A NOTE ON THE LATTICE STRUCTURE FOR MATCHING MARKETS VIA LINEAR PROGRAMMING

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Abstract. Given two stable matchings in a many-to-one matching market with $q$-responsive preferences, by manipulating the objective function of the linear program that characterizes the stable matching set, we compute the least upper bound and greatest lower bound between them.

1. Introduction. Stable matchings have been studied for several decades, beginning with Gale and Shapley’s pioneering paper, see Gale and Shapley [2]. They introduce the notion of stable matchings and provide an algorithm for finding them. Since then, a considerable amount of work was carried out on both theory and applications of stable matchings. In this paper, we focus on a many-to-one matching market with $q$-responsive preferences. For a very complete survey on these markets, see Roth and Sotomayor [6].

One of the most important results in the matching literature is that the set of stable matchings has a dual lattice structure. This is important for at least two reasons. First, it shows that even if agents of the same side of the market compete for agents of the other side, the conflict is attenuated since, on the set of stable matchings, agents on the same side of the market have a coincidence of interests. Second, many algorithms are based on this lattice structure. For example, algorithms that yield stable matchings in centralized markets. In this paper, we present an alternative method to compute the binary operations using linear programming. Each stable matching is represented by an assignment matrix, called the incidence vector of the stable matching.

For many-to-one matching markets with $q$-responsive preferences, Baïou and Balinski [1] characterize the set of stable matchings as extreme points of a convex polytope generated by a linear inequalities system. For a more restrictive market, the marriage market, Roth et al. [5] introduce a linear program, where the objective function is the sum of all entries of an incidence vector. They characterize the stable
matchings as the integer solutions of this linear program. Setting a similar linear program with the same objective function, the sum of all entries of an incidence vector, but Baiou and Balinski’s linear inequalities as constraints, the set of stable matching of a many-to-one matching market can be characterized as well. Furthermore, the optimal solutions of this linear program do not distinguish among stable matchings (all stable matchings are optimal solutions). This is because, since the set of agents matched in each stable matching is always the same, the objective value in each stable matching is also the same.

Given that each extreme point of the convex polytope generated by the constraints of the linear program is a stable matching, for any modification of the objective function we can assure that the optimal solution is always a stable matching. In this paper, given two stable matchings, we present a modification of the objective function depending on these stable matchings. This modification, assures that the optimal solution of the new linear program is the least upper bound (l.u.b.), for one side of the market, between the two stable matchings. Analogously, we compute the greatest lower bound (g.l.b.) for the same side of the market. Moreover, the same technique can be used to compute the l.u.b. and g.l.b. for the other side of the market.

**Related literature.** For the marriage market, Vande Vate [10] and Rothblum [7] present a linear inequalities system, in which the extreme points of the generated convex polytope corresponds with the incidence vector stable matchings. Roth et al. [5] present a linear program that characterizes the stable matchings as optimal solutions. Using linear programming techniques, they give alternative proofs to many known results of matching theory.

Baïou and Balinski [1] present a linear inequality system that characterizes the stable matchings as extreme points of a convex polytope for many-to-one matching markets with q-responsive preferences. They use different techniques than the ones used in Vande Vate [10], Rothblum [7], and Roth et al. [5]. Sethuraman et al. [9] present an alternative proof to the characterization of stable matchings as extreme points.

Concerning the related literature on lattice structures, Roth and Sotomayor [6] prove that the stable matching set for a many-to-one matching market with q-responsive preferences has a dual lattice structure. They define binary operations to compute the l.u.b. and g.l.b. for each side of the market.

The rest of the paper is organized as follows. In Section 2, we present the matching market. In Section 3, we present our main result that shows how to compute the l.u.b. and g.l.b. via linear programming.

### 2. Preliminaries.

A many-to-one matching market consists of two sets of agents, the set \( F = \{f_1, \ldots, f_n\} \) of firms and the set \( W = \{w_1, \ldots, w_m\} \) of workers. Each firm \( f \) has a maximum number of positions to fill: its quota, denoted by \( q_f \). Furthermore, each agent has a strict, transitive, preferences ordering of the acceptable agents on the other side of the market (\( \succ_f \) and \( \succ_w \)), i.e., those agents on the other side of the market that it prefers to remain unmatched (an acceptable pair fulfills that \( w \succ_f f \) and \( f \succ_w w \)). Let \( A \subseteq F \times W \) denote the set of acceptable pairs.

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1. This result is known as the Rural Hospital Theorem, see Roth [4, 3] among others.
2. Vande Vate [10] consider a marriage market with the same cardinality for the set of men and women, and all mutually acceptable, whereas Rothblum [7] consider a marriage market with different cardinality and possibly non-acceptable agents.
Preference profiles are \((n + m)\)-tuples of preference relations and they are denoted by \(P = (\succ_{f_1}, \ldots, \succ_{f_n}, \succ_{w_1}, \ldots, \succ_{w_m})\). The matching market for the sets \(F\) and \(W\) with the preference profile \(P\) and vector of quotas \(q\) is denoted by \((F, W, P, q)\). We consider the somewhat restrictive market of responsive preferences in which the firms have preferences over individual workers, not over groups of workers. That is to say, each firm \(f\) orders subsets of workers in a responsive manner when adding “good” workers to a set leads to a better set, whereas adding “bad” workers to a set leads to a worse set. In addition, for any two subsets that differ in only one worker, the firm prefers the subset containing the most preferred worker.\(^3\) We refer the reader to Roth and Sotomayor [6] for an interesting discussion on this issue.

A matching \(\mu\) is a mapping from the set \(F \cup W\) into the set of all subsets of \(F \cup W\) such that, for all \(w \in W\) and all \(f \in F\),

1. \(|\mu(w)| = 1\) and if \(\mu(w) \neq \{w\}\), then \(\mu(w) \subseteq F\).
2. \(\mu(f) \in 2^W\) and \(|\mu(f)| \leq q_f\).
3. \(\mu(w) = \{f\}\) if and only if \(w \in \mu(f)\).

Usually, we will omit the curly brackets, for instance, instead of condition 1. and 3., we will write: “1. \(|\mu(w)| = 1\) and if \(\mu(w) \neq w\), then \(\mu(w) \subseteq F\),” and “3. \(\mu(w) = f\) if and only if \(w \in \mu(f)\).”

Let \(\mu \succ_f \mu'\) denote that all firms like \(\mu\) at least as well as \(\mu'\) with at least one firm strictly preferring \(\mu\) to \(\mu'\), that is, \(\mu(f) \succeq_f \mu'(f)\) for all \(f \in F\) and \(\mu(f') \succ_f \mu'(f')\) for at least one firm \(f' \in F\). We say that \(\mu \succeq_F \mu'\) means that either \(\mu \succ_F \mu'\) or \(\mu = \mu'\). Analogously, define \(\mu \succeq_W \mu'\) and \(\mu \succeq_W \mu'\).

We say that matching \(\mu\) is individually rational if \(\mu(w) = f\) for some worker \(w\) and firm \(f\), then \((f, w)\) is an acceptable pair. Similarly, a pair \((f, w)\) is a blocking pair for matching \(\mu\), if the worker \(w\) is not employed by the firm \(f\), but they both prefer to be matched to each other. That is, matching \(\mu\) is blocked by a firm-worker pair \((f, w)\):

1. If \(|\mu(f)| = q_f\), \(f \succeq_w \mu(w)\) and \(w \succ_f w'\) for some \(w' \in \mu(f)\).
2. If \(|\mu(f)| < q_f\), \(\mu(w) \neq f\) and \(f \succeq_w \mu(w)\) and \(w \succ_f \emptyset\).

Thus, a matching \(\mu\) is stable if it is individually rational and has no blocking pairs. We denote by \(S(P)\) to the set of all stable matchings at the preference profile \(P\).

Roth and Sotomayor [6] proved that the set of stable matchings have a dual lattice structure. To do so, given two stable matchings \(\mu\) and \(\mu'\), they define \(\mu \vee_F \mu'\) as the following matching:

\[
\mu \vee_F \mu'(f) = \max_{\succ_f} \{\mu(f), \mu'(f)\},
\]

for each \(f \in F\). In words, each firm \(f\) is matched with the most preferred set of workers between \(\mu(f)\) and \(\mu'(f)\). Similarly, they define \(\mu \wedge_F \mu'\) as the following matching:

\[
\mu \wedge_F \mu'(f) = \min_{\succ_f} \{\mu(f), \mu'(f)\},
\]

\(^3\)The preference relation \(\succ_f\) over \(2^W\) is \(q_f\)-responsive if it satisfies the following conditions:

1. For all \(T \subseteq W\) such that \(|T| > q_f\), we have that \(\emptyset \succ_f T\).
2. For all \(T \subseteq W\) such that \(|T| < q_f\) and \(w \notin T\), we have that \(T \cup \{w\} \succ_f T\) if and only if \(w \succ_f \emptyset\).
3. For all \(T \subseteq W\) such that \(|T| < q_f\) and \(w, w' \notin T\), we have \(T \cup \{w\} \succ_f T \cup \{w'\}\) if and only if \(w \succ_f w'\).
for each $f \in F$. In words, each firm $f$ is matched with the less preferred set of workers between $\mu(f)$ and $\mu'(f)$. Analogously, they define the matchings $\mu \lor_F \mu'$ and $\mu \land_F \mu'$, concerning the preference lists of the workers. Roth and Sotomayor [6] proved that $\mu \lor_F \mu'$, $\mu \land_F \mu'$, $\mu \land_W \mu'$ and $\mu \lor_W \mu'$ are stable matchings. Moreover,

$$l.u.b. \{ \mu, \mu' \} = \mu \lor_F \mu', \quad g.l.b. \{ \mu, \mu' \} = \mu \land_F \mu',$$

$$l.u.b. \{ \mu, \mu' \} = \mu \lor_W \mu', \quad g.l.b. \{ \mu, \mu' \} = \mu \land_W \mu'.$$

In the rest of the paper, we only present our result for one side of the market, the firms’ side. The workers’ side is a particular case.

Let $\mu$ and $\mu'$ be two stable matchings. Define $C^0_f(\mu, \mu') = \{ w \in W : w \in \mu(f) \cup \mu'(f) \}$, be the set of all workers assigned with firm $f$ either under $\mu$ or $\mu'$. Also, let $\tilde{C}^1_f(\mu, \mu') = \{ w \in C^0_f(\mu, \mu') : w = \max_{x \in \mu} \{ w' : w' \in C^0_f(\mu, \mu') \} \}$, be the set (singleton) of the most preferred worker in $C^0_f(\mu, \mu')$. For $k \geq 2$, define $\tilde{C}^k_f(\mu, \mu') = \{ w \in C^0_f(\mu, \mu') : \text{there is no } w' \in C^0_f(\mu, \mu') \setminus \tilde{C}^{k-1}_f(\mu, \mu'), w >_f w' \}$. In words, $\tilde{C}^k_f(\mu, \mu')$ is the set of the $k$-best workers in $C^0_f(\mu, \mu')$. Analogously, let $\tilde{C}_f^1(\mu, \mu') = \{ w \in C^0_f(\mu, \mu') : w = \min_{x \in \mu} \{ w' : w' \in C^0_f(\mu, \mu') \} \}$, be the set (singleton) of the least preferred worker in $C^0_f(\mu, \mu')$. For $k \geq 2$, define $\tilde{C}_f^k(\mu, \mu') = \{ w \in C^0_f(\mu, \mu') : \text{there is no } w' \in C^0_f(\mu, \mu') \setminus \tilde{C}_f^{k-1}(\mu, \mu'), w >_f w' \}$. In words, $\tilde{C}_f^k(\mu, \mu')$ is the set of the $k$-worst workers in $C^0_f(\mu, \mu')$.

The following lemma presents an important fact of the responsive property of the preferences.

**Lemma 2.1** (Roth and Sotomayor [6]). Let $\mu$ and $\mu'$ be two stable matchings. If $\mu(f) >_f \mu'(f)$ for some $f \in F$, then $w >_f w'$ for each $w \in \mu(f)$ and $w' \in \mu'(f) \setminus \mu(f)$. That is, $f$ prefers every worker assigned under $\mu$ to every worker who is assigned under $\mu'$ but not under $\mu$.

The following remark, derived from the previous lemma, states that for each firm $f$, $\mu \lor_F \mu'(f)$ is not only the best subset of workers between $\mu(f)$ and $\mu'(f)$ but also coincide with the $q_f$-best workers among those that are assigned either $\mu(f)$ or $\mu'(f)$. Analogously, for $\mu \land_F \mu'(f)$.

**Remark 1.** Given two stable matchings $\mu$ and $\mu'$, following Lemma 2.1 we have

i) $\mu \lor_F \mu'(f) = \tilde{C}^q_f(\mu, \mu')$ for each $f \in F$.

ii) $\mu \land_F \mu'(f) = \tilde{C}^0_f(\mu, \mu')$ for each $f \in F$.

3. **Main result.** Baïou and Balinski [1] characterize the convex hull of all stable matchings for a many-to-one matching market, where firms have $q$-responsive preferences. That is, the extreme points of the convex polytope generated by Baïou and Balinski’s linear inequalities are exactly the stable matchings of the many-to-one matching market. We use the Baïou and Balinski formulation as constraints of two linear programs, $LP(l.u.b.)$ and $LP(g.l.b.)$, to compute the $l.u.b.$ and the $g.l.b.$ between two stable matchings respectively. Let us denote by $B - B$ to be the convex polytope generated by Baïou and Balinski’s characterization.

A matching $\mu$, is represented by an **incidence vector**, i.e., a vector $x^\mu \in \{0, 1\}^{F \times |W|}$ holding that $x^\mu_{f,w} = 1$ if and only if $\mu(w) = f$ and $x^\mu_{f,w} = 0$ otherwise.

To compute the binary operations via linear programming, first we need to define a weight correlated with the firms’ preferences.
Definition 3.1. Let \((F, W, P, q)\) be a many-to-one matching market. Denote by \(\alpha_f\) a weight correlated with the preferences of the firm \(f\) as follows:

i) \(\alpha_{f,w} > 0\) if and only if \(w\) is acceptable for \(f\).

ii) \(\alpha_{f,w} > \alpha_{f,w'}\) if and only if \(w \succ_f w'\), for each pair \((f, w), (f, w') \in A\).

Notice that for \(\mu \in S(P)\), the following holds:

\[
\sum_{j \in W} \alpha_{f,j} x_{f,j}^\mu = \sum_{j \in \mu(f)} \alpha_{f,j} \quad \text{and} \quad \sum_{i \in F} \alpha_{i,w} x_{i,w}^\mu = \alpha_{\mu(w),w}.
\]

From now on, \(\mu\) and \(\mu'\) represent any two stable matchings. Let \(\text{supp}(\mu)\) denote the support of the stable matching \(\mu\), i.e., \(\text{supp}(\mu) = \{(i, j) \in F \times W : x_{i,j}^\mu > 0\}\). Moreover, denote \(\text{supp}(\mu, \mu') = \text{supp}(\mu) \cup \text{supp}(\mu')\). Let us define the following two linear programs:

\[
\begin{align*}
\text{LP}(l.u.b.) & \quad \max_{(i,j) \in \text{supp}(\mu, \mu')} \alpha_{i,j} x_{i,j} \\
\text{st} & \quad x \in B - B
\end{align*}
\]

\[
\begin{align*}
\text{LP}(g.l.b.) & \quad \max_{(i,j) \in \text{supp}(\mu, \mu')} \frac{1}{\alpha_{i,j}} x_{i,j} \\
\text{st} & \quad x \in B - B
\end{align*}
\]

Notice that both objective functions depend on the two stable matchings \(\mu\) and \(\mu'\). Recall that, by Baïon and Balinski’s characterization, the extreme points of the convex polytope \(B - B\) are exactly the stable matchings. Thus, by using linear programming results,\(^4\) we can state the following remark.

Remark 2. There is a stable matching such that its incidence vector is an optimal solution of \(\text{LP}(l.u.b.)\) or \(\text{LP}(g.l.b.)\).

Remark 3. Notice that, by definition of the binary operations \(\vee_F\) and \(\wedge_F\), and Remark 1, we have that and for each \(f \in F\), the following holds:

i) \(\sum_{j \in \mu \vee_F \mu'(f)} \alpha_{f,j} = \sum_{j \in \wedge^\mu_{f} (\mu, \mu')} \alpha_{f,j}\),

ii) \(\sum_{j \in \mu \wedge_F \mu'(f)} \alpha_{f,j} = \sum_{j \in \wedge^\mu_{f} (\mu, \mu')} \alpha_{f,j}\),

iii) \(\text{supp}(\mu \vee_F \mu') \subseteq \text{supp}(\mu, \mu')\) and \(\text{supp}(\mu \wedge_F \mu') \subseteq \text{supp}(\mu, \mu')\).

Condition ii) implies that

\[
\sum_{j \in \mu \wedge_F \mu'(f)} \frac{1}{\alpha_{f,j}} = \sum_{j \in \wedge^\mu_{f} (\mu, \mu')} \frac{1}{\alpha_{f,j}}.
\]

Our main result states that the incidence vectors of \(\mu \vee_F \mu'\) and \(\mu \wedge_F \mu'\) are the unique optimal solutions for \(\text{LP}(l.u.b.)\) and \(\text{LP}(g.l.b.)\) respectively.

Theorem 3.2. Let \(\mu\) and \(\mu'\) be two stable matchings. The incidence vector of the stable matching \(\mu \vee_F \mu'\) (\(\mu \wedge_F \mu'\)) is the unique optimal solution for the linear program \(\text{LP}(l.u.b.)\) (\(\text{LP}(g.l.b.)\)).

Proof. Let \(\mu\) and \(\mu'\) be two stable matchings. We prove that the incidence vector of the stable matching \(\mu \vee_F \mu'\) is the unique optimal solution for the linear program \(\text{LP}(l.u.b.)\). First, we prove that \(\mu \vee_F \mu'\) is an optimal solution of \(\text{LP}(l.u.b.)\), and then we prove that this solution is unique. Thus, by Remark 2, let \(\lambda\) be a stable matching such that \(x^\lambda\) is an optimal solution for the linear program \(\text{LP}(l.u.b.)\).

\(^4\)There is an extreme point of the convex polytope that is an optimal solution of the linear program, see Schrijver [8] for more detail.
Since \( x^\lambda \) is the optimal solution of \( LP(l.u.b.) \) and \( \text{supp}(\mu \vee_F \mu') \subseteq \text{supp}(\mu, \mu') \), then
\[
\sum_{(i,j) \in \text{supp}(\mu, \mu')} \alpha_{i,j} x^\lambda_{i,j} \leq \sum_{(i,j) \in \text{supp}(\mu, \mu')} \alpha_{i,j} x^\lambda_{i,j}.
\]
(1)

To prove that (1) is satisfied with equality, we proceed as follows: denote by \( \text{supp}_F(\mu) = \{ i \in F : x^\mu_{i,\mu(i)} > 0 \} \), and \( \text{supp}_F(\mu, \mu') = \text{supp}_F(\mu) \cup \text{supp}_F(\mu') \).
\[
\sum_{(i,j) \in \text{supp}(\mu, \mu')} \alpha_{i,j} x^\lambda_{i,j} = \sum_{i \in \text{supp}_F(\mu, \mu')} \sum_{j \in \lambda(i)} \alpha_{i,j} \leq \sum_{i \in \text{supp}_F(\mu, \mu')} \sum_{j \in \text{supp}(\mu, \mu')} \alpha_{i,j}.
\]
By Remark 3 (i),
\[
\sum_{i \in \text{supp}_F(\mu, \mu')} \sum_{j \in \text{supp}(\mu, \mu')} \alpha_{i,j} = \sum_{i \in \text{supp}(\mu \vee_F \mu')} \sum_{j \in \text{supp}(\mu \vee_F \mu')} \alpha_{i,j}.
\]
Thus,
\[
\sum_{(i,j) \in \text{supp}(\mu, \mu')} \alpha_{i,j} x^\lambda_{i,j} \leq \sum_{(i,j) \in \text{supp}(\mu, \mu')} \alpha_{i,j} x^\lambda_{i,j}.
\]
(2)

By (1) and (2),
\[
\sum_{(i,j) \in \text{supp}(\mu, \mu')} \alpha_{i,j} x^\lambda_{i,j} = \sum_{(i,j) \in \text{supp}(\mu, \mu')} \alpha_{i,j} x^\lambda_{i,j}.
\]
(3)

By definition of \( \hat{C}^\text{Rf}_f(\mu, \mu') \), we have \( \hat{C}^\text{Rf}_f(\mu, \mu') \subseteq C^\text{Rf}_f(\mu, \mu') \) for each \( f \in F \).

Moreover, by Remark 3 (i) we have that \( \mu \vee_F \mu' \) assign to each \( f \) the \( q_f \)-best workers in \( C^\text{Rf}_f(\mu, \mu') \) and therefore \( \mu \vee_F \mu' \) is an optimal solution of \( LP(l.u.b.) \). Assume now that \( \lambda \neq \mu \vee_F \mu' \). That is, there is a firm \( \hat{f} \in \text{supp}(\mu, \mu') \) such that \( \lambda(\hat{f}) \neq \mu \vee_F \mu'(\hat{f}) \). Recall that by Remark 3 \( \hat{C}^\text{Rf}_f(\mu, \mu') = \mu \vee_F \mu'(\hat{f}) \), thus \( \sum_{j \in \mu \vee_F \mu'(\hat{f})} \alpha_{f,j} \geq \sum_{j \in \lambda(\hat{f})} \alpha_{f,j} \). This implies, by (3), that there is \( f' \in \text{supp}(\mu, \mu') \) such that \( \sum_{j \in \mu \vee_F \mu'(f')} \alpha_{f',j} < \sum_{j \in \lambda(f')} \alpha_{f',j} \), which in turns is a contradiction of \( \mu \vee_F \mu'(f') = \hat{C}^\text{Rf}_f(\mu, \mu') \). Therefore, \( \lambda = \mu \vee_F \mu' \).

Analogous argument shows that the incidence vector of \( \mu \wedge_F \mu' \) is the unique optimal solution of the linear program \( LP(g.l.b.) \).

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