WEAK-PAINLEVÉ PROPERTY AND INTEGRABILITY OF GENERAL DYNAMICAL SYSTEMS

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Abstract. The purpose of this paper is to investigate the connection between singular property and integrability for general dynamical systems. We will firstly present some methods to test the Painlevé property and weak-Painlevé property, then we will show the equivalence between the weak-Painlevé property and certain formal integrability for general dynamical systems.

1. Introduction. Investigating the singular structure of general solutions and the integrability are two of the basic and important approaches to studying the solvability of general dynamical systems. Since these two theories were established, to find the connection between them has naturally become a more and more interesting and important topic in the field of differential equations. Maybe the pioneer work can be traced to Kowalevskaya’s study of the Heavy top problem, i.e., the motion under gravity of a rigid body about a fixed point [13], the remarkable insight of Kowalevskaya is that the concerned problem can be solved explicitly whenever the parameters are taken such that the dependent variables are meromorphic with respect to the time $t$ in complex plane, and it is now well known that the solvable cases she got are the very four integrable cases of the problem [21]: isotropic, Euler, Lagrange and Kowalevskaya. The idea of Kowalevskaya was the first trying to detect the regular property of dynamical systems by study the singular structure of their solutions. So far, one of the famous results about singular analysis is the so called Painlevé analysis which means, in general, the procedure to obtain necessary conditions for Painlevé property of given system. Generally speaking, here a dynamical system has Painlevé property if its general solution has no movable critical

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singular points on complex plane, i.e., the general solution is single-valued in the complex plain. Since 80th of the last century, there have been numerous references and reports for successful developments and applications of Painlevé analysis, such as [1, 9, 10, 11, 15, 31, 32, 33, 34, 35] and so on.

About several years after the Kovalevskaya’s work, Liapounov proved the only cases of the family of Heavy top problem with general solution uniform over the whole complex time plane are the four integrable cases [21], in this way Liapounov improved the Kovalevskaya’s result, for the first time, he considered variational equation of the concerned system, with respect to the initial conditions, in complex time along a suitable particular solution. In 1982, Ziglin [44] got a non-integrability result for complex analytical Hamiltonian systems by using the constrains imposed by the existence of some first integrals on the monodromy group of the normal variational equation along some complex integral curve. This idea was later developed by Morales-Ruiz, Ramis, Simó [24, 26, 25], and Baider, Churchill, Rod and Singer [3] in the end of last century. By using the differential Galois theory and investigating the relationship between the integrability and the structure of the identity component of the differential Galois group of the variational equation along some complex integral curve, they obtained a stronger integrability condition for analytic Hamiltonian systems, the corresponding method and results are usually called Morales-Ramis theory. So far, many developments and applications of both the Ziglin theory and Morales-Ramis theory have been obtained in numerous references, such as [18, 20, 22, 25, 26, 27, 29, 30, 37, 38, 43, 44] and so on. In recent, Nguyen Tien Zung [2] and the authors [19] also considered the similar problem for general dynamical systems, and obtained some similar results as Morales-Ramis theory.

There had been an interesting phenomenon from some early work about Painlevé property and integrability, that is many concerned systems passing the Painlevé test always turned to be integrable. Here a dynamical systems is integrable if it has sufficient number of first integrals such that the general solution can be solved by quadrature of known functions. This interesting phenomenon had led an open problem for a long time: is a given system possessing Painlevé property integrable? further, does any integrable system have Painlevé property? Unfortunately, the answers to both questions are “No”. A simple counter example for the later one can be found in [33]; and by using differential Galois theory, Morales-Ruiz [28] proved that Painlevé II equation with a large number of parameters is not rationally integrable (in Liouville sense). Then an natural question should be answered: what is the exact connection between the integrability and the singular property for general dynamical systems? One of the important results is is due to Yoshida [41, 42], by considering the connection between the integrability of dynamical systems and corresponding Kowalveskaya exponents with respect to a balance, he showed that if a similarity invariant system is algebraically integrable, then the eigenvalues of corresponding Kowalveskaya matrix are all rational. Following Yoshida’s work, many results have been obtained about the connections between the integrability, partial integrability, non-integrability of dynamical systems and the singular property, see [8, 16] for example.

From above observations, one can find that it is hard to investigate the relationship between the Painlevé property and integrability in general, while there are some close connection between the singular property and integrability indeed.
These facts lead us to consider a “weak form” of above relationship, i.e., the connection between the weak-Painlevé property (a weak form of Painlevé property instead of the classical Painlevé property) and certain formal integrability of general analytic dynamical systems. Here a dynamical system is said to have weak-Painlevé property if its general solution can be expanded in Puiseux series, i.e., its general solution can be expanded as a Laurent series with respect to \((t - t_*)^\frac{1}{l}\), for some \(l \in \mathbb{Z}^+\) around the movable singularity \(t_*\).

The structure of this paper is as follows. In section 2, we briefly review some preliminary notions and results of classical Painlevé analysis (for polynomial systems, in fact). Then we will present a Painlevé analysis for general analytic systems in section 3. In last section, we will turn to consider weak-Painlevé property for general systems, further, we will show that weak-Painlevé property is indeed equivalent to certain formal integrability.

2. Classical Painlevé analysis. Before we start our main topics, let us firstly recall some basic notions and results about the classical Painlevé analysis for polynomial systems, we follows [8], for more references see [17, 36] and so on.

Consider a polynomial system of differential equation

\[
\dot{x} = f(x),
\]

where \(x \in \mathbb{C}^n, t \in \mathbb{C}\) and each component of \(f\) is polynomial in \(x\). Let

\[
t = \alpha^{p_1}T + \alpha^{p_2}a, \quad x = \alpha^{q_1}X + \alpha^{q_2}b,
\]

where \(p_1, p_2, q_1, q_2 \in \mathbb{Z}\), \(\alpha\) is a parameter, \(a\) and all components of \(b\) are constants. By a convenient choice of \(p_1, p_2, q_1, q_2\), system (1) is changed to

\[
\dot{X} = F(X; T; \alpha).
\]

Expand the general solution \(X\) of (2) as follows:

\[
X(T) = \sum_{i=0}^{\infty} X^i(T)\alpha^i.
\]

The following result due to Painlevé [7, 15] is crucial for Painlevé analysis.

**Lemma 1. (Painlevé’s \(\alpha\)-lemma)** Let \(F(X; T; \alpha)\) be analytic in a path-connected domain \((X, T, \alpha) \in D_X \times D_T \times D_\alpha \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}\). For given initial condition \((X(T_0), T_0) \in D_X \times D_T\) and \(\epsilon \in \mathbb{R}_0^+\) sufficiently small, the general solution of (1) is single-valued for all \(\alpha\) such that \(|\alpha| < \epsilon\) if and only if \(X(T)\) is single-valued for all \(i\) and \(X^i\) defined in (3).

Assume that there exists a decomposition

\[
f = f^{(0)} + f^{(1)} + \cdots + f^{(m)}
\]

with \(m + 1\) weight-homogeneous components such that the dominant part of vector field \(f^{(0)}\) is a scale system, i.e., the system

\[
\dot{x} = f^{(0)}(x)
\]

has the exact scale-invariant solution

\[
x^{(0)} = \xi \tau^p,
\]
where $\tau = t - t_*, p \in \mathbb{Q}^n, \xi \in \mathbb{C}^n, |\xi| \neq 0, t_*$ is an arbitrary movable singularity. Then, there exist $q^{(1)}, \ldots, q^{(m)} \in \mathbb{Q}$ with $0 < q^{(i)} < q^{(j)}, \forall i < j$, such that

$$f_i^{(j)}(\mu p x) = \mu^{p_i + q^{(j)} - 1} f_i^{(j)}(x), \; i = 1, \ldots, n, \; j = 1, \ldots, m, \; \forall \mu \in \mathbb{C}.$$ 

Here, we call the pair $(\xi, p)$ a dominator balance (or balance) for the vector field (1), and denote all balances satisfying above conditions by $\mathcal{F}$. For a given balance $(\xi, p) \in \mathcal{F}$, rewrite the general solution of system (5) around $\xi \tau^p$ as $\mathbf{x} = (\xi + \mathbf{u}) \tau^p$, then we have

$$\tau \dot{\mathbf{u}} = K \mathbf{u} + \tilde{f}^{(0)}(\mathbf{u}), \; \mathbf{u} \in \mathbb{C}^n,$n

where $K = Df^{(0)}(\xi) - \text{diag}(p)$ is called Kovalevskaya matrix, whose eigenvalues are called Kovalevskaya exponents, $\tilde{f}^{(0)}(\mathbf{u}) = f^{(0)}(\mathbf{u} + \xi) - Df^{(0)}(\xi) \mathbf{u}$. Denote by $\lambda_1, \ldots, \lambda_n$ the Kovalevskaya exponents, it is well-known that one of them equals $-1$, without loss of generality, we assume $\lambda_n = -1$.

**Theorem 2.1. (Painlevé’s test #1 [8])** If system (1) has Painlevé property. Then, for any balance $(\xi, p) \in \mathcal{F}$, all the dominant exponents $p_1, \ldots, p_n$ are integers, the Kovalevskaya matrix $K$ is semi-simple and Kovalevskaya exponents $\lambda_1, \ldots, \lambda_n$ are integers.

Now, in order to obtain more information about the Painlevé property, one can make the change of variables

$$\mathbf{x} = \tau^p (\xi + \mathbf{X}(\tau)),$n

then system (1) becomes

$$\tau \dot{\mathbf{X}} = K \mathbf{X} + \tilde{f}(\tau, \mathbf{X}), \; \mathbf{X} \in \mathbb{C}^n,$n

where

$$\tilde{f}(\tau, \mathbf{X}) = \tilde{f}^{(0)}(\mathbf{X}) + \sum_{i=1}^m \tau^{q_i} f^{(i)}(\mathbf{X} + \xi).$$n

Further let $\frac{1}{q}$ be the least common denominator of $q_1, \ldots, q_m, \tau = e^s, X_{n+1} = e^{qs}$, we have

$$\dot{\mathbf{X}}' = \tilde{K} \mathbf{X} + \tilde{f}(\mathbf{X}), \; \mathbf{X} \in \mathbb{C}^{n+1},$$n

where $\mathbf{X} = (X_1, \ldots, X_n, X_{n+1})$, $'$ is the differentiation with respect to $s$,

$$\tilde{K} = \begin{pmatrix} K & \delta f^{(1)}(\xi) \\ 0 & q \end{pmatrix}, \; \delta = \begin{cases} 1, & \text{if } \frac{q_i}{q} = 1; \\ 0, & \text{if } \frac{q_i}{q} > 1. \end{cases}$$n

$$\tilde{f}_j(\mathbf{X}) = \tilde{f}_j^{(0)}(\mathbf{X}) + (1 - \delta) X_n^{q_i} f_j^{(1)}(\mathbf{X} + \xi) + \sum_{i=2}^m X_i^{q_i} f_j^{(i)}(\mathbf{X} + \xi), \; j = 1, \ldots, n,$n

$$\tilde{f}_{n+1} = 0,$n

By simple calculation, one can find that the eigenvalues of $\tilde{K}$ are $\Lambda = \{\lambda_1, \ldots, \lambda_n, \lambda_{n+1}\}$ with $\lambda_{n+1} = q$, and $\tilde{f}(\mathbf{X}) = O(|\mathbf{X}|^2)$. Furthermore, we can obtain the local formal solution of system (8) around the original point as the form

$$\dot{\mathbf{X}} = \sum_{i \in \mathbb{N}^{n+1}, |i| = 1} C_i s^i e^{<\Lambda, i>*s},$$n

where $\Lambda = \{\lambda_1, \ldots, \lambda_n, \lambda_{n+1}\}$. For any balance $(\xi, p)$ \textup{cfr.} subsections 2.2 and 3.2 for more details.
where $C_i(s)$ are polynomial valued vectors with respect to $s$, $a = (a_1, \cdots, a_n, a_{n+1})$ are free parameters, $a^i = a_1^i a_2^i \cdots a_{n+1}^i$, and $(\Lambda, i) = \sum_{j=1}^{n+1} \lambda_j i_j$. Then, one can conclude easily that the local formal solution of system (1) around the singularity $t_*$ can be represented as

$$x = \tau^p(\xi + \sum_{i \in \mathbb{N}^{n+1}, |i|=1} \mathcal{C}_i(\log \tau) a^i \tau^{<(\Lambda, i)>}),$$

where $a_{n+1} = 1$, $\mathcal{C}_i(s)$ is the $n$-dimensional vector obtained by the first $n$ components of $C_i(s)$.

**Theorem 2.2.** (Painlevé’s test #2, [3][8]) If system (1) has Painlevé property. Then, for any balance $(\xi, p) \in \mathcal{F}$, all the dominant exponents $p_1, \cdots, p_n$ are integers, the Kovalevskaya matrix $K$ is semi-simple and Kovalevskaya exponents $\lambda_1, \cdots, \lambda_n$ are integers, and the local solution can be represented as formal Laurent series

$$x = \tau^p(\xi + \sum_{i=-\infty}^{+\infty} B_i \tau^i)$$

with $n-1$ arbitrary constants, where $B_i$ are constant valued vectors. Furthermore, system (1) has local Laurent solution

$$x = \tau^p(\xi + \sum_{i=1}^{+\infty} B_i \tau^i)$$

with as many arbitrary constants as the number of positive Kovalevskaya exponents.

**Remark 1.** A large number of results ([9, 10, 11] etc.) have been obtained manifest that the Painlevé analysis given above [8] is powerful in the study of differential equations. However, it also has some limitations and insufficiencies, in fact, one can find that the above methods can be only used to analyze polynomial systems, but not suitable for general analytic systems.

### 3. Painlevé analysis for analytic systems.

In this section, we develop a new Painlevé analysis for analytic system

$$\dot{x} = f(x),$$

where $x \in \mathbb{C}^n$, $t \in \mathbb{C}$ and $f$ is analytic in $x$. The method we will use looks alike the classical one for the polynomial systems referred in above section, however, there exist some clearly distinctions as we propose below.

Let us decompose the vector field $f(x)$ as

$$f = f^{(0)} + f^{(1)} + \cdots + f^{(m)} + \cdots,$$

such that

$$\dot{x} = f^{(0)}(x)$$

has an exact scale-invariant solution

$$x^{(0)} = \xi \tau^p,$$

where $\tau = t - t_*$, $p \in \mathbb{Q}^n$, $\xi \in \mathbb{C}^n$, $|\xi| \neq 0$, $t_*$ is an arbitrary movable singularity. Furthermore, there exist $q^{(m)} \in \mathbb{Q}, m = 1, \cdots$, with $q^{(j)} < q^{(i)} < 0, \forall i < j$, such that

$$f^{(j)}(\mu^p x) = \mu^{p_{j}+q^{(j)}-1} f^{(j)}(x), \quad i = 1, \cdots, n, \quad j = 1, \cdots, m, \cdots, \forall \mu \in \mathbb{C}.$$
Here, we also call the pair \((\xi, p)\) a **dominator balance** (or balance) for the vector field \((10)\), and we denote all balances satisfying above conditions by \(\tilde{F}\).

For a given balance \((\xi, p)\in \tilde{F}\), rewrite the general solution of system \((11)\) around \(\xi \tau^p\) as \(x = (\xi + u) \tau^p\), then we also have

\[
\tau \dot{u} = Ku + \tilde{f}^{(0)}(u), \quad u \in \mathbb{C}^n
\]  

(12)

with Kovalevskaya matrix \(K = D\tilde{f}^{(0)}(\xi) - \text{diag}(p)\), Kovalevskaya exponents \(\lambda_1, \cdots, \lambda_n\), and \(\tilde{f}^{(0)}(u) = f^{(0)}(u + \xi) - f^{(0)}(\xi) - D\tilde{f}^{(0)}(\xi)u = O(|u|^2)\), without loss of generality, we also assume \(\lambda_n = -1\).

**Lemma 2.** If system \((10)\) has Painlevé property. Then, for any balance \((\xi, p)\in \tilde{F}\), all the components of \(p\) are integers, the Kovalevskaya matrix \(K\) is semi-simple and Kovalevskaya exponents \(\lambda_1, \cdots, \lambda_n\) are integers.

**Proof.** Make the scale transformation \(x \to e^{-p}x, t \to e^{-1}t\) to system \((10)\), we get

\[
\dot{x} = f(x, \epsilon) = \sum_{m=0}^{\infty} \epsilon^{-q(m)} f^{(m)}(x),
\]  

(13)

where \(q^{(0)} = 0\) and \(-q(m) \in \mathbb{Q}^+, \forall m > 0\). Then if system \((10)\) has Painlevé property, system \((13)\) has it too, that is to say, the general solution of \((13)\) is single-valued. Expand it as a series with respect to \(\epsilon\)

\[
x(t, \epsilon) = \sum_{m=0}^{\infty} \epsilon^m x^{(m)},
\]

where \(x^{(0)}\) is a solution of \((11)\). From Painlevé’s \(\alpha\)-lemma, we know that \(x^{(0)}\) is single-valued. Then it is consequence that, system \((11)\) has Painlevé property.

Since \(x^{(0)} = \xi \tau^p\) is an exact scale-invariant solution of system \((11)\), it must be single-valued, hence, we get that each component of \(p\) must be integer, i.e., \(p \in \mathbb{Z}^n\). Then, we conclude that system \((12)\) also has Painlevé property.

Expand the general solution of \((12)\) as a series with respect to \(\epsilon\)

\[
u(t, \epsilon) = \sum_{i=0}^{\infty} \epsilon^i u^{(i)},
\]

then \(u^{(0)}\) is a solution of the homogeneous linear system \(\tau \dot{u} = Ku\). From Painlevé’s \(\alpha\)-lemma, we get that \(\tau K\) must be single-valued, which leads to that Kovalevskaya matrix \(K\) must be semi-simple and Kovalevskaya exponents are integers.

Now, let us introduce the change of variables

\[x = \tau^p(\xi + X(\tau)),\]

then system \((10)\) becomes

\[\tau \dot{X} = KX + \tilde{f}(\tau, X), \quad X \in \mathbb{C}^n,\]

(14)

where

\[\tilde{f}(\tau, X) = \tilde{f}^{(0)}(X) + \sum_{i=1}^{\infty} \tau^{q(m)} f^{(m)}(X + \xi)\]

Assume that there exists the least common denominator of \(q_1, \cdots, q_m, \cdots\), denoted by \(\frac{1}{q}\). Let

\[\tau = e^{-s}, X_{n+1} = e^{qs}.$
Then we have
\[ \vec{X}' = \vec{K} \vec{X} + \vec{f}(\vec{X}), \quad \vec{X} \in \mathbb{C}^{n+1}, \]  
(15)
where \( \vec{X} = (X_1, \ldots, X_n, X_{n+1}) \), \( ' \) is differentiation with respect to \( s \),
\[ \vec{K} = \begin{pmatrix} -K & -\delta f^{(1)}(\xi) \\ 0 & q \end{pmatrix}, \quad \delta = \begin{cases} 1, & \text{if } \frac{m}{q} = -1; \\ 0, & \text{if } \frac{m}{q} < -1. \end{cases} \]

\[ -f_j(\vec{X}) = f_j^{(0)}(\vec{X}) + (1 - \delta)X_{n+1} f_j^{(1)}(\vec{X} + \xi) + \sum_{m=2}^{\infty} X_{n+1}^{m} f_j^{(m)}(\vec{X} + \xi), \]
\[ \tilde{f}_{n+1} = 0, \]
with \( j = 1, \ldots, n \). One can note that the eigenvalues of \( \vec{K} \) are \( \tilde{\Lambda} = \{-\lambda_1, \ldots, -\lambda_n, \lambda_{n+1}\} \) with \( \lambda_{n+1} = q \), and \( \vec{f}(\vec{X}) = O(|\vec{X}|^2) \).

**Remark 2.** We remark that, in general, for given balance \( (\xi, p) \in \tilde{\mathcal{F}} \), there may not exist the least common denominator \( \frac{m}{q} \) of \( q_1, \ldots, q_m, \ldots \). However, it is easy to note the fact that, for any analytic system without linear terms, there exists at least one balance, such that the least common denominator of corresponding \( q_1, \ldots, q_m, \ldots \) exists.

**Lemma 3.** The local formal solution of system (15) around the original point can be represented as
\[ \vec{X} = \sum_{i \in \mathbb{N}^{n+1}, |i|=1}^{\infty} C_i(s) a^i e^{(\tilde{\Lambda}, \tilde{\xi})s}, \]  
(16)
where \( C_i(s) \) are polynomial valued vectors with respect to \( s \), \( a = (a_1, \ldots, a_n, a_{n+1}) \) are free parameters with \( a^i = a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} a_{n+1}^{i_{n+1}} \), and \( (\tilde{\Lambda}, \tilde{\xi}) = -\sum_{j=1}^{n} \lambda_j i_j + \lambda_{n+1} i_{n+1} \).

**Proof.** Consider the general solution
\[ \vec{X} = \sum_{k=0}^{\infty} \epsilon^k \vec{X}^{(k)}(s), \]  
(17)
where \( \epsilon \) is an arbitrary parameter that does not appear in the vector field. Then one can find that for each \( k = 0, 1, \ldots, \vec{X}^{(k)}(s) \) satisfies
\[ \vec{X}^{(k)} = \vec{K} \vec{X}^{(k)} + P_k(\vec{X}^{(0)}, \ldots, \vec{X}^{(k-1)}), \]  
(18)
where \( P_0 = 0, P_k(\vec{X}^{(0)}, \ldots, \vec{X}^{(k-1)}) \) is polynomial with respect to \( \vec{X}^{(0)}, \ldots, \vec{X}^{(k-1)} \) obtained by Taylor expanding the vector \( \vec{f} \) with respect to \( \epsilon \). Then we have
\[ \vec{X}^{(0)} = e^{K s} a, \]
for some free parameters \( a = (a_1, \ldots, a_n, a_{n+1})^T \), and
\[ \vec{X}^{(k)} = \sum_{i \in \mathbb{N}^{n+1}, |i|=1}^{N_k} C_i^{(k)}(s) a^i e^{(\tilde{\Lambda}, \tilde{\xi})s}, \]
with some integer \( N_k \) depending \( k \), and \( C_i^{(k)}(s) \) polynomial with respect to \( s \). Now let \( \epsilon = 1 \), we can get the result. \( \square \)
Then, one can conclude easily that the local solution of system (10) around the singularity $t^*$ can be represented as
\[
x = \tau^p(\xi + \sum_{i \in \mathbb{N}^{n+1}, |i| = 1} \hat{C}_i(-\log \tau)a^i \tau^{-\langle \Lambda, i \rangle}),
\]
(19)
where $a_{n+1} = 1$, and $\hat{C}_i(s)$ is the $n$-dimensional vector with the first $n$ components of $C_i(s)$. Additionally, the above solution is the general solution of system (10) with $n$ arbitrary constants.

**Theorem 3.1.** If system (10) has Painlevé property. Then, for any balance $(\xi, p) \in \tilde{F}$,

1. all the components of $p$ are integers, the Kovalevskaya matrix $K$ is semi-simple and Kovalevskaya exponents $\lambda_1, \ldots, \lambda_n$ are integers;
2. the local solution can be represented as formal Laurent series
\[
x = \tau^p(\xi + \sum_{i = -\infty}^{+\infty} B_i \tau^i)
\]
(20)
with $n - 1$ arbitrary constants, where $B_i$ are constant valued vectors;
3. there are local convergent Laurent solutions
\[
x = \tau^p(\xi + \sum_{i = 1}^{+\infty} B_{-i} \tau^{-i}),
\]
(21)
around the infinity with as many arbitrary constants as the number of negative Kovalevskaya exponents.

**Proof.** From lemma 2, we need only to prove the last two statements.

If system (10) has Painlevé property, then for any balance $(\xi, p) \in \tilde{F}$, its general solution around the singularity $t^*$ should be single-valued. we expand the general solution as
\[
x = \tau^p(\xi + \sum_{k=0}^{+\infty} \epsilon^k X^{(k)}(\tau)),
\]
with small free parameter $\epsilon$. By lemma 3 and its proof, one can easily conclude
\[
X^{(k)}(\tau) = \sum_{i \in \mathbb{N}^{n+1}, |i| = 1} N_k \hat{C}_i^k(-\log \tau)a^i \tau^{-\langle \Lambda, i \rangle}, k = 0, 1, \ldots
\]
with some integers $N_k$ depending $k$, and $\hat{C}_i^k(s)$ polynomial with respect to $s$. On the other hand, by Painlevé’s $\alpha$-lemma, for every $k = 0, 1, \ldots, X^{(k)}(\tau)$ must be also single-valued, then $t^*$ can not be the logarithmic branch point or algebraic branch point. we have that $\hat{C}_i^k(s), i \in \mathbb{N}^{n+1}$ are constant vectors, and for $i \in \mathbb{N}^{n+1}$ with $\langle \Lambda, i \rangle \notin \mathbb{Z}$, there must be $\hat{C}_i^k(-\log \tau)a^i = 0$. Therefore, the general solution can be represented as formal Laurent series as the form of (20) with $n - 1$ arbitrary constants, we get the second result.

To prove the last statement, without loss of generality, assume the negative Kovalevskaya exponents are $\lambda_l, \ldots, \lambda_n$ with $\lambda_n = -1$. Then $-\lambda_l, \ldots, -\lambda_n$ and $\lambda_{n+1}$ are all the eigenvalues of $K$ with positive real part. Then by lemma 3, the
local formal solution of (15) around the origin on the unstable manifold can be expressed as

\[ \tilde{X}_u = \sum_{i \in \mathbb{N}^{n+2-l}, |i|=1} C_i(s) a_i e^{(-\lambda_{i1} - \cdots - \lambda_{i(n-l+1)} + \lambda_{n+1} i_{n-l+2})s}, \]  

(22)

with \( a_1 = a_2 = \cdots = a_{l-1} = 0 \). Further, by unstable manifold theorem of general dynamical systems [6], there exist \( a_0, s_0, \lambda_0 \in \mathbb{R}, \) with \( 0 < \lambda_0 < \min\{-\lambda_l, \cdots , -\lambda_n, \lambda_{n+1}\} \), and \( \delta \) small enough such that for all \( |s| < s_0, |a| < \delta \) with \( a_1 = a_2 = \cdots = a_{l-1} = 0 \), the solution \( \tilde{X}_u \) can be bounded as

\[ |\tilde{X}_u| < a_0 |a| e^{\lambda_0 |s|}. \]  

(23)

This implies the system (10) has local convergent solution of the following form around the infinity,

\[ x_u = \tau^p (\xi + \sum_{i \in \mathbb{N}^{n+2-l}, |i|=1} \tilde{C}_i (-\log \tau) a_i \tau^{\lambda_{i1} + \cdots + \lambda_{i(n-l+1)} - \lambda_{n+1} i_{n-l+2}}), \]

\[ (24) \]

with \( a_1 = a_2 = \cdots = a_{l-1} = 0, a_{n+1} = 1 \). Then we can easily get the last statement by noting the proof of the second one.

4. Weak-Painlevé property and integrability. Now we turn to investigate the relationship between singular property and integrability of general dynamical systems.

Assume that system (1) has weak-Painlevé property, then there exists \( l \in \mathbb{Z}^+ \) such that the general solution of (1) can be expanded as

\[ x = \sum_{i = -\infty}^{+\infty} D_i \tau^i, \]

(25)

where \( D_i \) are constant vectors, \( \tau = t - t_* \). Let \( s = \tau^l \), then system (1) becomes

\[ \frac{d}{ds} x = ls^{l-1} f(x), \]

(26)

and the general solution (25) can be expanded as

\[ x = \sum_{i = -\infty}^{+\infty} D_i s^i. \]

We have the following similar results as the case of Painlevé property.

**Theorem 4.1.** If system (1) has weak-Painlevé property. Then, for any balance \((\xi, p) \in \mathcal{F},\) all the components of \( p \) are rational numbers; the Kovalevskaya matrix \( K \) is semi-simple and the Kovalevskaya exponents \( \lambda_1, \cdots, \lambda_n \) are rational numbers; the local solution of (1) can be represented as

\[ x = \tau^p (\xi + \sum_{i = -\infty}^{+\infty} B_i \tau^i), \]

(27)

with \( n - 1 \) arbitrary constants, and some \( l \in \mathbb{Z}^+, B_i \) are constant valued-vectors; (1) has local convergent Puiseux series solutions

\[ x = \tau^p (\xi + \sum_{i = 1}^{+\infty} B_i \tau^i), \]

(28)
with as many arbitrary constants as the number of positive Kovalevskaya exponents.

Proof. Since system (1) has a balance \((\xi, p)\), system
\[
\begin{align*}
\frac{dx}{ds} &= lx^{l-1}f(x), \\
\frac{dx_{n+1}}{ds} &= 1
\end{align*}
\]
(29)
has a balance \((\tilde{\xi}, \tilde{p})\), where \(\tilde{\xi} = (\xi, 1)\), \(\tilde{p} = (lp, 1)\) (29) has the following decomposition,
\[
\begin{align*}
\frac{dx}{ds} &= lx^{l-1}f(x) = lx^{l-1}f^{(0)}(x) + \cdots + lx^{l-1}f^{(m)}(x), \\
\frac{dx_{n+1}}{ds} &= 1
\end{align*}
\]
(30)
where the leading term
\[
\begin{align*}
\frac{dx}{ds} &= lx^{l-1}f^{(0)}(x), \\
\frac{dx_{n+1}}{ds} &= 1
\end{align*}
\]
(31)
is a weight homogeneous system with the scale-invariant solution \(x^{(0)} = \xi s^p, x_{n+1}^{(0)} = s\). By simple calculation, we get the corresponding Kovalevskaya matrix
\[
\hat{K} = \begin{pmatrix}
lp & 0 \\
0 & -1
\end{pmatrix},
\]
where \(K\) is the Kovalevskaya matrix corresponding system (1) with balance \((\xi, p)\).

If system (1) has weak-Painlevé property, we can conclude easily that the general solution of system (29) should be expanded as a Laurent series. With the same method as Painlevé analysis, the general solution of system (31) and hence, the general solution of variational equations of (31) along the \(x^{(0)} = \xi s^p, x_{n+1}^{(0)} = s\)
\[
\frac{dy}{ds} = \hat{K} \frac{y}{s}
\]
should be also expanded as a Laurent series, which implies that \(lp_1, \cdots, lp_n\) must be integers, \(\hat{K}\) must be semi-simple, and all its eigenvalues must be integers. Therefore, \(p_1, \cdots, p_n\) must be rational numbers, \(K\) must be semi-simple, and all the Kovalevskaya exponents must be rational numbers.

Now, based on the above discussions as well as the similar idea in the proof of theorem 2.2[8] and theorem 3.1, one can easily conclude that the general solution of system (4.1) can be expanded as the Laurent series
\[
\begin{align*}
x &= s^p(\xi + \sum_{i=-\infty}^{+\infty} B_i s^i), \\
x_{n+1} &= s
\end{align*}
\]
(32)
with \(n\) arbitrary constants, \(B_i\) is constant valued-vector, and system (4.1) has local Laurent solution
\[
\begin{align*}
x &= \tau^p(\xi + \sum_{i=-\infty}^{+\infty} B_i s^i), \\
x_{n+1} &= s
\end{align*}
\]
(33)
with as many arbitrary constants as the number of positive Kovalevskaya exponents. Then by replacing \(s^i\) by \(\tau = t - t_*\), one can get the proof ultimately.

We remark that the above results can be found in many other references without strict proof, such as [8, 10, 36] for example. And it is more important that the similar results as 4.1 also hold for analytic systems (10) with a little modification. In fact, we have the following results.
Theorem 4.2. If system (10) has weak-Painlevé property. Then, for any balance \((\xi, p) \in \tilde{F}\),
1. all the components of \(p\) are rational numbers, the Kovalevskaya matrix \(K\) is semi-simple and Kovalevskaya exponents \(\lambda_1, \cdots, \lambda_n\) are rational numbers;
2. the local solution can be represented as
\[
x = \tau^p(\xi + \sum_{i=-\infty}^{+\infty} B_i \tau^i),
\]
with \(n-1\) arbitrary constants, and some \(l \in \mathbb{Z}^+, B_i\) are constant valued vectors
3. furthermore, there are local convergent Puiseux series solutions
\[
x = \tau^p(\xi + \sum_{i=1}^{+\infty} B_{-i} \tau^{-i}),
\]
around the infinity with as many arbitrary constants as the number of negative Kovalevskaya exponents.

Proof. The proof can be given easily as in the proof of theorem 3.1 and 4.1, we omit it here.

At last, based on the above preparations, we get the following result by investigating the connection between weak-Painlevé property and integrability of general analytic systems.

Theorem 4.3. Assume \(\tilde{F} \neq \emptyset\) for system (10). Then (10) has weak-Painlevé property if and only if it is completely integrable in sense that it has \(n\) functionally independent time-dependent first integrals \(I_1(x, t), \cdots, I_n(x, t)\) which are formal series with respect to \((x, t^\dag)\) for some \(l \in \mathbb{N}\).

Proof. If system (10) has weak-Painlevé property, the general solution can be expanded as (34). Furthermore, by theorem 4.2 and the proofs of theorem 3.1 and 4.1, it is not difficult to get the general solution of the following form
\[
x = \tau^p(\xi + \sum_{i \in \mathbb{N}^{n+1}, |i|=1} C_i a_i^i \tau^{<\hat{\Lambda}, i>}),
\]
where \(C_i\) are constant vectors, \(a_i\) are free parameters with \(a_{n+1} = 1\), and \(\hat{\Lambda} = \{-\lambda_1, \cdots, -\lambda_n, \frac{q}{l}\}\) with some positive rational number \(q\), and eigenvalues of Kovalevskaya exponents \(\lambda_1, \cdots, \lambda_n\). Let
\[
\hat{Y} = \tau^{\hat{\Lambda}}.
\]
By theorem 4.2, all the eigenvalues \(\lambda_1, \cdots, \lambda_n\) are rational integers, therefore, (37) has \(n\) independent rational first integrals \(F_1(\hat{Y}), \cdots, F_n(\hat{Y})\). Let
\[
I_i(x, t) = F_i(\Phi^{-1}(\tau^{-p}x - \xi, \tau^q)), i = 1, \cdots, n
\]
with \(\Phi(\hat{Y}) = \sum_{i \in \mathbb{N}^{n+1}, |i|=1} C_i a_i^i \hat{Y}^i\). It is easy to see that they are formal Laurent series with respect to \(x\) and \(\tau^q\), and they are independent and invariant along any solution of system (1), i.e., they are time-dependent first integrals of system (1).

Conversely, let \((\xi, \hat{p}) \in \tilde{F}\) such that system (10) has the decomposition (11). Assume that the system (10) has \(n\) functionally independent first integrals \(I_1(x, \tau), \cdots, \)
$I_n(x, \tau)$ which are formal series with respect to $(x, \tau^\frac{1}{n})$ for some $l \in \mathbb{N}$. Then the following system

\[
\begin{aligned}
\frac{d}{d\tau} x &= I_{n-1} f(x), \\
\frac{d}{d\tau} x_{n+1} &= 1,
\end{aligned}
\]  

(38)

with $x_{n+1} = s = \tau^\frac{1}{n}$, has $n$ functionally independent first integrals $I_1(x, x_{n+1}), \ldots, I_n(x, x_{n+1})$ which are formal series with respect to $(x, x_{n+1})$. System (38) has a balance $((\xi_1, \ldots, \xi_n, 1), (l\tilde{p}_1, \ldots, l\tilde{p}_n, 1))$ with the following decomposition

\[
\begin{pmatrix}
\frac{d}{d\tau} x \\
\frac{d}{d\tau} x_{n+1}
\end{pmatrix} = 
\begin{pmatrix}
I_{n-1} f^0(x) \\
0
\end{pmatrix} + \cdots.
\]  

Obviously, the following system

\[
\begin{pmatrix}
\frac{d}{d\tau} x \\
\frac{d}{d\tau} x_{n+1}
\end{pmatrix} = 
\begin{pmatrix}
I_{n-1} f^0(x) \\
0
\end{pmatrix}
\]  

(40)

has $n$ functional independent first integrals $I_1(x, x_{n+1}), \ldots, I_n(x, x_{n+1})$ which are rational of $(x, x_{n+1})$. By the main theorem in [18], the identity component of the differential Galois group of the variational equation of system (40) along its scale-invariant solution $(\xi_1 s^{l\tilde{p}_1}, \ldots, \xi_n s^{l\tilde{p}_n}, s)$ is abelian, and the identity component of the differential Galois group of the corresponding normal variational equation should be trivial.

Make a change of variables $(x^{(1)}(t), x_{n+1}^{(1)}(t)) \to (s^{\tilde{p}} x^{(1)}, s x_{n+1}^{(1)})$, the variational equation is equivalently transformed to

\[
\begin{pmatrix}
\frac{d}{ds} x^{(1)}(t) \\
\frac{d}{ds} x_{n+1}^{(1)}(t)
\end{pmatrix} = \frac{1}{s} \tilde{K} \begin{pmatrix}
x^{(1)}(t) \\
x_{n+1}^{(1)}(t)
\end{pmatrix} = \frac{1}{s} \begin{pmatrix}
I K & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
x^{(1)}(t) \\
x_{n+1}^{(1)}(t)
\end{pmatrix},
\]  

(41)

and the corresponding normal variational equation is transformed to

\[
\frac{d}{ds} x^{(1)} = \frac{1}{s} IK x^{(1)},
\]  

(42)

where $K$ is the Kovalevskaya matrix of system $x = f^0(x)$ with the balance $(\tilde{\xi}, \tilde{p})$, and $\tilde{K}$ is the Kovalevskaya matrix of system (40) with the balance $((\xi_1, \ldots, \xi_n, 1), (l\tilde{p}_1, \ldots, l\tilde{p}_n, 1))$. It is easy to see that the Galois group of this linear equation is isomorphic to the normal variational equation. Then the identity component of the Galois group of (42) is trivial, this leads to that $K$ should be semi-simple and all of its eigenvalues should be rational numbers. Then by theorem 5.9 and theorem 5.10 in [8], system (10) has weak-Painlevé property.

\[\square\]

**Remark 3.** We remark that the theoretical value of Theorem 4.3 is, in fact, greater than the practical application meaning. On one hand, the above theorem implies the integrability of a given dynamical system has, indeed, deep relations with the singularity structure of its general solutions. However, the kinds of the first integrals we got are not so nice. One of the main reasons may be due to the unknown convergence and smoothness of the solutions in the singular form, which may imply that, to make a system integrable in good kinds of integrals, additional conditions should be given beside the (weak-)Painlevé property, for example, proper sign of the Kovalevskaya exponents may be necessary from Theorem 4.2.

Nevertheless, we can still conclude from Theorem 4.3 that, maybe one can find more integrable systems by investigating the weak Painlevé property rather than Painlevé property, just as illustrated in the following example.
Example 1. Consider the system
\[
\begin{align*}
\dot{x}_1 &= x_3 + f_1(x_1, x_2, x_3, x_4), \\
\dot{x}_2 &= x_4 + f_2(x_1, x_2, x_3, x_4), \\
\dot{x}_3 &= -2bx_1x_2 + f_3(x_1, x_2, x_3, x_4), \\
\dot{x}_4 &= -bx_1^2 + x_2^2 + f_4(x_1, x_2, x_3, x_4),
\end{align*}
\] (43)
where $b$ is a non-zero parameter, $f_1, f_2$ are analytical functions with lowest order of two, and $f_3, f_4$ are analytical functions with lowest order of three.

Let us consider the integrability of this system based on the Theorem 4.3. Then, we should look for conditions on the parameter $b$ such that the system (43) satisfies the first necessary conditions of Theorem 4.2. It is not difficult to find that there is a weight-homogeneous decomposition of the vector field provided by the exponents $p = (-2, -2, 3, -3)$, and corresponding first weight-homogeneous term is
\[
f^{(0)} = \begin{pmatrix} x_3 \\ x_4 \\ -2bx_1x_2 \\ -bx_1^2 + x_2^2 \end{pmatrix}.
\]
By simple computations, one can solve the equation $p\xi = f^{(0)}(\xi)$, and get three balances,
\[
\xi^{(1)} = (0, 6, 0, -12), \\
\xi^{(2)} = (\frac{2}{b}\sqrt{b(2b-1)}, \frac{3}{b}, \frac{6}{b}\sqrt{b(2b-1)}, -\frac{6}{b}), \\
\xi^{(3)} = (-\frac{3}{b}\sqrt{b(2b-1)}, \frac{3}{b}, -\frac{6}{b}\sqrt{b(2b-1)}, -\frac{6}{b})
\]
and corresponding Kowalevskaya exponents,
\[
\Lambda^{(1)} = \{-1, 6, \frac{1}{2}(5 \pm \sqrt{1 + 48b})\}, \\
\Lambda^{(2)} = \Lambda^{(3)} = \{-1, 6, \frac{1}{2}(5 \pm \sqrt{-23 + 24/b})\}.
\]

Then we can conclude that, to make system (43) integrable in sense as referred in Theorem 4.3, the parameter $b$ should be in the set
\[
B = \{b \in \mathbb{R} | \sqrt{1 + 48b}, \sqrt{-23 + 24/b} \in \mathbb{Q}\} = \{\frac{k^2 - l^2}{48l^2} = \frac{24m^2}{n^2 + 23m^2} | k, l, m, n \in \mathbb{N}\}.
\]

We remark that, in considering general integrability, one should also make above discussions. For example, let us consider the system (43) with $f_1 = f_2 = f_3 = f_4 \equiv 0$, i.e., we consider the weight-homogeneous system
\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = f^{(0)} = \begin{pmatrix} x_3 \\ x_4 \\ -2bx_1x_2 \\ -bx_1^2 + x_2^2 \end{pmatrix},
\] (44)
which is, in fact, a Hénon-Heiles Hamiltonian system with Hamiltonian $H = \frac{1}{2}(x_3^2 + x_4^2) + bx_1^2x_2 - \frac{1}{2}x_2^3$.

Firstly, we choose $b$ to make system (44) pass the Painlevé test (Theorem 3.1), then $\sqrt{1 + 48b}, \sqrt{-23 + 24/b}$ must be odd integers, by simple calculation, one can find that $b \in \{1, \frac{1}{2}, \frac{1}{4}\}$. Secondly, let us choose $b$ in a weaker case such that $\sqrt{1 + 48b}$,
\( \sqrt{-23 + 24/b} \) are integers, then (44) surely passes the first step of weak-Painlevé test (Theorem 4.2) and one can conclude that \( b \in \{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{16}\} \).

In fact, from [18], one can get that (44) is integrable (in Liouville sense) only on three cases, i.e.,

\[ b \in \{1, \frac{1}{6}, \frac{1}{16}\}. \]

While \( b = \frac{1}{2} \) with corresponding Kovalevskaya exponents \( \{-1, 0, 5, 6\} \) is not the parameter to make system (44) integrable.

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