FIXED POINTS FOR ACTIONS OF \text{AUT}(F_n) ON CAT(0) SPACES

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Abstract. For \(n \geq 3\) we study global fixed points for isometric actions of the automorphism group of a free group of rank \(n\) on complete \(d\)-dimensional CAT(0) spaces. We prove that whenever the automorphism group of a free group of rank \(n\) acts by isometries on complete \(d\)-dimensional CAT(0) space such that \(d < 2 \cdot \left\lfloor \frac{n}{3} \right\rfloor\) it must fix a point. This property has implications for irreducible representations of the automorphism group of a free group of rank \(n\). For \(n \geq 4\) we obtain similar results for the unique subgroup of index two in the automorphism group of a free group.

1. Introduction

The background for our results are the articles “A condition that prevents groups from acting nontrivially on trees” [3] and “On the dimension of CAT(0) spaces where mapping class groups act” [4] by Bridson.

We consider isometric actions of \text{Aut}(F_n) on complete \(d\)-dimensional CAT(0) spaces. We denote by \text{Aut}(F_n) the automorphism group of a free group of rank \(n\) and by \text{SAut}(F_n) the unique subgroup of index two in \text{Aut}(F_n). The abelianization map \(F_n \to \mathbb{Z}^n\) gives a natural map \text{Aut}(F_n) \to \text{GL}_n(\mathbb{Z}). The special automorphism group of a free group of rank \(n\), \text{SAut}(F_n), is defined as the preimage of \text{SL}_n(\mathbb{Z}). A CAT(0) space is a geodesic metric space in which no geodesic triangle is fatter than a triangle in the euclidean plane. Here we mean by dimension the covering dimension of a topological space.

A group has (Serre’s) property \(\text{FA}\) if every action by isometries on a tree has a global fixed point. A generalization of property \(\text{FA}\) is property \(\text{FA}_d\). A group has property \(\text{FA}_d\) if every action by isometries on a complete \(d\)-dimensional CAT(0) space has a global fixed point.

We prove the following theorems.

**Theorem A.** Let \(X\) be a \(d\)-dimensional complete CAT(0) space and \(\Phi: \text{Aut}(F_n) \to \text{Isom}(X)\) be a homomorphism. If \(n \geq 3\) and \(d < 2 \cdot \left\lfloor \frac{n}{3} \right\rfloor\), then \text{Aut}(F_n) has a global fixed point in \(X\).
Theorem B. Let $X$ be a $d$-dimensional complete CAT(0) space and 
\[ \Phi : \text{SAut}(F_n) \to \text{Isom}(X) \]
be a homomorphism. If $n \geq 4$ and $d < 2 \cdot \lfloor \frac{2n-1}{3} \rfloor$, then $\text{SAut}(F_n)$ has a global fixed point in $X$.

Here we denote by $\lfloor x \rfloor$ the largest integer not greater than $x$.

The result of Theorem A was announced in “Automorphism groups of free groups, surface groups and free abelian groups” by Bridson and Vogtmann and in personal communication by Bridson.

The following result is proved in “Group actions and Helly’s theorem” by Farb.

Theorem. (\[7, 1.8\]) Let $K$ be an algebraically closed field and $G$ be a group. If $G$ has property $\text{FA}_d$, then there are only finitely many conjugacy classes of irreducible representations 
\[ \rho : \text{Aut}(F_n) \to \text{GL}_{d+1}(K). \]

Using this fact we note the following results in the representation theory of $\text{Aut}(F_n)$ and $\text{SAut}(F_n)$.

Corollary C. Let $K$ be an algebraically closed field. If $n \geq 3$ and $d \leq 2 \cdot \lfloor \frac{n}{3} \rfloor$, then there are only finitely many conjugacy classes of irreducible representations 
\[ \rho : \text{Aut}(F_n) \to \text{GL}_d(K). \]

Corollary D. Let $K$ be an algebraically closed field. If $n \geq 4$ and $d \leq 2 \cdot \lfloor \frac{2n-1}{3} \rfloor$, then there are only finitely many conjugacy classes of irreducible representations 
\[ \rho : \text{SAut}(F_n) \to \text{SL}_d(K). \]

We prove Theorem A in three steps. The first goal is to construct a generating set of $\text{Aut}(F_n)$ with the property that each group generated by two elements of this set is finite.

The second goal is to prove the following Theorem, see Corollary 7.3. The parts of the proof of this Theorem can be found in [4, 3.6]. We include a proof in this article for the sake of completeness.

Theorem. (\[4, 3.6\]) Let $k$ and $l$ be in $\mathbb{N}_{>0}$ and let $X$ be a complete $d$-dimensional CAT(0) space with $d < k \cdot l$. Let $S$ be a subset of $\text{Isom}(X)$ and $S_1, \ldots, S_l$ be conjugates of $S$ such that $[S_i, S_j] = 1$ for $i \neq j$. If each $k$-element subset of $S$ has a fixed point in $X$, then each finite subset of $S$ has a fixed point in $X$.

In the last step we combine Helly’s Theorem for isometric actions on complete CAT(0) spaces and the theorem above to prove Theorem A.

The structure of the proof of Theorem B is similar.

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2. A Generating Set of the Automorphism Group of a Free Group

Let $F_n$ be a free group of rank $n$ with basis $\{x_1, \ldots, x_n\}$. We denote by $\text{Aut}(F_n)$ the automorphism group of $F_n$.

In this section we construct a generating set of $\text{Aut}(F_n)$ such that each subgroup which is generated by two elements of this set is finite.

Let $\alpha, \beta$ be in $\text{Aut}(F_n)$, then the automorphism $\alpha \beta$ is the composite

$$\alpha \beta : F_n \to F_n, \quad x \mapsto x^{\alpha \beta},$$

where $\alpha$ acts before $\beta$. Let $i, j$ be in $\{1, \ldots, n\}$ with $i \neq j$. We define the Right Nielsen automorphism $\rho_{ij}$, the involutions $(x_i, x_j)$ and $e_i$ as follows:

$$\rho_{ij}(x_k) := \begin{cases} x_i x_j, & k = i \\ x_k, & k \neq i \end{cases} \quad (x_i, x_j)(x_k) := \begin{cases} x_j, & k = i \\ x_i, & k = j \\ x_k, & k \neq i, j \end{cases} \quad e_i(x_k) := \begin{cases} x_i^{-1}, & k = i \\ x_k, & k \neq i. \end{cases}$$

The group $\text{Aut}(F_n)$ has a finite generating set.

**Proposition 2.1.** (II III 1.5) Let $n \geq 3$, then the group $\text{Aut}(F_n)$ is generated by the following set of automorphisms

$$Y_1 := \{\rho_{ij}, e_i \mid i, j \in \{1, \ldots, n\}, i \neq j\}.$$  

Our first goal is to modify the set $Y_1$ in such a way that each group which is generated by two elements of the new generating set of $\text{Aut}(F_n)$ is finite. Compare [3, 1.1, 1.2].

**Proposition 2.2.** Let $n \geq 3$.

i) The group $\text{Aut}(F_n)$ is generated by the following set of automorphisms

$$Y_2 := \{(x_1, x_2)e_1e_2, (x_2, x_3)e_1, (x_i, x_{i+1}), \rho_{12}e_2, e_n \mid i \in \{3, \ldots, n-1\}\}.$$  

ii) Let $\{y_1, y_2\}$ be a subset of $Y_2$. Then the subgroup $\langle \{y_1, y_2\}\rangle$ of $\text{Aut}(F_n)$ is finite.

iii) The group generated by $Y_2 - \{\rho_{12}e_2\}$ is finite.

**Proof.** We first prove that $Y_2$ is a generating set of $\text{Aut}(F_n)$. We denote by $\Sigma(X) \subseteq \text{Aut}(F_n)$ the group of automorphisms which permute the basis $\{x_1, \ldots, x_n\}$. Let $\sigma \in \Sigma(X)$. The conjugation by $\sigma$ sends $\rho_{ij}$ to $\rho_{\sigma(i)\sigma(j)}$ and $e_i$ to $e_{\sigma(i)}$:

$$\sigma \rho_{ij} \sigma^{-1} = \rho_{\sigma(i)\sigma(j)} \quad \sigma e_i \sigma^{-1} = e_{\sigma(i)}.$$  

Therefore $\text{Aut}(F_n)$ is generated by the set $\langle \Sigma(X), \rho_{12}, e_n\rangle$. The group $\Sigma(X)$ is generated by the involutions $(x_i, x_{i+1})$ with $i \in \{1, \ldots, n-1\}$. We can replace $\rho_{12}$ by the involution $\rho_{12}e_2$. In particular $\text{Aut}(F_n)$ is generated by the set

$$Y_3 := \{(x_i, x_{i+1}), \rho_{12}e_2, e_n \mid i \in \{1, \ldots, n-1\}\}.$$
We want to show that the involutions \((x_1, x_2)\) and \((x_2, x_3)\) are in \(Y_2\). We consider the relations:

\[
(x_2, x_3)e_1 (x_3, x_n) (x_2, x_3)e_1 = (x_2, x_n),
\]

\[
\epsilon_{Y_2} (x_2, x_n)e_n (x_2, x_n)e_{Y_2} = e_2.
\]

Thus the involutions \((x_2, x_n)\) and \(e_2\) are in \(Y_2\). By the relation

\[
(x_2, x_n) (x_3, x_n) (x_2, x_n) = (x_2, x_3)
\]

we have that \((x_2, x_3)\) is in \(Y_2\). We see that \(e_1 = (x_2, x_3)(x_2, x_3)e_1\) is in \(Y_2\). We have

\[
(x_1, x_2) = (x_1, x_2)e_1 e_2 e_2 e_1
\]

and therefore the involution \((x_1, x_2)\) is in \(Y_2\) and \(Y_2\) is a generating set of \(\text{Aut}(F_n)\).

Next we prove the second statement of the proposition. For \(y_1, y_2\) let \(m_{y_1, y_2} = \text{ord}(y_1y_2)\). This defines a Coxeter matrix whose Coxeter diagram looks as follows.

![Coxeter diagram](image)

**Figure 1**

The corresponding Coxeter group \(W\) has a natural epimorphism onto \(\langle Y_2 \rangle\), compare [9]. In particular, all subgroups of \(\langle Y_2 \rangle\) generated by two elements of \(Y_2\) are finite.

Now we consider the subset \(Y_2 - \{\rho \rho_2 \rho_2\}\). The subgroup \(\langle Y_2 - \{\rho \rho_2 \rho_2\} \rangle\) of \(\text{Aut}(F_n)\) is isomorphic to a subgroup of \(\text{Sym}(n) \ltimes \mathbb{Z}_2^n\) and therefore finite. \(\square\)

3. **A generating set of the special automorphism group of a free group**

We define the Left Nielsen automorphism \(\lambda_{ij}\) as follows:

\[
\lambda_{ij}(x_k) := \begin{cases} x_j x_i, & k = i \\ x_k, & k \neq i. \end{cases}
\]
Proposition 3.1. Let \( n \geq 3 \), then the group \( \text{SAut}(F_n) \) is generated by the following set of automorphisms

\[
Y_3 := \{ \rho_{i, i+1}, \rho_{n, 1}, \lambda_{i, i+1}, \lambda_{n, 1} \mid i \in \{1, \ldots, n-1\} \}.
\]

Proof. It is known that \( \text{SAut}(F_n) \) is generated by \( \{ \rho_{ij}, \lambda_{ij} \mid i, j \in \{1, \ldots, n\} \} \), see [8]. From the relations \([\rho_{ij}, \rho_{jk}] = 1\) and \([\lambda_{ij}, \lambda_{jk}] = 1\) for \( i, j, k \) distinct, it follows that \( \text{SAut}(F_n) = \langle Y_3 \rangle \).

\[ \square \]

Proposition 3.2. Let \( n \geq 4 \).

i) The group \( \text{SAut}(F_n) \) is generated by the following set of automorphisms

\[
Y_4 := \{ (x_1, x_2)e_1e_2e_3, (x_2, x_3)e_1(x_i, x_{i+1})e_3, \rho_{12}e_2e_3, e_3e_4 \mid i \in \{3, \ldots, n-1\} \}.
\]

ii) The group generated by \( Y_4 - \{ \rho_{12}e_2e_3 \} \) is finite.

iii) Let \( \{y_1, y_2\} \) be a subset of \( Y_4 \). Then the subgroup \( \langle \{y_1, y_2\} \rangle \) of \( \text{SAut}(F_n) \) is finite.

Proof. Let \( i, j \) be in \( \{1, \ldots, n\} \). We have the relation

\[
e_i e_j \rho_{ij} e_j = \lambda_{ij}
\]

and therefore \( \text{SAut}(F_n) \) is generated by the set \( \{ e_i e_{i+1}, e_n e_1, \rho_{i, i+1}, \rho_{n, 1} \mid i \in \{1, \ldots, n-1\} \} \).

From the relations \( (x_2, x_3)e_1e_2e_3 = e_2e_4 \) and \( (x_3, x_4)e_2e_4e_3 = e_2e_3 \), we have that \( \rho_{12} \) is in \( \langle Y_4 \rangle \). We see that the automorphisms \( (x_{i+1}, x_{i+2})(x_i, x_i+1) \) for \( i \in \{1, \ldots, n-2\} \) are in \( \langle Y_4 \rangle \) and therefore \( \rho_{i, i+1}, \rho_{n, 1}, e_i e_{i+1}, e_n e_1 \) for \( i \in \{1, \ldots, n-1\} \) are in \( \langle Y_4 \rangle \) and \( Y_4 \) is a generating set of \( \text{SAut}(F_n) \).

Next we note that each element in the set \( Y_4 - \{ (x_3, x_4)e_3 \} \) is an involution and the order of \( (x_3, x_4)e_3 \) is 4. The subgroup \( \langle Y_4 - \{ \rho_{12}e_2e_3 \} \rangle \) of \( \text{SAut}(F_n) \) is isomorphic to a subgroup of \( \text{Sym}(n) \times \mathbb{Z}_2^n \) and therefore finite.

Next we prove the last statement of the proposition. Let \( \{y_1, y_2\} \) be a subset of \( Y_4 \). If \( \{y_1, y_2\} \) is a subgroup of \( \langle Y_4 - \{ \rho_{12}e_2e_3 \} \rangle \), then the statement is obvious. Now we consider the subgroup \( \{ \rho_{12}e_2e_3, y \} \). If the commutator is \( [\rho_{12}e_2e_3, y] = 1 \), then the subgroup \( \{ \rho_{12}e_2e_3, y \} \) is finite. If \( \rho_{12}e_2e_3 \) does not commute with \( y \), then we consider the following cases: If \( y \) is in \( Y_4 - \{ (x_3, x_4)e_3 \} \), then with similar Coxeter groups arguments as in Proposition 2.2 it follows that the subgroup \( \{ \rho_{12}e_2e_3, y \} \) is finite. On the other hand, if \( y \) is equal to \( (x_3, x_4)e_4 \), then the subgroup \( D_4 := \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1} \rangle \) has an epimorphism onto \( \{ \rho_{12}e_2e_3, (x_3, x_4)e_3 \} \) and therefore finite. \[ \square \]

4. SOME FACTS ABOUT CAT(0) SPACES

We recall key definitions and important properties of CAT(0) spaces that we will need. See “Metric spaces of non-positive curvature” [6] by Bridson and Haefliger for details.

Let \((X, d)\) be a metric space and \(x, y \) in \(X\). A geodesic joining \( x \) to \( y \) is a map \( c_{xy} : [0, l] \to X \), such that \( c(0) = x \), \( c(l) = y \) and \( d(c(t), c(t')) = |t - t'| \) for all \( t, t' \in [0, l] \). The image of \( c_{xy} \), denoted by \( [x, y] := \text{im}(c_{xy}) \), is called a geodesic segment. A metric space \( X \) is said to be a geodesic space if every two points \( x, y \) in \( X \) are joined by a geodesic.
A geodesic triangle in $X$ consists of three points $p_1, p_2, p_3$ in $X$ and a choice of three geodesic segments $[p_1, p_2]$, $[p_2, p_3]$, $[p_3, p_1]$. Such a geodesic triangle will be denoted by $\Delta(p_1, p_2, p_3)$.

A triangle $\Delta(p_1, p_2, p_3) \subseteq \mathbb{R}^2$ is called an euclidian comparison triangle for $\Delta(p_1, p_2, p_3)$ if it is a geodesic triangle in $\mathbb{R}^2$ and if $d(p_i, p_j) = d(\overline{p_i}, \overline{p_j})$ for $i, j$ in $\{1, 2, 3\}$. Let $i, j$ be in $\{1, 2, 3\}$. A point $\overline{x}$ in $[\overline{p_i}, \overline{p_j}]$ is called a comparison point for $x$ in $[p_i, p_j]$ if $d(x, p_j) = d(\overline{x}, \overline{p_j})$.

Let $X$ be a metric space. The geodesic triangle in $X$ is said to satisfy the CAT(0) inequality if for all $x, y$ in the geodesic triangle and all comparison points $\overline{x}, \overline{y}$ the inequality $d(x, y) \leq d(\overline{x}, \overline{y})$ holds. A metric space $X$ is called a CAT(0) space if $X$ is a geodesic space and all of its geodesic triangles satisfy the CAT(0) inequality.

Let $X$ be a complete CAT(0) space. We denote by $\text{Isom}(X)$ the group of isometries of $X$. The diameter of $Y$ is defined by

$$\text{diam}(Y) = \sup \{ d(x, y) \mid x, y \in Y \}.$$ 

The subset $Y$ is called bounded if $\text{diam}(Y)$ is finite.

The following version of the Bruhat-Tits Fixed Point Theorem, compare [6, II 2.8], will play an important role in the proof of the main theorem.

**Proposition 4.1.** Let $G$ be a group, let $X$ be a complete CAT(0) space and $\Phi : G \to \text{Isom}(X)$ be a homomorphism. Then the following conditions are equivalent:

i) The group $G$ has a global fixed point.

ii) Each orbit of $G$ is bounded.

iii) The group $G$ has a bounded orbit.

If the group $G$ satisfies one of the conditions above, then $G$ is called bounded on $X$.

**Proof.** The proof of i) $\Rightarrow$ ii) and ii) $\Rightarrow$ iii) is obvious, and iii) $\Rightarrow$ i) follows from [6, II 2.8]. □

Using proposition [4.1] we prove the following result.

**Corollary 4.2.** Let $G_1, G_2$ be groups, $X$ be a complete CAT(0) space and $\Phi_1 : G_1 \to \text{Isom}(X)$, $\Phi_2 : G_2 \to \text{Isom}(X)$ be homomorphisms. If $G_1, G_2$ are bounded groups on $X$ and $\Phi_1(g_1) \circ \Phi_2(g_2) = \Phi_2(g_2) \circ \Phi_1(g_1)$ for all $g_1$ in $G_1$ and $g_2$ in $G_2$, then we have:

i) The map

$$\Phi_1 \times \Phi_2 : G_1 \times G_2 \to \text{Isom}(X)$$

$$(g_1, g_2) \mapsto \Phi_1(g_1) \circ \Phi_2(g_2)$$

is a homomorphism.

ii) The group $G_1 \times G_2$ has a fixed point in $X$. 
Proof. The actions \( \Phi_1 \) and \( \Phi_2 \) commute and therefore the map \( \Phi_1 \times \Phi_2 \) is a homomorphism.

Next we show that the group \( G_1 \times G_2 \) has a bounded orbit. Let \( x \) in \( X \) be a fixed point of \( G_2 \) in \( X \). We have

\[
(G_1 \times G_2)(x) = \{ \Phi(g_1) \circ \Phi(g_2)(x) \mid g_1 \in G_1, g_2 \in G_2 \} = \{ \Phi(g_1)(x) \mid g_1 \in G_1 \}.
\]

The group \( G_1 \) is bounded on \( X \) and thus the orbit \((G_1 \times G_2)(x)\) is bounded. It follows from proposition 4.1 that \( G_1 \times G_2 \) has a fixed point in \( X \).

\( \square \)

5. Some facts about simplicial complexes and nerves

We recall some definitions and basic facts about (finite) simplicial complexes. See “Metric spaces of non-positive curvature” by Bridson and Haefliger [6, I 7A] for details.

A simplicial complex \( \Delta \) with a non-empty vertex set \( \mathcal{V} \) is a collection of finite subsets of \( \mathcal{V} \), called simplices, such that every one element subset of \( \mathcal{V} \) is a simplex and \( \Delta \) is closed under taking subsets. Let \( A \) be a simplex in \( \Delta \) be a simplex. The cardinality \( r \) of \( A \) is called the rank of \( A \) and \( r - 1 \) is called the dimension of \( A \). The dimension of \( \Delta \) is defined as follows:

\[
\dim(\Delta) := \sup \{ \dim(A) \mid A \in \Delta \}.
\]

Let \( \Delta \) be a simplicial complex. The geometric realization \(|\Delta|\) of \( \Delta \) is a topological space partitioned into simplices of \( \Delta \). Let \( \mathcal{V} \) be a real vector space with basis \( \mathcal{V} \) and \( |A| \) be the interior of the simplex in this vector space spanned by the vertices of \( A \). We define

\[
|\Delta| := \bigcup_{A \in \Delta} |A| \subseteq \mathbb{R}^{|\mathcal{V}|}.
\]

The topology of \(|\Delta|\) is the weak topology.

Next we need some more definitions.

**Definition 5.1.** Let \( \Delta_1, \Delta_2 \) be simplicial complexes with vertex sets \( \mathcal{V}_1, \mathcal{V}_2 \). The join \( \Delta_1 \star \Delta_2 \) of \( \Delta_1 \) and \( \Delta_2 \) is a simplicial complex with vertex set equal to the disjoint union \( \mathcal{V}_1 \cup \mathcal{V}_2 \) and \( A \subseteq \mathcal{V}_1 \cup \mathcal{V}_2 \) is a simplex in \( \Delta_1 \star \Delta_2 \) if and only if \( A = A_1 \cup A_2 \) where \( A_1 \) is a simplex in \( \Delta_1 \) and \( A_2 \) is a simplex in \( \Delta_2 \).

**Definition 5.2.** Let \( X \) be a set and \( \mathcal{F} \) a collection of subsets of \( X \). The nerve \( \mathcal{N}(\mathcal{F}) \) is the simplicial complex whose vertex set is \( \mathcal{F} \) and whose non-empty simplices are finite subsets \( \{ F_1, \ldots, F_k \} \subseteq \mathcal{F} \) with \( F_1 \cap \ldots \cap F_k \neq \emptyset \).

**Proposition 5.3.** [4, 3.3]) Let \( X \) be a complete CAT(0) space and let \( S_1, \ldots, S_l \) be subsets of \( \text{Isom}(X) \) such that \( [S_i, S_j] = 1 \) holds for all \( 1 \leq i < j \leq l \). Let \( \mathcal{F}_i = \{ \text{Fix}(s) \mid s \in S_i \} \) and \( \mathcal{N}_i = \mathcal{N}(\mathcal{F}_i) \). Put \( \mathcal{N} = \mathcal{N}(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_l) \). Then we have

\[
\mathcal{N} = \mathcal{N}_1 \star \ldots \star \mathcal{N}_l.
\]

**Proof.** We first note that the vertex sets of \( \mathcal{N} \) and \( \mathcal{N}_1 \star \ldots \star \mathcal{N}_l \) are equal.

Let \( A = \{ \text{Fix}(s_1), \ldots, \text{Fix}(s_k) \} \) be a simplex in \( \mathcal{N} \). We write the set \( A \) as \( A = A_1 \cup \ldots \cup A_l \) with \( A_i \subseteq \mathcal{F}_i \) for \( i \) in \( \{ 1, \ldots, l \} \). The intersection \( \cap A_i \) is non-empty for all \( i \) in \( \{ 1, \ldots, l \} \) and therefore the set \( A_i \) is a simplex in \( \mathcal{N}_i \) for every \( i \) in \( \{ 1, \ldots, l \} \). It follows that the subset \( A \) is a union of simplices in \( \mathcal{N}_i \) and therefore a simplex in \( \mathcal{N}_1 \star \ldots \star \mathcal{N}_l \). We have shown that \( \mathcal{N} \subseteq \mathcal{N}_1 \star \ldots \star \mathcal{N}_l \).
Now we prove the other inclusion. Let $B$ be a simplex in $N_1 \ast \ldots \ast N_l$. We know that $B = B_1 \cup \ldots \cup B_l$ where $B_i$ is a simplex in $N_i$ for $i \in \{1, \ldots, l\}$. Now we have to show that $\cap B$ is non-empty. We consider the set $S_i' := \{s \in S_i \mid \text{Fix}(s) \in B_i\}$ and the group which is generated by $S_i'$ for $i \in \{1, \ldots, l\}$. The subset $B_i$ is a simplex in $N_i$ and therefore the fixed point set $\text{Fix}(S_i')$ is non-empty for all $i \in \{1, \ldots, l\}$. Next we note that $[\langle S_i' \rangle, \langle S_j' \rangle] = 1$ for $i \neq j$. It follows from Corollary 4.2 that the set $\bigcap_{i=1}^l \text{Fix}(S_i') = \text{Fix}(\bigcup_{i=1}^l (S_i'))$ is non-empty. In particular the set $\cap B$ is non-empty and therefore the subset $B$ is a simplex in $N$. This completes the proof. □

6. Helly’s Theorem for isometric actions on complete CAT(0) spaces

In this section we state Helly’s Theorem for actions on complete CAT(0) spaces, compare “The FA$_n$ conjecture for coxeter groups”[1] by Barnhill.

**Theorem 6.1.** ([1] 5.10) Let $G$ be a group, $Y$ be a generating set of $G$ and let $X$ a complete $d$-dimensional CAT(0) space. Let

$$\Phi : G \rightarrow \text{Isom}(X)$$

be a homomorphism. If each $(d + 1)$-element subset of $Y$ has a fixed point in $X$, then $G$ has a fixed point in $X$.

7. Homological properties of nerves and step two of the proof of Theorem A

Our goal is to show the last theorem which we state in the introduction. For the proof we need the following homological property of nerves.

**Proposition 7.1.** ([1] 5.8) Let $G$ be a group, let $G_1, \ldots, G_l$ be subgroups of $G$, let $X$ be a $d$-dimensional complete CAT(0) space and let

$$\Phi : G \rightarrow \text{Isom}(X)$$

be a homomorphism. If $m \geq d$ and $d > 0$, then

$$H_m(\langle N(\text{Fix}(\Phi(G_1)), \ldots, \text{Fix}(\Phi(G_l)))\rangle) \simeq \{0\}.$$

The ingredients of the proof of the following theorem can be found in [1]. We prove it using Proposition 7.1.

**Theorem 7.2.** ([1] 3.4) Let $k_1, \ldots, k_l$ be in $\mathbb{N}_{>0}$ and $X$ be a $d$-dimensional complete CAT(0) space with $0 < d < k_1 + \ldots + k_l$. Let $S_1, \ldots, S_l$ be subsets of Isom$(X)$ such that $[S_i, S_j] = 1$ for $i \neq j$. If each $k_i$-element subset of $S_i$ has a fixed point in $X$ for all $i$ in $\{1, \ldots, l\}$, then for some $j$ in $\{1, \ldots, l\}$ every finite subset of $S_j$ has a fixed point.

**Proof.** We assume this is false. Let $i$ be in $\{1, \ldots, l\}$. We fix a minimal $k'_i \geq k_i$ such that there exist a $(k'_i + 1)$-element subset $T_i = \{s_{i,1}, \ldots, s_{i,k'_i+1}\} \subseteq S_i$ with empty fixed point set. We know that each $k'_i$-element subset of $T_i$ has a fixed point. Therefore the
nerve of $F_i = \{\text{Fix}(s_{i,1}), \ldots, \text{Fix}(s_{i,k_i+1})\}$ is the boundary of a $k'_i$-simplex. It follows from Proposition 5.3 that $\mathcal{N}(F_1 \cup \ldots \cup F_l) \cong \partial \Delta_{k'_1} \ast \ldots \ast \partial \Delta_{k'_l}$. The geometric realisation of this nerve is homeomorphic to a sphere,

$$|\mathcal{N}(F_1 \cup \ldots \cup F_l)| \cong S^{k'_1 + \ldots + k'_l - 1}.$$ 

Therefore the singular homology groups of the topological spaces above are isomorphic

$$H_*(|\mathcal{N}(F_1 \cup \ldots \cup F_l)|) \cong H_*(S^{k'_1 + \ldots + k'_l - 1}).$$

We have the inequality $k'_1 + \ldots + k'_l - 1 \geq d$ and from Corollary 7.1 we have that $H_{k'_1 + \ldots + k'_l - 1}(\mathcal{N}(F_1 \cup \ldots \cup F_l)) \cong \{0\}$. This contradicts

$$H_{k'_1 + \ldots + k'_l - 1}(S^{k'_1 + \ldots + k'_l - 1}) \cong \mathbb{Z}.$$

\[\square\]

The following is a consequence of Theorem 7.2.

**Corollary 7.3.** (\[1\] 3.6) Let $k$ and $l$ be in $\mathbb{N}_{>0}$ and let $X$ be a complete $d$-dimensional CAT(0) space, with $d < k' l$. Let $S$ be a subset of $\text{Isom}(X)$ and let $S_1, \ldots, S_l$ be conjugates of $S$ such that $[S_i, S_j] = 1$ for $i \neq j$. If each $k$-element subset of $S$ has a fixed point in $X$, then each finite subset of $S$ has a fixed point in $X$.

**Proof.** This is clear from Theorem 7.2 since the fixed point sets of the $S_i$ are conjugate. \[\square\]

8. **Proof of Theorem A**

We are now ready to prove Theorem A.

**Theorem A.** Let $X$ be a $d$-dimensional complete CAT(0) space and

$$\Phi : \text{Aut}(F_n) \to \text{Isom}(X)$$

be a homomorphism. If $n \geq 3$ and $d < 2 \cdot [\frac{n}{4}]$, then $\text{Aut}(F_n)$ has a global fixed point.

**Proof.** We consider the following generating set of $\text{Aut}(F_n)$

$$Y_2 := \{(x_1, x_2) e_1 e_2, (x_2, x_3) e_1, (x_i, x_{i+1}), \rho_{12} e_2, e_n | i \in \{3, \ldots, n - 1\}\}.$$ 

If $n = 3$, then the conclusion of Theorem A follows from Proposition 2.2 and from Theorem 6.1. We assume that $n \geq 4$.

First we explain the structure of the proof. As a first step we show the following by induction on $k$: If $k \leq d + 1$ and $d < 2 \cdot [\frac{n}{4}]$, then each $k$-element subset of $Y_2$ has a fixed point. Then it follows from Theorem 6.1 that $\text{Aut}(F_n)$ has a global fixed point.

We begin by proving that each 3-element subset of $Y_2$ has a global fixed point. Let $\{y_1, y_2, y_3\}$ be a subset of $Y_2$. If $\rho_{12} e_2$ is not in $\{y_1, y_2, y_3\}$, then it follows from Proposition 2.2 that $\{\{y_1, y_2, y_3\}\}$ is a finite subgroup of $\text{Aut}(F_n)$ and therefore each orbit of this group is bounded. It follows from Proposition 4.1 that this subgroup has a global fixed point.
Suppose now that \( \rho_{12}e_2 \) is in \( \{y_1,y_2,y_3\} \). We consider the Coxeter diagram, Figure 1, which we calculated in the proof of Proposition 2.2. We define
\[
S := \{(x_1,x_2)e_1 e_2, (x_2, x_3)e_1, \rho_{12}e_2\}.
\]

It follows from the Coxeter diagram that for \( \{\rho_{12}e_2, y_2, y_3\} \) not equal to \( S \), the subgroup \( \{\rho_{12}e_2, y_2, y_3\} \) is finite and has a global fixed point by Proposition 4.1.

Now we have to show that \( S \) has a fixed point. We define for \( i \in \{1, \ldots, \lfloor \frac{n}{3} \rfloor \} \) the permutations \( \sigma_i := (x_1, x_3(i-1)+1)(x_2, x_3(i-1)+2)(x_3, x_3(i-1)+3) \) and the sets
\[
S_i := \sigma_i S \sigma_i^{-1}.
\]
The sets \( S_1, \ldots, S_{\lfloor \frac{n}{3} \rfloor} \) have the property that \( [S_i, S_j] = 1 \) for \( i \neq j \). In Proposition 2.2 we proved that each subgroup of \( Aut(F_n) \) which is generated by a 2-element subset of \( Y_2 \) is finite. It follows from Proposition 4.1 that these groups are bounded on \( X \) and by Corollary 7.3 the set \( S \) has a fixed point.

We proceed by induction on \( k \). We assume that \( k + 1 \geq 4 \). Let \( Y' \) be a \((k + 1)\)-element subset of \( Y_2 \). We have the following cases:

i) If \( \rho_{12}e_2 \) is not in \( Y' \), then it follows from Proposion 2.2 that \( \langle Y' \rangle \) is a finite subgroup of \( Aut(F_n) \) and this subgroup has by Proposition 4.1 a fixed point.

ii) If \( \rho_{12}e_2 \) is in \( Y' \), then we consider the following cases:

a) If the Coxeter diagram of \( Y' \) is not connected, then it follows from the induction assumption and from Corollary 7.2 that \( Y' \) has a fixed point.

b) If the Coxeter diagram of \( Y' \) is connected, then we have
\[
Y' = \{\rho_{12}e_2, (x_1, x_2)e_1 e_2, (x_2, x_3), (x_3, x_4), \ldots, (x_k, x_{k+1})\} \text{ or }
Y' = \{\rho_{12}e_2, (x_2, x_3)e_1, (x_3, x_4), \ldots, (x_{k+1}, x_{k+2})\}.
\]
The involution \( e_3 \) is not in \( Y' \), because it follows from the inequalities \( k + 1 \leq d + 1 \) and \( d < 2 \cdot \lfloor \frac{n}{3} \rfloor \) that \( k + 1 < n \).

Let \( Y' \) be equal to \( \{\rho_{12}e_2, (x_1, x_2)e_1 e_2, (x_2, x_3), (x_3, x_4), \ldots, (x_k, x_{k+1})\} \). We define for \( i \in \{1, \ldots, \lfloor \frac{n}{k+1} \rfloor \} \) the permutations \( \tau_i := (x_1, x_3(i-1)+1)(x_2, x_3(i-1)+2)(x_3, x_3(i-1)+3) \) and the sets \( S_i := \tau_i Y' \tau_i^{-1} \). The sets \( S_1, \ldots, S_{\lfloor \frac{n}{k+1} \rfloor} \) have the property that \( [S_i, S_j] = 1 \) for \( i \neq j \). By the induction assumption each \( k \)-element subset of \( Y' \) has a fixed point and it follows from Corollary 7.3 that for \( d < k \cdot \lfloor \frac{n}{k+1} \rfloor \) the set \( Y' \) has a fixed point. Let \( Y' \) be equal to \( \{\rho_{12}e_2, (x_2, x_3)e_1, (x_3, x_4), \ldots, (x_{k+1}, x_{k+2})\} \). We have the following Coxeter diagram of \( Y' \):

\[
\begin{array}{c}
\rho_{12}e_2 & (x_2, x_3)e_1 & (x_3, x_4) & \cdots & (x_k, x_{k+1}) & (x_{k+1}, x_{k+2})
\end{array}
\]

\textbf{Figure 2}
It follows from the Coxeter diagram above that the group \( \langle Y' \rangle \) is finite and therefore this group has a fixed point by Proposition 4.1.

9. Proof of Theorem B

**Theorem B.** Let \( X \) be a \( d \)-dimensional complete \( \text{CAT}(0) \) space and
\[
\Phi : \text{SAut}(F_n) \to \text{Isom}(X)
\]
be a homomorphism. If \( n \geq 4 \) and \( d < 2 \cdot \left[ \frac{n-1}{3} \right] \), then \( \text{SAut}(F_n) \) has a global fixed point in \( X \).

**Proof.** We consider the following generating set of \( \text{SAut}(F_n) \)
\[
Y_4 := \{(x_1, x_2)e_1e_2e_3, (x_2, x_3)e_1, (x_i, x_{i+1})e_3, \rho_{12}e_2e_3, e_4 \mid i \in \{3, \ldots, n-1\}\}.
\]
If \( n \leq 6 \), then the conclusion of theorem B follows from Proposition 3.2 and from Theorem 6.1. We assume that \( n \geq 7 \).

We show again the following by induction on \( k \): If \( k \leq d + 1 \) and \( d < 2 \cdot \left[ \frac{n-1}{3} \right] \), then each \( k \)-element subset of \( Y_4 \) has a fixed point. Then it follows from Theorem 6.1 that \( \text{SAut}(F_n) \) has a global fixed point.

If \( k \) is equal to 2, then we know by Proposition 3.2 and by Proposition 4.1 that each 2-element subset of \( Y_4 \) has a fixed point.

We proceed by induction on \( k \). Let \( Y' \) be a \((k+1)\)-element subset of \( Y_4 \). We have the following cases:

i) If \( \rho_{12}e_2e_3 \) is not in \( Y' \), then it follows from Proposition 3.2 that \( \langle Y' \rangle \) is a finite subgroup of \( \text{SAut}(F_n) \) and this subgroup has by Proposition 4.1 a fixed point.

ii) If \( \rho_{12}e_2e_3 \) is in \( Y' \), then we consider the following cases:

a) If there exists a non-empty proper subset \( Y'' \) of \( Y' \) with the property \( [Y'', Y' - Y''] = 1 \), then it follows from the induction assumption and from Corollary 4.2 that \( Y' \) has a fixed point.

b) Otherwise we note that the element \((x_{n-1}, x_n)e_3\) is not in \( Y' \), because it follows from the inequalities \( k + 1 \leq d + 1 \) and \( d < 2 \cdot \left[ \frac{n-1}{3} \right] \) that \( k + 1 < n - 2 \).

We consider the determinant homomorphism
\[
\det : \text{Aut}(F_{n-1}) \to \text{GL}_n(\mathbb{Z}) \to \mathbb{Z}_2
\]
and we define
\[
\Psi : \text{Aut}(F_{n-1}) \to \text{SAut}(F_n)
\]
as follows
\[
f \mapsto f'
\]
\[
f'(x_k) := \begin{cases} f(x_k), & k \in \{1, \ldots, n-1\} \\ x_k^{\det(f)}, & k = n. \end{cases}
\]
The homomorphism \( \Psi \) is injective and \( Y' \) is contained in \( \text{im}(\Psi) \). By Theorem A the group \( \text{im}(\Psi) \) has a global fixed point and therefore \( Y' \) has a fixed point.
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