WEIERSTRASS CYCLES AND TAUTOLOGICAL RINGS IN VARIOUS MODULI SPACES OF ALGEBRAIC CURVES

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Abstract. We analyze Weierstrass cycles and tautological rings in moduli spaces of smooth algebraic curves and in moduli spaces of integral algebraic curves with embedded disks with special attention to moduli spaces of curves having genus \(\leq 6\). In particular, we show that our general formula gives a good estimate for the dimension of Weierstrass cycles for low genera.

1. Introduction

A numerical semigroup \(H\) is a subsemigroup of nonnegative integers such that the greatest common divisor of the elements of \(H\) is 1 and \(\mathbb{N}_0 \setminus H\) is a finite set. Elements of \(\mathbb{N}_0 \setminus H\) are called gaps and the number \(g(H)\) of gaps is called the genus of \(H\). Let \(C\) be a smooth complex curve of genus \(g\) and \(p \in C\). Let \(H_p\) be the set consisting of 0 and such integers \(n\) that there exists function on \(C\) holomorphic everywhere except \(p\) and having pole of order \(n\) at \(p\). Then \(H_p\) is a numerical semigroup of genus \(g\); it is called the Weierstrass semigroup of \(p\). If \(H\) is a numerical semigroup of genus \(g\) such that there exists a smooth curve \(C\) with a point \(p\) having Weierstrass semigroup \(H\), we say that \(H\) is a Weierstrass semigroup.

The moduli space of complex smooth pointed curves \(1\) of genus \(g\) is denoted by \(\mathcal{M}_{g,1}\) and has dimension \(3g - 2\) for \(g > 0\). If \(H\) is a numerical semigroup of genus \(g\), we denote \(\mathcal{M}_H\) the locally closed subscheme of \(\mathcal{M}_{g,1}\) consisting of points \((C,p)\) such that \(p\) has Weierstrass semigroup \(H\). One can say that \(\mathcal{M}_H\) is the moduli space of smooth pointed curves with prescribed Weierstrass semigroup \(H\). Some interesting questions arise when studying \(\mathcal{M}_H\). When is the space \(\mathcal{M}_H\) nonempty? If it is nonempty, what is its dimension (or codimension in \(\mathcal{M}_{g,1}\)?)? In [12], Eisenbud and Harris obtained an lower bound of \(\dim \mathcal{M}_H\) for any irreducible component of \(\mathcal{M}_H\) (see (2.3) below); in [1], Deligne obtained the estimate (2.4) which gives the upper bound of \(\dim \mathcal{M}_H\). It was shown in [2] that any numerical semigroup of genus \(\leq 7\) is a Weierstrass semigroup. In [8] and [9], Nakano and Mori studied the dimension of \(\mathcal{M}_H\) for \(g \leq 6\).

The closure of \(\mathcal{M}_H\), denoted by \(W_H\), is called a Weierstrass cycle of semigroup \(H\) (or \(W\)-cycle for short). In [6], using Krichever map, we have found a general estimate from below for the dimension of Weierstrass cycles. In the case when this estimate is precise we expressed the cohomology classes of \(W_H\) in terms of \(\lambda\) and \(\psi\)-classes. In present paper we apply these results to low genera. In section one, we discuss the estimate for the dimension of Weierstrass cycles in more detail; we show that it is precise for \(g \leq 5\) and for \(g = 6\) the error is \(\leq 1\). In section two, we list the cohomology classes of \(W_H\) for \(g \leq 6\).

The definition of Weierstrass point can be applied also to singular curves if \(p\) is a smooth point. Moreover, all our constructions can be generalized to the case of integral curves (= irreducible reduced projective curves). In particular, we can define Weierstrass cycles in

\(^1\)All the curves in this paper are assumed to be irreducible projective curves.

\(^2\)The dimensions and codimensions of spaces considered in this paper are over \(\mathbb{C}\).
moduli spaces of integral curves. We will show that for small genera the codimensions of these cycles coincide with codimensions of Weierstrass cycles in the moduli spaces of smooth curves.

In [7] we have used the identification of the moduli space \( \hat{\mathcal{CM}}_g \) of complex integral curves of genus \( g \) with embedded disks with Krichever locus of the Segal-Wilson version the Sato Grassmannian. This moduli space can be represented also by a closed subscheme of the Sato Grassmannian \( \text{Gr}(\mathcal{H}) \), defined in the framework of algebraic geometry [10, 11]. The circle \( T = U(1) \) acts on both of \( \hat{\mathcal{CM}}_g \) and \( \text{Gr}(\mathcal{H}) \); the Krichever embedding \( k \) of \( \hat{\mathcal{CM}}_g \) into \( \text{Gr}(\mathcal{H}) \) induces a homomorphism \( k^* \) of equivariant cohomology. The tautological ring of \( \hat{\mathcal{CM}}_g \) can be defined as the image of \( k^* \) or as the smallest graded complex subalgebra \( R = \bigoplus_{i \geq 0} R_i \) of \( H^*_T(\hat{\mathcal{CM}}_g) \) generated by the equivariant \( \lambda \) and \( \psi \)-classes (see [7] for the definition of these classes). In section 4, we give explicit formulas for relations in tautological ring of \( \hat{\mathcal{CM}}_g \) for \( g \leq 6 \). We do not claim that all relations in this ring follow from our relations. We give only an estimate from above for the size of this ring. We obtain an estimate from below considering the restriction of cohomology classes to fixed points of \( T \)-action. (These fixed points correspond to monomial curves.) We can define the set \( \hat{\mathcal{CM}}'_H \) in \( \hat{\mathcal{CM}}_g \) fixing the Weierstrass semigroup \( H \) at the marked point and its closure, the Weierstrass cycle \( \hat{W}_H \). In the subspace of smooth curves with disks \( \tilde{\mathcal{M}}_g \subset \hat{\mathcal{CM}}_g \) we have corresponding objects \( \tilde{\mathcal{M}}_H \) and \( \tilde{W}_H \). The equivariant cohomology classes corresponding to \( T \)-invariant cycles \( \tilde{W}_H \) can be identified with cohomology classes in \( \mathcal{M}_g,1 \) corresponding to Weierstrass cycles \( W_H \) (see [7]). We study classes of Weierstrass cycles in equivariant cohomology; for smooth curves this gives us information about classes of Weierstrass cycles in ordinary cohomology.

In [10], Stöhr constructed the moduli spaces \( \mathcal{M}^\text{Gor}_H \) of Gorenstein curves with symmetric Weierstrass semigroup \( H \); in [11], Contiero and Stöhr studied the dimension of \( \mathcal{M}^\text{Gor}_H \).

2. Estimates

The Krichever map allows us to estimate from below of the dimension of Weierstrass cycles (simply \( W \)-cycles). We compare our estimate with the results obtained from the moduli space of curves of low genera in [8], [9] and [11]. For genus \( \leq 5 \), we see that our estimate is precise; together with the irreducibility of all \( W \)-cycles (except one) proven in [9], this permits us to calculate the corresponding cohomology classes (sometimes up to a constant factor). For genus 6, Nakano calculated the dimension of \( W \)-cycles and proved irreducibility of 16 \( W \)-cycles out of 23. Our estimate together with Deligne estimate from above gives the exact answer for 6 cycles that cannot be analyzed by methods of [9]. There is one \( W \)-cycle whose dimension cannot be found exactly neither by methods of [9], nor by our methods. We were able to prove that the codimension is either 2 or 3. In 22 cases when the dimension is known, our estimate is precise in 16 cases; in the remaining cases, the difference between the exact dimension of \( W \)-cycles and our estimate is equal to 1.

The \( W \)-cycle \( W_H \) is related to the Schubert cycle \( \Sigma_S \) on the Grassmannian via the Krichever map. Some basic facts about Krichever map in algebraic approach are reviewed as follows. (For details, see [17] and [16].) The field of Laurent series \( \mathbb{C}((z)) \) is a complete complex topological vector space (field) with respect to the filtration \( \{z^n\mathbb{C}[[z]] : n \in \mathbb{Z}\} \). Let \( \mathcal{H} \) be the closed linear subspace of \( \mathbb{C}((z)) \) generated by \( \{z^i : i \neq -1\} \) together with a direct sum decomposition \( \mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+ \), where \( \mathcal{H}_+ = \mathbb{C}[[z]] \) and \( \mathcal{H}_- = \mathcal{H}^* \mathcal{H}_+ \) is Fredholm whose index \( \dim \ker \pi_W - \dim \operatorname{coker} \pi_W \) is \( g \).
The Schubert cells $\Sigma_Z$ of $\text{Gr}(\mathcal{H})$ are labeled by sequences of virtual cardinality $g$. Here a sequence of virtual cardinality $g$ is a decreasing sequence of integers $Z$ with $z_i \neq -1$ for all $i$ such that both of $Z \cap \mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{<0} - Z$ are finite sets, and $\#(Z \cap \mathbb{Z}_{\geq 0}) - \#(\mathbb{Z}_{<0} - Z) = g$. One can check that if $Z = (z_i)$ is a sequence of virtual cardinality $g$, then $z_i = -i + g - 1$ for $i \gg 0$. In this case, the codimension of the Schubert cell $\Sigma_Z$ in $\text{Gr}(\mathcal{H})$ is given by the formula

$$w(Z) = \sum_{i=1}^{i_0} (z_i + i - g) + \sum_{i=i_0}^{\infty} (z_i + i - g + 1),$$

where $i_0$ is the index such that $z_{i_0} \geq 0$ and $z_{i_0+1} < 0$. The Kronecker map

$$k: \widehat{CM}_g \rightarrow \text{Gr}(\mathcal{H})$$

is a closed immersion defined by $(C, p, z) \mapsto Z(H^0(C - p, \omega_C))$, where $\omega_C$ is the dualizing sheaf of $C$ on $\omega_C$. Moreover, the moduli space $\widehat{M}_g$ is contained in $\widehat{CM}_g$ as an open subscheme.

Let $F: \widehat{M}_g \rightarrow M_{g,1}$ be the forgetful map $(C, p, z) \mapsto (C, p)$. We see that

$$(2.1) \quad \text{codim}(\widehat{W}_H; \widehat{M}_g) = \text{codim}(W_H; M_{g,1}).$$

Here we use the notation $\widehat{W}_H = F^{-1}W_H$ for any Weierstrass semigroup $H$. Later we will denote $\text{codim}(W_H; M_{g,1})$ by $\text{codim} W_H$ for simplicity.

Let $\zeta: Z \rightarrow \mathbb{Z}$ be the translation operator $n \mapsto n + 1$ and set $S = \zeta^{-1}(Z - H)$ for every Weierstrass semigroup $H$. The set $S$ can be considered as a decreasing sequence of integers and is called the Weierstrass sequence corresponding to $H$. It follows from the Riemann-Roch theorem that every Weierstrass sequence $S$ is a sequence of virtual cardinality $g$ with $s_0 = 0$. One can check that the $W$-cycle $\widehat{W}_H$ is the preimage of $\Sigma_S$ under the Kronecker map, i.e. $\widehat{W}_H = k^{-1}(\Sigma_S)$. In other words the Weierstrass cycles are intersections of Schubert cells with the Kronecker locus. Notice that $\widehat{W}_H$ is a closure of $\widehat{M}_H = k^{-1}(\Sigma_S)$ that can be regarded as a moduli space of smooth curves with embedded disks and with prescribed Weierstrass semigroup $H$ at the center of the disk.

Let $\Sigma_Z$ be a Schubert cell in $\text{Gr}(\mathcal{H})$. The $W$-cycle $\widehat{W}_Z$ is a union of $W$ cycles $\widehat{W}_S$ with $S \geq Z$. It is obvious that this cycle (an intersection of $\Sigma_S$ with the Kronecker locus) has codimension $\leq w(Z)$. By (2.1) and taking into account that the codimension of a union is the minimal codimension of components, we obtain the following statement:

**Theorem 2.1.** If $S$ runs over Weierstrass sequences obeying $S \geq Z$, then

$$(2.2) \quad \min \text{codim} W_S \leq w(Z).$$

The codimension of the Schubert cycle $\Sigma_S$ labeled by a Weierstrass sequence $S$ is $\sum_{i=1}^{\infty} (s_i + i - g)$. We see that (2.2) is a stronger statement than Eisenbud-Harris (EH) estimate

$$(2.3) \quad \text{codim} W_S \leq w(S)$$

(see [12]). Note that the relation between the Weierstrass sequence $S$ and the Weierstrass gap sequence $\Gamma = \{1 = n_1 < \cdots < n_g\}$ of a Weierstrass semigroup $H$ is given by $s_i = n_{g-i+1} - 1$ for $1 \leq i \leq g$. Since a Weierstrass sequence $S$ is determined by its first $g$-values and the set of its first $g$-values is in one-to-one correspondence with the Weierstrass gap sequence $\Gamma$, we denote $W_S$ by $W_{\Gamma}$.

3Let $s_1$ be the largest integer in $S$ and define $s_j$ recursively by requiring that $s_j$ is the largest integer in $S \setminus \{s_1, \cdots, s_{j-1}\}$.
If $S, Z$ are two decreasing sequences of integers of virtual cardinality $g$, we say that $S \leq Z$ if $s_i \leq z_i$ for all $i$. Hence if $S$ is a Weierstrass sequence and $Z$ a sequence of virtual cardinality $g$ obeying $Z \leq S$, then $z_i = g - i - 1$ for $i \geq g + 1$. From now on, we only consider sequences $Z$ obeying $z_i = g - i - 1$ for $i \geq g + 1$. In this case, the finite increasing sequence $\{z_g + 1 \leq \cdots \leq z_1 + 1\}$ is called the “gap sequence” for $Z$.

We are using the description of Weierstrass gap sequence for $g \leq 6$ given in [9] and [8].

For $g \leq 3$, the only case when the EH estimate is not exact is the case of Weierstrass sequence $S$ with the gap sequence $\{1, 3, 5\}$. We take $Z$ with the gap sequence $\{1, 3, 4\}$. The Weierstrass sequence obeying $S \geq Z$ is unique. We obtain $\text{codim}W_{1,3,5} \leq 2$.

For $g = 4$, the EH estimate is not exact only in the cases of gap sequences $\{1, 3, 5, 7\}$ and $\{1, 2, 4, 7\}$; we use $Z$ with gap sequences $\{1, 3, 4, 5\}$ and $\{1, 2, 4, 6\}$ and obtain that $\text{codim} W_{1,3,5,7} \leq 3$ and $\text{codim} W_{1,2,4,7} \leq 3$.

For $g = 5$ the EH estimate is not exact only in the cases of gap sequences $\{1, 3, 5, 7, 9\}$ and $\{1, 2, 3, 5, 9\}$; we use $Z$ with gap sequences $\{1, 3, 4, 5, 6\}$ and $\{1, 2, 3, 5, 8\}$; we see that $\text{codim} W_{1,3,5,7,9} \leq 4$ and $\text{codim} W_{1,2,3,5,9} \leq 4$.

For $g = 6$, the EH estimate is not exact for the following gap sequences: $\{1, 2, 3, 5, 7, 11\}$, $\{1, 2, 3, 6, 7, 11\}$, $\{1, 2, 4, 5, 7, 8\}$, $\{1, 2, 4, 5, 7, 10\}$, $\{1, 2, 4, 5, 8, 11\}$, $\{1, 3, 5, 7, 9, 11\}$. Together with Deligne’s estimate,

\[
\dim \mathcal{M}_H \leq 2g - \left[\text{End} H : H\right] - 2,
\]

we determine the dimensions of $\mathcal{M}_H$ which were not found in [9]. The results are listed in the following table.

| gap sequence of $S$ | E-H | our estimate | gap sequence of $Z$ | exact codim |
|---------------------|-----|--------------|---------------------|-------------|
| $\{1, 2, 3, 5, 7, 11\}$ | 8   | 7            | $\{1, 2, 3, 4, 7, 11\}$ | 6           |
| $\{1, 2, 3, 6, 7, 11\}$ | 9   | 6            | $\{1, 2, 3, 6, 7, 8\}$ | 6           |
| $\{1, 2, 4, 5, 7, 8\}$ | 6   | 4            | $\{1, 2, 4, 5, 6, 7\}$ | 4           |
| $\{1, 2, 4, 5, 7, 10\}$ | 8   | 6            | $\{1, 2, 4, 5, 6, 9\}$ | 5           |
| $\{1, 2, 4, 5, 8, 11\}$ | 10  | 7            | $\{1, 2, 3, 5, 8, 9\}$ | 6           |
| $\{1, 3, 5, 7, 9, 11\}$ | 15  | 5            | $\{1, 3, 4, 5, 6, 7\}$ | 5           |

3. Cohomology Classes of $W$-cycles

The circle $T = U(1)$ acts on both of $\hat{\mathcal{CM}}_g$ and $\text{Gr}(\mathcal{H})$. Denote $u = c_1(O_{\mathbb{P}^\infty}(1)) = c_1(B_T)$ the first Chern class of the tautological line bundle over $\mathbb{P}^\infty$ (over the classifying space $B_T$). Then both of $H^*_T(\text{Gr}(\mathcal{H}))$ and $H^*_T(\hat{\mathcal{CM}}_g)$ are modules over $\mathbb{C}[u]$. The Krichever map $k : \hat{\mathcal{CM}}_g \to \text{Gr}(\mathcal{H})$ is equivariant and induces a morphism $k^* : H^*_T(\text{Gr}(\mathcal{H})) \to H^*_T(\hat{\mathcal{CM}}_g)$.

The class $\psi = -k^* u$ in $H^*_T(\hat{\mathcal{CM}}_g)$ is called the equivariant $\psi$-class. The Hodge bundle $E$ over $\hat{\mathcal{CM}}_g$ is a $T$-equivariant vector bundle of rank $g$ whose fiber over $(C, p, z)$ is the vector space $z(H^0(C, \omega_C))$. The $i$-th equivariant Chern class of $E$ is denoted by $\lambda_i$ for $1 \leq i \leq g$. (We will keep the same notation for $\psi$ and $\lambda$-classes when we consider their images in $H^*_T(\hat{\mathcal{CM}}_g)$).

Let $X$ be a nonsingular variety and $Y$ be a subvariety of $X$. Given a subvariety $V$ of codimension $n$ in $X$, we say that $V$ and $Y$ are in the general position if the intersection of the tangent space to $V$ and the the tangent space to $Y$ at every common point has codimension $n$ in the tangent space to $Y$. Denote $\iota : Y \hookrightarrow X$ the inclusion map. If $V$ and $Y$ are in the general position, $\iota^*[V] = [V \cap Y]$. Here $[V]$ is the cohomology class dual to $V$ in $H^n(X)$. We also apply similar consideration to equivariant cohomology when a group $G$ acts on $X$ and $V$ and $Y$ are $G$-invariant.
If \( V \) and \( Y \) are in general position then \( \text{codim}(V \cap Y) = \text{codim}(V; X) \) or equivalently \( \dim(V \cap Y) = \dim V + \dim Y - \dim X \). However, this equality can be valid also in the case when \( V \) and \( Y \) are not in general position; then we say that \( V \cap Y \) has expected dimension. In this case \( \star[V] \) is a linear combination of the cohomology classes corresponding to the irreducible components of \( V \cap Y \).

In this section, we take \( X = \text{Gr}(\mathcal{H}) \) and \( Y = \tilde{C}\mathcal{M}_g \); the role of \( \iota \) is played by the Krichever map \( k : \tilde{C}\mathcal{M}_g \to \text{Gr}(\mathcal{H}) \). We identify \( C\mathcal{M}_g \) with its image via \( k \) (with the Krichever locus).

If \( \Sigma_S \) and the Krichever locus \( \tilde{C}\mathcal{M}_g \) are in the general position, the cohomology class corresponding to the \( W \)-cycle \( W_S \) is given by the formula (3.7) of [7].

Similar formula is correct up to a constant factor if the \( W \)-cycle \( W_S \) is irreducible and our estimate of its dimension is precise. More rigorously we can formulate this statement in the following way. We assume that we have an equality in (2.2). In other words, we assume that \( \text{codim} W_S = w(Z) \) and for all Weierstrass sequences \( \tilde{S} \) such that \( \tilde{S} \geq Z \) we have \( \text{codim} W_{\tilde{S}} > w(Z) \). (Notice, that we have made an additional assumption that the sequence obeying \( \text{codim} W_S = w(Z) \) is unique.) We can say that in this situation the intersection of Schubert cycle \( \Sigma_Z \) with the Krichever locus is equal to the \( W \)-cycle \( W_S \) and the intersection has expected codimension. We obtain the following

**Theorem 3.1.**

\[
(3.1) \quad [W_S] = \text{const} \cdot \det \left[ \sum_{a+b=\mu+i-j} h_a(x_1, \cdots, x_g)e_b(1,2,\cdots, \mu_i - i + g)\psi^j \right]_{1 \leq i,j \leq l(\mu)},
\]

where \( \mu \) is the partition corresponding to \( Z \) (it is defined by \( \mu_i = z_i + i - g \) for \( 1 \leq i \leq g \)) and \( \{x_1, \cdots, x_g\} \) are the equivariant Chern roots of the Hodge bundle \( E \).

Here \( h_a(x) \) and \( e_b(x) \) stand for the \( a \)-th complete symmetric functions and the \( b \)-th symmetric functions in \( x \) respectively.

Notice that all of our considerations go through for Weierstrass cycles in \( \tilde{C}\mathcal{M}_g \). The only difference is that instead of Weierstrass sequences we should use sequences corresponding to numerical semigroups. (Every numerical semigroup corresponds to a Weierstrass point of a singular curve; a similar statement is wrong for smooth curves. However, for small genera \( g < 7 \) this statement is correct [2].)

### 4. Tautological Ring of \( \tilde{C}\mathcal{M}_g \)

The tautological ring \( R = R(\tilde{C}\mathcal{M}_g) \) is the \( \mathbb{Q} \)-subalgebra of \( H^*(\tilde{C}\mathcal{M}_g) \) generated by \( \lambda_i, 1 \leq i \leq g \), and \( \psi \). We assign \( \text{deg} \lambda_i = i \) and \( \text{deg} \psi = 1 \). Then \( R \) is a graded \( \mathbb{Q} \)-algebra \( \bigoplus_{i \geq 0} R_i \). Set \( h_i(R) = \dim \mathbb{Q} R_i \) for all \( i \). The Hilbert Poincare series of \( R \) is denoted by \( P_R(t) = \sum h_i(R)t^i \in \mathbb{Z}[\![t]\!] \).

Consider the free polynomial algebra \( A = \mathbb{Q}[\lambda_1, \cdots, \lambda_g, \psi] \) generated by the symbols \( \lambda_1, \cdots, \lambda_g \), and \( \psi \) with \( \text{deg} \lambda_i = i \) and \( \text{deg} \psi = 1 \). Then \( R = A/I_{t\text{au}} \) for the ideal \( I_{t\text{au}} \) called the ideal of tautological relation.

Since the monomial curves are in one-to-one correspondence with the \( T \)-fixed points on \( \tilde{C}\mathcal{M}_g \), we obtain a ring homomorphism \( \text{ev} \) from \( R \) to \( \bigoplus S \mathbb{C}[\![\psi]\!] \). Here \( S \) runs over all Weierstrass sequence. The ideal \( I_{t\text{au}} \) is contained in the kernel \( I_{\text{ev}} \) of \( \text{ev} \). This gives us a surjective ring homomorphism \( R \to A/I_{t\text{au}} \). On the other hand, if the intersection of \( \Sigma_\mu \) and \( k(\tilde{C}\mathcal{M}_g) \) is empty, then \( k^*\Omega^*_\mu = 0 \) in \( R \). We denote by \( I \) the ideal of \( A \) generated by
$k \ast \Omega^T_{\mu}$. Then $I \subset I_{\text{ev}}$. Then we obtain another surjective ring homomorphism $A/I \to R$. As a consequence, we have the following estimates

$$h_i(A/I_{\text{ev}}) \leq h_i(R) \leq h_i(A/I).$$

The Hilbert Poincare series of $A/I_{\text{ev}}$ and $A/I$ are given by the following table for $2 \leq g \leq 6$:

| $g$ | $P_{A/I_{\text{ev}}}$                  | $P_{A/I}$                  |
|-----|----------------------------------------|----------------------------|
| 2   | $1 + t$                                | $(1 + t + 2t^2 - 2t^3 - t^4)(1 - t)^{-1}$ |
| 3   | $1 + 2t + 2t^2 + t^3$                  | $1 + 2t + 4t^2 + 7t^3 + 9t^4 + 9t^5 + 6t^6 + t^7$ |
| 4   | $1 + 2t + 4t^2 + 3t^3 + t^4$           | $1 + 2t + 4t^2 + 7t^3 + 12t^4 + 16t^5 + 20t^6$  |
|     |                                        | $+ 22t^7 + 21t^8 + 15t^9 + 9t^{10} + 2t^{11}$ |
| 5   | $1 + 2t + 4t^2 + 7t^3 + 2t^4 + t^5$    | $1 + 2t + 4t^2 + 7t^3 + 12t^4 + 19t^5 + 27t^6$  |
|     |                                        | $+ 35t^7 + 43t^8 + 51t^9 + 54t^{10} + 54t^{11}$ |
|     |                                        | $+ 49t^{12} + 41t^{13} + 27t^{14} + 12t^{15} + 2t^{16}$ |
| 6   | $1 + 2t + 4t^2 + 7t^3 + 11t^4 + 6t^5 + 3t^6$ | $1 + 2t + 4t^2 + 7t^3 + 12t^4 + 19t^5 + 30t^6$  |
|     |                                        | $+ 42t^7 + 57t^8 + 73t^9 + 92t^{10} + 110t^{11}$ |
|     |                                        | $+ 127t^{12} + 138t^{13} + 149t^{14} + 151t^{15} + 144t^{16}$ |
|     |                                        | $+ 129t^{17} + 106t^{18} + 75t^{19} + 41t^{20} + 15t^{21} + 2t^{22}$ |

5. Appendix

In the appendix, we use the method in [13] and [14] to give another proof of (3.1) in the case of moduli space of pointed smooth curves and list the classes $\mathbb{W}_\mu$ for $2 \leq g \leq 5$. Let us briefly recall the result we need from [13].

Suppose that we are given vector bundles

$$A_1 \subset A_2 \subset \cdots \subset A_k$$

and $B_1 \to B_2 \to \cdots \to B_k$ on a scheme $X$ of ranks $a_1 \leq \cdots \leq a_k$ and $b_1 \geq b_2 \geq \cdots \geq b_k$ together with a morphism $h : A_k \to B_1$ of bundles. Let $r = (r_1, \cdots, r_k)$ be a $k$-tuple of nonnegative integers such that $0 < a_i - r_i < a_{i+1} - r_{i+1}$ and $b_i - r_i > b_{i+1} - r_{i+1} > 0$ for $1 \leq i \leq k - 1$. Define $\Omega_r(h)$ to be the subscheme by the conditions that the rank of the map from $A_i \to B_i$ is at most $r_i$ for $1 \leq i \leq k$.

Let $\lambda$ be the partition $(p_1^{m_1}, \cdots, p_k^{m_k})$, where $p_1 = a_k - r_k$, $p_2 = a_{k-1} - r_{k-1}, \cdots, p_k = a_1 - r_1; m_1 = b_k - r_k, m_2 = (b_{k-1} - r_{k-1}) - (b_k - r_k), \cdots, m_k = (b_1 - r_1) - (b_2 - r_2)$. Let $m = m_1 + \cdots + m_k$ and $\rho(i) = \max\{s : 1 \leq s \leq k, i \leq b_s - r_s\}$. Similarly, we consider the conjugate partition $\mu = (q_1^{n_1}, \cdots, q_k^{n_k})$ of $\lambda$. Let $q_1 = b_1 - r_1, \cdots, q_k = b_k - r_k$; $n_1 = a_1 - r_1, \cdots, n_k = (a_k - r_k) - (a_{k-1} - r_{k-1})$, and put $n = n_1 + \cdots + n_k$. Set $\rho'(i) = \min\{s : 1 \leq s \leq k, i \leq a_s - r_s\}$ and $d(r) = |\mu|$. Denote

$$P_r = \det(c_{\lambda-i+j}(A_{\rho(i)}^\vee - B_{\rho(i)}^\vee))_{1 \leq i,j \leq m}$$

$$= \det(c_{\mu-i+j}(B_{\rho'(i)}^\vee - A_{\rho'(i)}^\vee))_{1 \leq i,j \leq n}.$$

**Theorem 5.1.** [13] If $X$ is purely $d$-dimensional, there exists a class $\mathbb{W}_r$ in $A_{d-d(r)}(\Omega_r(h))$ such that the image of $\mathbb{W}_r$ in $A_{d-d(r)}(X)$ is $P_r \cap [X]$. Moreover, we have the following properties.

1. Each component of $\Omega_r(h)$ has codimension at most $d(r)$ in $X$.
2. If $\text{codim}(\Omega_r(h), X) = d(r)$, then $\mathbb{W}_r$ is a positive cycle whose support is $\Omega_r(h)$.
3. If $\text{codim}(\Omega_r(h), X) = d(r)$ and $X$ is Cohen-Macaulay, then $\Omega_r(h)$ is also Cohen-Macaulay and $\mathbb{W}_r = [\Omega_r(h)]$. 

Let $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ be the universal curve whose relative dualizing sheaf is denoted by $\omega_\pi$. Denote $\pi_1, \pi_2 : \mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \to \mathcal{M}_g$ the projection of $\mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,1}$ to its first and second component respectively. The diagonal embedding $\mathcal{M}_{g,1} \to \mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,1}$ is denoted by $\Delta$. The r-th jet bundle $J^r_\pi \omega_\pi$ over $\mathcal{M}_{g,1}$ is defined to be

$$
J^r_\pi \omega_\pi = (\pi_1)_* \left( \pi_2^* \omega_\pi \otimes \mathcal{O}_{\mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,1}}/I^r_\Delta \right).
$$

Let $\mu = (\mu_i)$ be a partition of length at most $g-1$ and $W_\mu$ be the set of points $(C,p)$ in $\mathcal{M}_{g,1}$ such that $s_i(p) \geq \mu_i - i + g$ for $1 \leq i \leq g$, i.e.

$$
W_\mu = \{(C,p) \in \mathcal{M}_{g,1} : s_i(p) \geq \mu_i - i + g, \ 1 \leq i \leq g\}.
$$

Here $(s_i(p))_i$ is the Weierstrass sequence of $p \in C$. We call $W_\mu$ the $W$-cycle of partition $\mu$. If $S$ is a Weierstrass sequence and $\mu$ is the partition defined by $\mu_i = s_i + i - g$ for $i \geq 1$, then $W_\mu = W_S$

Assume that the partition $\mu$ is of the form $q_1^{n_1} \cdots q_k^{n_k}$, where $q_1 > \cdots > q_1 \geq 1$ and $n_1, \cdots, n_k$ are natural numbers. The weight of $\mu$ is defined to be

$$
|\mu| = \sum_i \mu_i = \sum_{i=1}^k n_i q_i.
$$

Denote $r_i = \sum_{j=1}^i n_j - g$ and $r_i = q_i - r_i$ for $1 \leq i \leq k$. Let $A_i = \mathbb{E}$ be the Hodge bundle and $B_i = J^{r_i-1}_\pi \omega_\pi$ for $1 \leq i \leq k$. We consider the natural maps $\varphi_i : A_i \to B_i$ for $1 \leq i \leq k$.

**Proposition 5.1.** The $W$-cycle $W_\mu$ coincides with the degenerating set

$$
\Omega_\mu(\varphi) = \{(C,p) \in \mathcal{M}_{g,1} : \text{rank } \varphi_i(C,p) \leq r_i, \ 1 \leq i \leq k\}.
$$

**Proof.** Let $(C,p)$ be a point in $W_\mu$. Then $s_i(p) \geq \mu_i - i + g$ for $1 \leq i \leq g$. Therefore $s_j(p) \geq q_j - i + g$ for $1 \leq j \leq n_1$. We find that

$$
\#\{s_i(p) : s_i(p) \leq q_i - n_1 + g - 1\} \leq g - n_1.
$$

This shows that $\text{rank } \varphi_1(C,p) \leq g - n_1 - 1$. Inductively, we can show that

$$
\#\{s_j(p) : s_j(p) \leq b_i - 1\} \leq g - (n_1 + \cdots + n_i) = r_i
$$

which implies that $\text{rank } \varphi_i(C,p) \leq r_i$ for all $i$. We find $(C,p) \in \Omega_\mu(\varphi)$. Conversely, assume $(C,p) \in \Omega_\mu(\varphi)$. Since $\text{rank } \varphi_1(C,p) \leq r_1 = n_1 - g$, we find that $s_n(p) \geq q_1 - r_1 = q_1 - n_1 + g$. Since $(s_i(p))$ is decreasing, $s_i(p) \geq q_i - i + g$ for $1 \leq i \leq n_1$. Inductively, we can show that $s_i(p) \geq \mu_i - i + g$ for $1 \leq i \leq g$. Hence $(C,p) \in W_\mu$. We complete the proof. \[\square\]

Let $n = n_1 + \cdots + n_k$, and $\rho(i) = \min\{s : 1 \leq s \leq k, \ i \leq n_1 + \cdots + n_k\}$, and

$$
P_\mu = \det (c_{\mu_i-i+j}(B_{\rho(i)} - \mathbb{E}))_{1 \leq i,j \leq n}.
$$

By theorem [5.1], we know that there is a class $W_\mu$ in $A_{|\mu|}(W_\mu)$ such that each irreducible component of $W_\mu$ has codimension at most $|\mu|$ in $\mathcal{M}_{g,1}$ and the image of $W_\mu$ in $A_{3g-2-|\mu|}(\mathcal{M}_{g,1})$ is $W_\mu \cap [\mathcal{M}_{g,1}]$. Since $\mathcal{M}_{g,1}$ is smooth, it is Cohen-Macaulay. If $\text{codim}(W_\mu, \mathcal{M}_{g,1}) = |\mu|$, then $W_\mu$ is Cohen-Macaulay and $[W_\mu] = W_\mu$.

Moreover, using the fact that [6.1] can be expressed in terms of double Schur function, one can show $W_\mu$ equals the right hand side of [3.1]; in other words, one sees that $W_\mu = k^*\Omega^T_\mu$, where $\Omega^T_\mu$ is the $T$-equivariant cohomology class of the Schubert cycle $\Sigma_\mu$.

When $g = 2, 3, 4$ all the Weierstrass cycles have the expected codimension. When $g = 5$, the Weierstrass cycles $W_{(2,1,1)}$, $W_{1,2,4,5,8} = W_{(3,1,1)}$ and $W_{1,2,3,5,9} = W_{(4,1)}$ do not have the

---

4In this case, we say that $W_\mu$ has the expected codimension
expected codimension. Moreover, $W_{(2,1,1)} = W_{(1,1,1)}$. We list results in the case when $W_\mu$ has the expected codimension in the following table.

| genus | $W_\mu$     | codim | $|\mu|$ | $\mathbb{W}_\mu$                 |
|-------|-------------|-------|--------|-----------------------------------|
| 2     | $W_{(1)}$   | 1     | 1      | $3\psi - \lambda_1$             |
|       | $W_{(1,1)}$ | 2     | 2      | $7\psi^2 - 3\psi \lambda_1 + \lambda_1^2 - \lambda_2$ |
|       | $W_{(2)}$   | 2     | 2      | $35\psi^2 - 10\psi \lambda_1 + \lambda_1^2 - \lambda_2$ |
| 3     | $W_{(1)}$   | 1     | 1      | $10\psi - \lambda_1$            |
|       | $W_{(1,1)}$ | 2     | 2      | $25\psi^2 - 6\psi \lambda_1 + \lambda_1^2 - \lambda_2$ |
|       | $W_{(2)}$   | 2     | 2      | $85\psi^2 - 15\psi \lambda_1 + \lambda_2$ |
|       | $W_{(1,1,1)}$ | 3     | 3      | $15\psi^3 + 3\psi \lambda_1^2 - \lambda_1^4 - 3\psi \lambda_2 + \lambda_1(-7\psi^2 + 2\lambda_2) - \lambda_3$ |
|       | $W_{(2,1)}$ | 3     | 3      | $285\psi^3 + 15\psi \lambda_1^2 - 9\psi \lambda_2 - \lambda_1(90\psi^2 + \lambda_2) + \lambda_3$ |
|       | $W_{(3)}$   | 3     | 3      | $735\psi^3 - 175\psi^2 \lambda_1 + 21\psi \lambda_2 - \lambda_3$ |
| 4     | $W_{(1)}$   | 1     | 1      | $15\psi - \lambda_1$            |
|       | $W_{(1,1)}$ | 2     | 2      | $65\psi^2 - 10\psi \lambda_1 + \lambda_1^2 - \lambda_2$ |
|       | $W_{(2)}$   | 2     | 2      | $175\psi^2 - 21\psi \lambda_1 + \lambda_2$ |
|       | $W_{(1,1,1)}$ | 3     | 3      | $90\psi^3 + 6\psi \lambda_1^2 - \lambda_1^4 - 6\psi \lambda_2 + \lambda_1(-25\psi^2 + 2\lambda_2) - \lambda_3$ |
|       | $W_{(2,1)}$ | 3     | 3      | $1015\psi^3 + 21\psi \lambda_1^2 - 11\psi \lambda_2 - \lambda_1(210\psi^2 + \lambda_2) + \lambda_3$ |
|       | $W_{(3)}$   | 3     | 3      | $1960\psi^3 - 322\psi^2 \lambda_1 + 28\psi \lambda_2 - \lambda_3$ |
|       | $W_{(1,1,1,1)}$ | 4     | 4      | $31\psi^4 - 15\psi \lambda_1^3 + 7\psi^2 \lambda_1^2 - 3\psi \lambda_1^4 + \lambda_1 - 7\psi^2 \lambda_2$ |
|       | $W_{(2,2)}$ | 4     | 4      | $3850\psi^4 - 1050\psi^3 \lambda_1 + 140\psi^2 \lambda_1^2 - 55\psi^2 \lambda_2 - 15\psi \lambda_1 \lambda_2$ |
|       | $W_{(3,1)}$ | 4     | 4      | $12831\psi^4 - 3220\psi^3 \lambda_1 + 322\psi^2 \lambda_1^2 - 42\psi^2 \lambda_2$ |
|       | $W_{(4)}$   | 4     | 4      | $22449\psi^4 - 4536\psi^3 \lambda_1 + 546\psi^2 \lambda_2 - 36\psi \lambda_3 + \lambda_4$ |

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