BERNSTEIN-SATO IDENTITIES AND CONFORMAL SYMMETRY BREAKING OPERATORS

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Abstract. We present Bernstein-Sato identities for scalar-, spinor- and differential form-valued distribution kernels on Euclidean space associated to conformal symmetry breaking operators. The associated Bernstein-Sato operators lead to partially new formulae for conformal symmetry breaking differential operators on functions, spinors and differential forms.

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1. Introduction

Many aspects of harmonic analysis on Euclidean $n$-space, the $n$-sphere and $n$-hyperbolic space are related to the action of the conformal group. This is true not only for functions, but also for spinors and differential forms. Motivated partly by representation theory and partly by conformal differential geometry, there has recently been much

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2010 Mathematics Subject Classification. 42B37; 14F10, 47B06, 53A30.

Key words and phrases. Riesz distribution, Intertwining operator, Conformal symmetry breaking (differential) operator, Bernstein-Sato operator.

M. Fischmann and B. Ørsted were supported by the Department of Mathematics (Aarhus University) and the Danish Research Council, P. Somberg was supported by GA P201/12/G028.
progress in establishing concrete and explicit formulas for natural integral and differential operators exhibiting some form of conformal invariance. Here a key concept is that of A. Juhl’s residue families \cite{J99} in the framework of conformal geometry and T. Kobayashi’s symmetry-breaking operators \cite{KS15} in representation theory.

In this paper we collect and extend many formulas related to distribution kernels for both integral and differential operators in the three basic situations of functions, spinors, and differential forms. We shall treat both the absolute case of the conformal group, as well as the relative case of the conformal groups of Euclidean space and a coordinate hyperplane, i.e. the case of conformal symmetry-breaking operators. As it turns out, there is a series of natural identities based on some partly new second-order operators which are termed in the present article Bernstein-Sato operators.

Among the very useful and important ingredients in the theory of meromorphic continuation of families of distributions and the closely related theory of $\mathcal{D}$-modules are the Bernstein-Sato identities, see the seminal paper \cite{Ber71} by J. Bernstein. In particular, they allow to find the precise position of the corresponding poles as well as to encode a recurrence structure for the residues in the distribution family under consideration.

In recent years there appeared several approaches to a classification scheme for conformally covariant differential operators $P_{2N}$, $\mathcal{D}_{2N+1}$ and $L_{2N}^{(p)}$ (acting on functions, spinors and differential $p$-forms) on semi-Riemannian (spin-)manifolds, cf. \cite{GJMS92, GMP12, BG05}. Furthermore, the operators $P_{2N}$ and $\mathcal{D}_{2N+1}$ were extended to a theory of 1-parameter families of conformally covariant differential operators \cite{J99, FS14}, nowadays known and termed as the residue families. These correspond to the relative case. Concerning differential forms, not much is so far known and available in the literature.

The above mentioned operators ($P_{2N}$, $\mathcal{D}_{2N+1}$ and $L_{2N}^{(p)}$) on Euclidean space $\mathbb{R}^n$ arise as residues of Knapp-Stein intertwining integral operators for certain families of induced representations of conformal Lie groups, cf. \cite{KS71}. These operators are 1-parameter families of pseudo-differential convolution operators with respect to Riesz distributions on functions \cite{Rie49}, spinors \cite{CØ14} and differential forms \cite{FO17}, respectively.

One may also study Knapp-Stein type operators associated to a pair of conformal Lie groups (the relative case); these form a 2-parameter family of distributions, see \cite{KS15, MO17, K17}. They are termed conformal symmetry breaking operators, and are intertwining integral operators acting on principal series representations (realized for example in the non-compact picture) for the action of conformal Lie algebras on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. Their residues are given by 1-parameter families of equivariant differential operators termed conformal symmetry breaking differential operators. Note that these conformal symmetry breaking differential operators are just specific cases of residue family operators. The conformal symmetry breaking operators were studied in a more general context, e.g. the case of conformal Lie groups for $\mathbb{R}^n$ and $\mathbb{R}^{n-m}$ with $1 \leq m \leq n-1$ is discussed in \cite{MO15}. However, curved generalizations of the residue families, both of co-dimension one, and of higher co-dimensions, are not yet properly understood. One of the motivations for the present paper is to gain a better understanding of the model (flat) case in order to undertake a subsequent study of the curved generalizations, related to AdS/CFT and the corresponding Poincare-Einstein geometry.

Thus in the present paper, we prove Bernstein-Sato identities for distribution kernels of the three basic types of conformal symmetry breaking operators: scalar-valued
(acting on $\mathcal{D}'(\mathbb{R}^n)$)

\begin{align*}
K^+_{\lambda,\nu}(x',x_n) &= |x_n|^\lambda + v - n |(x')^2 + x_n^2|^{-\nu}, \\
K^-_{\lambda,\nu}(x',x_n) &= \text{sgn}(x_n)|x_n|^\lambda + v - n |(x')^2 + x_n^2|^{-\nu}, \\
\end{align*}

(1.1)

spinor-valued (acting on $\mathcal{D}'(\mathbb{R}^n)$)

\begin{align*}
K^{\pm}_{\lambda,\nu}(x',x_n) &= K^{\pm}_{\lambda,\nu+\frac{1}{2}}(x',x_n)x',
\end{align*}

(1.2)

and differential form-valued (acting on $\mathcal{D}'\Omega(\mathbb{R}^n)$)

\begin{align*}
K^{(p),\pm}_{\lambda,\nu}(x',x_n) &= K^{\pm}_{\lambda,\nu+1}(x',x_n)(i_x \varepsilon_x - \varepsilon_x i_x)i_{e_n} \varepsilon_{e_n}.
\end{align*}

(1.3)

We use the notation $\lambda, \nu \in \mathbb{C}, x = (x',x_n) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$, $x \cdot$ for the Clifford multiplication with the vector $x \in \mathbb{R}^n$, and $i_x, \varepsilon_x$ are the interior and exterior products with respect to $x$ on differential forms. Our convention for the distribution kernels are as duals to what they are considered as in the standard literature mentioned above. This choice is justified by the use of Fourier transform, which is applied to these distribution kernels directly without further dualization and leads to generalizations of singular vectors studied in [KÖSS15, FJS16, KKP16].

The proposal to find Bernstein-Sato operators of interest in the context of conformal symmetry breaking operators was initiated in [PS15], where certain shift operators for Gegenbauer polynomials regarded as the residues of Fourier transformed $K^\pm_{\lambda,\nu}(x',x_n)$ are studied. Later on, a more sophisticated approach was suggested in a private communication by J.L. Clerc. Moreover, the Bernstein-Sato operators are themselves intertwining operators for the relevant conformal Lie groups. Concerning detailed notation and intertwining results we refer to [C16, C17].

Our paper is structured as follows. In Section 2 we fix the notation and recall fairly standard results related to Riesz distributions on functions, spinors and differential forms.

In Section 3 we present a construction of Bernstein-Sato type operators for functions $P(\lambda)$, (3.5), spinors $P(\lambda)$, (3.12), and differential forms $P^p(\lambda)$, (3.21). Furthermore, we show that they satisfy a Bernstein-Sato identity on the space of distribution kernels for functions in Theorem 3.5 spinors in Theorem 3.12 and differential forms in Theorem 3.19 respectively.

In Section 4 we comment on the origin of the constructed Bernstein-Sato operators (it is interesting that they have several, a priori quite different definitions) and discuss some applications to the conformal symmetry breaking differential operators on functions, spinors and differential forms. The construction results in known formulas for conformal symmetry breaking differential operators on functions, cf. Theorem 4.2 and new formulas for conformal symmetry breaking differential operators on spinors and differential forms, see Theorems 4.6 and 4.9.

Acknowledgment: We would like to express our thanks to J.L. Clerc for the private discussion leading to the present paper.
2. Preliminaries

Let \( \mathbb{R}^n \) be equipped with the canonical flat metric \( \langle \cdot, \cdot \rangle \). We collect some basic known facts concerning tree types of Riesz distributions: for scalars [Rie49, GS64], spinors [CØ14] and differential forms [FØ17].

We denote by \( S(\mathbb{R}^n) \) the algebra of Schwartz functions on \( \mathbb{R}^n \) and follow the convention for the Fourier transform [GS64] on Schwartz functions \( f \in S(\mathbb{R}^n) \):

\[
F(f)(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^n} f(x) e^{i\langle x, \xi \rangle} dx,
\]

which also extends to the space of tempered distributions \( S'(\mathbb{R}^n) \). Note the identity

\[
F(f * g)(\xi) = F(f)(\xi)F(g)(\xi),
\]

where \((f * g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^n} f(x - y) g(y) dy\) denotes the convolution of Schwartz functions \( f \) and \( g \). This normalization of the Fourier transform is chosen in such a way that \( F(\delta_0) = 1 \), where \( \delta_0 \) is the Dirac distribution centered at the origin. Recall that for a polynomial \( P \) in \( n \) variables we have the identities

\[
F(P(\partial_{x_1}, \ldots, \partial_{x_n})f)(\xi) = F(P(ix_1, \ldots, ix_n)f)(\xi),
\]

\[
F(P(\partial_{x_1}, \ldots, \partial_{x_n})f)(\xi) = P(-i\xi_1, \ldots, -i\xi_n)F(f)(\xi)
\]

for \( f \in S(\mathbb{R}^n) \).

The Fourier transform \( F \) extends to the space of spinor-valued and differential forms-valued Schwartz functions (as well as the tempered distributions), i.e., \( S(\mathbb{R}^n) \) and \( S^p(\mathbb{R}^n) \) (\( S'(\mathbb{R}^n) \) and \( S'^p(\mathbb{R}^n) \)), and will be denoted by \( F \) as well.

2.1. Riesz distribution. Let \( x \in \mathbb{R}^n \). The classical **Riesz distribution** [Rie49, GS64] is defined by

\[
r^{\lambda}(x) \overset{\text{def}}{=} (x_1^2 + \ldots + x_n^2)^{\frac{\lambda}{2}} = |x|^\lambda,
\]

where \( \lambda \in \mathbb{C} \). It is an analytic function in the complex half-plane \( \Re(\lambda) > -n \). Due to the Bernstein-Sato identity

\[
\Delta r^{\lambda+2}(x) = (\lambda + 2)(\lambda + n)r^{\lambda}(x),
\]

where \( \Delta = \sum_{k=1}^{n} \partial_{x_k}^2 \), the meromorphic continuation (with simple poles at \( \lambda = -n - 2k \) for \( k \in \mathbb{N}_0 \)) of \( r^{\lambda}(x) \) to \( \lambda \in \mathbb{C} \) follows. Let us introduce a meromorphic function

\[
c_\lambda \overset{\text{def}}{=} 2^{\lambda+n}\pi^{\frac{\lambda}{2}}\Gamma\left(\frac{\lambda + n}{2}\right)\Gamma\left(-\frac{\lambda}{2}\right)^{-1},
\]

and the standard notation for the Pochhammer symbol

\[(a)_n \overset{\text{def}}{=} a \cdot (a + 1) \cdot \ldots \cdot (a + n - 1)\]

for \( n \in \mathbb{N} \) and \( a \in \mathbb{C} \). Then a classical result states

**Proposition 2.1.** The Fourier transformation of \( r^{\lambda}(x) \) is given by

\[
F(r^{\lambda})(\xi) = c_\lambda r^{-\lambda-n}(\xi).
\]
Based on the Bernstein-Sato identity \((2.3)\) and a knowledge of the residue for \(r^\lambda(x)\) at \(\lambda = -n\), see [GS64], we get immediately

**Corollary 2.2** The residue of \(r^\lambda(x)\) at \(\lambda = -n-2k\) for \(k \in \mathbb{N}_0\) is given by

\[
\text{Res}_{\lambda=-n-2k}(r^\lambda(x)) = \frac{2\pi^{\frac{n}{2}}}{4^k k! \Gamma\left(\frac{n}{2}\right)} \Delta^k \delta_0(x).
\]

Consequently, the residues of \(r^\lambda(x)\) are related to GJMS operators \(P_{2N} = \Delta^N\) on \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\) for any \(N \in \mathbb{N}_0\).

### 2.2. Riesz distribution for spinors

We proceed with the Riesz distribution for spinors on \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\), see [CO14]. We write \(S_n^\pm\) for the irreducible half-spin representations for even \(n\) and \(S_n\) for the irreducible spin representation in the case of odd \(n\). Then it holds that \(S_n^\pm \simeq S_{n-1}^\pm\) for even \(n\), while \(S_n \simeq \bigoplus_{\lambda \in \mathbb{Z}} S_{\lambda-1}^+ \oplus S_{\lambda-1}^-\) for odd \(n\). Let \(\Sigma_n\) be the spinor bundle of \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\) associated to the spin representation \(S_n\) for odd \(n\), respectively \(S_n^\pm\) for even \(n\). The Clifford multiplication \(\cdot\) is normalized by \(x \cdot y + y \cdot x = -2\langle x, y \rangle\) for \(x, y \in \mathbb{R}^n\). The action of the Dirac operator on spinor fields \(\varphi \in \Gamma(\Sigma_n)\) is locally, with respect to the standard basis \(\{e_k\}\) of \(\mathbb{R}^n\), given by

\[
\mathcal{D}\varphi = \sum_{k=1}^{n} e_k \cdot \partial_k \varphi.
\]

We use the identification of a point \(x \in \mathbb{R}^n\) with a vector in \(\mathbb{R}^n\), and define the \(\text{End}(\Sigma_n)\)-valued distribution called the **Riesz distribution for spinors**:

\[
f^{\lambda}(x) \overset{\text{def}}{=} r^{\lambda-1}(x)x \cdot r^{\lambda}(x) \frac{x}{|x|}.
\]

In the region \(\Re(\lambda) > -n\), \((2.5)\) is an analytic function and satisfies the Bernstein-Sato identity

\[
\Delta f^{\lambda+2}(x) = (\lambda + 1)(\lambda + n + 1)f^{\lambda}(x).
\]

This follows from \(\mathcal{D} f^{\lambda}(x) = -(\lambda + n - 1)r^{\lambda-1}(x)\), \(\mathcal{D} r^{\lambda-1}(x) = (\lambda - 1)r^{\lambda-2}(x)\) and \(\mathcal{D}^2 = -\Delta\). In turn, the equation \((2.6)\) implies meromorphic continuation of \(f^{\lambda}(x)\) to the complex plane \(\mathbb{C}\) with simple poles at \(\lambda = -n-1-2k\) for \(k \in \mathbb{N}_0\).

The Fourier transform preserves the family of Riesz distributions for spinors.

**Proposition 2.3** The Fourier transform of \(f^{\lambda}(x)\) is given by

\[
F(f^{\lambda})(\xi) = \check{f}_\lambda r^{-\lambda-n}(\xi),
\]

where \(\check{f}_\lambda \overset{\text{def}}{=} -i\frac{\epsilon_{\lambda+1}}{\lambda+1}\).

Since we could not find its proof in the literature, we shall supply it here.

**Proof.** Starting on the right side of \((2.7)\) and using Proposition 2.1 together with the fact that \(\xi \cdot F(\varphi)(\xi) = iF(\mathcal{D}\varphi)(\xi)\), we compute

\[
F(r^{-\lambda-n}(\xi))F(\varphi)(\xi) = r^{-\lambda-n-1}(\xi)\xi \cdot F(\varphi)(\xi)
= i(\epsilon_{\lambda+1})^{-1}F(r^{\lambda+1})(\xi)F(\mathcal{D}\varphi)
\]
\[ = i(c_{\lambda+1})^{-1}F(\int_{\mathbb{R}^n} r^{\lambda+1}(x - y) \slashed{D} \varphi dy). \]

We choose a scalar product on \( \langle \cdot, \cdot \rangle_{\Sigma_n} \) on spinors, and also a constant spinor \( \phi \). Then we have

\[
\langle \phi, r^{\lambda+1}(x - y) \slashed{D} \varphi \rangle_{\Sigma_n} = \langle \phi, (\lambda + 1) r^{\lambda-1}(x - y) \sum_{j=1}^n (x_j - y_j) e_j \cdot \varphi \rangle_{\Sigma_n},
\]

and therefore

\[
F(\int_{\mathbb{R}^n} r^{\lambda+1}(x - y) \slashed{D} \varphi dy) = (\lambda + 1) F(\int_{\mathbb{R}^n} r^{\lambda-1}(x - y) \sum_{j=1}^n (x_j - y_j) e_j \cdot \varphi dy).
\]

The proof is complete. \( \square \)

Finally, we recall the residues of \( r^\lambda(x) \), see [CØ14, Proposition 6.3], which correspond to (odd) conformal powers of the Dirac operator \( \slashed{D}_{2N+1} = \slashed{D}_{2N+1}^2 \) on \( (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \) for any \( N \in \mathbb{N}_0 \).

**Proposition 2.4** The residue of \( r^\lambda(x) \) at \( \lambda = -n - 2k - 1 \), for \( k \in \mathbb{N} \), is given by

\[
\text{Res}_{\lambda=-n-1-2k}(r^\lambda(x)) = \frac{2\pi^n}{4k! \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{2}{2}\right)} \slashed{D}^{2k+1} \delta_0(x).
\]

### 2.3. Riesz distribution for differential forms

We consider differential forms on \( \mathbb{R}^n \). As in the previous section a point \( x \in \mathbb{R}^n \) is also regarded as a vector. The inner and exterior products with respect to the vector \( x \) are denoted by

\[
i_x \overset{\text{def}}{=} \sum_{k=1}^n x_k i_{e_k}, \quad \varepsilon_x \overset{\text{def}}{=} \sum_{k=1}^n x_k \varepsilon_{e_k},
\]

respectively. The exterior differential, its co-differential and the form Laplacian act on differential \( p \)-forms \( \Omega^p(\mathbb{R}^n) \) by

\[
d \overset{\text{def}}{=} \sum_{k=1}^n \varepsilon_{e_k} \partial_k, \quad \delta \overset{\text{def}}{=} -\sum_{k=1}^n i_{e_k} \partial_k, \quad \Delta_p \overset{\text{def}}{=} d\delta + \delta d = -\Delta,
\]

while similar operators on \( \Omega^p(\mathbb{R}^{n-1}) \) are denoted by \( d' \), \( \delta' \) and \( \Delta'_p \), respectively.

Now, the Riesz distribution on differential forms [FO17] is defined by

\[
R^\lambda_p(x) \overset{\text{def}}{=} r^{\lambda-2}(x) (i_x \varepsilon_x - \varepsilon_x i_x).
\]

In the region \( \Re(\lambda) > -n \) of \( \mathbb{C} \) it is an analytic function and satisfies the following Bernstein-Sato identity

\[
\left[(\lambda + 2n - 2p)(\lambda + 2p - 2)\delta d + (\lambda + 2p)(\lambda + 2n - 2p - 2)d\delta\right] R^\lambda_p(x) = -(\lambda - 2)(\lambda + n - 2)(\lambda + 2p)(\lambda + 2n - 2p)R^{\lambda-2}_p(x).
\]
This implies the meromorphic continuation of $R_p^\lambda(x)$ to $\lambda \in \mathbb{C}$ with simple poles at $\lambda = -n - 2k$ for $k \in \mathbb{N}_0$. We shall introduce
\begin{equation}
\alpha_\lambda \overset{\text{def}}{=} \frac{n}{2} - p + \lambda, \quad \beta_\lambda \overset{\text{def}}{=} \frac{n}{2} - p - \lambda \quad \text{for} \quad \lambda \in \mathbb{C},
\end{equation}
which are related to Branson-Gover operators $L_{2N}^{(p)} = \alpha_N(\delta d)^N + \beta_N(d\delta)^N$ on $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ for any $N \in \mathbb{N}$.

**Proposition 2.5** The Fourier transform of $R_p^\lambda(x)$ is given by
\[ \mathcal{F}(R_p^\lambda)(\xi) = \overline{c}_\lambda r^{-\lambda-n-2}(\xi)(\alpha_\lambda \frac{n-\alpha}{2} i\xi x + \beta_\lambda \frac{n-\beta}{2} \epsilon x) \]
where $\overline{c}_\lambda \overset{\text{def}}{=} (\lambda - 1)(\lambda - 2)c\lambda$.

Finally, we recall that the residues of $R_p^\lambda(x)$ correspond to the Branson-Gover operators on $\mathbb{R}^n$.

**Proposition 2.6** Let $k \in \mathbb{N}_0$. Then the residue of $R_p^\lambda(x)$ at $\lambda = -n - 2k$ is given by
\[ \text{Res}_{\lambda=-n-2k}(R_p^\lambda(x)) = \frac{(-1)^k 2n^{2k}}{4^k k!} \Gamma(\frac{\alpha_\lambda}{2} + k + 1) \text{Id}^{(\lambda-n-2k)} \delta_0(x). \]

### 3. Bernstein-Sato identity and operator

In the present section we shall prove some Bernstein-Sato identities for distribution kernels associated to conformal symmetry breaking operators [KS15, MØ17, K17]. By an abuse of notation, we introduce these distribution kernels as adjoints to those appearing in the references. The main impact of this choice is that taking Fourier transform of these distribution kernels leads to a direct contact (without any further dualisation) with a generalized version of singular vectors studied in [KÖSS15, FJS16, KKP16].

#### 3.1. Bernstein-Sato identity and operator in the scalar case

In this section we prove Bernstein-Sato identity for the distribution kernels associated to conformal symmetry breaking operators acting on functions:
\begin{align}
K_{\lambda,\nu}^+(x', x_n) & \overset{\text{def}}{=} |x_n|^\lambda + \nu - n (|x'|^2 + x_n^2)^{-\nu}, \\
K_{\lambda,\nu}^-(x', x_n) & \overset{\text{def}}{=} \text{sgn}(x_n)|x_n|^\lambda + \nu - n (|x'|^2 + x_n^2)^{-\nu} = x_n K_{\lambda-1,\nu}^+(x', x_n). \tag{3.1}
\end{align}

For a detailed analysis of their meromorphic behavior with respect to $(\lambda, \nu) \in \mathbb{C}^2$, see [KS15, MØ17].

A method of finding Bernstein-Sato operators, which we follow and which we briefly recall, is based on the discussion in [C16, C17]. The Knapp-Stein intertwining operator for conformal Lie group, acting on density bundle induced from the character $\gamma$, is given by
\[ (I_{\gamma} f)(x) \overset{\text{def}}{=} (r^{-2\gamma} \ast f)(x) = \int_{\mathbb{R}^n} r^{-2\gamma}(x - y)f(y)dy , \tag{3.2} \]
where $f \in \mathcal{S}(\mathbb{R}^n)$. It follows from Proposition 2.1 that
\[ I_{n-\lambda} \circ I_{\lambda} = c_{2\lambda-2n}c_{-2\lambda} \text{Id}. \]
Furthermore, we define the multiplication operator

\[(M_{x_n} f)(x) \overset{\text{def}}{=} x_n f(x). \quad (3.3)\]

**Remark 3.1** We note that both \(I_\gamma\) and \(M_{x_n}\) are intertwining operators for the conformal Lie groups on \(\mathbb{R}^n\) and \(\mathbb{R}^{n-1}\), respectively. For more details we refer to [C17].

Now we define the operator

\[D(\lambda) \overset{\text{def}}{=} I_{\lambda+1} \circ M_{x_n} \circ I_{n-\lambda} \quad (3.4)\]

The next statement is remarkable due to the fact that \((3.4)\) is a composition of pseudo-differential operators, cf. Clerc [C17].

**Proposition 3.2** The operator \(D(\lambda)\) in \((3.4)\) is a differential operator of order 2, i.e.,

\[D(\lambda)f = -\tilde{c}_\lambda \left[ (2\lambda - n) \partial_n f + \Delta(x_n \cdot f) \right] \]

for any \(f \in S(\mathbb{R}^n)\) and the multiple \(\tilde{c}_\lambda \overset{\text{def}}{=} c_{-2\lambda-2c_{2\lambda-2n}}\) (cf., \((2.4)\)).

We recall its proof.

**Proof.** In the Fourier image, we compute

\[\mathcal{F}(D(\lambda)f)(\xi) = \mathcal{F}(I_{\lambda+1} \circ M_{x_n} \circ I_{n-\lambda} f)(\xi) = -i\tilde{c}_{-2\lambda-2c_{2\lambda-2n}} r^{2\lambda+2-n}(\xi) \partial_n \left[ r^{n-2\lambda}(\xi) \mathcal{F}(f)(\xi) \right].\]

The identity

\[\partial_n r^{n-2\lambda}(\xi) = (n - 2\lambda) \xi_n r^{n-2\lambda-2}(\xi)\]

then implies

\[\mathcal{F}(D(\lambda)f)(\xi) = -i\tilde{c}_\lambda r^{2\lambda+2-n}(\xi) \left[ (n - 2\lambda) \xi_n r^{n-2\lambda-2}(\xi) + r^{n-2\lambda}(\xi) \partial_n \right] \mathcal{F}(f)(\xi) = \tilde{c}_\lambda \mathcal{F}((n - 2\lambda) \partial_n f - \Delta(x_n \cdot f))(\xi),\]

which completes the proof. \(\square\)

By virtue of Proposition 3.2, we define the second order differential operator on tempered distributions (notice the shift of the parameter \(\lambda\))

\[P(\lambda) : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n) \quad f \mapsto \Delta(x_n \cdot f) + (n - 2\lambda) \partial_n f.\]

Note that by Leibniz’s rule we have

\[P(\lambda) = x_n \Delta - (2\lambda - n - 2) \partial_n. \quad (3.5)\]

**Remark 3.3** The operator \(P(\lambda)\), acting on tempered distributions on \(\mathbb{R}^n\), is an intertwining differential operator for the conformal Lie group on \(\mathbb{R}^{n-1}\), cf. Remark 3.1. The same holds for its iterations used in later sections.
By the identities in [GS64],
\[ \partial_n(|x_n|^\lambda) = \lambda \text{sgn}(x_n)|x_n|^{\lambda-1}, \]
\[ \partial_n(\text{sgn}(x_n)|x_n|^\lambda) = \lambda|x_n|^{\lambda-1}, \]
a straightforward computation reveals the following result.

**Lemma 3.4** The distributions \( K_{\lambda,\nu}^\pm(x', x_n) \) satisfy

1. \[ x_n K_{\lambda,\nu}^\pm(x', x_n) = K_{\lambda+1,\nu}^\pm(x', x_n), \]  
2. \[ \partial_n(K_{\lambda,\nu}^\pm(x', x_n)) = (\lambda + \nu - n)K_{\lambda-1,\nu}^\pm(x', x_n) - 2\nu K_{\lambda,\nu+1}^\pm(x', x_n), \]
3. \[ \partial_i(K_{\lambda,\nu}^\pm(x', x_n)) = -2\nu x_i K_{\lambda-1,\nu+1}^\pm(x', x_n) \quad 1 \leq i \leq n - 1, \]
4. \[ \Delta(K_{\lambda,\nu}^\pm(x', x_n)) = (\lambda + \nu - n - 1)2K_{\lambda-2,\nu}^\pm(x', x_n) - 2\nu(2\lambda - n - 2)K_{\lambda-1,\nu+1}^\pm(x', x_n). \]

From this Lemma we may conclude

**Theorem 3.5** The operator \( P(\lambda) \) is a spectral shift operator for distribution kernels \( K_{\lambda,\nu}^\pm(x', x_n) \), i.e.,

\[ P(\lambda)K_{\lambda,\nu}^\pm(x', x_n) = (\lambda + \nu - n)(\nu - \lambda + 1)K_{\lambda-1,\nu}^\pm(x', x_n), \quad (3.6) \]

and

\[ P\left(\frac{\lambda + \nu + 1}{2}\right)K_{\lambda,\nu}^\pm(x', x_n) = 2\nu(\nu - \lambda + 1)K_{\lambda,\nu+1}^\pm(x', x_n). \quad (3.7) \]

**Proof.** From Lemma 3.4 we obtain

\[ (n - 2\lambda')\partial_n(K_{\lambda,\nu}^\pm(x', x_n)) = (n - 2\lambda')(\lambda + \nu - n)K_{\lambda-1,\nu}^\pm(x', x_n) - 2\nu(n - 2\lambda')K_{\lambda,\nu+1}^\pm(x', x_n), \]
\[ \Delta(x_n K_{\lambda,\nu}^\pm(x', x_n)) = (\lambda + \nu - n)K_{\lambda-1,\nu}^\pm(x', x_n) - 2\nu(2\lambda - n)K_{\lambda,\nu+1}^\pm(x', x_n), \]

hence

\[ P(\lambda')K_{\lambda,\nu}^\pm(x', x_n) = (\lambda + \nu - n)(-2\lambda' + \lambda + \nu + 1)K_{\lambda-1,\nu}^\pm(x', x_n) \]
\[ - 2\nu(2\lambda + 2\lambda' - 2n)K_{\lambda,\nu+1}^\pm(x', x_n). \]

Then for \( \lambda' \overset{\text{def}}{=} \lambda \) we conclude

\[ P(\lambda)K_{\lambda,\nu}^\pm(x', x_n) = (\lambda + \nu - n)(\nu - \lambda + 1)K_{\lambda-1,\nu}^\pm(x', x_n), \]

while for \( \lambda' \overset{\text{def}}{=} \frac{\lambda + \nu + 1}{2} \) we get

\[ P\left(\frac{\lambda + \nu + 1}{2}\right)K_{\lambda,\nu}^\pm(x', x_n) = 2\nu(\nu - \lambda + 1)K_{\lambda,\nu+1}^\pm(x', x_n). \]
The proof is complete. \(\Box\)

**Remark 3.6** Assuming the coefficients \(A, B\) by \(\partial_n\) and \(\Delta(x_n)\) in the formula \((3.5)\) are not known, i.e., \(\tilde{P}(\lambda) \overset{\text{def}}{=} A\partial_n + B\Delta(x_n)\). Then the system of equations
\[
(\lambda + \nu - n)A + (\lambda + \nu - n)_2B = (\lambda + \nu - n)(\nu - \lambda + 1),
-2\nu A - 2\nu(2\lambda - n)B = 0,
\]
which is equivalent to \(\tilde{P}(\lambda)K_{\lambda,\nu}^\pm(x', x_n) = (\lambda + \nu - n)(\nu - \lambda + 1)K_{\lambda-1,\nu}^\pm(x', x_n)\), has a unique solution given by
\[
A \overset{\text{def}}{=} n - 2\lambda, \quad B \overset{\text{def}}{=} 1.
\]
This agrees with Theorem 3.3, i.e., \(\tilde{P}(\lambda) = P(\lambda)\).

**Remark 3.7** We notice that \(K_{\lambda,\nu}^\pm(x', x_n)\) generalizes the Riesz distribution \(r^\lambda(x)\), \((2.2)\), as follows. Once we set \(\lambda \overset{\text{def}}{=} \frac{\mu}{2} + n\) and \(\nu \overset{\text{def}}{=} -\frac{\mu}{2}\), for \(\mu \in \mathbb{C}\) such that \(\Re(\mu) > -n\), we get
\[
K_{\frac{\mu}{2} + n, -\frac{\mu}{2}}^\pm(x', x_n) = |x_n|^\frac{\mu}{2} + n - \frac{\mu}{2} - n(|x'|^2 + x_n^2)^{\frac{\mu}{2}} = r^\mu(x).
\]
Then Theorem (3.7) implies a distributional identity
\[
P(\frac{\mu}{2} + n)r^\mu(x) = 0,
\]
which is equivalent in the light of \(P(\frac{\mu}{2} + n) = -(\mu + n - 2)\partial_n + x_n\Delta\) and \(\partial_n(r^\mu(x)) = \mu x_n r^{\mu-2}(x)\) to
\[
x_n(\Delta(r^\mu(x)) - \mu(\mu + n - 2)r^{\mu-2}(x)) = 0.
\]
Hence we recover the Bernstein-Sato identity \((2.3)\) for the Riesz distribution \(r^\mu(x)\), i.e.,
\[
\Delta(r^\mu(x)) = \mu(\mu + n - 2)r^{\mu-2}(x).
\]

### 3.2. Bernstein-Sato identity and operator in the spinor case

In the present section we prove a Bernstein-Sato identity for distribution kernels associated to conformal symmetry breaking operators acting on spinors:
\[
K_{\lambda,\nu}^\pm(x', x_n) \overset{\text{def}}{=} K_{\lambda - \frac{1}{2}, \nu + \frac{1}{2}}^\pm(x', x_n)x_n.
\]
Similarly to the scalar case, we introduce a Bernstein-Sato operator for \(K_{\lambda,\nu}^\pm(x', x_n)\). First, we recall the Knapp-Stein intertwining operator in the non-compact realization of the induced representation of conformal Lie group on spinors [CO14]:
\[
(I_\gamma \varphi)(x) \overset{\text{def}}{=} (y^{-2\gamma} \star \varphi)(x) = \int_{\mathbb{R}^n} y^{-2\gamma}(x - y)\varphi(y)dy
\]
where \(\varphi \in \mathcal{S}(\mathbb{R}^n)\). By Proposition 2.3 it follows
\[
I_{n-\lambda} \circ I_\lambda = \mathcal{F}_{2\lambda-2n} \mathcal{F}_{-2\lambda} \text{Id},
\]
and as in the scalar case we define the operator
\[ \mathcal{D}(\lambda) \overset{\text{def}}{=} F_{\lambda+1} \circ M_{x_n} \circ F_{n-\lambda} \]  
(3.10)
with \( M_{x_n} \) acting by the scalar multiplication.

**Theorem 3.8** The operator \( \mathcal{D}(\lambda) \) in (3.10) is a differential operator of order 2, i.e.,
\[ \mathcal{D}(\lambda) \varphi = \tilde{\mathcal{D}}_{\lambda} \{ (2\lambda - n + 1)\partial_n \varphi + \mathcal{D}(e_n \cdot \varphi) + \Delta(x_n \varphi) \} , \]  
(3.11)
where \( \varphi \in S(\mathbb{R}^n) \) and \( \tilde{\mathcal{D}}_{\lambda} \) is a differential operator of order\( 2 \).

**Proof.** As in the scalar case, we need to understand the right hand side of
\[ \mathcal{F}(\mathcal{D}(\lambda) \varphi) = -i \tilde{\mathcal{D}}_{2\lambda-2} \tilde{\mathcal{D}}_{2\lambda-2} t^{2\lambda-n+2} (\xi) \partial_n [ t^{n-2\lambda} (\xi) \mathcal{F}(\varphi)(\xi) ] . \]
By
\[ \partial_n (t^{n-2\lambda}) (\xi) = (n - 2\lambda - 1) \xi_n t^{n-2\lambda-3} (\xi) \xi_n + r^{n-2\lambda-1} (\xi) e_n , \]
it equals to
\[ -i \tilde{\mathcal{D}}_{2\lambda-2} \tilde{\mathcal{D}}_{2\lambda-2} [(n - 2\lambda - 1) \xi_n r^{-2} (\xi) \xi_n + \mathcal{F}(\varphi) + \xi_n \mathcal{F}(e_n \cdot \varphi) + \xi_n \partial_n \mathcal{F}(\varphi)] \]
\[ = -i \tilde{\mathcal{D}}_{2\lambda-2} \tilde{\mathcal{D}}_{2\lambda-2} [(n - 2\lambda - 1) \xi_n r^{-2} (\xi) |\xi|^2 \mathcal{F}(\varphi) + \xi_n \mathcal{F}(e_n \cdot \varphi) - |\xi|^2 \partial_n \mathcal{F}(\varphi)] \]
\[ = \tilde{\mathcal{D}}_{2\lambda-2} \tilde{\mathcal{D}}_{2\lambda-2} \mathcal{F} ((2\lambda - n + 1) \partial_n \varphi + \mathcal{D}(e_n \cdot \varphi) + \Delta(x_n \varphi)) \]
and the proof is complete. \( \square \)

Inspired by the previous Theorem we define, by a shift of the parameter \( \lambda \), the operator
\[ \check{P}(\lambda) : S' (\mathbb{R}^n) \to S'(\mathbb{R}^n) \]
\[ \varphi \mapsto (n - 2\lambda + 1) \partial_n \varphi + \check{D}(e_n \cdot \varphi) + \Delta(x_n \varphi) . \]  
(3.12)

**Remark 3.9** Using the identity
\[ \check{D}(e_n) = -e_n \cdot \check{D}' - \partial_n , \]
we can write
\[ \check{P}(\lambda) \varphi = P(\lambda) \varphi - e_n \cdot \check{D}' \varphi . \]
Here \( \check{D}' \) is the tangential Dirac operator and \( P(\lambda) \) is the scalar Bernstein-Sato operator, see (3.5).

**Remark 3.10** Similarly to the scalar case, the operator \( \check{P}(\lambda) \) is an intertwining differential operator for the conformal Lie group on \( \mathbb{R}^{n-1} \), and so is true for its iterations used in later sections.

Now we collect a few basic properties of the distribution kernels \( K_{\lambda,\nu}^\pm (x', x_n) \) with respect to certain algebraic and differential actions.
Lemma 3.11  The distribution kernels $K_{\lambda,\nu}^\pm(x', x_n)$ satisfy the following algebraic and differential identities:

\begin{align}
(1) & \quad x_n K_{\lambda,\nu}^\pm(x', x_n) = K_{\lambda+1,\nu}^\pm(x', x_n), \\
(2) & \quad \partial_n(K_{\lambda,\nu}^\pm(x', x_n)) = (\lambda + \nu - n)K_{\lambda-1,\nu}^\pm(x', x_n) - 2(\nu + \frac{1}{2})K_{\lambda,\nu+1}^\pm(x', x_n) \\
& \quad \quad + K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n)e_n, \\
(3) & \quad \mathcal{D}(K_{\lambda,\nu}^\pm(x', x_n)) = 2(\nu + \frac{1}{2})(\lambda - n - 1)K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n) \\
& \quad \quad + (\lambda + \nu - n)e_n \cdot K_{\lambda-1,\nu}^\pm(x', x_n), \\
(4) & \quad \Delta(K_{\lambda,\nu}^\pm(x', x_n)) = (\lambda + \nu - n - 1)2K_{\lambda-2,\nu}^\pm(x', x_n) \\
& \quad \quad - 2(\nu + \frac{1}{2})(2\lambda - n - 1)K_{\lambda-1,\nu+1}^\pm(x', x_n) \\
& \quad \quad + 2(\lambda + \nu - n)K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n)e_n. 
\end{align}

Proof. The proof is based on Lemma 3.3 and the identities

\[ |x|^2 = -x \cdot x, \quad e_k \cdot x = -x \cdot e_k \cdot -2x_k, \quad \partial_k(x) = e_k, \quad k = 1, \ldots, n, \]

with $x = \sum_{k=1}^{n} x_k e_k$.

To be more concrete, the first result is obvious by definition of $K_{\lambda,\nu}^\pm(x', x_n)$. The remaining claim relies on Lemma 3.3 and the Leibniz-rule. For example, we have

\[
\partial_n(K_{\lambda,\nu}^\pm(x', x_n)) = \partial_n(K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n))x \cdot + K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n)\partial_n(x).
\]

\[
= (\lambda + \nu - n)K_{\lambda-1,\nu}^\pm(x', x_n) - 2(\nu + \frac{1}{2})K_{\lambda,\nu+1}^\pm(x', x_n) \\
+ K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n)e_n, 
\]

while for $1 \leq k \leq n - 1$ we get

\[
\partial_k(K_{\lambda,\nu}^\pm(x', x_n)) = \partial_k(K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n))x \cdot + K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n)\partial_k(x).
\]

\[
= -2(\nu + \frac{1}{2})x_k K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n)x \cdot + K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n)e_k. \\
= -2(\nu + \frac{1}{2})x_k K_{\lambda-1,\nu+1}(x', x_n) + K_{\lambda-\frac{1}{2},\nu+\frac{1}{2}}^\pm(x', x_n)e_k. 
\]

The remaining assertions then follow easily. \qed
Consequently, the previous Lemma implies the following Bernstein-Sato identity for the distribution kernels \( K_{\lambda,\nu}(x', x_n) \).

**Theorem 3.12**  The operator \( \mathcal{P}(\lambda) \) is a spectral shift operator for the distribution kernels \( K_{\lambda,\nu}(x', x_n) \), i.e.,

\[
\mathcal{P}(\lambda) K_{\lambda,\nu}(x', x_n) = (\lambda + \nu - n)(\nu - \lambda + 1) K_{\lambda-1,\nu}(x', x_n).
\]  

(3.17)

**Proof.** The proof is based on Lemma 3.11 and the identity

\[
\partial_n (e_n \cdot \phi) = -e_n \cdot \partial_n \phi - 2 \partial_n \phi.
\]

A straightforward computation shows

\[
\mathcal{P}(\lambda) K_{\lambda,\nu}(x', x_n) = (\lambda + \nu - n)(\nu - \lambda + 1) K_{\lambda-1,\nu}(x', x_n).
\]

The proof is complete. \( \square \)

**Remark 3.13**  Regarding the coefficients in equation (3.12) by \( \partial_n, \mathcal{D}(e_n \cdot) \) and \( \Delta(x_n \cdot) \) as unknown, the ansatz for the operator \( \tilde{\mathcal{P}}(\lambda) \)

\[
\tilde{\mathcal{P}}(\lambda) \overset{\text{def}}{=} A \partial_n + B \mathcal{D}(e_n \cdot) + C \Delta(x_n \cdot)
\]

leads to the system of equations

\[
(\lambda + \nu - n)A - (\lambda + \nu - n)B + (\lambda + \nu - n)C = (\lambda + \nu - n)(\nu - \lambda + 1),
\]

\[-2(\nu + \frac{1}{2})A + 4(\nu + \frac{1}{2})B - 2(\nu + \frac{1}{2})(2\nu - n + 1)C = 0,
\]

\[A - 2(\nu - \frac{n - 3}{2})B + 2(\lambda + \nu - n + 1)C = 0,
\]

equivalent to \( \tilde{\mathcal{P}}(\lambda) K_{\lambda,\nu}(x', x_n) = (\lambda + \nu - n)(\nu - \lambda + 1) K_{\lambda-1,\nu}(x', x_n) \). The unique solution of this system is given by

\[
A \overset{\text{def}}{=} n - 2\lambda + 1, \quad B \overset{\text{def}}{=} 1, \quad C \overset{\text{def}}{=} 1,
\]

which agrees with Theorem 3.12, i.e., \( \tilde{\mathcal{P}}(\lambda) = \mathcal{P}(\lambda) \).

**Remark 3.14**  The distribution kernels \( K_{\lambda,\nu}(x', x_n) \) generalize the Riesz distribution \( \mathcal{R}_\lambda(x) \), see (2.5), in the sense that for \( \mu \in \mathbb{C} \) with \( \Re(\mu) > -n \) it holds

\[
K_{\frac{\mu}{2}+n,-\frac{\mu}{2}}(x', x_n) = \mathcal{R}_\mu(x).
\]  

(3.18)

The last theorem implies that

\[
\mathcal{P}(\frac{\mu}{2} + n) \mathcal{R}_\mu(x) = 0
\]

as a distributional identity, which is equivalent to

\[-(\mu + n - 1) \partial_n(\mathcal{R}_\mu(x)) - e_n \mathcal{D}(\mathcal{R}_\mu(x)) + x_n \Delta(\mathcal{R}_\mu(x)) = 0.
\]
By
\[\partial_n(\tilde{\varphi}^\mu(x)) = (\mu - 1)\tilde{\varphi}^{\mu-2}(x) + r^{\mu-1}(x)e_n,\]
\[\mathcal{P}(\tilde{\varphi}^\mu(x)) = - (\mu + n - 1)r^{\mu-1}(x),\]
we obtain
\[\Delta(\tilde{\varphi}^\mu(x)) = (\mu - 1)(\mu + n - 1)\tilde{\varphi}^{\mu-2}(x).\]
This is the Bernstein-Sato identity for \(\tilde{\varphi}^\mu(x)\), see (2.6). Independently this follows from \(\mathcal{P}(r^{\lambda-1}(x)) = (\lambda - 1)\tilde{\varphi}^{\lambda-2}(x)\), a coupled Bernstein-Sato identity for scalars and spinors, and \(\mathcal{P}^2 = -\Delta\).

3.3. Bernstein-Sato identity and operator in the form case. In the present section we prove a Bernstein-Sato identity for distribution kernels associated to conformal symmetry breaking operators on differential forms:

\[K_{\lambda,\nu}^{(p),\pm}(x', x_n) \overset{\text{def}}{=} K_{\lambda-1,\nu+1}^{\pm}(x', x_n)(i_x \varepsilon_x - \varepsilon_x i_x)i_{\varepsilon_n} \varepsilon_{\varepsilon_n}, \quad (3.19)\]

Similarly to the scalar case, we introduce a Bernstein-Sato operator for \(K_{\lambda,\nu}^{(p),\pm}(x', x_n)\). First, we recall the Knapp-Stein intertwining operator in the non-compact realization of the induced representation of conformal Lie group on differential forms [FØ17]:

\[(I^p_{\lambda,\nu})(x) \overset{\text{def}}{=} (R_p^{-2\gamma} * \omega)(x) = \int_{\mathbb{R}^n} R_p^{-2\gamma}(x - y)\omega(y)dy.\]

By Proposition 2.5 we obtain
\[I^p_{n-\lambda} \circ I^p_{\lambda} = c_{2\lambda-2n}c_{-2\lambda}(\lambda - p)(n - p - \lambda)\text{ Id}.\]
As in the scalar case we define the operator
\[D^p(\lambda) \overset{\text{def}}{=} I^p_{\lambda+1} \circ M_{x_n} \circ I^p_{n-\lambda}, \quad (3.20)\]
with \(M_{x_n}\) acting by the scalar multiplication.

**Theorem 3.15** The operator \(D^p(\lambda)\) in (3.20) is a differential operator of order 2, i.e.,
\[D^p(\lambda)\omega = \tilde{c}_\lambda \left[ (2\lambda - n)(\lambda - p + 1)(\lambda - n + p + 1)\partial_n \omega \right. \]
\[\left. + (2\lambda - n)[(\lambda - p + 1)\delta(\varepsilon_n \omega) - (\lambda - n + p + 1)d(i_{\varepsilon_n} \omega)] \right. \]
\[\left. + [(\lambda - p + 1)(n - \lambda - p)\delta \delta + (\lambda - p)(n - \lambda - p - 1)d\delta](x_n \cdot \omega), \right] \]
where \(\omega \in \Omega^p(\mathbb{R}^n)\) and \(\tilde{c}_\lambda \overset{\text{def}}{=} c_{-2\lambda-2n}c_{2\lambda-2n}\).

**Proof.** In the Fourier image it follows that
\[\mathcal{F}(D^p(\lambda)\omega)(\xi) = \mathcal{F}(I^p_{\lambda+1} \circ M_{x_n} \circ I^p_{n-\lambda})(\xi) \]
\[= -i\tilde{c}_{-2\lambda-2n}c_{2\lambda-2n}r^{2\lambda-n}(\xi)(\alpha_{\lambda+1-\frac{n}{2}}i_\xi \varepsilon_\xi + \beta_{\lambda+1-\frac{n}{2}}\varepsilon_\xi i_\xi) \times \]
\[\times \partial_n \left[ r^{n-2\lambda-2}(\xi)(\alpha_{\frac{n}{2}-\lambda}i_\xi \varepsilon_\xi + \beta_{\frac{n}{2}-\lambda}\varepsilon_\xi i_\xi)\mathcal{F}(\omega)(\xi) \right]. \]
The identities
\[
\partial_n(r^{n-2\lambda-2}(\xi)) = (n - 2\lambda - 2)\xi_n r^{n-2\lambda-4}(\xi), \\
\partial_n(i\xi\varepsilon_i\xi) = i\varepsilon_n \varepsilon_i \xi + i\xi \varepsilon_i e_n = \xi_n - \varepsilon_i i e_n + i\xi \varepsilon_i e_n, \\
\partial_n(\varepsilon_i i e_i) = \varepsilon_{e_n} i \xi + \varepsilon_i i e_i = \xi_n - i\xi \varepsilon_i e_n + \varepsilon_i \xi e_n
\]
imply
\[
\mathcal{F}(D^p(\lambda)\omega)(\xi) = -i\tilde{\omega}_\lambda r^{2n-2}(\xi)(\alpha_{\lambda+1-\frac{n}{2}} \xi \varepsilon_i \xi \varepsilon_i + \beta_{\lambda+1-\frac{n}{2}} \xi \varepsilon_i \xi) \times \\
\times [(n - 2\lambda - 2)\xi_n r^{n-2\lambda-4}(\xi)(\alpha_{\frac{n}{2}-\lambda} \eta \varepsilon_i \xi \varepsilon_i + \beta_{\frac{n}{2}-\lambda} \xi \varepsilon_i \xi) \mathcal{F}(\omega)(\xi) \\
+ (\alpha_{\frac{n}{2}-\lambda} + \beta_{\frac{n}{2}-\lambda})\xi_n r^{n-2\lambda-2}(\xi) \mathcal{F}(\omega)(\xi) \\
+ (\beta_{\frac{n}{2}-\lambda} - \alpha_{\frac{n}{2}-\lambda})\xi_n r^{n-2\lambda-2}(\xi) (\varepsilon_i i e_n - i\xi \varepsilon_i e_n) \mathcal{F}(\omega)(\xi) \\
+ r^{n-2\lambda-2}(\xi)(\alpha_{\frac{n}{2}-\lambda} \xi \varepsilon_i \xi \varepsilon_i + \beta_{\frac{n}{2}-\lambda} \xi \varepsilon_i \xi) \partial_n \mathcal{F}(\omega)(\xi)].
\]
The substitution
\[
(\alpha_{\frac{n}{2}-\lambda} \xi \varepsilon_i \xi + \beta_{\frac{n}{2}-\lambda} \xi \varepsilon_i i \xi) = (\alpha_{\frac{n}{2}-\lambda} - \xi \varepsilon_i \xi \xi + \beta_{\frac{n}{2}-\lambda} - \xi \varepsilon_i i \xi \xi) + (i\xi \varepsilon_i - \xi \varepsilon_i i \xi)
\]
together with
\[
\alpha_{\lambda+1-\frac{n}{2}} \alpha_{\frac{n}{2}-\lambda} - 1 = \beta_{\lambda+1-\frac{n}{2}} \beta_{\frac{n}{2}-\lambda}, \\
i\xi \varepsilon_i + \varepsilon_i i \xi = |\xi|^2, \quad (i\xi)^2 = 0 = (\varepsilon_i)^2
\]
give
\[
\mathcal{F}(D^p(\lambda)\omega)(\xi) = -i\tilde{\omega}_\lambda [(n - 2\lambda - 2)\alpha_{\lambda+1-\frac{n}{2}} \alpha_{\frac{n}{2}-\lambda} \xi_n \mathcal{F}(\omega)(\xi) \\
+ (n - 2\lambda - 2)\xi_n r^{2n-2}(\xi)(\alpha_{\lambda+1-\frac{n}{2}} \xi \varepsilon_i \xi \varepsilon_i - \beta_{\lambda+1-\frac{n}{2}} \xi \varepsilon_i \xi) \mathcal{F}(\omega)(\xi) \\
+ (\alpha_{\frac{n}{2}-\lambda} + \beta_{\frac{n}{2}-\lambda})\xi_n r^{n-2\lambda-2}(\xi)(\alpha_{\lambda+1-\frac{n}{2}} \xi \varepsilon_i \xi \varepsilon_i + \beta_{\lambda+1-\frac{n}{2}} \xi \varepsilon_i \xi) \mathcal{F}(\omega)(\xi) \\
+ (\beta_{\frac{n}{2}-\lambda} - \alpha_{\frac{n}{2}-\lambda})\xi_n r^{n-2\lambda-2}(\xi)(\beta_{\lambda+1-\frac{n}{2}} \xi \varepsilon_i \xi \varepsilon_i \varepsilon e_n - \alpha_{\lambda+1-\frac{n}{2}} \xi \varepsilon_i \xi \varepsilon e_n) \mathcal{F}(\omega)(\xi) \\
+ (\alpha_{\lambda+1-\frac{n}{2}} \alpha_{\frac{n}{2}-\lambda} \xi \varepsilon_i \xi \xi + \beta_{\lambda+1-\frac{n}{2}} \beta_{\frac{n}{2}-\lambda} \xi \varepsilon_i \xi \xi) \partial_n \mathcal{F}(\omega)(\xi)].
\]
Now, recalling the explicit form of the coefficients \(\alpha_\lambda, \beta_\lambda,\) cf. \(2.11,\)
\[
\alpha_{\lambda+1-\frac{n}{2}} = (\lambda - p + 1), \quad \alpha_{\frac{n}{2}-\lambda} = (n - \lambda - p), \quad \alpha_{\frac{n}{2}-\lambda} = (n - \lambda - p - 1), \\
\beta_{\lambda+1-\frac{n}{2}} = (n - \lambda - p - 1), \quad \beta_{\frac{n}{2}-\lambda} = (\lambda - p), \quad \beta_{\frac{n}{2}-\lambda} = (\lambda - p + 1),
\]
allows to conclude
\[
(n - 2\lambda - 2)\alpha_{\lambda+1-\frac{n}{2}} + (\alpha_{\frac{n}{2}-\lambda} + \beta_{\frac{n}{2}-\lambda})\alpha_{\lambda+1-\frac{n}{2}} = 2(n - \lambda - p - 1)(\lambda - p + 1), \\
-(n - 2\lambda - 2)\beta_{\lambda+1-\frac{n}{2}} + (\alpha_{\frac{n}{2}-\lambda} + \beta_{\frac{n}{2}-\lambda})\beta_{\lambda+1-\frac{n}{2}} = 2(\lambda - p + 1)(n - \lambda - p - 1), \\
\varepsilon i \xi e \xi e i e_n = |\xi|^2 \xi i e_n, \quad \xi \varepsilon_i \xi \varepsilon_i e_n = |\xi|^2 i \xi e_n.
\]
This finally implies
\[
\mathcal{F}(D^p(\lambda)\omega) = \tilde{\omega}_\lambda \mathcal{F}((2\lambda - n)(\lambda - p + 1)(\lambda - n + p + 1) \partial_n \omega \\
+ (2\lambda - n)(\lambda - p + 1)\delta(\xi e_n \omega) - (\lambda - n + p + 1)d(i e_n \omega) \\
+ [(\lambda - p + 1)(n - \lambda - p)\delta d + (\lambda - p)(n - \lambda - p - 1)d\delta](x_n \cdot \omega),
\]
which completes the proof.
Let us renormalize the operator $D^p(\lambda)$ and shift the parameter $\lambda$ to $n - \lambda$:

$$P^p(\lambda) : S^{\prime,p}(\mathbb{R}^n) \rightarrow S^{\prime,p}(\mathbb{R}^n)$$

$$\omega \mapsto -(2\lambda - n)(\lambda - n + p - 1)(\lambda - p - 1)\partial_n \omega$$

$$+ (2\lambda - n)[(\lambda - n + p - 1)\delta(\varepsilon_{e_n}\omega) - (\lambda - p - 1)d(i_{e_n}\omega)]$$

$$- [(\lambda - n + p - 1)(\lambda - p)\delta d - (\lambda - n + p)(\lambda - p - 1)d\delta](x_n \cdot \omega)$$

Remark 3.16 The identities of the form

$$-\Delta = -\sum_{k=1}^n \partial_k^2 = d\delta + d\delta, \quad \delta \varepsilon_{e_n} = -\varepsilon_{e_n} \delta - \partial_n,$$

$$d\delta(x_n \cdot) = \varepsilon_{e_n} \delta - di_{e_n} + x_n d\delta, \quad \delta d(x_n \cdot) = -\varepsilon_{e_n} \delta + di_{e_n} - 2\partial_n + x_n d\delta,$$

allow to write

$$P^p(\lambda) = -(2\lambda - n - 2)(\lambda - n + p - 1)(\lambda - p)\partial_n$$

$$- (2\lambda - n - 2)[(\lambda - n + p)\varepsilon_{e_n}\delta + (\lambda - p)d\varepsilon_{e_n}]$$

$$- (\lambda - n + p - 1)(\lambda - p)x_n \delta d - (\lambda - n + p)(\lambda - p - 1)x_n d\delta.$$  \hspace{1cm} (3.22)

In terms of $P(\lambda)$, see (3.5), it holds

$$P^p(\lambda) = (\lambda - n + p - 1)(\lambda - p)P(\lambda) - (n - 2p)x_n d\delta$$

$$- (2\lambda - n - 2)[(\lambda - n + p)\varepsilon_{e_n}\delta + (\lambda - p)d\varepsilon_{e_n}].$$ \hspace{1cm} (3.23)

Remark 3.17 Similarly to the scalar case, the operator $P^p(\lambda)$ is an intertwining differential operator for the conformal Lie group on $\mathbb{R}^{n-1}$, and so is true for its iterations used in later sections.

Now we present some basic properties of $K_{\lambda,\nu}^{(p),\pm}(x', x_n)$. First note that

$$K_{\lambda,\nu}^{(p),\pm}(x', x_n) = K_{\lambda,\nu}^{(p),\pm}(x', x_n)i_{e_n} \varepsilon_{e_n} - 2K_{\lambda-1,\nu+1}^{\pm}(x', x_n)i_{e_n} \varepsilon_{e_n}.$$ \hspace{1cm} (3.24)

Lemma 3.18 The distribution kernels $K_{\lambda,\nu}^{(p),\pm}(x', x_n)$ satisfy

(1)

$$x_n K_{\lambda,\nu}^{(p),\pm}(x', x_n) = K_{\lambda+1,\nu}^{(p),\mp}(x', x_n),$$

(2)

$$\partial_n(K_{\lambda,\nu}^{(p),\pm}(x', x_n)) = (\lambda + \nu - n)K_{\lambda+1,\nu}^{(p),\mp}(x', x_n) - 2\nu K_{\lambda+1,\nu+1}^{\mp}(x', x_n)i_{e_n} \varepsilon_{e_n}$$

$$+ 4(\nu + 1)K_{\lambda-1,\nu+2}^{\mp}(x', x_n) \varepsilon_{e_n}i_{e_n} \varepsilon_{e_n} - 2K_{\lambda-1,\nu+1}^{\pm}(x', x_n)i_{e_n} \varepsilon_{e_n} \varepsilon_{e_n};$$

(3)

$$\varepsilon_{e_n}(K_{\lambda,\nu}^{(p),\pm}(x', x_n)) = 2(\lambda - p)K_{\lambda-1,\nu+1}^{\pm}(x', x_n)i_{e_n} \varepsilon_{e_n} \varepsilon_{e_n},$$

\varepsilon_{e_n}(K_{\lambda,\nu}^{(p),\pm}(x', x_n)) = 2(\lambda - p)K_{\lambda-1,\nu+1}^{\pm}(x', x_n)i_{e_n} \varepsilon_{e_n} \varepsilon_{e_n},$$
We start with some observations. First compute, using Lemma 3.4 and Leibniz’s rule, use Equation (3.24) and compute the differential actions on both summands separately. The first claim follows from definition (3.19). As for the remaining properties, we

Proof. The first claim follows from definition (3.19). As for the remaining properties, we use Equation (3.24) and compute the differential actions on both summands separately. We start with some observations. First compute, using Lemma 3.4 and Leibniz’s rule, the identities

\[
\partial_n(K^\pm_{\lambda,\nu}(x', x_n)) = (\lambda + \nu - n)K^\pm_{\lambda-1,\nu}(x', x_n)H - 2\nu K^\pm_{\lambda,\nu+1}(x', x_n)H + K^\pm_{\lambda,\nu}(x', x_n)\partial_n(H)
\]

\[
\partial_k(K^\pm_{\lambda,\nu}(x', x_n)) = -2\nu x_k K^\pm_{\lambda-1,\nu+1}(x', x_n)H + K^\pm_{\lambda,\nu}(x', x_n)\partial_k(H), \quad 1 \leq k \leq n - 1,
\]

and conclude

\[
\delta(K^\pm_{\lambda,\nu}(x', x_n)) = 2\nu K^\pm_{\lambda-1,\nu+1}(x', x_n)\varepsilon x - (\lambda + \nu - n)K^\pm_{\lambda-1,\nu}(x', x_n)i_{\varepsilon x}H + K^\pm_{\lambda,\nu}(x', x_n)\delta(H),
\]

\[
d(K^\pm_{\lambda,\nu}(x', x_n)) = -2\nu K^\pm_{\lambda-1,\nu+1}(x', x_n)\varepsilon x H + (\lambda + \nu - n)K^\pm_{\lambda-1,\nu}(x', x_n)\varepsilon_{\varepsilon x}H + K^\pm_{\lambda,\nu}(x', x_n)\delta(H),
\]

\[
\delta(x_{\varepsilon x}K^\pm_{\lambda,\nu}(x', x_n)) = - (n - p + E) K^\pm_{\lambda,\nu}(x', x_n)H - \varepsilon x \delta(K^\pm_{\lambda,\nu}(x', x_n)H),
\]

\[
\delta(x_{\varepsilon x}K^\pm_{\lambda,\nu}(x', x_n)) = - \partial_n(K^\pm_{\lambda,\nu}(x', x_n)H) - \varepsilon x \delta(K^\pm_{\lambda,\nu}(x', x_n)H),
\]

\[
d(i_{\varepsilon x}K^\pm_{\lambda,\nu}(x', x_n)) = (p + E) K^\pm_{\lambda,\nu}(x', x_n)H - i_{\varepsilon x}d(K^\pm_{\lambda,\nu}(x', x_n)H),
\]

\[
d(i_{\varepsilon x}K^\pm_{\lambda,\nu}(x', x_n)) = \partial_n(K^\pm_{\lambda,\nu}(x', x_n)H) - i_{\varepsilon x}d(K^\pm_{\lambda,\nu}(x', x_n)H)
\]

for some endomorphism \(H\) of differential forms. Here we denote by \(E \equiv \sum_{k=1}^{n} x_k \partial_k\) the Euler operator and close our observations with

\[
E(K^{(p),\pm}_{\lambda,\nu}(x', x_n)) = (\lambda - \nu - n)K^{(p),\pm}_{\lambda,\nu}(x', x_n).
\]
Now it is straightforward to compute
\[
\partial_n(K_{\lambda,\nu}^{\pm}(x', x_n) i_{en} e_{en}) = (\lambda + \nu - n)K_{\lambda-1,\nu}^{\pm}(x', x_n) i_{en} e_{en} - 2\nu K_{\lambda,\nu+1}^{\pm}(x', x_n) i_{en} e_{en},
\]
\[
-2\partial_n(K_{\lambda-1,\nu+1}^{\pm}(x', x_n) e_{x} i_{en} e_{en}) = -2(\lambda + \nu - n)K_{\lambda-2,\nu+1}^{\pm}(x', x_n) e_{x} i_{en} e_{en} \\
+ 4(\nu + 1)K_{\lambda-1,\nu+2}^{\pm}(x', x_n) e_{x} i_{en} e_{en} \\
- 2K_{\lambda-1,\nu+1}^{\pm}(x', x_n) e_{en} i_{en} e_{en},
\]
\[
\epsilon_{en} \delta(K_{\lambda,\nu}^{\pm}(x', x_n) i_{en} e_{en}) = 2\nu K_{\lambda-1,\nu+1}^{\pm}(x', x_n) e_{en} i_{en} e_{en},
\]
\[
-2\epsilon_{en} \delta(K_{\lambda-1,\nu+1}^{\pm}(x', x_n) e_{x} i_{en} e_{en}) = -(2(\nu + \lambda + p)K_{\lambda-1,\nu+1}^{\pm}(x', x_n) e_{en} i_{en} e_{en},
\]
\[
d(i_{en} K_{\lambda,\nu}^{\pm}(x', x_n) i_{en} e_{en}) = 0,
\]
\[
-2d(i_{en} K_{\lambda-1,\nu+1}^{\pm}(x', x_n) e_{x} i_{en} e_{en}) = 4(\nu + 1)K_{\lambda-1,\nu+2}^{\pm}(x', x_n) e_{x} i_{en} e_{en} \\
- 2(\lambda + \nu - n + 1)K_{\lambda-1,\nu+1}^{\pm}(x', x_n) e_{en} i_{en} e_{en} \\
- 2pK_{\lambda-1,\nu+1}^{\pm}(x', x_n) i_{en} e_{en},
\]
\[
d\delta(K_{\lambda,\nu}^{\pm}(x', x_n) i_{en} e_{en}) = -4(\nu) K_{\lambda-2,\nu+2}^{\pm}(x', x_n) e_{x} i_{en} e_{en} \\
+ 2\nu(\lambda + \nu - n)K_{\lambda-2,\nu+1}^{\pm}(x', x_n) e_{en} i_{en} e_{en} \\
+ 2\nu(\lambda - n - p - 2)K_{\lambda-1,\nu+1}^{\pm}(x', x_n) i_{en} e_{en},
\]
\[
-2d\delta(K_{\lambda-1,\nu+1}^{\pm}(x', x_n) e_{x} i_{en} e_{en}) = 4(\nu + 1)(\nu - \lambda + p)K_{\lambda-2,\nu+2}^{\pm}(x', x_n) e_{x} i_{en} e_{en} \\
- 2(\lambda + \nu - n)(\nu - \lambda + p)K_{\lambda-2,\nu+1}^{\pm}(x', x_n) e_{en} i_{en} e_{en} \\
- 2p(\nu - \lambda + p)K_{\lambda-1,\nu+1}^{\pm}(x', x_n) i_{en} e_{en},
\]
and
\[
\delta d(K_{\lambda,\nu}^{\pm}(x', x_n) i_{en} e_{en}) = 4(\nu) K_{\lambda-2,\nu+2}^{\pm}(x', x_n) e_{x} i_{en} e_{en} \\
- 2\nu(\lambda + \nu - n)K_{\lambda-2,\nu+1}^{\pm}(x', x_n) e_{en} i_{en} e_{en} \\
+ 2\nu(\lambda - n - p - 2)K_{\lambda-1,\nu+1}^{\pm}(x', x_n) i_{en} e_{en},
\]
\[
-2\delta d(K_{\lambda-1,\nu+1}^{\pm}(x', x_n) e_{x} i_{en} e_{en}) = -4(\nu + 1)(\lambda + \nu - n + p)K_{\lambda-2,\nu+2}^{\pm}(x', x_n) e_{x} i_{en} e_{en} \\
+ 2(\lambda + \nu - n - 1)K_{\lambda-3,\nu+1}^{\pm}(x', x_n) e_{x} i_{en} e_{en} \\
- 2(\lambda + \nu - n)(\lambda - \nu - p - 2)K_{\lambda-2,\nu+1}^{\pm}(x', x_n) e_{en} i_{en} e_{en} \\
- 2p(\lambda - \nu - p - 2)K_{\lambda-1,\nu+1}^{\pm}(x', x_n) i_{en} e_{en}.
\]
This completes the proof. \(\square\)

**Theorem 3.19** The distribution kernels \(K_{\lambda,\nu}^{(p),\pm}(x', x_n)\) satisfy
\[
P^{(p)}(\lambda)K_{\lambda,\nu}^{(p),\pm}(x', x_n) = (\lambda + \nu - n)(\nu - \lambda + 1)(\lambda - n + p - 1)(\lambda - p)K_{\lambda-1,\nu}^{(p),\pm}(x', x_n).
\]
Proof. We use Equation (3.22) for the operator $P^p(\lambda)$ and Lemma 3.18. The statement is based on collecting terms contributing to $K_{\lambda-1,\nu}^{(p)}(x', x_n)$, $K_{\lambda,\nu}^{+}(x', x_n)i_{\epsilon_n\epsilon_n}$, $K_{\lambda-1,\nu+2}^{\pm}(x', x_n)\epsilon_{x}\epsilon_{x}\epsilon_{\epsilon_n}$, and $K_{\lambda-1,\nu+1}^{\pm}(x', x_n)i_{\epsilon_n\epsilon_n}$, respectively. This gives

$-(2\lambda-n-2)(\lambda-n+p-1)(\lambda-p)(\lambda+\nu-n)+(\lambda-n+p-1)(\lambda-p)(\lambda+\nu-n-1)\lambda = (\lambda+\nu-n)(\nu-\lambda+1)(\lambda-n+p-1)(\lambda-p)$,

$2\nu(2\lambda-n-2)(\lambda-n+p-1)(\lambda-p)+2p(2\lambda-n-2)(\lambda-p)-2p(\lambda-n+p)(\lambda-p-1)\lambda = 0$,

$-4(\nu+1)(2\lambda-n-2)(\lambda-n+p-1)(\lambda-p)-4(\nu+1)(2\lambda-n-2)(\lambda-p)+4(\nu+1)(\lambda-n+p)(\lambda-n+p-1)2(\lambda-p) = 0$,

$2(2\lambda-n-2)(\lambda-n+p-1)(\lambda-p)-2(2\lambda-n-2)(\lambda-n+p)(\lambda-p)+2(2\lambda-n-2)(\lambda-p)(\lambda+\nu-n+1)2(\lambda-n+p)(\lambda-p-1)2(\lambda-n+p-1)(\lambda-n+p)(\lambda-n+p-1)(\lambda-p) = 0$,

and the proof is complete. □

Remark 3.20 Let us define the operator

$\tilde{P}^p(\lambda) \overset{\text{def}}{=} A\partial_n + B\epsilon_n\delta + C\delta i_n + D\partial_x\delta + Ex\delta d$

for some unknown $A, B, C, D, E$. The equation

$\tilde{P}^p(\lambda)K_{\lambda,\nu}^{\pm}(x', x_n) = (\lambda+\nu-n)(\nu-\lambda+1)(\lambda-n+p-1)(\lambda-p)K_{\lambda+1,\nu}^{\pm}(x', x_n)$

is equivalent to the following system for $A, B, C, D, E$ and $c \overset{\text{def}}{=} (\lambda+\nu-n)(\nu-\lambda+1)(\lambda-n+p-1)(\lambda-p)$:

$(\lambda+\nu-n)A-(\lambda+\nu-n-1)2E = c$,

$-2\nu A + 2\nu p D + 2\nu(2\lambda-n-p-2)E - 2p C$,

$-2(\nu-\lambda+p)p D - 2(\lambda-n+p-2)p E = 0$,

$2\nu B + 2\nu(\lambda+\nu-n)D - 2\nu(\lambda+\nu-n)E - 2A - 2(\nu-\lambda+p)B$,

$-2(\lambda+\nu-n+1)C - 2(\lambda+\nu-n)(\nu-\lambda+p)D$,

$-2(\lambda+\nu-n)(\lambda-\lambda+p-2)E = 0$,

$-4(\nu^2)D + 4(\nu^2)E + 4(\nu+1)A + 4(\nu+1)C + 4(\nu+1)(\nu-\lambda+p)D$,

$-4(\nu+1)(\lambda+\nu-n+p)E = 0$,

$-2(\lambda+\nu-n)A + 2(\lambda+\nu-n-1)2E = -2c$.

Here the contributions are sorted again according to $K_{\lambda-1,\nu}^{\pm}(x', x_n)i_{\epsilon_n\epsilon_n}$, $K_{\lambda,\nu}^{\pm}(x', x_n)i_{\epsilon_n\epsilon_n}$, $K_{\lambda-1,\nu+2}^{\pm}(x', x_n)\epsilon_{x}\epsilon_{x}\epsilon_{\epsilon_n}$, and $K_{\lambda-1,\nu+1}^{\pm}(x', x_n)i_{\epsilon_n\epsilon_n}$. Note that in this case the system does not have a unique solution, we could choose either $B, C$ or $D$ as a free parameter. Choosing one of them as in Equation (3.22) will determine the other two and we will recover $P^p(\lambda)$. This observation is a reflection of the fact that
in general Bernstein-Sato operators are not unique in contrast with the uniqueness of the Bernstein-Sato polynomial.

4. Applications of Bernstein-Sato identities and operators

In the final section we highlight different origins of Bernstein-Sato operators. Furthermore, we discuss several applications related to conformal symmetry breaking differential operators [KÖSS15, FJS16, KKP16]. As a consequence we shall observe that the Bernstein-Sato operators recover conformal symmetry breaking differential operators for functions, spinors and differential forms by partially new formulas. They differ from the known formulas in their product structure expansion, which gives a nice symmetric way of organizing their rather complicated structure.

4.1. Origins of the Bernstein-Sato operator - the scalar case. We discuss some other origins of the Bernstein-Sato operator \( P(\lambda) \), see (3.5), for functions (less is known for \( \bar{P}(\lambda) \) and \( P^p(\lambda) \)). To our best knowledge, there are three other approaches to construct the Bernstein-Sato operator [MØZ16, GZ01, GW15].

**Representation theory:** The differential action on the induced representation \( \pi_\nu \) of the Casimir element \( C \) for conformal Lie algebra \( \mathfrak{o}(1, n+1) \) is given in [MØZ16, Equation 5.1]:

\[
C(\mu) \overset{\text{def}}{=} d\pi_\mu(C) = x_n^2 \Delta + 2(\mu + 1)x_n \partial_n + (\mu + \frac{n}{2})(\mu - \frac{n}{2} + 1).
\]

The relationship to \( P(\lambda) \) is

\[
x_n P(\lambda) = C(-\lambda + \frac{n}{2}) - (\lambda - n)(\lambda - 1).
\]

(4.1)

Note that by Theorem 3.5 we obtain

\[
C(-\lambda + \frac{n}{2})K_{\lambda,\nu}^\pm(x', x_n) = -\nu(n - 1 - \nu)K_{\lambda,\nu}^\pm(x', x_n),
\]

hence \( K_{\lambda,\nu}^\pm(x', x_n) \) is an eigendistribution of \( C(-\lambda + \frac{n}{2}) \).

**Hyperbolic metric:** The eigen-equation associated to the Laplace operator for the hyperbolic metric \( g_{hyp} = x_n^{-2}(dx_1^2 + \cdots + dx_n^2) \) on the hyperbolic space also induces the operator \( P(\lambda) \), see [GZ01]. In particular,

\[
x_n^{n-s} P(s - 1) = (\Delta_{g_{hyp}} - s(n - 1 - s))x_n^{n-1-s}.
\]

(4.2)

Here we used \( \Delta_{g_{hyp}} = x_n^2 \Delta - (n - 2)x_n \partial_n \), where \( \Delta = \sum_{k=1}^n \partial_k^2 \) is the Laplace operator associated to the metric \( x_n^2 g_{hyp} \).

**Tractor calculus:** The invariant pairing of the tractor-D operator \( D_A \) with the scale tractor \( I_A \) in the Poincaré-metric gives, see [GW15 Section 5.6],

\[
P(s) = -I \cdot D.
\]

(4.3)

Here the parameter \( s \) on the right hand side of the previous equation corresponds to the weight of weighted tractor bundle. Note that the degenerate Laplacian \( I \cdot D \) is of more general nature then our \( P(\lambda) \) requires.
4.2. Origins of the Bernstein-Sato operator - the spinor case. In comparison to the scalar case much less is known for the operator $\mathcal{P}(\lambda)$, see \eqref{eq:3.12}, in the available literature.

Hyperbolic metric: The eigen-equation associated to the square of the Dirac operator \cite{GMP12} for the hyperbolic metric $g_{hyp} = x_n^{-2}(dx_1^2 + \cdots + dx_n^2)$ also induces the operator $\mathcal{P}(\lambda)$. In particular, the eigen-equation $D^2 g_{hyp} - i\lambda = 0$ on $\Gamma(\Sigma_{n}^{g_{hyp}})$, where $\Sigma_n^{g_{hyp}}$ denotes the spinor bundle associated to the hyperbolic space, is equivalent, via conformal covariance ($\bar{g} = dx_1^2 + \cdots + dx_n^2$) of the Dirac operator, to $D(\bar{g}) - i\lambda = 0$ on $\Gamma(\Sigma_n)$. We have

$$D(\bar{g}) \equiv x_n \mathcal{P}' + x_n D_n - \frac{n-1}{2} e_n',$$

where $\mathcal{P}' = \sum_{k=1}^{n-1} e_k \cdot \partial_k$ and $D_n \equiv e_n \cdot \partial_n$. Now, we define the operator $D(\mu)$ by the operator equation $[D(\bar{g})^2 + (\mu - \frac{n-1}{2})]x_n^\mu = -x_n^\mu D(\mu)$. It then follows

$$D(\mu) = x_n^2 \Delta - 2(\mu - \frac{n-1}{2})x_n \partial_n - x_n e_n \cdot \mathcal{P}' = x_n \mathcal{P}(\mu + \frac{3}{2}). \quad (4.4)$$

4.3. Origins of the Bernstein-Sato operator - the form case. In comparison to the scalar and spinor cases even less is known about the operator $P^p(\lambda)$, see \eqref{eq:3.21}, in the available literature. A potential origin of $P^p(\lambda)$ in the construction of the hyperbolic metric \cite{AG11} seems not to be well established; to our best knowledge this approach leads to an operator different from $P^p(\lambda)$. It is not clear to the authors if this discrepancy can be explained by the non-uniqueness phenomenon mentioned in Remark 3.20.

4.4. Conformal symmetry breaking differential operators for functions. The operator $P(\lambda)$, see \eqref{eq:3.5}, recovers conformal symmetry breaking differential operators \cite{J09, KS15, Koss15}

$$D_N(\lambda) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-1}), \quad (4.5)$$

which are given by

$$D_{2N}(\lambda) = \sum_{k=0}^{N} a_k^{(N)}(-\lambda)(\Delta')^k t^* \partial_n^{2N-2k},$$

$$D_{2N+1}(\lambda) = \sum_{k=0}^{N} b_k^{(N)}(-\lambda)(\Delta')^k t^* \partial_n^{2N-2k+1} \quad (4.6)$$

with

$$a_j^{(N)}(\lambda) \equiv \frac{(-2)^{N-j} N!}{j!(2N-2j+1)!} \prod_{k=j}^{N-1}(2\lambda - 4N + 2k + n + 1),$$

$$b_j^{(N)}(\lambda) \equiv \frac{(-2)^{N-j} N!}{j!(2N-2j)!} \prod_{k=j}^{N-1}(2\lambda - 4N + 2k + n - 1). \quad (4.7)$$
Here \( \Delta' \defeq \sum_{k=1}^{n-1} \partial_k^2 \) denotes the tangential Laplacian for the embedding \( \iota : \mathbb{R}^{n-1} \to \mathbb{R}^n \), \( \mathbb{R}^{n-1} \ni x' \mapsto (x', 0) \in \mathbb{R}^n \). As mentioned in the introduction, the families (conformal symmetry breaking differential operators) \( D_{2N}(\lambda) \) interpolate between GJMS-operators on \( \mathbb{R}^{n-1} \) and \( \mathbb{R}^n \) equipped with the flat Euclidean metric, respectively. More precisely it holds

\[
D_{2N}(-\frac{n-1}{2} + N) = (\Delta')^N \iota^* ,
\]
\[
D_{2N}(-\frac{n}{2} + N) = \iota^*(\Delta)^N. \tag{4.8}
\]

This appearance of GJMS-operators is part of a sequence of factorization identities for \( D_N(\lambda) \), see \[j09\].

Now let us define a family \( (N \in \mathbb{N}_0, \lambda \in \mathbb{C}) \) of differential operators on functions (termed Bernstein-Sato family for functions)

\[
D_{BS}^N(\lambda) \defeq \iota^* P(\lambda - N + 1) \circ \cdots \circ P(\lambda). \tag{4.9}
\]

This definition, although used in a different setting, already appeared in \[c17\].

**Example 4.1** We present the first- and second-order relations between \( D_{BS}^N(\lambda) \) and \( D_N(\lambda) \). First of all, recall from \[4.6\] that

\[
D_1(\lambda) = \iota^* \partial_n , \\
D_2(\lambda) = \Delta' \iota^* + (2\lambda - n + 3)\iota^* \partial_n^2.
\]

A direct computation shows

\[
D_{BS}^1(\lambda) = \iota^* P(\lambda) = (n - 2\lambda + 2)\iota^* \partial_n \\
= (n - 2\lambda + 2)D_1(n - \lambda) ,
\]
\[
D_{BS}^2(\lambda) = \iota^* P(\lambda - 1) \circ P(\lambda) \\
= (n - 2\lambda + 4)\iota^* \partial_n [(n - 2\lambda + 2)\partial_n + x_n \Delta] \\
= (n - 2\lambda + 4)[(n - 2\lambda + 3)\iota^* \partial_n^2 + \Delta' \iota^*] \\
= (n - 2\lambda + 4)D_2(n - \lambda).
\]

Now we shall discuss the relationship between differential operators \( D_{BS}^N(\lambda) \) and \( D_N(\lambda) \), see also \[c17\] where the proof follows from the property of uniqueness of intertwining differential operators. We present a direct proof of this fact.

**Theorem 4.2** Let \( N \in \mathbb{N} \). Then we have

\[
D_{BS}^{2N}(\lambda) = (-2)^N(\lambda - \frac{n}{2} - 2N)_N(2N - 1)!!D_{2N}(n - \lambda),
\]
\[
D_{BS}^{2N+1}(\lambda) = (-2)^{N+1}(\lambda - \frac{n}{2} - 2N - 1)_{N+1}(2N + 1)!!D_{2N+1}(n - \lambda).
\]
Proof. The proof goes by induction. Recall Example 4.1 for the lowest order relationship. By induction we compute

\[ D_{2N}^{BS}(\lambda) = D_{2N-1}^{BS}(\lambda - 1) \circ P(\lambda) \]
\[ = (-2)^N(\lambda - \frac{n}{2} - 2N)N(2N - 1)!!D_{2N-1}(n - \lambda + 1) \circ P(\lambda). \]

Since

\[ i^* \partial_n^{2N-2k-1}[(2\lambda - n + 2)\partial_n + x_n\Delta] = (2\lambda - n + 2)i^* \partial_n^{2N-2k} \]
\[ + (2N - 2k - 1)[\Delta i^* \partial_n^{2N-2k-2} + i^* \partial_n^{2N-2k}], \]
we may conclude

\[ D_{2N}^{BS}(\lambda) = (-2)^N(\lambda - \frac{n}{2} - 2N)N(2N - 1)!! \left[ A_0(\lambda)i^* \partial_n^{2N} + A_N(\lambda)(\Delta')^N \right] \]
\[ + \sum_{k=1}^{N-1} A_k(\lambda)(\Delta')^k i^* \partial_n^{2N-2k} \]

with

\[ A_0(\lambda) \overset{\text{def}}{=} (n - 2\lambda + 2N + 1)b_0^{(N-1)}(\lambda - n - 1), \]
\[ A_N(\lambda) \overset{\text{def}}{=} b_{N-1}^{(N-1)}(\lambda - n - 1), \]
\[ A_k(\lambda) \overset{\text{def}}{=} (n - 2\lambda + 2N - 2k + 1)b_k^{(N-1)}(\lambda - n - 1) + (2N - 2k + 1)b_{k-1}^{(N-1)}(\lambda - n - 1). \]

It follows from (4.7) that for \( 0 \leq k \leq N \) holds

\[ A_k(\lambda) \overset{\text{def}}{=} a_k^{(N)}(\lambda - n), \]

hence we get

\[ D_{2N}^{BS}(\lambda) = (-2)^N(\lambda - \frac{n}{2} - 2N)N(2N - 1)!!D_{2N}(n - \lambda). \]

The remaining statement is proved analogously. The proof is complete. \( \square \)

An immediate consequence of the last result is

**Corollary 4.3** Assuming \( N \in \mathbb{N}_0 \), we have

\[ D_{2N-1}(n - \lambda + 1) \circ P(\lambda) = D_{2N}(n - \lambda), \]
\[ D_{2N}(n - \lambda + 1) \circ P(\lambda) = -(2N + 1)(2\lambda - n - 2N - 2)D_{2N+1}(n - \lambda). \]

**Remark 4.4** To our best knowledge there are two other ways to compute \( D_N^{BS}(\lambda) \). The first one is based on Fourier transform, cf. proof of Proposition 3.2 for \( N = 1 \). To this aim one needs the following identities:

\[ \partial_n^{2N}(r^{n-2\lambda}(\xi)) = (2N - 1)!!2^N(\frac{n}{2} - \lambda - N + 1)N!r^{n-2\lambda-4N}(\xi) \times \]
\[ \times \sum_{k=0}^{N} \left( a_{N-k}^{(N)}(\lambda - n + 2N)N^{2N-2k}(\xi')_n^{2k} \right), \]
\[ \partial_n^{2N+1}(r^{n-2\lambda}(\xi)) = (2N-1)!!2^{N+1}(\frac{n}{2} - \lambda - N)_{N+1} r^{n-2\lambda-4N-2}(\xi) \times \]
\[ \times \sum_{k=0}^{N} b_{N-k}(\lambda - n + 2N + 1) r^{2N-2k}(\xi) e_n^{2k}. \]

We note the appearance of coefficients (4.7).

Another way is based on commutators
\[ [P(\lambda), \partial_n] = -\Delta, \quad [P(\lambda), \Delta] = -2\partial_n \Delta \]
and the fact that \( i^* P(\lambda) = (n - 2\lambda + 2)i^* \partial_n \).

Both approaches computing \( D^{BS}_{2N}(\lambda) \) are computationally rather tedious and we will not give any more detail here.

4.5. Conformal symmetry breaking differential operators for spinors. Here we discuss how the operator \( P(\lambda) \), see (3.12), by its iterations recovers conformal symmetry breaking differential operators for spinors
\[ \mathcal{D}_N(\lambda) : C^\infty(\mathbb{R}^n, \Sigma_n) \to \begin{cases} C^\infty(\mathbb{R}^{n-1}, \Sigma_{n-1}) & , n \text{ even} , \\ C^\infty(\mathbb{R}^{n-1}, \Sigma_{n-1}^+ \oplus \Sigma_{n-1}^-) & , n \text{ odd} \end{cases} \]
introduced in [KÖSS15] (note the wrong sign of \( \lambda \) in the pre-factor in the reference), and later appearing in [MØ17]. They are given by
\[ \mathcal{D}_{2N}(\lambda) \overset{\text{def}}{=} D_{2N}(\lambda + \frac{1}{2}) + 2ND_{2N-1}(\lambda + \frac{1}{2})\mathcal{D}'(e_n \cdot), \]
\[ \mathcal{D}_{2N+1}(\lambda) \overset{\text{def}}{=} (2\lambda - n + 2N + 2)D_{2N+1}(\lambda + \frac{1}{2})(e_n \cdot) + D_{2N}(\lambda + \frac{1}{2})\mathcal{D}', \]
where \( D_N(\lambda) \) (note that we will mean by \( i^* \) just restriction) are the conformal symmetry breaking differential operators (1.6) and \( \mathcal{D}' = \sum_{k=1}^{n-1} e_k \cdot \partial_k \) is the tangential Dirac operator.

The family \( \mathcal{D}_{2N+1}(\lambda) \) interpolates between conformal powers of the Dirac operator on \( \mathbb{R}^{n-1} \) and \( \mathbb{R}^n \) equipped with the flat Euclidean metric, respectively. More precisely, it holds
\[ \mathcal{D}_{2N+1}(\frac{n-1}{2} - \frac{1}{2} - N) = (\mathcal{D}')^{2N+1}i^*, \]
\[ \mathcal{D}_{2N+1}(\frac{n}{2} - \frac{1}{2} - N) = (-1)^N i^* \mathcal{D}_{2N+1}^{2N+1} \]
This appearance of conformal powers of the Dirac operators is part of a sequence of factorization identities for \( \mathcal{D}_N(\lambda) \), see [FS14].

Now we define the family for \( N \in \mathbb{N}_0 \) and \( \lambda \in \mathbb{C} \) of differential operators on spinors (termed Bernstein-Sato family for spinors) by the composition
\[ \mathcal{D}_N^{BS}(\lambda) \overset{\text{def}}{=} i^* P(\lambda - N + 1) \circ \cdots \circ P(\lambda) \]
This definition goes again in the spirit of [C17].
Example 4.5 We present the relationship for the first and the second-order families $\mathcal{D}_{N}^{BS}(\lambda)$ and $\mathcal{D}_{N}(\lambda)$. Firstly, via (4.6) we recall
\[
\mathcal{D}_{1}(\lambda) = (2\lambda - n + 2)\partial_{n}(e_{n}^{•}) + \mathcal{D}^{\prime},
\]
\[
\mathcal{D}_{2}(\lambda) = D_{2}(\lambda + \frac{1}{2}) + 2D_{1}(\lambda + \frac{1}{2})\mathcal{D}^{\prime}(e_{n}^{•})
= \Delta^{\prime} + (2\lambda - n + 4)\iota^{*}\partial_{n}^{2} + 2\iota^{*}\partial_{n}\mathcal{D}^{\prime}(e_{n}^{•})
\]
The Bernstein-Sato families of first and second-order, respectively, read as
\[
\mathcal{D}_{1}^{BS}(\lambda) = \iota^{*}\mathcal{P}(\lambda) = \iota^{*}[(n - 2\lambda + 2)\partial_{n} + n\Delta - e_{n}\mathcal{D}^{\prime}]
= -e_{n} \cdot \mathcal{D}_{1}(n - \lambda),
\]
\[
\mathcal{D}_{2}^{BS}(\lambda) \overset{\text{def}}{=} \iota^{*}\mathcal{P}(\lambda - 1) \circ \mathcal{P}(\lambda)
= [(n - 2\lambda + 4)\iota^{*}\partial_{n} - e_{n} \cdot \mathcal{D}^{\prime}\iota^{*}][(n - 2\lambda + 2)\partial_{n} + n\Delta - e_{n} \cdot \mathcal{D}^{\prime}]
= (n - 2\lambda + 3)[(n - 2\lambda + 4)\iota^{*}\partial_{n}^{2} - \Delta^{\prime} + 2\mathcal{D}^{\prime}\partial_{n}(e_{n})]
= -(n - 2\lambda + 3)\mathcal{D}_{2}(n - \lambda).
\]

Now we explain a general relationship between the families $\mathcal{D}_{N}^{BS}(\lambda)$ and $\mathcal{D}_{N}(\lambda)$.

Theorem 4.6 For $N \in \mathbb{N}_{0}$ we have
\[
\mathcal{D}_{2N}^{BS}(\lambda) = (-2)^{N}(\lambda - n \frac{n}{2} - 2N + \frac{1}{2})N(2N - 1)!! \mathcal{D}_{2N}(n - \lambda),
\]
\[
\mathcal{D}_{2N+1}^{BS}(\lambda) = -(n - 2\lambda + 3)N(2N + 1)!!e_{n} \cdot \mathcal{D}_{2N+1}(n - \lambda).
\]

Proof. We recall Example 4.5. Then by induction on the order we have
\[
\mathcal{D}_{2N}^{BS}(\lambda) = \mathcal{D}_{2N}^{BS}(\lambda - 1) \circ \mathcal{P}(\lambda)
= (-1)^{N}2^{N-1}(\lambda - n \frac{n}{2} - 2N + \frac{1}{2})N-1(2N - 1)!! \mathcal{D}_{2N-1}(-\lambda + n + 1) \circ \mathcal{P}(\lambda)
= (-1)^{N-1}2^{N-1}(\lambda - n \frac{n}{2} - 2N + \frac{1}{2})N-1(2N - 1)!![A_{0}\iota^{*}\partial_{n}^{2N}]
+ \sum_{k=1}^{N-1}(-1)^{k}A_{k}(\mathcal{D}^{\prime})^{2k}\iota^{*}\partial_{n}^{2N-2k} + A_{N}(\mathcal{D}^{\prime})^{2N}\iota^{*}
+ B_{0}\iota^{*}\partial_{n}^{2N-1}\mathcal{D}^{\prime}(e_{n}^{•}) + \sum_{k=1}^{N-1}(-1)^{k}B_{k}(\mathcal{D}^{\prime})^{2k}\iota^{*}\partial_{n}^{2N-2k-1}\mathcal{D}^{\prime}(e_{n}^{•})],
\]
where
\[
A_{0} \overset{\text{def}}{=} (-2\lambda + n + 2N + 2)(n - 2\lambda + 2N + 1)b_{0}^{(N-1)}(\lambda - n - \frac{3}{2}),
A_{k} \overset{\text{def}}{=} (-2\lambda + n + 2N + 2)(n - 2\lambda + 2N - 2k + 1)b_{k}^{(N-1)}(\lambda - n - \frac{3}{2})
\]
+ (2N - 2k + 1)b_{k-1}^{(N-1)}(\lambda - n - \frac{3}{2}) - a_{k-1}^{(N-1)}(\lambda - n - \frac{3}{2}),
A_N \overset{\text{def}}{=} (-2\lambda + n + 2N + 2) - 1,
B_0 \overset{\text{def}}{=} (-2\lambda + n + 2N + 2)b_0^{(N-1)}(\lambda - n - \frac{3}{2}) + (n - 2\lambda + 2N)a_0^{(N-1)}(\lambda - n - \frac{3}{2}),
B_k \overset{\text{def}}{=} (-2\lambda + n + 2N + 2)b_k^{(N-1)}(\lambda - n - \frac{3}{2})
+ (n - 2\lambda + 2N - 2k)a_k^{(N-1)}(\lambda - n - \frac{3}{2}) + (2N - 2k)a_{k-1}^{(N-1)}(\lambda - n - \frac{3}{2}).

Then it follows that for \( k = 0, \ldots N \)
\[ A_k = (-2\lambda + n + 2N + 1)a_k^{(N)}(\lambda - n - \frac{1}{2}), \]
while for \( k = 0, \ldots N - 1 \) it holds
\[ B_k = (-2\lambda + n + 2N + 1)b_k^{(N-1)}(\lambda - n - \frac{1}{2}). \]

This proves the even-order case. The odd-order case is completely analogous and will be omitted.

\[ \square \]

**Remark 4.7** The facts analogous to Remark 4.4 constitute different proofs of Theorem 4.6.

4.6. Conformal symmetry breaking differential operators for differential forms.

Conformal symmetry breaking differential operators acting on differential forms [FJS16, KKP16]

\[ D_N^{(p \rightarrow p)}(\lambda) : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\mathbb{R}^{n-1}) \] (4.10)

are given by
\[ D_{2N}^{(p \rightarrow p)}(\lambda) = (p - \lambda - 2N)D_{2N}(\lambda) + 2N(2\lambda - n + 2N + 1)D_{2N-1}(\lambda + 1)d'i_{e_n} \]
\[ - 2ND_{2N-2}(\lambda + 1)d'\delta', \]
\[ D_{2N+1}^{(p \rightarrow p)}(\lambda) = (p - \lambda - 2N - 1)D_{2N+1}(\lambda) + D_{2N}(\lambda + 1)d'i_{e_n} - 2ND_{2N-1}(\lambda + 1)d'\delta'. \]

Note the opposite sign convention for the family parameter \( \lambda \) and a clash of notation for \( \Delta \) when compared to [FJS16], i.e., \( D_{2N+1}^{(p \rightarrow p)}(-\lambda) \) and \( D_{2N+1}^{(p \rightarrow p)}(-\lambda) \) introduced in [FJS16] correspond to \((-1)^N D_{2N}^{(p \rightarrow p)}(\lambda) \) and \((-1)^N D_{2N}^{(p \rightarrow p)}(\lambda) \) defined in (4.10), respectively. As for the definition of \( D_N(\lambda) \), see Equation (4.6). Also note that \( i' \) in the definition of \( D_N(\lambda) \) denotes the pull-back of differential forms. The operators \( D_N^{(p \rightarrow p)}(\lambda) \) interpolate between Branson-Gover operators for \( \mathbb{R}^{n-1} \) and \( \mathbb{R}^n \) equipped with the flat Euclidean metric, respectively. More precisely, it holds
\[ D_{2N}^{(p \rightarrow p)}(\frac{n-1}{2} - N) = (-1)^{N+1} \left[ \left( \frac{n-1}{2} - p - N \right)(d'\delta)^N + \left( \frac{n-1}{2} - p + N \right)(\delta d')^N \right], \]
\[ D_{2N}^{(p \rightarrow p)}(\frac{n}{2} - N) = (-1)^{N+1} \left[ \left( \frac{n}{2} - p - N \right)(d\delta)^N + \left( \frac{n}{2} - p + N \right)(\delta d)^N \right]. \]
Again, this appearance of Branson-Gover operators is a part of the sequence of factorizations identities for \( D_{2N}^{(p \to p)}(\lambda) \), see \([FJS16]\).

Let us introduce a family (termed Bernstein-Sato family for differential forms of first type)

\[
D_{N}^{BS,(p \to p)}(\lambda) := \iota^* P^p(\lambda - N + 1) \circ \cdots \circ P^p(\lambda) : \Omega^p(\mathbb{R}^n) \to \Omega^p(\mathbb{R}^{n-1}).
\]

This definition is again inspired by \([C17]\).

Now we state some low-order relations between \( D_{N}^{BS,(p \to p)}(\lambda) \) and \( D_{N}^{(p \to p)}(\lambda) \).

**Example 4.8** By definitions (4.10) and (4.6) we have

\[
\begin{align*}
D_1^{(p \to p)}(\lambda) &= (p - \lambda - 1)\iota^* \partial_n + d'\iota^* i_{\epsilon_n}, \\
D_2^{(p \to p)}(\lambda) &= (p - \lambda - 2)\Delta' \iota^* + (2\lambda - n + 3)(p - \lambda - 2)\iota^* \partial_n^2 \\
&\quad + 2(2\lambda - n + 3)d'\iota^* i_{\epsilon_n} \partial_n - 2d' \delta' \iota^*, \\
D_3^{(p \to p)}(\lambda) &= (p - \lambda - 3)\Delta' \iota^* \partial_n + \frac{1}{3}(2\lambda - n + 5)(p - \lambda - 3)\iota^* \partial_n^3 \\
&\quad + (2\lambda - n + 5)d' \iota^* i_{\epsilon_n} \partial_n^2 + \Delta' d' \iota^* i_{\epsilon_n} - 2d' \delta' \iota^* \partial_n.
\end{align*}
\]

It is then straightforward to verify

\[
\begin{align*}
D_1^{BS,(p \to p)}(\lambda) &= -(-2\lambda - n - 2)(\lambda - p)D_1^{(p \to p)}(n - \lambda), \\
D_2^{BS,(p \to p)}(\lambda) &= -(-2\lambda - n - 4)(\lambda - n + p - 1)(\lambda - p - 1)D_2^{(p \to p)}(n - \lambda), \\
D_3^{BS,(p \to p)}(\lambda) &= 3(2\lambda - n - 6)(2\lambda - n - 4)(\lambda - p - 2)D_3^{(p \to p)}(n - \lambda).
\end{align*}
\]

Now we state a general relationship between the families \( D_{N}^{BS,(p \to p)}(\lambda) \) and \( D_{N}^{(p \to p)}(\lambda) \).

**Theorem 4.9** For \( N \in \mathbb{N} \) holds

\[
\begin{align*}
D_{2N}^{BS,(p \to p)}(\lambda) &= (-2)^N(\lambda - \frac{n}{2} - 2N)_N(2N - 1)!! \times \\
&\quad \times (\lambda - n + p - 2N + 1)_{2N-1}(\lambda - p - 2N + 1)_{2N}D_{2N}^{(p \to p)}(n - \lambda), \\
D_{2N+1}^{BS,(p \to p)}(\lambda) &= (-2)^{N+1}(\lambda - \frac{n}{2} - 2N - 1)_{N+1}(2N + 1)!! \times \\
&\quad \times (\lambda - n + p - 2N)_{2N}(\lambda - p - 2N)_{2N+1}D_{2N+1}^{(p \to p)}(n - \lambda).
\end{align*}
\]

**Proof.** The proof goes by induction and starts with Example 4.8. By definition (4.11) and induction hypothesis it follows

\[
D_{2N}^{BS,(p \to p)}(\lambda) = D_{2N-1}^{BS,(p \to p)}(\lambda - 1) \circ P^p(\lambda) = c(2N - 1, \lambda - 1)D_{2N-1}^{(p \to p)}(n - \lambda + 1) \circ P^p(\lambda)
\]

for
\[ c(2N - 1, \lambda - 1) = (-2)^N(\lambda - \frac{n}{2} - 2N)_N(2N - 1)!! \times (\lambda - p - 2N + 1)_{2N-1}(\lambda - n + p - 2N + 1)_{2N-2}. \]

Now (4.10) gives that
\[
D_{2N-1}^{(p\rightarrow p)}(n - \lambda + 1) \circ P^p(\lambda) = \left[ (\lambda - n + p - 2N)D_{2N-1}(n - \lambda + 1) \circ P^p(\lambda) + D_{2N-2}(n - \lambda + 2)d^*i_{en}P^p(\lambda) - (2N - 2)D_{2N-3}(n - \lambda + 2)d^*d^*P^p(\lambda) \right]. \tag{4.12}
\]

The individual summands on the right hand side of the last display simplify by (3.23) and the identities
\[
\iota^*\partial_n^k(x_nF) = k\iota^*\partial_n^{k-1}F, \quad \iota^*d\delta = d\delta'\iota^* - d^*i_{en}\partial_n, \quad \iota^*\varepsilon_{en}\delta = 0, \quad \iota^*di_{en} = d^*i_{en},
\]
where \( k \in \mathbb{N} \) and \( F \) is a differential operator. Hence we see by Corollary 4.3
\[
D_{2N-1}(n - \lambda + 1)P^p(\lambda) = (\lambda - n + p - 1)(\lambda - p)\sum_{k=0}^{N-1}a_k^{(N-1)}(\lambda - n)(\Delta')^k\iota^*\partial_n^{2N-2k}
\]
\[
+ \sum_{k=0}^{N-1}[(n - 2p)(2N - 2k - 1) - (2\lambda - n - 2)(\lambda - p)]b_k^{(N-1)}(\lambda - n - 1) \times
\]
\[
(\Delta')^k\iota^*\partial_n^{2N-2k-1}d^*i_{en}
\]
\[-(n - 2p)\sum_{k=0}^{N-1}(2N - 2k - 1)b_k^{(N-1)}(\lambda - n - 1)(\Delta')^k\iota^*\partial_n^{2N-2k-2}d^*\delta'.
\]

Similarly, the identities
\[
P(\lambda) = P(\lambda - 1) - 2\partial_n, \quad \iota^*d^*i_{en}d\delta = d\delta'\iota^*\partial_n - d^*i_{en}\partial_n^2,
\]
\[
\iota^*d^*i_{en}\varepsilon_{en}\delta = d\delta'\iota^* - d^*i_{en}\partial_n, \quad \iota^*di_{en} = d^*i_{en}\partial_n
\]
and Corollary 4.3 allow to conclude
\[
D_{2N-2}(n - \lambda + 2)d^*i_{en}P^p(\lambda)
\]
\[
= \sum_{k=0}^{N-1} - (2N - 1)(2\lambda - n - 2N - 2)(\lambda - n + p - 1)(\lambda - p)b_k^{(N-1)}(\lambda - n - 1)
\]
\[
+ \left[ - 2(\lambda - n + p - 1)(\lambda - p) + (n - 2p)(2N - 2k - 2)
\right. \]
\[
+ (2\lambda - n - 2)(\lambda - n + p) - (2\lambda - n - 2)(\lambda - p)]a_k^{(N-1)}(\lambda - n - 2)
\]
\[
\times (\Delta')^k\iota^*\partial_n^{2N-2k-1}d^*i_{en}
\]
\[
+ \sum_{k=0}^{N-1} - (n - 2p)(2N - 2k - 2) - (2\lambda - n - 2)(\lambda - n + p)a_k^{(N-1)}(\lambda - n - 2)
\]
\[
\times (\Delta')^k\iota^*\partial_n^{2N-2k-2}d^*\delta'.
\]
Finally, by
\[ d'\delta'i^*e_n = d'\delta'i^*i_{e_n}, \quad d'\delta'i^*(\varepsilon_{e_n}\delta) = 0, \quad d'\delta'i^*i_{e_n} = d'\delta'd'i^*i_{e_n} \]
and Corollary 4.3 we have
\[ D_{2N-2}(n - \lambda + 2)d'\delta'P^p(\lambda) \]
\[ = \sum_{k=0}^{N-1} \left[ -(n - 2p)(2N - 2k - 1) + (2\lambda - n - 2)(\lambda - p) \right] b_k^{(N-2)}(\lambda - n - 2) \times \]
\[ \times (\Delta')^k i^* \partial_n^{2N-2k-1} d'i_{e_n} \]
\[ + \sum_{k=0}^{N-1} \left[ (\lambda - n + p - 1)(\lambda - p) \right] a_k^{(N-1)}(\lambda - n - 1) - 2b_k^{(N-2)}(\lambda - n - 2) \]
\[ + (n - 2p)(2N - 2k - 1)b_k^{(N-2)}(\lambda - n - 2) \]
\[ (\Delta')^k i^* \partial_n^{2N-2k-2} d'\delta'. \]
Consequently, we see that Equation (4.12) simplifies to
\[ D_{2N-1}^{(p\rightarrow p)}(n - \lambda + 1) \circ P^p(\lambda) \]
\[ = (\lambda - n + p - 1)(\lambda - p)(\lambda - n + p - 2N)D_{2N}(n - \lambda) \]
\[ - (\lambda - n + p - 1)(\lambda - p)(2N)(2\lambda - n - 2N - 1)D_{2N-1}(n - \lambda + 1)d'i_{e_n} \]
\[ - (\lambda - n + p - 1)(\lambda - p)(2N - 2)D_{2N-2}(n - \lambda + 1)d'\delta' \]
\[ = (\lambda - n + p - 1)(\lambda - p)D_{2N}^{(p\rightarrow p)}(n - \lambda) \]
and hence we have
\[ D_{2N}^{BS, (p\rightarrow p)}(n - \lambda) = (-2)^N(\lambda - n/2 - 2N)_N(2N - 1)!! \times \]
\[ \times (\lambda - n + p - 2N + 1)_{2N-1}(\lambda - p - 2N + 1)_{2N}D_{2N}^{(p\rightarrow p)}(n - \lambda). \]
The odd-order families follow by a similar argument. The proof is complete. \(\square\)

**Corollary 4.10**  
Let \( N \in \mathbb{N}_0 \). Then it holds
\[ D_{2N-1}^{(p\rightarrow p)}(n - \lambda + 1) \circ P^p(\lambda) = (\lambda - n + p - 1)(\lambda - p)D_{2N}^{(p\rightarrow p)}(n - \lambda), \]
\[ D_{2N}^{(p\rightarrow p)}(n - \lambda + 1) \circ P^p(\lambda) = - (\lambda - n + p - 1)(\lambda - p)(2N + 1) \times \]
\[ \times (2\lambda - n - 2N - 2)D_{2N+1}^{(p\rightarrow p)}(n - \lambda). \]

**Remark 4.11**  
The Bernstein-Sato families of the first type \( D_N^{BS, (p\rightarrow p)}(\lambda), N \in \mathbb{N}_0, \) induce the full classification list for conformal symmetry breaking differential operators on differential forms, cf. [PJS16 Theorem 3]. For example, the Bernstein-Sato families of the second type arise by post- and pre-composition of \( D_N^{BS, (p\rightarrow p)}(\lambda), N \in \mathbb{N}_0, \) with the Hodge-star operators on \( \mathbb{R}^{n-1} \) and \( \mathbb{R}^n \), respectively.
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