Quantum Hall states and conformal field theory on a singular surface

T Can¹ and P Wiegmann²,³

¹ Initiative for the Theoretical Sciences, The Graduate Center, CUNY, New York, NY 10012, United States of America
² Kadanoff Center for Theoretical Physics, University of Chicago, 5640 South Ellis Ave, Chicago, IL 60637, United States of America

E-mail: tcan@gc.cuny.edu

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Abstract
In Can et al (2016 Phys. Rev. Lett. 117), quantum Hall states on singular surfaces were shown to possess an emergent conformal symmetry. In this paper, we develop this idea further and flesh out details on the emergent conformal symmetry in holomorphic adiabatic states, which we define in the paper. We highlight the connection between the universal features of geometric transport of quantum Hall states and holomorphic dimension of primary fields in conformal field theory. In parallel we compute the universal finite-size corrections to the free energy of a critical system on a hyperbolic sphere with conical and cusp singularities, thus extending the result of Cardy and Peschel for critical systems on a flat cone (Cardy and Peschel 1988 Nucl. Phys. B 300 377–92), and the known results for critical systems on polyhedra and flat branched Riemann surfaces.

Keywords: adiabatic transport, fractional quantum Hall effect, singular geometry, conformal field theory

(Some figures may appear in colour only in the online journal)

1. Introduction
This paper is devoted to three subjects: (i) geometric transport of quantum Hall states (QH) on singular surfaces. These are surfaces with multiple conical and parabolic singularities; (ii) finite-size corrections to the free energy of critical systems on such surfaces; (iii) connection between the quantum Hall effect (QHE) and critical systems in two dimensions.
1.1. Finite-size corrections in critical systems on singular surfaces

In the seminal paper [2], Cardy and Peschel computed the finite-size correction to the free energy of critical systems on a flat conical surface. The free energy of a critical system consists of the extensive part which grows with system size, and a finite-size correction, the intrinsic part, which grows at most logarithmically with the system size.

The extensive part typically depends on details of the system at the smallest (say, lattice) scale, and is known to be non-universal. However, the finite-size correction is geometrical in nature, independent of microscopic details, and hence a universal characteristic of the system. It also represents the Casimir effect of one dimensional conformal invariant quantum field theory, as well as its specific heat.

In [2], the free energy was argued to scale logarithmically in volume

\[(V \partial V) f = -\frac{c}{12} \chi,\]

with a scale-free coefficient given by the Euler characteristic \(\chi\), a topological invariant of the surface, and \(c\) the central charge of the critical system.

Cardy and Peschel [2] also computed the finite-size correction of an isolated conical, or elliptic, singularity. We denote the deficit angle by \(2\pi \alpha\), and the total opening angle, or simply cone angle, by \(2\pi \gamma\), with \(\gamma = 1 - \alpha\) (see figure A1). Then, the intrinsic part of the free energy at \(L \to \infty\) was found to be

\[(V \partial V) f = -\frac{c}{24} \frac{h_\gamma}{\gamma}, \quad h_\gamma = 1 - \gamma^2 = \alpha(2 - \alpha).\]

This formula is valid for all values of \(\gamma > 0\). When \(0 < \gamma \leq 1\), the conical singularity has positive net curvature concentrated near the tip, and it can be embedded in 3D as a ‘party hat’ (see figure A1). For \(\gamma > 1\), the curvature is negative and the opening angle exceeds \(2\pi\). This is the case of a branched Riemann surface. A branched Riemann surface can be described by a metric with integer \(\gamma = n\) singularities at branch points, since the opening angle indicates that only after \(2\pi n\) traversals around the origin does one return to the starting point. In this case, \(\alpha = 1 - n\) and the sign of the free energy correction changes.

Conformal field theory on flat branched Riemann surfaces has been studied in early works by Knizhnik [3] and related work [4–7].

In [8], Cardy and Calabrese used an extension of the formula (2) for \(\alpha = 1 - n\) to compute the entanglement entropy in one-dimensional quantum conformal invariant systems, by identifying the Renyi entropy with the free energy correction of a system on the \(n\)-sheeted, also flat, Riemann surface. The entanglement entropy follows from the limit \(n \to 1\).

In these works, the singularities or branch points have an interpretation as primary operators of a conformal field theory, with a holomorphic conformal dimension given by \(h_\gamma\). More precisely, this observation implies that a critical system on a single cone of the linear size \(L\) is equivalent to a system on a disk of the size \(L^2\), with a special vertex operator of conformal dimension \(h_\gamma\) inserted in the center of the disk, figure A1.

The origin of the finite-size energy, or Casimir effect, is the gravitational, or trace, anomaly: despite an apparent scale invariance, the trace of the stress tensor does not vanish.

Critical systems on surfaces with multiple singularities are less studied. Comprehensive results are available for polyhedra, flat surfaces whose vertices represent conical singularities. In this case, the essential part of the spectral determinant has been found explicitly in [9–12].

In the present work, we will elaborate further on the properties of critical systems on constant curvature (not necessarily flat) surfaces with many singularities, elliptic and parabolic alike.
Extension of the results for critical systems on flat surfaces to singular positively and negatively curved surfaces meets difficulties related to a lack of explicit formulas in the uniformization theory of Riemann surfaces. Nevertheless, the critical exponents and the leading singular behavior of the free energy on conical singularities happen to be closely related to that of polyhedra. We will show this in the paper.

Hyperbolic geometry also introduces an especially interesting type of singularity which is not sensible for a flat or elliptic metrics. These are hyperbolic funnels, or cusps, also called parabolic singularities, where singular points are infinitely far away and do not belong to the surface (figure A2). Formally, cusps can be seen as a limit $\alpha \to 1$, or $\gamma \to 0$ of a conical singularity on a hyperbolic surface. However, this limit is singular and can not be blindly taken from formulas for conical singularities.

Cusps are topologically equivalent to a cylinder. We will see that the finite-size energy (in units of central charge) of the hyperbolic punctured disk (which comprises half of the pseudosphere in figure A2) with the circumference $L$ is equal to that of a cylinder with the same circumference. If $L = \frac{4\pi}{\log |p|}$ then we will have

$$\text{cusp} : f = -\frac{c}{6L}, \quad (\partial_p p)f = -\frac{c}{24}. \quad (3)$$

We present some results about critical systems on surfaces with multiple cusps.

The scaling dimension with respect to rescaling the volume is not the only critical exponent. On a surface with multiple singularities new exponents emerge. If two or more singularities merge the free energy scales with the short distance between the merging singularities and with the large distance if one group of singularities are far separated from another group.

Among many critical exponents we are primarily concerned with two distinguished exponents. One corresponds to a merging of two singularities, another to a separation of one singularity from an aggregate of others. A genus-0 surface with a discrete set of $n$ curvature singularities is described by the Riemann sphere $\hat{C}$. We denote the coordinates of singularities by $p_1, p_2, \ldots, p_n$.

Then, if two singularities with angles $\gamma_k$ and $\gamma_j$ merge, the free energy scales as

$$(p_k - p_j)^{\partial_p f} \big|_{p_k \to p_j} = \Lambda_{kj}. \quad (4)$$

But if one conical singularity, say with the angle $\gamma_k$, is separated from others by a large distance $d(p_k, \{p\})$, this configuration can be also viewed as a result of merging $n - 1$ singularities, all except one. Then the free energy scales as

$$p_k^{\partial_p f} \big|_{d(p_k, \{p\}) \to \infty} = -\Delta_k. \quad (5)$$

Other exponents correspond to merging three or more cones. We compute the two exponents (4) and (5) and discuss the relations between them, assuming that all conical singularities have generic angles, i.e. the result of merging is a generic conical singularity again. This means that a sum of any number of angles is not an integer. This condition excludes cases when the result of the merging is either a cusp or an orbifold, or both. In this case the critical exponent of multiple merging can be computed in a similar manner.

Symbolically the free energy can be represented as a string of primary operators $V_{\alpha_k}(p_k)$ located at $p_k$ on a surface without singularities

$$e^{-f} \propto \prod_m V_{\alpha_k}(p_m). \quad (6)$$

This interpretation has been realized by Knizhnik [3] for a flat branched Riemann surface with identical singularities.
In this representation the dimensions $\Lambda_{kj}$ appear as the holomorphic conformal dimensions and the operator product expansion (OPE)

$$\prod_m V_{\alpha_m}(p_m) \sim |p_k - p_j|^{2\Lambda_{kj}} \prod_{m \neq k,j} V_{\alpha_m}(p_m).$$

(7)

If the result of the merging of two cones is a cusp or an orbifold, the OPE consists of more terms.

In the case of a punctured sphere, there is only one exponent (3)

$$(p_k - p_j)\partial_{\rho_k} f|_{p_k \rightarrow p_j} = \frac{c}{24}.$$ 

(8)

This is due to a special property of cusps: merging two or more cusps is again a cusp. We will extensively explore this property. In this case, the OPE possesses descendants which contribute the logarithmic corrections to the free energy.

Perhaps the simplest example of a critical system in 2D is the free boson, whose free energy is given by the regularized spectral determinant of the Laplace operator. Since, in the critical regime the finite-size part of the free energy is universal, we can thus identify it with the logarithm of spectral determinant up to terms independent of positions of singularities

$$f = -\frac{c}{2} \log \text{Det}' (-\Delta) + \text{metric independent terms.}$$

(9)

This formula leaves the details of the specific critical system to the central charge, while dependence of geometry is captured by the spectral determinant. The determinant is primed to indicate that zero modes have been excluded from the determinant. For one and two conical singularities, the determinant can be computed by various methods [13–16].

Hence our results for critical exponents could be understood as the limiting behavior of the spectral determinant as singularities merge. In a more formal language we will obtain the limiting behavior of the spectral determinant on the boundary of the moduli space $\mathcal{M}_{0,n}$.

1.2. Adiabatic quantum states and quantum Hall effect

These ideas and results from critical systems have recently found new life in the study of adiabatic quantum states. By an adiabatic quantum state, we mean a system which remains in its instantaneous ground state under the adiabatic evolution of the parameters of the system. Varying these parameters, such as the gauge field and metric, will deform the ground state, but will not drive transitions to excited states. As a specific example, consider model fractional quantum Hall wave functions. These are states which evolve adiabatically under slowly changing parameters of the system. Crucially, the evolution of the states does not drive transitions to excited states.

Specifically, the position of the singularities, as well as the degree of the singularity (e.g. the opening angle of the cone), can serve as the adiabatically varying parameters. For instance, a singularity can be made sharper, or it can be braided around another singularity. This is the subject of geometric adiabatic transport, and will be the main focus of this paper.

The object of interest in adiabatic quantum states is the holonomy, or the phase the states acquire when the adiabatic parameters are taken along a closed contour. Adiabatic transport along non-contractible contours in parameter space are of a special interest. In this case, the adiabatic phases are topological in nature and are related to quantized transport coefficients.

We will consider the geometric transport of quantum Hall states on a sphere with elliptic (cone) and parabolic (cusp) singularities. The space of parameters in this case is naturally
identified with the moduli space $\mathcal{M}_{0,n}$ of an $n$-marked or punctured sphere (see section 3.1 for a definition and further references), which is isomorphic to the configuration space of the coordinates of the singularities.

On a sphere the state $|\Psi\rangle$ is not degenerate, and the holonomy is just a phase. Symbolically, the adiabatic phase reads

$$\Phi_\Gamma = \oint_\Gamma A, \quad A = i\langle\Psi|d\Psi\rangle,$$

where $A$ is the adiabatic connection for the quantum state $|\Psi\rangle$, and the integral goes along a closed path in the space of parameters.

The integer and fractional quantum Hall effects are the most studied examples of adiabatic quantum systems, with a body of knowledge that goes nearly as deep experimentally as theoretically. Quantized electromagnetic adiabatic transport has been measured in quantum Hall (QH) states to metrology precision [17].

Importantly, QH states do not possess conformal symmetry. In contrast to critical systems, QH states feature a scale given by the magnetic length $\ell = \sqrt{\hbar/eB}$. However, QH states are holomorphic in the space of complex adiabatic parameters, and it is through this property that they are connected to conformal field theory (CFT). We will clarify the relation between QH states and CFT in this paper.

Holomorphic adiabatic states are states which depend holomorphically on the complex valued adiabatic parameters (not coordinates of particles). The holomorphic property is the governing property which in the end reduces the problem to conformal field theory. Specifically, we will show that the operators representing singularities transform as conformal primaries. The holomorphic property endows states with robust transport characteristics. This observation links geometric transport to critical systems and conformal field theory. This is the major result of this paper. We also conjecture that this is a general property of any holomorphic adiabatic state, not just QH states. For a recent review on developments in the geometry of QH states we suggest [18].

As indicated in [1, 19], the adiabatic phase has an intensive part which is topological in nature, whose contribution depends only on the homology class of the path $\Gamma$ in the parameter space, not on its shape or area. They are the analog of theta terms of quantum field theory, elusive but important. This part of the phase survives as a non-contractible contour $\Gamma$ is shrunk to zero area about special points in parameter space, and is directly related to precisely quantized transport coefficients. It is in the focus of this paper.

We will show that (the topological part of) the adiabatic phases of geometric transport of QH states are directly related to the conformal dimensions of a critical system discussed in the previous section.

Specifically let us denote by $p = p_1, \ldots, p_n$ the complex coordinates of the singularities and move in such manner that the conformal factor of the metric and the magnetic potential are kept fixed (see a detailed definition in section 4). Then we show that the adiabatic connection differs from $df$ by an exact form which does not contribute to the phase

$$A_\rho = df + \text{exact form}.$$  

Here $d_\rho$ is the Dolbeault operator (see section 3.4 for a definition), and $f$ is the finite-size free energy of the critical system which corresponds to the QH state.

Equivalently this means that the adiabatic phase with respect to a path $\Gamma$ in the moduli space is
\[ \Phi_\Gamma = -\operatorname{Im} \oint_\Gamma A_p = -\operatorname{Im} \oint_\Gamma d_pf. \] (12)

We will also show that the adiabatic phases (12) (in units of $2\pi$) and appropriate contours $\Gamma$ are the conformal dimensions (5) and (4). Specifically, if $\Gamma$ represents a rotation of the entire system about a singularity with the cone angle $\gamma$, then $\Phi_\gamma = -2\pi \Delta_\gamma$, and if the adiabatic process moves a singularity $\gamma$ around a singularity $\gamma'$, then the exchange phase is $\Phi_{\gamma\gamma'} = 2\pi \Lambda_{\gamma\gamma'}$.

The formula (12) establishes a formal relation between QH states and conformal field theory. It allows to read the adiabatic phase of geometric transport from the finite-size energy of the corresponding critical system and to apply the methods of conformal field theory to interference and transport phenomena in a broad class of quantum systems.

This relatively new application of conformal field theory addresses richer physical phenomena than just thermodynamics of critical systems.

The formula (12) also defines the notion of ‘central charge’, a transport coefficient $c$, for states which generally are not conformal. In the case of Laughlin’s series of states with the filling fraction $\nu$ and spin $j$ (see section 4.1 for the definition of spin in QHE), the corresponding critical system is characterized by the central charge

\[ c = 1 - \frac{12}{\nu} \left( \frac{1}{2} - j\nu \right)^2. \] (13)

Geometric adiabatic transport of QH states on smooth surfaces with genus two or higher has been discussed in [19, 20]. The geometric transport on a torus, the genus one surface, is yet to be studied. In this paper we consider the geometric transport on a genus zero surface. In this case, the complex structure moduli are constructed from the positions of singularities. We look at adiabatic phases when the surface is deformed in such manner that positions of singularities move along non-contractible closed paths encircling other singularities. Treating positions of singularities as adiabatic parameters was suggested in recent papers [1, 21, 22].

In addition to the quantized transport coefficients, the adiabatic phase describes the response to small changes of geometry. Moving singular points around forces the electronic fluid to gyrate. It changes the angular momentum of the fluid and exerts a torque. These aspects have been discussed in [1, 22]. We do not discuss it here.

The response of QH states to smooth changes of geometry, as well as the effects of the gravitational anomaly in the QHE have been discussed in many recent papers [18, 23–30]. In contrast to these papers we focus on singularities.

Conical singularities naturally appear in some physical settings. Disclination defects in a crystalline lattice are equivalent to conical singularities, and they are common in graphene [31]. There, two disclinations with the degree $\alpha = \pm 1/3$ are pentagons and heptagons, respectively, embedded into the honeycomb lattice of graphene. Conical geometry has also been simulated in photonic systems [32]. Cusps can be seen as contact leads, whose end points extend to infinity and thus are excluded from the sample [33, 34].

We mention the early paper on Landau levels in a singular geometry [35] and recent papers on the subject [1, 16, 21, 22, 36] and the study of QH electric transport on a surface with a cusp [33, 34].

Similar to the finite-size energy in critical systems, the adiabatic phases do not depend on details of the system. But they also do not depend on details of geometry of the surface. Conformal transformation of the metric leaves the adiabatic phase invariant, and we can consider motion in the space of metrics in a fixed conformal class. The Uniformization Theorem asserts that every surface is conformally equivalent to a surface with constant curvature: positive, negative, or zero. What remains of the space of metrics is the so-called moduli space.
\( \mathcal{M}_{0,n} \), which is the finite dimensional space of complex structure moduli. In light of these facts, we consider constant curvature surfaces with a finite number of prescribed singularities, with special attention paid to negative curvature surfaces.

The purpose of this paper is to give an expository account of the geometric transport of QH states on \( \mathcal{M}_{0,n} \), the moduli space of a sphere with \( n \)-singularities (cones and cusps). This involves some review and rephrasing of the main results of the previous works [1, 22], as well as some novel generalizations.

2. Main results and organization of the paper

The central formula (12) requires us to make a detour into the separate topic of critical systems on singular surfaces, so that we may determine the finite-size correction and compute the adiabatic phases. Results in this domain are scattered and not easily adaptable to our purposes. A part of this paper is devoted to discussing conformal field theory on singular surfaces, and could be read independently from the part on QHE.

The purpose of this paper is thus two-fold:

(i) to connect geometric transport of holomorphic adiabatic states (QH states in particular) to critical systems (sections 3–5), and to express topological adiabatic phases in terms of conformal dimensions of primary CFT operators;

(ii) to compute dimensions (critical exponents) of singularities in critical systems on a singular surface (section 6 and appendix B). curvature transport of QH states.

Having established item (i), the details of geometric transport can be extracted from item (ii), which concerns only critical systems.

We deal exclusively with the Laughlin wave function, though we try to present our results in a manner that makes potential generalization to other states possible.

Before turning to the QHE, we start with a general discussion of holomorphic adiabatic states and introduce the generating functional in section 3. We assert that the generating functional is quasi-primary with respect to the Möbius transformation of the positions of the singularities, and that their dimensions determine the topological parts of the adiabatic phase.

In section 4, we turn to the example of QH states, reviewing the construction of holomorphic wave functions on a Riemann surface with curvature singularities. Then in section 5, we use the vertex construction for the Laughlin wave function and obtain the central charge (13) of the critical system corresponding to Laughlin’s QH states. In this part, we introduce the Quillen metric for the fractional QHE generalizing the Quillen metric for the integer QHE of [20, 30, 37].

Finally, in section 6, we compute the dimensions of conformal field theory on a sphere with singularities. We connect the dimensions to the asymptotes of accessory parameters and what we call (for lack of an existing name) ‘auxiliary’ parameters of the uniformization theory of singular surfaces. We present the explicit calculation of the accessory and auxiliary parameters and the free energy of critical system on polyhedra in appendix B and review the metrics of singular surfaces of revolution in appendix A and section 6.6.

The building block of adiabatic phases is a local angular momentum of the quantum state about a given singularity. Electrons located at the vicinity of a singularity gyrate faster if the curvature is positive (or slower if the curvature is negative) than the bulk electrons. As it was shown in [1], the excess (or the deficit) of the angular momentum of a conical singularity (in units of the Planck constant \( \hbar \)) is
cone : \[ L_{\gamma} = \frac{c}{24} \frac{h_{\gamma}}{\gamma} = \frac{c}{24} \left( \frac{1}{\gamma} - \gamma \right). \] (14)

This result is closely related to the formula of Cardy and Peschel (2).

If we replace the rescaling of the volume \( V \tilde{\partial} \), where \( L \) is the linear scale system, and further replace it by a holomorphic complexification \( L \rightarrow L \partial^{\theta} \) and identify \(-i \partial^{\theta}\) with the rotation operator.

The result for the cusp is shown in [22] to be given by

\[ cusp : \quad L_{\text{cusp}} = \frac{c}{24}. \] (15)

We will show how to obtain these formulas in section 3.11.

We mainly focus on surfaces with a constant curvature \( R_0 \), but some formulas remain valid for surfaces with variable curvature. In this case \( R_0 \) will be the mean curvature. We denote

\[ \alpha_0 = \left( \frac{V}{2\pi} \right) R_0, \] (16)

and call it a ‘background charge’. It is zero for polyhedral surfaces.

If all singularities are conical (a marked sphere), the surface is compact and \( \alpha_0 = 2(\chi - \sum \alpha_j)\), where \( \chi \) is the Euler characteristic (\( \chi = 2 \) for the sphere). Throughout the paper we label orders of conical singularities (the deficit angles in units of \( 2\pi \)) as \( \alpha_1, \ldots, \alpha_n \), the opening angles (in units of \( 2\pi \)) as \( \gamma_1, \ldots, \gamma_n \), the dimensions in (2) as, \( h_1, \ldots, h_n \), and the angular momenta (14) as \( L_1, \ldots, L_n \).

The first adiabatic phase \( \Phi_k \) appears when we rotate the entire system about a chosen conical singularity, say \( \alpha_k \), or equivalently rotate a chosen conical singularity around a conglomeration of the other cones. We can obtain it by studying the scaling behavior (at a fixed conformal factor and magnetic potential, see section 4.1) when one singularity is sent to infinity \( \alpha_k \rightarrow \infty \), while the others stay fixed. The result is

\[ \text{cone} : \quad \Phi_k = -2\pi \Delta_k, \quad \Delta_k = -\alpha_k \sum_{j=1}^{n} L_j + \frac{1}{2} \alpha_0 L_k. \] (17)

This phase is due to the geometric analogue of the Aharonov–Bohm phase in which a particle with an angular momentum \( L_k \) taken along a closed path picks up a phase proportional to the enclosed curvature \( \frac{1}{2} L_k \times \text{enclosed curvature} \). The \( k \)th singularity will encircle the total curvature \( (V/4\pi)R_0 - \alpha_k = \frac{1}{2} \alpha_0 - \alpha_k \), since it will not see its own \( \alpha_0 \) curvature. Furthermore, during this process the other singularities of angular momentum \( L_j \) will encircle the curvature \( \alpha_j \) but in the opposite orientation. These effects together give (17).

Another adiabatic phase occurs upon braiding of singularities. Concretely, this is the process in which two singularities exchange their positions. Assume that we can deform the surface in such manner that a conical point with the deficit angle \( \alpha_k \) adiabatically encircles another conical point with a deficit angle \( \alpha_j \) around an infinitesimal small circle. Also assume that \( \alpha_j + \alpha_k < 1 \) (this condition excludes the case when the result of merging two conical points is a cusp). We will see that the state acquires the phase

\[ \text{cones} : \quad \Phi_{jk} = 2\pi \Lambda_{jk}, \quad \Lambda_{jk} = -\frac{c}{12} \alpha_k \alpha_j + \alpha_j L_k + \alpha_k L_j = \frac{c}{24} \alpha_k \alpha_j \left( \frac{1}{\gamma_j} + \frac{1}{\gamma_k} \right). \] (18)

This result was found in [1]. The last two terms in (18) represent the geometric analog of the Aharonov–Bohm (AB) phase, in which a particle with spin \( L_j \) (or \( L_k \)) encircles a curvature singularity with total integrated curvature \( 4\pi \alpha_j \) (or \( 4\pi \alpha_k \)). The first term is \( \frac{1}{12} \alpha_k \alpha_j \) (in units of
(2π) is the braiding phase. It gives the mutual (or exchange) statistics to conical singularities. When they exchange places, the state acquires a phase equal to $-\frac{n}{24} \alpha_k \alpha_j$ plus the AB phase.

The two types of adiabatic phases are connected via the sum rule

$$\Phi_k = \sum_{j \neq k} \Phi_{j,k} - 2\pi c_1(p_k),$$

where $c_1(p_k)$ is the first ‘Chern number’ on the moduli space equal to the integral of the adiabatic curvature over the $k$th hyperplane of the moduli space (excluding its boundary). It is computed in section 3.12. The result is

$$c_1(p_k) = c_{24} \alpha_0 \alpha_k,$$

(20)

The relation between the dimensions then is the sum rule

$$\Delta_k = -\sum_{j \neq k} \Lambda_{j,k} + c_{24} \alpha_0 \alpha_k.$$

(21)

This sum rule could be interpreted as follows. The integral of the adiabatic curvature over the compactified moduli space $\mathcal{M}_{0n}$ which includes its boundary $\{p_k = \infty, p_k = p_j\}$ vanishes (mod 2π). In this integral the contribution of the boundary of the moduli space $\partial \mathcal{M}_{0n}$ is $\Phi_k = \sum_{j \neq k} \Phi_{j,k}$. It is balanced by the the total adiabatic phase over the moduli space $\mathcal{M}_{0n}$ which excludes the boundary. This is the first Chern number $c_1(p_k)$. One can interpret $c_1(p_k)$ as an exchange phase between a $k$th particle and the background charge $\alpha_0$.

Now let us turn to a punctured sphere. In this case, there is only one independent phase, the braiding phase (22), due to a special property of cusps: merging two or more cusps is again a cusp. Hence when we rotate the system about $p_k$ the acquired phase $\Phi_k$ is $2\pi$ times the angular momentum of one cusp $L_{\text{cusp}}$. In [22] it was shown that the angular momentum is also the braiding phase of two cusps $\Lambda_{\text{cusps}} = L_{\text{cusp}} = \frac{c}{24}$.

(22)

This result does not follow adiabatically as a limit of sharpening angle of a hyperbolic cone in (18) $\alpha \to 1$ (see. Figure A2). This is an important conclusion. The process $\alpha \to 1$, or $\gamma \to 0$ on a hyperbolic surface which formally yields to a cusp is not adiabatic. For instance, as a hyperbolic cone is sharpened to become a cusp, the localized fraction of particles at the cone tip are removed suddenly from the system when $\alpha = 1$.

These are some results we obtain in the text of the paper. They have applications for the limiting behavior of the free energy of a critical system (4), (5) and (8), due to the relation (9).

The overall rescaling $p \to \lambda^{1/2} p$ on a marked surface with a constant curvature is equivalent to the rescaling of the volume $V \to \lambda^{-(2-\alpha_0)} V$. The rescaling yields the dimension

$$-\sum_k p_k \partial_p f = \Delta_0 + \sum_k \Delta_k, \quad \Delta_0 = -\frac{c}{24} \alpha_0 (2 - \alpha_0),$$

(23)

where $\Delta_0$ is the dimension of the background charge.

With the help of the identity $\sum_k \Delta_k = -(2 - \alpha_0) \sum_k L_k$ due to (17), we obtain

$$-(V \partial_V) f = \frac{c}{24} \alpha_0 + \sum_k L_k.$$

(24)
This formula generalizes the known results for particular surfaces with constant curvature: regular surfaces of arbitrary genus (1), polyhedral surfaces (see appendix B), the result (2) of [2] for a single flat cone, and the results for singular surfaces surfaces of revolution (see [16] and references therein).

These results (due to (9)) can be expressed in terms of the limiting behavior of the spectral determinant of the Laplace operator near the boundary of moduli space. At a fixed conformal factor and regardless of the sign of the curvature the limiting behavior of the spectral determinant reads

\[
\text{cone: } \log \text{Det}'(-\Delta) = \begin{cases} 
  p_k \to p_j : & \frac{1}{6} \alpha_k \alpha_j \left( \frac{1}{\gamma_k} + \frac{1}{\gamma_j} \right) \log |p_k - p_j|, \\
  d(p_k, p) \to \infty : & \frac{1}{8} \left( \alpha_k \sum_j \frac{h_j}{\gamma_j} - \frac{1}{2} \alpha_0 \frac{h_0}{\gamma_0} \right) \log |p_k|.
\end{cases}
\]  

(25)

\[
\text{cusp: } \log \text{Det}'(-\Delta)|_{p_k \to p_j} = \frac{1}{6} \log |p_k - p_j|.
\]  

(26)

Also we rewrite the scaling formula (24) for the surface with conical singularities as

\[
-(V \partial V) \log \text{Det}'(-\Delta) = \frac{1}{6} \chi + \frac{1}{12} \sum_k \left( \sqrt{\gamma_k} - \frac{1}{\sqrt{\gamma_k}} \right)^2.
\]  

(27)

The best of our knowledge, equations (25)–(27) are new results. We start from some basic properties of adiabatic holomorphic quantum Hall states and emphasize their common features with critical systems. The central property is a transformation law for the adiabatic connection under \( SL(2, \mathbb{C}) \) (i.e. a Möbius transformation of the position of singularities). This property, plus fusion rules for merging singularities and the value of the central charge appears to be sufficient to completely describe the geometric transport.

3. Geometric transport of holomorphic adiabatic states

In this section we introduce the concept of holomorphic adiabatic states and show that much of the key features of geometric transport follow from some simple defining properties of these states. Throughout, we have in mind quantum Hall states as the prototypical example, but we keep the discussion less specific to stress what we believe is a broader class of many-body quantum states.

We begin with a lightning review of the moduli space of singular metrics, followed by a discussion of adiabatic transport on such spaces.

This section is the conceptual heart of the paper, with the key concepts and connections presented, and without derivation. We save the derivation to later sections where we deal with Laughlin’s series of QH states.

3.1. Moduli space of a sphere with singularities

We begin with a lightning review of the moduli space of singular metrics, mainly to introduce nomenclature. For more details, we suggest [39-41]. In section 4, we will describe the construction of metrics on punctured spheres.

We are primarily concerned with constant curvature \( R_0 \) metrics on genus-0 surfaces with a discrete set of curvature singularities at the points \( p = \{ p_1, \ldots, p_n \} \). Such a Riemann surface

\[ \text{The formula (25) was known before for polyhedra surfaces (see appendix B and references therein), the formula similar to (76) for compact surfaces in the Schottky space was quoted in [38].} \]
\[ \Sigma \] is described by the Riemann sphere \( \hat{\mathbb{C}} \) with marked or punctured points \( p \). In the case of conical singularities, the points belong to the surface. We refer to it as a marked sphere. Cusp points can be viewed as the limit in which hyperbolic cones are sharpened such that their tip is pushed off to infinity. Such a surface is non-compact. We refer to it as a punctured sphere.

The choice of complex coordinates is determined by the complex structure moduli, which define an equivalence class of conformally equivalent metrics. The sphere has a unique choice of complex structure, so the moduli space is a single point. The singular sphere, however, has a larger moduli space related to the space of configurations of punctures \( C_n = \{ (p_1, ..., p_n), p_i \neq p_j \} \). Upon identifying points which are equivalent under Möbius transformations \( SL(2, \mathbb{C}) \) and permutation of the indices (action by the symmetric group on \( n \) elements \( \text{Symm}(n) \)), we obtain the moduli space \( M_{0,n} \) for the genus-0 \( n \)-punctured sphere.

\[ M_{0,n} = C_n / SL(2, \mathbb{C}) \times \text{Symm}(n). \] (28)

Since a Möbius transformation can be used to fix the position of three points on the Riemann sphere, the moduli space ends up having complex dimension \( n - 3 \). For \( n \geq 3 \), the uniformization theorem guarantees that there exists a meromorphic function which maps \( \Sigma \) to one of three surfaces \( S \): the sphere (for \( R_0 > 0 \)), the plane (for \( R_0 = 0 \)), or the upper half plane \( \text{Im} \ w > 0 \) (for \( R_0 < 0 \)). This map \( w(z) : \Sigma \to S \) is known as the developing map, while the inverse is a covering map often called the Klein map. The covering map is always explicitly available for three singularities, essentially due to the fact that in this case the moduli space shrinks to a point [42]. The developing map for \( R_0 = 0 \) can also be constructed explicitly [43] for an arbitrary number of singularities, and is formally identical to the Schwarz–Christoffel map for polygonal domains.

A non-contractible closed path on a punctured sphere is not closed in the \( w \)-plane, but its ends can be brought together by a modular transformation. These transformations generate the Fuchsian group, isomorphic to the fundamental group of the punctured Riemann sphere. The quotient of the upper half plane with the Fuchsian group is the fundamental domain of the multi-valued developing map \( w(z) : \Sigma \to S \).

3.2. Holomorphic adiabatic states and generating functional

The central point of the theory of quantized transport in the QHE is that QH states are holomorphic adiabatic states. A holomorphic adiabatic state is a holomorphic section of a line bundle fibered over the space of complex-valued adiabatic parameters. This property holds when the adiabatic parameters are magnetic fluxes threading handles of a multiply-connected surface. And it is also true for geometric transport, where the adiabatic parameters are complex structure moduli \( M_{0,n} \).

This seemingly benign definition of holomorphic adiabatic states has profound consequences, as we will now show.
We adopt the inner product of states with respect to a measure $m(z, \bar{z})dzd\bar{z}$

$$\langle \Psi' | \Psi \rangle = \int \psi'(z_1, \ldots, z_N) \psi(z_1, \ldots, z_N) \prod_{i=1}^{N} m(z_i, \bar{z}_i)dz_i d\bar{z}_i.$$  

(29)

chosen such that it stays unchanged in the adiabatic process. We will specify the measure for the states on the lowest Landau level (LLL) in section 4.1.

Specifically, the holomorphic state reads

$$\psi(z_1, \ldots, z_N | p) = \frac{\chi(z_1, \ldots, z_N | p)}{\sqrt{Z(p, \bar{p})}},$$  

(30)

where the non-normalized state $\chi$ is a multi-valued holomorphic function in $p$. The dependence of anti-holomorphic $\bar{p}$ is found only in the real normalization factor $Z$

$$Z(p, \bar{p}) = \int |\chi(z_1, \ldots, z_N | p)|^2 \prod_{i=1}^{N} m(z_i, \bar{z}_i)dz_i d\bar{z}_i.$$  

(31)

We emphasize that the state is holomorphic with respect to the adiabatic parameters, in our case the positions of singularities, and not necessarily the particle coordinates.

### 3.3. Möbius transformation of holomorphic states

Adiabatic quantum states on a closed surface have no physical boundaries, hence are invariant under a simultaneous Möbius transformation of coordinates of particles $z_1, \ldots, z_N$ and coordinates of singularities $p$. For a finite number of particles, the numerator and the denominator are simultaneously Möbius invariant. This, is no longer true when the number of particles $N$ is sent to infinity. In this limit, the generating functional $Z$, hence the non-normalized state $\chi$, are transformed under Möbius transformation (at a fixed measure $m(z, \bar{z})$), in such manner that the normalized state $\psi$ remains invariant.

A basic property of such states which we may take as a definition of holomorphic states is that the generating functional transforms as a quasi-primary

$$p_k \rightarrow g(p_k) = \frac{ap_k + b}{cp_k + d} : Z(g(p)) = \prod_k |g'(p_k)|^{-\Delta_k} Z(p).$$  

(32)

The property (32) appears to be a governing principle. We prove it in equation (10) for QH states, but would like to emphasize that it represents a minimal assumption, combined with holomorphicity, which gives rise to conformal symmetry of adiabatic states.

It is tempting to assume that not just QH states, where we checked the transformation (32) directly, but a broad class of holomorphic adiabatic states features a relation to conformal field theory.

Before discussing the emergent conformal symmetry, we review the consequences of the Möbius transformation on the adiabatic connection, following [44].

### 3.4. Adiabatic connection

The holomorphic property has an important consequence. All information of the adiabatic transport is encoded in the normalization factor $Z(p)$ referred to as the generating functional. Proceeding forward, it is convenient to express the adiabatic connection (10)
in the holomorphic basis $A = \frac{i}{2}(A_p + \bar{A}_p)$, where $A_p = 2\langle \Psi | dp | \Psi \rangle = \sum_{k=1}^{n} A_k dp_k$ and $\bar{A}_p = -2\langle dp | \Psi | \Psi \rangle = \sum_{k=1}^{n} \bar{A}_k d\bar{p}_k$, where

$$d_p = \sum_{k=1}^{n} dp_k \partial_{p_k}$$  \hspace{1cm} (33)

is the Dolbeault operator$^5$.

A straightforward calculation using (30) shows that the adiabatic connection is determined by the generating functional

$$A_p = dp \log Z, \quad \bar{A}_p = -d\bar{p} \log Z.$$  \hspace{1cm} (34)

In other words, the adiabatic curvature is a Kähler form, and the generating functional is the Kähler potential. Likewise it follows that the adiabatic phase is expressed entirely in terms of the generating functional

$$\Phi \Gamma = -\text{Im} \oint \Gamma dp \log Z.$$  \hspace{1cm} (35)

For QH states, the Kähler property of the adiabatic curvature was known for a long time, see e.g. [45], and has since become standard lore in the literature. For a more general treatment in the QH setting, along with a formal proof of this result, see e.g. [30].

The generating functional contains much more information about the system than just the adiabatic phase. Consider for example conductances associated with the adiabatic parameters $p$. According to the Kubo formula conductances are components of the adiabatic curvature 2-form (see, e.g. [37])

$$dA = i(d\Psi | d\Psi) = \sigma_{pp}(\frac{i}{2} dp \wedge d\bar{p}).$$  \hspace{1cm} (36)

We see that the generating functional describes the conductance matrix

$$\sigma_{ij} = \partial_{p_i} \partial_{p_j} \log Z.$$  \hspace{1cm} (37)

A regular part of the generating functional yields an exact part of the adiabatic connection 1-form, which does not contribute to the adiabatic phase. This part, however, contributes to the conductance, and as suggested in [37] could be regarded as mesoscopic fluctuations.

Equipped with the adiabatic connection, we note that normalized holomorphic adiabatic states satisfy

$$\left(\partial_p - \frac{1}{2} A_p \right) \Psi = 0.$$  \hspace{1cm} (38)

This property could also serve as a definition of holomorphic adiabatic states.

3.5. Quantized transport and topological part of the adiabatic phase

The adiabatic phase consists of two distinct contributions, a geometric and a topological part. The geometric part depends on the shape of the path, whereas the topological part depends only on the homology of the path.

The two kinds of phases can be distinguished by their adiabatic curvature. The geometric part of the adiabatic curvature is a smooth function of adiabatic parameters. In contrast, the

$^5$ Although the sum here extends over $n$ complex dimensions, $SL(2, \mathbb{C})$ symmetry will reduce the number of independent dimensions to $n-3$. 

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topological part is accumulated on a finite set of points where the adiabatic curvature (36) is a delta-function. These points occur on the boundary of the moduli space, when singularities merge. The weights of the delta functions are quantized conductances (37), intrinsic characteristics of the quantum state. They can not change continuously and, therefore, are not affected by small perturbations.

We focus on the topological part of the phases. They are determined by the limiting behavior of the adiabatic connection

\[ A_k \big|_{d(p_k, p) \to \infty} \sim \frac{\Phi_k}{2\pi p_k}, \quad A_k \big|_{p_k \to p_j} \sim \frac{\Phi_{kj}}{2\pi (p_k - p_j)}. \]

One corresponds to the process when the system rotates by \(2\pi\) about a point \(p_k\). The second corresponds to the braiding of singularities \(p_k\) and \(p_j\). In the rest of the paper we derive the formulas for the adiabatic phases \(\Phi_k\) and \(\Phi_{kj}\) quoted in section 3.2.

3.6. Transformation law and Möbius sum rules for adiabatic connection

If we assume the transformation property (32), then the adiabatic connection \(A_p\) transforms as

\[ A_p \to A_p - \frac{1}{2} \sum_{k=1}^{n} \Delta_k \partial p_k \log g'(p_k) dp_k, \]

Explicitly, the transformation rule for the components of the connection reads

\[ A_k(g(p)) = (ad - cb)^{-1}(cp_k + d)^2 \left(A_k(g(p)) + \frac{\Delta_k}{cp_k + d}\right). \]

The invariance under an overall translation of singularities \(\sum_k A_k(g(p)) = 0\) yields the two additional sum rules

\[ \sum_k A_k = 0, \quad \sum_k (\Delta_k + 2A_k p_k) = 0, \quad \sum_k (\Delta_k p_k + A_k p_k^2) = 0. \]

These formulas can be illustrated by the familiar expression of the three point correlation functions of quasi-primary operators. If there are only three singularities, the sum rules alone determine the adiabatic connections

\[ A_1 = \frac{\Delta_3 - \Delta_1 - \Delta_2}{2(p_1 - p_2)} + \frac{\Delta_2 - \Delta_3 - \Delta_1}{2(p_1 - p_3)}. \]

Other components are obtained by a permutation.

Then the generating functional is determined up to an overall constant \([44]\)

\[ \mathcal{Z}(p_1, p_2, p_3) = |p_1 - p_2|^{\Delta_1 - \Delta_3 - \Delta_2} |p_2 - p_3|^{\Delta_2 - \Delta_1 - \Delta_3} |p_3 - p_1|^{\Delta_3 - \Delta_2 - \Delta_1} C(\alpha_1, \alpha_2, \alpha_3). \]

The constant \(C(\alpha_1, \alpha_2, \alpha_3)\) depends on the orders of singularities.

3.7. Fusion rules and dimensions from Möbius sum rules

The symmetry under the Möbius transformation is weaker than conformal symmetry, but in practice it helps one compute the dimensions and the geometric transport.

In the case of more than three singularities, explicit formulas for the connection are only available for the flat polyhedral surfaces (appendix B). However, the asymptotes as singularities
merge can be obtained using the sum rules (42), along with some additional assumption about the result of the merging.

When two singularities \( p_k \) and \( p_j \) merge, a new singularity occurs. We denote the dimension of this singularity \( \Delta_{ij} \), and the new adiabatic connection as \( A_{ij} \). The original components of the adiabatic connection \( A_k \) and \( A_j \) diverge upon merging these singularities. However, the first sum rule requires that the sum \( A_k + A_j \), is regular. The second sum rule implies that the original connections diverge as

\[
A_k \bigg|_{p_k \rightarrow p_j} = -A_j \bigg|_{p_k \rightarrow p_j} = \frac{\Delta_{ij} - \Delta_k - \Delta_j}{2(p_k - p_j)}.
\]

From which we learn that the mutual statistics is given by

\[
\Lambda_{ij} = \frac{1}{2} (\Delta_{ij} - \Delta_k - \Delta_j).
\]

Therefore, the mutual statistics is completely fixed by the dimensions \( \Delta_k \) and the fusion rule which determines \( \Delta_{ij} \). Further relations for conical singularities follow from (17), (18) and (21). For example \( \Delta_{ij} = -\sum_{i \neq k,j} (\Lambda_k + \Lambda_{ij}) + e_k(p_k) + e_j(p_j) \). The sum rules (42) with a combination of fusion rules appear to be useful for computing the dimensions. We explore similar sum rules in section 6.5.

3.8. Large N expansion and conformal field theory

The transformation formula for the generating functional (32) already implies a connection to conformal field theory. We will now make this connection precise, and argue that in fact the geometric transport is captured by the finite-size correction to the free energy of a critical system. Importantly, this connection appears in the \( 1/N \) expansion. For a finite number of particles, the adiabatic connection is a regular function of the adiabatic parameters. The singularities of the connection, and hence the quantized transport, are only strictly seen in the limit of large \( N \) number of particles.

The large \( N \) expansion of the generating functional of QH states has been studied in [23, 26, 27]. It has the form

\[
Z = Z_2^N \cdot Z_1^N \cdot N^D \cdot e^{\int f} \cdot (1 + O(1/N))\),
\]

where \( Z_2 \) and \( Z_1 \) are regular functions on moduli space, and \( D \) is moduli independent, but universal \( \alpha \) dependent factor related to the dimensions. We will obtain this expansion in the next section, where we will argue that \( \log Z - f \) is a regular function on moduli space\(^6\). Hence, the adiabatic connection (34) is a differential of \( f \), equations (11) and (34), and the phase, equation (35) read

\[
\Phi_\Gamma = -\text{Im} \int_\Gamma d\phi f,
\]

as quoted in the introduction (12).

This observation (46), could be traced to papers [23, 46–48]. It formalizes the relation between QH states and conformal field theory:

the topological part of the adiabatic phase of holomorphic states in moduli space is determined by scaling dimensions of the corresponding conformal field theory.

\(^6\)The sign \( + \) in front of \( f \) in (46) is not a misprint. The generating functional is the inverse of the partition function of the relevant critical system (see (82)).
There are several methods to obtain this result for the QH states. The most powerful method where every step is under control is based on the Ward identity, developed in [1, 22, 23, 47] (see also [27]). Alternative methods are based on collective field theory approach [29], and on the vertex construction [26], see also [19]. We also mention related approaches of [25] and [28]. Among them the vertex construction seems the most economical. We adopt it in this paper.

Let us assume the formula (47) for now, and walk one more step before turning to the specific example of QH states.

3.9. Adiabatic connection, quasi-conformal map and the Schwarzian

When we move singularities we change the surface metric. Under a variation of the metric the free energy of a critical system changes. The rate of change is the stress tensor of the critical system. Hence, the 1-form \( df \), could be expressed through the (holomorphic component of the) stress tensor of the corresponding critical system. It is known to be proportional to the Schwarzian of the metric. In this section we use these facts to write the connection in terms of the Schwarzian and a quasi-conformal map describing the displacement of singularities. The main tool is the simple equation which connects the quasi-conformal transformation of coordinates to the transformation of the position of singularities \( p \) (see, e.g. [40]).

We start with the metric \( ds^2 = e^{\phi} |dz|^2 \) expressed in complex coordinates. The conformal factor \( e^{\phi} \) is a function of the positions of the singularities \( p \). When we move singularities the metric changes \( ds^2 \rightarrow ds^2 + dp(dz^2) \). The new metric is no longer diagonal. Its general form reads \( dp(dz^2) = e^{\phi}(dp|dz|^2 + \mu|dz|^2 + \bar{\mu}|dz|^2) \), where \( \mu|dz|^2 \) is a harmonic Beltrami differential. It obeys the condition \( \nabla z \mu = 0 \), where \( \nabla z = \partial_z + \partial_{z}\phi \).

The new metric can be brought into the diagonal form by an appropriate choice of coordinates \( z'(z, \bar{z}) = z + \xi(z, \bar{z}) \) determined by the Beltrami equation \( \partial_z \xi = \mu \) and the condition \( \nabla z \xi = dp \phi \).

In terms of a basis of displacements of singularities \( \xi = \sum \xi_k dp_k \), this relation (48) explicitly reads

\[
\partial_k \phi - \xi_k \partial_z \phi - \partial_z \xi_k = 0.
\]

It expresses the transformation of the metric under a motion of singularities.

Under a general change of the metric, which includes a change of the complex structure and also a change of the conformal factor, a change of the free energy is expressed through components of the stress tensor

\[
df = -\frac{1}{\pi} \int T_{\mu} dz d\bar{z}, \quad df = \frac{1}{\pi} \int \Theta d\phi dV.
\]

Here \( T \) is the holomorphic component of the stress tensor, \( dV = e^{\phi} dz d\bar{z} \) is the volume element, and \( \Theta \) is the trace of the stress tensor.

In the conformal field theory with the central charge less than one (the case corresponding to the Laughlin states we consider) the holomorphic component of the stress tensor is proportional to the Schwarzian

\[
T = \frac{c}{12} S[\phi],
\]

\[
S[\phi] = -\frac{1}{2} (\partial_z \phi)^2 + \partial_{\bar{z}}^2 \phi.
\]
The trace of the stress tensor is singular at singularities, but away from singular points it is proportional to the curvature (the trace anomaly).

A classical result of Schwarz asserts that the Schwarzian for surfaces with constant curvature is the meromorphic function with poles of the second and the first degree at singularities

\[ S(\phi) = \sum_j \left[ \frac{1}{2} \frac{h_j}{(z - p_j)^2} - \frac{C_j}{(z - p_j)} \right]. \tag{53} \]

Here, \( h_j = \alpha_j(2 - \alpha_j) \) are the dimensions, equation (2), and \( C_j \) are called accessory parameters, further discussed in section 6.5. We comment on the derivation of this result in section 6.1.

This property allows one to reduce the volume integral in the first equation of (50) to a contour integral encircling singularities

\[ df = \frac{c}{12} \sum_j \frac{1}{2\pi i} \left[ \frac{1}{2} h_j \oint_{\infty} \frac{\xi dz}{(z - p_j)^2} + C_j \oint_{\infty} \frac{\xi dz}{z - p_j} \right]. \tag{54} \]

Summing up, the connection \( df \) is expressed through its central charge and the data of uniformization theory. These are two independent problems. We compute the former in section 5.4, and the latter in section 6.

### 3.10. Scaling

The second formula of (50) can be used to obtain the overall scaling (24).

A uniform change of the conformal factor is equivalent to a change of the volume

\[ \int \delta f \delta \phi dV = -V \frac{\partial V}{\partial f}. \]

Then equation (50) yields

\[ -V \frac{\partial V}{\partial f} = \frac{1}{2\pi} \int \Theta dV. \]

The trace of the stress tensor consists of the regular part (the trace anomaly) proportional to the scalar curvature

\[ \Theta_{\text{reg}} = \frac{c}{48} R_0, \tag{55} \]

and the singular contributions supported only at the set of singular points \( z = p_j \). With the help of the conservation law \( \partial T + e^\phi \partial \Theta = 0 \) we express the trace of the stress tensor through its holomorphic component

\[ \frac{1}{\pi} \int \Theta dV = -\frac{1}{\pi} \oint \frac{\xi dz}{z - z'} T(z') dz' dz \]

Taking into account the asymptote of the metric \( \phi|_{z \to p_k} \sim -2\alpha_k \log |z - p_k| \) (see section 6.1) and (52), (51) and (53) we obtain

\[ \frac{1}{\pi} \int \Theta dV = \frac{1}{\pi} \int \Theta_{\text{reg}} dV + \sum_k L_k = \frac{c}{24} \alpha_0 + \sum_k L_k. \tag{56} \]

Equation (24) follows.

We comment, that the term with the accessory parameters in (54) does not contribute to the scaling. Rather, it describes the polarization of the stress tensor

\[ C_k = (\pi \gamma_k)^{-1} \oint_{p_k} (z - p_k) \Theta dz. \tag{57} \]

### 3.11. Angular momentum

Equation (50) could be interpreted in another manner. We can consider \( \xi(z) \) as a displacement of a fluid particle located at \( z \). A choice \( \xi'|_{z \to p_k} = (z - p_k) e^{i\delta \theta} \) or \( \xi(z)|_{z \to p_k} = i(z - p_k) \delta \theta \) represents a local rotation of a fluid particle about a conical point \( p_k \) by angle \( \delta \varphi = \delta \theta / \gamma_k \)
for a cone, or by the angle \( \delta \varphi = \delta \theta \) for a cusp. We see it by expressing the transformation in terms of the developing map (see (89) and (90) below) in the fundamental domain \( z'_{|z \rightarrow pk} \sim e^{i \varphi} (w(z) - w(p_k))^{1/n} \). The adiabatic phase obtained under the rotation is the angular momentum times the angle of the rotation \( L_k \delta \varphi \). Computing it with the help of (54), we obtain

\[
L_k \delta \varphi = \delta f = \frac{c}{24} \frac{h_k}{\gamma_k} \int_{p_k} \frac{\delta \theta}{z - p_k} \frac{dz}{2\pi i} = \frac{c}{24} \frac{h_k}{\gamma_k} \delta \theta.
\]

This yields the equation (14) \( L_k = \frac{c}{24} \frac{h_k}{\gamma_k} \).

The same transformation in case of the cusp reads \( z'_{|z \rightarrow pk} \sim e^{i \theta} e^{2\pi i \zeta} \), where \( \zeta \) is the coordinate in the upper half plane (see (71) below). In this case the angle of the rotation is just \( \varphi = \theta \). Repeating the calculations we obtain (15) \( L_{\text{cusp}} = \frac{c}{24} \).

Equation (56) gives an interpretation of the trace of the stress tensor \( \Theta \) as a density of angular momentum [1].

### 3.12. Chern number: integrated adiabatic curvature

In this section we compute the first Chern number on the moduli space\(^7\) of a surface with conical singularities, and obtain a sum rule (quoted above in equation (19)) connecting the dimensions and the exchange statistics for conical singularities.

The first Chern number is the total flux of the adiabatic curvature over a closed 2-cycle in the moduli space. Let us first discuss conical singularities. We choose the \( k \)th singularity and consider the 2-cycle swept out by the space of the complex parameter \( p_k \). In the case of conical singularities it is a complex hyperplane \( M_k \), which excludes the points \( \partial M_k = \{ p_j, j \neq k, \infty \} \) occurring at the boundary of the moduli space.

The first Chern number \( c_1(p_k) \) is equal to the adiabatic curvature integrated over \( M_k \). It picks only the geometric part of the adiabatic phase. However, since the curvature is a Kähler form, application of Green’s theorem relates the first Chern number to the topological part of the adiabatic phase via

\[
c_1(p_k) = \sum_{j \neq k} \Lambda_{kj} + \Delta_k. \tag{58}
\]

Here, we compute the Chern number by integrating the adiabatic curvature over \( M_k \), i.e. computing the total geometric phase,

\[
c_1(p_k) = -\frac{1}{\pi} \int_{M_k} \partial_{\bar{p}_k} \partial_{p_k} \Theta \, d\phi \, d\bar{\phi}_k = -\frac{1}{\pi} \int_{M_k} \partial_{\bar{p}_k} \left( \frac{1}{\pi} \int_{M_k} \Theta \partial_{p_k} \phi \, dV \right) \, d\phi \, d\bar{\phi}_k. \tag{59}
\]

To get the second equality we have utilized the variational formula for the free energy equation (50).

The contribution of the singular part of the trace of the stress does not enter (59), since singular points are excluded from the integral over \( p_k \). Then keeping only the regular part of the trace of the stress tensor \( \Theta^{\text{reg}} \), and exchanging the order of integrals, we are left to

\[\ldots\]

\(^7\)More accurately, the Chern class and Chern number are defined for the bundles on a nonsingular manifolds, where it is an integer. The moduli space is an orbifold with boundary points. Nevertheless, we still call this topological characteristic the Chern number. There will be no integer quantization of this number. Rather, the ‘Chern number’ of an orbifold is a rational number. A rational quantization places a constraint on the geometry which supports completely filled LLL and fractional QH states.
evaluate \( \int dV \Theta_{\text{reg}} \int_{M_{p_k}} \partial_{p_k} \phi \partial_{p_k} \phi \). The integral over \( M_{p_k} \) is dominated by the singularity at \( p_k \to z \), where the metric approaches \( \partial_{p_k} \partial_{\bar{p}_k} \phi = -\pi \alpha_k \delta(p_k - z) \). Other singularities of \( \partial_{p_k} \partial_{\bar{p}_k} \phi \) occur when \( p_k = p_j \), which are excluded from the domain \( M_{p_k} \). Hence

\[
c_1(p_k) = \frac{\alpha_k}{\pi} \int_{C/(p)} \Theta_{\text{reg}} dV = \frac{c_k}{48} R_0 V = \frac{c}{24} \alpha_k \alpha_k.
\]

(60)

Combined with (58), we reproduce (19) and (20) quoted in section 3.2.

We comment that in the case of cusps the \( p_k = \infty \) does not belong to the boundary of \( M_k \). In this case (58) reads

\[
c_1 = (n - 1) \Lambda_{\text{cusp}} = (n - 1) \frac{\alpha_k}{2 \pi} \text{[22]}
\]

4. Quantum Hall states on a singular surface

Here we give a brief account of states in the lowest Landau level (LLL) on curved surfaces. For more details we refer to [27], and for a mathematically oriented reader we recommend [18].

4.1. Lowest Landau levels on curved surfaces

Electrons reside on a surface threaded by a uniform magnetic field \( B \) normal to the surface \((eB > 0)\). In units \( e = \hbar = 1 \) the total flux through the surface is \( N \Phi = BV / (2\pi) \). We will define the magnetic potential \( Q \) as the solution of the Poisson equation

\[
2B = -\Delta Q.
\]

(61)

The magnetic field is the Laplacian of the magnetic potential

\[
B = -2e^{-\phi} \partial_{\bar{z}} \partial_{z} Q.
\]

The spin \( j \)-states in the LLL with \( N \) particles are tensors of rank \( j \) defined as zero modes of the anti-holomorphic momentum operators on the space of \( J \)-differentials

\[
\nabla_{i}^{(j)} \psi_{j}(z_{1}, \ldots, z_{N}) = 0, \quad i = 1, \ldots, N.
\]

(62)

The \( d \)-bar operator in complex coordinates and with respect to the quantum mechanical measure \( L^2 \), reads

\[
\nabla_{i}^{(j)} = e^{-\phi/2} (\bar{\partial}_{i} - iA_{i}) e^{\phi/2}.
\]

The spin of the state could be an integer or half integer. For QH states the spin has been introduced in [27] and is an important characteristic of states, often omitted in the literature.

A general solution of the set of equation (62) on a genus zero surface reads

\[
\Psi(z_{1}, \ldots, z_{N}) = Z^{-1/2} X(z_{1}, \ldots, z_{N}) \prod_{i=1}^{N} \exp \left( \frac{1}{2} Q(z_{i}, \bar{z}_{i}) - \frac{1}{2} j \phi(z_{i}, \bar{z}_{i}) \right).
\]

(63)

where \( X \) is a symmetric or antisymmetric holomorphic polynomial, \( Z \) is the normalization factor.
The normalization condition involves the volume integral over the Riemann surface $dV = e^{\phi} dz \, d\bar{z}$, and results in the normalization factor (31)

$$Z = \int |\mathcal{X}(z_1, \ldots, z_N)|^2 \prod_{i=1}^{N} m(z_i, \bar{z}_i) dz_i \, d\bar{z}_i,$$

(64)

where the measure

$$m = \exp (Q - (j - 1)\phi),$$

(65)

defines the inner product of holomorphic sections $\mathcal{X}$.

The polynomial $\mathcal{X}$ has further constraints. Convergence of the integral (64) limits the number of admissible particles. Let us denote by $h_N$ the degree of the polynomial. The polynomial grows as $\mathcal{X} \sim z_i^{h_N}$ as a given variable, $z_i$, tends to infinity. The growth must be compensated by the measure (65). The conformal factor and the magnetic potential behave as $\phi \sim -4 \log |z|$, $Q \sim -2N_0 \log |z|$, so the measure (65) falls as $z_i^{-2(N_0 - 2j)}$. Therefore, the state can be normalized if $h_N - N_0 + 2j < 0$. This is the Riemann–Roch–Hirzebruch condition which limits the number of holomorphic sections of the Riemann surface.

A physical assumption is that the degree $h_N$ grows with $N$ at most linearly $h_N = \beta N + h_0$, where $\beta$ and $h_0$ are integer valued parameters. This follows from the assumption that the interaction between particles is pairwise. Therefore, the largest admissible number of particles is the integer part of $\beta^{-1}(-N_0 + 2j + h_0)$. This is the scenario we consider. The offset $h_0$ is called the shift [49].

The maximal admissible state uniformly occupies the surface with a density $N/V \approx \nu(N_0/V) = \nu(eB/2\pi \hbar)$. The parameter $\nu \equiv \beta^{-1} = N_0/N$ is the filling fraction. Such a state has no boundaries, and therefore is invariant under the Möbius transformation of the Riemann sphere.

The Möbius invariance imposes strong restrictions on admissible polynomials $\mathcal{X}$. One series of states, the Laughlin states, is singled out by the condition $h_N = \beta(N - 1)$. In this case the number of admissible particles is

$$\beta(N - 1) = N_0 - 2j,$$

(66)

provided that $N_0 - 2j$ is a multiple of $\beta$. This condition is fulfilled by the polynomial

$$\mathcal{X}(z_1, \ldots, z_N) = \prod_{i>j} (z_i - z_j)^\beta.$$

(67)

At $\beta = 1$, the completely filled LLL is the Slater determinant of single-particle states.

Under Möbius transformation $z_i \rightarrow g(z_i)$, $p_k \rightarrow g(p_k)$ the polynomial (67) and the measure (65) transforms as

$$\mathcal{X} \rightarrow \prod_{l} |g'(z_l)|^{\frac{1}{2} \beta(N-1)} \mathcal{X}, \quad \prod_{l} m(\bar{z}_l) d\bar{z}_l \rightarrow \prod_{l} [g'(z_l)]^{-N_0 + 2j} \prod_{l} m(\bar{z}_l) d\bar{z}_l.$$

They compensate each other under the condition (66).

In order to further specify the form of the wave function, we need more knowledge of the magnetic potential $Q$ and the metric $\phi$ on singular surfaces. We discuss it in the next section.

4.2. Metric of a sphere with singularities

In section 3.1, we discussed the moduli space of singular metrics. Here we review the construction of such metrics using the tools from uniformization theory.
We refer to a Riemann sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) with a set of marked points \( \mathbf{p} = \{ p_1, \ldots, p_n \} \) with a concentration of curvature (conical singularities) as a marked sphere \( \Sigma \), and a punctured sphere \( \Sigma = \hat{\mathbb{C}} / \mathbf{p} \), where the points \( \mathbf{p} \) are removed. The singularities of a punctured sphere are cusps.

In complex coordinates, where the metric is diagonal \( ds^2 = e^{\phi}|dz|^2 \), the one-component Ricci tensor \( R_{z\bar{z}} = \frac{1}{4} e^{\phi} R = -\partial_z \partial_{\bar{z}} \phi \) for the marked surface with singularities of order \( \alpha_1, \ldots, \alpha_n \) reads

\[
4R_{z\bar{z}} = e^{\phi} R_0 + 4\pi \sum_k \alpha_k \delta^{(2)}(z - p_k). \tag{68}
\]

The curvature of the punctured sphere is given by the same formula with \( \alpha_k = 1 \). Away from the singularities, the scalar curvature \( R_0 \) is a smooth function.

Surfaces with given singularities are conformally equivalent to that of constant curvature. As we mentioned above, adiabatic phases depend only on the conformal classes, so it is sufficient to consider only surfaces with a constant curvature: positive, negative and zero. We normalize the curvature to be \( R_0 = 0, \pm 2 \).

Metrics of surfaces with constant curvature are solutions of the Liouville equation with prescribed behavior at singularities

\[
4\partial_z \partial_{\bar{z}} \phi = -R_0 e^{\phi}, \quad \phi|_{z=p_j} \sim -\alpha_k \log |z - p_j|^2. \tag{69}
\]

We assume that infinity is a regular point without any singularities.

The solution of the Liouville equation can be formally written in terms of a developing map \( w(z) : \Sigma \rightarrow S \) from \( \Sigma \) to the fundamental domain

\[
e^{\phi} = \frac{4|w'(z)|^2}{(1 + \frac{1}{2} R_0 |w(z)|^2)^2}, \tag{70}
\]

with a prescribed behavior at singular points following from (69). For \( R_0 = +2 \), the fundamental domain is a finite region of the Riemann sphere. For \( R_0 = 0 \), it will be a polygon on the complex plane. For \( R_0 = -2 \), the range of the developing map is a finite volume region on the Poincaré disk.

Hyperbolic surfaces with \( R_0 = -2 \) are also represented by the Poincaré metric on the upper half plane. In coordinates \( \zeta = -i \frac{w+1}{w-1} \), the metric (70) reads

\[
ds^2 = \frac{|d\zeta|^2}{(\text{Im } \zeta)^2}, \quad \text{Im } \zeta > 0. \tag{71}
\]

For parabolic (cusp) singularities, it turns out that the metric (71) is more convenient.

If \( \alpha < 1 \), the singularity is conical. Locally it is equivalent to an embedded cone with the apex angle \( 2 \arcsin \gamma \), where \( 2\pi \gamma = 2\pi(1 - \alpha) \) is called a cone angle. Non-convex surfaces can contain cone points with order \( \alpha < 0 \). The branch point of a multi-sheeted Riemann surface can be described by a local metric with negative integer \( \alpha \).

An especially interesting case occurs when \( \gamma \) or \( 1/\gamma \) is an integer. In this case, the puncture is an orbifold point, a fixed point of the action of a discrete group of automorphisms [50]. Though interesting and worth noting, this fact does not ultimately make a difference in our final results.

The Gauss–Bonnet formula for a compact surface of a constant curvature implies that the integer valued Euler characteristic receives local contributions \( \alpha_k \) from each singularity.
The Gauss–Bonnet formula limits the total order of singular points $\sum \alpha_k$ if the volume is finite. If the curvature $R_0$ is positive, then $\sum \alpha_k < \chi$. If all conical points are sharp $\alpha > 0$ they exist only on a sphere, where $\chi = 2$. In this case, the degrees of the singularities are restricted by the condition $0 < 2 - \sum \alpha_k < 2 \min(1, \min \gamma)$, where $\min \gamma$ is the smallest cone angle $\gamma_k$ [51, 52]. For example, unless $\alpha$ is a negative integer, the only spherical surface with two isolated singularities is the spindle, where conical points are necessarily the same degree and antipodal [51] (see appendix A).

In the case of the surface with negative curvature $R_0 < 0$, the bound is reversed $\sum \alpha_k > \chi$ (we refer to [39, 53] for a review of hyperbolic geometry). In this case the number of cones or cusps is limited only from below. On a punctured sphere, where all singularities are cusps $R_0 V = 4\pi(2 - n)$. This condition excludes a pseudosphere, a hyperbolic surface of revolution with two singularities and an edge (see appendix A).

A flat ($R_0 = 0$) compact surface is a polyhedron, and can be constructed by gluing together flat triangles. The vertices of the polyhedron are conical singularities with a conical angle equal to the sum of the angles of the triangles adjacent to the vertex. The total degrees of singularities is equal to Euler characteristic $\sum \alpha_k = \chi$.

Polyhedra of genus zero are the only surfaces where the developing map is explicit beyond three singularities. It takes the form of the Schwarz–Christoffel conformal map [43, 50], expressed conveniently in terms of its first derivative $w'(z) = \partial_z w$ as

$$w'(z) = e^{\frac{a_0}{2}} \prod_{j=1}^{n} (z - p_j)^{-\alpha_j}, \quad \sum \alpha_j = 2. \quad (73)$$

The conformal factor $e^{a_0}$ in (73) fixes the volume. At a fixed volume the conformal factor depends on the moduli $p$.

4.3. QH states on singular surfaces of constant curvature

The Laughlin state (65) for genus-zero surfaces with a constant curvature are explicit in terms of the developing map $w(z)$. In a uniform magnetic field, the magnetic potential (61) can be written explicitly as

$$Q = -\frac{B}{2} |w(z)|^2, \quad R_0 = 0, \quad \text{polyhedra}$$

$$Q = -\text{sign}(R_0) k \log \left(1 + \frac{|R_0|}{2} |w(z)|^2 \right), \quad R_0 \neq 0,$$

where we denoted for $R_0 \neq 0$

$$k = \frac{4B}{|R_0|} = \frac{4\Phi}{|\alpha_0|}.$$

Using these formulas we write the most explicit form of the non-normalized state (30) as a function on the fundamental domain

$$\Psi = Z^{-1/2} \prod_{i<j} (z(w_i) - z(w_j))^\beta \times \begin{cases} e^{-\frac{1}{2} |w_i|^2} \prod_{k=1}^{n} (z(w_i) - p_k)^{(j-1)\alpha_k}, & R_0 = 0, \quad \text{polyhedra} \\ (1 + |w_j|^2)^{-\frac{j+1}{2}} \zeta'(w_j)^{-j-1}, & R_0 = 2, \quad \text{elliptic} \\ (1 - |w_j|^2)^{-\frac{j+1}{2}} \zeta'(w_j)^{-j-1}, & R_0 = -2, \quad \text{hyperbolic} \end{cases}$$
Despite a lack of explicit formulas for the metric beyond the thrice punctured sphere and flat polyhedra, the asymptotes near singularities turn out to be sufficient to determine the adiabatic phases.

5. Quantum Hall states and conformal field theory

In this section, we obtain the relations discussed above for QH states. In particular, we prove the connection between geometric transport and critical systems highlighted in section 3.8.

5.1. Vertex construction

The vertex construction represents the QH state as an expectation value of a string of vertex operators of a Gaussian free field. Initially proposed in [54], the method was significantly extended by Ferrari and Klevtsov [26], see also [19].

The electrons in the Laughlin states are represented by the vertex operator $e^{iX}$ of the Gaussian field $X$ of spin $j$ with the charge equal to the filling fraction. The field is compactified as $X \sim X + 2\pi$. Its action reads

$$S[X] = \frac{\nu}{\pi} \int \left( \frac{1}{2} |\partial_z X|^2 - i j (\partial_z \phi \partial_{\bar{z}} + \partial_{\bar{z}} \phi \partial_z) X \right) dz d\bar{z} + \frac{i}{2\pi} \int \left( \nu B + \frac{1}{4} R \right) X dV. \quad (74)$$

The first term in (74) implies the operator product expansion $e^{iX(z_1)} e^{iX(z_2)} \sim |z_1 - z_2|^{2\beta}$ as $z_1 \to z_2$. The second term reflects the spin of the field. The last terms describe the coupling to the magnetic field and curvature. Equivalently, the action may be written

$$S[X] = \frac{\nu}{2\pi} \int |\partial_z X|^2 dz d\bar{z} + \frac{i}{2\pi} \int \left( \nu B + \frac{1}{2} \mu_H R \right) X dV, \quad (75)$$

where $\mu_H$, the geometric transport coefficient obtained by the combination of the second and the fourth terms

$$\mu_H = \frac{1}{2} - j \nu. \quad (76)$$

We will now show that the correlator of a string of vertex operators $e^{iX}$ localized at positions of particles

$$\langle e^{i \sum_{i=1}^N X(z_i)} \rangle = Z_G^{-1} \int e^{-S[X]} e^{i \sum_{i=1}^N X(z_i)} DX,$$

represents the square of the amplitude of the non-normalized QH state (63) and (67)

$$\prod_{i=1}^N \langle e^{iX(z_i)} \rangle dV = |X(z_1, \ldots, z_N)|^2 \prod_i m(z_i, \bar{z}_i) dz_i d\bar{z}_i. \quad (77)$$

Here

$$Z_G = \int e^{-S[X]} DX \quad (78)$$

is the partition function of the Gaussian field coupled to a magnetic field.

First we separate the constant part (the zero mode) $X_0$ of the field $X(z, \bar{z}) = X_0 + \bar{X}(z, \bar{z})$, such that $\int \bar{X} = 0$. The integration over the zero mode gives the condition between the number of particles and the magnetic flux.
\[ N = \nu N_\Phi + \mu \nu \chi . \]
equivalent to (66). Here \( \chi = \frac{1}{4\pi} \int R \) is the Euler characteristic, and \( \chi = 2 \) for the sphere.

The integration over the remaining modes \( \tilde{X} \) gives
\[
\langle e^{\int \sum \lambda_i (z_i \tilde{z}_i)} \rangle = e^{-\frac{1}{2} \sum_{i=1}^{N} \left[ \sum_{j=1}^{N} G(z_i, z_j) + G^R(z_i) \right]} e^{\sum_{i=1}^{N} \Theta(z_i) + \frac{i}{2\pi} \phi(z_i),}
\]
Here \( G(z, z') = -\frac{\nu}{2\pi} (X(z)X(z'))_c \) is the Green function of the Laplace operator \(-\Delta G = \delta(z - z') - \frac{1}{\nu} \), and \( G^R(z) = -\frac{\nu}{2\pi} (X^2)_c \) is the regularization of the Green function at merging points defined by using the geodesic distance between the points \( d(z, z') \) as the limit \( G^R = \lim_{\nu \rightarrow \infty} \left[ G(z, z') + \frac{1}{2\nu} \log d(z, z') \right] \). Up to additive constant terms \( G^R = \frac{\delta}{2\pi} + \text{const.} \) For more details, see [26]. Comparing with (65) and (67) we obtain the vertex representation of the Laughlin \( j \)-spin state (77).

Our goal is to compute the normalization factor of the state (31). From (77) it follows
\[
Z = \int \prod_{i=1}^{N} e^{\int \sum \lambda_i (z_i \tilde{z}_i)} dV_i = Z_0^{-1} e^{\mathcal{F}},
\]
where
\[
e^{\mathcal{F}} = \int \left[ \int e^{\lambda \sum \lambda_i (z_i \tilde{z}_i)} dV \right]^N e^{-S[X]} DX.
\]

5.2. Quillen metric

So far the vertex construction is merely rewriting the states in terms of the Gaussian field coupled to a magnetic field. As such it does not bring any additional information. The representation became helpful when the authors of [26] observed that in the large \( N \) limit the functional \( \mathcal{F} \) in (80) is a local functional of the curvature, and therefore, \( d\mathcal{F} \) is an exact form. It does not contribute to the adiabatic phase, as \( \oint d\mathcal{F} = 0 \). A physical reason for this is that \( e^{\mathcal{F}}(\xi) \) does not contain zero modes, which are solely responsible for adiabatic phases.

In the case of the integer QHE (\( \beta = 1 \)) \( e^{\mathcal{F}} \) was identified with the spectral determinant of the Laplace operator in magnetic field \( \Delta_R = (\nabla^{(j)})^2 \nabla^{(j)} \),
\[
e^{\mathcal{F}} = \text{Det} \left(-\Delta_R^j\right), \quad \beta = 1,
\]
where the prime indicates that the LLL states are excluded. This follows from the fermionic version of the integral (80). At \( \beta = 1 \) the integral (80) over Bose fields is equivalent to the integral
\[
e^{\mathcal{F}} = \int e^{-\int \psi^{\dagger} (-\Delta_R^j) \psi} D\psi D\psi^{\dagger},
\]
over the Fermi field which does not contain the modes in the LLL. If \( \psi^{(n)}_k \) are the wave functions of the \( n \)th Landau level, and \( E^{(n)}_k \) are their energies, then the integration in (81) goes over \( \psi = \sum_{n>0} \sum_{k} c_k^{(n)} \psi^{(n)}_k \), where \( c_k^{(n)} \) are Grassmann variables and the modes in the LLL \( (n = 0) \) are excluded [18]:
\[
e^{\mathcal{F}} = \prod_{n>0} \prod_{k \leq N_n} E^{(n)}_k, \quad \text{where } N_n \text{ is the number of states on the } n \text{th level.}
\]
We comment that in general only the LLL is degenerate on a surface with a finite volume.

It is obvious that the spectral determinant of the Laplace operator where LLL states are excluded is a regular function. For smooth surfaces the expansion of \( \mathcal{F} \) in gradients of curvature had been computed in [27, 30, 48].
We conclude that \( \log Z \) and \( \log Z_H = \log Z - F \) yield the same adiabatic phase, and that \( \log Z_H \) is equal and opposite in sign to the free energy of the Gaussian field coupled to the magnetic field

\[
Z_H = e^{-F} = Z_G^{-1}. \tag{82}
\]

In the case of the integer QHE the ratio

\[
Z_H = \frac{Z}{\operatorname{Det}'(-\Delta_B)}
\]

is called the Quillen metric (see [30] and references therein). The Quillen metric singles out the anomalous part of the generating functional solely responsible for the quantized transport and the generating functional \( F \) of mesoscopic fluctuations. The advantage of the Quillen metric is that it is exactly computable, as it is equal to the inverse of the partition function of the free Gaussian field (82). This fact is referred to as a local index theorem (see e.g. [55]). The ratio (82) extends the notion of the Quillen metric to the case of fractional QH states which essentially differ from Slater determinants. But, like in the integer case, the ratio (82) is also exactly computable. The formula (82) can be regarded as the extension of the Quillen metric and local index theorem to the fractional QHE. In this form it has been introduced in [19].

The exact form \( d_p F \) does not contribute to the adiabatic phase. However, it contributes to the conductance

\[
\sigma_{\bar{p}p} = \sigma_{\bar{p}p}^H + \partial_{\bar{p}} \partial_p \bar{F}, \quad \text{where} \quad \sigma_{\bar{p}p}^H = \partial_{\bar{p}} \partial_p \log Z_H.
\]

Following [37] we interpret \( \partial_{\bar{p}} \partial_p \bar{F} \) as mesoscopic fluctuations of the conductance, subject to details of the system, versus \( \sigma_{\bar{p}p}^H \), a universal part of the conductance determined solely by geometric characteristics.

### 5.3. Laughlin states and Gaussian free field

Now let us turn to the partition function of the Gaussian field (78). It is equal to the inverse of the Quillen metric, the only object we need to examine.

We compute it by shifting the field \( X = Q + \varphi \). Then the partition function of the Gaussian field is (78) reduces to

\[
- \log Z_G = \frac{1}{2\pi} \int \left( \nu |\partial_z Q|^2 + \mu_H \partial_z Q \partial_{\bar{z}} \varphi + \partial_{\bar{z}} Q \partial_z \varphi \right) d\bar{z} dz + f, \tag{83}
\]

where

\[
e^{-f} = \int e^{-\frac{1}{8\pi} \int (\nabla \varphi)^2 dV - \frac{i}{4\pi} \int \bar{\varphi} \nabla \varphi \cdot D \varphi}. \tag{84}
\]

The first factor in (83) is the extensive part growing with the number of magnetic flux quanta. It consists of two different contributions. The first is of the order of \( N_B^2 \). It does not depend on the metric, hence does not contribute to the adiabatic phase. The second term yields the extensive contribution to the angular momentum \( \mu H N_B \) but does not depend on \( p_k \). In a uniform magnetic field it is simplified

\[
- \log Z_G = \frac{B}{4\pi} \int (\nu Q + 2\mu_H \phi) dV + f. \tag{85}
\]

The second factor, is the partition function of the Gaussian field with the background charge \( \mu_H \). It encapsulates the geometric transport. This part does not depend on the magnetic field, and can be computed independently from the QHE.
If the surface is smooth, \( f \) can be expressed as
\[
f = \frac{\mu_H^2}{2\pi \nu} \int |\partial_z \phi|^2 \, dz \, d\bar{z} + \frac{1}{2} \log \text{Det}(-\Delta).
\]
(86)

But on a singular surface, the first term in (86) diverges as a logarithm of a distance between singular points when they merge, and requires a regularization. A proper regularization is a removal of a small ball of the area \( \varepsilon \) around each singularity, and to ensure that the volume of this ball is the same at all singularities. This can be done, of course, but is not necessary for a computation of the adiabatic phase. To this end, we need only the variation of the free energy, or stress tensor. We compute it in the next section.

### 5.4. ‘Central charge’ for Laughlin states

As we have seen, the problem is reduced to the computation of the Gaussian field on a Riemann surface. The central charge of the Gaussian field \( c \) could be read from (84). It is determined by the stiffness \( \nu \), and the background charge. We obtain
\[
c = 1 - 12\mu_H^2 \nu^{-1}.
\]
(87)

where \( \mu_H \) is given in (76). This result is standard. The easiest way to see it is to assume that the surface is smooth and compute the stress tensor from (86). We write only the holomorphic component of the stress tensor. The contribution of the term \( \frac{\mu_H^2}{2\pi \nu} \int |\nabla \phi|^2 \, dV \) to the stress is \( \mathbf{T}^{(1)} = -\frac{\mu_H^2}{2\pi} \mathcal{S}[\phi] \). The stress tensor of the spectral determinant is controlled by the gravitational anomaly \( \mathbf{T}^{(2)} = \frac{1}{12\pi} \mathcal{S}[\phi] \). Together \( \mathbf{T} = \mathbf{T}^{(1)} + \mathbf{T}^{(2)} = \frac{c}{12} \mathcal{S}[\phi] \).

The remaining task is to compute the entries of (53) and to evaluate (54). These problems belong to the uniformization theory.

### 5.5. Möbius transformation and the emergent conformal symmetry

The vertex construction we have outlined in this section suggests that the generating functional consists of two parts: (i) the partition function of a free field theory coupled to a magnetic field, and (ii) a regular function of the moduli \( e^F \) (82). The partition function of the Gaussian field is further decomposed into the extensive part proportional to the magnetic field and the intensive part \( e^{-f} \) (84), the free energy of the critical field theory. Apart from \( f \), all contributions are regular functions of the moduli, and consequently do not contribute to the geometric transport. Hence, we are left with a critical system, or CFT, on a singular surface. It fully determines the geometric transport.

We observe the emergence of global conformal symmetry, the symmetry with respect to the Möbius transformations of coordinates of singularities \( p \). It follows from the representation of the partition function as a string of primary fields evaluated at the singularity coordinates, in the spirit of [3, 8, 56]. The intensive part of the partition function \( f \) posses a local conformal symmetry, but the extensive term \( e^{-f} \) (84) and the term \( e^F \) do not. Nevertheless, they are invariant under the Möbius transformations. We conclude that the generating functional is a quasi-primary in the singularity coordinates, as it was stated in section 3.3.

### 6. Conformal field theory on a singular surface

Now, we return to the formula (54) expressing the the variation of the free energy in terms of the quasi-conformal map and the accessory parameters. To evaluate it we need detailed
information about the sub-leading terms of the metric in the expansion about a singular point
\( z \to p_j \).

### 6.1. Sub-leading asymptotes of the metric

The expansion of the developing map around a conical point reads

\[
\begin{align*}
  w(z)\big|_{z \to p_j} &\sim \left( \frac{z - p_j}{l_j} \right)^\gamma \left( 1 + \frac{\gamma_j}{l_j} C_j(z - p_j) + \ldots \right), \\
  w'(z)\big|_{z \to p_j} &\sim \frac{\gamma_j}{l_j} \left( \frac{z - p_j}{l_j} \right)^{-\alpha_j} \left( 1 + \frac{C_j}{\alpha_j} (z - p_j) + \ldots \right).
\end{align*}
\]

(88)

(89)

The scaling coefficient \( l_j \), and the coefficients \( C_j \), are each functions of all singularity coordinates. The coefficients \( C_j \) are called *accessory parameters*.

To deal with cusps we use a Poincaré mapping of a punctured sphere to the upper half plane
\((71)\), (see, e.g. \([40]\))

\[
\begin{align*}
  2\pi \zeta(z)\big|_{z \to p_j} &\sim \log \left( \frac{z - p_j}{l_j} \right) + C_j(z - p_j) + \ldots,
  \\
  2\pi \zeta'(z)\big|_{z \to p_j} &\sim \frac{1}{z - p_j} + C_j + \ldots
\end{align*}
\]

(90)

The asymptotes (88) and (89) ensure that near a conical singularity, the metric appears locally
as

\[
e^\phi |dx|^2 = \frac{4\gamma^2 |x_j| - 2\alpha_i}{\left( 1 + \frac{2\gamma}{\alpha_j} |x_j|^2 \right)^2} |dx|^2, \quad x_i = \frac{z - p_i}{l_i}.
\]

(91)

and that the limit \( \gamma \to 0 \) reproduces the cusp metric for a hyperbolic surface \( R_0 = -2 \) as the
punctured disk

\[
e^\phi |dx|^2 = \frac{|dx|^2}{|x_j|^2 \log^2 |x_j|}.
\]

(92)

The sub-leading terms of (70), (90) and (89) determine the sub-leading asymptotes of the
conformal factor

\[
\begin{align*}
  \phi\big|_{z \to p_j} &\sim -2\alpha_j \log |z - p_j| - 2\gamma_j \log l_j + 2\text{Re} \left( \frac{C_j}{\alpha_j} (z - p_j) \right) + \ldots, \\
  \partial_z \phi\big|_{z \to p_j} &\sim -\frac{\alpha_j}{z - p_j} + \frac{C_j}{\alpha_j} + \ldots
\end{align*}
\]

(93)

The metric for cusps follows simply by the \( \gamma \to 0 \) limit of the hyperbolic cone metric

\[
\begin{align*}
  \phi\big|_{z \to p_j} &\sim -2 \log |z - p_j| - \log \log \frac{|z - p_j|}{l_j} + 2\text{Re} \left( C_j(z - p_j) \right) + \ldots, \\
  \partial_z \phi\big|_{z \to p_j} &\sim -\frac{1}{z - p_j} \left( 1 + \frac{1}{\log |z - p_j|} \right) + C_j + \ldots
\end{align*}
\]

(94)

This is to be contrasted with the degeneration of \( w(z) \) as \( \gamma \to 0 \), which does not produce \( \zeta(z) \).
In order to compute the free energy we will need the derivative of the metric with respect to the position of the singularity at a fixed coordinate. For conical points it follows from (93)\[
(\nabla_{p_k} + \delta_k \partial_z) e^\phi \bigg|_{z \to p_j} = 0, \quad \nabla_{p_k} = \partial_{p_k} + 2 \gamma_j \partial_{p_k} \log l_j.
\]
The formula for the cusp metric is different. In this case the effect of the difference between translation in $p_k$ and in $z$ vanishes as $z \to p_k$\[
(\partial_{p_k} + \delta_k \partial_z) \phi \bigg|_{z \to p_j} = \frac{2 \partial_{p_k} \log l_j}{\log \left| \frac{z - p_j}{l_j} \right|} \to 0.
\]
In order to clarify the meaning of the scales $l_j$ consider small balls of the area $\varepsilon_j^2$ surrounding the conical points as a short distance cut-off. Then the cut-off in the $z$-plane (the punctured sphere) for each ball $|z - p_j| < \delta_j$ is determined by the area of each ball $\varepsilon_j^2 \propto \int_{|z - p_j| < \delta_j} e^\phi \, dz d\bar{z} \approx 4 \pi \gamma_j (\delta_j/l_j)^{2\gamma_j}$. Thus the scale factor $l_j$ relates the cut-off in $z$-plane (uniformization space) to the physical cut-off $\varepsilon_j$\[
\varepsilon_j \sim (\delta_j/l_j)^{\gamma_j}.
\]
In the case of cusps the relation is $\varepsilon_j^2 \sim 1 / \log(l_j/\delta_j)$, as it follows from the formula for a cone as $\gamma_j \to 0$.

The physically meaningful cutoff requires $\varepsilon_j$ to be same for all cones, equal to some short scale cut-off $\varepsilon$ to every singularity, whereas $\delta_j$, a cut-off in the abstracted complex plane may vary. Hence,

$$l_j \sim \delta_j.$$ 

### 6.2. Auxiliary parameters

With the help of formulas of the section 6.1 we can solve the equation (48) for the displacements $\xi$. Using (93) we find

$$\xi |_{z \to p_j} = -d p_j - (z - p_j) d p_k \log l_j^2 + \ldots$$

Now we substitute this into (54), and use (95) and (96). We obtain the expression for the adiabatic connection in terms of sub-leading metric asymptotes

$$\partial_{p_k} f = \frac{c}{12} (C_k + D_k).$$

where we denoted

$$D_k = \frac{1}{2} \partial_{p_k} \log H, \quad H = \prod_j l_j^{2h_j}.$$ 

The parameters $D_k$ play equally important role as the accessory parameters. For lack of a name in the literature, we will call them auxiliary parameters.

When all singularities are cusps, the function $H$ is identical to that defined in [41, 57] as providing a Kähler potential for the local index theorem. The factor $H$ can be better understood if we express it with the help of cut-offs of singularities (97)\[
\log \sqrt{H} = \sum_j \frac{h_j}{\gamma_j} (\gamma_j \log \delta_j - \log \varepsilon_j).
\]
As we already said, the regularization of singularities in critical phenomena requires that the volume of the cut-off of different singularities is the same $\varepsilon_j = \text{const}$ for all singularities. In this case cut-offs $\delta_j$ of different singularities in
the uniformization plane are different. We comment that, in contrast, in 2D quantum gravity the cut-off is uniform in the uniformization plane $\delta_j = \text{const}$.

### 6.3. Liouville functional

An important consequence of this formula is that the accessory parameters are generated by the functional $Z_L$

$$ C_k = \partial_{p_k} \log Z_L. \tag{100} $$

defined via

$$ f = \frac{c}{12} \log(\sqrt{H} Z_L). \tag{101} $$

In [40, 53], where the relation (100) has been proven, $-2\pi \log Z_L$ is referred to as the Liouville action or functional.

### 6.4. Metrics on the moduli space

Each term of (101) which enters the adiabatic curvature

$$ dA = -\frac{c}{12} \sum_{k,l} \left[ \partial_{p_k} \partial_{\bar{p}_l} \log Z_L + \partial_{\bar{p}_k} \partial_{p_l} \log \sqrt{H} \right] dp_k \wedge d\bar{p}_l \tag{102} $$

is a geometric invariant regarded as a Kähler metric on moduli space [40, 53]. The first term in brackets in (102) is the Weil–Petersson metric. The second is called the Takhtajan-Zograf metric. Correspondingly, $\log Z_L$ and $\log H$ are the respective Kähler potentials for these metrics on moduli space. The product $Z_L \sqrt{H}$ is related to the Quillen metric (82)

$$ Z_{\mathcal{H}} = e^{\frac{c}{2\pi} \int (\nu Q + 2\mu \phi) (Z_L \sqrt{H})}. $$

### 6.5. Accessory parameters

The Schwarzian (52) of the metric which enters (54) is the Schwarz derivative of the developing map

$$ S[w] = -\frac{1}{2} (\partial_z \log w')^2 + \partial_z^2 \log w'. \tag{103} $$

We already know that it is a meromorphic function with poles of the second degree at singularities (regardless of the sign of $R_0$). The asymptotes of the developing map (88) and (90), or the metric (93) and (94) determine the residues

$$ S[w, z] = \sum_j \left[ \frac{1}{2} \frac{h_j}{(z-p_j)^2} + \frac{C_j}{(z-p_j)} \right], \tag{104} $$

where the accessory parameters $C_j$ encode the isometry group of the metric, or equivalently the monodromy group of the developing map.

It follows from (104) that under the Möbius transformation $z \to g(z) = \frac{w + p}{w + q}$ the Schwarzian derivative transforms as $S[w, z] \to (g'(z))^2 S[w(g(z)), g(z)]$. Treated as a function of $z$, $p$ and $C_k$, the Schwarzian (104) obeys the property
\[ S[z; p; C(p)] = (g'(z))^2 S[g(z); g(p); C(g(p))]. \]

A direct consequence of this is that the accessory parameters transform under Möbius transformation as
\[
C_k(g(p)) = \frac{1}{g'(p_k)} \left( C_k(p) - \frac{1}{2} h_k \partial_{p_k} \log g'(p_k) \right). \tag{105}
\]
This implies that the 1-form \( C = \sum_k C_k dp_k \) transforms as
\[
C \to C - \frac{1}{2} \sum_k h_k dp_k \log g'(p_k) \tag{106}
\]
in a manner similar to the adiabatic connection (40).

Consequences of the Möbius symmetry are described in section 3.5. We relist them here.

(i) The Liouville functional \( Z_L(p) \) transforms as a quasi-primary field
\[
Z_L(g(p)) = \prod_k |g'(p_k)|^{-h_k} Z_L(p), \tag{107}
\]
(ii) the sum rules
\[
\sum_k C_k = 0, \quad \sum_k (h_k + 2 C_k p_k) = 0, \quad \sum_k (h_k p_k + C_k p_k^2) = 0; \tag{108}
\]
(iii) the limiting behavior
\[
C_k \bigg|_{d(p_k, p_j) \to \infty} \sim -\frac{\delta_{kj}}{p_k} h_k, \quad C_k \bigg|_{p_k \to p_j} \sim -\frac{1}{2} \frac{\lambda_{kj}}{p_k - p_j}, \quad \lambda_{kj} = h_k + h_j - h_{kj}, \tag{109}
\]
where \( h_{kj} \) is the dimension of the singularity obtained as a result of merging two singularities
\[
Z_L(p_1, p_2, p_3, \ldots p_n) \bigg|_{p_1 \to p_2} \sim |p_1 - p_2|^{-\lambda_{12}} Z_L(p_1, p_3, \ldots p_n). \tag{110}
\]
Merging two singularities determines \( \lambda_{kj} \). In the case of merging cones of the order \( \alpha_k \) and \( \alpha_j \), we obtain a cone of order \( \alpha_j + \alpha_k \) assuming that \( \alpha_j + \alpha_k < 1 \). Then \( h_{kj} = (\alpha_k + \alpha_j)(2 - \alpha_k - \alpha_j) \), and \( \lambda_{kj} = 2 \alpha_k \alpha_j \).

The fusion rule of cusps is different. Whereas merging two cones \( \alpha_1 \) and \( \alpha_2 \) will produce a new cone of the order \( \alpha_1 + \alpha_2 \), merging two cusps merely yields another cusp. Therefore, in (109) we set \( h_k = 1 \) and, also, \( h_{kj} = 1 \). Hence, the asymptote of the accessory parameters at merging cusps is half of what follows from the formula (111) for cones by setting \( \alpha = 1 \).

Summing up
\[
\text{cones : } \quad C_k \sim \begin{cases} -\frac{\alpha_k \alpha_j}{p_k - p_j}, & \text{if } p_k \to p_j \\ -\frac{h_k}{p_k}, & \text{if } d(p_k, p_j) \to \infty \end{cases}, \quad \text{cusps : } \quad C_k \bigg|_{p_k \to p_j} \sim -\frac{1}{2(p_k - p_j)}. \tag{111}
\]
A more refined asymptote for cusps \([38, 39]\) is
\[
C_k \bigg|_{p_k \to p_j} \sim -\frac{1}{2} \frac{1}{p_k - p_j} \left( 1 - \frac{\pi^2}{(\log |p_k - p_j|)^2} + \ldots \right). \tag{112}
\]
We comment that we were able to obtain the limiting behavior of accessory parameters solely based on Möbius invariance and fusion properties of singularities. These arguments are applicable to a surface with positive, zero or negative curvature. For cones the formulas could be checked against the explicit formula for polyhedral surfaces of appendix B.

Another comment is that the relation (98) suggests that the auxiliary parameters $D_k$ transform under Möbius similar to the accessory parameters (105), and that not only $Z_L$, but also the functional $H$ is a quasi-primary.

The limiting behavior of accessory parameters (111) is rather obvious. Nevertheless it is not easy to find them in the literature (see however [38, 39, 58]). In section 6.6 we give an alternative derivation of equation (111).

### 6.6. Limiting behavior of accessory and auxiliary parameters

The accessory and auxiliary parameters $C_k$ and $D_k$ are singular at the boundary of the moduli space. This means that as singularities merge, we expect simple poles in $p_k$ to appear. Their asymptotes as singularities merge determine the critical exponents and consequently the topological part of the adiabatic phase. Here we present heuristic arguments for how to obtain these asymptotes.

We start from cones and assume that the sum of their orders is not a positive integer.

Consider merging singularities $(p_k, \alpha_k)$ and $(p_j, \alpha_j)$. When the points are close, the asymptotes of the metrics given by (93) $\phi|_{z\rightarrow p_j}$ and $\phi|_{z\rightarrow p_k}$ are valid in a common domain. Then the expansions of both metrics about the middle point $z = \frac{1}{2}(p_j + p_k) + x$ must give the same result. Hence,

$$
\left(\phi|_{z\rightarrow p_j}\right)|_{z\rightarrow p_j} \sim 2\alpha_k \log|p_j - p_k| - 2\gamma_k \log l_k + \frac{C_k}{2\alpha_k + \frac{\alpha_j}{p_k - p_j}}.
$$

must stay the same under a permutation of $j$ and $k$. This condition yields the asymptotes

$$l_k^\alpha \sim |p_j - p_k|^{\alpha_j},$$

cones: $p_k \rightarrow p_j$

$$C_k \sim -\frac{\alpha_k \alpha_j}{p_k - p_j},$$

$$D_k \sim \left(\alpha_j \frac{h_k}{\gamma_k} + \alpha_k \frac{h_j}{\gamma_j}\right) \frac{1}{2(p_k - p_j)}.$$

The limiting behavior of the accessory parameters matches that obtained from the sum rules in section 6.5 (equation (111)). Essentially these calculations mean that a fusion of two cones of degrees $\alpha_k$ and $\alpha_j$ results into the cone of the degree $\alpha_k + \alpha_j$.

The asymptote of $D_k$ as $d(p_k, p) \rightarrow \infty$ follows from the asymptote of $C_k$ (111) and the sum rule (21). However, it is instructive to obtain it by matching asymptotes.

We evaluate the metric at three points. One is $|z| \gg |p|$, a regular point far separated from all singularities. The asymptote of the metric there is $\phi|_{z\rightarrow \infty} \sim -2\sum_k \alpha_k \log|z| + O(z^{-1})$. The second point is close to $z \rightarrow p_k$. The metric there is $\phi|_{z\rightarrow p_k} \sim -2\alpha_k \log|z - p_k| - 2\log \tilde{l}_k$. The third point is close to the antipode, where singularities aggregate. The metric there is $\phi|_{z\rightarrow 0} \sim -2\sum_{j \neq k} \alpha_j \log|z| - 2\sum_{j \neq k} \log \tilde{l}_j$. These asymptotes must match in the common domain

$$(\phi|_{z\rightarrow \infty})|_{z\rightarrow p_k} = (\phi|_{z\rightarrow p_k})|_{z\rightarrow 0} = (\phi|_{z\rightarrow 0})|_{z\rightarrow p_k}.$$
This condition gives the asymptote of the scaling factors and the asymptote of the accessory and auxiliary parameters

$$f_j^{(\alpha)} \sim \delta_{ij} |p_i| \left| \sum \alpha_j \right| + (1 - \delta_{ij}) |p_k| \alpha_k, \quad (116)$$

cones: \( \text{d}(p_k, p) \rightarrow \infty \quad C_j \sim -\delta_{jk} \frac{h_k}{p_k}, \quad (117) \)

$$D_j \sim \frac{\delta_{jk}}{2p_k} \sum_{j \neq k} \left( \alpha_k \frac{h_j}{\gamma_j} + \frac{h_k}{\gamma_k} \alpha_j \right), \quad (118)$$

The limiting behavior of the parameters for conical points is well illustrated by the explicit formulas for polyhedra surfaces collected in appendix B.

Equations (115) and (118) determines the limiting behavior, and transformation properties of the potential \( H \) under the Möbius transformation (see (107)) and (120) for the Liouville functional

$$H(g(p)) = \prod_k |g'(p_k)|^{\alpha_k \alpha_j \gamma_j \left( 2 + \frac{1}{\gamma_j} + \frac{1}{\gamma_k} \right)} H(p), \quad (119)$$

$$H(p_1, p_2, p_3, \ldots, p_n) |_{p_1 \rightarrow p_2} \sim |p_1 - p_2|^{\alpha_1 \alpha_2 \gamma_1 \left( 2 + \frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right)} H(p_1, p_3, \ldots, p_n). \quad (120)$$

The short distance asymptote of auxiliary parameters for cusps can be deduced from that for cones, by replacing \( f_j^{(\alpha)} \rightarrow l_k \) and setting \( \alpha_k = 1 \). Then equation (113) yields

$$l_k \sim |p_j - p_k|, \quad (121)$$

cusps: \( p_k \rightarrow p_j \quad C_k \sim -\frac{1}{2(p_k - p_j)}, \quad (122) \)

$$D_k \sim \frac{1}{p_k - p_j}. \quad (123)$$

These formulas and the formulas (111) complete the calculations of the limiting behavior of the parameters of uniformization theory which determine the dimensions of critical systems and geometric transport coefficients of QH states on genus zero singular surfaces.

Combining these formulas with (98) and (39) we obtain the asymptotes of the free energy (4) and (5) and (8) and the adiabatic phase (11) and (17) quoted in section 3.2, the dimension and the OPE exponents (14) and (18) for cones and cusps (15) and (22). We see that the accessory parameters determine the exchange statistics \(-\frac{\pi}{2} \alpha_k \alpha_j\), the first term in (18), and auxiliary parameters determine the AB phases \( \alpha_k L_k + \alpha_j L_j \) of (18). In the case of cusps the exchange phase is half of what would be obtained from cones at \( \alpha = 1 \), due to the factor 1/2 in the asymptote of the accessory parameter in (111).

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Appendix A. Singular surfaces of revolution

Here we summarize the metric of surfaces of constant curvature with two punctures. These are surfaces of revolution immersed in 3D Euclidean space. In this case the developing map is either a power law or a logarithmic function.

If the cone is flat ($R_0 = 0$ in (68)), a singular conformal map $z \rightarrow w(z) = z^\gamma$ brings the metric (70) to the Euclidean form

$$ds^2 = |dw|^2, \quad w(z) = z^\gamma.$$  \hspace{1cm} (A.1)

It maps a punctured disk centered at the conical point to a wedge of a plane $0 \leq \arg w < 2\pi\gamma$, whose sides are isometrically glued together as ‘wedge-periodic’ condition (see the figure A1).

In polar coordinates $w = \frac{1}{\gamma} r e^{i\gamma\theta}$, where $\theta$ runs the full circle ($0 \leq \theta < 2\pi$), and $r = \gamma |z|^\gamma$ the metric reads

$$ds^2 = r^2 d\theta^2 + \gamma^{-2} dr^2.$$  \hspace{1cm} (A.2)

A model for the metric with a constant positive curvature $R_0 = +2$ is the spindle, or ‘football’, a surface of revolution of a spherical arch. Its metric is the singular conformal map of a sphere with the removed sector to a twice punctured plane

$$ds^2 = \left( \frac{2|dw|}{1 + |w|^2} \right)^2, \quad w(z) = z^\gamma.$$  \hspace{1cm} (A.3)

The spindle has two identical antipodal elliptic cones at $z = 0$ and $z = \infty$. It is the only surface of constant curvature with two isolated singularities [51].

In polar coordinates $w = \frac{r}{i\sqrt{\gamma^2 - r^2}} e^{i\gamma\theta}$ the spindle metric reads

$$ds^2 = r^2 d\theta^2 + \frac{dr^2}{\gamma^2 - r^2}.$$  \hspace{1cm} (A.4)

Another useful form of the metric $ds^2 = dt^2 + \gamma^2 \sin^2 t \, dt^2$ was found in coordinates $r = \gamma \sin t$.

A model for the negative constant curvature $R_0 = -2$ are Minding surfaces, figure A2. They are the surfaces of revolution of a curve $y(x)$ defined by the equation

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{FigureA1.png}
\caption{Schematic diagram of a cone and its 3D embedding.}
\end{figure}
In polar coordinates the metric of Minding surfaces reads
\[ ds^2 = r^2 d\theta^2 + \frac{dr^2}{\gamma^2 + r^2}. \]
If \( \gamma^2 < 0 \) the singularity is smoothed. That curve is called Minding tubular.
If \( \gamma^2 > 0 \) the surface has two conical singularities. These are the only surfaces of constant negative curvature with conical singularities immersed in 3D.

Another form of the metric of the Minding cones is \( ds^2 = dr^2 + \gamma^2 \sinh^2 t \, d\theta^2 \) in coordinates \( r = \gamma \sinh t \).

In complex coordinates \( w = \sqrt{\gamma^2 + r^2} e^{i\gamma \theta} \) the Minding cones \( (\gamma^2 > 0) \) is a singular map of the Poincaré disk which removes a sector of the disk
\[ ds^2 = \left( \frac{2|dw|}{1-|w|^2} \right)^2, \quad w(z) = z^\gamma. \] (A.5)
The map \( \zeta = -i \frac{w+1}{w-1} \) brings this metric to Poincaré half plane model
\[ ds^2 = \frac{|d\zeta|^2}{(\Im \zeta)^2}, \quad \Im \zeta > 0. \] (A.6)

Conical singularities on surfaces of negative curvature have an important degeneration, the cusp, or the parabolic singularity. At \( \alpha \to 1 \), or \( \gamma \to 0 \) a hyperbolic cone degenerates into a cusp, whose tip is sent to infinity. In this case the curve \( y(x) \) (A.4) becomes the tractrix
\[ y' = -\frac{1}{x} \sqrt{1-x^2}. \]
Such surface is called the pseudosphere. As it follows from (A.5), the metric of the cusp in polar coordinates reads
\[ ds^2 = r^2 d\theta^2 + r^{-2} dr^2. \] (A.7)
The metric (A.7) is conformal in the coordinate \( z = e^{-\frac{1}{2} - i\theta} \).
\[ ds^2 = \frac{|dz|^2}{|z|^2 \log^2 |z|}, \quad |z| < 1. \]  
(A.8)

In the coordinate \( \zeta = i \log z \) the metric becomes Poincaré model (A.6).

A distinguished feature of surfaces with constant negative curvature immersed in 3D Euclidean space is an unavoidable edge (a consequence of Hilbert’s theorem: complete isometric immersions of surfaces with constant negative curvature do not exist).

**Appendix B. Explicit formulas for polyhedra surfaces**

The explicit formulas for polyhedra surfaces \( \sum \alpha_k = 2, \quad \alpha_0 = 0 \) illustrate the limiting behavior of the accessory and auxiliary parameters.

The developing map for a polyhedra surface is the Schwarz–Cristoffel map

\[ w'(z) = e^{\frac{\phi_0}{2} \prod_{j=1}^{n} (z - p_j)^{-\alpha_j}}, \]  
(B.1)

where \( p \) are images of polyhedra vertices, and the conformal factor \( \phi_0 \) is fixed by the volume

\[ e^{-\phi_0} = \frac{1}{V} \int \prod_{j=1}^{n} |z - p_j|^{-2\alpha_j} |dz|^2. \]  

The metric is \( ds^2 = e^{\phi} |dz|^2 \) with

\[ \phi = \log |w'|^2 = \phi_0 - 2 \sum_j \alpha_j \log |z - p_j|. \]  
(B.2)

Then the explicit formulas for the accessory and auxiliary parameters follow from their definitions (89) and (99)

\[ C_k = \partial_{p_k} \log Z_L = - \sum_{k \neq j} \frac{\alpha_j \alpha_k}{p_k - p_j}, \]  
(B.3)

\[ l_j^{-2\gamma_j} = e^{\phi_0} \prod_{l \neq j} |p_j - p_l|^{-2\alpha_j}, \]  
(B.4)

\[ D_k = \frac{1}{2} \partial_{p_k} \log H = \sum_{j \neq k} \frac{1}{2(p_k - p_j)} \left( \frac{h_k}{\gamma_k} + \frac{h_j}{\gamma_j} \right) \alpha_k \gamma_k + \alpha_j \gamma_j \right) - \frac{1}{2} \partial_{p_k} \phi_0 \sum_j \frac{h_j}{\gamma_j}. \]  
(B.5)

The Liouville functional \( Z_L \), and the functional \( H \), the Kähler potentials for the Weil–Petersson and Takhtajan-Zograf metrics on the moduli space, follow from (100) and (99)

\[ Z_L \propto \prod_{j > k} |p_j - p_k|^{-2\alpha_0 \alpha_j}, \]  

\[ H = \prod_j l_j^{2h_j} = e^{-\phi_0 \sum_j \frac{h_j}{\gamma_j} \prod_{j \neq k} |p_j - p_k|^{2(\alpha_j \frac{h_k}{\gamma_k} + \alpha_k \frac{h_j}{\gamma_j})}. \]  
(B.6)

The symbol \( \propto \) means ‘up to moduli independent terms’. These formulas determine the free energy and the spectral determinant via the relation \( \partial_{p_k} f = \frac{c_1}{2} (C_i + D_i) \). We write the free energy in various suggestive forms
\[ e^{-f} \propto (\text{Det}(-\Delta))^{-\frac{\chi}{2}} \propto (Z_k \cdot \sqrt{H})^{-\frac{\chi}{2}} = e^{\phi_0} \sum_{\alpha} \prod_{k<j} (p_j - p_k)^{-\frac{\alpha}{2} \alpha_k (\frac{1}{\pi} + \frac{1}{\gamma})} \]

\[ = \prod_j l_j^{-\phi_0} \prod_{k<j} (p_j - p_k)^{\alpha_k \alpha_j} \]

\[ = e^{-\phi_0} \prod_j l_j^{-\phi_0} \]  

Formulas (B.7)–(B.9) is the result of [9, 10], and also rigorously proven results of [11, 12].

Equation (B.9) represents the decomposition (101) of the partition function on the Liouville-Polyakov action and the functional \( H \). Equation (B.9) presents the multiplicative form of the partition function as a product of scales of individual conical points.

The formula (B.9) is readily generalized for higher genus surfaces. The first factor there represents the overall scaling \( e^{-\phi_0} \) of surfaces with the Euler characteristic \( \chi \). For more details see [12]. The most general formula is (24).

The relation between the adiabatic connection and the free energy (11) yields the application to QHE. The adiabatic connection of Laughlin states in the moduli space of a polyhedra is

\[ A_k = \partial_{p_k} f = \sum_{j \neq k} \frac{\Lambda_{kj}}{p_k - p_j} - \partial_{\phi_0} \phi_0 \sum_j L_j, \]  

where \( \Lambda_{kj} = \frac{\gamma}{\pi} \alpha_k \alpha_j (\frac{1}{\gamma} + \frac{1}{\gamma}) \) is the mutual statistics (18), and \( L_j = \frac{\phi_0}{\gamma} \) is the angular momentum (14).

We see that the adiabatic connection for the polyhedra is fulfilled by its asymptotes (39). In this case the parameters \( C \) and \( D \) and the adiabatic connection are rational holomorphic functions of the moduli and the partition function is the square of a holomorphic function. This is no longer true if the surface is not flat. However, as we had seen the limiting behavior at merging conical points does not depend on the background metric, and could be read from the flat case. However, the dimensions \( \Delta_k \) (17) explicitly depend on \( \alpha_0 \). That part is missed in polyhedra. Polyhedra are flat surfaces. Their first Chern number vanish \( c_1(p_k) = 0 \). The adiabatic phase of polyhedra possesses only the topological part of the phase.

The overall scaling is encoded by the first factor in (B.7): because the surface is flat the scaling \( p \to \lambda^{1/2} p \) at a fixed volume yields the transformation of the conformal factor \( e^{\phi_0} \to \lambda^{2\alpha_k} e^{\phi_0} \), or, at a fixed conformal factor to the scaling of the volume \( V \to \lambda^{-2} V \). Under the scaling the partition function transforms as \( e^{-f} \to \lambda^{2 \sum \alpha_k} e^{-f} \).

If the conformal factor \( \phi_0 \) is fixed, the partition function is the quasi-primary

\[ p_k \to g(p_k) : e^{-f} \to \prod_j |g'(p_j)|^{-2 \alpha_k} e^{-f}, \]  

where \( \Delta_k \) is given by (17).

At a fixed volume the partition function is invariant under the Möbius transformation \( p_k \to g(p_k) \). This follows from (B.10), the transformation property of the conformal factor \( e^{-\phi_0} \to \prod_k |g'(p_k)|^{-2 \alpha_k} e^{-\phi_0} \), and the identity \( \alpha_k \sum_j L_j = \sum_{k \neq j} \Lambda_{kj} \), rested on the polyhedra condition \( \alpha_0 = 0 \). This is the version of the sum rule (21) for the polyhedra surface.
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