On the Yang-Baxter Poisson algebra in non-ultralocal integrable systems

Vladimir V. Bazhanov$^1$, Gleb A. Kotousov$^{1,2}$

and Sergei L. Lukyanov$^{2,3}$

$^1$Department of Theoretical Physics
Research School of Physics and Engineering
Australian National University, Canberra, ACT 2601, Australia

$^2$NHETC, Department of Physics and Astronomy
Rutgers University
Piscataway, NJ 08855-0849, USA

and

$^3$Kharkevich Institute for Information Transmission Problems
Moscow, 127994, Russia

Abstract

A common approach to the quantization of integrable models starts with the formal substitution of the Yang-Baxter Poisson algebra with its quantum version. However it is difficult to discern the presence of such an algebra for the so-called non-ultralocal models. The latter includes the class of non-linear sigma models which are most interesting from the point of view of applications. In this work, we investigate the emergence of the Yang-Baxter Poisson algebra in a non-ultralocal system which is related to integrable deformations of the Principal Chiral Field.
1 Introduction

Throughout the development of integrability, there has been a fruitful exchange of ideas and methods centered around the mathematical structure commonly known as the Yang-Baxter algebra

\[ R(\lambda_2/\lambda_1) \left( M(\lambda_1) \otimes 1 \right) \left( 1 \otimes M(\lambda_2) \right) = \left( 1 \otimes M(\lambda_2) \right) \left( M(\lambda_1) \otimes 1 \right) R(\lambda_2/\lambda_1). \]  

(1.1)

It appeared in the context of lattice systems \[1\] with \( M \) being a matrix built from the local statistical weights which satisfy a local Yang-Baxter equation (see fig. 1). The fundamental role of the Yang-Baxter algebra in the context of 1 + 1 dimensional classically integrable field theory was first pointed out by Sklyanin \[2\] and further developed in the works of the Leningrad school \[3\]. It was observed that for many partial differential equations admitting the zero curvature representation, the canonical Poisson structure yields the equal-time Poisson brackets

\[ \{ A_x(x|\lambda_1), A_x(y|\lambda_2) \} = \left[ A_x(x|\lambda_1) \otimes 1 + 1 \otimes A_x(y|\lambda_2), r(\lambda_1/\lambda_2) \right] \delta(x-y) \]  

(1.2)

for the \( x \)-component of the flat connection. The “ultralocal” relations (1.2) imply that the monodromy matrix,

\[ M(\lambda) = \mathcal{P} \exp \left( \int_0^R dx \, A_x(x|\lambda) \right), \]  

(1.3)

obeys

\[ \{ M(\lambda_1), M(\lambda_2) \} = \left[ M(\lambda_1) \otimes M(\lambda_2), r(\lambda_1/\lambda_2) \right] , \]  

(1.4)

which can be thought of as the classical limit of eq. (1.1) with \( r(\lambda) \) being the classical counterpart to the \( R \)-matrix. The Poisson algebra (1.4) is key in the Hamiltonian treatment of the integrable field theory as it immediately implies the existence of a commuting family of conserved charges generated by the trace of the monodromy matrix.

To see how (1.2) leads to the classical Yang-Baxter Poisson algebra (1.4), one can discretize the path-ordered integral in (1.3) on a finite number of segments so that \( M(\lambda) \) is given by an ordered product of elementary transport matrices

\[ M_n = \mathcal{P} \exp \left( \int_{x_{n+1}}^{x_n} dx \, A \right). \]

Since the r.h.s. of (1.2) is proportional to the \( \delta \)-function, the Poisson brackets of \( M_n \) corresponding to different segments of the path vanish. Then the proof of eq. (1.4) becomes practically equivalent to the “lattice derivation” of the quantum relation (1.1) pictured in fig. 1.
For many interesting integrable systems, the Poisson brackets of the flat connection are “non-ultralocal”: they are modified from (1.2) by the presence of a term proportional to $\delta'(x-y)$. This results in ambiguities in the calculation of the Poisson brackets of the monodromy matrix which come from contact terms arising from the integration of the derivative of the $\delta$-function. In the work [4] a certain “equal-point” limiting prescription was put forward to handle such ambiguities which enables the introduction of a commuting family of conserved charges. However the fundamental relations (1.4) are modified in this approach and it is unclear how to proceed with the quantization of the model even at the formal algebraic level. The natural question arises of whether it is possible to find a way of handling the contact terms such that (1.4) is unchanged. For the case of the Principal Chiral Field such a procedure was proposed in the work [5]. In these notes, we will tackle this question differently by using an explicit realization of the quantum Yang-Baxter algebra (1.1) and taking its classical limit. We’ll discuss the implications of our results for the two parameter deformation of the $SU(2)$ Principal Chiral Field introduced in [6].

2 From quantum universal $R$-matrix to $U(1)$ current algebra realization of Yang-Baxter Poisson structure

The algebraic structure underlying eq. (1.1) was clarified within the theory of quasi-triangular Hopf algebras by Drinfeld [7]. A basic example is when the rôle of the Hopf algebra is played by $U_q(\hat{\mathfrak{g}})$ – the quantum deformation of the universal enveloping algebra of the affine algebra $\mathfrak{g}$. In this case a crucial element is the universal $R$-matrix which lies in the tensor product $U_q(\hat{\mathfrak{b}}) \otimes U_q(\hat{\mathfrak{b}})$ and satisfies the relation

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}. \quad (2.1)$$

An important feature of $\mathcal{R}$ is that it is decomposed as $\mathcal{R} \in U_q(\hat{\mathfrak{b}}_+) \otimes U_q(\hat{\mathfrak{b}}_-)$ where $U_q(\hat{\mathfrak{b}}_{\pm})$ stand for the Borel subalgebras of $U_q(\hat{\mathfrak{g}})$. If we consider now the evaluation homomorphism of $U_q(\hat{\mathfrak{g}})$ to the loop algebra $U_q(\mathfrak{g})[\lambda, \lambda^{-1}]$ and specify an $N$-dimensional matrix representation $\pi$ of $U_q(\mathfrak{g})$, then

$$L(\lambda) = (\pi(\lambda) \otimes 1)[\mathcal{R}], \quad (2.2)$$

is a $U_q(\hat{\mathfrak{b}}_-)$-valued $N \times N$ matrix whose entries depend on an auxiliary parameter $\lambda$. In its turn, the formal algebraic relation (2.1) becomes the Yang-Baxter algebra (1.1) with $M$ substituted by $L$ while

$$R(\lambda_2/\lambda_1) = (\pi(\lambda_1) \otimes \pi(\lambda_2))[\mathcal{R}].$$

These notes will mostly focus on $\mathfrak{g} = \mathfrak{sl}_2$. In this case, the Borel subalgebra $U_q(\hat{\mathfrak{b}}_{+) is generated by four elements, \{y_0, y_1, h_0, h_1\}$ and its evaluation homomorphism is defined by

$$y_0 \mapsto \lambda q^{-\frac{\lambda}{2}} e_+, \quad y_1 \mapsto \lambda q^\frac{\lambda}{2} e_-, \quad h_0 \mapsto h, \quad h_1 \mapsto -h, \quad (2.3)$$

where $\{h, e_{\pm}\}$ are the generators of $U_q(\mathfrak{sl}_2)$, subject to the commutation relations

$$[h, e_{\pm}] = \pm 2 e_{\pm}, \quad [e_+, e_-] = \frac{q^h - q^{-h}}{q - q^{-1}}. \quad (2.4)$$
Below, with some abuse of notation, we will not distinguish between the formal generators of $U_q(\mathfrak{sl}_2)$ and their matrices in a finite dimensional representation. Explicitly, using the formula for the universal $R$-matrix given in [9], one can obtain $L(\lambda)$ as a formal series expansion in powers of the spectral parameter $\lambda$.\footnote{In fact, eq. (2.5) follows from an expression of the $R$-matrix which is equivalent to the one in [9] (and used in [10]) upon the substitution $q \mapsto q^{-1}$ (see eq. (2.12)). This is to keep with the conventions of the recent work [11].}

$$L(\lambda) = \left[ 1 + \lambda (q - q^{-1}) (x_0 q^{\frac{b}{2}} e_+ + x_1 q^{-\frac{b}{2}} e_-) + \lambda^2 \frac{(q - q^{-1})^2}{1 + q^2} \left( x_0^2 (q^{\frac{b}{2}} e_+)^2 + x_1^2 (q^{-\frac{b}{2}} e_-)^2 + \frac{q^2 x_0 x_1 - x_1 x_0}{1 - q^{-2}} (q^{\frac{b}{2}} e_+) (q^{-\frac{b}{2}} e_-) \right) + \ldots \right] q^{-\frac{1}{2} h_0}. \quad (2.5)$$

The expression in the square brackets contains only the generators $x_0, x_1 \in U_q(\mathfrak{b}_-)$ satisfying the Serre relations

$$x_i^3 x_j - [3]_q x_i^2 x_j x_i + [3]_q x_i x_j x_i^2 - x_j x_i^3 = 0 \quad (i, j = 0, 1), \quad (2.6)$$

where $[n]_q \equiv (q^n - q^{-n})/(q - q^{-1})$. Note that the two remaining generators $h_0, h_1$, which obey

$$[h_0, x_0] = -[h_1, x_0] = -2x_0, \quad [h_0, x_1] = -[h_1, x_1] = 2x_1, \quad [h_0, h_1] = 0, \quad (2.7)$$

appear only in an overall factor multiplying the square bracket $[\ldots]$ in (2.5). In fact, since $h_0 + h_1$ is a central element, for our purposes and without loss of generality we have set it to be zero.

Until this point there was no need to specify a representation of $U_q(\mathfrak{b}_-)$ – the Yang-Baxter relation (11) is satisfied identically provided (2.6), (2.7) hold true. In ref. [10], a representation of $U_q(\mathfrak{b}_-)$ was considered in the (extended) Fock space of a single bosonic field. The Borel generators $x_0, x_1$ were given by the integral expressions

$$x_0 = \frac{1}{q - q^{-1}} \int_0^R \int_0^R dz V^+(z), \quad x_1 = \frac{1}{q - q^{-1}} \int_0^R \int_0^R dz V^-(z). \quad (2.8)$$

Here the vertex operators

$$V^\pm(z) = e^{\mp 2i\beta\varphi(z)}$$

are built from the bosonic field

$$\varphi(z) = \varphi_0 + \frac{2\pi z}{R} \hat{p} + i \sum_{n \neq 0} \frac{a_n}{n} e^{-2\pi i n \beta z} \quad (2.9)$$

whose Fourier coefficients satisfy the commutations relations of the Heisenberg algebra

$$[a_n, a_m] = \frac{n}{2} \delta_{n+m, 0}, \quad [\varphi_0, \hat{p}] = \frac{i}{2}. \quad (2.10)$$

The remaining generator $h_0 = -h_1$ coincides with the zero mode momentum $\hat{p}$ up to a simple factor:

$$h_0 = \frac{2}{\beta} \hat{p}. \quad (2.11)$$
The parameter \( \beta \) appearing in the above formulae is related to the deformation parameter \( q \) as

\[
q = e^{-i\pi \beta^2} .
\]  

(2.12)

Defining the Fock space \( \mathcal{F}_p \) as the highest weight module of the Heisenberg algebra with highest weight vector \( |p\rangle \): \( \hat{p}\,|p\rangle = p\,|p\rangle \), it easy to see that the generators (2.8) act as

\[
x_0 : \mathcal{F}_p \mapsto \mathcal{F}_{p-\beta} , \quad x_1 : \mathcal{F}_p \mapsto \mathcal{F}_{p+\beta}
\]

and hence that the matrix elements of \( L(\lambda) \) (2.5) are operators in the extended Fock space \( \oplus_{n=-\infty}^{\infty} \mathcal{F}_{p+n\beta} \).

It was observed in [10] that using the commutation relations,

\[
V_1^{\sigma_1}(z_1) V_2^{\sigma_2}(z_2) = q^{2\sigma_1\sigma_2} V_2^{\sigma_2}(z_2) V_1^{\sigma_1}(z_1) , \quad z_2 > z_1 \quad (\sigma_{1,2} = \pm) \tag{2.13}
\]

the monomials built from the generators \( x_0, x_1 \) can be expressed in terms of the ordered integrals

\[
J(\sigma_1, \ldots, \sigma_m) = \int_{R>z_1>z_2>\ldots>z_m>0} dz_1 \ldots dz_m V_1^{\sigma_1}(z_1) \ldots V_m^{\sigma_m}(z_m) , \tag{2.14}
\]

which yields the following expression for \( L(\lambda) \)

\[
L(\lambda) = \sum_{m=0}^{\infty} \lambda^m \sum_{\sigma_1 \ldots \sigma_m = \pm} (q^{\frac{\pi}{2} \sigma_1} e_{\sigma_1}) \ldots (q^{\frac{\pi}{2} \sigma_m} e_{\sigma_m}) J(\sigma_1, \ldots, \sigma_m) e^{i\pi \beta \hat{p} \hbar} . \tag{2.15}
\]

The latter is recognized as the path ordered exponent

\[
L(\lambda) = \mathcal{P} \exp \left( \lambda \int_0^R dz \left( V^+ q^{\frac{\pi}{2}} e_+ + V^- q^{\frac{\pi}{2}} e_- \right) \right) e^{i\pi \beta \hat{p} \hbar} . \tag{2.16}
\]

It should be emphasized that since the OPE of the vertex operators is singular,

\[
V^\pm(z_2) V^\mp(z_1) \big|_{z_2 \to z_1+0} \sim (z_2 - z_1)^{-2\beta^2} ,
\]

the ordered integrals are well defined only for \( 0 < \beta^2 < \frac{1}{2} \). However, each term in the formal series expansion (2.5), being expressed in terms of the basic contour integrals \( x_0, x_1 \), is well defined for all values of \( \beta \) except the cases when \( \beta^2 = 1 - \frac{1}{2n} \) with \( n = 1, 2, 3, \ldots \). In fact, the series expansion (2.5) can be thought of as an analytic regularization of the divergent path-ordered exponent (2.16) within the domain \( \frac{1}{2} < \beta^2 < 1 \).

Let’s consider the classical limit where \( \beta \to 0 \) so that the deformation parameter \( q \) tends to one. The commutation relations (2.4) turn into

\[
[h, e_{\pm}] = \pm 2e_{\pm} , \quad [e_{\pm}, e_{\mp}] = h , \tag{2.17}
\]

while \( \phi \equiv \beta \varphi \) becomes a classical quasiperiodic field,

\[
\phi(R) - \phi(0) = 2\pi P , \tag{2.18}
\]
satisfying the Poisson bracket relations
\[ \{ \phi(z_1), \phi(z_2) \} = -\frac{i}{2} \epsilon(z_1 - z_2) \quad (2.19) \]
with \( \epsilon(z) = 2m + 1 \) for \( mR < z < (m+1)R \) \((m \in \mathbb{Z})\). Since for small \( \beta \) there is no convergence issue the \( \beta \to 0 \) limit of (2.16) is straightforward, yielding the classical path-ordered exponent of the form
\[ L_{\text{cl}}(\lambda) = \mathcal{P} \exp \left( \lambda \int_0^R dz \left( e^{-2i\phi} e_+ + e^{2i\phi} e_- \right) \right) e^{i\pi P \hbar} . \]
Here, abusing notation for the sake of readability, we denote the classical counterparts to the quantum operators by the same symbols, in particular, \( e_\pm \) now fulfill relations (2.17) and \( \phi \) is the classical field satisfying (2.18), (2.19).

The matrix \( L_{\text{cl}}(\lambda) \) essentially coincides with the monodromy matrix for the linear differential equation
\[ (\partial_z - A) \Psi(z) = 0 , \quad (2.20) \]
where
\[ A(z|\lambda) = j(z) \hbar + \lambda (e_+ + e_-) , \quad j(z) = i \partial \phi(z) . \quad (2.21) \]
Indeed, a simple calculation leads to
\[ L_{\text{cl}}(\lambda) e^{i\pi P \hbar} = \Omega^{-1} \left[ \mathcal{P} \exp \left( \int_0^R dz A(z|\lambda) \right) \right] \Omega \quad (2.22) \]
with \( \Omega = e^{i\phi(R)\hbar} \). We now observe that the ordinary differential equation (2.20) is the auxiliary linear problem for the classically integrable mKdV hierarchy, while
\[ \{ j(z_1), j(z_2) \} = -\frac{1}{2} \delta'(z_1 - z_2) \quad (2.23) \]
(which follows from (2.19)) is its first Hamiltonian structure. The above formula implies that the Poisson brackets of \( A \) do not have the ultralocal structure
\[ \mathcal{P} e^{i\pi P \hbar} := \Omega^{-1} \left[ \mathcal{P} \exp \left( \int_0^R dz A(z|\lambda) \right) \right] \Omega \]
and, as it was mentioned earlier, the computation of the Poisson brackets for the path-ordered exponent \( \mathcal{P} \exp \left( \int_0^R dz A \right) \) is inevitably met with ambiguities in treating the contact terms. Nonetheless, the classical limit of the Yang-Baxter algebra (1.1) unambiguously yields that (1.4) is satisfied with \( M(\lambda) \) substituted by \( L_{\text{cl}}(\lambda) \) from (2.22) while \( r(\lambda) = r_{-}(\lambda) \), where
\[ r_{-}(\lambda) = -\frac{1}{\lambda - \lambda^{-1}} \left( e_+ \otimes e_- + e_- \otimes e_+ + \frac{1}{4}(\lambda + \lambda^{-1}) \hbar \otimes \hbar \right) . \quad (2.24) \]
Thus we see that starting from an explicit realization of the quantum algebra (1.1) and taking the classical limit is a clear-cut way of obtaining the monodromy matrix satisfying the classical Yang-Baxter Poisson algebra for a non-ultralocal flat connection.

3 From quantum universal \( R \)-matrix to \( SU(2) \) current algebra realization of Yang-Baxter Poisson structure

It is known \([12,13]\) that the Borel subalgebra \( U_q(\hat{\mathfrak{sl}}_2) \subset U_q(\hat{\mathfrak{su}}_2) \) admits a realization with \( x_0 \) and \( x_1 \) given by (2.8), where the vertices \( V^\pm \) are built from three bosonic fields \( \varphi_1, \varphi_2, \varphi_3 \):
\[ V^\pm = \frac{1}{2b^2} \left( i b \partial \varphi_3 + \alpha_2 \partial \varphi_2 \pm \alpha_1 \partial \varphi_1 \right) e^{x^\pm \varphi_3} . \quad (3.1) \]
The expansion coefficients of $\phi_i$, defined by the formula similar to (2.9), generate three independent copies of the Heisenberg algebra (2.10). The relation (2.11) is replaced now by

$$h_0 = -h_1 = -4ib\hat{p}_3,$$

(3.2)

where $\hat{p}_3$ is the zero mode momentum of the field $\phi_3$. It should be highlighted that the parameters $\alpha_1, \alpha_2, b$ appearing in eq. (3.1) are subject to the constraint

$$\alpha_1^2 + \alpha_2^2 - b^2 = \frac{1}{2},$$

(3.3)

and $b$ is related to the deformation parameter $q$ as

$$q = e^{i\hbar} \text{ with } \hbar = \frac{\pi}{2b^2}.$$  

(3.4)

The set of generators $\{x_0, x_1, h_0, h_1\}$ defined by (2.8), (3.1), (3.2) fulfill the Serre and commutation relations (2.6), (2.7). In consequence, $L(\lambda)$ (2.12) derived from the universal $R$-matrix by taking this realization of $U_q(\mathfrak{sl}_2)$ satisfies the Yang-Baxter algebra (1.1). The formal power series expansion in $\lambda$ (2.5) is still applicable however eq. (2.15), which expresses $L(\lambda)$ in terms of the ordered integrals, turns out to be problematic because of an issue with convergence. Thus the path ordered exponent expression for $L(\lambda)$ (2.16) that was obtained from recasting the contour integrals into the ordered integrals using the commutation relations (2.13) (which are still valid) is ill defined. When taking the classical limit $b \to \infty$ it is essential to keep this in mind.

To study the classical limit, it is convenient to work with $\phi_i \equiv \phi_i/(2b)$ which become classical quasi-periodic fields

$$\phi_i(R) - \phi_i(0) = 2\pi P_i \quad (i = 1, 2, 3)$$

(3.5)

satisfying equations similar to (2.19). As it follows from (2.8), (3.1), (3.3) the classical counterparts of $x_0$ and $x_1$ are given by

$$\chi_0 = \lim_{b \to \infty} (q - q^{-1}) x_0 = \int_0^R \text{d} z V^+_{cl}(z), \quad \chi_1 = \lim_{b \to \infty} (q - q^{-1}) x_1 = \int_0^R \text{d} z V^-_{cl}(z),$$

(3.6)

where

$$V^\pm_{cl} = \left( i \partial \phi_3 + \frac{1}{\sqrt{1+\nu^2}} \partial \phi_2 \pm \frac{\nu}{\sqrt{1+\nu^2}} \partial \phi_1 \right) e^{\pm 2\phi_3}$$

(3.7)

and

$$\nu \equiv \lim_{b \to \infty} \frac{\alpha_1}{\alpha_2}.$$
principle \([\ldots, \ldots] \mapsto i\hbar \{\ldots, \ldots\}\). It is easy to see that with this procedure the first few terms shown in (2.5) become

\[
\lim_{\hbar \to 0} L(\lambda) = \left[ 1 + \lambda (\chi_0 e_+ + \chi_1 e_-) + \ldots \right] e^{-\pi P_3 \hbar}
\]

where \(\hbar, e_\pm\) satisfy the commutation relations of the \(\mathfrak{sl}_2\) algebra (2.17).

The calculation for higher order coefficients quickly becomes cumbersome. For example, the formal expansion of \(R q^{-\hbar^2/2} \in U_q(\mathfrak{g}_+) \otimes U_q(\mathfrak{g}_-)\) contains the term \(y_1 y_0^2 y_1 \otimes P_{4(1001)}(x_0, x_1)\) with

\[
P_{4(1001)}(x_0, x_1) = \frac{q^6(q - q^{-1})^2}{[4]_q [2]_q} \left( x_0^2 x_1^2 - [3]_q x_0 x_1 x_0 x_1 + x_0 x_1^2 x_0 + [3]_q x_1 x_0^2 x_1 - [3]_q x_1 x_0 x_1 x_0 + x_1^2 x_0^2 \right)
\]

which makes a fourth order contribution to the series (2.5) once the evaluation homomorphism (2.3) of \(y_0, y_1\) is taken. Expanding \(q\) for small \(\hbar\) in \(P_{4(1001)}(x_0, x_1)\) yields

\[
P_{4(1001)}(x_0, x_1) = -\frac{1}{8} \hbar^2 \left( 1 + O(\hbar) \right) \times \left( [x_0, [x_0, x_1]] x_1 + x_1 [x_0, [x_0, x_1]] - [x_0, x_1]^2 + \hbar^2 (x_0 x_1 x_0 x_1 + x_1 x_0 x_1 x_0 - x_1^2 x_0) + O(\hbar^3) \right).
\]

Now, replacing \(x_0, x_1\) by their classical counterparts (3.6), using the correspondence principle and taking the limit \(\hbar \to 0\) gives

\[
\lim_{\hbar \to 0} P_{4(1001)}(x_0, x_1) = \frac{1}{8} \left( 2 \chi_1 \{\chi_0, \{\chi_0, \chi_1\}\} - \{\chi_0, \chi_1\}^2 + \chi_0^2 \chi_1^2 \right).
\]

For the full contribution to the fourth order of (3.8) one should take into account all sixteen polynomials \(P_{4(i_1 i_2 i_3 i_4)}(x_0, x_1)\) with \(i_1, i_2, i_3, i_4 = 0, 1\) corresponding to the terms \(y_{i_1} y_{i_2} y_{i_3} y_{i_4} \otimes P_{4(i_1 i_2 i_3 i_4)}(x_0, x_1)\) in the expansion of the universal \(R\)-matrix.

Our calculations to fifth order in \(\lambda\) support the existence of the limit

\[
\lim_{\hbar \to 0} L = L_{cl}.
\]

By construction, \(L_{cl}\) is a formal series expansion in \(\lambda\) whose coefficients are built from \(\chi_0, \chi_1\) and their Poisson brackets\(^2\). To proceed further, the latter need to be computed explicitly. This can be carried out along the following lines. Starting from the relations

\[
\{\phi_i(z_1), \phi_j(z_2)\} = -\frac{1}{4} \delta_{ij} \epsilon(z_1 - z_2)
\]

it is easy to show that \(V_{cl}^\pm\) (3.7) and

\[
V_{cl}^0 = -2 \left( \frac{1}{\sqrt{1 + \hbar^2}} \partial \phi_3 - i \partial \phi_2 \right)
\]

\(^2\text{Note that the elements } \chi_0 \text{ and } \chi_1 \text{ satisfy the classical analogs of the Serre relations (2.6),}
\]

\[
\{\chi_i, \{\chi_i, \chi_j\}\} = \chi_j^2 \{\chi_i, \chi_j\} \quad (i, j = 0, 1).
\]
form a closed Poisson algebra

\[
\{ V_{cl}^0(z_1), V_{cl}^0(z_2) \} = -\frac{2\nu^2}{1+\nu^2} \delta'(z_1 - z_2)
\]

\[
\{ V_{cl}^0(z_1), V_{cl}^\pm(z_2) \} = \pm \frac{2}{\sqrt{1+\nu^2}} V_{cl}^\pm(z_1) \delta(z_1 - z_2)
\]

\[
\{ V_{cl}^\pm(z_1), V_{cl}^\mp(z_2) \} = -\frac{\nu^2}{1+\nu^2} \delta'(z_1 - z_2) + \frac{V_{cl}^0(z_1)}{\sqrt{1+\nu^2}} \delta(z_1 - z_2) + V_{cl}^\pm(z_1) V_{cl}^\mp(z_2) \epsilon(z_1 - z_2)
\]

\[
\{ V_{cl}^\pm(z_1), V_{cl}^\pm(z_2) \} = -V_{cl}^\pm(z_1) V_{cl}^\pm(z_2) \epsilon(z_1 - z_2).
\] (3.12)

Recall that \( \chi_0 \) and \( \chi_1 \) are given by integrals over the classical vertices \( (3.6) \) so that these relations are sufficient for the explicit calculation of any of the Poisson brackets occurring in the r.h.s of \( (3.8) \). However, due to the presence of the derivative of the \( \delta \)-function in \( (3.12) \), ambiguous integrals occur in the computations. For instance: \( \{ \chi_0, \chi_1 \} = c_1 \nu^2/(1+\nu^2) + \ldots \) with

\[
c_1 = -\int_0^R dz_1 dz_2 \delta'(z_1 - z_2) = \int_0^R dz \left( \delta(z - R) - \delta(z) \right). \tag{3.13}
\]

In general, one is faced with many other sorts of integrals involving \( \delta'(z_1 - z_2) \). However, they are not all independent and their number can be reduced if, before performing explicit calculations, one uses the Jacobi identity and skew-symmetry to bring the Poisson brackets to the form

\[
\{ \chi_{\sigma_1}, \{ \chi_{\sigma_2}, \{ \ldots, \{ \chi_{\sigma_m-1}, \chi_{\sigma_m} \ldots \} \} \} \} \quad (\sigma_1, \ldots, \sigma_m = 0, 1) \tag{3.14}
\]

(e.g., \( \{ \chi_0, \chi_1 \}, \{ \chi_0, \chi_1 \} = \{ \chi_0, \{ \chi_1, \chi_0 \} \} + \{ \chi_1, \{ \chi_0, \chi_0 \} \} \)). This way, in our fifth order computations we were met with only two more types of ambiguous integrals. The first is of the form

\[
I_1 = \int_0^R dz_1 \ldots dz_4 \delta'(z_1 - z_3) \epsilon(z_2 - z_3) \epsilon(z_3 - z_4) F(z_2) G(z_4),
\]

where \( F \) and \( G \) are some functions. Formal integration by parts w.r.t. \( z_3 \) yields

\[
I_1 = c_1 \int_0^R dz_1 dz_2 F(z_1) G(z_2)
\]

with \( c_1 \) as in \( (3.13) \). The other ambiguous integral is

\[
I_2 = \int_0^R dz_1 dz_2 dz_3 F(z_2) \epsilon(z_2 - z_3) \delta'(z_1 - z_3).
\]

In this case, integration by parts leads to

\[
I_2 = 2(c_2 - 1) \int_0^R dz F(z) \quad \text{with} \quad c_2 = \frac{1}{2} \int_0^R dz \left( \delta(z - R) + \delta(z) \right). \tag{3.15}
\]

We explicitly computed the expansion of \( L_{cl} \) to fifth order and found that all the ambiguities are absorbed in the two constants \( c_1 \) and \( c_2 \). Furthermore, if \( c_1 = 0 \) and \( c_2 \) is arbitrary, the series can be collected into a path-ordered exponent

\[
L_{cl} = \mathcal{P} \exp \left( \int_0^R dz B \right) e^{-\pi P_{\text{th}}}. \tag{3.16}
\]
with
\[ B = f \left( V_{cl}^+(z) \ e_+ + V_{cl}^-(z) \ e_- \right) + \frac{1}{2} g V_{cl}^0(z) \ h \] (3.17)
and
\[ f = \frac{\rho \sqrt{1 + \mu^2}}{1 - \rho^2} \left( 1 + (1 + \nu^2 (c_2 - 1)) \lambda^2 + (1 + 4 \nu^2 (c_2 - 1) + 2 \nu^4 (c_2 - 1)^2) \lambda^4 + O(\lambda^6) \right) \]
\[ g = \frac{\lambda^2 \sqrt{1 + \nu^2}}{1 - \nu^2} \left( 1 + (2 \nu^2 (c_2 - 1) + 1) \lambda^2 + O(\lambda^4) \right) . \]

That \( c_1 \) vanishes seems to be a natural requirement as, in the problem at hand, the \( \delta \)-function should be understood as the formal series \( \frac{1}{R} \sum_{m=-\infty}^{\infty} e^{2\pi i m z} \) and hence \( \delta(z - R) = \delta(z) \). Note that for the periodic \( \delta \)-function the constant \( c_2 \) in (3.13) becomes
\[ c_2 = \int_0^R dz \delta(z) . \] (3.18)

Unfortunately there is no proof that the limit (3.9) exists and can be represented by eq. (3.16) and (3.17) with some functions \( f \) and \( g \) — this has been checked perturbatively to fifth order only. However, if this is accepted as a conjecture then \( f \) and \( g \) should have the form
\[ f = \frac{\rho \sqrt{1 + \mu^2}}{1 - \rho^2} , \quad g = \frac{\rho^2 \sqrt{1 + \mu^2}}{1 - \rho^2} , \] (3.19)
where \( \rho = \rho(\lambda) \) solves the equation
\[ \lambda = \frac{\rho (1 - \rho^2)}{1 - (1 + (1 - c_2) \nu^2) \rho^2} . \] (3.20)

This follows from an analysis of the simplest matrix element of \( L_{cl} \) for which the series (3.8) can be obtained to all orders in \( \lambda \).

To summarize, we expect that the limit (3.9) exists and results in (3.16), where \( B \) is given by
\[ B(z|\rho) = \frac{\sqrt{1 + \mu^2}}{1 - \rho^2} \left( \rho \left( V_{cl}^+(z) \ e_+ + V_{cl}^-(z) \ e_- \right) + \frac{1}{2} \rho^2 V_{cl}^0(z) \ h \right) \] (3.21)
and with \( \rho = \rho(\lambda) \) defined through the relation (3.20). By construction \( L_{cl} \) must satisfy the classical Yang-Baxter Poisson algebra,
\[ \{ L_{cl}(\rho_1) \otimes L_{cl}(\rho_2) \} = [ L_{cl}(\rho_1) \otimes L_{cl}(\rho_2) , r(\lambda_1/\lambda_2) ] \] (3.22)
with \( \rho_{1,2} = \rho(\lambda_{1,2}) \) and\(^3\)
\[ r(\lambda) = + \frac{1}{\lambda - \lambda^{-1}} \left( e_+ \otimes e_- + e_- \otimes e_+ + \frac{1}{4} (\lambda + \lambda^{-1}) \ h \otimes h \right) . \] (3.23)

Eq. (3.12) implies that the Poisson brackets of \( B \) (3.17) are not local in the sense that apart from the \( \delta \)-function and its derivative they contain terms with the \( \epsilon \)-function. Nevertheless, a simple calculation shows that the Lie algebra valued 1-form \( B(z|\rho) \) is gauge equivalent to
\[ A(z|\rho) = \frac{\rho \sqrt{1 + \mu^2}}{1 - \rho^2} \left( j^+(z) \ e_+ + j^-(z) \ e_- \right) + \frac{1}{2} \left( \frac{\rho^2 \sqrt{1 + \mu^2}}{1 - \rho^2} + \xi \right) j^0(z) \ h \] (3.24)
\(^3\)Note that here the classical \( r \)-matrix differs from the one in (3.24) by an overall sign.
and the fields
\[ j^\pm = (i \partial \phi_3 + \frac{1}{\sqrt{1+\nu^2}} \partial \phi_2 \pm \frac{\nu}{\sqrt{1+\nu^2}} \partial \phi_1) e^{\pm 2\xi(\phi_3+i\phi_2)} \]
\[ j^0 = -2 \left( \frac{1}{\sqrt{1+\nu^2}} \partial \phi_3 - i \partial \phi_2 \right) \]
satisfy the classical current algebra
\[ \{ j^+(z_1), j^-(z_2) \} = -\frac{\nu^2}{1+\nu^2} \delta'(z_1 - z_2) + j^0(z_1) \delta(z_1 - z_2) \]
\[ \{ j^0(z_1), j^+(z_2) \} = \pm 2 j^\pm(z_1) \delta(z_1 - z_2) \]
\[ \{ j^0(z_1), j^0(z_2) \} = -\frac{2\nu^2}{1+\nu^2} \delta'(z_1 - z_2) \]
\[ \{ j^\pm(z_1), j^\pm(z_2) \} = 0 . \]

The constant \( \xi \) in the above formulae is given by
\[ \xi = \frac{\sqrt{1+\nu^2}}{1+\sqrt{1+\nu^2}} . \]

It follows from eq. (3.25) that the \( \epsilon \)-function is not present in the Poisson brackets of \( A \) \( (3.24) \) so they are local, although not ultralocal. In terms of the 1-form \( A \), eq. (3.16) can be re-written as
\[ L_{cl}(\rho) e^{((2\xi-1)P_3 + 2i\xi P_2)\pi h} = \Omega^{-1} \left[ P \exp \left( \int_0^R dz A(z|\rho) \right) \right] \Omega , \]
where \( \Omega = \exp \left( (\xi - 1) \phi_3(R) \hbar + i \xi \phi_2(R) \hbar \right) \) and \( P_1 \) are defined by eq. (3.5). The r.h.s. of (3.26) is the monodromy matrix for the linear problem (2.20) with \( A \) given by (3.24) and \( \rho \) playing the rôle of the auxiliary spectral parameter.

Despite that the Poisson brackets of the 1-form \( A \) are non-ultralocal for \( \nu \neq 0 \), \( L_{cl}(\rho) \) in (3.26) obeys the classical Yang-Baxter Poisson algebra (3.22). The \( \delta' \)-terms introduce an ambiguity in taking the classical limit which is manifest in the arbitrary constant \( c_2 \) (3.18). The effect of this is observed in the finite renormalization of the spectral parameter \( \lambda \mapsto \rho(\lambda) \) (3.20). Notice that for the ultralocal case, i.e., \( \nu = 0 \), the dependence on \( c_2 \) drops out and \( \rho = \lambda \).

### 4 Some facts about the Klimčík model

The Principal Chiral Field (PCF) is one of the keystone models of integrable field theory in \( 1+1 \) dimensions. In the simplest setup, where the model is associated with a simple Lie algebra \( \mathfrak{g} \) equipped with the Killing form \( \langle \cdot, \cdot \rangle \), the Lagrangian is given by
\[ L_{PCF} = -\frac{4}{g^2} \left\langle U^{-1} \partial_+ U, U^{-1} \partial_- U \right\rangle . \]
Here the field \( U(t, x) \) takes values in the Lie group \( \mathcal{G} \) corresponding to the Lie algebra so that \( U^{-1} \partial_\pm U \in \mathfrak{g} \), and the subscripts \( \pm \) label the light-cone co-ordinates
\[ x_\pm = t \pm x , \quad \partial_\pm = \frac{1}{2} (\partial_t \pm \partial_x) . \]
In Ref. [14], Klimčík introduced a two parameter deformation of the PCF. The construction uses the so-called Yang-Baxter operator – a linear operator \( \hat{R} \) acting in \( \mathfrak{g} \) which is defined through the root decomposition of the Lie algebra, \( \mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \), w.r.t. the Cartan subalgebra \( \mathfrak{h} \). Namely, for any element \( e_\pm \) from the nilpotent subalgebras \( \mathfrak{n}_\pm \): \( \hat{R}(e_\pm) = \mp i e_\pm \), while \( \hat{R}(h) = 0 \) for \( \forall h \in \mathfrak{h} \). The Lagrangian of the Klimčík model with deformation parameters \( \varepsilon_1, \varepsilon_2 \) is given by

\[
L_K = -\frac{4}{g^2} \left\langle U^{-1} \partial_+ U, \left( \mathbf{1} - i\varepsilon_1 \hat{R}_U - i\varepsilon_2 \hat{R} \right)^{-1} (U^{-1} \partial_- U) \right\rangle, \tag{4.3}
\]

where the action of \( \hat{R}_U \) is defined as

\[
\hat{R}_U(a) = U^{-1} \hat{R}(U a U^{-1}) U \quad \text{for} \quad \forall a \in \mathfrak{g} \tag{4.4}
\]

(4.4)

(4.5)

(4.6)

(4.7)

(4.8)

Also, \( R^b_a \) in the above formulae stands for the matrix elements of the Yang-Baxter operator

\[
\hat{R}(t_a) = t_b R^b_a.
\]
In order to clarify the Poisson bracket relations \(4.7\), let us mention that \(I_\pm\) are related by a linear transformation to the currents

\[
J_\pm(x) = \sum_a J_a^\pm(x) t_a , \quad [t_a, t_b] = i f_{abc} t_c ,
\]

which generate two independent copies of the classical current algebra:

\[
\{ J_a^\sigma(x), J_b^\sigma(y) \} = \frac{1}{g^2 \varepsilon_1} \delta_{\sigma\sigma'} q^{ab} \delta'(x - y) + \delta_{\sigma\sigma'} f^{abc} q_{cd} J_c^d(y) \delta(x - y) .
\]

Here \(\sigma, \sigma' = \pm\) and

\[
q_{ab} = -\frac{1}{4} f_{ac}^d f_{bd}^c = \langle t_a, t_b \rangle .
\]

For an explicit description of the relation between \(I_\sigma\) and \(J_\sigma\) \((\sigma = \pm)\), it is convenient to use the root decomposition of the Lie algebra and represent the currents in the form

\[
I_\sigma(x) = I_\sigma^+(x) + I_\sigma^0(x) + I_\sigma^-(x) : I_\sigma^\pm(x) \in n_\pm , \quad I_\sigma^0(x) \in h
\]

and similarly for \(J_\pm\). Then it turns out that

\[
I_\sigma^A = \frac{g^2}{2} \sum_{\sigma' = \pm} (1 + \sigma' \varepsilon_1 - A \sigma \varepsilon_2) J_{\sigma'}^A \quad \text{with} \quad A = \pm, 0 .
\]

Note that the Hamiltonian of the Klimčík model \(4.6\) is expressed in terms of the currents \(J_\pm\) as

\[
H = \frac{g^2}{4} \int dx \sum_{\sigma, \sigma' = \pm} \left( A_{\sigma\sigma'}^\parallel \langle J_\sigma^0, J_{\sigma'}^0 \rangle + 2 A_{\sigma\sigma'}^\perp \langle J_\sigma^+, J_{\sigma'}^- \rangle \right) ,
\]

where

\[
A_{\pm\pm}^\parallel = 1 + \varepsilon_1^2 , \quad A_{\pm\mp}^\parallel = 1 - \varepsilon_1^2 ,
\]

\[
A_{\pm\pm}^\perp = 1 + \varepsilon_1^2 - \varepsilon_2^2 , \quad A_{\pm\mp}^\perp = (1 + \varepsilon_1 \mp \varepsilon_2)(1 - \varepsilon_1 \pm \varepsilon_2) .
\]

### 4.2 Classical integrability

A remarkable feature of the two parameter deformation of the PCF \(4.3\) is that it preserves the integrability of the original model \(14\). The flat connection appearing in the zero curvature representation

\[
[\partial_+ - A_+, \partial_- - A_-] = 0
\]

is expressed in terms of the currents \(J_\pm\) as

\[
A_\sigma = -\frac{i \varepsilon_2}{1 - \rho^2_\sigma} \left( (\rho_\sigma)^{1-\sigma} I_\sigma^+ + (\rho_\sigma)^{1+\sigma} I_\sigma^- + \frac{1}{2} (1 + \rho^2_\sigma) I_\sigma^0 \right) \quad (\sigma = \pm) ,
\]
where the auxiliary parameters $\rho_{\pm}^2$ are subject to the single constraint

$$(\rho_{\pm}^2)^2 = \frac{(1 + \varepsilon_1 - \varepsilon_2)(1 - \varepsilon_1 - \varepsilon_2)}{(1 - \varepsilon_1 + \varepsilon_2)(1 + \varepsilon_1 + \varepsilon_2)}.$$  \hspace{1cm} (4.18)

For our purposes, we will also use a slightly different gauge $A_\pm^{(\omega)}$ which is defined as follows.
The equations of motion imply the conservation of the current $I_0 \sigma$

$$\partial_+ I_0^\sigma + \partial_- I_0^\sigma = 0,$$  \hspace{1cm} (4.19)

which allows one to introduce the dual field $\omega$

$$\partial_+ \omega = -\frac{1}{2} \varepsilon_2 I_0^\sigma, \quad \partial_- \omega = \frac{1}{2} \varepsilon_2 I_0^\sigma,$$  \hspace{1cm} (4.20)

taking values in the Cartan subalgebra $\mathfrak{h}$. Then,

$$\partial_{\pm} - A_{\pm}^{(\omega)} = e^{i\omega} (\partial_{\pm} - A_{\pm}) e^{-i\omega}.$$  \hspace{1cm} (4.21)

Apart from the local integrability condition – the zero curvature representation – proper
global requirements should be imposed to ensure integrability of the model. We consider
the Klimčík model with the space co-ordinate restricted to the segment $x \in [0, R]$. Since
the Lagrangian (4.3) is invariant under the transformation $U \mapsto H_1 U H_2$ where $H_1, H_2$ are
elements of the Cartan subgroup $\mathfrak{h} \subset \mathfrak{g}$, a natural choice for the boundary conditions is

$$U(t, x + R) = H_1 U(t, x) H_2, \quad H_1, H_2 \in \mathfrak{h}.$$  \hspace{1cm} (4.22)

With these conditions, the flat connection (4.17) becomes a quasiperiodic 1-form:

$$A_{\sigma}(t, x + R) = H_2^{-1} A_{\sigma}(t, x) H_2.$$  \hspace{1cm} (4.23)

Let us define the monodromy matrix at the time slice $t_0$ by

$$M(\rho) = \mathcal{P} \exp \left( \int_0^R dx \, A_x \right) \bigg|_{t= t_0} \quad (\rho \equiv \rho_+),$$  \hspace{1cm} (4.24)

with $A_x = A_+ - A_-$. Here the dependence on $\rho \equiv \rho_+$ is indicated explicitly though, of
course, the monodromy matrix also depends on $\varepsilon_1, \varepsilon_2$, while $\rho_-$ is expressed in terms of these
parameters using (4.18). Then a textbook calculation shows that

$$T(\rho) \equiv \text{Tr} \left[ H_2 M(\rho) \right]$$  \hspace{1cm} (4.25)

is independent of the choice of the time slice $t_0$ so that it can be thought of as the generating
function of a continuous family of conserved charges. In the contemporary paradigm of inte-
grability in 1 + 1 dimensional field theory it is crucial to prove that these conserved charges
mutually Poisson commute, i.e.,

$$\{ T(\rho_1), T(\rho_2) \} = 0$$  \hspace{1cm} (4.26)

---

4Eq. (20) from ref. [14] is equivalent to (4.17) with $L_{+\beta}^{\alpha} (\zeta) = A_\pm$ provided the following identifications are
made $\alpha = i \varepsilon_1, \beta = i \varepsilon_2$ and the spectral parameter $\zeta = \frac{\rho_+^2 - \rho_-^2}{\rho_+^2 - \rho_-^2}$.  

5In the limit $\rho_+ \to \infty$ and $\rho_- \to 0$ the connection (4.17) becomes upper triangular, $A_{\sigma} \in \mathfrak{n}_+ \oplus \mathfrak{h}$, so that
eq (4.19) immediately follows from the zero curvature representation.
for arbitrary \( \rho_1 \neq \rho_2 \). Owing to the complicated and non-ultralocal form of the Poisson brackets \( \{ A_x(x_1), A_x(x_2) \} \), the relations (4.26) are far from evident (see e.g. [15]).

For \( \varepsilon_1 = \varepsilon_2 = 0 \) (which corresponds to the PCF) the computation of the Poisson brackets of the monodromy matrix was discussed in ref. [5]. In this case, the formula (4.5) for the currents becomes

\[
I_\pm = -2i U^{-1} \partial_\pm U.
\]

Assuming that \( \rho_\pm = 1 - \varepsilon_2 \zeta_\pm \) and \( \zeta_\pm \) are kept fixed as \( \varepsilon_{1,2} \to 0 \), eq. (4.17) turns into the Zakharov-Mikhailov connection [16]

\[
\lim_{\varepsilon_1,\varepsilon_2 \to 0} A_\pm = -\zeta_\pm^{-1} U^{-1} \partial_\pm U, \tag{4.27}
\]

while the constraint (4.18) boils down to the relation \( \zeta_+ + \zeta_- = 2 \). The monodromy matrix for the PCF can be defined by taking the limit of (4.24):

\[
M^{(0)}(\zeta) = \lim_{\varepsilon_1,\varepsilon_2 \to 0} M(\rho)|_{\rho = 1 - \varepsilon_2 \zeta_+}, \quad \text{where} \quad \zeta_\pm \equiv 1 \pm \zeta. \tag{4.28}
\]

In ref. [5], for overcoming the non-ultralocality problem, the authors proposed a certain formal regularization procedure which results in the Yang-Baxter Poisson algebra

\[
\{ M^{(0)}(\zeta_1) \otimes M^{(0)}(\zeta_2) \} = \left[ M^{(0)}(\zeta_1) \otimes M^{(0)}(\zeta_2), r^{(0)}(\zeta_1 - \zeta_2) \right] \tag{4.29}
\]

with

\[
r^{(0)}(\zeta_1 - \zeta_2) = -\frac{g^2}{2} x^{ab} t_a \otimes t_b \frac{\zeta_1 - \zeta_2}{\zeta_1 - \zeta_2}. \tag{4.30}
\]

Of course, eq. (4.29) complemented by \( [H_2 \otimes H_2, r^{(0)}(\zeta)] = 0 \), immediately implies the desired commutativity conditions (4.26) specialized to the PCF. However, for the general Klimčík model it is uncertain whether the classical Yang-Baxter Poisson algebra emerges, even at the formal level. Below we’ll try to unravel this problem for \( \mathfrak{g} = SU(2) \) by using results obtained in Section 3. As before our considerations are inspired by the quantum case and it will be useful to keep the following few aspects of the quantum model in mind.

### 4.3 RG flow

Similar to the PCF, there is strong evidence to suggest that the integrability of the Klimčík model extends to the quantum level. Among other things, this implies the perturbative renormalizability of the model. In fact, one loop renormalizability was demonstrated for a more general class of field theories in the work [17]. The RG flow equations describing the cutoff dependence of the bare coupling constants are given by [6, 18] (see also Appendix A for some details)

\[
\partial_\tau \varepsilon_1 = -\frac{1}{2} \hbar g^2 \varepsilon_1 \left( 1 - (\varepsilon_1 - \varepsilon_2)^2 \right) \left( 1 - (\varepsilon_1 + \varepsilon_2)^2 \right) + O(\hbar^2)
\]

\[
\partial_\tau (\varepsilon_2 / \varepsilon_1) = O(\hbar^2)
\]

\[
\partial_\tau (g^2 / \varepsilon_1) = O(\hbar^2). \tag{4.31}
\]
with \( \partial_r \equiv 2\pi \Lambda \frac{\partial}{\partial \Lambda} \). The second equation in (4.31) shows that

\[
\nu^2 = \frac{\varepsilon_2}{\varepsilon_1} \tag{4.32}
\]
is an RG invariant and the third equation is fulfilled if we choose

\[
g^2 = \left| \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 \varepsilon_2} \right|. \tag{4.33}
\]

This way in the quantum theory there is only one \( \Lambda \)-dependent bare coupling. Within the domain

\[
0 < \varepsilon_1, \varepsilon_2 < 1
\]
which will be considered in these notes, it is convenient to use the parameterization

\[
\varepsilon_1 = \frac{1}{\sqrt{(1 + \kappa^{-1} \nu^2)(1 + \kappa \nu^2)}}, \quad \varepsilon_2 = \frac{\nu^2}{\sqrt{(1 + \kappa^{-1} \nu^2)(1 + \kappa \nu^2)}} \tag{4.34}
\]
where \( \nu^2 > 0 \) and

\[
\kappa = \kappa(\Lambda) : 0 < \kappa < 1. \tag{4.35}
\]

It follows from the RG flow equations (4.31) that a consistent removal of the UV cutoff \( \Lambda \) requires that

\[
\lim_{\Lambda \to \infty} \kappa(\Lambda) = 1^- . \tag{4.36}
\]
Thus in the high energy limit the renormalized running coupling will tend to one from below.

5 Monodromy matrix for the Fateev model

Choosing a local co-ordinate frame \( \{ X^\mu \} \) on the group manifold \( \mathfrak{g} \), the Klimčík Lagrangian can be written in the form

\[
\mathcal{L} = 2 G_{\mu\nu}(X) \partial_+ X^\mu \partial_- X^\nu - B_{\mu\nu}(X) \left( \partial_+ X^\mu \partial_- X^\nu - \partial_- X^\mu \partial_+ X^\nu \right). \tag{5.1}
\]

Field theories of this type are known as non-linear sigma models and describe the propagation of a string on a Riemannian manifold (the target space). Interested readers can find some details concerning the target space background for the general model in Appendix \( A \). Below we will focus on the simplest case with group \( \mathfrak{g} = SU(2) \) where the target space is topologically equivalent to the three sphere. With this choice, the \( B \)-term in (5.1) is a total derivative and can be ignored \( [19] \) and the theory coincides with the model originally introduced by Fateev in \( [6] \). The zero curvature representation for the Fateev model was found in \( [20] \) in a gauge which is different but equivalent to that of \( (4.17) \) specialized to the case \( \mathfrak{g} = SU(2) \) (the exact relation can be found in Appendix \( B \)). In both gauges, the Poisson brackets of the connection do not possess the ultralocal property and it is unknown whether an “ultralocal” gauge actually exists except for the cases with \( \varepsilon_2/\varepsilon_1 = 0, \infty \) considered in \( [11] \). Thus, with a view towards first principles quantization, the Poisson algebra generated by the monodromy matrices is of prime interest for the Fateev model and more generally the Klimčík one.
In the context of quantization, the target space with \( \kappa \to 1^- \) deserves special study. For this purpose, we introduce a co-ordinate frame based on the Euler decomposition for the group element

\[
U = e^{-\frac{i}{2}h^a} e^{-\frac{i}{2}(e_+ + e_-)} e^{-\frac{i}{2}h^b},
\]

(5.2)

where \( h, e_\pm \) are the generators of the Lie algebra \( g = sl_2 \) \((2.17)\). In fact, it is useful to substitute the angle \( \theta \in (0, \pi) \) for \( \phi \in (-\infty, \infty) \) such that

\[
\tan\left(\frac{\theta}{2}\right) = e^{\phi - \phi_0}, \quad e^{\phi_0} = \sqrt{\frac{1 + \kappa}{1 - \kappa}}.
\]

(5.3)

In this frame, the symmetry \( U \mapsto H_1 U H_2 (H_1, H_2 \in \mathfrak{h}) \) of the general Klimčík model is manifested as the invariance of the Fateev model w.r.t. the constant shifts

\[
v \mapsto v + v_0, \quad w \mapsto w + w_0.
\]

(5.4)

The corresponding Noether currents will be denoted by \( j^{(v)} \) and \( j^{(w)} \) respectively. With the continuity equations

\[
\partial_+ j^{(A)}_+ + \partial_- j^{(A)}_- = 0 \quad (A = v, w)
\]

(5.5)

one can introduce the dual fields \( \tilde{v}, \tilde{w} \) through the relations

\[
\tilde{j}^{(v)}_\pm = \pm \partial_\pm \tilde{v}, \quad \tilde{j}^{(w)}_\pm = \pm \partial_\pm \tilde{w}.
\]

(5.6)

It turns out that the dual field \( \omega \) defined by eq. (4.20) coincides with

\[
\omega = \frac{1}{2} \left[ \sqrt{1 + \nu^2} \tilde{w} + i \cdot \frac{1}{2} \log \left( \frac{\cosh(\phi_0 + \phi)}{\cosh(\phi_0 - \phi)} \right) \right] h.
\]

(5.7)

The boundary conditions \((4.22)\) specialized for the \( SU(2) \) case with

\[
H_1 = e^{-i\pi k_1 h}, \quad H_2 = e^{-i\pi k_2 h},
\]

(5.8)

imply the following conditions imposed on the fields \((\phi, v, w)\):

\[
\phi(t, x + R) = \phi(t, x), \quad v(t, x + R) = v(t, x) + 2\pi k_1, \quad w(t, x + R) = w(t, x) + 2\pi k_2.
\]

(5.9)

Also we will focus on the neutral sector of the model, which means periodic boundary conditions for the dual fields

\[
\tilde{v}(t, x + R) = \tilde{v}(t, x), \quad \tilde{w}(t, x + R) = \tilde{w}(t, x).
\]

(5.10)

Taking into account that

\[
\hat{R}(h) = 0, \quad \hat{R}(e_\pm) = \mp i e_\pm
\]

and using the parameterization \((5.2), (5.3)\) the Lagrangian \((4.3)\) with \( g^2 \) as in \((4.33)\) can be expressed in terms of three real fields \((\phi, v, w)\) and two real parameters \( \kappa \) and \( \nu \) \((4.34)\). Here there is no need to present the explicit formula, we just note that for \( |\phi| \ll \phi_0 \) the Fateev Lagrangian can be approximated by (up to a total derivative)

\[
\mathcal{L}_F \approx 2 \left( \partial_+ \phi \partial_- \phi + \frac{1}{1 + \nu^2} \partial_+ v \partial_- v + \frac{1}{1 + \nu^2} \partial_+ w \partial_- w \right).
\]

(5.11)
of eqs. (5.14), (5.16) we express $M$ by (3.21), (3.7), (3.11) and

This implies that as $\kappa \to 1^−$, i.e., $\phi_0 \to \infty$ most of the target manifold asymptotically approaches the flat cylinder with metric $G_{\alpha\beta}dX^\alpha dX^\beta = (d\phi)^2 + (1+\nu^{-2})^{-1}(dv)^2 + (1+\nu^2)^{-1}(dw)^2$ while the curvature is concentrated at the tips corresponding to $\phi = \pm \infty$. In the asymptotically flat domain, the general solution to the equations of motion can be expressed in terms of six arbitrary functions $\phi_1$ and $\phi_2$:

$$v(t, x) \simeq \sqrt{1 + \nu^{-2}} (\phi_1(x_+) + \phi_2(x_-)),$$

$$w(t, x) \simeq \sqrt{1 + \nu^2} (\phi_2(x_+) + \phi_2(x_-))$$

with

$$\phi(t, x) \simeq \phi_3(x_+) + \phi_3(x_-),$$

while for the dual fields one has

$$\tilde{v}(t, x) \simeq \phi_1(x_+) - \phi_1(x_-),$$

$$\tilde{w}(t, x) \simeq \phi_2(x_+) - \phi_2(x_-).$$

Having clarified the geometry of the target manifold for $\kappa \to 1^-$, one can turn to the form of the flat connection (4.17) in this limit. We assume that the co-ordinates $(\phi, v, w)$ are kept within the asymptotic domain where eqs. (5.12), (5.13) are valid. Also, since the product $\rho_+\rho_- (4.18)$ vanishes as $1 - \kappa$, we keep $\rho_+$ fixed while $\rho_- \to 0$. Then a direct calculation shows that

$$\lim_{\kappa \to 1^-} \left( \partial_+ - (\rho_+ / \rho_-)^{\pm h} A_+^{(\omega)} (\rho_+ / \rho_-)^{-\frac{h}{2}} \right) = e^{2i\omega_+(x_+) (\partial_+ - B(x_+ | \rho_+))} e^{-2i\omega_+(x_+)} ,$$

where we have used the gauge $A_+^{(\omega)}$ from eq. (4.21). The 1-form $B$ in this equation is defined by (3.21), (3.7), (3.11) and

$$\omega_+(x_+) = \frac{1}{2} \left( \sqrt{1 + \nu^2} \phi_2(x_+) + i \phi_3(x_+) \right) h .$$

For the other connection component one finds

$$\lim_{\kappa \to 1^-} (\rho_+ / \rho_-)^{\pm h} A_-^{(\omega)} (\rho_+ / \rho_-)^{-\frac{h}{2}} = 0 .$$

We now turn to the monodromy matrix that was introduced previously in (4.24). In light of eqs. (5.14), (5.16) we express $M(\rho)$ in terms of $A_+^{(\omega)}$:

$$M(\rho) = e^{-i\omega(t_0, R)} \tilde{P} \exp \left( \int_0^R dx A_+^{(\omega)} \right) \bigg|_{t=t_0} e^{i\omega(t_0, 0)} (\rho \equiv \rho_+) .$$
Since the connection $A_{\sigma}^{(\omega)}$ is flat, the integral over the segment $(0, R)$ can be transformed into the piecewise integral over the light cone segments as shown in fig. 2. The monodromy matrix is then expressed in terms of the light cone values of the connection as

$$M(\rho) = e^{-i\omega(t_0, R)} \frac{\mathcal{P}}{\mathcal{P}} \exp \left( \int_{t_0}^{t_0-R} A_{\sigma}^{(\omega)}(x_-) \, dx_- \right) \frac{\mathcal{P}}{\mathcal{P}} \exp \left( \int_{t_0}^{t_0+R} A_{\sigma}^{(\omega)}(x_+) \, dx_+ \right) e^{i\omega(t_0, 0)}$$  

(5.18)

where

$$A_{\sigma}^{(\omega)}(x_+) = A_{\sigma}^{(\omega)}(t, x)|_{x_+ = t_0} , \quad A_{\sigma}^{(\omega)}(x_-) = A_{\sigma}^{(\omega)}(t, x)|_{x_- = t_0+R} .$$  

(5.19)

For $\kappa$ close to 1 the instant $t_0$ can be chosen such that the values of the fields lie in the asymptotically flat region of the target manifold where formulae (5.12), (5.13) are applicable. Then with eqs. (5.14), (5.16) at hand, it is straightforward to show that the following limit exists

$$\lim_{\rho_+ \rightarrow \text{fixed}} \left( \rho_+/\rho_- \right)^{\pm \frac{1}{2}} M(\rho) \left( \rho_+/\rho_- \right)^{-\frac{1}{2}} = M^{(1)}(\rho) .$$  

(5.20)

Explicitly, $M^{(1)}(\rho)$ can be expressed in terms of $L_{cl}(\rho)$ previously defined in (3.16) and (3.21):

$$M^{(1)}(\rho) = \Omega^{-1} L_{cl}(\rho) e^{\pi(2i\sqrt{1+\nu^2}P_3-P_5)\hbar} \Omega .$$  

(5.21)

Here we take into account that $\phi(t_0, x + R) = \phi(t_0, x)$, $w(t, x + R) = w(t, x)$ and use

$$P_3 \equiv \frac{1}{2\pi} \left( \phi_3(t_0 + R) - \phi_3(t_0) \right) = -\frac{1}{2\pi} \left( \phi_3(t_0 - R) - \phi_3(t_0) \right)$$  

(5.22)

$$P_2 \equiv \frac{1}{2\pi} \left( \phi_2(t_0 + R) - \phi_2(t_0) \right) = +\frac{1}{2\pi} \left( \phi_2(t_0 - R) - \phi_2(t_0) \right)$$

and

$$\Omega = e^{-\frac{i}{2}\omega_0} : \omega_0 = w(t_0, R) + i \left( \phi_3(t_0 + R) - \phi_3(t_0 - R) \right) .$$  

(5.23)

It follows from the Lagrangian that the chiral fields $\phi_i$ can be chosen to satisfy the Poisson bracket relations

$$\{ \phi_i(x_+), \phi_j(x'_+) \} = -\frac{1}{4} \delta_{ij} \epsilon(x_+ - x'_+)$$  

(5.24)

and hence, using the results of the previous section, $L_{cl}(\rho)$ obeys the Yang-Baxter Poisson algebra (3.22). In the Hamiltonian picture the boundary condition $w(t, x + R) = w(t, x) + 2\pi k_2$ with $k_2$ a non-dynamical constant is a constraint of the first kind à la Dirac which should be supplemented by a gauge fixing condition. Considering the fields in the asymptotically flat domain where formulae (5.12), (5.13) hold true leads to the relation

$$P_2 = \frac{k_2}{2\sqrt{1+\nu^2}}$$  

(5.25)

and the gauge fixing condition can be chosen as $w(t_0, R) = 0$. This way $\omega_0$ in (5.23) becomes

$$\omega_0 = i \left( \phi_3(t_0 + R) - \phi_3(t_0 - R) \right).$$

Similarly, we supplement the periodic boundary condition $\phi(t_0, x + R) = \phi(t_0, x)$ by the constraint $\phi_3(t_0 - R) = 0$, so that

$$\omega_0 = i \phi_3(t_0 + R) .$$  

(5.26)

The Poisson brackets of $M^{(1)}(\rho) = \Omega^{-1} L_{cl}(\rho) e^{\pi ik_2-P_3} \hbar \Omega$ are obtained by using (3.22) and the simple relations

$$\{ L_{cl}(\rho), \pi P_3 \} = \frac{1}{4} [ \hbar, L_{cl}(\rho) ] , \quad \{ L_{cl}(\rho), \omega_0 \} = \frac{1}{4} \hbar L_{cl}(\rho) , \quad \{ \omega_0, \pi P_3 \} = \frac{1}{4} .$$  

(5.27)
The latter follow from eqs. (5.22), (5.24), (5.26). Also, taking into account that
\[
[1 \otimes h + h \otimes 1, r(\lambda)] = 0,
\]
(5.28)
one arrives at
\[
\{ M^{(1)}(\rho_1) \otimes M^{(1)}(\rho_2) \} = \left[ M^{(1)}(\rho_1) \otimes M^{(1)}(\rho_2), r(\lambda_1/\lambda_2) \right],
\]
(5.29)
where recall that \( \rho_{1,2} \) depend on \( \lambda_{1,2} \) via the relation (3.20).

It should be highlighted that the Poisson algebra (5.29) was obtained for a certain choice of the time slice \( t_0 \) when the fields take values in the asymptotic region. The validity of this equation for an arbitrary choice of \( t_0 \) is debatable, since the monodromy matrix itself is not a conserved quantity. However, that eq. (5.29) holds true even for a particular value of \( t_0 \) is sufficient to prove the commutativity condition
\[
\{ T^{(1)}(\rho_1), T^{(1)}(\rho_2) \} = 0
\]
with
\[
T^{(1)}(\rho) = \text{Tr}[e^{-ir\kappa^2 h} M^{(1)}(\rho)] = \lim_{\rho_+ \to 1, \rho_+ \text{fixed}} \text{Tr}[e^{-ir\kappa^2 h} M^{(1)}(\rho)].
\]
(5.30)

In view of the above, it makes sense to reconsider our definition of the monodromy matrix for the Fateev model and introduce
\[
M^{(\kappa)}(\rho) = (\rho_+ / \rho_-)^{\frac{\kappa}{\mathfrak{h}}} M^{(1)}(\rho) (\rho_+/\rho_-)^{-\frac{\kappa}{\mathfrak{h}}} \quad (\rho \equiv \rho_+).
\]
(5.31)
We’ve just seen that in the \( \kappa \to 1^- \) limit, the matrix \( M^{(\kappa)}(\rho) \) obeys the Yang-Baxter Poisson algebra (5.29). On the other hand, the redefinition (5.31) has no effect on the monodromy matrix as \( \kappa \to 0 \) and both \( \rho_\pm \to 1 \) so that the Yang-Baxter algebra is still satisfied but in the form (4.29). Finally, the case \( \nu = 0 \) with \( \kappa \in (0, 1) \) was already considered in the work [11] where it was shown that
\[
\{ M^{(\kappa)}(\rho_1) \otimes M^{(\kappa)}(\rho_2) \} = \left[ M^{(\kappa)}(\rho_1) \otimes M^{(\kappa)}(\rho_2), r(\lambda_1/\lambda_2) \right] \quad (\nu \to 0)
\]
(5.32)
with \( \rho_{1,2} = \lambda_{1,2} \). All this suggests that the key relations (5.32) may extend to the parametric domain \( \nu^2 > 0 \) and \( \kappa \in (0, 1) \) with some function \( \rho = \rho(\lambda|\nu, \kappa) \) (which is unknown in general).

6 Conclusion

For classically integrable field theories, the Yang-Baxter Poisson algebra plays a rôle similar to that of the canonical Poisson bracket relations for a general mechanical system. Whereas the correspondence principle prescribes the replacement of the canonical Poisson brackets with commutators, the “first principles” quantization in integrable models starts with the formal substitution of the Yang-Baxter Poisson algebra by the quantum Yang-Baxter algebra. However, many interesting models possessing the zero curvature representation belong to the non-ultralocal class of theories where it is difficult to ascertain the emergence of the Yang-Baxter Poisson algebra. This makes the quantization of such models problematic.

In this work, we investigated the emergence of the Yang Baxter Poisson algebra in a non-ultralocal system. Our considerations are inspired by the age-old observation that the quantum monodromy operator is somehow better behaved than its classical counterpart. In our central example we recovered the Yang-Baxter Poisson algebra in a non-ultralocal system based on
the $SU(2)$ current algebra by starting with an explicit quantum field theory realization of the Yang-Baxter relation and then taking the classical limit. As a result of the entangled interplay between the classical limit and the scaling one, which required ultraviolet regularization of the model, we found that the classical monodromy matrix is somewhat more cumbersome than its quantum counterpart. It turned out that the net result of the non-ultralocal structure for the Yang-Baxter Poisson algebra is the non-universal renormalization of the spectral parameter which occurs even at the classical level. This is somewhat in the spirit of Faddeev and Reshetikhin \[21\] who proposed to ignore the problem of non-ultralocality, arguing that it is a consequence of choosing the “false vacuum”, and to restore the ultralocality of the current algebra by hand.

The example we elaborated is relevant to the Fateev model, an integrable two parameter deformation of the $SU(2)$ Principal Chiral Field. It provides evidence for the existence of the Yang-Baxter Poisson structure for this remarkable non-linear sigma model, which was shown for several particular cases in the parameter space. We believe that unraveling the Yang-Baxter Poisson algebra for non-ultralocal systems is important in many respects. Of special interest is the Klimčík model and its reductions \[22\] which have recently attracted a great deal of attention in the context of the AdS/CFT correspondence \[23, 24\]. We supplement these notes by two appendices which collect a number of facts about the Klimčík model that, in our opinion, fill some gaps in the current literature.

**Note added.** In the previous version in Appendix A, a formula was presented relating the currents $I_\sigma$ and $J_\sigma$. It turns out to admit a significant simplification. This has allowed us to shorten the presentation by transferring the simplified formula to the main body of the text, see eq. (4.13), and removing Appendix A entirely. Parts of what used to be Appendix A have been incorporated into sec.4 and some accompanying minor stylistic changes were made, e.g., the splitting of sec.4 into subsections and the removal of some redundant formulae.

7 Acknowledgments

Part of the work was done during the third author’s visit to the International Institute of Physics at Natal and KITP at Santa Barbara in the fall of 2017. S.L. would like to thank these institutes for the support and hospitality he received during his stay. The authors are grateful to Vladimir Mangazeev for useful suggestions as well as his interest in this work.

A Appendix

Here we discuss some geometrical aspects of the Klimčík non-linear sigma model. The target space is topologically the same as $\mathcal{G}$ (which below is assumed to be a compact simple Lie group) but equipped with a certain anisotropic metric $G_{\mu\nu}$. The latter can be thought of as a two-parameter deformation of the left/right invariant metric on the group manifold. In fact, the form of the Lagrangian (5.1) suggests that the target manifold is equipped with the affine connection $\Gamma$ such that the metric is covariantly constant w.r.t. $\Gamma$, while its torsion is defined by the antisymmetric tensor $B_{\mu\nu}$. To be precise, the covariant torsion tensor

$$H_{\lambda\mu\nu} = G_{\lambda\rho} \left( \Gamma_\rho_{\mu\nu} - \Gamma_\rho_{\nu\mu} \right)$$

(A.1)
(here $\Gamma^\rho_{\mu\nu}$ stands for the Christoffel symbol), is a closed 3-form with $B_{\mu\nu}$ playing the rôle of the torsion potential:

$$H_{\lambda\mu\nu} = \partial_{\lambda}B_{\mu\nu} + \partial_{\nu}B_{\lambda\mu} + \partial_{\mu}B_{\nu\lambda}.$$  

A remarkable feature of the Klimčík target space background is that it admits a set of 1-forms which can be thought of as deformations of the Maurer-Cartan forms. Introduce two sets $\{e^a_\mu(\sigma)\}_{a=1}^D$ ($D = \dim G$):

$$\mathfrak{t}_a \ e^a_\mu(\sigma) \ dX^\mu = -2i \hat{\Omega}^{-1} (\mathbf{U}^{-1} \ d\mathbf{U}).$$  

(A.2)

Here $\hat{\Omega}_\sigma$ stands for the linear operator acting in $\mathfrak{g}$,

$$\hat{\Omega}_\sigma = \hat{1} + i \sigma \varepsilon_1 \hat{R}_{\mathbf{U}} + i \sigma \varepsilon_2 \hat{R}$$  

(A.3)

and $\sigma$ takes two values $\pm$. It is not difficult to show that the metric can be written as

$$G_{\mu\nu} = \frac{1}{2g^2} q_{ab} \ e^a_\mu(+) \ e^b_\nu(+) = \frac{1}{2g^2} q_{ab} \ e^a_\mu(-) \ e^b_\nu(-),$$  

(A.4)

i.e., $\{e^a_\mu(+)^{D=1}\}$ and $\{e^a_\mu(-)^{D=1}\}$ are two vielbein sets in the cotangent space of the target manifold. Notice the following simple relations

$$G^{\mu\nu} \ e^a_\mu(+) \ e^b_\nu(+) = G^{\mu\nu} \ e^a_\mu(-) \ e^b_\nu(-) = 2g^2 \ q_{ab}$$

and

$$\sqrt{\det G_{\mu\nu}} = (\det \hat{\Omega}_\sigma)^{-1} \times \sqrt{\det G^{(0)}_{\mu\nu}},$$  

(A.5)

where $G^{(0)}_{\mu\nu} = G_{\mu\nu}|_{\varepsilon_1=\varepsilon_2=0}$.

It turns out that the torsion also admits simple expressions involving $e^a_\mu(\sigma)$ and the structure constants $F^{abc}(\sigma, \sigma'|\sigma'')$ appearing in the Poisson algebra (4.7):

$$H_{\lambda\mu\nu} = + \frac{1}{4g^2} (F_{abc}(-+|+) \ e^c_\lambda(+) \ e^a_{\mu}(+) \ e^b_{\nu}(+) - 2F_{abc}(+|+) \ e^c_\lambda(+) \ e^a_{\mu}(+) \ e^b_{\nu}(+))$$  

(A.6a)

and

$$H_{\lambda\mu\nu} = - \frac{1}{4g^2} (F_{abc}(+-|-) \ e^c_\lambda(-) \ e^a_{\mu}(+) \ e^b_{\nu}(-) - 2F_{abc}(-|-) \ e^c_\lambda(-) \ e^a_{\mu}(-) \ e^b_{\nu}(-)).$$  

(A.6b)

Here the symbol $[\lambda\mu\nu]$ denotes the alternating summation over all possible permutations of the indices $\lambda, \mu$ and $\nu$.

Before discussing the origin of the above formulæ for the metric and torsion, let us first inspect the reality condition for the target space background. Consider the metric and the torsion as a function of $\varepsilon_1$ with the ratio $\varepsilon_2/\varepsilon_1$ a fixed real number. First of all it is easy to see that the determinant $\det \hat{\Omega}_\sigma$ which appears in the formula (A.5) does not depend on the choice of the sign factor $\sigma$ – it is a polynomial in the variable $\varepsilon_2^2$ of degree coinciding with the integer part of half of $D \equiv \dim(\mathfrak{g})$:

$$\det \hat{\Omega}_\sigma = 1 + \sum_{n=1}^{[\frac{D}{2}]} \omega^{(n)} \ \varepsilon_1^{2n},$$

22
where the coefficients \( \omega^{(n)} \) are real as \( \Im m (\varepsilon_2 / \varepsilon_1) = 0 \). In their turn, the components of the metric tensor and the torsion are rational functions of \( \varepsilon_1 \) of the form

\[
G_{\mu\nu} = \frac{1}{\det \Omega_\sigma} \sum_{n=0}^{D-1} g^{(n)}_{\mu\nu} \varepsilon_1^{2n} \tag{A.7}
\]

\[
H_{\lambda\mu\nu} = \frac{i \varepsilon_1}{(\det \Omega_\sigma)^2} \sum_{n=0}^{D-1} h^{(n)}_{\lambda\mu\nu} \varepsilon_1^{2n}.
\]

For pure imaginary \( \varepsilon_1 \), the 1-forms \( e^a_\mu (\sigma) \) are real and, as it follows from (A.4), the metric is positive definite. Formula (A.7) implies that it remains positive definite for sufficiently small real \( \varepsilon_1 \). At the same time, as it follows from (A.6), (4.8) the torsion is real for pure imaginary \( \varepsilon_1 \). Therefore the expansion coefficients \( h^{(n)}_{\lambda\mu\nu} \) turn out to be real as \( \Im m (\varepsilon_2 / \varepsilon_1) = 0 \). However, \( H_{\lambda\mu\nu} \) takes pure imaginary values for real \( \varepsilon_1 \) and \( \varepsilon_2 \), in particular for \( 0 < \varepsilon_1 < 1, 0 < \varepsilon_2 < 1 - \varepsilon_1 \).

Vielbeins

To clarify the special rôle of the 1-forms (A.2) for the Klimčík target space background let us make the following observations.

First we point out that the 1-forms \( e^a_\mu (\sigma) \) are covariantly constant w.r.t. the spin-connection

\[
\omega_{\nu, b}^a (\pm) = F_{ac}^b (\pm \mid +) e^c_\nu (\pm),
\]

i.e.,

\[
\partial_\nu e^a_\mu (\pm) - \Gamma^\lambda_{\mu\nu} e^a_\lambda (\pm) + \omega_{\nu, b}^a (\pm) e^b_\mu (\pm) = 0. \tag{A.8}
\]

A simple consequence of this fact is that the covariant derivative of the metric (A.4) is zero, as it should be. In a similar manner, the 1-forms \( e^a_\mu (\sigma) \) satisfy the covariant constant condition

\[
\partial_\nu e^a_\mu (\pm) - \Gamma^\lambda_{\nu\mu} e^a_\lambda (\pm) + \omega_{\nu, b}^a (\pm) e^b_\mu (\pm) = 0 \tag{A.9}
\]

which involves another spin-connection

\[
\omega_{\nu, b}^a (\sigma) = F_{ac}^b (\pm \mid -) e^c_\nu (\sigma). \]

Finally, the covariantly constant 1-forms obey the Maurer-Cartan type equations:

\[
\partial_\nu e^a_\mu (\pm) - \frac{1}{2} \left( q^{aa'} F_{a'b c} (\pm \mid +) - \Theta^{aa'} F_{a'b c} (\pm \mid +) \right) e^b_\nu (\pm) e^c_\mu (\pm) = 0
\]

\[
\partial_\nu e^a_\mu (\pm) - \frac{1}{2} \left( q^{aa'} F_{a'b c} (\pm \mid -) - \Theta^{aa'} F_{a'b c} (\pm \mid -) \right) e^b_\nu (\pm) e^c_\mu (\pm) = 0 \tag{A.10}
\]

\(^7\)Presumably the metric remains positive definite in the parameter domain \( 0 < \varepsilon_1 < 1, 0 < \varepsilon_2 < 1 - \varepsilon_1 \).
with
\[ \Theta^{aa'} : \quad e^a_{\mu}(+) = \Theta^{ab}_{\mu} e^b_{\mu}(-) \quad \text{and} \quad e^{a'}_{\nu}(-) = \frac{1}{2g^2} G^{\mu\nu} e^a_{\mu}(+) e^{a'}_{\nu}(-), \quad \Theta^a_{\nu} q^{cd} \Theta^b_{\mu} = q^{ab}. \]

Relations (A.8), (A.9) allow one to express the torsion in terms of \( e^a_{\mu}(\sigma) \). Namely, a simple calculation yields
\[
\Gamma_{\lambda\mu\nu} = \frac{1}{2g^2} q_{ab} \left( \omega_{\nu,c} a^a b \left( \omega_{\mu,b} c e^a_{\lambda}(+) e^b_{\mu}(+) \right) + e_{\lambda}^a(+) \partial_\nu e^b_{\mu}(+) \right)
\]
\[
\Gamma_{\lambda\mu\nu} = \frac{1}{2g^2} q_{ab} \left( \omega_{\mu,c} a^a b \left( \omega_{\nu,b} c e^a_{\lambda}(-) e^b_{\nu}(-) \right) + e_{\lambda}^a(-) \partial_\mu e^b_{\nu}(-) \right).
\]

These formulae, combined with (A.1) imply
\[ H_{\lambda\mu
u} = \frac{1}{2g^2} q_{ab} \left( \omega_{\nu,c} a^a b \left( \omega_{\mu,b} c e^a_{\lambda}(+) e^b_{\mu}(+) \right) + e_{\lambda}^a(+) \partial_\nu e^b_{\mu}(+) \right) + e_{\lambda}^a(-) \partial_\mu e^b_{\nu}(-) \]
\[ \sum_{\sigma, \sigma', \sigma''} = \frac{1}{2} \sum_{\sigma = \pm} F_{abc}(\sigma, \sigma', \sigma'') e^a_{\lambda}(\sigma) e^b_{\mu}(\sigma) e^c_{\nu}(\sigma'). \]

In the case under consideration, the torsion is a 3-form and the more elegant expressions (A.6) can be achieved by anti-symmetrizing w.r.t. the Greek indices and using the formula
\[ \tilde{F}_{abc}(\sigma, \sigma', \sigma'') = - \tilde{F}_{bac}(\sigma', \sigma, \sigma'') = - \tilde{F}_{acb}(\sigma, \sigma'', \sigma'). \]

Then (A.4), (A.6) can be re-written as
\[ G_{\mu
u} = \frac{1}{4g^2} \sum_{\sigma = \pm} \sigma q_{ab} T^a_{\lambda}(\sigma) T^b_{\mu}(\sigma) \]
\[ H_{\lambda\mu
u} = \frac{1}{4g^2} \sum_{\sigma, \sigma', \sigma'' = \pm} \text{sgn}(\sigma + \sigma' + \sigma'') \tilde{F}_{abc}(\sigma, \sigma', \sigma'') T^a_{\lambda}(\sigma) T^b_{\mu}(\sigma') T^c_{\nu}(\sigma''), \]

where we also use
\[ T^a_{\mu}(\sigma) \equiv e^{\frac{i\pi}{4\sigma}} e^a_{\mu}(\sigma). \]
Ricci tensor

Let $R_{\mu\nu}$ be the Ricci tensor built from the affine connection $\Gamma$ (A.11). For practical purposes, it is useful to express it in terms of the symmetric Ricci tensor $\bar{R}_{\mu\nu}$ associated with the Levi-Civita connection.\footnote{Below, the Ricci tensor is defined as $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ where $R^\alpha_{\beta\mu\nu}$ is the Riemann tensor $R^\lambda_{\mu\rho\nu} = \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\lambda} \Gamma^\lambda_{\rho\nu} - \Gamma^\lambda_{\rho\lambda} \Gamma^\lambda_{\mu\nu}$ and $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$ stands for the Christoffel symbols for the Levi-Civita connection.} Using the results from the work [17] one can show that

$$\frac{1}{2}R_{(\mu\nu)} = R_{\mu\nu} - \frac{1}{4}H^{\alpha\beta\gamma}H_{\alpha\beta\rho\nu} = \frac{3}{8} (1 - (\varepsilon_1 - \varepsilon_2)^2) (1 - (\varepsilon_1 + \varepsilon_2)^2) \sum_{\sigma = \pm} q_{ab} e^a_{\mu}(\sigma) e^b_{\nu}(-\sigma)$$

$$- \nabla_\mu W_\nu - \nabla_\nu W_\mu$$

(A.12)

$$\frac{1}{2}R_{[\mu\nu]} = \frac{1}{2} \nabla_\lambda H^\lambda_{\mu\mu} = \frac{3}{8} (1 - (\varepsilon_1 - \varepsilon_2)^2) (1 - (\varepsilon_1 + \varepsilon_2)^2) \sum_{\sigma = \pm} q_{ab} \sigma e^a_{\mu}(\sigma) e^b_{\nu}(-\sigma)$$

$$+ W_\lambda H^\lambda_{\mu\mu} + \partial_\mu W_\nu - \partial_\nu W_\mu .$$

Here

$$W_\mu = -\frac{1}{2} \partial_\mu \log (\det \Omega_\sigma) + w_\mu$$

(A.13)

with $\Omega_\sigma$ given by (A.3) and

$$w_\mu = \pm \frac{i}{4} e^a_{\mu}(\pm) f_{ab}^c (\varepsilon_1 \bar{R} - \varepsilon_2 R)^b_c .$$

The last formula holds true for any choice of the sign $\pm$ and we use the notation

$$\bar{R}^b_c = (U^{-1} R U)^b_c = (U^{-1})^b_{\tilde{c}} \tilde{R}_{\tilde{c} \tilde{b}} U^\tilde{c} \tilde{b} ,$$

where $U_{a b}$ stands for the $D \times D$ matrix of the group element $U$ in the adjoint representation:

$$U_{t_a} U^{-1} = t_b U^b_a .$$

1-loop renormalization of the Klimčík NLSM

In the path-integral quantization, the general NLSM (5.1) should be equipped with a UV cutoff. A consistent removal of the UV divergences requires that the “bare” target space metric and torsion potential be given a certain dependence on the cutoff momentum $\Lambda$. To the first perturbative order in the Planck constant $\hbar$ the RG flow equations are given by [25–27]

$$\partial_t G_{\mu\nu} = -\hbar \left( R_{\mu\nu} - \frac{1}{4} H^{\alpha\beta\gamma}H_{\alpha\beta\rho\nu} + \nabla_\mu V_\nu + \nabla_\nu V_\mu \right) + O(\hbar^2)$$

$$\partial_t B_{\mu\nu} = -\hbar \left( -\frac{1}{2} \nabla_\lambda H^\lambda_{\mu\nu} + V_\lambda H^\lambda_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \right) + O(\hbar^2) ,$$

(A.14)

where $\partial_t \equiv 2\pi \Lambda \frac{\partial}{\partial \Lambda}$. The infinitesimal variation of the Klimčík metric and torsion potential, assuming that the combinations of the couplings $\frac{\partial}{\partial \varepsilon_1}, g^2 \varepsilon_1$ are kept fixed, can be expressed as

$$\delta G_{\mu\nu} = \frac{\delta \varepsilon_1}{4g^2 \varepsilon_1} \sum_{\sigma = \pm} q_{ab} e^a_{\mu}(\sigma) e^b_{\nu}(-\sigma)$$

$$\delta B_{\mu\nu} = -\frac{\delta \varepsilon_1}{4g^2 \varepsilon_1} \sum_{\sigma = \pm} q_{ab} \sigma e^a_{\mu}(\sigma) e^b_{\nu}(-\sigma) .$$


With the explicit formulae for the Ricci tensor \((A.12)\), it is easy to see that the general RG flow equations \((A.14)\) are satisfied if \(V_\mu = \Lambda_\mu = W_\mu\) with \(W_\mu\) given by \((A.13)\). Also it follows that the evolution of the bare couplings under a change in \(\Lambda\) is described by the system of ordinary differential equations \((4.31)\).

**B Appendix**

In this Appendix we provide the explicit relation between the flat connection \((4.17)\) for the case of the Fateev model \((\mathfrak{g} = SU(2))\) and that given in the work \([20]\).

In that work a more general four parameter deformation of the \(SU(2)\) principal chiral field is considered which contains the Fateev model as a two-parameter subfamily. The deformation parameters were denoted by \((\eta, \nu^{(L)}, \sigma, q)\) and, for the case of the Fateev model, \(\nu^{(L)}\) together with \(\sigma\) should be set to zero:

\[
\nu^{(L)} = \sigma = 0.
\]

Here the superscript \(L\) has been used to distinguish the parameter \(\nu\) in ref. \([20]\) with the one from this work. The remaining two parameters \(\eta\) and \(q\) are related to \(\kappa\) and \(\nu\) in \((4.34)\) as

\[
\kappa = \frac{\vartheta_2^2(0,q^2)}{\vartheta_3^2(0,q^2)}, \quad \nu = -i \frac{\vartheta_1(i\eta,q^2)}{\vartheta_4(i\eta,q^2)},
\]

where \(\vartheta_a\) stand for the conventional theta functions. In ref. \([20]\) the same co-ordinates \(v\) and \(w\) that appear in the Euler decomposition \((5.2)\) are used, while \(\phi\) from \((5.3)\) is replaced by \(u\), such that

\[
\tanh(\phi) = \frac{\vartheta_2(u,q^2)\vartheta_3(0,q^2)}{\vartheta_3(u,q^2)\vartheta_2(0,q^2)} (0 < u < \pi).
\]

The flat connection \(A^{(L)}_{\pm}\) found in \([20]\) is defined by eqs. \((1.6)\), \((2.7)\) and \((2.10)\)–\((2.14)\) from that work, where \(\lambda\) is the spectral parameter and, for the Fateev model, \(\eta_+ = \eta_- = \eta\) and \(\phi_\pm = 0\). Formulae \((2.7)\), \((2.10)\) involve the vielbein \(e^a_\mu\) \((\mu = u, v, w)\), which in turn are given by eqs. \((2.28)\)–\((2.32)\). Here, for the convenience of the reader, we reproduce the main equations needed for the computation of \(A^{(L)}_{\pm}\) specializes to the Fateev model.

The non-vanishing components of the vielbein are given by

\[
e^3_u = \frac{i}{g} \frac{\vartheta_2(i\eta,q) \vartheta_1'(0,q)}{\vartheta_1(i\eta,q) \vartheta_2(0,q)},
\]

\[
e^\pm_v = \mp \frac{i}{g} \frac{\vartheta_2(0,q^2) \vartheta_4(i\eta \pm u,q^2)}{\vartheta_4(u,q^2) \vartheta_4(i\eta,q^2)},
\]

\[
e^\pm_w = \pm \frac{i}{g} \frac{\vartheta_2(0,q^2) \vartheta_1(i\eta \pm u,q^2)}{\vartheta_4(u,q^2) \vartheta_1(i\eta,q^2)}.
\]

Note that, with these expressions at hand, it is simple to re-write the Lagrangian of the Fateev model in terms of the parameters \((\eta, q)\) and the co-ordinates \(X^\mu = (u, v, w)\) since

\[
\mathcal{L}_F = 2G_{\mu\nu} \partial_+ X^\mu \partial_- X^\nu
\]
and the non-zero components of the metric tensor $G_{\mu\nu}$ are

$$G_{uu} = (e_u^2)^2, \quad G_{vv} = e_v^+ e_v^-, \quad G_{ww} = e_w^+ e_w^-, \quad G_{vw} = \frac{1}{2} (e_v^+ e_w^- + e_w^+ e_v^-).$$

The connection is constructed from the matrix valued 1-form $\zeta_\mu(\lambda)$ defined by

$$\zeta_\mu(\lambda) = f_3(\lambda) e_\mu^3 \sigma^3 + f_+(\lambda) e_\mu^+ \sigma^+ + f_-(\lambda) e_\mu^- \sigma^-,$$

where $\sigma^3$ and $\sigma^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$ are the standard Pauli matrices, while

$$f_+(\lambda) = -f_-(\lambda) = -\frac{g}{2} \frac{\partial_1(u - \frac{i}{2}, q) \partial_1(i\eta, q) \partial_2(0, q)}{\partial_1(u, q) \partial_2(i\eta, q) \partial_1(\frac{i}{2}, q)}$$

$$f_3(\lambda) = -\frac{g}{2} \frac{\partial_1(i\eta, q) \partial_2(0, q) \partial_1'(\frac{i}{2}, q)}{\partial_2(i\eta, q) \partial_1'(0, q) \partial_1(\frac{i}{2}, q)}.$$

In terms of this 1-form, the connection components $A^{(L)}_\pm$ are expressed as

$$A^{(L)}_+ = \frac{1}{2i} \sum_\mu \left( \zeta_\mu(i\eta + \lambda) + \sigma^2 \zeta_\mu(i\eta - \lambda) \sigma^2 \right) \partial_+ X^\mu,$$

$$A^{(L)}_- = \frac{1}{2i} \sum_\mu \left( \zeta_\mu(i\eta - \lambda - \pi) + \sigma^2 \zeta_\mu(i\eta - \lambda + \pi) \sigma^2 \right) \partial_- X^\mu,$$

where $X^\mu = (u, v, w)$. One should keep in mind that the zero curvature representation in [20] is

$$[\partial_+ + A^{(L)}_+, \partial_- + A^{(L)}_-] = 0,$$

which differs from the convention used in this work (4.16) by the overall sign of $A_\pm$.

The gauge transformation that maps the flat connection $A^{(L)}_\pm$ to the one in (4.17), (4.5) with $U$ understood as a matrix in the fundamental representation of $SU(2)$ (i.e., $h = \sigma^3$, $e_\pm = \sigma^\pm$), is described as follows:

$$\partial_\pm - A_\pm = S \left( \partial_\pm + A^{(L)}_\pm \right) S^{-1},$$

where

$$S = \sqrt{\frac{\partial_1(\lambda, q^2) \partial_1(0, q^2)}{2 \partial_1(\lambda, q^2) \partial_1(u, q^2)}} \left( \begin{array}{ccc} \frac{i e^{i\omega}}{\partial_2(i\lambda, q^2)} & \frac{i e^{i\omega}}{\partial_3(i\lambda, q^2)} & \frac{i e^{i\omega}}{\partial_1(i\lambda, q^2)} \\ \frac{i e^{-i\omega}}{\partial_1(i\lambda, q^2)} & \frac{i e^{i\omega}}{\partial_3(i\lambda, q^2)} & \frac{i e^{-i\omega}}{\partial_1(i\lambda, q^2)} \\ \frac{i e^{-i\omega}}{\partial_1(i\lambda, q^2)} & \frac{i e^{i\omega}}{\partial_3(i\lambda, q^2)} & \frac{i e^{-i\omega}}{\partial_1(i\lambda, q^2)} \end{array} \right)$$

and $S^{-1} = \sigma_2 S^T \sigma_2$ (det $S = 1$). The parameters $\rho_\pm$ are expressed in terms of the spectral parameter $\lambda$ as

$$\rho_+ = \frac{\partial_1^2(\frac{i}{2}, q)}{\partial_1^2(\frac{i}{2}, q)}, \quad \rho_- = \frac{\partial_1^2(\frac{i}{2}, q)}{\partial_1^2(\frac{i}{2}, q)}.$$

Finally note that the original deformation parameters $\varepsilon_1, \varepsilon_2$ in the Lagrangian (4.3) are related to $q$ and $\eta$ as

$$\varepsilon_1 = \frac{\partial_2^2(i\eta, q^2) \partial_3(0, q^2) \partial_2(0, q^2)}{\partial_1^2(0, q^2) \partial_3(i\eta, q^2) \partial_2(i\eta, q^2)}, \quad \varepsilon_2 = -\frac{\partial_2^2(i\eta, q^2) \partial_3(0, q^2) \partial_2(0, q^2)}{\partial_1^2(0, q^2) \partial_3(i\eta, q^2) \partial_2(i\eta, q^2)}.$$
References

[1] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London (1982).

[2] E. K. Sklyanin, *On the complete integrability of the Landau-Lifshitz equation*, preprint LOMI E-3-79 (1979).

[3] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer-Verlag, Berlin (1987).

[4] J. M. Maillet, *New integrable canonical structures in two-dimensional models*, Nucl. Phys. B 269, 54 (1986).

[5] A. Duncan, H. Nicolai and M. Niedermaier, *On the Poisson bracket algebra of monodromy matrices*, Z. Phys. C 46, 147 (1990).

[6] V. A. Fateev, *The sigma model (dual) representation for a two-parameter family of integrable quantum field theories*, Nucl. Phys. B 473, 509 (1996).

[7] V. G. Drinfel’d, *Quantum Groups*, Proceedings of the International Congress of Mathematics, Berkeley 1986, 1, 798-820, Academic Press, California (1987).

[8] M. Jimbo, *A q-difference analogue of U(g) and the Yang-Baxter equation*, Lett. Math. Phys. 10, 63 (1985).

[9] S. M. Khoroshkin, A. A. Stolin and V. N. Tolstoi, *Gauss decomposition of trigonometric R matrices*, Mod. Phys. Lett. A 10, 1375 (1995) [arXiv:hep-th/9404038].

[10] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable structure of conformal field theory. 3. The Yang-Baxter relation*, Commun. Math. Phys. 200, 297 (1999) [arXiv:hep-th/9805008].

[11] V. V. Bazhanov, G. A. Kotousov and S. L. Lukyanov, *Quantum transfer-matrices for the sausage model*, JHEP 1801, 021 (2018) [arXiv:1706.09941].

[12] B. L. Feigin and A. M. Semikhatov, *The $\hat{s}\mathfrak{l}(2)\oplus \hat{s}\mathfrak{l}(2)/\hat{s}\mathfrak{l}(2)$ coset theory as a Hamiltonian reduction of $\hat{D}(2|1;\alpha)$*, Nucl. Phys. B 610, 489 (2001) [arXiv:hep-th/0102078].

[13] V. V. Bazhanov and S. L. Lukyanov, *Integrable structure of quantum field theory: classical flat connections versus quantum stationary states*, JHEP 1409, 147 (2014) [arXiv:1310.4390].

[14] C. Klimcik, *Integrability of the bi-Yang-Baxter sigma-model*, Lett. Math. Phys. 104, 1095 (2014) [arXiv:1402.2105].

[15] F. Delduc, S. Lacroix, M. Magro and B. Vicedo, *On the Hamiltonian integrability of the bi-Yang-Baxter sigma-model*, JHEP 1603, 104 (2016) [arXiv:1512.02462].

[16] V. E. Zakharov and A. V. Mikhailov, *Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique*, Sov. Phys. JETP 47, 1017 (1978) [Zh. Eksp. Teor. Fiz. 74, 1953 (1978)].
[17] G. Valent, C. Klimcik and R. Squellari, *One loop renormalizability of the Poisson-Lie sigma models*, Phys. Lett. B 678, 143 (2009) [arXiv:0902.1459].

[18] K. Sfetsos, K. Siamps and D. C. Thompson, *Generalized integrable λ- and η-deformations and their relation*, Nucl. Phys. B 899, 489 (2015) [arXiv:1506.05784].

[19] B. Hoare and A. A. Tseytlin, *On integrable deformations of superstring sigma models related to AdS_n × S^n supercosets*, Nucl. Phys. B 897, 448 (2015) [arXiv:1504.07213].

[20] S. L. Lukyanov, *The integrable harmonic map problem versus Ricci flow*, Nucl. Phys. B 865, 308 (2012) [arXiv:1205.3201].

[21] L. D. Faddeev and N. Y. Reshetikhin, *Integrability of the principal chiral field model in 1+1 dimension*, Ann. Phys. 167, 227 (1986).

[22] F. Delduc, M. Magro and B. Vicedo, *On classical q-deformations of integrable sigma-models*, JHEP 1311, 192 (2013) [arXiv:1308.3581].

[23] F. Delduc, M. Magro and B. Vicedo, *Integrable deformation of the AdS_5 × S^5 superstring action*, Phys. Rev. Lett. 112, 051601 (2014) [arXiv:1309.5850].

[24] G. Arutyunov, R. Borsato and S. Frolov, *S-matrix for strings on η-deformed AdS_5 × S^5*, JHEP 1404, 002 (2014) [arXiv:1312.3542].

[25] D. Friedan, *Nonlinear models in 2 + ε dimensions*, Phys. Rev. Lett. 45, 1057 (1980); Ann. Phys. 163, 318 (1985).

[26] E. S. Fradkin and A. A. Tseytlin, *Quantum string theory effective action*, Nucl. Phys. B 261, 1 (1985).

[27] C. G. Callan, Jr., E. J. Martinec, M. J. Perry and D. Friedan, *Strings in background fields*, Nucl. Phys. B 262, 593 (1985).