Thermodynamics of modified black holes from gravity’s rainbow

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We study the thermodynamics of modified black holes proposed in the context of gravity’s rainbow. A notion of intrinsic temperature and entropy for these black holes is introduced. In particular for a specific class of modified Schwarzschild solutions, their temperature and entropy are obtained and compared with those previously obtained from modified dispersion relations in deformed special relativity. It turns out that the results of these two different strategies coincide, and this may be viewed as a support for the proposal of deformed equivalence principle.

I. INTRODUCTION

Doubly special relativity is a deformed formalism of special relativity to preserve the relativity of inertial frames while at the same time keep Planck energy as an invariant scale, namely a universal constant for all inertial observers\textsuperscript{[1, 2, 3, 4, 5, 6]}. This can be accomplished by a non-linear Lorentz transformation in momentum space, which leads to a deformed Lorentz symmetry such that the usual energy-momentum relations or dispersion relations in special relativity may be modified with corrections in the order of Planck length. Modified dispersion relations (MDR) can also be derived in the study of semi-classical limit of loop quantum gravity\textsuperscript{[7, 8, 9, 10]}. Experimentally such modifications may be responsible for threshold anomalies of ultra high energy cosmic rays and TeV photons\textsuperscript{[11, 12, 13, 14, 15, 16]}.

Recently in\textsuperscript{[17]} Magueijo and Smolin argued that this formalism may be generalized to incorporate the curvature of spacetime. They proposed a deformed equivalence principle of general relativity, stating that the free falling observers who make measurements with energy $E$ will observe the same laws of physics as in modified special relativity. One important implication of this idea is that there is no single classical geometry of spacetime probed by a particle moving in it when the effects of the probe itself are taken into account. On the contrary, the spacetime is described by a one parameter family of metrics which may depend on the energy of this particle, forming a “rainbow” metric. Consequently the connection and curvature are energy dependent such that the usual Einstein’s equations is replaced by a one parameter family of equations. As specific examples the modified version of FRW solution and Schwarzschild solution to these equations have been presented in\textsuperscript{[17]} as well.

In this paper we intend to study the thermodynamics of modified black hole solutions, aiming to test the idea of deformed equivalence principle proposed above. As in\textsuperscript{[18]} we have investigated the impacts of modified dispersion relations on black holes, we find that the temperature as well as the entropy of black holes receive corrections due to the modification of energy-momentum relations of photons. Then the question is whether these results are consistent with the proposal of gravity’s rainbow. We present an affirmative answer to this question in this paper. Starting from the modified Schwarzschild black hole solutions, we firstly obtain an energy dependent temperature through the calculation of surface gravity, then we propose the notion of intrinsic temperature as well as intrinsic entropy for these black holes. Comparing these results with the ones obtained in\textsuperscript{[15, 16]} we argue that these quantities obtained through gravity’s rainbow and modified dispersion relations are exactly the same and this coincidence may be viewed as a support for the proposal of deformed equivalence principle.

II. RAINBOW METRIC AND MODIFIED BLACK HOLE SOLUTIONS

In this section we briefly review the rainbow metric proposed in\textsuperscript{[17]}. In the context of deformed or doubly special relativity\textsuperscript{[6]}, the invariant of energy and momentum in general may be modified as

$$E^2 f_1^2(E, \eta) - p^2 f_2^2(E, \eta) = m_0^2,$$

where $f_1$ and $f_2$ are two functions of energy from which a specific formulation of boost generator can be constructed and $\eta$ is a dimensionless parameter. The correspondence principle requires that $f_1$ and $f_2$ approach to unit as $E/M_p \ll 1$. Since these theories are typically formulated in momentum space and the transformation laws are no longer linear, the definition of a dual space or position space is non-trivial. One possible strategy is to require that the contraction between momenta and infinitesimal displacement be a linear invariant.

$$dx^\mu p_\mu = dt E + dx^i p_i.$$
As a result, the dual space is endowed with an energy dependent invariant which is called a rainbow metric
\[ ds^2 = -\frac{1}{f_1^2(E, \eta)} dt^2 + \frac{1}{f_2^2(E, \eta)} dx^2. \] (3)

The rainbow metrics lead to a one parameter family of connections and curvature tensors such that Einstein’s field equations are modified as
\[ G_{\mu\nu}(E) = 8\pi G(E)T_{\mu\nu}(E) + g_{\mu\nu}(E)\Lambda(E). \] (4)

Furthermore, in [17] the modified Schwarzschild solution to (4) has also been demonstrated in terms of energy dependent invariant which is called a rainbow metric
\[ G_{\mu\nu}(E) = 8\pi G(E)T_{\mu\nu}(E) + g_{\mu\nu}(E)\Lambda(E). \] (4)

From this solution we see that the position of horizon is the same for all observers as well. First we consider the temperature of modified black holes (5), namely \( T = E \) such that the horizon may be a function of \( f_2^2 \) so as to be energy dependent.

### III. THERMODYNAMICS OF MODIFIED BLACK HOLES

Now we turn to investigate the thermodynamics of modified Schwarzschild black holes. For explicitness, we adopt a modified dispersion relation (MDR) by taking \( f_1^2 = [1 - (\ell_p E)^2] \) and \( f_2^2 = 1 \), where Planck length \( \ell_p \equiv \sqrt{8\pi G} \equiv 1/M_p \). For this specific solution the area of the horizon is the same for all observers as well. First we consider the temperature of the modified black holes [15], which can usually be obtained by calculating the surface gravity \( \kappa \) on horizons, namely
\[ T = \frac{\kappa}{2\pi}, \] (6)

where \( \kappa \) is related to the metric by
\[ \kappa = -\frac{1}{2} \lim_{r \to R} \sqrt{-g^{11}(g^{00})'} g^{00} \left( g^{00} \right)'. \] (7)

Now for modified Schwarzschild black holes it is straightforward to obtain the surface gravity as
\[ \kappa = \frac{1}{\sqrt{1 - (\ell_p E)^2}} \frac{1}{4GM}. \] (8)

Therefore, the temperature is
\[ T = \frac{\kappa}{2\pi} = \frac{1}{\sqrt{1 - (\ell_p E)^2}} \frac{1}{8\pi GM}. \] (9)

which is also energy dependent.

Now we propose to define an intrinsic temperature for large modified black holes by taking the photons in the vicinity of black hole horizon as an ensemble. Suppose we make the measurement with the use of photons with average energy \( E = \langle E \rangle \), then we expect that the temperature of black holes can be identified with the energy of photons emitted from black holes [18] [20], namely \( T = E \) such that we have
\[ l_p^2 T^4 - T^2 + \frac{M^2}{M^2} = 0. \] (10)

Thus we obtain the intrinsic temperature of modified Schwarzschild black holes as
\[ T = \left[ \frac{M^2_e}{2} \left( 1 - \sqrt{1 - \frac{4M^2_e}{M^2}} \right) \right]^{1/2}. \] (11)

It requires that the mass of black holes \( M \geq 2M_p \), and correspondingly the temperature \( T \leq M_p/\sqrt{2} \). For large black holes with \( M \gg 2M_p \), it goes back to the ordinary form,
\[ T = \frac{M^2_e}{M}. \] (12)

The intrinsic temperature [11] is exactly the same as the one we have obtained in [18] through the modified dispersion relation and uncertainty principle. The derivation there can be summarized as follows. Given \( f_1^2 = [1 - (\ell_p E)^2] \) and \( f_2^2 = 1 \), the corresponding modified dispersion relation reads as
\[ l_p^2 E^4 - E^2 + (\mu^2 + m_0^2) = 0, \] (13)

thus the energy can be non-perturbatively solved as
\[ E^2 = \frac{1}{2l_p^2} \left[ 1 - \sqrt{1 - 4l_p^2(\mu^2 + m_0^2)} \right]. \] (14)

Applying this relation to the photons emitted from black holes and identifying the characteristic temperature of this black hole with the photon energy \( E \), we may have
\[ T = \left[ \frac{M^2_e}{2} \left( 1 - \sqrt{1 - 4l_p^2(\mu^2 + m_0^2)} \right) \right]^{1/2}. \] (15)

Moreover, we apply the ordinary uncertainty relation to photons in the vicinity of black hole horizons [18] [20],
\[ p \sim \delta p \sim \frac{1}{\delta x} \sim \frac{1}{4\pi R}, \] (16)

where a “calibration factor” \( 4\pi \) is introduced. Plugging this relation into (15) we easily find the temperature of Schwarzschild black holes is the same form as (11).

From (14) it is very interesting to notice that a single particle has a maximum energy \( E_{\max} = M_p/\sqrt{2} \), rather than \( M_p \). This is important as it also provides a cutoff for the temperature of modified black holes possibly probed by any particle with energy \( E \), which is [20]. From this equation we notice that for a modified black hole with
fixed mass $M$ there exists a maximum but finite value possibly probed any particle, which is $T_{max} = \sqrt{2M^2/g}$, rather than a divergent number as one intuitively requires the energy of particles approaches the Planck mass $M_p$.

Next assuming the first thermodynamical law holds for modified black holes, namely $dM = TdS$, we may obtain the intrinsic entropy by plugging the intrinsic temperature \( t \) into this relation and then taking integrations. The result reads as

\[
S = \frac{1}{\sqrt{2}} \left[ \frac{A}{8G} (1 + t)^{3/2} - \frac{1}{\sqrt{2}} \ln \left( \frac{A}{8G} (1 + t) \right) \right],
\]

where $t = \sqrt{1 - 8G/A}$. When $A \gg 8G$, it becomes

\[
S = \frac{A}{4G} - \frac{1}{2} \ln \frac{A}{4G} + ...
\]

Therefore, we find the total entropy of modified black holes contains a leading term which is nothing but the familiar Bekenstein-Hawking entropy and a logarithmic correction term.

**IV. CONCLUSIONS**

In this paper we have considered the thermodynamics of modified black holes in the formulation of deformed general relativity. Taking an specific rainbow metric as example, we find that the temperature of black holes probed by particles is energy dependent. But still we may define an intrinsic temperature as well as entropy for such black holes by considering the ensemble composed of photons in the vicinity of horizons. In contrast to the ordinary ones, the mass of modified black holes is bounded from below such that the temperature will reach a maximum but finite value with the evaporation of black holes. In particular they will cease radiation as the size of black holes approaches the Planck scale, providing a mechanism to treat black hole remnants as a candidate for cold dark matter. The results obtained in this paper are consistent with those obtained through the modified dispersion relations, and this coincidence supports the deformed equivalence principle proposed in [17].

We point out that the scheme presented here can be generalized to other sorts of modified black hole solutions and their implications to cosmology are under investigation and will be discussed elsewhere.

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[1] G. Amelino-Camelia, Phys. Lett. B 510, 255 (2001) arXiv:hep-th/0012235.
[2] G. Amelino-Camelia, Int. J. Mod. Phys. D 11, 35 (2002) arXiv:gr-qc/0012051.
[3] G. Amelino-Camelia, J. Kowalski-Glikman, G. Man-danici and A. Procaccini, “Phenomenology of doubly special relativity,” arXiv:gr-qc/0312121.
[4] G. Amelino-Camelia, “The three perspectives on the quantum-gravity problem and their implications for the fate of Lorentz symmetry,” arXiv:gr-qc/0309054.
[5] J. Magueijo and L. Smolin, Phys. Rev. Lett. 88, 190403 (2002) arXiv:hep-th/0112090.
[6] J. Magueijo and L. Smolin, Phys. Rev. D 67, 044017 (2003) arXiv:gr-qc/0207085.
[7] R. Gambini and J. Pullin, Phys. Rev. D 59, 124021 (1999) arXiv:gr-qc/9806038.
[8] L. Smolin, [arXiv:hep-th/0290797].
[9] H. Sahlmann and T. Thiemann, [arXiv:gr-qc/0207031].
[10] L. Smolin, “Falsifiable predictions from semiclassical quantum gravity,” arXiv:hep-th/0501091.
[11] D. Colladay and V. A. Kostelecky, Phys. Rev. D 58, 116002 (1998) arXiv:hep-ph/9809521.
[12] S. R. Coleman and S. L. Glashow, Phys. Rev. D 59, 116008 (1999) arXiv:hep-ph/9812411.
[13] G. Amelino-Camelia and T. Piran, Phys. Rev. D 64, 036005 (2001) arXiv:astro-ph/0008107.
[14] T. Jacobson, S. Liberati and D. Mattingly, Phys. Rev. D 66, 081302 (2002) arXiv:hep-th/0112207.
[15] R. C. Myers and M. Pospelov, Phys. Rev. Lett. 90, 211601 (2003) arXiv:hep-th/0301124.
[16] T. A. Jacobson, S. Liberati, D. Mattingly and F. W. Stecker, Phys. Rev. Lett. 93, 021101 (2004) arXiv:astro-ph/0309681.
[17] J. Magueijo and L. Smolin, Class. Quant. Grav. 21, 1725 (2004) arXiv:gr-qc/0305053.
[18] Y. Ling, B. Hu and X. Li, “Modified dispersion relations and black hole physics”, arXiv:gr-qc/0512083.
[19] R. J. Adler, P. Chen and D. I. Santiago, Gen. Rel. Grav. 33, 2101 (2001) arXiv:gr-qc/0106080.
[20] P. Chen and R. J. Adler, Nucl. Phys. Proc. Suppl. 124, 103 (2003) arXiv:gr-qc/0205106.
[21] J.H.MacGibbon, Nature 329, 308 (1987). J. D. Barrow, E. J. Copeland and A. R. Liddle, Phys. Rev. D 46, 645 (1992). B. J. Carr, J. H. Gilbert and J. E. Lidsey, Phys. Rev. D 50, 4853 (1994) arXiv:astro-ph/9405027.