Abstract

This paper studies sparse super-resolution in arbitrary dimensions. More precisely, it develops a theoretical analysis of support recovery for the so-called BLASSO method, which is an off-the-grid generalisation of $\ell^1$ regularization (also known as the LASSO). While super-resolution is of paramount importance in overcoming the limitations of many imaging devices, its theoretical analysis is still lacking beyond the 1-dimensional (1-D) case. The reason is that in the 2-dimensional (2-D) case and beyond, the relative position of the spikes enters the picture, and different geometrical configurations lead to different stability properties. Our first main contribution is a connection, in the limit where the spikes cluster at a given point, between solutions of the dual of the BLASSO problem and Hermite polynomial interpolation ideals. Polynomial bases for these ideals, introduced by De Boor, can be computed by Gaussian elimination, and lead to an algorithmic description of limiting solutions to the dual problem. With this construction at hand, our second main contribution is a detailed analysis of the support stability and super-resolution effect in the case of a pair of spikes. This includes in particular a sharp analysis of how the signal-to-noise ratio should scale with respect to the separation distance between the spikes. Lastly, numerical simulations on different classes of kernels show the applicability of this theory and highlight the richness of super-resolution in 2-D.

1 Introduction

Sparse super-resolution is a fundamental problem of imaging sciences, where one seeks to recover the positions and amplitudes of pointwise sources (so-called “spikes”) from linear measurements against smooth functions. A typical example is deconvolution, where the measurements correspond to a low-pass filtering (or equivalently the observation of low-frequency Fourier coefficients), and is at the heart of fluorescent microscopy techniques such as PALM and STORM [5, 49]. More generally, the measurements need not to be translation-invariant, and this is for instance the case in MEG/EEG [3] where the location of pointwise activity sources is crucial [32].

The fundamental question in all these fields is to understand the super-resolution limit (often called the “Rayleigh limit”) of some computational method. This corresponds, for a given signal-to-noise ratio, to the minimum allowable separation limit between spikes so that their locations can be estimated. Certifying whether (or not) this limit goes to zero as the noise level drops, and at which speed, is a difficult problem, which until now, has mostly been addressed in 1-D. These questions are even more involved in higher-dimensions, because the geometry of the spikes configurations is much richer, and in contrary to the 1-D setting, these geometric configurations (e.g. whether 3 spikes are aligned or not) are expected to impact the super-resolution ability. It is the purpose of this paper to shed some light on higher-dimensional super-resolution phenomena.
1.1 BLASSO and Super-resolution

In this work, we focus on inverse problems for arbitrary dimension $d$. To this end, we consider the underlying domain to be $\mathcal{X} = \mathbb{R}^d$ or $\mathcal{X} = \mathbb{R}^d / \mathbb{Z}^d$ the $d$-dimensional torus. The sparse recovery methods we consider are framed as optimization problems on the Banach space $\mathcal{M}(\mathcal{X})$ of bounded Radon measures on $\mathcal{X}$. The unknown data to recover is thus a measure $m_0 \in \mathcal{M}(\mathcal{X})$, and one has access to linear measurements $y$ in some Hilbert space $\mathcal{H}$, with

$$y = \Phi m_0 + w \in \mathcal{H}$$

where $w \in \mathcal{H}$ models the acquisition noise, and $\Phi : \mathcal{M}(\mathcal{X}) \to \mathcal{H}$ is an operator of the form

$$\Phi : m \in \mathcal{M}(\mathcal{X}) \mapsto \int_{\mathcal{X}} \varphi(x) \eta(x),$$

defined through its kernel $\varphi : x \in \mathcal{X} \mapsto \varphi(x) \in \mathcal{H}$. This kernel is assumed to be smooth, and it is indeed this smoothness that makes the operator “coherent” and ill-posed, thus requiring some sort of regularization technique to invert the forward imaging model $\Phi \|y\|$. 

A typical example is deconvolution, which corresponds to the translation invariant problem, where $\varphi(x) = \psi(x - \cdot)$ for some $\psi \in L^2(\mathcal{X})$ and $\mathcal{H} = L^2(\mathcal{X})$. On $\mathcal{X} = \mathbb{R}^d / \mathbb{Z}^d$, when $\psi$ has a finitely supported Fourier spectrum $(\hat{\psi}(\omega))_{\omega \in \Omega}$, up to a rescaling of the measurement, this is equivalent to consider directly a sampling of the Fourier frequencies, i.e. $\varphi(x) = (\hat{\psi}(\omega))(x)$ with $\mathcal{H} = \mathbb{C}^{|\Omega|}$. Non-translation invariant operators also ubiquitous in imaging, for instance non-stationary blurs or indirect observation on the boundary of the domain as for instance in EEG/MEG. Denoting $S \subset \mathcal{X}$ the domain of interest (for instance the boundary of the skull for brain imaging), these techniques can be modelled as a sub-sampled convolution, i.e. $\varphi(x) = (\psi(x-z))_{z \in \partial S}$, where typically $\psi$ is some kernel associated to a stationary electric or magnetic field. Another example of non-convolution problems routinely encountered in imaging is the Laplace transform $\varphi(x) = e^{-\langle x, \cdot \rangle}$. Lastly, let us mention the problem of mixture estimation, which can also be framed as a multi-dimensional sparse recovery problem [33].

This work focuses on the recovery of a superposition of point sources, which are measures of the form $m = \sum_{i=1}^N a_i \delta_{z_i}$, where $a_i \delta_{z_i}$ is a Diracs (or a “spike”) at position $z i \in \mathcal{X}$ and with amplitude $a_i \in \mathbb{R}$. Following several recent works (reviewed in Section 1.2 below), we regularize using the total variation norm $|m|(\mathcal{X})$ of $m \in \mathcal{M}(\mathcal{X})$, which is defined as

$$|m|(\mathcal{X}) \overset{\text{def}}{=} \sup \left\{ \int_{\mathcal{X}} \eta(x) \mathrm{d}m(x) : \eta \in C(\mathcal{X}), \|\eta\|_{L^\infty} \leq 1 \right\}.$$ 

This corresponds to the generalization of the $l^1$ and $L^1$ norms of vectors and functions, since one has $|\sum a_i \delta_{z_i}|(\mathcal{X}) = \|a\|_{l^1}$ and if $m$ has a density $f = \frac{m}{\mathcal{X}}$ with respect to the Lebesgue measure $\mathcal{X}$, then $|m|(\mathcal{X}) = \|f\|_{L^1(\mathcal{X})}$.

We thus consider a least-squares total-variation regularization

$$\arg\min_{m \in \mathcal{M}(\mathcal{X})} |m|(\mathcal{X}) + \frac{1}{2\lambda} \|\Phi m - y\|^2_{\mathcal{H}}, \quad (P_\lambda(y))$$

where $\lambda > 0$ is the regularization parameter, which should be adapted to the noise level $\|w\|$ (theoretical results such as ours advocating a linear scaling between $\lambda$ and $\|w\|_{\mathcal{H}}$). In the case of noiseless measurements $y$ (i.e. $w = 0$), the limit of $(P_\lambda(y))$ as $\lambda \to 0$ is the constrained problem

$$\arg\min_{m \in \mathcal{M}(\mathcal{X})} |m|(\mathcal{X}) \text{ subject to } \Phi m = y \quad (P_0(y))$$

The purpose of this paper is to study the structure (and in particular the support) of the solutions to $(P_\lambda(y))$ when $\|w\|$ and $\lambda$ are small, and the positions of the spikes $(z_i)_i$ of the sought after measure $m_0$ cluster around a fixed point.
1.2 Previous Works

On-the-grid LASSO There is a long history on the use of convex methods, and in particular the $\ell^1$ norm, to recover sparse signals on a grid. This was pioneered by geophysicists [16, 40, 51] and then became mainstream in signal processing [15] and for model selection in statistics [58]. General theoretical approaches (only constraining the sparsity of the signal), such as those used to study compressed sensing [27, 14], are however ineffective for super-resolution, where the linear measurement operator is highly coherent. The theory of super-resolution requires to impose constraints on the minimum separation distance between the spikes in place of the total number of those spikes.

Off-the-grid BLASSO, minimum-separation condition. In order to avoid introducing a-priori a fixed grid, which might be detrimental both theoretically and computationally, it makes sense to consider the “off-the-grid” setting introduced in the previous section 1.1, as exposed by several authors, including [22, 10, 13]. The ground breaking work of Candès and Fernandez-Granda [13], for the low-pass filter, proved that under a $\mathcal{O}(1/f_c)$ minimum separation condition between the spikes, exact recovery is achieved. This initial result has been extended to include noise stability [12, 2]. Further refinements have been proposed, for instance statistical bounds [6] and exact support recovery condition [24] for more general (not necessarily translation invariant) measurements. This line of theoretical study works in arbitrary dimensions, but this does not correspond to a super-resolution regime ($\mathcal{O}(1/f_c)$ being often considered as the natural Rayleigh). For spikes with arbitrary signs, it is easy to construct counter example below the Rayleigh limit where the BLASSO does not work even when there is no noise (this is to be contrasted with Prony-type approaches, see below).

BLASSO, positive spikes in 1-D. Going below the Rayleigh limit requires an analysis of the signal-to-noise scaling, i.e. at which speed the signal-to-noise ratio SNR should drop to zero as a function of the spikes separation distance. This scaling is typically polynomial, and the exponent depends on the number of spikes that cluster around a given location. The study of this scaling was initiated by [20], for the combinatorial search method (thus not numerically tractable). With such non-convex methods, the optimal scaling is obtained with no constraint on the sign pattern of the spikes, see also [24] for a refined analysis.

As noticed above, BLASSO $\{P_\lambda(y)\}$ in general cannot go below the Rayleigh limit for measures with arbitrary signs. For positive spikes, BLASSO achieves super-resolution for some classes of measurement operators $\Phi$. This was initially studied for the LASSO problem (on a grid) [28, 31], and for the BLASSO, this holds for low-pass Fourier measurements and polynomial moments [22]. This exact recovery of positive spikes can also cope with sub-sampling [53], but this is restricted to the 1-D setting.

Stability to noise for the LASSO (on discrete grid) is studied in [45], with a signal-to-noise scaling $\|w\| = \mathcal{O}(t^N)$ where $N$ is the number of spikes clustered in a radius of $t$. This result holds in 1-D and 2-D, see also [4]. In the 1-D BLASSO case, a signal-to-noise scaling $\|w\| = \mathcal{O}(t^{N-1})$ actually leads to exact support recovery under a non-degeneracy condition (which is true for the Gaussian convolution kernel) [25]. Our work proposes extensions of this last result in arbitrary dimensions.

BLASSO, positive spikes in multiple-dimension. Only few theoretical works have studied sparse super-resolution in more than one dimension. Let us single out the work of [15], which proves that 2-D super resolution on a discrete grid has a similar signal-to-noise scaling with spikes separation distance $t$ as in 1-D. This however does not provide an understanding of how super-resolution and support recovery operates in multiple dimensions, which in turn requires to work off-the-grid, using the BLASSO, as we do in this paper. Let us also note the work of [53] which studies scaling of statistical decision-theoretic bounds for pairs of spikes. This is inline with our study in Section 5 which shows that the BLASSO achieves the same signal-to-noise scaling.

Prony’s type methods. While this paper is dedicated to convex $\ell^1$-type methods, there is a large body of methods and analysis that use non-convex or non-variational approaches. These methods are very often generalizations of the initial idea of Prony [47] which encodes the spikes positions as the zeros of some
polynomial, whose coefficients are derived from the measurements, see [56] and the review paper [38]. Let us for instance cite MUSIC [54], matrix pencil [33], ESPRIT [48], finite rate of innovation [7], Cadzow’s denoising [11, 19].

It is not the purpose of this paper to advocate for or against the use of convex methods, and there are pros-and-cons both in term of both practical performances and theoretical understanding. An important advantage of Prony based approaches is that, in the noiseless setting \( w = 0 \), they achieve exact recovery without any condition on the sign of the spikes (whereas BLASSO requires either a minimum separation distance or positivity). The theoretical analysis of these approaches in the presence of noise is however more intricate, and only partial results exist. Cramer-Rao statistical bounds can be derived [18] and non-asymptotic bounds have been proposed under minimum-separation conditions [11, 44]. A second difficulty with Prony based approaches is that they are non-trivial to extend to higher dimensions, and there is no general agreement on a canonical formulation even in 2-D. We refer for instance to [46, 39, 52, 50, 17, 1] and the references therein for several such extensions. Furthermore, in the noiseless setting, in contrary to the 1-D setting, the number of recovered spikes does not scale linearly with the number of measurements [37].

**Numerical Methods for the BLASSO.** While it is not the purpose of this paper to develop numerical solvers, let us sketch some pointers to solvers for the BLASSO problem. The BLASSO is computationally challenging because it is an infinite dimensional optimization problem. The most straightforward approach is to approximate the problem on a grid, which then becomes a finite dimensional LASSO (which itself is a linear program). This however leads to quantization artifacts, typically doubling the number of observed spikes in 1-D [30].

In the case of a finite number of Fourier frequencies, the dual optimization problem is finite dimensional. In 1-D, it can be solved exactly by lifting to a semi-definite program in \( O(f_c^2) \) variables [13]. In 2-D and on more general semi-algebraic domains, this lifting is more involved numerically, and requires the use of a Lasserre hierarchy [23]. For general measurements, [10] proposed to use the Frank-Wolfe algorithm, which operates by adding in a greedy-manner new spikes, and is a convex counterpart of the celebrated matching pursuit algorithm [43] over a continuous dictionary [35]. It has a slow convergence rate, which is improved by interleaving non-convex optimization steps, also used in [9]. This algorithm is in practice very efficient, and leads to state of the art results in 2-D and 3-D resolution, for instance with application to single-molecule imaging [9].

### 1.3 Contributions

We begin Section 2 with a recap on the link between support stability and the dual solutions of \( \{P_\lambda(y)\} \). As already noted for instance in [29], support stability of \( \{P_\lambda(y)\} \) when \((w, \lambda)\) are small, is governed by a minimal norm dual solution associated with \( \{P_\lambda(y)\} \), and as shown in [25] for the 1-D case, analysis of the limit (as the minimum separation distance \( t \) tends to 0) of these minimum norm dual solutions lead to a clear understanding of the behaviour of support stability. As a contribution, we also provide a closed form expression for this limiting certificate in the case of the ideal low pass filter with \( N = f_c \), thus complementing some of the numerical observations from [25].

Our first main contribution is detailed in Section 2.3.4 where we provide a characterisation of the limiting certificate in the multi-dimensional setting. Furthermore, we provide a closed form expression for this limiting certificate in the case of the Gaussian filter. Due to the necessity of this certificate, our analysis thus sheds light on the behaviour of support stability in arbitrary dimensions.

Our second main contribution, detailed in Section 3, is a detailed analysis of the structure of the solution to \( \{P_\lambda(y)\} \) in the case of \( N = 2 \) spikes. Under a non-degeneracy condition on the limiting dual certificate, we show that that the solution to the BLASSO is composed of two spikes, and we precisely characterise how the signal-to-noise should scale with the separation between the spikes for this recovery to hold.

Lastly, Section 4 showcases numerical illustrations of these theoretical advances. The code to reproduce the results of this paper is available online[1].

[1]: https://github.com/gpeyre/2017-MSL-super-resolution
2 Asymptotics of Dual Certificates

2.1 Vanishing Pre-certificate \( \eta_{V,Z} \)

A positive discrete measure \( m_0 = \sum_i a_i \delta_{z_i} \in \mathcal{M}(\mathcal{X}) \) is solution to \( \{P_0(y)\} \) if and only if the set of Lagrange multipliers of the constraint, often referred to in the literature as “dual certificates”

\[
\mathcal{D}(Z) = \{ \eta \in \text{Im}(\Phi^*) : \|\eta\|_\infty \leq 1, \forall i, \eta(z_i) = 1 \}
\]

is non-empty, where we denote by \( Z = (z_i)_{i=1}^N \in \mathcal{X}^N \) the spikes locations. Proving that \( \mathcal{D}(Z) \neq \emptyset \) is thus a Lagrange interpolation problem using continuous interpolating functions in \( \text{Im}(\Phi^*) \), and with the additional constraint that \( \eta \) should be bounded by 1. In the following, we say that a smooth function \( \eta \) is non-degenerate for the positions \( Z \) if

\[
\forall x \notin Z, \quad \eta(x) < 1, \quad \text{and} \quad \forall i = 1,\ldots,N, \quad \nabla^2 \eta(z_i) < 0. \tag{ND(Z)}
\]

Here, we denote by \( \nabla^2 \eta(x) \in \mathbb{R}^{d \times d} \) the Hessian matrix of \( \eta \) at \( x \in \mathcal{X} \), and \( A \prec 0 \) means that the matrix \( A \in \mathbb{R}^{d \times d} \) is negative definite. Throughout this paper, we shall use \( \nabla^j \) to denote the full gradient of order \( j \), for \( \alpha \in \mathbb{N}_0^d, \partial \alpha \overset{\text{def.}}{=} P(\partial) \) where \( P(X) = X^\alpha \), and given \( x, z \in \mathcal{X} \) and \( f \in \mathcal{C}^3(\mathcal{X}) \), let \( \partial^j f(x) \overset{\text{def.}}{=} \frac{d^j}{d\tau^j} |_{\tau=0} f(x + tz) \). The condition \( \text{[ND(Z)]} \) is a strengthening of the condition of being a dual certificate, and is reminiscent of the non-degeneracy condition on the relative interior of the sub-differential which is standard in sensitivity analysis in finite dimension \([8]\) (recall that we are here dealing with infinite-dimensional optimization problems).

As initially shown in \([29]\), the support stability of the solution to \( \{P_\lambda(y)\} \) with small \((w, \lambda)\) is governed by a specific dual certificate with minimum norm

\[
\eta_{0,Z} \overset{\text{def.}}{=} \Phi^* p_{0,Z} \quad \text{where} \quad p_{0,Z} \overset{\text{def.}}{=} \text{argmin}_{p \in \mathcal{H}} \{ \|p\|_{\mathcal{H}} : \Phi^* p \in \mathcal{D}(Z) \}.
\]

More precisely, if \( \eta_{0,Z} \) is non-degenerated (i.e. satisfies \( \text{[ND(Z)]} \)), then for \((\|w\|/\lambda, \lambda) = O(1)\), then \([29]\) shows that the solution of \( \{P_\lambda(y)\} \) is unique and composed of \( N \) spikes, whose positions and amplitudes converge smoothly toward \((a_i, Z)\) as \( \lambda \to 0 \).

Direct analysis of this \( \eta_{0,Z} \), and in particular proving that it satisfies condition \( \text{[ND(Z)]} \), is difficult, mainly because of the non-linear constraint \( \|\eta_{0,Z}\|_{\infty} \leq 1 \). Fortunately, \([29]\) introduces a simpler proxy, which is defined by replacing this constraint by the linear one of having vanishing derivatives at the spikes positions

\[
\eta_{V,Z} \overset{\text{def.}}{=} \Phi^* p_{V,Z} \quad \text{where} \quad p_{V,Z} \overset{\text{def.}}{=} \text{argmin}_{p \in \mathcal{H}} \{ \|p\|_{\mathcal{H}} : \forall i, (\Phi^* p)(z_i) = 1, \nabla (\Phi^* p)(z_i) = 0_d \} \tag{2}
\]

where \( \nabla \eta(x) \in \mathbb{R}^d \) is the gradient vector of \( \eta \) at \( x \in \mathcal{X} \). The interest of this vanishing pre-certificate \( \eta_{V,Z} \) stems from the fact that

\( \eta_{V,Z} \) satisfies \( \text{[ND(Z)]} \) \( \implies \) \( \eta_{0,Z} \) satisfies \( \text{[ND(Z)]} \) and \( \eta_{0,Z} = \eta_{V,Z} \).

(and the converse is also true), which means that one can simply check the non-degeneracy of \( \eta_{V,Z} \) to guarantee support stability in the small noise regime.

Remark 1 (Computation of \( \eta_{V,Z} \)). It is important to note that \( p_{V,Z} \) can be computed easily by solving a linear system of size \( Q \times Q \) where \( Q \overset{\text{def.}}{=} (d + 1)N \). Indeed, introducing the correlation kernel

\[
\forall (x, x') \in \mathcal{X}^2, \quad C(x, x') \overset{\text{def.}}{=} \langle \varphi(x), \varphi(x') \rangle_{\mathcal{H}} \in \mathbb{R}, \tag{3}
\]

one has that

\[
\eta_{V,Z}(x) = \sum_{i=1}^N \sum_{k=0}^d \alpha_{i,k} \partial_{1,k} C(z_i, x) \quad \text{where} \quad \alpha = M^{-1} u_d, N \quad \text{and} \quad M = (\partial_{1,k} \partial_{2,\ell} C(z_i, z_j))_{i,j=1,\ldots,N},
\]

where \( u_d \) is the standard basis of \( \mathbb{R}^d \).
where we denoted $\partial_{1,k}\partial_{2,\ell}C(x,x') \in \mathbb{R}$ the derivative at $(x,x') \in \mathcal{X}^2$ with respect to the $k^{th}$ coordinate of $x$ and the $\ell^{th}$ coordinate of $x'$. The vector $u_{d,N} \in \mathbb{R}^Q$ is defined by

$$\forall (i,k) \in \{1, \ldots, N\} \times \{0, \ldots, d\}, \quad (u_{d,N})_{i,k} \overset{\text{def.}}{=} \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $M \in \mathbb{R}^{Q \times Q}$ operates as $M\alpha = (\sum_{j,\ell} M_{(i,k),(j,\ell)} \alpha_{j,\ell})_{(i,k)}$.

This simple expression [1] for $\eta_{V,Z}$, which only requires the inversion of a $Q \times Q$ linear system, has been used in 1-D to show that $\eta_{V,Z}$ is non-degenerate, even for measure with arbitrary sign, under a minimum separation condition and for a class of convolution operator with fast decay (including the Cauchy kernel $\varphi(x) = (1 + (x - \cdot)^2) \in L^2(\mathbb{R})$), see [27].

The main goal of this paper is to study the non-degeneracy of $\eta_{V,Z}$ in multiple dimensions. Of particular interest for us is the case where the spike locations cluster around a point (which we set without loss of generality to be 0). So, we consider spike locations of the form $tZ = (t\xi_i)_{i=1}^N$ where the scaling parameter $t > 0$ controls the minimum separation distance between the spikes. An important question is to understand whether $\eta_{V,tZ}$ converges as $t \to 0$ and to derive an explicit formula for this limit (which we denote by $\eta_{W,Z}$).

### 2.2 Asymptotic Pre-certificate in 1-D

We now summarise the results of [25], which hold in the 1-D setting, $d = 1$. First one has the convergence of $\eta_{V,tZ}$, as $t \to 0$, towards

$$\eta_W \overset{\text{def.}}{=} \Phi^* p_W \quad \text{where} \quad p_W \overset{\text{def.}}{=} \arg\min_{p \in \mathcal{H}} \left\{ \|p\|_\mathcal{H} \mid (\Phi^* p)(0) = 1, \forall s = 1, \ldots, 2N - 1, (\Phi^* p)^{(s)}(0) = 0 \right\}, \quad (4)$$

where $\eta^{(k)}$ denotes the $k^{th}$ derivative of $\eta$. Note in particular that the limit $\eta_W$ is independent of $Z$, which should be contrasted with the higher dimensional case, see Section 2.3 below.

The main result of [25] is that, if $\eta_W$ is non-degenerate, in the sense that

$$\forall x \neq 0, \quad \eta_W(x) < 1, \quad \text{and} \quad \eta_W^{(2N)}(0) < 0$$

then for $t$ small enough, $\eta_{V,tZ}$ is also non-degenerate, and one can compute a sharp estimate of the support stability constant involved as a function of $t$. More precisely, one should have $\|w\|_\mathcal{H}/\lambda = O(1)$ and $\lambda = O(t^{2N-1})$ in order for $P_\lambda(y)$ to recover the correct number $N$ of spikes and to have smoothly converging positions and amplitudes as $\lambda \to 0$.

**Remark 2** (Computation of $\eta_W$). Similarly to (3), $\eta_W$ is conveniently computed by solving a finite dimensional linear system of size $2N \times 2N$

$$\eta_W(x) = \sum_{s=0}^{2N-1} \beta_s \partial_1^{(r)} C(0,x) \quad \text{where} \quad \beta = R^{-1} \delta_{2N} \quad \text{and} \quad R = (\partial_1^{(r)} \partial_2^{(s)} C(z_i, z_j))_{r,s=0,\ldots,2N-1} \in \mathbb{R}^{2N \times 2N}, \quad (5)$$

where $\partial_1^{(r)}$ and $\partial_2^{(s)}$ are the $r^{th}$ and $s^{th}$ order derivatives with respect to the first and second variables (with the convention $\partial_1^{(0)} C = C$), and $\delta_{2N} \overset{\text{def.}}{=} (1, 0, \ldots, 0)^* \in \mathbb{R}^{2N}$.

The simple expression [5] allows easily to study $\eta_W$, and numerical computations shows that it is indeed non-degenerate for many low pass filters [25]. It can even be computed in closed form in the case of the Gaussian filter, see also Section 2.5.1 below.

The following proposition, which is new and thus a contribution of our paper, shows that one can also compute $\eta_W$ in closed form in the case of an ideal low-pass filter for the special case $N = f_c$, and that both $\eta_{V,Z}$ and $\eta_W$ are non-degenerate. The general case of $f_c \neq N$ is still an open problem.
\textbf{Theorem 1.} Let $\mathcal{X} = \mathbb{Z}/\mathbb{R}$. Let $\tilde{\varphi}_D(x) = \sum_{|k| \leq f_c} e^{2\pi ikx}$ where $f_c \in \mathbb{N}$ is the cutoff frequency. Let $\varphi(x) = \tilde{\varphi}_D(-x)$. If $f_c = N$, then $\eta_{W,Z}(x) < 1$ for all $x \notin z$, and one has

$$\eta_{W}(x) = 1 - C \sin 2^N(\pi x),$$

for some $C > 0$ (whose explicit form can be found in \cite{35}). So in particular, $\eta_{W}(x) < 1$ for all $x \in \mathcal{X} \setminus \{0\}$.

The proof of this theorem can be found in Appendix \[B\.

### 2.3 Asymptotic Pre-certificate in Arbitrary Dimension

We now aim at generalizing the expression (5) to the case $d > 1$.

#### 2.3.1 General Definition of Vanishing Pre-Certificates

The first difficulty is that the limit of $\eta_{W,Z}$ as $t \to 0$, provided that it exists, depends on the direction of convergence $Z \in \mathcal{X}^N$, as illustrated by Figure \[1\] and we will thus denote it $\eta_{W,Z}$. Intuitively, the difficulty is to identify which derivatives of $\eta_{W,Z}$ should vanish in the limit. They should span a space of dimension $Q \overset{\text{def.}}{=} N(d + 1)$, since this matches the number of constraints appearing in (2).

We denote $\Pi^d$ the space of polynomials in $d$ variables $(X_1, \ldots, X_d)$ and for a $d$-tuple $k = (k_1, \ldots, k_d)$, the associated monomial $X^k \overset{\text{def.}}{=} X_1^{k_1} \cdots X_d^{k_d}$. To ease the description of these constraints, for a polynomial $P = \sum_k a_k X^k \in \Pi^d$, we denote $P(\partial)$ the differential operator

$$P(\partial) \overset{\text{def.}}{=} \sum_k a_k \partial_1^{k_1} \cdots \partial_d^{k_d}$$

where $\partial_s$ is the derivative with respect to the $s$th variable.

The construction of the limiting certificate requires to identify a linear subspace $S_z \subset \Pi^d$ which encodes the vanishing derivative constraints (which should have dimension $Q$) and solve

$$\eta_{W,Z} \overset{\text{def.}}{=} \Phi^* p_{W,Z} \quad \text{where} \quad p_{W,Z} \overset{\text{def.}}{=} \arg\min_{p \in \mathcal{H}} \left\{ \|p\|_\mathcal{H} ; (\Phi^* p)(0) = 0, \forall P \in \tilde{S}_z, (P(\partial)[\Phi^* p])(0) = 0 \right\},$$

where $\tilde{S} \subset S$ is the linear subspace of polynomials $P$ such that $P(0) = 0$. We show in Section \[2.3.2\] below that indeed such a space $S_z$ exists, and that it can be computed using a simple Gaussian elimination algorithm.

\textbf{Remark 3 (Computation of $\eta_{W}$).} Once again, $\eta_{W,Z}$ defined by (5) can be computed by solving a $Q \times Q$ linear system. Indeed, the linear space $S_z$ is described using a basis of polynomials

$$S_z = \text{Span} \{P_r ; r = 0, \ldots, Q - 1\}$$

to which we impose for notation convenience $P_0 = 1$ (so that $P_0(\partial)[\eta] = \eta$). Then, denoting the various derivatives of the covariance as

$$\forall (r,s) \in \{0, \ldots, Q - 1\}^2, \quad C_{r,s} \overset{\text{def.}}{=} P_r^{(1)}(\partial) P_s^{(2)}(\partial)[C]$$

where here we have use the notations $P_r^{(1)}(\partial)$ and $P_s^{(2)}(\partial)$ to indicate whether the polynomial should be used to differentiate on the first variable $x$ or the second variable $x'$ of $C(x,x')$ (in particular $C_{0,0} = C$), one has

$$\eta_{W,Z}(x) = \sum_{r=0}^{Q-1} \beta_r C_{r,0}(0,x) \quad \text{where} \quad \beta = R^{-1}\delta_Q$$

and $R = \left(C_{r,s}(0,0)\right)_{r,s=0,\ldots,Q-1} \in \mathbb{R}^{Q \times Q}$.

Note that in dimension $d = 1$, the expressions (5) and (6) are equivalent to those already given in (4) and (5) when using the monomial basis $S_z = \text{Span}\{X_1^r \}_{r=0}^{2N}$, with $Q = 2N$. 

7
Figure 1: Display of the evolution of $\eta_{\nu,tZ}$ for $t \to 0$. Top: 1-D Gaussian convolution. Middle: 2-D Gaussian convolution, $Z$ in generic positions. Bottom: same but with aligned positions $Z$. 
2.3.2 The Least Interpolant Space $L_\mathcal{F}$

The definition (2) of $\eta_{V,Z}$ involves a Hermite interpolation problem at nodes $Z = (z_i)_{i=1}^N$. Considering the asymptotic $t Z$ with $t \to 0$ of the associated interpolation problem naturally leads to the analysis (through Taylor expansion) of the behavior of polynomial interpolation. Polynomial interpolation in arbitrary dimension is notoriously difficult, and we refer to the monograph [42] for a detailed account on this topic. This is due in large part to the fact that finding suitable polynomial spaces so that the interpolation problem is regular (has a unique solution) is non trivial, and that, in contrary to the 1-D case, such a space depends on the interpolating positions $Z$. As we now explain, solving this issue is at the heart of the description of $\eta_{V,Z}$, and can be achieved in a canonical way using a construction of de Boor. Although we only use it for a specific interpolation problem (Hermite interpolation with first order derivatives only), we describe here in more generality.

An interpolation problem over a space $S \subset \Pi^d$ looks for a polynomial $P \in S$ solution of a system of equations

$$F_i(P) = c_r \quad \text{for} \quad r = 1, \ldots, Q \quad (8)$$

for some $c \in \mathbb{R}^Q$, and where $\mathcal{F} = (F_r)_{r=1}^Q$ are linear forms. Of interest for us are differential forms evaluated at the positions $(z_i)_{i=1}^N$

$$\forall i = 1, \ldots, N, \quad \forall j = 1, \ldots, k_i, \quad F_{(i,j)}(P) = \left( P_{i,j}(\partial)[P] \right) (z_i) \quad (9)$$

where we denoted $r = (i, j)$ the index, $Q = \sum_i k_i$, and $(P_{i,j})_{i,j}$ are given polynomials.

As an example, the Hermite interpolation problem in dimension $d$ uses the monomials $\{P_{i,j}\}_{j=1} = \{X^\alpha\}_{|\alpha| \leq k}$, up to a fixed degree $k$, where $|\alpha| = \sum_{s=1}^d a_s$ is the degree of the monomial. Lagrange interpolation is the special case where $n_i = 0$, and to account for the constraints appearing in (2), we need to set $k_i = 1$, so that $Q = (d+1)N$.

An important question is how one should choose the subspace $S$ for this problem to be regular, i.e. have an unique solution for any choice of right hand side $c$ in (8). In the univariate case $d = 1$, one can always choose $S = \Pi^d_n$ where the degree is $n = Q$. However, the situation is much more complicated in the multivariate case because one cannot always choose $S = \Pi^d_n$ for some $n \in \mathbb{N}_0$. To understand the issues here, first note that since $\dim \Pi^d_n = \binom{d+n}{d}$ and the number of partial derivatives to interpolate at $z_i$ is $\binom{d+k_i}{k_i}$, we would need $n$ to satisfy

$$\binom{d+n}{d} = \sum_{i=1}^N \binom{d+k_i}{d} \quad (10)$$

For instance, for the Lagrange interpolation at $Z = \{z_1, z_2\} \subset \mathbb{R}^2$ (so $d = 2$), there does not exist an integer $n$ such that (10) holds since the number of interpolation conditions is 2 while $\binom{2}{2} = 1$ and $\binom{3}{2} = 3$. Furthermore, in the case of Hermite interpolation at $Z$ with $k_i = 1$, although choosing $S = \Pi^d_2$ would satisfy (10), interpolation with this space is in fact singular for all choices of $z_1$ and $z_2$ [12].

In [24], given a finite set of linear functionals $\mathcal{F} = \{F_g\}_g$, de Boor and Ron established a general technique for finding an appropriate polynomial space $S = S_\mathcal{F}$, so that the interpolation problem is regular, i.e. such that for any $c \in \mathbb{C}^Q$, there exists a unique element $P \in S_\mathcal{F}$ such that (8) holds. This space is defined through the use of the least term in formal expansion of exponential forms.

**Definition 1** (Least term). Let $g$ be a real-analytic function on $\mathbb{R}^d$ (or at least analytic at $x = 0$, so that $g(x) = \sum_{|\alpha| = 0}^\infty a_\alpha x^\alpha$). Let $\alpha_0$ be the smallest integer $|\alpha|$ such that $a_\alpha \neq 0$. Then the least term $g_\alpha$ of $g$ is $g_\alpha = \sum_{|\alpha| = \alpha_0} a_\alpha x^\alpha$.

**Definition 2** (Exponential space). For a linear functional $F \in (\Pi^d)'$, we define the formal power series

$$g_F(x) \overset{\text{def}}{=} F(e^x) \overset{\text{def}}{=} \sum_{|\alpha| = 0}^\infty \frac{F(p_\alpha)}{j!} x^\alpha.$$  

9
where \( p_\alpha \overset{\text{def}}{=} x \mapsto x^\alpha \). Given functionals \( \mathcal{F} = \{ F_q \}_{q=1}^Q \) of the form \((9)\), we define the space

\[
\exp_{\mathcal{F}} \overset{\text{def}}{=} \text{Span}\{ g_F : F \in \mathcal{F} \},
\]

\[
\mathcal{L}_{\mathcal{F}} \overset{\text{def}}{=} \text{Span}\{ g_i : g \in \exp_{\mathcal{F}} \}.
\]

The polynomial space \( \mathcal{L}_{\mathcal{F}} \) is called the least interpolant space.

The main theorem of \([21]\) asserts that this space \( \mathcal{L}_{\mathcal{F}} \) defines regular interpolation problems. Note that of course \( \mathcal{L}_{\mathcal{F}} \) depends on the positions \( Z \). We recall below this theorem, stated as in \([42]\).

**Theorem 2** \((42)\). Let \( \mathcal{F} = \{ F_q \}_{q=1}^Q \) be functionals of the form \((9)\). Then, for any \( c \in \mathbb{R}^Q \), there exists a unique \( P \in \mathcal{L}_{\mathcal{F}} \) with

\[
\forall r = 1, \ldots, Q, \quad F_r(P) = c_r.
\]

**Example 1.** Let us consider an example presented in \([42]\), which is also serves as an explanatory example throughout this article.

Consider the problem of Hermite interpolation at \( Z = \{ z_1 \overset{\text{def}}{=} (0,0), z_2 \overset{\text{def}}{=} (0,1) \} \). The 6 linear functionals are

\[
\mathcal{F} = \{ F_{0,i} : p \mapsto p(z_i), \; F_{1,i} : p \mapsto \partial_x p(z_i), \; F_{2,i} : p \mapsto \partial_y p(z_i) \}_{i=1,2}.
\]

Then, the corresponding exponential functions are

\[
g_{F_{0,1}}(x,y) = 1, \quad g_{F_{1,1}}(x,y) = x, \quad g_{F_{2,1}}(x,y) = y,
\]

\[
g_{F_{0,2}}(x,y) = e^y, \quad g_{F_{1,2}}(x,y) = xe^y, \quad g_{F_{2,2}}(x,y) = ye^y,
\]

and \( \mathcal{L}_{\mathcal{F}} = \text{Span}\{1, x, y, xy, y^2, y^3\} \).

The polynomial space constructed by de Boor and Ron preserves many properties of univariate interpolation, but notably, \( \mathcal{L}_{\mathcal{F}} \) is of least degree in the sense that for any other space \( S \) leading to a regular interpolation problem,

\[
\dim(S \cap \Pi_n^d) \leq \dim(\mathcal{L}_{\mathcal{F}} \cap \Pi_n^d), \quad \forall n \in \mathbb{N}_0.
\]

Furthermore, if we write \( S_Z \overset{\text{def}}{=} \mathcal{L}_{\mathcal{F}} \) to be the least interpolant spaced associated to \( \mathcal{F} \) defined in \((9)\) (to highlight the dependency on \( Z \)), then we have that for all invertible matrices \( A \in \mathbb{R}^{d \times d} \) and \( x \in \mathbb{R}^d \),

\[
S_{AZ+x} = S_Z \circ A^\top = \{ P \circ A^\top : P \in S_Z \},
\]

where we denoted \( AZ + x = (AZ_i + x)_i \), \( A^\top \) the transpose matrix.

**2.3.3 The de Boor Basis \( B_\mathcal{F} \) of \( \mathcal{L}_{\mathcal{F}} \)**

As highlighted already in Remark \([3]\) to be useful from a computational point of view, it is important to describe an interpolation space \( S \) using a basis of polynomial \( \{ P_r \}_{r=0,\ldots,Q-1} \). In the case of the least interpolant space \( S_Z = \mathcal{L}_{\mathcal{F}} \) for Hermite functionals \( \mathcal{F} \) defined in \((9)\), algorithms for finding such a basis are presented in \([20]\). For simplicity, we describe one of the proposed algorithms in the special case of 2-D Hermite interpolation with the first order derivative, i.e. the case \( k_1 = 1 \), as this is the one of interest for us. However, it extends to verbatim in the general setting with interpolation conditions defined via general differential forms \((9)\), see \([20]\) for further details.

The basic procedure of computing a basis of \( S_Z \) for \( Z \in \mathcal{X}^N \) can be summarised as follows.

**Procedure 1.** \([20] \text{ Thm. } 2.7\]

1. By identifying each polynomial with its coefficients, define the Hermite interpolation operator \( V_Z : \Pi_n \to \mathbb{R}^{d N} \) as an infinite dimensional matrix (with infinitely many columns indexed by \( \mathbb{N}_0^2 \) and \( Q \) columns indexed by \( \{1, \ldots, Q\} \))

\[
V_Z : a \mapsto \{ (P_a(z_i))_{i=1}^N, (\partial_{x_1} P_a(z_i))_{i=1}^N, (\partial_{x_2} P_a(z_i))_{i=1}^N \}_{i=1}^Q \in \mathbb{R}^Q,
\]

where \( P_a = \sum_{\alpha \in \mathbb{N}_0^2} a_\alpha X^\alpha \).
2. Perform Gaussian elimination with partial pivoting \cite{49} to obtain the decomposition $V = LW$, where $L \in \mathbb{R}^{Q \times Q}$ is an invertible matrix and $W \in \mathbb{R}^{3N \times N}$ is in row reduced echelon form.

3. For each row $j$ of $W$, let $\beta_j$ be the first index of $W_j$. such that $W_{j,\alpha} \neq 0$. Define

$$\forall j = 0, \ldots, Q - 1, \quad P_j(X) \overset{\text{def.}}{=} \sum_{|\alpha| = |\beta_j|} \frac{1}{\alpha!} W_{j,\alpha} X^\alpha.$$ 

Then, $\{P_j\}_{j=1}^{Q-1}$ defines a basis of $S_Z$.

Remark 4. It is in fact sufficient to restrict $V_Z$ to the polynomial space $\Pi_{2N-1}$ because Hermite interpolation on $N$ nodes is always regular on $\Pi_{2N-1}$ \cite[Theorem 19]{12}. Therefore, since $|\Pi_{2N-1}| = 2N^2 + N$, a basis of $S_Z$ can be computed in $O(Q^2 N^2)$ operations. In particular, we can replace $V_Z$ by $\tilde{V}_Z \in \mathbb{R}^{Q \times (2N^2+N)}$ where

$$\tilde{V}_Z \overset{\text{def.}}{=} \begin{pmatrix} \binom{a_0}{l} & \binom{a_1}{l} & \binom{a_2}{l} \\ \binom{b_0}{l} & \binom{b_1}{l} & \binom{b_2}{l} \\ \binom{c_0}{l} & \binom{c_1}{l} & \binom{c_2}{l} \end{pmatrix} \in \mathbb{R}^{Q \times (2N^2+N)}$$

Note that as a result, in Step 2, $W$ is a $Q \times (2N^2 + N)$ matrix.

Remark 5. The main result of the paper \cite{20} also presents a more sophisticated construction of a basis $S_Z$, based on Gaussian elimination after appropriately grouping together columns of $V_Z$. That approach has the advantage that the resultant basis is numerically more stable and is orthogonal with respect to the product $\langle P, Q \rangle_B \overset{\text{def.}}{=} (P(\partial)||Q)(0)$. This basis will be referred to as the de Boor basis. In the following section, we shall establish a precise link between the least interpolant space and $\eta_W$ using Theorem \[1\]. We emphasize, however, that although an explicit basis is useful for computational purposes, it is rather the space $S_Z$ which determines $\eta_W$.

Example 2. Returning to Example \[1\] where $Z = \{(0,0), (0,1)\}$, we have that $V_Z = LW$, where

$$L \overset{\text{def.}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1/2 & 0 & 1/6 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1/2 \end{pmatrix}$$

and $W : a \mapsto \begin{pmatrix} a_{0,0} \\ a_{0,1} \\ a_{1,0} \\ a_{0,2} + \sum_{j \geq 4} \binom{6-2j}{j} a_{j,0} \\ a_{1,1} + \sum_{j \geq 2} \frac{1}{j} a_{1,j} \\ a_{0,3} + \sum_{j \geq 4} \frac{6j-12}{j^2} a_{0,j} \end{pmatrix}$.

By Theorem \[1\] a basis of $\mathcal{L}_x$ is therefore $\mathcal{B}_x = \{1, x, y, x^2, xy, y^3\}$.

Example 3 (Further examples). In the following, we write $x^i y^j$ for the polynomial $(x, y) \mapsto x^i y^j$.

- When $Z = \{(a_j, 0) ; j = 1, \ldots, N\}$

  $$\mathcal{B}_x = \{1, x, x^2, \ldots, x^{2N-1}, y, xy, \ldots, x^{N-1} y\}$$

  is a basis for $S_Z$.

- When $Z = \{(0,1), (0,0), (1,0)\}$

  $$\mathcal{B}_x = \{1, x, y, x^2, y^2, xy, x^3, y^3, x^2 y - y^2 x\}.$$  

- When $Z = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$

  $$\mathcal{B}_x = \{1, x, y, x^2, y^2, xy, x^3, y^3, x^2 y, x^3 y, y^3 x\}.$$
2.3.4 The Limiting Certificate

With the construction of the least interpolant space at hand, we are now ready to explicitly define the limit \( \eta_{W,Z} \) of \( (\eta_{V,tZ})_{t > 0} \) as \( t \to 0 \). We use for this the following interpolation space for spikes \( Z = (z_i)_{i=1}^N \)

\[
S_Z \overset{\text{def.}}{=} \mathcal{L}_{\mathcal{F}} \quad \text{where } \mathcal{F} \text{ is defined in } [9] \text{ with } \forall i, n_i = 1. \tag{11}
\]

To begin with, let us define an operator which will be useful for establishing the technical results of this paper. Let

\[
\Gamma_{tZ} : \mathbb{R}^{3N} \to \mathcal{H}, \quad \Gamma_{tZ} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \overset{\text{def.}}{=} \sum_{j=1}^N a_j \varphi(tz_j) + \sum_{j=1}^N b_j \partial_x \varphi(tz_j) + \sum_{j=1}^N c_j \partial_{x_2} \varphi(tz_j) \tag{12}
\]

Note that the precertificates can be written as \( p_{V,tZ} = \Gamma_{tZ}^* \frac{1}{2} \). Moreover, observe that given \( p \in \mathcal{H} \), by identifying \( p \) with the coefficients of the Taylor expansion of \( (\Phi^* p)(t \cdot) \) around 0, that is \( P = \left( \langle p, \frac{\partial^p}{\partial \varphi(0)} \rangle \right)_{\alpha \in \mathbb{N}^N_0} \), we can associate \( \Gamma_{tZ} \) with the Hermite interpolation matrix \( V_Z \) (as defined in Procedure [1]) via

\[
\Gamma_{tZ}^* p = \text{Diag}\left((1_N, t^{-1}1_N) V_Z P \right). \tag{13}
\]

**Theorem 3.** Let \( S_Z \) be the least interpolant space defined in [11] and suppose that \( \{h(\partial)\varphi(0) : h \in S_Z\} \) is of dimension \( 3N \). Then, for \( Z \in \mathcal{X}^N \), one has

\[
\left\| p_{V,tZ} - p_{W,Z} \right\|_{\mathcal{H}} = O(t) \quad \text{and} \quad \left| \eta_{V,tZ} - \eta_{W,Z} \right|_{\mathcal{L}^\infty(\mathcal{X})} = O(t)
\]

where \( p_{W,Z} \) and \( \eta_{W,Z} \) are defined in [6] using \( S_Z \).

**Remark 6.** In this article, we are interested in the limit of \( \eta_{V,tZ} \) which are defined using Hermite interpolation conditions at \( tZ \). Note however that this result holds also for the limit of certificates defined via other differential forms. In particular, given any linear subspace of polynomials \( \mathfrak{S} \) such that \( P \in \mathfrak{S} \) implies that \( P(0) = 0 \), if

\[
\tilde{p}_{V,tZ} \overset{\text{def.}}{=} \left\{ \|p\| : (\Phi^* p)(t z) = 1, P(\partial) (\Phi^* p)(t z) = 0, \forall x \in Z, P \in \mathfrak{S} \right\},
\]

then \( \|\tilde{p}_{V,tZ} - \tilde{p}_{W,Z}\|_{\mathcal{H}} = O(t) \), where \( \tilde{p}_{W,Z} \) is defined through [6] using the least interpolation space associated with \( \mathfrak{S} \cup \{x \mapsto 1\} \).

**Proof.** Suppose that we decompose \( V_Z \) via Gaussian elimination so that \( V_Z = LW \), where \( L \) is invertible and \( W \) is in row-reduced echelon form. Let \( (\beta_j)_{j=0}^{Q-1} \) be as in Step 3 of Procedure [1]. Then, using the representation of \( \Gamma_{tZ} \) from [13], we have that

\[
\Gamma_{tZ}^* P = \begin{pmatrix} 1_N \\ 0_{2N} \end{pmatrix} \iff V_Z P = \begin{pmatrix} 1_N \\ 0_{2N} \end{pmatrix} \iff WP = L^{-1} \begin{pmatrix} 1_N \\ 0_{2N} \end{pmatrix}.
\]

Note that since the first \( N \) entries of the first column of \( V_Z \) are all 1’s, we have that \( L^{-1} \begin{pmatrix} 1_N \\ 0_{2N} \end{pmatrix} = \delta_{3N} \). By definition of the \( \beta_i \)'s, we have that

\[
WP = \left( 1^{[\beta]} \sum_{|\alpha|=|\beta|} W(z_i, \alpha) (\partial^\alpha \varphi(0), p)/\alpha! + O(1^{[\beta]+1}) \right)_{i=1}^{3N}.
\tag{14}
\]

Let \( \Psi^* : \mathcal{H} \to \mathbb{R}^{3N} \) be defined by

\[
\Psi^* p = (h_i(\partial) [\Phi^* p](0))_{i=1}^{3N},
\]
where
\[ \{h_i\}_{i=1}^{3N} \overset{\text{def}}{=} \left\{ X \mapsto \sum_{|\alpha|=|\beta_i|} \frac{W_{\alpha \beta_i}}{\alpha!} X^\alpha : i = 1, \ldots, 3N \right\} \]
is known to be a basis of \( \mathcal{S}_Z \) from Theorem 3. Therefore, from (14), there exists an operator \( \tilde{\Psi}_t : \mathcal{H} \to \mathbb{R}^{3N} \) with \( \Psi_t = \mathcal{O}(t) \) such that
\[ WP = \text{diag}\left(\{t^{|\beta_i|}\}_{i=1}^{3N}\right) \left(\Psi^*p + \tilde{\Psi}_t^*p\right). \]
Therefore,
\[ \Gamma^*_{t^2} p \begin{pmatrix} 1_N \\ 0_{2N} \end{pmatrix} \iff \Psi^*p + \tilde{\Psi}_t^*p = \delta_{3N} \]
and \( p_{V_{t^2}} = (\Psi^* + \tilde{\Psi}_t^*)^\dagger \delta_{3N} = \Psi^* \delta_{3N} + \mathcal{O}(t) \) whenever \( \Psi \) is full rank. Finally, since the first row of \( V_Z \) coincides with the first row of \( W \) (thanks to the fact that the top left entry of \( V_Z \) is 1), \( h_1 \equiv 1 \). Therefore, \( p_{V_{t^2}} \) is as defined in (6) using \( \mathcal{S}_Z \). The final claim of this theorem is due to the following inequality:
\[ \|\eta_{V,Z} - \eta_{V,t^2}\|_{L^\infty(\mathcal{X})} = \sup_{x \in \mathcal{X}} \|\varphi(x), p_{W,Z} - p_{V,t^2}\|_H \leq \sup_{x \in \mathcal{X}} \|\varphi(x)\|_H \|p_{W,Z} - p_{V,t^2}\|_H. \]

\[ \square \]

Remark 7 (Computation of \( \eta_{V,Z} \)). With this theorem, it is now simple to compute \( \eta_{V,Z} \) using the scheme detailed in Remark 3 and the de Boor basis \( B = \{P_r\}_{r=0}^{2N} \) in formula (7).

A key assumption of Theorem 3 is on the dimension of the space \( \{h(\partial)\varphi(0) : h \in \mathcal{S}_Z\} \). The following proposition shows that this is satisfied for convolution kernels of sufficiently large bandwidth.

**Proposition 1.** Let \( \Phi \) be a convolution operator with \( \varphi(x) = \psi(-y) \). Let \( L \) be the smallest integer such that \( \Pi^2_L \supseteq \mathcal{S}_Z \). Suppose that \( \hat{\psi}(\alpha) \neq 0 \) for all \( \alpha \in \mathbb{N}_0^2 \) such that \( |\alpha| \leq L \). Then, given any \( z \in \mathcal{X}^N \), \( \{h(\partial)\varphi(0) : h \in \mathcal{S}_Z\} \) is of dimension \( 3N \). Furthermore, \( L \leq 2N - 1 \).

The proof of this proposition can be found in Appendix A.

### 2.4 Necessity of \( \eta_W \)

The following result shows that if there is support stability for \( m_{a,tZ} \) for all \( t \) sufficiently small and \( Z \) sufficiently close to \( Z_0 \), then \( \eta_{V,Z} \) must be a valid certificate. In particular, if \( \eta_{V,Z}(x) > 1 \) for some \( x \in \mathcal{X} \), then under arbitrarily small noise and regularization parameter \( \lambda, [P_\lambda(y)] \) will produce solutions with additional small spikes (see Section 4).

**Theorem 4.** Suppose that \( \{h(\partial)\varphi(0) : h \in \mathcal{S}_Z\} \) is of dimension \( 3N \) and that \( \varphi \in \mathcal{C}^2(\mathcal{X}) \). Suppose that there exists \( t_n \to 0 \) and \( (a_n, Z_n) \in \mathbb{R}^N \times \mathcal{X}^N \) with \( Z_n \to Z_0 \) such that \( m_{a_n,t_n,Z_n} \) is support stable: i.e. For each \( n \), there exists a neighbourhood \( V_n \subset \mathbb{R} \times \mathcal{H} \) of 0 a continuous path \( g_n : (\lambda, w) \in V_n \mapsto (a, Z) \in \mathbb{R}^N \times \mathcal{X}^N \) such that \( m_{a,Z} \) solves \( P_\lambda(y_n + w) \) with \( y_n = \Phi m_{a_n,t_n,Z_n} \). Then, \( \|\eta_{W,Z_0}\|_{L^\infty} = 1 \).

**Proof.** First note that since \( \{h(\partial)\varphi(0) : h \in \mathcal{S}_Z\} \) is of dimension \( 3N \), we have that \( \Gamma_{t_n,Z_n} \) is full rank for all \( n \) sufficiently large.

We first show that the path \( g_n \) coincides with a \( \mathcal{C}^1 \) function in a neighbourhood of 0: For each \( n \), define for \( u = (\lambda, Z) \) and \( v = (\lambda, w) \),
\[ f_n(u, v) \overset{\text{def}}{=} \Gamma^*_{t_n,Z} (\Phi_{t_n,Z} a - \Phi_{t_n,Z_n} a_n - w) - \lambda \begin{pmatrix} 1_N \\ 0_{2N} \end{pmatrix}. \]
For all \( (\lambda, w) \in V_n \), optimality of \( m_{a,Z} \) for \( (\lambda, Z) \) implies that \( f_n(g(\lambda, w), (\lambda, w)) = 0 \). Since \( f_n \) is \( \mathcal{C}^1 \), \( f_n((a_n, Z_n), 0) = 0 \), and \( f_n((a_n, Z_n), 0) \) is invertible, we can apply the implicit function theorem to deduce that there exists \( g_n \in \mathcal{C}^1 \), a neighbourhood \( V \subset \mathbb{R} \times \mathcal{H} \) of 0 and a neighbourhood \( U \subset \mathbb{R}^N \times \mathcal{X}^N \) of \( (a_n, Z_n) \)
such that \( g_* : V \rightarrow U \) and \( g_*(\lambda, w) = (a, Z) \) if and only if \( f_n((a, Z), (\lambda, w)) = 0 \). Now, by continuity of \( g_n \), there exists \( \tilde{V} \subset V \cap V \) such that \( g_n(\tilde{V}) \subset U \). Moreover, since \( f_n(g(\lambda, w), (\lambda, w)) = 0 \) for all \( (\lambda, w) \in \tilde{V} \), we have that \( g_n(\lambda, w) = g_*(\lambda, w) \) for all \( (\lambda, w) \in \tilde{V} \). Therefore, \( m_{a_n, t_n} \) is support stable with a \( \mathcal{C}^1 \) function.

We may now apply [29, Proposition 8] to conclude that \( \|\eta_{V, t_n}Z_n\|_{L^\infty} \leq 1 \). It remains to show that

\[
\eta_{V, t_n}Z_n = \Phi^t \Gamma_{t_n}^{* \dagger} \left( \begin{array}{c} 1_N \\ 0_{2N} \end{array} \right) \rightarrow \eta_WZ_0
\]

as \( n \rightarrow \infty \). To show this, note that if \( \lim_{n \rightarrow \infty} Z_n = Z_0 \), then for \( n \) sufficiently large, there exist smooth bijections \( T_n : X \rightarrow X \) such that \( T_nZ_n = Z_n \) and \( \lim_{n \rightarrow \infty} \|T_n - \text{Id}\| = 0 \). Let \( \hat{T}_n = t_nT_n \circ (t_n^{-1}\text{Id}) \). Then, \( \hat{T}_n(t_nZ_0) = t_nZ_n \). So, by Corollary 2

\[
\eta_{V, t_n}Z_n = \eta_{V, t_n}Z_0 + \mathcal{O}(\|\text{Id} - \hat{T}_n\|) = \eta_{V, t_n}Z_0 + \mathcal{O}(\|\text{Id} - T_n\|),
\]

thus yielding (15) by letting \( n \rightarrow \infty \).

\[\square\]

2.5 Special cases

2.5.1 Explicit formula of \( \eta_{W,Z} \) for Gaussian convolution

We consider the Gaussian convolution measurement operator \( \varphi(x) = \psi(x - \cdot) \in H = L^2(\mathbb{R}^d) \) on \( X = \mathbb{R}^d \) where

\[
\psi(x_1, x_2) \overset{\text{def.}}{=} (y_1, y_2) \mapsto e^{-((|x_1 - y_1|^2 + |x_2 - y_2|^2)}
\]

e.i. \( \Phi m = m \ast \psi \) is the convolution against kernel \( \psi \).

**Proposition 2.** Suppose that the de Boor basis \( B \) for \( S_Z \) is of the form

\[
\{ P_\alpha \}_{\alpha \in J} \overset{\text{def.}}{=} \{ X \mapsto X^\alpha \colon |\alpha| \leq L \} \cup \{ P_\alpha \}_{j \in J_{L+1}}
\]

where \( P_\alpha, j \in J_{L+1} \) are homogeneous polynomials of degree \( L + 1 \). Define the inner product \( \langle P, Q \rangle_B \overset{\text{def.}}{=} (P(\partial)Q)(0) \). Then,

\[
\eta_{W,Z} = e^{-(x_1^2 + x_2^2)/2} \left( \sum_{\alpha \in J} \frac{\langle P_\alpha, \tilde{\psi} \rangle_B}{\langle P_\alpha, P_\alpha \rangle_B} P_\alpha(x) \right)
\]

where \( \tilde{\psi}(x) \overset{\text{def.}}{=} \exp(\|x\|^2/2) \).

In particular, if \( B = \{ X \mapsto X^\alpha \colon |\alpha| \leq L \} \), then

\[
\eta_{W,Z}(x) = \exp\left(-\|x\|^2/2\right) \sum_{0 \leq 2\alpha_1 + 2\alpha_2 \leq L} \frac{1}{\alpha_1!^2 \alpha_2!^2} x_1^{2\alpha_1} x_2^{2\alpha_2}.
\]

In the case where \( Z \) consists of \( N \) points, all aligned along the first axis,

\[
\eta_{W,Z}(x) = \exp\left(-\|x\|^2/2\right) \sum_{0 \leq j \leq 2N-1} \frac{x_j^2}{j!}.
\]

**Proof.** First note that \( \eta_{W,Z} \) is of the unique function of the form

\[
\eta_{W,Z}(x) = \sum_{|\alpha| \leq L} a_\alpha \partial^n [\psi \ast \psi](x) + \sum_{\alpha \in J_{L+1}} a_\alpha (P_\alpha(\partial)(\psi \ast \psi))(x),
\]

with \( \eta_{W,Z}(0) = 1 \), and \( P_\alpha(\partial)\eta_{W,Z}(0) = 0 \) for \( \alpha \in J \setminus \{(0,0)\} \). Note that

\[
[\psi \ast \psi](x_1, x_2) = \frac{\pi}{2} \exp\left(-\frac{(|x_1 - y_1|^2 + |x_2 - y_2|^2)}{2}\right).
\]
Moreover,
\[
\frac{dn}{dx^n} \exp(-x^2/2) = (-1)^n \exp(-x^2/2) H_n(x)
\]
where \( H_n \) is a monic polynomial of degree \( n \) (called the Hermite polynomial of degree \( n \)).

Since \( P_\alpha \) is a homogeneous polynomial of degree \( |\alpha| \), it follows that
\[
(P_\alpha(\partial)[\psi \ast \psi])(x) = (-1)^{|\alpha|} \exp(-\|x\|^2/2)(P_\alpha(x) + f_\alpha(x))
\]
where \( f_\alpha \) is a polynomial of degree of at most \( |\alpha| - 1 \). Therefore,
\[
F(x) = \exp(\|x\|^2/2) \cdot \eta_{W,Z}(x) = \sum_{|\alpha| \leq L} a_\alpha x^\alpha + \sum_{|\alpha| = -L+1} a_\alpha P_\alpha(x).
\]
is a polynomial of degree \( L + 1 \) and and it remains to determine the coefficients \( a \). Since the de Boor basis is orthogonal w.r.t. the product \( \langle \cdot, \cdot \rangle_B \) (see Remark \[\ref{remark:deboor_basis} \]), we have that
\[
P_\alpha(\partial)F(0) = a_\alpha P_\alpha(\partial)P_\alpha(0) = [P_\alpha(\partial)\tilde{G}](0).
\]
Therefore,
\[
\eta_{W,Z} = e^{-(a_1^2 + a_2^2)} \left( \sum_{\alpha \in J} \frac{(P_\alpha, \tilde{\psi})_B}{(P_\alpha, P_\alpha)_B} P_\alpha(x) \right).
\]
The case where \( Z \) consists of aligned points can be dealt with in a similar manner. \( \square \)

### 2.5.2 Convolution Operators and Vanishing Odd Derivatives

The following proposition shows that convolution operators enjoy the property that the odd derivatives of \( \eta_{W,Z} \) vanish. This typically leads to better behaved (e.g. non-degenerate) certificates, as illustrated in Section \[\ref{section:eg_non_deg_cert} \]. More generally, this proposition shows that, if the correlation kernel of \( \Phi \) satisfies \( C(x, x') = C(-x, -x') \), then the vanishing of consecutive derivatives up to some \( N \in \mathbb{N} \) will imply the vanishing of all odd derivatives.

**Proposition 3.** Let \( C : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be such that \( C(x, x') = C(-x, -x') \). Let \( C_{\alpha,\beta} \stackrel{\text{def.}}{=} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \partial_{\bar{x}_1}^{\beta_1} \cdots \partial_{\bar{x}_d}^{\beta_d} C \).

For \( N \in \mathbb{N} \) we define \( \eta(x) \stackrel{\text{def.}}{=} \sum_{|\alpha| \leq N} b_\alpha C_{\alpha,0}(0, x) \), where
\[
b_\alpha \stackrel{\text{def.}}{=} R^{-1}(1, 0, \cdots, 0)^T, \quad R \stackrel{\text{def.}}{=} (C_{\alpha,\beta}(0, 0))_{\alpha,\beta \in \mathbb{N}_0^d, |\alpha|, |\beta| \leq N} \in \mathbb{R}^{K \times K} \quad \text{and} \quad K \stackrel{\text{def.}}{=} \binom{N + d}{N}.
\]

Then, \( \nabla^j \eta(0) = 0 \) for all odd integers \( j \in \mathbb{N} \).

**Proof.** Since \( C(x, x') = C(-x, -x') \), \( C_{\alpha,\beta}(0, 0) = 0 \) whenever \( |\alpha + \beta| \) is odd. Therefore, we may rearrange the row and columns of \( R \) so that
\[
R = \begin{pmatrix}
R_1 & 0 \\
0 & R_2
\end{pmatrix}, \quad R_1 = (C_{\alpha,\beta}(0, 0))_{\alpha,\beta \in \mathbb{N}_0^d, |\alpha|, |\beta| \text{ both even}}, \quad R_2 = (C_{\alpha,\beta}(0, 0))_{\alpha,\beta \in \mathbb{N}_0^d, |\alpha|, |\beta| \text{ both odd}}.
\]

So,
\[
R^{-1}(1, 0, \cdots, 0)^T = \begin{pmatrix}
R_1^{-1} & 0 \\
0 & R_2^{-1}
\end{pmatrix} (1, 0, \cdots, 0)^T = R_1^{-1}(1, 0, \cdots, 0)^T.
\]

Therefore,
\[
\eta = \sum_{|\alpha| \leq N} b_\alpha C_{\alpha,0}(0, x)
\]
is an even function, and therefore, all odd derivatives of \( \eta \) must vanish. \( \square \)
Remark 8. As a consequence of this lemma, consider

\[ p_{\ast} \overset{\text{def}}{=} \text{argmin} \left\{ \|p\|_{L^2} : (\Phi^*p)(0) = 1, \ \nabla^k(\Phi^*p)(0) = 0, \ k = 1, \ldots, 2k, \ \partial^\alpha(\Phi^*p)(0) = 0, \alpha \in J \right\}, \]

where \( J \) consists of multi-indices such that for \( j \in J, |j| = 2k + 1 \). For \( N \in \mathbb{N} \), let

\[ p_N \overset{\text{def}}{=} \text{argmin} \left\{ \|p\|_{L^2} : \Phi^*p(0) = 1, \ \nabla^k(\Phi^*p)(0) = 0, \ k = 1, \ldots, N \right\}. \]

and consider \( p_{2k} \) and \( p_{2k+1} \) for some \( k \in \mathbb{N} \). Then, \( \|p_{2k}\| \leq \|p_{\ast}\| \leq \|p_{2k+1}\| \). However, by the above lemma, \( p_{2k} = p_{2k+1} \). Therefore, \( p_{\ast} = p_{2k+1} \) and in particular, \( \nabla^{2n+1}(\Phi^*p_{\ast})(0) = 0 \) for all \( n \in \mathbb{N} \).

3 Pair of Spikes

In this section, we consider the problem of recovering a superposition of two spikes at positions \( tZ_0 \), \( m_0 = \sum_{j=1}^2 a_{0,j} \delta_{z_0,j} \), via BLASSO minimization \( \{P_x(y)\} \) with \( y = \Phi m_0 + w \) for small noise \( w \in \mathcal{H} \). We will show that for small \( t \), provided that the limiting certificate \( \eta_{W,Z_0} \) is non-degenerate (see Definition 3), the solution of \( \{P_x(y)\} \) is support stable with respect to \( m_0 \). By support stable, we mean that the solution is unique, has exactly 2 spikes and that the positions and amplitude of the recovered measure converge to \( m_0 \) whenever \( (\lambda, w) \) converge to 0 sufficiently fast. The main result of this section will not only establish support stability, but also give precise bounds on how fast \( (\lambda, w) \) should converge to 0.

In the 1-D case, the limiting certificate is said to be non-degenerate if \( \eta_{W}(x) < 1 \) for all \( x \neq 0 \), and its first derivative which has not been imposed to vanish at zero is negative at zero. In the case where \( \eta_{W} \) is defined on \( N \) spikes, this is the derivative of order \( 2N \). In 2-D, the behaviour of \( \eta_{W,Z} \) is in general non-isotropic, and in general, full derivatives are not imposed to vanish completely. When \( Z = (z_1, z_2) \in \mathcal{X}^2 \), recalling that \( \eta_{W,Z}(0) = 1 \),

\[ \partial^j_{d_z} \eta_{W,Z}(0) = 0, \ j = 1, 2, 3, \ \partial^2_{d_z} \eta_{W,Z}(0) = 0, \ \partial_{d_z} \partial^2_{d_z} \eta_{W,Z}(0) = 0, \]

where \( d_z = z_1 - z_2 \), the analogous notion of non-degeneracy for \( \eta_{W,Z_0} \) is as follows.

Definition 3. Let \( Z \overset{\text{def}}{=} \{z_1, z_2\} \in \mathcal{X}^2 \) and let \( d_z = z_2 - z_1 \), we say that \( \eta_{W,Z} \) is non-degenerate if \( \eta_{W,Z}(x) < 1 \) for all \( x \neq 0 \) and

\[ \begin{pmatrix} \partial^2_{d_z} \eta_{W,Z}(0) & \frac{1}{2} \partial^2_{d_z} \partial^2_{d_z} \eta_{W,Z}(0) \\ \frac{1}{2} \partial^2_{d_z} \partial^2_{d_z} \eta_{W,Z}(0) & \frac{1}{12} \partial^4_{d_z} \eta_{W,Z}(0) \end{pmatrix} < 0. \]

The main result of this section is as follows.

Theorem 5. Let \( Z_0 \in \mathcal{X}^2 \). Suppose that \( \Psi_{Z_0} \) is full rank and that \( \eta_{W,Z_0} \) is non-degenerate. Then, there exists constants \( t_0, c_1, c_2, M, \) such that for all \( t \in (0, t_0) \), all \( (\lambda, w) \in B(0, c_1 t^3) \) and \( \|w/\lambda\| \leq c_2 \),

- \( P_\lambda(y_t + w) \) has a unique solution.
- the solution has exactly \( N \) spikes and is of the form \( m_{a_tZ} \) where \( (a, Z) = g^*_t(\lambda, w) \), a continuously differentiable function defined on \( B(0, c_1 t^4) \).
- The following inequality holds:

\[ |(a, Z) - (a_0, Z_0)|_\infty \leq M \left( \frac{\|\lambda\| + \|w\|}{t^3} \right). \] (16)
Reading guide. We begin with some preliminary bounds in Section 3.1. In Section 3.2, we show that this stability is a direct consequence of the non-degeneracy transfer of \( \eta_{V,Z} \) to \( \eta_{V,Z_0} \). Observe that support stability is then a direct consequence of this non-degeneracy transfer, Theorem \( 3 \) which shows that \( \eta_{V,Z_0} \) converges to \( \eta_{V,Z} \) and the main result of [29] which shows that non-degeneracy of \( \eta_{V,Z_0} \) implies support stability. The remainder of this section is then devoted to establishing precisely how fast \( (\lambda, \|w\|_H) \) need to converge to 0 to ensure support stability. In Section 3.1, we derive more precise bounds on the convergence of \( \eta_{V,Z} \) to \( \eta_{V,Z_0} \) in the case where \( Z \in X^2 \). In Section 3.3.1 we construct a \( \mathcal{C}^1 \) mapping \( (\lambda, w) \to (a, Z) \), and show that the associated measure \( m_{a,Z} \) is indeed a solution to \( \{P_\lambda(y)\} \). Similarly to the approach of [29], this is achieved via the Implicit Function Theorem. Section 3.3.2 is then devoted to analysis of the size of the region for which this function \( g \) is defined, establishing bounds on the differential of \( g \) (which eventually leads to the convergence bounds of Theorem 5 and Section 3.3.3) proves that the measure \( m_{a,Z} \) is indeed a solution of \( \{P_\lambda(y)\} \).

3.1 Preliminaries

We have already seen from Theorem 5 that \( \eta_{V,Z} \) converges to \( \eta_{V,Z_0} \) as \( t \to 0 \). For the purpose of deriving precise estimates on the speed of convergence of \( (\lambda, w) \) for support stability, we write explicitly in this section the relationship between \( \eta_{V,Z} \) and \( \eta_{V,Z_0} \).

Lemma 1. Let \( Z = \{(u_1, u_2), (v_1, v_2)\} \). Then,

\[
\Gamma_{tZ} = \Psi_{tZ}H_{tZ}
\]

where \( \Psi_{tZ} \coloneqq \Psi_Z + \Lambda_{tZ} \),

\[
\Psi_{tZ}^* \coloneqq \begin{pmatrix}
0,0 & a_{0,0} \\
0,1 & a_{0,1} \\
a_{1,0} & a_{1,1} + (u_1 - v_1) a_{2,0} / (u_2 - v_2) \\
a_{0,3} + a_{3,0} (u_1 - v_1)^2 / (u_2 - v_2) + 3 a_{1,2} (u_1 - v_1) / (u_2 - v_2) \\
2 & t u_2 & t^2 u_2^2 / 2 & t^2 u_1 u_2 & t^3 u_2^3 / 6 \\
1 & t v_2 & t^2 v_2^2 / 2 & t^2 v_1 v_2 & t^3 v_2^3 / 6 \\
0 & 0 & 1 & 0 & t u_2 & 0 \\
0 & 0 & 1 & 0 & t v_2 & 0 \\
0 & 1 & 0 & t u_2 & t v_1 & t^2 v_2^2 / 2 \\
0 & 1 & 0 & t v_2 & t v_1 & t^2 v_2^2 / 2
\end{pmatrix},
\]

\[
H_{tZ}^* \coloneqq \begin{pmatrix}
1 & t u_2 & t u_1 & t^2 u_2^2 / 2 & t^2 u_1 u_2 & t^3 u_2^3 / 6 \\
1 & t v_2 & t v_1 & t^2 v_2^2 / 2 & t^2 v_1 v_2 & t^3 v_2^3 / 6 \\
0 & 0 & 1 & 0 & t u_2 & 0 \\
0 & 0 & 1 & 0 & t v_2 & 0 \\
0 & 1 & 0 & t u_2 & t v_1 & t^2 v_2^2 / 2 \\
0 & 1 & 0 & t v_2 & t v_1 & t^2 v_2^2 / 2
\end{pmatrix}.
\]

and \( \Lambda_{tZ} : \mathcal{H} \to \mathbb{R}^6 \) satisfies the following properties:

- \( \text{diag}(t^{-2}, t^{-1}, t^{-1}, t^{-1}, t^{-1}, t^{-1}) \Lambda_{tZ} = \mathcal{O}(1) \), in particular, \( \Lambda_{tZ} = \mathcal{O}(t) \).
- For any \( Z_0 \in X^2 \), \( \text{diag}(t^{-2}, t^{-1}, t^{-1}, t^{-1}, t^{-1}, t^{-1}) (\Lambda_{tZ} - \Lambda_{tZ}^*) = \mathcal{O}(\|Z - Z_0\|) \).

Furthermore, given \( Z_0 \in X^2 \), we have that

- \( \Psi_Z = \Psi_{Z_0} + \mathcal{O}(\|Z - Z_0\|) \)
- If \( \Psi_{Z_0} \) is full rank, then \( \Gamma_{tZ}^* \Gamma_{tZ}^{-1} (1_{0}^t) = \Psi_{Z_0}^* 1_{0}^t + \mathcal{O}(\|Z - Z_0\|) + \mathcal{O}(t) \).

Proof. We first recall that for \( p \in \mathcal{H} \), we can define a vector \( a = (a_{j,k})_{(j,k)\in \mathbb{N}^2_0} \coloneqq (\langle \partial^\alpha \varphi(0), p \rangle)_{\alpha \in \mathbb{N}^6_0} \) and write

\[
\Gamma_{tZ}^* p = V_{tZ} a,
\]

17
where $V_{tZ}$ is the Hermite interpolation matrix at $tZ$. Then, by performing Gaussian elimination on the matrix $V_{tZ}$, we obtain the following decomposition:

$$
\Gamma_{tZ}^* = H_{tZ}^* + \Psi_{tZ}^* P
$$

Note that $H_{tZ}^{-1} = \text{diag}(1, 1/t, 1/t^2, 1/t^3) \text{diag}(1, 1, t, t, t)$, and $(u_2 - v_2)^3 H_{tZ}^{-1}$ is

$$
\begin{pmatrix}
-v_2^3 + 3u_2v_2^2 - 2v_2 + v_2^3 & u_2^3 - 3u_2v_2^2 & u_2v_2^2 - 2v_2u_2^2 + v_2^3 & -u_2v_2^2(u_2 - v_2)^2 & -u_2^2(v_2 - u_2)^2 & 0 \\
6(u_2 + v_2) & -6(u_2 + v_2) & 6(u_2 + v_2) & 6(u_2 + v_2) & 6(u_2 - v_2) & 6(u_2 - v_2) \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

By inspection of $H_{tZ}^{-1} V_{tZ}$, we see that

- $h_1 = O(t^2)$ and $h_j = O(t)$ for all $j = 2, \ldots, 6$.
- The terms $h_1/t^2$ and $h_j/t$ for $j \geq 2$ are uniformly bounded in $t$ for $|t| \leq t_0$, and when considered as functions of $u_1, u_2, v_1, v_2$, they are continuous and differentiable everywhere except at $u_2 = v_2$. So, $\text{diag}(1/t^2, 1/t, \cdots, 1/t)(\Lambda_{tZ}^* - \Lambda_{tZ}) = O(||Z - Z_0||)$ provided that $Z_0 = \{(a, b), (c, d)\}$ is such that $b \neq d$. Note that the case where $b = d$ can be dealt with similarly by changing the order of Gaussian elimination.

To see that $\Psi_{tZ}^* = \Psi_{tZ}^* + O(||Z - Z_0||)$, observe that when considering $\Psi_{tZ}^* P$ as a function of $Z$, it is differentiable everywhere except at $u_2 = v_2$ and provided that $Z_0 = \{(a, b), (c, d)\}$ is such that $b \neq d$. Again, the case where $b = d$ can be dealt with similarly by changing the order of Gaussian elimination.

For the last claim, note that $L_{tZ}(\frac{12}{a_0}) = \delta_6$. So, $\Gamma_{tZ}^* = \frac{12}{a_0}$ if and only if $(\Psi_{tZ}^* + \Lambda_{tZ})^* = \delta_6$. From

$$
\Psi_{tZ} + \Lambda_{tZ} = \Psi_{tZ} + O(||Z - Z_0||) + O(t),
$$

we see that $\Psi_{tZ}$ is full rank whenever $\Psi_{tZ}$ is full rank and provided that $||Z - Z_0|| + |t|$ is sufficiently small. Therefore,

$$
(\Psi_{tZ}^* + \Lambda_{tZ})^* = \Psi_{tZ}^* + O(||Z - Z_0||) + O(t).
$$

In the case of $Z_0 = \{(0, 0), (0, 1)\}$, the de Boor basis associated with $Z_0$ is $B_{Z_0} = \{1, y, x, y^2, xy, y^3\}$. Moreover, in this case, by writing $a_{j,k} = \langle \partial_2^j \partial_y^k \varphi(0), p \rangle$ for $p \in H$,

$$
\Psi_{tZ}^* P = \begin{pmatrix}
a_{0,0} & a_{1,0} & a_{2,0} & a_{3,0} \\
a_{0,1} & a_{1,1} & a_{2,1} & a_{3,1} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

$$
\Lambda_{tZ}^* P = \begin{pmatrix}
\sum_{j \geq 4} \frac{t_{j-2}}{t_{j-2}} a_{j,0} (6 - 2j) \\
\sum_{j \geq 2} \frac{t_{j-2}}{t_{j-2}} a_{1,j} \\
\sum_{j \geq 4} \frac{t_{j-2}}{t_{j-2}} a_{0,j} (6j - 12)
\end{pmatrix},
$$

(17)
and
\[
H_{tZ}^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-6/t^2 & 6/t^2 & 0 & -4/t & -2/t & 0 \\
0 & 0 & -1/t & 1/t & 0 & 0 \\
12/t^3 & -12/t^3 & 0 & 0 & 6/t^2 & 6/t^2
\end{pmatrix}, \quad \Phi = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
t & 0 & t^2/2 & 0 & t^3/6 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
\]

In the following, let \( \Pi_{tZ} = P(\text{Im} \Gamma_{tZ}) = \text{Id} - \Gamma_{tZ}^\dagger \Gamma_{tZ} \) be the orthogonal projection of \( \text{Im} \Gamma_{tZ} \).

**Proposition 4.** Let \( Z_0 = \{(0, 0), (0, 1)\} \), \( a \in \mathbb{R}^2 \) and \( Z \in X^2 \). Then, there exists a constant \( C \) depending only on \( \varphi \) such that
\[
\left\| \Pi_{tZ} \Gamma_{tZ_0} \left( \begin{array}{c} a \\ 0 \\ 0 \end{array} \right) \right\|_{\mathcal{H}} \leq C \max \{t^2 \|Z - Z_0\|^2, t^3 \|Z - Z_0\|\}.
\]

**Proof.**
\[
\Pi_{tZ} \Gamma_{tZ_0} = \Pi_{tZ} (\Psi_{Z_0} + \Lambda_{tZ_0}) H_{tZ_0}
= \Pi_{tZ} (\Psi_{Z_0} + \Lambda_{tZ} + (\Psi_{Z_0} - \Psi_{Z}) + (\Lambda_{tZ_0} - \Lambda_{tZ})) H_{tZ_0}
= \Pi_{tZ} ((\Psi_{Z_0} - \Psi_{Z}) + (\Lambda_{tZ_0} - \Lambda_{tZ})) H_{tZ_0}
\]

Note that
\[
H_{tZ_0} \left( \begin{array}{c} a \\ 0 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c}
a_1 + a_2 \\
ta_2 \\
0 \\
t^2 a_2/2 \\
0 \\
t^3 a_2/6
\end{array} \right).
\]

Let \( B_Z \) be the de Boor basis associated with Hermite interpolation at \( Z \). Note that \( B_Z \supset \{1, x, y\} \) for all \( Z \). So, the first 3 entries of \( \Psi_{Z_0} - \Psi_Z \) are all zero. Let \( Z = \{(u_1, u_2), (v_1, v_2)\}, p \in \mathcal{H}, \) and let \( a_{j, k} = \langle \partial_x^j \partial_y^k \varphi(0), p \rangle. \)

Then, the 4th entry of \( (\Psi_{Z} - \Psi_{Z_0}) p \) is
\[
a_{2,0} (u_1 - v_1)^2 / (u_2 - v_2)^2 \lesssim \|Z - Z_0\|^2
\]
and the 6th entry of \( (\Psi_{Z} - \Psi_{Z_0}) p \) is
\[
a_{3,0} (u_1 - v_1)^3 / (u_2 - v_2)^3 + 3a_{1,2} (u_1 - v_1) / (u_2 - v_2) + 3a_{2,1} (u_1 - v_2)^2 / (u_2 - v_2)^2 \lesssim \|Z - Z_0\|.
\]

Therefore,
\[
\| \Pi_{tZ} (\Psi_{Z_0} - \Psi_{Z}) H_{tZ_0} \|_{\mathcal{H}} \leq C \max \{t^2 \|Z - Z_0\|_{\infty}^2, t^3 \|Z - Z_0\|\},
\]
where \( C \) depends only on \( \varphi \).

For any \( Z \), the 4th and 6th entries of \( \Lambda_{tZ} \) are \( \mathcal{O}(t) \), the 1st entry of \( (\Lambda_{tZ_0} - \Lambda_{tZ}) p \) is
\[
-t^2 (u_1 v_2 - u_2 v_1)^2 / 2(u_2 - v_2)^2 a_{2,0} + t^3 P_1(u_1, u_2, v_1, v_2)/(u_2 - v_2)^2,
\]
and the 2nd entry of \( (\Lambda_{tZ_0} - \Lambda_{tZ}) p \) is
\[
t(u_1 - v_2)(u_1 v_2 - u_2 v_1)/(u_2 - v_2)^2 a_{2,0} + t^2 P_2(u_1, u_2, v_1, v_2)/(u_2 - v_2)^2,
\]
where \( P_1 \) and \( P_2 \) are 4-variate polynomials. Therefore,
\[
\| \Pi_{tZ} (\Lambda_{Z_0} - \Lambda_{Z}) H_{tZ_0} \|_{\mathcal{H}} \leq C' \max \{t^2 \|Z - Z_0\|^2, t^3 \|Z - Z_0\|\},
\]
where \( C' \) depends only on \( \varphi \). Combining this bound with (18) gives the required result. \( \square \)
3.2 Non-degeneracy Transfer

We have already seen that \( \eta_{W,Z_0} \) converges to \( \eta_{W,Z_0} \) as \( t \to 0 \). In this section, we show in Proposition 5 that non-degeneracy of \( \eta_{W,Z_0} \) in the sense of Definition 9 implies that any certificate defined via Hermite interpolation conditions at \( Z \) and sufficiently close to \( \eta_{W,Z_0} \) will also be a valid certificate saturating only at \( Z \). Furthermore, we show that in Proposition 6 that \( \eta_{W,Z_0} \) is non-degenerate and as a direct consequence of the main result of [29], the solution of (\( P_{\lambda}(y) \)) is stable with respect to \( m_0 \).

**Proposition 5.** Let \( Z_0 = (z_1^*, z_2^*) \in \mathcal{X}^2 \) and let \( dz_0 = z_1^* - z_2^* \). Suppose that \( \eta_{W,Z_0}(x) \) is non-degenerate. Then, there exists \( c_1, c_2, c_3 > 0 \) such that given any \( \eta \in C^\infty \), \( t \in (0, c_2) \) and \( Z = (z_1, z_2) \in B(Z_0, c_3) \) satisfying

(i) \( \eta(tz_i) = 1, \nabla \eta(tz_i) = 0 \) for \( i = 1, 2 \),

(ii) \( \| \nabla^j \eta - \nabla^j \eta_{W,Z_0} \|_\infty \leq c_1 \) for \( |j| \leq 5 \),

we have that \( \eta(x) < 1 \) for all \( x \not\in tZ \).

**Proof.** For a contradiction, suppose that for all \( n > 0 \), there exists \( \eta_n, Z_n \in \mathcal{X}^2 \) with \( \| Z_n - Z_0 \| \leq 1/n \) and \( t_n \in (0, 1/n) \) such that for \( j = 1, 2 \), \( \eta_n(t_n z_n,j) = 1, \nabla \eta_n(t_n z_n,j) = 0 \) with \( \| \eta_n - \eta_{W,Z_0} \|_\infty \leq 1/n \) and \( x_n \not\in t_n Z_n \) such that \( \eta_n(x_n) = 1 \). Note that since \( \eta_{W,Z_0}(x) < 1 \) for all \( x \not= 0 \), we must have that \( x_n \to 0 \) as \( n \to \infty \). Let \( x_n = (f(t_n)u_n, g(t_n)v_n) \), where \( \lim_{t \to 0} f(t) = 0 \) and \( \lim_{t \to 0} g(t) = 0 \).

We first fix \( n \) and derive some equations satisfied by \( \eta_n \) and its derivatives. Without loss of generality, let \( z_n,1 = (1, 0) \) and \( z_n,2 = (-1, 0) \) (otherwise, we simply consider derivatives with respect to direction \( d_Z \) and \( d_Z^\perp \) instead of the canonical directions). To simplify notation, let us drop the subscript \( n \) and simply write \( \eta, t, c_1, c_2, u, v, Z \) for \( \eta_n, t_n, c_{n,1}, c_{n,2}, u_n, v_n, Z_n \). By expanding \( \eta \) about 0, we obtain

\[
\eta(X) = \sum_{|\alpha| \leq 3} b_\alpha X_\alpha + R_{0,4}(X)X_1^4 + R_{1,3}(X)X_1X_3^3 + R_{2,2}(X)X_2^2X_3^2 + R_{3,1}(X)X_3^3X_2
\]

\[
+ b_{4,0}X_4^4 + R_{4,1}(X)X_1^4X_2 + R_{5,0}(X)X_1^5,
\]

where given \( \alpha \in \mathbb{N}_0^2 \),

\[
b_\alpha \overset{\text{def.}}{=} \frac{\partial^\alpha \eta(0)}{\alpha!}, \quad R_\alpha(X) \overset{\text{def.}}{=} |\alpha|! \int_0^1 (1 - s)^{|\alpha| - 1} \partial^\alpha \eta(sX)ds.
\]

To simplify notation, in the following, we write \( b_\alpha = R_\alpha \) and note that thanks to assumption (ii), each of these terms is uniformly bounded in \( n \). Let

\[
t_0 \overset{\text{def.}}{=} \eta((f(t)u, g(t)v)),
\]

\[
t_1 \overset{\text{def.}}{=} \eta(tz_1), \quad \nabla X_1 t_1 \overset{\text{def.}}{=} \partial X_1 \eta(tz_1), \quad \nabla X_2 t_1 \overset{\text{def.}}{=} \partial X_2 \eta(tz_1),
\]

\[
t_2 = \eta(tz_2), \quad \nabla X_1 t_2 \overset{\text{def.}}{=} \partial X_1 \eta(tz_2), \quad \nabla X_2 t_2 \overset{\text{def.}}{=} \partial X_2 \eta(tz_2).
\]

Then, \( \gamma_1 = \frac{t_1 + t_2}{2} = b_{0,0} + t^4 b_{4,0} = 1 \), \( \gamma_2 = \frac{t_1 - t_2}{2t} = b_{1,0} + t^2 b_{3,0} + t^4 b_{5,0} = 0 \), \( \gamma_3 = \frac{\partial X_2 t_1 + \partial X_1 t_2}{2} = b_{0,1} + t^2 b_{2,1} + t^4 b_{4,1} = 0 \), \( \gamma_4 = \frac{\partial X_2 t_1 - \partial X_1 t_2}{2t} = b_{1,1} + t^2 b_{3,1} = 0 \), \( \gamma_5 = \frac{\partial X_1 t_1 - \partial X_2 t_2}{4t} = b_{2,0} + 2t^2 b_{4,0} = 0 \), \( \gamma_6 = \frac{1}{2t^2} (\partial X_1 t_1 - \partial X_2 t_2) = b_{3,0} + 2t^2 b_{5,0} = 0 \).
By subtracting appropriate multiples of \( \{\gamma_j\}_{j=1}^6 \) from \( u_0 \), we obtain

\[
0 = \gamma \overset{\text{def}}{=} u_0 - \gamma_1 - g(t)u \gamma_2 - g(t)v \gamma_3 - f(t)g(t)uv \gamma_4 - f(t)^2u^2 \gamma_5 - (f(t)^3u^3 - t^2f(t)u) \gamma_6,
\]

and so,

\[
0 = g(t)^2v^2b_{0.2} + g(t)^2(v^2b_{0.3} + f(t)uv^2b_{1.2} + g(t)^2v^2b_{0.4} + f(t)g(t)uv^3b_{1.3} + f(t)^2u^2v^2b_{2.2})
+ (f(t)^2g(t)u^2v - t^2g(t)v)b_{2.1} + uf(t)(f(t)^3v)u^2v - t^2g(t)v)b_{3.1}
+ (f(t)^2u^2 - t^2)^2b_{4.0} + (f(t)^2u^2 - t^2)^2b_{5.0} + g(t)v(f(t)^4u^4 - t^4)b_{4.1}.
\]

(19)

Note that since \( Z_n \to Z \), we have \( \lim_{n \to \infty} \partial_{d_{x_n}}^2 \partial_{d_{z_n}}^2 \eta_n(0) = \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) \). By possible extracting a subsequence, assume that \( (u_n, v_n) \to (u, v) \). Then, by considering the limit of (19), we arrive at one of the 7 following cases:

(i) If \( \lim_{t \to 0} f(t)/t \leq 1 \) and \( \lim_{t \to 0} g(t)/t^2 = 0 \), then \( \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) = 0 \).

(ii) If \( \lim_{t \to 0} f(t)/t \leq 1 \) and \( \lim_{t \to 0} g(t)/t^2 = 0 \), then \( \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) = 0 \).

(iii) If \( \lim_{t \to 0} f(t)/t \leq 1 \) and \( \lim_{t \to 0} t^2/g(t) = 0 \), then \( \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) = 0 \).

(iv) If \( \lim_{t \to 0} f(t)/t = 0 \) and \( \lim_{t \to 0} g(t)/t^2 = 1 \), then \( v^2 \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) = 0 \).

(v) If \( \lim_{t \to 0} f(t)/t = 1 \) and \( \lim_{t \to 0} g(t)/t^2 = 1 \), then \( v^2 \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) = 0 \).

(vi) If \( \lim_{t \to 0} f(t)/t = 0 \) and \( \lim_{t \to 0} g(t)/t^2 = 1 \), then \( v^2 \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) = 0 \).

(vii) If \( \lim_{t \to 0} f(t)/t \leq 1 \) and \( \lim_{t \to 0} f(t)^2/g(t) = 0 \), then \( \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) = 0 \).

In the above, the conclusion of cases (i)-(v) is obtained by taking the limit of (19) after dividing by \( t^4 \), and the conclusion of cases (vi)-(vii) is obtained by taking the limit of (19) after dividing by \( g(t)^2 \).

Therefore, it follows that there exists \((a, b) \in \mathbb{R}^2\) such that

\[
a^2 \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) + ab \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) + \frac{1}{12} b^2 \partial_{d_{x_0}}^2 \partial_{d_{z_0}}^2 \eta W,Z_0(0) = 0,
\]

which is a contradiction to the assumption that \( \eta W,Z_0 \) is non-degenerate. \( \square \)

Remark 9. From Proposition 5 it follows that \( \eta_{V,x} \) is a valid certificate for all \( t \) sufficiently small. The following result shows that \( \eta_{V,x} \) is in fact non-degenerate, and therefore, as a direct consequence of the main result of [29], the solution of \( \mathcal{P}_\chi(y) \) is support stable with respect to \( m_0 \).

Proposition 6. Let \( Z \in \mathcal{X}^2 \). Assume that \( \partial_{d_z}^2 \partial_{dZ} \eta_{W,Z}(0) = 0 \). Then,

\[
\partial_{d_z}^2 \Phi_{tZ} = \frac{t^2}{12} \partial_{d_x}^2 \eta W,Z(0) \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + \mathcal{O}(t^3), \quad \partial_{d_z} \partial_{d_x} \Phi_{tZ} = \frac{t}{2} \partial_{d_x}^2 \partial_{d_z} \eta W,Z(0) \left( \begin{array}{c} -1 \\ 1 \end{array} \right) + \mathcal{O}(t^2),
\]

\[
\partial_{d_z}^2 \Phi_{tZ} = \partial_{d_z}^2 \eta W,Z(0) \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + \mathcal{O}(t).
\]

Therefore, provided that \( \eta_{W,Z} \) is non-degenerate, then for all \( t \) sufficiently small, \( \nabla^2 \eta_{V,Z}(z) < 0 \) for all \( z \in Z \).
Proof. Without loss of generality, let $Z = \{(0,0), (0,1)\}$. We shall show that
\[
\partial^2_y \Phi^*_t Z p_{V,tZ} = \frac{t^2}{12} \partial^4_y \eta_{W,Z}(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(t^3), \quad \partial_x \partial_y \Phi^*_t Z p_{V,tZ} = \frac{t}{2} \partial^2_y \partial_x \eta_{W,Z}(0) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + O(t^2),
\]
\[
\partial^2_z \Phi^*_t Z p_{V,tZ} = t \partial^2_z \eta_{W,Z}(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(t).
\]

Let $a \in \mathbb{R}^2$. By Taylor expanding about 0, we obtain
\[
\partial^2_y \Phi^*_t Z a = \Psi Z \tilde{V}^{(1)}_{tZ} a + a_2 \sum_{j=2}^{\infty} \frac{t^j}{j!} \partial^{j+2} \varphi(0), \quad \partial_x \partial_y \Phi^*_t Z a = \Psi Z \tilde{V}^{(2)}_{tZ} a + a_2 \sum_{j=1}^{\infty} \frac{t^j}{j!} \partial^{j+1} \partial_x \varphi(0)
\]
where
\[
\tilde{V}^{(1)}_{tZ} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{V}^{(2)}_{tZ} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

and
\[
\partial^2_z \Phi^*_t Z a = a_1 \partial^2_z \varphi(0) + a_2 \left( \sum_{j=0}^{\infty} \frac{t^j}{j!} \partial^2_y \partial^2_z \varphi(0) \right).
\]

We first consider $\partial^2_y \Phi^*_t Z p_{V,tZ}$: Recall the definition of $\Lambda_{tZ}$ from [17] and observe that
\[
\tilde{V}^{(1),*}_{tZ} \left( \Psi^* Z + \Lambda_{tZ} \right) \Gamma_{tZ}^{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tilde{V}^{(1),*}_{tZ} H_{tZ}^{\frac{1}{2}} \Gamma_{tZ}^{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tilde{V}^{(1),*}_{tZ} \delta_0 = 0. \tag{20}
\]

Moreover, using the fact that $p_{V,tZ} = p_{W,Z} + O(t)$,
\[
\left( \sum_{j=2}^{\infty} \frac{t^j}{j!} \partial^{j+2} \varphi(0) \right) p_{V,tZ} = t^2 \left( p_{V,tZ}, \partial^4_y \varphi(0) \right) + O(t^3) = \left( t^2 \partial^4_y \eta_{W,Z}(0) + O(t^3) \right). \tag{21}
\]

Also,
\[
\tilde{V}^{(1),*}_{tZ} \Lambda^{\frac{1}{2}}_{tZ} p_{V,tZ} = \tilde{V}^{(1),*}_{tZ} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} t^2 \partial^4_y \eta_{W,Z}(0) + O(t^3) \\ t \partial^2_y \partial_x \eta_{W,Z}(0) + O(t^2) \end{pmatrix}. \tag{22}
\]

Summing (20), (21), and (22) gives the required bound on $\partial^2_y \Phi^*_t Z p_{V,tZ}$.

For the second bound,
\[
\partial_x \partial_y \Phi^*_t Z = \Psi Z \tilde{V}^{(2)}_{tZ} + \left( \sum_{j=1}^{\infty} \frac{t^j}{j!} \partial^{j+1} \partial_x \varphi(0) \right) = \left( \Psi Z + \Lambda_{tZ} \right) \tilde{V}^{(2)}_{tZ} - \Lambda_{tZ} \tilde{V}^{(2)}_{tZ} + \left( \sum_{j=1}^{\infty} \frac{t^j}{j!} \partial^{j+1} \partial_x \varphi(0) \right).
\]

22
As before, \( \tilde{\nu}_{1Z}^{(2),*}(\Psi_z + \Lambda_{1Z})^* p_{V,1Z} = 0 \),

\[
(\sum_{j \geq 1} \frac{t^j}{j!} \partial_y^{j+1} \partial_\nu \varphi(0), p_{V,1Z}) = t \partial_x^2 \partial_\nu \eta_{W,Z}(0) + O(t^2),
\]

and

\[
\tilde{\nu}_{1Z}^{(2),*} \Lambda^{*}_{1Z} p_{V,1Z} = \frac{t}{2} \partial_y^2 \partial_\nu \eta_{W,Z}(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(t^2).
\]

So, \( \partial_\nu \partial_\x \Phi_{1Z} p_{V,1Z} = \frac{t}{2} \partial_y^2 \partial_\nu \eta_{W,Z}(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(t^2). \)

The proof of the last bound follows because

\[
\partial_x^2 \Phi_{1Z}^* p_{V,1Z} = \partial_x^2 \Phi_{1Z}(p_W + O(t)) = \partial_x^2 \eta_{W,Z}(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(t).
\]

\[\square\]

### 3.3 Proof of Theorem 5

First note that if Theorem 5 is true for \( Z_0 \overset{\text{def}}{=} Z \) for some fixed \( Z \in \mathcal{X}^2 \), then given any \( c \in \mathcal{X} \), the result is also true for \( Z_0 \overset{\text{def}}{=} c + Z \). Let \( \Phi_{\lambda,\nu}^{\Phi,\nu} \) be the solution to the dual formulation of \( (P_\lambda(y)) \), and let \( \eta_{W}^{\Phi,y} \) be the associated precertificate. Then, by letting \( T : \mathcal{X} \to \mathcal{X}, z \mapsto z + tc \) and thanks to the reparametrization observations of Appendix C, we have

\[
\Phi_{\lambda,\nu}^{\Phi_{m_0,a}(Z_0+c)} = \Phi_{\lambda,\nu}^{\Phi_{T_{1}}}, \Phi_{\lambda,\nu}^{\Phi_{T_{2}}}, m_0,a(Z_0, \cdot - t)c).
\]

Therefore \( \eta_{W}^{\Phi_{m_0,a}(c)} = \Phi_{T_{1}}^{\nu} \). So, \( \eta_{W}^{\Phi_{m_0,a}(c)} \) is non-degenerate if and only if \( \Phi_{T_{1}}^{\nu} \) is non-degenerate. Moreover, since (provided that \( \lambda \) and \( w \) satisfies the conditions of Theorem 5) non-degeneracy of \( \Phi_{T_{1}}^{\nu} \) implies that \( \eta_{W}^{\Phi_{T_{1}}^{\nu}} \) is also true for \( Z \in \mathcal{X}^2 \), and \( m_0,a \) is the unique solution with \( a \) and \( Z \) satisfying \( 16 \), we know that \( \Phi_{T_{1}}^{\nu} \) with \( \tilde{\gamma} = \Phi_{m_0,a}(Z_0+c) + w \), saturates only at \( Z + tc \) and \( m_0,a(Z_0+tc) \) is the unique solution. Therefore, without loss of generality, it suffices to prove Theorem 5 for \( Z_0 = \{(0,0), (a,b)\} \) for some \( (a,b) \in \mathcal{X} \). Furthermore, we simply consider \( Z_0 = \{(0,0), (0,1)\} \), since otherwise, in the following, we can simply consider derivatives with respect to \( d_{Z_0} \) instead of the canonical directions.

#### 3.3.1 Implicit Function Theorem

From the first order optimality conditions of \( (P_\lambda(y)) \), we have that \( m_0,a \) solves \( (P_\lambda(y)) \) if and only if

\[
\Phi_{\lambda,\nu}^{\Phi_{m_0,a}(Z_0+c)} \in \partial |\cdot| \left( m_0,a \right).
\]

Therefore, we aim to construct a \( \mathfrak{C}^1 \) mapping \( g : (\lambda, w) \in \mathbb{R} \times \mathcal{H} \mapsto (a, Z) \in \mathbb{R}^2 \times \mathcal{X}^2 \) such that \( (a, Z) \) satisfies \( 23 \). Furthermore, bounds on the derivatives of \( g \) will provide conditions on the required speed at which \( (\lambda, w) \) converge to 0. To this end, following \( 29 \) and \( 32 \), let \( u = (a, Z) \) and \( v = (\lambda, w) \), and define

\[
f_t(u,v) \overset{\text{def}}{=} \Gamma_{1Z}^*(\Phi_{1Z}(a - \Phi_{1Z,a_0} - w) + \lambda \begin{pmatrix} 1_N \\ 0_{2N} \end{pmatrix} + \lambda \begin{pmatrix} 1_N \\ 0_{2N} \end{pmatrix}).
\]

To construct candidate solutions to \( (P_\lambda(y)) \), we search for parameters \( u \) and \( v \) for which \( f_t(u,v) = 0 \).

For \( a \in \mathbb{R}^2 \) and \( \alpha \in \mathbb{N}_0^2 \), let \( \partial^\alpha (\Phi_{1Z}) a = \sum_{j=1}^2 a_j \partial^\alpha \varphi(t_{jZ}) \). Then, the derivatives of \( f_t \) are

\[
\partial_a f_t(u,v) = \Gamma_{1Z}^* \Gamma_{1Z} J_{ta} + t \begin{pmatrix} 0 & \text{diag}(\partial_\lambda \Phi_{1Z}^* A) \\
0 & \text{diag}(\partial_\nu \Phi_{1Z}^* A) \end{pmatrix}
\]

\[
\partial_v f_t(u,v) = \begin{pmatrix} \begin{pmatrix} 1_{2N} \\ 0_{2N} \end{pmatrix} & -\Gamma_{1Z}^* \end{pmatrix}
\]

23
3.3.2 Bounds on $V_i$ rely on purely 1-D tools.

Technical details differ due to the anisotropy of the limiting certificate $\eta_v$ that follows:

So, $f_t$ is a continuously differentiable function, $\partial_a f_t(u_0, 0) = \Gamma_{tZ_0}^* \Gamma_{tZ_0} J_{t_a}$ is invertible by Proposition 1 and $f_t(u_0, 0) = 0$. Therefore, we may apply the Implicit Function Theorem to deduce that there exists a neighbourhood $V_t$ of $0$ in $\mathbb{R} \times \mathcal{H}$, a neighbourhood $U_t$ of $u_0$ in $\mathbb{R}^2 \times \mathcal{X}^2$ and a $\mathcal{C}^1$ function $g_t : V_t \to U_t$ such that for all $(u, v) \in U_t \times V_t$, $f_t(u, v) = 0$ if and only if $u = g_t(v)$. Furthermore, the derivative of $g_t$ is

$$d g_t(v) = - (\partial_a f_t(g_t(v), v))^{-1} \partial_v f_t(g_t(v), v).$$

So, to prove Theorem 5, given $(\lambda, w)$, for $(a, Z) = g((\lambda, w))$, we simply need to establish the following two facts.

1. $g_t$ is well defined on a region $V_t$ which contains a ball of radius on the order of $t^4$.
2. $m_a, Z$ is a solution of $(P_{\lambda}(y))$, i.e. it satisfies (23). To this end, we define the associated certificate as

$$p_{\lambda, t} \equiv \frac{\Phi^* (\Phi_Z a - \Phi_{tZ_0} a_0 - w)}{\lambda}, \quad \eta_{\lambda, t} \equiv \Phi^* p_{\lambda, t} \quad (26)$$

and show (Proposition 8) that $p_{\lambda}$ converges to $p_{V_t}$ as $(\lambda, w) \to 0$. Therefore, by Lemma 1 and Theorem 5 $p_{\lambda}$ must satisfy (23).

We remark that although the key steps of this proof are the same as the 1-D proof presented in [25], the technical details differ due to the anisotropy of the limiting certificate $\eta_{V, Z_0}$ and since some of the proofs in [25] rely on purely 1-D tools.

### 3.3.2 Bounds on $V_i$

For $r > 0$, let $B(0, r) \subset \mathbb{R} \times \mathcal{H}$ be defined as $B(0, r) \equiv \{ (\lambda, w) ; \lambda \in [0, r], \|w\|_{\mathcal{H}} < r \}$. To show that we can construct a function $g_t^*$ which is defined on a ball of radius $t^4$, let $V_t^*$ be defined as follows: $V_t^* \equiv \bigcup_{v \in V} V$, where $V$ is the collections of all open sets $V \subset \mathbb{R} \times \mathcal{H}$ such that

- $0 \in V$,
- $V$ is star-shaped with respect to $0$,
- $V \subset B(0, c_x t^4)$ where $c_x > 0$ is the constant defined in Lemma 2,
- there exists a $\mathcal{C}^1$ function $g : V \to \mathbb{R}^2 \times \mathcal{X}^2$ such that $g(0) = u_0$, $f_t(g(v), v) = 0$ for all $v \in V$,
- $g(V) \subset B_{c_x}(a_0) \times B_{c_x}(Z_0)$.

Note that the definition of this set $V_t^*$ is the same as in [25], except for the last condition, where we require that $\|Z - Z_0\| \leq c_s t$ for all $Z$ such that $(a, Z) \in g(V)$. This is natural, since we eventually require that the distance between $p_{\lambda, t}$ and $p_{V, Z_0}$ is $O(t)$. As explained in [25 Section 4.3], this set $V_t^*$ is well defined and non-empty. We may therefore define a function $g_t^* : V_t^* \to \mathbb{R}^2 \times \mathcal{X}^2$ where

$$g_t^*(v) \equiv g(v), \quad \text{if } v \in V, \quad V \in V, \quad \text{and } g \text{ is the corresponding function.}$$

The goal of the remainder of this subsection is to show that $V_t^*$ contains a ball of radius $t^4$. 

24
Lemma 2. Let $$G_{tZ}(\lambda, w) \overset{def}{=} \Psi_{tZ}^* \Psi_{tZ} + tH_{tZ}^{-1} F_{tZ} J_{ta}^{-1} H_{tZ}^{-1}$$ where

$$F_{tZ} \overset{def}{=} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \text{diag}(\partial^2 \Phi_{tZ}^*(q_{tZ})) & \text{diag}(\partial_y \partial_x \Phi_{tZ}^*(q_{tZ})) \\ 0 & \text{diag}(\partial_y \partial_x \Phi_{tZ}^*(q_{tZ})) & \text{diag}(\partial_y^2 \Phi_{tZ}^*(q_{tZ})) \end{pmatrix}$$

and

$$q_{tZ} \overset{def}{=} \lambda \Gamma_{tZ}^* \begin{pmatrix} 1_N \\ 0 \end{pmatrix} + \Pi_{tZ} w + \Pi_{tZ} \Gamma_{tZ0} \begin{pmatrix} a_0 \\ 0 \end{pmatrix}.$$  

There exists $$c_s > 0$$ such that for all $$Z \in \mathcal{X}^2$$ and $$w \in \mathcal{H}$$ with $$\|Z - Z_0\|_\infty \leq c_s t$$, $$\lambda \leq c_s t^4$$ and $$\|w\| \leq c_s t^4$$, $$G_{tZ}(\lambda, w)$$ is invertible and has inverse bounded by $$3\|\Psi_{tZ}^* \Psi_{tZ0}^{-1}\|$$. Moreover, if $$f_t(u, v) = 0$$, then $$\partial_u f_t(u, v) = H_{tZ}^* G_{tZ}(\lambda, w) H_{tZ} J_{ta}$$.

Proof. First observe that since for $$|\alpha| = 2$$, $$\partial^\alpha \Phi \Gamma_{tZ}^* \begin{pmatrix} 1_N \\ 0 \end{pmatrix}$$ converges uniformly to $$\partial^\alpha \Phi \psi_{w_{tZ}}$$ and $$\|Z - Z_0\|_\infty \leq c_0 t$$, $$\|\partial^\alpha \Phi \Gamma_{tZ}^* \begin{pmatrix} 1_N \\ 0 \end{pmatrix}\|$$ is uniformly bounded. Therefore, from Proposition 4 for $$|\alpha| = 2$$,

$$\|\partial^\alpha \Phi \psi_{w_{tZ}}\|_{\infty} \leq \lambda + \|w\| + t^2 |Z - Z_0|^2 + t^3 |Z - Z_0|_{\infty}.$$  

Therefore, $$\|F_{tZ}^{-1}\| \leq C c_0 t^4$$ for some constant $$C$$ which depends only on $$\varphi$$.

Recalling the definition of $$H_{tZ}$$ and $$J_{ta}$$, we have that

$$\left\| tH_{tZ}^{-1} F_{tZ} J_{ta}^{-1} H_{tZ}^{-1} \right\|_{t} \leq \left\| t^2 \text{diag}(1, \frac{1}{t}, \frac{1}{t^2}, \frac{1}{t^3}) \right\| \left\| H_{tZ}^{-1} F_{tZ} J_{ta}^{-1} H_{tZ}^{-1} \right\| \leq C \left\| H_{tZ}^{-1}\right\|_{t} \left\| J_{ta}^{-1}\right\|_{t} c_0.$$  

Therefore, by the above bound and by Lemma 1, we have that

$$\|G_{tZ} - \Psi_{tZ0} \Psi_{tZ0}\| \leq C' |Z - Z_0|_{\infty} + C \left\| H_{tZ}^{-1}\right\|_{t} \left\| J_{ta}^{-1}\right\|_{t} c_0,$$

where $$C'$$ depends only on $$\varphi$$ and the required result follows by choosing $$c_0$$ to be sufficiently small.

From (24) and since $$\Gamma_{tZ} = \Psi_{tZ} H_{tZ}$$ by Lemma 1, if $$f_t(u, v) = 0$$, then

$$\partial_u f_t(u, v) = H_{tZ}^* \begin{pmatrix} \Psi_{tZ} \Psi_{tZ} + tH_{tZ}^{-1} q_{tZ}^* A & 0 \\ 0 & \text{diag}(\partial^2 \Phi_{tZ}^* A) \end{pmatrix} J_{ta}^{-1} H_{tZ}^{-1} H_{tZ} J_{ta}$$

where $$A = \Phi_{tZ} a - \Phi_{tZ} a_0 - w$$. Therefore, it is enough to show that $$A = -q_{tZ}$$.

Since $$f_t(u, v) = 0$$,

$$\Gamma_{tZ}^* (\Phi_{tZ} a - \Phi_{tZ} a_0 - w_n) + \lambda \begin{pmatrix} 1_N \\ 0 \end{pmatrix} = 0.$$  

(27)

We can rewrite (27) as

$$-\Gamma_{tZ}^* A \begin{pmatrix} a_n \\ 0 \end{pmatrix} = -\Gamma_{tZ}^* \Gamma_{tZ0}^* \begin{pmatrix} a_0 \\ 0 \end{pmatrix} - \Gamma_{tZ}^* w_n + \lambda \begin{pmatrix} 1_N \\ 0 \end{pmatrix}.$$  

(28)

By applying $$\Gamma_{tZ}(\Gamma_{tZ}^* \Gamma_{tZ})$$ to both sides, we obtain

$$-\Gamma_{tZ} \begin{pmatrix} a_n \\ 0 \end{pmatrix} = -\Gamma_{tZ} \Gamma_{tZ0} \begin{pmatrix} a_0 \\ 0 \end{pmatrix} - \Gamma_{tZ} \Gamma_{tZ}^* w + \lambda \Gamma_{tZ}^* \begin{pmatrix} 1_N \\ 0 \end{pmatrix}.$$

25
Therefore,

\[ A = \Gamma_{tZ} \left( \frac{a}{0_{2N}} \right) - \Gamma_{tZ_0} \left( \frac{a_0}{0_{2N}} \right) - w = - \left( \lambda \Gamma_{tZ}^{-1} \left( \frac{1}{0_{2N}} \right) + \Pi_{tZ} w + \Pi_{tZ} \Gamma_{tZ_0} \left( \frac{a_0}{0_{2N}} \right) \right) = -q_{tZ}, \]

as required.

\[ \square \]

Corollary 1. Let \( c_0 \leq c_* \) where \( c_* \) is as in Lemma 2. Suppose that \( |Z - Z_0|_\infty \leq c_0 t, \lambda \leq c_0 t^4 \) and \( \|w\| \leq c_0 t^4 \). Then, there exists a constant dependent only on \( \varphi, a_0, Z_0 \) such that

\[ \|\partial_w g_t^*(v)\| \leq \frac{C}{t^3}. \]

and

\[ \|\partial_{\lambda} g_t^*(v)\| \leq C(c_0 t^{-3} + t^{-2}), \]

Proof.

\[ dg_t^*(v) = -J_{t_{a}}^{-1} H_{t_{a}}^{-1} G_{t_{a}}(\lambda, w)^{-1} H_{t_{a}}^{-1} \left( \frac{1}{0_{2N}} \right) - H_{t_{a}}^{-1} \Psi_{t_{a}} \]

\[ = J_{a}^{-1} H_{a}^{-1} \operatorname{diag}(1, t^{-1}, t^{-1}, t^{-2}, t^{-2}, t^{-3}) \Gamma_{a}(\lambda, w)^{-1} (\delta_{a}, \Psi_{a}). \]

Therefore,

\[ \partial_w g_t^*(v) = O(t^{-3}). \]

Recall from Lemma 2 that \( G_{t_{Z}} = \Psi_{t_{Z}}^{*} \Psi_{Z_0} + C' |Z - Z_0|_\infty + C \|H_{Z_0}^{-1}\|^2 \|J_{a}^{-1}\| c_0. \)

Note that by ordering \( \Psi_{Z_0} \) as \( \{\varphi_{0,0}, \varphi_{0,1}, \varphi_{0,2}, \varphi_{1,0}, \varphi_{1,1}, \varphi_{0,3}\} \), \( \Psi_{t_{Z}}^{*} \Psi_{Z_0} \) is a checkerboard matrix and its \((6,1)^{th}\) entry is zero. Therefore, \( (\Psi_{t_{Z}}^{*} \Psi_{Z_0})^{-1} \) is also a checkerboard matrix with zero as its \((6,1)^{th}\) entry. So, \( G_{t_{Z}}^{-1} = (\Psi_{t_{Z}}^{*} \Psi_{Z_0})^{-1} + C'' c_0 \), where \( C'' \) is a constant dependent only on \( \varphi, a_0 \) and \( Z_0 \). So,

\[ \|\partial_{\lambda} g_t^*(v)\| \leq C(c_0 t^{-3} + t^{-2}), \]

where \( C \) is a constant dependent only on \( \varphi \) and \( (a_0, Z_0) \). \[ \square \]

We are finally ready to show that \( V_t^* \) contains a ball with radius on the order of \( t^4 \):

Proposition 7. There exists \( C > 0 \) such that for all \( t \in (0, t_0) \),

\[ V_t^* \supset B(0, Ct^4), \]

where \( C \sim c_* \).

Proof. Let \( v \in \mathbb{R} \times \mathcal{H} \) be such that \( \max(\lambda, \|w\|) = 1 \). Let

\[ R_v = \sup \{ r \geq 0 ; rv \in V_t^* \}. \]

First note that \( R_v \in (0, Ct^4) \), and since \( g_t^* \) is uniformly continuous on \( V_t^* \), \( g_t^*(R_v v) \) is defined. Moreover, \( f_t(g_t^*(R_v v), R_v v) = 0. \)

By maximality of \( V_t^* \), it is necessarily the case that \( g_t^*(R_v v) \in \partial(\mathcal{B}_{c_0}(a_0) \times \mathcal{B}_{c_0}(Z_0)) \) (otherwise, we can apply the implicit function theorem to construct a neighbourhood \( V \in \mathcal{V} \) such that \( V_t^* \not\subset V \).

Suppose that \( g_t^*(R_v v) \in \mathcal{B}_{c_0}(a_0) \times \partial(\mathcal{B}_{c_0}(Z_0)) \). Then, for \( (a, Z) = g_t^*(R_v v) \),

\[ c_* t = \|Z - Z_0\| \leq \int_0^1 |dg_t^*(sv) \cdot R_v v|_\infty \, ds \leq \frac{M}{r^2} R_v \quad \Rightarrow \quad R_v \geq \frac{c_* t^4}{M}. \]

On the other hand, if \( g_t^*(R_v v) \in \partial(\mathcal{B}_{c_0}(a_0) \times \mathcal{B}_{c_0}(Z_0)) \), then \( R_v \geq \frac{c_* t^3}{M} \). Repeating this for all \( v \in \mathbb{R} \times \mathcal{H} \) with unit norm yields the required result.

\[ \square \]
3.3.3 Use of Non-degeneracy

Throughout this section, given \((a, Z) = g_t(\lambda, w)\), recall the definition of \(p_{\lambda, t}\) and \(\eta_{\lambda, t}\) from \((20)\).

**Proposition 8.** Let \(\varepsilon > 0\). Then, there exists \(c_0 > 0\) and \(t_0 > 0\) such that for all \(Z, \lambda, w, t\) with \(0 < t < t_0\), \(\lambda \leq c_0 t^4\) and \(\|w\| \leq c_0 t^4\) and \(\|w\| / \lambda \leq c_0\), we have that

\[
\|p_{\lambda, t} - p_{W,Z_0}\| \leq \varepsilon.
\]

**Proof.** By Proposition \(7\) there exists \(c\) such that for all \((\lambda, w) \in B(0, c_0 t^4)\), with \(c_0 \leq c\), \(g_t^*\) is well defined. For \((a, Z) = g_t^*(\lambda, w)\),

\[
p_{\lambda, t} = \Gamma_{1, t}^* \left( \frac{1}{N} + \Pi_{\partial Z_0} \frac{w}{\lambda} + \frac{1}{\lambda} \Pi_{\partial Z_0} \Gamma_{1, t} \left( \frac{a_0}{0} \right) \right) = p_{W,Z_0} + \mathcal{O}(t) + \mathcal{O}\left( \frac{\|w\|}{\lambda} \right) + \frac{1}{\lambda} \Pi_{\partial Z_0} \Gamma_{1, t} \left( \frac{a_0}{0} \right).
\]

To bound the last term on the RHS,

\[
\left\| \frac{1}{\lambda} \Pi_{\partial Z_0} \Gamma_{1, t} \left( \frac{a_0}{0} \right) \right\| \leq C \max\{t^2 |Z - Z_0|^2, t^3 |Z - Z_0|_\infty \}
\]

\[
\leq C \max\{\frac{\|w\|^2}{t^4}, \|w\|\} + \frac{C}{\lambda} \max\{t^2 (L^2 c_0^2 t^{-6} + t^{-4}) \lambda^2, L c_0 \lambda \}
\]

\[
\leq C \left( \frac{\|w\|}{\lambda} + t^{-4} c_0^2 \lambda + c_0 \right),
\]

where the first inequality follows from Proposition \(3\) and the second inequality follows from Corollary \(1\). The result now follows by choosing \(c_0\) sufficiently small. \(\square\)

**Proof of Theorem \(2\).** By Proposition \(8\) if \((a, Z) = g_t^*(\lambda, w)\), then since \(p_{\lambda, t}\) can be made arbitrarily close to \(p_{W,Z_0}\), we can apply Proposition \(5\) to conclude that \(p_{\lambda, t}\) is a valid certificate and hence the (unique) solution to the dual problem of \((P_\lambda(y))\). Moreover, \(\eta_{\lambda, t}\) attains the value 1 only at the points in \(Z\). Therefore, the support of any solution of \((P_\lambda(y))\) is contained in \(Z\) and by invertibility of \(\Phi^*_2\), it follows that \(m_{a,z}\) is the unique solution of \((P_\lambda(y))\). Finally, the bounds on \(||(a, Z) - (a_0, Z_0)||\) is a direct consequence on the bounds on the differential \(d g_t^*\). \(\square\)

3.4 Limitations

The key idea behind the stability result of Theorem \(5\) is Proposition \(3\) any certificate which is sufficiently close to \(\eta_{W,Z_0}\) is also a valid certificate. We have only proved this result in the case of a pair of spikes, although a similar proof technique can be applied to the case where \(Z_0\) consists of \(N\) aligned points in direction \(d_{z_0}\), with the natural extension of the non-degeneracy condition (c.f. construction of \(\eta_{W,Z_0}\) from Example \(3\)) being:

\[
\begin{pmatrix}
\frac{\partial^2}{\partial z^2_0} \eta_{W,Z_0}(0) & \frac{1}{\sqrt{N}} \frac{\partial}{\partial z_0} \frac{\partial^N}{\partial z_0^N} \eta_{W,Z_0}(0) \\
\frac{1}{\sqrt{N}} \frac{\partial}{\partial z_0} \frac{\partial^N}{\partial z_0^N} \eta_{W,Z_0}(0) & \frac{2}{(2N)!} \frac{\partial^N}{\partial z_0^N} \eta_{W,Z_0}(0)
\end{pmatrix} < 0.
\]

However, Proposition \(3\) is in general not valid and therefore, the question of whether there is support stability in the case of more than 2 spikes remains open. The purpose of this section is to present some examples to illustrate this phenomenon. Note also that there exists examples (see the Gaussian mixture example from Section \(4\)) where one can numerically observe support stability when recovering a pair of spikes, but not in the case of 3 or more spikes.

In the following examples, consider let \(\Phi\) be a convolution operator, i.e. \(\varphi(x) = \tilde{\varphi}(x - \cdot)\).

**Proposition 9 (Case \(N = 3\)).** Let \(Z = \{z_1, z_2, z_3\} \in \mathbb{R}^4\) be 3 points which are not colinear. Let \(x\) is any point in the interior of the convex hull of \(Z\). Let

\[
p_t = \text{argmin} \left\{ \|p\| : (\Phi^* p)(tv) = 1, \nabla (\Phi^* p)(tv) = 0, \forall v \in Z, \Phi^* p(tx) = 1 \right\}.
\]

Then, \(\lim_{t \to 0} \|p_t - p_{W,z}\| = 0\).
Proof. The least interpolant space associated to Hermite interpolation at \( Z \) contains the polynomial space of degree 2, \( \Pi^2_2 \). Moreover, by Lemma 3 since \( \Phi \) is a convolution operator, \( \nabla^k \eta_{W,Z}(0) = 0 \) for all odd integers \( k \). Therefore, \( \nabla^3 \eta_{W,Z}(0) = 0 \) for \( k = 1, 2, 3 \). On the other hand, the least interpolant space associated to Hermite interpolation at \( Z \) plus Lagrange interpolation at \( x \) is \( \Pi^2_3 \). Therefore, \( p_{W,Z} = p_t + O(t) \). \( \square \)

**Proposition 10.** Let \( \tilde{\phi} \) be the Gaussian kernel. Let \( Z = \{(1,1),(-1,1),(1,-1),(-1,-1)\} \). Let \((u,v) \in \mathbb{R}^2 \) be such that \( u^2 + v^2 = 1 \) and let \( \tilde{Z} = \{(u,v)\} \cup Z \).

\[
p_t = \arg\min \|p\| \quad ; \quad (\Phi^* p)(tx) = 1, \nabla(\Phi^* p)(tx) = 0, \forall x \in Z, \Phi^* p(t(u,v)) = 1 \}
\]

Then, \( p_{W,z} = \lim_{t \to 0} p_t \).

**Proof.** First note that the least interpolant space associated with Hermite interpolation at \( Z \) is spanned by the following basis:

\[
B_Z = \{X^\alpha : |\alpha| \leq 3\} \cup \{X^\beta : \beta \in \{(1,3), (3,1)\}\}.
\]

Let \( \tilde{Z} = \{(1,1),(-1,1),(1,-1)\} \). Then, we have that \( \nabla^j \eta_{W,\tilde{Z}}(0) = 0 \) for \( j = 1, 2, 3 \) and \( \partial^2_x \partial_y \eta_{W,\tilde{Z}}(0) = \partial^3_y \eta_{W,\tilde{Z}}(0) = 0 \). Therefore, \( \eta_{W,\tilde{Z}} = \eta_{W,z} \).

Observe now that the de Boor basis associated with Hermite interpolation on \( Z \) and Lagrange interpolation on \((u,v)\) is

\[
B_Z \cup \left\{p(x,y) = y^4 + 6x^2y^2 \left(\frac{u^2 - 1}{v^2 - 1}\right) + x^4 \left(\frac{u^2 - 1}{v^2 - 1}\right)^2\right\}
\]

Moreover, by the explicit formula given in Proposition 2, we have that \( \partial^4_x \eta_{W,z}(0) = \partial^4_y \eta_{W,z}(0) = -48 \) and \( \partial^2_x \partial^2_y \eta_{W,z}(0) = -16 \). Therefore, \( p(\partial_x, \partial_y) \eta_{W,z}(0) = 0 \) whenever

\[
-48 - 96 \left(\frac{u^2 - 1}{v^2 - 1}\right) - 48 \left(\frac{u^2 - 1}{v^2 - 1}\right)^2 = 0.
\]

i.e. \( u^2 + v^2 = 2 \). So, provided that \( u^2 + v^2 = 2 \), then \( p_{W,z} = \lim_{t \to 0} p_t \). \( \square \)

4 Numerical Study

4.1 Considered Setups

We consider three different imaging operators \( \Phi \), intended to be representative of three different setups routinely encountered in imaging or machine learning. For each setup, in order to perform the computations of \( \eta_{W,Z}, \eta_{W,Z} \) and to implement the Frank-Wolfe algorithm detailed in Section 4.3, the only requirement is to be able to evaluate the correlation kernel \( C \) defined in [3] and its derivatives.

In these examples, we consider the clustering of the spikes positions at a fixed point \( z_0 \in \mathcal{X} \), i.e. consider for \( t > 0 \) the positions \( Z_t = (z_0 + t(z_i - z_0))_{i=1}^N \in \mathcal{X}^N \). For the purpose of simplifying notation, the previous sections detailed only the case of \( z_0 = 0 \), i.e. \( Z_t = tZ \), however, all previous results also hold in this more general setting by a change of variable \( x \in \mathcal{X} \to x - z_0 \in \mathcal{X} \). Note that if \( \mathcal{X} \) is not translation invariant, one should restrict the translation around \( z_0 \) and extend it into a smooth diffeomorphism on \( \mathcal{X} \), see Appendix C for a proof of the reparametrization invariance of \( \eta_{W,Z} \).

- **Gaussian convolution:** this corresponds to a translation invariant setup, which is typical in the modelling of acquisition blur in image processing. We consider \( \varphi(x) = e^{-\frac{\|x - x'\|^2}{2\sigma^2}} \in \mathcal{H} = L^2(\mathbb{R}^2) \) on \( \mathcal{X} = \mathbb{R}^2 \), and one has

\[
C(x,x') = e^{-\frac{\|x - x'\|^2}{4\sigma^2}}.
\]

In this case, the clustering point is set to be \( z_0 = 0 \).
• **Gaussian mixture estimation**: In machine learning, an important problem is to estimate the parameters \((z_i)_{i=1}^N \in \mathcal X^N\) of a mixture \(\sum_{i=1}^N a_i \varphi(z_i)\) of \(N\) elementary distributions parameterized by \(\varphi\) from samples or moments observations, see [33] for an overview of this problem. This problem can be recast as a super-resolution problem, where one seeks to recover the measure \(m_0 = \sum_i a_i \delta_{z_i}\) from observations of the form \((x)\) where the noise \(w\) accounts for the sampling scheme (in a real-life machine learning setup, the operator \(\Phi\) itself is noisy to account for the sampling scheme). We consider here a classical instance of this setup, where one looks for a mixture of 1-D Gaussians, parameterized by mean \(m \in \mathbb R\) and standard deviation \(s \in \mathbb R_+\), i.e. \(x = (m, s) \in \mathcal X = \mathbb R \times \mathbb R_+\), so that \(\varphi(x) = \frac{1}{s} e^{-\frac{(x-m)^2}{2s^2}} \in \mathcal H = L^2(\mathbb R)\) and the correlation operator reads

\[
C((m, s), (m', s')) = \frac{1}{\sqrt{s^2 + s'^2}} e^{-\frac{(m-m')^2}{2(s^2+s'^2)}}.
\]

In this case, the clustering point is set to be \(z_0 = (m_0, s_0) = (0, 2)\).

• **Neuro-imaging**: for medical and neuroscience imaging applications, a standard goal is to estimate pointwise sources inside some domain \(\mathcal X \subset \mathbb R^d\) (where \(d = 2\) or \(3\)) from measurements on the boundary \(\partial \mathcal X\). The operator is thus of the form \(\varphi(x) = (\psi(x, u))_{u \in \partial \mathcal X} \in \mathcal H = L^2(\partial \mathcal X)\) (equipped with the uniform measure on the boundary) where the kernel \(\psi(x, u)\) corresponds to the impulse response of the measurement operator. To model MEG or EEG acquisition [32], we consider a singular kernel \(\psi(x, u) = ||x - u||^{-2}\) which accounts for the decay of the electric or magnetic field in a stationary regime. We consider a disk domain \(\mathcal X = \{x \in \mathbb R^2 : ||x|| < 1\}\) which could model a slice of a head. The correlation function associated to this problem is

\[
C(x, x') = 2\pi \frac{1 - ||x||^2 ||x'||^2}{(1 - ||x||^2)(1 - ||x'||^2)((1 - \langle x, x' \rangle)^2 + ||x \wedge x'||^2)},
\]

see Appendix [3] for a proof. In this case, the clustering point is set to be \(z_0 = (0.4, 0.3) \in \mathcal X\).

As it is customary for sparse regularization, we perform the BLASSO recovery using an \(L^2\) normalized operator, i.e. perform the replacement

\[
\varphi(x) \leftarrow \frac{\varphi(x)}{||\varphi(x)||_\mathcal H} \implies C(x, x') \leftarrow \frac{C(x, x')}{\sqrt{C(x, x') C(x', x')}}.
\]

Note that for translation invariant operators (i.e. convolutions), the kernels are already normalized.

### 4.2 Asymptotic Certificate \(\eta_{W,Z}\)

Figure 2 explores the behaviour of \(\eta_{W,Z}\) in the three considered cases:

• **Gaussian convolution** [30]: we found numerically that \(\eta_{W,Z}\) is always non-degenerate, for any \(N\) and spikes configuration \(Z\). This is inline with the theoretical results of Section 2.5.1. This implies that one can hope (and provably do so for \(N = 2\) according to Theorem 3) to achieve super-resolution for Gaussian deconvolution (provided, of course, that the signal-to-noise ratio is large enough).

• **Neuro-imaging** [32]: we observed numerically that \(\eta_{W,Z}\) is always non-degenerate for \(N = 2\) and more generally for aligned spikes. In contrast, for three non-aligned spikes, \(\eta_{W,Z}\) is not a valid certificate (\(||\eta_{W,Z}||_\infty > 1\)) which means that in the presence of noise, one cannot stably super-resolve 3 close spikes.

• **Gaussian mixture estimation** [31]: here, the situation is more complicated, and for \(N = 2\) spikes, \(\eta_{W,Z}\) is non degenerate if \(|m_2 - m_1| \leq |s_2 - s_1|\). This means that one can super-resolve with BLASSO a mixture of two Gaussians provided that the variation in the means is not too large with respect to the
Figure 2: Display of the evolution of $\eta_{W,Z}$ for the three different operators $\Phi$. The dashed red line shows the directions $(Z_t)_{t>0}$ along which the spikes are converging. Red color indicates regions where $\eta_{W,Z}(x) > 1$, i.e. it is degenerated. Top: Gaussian convolution [30]. Middle: Gaussian mixture [31], here the horizontal axis is the standard deviation $s \in [0.5, 6]$ and the vertical axis is the mean $m \in [-3, 3]$. Bottom: neuro-imaging like [32].
Figure 3: Display of the evolution of the solution $m_\lambda$ of $(P_\lambda(y))$ (computed using Frank-Wolfe algorithm) as a function of $\lambda$ for two different operators $\Phi$, in cases where $\eta_{V,Z}$ is non-degenerated. The settings are the same as for Figure 2 and the bottom row is a zoom in the dashed rectangular region indicated on the top row. A spike $a_\delta x$ of $m_\lambda$ is indicated with a disk centered at $x$ of radius proportional to $a$, and the color ranges between blue for $\lambda = 0$ and red for $\lambda = \lambda_{\text{max}}$. The background color image shows $\eta_{V,Z}$ where $z$ are the spikes positions of $m_0$ (plotted in black).

variation in standard deviations. Note also that in the special 1-D case where either the means or the standard deviation are equal and known (which leads to a 1-D super resolution problem along the $m$ or $s$ axis) then the resulting 1-D $\eta_{W}$ is non-degenerate. It is the interplay between means and standard deviation that makes the super-resolution possibly problematic.

An important aspect to consider, which explains partly the above observations, is that, as explained in Section 2.5.2, convolution operators tend to have much better behaved $\eta_{W,Z}$ than arbitrary operators (such as the neuro-imaging and the Gaussian mixture), because their odd derivatives always vanish. In contrast, the vanishing of odd derivatives for a generic operator only occur for particular values of $N$ and spikes configuration (e.g. aligned spikes). Without having its odd derivatives vanishing, $\eta_{W,Z}$ cannot be expected to be smaller than 1 near the spikes position $(z_i)_i$.

### 4.3 Spikes Recovery with Frank-Wolfe

In order to solve numerically the BLASSO problem $(P_{\lambda}(y))$, we follow \cite{10, 9} and use the Frank-Wolfe algorithm (also known as conditional gradient) with improved non-convex updates. The algorithm starts with the initial zero measure $m^{(0)} = 0$, and alternates between a “matching pursuit” step which generates a new spike location

$$\tilde{x} \triangleq \arg \max_{x \in \mathcal{X}} |\eta^{(\ell)}(x)| \quad \text{where} \quad \eta^{(\ell)}(x) \triangleq \frac{1}{\lambda} \langle \varphi(x), y - \Phi m^{(\ell)} \rangle_H, \quad (33)$$

with associated amplitude $\tilde{a} \triangleq \lambda \eta^{(\ell)}(\tilde{x}_{\ell+1})$, and a local non-convex minimization step, initialized with

$$r \leftarrow (x_1^{(\ell)}, \ldots, x_k^{(\ell)}, \tilde{x}) \in \mathcal{X}^{\ell+1} \quad \text{and} \quad b \leftarrow (a_1^{(\ell)}, \ldots, a_k^{(\ell)}, \tilde{a}) \in \mathbb{R}^{\ell+1}$$

$$\begin{align*}
(x^{(\ell+1)}, a^{(\ell+1)}) & \triangleq \arg \min_{(r,b) \in \mathcal{X}^{\ell+1} \times \mathbb{R}^{\ell+1}} \frac{1}{2\lambda} \left\| y - \sum_{i=1}^{\ell+1} b_i \varphi(r_i) \right\|_2^2 + \|b\|_1.
\end{align*} \quad (34)
$$

After each iteration, the measure is updated as

$$m_{\ell+1} \triangleq \sum_{i=1}^{\ell+1} a_i^{(\ell+1)} \delta_{x_i^{(\ell+1)}}.$$
η, V, Z and m₀
Gaussian mixture N = 2

η(ℓ) and m(ℓ)

η, V, Z and m₀
Neuro-imaging N = 3

η(ℓ) and m(ℓ)

Figure 4: Display of the solution m(ℓ) computed using ℓ = 40 Frank-Wolfe iterations, in cases where η_{V,Z} is degenerated (as indicated by red regions). The settings are the same as for Figure 2. The light blue dots indicate the support of m(ℓ) (which thus allows one to locate spikes with very small amplitude) while blue dots are displayed with a size propositional to the amplitude of the corresponding spike.

The termination criterion is |η(ℓ)(x)| ≲ 1, which means that m(ℓ) is a solution to (P_λ(y)) because η(ℓ) is a valid dual certificate of optimality for m(ℓ). The algorithm is known to converge in the sense of the weak topology of measures to a solution of (P_λ(y)), see [10]. Without the non-convex update, convergence is slow (the rate on is only O(1/ℓ) on the BLASSO functional being minimized [36]). However, as we illustrate next, empirical observations suggest that by applying the non-convex update (34), convergence is often reached in a finite number of iteration.

Numerically, the low-dimensional optimization problems (33) and (34) are solved using a quasi-Newton (L-BFGS) solver. Computing the gradient of the involved functionals only require the evaluation of the correlation operator C and its derivative, assuming the measure m(ℓ) are stored using a list of (positions, amplitudes).

Figure 3 explores the behaviour of the solution m_λ of (P_λ(y)) as (λ, w) → 0, in cases where η_{V,Z} is non-degenerate, so that support is stable in this low-noise regime. Inline with support stability theorems, we scale the noise linearly with λ, y = Φm₀ + λw, and set the noise w to be of the form w = Φ ¯{m} where ¯{m} is a random measure ∑ b_j δ_{u_j} of Q = 20 random points (u_j)_{j=1}^Q ∈ Λ^Q where (b_j) is white noise with standard deviation 10^{-3}. Numerically, we found that in these cases where η_{V,Z} is non-degenerate, Frank-Wolfe with non-convex update converges in a finite number of steps. The color (from blue to red) allows to track the evolution with λ of the solution, which highlight the smoothness of the solution path.

Figure 4 shows, in contrast, cases where η_{V,Z} is degenerate. According to Section 2.4, in this case, the support of the solution m_λ is not stable for small λ, and one expects this solution to be composed of more than N diracs. Numerically, in these case, Frank-Wolfe does not converge in a finite number of steps, and it keeps creating new spikes of very small amplitudes. The figure shows how these additional spikes are added to force |η(ℓ)| to be smaller, while η_{V,Z} is not.

Aknowlegements

We would like to thank Vincent Beck stimulating discussions about polynomial interpolation.

5 Conclusion

This article presented a study of the multivariate BLASSO problem in the case when recovering positive spikes positioned very close together. In particular, we focussed on the question of support stability. Previous studies [29, 25] have highlighted the importance of the precertificate for this question, and as a first contribution, we presented a procedure for computing the limit η_{W,Z} of the associated precertificates as the
point sources converge towards a limit point. Since a necessary condition for support stability is that this certificate is valid (it is uniformly bounded by 1), one can quickly check whether this certificate is valid before proceeding with more detailed analysis. Our second main contribution is a detailed analysis in the case of recovering a superposition of 2 spikes. Here, we showed that under a nondegeneracy condition on \( \eta_{W,Z} \), support stability can be achieved provided that the norm of the additive noise \( \|w\| \) and the regularization parameter \( \lambda \) decays like \( t^4 \), where \( t \) is the spacing between the 2 spikes. The question of which conditions are necessary for support stability when recovering more than 2 spikes remains open. The final part of this paper presented numerical examples related to 3 different imaging situations, it is perhaps interesting to observe that breakdown of support stability in the Gaussian mixture case and the neuro-imaging case, and this is potentially an interesting area for further investigation.

A Proof of Proposition 1 (Linear Independence)

**Step I.** Let us first show that \( \Psi_L \overset{\text{def.}}{=} \{ \partial^\alpha \varphi(0) : |\alpha| \leq L, \alpha \in \mathbb{N}_0^2 \} \) is linearly independent provided that \( \hat{\psi}(\alpha) \neq 0 \) for all \( \alpha \in \mathbb{N}_0^2 \) with \( |\alpha| \leq L \).

Observe that \( \partial^\alpha \varphi(0) = \partial^\alpha \psi \), and that \( \Psi_L = \{ \partial^\alpha \psi : |\alpha| \leq L, \alpha \in \mathbb{N}_0^2 \} \) is linearly independent if and only if the Fourier coefficients of the elements in \( \Psi_L \) are linearly independent. The Fourier Transform of \( \partial^\alpha \hat{\varphi} \) evaluated at frequencies \( \xi \overset{\text{def.}}{=} (\xi_j)_{j=1}^M \) are \( \left( \hat{\psi}(\xi_j)(2\pi i \xi_j)^\alpha \right)_{j=1}^M \). Therefore, \( \Psi_L \) is linearly independent if the columns of the matrix \( \text{diag}(\hat{\psi}(\xi_j)\text{1}_{1 \leq j \leq N})M_\xi \) are linearly independent, where \( M_\xi \overset{\text{def.}}{=} \left( (2\pi i \xi_j)^\alpha \right)_{|\alpha| \leq L} \) is the Lagrange interpolation matrix, with evaluation at points \( \xi \) and using the polynomial basis \( (X^\alpha)_{|\alpha| \leq L} \). From [42, Theorem 1], we know that \( M_\xi \) is invertible for almost every choice of \( \xi \) where \( M = \Pi_L^2 \). Furthermore, one possible choice of \( \xi \) is

\[
\{ \alpha \in \mathbb{N}_0^2 : |\alpha| \leq L \}.
\]

Therefore, to ensure linear independence of \( \Psi_L \), it is enough to check that \( \hat{\psi}(\alpha) \neq 0 \) for all \( \alpha \in \mathbb{N}_0^2 \) such that \( |\alpha| \leq L \).

**Step II.** We are now ready to show that \( \{ h(\partial)\varphi(0) : h \in S_\xi \} \) is of dimension \( 3N \).

Recall from Remark 4 that \( S_\xi \) associated with Hermite interpolation at \( N \) points \( Z \) satisfies \( S_\xi \subset \Pi_L^2 \), where \( L \overset{\text{def.}}{=} 2N - 1 \). Let \( B \) be the coefficient matrix such that \( B(X^\alpha)_{|\alpha| \leq L} = (g_i(X))_{1 \leq i \leq 3N} \). Note since \( B \) is a basis, given any \( a \in \mathbb{R}^{3N}, B^T a = 0 \) if and only if \( a = 0 \), since otherwise, there would be an \( a \neq 0 \) such that

\[
0 = \langle a, \ B(X^\alpha)_{|\alpha| \leq L} \rangle
\]

which would contradict the assumption that \( B \) is a basis. Therefore, if \( \{ g_i(\partial)\varphi(0) : i = 1, \ldots, 3N \} \) is linearly dependent, then there exists \( 0 \neq a \in \mathbb{R}^{3N} \) such that

\[
0 = \langle a, \ B(\partial^\alpha \varphi(0))_{|\alpha| \leq L} \rangle = \langle B^T a, \ (\partial^\alpha \varphi(0))_{|\alpha| \leq L} \rangle.
\]

This leads to the required contradiction, we have shown in the first step that \( \Psi_L \) is linearly independent, and therefore, \( B^T a = 0 \) and hence \( a = 0 \).
B Proof of Theorem \[ \text{1} \]

B.1 Part 1: \( \eta_N(x) < 1 \) for all \( x \not\in \mathbb{Z} \)

Recall that \( \eta_N \) is of the form

\[
\eta_N(x) = \sum_{j=1}^{N} \alpha_j \langle \varphi(z_j), \varphi(x) \rangle + \sum_{j=1}^{N} \beta_j \langle \varphi'(z_j), \varphi(x) \rangle
\]

\[
= \sum_{j=1}^{N} \alpha_j \langle v(z_j), v(x) \rangle + \sum_{j=1}^{N} \beta_j \langle v'(z_j), v(x) \rangle,
\]

where \( v : \mathbb{T} \to \mathbb{C}^{2f_r+1} \) is defined by \( v(x) = (e^{2\pi i k x})_{|k| < f_r} \). If there exists \( \tau \not\in \mathbb{Z} \) such that \( \eta_N(\tau) = 1 \), \( L^* R \) is a singular matrix, where

\[
R \overset{\text{def.}}{=} \begin{pmatrix}
e^{2\pi i f z_1} - e^{2\pi i f \tau} & \cdots & e^{2\pi i f z_N} - e^{2\pi i f \tau} & (2\pi i f)e^{2\pi i f z_1} & \cdots & (2\pi i f)e^{2\pi i f z_N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
e^{-2\pi i z_1} - e^{-2\pi i \tau} & \cdots & e^{-2\pi i z_N} - e^{-2\pi i \tau} & (2\pi i)e^{-2\pi i z_1} & \cdots & (2\pi i)e^{-2\pi i z_N} \\
e^{-2\pi i f z_1} & \cdots & e^{-2\pi i f z_N} - e^{-2\pi i f \tau} & (2\pi i)e^{2\pi i f z_1} & \cdots & (2\pi i)e^{2\pi i f z_N}
\end{pmatrix}
\]

and

\[
L \overset{\text{def.}}{=} \begin{pmatrix}
e^{2\pi i z_1} & \cdots & e^{2\pi i z_N} & (2\pi i)e^{2\pi i z_1} & \cdots & (2\pi i)e^{2\pi i z_N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
e^{2\pi i z_1} & \cdots & e^{2\pi i z_N} & (2\pi i)e^{2\pi i z_1} & \cdots & (2\pi i)e^{2\pi i z_N} \\
e^{-2\pi i z_1} & \cdots & e^{-2\pi i z_N} & (2\pi i)e^{-2\pi i z_1} & \cdots & (2\pi i)e^{-2\pi i z_N} \\
e^{-2\pi i f z_1} & \cdots & e^{-2\pi i f z_N} & (2\pi i)e^{2\pi i f z_1} & \cdots & (2\pi i)e^{2\pi i f z_N}
\end{pmatrix}.
\]

To show that this is impossible, first observe that the matrix \( L \) has the same determinant as the following \((2N+1) \times (2N+1)\) matrix:

\[
\begin{pmatrix}
0 & e^{2\pi i f z_1} & \cdots & e^{2\pi i f z_N} & (2\pi i f)e^{2\pi i f z_1} & \cdots & (2\pi i f)e^{2\pi i f z_N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & e^{2\pi i z_1} & \cdots & e^{2\pi i z_N} & (2\pi i)e^{2\pi i z_1} & \cdots & (2\pi i)e^{2\pi i z_N} \\
1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & e^{-2\pi i z_1} & \cdots & e^{-2\pi i z_N} & (2\pi i)e^{-2\pi i z_1} & \cdots & (2\pi i)e^{-2\pi i z_N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & e^{-2\pi i f z_1} & \cdots & e^{-2\pi i f z_N} & (2\pi i f)e^{2\pi i f z_1} & \cdots & (2\pi i f)e^{2\pi i f z_N}
\end{pmatrix}.
\]

So, if \( \det(L) = 0 \), then there exists \( \alpha \neq 0 \) such that

\[
F(x) = 1 + \sum_{j=1}^{f} \alpha_j x^j + \sum_{j=-f}^{1} \alpha_j x^j
\]

has roots at \( e^{2\pi i z_l} \) for \( l = 1, \ldots, N \) and at 0. Moreover, \( F'(e^{2\pi i z_l}) = 0 \) for all \( l = 1, \ldots, N \). However, this would imply that \( x^f F(x) \) has \( 2N + 1 \) roots, which is a contradiction to the fact that this is a polynomial of degree \( 2f = 2N \). Therefore, \( \det(L) \neq 0 \).
So, to prove this theorem, it suffices to show that \(R\) is nonsingular. The determinant of \(R\) is equal to that of the following \((2N + 1) \times (2N + 1)\) matrix:

\[
\begin{pmatrix}
 e^{2\pi if_x} & e^{2\pi if_z} & \cdots & e^{2\pi if_z} & (2\pi i) e^{2\pi if_z} & \cdots & (2\pi i) e^{2\pi if_z} \\
 e^{2\pi if_x} & e^{2\pi if_z} & \cdots & e^{2\pi if_z} & (2\pi i) e^{2\pi if_z} & \cdots & (2\pi i) e^{2\pi if_z} \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
e^{-2\pi i f_x} & e^{-2\pi if_z} & \cdots & e^{-2\pi if_z} & (-2\pi i) e^{-2\pi if_z} & \cdots & (-2\pi i) e^{-2\pi if_z} \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
e^{-2\pi i f_x} & e^{-2\pi if_z} & \cdots & e^{-2\pi if_z} & (-2\pi i) e^{-2\pi if_z} & \cdots & (-2\pi i) e^{-2\pi if_z}
\end{pmatrix}.
\]

We must have \(\det(R) \neq 0\) because otherwise, by the same argument as before, we would construct a polynomial of degree \(2N\) with at least \(2N + 1\) roots (at least double roots at \(e^{2\pi iz}\) for \(l = 1, \ldots, N\) and a single root at \(e^{2\pi i r}\)).

So, if \(f_c = N\), then \(\eta_V(x) \neq 1\) for all \(x \notin \mathbb{Z}\). Therefore, since \(\eta_V(z_j) = 1\), either \(\eta_V(x) \geq 1\) for all \(x\) or \(\eta_V(x) \leq 1\) for all \(x\). Note that both \(\eta_V\) and \(2 - \eta_V\) satisfy the vanishing derivatives constraints. Suppose that \(\eta_V(x) \geq 1\) for all \(x\). Then, the equations \(\eta_V(0) - \eta_V(\tau) = 0\), \(\partial_j \eta_V(0) = 0\) for \(j = 1, \ldots, 2N - 1\) can be written as the linear system \(M_\tau a = 0\), where

\[
M_\tau \overset{\text{def.}}{=} \begin{pmatrix}
\langle v(0), v(0) - v(\tau) \rangle & \langle v(1), v(0) - v(\tau) \rangle & \cdots & \langle v(2N-1), v(0) - v(\tau) \rangle \\
\langle v(0), v(1) \rangle & \langle v(1), v(1) \rangle & \cdots & \langle v(2N-1), v(1) \rangle \\
\langle v(0), v(2) \rangle & \langle v(1), v(2) \rangle & \cdots & \langle v(2N-1), v(2) \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle v(0), v(2N-1) \rangle & \langle v(1), v(2N-1) \rangle & \cdots & \langle v(2N-1), v(2N-1) \rangle
\end{pmatrix}.
\]

We will now proceed to show that \(\det(M_\tau) \neq 0\) for all \(\tau \neq 0\), and therefore, \(\eta_V(\tau) < 1\) for all \(\tau \neq 0\). Note that \(M_\tau^* = L^* R\) where

\[
R \overset{\text{def.}}{=} \begin{pmatrix}
(v(0) - v(\tau), v(1), v(2), \cdots, v(2N-1))
\end{pmatrix},
\]

\[
L \overset{\text{def.}}{=} \begin{pmatrix}
(v(0), v(1), v(2), \cdots, v(2N-1))
\end{pmatrix}.
\]

Let \(n = 2f_c\). Since the row corresponding to frequency \(k = 0\) for the matrix \(R\) is zero, we can write \(M_\tau^* = L^* R\), where

\[
\tilde{R} \overset{\text{def.}}{=} \begin{pmatrix}
1 - e^{2\pi if_c \tau} & 2\pi i f_c & (2\pi i f_c)^2 & \cdots & (2\pi i f_c)^{n-1} \\
1 - e^{2\pi if_c (-1)^{l-1} \tau} & 2\pi i (f_c - 1) & (2\pi i (f_c - 1))^2 & \cdots & (2\pi i (f_c - 1))^{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 - e^{2\pi i \tau} & 2\pi i & (2\pi i)^2 & \cdots & (2\pi i)^{n-1} \\
1 - e^{-2\pi i \tau} & -2\pi i & (-2\pi i)^2 & \cdots & (-2\pi i)^{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 - e^{-2\pi if_c \tau} & -2\pi i f_c & (-2\pi i f_c)^2 & \cdots & (-2\pi i f_c)^{n-1}
\end{pmatrix} \in \mathbb{C}^{n \times n},
\]
and

$$\hat{L}^{\text{def.}} = \begin{pmatrix}
1 & 2\pi i f_c & (2\pi i f_c)^2 & \cdots & (2\pi i f_c)^{n-1} \\
1 & 2\pi i (f_c - 1) & (2\pi i (f_c - 1))^2 & \cdots & (2\pi i (f_c - 1))^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2\pi i & (2\pi i)^2 & \cdots & (2\pi i)^{n-1} \\
1 & -2\pi i & (-2\pi i)^2 & \cdots & (-2\pi i)^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & -2\pi i f_c & (-2\pi i f_c)^2 & \cdots & (-2\pi i f_c)^{n-1}
\end{pmatrix} \in \mathbb{C}^{n \times n}.$$ 

Since $\hat{L}$ is a Vandermonde matrix generated by $n$ distinct points $\{(2\pi ik)\}_{|k| \leq f_c, k \neq 0}$,

$$\det(\hat{L}) = \prod_{l \neq j} (2\pi i(l - j)) = (2\pi i)^{2f_c^2 - f_c} \prod_{l \neq j} (l - j) \neq 0.$$ 

So, it remains to show that $\det(\tilde{R}) \neq 0$ for all $\tau \neq 0$.

$$\det(\tilde{R}) = \sum_{k=1}^{f_c} (-1)^{k+f_c} (1 - e^{2\pi i k \tau}) \prod_{l \neq j} (2\pi i(l - j)) \prod_{l = -f_c, l \neq 0, k} (2\pi il)$$

$$+ \sum_{k = -f_c}^{-1} (-1)^{k+f_c+1} (1 - e^{2\pi i k \tau}) \prod_{l \neq j} (2\pi i(l - j)) \prod_{l = -f_c, l \neq 0, k} (2\pi il)$$

$$= -\det(\hat{L}) (f_c!)^2 \left( \sum_{k=1}^{f_c} \frac{(-1)^k (2 - e^{2\pi i k \tau} - e^{-2\pi i k \tau})}{(f + k)! (f - k)!} \right)$$

Let $x = e^{-2\pi i \tau}$, then

$$F(x) \overset{\text{def.}}{=} (2f_c)! \sum_{k=1}^{f_c} \frac{(-1)^k (2 - e^{2\pi i k \tau} - e^{-2\pi i k \tau})}{(f + k)! (f - k)!} = \sum_{k=1}^{f_c} \left( \frac{2f_c}{f_c - k} \right) (-1)^k (2 - x^k - x^{-k}).$$

Observe that

$$\sum_{k=1}^{f_c} (-1)^k \left( \frac{2f_c}{f_c - k} \right) (x^k + x^{-k}) = (-1)^{f_c} (1 - x)^{2f_c} x^{f_c} - \left( \frac{2f_c}{f_c} \right),$$

and $2 \sum_{k=1}^{f_c} (-1)^k \left( \frac{2f_c}{f_c - k} \right) = -\left( \frac{2f_c}{f_c} \right)$. Therefore,

$$F(x) = \frac{(-1)^{f_c+1} (1 - x)^{2f_c}}{x^{f_c}} = -2^{2f_c} \sin^{2f_c}(\pi \tau).$$

So, $\det(\tilde{R}) = 0$ if and only if $\tau = 0$. In particular,

$$\det(M) = \det(\tilde{R}) \det(\hat{L}) = \frac{2^{2f_c} \det(\hat{L})^2}{(2f_c)^2} \sin^{2f_c}(\pi \tau) > 0$$

for all $\tau \neq 0$. 

36
Finally, for the explicit formula of $\eta_W$, note that

$$\det(P_T) = \det(P_0) - \det(M_0) \in \text{Span} \left\{ (\partial^j \varphi(0), \varphi(x)) : j = 0, \ldots, 2N - 1 \right\},$$

where $P_T$ is the cross-Grammian matrix between the vectors $\{\nu^{(j)}(0)\}^{2N-1}_{j=1}$ and $\{\nu(\tau)\} \cup \{\nu^{(j)}(0)\}^{2N-1}_{j=1}$. Moreover, $\det(P_0) > 0$ since $P_0$ is positive definite. Therefore, since the function

$$g(\tau) \overset{\text{def.}}{=} 1 - C \sin^{2f_c}(\pi \tau), \quad C \overset{\text{def.}}{=} \frac{24f_c^2 \pi (4f_c^2 - 2f_c)}{(2f_c)} \prod_{l > j} (l - j)^2,$$  \tag{35}

satisfies $g(0) = 1$ and $\partial^j g(0) = 0$ for $j = 1, \ldots, 2N - 1$, we have that $\eta_W = g$.

### C Reparameterization Invariance

In the following, given a Borel map $T : \mathcal{X} \to \mathcal{Y}$, and a Borel measure $\mu$ defined on $\mathcal{X}$, $T_\sharp \mu$ is the pushforward measure of $\mu$, so that for all integrable $f \in L^1(\mathcal{Y})$,

$$\int_{\mathcal{Y}} f(x) d(T_\sharp \mu)(x) = \int_{\mathcal{X}} f(T(x)) d\mu(x).$$

**Proposition 11.** Let $T : \mathcal{X} \to \mathcal{Y}$ be a bijection such that the Jacobian of $T^{-1}$ is invertible. Consider the following minimization problems:

$$\min_m \frac{1}{2} \|y - \Phi m\|^2_H + \lambda |m| (\mathcal{X}).$$  \tag{36}

$$\min_m \frac{1}{2} \|y - (\Phi \circ T_\sharp^{-1})m\|^2_H + \lambda |m| (\mathcal{Y}).$$  \tag{37}

If $\mu$ solve \textbf{[36]}, then $\nu \overset{\text{def.}}{=} T_\sharp \mu$ solves \textbf{[37]}. Let $\eta^{\Phi, y}_{\mathcal{D}}$ and $\eta^{\Phi \circ T_\sharp^{-1}, y}_{\mathcal{D}}$ be the dual certificates and let $\eta^{\Phi, y}_{\mathcal{V}}$ and $\eta^{\Phi \circ T_\sharp^{-1}, y}_{\mathcal{V}}$ be the precertificates associated to \textbf{[36]} and \textbf{[37]} respectively. Then, $\eta^{\Phi \circ T_\sharp^{-1}, y}_{\mathcal{D}} = \eta^{\Phi, y}_{\mathcal{D}} \circ T^{-1}$ and $\eta^{\Phi \circ T_\sharp^{-1}, y}_{\mathcal{V}} = \eta^{\Phi, y}_{\mathcal{V}} \circ T^{-1}$.

**Proof.** The dual certificate of \textbf{[37]} is

$$\eta^{\Phi \circ T_\sharp^{-1}, y}_{\mathcal{D}} = \frac{1}{\lambda} (\Phi \circ T_\sharp^{-1})^* (\Phi \circ T_\sharp^{-1} \nu - y) = \frac{1}{\lambda} (\Phi \circ T_\sharp^{-1})^* (\Phi \mu - y) = \eta^{\Phi, y}_{\mathcal{D}} \circ T^{-1}.$$

For the precertificates, suppose that $y = \Phi m_{a,Z} = (\Phi \circ T_\sharp^{-1}) m_{a,Z}$. Then,

$$\Phi \circ T_\sharp^{-1} \nu_{P_V} = \arg\min \{ ||p|| : (\Phi \circ T_\sharp^{-1})^* p(Tz_j) = 1, \ |\nabla((\Phi \circ T_\sharp^{-1})^* p)(Tz_j) = 0 \} = \arg\min \{ ||p|| : (\Phi^* p)(z_j) = 1, \ |\nabla(\Phi^* p)(z_j) = 0 \} = \nu_{P_V}^{\Phi, y}.$$

where we have used the fact that, by letting $J_{T^{-1}}$ denote the Jacobian of $T^{-1}$,

$$0 = |\nabla((\Phi \circ T_\sharp^{-1})^* p)(Tz_j) = \nabla(\Phi^* p(T^{-1})^*)(Tz_j) = |J_{T^{-1}}(Tz_j)| \nabla(\Phi^* p)(z_j)$$

implies that $\nabla(\Phi^* p)(z_j)$ since the Jacobian of $T^{-1}$ is invertible. Therefore,

$$\eta^{\Phi \circ T_\sharp^{-1}, y}_{\mathcal{V}} = (\Phi \circ T_\sharp^{-1})^* \nu_{P_V}^{\Phi, y} = (\Phi \circ T_\sharp^{-1})^* \nu_{P_V}^{\Phi, y} = \eta^{\Phi, y}_{\mathcal{V}} \circ T^{-1}.$$  

\qed
Remark 10. Let $T$ be as in Proposition [11] let $m_{a, Z} = \sum_j a_j \delta_{z_j}$ and $y = \Phi m_{a, Z}$. Let $\tilde{\Phi} \overset{\text{def.}}{=} (\Phi \circ T^{-1})$ and $\tilde{y} \overset{\text{def.}}{=} \tilde{\Phi} m_{a, Z}$. Then, $\tilde{y} = \Phi m_{a, Z}$. So, (37) can be rewritten as

$$\min_{m} \frac{1}{2} \left\| \tilde{y} - \tilde{\Phi} m \right\|^2 + \lambda \left\| m \right\| (\mathcal{V}).$$

Therefore, to check that the dual certificate of this problem $\eta^\Phi\tilde{y}$ is nondegenerate, it is enough to show that $\eta^\Phi\tilde{y}$ is nondegenerate.

Corollary 2. Let $T$ be as in Proposition [11]. Then,

$$\eta^\Phi m_{a, Z} = \eta^\Phi m_{a, Z} + \mathcal{O}(\left\| \text{Id} - T \right\|).$$

Proof. Given $\varphi : \mathcal{X} \to \mathcal{H}$, let $\Gamma_{\varphi, Z} : \mathbb{R}^{3N} \to \mathcal{H}$ be defined as in [12], where the subscript $\varphi$ makes explicit the associated kernel. By Proposition [11] we have that

$$\eta^\Phi_{\varphi} m_{a, Z} = \eta^\Phi_{\varphi} m_{a, Z} \circ T^{-1} = \Phi^* \Gamma_{\psi, Z}^{\dagger \dagger} \left( \mathcal{O} \right),$$

where $\psi = \varphi \circ T$. On the other hand, $\eta^\Phi m_{a, Z} = \Phi^* \Gamma_{\varphi, Z}^{\dagger \dagger} \left( \mathcal{O} \right)$. Therefore,

$$\eta^\Phi m_{a, Z} = \eta^\Phi m_{a, Z} + \mathcal{O}(\left\| \text{Id} - T \right\|).$$

\[\square\]

D Proof of Correlation Function (32)

Let $\mathcal{X} \subset \mathbb{R}^2$ denote the open unit disc. Then, for $x = (x_1, x_2), x' = (x'_1, x'_2) \in \mathcal{X}$, we let $p = M_1 + iM_2, p_1 = A_1 + iA_2$ and $z = e^{it}$. Interpreting $\mathcal{X}$ as the unit disc on the complex plane, one has

$$C(x, x') = \langle \varphi(x), \varphi(x') \rangle = \int_{\partial \mathcal{X}} \frac{dz}{iz} \frac{|z - p|^2}{|z - p_1|^2}$$

$$= \int_{\partial \mathcal{X}} \frac{dz}{(z - p)(1 - z\overline{p})(z - p_1)(1 - z\overline{p}_1)}.$$  

When $p \neq p_1$, there are 2 poles inside $\mathcal{X}$: $z = p, p_1$, so by the Cauchy residue theorem,

$$C(x, x') = 2\pi \left( \frac{P}{(1 - |p|^2)(p - p_1)(1 - p\overline{p}_1)} + \frac{P_1}{(p_1 - p)(1 - p\overline{p})(1 - |p|^2)} \right)$$

When $p = p_1$, there is 1 pole inside $\mathcal{X}$: $z = p$, so,

$$C(x, x') = 2\pi \left( \frac{d}{dz} \left( \frac{iz}{(1 - z\overline{p})^2} \right) \bigg|_{z=p} \right) = 2\pi \left( \frac{1}{(1 - |p|^2)^2} + \frac{2|p|^2}{(1 - |p|^2)^3} \right)$$

One can then check that both expressions simplify to

$$C(x, x') = 2\pi \frac{1 - |p|^2|p_1|^2}{(1 - |p|^2)(1 - |p|^2)(1 - p\overline{p}_1)^2}.$$
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