ON MATHIEU-TYPE SERIES FOR THE UNIFIED GAUSSIAN HYPERGEOMETRIC FUNCTIONS

Rakesh K. Parmar, Tibor K. Pogány

Academician Gradimir V. Milovanović is septuagenarian

The main purpose of this paper is to present closed integral form expressions for the Mathieu-type $a$-series and for the associated alternating versions whose terms contain a generalized $p$-extended Gauss’ hypergeometric function. Related bounding inequalities for the $p$-generalized Mathieu-type series are also obtained. Finally, a set of various (known or new) special cases and consequences of the results earned are presented.

1. INTRODUCTION AND MOTIVATION

Various extensions of Gauss’ hypergeometric function and other special functions were investigated recently by several authors, consult for instance [5]-[10], [15, 16], [19]-[22]. The importance of these functions is that they inherit most of the properties of the original functions and provide new relations between different special functions. In particular, the generalized Gauss hypergeometric function [29, p. 350, Eq. (1.13)] (see also, [14, p. 631, Eq. (1)]) and generalized confluent hypergeometric function [2, p. 3695, Eq. (9)] are defined as follows:

\[ P^{(\alpha,\beta;\kappa,\mu)}_p(a, b; c; z) = \sum_{n \geq 0} \left( \begin{array}{c} a \\ n \end{array} \right)_n \frac{B_{p,\kappa,\mu}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \]

*Corresponding author. Tibor K. Pogány
2010 Mathematics Subject Classification. 33C20, 33B15, 33C05.
Keywords and Phrases. Generalized $p$-extended Beta function, Generalized $p$-extended Gauss’ hypergeometric function, integral representations, Mathieu–type series, Cahen formula, bounding inequality.
\( (p, \kappa, \mu \geq 0; \min\{\Re(\alpha), \Re(\beta)\} > 0; \Re(c) > \Re(b) > 0, |z| < 1) \):

\[
\Phi_p^{(\alpha, \beta; \kappa, \mu)}(b; c; z) = \sum_{n \geq 0} \frac{B_{p, \kappa, \mu}^{(\alpha, \beta)}(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!},
\]

\( (p, \kappa, \mu \geq 0; \min\{\Re(\alpha), \Re(\beta)\} > 0; \Re(c) > \Re(b) > 0, |z| < \infty), \)

where \( B_{p, \kappa, \mu}^{(\alpha, \beta)}(x, y) \) is the generalized Beta function \([29]\) (see also, \([14]\)) defined by

\[
B_{p, \kappa, \mu}^{(\alpha, \beta)}(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} \, _1F_1 \left( \alpha; \beta; -\frac{p}{t^n (1 - t)^\mu} \right) dt,
\]

\( (\Re(p), \kappa, \mu \geq 0; \min\{\Re(\alpha), \Re(\beta)\} > 0, \Re(x) > -\Re(\kappa \alpha), \Re(y) > -\Re(\mu \alpha)). \)

Here

\[
_1F_1(a, b; z) = \sum_{n \geq 0} \frac{(a)_n}{(b)_n} \frac{z^n}{n!},
\]

stands for the Kummer’s function or the confluent hypergeometric function, see \([1, p. 509]\)^1. The cases of (1.1) when \( \kappa = \mu \) correspond to the generalized hypergeometric function introduced by Parmar \([18, p. 44]\):

\[
F_p^{(\alpha, \beta; \kappa, \mu)}(a, b; c; z) = \sum_{n \geq 0} \frac{(a)_n}{(b)_n} \frac{B_{p, \kappa, \mu}^{(\alpha, \beta)}(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!},
\]

\( (p \geq 0, \mu \geq 0; \min\{\Re(\alpha), \Re(\beta)\} > 0; \Re(c) > \Re(b) > 0, |z| < 1), \)

which again, in case \( \alpha = \beta \) and \( \kappa = \mu \) in (1.1), reduces to the definition by Lee et al. \([12]\):

\[
F_p^{(\mu)}(a, b; c; z) = \sum_{n \geq 0} \frac{(a)_n}{(b)_n} \frac{B_p^{(\mu)}(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!},
\]

\( (p, \mu \geq 0; \Re(c) > \Re(b) > 0, |z| < 1). \)

Yet another case \( \kappa = \mu = 1 \) in (1.1) was studied by Özergin et al. \([17]\]

\[
F_p^{(\alpha, \beta)}(a, b; c; z) = \sum_{n \geq 0} \frac{(a)_n}{(b)_n} \frac{B_p^{(\alpha, \beta)}(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!},
\]

\( (p \geq 0; \min\{\Re(\alpha), \Re(\beta)\} > 0; \Re(c) > \Re(b) > 0, |z| < 1). \)

In a recent article, Choi et al. \([9]\) considered the general Mathieu–type series and their alternating variants whose terms contain the \((p, q)\)-extended Gaussian hypergeometric function \( F_{p, q}(z) \), and in turn, when \( p = q \) the \( p \)-extended Gaussian hypergeometric function \( F_p(z) \) and obtained the closed integral form expressions

---

1 We point out that there is a wide class of elementary and special functions covered by Kummer’s \(_1F_1\), consult for instance \([1, pp. 509-510, §13.6. Special cases]\).
Theorem 1. Let two principal integrals. Then we list certain special cases of our first main result. Here, and in what follows, the real sequence \(a\) for these functions in the widest possible range of the parameters involved.

This paper are to obtain integral representations and allied bounding inequalities \(\tilde{\mathcal{F}}(1.4)\) increases and tends to \(\lambda,\eta, r>\) where \(\lambda, \eta, r>\) and \(\min\{\Re\alpha, \Re\beta\} > 0\), we define the Mathieu–type \(a\)–series \(\mathfrak{S}_{\lambda, \eta}\) and its alternating variant \(\tilde{\mathfrak{S}}_{\lambda, \eta}\) in the form of the series

\[
\tilde{\mathfrak{S}}_{\lambda, \eta}(\mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}; a; r) := \sum_{n \geq 1} \frac{F_p^{(\alpha, \beta; \kappa, \mu)}(\alpha, \beta; c; \frac{r^n}{a_n})}{a_n^\lambda(a_n + r^n)\eta}
\]

where \(\lambda, \eta, r > 0; \Re (c) > \Re (b) > 0\), and in the same range of parameters:

\[
\tilde{\mathfrak{S}}_{\lambda, \eta}(\mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}; a; r) := \sum_{n \geq 1} (-1)^{n-1} F_p^{(\alpha, \beta; \kappa, \mu)}(\lambda, \beta; c; \frac{r^n}{a_n})
\]

Here, and in what follows, the real sequence \(a = (a_n)_{n \geq 1}\) is the restriction of an increasing function \(a: \mathbb{R}^+ \mapsto \mathbb{R}^+\) such that \(a(x)|_{x \in \mathbb{N}} = a\). The main purposes of this paper are to obtain integral representations and allied bounding inequalities for these functions in the widest possible range of the parameters involved.

2. INTEGRAL REPRESENTATIONS OF \(\tilde{\mathfrak{S}}_{\lambda, \eta}(\mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)})\) AND \(\tilde{\mathfrak{S}}_{\lambda, \eta}(\mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)})\)

In this section, we first give the closed integral form expressions for the series \(\tilde{\mathfrak{S}}_{\lambda, \eta}(\mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}; a; r)\) and \(\tilde{\mathfrak{S}}_{\lambda, \eta}(\mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}; a; r)\) in the form of linear combinations of two principal integrals. Then we list certain special cases of our first main result.

Theorem 1. Let \(\lambda > 0, \eta > 0, r > 0\) and the real sequence \(a = (a_n)_{n \geq 1}\) monotone increases and tends to \(\infty\). Then for \(p \geq 0, \kappa \geq 0, \mu \geq 0\) and \(\min\{\Re\alpha, \Re\beta\} > 0\), we have

\[
\tilde{\mathfrak{S}}_{\lambda, \eta}(\mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}; a; r) = \lambda \mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}(\lambda + 1, \eta, a_1) + \eta \mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}(\lambda + 1, \eta, a_1)
\]

\[
\tilde{\mathfrak{S}}_{\lambda, \eta}(\mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}; a; r) = \lambda \mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}(\lambda + 1, \eta, a_1) + \eta \mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}(\lambda + 1, \eta, a_1)
\]
where for all $\Re(c) > \Re(b) > 0$

\begin{equation}
(2.7) \quad \mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}(\lambda, \eta, a_1) = \int_{a_1}^\infty \frac{F_p^{(\alpha, \beta; \kappa, \mu)}(\lambda, b; c; -\frac{x^2}{z}) [a^{-1}(x)]}{x^\lambda (x + r^2)^\eta} \, dx
\end{equation}

\begin{equation}
(2.8) \quad \mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}(\lambda, \eta, a_1) = \int_{a_1}^\infty \frac{F_p^{(\alpha, \beta; \kappa, \mu)}(\lambda, b; c; -\frac{x^2}{z}) \sin^2 \left( \frac{\pi}{4}[a^{-1}(x)] \right)}{x^\lambda (x + r^2)^\eta} \, dx
\end{equation}

and $a^{-1}$ denotes the inverse of a while $[a^{-1}]$ stands for the integer part of $a^{-1}$.

**Proof.** Consider the Laplace transform of the function $t^{\lambda-1} \Phi_p^{(\alpha, \beta; \kappa, \mu)}(b; c; t)$ \cite[p. 3695, Eq. (9)]{2} by using the definition \eqref{1.1}. For all real $\omega$ it equals

\begin{equation}
(2.9) \quad F_p^{(\alpha, \beta; \kappa, \mu)}(\lambda, b; c; \frac{\omega}{z}) = \frac{z^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-z t} t^{\lambda-1} \Phi_p^{(\alpha, \beta; \kappa, \mu)}(b; c; \omega t) \, dt.
\end{equation}

Taking $\xi = a_n + r^2$ in the gamma function formula

$$\Gamma(\eta) \xi^{-\eta} = \int_0^\infty e^{-\xi t} t^{\eta-1} \, dt, \quad (\Re(\xi), \Re(\eta) > 0),$$

after rearrangement $\omega = -r^2$, $z = a_n$ in \eqref{2.9}, the integral \eqref{2.7} becomes

$$\mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}(\lambda, \eta) = \int_0^\infty \int_0^\infty \frac{e^{-r^2 s t^{\lambda-1} s^{\eta-1}}}{\Gamma(\lambda) \Gamma(\eta)} \sum_{n \geq 1} e^{-a_n(t+s)} \Phi_p^{(\alpha, \beta; \kappa, \mu)}(b; c; -r^2 t) \, dt \, ds.
$$

Using the Cahen formula \cite{4} for summing up the resulting Dirichlet series by virtue of the technique developed in \cite{22, 24}, we conclude

$$\mathcal{D}_n(t + s) = \sum_{n \geq 1} e^{-a_n(s+t)} = (s + t) \int_{a_1}^\infty e^{-(t+s)z} [a^{-1}(x)] \, dx.$$

This gives

\begin{equation}
(2.10) \quad \mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}(\lambda, \eta) = \frac{1}{\Gamma(\lambda) \Gamma(\eta)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-(r^2+s)x} (t + s) t^{\lambda-1} s^{\eta-1} [a^{-1}(x)]
\end{equation}

\begin{equation}
\times \Phi_p^{(\alpha, \beta; \kappa, \mu)}(b; c; -r^2 t) \, dt \, ds \, dx =: \mathcal{I}_t + \mathcal{I}_s,
\end{equation}

where

\begin{equation}
\mathcal{I}_t = \int_0^\infty \left( \int_{a_1}^\infty \left( \int_0^\infty \frac{\Phi_p^{(\alpha, \beta; \kappa, \mu)}(b; c; -r^2 t)}{\Gamma(\eta) \Gamma(\lambda) e^{zt}} \, dt \right) e^{-xs}[a^{-1}(x)] \, dx \right) e^{-rs} s^{\eta-1} \, ds
\end{equation}

\begin{equation}
= \frac{\lambda}{\Gamma(\eta)} \int_{a_1}^\infty \left( \int_0^\infty e^{-(x+r^2)s} s^{\eta-1} \, ds \right) \frac{[a^{-1}(x)]}{x^{\lambda+1}} F_p^{(\alpha, \beta; \kappa, \mu)} \left( \lambda + 1, b; c; -\frac{r^2}{x} \right) \, dx
\end{equation}

\begin{equation}
(2.11) \quad = \lambda \int_{a_1}^\infty F_p^{(\alpha, \beta; \kappa, \mu)} \left( \lambda + 1, b; c; -\frac{r^2}{x} \right) \frac{[a^{-1}(x)] \, dx}{x^{\lambda+1}(x + r^2)^\eta} = \lambda \mathcal{F}_p^{(\alpha, \beta; \kappa, \mu)}(\lambda + 1, \eta).
\end{equation}
In a similar way, we get
\[ I_s = \eta \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{(x + r^2)^{\eta + 1}} \left( \int_{0}^{\infty} e^{-\lambda x} \frac{1}{\Gamma(\lambda)} \Phi_p(\alpha, \beta, \kappa, \mu)(b, c; -r^2 t) \, dt \right) \, dx \]
(2.12)
\[ = \eta \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{x^{\lambda}(x + r^2)^{\eta + 1}} F_p^{(\alpha, \beta, \kappa, \mu)} \left( \lambda, b; c; -\frac{r^2}{x} \right) \, dx = \eta F_p^{(\alpha, \beta, \kappa, \mu)}(\lambda, \eta + 1). \]

Now, applying (2.11) and (2.12) to (2.10) we get the representation (2.5).

The derivation of (2.6) is similar to this proving procedure. As to the alternating Dirichlet series \( D_\nu(x) \) integral form, having in mind again the Cahen formula, we have [24, 27]
\[ \tilde{D}_\nu(x) = \sum_{n \geq 1} (-1)^{n-1} e^{-a_n(x)} = x \int_{a_1}^{\infty} e^{-xt} \tilde{A}(t) \, dt, \]
and therefore
\[ \tilde{D}_\nu(x) = x \int_{a_1}^{\infty} e^{-xt} \sin^2 \left( \frac{\pi}{2} [a^{-1}(x)] \right) \, dt, \]
since the counting function turns out to be
\[ \tilde{A}(t) = \sum_{n: a_n \leq t} (-1)^{n-1} = \frac{1 - (-1)^{[a^{-1}(t)]}}{2} = \sin^2 \left( \frac{\pi}{2} [a^{-1}(t)] \right). \]

Hence, because
\[ \tilde{D}_\nu(t + s) = (t + s) \int_{a_1}^{\infty} e^{-(t+s)x} \sin^2 \left( \frac{\pi}{2} [a^{-1}(t)] \right) \, dx, \]
we conclude (2.6) by the obvious remaining steps.

Now, in the case \( \kappa = \mu \), Theorem 1 reduces to the following

**Corollary 1.1.** Let \( \lambda > 0, \eta > 0, r > 0 \), and let the real sequence \( a \) monotone increases and tends to \( \infty \). Then for \( p \geq 0, \mu \geq 0 \) and \( \min \{ \mathbb{R}(\alpha), \mathbb{R}(\beta) \} > 0 \), we have
\[ \tilde{\Phi}_{\lambda, \eta}(F_p^{(\alpha, \beta, \mu)}; a; r) = \lambda \int_{a_1}^{\infty} F_p^{(\alpha, \beta, \mu)}(\lambda, t, a; r) \, dt \]
and
\[ \tilde{\Phi}_{\lambda, \eta}(F_p^{(\alpha, \beta, \mu)}; a; r) = \lambda \int_{a_1}^{\infty} F_p^{(\alpha, \beta, \mu)}(\lambda, t, a; r) \, dt. \]
Again, in the case \( \alpha = \beta \) and \( \kappa = \mu \), Theorem 1 reduces to the following

**Corollary 1.2.** Let \( \lambda > 0, \eta > 0, r > 0 \), and let the real sequence \( a \) monotone increases and tends to \( \infty \). Then for \( p \geq 0, \mu \geq 0 \), we have

\[
\tilde{F}_{\lambda, \eta}(F_p^{(\mu)}; a; r) = \lambda \int_{a_1}^{\infty} \frac{F_p^{(\mu)}(\lambda; b; c; -\frac{r^2}{2}) [a^{-1}(x)]}{x^{\lambda}(x + r^2)^{\eta}} \, dt
\]

where

\[
F_p^{(\mu)}(\lambda, \eta, a_1) = \int_{a_1}^{\infty} \frac{F_p^{(\mu)}(\lambda; b; c; -\frac{r^2}{2}) [a^{-1}(x)]}{x^{\lambda}(x + r^2)^{\eta}} \, dt
\]

Furthermore, in the case \( \kappa = \mu = 1 \), Theorem 1 becomes

**Corollary 1.3.** Let \( \lambda > 0, \eta > 0, r > 0 \), and let the real sequence \( a \) monotone increases and tends to \( \infty \). Then for \( p \geq 0 \) and \( \min\{\Re(\alpha), \Re(\beta)\} > 0 \), we have

\[
\tilde{F}_{\lambda, \eta}(F_p^{(\alpha, \beta)}; a; r) = \lambda \int_{a_1}^{\infty} \frac{F_p^{(\alpha, \beta)}(\lambda; b; c; -\frac{r^2}{2}) [a^{-1}(x)]}{x^{\lambda}(x + r^2)^{\eta}} \, dt
\]

where

\[
F_p^{(\alpha, \beta)}(\lambda, \eta, a_1) = \int_{a_1}^{\infty} \frac{F_p^{(\alpha, \beta)}(\lambda; b; c; -\frac{r^2}{2}) [a^{-1}(x)]}{x^{\lambda}(x + r^2)^{\eta}} \, dt
\]

**Remark 1.** The special case for \( \alpha = \beta \) and \( \kappa = \mu = 1 \) reduces to the known result for the \( p \)-extended Gauss hypergeometric function \( F_p \) [9]. When \( p = 0 \) we have the claim of Theorem 1 for the Gaussian \( _2F_1 \) which is studied in [22].

### 3. BOUNDING INEQUALITIES FOR THE \( p \)-GENERALIZED MATHIEU–TYPE SERIES

Very recently Luo et al. [14, Remark 2.6] have established an upper bounds for the generalized \( p \)-extended Beta function \( B_{p, \kappa, \mu}^{(\alpha, \beta)}(x, y) \). Namely, we have that for all real parameters \( p, \kappa, \mu, \alpha, \beta > 0 \), and \( x, y > 0 \), we have

\[
B_{p, \kappa, \mu}^{(\alpha, \beta)}(x, y) \leq \Omega_{\kappa, \mu}^{(\alpha, \beta)}(p) B(x, y),
\]
Moreover, for all \(\lambda \in (0, 1]\), \(\eta > 0\) and let the real sequence \(a = (a_n)_{n \geq 1}\) monotone increases and tends to \(\infty\). Then for all \(r \in (0, \sqrt{\lambda})\), \(p, \kappa, \mu, \alpha, \beta > 0\) and \(c > b > 0\), we have

\[
\Omega^\alpha_{\kappa, \mu} (p) = _1 F_1 (\alpha; \beta; \frac{(\kappa + \mu)^{\kappa + \mu}}{\kappa^\mu \mu^\mu} p).
\]

We report here the following results [14, Corollary 2.7]

\[
\begin{align*}
F_p^{(\alpha, \beta; \kappa, \mu)} (a, b; c; z) & \leq \Omega^\alpha_{\kappa, \mu} (p) \ _2 F_1 (a, b; c; z) \\
\Phi_p^{(\alpha, \beta; \kappa, \mu)} (b; c; z) & \leq \Omega^\alpha_{\kappa, \mu} (p) \Phi (b; c; z),
\end{align*}
\]

where \(p, \kappa, \mu, \alpha, \beta > 0\), and \(x, y > 0; \ c > b > 0\) and \(|z| < 1\). Next, we need also a certain Luke’s upper bound exposed in [13] for the Gaussian hypergeometric function. Precisely, there holds [13, p. 52, Eq. (4.7)]

\[
\begin{align*}
_2 F_1 (a; b; c; -z) & < 1 - \frac{2ab(c + 1)}{c(a + 1)(b + 1)} \left( 1 - \frac{2(c + 1)}{2(c + 1) + (a + 1)(b + 1)z} \right),
\end{align*}
\]

\((b \in (0, 1], c \geq a > 0; z > 0)\).

For the sake of simplicity we introduce the shorthand notation

\[
\mathcal{U}_a (\lambda, \eta) := \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{x^{a}(x + r^2)^\eta} \, dx.
\]

In the sequel we consider a class of Mathieu–type series (1.3) and (1.4) in which the defining functions \(a : \mathbb{R}_+ \mapsto \mathbb{R}_+\) behave so that \(\mathcal{U}_a (\lambda, \eta, s)\) converges.

**Theorem 2.** Let \(\lambda \in (0, 1]; \ \eta > 0\) and let the real sequence \(a = (a_n)_{n \geq 1}\) monotone increases and tends to \(\infty\). Then for all \(r \in (0, \sqrt{\lambda})\), \(p, \kappa, \mu, \alpha, \beta > 0\) and \(c > b > 0\), we have

\[
\begin{align*}
\mathfrak{g}_{\lambda, \eta} (F_p^{(\alpha, \beta; \kappa, \mu)}; a; r) & \leq \lambda \Omega^\alpha_{\kappa, \mu} (p) \left\{ \begin{array}{c}
\left( 1 - \frac{2(\lambda + 1)b(c + 1)}{c(\lambda + 2)(b + 1)} \right) \mathcal{U}_a (\lambda + 1, \epsilon) \\
+ \frac{4(\lambda + 1)b(c + 1)^2}{c(\lambda + 2)(b + 1) [(\lambda + 2)(b + 1)r^2 + 2(c + 1)a_1]} \end{array} \right. \\
+ \eta \Omega^\alpha_{\kappa, \mu} (p) \left\{ \begin{array}{c}
\left( 1 - \frac{2b(c + 1)}{c(\lambda + 1)(b + 1)} \right) \mathcal{U}_a (\lambda, \epsilon + 1) \\
+ \frac{4b(c + 1)^2}{c(\lambda + 1)(b + 1) [(\lambda + 1)(b + 1)r^2 + 2(c + 1)a_1]} \end{array} \right.
\end{align*}
\]

Moreover, for all \(\lambda + \eta > 1; \ r \in (0, \sqrt{\lambda})\), \(p, \kappa, \mu, \alpha, \beta > 0\) and \(c > b > 0\) we have

\[
\begin{align*}
\mathfrak{g}_{\lambda, \eta} (F_p^{(\alpha, \beta; \kappa, \mu)}; a; r) & \leq \lambda \Omega^\alpha_{\kappa, \mu} (p) \left\{ \begin{array}{c}
\left( 1 - \frac{2(\lambda + 1)b(c + 1)}{c(\lambda + 2)(b + 1)} \right) \frac{a_1^{\lambda - \eta}}{\lambda + \eta} \ _2 F_1 (\eta, \lambda + \eta; \eta + 1; -\frac{r^2}{a_1}) \\
+ \frac{4b(c + 1)^2}{c(\lambda + 1)(b + 1) [(\lambda + 1)(b + 1)r^2 + 2(c + 1)a_1]} \end{array} \right.
\end{align*}
\]
More precisely here and in what follows the parameters involved become real in the definition (1.1)

\[ F_p^{(\alpha,\beta,\kappa,\mu)}(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} F_1(\alpha; \beta; -\frac{p}{t^\kappa(1-t)^\mu}) \, dt. \]

More precisely here and in what follows the parameters \( p, \kappa, \mu \geq 0; \min\{\alpha, \beta\} > 0; c > b > 0 \) and \(-1 < z < 1\). Therefore, by virtue of (3.14) and (3.16) it follows

\[ \mathcal{J}_p(\lambda, \eta, a_1) = \int_{a_1}^\infty \frac{F_p^{(\alpha,\beta,\kappa,\mu)}(\lambda, b; c; x^{-1}(x))}{x^{\lambda}(x + r^2)^{\eta}} \, dx \]

\[ \leq \Omega^{\alpha,\beta}_{\kappa,\mu}(p) \int_{a_1}^\infty \frac{2F_1(\lambda, b; c; x^{-1}(x))}{x^{\lambda}(x + r^2)^{\eta}} \, dx \]

\[ \leq \Omega^{\alpha,\beta}_{\kappa,\mu}(p) \left\{ \left( 1 - \frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)} \right) \int_{a_1}^\infty \frac{[a^{-1}(x)]}{x^{\lambda}(x + r^2)^{\eta}} \, dx \right. 

\[ + \frac{4\lambda b(c+1)^2}{c(\lambda+1)(b+1)} \int_{a_1}^\infty \frac{x^{1-\lambda}[a^{-1}(x)]}{(x + r^2)^{\eta}([\lambda+1](b+1)\lambda^2 + 2(c+1)x)} \right\} \]

\[ \leq \Omega^{\alpha,\beta}_{\kappa,\mu}(p) \left\{ \left( 1 - \frac{2\lambda b(c+1)}{c(\lambda+1)(b+1)} \right) \mathcal{J}_a(\lambda, \epsilon) 

\[ + \frac{4\lambda b(c+1)^2}{c(\lambda+1)(b+1)} \mathcal{J}_a(\lambda - 1, \epsilon) \right\} \].

The rest is obvious. Next, we recall (2.6):

\[ \bar{F}_p^{(\alpha,\beta,\kappa,\mu)}(a; r) = \lambda \mathcal{J}_p(\lambda + 1, \eta, a_1) + \eta \mathcal{J}_p(\lambda, \eta + 1, a_1). \]
By the positivity of the integrand of (2.8) and in view of (3.14) we have
\[
\tilde{F}_p(\lambda, \eta) \leq \int_{a_1}^{\infty} \frac{f_p^{(\alpha, \beta, \kappa, \mu)}}{x^\lambda(x + r^2)\eta} \, dx \leq \Omega_n^{\alpha, \beta}(p) \int_{a_1}^{\infty} \frac{\text{2F}_1(\lambda, b; c; -\frac{x^2}{r^2})}{x^\lambda(x + r^2)\eta} \, dx.
\]
In turn, with the aid of (3.16) we conclude
\[
\tilde{F}_p(\lambda, \eta) \leq \Omega_n^{\alpha, \beta}(p) \left\{ \left( 1 - \frac{2\lambda b(c + 1)}{c(\lambda + 1)(b + 1)} \right) \int_{a_1}^{\infty} \frac{dx}{x^\lambda(x + r^2)\eta} + \frac{4\lambda b(c + 1)^2}{c(\lambda + 1)(b + 1)} \int_{a_1}^{\infty} \frac{dx}{x^{\lambda-1}(x + r^2)r([\lambda + \eta/(b+1)](b + 1)r^2 + 2(c + 1)x)} \right\}.
\]
Using [10, p. 313, Eq. 3.194 1.] for \( \lambda + \eta > 1 \) we have
\[
\int_{a_1}^{\infty} \frac{dx}{x^\lambda(x + r^2)\eta} = \frac{\Gamma\left(\lambda + \eta - 2\right)}{\Gamma\left(1 + \frac{\nu}{2}\right)\Gamma\left(\lambda + \eta - 1\right)} \left( \frac{1}{\eta} \right),
\]
which for \( \lambda + \eta > 2 \) implies
\[
\int_{a_1}^{\infty} \frac{dx}{x^{\lambda-1}(x + r^2)r([\lambda + \eta/(b+1)](b + 1)r^2 + 2(c + 1)x)} \leq \frac{a_1^{-\lambda-\eta}2F_1\left(\eta, \lambda + \eta - 2; \eta + 1; -\frac{r^2}{a_1}\right)}{(\lambda + \eta - 2)(\lambda + \eta - 1)(b + 1)r^2 + 2(c + 1)a_1].
\]
Collecting these formulae we get the upper bound
\[
\tilde{F}_p(\lambda, \eta) \leq \Omega_n^{\alpha, \beta}(p) \left\{ \left( 1 - \frac{2\lambda b(c + 1)}{c(\lambda + 1)(b + 1)} \right) \frac{1}{a_1^{\lambda+\eta-1}(\lambda + \eta - 1)} \times 2F_1\left(\eta, \lambda + \eta - 1; \eta + 1; -\frac{r^2}{a_1}\right) + \frac{4\lambda b(c + 1)^2}{c(\lambda + 1)(b + 1)} \frac{a_1^{2-\lambda-\eta}2F_1\left(\eta, \lambda + \eta - 2; \eta + 1; -\frac{r^2}{a_1}\right)}{(\lambda + \eta - 2)(\lambda + \eta - 1)(b + 1)r^2 + 2(c + 1)a_1].
\right\}
\]
Now, obvious steps lead to the asserted upper bound (3.17).

**Remark 2.** Specifying the parameters in (3.13), we arrive at corollaries of Theorem 2. However, the upper bound expressions for related Mathieu-type series and its alternating variants \( \tilde{\gamma}_{\lambda, \eta}(F_p^{(\alpha, \beta, \mu)}; \alpha; r), \tilde{\gamma}_{\lambda, \eta}(F_p^{(\alpha, \beta, \kappa)}; \alpha; r), \tilde{\gamma}_{\lambda, \eta}(F_p^{(\alpha, \beta, \kappa)}; \alpha; r), \tilde{\gamma}_{\lambda, \eta}(F_p^{(\alpha, \beta, \kappa)}; \alpha; r), \tilde{\gamma}_{\lambda, \eta}(F_p^{(\alpha, \beta, \kappa)}; \alpha; r), \tilde{\gamma}_{\lambda, \eta}(F_p^{(\alpha, \beta, \kappa)}; \alpha; r) \) we leave to the interested reader.

4. DISCUSSION
Our research methodology is based on the following steps. We consider a Mathieu–type series with terms containing special functions (Gaussian hypergeometric function \[ _2F_2 \], Fox–Wright generalized hypergeometric \( _p\Psi_q \) function \[ _{26} \], generalized hypergeometric function \( _pF_q \) and Meijer G function \[ _{27} \], Fox’s H function \[ _{24} \]) which all possess integral representations. The parameters and the constitutional coefficients families permit either series convergence and summation – integration interchange. *Mutatis mutandis*, by this procedure the ‘inner’ Dirichlet–series sum in the integrand becomes summable. Moreover, with the help of the Cahen–formula we next deduce a Laplace–integral expression; for instance, displays (2.7), (2.8) in Theorem 1 are illustrative examples. Finally, the resulting integrand’s structure enables to construct contiguous relations by the related output integrals, see (2.5) and (2.6).

It is worth to mention the Mathieu–type series of more general structure like Mathieu \((a, \lambda)\)–series introduced by Pogány in \[ _{23} \] and the \( a \)–series in \[ _{28} \] which integral expressions were obtained by similar derivation process.

An open problem can be posed concerning the existence of a generic (appropriately convergent) power series instead of \( F_p^{(\alpha,\beta;\kappa,\mu)} \) precised by (1.1) in (1.3) and subsequently in (1.4) which use could lead to general formulæ similar to (2.7), (2.8). By these efforts the re–formulated results in terms of generic series \( S(x) = \sum g_n x^n \), say, would contain among others the case of \( F_p^{(\alpha,\beta;\kappa,\mu)} \) considered in Theorem 1 as an obvious corollary. Unfortunately, the use of a generic power series needs at least the highly strong assumption by which \( S(x) \) should have an suitable integral expression by which we could follow our consideration methodology exposed above.

However, these goals heavily overgrow the purposes and tasks of the recent article, we leave it for another address.

**Acknowledgements.** The authors are grateful to the unknown referees for helpful comments, which finally encompass the article. The research of T. K. Pogány has been supported in part by the University of Rijeka, Croatia under the project number uniri-pr-prirod-19-16.

**REFERENCES**

1. M. Abramowitz, I. A. Stegun (Eds.), "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables". Applied Mathematics Series 55, National Bureau of Standards, Washington, D. C., 1964; 9th Reprinted edition. Dover Publications, New York, 1972.
2. R. P. Agarwal, M. J. Luo, P. Agarwal, *On the extended Appell-Lauricella hypergeometric functions and their applications*, Filomat 31 (2017), No. 12, 3693–3713.
3. Á. Baricz, P. L. Butzer, T. K. Pogány, *Alternating Mathieu series, Hilbert - Eisenstein series and their generalized Omega functions*, in T. Rassias, G. V. Milovanovic (Eds.), "Analytic Number Theory, Approximation Theory, and Special Functions - In Honor of Hari M. Srivastava", 775. Springer, New York, 2014.
4. E. Cahen, *Sur la fonction \( \zeta(s) \) de Riemann et sur des fontions analogues*, Ann. Sci. l'École Norm. Sup. Sér. Math. 11 (1894), 75–164.
Mathieu-type series of unified Gaussian functions

5. M. A. Chaudhry, A. Qadir, M. Rafique, S. M. Zubair, *Extension of Euler’s Beta function*, J. Comput. Appl. Math. 78 (1997), 19–32.
6. M. A. Chaudhry, A. Qadir, H. M. Srivastava, R. B. Paris, *Extended hypergeometric and confluent hypergeometric functions*, Appl. Math. Comput. 159 (2004), 589–602.
7. M. A. Chaudhry, S. M. Zubair, ”On a Class of Incomplete Gamma Functions with Applications”, CRC Press (Chapman and Hall), Boca Raton, FL, 2002.
8. J. Choi, A. K. Rathie, R. K. Parmar, *Extension of extended beta, hypergeometric and confluent hypergeometric functions*, Honam Math. J. 36 (2014), No. 2, 339–367.
9. J. Choi, R. K. Parmar, T. K. Pogány, *Mathieu-type series built by \((p, q)\)-extended Gaussian hypergeometric function*, Bull. Korean Math. Soc. 54 (2017), No. 3, 789–797.
10. I. S. Gradshteyn, I. M. Ryzhik, ”Table of integrals, series, and products”. Translated from the Russian. Sixth edition. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. Academic Press, Inc., San Diego, CA, 2000.
11. D. Jankov Maširević, R. K. Parmar, T. K. Pogány, \((p, q)\)-extended Bessel and modified Bessel functions of the first kind, Results Math. 72 (2017), No. 1–2, 617–632.
12. D. M. Lee, A. K. Rathie, R. K. Parmar, Y. S. Kim, *Generalization of extended beta function, hypergeometric and confluent hypergeometric functions*, Honam Math. J. 33 (2011), No. 2, 187–206.
13. Y. L. Luke, *Inequalities for generalized hypergeometric functions*, J. Approx. Theory 5 (1974), 41–65.
14. M. J. Luo, G. V. Milovanović, P. Agarwal, *Some results on the extended beta and extended hypergeometric functions*, Appl. Math. Comput. 248 (2014), 631–651.
15. M. J. Luo, R. K. Parmar, R. K. Raina, *On extended Hurwitz–Lerch zeta function*, J. Math. Anal. Appl. 448 (2017), 1281–1304.
16. K. Mehrez, S. M. Sitnik, *Generalized Volterra functions, its integral representations and applications to the Mathieu-type series*, Appl. Math. Comput. 347 (2019), 578–589.
17. E. Özergin, M. A. Özarslan, A. Altin, *Extension of gamma, beta and hypergeometric function*, J. Comp. Appl. Math. 235 (2011), No. 16, 4601–4610.
18. R. K. Parmar, *A new generalization of Gamma, Beta, hypergeometric and confluent hypergeometric functions*, Matematiche (Catania) 68 (2013), 33–52.
19. R. K. Parmar, R. B. Paris, P. Chopra, *On an extension of extended beta and hypergeometric functions*, J. Classical Anal. 11 (2017), No. 2, 91–106.
20. R. K. Parmar, T. K. Pogány, *Extended Srivastava’s triple hypergeometric \(H_{A,p,q}\) function and related bounding inequalities*, J. Contemp. Math. Anal. 52 (2017), No. 6, 261–272.
21. R. K. Parmar, R. K. Saxena, T. K. Pogány, *On properties and applications of \((p, q)\)-extended \(\tau\)-hypergeometric functions*, C. R. Acad. Sci. Paris, Ser. I 356 (2018), No. 3, 278–282.
22. T. K. Pogány, *Integral representation of a series which includes the Mathieu \(a\)-series*, J. Math. Anal. Appl. 296 (2004), 309–313.
23. T. K. Pogány, Integral representation of Mathieu \((a, \lambda)\)-series, Integral Transforms Spec. Funct. 16 (2005), No. 2, 685–689.

24. T. K. Pogány, Integral expressions of Mathieu-type series whose terms contain Fox’s \(H\)-function, Appl. Math. Lett. 20 (2007), 764–769.

25. T. K. Pogány, R. K. Parmar, On \(p\)-extended Mathieu series, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. 22 (2018), No. 534, 107–117.

26. T. K. Pogány, H. M. Srivastava, Some Mathieu-type series associated with the Fox–Wright function, Comput. Math. Appl. 57 (2009), No. 1, 127–140.

27. T. K. Pogány, Ž. Tomovski, On Mathieu-type series whose terms contain generalized hypergeometric function \(\mathbf{pF}_{q}\) and Meijer’s \(G\)-function, Math. Comput. Modell. 47 (2008), No. 9–10, 952–969.

28. T. K. Pogány, H. M. Srivastava, Ž. Tomovski, Some families of Mathieu \(a\)-series and alternating Mathieu \(a\)-series, Appl. Math. Comput. 173(1) (2006), 69–108.

29. H. M. Srivastava, P. Agarwal, S. Jain, Generating functions for the generalized Gauss hypergeometric functions, Appl. Math. Comput. 247 (2014), 348–352.

30. H. M. Srivastava, R. K. Parmar, P. Chopra, A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, Axioms 1 (2012), 238–258.

Rakesh K. Parmar
Department of Mathematics, University College of Engineering and Technology, Bikaner (Bikaner Technical University), Bikaner-334004, Rajasthan, India,
E-mail: rakeshparmar27@gmail.com

Tibor K. Pogány
Faculty of Maritime Studies, University of Rijeka, Croatia
Institute of Applied Mathematics, Óbuda University, H-1034 Budapest, Hungary
E-mail: pogany@pfri.hr