Multiplicative order compact operators between vector lattices and $l$-algebras

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Abstract: In the present paper, we introduce and investigate the multiplicative order compact operators from vector lattices to $l$-algebras. A linear operator $T$ from a vector lattice $X$ to an $l$-algebra $E$ is said to be $\text{omo}$-compact if every order bounded net $x_\alpha$ in $X$ possesses a subnet $x_{\alpha_\beta}$ such that $T x_{\alpha_\beta} \xrightarrow{\text{omo}} y$ for some $y \in E$. We also introduce and study $\text{omo}$-$M$- and $\text{omo}$-$L$-weakly compact operators from vector lattices to $l$-algebras.

Keywords: vector lattice, $l$-algebra, $\text{omo}$-convergence, $\text{omo}$-continuous, $\text{omo}$-compact, $\text{omo}$-$M$-, and $\text{omo}$-$L$-weakly compact operator.

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1 Introduction

Compact operators play significant role in the operator theory and its applications. Various kinds of classical convergences, like order and relatively
uniform convergences, in vector lattices are not topological [14, Thm.2], [12, Thm.5], [13, Thm.2.2]. Fortunately, even without any topology, several natural types of compact operators can be investigated (see, e.g. [10]). In the present paper, we introduce and investigate \( \omega \)-weak-compact operators from vector lattices to \( l \)-algebras. Throughout the paper, all vector lattices are assumed to be real and Archimedean, and all operators to be linear. We denote by letters \( X \) and \( Y \) vector lattices, and by \( E \) and \( F \) \( l \)-algebras.

A net \( x_\alpha \) in \( X \):

- \( \omega \)-converges to \( x \in X \) (shortly, \( x_\alpha \xrightarrow{\omega} x \)), if there exists a net \( y_\beta \downarrow 0 \) such that, for any \( \beta \), there exists \( \alpha_\beta \) satisfying \( |x_\alpha - x| \leq y_\beta \) for all \( \alpha \geq \alpha_\beta \);
- \( r \)-converges to \( x \in X \) (shortly, \( x_\alpha \xrightarrow{r} x \)) if, for some \( u \in X_+ \), there exists a sequence \( \alpha_n \) of indexes such that \( |x_\alpha - x| \leq \frac{1}{n} u \) for all \( \alpha \geq \alpha_n \) (see, e.g. [16, 1.3.4, p.20]).

An operator \( T : X \to Y \) is called:

- \( \omega \)-bounded, if \( T \) takes order bounded sets to order bounded ones.
- \( r \)-regular, if \( T = T_1 - T_2 \) with \( T_1, T_2 \geq 0 \);
- \( \omega \)-continuous, if \( T x_\alpha \xrightarrow{\omega} 0 \) whenever \( x_\alpha \xrightarrow{\omega} 0 \);
- \( r \)-continuous, if \( T x_\alpha \xrightarrow{r} 0 \) whenever \( x_\alpha \xrightarrow{r} 0 \);

The set \( \mathcal{L}_b(X, Y) \) of \( \omega \)-bounded operators from \( X \) to \( Y \) is a vector space. Every regular operator is \( \omega \)-bounded. The set \( \mathcal{L}_r(X, Y) \) of all regular operators from \( X \) to \( Y \) is an ordered vector space with respect to the order: \( T \geq 0 \) if \( T x \geq 0 \) for all \( x \in X_+ \), we write \( \mathcal{L}_r(X) := \mathcal{L}_r(X, X), \mathcal{L}_b(X) = \mathcal{L}_b(X, X) \) etc. If \( Y \) is Dedekind complete then \( \mathcal{L}_b(X, Y) \) coincides with \( \mathcal{L}_r(X, Y) \) and is a Dedekind complete vector lattice [1 Thm.1.67] containing the set \( \mathcal{L}_n(X, Y) \) of all \( \omega \)-continuous operators from \( X \) to \( Y \) as a band [1 Thm.1.73]. It is clear that each positive and hence each regular operator is \( r \)-continuous.

Assume that vector lattices \( X \) and \( Y \) are equipped with linear convergences \( c_1 \) and \( c_2 \) respectively. An operator \( T : X \to Y \) is called
- \( c_1 c_2 \)-continuous (cf. [7 Def.1.4]), whenever \( x_\alpha \xrightarrow{c_1} 0 \) in \( X \) implies \( T x_\alpha \xrightarrow{c_2} 0 \) in \( Y \).

In the case when \( c_1 = c_2 \), we say that \( T \) is \( c_1 \)-continuous. The collection of all \( c_1 c_2 \)-continuous operators from \( X \) to \( Y \) is denoted by \( \mathcal{L}_{c_1 c_2}(X, Y) \), and if \( c_1 = c_2 \), we denote \( \mathcal{L}_{c_1}(X, Y) \) by \( \mathcal{L}_{c_1}(X, X) \).

A vector lattice \( X \) is called:

- \( l \)-algebra, if \( X \) is an associative algebra such that \( x \cdot y \in X_+ \) whenever \( x, y \in X_+ \).

An \( l \)-algebra \( E \) is called:

- \( d \)-algebra if \( u \cdot (x \land y) = (u \cdot x) \land (u \cdot y) \) and \( (x \land y) \cdot u = (x \cdot u) \land (y \cdot u) \) for all \( x, y \in E \) and \( u \in E_+ \);
- \( f \)-algebra if \( x \land y = 0 \) implies \( (u \cdot x) \land y = (x \cdot u) \land y = 0 \) for all \( u \in E_+ \);
- semiprime whenever the only nilpotent element in \( E \) is 0;
- unital if \( E \) has a positive multiplicative unit.

Any vector lattice \( X \) is a commutative \( f \)-algebra with respect to the trivial algebra multiplication \( x \ast y = 0 \) for all \( x, y \in X \).

Let \( \varepsilon \) be a linear convergence on \( E \) (see, [7 Def.1.6]). The algebra multiplication in \( E \) is called

- right \( \varepsilon \)-continuous (resp., left \( \varepsilon \)-continuous) if \( x_\alpha \xrightarrow{\varepsilon} x \) implies \( x_\alpha \cdot y \xrightarrow{\varepsilon} x \cdot y \) (resp., \( y \cdot x_\alpha \xrightarrow{\varepsilon} y \cdot x \)) every \( y \in E \) (cf. [7 Def.5.3]).

- The right \( \varepsilon \)-continuous algebra multiplication will be referred to as \( \varepsilon \)-continuous multiplication.

**Example 1.1.** Consider \( T, T_k \in \mathcal{L}_r(\ell^\infty) \) defined as follows: \( T x := l(x) \cdot 1_N \) and \( T_k x = x \cdot 1_{\{m \in \mathbb{N} : m \geq k\}} \) for all \( x \in \ell^\infty \) and all \( k \in \mathbb{N} \), where \( l \) is a positive extension to \( \ell^\infty \) of the functional \( l(x) = \lim_{n \to \infty} x_n \) on the space \( c \) of all
convergent real sequences. Clearly, $T_k \downarrow \geq 0$. If $T_k \geq S \geq 0$ in $\mathcal{L}_r(\ell^\infty)$ for all $k \in \mathbb{N}$ then, for every $p \in \mathbb{N}$

$$T_k e_p \geq S e_p \geq 0 \quad (\forall k \in \mathbb{N}),$$

where $e_p = \mathds{1}_{\{p\}} \in \ell^\infty$. Since $T_k e_p = 0$ for all $k > p$ then $S e_p = 0$ for all $p \in \mathbb{N}$. As $\ell^\infty = \ker(l) \oplus \mathbb{R} \cdot \mathds{1}_\mathbb{N}$, $S = s \cdot T$ for some $s \in \mathbb{R}_+$, and hence

$$T_2 \mathds{1}_\mathbb{N} = \mathds{1}_{\{m \in \mathbb{N} : m \geq 2\}} \geq s \cdot T \mathds{1}_\mathbb{N} = s \cdot \mathds{1}_\mathbb{N},$$

which implies $s = 0$, and hence $S = 0$. Thus, $T_k \downarrow 0$. However, the sequence $T \circ T_k = T$ does not $\sigma$-converge to 0, showing that the algebra multiplication in $\mathcal{L}_r(\ell^\infty)$ is not left $\sigma$-continuous. This also shows that, in unital $l$-algebras, $\sigma$-convergence can be properly weaker than $m\sigma$-convergence.

A net $x_\alpha$ in $E$ $m\sigma$-converges ($m\ell$-converges) to $x$ whenever

$$|x_\alpha - x| \cdot u \xrightarrow{\ell} 0 \quad (\text{respectively } u \cdot |x_\alpha - x| \xrightarrow{\ell} 0) \quad (\forall u \in E_+),$$

briefly $x_\alpha \xrightarrow{m\sigma} x$ and $x_\alpha \xrightarrow{m\ell} x$. In commutative algebras $m\sigma \equiv m\ell$. Since $m\ell$-convergence turns to $m\sigma$-convergence and vice versa, if we replace the algebra multiplication in $E$ by “$\cdot$”, defined as follows: $x \cdot y := y \cdot x$, we restrict ourselves to $m\sigma$-convergence, denoting it by $m\sigma$-convergence (cf. [4, 5, 7]).

Let $X$ be a Dedekind complete vector lattice. Then $\mathcal{L}_r(X)$ is an unital Dedekind complete $l$-algebra under the operator multiplication, containing $\mathcal{L}_n(X)$ as an $l$-subalgebra. The algebra multiplication is: right $m\sigma$-continuous in $\mathcal{L}_r(X)$; and is both left and right $m\sigma$-continuous in $\mathcal{L}_n(X)$ [6 Thm.2.1].

Example 1.2. (cf. [3 Ex.3.1]) Let $E$ be an $f$-algebra of all bounded real functions on $[0, 1]$ which differ from a constant on at most countable set of $[0, 1]$. Let $T : E \to E$ be an operator that assigns to each $f \in E$ the constant function $Tf$ on $[0, 1]$ such that the set $\{x \in [0, 1] : f(x) \neq (Tf)(x)\}$ is at most countable. Then $T$ is a rank one continuous in $\|\cdot\|_\infty$-norm positive operator. Consider the following net indexed by finite subsets of $[0, 1]$:

$$f_\alpha(x) = \begin{cases} 1 & \text{if } x \notin \alpha \\ 0 & \text{if } x \in \alpha \end{cases}$$

4
Then $f_\alpha \downarrow 0$ in $E$, yet $\|f_\alpha\|_\infty = 1$ for all $\alpha$. Thus, $T$ is neither $\text{omo}$- nor $\text{mo}$-continuous. However, $T$ is $\tau$-continuous and since $E$ is unital, $T$ is $\text{mr}$-continuous.

The structure of the paper is as follows. In Section 2, we introduce $\text{omc}$-compact operators from a vector lattice to an $l$-algebra and investigate their general properties with an emphasis on $\text{omo}$- and $\text{omr}$-cases. In Section 3, we investigate the domination problem for $\text{omc}$-compact operators; we define and study $\text{omo}$-$M$- and $\text{omo}$-$L$-weakly compact operators. For further unexplained terminology and notations, we refer to [1, 2, 6, 7, 9, 10, 15, 16, 17, 18, 19].

2 The properties of $\text{omc}$-compact operators

We begin with the following definition (cf. [6 Def.2.12]).

**Definition 2.1.** A subset $A$ of an $l$-algebra $E$ is called $\text{mr}_\circ$-bounded (resp., $\text{ml}_\circ$-bounded) if the set $A \cdot u$ (resp., $u \cdot A$) is order bounded for every $u \in E_+$. An operator $T$ from a vector lattice $X$ to an $l$-algebra $E$ is called $\text{mr}_\circ$-bounded (resp., $\text{ml}_\circ$-bounded) if $T$ maps order bounded subsets of $X$ into $\text{mr}_\circ$-bounded (resp., $\text{ml}_\circ$-bounded) subsets of $E$.

As usual, we restrict to $\text{mr}_\circ$-bounded subsets and operators, and refer to them as $\text{mo}$-bounded. In any $l$-algebra $E$ with trivial multiplication, $x * y = 0$ for all $x, y \in E$, each subset $A$ of $E$ is $\text{mo}$-bounded and as result, every operator from a vector lattice $X$ to such an $l$-algebra $E$ is $\text{mo}$-bounded. For elementary properties of $\text{mo}$-bounded operators in $l$-algebras, we refer the reader to the paper [6].

**Example 2.2.** (cf. [7 Ex.6]). Take a free ultrafilter $\mathcal{U}$ on natural numbers $\mathbb{N}$. Then a sequence $\lambda_n$ of reals converges along $\mathcal{U}$ to $\lambda$ whenever $\{k \in \mathbb{N} : |\lambda_k - \lambda| \leq \varepsilon\} \in \mathcal{U}$ for every $\varepsilon > 0$. Hence, for any element $x := (x_n)_{n=1}^\infty \in \ell^\infty$, the sequence $x_n$ converges along $\mathcal{U}$ to $x_\mathcal{U} := \lim_\mathcal{U} x_n$. In that case, one can define an $l$-algebra multiplication $*$ in $\ell^\infty$ by $x * y := (\lim_\mathcal{U} x_n) \cdot (\lim_\mathcal{U} y_n) \cdot 1$, where $1$ is a sequence of reals identically equal to $1$. It is easy to see that $(\ell^\infty, *)$ is a $d$-algebra. Then the set $A = \{k e_k : k \in \mathbb{N}\}$ is $\text{mo}$-bounded yet not $\sigma$-bounded.
Remark 2.3. Let $T$ be an operator from a vector lattice $X$ to an $l$-algebra $E$. Then

(i) If $T$ is $\circ$-bounded (in particular if $T$ is regular) then $T$ is $\mathfrak{m}_\circ$- and $\mathfrak{m}_r\circ$-bounded.

(ii) If $T$ is $\mathfrak{m}_\circ$- or $\mathfrak{m}_r\circ$-bounded operator and $E$ is unital $l$-algebra then $T$ is order bounded.

(iii) By [6, Thm.2.6], every $r$-continuous operator $T$ from an Archimedean vector lattice to an Archimedean $l$-algebra is $\mathfrak{r}_\circ\mathfrak{m}$-continuous and then, by [6, Thm.2.15], $T$ is $\mathfrak{m}_\circ$-bounded.

(iv) It follows from [1, Lem.1.4] that every order continuous operator is order bounded and hence $\mathfrak{m}_\circ$- and $\mathfrak{m}_r\circ$-bounded.

(v) Every $\mathfrak{m}_\circ$, $\mathfrak{om}_\circ$, or $\mathfrak{rm}_\circ$-continuous is $\mathfrak{m}_\circ\circ$-bounded and $\mathfrak{m}_r\circ$-bounded.

Moreover, every $\mathfrak{m}_\circ\circ$, $\mathfrak{om}_\circ\circ$, or $\mathfrak{rm}_\circ\circ$-continuous (resp., $\mathfrak{m}_r\circ$, $\mathfrak{om}_r\circ$, or $\mathfrak{rm}_r\circ$-continuous) is $\mathfrak{m}_\circ\circ$-bounded (resp., $\mathfrak{m}_r\circ$-bounded) [6, Thm.2.14].

The converse of Remark 2.3 (i) need not to be true in general. Indeed, in any $l$-algebra with trivial multiplication, every operator is $\mathfrak{m}_\circ$- and $\mathfrak{m}_r\circ$-bounded. A more interesting example is given below.

Example 2.4. Consider an operator $T$ from the vector lattice $X := c$ the set of all convergent real sequences to the $f$-algebra $E := c_0$ of real sequence converging to zero, defined by

$$T(x_1, x_2, x_3, \cdots) = (x, x - x_1, x - x_2, x - x_3, \cdots),$$

where $x = \lim_{n \to \infty} x_n$. Then $T$ is an $\mathfrak{m}_\circ$- and $\mathfrak{m}_r\circ$-bounded operator. However, it follows from $T(0, \cdots, 0, 1, 1, \cdots) = (1, \cdots, 1, 0, 0, \cdots)$ that $T([0, 1])$ is not order bounded in $E$, and so, $T$ is not order bounded.

The converse of Remark 2.3 (iv) need not to be true in general. To see this, we include the following example.
Example 2.5. (cf. [6, Ex.2.8]). Let \((\ell^\infty, \ast)\) be as in Example 2.2. Now, the identity operator \(I : (\ell^\infty, \ast) \to (\ell^\infty, \ast)\) is order bounded, but not \(\omega_0\)-continuous. Indeed, take the characteristic functions \(h_n = 1_{\{k \in \mathbb{N} : k \geq n\}} \in \ell^\infty\). Then \(h_n \omega_0 \to 0\) in \(\ell^\infty\) yet the sequence \(|I(h_n) - I(0)| \ast 1 = h_n \ast 1 = 1\) does not \(\omega\)-converge to 0. Thus, the sequence \(I(h_n)\) does not \(\omega_0\)-converge to 0 and hence \(I\) is not \(\omega_0\)-continuous.

Remind that an operator between normed spaces is called \textit{compact} if it maps the closed unit ball to a relatively compact set. Equivalently, the operator is compact if, for each norm bounded sequence, there exists a subsequence such that the image of it is convergent. Motivated by this, we introduce the following notions.

\textbf{Definition 2.6.} An operator \(T\) from a vector lattice \(X\) to an \(l\)-algebra \(E\) is called

\begin{enumerate}[(a)]
\item \(\omega_0\)-\textit{compact} (resp., \(\omega_1\)-\textit{compact}) if every order bounded net \(x_\alpha\) in \(X\) possesses a subnet \(x_{\alpha_\beta}\) such that \(Tx_{\alpha_\beta} \omega_0 \to y\) (resp., \(Tx_{\alpha_\beta} \omega_1 \to y\)) for some \(y \in E\);
\item \(\omega_0\)-\textit{compact} if \(T\) is both \(\omega_0\)- and \(\omega_1\)-compact;
\item \textit{sequentially} \(\omega_0\)-\textit{compact} (resp., \(\omega_1\)-\textit{compact}) if every order bounded sequence \(x_n\) in \(X\) possesses a subsequence \(x_{n_k}\) such that \(Tx_{n_k} \omega_0 \to y\) (resp., \(Tx_{n_k} \omega_1 \to y\)) for some \(y \in E\);
\item \textit{sequentially} \(\omega_0\)-\textit{compact} if \(T\) is both sequentially \(\omega_0\)- and \(\omega_1\)-compact.
\end{enumerate}

\textbf{Example 2.7.} Define an operator \(T : c_0 \to c_0\) by

\[ T \left( \sum_{k=1}^\infty a_k e_k \right) = \sum_{k=1}^\infty \frac{a_k}{k} e_k, \]

where \(e_k = 1_{\{n\}}\) and \(a_k\) is a real sequence converging to zero. Then \(T\) is compact on the \(f\)-algebra \((c_0, \| \cdot \|_\infty)\), and is \(\omega_0\)-compact.

In the next example, we show that there is an operator that is neither \(\omega_0\)-compact nor sequentially \(\omega_0\)-compact.
Example 2.8. The identity operator on the $l$-algebra $L_\infty[0,1]$ with pointwise multiplication is neither $\omega\omega$-compact nor sequentially $\omega\omega$-compact. Indeed, take the sequence of Rademacher function $r_n(t) = \text{sgn}(\sin(2^n\pi t))$ on $[0,1]$. Clearly, $r_n$ is order bounded. Now, assume that $r_n$ has a $\omega$-convergent subnet $r_{n\alpha}$, say $r_{n\alpha} \omega \rightarrow f$ for some $f \in L_\infty[0,1]$. Then $r_n \omega \rightarrow f$ and hence $r_n(t) \rightarrow f(t)$ almost everywhere violating that $r_n(t)$ diverges on $[0,1]$ except countably many points of form $\frac{k}{m}$ for $k, m \in \mathbb{N}$.

An $\omega\omega$-compact operator need not be sequentially $\omega\omega$-compact. To see this, we consider [11] Ex.7 for the next example.

Example 2.9. Consider the set $E := \mathbb{R}^X$ of all real-valued functions on $X$ equipped with the product topology, where $X$ is the set of all strictly increasing maps from $\mathbb{N}$ to $\mathbb{N}$. It follows from [17] Ex.3.10(i) that $E$ is a unital Dedekind complete $f$-algebra with respect to the pointwise operations and ordering.

(i) The identity map $I$ on $E$ is an $\omega\omega\omega$-compact operator. Indeed, assume that $f_\alpha$ is an order bounded net in $E$. It follows from [11] Ex.7(1)] that there exists a subnet $f_{\alpha\beta}$ such that $f_{\alpha\beta} \omega \rightarrow f$ for some $f \in E$. Since every $f$-algebra has $\omega$-continuity algebra multiplication, it follows from [7] Lm.5.5] that $f_{\alpha\beta} \omega \rightarrow f$. Therefore, $I$ is $\omega\omega\omega$-compact.

(ii) The identity map $I$ on $E$ is not sequentially $\omega\omega\omega$-compact. Consider a sequence $f_n$ in $\{-1,1\}^X$. Then $f_n$ is order bounded in $E$ and $f_n$ has no $\omega$-convergent subsequence [11] Ex.7(2)]. Thus, every subsequence does not $\omega\omega\omega$-converge because the $f$-algebra $E$ has a unit element.

Remark 2.10. It is known that any compact operator is norm continuous, but in general we may have a $\omega\omega\omega$-compact operator which is not $\omega\omega\omega$-continuous. Indeed, denote by $\mathcal{B}$ the Boolean algebra of the Borel subsets of $[0,1]$ equals up to measure null sets. Let $\mathcal{U}$ be any ultrafilter on $\mathcal{B}$. Then it can be shown that the linear operator $\varphi_\mathcal{U} : L_\infty[0,1] \rightarrow \mathbb{R}$ defined by

$$\varphi_\mathcal{U}(f) := \lim_{A \in \mathcal{U}} \frac{1}{\mu(A)} \int_A f \, d\mu$$

8
is $\omega^\mathfrak{m}$-compact (see [7, Lem. 5.5]) because the algebra multiplication in $\mathbb{R}$ is order continuous (cf. [15, 17]). However, it is not $\omega^\mathfrak{m}$-continuous.

In any $l$-algebra $E$, $x \geq y$ implies $x \cdot u \geq y \cdot u$ for all $u \in E_+$. But, in general, the inequality $x \cdot u \geq 0$ for all $u \in E_+$ does not imply $x \geq 0$.

**Definition 2.11.** An $l$-algebra $E$ is called right straight $l$-algebra (resp., left straight $l$-algebra) if $x \in E_+$ whenever $x \cdot u \in E_+$ (resp., $x \cdot u \in E_+$) for all $u \in E_+$. If an $l$-algebra $E$ is both left and right straight $l$-algebra, we say that $E$ is a straight $l$-algebra.

Clearly every unital $l$-algebra is straight. An algebra in [6, Ex. 2.8]) gives an example of a $d$-algebra which is not a straight $l$-algebra. The following theorem is an $\omega^\mathfrak{m}$-version of [11, Thm. 2].

**Theorem 2.12.** Let $T$ be an operator from a vector lattice $X$ to an $l$-algebra $E$. Then

(i) if $E$ is right straight $l$-algebra (resp., left straight $l$-algebra) and $T$ is $\omega^\mathfrak{m}$-compact (resp., $\omega^\mathfrak{m}$-compact) operator then it is order bounded;

(ii) if $T$ is $\omega^\mathfrak{m}$-compact (resp., $\omega^\mathfrak{m}$-compact) operator then it is $\mathfrak{m}$-bounded (resp., $\mathfrak{m}$-bounded) operator.

**Proof.** (i) Suppose that $T$ is an $\omega^\mathfrak{m}$-compact operator, but not order bounded. So, there is an order bounded subset $B$ of $X$ such that $T(B)$ is not order bounded in $E$. Hence, for every $y \in E_+$, there exists some $x_y \in B$ such that $|T(x_y)| \nless y$.

Since the net $(x_y)_{y \in E_+}$ is order bounded, there exists a subnet $(y_v)_{v \in \mathcal{V} \subseteq E_+} = (x_{\phi(v)})_{v \in \mathcal{V} \subseteq E_+}$ of $(x_y)_{y \in E_+}$ such that $T(y_v)$ is $\mathfrak{m}$-converges to some $z \in E$, i.e., for each positive element $w \in E_+$, $|T(y_v) - z| \cdot w \xrightarrow{\omega} 0$ because $T$ is $\omega^\mathfrak{m}$-compact operator. So, $|T(y_v) - z| \cdot w$ has an order bounded tail, which means that for an arbitrary positive element $w \in E_+$ there exist some indexes $v_0 \in \mathcal{V}$ and elements $e \in E_+$ such that

$$|T(y_v) - z| \cdot w \leq e$$
for each $v \geq v_0$. It follows from the inequality $|T(y_v)| \leq |T(y_v) - z| + |z|$ that we have $|T(y_v)| \cdot w \leq |T(y_v) - z| \cdot w + |z| \cdot w \leq e + |z| \cdot w$ for every $v \geq v_0$. Now, fix $t := e + |z| \cdot w \in E_+$. Then we have

$$|T(x_{\phi(v)})| \cdot w \leq t$$

for all $v \geq v_0$. Now, take an index $v_1 \in V \subset E_+$ so that $\phi(v) \cdot w \geq t$ holds for all $v \geq v_1$. Then, for any $v \geq v_0 \lor v_1$, we have $|T(x_{\phi(v)})| \nleq \phi(v)$, and so, $|T(x_{\phi(v)})| \cdot w \nleq \phi(v) \cdot w$ because $E$ is right straight $l$-algebra. Therefore, we have

$$|T(x_{\phi(v)})| \cdot w \nleq t,$$

which is a contradiction with (1). Therefore, we obtain the desired result.

(ii) The proof is a modification of the proof (i).

The idea in Theorem 2.12 need not to be true in the case of sequentially $\omega\sigma$-compactness. To see this, we consider [11, Lm.4 and Ex.6] for the following example.

Example 2.13. Let $F$ be the $l$-algebra of all bounded real-valued functions defined on the real line with countable support, and $E$ be the directed sum $\mathbb{R}1 \oplus F$, where $1$ denotes the constant function taking the value 1. Define an operator $T$ from $E$ to $F$ as a projection such that the range is $F$ and the kernel is $\mathbb{R}1$. Then $T$ is a sequentially $\omega\sigma$-compact, but not order bounded. Indeed, take an order bounded sequence $(f_n)$ in $E$. Then there exist some scalars $\lambda > 0$ such that $|f_n| \leq \lambda$ for all $n$. On the other hand, we can write $f_n = \beta_n + g_n$ with real numbers $\beta_n$ and functions $g_n$ in $F$. It follows from [11, Ex.6] that $(g_n)$ is order convergent in $F$. Then it is clear from [13, Thm.VIII.2.3] that $(g_n)$ is also $\omega\sigma$-convergent in $F$. On the other hand, it is clear that the image of the net $(1_x)_{x \in [0,1]}$ is not order bounded in $F$. Therefore, the operator $T$ is not order bounded.

Proposition 2.14. Let $E$ be an $l$-algebra and $R, T, S$ are operators on $E$.

(i) If $T$ is (sequentially) $\omega\sigma$-compact (resp., $\omega\sigma\omega$-compact) and $S$ is (sequentially) $\omega\sigma$-continuous (resp., $\omega\sigma\omega$-continuous) then the operator $S \circ T$ is (sequentially) $\omega\sigma$-compact (resp., $\omega\sigma\omega$-compact).
(ii) If $T$ is (sequentially) $\omega_m, o$-compact (resp., $\omega_m o$-compact) and $R$ is order bounded, then $T \circ R$ is (sequentially) $\omega_m, o$-compact (resp., $\omega_m o$-compact).

(iii) If $T$ is a positive, order continuous and $\omega_m, o$-compact (resp., $\omega_m o$-compact) operator, $F$ is right straight $l$-algebra (resp., left straight $l$-algebra), and $S_\alpha \downarrow 0$ is a decreasing net of order bounded operators then $T \circ S_\alpha \downarrow 0$ is a decreasing net of $\omega_m, o$-compact (resp., $\omega_m o$-compact) operators.

Proof. (i) Let $x_\alpha$ be an order bounded net in $E$. Since $T$ is $\omega_m, o$-compact, there are a subnet $x_{\alpha \beta}$ and $x \in E$ such that $T x_{\alpha \beta} \overset{m.o.}{\longrightarrow} x$. It follows from the $m.o.$-continuity of $S$ that $S(T x_{\alpha \beta}) \overset{m.o.}{\longrightarrow} S(x)$. Therefore, $S \circ T$ is $\omega_m, o$-compact.

(ii) Assume $x_\alpha$ to be an order bounded net in $E$. Since $R$ is order bounded, the net $R x_\alpha$ is order bounded. Now, the $\omega_m, o$-compactness of $T$ implies that there are a subnet $x_{\alpha \beta}$ and $z \in E$ such that $T R x_{\alpha \beta} \overset{m.o.}{\longrightarrow} z$. Therefore, $T \circ R$ is $\omega_m, o$-compact.

(iii) Let $S_\alpha \downarrow 0$ be a net of order bounded operators. Then it follows from Theorem 2.12 that $T \circ S_\alpha$ is an order bounded operator, and also, $\omega_m, o$-compact operator for each index $\alpha$ by (ii). Moreover, since $T \geq 0$, we have $T \circ S_\alpha \downarrow$. On the other hand, by [18, Thm.VIII.2.3], $T \circ S_\alpha \downarrow 0$ if and only if $T \circ S_\alpha x \downarrow 0$ for each $x \in E_+$. The result follows from $S_\alpha \downarrow 0$.

The sequential and $\omega_m o$-compact cases are analogous. □

Proposition 2.15. Every order continuous finite rank operator on an $l$-algebra $E$ with $o$-continuous multiplication is $\omega_o$-compact.

Proof. Let $T : E \rightarrow E$ be order continuous and $\dim(TE) < \infty$. Then

$$T = \sum_{k=1}^{m} x_k \otimes f_k$$

for $x_1, \ldots, x_m \in E$ and $f_1, \ldots, f_m \in E'_n$.

Without lost of generality, we may assume that $T = x_1 \otimes f_1$. Since $E'_n$ is Dedekind complete, $f_1$ is regular, and $T$ is also regular. Without lost of generality, suppose $x_1 \geq 0$ and $f_1 \geq 0$. Let $z_\alpha$ be an order bounded net in $E$. Then $T z_\alpha = (x_1 \otimes f_1)(z_\alpha) = f_1(z_\alpha) x_1$ is order bounded since every order
continuous functional is order bounded. Since \( \dim(TE) = 1 \), there exists a subnet \( z_{\alpha, \beta} \) such that \( Tz_{\alpha, \beta} \xrightarrow{m} y \in T(E) \). Using \( \dim(TE) = 1 \) again, we obtain \( Tz_{\alpha, \beta} \xrightarrow{m} y \). Therefore \( T \) is \( \omega \)-compact.

The following result is an extension of Example 2.7.

**Proposition 2.16.** Let \( E \) be an \( l \)-algebra with \( \phi \)-continuous algebra multiplication. Then the algebra \( \mathcal{L}_{rc}(E) \) of regular order compact operators is a subspace of \( \mathcal{L}_{romo}(E) \), which is itself an right algebraic ideal of \( \mathcal{L}_r(E) \).

**Proof.** Suppose that \( T \) is a regular order compact operator on a right \( \phi \)-continuous \( l \)-algebra \( E \), and \( x_\alpha \) is an order bounded net in \( E \). Then there exist a subnet \( x_{\alpha, \beta} \) and some \( y \in E \) such that \( Tx_{\alpha, \beta} \xrightarrow{m} y \). It follows from [7, Lm.5.5] that \( Tx_{\alpha, \beta} \xrightarrow{m} y \). Thus, we obtain that \( T \) is \( \omega \)-compact. As the proof of \( \omega \)-compactness is analogous, \( \mathcal{L}_{rc}(E) \) is subspace of \( \mathcal{L}_{romo}(E) \). On the other hand, it is well known that \( \mathcal{L}_r(E) \) is a subspace of \( \mathcal{L}_r(E) \). It follows from Theorem 2.14 (ii) that \( \mathcal{L}_{romo}(E) \) is an right algebraic ideal of \( \mathcal{L}_r(E) \).

## 3 Domination problem for compact operators

In this section, we study the domination problem for \( \omega \)-compact operators, and we introduce the \( \omega \)-\( M \) and \( \omega \)-\( L \)-weakly compact operators. Now, we consider the domination problem for positive \( \omega \)-\( (\omega \omega) \)-continuous and \( \omega \)-\( \bowtie \omega \)-compact operators. We have a positive answer for \( \omega \)-\( (\omega \omega) \)-continuous operators in the next lemma.

**Lemma 3.1.** Let \( T \) and \( S \) be positive operators between \( l \)-algebras \( E \) and \( F \) satisfying \( 0 \leq S \leq T \). If \( T \) is an \( \omega \)-\( \phi \)-continuous (resp., \( \omega \)-\( \bowtie \), \( \omega \)-\( \phi \)- and \( \omega \)-\( \bowtie \)\( \phi \)-continuous) operator imply then \( S \) has the same property.

**Proof.** Suppose that \( T \) is an \( \omega \)-\( \phi \)-continuous operator and \( x_\alpha \) is \( \omega \)-\( \phi \)-convergent to \( x \in E \). Then we have \( Tx_\alpha \xrightarrow{m} T x \) in \( F \). It follows from [2, Lem.1.6] that

\[
0 \leq |Sx_\alpha - Sx| \leq S(|x_\alpha - x|) \leq T(|x_\alpha - x|)
\]
holds for all $\alpha$ because $S$ is a positive operator. Hence, we get

$$|Sx_\alpha - Sx| \cdot u \leq T(|x_\alpha - x|) \cdot u$$

(2)

for all $u \in F_+$. On the other hand, it follows from [4 Prop.2.4] that $x_\alpha \overset{\text{m}_\alpha}\rightarrow x$ implies $|x_\alpha - x| \overset{\text{m}_\alpha}\rightarrow 0$, and so, we obtain $T(|x_\alpha - x|) \overset{\text{m}_\alpha}\rightarrow 0$ by the $\text{m}_\alpha$-continuity of $T$, i.e., $T(|x_\alpha - x|) \cdot u \overset{\delta}\rightarrow 0$ for all $u \in F_+$. Hence, the desired result raises from the inequality (2), $Sx_\alpha \overset{\text{m}_\alpha}\rightarrow Sx$ in $F$. The proof for the cases of $\text{m}_\alpha$-, $\text{om}_\alpha$-, and $\text{om}_\alpha$-continuity are similar.

Recall that a net $(x_\alpha)_{\alpha \in A}$ in an $l$-algebra is called $\text{om}_\alpha$-Cauchy if the net $(x_\alpha - x_{\alpha'})_{(\alpha, \alpha') \in A \times A}$ is $\text{om}_\alpha$-convergent to 0. Moreover, an $l$-algebra is called $\text{om}_\alpha$-complete if every $\text{om}_\alpha$-Cauchy net is $\text{om}_\alpha$-convergent; see [4 Def.2.11].

**Theorem 3.2.** Let $X$ be a vector lattice and $E$ be a Dedekind and sequentially $\text{om}_\alpha$-complete $l$-algebra with $\alpha$-continuous algebra multiplication. If $T_m : X \rightarrow E$ is a sequence of sequential $\text{om}_\alpha$-compact operators and $T_m \overset{\delta}_o \rightarrow T$ in $\mathcal{L}_b(X, E)$ then $T$ is sequentially $\text{om}_\alpha$-compact.

**Proof.** Let $x_n$ be an order bounded sequence in $X$, $T_m$ be a sequence of sequential $\text{om}_\alpha$-compact operators and $E$ be sequentially $\text{om}_\alpha$-complete. Then there is $w \in X_+$ such that $|x_n| \leq w$ for all $n \in N$. Also, by a standard diagonal argument, there exists a subsequence $x_{n_k}$ such that for any $m \in N$, $T_m x_{n_k} \overset{\text{m}_\alpha}\rightarrow y_m$ for some $y_m \in E$. Let’s show that $y_m$ is a $\text{om}_\alpha$-Cauchy sequence in $E$. Fix an arbitrary $u \in E_+$. Then we have

$$|y_m - y_j| \cdot u \leq |y_m - T_m x_{n_k}| \cdot u + |T_m x_{n_k} - T_j x_{n_k}| \cdot u + |T_j x_{n_k} - y_j| \cdot u.$$

Then the first and third terms in the last inequality both order converge to zero as $m \rightarrow \infty$ and $j \rightarrow \infty$, respectively. Since $T_m \overset{\delta}_o \rightarrow T$ in vector lattice $\mathcal{L}_b(X, E)$, we have $|T_m - T_j| \overset{\delta}_o \rightarrow 0$, and so, it follows from [18 Thm.VIII.2.3] that $|T_m - T_j|(x) \overset{\delta}_o \rightarrow 0$ for all $x \in X$. Then, by using [1 Thm.1.67(a)], we obtain the inequality

$$|T_m x_{n_k} - T_j x_{n_k}| \cdot u \leq |T_m - T_j|(x_{n_k}) \cdot u \leq |T_m - T_j|(w) \cdot u.$$

Since $E$ has $\alpha$-continuous algebra multiplication, it follows from [7 Lem.5.5] that $|T_m - T_j|(x) \overset{\delta}_o \rightarrow 0$ implies $|T_m - T_j|(w) \cdot u \overset{\delta}_o \rightarrow 0$. Hence, we obtain that
\[ |T_m x_{n_k} - T_j x_{n_k}| \cdot u \xrightarrow{\mathfrak{o}} 0. \] Therefore, \( y_m \) is an \( \mathfrak{o} \)-Cauchy. Now, by sequentially \( \mathfrak{o} \)-completeness of \( E \), there is \( y \in E \) such that \( y_m \xrightarrow{\mathfrak{o}} y \) in \( E \) as \( m \to \infty \).

Hence,
\[
|T x_{n_k} - y| \cdot u \leq |T x_{n_k} - T_m x_{n_k}| \cdot u + |T_m x_{n_k} - y_m| \cdot u + |y_m - y| \cdot u \\
\leq |T_m - T|(|x_{n_k}|) \cdot u + |T_m x_{n_k} - y_m| \cdot u + |y_m - y| \cdot u \\
\leq |T_m - T|(|w|) \cdot u + |T_m x_{n_k} - y_m| \cdot u + |y_m - y| \cdot u.
\]

Now, for fixed \( m \in \mathbb{N} \), and as \( k \to \infty \), we have
\[
\limsup_{k \to \infty} |T x_{n_k} - y| \cdot u \leq |T_m - T|(|w|) \cdot u + |y_m - y| \cdot u.
\]

But \( m \in \mathbb{N} \) is arbitrary, so \( \limsup_{k \to \infty} |T x_{n_k} - y| \cdot u = 0 \). Thus, \( |T x_{n_k} - y| \cdot u \xrightarrow{\mathfrak{o}} 0 \), i.e., \( T x_{n_k} \xrightarrow{\mathfrak{o}} y \). Therefore, \( T \) is sequentially \( \mathfrak{o} \)-compact.

The sequentially \( \mathfrak{o} \)-compact case is analogous. \( \square \)

In the rest of the section, we discuss \( \mathfrak{o} \)-\( M \)- and \( \mathfrak{o} \)-\( L \)-weakly compact operators. Remind that a norm bounded operator \( T \) from a normed lattice \( X \) into a normed space \( Y \) is called \( M \)-weakly compact if \( T x_n \xrightarrow{\|\|} 0 \) holds for every norm bounded disjoint sequence \( x_n \) in \( X \). Also, a norm bounded operator \( T \) from a normed space \( Y \) into a normed lattice \( X \) is called \( L \)-weakly compact whenever \( \lim ||x_n|| = 0 \) holds for every disjoint sequence \( x_n \) in the solid hull \( \text{sol}(T(B_Y)) := \{ x \in X : \exists y \in T(B_Y) \text{ with } |x| \leq |y| \} \) of \( T(B_Y) \), where \( B_Y \) is the closed unit ball of \( Y \). Similarly we have the following notions.

**Definition 3.3.** Let \( T : X \to E \) be a sequentially \( \mathfrak{o} \)-continuous operator.

1. If \( T x_n \xrightarrow{\mathfrak{o}} 0 \) for every order bounded disjoint sequence \( x_n \) in \( X \) then \( T \) is said to be \( \mathfrak{o} \)-\( M \)-weakly compact.

2. If \( y_n \xrightarrow{\mathfrak{o}} 0 \) for every disjoint sequence \( y_n \) in \( \text{sol}(T(A)) \), where \( A \) is any order bounded subset of \( X \), then \( T \) is said to be \( \mathfrak{o} \)-\( L \)-weakly compact.

**Proposition 3.4.** Let \( T \) be an order bounded \( \sigma \)-order continuous operator from a normed lattice \( X \) to an \( l \)-algebra \( E \) with \( \sigma \)-continuous algebra multiplication. Then \( T \) is \( \mathfrak{o} \)-\( M \)- and \( \mathfrak{o} \)-\( L \)-weakly compact.
Proof. Clearly, $T$ is sequentially $\omega\sigma$-continuous operator because $E$ has $\omega$-continuous algebra multiplication; see [6, Lem.5.5]. Let $x_n$ be an order bounded disjoint sequence in $X$. Then it follows from [9, Rem.10] that we get $x_n \overset{o}{\to} 0$. Thus, we have $Tx_n \overset{\omega o}{\to} 0$. Therefore, $T$ is $\omega\sigma\omega$-$M$-weakly compact.

Now, we show that $T$ is $\omega\sigma\omega$-$L$-weakly compact. Let $A$ be an order bounded set in $X$. Thus, $T(A)$ is order bounded, and so, $\text{sol}(T(A))$ is an order bounded set in $E$. Take an arbitrary disjoint sequence $y_n$ in $\text{sol}(T(A))$. Then, using [9, Rem.10], we have $y_n \overset{o}{\to} 0$, and so, $y_n \overset{\omega o}{\to} 0$ since $E$ has $\omega$-continuous algebra multiplication; see [6, Lem.5.5]. Thus, $T$ is $\omega\sigma\omega$-$L$-weakly compact.

Similarly to [3, Cor.2.3], we obtain the following result.

**Proposition 3.5.** Let $T, S : X \to E$ be two linear operators from a normed lattice $X$ to an $l$-algebra $E$ such that $0 \leq S \leq T$. If $T$ is $\omega\sigma\omega$-$M$- or $\omega\sigma\omega$-$L$-weakly compact then $S$ has the same property.

**Proof.** Suppose that $T$ is an $\omega\sigma\omega$-$M$-weakly compact operator. Thus, it follows from Lemma 3.1 that $S$ is an $\omega\sigma$-continuous operator. Let $x_\alpha$ be an order bounded disjoint net in $X$. So, $|x_n|$ is also order bounded and disjoint. Since $T$ is $\omega\sigma\omega$-$M$-weakly compact, $T(|x_n|) \overset{\omega o}{\to} 0$ in $E$. Following from the inequality

$$0 \leq |Sx_n| \cdot u \leq S(|x_n|) \cdot u \leq T(|x_n|) \cdot u$$

for all $n \in \mathbb{N}$ and for every $u \in E_+$ (cf. [2, Lem.1.6]), we get $Sx_n \overset{\omega o}{\to} 0$ in $E$. Thus, $S$ is $\omega\sigma\omega$-$M$-weakly compact.

Next, we show that $S$ is $\omega\sigma\omega$-$L$-weakly compact. Let $A$ be an order bounded subset of $X$. Put $|A| = \{|a| : a \in A\}$. Clearly, $\text{sol}(S(A)) \subseteq \text{sol}(S(|A|))$ and since $0 \leq S \leq T$, we have $\text{sol}(S(|A|)) \subseteq \text{sol}(T(|A|))$. Let $y_n$ be a disjoint sequence in $\text{sol}(S(|A|))$ then $y_n$ is in $\text{sol}(T(|A|))$ and, since $T$ is $\omega\sigma\omega$-$L$-weakly compact then $T(|x_n|) \overset{\omega o}{\to} 0$ in $E$. Therefore, by inequality (3), $S$ is $\omega\sigma\omega$-$L$-weakly compact. □

**Proposition 3.6.** If $T : X \to E$ is an $\omega\sigma\omega$-$L$-weakly compact lattice homomorphism then $T$ is $\omega\sigma\omega$-$M$-weakly compact.
Proof. Take an order bounded disjoint sequence $x_n$ in $X$. Since $T$ is lattice homomorphism, we have that $Tx_n$ is disjoint in $E$. Clearly $Tx_n \in \text{sol}(\{Tx_n : n \in \mathbb{N}\})$. By $\omega\omega-L$-weakly compactness of $T$, we have $Tx_n \overset{\text{me}}{\to} 0$ in $E$. Therefore, $T$ is $\omega\omega-M$-weakly compact. 

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