New separation theorems and sub-exponential time algorithms for packing and piercing of fat objects

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1 Introduction and Summary

An effective tool in the design of divide and conquer graph algorithms is separation. The separator theorem of Lipton and Tarjan \[15,16\] asserts that any \(n\) vertex planar graph can be separated into two subgraphs with a splitting ratio of \(1/3 - 2/3\), that is, each subgraph having at most \(2n/3\) vertices, by removing \(O(\sqrt{n})\) vertices. Additional results for graphs and geometric objects have been obtained by Alon et al. \[3\], Alber and Fiala \[2\], Miller et al \[17\], Smith and Wormald \[18\], Chan \[1\], Fox and Pach \[8,9,10,12\], Fox, Pach and Toth \[11\], and Shahrokhi \[19\].

1.1 Past Related Results

Throughout this paper \(C\) denotes a finite collection of subsets of \(R^d\) of cardinality \(n\). Miller et al \[17\] proved that given \(C\) a set of spheres in \(R^d\), where \(d\) is fixed and each point is common to only a constant number of spheres, there is a sphere \(S\) in \(R^d\), so that at most \(\frac{d+1}{d+2}n\) of the spheres in \(C\) are entirely inside of \(S\), at most \(\frac{d+1}{d+2}n\) are entirely outside, and at most \(O_d(n^{\frac{d+1}{d+2}})\) intersect the boundary of \(S\). Smith and Wormald \[18\], among other results, derived variations of this fundamental result, where the separator is a box, and hence, reduced the splitting ratio to \(1/3 - 2/3\) in any dimension \(d\). Nonetheless, their result required *disjointness* assumption of the spheres, which could be weakened to the assumption that there is a very small overlapping among the objects. Since the intersection graphs of many geometric objects exhibit a large vertex connectivity, it is impossible to separate them with the removal of a small number of vertices. Consequently, researchers, have focused on separating intersection graphs of geometric objects, including spheres, with respect to other measures.

Alber and Fiala \[2\] studied the separation properties of unit discs in \(R^2\) with respect to the area which coincides with the packing number, in case of unit discs. For a set \(C\) of unit discs in \(R^2\), they derived a separator theorem with a constant splitting ratio, used it to compute the \(Pack(C)\) in \(nO(\sqrt{Pack(C)})\) time, and also derived a PTAS for computing the packing number.

Chan \[1\] studied the packing and piercing numbers of fat objects of arbitrary sizes in \(R^d\). He introduced the concept of a measure on fat objects that coincides with the packing and piercing numbers, and generalized the separation result of Smith and Wormald for this measure. In simple words, Chan’s beautiful separation theorem asserts that given a set \(C\) of \(n\) fat objects in \(R^d\), with a sufficiently large measure \(\mu(C)\), there is a cube \(R\) so that the measure of objects inside of \(R\) is at most \((1 - \alpha)\mu(C)\), the measure of objects outside of \(R\) is at most \((1 - \alpha)\mu(C)\), and the measure of objects intersecting \(R\) is \(nO_d(\mu(C)^{\frac{d+1}{d+2}})\), where \(\alpha\) is a constant whose value is about \(\frac{\sqrt{d} + 1}{\sqrt{d} + 2}\). Chan then used his separation theorem to design Polynomial time Approximation Schemes (PTAS) for packing and piercing problems that run in \(nO((\frac{d}{\alpha})^\varepsilon)\) time and \(O(n)\) space. Specifically, he improved the running time of the best previous PTAS for packing problem that was due to Erlebach et al \[5\].

In \[19\], we obtained a combinatorial measure sep-
aration theorem for a class of graphs containing the intersection graphs of nearly fat objects in \( \mathbb{R}^d \), and obtained sub-exponential algorithms and PTAS for packing and piercing number of unit lght rectangles, unit discs, and other related problems.

Fox and Pach [8], [9], [10] have developed a series of new separator theorems for planar curves and string graphs. These results highly generalize the planar separator theorem and have striking applications in combinatorial geometry. Very recently, they discovered that their methods can be used to solve the maximum independent set problem, in sub-exponential time, for a variety of geometric graphs [12]. Specifically, they have just reported a sub-exponential time algorithm with the running time of \( 2^{O_d(\sqrt{n \log n})} \) (or \( n^{O_d(\frac{n}{\log n})} \)), for computing the packing number of spheres.

1.2 Our Results

We present several main results. First, we present an improvement to Chan’s separation theorem which is obtained by combining some of his ideas with the original work of Smith and Wormald. Specifically, using a box of aspect ratio \( \frac{2}{3} \) in \( \mathbb{R}^d \), as the separator, we obtain a parametric splitting ratio of \( 1/3 - 2/3(1 + \epsilon) \), for any \( 0 < \epsilon < \frac{1}{2} \), for separating the measure, independent of the dimension. Moreover, we provide an explicit (constant) lower bound for \( \mu(C) \), in terms of \( d \) and \( \epsilon \), for obtaining such a suitable separation, contrasting the result in [4] that was obtained for *sufficiently large values*. We use this separation theorem to derive sub-exponential time algorithms for packing and piercing problems, running in \( n^{O_d(\text{Pack}(C) \frac{d}{d-1})} \) and \( n^{O_d(\text{Pierce}(C) \frac{d}{d-1})} \) time, respectively, and \( O(n \log n) \) storage. Finally, we convert these algorithms to PTAS that run in \( n^{O_d(\epsilon^{d-1})} \) time and \( O(n \log n) \) storage.

2 Preliminaries

For a closed set \( B \) in \( \mathbb{R}^d \), let \( \delta_B \) denote the boundary of \( B \), and note that \( \delta_B \subseteq B \). Let \( \bar{B} \) denote \( \mathbb{R}^d - B \), that is, \( \bar{B} \) is the set of points outside of \( B \). A collection \( C \) of objects in \( \mathbb{R}^d \) is fat, if for every \( r \) and size \( r \) box \( R \), we can choose a constant number \( c \) of points such that every object in \( C \) that intersects \( R \) and has size at least \( r \) contains one of these points [4]. It should be noted that the class of fat objects contains spheres, cubes, and boxes with bounded aspect ratios.

Let \( \mathcal{C} \) be a collection of subsets of \( \mathbb{R}^d \), and let \( \mu \) be a mapping that assigns non-negative values to subsets of \( \mathcal{C} \). Chan [4] calls \( \mu \) a measure, if for any \( \mathcal{A}, \mathcal{B} \subseteq \mathcal{C} \) the following hold.

(i) \( \mu(\mathcal{A}) \leq \mu(\mathcal{B}) \), if \( \mathcal{A} \subseteq \mathcal{B} \).

(ii) \( \mu(\mathcal{A} \cup \mathcal{B}) \leq \mu(\mathcal{A}) + \mu(\mathcal{B}) \).

(iii) \( \mu(\mathcal{A} \cup \mathcal{B}) = \mu(\mathcal{A}) + \mu(\mathcal{B}) \), if no object in \( \mathcal{A} \) intersects an object in \( \mathcal{B} \).

(iv) Given any \( r > 0 \) and any size-\( r \) box \( R \) in \( \mathbb{R}^d \), if every object in \( \mathcal{A} \) has size at least \( R \) and intersects \( R \), then \( \mu(\mathcal{A}) \leq c \), for a constant \( c \).

(v) A constant-factor approximation to \( \mu(\mathcal{A}) \) can be computed in \( |\mathcal{A}|^{O(\epsilon)} \) time. Moreover, if \( \mu(\mathcal{A}) \leq b \), then, \( \mu(\mathcal{A}) \) can be computed exactly in \( |\mathcal{A}|^{O(b)} \) time.

Note that feasible solutions to the packing and piercing problems gives rise to measures.

Let \( \mathcal{A} \subseteq \mathcal{C} \), and let \( B \) be a closed subset of \( \mathbb{R}^d \). Let \( \mathcal{A}_{R-\delta_B} \) and \( \mathcal{A}_B \) denote, respectively, the set of all objects in \( \mathcal{A} \) that are contained in \( B - \delta_B \), or are completely inside of \( B \), and the set of all objects in \( \mathcal{A} \) that are contained in \( B \), or are completely outside of \( B \), respectively. Let \( \mathcal{A}_B \) and \( \mathcal{A}_{\delta_B} \) denote the set of all objects in \( \mathcal{A} \) that have their centers in \( B \), and the set of all objects in \( \mathcal{A} \) that have a point in common with \( \delta_B \).

Aspect ratio of a box in \( \mathbb{R}^d \) is the ratio of its longest side to its shortest side. Chan [4] proved the following separation theorem.

Theorem 1 Given a measure \( \mu \) satisfying (i) – (iv) and a collection \( \mathcal{C} \) of \( n \) objects in \( \mathbb{R}^d \) with sufficiently large \( \mu(\mathcal{C}) \) there is a box \( R \) with \( \mu(\mathcal{C}_{R-\delta_B}), \mu(\mathcal{C}_R) \geq \alpha \mu(\mathcal{C}), \) and \( \mu(\mathcal{C}_{\delta_B}) = O_d(n^{\alpha(\mu(C))^{\frac{d-1}{d}}}) \), where \( \alpha \) is some fixed constant. Moreover, if (v) is satisfied then \( R \) can be computed in polynomial time and linear space.

3 The Separation Theorem

Theorem 2 Let \( \mathcal{C} \) be a set of \( n \) objects in \( \mathbb{R}^d \), \( d \geq 2 \) and let \( \mu \) be a measure on \( \mathcal{C} \) satisfying (i) – (iv), and let \( 0 < \epsilon < \frac{1}{2} \). If \( \mu(\mathcal{C}) \geq (\frac{\delta_B}{\delta})^d \), then, there is a box \( R \) so that

\[
\mu(\mathcal{C}_{R-\delta_B}) \leq \frac{2}{3}(1 + \epsilon)\mu(\mathcal{C}), \mu(\mathcal{C}_R) \leq \frac{2}{3}(1 + \epsilon)\mu(\mathcal{C}), \text{ and}
\]

\[
\mu(\mathcal{C}_{\delta_B}) = O_d(n^{\mu(\mathcal{C})^{\frac{d-1}{d}}}).
\]

Proof. Let \( B \) be a minimum volume box with aspect ratio at most 2 with \( \mu(\mathcal{C}_B) \geq (\frac{\|\mathcal{C}\|}{\mathcal{C}})^d \mu(\mathcal{C}) \), whose side lengths are \( l_1 \leq l_2 \leq \ldots \leq l_d, l_d \leq 2l_1 \). Let \( s \) denote the center of \( B \). For any \( m \) with \( 1 \leq m < 2^\frac{d}{2} \), let \( B_m \) be the box that is the magnified version of \( B \) by the magnification factor \( m \). Thus, \( B_m \) is a box of side lengths \( l_i^m \leq l_i^m \leq \ldots \leq l_d^m \), \( l_d^m \leq 2l_1^m \), that has center \( s \), and contains \( B \). Note that \( l_i^m = ml_i < 2^\frac{d}{2}l_i \), for \( i = 1, 2, \ldots, d \), and hence the volume of \( B_m \) is strictly less than 2 times the volume of \( B \). By cutting \( B_m \),
in the middle of its longest side, we can decompose $B$ into two boxes $B_{ni}, i = 1, 2$, of aspect ratio at most two, each having a volume strictly smaller than volume of $B$. Thus, $\mu(C_{B_{ni}}) < \frac{1}{2}(1 + \epsilon)\mu(C)$, $i = 1, 2$, since $B$ has the minimum volume. Consequently, using (ii), we deduce that $\mu(C_{B_m}) < \frac{1}{2}(1 + \epsilon)\mu(C)$. Therefore, $\mu(C_{B_{m-1}}) < \frac{1}{2}(1 + \epsilon)\mu(C)$. Next, let $C_m$ be a cube of side length $\frac{d}{2^{m-1}}$ having center $s$, and note that area of $\delta C_m$ is $2d^2 \frac{d^{m-1}}{2^{m-1}} < \frac{8d^2}{d^{m-1}} < 4d^2 \frac{d^{m-1}}{2^{m-1}}$. Now, let $l = \frac{d}{2^{m-1}}$, and note that $\delta C_m$, and hence $\delta B_m$ can be covered with at most $dS^d, \mu(C)^{\frac{d+1}{d}}$ cubes of size $l$. Let $C_1^1$ denote the set of all objects in $C$ of size at least $l$, and let $1 \leq m < 2^\frac{l}{d}$. Then, by (iv), $\mu(C_{B_m}) \leq c.d.S^d, \mu(C)^{\frac{d+1}{d}}$. Similarly, let $C_2^2$ denote the set of all objects in $C$ of size strictly smaller than $l$. We need the following claim to finish the proof.

**Claim.** Let $1 \leq m_1 < m_2 < 2^\frac{l}{d}$, with $m_2 - m_1 \geq \frac{1}{\mu(C)^2}$, let $A_1 \in C_{B_m_1}^1$, and let $A_2 \in C_{B_m_2}^2$. Then, $A_1 \cap A_2 = \emptyset$.

**Justification.** For any $x, y \in R^d$, let $\text{distance}(x, y)$ denote the distance between $x$ and $y$. Note that $m_2 - m_1 \geq \frac{1}{\mu(C)^2}$ implies that for any two points $a \in \delta B_{m_2}$ and $b \in \delta B_{m_1}$, we must have $\text{distance}(a, b) \geq \frac{1}{2\mu(C)^2} \geq \frac{1}{4\mu(C)^2} \geq 2l$. Assume to the contrary that $A_1 \cap A_2 \neq \emptyset$, and let $x \in A_1 \cap A_2$. Let $x_1 \in A_1 \cap \delta B_{m_1}$, and let $x_2 \in A_2 \cap \delta B_{m_2}$. Note that $\text{distance}(x_1, x_2) < l$ and $\text{distance}(x_2, x_1) < l$, and thus, $\text{distance}(x_1, x_2) < 2l$ which is a contradiction.

Now use the claim and employ (ii) to conclude that

$$\sum_{j=0}^{[(2^\frac{l}{d} - 1)\mu(C)^{-1}]} \mu(C_{B_{m+1}^{j+1}}) \leq \mu(C).$$

It follows that there is a $j$ so that for $m^* = 1 + \frac{1}{\mu(C)^{\frac{d}{d-1}}}$ we have $\mu(C_{B_m^*}) \leq \frac{1}{4\mu(C)^2} \leq d^2 \mu(C)^{\frac{d+1}{d}},$ since, $\frac{1}{4\mu(C)^2} \geq \frac{1}{4\mu(C)^2}$. We conclude that $\mu(C_{B_{m^*}}) \leq c.d.S^d, n\mu(C)^{\frac{d+1}{d}}$. Now let $R = B_{m^*}$, and to finish the proof note that for $\mu(C) \geq (\frac{2c.d.S^d}{\epsilon})^d$, we have $\mu(C_{SR}) \leq \frac{\mu(C_{SR})}{\epsilon^\frac{d}{d-1}}$.  

**4 Sub-Exponential Time Algorithms**

Our next result is the following.

**Theorem 3** Let $C$ be a set of $n$ fat objects in $R^d$, $d \geq 2$, then $\text{Pack}(C)$ and $\text{Pierce}(C)$ can be computed in $O(Pack(C)^{\frac{d}{d-1}})$ and $O(Pierce(C)^{\frac{d}{d-1}})$, respectively, and $O(n \log n)$ storage.

**Proof.** We provide the details for computing packing number; The details relevant to the computation of piercing number are similar, but slightly different. We use the following recursive algorithm adopted from [16] and tailored to our needs, which we previously utilized in [19].

**Step 1.** Determine an approximate solution $\mu$ to the packing number using (iv). Choose a separation parameter $\alpha = (1 + \epsilon)$ for Theorem 2, and let $C = C(d)$ be the Lower bound on $\mu(C)$ in Theorem 2. Test using (v) to determine if $Pack(C) < C$. If so, then compute a solution in $O(n^C)$ time and return it. If not, proceed to the recursive step.

**Step 2 (Recursive Step).** Find box $R$ by applying Theorem 3 to $\mu$. For any independent set of objects $I$ in $C_{SR}$ compute $\text{Pack}(C_B - N(I))$, $\text{Pack}(C_B - N(I))$, and return

$$\max_{I} \{\text{Pack}(C_B - N(I) + \text{Pack}(C_B - N(I)) + |I|\}$$

where the maximum is taken over all independent sets of objects $I$ in $C_{SR}$.

It is easy to verify that the algorithm computes $\text{Pack}(C)$ correctly. For the running time, let $T(n, p)$ denote the execution time of the algorithm on an instance of the problem with $\text{Pack}(C) = p$. Note that, $T(n, p) \leq \frac{n^{O(\mu^{\frac{d}{d-1}})}}{T(n, (1 - \alpha)p)$, if $p \geq b$; Otherwise $T(n, p) \leq O(n^b)$. It is not difficult to verify that $T(n, p) = n^{O(\mu^{\frac{d}{d-1}})}$ as claimed. □

**5 Approximation Algorithms**

The algorithms in the previous section gives rise to polynomial time approximation schemes (PTAS) for both of stated problems with $n^{O(\mu^{\frac{d}{d-1}})}$ time and $O(n \log n)$ storage. For the PTAS, one can slightly modify the original divide and conquer approach in [15].

Specifically, one must first obtain an constant time approximation solution to the problem using (v), use it to define the measure $\mu$, and apply the separation theorem, or Theorem 2, to this $\mu$. In the recursive step, the divide and conquer algorithm stops when the value of the measure is *small*. That is, if the value of the measure is $O((\frac{1}{2})^2)$, then an exact solution is computed by the application of our sub-exponential time algorithm. The claims concerning the running time, storage, and quality of the approximation are easy to verify.

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