Spherical character of a supercuspidal representation as weighted orbital integral

P. Delorme, P. Harinck

Abstract

Let $E/F$ be an unramified quadratic extension of local non archimedean fields of characteristic 0. Let $H$ be an algebraic reductive group, defined and split over $F$. We assume that the split connected component of the center of $H$ is trivial. Let $(\tau, V)$ be a $H(F)$-distinguished supercuspidal representation of $H(E)$. Using the recent results of C. Zhang [Z], and the geometric side of a local relative trace formula obtained by P. Delorme, P. Harinck and S. Souaifi [DHS], we describe spherical characters associated to $H(F)$-invariant linear forms on $V$ in terms of weighted orbital integrals of matrix coefficients of $\tau$.

Mathematics Subject Classification 2000: 11F72, 22E50.

Keywords and phrases: $p$-adic reductive groups, symmetric spaces, truncated kernel, spherical character, weighted orbital integral.

1 Introduction

Let $E/F$ be an unramified quadratic extension of local non archimedean fields of characteristic 0. Let $H$ be an algebraic reductive group, defined and split over $F$. We denote by $G := \text{Res}_{E/F} H_E$ the restriction of scalars of $H_E$. Then $G := G(F)$ is isomorphic to $H(E)$. We set $H := H(F)$. We denote by $\sigma$ the involution of $G$ induced by the nontrivial element of the Galois group of $E/F$.

An unitary irreducible admissible representation $(\pi, V)$ of $G$ is $H$-distinguished if the space $V^H = \text{Hom}_H(\pi, \mathbb{C})$ of $H$-invariant linear forms on $V$ is nonzero. In that case, a distribution $m_{\xi, \xi'}$, called spherical character, can be associated to two $H$-invariant linear forms $\xi, \xi'$ on $V$ (cf. (2.1)). By ([Ha] Theorem 1), spherical characters are locally integrable functions on $G$, which are $H$ biinvariant and smooth on the set $G^{\sigma-\text{reg}}$ of elements $g$, called $\sigma$-regular points, such that $g$ is semisimple and $g^{-1}\sigma(g)$ is regular in $G$ in the usual sense.

*The first author was supported by a grant of Agence Nationale de la Recherche with reference ANR-13-BS01-0012 FERPLAY.
We assume that the split component of the center of $H$ is trivial. Let $(\tau, V)$ be a $H$-distinguished supercuspidal representation of $G$.

The aim of this note is to give the value of a spherical character $m_{\xi_1, \xi_2}(g)$, when $g \in G$ is a regular point for the symmetric space $H \backslash G$ and $\xi, \xi' \in V^{*H}$, in terms of weighted orbital integrals of a matrix coefficient of $\tau$ (cf. Theorem 3.1). This result is analogous to that of J. Arthur in the group case ([Ar1]). Notice that this result of J. Arthur can be deduced from his local trace formula ([Ar2]) which was obtained later.

We use the recent work of C. Zhang [Z], which describes the space of $H$-invariant linear forms of supercuspidal representations, and the geometric side of a local relative trace formula obtained by P. Delorme, P. Harinck and S. Souaifi [DHS].

2 Spherical characters

We denote by $C_c^\infty(G)$ the space of compactly smooth functions on $G$. We fix a $H$-distinguished supercuspidal representation $(\tau, V)$ of $G$. We denote by $d_{\tau}$ its formal degree.

Let $(\cdot, \cdot)$ be a $G$-invariant hermitian inner product on $V$. Since $\tau$ is unitary, it induces an isomorphism $\iota: v \mapsto (\cdot, v)$ from the conjugate complex vector space $\overline{V}$ of $V$ and the smooth dual of $V$, which intertwines the complex conjugate of $\tau$ and its contragredient $\overline{\tau}$. If $\xi$ is a linear form on $V$, we define the linear form $\overline{\xi}$ on $V$ by $\overline{\xi}(v) := \overline{\xi(v)}$.

For $\xi_1$ and $\xi_2$ two $H$-invariant linear forms on $V$, we associate the spherical character $m_{\xi_1, \xi_2}$ defined to be the distribution on $G$ given by

$$m_{\xi_1, \xi_2}(f) := \sum_{u \in B} \xi_1(\tau(f)u)\overline{\xi_2(u)}, \quad (2.1)$$

where $B$ is an orthonormal basis of $V$. Since $\tau(f)$ is of finite rank, this sum is finite. Moreover, this sum does not depend on the choice of $B$. Indeed, let $(\tau^*, V^*)$ be the dual representation of $\tau$. For $f \in C_c^\infty(G)$, we set $\hat{f}(g) := f(g^{-1})$. By ([R] Théorème III.3.4 and I.1.2), the linear form $\tau^*(\hat{f})\xi$ belongs to $\overline{V}$. Hence we can write $\iota^{-1}(\tau^*(\hat{f})\xi) = \sum_{v \in B} (\tau^*(\hat{f})\xi)(v) \cdot v$ where $(\lambda, v) \mapsto \lambda \cdot v$ is the action of $\mathbb{C}$ on $\overline{V}$. Therefore we deduce easily that one has

$$m_{\xi_1, \xi_2}(f) = \overline{\xi_2(\iota^{-1}(\tau^*(\hat{f})\xi_1))}. \quad (2.2)$$

Since $\tau$ is a supercuspidal representation, we can define the $H \times H$-invariant pairing $\mathcal{L}$ on $V \times \overline{V}$ by

$$\mathcal{L}(u, v) := \int_H (\tau(h)u, v)dh.$$  

By ([Z] Theorem 1.5),

$$\text{the map } v \mapsto \xi_v : u \mapsto \mathcal{L}(u, v) \text{ is a surjective linear map from } \overline{V} \text{ onto } V^{*H}. \quad (2.3)$$

For $v, w \in V$, we denote by $c_{v, w}$ the corresponding matrix coefficient defined by $c_{v, w}(g) := (\tau(g)v, w)$ for $g \in G$.  

2
2.1 Lemma. Let $\xi_1, \xi_2 \in V^*H$ and $v, w \in V$. Then we have

$$m_{\xi_1, \xi_2}(\bar{c}_{v,w}) = d(\tau)^{-1}\xi_1(v)\xi_2(w).$$

Proof:
By (2.3), there exist $v_1$ and $v_2$ in $V$ such that $\xi_j = \xi_{v_j}$ for $j = 1, 2$. By definition of the spherical character, for $f \in C^c_c(G)$ and $B$ an orthonormal basis of $V$, one has

$$m_{\xi_1, \xi_2}(f) = \sum_{u \in B} \int_H (\tau(h) \tau(f)u, v_1)dh \int_H (\tau(h)u, v_2)dh$$

$$= \sum_{u \in B} \int_{H \times H} (u, \tau(f)\tau(h_1)v_1)\tau(h_2)v_2, u)dh_1dh_2$$

$$= \int_{H \times H} (\tau(h_2)v_2, \tau(f)\tau(h_1)v_1)dh_1dh_2$$

Hence we obtain

$$m_{\xi_1, \xi_2}(f) = \int_{H \times H} \int_G f(g)(\tau(h_1gh_2)v_2, v_1)dgdh_1dh_2. \quad (2.4)$$

Let $f(g) := \bar{c}_{v,w}(g) = (\tau(g)w, v)$. By the orthogonality relation of Schur, for $h_1, h_2 \in H$, one has

$$\int_G (\tau(g)\tau(h_2)v_2, \tau(h_1)v_1)(\tau(g)w, v)dg = d(\tau)^{-1}(\tau(h_2)v_2, w)(v, \tau(h_1)v_1).$$

Thus, we deduce that

$$m_{\xi_1, \xi_2}(f) = d(\tau)^{-1}\xi_w(v_2)\xi_v(v) = d(\tau)^{-1}\xi_1(v)\xi_2(w). \quad \square$$

3 Main result

We first recall some notations of [DHS] to introduce weighted orbital integrals.

We refer the reader to ([RR] §3) and ([DHS] §1.2 and 1.3) for the notations below and more details on $\sigma$-regular points. Let $D_G$ be the usual Weyl discriminant function of $G$. By ([RR] Lemma 3.2 and Lemma 3.3), an element $g \in G$ is $\sigma$-regular if and only if $D_G(g^{-1}\sigma(g)) \neq 0$. The set $G^{\sigma\text{-reg}}$ of $\sigma$-regular points of $G$ is described as follows. Let $\underline{S}$ be a maximal torus of $\underline{H}$. We denote by $\underline{S}_\sigma$ the connected component of the set of points $\gamma \in \text{Res}_{F}\underline{S}_E$ such that $\sigma(\gamma) = \gamma^{-1}$. We set $S_\sigma := \underline{S}_\sigma(F)$. By Galois cohomology, there exists a finite set $\kappa_S \subset G$ such that $H \underline{S}_{\sigma} \cap G = \cup_{x \in \kappa_S} H x S_\sigma$.

By ([RR] Theorem 3.4) and ([DHS] (1.30)), if $g \in G^{\sigma\text{-reg}}$, there exist a unique maximal torus $\underline{S}$ of $\underline{H}$ defined over $F$ and 2 unique points $x \in \kappa_S$ and $\gamma \in S_\sigma$ such that $g = x\gamma$. We denote by $M$ the centralizer of the split connected component of $\underline{S} := \underline{S}(F)$. Then $M$ is
Levi subgroup, that is the Levi component of a parabolic subgroup of $H$. We define the weight function $w_M$ on $H \times H$ by

$$w_M(1, y_1, y_2) := \tilde{w}_M(1, y_1, y_2),$$

where $\tilde{w}_M$ is the weight function defined in ([DHS] Lemma 2.10) and 1 is the neutral element of $H$.

For $x \in \kappa_S$, we set $d_{M, S, x} := c_M c_{S, x}$ where the constants $c_M$ and $c_{S, x}$ are defined in ([DHS] (1.33)).

For $f \in C_c^\infty(G)$, we define the weighted orbital integral of $f$ on $G^{\sigma - \text{reg}}$ as follows. Let $g \in G^{\sigma - \text{reg}}$. We keep the above notations and we write $g = x \gamma$ with $x \in \kappa_S$ and $\gamma \in S_\sigma$.

We set

$$W\mathcal{M}(f)(g) := \frac{1}{d_{M, S, x}} |D_G(g^{-1} \sigma(g))|^{1/2} \int_{H \times H} f(y_1 y_2) w_M(1, y_1, y_2) dy_1 dy_2.$$

### 3.1 Theorem.

For $v, w \in V$, we have

$$W\mathcal{M}(c_{v, w})(g) = m_{x, x}(g), \quad g \in G^{\sigma - \text{reg}}.$$

**Proof:**

Let $f_1$ be a matrix coefficient of $\tau$ and $f_2 \in C_c^\infty(G)$. We set $f := f_1 \otimes f_2$. Let $R$ be the regular representation of $G \times G$ on $L^2(G)$ given by $[R(x_1, x_2) \Psi](g) = \Psi(x^{-1}_1 g x_2)$. Then $R(f)$ is an integral operator with smooth kernel $K_f$ given by $K_f(x, y) = \int_G f_1(x u) f_2(u y) du$. As in ([DHS] §2.2), we introduce the truncated kernel

$$K^T(f) := \int_{H \times H} K_f(x, y) u(x, T) u(y, T) dxdy$$

where $u(x, T)$ is the truncated function of J. Arthur on $H$ (cf. [DHS] (2.7)). It is the characteristic function of a compact subset of $H$, depending on a parameter $T$ in a finite dimensional vector space, which converges to the function equal to 1 when $\|T\|$ approaches $+\infty$. We will give the spectral asymptotic expansion of $K^T(f)$.

For $x \in G$, we define

$$h(g) := \int_G f_1(x u) f_2(u g x) du,$$

so that

$$K_f(x, y) = [\rho(y x^{-1}) h](e),$$

where $\rho$ is the right irreducible regular representation of $G$.

If $\pi$ is a unitary irreducible admissible representation of $G$, one has

$$\pi(\rho(y x^{-1}) h) = \int_{G \times G} f_1(x u) f_2(u g y) \pi(g) dudg$$

$$= \int_{G \times G} f_1(x u) f_2(u_2) \pi(u^{-1} u_2 y^{-1}) dudu_2 = \int_{G \times G} f_1(u_1^{-1}) f_2(u_2) \pi(u_1 x u_2 y^{-1}) dudu_2$$

$$= \pi(f_1) \pi(x) \pi(f_2) \pi(y^{-1}).$$

4
Since $\tau$ is supercuspidal and $f_1$ is a matrix coefficient of $\tau$, we deduce that $\pi(\rho(yx^{-1})h)$ is equal to 0 if $\pi$ is not equivalent to $\tau$. Therefore, applying the Plancherel formula ([W2 Théorème VIII.1.1.) to $[\rho(yx^{-1})h]$, we obtain

$$K_f(x, y) = d(\tau) \text{tr}(\tau(\tilde{f}_1)\tau(x)\tau(f_2)\tau(y)).$$

We identify $\tilde{V} \otimes V$ with a subspace of Hilbert-Schmidt operators on $V$. Taking an orthonormal basis $B_H(V)$ of $\tilde{V} \otimes V$ for the scalar product $(S, S') := \text{tr}(SS'^*)$, one obtains

$$K_f(x, y) = d(\tau) \text{tr}(\tau(\tilde{f}_1)\tau(x)\tau(f_2)\tau(y)^*) = d(\tau)\text{tr}(\tau(\tilde{f}_1)\tau(x)\tau(f_2), \tau(y))$$

$$= d(\tau) \sum_{S \in B_{HS}(V)} (\tau(\tilde{f}_1)\tau(x)\tau(f_2), S^*) (\tau(y), S^*)$$

$$= d(\tau) \sum_{S \in B_{HS}(V)} \text{tr}(\tau(x)\tau(f_2)S\tau(\tilde{f}_1))\text{tr}(\tau(y)S),$$

where the sums over $S$ are finite since $\tau(f_2)$ and $\tau(\tilde{f}_1)$ are of finite rank. Therefore, the truncated kernel is equal to

$$K^T(f) = d(\tau) \sum_{S \in B_{HS}(V)} P^T_{\tau}(\tau \otimes \tau(f)S)P^T_{\tau}(S)$$

where

$$P^T_{\tau}(S) = \int_H \text{tr}(\tau(h)S)u(h, T)dh, \quad S \in \tilde{V} \otimes V.$$ 

For $\tilde{v} \otimes v \in \tilde{V} \otimes V$, one has $\text{tr}(\tau(h)(\tilde{v} \otimes v)) = c_{\tilde{v}, v}(h)$. Since $c_{\tilde{v}, v}$ is compactly supported, the truncated local period $P^T_{\tau}(S)$ converges when $\|T\|$ approaches infinity to

$$P_{\tau}(S) = \int_H \text{tr}(\tau(h)S)dh.$$ 

Therefore, we obtain

$$\lim_{\|T\| \to +\infty} K^T(f) = d(\tau)m_{P_{\tau}, P_{\tau}}(f), \quad (3.1)$$

where $m_{P_{\tau}, P_{\tau}}$ is the spherical character of the representation $\tau \otimes \tau$ associated to the $H \times H$-invariant linear form $P_{\tau}$ on $\tilde{V} \otimes V$.

By ([DHS] Theorem 2.15), the truncated kernel $K^T(f)$ is asymptotic to a distribution $J^T(f)$ as $\|T\|$ approaches $+\infty$ and the constant term $\tilde{J}(f)$ of $J^T(f)$ is explicitly given in ([DHS] Corollary 2.11). Therefore, we deduce that

$$d(\tau)m_{P_{\tau}, P_{\tau}}(f) = \tilde{J}(f). \quad (3.2)$$

We now express $m_{P_{\tau}, P_{\tau}}$ in terms of $H$-invariant linear forms on $V$. Let $V_H$ be the orthogonal of $V^*H$ in $V$. Since $\xi_u(v) = \overline{\xi_\nu(u)}$ for $u, v \in V$, the space $\overline{V}_H$ is the kernel of $v \mapsto \xi_u$. Let $W$ be a complementary subspace of $V_H$ in $V$. Then, the map $v \mapsto \xi_v$ is an isomorphism from $\overline{W}$ to $V^*H$ and $(u, v) \mapsto \xi_u(u)$ is a nondegenerate hermitian form on
We identify $\nu$ and $\tilde{V}$ by the isomorphism $\iota$. We claim that

$$P_\tau = \sum_{i=1}^{\infty} \frac{1}{\xi_i(\epsilon_i)} \overline{\xi_i} \otimes \xi_i$$

(3.3)

Indeed, we have $P_\tau(v \otimes u) = \xi_u(u) = \xi_v(v)$. Hence, the two sides are equal to 0 on $\tilde{V} \otimes V = V_H \otimes V + \tilde{V}_H \otimes V_H$ and take the same value $\xi_k(\epsilon_l)$ for $k, l \in \{1, \ldots, n\}$. Hence, by definition of spherical characters, we deduce that

$$m_{P_\tau, P_\tau}(f_1 \otimes f_2) = \sum_{u \otimes \nu \in o.b. (\tilde{V} \otimes V)} P_\tau \left( \overline{\tau(f_1)} \otimes \tau(f_2)(u \otimes v) \right) P_\tau(u \otimes v)$$

$$= \sum_{u \otimes \nu \in o.b. (\tilde{V} \otimes V)} P_\tau \sum_{i,j=1}^{n} \frac{1}{\xi_i(\epsilon_i) \xi_j(\epsilon_j)} \overline{\xi_i(\bar{\tau}(f_1)u)\xi_j(\tau(f_2)v)} \xi_j(\nu),$$

where $o.b. (\tilde{V} \otimes V)$ is an orthonormal basis of $\tilde{V} \otimes V$. By definition of $\hat{\xi}$ for $\xi \in V^*H$, one has $\hat{\xi}(\bar{\tau}(f_1)u) = \xi(\tau(f_1))$. Therefore, we obtain

$$m_{P_\tau, P_\tau}(f_1 \otimes f_2) = \sum_{i,j=1}^{n} \frac{1}{\xi_i(\epsilon_i) \xi_j(\epsilon_j)} \overline{m_{\xi_i, \xi_j}(f_1)} m_{\xi_i, \xi_j}(f_2).$$

(3.4)

Let $v$ and $w$ in $V$. Let $f_1 := c_{v,w}$ so that $f_1 = c_{v,w}$. If $v \in V_H$ or $w \in V_H$, it follows from Lemma 2.1 that $m_{\xi_i, \xi_j}(f_1) = 0$ for $i, j \in \{1, \ldots, n\}$, hence $m_{P_\tau, P_\tau}(f_1 \otimes f_2) = 0$. Thus, we deduce from (3.2) that

$$\tilde{J}(c_{v,w} \otimes f_2) = 0, \quad v \in V_H \text{ or } w \in V_H.$$ 

(3.5)

Let $i, j \in \{1, \ldots, n\}$. We set $f_1 := c_{\epsilon_i, \epsilon_j}$, hence $f_1 = c_{\epsilon_i, \epsilon_j}$. By Lemma 2.1, one has $m_{\xi_i, \xi_j}(f_1) = d(\tau)^{-1} \xi_i(\epsilon_i) \xi_j(\epsilon_j)$. Therefore, by (3.2) and (3.4) we obtain

$$\tilde{J}(c_{\epsilon_i, \epsilon_j} \otimes f_2) = m_{\xi_i, \xi_j}(f_2).$$

(3.6)

By sesquilinearity, ones deduces from (3.5) and (3.6) that one has

$$\tilde{J}(c_{v,w} \otimes f_2) = m_{\xi_v, \xi_w}(f_2) \quad v, w \in V.$$ 

(3.7)

Let $g \in G^{\sigma-reg}$. Let $(J_n)_{n}$ be a sequence of compact open sugroups whose intersection is equal to the neutral element of $G$. The characteristic function $\phi_n$ of $J_n g J_n$ approaches the Dirac measure at $g$ as $n$ approaches $+\infty$. Thus, if $v, w \in V$ then $m_{\xi_v, \xi_w}(\phi_n)$ converges to $m_{\xi_v, \xi_w}(g)$. By ([DHS] Corollary 2.11) the constant term $\tilde{J}(c_{v,w} \otimes \phi_n)$ converges to $W_M(c_{v,w})(g)$. We deduce the Theorem from (3.7).
References

[Ar1] J. Arthur, The characters of supercuspidal representations as weighted orbital integrals, Proc. Indian Acad. Sci. Math. Sci., 97 (1987), 3-19.

[Ar2] J. Arthur, A Local Trace formula, Publ. Math. Inst. Hautes Études Sci., 73 (1991), 5 - 96.

[DHS] P. Delorme, P. Harinck and S. Souaifi, Geometric side of a local relative trace formula, arXiv:1506.09112 (47 p.),

[Ha] J. Hakim, Admissible distributions on p-adic symmetric spaces, J. Reine Angew. Math. 455 (1994), 119.

[RR] C. Rader, S. Rallis, Spherical characters on p-adic symmetric spaces, Amer. J. Math., Vol 118, N° 1 (5 Feb. 1996), 91-178.

[R] D. Renard, Représentations des groupes réductifs p-adiques, Cours spécialisés, volume 17, SMF.

[W2] J.-L. Waldspurger, La formule de Plancherel pour les groupes p-adiques (d’après Harish-Chandra), J. Inst. Math. Jussieu 2 (2003) 235 - 333.

[Z] C. Zhang, Local periods for discrete series representations, Preprint, arXiv:1509.06166.