Ptolemy's Methodological Principles in the Creation of His Map Projections

Gyula Pápay

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Abstract
Ptolemy developed instructions for three projections, but only the first two projections were used in practice. These instructions, which contributed significantly to the development of theoretical cartography, did not explicitly describe the methodological basis on which they were elaborated. The derivation of the data used in the instructions thus remained partly unclear, such as the localisation of the centre of the latitude circles in the first projection. Here, an attempt was made to reconstruct Ptolemy's methodological principles. It turned out that he used a simple but ingenious method that made it possible to create all projections without trigonometric calculations, only with the use of compass (instrument to construct circles) and ruler.

Keywords History of theoretical cartography · Projection · Geometry · Rectification of the circle

Methodischen Grundlagen von Ptolemäus bei der Erstellung seiner Kartenprojektionen

Zusammenfassung
Ptolemäus erstellte Anleitungen zu drei Projektionen, wobei nur die ersten beiden Projektionen zur praktischen Anwendung kamen. In diesen Anleitungen, die zur Herausbildung der theoretischen Kartographie maßgeblich beitrugen, wurde nicht explizit dargelegt, auf welchen methodischen Grundlagen sie erarbeitet wurden. Die Herleitung der in der Anleitung verwendeten Angaben blieb somit teilweise ungeklärt, wie z. B. die Lokalisierung des Mittelpunktes der Breitenkreise in der ersten Projektion. Hier wurde der Versuch unternommen, die methodischen Grundlagen von Ptolemäus zu rekonstruieren. Es stellte sich dabei heraus, dass er eine einfache, aber geniale Methode verwendete, die es ermöglichte, sämtliche Projektionen ohne trigonometrische Berechnungen, nur mit Anwendung von Zirkel und Linear zu erstellen.

Schlüsselwörter Geschichte der theoretischen Kartographie · Projektion · Geometrie · Rektifikation des Kreises

1 Introduction

The “Geography” (“Geographike hyphegesis”), written by Klaudius Ptolemy around 150 AD, is the oldest completely preserved theoretical treatise on cartography. This treatise contains instructions on the construction of map projections. It thus made a significant contribution to the development of theoretical cartography. Ptolemy created three map projections. In his first projection, the latitude circles are rendered as circular arcs for the first time. In his second projection, both the latitude circles and the meridians are designed as circular arcs. The highest level of the principle of similarity is achieved in his third projection, which is based on a perspective representation of the globe. However, only the first two projections were used for the cartographic representation, as they were optimised for the representation of the oikumene. The term “oikumene” is interpreted in different ways. Mostly it is understood to mean the inhabited area of the earth (cf. Berggren and Jones 2000 et al.). A. Stückelberger pointed out that Ptolemy often used the expression "he kath' hemas oikumene" ("the oikumene concerning us") (Stückelberger and Mittenhuber 2009, p. 254). This term

1 Thanks are due to Frank Dickmann for the suggestion to elaborate this study in this detailed form.
could thus be interpreted as referring to the world known in the age of that time. Ptolemy described in detail how to construct the projections he developed, but he gave no explanation for what method he used to conceive his first and second projections.

In the last centuries, numerous studies have dealt with Ptolemy’s projection theory, for example Mollweide 1805, Wilberg 1834, Schöne 1909, Schnabel 1930, Kubitschek 1935, Hopfner 1938, Neugebauer 1959, Polaschek 1965, Hövermann 1980, Dürst 1983 and Dilke 1987. The theory of Ptolemy was rediscovered in the thirteenth century. However, it only became really well known at the beginning of the fifteenth century, after it had been translated into Latin. The true character of the Ptolemaic projections nevertheless remained unrecognised. This was mainly caused by the great similarity of the first two projections to cone projections.

The last complete edition of the “Geography” was published in 2006 (Stückelberger and Grasshoff 2006). This edition is based on the manuscript from the Sultan’s Library (Codex Seragliensis GI 57), which was rediscovered in 1927 and is kept in the Topkapi Museum in Istanbul. This edition was complemented in 2009 with a supplement volume edited by A. Stückelberger and F. Mittenhuber (Stückelberger and Mittenhuber 2006). The importance of these editions is that they contain the complete text of Ptolemy’s “Geography” with an excellent commentary. The previous complete edition was published in 1843/45 (Nobbe 1843–1845). The facsimile edition of the Codex Seragliensis was published in 2017 (Stückelberger, Mittenhuber, Fuchs and Şengör 2017). These editions initiated further publications concerning Ptolemaic geography (cf. et Graßhoff, Mittenhuber and Rinner 2017 as well as Stückelberger and Rohner 2012) and including the present contribution.

2 The Cone Projection Thesis

The view that Ptolemy’s first and second projections are cone projections was present in the literature for a very long time. The following examples outlines the time frame of the prevalence of this view: Mollweide (1805), Tissot (1881/1887), Kubitschek (1935), Hopfner (1938), Neugebauer (1959), Polaschek (1965) and Stückelberger and Grasshoff (2006). The second projection derived from the first projection has been considered a modified cone projection in some studies. However, a terminological distinction was not always made as clearly as by Berggen and Jones. The first projection is referred to here as a “simple conical projection”, the second projection on the other hand as a “pseudoconical projection” (Berggren and Jones 2000, p. 36, p38).

Kuening criticised this entrenched view as early as 1955. Regarding the first projection, which bears the greatest resemblance to a cone projection, he made the following statement: “This first projection of Ptolemy is often regarded as a projection on a cone touching the globe on 36° N.lat., but a cone which has its apex at 115° from the equator touches the globe at 51° and not 36°, or a cone which touches the globe at 36° has its apex at a good 153° and not at 115°. Moreover, Ptolemy does not speak of projecting on a cone” (Keuning 1955, S. 10). The cone projection thesis was also questioned by Pápay (1994, 2005). Stückelberger modified his earlier view by noting that the first projection was not a cone projection from a mathematical point of
view (Stückelberger 2012, p. 70, Stückelberger and Rohner 2012, p. 45). The true character of the first projection was not determined by Keuning (1955) and thus it could not be explained why the distance of the apex from the equator is 115 units. Ptolemy did not explain in his instructions how he determined this distance.

3 Construction Phases of Ptolemy's First Projection

Figures 1, 2, 3 and 4 demonstrate the phases of the construction of this projection. The explanations for the figures were derived from Ptolemy’s instructions.
4 Attempts to Explain the Position of Point \(H\) in the First Projection

The point \(H\) is the apex of the ray bundles of the meridians. Undoubtedly, the position of point \(H\) is a hidden key to clarifying the exact type of projection of Ptolemy’s first projection. So far, very few attempts have been made in this regard. This circumstance can probably be explained by the assumption that Ptolemy determined this position in an empirical way that is no longer comprehensible. For example, Keuning recognised that the point \(H\) in the first projection has cardinal significance, but no attempt was made by him to explain it. It was assumed that Ptolemy took over the position determination of \(H\) from an earlier projection (Pápay 1994, p. 105). This assumption was considered superfluous by Stückelberger and Rohner (2012, p. 45). They provided a mathematical explanation for the projection method. Their starting base was the values of Ptolemy’s chord table, which he cited in the “Almagest” (Syntaxis 1.10f.). “The values represent a sine function, the knowledge of which (and thus also of the cosine function) is thus assured for Ptolemy, although he does not yet know a term for it and does not yet use the graphic representation of a sine curve” (Stückelberger and Rohner 2012, p. 39). To determine the distance between the two points \(H\) and \(O\), a mathematical formula with cosine function was created (Stückelberger and Rohner 2012, p. 41):

\[
0.4540 = \cos(63^\circ) = \frac{\text{radius PK Thule}}{\text{radius equator}} = \frac{\text{HO}}{\text{HS}} = \frac{x}{x + 63\text{ units}}
\]

The results are correct, also with regard to the other data calculated with the cosine function, also with regard to the precise numbers in book 8. Here the relation between the mean parallel and the meridian is given for each map. Nevertheless, there is a reasonable doubt that Ptolemy used such mathematical tools. In ancient Greek mathematics, the compass and ruler were the preferred tools for solving geometric problems rather than mathematical formulas. It was already demanded by Plato to use only compass and ruler as far as possible and other aids should only be allowed if one does not make any other progress (Steele 1965, p 174).

5 The Transformed Great Circle Method

In the “Geographike” there are several, partly hidden “mosaic stones” that can be put together to reconstruct Ptolemy’s method, when the cartographic approach is given priority. The primary basis for Ptolemy was undoubtedly the comparison with the globe. In his instructions, the globe is mentioned twelve times. It can be assumed that Ptolemy initially viewed the globe in such a way that the central meridian and the equator appeared as straight lines (Fig. 5). However, he did not transfer the image of the globe onto
the flat surface with projecting, but pressed it onto the flat surface virtually.

The great circle of the globe was thereby transformed into a larger circle in which the length of the equator and the middle meridian were each 180 units. In Ptolemy’s instruction, the term "great circle" has two different meanings. First, it refers to the equator, the meridians and in particular the meridians bounding one half of the globe (i.e. the prime meridian together with the meridian of 180°). In a broader sense, it denotes a circle whose radius is 90 units. Such a circle is created by straightening the curved half of the central meridian of 90°, which corresponds to 90 units. In Ptolemy's instruction for his first projection, the transformed great circle does not appear. Why Ptolemy did not describe his procedure and only communicated the results could have several reasons. A very probable reason could be that Ptolemy wanted his instruction to be easy to follow and wanted to avoid any complication.

Based on the structure shown in Fig. 5, it would actually have been possible to create a map network with parallels and meridians. In such a map, however, the oikumene would not be optimally represented. To achieve a better representation of the oikumene, the central parallel of the oikumene had to be moved to the centre. To achieve such a resemblance to the globe, Ptolemy's basic conception meant that the transformation had to be modified. The rectilinear parallels were to be transformed into circular form. In addition, the representation was to be "tilted" in such a way that the parallel of Rhodes, which in the later phase was also to be made true to the distance, was moved more into the centre of the map. Ptolemy solved these conceptual requirements in such a way that he drew a circle in the transformed great circle for the parallel of Rhodes, which connected the centre of the great circle with the end point of the parallel of Rhodes. Figure 6 shows such a turning of the globe, which was necessary to create or understand the first projection.

To achieve a greater resemblance to the globe, Ptolemy's basic conception meant that the transformation had to be modified. The rectilinear parallels were to be transformed into circular form. In addition, the representation was to be "tilted" in such a way that the parallel of Rhodes, which in the later phase was also to be made true to the distance, was moved more into the centre of the map. Ptolemy solved these conceptual requirements in such a way that he drew a circle in the transformed great circle for the parallel of Rhodes, which connected the centre of the great circle with the end point of the parallel of Rhodes. Figure 7 presents this method and at the same time provides evidence that Ptolemy determined the distance of the centre of the circles for the parallels from the equator in this way: the cardinal

\[4\] "Wenn der Blick auf die Mitte des nördlichen Viertels des Globus fällt, auf welchem der grösste Teil der Oikumene eingezeichnet wird, dann können die Meridiane den Eindruck von Geraden erwecken, wenn sie jeweils beim Drehen [des Globus] dem Betrachter genau gegenüber zu liegen kommen und ihre Ebene durch den Blickpunkt fällt. Bei den Parallelkreisen ist dies jedoch nicht der Fall wegen der Lage des Nordpols abseits [von der Blickachse]. Sie zeigen deutlich Teile von Kreisen, welche sich nach Süden ausbuchen."

(English translation by Stückelberger and Grasshoff 2006, p. 111).
point $H$ is 115 units from the equator. It is very remarkable how ingenious and also elegant was Ptolemy’s method in working out his theory of projection.

Very clearly confirms the assumption regarding Ptolemy’s methodology the Fig. 8, in which the transformed great circle is combined with the degree network of the first projection.

In the transformed great circle (Fig. 7) only the central meridian and the equator are equidistant. The lengths of the rectilinear parallels are distorted. Figure 9 demonstrates how the exact lengths of the parallels can be determined using the example of Rhodes and Thule.

The diameters of the parallels of Rhodes and Thule on the globe are denoted by $a$ and $b$. The lengths of the straightened parallels on the half of the globe are designated by $a_1$ and $b_1$. The equations at the bottom right confirm the correctness of the graphical ascertainment of lengths. The following calculations confirm the correctness of the graphical ascertainment of lengths. The length of a multiplied by $\pi$ and divided by 2 is equal to the length of $a_1$. The length of $b$ multiplied by $\pi$ and divided by 2 is equal to the length of $b_1$.

The determination of the length of the parallel is actually connected with a geometrical problem, with the rectification of the circle, i.e. with the transformation of the circle into a square of equal circumference. Ptolemy’s method offered an approximate but very practical solution to this problem. This can be demonstrated with Fig. 9, because half the length of the line $a_1$ corresponds to one side of the square that is circumferentially equal to a circle whose radius is half the length of $a$. This is a conditional or approximate solution to the otherwise unsolvable geometric problem, but it was perfectly sufficient to create a map projection. It is therefore a conditional or approximate rectification, since the starting point of it is the already rectified great circles, the equator and the central meridian.

The transformed great circle method allows the ascertainment of correct lengths for each parallel (Fig. 10). The longitude shortens from the equator to the pole in a cosine function. Ptolemy gives the values of the cosine function in his “Almagest”, but without using this term and also without a graphical representation (Stückelberger and Rohner 2012).
6 Ptolemy’s Second Projection

With his second projection, Ptolemy intended to achieve an even greater similarity with the globe. For this reason, the globe was “… so placed that the axis of visual rays passes through both the intersection nearer the eye of the meridian that bisects it’s the longitudinal dimension of the known world and the parallel that bisects the latitudinal dimension, and also the globe’s center” (English translation by Berggren and Jones 2000, p. 88) Using this view of the globe as a starting point, Ptolemy described a supporting construction (Geography 1.24.13) shown in Fig. 11. The second projection is based on the first projection and thus also on the transformed great circle method. Ptolemy’s instructions here are the first explicit reference to this method: “Let us imagine the great circle ABGD that delimits the visible hemisphere, the semicircle AEG of the meridian that bisects the hemisphere…” (English translation by Berggren and Jones 2000, p. 89). The fact that this great circle is not the actual great circle of the globe is made obvious by the further details. Ptolemy indicates that the line BE relates to the line in the same way as 90 to 235/6°. This great circle is therefore a transformed great circle.

However, this transformed great circle is not quite identical to the transformed great circle in the first projection because the latitude of Syene has been moved to the height of the equator. Figure 12 demonstrates the globe rotated in this way in the transformed great circle. Here the transformed great circle has also been rotated, so the latitude of Syene is in the centre of the great circle. Consequently, the
transformed great circle of the second projection is a modified form of the great circle of the first projection.

Figure 13 shows that Ptolemy used the same method to determine the centre of the parallel circles as in the first projection, and Fig. 14 shows the degree grid of the finished projection.

7 Ptolemy's Third Projection

The creation of another projection was reasoned by Ptolemy as follows: “The caption for the general exposition [of the world map] would be of suitable length if carried out to this extent. It would not be out of place, however, to add a method of drawing the hemisphere containing the oikoumene on a plane as it is seen [on a globe], and surrounded by a ringed globe.”7 (English translation by Berggen, Jones, p. 112) This projection differs from the first two projections firstly in that it has not been used in practical cartography, secondly in that it has a perspective structure, whereby the perspective structure is curiously not consequently respected. The armillary sphere is seen in perspective, but not the degree network structure of the oikumene map (Dilke 1987, p. 188). The division of longitude and latitude was done similarly to the first and second projections (Stückelberger and Grasshoff 2006, p. 755) (Fig. 15).

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7 “So viel dürfte genügen für die Beschreibung einer proportionsgerechten Gesamtanordnung [der Weltkarte]. Es scheint aber nicht unangebracht zu sein, noch die Methode hinzufügen, wie die dem Betrachter sich zeigende Hemisphäre [des Globus], auf der unsere Oikumene liegt, auf einer ebenen Fläche gezeichnet werden kann, wenn sie von einer Armillarsphäre umgeben ist.” (Geography, 7.6.1, German translation by Stückelberger and Grasshoff 2006, p. 753).
8 Network Structure of the Projections in Ptolemy’s Regional Maps

In Ptolemy’s Book 8 there is a guide to the regional maps, i.e. to the maps of the different parts of the oikumene. They were drawn on the rectangular projection, which was already employed by Marynus. Here both the meridians and the parallels are rectilinear. The relationship between the degree lengths on the meridians and on the parallels changes due to the different latitudes. Consequently, this relationship changes from map to map. These relations are then listed:

- in Europe: 1. map 11:20, 2. map 3:4, 3. map 2:3, 4. map 3:5, 5. map 43:60, 6. map 3:4, 7. map 4:5, 8. map 11:20, 9. map 43:60, 10. map 7:9, in Africa: 1. map 13:15, 2. map 13:15, 3. map 53:60, 4. map 1:1, in Asia: 1. 3:4, 2. map 7:12, 3. map 11:15, 4. map 5:6, 5. map 4:5, 6. map 11:12, 7. map 2:3, 8. map 2:3, 9. map 13:15, 10. map 11:12, 11. map 1:1 (Geography, 8.3.1, Stückelberger and Grasshoff 2006, pp. 775–905).

These are the relationships between the mid-parallels and the meridians regarding lengths. Here, too, the question arises whether these relations can also be detected only using compass and ruler? It is very remarkable that with the application...
of the transformed great circle method this task is also solvable. Figure 16 demonstrates this efficient possibility using the example of the mid-parallel of 45°, where the ratio is 43:60. To determine the meridian-parallel relation, first find the quarter length of the parallel and transfer it to the equator line. The end point is connected to the upper end of the meridian. With this, the relation is actually also found, but should be converted into a simpler formulation. To achieve this, the parallel line of the original relation line is shifted until round or lower numbers are obtained. In this case, the relation is 43:60. It can be assumed that Ptolemy ascertained the different relationships by applying this method.
9 Alternative Application of the Transformed Great Circle

Based on the method of the transformed globe or the transformed great circle (Fig. 5), it would theoretically have been possible to create different projections. The simplest possibility would have been to leave the parallel straight and divide it equally. This would have created the so-called "globular projection" (Fig. 17). Although this projection would have very well fulfilled the similarity to the globe that Ptolemy was aiming for, it could not be considered by him. He did not want to reach the globe similarity globally at all, but only to the reference of the oikumene, which comprised about only a quarter of the earth. From a European point of view, the European discoveries enormously expanded the space of the known world, the cartographic representation of which required other projections. Paradoxically, the detachment from the Ptolemaic heritage began at the beginning of the sixteenth century with the "globular projection" created by Amerigo Vespucci around 1500 (Pápay 2021).
**Fig. 16** Determine the relations between the parallels and meridians for the rectangular projection

**Fig. 17** The globular projection of Vespucci

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