MULTIPLE POSITIVE SOLUTIONS FOR A $p$-LAPLACE BENCI-CERAMI TYPE PROBLEM ($1 < p < 2$), VIA MORSE THEORY

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Abstract. Let us consider the quasilinear problem

\[
(P_{\varepsilon}) \begin{cases}
-\varepsilon p \Delta_p u + u^{p-1} = f(u) & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geq 2$, $1 < p < 2$, $\varepsilon > 0$ is a parameter and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) = 0$, having a subcritical growth. We prove that there exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$, $(P_{\varepsilon})$ has at least $2P_1(\Omega) - 1$ solutions, possibly counted with their multiplicities, where $P_1(\Omega)$ is the Poincaré polynomial of $\Omega$. Using Morse techniques, we furnish an interpretation of the multiplicity of a solution, in terms of positive distinct solutions of a quasilinear equation on $\Omega$, approximating $(P_{\varepsilon})$.

1. Introduction

Let us consider the quasilinear elliptic problem

\[
(P_{\varepsilon}) \begin{cases}
-\varepsilon p \Delta_p u + u^{p-1} = f(u) & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geq 2$, $1 < p < 2$, $\varepsilon > 0$ is a parameter and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) = 0$, having a subcritical growth.

In [6] Benci and Cerami studied $(P_{\varepsilon})$ for $p = 2$, proving via Morse theory that the number of solutions to $(P_{\varepsilon})$ is related to the topology of $\Omega$. In [18] the previous result was extended to the case $2 \leq p < N$. In both cases it was proved that $(P_{\varepsilon})$ has at least $2P_1(\Omega) - 1$ solutions, counted with their multiplicities (see Definition 2.6). Let us denote by $I_{\varepsilon}$ the energy functional of $(P_{\varepsilon})$.

When $p = 2$, $I_{\varepsilon}$ is defined on the Hilbert space $W^{1,2}_0(\Omega)$, so that the multiplicity of a solution $u_0$ is exactly one if $u_0$ is a nondegenerate critical point of $I_{\varepsilon}$, i.e. if $I'_{\varepsilon}(u_0)$ is an isomorphism. Moreover the nondegeneracy condition is generally verified, thanks to the celebrated result proved by Marino and Prodi [25].

When $p \neq 2$, as $I_{\varepsilon}$ is defined on $W^{1,p}_0(\Omega)$ which is a Banach space, a lot of difficulties arise in order to relate hessian notions to topological objects. In fact, it is not clear what can be a reasonable definition of nondegenerate critical point, as it makes no sense to require that the second derivative of the energy functional in a critical point is invertible.

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since a Banach space, in general (and \(W^{1,p}_0(\Omega)\) in particular), is not isomorphic to its dual space. Furthermore it can be proved that \(I''_{\varepsilon}(u_0)\) cannot be even a Fredholm operator and Marino-Prodi perturbation type results \([25]\) do not hold (see also \([27, 9, 16]\) for further details). The multiplicity result in \([18]\), where \(p \geq 2\), are proved exploiting critical groups estimates in the spirit of differential Morse relation, using a new definition in which a critical point is nondegenerate if the second derivative of the energy functional is injective (see \([16]\)).

Moreover in \([18]\) a further perturbation result was proved, showing that \((P_{\varepsilon})\) is always close to a differential problem having at least \(2P_1(\Omega) - 1\) distinct positive solutions, which is an interpretation of the notion of the multiplicity of each solution to \((P_{\varepsilon})\).

In this work we consider \((P_{\varepsilon})\) when \(p \in (1, 2)\), which brings additional delicate difficulties. In fact, we can see that \(u \in W^{1,p}_0(\Omega) \mapsto \int_{\Omega} |\nabla u|^p\, dx \in \mathbb{R}\) is not \(C^2\), thus also the energy functional \(I_{\varepsilon}\) is not twice differentiable. Moreover, as \(p < 2\), even the nonlinearity \(f\) could not be \(C^1\) (see Remark \([14]\)), so that this further problem should also be managed. Despite these difficulties, we extend the previous results when \(1 < p < 2\), preserving the generality of a quite large class of nonlinearities \(f\). In order to do that, we build a convenient \(C^1\) perturbation of \(f\) and a class of problems approximating \((P_{\varepsilon})\), so that the corresponding energy functionals are arbitrarily close to \(I_{\varepsilon}\), according to a suitable norm (see Lemma \([5, 1]\)).

In this work we take advantage of recent results proved in \([14]\), introducing some bilinear forms defined on a Hilbert space, which are inspired by the formal second derivatives of the approximating functionals.

The critical case of the problem, introduced by Brezis-Nirenberg \([17]\) in the semilinear case \(p = 2\) and extended to the quasilinear case \(p \neq 2\) by Azorero-Peral \([3, 4]\) and Guedda-Veron \([23]\), was studied in \([19]\) for \(p \in (1, 2)\), where we proved a multiplicity result when \(f\) is a homogeneous critical nonlinearity, so that the \((P.S.)\) condition at any level fails.

In this work, denoting by \(p^* = \frac{Np}{N-p}\), we assume that \(f \in C^0(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})\) satisfies the following conditions:

- \((f_1)\) there exists \(q \in (p, p^*)\) such that
  \[
  \frac{d}{dt} \frac{f(t)}{t^{q-1}} < 0 \quad \forall \; t > 0;
  \]
- \((f_2)\) there exists \(\theta \in (0, 1/p)\) such that
  \[
  F(t) \leq \theta tf(t) \quad \forall \; t \geq 0
  \]
  where \(F(t) = \int_0^t f(s)\, ds\);
- \((f_3)\) \(\frac{d}{dt} \frac{f(t)}{t^{p-1}} > 0 \quad \forall \; t > 0\);
- \((f_4)\) \(\lim_{t \to 0^+} t^{2-p} f'(t) = 0\);
- \((f_5)\) \(f(t) = 0 \quad \forall \; t < 0\).

Continuity of \(f\) and assumption \((f_3)\) give that \(f(0) = 0\).
Remark 1.1. We observe that the functions satisfying the previous assumptions may not be $C^1$ in 0. For example, this is the case for $f(t) = (t^+)^{p-1}$, where $p < r < \min\{2, p^*\}$. In particular, note that if $N \geq 3$ and $p \in (1, \frac{2N}{N+2})$, then $p^* < 2$.

Conversely, if we assume that $f$ is $C^1$ on $\mathbb{R}$, then $f'(0) = 0$, which directly gives $(f_4)$.

Remark 1.2. Let us note that the assumptions $(f_1) - (f_5)$ are satisfied also by non-homogeneous functions. For instance, we may think of

$$f(t) = a_1(t^+)^{r_1-1} + a_2(t^+)^{r_2-1} + \ldots + a_m(t^+)^{r_m-1}$$

where $a_1, a_2, \ldots, a_m > 0$ and $p < r_1 < r_2 < \ldots < r_m < p^*$.

Another example is given by

$$f(t) = \frac{d}{dt} \left((t^+)^r \log(a + t^+)\right)$$

where $r \in (p, p^*)$ and $a$ is big enough.

In this work, inspired by the ideas in [6], we want to prove multiplicity results, related to the topology of $\Omega$. In order to establish the first one, we denote by $\text{cat}_{\Omega}(\Omega)$ the Lusternick–Schnirelmann category of $\Omega$ in itself.

**Theorem 1.3.** If $\text{cat}_{\Omega}(\Omega) > 1$, there exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$, $(P_\varepsilon)$ has at least $\text{cat}_{\Omega}(\Omega) + 1$ distinct solutions.

In order to state the following result, which will be proved exploiting Morse Theory, let us recall a classical topological definition.

**Definition 1.4.** Let $\mathbb{K}$ be a field. For any pair of topological spaces $(A, B)$ with $B \subset A$, we denote by $\mathcal{P}_t(A, B)$ the Poincaré polynomial of $(A, B)$, defined as

$$\mathcal{P}_t(A, B) = \sum_{q=0}^{+\infty} \dim \, H^q(A, B) \, t^q$$

where $H^q(A, B)$ stands for the $q$-th Alexander-Spanier relative cohomology group of $(A, B)$, with coefficients in $\mathbb{K}$; we also define the Poincaré polynomial of $A$ as

$$\mathcal{P}_t(A) = \mathcal{P}_t(A, \emptyset).$$

**Theorem 1.5.** There exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$, $(P_\varepsilon)$ has at least $2\mathcal{P}_1(\Omega) - 1$ solutions, possibly counted with their multiplicities.

The definition of multiplicity of a solution is given in Definition 2.6. Note that, as showed in [6], $2\mathcal{P}_1(\Omega) - 1$ is bigger than $\text{cat}_{\Omega}(\Omega) + 1$, if we assume that $\Omega$ is topologically rich. However the last theorem, proved applying a topological version of Morse theory, does not guarantee the existence of $2\mathcal{P}_1(\Omega) - 1$ distinct solutions, so it is crucial to understand more deeply what the notion of multiplicity of a solution means. Indeed we prove here that there is a sequence of quasilinear problems approaching $(P_\varepsilon)$, each of them having at least $2\mathcal{P}_1(\Omega) - 1$ distinct solutions, which are close to the solutions of $(P_\varepsilon)$. 
More precisely, we prove the following perturbation result, in which we say that $\partial \Omega$ satisfies the interior sphere condition if for each $x_0 \in \partial \Omega$ there exists a ball $B_R(x_1) \subset \Omega$ such that $B_R(x_1) \cap \partial \Omega = \{x_0\}$.

**Theorem 1.6.** Assume that $\partial \Omega$ satisfies the interior sphere condition. There exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$, either $(P_\varepsilon)$ has at least $2 \mathcal{P}_1(\Omega) - 1$ distinct solutions or, for every $\alpha_n \to 0^+$, there exist a sequence $f_{\alpha_n}$ suitably approximating $f$ and a sequence $h_n \subset C^1(\overline{\Omega})$ with $\|h_n\|_{C^1(\overline{\Omega})} \to 0$ such that problem

$$(P_n) \begin{cases}
-\varepsilon^p \text{div}((|\nabla u|^2 + \alpha_n)^{(p-2)/2} \nabla u) + u (\alpha_n + u^2)^{(p-2)/2} = f_{\alpha_n}(u) + h_n & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

has at least $2 \mathcal{P}_1(\Omega) - 1$ distinct solutions, for $n$ large enough.

**Remark 1.7.** Considering the case in which $(P_\varepsilon)$ has fewer than $2 \mathcal{P}_1(\Omega) - 1$ distinct solutions, we will see that all the solutions of $(P_n)$ are arbitrarily close to solutions of $(P_\varepsilon)$. More precisely, if $\bar{u}$ is a solution of $(P_\varepsilon)$ and the multiplicity of $\bar{u}$ is $\bar{k}$, then, for any fixed $R > 0$, each problem $(P_n)$ has at least $\bar{k}$ distinct solutions in $B_R(\bar{u})$ and besides these solutions converge to $\bar{u}$ in $C^1(\overline{\Omega})$-norm, for $n \to \infty$.

We mention that in [2], using Ljusternik–Schnirelman category, Alves has proved the existence of $\text{cat}(\Omega)$ solutions to $(P_\varepsilon)$, when $p \geq 2$.

Perturbation results in Morse theory for quasilinear problem having a right-hand side subcritically at infinity have been obtained in [17, 15] (see also [12, 20]).

### 2. Proofs of Theorems 1.3 and 1.5

Combining $(f_1)$ and $(f_4)$, we see that there are $q \in (p, p^*)$ and $c > 0$ such that, for every $t > 0$,

(2.1) \[ tf'(t) \leq \frac{p-1}{2} t^{p-1} + ct^{p-1} \]

(2.2) \[ f(t) \leq \frac{1}{2} t^{p-1} + ct^{q-1} \]

(2.3) \[ F(t) \leq \frac{1}{2p} t^p + ct^q. \]

Standard arguments prove that the solutions to $(P_\varepsilon)$ correspond to critical points of the $C^1$ functional $I_\varepsilon : W^{1,p}_0(\Omega) \to \mathbb{R}$ defined by setting

$$I_\varepsilon(u) = \frac{\varepsilon^p}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{p} \int_{\Omega} |u|^p \, dx - \int_{\Omega} F(u) \, dx.$$ 

We define an equivalent norm on $W^{1,p}_0(\Omega)$ as

$$\|u\|_\varepsilon = \left( \frac{\varepsilon^p}{p} \int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}}$$

while $\langle \cdot , \cdot \rangle : W^{-1,p'}(\Omega) \times W^{1,p}_0(\Omega) \to \mathbb{R}$ denotes the duality pairing.
Denoting by $A_\varepsilon : W^{1,p}_0(\Omega) \to \mathbb{R}$
\[
A_\varepsilon(u) = \langle I'_\varepsilon(u), u \rangle,
\]
we introduce the Nehari manifold
\[
\Sigma_\varepsilon(\Omega) = \{ u \in W^{1,p}_0(\Omega) : u \neq 0, A_\varepsilon(u) = 0 \}.
\]

Naturally each nontrivial critical point of $I_\varepsilon$ is a nonnegative function which belongs to $\Sigma_\varepsilon(\Omega)$.

Although $f$ may be not $C^1$ in $0$, the assumptions on $f$, through (2.1) and (2.2), assure that $A_\varepsilon$ is still a $C^1$ functional and
\[
\langle A'_\varepsilon(u), v \rangle = p \varepsilon^p \int_\Omega \left( \frac{\nabla u}{|\nabla u|^{2-p}} \right) \frac{uv}{|u|^{2-p}} dx + p \int_\Omega \frac{uv}{|u|^{2-p}} dx - \int_\Omega f'(u)uv + f(u)v \ dx.
\]

The following Lemma lists some useful properties about $\Sigma_\varepsilon(\Omega)$, which hold also when $\Omega$ is replaced by another bounded set or by $\mathbb{R}^N$. The proof is strongly inspired by [6], even if with some slightly new arguments.

**Lemma 2.1.** For every $\varepsilon > 0$, $\Sigma_\varepsilon(\Omega)$ is a 1-codimensional submanifold of $W^{1,p}_0(\Omega)$, which is $C^1$-diffeomorphic to
\[
\mathcal{S}_\varepsilon = \{ u \in W^{1,p}_0(\Omega) : \|u\|_\varepsilon = 1 \} \setminus \{ u \in W^{1,p}_0(\Omega) : u \leq 0 \text{ a.e. in } \Omega \}.
\]
Furthermore there exist $\sigma_\varepsilon > 0$ and $K_\varepsilon > 0$ such that
\[
\|u\| \geq \sigma_\varepsilon, \quad I_\varepsilon(u) \geq K_\varepsilon \quad \forall u \in \Sigma_\varepsilon(\Omega).
\]

**Proof.** Taking account of (2.3), there is $c_\varepsilon > 0$ such that
\[
I_\varepsilon(u) \geq \frac{1}{2p} \|u\|_p^p - c_\varepsilon \|u\|_\varepsilon^q \quad \forall u \in W^{1,p}_0(\Omega),
\]
therefore $0$ is a local minimum for $I_\varepsilon$.

Let us denote by
\[
\Omega_{u,\delta} = \{ x \in \Omega : u(x) > \delta \}
\]
for any $u \in W^{1,p}_0(\Omega)$ and $\delta \in \mathbb{R}$. From $(f_3)$ we infer that
\[
\Omega_{u,0} = \{ x \in \Omega : u(x) > 0 \},
\]
hence, by $(f_3)$,
\[
\langle A'_\varepsilon(u), u \rangle < 0 \quad \forall u \in \Sigma_\varepsilon(\Omega).
\]

For every fixed $v \in \mathcal{S}_\varepsilon$, let us consider the map $t \in [0, +\infty) \mapsto I_\varepsilon(tv)$. We start proving that
\[
\lim_{t \to +\infty} I_\varepsilon(tv) = -\infty
\]
Actually, there is $\delta_0 > 0$ such that $|\Omega_{u,\delta_0}| > 0$. Indeed, if not, for every $n \in \mathbb{N}$, it should be $|\Omega_{u,\frac{1}{n}}| = 0$, so that
\[
|\Omega_{u,0}| = \lim_{n \to \infty} |\Omega_{u,\frac{1}{n}}| = 0
\]
which gives a contradiction, as \( v \in \mathcal{S}_\varepsilon \).

By \((f_2)\) we infer that \( F(s) \geq s^\frac{p}{p-\theta} F(\delta_0)/\delta_0^{\frac{1}{p}} \), for any \( s \geq \delta_0 \). Hence, for any \( t \geq 1 \)

\[
I_\varepsilon(tv) \leq \frac{tp}{p} - \int_{\Omega_{\varepsilon, \delta_0}} F(tv(x)) \, dx \leq \frac{tp}{p} - \frac{1}{tp} F(\delta_0)|\Omega_{\varepsilon, \delta_0}|
\]

which proves \((2.7)\).

As a consequence, there is \( \xi > 0 \) such that

\[
I_\varepsilon(\xi v) = \max_{t \geq 0} I_\varepsilon(tv).
\]

Clearly \( \xi v \) belongs to \( \Sigma_\varepsilon(\Omega) \) and \( \int_\Omega f(\xi v)v^{p-1} \, dx = 1 \), hence we deduce by \((f_3)\) that
\( \xi = \xi_\varepsilon(v) \) is unique and \( \Sigma_\varepsilon(\Omega) \) is the image of the function \( \psi_\varepsilon : \mathcal{S}_\varepsilon \to W^{1,p}_0(\Omega) \) defined by \( \psi_\varepsilon(v) = \xi_\varepsilon(v)v \).

Taking account of \((2.6)\), the implicit function theorem assures that \( \xi_\varepsilon \) and \( \psi_\varepsilon \) are \( C^1 \) functions.

Finally, for each \( u \in \Sigma_\varepsilon(\Omega), \ v = u/\|u\|_\varepsilon \) belongs to \( \mathcal{S}_\varepsilon \) and, using \((2.5)\),

\[
I_\varepsilon(u) = \max_{t \geq 0} I_\varepsilon(tv) \geq \max_{t \geq 0} \left( \frac{1}{2p} t^p - c_\varepsilon t^q \right) = K_\varepsilon > 0.
\]

As \( I_\varepsilon(0) = 0 \), by continuity we complete the proof. \( \square \)

The following Lemma shows how \( \Sigma_\varepsilon(\Omega) \) is a natural constraint for problem \((P_\varepsilon)\).

**Lemma 2.2.** \( u \) is a nontrivial critical point of \( I_\varepsilon \) if and only if it is a critical point of \( I_\varepsilon \) on \( \Sigma_\varepsilon(\Omega) \), moreover \( (I_\varepsilon) \) and \( (I_\varepsilon)_{\Sigma_\varepsilon(\Omega)} \) satisfy \((P.S.)_c\) for all \( c \in \mathbb{R} \).

**Proof.** The first statement comes directly from \((2.6)\).

In Corollary 3.4, it will be proved that \( I_\varepsilon \) satisfies \((P.S.)_c\) for any \( c \in \mathbb{R} \).

Let \( c \in \mathbb{R} \) and \( u_k \subset \Sigma_\varepsilon(\Omega), \ \lambda_k \subset \mathbb{R} \) be sequences such that \( I_\varepsilon(u_k) \to c \) and

\[
(2.8) \quad I'_\varepsilon(u_k) - \lambda_k A'_\varepsilon(u_k) \to 0.
\]

Since \( \left( \frac{1}{p} - \theta \right) \|u_k\|_\varepsilon^p \leq I_\varepsilon(u_k) \), the sequence \( u_k \) is bounded, so that

\[
(2.9) \quad -\lambda_k \langle A'_\varepsilon(u_k), u_k \rangle \to 0.
\]

Moreover, there is \( \bar{u} \in W^{1,p}_0(\Omega) \) such that \( u_k \) converges to \( \bar{u} \), weakly in \( W^{1,p}_0(\Omega) \) and strongly in \( L^r(\Omega) \), if \( r \in (p, p^*) \). Therefore, through \((2.1)\) and \((2.2)\),

\[
\int_\Omega f'(u_k)u_k^2 - (p-1)f(u_k)u_k \to \int_\Omega f'(\bar{u})\bar{u}^2 - (p-1)f(\bar{u})\bar{u}
\]

where, by \((f_3), \ a_0 = \int_\Omega f'(\bar{u})\bar{u}^2 - (p-1)f(\bar{u})\bar{u} \geq 0 \). If \( a_0 = 0 \), then \( \bar{u}(x) \leq 0 \) almost everywhere in \( \Omega \), and in particular

\[
\|u_k\|_\varepsilon^p = \int_\Omega f(u_k)u_k \, dx \to \int_\Omega f(\bar{u})\bar{u} \, dx = 0
\]

which contradicts \((2.3)\).

So, taking account of \((2.9)\) and \((2.8)\),

\[
-\langle A'_\varepsilon(u_k), u_k \rangle \to a_0 > 0 \Rightarrow \lambda_k \to 0 \Rightarrow I'_\varepsilon(u_k) \to 0
\]

which, as \( I_\varepsilon \) satisfies \((P.S.)_c\), concludes the proof. \( \square \)
Since $I_\varepsilon$ satisfies (P.S.) on $\Sigma_\varepsilon(\Omega)$, the infimum is achieved. Let us denote

$$m(\varepsilon, \Omega) = \inf\{I_\varepsilon(u) : u \in \Sigma_\varepsilon(\Omega)\}.$$

Without any loss of generality, we shall assume that $0 \in \Omega$. Moreover we denote by $r > 0$ a number such that $\Omega^r = \{x \in \mathbb{R}^N \mid d(x, \Omega) < r\}$ and $\Omega^r = \{x \in \Omega \mid d(x, \partial \Omega) > r\}$ are homotopically equivalent to $\Omega$ and $B_r(0) \subset \Omega$.

We notice that if $\Omega = B_r(y)$, the number $m(\varepsilon, B_r(y))$ does not depend on $y$, so we set $m(\varepsilon, r) = m(\varepsilon, B_r(y))$.

We also set $\Sigma_m(\varepsilon, r) = \{u \in \Sigma_\varepsilon(\Omega) : I_\varepsilon(u) \leq m(\varepsilon, r)\}$.

Now we can reason as in [18] so that, relying also on [21, 22] which still hold when $p \in (1, 2)$, we infer the following two results (cf. Proposition 4.4 and 4.6 in [18]).

**Proposition 2.3.** There exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$

$$\dim H^k(\Sigma_m^{(\varepsilon, r)}) \geq \dim H^k(\Omega).$$

**Proposition 2.4.** There exists $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*)$ there are $\alpha > m(\varepsilon, \Omega)$ and $c \in (0, m(\varepsilon, \Omega))$ such that

$$P_t(I_\alpha^\varepsilon, I_c^\varepsilon) = tP_t(\Omega) + tZ(t)$$

$$P_t(W^{1,p}_0(\Omega), I_\alpha^\varepsilon) = t^2(P_t(\Omega) - 1) + t^2Z(t)$$

where $Z(t)$ is a polynomial with nonnegative integer coefficients.

**Proof of Theorem 1.3.** By Proposition 2.3, we infer that $cat_{\Sigma_m^{(\varepsilon, r)}}(\Sigma_m^{(\varepsilon, r)}) \geq cat_\Omega(\Omega)$, so that, applying classical results of Ljusternick-Schnirelmann theory, $I_\varepsilon : \Sigma_\varepsilon^{m(\varepsilon, r)} \to \mathbb{R}$ has at least $cat_\Omega(\Omega)$ critical points. Moreover, having assumed $cat_\Omega(\Omega) > 1$, we have that $\Sigma_m^{m(\varepsilon, r)}$ is not contractible, while $\Sigma_\varepsilon(\Omega)$ is, hence there is a further critical point $u$ with $I_\varepsilon(u) > m(\varepsilon, r)$. \hfill $\Box$

In order to prove Theorem 1.3, which involves Morse theory, we recall some notions (see [10, 11]).

**Definition 2.5.** Let $\mathbb{K}$ be a field, $X$ a Banach space and $f$ a $C^1$ functional on $X$. Let $u$ be a critical point of $f$, $c = f(u)$ and $U$ be a neighborhood of $u$. We call

$$C_q(f, u) = H^q(f^c \cap U, (f^c \setminus \{u\}) \cap U)$$

the $q$-th critical group of $f$ at $u$, $q = 0, 1, 2, \ldots$, where $f^c = \{v \in X : f(v) \leq c\}$, $H^q(A, B)$ stands for the $q$-th Alexander-Spanier cohomology group of the pair $(A, B)$ with coefficients in $\mathbb{K}$. By the excision property of the singular cohomology theory, the critical groups do not depend on a special choice of the neighborhood $U$. 
**Definition 2.6.** We introduce the Morse polynomial of $f$ in $u$, defined as
\[
i(f,u)(t) = \sum_{q=0}^{+\infty} \dim C_q(f,u) t^q.
\]
We call *multiplicity* of $u$ the number $i(f,u)(1) \in \mathbb{N} \cup \{+\infty\}$.

The following theorem is a topological version of the classical Morse relation (cf. Theorem 4.3 in [10]).

**Theorem 2.7.** Let $X$ be a Banach space and $f$ be a $C^1$ functional on $X$. Let $a,b \in \mathbb{R}$ be two regular values for $f$, with $a < b$. If $f$ satisfies the $(P.S.)_c$ condition for all $c \in (a,b)$ and $u_1, \ldots, u_l$ are the critical points of $f$ in $f^{-1}(a,b)$, then
\[
\sum_{j=1}^{l} i(f,u_j)(t) = P_1(f^b,f^a) + (1 + t)Q(t)
\]
where $Q(t)$ is a formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$.

**Proof of Theorem 1.5.** Choosing $\varepsilon^*$ as required by Proposition 2.4 the proof comes from (2.12), (2.10) and (2.11). In particular, denoting by $m(u)$ the multiplicity of any critical point $u$ of $I_\varepsilon$ (see Definition 2.6), we get
\[
\sum_{I_\varepsilon(u)<\alpha} m(u) = P_1(\Omega) + Z(1) + 2Q_-(1) \geq P_1(\Omega)
\]
\[
\sum_{I_\varepsilon(u)\geq\alpha} m(u) = P_1(\Omega) - 1 + Z(1) + 2Q_+(1) \geq P_1(\Omega) - 1
\]
where, by Theorem 2.7 $Q_-(t)$ and $Q_+(t)$ are suitable formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$. □

**3. Approximating functionals**

In order to obtain a further result which provides at least $2P_1(\Omega) - 1$ distinct solutions for a sequence of problems approaching $(P_\varepsilon)$, we build here some functionals which approximate $I_\varepsilon$.

Let us fix
\[
s > \max\{2, 2q - 1\}
\]
where $q$ is introduced by $(f_1)$. For every $\alpha \geq 0$ we set
\[
F_\alpha(t) = F\left(\left(\alpha + (t^+)^s\right)^{1/s}\right),
\]
\[
f_\alpha(t) = F'_\alpha(t),
\]
\[
G_\alpha(t) = \frac{1}{p} \left(\alpha + t^2\right)^{\frac{2}{p}}.
\]
For every $\varepsilon, \alpha > 0$ and $h \in C^1(\overline{\Omega})$ we define
We observe that, for any $a, b > 0$,
\begin{equation}
(3.2) \quad J_{\varepsilon, a}(u) = \frac{\varepsilon^p}{p} \int_{\Omega} (\alpha + |\nabla u|^2)^{\frac{p}{2}} dx + \frac{1}{p} \int_{\Omega} (\alpha + u^2)^{\frac{p}{2}} dx - \int_{\Omega} F_{a}(u) dx
\end{equation}
and
\begin{equation}
(3.3) \quad J_{\varepsilon, a,h}(u) = J_{\varepsilon, a}(u) - \int_{\Omega} h(x)u(x) \, dx.
\end{equation}

It is immediate that
\begin{equation}
I_{\varepsilon}(u) = \int_{\Omega} \varepsilon^p G_{\alpha}(|\nabla u|) + G_{\alpha}(u) - F_{0}(u) \quad J_{\varepsilon, a}(u) = \int_{\Omega} \varepsilon^p G_{\alpha}(|\nabla u|) + G_{\alpha}(u) - F_{a}(u).
\end{equation}

Note that $G_{\alpha}, F_{a} \in C^{2}(\mathbb{R}, \mathbb{R})$ when $\alpha > 0$, while $G_{0}$ is just $C^{1}$ and so it could be about $F_{0}$ (see Remark [1.1]).

Nevertheless, the functional $u \mapsto \int_{\Omega} G_{\alpha}(|\nabla u|)$ and, consequently, $J_{\varepsilon, a}$ are still just $C^{1}$ in $W_{0}^{1,p}(\Omega)$.

**Lemma 3.1.** For any bounded $B \subset W_{0}^{1,p}(\Omega)$
\begin{equation}
(3.3) \quad \lim_{\alpha \to 0} \|J_{\varepsilon, a} - I_{\varepsilon}\|_{C^{1}(B)} = 0
\end{equation}
\begin{equation}
(3.4) \quad \lim_{\|b\|_{C^{1}(\Omega)} \to 0} \|J_{\varepsilon, a,h} - J_{\varepsilon, a}\|_{C^{1}(B)} = 0.
\end{equation}

**Proof.** We observe that, for any $t \in \mathbb{R}$
\begin{equation}
|G_{\alpha}(t) - G_{0}(t)| = \left| \frac{1}{p} (\alpha + t^2)^{\frac{p}{2}} - \frac{1}{p} |t|^p \right| \leq \frac{\alpha^{p/2}}{p},
\end{equation}
\begin{equation}
|G'_{\alpha}(t) - G'_{0}(t)| = \left| t^{p-1} (\alpha + t^2)^{\frac{2-p}{2}} - |t|^{2-p} \right| \leq \frac{\alpha^{p-1}}{(\alpha + t^2)^{\frac{2-p}{2}}} \leq \frac{\alpha^{p-1}}{\alpha^{\frac{2-p}{2}}} |t|^{2-p} \quad \text{if } p \in (1, \frac{3}{2}] \quad \text{and} \quad \frac{\alpha^{p-1}}{\alpha^{\frac{2-p}{2}}} |t|^{2-p} \quad \text{if } p \in (\frac{3}{2}, 2).
\end{equation}

Moreover, by $(f_{2})$ $F(\alpha^{1/s}) \leq \theta f(\alpha^{1/s})\alpha^{1/s}$, hence taking account of $(2.2)$ we get
\begin{equation}
(3.7) \quad |F_{a}(t) - F(t)| \leq \alpha^{1/s} f(|t| + \alpha^{1/s}) \leq O(\alpha^{1/s})(1 + |t|^{q-1}) \quad \forall t \in \mathbb{R}.
\end{equation}

Having chosen $s > 2q - 1$, we have $\frac{s-1}{2} > q - 1$, so by $(f_{1})$
\begin{equation}
(3.8) \quad t \in (0, +\infty) \mapsto \frac{f^{2}(t)}{t^{s-1}} \quad \text{is decreasing.}
\end{equation}

Moreover, introducing $k : (0, +\infty) \to \mathbb{R}$ defined as $k(t) = \frac{t^{\frac{s-1}{2}}}{f(\alpha^{1/s})}$, it is immediate that
\begin{equation}
k(a) < k(a + b) < k(a) + k(b)
\end{equation}
for any $a, b > 0$, hence by $(3.8)$ we get
\[ |F'(t) - F'_\alpha(t)| = \left| f(t) - f \left( (\alpha + t^s)^{1/s} \right) \frac{t^{s-1}}{(\alpha + t^s)^{\frac{s-1}{s}}} \right| \]

\[ = f \left( (\alpha + t^s)^{1/s} \right) f(t) \left| \frac{\alpha + t^s}{(\alpha + t^s)^{\frac{s-1}{s}}} - \frac{t^{s-1}}{f(t)} \right| \]

\[ = f \left( (\alpha + t^s)^{1/s} \right) \frac{f(t)}{f(t)} \left( k(\alpha + t^s) - k(t^s) \right) \leq f \left( (\alpha + t^s)^{1/s} \right) \frac{f(t)}{f(t)} k(\alpha) \]

\[ \leq \frac{f^2 \left( (\alpha + t^s)^{1/s} \right)}{(\alpha + t^s)^{\frac{s-1}{s}}} k(\alpha) \leq \frac{f^2 (\alpha^{1/s})}{\alpha^{\frac{s-1}{s}}} \frac{\alpha^{\frac{s-1}{s}}}{f(\alpha^{1/s})} = f \left( \alpha^{1/s} \right). \]

So, from (3.3), (3.6), (3.7) and (3.9), we infer (3.3), while (3.4) is trivial.

\[ \square \]

We now aim to prove that, for every \( \varepsilon > 0 \), \( \alpha \in [0,1] \) and \( h \in C^1(\bar{\Omega}) \), \( J_{\varepsilon,\alpha,h} \) satisfies a compactness condition. We begin to recall a classical definition in a reflexive Banach space, taken from [8, 26].

**Definition 3.2.** Let \( X \) be a reflexive Banach space and \( D \subset X \). A map \( H : D \to X' \) is said to be of class \( (S)_+ \), if, for every sequence \( u_k \) in \( D \) weakly convergent to \( u \) in \( X \) with

\[ \limsup_{k \to \infty} \langle H(u_k), u_k - u \rangle \leq 0, \]

we have \( \|u_k - u\| \to 0 \).

The following result provides a compactness property about the approximating functionals \( J_{\varepsilon,\alpha,h} \). It is based on [11, Theorem 3.5] (see also [12, Theorem 2.1]). For reader’s convenience, we sketch the proof.

**Theorem 3.3.** For every \( \varepsilon > 0 \), \( p \in (1,2) \), \( \alpha \in [0,1] \), \( h \in C^1(\bar{\Omega}) \), the functional \( J'_{\varepsilon,\alpha,h} \) is of class \( (S)_+ \).

**Proof.** Let us fix \( \varepsilon > 0 \). For every \( \alpha \in [0,1] \), let \( \Psi_{\varepsilon,\alpha} : \mathbb{R}^N \to \mathbb{R} \), \( b_\alpha : \mathbb{R} \to \mathbb{R} \) and \( H_\alpha : W^{1,p}_0(\Omega) \to W^{-1,p'} \) be the maps

\[ \Psi_{\varepsilon,\alpha}(\xi) = \varepsilon^p G_\alpha(\|\xi\|) \]

\[ b_\alpha(t) = C'_\alpha(t) - F'_\alpha(t) \]

\[ H_\alpha(u) = \langle J'_{\varepsilon,\alpha}(u), \cdot \rangle \]

so that \( H_\alpha(u) = -\div (\nabla \Psi_{\varepsilon,\alpha}(\nabla u)) + b_\alpha(u) \).

We start by showing that there is \( C > 0 \) and, for every \( \delta > 0 \), there is a suitable \( c(\delta) \in \mathbb{R} \) such that

\[ |\nabla \Psi_{\varepsilon,\alpha}(\xi)| \leq \varepsilon^p |\xi|^{p-1} \]

\[ |b_\alpha(s)| \leq C + C|s|^{p^* - 1} \]
For every Corollary 3.4.

(3.13) \[ \nabla \Psi_{\varepsilon, \alpha}(\xi) \cdot \xi \geq \frac{\varepsilon p}{2} |\xi|^p - C \]

(3.14) \[ b_\alpha(s)s \geq -\delta |s|^p + c(\delta) \]

(3.15) \[ (\nabla \Psi_{\varepsilon, \alpha}(\xi) - \nabla \Psi_{\varepsilon, \alpha}(\eta)) \cdot (\xi - \eta) > 0 \]

for every \( \alpha \in [0, 1] \), \( \xi, \eta \in \mathbb{R}^N \), \( \eta \neq \xi \in \mathbb{R}^N \), \( s \in \mathbb{R} \).

As (3.11) and (3.12) are trivial, let us prove (3.13). Denoting by \( \gamma_\alpha = \left( \frac{\alpha}{2(p - 1)} \right)^{1/2} \),

\[ |\xi| \geq \gamma_\alpha \Rightarrow \left( \frac{|\xi|^2}{\alpha + |\xi|^2} \right)^{\frac{\alpha}{2}} \geq \frac{1}{2} \Rightarrow \frac{\varepsilon p |\xi|^2}{(\alpha + |\xi|^2)^{\frac{\alpha}{p}}} \geq \frac{\varepsilon p}{2} |\xi|^p \]

\[ |\xi| < \gamma_\alpha \Rightarrow \frac{\varepsilon p}{2} |\xi|^p \leq \frac{\varepsilon p}{2} \gamma_\alpha^p \leq \frac{\varepsilon p}{2} \gamma_1^p, \]

so we infer (3.13), choosing \( C \geq \frac{\varepsilon p}{2} \gamma_1^p \).

Now we immediately see that there is \( c_1 > 0 \) such that

\[ |tb_\alpha(t)| \leq c_1 (1 + |t|^p + |t|^q) \quad \forall \alpha \in [0, 1], \ t \in \mathbb{R}. \]

So we get (3.14) putting

\[ c(\delta) = \min_{t \geq 0} \left( \delta |t|^p - c_1 (1 + |t|^p + |t|^q) \right). \]

Let us consider \( \eta \neq \xi \in \mathbb{R}^N \). If \( |\xi| = |\eta| \), then

\[ (\nabla \Psi_{\varepsilon, \alpha}(\xi) - \nabla \Psi_{\varepsilon, \alpha}(\eta)) \cdot (\xi - \eta) = \varepsilon p \frac{|\xi - \eta|^2}{(\alpha + |\eta|^2)^{\frac{\alpha}{p}}} > 0. \]

Otherwise, if \( |\xi| \neq |\eta| \), by the monotonicity of the real function \( t \in \mathbb{R} \mapsto t (\alpha + t^2)^{\frac{\alpha}{p}} \), we get

\[ |\xi| |\eta| \left( \frac{1}{(\alpha + |\xi|^2)^{\frac{\alpha}{p}}} + \frac{1}{(\alpha + |\eta|^2)^{\frac{\alpha}{p}}} \right) < \frac{|\xi|^2}{(\alpha + |\xi|^2)^{\frac{\alpha}{p}}} + \frac{|\eta|^2}{(\alpha + |\eta|^2)^{\frac{\alpha}{p}}} \]

which gives (3.15).

As (3.11) – (3.13) hold, \( \nabla \Psi_{\varepsilon, \alpha} \) and \( b_\alpha \) satisfy the assumptions required by Theorem 3.5 in [1], so that \( J'_{\varepsilon, \alpha} \) is of class \((S)_+\). Moreover, for every \( h \in C^1(\overline{\Omega}) \), it is immediate that \( J'_{\varepsilon, \alpha, h} \) is of class \((S)_+\) too.

\textbf{Corollary 3.4.} For every \( \varepsilon > 0 \), \( p \in (1, 2) \), \( \alpha \in [0, 1] \), \( h \in C^1(\overline{\Omega}) \), the functional \( J_{\varepsilon, \alpha, h} \) satisfies \((P.S.)_c\) for all \( c \in \mathbb{R} \). 

Proof. By (3.1) and (f₁), \( t \in (0, +\infty) \rightarrow \frac{f(t)}{t} \) is a decreasing function, hence (f₂) implies that
\[
F_\alpha(t) - \theta F'_\alpha(t)t \leq \theta \alpha \frac{1}{2} f(\alpha \frac{1}{2}) \quad \text{for any } t \in \mathbb{R}.
\]
Moreover, there is \( c_1 > 0 \) such that \( \int \h u \, dx \leq c_1 \| u \|_\varepsilon \) and
\[
\frac{1}{p} \| u \|_\varepsilon^p \leq J_{\varepsilon,\alpha,h}(u) - \theta \langle J'_{\varepsilon,\alpha,h}(u), u \rangle + \theta \alpha \frac{1}{2} f(\alpha \frac{1}{2}) |\Omega| + (1 - \theta) c_1 \| u \|_\varepsilon.
\]
Let \( c \in \mathbb{R} \) and \( \{ u_k \} \) be a sequence such that \( J_{\varepsilon,\alpha,h}(u_k) \to c \) and \( J'_{\varepsilon,\alpha,h}(u_k) \to 0 \). If there is \( \beta > 0 \) such that, up to subsequences, \( \| u_k \|_\varepsilon \geq \beta \), then by (3.16)
\[
\frac{1}{p} \| u_k \|_\varepsilon^{p-1} \leq \frac{c + \theta \alpha \frac{1}{2} f(\alpha \frac{1}{2}) |\Omega|}{\beta} + (1 - \theta) c_1 + o(1)
\]
so \( \{ u_k \} \) is bounded and the previous Theorem completes the proof.

Let us state a regularity result (see [23, 24, 14] and references therein).

**Theorem 3.5.** Let \( B \) be a bounded subset of \( W_0^{1,p}(\Omega) \) and \( \varepsilon > 0 \). There exist \( \eta \in (0, 1) \) and \( K > 0 \) such that, for any \( \alpha \in [0, 1] \) and \( h \in C^1_0(\Omega) \) with \( \| h \|_{C^1(\Omega)} \leq 1 \), if \( u \in B \) solves
\[
-\varepsilon^p \text{div}((|\nabla u|^2 + \alpha)^{(p-2)/2} \nabla u) + u (\alpha + u^2)^{(p-2)/2} = f_\alpha(u) + h(x)
\]
then \( u \in C^{1,\eta}(\Omega) \) and \( \| u \|_{C^{1,\eta}(\Omega)} \leq K \).

Now let us consider a critical point \( u_0 \) of \( J_{\varepsilon,\alpha,h} \). Assume that \( \alpha \) and \( h \) satisfy the assumptions of the previous theorem, so that \( u_0 \in C^{1,\eta}(\Omega) \), for some \( \eta \in (0, 1) \). It is crucial to give a notion of Morse index, which is not standard, as \( J_{\varepsilon,\alpha,h} \) is not \( C^2 \). If \( \alpha > 0 \) and \( u \in W^{1,\infty}(\Omega) \), let us define on \( W_0^{1,2}(\Omega) \) the following bilinear form
\[
B_\alpha(u)(z_1, z_2) = \int_\Omega \Psi''_{\varepsilon,\alpha}(\nabla u)[\nabla z_1, \nabla z_2] + \int_\Omega b'_\alpha(u) z_1 z_2
\]
where \( \Psi_{\varepsilon,\alpha} \) and \( b_\alpha \) are defined by (3.10), hence
\[
B_\alpha(u)(z_1, z_2) = \varepsilon^p \int_\Omega \frac{(\nabla z_1/\nabla z_2)}{(\alpha + |\nabla u|^2)^{\frac{p-2}{2}}} - \varepsilon^p (2 - p) \int_\Omega \frac{(\nabla u/\nabla z_1)(\nabla u/\nabla z_2)}{(\alpha + |\nabla u|^2)^{\frac{p-2}{2}}}
\]
\[
+ \int_\Omega \frac{\alpha + (p-1)u^2}{(\alpha + u^2)^{\frac{p-2}{2}}} z_1 z_2
\]
\[
- \int_\Omega f'(u^+) \left( (\alpha + u^+)^{1/s} \right) \frac{(u^+)^{2s-2}}{(\alpha + (u^+)^s)^{\frac{2s-2}{2}}} z_1 z_2
\]
\[
- \int_\Omega f((\alpha + u^+)^{1/s}) \alpha (s-1)(u^+)^{s-2} \left( \alpha + (u^+)^s \right)^{\frac{s-2}{2}} z_1 z_2.
\]
In addition, we introduce $Q^\alpha_u : W^{1,2}_0(\Omega) \to \mathbb{R}$ defined by

$$Q^\alpha_u(z) = B^\alpha(u)(z, z).$$

The definition of $B^\alpha(u)$ is inspired by the formal second derivative of $J_{\varepsilon, \alpha, h}$ in $u$. Let us point out that, as $p < 2$, for any $u \in W^{1,\infty}(\Omega)$, $B^\alpha(u)$ and $Q^\alpha_u$ are well defined on $W^{1,2}_0(\Omega)$, but not on $W^{1,p}_0(\Omega)$.

In particular, $Q^\alpha_{u_0}$ is a smooth quadratic form on $W^{1,2}_0(\Omega)$ and we define the Morse index of $J_{\varepsilon, \alpha, h}$ at $u_0$ (denoted by $m(J_{\varepsilon, \alpha, h}, u_0)$) as the supremum of the dimensions of the linear subspaces of $W^{1,2}_0(\Omega)$ where $Q^\alpha_{u_0}$ is negative definite. Analogously, the large Morse index of $J_{\varepsilon, \alpha, h}$ at $u_0$ (denoted by $m^*(J_{\varepsilon, \alpha, h}, u_0)$) is the supremum of the dimensions of the linear subspaces of $W^{1,2}_0(\Omega)$ where $Q^\alpha_{u_0}$ is negative semidefinite. We clearly have $m(J_{\varepsilon, \alpha, h}, u_0) \leq m^*(J_{\varepsilon, \alpha, h}, u_0) < +\infty$. This notion of Morse index is crucial in order to get estimates of the critical groups.

Indeed, the following result gives a description of the critical group of the functional $J_{\varepsilon, \alpha, h}$ at $u_0$ in terms of the Morse index. The proof derives directly from [14, Theorem 2.3] (see also [13, Theorem 1.3]).

**Theorem 3.6.** Let $h \in C^1(\overline{\Omega})$ and $\varepsilon, \alpha > 0$. If $u_0$ is a critical point of $J_{\varepsilon, \alpha, h}$ and

$$m(J_{\varepsilon, \alpha, h}, u_0) = m^*(J_{\varepsilon, \alpha, h}, u_0),$$

then $u_0$ is an isolated critical point of $J_{\varepsilon, \alpha, h}$ and

$$\begin{cases} 
C_m(J_{\varepsilon, \alpha, h}, u_0) \cong \mathbb{K} & \text{if } m = m(J_{\varepsilon, \alpha, h}, u_0), \\
C_m(J_{\varepsilon, \alpha, h}, u_0) = \{0\} & \text{if } m \neq m(J_{\varepsilon, \alpha, h}, u_0).
\end{cases}$$

**Remark 3.7.** If the assumptions of the previous theorem are satisfied, the multiplicity of $u_0$ is one, namely, according to Definition 2.6, $i(J_{\varepsilon, \alpha, h}, u_0)(1) = 1$.

In order to prove Theorem 1.6, we recall an abstract theorem, proved in [15] (see also [5] and [10]).

**Theorem 3.8.** Let $A$ be an open subset of a Banach space $X$. Let $f$ be a $C^1$ functional on $A$ and $u \in A$ be an isolated critical point of $f$. Assume that there exists an open neighborhood $U$ of $u$ such that $\overline{U} \subset A$, $u$ is the only critical point of $f$ in $\overline{U}$ and $f$ satisfies the Palais–Smale condition in $\overline{U}$.

Then there exists $\bar{\mu} > 0$ such that, for every $g \in C^1(A, \mathbb{R})$ satisfying

- $\|f - g\|_{C^1(A)} < \bar{\mu},$
- $g$ satisfies the Palais–Smale condition in $\overline{U},$
- $g$ has a finite number $\{u_1, u_2, \ldots, u_m\}$ of critical points in $U$,

we have

$$\sum_{j=1}^m i(g, u_j)(t) = i(f, u)(t) + (1 + t)Q(t),$$

where $Q(t)$ is a formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$. 
4. INTERPRETATION OF MULTIPlicity: NUMBER OF DISTINCT SOLUTIONS OF APPROXIMATING PROBLEMS

Let \( \varepsilon^* \) be defined by Theorem 1.5 and \( \varepsilon \in (0, \varepsilon^*) \). If \( (P_\varepsilon) \) has at least \( 2\mathcal{P}_1(\Omega) - 1 \) distinct solutions, then the assert is proved, otherwise \( I_\varepsilon \) has a finite number of isolated critical points \( \overline{u}_1, \ldots, \overline{u}_k \) having multiplicities \( \overline{m}_1, \ldots, \overline{m}_k \) where

\[
2 \leq k < 2\mathcal{P}_1(\Omega) - 1 \quad \text{and} \quad \sum_{j=1}^{k} \overline{m}_j \geq 2\mathcal{P}_1(\Omega) - 1.
\]

Let \( \alpha_n \to 0^+ \) and \( R > 0 \) be such that \( \overline{B}_R(\overline{u}_i) \cap \overline{B}_R(\overline{u}_j) = \emptyset \), when \( i \neq j \). We set

\[
(4.1) \quad A = \bigcup_{j=1}^{k} \overline{B}_R(\overline{u}_j).
\]

If \( J_{\varepsilon,\alpha_n} \), defined by (3.2), has at least \( 2\mathcal{P}_1(\Omega) - 1 \) critical points, then we choose \( h_n = 0 \), otherwise \( J_{\varepsilon,\alpha_n} \) has \( k_n < 2\mathcal{P}_1(\Omega) - 1 \) isolated critical points \( u_1, \ldots, u_{k_n} \), having multiplicities \( m_1, \ldots, m_{k_n} \). For simplicity, we will often omit the dependence on \( n \) of \( u_i \) and their related objects. If \( n \) is sufficiently large, by (3.3) and Theorem 3.8, \( k_n \geq k \) and

\[
(4.2) \quad \sum_{i=1}^{k_n} m_i \geq \sum_{j=1}^{k} \overline{m}_j \geq 2\mathcal{P}_1(\Omega) - 1.
\]

Reasoning as in [19] (p.11-12), we obtain that, for any \( i = 1, \ldots, k_n \), there are \( V_i \) and \( W_i \) subspaces of \( W_0^{1,p}(\Omega) \) such that

1. \( W_0^{1,p}(\Omega) = V_i \oplus W_i \);
2. \( V_i \subset C^1(\Omega) \) and \( \dim V_i = m^*(J_{\varepsilon,\alpha_n}, u_i) < +\infty \);
3. \( V_i \) and \( W_i \) are orthogonal in \( L^2(\Omega) \);
4. \( Q_n^\alpha w > 0 \) for any \( w \in V_i \setminus \{0\} \).

Setting

\[
(4.3) \quad V^n = V_1 + V_2 + \cdots + V_{k_n} \quad \text{and} \quad W^n = \bigcap_{i=1}^{k_n} W_i,
\]

we still have:

- \( W_0^{1,p}(\Omega) = V^n \oplus W^n \);
- \( V^n \subset C^1(\Omega) \) is finite dimensional and \( W^n \subset W_i \) for any \( i = 1, \ldots, k_n \);
- \( \int_{\Omega} vw = 0 \) for any \( v \in V^n, w \in W^n \).

**Remark 4.1.** We see that \( \dim V^n \geq 1 \), otherwise, for each \( i = 1, \ldots, k_n \), it should be \( 0 = m^*(J_{\varepsilon,\alpha_n}, u_i) = m(J_{\varepsilon,\alpha_n}, u_i) \) \( \Rightarrow \) \( m_i = 1 \), so by (1.2.2) \( k_n \geq 2\mathcal{P}_1(\Omega) - 1 \), while we are supposing \( k_n < 2\mathcal{P}_1(\Omega) - 1 \).

In this setting the following two results hold (see Theorem 3.8 and Lemma 3.9 in [19]).
Theorem 4.2. There exist $r, \delta, M > 0$, $\beta \in (0,1]$ and $\varrho \in (0,r]$ such that for any $i \in \{1, \ldots, k_n\}$ and $v \in V^n \cap \overline{B_\varrho(0)}$ there exists one and only one $\psi_i(v) \in W^n \cap B_r(0)$ such that
\begin{equation}
\langle J'_{\varepsilon,\alpha}(u_i + v + \psi_i(v)), w \rangle = 0 \quad \forall w \in W^n.
\end{equation}

Moreover $v + \psi_i(v) \in C^{1,2}(\Omega)$, $\|v + \psi_i(v)\|_{C^{1,2}(\Omega)} \leq M$ and, denoting by
\[U_i = u_i + (V^n \cap B_\varrho(0)) + (W^n \cap B_r(0)),\]
we have $U_i \cap U_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{k_n} U_i \subset A$, where $A$ is the open bounded set defined by (4.9).

Finally
\begin{equation}
B_{\alpha_n}(u_i + v + \psi_i(v))(w, w) \geq \delta \int_\Omega |\nabla w|^2 \, dx
\end{equation}
for every $i \in \{1, \ldots, k_n\}$, $v \in V^n \cap \overline{B_\varrho(0)}$ and $w \in W^n \cap W_0^{1,2}(\Omega)$.

Lemma 4.3. For any $i = 1, \ldots, k_n$, $\psi_i$ is continuous from $V^n \cap \overline{B_\varrho(0)}$ in $W^n \cap C^1(\Omega)$ and of class $C^1$ into $W_0^{1,2}(\Omega)$. In addition,
\begin{equation}
B_{\alpha_n}(u_i + z + \psi_i(z))(h + \langle \psi_i'(z), h \rangle, w) = 0
\end{equation}
for any $z \in V^n \cap \overline{B_\varrho(0)}$, $h \in V^n$ and $w \in W_0^{1,2}(\Omega)$.

Moreover the function $\varphi_i : V^n \cap \overline{B_\varrho(0)} \to \mathbb{R}$ defined by
\[\varphi_i(v) = J'_{\varepsilon,\alpha}(u_i + v + \psi_i(v))\]
is of class $C^2$ and
\begin{equation}
\langle \varphi_i'(z), h \rangle = \langle J'_{\varepsilon,\alpha}(u_i + z + \psi_i(z)), h \rangle
\end{equation}
\begin{equation}
\langle \varphi_i''(z)h, v \rangle = B_{\alpha_n}(u_i + z + \psi_i(z))(h + \psi_i'(z)h, v)
\end{equation}
for any $z \in V^n \cap \overline{B_\varrho(0)}$ and $h, v \in V^n$.

Our aim is to build a suitable perturbation of $J_{\varepsilon,\alpha_n}$, such that all its critical points have multiplicity one.

Let us denote by $V = V^n$ and $W = W^n$ the spaces introduced in (4.3) and let $\{e_1, \ldots, e_\pi\}$ be an $L^2$-orthonormal basis of $V$, where $\pi = \dim V \geq 1$, as seen in Remark 4.1. Denoting by $V'$ the dual space of $(V, \|\cdot\|_{L^2})$, for any $v' \in V'$ we introduce $l_{v'} = \sum_{k=1}^{\pi} \langle v', e_k \rangle e_k \in V$ and $L_{v'} : W_0^{1,p}(\Omega) \to \mathbb{R}$ the functional defined by
\[L_{v'}(u) = \int_\Omega l_{v'} u \, dx.
\]

By construction, for any $i \in \{1, \ldots, k_n\}$ $u_i$ is the only critical point of $J_{\varepsilon,\alpha_n}$ in $U_i$ and $U_i \subset A$. So let $\bar{\mu}_i$ be defined by Theorem 3.3 and put $\mu = \min\{\bar{\mu}_1, \ldots, \bar{\mu}_{k_n}\}$.

We prove that there is $\gamma > 0$ such that
\begin{equation}
v' \in V' \text{ and } \|v'\|_{V'} < \gamma \quad \Rightarrow \quad \|l_{v'}\|_{C^1(\Omega)} < 1/n \quad \text{and} \quad \|L_{v'}\|_{C^1(A)} < \mu.
\end{equation}
In fact, as \( V \) is finite dimensional, there is \( c_n > 0 \) such that
\[
\| v \|_{C^1(\Omega)} \leq c_n \| v \|_{L^2} \quad \forall v \in V.
\]
As \( v' \in V' \), we have
\[
\| L_{v'} \|_{C^1(\Omega)} = \left\| \sum_{k=1}^{\pi} (v', e_k) e_k \right\|_{C^1(\Omega)} \leq \sum_{k=1}^{\pi} \| v' \|_{L^2} \| e_k \|_{C^1(\Omega)} \leq \sum_{k=1}^{\pi} \| v' \|_{L^2} c_n \| e_k \|_{L^2} \leq \| v' \|_{L^2} c_n.
\]
Moreover, \( A \) being bounded, there is \( c_A > 0 \) such that
\[
\| L_{v'} \|_{C^1(A)} \leq c_A \| L_{v'} \|_{C^1(\Omega)}.
\]
Therefore we get \( (4.9) \) by choosing \( \gamma = \frac{1}{n} \min \left\{ \frac{1}{n}, \frac{\mu}{n} \right\} \).

Set \( \gamma_1 = \gamma / k_n \). Applying Sard’s Lemma to \( \varphi' : V \to V' \), there exists \( v'_1 \in V' \) such that \( \| v'_1 \|_{V'} < \gamma_1 \) and if \( \varphi'_1(v) = v'_1 \), then \( \varphi''_1(v) \) is an isomorphism. Moreover there is \( \beta_1 > 0 \) such that if \( v' \in V' \), \( \| v' \|_{V'} \leq \beta_1 \) and \( \varphi'_1(v) = v'_1 + v' \), then \( \varphi''_1(v) \) is an isomorphism.

Analogously, for \( i = 2, \ldots, k_n \), there exist \( \beta_i > 0 \), \( \gamma_i = \min \{ \gamma_{i-1}, \beta_{i-1} / (k_n - i + 1) \} \) and \( v'_i \in V' \) such that \( \| v'_i \|_{V'} < \gamma_i \) and if \( v' \in V' \), \( \| v' \|_{V'} \leq \beta_i \) and \( \varphi'_i(v) = v'_1 + \ldots + v'_i + v' \), then \( \varphi''_i(v) \) is an isomorphism.

So, denoting by \( \overline{\nu}_n = v'_1 + \ldots + v'_{k_n} \), \( h_n = l_{\overline{\nu}_n} \) and \( J_n = J_{\varepsilon, \alpha_n, h_n} \), \( (4.9) \) shows that
\[
\| \overline{\nu}_n \| < \gamma \Rightarrow \| h_n \|_{C^1(\Omega)} < \frac{1}{n} \quad \text{and} \quad \| J_n - J_{\varepsilon, \alpha_n} \|_{C^1(A)} < \mu.
\]
Solutions to
\[
(P_n) \begin{cases}
-\varepsilon^p \text{div}((|\nabla u|^2 + \alpha_n)^{(p-2)/2} \nabla u) + u \left( \alpha_n + u^2 \right)^{(p-2)/2} = f_{\alpha_n}(u) + h_n, & \text{in } \Omega \\
u > 0, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega
\end{cases}
\]
are critical points of the functional \( J_n \).

We will prove that, when \( \bar{u} \in W_0^{1,p}(\Omega) \) and \( \bar{z} \in W_0^{1,2}(\Omega) \), we have
\[
(4.10) \quad J'_{\alpha_n}(\bar{u}) = 0 \quad \text{and} \quad B_{\alpha_n}(\bar{u})(\bar{z}, \cdot) = 0 \Rightarrow \bar{z} = 0.
\]

Indeed, since \( \bar{u} \) is a critical point of \( J_n \), \( (3.4) \) and Corollary \( 3.4 \) assure that \( \bar{u} \in U_i \), for a suitable \( i \in \{1, \ldots, k_n\} \) and for \( n \) large enough. In particular, there are \( \tilde{v} \in V \cap B_{\varepsilon}(0) \) and \( \tilde{w} \in W \cap B_{\varepsilon}(0) \) such that \( \bar{u} = u_i + \tilde{v} + \tilde{w} \).

Considering that \( V \) and \( W \) are orthogonal in \( L^2(\Omega) \),
\[
\langle J'_{\varepsilon, \alpha_n}(u_i + \tilde{v} + \tilde{w}), w \rangle = \langle J'_{\alpha_n}(\bar{u}), w \rangle + \int_{\Omega} h_n w = 0 \quad \forall w \in W.
\]
Therefore, by \( (4.1) \), \( \tilde{w} = \psi_{\varepsilon}(\tilde{v}) \).

For each arbitrary \( v \in V \)
\[
\int_{\Omega} h_n v = \int_{\Omega} \sum_{k=1}^{\pi} (\overline{\nu}_n', e_k) v e_k = \langle \overline{\nu}_n', \sum_{k=1}^{\pi} (v') \rangle = \langle \overline{\nu}_n', v \rangle
\]
thus, by \( (4.7) \), \( \langle \varphi'_i(\tilde{v}), v \rangle = \langle J'_{\varepsilon, \alpha_n}(\bar{u}), v \rangle = \langle J'_{\alpha_n}(\bar{u}), v \rangle + \int_{\Omega} h_n v = \langle \overline{\nu}_n', v \rangle \). Hence \( \varphi'_i(\tilde{v}) = \overline{\nu}_n = v'_1 + \ldots + v'_{k_n} \) and, by construction,
\[
(4.11) \quad \varphi''_i(\tilde{v}) \text{ is an isomorphism.}
\]
Let us write \( \bar{z} = \bar{v} + \bar{w} \), where \( \bar{v} \in V \) and \( \bar{w} \in W \). If \( h \in V \), combining (4.8) and (4.6), we get
\[
\langle \varphi''_i(\hat{\bar{v}})h, \bar{v} \rangle = B_{\alpha_n}(\hat{\bar{u}})(h + \langle \psi'_i(\hat{\bar{v}}), h \rangle, \bar{v}) = B_{\alpha_n}(\hat{\bar{u}})(h + \langle \psi'_i(\hat{\bar{v}}), h \rangle, \bar{z}).
\]
As we are assuming that \( B_{\alpha_n}(\hat{\bar{u}})(\bar{z}, \cdot) = 0 \),
\[
\langle \varphi''_i(\hat{\bar{v}})h, \bar{v} \rangle = 0 \quad \forall h \in V
\]
then, by (4.11), \( \bar{v} \) must be 0, so that \( \bar{z} = \bar{w} \in W \).

Recalling (4.5), we see that
\[
0 = B_{\alpha_n}(\hat{\bar{u}})(\bar{w}, \bar{w}) = \delta \int_{\Omega} |\nabla \bar{w}|^2,
\]
hence \( \bar{w} = 0 \) and this proves (4.10).

In other words, if \( \hat{\bar{u}} \) is a critical point of \( J_n \), then \( m^*(J_n, \hat{\bar{u}}) = m(J_n, \hat{\bar{u}}) \). Consequently we can apply Theorem 3.3, getting that the multiplicity of every critical point of \( J_n \) is one. Hence, by Theorem 3.8 and (1.2), \( J_n \) has at least \( \hat{\bar{u}}^1_n, \ldots, \hat{\bar{u}}^2_{2^p_n(\Omega)} \) distinct critical points. It remains to be proved that any \( \hat{\bar{u}}^i_n \) is positive. From Theorem 3.5 the critical points of \( J_n \) are uniformly bounded in \( C^1_{\eta}(\Omega) \), thus, up to subsequences, \( \hat{\bar{u}}^i_n \) converges in \( C^1(\Omega) \) to \( \hat{\bar{u}}_j \), for \( n \to +\infty \). As \( \hat{\bar{u}}_j \) solves \( (P_{\varepsilon}) \), by Theorem 5 in [28] we infer that \( \hat{\bar{u}}_j > 0 \) and \( \frac{\partial \hat{\bar{u}}_j}{\partial \nu}(x_0) > 0 \), where \( x_0 \in \partial \Omega \) and \( \nu \) is the interior normal to \( \partial \Omega \) at \( x_0 \). This implies \( \hat{\bar{u}}^i_n > 0 \) on \( \Omega \), for \( n \) sufficiently large.

\[ \square \]

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