Chiral SU(2)\(_k\) currents as local operators in vertex models and spin chains

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Abstract

The six-vertex model and its spin-\(S\) descendants obtained from the fusion procedure are well-known lattice discretizations of the SU(2)\(_k\) WZW models, with \(k = 2S\). It is shown that, in these models, it is possible to exhibit a local observable on the lattice that behaves as the chiral current \(J^a(z)\) in the continuum limit. The observable is built out of generators of the \(su(2)\) Lie algebra acting on a small (finite) number of lattice sites. The construction works also for the multi-critical quantum spin chains related to the vertex models, and is verified numerically for \(S = 1/2\) and \(S = 1\) using Bethe ansatz and form factors techniques.

Keywords: conformal field theory, quantum integrable models, tensor product states

(Some figures may appear in colour only in the online journal)

1. Introduction: discretizing conformal blocks

Two-dimensional conformal field theory (CFT) has proved to be an extremely powerful tool in the study of many problems in theoretical physics ranging from condensed matter to string theory. Its effectiveness is rooted in the infinite dimensional algebra of conformal transformations, which is typically generated by two mutually commuting copies of the Virasoro algebra, one holomorphic and the other anti-holomorphic. In a CFT the operator product expansion (OPE) of two fields decomposes generically into a direct sum of conformal
families indexed by primary fields \([BPZ84]\). This fact leads to the notion of conformal blocks which represent holomorphic (or chiral) contributions to correlation functions

\[
\{ \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) \ldots \Phi_n(z_n, \bar{z}_n) \}
\]

(1)
of primary fields \(\Phi_i(z_i, \bar{z}_i)\). A conformal block is specified by a choice of intermediate fusion channels, and can be encoded in the following diagram:

\[
\begin{array}{cccccc}
& & h_2 & & h_{n-2} & h_{n-1} \\
& h_1 & & & & h_n \\
\end{array}
\]

The conformal block \(\mathcal{F}(\{z_i\}||\{h_i\}, \{h'_i\})\) is a function of the holomorphic coordinates \(z_1, \ldots, z_n\), and depends on external chiral conformal dimensions \(h_1, \ldots, h_n\), the intermediate dimensions \(h'_1, \ldots, h'_{n-3}\), and the central charge of the theory. The correlation function (1) is reconstructed by gluing the holomorphic block with its anti-holomorphic counterpart, \(\mathcal{F}(\{z_i\}||\{\bar{h}_i\}, \{\bar{h}'_i\})\), and summing over the intermediate channels weighted by the OPE coefficients. (For standard textbooks on CFT, see e.g., \([DFMS97, \text{Gin90, Hen99, Mus10}]\).)

For generic sets of primary operators \(\Phi_i(z_i, \bar{z}_i)\), the blocks have non-trivial monodromy. But, if the operators \(\Phi_i(z_i, \bar{z}_i)\) are mutually local, then \(\mathcal{F}(\{z_i\}||\{h_i\}, \{h'_i\})\) (respectively \(\mathcal{F}(\{z_i\}||\{\bar{h}_i\}, \{\bar{h}'_i\})\)) is a meromorphic (respectively anti-meromorphic) function of \(z_i\), with poles located at the positions \(z_j, j \neq i\). When this is the case, it is a natural question to ask whether \(\mathcal{F}(\{z_i\}||\{h_i\}, \{h'_i\})\) can itself be realized as a correlator of local observables, without its anti-holomorphic counterpart. One can further wonder whether it is possible to construct local observables in some lattice model, whose correlators would then converge to this conformal block in the continuum limit. This is the basic question that is motivating this paper.

Perhaps the simplest situation where this question can be asked is when the operators \(\Phi_i(z_i, \bar{z}_i)\) are all chiral currents \(J^a(z_i)\) arising in a Wess–Zumino–Witten (WZW) model. These currents are primary operators with respect to the Virasoro algebra, but not with respect to the full chiral algebra, which they themselves generate—typically, a Kac–Moody algebra. In this paper we restrict ourselves to correlators of the form

\[
\{ J^{a_1}(z_1) J^{a_2}(z_2) \ldots J^{a_n}(z_n) \}
\]

(2)
in \(\text{SU}(2)_k\) WZW models. We believe that the extension to other WZW models is relatively straightforward. For simplicity, we assume that we are on a surface of genus zero, namely the Riemann sphere; on surfaces of higher genus, the correlators (2) would depend on the boundary conditions around the different cycles of the surface (see e.g., \([\text{Ber88a, Ber88b}]\)). The question we wish to answer is the following: is it possible to find a two-dimensional lattice model, and a set of local observables in this lattice model, such that the continuum limit of their correlator is the conformal block of equation (2)?

The reason why this seems non-trivial to us is that lattice models at criticality are described by non-chiral CFTs in the continuum limit, so the correlators of local observables on the lattice typically become field theory correlators of the form (1), involving a sum of products of chiral and anti-chiral blocks. Separating the chiral from the anti-chiral part of local operators in lattice models appears to be difficult in general, and typically leads to non-local operators attached to defect lines, usually called parafermionic observables \([\text{FK80, KC71}]\). The latter do not appear in this paper though, since we are dealing with currents only. Notice, however, that parafermionic observables would appear if one wanted to construct lattice
versions of the holomorphic SU(2)_k primary fields (primary with respect to the full chiral algebra). We hope to come back to this question in the near future. We also note that essentially the same program has been carried out independently by Mong et al for a three-state Potts quantum spin chain [MCA+14].

In this paper, we consider the family of (integrable) spin-k/2 vertex models which descend from the six-vertex model [Bab82, Bab83, KRS81, Tak82]. These have long been known to be multi-critical points of spin-k/2 models, whose continuum limit is the SU(2)_k WZW model [Aff85, Aff86, AH87, AGSZ89, DFSZ88, Wit84]. We construct a set of local lattice observables J\_x^a — where x is the position of a point on the lattice, and a labels the three generators of the su(2) Lie algebra—which has the following property. As one sends the lattice spacing a\_0 to zero, our lattice observables become the holomorphic currents generating su\_\partial(2)_k:

\[ J^a_x \sim J^a(z) + O(a_0^\beta), \]  

for some exponent \( \beta > 0 \). As usual, this type of identity is meaningful only when the observables are inserted in correlators, namely

\[ \langle J_{x_1}^a \ldots J_{x_n}^a \rangle_{\text{lattice}} = \langle J_{z_1}^a \ldots J_{z_n}^a \rangle_{\text{CFT}} + O(a_0^\beta). \]  

Throughout the paper, the relation between the lattice position \( x = (x, y) \) and the complex coordinate \( z \) is fixed as

\[ z = x + e^{iy} \]  

as illustrated in figure 1. We will obtain our lattice observables (3) from conserved currents in the vertex models, using a trick to isolate the holomorphic part.

The original motivation for the present paper comes from the analogy between two classes of variational wave functions for quantum systems in two dimensions: Tensor Network States (or Tensor Product States or Projected Entangled Paired States) on the one hand, and the Moore–Read class of trial wave functions [MR91] for chiral topological phases—e.g. quantum Hall systems—that are expressed by conformal blocks on the other hand. (See also the discussion in section 5 of [DRR12] about this analogy.) In this spirit, the case of \( \hat{su}(2)_k \) ‘lattice conformal blocks’ is related to the Read–Rezayi states of fractional quantum Hall systems [RR99]. Applications of our work to Tensor Network States and related topics will be
discussed elsewhere. We note that lattice models related to $\tilde{su}(2)_k$ conformal blocks have been investigated recently in [Gre11, NCS11, TRSG12], using the original Moore–Read construction to produce wave functions that are the ground states of long-range spin systems of the Haldane-Shastry type. In contrast to the present paper, these references do not aim at the discretization of the blocks themselves. Finally, it is also worth mentioning that, apart from condensed matter applications, renewed interest in conformal blocks has been triggered recently by the AGT conjecture [AGT10], which relates conformal blocks to partition functions of $\mathcal{N} = 2$ four-dimensional supersymmetric gauge theories.

The paper is organized as follows. In section 2 we review the necessary background material that is used later in the paper. In section 3 we analyze the lattice spin operator $S^a_x$ and its $a_0$-expansion in terms of the local fields in the continuum. We identify the first few coefficients in this expansion by symmetry arguments. The knowledge of these coefficients allows us to construct our lattice observable (3) for all the descendants of the six-vertex model, and the corresponding critical spin chains. This is explained in section 4. Section 5 contains a few checks of our predictions for the coefficients, which we obtain from numerical evaluation of the form-factors for $k = 1$ (spin-1/2) and $k = 2$ (spin-1). We conclude in section 6. We also provide two appendices. In the first one we discuss logarithmic corrections and explain why they do not appear at the leading order in correlations functions of the chiral current. The second appendix contains some details about the calculation of the form factors and other technical aspects of the Bethe ansatz solution for general $k \geq 1$.

2. Background material

2.1. The $\tilde{su}(2)_k$ current algebra

Let us start by collecting the piece of information about the $\tilde{su}(2)_k$ current algebra that will be needed in the rest of the paper. We refer the reader to [DFMS97, KZ84, Wit84] for further details.

$\tilde{su}(2)_k$ is an affine Lie algebra generated by the modes $J^a_n$ of the holomorphic currents $J^a(z)$, defined by $J^a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J^a_n$. The index $a$ refers to a generator of the underlying Lie algebra $su(2)$. The OPEs between the currents involve the structure constants $f_{abc}$ and the Killing form $\kappa_{ab}$ of $su(2)$, as well as the level $k$:

$$J^a(z)J^b(w) = \frac{k}{(z-w)^2} + \frac{i f_{ab}}{z-w} J^c(w) + \text{regular terms}. \quad (6)$$

It is sometimes convenient to fix a basis of $su(2)$. We chose the one given by the Pauli matrices, $\frac{1}{2}\sigma^a$, $a = 1, 2, 3$. Then the structure constants are the completely anti-symmetric tensor $f^{abc} = \epsilon^{abc}$, and the Killing form coincides with the Kronecker delta $\kappa^{ab} = \delta^{ab}$. The Lie algebra $su(2)$ is embedded as the zero-modes subalgebra: $[J^a_0, J^b_0] = i f^{abc} J^c_0$.

The primary operators $\phi^a_j(w)$ for the affine Lie algebra (which should not be confused with primaries for the Virasoro algebra) are local fields with respect to the $\tilde{su}(2)_k$ currents that satisfy the following OPEs:

$$J^a(z)\phi^b_j(w) = \frac{S^a \cdot \phi^b_j(w)}{z-w} + \text{regular terms}. \quad (7)$$

In this formula we think of $\phi^a_j$ as vector-valued, with components $\phi^a_{j_3}$, with $j_3 = -j, -j+1, ..., j-1, j$. $S^a$ is the spin-$j$ representation matrix. The component of $\phi^a_j$ with $j_3 = j$ is a highest weight vector for the affine Lie algebra and and the corresponding representation is generated
by acting on it with the lowering operators $J_0^1 - iJ_0^2$ and $J_{-n_1}^a \cdots J_{-n_k}^a$, $n_i > 0$. (This procedure produces null vectors which have to be removed to obtain an irreducible representation.)

In order for the theory to be unitary, $k$ must be a positive integer, and the spin $j$ must be integer or half-integer, with the additional restriction

$$j \in \{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}. \quad (8)$$

Thus, at level $k$, there are exactly $k + 1$ primary operators, and $\phi_0$ is the identity field. The Sugawara construction realizes the stress-tensor of the theory as a bilinear in the currents

$$T(z) = \frac{1}{k + 2} : J^a(z) J^a(z) :,$$

and the following OPEs can be computed from (6), (7) and (9):

$$T(z) J^a(w) = \frac{1}{(z - w)^2} J^a(w) + \frac{1}{z - w} \partial J^a(w) + \text{regular terms} \quad (10a)$$

$$T(z) \phi_j(w) = \frac{h_j}{(z - w)^2} \phi_j(w) + \frac{1}{z - w} \partial \phi_j(w) + \text{regular terms} \quad (10b)$$

$$T(z) T(w) = \frac{c/2}{(z - w)^2} + \frac{2}{(z - w)^2} T(w) + \frac{1}{z - w} \partial T(w) + \text{regular terms} \quad (10c)$$

with

$$h_j = \frac{j(j + 1)}{k + 2} \quad c = \frac{3k}{k + 2}. \quad (11)$$

We see that $J^a(z)$ has conformal dimension 1, as expected for a current, that $\phi_j$ has conformal dimension $h_j$, and that $T(z)$ is the holomorphic stress-tensor of a CFT with central charge $c$ given in formula (11). In a similar way, $\bar{\mathfrak{su}}(2)_k$ is generated by the anti-holomorphic current $\bar{J}^a(\bar{z})$ satisfying the anti-holomorphic counterpart of the above relations.
2.2. The six-vertex model and its continuum limit

Next, we review a few useful facts about the six-vertex model at the SU(2) invariant point, which is well known to be a lattice discretization of the diagonal $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ CFT. We consider the six-vertex model defined on the lattice $\mathbb{Z}^2$, see figure 2, where each edge carries one spin-1/2 degree of freedom. The Boltzmann weights of the different spin configurations are obtained from the $R$-matrix, which is a tensor associated to every site $x$ of the lattice, involving only the four spin 1/2 representations living on the adjacent edges at positions $x \pm \frac{1}{2} u$, $x \pm \frac{1}{2} v$:

$$R_x \in \left( \frac{1}{2} \right)_{x+\frac{1}{2} y} \otimes \left( \frac{1}{2} \right)_{x+\frac{1}{2} u} \otimes \left( \frac{1}{2} \right)^*_{x-\frac{1}{2} u} \otimes \left( \frac{1}{2} \right)^*_{x-\frac{1}{2} y}. \quad (12)$$

Here $\left( \frac{1}{2} \right)$ stands for the fundamental of $\mathfrak{su}(2)$ attached to the edge $e$ and $\left( \frac{1}{2} \right)^*$ for its dual (which is isomorphic to the fundamental representation). We represent graphically the $R$-matrix $(R_x)_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}$ as

$$\sigma_1 \in \left( \frac{1}{2} \right) \quad \sigma_3 \in \left( \frac{1}{2} \right)^* \quad \sigma_2 \in \left( \frac{1}{2} \right) \quad \sigma_4 \in \left( \frac{1}{2} \right)^* \quad (13)$$

The total weight of a global configuration of spins is obtained by contracting all the $R$-matrices, using the canonical pairing on each edge. It is customary to use the canonical isomorphism $W \otimes V^* \approx \text{Hom}(V, W)$ to view the $R$-matrix as the linear operator (i.e. as the complex matrix)

$$R_x \in \left( \frac{1}{2} \right)_{x+\frac{1}{2} u} \otimes \left( \frac{1}{2} \right)_{x-\frac{1}{2} u} \quad \left( \frac{1}{2} \right)_{x+\frac{1}{2} y} \otimes \left( \frac{1}{2} \right)^*_{x-\frac{1}{2} y} \quad (14)$$

acting from the south-west to the north-east direction. Global SU(2) symmetry is imposed by requiring that the $R$-matrix is an element of the spin-0 (i.e. invariant) subspace of the tensor product (12). This subspace is two-dimensional, so the $R$-matrix of equation (14) can be written as a linear combination of the identity $I$ and the projector $P_0$ onto the singlet contained in $\left( \frac{1}{2} \right) \otimes \left( \frac{1}{2} \right)$. Up to normalization, this leaves us with one free parameter, which is the relative weight of $I$ and $P_0$. For the purposes of this paper, it is convenient to parametrize the $R$-matrix directly by the geometric angle $\alpha$, see figure 2. ($\alpha$ is related to the spectral parameter as given in appendix B.) Throughout the paper, the $R$-matrix will be the same for every vertex $x$, so we will often drop the subscript $x$. The explicit expression of $R$ together with its graphical representation is:
This is related to the usual parametrization of the weights of the six-vertex model by 
\[ a(\alpha) = \frac{\alpha}{\pi}, \quad b(\alpha) = \frac{\alpha}{\pi} - 1, \quad c(\alpha) = 1. \]
The R-matrix satisfies the Yang–Baxter equation

\[ R(\alpha) = \frac{\alpha}{\pi} 1 + 2 \left( 1 - \frac{\alpha}{\pi} \right) P_0. \]  
(15)

and the inversion relation

\[ \alpha(2\pi - \alpha) = \frac{\alpha}{\pi} \left( \frac{2\pi - \alpha}{\pi} \right). \]  
(16)

Now we replace the infinite lattice by a cylinder with \( N \) sites in the periodic direction. The transfer matrix acting on the space of spins at fixed y-coordinate is:

\[ T_L(\alpha) = \frac{\alpha}{\pi} \left( \frac{2\pi - \alpha}{\pi} \right). \]  
(18)

Our convention is that the transfer matrix acts from bottom to top. The length of the system is \( L = Na_0 \), where \( N \) is an integer. It is always a pleasure to observe that the Yang–Baxter equation (16) and the inversion relation (17) imply that, for any values of the spectral parameters \( \alpha \) and \( \beta \),

\[ [ T_L(\alpha), T_L(\beta) ] = 0. \]  
(19)

The integrals of motion are defined as the logarithmic derivatives of \( T_L \) at \( \alpha = \pi \). Since \( R(\pi) = 1 \), the first integral of motion is the operator \( e^{-i\alpha P} \) translating one site to the right:

\[ T_L(\pi) = e^{-i\alpha P}. \]  
(20)

The first derivative of the R-matrix is the Hamiltonian density \( h \equiv R'(\sigma) = \frac{i}{\sigma} 1 - \frac{2}{\sigma} P_0 \). For two spins-1/2, \( S_1 \) and \( S_2 \), we have \( P_0 = \frac{1}{4} 1 - S_1 \cdot S_2 \). Hence the next integral of motion is the
spin-$\frac{1}{2}$ antiferromagnetic Heisenberg Hamiltonian

$$H^{(1/2)} = T'_L (\pi) \cdot T^{-1}_L (\pi) = \sum_{x, a_0 = 1}^N h_{x, x+a_0} = \frac{2}{\pi} \sum_{x, a_0 = 1}^N \left( S_x \cdot S_{x+a_0} + \frac{1}{4} \right).$$

with periodic boundary conditions $S_{x+a_0} = S_{x}$. The spectrum of the Heisenberg Hamiltonian, and more generally the one of the transfer matrix $T_L (\alpha)$ for any value of $\alpha$, can be obtained from the Bethe ansatz (for a review, see [GRAS05], or the information given in appendix B).

It is known that, in the thermodynamic limit, the low-lying eigenvalues of $\log T_L (\alpha)$ match the ones of the CFT Hamiltonian:

$$T_L (\alpha) \simeq \sum_{L \geq a_0} \exp \left( -a_0 \sin \alpha \left( E_0 (\alpha) L + H_{\text{CFT}} \right) + i a_0 \cos \alpha P_{\text{CFT}} \right).$$

$$H_{\text{CFT}} = \frac{2\pi}{L} \left( L_0 + L_0 - \frac{c}{12} \right) \quad P_{\text{CFT}} = \frac{2\pi}{L} \left( L_0 - L_0 \right).$$

Here $E_0 (\alpha)$ is the free energy per unit area in the thermodynamic limit; it varies continuously with $\alpha$, and in particular, $\sin \alpha \times E_0 (\alpha)$ vanishes when $\alpha \to 0$ or $\alpha \to \pi$. $L_0 (L_0)$ is the zero mode of the holomorphic (anti-holomorphic) component of the Sugawara stress-tensor $\mathcal{H}$ for $su(2)$:

$$T (\tau) = \sum_{n \in \mathbb{Z}} \tau^{-n-2} L_n \quad \mathcal{T} (\tau) = \sum_{n \in \mathbb{Z}} \tau^{-n-2} \bar{L}_n.$$

We have also introduced an operator $\Sigma$, which satisfies $\Sigma^2 = 1$. Such an operator is needed because of the staggering: some of the low-energy states in the spectrum of the Heisenberg Hamiltonian have a momentum close to $\pi$, rather than to 0. For instance, when $N \in 4\mathbb{N} + 2$, the ground state itself has momentum $\pi$, and the corresponding eigenvalue of $T_L$ is a negative real number. To match the lattice momentum $P$ with the CFT momentum operator $P_{\text{CFT}}$, one needs to take this sign into account; this is precisely what the operator $\Sigma$ does. Notice that if, instead of $T_L (\alpha)$, we were focusing on the double-row transfer matrix $T_2 (\alpha)$, then an identification of the form (22) would still hold, and this time no operator $\Sigma$ would be needed.

The identification of the spectrum of $T_L (\alpha)$ with the one of $H_{\text{CFT}}$ and $P_{\text{CFT}}$ is valid for chains with an even number of sites $N$. Then the spectrum is the one of the operator in (22) acting on

$$H_{\text{CFT}} = \left[ \phi_0 \right] \otimes \left[ \bar{\phi}_0 \right] \oplus \left[ \phi_1 \right] \otimes \left[ \bar{\phi}_1 \right].$$

(Recall that there are only two primary fields at level $k = 1$, $\phi_0$ and $\phi_1$.) In other words, the continuum limit of the six-vertex model is the diagonal $su(2)_k \otimes \bar{su}(2)_k$ CFT. We will use this well-known result as our starting point in the construction of the lattice holomorphic $su(2)_k$ current in section 4.

2.3. Spin-$k/2$ descendants of the six-vertex model and $\bar{su}(2)_k$

In this paper, we are interested in constructing a lattice version of the holomorphic $\bar{su}(2)_k$ current for general $k \gg 1$; thus, we need lattice discretizations of the $\bar{su}(2)_k \otimes \bar{su}(2)_k$ CFT. As
reviewed in the previous section, a discretization at level \( k = 1 \) is given to us by the six-vertex model. For \( k > 1 \), it turns out that the lattice discretizations existing in the literature are also related to the six-vertex model: they are spin-\( k/2 \) descendants of the six-vertex model obtained from the fusion procedure of Kulish–Reshetikhin–Sklyanin [KRS81]. One starts from the \( \text{SU}(2) \)-invariant \( R \)-matrix (12)–(15), and constructs a new \( \text{SU}(2) \)-invariant \( R \)-matrix for higher spin representations as follows. The new \( R \)-matrix is a tensor

\[
R^{(k/2)}_x \in \left( \frac{k}{2} \right)_{x + \frac{1}{2} y} \otimes \left( \frac{k}{2} \right)_{x + \frac{1}{2} n} \otimes \left( \frac{k}{2} \right)_{x - \frac{1}{2} n} \otimes \left( \frac{k}{2} \right)_{x - \frac{1}{2} y},
\]

which is best represented pictorially as

where the big ellipses stand for the projector onto the spin-\( k/2 \) representation of \( \text{SU}(2) \), namely the full symmetrizer in \( \bigotimes_{i=1}^k \mathbb{C}^2 \). Here \( \alpha \) is again the geometric angle. The parameters \( \alpha_n \) all depend on \( \alpha \) in a specific way. The choice of these parameters is a crucial step in the fusion procedure. It turns out that the correct choice is

\[
\alpha_n = \alpha + (k - n) \pi.
\]

(27)

It ensures that the following two identities hold:

\[
\begin{align*}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4
\end{align*}
\]

(28a)

\[
\begin{align*}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4
\end{align*}
\]

(28b)

These two identities may be proved inductively. It is yet another pleasant exercise to check that the above two relations, together with (16), imply that \( R^{(k/2)} \) is itself a solution to the Yang–Baxter equation for higher spin:
Similarly, one can check that the relation (28) and the inversion relation (17) for the original $R$-matrix, imply that $R^{(k/2)}$ satisfies the same relation, up to a global coefficient:

$$
2\pi - \alpha \pi = \prod_{p=-k+1}^{k-1} \left[ \left( \frac{\alpha}{\pi} + p \right) \left( \frac{2\pi - \alpha}{\pi} + p \right) \right]^{-1}. 
$$

(30)

As in the spin-1/2 case, the $R$-matrix is used to construct the one-parameter family of commuting transfer matrices $T^{(k/2)}_L(\alpha)$ on $N = L a_0$ sites, with periodic boundary conditions; the spectrum of the transfer matrix can again be obtained from the Bethe ansatz (see [GRAS05] or appendix B). The low-lying spectrum of $-\log T^{(k/2)}_L(\alpha)$ has been identified with the spectrum of the CFT Hamiltonian (22), where $L_0$ and $\bar{L}_0$ are the zero modes of the Sugawara stress-tensor in the $\widehat{su}(2)_k \otimes \widehat{su}(2)_k$ theory (for more details about the general $k$ case, see e.g. [AGSZ89]). Like in the spin-1/2 case, this identification holds when the number of sites $N$ is even, and when the Hilbert space of the CFT is the diagonal module

$$
H_{\text{CFT}} = \bigoplus_{j=0}^{k/2} \left[ \phi_j \right] \otimes \left[ \phi_j \right].
$$

(31)

Again, the known identification of the diagonal $\widehat{su}(2)_k \otimes \widehat{su}(2)_k$ theory as the continuum limit of the spin-$k/2$ descendant of the six-vertex model is the starting point of this paper.

Let us conclude this section with the explicit formulas for the $R$-matrix and for the critical Hamiltonian, which are often useful in calculations. A tedious but straightforward computation leads to

$$
R^{(k/2)}(\alpha) = \frac{1}{k!} \sum_{j=0}^{k} \prod_{p=0}^{l-1} \left( p + \frac{\alpha}{\pi} \right) \prod_{q=0}^{k+1} \left( q - \frac{\alpha}{\pi} \right) P_j,
$$

(32)

where $P_j$ is the projector on spin $j$. $P_j$ can be expressed as a polynomial in the Heisenberg coupling $S^x \otimes S^y$ in $\left( \frac{1}{2} \right) \otimes \left( \frac{1}{2} \right)$ as

$$
P_j = \prod_{\ell=0}^{k} \prod_{\ell \neq j} S^x \otimes S^y - x_\ell,
$$

(33)

with $x_\ell = \frac{1}{2} (\ell + 1) - \frac{k}{2} \left( \frac{1}{2} + 1 \right)$. When $\alpha \to \pi$, we have

$$
R^{(k/2)}(\alpha) = \frac{1}{k!} \sum_{j=0}^{k} \left[ \prod_{p=0}^{l-1} \left( p + \frac{\alpha}{\pi} \right) \prod_{q=0}^{k+1} \left( q - \frac{\alpha}{\pi} \right) P_j \right] + O((\alpha - \pi)^2),
$$

(34)

with $c_j = \sum_{p=1}^{k} \frac{1}{p}$ the harmonic number. As in the $k = 1$ case, it is convenient to first define the Hamiltonian density $h \equiv R'(\pi)$, and then observe that
\[ H^{(k/2)} = \left( \frac{\partial}{\partial \alpha} \log T_N \right) \bigg|_{\alpha = \pi} = \sum_{x,y \in \mathbb{Z}} R_{x,x+u_0} = \frac{2}{\pi} \sum_{x,y \in \mathbb{Z}} \left( \sum_{j=1}^{k} C_j \left( P_j \right)_{x,x+u_0} - \frac{1}{2} \tau_k \right). \] (35)

3. Expansion of the lattice spin operator

In the thermodynamic limit all observables in the vertex model can be expressed as polynomials in the spin operators \( S^a_x \), which act on the spin-\( k/2 \) representations of \( SU(2) \) that live on the edges of the lattice. \( S^a_x \) itself can be expanded in terms of the primary fields and their descendants

\[ S^a_x = \sum_{a=0}^{k} \sum_{j \in \{0,1,2,...,4\}} C^{(j)}_x a^{2j}_0 R_{j, j} \left[ \phi_j(z) \otimes \bar{\phi}_j(z) \right] + \text{descendants}, \] (36)

where, again, \( x = (x, y) \) and \( z = x + \epsilon^a y \). As remarked in the introduction, this type of identity makes sense when it is inserted inside correlators. Note that the currents \( J^a(z) \) and \( J^a(z) \) contribute to the descendants. The notation \( R[.] \) requires some explanation. By definition, the left-hand side transforms in the adjoint (spin-1) representation under the action of \( su(2) \). The terms that appear in the right-hand side must therefore transform in the adjoint representation as well. Now note that the field \( \phi_j(z) \otimes \bar{\phi}_j(z) \) is a matrix-valued primary field, with \( (2j + 1) \times (2j + 1) \) entries, which transforms in the representation \( (j) \otimes (j) \) under the action of the diagonal \( su(2) \) subalgebra generated by the zero modes \( \epsilon_j^0 + \bar{\epsilon}_j^0 \). Then \( R[.] \) stands for the projector onto the unique spin-1 irreducible representation occurring in \( (j) \otimes (j) = \oplus_{p=0}^{2j} (p) \) if \( j > 0 \). For \( j = 0 \), \( R[.] \) just vanishes. Now, if \( j > 0 \), \( R[\phi_j \otimes \bar{\phi}_j] \) has three components transforming in the adjoint representation, so one may identify them as the three components \( a = 1, 2, 3 \) in a unique way (up to a global factor that is irrelevant for our purposes).

3.1. The coefficients \( C^{(j)}_x \)

The coefficients \( C^{(j)}_x \) are non-universal; yet, they are crucial when one tries to match the correlation functions computed on the lattice with the ones in the continuum limit. In general, computing these factors is a difficult task. It has nevertheless been carried out up to some extent in the literature, at least for \( k = 1 \) [Aff98, Luk98, LT03]. To our knowledge, no explicit form is known for arbitrary \( k \) and arbitrary \( j \). Fortunately though, the explicit values of these coefficients won’t be needed for the purposes of this paper. The only thing we need is the fact that roughly half of the coefficients are staggered, and that the remaining half vanishes, as we now argue.

It is known (see for instance [Aff85, Aff88, AH87]) that one site lattice translations by either of the two vectors \( u \) or \( v \) corresponds to changing the sign of the matrix-valued WZW field \( g(z, \bar{z}) = \phi_z(z) \otimes \bar{\phi}_z(z) \):

\[ g(z, \bar{z}) \mapsto -g(z, \bar{z}). \] (37)
(As before we identify $z = x + e^{i\alpha} y, \bar{z} = x + e^{-i\alpha} y$ for $(x, y)$ a point of the lattice.) In the $k = 1$ case, this fact in particular prevents the relevant SU(2)-symmetric perturbation $\text{Tr}[g(z, \bar{z})]$ to appear in the effective action that describes the spin-$\frac{1}{2}$ Heisenberg chain. This perturbation would drive the system away from criticality but since it breaks translation-invariance, which is a symmetry of the Heisenberg Hamiltonian, its appearance in the effective theory is prohibited. When instead translation symmetry is explicitly broken, the spin-$\frac{1}{2}$ chain typically dimerizes.

Now, since the matrix-valued field $\phi_j(z) \otimes \bar{\phi}_j(\bar{z})$ can be obtained by fusing $\phi(z) \otimes \bar{\phi}(\bar{z})$ with itself $2j$ times, we see that a translation by $u$ or by $v$ on the lattice must act as

$$\phi_j(z) \otimes \bar{\phi}_j(\bar{z}) \mapsto ( -1 )^{j} \phi_j(z) \otimes \bar{\phi}_j(\bar{z}).$$

This means that all the coefficients $C_j$ with half-integer $j$ are staggered:

$$C_j^{(j)} = \begin{cases} C_{j,\text{horz}}^{(j)} ( -1 )^{j} & \text{(horizontal edges)} \\ C_{j,\text{vert}}^{(j)} ( -1 )^{j} & \text{(vertical edges)} \end{cases}$$ (39)

while the coefficients for integer $j$ are not. Actually, the latter simply vanish:

$$C_j^{(j)} = 0.$$ (40)

This may be justified as follows. First, we observe that since for integer $j$ the coefficient $C_j^{(j)}$ is not staggered, it must be independent of $x$. Then, without loss of generality, one may focus on this coefficient at a specific point $x = (x, y)$ with $y = 0$, such that $x$ is a fixed point of the spatial inversion $z = x + e^{i\alpha} y \mapsto \bar{z} = x + e^{-i\alpha} y$. Under this transformation, $S^a_x$ is mapped to itself, so any non-zero term appearing in the right-hand side of (36) must be invariant. On the other hand, spatial inversion is a symmetry of the continuous euclidean field theory, which exchanges the chiral and anti-chiral sectors $\phi_j$ and $\bar{\phi}_j$ in $R[\phi_j \otimes \bar{\phi}_j]$. Since the spin-(1) representation appearing in the decomposition $(j) \otimes (j)$ is always anti-symmetric for integer spin $j$, then $R[\phi_j \otimes \bar{\phi}_j]$ must be odd under spatial inversion. Therefore it cannot contribute to $S^a_x$ in the continuum limit.

The structure (39) and (40) of the expansion of $S^a_x$ is related to properties of the Bethe states and of the associated form factors. These are discussed for $k = 1$ and $k = 2$ in [HC07, VC14] which also make crucial use of spatial inversion. More details about this point are given in appendix B.

### 3.2. Contribution of the current to $S^a_x$

Next, we analyze the second part of the right-hand side of (36), namely the contribution of the descendants. The latter are generated by the action of the chiral and anti-chiral currents on the primary fields. They always have a scaling dimension that is the one of the primary operator they descend from, plus some positive integer number. The currents $J^c(z)$ and $T^a(z)$ themselves are descendants of the identity, and have scaling dimension one. All other descendants appearing in the rhs of (36) have strictly larger scaling dimensions; the smallest possible one being $\beta \equiv 2h_1 + 1 > 1$, for the first descendants of the primary field $\phi_{\frac{1}{2}}(z) \otimes \bar{\phi}_{\frac{1}{2}}(\bar{z})$. Thus...
The next step is to fix the coefficients $C^J_a$ and $C^J_x$, using the SU(2) symmetry of the vertex model. By construction, the $R$-matrix $R^{(k)}_x$ is an SU(2)-invariant tensor, see equation (25), and as such it is annihilated by the total spin operator

$$S^a_{x^+\frac{1}{2}a} + S^a_{x+\frac{1}{2}a} - \left(S^a_{x-\frac{1}{2}a}\right)' - \left(S^a_{x-\frac{1}{2}a}\right)'; \quad (42)$$

for any generator $S^a$ of su(2). One may interpret this as the fact that the following discrete contour integral around a vertex at position $x$ vanishes:

$$\sum_{\alpha} \equiv \sum_{\alpha \in E(\Gamma)} S^a_{\alpha} = 0. \quad (43)$$

In equation (42), $-\langle S'\rangle$ corresponds to the dual action on $\left(\frac{k}{2}\right)^{\bar{y}}$. If instead we regarded the $R$-matrix as the linear operator of equation (14) (generalized for arbitrary $k$), then SU(2) invariance would read as the vanishing of

$$S^a_{x+\frac{1}{2}a} + S^a_{x+\frac{1}{2}a} - S^a_{x-\frac{1}{2}a} - S^a_{x-\frac{1}{2}a}; \quad (44)$$

when inserted in a correlator. Note that the presence of the transpose in equation (42) is due to the fact that vectors in the dual representation are regarded as column vectors on which the matrix acts on the left. However, when the $R$-matrix is regarded as an operator on the Hilbert space of the vertex model, as in equation (44), the action by the commutator has to be used. In particular this second point of view has to be used when we consider equation (41) in a correlator.

The observation (43) extends to larger contours $\Gamma$ on the dual lattice, as illustrated in figure 3(a). Let us introduce the following notation for this discrete contour sum:

$$S^a_{\Gamma} \equiv \sum_{x \in E(\Gamma)} S^a_{x}, \quad (45)$$

where $E(\Gamma)$ is the set of dual edges visited by the contour and $S^a_{\Gamma}$ is understood as acting either on $\left(\frac{k}{2}\right)^{\bar{y}}$ or as its dual representation on $\left(\frac{k}{2}\right)^y$ at $x$. As long as $\Gamma$ does not enclose any operator (see figure 3(a)), the discrete contour integral $S^a_{\Gamma}$ vanishes, as a direct consequence of global SU(2) symmetry. More generally, SU(2) invariance allows us to deform the contour $\Gamma$ without changing the correlators in which $S^a_{\Gamma}$ is inserted. This implies that, for two contours $\Gamma$ and $\Gamma'$ without operators between them (as illustrated in figure 3(b)), one can deform $\Gamma'$ to $\Gamma$ and obtain the relation

$$\left[ S^a_{\Gamma}, S^b_{\Gamma'} \right] = i f^{abc} S^c_{\Gamma}, \quad (46)$$

which, again, holds when it is inserted inside a correlator.

It is known that some version of this actually survives when the Lie algebra su(2) is replaced by a quantum group. This was discussed in full generality by Bernard and Felder [BF91]; more recently, the existence of such conserved (non-local) currents on the lattice was...
used to enlighten the topic of ‘lattice holomorphicity’ [IWWZJ13]. We note that, in our case, everything is much simpler than in the quantum group case, and that the very existence of vanishing discrete contour integrals boils down to the SU(2) symmetry of the model. These exist even when the model is not integrable—one could choose any other SU(2)-invariant R-matrix that would not satisfy the Yang–Baxter relation—and independently of criticality of the model in the continuum limit.

However, if one knows already that the model is critical—as is the case here, thanks to integrability results—then the existence of the vanishing discrete contour integrals does have

Figure 3. (a) The discrete contour $\Gamma$ is a closed curve on the edges of the dual lattice. (b) A configuration with two contours $\Gamma$ and $\Gamma'$, and some local operators $\mathcal{O}_{\kappa_1}, \mathcal{O}_{\kappa_2}, \ldots$. None of these operators lies in the annular region between $\Gamma$ and $\Gamma'$. 
some consequences. One of the consequences is that those discrete contour integral must have a continuous counterpart in the CFT describing the continuum limit. The relation between the lattice observables and the fields in the continuum is constrained. Indeed then equation (46) strongly suggests that, in the continuum limit $a_0 \to 0$, we should identify the discrete contour integral $\Gamma$ with the contour integral

$$\frac{1}{2\pi i} \oint C d\zeta J^\alpha (\zeta) = \frac{1}{2\pi i} \oint C d\bar{\zeta} \bar{J}^\alpha (\bar{\zeta}) = J^\alpha_0 + \bar{J}^\alpha_0,$$  

(47)

which generates the $su(2)$ subalgebra in $\bar{su}(2) \otimes \bar{su}(2)$. The discretized line element $dz$ around a vertex is $\{a_0, a_0 e^{i\alpha}, -a_0, -a_0 e^{i\alpha}\}$, where the contour is read counterclockwise starting from the edge below the vertex. Then the identification of $S^\alpha_0$ with (47) is possible if the contribution of the currents to the spin operator on the lattice is

$$S^\alpha_x = \cdots + \frac{a_0}{2\pi i} \left[ e^{i\phi} J^\alpha_x (z) - e^{-i\phi} \bar{J}^\alpha (\bar{z}) \right] + \cdots,$$

(48)

where the factor $e^{i\phi}$ is a phase that depends on the orientation of the edge $x$:

$$e^{i\phi} = \begin{cases} -1 & (x \text{ vertical edge}) \\ e^{i\alpha} & (x \text{ horizontal edge}) \end{cases}$$

(49)

Indeed, if the currents $J^\alpha$ and $\bar{J}^\alpha$ appear in the expansion of the lattice spin operator in the form (48), then they contribute to $S^\alpha_0$ as a Riemann sum, giving precisely the contour integral (47) in the limit $a_0 \to 0$. The contribution of all the other terms appearing in the expansion (41) vanishes in this limit. This is because, as we have seen previously, the non-zero terms are either staggered (primary operators with half-integer spin $j$), leading to an alternating sum, or subleading (descendants that are not the currents themselves). Either way, their contribution to the discrete contour integral goes to zero when $a_0 \to 0$.

In summary, the argument can be formulated as follows: the discrete contour integral should become the contour integral (47) in the continuum limit, and this allows us to fix the coefficients $C^J_x$ and $C^\bar{J}_x$. We end up with the following expression of the lattice spin operator:

$$S^\alpha_x = \sum_{j \text{ half-int.}} C^J_x \left[ a_0^{-2j} R \{ \phi_j (z) , \phi_j (\bar{z}) \} \right]^w$$

$$+ \frac{a_0}{2\pi i} \left[ e^{i\phi} J^\alpha_x (z) - e^{-i\phi} \bar{J}^\alpha (\bar{z}) \right] + O\left(a_0^0\right).$$

(50)

We note that the amplitude of the coefficients $C^J_x$ and $C^\bar{J}_x$ was known previously in the literature: it appeared already in [AGSZ89], and perhaps in earlier references; there, the argument to fix this amplitude is similar to the one we just used. However, the position-dependent complex phase of these coefficients is, to our knowledge, a new result. It is the key point of this paper, which allows us to cook up a lattice observable that behaves as the chiral current in the continuum limit.

4. Chiral currents as local lattice operators

After the preparatory steps of the previous section, we are finally ready to construct a lattice observable $J^\alpha_x$ which has the behavior of equation (3). We first do it for the vertex model with an arbitrary geometric angle $\alpha$, and then compute an expression for the chiral current in the spin chain, by considering the anisotropic limit $\alpha \to \pi$. 


4.1. Chiral current in the vertex model

We have reached the expression (50) for the expansion of the lattice spin operator in terms of the CFT operators. Clearly, we can take linear combinations of this expression at different sites \( x \) in order to cook up new expressions where the leading contribution in the limit \( a_0 \to 0 \) is nothing but the chiral current \( J^a(z) \) itself. One simple possibility is

\[
\alpha \pi \begin{bmatrix}
S^a_{x+\frac{1}{2}a} + S^a_{x-\frac{1}{2}a} + e^{-i\alpha} S^a_{x+\frac{1}{2}y} + e^{-i\alpha} S^a_{x-\frac{1}{2}y} \\
\end{bmatrix}
\] (51)

This expression is chosen such that all the primary fields in (50) cancel because of the staggering—more precisely, our expression picks up their first derivative, which has scaling dimension \( \beta = h + 1 \), so it is less relevant than the current—and the same happens for the anti-chiral current \( J^a(\bar{z}) \), thanks to the complex phases in (50). We have normalized our expression such that

\[
J^a(a_0) = J^a(z) + O(a_0^\beta). \quad (52)
\]

(Once again, this identification makes sense when inserted in correlation functions.) More complicated linear combinations, involving more lattice sites, could be used as well; the only constraint is that the terms that could be more relevant (in the RG sense) than, or as relevant as, \( J^a(z) \), cancel. Similarly, the anti-chiral current on the lattice can be identified as

\[
J^a(a_0) = J^a(z) + O(a_0^\beta). \quad (53)
\]

Equations (51) and (53) are the main result of this paper. They answer the question whether or not it is possible to realize the CFT chiral current in the lattice model (see the introduction) in the affirmative way, and tell us explicitly how one can do it.

4.2. Chiral current in the spin chain from the anisotropic limit

We just exhibited a lattice observable \( J^a(\alpha) \) which becomes the holomorphic current \( J^a(z) \) in the continuum limit. It is then natural to ask whether one can construct a similar operator \( J^a_x(\alpha) \) acting directly on the Hilbert space \( \bigotimes_{k=1}^\infty \mathbb{C} \) of the spin chain. To answer this question, we go back to the transfer matrix formulation and to the description of the \( R \)-matrix as an operator \( R_{x,x} \)—we drop the superscript \( \bigotimes_{k=1}^\infty \) in this section—acting on spaces \( \left( \bigotimes_{k=1}^\infty \mathbb{C} \right)_j \).

Before we take the anisotropic limit, it is convenient to do a small manipulation, and replace the above expression of \( J^a_x(\alpha) \) by \( i \) times this expression. The reason is the following. The conformal mapping from the plane (with complex coordinate \( z \)) to the cylinder (complex coordinate \( w = x + iy \), with \( x \) defined modulo \( L \)) is \( z = \exp \left( -i \frac{\pi}{L} \right) \); the Jacobian of this transformation leads to \( J(x,y) = J(w) = \frac{2\pi i}{L} J(z) = \frac{1}{i} \sum_n e^{-i\alpha} J_n \). Because of the factor \( \frac{1}{i} \) coming from the Jacobian, one sees that the operator \( J(x,y) \) (in Schrödinger picture) is anti-Hermitian. Since we find it more natural to work with a Hermitian operator \( J(x) \), we simply make the replacement \( J(x) \to i J(x) \) when we work on the vertical cylinder. Taking this additional factor \( i \) into account, our expression for the lattice chiral current (51) at position \( x \), becomes, in the transfer matrix formalism:
The operator \( J^a_\alpha(x) \) is a mixture of the transfer matrix of the vertex model with the lattice current operator. As such, it is usually not a local operator acting on the spin chain. This is no different, of course, from the standard observation that the transfer matrix itself is not a local operator acting on the spin chain. At this point, we simply hope that the expression becomes a genuinely local operator acting on a finite number of sites in the spin chain when one takes the anisotropic limit \( \alpha \to \pi \). However, this clearly cannot hold, since the transfer matrix itself does not become the identity in that limit; instead, it becomes the translation operator (20), which is non-local. Therefore, if we took the limit \( \alpha \to \pi \) directly in (54), we would get a result which is a mixture of our lattice current operator with the translation operator, and this has to be non-local. This is not quite what we are looking for. So, before we take the anisotropic limit, we multiply \( J^a_\alpha(x) \) by the inverse of the transfer matrix \( T_N^{-1}(\alpha) \), and this naturally compensates all the unwanted part of the operator (54). Thus, we define our local current in the spin chain as the anisotropic limit \( \alpha \to \pi \) of the following combination

\[
\alpha \equiv \alpha \to \pi \lim \left( J^a_\alpha(x) \cdot T_N^{-1}(\alpha) \right). (56)
\]

The explicit evaluation of the limit shows that this operator is indeed local, as expected:

\[
\alpha \to \pi \lim \left( J^a_\alpha(x) \cdot T_N^{-1}(\alpha) \right) = \left( \sigma_x^a + \sigma_y^a \right) R_{x,x'}(\alpha) + R_{x,x'}(\alpha) \left( \sigma_x^a + e^{-i\alpha} \sigma_y^a \right). (55)
\]

where

\[
J^a_\alpha(x) = \left( \sigma_x^a + \sigma_y^a \right) R_{x,x'}(\alpha) + R_{x,x'}(\alpha) \left( \sigma_x^a + e^{-i\alpha} \sigma_y^a \right). (55)
\]

The explicit form of the chiral current in the spin chain is the second main result of this paper. It is the spin chain counterpart of the more general expression (51). The latter is more general in the sense that it is valid for an arbitrary angle \( \alpha \) in the vertex model, while the expression (57) involves only the limit \( \alpha \to \pi \). It is possible to get to the expression for the spin chain in a more direct way, without relying on the result for the vertex model; we will explain this alternative derivation shortly. But, before we do so, let us give the explicit form of the current operator.

Plugging the explicit form of the Hamiltonian density \( h_{x,x+1} \) leads to the following expressions. In the spin-1/2 case, \( h_{x,x+1} \) leads to the following expressions. In the spin-1/2 case, \( h_{x,x+1} = \frac{1}{2} (S_x^a \cdot S_{x+1}^a + \frac{1}{2} I) \), so one finds

\[
J^a_\alpha(x) = \frac{i \pi}{2a_0 \sin \alpha} \times \alpha \equiv \alpha \to \pi \lim \left( J^a_\alpha(x) \cdot T_N^{-1}(\alpha) \right). (56)
\]

The explicit evaluation of the limit shows that this operator is indeed local, as expected:

\[
\alpha \to \pi \lim \left( J^a_\alpha(x) \cdot T_N^{-1}(\alpha) \right) = \left( \sigma_x^a + \sigma_y^a \right) R_{x,x'}(\alpha) + R_{x,x'}(\alpha) \left( \sigma_x^a + e^{-i\alpha} \sigma_y^a \right). (55)
\]
\[ k = 1 \quad J_k^a = \frac{\pi}{2a_0} \left( S_k^a + S_{k+a_0}^a + \frac{4}{\pi} \left( S_k \times S_{k+a_0} \right)^a \right). \]  

(58)

where \((S_x \times S_{x+a_0})^a \equiv f^a_{bc} S_b^x S_c^{x+a_0}\). For spin-1, one has

\[ h_{x,x+a_0} = \frac{1}{2\pi} (S_x \cdot S_{x+a_0} - (S_x \cdot S_{x+a_0})^2 + 3 I), \]

which gives

\[ k = 2 \quad J_2^a = \frac{\pi}{2a_0} \left( S_2^a + S_{2+a_0}^a + \frac{1}{\pi} \left( S_2 \times S_{2+a_0} \right)^a \right) \]

\[- \frac{1}{\pi} \left( (S_2 \cdot S_{2+a_0})^a (S_2 \times S_{2+a_0})^a + (S_2 \times S_{2+a_0})^a (S_2 \cdot S_{2+a_0})^a \right) \].

(59)

For spin-3/2,

\[ h_{x,x+a_0} = \frac{1}{2\pi} \left( -\frac{2}{3} S_x \cdot S_{x+a_0} + \frac{1}{27} (S_x \cdot S_{x+a_0})^2 + \frac{2}{27} (S_x \cdot S_{x+a_0})^3 + \frac{1}{12} I \right). \]

One gets

\[ k = 3 \quad J_3^a = \frac{\pi}{2a_0} \left( S_3^a + S_{3+a_0}^a - \frac{1}{4\pi} \left( S_3 \times S_{3+a_0} \right)^a \right) \]

\[ + \frac{2}{27\pi} \left( (S_3 \cdot S_{3+a_0})^a (S_3 \times S_{3+a_0})^a + (S_3 \times S_{3+a_0})^a (S_3 \cdot S_{3+a_0})^a \right) \]

\[ + \frac{4}{27\pi} \left( (S_3 \cdot S_{3+a_0})^2 (S_3 \times S_{3+a_0})^a \right) \]

\[ + \left( S_3 \cdot S_{3+a_0} \right)^a (S_3 \times S_{3+a_0})^a (S_3 \cdot S_{3+a_0}) \]

\[ + \left( S_3 \times S_{3+a_0} \right)^a (S_3 \cdot S_{3+a_0})^a \].

(60)

4.3. Alternative derivation

There is an alternative way of getting to the formula (57). We adapt an idea suggested to us by Hubert Saleur (private communication), which builds upon earlier work of Koo and Saleur about the lattice realization of the stress-tensor [KS94]. It is much more straightforward than going first through the derivation of the main result (51), and then extracting the spin chain expression through the limiting procedure exposed in section 4.2. On the other hand, as we will see, the different steps in this alternative derivation are perhaps not under as good a control as in the route we took to arrive at the main result (51). But we view this alternative route as complementary, and as a further evidence that our scheme for taking the anisotropic limit is meaningful. Besides, we find this way of looking at formula (57) physically illuminating.

One proceeds as follows. First, we know that the critical Hamiltonian for the spin chain is a finite-size version of the CFT Hamiltonian. Namely

\[ \frac{2\pi a_0}{L} \left( L_0 + \frac{L_0}{2} - \frac{c}{12} \right) \approx H^{k/2} - E_{\infty} L = \sum_{x} h_{x,x+a_0} - E_{\infty} L, \]

(61)

where \( h_{x,x+a_0} \) is the Hamiltonian density. Similarly, it is natural to make an identification of the form

\[ J_k^a + T_k^a \approx \sum_{x} e^{-i2\pi x} S_k^a. \]

(62)

Notice that there is some freedom in this identification, however. For instance, the discrepancy between the following formula

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\[ J^a_n + J^a_n \simeq \sum_x e^{-i \frac{\pi}{4} (x+a_0/2)} \frac{S_x^a + S_{x+a_0}^a}{2}, \tag{63} \]

and (62) vanishes as \( a_0/L \) when \( L/a_0 \rightarrow \infty \). Different identifications may lead to different results in the end, so one needs to be careful here. We find that it is best to use (63). This may be justified in a way that is similar to the discussion of section 3: the spin operator has an expansion in terms of the CFT fields, in particular in terms of the primary fields \( \phi_j \otimes \bar{\phi}_j \) and the currents. In order to get rid of the unwanted primary operators, one can make use of the staggering, and average the spin operator over two neighboring sites. So we believe that (63) is the safer starting point here.

From the two expressions (61) and (63), one constructs a third one, that is a lattice version of the combination of modes \( J_n^+ - J_n^- \):

\[
- n \left( J_n^a - J_n^a \right) = \left[ L_0 + \sum \phi^a \left( S_x^a + S_{x+a_0}^a \right) / 2 \right] \\
\simeq - \frac{L}{2 \pi a_0} \sum_{x \neq 0} e^{-i \frac{\pi}{4} (x+a_0/2)} \left[ h_{x,x+a_0}, S_x^a + S_{x+a_0}^a \right] / 2 \\
\simeq - i \frac{n}{2} \sum_x e^{-i \frac{\pi}{4} (x+a_0/2)} \left[ h_{x,x+a_0}, S_x^a + S_{x+a_0}^a \right]. \tag{64} \]

In the last line we have used \( e^{-i \frac{\pi}{4} (x+a_0/2)} \simeq 1 - i \frac{2 \pi a_0}{L} \), which is valid if one fixes \( n \) and then take \( L \gg a_0 \), and then \( h_{x,x+a_0}, S_x^a + S_{x+a_0}^a \right] = 0 \), which is simply stating that \( h_{x,x+a_0} \) is SU(2) invariant. Combining (63) and (64), one gets the following approximation for \( J_n^a \):

\[ J_n^a \simeq \frac{a_0}{2 \pi} \sum_x e^{-i \frac{\pi}{4} (x+a_0/2)} \frac{\pi}{2a_0} \left( S_x^a + S_{x+a_0}^a + i \left[ h_{x,x+a_0}, S_x^a + S_{x+a_0}^a \right] \right). \tag{65} \]

which is nothing but the Fourier mode of \( J_n^a \) obtained from the anisotropic limit. So one recovers (57) as claimed. Notice that it is important to use (63) instead of (61) when we recombine the expressions for \( J_n^+ + J_n^- \) and \( J_n^+ - J_n^- \); otherwise we would actually get a different result. Again, this is a crucial point, in light of the discussion in section 3, since what we need is to suppress the contribution of the primary operators \( \phi_j \otimes \bar{\phi}_j \) with \( j \) half-integer. But, apart from this point, which was already encountered in the construction of the chiral current in the vertex model, what is interesting here is the way the chiral component of the current is isolated. In the vertex model, this was done by fine-tuning the phases of the spin operator on neighboring sites, depending on the geometric angle \( \alpha \); in that sense, the chiral and anti-chiral components were separated thanks to the geometry of the two-dimensional model. Here, the left- and right-movers are separated by their dynamic properties, which one probes with the commutator of the Hamiltonian with the spin operator. We find that, even though one should certainly expect it, it is quite a non-trivial check that the two viewpoints match so perfectly in the anisotropic limit.

Let us finally emphasize that, in the construction of the lattice operator \( J_n^a \) for the spin chain, we did not use much, apart from the fact that we have a critical Hamiltonian, which we normalized such that the velocity of the excitations is one, and the identification of the modes of the Lie algebra generators \( S_x^a \) with the Kac–Moody modes in the continuum. The latter is the crucial step, and one needs to be careful in this identification, as the most obvious guess for the lattice analogue of the Kac–Moody modes may not be the correct expression, and might lead to wrong results. This step is then a bit subtle, and involves a discussion of the relation between the lattice observables and the field theory operators similar to the one in section 3. But, apart from these subtleties, we only used the fact that the commutator of \( L_0 \)
with $J_x^J$ is proportional to $n$, which is a completely general feature of the $L_0$-operator, and in particular, we didn’t make use of the existence of a Sugawara construction here. This means that such an approach might be generalizable to many different critical spin chains including those with a Lie (super-)group symmetry and non-integrable ones.

5. Numerical checks

The coefficients appearing in the expansion of the lattice spin operator 36 can in principle be related to form factors. This is what we discuss in this section. The relation with form factors gives us a way of checking numerically that the coefficients $C_x^J$ are indeed given by

$$C_x^J = \begin{cases} \frac{a_0}{2\pi i} & (x \text{ vertical edge}) \\ \frac{a_0}{2\pi i} e^{i\alpha} & (x \text{ horizontal edge}), \end{cases}$$

as we argued in equation (50). The numerical results are plotted in figure 4 for $k = 1$ and in figure 5 for $k = 2$. We find that they are in perfect agreement with (66).

The relation between form factors and the coefficients $C_x^J$ goes as follows. Consider once again the infinitely long cylinder of circumference $L = a_0 N$ generated by the transfer matrix $T_L^{1/2}(\alpha)$. Let $|0\rangle$ be the eigenstate of $T_L^{1/2}(\alpha)$ with the largest eigenvalue (in absolute value).
This state is also the ground state of the Hamiltonian $H^{k/2}$. Let $|s\rangle$ be some other eigenstate of $T_L^{k(2)}$ or, equivalently, of $H^{k(2)}$. One can always choose $|s\rangle$ to be a state with the following fixed quantum numbers:

- momentum $P$,
- total SU(2) spin $(\sum_i S_i)^2 = S(S + 1)$,
- magnetization $\sum_i S_i^z = S^z$.

In addition to these three quantum numbers, the state $|s\rangle$ has some energy $E_s$, which is the eigenvalue of $H^{k(2)}$. These four quantities uniquely identify an eigenstate of the transfer matrix. The ground state itself is identified as follows: it has energy $E_0$, total spin and magnetization zero, and momentum $P_0 = 0$ or $P_0 = \pi/a_0$, depending on the parity of $k N/2$ (recall that $L = a_0 N$ and that we assume that $N$ is even throughout the paper).

Next, we focus on the unique eigenstate characterized by:

- $P = R_0 + \frac{2\pi}{T}$
- total spin $S = 1$
- magnetization $S^z = 0$
- it is the lowest-energy state with the above three quantum numbers.

We call this state $|J^3\rangle$. This notation comes from the fact that one wants to identify the spectrum of $H^{k(2)}$ with the one of $E_{\text{CFT}} + H_{\text{CFT}} = E_{\text{CFT}} + \frac{\alpha}{L} \left(L_0 + L_0^c - \frac{c}{12} \right)$ in the continuum limit, and the lowest energy state with these quantum numbers in the CFT is $|J^3_0\rangle$.

In particular, this means that the energy of the state $|J^3\rangle$ must behave as

$$E_{J^3}(L) - E_0(L) \sim \frac{2\pi}{L} + o\left(L^{-1}\right),$$

if $E_0(L)$ is the energy of the ground state. Note also that, with our convention for the OPEs of the currents, the norm of the CFT state $|J^3_0\rangle$ is $\langle J^3_0 J^3_0 \rangle = k/2$ so we fix the normalization of our lattice state $|J^3\rangle$ to be

![Figure 5. Same as figure 4, this time for $k = 2$. Again, the results support our formula (66). We compute $C_{\text{vert}, N}$ and $C_{\text{horiz}, N}$ for $N = 8, 16, 32, 64, 256, 512.$](image-url)
as well. Of course, alternatively, one could just identify the eigenstate $|J^3\rangle$ by specifying its Bethe roots configuration; we give more details about this in appendix B.

At last, we arrive at the connection with the coefficients $C_{x}^{J}$. Let us focus first on a point $x = (x, y)$ on a vertical edge, so $(x, y) \in a_{0}\mathbb{Z}_{N} \times a_{0}(\mathbb{Z} + \frac{1}{2})$. We are interested in the form factor $\langle J^3 | S_{x}^{a} | 0 \rangle$, which is now almost well-defined for any finite size $L$. Namely, it is defined up to a phase, since the phases of the ground state $|0\rangle$ and of $|J^3\rangle$ are arbitrary. But we will come back to this question later. For now, we simply observe that, since $S_{x}^{a}$ admits an expansion of the form

$$S_{x}^{a} = \cdots + a_{0} \left[ C_{x}^{J} J^{a}(z) + C_{x}^{J} J^{a}(z) \right] + \cdots,$$

and since the mode expansion of $J^{a}(z)$ on the cylinder is

$$J^{a}(z) = \frac{2\pi}{L} \sum_{n} e^{i\frac{2\pi}{L} J^{a}_{n}},$$

we expect the following behavior of the form factor (up to an undetermined phase):

$$\langle J^3 | S_{x}^{a} \rangle_{L \to \infty} = \frac{2\pi a_{0}}{L} C_{x}^{J} \sum_{n} e^{i\frac{2\pi}{L} J^{a}_{n}} \langle 0 | J^3 J^{a}_{n} | 0 \rangle + o(L^{-1})$$

$$= \frac{2\pi a_{0}}{L} C_{x}^{J} e^{i\frac{2\pi}{L} J^{a}_{0}} \langle 0 | J^3 J^{a}_{0} | 0 \rangle + o(L^{-1}).$$

Thus, the following quantity

$$C_{x}^{J} \equiv \frac{2}{k} \frac{L}{2\pi a_{0}} e^{i\frac{2\pi}{L} J^{a}_{0}} \langle J^3 | S_{x}^{a} | 0 \rangle$$

is independent of $x$, and should converge (in amplitude) to the coefficient $C_{x}^{J}$.

$$\left| C_{x}^{J} \right| \rightarrow_{N \to \infty} \left| C_{x}^{J} \right|$$

The overall factor $2/k$ in (72) comes from the normalization (68). This quantity can be computed in finite size, and then extrapolated to the thermodynamic limit, which gives an estimate of the amplitude of the coefficient $C_{x}^{J}$ on vertical edges.

Second, consider the case of a point $x = (x, y)$ on a horizontal edge, so $(x, y) \in a_{0}\mathbb{Z}_{N} \times a_{0}(\mathbb{Z} + \frac{1}{2})$. The coefficient $C_{x}^{J}$ may still be related to a finite-size quantity involving the lattice ground state $|0\rangle$ and our state $|J^{3}\rangle$, but this quantity is not, strictly speaking, a form factor. Instead, it is the matrix element of the following operator (we do not add a superscript $(k/2)$ here, but this operator of course depends on $k$):

$$T_{L_{a}^{(x)}}^{(a_{0})}(\alpha) = \begin{pmatrix}
S_{x}^{a} & \alpha \bar{a} \\
\bar{a} \alpha & S_{x}^{a}
\end{pmatrix},$$

$$a_{0} \quad 2a_{0} \quad a_{0}N$$

(74)
which we use to form the ratio

$$C_{\text{horiz}.N}^J \equiv \frac{2}{k} \frac{L}{2\pi a_0} e^{i\frac{\pi}{2}} \frac{\langle J^1 \mid T^{(S)}_L(a) \mid 0 \rangle}{\langle 0 \mid T^{(S)}_L(a) \mid 0 \rangle}$$

(horizon edge). (75)

Again, the global factor $2/k$ comes from the normalization (68), and again, this quantity is defined only up to a phase, coming from the undetermined phases of the ground state $|0\rangle$ and of $|J^3\rangle$. The quantity $C_{\text{horiz}.N}^J$ does not depend on $x$, and its amplitude $|C_{\text{horiz}.N}^J|$ allows us to estimate the amplitude of the coefficient $C^j$ on horizontal edges.

Finally, note that, although $C_{\text{verti}.N}^J$ and $C_{\text{horiz}.N}^J$ are defined up to a phase, their relative phase is well-defined, and is a quantity which can be measured

$$\frac{C_{\text{horiz}.N}^J}{C_{\text{verti}.N}^J} = \frac{\langle 0 \mid \langle \frac{1}{2} \rangle \mid J^3 \mid S^3 \rangle \mid 0 \rangle}{\langle 0 \mid T^{(S)}_L(a) \mid 0 \rangle \langle J^3 \mid S^3 \rangle \mid 0 \rangle}, \quad x \in a_0\mathbb{Z}_N. \tag{76}$$

We thus have access to the relative phase between horizontal and vertical edges, if we compute this phase for finite $N$ and then extrapolate the results to $N \to \infty$.

Numerically, we compute the matrix elements $\langle J^3 \mid S^3 \rangle \mid 0 \rangle$, $\langle J^3 \mid T^{(S)}_L(a) \rangle \mid 0 \rangle$ using form factor techniques and the integrable structure of the six-vertex model. We make use of Slavnov’s determinant formula [Sla89] applied to the matrix elements of spin-1/2 [KMT99] and spin-1 [CAM07] chains. More details about these points are given in appendix B, where for convenience the discussion is carried out for $J^+, S^+$ instead of $J^3, S^3$. (Due to SU(2) invariance this change affects our formulas only in the normalization.) These techniques allow us to go to large system sizes, as opposed to naive exact diagonalization. Large system sizes are really needed here: the finite-size corrections decay very slowly because of sub-leading logarithmic corrections, see [AGSZ89] and appendix A. In figures 4 and 5, we plot our results for the finite-size observables (72) and (75), which converge toward the coefficient $C^j$ on horizontal and vertical edges.

6. Conclusion

We considered a class of lattice models that are known to be discretizations of the $SU(2)_k$ WZW model—the descendants of the six-vertex model—and identified local observables on the lattice that behave as the components of the chiral current $J^+(z)$ in the continuum limit. These observables are constructed using a combination of lattice spin operators on neighboring edges. We started by a careful analysis of the expansion of the lattice spin operator $S^a_x$ in terms of the fields in the continuum limit, and were able to put some constraints on the operators that can appear. We found that primary operators $\phi_j$ with half-integer spin $j$ all come with staggered coefficients—a fact that has long been known when $j = \frac{1}{2}$, but which holds in full generality—while the primary operators with integer spin $j$ are absent from the expansion. Most importantly, we argued that the chiral and anti-chiral components of the current appear in the expansion, with coefficients that can be determined by combining SU(2)-symmetry and some intuition about how quantities that are automatically conserved on the lattice become the zero modes of the currents in the continuum limit. We provided numerical checks that support the identification of these coefficients. The analysis of the expansion of $S^a_x$ was finally used to produce a new observable involving the spin operator on a few neighboring edges, and such that this observable itself admits an expansion in which the most relevant operator is the chiral current. This new observable therefore is one local lattice operator that achieves the goal we were aiming for. In particular, multi-point correlators of this observable do become
the correlators of the chiral currents (2), as we wanted. Clearly, many other observables with this property could be constructed, using more lattice sites. The one we exhibited here is minimal in the sense it involves only four neighboring lattice points. This observable was constructed for an arbitrary geometric angle $\alpha$. Looking at the anisotropic limit, we were able to obtain an expression for the chiral current in the spin-$\frac{1}{2}$ Heisenberg spin chain, and more generally, in the multi-critical spin-$\frac{k}{2}$ spin chain, in terms of a commutator of the local spin operator with the Hamiltonian density.

The present work can be extended to other lattice models possessing a continuous symmetry, in a way that seems relatively straightforward. One could, for instance, consider vertex models discretizing SU(N)$_k$ WZW models, and construct lattice versions of the chiral currents of such theories. Another interesting direction would be to study the chiral current in supersymmetric spin systems such as those investigated in [RS01]. Finally, as we already mentioned in the introduction, it would be interesting to have lattice versions of other (non-local) chiral observables, such as the chiral part of the primary fields, $\phi_j(z)$. These must be non-local, and should typically be associated with defects or local dislocations of the lattice. We hope to come back to this question soon.

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Appendix A. Current–current perturbation and logarithmic corrections

In the main text, we overlooked the role of logarithmic corrections which appear in all of the spin-$\frac{k}{2}$ descendants of the six-vertex. We were able to do so because such logarithmic corrections do not affect the correlations of the observable we constructed, at least not at the leading order. They appear only in subleading corrections. This is in strong contrast to the case of the spin–spin correlation, for example, which is well-known to pick a logarithmic prefactor at the leading order:

$$\epsilon \propto \log[z - z']$$

where $\epsilon_{x,x'} = \pm 1$ depends on the positions and takes care of the staggering. We have again used the notation $x = (x, y), z = x + \epsilon x'y$. The purpose of this appendix is to explain why this happens for the spin operator, but not for the lattice chiral observable we have constructed. We follow the beautiful treatment of Affleck, Gepner, Schulz and Ziman [AGSZ89].

The critical vertex models we are interested in can be described by the SU(2)$_k$ WZW model, with perturbations. Since we know that these models are at the critical point, no relevant perturbation is allowed. Irrelevant perturbations are certainly allowed, but also marginal ones. The right and left currents lead to a marginal perturbation
\( S_{WZW} \rightarrow S_{WZW} + g_0 \int \frac{d^2 z}{2\pi} J(z) \cdot \bar{J}(\bar{z}). \) (A.2)

One could also wonder whether terms like \( J \cdot J \) or \( \bar{J} \cdot \bar{J} \) could appear (we suppress the \( z, \bar{z} \) dependence when clear from the context), but these are just the chiral and the anti-chiral components of the stress-tensor, and these would change the metric, namely the geometric angle \( \alpha \) in the main text. We have already fixed our conventions such that \( \alpha \) is properly taken into account, so these two perturbations are actually not present here. Now let us come back to the \( J \cdot J \) term, which has much less trivial effects on the effective theory describing the critical point. The \( \beta \)-function for the renormalized coupling constant \( g \) can be computed as follows (for examples of such calculations, see e.g. [LW03]). We introduce an UV (IR) cutoff \( a_0 (L) \), and look at how \( g_0 \) is modified by higher order terms in the expansion of the exponential of the perturbation:

\[
\left\langle \ldots e^{g_0 \int \frac{d^2 z}{2\pi} J(z) \cdot \bar{J}(\bar{z})} \right\rangle_{WZW} = \left\langle \ldots \left( 1 + g_0 \int \frac{d^2 z}{2\pi} J(z) \cdot \bar{J}(\bar{z}) + \frac{g_0^2}{2} \int \frac{d^2 z}{2\pi} J(z) \cdot \bar{J}(\bar{z}) \int \frac{d^2 z'}{2\pi} J(z') \cdot \bar{J}(\bar{z'}) + O(g_0^3) \right) \right\rangle_{WZW}
\]

\[
= \left\langle \ldots \left( 1 + \left( g_0 - g_0^2 \log \left( L/a_0 \right) \right) \int \frac{d^2 z}{2\pi} J(z) \cdot \bar{J}(\bar{z}) \right) + O(g_0^3) \right\rangle_{WZW} . \quad (A.3)
\]

Here the dots stand for arbitrary operator insertions and the expectation values are computed in the unperturbed theory. From the first to the second line, we have used the OPE \( J^a(z) J^b(\bar{z}) J^c(\bar{z'}) J^c(\bar{z}) \approx (ie^{i\theta z})^2/|z-z'|^2 J^a(z) J^c(\bar{z}) J^c(\bar{z}) \), and we have evaluated the integral \( \int \frac{d^2 z}{2\pi} \frac{1}{|z-z'|} \) over \( a_0 < |z-z'| < L \), which gives \( \log \left( L/a_0 \right) \).

Thus, at the leading order, the coupling is renormalized from \( g_0 \) to \( g = g_0 - g_0^2 \log \left( L/a_0 \right) + O(g_0^3) \), which gives

\[
\beta(g) \equiv \frac{\partial g}{\partial \log a_0} = g^2 + O(g^3) . \quad (A.4)
\]

The perturbation \( J \cdot J \) is thus marginally irrelevant for \( g_0 > 0 \) and marginally relevant for \( g_0 < 0 \). For the vertex models we consider, it is known that we are in the marginally irrelevant case, so for small \( g_0 \) we find

\[
g \propto (\log a_0)^{-1} . \quad (A.5)
\]

Variations of the correlation functions with the UV cutoff \( a_0 \) can be estimated thanks to the Callan–Symanzik equation. For example, for the two-point function of a (Virasoro) primary field \( \phi(z, \bar{z}) \) with scaling dimension \( \Delta \), we have

\[
\left( \frac{\partial}{\partial \log a_0} + \beta(g) \frac{\partial}{\partial \log \left( L/a_0 \right)} \right) \left\langle \phi(z_1, \bar{z}_1) \phi(z_2, \bar{z}_2) \right\rangle = 0 . \quad (A.6)
\]

Importantly, the anomalous scaling dimension \( \Delta (g) \) is different from the scaling dimension \( \Delta \) in the pure CFT:

\[
\Delta (g) = \Delta - g \sum_{n \in \mathbb{Z}} \left\langle J_n \cdot \bar{J}_n \right\rangle_{\phi} + O(g^2) . \quad (A.7)
\]
(Above and below $\langle J_n \cdot J_n \rangle_\phi$ is the expectation value of this operator in the state $|\phi\rangle$ with scaling dimension $\Delta$.) This is because the Hamiltonian is affected by the current–current perturbation:

$$H_{\text{CFT}} \to H_{\text{CFT}} - g \int \frac{dx}{2\pi} J(x) \cdot e^{i\phi(x)}.$$  \hspace{1cm} (A.8)

For instance, the primary field $\phi(z, \bar{z}) = P_f [\phi \otimes \bar{\phi}]$ has the anomalous scaling dimension

$$\Delta(g) = \Delta - \frac{j(j' + 1) - 2j(j + 1)}{2} g + O(g^2).$$  \hspace{1cm} (A.9)

Plugging (A.7) into (A.6), dropping terms of order $O(g^2)$, and integrating the differential equation, one finds that the two-point function becomes

$$\langle \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2) \rangle \propto \left( \log \left( \frac{|z_1 - z_2|}{a_0} \right) \right)^{2j} \left( \frac{|z_1 - z_2|}{a_0} \right)^{-2\Delta}.$$  \hspace{1cm} (A.10)

In particular, since the leading contribution to the lattice spin operator $S^a_i$ is the primary operator $\Pi[\phi \otimes \bar{\phi}]$, we have $2 \langle J_0 \cdot J_0 \rangle_\phi = \frac{1}{2}$, in agreement with (A.1).

Finally, we see the reason why there are no multiplicative logarithmic corrections to the chiral current–current correlation $\langle J^a(z_1) J^b(z_2) \rangle$: it is because $\sum \langle J_n \cdot J_n \rangle_j = 0$, so there is no correction to the scaling dimension at order $O(g)$. Note that the structure factors are also affected by these logarithmic corrections. Since they are related to one-point functions, one can evaluate their variation with the system size $L/a_0$ from the Callan–Symanzik equation for the one-point function. For instance, the following structure factor must scale as

$$\langle \Pi[\phi \otimes \bar{\phi}] | S^a \rangle \propto (-1)^j \left( \frac{\log (L/a_0)}{L/a_0} \right)^{j(j+1)/2} \delta^{ab}.$$  \hspace{1cm} (A.11)

for half-integer $j$. We used the notation $\Delta_j = h_j + \bar{h}_j = \frac{j(j+1)}{2}$. For integer $j$, the structure factor vanishes, as discussed in the main text (see also the discussion in [VC14]). For the structure factor computed in the main text (see figures 4 and 5), we do not find a logarithmic correction, again because $\sum \langle J_n \cdot J_n \rangle_j = 0$.

Appendix B. Bethe ansatz and form factors

B.1. Bethe ansatz of the spin-1/2 Heisenberg chain

In this appendix, we prefer to parametrize the weights of the six-vertex model with $u$ rather than with the geometric angle $\alpha$:

$$a(u) = 1 \quad b(u) = -u \quad c(u) = 1.$$  \hspace{1cm} (B.1)

The spectral parameter $u$ is related to the geometric angle by $u = 1 - \frac{\alpha^2}{\pi}$. We use the standard notations for the elements of the monodromy matrix:
where a thin horizontal line carries a spin-1/2 representation with spectral parameter \( u \), and a thick vertical line carries an arbitrary representation, typically a tensor product of a spin-\( k/2 \) representations. These operators satisfy the ‘RTT’ relations, which directly follow from the Yang–Baxter equation. For instance:

\[
B(u)B(v) = B(v)B(u), \quad (B.3a)
\]

\[
A(u)B(v) = \frac{a(v-u)}{b(v-u)}B(v)A(u) - \frac{c(v-u)}{b(v-u)}B(u)A(v), \quad (B.3b)
\]

\[
D(u)B(v) = \frac{a(u-v)}{b(u-v)}B(v)D(u) - \frac{c(u-v)}{b(u-v)}B(u)D(v), \quad (B.3c)
\]

and so on. For a set of complex numbers \( \{\lambda_1, \ldots, \lambda_M\} \), we define the corresponding Bethe state as

\[
\left| \{\lambda_q\} \right\rangle = B(u_1) \cdots B(u_M) \left| \uparrow \uparrow \right\rangle, \quad (B.4)
\]

where

\[
u_q = -\frac{k}{2} + 1 - i\frac{\lambda_q}{2}. \quad (B.5)
\]

A Bethe state \( (B.4) \) is an eigenstate of the transfer matrix \( T_N^{(k/2)} \), and therefore of the Hamiltonian \( H_N^{(k/2)} \) of equation (35), only if the Bethe equations are satisfied:

\[
\prod_{\substack{q \neq j \\text{and} \ j \neq q}}^{N} \frac{\lambda_j + i k}{\lambda_j - i k} = \prod_{q \neq j}^{N} \frac{\lambda_j - \lambda_q + 2i}{\lambda_j - \lambda_q - 2i}. \quad (B.6)
\]

One can check that the momentum and energy—i.e. the eigenvalue of the Hamiltonian \( H_N^{(k/2)} \) —of a Bethe state \( \left| \{\lambda_q\} \right\rangle \) is:

\[
P\left( \{\lambda_q\} \right) = \frac{1}{\alpha_0} \sum_{q=1}^{M} \pi - 2 \arctan \frac{\lambda_q}{k} \quad \mod \frac{2\pi}{\alpha_0}, \quad (B.7)
\]

\[
E\left( \{\lambda_q\} \right) = \sum_{q=1}^{M} \frac{-4k}{k^2 + \lambda_q^2} + \epsilon_k. \quad (B.8)
\]

**B.2. Root configurations corresponding to some low-energy states of interest**

- For any even \( N \) the ground state is a singlet and corresponds to a configuration of \( N/2 \) \( k \)-strings, with Bethe–Takahashi numbers \([TS72,VC14]\)
\[
\left\{ -\frac{N-2}{4}, -\frac{N-2}{4} + 1, ..., -\frac{N-2}{4} + \frac{N-2}{4}\right\}. \]
The ground state has momentum \( P_0 = 0 \) if \( kN/2 \) is even, and momentum \( P_0 = \pi a_0 \) if \( kN/2 \) is odd.

- The Bethe state that would correspond to the CFT state \( |\phi_j \rangle \otimes |\phi_j \rangle \) in the continuum limit is the one with \( N/2 - 1 \) \( k \)-strings, and one \((k - 1)\)-string, with Bethe–Takahashi numbers \( \left\{ -\frac{N}{2} + 1, -\frac{N}{2} + 2, ..., -\frac{N}{2} - 2, -\frac{N}{2} - 1 \right\} \) and \( \{0\} \). This Bethe state is, as usual, a highest weight state. Acting on it with the \( su(2) \) generators, one generates a triplet of degenerate excited states. One can check that the momentum of these three eigenstates is \( P = \left( R_0 + \frac{k}{a_0} \right) \mod \frac{2\pi}{a} \).

- The above facts generalize to \( \phi_j \) for \( j > \frac{1}{2} \) as follows. We claim that the Bethe state which corresponds to the (highest weight) CFT state \( |\phi_j \rangle \otimes |\phi_j \rangle \) is the one that can be obtained from the ground-state by taking out two \( k \)-strings, and replacing them by one \((k - 2)\)-string, and one \((k + 2)\)-string. The Bethe–Takahashi numbers are \( \left\{ -\frac{N-2}{2} + 1, -\frac{N-2}{2} + 2, ..., -\frac{N-2}{2} - 2, -\frac{N-2}{2} - 1 \right\} \), \( \{0\} \) and \( \{0\} \) for the newly created strings. This gives us a triplet of degenerate states, as it should. Also, notice that such a Bethe state exists only if \( j \leq \frac{k}{2} \), which is consistent with the fact that the primary fields for the (chiral) Kac–Moody algebra must have \( SU(2) \)-spin \( \leq j \leq \frac{k}{2} \). The momentum \( P \) of these states can be checked to be

\[
a_0 \times (P - R_0) \equiv \begin{cases} \pi \mod 2\pi & \text{if } j \text{ is half-integer,} \\ 0 \mod 2\pi & \text{if } j \text{ is integer.} \end{cases} \tag{B.9}
\]

For integer \( j \), this follows from a subtle fact about the Bethe equations, which arises when one looks for solutions with a \( p \)-string, \( p \) odd and \( 2p \geq k \). In that case, it turns out that there are solutions to the Bethe equations with a pair of the Bethe roots that are \textit{exactly} equal to \( \pm i \frac{k}{2} \). This phenomenon is known in the literature: the corresponding Bethe states are dubbed ‘singular’ for instance in [HC07,VC14]. These Bethe states are still perfectly valid eigenstates of the Hamiltonian, however, some particular limiting procedure must be used when one calculates the corresponding energies or momenta. The fact that the Bethe states for integer spin \( j \) correspond precisely to these ‘singular’ states is crucial; this is what leads to (B.9), in agreement with what we claimed in the main parts of the paper. We note that essentially the same discussion appeared already in [AGSZ89].

- Finally, the Bethe state which we identify with the CFT state \( |J_j \rangle \) is the one with one \((k - 1)\)-string, and \( N/2 - 1 \) \( k \)-strings, and Bethe–Takahashi numbers \( \{0\} \) and \( \{0\} \) for \( \{0\} \). This state has momentum \( P = R_0 + \frac{2\pi}{a_0} \mod \frac{2\pi}{a_0} \), as required.

Figures B1 and B2 show some of the Bethe root configurations described above.

### B.3. Useful formulas for \( k = 1 \)

**Eigenvalue of the transfer matrix** \( T^{(1/2)}(u) \). In terms of the elements of the monodromy matrix, we have

\[
T^{(1/2)}(u) = A(u) + D(u). \tag{B.10}
\]

This, together with the ‘RTT’ relations and the Bethe equations, allows one to show that the eigenvalue of \( T^{(1/2)}(u) = (1 - i\mu)/2 \) corresponding to a Bethe state \( \{|\lambda_q \rangle \} \) is
The norm of a Bethe state $\{\lambda_q\}$ (where $\{\lambda_q\}$ is a solution of the Bethe equations) can be expressed in terms of a determinant

$$A^{(1/2)}(\mu) = \left[ \prod_{q=1}^{M} \lambda_q - \mu - 2i + \left( \frac{\mu + i}{\mu - i} \right)^N \prod_{q=1}^{M} \frac{\lambda_q - \mu + 2i}{\lambda_q - \mu} \right].$$  \(\text{(B.11)}\)

**Norms of Bethe states and overlaps.** The norm of a Bethe state $\{\lambda_q\}$ can be expressed in terms of a determinant

$$\langle \{\lambda_q\} | \{\lambda_q\} \rangle = (2i)^M \left( \prod_{a\neq b} \frac{\lambda_a - \lambda_b - 2i}{\lambda_a - \lambda_b} \right) \times \det G,$$  \(\text{(B.12)}\)

where the entries of the $M \times M$ Gaudin matrix $G$ are:

$$G_{ab} = \begin{cases} \frac{-2iN}{\lambda_a^2 + 1} + \sum_{k \neq a} \frac{4i}{(\lambda_a - \lambda_k)^2 + 4} & \text{for } a = b, \\ \frac{-4i}{(\lambda_a - \lambda_k)^2 + 4} & \text{for } a \neq b. \end{cases}$$  \(\text{(B.13)}\)

The overlap between a Bethe eigenstate $\{\lambda_q\}$ and an off-shell Bethe state $\{\mu_p\}$ (i.e. the $\mu_p$ are not assumed to satisfy the Bethe equations) also admits a determinant expression
$\lambda_{\mu} = \det T$, (B.14)

where $A$ and $T$ are $M \times M$ matrices with elements

$$A_{ab} = \frac{1}{\mu_b - \lambda_a}, \quad (B.15)$$

$$T_{ab} = \frac{2i}{(\lambda_a - \mu_b)^2} \left[ \prod_{k \neq a} \frac{\lambda_k - \mu_b - 2i}{\lambda_k - \mu_b} - \left( \frac{\mu_b + i}{\mu_b} \right)^N \prod_{k \neq a} \frac{\lambda_k - \mu_b + 2i}{\lambda_k - \mu_b} \right]. \quad (B.16)$$

**Form factor for $k = 1$.** The matrix element $\langle \{ \lambda \} \left| S_{\mu_0}^+ \right| \{ \mu \} \rangle$ between two (not normalized) Bethe states has been computed in [KMT99]. Here there are $M \lambda$-roots, and $M + 1 \mu$-roots. The result, which we use in our numerical evaluation, is:
where \( H \) is an \((M + 1) \times (M + 1)\) matrix with entries

\[
H_{ab} = \begin{cases} 
-2i 
& \text{if } b < M + 1 \\
\frac{-2i}{\mu^2 + 1} 
& \text{if } b = M + 1.
\end{cases}
\]

(B.18)

### B.4. Useful formulas for \( k = 2 \)

**Eigenvalues of the transfer matrix \( T^I(u) \).** The transfer matrix is expressed in terms of the entries of the monodromy matrix as

\[
T^{(1)}(u) = A_+ A_- + D_+ D_- + \frac{1}{2} (A_+ D_- + D_+ A_- + B_+ C_- + C_+ B_-),
\]

where we have used the notations \( A_\pm = A(u \pm 1/2) \), and similarly for \( B, C, D \). Setting \( u = -i\mu/2 \), the eigenvalue associated to the Bethe state \( |\{\lambda_q\}\rangle \) is

\[
A^{(1)}(u, \{\lambda_q\}) = A^{(1/2)}(u + 1/2, \{\lambda_q\}) A^{(1/2)}(u - 1/2, \{\lambda_q\}) - \left( \frac{\mu + 3i}{\mu - i} \right)^N.
\]

(B.20)

**Norms and overlaps.** The norm of a Bethe state \( |\{\lambda_q\}\rangle \) (where \( \{\lambda_q\} \) is a solution to the Bethe equations for \( k = 2 \)) is

\[
\langle \{\lambda_q\} \mid \{\lambda_q\} \rangle = (-2i)^N \left( \prod_{a \neq b} \frac{\lambda_a - \lambda_b - 2i}{\lambda_a - \lambda_b} \right) \times \det G,
\]

where the entries of the Gaudin matrix \( G \) are:

\[
G_{ab} = \begin{cases} 
\frac{-4iN}{\lambda_a^2 + 1} + \sum_{b \neq a} \frac{4i}{(\lambda_a - \lambda_b)^2 + 4} 
& \text{for } a = b, \\
\frac{-4i}{(\lambda_a - \lambda_b)^2 + 4} 
& \text{for } a \neq b.
\end{cases}
\]

(B.22)

The overlap between a Bethe eigenstate \( |\{\lambda_q\}\rangle \) and an off-shell Bethe state \( |\{\mu_p\}\rangle \) is:

\[
\langle \{\lambda_q\} \mid \{\mu_p\} \rangle = \frac{\det T}{\det A},
\]

where \( A \) and \( T \) are \( M \times M \) matrices with elements

\[
A_{ab} = \frac{1}{\mu_b - \lambda_a},
\]

(B.24)
\[ T_{ab} = -\frac{2i}{(\lambda_a - \mu_b)^2} \left[ \prod_{k \neq a} \frac{\lambda_k - \mu_b - 2i}{\lambda_k - \mu_b} - \left( \frac{\mu_b + 2i}{\lambda_b - 2i} \right)^N \prod_{k \neq a} \frac{\lambda_k - \mu_b + 2i}{\lambda_k - \mu_b} \right]. \] (B.25)

Form factor for \( k = 2 \). The expression for the matrix element \( \langle \{ \lambda \} | S^+_0 | \{ \mu \} \rangle \) between two Bethe states for \( k > 1 \) is given in [CAM07]. Here we adapt this result to our needs. Again, there are \( M \lambda \)-roots, and \( M + 1 \mu \)-roots. We have:

\[ \langle \{ \lambda \} | S^+_0 | \{ \mu \} \rangle = \prod_k (\mu_k + 2i) \prod_{j>1} \left( \lambda_j - \mu_j \right) \prod_{p<q} (\lambda_p - \lambda_q) \det H, \] (B.26)

where \( H \) is an \((M + 1) \times (M + 1)\) matrix with entries

\[ H_{ab} = \begin{cases} \frac{2i}{\mu_a - \lambda_b} \left[ \prod_{k \neq a} (\mu_k - \lambda_b - 2i) - \left( \frac{\lambda_b + 2i}{\lambda_b - 2i} \right)^N \prod_{k \neq a} (\mu_k - \lambda_b + 2i) \right] & \text{if } b < M + 1 \\ \frac{-4i}{\mu_a^2 + 1} & \text{if } b = M + 1. \end{cases} \] (B.27)

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