Matrix coordinate Bethe Ansatz: applications to XXZ and ASEP models

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Abstract

We present the construction of the full set of eigenvectors of the open asymmetric simple exclusion process (ASEP) and XXZ models with special constraints on the boundaries. The method combines both recent constructions of coordinate Bethe Ansatz and the old method of matrix Ansatz specific to the ASEP. This ‘matrix coordinate Bethe Ansatz’ can be viewed as a non-commutative coordinate Bethe Ansatz, the non-commutative part being related to the algebra appearing in the matrix Ansatz.

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1. Introduction

Questions of the integrability of models in statistical and quantum mechanics have been studied much more for periodic systems than for open systems, for which the numbers of particles and excitations may vary. However, open boundary conditions have become central in non-equilibrium physics, for which exactly solvable models are needed to explore new features. Fast advances were made in the 1980s to generalize Yang–Baxter equations to the open case (see [1]). However, although the boundaries which preserve the integrability have been classified quite easily [2, 3], the computation of the eigenvalues and the eigenvectors for the non-diagonal boundaries is a tricky problem. Indeed, no standard method exists to completely diagonalize Sklyanin-type transfer matrices with non-diagonal boundary matrices. Recently, progress has been made in this direction [4–10], but the problem is far from being solved. The present paper addresses this question for a simple model that is used both in statistical mechanics and in spin chains.
The asymmetric simple exclusion process (ASEP) is the simplest model of transport of hard-core particles along a one-dimensional lattice, which exhibits non-trivial behaviours. Each site of the lattice may be occupied by one particle or be empty. On the boundaries, two reservoirs add or remove particles at different rates and create a non-zero flux particle from one to the other. For a lattice of \( L \) sites, the state space is \( 2^L \)-dimensional and it can be mapped exactly to a system of \( L \) spin \( 1/2 \), the role of the Markov transition matrix being held by the XXZ spin chain Hamiltonian. The role of the reservoirs corresponds in this case to non-diagonal magnetic fields on the boundaries. Although all the coefficients have very different interpretations and may be complex or real depending on the model, the mathematical structure remains the same and integrability techniques can be applied in both cases. In this paper, ASEP notations are chosen, without any loss of generality if one assumes that the coefficients may become complex. The detailed model is described in section 2, as well as previous integrability results on this model.

Although the models are identical, there exists at least one method that has been developed and applied only to the ASEP case: the matrix Ansatz (or the DEHP method) [11]. The reason for this fact is that it requires the \textit{a priori} knowledge of one eigenvalue to build the corresponding eigenvector; this is precisely the case for stochastic models for which the existence of invariant measures is known. Attempts to establish parallels between Bethe and matrix Ansätze produced various partial results [12, 13] (see however [14] for the case of periodic boundary conditions) but never a full understanding of the exact relation between both approaches. This paper shows how to build coordinate Bethe Ansatz eigenvectors from a matrix Ansatz-type vacuum state in section 3.

The paper is structured as follows. We start in section 2 with a brief summary of open ASEP models. Then, in section 3, we present our method, which can be viewed as a non-commutative coordinate Bethe Ansatz. The non-commutative part is based on the algebra appearing in the matrix Ansatz. We present our conclusion in section 4. Appendices A and B gather technical results needed in our construction. Finally, appendix C is devoted to the study of finite-dimensional representations of the matrix Ansatz algebra.

2. ASEP Hamiltonians and constraints on boundaries

The Markov transition matrix for the open ASEP model is given by

\[
W = \hat{K}_1 + K_L + \sum_{j=1}^{L-1} w_{j,j+1},
\]

where the indices indicate the spaces on which the following matrices act non-trivially:

\[
w = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -q & p & 0 \\
0 & q & -p & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix}, \quad \hat{K} = \begin{pmatrix}
-\alpha & \gamma e^{-s} \\
\alpha e^s & -\gamma 
\end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix}
-\delta & \beta \\
\delta & -\beta 
\end{pmatrix}.
\]

(2.2)

It is well established [15–18] that this ASEP model is related by a similarity transformation to the so-called integrable open XXZ model [1]. Some care must be taken with the properties of the matrix \( W \). In the case of the ASEP with \( s = 0 \), the matrix is stochastic but not Hermitian. In the context of the ASEP, \( s \) is the parameter of the generating function of the current of particles and, for \( s \neq 0 \), the matrix \( W \) is even not stochastic. In the general case encompassing both ASEP and XXZ, there is no stochastic nor self-adjoint property for \( W \) and left and right eigenvectors are generically different.
the constraint: it is the maximal number of pseudo-excitations in the Ansatz.

Bethe Ansatz also provides one part of the spectrum, \( \omega \). An advantage of this method consists in giving an interpretation of the number of eigenvalues and the left or right eigenvectors corresponding to \( \omega \). However, it seems that this generalized coordinate Bethe Ansatz is not enough to obtain the missing cases.

only one type of Bethe equation via a generalization of the coordinate Bethe Ansatz \([20]\). It succeeded in computing the eigenvalues and the left or right eigenvectors corresponding to \( \omega \) for any \( \alpha, \beta, \gamma \) and \( s \) but also of the bulk parameters \( p \) and \( q \) \([17, 18, 9, 10]\):

\[
c_+(\alpha, \gamma)c_-(\beta, \delta) = e^\epsilon \left( \frac{p}{q} \right)^{L-1-n} \quad \text{for} \quad \epsilon, \epsilon' = \pm \quad \text{and} \quad n \in \{0, 1, 2, \ldots, L - 1\} \tag{2.3}
\]

\[
c_+(x, y) = y/x \quad \text{and} \quad c_-(x, y) = -1. \tag{2.4}
\]

The four possible choices for the signs \( \epsilon, \epsilon' \) and the corresponding constraint are summarized in Table 1, as well as the corresponding notation used for the Markov matrix \( W \) when the constraints are satisfied.

Similar to \( W \), we denote by \( K^\epsilon,\epsilon' \) and \( \tilde{K}^\epsilon,\epsilon' \) the boundary matrices \( K \) and \( \tilde{K} \) when the corresponding constraint (2.3) is satisfied. In the ASEP, the parameters \( \alpha, \beta, \gamma \) and \( \delta \) are positive. Then, only the first two constraints in Table 1 may be considered as relevant. However, we also treat the last two sets since they become relevant when we map the ASEP problem to the XXZ model.

By numerical investigations in \([19]\), it has been established that the whole spectrum of \( W^\epsilon,\epsilon' \) is given by two different types of Bethe equations. In previous papers \([9, 10]\), we succeeded in computing the eigenvalues and the left or right eigenvectors corresponding to only one type of Bethe equation via a generalization of the coordinate Bethe Ansatz \([20]\). It seems that this generalized coordinate Bethe Ansatz is not enough to obtain the missing cases. An advantage of this method consists in giving an interpretation of the number \( n \) entering in the constraint: it is the maximal number of pseudo-excitations in the Ansatz.

For \( W_n^{\epsilon,\epsilon'} \) (i.e. \( \epsilon' = 1 \)), a second method, now called the matrix Ansatz (or DEHP method), has been developed in \([11]\) to find the second part of the spectrum. We give the outlines of the historical method in subsection 3.1. Then, we present the new results in subsection 3.3 which consists in a generalization of the matrix Ansatz. It allows us to obtain the second part of the spectrum for any \( W_n^{\epsilon,\epsilon'} \), i.e. for any subset of constraints.

In \([9, 10]\), we also proved that for a new type of constraint, the generalized coordinate Bethe Ansatz also provides one part of the spectrum,

\[
c_+(\alpha, \gamma)c_-(\beta, \delta) = e^\epsilon \left( \frac{p}{q} \right)^n \quad \text{for} \quad \epsilon, \epsilon' = \pm \quad \text{and} \quad n \in \{0, 1, \ldots, L - 1\} \tag{2.5}
\]

\[
c_s^\epsilon(u, v) = \frac{p - q + v - u \pm \sqrt{(p - q + v - u)^2 + 4uv}}{2u}. \tag{2.6}
\]

Table 1. Possible values for the parameters and the constraints imposed by (2.3) where \( n \) takes integer values between 0 and \( L - 1 \).

| \( c_+(\alpha, \gamma) \) | \( c_-(\beta, \delta) \) | Constraints | \( W \) |
|---|---|---|---|
| \( \frac{z}{a} \) | \( \frac{z}{b} \) | \( \frac{a}{b} \) \( \frac{z}{x} \) \( L-1-n \) = 1 | \( W_{n}^{++} \) |
| \( -1 \) | \( -1 \) | \( e' \) \( \frac{z}{x} \) \( L-1-n \) = 1 | \( W_{n}^{-} \) |
| \( -1 \) | \( \frac{z}{a} \) | \( -\frac{a}{b} e' \) \( \frac{z}{x} \) \( L-1-n \) = 1 | \( W_{n}^{-+} \) |
| \( \frac{z}{a} \) | \( -1 \) | \( -\frac{a}{b} e' \) \( \frac{z}{x} \) \( L-1-n \) = 1 | \( W_{n}^{-} \) |
Table 2. Possible values for the parameters and the constraints imposed by (2.5), where \( n \) is an integer between 0 and \( L - 1 \).

| \( \epsilon \) | \( \epsilon' \) | Constraints | \( W \) |
|---|---|---|---|
| + | + | \( c_+^* (\alpha, \gamma) c_+^* (\beta, \delta) = e^\gamma \left( \frac{\epsilon}{\epsilon'} \right)^n \) | \( W_n^{++} \) |
| - | - | \( c_+^* (\alpha, \gamma) c_+^* (\beta, \delta) = e^\gamma \left( \frac{\epsilon}{\epsilon'} \right)^n \) | \( W_n^{--} \) |
| + | - | \( c_+^* (\alpha, \gamma) c_+^* (\beta, \delta) = e^\gamma \left( \frac{\epsilon}{\epsilon'} \right)^n \) | \( W_n^{+-} \) |
| - | + | \( c_+^* (\alpha, \gamma) c_+^* (\beta, \delta) = e^\gamma \left( \frac{\epsilon}{\epsilon'} \right)^n \) | \( W_n^{-+} \) |

The four possible choices for the signs \( \epsilon, \epsilon' \) and the corresponding constraints are summarized in table 2. We denote by \( W_{\epsilon, \epsilon'}^n \) the matrix \( W \) when one constraint (2.5) is satisfied.

As for the matrix \( W_{\epsilon, \epsilon'}^n \), in [10], we computed one part of the spectrum for \( W_{\epsilon, \epsilon'}^n \) Markov matrices thanks to the generalized coordinate Bethe Ansatz. We believe that a very similar procedure as the one presented in section 3.3 should hold in this case when we transform the matrix as in [10]. We will not give in detail the computations in the present paper.

3. Matrix coordinate Bethe Ansatz

3.1. Matrix Ansatz without excitation

In this subsection, we present the method introduced in [11] to obtain the eigenvector with vanishing eigenvalue (i.e. the steady state or the invariant measure) for the Markov process with \( s = 0 \). This state corresponds exactly to the part of eigenspace we do not obtain from the coordinate Bethe Ansatz for \( W_{-L}^{-L-1} \) in [9, 10].

Particular care is required since it involves various operators acting on different vector spaces. The physical state space, written all along the paper \( \mathcal{H} \), is \( 2^L \)-dimensional and the canonical basis can be indexed by the occupation numbers \( \tau_i \in \{0, 1\} \) (resp. spin \( s_i \in \{-\frac{1}{2}, \frac{1}{2}\} \) in the XXZ language) on each site \( i \). All the vectors of \( \mathcal{H} \) are written in the ket notation \(|·⟩⟩\) and all the dual vectors are written in the bra form \langle⟨·|\).

The matrix Ansatz states that the steady state \(|\Phi⟩\) of the ASEP with \( s = 0 \) has components given by

\[
⟨τ_1τ_2...τ_L|Φ⟩ = ⟨⟨V_1| \prod_{1≤i≤L} (τ_iD + (1 - τ_i)E) |V_2⟩⟩, \tag{3.1}
\]

where the arrow means that the product has to be built from left to right when the index \( i \) increases. One has for example \( ⟨011001|Φ⟩ = ⟨⟨V_1|EDDEED|V_2⟩⟩ \) for \( L = 6 \). The non-commuting matrices \( D \) and \( E \) act on an abstract auxiliary vector space \( \mathcal{V} \), which is different from \( \mathcal{H} \). The vector \(|V_1⟩⟩\) lies in the space \( \mathcal{V} \) (as any vector written with a ket notation \(|·⟩⟩\)), whereas the vector \(|V_2⟩⟩\) is in its dual \( \mathcal{V}^* \) (as any vector written with a bra notation \langle⟨·|\)).

Equation (3.1) expresses the components of \(|Φ⟩\) ∈ \( \mathcal{H} \) as a scalar product between vectors on the abstract auxiliary space \( \mathcal{V} \).

One checks that \(|Φ⟩\) is an eigenvector of \( W_{-L}^{-L-1} \) with vanishing eigenvalue if the two matrices \( D \) and \( E \) acting on \( \mathcal{V} \) and the two boundary vectors satisfy the commutation rules:

\[
qED - pDE = D + E, \tag{3.2a}
\]
\[ \langle \langle V_1 | (\alpha E - \gamma D + 1) = 0, \quad (3.2b) \]
\[ (\beta D - \delta E + 1)|V_2\rangle = 0. \quad (3.2c) \]

These three relations reduce quadratic relations in \( D \) and \( E \) to linear expressions in \( D \) and \( E \) in the bulk and linear relations in \( D \) and \( E \) to scalar expressions on the boundaries; it allows one to determine recursively all the components of the eigenvector \( |\Phi \rangle \) and it does not need an explicit representation of the algebra. An irrelevant minus sign relative to the standard notation for the matrix Ansatz has been introduced in (3.2) for later convenience. In particular, the matrix Ansatz gives easy access to correlation functions with standard transfer matrix techniques \[11\].

To easily prove that \( |\Phi \rangle \) is an eigenvector of \( W_{L-1}^{-} \), we first rewrite \( |\Phi \rangle \) as follows:
\[ |\Phi \rangle = \langle \langle V_1 | D \otimes D \otimes \ldots \otimes D |V_2\rangle \rangle. \quad (3.3) \]

The convention used in this notation will be used all through the paper and corresponds to the following interpretation: a vector \( (D \otimes D) \) has operator entries instead of scalar ones (two-dimensional module over the endomorphisms of \( V \)), the tensor product of such elements is an element of a \( 2^L \)-dimensional module over the endomorphisms of \( V \), whose components are products of size \( L \) of operators \( E \) and \( D \) with the usual tensor rule:
\[ \left( \begin{array}{c} a \\ b \end{array} \right) \otimes \left( \begin{array}{c} a' \\ b' \end{array} \right) = \left( \begin{array}{cc} aa' & ab' \\ ba' & bb' \end{array} \right). \quad (3.4) \]

Let us emphasize the order of the non-commuting operators in the vector of the rhs of the previous equation. The right action of \( |V_2\rangle \) \in \( V \) produces a \( 2^L \)-dimensional vector whose components are vectors of \( V \), which is further projected through the left action of \( \langle V_1 | \in V^* \) to a \( 2^L \)-dimensional vector whose components are complex numbers and is thus identified to \( \mathcal{H} \). For example, the vector
\[ \langle \langle V_1 | \left( \begin{array}{c} a \\ b \end{array} \right) \otimes \left( \begin{array}{c} a' \\ b' \end{array} \right) |V_2\rangle \rangle = \left( \begin{array}{c} \langle \langle V_1 | ad' |V_2\rangle \rangle \\ \langle \langle V_1 | ab' |V_2\rangle \rangle \\ \langle \langle V_1 | ba' |V_2\rangle \rangle \\ \langle \langle V_1 | bb' |V_2\rangle \rangle \end{array} \right) \quad (3.5) \]

is a 4-component vector with complex entries.

Secondly, using definitions (2.2) and equations (3.2), we show that
\[ \omega \left( \begin{array}{c} E \\ D \end{array} \right) \otimes \left( \begin{array}{c} E \\ D \end{array} \right) = \left( \begin{array}{c} E \\ D \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \otimes \left( \begin{array}{c} E \\ D \end{array} \right) \quad (3.6) \]
\[ \hat{K}_{L-1}^{-} \langle \langle V_1 | \left( \begin{array}{c} E \\ D \end{array} \right) \rangle \rangle = \langle \langle V_1 | \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \rangle \rangle \quad \text{and} \quad K_{L-1}^{-} \langle \langle V_1 | \left( \begin{array}{c} E \\ D \end{array} \right) |V_2\rangle \rangle = - \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \langle \langle V_2 | \rangle \rangle \quad (3.7) \]

The first equation encodes four relations on the endomorphisms \( D \) and \( E \) and each equation on the second line encodes two relations between vectors of \( V \) or \( V^* \). Since this type of result is central in this paper and to be pedagogical, we explain in detail the computation for the second relation in (3.7)
and the vectors

\[|\omega(u)\rangle = \left(\frac{E - [u]}{uD + [u]}\right) |t(u)\rangle = \begin{pmatrix} u \\ -1 \end{pmatrix}, \quad |\bar{t}(u)\rangle = (p - q) \begin{pmatrix} 0 \\ uD + [u] \end{pmatrix}, \quad |\bar{\omega}\rangle = |\bar{t}\rangle, \quad |\bar{\omega}\rangle = |\bar{t}\rangle, \quad |\bar{\omega}\rangle = |\bar{t}\rangle. \tag{3.12}\]

where the notation |\rangle (ket in bold) stands for 2-component vectors in which the entries are numbers as well as the commuting relation (3.11). Let us remark that 1 is the identity operator acting on \(V\).

Finally, remarking that the matrices \(\omega, K\) and \(\hat{K}\) do not act in the auxiliary space \(V\) where lie \((\langle V_1 \rangle | V_2\rangle\), we obtain telescopic terms that can be simplified to obtain

\[W_{L-1}^{-} |\Phi\rangle = 0. \tag{3.10}\]

The goal of this paper consists in generalizing (3.3) to also obtain eigenvalues and eigenvectors for all the \(A_{\Psi}\). For that purpose, we add some pseudo-excitations above the previous eigenvector as is done in the usual Bethe Ansatz. Before giving the matrix Ansatz with \(n\) pseudo-excitations, we need to introduce some vectors playing the role of these excitations as well as some of their properties.

3.2. Pseudo-excitations and properties

Let us introduce the notation

\[|u\rangle = \begin{pmatrix} 1 - u \\ q - p \end{pmatrix} \tag{3.11}\]

and the vectors

\[|\omega(u)\rangle = \left(\frac{E - [u]}{uD + [u]}\right) |t(u)\rangle = \begin{pmatrix} u \\ -1 \end{pmatrix}, \quad |\bar{t}(u)\rangle = (p - q) \begin{pmatrix} 0 \\ uD + [u] \end{pmatrix}, \quad |\bar{\omega}\rangle = |\bar{t}\rangle, \quad |\bar{\omega}\rangle = |\bar{t}\rangle, \quad |\bar{\omega}\rangle = |\bar{t}\rangle. \tag{3.12}\]

where the notation |\rangle (ket in bold) stands for 2-component vectors in which the entries are operators on \(V\) (to differentiate them from vectors |\rangle \in \mathcal{H}\) introduced in section 3.1, which contain complex numbers and are obtained after projection with \(\langle V_1 \rangle | V_2\rangle\).

If the non-commuting matrices \(D\) and \(E\) satisfy relation (3.2a), we obtain the following relations:

\[w|\omega(u)\rangle \otimes |\omega(u)\rangle = |\omega(u)\rangle \otimes |t(u)\rangle - |t(u)\rangle \otimes |\omega(u)\rangle \tag{3.13a}\]

\[w|\omega(v)\rangle \otimes |\omega(v)\rangle = |\omega(v)\rangle \otimes \left[\frac{u}{p}\right] - |\bar{t}(v)\rangle \otimes |\omega(v)\rangle \tag{3.13b}\]

\[w|\omega(u)\rangle \otimes |\omega(v)\rangle = -p|\omega(u)\rangle \otimes |\omega(v)\rangle + p|\omega(v)\rangle \otimes |\omega(u)\rangle \tag{3.13c}\]

\[w|\omega(v)\rangle \otimes \left[\frac{u}{p}\right] = -q|\omega(v)\rangle \otimes |\omega(u)\rangle \otimes |\omega(v)\rangle \tag{3.13d}\]

where \(u\) and \(v\) are still arbitrary numbers that will be fixed later. The proof of relations (3.13) is straightforward: one projects each relation on the four components, and then one uses the definition of the \([u]\) numbers as well as the commuting relation (3.2a). Let us remark that
relation (3.13a) for \( u = 1 \) is similar to relation (3.6). Relations (3.13) deal with the bulk part of \( W \). Now we study these vectors on the boundaries.

Let us introduce the following functions:

\[
\lambda_+(u, v) = 0 \quad \text{and} \quad \lambda_-(u, v) = -u - v. \tag{3.14}
\]

In the following section, we impose the following relation on the right boundary, for \( \epsilon \in \{-, +\}, \)

\[
K|\omega(u)|V_2) = \lambda_+(\beta, \delta)|\omega(u)|V_2) + |t(u)|V_2),\tag{3.15}
\]

where, as explained previously, \( |\omega(u)|V_2) \) means that \( |V_2) \) is right-applied to each entry of \( |\omega(u)| \). This relation provides generally two different constraints \( (K \) is a \( 2 \times 2 \) matrix) on vectors of \( V \) except if \( u = -1/c_{-\epsilon}(\beta, \delta) \). In that case, the remaining constraint becomes

\[
(\beta D - \delta E + 1)|V_2) = 0 \quad \text{for} \quad u = -1/c_{-\epsilon}(\beta, \delta). \tag{3.16}
\]

We recover relation (3.2c) for both values of \( \lambda_+(\beta, \delta) \).

Similarly, on the left boundary, we impose the following relation:

\[
\tilde{K}|\omega(u)|V_1) = \lambda_+(\alpha, \gamma)|\omega(u)|V_1) + |\omega(u)|V_1), \tag{3.17}
\]

where the notation \( |\omega(u)|V_1) \) means that \( |V_1) \) is left-applied to each operator entry of \( |\omega(u)| \).

With a special choice of \( u \), the two constraints reduce again to only one, given by

\[
\langle|V_1)|\omega^e(E - u) - \gamma u^e(uD + [u]) + 1\rangle = 0 \quad \text{for} \quad u = -e^{-s}c_{-\epsilon}(\alpha, \gamma). \tag{3.18}
\]

### 3.3. Matrix Ansatz with excitations

Let us define the tensor product (for modules defined on the endomorphisms of \( V \)) of vectors \( |\omega(u)| \) over the sites \( i \) to \( j \) and write it as

\[
|\Omega(u)|^j_i = |\omega(u)||\omega(u)||\omega(u)||\omega(u)||\omega(u)|j. \tag{3.19}
\]

We now fix an integer \( n \) and introduce the state with \( n - m \) excitations at the ordered positions \( 1 \leq x_m + 1 < \cdots < x_n \leq L \) defined as the \((C^2)^{\otimes L}\)-vector in \( \mathcal{H} \) with the projections

\[
|x_{m+1}, \ldots, x_n) = \frac{1}{\sqrt{p}} \prod_{k=m+1}^{n-1} (\kappa_{x_k+1} - \cdots - \kappa_{x_k-1} - 1)\langle|V_1)|\Omega(u_{m+1})|^{x_{m+1}-1}_{x_{m+1}-1} \times |\omega(u_{m+1})|^{x_{m+1}}_{x_{m+1}}|\Omega(u_{m+2})|^{x_{m+2}-1}_{x_{m+2}-1} \ldots |\omega(u_n)|^{x_n}_{x_n}|\Omega(u_{n+1})|^{L}_{L-1}|V_2)). \tag{3.20}
\]

The overall factor \( \sqrt{q/p} \) is introduced only in order to normalize the Bethe roots. The coefficients \( u_m \) and \( v_m \) are related through the recursions

\[
u_{m+1} = \frac{q}{p} u_m, \quad v_{m+1} = \frac{q}{p} v_m, \tag{3.21}
\]

and the initial coefficients \( u_1 \) and \( v_1 \) are still arbitrary. These vectors correspond to states where \( m \) excitations have left the system (through the left boundary). To clarify the notation, the state with no excitation corresponds to \( m = n \) and is given by

\[
|\emptyset\rangle = \langle|V_1)|\Omega(u_{n+1})|^{L}_{L-1}|V_2)), \tag{3.22}
\]

and, for \( u_{n+1} = 1 \), we recover the state \( |\Phi\rangle \) defined in (3.3).

We also need to introduce some other definitions concerning the set on which we are going to sum in our Ansatz. The set \( G_m \) is a full set of representatives of the coset \( BC_n/BC_m \) (\( G_0 = BC_n \), by convention) and \( BC_m \) is the \( B_m \) Weyl group, generated by transpositions \( \sigma_j, j = 1, \ldots, m - 1 \), and the reflection \( r_i \) (for details, see appendix A). It acts on a vector \( k = (k_1, \ldots, k_n) \) of \( \mathbb{C}^n \): \n
\[
k_r = (-k_1, k_2, \ldots, k_n), \quad k_{r_j} = (k_1, \ldots, k_{j+1}, k_j, \ldots, k_n). \tag{3.23}
\]
We also introduce the following truncated vector \( k^{(m)}_g \), for \( 0 \leq m \leq n \) and \( g \in G_m \):

\[
\begin{align*}
  k^{(m)}_g &= (k_{g(m+1)}, \ldots, k_{gm}).
\end{align*}
\]  

(3.24)

We are now in a position to state the main result of this paper which provides eigenvalues and eigenstates of \( W^{e,e'}_{L-1-n} \), i.e. the matrix \( W \) with the constraint \( c_x(\alpha, \gamma)c_x(\beta, \delta) = e^{(p/q)^n} \).

**Theorem 3.1.** The vector

\[
|\Phi_n\rangle = \sum_{m=0}^{n} \sum_{x_{m+1} < \cdots < x_n} \sum_{g \in G_m} A^{(m)}_g e^{i k^{(m)}_g} |x_{m+1}, \ldots, x_n\rangle
\]

(3.25)

is an eigenstate of \( W^{e,e'}_{L-1-n} \) with the eigenvalue

\[
\mathcal{E}^{e,e'}_{L-1-n} = \lambda_{++}(\alpha, \gamma) + \lambda_{--}(\beta, \delta) + \sum_{j=1}^{n} \Lambda(e^{i k^{(j)}_g}) \quad \text{where} \quad \Lambda(z) = \sqrt{p/q} \left( z + \frac{1}{z} \right) - p - q
\]

(3.26)

if the following relations are fulfilled.

(a) The coefficients \( u_1 \) and \( v_1 \) entering in definition (3.20) are

\[
u_1 = -e^{-i}c_x(\alpha,\gamma) \quad \text{and} \quad v_1 = \frac{1}{2 \delta} \left( \frac{p}{q} \right)^{m-1} (p - q + \delta - \beta \pm \sqrt{(\beta - \delta + q - p)^2 + 4\beta\delta}).
\]

(3.27)

(b) The non-commuting elements \( E \) and \( D \) obey

\[
qED - pDE = E + D
\]

(3.28)

\[
(\beta D - \delta E + 1) |V_1\rangle = 0
\]

(3.29)

\[
\langle V_1| (\alpha e^{\delta}(E - [u_1]) - \gamma e^{-\delta}(D - [1/u_1]) + 1) = 0.
\]

(3.30)

(c) The coefficients \( A^{(m)}_g \) verify

\[
A^{(0)}_g = S(e^{ik_{g1}}, e^{ik_{g(m+1)}})A^{(0)},
\]

(3.31)

\[
A^{(m)}_g = T^{(m)}(e^{ik_{g1}}, \ldots, e^{ik_{gm}})A^{(m-1)},
\]

(3.32)

where

\[
S(z_1, z_2) = -\frac{a(z_1, z_2)}{a(z_2, z_1)} \quad \text{with} \quad a(z_1, z_2) = \frac{i}{z_1 z_2} \left( \frac{q}{p} + \frac{p}{q} \right) z_2 - z_1 z_2 - 1.
\]

(3.33)

and

\[
T^{(m)}(z_1, \ldots, z_m) = \frac{D^{(m-1)}_1(z_m)}{p_1(z_m) V^{(m)}_1(z_m) \prod_{j=1}^{m-1} a(z_m, z_j) a(z_j, 1/z_m)}.
\]

(3.34)

\[
D^{(m-1)}_1 = \frac{v_m}{v_m - u_{m+1}} \left( \alpha^e v_m + \gamma - \alpha + p - q - \frac{\gamma}{e^v v_m} \right).
\]

(3.35)

\[
V^{(m)}_1(z) = \Lambda(z) + (\lambda_{++}(\alpha, \gamma) + \gamma \left( 1 - \frac{1}{z} \sqrt{p/q} \right) + (\lambda_{--}(\alpha, \gamma) + \alpha \left( 1 - \frac{1}{z} \sqrt{p/q} \right).
\]

(3.36)

\[
p_1(z) = z + \frac{1}{\sqrt{p/q}} \frac{pu_2 - qv_1}{v_1 - u_2}.
\]

(3.37)
(d) The pseudo-momentum $k_j$ must satisfy the following Bethe equations, for $1 \leq j \leq n$:

$$
\prod_{\ell=1}^{n} S(e^{ik_j}, e^{ik_\ell}) S(e^{-ik_\ell}, e^{ik_j}) = e^{2iL_k} \frac{V^+_1(e^{ik_j}) V^+_{L}(e^{ik_j})}{V^+_1(e^{-ik_j}) V^+_{L}(e^{-ik_j})},
$$

(3.38)

with

$$
V^\pm_L(z) = \Lambda(z) + (\lambda_\pm (\beta, \delta) + \beta) \left( 1 - \frac{1}{z} \sqrt{p/q} \right) + (\lambda_\pm (\beta, \delta) + \delta) \left( 1 - \frac{1}{z} \sqrt{q/p} \right).
$$

(3.39)

Before giving the proof of this theorem in subsection 3.4, we make some remarks on the theorem.

**Remark 3.1.** The first relation in (3.27) is equivalent, via constraint (2.3), to

$$
\tau_{n+1} = -1/c_c (\beta, \delta).
$$

(3.40)

The sign in the definition of $\tau_1$ is irrelevant.

**Remark 3.2.** The algebra generated by $E$ and $D$ is very close to the one introduced in [11]: the only difference lies in equation (3.30). We study the finite-dimensional representations of this algebra in appendix C, which gives intriguing relations with the second set of constraints (2.5).

**Remark 3.3.** A consequence of (3.32) (for $m = 1$) and $A^{(1)}_{g\gamma_1} = A^{(1)}_{\gamma_1}$ is

$$
A^{(0)}_{g\gamma_1} = \frac{T^{(1)}(e^{ik_1})}{T^{(1)}(e^{-ik_1})} A^{(0)}_{\gamma_1}.
$$

(3.41)

This relation with (3.31) allows us to express $A^{(0)}_g$ for any $g \in BC_n$ in terms of $A^{(0)}_1$ (where the subscript 1 stands for the unit of $BC_n$ group). Finally, using (3.32) recursively, we can express all the coefficients $A^{(m)}_g$ in terms of only $A^{(0)}_1$. This last coefficient is usually chosen such that the eigenfunction $|\Phi_1\rangle$ be normed.

**Remark 3.4.** In our previous work [10], we found via the coordinate Bethe Ansatz the eigenfunctions of $W_{n,\epsilon,\epsilon'}$ with eigenvalues

$$
\tilde{E}_{n,\epsilon,\epsilon'} = \lambda_\epsilon (\alpha, \gamma') + \lambda_\epsilon (\beta, \delta) + \sum_{j=1}^{n} \Lambda(e^{ip_j}).
$$

(3.42)

Note the change of signs in the index of both $\lambda$ in these eigenvalues (3.42) in comparison with (3.26). The pseudo-momentum $p_j$ satisfy the following Bethe equations:

$$
\prod_{\ell=1}^{n} S(e^{ip_j}, e^{ip_\ell}) S(e^{-ip_\ell}, e^{ip_j}) = e^{2iL_p} \frac{V^{-}_{1}(e^{ip_j}) V^{-}_{L}(e^{ip_j})}{V^{-}_{1}(e^{-ip_j}) V^{-}_{L}(e^{-ip_j})}, \quad j = 1, 2, \ldots, n.
$$

(3.43)

By numerical investigations [19], it has been conjectured that the set of eigenvalues given by (3.42), (3.43) with $n$ pseudo-excitations and by (3.26), (3.38) with $L-1-n$ pseudo-excitations give the complete spectrum of $W_{n,\epsilon,\epsilon'}$. Therefore, our previous results [10] with the results of this paper seem to provide the complete spectrum for $W_{n,\epsilon,\epsilon'}$ as well as the associated eigenstates.
3.4. Proof of the main theorem 3.1

To prove the theorem, we show that the following equation holds:

\[ W_{L-1-n}^{\epsilon, \epsilon'} |\Phi_n\rangle = e^{\beta \Phi} W_{L-1-n}^{\epsilon, \epsilon'} |\Phi_n\rangle. \tag{3.44} \]

The proof is very similar to the one we performed in [10] for the generalized coordinate Bethe Ansatz, except that entries of |\omega| terms are now non-commuting operators. It consists in projecting equation (3.44) on the different |x_1, \ldots, x_n\rangle and to prove that each projection is true if conditions (a)–(d) of the theorem hold. We write only the projections leading to independent relations (one can check that the remaining ones do not lead to new relations).

On |x_1, \ldots, x_n\rangle for (x_1, \ldots, x_n) generic. (i.e. 1 < x_1, x_n < L and 1 + x_j < x_{j+1}).

Before performing this projection, let us remark that, using relations (3.13), (3.15) and (3.17) as well as the conditions (b) of the theorem, we can show that

\[ W_{L-1-n}^{\epsilon, \epsilon'} |x_1, \ldots, x_n\rangle = (\lambda_{-\epsilon} (\alpha, \gamma) + \lambda_{-\epsilon} (\beta, \delta)) |x_1, \ldots, x_n\rangle + \sum_{j=1}^{n} [(-p - q) |x_1, \ldots, x_n\rangle \quad \left| \begin{array}{l} x_j \end{array} \right| + | \ldots, x_j - 1, \ldots |]. \tag{3.45} \]

We recall that relations (3.15) and (3.17) are valid only if

\[ u_1 = -e^{-\gamma} c_{\epsilon} (\alpha, \gamma) \quad \text{and} \quad u_{n+1} = -\frac{1}{c_{\epsilon} (\beta, \delta)} \tag{3.46} \]

which is the first relation in (a) (see also remark 3.1). In addition to that, the recursion relation between the (3.21) implies the constraint \( c_{\epsilon} (\alpha, \gamma) c_{\epsilon'} (\beta, \delta) = e^{\beta} (p)^{n} \) which is the constraint for \( W_{L-1-n}^{\epsilon, \epsilon'} \).

Finally, the projection on |x_1, \ldots, x_n\rangle (for generic x_1, \ldots, x_n) of equation (3.44) holds if the energy takes the form (3.26).

On |x_1, \ldots, x_n\rangle with x_{j+1} = 1 + x_j. (and x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_n generic).

Using (3.13b) and this projection, we obtain a relation (3.31) between \( A^{(0)} \) and \( A^{(0)}_{|\omega|_j} \).

As expected, the expression of the scattering matrix is similar to the periodic case since the boundaries are not involved in this process.

On |x_{m+1}, \ldots, x_n\rangle. (x_{m+1}, \ldots, x_n generic and m \geq 1).

Before performing this projection, we need to know how the left boundary matrix \( \hat{K}_L \) acts on the vectors |\omega(v_m)\rangle and |\omega(u_m)\rangle. By direct computation, using relation (3.30), we show that

\[ \hat{K}_{L-1-n}^{\epsilon, \epsilon'} |V_1| |\omega(v_m)\rangle = \hat{\Lambda}_1^{(m-1)} |V_1| |\omega(v_m)\rangle + D_1^{(m-1)} |V_1| |\omega(u_{m+1})\rangle + |V_1| \tilde{F}(v_m), \tag{3.47} \]

\[ \hat{K}_{L-1-n}^{\epsilon, \epsilon'} |V_1| |\omega(u_m)\rangle = \Lambda_1^{(m-1)} |V_1| |\omega(u_m)\rangle + C_1^{(m-1)} |V_1| |\omega(v_{m-1})\rangle + |V_1| \tilde{F}(v_m), \tag{3.48} \]

where the same notational convention is used as for (3.17) and where \( D_1^{(m-1)} \) is defined in (3.35) and

\[ \hat{\Lambda}_1^{(m-1)} = \frac{1}{u_{m+1} - v_m} (\alpha e^{\beta} v_m u_{m+1} + (\gamma + p - q) v_m - \alpha u_{m+1} + \gamma e^{-\beta}) \quad \tag{3.49} \]

\[ \Lambda_1^{(m-1)} = \frac{(1 - e^{\beta} u_m) (\alpha u_{m-1} + \gamma e^{-\beta})}{u_m - v_{m-1}} \quad \tag{3.50} \]
\[ C^{(m-1)}_1 = \frac{(1 - e^{iu_m})(\alpha u_m + \gamma e^{-j})}{v_{m-1} - u_m}. \] (3.51)

Using these relations, we finally obtain that the projection on \( |x_{m+1}, \ldots, x_n \rangle \) provides the following constraints, for any \( g \in G_n \):

\[ D^{(m-1)}_1 \sum_{h \in H_m} A_{gh}^{(m-1)} e^{ik_{jhn}} + \left( A_1^{(m)} - \lambda_{-\epsilon} (\alpha, \gamma) - \sum_{j=1}^{m} \Lambda (e^{ik_{j}}) \right) A_g^{(m)} = 0, \] (3.52)

where \( H_m = BC_m / BC_{m-1} \) (see appendix A).

We are going to demonstrate that this last constraint (3.52) holds if relations (3.31) and (3.32) are true\(^5\). We start by remarking that a consequence of the latter equations (3.31) and (3.32) is

\[ A_{gh}^{(m)} = A_g^{(m)} \times \begin{cases} 1 & 1 \leq \alpha \leq m - 1, \\ \frac{T^{(m)}(e^{ik_{hj}}, \ldots, e^{ik_{j-1}}, e^{ik_{j+1}})}{T^{(m)}(e^{ik_{hj}}, \ldots, e^{ik_{j-1}})} S(e^{ik_m}, e^{ik_{j+1}}) & \alpha = m, \\ S(e^{ik_m}, e^{ik_{j+1}}) & \alpha \geq m + 1. \end{cases} \] (3.53)

Then, using again (3.32) to express now \( A_{gh}^{(m)} \) in terms of \( A_{gh}^{(m-1)} \) and using (3.53) to express \( A_{gh}^{(m-1)} \) \((h \in H_m)\) in terms of \( A_{gh}^{(m-1)} \), relation (3.52) becomes the functional relation

\[
\sum_{j=1}^{m} \left[ z_j V_1(z_j) p_1(z_j) \prod_{\ell=j+1}^{m} a(z_{\ell}, z_{\ell}) a(z_{\ell}, \frac{1}{z_{\ell}}) \right] \\
+ \frac{z_j}{1 - z_j} V_1^{(z_j)} p_1 \left( \frac{1}{z_j} \right) \prod_{\ell=j+1}^{m} a(z_{\ell}, z_{\ell}) a(z_{\ell}, \frac{1}{z_{\ell}}) \\
= \lambda_{-\epsilon} (\alpha, \gamma) - \Lambda^{(m)}_1 + \sum_{j=1}^{m} \Lambda(z_j),
\] (3.54)

where \( z_j \) stands for \( \exp(ik_{j}) \) and the functions are defined in (3.33)–(3.36). To prove this last relation (3.54), let us introduce the following function:

\[ F^{(m)}(z) = \sqrt{pq} \frac{V_1(z) p_1(z)}{\Lambda(z)} \left( 2z - \sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}} \right) \prod_{\ell=1}^{m} a(z_{\ell}, z_{\ell}) a \left( z_{\ell}, \frac{1}{z_{\ell}} \right). \] (3.55)

Then, one can prove that (3.54) is equivalent to \( \sum_{\text{residues of } F^{(m)}(z) = 0} \) (see appendix B for the complete list of its residues), which finishes to demonstrate that constraint (3.52) is verified if relations (3.31) and (3.32) are true.

On \( |x_{m+1}, \ldots, x_n \rangle \) \((x_{m+1}, \ldots, x_n \text{ generic and } m \geq 1)\).

This projection provides a second relation between the coefficients from the levels \( m - 1 \) and \( m \). We obtain the following constraint, for any \( g \in G_n \):

\[
\sum_{h \in H_m} \left( \sqrt{pq} e^{ik_{jhn}} + \tilde{\Lambda}_1^{(m-1)} - \lambda_{-\epsilon} (\alpha, \gamma) - \sum_{j=1}^{m} \Lambda (e^{ik_{j}}) \right) A_{gh}^{(m-1)} e^{ik_{jhn}} + C_{11}^{(m)} A_g^{(m)} = 0.
\] (3.56)

\(^5\) This demonstration is similar to the one done in [10] but we give it here again for the completeness of the present proof.
We are going to prove that this constraint is satisfied if relations (3.31) and (3.32) hold, following a similar demonstration as previously. Using the previous projection (3.52) (already proven) and then relations (3.31) and (3.32), we prove that projection (3.56) becomes the following functional relation:

\[
\sqrt{pq} \sum_{j=1}^{m} \left[ \frac{z_j^2}{z_j^2 - 1} V_j(z_j) p_1(z_j) \prod_{\ell=1, \ell \neq j}^{m} a(z_\ell, z_j) a(z_\ell, z_j) \right] = \left( \sum_{j=1}^{m} \Lambda(z_j) + \lambda_{-\epsilon}(\alpha, \gamma) - \tilde{\Lambda}_1^{(m-1)} + q \left( \sum_{j=1}^{m} \Lambda(z_j) + \lambda_{-\epsilon}(\alpha, \gamma) - \Lambda_1^{(m)} \right) \right) - C_1^{(m)} D_1^{(m-1)}.
\]

(3.57)

The function to consider now is \( G^{(m)}(z) = \sqrt{pq} z F^{(m)}(z) \). Finally, we prove that the functional relation (3.57) is equivalent to \( \sum_{\text{residues of } G^{(m)}} G^{(m)}(z) = 0 \) (see appendix B for the computation of the residues).

**On \(|x_1, \ldots, x_{n-1}, L\rangle\).** (\(x_1, \ldots, x_{n-1}\) generic).

To perform this projection, we need to know the action of the right boundary on one pseudo-excitation

\[
K_{L-1 \rightarrow N}^{\epsilon, \epsilon'} |\omega(v_n)\rangle |V_2\rangle = \tilde{\Lambda}_L |\omega(v_n)\rangle |V_2\rangle - |\tilde{f}(v_n)\rangle |V_2\rangle
\]

(3.58)

where the same convention is used as for (3.15) and where \( \tilde{\Lambda}_L = \delta(v_n - 1) \) and \( v_n = \left( \frac{z}{p} \right)^{n-1} v_1 \) with \( v_1 \) given by (3.27).

Then we can prove that this last independent projection holds if the so-called Bethe equations (3.38) are satisfied.

**4. Conclusion**

The previous sections present the construction of the full set of eigenvectors of the ASEP and the XXZ spin chain with special constraints on the boundaries. The method combines both recent constructions of the coordinate Bethe Answer for the same set of constraints [9, 10] and the old method of the matrix Ansatz [11] specific to the ASEP. Although computations have been shown for only one set of special constraints, the construction should be transposed without effort to the second set of special constraints discovered in [10]. Left eigenvectors are also very simple to build using the same methods.

A first intriguing feature of the matrix coordinate Bethe Ansatz for the first set of special constraints (2.3) is presented in appendix C: finite-dimensional representations of the matrices \( D \) and \( E \) can be found only if the second set of constraints (2.5) is satisfied. Up to now, however, we do not have simple explanations of this fact.

The matrix Ansatz has proven to be useful, at least in the case of zero excitations, for the study of correlation functions [21] since it is reduced to standard transfer matrix techniques of one-dimensional statistical mechanics. The same question in the context of the Bethe Ansatz is notably difficult and has led to many different approaches such as the quantum inverse scattering method. It would be interesting to investigate whether the present formulation may simplify the study of correlation functions, either in the present case with boundaries or in the standard periodic case.

The proofs presented here seem to indicate that the matrix Ansatz state plays the role of a new vacuum state, although it is highly non-trivial and does not factorize, in the context of
the open XXZ spin chain. The standard coordinate Bethe Ansatz approach has been used but it would be worth knowing if the algebraic Bethe Ansatz could be adapted to obtain the same eigenvectors as in [4].

Numerical evidence tends to show that the set of eigenvectors is now complete and gives a synthetic view of the $BC_n$ structure of the eigenstates. The constructions for the two sectors of the spectrum are similar in their structure but very different from the point of view of the reference vacuum state, although a physical interpretation for the ASEP gives some hints [22, 23]. One may hope that a further understanding of the passage from one sector to the other may allow one to couple together both sectors and study ASEP and XXZ spin chains with boundaries out of the special constraints that allowed the present framework.

Finally, the matrix Ansatz approach was found to be useful for various stochastic models of particles with different types or impurities [24, 25], as well as for the study of tableaux in combinatorics [26], even if no Bethe Ansatz approach exists yet for these models. It would be interesting to know whether our approach may be helpful for these other models, so that integrable system methods can be extended to them.

Appendix A. Weyl group $BC_n$ and cosets

The Weyl group $BC_n$ is generated by the set $\{r_1, \sigma_1, \ldots, \sigma_{n-1}\}$ with the following constraints:

$$\sigma_j^2 = 1 = r_1^2, \quad \sigma_1 r_1 \sigma_1 = r_1 \sigma_1 r_1, \quad \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}. \quad (A.1)$$

The subgroup generated by $\{\sigma_1, \ldots, \sigma_{n-1}\}$ is the symmetric group. We now consider its subgroups generated by $\{r_1, \sigma_1, \ldots, \sigma_{m-1}\}$, $m \leq n$, which we identify with $BC_m$.

For $g \in BC_n$, we then define the class $gBC_m = \{gh ; h \in BC_m\}$, called a left coset. It is known that the set of all classes $gBC_m$, which is called $BC_m/BC_m$, forms a partition $BC_n$: we can thus define $G_m$ as a full set of representative of $BC_m/BC_m$, such that one has the unique decomposition $BC_n = \bigcup_{g \in G_m} gBC_m$. We set, by convention, $G_0 \sim BC_n$ and $G_n = \{1\}$.

The action of an element $g$ of $G_m$ on a vector $k^{(m)} = (k_{m+1}, \ldots, k_n)$ of $R^{n-m}$ is given by $k^{(m)}_g = (k_{g(m+1)}, \ldots, k_{g(n)})$. One checks that this action does not depend on the choice of the representative $g$, such that the action of $BC_n/BC_m$ is well defined on $R^{n-m}$ without further specifications. This definition is useful because the set $\{k^{(m)}_g | g \in G_m\}$ contains once and only once the vector $(\epsilon_{i_1} k_{i_1}, \ldots, \epsilon_{i_n} k_{i_n})$ for any choice $\epsilon_j = \pm 1, 1 \leq i_j \leq n$ and $i_j \neq i_l$. For example, $\{k^{(n-1)}_g | g \in G_{n-1}\} = \{(k_n), (-k_n), (k_{n-1}), (-k_{n-1}), \ldots, (k_1), (-k_1)\}$.

Finally, we introduce $H_m$ which is a full set of representatives of the coset $BC_m/BC_{m-1}$ which may chosen as follows:

$$[id, \sigma_{m-1}, \sigma_{m-2} \sigma_{m-1}, \ldots, \sigma_1 \ldots \sigma_{m-2} \sigma_{m-1}], \quad r_1 \sigma_1 \ldots \sigma_{m-2} \sigma_{m-1}, \quad \sigma_1 r_1 \sigma_1 \ldots \sigma_{m-2} \sigma_{m-1}, \ldots, \sigma_{m-1} \ldots \sigma_1 r_1 \sigma_1 \ldots \sigma_{m-2} \sigma_{m-1}] \quad (A.2)$$

Appendix B. List of the residues of the functions $F^{(m)}$ and $G^{(m)}$

We list in this appendix the residues of the functions $F^{(m)}$, defined by (3.55), and $G^{(m)}$. The residues of $F^{(m)}$ are

$$\text{Res}(F^{(m)}(z))\big|_{z=z_j} = \frac{z_j^{N^+_j}(z_j) p_1(z_j)}{z_j^2 - 1} \prod_{\ell=1}^{m} a(z_j, z_\ell) a\left(z_\ell, \frac{1}{z_j}\right), \quad (B.1)$$
\[\text{Res}(F^{(m)}(z))\bigg|_{z = 1/z_j} = \frac{1}{z_j((1/z_j)^2 - 1)} V_i^j \left( \frac{1}{z_j} \right) p_1 \left( \frac{1}{z_j} \right) \prod_{\ell = 1}^m \left( \frac{1}{z_j}, z_\ell \right) a(z_\ell, z_j), \quad (B.2)\]

\[\text{Res}(F^{(m)}(z))\bigg|_{z = \sqrt{q}} = \sqrt{pq} \left( \frac{q}{p} \right)^m \frac{\lambda_c(\alpha, \gamma) + \alpha}{p - q} p_1 \left( \frac{\sqrt{pq}}{q} \right), \quad (B.3)\]

\[\text{Res}(F^{(m)}(z))\bigg|_{z = \sqrt{\frac{q}{p}}} = \sqrt{pq} \left( \frac{p}{q} \right)^m \frac{\lambda_c(\alpha, \gamma) + \gamma}{q - p} p_1 \left( \frac{\sqrt{pq}}{p} \right). \quad (B.4)\]

\[\text{Res}(F^{(m)}(z))\bigg|_{z = \frac{p}{q}} = \sqrt{pq} \frac{(p - q + 2\alpha - 2\gamma) p_1 \left( \frac{p + q}{2\sqrt{pq}} \right)}{2(q - p)}. \quad (B.5)\]

\[\text{Res}(F^{(m)}(z))\bigg|_{z = \infty} = -\sum_{\ell = 1}^m \Lambda(z_\ell) - \sqrt{pq} \frac{(p + q)}{2\sqrt{pq}} + \frac{1}{2}(2\lambda_c(\alpha, \gamma) + \alpha + \gamma). \quad (B.6)\]

The residues of \(G^{(m)}(z)\) at the point \(z = z_0\) with \(z_0 \neq 0, \infty\) are easy to compute:

\[\text{Res}(G^{(m)}(z))\bigg|_{z = z_0} = \sqrt{pq} q_0 \text{Res}(F^{(m)}(z))\bigg|_{z = z_0}. \quad (B.7)\]

Since 0 is not a pole, it remains to compute the residue at infinity:

\[\text{Res}(G^{(m)}(z))\bigg|_{z = \infty} = \sum_{j,k=1}^m \Lambda(z_j) \Lambda(z_k) + 2 \left( \text{Res}(F^{(m)}(z))\bigg|_{z = \infty} - \frac{p + q}{4} \right) \sum_{j = 1}^m \Lambda(z_j)
- \sqrt{pq} p_1 \left( \frac{p + q}{2\sqrt{pq}} \right) \left( 2\lambda_c(\alpha, \gamma) + \alpha + \gamma + \frac{p + q}{2} \right)
- \frac{(p - q)(\alpha - \gamma)}{4} + \frac{(p + q)^2}{8}. \quad (B.8)\]

Appendix C. Representations of the algebra generated by \(E\) and \(D\)

In this appendix, we study the finite-dimensional irreducible representations of the algebra we used to construct the matrix Ansatz, since such representations emerge only for special constraints already encountered in the study of the open ASEP or XXZ spin chains. In the case without excitation, these types of representations have been studied previously in [16, 27]. We will follow similar proofs for the cases with excitations.

To be self-contained in this appendix, we recall that the algebra needed in these cases is defined by

\[qED - pDE = D + E, \quad (C.1a)\]

\[(\beta D - \delta E + 1)|V_2\rangle = 0 \quad (C.1b)\]

\[\langle V_1 \mid (\alpha \epsilon^*(E - [u]) - \frac{\gamma}{\epsilon \omega^*} (uD + [u]) + 1) = 0, \quad (C.1c)\]

where \(u = -\epsilon^{-\epsilon} c_-(\alpha, \gamma) = -(p/q)\epsilon^*/c_-(\beta, \delta), \epsilon, \epsilon' \in \{\pm\}, c_+(x, y) = y/x\) and \(c_-(x, y) = -1.\)
In \[27\], it is proved that, to satisfy the ‘bulk’ part (C.1a), the non-commuting elements \(E\) and \(D\) must take the following form (up to similarity transforms):

\[
D = \frac{1}{q-p} \sum_{j=1}^{N} \left( 1 + a \left( \frac{q}{p} \right)^{j-1} \right) E_{jj}
\]

\[
E = \frac{1}{q-p} \sum_{j=1}^{N} \left( 1 + \frac{1}{a} \left( \frac{p}{q} \right)^{j-1} \right) E_{jj} + \frac{1}{q-p} \sum_{j=1}^{N-1} E_{j+1,j},
\]

where \(N\) is the dimension of the representation, \(a\) is a free parameter and \(E_{jj}\) is the elementary matrix with 1 in the entry \((i, j)\) and 0 otherwise.

Following the arguments of \[27\], the ‘boundary’ conditions (C.1b) and (C.1c) imply that there exist the integers \(k\) and \(\ell\) between 1 and \(N\) such that

\[
\beta \left( 1 + a \left( \frac{q}{p} \right)^{k-1} \right) - \delta \left( 1 + \frac{1}{a} \left( \frac{p}{q} \right)^{\ell-1} \right) + q - p = 0
\]

\[
a e^\prime \left( u + \frac{1}{a} \left( \frac{p}{q} \right)^{\ell-1} \right) - \frac{\gamma}{u e^\prime} \left( 1 + au \left( \frac{q}{p} \right)^{\ell-1} \right) + q - p = 0.
\]

With \(N\) being the size of the irreducible representation, one obtains the constraint \(|k - \ell| = N - 1\). In the case \(k = N > 1\) and \(\ell = 1\), one has

\[
\langle V_1 \rangle (1, 0, \ldots, 0) \text{ and } \langle V_2 \rangle = (0, \ldots, 0, 1) \quad \Rightarrow \quad \langle \langle V_1 | D^j | V_2 \rangle \rangle = 0, \quad \forall j.
\]

Now, since (C.1a) implies that

\[
q^n E D^n = (1 + pD)^n E + \sum_{\ell=1}^{n} q^{\ell-1} (1 + pD)^{n-\ell} D^\ell \quad \forall n,
\]

(C.6) shows, together with (C.1b), that all words (in \(E\) and \(D\)) vanish. Thus, this case must be excluded.

It remains the case \(k = 1\) and \(\ell = N \geq 1\) for which equations (C.4) and (C.5) become

\[
\beta a^2 + (\beta - \delta + q - p)a - \delta = 0
\]

\[
a \left( \frac{e^\prime}{a} \left( \frac{p}{q} \right)^{N-1} \right)^2 + \left( \frac{\alpha e^\prime}{u a^2} + q - p \right) \left( \frac{e^\prime}{a} \left( \frac{p}{q} \right)^{N-1} \right) - \gamma = 0.
\]

Then, using definition (2.6), we obtain, for any choice for \(a\),

\[
a = c^*_\tau(\beta, \delta) \quad \text{and} \quad \frac{e^\prime}{a} \left( \frac{p}{q} \right)^{N-1} = c^*_\tau(\alpha, \gamma) \quad \text{with} \quad \tau, \tau' = \pm.
\]

Therefore, a finite-dimensional representation exists if there exist two signs of \(\tau\) and \(\tau'\) such that the following relation is true:

\[
c^*_\tau(\alpha, \gamma) c^*_\tau(\beta, \delta) = e^\prime \left( \frac{p}{q} \right)^{N-1},
\]

where the functions \(c^*_\tau\) have been defined in (2.6).

For real parameters and \(u, v > 0\), one has \(c^*_\tau(u, v) < 0\) and \(c^*_\tau(u, v) > 0\) (since for \(uv > 0\) one has \(\sqrt{(p-q+v-u)^2 + 4uv} > |p-q+v-u|\)). Thus, for ASEP models, only the constraints corresponding to \(\tau = \tau'\) have to be considered (as it is the case for the first choice). We recognize in the framework of the matrix Ansatz the same constraints (2.5) that
appear for the coordinate Bethe Ansatz, although the relations between both approaches are very different.

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