1. Introduction

A basic ingredient in the theories of baryogenesis is the anomaly in the baryon-current divergence,
\[ \partial_{\mu} j_B^\mu = 3q + \cdots, \quad q = \frac{1}{16\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}, \]
where \( q \) is the topological-charge density in the SU(2) gauge fields, which is the divergence of the Chern–Simons current, \( q = \partial_{\mu} j_{\text{CS}}^\mu \) (the \( \cdots \) denote the U(1) contribution which is supposed to be inessential). In electroweak baryogenesis the baryon number \( B \) is generated during the electroweak transition and given by
\[ B(t) = 3 \int_0^t dt' \int d^3x \langle q(x, t') \rangle = 3 \langle N_{\text{CS}}(t) \rangle, \]
where \( N_{\text{CS}} = \int d^3x j_{\text{CS}}^0 \) is the Chern-Simons number. We assumed the transition to start at \( t = 0 \) and set \( N_{\text{CS}}(0) = 0 \).

The computation of \( q \) is a ‘classical’ problem in lattice gauge theory. The ‘naive’ lattice transcription
\[ \text{tr} F(x) \tilde{F}(x) \propto \epsilon^{\lambda\mu_1\cdots\mu_n} \text{tr} U_{\lambda\lambda}(x) U_{\mu_1\cdots\mu_n}(x) \text{symmetrized} \]
suffers from short-distance fluctuations. Out of equilibrium the problems are worse in the quantum world.

Fortunately, the classical approximation can be used in case of large occupation numbers, e.g. at sufficiently high temperature. But even in this case there are short-distance problems because of the slow decrease of occupation numbers at large momentum, \( n_p \sim T/p \). However, if we have good reason to start with suppressed short-distance modes, then the problem may show up only at late times, because classical equipartition is a slow process. For times of interest to the problem we can then use a ‘naive’ lattice transcription of \( q \). An ideal case is the scenario of baryogenesis from a rapid cold electroweak transition.

2. Scenario

Suppose the universe is left cold at the end of low-scale inflation. Subsequently the effective mass \( m_{\text{eff}}^2 \) in the Higgs potential
\[ m_{\text{eff}}^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \]
changes sign, e.g. due to a coupling to the inflaton field. The negative \( m_{\text{eff}}^2 \) causes a (‘tachyonic’ or ‘spinodal’) instability in which the system is far out of equilibrium, and under influ-
ence of a CP bias one expects a net change in the Chern-Simons number, with a corresponding baryon asymmetry. Non-linear effects lead to rapid approximate thermalization to an effective temperature \( T \) that is expected (counting d.o.f.) to be much lower than the standard finite-temperature transition temperature \( T_c \), so subsequent sphaleron transitions are expected to be negligible.

The reason for the change of sign of \( m_{\text{eff}}^2 \) is uncertain. Not committing ourselves to any specific cause, we use a sudden quench:

\[
\mu_{\text{eff}}^2 = +\mu^2, \quad t < 0,
= -\mu^2, \quad t > 0.
\]

This maximal non-equilibrium setup presumably also maximizes the asymmetry for given CP violation.

3. Classical approximation

The effect of the quench can be studied by first neglecting all interactions. Let \( \hat{\phi} \) be a real component of the Higgs field operator. Its spatial Fourier transform behaves around time zero as

\[
\hat{\phi}_p = \frac{1}{\sqrt{2\mu_p}} \left( \hat{a}_p e^{-i\omega_p t} + \hat{a}^\dagger_p e^{i\omega_p t} \right), \quad t < 0,
= \hat{a}_p e^{-i\omega_p t} + \hat{\phi}_p e^{i\omega_p t}, \quad t > 0,
\]

\[
\omega^\pm_p = \sqrt{\pm\mu^2 + p^2}.
\]

Matching at \( t = 0 \) determines \( \hat{a} \) and \( \hat{\phi} \) in terms of \( \hat{a}^\dagger \) and \( \hat{\phi}_p \). The modes \( p < \mu \) are unstable and grow exponentially: \( \omega^- = i|\omega^-| = i\sqrt{\mu^2 - p^2} \),

\[
\phi_p \rightarrow \alpha_p e^{i|\omega^-| t} \quad \pi_p \rightarrow |\omega^-| \alpha_p e^{i|\omega^-| t} \approx |\omega^-| \phi_p
\]

So there is classical behavior for \( |\omega^-| t \gg 1 \). Indeed, generic quantum correlation functions in the initial state \( |0\rangle \) ( annihilated by the \( \hat{a} \) ) can be well approximated by a classical distribution [2],

\[
\exp \left[ -\frac{1}{2} \sum_{|p| < \mu} \left( \frac{\xi^+_p}{n_p + 1/2 + \tilde{n}_p} + \frac{\xi^-_p}{n_p + 1/2 - \tilde{n}_p} \right) \right],
\]

where \( \xi^\pm_p = (\omega_p \phi_p \pm \pi_p) / \sqrt{2\omega_p} \), and \( n \) and \( \tilde{n} \) are instantaneous particle numbers. In the unstable region, as \( \mu t \gg 1 \) [2],

\[
n_p + 1/2 + \tilde{n}_p \gg 1, \quad n_p + 1/2 - \tilde{n}_p \to 0,
\]

so \( \xi^+_p \) grows large whereas \( \xi^-_p \to 0 \). The canonical coordinates and momenta grow large and become correlated, an ideal case for making a classical approximation.

As soon as the unstable modes have grown large enough, but before non-linear interaction effects are expected to take over, we could sample the above distribution to provide initial conditions for subsequent classical evolution. However, we may as well sample the distribution at time zero, since this leads to exactly the same distribution later on in the free-field approximation. As \( n = \tilde{n} = 0 \) at \( t = 0 \), only the 1/2 remain in the denominators, which is why we call these ‘just a half’ initial conditions [2]. Note that only momenta with \( p < \mu \) are initialized.

4. Simulation

To study the emerging asymmetry we used the SU(2) Higgs model with effective CP violation, given by

\[
-\mathcal{L} = \frac{1}{2g^2} tr F_{\mu\nu} F^{\mu\nu} + D_\mu \phi \partial^\mu \phi
+ \frac{\mu^4}{4\lambda} - \frac{\mu^2}{\lambda} \phi \bar{\phi} + \lambda(\phi \bar{\phi})^2
+ \kappa \phi \bar{\phi} \frac{1}{2} e^{\mu_\rho \sigma} tr F_{\mu\nu} F_{\rho\sigma}.
\]

Here \( \kappa \) parametrizes the strength of an effective CP-violating term of dimension six. It is supposed to represent a leading effect induced by physics beyond the standard model, or even mimic CP violation in the standard model itself.

We discretized the action on a (lorentzian) space-time lattice, from which the field equations follow in the usual way. Initial conditions were chosen according to the ‘just a half’ scheme, and also using a thermal Bose-Einstein distribution with small \( T = 0.1 m_H \) for comparison. Since the initial field configurations are smooth on the lattice scale, we used ‘naive’ lattice transcriptions, even for \( tr F \bar{F} \).

The results to follow where obtained with a lattice spacing \( a m_H = 0.35 \), volume \( (L m_H)^3 = (60 \times 0.35)^3 = 21^3 \), couplings \( g = 2/3 \) and \( \lambda \) such that \( m_H / m_W = 1, \sqrt{2} \) and 2 (recall \( m_H = \sqrt{2\lambda} v = \sqrt{2} \mu, m_W = g v/2 \), with \( v = \mu / \sqrt{\lambda} \) the
vacuum expectation value of the Higgs field).

Figure 1 shows the behavior in time of the volume-averaged $\phi^\dagger \phi$, the Chern-Simons number and the winding number in the Higgs field, $N_w$, for one initial condition, for $\kappa = 0$. At first $\phi^\dagger \phi$ increases exponentially, then peaks at $m_H t = 7.5$ and subsequently executes a damped oscillation towards a value somewhat smaller than $v^2$. For small energies, in equilibrium, the winding number is expected to be near $N_{CS}$, as this diminishes the covariant derivative $D_\mu \phi$. Fig. 1 shows that $N_w$ behaves at first erratically and then settles near an integer (the smaller the lattice spacing, the closer to an integer). The early $N_{CS}$ is very small and subsequently it appears to approach $N_w$. We see no sphaleron transitions (jumps in $N_{CS}$ by approximately an integer) in $50 < m_H t < 500$. The effect of non-zero $\kappa$ on single trajectories is unpredictable, due to the chaotic nature of the dynamics, but an effect is present in the average over initial conditions (as needed for the quantum expectation value).

Figure 2 shows an example of averages for non-zero $\kappa$. After an initial rise of $\langle N_{CS} \rangle$ (which we understand semi analytically), there appears to be a sort of resonance with $\langle \phi^\dagger \phi \rangle$ that may even lead (for larger $\kappa$) to a $\langle N_{CS} \rangle$ of opposite sign. We found a similar resonant effect in 1+1 dimensions, where it leads to a strong dependence on $m_H/m_W$. Note that after $tm_H = 40$ there is no visible drift of $\langle N_{CS} \rangle$ towards zero due to sphaleron transitions.

5. Conclusion

Baryogenesis through tachyonic electroweak transition offers ideal case for lattice simulations using the classical approximation. Together with Jon-Ivar Skullerud we have also computed Higgs and W particle numbers [3]. As expected, they grow large at low momenta, while staying negligible for momenta $p > 1.5 m_H$. This supports our experience that the effective CP-violating interaction $-L_{CP} = \kappa \int \phi^\dagger \phi \, tr \, \tilde{F} \tilde{F}$ can be meaningfully implemented using a ‘naive’ lattice transcription. More details are in [4] (and to be published).

The resulting asymmetry appears to be substantial: using simple estimates the observed baryon to photon ratio can be reproduced with $\kappa \approx 1 \times 10^{-5} \text{TeV}^{-2}$, assuming $m_H = \sqrt{2} m_W$.

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