock portfolios (see Definition 2.1 of Karatzas & Ruf, 2017) the sum of the product between the trading strategy \( \theta_i \) and the stock price \( S_i \) is equal to the sum of stochastic integrals of each trading strategy with respect to the corresponding stock price, along with the initial capital 1, across all times. There are neither withdrawals nor infusions of capital; gains are reinvested, losses are absorbed.

Before we present the most important example of stock portfolios, we introduce the notation
\[
\Sigma := S_1 + \cdots + S_N, \quad (105)
\]
representing the total capitalization of whole equity market.

Example 4.3 (Market portfolio). Suppose that an investment strategy \( \theta \) is given as \( \theta \equiv 1/\Sigma(0) \equiv (1, 1, \ldots, 1)/\Sigma(0) \) with initial wealth \( x = 1 \). Then, its wealth process is just the total capitalization normalized by its initial value:
\[
X(\cdot; 1, \theta) = \Sigma(\cdot) / \Sigma(0). \quad (106)
\]
Whereas, from \((25)\), the corresponding portfolio \( \pi \equiv \mu \equiv (\mu_1, \ldots, \mu_N) \) can be expressed as
\[
\mu_i(\cdot) = S_i(\cdot)/\Sigma(\cdot) = S_i(\cdot) / S_1(\cdot) + \cdots + S_N(\cdot), \quad \text{for} \quad i = 1, \ldots, N. \quad (107)
\]
We call this special stock portfolio \( \mu \) the market portfolio, and its component processes in \((107)\) market weights; it is considered as the most important stock portfolio, as its wealth process gives the evolution of total market capitalization.

In an analogous manner, we define the top \( n \) market portfolio, which we denote by \( \tilde{\mu} \equiv (\tilde{\mu}_1, \ldots, \tilde{\mu}_N) \), with components
\[
\tilde{\mu}_i(\cdot) := \tilde{S}_i(\cdot) / \tilde{\Sigma}(\cdot) = \frac{\tilde{S}_i(\cdot)}{\tilde{S}_1(\cdot) + \cdots + \tilde{S}_N(\cdot)}, \quad \text{for} \quad i = 1, \ldots, N, \quad (108)
\]
where
\[
\tilde{\Sigma} := \sum_{i=1}^{N} \tilde{S}_i = S_{(1)} + \cdots + S_{(n)}, \quad \text{and} \quad \tilde{S}_i(\cdot) := 1_{[u_i(\cdot) \leq n]} S_i(\cdot), \quad \text{for} \quad i = 1, \ldots, N. \quad (109)
\]
The denominator $\tilde{\Sigma}$ of (108) represents the sum of the capitalizations of the top $n$ stocks; thus, $\tilde{\mu}_i(t)$ is the proportion of the capitalization of stock $i$, if this stock belongs to the top $n$, to the total capitalization of the top $n$ stocks at time $t$. In other words, $\tilde{\mu}_i$ can be interpreted as the “market weight” of $i$-th stock in the restricted market composed of the top $n$ stocks by capitalization. It is easy to check that $\tilde{\mu}$ is a stock portfolio among the top $n$ stocks, i.e., $\tilde{\mu} \in I(R) \cap \Delta^{N-1} \cap T(n)$.

### 4.2 Capital Asset Pricing Model

The CAPM posits that individual stocks cannot systematically outperform the market. In our open market setting, this requirement can be cast as saying that each individual stock, whenever it belongs to the top $n$ stocks, cannot outperform the top $n$ market. In this subsection, we briefly discuss this model for the top $n$ market. Recalling the top $n$ stock portfolio $\tilde{\mu}$ defined in (108), we have the next definition.

**Definition 4.4 (CAPM).** We say that the top $n$ market is in the realm of the CAPM if

$$\tilde{R}_i = \int_0^T \beta_i(t)dR_{\tilde{\mu}}(t) + N_i, \quad i = 1, \ldots, N, \quad (110)$$

hold for appropriate processes $\beta_i \in I(R_{\tilde{\mu}})$, $i = 1, \ldots, N$, and for continuous local martingales $N_i$ with $N_i(0) = 0$ which are orthogonal to $R_{\tilde{\mu}}$ for all $i = 1, \ldots, N$:

$$[N_i, R_{\tilde{\mu}}] \equiv 0.$$

The following result characterizes this property in terms of the local characteristics of the top market introduced in Section 2.4.

**Proposition 4.5 (Characterization of CAPM).** The top $n$ market is in the realm of the CAPM if, and only if, the following two conditions hold.

(A) There exists a scalar predictable process $b$, playing the role of “leverage,” such that

$$\sum_{i=1}^N \int_0^T |b(t)|1_{\{c_{\tilde{\mu}} > 0\}}|dC_{\tilde{\mu}}(t)| < \infty, \quad \text{for} \quad T \geq 0, \quad (111)$$

and the following equalities hold $(\mathbb{P} \otimes O)$-a.e.:

$$\tilde{\alpha}_i = b c_{\tilde{\mu}}, \quad \text{on} \quad \{c_{\tilde{\mu}} > 0\} \quad \text{for} \quad i = 1, \ldots, N. \quad (112)$$

(B) On the set $\{c_{\tilde{\mu}} = 0\}$, we have $(\mathbb{P} \otimes O)$-a.e.:

$$\alpha_{\tilde{\mu}} = 0 \iff \tilde{\alpha}_i = 0, \quad i = 1, \ldots, N. \quad (113)$$
When these conditions are satisfied, the process $b$ of (111) and the processes $\beta_i \in \mathcal{I}(R_{\tilde{\mu}})$ of (110) can be chosen, respectively, as

$$b = \frac{\alpha_{\tilde{\mu}}}{c_{\mu_\tilde{\mu}}} 1_{\{c_{\mu_\tilde{\mu}} > 0\}}, \quad (114)$$

$$\beta_i = \frac{c_{\tilde{\mu}}}{c_{\mu_\tilde{\mu}}} 1_{\{c_{\mu_{\tilde{\mu}}} > 0\}} + \frac{\alpha_{\tilde{\mu}}}{\alpha_{\mu}} 1_{\{c_{\mu_{\tilde{\mu}}} = 0, \alpha_{\tilde{\mu}} \neq 0\}}, \quad i = 1, \ldots, N. \quad (115)$$

**Proof.** Let us assume first that the top $n$ market is in the realm of the CAPM. Recalling the notation (46), we have

$$C_{\tilde{\mu}} = [\tilde{R}, R_{\tilde{\mu}}] = \int_0^\infty \beta_i(t) d[R, R_{\tilde{\mu}}](t) + [N, R_{\tilde{\mu}}] = \int_0^\infty \beta_i(t) dC_{\mu_\tilde{\mu}}(t),$$

which implies that $c_{\tilde{\mu}} = \beta_i c_{\mu_{\tilde{\mu}}}$ also hold (\(\mathbb{P} \otimes O\))-a.e., for $i = 1, \ldots, N$. On $\{c_{\mu_\tilde{\mu}} > 0\}$, it follows that

$$\beta_i = \frac{c_{\tilde{\mu}}}{c_{\mu_\tilde{\mu}}} \beta_i c_{\mu_{\tilde{\mu}}} \quad \text{for} \quad i = 1, \ldots, N.$$ Moreover, since $\tilde{R} - \int_0^\infty \beta_i(t) dR_{\tilde{\mu}}(t)$ is a local martingale, we obtain $\tilde{A}_i = \int_0^\infty \beta_i(t) dA_{\tilde{\mu}}(t)$, and also $\tilde{\alpha}_i = \beta_i \alpha_\tilde{\mu}$ holds (\(\mathbb{P} \otimes O\))-a.e. for $i = 1, \ldots, N$. As a consequence, the identities of (112)

$$\tilde{\alpha}_i = \frac{\alpha_{\tilde{\mu}}}{c_{\mu_\tilde{\mu}}} c_{\tilde{\mu}} = bc_{\tilde{\mu}}, \quad \text{hold for} \quad i = 1, \ldots, N, \quad (\mathbb{P} \otimes O) - \text{a.e. on} \quad \{c_{\mu_\tilde{\mu}} > 0\},$$

with $b$ given as in (114). Also, the (\(\mathbb{P} \otimes O\))-a.e. identities $\tilde{\alpha}_i = \beta_i \alpha_\tilde{\mu}$, combined with $\alpha_{\tilde{\mu}} = \tilde{\mu}' \alpha$, lead to the condition (B). Finally, $b = \tilde{\alpha}_i / c_{\tilde{\mu}}$ on $\{c_{\mu_\tilde{\mu}} > 0, c_{\tilde{\mu}} \neq 0\}$ implies that $|b| 1_{\{c_{\mu_\tilde{\mu}} > 0\}} |c_{\tilde{\mu}}| \leq |\tilde{\alpha}_i|$ hold for $i = 1, \ldots, N$, and thus the condition (111):

$$\sum_{i=1}^N \int_0^T |b(t)| 1_{\{c_{\mu_\tilde{\mu}} > 0\}} |dC_{\tilde{\mu}}(t)| \leq \sum_{i=1}^N \int_0^T |d\tilde{A}_i(t)| < \infty, \quad \text{for all} \quad T \geq 0.$$

Conversely, suppose that the conditions (A) and (B) are valid. For $i = 1, \ldots, N$, defining $\beta_i$ via (115), we have

$$\int_0^T |\beta_i(t)| |dA_{\tilde{\mu}}(t)| \leq \int_0^T |b(t)| 1_{\{c_{\mu_\tilde{\mu}} > 0\}} |dC_{\tilde{\mu}}(t)| + \int_0^T |d\tilde{A}_i(t)| < \infty,$$

as well as

$$\int_0^T |\beta_i(t)|^2 dC_{\mu_\tilde{\mu}}(t) = \int_0^T \frac{|c_{\tilde{\mu}}(t)|^2}{c_{\mu_\tilde{\mu}}(t)} 1_{\{c_{\mu_\tilde{\mu}} > 0\}} dO(t) \leq \int_0^T c_{\tilde{\mu}}(t) dO(t) = \tilde{C}_i(T) < \infty.$$ These inequalities imply that $\beta_i \in \mathcal{I}(R_{\tilde{\mu}})$ for $i = 1, \ldots, N$. Furthermore, recalling the semimartingale decomposition (35), we observe that

$$\int_0^T \beta_i(t) dR_{\tilde{\mu}}(t) = \int_0^T \tilde{\beta}_i(t) \tilde{\mu}'(t) d\tilde{A}(t) + \int_0^T \beta_i(t) \tilde{\mu}'(t) d\tilde{M}(t).$$
\[ \begin{align*}
&= \int_0^\beta(t) \mathbf{1}_{\{c_{\tilde{\mu}}(t) > 0\}} b(t) dC_{\tilde{\mu}}(t) + \int_0^\beta(t) \mathbf{1}_{\{c_{\tilde{\mu}}(t) = 0\}} \tilde{\mu}'(t) d\tilde{A}(t) \\
&\quad + \int_0^\beta(t) \tilde{\mu}'(t) d\tilde{M}(t),
\end{align*} \]  

from (114). The first two integrals on the right-hand side of (116) can be expressed as

\[ \int_0^\beta(t) \mathbf{1}_{\{c_{\tilde{\mu}}(t) > 0\}} b(t) dC_{\tilde{\mu}}(t) = \int_0^\beta(t) \mathbf{1}_{\{c_{\tilde{\mu}}(t) > 0\}} b(t) dC_{\tilde{\mu}}(t) = \int_0^\beta(t) \mathbf{1}_{\{c_{\tilde{\mu}}(t) > 0\}} d\tilde{A}(t), \]

and

\[ \int_0^\beta(t) \mathbf{1}_{\{c_{\tilde{\mu}}(t) = 0\}} \tilde{\mu}'(t) d\tilde{A}(t) = \int_0^\beta(t) \mathbf{1}_{\{c_{\tilde{\mu}}(t) = 0\}} d\tilde{A}(t), \]

for \( i = 1, \ldots, N \), on account of (115). Thus, we obtain

\[ \int_0^\beta(t) dR_{\tilde{\mu}}(t) = \tilde{A}_i + \int_0^\beta(t) \tilde{\mu}'(t) d\tilde{M}(t) = \tilde{R}_i - \int_0^\beta(t) (e^\beta - \beta(t) \tilde{\mu}(t))' d\tilde{M}(t) = \tilde{R}_i - N_i, \]

which is (110), where we define \( N_i = \int_0^\beta(t) (e^\beta - \beta(t) \tilde{\mu}(t))' d\tilde{M}(t) \) for \( i = 1, \ldots, N \). We observe that the identities \( (e^\beta - \beta(t) \tilde{\mu})' c_{\tilde{\mu}} = c_i - \beta_i c_{\tilde{\mu}} = 0 \) hold on the set \( \{c_{\tilde{\mu}} > 0\} \) from the definition (115), as well as on the set \( \{c_{\tilde{\mu}} = 0\} \) since \( c_i = 0 \) holds there. Finally, we obtain

\[ [N_i, R_{\tilde{\mu}}] = \int_0^\beta(t) (e^\beta - \beta(t) \tilde{\mu}(t))' c(t) \tilde{\mu}(t) dO(t) \equiv 0, \quad i = 1, \ldots, N, \]

which shows that the top \( n \) market is in the realm of the CAPM. \( \square \)

### 4.3 Functional generation of portfolios

Functionally generated portfolios were first introduced by Fernholz (1999). Given a function \( G : \Delta^N_{+1} \to (0, \infty) \) of class \( C^2 \) with the notation

\[ \Delta^N_{+1} := \Delta^N \cap \mathbb{R}^N_+ = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_i \geq 0 \text{ for } i = 1, \ldots, N, \sum_{i=1}^N x_i = 1 \right\}, \]

we can generate a portfolio \( \pi^G \) from \( G \), depending on the vector of market weights \( \mu \). The formula (11.2) of Fernholz and Karatzas (2009), colloquially known as the “master formula,” gives a simple way to compare the relative wealth process of \( \pi^G \) with respect to the “market,” namely, the market portfolio \( \mu \); see, Chapter III of Fernholz and Karatzas (2009) for an overview.

In what follows, we present a new way to generate portfolios from a function having the market portfolio among the top \( n \) stocks \( \tilde{\mu} \) in (108) as its input. We also derive a new “master formula” to compare the wealth of the so-generated portfolio, relative to \( \tilde{\mu} \), the market portfolio among the top \( n \) stocks.
For any stock portfolio \( \pi \in I(R) \cap \Delta^{N-1} \), we have \( \pi_0 \equiv 0 \) (no investing in the money market), thus

\[
\frac{dX^\tilde{\mu}_\pi(t)}{X^\tilde{\mu}_\pi(t)} = \sum_{i=1}^N \pi_i(t) dR^\tilde{\mu}_i(t) = \sum_{i=1}^N \frac{\pi_i(t)}{S_i^\tilde{\mu}(t)} dS_i^\tilde{\mu}(t),
\]

from (61). Here, we recall from (54) that \( S^\tilde{\mu}_i \) is the vector of stock prices denominated by the wealth process \( X^\tilde{\mu} \) of the market portfolio among the top \( n \) stocks.

We note at this point, that the market portfolio \( \mu \) in Example 4.3, has a very nice property: the denominated stock price \( S^\mu_i \) has a simple representation, namely \( S^\mu_i(\cdot) = \Sigma(0) \mu_i(\cdot) \), for \( i = 1, \ldots, N \). Thus, if we used \( \mu \) instead of \( \tilde{\mu} \) in deriving (118), the last integrator would be rewritten as \( dS^\mu_i(t) = \Sigma(0) d\mu_i(t) \). However, unlike \( \mu \), the components of \( \tilde{\mu} \) in (108) do not admit such a simple representation. For this reason, we will use the denominated stock price \( S^\tilde{\mu}_i(t) \) as integrators, and will let the generating function \( G \) depend on \( S^\tilde{\mu}_i(t) \) instead of \( \tilde{\mu}_i(t) \) in what follows.

For a given function \( G : (0, \infty)^N \rightarrow (0, \infty) \) of class \( C^2 \), we want to write the relative-log wealth as

\[
\log X^\tilde{\mu}_\pi(t) = \log \left( \frac{G(S^\tilde{\mu}(t))}{G(S^\tilde{\mu}(0))} \right) + J^\tilde{\mu}_\pi(t), \quad \text{for any } t \geq 0,
\]

for some function \( J^\tilde{\mu}_\pi(\cdot) \) of finite variation. In order to find \( J^\tilde{\mu}_\pi(\cdot) \), we apply Itô’s rule, to obtain

\[
\frac{dX^\tilde{\mu}_\pi(t)}{X^\tilde{\mu}_\pi(t)} = dJ^\tilde{\mu}_\pi(t) + \sum_{i=1}^N D_i G(S^\tilde{\mu}(t)) \frac{dS^\tilde{\mu}_i(t)}{G(S^\tilde{\mu}(t))} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N D^2_{ij} G(S^\tilde{\mu}(t)) \frac{d[S^\tilde{\mu}_i(t), S^\tilde{\mu}_j(t)](t)}{G(S^\tilde{\mu}(t))}.
\]

Comparing the two equations (118) and (120), suppose we can find a portfolio \( \pi \) such that

\[
\sum_{i=1}^N \frac{\pi_i(t)}{S^\tilde{\mu}_i(t)} dS^\tilde{\mu}_i(t) = \sum_{i=1}^N D_i G(S^\tilde{\mu}(t)) \frac{dS^\tilde{\mu}_i(t)}{G(S^\tilde{\mu}(t))},
\]

holds, then we have

\[
dJ^\tilde{\mu}_\pi(t) = -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{D^2_{ij} G(S^\tilde{\mu}(t))}{G(S^\tilde{\mu}(t))} d[S^\tilde{\mu}_i(t), S^\tilde{\mu}_j(t)](t).
\]

Now, a candidate portfolio \( \pi \) satisfying (121), is given as

\[
\pi_i(t) = S^\tilde{\mu}_i(t) \frac{D_i G(S^\tilde{\mu}(t))}{G(S^\tilde{\mu}(t))}, \quad i = 1, \ldots, N;
\]
but it need not belong to $\Delta^{N-1}$. Instead, we set

$$\pi_i(t) := S_{i}^{\tilde{\mu}}(t) \frac{D_i G(S_{i}^{\tilde{\mu}}(t))}{G(S_{i}^{\tilde{\mu}}(t))} + \tilde{\mu}_i(t) - \tilde{\mu}_i(t) \sum_{j=1}^N S_{j}^{\tilde{\mu}}(t) \frac{D_j G(S_{j}^{\tilde{\mu}}(t))}{G(S_{j}^{\tilde{\mu}}(t))}, \quad i = 1, \ldots, N,$$  \hfill (122)

then it is easy to show that $\pi \in \Delta^{N-1}$. To check the condition (121), we note

$$\sum_{i=1}^N \frac{\pi_i(t)}{S_{i}^{\tilde{\mu}}(t)} dS_{i}^{\tilde{\mu}}(t) = \sum_{i=1}^N \frac{D_i G(S_{i}^{\tilde{\mu}}(t))}{G(S_{i}^{\tilde{\mu}}(t))} dS_{i}^{\tilde{\mu}}(t) + \left\{ 1 - \sum_{j=1}^N S_{j}^{\tilde{\mu}}(t) \frac{D_j G(S_{j}^{\tilde{\mu}}(t))}{G(S_{j}^{\tilde{\mu}}(t))} \right\} \sum_{i=1}^N \frac{\tilde{\mu}_i(t)}{S_{i}^{\tilde{\mu}}(t)} dS_{i}^{\tilde{\mu}}(t),$$

and the last term vanishes because

$$\sum_{i=1}^N \frac{\tilde{\mu}_i(t)}{S_{i}^{\tilde{\mu}}(t)} dS_{i}^{\tilde{\mu}}(t) = \sum_{i=1}^N \bar{\mu}_i(t) dR_{i}^{\tilde{\mu}}(t) = dR_{\tilde{\mu}}(t) = 0.$$

Here, $R_{\tilde{\mu}} = \mathcal{L}(X_{\tilde{\mu}}) = \mathcal{L}(1) \equiv 0$, from Proposition 3.1.

The construction described above can be formulated as the following definition and proposition.

**Definition 4.6 (Functionally generated portfolio).** Let $G : (0, \infty)^N \rightarrow (0, \infty)$ be a twice continuously differentiable function. Then, the vector $\pi^G \equiv \pi = (\pi_1, \ldots, \pi_N)$ defined as in (122) is called the **stock portfolio generated by the function $G$ via the market portfolio among the top $n$ stocks**.

**Proposition 4.7 (Master Formula).** For the stock portfolio $\pi^G$ generated by $G$ with the market portfolio among the top $n$ stocks, we have the decomposition

$$\log \left( \frac{X_{\pi^G}}{X_{\tilde{\mu}}} \right) = \log \left( \frac{G(S_{\tilde{\mu}})}{G(S_{\tilde{\mu}}(0))} \right) - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \int_0^t \frac{D_{i,j}^2 G(S_{\tilde{\mu}}(t))}{G(S_{\tilde{\mu}}(t))} d[S_{i}^{\tilde{\mu}}, S_{j}^{\tilde{\mu}}](t).$$  \hfill (123)

The above arguments, leading to Definition 4.6 and Proposition 4.7, have two weaknesses. First, the functionally generated stock portfolio $\pi^G$ in (122) is not generally a portfolio among the top $n$ stocks, i.e., it can fail to belong to $\mathcal{T}(n)$. Thus, the Master formula (123) compares a portfolio $\pi^G$, which is not a portfolio among the top $n$ stocks, with $\mu$, which is a portfolio among the top $n$ stocks. We will fix this issue by restricting the class of generating functions $G$ in the next subsection.

Second, when we construct a portfolio via (122) or use the Master formula (123), we need to know at each time $t \geq 0$ the entire history of the process $X_{\tilde{\mu}}$, up to time $t$, because these equations require the values of the vector $S_{\tilde{\mu}} = S/X_{\tilde{\mu}}$. This issue is unfortunately inevitable in the open market, because of its own nature of $\tilde{\mu}$; as it is composed of the top $n$ stocks, we need to keep track of the ranks of $N$ stocks all the time, whereas computing the wealth $X_{\tilde{\mu}}$ generated by the market portfolio $\mu$ only requires current stock prices (and the stock prices at time $t = 0$), from its simple representation (106). Though we cannot resolve this second issue, we will give a representation of $X_{\tilde{\mu}}$ in the following subsection.
4.4 Functionally generated portfolios using ranks

Recalling the rank notation in Definition 2.2, we define the random permutation process \( p_k(t) \) of \( \{1, \ldots, N\} \) such that for \( k = 1, \ldots, N \),

\[
S_{p_k(t)}(t) = S_k(t), \tag{124}
\]

\[
p_k(t) < p_{k+1}(t) \quad \text{if} \quad S_k(t) = S_{k+1}(t).
\]

Here, \( p_k(t) \) represents the index (i.e., name) of the stock occupying rank \( k \) at time \( t \); we break ties using the lexicographic rule, so it is the inverse permutation of \( u_i(t) \) introduced in (4): \( u_i(t) = k \iff p_k(t) = i \), for all \( t \geq 0 \).

For any continuous semimartingale \( Y \), we denote the local time accumulated at the origin by \( Y(\cdot) \) up to time \( t \geq 0 \) by \( L_Y(t) \):

\[
L_Y(t) := \frac{1}{2} \left( |Y(t)| - |Y(0)| - \int_0^t \text{sign}(Y(s))dY(s) \right), \quad \text{where} \quad \text{sign}(x) = 2 \times \mathbb{1}_{(0,\infty)}(x) - 1.
\]

Then, \( L_{S(k)-S(\ell)}(t) \) can be interpreted as the “collision local time” accumulated up to time \( t \), whenever the \( k \)-th and \( \ell \)-th ranked processes of \( S \) collide. In order to simplify the local time terms throughout this section, we introduce the following definition, which prohibits the accumulation of local times of “triple collisions” between the stock prices.

**Definition 4.8.** The components of the price vector \( S = (S_1, \ldots, S_N) \) in Definition 2.1 are called **pathwise mutually nondegenerate**, if

(i) the set \( \{ t : S_i(t) = S_j(t) \} \) has Lebesgue measure zero, \( \mathbb{P} \)-a.e., for all \( i \neq j \); and if

(ii) \( L_{S(k)-S(\ell)}(t) \equiv 0 \) holds \( \mathbb{P} \)-a.e., for all \( |k - \ell| \geq 2 \).

**Proposition 4.9.** Suppose that the components of the price vector \( S \) are pathwise mutually nondegenerate. Then, with the notation (109), the wealth process \( X_{\bar{\mu}} \) generated by the portfolio \( \bar{\mu} \) of (108) admits the representation

\[
X_{\bar{\mu}}(\cdot) = \frac{\bar{\Sigma}(\cdot)}{\bar{\Sigma}(0)} \exp \left( -\frac{1}{2} \int_0^\cdot \frac{1}{\bar{\Sigma}(t)} dL_{S(n)-S(n+1)}(t) \right). \tag{125}
\]

**Proof.** From Proposition 3.1 and the fact that \( \bar{\mu} \) is a stock portfolio, we have

\[
X_{\bar{\mu}}(\cdot) = \mathcal{E} \left( \int_0^\cdot \sum_{i=1}^N \bar{\mu}_i(t) dR_i(t) \right) = \mathcal{E} \left( \int_0^\cdot \sum_{i=1}^N \frac{\bar{S}(t)}{\bar{\Sigma}(t)} dS_i(t) \right) = \mathcal{E} \left( \int_0^\cdot \sum_{i=1}^N \sum_{k=1}^n \frac{1_{[u_i(t)=k]}}{\bar{\Sigma}(t)} dS_i(t) \right).
\]
On the other hand, from Proposition 4.1.11 of Fernholz (2002), we have

$$\sum_{i=1}^{N} 1_{[\mu_i(t) = k]} dS_i(t) = dS_k(t) - \frac{1}{2} dL_{S_k} - S_{k+1}(t) + \frac{1}{2} dL_{S_{k-1} - S_k}(t),$$

for $k = 1, \ldots, N$ and $t \geq 0$, with the conventions $L_{S_0} - S_1 \equiv 0$ and $L_{S_N} - S_{N+1} \equiv 0$. Thus, we obtain

$$X_{\tilde{\mu}}(\cdot) = \mathcal{E}\left( \int_{0}^{\cdot} \frac{dS_k(t)}{\bar{S}(t)} - \frac{1}{2} \int_{0}^{\cdot} \frac{dL_{S_k}(t)}{\bar{S}(t)} \right) = \mathcal{E}\left( \int_{0}^{\cdot} \frac{d\bar{S}(t)}{\bar{S}(t)} - \frac{1}{2} \int_{0}^{\cdot} \frac{dL_{S_k}(t)}{\bar{S}(t)} \right).$$

The exponential term of (125) captures the “leakage,” the effect caused by stocks which cross over from the top $n$ league to the bottom. Due to this effect, we need to keep track of the collision local time $L_{S_n} - S_{n+1}$ in order to compute $X_{\tilde{\mu}}$, as we pointed out at the end of the previous subsection.

We next present Fernholz’s original method of constructing rank-dependent portfolios from generating functions. We write $\mu_k$ to represent the $k$-th ranked market weight among $\mu_1, \ldots, \mu_N$ for $k = 1, \ldots, N$, and introduce the vector $\mu = (\mu_1, \ldots, \mu_N)$ with components $\mu_k = S_k/\Sigma, k = 1, \ldots, N$, as in (2), (105). The following result is based on Theorem 4.2.1 of Fernholz (2002).

**Theorem 4.10** (Functionally generated portfolios based on ranked market weights). Suppose that the price vector $S$ is pathwise mutually nondegenerate. Let $p_k(\cdot), k = 1, \ldots, N$ be the random permutation process defined by (124) and let $G$ be a function defined on a neighborhood $U$ of $\Delta_{N-1}$. Suppose that there exists a positive $C^2$ function $G$ such that for $(x_1, \ldots, x_N) \in U$,

$$G(x_1, \ldots, x_N) = G(x_1, \ldots, x_N).$$

Then, $G$ generates the stock portfolio $\pi^G$ such that for $k = 1, \ldots, N$,

$$\pi_{p_k(t)}^G(t) = \left( \frac{D_k G(\mu(t))}{G(\mu(t))} + 1 - \sum_{\ell=1}^{N} \mu_\ell \frac{D_\ell G(\mu(t))}{G(\mu(t))} \right) \mu_k(t), \quad \text{for} \quad t \geq 0.$$  

The log-relative wealth process of $\pi^G$ with respect to the market portfolio $\mu$, can be expressed via the “master formula”:

$$\log \left( \frac{X_{\pi^G}}{X_\mu} \right) = \log \left( \frac{G(\mu)}{G(\mu(0))} \right) - \frac{1}{2} \int_{0}^{\cdot} \sum_{k=1}^{N-1} \left( \frac{\pi_{p_k(t)}^G(t)}{\mu_k(t)} - \frac{\pi_{p_{k+1}(t)}^G(t)}{\mu_{k+1}(t)} \right) dL^{\mu_k - \mu_{k+1}}(t)$$

$$- \frac{1}{2} \int_{0}^{\cdot} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{D_{k,\ell} G(\mu)}{G(\mu)} - d[\mu_k, \mu_\ell](t).$$  

□
The portfolio \( \pi^G \) generated via the recipe (127) is easily checked to be a stock portfolio, i.e., \( \pi^G \in I(R) \cap \Delta^{N-1} \); however, it is not generally a portfolio among the top \( n \) stocks, because \( \pi^G_{p_k(t)}(t) \) may have a nonzero value for \( k > n \) at some time \( t \geq 0 \). In order to make it a portfolio among the top \( n \) stocks, we need to impose two conditions on the function \( G \) in Theorem 4.10:

(A) \( G \) is “balanced,” i.e., satisfies the identity

\[
G(x_1, \ldots, x_N) = \sum_{j=1}^{N} x_j D_j G(x_1, \ldots, x_N), \quad \text{for any } x \in U, \tag{129}
\]

(B) \( G(x) \) depends only on the first \( n \) components of \( x \).

If the condition (A) is satisfied, then the portfolio \( \pi^G \) of (127) has a simpler representation as

\[
\pi^G_{p_k(t)}(t) = \frac{D_k G(\mu(t))}{G(\mu(t))} \mu_k(t), \quad \text{for } t \geq 0. \tag{130}
\]

Moreover, if the condition (B) holds as well, then \( D_k G(\mu) = 0 \) for \( k > n \), thus \( \pi^G_{p_k(t)}(t) = 0 \) for \( k > n \). This means that the portfolio \( \pi^G \) does not invest in the \( i = p_k(t) \)-th stock at time \( t \), if the rank \( k \) of this \( i \)-th stock is bigger than \( n \) at time \( t \).

**Definition 4.11** (Admissible generating function in open market). We call a function \( G \) in Theorem 4.10 an **admissible generating function of market consisting of the top \( n \) stocks**, if it satisfies conditions (A) and (B) above.

**Corollary 4.12.** If \( G \) in Theorem 4.10 is an admissible generating function of market consisting of top \( n \) stocks, then \( G \) generates the stock portfolio among the top \( n \) stocks \( \pi^G \in I(R) \cap T(n) \cap \Delta^{N-1} \), given as (130) for \( k = 1, \ldots, N \). In this case, we have the master formula

\[
\log \left( \frac{X_{\pi^G}}{X_\mu} \right) = \log \left( \frac{G(\mu)}{G(\mu(0))} \right) - \frac{1}{2} \int_0^t \sum_{k=1}^n \left( \frac{D_k G(\mu(t))}{G(\mu(t))} - \frac{D_{k+1} G(\mu(t))}{G(\mu(t))} \right) dL_{\mu(k)-\mu(k+1)}(t) \\
- \frac{1}{2} \int_0^t \sum_{k=1}^n \sum_{\ell=1}^n \frac{D_k^2 G(\mu(t))}{G(\mu(t))} d[\mu_k, \mu_\ell](t). \tag{131}
\]

**Example 4.13** (Balanced functions). By solving the partial differential equation of (129), a balanced function \( G \) can be shown to be homogeneous of degree 1, i.e, the identity

\[
G(ax) = aG(x) \tag{132}
\]

holds for any \( x \in U \) and \( a > 0 \). From this simple characterization of balanced functions, we illustrate three types of balanced functions here:

(i) \( G(x) = \frac{1}{c_1 + \cdots + c_N} \sum_{i=1}^{N} c_i x_i \),

(ii) \( G(x) = (\prod_{i=1}^{N} x_i)^{1/N} \),
These functions are closely related to “three Pythagorean means”; (i) and (ii) are just the weighted-arithmetic and geometric means of the components of \( x \), and (iii) becomes the harmonic mean when \( p = -1 \). A plethora of examples of these types can be found in the literature. The “capitalization-weighted portfolios” of large and small stocks from Example 6.2, Example 6.3 of Karatzas & Ruf (2017), or from Example 4.3.2 of Fernholz (2002) are special cases of (i). The “equal-weighted portfolio,” which holds equal weights across all assets, in Section 4.3 of Karatzas & Ruf (2017), is generated by (ii). The portfolio generated by (iii) for \( 0 < p < 1 \) is called “diversity-weighted portfolio,” and is discussed in detail in Example 3.4.4 and Section 6.2 of Fernholz (2002). Diversity-weighted portfolios with negative parameter \( p < 0 \) in (iii) are the main subject of Karatzas and Vervuurt (2015). We can slightly generalize and make these functions satisfy conditions (A) and (B) as well:

\[
\begin{align*}
(i') \quad & G(x) = \sum_{i=1}^{N} c_i x_i, \\
(ii') \quad & G(x) = \prod_{i=1}^{N} x_i^{c_i}, \quad \text{with} \quad \sum_{i=1}^{N} c_i = 1, \\
(iii') \quad & G(x) = \left( \sum_{i=1}^{N} x_i^p \right)^{1/p},
\end{align*}
\]

for some constants \( c_i \)'s and \( p \).

The following example further devolops Example 4.3.2 of Fernholz (2002), and shows that the top \( n \) market portfolio \( \tilde{\mu} \), defined in (108), can be generated functionally.

**Example 4.14 (Top \( n \) market portfolio).** Consider the function

\[
G(x) = G(x_{(1)}, \ldots, x_{(n)}) = \sum_{k=1}^{n} x_{(k)}
\]

satisfying the conditions (A) and (B) above. Corollary 4.12 implies that \( G \) generates the portfolio

\[
\pi_G^{\beta_k(\cdot)}(\cdot) = \frac{\mu_{(k)}(\cdot)}{\mu_{(1)}(\cdot) + \cdots + \mu_{(n)}(\cdot)} 1_{[k \leq n]} = \frac{S_{(k)}(\cdot)}{S_{(1)}(\cdot) + \cdots + S_{(n)}(\cdot)} 1_{[k \leq n]}.
\]

This coincides with the top \( n \) market portfolio \( \bar{\mu} \), because

\[
\frac{S_{(k)}(\cdot) 1_{[k \leq n]}}{S_{(1)}(\cdot) + \cdots + S_{(n)}(\cdot)} = \frac{S_{\beta_k(\cdot)}(\cdot) 1_{[k \leq n]}}{S(\cdot)} = \bar{\mu}_{\beta_k(\cdot)}(\cdot),
\]

holds for \( k = 1, \ldots, N \), from (108). The master formula (131) is then

\[
\log \left( \frac{X_{\bar{\mu}}}{X_{\mu}} \right) = \log \left( \frac{\mu_{(1)}(\cdot) + \cdots + \mu_{(n)}(\cdot)}{\mu_{(1)}(0) + \cdots + \mu_{(n)}(0)} \right) - \frac{1}{2} \int_0^t \frac{dL^{\mu_{(n)}(\cdot)} - \mu_{(n+1)}(t)}{\mu_{(1)}(t) + \cdots + \mu_{(n)}(t)}.
\]
In Corollary 4.12, the portfolio $\pi^G$ is indeed a stock portfolio among the top $n$ stocks; but the master formula (131) compares its performance with the market portfolio $\mu$, which is not a portfolio among the top $n$ stocks.

In the open market setting, since we only consider portfolios among the top $n$ stocks, it is more appropriate to compare a portfolio’s performance with respect to $\tilde{\mu}$, rather than $\mu$. This can be done by combining (131) and (133).

**Corollary 4.15** (Master formula in the top $n$ market). *For a functionally generated portfolio $\pi^G$ as in Corollary 4.12, the master formula, which compares the log-relative wealth of $\pi^G$ to that generated by the top $n$ market $\tilde{\mu}$, is given as*

$$
\begin{align*}
\log \left( \frac{X_{\pi^G}}{X_{\tilde{\mu}}} \right) &= \log \left( \frac{G(\mu)}{G(\mu(0))} \right) - \log \left( \frac{\mu(1)(\cdot) + \cdots + \mu(n)(\cdot)}{\mu(1)(0) + \cdots + \mu(n)(0)} \right) \\
&\quad - \frac{1}{2} \int_0^T \sum_{k=1}^n \left( \frac{D_k G(\mu(t))}{G(\mu(t))} - \frac{D_{k+1} G(\mu(t))}{G(\mu(t))} \right) dL_{\mu(k) - \mu(k+1)}(t) \\
&\quad + \frac{1}{2} \int_0^T \frac{dL_{\tilde{\mu}(k) - \tilde{\mu}(n)(t)}}{\mu(1)(t) + \cdots + \mu(n)(t)} - \frac{1}{2} \int_0^T \sum_{k=1}^n \sum_{\ell=1}^n D_{k,\ell}^2 G(\mu(t)) \frac{d[\mu(k), \mu(\ell)](t)}{G(\mu(t))}.
\end{align*}
$$

We call this formula of (134), the “master formula for the top $n$ market,” in order to distinguish it from the formula of (128), which we call the “master formula in the entire market.”

**Example 4.16** (Diversity-weighted portfolio). Consider a function

$$
G(x) = G(x_1, \ldots, x_N) = \left( \sum_{k=1}^n x^{p}_{(k)} \right)^{1/p}
$$

with a fixed constant $p \in (0, 1)$. Corollary 4.12 implies that $G$ generates the “diversity-weighted portfolio”

$$
\pi^G_{\mu_p}(\cdot) = \frac{\mu^p(\cdot)}{\mu_p(\cdot) + \cdots + \mu^p(n)(\cdot)} 1\{k \leq n\}, \quad k = 1, \ldots, N.
$$

The master formula in the top $n$ market in (134) is then given as

$$
\begin{align*}
\log \left( \frac{X_{\pi^G}}{X_{\tilde{\mu}}} \right) &= \frac{1}{p} \log \left( \frac{\mu^p(\cdot) + \cdots + \mu^p(n)(\cdot)}{\mu^p(0) + \cdots + \mu^p(n)(0)} \right) - \log \left( \frac{\mu(1)(\cdot) + \cdots + \mu(n)(\cdot)}{\mu(1)(0) + \cdots + \mu(n)(0)} \right) \\
&\quad - \frac{1}{2} \int_0^T \frac{\mu^p(\cdot)}{\mu(1)(t) + \cdots + \mu(n)(t)} dL_{\tilde{\mu}(n) - \tilde{\mu}(n+1)}(t) + \frac{1}{2} \int_0^T \frac{dL_{\tilde{\mu}(n) - \tilde{\mu}(n+1)}(t)}{\mu(1)(t) + \cdots + \mu(n)(t)}
\end{align*}
$$

(135)
\[- \frac{1 - p}{2} \int_0^t \sum_{k=1}^n \sum_{\ell=1}^n \left( \mu_{(k)}^p(t) \frac{\mu_{(\ell)}^{p-1}(t)}{\mu_{(1)}^p(t) + \cdots + \mu_{(n)}^p(t)} \right) d[\mu_{(k)}, \mu_{(\ell)}](t) \]
\[+ \frac{1 - p}{2} \int_0^t \sum_{k=1}^n \mu_{(k)}^{p-2}(t) \frac{\mu_{(k)}^p(t)}{\mu_{(1)}^p(t) + \cdots + \mu_{(n)}^p(t)} d[\mu_{(k)}, \mu_{(k)}](t). \]

Here, in the first integral of (135), we use the fact that the local time process $L_{\mu_{(k)}^p(s)} \mu_{(k+1)}^p(\cdot)$ is flat off the set $\{s \geq 0 : \mu_{(k)}(s) = \mu_{(k+1)}(s)\}$ for $k = 1, \ldots, n - 1$.

### 4.5 Universal portfolio

We explore in this subsection the universal portfolio in the context of open markets. This portfolio was introduced by Cover (1991) in discrete time, and its extension to continuous time was developed by Jamshidian (1992). More recent work on this project, and under the setting of Stochastic Portfolio Theory, can be found in Cuchiero, Schachermayer, and Wong (2019).

Recalling the notation $\Delta_+^{N-1}$ from (117), we need first the following notation:

\[ \Delta_+^{N-1, n} := \left\{ x \in \mathbb{R}^N \middle| x_k \geq 0 \text{ for } k = 1, \ldots, N, \sum_{k=1}^n x_k = 1, \ x_{n+1} = \cdots = x_N = 0 \right\} \]

throughout this subsection. Since we are only allowed to invest in the top $n$ stocks in an open market, the notion of Cover’s “constant rebalanced portfolio” needs to be amended, as follows.

**Definition 4.17 (constant rebalanced portfolio by rank).** If a stock portfolio $\pi \in I(R) \cap T(n) \cap \Delta^{N-1}$ among the top $n$ stocks satisfies

\[ \pi_p(t) = \xi_k \quad \text{for} \quad t \geq 0, \ k = 1, \ldots, N \]

with some $\xi = (\xi_1, \ldots, \xi_N) \in \Delta_+^{N-1, n}$, we call $\pi$ a constant rebalanced portfolio among the top $n$ stocks by rank. This portfolio rebalances at all times to maintain a constant proportion $\xi_k$ of current wealth invested in the $k$-th ranked stock, for $k \leq n$. We denote the collection of constant rebalanced portfolios among the top $n$ stocks by $CR^n$.

**Proposition 4.18.** Every constant rebalanced portfolio among the top $n$ stocks by rank is functionally generated.

*Proof.* For a fixed $\xi \in \Delta_+^{N-1, n}$, consider a function

\[ G(x) = G(x_1, \ldots, x_N) = \prod_{k=1}^n x_{\xi_k}^k. \]
It is easy to check that $G$ is an admissible generating function of market consisting of the top $n$ stocks, and it generates the portfolio via the recipe (130):

$$\pi_{p_k(t)}^G(t) = \xi_k, \quad t \geq 0, \quad k = 1, \ldots, N.$$ 

Since $\xi$ is chosen arbitrarily from $\Delta_{+}^{N-1,n}$, the claim follows.

Thanks to Proposition 4.18, for every $\xi \in \Delta_{+}^{N-1,n}$ there exists a corresponding portfolio $\pi \in CR^n$ as in Definition 4.17, and we write $X_\xi(t)$ to represent the wealth process of $\pi$ at time $t$ in the manner of (22), namely,

$$X_\xi(t) \equiv X_\pi(t) = \mathcal{E}\left(\int_0^t \sum_{i=1}^N \pi_i(s) dR_i(s)\right) = \mathcal{E}\left(\sum_{k=1}^n \xi_k \int_0^t \sum_{i=1}^N \mathbb{1}_{\{u_i(s)=k\}} dR_i(s)\right) \quad \text{for} \quad t \geq 0. \quad (139)$$

For $T > 0$ fixed, we define

$$X^*(T) := \sup_{\pi \in CR^n} X_\pi(T) = \sup_{\xi \in \Delta_{+}^{N-1,n}} X_\xi(T). \quad (140)$$

This $X^*(T)$ represents the maximal wealth at time $T$, achievable over all constant rebalanced portfolios among the top $n$ stocks by rank. We show in the following result that an $\mathcal{F}(T)$-measurable random vector of weights $\pi^*(T) \equiv \xi^*$ exists, which attains the supremum in (140), namely, such that $X^*(T) = X_{\pi^*(T)}(T) = X_{\xi^*}(T)$ holds.

**Lemma 4.19.** For a fixed $T > 0$, the mapping $\Delta_{+}^{N-1,n} \ni \xi \mapsto X_\xi(T) \in \mathbb{R}$ is continuous.

**Proof.** For $\xi, \zeta \in \Delta_{+}^{N-1,n}$, we have

$$\log X_\xi(T) - \log X_\zeta(T) = \log \left(\frac{X_\xi(T)}{X_\zeta(T)}\right) = \frac{1}{2} \int_0^T \sum_{k=1}^n \frac{\xi_k - \zeta_k}{\mu_k(t)}^2 d\mu_k(t)$$

In the last equality, we used the master formula (134) twice, and applied it to the functions of the form (138) for $\xi$ and $\zeta$, respectively. Since the functions $\mu_k(t)$, $L^{\mu_k(t)}$ are continuous for $1 \leq k \leq n$ on the right-hand side (141) are all continuous, they are bounded on the compact interval $[0, T]$. Thus, we obtain the estimate $| \log X_\xi(T) - \log X_\zeta(T) | \leq |\xi - \zeta| K_T$ for some positive
constant $K_T$, which depends on $\min_{0 \leq t \leq T} \mu(k)(t)$, $L^{\mu(k)-\mu(k+1)}(T)$, $[\mu(k), \mu(k)](T)$ for $k = 1, ..., n$, and this proves the continuity.

**Definition 4.20** (Best retrospectively chosen vector of weights). The continuity established in Lemma 4.19 shows that there exists a vector $\xi^* \equiv \pi^*(T) \in \Delta^{N-1,n}_+$ which attains the supremum in (140) for a fixed $T \in (0, \infty)$. We call this $\mathcal{F}(T)$-measurable, $\Delta^{N-1,n}_+$-valued random variable $\pi^*(T)$ the best retrospectively chosen vector of weights among the top $n$ stocks, for the given $T \in (0, \infty)$.

Even though $\pi^*(T)$ was meant to outperform all constant rebalanced portfolios among the top $n$ stocks by rank at $T > 0$, constructing it requires knowledge of stock prices over the entire interval $[0, T]$, that is, ahead of time. Cover (1991) introduced a remarkable way to construct a portfolio, called “universal portfolio,” depending only on past stock prices, whose long-run performance is almost as good as that of the best retrospectively chosen vector of weights. Cover’s idea of building the universal portfolio, was to determine its weights by averaging the performances of all constant portfolio weights, at any time $t \geq 0$.

**Definition 4.21** (Universal portfolio). With the notation $\Delta^{N-1,n}_+$ of (136), the portfolio $\hat{\pi}$ defined as

$$\hat{\pi}_{pk}(t) := \frac{\int_{\Delta^{N-1,n}_+} \xi_kX_\xi(t)d\xi}{\int_{\Delta^{N-1,n}_+} X_\xi(t)d\xi}, \quad \text{for } t \geq 0, \ k = 1, ..., N,$$

(142)
is called universal portfolio among the top $n$ stocks.

From the notation $\Delta^{N-1,n}_+$, we have $\hat{\pi}_{pk}(t) = 0$ for all $t \geq 0$ for $k > n$; i.e., $\hat{\pi}$ invests only in the top $n$ stocks, thus it belongs to $I(R) \cap \mathcal{T}(n) \cap \Delta^{N-1}_+$, the collection of stock portfolios among the top $n$ stocks. We next compute the wealth generated by the universal portfolio.

**Proposition 4.22.** The wealth process $X_{\hat{\pi}}$ is given as

$$X_{\hat{\pi}}(t) = \frac{\int_{\Delta^{N-1,n}_+} X_\xi(t)d\xi}{\int_{\Delta^{N-1,n}_+} X_\xi(t)d\xi}, \quad \text{for } t \geq 0.$$

(143)

**Proof.** Let $Z(t)$ denote the right-hand side of (143). We have

$$\frac{dZ(t)}{Z(t)} = \frac{\int_{\Delta^{N-1,n}_+} dX_\xi(t)d\xi}{\int_{\Delta^{N-1,n}_+} X_\xi(t)d\xi} = \frac{\int_{\Delta^{N-1,n}_+} X_\xi(t) \sum_{i=1}^{n} \sum_{k=1}^{n} \xi_k \mathbf{1}_{[\mu_i(t) = k]} dR_i(t)d\xi}{\int_{\Delta^{N-1,n}_+} X_\xi(t)d\xi}$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{n} \hat{\pi}_{pk}(t) \mathbf{1}_{[\mu_i(t) = k]} dR_i(t) = \sum_{i=1}^{N} \hat{\pi}_i(t)dR_i(t) = \frac{dX_{\hat{\pi}}(t)}{X_{\hat{\pi}}(t)}.$$

Here, the second, third and last equalities are from (139), (142), and (22), respectively. Since $X_{\hat{\pi}}(0) = Z(0) = 1$, the result follows. \qed
We are now ready to compare the long-run performance of the universal portfolio with the best retrospectively chosen vector of weights.

**Theorem 4.23.** Suppose that the portfolio \( \mu \), defined in (107), satisfies

\[
\mu(1)(t) \geq \cdots \geq \mu(n)(t) \geq \delta, \quad \text{for all} \quad t \geq 0 \quad \text{for some} \quad \delta > 0,
\]

(144)

\[
\limsup_{T \to \infty} \frac{1}{T} \left[ \mu(k), \mu(k) \right](T) < \infty, \quad \limsup_{T \to \infty} \frac{1}{T} L^{\mu(k) - \mu(k+1)}(T) < \infty, \quad \text{for} \quad k = 1, \ldots, n.
\]

(145)

Then, the best retrospectively chosen vector of weights and the universal portfolio have the same asymptotic growth rate, that is,

\[
\lim_{T \to \infty} \frac{1}{T} \left( \log X_{\pi^*(T)}(T) - \log X_{\hat{\pi}}(T) \right) = 0.
\]

(146)

where \( \pi^*(T) \) and \( \hat{\pi} \) are as in Definitions 4.20 and 4.21, respectively.

**Proof.** Since \( X_{\pi^*(T)}(T) \geq X_\xi(T) \) holds for every \( \xi \in \Delta_{+}^{N-1,n} \) for every \( T \geq 0 \), the inequality “\( \geq \)” of (146) is obvious from (143).

We now show the reverse inequality. Let \( \xi^* \in \Delta_{+}^{N-1,n} \) be the corresponding vector of weights \( \pi^*(T) \) as in Definition 4.20. For any \( \xi \in \Delta_{+}^{N-1,n} \) satisfying \( ||\xi^* - \xi|| \leq \eta \) for some \( \eta > 0 \), we have the estimate

\[
\frac{1}{T} \left( \log X_\xi(T) - \log X_{\xi^*}(T) \right) \geq -\eta \left( \frac{a_n}{\delta} \max_{1 \leq k \leq n} L^{\mu(k) - \mu(k+1)}(T) + \frac{b_n}{\delta^2} \max_{1 \leq k \leq n} \left[ \mu(k), \mu(k) \right](T) \right)
\]

\[= -\eta \frac{K_T}{T}, \]

in the same manner as in the proof of Lemma 4.19, for some positive constants \( a_n \) and \( b_n \) depending on \( n \). Due to the condition (145), we can take \( \eta \) sufficiently small such that \( \frac{\eta}{T} K_T \leq \epsilon \) holds for every \( T \geq 1 \), for any given \( \epsilon > 0 \). To summarize, for any given \( \epsilon > 0 \), there exists \( \eta > 0 \) such that

\[
\frac{1}{T} \left( \log X_\xi(T) - \log X_{\xi^*}(T) \right) \geq -\epsilon
\]

(147)

holds for every \( \xi \in B(\xi^*, \eta) \) and for every \( T \geq 1 \). Here, \( B(\xi^*, \eta) \) is the intersection of \( \Delta_{+}^{N-1,n} \) and \( || \cdot || \)-ball in \( \mathbb{R}^N \) centered at \( \xi^* \) with radius \( \eta \). We denote \( V_{B(\xi^*, \eta)} \) and \( V_{\Delta_{+}^{N-1,n}} \) the volume of \( B(\xi^*, \eta) \) and the volume of the subset \( \Delta_{+}^{N-1,n} \) of \( \mathbb{R}^N \), respectively.

From (143) and Jensen’s inequality, we have

\[
\left( \frac{X_{\hat{\pi}}(T)}{X_{\pi^*(T)}(T)} \right)^\frac{1}{T} \geq \left( \frac{\int_{\Delta_{+}^{N-1,n}} X_\xi(T)d\xi}{X_{\xi^*}(T) V_{\Delta_{+}^{N-1,n}}} \right)^\frac{1}{T} \geq \left( \frac{\int_{B(\xi^*, \eta)} X_\xi(T)d\xi}{X_{\xi^*}(T) V_{\Delta_{+}^{N-1,n}}} \right)^\frac{1}{T}
\]
\[
\left( V_B(\xi, \eta) \right)^{\frac{1}{T}} \int_{B(\xi, \eta)} X_{\xi}(T) \frac{1}{T} d\xi \geq \left( V_{\Delta}^{N-1, n} \right)^{\frac{1}{T}} \left( X_{\xi}(T) \right)^{\frac{1}{T}} \frac{1}{T} \left( V_{\Delta}^{N-1, n} \right)^{\frac{1}{T}} = \left( V_B(\xi, \eta) \right)^{\frac{1}{T}} \int_{B(\xi, \eta)} \left( X_{\xi}(T) \right)^{\frac{1}{T}} \frac{1}{T} d\xi \geq \left( V_B(\xi, \eta) \right)^{\frac{1}{T}} \left( V_{\Delta}^{N-1, n} \right)^{\frac{1}{T}} e^{-\epsilon},
\]

where the last inequality is from (147). Taking logarithms, then letting \( T \to \infty \) for any given \( \epsilon > 0 \), the desired inequality follows. \[\square\]

It is acknowledged that the condition (144) is quite a strong mathematical assumption, but it may be a weak requirement from an empirical point of view, especially when the size \( n \) of the open market is much smaller than \( N \), the size of the entire market. The assumptions (145) are also fairly weak empirical requirements; in particular, Figure 13.2 of Fernholz and Karatzas shows that the local times \( L^{\mu(k)-\mu(k+1)}(\cdot) \) increase linearly over time, i.e., \( L^{\mu(k)-\mu(k+1)}(t) \approx \lambda_{k,k+1} t \), and the positive rates \( \lambda_{k,k+1} \) grow with \( k \).

5 | CONCLUSION

Most of the results in Section 3, including the main Theorem 3.19, which is a foundational result for equity market structure and the study of arbitrage in open markets, can be formulated quite simply in terms of the local characteristics \( \tilde{\alpha} \) and \( \tilde{c} \) of the open market, defined in (40), (41). In particular, the supermartingale numéraire portfolio \( \rho \) in the top \( n \) open market, if it exists, should satisfy the equation \( \tilde{\alpha} = \tilde{c} \rho \) of (63). From this equation, we were able to conclude that the supermartingale numéraire portfolio \( \rho \) in the open market takes the form of \( \rho = D \tilde{c} \tilde{\alpha} \). Here, multiplying by the diagonal matrix \( D \) of (47) makes the portfolio invest only in the top \( n \) stocks, while maintaining its supermartingale numéraire property.

However, as foretold in the introductory part of Section 4, we cannot use this technique to deal with stock portfolios; multiplying by \( D \) a stock portfolio in order to make it invest only in the top \( n \) stocks, destroys its self-financing property. For example, a unit vector \( \pi := e^1 = (1, 0, \ldots, 0) \) is a stock portfolio which invests all capital into the first stock, but \( D \pi \) is not a stock portfolio as it invests all wealth into the money market whenever the first stock fails to belong to the top \( n \) market. Thus, for stock portfolios in open markets, a different approach is offered. Fernholz’s functional generation of stock portfolios with ranked market weights, under the extra conditions (A) and (B) of Definition 4.11, provides a systematic way to construct stock portfolios that invest only in the top \( n \) open market. This approach also yields in Corollary (4.15) the “master formula,” which allows comparing these portfolios with the top \( n \) market portfolio \( \tilde{\mu} \). As an application of this formula, we could prove that Cover’s result on the universal portfolio is also valid in open markets.

Nonetheless, there are a lot of limitations when considering stock portfolios in open markets. First, the balance condition (129) significantly restricts the class of generating functions in open markets. Moreover, the local time terms which appear on the right-hand side of the master formula (134), make it very difficult to find stock portfolios in open markets which
outperform $\tilde{\mu}$. These difficulties are an inevitable price to pay for dealing with stock portfolios in open markets.

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**REFERENCES**

Banner, A. D., & Ghomrasni, R. (2008). Local times of ranked continuous semimartingales. *Stochastic Processes and their Applications, 118*(7), 1244–1253.

Cover, T. M. (1991). Universal portfolios. *Mathematical Finance, 1*(1), 1–29.

Cuchiero, C., Schachermayer, W., & Wong, T.-K. L. (2019). Cover’s universal portfolio, stochastic portfolio theory, and the numéraire portfolio. *Mathematical Finance, 29*(3), 773–803.

Fernholz, R. (1999). Portfolio generating functions. In M. Avellaneda (Ed.), *Quantitative Analysis in Financial Markets* (pp. 344–367). River Edge, NJ: World Scientific.

Fernholz, R. (2002). Stochastic Portfolio Theory. In *Applications of mathematics, Stochastic modelling and applied probability*, Vol. 48. New York: Springer-Verlag. [https://doi.org/10.1007/978-1-4757-3699-1](https://doi.org/10.1007/978-1-4757-3699-1)

Fernholz, R., & Karatzas, I. (2009). Stochastic portfolio theory: An overview. In A. Bensoussan & Q. Zhang (Eds.), *Handbook of Numerical Analysis, Volume Mathematical Modeling and Numerical Methods in Finance*, (Vol. 15, pp. 89–167). North Holland, Amsterdam: Elsevier B.V.

Jamshidian, F. (1992). Asymptotically optimal portfolios. *Mathematical Finance, 2*(2), 131–150.

Karatzas, I., & Kardaras, C. (2007). The numéraire portfolio in semimartingale financial models. *Finance and Stochastics, 11*(4), 447–493.

Karatzas, I., & Kardaras, C. (2021). *Portfolio theory and arbitrage*. In *Graduate studies in mathematics*, Providence, RI: American Mathematical Society.

Karatzas, I., & Ruf, J. (2017). Trading strategies generated by Lyapunov functions. *Finance and Stochastics, 21*(3), 753–787.

Karatzas, I., & Shreve, S. E. (1991). Brownian Motion and Stochastic Calculus. In *Graduate Texts in Mathematics* (2nd ed., Vol. 113). New York: Springer-Verlag. [https://doi.org/10.1007/978-1-4612-0949-2](https://doi.org/10.1007/978-1-4612-0949-2)

Karatzas, I., & Vervuurt, A. (2015). Diversity-weighted portfolios with negative parameter. *Annals of Finance, 11*(3-4), 411–432.

Larsen, K., & Žitković, G. (2007). Stability of utility-maximization in incomplete markets. *Stochastic Processes and their Applications, 117*(11), 1642–1662.

Schweizer, M. (1995). On the minimal martingale measure and the Föllmer-Schweizer decomposition. *Stochastic Analysis and Applications, 13*(5), 573–599.

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