A SHORT PROOF OF REISNER’S THEOREM ON
COHEN-MACAULAY SIMPLICIAL COMPLEXES

SILVANO BAGGIO

Abstract. We present a short proof of Reisner’s Theorem, characterizing which
simplicial complexes have a Cohen-Macaulay face ring. In some cases, we can
also express some homological invariants of the face ring in terms of the reduced
homology of the complex.

Keywords: Stanley-Reisner rings, sheaves on simplicial complexes.

Introduction

Let Σ be a simplicial complex on the set
V = {v1, . . . , vs}, and k[Σ] the face ring
(or Stanley-Reisner ring) over the field k. A theorem by Reisner ([Reis Page 2 of 2],
here Proposition 1.3) states that k[Σ] is Cohen-Macaulay if and only if the reduced
homology of Σ, and that of all the links of its faces, is zero, except possibly in the
top degree (that is, a geometric realization |Σ| of Σ is a homology sphere).

The proof presented here avoids using local cohomology, neither directly as in
Reisner’s paper, or indirectly, via Hochster’s theorem as in [BH 5.3.9]. Here the
technical difficulty relies in some homological algebra, applied to certain sheaves on
Σ, with the topology induced by the (reverse) face order. In fact, in [Yuz] Reisner’s
Theorem is proven as a corollary of a more general theorem on the rings of sections of
sheaves on posets. Our method is quite similar to his, and also to that used in [Bac].
Our proof is more direct and self contained, and we treat separately the case where
|Σ| is a homology manifold: in that case, proposition 2.1 provides a description of
the homology invariants Tori R(k[Σ], k) in terms of the reduced cohomology of Σ.

1. Simplicial complexes and sheaves on them

1.1. Notations and basic facts. By a simplicial complex over the finite set (of
vertices) V = {v1, . . . , vs} we mean the pair (V, Σ), where Σ is a set of subsets of V
(the simplexes or faces), such that:

σ ∈ Σ, τ ⊂ σ ⇒ τ ∈ Σ;
{v} ∈ Σ ∀v ∈ V.

The dimension of a face σ is the number of vertices of σ minus one; so, by definition,
dim ∅ = −1. The dimension of Σ is max{dim σ | σ ∈ Σ}. A simplicial complex is
pure if all its maximal faces have the same dimension.

Definition 1. Let R be a ring, and Σ a simplicial complex with vertices V = {v1, . . . , vs} and faces S ⊂ P(V). The Stanley-Reisner algebra, or face ring on R
relative to Σ is the R-algebra

R[Σ] = R[X1, . . . , Xs]
I,

where I ⊂ R[X1, . . . , Xs] is the ideal generated by all monomials Xs i=1 Xs i with
{v1, . . . , vih} ∈ S.
Definition 2. (See [BH, 5.3]) Let \((V, \Sigma)\) be a simplicial complex of dimension \(n - 1\), and let \(V\) be given a total order. For each \(i\)-dimensional face \(\sigma\) we write \(\sigma = [v_0, \ldots, v_i]\) if \(\sigma = \{v_0, \ldots, v_i\}\) and \(v_0 < \cdots < v_i\).

The \emph{augmented chain complex} of \(\Sigma\) is:

\[
\mathcal{C}(\Sigma) : 0 \to \mathcal{C}_{n-1} \xrightarrow{\partial} \mathcal{C}_{n-2} \to \cdots \to \mathcal{C}_0 \xrightarrow{\partial} \mathcal{C}_{-1} \to 0
\]

where we set

\[
\mathcal{C}_i = \bigoplus_{\sigma \in \Sigma, \dim \sigma = i} \mathbb{Z}\sigma \quad \text{and} \quad \partial \sigma = \sum_{j=0}^{i} (-1)^j \sigma_j
\]

for all \(\sigma \in \Sigma\), and \(\sigma_j = [v_0, \ldots, \hat{v}_j, \ldots, v_i]\) for \(\sigma = [v_0, \ldots, v_i]\).

The \(i\)-th \emph{reduced simplicial homology} of \(\Sigma\) with values in an abelian group \(G\) is:

\[
\tilde{H}_i(\Sigma, G) = H_i(C(\Sigma) \otimes G) \quad i = -1, \ldots, n - 1.
\]

The dual cochain complex \(\operatorname{Hom}_\mathbb{Z}(\mathcal{C}(\Sigma), G)\) has differentials \(\bar{\partial}\), defined as: \((\bar{\partial}\phi)(\alpha) = \phi(\partial \alpha)\), for \(\phi \in \operatorname{Hom}_\mathbb{Z}(\mathcal{C}_i, G)\), \(\alpha \in \mathcal{C}_{i+1}\). The \(i\)-th group of \(\text{reduced simplicial cohomology}\) of \(\Sigma\) with values in \(G\) is:

\[
\tilde{H}^i(\Sigma, G) = H^i(\operatorname{Hom}_\mathbb{Z}(\mathcal{C}(\Sigma), G)) \quad i = -1, \ldots, n - 1.
\]

If \(\sigma\) is a face of the simplicial complex \(\Sigma\), the \emph{link} of \(\sigma\) in \(\Sigma\) is \(\operatorname{lk}_\Sigma \sigma = \{\tau \in \Sigma \mid \sigma \not\subset \tau, \sigma \cap \tau \in \Sigma\}\). It is easy to see that \(\operatorname{lk}_\Sigma \sigma\) is itself a simplicial complex over the set \(\{v \in V \mid v \in \tau \exists \tau \in \operatorname{lk}_\Sigma \sigma\}\).

We denote by \(\operatorname{St} \sigma = \{\tau \in \Sigma \mid \sigma \subset \tau\}\) the \emph{star} of \(\sigma\) in \(\Sigma\), and by \(\overline{\operatorname{St}} \sigma\) the least subcomplex of \(\Sigma\) containing \(\operatorname{St} \sigma\).

Lemma 1.1. \(\Sigma\) be a simplicial complex on the vertices \(\{v_1, \ldots, v_t\}\), \(\sigma = [v_1, \ldots, v_l]\) a face of \(\sigma\). Then we have an isomorphism of localizations of Stanley-Reisner rings:

\[
k[\Sigma](X_\sigma) \cong k[\overline{\operatorname{St}} \sigma](X_\sigma),
\]

where \(X_\sigma\) is the image of the monomial \(X_1 \cdots X_t\).

Proof. Let the vertices of \(\overline{\operatorname{St}} \sigma\) be \(\{v_1, \ldots, v_t\}\) \((1 \leq r \leq t)\). By definition \(k[\Sigma] = k[X_1, \ldots, X_t]/_{\Sigma}\), where \(I_{\Sigma} = (X_i, \ldots, X_k \mid [v_i, \ldots, v_k] \not\in \Sigma)\), while \(k[\overline{\operatorname{St}} \sigma] = k[X_1, \ldots, X_t]/_{\sigma}\), where \(I_{\sigma} = (X_i, \ldots, X_k \mid [v_i, \ldots, v_k] \not\in \overline{\operatorname{St}} \sigma)\). After we localize to the multiplicative system \(\{X_\sigma^n \mid n \geq 0\}\), all monomials \(X_{i_1} \cdots X_{i_k} \in k[\Sigma]\) such that \([v_{i_1}, \ldots, v_{i_k}] \not\in \overline{\operatorname{St}} \sigma\) vanish: so the inclusion \(k[X_1, \ldots, X_t] \hookrightarrow k[X_1, \ldots, X_t]\) can be lifted to a well defined ring homomorphism \(k[\Sigma](X_\sigma) \to k[\overline{\operatorname{St}} \sigma](X_\sigma)\) that is easily seen to be injective and surjective.

Lemma 1.2. With the same notation as above,

\[
k[\operatorname{lk} \sigma] = \frac{k[\overline{\operatorname{St}} \sigma]}{(X_\sigma)}.
\]

Proof. \(k[\operatorname{lk} \sigma] = k[X_{l+1}, \ldots, X_t]/_{I_{\operatorname{lk} \sigma}}\), with \(I_{\operatorname{lk} \sigma} = I_{\sigma} \cap k[X_{l+1}, \ldots, X_t]\).

Proposition 1.3. \textbf{Reis, Theorem 2].}

Let \((V, \Sigma)\) be a simplicial complex.

The ring \(k[\Sigma]\) is Cohen-Macaulay if and only if

(1) \(\tilde{H}_i(\operatorname{lk} \sigma, k) = 0 \quad \forall i < \dim(\operatorname{lk} \sigma) \quad \forall \sigma \in \Sigma\),

and

(2) \(\tilde{H}_i(\Sigma, k) = 0 \quad \forall i < \dim \Sigma\).
Remark 1. Condition (1) depends only on the topology of the support of $\Sigma$. In fact, by [Sta Prop. 4.3], it is equivalent to:

$$\bar{H}_i((\Sigma_1, |\Sigma| \setminus x, k) \quad \forall i < \dim(\Sigma), \forall x \in |\Sigma|,$$

where $|\Sigma|$ is a given geometric realization of $\Sigma$.

Moreover, we can replace condition (II) with

$$\bar{H}^i(lk \sigma, k) = 0 \quad \forall i < \dim(lk \sigma) \quad \forall \sigma \in \Sigma,$$

that is, replace (reduced) homology with cohomology. This is a consequence of the Universal Coefficient Theorems ([Mac], Thm III.4.1 and Thm V.11.1), and shall be used in the proof of Proposition 2.1.

### 1.2. Sheaves on simplicial complexes

Sheaves on posets have been often used to study properties of rings which can be expressed as global sections (See [Bac], [Yuz], and also [BBFK], [BreLu], [Bri]).

A simplicial complex $\Sigma$ can be considered as a topological space, where the open sets are the subcomplexes. More generally, every poset $(X, \leq)$ can be given a topology, where the open sets are the increasing subsets, that is, the subsets $Y \subset X$ satisfying: $y \in Y, x \in X, y \leq x \Rightarrow x \in Y$. These two topologies coincide on a simplicial complex, provided we take the the reverse face order.

Some obvious remarks: every point (face) $\sigma$ is contained in a least open subset, the subcomplex $\bar{\sigma}$ generated by $\sigma$. The closure of $\{\sigma\}$ is its star $\text{St} \sigma = \{\tau \in \Sigma \mid \sigma \subset \tau\}$. The empty set $\emptyset$ is the maximum element in $\Sigma$ for the reverse face order, and the closure of $\{\emptyset\}$ is the whole $\Sigma$; in particular $\Sigma$ is an irreducible topological space.

A sheaf $\mathcal{F}$ of abelian groups on $\Sigma$ is by definition the data of an abelian group $\mathcal{F}(\Sigma')$ (sections of $\mathcal{F}$ on $\Sigma'$) for every subcomplex $\Sigma' \subset \Sigma$, and a group homomorphism (restriction) $\phi^\Sigma_{\Sigma_1} : \mathcal{F}_{\Sigma_2} \to \mathcal{F}_{\Sigma_1}$ for every pair $\Sigma_1 \subset \Sigma_2 \subset \Sigma$, such that for $\Sigma_1 \subset \Sigma_2 \subset \Sigma_3$, we have $\phi^\Sigma_{\Sigma_1} \phi^\Sigma_{\Sigma_2} = \phi^\Sigma_{\Sigma_3}$.

For $\Sigma' = \cup_i \Sigma_i \subset \Sigma$, if $x_i \in \mathcal{F}(\Sigma_i)$ for every $i$, and $\phi^\Sigma_{\Sigma_i \cap \Sigma_j}(x_i) = \phi^\Sigma_{\Sigma_i \cap \Sigma_j}(x_j)$ for every $i, j$, then there exists a unique $x \in \mathcal{F}(\Sigma')$ such that $\phi^\Sigma_{\Sigma_i}(x) = x_i$ for every $i$.

It is clear that the stalk of $\mathcal{F}$ at $\sigma \in \Sigma$ is $\mathcal{F}_\sigma = \mathcal{F}(\bar{\sigma})$. The sections of $\mathcal{F}$ on the subcomplex $\Sigma'$ can be described as

$$\mathcal{F}(\Sigma') = \{x \in \prod_{\sigma \in \Sigma'_{\text{max}}} \mathcal{F}_\sigma \mid x_{\sigma | \sigma \cap \sigma'} = x_{\sigma' | \sigma \cap \sigma'}\},$$

where $\Sigma'_{\text{max}}$ are the maximal faces of $\Sigma'$. This implies that assigning a sheaf $\mathcal{F}$ on $\Sigma$ is the same as assigning the stalks $\mathcal{F}_\sigma$ for all $\sigma \in \Sigma$, and the restrictions $\phi^\Sigma_{\sigma_1} : \mathcal{F}_{\sigma_2} \to \mathcal{F}_{\sigma_1}$ for every pair of faces $\sigma_1 \subset \sigma_2$, with the only condition that, for $\sigma_1 \subset \sigma_2 \subset \sigma_3$, we have $\phi^\Sigma_{\sigma_2} \phi^\Sigma_{\sigma_1} = \phi^\Sigma_{\sigma_3}$.

The simplest sheaves on $\Sigma$ can be defined in the following way on the stalks. If $G$ is an abelian group, and $\sigma \in \Sigma$: $G(\sigma)_{\bar{\sigma}} = G$, while $G(\sigma)_\tau = 0$ if $\tau \neq \sigma$; all restrictions are zero. We call $G(\sigma)$ the simple sheaf with support in $\sigma$ and values in $G$. The cohomology of such sheaves can be described directly in terms of the reduced cohomology of the links in $\Sigma$.

**Lemma 1.4.** (See [Bac] Lemma 3.1)

(i) The global sections of the simple sheaf $G(\sigma)$ are

$$G(\sigma)(\Sigma) = H^0(\Sigma, G(\sigma)) = \begin{cases} G & \text{if } \sigma \text{ is a maximal cone,} \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If $i \geq 1$, then $H^i(\Delta, G(\sigma)) \cong \bar{H}^{i-1}(lk \sigma, G)$. 
Proof. (i) follows from [4]. Let us prove (ii). The map \( j_\sigma : \text{lk} \sigma \to \Sigma \), sending \( \tau \to \tau \cup \sigma \) is a continuous injection, and its image is \( \text{St} \sigma \), which is closed in \( \Sigma \). Since \( G(\sigma) \) is the push forward via \( j_\sigma \) of the sheaf \( G(\emptyset) \) on \( \text{lk} \sigma \), we have: \( H^i(\Sigma, G(\sigma)) = H^i(\text{lk} \sigma, G(\emptyset)) \). We need only to prove that \( H^i(\Sigma, G(0)) = \tilde{H}^{i-1}(\Sigma, G) \) for \( i > 0 \).

Let \( \tilde{G} \) the constant sheaf on \( \Sigma \), with values in \( G \). \( G(\emptyset) \) is a subsheaf of \( \tilde{G} \). Let \( G_\emptyset = \tilde{G}/G(\emptyset) \). Since \( \tilde{G} \) is acyclic, the short exact sequence \( 0 \to G(\emptyset) \to \tilde{G} \to G_\emptyset \to 0 \) induces the long exact sequence in cohomology (assume \( \dim \Sigma > 0 \):

\[
0 \to H^0(\Sigma, \tilde{G}) \to H^0(\Sigma, G_\emptyset) \to H^1(\Sigma, G(\emptyset)) \to H^1(\Sigma, \tilde{G}) = 0,
\]

which implies \( H^1(\Sigma, G(\emptyset)) \cong \tilde{H}^0(\Sigma, G) \); moreover, \( H^{i-1}(\Sigma, G_\emptyset) \cong H^i(\Sigma, G(0)) \) for \( i \geq 2 \). We can conclude by applying the following lemma. \( \square \)

**Lemma 1.5.** [Bac] Theorem 2.1

\[
H^i(\Sigma \setminus \{ \emptyset \}, \tilde{G}) \cong \tilde{H}^i(\Sigma, G),
\]

where \( \tilde{G} \) is the constant sheaf with values in \( G \), on \( \Sigma \setminus \{ \emptyset \} \).

**Proof.** The set \( \mathcal{S} = \{ C_j = \text{St}(v_j) \mid j = 1, \ldots, n \} \) is a closed covering of \( \Sigma \setminus \{ \emptyset \} \). For any \((p + 1)\)-tuple of indices \( i_0, \ldots, i_p \), the intersection \( C_{i_0} \cap \cdots \cap C_{i_p} \) is either the empty set (if \( \{ v_0 \ldots v_p \} \notin \Sigma \)), or the star \( \text{St}(v_0 \ldots v_p) \). Let \( G_{i_0 \ldots i_p} = \tilde{G}|_{C_{i_0} \cap \cdots \cap C_{i_p}} \); these (constant) sheaves are all flabby and then acyclic.

Consider the complex

\[
(5) \quad 0 \to \tilde{G} \to \bigoplus_{i=1}^{m} G_i \to \bigoplus_{1 \leq i_0 < i_1 \leq m} G_{i_0 i_1} \to \cdots \to \bigoplus_{1 \leq i_0 < \cdots < i_n \leq m} G_{i_0 \ldots i_n} \to 0,
\]

where differentials are defined as follows (indices with \( \sim \) are omitted):

\[
(6) \quad d((a_{i_0 \ldots i_k})_{i_0 \ldots i_k}) = \left( \sum_{h=0}^{k+1} (-1)^h \tilde{a}_{j_0 \ldots j_h \ldots j_{k+1}} \right)_{j_0 \ldots j_{k+1}}.
\]

The above notation means: if \( a \in G_{j_0 \ldots j_h \ldots j_{k+1}}(\Sigma') \), with \( \Sigma' \subset \Sigma \), then \( \tilde{a} \) is the image of \( a \) via the projection \( G_{j_0 \ldots j_h \ldots j_{k+1}}(\Sigma') \to G_{j_0 \ldots j_h \ldots j_{k+1}}(\Sigma') \).

The complex (5) is exact, and so it is an acyclic resolution of the sheaf \( \tilde{G} \) (To see this, check that the complex of the stalks relative to each face of \( \Sigma \) is exact). Since \( H^0(\text{St}(\sigma), \tilde{G}|_{\text{St}(\sigma)}) = G \) for any \( \sigma \), the complex of the global sections of (5) is

\[
0 \to \bigoplus_{\sigma \in \Delta_n} G(\sigma) \to \cdots \to \bigoplus_{\sigma \in \Delta_n} G(\sigma) \to 0.
\]

If we define differentials as in (6), this is the cochain complex of \( S_\Sigma \) with values in \( G \), and its cohomology is the reduced simplicial cohomology of \( S_\Sigma \). \( \square \)

1.3. **Definition of the sheaf \( A \).** Let \( \Sigma \) be a simplicial complex of dimension \( d \) over the set \( V = \{ v_1, \ldots, v_n \} \). From now on, let \( R = k[X_1, \ldots, X_n] \) be the ring of the polynomials in \( n \) indeterminates on the field \( k \).

Let us define the sheaf of \( R \)-algebras \( A \) over \( \Sigma \): its sections over the subcomplex \( \Sigma' \) are \( A(\Sigma') = R/I(\Sigma') \), where \( I(\Sigma') = (X_{i_1} \ldots X_{i_k} \mid [v_{i_1}, \ldots, v_{i_k}] \notin \Sigma') \). If \( \Sigma'_1 \subset \Sigma'_2 \), one can define a surjective homomorphism of \( R \)-algebras \( R/I(\Sigma'_2) \to R/I(\Sigma'_1) \), sending \( X_i \) to itself if \( v_i \in \Sigma'_1 \), to zero otherwise. This homomorphism is the restriction morphism \( A(\Sigma'_2) \to A(\Sigma'_1) \).

Equivalently, \( A \) can be defined as the only sheaf on \( \Sigma \), such that its stalk at \( \sigma = [v_{i_1}, \ldots, v_{i_k}] \) is \( A_\sigma = k[X_{i_1}, \ldots, X_{i_k}] \), and, if \( \tau = [v_{i_1}, \ldots, v_{i_j}] \), with \( j < h \) the map \( A_\sigma \to A_\tau \) (that is, \( A(\sigma) \to A(\tau) \)) sends \( X_{i_l} \) to itself for \( l \leq j \), to zero otherwise.
2. The main proof

First we prove Reisner’s Theorem in a particular case, when the simplicial complex is a homology manifold. The proof of the general case is at the end of this section.

**Proposition 2.1.** Let \( \Sigma \) be a pure \( d \)-dimensional simplicial complex on \( n \) vertices, satisfying condition \( \square \). Then

\[
\text{Tor}_i^R(k[\Sigma], k) = \bigoplus_{i=0}^n \widetilde{H}^{i-r-1}(\Sigma, \wedge^i k^n).
\]

In particular, the reduced homology of \( \Sigma \) vanishes in degree less than \( d \) if and only if

\[
\text{Tor}_i^R(k[\Sigma], k) = 0 \quad \forall i > n - d - 1,
\]

and this is equivalent to: \( k[\Sigma] \) is a Cohen-Macaulay ring.

**Proof.** Let \( K^*(X_1, \ldots, X_n) \) be the Koszul complex relative to \( X_1, \ldots, X_n \in R \). It is a free resolution of \( k = R/(X_1, \ldots, X_n) \) as an \( R \)-module. We can consider \( K^*(X_1, \ldots, X_n) \) as a complex of constant sheaves on \( \Sigma \), with negative degree, from \( -n \) to 0. Let \( K \) be the complex of sheaves of \( R \)-algebras, obtained by tensoring \( K^*(X_1, \ldots, X_n) \) with the sheaf \( A \). Every \( K^{-i} \cong \wedge^i A^n \) is flasque and therefore acyclic. We have: \( \text{Tor}_i^R(k[\Sigma], k) = H^{-i}(K^*(\Sigma)) = \mathbb{H}^{-i}(\Sigma, K^*) \) for \( i = 1, \ldots, n \), where \( \mathbb{H} \) denotes the hypercohomology functor.

Let us consider the decreasing sequence of open sets \( \Sigma^i = \{ \sigma \in \Sigma \mid \dim \sigma \leq d - i \} \) \( \longrightarrow \Sigma \). Define \( K^*_i \) as the complex \( f_i! (K^*_i |_{\Sigma^i}) \), that is, \( K \) restricted to \( \Sigma^i \) and then extended by zero to \( \Sigma \). We obtain a sequence of sheaf complexes \( 0 = K^*_{d+1} \hookrightarrow K^*_{d} \hookrightarrow \cdots \hookrightarrow K^*_0 = K^* \). Note that the quotient \( K^*_p/K^*_{p+1} \) is supported in \( (\Sigma_{p+1})^C \), which is a discrete topological space, so we can express \( K^*_p/K^*_{p+1} \) as a direct sum of sheaves \( \bigoplus_{\dim \sigma = d-p} K^*_\sigma \).

Standard arguments of homological algebra provide us with a spectral sequence \( E_1^{pq} = \mathbb{H}^{p+q}(\Sigma, K^*_p/K^*_{p+1}) \Rightarrow \mathbb{H}^{p+q}(\Sigma, K^*) \).

We can decompose the \( E_1 \)-terms as \( E_1^{pq} = \bigoplus_{\dim \sigma = d-p} \mathbb{H}^{p+q}(\Sigma, K^*_\sigma) \).

**Claim:** condition \( \square \) implies that \( E_1^{pq} = 0 \) for \( 0 \leq p \leq d \) and \( p + q < d - n + 1 \), or, equivalently, that, if \( \dim \sigma \geq 0 \), then \( \mathbb{H}^i(\Sigma, K^*_\sigma) = 0 \) for \( l < d - n + 1 \).

**Proof of the claim.** There is another standard spectral sequence, converging to the hypercohomology: \( E_2^{ij} = H^j(\Sigma, \mathcal{H}^i(K^*_\sigma)) \Rightarrow \mathbb{H}^{i+j}(\Sigma, K^*_\sigma) \), where \( \mathcal{H}^i \) denotes the \( j \)-th cohomology sheaf of a complex of sheaves. Lemma \[\text{Lemma 1.4}\] implies: \( H^j(\Sigma, \mathcal{H}^i(K^*_\sigma)) = \mathbb{H}^{j-\dim \sigma}(\Sigma, \mathcal{H}^i(K^*_\sigma)) \). Because of condition \( \square \), these groups vanish for \( 0 < i < \dim(\text{lk} \sigma) + 1 \Leftrightarrow 0 < i < d - \dim \sigma \) (and they vanish trivially for \( i > \dim(\text{lk} \sigma) + 1 = d - \dim \sigma \)). Two cases remain: \( i = 0 \) and \( i = d - \dim \sigma \).

If \( \sigma \) is maximal, \( \dim \sigma = d \), \( \text{projdim}_R(A_\sigma) = n - d - 1 \), and, by Lemma \[\text{Lemma 1.4}\]

\( H^0(\Sigma, \mathcal{H}^j(K^*_\sigma)) = H^{d-\dim \sigma}(\Sigma, \mathcal{H}^j(K^*_\sigma)) = \text{Tor}_j^R(A_\sigma, k) = 0 \) for \( j < d - n + 1 \).

If \( \sigma \) is not maximal, we know, by Lemma \[\text{Lemma 1.4}\] that \( H^0(\Sigma, \mathcal{H}^j(K^*_\sigma)) = 0 \). Since \( \text{Tor}_j^R(A_\sigma, k) = 0 \) for \( j < \text{projdim}_R(A_\sigma) = \dim \sigma - n + 1 \), we conclude that, also when \( i = d - \dim \sigma \), \( E_2^{ij} = 0 \) for \( i + j < (d - \dim \sigma) + (\dim \sigma - n + 1) = d - n + 1 \). **The claim is proved.**

So \( \mathbb{H}^*(\Sigma, K^*_\sigma) = \mathbb{H}^*(\Sigma, K^*_\sigma) \). Since the differentials of \( K^*_\sigma \) are zero, we can decompose \( K^*_\sigma = \bigoplus_{i=-n}^{d} K^*_\sigma \); here \( K^*_\sigma \) is the complex everywhere zero except in degree \( i \), where it is the constant sheaf \( K^*_\sigma \).

Since \( \mathbb{H}^*(\Sigma, K^*_\sigma) = H^{r-i}(\Sigma, K^*_\sigma) =\)
Let $R = k[X_1, \ldots, X_n]$; let $S$ be an $R$-algebra, finitely generated as an $R$-module, and such that $\text{ht} \mathfrak{M}$ is the same for all maximal ideals $\mathfrak{M} \subset S$. Then $S$ is Cohen-Macaulay if and only if $\text{Tor}_i^R(S, k) = 0$ for all $i > n - d$, where $d = \dim S$.

**Proof.** By definition, $S$ is CM if and only if $d = \dim S_{\mathfrak{M}} = \text{depth } S_{\mathfrak{M}}$ for all maximal ideals $\mathfrak{M} \subset S$. The Auslander-Buchsbaum formula ([BH, Theorem 1.3.3]), applied to $S_{\mathfrak{M}}$ as an $R_m$ module (with $m = \mathfrak{M} \cap R$) yields: $\text{depth}_{R_m}(S_{\mathfrak{M}}) + \text{projdim}_{R_m} S_{\mathfrak{M}} = \text{depth } R_m = n$. By [BH, Theorem 1.3.2], $\text{projdim}_{R_m} S_{\mathfrak{M}} = \sup \{i \mid \text{Tor}_i^{R_m}(S_{\mathfrak{M}}, k) \neq 0\}$. We conclude by noting that this last number equals $\sup \{i \mid \text{Tor}_i^{R}(S, k) \neq 0\}$, since $\text{Tor}_i^{R_m}(S_{\mathfrak{M}}, k) \cong R_m \otimes_R \text{Tor}_i^{R}(S, k)$; and $\text{depth}_{R_m}(S_{\mathfrak{M}}) = \text{depth}_{S_{\mathfrak{M}}}(S_{\mathfrak{M}})$ ([BH, Ex. 1.2.26]).

We give now the proof of Reisner’s Theorem for a generic simplicial complex $\Sigma$.

**Proof.** (of Proposition 1.3) First, we recall that a Cohen-Macaulay simplicial complex is pure ([BH, Cor. 5.1.5]). So, if $k[\Sigma]$ is Cohen-Macaulay, we can apply Proposition 2.1 and obtain conditions (1) and (2). Conversely, suppose first that condition (1) holds for $\Sigma$, but condition (2) does not. Then we can apply Proposition 2.1 to $k[\Sigma]$ is not Cohen-Macaulay. If condition (1) does not hold, then there exists a maximal face $\sigma \in \Sigma$ among the faces such that $\tilde{H}_i(lk \sigma, k) \neq 0$ for some $i < \dim (lk \sigma)$. Then $lk \sigma$ satisfies condition (1) but not condition (2). We can apply Proposition 2.1 to $lk \sigma$: $k[lk \sigma]$ is not Cohen-Macaulay. By Lemmas 1.1 and 1.2, $k[\Sigma]$ is not Cohen-Macaulay.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI BOLOGNA, PIAZZA DI PORTA SAN DONATO 5, 40126 BOLOGNA, ITALY

E-mail address: baggio@dm.unibo.it