Discrete Riccati equation, hypergeometric functions and circle patterns of Schramm type

Agafonov S.I.

Department of Mathematical Sciences
Loughborough University
Loughborough, Leicestershire LE11 3TU
United Kingdom
e-mail: SAgafonov@rusfund.ru

Abstract

Square grid circle patterns with prescribed intersection angles, mimicking holomorphic maps \( z^\gamma \) and \( \log(z) \) are studied. It is shown that the corresponding circle patterns are embedded and described by special separatrix solutions of discrete Painlevé and Riccati equations. General solution of this Riccati equation is expressed in terms of the hypergeometric function. Global properties of these solutions, as well as of the discrete \( z^\gamma \) and \( \log(z) \), are established.

1 Introduction

The theory of circle patterns is a rich fascinating area having its origin in classical theory of circle packings. Its fast development in recent years is caused by mutual influence and interplay of ideas and concepts from discrete geometry, complex analysis and the theory of integrable systems.

The progress in this area was initiated by Thurston’s idea \cite{23,16} about approximating the Riemann mapping by circle packings. Classical circle packings comprised of disjoint open disks were later generalized to circle patterns where the disks may overlap (see for example \cite{14}). Different underlying combinatorics were considered. Schramm introduced a class of circle patterns with the combinatorics of the square grid \cite{21}; hexagonal circle patterns were studied in \cite{6} and \cite{7}.

The striking analogy between circle patterns and the classical analytic function theory is underlined by such facts as the uniformization theorem concerning circle packing realizations of cell complexes of a prescribed combinatorics \cite{4}, discrete maximum principle, Schwarz’s lemma \cite{19} and rigidity properties \cite{16,14}, discrete Dirichlet principle \cite{21}.

The convergence of discrete conformal maps represented by circle packings was proven by Rodin and Sullivan \cite{20}. For a prescribed regular combinatorics this result was refined. He and Schramm \cite{13} showed that for hexagonal packings the convergence is \( C^\infty \). The uniform convergence for circle patterns with the combinatorics of the square grid and orthogonal neighboring circles was established by Schramm \cite{21}.

Approximation issue naturally leads to the question about analogs to standard holomorphic functions. Computer experiments give evidence for their existence \cite{12,15} however not very
much is known. For circle packings with the hexagonal combinatorics the only explicitly described examples are Doyle spirals [11,8] which are discrete analogs of exponential maps and conformally symmetric packings, which are analogs of a quotient of Airy functions [5]. For patterns with overlapping circles more explicit examples are known: discrete versions of exp(z), erf(z) [21], z^α, log(z) [2] are constructed for patterns with underlying combinatorics of the square grid; z^γ, log(z) are also described for hexagonal patterns [6, 7].

It turned out that an effective approach to the description of circle patterns with overlapping circles is given by the theory of integrable systems (see [8, 6, 7]). For example, Schramm’s circle patterns are governed by a difference equation which is the stationary Hirota equation [21] (see [21] for a survey). This approach proved to be especially useful for the construction of discrete z^γ and log(z) in [2, 6, 7] with the aid of some isomonodromy problem. Another connection with the theory of discrete integrable equations was revealed in [1, 2]: embedded circle patterns are described by special solutions of discrete Painlevé II equations, thus giving geometrical interpretation thereof.

This research was motivated by the attempt to carry the results of [1, 2] over square grid circle patterns with prescribed intersection angles giving Schramm patterns as a special case. Namely, we prove that such circle patterns mimicking z^γ and log(z) are embedded. This turned out to be not straightforward and lead to asymptotical analysis of solutions to discrete Riccati equation. As the Riccati differential equation is known as the only equation of the first order possessing Painlevé property though there is no satisfactory generalization thereof on discrete equations.

We use the following definition for square grid circle patterns, which is slightly modified version of one from [21].

**Definition 1** Let G be a subgraph of the 1-skeleton of the cell complex with vertices \( \mathbb{Z} + i\mathbb{Z} = \mathbb{Z}^2 \) and \( 0 < \alpha < \pi \). Square grid circle pattern for G with intersection angles \( \alpha \) is an indexed collection of circles on the complex plane

\[
\{ C_z : z \in V(G), \ C_z \in \mathbb{C} \}
\]

that satisfy:

1) if \( z, z + i \in V(G) \) then the intersection angle of \( C_z \) and \( C_{z+i} \) is \( \alpha \),
2) if \( z, z + 1 \in V(G) \) then the intersection angle of \( C_z \) and \( C_{z+1} \) is \( \pi - \alpha \),
3) if \( z, z + 1 + i \in V(G) \) (or \( z, z - 1 + i \in V(G) \)) then the disks, defined by \( C_z \) and \( C_{z+1+i} \) (\( C_z \) and \( C_{z-1+i} \) respectively) are tangent and disjoint,
4) if \( z, z_1, z_2 \in V(G) \), \( |z_1 - z_2| = \sqrt{2} \), \( |z - z_1| = |z - z_2| = 1 \) (i.e. \( C_{z_1}, C_{z_2} \) are tangent and \( C_z \) intersects \( C_{z_1} \) and \( C_{z_2} \)) and \( z_2 = z + i(z_1 - z) \) (i.e. \( z_2 \) is one step counterclockwise from \( z_1 \)), then the circular order of the triplet of points \( C_z \cap C_{z_1} \cap C_{z_2} \) agrees with the orientation of \( C_z \).

The intersection angle is the angle at the corner of disc intersection domain (Fig. 1).

To visualize the analogy between Schramm’s circle patterns and conformal maps, consider regular patterns composed of unit circles and suppose that the radii are being deformed to preserve the intersection angles of neighboring circles and the tangency of half-neighboring ones. Discrete maps taking the intersection points and the centers of the unit circles of the standard regular
patterns to the respective points of the deformed patterns mimic classical holomorphic functions, the deformed radii being analog of $|f'(z)|$ (see Fig. 2).

In section 2 we give definition of discrete $Z^\gamma$ as a solution to some integrable equation subjected to a non-autonomous constraint. Its geometrical properties of immersion and embeddedness are expressed in terms of solution for radii of corresponding circle patterns.

For this solution to be positive it is necessary that some discrete Riccati equation has positive solution. This equation is studied in section 3 where its general solution is expressed via the hypergeometric function.

Section 4 completes the proof of embeddedness, discrete equations of Painlevé type being the main tool. Possible generalizations for non-regular combinatorics and for non-circular patterns are discussed in section 6.

2 Discrete $Z^\gamma$ and square grid circle patterns of Schramm type.

**Definition 2** Discrete map $Z^\gamma$, $0 < \gamma < 2$ is the solution $f : \mathbb{Z}^2_+ \to \mathbb{C}$ of

$$ q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) = e^{-2i\alpha} $$

$$ \gamma f_{n,m} = 2n \frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{f_{n+1,m} - f_{n-1,m}} + 2m \frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{f_{n,m+1} - f_{n,m-1}}, $$

3
with $0 < \alpha < \pi$ and the initial conditions

$$f_{1,0} = 1, \quad f_{0,1} = e^{i\gamma \alpha}, \quad (3)$$

where $q$ stands for cross-ratio of elementary quadrilaterals:

$$q(f_1, f_2, f_3, f_4) = \frac{(f_1 - f_2)(f_3 - f_4)}{(f_2 - f_3)(f_4 - f_1)}.$$

Constraint (2) was obtained from some isomonodromy problem of Lax representation of (1), found for $\gamma = 1$ in [17] (see also [2],[7]). The definition of $Z^\gamma$ can be justified by the following properties:

- if one thinks of $f$ as defined on the vertices of the cell complex with diamond-shaped faces (see Fig.2) then (1) means that $f$ respects the cross-ratios of the faces and therefore is "locally conformal";
- asymptotics of (2) as $n, m \to \infty$ suggests that $f$ approximates $z^\gamma$.

**Remark.** Equation (1) with $\alpha = \pi/2$ was used in [8] to define discrete conformal maps. The motivation was that $f$ maps vertices of squares into vertices of the "conformal squares". Consider the surface glued of these conformal squares along the corresponding edges. This surface is locally flat but can have cone-like singularities in vertices. If the map is an immersion then the corresponding surface does not have such singularities. Therefore it is more consistent to define as discrete conformal an *immersion* map on the vertices of cell decomposition of $C$ which preserves cross-ratios of its faces.

**Proposition 1** [7, 2] Constraint (2) is compatible with (1).

Compatibility is understood as a solvability of some Cauchy problem. In particular a solution to (1), (2) in the subset $\mathbb{Z}^2_+$ is uniquely determined by its values $f_{1,0}, f_{0,1}$. Indeed, constraint (2) gives $f_{0,0} = 0$ and defines $f$ along the coordinate axis $(n, 0)$, $(0, m)$ as a second-order difference equation. Then all other $f_{k,m}$ with $(k, m) \in \mathbb{Z}^2_+$ are calculated via cross-ratios (1).

In this paper the following initial conditions for (1), (2) are considered:

$$f_{1,0} = 1, \quad f_{0,1} = e^{i\beta} \quad (4)$$

with real $\beta$. The solution $f$ defines a circle pattern with square grid combinatorics. For $\alpha = \pi/2$ such circle patterns were introduced by Schramm in [21]. For any $0 < \alpha < \pi$ we obtain a natural generalization of Schramm circle patterns.

In what follows we say that the triangle $(z_1, z_2, z_3)$ has *positive (negative) orientation* if

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_3 - z_1}{z_2 - z_1} e^{i\phi} \quad \text{with } 0 \leq \phi \leq \pi \quad (-\pi < \phi < 0).$$

**Lemma 1** Let $q(z_1, z_2, z_3, z_4) = e^{-2i\alpha}$, $0 < \alpha < \pi$.

- If $|z_1 - z_2| = |z_1 - z_4|$ and the triangle $(z_1, z_2, z_4)$ has positive orientation then $|z_3 - z_2| = |z_3 - z_4|$ and the angle between $[z_1, z_2]$ and $[z_2, z_3]$ is $(\pi - \alpha)$. 

Proposition 2

All the elementary quadrilaterals \((f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})\) for the solution of (1) with initial \(f_{0,1}\) are of kite form. Moreover, each elementary quadrilateral has one of the forms enumerated in lemma 1.

Proof: straightforward.

Proposition 2

All the elementary quadrilaterals \((f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})\) for the solution of (1), (2) with initial \(f_{0,1}\) are of kite form. Moreover, each elementary quadrilateral has one of the forms enumerated in lemma 1.

Proof: Given initial \(f_{0,1}\) and \(f_{1,0}\) constraint (2) gives \(f_{0,0}\) and \(f_{0,m}\) for all \(n,m \geq 1\). It is easy to check that \(f\) has the following equidistant property:

\[
f_{2n,0} - f_{2n-1,0} = f_{2n+1,0} - f_{2n,0}, \quad f_{0,2m} - f_{0,2m-1} = f_{0,2m+1} - f_{0,2m}
\]

(5)

for any \(n \geq 1, m \geq 1\). Lemma 1 and \(|f_{1,0} - f_{0,0}| = |f_{0,1} - f_{0,0}|\) allows one to apply induction in \(n,m\), starting with \(n = 0, m = 0\).

Proposition 2 implies that for \(n + m = 0 \pmod{2}\) the points \(f_{n+1,m}, f_{n,m+1}\) lie on the circle with the center at \(f_{n,m}\). For the most \(\beta\) (namely for \(\beta \neq \alpha\)) the behavior of thus obtained circle pattern is quite irregular: inner parts of different elementary quadrilaterals intersect.

Definition 3

A discrete map \(f_{n,m}\) is called an immersion if inner parts of adjacent elementary quadrilaterals \((f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})\) are disjoint.

Consider the sublattice \(\{n, m : n + m = 0 \pmod{2}\}\) and denote by \(V\) its quadrant

\[
V = \{ z = N + iM : N, M \in \mathbb{Z}^2, M \geq |N| \},
\]

where

\[
N = (n - m)/2, \quad M = (n + m)/2.
\]

We use complex labels \(z = N + iM\) for this sublattice. Denote by \(C(z)\) the circle of the radius

\[
R_z = |f_{n,m} - f_{n+1,m}| = |f_{n,m} - f_{n,m+1}| = |f_{n,m} - f_{n-1,m}| = |f_{n,m} - f_{n,m-1}|
\]

(6)

with the center at \(f_{n+M,M-N} = f_{n,m}\).

Let \(\{C(z)\}, z \in V\) be a square grid circle pattern on the complex plane. Define \(f_{n,m} : \mathbb{Z}^2 \to C\) as follows:

a) if \(n + m = 0 \pmod{2}\) then \(f_{n,m}\) is the center of \(C(\frac{n-m}{2} + i \frac{n+m}{2})\),

b) if \(n + m = 1 \pmod{2}\) then \(f_{n,m} := C(\frac{n-m-1}{2} + i \frac{n+m-1}{2}) \cap C(\frac{n-m+1}{2} + i \frac{n+m+1}{2} = C(\frac{n-m+1}{2} + i \frac{n+m+1}{2}) \cap C(\frac{n-m-1}{2} + i \frac{n+m-1}{2})\).

Since all elementary quadrilaterals \((f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})\) are of kite form equation (1) is satisfied automatically. In what follows the function \(f_{n,m}\), defined as above by a) and b) is called a discrete map corresponding to the circle pattern \(\{C(z)\}\).
Proposition 3 Let the solution \( f \) of (1), (2) with initial (4) is an immersion, then \( R(z) \) defined by (11) satisfies the following equations:

\[
-MR_z R_{z+1} + (N+1)R_{z+1} R_{z+1+i} + (M+1)R_{z+1+i} R_{z+i} - NR_z i R_z = \frac{\gamma}{2} (R_z + R_{z+1+i}) (R_{z+1} + R_{z+i})
\]  

(7)

for \( z \in \mathbf{V}_l := \mathbf{V} \cup \{-N + i(N-1)\} | N \in \mathbb{N} \) and

\[
(N + M)(R_{z+i} + R_{z+1})(R_z^2 - R_{z+i} R_{z-1} + \cos \alpha R_z (R_{z-1} - R_{z+1})) + \\
(M - N)(R_{z-1} + R_{z+1})(R_z^2 - R_{z+i} R_{z+1} + \cos \alpha R_z (R_{z+1} - R_{z+i})) = 0,
\]

(8)

for \( z \in \mathbf{V}_{\text{rint}} := \mathbf{V} \setminus \{\pm N + iN | N \in \mathbb{N} \} \).

Conversely let \( R(z) : \mathbf{V} \rightarrow \mathbb{R}_+ \) satisfy (7) for \( z \in \mathbf{V}_l \) and (8) for \( z \in \mathbf{V}_{\text{rint}} \). Then \( R(z) \) define a square grid circle patterns with intersection angles \( \alpha \), the corresponding discrete map \( f_{n,m} \) is an immersion and satisfies (11), (12).

Proof: Circle pattern is immersed if and only if all triangles \( (f_{n,m}, f_{n+1,m}, f_{n,m+1}) \) of elementary quadrilaterals of the map \( f_{n,m} \) have the same orientation (for brevity we call it orientation of quadrilaterals). Suppose that the quadrilateral \( (f_{0,0}, f_{1,0}, f_{1,1}, f_{0,1}) \) has positive orientation. Let the circle pattern \( f_{n,m} \) be an immersion. For \( n + m \equiv 1 \mod 2 \) points \( f_{n,m}, f_{n-1,m+1}, f_{n-2,m}, f_{n-1,m-1} \) lie on circle with the center at \( f_{n-1,m} \) and radius \( R_z \), where \( z = (n - m - 1)/2 + i(n + m - 1)/2 \) (See the left part of Fig. 6). Using (11) one can compute \( f_{n,m+1} \) and \( f_{n,m-1} \). Lemma 1 and proposition 2 imply that \( f_{n+1,m} \) is in line with \( f_{n-1,m}, f_{n,m} \) and that the points \( f_{n,m+1}, f_{n,m}, f_{n,m-1} \) are also collinear. Denote by \( R_{z+1}, R_{z+i} \) the radii of the circle at \( f_{n,m-1} \) and \( f_{n,m+1} \) respectively. Define \( R_{z+1+i} = R_z (f_{n+1,m} - f_{n,m}) \). If (2) is satisfied at \( (n - 1, m) \) then (2) at \( (n, m) \) is equivalent to (7), \( R_{z+1+i} \) being positive iff the quadrilaterals \( (f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) \) and \( (f_{n,m-1}, f_{n+1,m-1}, f_{n+1,m}, f_{n,m}) \) have positive orientation.

Similarly starting with (2) at \( (n, m - 1) \), where \( n + m \equiv 0 \mod 2 \) (see the right part of Fig. 6) one can determine evolution of the cross-like figure formed by \( f_{n,m-1}, f_{n+1,m-1}, f_{n,m}, f_{n-1,m-1}, f_{n,m-2} \) into \( f_{n+1,m}, f_{n+2,m}, f_{n+1,m+1}, f_{n,m}, f_{n+1,m-1} \). Equation (2) at \( (n + 1, m) \) is equivalent to (7) and (8) at \( z = (n - m)/2 + i(n + m)/2 \). \( R_{z+1} \) is positive only for immersed circle pattern.

Now let \( R_z \) be some positive solution to (7), (8). We can rescale it so that \( R_0 = 1 \). This solution is completely defined by \( R_0, R_i \). Consider solution \( f_{n,m} \) of (11), (2) with initial data (4) where \( \beta \) is chosen so that the quadrilateral \( (f_{0,0}, f_{1,0}, f_{1,1}, f_{0,1}) \) has positive orientation and satisfies the conditions \( R_0 = 1 = |f_{0,0} - f_{1,0}| \) and \( R_i = |f_{1,1} - f_{1,0}| \). The map \( f_{n,m} \) defines circle pattern due to proposition 2. It can be uniquely computed from (7), (8). To this end one have to resolve (7) with respect to \( R_{z+i+1} \) and use it to find \( f_{n+1,m} \) from \( R_{z+1+i} = R_z (f_{n+1,m} - f_{n,m}) \) and to resolve (8) for \( R_{z+i} \) and use it to find \( f_{n+1,m+1} \) from \( R_{z+i} = R_z (f_{n+1,m+1} - f_{n,m+1}) \). One can reverse the argument used in derivation of (7), (8) to show that \( f \) satisfies (11), (2). Moreover, since \( R_z \) is positive, at each step we get positively oriented quadrilaterals. Q.E.D.

Note that initial data (4) for \( f_{n,m} \) imply initial data for \( R_z \):

\[
R_0 = 1, \quad R_i = \frac{\sin \frac{\beta}{2}}{\sin (\alpha - \frac{\beta}{2})}.
\]

(9)

Definition 4 A discrete map \( f_{n,m} \) is called embedded if inner parts of different elementary quadrilaterals \( (f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) \) do not intersect.
Proposition 4: If for a solution $R_z$ of (7), (8) with $\gamma \neq 1$ and initial conditions (9) holds true:

$$R_z > 0, \quad (\gamma - 1)(R_z^2 - R_{z+1}R_{z-1} + \cos \alpha R_z(R_{z-1} - R_{z+1})) \geq 0 \quad (10)$$

in $V_{int}$, then the corresponding discrete map is embedded.

Proof: Since $R(z) > 0$ the corresponding discrete map is an immersion. Consider piecewise linear curve $\Gamma_n$ formed by segments $[f_{n,m}, f_{n,m+1}]$, where $n > 0$, $0 \leq m \leq n - 1$ and the vector $v_n(m) = (f_{n,m}, f_{n,m+1})$ along this curve. Due to Proposition 2 this vector rotates only in vertices with $n + m = 0 \pmod{2}$ as $m$ increases along the curve. The sign of the rotation angle $\theta_n(m)$, where $-\pi < \theta_n(m) < \pi$, $0 < m < n$ is defined by the sign of expression

$$R_z^2 - R_{z+1}R_{z-1} + \cos \alpha R_z(R_{z-1} - R_{z+1})$$

(note that there is no rotation if this expression vanishes), where $z = (n - m)/2 + i(n + m)/2$ is a label for the circle with the center in $f_{n,m}$. If $n + m = 1 \pmod{2}$ define $\theta_n(m) = 0$.

Now the theorem hypothesis and equation (5) imply that the vector $v_n(m)$ rotates with increasing $m$ in the same direction for all $n$, and namely, clockwise for $\gamma < 1$ and counterclockwise for $\gamma > 1$. Due to the following Lemma it is sufficient to prove (10) only for $1 < \gamma < 2$.

Lemma 2: If $R_z$ is a solution of (7), (8) for $\gamma$ then $1/R_z$ is a solution of (7,8) for $\tilde{\gamma} = 2 - \gamma$.

Consider the sector $B := \{z = re^{i\varphi} : r \geq 0, \ 0 \leq \varphi \leq \gamma \alpha/2\}$. The terminal points of the curves $\Gamma_n$ lie on the sector border.

Lemma 3: The curve $\Gamma_n$ has no self-intersection and lies in the sector $B$.

Lemma is proved by induction. For $n = 1$ it is obviously true since the curve $\Gamma_1$ is a line segment. Let it be true for $n > 1$. Note that for immersed $f$ the first and the last segments of $\Gamma_{n+1}$ lie in $B$ therefore the vector $v_{n+1}(m)$ rotates counterclockwise around convex domain of $B$ confined by $\Gamma_n$ and can not make a full circle. Lemma is proved.
Each curve $\Gamma_n$ cuts the sector $B$ into a finite part and an infinite part. Since the curve $\Gamma_n$ is convex and the borders of all elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ for imbedded $Z^\gamma$ have the positive orientation the segments of the curve $\Gamma_{n+1}$ lie in the infinite part. Now the induction in $n$ completes the proof of Proposition 4 for $1 < \gamma < 2$. The proof for $0 < \gamma < 1$ is similar. Q.E.D.

One can compute $Z_{\gamma,0}^\gamma$ from (2) and obtain the asymptotics $Z_{\gamma,0}^\gamma \simeq c(\gamma)n^\gamma$ from Stirling formula for large $x$:

$$\Gamma(x) \simeq \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}}.$$ (11)

Numerical experiments support the following conjecture.

**Conjecture.** $Z_{\gamma,m}^\gamma \simeq c(\gamma)(n + e^{i\alpha}m)^\gamma$.

### 3 Discrete Riccati equation and hypergeometric functions

Let $r_n$ and $R_n$ be radii of the circles with the centers at $f_{2n,0}$, $f_{2n+1,1}$ respectively (see Fig.4).

![Figure 4: Circles on the border.](image)

Constraint (2) and property (5) gives

$$r_{n+1} = \frac{2n + \gamma}{2(n+1) - \gamma} r_n.$$ (12)

From elementary geometric considerations one gets

$$R_{n+1} = \frac{r_{n+1} - R_n \cos \alpha}{R_n - r_{n+1} \cos \alpha} r_{n+1}$$

Define

$$p_n = \frac{R_n}{r_n}, \quad g_n(\gamma) = \frac{2n + \gamma}{2(n+1) - \gamma}$$

and denote $t = \cos \alpha$ for brevity. Now the equation for radii $R$, $r$ takes the form:

$$p_{n+1} = \frac{g_n(\gamma) - tp_n}{p_n - tg_n(\gamma)}.$$ (13)

**Remark.** Equation (13) is a discrete version of *Riccati* equation. This title is motivated by the following properties:
• cross-ratio of each four-tuple of its solutions is constant as \( p_{n+1} \) is Möbius transform of \( p_n \),
• general solution is expressed in terms of solution of some linear equation (see below this linearisation).

Below we call (13) d-Riccati equation.

**Proposition 5** Solution of discrete Riccati equation (13) is positive for \( n \geq 0 \) iff

\[
p_0 = \frac{\sin \frac{\gamma \alpha}{2}}{\sin \frac{(2-\gamma)\alpha}{2}}
\]

(14)

Proof is based on the closed form of the general solution of d-Riccati linearisation. It is linearised by the standard Ansatz

\[
p_n = \frac{y_{n+1}}{y_n} + tg_n(\gamma)
\]

(15)

which transforms it into

\[
y_{n+2} + tg_n(\gamma + 1)y_{n+1} + ((t^2 - 1)g_n(\gamma)y_n = 0.
\]

(16)

One can guess that there is only one initial value \( p_0 \) giving positive d-Riccati solution from the following consideration: \( g_n(\gamma) \to 1 \) as \( n \to \infty \), and the general solution of (16) with limit values of coefficients is \( y_n = c_1(-1)^n(1 + t)^n + c_2(1 - t)^n \). So \( p_n = \frac{y_{n+1}}{y_n} + tg_n(\gamma) \to -1 \) for \( c_1 \neq 0 \). However \( c_1, c_2 \) defines only asymptotics of a solution. To relate it to initial values one needs some kind of connection formulas. Fortunately it is possible to find the general solution to (16).

**Proposition 6** The general solution to (16) is

\[
y_n = \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1 - \frac{\gamma}{2})}(c_1\lambda_1 n + 1 - \gamma/2 F(\frac{3 - \gamma}{2}, \frac{\gamma - 1}{2}, \frac{1}{2} - n, z_1) +
\]

\[
+ c_2\lambda_2 n + 1 - \gamma/2 F(\frac{3 - \gamma}{2}, \frac{\gamma - 1}{2}, \frac{1}{2} - n, z_2)
\]

where \( \lambda_1 = -t-1, \lambda_2 = 1-t, z_1 = (t-1)/2, z_2 = -(1+t)/2 \) and \( F \) stands for the hypergeometric function.

Proof: Solution was found by slightly modified symbolic method (see [10] for method description). Substitution

\[
y_n = u_x\lambda^x, \ x = n + 1 - \gamma/2
\]

transforms (16) into

\[
\lambda^2(x + 1)xu_{x+2} + 2(t(x + \gamma + 1/2)xu_{x+1} + (t^2 - 1)(x + \gamma - 1)(x + 1)u_x = 0.
\]

(19)

We are looking for solution in the form

\[
u_x = \sum_{m=-\infty}^{\infty} a_m v_{x,m}
\]

(20)
where \( v_{x,m} \) satisfies
\[
(x + m)v_{x,m} = v_{x,m+1}, \quad xv_{x+1,m} = v_{x,m+1}. \tag{21}
\]

**Remark.** Note that the label \( m \) in (20) is running by step 1 but is not necessary integer therefore \( v_{x,m} \) is a straightforward generalization of \( x^{(m)} = (x + m - 1)(x + m - 2)\ldots(x+1)x \) playing the role of \( x^m \) in the calculus of finite differences. General solution to (21) is expressed in terms of \( \Gamma \)-function:
\[
v_{x,m} = c \frac{\Gamma(x + m)}{\Gamma(x)} \tag{22}
\]

Stirling formula (11) for large \( x \) gives the asymptotics for \( v_{x,m} \):
\[
v_{x,m} \simeq cx^m \quad \text{for} \quad x \to \infty. \tag{23}
\]

Substituting (20) into (19), making use of (21) and collecting similar terms one gets the following equation for coefficients:
\[
(\lambda^2 + 2t\lambda + t^2 - 1)a_{m-2} + 2\left(1 + \frac{\gamma}{2} - m\right)(t\lambda + t^2 - 1)a_{m-1} + (t^2 - 1)(1 - m)(\gamma - 1 - m)a_m = 0. \tag{24}
\]

Choice \( \lambda_1 = -t - 1 \) or \( \lambda_2 = 1 - t \) kills the term with \( a_{m-2} \). To make series (20) convergent we can use the freedom in \( m \) to truncate (20) on one side. The choice \( m \in \mathbb{Z} \) or \( m \in \gamma + \mathbb{Z} \) leads to divergent series. For \( m \in \frac{\gamma + 1}{2} + \mathbb{Z} \) equation (24) gives \( a_{\frac{\gamma + 1}{2} + k} = 0 \) for all non-negative integer \( k \) and
\[
a_{\frac{\gamma + 1}{2} - k - 1} = \frac{1 - t^2}{t\lambda + t^2 - 1} \frac{(k - \frac{\gamma - 1}{2})(k + 1 + \frac{\gamma - 1}{2})}{2k} a_{\frac{\gamma + 1}{2} - k} \tag{25}
\]

where \( \lambda = \lambda_1, \lambda_2 \). Substitution of solution of this recurrent relation in terms of \( \Gamma \)-functions and (22) yields
\[
y_x = \lambda^x \sum_{k=1}^{\infty} \left( \frac{1 - t^2}{2(t\lambda + t^2 - 1)} \right)^k \frac{\Gamma(k - \frac{\gamma - 1}{2})\Gamma(k - 1 + \frac{\gamma - 1}{2})\Gamma(x + \frac{\gamma + 1}{2} - k)}{\Gamma(k)\Gamma(x)}. \tag{26}
\]

**Lemma 4** For both \( \lambda = -t - 1, 1 - t \) series (26) converges for all \( x \).

**Proof of Lemma 4** Since \( z = \frac{1-t^2}{2(t\lambda + t^2 - 1)} = (t - 1)/2, -(1 + t)/2 \) for \( \lambda_1, \lambda_2 \) respectively and \( t = \cos \alpha < 1 \) the convergence of (26) depends on the behavior of \( \frac{\Gamma(k - \frac{\gamma - 1}{2})\Gamma(k - 1 + \frac{\gamma - 1}{2})\Gamma(x + \frac{\gamma + 1}{2} - k)}{\Gamma(k)} \).

Stirling formula (11) ensures that this expression is bounded by \( c k^{\phi(x, \gamma)} \) for some \( c \) an \( \phi(x, \gamma) \) which gives convergence.

Series (26) is expressed in terms of hypergeometric functions:
\[
y_x = \lambda^x \frac{\Gamma(x + \frac{\gamma - 1}{2})\Gamma(1 - \frac{\gamma - 1}{2})\Gamma(\frac{\gamma - 1}{2})}{\Gamma(x)} F\left(1 - \frac{\gamma - 1}{2}, \frac{\gamma - 1}{2}, 1 - \left(x + \frac{\gamma - 1}{2}\right), z\right)
\]

where
\[
F\left(1 - \frac{\gamma - 1}{2}, \frac{\gamma - 1}{2}, 1 - \left(x + \frac{\gamma - 1}{2}\right), z\right) = 1 + z \left(1 - \frac{\gamma - 1}{2}\right) \left(\frac{\gamma - 1}{2}\right) \left(1 - \left(x + \frac{\gamma - 1}{2}\right)\right) + ... \tag{27}
\]
(1 - \frac{\gamma - 1}{2})^{n+1} z^{n+\gamma/2} + ... + z^k \left[ (1 - \frac{\gamma - 1}{2})(2 - \frac{\gamma - 1}{2})...(k - \frac{\gamma - 1}{2}) \right] \left( \frac{\gamma - 1}{2} \right)^n (k + 1) + ... + z^k \left( \frac{\gamma - 1}{2} \right)^n (k - 1 + \frac{\gamma - 1}{2}) \right] + ...$$

Here we use the standard designation $F(a, b, c, z)$ for hypergeometric function as a holomorphic at $z = 0$ solution for equation

$$z(1 - z)F_{zz} + [c - (a + b + 1)z]F_z - abF = 0. \quad (28)$$

Now we can complete the proof of proposition 6. Due to linearity general solution of (16) is given by superposition of any two linear independent solutions. As was shown each summond at $z = 0$ solution for equation (29) substituted into (15). Let us define

$$s(z) = 1 + z \left[ (1 - \frac{\gamma - 1}{2})(2 - \frac{\gamma - 1}{2})...(k - \frac{\gamma - 1}{2}) \right] \left( \frac{\gamma - 1}{2} \right)^n (k - 1 + \frac{\gamma - 1}{2}) \right] k!(k - 1 + \frac{\gamma - 1}{2})...$$

It is the hypergeometric function $F\left(\frac{3-\gamma}{2}, \frac{\gamma - 1}{2}, \frac{1}{2} - n, z\right)$ with $n = 0$. A straightforward manipulation with series shows that

$$p_0 = 1 + \frac{2(\gamma - 1)}{2 - \gamma} z + \frac{4z(z - 1) s'(z)}{2 - \gamma s(z)} \quad (31)$$

where $z = \frac{1+\nu}{2}$. Note that $p_0$ as a function of $z$ satisfies some ordinary differential equation of first order since $s'(z)$ satisfies Riccati equation obtained by reduction of (28). Computation shows that $\frac{\sin \frac{\gamma}{2}}{\sin \frac{\gamma - 1}{2} s(z)}$ satisfies the same ODE. Since both expression (31) and (14) are equal to 1 for $z = 0$ they coincide everywhere. Q.E.D.

**Corollary 1** If there exists immersed $f_{n,m}$ satisfying (7), (8), (4) it is defined by initial data (3).

## 4 Embedded circle patterns and discrete Painlevé equations

Let $R_z$ be a solution of (7) and (8) with initial condition (9). For $z \in V_{int}$ define $P_{N,M} = P_z = \frac{R_z}{R_{z+1}}$, $Q_{N,M} = Q_z = \frac{P_z}{R_{z-1}}$. Then (5) and (6) are rewritten as follows

$$Q_{N,M+1} = \frac{(N - M)Q_{N,M}(1 + P_{N,M})(Q_{N,M} - P_{N,M} \cos \alpha) - (M + N)P_{N,M}S_{N,M}}{Q_{N,M}[(M + N)S_{N,M} - (M - N)(1 + P_{N,M})(P_{N,M} - Q_{N,M} \cos \alpha)]}, \quad (32)$$
\[ P_{N,M+1} = \frac{(2M + \gamma)P_{N,M} + (2N + \gamma)Q_{N,M}Q_{N,M+1}}{(2(N + 1) - \gamma)P_{N,M} + (2(M + 1) - \gamma)Q_{N,M}Q_{N,M+1}}, \]  

(33)

where

\[ S_{N,M} = Q_{N,M}^2 - P_{N,M} + Q_{N,M}(1 - P_{N,M}) \cos \alpha. \]

Property (10) for (32), (33) reads as

\[ (\gamma - 1)(Q_{N,M}^2 - P_{N,M} + Q_{N,M}(1 - P_{N,M}) \cos \alpha) \geq 0, \quad Q_{N,M} > 0, \quad P_{N,M} > 0. \]  

(34)

Equations (32), (33) can be considered as a dynamical system for variable \( M \), where due to (12)

\[ P_{N+1,N} = \frac{2N + \gamma}{2(N + 1) - \gamma}. \]  

(35)

**Proposition 7** For each \( N \geq 0 \) there exists \( q_N > 0 \) such that the solution of system (32), (33) subjected to (35) and \( Q_{N+1,N} = q_N \) has property (34) for all \( M > N \).

**Proof:** Due to Lemma 2 it is sufficient to prove (34) only for \( 0 < \gamma < 1 \). Define real function \( F(P) \) on \( \mathbb{R}_+ \) implicitly by \( F^2 - P + F(1 - P) \cos \alpha = 0 \) for \( 0 \leq P \leq 1 \) and by \( F(P) \equiv 1 \) for \( 1 \leq P \).

![Figure 5: The case \( \cos \alpha = -1/2 \).](image)

Designate

\[ D_u := \{(P,Q) : P > 0, Q > F(P)\}, \quad D_d := \{(P,Q) : Q < 0\}, \]

\[ D_0 := \{(P,Q) : P > 0, 0 \leq Q \leq F(P)\}, \quad D_f := \{(P,Q) : P \leq 0, Q \geq 0\} \]

as in Fig.5

System (32), (33) defines the map \( \Phi_N(M) : (P_{N,M},Q_{N,M}) \rightarrow (P_{N,M+1},Q_{N,M+1}) \). This map has the following properties:

- it is a continuous map on \( D_0 \). Values of \( \Phi_N(M) \) on the border of \( D_0 \) are defined by continuity in \( \mathbb{RP}^2 \).
- For \( (P_{N,M},Q_{N,M}) \in D_0 \) holds true \( \Phi_N(M,P_{N,M},Q_{N,M}) \in D_0 \cup D_u \cup D_d \), i.e. \( (P,Q) \) can not jump in one step from \( D_0 \) into \( D_f \).
Consider the solution to (32), (33) with initial conditions on the segment \( S(N) \) determined by (35) and \( Q_{N,N+1} = q \), where \( 0 \leq q \leq F(P_{N+1,N}) \) then \((P_{N,M},Q_{N,M}) = (P_{N,M}(q),Q_{N,M}(q))\).

Define \( S_n(N) = \{ q : (P_{N,M}(q),Q_{N,M}(q)) \in D_0 \} \). Then \( S_n(N) \) is a closed set as \( \Phi_N(M) \) is continuous on \( D_0 \) and \((P,Q)\) can not jump in one step from \( D_0 \) on the half-line \( P = 0, Q > 0 \). As a closed subset of a segment \( S(N) \) the set \( S_n(N) \) is a collection of disjoint segments \( \{ S_n(N) \} \).

**Lemma 6** There exists sequence \( \{ S_n(n) (N) \}_{n > N} \) such that:

- \( S_n(n)(N) \) is mapped by \((P_{N,n}(q),Q_{N,n}(q))\) onto some curve \( c_n \in D_0 \) with one terminal point on the curve \( Q = F(P) \) and the other on the line \( Q = 0 \).
- \( S_n(n+1)(N) \subset S_n(n)(N) \).

Lemma is proved by induction. For \( n = N + 1 \) it is trivial. Let it holds true for \( n \). \( \Phi_N(M) \) maps \((P,Q) \in D_0 \) with \( Q = F(P) \) into \( D_u \) and \((P,0) \) with \( P > 0 \) into \( D_d \). Thus \( \Phi_N(M) \) maps \( c_n \) into some curve \( \bar{c}_{n+1} \) lying in \( D_u \cup D_d \) with one terminal point in \( D_u \) and with the other in \( D_d \). Therefore at least one of the connected components of \( \bar{c}_{n+1} \cap D_0 \) has its terminal points on the border of \( D_0 \) as stated by Lemma.

As the segments of \( \{ S_n(n)(N) \} \) constructed in Lemma are nonempty there exits \( q_N \in S_n(N) \) for all \( n > N \). For this \( q_N \) holds true: \((P_{N,M}(q),Q_{N,M}(q)) \in D_0 \setminus \partial D_0 \). If \((P_{N,M}(q),Q_{N,M}(q)) \in \partial D_0 \) then \((P_{N,M+1}(q),Q_{N,M+1}(q)) \) would jump out of \( D_0 \). Q.E.D.

**Theorem 1** The discrete map \( Z^\gamma \), \( 0 < \gamma < 2 \) is embedded.

**Proof:** Proposition 6 ensures that for any \( N > 0 \) there exist \( R_{N+iN}, R_{N+i(N+1)} \) such that the solution to (32), (33) with these initial values is positive for \( z = N+iM, M \geq N \). Then equation (36) defines recursively \( R_{K+iK} > 0, K > N \) and implies that \( R_{n+iN} \) for all \( n, m : n \geq N, m \geq n \). Therefore asymptotics (29) implies that \( R_{N+iN}/R_{N+i(N+1)} = Q_{N,N+1} \) is exactly as defined by initial condition (9) for \( Z^\gamma \). Proposition 4 completes the proof.

**Remark 1.** For \( N = 0 \) system (32), (33) for \((P_{N,M},Q_{N,M}) \) reduces to the special case of discrete Painlevé equation (the case \( \alpha = \pi/2 \) was studied in [11]):

\[
(n + 1)(x_n^2 - 1) \left( \frac{x_{n+1} + x_n}{\varepsilon} \right) - n(1 - x_n^2/\varepsilon^2) \left( \frac{x_{n-1} + \varepsilon x_n}{\varepsilon + x_{n-1} x_n} \right) = \gamma x_n \frac{\varepsilon^2 - 1}{2 \varepsilon^2}, \tag{36}
\]

where \( \varepsilon = e^{i \alpha} \). This equation allows to represent \( x_{n+1} \) as a function of \( n, x_{n-1} \) and \( x_n: x_{n+1} = \Phi(n,x_{n-1},x_n) \). \( \Phi(n,u,v) \) maps the torus \( T^2 = S^1 \times S^1 = \{(u,v) \in \mathbb{C} : |u| = |v| = 1 \} \) into \( S^1 \) and has the following properties:

- \( \forall n \in \mathbb{N} \) it is a continuous map on \( A_I \times A_I \) where \( A_I = \{ e^{i \beta} : \beta \in [0,\alpha) \} \).
- For \( (u,v) \in A_I \times A_I \) holds true \( \Phi(n,u,v) \in A_I \cup A_{II} \cup A_{IV} \), where \( A_{II} = \{ e^{i \beta} : \beta \in (\alpha, \pi] \} \) and \( A_{IV} = \{ e^{i \beta} : \beta \in [\alpha - \pi, 0) \} \), i.e. \( x \) can not jump in one step from \( A_I \) into \( A_{II} = \{ e^{i \beta} : \beta \in (-\pi, \alpha - \pi) \} \).

These properties guarantees that there exists the unitary solution \( x_n = e^{i \alpha_n} \) of this equation with \( x_0 = e^{i \alpha_0/2} \) in the sector \( 0 < \alpha_n < \alpha \). This solution corresponds to embedded \( Z^\gamma \). Equation (36) is a special case of more general reduction of cross-ratio equation (see [18], [2] for the detail).
5 Circle patterns $Z^2$ and Log

For $\gamma = 2$ formula (12) gives infinite $R_{1+i}$. The way around this difficulty is re-normalization $f \rightarrow (2 - \gamma)f/\gamma$ and limit procedure $\gamma \rightarrow 2 - 0$, which leads to the re-normalization of initial data (see [7]). As follows from (14) this re-normalization gives:

$$R_0 = 0, \quad R_{1+i} = 1, \quad R_i = \frac{\sin \alpha}{\alpha}. \quad (37)$$

**Definition 5** Circle pattern $Z^2$ has radius function specified by (7),(8) with initial data (37).

The symmetry

$$R \rightarrow \frac{1}{R}, \quad \gamma \rightarrow 2 - \gamma. \quad (38)$$

of (7),(8) stated in Lemma 2 is the duality transformation (see [9]). Smooth analog $f \rightarrow f^*$ for holomorphic functions $f(w), f^*(w)$ is:

$$\frac{df(w)}{dw} \frac{df^*(w)}{dw} = 1.$$ 

Note that $\log^*(w) = w^2/2$.

**Definition 6** [7] Circle pattern Log is a circle pattern dual to $Z^2$.

**Theorem 2** Discrete maps corresponding to circle patterns $Z^2$ and Log are embedded.

**Proof:** The circle radii for $Z^2$ and Log are subject to (7),(8) with $\gamma = 2$ and $\gamma = 0$ respectively. For these values of $\gamma$ Proposition 4 is true: the proof is the same since $Z^2$ and Log are immersed. Due to Lemma 2 it suffices to prove the property (10) only for $Z^2$.

Consider the discrete conformal map $2^{-\gamma}Z^\gamma$ with $0 < \gamma < 2$. The corresponding solution $R_z$ of (7),(8) is a continuous function of $\gamma$. So there is a limit as $\gamma \rightarrow 2 - 0$, of this solution with the property (11), which is violated only for $z = 0$ since $R_0 = 0$.

Q.E.D.

6 Concluding remarks

Further generalizations of discrete $Z^\gamma$ and Log are possible.

I. One can relax the unitary condition for cross-ratios and consider solutions to

$$q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) = \kappa^2 e^{-2i\alpha} \quad (39)$$

subjected to the same constraint (2) with the initial data

$$f_{1,0} = 1, \quad f_{0,1} = \frac{e^{i\gamma\alpha}}{\kappa}. \quad (40)$$

This solution is a discrete analog of $Z^\gamma$ defined on the vertices of regular parallelogram lattice (see Fig. 6). However, thus obtained mappings are deprived of geometrical flavor as they do not define circle patterns.
Another possibility is to de-regularize prescribed combinatorics by projection of $\mathbb{Z}^n$ on a plane as follows (see [22]). Consider $\mathbb{Z}_+^n \subset \mathbb{R}^n$. For each coordinate vector $e_i = (e_i^1, ..., e_i^n)$ where $e_i^j = \delta_i^j$ define unit vector $\xi_i$ in $C = \mathbb{R}^2$ so that for any pair of indexes $i, j$ vectors $\xi_i, \xi_j$ form a basis in $\mathbb{R}^2$. Let $\Omega \in \mathbb{R}^n$ be some 2-dimensional connected simply connected cell complex with vertices in $\mathbb{Z}_+^n$. Suppose $0 \in \Omega$. (We denote by the same symbol $\Omega$ the set of the complex vertices.) Define the map $P: \Omega \to C$ by the following conditions:

1) $P(0) = 0$,
2) if $x, y$ are vertices of $\Omega$ and $y = x + e_i$ then $P(y) = P(x) + \xi_i$.

It is easy to see that $P$ is correctly defined and unique.

We call $\Omega$ a projectable cell complex iff its image $\omega = P(\Omega)$ is embedded, i.e. intersections of images of different cells of $\Omega$ do not have inner parts. Using projectable cell complexes one can obtain not only regular square grid and hexagonal combinatorics but more complex ones, i.e. combinatorics of Penrose tilings.

It is natural to define discrete conformal map on $\omega$ as a discrete complex immersion function $f$ on vertices of $\omega$ preserving cross-ratios of $\omega$-cells. The argument of $f$ can be labeled by the vertices $x$ of $\Omega$. Hence for any cell of $\Omega$, constructed on $e_k, e_j$ the function $f$ satisfies the following equation for cross-ratios:

$$q(f_x, f_{x+e_k}, f_{x+e_k+e_j}, f_{x+e_j}) = e^{-2i\alpha_{k,j}},$$

(41)

where $\alpha_{k,j}$ is the angle between $\xi_k$ and $\xi_j$, taken positive if $(\xi_k, \xi_j)$ has positive orientation and taken negative otherwise. Now suppose that $f$ is a solution to (41) defined on the whole $\mathbb{Z}_+^n$.

Conjecture 1. Equation (41) is compatible with the constraint

$$\gamma f_x = \sum_{s=1}^n 2x_s \frac{(f_{x+e_s} - f_x)(f_x - f_{x-e_s})}{f_{x+e_s} - f_{x-e_s}}$$

(42)

For $n = 3$ this conjecture was proven in [7].

Now we can define discrete $\mathbb{Z}^n : \omega \to C$ for projectable $\Omega$ as solution to (41), (42) restricted on $\Omega$. Initial conditions for this solution are of the form (3) so that the restrictions of $f$ on each two-dimensional coordinate lattices is an immersion defining circle pattern with prescribed intersection angles.
This definition naturally generalizes the definition of discrete $Z^\gamma$ given in [7] for $\Omega = \{(k, l, m) : k + l + m = 0, \pm 1\}$.

**Conjecture 2.** $\text{Discrete } Z^\gamma : \omega \to \mathbb{C} \text{ is an immersion.}$

Schramm [21] showed that there is a square grid circle pattern mimicking $\text{Erf}(\sqrt{i}z)$ but an analog of $\text{Erf}(z)$ does not exist. The obstacle is purely combinatorial. There is a hope that combinatorics of projectable cells can give more examples of discrete analogs of classical functions.

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