Triply charmed and bottom dibaryons in the one boson exchange model

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Abstract

The pentaquark states, $P_{c}(4312)$, $P_{c}(4440)$ and $P_{c}(4457)$, could be nicely arranged into a multiplet of seven molecules of $\bar{D}^{(*)}\Sigma_{c}^{(*)}$ dictated by heavy quark spin symmetry, while the $\Xi_{cc}^{(*)}\Sigma_{c}^{(*)}$ system can be related to the $\bar{D}^{(*)}\Sigma_{c}^{(*)}$ system via heavy antiquark diquark symmetry. In this work we employ the one boson exchange model to study the interactions between $\Xi_{cc}^{(*)}$ and $\Sigma_{c}^{(*)}$ with constraints from the pentaquark system. We show that a multiplet of ten triply charmed dibaryons emerge naturally in the isospin 1/2 sector, where only 3 appear in the isospin 3/2 sector. In addition, we study their heavy quark flavor partners. Ten triply bottom diybaryons are found in the isospin 1/2 sector while only 7 are likely in the isospin 3/2 sector. Furthermore, the predicted mass splitting between the $0^{+}\Xi_{cc}\Sigma_{c}$ state and its $1^{+}$ counterpart is found to be correlated with that between the $P_{c}(4457)$ and the $P_{c}(4440)$, dictated by heavy antiquark diquark symmetry as recently pointed out in Ref. [e-Print: 1907.11220].
I. INTRODUCTION

In 2019 the LHCb collaboration surprised the hadron physics community by reporting the observation of three pentaquark states, $P_c(4312)$, $P_c(4440)$ and $P_c(4457)$ \cite{1}, updating their 2015 study \cite{2}. Given the fact that the $P_c(4312)$ is located 9.8 MeV below the $\bar{D}\Sigma_c$ threshold, and the $P_c(4440)$ and $P_c(4457)$ are 21.8 and 4.8 MeV below the $\bar{D}^*\Sigma_c$ one, they provide the most robust candidates so far for hadronic molecules.\footnote{1 It should be noted that the existence of $\bar{D}^{(*)}\Sigma_c^{(*)}$ molecules had been predicted before the first LHCb results \cite{3-8} and although at present the molecular interpretation is the most favored one, there exist other explanations, e.g., hadro-charmonium \cite{9}, compact pentaquark states \cite{10,16}, or virtual states \cite{17}. See Refs. \cite{18,20} for some latest reviews.} Indeed, a large amount of theoretical works have been performed within the molecular picture, focusing on various aspects ranging from mass spectroscopy \cite{21-34}, decays \cite{35-42}, to production mechanisms \cite{43-47}. It is interesting to note that these three pentaquark states might be part of a multiplet of seven molecular states \cite{21,25,27,28,30,48}, dictated by heavy quark spin symmetry (HQSS) \cite{49,50}.

Since the Belle collaboration discovered the $X(3872)$ in 2003 \cite{51} and the BESIII collaboration discovered the $Z_c(3900)$ \cite{52} and the $Z_c(4020)$ \cite{53}, many studies have been performed to explore the existence of multiplets of hadronic molecules in the meson-meson sector. For instance, assuming that the $X(3872)$ is a $1^{++} \bar{D}D^*$ state, a $2^{++} \bar{D}^*D^*$ state has been predicted \cite{54,55}. With further assumptions on the nature of some other states, such as $X(3915)$ \cite{54} or $Z_b(10610)$ \cite{55} or adoption of more model dependent approaches, such as the one-boson-exchange (OBE) model \cite{56}, more states can be predicted. Nevertheless, whether or not a complete multiplet of molecules exists remains unsettled in the meson-meson sector, partly due to the fact that the weak attraction between $\bar{D}$ and $D^*$ deduced from the small binding of the $X(3872)$ could easily disappear or even turn repulsive away from the strict HQS limit, which might explain the still absence of its spin 2 partner, $X(4012)$.

The new LHCb data provide us an opportunity to study multiplets of hadronic molecules in the meson-baryon system. In our previous work we adopted the contact range effective field theory (EFT) to describe the three pentaquark states in a hadronic molecular picture, and a complete multiplet of hadronic molecules emerges \cite{21}, which has been later corroborated by many studies \cite{25,27,28}.

Heavy antiquark diquark symmetry (HADS) dictates that an heavy antiquark behaves the same as a heavy di-quark form the perspective of the strong interaction in the limit of heavy quark
masses \cite{57,58}. Therefore, the strong interaction experienced by a heavy anti-charm meson is the same as that of a doubly charmed baryon. As a result, the interaction between $\bar{D}^{(*)}$ and $\Sigma_c^{(*)}$ is the same as that between $\Xi_{cc}^{(*)}$ and $\Sigma_c^{(*)}$, as shown in Fig. 1, up to the breaking of HADS.

In Ref. \cite{59}, HADS was utilized in the contact range EFT to explore the likely existence of $\Xi_{cc}^{(*)}\Sigma_c^{(*)}$ dibaryons implied by the existence of the pentaquark states, and a complete multiplet of ten dibaryons emerges even taking into account the breaking of HADS at the level of 25%. Another interesting finding of Ref. \cite{59} is the existence of a strong correlation between the mass splitting of any doublet in the dibaryon system and the spin assignments of the $P_c(4440)$ and $P_c(4457)$. Though an lattice QCD study of the pentaquark system is difficult \cite{60,61}, the study of the dibaryon systems seems to be relatively easier and has recently been performed in Ref. \cite{62}. However, unfortunately, only the deuteron like $1^+$ states were studied. If the spectrum of the $0^+$ states can be obtained, one could determine the spins of the $P_c(4440)$ and $P_c(4457)$ model independently using the correlation dictated by HADS and first observed in Ref. \cite{59}.

FIG. 1. From hidden charm pentaquark states to triply charmed Hexaquark dibaryons through HADS

In this work we adopt the OBE model to study the interactions between $\Xi_{cc}^{(*)}$ and $\Sigma_c^{(*)}$. The OBE model is one of the most widely used theoretical tools in studying heavy hadron molecules, see, e.g., Refs. \cite{63,64}. According to this model, the potential between two hadrons is generated by the exchange of a series of light mesons, such as the $\pi, \sigma, \rho$ and $\omega$. Though physically intuitive and numerically clean, the OBE model suffers from the need of regulators (form factors and cutoffs) to mimic the finite sizes of hadrons involved. Given the fact there is no a priori information which cutoff to use\footnote{We implicitly assume that different regulators should yield the same physical results, if the theory is properly normalized, and focus on the impact of the value of the cutoff in a cutoff regularization scheme.}, though according to the size of typical hadrons, a value of 0.5 to 1 GeV is more preferred \cite{65,66}. In the nucleon-nucleon system, because of the large nucleon-nucleon scattering data set, the form factors and related cutoffs can be determined by fitting to the large amount of data \cite{67}. While in the heavy sector, data are scarce, as a result, in most cases the cutoff has to be
determined by fitting to the same data that one wants to describe, therefore giving the impression that the model has no predictive power. In this sense, the OBE model should better be used to estimate the interaction between two hadrons by varying the cutoffs within a reasonable range. On the other hand, in a few cases where there is a clean bound state candidate of two hadrons, one can fix the cutoff by reproducing such a state first and then make prediction for other related systems using symmetry arguments. This is the case for the present study. In this work, following Ref. [28] and assuming that the \( P_c(4312) \) is a bound state of \( \bar{D}\Sigma_c \), we can fix the cutoff, i.e., \( \Lambda = 1.119 \text{ GeV} \). With this cutoff and the OBE potential we study the \( \Xi^{(*)}_{cc} \Sigma_c^{(*)} \) system.

The manuscript is structured as follows. In Section II we present the details of the OBE model as applied to the baryon-baryon system containing heavy quarks \( c \) and \( b \). In Section III we determine the cut-off in the OBE model by reproducing the \( P_c(4312) \) as a hadronic molecule and from this we predict the full spectrum of triply heavy baryon molecules. Finally we present the conclusions in Section IV.

II. FORMULISM

In this section we derive the OBE potentials that we use in this work. The OBE interaction of two heavy hadrons is generated by the exchange of \( \pi \), \( \sigma \), \( \rho \) and \( \omega \). Among them, the vector mesons, \( \rho \) and \( \omega \), provide the short-range interaction, the scalar meson \( \sigma \) provides the middle-range interaction, and the \( \pi \) meson provides the long-range interaction. Given the exploratory nature of the present work, the contributions of other mesons are neglected [28, 68].

A. Interaction Lagrangian

The Lagrangians describing the interaction between a doubly heavy charmed baryon and a light meson, \( \rho, \omega, \sigma \), and \( \pi \), can be written as [56]

\[
\mathcal{L}_{T_{cc}T_{cc}\pi} = \frac{ig_1}{\sqrt{2}f_\pi} \bar{T}_{cc}^\dagger \cdot (\bar{\pi} \cdot \vec{\nabla} \times \vec{T}_{cc}),
\]

\[
\mathcal{L}_{T_{cc}T_{cc}\sigma} = g_{\sigma 1} \bar{T}_{cc}^\dagger \bar{T}_{cc},
\]

\[
\mathcal{L}_{T_{cc}T_{cc}\rho} = g_{\rho 1} \bar{T}_{cc}^\dagger (\bar{\rho} \cdot \vec{\nabla} \cdot \vec{T}_{cc}) - \frac{f_{\rho 1}}{4M_1} \bar{T}_{cc}^\dagger \cdot (\partial^i \bar{\rho}^j - \partial^j \bar{\rho}^i) \vec{T}_{cc},
\]
\[ \mathcal{L}_{\tilde{T}_{cc}T_{cc}\omega} = g_{\omega 1} \tilde{T}_{cc}^\dagger \omega^0 \cdot \tilde{T}_{cc}^\dagger \]
\[ - \frac{f_{\omega 1}}{4 M_1} \tilde{T}_{cci}^\dagger \left( \partial^j \omega^i - \partial^i \omega^j \right) \tilde{T}_{cj}^\dagger, \]

where \( \tilde{T}_{cc} = \left( \frac{1}{\sqrt{3}} \Xi_{cc} \bar{\bar{\sigma}} + \Xi_{cc}^* \right) \) is a superfield of \( \Xi_{cc} \) and \( \Xi_{cc}^* \) constrained by HQSS. With HADS these Lagrangians can also be derived from the Lagrangians describing the interaction between a heavy meson and a light meson [69]. The \( g_1 \) and \( g_\sigma \) are the couplings to the \( \pi \) and \( \sigma \) meson, respectively, while the \( g_v \) and \( f_v \) with \( V = \rho, \omega \) are the electro- and magnetic-type couplings to the vector mesons, and \( M_1 \) is a mass scale rendering \( f_v \) dimensionless.

For the Lagrangians describing the interaction between a singly charmed baryon and a light meson, we have [28]

\[ \mathcal{L}_{S_cS_c\pi} = \frac{ig_2}{\sqrt{2} f_{\pi}} \tilde{S}_c^\dagger \cdot (\tilde{T} \cdot \pi \bar{\bar{\nabla}} \times \tilde{S}_c), \]
\[ \mathcal{L}_{S_cS_c\sigma} = g_{\sigma 2} \tilde{S}_c^\dagger \cdot \tilde{S}_c, \]
\[ \mathcal{L}_{S_cS_c\rho} = g_{\rho 2} \tilde{S}_c^\dagger (\tilde{T} \cdot \rho^0) \cdot \tilde{S}_c \]
\[ - \frac{f_{\rho 2}}{4 M} \tilde{S}_ci^\dagger \tilde{T} \cdot \left( \partial^j \rho^i - \partial^i \rho^j \right) \tilde{S}_{cj}, \]
\[ \mathcal{L}_{S_cS_c\omega} = g_{\omega 2} \tilde{S}_c^\dagger \omega^0 \cdot \tilde{S}_c \]
\[ - \frac{f_{\omega 2}}{4 M} \tilde{S}_ci^\dagger \left( \partial^j \omega^i - \partial^i \omega^j \right) \tilde{S}_{cj}, \]

where \( \tilde{S}_c = \left( \frac{1}{\sqrt{3}} \Sigma_c \bar{\bar{\sigma}} + \Sigma_c^* \right) \) denote the superfield of \( \Sigma_c \) and \( \Sigma_c^* \) dictated by HQSS, and \( g_2, g_{\sigma 2}, g_{\rho 2} \) and \( f_{\omega 2} \) are the corresponding coupling constants.

B. The OBE potentials

With the Lagrangians for the doubly charmed and singly charmed baryons, the OBE potentials can be easily derived as follows,

\[ V = \zeta V_\pi + V_\sigma + V_\rho + \zeta V_\omega, \]

where \( \zeta = \pm 1 \), for which our convention is

\[ \zeta = +1 \quad \text{for } T_{cc}S_c, \]
\[ \zeta = -1 \quad \text{for } \bar{T}_{cc}^\dagger S_c . \]
and $V_{\pi,\sigma,\rho,\omega}$ in the momentum space read

\begin{align*}
V_\pi(q) &= \vec{\tau} \cdot \vec{T} \frac{g_1 g_2 \bar{\sigma}_1 \cdot \bar{q} \bar{\sigma}_2 \cdot \bar{q}}{2f_\pi^2 \vec{q}^2 + \mu_\pi^2}, \\
V_\sigma(q) &= -\frac{g_\sigma g_{\sigma 2}}{\vec{q}^2 + m_\sigma^2} \\
V_\rho(q) &= \vec{\tau} \cdot \vec{T} \left[ \frac{g_\rho g_{\rho 2}}{\vec{q}^2 + m_\rho^2} \right. \\
&\quad \left. - \frac{f_\rho_1 f_\rho_2}{2M_1 2M_2} \frac{(\bar{\sigma}_1 \times \bar{q}) \cdot (\bar{\sigma}_2 \times \bar{q})}{\vec{q}^2 + \mu_\rho^2} \right], \\
V_\omega(q) &= -\frac{g_\omega g_{\omega 2}}{\vec{q}^2 + m_\omega^2} \\
&\quad + \frac{f_\omega_1 f_\omega_2}{2M_1 2M_2} \frac{(\bar{\sigma}_1 \times \bar{q}) \cdot (\bar{\sigma}_2 \times \bar{q})}{\vec{q}^2 + \mu_\omega^2},
\end{align*}

The coordinate space potentials can be obtained by Fourier transforming the momentum space potentials, and read

\begin{align*}
V_\pi(r) &= \vec{\tau} \cdot \vec{T} \frac{g_1 g_2}{6f_\pi^2} \left[ - \bar{\sigma}_1 \cdot \bar{\sigma}_2 \delta(\vec{r}) \\
&\quad + \bar{\sigma}_1 \cdot \bar{\sigma}_2 m_\pi^3 W_Y(\mu_\pi r) \\
&\quad + S_{12}(\bar{\vec{r}}) m_\pi^3 W_T(m_\pi r) \right], \\
V_\sigma(r) &= -g_\sigma g_{\sigma 2} m_\sigma W_Y(m_\sigma r), \\
V_\rho(r) &= \vec{\tau} \cdot \vec{T} \left[ g_\rho g_{\rho 2} m_\rho W_Y(m_\rho r) \\
&\quad + \frac{f_\rho_1 f_\rho_2}{2M_1 2M_2} \left( - \frac{2}{3} \bar{\sigma}_1 \cdot \bar{\sigma}_2 \delta(\vec{r}) \\
&\quad + \frac{2}{3} \bar{\sigma}_1 \cdot \bar{\sigma}_2 m_\rho^3 W_Y(m_\rho r) \\
&\quad - \frac{1}{3} S_{12}(\bar{\vec{r}}) m_\rho^3 W_T(m_\rho r) \right) \right], \\
V_\omega(r) &= g_\omega g_{\omega 2} m_\omega W_Y(m_\omega r) \\
&\quad + \frac{f_\omega_1 f_\omega_2}{2M_1 2M_2} \left( - \frac{2}{3} \bar{\sigma}_1 \cdot \bar{\sigma}_2 \delta(\vec{r}) \\
&\quad + \frac{2}{3} \bar{\sigma}_1 \cdot \bar{\sigma}_2 m_\omega^3 W_Y(m_\omega r) \\
&\quad - \frac{1}{3} S_{12}(\bar{\vec{r}}) m_\omega^3 W_T(\mu_\omega r) \right),
\end{align*}

where the dimensionless functions $W_Y(x)$ and $W_T(x)$ are defined as

\begin{align*}
W_Y(x) &= e^{-x} \quad (20) \\
W_T(x) &= \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) e^{-x} \quad (21)
\end{align*}
with $S_{12}$ the standard tensor operator

$$S_{12}(\hat{r}) = 3\hat{\sigma}_1 \cdot \hat{r} \hat{\sigma}_2 \cdot \hat{r} - \hat{\sigma}_1 \cdot \hat{\sigma}_2$$  \hspace{1cm} (22)

C. Form factors

We have derived the OBE potentials assuming that hadrons are point-like particles, but in reality they are not. To take into account the finite sizes of hadrons, we introduce a monopolar form factor for each vertex, and then the resulting potentials in momentum space can be written as

$$V_M(q, \Lambda) = V_M(\vec{q}) F_1(\vec{q}, \Lambda_1) F_2(\vec{q}, \Lambda_2) ,$$ \hspace{1cm} (23)

where the monopolar form factor is

$$F(q, \Lambda) = \frac{\Lambda^2 - m^2}{\Lambda^2 - q^2} .$$ \hspace{1cm} (24)

For the sake of simplicity, we adopt the same cutoff for both vertices appearing in the one-boson exchange. With the consideration of form factors, the coordinate space potentials can be easily obtained by the following substitutions from those of Eqs.(20-21)

$$\delta(r) \rightarrow m^3 d(x, \lambda) ,$$ \hspace{1cm} (25)

$$W_Y(x) \rightarrow W_Y(x, \lambda) ,$$ \hspace{1cm} (26)

$$W_T(x) \rightarrow W_T(x, \lambda) ,$$ \hspace{1cm} (27)

where $\lambda = \Lambda/m$, $d$, $W_Y$ and $W_T$ read

$$d(x, \lambda) = \frac{(\lambda^2 - 1)^2}{2\lambda} \frac{e^{-\lambda x}}{4\pi} ,$$ \hspace{1cm} (28)

$$W_Y(x, \lambda) = W_Y(x) - \lambda W_Y(\lambda x) \left[ -\frac{(\lambda^2 - 1)}{2\lambda} \frac{e^{-\lambda x}}{4\pi} \right] ,$$ \hspace{1cm} (29)

$$W_T(x, \lambda) = W_T(x) - \lambda^3 W_T(\lambda x) \left[ -\frac{(\lambda^2 - 1)}{2\lambda} \lambda^2 \left( 1 + \frac{1}{\lambda x} \right) \frac{e^{-\lambda x}}{4\pi} \right] .$$ \hspace{1cm} (30)

Note that in the present work, following Ref. [28], we remove the dirac delta potential, $\delta(\vec{r})$, in the spin-spin component of the OBE potential. The delta potential, as can be seen in Fig. 2 of Ref. [28], distorts the long-range one pion exchange potential at a distance of about 1 fm, thus obscuring the physics of the pion exchange, which given its long range is expected to be model
independent and should not be distorted. In Ref. [28], it was shown that only by removing this
delta potential, one can achieve a simultaneous description of the three pentaquark states with a
single cutoff fixed by reproducing either of the three states. Otherwise, one will be forced to adopt
different cutoffs for different pentaquark states. We note that the RCNP group also follows such a
procedure in their use of the OBE model [70,72].

D. Coupling constants

| Coupling Value for $\Xi_{cc}/\Xi_{cc}^*$ | Coupling Value for $\Sigma_{c}/\Sigma_{c}^*$ |
|------------------------------------------|------------------------------------------|
| $g_1$                                    | $g_2$                                    |
| -0.2                                     | 0.84                                     |
| $g_{\sigma_1}$                           | $g_{\sigma_2}$                           |
| 3.4                                      | 6.8                                      |
| $g_{\rho_1}$                             | $g_{\rho_2}$                             |
| 2.6                                      | 5.8                                      |
| $g_{\omega_1}$                           | $g_{\omega_2}$                           |
| 2.6                                      | 5.8                                      |
| $\kappa_{\rho_1}$                        | $\kappa_{\rho_2}$                        |
| -0.8                                     | 1.7                                      |
| $\kappa_{\omega_1}$                      | $\kappa_{\omega_2}$                      |
| -0.8                                     | 1.7                                      |
| $M_1$                                    | $M_1$                                    |
| 940                                      | 940                                      |

The OBE potentials are determined by the couplings of the exchanged bosons to the heavy
baryons and mesons. In this section, we explain how they are fixed and estimate how uncertain
they are. The coupling of the $\pi$ to the doubly charmed baryon can be derived from either the quark
model or HADS. Both approaches yield a similar value: $g_1 = -0.25$ [73] from the quark model
and $g_1 = -0.2$ [74] from HADS, and in this work we adopt $-0.2$. The coupling of the $\pi$ to $\Sigma_c$
and $\Sigma_{c}^*$, $g_2$, was extracted to be 0.84 in lattice QCD [75], which is smaller than the prediction
of the quark model [76].

For the couplings to the $\sigma$ meson, we estimate them using the quark model. From the nucleon
and $\sigma$ meson coupling of the linear sigma model, $g_{\sigma NN} = 10.2$, the corresponding couplings are
determined to be $g_{\sigma_1} = \frac{1}{3}g_{\sigma NN} = 3.4$ and $g_{\sigma_2} = \frac{2}{3}g_{\sigma NN} = 6.8$ [77].
The couplings of the light vector mesons contain both electric-type \((g_v)\) and magnetic-type \((f_v)\) ones, which are related via \(f_v = \kappa_v g_v\). For the \(\rho\) and \(\omega\) couplings, from SU(3)-symmetry and the OZI rule, we obtain \(g_\omega = g_\rho\) and \(f_\omega = f_\rho\). The electric-type coupling of the doubly charmed baryon to the vector mesons is estimated to be \(g_{v1} = 2.6\) [78], and the corresponding coupling to the singly charmed baryon is determined to be \(g_{v2} = 5.8\) [76]. The magnetic-type coupling of the \(\Sigma_c^{(*)}\) baryon is estimated to be \(\kappa_{v2} = 1.7\) [79], and the corresponding coupling of the \(\Xi_{cc}^{(*)}\) baryon is \(\kappa_{1v} = -0.8\). All the couplings are given in Table I for easy reference.

E. Wave functions and partial wave decomposition

The generic wave function of a baryon-baryon system reads

\[
|\Psi \rangle = \Psi_{JM}(\vec{r})|IM_I\rangle,
\]

where \(|IM_I\rangle\) denotes the isospin wave function and \(\Psi_{JM}(\vec{r})\) the spin and spatial wave function. The dynamics in isospin space is embodied in the isospin factor \(\tau \cdot T\). In this work the total isospin is either \(1/2\) or \(3/2\) for the \(\Xi_{cc}^{(*)}\Sigma_{cc}^{(*)}\) system, and the corresponding isospin factor are \(-2\) and \(1\), respectively.

The spin wave function can be written as a sum over partial wave functions, which can be written as (in the spectroscopic notation)

\[
|^{2S+1}L_J\rangle = \sum_{M_S, M_L} \langle LM_L S M_S | JM \rangle |S M_S\rangle Y_{LM_L}(\hat{r}) ,
\]

where \(\langle LM_L S M_S | JM \rangle\) are the Clebsch-Gordan coefficients, \(|S M_S\rangle\) the spin wavefunction, and \(Y_{LM_L}(\hat{r})\) the spherical harmonics.

In the partial wave decomposition of the OBE potential, we encounter both spin-spin and tensor components

\[
C_{12} = \vec{\sigma}_1 \cdot \vec{\sigma}_2 ,
\]

\[
S_{12} = 3\vec{\sigma}_1 \cdot \hat{r} \vec{\sigma}_2 \cdot \hat{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2 ,
\]

In the present study, we consider both \(S\) and \(D\) waves. The relevant matrix elements are listed in Table II.
TABLE II. Relevant partial wave matrix elements for the $\Xi_{QQ}^{(*)}\Sigma_{Q}^{(*)}$ system with $Q = c$ or $b$.

| State | $J^P$ | Partial wave | $\langle \vec{a}_1 \cdot \vec{a}_2 \rangle$ | $S_{12}(\vec{a}_1, \vec{a}_2, \vec{r})$ |
|-------|-------|--------------|---------------------------------|---------------------------------|
| $\Xi_{QQ}^{(*)}\Sigma_{Q}^{(*)}$ | $J = 0$ | $^1S_0$ | -3 | 0 |
| $\Xi_{QQ}^{(*)}\Sigma_{Q}^{(*)}$ | $J = 1$ | $^3S_1^{-3}D_1$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $0 \sqrt{8}$ |
| $\Xi_{QQ}^{(*)}\Sigma_{Q}^{(*)}$ | $J = 1$ | $^3S_1^{-3}D_1$ | $\begin{pmatrix} -\frac{5}{2} & 0 & 0 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}$ | $0 \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \end{pmatrix}$ |
| $\Xi_{QQ}^{(*)}\Sigma_{Q}^{(*)}$ | $J = 1$ | $^5S_2^{-3}D_2^{-5}D_2$ | $\begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}$ | $0 \begin{pmatrix} 3\frac{3}{10} & 3\frac{7}{7} \end{pmatrix}$ |
| $\Xi_{QQ}^{(*)}\Sigma_{Q}^{(*)}$ | $J = 0$ | $^1S_0^{-5}D_0$ | $\begin{pmatrix} -\frac{15}{4} & 0 \\ 0 & -\frac{3}{4} \end{pmatrix}$ | $0 \begin{pmatrix} -3 \\ -3 \end{pmatrix}$ |
| $\Xi_{QQ}^{(*)}\Sigma_{Q}^{(*)}$ | $J = 1$ | $^3S_1^{-3}D_1^{-5}D_1^{-7}D_1$ | $\begin{pmatrix} -\frac{11}{4} & 0 & 0 & 0 \\ 0 & -\frac{11}{4} & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{9}{4} \end{pmatrix}$ | $0 \begin{pmatrix} \frac{17}{5\sqrt{2}} & 0 & -\frac{3\sqrt{7}}{5} \end{pmatrix}$ |
| $\Xi_{QQ}^{(*)}\Sigma_{Q}^{(*)}$ | $J = 2$ | $^5S_2^{-1}D_2^{-3}D_2^{-5}D_2^{-7}D_2$ | $\begin{pmatrix} -\frac{3}{4} & 0 & 0 & 0 & 0 \\ 0 & -\frac{15}{4} & 0 & 0 & 0 \\ 0 & 0 & -\frac{11}{4} & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{9}{4} \end{pmatrix}$ | $0 \begin{pmatrix} -\frac{3}{\sqrt{5}} & 0 & 3\frac{7}{10} & 0 \end{pmatrix}$ |
| $\Xi_{QQ}^{(*)}\Sigma_{Q}^{(*)}$ | $J = 3$ | $^7S_3^{-3}D_3^{-5}D_3^{-7}D_3$ | $\begin{pmatrix} \frac{9}{4} & 0 & 0 & 0 \\ 0 & \frac{11}{4} & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{9}{4} \end{pmatrix}$ | $0 \begin{pmatrix} -\frac{3\sqrt{3}}{5} & 0 & \frac{5\sqrt{3}}{9} \end{pmatrix}$ |
TABLE III. Binding energies of the triply charmed and bottomed dibaryons with $I = 1/2$, with uncertainties originating from the breaking of HADS, and the corresponding masses, for which only central values are given.

| Molecule | $I$ | $J^P$ | $B_{H_{exc}}$ (MeV) | $M_{H_{exc}}$ (MeV) | $B_{H_{bb}}$ (MeV) | $M_{H_{bb}}$ (MeV) |
|----------|-----|-------|----------------------|-------------------|---------------------|-------------------|
| $\Sigma_Q \Xi_Q$ | $\frac{1}{2}$ | $0^+$ | $29.4^{+24.1}_{-18.4}$ | $6046$ | $89.7^{+32.8}_{-30.1}$ | $15850$ |
| $\Sigma_Q \Xi_Q$ | $\frac{1}{2}$ | $1^+$ | $21.6^{+20.1}_{-14.6}$ | $6053$ | $76.1^{+29.9}_{-26.5}$ | $15864$ |
| $\Sigma^*_Q \Xi_Q$ | $\frac{1}{2}$ | $1^+$ | $29.8^{+24.2}_{-18.6}$ | $6109$ | $89.3^{+32.8}_{-30.1}$ | $15872$ |
| $\Sigma^*_Q \Xi_Q$ | $\frac{1}{2}$ | $2^+$ | $21.4^{+19.9}_{-14.5}$ | $6118$ | $74.5^{+28.6}_{-26.0}$ | $15886$ |
| $\Sigma^*_Q \Xi_Q$ | $\frac{1}{2}$ | $1^+$ | $15.6^{+16.7}_{-11.5}$ | $6167$ | $64.3^{+26.1}_{-23.3}$ | $15900$ |
| $\Sigma_Q \Xi_Q$ | $\frac{1}{2}$ | $0^+$ | $34.5^{+27.0}_{-21.0}$ | $6148$ | $99.6^{+36.1}_{-33.1}$ | $15861$ |
| $\Sigma^*_Q \Xi_Q$ | $\frac{1}{2}$ | $0^+$ | $13.9^{+15.2}_{-10.2}$ | $6237$ | $58.6^{+24.1}_{-21.7}$ | $15926$ |
| $\Sigma^*_Q \Xi_Q$ | $\frac{1}{2}$ | $1^+$ | $16.5^{+17.3}_{-12.0}$ | $6230$ | $65.2^{+26.0}_{-23.6}$ | $15920$ |
| $\Sigma^*_Q \Xi_Q$ | $\frac{1}{2}$ | $2^+$ | $24.7^{+21.8}_{-16.2}$ | $6222$ | $80.7^{+30.7}_{-27.9}$ | $15904$ |
| $\Sigma^*_Q \Xi_Q$ | $\frac{1}{2}$ | $3^+$ | $35.7^{+27.0}_{-21.3}$ | $6211$ | $98.5^{+35.3}_{-32.6}$ | $15886$ |

III. RESULTS AND DISCUSSION

The main ambiguity of the OBE model is the value of the cutoff in the form factor, which in principle is a free parameter in the range of $0.5 \sim 2$ GeV. However, cutoffs in this range heavily affect the resulting interaction. For the case of heavy hadron molecules, this means that there is either a bound state or there is not. One way to make concrete predictions is to fix the cutoff using an established molecular state as a reference, and then with this value one predicts interactions in other related systems and study the likely existence of molecular states. Such an approach has been adopted in a number of studies. For instance, assuming that the $X(3872)$ is a hadronic molecule of $\bar{D}D^*$, the cutoff was determined to $\Lambda = 1.04$ GeV [56]. In Ref [28], the cutoff was determined to be $\Lambda = 1.119$ GeV assuming that $P_c(4312)$ is a bound state of $\bar{D}\Sigma_c$. According to HADS, the interactions between $\bar{D}^{(*)}$ and $\Sigma_c^{(*)}$ are the same as those between $\Xi^{(*)}_{cc}$ and $\Sigma_c^{(*)}$ [69], up to corrections of the order of 25% (see below for more discussions). Thus in this work we adopt $\Lambda = 1.119$ GeV to study the triply charmed and bottom dibaryon systems.
Since we have used HADS in deriving the potentials of the $\Xi_{cc}^{(*)}\Sigma_c^{(*)}$ system, we should study the breaking of the HADS, which is only exact in the infinite heavy quark mass limit. The breaking of HADS can be estimated, as a rule of thumb, by calculating $\Lambda_{QCD}/m_Q v$ with $v$ the velocity of the heavy diquark pair \[57\], which yields a value of $0.25 - 0.4$ for the charm system. In the present work, following Ref.\[59\], we take 25\% as an educated guess. The change of our potentials induced by such a breaking can be taken into account by modifying the OBE potentials in the following way:

$$V = V_{OBE}(1 + \delta_{HADS}),$$  \hspace{1cm} (35)$$

where $V_{OBE}$ is the central value of the OBE potentials derived above and $\delta_{HADS} = 0.25$ is the uncertainty induced by HADS.

With all the ingredients explained above we can solve the Schrödinger equation in coordinate space, and study the spectrum of the $\Xi_{cc}^{(*)}\Sigma_c^{(*)}$ system. Those with $I = 1/2$ are presented in Table \[III\]. Interestingly we find ten triply charmed dibaryons with binding energies of $10 \sim 40$ MeV. In our previous study we employed the contact range EFT constrained by HQSS and HADS to predict ten triply charmed dibaryons from the LHCb pentaquark states \[59\]. Our current results are consistent with those of Scenario B of that study. In addition, the predicted binding energy of the $1^+ \Xi_{cc}\Sigma_c$ state, $B = 21.6^{+20.1}_{-14.6}$ MeV, is consistent with the lattice QCD result of Ref. \[62\], which is $8 \pm 17$ MeV.

The spins of $P_c(4440)$ and $P_c(4457)$ are not determined experimentally yet, while in Ref. \[59\] we discovered that the mass splittings between any doublets of the triply charmed dibaryons are correlated with that of the $P_c(4440)$ and $P_c(4457)$. For instance, if the $0^+ \Xi_{cc}\Sigma_c$ state has a larger mass than its $1^+$ counterpart, then the spin of the $P_c(4457)$ is $3/2$ and that of the $P_c(4440)$ is $1/2$. On the other hand, if the $0^+ \Xi_{cc}\Sigma_c$ state mass is smaller than $1^+$, then the opposite assignment would be favored. The present OBE results for the $0^+ \Xi_{cc}\Sigma_c$ and the $1^+ \Xi_{cc}\Sigma_c$ seem to prefer Scenario B, which indicates that the spins of $P_c(4440)$ and $P_c(4457)$ should be $3/2$ and $1/2$, respectively.

Unlike the EFT approach, the isospin 1/2 and 3/2 sectors are correlated in the OBE model. Thus we can calculate the spectrum of the $\Xi_{cc}^{(*)}\Sigma_c^{(*)}$ system with $I = 3/2$, and the results are displayed in Table \[IV\]. Unlike the $I = 1/2$ case, not all the possible combinations have attraction strong enough to bind. As a matter of fact, only three states, $1^+ \Sigma_c\Xi_{cc}^*$, $0^+ \Sigma_c^*\Xi_{cc}^*$ and $1^+ \Sigma_c^*\Xi_{cc}^*$, bind within the uncertainties induced by the breaking of HADS.
TABLE IV. Binding energies of the triply charmed and bottomed dibaryons (if exist) with $I = 3/2$, with uncertainties originating from the breaking of HADS, and the corresponding masses, for which only central values are given. The † indicates that the particular channel does not bind, and ? denotes the likely existence of a bound state.

| Molecule          | $I$ $J^P$ | $B_{H_{cc}}$ (MeV) | $M_{H_{cc}}$ (MeV) | $B_{H_{bb}}$ (MeV) | $M_{H_{bb}}$ (MeV) |
|-------------------|-----------|--------------------|--------------------|--------------------|--------------------|
| $\Sigma_Q \Xi_{QQ}$ $3/2$ 0+ | †         | †                  | 1.8$^{+2.9}_{-1.6}$ | 15938              |
| $\Sigma_Q \Xi_{QQ}$ $3/2$ 1+ | †         | †                  | 11.8$^{+9.0}_{-7.0}$ | 15928              |
| $\Sigma^*_Q \Xi_{QQ}$ $3/2$ 1+ | †         | †                  | 2.8$^{+2.5}_{-2.3}$ | 15958              |
| $\Sigma^*_Q \Xi_{QQ}$ $3/2$ 2+ | 0.1$^{+2.0}_{-2.0}$ | ?                  | 16.7$^{+11.4}_{-9.3}$ | 15944              |
| $\Sigma_Q \Xi^*_{QQ}$ $3/2$ 1+ | 3.1$^{+6.3}_{-3.0}$ | 6180               | 28.1$^{+16.7}_{-14.1}$ | 15936              |
| $\Sigma_Q \Xi^*_{QQ}$ $3/2$ 2+ | †         | †                  | 3.1$^{+3.9}_{-2.5}$ | 15961              |
| $\Sigma^*_Q \Xi^*_{QQ}$ $3/2$ 0+ | 7.8$^{+10.3}_{-6.4}$ | 6237               | 40.9$^{+22.2}_{-19.2}$ | 15944              |
| $\Sigma^*_Q \Xi^*_{QQ}$ $3/2$ 1+ | 4.2$^{+7.4}_{-3.9}$ | 6230               | 30.1$^{+17.6}_{-14.9}$ | 15935              |
| $\Sigma^*_Q \Xi^*_{QQ}$ $3/2$ 2+ | 0.3$^{+2.3}_{-1.8}$ | ?                  | 14.4$^{+10.3}_{-8.3}$ | 15971              |
| $\Sigma^*_Q \Xi^*_{QQ}$ $3/2$ 3+ | †         | †                  | 0.5$^{+1.6}_{-1.2}$ | ?                  |

Heavy quark flavor symmetry dictates that in our above study, we can replace the charm quark with its bottom counterpart, and the interactions will remain the same, again up to corrections of $1/m_b - 1/m_c$. As a result, we can study the spectrum of the $\Xi^{(*)}_{bb}\Sigma^{(*)}_b$ system with both $I = 1/2$ and $I = 3/2$. Naively, as the constituents of the system become heavy, the binding energy becomes larger, because the interaction remains the same but the kinetic energy of the system decreases. This can be clearly seen in Tables III and IV. All the ten states in $I = 1/2$ bind, but not of the $I = 3/2$ states. For instance, it is quite likely that the $3^+ \Sigma^*_b \Xi^*_b$ state does not bind.

IV. SUMMARY AND CONCLUSION

Motivated by the experimental discovery of the $P_c(4312)$, $P_c(4440)$ and $P_c(4457)$ by the LHCb collaboration and the fact that they can well be part of a complete multiplet of $\bar{D}^{(*)}_{cc}\Sigma^{(*)}_c$ molecules, we studied the $\Xi^{(*)}_{cc}\Sigma^{(*)}_c$ system in the one-boson exchange model. The model parameters are
related to those of the $\bar{D}^{(*)}\Sigma_c^{(*)}$ system via heavy antiquark diquark symmetry by reproducing the pentaquark states as meson-baryon molecules of $\bar{D}^{(*)}\Sigma_c^{(*)}$. Ten triply charmed dibaryons with isospin 1/2 are found to bind, consistent with scenario B of the previous EFT study [59]. In addition, ten triply bottom dibaryons are predicted in the isospin 1/2 sector. As for the isospin 3/2 sector, only 3 molecular states are likely in the charm sector while 9 are possible in the bottom sector.

It should be noted that the $1^+ \Xi_{cc}\Sigma_c$ state has a larger mass than its $0^+$ counterpart in the OBE model, consistent with HADS as predicted in the previous EFT study [59]. Future lattice QCD studies of these systems will provide a nontrivial test not only of HADS but also of the molecular nature of the pentaquark states.

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