The cohomology of biquotients
via a product on the two-sided bar construction
(expository version)

Jeffrey D. Carlson

with an appendix by Jeffrey D. Carlson and Matthias Franz

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Abstract

We compute the Borel equivariant cohomology ring of the left $K$-action on a homogeneous space $G/H$, where $G$ is a connected Lie group, $H$ and $K$ are closed, connected subgroups and 2 and the torsion primes of the Lie groups are units of the coefficient ring. As a special case, this gives the singular cohomology rings of biquotients $H\backslash G/K$. This depends on a version of the Eilenberg–Moore theorem developed in the appendix, where a novel multiplicative structure on the two-sided bar construction $B(A'', A, A')$ is defined, valid when $A'' \leftarrow A \rightarrow A'$ is a pair of maps of homotopy Gerstenhaber algebras.

Homogeneous spaces, which can be realized as coset spaces $G/H$ for $G$ a transitively acting Lie group and $H$ the stabilizer of a point, are arguably the most highly symmetric, most canonical, and most thoroughly investigated objects of study in differential geometry after Lie groups. A generalization of perennial interest, which offers many interesting examples in positive–curvature geometry, is the class of biquotients, the orbit spaces $K\backslash G/H$ of $G$ under free left–right actions by products $K \times H$ of two closed subgroups.

The cohomology rings of a Lie group $G$ over $\mathbb{Q}$ and those finite fields $\mathbb{F}_p$ for which $H^*(G; \mathbb{Z})$ lacks $p$-torsion have been known to be exterior algebras since the fundamental 1941 work of Hopf [Hopf41], and that of a homogeneous space $G/H$ with connected stabilizer $H$ has been known over $\mathbb{R}$ since work of Henri Cartan from 1950 [Ca51]:

$$H^*(G/H) \cong \text{Tor}_{H^*(BG)}(\mathbb{R}, H^*(BH)).$$

In his 1952 dissertation [Bor53, §30], Borel used the Serre spectral sequence of the Borel fibration $G \rightarrow G/H \rightarrow BH$ to show the same ring isomorphism also holds when $H$ is of maximal rank, with $\mathbb{F}_p$ coefficients if $H^*(G; \mathbb{Z})$ and $H^*(H; \mathbb{Z})$ lack $p$-torsion and with $\mathbb{Z}$ coefficients if they are torsion-free. Starting with Baum’s 1962 dissertation [Baum68], a program began to obtain a more general result using the then-new Eilenberg–Moore spectral sequence of the fibration $G/H \rightarrow BH \rightarrow BG$. Cartan’s result implies this spectral sequence’s collapse over $\mathbb{R}$, and Baum proved collapse under hypotheses covering the best-studied homogeneous spaces. Subsequent work of

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authors including Baum, Larry Smith, Gugenheim, May, Munkholm, and Joel Wolf in the 1960s–70s proved collapse under substantially more general hypotheses. However, almost all of these proofs provided only the additive structure of $H^*(G/H)$, and those which in special cases gave the ring structure could be factored through Borel’s theory. A collapse result guaranteeing a multiplicative isomorphism was provided by Franz only in 2019 [Fr19].

Arguably the most general of the Eilenberg–Moore collapse results actually applies to a more general case than homogeneous spaces: Munkholm proves the collapse of the spectral sequence corresponding to the total space of a pullback bundle

$$X \times_B E \to E \to X \to B$$

when $X$, $B$, and $E$ have polynomial cohomology on countably many generators, the fundamental group $\pi_1(B)$ is trivial, and 2 is a unit of the coefficient ring [Mu74]. A biquotient—and hence, in particular, a homogeneous space—fits into this setting when $X = BK$, $B = BG$, and $E = BH$ are models of classifying spaces chosen in such a way that $BH \to BG$ is a fiber bundle, and Singhof [Sio93] used Munkholm’s result in this case to determine that $H^*(K\backslash G/H)$ is additively isomorphic to $\text{Tor}^*_{H^*(BG)}(H^*(BK), H^*(BH))$, and simply to $\text{Tor}^0$ when the rank of $G$ is the sum of the ranks of $K$ and $H$. Because the Eilenberg–Moore spectral sequence is concentrated in the 0th column in this case, there is no extension problem, so this is in fact a ring isomorphism. One might hope Singhof’s and Franz’s theorems were special cases of a more general result, and the initial motivation of this paper was to show this hope is justified.

**Theorem 0.1.** Let $G$ be a connected Lie group, $H$ and $K$ closed, connected subgroups, and $k$ a principal ideal domain in which 2 is a unit and the torsion primes of $G$, $H$, and $K$ are invertible. Then the Borel equivariant cohomology ring of the left translation action of $K$ on $G/H$, or, equivalently, of the two-sided action of $K \times H$ on $G$ by $(x, h) \cdot g := xgh^{-1}$, is

$$H^*_k(G/H; k) \cong H^*_{K \times H}(G; k) \cong \text{Tor}^*_{H^*(BG,k)}(H^*(BK; k), H^*(BH; k)).$$

In particular, if the two-sided action of $K \times H$ on $G$ is free, the cohomology ring of the biquotient $K\backslash G/H$ is given by

$$H^*(K\backslash G/H; k) \cong \text{Tor}^*_{H^*(BG,k)}(H^*(BK; k), H^*(BH; k)).$$

In broadest outline, our proof uses the Eilenberg–Moore theorem to show the cohomology of the homotopy pullback of the diagram $BK \to BG \leftarrow BH$ is $\text{Tor}^*_{C^*(BG)}(C^*(BK), C^*(BH))$ and constructs maps between $H^*(BG)$ and $C^*(BG)$ for $\Gamma \in \{G, K, H\}$ which then induce an isomorphism with $\text{Tor}^*_{H^*(BG)}(H^*(BK), H^*(BH))$. This much it has in common with many collapse results, but our result also shows that this map takes the classical product on the Tor of cochain algebras to that on the Tor of cohomology rings. The way this multiplicativity is established closely follows Franz’s proof, but among the technical underpinnings necessary to extend his approach from a fibration to a general pullback is one substantial innovation.

A Tor of $\text{dga}$s in full generality is not endowed with a product, and the product defined on $\text{Tor}^*_{C^*(BG)}(C^*(BK), C^*(BH))$ is synthetic, arising from the homological external product rather than a multiplicative structure on the resolution itself. As a consequence, in previous collapse
results, one could not say whether the isomorphisms shown were multiplicative. But there is a
cochain complex \( \mathbf{B}(A', A, A'') \), the two-sided bar construction, functorial in spans \( A' \leftarrow A \rightarrow A'' \) of
dga maps, whose cohomology is \( \text{Tor}_A(A', A'') \) under mild conditions. In Theorem A.1, assuming
the maps in the span are actually maps of so-called homotopy Gerstenhaber algebras, a type of
dga with extra structure (of which cochain algebras are the main examples), we are able to
define a product on \( \mathbf{B}(A', A, A'') \) inducing a product on \( \text{Tor}_A(A', A'') \) which specializes to the
known products when \( A', A, A'' \) are cochain algebras or cohomology rings. This product may
be the point of greatest interest in this paper; it is certainly the most difficult. Then the maps
between \( H^*(B\Gamma) \) and \( C^*(B\Gamma) \) alluded to in the previous paragraph, chosen with sufficient care,
and assuming 2 is a unit of the coefficient ring \( k \), will preserve this novel product in the transition
from \( \mathbf{B}(H^*(BK), H^*(BG), H^*(BH)) \) to \( \mathbf{B}(C^*(BK), C^*(BG), C^*(BH)) \).

Outline. The layout of the paper is as follows.

0.2.1. In Section 1 we recall algebraic conventions and the two-sided bar construction, taking
some pains to convincingly justify a class of maps whose existence is asserted in the literature.

0.2.2. In Section 2 we discuss extended homotopy Gerstenhaber algebras and recall results showing
normalized cochain algebras and cohomology rings are examples.

0.2.3. In Section 3 we recall the quasi-isomorphisms we need, \( A_{\infty} \)-algebra maps \( \lambda \) from \( H^*(B\Gamma) \)
to \( C^*(B\Gamma) \) for \( \Gamma \in \{ K, G, H \} \) and dga maps \( f: C^*(BT) \longrightarrow H^*(BT) \) for \( T \in \{ T_K, T_H \} \) maximal
tori in \( K \) and \( H \). These \( f \) will be defined so as to annihilate the error terms distinguishing \( \lambda \)
from a genuine dga map, so that the composites \( H^*(BK) \rightarrow C^*(BK) \rightarrow C^*(BT_K) \rightarrow H^*(BT_K) \)
and \( H^*(BH) \rightarrow H^*(BT_H) \) are just the functorially induced \( H^*(B(T_K \hookrightarrow K)) \) and \( H^*(B(T_H \hookrightarrow H)) \).
The dga quasi-isomorphisms \( f \) do not necessarily exist if \( T \) is replaced by a more general
Lie group, and to construct them we need a simplicial model for \( BT \), which necessitates the
replacements discussed in Section 4.

It is an important technical point in showing the three \( A_{\infty} \)-maps \( \lambda \) from \( H^*(B\Gamma) \) to \( C^*(B\Gamma) \)
induce a map of Tors that they are essentially functorial up to homotopy. This fact, and our control
over the error term annihilated by \( f \), come from the existence of a certain auxiliary structure on
extended homotopy Gerstenhaber algebras, a so-called strongly homotopy commutative algebra
structure \( \Phi \) whose existence was proven by Franz. The explicit formula for \( \Phi \) plays a key role
in Appendix A. The fact that \( \Phi \), so defined, is a structure of the type sought follows from an
extremely involved cochain-level computation [Fr30a].

0.2.4. In Section 4, we show a version of the Eilenberg–Moore theorem where the product on the
cohomology of the pullback of \( X \rightarrow B \leftarrow E \) arises from the product on \( \mathbf{B}(C^*(X), C^*(B), C^*(E)) \).
We then do some topological massaging to show that this Eilenberg–Moore theorem applies to
the particular simplicial model of \( BK \rightarrow BG \leftarrow BH \) we will need to use to apply the dga formality
maps discussed in the previous paragraph.

0.2.5. In Section 5, we weave these threads together to prove Theorem 0.1. We first construct a
quasi-isomorphism \( \Theta: \mathbf{B}(H^*(BK), H^*(BG), H^*(BH)) \longrightarrow \mathbf{B}(C^*(BK), C^*(BG), C^*(BH)) \), roughly
to be thought of as \( \mathbf{B}(\lambda_K, \lambda_G, \lambda_H) \), which is not necessarily multiplicative. Letting \( T_K \leq K \) and \( T_H \leq H \)
be maximal tori, we then map the codomain into \( \mathbf{B}(C^*(BT_K), C^*(BG), C^*(BT_H)) \), using the
maps \( \rho: C^*(B\Gamma) \longrightarrow C^*(BT_{\Gamma}) \) for \( \Gamma \in \{ K, H \} \) to define a strictly multiplicative

\[ \Psi = \mathbf{B}(fp, \text{id}, fp): \mathbf{B}(C^*(BK), C^*(BG), C^*(BH)) \longrightarrow \mathbf{B}(H^*(BT_K), C^*(BG), H^*(BT_H)) \]
inducing an injection in cohomology. Because we have chosen \( f \) and \( \lambda \) to compose nicely, the composite \( \Psi \Theta \) is multiplicative up to homotopy, and because the multiplicative map \( H^*(\Psi) \) is injective, this shows \( H^*(\Theta) \) is multiplicative as well, concluding the proof.

Inspiration for this proof comes from the work of many authors, and we give credit to the originators of these ideas in the Historical Remarks 5.17.

0.2.6. Finally, in Appendix A, the technical core of the present work, we construct the product on the two-sided bar construction used in the proof of our variant of the Eilenberg–Moore theorem.

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1. Algebras, twisting cochains, and bar constructions

In this background section we establish notational conventions and recount some foundational lemmas. Nothing is original here save, possibly, some of the lemmas on two-sided twisted tensor products and bar constructions, which if not published are likely still known.

Algebras, coalgebras, and the cup product

Fix forever a commutative base ring \( k \) with unity, with respect to which all tensor products and Hom-modules will be taken. We will take as understood the notions of derivation, coederivation, differential graded \( k \)-algebra (henceforth \( \text{dga} \)), and differential graded \( k \)-coalgebra (\( \text{dgc} \)), as well as their tensor products and Hom-modules. A commutative \( \text{dga} \) is a \( \text{cdga} \).

All algebras we consider are nonnegatively-graded, associative, and connected unless otherwise noted, and all coalgebras nonnegatively-graded, coassociative, and cocomplete (the notion of cocompleteness is reviewed in Definition 1.3). All \( \text{dga} \) maps preserve unit and augmentation, and all \( \text{dgc} \) maps preserve counit and coaugmentation. All ideals of \( \text{dga} \)s will be two-sided differential ideals.

We allow general cochain maps to be of nonzero degree and will explain our conventions for this case momentarily in Definition 1.5.1. The base ring \( k \) itself will be considered as a \( \text{dg} \) Hopf algebra concentrated in degree zero, in the unique way possible.

Notation 1.1.1. A \( \text{dga} \) \( A \) is a list comprising, besides the underlying graded \( k \)-module \( A \), a canonically-named multiplication \( \mu_A : A \otimes A \to A \), unit \( \eta_A : k \to A \), augmentation \( \varepsilon_A : A \to k \), and differential \( d_A : A \to A \), the clarifying decorations suppressed when practicable.

1.1.2. A \( \text{dgc} \) comprises a graded \( k \)-module \( C \), comultiplication \( \Delta_C : C \to C \otimes C \), counit \( \varepsilon_C : C \to k \), coaugmentation \( \eta_C : k \to C \), and differential \( d_C : C \to C \).

1.1.3. We write \( |x| \) for the degree \( n \) of a homogeneous element \( x \in M_n \) of any graded \( k \)-module \( M = \bigoplus M_n \). As we assume \( A \) is connected, the augmentation ideal \( \bar{A} := \ker \varepsilon \) is \( \bigoplus_{n \geq 1} A_n \), and as \( \text{dgc}s C \) are cocomplete, the coaugmentation coideal \( \bar{C} := \text{coker} \eta \) is identified with \( \bigoplus_{n \geq 1} C_n \).
Notation 1.2.1. We use \( \bullet \) to abbreviate indices or exponents representing an indefinite number of tensor factors to be determined from context; for instance, if \( A \) is a graded algebra, \( a_\bullet \) denotes a pure tensor \( a_1 \otimes \cdots \otimes a_n \in A^{\otimes n} \), where \( n \) is to be gleaned contextually. A repeated \( \bullet \) implies summation unless explicitly stated otherwise, so that for example \( \text{id}^\bullet \otimes d_A \otimes \text{id}^\bullet : A^{\otimes n} \to A^{\otimes n} \) represents the sum of the \( n \) maps \( \text{id}^\bullet \otimes d_A \otimes \text{id}^{\otimes n-\ell} \) applying \( d_A \) to one tensor factor. Given a pure tensor \( a_\bullet \in A^{\otimes n} \), an expression involving the string \( a_\bullet \) multiple times represents a sum over order-preserving subdivisions of \( a_\bullet \) into tensor-factors; for example, \( f(a_\bullet \otimes b \otimes a_\bullet \otimes c \otimes a_\bullet) \) denotes the sum, for \( 0 \leq \ell \leq m \leq n \), of all terms

\[
f(a_1 \otimes \cdots \otimes a_\ell \otimes b \otimes a_{\ell+1} \otimes \cdots \otimes a_m \otimes c \otimes a_{m+1} \otimes \cdots \otimes a_n).
\]

1.2.2. Associativity of a DGA \( A \) implies all the various iterated compositions \( A^{\otimes n} \to A \) of any of the maps \( \text{id}^\bullet \otimes \mu \otimes \text{id}^\bullet : A^{\otimes \ell} \to A^{\otimes \ell-1} \) (no summation) for \( \ell \leq n \) amount to the same \( k \)-linear map \( \mu^{[n]} \). Specifically, in this notation, we have \( \mu^{[2]} = \mu \), and \( \mu^{[1]} = \text{id}_A \), and \( \mu^{[0]} = \eta_A \). Likewise, coassociativity of a graded coalgebra \( C \) implies a well-defined iterated comultiplication \( \Delta^{[n]} : C \to C^{\otimes n} \) with \( \Delta^{[2]} = \Delta \), and \( \Delta^{[1]} = \text{id}_C \), and \( \Delta^{[0]} = \epsilon_C \).

Definition 1.3. The coaugmentation ideal \( \mathcal{C} \) of a DGCA \( C \) carries a reduced comultiplication \( \bar{\Delta} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \) given by \( c \mapsto \Delta c - 1 \otimes c - c \otimes 1 \). Coconnectedness implies every element of \( \mathcal{C} \) is annihilated by some iterate \( \Delta^{[n]} : \mathcal{C} \to \mathcal{C}^{\otimes n} \) of this map. This nilpotence property is called \textit{coconnectedness}.

Definition 1.4. Given any two graded \( k \)-modules \( A \) and \( B \), we grade the tensor product \( A \otimes B \) in the standard way by

\[
(A \otimes B)_n := \bigoplus_{p+q=n} A_p \otimes B_q.
\]

There is a graded \( k \)-linear isomorphism \((1 2) : A \otimes B \to B \otimes A \) given on homogeneous pure tensors by \( a \otimes b \mapsto (-1)^{|a||b|} b \otimes a \). Composing such transpositions, we associate to any finite list \( A^{(1)}, A^{(2)}, \ldots, A^{(n)} \) of graded \( k \)-modules and any permutation \( \pi \) of \( \{1, \ldots, n\} \) a unique graded \( k \)-linear isomorphism \( A^{(1)} \otimes \cdots \otimes A^{(n)} \to A^{(\pi(1))} \otimes \cdots \otimes A^{(\pi(n))} \) which we again denote \( \pi \). The sign inflicted by moving homogeneous elements past one another (e.g., \(-1)^{|a||b|}\) for the transposition \((1 2)\) above) is known as the \textit{Koszul sign}.

When we explicitly write out permutations, we tend to use cycle notation, so that \((1 2 3) = (2 3)(1 2)\) sends \( a \otimes b \otimes c \) to \((-1)^{|a||b||c|} c \otimes a \otimes b \) and so on. We may at times also abusively write this as \( \tau_{a \otimes b \otimes c} \) to indicate the arguments being switched. Of particular interest to us are \textit{shuffles}: a \((p, q)\)-shuffle is a permutation \( \pi \) of \( \{1, \ldots, p+q\} \) such that \( \pi(i) < \pi(j) \) whenever \( i < j \) both lie in \( \{1, \ldots, p\} \) or both lie in \( \{p+1, \ldots, p+q\} \). Thus shuffles interleave two blocks, leaving the order within each block unaffected.

Definition 1.5.1. Given two graded \( k \)-modules \( C \) and \( A \), we write \( \text{Hom}_n(C, A) \) for the \( k \)-module of \( k \)-linear maps \( f \) sending each \( C_j \) to \( A_{j+n} \), and set the degree \( |f| \) to \( n \) for such a map. We write \( \text{Hom}(C, A) \) for the direct sum \( \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(C, A) \) equipped with this grading. If \( C \) and \( A \) are cochain complexes, then \( \text{Hom}(C, A) \) becomes a cochain complex under the differential \( D = d_{\text{Hom}(C, A)} \) given by

\[
Df := d_A f - (-1)^{|f|} f d_C.
\]

An element \( f \in \text{Hom}(C, A) \) is a \textit{cochain map} if \( Df = 0 \).
1.5.2. Given a cochain complex \( B \), its \textit{desuspension} \( s^{-1}B = \{ s^{-1}b : b \in B \} \) is the graded \( k \)-module given by \( (s^{-1}B)_n := B_{n+1} \), equipped with the differential \( d_{s^{-1}B} : s^{-1}b \mapsto -s^{-1}db \) making the defining graded \( k \)-module isomorphism \( s^{-1} : b \mapsto s^{-1}b \) a cochain map. The inverse cochain isomorphism \( s : s^{-1}B \to B \) is called the \textit{suspension}.

1.5.3. Given graded \( k \)-modules \( C, C', A, A' \), the sign for the tensor product of maps \( f \in \text{Hom}(C, A) \) and \( f' \in \text{Hom}(C', A') \) is given by the Koszul rule: the composition

\[
\text{Hom}(C, A) \otimes \text{Hom}(C', A') \otimes C \otimes C' \xrightarrow{(2,3)} \text{Hom}(C, A) \otimes C \otimes \text{Hom}(C', A') \otimes C' \xrightarrow{\text{ev} \otimes \text{ev}} A \otimes A'
\]
determines the element \( f \otimes f' \) of \( \text{Hom}(C \otimes C', A \otimes A') \) taking \( c \otimes c' \) to \( (-1)^{|f'||c|} f(c) \otimes f'(c') \). If \( A \) and \( A' \) are graded algebras, then the composition

\[
\mu_{A \otimes A'} := A \otimes A' \otimes A \otimes A' \xrightarrow{(2,3)} A \otimes A \otimes A' \otimes A' \xrightarrow{\mu_A \otimes \mu_A'} A \otimes A'
\]
makes \( A \otimes A' \) a graded algebra; on the level of elements, \( (a \otimes a') \cdot (b \otimes b') := (-1)^{|a'||b} ab \otimes a' b' \).

Dually, if \( C \) and \( C' \) are graded coalgebras, then \( C \otimes C' \) becomes a graded coalgebra under \( \Delta_C \otimes \Delta_{C'} := (2,3)(\Delta_C \otimes \Delta_{C'}) \). If \( f \) and \( f' \) are both maps of \( \text{dgc} \)s (resp. \( \text{dgcs} \)), then \( f \otimes f' \) too is a \( \text{dga} \) map (resp. \( \text{dgcc} \) map).

1.5.4. Given two cochain complexes \( B \) and \( B' \), the tensor product \( B \otimes B' \) becomes a cochain complex with differential

\[
d_{\otimes} = d_B \otimes B' := d_B \otimes d_{B'} + \text{id}_B \otimes d_{B'}.
\]
This gives the familiar formula \( d(b \otimes b') = db \otimes b' + (-1)^{|b|} b \otimes db' \) since \( d_B \) is an element of \( \text{Hom}_1(B, B) \). If \( B \) and \( B' \) are \( \text{dgc} \)s or \( \text{dgcs} \), the same differential makes \( B \otimes B' \) a \( \text{dga} \) or a \( \text{dgcc} \) respectively. If \( B = B' = A \) is a \( \text{dga} \), this prescription makes \( \mu : A \otimes A \to A \) a cochain map, which is a \( \text{dga} \) map if and only if \( A \) is a cdga; if \( B = B' = C \) is a \( \text{dgc} \), then \( \Delta : C \to C \otimes C \) is a cochain map. Given a third cochain complex \( B'' \), composition of linears maps induces a graded linear map

\[
\circ : \text{Hom}(B', B'') \otimes \text{Hom}(B, B') \to \text{Hom}(B, B'')
\]
which is itself a cochain map, meaning \( D(f' \circ f) = Df' \circ f + (-1)^{|f'||f|} f' \circ Df \). In particular, taking \( B = B' = B'' \), we see \( \text{End} B := \text{Hom}(B, B) \) is a (possibly non-augmented) \( \text{dga} \) under composition, with unity \( \text{id}_B \).

1.5.5. If \( C \) is a \( \text{dgc} \) and \( A \) a \( \text{dga} \), then \( \text{Hom}(C, A) \) is again a \( \text{dga} \) with respect to the differential \( D \) of Definition 1.5.1 and the \textit{cup product}

\[
f \sim g := \mu_A(f \otimes g)\Delta_C,
\]
with unity \( \eta_A \varepsilon_C \) and augmentation sending \( f : C \to A \) to \( \eta_A f \varepsilon_C \).

If \( h \in \text{Hom}_0(C, A) \) satisfies \( h \eta_C = \eta_A \), then \( (\eta_A \varepsilon_C - h)\eta_C = 0 \), so the cup-power \( (\eta_A \varepsilon_C - h)^{-\ell} = \mu_A^{[\ell]}(\eta_C \varepsilon_A - h) \otimes \Delta_C^{[\ell]} \) annihilates the kernel of \( \Delta_C^{[\ell]} \). As \( C \) is assumed cocomplete, these kernels exhaust it, so the sum

\[
h^{-1} := \sum_{\ell=0}^{\infty} (\eta_A \varepsilon_C - h)^{-\ell}
\]
is finite on every element of \( C \) and gives a cup-inverse to \( h \).

\(^2\) These definitions arise from the Koszul sign formalism for tensor products if we let \( k\{s^{-1}\} \) be the free \( k \)-module on one generator concentrated in degree \( -1 \) and treat \( s^{-1}B \) as an alternative notation for \( k\{s^{-1}\} \otimes B \).

\(^3\) To see this, note that precomposition with \( \Delta_C \) is a cochain map since \( d_C \otimes_C \Delta_C = \Delta_C d_C \) and postcomposition with \( \mu_A \) is a cochain map since \( d_A \mu_A = \mu_A d_A \otimes A \), so that \( d(\sim) = 0 \).
Twisting cochains and the bar construction

Of central importance in all that follows is the bar construction.

**Definition 1.6.1.** An augmentation of a graded $k$-module $M$ will mean a decomposition $M \cong k \oplus \overline{M}$, with $k$ in degree 0. Given an augmented $k$-module $M$, we define a new graded module by

$$BM := \bigoplus_{n \geq 0} (s^{-1}M)^\otimes n.$$ 

We write a pure tensor $s^{-1}x_1 \otimes \cdots \otimes s^{-1}x_n \in BM$ as $[x_1] \cdots [x_n]$ or $[x_*]$. This convention implies the empty bar-word $[]$ must denote the element of $(s^{-1}M)^\otimes 0 = k$ corresponding to $1 \in k$.

**1.6.2.** The module $BM$ carries two relevant gradings. One is the length $\ell$, assigning the summand $B_nM := (s^{-1}M)^\otimes n$ grade $n$. The associated projection $BM \to B_nM$ is denoted $pr_n$. In particular, the counit $\varepsilon_{BM} : BM \to k$ is $pr_0$, and the coaugmentation $\eta_{BM}$ is the inclusion $B_0M \to BM$. The other grading we will need is the total grading given by

$$|[x_*]| = \sum |[x_j]| = \sum (|x_j| - 1) = -\ell(x_*) + \sum |x_j|.$$ 

**1.6.3.** There is a natural associative multiplication $BM \otimes BM \to BM$, concatenation of words, given by the rebracketing

$$\Delta_- : [x_1] \cdots [x_p] \otimes [y_1] \cdots [y_q] \mapsto [x_1] \cdots [x_p] |y_1| \cdots |y_q];$$

it will be useful to have a notation for this product (which makes $BM$ the tensor algebra on $s^{-1}M$) but we will not normally consider it as part of the structure of $BM$.

**1.6.4.** The comultiplication on $BM$ is given by the deconcatenation

$$\Delta_{BM} : [x_*] \mapsto [x_1] \otimes [x_*] := \sum_{p=0}^{\ell(x_*)} [x_1] \cdots [x_p] \otimes [x_{p+1}] \cdots [x_{\ell(x_*)}],$$

where we are using the abbreviation convention set forth in Notation 1.2.1. We will also at times adopt Einstein–Sweedler notation for the values of $\Delta_{BM}$ and other comultiplications; e.g.,

$$\Delta_{BM}^{[n]}[x_*] := [x_{(1)}] \otimes \cdots \otimes [x_{(n)}],$$

implicitly summing over partitions of $[x_*] = [x_1] \cdots [x_\ell]$ into $n$ bar-words. With this comultiplication and the counit and coaugmentation of Definition 1.6.2, $BM$ is a graded coalgebra, cocomplete with respect to the length grading and also, if $\overline{M}$ is positively graded, with respect to the total grading.

Note that $\Delta_{BM}^{[n]}$ takes $B_nM$ injectively to $(B_1M)^\otimes n$ and annihilates $B_{\leq n-1}M$, so that $BM$ is cocomplete.

**Proposition 1.7.** The bar construction $BM$ is the cofree cocomplete coalgebra on $s^{-1}\overline{M}$ in the sense that given another cocomplete graded coalgebra $C$, graded coalgebra maps $F : C \to BM$ correspond bijectively to degree-0 graded linear maps $\bar{F} : C \to B_1M$, or equivalently degree-1 maps $C \to \overline{M}$.

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4 It is more conventional to call this the tensor coalgebra, (notating it, e.g., as $T(M)$) and grace $BA$ with the title of “bar construction” only when $A$ is a dga, as in the coming Definition 1.8.1, but we prefer to economize on notation.
Proof. Given such a coalgebra map $F$, we truncate it to $\tilde{f} = \text{pr}_1 F : C \longrightarrow B_1 M$. Conversely, if such a linear map $\tilde{f}$ extends to a coalgebra map $F : C \longrightarrow BM$, then we have $\Delta_B M = (F \otimes F)\Delta_C$ and by iteration $\Delta_B^n M = F^{\otimes n} \Delta_C^n$. Now $\Delta_B^n$ restricts to an injective map from $B_n M = \text{pr}_n B M$ to $(B_1 M)^{\otimes n} = \text{pr}_1^{\otimes n} \Delta_B^n B M$, so that $\text{pr}_n F$ determines $\text{pr}_n F$ uniquely; one just applies the iterated concatenation $\Delta[n] : (B_1 M)^{\otimes n} \longrightarrow B_n M$ sending $[x_1] \otimes \cdots \otimes [x_n]$ to $[x_1] \cdots [x_n]$. Thus
\[
\text{pr}_n F = \Delta[n] \circ \text{pr}_1^{\otimes n} \circ \Delta_B^n \circ F = \Delta[n] \circ \text{pr}_1^{\otimes n} \circ F^{\otimes n} \circ \Delta_C^n = \Delta[n] \circ \tilde{f}^{\otimes n} \circ \Delta_C^n,
\]
so that $\text{pr}_n F(c)$, a bar-word of length $n$, is obtained by concatenating the length-1 bar words $\tilde{f}(c(i))$, where $\Delta[i] C = \sum c(1) \otimes \cdots \otimes c(n)$, and summing. Then $F(c) = \sum_{n \geq 0} \text{pr}_n F(c)$ gives well-defined elements of the direct sum $BM = \bigoplus_{n \geq 0} B_n M$ because by completeness each $\Delta_C^n$ vanishes for $n$ sufficiently large. By definition, $\text{pr}_1 F = \tilde{f}$.

**Definition 1.8.1.** Suppose now $A$ is a dga. Then $s^{-1}\bar{A}$ is a cochain complex with respect to the differential $d_{s^{-1}A} = -s^{-1}d_A s$ and we assign $(s^{-1}A)^{\otimes n}$ the differential $d_{\otimes}$. The sum of these differentials over $n$ is again a differential on $A$, the **internal differential** $d_{\text{int}}$.

The concatenation product $\Delta_{\text{int}}$ of 1.6.3 makes $BA$ a dga with respect to this differential; though we will not use this dga structure, we will at one point use the fact that $\Delta_{\text{int}}$ is a cochain map $BA \otimes BA \longrightarrow BA$ when $BA$ is given the internal differential.

**1.8.2.** There is also an **external differential** $B_n A \longrightarrow B_{n-1} A$ on $BA$ given by
\[
d_{\text{ext}} = \text{id}^* \otimes s^{-1} \mu_{BA}^{\otimes 2} \otimes \text{id}^*.
\]
for $n \geq 2$ and vanishing on $B_1 A = s^{-1}\bar{A}$ and $B_0 A = k$. Recall that under our conventions from Notation 1.2.1, this expression denotes a sum; more explicitly
\[
d_{\text{ext}} : [a_1] \cdots [a_n] \longmapsto \pm [a_1 a_2 | a_3 | \cdots | a_n] \\
\pm \cdots \\
\pm [a_1 | a_2 | \cdots | a_{n-1} a_n]
\]
where each “$\pm$” is the Koszul sign from Definition 1.4.

**1.8.3.** Finally, the sum $d_{BA} := d_{\text{int}} + d_{\text{ext}}$ is also a differential, and makes $BA$ a dgc. When $A$ is a dga, this dgc structure on $BA$ is always understood.

The bar construction is functorial in dga homomorphisms.

**Definition 1.9.** Given a dga map $f : A \longrightarrow B$, there is an induced dgc map $BF : BA \longrightarrow BB$ given on $B_n A$ by $(s^{-1}fs)^{\otimes n}$, which is to say $(BF)[a_*] := [f a_1] \cdots [f a_n] =: [f a_*]$. The map $BF$ preserves both the length and the total grading.

A dgc map into the bar construction of a dga also admits another important description.

**Definition 1.10.** For each connected dga $A$, the composition
\[
t_A : BA \xrightarrow{\text{pr}_1} s^{-1}\bar{A} \xrightarrow{s} \bar{A} \longrightarrow A
\]
of degree 1 is called the **tautological twisting cochain**. Viewed as a transformation of functors between the underlying graded module of $BA$ and that of $A$ itself, $t_{(-)}$ is natural in dga maps.
Given another dgc $C$ and a dgc map $F: C \to BA$, the cofreedom of $BA$ from Proposition 1.7 implies that $F$ is uniquely determined by the composition $pr_1 F: C \to BA \to B_1 A$, or equivalently by $t := t_A F = s pr_1 F: C \to A$. By construction, this map $t$ satisfies $\varepsilon_A t = 0 = t \eta_C$. Moreover, since $t_A$ annihilates everything but $B_1 A$, we have

$$td_C = t_A F d_C = t_A d_{BA} F = s (d_{int}|_{B_1 A} + d_{ext}|_{B_2 A}) F$$

$$= s (-s^{-1} d_A s pr_1 F + s^{-1} \mu_A s^{\otimes 2} (pr_1 F)^{\otimes 2} \Delta_C)$$

$$= -d_A t + \mu_A (t \otimes t) \Delta_C,$$

which can be rewritten as the Maurer–Cartan identity $Dt = t \sim t = 0$.

**Definition 1.12 (Brown [Br59, §3])**. Let $C$ be a dgc and $A$ a dga. An element $t \in \operatorname{Hom}_1 (C, A)$ satisfying the three conditions

$$\varepsilon_A t = 0 = t \eta_C, \quad Dt = t \sim t$$

is called a **twisting cochain**.

Any twisting cochain $C \to A$ extends uniquely to a dgc map $C \to BA$, as noted in Proposition 1.7, and inversely, we have seen that given a dgc map $F: C \to BA$, postcomposing the tautological twisting cochain $t_A : BA \to A$ gives a twisting cochain $t_A F : C \to A$, so these definitions describe a bijection between twisting cochains $C \to A$ and dgc maps $C \to BA$.

**Example 1.13**. Given a dga $A$, one can check the map

$$t \triangleright = \mu_A (t_A \otimes \varepsilon_A + \varepsilon_A \otimes t_A) : BA \otimes BA \to A,$$

which takes $[a] \otimes 1$ and $1 \otimes [a]$ to $a$ and annihilates all other pure tensors, is a twisting cochain if and only if $A$ is commutative. In this case, the uniquely induced dgc map

$$\mu \triangleright : BA \otimes BA \to BA,$$

called the **shuffle product**, is a product making $BA$ a dga. The shuffle product takes $[a_*] \otimes [b_*] \in B_p A \otimes B_q A$ to the sum of all $(p, q)$-shuffles (with Koszul sign) of $[a_*] [b_*]$.

More generally, let $A$ and $B$ be graded modules. Then the **shuffle map**

$$\triangledown : BA \otimes BB \to B (A \otimes B)$$

is the direct sum of the maps $B_p A \otimes B_q B \to B_{p+q} (A \otimes B)$ sending $[a_*] \otimes [b_*]$ to the sum of all tensor $(p, q)$-shuffles of $[a_1 \otimes 1] \cdots [a_p \otimes 1] \otimes [b_1] \cdots [1 \otimes b_q]$. If $A$ and $B$ are dgas, then $\triangledown$ is a dgc map. If $A = B$ is a cdga, then by the converse of the Eckmann–Hilton argument, $\mu : A \otimes A \to A$ is itself a dga map, and the composition $B \mu \circ \triangledown$ is the product $\mu \triangleright$ of the previous paragraph.

The bijection between dgc maps and twisting cochains preserves suitable homotopy notions.

---

5 Breaking $[a_*] \otimes [b_*]$ into two pieces $(-1)^{|b|} [b_1] [a_*] [b_*] \otimes [a_1] [b_2] [b_*]$ via $\Delta_{BA} \otimes BA$ and then applying $\triangledown \otimes \triangledown$ to shuffle both into lists of terms $a_1 \otimes 1$ and $1 \otimes b_1$ is the same as first shuffling into such terms via $\triangledown$ and then deconcatenating the result into two lists of such terms.
Definition 1.14. A dgc homotopy from one dgc map $F: C \to B$ to another such map $G$ is a degree$(-1)$ map $H: C \to B$ such that
\[ \varepsilon_B H = 0, \quad H\eta_C = 0, \quad DH = G - F, \quad \Delta_B H = F \otimes H + H \otimes G. \]

We write $H: F \simeq G$ in this situation. Taking $B = BA$ for an augmented dga $A$, we may translate this to an appropriate notion of homotopy for twisting cochains. Given two twisting cochains $t, u \in \text{Hom}_1(C, A)$, a twisting cochain homotopy from the former to the latter is a map $h \in \text{Hom}_0(C, A)$ satisfying the three conditions
\[ \varepsilon_C h = \varepsilon_A, \quad h\eta_A = \eta_C, \quad Dh = t - h - u. \]

We again write $h: t \simeq u$.

Proposition 1.15. Given a dgc homotopy $H: C \to BA$, the map $h = \eta_A \varepsilon_C + t_A H: C \to A$ is a twisting cochain homotopy, and the assignment $H \mapsto h$ is a bijection from the set of dgc homotopies $F \simeq G$ to the set of twisting cochain homotopies $t_A F \simeq t_A G$.

Twisted tensor products and the two-sided bar construction

Aside from providing a more tractable encoding of dgc maps $C \to BA$ and homotopies therebetween, twisting cochains $C \to A$ can also be harnessed to produce new differentials on $C \otimes A$, which we will use to define the two-sided bar construction.

Definition 1.16. Let $C$ be a dgc and $A$ a dga, $M$ a differential right $C$-comodule, and $N$ a differential left $A$-module. One defines the cap product with an element $\varphi \in \text{Hom}(C, A)$ by
\[ \delta^R_\varphi := (\text{id}_N \otimes \mu_M) (\text{id}_N \otimes \varphi \otimes \text{id}_M) (\Delta_N \otimes \text{id}_M): N \otimes M \to N \otimes M, \quad x \otimes y \mapsto \pm x_{(1)} \otimes \varphi(x_{(2)}) y, \]
where we have adopted the Einstein–Sweedler notation of Definition 1.6.4.\(^6\)

It is then an exercise [Gu60, Lem. 2.2; (1), p. 95] that the assignment $\varphi \mapsto \delta^R_\varphi$ is a dga homomorphism $\text{Hom}(C, A) \to \text{End}(N \otimes M)$ with respect to the dga structures defined in Definitions 1.5.5 and 1.5.4. As a consequence, if $t \in \text{Hom}_1(C, A)$ is a twisting cochain as defined in Definition 1.12, then $d \otimes - \delta^R_t$ is a differential on $N \otimes M$, for its square $-D \delta^R_t + (\delta^R_t)^2 = \delta^R_{-D t + t} \delta^R_t$ is $\delta^R_0 = 0$.

Definition 1.17 (Brown [Br59, §3]). Let $C$ be a dgc, $A$ a dga, $t: C \to A$ a twisting cochain, $N$ a differential right $C$-comodule, and $M$ a differential left $A$-module. We call $N \otimes M$, equipped with the differential $d \otimes - \delta^R_t$, a twisted tensor product, and denote it by $N \otimes^R_M$.\(^5\)

---

\(^5\) Compare the definition of a cap product between the homology and cohomology theories represented by a ring spectrum to see the analogy.

\(^6\) Twisting cochains originated as a way to encode a model for cochains on the total space of a fiber bundle $F \to E \to B$ in terms of cochains on $B$ and $F$ and the monodromy action of loops in $B$ on $C^*(F)$, morally using the representation $t: C_\ast(B) \to \text{Aut} C^*(F)$ to make a model $C^*(B; C^*(F))$ for $C^*(E)$ preserving the interaction that is usually lost in the early pages $E_0 = C_\ast(B) \otimes C^*(F)$, $E_1 = C^*(B; H^\ast(F))$, $E_2 = H^\ast(B; H^\ast(F))$ of the Serre spectral sequence.

For $G$ a group and $A = C^*(G)$, the associated model for the total space of the universal bundle $G \to EG \to BG$ is the so-called one-sided bar construction $BA \otimes^R_M A$ given by modifying the expected tensor differential $d_{\text{int}}$ using the tautological twisting cochain $I^A: BA \to A$. We will meet the two-sided generalization of the one-sided bar construction presently.
If \( N = C \) and \( M = A \), one can check that \( \Delta_C \otimes \text{id}_A \) makes \( C \otimes I \) a differential left \( C \)-comodule and \( \text{id}_C \otimes \mu_A \) makes it a differential right \( A \)-module.\(^8\)

**Definition 1.18.1.** There is an analogous construction producing an endomorphism \( \delta^L_\varphi \) of \( M \otimes N \) from a graded linear \( \varphi: C \to A \), a differential left \( C \)-comodule \( N \), and a differential right \( A \)-module \( M \). The map \( \varphi \mapsto \delta^L_\varphi \) is again a cochain map, but is anticommutative, in the sense that \( \delta^L_\varphi \circ \delta^L_\varphi = (-1)^{|\varphi||\psi|} \delta^L_\varphi \circ \varphi \). Thus, given a twisting cochain \( t: C \to A \), the appropriate differential to define a twisted tensor product \( M \otimes_t N \) is \( \delta^L_1 + d_b \)—note the opposite sign [HuMS74, Prop. 1.2].

**1.18.2.** In the case of particular interest to us, we will have a coaugmented dgc \( C \), two augmented dgas \( A', A'' \), and two twisting cochains \( t': C \to A' \) and \( t'': C \to A'' \), and we may combine the constructions to define a **two-sided twisted tensor product** [HuMS74, Rmks. II.5.4]

\[
A' \otimes_t C \otimes_t A'' := A' \otimes_t (C \otimes_t A'') \cong (A' \otimes_t C) \otimes_t A''.
\]

This is an \( (A', A'') \)-bimodule, and given ideals \( a' \triangleleft A' \) and \( a'' \triangleleft A'' \), we write \( (a', a'') \) for the \( (A', A'') \)-subbimodule \( a' \otimes C \otimes A'' + A' \otimes C \otimes a'' \).

We will need some lemmas describing when (homotopy-)commutative squares of maps

\[
\begin{array}{ccc}
A'_0 & \xrightarrow{t'_0} & C_0 & \xrightarrow{t''_0} & A''_0 \\
\downarrow & & \downarrow & & \downarrow \\
A'_1 & \xrightarrow{t'_{1}} & C_1 & \xrightarrow{t''_{1}} & A''_1
\end{array}
\]

induce a map \( A'_0 \otimes_{t'_0} C_0 \otimes_{t''_0} A''_0 \to A'_1 \otimes_{t'_{1}} C_1 \otimes_{t''_{1}} A''_1 \) of twisted tensor products. These properties seem to be well known in the one-sided case (see, e.g., Huebschmann [Hu89, p. 360] and Franz [Fr19, §7]) and we will need their easy-guessed generalizations to the two-sided case.

**Lemma 1.20.** Let \( G: C_0 \to C_1 \) be a dgc map, \( f': A'_0 \to A'_1 \) and \( f'': A''_0 \to A''_1 \) dga maps, and \( t'_j: C_j \to A'_j \) and \( t''_j: C_j \to A''_j \) twisting cochains (\( j \in \{0,1\} \)). Then \( f' \otimes G \otimes f'' \) is a cochain map

\[
A'_0 \otimes_{t'_0} C_0 \otimes_{t''_0} A''_0 \to A'_1 \otimes_{t'_{1}} C_1 \otimes_{t''_{1}} A''_1
\]

if \( t'_1 G = f't'_0 \) and \( t''_1 G = f''t''_0 \). If \( G \) is one such dgc map and \( \tilde{G} \) another, and \( H: C_0 \to C_1 \) is a dgc homotopy \( G \simeq \tilde{G} \) such that \( t'_1 H \) and \( t''_1 H \) are zero, then \( f' \otimes H \otimes f'' \) is a cochain homotopy \( f' \otimes G \otimes f'' \simeq f' \otimes \tilde{G} \otimes f'' \).

**Proof:** That the equations imply \( f'' \otimes G \otimes f'' \) is a cochain map follows on expanding out the objective \( D(f' \otimes G \otimes f'') = 0 \). To see the cochain homotopy, first note that \( d_b((f' \otimes H \otimes f'') - (f' \otimes H \otimes f''))d_b = f \otimes (G - \tilde{G}) \otimes f'' \) since \( Df' \) and \( Df'' \) are zero. To see \( \delta^R_1 (H \otimes f'') + (H \otimes f'') \delta^R_1 \) vanishes, recall that \( \Delta H = G \otimes H + H \otimes \tilde{G} \), so that \( (\text{id} \otimes t''_1) \Delta H = -H \otimes t''_1 \tilde{G} = -H \otimes f''t''_0 \); then

\[
\delta^R_1 (H \otimes f'') = -(\text{id} \otimes \mu) (H \otimes f''t''_1 \otimes f'')(\Delta \otimes \text{id}) = -(H \otimes f'') \delta^R_1.
\]

The proof that \( \delta^L_1 (f' \otimes H) + (f' \otimes H) \delta^L_1 \) vanishes is symmetric. \( \square \)

---

\(^8\) Conversely, any differential \( d \) on \( C \otimes A \) restricting to the given differentials on \( C \) and \( A \) is given by \( d_b + \delta^R_1 \) for \( t = \mu_A(\eta_C \otimes \text{id}_A)d(id_C \otimes \eta_A) \Delta_C \) [Gu60, §3].
Corollary 1.21. Let $C_0$ and $C_1$ be dgc, $A'$ and $A''$ dgas, $t': C_1 \rightarrow A'$ and $t'': C_1 \rightarrow A''$ twisting cochains, and $G: C_0 \rightarrow C_1$ a dgc homomorphism. Then $t' \circ G: C_0 \rightarrow A'$ and $t'' \circ G: C_0 \rightarrow A''$ are twisting cochains and

$$\text{id}_{A'} \otimes G \otimes \text{id}_{A''}: A' \otimes C_0 \otimes A'' \rightarrow A' \otimes C_1 \otimes A''$$

a cochain map.

Proof. Take $f' = \text{id}_{A'}$ and $f'' = \text{id}_{A''}$ in Lemma 1.20. \qed

Lemma 1.22. Let $C$ be a dgc, $A'$ and $A''$ dgas, $t'_j: C \rightarrow A'$ and $t''_j: C \rightarrow A''$ twisting cochains for $j \in \{0,1\}$, and $a' \leq A'$ and $a'' \leq A''$ ideals, and suppose there exist homotopies $h': t'_0 \approx t'_1$ and $h'': t'_1 \approx t''_0$ with $h'C \leq a'$ and $h''C \leq a''$. Then the composition

$$(\delta_{t'_{0}} \otimes \text{id}_{A''})(\text{id}_{A'} \otimes \delta_{t''_{0}}): A' \otimes C \otimes A'' \rightarrow A' \otimes C \otimes A''$$

is a cochain isomorphism $A' \otimes C \otimes A'' \rightarrow A' \otimes C \otimes A''$ congruent to the identity modulo $(a', a'')$.

Proof. On expanding out the definitions and subtracting off the tensor differential, one sees that $\text{id}'_A \otimes \delta_{t'_{0}}$ commutes with $\delta_{t'_2} \otimes \text{id}_{A''}$ and $\delta_{t''_{0}} \otimes \text{id}'_A$ with $\text{id}'_A \otimes \delta_{t''_{0}}$, so the conclusion will follow from two applications of the one-sided result (see [Fr19, Lem. 7.1]). The left-handed version of that result shows $D\delta_{t'_{0}}$ vanishes as follows:

$$(d_{\otimes} + \delta_{t'_1})\delta_{t'_{0}} - \delta_{t''_{1}}(d_{\otimes} + \delta_{t'_1}) = D(\delta_{t''_{0}}) + \delta_{t'_2} \delta_{t'_{0}} - \delta_{t''_{0}} \delta_{t'_1} = D_{\otimes} - \delta_{t''_{1}} - \delta_{t''_{1}} = 0;$$

the right-handed proof is similar. The congruence follows from the definition of the cap product and the facts $h'C \leq a'$ and $h''C \leq a''$. The inverse is $(\text{id}'_A \otimes \delta_{h''C})^{-1}(\delta_{h'C})^{-1} \otimes \text{id}_{A''}$ where the cup-inverses make sense by the assumed cocompleteness of $C$, per Definition 1.5.5. \qed

We now define the twisted tensor products of greatest interest to us.

Definition 1.23. Let $A$, $A'$, $A''$ be augmented dgas and $F': BA \rightarrow BA'$ and $F'': BA \rightarrow BA''$ dgc maps.\footnote{Note this second homotopy goes the opposite direction to the one one might expect.} Associated to $F'$ and $F''$ are twisting cochains $t' = t_A F'$ and $t'' = t_A F''$ given by postcomposition with the tautological twisting cochains, and we define the twisting tensor product

$$B(A', A, A'') := A' \otimes_{t_A F'} BA \otimes_{t_AF''} A''$$

As a matter of notation, we will write a pure tensor $a' \otimes [a_\ast] \otimes a'' \in B(A', A, A'')$ as $a' [a_\ast] a''$. (More generally, we may write $b[a_\ast] b'[a'_\ast] b''$ for a pure tensor in $B \otimes BA \otimes B' \otimes BA' \otimes B''$, and so on.) The differential of $B(A', A, A'')$ preserves the filtration $B_{\preceq l}(A', A, A'') := A' \otimes B_{\preceq l} A \otimes A''$ by length.

Example 1.24. This two-sided bar construction, when we take $F' = B f'$ and $F'' = B f''$ for dga maps $f': A \rightarrow A'$ and $f'': A \rightarrow A''$, coincides with the classical two-sided bar construction of differential graded algebras. In this case, the twisting cochains $t'$ and $t''$ defining the twisted tensor product $B(A', A, A'') = A' \otimes_{t'} BA \otimes_{t''} A''$ are simply $t_A B f' = f' t_A$ and $f'' t_A$, respectively.\footnote{These are $A_\ast$-algebra maps from $A$ to $A'$ and $A''$ in the language to be introduced in Section 3.}
We see already from the preceding lemmas that dga maps from $A'$ and $A''$ and dgc maps from $BA$ induce maps of two-sided bar constructions, and want to extend this functoriality to commutative diagrams of dgc maps of the form

$$
\begin{array}{ccc}
BA_0' & \longrightarrow & BA_0'' \\
\downarrow & & \downarrow \\
BA_0' & \longrightarrow & BA_1''.
\end{array}
$$

We write $G'_{(1)} : \overline{A_0'} \longrightarrow \overline{A_1'}$ for the composite $t_{A_1'} G' \circ s_{A_0'}^{-1}$ and similarly for $G''_{(1)}$; later, following Definition 3.2, we will see $\eta \varepsilon + G'_{(1)} : A_0' \longrightarrow A'_1$ and $\eta \varepsilon + G''_{(1)} : A_0'' \longrightarrow A''_1$ can be understood as a sort of approximate dga maps.

**Proposition 1.26** (Cf. Wolf [Wolf77, Thm. 7]). Suppose given a strictly commuting diagram of dgc maps as in (1.25). Then we define a cochain map $B(A_0', A_0, A_0'') \longrightarrow B(A_1', A_1, A_1'')$ by

$$
B(G', G, G'') = (Y' \otimes G \otimes Y'') \circ (id_{A_0} \otimes \Delta^{[3]}_{BA_0} \otimes id_{A_0''}),
$$

where $\Delta^{[3]}_{BA}[a_*] = [a_{(1)}] \otimes [a_{(2)}] \otimes [a_{(3)}]$ as per Definition 1.64 and

$$
\begin{align*}
\Delta'_{|V_{BA_0}} & : \varepsilon_{BA_0} \otimes id_{k} : \varepsilon_{BA_0} \otimes a_{(1)} \longrightarrow \varepsilon_{BA_0} \otimes \varepsilon_{[a_{(1)}]}, \\
\Delta''_{|V_{BA_0}} & : \varepsilon_{BA_0} \otimes id_{k} : \varepsilon_{BA_0} \otimes a_{(3)} \longrightarrow \varepsilon_{BA_0} \otimes \varepsilon_{[a_{(3)}]} c'', \\
\Delta_{|V_{BA_0}} & : \varepsilon_{BA_0} \otimes id_{k} : \varepsilon_{BA_0} \otimes a_{(3)} \longrightarrow \varepsilon_{BA_0} \otimes \varepsilon_{[a_{(3)}]} c''.
\end{align*}
$$

where $\Delta_{|V_{BA_0}}$ is the concatenation operation of 1.64. If $t_{[A_1]} G'$ takes $B_{\geq 2} A'_{0}$ into an ideal $a' \subseteq A'$ and $t_{[A_1]} G''$ takes $B_{\geq 2} A''_{0}$ into $a'' \subseteq A''_{1}$, then

$$
B(G', G, G'') = (\eta \varepsilon + G'_{(1)}) \otimes G \otimes (G''_{(1)} + \eta \varepsilon) \quad \text{(mod } (a', a'').\text{)}
$$

We include a proof since the only other written proof we are aware of is in Joel Wolf’s unpublished dissertation [Wolf].

**Proof.** The final congruence is clear from (1.27). Factoring the diagram (1.25) as

$$
\begin{array}{ccc}
BA_0' & \longrightarrow & BA_0'' \\
\downarrow & & \downarrow \\
BA_0' & \longrightarrow & BA_1''.
\end{array}
$$

\[\text{But cf. Gugenheim–Munkholm [GuMu74, Thm. 3.5a] for a closely related construction.}\]
and taking symmetry into account, it is enough to show $B(id, G, id)$ and $B(G', id, id)$ are cochain maps. 

The former is just $id \otimes G \otimes id$. To see this, note that $t'_{A'_1}([a'] \otimes F'_0[a(1)])$ is zero unless $F'_0[a(1)]$ lies in $k = B_0 A'_1$, and this only happens when $[a(1)] = []$, since $\varepsilon_{A'_1}F'_0 = \varepsilon_{A_0}$ by the definition of a map of coaugmented dgc; and similarly for $t''_{A''_1'}$ and $[a(3)]$. Thus Corollary 1.21 applies.

As for $B(G', id, id)$, by the same considerations, all terms have $[a(3)] = []$, so the map leaves the $A''_1$ tensor factor inert and we need only show $\gamma'(id_{A'_1} \otimes \Delta_{BA_0})$ commutes with $\delta^L + \delta^R$. We state this as a separate lemma.

\[ \square \]

**Lemma 1.28** (Cf. Franz [Fr19, (7.6), Lem. 7.4]). Let $A, B, X$ be dgas and $BA \xrightarrow{F} BB \xrightarrow{G} BX$ dgc maps. Then we may define a cochain map $B \otimes BA \rightarrow X \otimes BA$ by

$$\Gamma = \begin{cases} 
\text{id}_{kBBA} & \text{on } k \otimes BA, \\
(t_X G \Delta_-(s^{-1}_B \otimes F) \otimes \text{id}_{BA})(\text{id}_{F} \otimes \Delta_{BA}) & \text{on } B \otimes BA,
\end{cases} \tag{1.29}$$

where $\Delta_-$ is the concatenation operation of 1.6.4.

**Proof.** On $k \otimes B_0 A$, the composites $\Gamma(d_\otimes + \delta^L_{GF})$ and $(d_\otimes + \delta^L_{GF}) \Gamma$ both vanish, and hence agree.

On $k \otimes B_{\geq 1} A$, we have

$$\Gamma(d_\otimes + \delta^L_{GF}) = \Gamma(id \otimes d_{BA} + \delta^L_{GF}) = id \otimes d_{BA} + \delta^L_{GF},$$

where the last step is not obvious. Making the identification $k \otimes BA = BA$ for brevity, one has

$$\Gamma_{\delta^L_{GF}|k \otimes B_{\geq 1} A} = (tG\Delta_-(s^{-1}_B \otimes F) \otimes \text{id})(\text{id} \otimes \Delta)(tF \otimes \text{id})\Delta
= (tG\Delta_-(s^{-1}tF \otimes F) \otimes \text{id})\Delta^3
= (tG\Delta_-(s^{-1}t \otimes \text{id})\Delta F \otimes \text{id})\Delta
= (tGF \otimes \text{id})\Delta,$$

using the fact

$$\Delta_-(s^{-1}t \otimes \text{id})\Delta^3 = \Delta_-(s^{-1}t \otimes \text{id})\Delta F \otimes \text{id})\Delta = (F \otimes \text{id})\Delta,$$

which in turn depends on the identity $\Delta F = (F \otimes F)\Delta$ and the computation

$$\Delta_-(s^{-1}t \otimes \text{id})\Delta[b_*] = \Delta_-([b_1] \otimes [b_2] \ldots [b_d]) = [b_*].$$

On the other hand, on $k \otimes B_{\geq 1} A$ we also have

$$(d_\otimes + \delta^L_{GF}) \Gamma = (d_\otimes + \delta^L_{GF}) = id \otimes d_{BA} + \delta^L_{GF}.$$
We recall from (1.11) that \( D t_X = t_X - t_Y \). As for \( D \Delta_- \), the restriction to \( s^{-1} \mathcal{B} \otimes \mathcal{B}_0 \mathcal{B} \) of the concatenation \( \Delta_- : s^{-1} \mathcal{B} \otimes \mathcal{B} \longrightarrow \mathcal{B} \mathcal{B} \) is simply \( s^{-1} \mathcal{B} = \mathcal{B}_1 \mathcal{B} \). So there one has \( D \Delta_- = 0 \). The differential on \( s^{-1} \mathcal{B} \otimes \mathcal{B} \) is \( d_\otimes + d_{\text{ext}} \), whereas the differential on \( \mathcal{B} \) is \( d_\otimes + d_{\text{ext}} \). Since \( \Delta_- \) commutes with \( d_\otimes \) by the observation in Definition 1.8.1, one has \( D \Delta_- = d_{\text{ext}} \Delta_- - \Delta_- (d_{\text{ext}} \otimes \Delta_{\text{ext}}) \). From the definition of \( d_{\text{ext}} \), this difference is

\[
\Delta_-(s^{-1} \mu \otimes \mathcal{B}_0 \mathcal{B})(d_{\text{ext}} \otimes \Delta_{\mathcal{B}}) : s^{-1} \mathcal{B} \otimes [b_*] \mathcal{B} \longrightarrow \pm [bb_1 b_{(2)}].
\]

Since \( t_B \) annihilates \( \mathcal{B}_0 \mathcal{B} \), the same expression is equally valid on \( s^{-1} \mathcal{B} \otimes \mathcal{B}_0 \mathcal{B} \) despite the initial case distinction. Using the manipulations

\[
(t \otimes t \otimes \mathcal{I})(\mathcal{I} \otimes \Delta)(s^{-1} \otimes F) = -(ts^{-1} \otimes t \otimes \mathcal{I})(\mathcal{I} \otimes \Delta F) = -(\mathcal{I} \otimes t \otimes \mathcal{I})(\mathcal{I} \otimes \Delta F),
\]

we see \( D \otimes \Gamma \) is the sum of two terms

\[
\left( (t \otimes t) G \Delta_-(s^{-1} \otimes F) \otimes \mathcal{I}(\mathcal{B} \mathcal{A}) : (\mathcal{I} \otimes \Delta)(\mathcal{B} \mathcal{A}) : \right)
\]

\[
b[a_*] \longrightarrow -tG \Delta_-(\{b] \otimes F[a_1]\}) tGF[a_2] \otimes [a_3],
\]

\[
\left( tG \Delta_-(s^{-1} \mu (\mathcal{I} \otimes t) \otimes \mathcal{B} \mathcal{B})(\mathcal{I} \otimes \Delta F) \otimes \mathcal{I}(\mathcal{B} \mathcal{A}) \right) : (\mathcal{I} \otimes \Delta)(\mathcal{B} \mathcal{A}) :
\]

\[
b[a_*] \longrightarrow tG \Delta_-(\{btF[a_1]\}) \otimes F[a_2] \otimes [a_3].
\]

Expanding the definitions and rearranging shows (1.31) is just \( \Gamma \delta_{\text{II}}^{\text{I}} \). Through standard rearrangements, it is also possible to express (1.30) and \( -\delta_{\text{I}G \text{F}}^{\text{II}} \) in such a way as to be identical except that \( (tG \otimes tG) \Delta_\Delta_- \) appears as a composition factor in (1.30) in the position where \( (tG \otimes tG)(\Delta_\Delta_- \otimes \mathcal{I})(\mathcal{I} \otimes \Delta) \) appears in \( \Gamma \delta_{\text{II}}^{\text{I}} \). But these two agree as well, for

\[
\Delta \Delta_- ([b] \otimes [b_*]) = ([b] \otimes [b_*]) + [b] [b_{(1)}] \otimes [b_{(2)}];
\]

\[
(\Delta_\Delta_- \otimes \mathcal{I})(\mathcal{I} \otimes \Delta)([b] \otimes [b_*]) = [b] [b_{(1)}] \otimes [b_{(2)}],
\]

and the extra term \( [b] \otimes [b_*] \) is annihilated by \( tG \otimes tG \) since \( tG[b] \] \( tG[b] = 0 \). Thus, as claimed, \( D \Gamma = D \otimes \Gamma + \delta_{\text{I}G \text{F}}^{\text{II}} \Gamma - \delta_{\text{II}}^{\text{II}} \Gamma \) vanishes.

**Remark 1.32.** The two-sided bar construction of Definition 1.23 can be generalized, with \( A' \) and \( A'' \) replaced with cochain complexes \( M' \) and \( M'' \) equipped with dgc maps \( \mathcal{B} \mathcal{A} \longrightarrow \mathcal{B} \text{ End } M' \) and \( \mathcal{B} \mathcal{A} \longrightarrow \mathcal{B} \text{ End } M'' \) [Wolf77, p. 321] and analogous maps of two-sided bar constructions à la Proposition 1.26 can then be defined, but we will not require this level of generality.

**Notation 1.33.** In the event that in the situation of (1.25), the maps \( G, G', G'' \) are respectively \( B_g, B_{g'}, B_{g''} \) for \( g : A_0 \longrightarrow A_1, g' : A_0' \longrightarrow A_1', g'' : A_0'' \longrightarrow A_1'' \), we abuse notation by writing the map of Proposition 1.26 as \( B(g', g, g'') : B(G', G, G'') \). Note this is also a special case of the map of Lemma 1.20.

We are interested in two-sided bar constructions because they provide functorial resolutions.

**Proposition 1.34** (See Barthel—May—Riehl [BaMR14, after Prop. 10.19]). Suppose \( k \) is a principal ideal domain and \( A \longrightarrow A' \) a map of \( k \)-DGAs flat over \( k \). Then given another \( k \)-DGA map \( A \longrightarrow A'' \), the cohomology of the classical two-sided bar construction \( B(A', A, A'') = B(A', A, A) \otimes A'' \) of Example 1.24 is \( \text{Tor}_A(A', A'') \).
These hypotheses will hold in all the cases we consider, and the result suggests Tor might be defined to be the cohomology of the two-sided bar construction in the more general setting.

**Definition 1.35** (Wolf [Wolf77, p. 322]). Let $A, A', A''$ be augmented dgas and $F': BA \to BA'$ and $F'': BA \to BA'\prime'$ dgc maps. We define $\text{Tor}_A(A', A'\prime')$ to be $H^*(BA, A, A'\prime')$, and in the situation of Proposition 1.26, we define $\text{Tor}_b(G', G''\prime')$ to be $H^*(G', G, G'\prime')$.

**Remark 1.36.** This is not the only reasonable choice. Writing $\Omega$ for the cobar construction, in the situation of Definition 1.23 we have induced dga maps $\Omega BA' \leftarrow \Omega BA \to \Omega BA''$, and Munkholm [Mu74, Prop. 5.3] instead sets $\text{Tor}_A(A', A'\prime') := \text{Tor}_{\Omega \Omega A}(\Omega BA', \Omega BA'\prime')$. One can check, using the natural transformation $\Omega B \to \text{id}$, Lemma 1.20, and a spectral sequence argument, that under sufficient flatness hypotheses, for example if $A, A'$, and $H^*(A)$ are all flat over $k$, this definition agrees with Definition 1.35.

**Discussion 1.37.** For the proof of Theorem A.13 later on, we will need a more explicit expression for the differential on the classical two-sided bar construction $B(A', A, A''\prime')$ of Example 1.24. Note that the total differential is the sum of the tensor differential $d_{\otimes}$ on $A' \otimes BA \otimes A''\prime'$ and two cap products. Following Definition 1.8.3, in the summand $\text{id}_{A'} \otimes d_{BA} \otimes \text{id}_{A''}$ of the tensor differential, the factor $d_{BA}$ is the sum of the internal differential $d_{\text{int}}$ given by the tensor differential $d_{\otimes}$ and an external differential $d_{\text{ext}}$, so on each $B_{*}(A', A, A''\prime')$, the total differential $d_{B_{*}(A', A, A''\prime')}$ is the sum of the tensor differential on $A' \otimes (s^{-1}A) \otimes A''\prime'$ and a non-tensor component given by the sum $\delta^L + d_{\text{ext}} + \delta^R$ of the external differential and two cap products:

$$a'_{*}a''_{*} \mapsto \pm a'_{f'(a_1)}[a_{2}a_{*}]a''_{*}$$

$$\pm a'_{*}[a_{p_{2}a_{p_{1}+1}a_{*}}]a''_{*}$$

$$\pm a'_{*}[a_{1}a_{*}]a_{_{l-1}}[f'(a_{l})a''_{*}].$$

To be explicit about signs, this non-tensor component of $d_{B_{*}(A', A, A''\prime')}$ is

$$d_{B_{*}(A', A, A''\prime')}^\text{ext} := \mu_{A'}(\text{id}_{A'} \otimes f's) \otimes \text{id}_{s^{-1}A} \otimes \text{id}_{A''}$$

$$+ \text{id}_{A'} \otimes d_{\text{ext}} \otimes \text{id}_{A''}$$

$$- \text{id}_{A'} \otimes \text{id}_{s^{-1}A} \otimes \mu_{A'}(f''s \otimes \text{id}_{A''}).$$

**Remark 1.40.** As in Definition 1.18.2, the minus sign in the last line of (1.39) is important, and there is some confusion on this sign in the literature.

2. Extended Homotopy Gerstenhaber algebras

We ultimately want to describe the multiplicative structure on $\text{Tor}_{\omega}(B)(C^{*}(X), C^{*}(E))$ in terms of a product on the two-sided bar construction. The special property of a normalized singular cochain algebra $A$ enabling us to do so will turn out to be the existence of a dgc Hopf algebra structure on $BA$, or in other words a dgc map $BA \otimes BA \to BA$ making it a dga.\footnote{Under our blanket assumption that $A$ is graded and connected, a bialgebra structure on $BA$ admits a unique antipode defined by a formula analogous to the one defining the cup-inverse in 1.5.5 [Ta71][GrR14, Props. 1.4.14.24].} We have seen that such a dgc map is determined by its composition with the tautological twisting cochain $t_A: BA \to A$. 

Definition 2.1. Let $A$ be a dga such that $BA$ admits a multiplication making it a dg Hopf algebra. If the twisting cochain $E := t_A \mu_{BA} : BA \otimes BA \to A$ satisfies $E_{j,\ell} := E_{[1] \otimes [1], A} = 0$ for $j \geq 2$, we call $A$, equipped with $\mu_{BA}$ (or equivalently, with $E$), a homotopy Gerstenhaber algebra (hga).

We will on occasion write the values of the product $\mu_{BA}$ in infix notation as $[a] * [b]$.

Remark 2.2. We note that unitality of $\mu_{BA}$ implies $E_{0,1}$ and $E_{1,0}$ must both be $s : s^{-1}A \to A$ and $E_{j,\ell}$ and $E_{0,\ell}$ must be 0 for $j, \ell \geq 2$. The “$\eta = 0$” clause of Definition 1.12 implies $E_{0,0} = E\eta_{BA} = 0$. For notational convenience, we will extend $E_{0,1}$ and $E_{1,0}$ to both be $s : s^{-1}A \to A$ (thus respectively sending $[1] \otimes [1]$ and $[1] \otimes [1]$ to 1) and extend the operations $E_{1,\ell}$ to $(s^{-1}A)^{\otimes 1+\ell}$ for $\ell \geq 1$ by setting them to 0 on pure tensors any of whose factors lies in $s^{-1}$ im $\eta_A$. That is, we will sometimes consider bar-words containing a letter $c$ in the coefficient ring $k$, but such words are to be annihilated by $E_{j,\ell}$ unless $(j, \ell) = (1, 0)$ or $(0, 1)$.

Notation 2.3. It will be useful later to translate $E_{1,\bullet}$ into a degree-zero map on $A \otimes BA$ by taking
\[
E := E_{1,\bullet}(s^{-1} \otimes \text{id}_{BA}) : A \otimes BA \to A,
\]
with $a[b] \to E([a] \otimes [b])$.

An hga structure on $A$ can be equivalently phrased as a list of conditions [Fr19, (6.2)–(6.4)] on the operations $E_{\ell} = E_{1,\ell} \circ (s^{-1})^{\otimes 1+\ell} : A \otimes A^{\otimes \ell} \to A$. To agree with the literature, we denote the values of the operations $E_{\ell}$ on pure tensors by $E_{\ell}(a; b_{\bullet}) = E_{\ell}(a; b_1, \ldots, b_{\ell})$ in place of the otherwise-expected $E_{\ell}(a \otimes b_1 \otimes \cdots \otimes b_{\ell})$.

Example 2.4. If $A$ is a commutative dga, then the shuffle product $\mu_{\mathbf{C}}$ of Example 1.13 makes the bar construction $BA$ a dg Hopf algebra, so that $A$ becomes an hga. The corresponding twisting cochain $t_{\mathbf{C}} = Y$ satisfies $E = E_{0,1} + E_{1,0}$, so the operations $E_{\ell}$ vanish for $\ell \geq 1$.

When the Maurer–Cartan identity $DE = E \circ Y$ for the twisting cochain $E : BA \otimes BA \to A$ defining an hga structure on $A$ is restricted to $B_2A \otimes BA$, it yields
\[
E_{1, \bullet}(s^{-1} \mu_{BA} \otimes \text{id}_{BA}) = \mu_A(E_{1, \bullet} \otimes E_{1, \bullet}) \Delta_{BA} \otimes BA,
\]
where the comultiplication $\Delta_{BA} \otimes BA$ is the sum of shuffles $\pi : [a_1]a_2 \otimes [b_{\bullet}] \longmapsto \pm [a_1][b_{\bullet}] \otimes [a_2][b_{\bullet}]$. Rephrased in terms of the degree-0 map $E$ of Notation 2.3, the preceding display becomes
\[
E(\mu_A \otimes \text{id}_{BA}) = \sum_{\pi} \mu_A(E \otimes E) : \bar{A} \otimes \bar{A} \otimes BA \to A
\]
(2.5)
for the same tensor shuffles $\pi$. Evaluating (2.5) on a pure tensor $a_1 \otimes a_2 \otimes [b_{\bullet}]$ yields a sort of Cartan formula
\[
E(a_1a_2[b]) = \sum \pm E(a_1[b])E(a_2[b]).
\]
(2.6)
Restricting the Maurer–Cartan identity for $E$ to $B_1A \otimes B_1A$ instead yields
\[
d_A E_{1,\ell} + E_{1,\ell} d_\otimes + E_{1,\ell-1} (id_{s^{-1}} \otimes d_{ext}) = \mu_A(s \otimes E_{1,\ell-1})(1, 2) + \mu_A(E_{1,\ell-1} \otimes s),
\]
where $d_\otimes$ refers to the “internal” differential $d_{\bar{A}} \otimes \text{id}_{BA} + \text{id}_{\bar{A}} \otimes d_{\otimes}$ on $\bar{A} \otimes BA$, omitting the external differential on the $BA$ factor. If we write $D_{\otimes}$ for the derivation on $\text{Hom}(\bar{A} \otimes BA, A)$ defined with respect to this internal differential and $d_{\otimes}$, this last equation can be rearranged to
\[
D_{\otimes}E = d_A E - E d_\otimes = \mu_A(s \otimes E)(1, 2) + E(\text{id}_A \otimes d_{ext}^B_{A, A, A}) - \mu_A(E \otimes s).
\]
(2.7)

13 Compare Franz [Fr20b, Lem. 3.1] for conditions for a map $C \to \Omega C \otimes \Omega C$ to be a twisting cochain inducing a dgc structure on the cobar construction $\Omega C$ of a dgc $C$. 
Evaluated on an element \( a[b_\bullet] \in \tilde{A} \otimes B_tA \), this formula becomes
\[
(D \otimes \mathcal{E})(a[b_\bullet]) = \pm b_1 \mathcal{E}(a[b_2[b_\bullet b_\ell]]) \\
\pm \mathcal{E}(a[b_1[b_\bullet [b_\bullet b_{p+1}]b_\ell]) \\
\pm \mathcal{E}(a[b_1[b_\bullet b_{\ell-1}])b_\ell
\]
for \( \ell \geq 2 \), with the middle term omitted if \( \ell = 1 \), and if \( \ell = 0 \), then we get instead \( D(\text{id}_A) = 0 \).

The case \( \ell = 1 \) gives, on precomposing \( s \otimes s \) on the right of (2.7),
\[
d_{\text{Hom}(A \otimes A)}(E_1) = \mu_A(1 \ 2) - \mu_A.
\]

In other words, \(-E_1\) is a cochain homotopy from multiplication to the transposed multiplication \((a, b) \mapsto (-1)^{|a||b|}ba\), witnessing the homotopy-commutativity of \( \mu_A \), or what is called a \textit{cup-1 product} after the initial example \( \sim_1 \), due to Steenrod in the case where \( A = C^*(X) \) is the singular cochain algebra of a topological space [Ste47, §§2, 5]. The Steenrod cup-1 product on \( C^*(X) \) is again commutative up to a homotopy witnessed by an operation \( \sim_2 \), and inductively Steenrod finds a sequence of binary operations \( \sim_i \) of degree \(-i\), each commutative up to a homotopy witnessed by \( \sim_{i+1} \).

**Definition 2.8.** A dga \( A \) is said to \textit{admit cup-1 products} if for \( 0 \leq j \leq i \) there exist degree-(\(-j\)) operations \( \mu_j: A \otimes A \to A \), starting with \( \mu_0 = \mu_A \) the multiplication of \( A \), such that
\[
d(\mu_{j+1}) = \mu_j - \mu_j(1 \ 2) \quad (0 \leq j < i).
\]

The gcAs \( A \) we consider will admit cup-2 products and a bit more. Unlike the maps \( \mu_{BA} \) and \( \mathcal{E} \), which will be intimately involved in our computations, this extra structure only serves as a hypothesis to results we cite, so we can afford not to be explicit about it, but for motivation we nevertheless expend a few words on its origins.

The value of the cup product of two singular cochains \( c' \in C^p(X; k) \) and \( c'' \in C^q(X; k) \) on a singular simplex \( \sigma: \Delta^{p+q} \to X \) is given by evaluating \( c' \) on the restriction \( \sigma' \) of \( \sigma \) to the convex hull \( \Delta^{[0, p]} \subseteq \Delta^{[p, p+q]} \) of the first \( p + 1 \) vertices, evaluating \( c'' \) on the restriction \( \sigma'' \) to \( \Delta^{[p, p+q]} \), and multiplying the two resulting values. This really amounts to evaluating \( \mu_k(c' \otimes c'') \) on the chain \( \Sigma_{j=0}^{p+q} \sigma|_{\Delta^{[0, j]} \otimes \Delta^{[p, p+q]}} \) (the value of the Alexander–Whitney diagonal making \( C_*(X) \) a dcc), and one can imagine other subdivision-type operations on cochains apportioning the vertices differently. This is in fact how the higher Steenrod cup-i products are defined; for example, \( c' \sim_3 c'' \) is defined by applying \( \mu_k(c' \otimes c'') \) to a sum of terms \( \pm \sigma|_{\Delta' \otimes \Delta''} \) for \( S' = [0, p] \cup [q, r] \cup [s, t] \) and \( S'' = [p, q] \cup [r, s] \) ranging over all values \( 0 \leq p < q \leq r \leq s \leq r \).

More generally, one can define higher-arity operations by applying \( \mu_k^{[m]}(c^{(1)} \otimes \cdots \otimes c^{(m)}) \) to the image of a chain under a chain operation \( C_\ell(X) \to C_*(X)^{\otimes n} \) of this sort. These chain (co)operations, sometimes termed \textit{interval-cut operations}, are defined by breaking the vertex set \( \{0, 1, \ldots, \ell\} \) of \( \Delta^\ell \) in all possible ways into some number \( m \geq n \) of subintervals \( I^{(i)} \) overlapping at their endpoints and assigning a union of some of the \( I^{(i)} \) to each position \( i \in \{1, \ldots, n\} \). Such an assignment gives an operation, uniformly defined in this way for every \( \ell \), parameterized by the sequence of length \( m \) with entries between 1 and \( n \) that lists, for \( 1 \leq j \leq m \), which argument \( I^{(i)} \) contributes to.

**Example 2.9.** Up to sign, the cup product corresponds to the sequence \((1, 2)\), the cup-1 product to \((1, 2, 1)\), and the cup-i product in general to an alternating sequence \((1, 2, 1, 2, \ldots)\) of length \( i + 2 \).
One considers only sequences with no two consecutive entries equal, to avoid producing degenerate simplices, and redefines the notion of cochain accordingly.

Definition 2.10. The normalized cochain algebra \( C^*(X_\bullet;k) \) on a simplicial set \( X_\bullet \) is the \( \mathfrak{dg} \) subalgebra containing all and only cochains vanishing on each degenerate simplex. It is augmented with respect to the map \( C^*(X_\bullet;k) \to C^*(x_0;k) \overset{\sim}{\to} k \) induced by the inclusion of the simplicial subset associated to any chosen basepoint \( x_0 \in X_0 \).

It can be shown that the differential of an interval-cut operation restricted to nondegenerate chains yields a linear combination of other such operations and the interval-cut operations are closed under the action of the symmetric group and composition [McS03, Props. 2.18, 19, 26][BF04, Prop. 1.2.7] and thus form a symmetric \( \mathfrak{dg} \)-operad \( \mathcal{K}_\bullet \), called the sequence operad [McS03] or surjection operad [BF04].

Theorem 2.11 (McClure–Smith [McS03, Thm. 2.15]). The normalized cochain algebra of a pointed simplicial set is a naturally a functor valued in \( \mathcal{K}_\bullet \)-algebras.

It is also a theorem of Berger–Fresse [BF04, Thm. 1.3.2] that the sequence operad is a quotient of the \( \mathfrak{dg} \)-operad \( \mathcal{E} \) associated to the classical Barratt–Eccles simplicial operad, and hence restriction along the quotient map \( \mathcal{E} \to \mathcal{K}_\bullet \) makes the normalized cochain algebra of a simplicial set an \( E_{\mathcal{K}_\bullet} \)-algebra [McS03, Thm. 2.15(c)]. The Barratt–Eccles operad is filtered by an increasing sequence of suboperads \( F_n \mathcal{E} \), whose geometric realizations are equivalent to the little \( n \)-cubes operads, and the sequence operad is accordingly filtered by quotients \( F_n \mathcal{K}_\bullet \) [BF04, Lem. 1.6.1].

Proposition 2.12 (McClure–Smith [McS03, Thm. 4.1][BF04, §1.6.6], Franz for the sign of the operation; cf. Franz [Fr2ob, (3.13)]). An \( \mathfrak{hga} \) structure on a \( \mathfrak{dga} \) \( A \) is equivalent to an \( F_2 \mathcal{K}_\bullet \)-algebra structure on \( A \), the operation \( E_\ell \) corresponding to the sequence \((1,2,1,3,1,\ldots,1,\ell,1)\).

Example 2.13. Strengthening the statement of Example 2.4, a \( \mathfrak{dga} \) \( A \) is canonically an \( \mathcal{K}_\bullet \)-algebra with all sequences in \( F_\ell \mathcal{K}_\bullet \) acting identically as 0 for \( \ell \geq 2 \).

Kadeishvili [Ko03] investigated conditions for the bar construction \( B A \) of an \( \mathfrak{hga} \) \( A \) over \( \mathbb{F}_2 \), viewed as a \( \mathfrak{dga} \) via the \( \mathfrak{hga} \) product \( \mu_{BA} \), to admit cup-\( i \) products, and gave a characterization in terms of operations on \( A \) amounting to algebra structures over certain symmetric \( \mathfrak{dg} \) suboperads of \( \mathcal{K}_\bullet \). For \( i = 1 \), the relevant suboperad is generated by \( F_2 \mathcal{K}_\bullet \) together with the sequences

\[
(1, p + 1, 1, p + 2, 1, p + 3, \ldots, 1, p + q, 1, p + q, 2, p + q, 3, p + q, \ldots, p, p + q)
\]

in \( F_3 \mathcal{K}_\bullet \), for \( p, q \geq 1 \), giving operations \( F_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A \). Particularly, \( F_{1,1} \) yields a cup-2 product on \( A \) itself, up to sign. Franz [Fr2oa] makes the following definition directly in terms of these operations and the associated identities.

\footnote{Sequences with 0 or 1 value lie in \( F_0 \mathcal{K}_\bullet \) and other sequences lie in \( F_\ell \mathcal{K}_\bullet \), where \( \ell \) is the maximum number of alternations in a subsequence with only two values. For example, \( \ell = 1 \) for the sequence \((1,2)\) and \( \ell = 3 \) for \((3,1,4,3,4,2,1,2)\), with maximum alternation number attained by the subsequence \((3,1,3,1)\).}

\footnote{This presentation is logically but not historically accurate; the filtrands \( F_n \mathcal{K}_\bullet \) were identified and shown to be equivalent to the little \( n \)-cubes operad by other methods [McS03] before the elucidating surjection from the Barratt–Eccles operad was found. McClure and Smith had already shown in 1999 that \( F_2 \mathcal{K}_\bullet \) is equivalent to the little squares operad [McS02], proving Deligne’s conjecture that the Hochschild cohomology of a ring is naturally an \( E_2 \)-algebra.}
Definition 2.15 (Franz [Fr20a, §3.2]; Kadeishvili over $\mathbb{F}_2$ [Kao3, §2.1]). Let $F_2^e \mathcal{X}$ be the symmetric dgc suboperad of $\mathcal{X}$ generated by $F_2 \mathcal{X}$ and the sequences (2.14). An extended homotopy Gerstenhaber algebra structure on a dga $A$ is an $F_2^e \mathcal{X}$-algebra structure. An extended HGA homomorphism $f: A \to B$ is a dga map which is simultaneously an $F_2^e \mathcal{X}$-algebra homomorphism; equivalently, such an $f$ is a dga map distributing over the operations $E_{\ell}$ and $F_{p,q}$.

Thus by Theorem 2.11, Example 2.13, and Definition 2.15, we a fortiori have the following.

Corollary 2.16. For any pointed simplicial set $X_\bullet$, its algebra $C^*(X_\bullet)$ of normalized cochains is naturally an extended HGA. Any cdga $A$ is naturally an extended HGA.

Convention 2.17. All cdgas in this work, particularly cohomology rings, come equipped with this trivial extended HGA structure. Consequently, if $A$ is an extended HGA and $B$ is a cdga, then an extended HGA map $f: A \to B$ annihilates the values of $F_{p,q}$ and of $E_{\ell}$ for $\ell \geq 1$.

3. $A_{\infty}$-maps and strong homotopy commutativity

If $A$ is a cochain complex, then the internal differential $d_{\text{int}}$ is defined on $BA$ as in Definition 1.8.1, and if $A$ is a dga, then the exterior and total differentials $d_{\text{ext}}$ and $d_B$ are also defined as in Definitions 1.8.2 and 1.8.3, but more generally, if $A$ is merely a graded $k$-module, then $BA$ is still naturally a cofree, coassociative graded coalgebra, and it is possible to consider potential coderivations on the graded coalgebra $BA$ such that it might become a dgc.

Definition 3.1. An $A_{\infty}$-algebra structure on an augmented graded module $A$ is a coderivation on $BA$ rendering it a dgc.

By cofreeness of $BA$, a coderivation $\tilde{m}$ on $BA$ is determined by the compositions

$$\tilde{m}_n: B_n A \to B A \xrightarrow{m} B A \to B_1 A$$

with the projection $\text{pr}_1: B A \to B_1 A$. Thus if we write

$$m_n = t_A \circ \tilde{m}_n \circ (s^{-1})^\otimes n: \bar{A}^\otimes n \to \bar{A},$$

it seems plausible that $F_2^e \mathcal{X}$ should equal $F_2 \mathcal{X}$, but we do not know this.

To see this, note that by iteration of the definition of a coderivation, one has $\Delta^{[\ell+1]} m = (\text{id} \otimes m \otimes \text{id}^*) \Delta^{[\ell+1]}$ for $\ell \geq 0$. To know $m(x)$ for some $x \in B A$ is to know each projection $\text{pr}_{\ell+1} m(x) \in \bigoplus B_{\ell+1} A$, and the inverse image of $(B_1 A)^{\otimes \ell+1}$ under $\tilde{\Delta}^{[\ell+1]}$ is $B_{\ell+1} A$, so

$$\text{pr}_{\ell+1} m = \Delta^{[\ell+1]} \text{pr}_{\ell+1} (\text{id}^* \otimes \text{pr}_1 m \otimes \text{id}^*) \Delta^{[\ell+1]},$$

where the iterated concatenation operation $\Delta^{[\ell+1]}$ is as defined in the proof of Proposition 1.7. In words, to compute $\text{pr}_{\ell+1} m(x)$ for $x \in B_n A$ a pure tensor, one rebrackets $x$ into $\ell$ undistinguished words of length 1 and one distinguished word of length $n - \ell$, applies $\text{pr}_1 m$ to the distinguished word, preserving the others, and rebrackets the result back into an element of $B A$. To compute $m(x)$, one sums these over $\ell$, or equivalently over $q = n - \ell$, and gets the sum of terms

$$(\text{id}^p \otimes (\text{pr}_1 m) \Delta^{[q]} \otimes \text{id}^{\otimes r}) \Delta^{[n]}$$

for $p + q + r = n$. That is, one views any subword of $x$ of any length $q$ as an element of $B_q A$, applies $\tilde{m}_q = \text{pr}_1 m|_{B_q A}$ to this subword alone, preserving the other letters of $x$, and rebrackets the result.
then \((A, \tilde{m})\) is determined by the list \((A, m_n)_{n \geq 1}\). It can be seen that \(BA\) being a dgc under \(\tilde{m}\) is equivalent to the relations

\[
\sum_{p+q+r=n} (-1)^{p+q} m_{p+1+r}(\text{id}^\otimes p \otimes m_q \otimes \text{id}^\otimes r) = 0: \tilde{A}^\otimes n \rightarrow \tilde{A} \quad (n \geq 0).^{18}
\]

For \(n = 1\), the relation is \(m_1^2 = 0\), so that \(d_A \circ d_A = 0\) is a differential making \(A\) a cochain complex. For \(n = 2\), the relation is \(m_1 m_2 - m_2 (m_1 \otimes \text{id}) - m_2 (\text{id} \otimes m_1) = 0\). Writing \(m_2(a \otimes b) = ab\), this says \(d_A(ab) = d_A(a)b + (-1)^{|a|} a d_A(b)\), so that \(d_A\) is a derivation with respect to \(m_2\). For \(n = 3\), the relation is

\[
m_1 m_3 - m_2 (\text{id} \otimes m_2) + m_2 (m_2 \otimes \text{id}) + m_3 (m_1 \otimes \text{id} \otimes \text{id} + \text{id} \otimes m_1 \otimes \text{id} + \text{id} \otimes \text{id} \otimes m_1) = 0.
\]

If we view \(m_3\) as an element of \(\text{Hom}(A^\otimes 3, A)\), equipping \(A\) with \(d_A\) and \(A^\otimes 3\) with \(d_{\otimes}\), then this can be rewritten as

\[
m_2 (m_2 \otimes \text{id}) - m_2 (\text{id} \otimes m_2) = D m_3,
\]

so that \(m_2\) is associative up to a homotopy witnessed by \(m_3\). Particularly, \(m_2\) induces an associative multiplication on the cohomology \(H^* (A, d_A)\). We extend \(d_A\) to \(A = k \oplus \tilde{A}\) by setting it to 0 on \(k\) and write \(d_A\) for the resulting differential. In this way an \(A_{\infty}\text{-algebra}\) can be seen as a generalization of a dga, the degree-0 factor \(k\) determining the unit and augmentation. Indeed, if \(m_3 = 0\) or \(m_1 = 0\), we see \(m_2\) is associative, so that then \(A\) really is a dga.

**Definition 3.2.** Given two \(A_{\infty}\text{-algebras}\) \(A\) and \(B\), an \(A_{\infty}\text{-map}\) from the former to the latter is defined to be a dgc homomorphism \(F: BA \rightarrow BB\). By cofreedom of \(BA\), such a map \(F\) is determined by the sequence of compositions

\[
F(n): A^\otimes n \xrightarrow{\sim} B_n A \xrightarrow{\text{pr}_1} BA \xrightarrow{F} BB \xrightarrow{\text{pr}_B} B.
\]

On expanding out the prescription \(\text{pr}_1 (m \circ F)|_{B_n A} = \text{pr}_1 (F \circ m)|_{B_{n} A}\), one finds the \(F(n)\) subject to the conditions

\[
\sum_{\sum i_1 + \cdots + i_n = n} (-1)^{\binom{i_1}{2} + \cdots + \binom{i_n}{2}} m_k (F(i_1) \otimes \cdots \otimes F(i_n)) = \sum_{p+q+r=n} (-1)^{p+q} F(p+1+r)(\text{id}^\otimes p \otimes m_q \otimes \text{id}^\otimes r),^{19}
\]

(3.3)

conversely, a sequence of maps \(F(n)\) satisfying these conditions assembles to a twisting cochain giving rise to an \(A_{\infty}\text{-map}\) \(F\).

Setting \(n = 1\) in (3.3), we see \(F(1)\) preserves \(m_1\), so that an \(A_{\infty}\text{-map}\) amounts to a cochain map \(F(1): (A, m_1) \rightarrow (B, m_1)\) and some additional homotopy data. Setting \(n = 2\) and viewing \(F(2)\) as

\[
0 = \tilde{m} \circ \tilde{m}|_{B_n A} = \tilde{m} A[p+1+r] \sum_{p+q+r=n} (\text{id}^\otimes p \otimes \tilde{m}_q \otimes \text{id}^\otimes r).
\]

Rearranging suspensions in \(s \circ \tilde{m} \circ \tilde{m}|_{B_n A} \circ (s^{-1})^\otimes n\) then gives the desired expressions.

\[
\sum m_k (\tilde{f}_i \otimes \cdots \otimes \tilde{f}_i) = \sum \tilde{f}_{p+1+r} (\text{id}^\otimes p \otimes \tilde{m}_q \otimes \text{id}^\otimes r).
\]

Composing the equation with \(s\) on the left and with \((s^{-1})^\otimes n\) on the right and rearranging gives the signs.

---

18 Note that \(m^2 = 0\) is equivalent to the simultaneous satisfaction, for all \(n \geq 0\), of the relations

\[
0 = \tilde{m} \circ \tilde{m}|_{B_n A} = \tilde{m} A[p+1+r] \sum_{p+q+r=n} (\text{id}^\otimes p \otimes \tilde{m}_q \otimes \text{id}^\otimes r).
\]

19 Writing \(\tilde{F}_n = \text{pr}_1 F|_{B_n A}\), the nice version of this is that

\[
\sum m_k (\tilde{f}_i \otimes \cdots \otimes \tilde{f}_i) = \sum \tilde{f}_{p+1+r} (\text{id}^\otimes p \otimes \tilde{m}_q \otimes \text{id}^\otimes r).
\]
an element of \( \text{Hom}(A \otimes \B, B) \) we see \( F_{(1)} m_2 = m_2(F_{(1)} \otimes F_{(1)}) + DF_{(2)} \), so \( F_{(1)} \) is multiplicative up to homotopy. If \( A \) and \( B \) are dgas, this means \( F_{(1)} + \eta_B \varepsilon_A \) is a dga map up to homotopy, and an ideal \( b \leq B \) containing the image of \( t_B F(B_{\geq 2}A) \) can be seen as a measure of this deviation, for \( F_{(1)} + \eta \) is a dga map just if \( t_B F(B_{\geq 2}A) = 0 \); and if so, then \( F = B(F_{(1)} + \eta \varepsilon) \).

**Definition 3.4.** Given an \( A_{\infty} \)-map \( F: BA \to BB \), we write \( F^+_{(1)} = F_{(1)} + \eta_B \varepsilon_A \) for the unital extension of its 1-component, by analogy with one-point compactification. If \( H^*(F_{(1)}): H^*(A) \to H^*(B) \) is an isomorphism, we call \( F \) an \( A_{\infty} \)-quasi-isomorphism.

We now want to see it is possible to construct an \( A_{\infty} \)-quasi-isomorphism from \( H^*(A) \) to \( A \) when \( A \) is a dga with polynomial cohomology. We will need two auxiliary notions for this.

**Proposition 3.5** ([Mu74, Prop. 3.3]). There exists an internal tensor product of \( A_{\infty} \)-maps, which, given dgas \( A, A', B, B' \) and dgc maps \( F: BA \to BA' \) and \( G: BB \to BB' \), produces a dgc map

\[
F \otimes G: B(A \otimes B) \to B(A' \otimes B').
\]

The operation \( (F, G) \mapsto F \otimes G \)

- extends the tensor product of dga maps in the sense that if \( f: A \to A' \) and \( g: B \to B' \) are dga maps, then \( Bf \otimes Bg = B(f \otimes g) \).
- is functorial in each variable separately,
- is unital in the sense that \( F \otimes \text{id}_k \) and \( \text{id}_k \otimes F \) can be identified with \( F \), and
- is associative in the sense that given a third pair \( C, C' \) of dgas and a dgc map \( H: BC \to BC' \), the iterated products \( (F \otimes G) \otimes H \) and \( F \otimes (G \otimes H) \): \( B(A \otimes B \otimes C) \to B(A' \otimes B' \otimes C') \) agree.

This notion of internal tensor product is connected to the ordinary external one by the shuffle map \( \nabla \) of Example 1.13.

**Proposition 3.6** ([Fr19, Lem. 4.4]). Let dgas \( A_{0}', A_{1}', A_{0}'' \) and \( A_{1}'' \) and dgc maps \( G': BA_{0}' \to BA_{1}' \) and \( G'': BA_{0}'' \to BA_{1}'' \) be given. Then the following square commutes:

\[
\begin{array}{ccc}
BA_{0}' \otimes BA_{0}'' & \xrightarrow{\nabla} & B(A_{0}' \otimes A_{0}'') \\
G' \otimes G'' & \downarrow & G' \otimes G'' \\
BA_{1}' \otimes BA_{1}'' & \xrightarrow{\nabla} & B(A_{1}' \otimes A_{1}'').
\end{array}
\]

The internal tensor product allows us to define another notion of homotopy-commutativity \textit{a priori} unrelated to hgas.

**Definition 3.7.** A \textbf{strongly homotopy commutative algebra} (henceforth \textit{shc-algebra}) is an augmented dga \( A \) equipped with an \( A_{\infty} \)-map from \( A \otimes A \) to \( A \) (i.e., a dgc map \( \Phi: B(A \otimes A) \to BA \)), satisfying the following conditions:

1. Its 1-component \( \Phi_{(1)}: A \otimes A \to A \) is the restriction \( \mu_{A|\widehat{A} \otimes \widehat{A}} \) of the given product on \( A \).
2. It is strictly unital in the sense that \( \Phi \circ B(\text{id}_A \otimes \eta_A) = \text{id}_B = \Phi \circ B(\eta_A \otimes \text{id}_A) \).

3. It is homotopy-associative: there is a homotopy between \( \Phi(\Phi \otimes \text{id}_A) \): \( B(A \otimes A \otimes A) \rightarrow B A \) and \( \Phi(\text{id}_A \otimes \Phi) \).

4. It is homotopy-commutative: there is a homotopy between \( \Phi \) and \( \Phi \circ B(1, 2) \), where \((1, 2)\) is the tensor-factor transposition \( A \otimes A \simeq A \otimes A \) of Definition 1.4.

We define the iterates \( \Phi^{[n]} : B(A_\otimes^n) \rightarrow B A \) of the structure map \( \Phi \) by \( \Phi^{[2]} := \Phi \) and \( \Phi^{[n+1]} := \Phi(\Phi^{[n]} \otimes \text{id}_A) \).

An shc-algebra structure \( \Phi \) on a dga \( A \) allows us to combine maps in a useful way: given sequences \((A_i)_{i=1}^n \) of dgas and \((F_j : B A_i \rightarrow B A)_{j=1}^n \) of dgc maps, the composite

\[
B(\otimes A_j) \xrightarrow{\otimes F_j} B(A_\otimes^n) \xrightarrow{\Phi^{[n]}} B A
\]

is guaranteed to be another dgc map. We will say this map is compiled from the \( F_j \).

Associated to each homogeneous element \( a \in A \) is a map from the free dga on one generator of degree \( |a| \) taking this generator to \( a \). If \( a \in A \) is a cocycle of even degree, this map factors through the map \( \lambda_a : k[x] \rightarrow A \) taking \( x \) to \( a \), where the differential on \( k[x] \) is trivial. Thus, given a list \( \vec{a} = (a_i) \) of even-degree cocycles of \( A \) and taking \( A_j := k[x_j] \) for \( |x_j| = |a_i| \) and \( F_j := B \lambda_{a_j} \), the compilation procedure (3.8) yields a dgc map

\[
\lambda_{\vec{a}} : B \left( \bigotimes_{j=1}^n k[x_j] \right) \xrightarrow{B(\otimes \lambda_{a_j})} B(A_\otimes^n) \xrightarrow{\Phi^{[n]}} B A.
\]

Then \( (\lambda_{\vec{a}})^+ \) is easily seen to be the optimist’s candidate for a ring homomorphism,

\[
k[\vec{x}] := k[x_1, \ldots, x_n] \rightarrow A,
\]

\[
x_1^{p_1} \cdots x_n^{p_n} \mapsto a_1^{p_1} \cdots a_n^{p_n}.
\]

Though this map is in fact almost never multiplicative, it is at least a quasi-isomorphism.

**Proposition 3.10** (Stasheff–Halperin [StaH70, Thm. 9][Mu74, 7.2]). If \( A \) is an shc-algebra whose cohomology ring \( H^*(A) \) is polynomial on classes represented by even-degree elements \( a_i \in A \) and \( \lambda_{\vec{a}} \) is defined as in (3.9), then \( H^*(\lambda_{\vec{a}})^+ : k[\vec{x}] \rightarrow H^*(A) \) is an isomorphism.

In the target application, \( A \) is the normalized cochain algebra on a classifying space \( BG \). Munkholm [Mu74, Prop. 4.7]20 showed that the cochain algebra of a simplicial set admits a natural shc-algebra structure, but to define the product on the two-sided bar construction we will need later for our variant \( A.27 \) of the Eilenberg–Moore theorem, we will use a result of Franz defining an shc-algebra structure in terms of extended HGA operations.

**Theorem 3.11** (Franz [Fr20a, Thm. 1.1, (4.2)]). An extended HGA \( A \) admits an shc-algebra structure whose structure map \( \Phi \) and associativity and commutativity homotopies are defined in terms of the extended HGA operations on \( A \) and hence are natural in extended HGA maps. Moreover, the composite \( \Phi \circ \nabla : B A \otimes B A \rightarrow B(A \otimes A) \rightarrow B A \) of the shuffle map of Example 1.13 and this structure map is the given product \( \mu_{BA} \) making \( A \) an HGA.

---

20 and stating a bit less, Gugenheim and Munkholm [GuMu74, Prop. 4.2]
We will require an explicit formula for \( t_A \Phi_A \) in the proof of Theorem A.5, but hold off on stating it until then. From Corollary 2.16 and Theorem 3.11, we immediately have the following.

**Corollary 3.12.** The normalized cochain algebra \( C^*(X_\bullet;k) \) of a pointed simplicial set is naturally an \( \mathsf{shc} \)-algebra.

**Corollary 3.13.** Given a pointed simplicial set \( X_\bullet \) with polynomial cohomology and any list \( \bar{a} \) in \( C^*(X_\bullet;k) \) of representatives for \( k \)-algebra generators of \( H^*(X_\bullet;k) \), there is a dgc map \( \lambda_\bar{a} : BH^*(X_\bullet;k) \to BC^*(X_\bullet;k) \), given as in (3.9), such that \( H^*(\lambda_\bar{a})_ {(\bar{a})} \) is the identity map of \( H^*(X_\bullet;k) \).

We have observed that though it is a quasi-isomorphism in the cases that interest us, the extended \( \mathsf{HGA} \) structure and the resulting \( \mathsf{shc} \) structure on \( C^*(X_\bullet) \) guarantee that, loosely speaking, it is a \( \mathsf{dga} \) map up to an error term contained in an ideal \( \epsilon X_\bullet \) of \( C^*(X_\bullet) \) functorial in \( X_\bullet \), independently of the choice of representatives \( \bar{a} \) in \( C^*(X_\bullet) \); thinking of \( \epsilon X_\bullet \) as a neighborhood of 0, we may consider it as a sort of uniform bound on failure to be a \( \mathsf{dga} \) map. It will be an important point in the proof of our main result that the bounding ideal \( \epsilon X_\bullet \) lies in the kernel of the formality map \( f \) to be described in Theorem 3.19, and hence \( f \) annihilates the error term.

**Definition 3.14** (Franz [Fr19, (10.2)]). Given a pointed simplicial set \( X_\bullet \), viewing its normalized cochain algebra \( C^*(X_\bullet) \) as an \( \mathsf{shc} \)-algebra via Corollary 3.12, we denote by \( \epsilon = \epsilon X_\bullet \subset C^*(X_\bullet) \) the ideal generated by the following elements, where \( a, b, b_\bullet, c_\bullet \) range over \( C^*(X_\bullet) \):

1. coboundaries,
2. elements of odd degree,
3. elements of the form \( E_\ell(a; b_\bullet) \) for \( \ell = \ell(b_\bullet) \geq 1 \),
4. elements of the form \( F_{p,q}(b_\bullet; c_\bullet) \) with \( (p,q) \neq (1,1) \),
5. elements of the form \( a \sim_2 E_\ell(b; c_\bullet) \) with \( \ell \geq 2 \),
6. elements of the form \( a \sim_2 (\cdots ((b_0 \sim_1 b_1) \sim_1 b_2) \sim_1 \cdots) \) for cocycles \( a \) and \( b_\bullet \).

From the naturality of the extended \( \mathsf{HGA} \) structure on a cochain algebra, the following functoriality property of \( \epsilon \) is immediate.

**Lemma 3.15** (Franz [Fr19, Prop. 10.1]). If \( \varphi : Y_\bullet \to X_\bullet \) is a map of pointed simplicial sets, then the ideals of Definition 3.14 satisfy \( \varphi^* \epsilon Y_\bullet \leq \epsilon Y_\bullet \).

Morally speaking, then, our maps \( \lambda_\bar{a} \) are functorial and multiplicative modulo \( \epsilon \).

**Theorem 3.16** (Franz [Fr20a, Prop. 7.2][Fr19, Prop. 11.5, Thm. 11.6]). Suppose 2 is a unit of \( k \). Then the maps \( \lambda_\bar{a} : k[\bar{x}] \to C^*(X_\bullet) \) of (3.9) are functorial modulo \( \epsilon \) in the sense that given a map \( \varphi : Y_\bullet \to X_\bullet \) of simplicial sets with polynomial cohomology and sequences \( \bar{a} \) in \( C^*(X_\bullet) \) and \( \bar{b} \) in \( C^*(Y_\bullet) \) representing generators of \( H^*(X_\bullet) \) and \( H^*(Y_\bullet) \) respectively, the left diagram of (3.17) commutes up to a \( \mathsf{dgc} \) homotopy \( H_\varphi : BH^*(X_\bullet) \to BC^*(Y_\bullet) \) whose associated twisting cochain homotopy \( \eta \varphi + t_{C^*(Y_\bullet)} H_\varphi \) sends \( B \varphi H^*(X_\bullet) \) into \( \epsilon Y_\bullet \).

\[
\begin{array}{cc}
BH^*(X_\bullet) & \xrightarrow{B H^* \varphi} & BH^*(Y_\bullet) \\
\lambda_\bar{a} & \downarrow & \lambda_\bar{b} \\
BC^*(X_\bullet) & \xrightarrow{BC^* \varphi} & BC^*(Y_\bullet)
\end{array}
\quad
\begin{array}{cc}
B(k[\bar{x}]) & \xrightarrow{B \mu} & B(k[\bar{x}]) \\
\lambda_\bar{a} \otimes \lambda_\bar{b} & \downarrow & \lambda_\bar{d} \\
B(C^*(X_\bullet) \otimes C^*(X_\bullet)) & \xrightarrow{\Phi} & BC^*(X_\bullet)
\end{array}
\]
Moreover $\lambda_\parallel$ is multiplicative modulo $\mathfrak{k}$ in the sense that $\iota_{C^*(X_\ast)}\lambda_\parallel$ takes $B_{\geq 2}(k[\mathfrak{x}])$ into $\mathfrak{k}_{X_\ast}$, and the right diagram of (3.17) commutes up to a dgc homotopy $H_\mu: B(k[\mathfrak{x}]^{\otimes 2}) \to BC^*(X_\ast)$ whose associated twisting cochain homotopy $\eta \epsilon + \iota_{C^*(X_\ast)}H_\mu$ sends $B_{\geq 1}(k[\mathfrak{x}]^{\otimes 2})$ into $\mathfrak{k}_{X_\ast}$.

We remark that the mere existence of such homotopies, without the bound in terms of $\mathfrak{k}$, is the technical heart of Munkholm’s collapse result [Mu74]. To make precise our claim that there is a map $C^*(BT) \to H^*(BT)$ in the other direction annihilating $\mathfrak{k}$, we recall the classical construction of the simplicial classifying space:

**Proposition 3.18** (See, e.g., May [May67, §21]). Let $G_\ast$ be a simplicial group. Then there exists a contractible simplicial $G_\ast$-space $W G_\ast$, functorial in $G_\ast$; That is, if $\varphi: G_\ast \to H_\ast$ is a homomorphism of simplicial groups, then $W \varphi: WG_\ast \to WH_\ast$ is $\varphi$-equivariant in the sense that $(W \varphi)(x \cdot g) = (W \varphi)(x) \cdot \varphi(g)$ for $x \in (WG_\ast)_n$ and $g \in G_n$. The projection $WG_\ast \to \overline{W}G_\ast := WG_\ast/G_\ast$ is a principal $G_\ast$-bundle, and the base $\overline{W}G_\ast$ is the classifying simplicial set for simplicial principal $G_\ast$-bundles.

The promised map is then provided by the following result.

**Theorem 3.19** (Franz [Fr19, Thm. 9.6, Prop. 9.7]). Let $T_\ast$ be a simplicial abelian group, pointed at 1 $\in T_0$ and such that the cohomology $H^*(T_\ast; \mathbb{Z})$ of the normalized cochain complex $C^*(T_\ast)$ is an exterior algebra on finitely many degree-1 generators. Then there exists a quasi-isomorphism $f = f_T: C^*(\overline{W}T_\ast) \to H^*(\overline{W}T_\ast)$ of extended hgas, called the **formality map**, which annihilates all extended hga operations $F_{p,q}$ except for $F_{1,1} = -\sim_2$. If 2 is a unit of $k$, the formality map can be chosen so as to annihilate all $\sim_2$-products of cocycles and such that its kernel contains the ideal $\mathfrak{k}_{\overline{W}T_\ast}$ of Definition 3.14.

### 4. Simplicial substitution

In this section we introduce a simplicial form of the homotopy pullback whose cohomology we will ultimately compute. We first realize $EK \otimes_K G/H$ as a pullback.

**Discussion 4.1.** Let $G$ be a Lie group and $H$ and $K$ closed subgroups. Writing $i: K \hookrightarrow G$ for the inclusion, the map

\[
EK \times G/H \to BK \times_{BG} EG/H,
\]

\[
(e, gH) \to (eK, (Ei)(e)gH)
\]

is constant on orbits of the diagonal $K$-action $k \cdot (e, gH) = (ek^{-1}, kgH)$ and descends to a homeomorphism from the orbit space $EK \otimes_K G/H$, inducing a pullback diagram

\[
\begin{array}{ccc}
EK \otimes_K G/H & \to & EG/H \\
\downarrow & & \downarrow \\
BK & \to & BG
\end{array}
\]

(4.2)

where the right vertical map is the projection $eH \to eG$ of a fiber bundle with fiber the homogeneous space $G/H$. When the left $K$-action by translation on $G/H$ is free (equivalently, when the two-sided action of $K \times H$ on $G$ is free), the homotopy quotient in the upper left is homotopy equivalent to the biquotient $K \backslash G/H$, but more generally, we see the homotopy quotient...
$E K \otimes_K G / H$ whose cohomology we are ultimately to compute is the homotopy pullback of the diagram $B K \to B G \leftarrow B H$. For another proof that $K \backslash G / H$ is the homotopy pullback when $K \times H$ acts freely, cf. Singhof [Sig93, §2].

We will also need to induce maps of bar constructions from the formality maps $C^*(B T) \to H^*(B T)$ from Theorem 3.19, for $T$ maximal tori of $K$ and $H$, and our definition of this map requires a simplicial model for $B T$. This will require us to replace $B G$, $B K$, and $B H$ with simplicial models as well.

**Definition 4.3.** Given a simplicial set $X_\bullet$, we write $|X_\bullet|$ for its geometric realization. Given a topological space $X$, we write $\text{Sing} X$ for the simplicial set with level $(\text{Sing} X)_n := \text{Map}(\Delta^n, X)$ the set of singular $n$-simplices, and faces and degeneracies respectively induced by face inclusions and vertex-order–preserving affine surjections on the domains. If $G$ is a topological group, we write $\overline{G}$ for $\text{Sing} G$ made a simplicial group by equipping each level $(\text{Sing} G)_n$ with the valuewise multiplication of maps $\Delta^n \to G$.

Note that $\overline{G}$ is merely a simplicial group, not a simplicial topological group. It nevertheless encodes the topology of $G$ owing to the adjunction $|-| \dashv \text{Sing}$.

**Lemma 4.4** (See, e.g., May [May67, Thm. 16.6(ii)]). For connected Lie groups $G$, the counit $\overline{|G|} \to G$ of the standard adjunction is a homomorphism and a homotopy equivalence, natural in continuous homomorphisms.

We will use $\overline{G}$ to define functorial simplicial replacements for classifying spaces. A particularly strong statement will be possible for a specific model.

**Definition 4.5** (Milgram, Steenrod, Segal [Mi67, Ste68, Se68]). Given a topological group $G$, let $\overline{C_G}$ be the one-object topological category with morphism space $G$, and let $C_G$ be the topological category with morphism space $G \times G$, object space $G$, and source and target maps the second and first projections respectively. The right diagonal action of $G$ is free and continuous and induces a continuous quotient functor $C_G \to \overline{C_G}$, and hence a map of simplicial spaces $\mathcal{N}(C_G) \to \mathcal{N}(\overline{C_G})$ between the nerves of these topological categories. The geometric realization $E G \to B G$ of this map is the Milgram model of the universal principal $G$-bundle. If $G$ is a Lie group, then $E G$ is a $G$–CW complex.

This model is closely related to the Eilenberg–Mac Lane $W$-construction from Proposition 3.18.

**Lemma 4.6.** For any connected topological group $G$, the geometric realization $|\overline{W G}|$ is $G$-equivariantly homotopy equivalent to the total space $E G$ of the universal principal $G$-bundle $EG \to BG$, and for any closed subgroup $H$ of $G$, there is an induced weak homotopy equivalence $|\overline{W G} / \overline{H}| \to E G / H$ natural in pairs $(G, H)$ of topological groups. In particular, $|\overline{W G}|$ is weakly homotopy equivalent to $BG$. If $G$ is a Lie group, these are homotopy equivalences.

**Proof.** For any simplicial group $G_\bullet$, there is a natural $|G_\bullet|$-equivariant homeomorphism $|\overline{W G_\bullet}| \to \overline{E G_\bullet}$ descending to a natural homeomorphism $|\overline{W G}| \to B |G_\bullet|$ [BH98]. Taking $G_\bullet = \overline{G}$ and using Lemma 4.4, we get a composite weak homotopy equivalence $|\overline{W G}| \to \overline{E G}$ equivariant with respect to the homomorphism $\overline{|G|} \to G$ and hence with respect to the restricted homomorphism $\overline{|H|} \to H$. Maps of long exact sequences of fibrations then show the maps $|\overline{W G} / \overline{H}| \to \overline{E G} / \overline{H} \to (E G) / H$ are also weak homotopy equivalences. But geometric realization preserves products and also, since it is a left adjoint, coequalizers, and $|\overline{W G} / \overline{H}|$ is the
coequalizer of the projection and action maps \( W\overline{G} \times \overline{H} \to W\overline{G} \), so there is a natural homeomorphism \( \left| W\overline{G} / \overline{H} \right| \approx \left| W\overline{G} / H \right| \). If \( G \) is compact, then since \( EG \) is a \( G \)-CW complex with free \( G \)-cells and the homogeneous manifold \( G/H \) is a \( \mathbb{C} \) complex, \( EG/H \approx EG \otimes_{G} G/H \) is a \( \mathbb{C} \) complex, and as geometric realization produces simplicial complexes, so also is \( \left| W\overline{G} / \overline{H} \right| \). Thus, by Whitehead’s theorem, \( \left| W\overline{G} / \overline{H} \right| \to EG/H \) is a homotopy equivalence.

**Lemma 4.7.** Let \( G \) be a compact, connected Lie group and \( H \) a closed subgroup. Then the geometric realization of the map \( W\overline{G} / \overline{H} \to W\overline{G} \) is homotopy equivalent to \( EG/H \to BG \).

**Proof.** The horizontal homotopy equivalences in the square

\[
\begin{array}{ccc}
\left| W\overline{G} / \overline{H} \right| & \to & EG/H \\
\downarrow & & \downarrow \\
\left| W\overline{G} \right| & \to & BG.
\end{array}
\]

come from Lemma 4.6, and the commutativity of the square connecting them comes from the inclusion \( (G, H) \to (G, G) \) by naturality in pairs of groups.

**Proposition 4.8.** Let \( G \) be a compact, connected Lie group and \( H \) and \( K \) closed subgroups, and suppose the coefficient ring \( k \) is a principal ideal domain. Then there are isomorphisms of graded algebras

\[
H^*_K(G/H) \cong \text{Tor}_{C^*(W\overline{K})}(C^*(W\overline{K}), C^*(W\overline{H})) \cong H^* B(C^*(W\overline{K}), C^*(W\overline{G}), C^*(W\overline{H})),
\]

natural with respect to the diagram \( K \to G \to H \).

**Proof.** Lemmas 4.6 and 4.7 give the bottom squares of the following commutative diagram of pointed simplicial sets.

\[
\begin{array}{ccc}
W\overline{K} & \to & W\overline{G} & \to & W\overline{H} \\
\downarrow & & \downarrow & & \downarrow \\
W\overline{K} & \to & W\overline{G} & \to & W\overline{G} / \overline{H} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Sing } W\overline{K} & \to & \text{Sing } W\overline{G} & \to & \text{Sing } W\overline{G} / \overline{H} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Sing } BK & \to & \text{Sing } BG & \to & \text{Sing } EG/H
\end{array}
\]

(4.9)

The middle row of vertical maps comes from the unit \( X_* \to \text{Sing } X_* \) of the adjunction. To know these maps are weak equivalences, it is enough [May67, Thm. 16.6(i)] to see \( W\overline{K}, W\overline{G}, \) and \( W\overline{G} / \overline{H} \) are Kan complexes, but we already know the first two are [May67, Lem. 21.3]. As for the last, on expanding out the definitions, we see \( \overline{G} / \overline{H} = \text{Sing}(G/H) \), which is a Kan complex, and \( W\overline{G} / \overline{H} \cong W\overline{G} \times_{\overline{G}} \overline{G} / \overline{H} \) is a twisted product, so \( W\overline{G} / \overline{H} \to W\overline{G} \) is a Kan fibration [May67, Prop. 18.4] and hence so is \( W\overline{G} / \overline{H} \to W\overline{G} \to * \).
The first row of vertical maps comes about because the functorially induced homotopy equivalence $\overline{W G/H} \to \overline{W G/G}$ of contractible simplicial sets is $\overline{\Pi}$-equivariant, the induced map $\overline{W G/H} \to \overline{W G/G}$ is a homotopy equivalence by the map of long exact fibration sequences, since everything is Kan, and moreover the composition with the quotient map $\overline{W G/G} \to \overline{W G/G}$ is the functorially induced map $\overline{W G/H} \to \overline{W G/G}$.

Taking normalized cochains, the vertical maps all become dga quasi-isomorphisms, inducing isomorphisms of Tors multiplicative with respect to the product of Theorem A.1. Precomposing the inverse of the Eilenberg–Moore isomorphism of Theorem A.27, we obtain a natural composite ring isomorphism

$$H^*_K(G/H) \overset{\sim}{\to} H^*|Y_*| \overset{\sim}{\to} H^*(Y_*) \overset{\sim}{\to} H^*\mathbf{B}(C^*(\overline{W K}), C^*(\overline{W G}), C^*(\overline{W H})).$$

Naturality of the composite follows from functoriality of $W, \text{Sing, } |\cdot|, \text{ and } C^*$, expanding (4.9) to an array of commutative cubes, and from the naturality of the Eilenberg–Moore isomorphism.

Remark 4.10. An alternative, more simplicial proof takes $\overline{W}$ instead of $\overline{\Pi}$, and now $\overline{W G}$ is 1-reduced. One argues $\overline{\Pi}$ is still a Kan complex with this definition of $\overline{\Pi}$, so $W G/H \to \overline{W G}$ is a Kan fibration, and constructs the pullback $Y_*$ of $\overline{W K} \to \overline{W G} \leftarrow W G/H$. Then one notes $|Y_*|$ is the pullback of $|\overline{W K}| \to |\overline{W G}| \leftarrow |W G/H|$, as geometric realization preserves pullbacks [May72, Cor. 11.6]. Since the geometric realization of a Kan fibration is a Serre fibration [Qu68], this pullback is also homotopy pullback, and it follows the map $|Y_*| \to EK \otimes_K G/H$ induced by the universal property of pullbacks from the maps of simplicial models to (4.2) is a weak homotopy equivalence. There is a simplicial version of our Eilenberg–Moore theorem A.27 applying to pullbacks of Kan fibrations over 1-reduced bases, so one can show natural isomorphisms

$$H^*_K(G/H) \overset{\sim}{\to} H^*|Y_*| \overset{\sim}{\to} H^*(Y_*) \overset{\sim}{\to} H^*\mathbf{B}(C^*(\overline{W K}), C^*(\overline{W G}), C^*(\overline{W H})).$$

Remark 4.11. May and Neumann observed [May82, Thm. 4.3] behind the Gugenheim–May computation of the cohomology groups of a homogeneous space $G/H$ applies equally in many cases to a generalized homogenous space, meaning the homotopy fiber $F$ of a map $B_H \to B_G$ of path-connected spaces. Here $F$ is to be thought of as $G/H$ for $G = \Omega B_G$ and $H = \Omega B_H$. The additional hypothesis, beyond a trivial $\pi_1(B_G)$-action on $H^*(F)$ and that $H^*(B_G)$ is a polynomial algebra over the principal ideal domain $k$ finitely generated in each degree, is that there be a map $B_T \to B_H$, where $B_T$ is homotopy equivalent to a $K(\mathbb{Z}^n, 2)$ for some natural number $n$, making $H^*(B_T)$ a free module of finite rank over $H^*(B_T)$; then $T = \Omega B_T$ is called a maximal torus of $H$.

Our result similarly generalizes. Given three path-connected spaces $B_G, B_K, B_H$ such that $K$ and $H$ admit maximal tori $T_k$ and $T_H$ and $H^*(B_G), H^*(B_K)$, and $H^*(B_H)$ are polynomial algebras finitely generated in each degree over the principal ideal domain $k$, replacing $\text{Sing} BT_K$ and $\text{Sing} BT_H$ by $\overline{W T K}$ and $\overline{W T H}$ respectively, the same proof as in the following Section 5 shows that the cohomology of the homotopy pullback $Y$ of the span $B_K \to B_G \leftarrow B_H$ is given as a graded group by $\text{Tot} H^*(B_T)(H^*(B_K), H^*(B_H))$, and as a ring if 2 is a unit of $k$.

We are not aware of great interest in such spaces $Y$ in general, but it may be worth observing that if $B_K$ and $B_H$ are points, we recover that $H^*(\Omega B_G)$ is an exterior algebra, and for the span $B_K \to B_K \times B_K \leftarrow B_K$, with both maps the diagonal, we find the cohomology ring of the free loop space $LB_K$ is the tensor product of $H^*(B_K)$ and $H^*(K)$. This is an instance of a more general
result of Saneblidze [Sa09], derived using Hochschild homology and a model of the total space of the fibration $\Omega Y \to LY \to Y$ equipped with a cochain-level product related to the product described in Theorem A.5 and the hga structure on $C^*(Y)$. His result requires only that the cup-one squares $y \cup_1 y$ of a set of polynomial generators $y \in H^*(Y)$ are zero, without any maximal torus assumption, and requires of $k$ only that it be a commutative ring.

5. The quasi-isomorphisms

We have finally assembled the necessary ingredients to prove Theorem 0.1.

**Notation 5.1.** In the calculation that follows, the base ring $k$ is now a principal ideal domain, still usually suppressed in the notation. We will not require that 2 be a unit initially. Let $G$ be a connected Lie group and $H$ and $K$ closed, connected subgroups such that $BG$, $BK$, and $BH$ have polynomial cohomology over $k$ (equivalently, such that the torsion primes of $G, H, K$ are units of $k$). We will work with normalized cochains on the simplicial models $\overline{\omega G}$ of Section 4, but in the notation for cohomology identify $H^*(BG)$ with $H^*(\overline{\omega G})$ and so on, suppressing the natural isomorphisms induced by the simplicial weak equivalences $\overline{\omega G} \to \text{Sing} |\overline{\omega G}| \to \text{Sing} BG$.

From Proposition 4.8 we have a natural isomorphism
\[ H^*_K(G/H) \cong \text{Tor}_{C^*(\overline{\omega G})}(C^*(\overline{\omega K}), C^*(\overline{\omega H})), \tag{5.2} \]
of graded algebras, and our goal is to use the maps $\lambda$ of (3.9) and $f$ of Theorem 3.19 to induce a cga isomorphism
\[ H^*_K(G/H) \cong \text{Tor}_{H^*(BG)}(H^*(BK), H^*(BH)) \tag{5.3} \]
natural in inclusion diagrams $K \hookrightarrow G \hookrightarrow H$. On the level of graded modules, the isomorphism (5.3) particularly means the Eilenberg–Moore spectral sequence of (4.2) collapses, a theorem of Munkholm [Mu74] we will reprove as Proposition 5.7. To improve this result to a ring isomorphism, as both Tor arise as the cohomology of a two-sided bar construction, we will connect the argument DGAs of the two via maps preserving enough structure to guarantee our novel product on the two-sided bar construction is preserved up to homotopy. The structure of this section closely follows that of Section 12 in Franz’s paper [Fr19] and specializes to it the case $K = 1$. The ideas in the proof originate in the work of many authors and we summarize these inspirations in the Historical Remarks 5.17.

**Discussion 5.4.** We begin by constructing an additive cochain map between the two bar constructions. Selecting arbitrarily and fixing a list $\overline{a}$ of cocycles representing irredundant generators of $H^*(BG)$, and similarly for $H^*(BK)$ and $H^*(BH)$, we may use (3.9) to construct $A_x$-quasi-isomorphisms $\lambda_G : BH^*(BG) \to BC^*(\overline{\omega G})$, $\lambda_K$, and $\lambda_H$. Recall that these maps are selected so that if $x_j$ denotes the cohomology class of $a_j \in C^*(\overline{\omega G})$ and $\text{rk} G = n$, then the extension $(\lambda_G)_{(1)} = \eta e + (\lambda_G)_{(1)}$ of the 1-component is the additive quasi-isomorphism
\[ k[\overline{x}] \cong H^*(BG) \to C^*(\overline{\omega G}), \]
\[ x_1^p_1 \cdots x_n^p_n \mapsto a_1^{p_1} \cdots a_n^{p_n} \]
and similarly for $\lambda_K$ and $\lambda_H$. We write the canonical twisting chains $BH^*(BG) \to H^*(BG)$ and $BC^*(\overline{\omega G}) \to C^*(\overline{\omega G})$ respectively as $t_H$ and $t_C$, and for any homomorphism $L' \to L$ of
topological groups will write $\rho = \rho_1^+$ for the functorially induced dga maps $C^*(WL) \to C^*(WL)$ and $H^*(BL) \to H^*(BL')$. Then the candidate quasi-isomorphism

$$\Theta : B(H^*(BK), H^*(BG), H^*(BH)) \to B(C^*(WK), C^*(WG), C^*(WH))$$

is defined as the composition of the cochain maps

$$\begin{array}{ccc}
H^*(BK) \otimes H^*(BG) \otimes H^*(BH) & \overset{B(\lambda, id, \lambda_H)}{\longrightarrow} & C^*(WK) \otimes B(H^*(BG) \otimes C^*(WH)) \\
C^*(WK) \otimes B(H^*(BG)) \otimes C^*(WH) & \overset{B(id, \lambda, id)}{\longrightarrow} & C^*(WK) \otimes B(C^*(WG) \otimes C^*(WH))
\end{array}$$

given respectively, from beginning to end, by Lemmas 1.26, 1.22, and 1.21, where the twisting cochain homotopies $h^K : BH^*(BG) \to C^*(WK)$ and $h^H : BH^*(BG) \to C^*(WH)$ figuring in the middle map come from Theorem 3.16.

**Lemma 5.6.** The map $\Theta$ defined in (5.5) satisfies

$$\Theta = (\lambda_K^+) \otimes \lambda_G \otimes (\lambda_H^+) \pmod{(t_{WK}, t_{WH})}.$$

**Proof.** As $t_{WK}$ and $t_{WH}$ by Theorem 3.16, the first factor $B(\lambda_K, id, \lambda_H)$ is congruent to $(\lambda_K^+) \otimes id \otimes (\lambda_H^+)$ modulo $(t_{WK}, t_{WH})$ by Proposition 1.26. Next, by Theorem 3.16 again, the twisting cochain homotopies $h^K$ and $h^H$ respectively send $B_{\geq 1}(BG)$ into $t_{WK}$ and $t_{WH}$, so $(\delta^L_{KH}) \otimes (\delta^R_{KH})$ is congruent to the identity modulo $(t_{WK}, t_{WH})$ by Lemma 1.22. Finally, $B(id, \lambda_C, id)$ is id $\otimes \lambda_C \otimes id$.

**Proposition 5.7.** We retain the notations $G, K, H$ from Notation 5.1 and $\Theta$ from (5.5).

(i) The induced map

$$H^*(\Theta) : Tor^*_{H^*(BG)}(H^*(BK), H^*(BH)) \to Tor^*_{C^*(WK)}(C^*(WK), C^*(WH))$$

of graded $k$-modules is an isomorphism.

(ii) The Eilenberg–Moore spectral sequence for the pullback square (4.2) collapses at the $E_2$ page.

**Proof.** (i) Because $H^*(BG)$ and $H^*(BK)$ are flat over $k$, under the length filtration of two-sided bar constructions discussed in Definition 1.23, the $E_2$ page of the target under the map of associated filtration spectral sequences induced by $\Theta$ is again $Tor^*_{H^*(BG)}(H^*(BK), H^*(BH))$, as noted in Remark A.29. Since $\lambda_G$, $(\lambda_K^+)$, and $(\lambda_H^+)$ are each quasi-isomorphisms, by Lemma 5.6, the map of $E_2$ pages is the identity map. These are half-plane spectral sequences with exiting differentials and in the associated exact couples $(E_1, A_1)$ one has $A_1^p = 0$ for $p > 0$, and particularly $\lim_{\longrightarrow} A_1^0 = 0$, so they are strongly convergent [Bo99, Thm. 6.1(a)] and hence $H^*(\Theta)$ is a graded $k$-linear isomorphism [Bo99, Thm. 5.3].

(ii) In the map of spectral sequences, the codomain is the Eilenberg–Moore spectral sequence of the homotopy pullback of the diagram $BK \leftarrow BG \to BH$. We have seen the spectral sequence map is a $k$-linear graded isomorphism from $E_2$ on, so it is enough to show the domain spectral sequence collapses, but $E_2$ of this domain sequence is already isomorphic to the sequence's target $H^*B(H^*(BK), H^*(BG), H^*(BH))$ as a graded $k$-module. 

\[\square\]
To show $H^*(\Theta)$ is multiplicative and natural will involve the formality map of Section 4, and particularly from here on out, we will need 2 to be a unit in $k$. We will soon specialize to maximal tori, but for now let arbitrary compact tori $T_K$ and $T_H$ and simplicial group homomorphisms $\alpha_K: \overline{T_K} \to \overline{K}$ and $\alpha_H: \overline{T_H} \to \overline{H}$ be given, and choose formality maps $f_{T_K}: \text{C}^*(\overline{W T_K}) \to H^*(BT_K)$ and $f_{T_H}: \text{C}^*(\overline{W T_H}) \to H^*(BT_H)$ as guaranteed by Theorem 3.19, recalling that these maps respectively annihilate the ideals $t_{W T_K}$ and $t_{W T_H}$ defined in Definition 3.14. As $f$ and $\rho$ are hga and hence dga maps, Lemma 1.20 provides a cochain map of two-sided bar constructions

$$\Psi_{\alpha_K, \alpha_H}: \text{C}^*(\overline{W K}) \otimes \text{BC}^*(\overline{W G}) \otimes \text{C}^*(\overline{W H}) \xrightarrow{f^p \otimes \text{id} \otimes f^q} H^*(BT_K) \otimes \text{BC}^*(\overline{W G}) \otimes H^*(BT_H). \quad (5.8)$$

**Lemma 5.9.** Let $G, K, H$ be as in Notation 5.1, $\Theta$ as in (5.5), and $\Psi$ as in (5.8).

(i) The cochain map $\Psi$ is multiplicative with respect to the product $\tilde{\mu}$ of Theorem A.1.

(ii) The composite cochain $\Psi \Theta$ is equal to

$$\rho^K_{T_K} \otimes \lambda_G \otimes \rho^H_{T_H}: H^*(BK) \otimes \text{BH}^*(BG) \otimes H^*(BH) \to H^*(BT_K) \otimes \text{BC}^*(\overline{W G}) \otimes H^*(BT_H).$$

(iii) If $\alpha_K$ and $\alpha_H$ are inclusions of maximal tori, the induced map $H^*(\Psi)$ in cohomology is injective.

**Proof.** (i) Multiplicativity of $\Psi$ follows from naturality of $\tilde{\mu}$ since $f$ and $\rho$ are hga maps.

(ii) The restriction $\rho^K_{T_K}$ sends $t_{W K}$ to $\overline{W T_K}$ by Lemma 3.15, while $f_K$ annihilates $t_{W T_K}$ by Theorem 3.19, and similarly $f_H\rho^H_{T_H}$ annihilates $t_{W T_H}$. As $\Theta$ is congruent to $(\lambda_K)^+ \otimes \lambda_G \otimes (\lambda_H)^+$ modulo $(t_{W K}, t_{W T})$ by Lemma 5.6, we then have

$$\Psi \Theta = (f^p \otimes \text{id} \otimes f^q) \Theta = f^p(\lambda_K)^+ \otimes \lambda_G \otimes f^q(\lambda_H)^+.$$

But taking cohomology of the cochain maps

$$H^*(BK) \xrightarrow{(\lambda_K)^+} \text{C}^*(\overline{W K}) \xrightarrow{\rho} \text{C}^*(\overline{W T_K}) \xrightarrow{f} H^*(BT_K)$$

yields

$$H^*(BK) \xrightarrow{\text{id}} H^*(BK) \xrightarrow{\rho} \text{C}^*(BT_K) \xrightarrow{\text{id}} H^*(BT_K)$$

and the differentials on $H^*(BK)$ and $H^*(BT_K)$ are zero, implying the cochain map $f_K \circ \rho^K_{T_K} \circ (\lambda_K)^+$ is itself $\rho^H_{T_K}$, and similarly for $\rho^H_{T_H}$.

(iii) We factor $\Psi$ as $(f \otimes \text{id} \otimes f)(\rho \otimes \text{id} \otimes \rho)$ and show both factors induce injections in cohomology. As $f \otimes \text{id} \otimes f$: $\text{B}(\text{C}^*(\overline{W T_K}), \text{C}^*(\overline{W G}), \text{C}^*(\overline{W T_H})) \to \text{B}(H^*(BT_K), C^*(\overline{W G}), H^*(BT_H))$ induces the identity map between the $E_2$ pages of the associated filtration spectral sequences, it is a quasi-isomorphism. As for $\rho \otimes \text{id} \otimes \rho$, consider the map of Serre spectral sequences induced by the map of (vertical) fibrations

$$
\begin{array}{ccc}
T_K G_{T_H} & \to & BT_K \times BT_H \\
\downarrow & & \downarrow \\
K G_H & \to & BK \times BH.
\end{array}
$$
The homotopy fiber of both fibrations is $K/T_K \times H/T_H$. Thus the $E_2$ page of the right spectral sequence is concentrated in even degree, implying the sequence collapses, and the map of spectral sequences implies the left spectral sequence collapses as well, and it follows that the left fibration induces an injection in cohomology. But by the naturality clause of Proposition 4.8 this injection is $H^*(\rho \otimes \text{id} \otimes \rho)$.

**Theorem 5.10.** The isomorphism $H^*(\Theta)$ of Proposition 5.7 is multiplicative.

**Proof.** Since we know from Lemma 5.9 that $H^*(\Psi)$ is injective and multiplicative it will be enough to show the map $\Psi \Theta = \rho \otimes \lambda_G \otimes \rho$ is multiplicative up to homotopy. As $H^*(BK)$ and $H^*(BH)$ are cdgas, the hga operations $E_k$ are zero for $k \geq 0$ by Convention 2.17, so the product $\tilde{\mu}$ of (A.6) reduces to $d'[a]a'' \cdot b'[b]b'' = \pm d'b' \otimes [a] \otimes a''b''$, which is just the tensor permutation $\Pi$ of (A.4) rearranging the factors in the correct order followed by the coordinatewise product $\mu^{\otimes 3}$. Recalling from the proof of Theorem A.1 that $\Pi$ gives a natural cochain isomorphism from the tensor-square of the two-sided bar construction to the two-sided twisted tensor product with respect to the twisting cochains $\ell'$ and $\ell''$ given by $\rho^{\otimes 2}(t \otimes \eta \varepsilon + \eta \varepsilon \otimes t)$, we can transfer the desired multiplicity of $\Psi \Theta$ up to homotopy to a question about maps

$$H^*(BK)^{\otimes 2} \otimes (BH^*(BG))^{\otimes 2} \otimes H^*(BH)^{\otimes 2} \longrightarrow H^*(BK) \otimes \text{BC}^*(\text{WG}) \otimes H^*(BT_B);$$

we want to find a homotopy between the cochain maps

$$(\rho \otimes \lambda_G \otimes \rho)(\mu \otimes \mu_{BH^*(BG)} \otimes \mu) = \rho \mu \otimes \lambda_G \mu_{BH^*(BG)} \otimes \rho \mu$$

and

$$(\mu \otimes \mu_{\text{BC}^*(\text{WG})} \otimes \mu)(\rho^{\otimes 2} \otimes \lambda_G^{\otimes 2} \otimes \rho^{\otimes 2}) = \mu \rho^{\otimes 2} \otimes \mu_{\text{BC}^*(\text{WG})} \lambda_G^{\otimes 2} \otimes \mu \rho^{\otimes 2}.$$ 

On tensor factors, we have $\rho \mu = \mu \rho^{\otimes 2}$ because $\rho$ is a ring map, while Theorem 3.16 provides a coalgebra homotopy $H_\mu : \lambda_G \beta_{H^*(BG)} \simeq \Phi(\lambda_G \otimes \lambda_G)$ such that $t_{\text{C}H_\mu}$ takes $B_{\geq 1}(H^*(BG)^{\otimes 2})$ into $t_{\text{WG}}$, so that the twisting cochains $f^\rho t_{\text{C}}$ annihilate the image of $H_\mu$. As the shuffle map $\nabla$ is a natural transformation $\otimes \longrightarrow \otimes$ of bifunctors, we may append the square of Proposition 3.6:

$$\begin{array}{ccc}
(BH^*(BG))^{\otimes 2} & \longrightarrow & B(H^*(BG)^{\otimes 2}) \longrightarrow B(H^*(BG)) \\
\lambda_G \otimes \lambda_G & \searrow & \lambda_G \otimes \lambda_G & \searrow & \lambda_G \\
(B\text{C}^*(\text{WG}))^{\otimes 2} & \longrightarrow & B(C^*(\text{WG})^{\otimes 2}) \longrightarrow \text{BC}^*(\text{WG}).
\end{array}$$

Recalling from Theorem 3.11 that $\Phi \circ \nabla = \mu_{\text{BC}^*(\text{WG})}$ and from Example 1.13 that $B_{\mu} \circ \nabla = \mu_{BH^*(BG)}$, we then see $H_\mu \nabla$ is a dgc homotopy from $\lambda_G \mu_{BH^*(BG)}$ to $\mu_{\text{BC}^*(\text{WG})}^{\lambda_G^{\otimes 2}}$. As $t_{\text{C}H_\mu} \nabla$ takes the coaugmentation coideal of $(BH^*(BG))^{\otimes 2}$ into $t_{\text{WG}}$, we may apply Lemma 1.20 to see that $f^\rho \otimes H_\mu \nabla \otimes f^\rho$ is a homotopy doing what we wanted.

This establishes the ring isomorphism of Theorem 0.1. It remains to show this isomorphism is natural in inclusion diagrams.
**Theorem 5.11.** Given a diagram

\[
\begin{array}{ccc}
K_1 & \longrightarrow & G_1 \\
\downarrow & & \downarrow \\
K_0 & \longrightarrow & G_0 \\
\end{array}
\]

of continuous group homomorphisms such that the cohomology of each of the groups’ classifying spaces is a polynomial ring over \(k\) and given cocycle representatives for an irredundant set of polynomial generators of each cohomology ring, so as to define quasi-isomorphisms \(\Theta_1\) and \(\Theta_0\) as in (5.5), the following induced square commutes:

\[
\begin{array}{ccc}
\text{Tor}^\bullet_{H^\bullet(BG_0)}(H^\bullet(BK_0), H^\bullet(BH_0)) & \longrightarrow & \text{Tor}^\bullet_{H^\bullet(BG)}(H^\bullet(BK_1), H^\bullet(BH_1)) \\
\downarrow & & \downarrow \\
H^\bullet(G_0/H_0) & \longrightarrow & H^\bullet(K_1/(G_1/H_1))
\end{array}
\]

**Proof.** Let \(i_{K_1} : T_{K_1} \longrightarrow K_1\) and \(i_{H_1} : T_{H_1} \longrightarrow H_1\) be inclusions of maximal tori. We expand the description of \(\Psi \Theta\) in Lemma 5.9 to a diagram of two-sided bar constructions

\[
\begin{array}{ccc}
H^\bullet(BG_0) \otimes BH^\bullet(BG_0) \otimes H^\bullet(BH_0) & \longrightarrow & H^\bullet(BK_1) \otimes BH^\bullet(BG_1) \otimes H^\bullet(BH_1) \\
\downarrow & & \downarrow \\
\Theta_0 & \longrightarrow & \Theta_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
C^\bullet(WK_0) \otimes BC^\bullet(WG_0) \otimes C^\bullet(WH_0) & \longrightarrow & C^\bullet(WK_1) \otimes BC^\bullet(WG_1) \otimes C^\bullet(WH_1) \\
\downarrow & & \downarrow \\
C^\bullet(WT_{K_1}) \otimes BC^\bullet(WT_{G_1}) \otimes C^\bullet(WT_{H_1}) & \longrightarrow & C^\bullet(WT_{K_1}) \otimes BC^\bullet(WT_{G_1}) \otimes C^\bullet(WT_{H_1}) \\
\downarrow & & \downarrow \\
H^\bullet(BT_{K_1}) \otimes BC^\bullet(WT_{G_1}) \otimes H^\bullet(BT_{H_1}) & \longrightarrow & H^\bullet(BT_{K_1}) \otimes BC^\bullet(WT_{G_1}) \otimes H^\bullet(BT_{H_1})
\end{array}
\]

Note carefully the asymmetry in the bottom square: on the left we have \(G_0\) rather than \(G_1\) but the tori \(T_{K_1}\) and \(T_{H_1}\) are subtori of \(G_1\) rather than \(G_0\). The composite \(\rho \otimes \text{id} \otimes \rho\) after \(\Theta\) in the right column is the map \(\Psi_{j_{K_1},j_{H_1}}\) of (5.8) and the composite after \(\Theta\) the first column is \(\Psi_{j_{K_1},j_{H_1}}\), where \(j_K\) and \(j_H\) are respectively the compositions \(T_{K_1} \hookrightarrow K_1 \rightarrow K_0\) and \(T_{H_1} \hookrightarrow H_1 \rightarrow H_0\).

Our goal is to show the top square commutes in cohomology, but as \(H^\bullet(\Psi_{j_{K_1},j_{H_1}})\) is injective by Lemma 5.9(iii), it will suffice to postcompose \(\Psi_{j_{K_1},j_{H_1}}\) to the two paths around the square and show the resulting maps induce the same map in cohomology. But one confirms tensor factor by tensor factor that the bottom two squares commute, so it will be enough to find a cochain homotopy between the two paths along the outer rectangle. By Lemma 5.9(ii), the composite along the upper right is

\[
(\rho_{T_{K_1}} \otimes \lambda_{G_1} \otimes \rho_{T_{H_1}})(\rho_{T_{K_1}} \otimes B\rho_{G_0} \otimes \rho_{T_{H_1}}) = \rho_{T_{K_1}} \otimes \lambda_{G_1} B\rho_{G_0} \otimes \rho_{T_{K_1}}
\]
and the composite along the lower left is

$$
(id \otimes B\rho_{G_1}^G \otimes id)(\rho_{T_{G_1}}^{K_0} \otimes \lambda_{G_0} \otimes \rho_{T_{H_1}}^{H_0}) = \rho_{T_{G_1}}^{K_0} \otimes B\rho_{G_1}^G \lambda_{G_0} \otimes \rho_{T_{H_1}}^{H_0}.
$$

(5.15)

But Theorem 3.16 provides a dgc homotopy $H_t$ between $\rho_{G_1}^G \lambda_{G_0}$ and $\lambda_{G_1} \rho_{G_1}^G$ such that $tCH_0$ takes $B_{g_1}^H(BG_0)$ into $\mathbb{W}_{G_1}$, so that the two twisting cochains $f \rho \in C$ defining the common codomain of (5.14) and (5.15) annihilate the image of $H_p$. Applying Lemma 1.20, we see $f \rho \otimes H_0 \otimes f \rho$ is the sought-after homotopy between (5.14) and (5.15).

**Corollary 5.16.** The isomorphism $H^*(\Theta)$ of Theorem 0.1 does not depend on the choice of representatives defining $\Theta$ in (5.5).

**Proof.** One can take the vertical maps to each be the identity in (5.12), defining $\Theta_0$ in terms of one set of representatives and $\Theta_1$ in terms of another. \hfill \square

**Historical Remarks 5.17.** The proof in this section takes its inspiration from many sources. Paul Baum’s thesis [Baum] was the first work to attempt to compute the cohomology of a homogeneous space in terms of the Eilenberg–Moore spectral sequence, but suffered from a gap that required a downsizing of the original statement. Baum’s proof made critical use of the fact that the Eilenberg–Moore spectral sequence computing $G/H$ collapsed if and only if that of $G/T$ did, for $T$ a maximal torus of $H$ [Baum68, Lem. 7.2], a recurring feature in subsequent collapse proofs.

For real coefficients, existing results of Cartan [Ca51] and Borel [Bor53, Thm. 25.1] proved the collapse through use of functorial commutative cochain models allowing dga maps between cochains and cohomology. Baum and Smith [BaumS67] and later Wolf [Wolf78] were able to obtain further collapse results over $\mathbb{R}$ along these lines, but such models cannot exist in positive characteristic [Bor, Thm. 7.1].

In 1967, May announced a proof of the collapse of the Eilenberg–Moore spectral sequence associated to a fibration $X \to BH \to B$ where $B$ has polynomial cohomology and $H^*(BH)$ is polynomial on even-degree generators [May68], with a weaker but fairly inclusive variant result when $k = \mathbb{F}_2$. The proof relied on an explicit description of the differentials in the algebraic Eilenberg–Moore spectral sequence in terms of matric Massey products and a dga quasi-isomorphism as described in the following paragraph. No resolution of the additive extension problem recovering the graded $k$-module $H^*(X)$ from $E_\infty$ was claimed. Owing to its technical complexity relative to later versions of the proof, this original version was never published.

In the published version [GuM], Gugenheim and May describe a dga quasi-isomorphism $f: C^*(BT) \to H^*(BT)$ annihilating cup-1 products and note that this $f$ induces an isomorphism $\text{Tor}_{C^*(BG)}(k, C^*(BT)) \to \text{Tor}_{C^*(BG)}(k, H^*(BT))$. They then define a differential $C^*(BG)$-module resolution of $k$ roughly as follows. First, they select cochains $x_j$ in $C^*(BG)$ representing polynomial generators of $H^*(BG)$ and form an impostor Koszul complex $C^*(BG) \otimes \Lambda[\mathbb{Z}]$ equipped with an attempted differential restricting to the traditional $\delta$ on $C^*(BG)$, sending exterior generators $z_j$ to $x_j$, and otherwise extended to be a derivation. Owing to the noncommutativity of $C^*(BG)$, this candidate does not satisfy $d^2 = 0$, but they rectify this by modifying $d$ in exterior degree $\geq 2$, using cup-1 products to compensate for the lacking commutativity.\footnote{This is a bit like defining an $A_\infty$-map $\lambda: H^*(BG) \to C^*(BG)$ and using $\lambda_{(1)}$ to consider the complex $C^*(BG) \otimes H^*(BG) \otimes H^*(BG) \otimes \Lambda[\mathbb{Z}]$, which we will see is essentially what Joel Wolf does.}

Then $\text{Tor}_{C^*(BG)}(k, H^*(BT))$ can be computed as the cohomology of

$$
H^*(BT) \otimes_{C^*(BG)} C^*(BG) \otimes \Lambda[\mathbb{Z}] \cong H^*(BT) \otimes \Lambda[\mathbb{Z}],
$$

\hfill \square
where $H^*(BT)$ is a $C^*(BG)$-module via $C^*(BG) \xrightarrow{\ell} C^*(BT) \xrightarrow{f} H^*(BT)$. But because the differential on $C^*(BG) \otimes \Lambda[Z]$ differs from the Koszul differential only by cup-1 products and $f$ annihilates cup-1 products, the differential on $H^*(BT) \otimes \Lambda[Z]$ agrees with the differential on the Koszul complex $H^*(BT) \otimes H^*(BG) \otimes \Lambda[Z]$ computing $\text{Tor}_{H^*(BG)}(k, H^*(BG))$. May and Neumann later observed that the same proof applies equally to yield the cohomology groups of a so-called generalized homogeneous space [MayNo2]; see Remark 4.11.

Of course maps in the other direction, $H^*(BG) \rightarrow C^*(BG)$, cannot be dga maps when $H^*(BG)$ requires more than one algebra generator. Up-to-homotopy multiplication operations arose in the topological setting in work of Sugawara [Su60] determining when the classifying space of a topological monoid is again an H-space in terms of a notion of strong homotopy commutativity, and were transmuted into the notion of an $A_{\infty}$-algebra map by Clark [Cl65] and connected to $A_{\infty}$- and more generally $A_n$-algebras by Stasheff [Sta63a, Sta63b]. The notion of an $s\text{hc}$-algebra is due to Stasheff–Halperin [StaH70] who used it to construct the $\lambda$ of (3.9). Using these maps they could propose a strategy to demonstrate the collapse, but they could not carry this program through, essentially because they were unable to conclude (p. 573) that the left square of (3.17) was homotopy-commutative for $X_* = \text{Sing } BG$ and $Y_* = \text{Sing } BH$.

Munkholm solved this problem [Mu74, §7.4], as noted in the introduction, further developed the notion of $s\text{hc}$-algebras, defined the internal tensor product $\boxtimes$ of maps, and used the Eilenberg–Zilber theorem to produce a natural $s\text{hc}$-algebra structure on singular cochains. His approach makes no special appeal to tori and requires no formality map, and applies not just to the Eilenberg–Moore spectral sequence of the fibration $G/H \rightarrow BH \rightarrow BG$ but that computing the cohomology of a pullback $X \times_B E$. The homotopy for the left square of (3.17) is created using the $s\text{hc}$-algebra structure to combine homotopies for subalgebras on one generator, which Munkholm classifies completely, showing there is no obstruction to their existence except in characteristic 2, where one needs $x \sim_{-1} x$ to vanish on polynomial generators $x$ of $H^*(BH)$ and $H^*(BK)$. Munkholm uses his homotopy to construct a map of Tors much like our $\Theta$, given first by left/right maps, then a homotopy, then the central map [Mu74, Thm. 5.4]. (Another map of Tors factored in this order occurs in work of Gugenheim–Munkholm [GuMu74, Thm. 3.7.2].)

The most direct spiritual ancestor of our proof may be that of Wolf [Wolf77, p. 331], who examines the composite of maps

$$
\begin{array}{ccc}
\text{BH}^*(BG) & \xrightarrow{\lambda_G} & \text{BH}^*(BT), \\
\downarrow \lambda_C & & \downarrow Bf \\
\text{BC}^*(BG) & \xrightarrow{B\rho} & \text{BC}^*(BT)
\end{array}
$$

which should be compared with the left square of (3.17). Wolf constructs $\lambda_C$ so that the terms $(\lambda_C)_{(n)}$ for $n \geq 2$ are cup-1 products annihilated by $f$, so the twisting cochain associated to the dotted composite equals $I_{H^*(BT)} \text{BH}^*(\rho)$, and by uniqueness, that composite is $\text{BH}^*(\rho)$; compare the proof of Lemma 5.9(ii). Thus pulling back along $\lambda_C$ and then pushing forward along $Bf$ gives an isomorphism $\text{Tor}_{C^*(BG)}(k, C^*(BT)) \rightarrow \text{Tor}_{H^*(BG)}(k, H^*(BT))$. Unlike Gugenheim–May, Wolf considers maps of Tors defined by maps of two-sided bar constructions induced by $A_{\infty}$-maps, but as with Gugenheim–May, the complications caused by pulling back along $\lambda_C$ in transitioning from $\text{Tor}_{C^*(BG)}(k, C^*(BT))$ to $\text{Tor}_{H^*(BG)}(k, C^*(BT))$ are erased by a formality map simplifying
the resolution used to compute $\text{Tor}_{H^*(BG)}(k, H^*(BT))$. Morally speaking, the same commutative square underlies both proofs.

Our product (A.6) on a two-sided bar construction of extended hgas is inspired by a similar product of Kadeishvili–Saneblidze inducing a dga structure on the one-sided bar construction, as applied to fibrations [KaS05]. Franz [Fr19] had the idea to combine (1) Wolf’s strategy of obtaining a multiplicative map by following a nonmultiplicative map with a formality map annihilating the error and (2) Kadeishvili and Saneblidze’s product on the one-sided bar construction showing hga maps induce multiplicative maps of Tors. The key technical points in that project were to define the shc-algebra structure and the formality map in such a way that Wolf’s strategy is applicable; the direct verification that this shc-algebra structure satisfies the associativity axiom is particularly onerous.

The present paper was motivated by the desire to do for homotopy biquotients what Franz did for homogeneous spaces, and particularly generalize the result of Singhof [Si93, Thm. 2.4], which uses Munkholm’s collapse result to compute the additive structure of $H^*(K\backslash G/H)$.

Appendix (with Matthias Franz)

A. A product on the two-sided bar construction

Given that an hga structure defines a dga structure on the bar construction of a dga, one might hope hga homomorphisms $A' \leftarrow A \rightarrow A''$ similarly induce a dga structure on the two-sided bar construction. We cannot assert this, but can at least define a non-associative product sufficient to prove the variant Eilenberg–Moore theorem A.27 needed for the proof of Theorem 0.1.

Theorem A.1. Let $A' \xleftarrow{f'} A \xrightarrow{f''} A''$ be hga homomorphisms. Then there exists a cochain map

$$\mu: B(A', A, A'') \otimes B(A', A', A'') \longrightarrow B(A', A, A''),$$

which we think of as a product, natural in the sense that given a commutative diagram

$$\begin{array}{ccc}
A' & \leftarrow & A \\
\downarrow{g'} & & \downarrow{g} \\
B' & \leftarrow & B
\end{array}
\quad
\begin{array}{ccc}
A'' & \rightarrow & A \\
\downarrow{g''} & & \downarrow{g} \\
B'' & \rightarrow & B
\end{array}
\quad
\begin{array}{ccc}
A' & \leftarrow & A'' \\
\downarrow{g'} & & \downarrow{g''} \\
B' & \leftarrow & B''
\end{array}
$$

of hga homomorphisms, the induced map $B(g', g, g''): B(A', A, A'') \longrightarrow B(B', B, B'')$ of Notation 1.33 is multiplicative.

Remark A.2. It requires the extended hga operations $F_{k, \ell}$ to define the natural homotopy $h^c$ witnessing the homotopy-commutativity axiom for Franz’s natural shc-algebra structure map $\Phi_A$ described in Theorem 3.11, but to define $\Phi_A$ itself and show that $\Phi_A \circ \nabla = \mu_{BA}$, which is all we require here, one needs only that $A$ be an hga.

Proof of Theorem A.1. By Theorem 3.11, there exists a dgc map $\Phi_A: B(A \otimes A) \longrightarrow BA$ giving an shc-algebra structure on $A$, and similarly for $A'$ and $A''$. We set

$$\mu := B(\Phi_A', \Phi_A, \Phi_A'') \circ (\text{id}_{A' \otimes A'} \otimes \nabla \otimes \text{id}_{A'' \otimes A''}) \circ \Pi,$$

(A.3)
where the $\mathbf{B}(-,-,-)$ maps are as defined in Proposition 1.26 and the initial map

$$\Pi: \mathbf{B}(A', A, A'') \otimes \mathbf{B}(A', A, A'') \longrightarrow (A' \otimes A') \otimes (\mathbf{B}A \otimes \mathbf{B}A) \otimes (A'' \otimes A'')$$

(A.4)

is the tensor permutation (2 3 5 4). Here the two-sided twisted tensor product is determined by the twisting cochains

$$t' = (f')^{\otimes 2}t_A \otimes \nabla = (f')^{\otimes 2}(t_A \otimes \eta_{A\varepsilon_B} + \eta_{A\varepsilon_B} \otimes t_A): (\mathbf{B}A)^{\otimes 2} \longrightarrow (A')^{\otimes 2},$$

$$t'' = (f'')^{\otimes 2}t_A \otimes \nabla = (f'')^{\otimes 2}(t_A \otimes \eta_{A\varepsilon_B} + \eta_{A\varepsilon_B} \otimes t_A): (\mathbf{B}A)^{\otimes 2} \longrightarrow (A'')^{\otimes 2}$$

for $\nabla$ the shuffle map of Example 1.13, and it is not hard to check $\Pi$ is a cochain map. To see

$$\text{id} \otimes \nabla \otimes \text{id}: (A' \otimes A')_{t'_{\otimes 2}}(\mathbf{B}A \otimes \mathbf{B}A) \otimes (A'' \otimes A'') \longrightarrow (A' \otimes A')_{t''_{\otimes 2}} \mathbf{B}(A \otimes A) \otimes (A'' \otimes A'')$$

is a cochain map, it is enough by Lemma 1.20 to observe $t' = (f' \otimes f')t_A \otimes A \nabla = t_A \otimes A_B (f' \otimes f') \nabla$ and similarly $t'' = t_A \otimes A_B (f'' \otimes f'') \nabla$. To see

$$\mathbf{B}(\Phi_{A'}, \Phi_A, \Phi_{A''}) : \mathbf{B}(A' \otimes A', A \otimes A, A'' \otimes A'') \longrightarrow \mathbf{B}(A', A, A'')$$

is a well-defined cochain map, by Proposition 1.26 it is enough to note the diagram

commutes by the naturality of $\Phi$ in hgas.

Naturality follows because $\Pi$ is natural in sextuples of cochain complexes, $\nabla$ is natural in pairs of cochain complexes, $\Phi$ is natural in extended hga maps, and $\mathbf{B}(-,-,-)$ is functorial in triples of dga maps. $\square$

To make real use of this product, we will require a more explicit formula.

**Theorem A.5.** The cochain map $\tilde{\mu}$ of Theorem A.1 is given in terms of the notations set in Definition 2.1 and Notation 2.3 by

$$a' \otimes a'' \otimes b'[b_\bullet]b'' \longrightarrow \pm a' b' \otimes [a_\bullet] \otimes \mathbb{C}(a''[f''b(3)]) b'' + \pm a' \mathbb{C}(f' a_1 [b'[f'(1)]) \otimes [a_\bullet] \otimes [b_\bullet] \otimes \mathbb{C}(a''[f''b(3)]) b''$$

(A.6)

where the first term is the sum over all decompositions $[b_\bullet] = [b(2)] \otimes [b(3)]$ into two tensor factors and the second is the double sum over decompositions $[a_\bullet] = [a(1)] \otimes [a(2)]$ where the first tensor factor is of length 1 and decompositions $[b_\bullet] = [b(1)] \otimes [b(2)] \otimes [b(3)]$ into three tensor factors, and where we recall...
from Remark 2.2 our convention that \( \mathcal{E}(f' a_1 | b' | f b_1) = 0 \) when \( b' = 1 \). The signs are those imposed by the Koszul convention; explicitly, the first sum is

\[
T_0 = \left( \mu_{A'} \otimes \mu_{BA} \otimes \mu_{A''} \left( \mathcal{E}(id_{A''} \otimes B f'^{\prime} \otimes id_{A'}) \right) \tau_{[b(3)];a''} \right) \circ \left( id_{A''} \otimes A' \otimes id_{BA} \otimes \Delta_{BA} \otimes id_{A''} \otimes A'' \right) \Pi,
\]

for \( \tau_{[b(3)];a''} \) the tensor shuffle taking the factors \( [b(3)] \otimes a' \) to \((-1)^{|[b(3)]| - |a'|} a' \otimes [b(3)]\) and \( \Pi \) the permutation \((2 \ 3 \ 5 \ 4)\) from (A.4), and the second is

\[
T_1 = \left( \mu_{A'} \left( \id_{A'} \otimes \mathcal{E}(f' \otimes s^{-1} \otimes B f') \right) \tau_{[a(1)]} \otimes \mu_{BA} \otimes \mu_{A''} \left( \mathcal{E}(id_{A''} \otimes B f'' \otimes id_{A'}) \right) \tau_{[b(3)];a''} \right)
\]

\[
\circ \left( id_{A'} \otimes \id_{A'} - \epsilon_{A'} \right) \otimes \left( \left( \pr_1 \otimes id_{BA} \right) \otimes \left( id_{BA} \otimes id_{BA} \right) \otimes \left( \epsilon_{BA} \otimes id_{BA} \right) \right) \Delta_{BA} \otimes \Delta_{BA} \otimes id_{A''} \otimes A'' \right) \Pi,
\]

where \( \tau_{[a(1)]} \) is again a tensor transposition.

Concatenations \( \Delta_\sigma \) (as defined in 1.6.3) have been omitted in the statement and proof to aid legibility (so far as possible).

**Proof.** We first establish the formula modulo 2. The composition of the first two maps in (A.3) takes a pure tensor \( a'[a_\bullet] \otimes b'[b_\bullet] \) to \((a' \otimes b') \otimes \nabla ([a_\bullet] \otimes [b_\bullet]) \otimes (a'' \otimes b'')\). Recall \( \nabla ([a_\bullet] \otimes [b_\bullet]) \) is the signed sum of terms \([a \otimes \beta]_* := [a_1 \otimes \beta_1] \cdots [a_\ell \otimes \beta_\ell] \) where each \( a_j \otimes \beta_j \) is either \( a_m \otimes 1 \) or \( 1 \otimes b_m \) for some \( m \) and the indices of the \( a \)-arguments appear in increasing order as one encounters them from left to right, as do the indices of the \( b \)-arguments. The map \( B(\Phi_{A'}, \Phi_{A''}, \Phi_{A''}) \) first breaks each term \([a \otimes \beta]_* \) into three factors \([a \otimes \beta]_{(1)} = [a_1 \otimes \beta_1] \cdots [a_p \otimes \beta_p], [a \otimes \beta]_{(2)}, [a \otimes \beta]_{(3)} \) via \( \Delta_{BA}^{[3]} \), and applies \( f' \) to each \( a \) and \( \beta \) in the first block and \( f'' \) to each in the third block. The verification for each of the three tensor factors will run in parallel.

Now there is a case distinction to be made. Write \( a'_j = f' a_j \) and so on. If \( a' \otimes b' \in A' \otimes A' \) and \( a'' \otimes b'' \in A'' \otimes A'' \), then by (1.27), \( B(\Phi_{A'}, \Phi_{A''}, \Phi_{A''}) \) takes \((a' \otimes b') \otimes \nabla ([a_\bullet] \otimes [b_\bullet]) \otimes (a'' \otimes b'')\) to

\[
\pm t_{A'} \Phi_{A'}[a' \otimes b' | a'_1 \otimes \beta'_1 | \cdots | a'_p \otimes \beta'_p] \otimes [a \otimes \beta]_{(2)} \otimes t_{A''} \Phi_{A''}[a'' \otimes \beta''_1 | \cdots | a''_q \otimes \beta''_q] \otimes (a'' \otimes b''),
\]

where \( p + \ell([a \otimes \beta]_{(2)}) + q = \ell(a_\bullet) + \ell(b_\bullet) \). If instead \( a' \otimes b' \in \im \eta_{A' \otimes A'} = k \), then the associated clause \( \Upsilon' = \id_k \otimes \epsilon \) from (1.27) applies instead, so the first tensor factor in (A.9) is replaced with \( a' \otimes b' \), and \( p = 0 \); similarly, if \( a'' \otimes b'' \in \im \eta_{A'' \otimes A''} \), then instead the third tensor factor in (A.9) is \( a'' \otimes b'' \), and \( q = 0 \). In either case, these simpler factors agree with those given in (A.6), (A.7), and (A.8) by our observations and conventions from Remark 2.2 on the vanishing of \( E \), so for the remainder of the verification we may assume \( a' \otimes b' \in A' \otimes A' \) and \( a'' \otimes b'' \in A'' \otimes A'' \).

At this point it becomes necessary to recall the unsigned formula [Fr20a, (4.2)] for \( t_{A'} \Phi_{A'} \):

\[
t_{A'} \Phi_{A'}[a \otimes \beta] = \sum_{\sigma} \pm a_1 E[a_2 | \beta_\bullet] \cdots E[a_n | \beta_\bullet] \beta_n,
\]

where the sum is over those permutations \( \sigma \) separately preserving the orders of the indices of the \( a \)-arguments and those of the \( \beta \)-arguments, and such that, moreover, at every point, reading
left to right, one has encountered the symbol \( \alpha \) more often than \( \beta \). The key point in simplifying this formula in the present context is that \( E[a_1|\beta_\ast] \) vanishes whenever one of the \( \beta \)-letters other than \( \beta_n \) is 1 by the convention set in Remark 2.2.

Thus in order for \( t_A\Phi_A'(\{a' \otimes b'\} \otimes [\alpha' \otimes \beta'_\ast]_\ast) \) to be nonzero, one of two things must happen. The first possibility is that \( p = \ell[\alpha' \otimes \beta'], \ast = 0 \), so that \( t_A\Phi_A'(a' \otimes b') = \pm a'b' \). The other possibility is that \( b', \beta'_1, \ldots, \beta'_{p-1} \) are all non-1. Particularly, if \( b' = 1 \), then \( \pm a'b' \) is again the only term, agreeing with (A.6), (A.7), and (A.8), so from now on we will assume \( b' \in \mathbb{A} \). Since for each \( j \) one of \( \alpha_j' \) and \( \beta_j' \) is 1, it follows that \( \alpha_1' = \cdots = \alpha_{p-1}' = 1 \). As \( E_{1,0}(\{1\} \otimes []) \) = 1 and otherwise \( E_{1,\ast}(\{1\} \otimes [\beta_\ast]) = 0 \) by the conventions of Remark 2.2, for a term in (A.10) to be nonzero one needs \( \alpha_p' \neq 1 \), so that \( \beta_p' = 1 \). Thus the only relevant terms have

\[
[a' \otimes \beta']_\ast = [1 \otimes f'b_1] \cdots [1 \otimes f'b_{p-1}] |f'a_1 \otimes 1],
\]

\[
t_A\Phi_A'(\{a' \otimes b'\} \otimes [\alpha' \otimes \beta'_\ast]_\ast) = \pm a'E_{1,p}(\{f'a_1 \otimes [b' |f'b_{p-1}]\}),
\]

where \([f'b_{p-1}] = [f'b_1] \cdots [f'b_{p-1}] \).

Similarly, in order that \( t_A\Phi_A'(\{a'' \otimes [\alpha'' \otimes \beta''_\ast]_\ast \otimes [a'' \otimes b'']\} = 0 \) by the convention if \( a'' = 1 \). As for the middle tensor factor, we know from Theorem 3.11 that \( \Phi \circ \nabla = \mu_{BA} \). Combining our descriptions of the three tensor factors, we have established (A.6).

It remains to determine the signs. The sign for the permutation \( \Pi \) of (A.4) is by definition the Koszul permutation sign. For the composition of the second two maps, recall from (1.27) that before involving \( \Phi_A' \) and \( \Phi_A'' \), the map \( B(\Phi_A', \mu_{BA}, \Phi_A'') \) first breaks \([a \otimes \beta]_\ast = \nabla([a_\ast] \otimes [b_\ast]) \) into three chunks via \( \Delta_{BA \otimes \mathbb{A}}^{[3]} \). As \( \nabla \) is a dcc map \( BA \otimes BA \rightarrow B(A \otimes A) \) by Example 1.13, we equally well have \( \Delta_{BA \otimes \mathbb{A}}^{[3]} \nabla = \nabla \otimes \Delta_{BA \otimes \mathbb{A}}^{[3]} \). Applying the definition (1.27), suppressing concatenations \( \Delta_{\cdots} \), gives

\[
B(\Phi_A', \Phi_A, \Phi_A'')(id_{A' \otimes A'} \otimes \nabla \otimes id_{A'' \otimes A''})
= \left(t_A\Phi_A'(s_{A' \otimes A'}^{-1} \otimes B(f' \otimes f') \nabla) \otimes \Phi_A \nabla \otimes t_A\Phi_A''(B(f'' \otimes f'') \nabla \otimes s_{A'' \otimes A''}^{-1}) \right)
\circ (id_{A' \otimes A'} \otimes \Delta_{BA \otimes BA}^{[3]} \otimes id_{A'' \otimes A''})
\tag{A.11}
\]

We will evaluate \( Q \) on terms

\[
(a' \otimes b' \otimes [a_{(1)}] \otimes [b_{(1)}]) \otimes ([a_{(2)}] \otimes [b_{(2)}]) \otimes ([a_{(3)}] \otimes [b_{(3)}]) \otimes a'' \otimes b''
\]

and compare with the evaluations of \( T_0 \) and \( T_1 \) from (A.7) and (A.8), recalling from the discussion of nonzero values of \( t_A\Phi_A' \) and \( t_A\Phi_A'' \) in the previous paragraph that the only nonzero terms in (A.11) arise when \([a_{(3)}] = []\) is of length 0 and either we have \([a_{(1)}] = [b_{(1)}] = []\) of length 0 as well or else \([a_{(1)}] = [a_1]\) is of length 1.

As each of the three tensor factors of \( Q \), of \( T_0 \), and of \( T_1 \) is of degree zero, it will be enough to compare these factors separately. The middle tensor factor \( \Phi_A \nabla \) of \( Q \) is \( \mu_{BA} \) by Theorem 3.11, as
in $T_0$ and $T_1$. For the first and third factors, we will need to be explicit about the sign for $\Phi$. The signed version of formula (A.10) for $(\Phi_A)_{(n)} = t_A \circ (\Phi_A)|_{B_n A} \circ (s_A^{-1} \otimes A)^{\otimes n}$ is

$$(\Phi_A)_{(n)} := (-1)^{n-1} \sum_{\sigma} \mu_A^{|\sigma|} (id_A \otimes \bigotimes_m E_{\ell_m} \otimes id_A) \sigma,$$  \hspace{1cm} (A.12)$$

where the sum is over shuffles $\sigma$ as in (A.10), so that in particular the sequence of $(\ell_m)$ gives a partition of $n - 1$, apportioning subsequences of $\beta_1, \ldots, \beta_{n-1}$ as arguments of $E[a_2|-$] through $E[a_n|-$], and $E_\ell$ is $E_{1,\ell} (s^{-1})^{\otimes 1 + \ell}$ as in Notation 2.3; and similarly for $\Phi_{A'}$ and $\Phi_{A''}$.

To interpret Q1, note that $\ell[a(1)]$ is either 0 or 1. In the case $[a(1)] = [1]$, the only term we need to worry about is $t_{A'} \Phi_{A'} s_{A'}^{-1} \otimes A = (\Phi_{A'})_{(1)}$, which is $\mu_{A'}$ by Definition 3.7.1. This is the first factor of $T_0$ from (A.7). When $[a(1)] = [a_1]$, we have seen from the discussion above that the only terms on which $t_{A'} \Phi_{A'}$ will potentially not vanish are those of the form $[a' \otimes b'|1 \otimes f'b]_{(1)} \otimes [f'a_1 \otimes 1]$. Here the notation $[a' \otimes b'|1 \otimes f'b]_{(1)}$ means some initial subword $[a' \otimes b'|1 \otimes f'b]_1 \cdots [1 \otimes f'b]_l$, which is $[a' \otimes b']$ if $\ell = 0$.

A term of $\nabla([a_1] \otimes [b(1)])$ leading to a term of this form is created by the operation

$$u: [a_1] \otimes [b(1)] \longrightarrow [a_1 \otimes 1 | 1 \otimes b(1)]$$

followed by the shuffle $\tau = \tau[a_1,1|1 \otimes b(1)]$ moving $[a_1 \otimes 1]$ past each $[1 \otimes b_j]$, resulting in

$$x := \pm [a' \otimes b'] \otimes [1 \otimes f'b]_{(1)} \otimes [f'a_1 \otimes 1].$$

To determine the sign explicitly, (A.12) tells us we should rewrite things in terms of the operations $E_\ell$. When we apply $\Phi_{A'}$ to $x$, we have seen above that the only nonzero term is $\pm a'E_p(a_1; b', b_{(1)})$, which comes from the summand in (A.12) corresponding to the shuffle

$$\phi': \ (a' \otimes b') \otimes (1 \otimes f'b) \otimes (f'a_1 \otimes 1) \longrightarrow (-1)^{|a_1| (|b'| + |b_{(1)}|)} a' \otimes 1^{p-1} \otimes f'a_1 \otimes b' \otimes f'b_{(1)} \otimes 1.$$ 

If we write $\chi$ for the map omitting tensor factors 1, then this nonzero term can be written as

$$\mu_A' (id_{A'} \otimes E_p) \tau_{\phi' \otimes b_{(1)}; a_1} \chi(s_{A'}^{-1} \otimes A')^{\otimes p + 1} x,$$

where $\tau_{\phi' \otimes b_{(1)}; a_1}$ is the tensor shuffle moving $a_1$ past $b' \otimes b_{(1)}$. Since $(s^{-1})^{\otimes n} s^{\otimes n}$ is multiplication by $(-1)^{(\ell^2)}$, we conclude from (A.12) that on $(A' \otimes A') \otimes B_1 A \otimes B_{p, A}$, we have

$$Q_1 = t_{A'} \Phi_{A'} \circ (s_{A'}^{-1} \otimes A') \otimes B(f')^{\otimes 2} \tau \nu$$

$$= (-1)^{p(\ell^2)} (\Phi_{A'}(p+1))^{\otimes 1 + \ell} \circ (s_{A'}^{-1} \otimes A') \otimes B(f')^{\otimes 2} \tau \nu$$

$$= (-1)^{p(\ell^2)} \left((-1)^p \mu_A' (id_{A'} \otimes E_{1,\ell} (s^{-1})^{\otimes 1 + \ell}) \tau_{\phi' \otimes b(1); a_1} \chi(-1)^p (id_{A'} \otimes A') \otimes s_{A'}^{\otimes p} B(f')^{\otimes 2} \tau \nu) \right)$$

$$= (-1)^{p(\ell^2)} \mu_A' (id_{A'} \otimes E_{\ell} (id_{A'} \otimes s_{A'}^{-1}^{\otimes p})) \tau_{\phi' \otimes b(1); a_1} \chi(id_{A'} \otimes A') \otimes s_{A'}^{\otimes p} B(f')^{\otimes 2} \tau \nu),$$

where the accounting of signs is as follows: the $(-1)^{p(\ell^2)}$ comes from $(\Phi_{A'})_{(1+p)} := t_{A'} \Phi_{A'} s_{A'}^{\otimes 1 + p}$ in (A.12), the first $(-1)^p$ is the leftmost term of the right-hand side of (A.12), and the second $(-1)^p$ comes from moving $s_{A'}^{\otimes p} A'$ past $s_{A'}^{-1}$. 


We must compare the sign of the value of $Q_1$ on $([a' \otimes b'] \otimes [a_1 | b_{(1)}])$ with that of the first factor $T_{1,1} = \mu_{A'} (\text{id}_{A'} \otimes \mathcal{E}(f' s \otimes s^{-1} \otimes B f')) \tau_{b_{(1)}}$ from (A.8). This is done symbolically by determining which tensor factors are moved past which others in the computation. For these purposes, the factors $\mu_{A'}, \text{id}_{A'}, \mathcal{E}, \tau, f', B (f' \otimes f')$, and $\nu$ of degree 0 are invisible, as are the tensor factors 1 of degree 0 produced by $\nu$ and deleted by $\tau$. In the below diagram $Q_1$ is in the left column and $T_{1,1}$ in the right, and in each row of each column, operations are listed on the left and arguments on the right, the result of applying the operations appearing on the right in the following row. As usual, moving two symbols $y$ and $z$ past each other incurs the sign $(-1)^{|y||z|}$, and the rule goes equally for $y = s^{\pm 1}$.

\[
\begin{array}{c}
\tau: \quad a' b' s^{-1} \quad s \otimes p^{-1} \quad a_1 \quad [b_{(1)}] \\
\tau_{b_{(1)}}: \quad a' b' \quad s^{-1} \quad [b_{(1)}] \\
\tau_{b_{(1)}} \otimes b_{(1)}: \quad a' b' \quad s^{-1} \quad [b_{(1)}] \\
\tau_{b_{(1)}} \otimes b_{(1)}: \quad a' b' \quad s^{-1} \quad [b_{(1)}] \\
(\ast)^{(p+1)}: \quad a' b' s^{-1} \quad \ast \quad (s^{-1}) \otimes p^{-1} \quad [b_{(1)}] \\
(\ast)^{(p+1)}: \quad a' b' s^{-1} \quad \ast \quad (s^{-1}) \otimes p^{-1} \quad [b_{(1)}] \\
\end{array}
\]

In determining whether the signs agree, we can remove cancelling signs within one diagram or pairs of matching signs in both diagrams, referring to this as “cancellation” either way. Then two crossings cancel if both involve transposing the same symbols $y$ and $z$ or if one crossing transposes $s$ and $z$ while the other transposes $s^{-1}$ and $z$. After cancellation, the only signs remaining are $(-1)^{|s^{-1}||s-1|} = -1$ arising from the red crossing on the right and the underlined green  

\[
(\ast)^{(p+1)}: |(s^{-1}) \otimes p^{-1} s^{-1} | = (\ast)^{(p+1)} + (\ast)^{-1} = (\ast)^{(p+1)} = -1 
\]

at the bottom on the left, so the operations in question are indeed the same.

As for $Q_3$ and $T_{0,3}$ (from (A.11), (A.7), and (A.8)), we have seen above the terms on which the $\Phi_{A^n}$ in $Q_3$ is potentially nonzero are those of the form $[1 \otimes f'' b]_{(3)} \otimes [a'' \otimes b''']$. In particular, $[a_{(3)}] = []$. Recall that we write $q = \ell[b_{(3)}]$. We see the many 1 factors arising because $\nabla ([\otimes [b_{(3)}])] = [1 \otimes b]_{(3)}$ contribute nothing to the iterated $\mu_{A^n}$-product, so we may omit them with the map $\nu$. When they are omitted, the shuffle involved in the nonvanishing summand of (A.12) simplifies to the shuffle $\tau_{f'' b_{(3)} a''}$ switching the $A^n$ tensor factors containing $a''$ and $f'' b_{(3)}$ in $f'' b_{(3)} \otimes a'' \otimes b''$. Thus on $B_{q} A \otimes A^n \otimes A^n$, we can expand $Q_3$ as

\[
Q_3 = t_{A^n} \Phi_{A^n} \circ (B (f'' \otimes 2) \otimes s_{A^n \otimes A^n}) \\
= (-1)^{(q+1)} (\Phi_{A^n})_{(1+q)} \otimes s_{A^n \otimes A^n} \circ (B (f'' \otimes 2) \nabla \otimes s_{A^n \otimes A^n}) \\
= (-1)^{(q+1)} (-1)^q \mu_{A^n} (\mathcal{E}_1 (s_{A^n}) \otimes 1 + q \otimes \text{id}_{A^n}) \tau_{f'' b_{(3)} a''} \nu \circ (s_{A^n \otimes A^n} \otimes B (f'') \nabla \otimes \text{id}_{A^n \otimes A^n}) \\
= (-1)^{(q+1)} (-1)^q \mu_{A^n} \mathcal{E}_1 (s_{A^n \otimes (s_{A^n})} \otimes \text{id}_{A^n}) \tau_{f'' b_{(3)} a''} \nu \circ (s_{A^n \otimes A^n} \otimes B (f'') \nabla \otimes \text{id}_{A^n \otimes A^n}),
\]

where as in the analysis of $Q_1$, the $(-1)^{(q+1)}$ comes from $(\Phi_{A^n})_{q+1} = t_{A^n} \Phi_{A^n} s_{A^n \otimes A^n}^{q+1}$ and the $(-1)^q$ is the leftmost factor of the right-hand side of (A.12). We may simplify the exponent of $-1$ by
observing that
\[
\binom{q+1}{2} - q = \left(\binom{q+1}{2} - \binom{q}{1}\right) = \binom{q}{2}.
\]

In the diagram below the crossings for \(Q_3\) appear on the left and that for
\[
T_{0,3} = T_{1,3} = \mu_{A''}(\mathcal{C}(\text{id}_{A''} \otimes Bf'') \otimes \text{id}_{A''}) \tau_{[b(3)];a''}
\]
appears on the right.

When matching crossings are cancelled, the only signs remaining are the underlined \((-1)^{\frac{q}{2}}\) and \(|(s^{-1})^{\otimes q} (s^{\otimes q})| = (-1)^{\frac{q}{2}}\) on the left, so the operations are equal, concluding the proof. \(\square\)

With this more explicit formulation of \(\hat{\mu}\), we are able to relate it to the product on an HGA.

**Theorem A.13.** Given a commutative diagram

\[
\begin{align*}
A' & \xrightarrow{\phi} A' \xrightarrow{\psi} A'' \\
A & \xrightarrow{\phi} A & \xrightarrow{\psi} A''
\end{align*}
\]

of HGA homomorphisms, the natural cochain map
\[
\xi := \mu_{A}^{[3]}(g' \otimes \eta A \epsilon_{BA} \otimes g'') : B(A', A, A'') \rightarrow \tilde{A},
\]
\[
a'[a]a'' \mapsto g'(a')\eta A \epsilon_{BA}[a_*]g''(a'').
\]

induces a map in cohomology multiplicative with respect to the product on \(H^*B(A', A, A'')\) induced by the product defined by Theorem A.1 and the expected product on \(H^*(\tilde{A})\).

**Proof.** By Theorem A.1, \(B(g', g, g'')\) is a multiplicative cochain map \(B(A', A, A'') \rightarrow B(\tilde{A}, \tilde{A}, \tilde{A})\), so it will be enough to show \(\xi : B(\tilde{A}, \tilde{A}, \tilde{A}) \rightarrow \tilde{A}\) is a cochain map inducing a multiplicative map in cohomology. In other words, we may as well start by assuming \(A' = A = A'' = \tilde{A}\) and \(f' = f'' = g' = g'' = \text{id}_A\). Let us then agree to write \(\mu = \mu_A\) and \(\text{id} = \text{id}_A\) and \(\epsilon = \eta_A \epsilon_{BA}\), so we can restate (A.14) as

\[
\xi = \mu_{A}^{[3]}(\text{id} \otimes \epsilon \otimes \text{id}) : c'[c_*]c'' \mapsto c'\epsilon[c_*]c''.
\]
It is trivial to check this is a cochain map.

To show the induced map \( H^*(\tilde{\epsilon}) \) in cohomology is multiplicative, we show that

\[
Dh = \tilde{\zeta}\tilde{\mu} - \mu\tilde{\zeta}^2 : B(A, A, A)^{\otimes 2} \rightarrow A
\]

for a certain homotopy \( h : B(A, A, A)^{\otimes 2} \rightarrow A \) to be produced momentarily. It will help to adapt (A.6), (A.7), (A.8) to the present set-up:

\[
\tilde{\mu} = \left( \mu \otimes \mu_{BA} \otimes \mu(\epsilon(\mathrm{id} \otimes \mathrm{id}_{BA}) \otimes \mathrm{id}) \right) \circ \Pi' \\
+ \left( \mu(\mathrm{id} \otimes \epsilon(s \otimes s^{-1} \otimes \mathrm{id}_{BA})) \otimes \mu_{BA} \otimes \mu(\epsilon(\mathrm{id} \otimes \mathrm{id}_{BA}) \otimes \mathrm{id}) \right) \circ \Pi'',
\]

(A.17)

where \( \Pi' \) and \( \Pi'' \) are preparatory tensor permutations. The permutations \( \tau \) in particular have been absorbed into these without sign change because the operators between \( \tau \) and \( \Pi \) in (A.7) and (A.8) are of degree zero. Here the value, up to sign (determined by the Koszul convention, but not written out), on the standard pure tensor \( a'[a_\bullet]a''b'[b_\bullet]b'' \) is displayed below the function, to make the argument easier to follow, though in principle these values are optional. We will continue with these notations throughout the proof. Recall that if \( b' \) is in the image of the unit map \( \eta_A \), then the \( \epsilon \) factor in the second term is defined to vanish on bar-words containing it by Remark 2.2.

It is evident from (A.15) that \( -\mu\tilde{\zeta}^2 \) vanishes unless \( \ell = \ell(a_\bullet) \) and \( r = \ell(b_\bullet) \) are both zero, in which case it is

\[
-\mu^{[6]}(\mathrm{id} \otimes \eta_A \otimes \mathrm{id} \otimes \mathrm{id} \otimes \eta_A \otimes \mathrm{id}) = - \mu^{[4]}.
\]

As \( \mu_{BA} \) is of degree 0 and \( \epsilon_{BA} \) annihilates \( B_{>1}A \), the first term of \( \tilde{\zeta}\tilde{\mu} \) vanishes, according to (A.15) and (A.17) unless \( \ell(a_\bullet) = \ell(b_\bullet) = 0 \), in which case it reduces to

\[
\mu^{[3]} \left( \mu \otimes \eta_A \otimes \mu(\epsilon(\mathrm{id} \otimes \mathrm{id}_{BA}) \otimes \mathrm{id}) \right) \circ \tau_{a''b'}.
\]

If additionally \( \ell(b_{(3)}) = 0 = r \), this further reduces to

\[
\mu^{[3]} \left( \mu \otimes \eta_A \otimes \mu(\mathrm{id} \otimes \mathrm{id}) \right) \circ \tau_{a''b'} = \mu^{[4]} \circ \tau_{a''b'}.
\]

The second term of \( \tilde{\zeta}\tilde{\mu} \) vanishes unless \( [a_\bullet] = [a] \) is of length 1 and \( [b_{(2)}] \) is of length 0, in which case it contributes

\[
\mu^{[4]}(\mathrm{id} \otimes \epsilon(s \otimes s^{-1} \otimes \mathrm{id}_{BA}) \otimes \epsilon(\mathrm{id} \otimes \mathrm{id}_{BA}) \otimes \mathrm{id}) \circ \pi''
\]

\[
a' \otimes \epsilon(a_1 \mid [b' \mid b_{(1)}]) \otimes \epsilon(a'' \mid [b_{(3)}]) \otimes b''
\]

for \( \pi'' \) running over tensor permutations \( a'[a]a''b'[b_{(1)}]b'' \rightarrow a' \otimes [a]b'[b_{(1)}]a''[b_{(3)}] \otimes b'' \). These \( \pi'' \) are the specialization of \( \Pi'' \) from (A.17) to the case \( \ell(a_{(2)}) = 0 \).
The promised homotopy is given as
\[
\xi - \mu_{A_2} = \begin{cases} 
\mu^4[\tau_{a''}, b'] - \mu^4[a'']b' & \text{if } \ell = 0 = r, \\
\mu^4[\text{id} \otimes \text{id} \otimes \text{C} \otimes \text{id}) \tau_{a''}, b'] & \text{if } \ell = 0 < r, \\
\mu^4[\text{id} \otimes \text{C}(s \otimes s^{-1} \otimes \text{id}_{BA}) \otimes \text{id} \otimes \text{id}_{BA} \otimes \text{id} \circ \tau'' & \text{if } \ell = 1, \\
\mu^4[\text{id} \otimes \text{C}(a''[b_*]) \otimes \text{C}(a''[b_[3]]) \otimes \text{id}) & \text{if } \ell \geq 2.
\end{cases}
\] (A.18)

The promised homotopy is given as
\[
h := \mu^4[\text{id} \otimes \varepsilon \otimes \text{C}(\text{id} \otimes s^{-1} \otimes \text{id}_{BA}) \otimes \text{id})
\]
\[
\hspace{1cm} a' \; \varepsilon[a_*] \quad \text{C}(a''[b' \mid b_*]) \quad b''
\] (A.19)

To show \(Dh\) agrees with \(\xi - \mu_{A_2} \otimes \text{id}\), we follow the same case distinctions.

If \(\ell = \ell(a_*) \geq 2\), then \(h\) and hence \(dh\) vanish since \(\varepsilon[a_*] = 0\), and since the differential on \(B(A, A, A)\) takes \(B_{\geq 2}(A, A, A)\) into \(B_{\geq 1}(A, A, A)\), so does \(hd_{B(A,A,A)} \otimes \text{id}\).

If \(\ell = 1\), then \(h\) and hence \(dh\) vanishes as before, but \(hd\) need not, by the formula for the “external” differential \(d_{B(A,A,A)}^{\text{ext}} = d_{B(A,A,A)} - d_{\otimes}\) given in (1.39), since the outer two operators reduce the length of the bar-word in \(a'[a]a''\) to zero, taking it respectively to \(\pm a'[a]a''\) and \(\pm a'[a]'a''\). Thus \(hd_{\otimes}\) and \(h(\text{id}_{B(A,A,A)} \otimes d_{B(A,A,A)})\) vanish, but for \(h(d_{B(A,A,A)} \otimes \text{id}_{B(A,A,A)})\), plugging (1.39) into (A.19), we get
\[
\mu^4[\text{id} \otimes \varepsilon \otimes \text{C}(\text{id} \otimes s^{-1} \otimes \text{id}_{BA}) \otimes \text{id})
\]
\[
\hspace{1cm} o\left(\left(\mu(\text{id} \otimes s) \otimes \eta_{BA} \otimes \text{id}\right) \otimes \text{id}_{B(A,A,A)}
\]
\[
\hspace{2cm} a' \quad a'' \quad b'[b_*]b''
\]
\[
- \text{id} \otimes \eta_{BA} \otimes \mu(\text{id} \otimes \text{id}) \otimes \text{id}_{B(A,A,A)}
\]
\[
\hspace{1cm} a' \quad b'[b_*]b''
\] (A.20)

the change in signs coming in both terms from the factor \(s\) of degree 1 coming from \(d_{B(A,A,A)}^{\text{ext}}\) being moved past the factor \(s^{-1}\) of degree \(-1\) in the argument of \(\varepsilon\). By the Cartesianesque formula (2.5), the second term of (A.20) can be replaced by
\[
\mu^4[\text{id} \otimes (\varepsilon \otimes \text{C}) \varepsilon(s \otimes \text{id} \otimes s^{-1} \otimes \text{id}_{BA}) \otimes \text{id})
\]
\[
\hspace{1cm} a' \quad (\varepsilon \otimes \text{C}) \varepsilon(a'' \mid b' \mid b_[3]) \quad b''
\]
where \(\pi\) runs over shuffles \(a \otimes a'' \otimes [b' \mid b_*] \rightarrow a[b'[b_*]_{1} \otimes a''[b'[b_*]_{2}] \) and \([b' \mid b_*]_{1} \otimes [b' \mid b_*]_{2}\) is a deconcatenation of \([b' \mid b_*]\). There is a case distinction here: the deconcatenation yielding \([\otimes [b' \mid b_*]\) recovers
\[
\mu^4[\text{id} \otimes s \otimes \text{C}(\text{id} \otimes s^{-1} \otimes \text{id}_{BA}) \otimes \text{id})
\]
\[
\hspace{1cm} a' \quad a'' \quad [b' \mid b_*] \quad b''
\]
cancelling the first term of (A.20). Otherwise, $b'$ is assigned to the first word in the deconcatenation, so the value is $[b'|b_{(1)}] \otimes [b_{(3)}]$, and one gets operations

\[
\begin{align*}
&(\text{id} \otimes \mathfrak{C}(s \otimes s^{-1} \otimes \text{id}_{B, A}) \otimes \mathfrak{C}(\text{id} \otimes \text{id}_{B, A}) \otimes \text{id}) \circ \pi'' \\
&\quad a' \quad \mathfrak{C}(a_1) \quad [b' | b_{(1)}]) \quad \mathfrak{C}(a'' [b_{(3)}]) \quad b''
\end{align*}
\]

agreeing with the case $\ell = 1$ of (A.18). Because $\pi$ preserves the relative positions of $s$ and $s^{-1}$, it does not incur a sign change in moving past them to become $\pi''$.

For the $\ell = 0$ cases, we write the differential on $B(A, A, A)^{\otimes 2}$ as the sum of the "internal" tensor differential $d_\otimes$ and the "external" differentials $\text{id}_{B}(A, A, A) \otimes d_{\otimes}^{\text{ext}}(A, A, A)$ and $d_{\otimes}^{\text{ext}}(A, A, A) \otimes \text{id}_{B}(A, A, A)$. We may omit the second external differential in consideration of $hd_{B}(A, A, A)$ since $d_{\otimes}(A, A, A)$ vanishes on $B_0(A, A, A)$, and when $r = 0$, we may omit the first external differential for the same reason. Write $D_\otimes$ for the differential on Hom $((A \otimes B A) \otimes A) \otimes 2$, where the domain is $B(A, A, A)^{\otimes 2}$ equipped with the tensor differential $d_\otimes$. Thus $D_\otimes h = d_{\otimes} h + h d_{\otimes}$. Because composition is a cochain map, as are $\mu_{A, id_{A}}, \epsilon_{B, A}$, one gets, $s^{-1}$, we have

\[
D_\otimes h = \mu[4](\text{id} \otimes \mathfrak{C}(A) \otimes (D_\otimes \mathfrak{C})(\text{id} \otimes s^{-1} \otimes \text{id}_{B, A}) \otimes \text{id}).
\]

Equation (2.7) shows that for fixed $r$, the function $D_\otimes \mathfrak{C}: \tilde{A} \otimes B_{1+r} A \rightarrow A$ is given by

\[
D_\otimes \mathfrak{C} = \mu(s \otimes \mathfrak{C})(1 2) + \mathfrak{C}(\text{id} \otimes d_{\otimes}) - \mu(\mathfrak{C} \otimes s).
\]

The second term also vanishes when $r = 0$, and then plugging (A.22) into (A.21) yields

\[
D_\otimes h = \mu[4](\text{id} \otimes (s \otimes \mathfrak{C}) \tau_{\alpha''; [b']}(\text{id} \otimes s^{-1} \otimes \mathfrak{C}(A) \otimes \text{id})) = \mu[4] \circ \tau_{\alpha''; [b']}
\]

Here we use that $\mathfrak{C} = \text{id}$ on $A \otimes B_0 A \cong A$ and the further simplification in the first term comes from the calculation

\[
(s \otimes \mathfrak{C}) \tau_{\alpha''; [b']}(\text{id} \otimes s^{-1}) = \tau_{\alpha''; [b']}(\text{id} \otimes s)(\text{id} \otimes s^{-1}) = \tau_{\alpha''; [b']}.
\]

The $\ell = 0 = r$ case of (A.18) agrees with (A.23), concluding that case.

For $\ell = 0 < r$, substituting (A.22) into (A.21) now gives instead

\[
D_\otimes h = \mu[4](\text{id} \otimes (s \otimes \mathfrak{C}) \tau_{\alpha''; [b']}(\text{id} \otimes s^{-1} \otimes \text{id}_{B, A}) \otimes \text{id})
\]

\[
+ \mu[3](\text{id} \otimes \mathfrak{C}(\text{id} \otimes s^{-1} \otimes \text{id}_{B_{r-1} A}) \otimes (\text{id} \otimes s^{-1} \otimes \text{id}_{s_{r-1} A} \otimes \text{id}_{B_{r-1} A}) \otimes \text{id})
\]

\[
+ \mu[3](\text{id} \otimes \mathfrak{C}(\text{id} \otimes \text{id} \otimes d_{\otimes}^{\text{ext}}|A, A) \otimes (\text{id} \otimes s^{-1} \otimes \text{id}_{B_{r-1} A} \otimes \text{id})
\]

\[
- \mu[4](\text{id} \otimes \mathfrak{C}(s \otimes s^{-1} \otimes \text{id}_{B_{r-1} A} \otimes \text{id})).
\]
As for \( h(\text{id}_{B(A,A,A)} \otimes d_{B_{\text{ext}}(A,A,A)}) \), composing its definition from (1.39) with the expression for \( h \) from (A.19) yields
\[
\mu^3 \left( \text{id} \otimes \mathcal{E} \left( \text{id} \otimes s^{-1} \mu(\text{id} \otimes s) \otimes \text{id}_{B_{\text{ext}}(A,A)} \right) \otimes \text{id} \right) \\
\quad \quad \quad a' \quad \mathcal{E}(a'' \quad [b' \ b_1 \ \cdots \ b_1]) \quad b'' \\
+ \mu^3 \left( \text{id} \otimes \mathcal{E} \left( \text{id} \otimes s^{-1} \otimes d_{\text{ext}|B_{A}}(A,A) \right) \otimes \text{id} \right) \\
\quad \quad \quad a' \quad \mathcal{E}(a'' \quad [b' \ d_{\text{ext}}[b_1]]) \quad b'' \\
- \mu^3 \left( \text{id} \otimes \mathcal{E} \left( \text{id} \otimes s^{-1} \otimes \text{id}_{B_{\text{ext}}(A,A)} \right) \otimes \mu(s \otimes \text{id}) \right) \\
\quad \quad \quad a' \quad \mathcal{E}(a'' \quad [b' \ b_1 \ \cdots \ b_1]) \quad b_r \quad b''
\]
\( (A.25) \)

These three terms are the respective opposites of the second through fourth terms of (A.24), as transitioning from one ordering of symbols to the other involves moving \( s^{-1} \) past \( s \) in the second and fourth terms and past \( d_{\text{ext}} \) in the third, incurring each time a sign change of \((-1)^{1-1} \). Thus these three pairs of terms cancel. It remains only to check the uncancelled first term of (A.24) agrees with the \( \ell = 0 < r \) clause in (A.18). For this, note that \( \text{id} \) has degree 0, so we have \( \tau_{a'',[b]}(\text{id} \otimes s^{-1}) = (s^{-1} \otimes \text{id})\tau_{a'';b'} \), and \( \mathcal{E} \) has degree zero as well, so
\[
\mu^4(\text{id} \otimes (s \otimes \mathcal{E})\tau_{a'',[b]}(\text{id} \otimes s^{-1} \otimes \text{id}_{B_{A}}(A,A)) \otimes \text{id}) = \mu^4(\text{id} \otimes (s \otimes \mathcal{E})(s^{-1} \otimes \text{id} \otimes \text{id}_{B_{A}}(A,A)) \otimes \text{id}) \tau_{a'';b'} \\
= \mu^4(\text{id} \otimes \text{id} \otimes \mathcal{E} \otimes \text{id}) \tau_{a'';b'}.
\]
\( \square \)

**Remark A.26.** We expect that if \( A', A, A'' \) are extended HGAs, the product \( \tilde{\mu} \) of Theorem A.1 is the 2-component of a differential on \( BB(A', A, A'') \) making \( B(A', A, A'') \) an \( A_{	ext{ext}} \)-algebra and making the map \( \tilde{\xi} \) of Theorem A.13 the extended 1-component \( \Xi_{(1)} \) of an \( A_{	ext{ext}} \)-map \( \Xi \) from \( B(A', A, A'') \) to \( \tilde{A} \), but we will leave the exploration of this possibility for another occasion.

We now can use the new product to establish the particular version of the Eilenberg–Moore theorem we will need.

**Theorem A.27 (Eilenberg–Moore with induced product).** Suppose the coefficient ring \( k \) is a principal ideal domain and suppose given a pullback square
\[
\begin{array}{ccc}
Y & \xrightarrow{\beta} & E \\
\downarrow{\alpha} & & \downarrow{\beta} \\
X & \longrightarrow & B
\end{array}
\]
of pointed topological spaces in which \( E \to B \) is a Serre fibration, \( B \) and \( X \) are path-connected, the action of \( \pi_1(B) \) on the cohomology of the fiber \( F \) is trivial, and (1) each \( H^n(F) \) is a finitely generated \( k \)-module or (2) each \( H^n(B) \) and each \( H^n(X) \) is a finitely generated \( k \)-module. Then there is a natural quasi-isomorphism
\[
\tilde{\xi} : B(C^*(X), C^*(B), C^*(E)) \longrightarrow C^*(Y),
\]
\[
\quad \quad x[b_*]e \longmapsto \alpha^*(x) \longmapsto \eta_{C^*(Y)}\tilde{\xi}_{B_{C^*(E)}}[b_*] \longmapsto \beta^*(e),
\]
inducing a ring isomorphism
\[
\text{Tor}_{C^*(B)}^* \left( C^*(X), C^*(E) \right) \xrightarrow{\sim} H^*(Y)
\]
with respect to the product on the domain induced by the product given in Theorem A.1.
We rely on the presentation of Gugenheim and May [GuM, Thm. 3.3] with some modification of hypotheses.

Proof. We assume $B$ and $X$ are path-connected to guarantee that the homotopy type of the fiber $F$ be well defined and we only have to discuss one group $\pi_1(B)$; one otherwise needs a separate argument for each path-component. This assumption then also implies $\pi_1(X)$ acts trivially on $H^*(F)$. Given any proper projective resolution $P\ast$ of $C^\ast(X)$ as a $C^\ast(B)$-module (or resolution in the more general sense of Gugenheim–May [GuM, Defs. 1.1]), then, Gugenheim–May show the expected composite filtered map $\vartheta': P\ast \otimes_{C^\ast(B)} C^\ast(E) \longrightarrow C^\ast(X) \otimes_{C^\ast(B)} C^\ast(E) \to C^\ast(Y)$ is a quasi-isomorphism [GuM, Thm. 3.3, Cor. 3.5] under slightly different finiteness hypotheses, namely that $k$ is Noetherian and the groups $H_n(X;\mathbb{Z})$ and $H_n(B;\mathbb{Z})$ are all finitely generated. These hypotheses are used only to see the canonical maps $C^\ast(B) \otimes H\ast(F) \longrightarrow C^\ast(B;H\ast(F))$ and $C^\ast(X) \otimes H\ast(F) \longrightarrow C^\ast(X;H\ast(F))$ are quasi-isomorphisms, using a lemma [GuM, Lem. 3.2] asserting that if $k$ is a commutative Noetherian ring, $G$ an $k$-module, and $C$ a chain complex over $\mathbb{Z}$ with each $H_n(C)$ finite, then $\text{Hom}_\mathbb{Z}(C,k) \otimes_k G \longrightarrow \text{Hom}_\mathbb{Z}(C,G)$ is a quasi-isomorphism. The proof of this lemma uses only that $Z$ is a principal ideal domain, and hence the same argument shows that if $k$ is a principal ideal domain, $C$ now a differential graded $k$-module, and $G$ a $k$-module, then $\text{Hom}_k(C,k) \otimes_k G \longrightarrow \text{Hom}_k(C,G)$ is a quasi-isomorphism. Thus, assuming $k$ is a principal ideal domain, we can replace Gugenheim–May’s hypothesis that each integral homology group $H_n(B;\mathbb{Z})$ and $H_n(X;\mathbb{Z})$ is finitely generated with the weaker hypothesis that the (co)homology groups with coefficients in $k$ are. If, alternatively, we assume (1) that the $H^n(F)$ is finitely generated over $k$, then again assuming $k$ is a principal ideal domain, the decomposition of each $H^n(F)$ as a finite product of cyclic $k$-modules shows $C^\ast(B) \otimes H^n(F) \longrightarrow C^\ast(B;H^n(F))$ and $C^\ast(X) \otimes H^n(F) \longrightarrow C^\ast(X;H^n(F))$ are isomorphisms of differential graded $k$-modules.

To see the suppressed details in the proof of the multiplicativity of $H\ast(\vartheta)$ with respect to the classical product on $\text{Tor}$ [GuM, p. 26], subdivide the vertical cohomological Eilenberg–Zilber map featuring along the upper-right of McCleary’s diagram gesturing at such a proof [Mc, p. 255] as

$$H^\ast(Y) \otimes H^\ast(Y) \xrightarrow{H^\ast(\vartheta)} H^\ast(C^\ast(Y) \otimes C^\ast(Y)) \xrightarrow{i} H^\ast\left(\left(C^\ast(Y) \otimes C^\ast(Y)\right)^\ast\right) \xrightarrow{\text{EZ}} H^\ast(Y \times Y).$$

Then there are evident horizontal maps subdividing the region into three rectangles that can be seen to commute on choosing maps between resolutions [GuM, Thm. 1.7] and expanding out the definition of the external product.

As $k$ is a principal ideal domain, Proposition 1.34 implies that $\tilde{C} = B(C^\ast(X),C^\ast(B),C^\ast(B))$ gives a resolution $\tilde{C} \longrightarrow C^\ast(X)$ of $C^\ast(X)$ as a differential $C^\ast(B)$-module and $H\ast(\tilde{C})$ is the desired Tor. The map $\vartheta$ then specializes to our $\xi$. As $C^\ast$ is a functor valued in $\text{hgas}$, by Theorems A.1 and A.13 there is a natural cochain map $\tilde{\mu}: \tilde{C} \otimes \tilde{C} \longrightarrow \tilde{C}$ such that the quasi-isomorphism $\tilde{\xi}$ is multiplicative up to homotopy in the sense that if $\mu$ is the cup product on $C^\ast(Y)$, then there exists $h: \tilde{C} \longrightarrow C(Y)$ with $Dh = \tilde{\xi}\tilde{\mu} - \mu\tilde{\xi} \otimes \tilde{\xi}$, and $\mu \otimes \tilde{\xi} - \tilde{\xi} \otimes h \mu$ is the standard product. Thus the $k$-module isomorphism $H^\ast(\tilde{\xi}) : H^\ast(\tilde{C}) \longrightarrow H^\ast(Y)$ takes the induced product

$$H^\ast(\tilde{C}) \otimes H^\ast(\tilde{C}) \xrightarrow{\times} H^\ast(\tilde{C} \otimes \tilde{C}) \xrightarrow{H^\ast(\vartheta)} H^\ast(\tilde{C})$$

on Tor to the cup product on $H^\ast(Y)$. Moreover, since the classical proof shows $H^\ast(\tilde{\xi})$ is multiplicative with respect the standard product on Tor and the cup product on $H^\ast(Y)$, we conclude that $H^\ast(\tilde{\mu}) \circ \times$ is the standard product. \qed
Remark A.28. The classical Eilenberg–Moore theorem obtains the ring structure on the domain through the external product and the Eilenberg–Zilber theorem, without reference to a cochain-level product on any complex computing Tor. We go to the trouble of showing this is the case because in Section 5, we need the fact that the product on Tor is induced by such a product in order show certain maps of Tors are multiplicative and finally obtain Theorem 0.1.

Remark A.29. The preceding proof is really only about the cohomology of the pullback, and does not make any statement about the Eilenberg–Moore spectral sequence. There is a subtlety here [BaMR14, Prop. 10.19 et seq.]: while the two-sided bar construction computes the Tor with the hypotheses of Theorem A.27, it cannot necessarily be used in computing the Eilenberg–Moore spectral sequence because we do not know without further hypotheses that the $E_2$ page of the resulting filtration spectral sequence will be $\text{Tor}^{\ast}_{H^\ast(B)}(H^\ast(X), H^\ast(E))$. It will, however, be if $H^\ast(B)$ and $H^\ast(X)$ are also flat over the coefficient ring $k$, which is true in the case of interest in the broader paper, where the spaces have polynomial cohomology. It actually follows from model-categorical considerations that there exists some filtration of the two-sided bar construction such that the $E_2$ page of the associated spectral sequence is the desired Tor, but in general we cannot be explicit about what this filtration is.

The (strong) convergence of the spectral sequence, however, is not in doubt. The primary sources tend to go to some effort to use only simply-connected algebras in order to make the filtrations finite, but this is not necessary to prove convergence [Bo99, Thm. 6.1(a)].

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DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
j.carlson@imperial.ac.uk

DEPARTMENT OF MATHEMATICS
WESTERN UNIVERSITY
mfranz@uwo.ca