Abstract

We study sets of mutually orthogonal Latin rectangles (MOLR), and a natural variation of the concept of self-orthogonal Latin squares which is applicable on larger sets of mutually orthogonal Latin squares and MOLR, namely that each Latin rectangle in a set of MOLR is isotopic to each other rectangle in the set. We call such a set of MOLR homogeneous.

In the course of doing this, we perform a complete enumeration of non-isotopic, and non-paratopic, sets of \(t\) mutually orthogonal \(k \times n\) Latin rectangles for \(k \leq n \leq 7\), for all \(t < n\). Additionally, for larger \(n\) we have enumerated homogeneous sets of MOLR, as well as sets of MOLR where the autotopism group acts transitively on the rectangles, and we call such sets of MOLR transitive.

We build the sets of MOLR row by row, and in this process we also keep track of which of the MOLR are homogeneous and/or transitive in each step of the construction process. We use the prefix stepwise to refer to sets of MOLR with this property.

Sets of MOLR are connected to other discrete objects, notably finite geometries and certain regular graphs. Here we observe that all projective planes of order at most 9 except the Hughes plane can be constructed from a stepwise transitive MOLR.
1 Introduction

Two Latin squares $L_A = (a_{ij})$ and $L_B = (b_{ij})$ of order $n$, see Section 2 for background definitions, are said to be orthogonal if $|\{(a_{ij}, b_{ij}) : 1 \leq i, j \leq n\}| = n^2$, that is, if when superimposing $L_A$ and $L_B$, we see each of the possible $n^2$ ordered pairs of symbols exactly once. Pairs of orthogonal Latin squares show up in many different areas of combinatorics, both applied and pure. On the applied side, Latin squares are well known as the source of statistical designs for experiments. For more involved experiments one may instead switch to a design based on a pair of orthogonal Latin squares, or more generally one can use a set of pairwise orthogonal Latin squares, referred to as mutually orthogonal Latin squares (MOLS). Here it is also possible to use mutually orthogonal $k \times n$ Latin rectangles (MOLR) when that suits the requirements of the statistical study. A less applied application of orthogonal Latin squares comes from the study of finite geometries. Here it is well known that a finite projective or affine plane of order $n$ exists if and only if a set of $(n - 1)$ pairwise orthogonal $n \times n$ Latin squares exists. Motivated by connections like these, the first aim of this paper has been to perform a complete enumeration of distinct, in the suitable sense, sets of $t$ mutually orthogonal $k \times n$ Latin rectangles, for as large values of the different parameters as possible. Our second aim has been to push this enumeration further for a few special classes of such sets of MOLR. Our computational results regarding MOLR here extend those in [19]. Earlier work on MOLS is surveyed in [13].

A particular type of orthogonal Latin square which has seen use as a design is self-orthogonal squares. A Latin square $L$ is said to be self-orthogonal if $L$ and its transpose $L^T$ are orthogonal. The existence of $n \times n$ self-orthogonal Latin squares (SOLS) for a number of values of $n$, not including $n = 10$ was established through many different constructions of infinite families of SOLS, e.g. in the seminal work by Mendelsohn [29]. In [20], Hedayat presented the first example of a self-orthogonal Latin square of order 10, and in [7], Brayton et al. showed that $n \times n$ self-orthogonal Latin squares exist for each $n \neq 2, 3, 6$. Self-orthogonal Latin squares of orders up to and including $n = 9$ were completely enumerated by Burger et al. in [12], and by the same authors for order 10 in [11].
Being self-orthogonal is a special case of the concept of being conjugate orthogonal, a concept introduced by Stein in [36]. Here a conjugate is defined by a permutation \( \sigma \in S_3 \) which interchanges the roles of rows, columns, and symbols in the square. The transpose corresponds to the \( \sigma \) which interchanges the roles of rows and columns. As Stein showed, for every \( \sigma \) except the identity there are Latin squares which are orthogonal to their \( \sigma \)-conjugate, and Phelps [33] later investigated which \( n \) are possible for a given \( \sigma \), settling the question with a handful of exceptions. These exceptions were later settled in [4].

Regarding conjugate orthogonality, one may sometimes find Latin squares which have several pairwise orthogonal conjugates, and it is possible to find examples where all six conjugates are pairwise orthogonal [2, 3]. This construction can of course not produce larger sets than six, and produces even smaller sets if we want to apply it to Latin rectangles. However, if transposition is viewed as one of many possible ‘equivalence transformations’ \( \tau \), \( \tau(L) = L^T \), then replacing \( \tau \) with some other form of ‘equivalence’ gives rise to a similar concept, where potentially a set of mutually orthogonal objects can be larger.

In the present paper, we employ the equivalence notion of isotopism, where two Latin squares \( L_A \) and \( L_B \) are said to be isotopic if there exists a permutation \( \pi_s \) of the symbols, a permutation \( \pi_r \) of the rows and a permutation \( \pi_c \) of the columns such that when applying all three permutations to \( L_A \), we get \( L_B \). We also consider paratopism, in which one also allows the interchange of the roles of columns and symbols.

The equivalence concept isotopism is a natural one when studying Latin rectangles. In particular, taking transposes does not map \( k \times n \) Latin rectangles to \( k \times n \) Latin rectangles. In the special case \( k = n \), where the Latin rectangles are in fact Latin squares, taking transposes may be taken into account, so as to study a proper generalization of the concept of self-orthogonality.

For small orders, we enumerate all sets of mutually orthogonal rectangles. For larger orders, where full enumeration of all MOLR is not feasible, we have proceeded using several natural subclasses of sets of MOLR. The first such class consists of those MOLR where each constituent Latin rectangle is isotopic to every other Latin rectangle in the set. We call such a set of MOLR homogeneous. The next class are the transitive sets of MOLR. A set of MOLR is transitive if the
autotopism group of the set of MOLR acts transitively on the set of rectangles. So, every transitive set of MOLR is homogeneous, but the reverse is not always true. Finally we consider stepwise transitive and stepwise homogeneous sets of MOLR, which are even more restricted classes, where we require that the set of MOLR can be constructed by adding one row at a time in such way that each set of MOLR along the way is transitive or homogeneous, respectively. These conditions are very restrictive, but e.g. all but one of the finite projective and affine planes of small orders turn out to have a corresponding stepwise transitive $(n − 1)$-MOLR.

We will also discuss how our results lead to a complete enumeration of certain finite geometries, in particular, resolvable projective, affine and hyperbolic planes.

The work in the present paper builds on and extends results from a previous paper of the current authors [22], where our focus was only on triples of MOLR. One distant goal is to approach the question of how large a maximum set of MOLS of order 10 is.

The paper is structured as follows. In Section 2 we give basic notation and formal definitions. In Section 3 we state the questions guiding our investigation, describe briefly the algorithm used to find all sets of MOLR and give some practical information regarding the computer calculations. In Section 4 we give further background on finite geometries.

In Section 5 we present the data our computer search resulted in, together with further analysis and results. In particular, in Section 5.1 we give the number of non-isotopic $t$-MOLR for each order we treated, and as our enumeration is complete up to order $n = 7$, in Section 5.2 we discuss our enumeration of subclasses of sets of larger order, in Section 5.3 we discuss the autotopism groups of the sets of MOLR, and finally in Section 5.4 we phrase some of our results in the language of finite geometries.

Section 6 concludes the main text, and is followed by a number of appendices containing more detailed data on autotopism groups sizes and all non-isotopic stepwise transitive $8$-MOLR with $n = 9$. 

4
2 Basic Notation and Definitions

A Latin square of order \( n \) is an \( n \times n \) matrix with cells filled by \( n \) symbols such that each row and each column contains each symbol exactly once. For \( k \leq n \) a matrix with \( k \) rows and \( n \) columns whose cells are filled by \( n \) symbols such that each row contains each symbol exactly once and each column contains each symbol at most once is called a \( k \times n \) Latin rectangle. In the following we use as symbol set \{0, 1, \ldots, n-1\}. We denote the \( t \)-th row of a \( k \times n \) rectangle \( A \) by \( A_t \) for \( t \in \{1, 2, \ldots, k\} \).

With mutually orthogonal Latin squares defined as above, we can extend the orthogonality condition to Latin rectangles. We say that two \( k \times n \) Latin rectangles \( A = (a_{i,j}) \) and \( B = (b_{i,j}) \) are orthogonal if each ordered pair \((a_{i,j}, b_{i,j})\) appears at most once. Also, a set of pairwise orthogonal Latin rectangles is called a set of Mutually Orthogonal Latin Rectangles (MOLR). A set of \( t \) pairwise orthogonal Latin rectangles is called a \( t \)-MOLR for short.

Let \((A_1, A_2, \ldots, A_t)\) be a \( t \)-MOLR of size \( k \times n \). The following group of isotopisms acts on the set of \( t \)-MOLR: \( G_{n,k,t} = S_t \times S_k \times S_n \times [S_n \times S_n \times \ldots \times S_n] \), where \( S_t \) corresponds to a permutation of the rectangles, \( S_k \) corresponds to a permutation of the rows, \( S_n \) corresponds to a permutation of the columns, and each of the last \( t \) \( S_n \) correspond to a permutation of the symbols in a single rectangle. Two \( t \)-MOLR \( A \) and \( A' \) of size \( k \times n \) are isotopic if there exists a \( g \in G_{n,k,t} \) such that \( g(A) = A' \). The autotopism group of a \( t \)-MOLR \( A \) is defined as \( \text{Aut}(A) := \{g \in G_{n,k,t} | g(A) = A\} \). The isotopism group is also a subgroup of the paratopism group, which is obtained from the isotopism group by making it possible to interchange the roles of column indices and symbol names. We refer the reader to Section 2 of [15] for an in depth discussion of the different symmetry and equivalence concepts for MOLS and MOLR.

We say that a \( t \)-MOLR is normalized if it satisfies the following conditions:

(S1) (Ordering among columns) The first row of each rectangle is the identity permutation.

(S2) (Ordering among rectangles) The second row of the \( i \)-th rectangle is lexicographically larger than the second row of the \((i+1)\)-th
rectangle. In other words, if \(a_1, a_2, \ldots, a_t\) are symbols in the position \((2,1)\) in the \(t\)-MOLR, seen as an ordered \(t\)-tuple, then it holds that \(a_1 > a_2 > \ldots > a_t\).

(S3) (Ordering among rows) The second row in the first rectangle is lexicographically larger than the third row, the third row is larger than the fourth row, and so on.

We use normalization to reduce the search space in our computer runs. For details, see [22]. Finally, as our computations proceed by adding consecutive rows to \(t\)-MOLR, we will have use for the following term: An extension of size \(k \times n\) is a \(t\)-MOLR which results from a \(t\)-MOLR of size \((k - 1) \times n\) by adding one more row to each rectangle.

3 Generation of \(t\)-MOLR

The basic method for our generation routine is quite simple. We start with the set of all \(1 \times n\) \(t\)-MOLR and find all possible extensions to \(2 \times n\) \(t\)-MOLR, followed by an isotopy reduction where only one representative for each isotopy class is kept. As part of the isotopy reduction, the autotopism group of each representative is also determined and its size stored. The extension step is then repeated until the desired number of rows is reached. The algorithms and methods used to generate the \(t\)-MOLR are rather straightforward extensions of those used in [22]. They were implemented in C++ and run in a parallelized version on the Kebnekaise and Abisko supercomputers at High Performance Computing Centre North (HPC2N). The total run time for all the data in the paper was a few hundred core-years.

In order to safeguard the correctness of our computational results several steps were taken. We wrote separate implementations in Mathematica of both the main algorithm and another much simpler method with which we performed an independent generation of the data for smaller sizes, in order to help verify the correctness of the C++ implementation. All data has been compared with the known classifications of MOLS and MOLR in the literature, and agrees with them.

The method described was applied directly in order to first generate all \(t\)-MOLR of a given size when this was possible, and after each
generation step we also classified the \( t \)-MOLR which belonged to one of the following two classes.

**Definition 3.1.** (a) A \( t \)-MOLR \( A = \{A_1, A_2, \ldots, A_t\} \) is homogeneous if any pair of rectangles \( A_i, A_j \in A \) are isotopic.

(b) A \( t \)-MOLR \( A \) is transitive if \( \text{Aut}(A) \) acts transitively on the set of rectangles in \( A \). That is, for any \( A_i, A_j \in A \) there exists \( \phi \in \text{Aut}(A) \) which maps \( A_i \) to \( A_j \).

In order to verify that a \( t \)-MOLR is homogeneous or transitive we require the entire \( t \)-MOLR, which becomes a problem when generating \( t \)-MOLR by adding rows one by one. In order to find all homogeneous \( t \)-MOLR we have to first generate all \( t \)-MOLR and then test them for homogeneity and transitivity. We are therefore also interested in two special classes of \( t \)-MOLR which allow for more efficient generation.

**Definition 3.2.** (a) An \( k \times n \) \( t \)-MOLR \( A \) is stepwise homogeneous if \( A \) is homogeneous and either \( k = 1 \), or \( k > 2 \) and \( A \) is the extension of a stepwise homogeneous \( (k-1) \times n \) \( t \)-MOLR.

(b) An \( k \times n \) \( t \)-MOLR \( A \) is stepwise transitive if \( A \) is transitive and either \( k = 1 \), or \( k > 2 \) and \( A \) is the extension of a stepwise transitive \( (k-1) \times n \) \( t \)-MOLR.

As we shall demonstrate below, the classical sets of \( (n-1) \) MOLS corresponding to a projective plane over a finite field of order \( n \) are, in fact, always stepwise transitive.

For both of the classes in Definition 3.2 we can generate the \( k \times n \) \( t \)-MOLR by starting out with the corresponding class of \( (k-1) \times n \) \( t \)-MOLR, finding all non-isotopic extensions of these and then discarding those which are not homogeneous or transitive, respectively. The restrictions typically lead to a far smaller set of \( t \)-MOLR to extend to the next value of \( k \), and thanks to this we were able to do complete enumeration of these classes for larger values of \( n \) than in the general case.

In addition to the objects themselves, we also calculated the size of the autotopism group of each object. With some exceptions due to size restrictions, all the data we generated is available for download at [1]. Further details about the organization of the data are given there.
We also include the size of the paratopism classes of MOLR in our tables. In order to obtain these we used the Nauty package \cite{28} to reduce the set of isotopism classes to paratopism classes. In order to do this the MOLR were encoded as graphs, using to the method described in \cite{15}.

4 Finite Geometries

As mentioned in the introduction, sets of \(n-1\) MOLS of size \(n\) have a well known connection to finite projective and affine planes. Here we will recollect some facts from finite geometry and demonstrate how a general \(t\)-MOLR can be translated into the finite geometry setting.

**Definition 4.1.** A pair \(\mathcal{P} = (V, L)\), where \(V\) is a finite set of points and \(L\) is a set of subsets of \(V\), which are called lines, is a finite plane if the following conditions are satisfied.

1. Each line has size at least 2.
2. Any pair of points is contained in exactly one line.
3. There exists a point \(p\) and a line \(\ell\), where \(\ell\) does not contain \(p\).
4. There exists a set of four points such that no three of them lie on the same line.

Additionally, the plane may satisfy one of the following parallelity properties:

\((P1)\) Every pair of lines have non-empty intersection.

\((P2)\) Given a point \(p\) and a line \(\ell\), which does not contain \(p\), there exists exactly one line through \(p\) which does not intersect \(\ell\).

\((P3)\) Given a point \(p\) and a line \(\ell\), which does not contain \(p\), there exist at least two lines through \(p\) which do not intersect \(\ell\).

A plane which satisfies \((P1)\) is called a finite projective plane, a plane which satisfies \((P2)\) is called a finite affine plane, and a plane which satisfies \((P3)\) is called a finite hyperbolic plane.
A collection $P$ of non-intersecting lines from $\mathcal{P}$ which form a partition of $V$ is called a parallel class. A partition of the lines of $\mathcal{P}$ into parallel classes is called a resolution of $\mathcal{P}$, and a geometry which has at least one resolution is called a resolvable geometry.

A set of $n-1$ MOLS of order $n$ is equivalent to a finite projective plane of order $n$, as pointed out by Bose [5]. However, the existence of a set of $(n-1)$ MOLS, for $n = p^r$, for a prime $p$, was demonstrated in a somewhat forgotten paper from 1896 by Moore [30]. See [16] for a historical discussion of that paper and the wider context of 19th-century design theory.

To put our results on Latin rectangles and squares in the context of finite geometries, we will first recall the explicit correspondence between finite projective geometries on the one hand, and complete sets of mutually orthogonal Latin squares on the other hand. We will begin with noting that a finite projective plane has $n^2 + n + 1$ points and $n^2 + n + 1$ lines each of cardinality $n + 1$, and that each point in a finite projective plane of order $n$ belongs to $n + 1$ lines.

To get the correspondence between a projective plane of order $n$ and a set of $n-1$ MOLS of order $n$, we first pick a line of the projective plane which we call the line at infinity, $L_\infty$. Next, we fix two (arbitrary) points $x_\infty$ and $y_\infty$ on $L_\infty$, and label the remaining $n-1$ points on $L_\infty$ by $\ell_1, \ell_2, \ldots, \ell_{n-1}$. Also, label the $n$ lines containing $x_\infty$ by $X_1, X_2, \ldots, X_n$, and likewise for the $n$ lines containing $y_\infty$, that is, lines $Y_1, Y_2, \ldots, Y_n$.

The points $\ell_1, \ell_2, \ldots, \ell_{n-1}$ will correspond to the $n-1$ Latin squares in the set of MOLS, and the point $p_{i,j}$ at the intersection between $X_i$ and $Y_j$ will correspond to the cell with $(i, j)$ in the Latin squares. The symbol that goes in cell $(i, j)$ in $L_k$ is given by fixing a labelling using symbols $s_1, s_2, \ldots, s_n$ of the $n$ lines (excluding $\ell_\infty$) that contain $\ell_k$, and checking which of the symbols was assigned to the unique line passing through $p_{i,j}$.

Note that the ample room for arbitrary choices of labelings gives different sets of MOLS, and that the construction can easily be reversed, so as to give a finite projective plane from a set of MOLS.

When $\mathcal{P}_n$ is the classical Galois plane over the finite field $GF(n)$ one can give a simple explicit form for a set of MOLS derived from that plane. Let $x \in GF(n)$ be a generator for the multiplicative group of $GF(n)$ and set $a_0 = 0, a_1 = 1, a_2 = x, \ldots, a_i = x^{i-1}$ for
$i = 0, 1, \ldots, n - 2$. Now, for $1 \leq k \leq n - 1$ define a Latin square $L_k$ by setting position $(i, j)$ equal to $a_i + a_k \times a_j$. As shown in [23] this defines a set of $(n - 1)$ MOLS, which also define the classical Galois plane over $GF(n)$.

For prime power $n$, finite projective planes can always be constructed using the Galois field $GF(n)$, but there are projective planes that do not arise in this way. For example, as early as 1907, Veblen and MacLagan-Wedderburn [38] constructed 3 projective planes of order 9, not isomorphic to the standard projective plane of order 9 arising from $GF(9)$. Considerably later, Lam et al. [25] showed by an exhaustive computer search that these 4 projective planes of order 9 are, in fact, the only ones.

For some other $n$, notably $n = 6$ and $n = 14$, the Bruck-Ryser theorem [10] excludes the possibility of a projective plane. For $n = 6$, non-existence was already known, since Tarry [37] had proven that Euler’s 36 officers problem (which asked for a set of just 2 MOLS of order 6) had no solution, leaving $n = 10$ the smallest open case. A delightful account of the search for a projective plane of order 10 can be found in [24], and the non-existence was settled in [26]. Combining this computational non-existence result with a result of Shrikhande [35], one gets that there does not exist a set of 7 or more MOLS of order 10. Since there are examples of pairs of orthogonal Latin squares of order 10, the maximum number $t_{10}$ of MOLR for $n = 10$ lies in the interval $2 \leq t_{10} \leq 6$.

Given a projective plane $\mathcal{P}$ of order $n$ we can construct an affine plane of the same order by deleting a line and all the points on it from $\mathcal{P}$. It is also well known that any affine plane can be obtained from a projective plane in this way. So, the existence of an affine or a projective plane of a given order are equivalent, and in turn equivalent to the existence of a set of $n - 1$ MOLS of order $n$. Finite hyperbolic geometries have not been studied in as great detail as the projective ones. The first axiomatization for finite hyperbolic planes was given in [18] and a few early constructions and structural theorems were given in [14, 21, 32]. Sandler [34] also noted that one will obtain a finite hyperbolic plane from a projective plane by deleting three lines that do not intersect in a single point, together with all points on these lines, or equivalently by deleting two parallel lines from an affine plane.
Another well studied class of finite geometries are nets. They were introduced by Bruck [8, 9] who also showed that they, analogous to the situations for projective and affine planes, are equivalent to $t$-MOLS. A net can be constructed in the following way. Let $A = \{A^1, A^2, \ldots, A^t\}$ be a set of mutually orthogonal $n \times n$ Latin squares, and let $V = \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq n\}$ be the point set. For each square $A^s$ and symbol $r$ we let the set of pairs such that $A^s_{i,j} = r$ be a line in $L$ and set $P(A) = (V, L)$. We now note that each line in $P(A)$ has size $n$, each point lies in $t$ lines, two lines defined by different squares intersect in exactly one point (since the squares are orthogonal), the set of lines defined by a single square are pairwise disjoint, and any pair of points lies in exactly one line (again by orthogonality). Additionally, the set of lines defined by a single square form a partition of $V$, i.e., a parallel class, and the parallel classes defined by the different squares form a partition of $L$, so $P(A)$ is a resolvable geometry. The resulting geometry here is a net of order $n$ and degree $t$. Note that if $t = n - 1$, then the net is a projective plane, and in general, nets are a particular class of partial geometries as defined by Bose [6].

If we apply the same construction as for nets to a $k \times n$ $t$-MOLR we get a weaker geometry which we will here call a partial net. When $k < n$ we lose the property that every pair of lines from different parallel classes intersect. Of particular interest here are $t$-MOLR which are maximal with respect to either $t$ or $k$, since these give rise to geometries which cannot be embedded in a larger partial net with the same value of $n$.

If the geometry $P(A)$ has the property that any pair of points lies in a line, then $P(A)$ is a hyperbolic geometry with curvature $c(P(A)) = t - k$.

Our more symmetric classes of $t$-MOLR here correspond to geometries with certain symmetries. A transitive $t$-MOLR gives rise to a geometry with the property that the automorphism group of the geometry acts transitively on the set of parallel classes in a specific resolution of the geometry. A stepwise transitive set in turn gives us a geometry with the property that the geometry can be built by stepwise adding new points in a way which preserves the transitivity of the resolution.
5 Results and Analysis

We now turn to the results and analysis of our computational work.

5.1 The Number of t-MOLR

In Tables 1–7 we present data on the number of non-isotopic and non-paratopic $k \times n$ t-MOLR.

|     | $2 \times 4$ | $3 \times 4$ | $4 \times 4$ |
|-----|-------------|-------------|-------------|
| $t = 2$ | 3           | 2           | 1           |
| $t = 3$ | 2           | 1           | 1           |

Table 1: The number of non-isotopic t-MOLR for $n = 4$.

|     | $2 \times 5$ | $3 \times 5$ | $4 \times 5$ | $5 \times 5$ |
|-----|-------------|-------------|-------------|-------------|
| $t = 2$ | 5           | 14          | 2           | 2           |
| $t = 3$ | 4           | 1           | 1           | 1           |
| $t = 4$ | 3           | 1           | 1           | 1           |

Table 3: The number of non-isotopic t-MOLR for $n = 5$. 
|       | 2×5 | 3×5 | 4×5 | 5×5 |
|-------|-----|-----|-----|-----|
| t = 2 | 3   | 9   | 1   | 1   |
| t = 3 | 3   | 1   | 1   | 1   |
| t = 4 | 2   | 1   | 1   | 1   |

Table 4: The number of paratopism classes of t-MOLR for \( n = 5 \).

|       | 2×6 | 3×6 | 4×6 | 5×6 | 6×6 |
|-------|-----|-----|-----|-----|-----|
| t = 2 | 28  | 1526| 2036| 85  | 0   |
| t = 3 | 103 | 2572| 513 | 7   | 0   |
| t = 4 | 92  | 118 | 12  | 8   | 0   |
| t = 5 | 33  | 0   | 0   | 0   | 0   |

Table 5: The number of non-isotopic t-MOLR for \( n = 6 \).

|       | 2×6 | 3×6 | 4×6 | 5×6 | 6×6 |
|-------|-----|-----|-----|-----|-----|
| t = 2 | 14  | 575 | 745 | 44  | 0   |
| t = 3 | 44  | 745 | 179 | 5   | 0   |
| t = 4 | 33  | 44  | 5   | 5   | 0   |
| t = 5 | 17  | 0   | 0   | 0   | 0   |

Table 6: The number of paratopism classes of t-MOLR for \( n = 6 \).

|       | 2×7 | 3×7  | 4×7  | 5×7  | 6×7  | 7×7  |
|-------|-----|------|------|------|------|------|
| t = 2 | 100 | 514162| 49415812| 21290125| 11582| 20    |
| t = 3 | 2858| 65883453| 323112477| 55545| 16   | 4     |
| t = 4 | 17609| 35469948| 68659| 204  | 7    | 3     |
| t = 5 | 10626| 22982| 19   | 5    | 5    | 1     |
| t = 6 | 1895 | 23   | 2    | 1    | 1    | 1     |

Table 7: The number of non-isotopic t-MOLR for \( n = 7 \).
Table 8: The number of paratopism classes of $t$-MOLR for $n = 7$.

In the data for $n \leq 7$ some patterns can be observed, somewhat interrupted by the exceptional behavior for $n = 6$. If we consider fixed values of $t$ and $n$ and increase $k$ we always see a unimodal sequence, and the peak of the sequence appears at a lower value of $k$ when $t$ is increased. The patterns conform well with the number of constraints on the symbols, as a function of $t$ and $k$. If we instead keep $n$ and $k$ fixed and increase $t$ we see a similar pattern, though here there is an exception for $n = 6$, $k = 5$, where there are fewer 3-MOLR than 4-MOLR. These observations motivate the following questions.

**Question 5.1.** *Is the number of $t$-MOLR for fixed $n$ and $t$ a unimodal sequence in $k$?*

**Question 5.2.** *Is the number of $t$-MOLR for fixed $n$ and $k$ a unimodal sequence in $t$ for $n \geq 7$?*

Additionally we see in e.g. Table 8 that for MOLR which are close to being maximum, either in terms of $k$ or $t$, there is symmetry among the numbers obtained by interchanging $t$ and $k$. We may also note that for $t = 6$, $k = 2$ we have 324 MOLR, which is the same as the number of trisotopism classes of Latin squares of order 7 [15]. Here two latin squares are trisotopic if the first square is isotopic either to the second square or its transpose. That these numbers are equal follows by considering the mapping given in the second paragraph of Section 5.3. Here it would be interesting to find similar explanations for the rest of the partial symmetry for these numbers.

We have also classified the small sets of MOLR according to their degree of regularity. In Tables 9, 11 for $n = 4, 5, 6$, we give the number of $t$-MOLR that are A) homogenous, B) transitive, C) stepwise
homogenous and D) stepwise transitive. In a sense, these four classes are gradually more regular, and the data in the tables gives the total numbers from each such class in the form A, B, C, D in each cell. In Table 12, data is presented in the form A, B, and in Table 13, data is in the form C, D. Comparing the data in Tables 11-17 with the data in Tables 9-13 it is clear that when \( k \) or \( t \) is small compared to \( n \), most \( t \)-MOLR have none of these stronger regularity properties, but whenever a \( t \)-MOLR exists, we also have a homogenous \( t \)-MOLR with the same parameters.

Whenever there exists a \( t \)-MOLR, we also find transitive \( t \)-MOLR for most parameters. The exception is \( n = 7 \), where there exist \( t \)-MOLS (\( k = 7 \), that is) with \( t = 4, 5 \) but no corresponding transitive \( t \)-MOLR, demonstrating that the autotopism group for the 6-MOLR does not have orbits of length 4 and 5. As a further example of observations from the data, for \( n = 7, k = 4 \) (see Table 12) there exist homogenous 5-MOLR, but no transitive 5-MOLR, and \( a \) fortiori, no stepwise transitive 5-MOLR.

|     | 2×4 | 3×4 | 4×4 |
|-----|-----|-----|-----|
| \( t = 2 \) | 2, 2, 2, 2 | 2, 2, 1, 1 | 1, 1, 1, 1 |
| \( t = 3 \) | 1, 1, 1, 1 | 1, 1, 1, 1 | 1, 1, 1, 1 |

Table 9: The number of non-isotopic \( t \)-MOLR for \( n = 4 \) sorted by increasing regularity.

|     | 2×5 | 3×5 | 4×5 | 5×5 |
|-----|-----|-----|-----|-----|
| \( t = 2 \) | 4, 3, 4, 3 | 11, 9, 7, 6 | 2, 1, 2, 1 | 2, 1, 2, 1 |
| \( t = 3 \) | 3, 2, 3, 2 | 1, 0, 0, 0 | 1, 0, 0, 0 | 1, 0, 0, 0 |
| \( t = 4 \) | 2, 2, 2, 2 | 1, 1, 1, 1 | 1, 1, 1, 1 | 1, 1, 1, 1 |

Table 10: The number of non-isotopic \( t \)-MOLR for \( n = 5 \) sorted by increasing regularity.
|     | 2×6         | 3×6         | 4×6         | 5×6         |
|-----|-------------|-------------|-------------|-------------|
| \( t = 2 \) | 12, 11, 11, 11 | 280, 170, 158, 103 | 229, 160, 66, 50 | 43, 36, 13, 12 |
| \( t = 3 \) | 16, 6, 16, 6 | 115, 29, 32, 4 | 62, 39, 4, 1 | 4, 3, 0, 0 |
| \( t = 4 \) | 9, 8, 9, 8 | 19, 17, 15, 15 | 4, 3, 0, 0 | 4, 4, 0, 0 |
| \( t = 5 \) | 2, 2, 2, 2 | 0, 0, 0, 0 | 0, 0, 0, 0 | 0, 0, 0, 0 |

Table 11: The number of non-isotopic \( t \)-MOLR for \( n = 6 \) sorted by increasing regularity.

|     | 2×7         | 3×7         | 4×7         | 5×7         | 6×7         | 7×7         |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| \( t = 2 \) | 42, 29 | 14 464, 3549 | 65 156, 27 299 | 22 432, 18 836 | 409, 392 | 9, 6 |
| \( t = 3 \) | 318, 15 | 49 370, 647 | 2985, 1578 | 111, 36 | 11, 6 | 4, 1 |
| \( t = 4 \) | 691, 21 | 1622, 110 | 84, 67 | 67, 53 | 7, 3 | 3, 0 |
| \( t = 5 \) | 176, 6 | 49, 42 | 2, 0 | 4, 2 | 5, 3 | 1, 0 |
| \( t = 6 \) | 26, 5 | 7, 7 | 2, 2 | 1, 1 | 1, 1 | 1, 1 |

Table 12: The number of non-isotopic homogeneous and transitive \( t \)-MOLR for \( n = 7 \).

|     | 2×7         | 3×7         | 4×7         | 5×7         | 6×7         | 7×7         |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| \( t = 2 \) | 42, 29 | 7 423, 2175 | 14 960, 10 029 | 4163, 3923 | 91, 84 | 6, 4 |
| \( t = 3 \) | 318, 15 | 13 975, 185 | 283, 160 | 8, 5 | 4, 1 | 4, 1 |
| \( t = 4 \) | 691, 21 | 585, 48 | 12, 1 | 3, 0 | 3, 0 | 3, 0 |
| \( t = 5 \) | 176, 6 | 48, 42 | 2, 0 | 1, 0 | 1, 0 | 1, 0 |
| \( t = 6 \) | 26, 5 | 6, 4 | 2, 2 | 1, 1 | 1, 1 | 1, 1 |

Table 13: The number of non-isotopic stepwise homogeneous and stepwise transitive \( t \)-MOLR for \( n = 7 \).

The number of stepwise homogenous and stepwise transitive \( t \)-MOLR are by definition smaller than (or equal to) the number of homogeneous and transitive \( t \)-MOLR respectively, but we again find stepwise homogenous examples for most parameter values. For \( n = 7 \), the number of...
$k = 4$, we have no stepwise transitive sets of MOLR, and additionally, there are no stepwise transitive 5-MOLR for $n = 7$, $k = 5, 6$.

Here we note that for each $n \leq 7$ the largest sets of MOLS are all stepwise transitive. Since in each case the maximum set of MOLS for these $n$ correspond to a Galois projective plane (that is, constructed from the corresponding Galois field) this reflects the high degree of symmetry of these planes. For $n = 9$ there are several projective planes, some of which are not Galois projective planes, and we will investigate that case below.

5.2 Larger Orders

For $n \geq 8$ we have not generated all $t$-MOLR, even though our programs are in principle able to do so. The problem here is that the number of $t$-MOLR becomes so large that several peta-byte would be required to store them on disc, and any kind of analysis of the whole set would become impractical. Instead, we have focused on two interesting subclasses, the stepwise homogenous and the stepwise transitive $t$-MOLR. These classes are restrictive enough to let us push the generation program a few more steps, and we have already seen that they contain a number of interesting examples.

In Table 14 we give the number of stepwise homogeneous and stepwise transitive $t$-MOLR for $n = 8$ and in Table 15 we give the number of stepwise transitive $t$-MOLR for $n = 9$. For $n = 8$ it is clear that for small parameters $k$ and $t$, the stepwise homogenous $t$-MOLR far outnumber the stepwise transitive ones. We also find stepwise homogenous $t$-MOLR for all parameters, but not stepwise transitive ones. This motivates the following question.

**Question 5.3.** For $n \geq 7$, is there a stepwise homogenous $t$-MOLR for every pair $t, k$ that allows a $t$-MOLR?

For $n = 9$ we only have data for the stepwise transitive class, since the number of stepwise homogenous $t$-MOLR is too large. Here it is clear that the possible values of $t$ are quite restricted. We note two interesting facts. First, there is a unique stepwise transitive 6-MOLS, which in turn has unique stepwise transitive restrictions to 8 and 7 rows. We present this object in Figure 1. Second, there are 5 stepwise transitive 8-MOLS, which are presented in Appendix G.
| \( t \) | 2×8      | 3×8      | 4×8      | 5×8      | 6×8      | 7×8      | 8×8      |
|------|----------|----------|----------|----------|----------|----------|----------|
| 2    | 186,99   | 446,443  | 4,432,284| 10,976,55| 3,826,527| 2,569,679| 2,427,32  |
| 3    | 11,565,66| 9,144,025| 7,627    | 17,850,2 | 41,505   | 628,75   | 111,32   |
| 4    | 216,950,152| 1,648,723| 4,284    | 3,547,712| 58,20    | 4,0      | 3,0      |
| 5    | 509,622,19| 2,652,0  | 267,0    | 2,0      | 1,0      | 1,0      | 1,0      |
| 6    | 91,013,109| 975,908  | 155,146  | 1,0      | 1,0      | 1,0      | 1,0      |
| 7    | 4538,5   | 2,2      | 2,2      | 1,1      | 1,1      | 1,1      | 1,1      |

Table 14: The number of non-isotopic stepwise homogenous and stepwise transitive \( t \)-MOLR for \( n = 8 \).

| \( t \) | 2×9     | 3×9     | 4×9     | 5×9     | 6×9     | 7×9     | 8×9     | 9×9     |
|------|---------|---------|---------|---------|---------|---------|---------|---------|
| 2    | 126     | 1,418,577| 560,524,587| 200,019,499,500| 6,748,036,463,7 | 5,872,237,985 | 14,940,988 | 28,955    |
| 3    | 202     | 72,836  | 1,746,912| 0       | 0       | 0       | 0       | 0       |
| 4    | 1067    | 356,680 | 2,640,163| 645,453 | 1816    | 31      | 7       | 5       |
| 5    | 17      | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| 6    | 543     | 21,620  | 244     | 33      | 16      | 1       | 1       | 1       |
| 7    | 39      | 1532    | 300     | 0       | 0       | 0       | 0       | 0       |
| 8    | 54      | 48      | 27      | 22      | 16      | 9       | 7       | 5       |

Table 15: The number of non-isotopic stepwise transitive \( t \)-MOLR for \( n = 9 \).
As mentioned earlier, it is known that there are exactly 4 projective planes of order 9. The Galois plane corresponds to the 8-MOLS with autopism group of order $10^{368}$, see Table 42 in Appendix F. The other four 8-MOLS can be divided into two pairs, such that both 8-MOLS in one pair correspond to the Hall plane, and those in the other pair correspond to the dual of the Hall plane. In Appendix G we include data on this pairing. This leaves the Hughes plane of order 9 as the smallest projective plane which is cannot be defined by a stepwise transitive MOLS.

With this in mind one may ask about the situation for larger orders as well. Wanless [40] has found that 8 of the 22 projective planes of order 16 cannot be constructed via a homogenous MOLS, and hence they can’t be constructed from a stepwise transitive set of MOLS either. In the online catalogue provided by Gordon Royle these are the planes labelled JOHN, BBS4, BBH2 and their duals, BBH1 (which is self-dual), and either MATH or its dual (the test did not identify which).

On the other hand, we can prove the following result.

**Theorem 5.4.** The set of $(n - 1)$ MOLS $L_1, \ldots, L_{n-1}$ corresponding to a projective plane constructed in the standard way from the finite field $GF(n)$ is stepwise transitive.

By ‘the standard way’, we refer to the construction given in Section 4.

**Proof.** To see that the theorem holds, note that multiplying the entries of $L_1$ by $x^t$ will map the $i$-th column of $L_1$ onto the $(t+1)$-th column of $L_{t+t}$, and so this mapping defines a transitive autotopism of this set of MOLS, acting cyclically on both the set of squares and their sets of columns. If we delete the first column from $L_1$ and the orbit of that column in the other squares we get a new set of $(n - 1) \times n$ MOLR, which is still transitive. This can be repeated, thus leading to a stepwise transitive construction of these sets of MOLS.

A full characterization of the family of projective planes which correspond to stepwise transitive sets of MOLS would of course be interesting, but even simpler questions are left open.
Question 5.5. For large $n$, what proportion of the projective planes of order $n$ correspond to homogenous or stepwise transitive sets of MOLS?

It would be of interest to use the existing catalogues of finite projective planes, the currently most extensive being that of Moorhouse [31] which now contains several hundred thousand examples, to check how common these properties are among the known non-Galois examples.

5.3 Autotopism Group Sizes of $t$-MOLR

We have computed the order of the autotopism group for all sets of MOLR discussed so far in the paper. Detailed statistics of these orders are given in appendices A to F. We will here discuss some of the symmetry properties of sets of MOLR in general, and some additional observations based on our data.

First let us note that the case of $2 \times n$ sets of MOLR is somewhat special. If we follow the construction for a partial net using a $2 \times n$ $t$-MOLR $A$, each of the lines which do not correspond to a row has size 2, and no such line connects two vertices in the same row. This means that if we delete the two lines of size $n$ we have a bipartite graph $g(A)$, where the two rows give us the bipartition. Additionally, the edges coming from each rectangle add a perfect matching, as do the edges coming from the columns, so we have a $(t + 1)$-regular bipartite graph with a natural edge colouring given by these matchings. The autotopism group of $A$ now corresponds to automorphisms of this edge-coloured graph which map the matching given by the column lines to itself. If we assume that $A$ was normalized we can also invert this construction and reconstruct the $t$-MOLR $A$. Now, for $t = n - 1$ this defines a proper edge colouring of $K_{n,n}$, i.e., a Latin square, so here we obtain a mapping from $2 \times n$ $(n - 1)$-MOLR to a Latin square $L(A)$, and an autotopism of $A$ defines an autotopism of $L(A)$ which fixes one symbol, again corresponding to the column lines in the partial net.

Given a regularity property we may also look at how it interplays with restrictions of a set of MOLR. First let us note that any subset of rectangles from a homogenous $t$-MOLR is homogenous, so this case is trivial, and the same is true for a stepwise homogenous $t$-MOLR. For
|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 8 | 7 | 6 | 5 | 3 | 2 | 4 | 1 | 0 |
| 7 | 0 | 8 | 4 | 1 | 6 | 5 | 3 | 2 |
| 6 | 2 | 0 | 1 | 8 | 7 | 3 | 5 | 4 |
| 5 | 8 | 4 | 6 | 2 | 3 | 7 | 0 | 1 |
| 4 | 3 | 5 | 7 | 0 | 1 | 8 | 2 | 6 |
| 3 | 4 | 1 | 0 | 7 | 8 | 2 | 6 | 5 |
| 2 | 6 | 7 | 8 | 5 | 0 | 1 | 4 | 3 |
| 1 | 5 | 3 | 2 | 6 | 4 | 0 | 8 | 7 |

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 6 | 5 | 8 | 0 | 7 | 3 | 1 | 4 | 2 |
| 8 | 4 | 7 | 6 | 2 | 1 | 3 | 5 | 0 |
| 4 | 7 | 5 | 8 | 1 | 2 | 0 | 6 | 3 |
| 7 | 3 | 0 | 2 | 6 | 8 | 4 | 1 | 5 |
| 5 | 6 | 4 | 1 | 3 | 0 | 2 | 8 | 7 |
| 2 | 8 | 6 | 7 | 0 | 4 | 5 | 3 | 1 |
| 1 | 0 | 3 | 5 | 8 | 6 | 7 | 2 | 4 |
| 3 | 2 | 1 | 4 | 5 | 7 | 8 | 0 | 6 |

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 6 | 4 | 7 | 1 | 8 | 0 | 3 | 5 |
| 3 | 8 | 0 | 5 | 6 | 7 | 4 | 2 | 1 |
| 5 | 4 | 7 | 2 | 3 | 0 | 8 | 1 | 6 |
| 8 | 5 | 3 | 1 | 7 | 6 | 2 | 4 | 0 |
| 1 | 0 | 6 | 8 | 2 | 4 | 7 | 5 | 3 |
| 4 | 7 | 8 | 6 | 5 | 1 | 3 | 0 | 2 |
| 6 | 2 | 1 | 4 | 0 | 3 | 5 | 8 | 7 |
| 7 | 3 | 5 | 0 | 8 | 2 | 1 | 6 | 4 |

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 1 | 4 | 0 | 3 | 5 | 8 | 7 | 1 |
| 5 | 7 | 8 | 6 | 3 | 4 | 0 | 1 | 2 |
| 6 | 0 | 4 | 1 | 2 | 8 | 7 | 3 | 5 |

Figure 1: The unique stepwise transitive 6-MOLS of order 9.
transitivity, the group structure comes into play. Given a transitive $t$-MOLR $A$ with autotopism group $G$ we get a subgroup $G'$ which describes the action of $G$ on the set of rectangles. If $G'$ has an element $g$ of order $r$ we will obtain a transitive $r$-MOLR from $A$ by taking the orbit of a single rectangle from $A$ under $g$. Whenever we have a transitive $t$-MOLR with autotopism group of order $t$ this implies the existence of $p$-MOLR for the same size $k \times n$ for every prime factor $p$ of $t$. For general parameters we observe that when $t$ or $k$ is small, transitive $t$-MOLR with autotopism group of order exactly $t$ are common.

Following this, we say that a transitive $t$-MOLR $A$ is $G$-complete if there does not exist a $t' > t$ and a $t'$-MOLR $B$ such that $A$ is the orbit of the rectangle in $B$ under an element $g \in \text{Aut}(B)$, and otherwise we say that $A$ is $G$-incomplete. As noted, there are many examples of $t$-MOLR with autotopism groups of size $t$, and hence we will also have many incomplete $r$-MOLR of the same size and $r$ a divisor of $t$. In our data we found the following:

**Observation 5.6.**

1. Among the stepwise transitive sets of MOLR for $n = 8$ there is one 3-MOLS with autotopism group of order 48, which is $G$-complete. We display that example in Figure 2.

2. For $n = 9$ none of the stepwise transitive 4-MOLR are $G$-complete. The 4-MOLR with autotopism groups of orders 5184 and 2592 both correspond to the 8-MOLS with autotopism group of order 10368. The two 4-MOLR with autotopism group of order 64 correspond to the two 8-MOLR with autotopism group of order 384.

3. For $n = 9$ the 8-MOLS with autotopism group of order 31104 does not correspond to any $G$-incomplete stepwise transitive 4-MOLS.

4. For $n = 9$ the stepwise transitive set of 6 MOLS is $G$-complete.

For stepwise transitive sets of MOLR, restrictions become far less well-behaved. Given a transitive $t$-MOLR $A$ and an autotopism $g$ which has order $r$ on the set of rectangles, we know that we will obtain a transitive $r$-MOLR $A'$ from $A$. However, assuming that $A$ is stepwise transitive does not necessarily lead to stepwise transitivity.
for $A'$. In order for this to happen it must also be the case that each of the stepwise transitive sets of MOLR which are used to construct $A$ have autotopisms with the same orbit as $g$, and this is not always the case. We see one such example at $n = 9$, where a stepwise transitive 6-MOLS exists, but no stepwise transitive triple.

### 5.4 Classification of Finite Geometries Based on $t$-MOLR

As described above, each $t$-MOLS gives rise to a net, and each $t$-MOLR gives rise to a partial net. In this language, then, we have enumerated all nets and partial nets for these parameter ranges.

Though we have not done so, it would be interesting to classify these geometries further. In particular, the connection to hyperbolic planes should be investigated. Here the construction given by Sandler [34], which deletes lines from a projective plane, can be applied to an $(n - 1)$-MOLR of size $k \times n$ and potentially lead to new hyperbolic geometries. In Sandler’s construction, a set of lines and the
points in them are deleted from a projective plane.

Expanding on Sandler’s basic construction, we note that given an \( n \times k \) t-MOLR, each point in the geometry will be collinear with \( (n-1) + (k-1) + t(k-1) \) other points. For \( t = n - 1 \) this simplifies to \( nk - 1 \), that is, the point is collinear with each of the other \( nk - 1 \) points, and so the geometry is a plane. For \( k = n \) this is an affine plane, but for \( k < n \) we have a plane with line sizes \( n \) and size \( k \), and this plane will not be of one of the three classical types. However, if we now delete a line (together with its points) which does not correspond to a row from this plane we will get a hyperbolic geometry analogous to Sandler’s original construction. If the set of MOLR can be extended to a set of MOLS, this new hyperbolic plane will be one of those given by Sandler’s construction, but for a set of MOLR which cannot be completed to a set of MOLS, we will get a distinct class of hyperbolic geometries. For our range of sizes, and the cases we have covered, the only maximal sets of MOLR are \((n-1)\)-MOLS, or are very small, e.g. single squares for \( n = 6 \). In particular:

**Observation 5.7.** The only hyperbolic geometries with lines of size at least 3 which can be constructed by the generalised Sandler method from MOLR with \( n \leq 7 \), or stepwise transitive MOLR with \( n \leq 9 \), can also be constructed from a projective plane.

These considerations motivate the following question.

**Question 5.8.** For what orders are there hyperbolic geometries that do not arise from Sandler’s construction?

### 6 Concluding Remarks

In this paper we have focused on enumeration of t-MOLR up to \( n = 9 \), coming tantalisingly close to the, in this setting, special value \( n = 10 \). As we have mentioned, a significant theoretical and computational effort led to the result that there is no set of 9 mutually orthogonal Latin squares of order 10, and hence no projective plane of order 10. However, an even more basic question remains:

**Question 6.1.** Is there a triple of mutually orthogonal Latin squares of order 10?
A large number of pairwise orthogonal Latin squares of order 10 are known, and as mentioned above, the self-orthogonal Latin squares of order 10 have been completely enumerated. An exhaustive search by McKay, Meynert, and Myrvold [27] proved that no square of order 10 with non-trivial autoparatopism group is part of an orthogonal triple. They also generated all Latin squares of order 10, but the number was too large for a complete search for orthogonal triples.

Note that these results do not immediately exclude the existence of transitive, or even stepwise transitive, triples of MOLS or MOLR, since the autotopism group of a single square or rectangle in such a triple can be trivial. Our data provides several such examples for rectangles. However, as the anonymous reviewer pointed out, from a transitive triple of MOLS one can, by translation to an orthogonal array, construct a triple of MOLS where at least one square has non-trivial autoparatopisms. So the result of McKay, Meynert, and Myrvold [27] does in fact rule out the existence of transitive triples of MOLS as well.

There are other restricted versions of Question 6.1 which remain.

**Question 6.2.** Is there a homogenous 3-MOLS of order 10?

**Question 6.3.** Is there a stepwise homogenous 3-MOLS of order 10?

A negative answer to the first of these two questions would lead to another extension of the result from [27]. For the second question, a more specialised version of the type of search we have performed might be able to handle the case $t = 3$ for $n = 10$ as well.

In [15] the authors tried to find example of three Latin squares that come as close to being mutually orthogonal as possible. They presented an example of three squares such that the first is orthogonal to the other two, and the final two have 7 transversals, thus limiting the number of repeated symbol pairs. There have also been earlier examples of MOLR and almost orthogonal Latin squares for $n = 10$. In [17], examples of triples of orthogonal $9 \times 10$ rectangles were constructed, in order to be used in the construction of designs, and in [39] a set of 4 such rectangles was constructed.

Using our program we performed a partial search for stepwise transitive MOLR with $n = 10$. We found several examples of stepwise transitive triples of $8 \times 10$ MOLR, and some of these could be extended...
Figure 3: A 3-MOLR of size $9 \times 10$.

to triples of $9 \times 10$ MOLR, but not while preserving transitivity. In Figure 3 we give one such example. This example can be uniquely extended to three Latin squares, such that all positions which break orthogonality lie in the last last row. Unfortunately, none of the examples we found could be extended to a triple of MOLS.

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A  Sizes of Autotopism Groups of MOLR for $n = 4$

Table 16: 2-MOLR for $n = 4$.

|        | $\Sigma$ | 8 | 16 |
|--------|----------|---|----|
| non-ISO| 3        | 1 | 2  |
| homog. | 2        | 0 | 2  |
| transitive | 2    | 0 | 2  |
| st. homog. | 2   | 0 | 2  |
| st. trans. | 2 | 0 | 2  |

|        | $\Sigma$ | 8 | 24 |
|--------|----------|---|----|
| non-ISO| 2        | 1 | 1  |
| homog. | 2        | 1 | 1  |
| transitive | 2    | 1 | 1  |
| st. homog. | 1   | 0 | 1  |
| st. trans. | 1 | 0 | 1  |

|        | $\Sigma$ | 96 |
|--------|----------|----|
| non-ISO| 1        | 1  |
| homog. | 1        | 1  |
| transitive | 1    | 1  |
| st. homog. | 1   | 1  |
| st. trans. | 1 | 1  |

Table 17: 3-MOLR for $n = 4$.

|        | $\Sigma$ | 16 | 48 |
|--------|----------|----|----|
| non-ISO| 2        | 1  | 1  |
| homog. | 1        | 0  | 1  |
| transitive | 1    | 0 | 1  |
| st. homog. | 1   | 0 | 1  |
| st. trans. | 1 | 0 | 1  |

|        | $\Sigma$ | 72 |
|--------|----------|----|
| non-ISO| 1        | 1  |
| homog. | 1        | 1  |
| transitive | 1    | 1  |
| st. homog. | 1   | 1  |
| st. trans. | 1 | 1  |

|        | $\Sigma$ | 288 |
|--------|----------|-----|
| non-ISO| 1        | 1  |
| homog. | 1        | 1  |
| transitive | 1    | 1  |
| st. homog. | 1   | 1  |
| st. trans. | 1 | 1  |
### B Sizes of Autotopism Groups of MOLR for $n = 5$

|   | 2×5 | $\Sigma$ | 2 | 4 | 10 | 20 |
|---|-----|---------|---|---|----|----|
| non-ISO | 5 | | 1 | 2 | 1 | 1 |
| homog. | 4 | | 0 | 2 | 1 | 1 |
| transitive | 3 | | 0 | 2 | 0 | 1 |
| st. homog. | 4 | | 0 | 2 | 1 | 1 |
| st. trans. | 3 | | 0 | 2 | 0 | 1 |

|   | 3×5 | $\Sigma$ | 1 | 2 | 4 | 10 | 20 |
|---|-----|---------|---|---|----|----|----|
| non-ISO | 14 | | 1 | 7 | 3 | 2 | 1 |
| homog. | 11 | | 1 | 4 | 3 | 2 | 1 |
| transitive | 9 | | 0 | 4 | 3 | 1 | 1 |
| st. homog. | 7 | | 0 | 1 | 3 | 2 | 1 |
| st. trans. | 6 | | 0 | 1 | 3 | 1 | 1 |

|   | 4×5 | $\Sigma$ | 20 | 40 |
|---|-----|---------|-----|----|
| non-ISO | 2 | | 1 | 1 |
| homog. | 2 | | 1 | 1 |
| transitive | 1 | | 0 | 1 |
| st. homog. | 2 | | 1 | 1 |
| st. trans. | 1 | | 0 | 1 |

|   | 5×5 | $\Sigma$ | 100 | 200 |
|---|-----|---------|------|------|
| non-ISO | 2 | | 1 | 1 |
| homog. | 2 | | 1 | 1 |
| transitive | 1 | | 0 | 1 |
| st. homog. | 2 | | 1 | 1 |
| st. trans. | 1 | | 0 | 1 |

Table 18: 2-MOLR for $n = 5$. 

32
| $n \times 5$ | $\Sigma$ | 2 | 6 | 10 |
|----------|--------|---|---|----|
| non-ISO  | 4      | 1 | 2 | 1  |
| homog.   | 3      | 0 | 2 | 1  |
| transitive | 2      | 0 | 2 | 0  |
| st. homog. | 3      | 0 | 2 | 1  |
| st. trans. | 2      | 0 | 2 | 0  |
| $3 \times 5$ | $\Sigma$ | 10 |
| non-ISO  | 1      | 1 |
| homog.   | 1      | 1 |
| st. homog. | 1      | 1 |
| $4 \times 5$ | $\Sigma$ | 20 |
| non-ISO  | 1      | 1 |
| homog.   | 1      | 1 |
| st. homog. | 1      | 1 |
| $5 \times 5$ | $\Sigma$ | 100 |
| non-ISO  | 1      | 1 |
| homog.   | 1      | 1 |

Table 19: 3-MOLR for $n = 5$. 

33
| $2 \times 5$ | $\Sigma$ | 6   | 24  | 40  |
|------------|----------|-----|-----|-----|
| non-ISO    | 3        | 1   | 1   | 1   |
| homog.     | 2        | 0   | 1   | 1   |
| transitive | 2        | 0   | 1   | 1   |
| st. homog. | 2        | 0   | 1   | 1   |
| st. trans. | 2        | 0   | 1   | 1   |
| $3 \times 5$ | $\Sigma$ | 40  |     |     |
| non-ISO    | 1        | 1   |     |     |
| homog.     | 1        | 1   |     |     |
| transitive | 1        | 1   |     |     |
| st. homog. | 1        | 1   |     |     |
| st. trans. | 1        | 1   |     |     |
| $4 \times 5$ | $\Sigma$ | 80  |     |     |
| non-ISO    | 1        | 1   |     |     |
| homog.     | 1        | 1   |     |     |
| transitive | 1        | 1   |     |     |
| st. homog. | 1        | 1   |     |     |
| st. trans. | 1        | 1   |     |     |
| $5 \times 5$ | $\Sigma$ | 400 |     |     |
| non-ISO    | 1        | 1   |     |     |
| homog.     | 1        | 1   |     |     |
| transitive | 1        | 1   |     |     |
| st. homog. | 1        | 1   |     |     |
| st. trans. | 1        | 1   |     |     |

Table 20: 4-MOLR for $n = 5$. 
### C Sizes of Autotopism Groups of MOLR for \( n = 6 \)

| 2×6 | \( \Sigma \) | 1 | 2 | 4 | 6 | 8 | 12 | 24 | 72 |
|-----|---------------|---|---|---|---|---|----|----|----|
| non-ISO | 28 | 1 | 7 | 7 | 1 | 3 | 6 | 2 | 1 |
| homog. | 12 | 0 | 3 | 2 | 0 | 3 | 1 | 2 | 1 |
| transitive | 11 | 0 | 2 | 2 | 0 | 3 | 1 | 2 | 1 |
| st. homog. | 12 | 0 | 3 | 2 | 0 | 3 | 1 | 2 | 1 |
| st. trans. | 11 | 0 | 2 | 2 | 0 | 3 | 1 | 2 | 1 |

| 3×6 | \( \Sigma \) | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 | 216 |
|-----|---------------|---|---|---|---|---|----|----|-----|------|
| non-ISO | 1526 | 1155 | 252 | 18 | 59 | 19 | 8 | 11 | 3 | 1 |
| homog. | 280 | 89 | 117 | 1 | 40 | 13 | 8 | 8 | 3 | 1 |
| transitive | 170 | 0 | 100 | 0 | 40 | 10 | 8 | 8 | 3 | 1 |
| st. homog. | 158 | 43 | 63 | 1 | 26 | 9 | 8 | 4 | 3 | 1 |
| st. trans. | 103 | 0 | 52 | 0 | 26 | 9 | 8 | 4 | 3 | 1 |

| 4×6 | \( \Sigma \) | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 24 | 48 |
|-----|---------------|---|---|---|---|---|----|----|-----|-----|-----|
| non-ISO | 2036 | 1425 | 425 | 30 | 78 | 35 | 16 | 21 | 1 | 3 | 2 |
| homog. | 229 | 36 | 112 | 5 | 31 | 11 | 16 | 12 | 1 | 3 | 2 |
| transitive | 160 | 0 | 92 | 0 | 29 | 9 | 15 | 9 | 1 | 3 | 2 |
| st. homog. | 66 | 7 | 27 | 1 | 11 | 3 | 8 | 6 | 0 | 2 | 1 |
| st. trans. | 50 | 0 | 22 | 0 | 10 | 3 | 6 | 6 | 0 | 2 | 1 |

| 5×6 | \( \Sigma \) | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 24 | 48 |
|-----|---------------|---|---|---|---|---|----|----|-----|-----|-----|
| non-ISO | 85 | 5 | 25 | 2 | 26 | 4 | 11 | 7 | 2 | 1 | 2 |
| homog. | 43 | 0 | 10 | 0 | 13 | 3 | 6 | 7 | 2 | 0 | 2 |
| transitive | 36 | 0 | 6 | 0 | 11 | 2 | 6 | 7 | 2 | 0 | 2 |
| st. homog. | 13 | 0 | 2 | 0 | 0 | 1 | 4 | 2 | 2 | 0 | 2 |
| st. trans. | 12 | 0 | 1 | 0 | 0 | 1 | 4 | 2 | 2 | 0 | 2 |

Table 21: 2-MOLR for \( n = 6 \).
\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\text{non-ISO} & \Sigma & 1 & 2 & 3 & 4 & 6 & 8 & 12 & 24 & 36 & 72 \\
\hline
\text{homog.} & 16 & 3 & 4 & 3 & 0 & 0 & 0 & 3 & 1 & 1 & 1 \\
\text{transitive} & 6 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 & 1 \\
\text{st. homog.} & 16 & 3 & 4 & 3 & 0 & 0 & 0 & 3 & 1 & 1 & 1 \\
\text{st. trans.} & 6 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\text{3×6} & \Sigma & 1 & 2 & 3 & 4 & 6 & 12 & 18 & 36 \\
\hline
\text{non-ISO} & 2572 & 1980 & 442 & 54 & 27 & 55 & 6 & 4 & 4 \\
\text{homog.} & 115 & 41 & 32 & 11 & 2 & 18 & 3 & 4 & 4 \\
\text{transitive} & 29 & 0 & 0 & 6 & 0 & 13 & 2 & 4 & 4 \\
\text{st. homog.} & 32 & 11 & 11 & 2 & 2 & 2 & 1 & 1 & 2 \\
\text{st. trans.} & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\text{4×6} & \Sigma & 1 & 2 & 3 & 4 & 6 & 8 & 9 & 12 & 18 & 24 & 36 \\
\hline
\text{non-ISO} & 513 & 93 & 194 & 96 & 37 & 64 & 3 & 2 & 11 & 9 & 1 & 3 \\
\text{homog.} & 62 & 1 & 8 & 11 & 1 & 23 & 0 & 2 & 3 & 9 & 1 & 3 \\
\text{transitive} & 39 & 0 & 0 & 3 & 0 & 18 & 0 & 2 & 3 & 9 & 1 & 3 \\
\text{st. homog.} & 4 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \\
\text{st. trans.} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{5×6} & \Sigma & 3 & 6 & 9 & 18 \\
\hline
\text{non-ISO} & 7 & 2 & 2 & 1 & 2 \\
\text{homog.} & 4 & 0 & 1 & 1 & 2 \\
\text{transitive} & 3 & 0 & 0 & 1 & 2 \\
\hline
\end{array}
\]

Table 22: 3-MOLR for \( n = 6 \).
Table 23: 4-MOLR for \( n = 6 \).

| 2×6 | Σ  | 1 | 2 | 3 | 4 | 8 | 12 | 16 | 18 | 24 | 36 | 48 |
|-----|----|---|---|---|---|---|----|----|----|----|----|----|
| non-ISO | 92 | 14 | 18 | 1 | 28 | 8 | 10 | 2 | 1 | 4 | 4 | 2 |
| homog. | 9  | 0  | 1  | 0  | 2  | 2 | 0  | 2 | 0 | 0 | 0 | 2 |
| transitive | 8  | 0  | 0  | 0  | 2  | 2 | 0  | 2 | 0 | 0 | 0 | 2 |
| st. homog. | 9  | 0  | 1  | 0  | 2  | 2 | 0  | 2 | 0 | 0 | 0 | 2 |
| st. trans. | 8  | 0  | 0  | 0  | 2  | 2 | 0  | 2 | 0 | 0 | 0 | 2 |

| 3×6 | Σ  | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 18 | 24 | 36 | 48 | 144 |
|-----|----|---|---|---|---|---|----|----|----|----|----|----|----|----|
| non-ISO | 118 | 16 | 38 | 5 | 22 | 10 | 6 | 9 | 2 | 4 | 1 | 3 | 1 | 1 |
| homog. | 19  | 0  | 0  | 0  | 5  | 1  | 6  | 2 | 2 | 0 | 1 | 0 | 1 | 1 |
| transitive | 17  | 0  | 0  | 0  | 5  | 0  | 6  | 1 | 2 | 0 | 1 | 0 | 1 | 1 |
| st. homog. | 15  | 0  | 0  | 0  | 3  | 0  | 6  | 1 | 2 | 0 | 1 | 0 | 1 | 1 |
| st. trans. | 15  | 0  | 0  | 0  | 3  | 0  | 6  | 1 | 2 | 0 | 1 | 0 | 1 | 1 |

| 4×6 | Σ  | 3 | 6 | 9 | 12 | 18 | 24 | 72 |
|-----|----|---|---|---|----|----|----|----|
| non-ISO | 12 | 2  | 2  | 2  | 1  | 3  | 1  | 1 |
| homog. | 4   | 0  | 0  | 1  | 1  | 0  | 1  | 1 |
| transitive | 3   | 0  | 0  | 0  | 1  | 0  | 1  | 1 |

| 5×6 | Σ  | 6 | 9 | 18 | 24 | 36 | 72 |
|-----|----|---|---|----|----|----|----|
| non-ISO | 8  | 1  | 2  | 1  | 1  | 1  | 2 |
| homog. | 4   | 0  | 0  | 0  | 1  | 1  | 2 |
| transitive | 4   | 0  | 0  | 0  | 1  | 1  | 2 |

Table 24: 5-MOLR for \( n = 6 \).

| 2×6 | Σ  | 1 | 2 | 4 | 6 | 8 | 12 | 16 | 20 | 24 | 36 | 48 | 72 | 240 |
|-----|----|---|---|---|---|---|----|----|----|----|----|----|----|----|
| non-ISO | 33 | 1  | 5  | 6  | 1  | 7  | 1  | 2  | 1  | 3  | 2  | 1  | 2  | 1 |
| homog. | 2   | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 1 |
| transitive | 2   | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 1 |
| st. homog. | 2   | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 1 |
| st. trans. | 2   | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 1 |
## Sizes of Autotopism Groups of MOLR for \( n = 7 \)

| \( 2 \times 7 \) | \( \Sigma \) | 1 | 2 | 4 | 14 | 24 | 28 | 48 |
|------------------|----------|---|---|---|----|----|----|----|
| non-ISO          |          | 100 | 21 | 55 | 18 | 2 | 1 | 1 | 2 |
| homog.           |          | 42  | 3  | 16 | 18 | 2 | 0 | 1 | 2 |
| transitive       |          | 29  | 0  | 8  | 18 | 0 | 0 | 1 | 2 |
| st. homog.       |          | 42  | 3 | 16 | 18 | 2 | 0 | 1 | 2 |
| st. trans.       |          | 29  | 0  | 8  | 18 | 0 | 0 | 1 | 2 |

| \( 3 \times 7 \) | \( \Sigma \) | 1 | 2 | 3 | 4 | 6 | 7 | 14 | 21 | 24 | 28 | 42 | 72 |
|------------------|----------|---|---|---|---|---|---|----|----|----|----|----|----|
| non-ISO          |          | 514 162 | 508 132 | 5880 | 48 | 65 | 23 | 1 | 4 | 2 | 1 | 1 | 1 | 1 |
| homog.           |          | 14 464 | 10 835 | 3524 | 6 | 65 | 23 | 1 | 4 | 2 | 1 | 1 | 1 | 1 |
| transitive       |          | 3549 | 0 | 3455 | 0 | 65 | 23 | 0 | 2 | 0 | 1 | 1 | 1 | 1 |
| st. homog.       |          | 7423 | 5017 | 2302 | 6 | 65 | 23 | 1 | 4 | 2 | 0 | 1 | 1 | 1 |
| st. trans.       |          | 2175 | 0 | 2082 | 0 | 65 | 23 | 0 | 2 | 0 | 1 | 1 | 1 | 1 |

| \( 4 \times 7 \) | \( \Sigma \) | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 14 | 21 | 24 | 28 | 42 |
|------------------|----------|---|---|---|---|---|---|---|----|----|----|----|----|
| non-ISO          |          | 49 415 812 | 49 363 791 | 51 060 | 428 | 444 | 54 | 11 | 14 | 6 | 2 | 1 | 1 |
| homog.           |          | 65 156 | 37 639 | 27 054 | 16 | 365 | 54 | 4 | 14 | 6 | 2 | 1 | 1 |
| transitive       |          | 27 299 | 0 | 26 867 | 0 | 361 | 54 | 0 | 14 | 1 | 0 | 1 | 1 |
| st. homog.       |          | 14 960 | 4249 | 10 418 | 16 | 205 | 54 | 0 | 11 | 3 | 2 | 1 | 1 |
| st. trans.       |          | 10 029 | 0 | 9775 | 0 | 187 | 54 | 0 | 11 | 0 | 0 | 1 | 1 |

| \( 5 \times 7 \) | \( \Sigma \) | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 14 | 21 | 28 | 42 |
|------------------|----------|---|---|---|---|---|---|---|----|----|----|----|
| non-ISO          |          | 21 290 125 | 21 243 988 | 45 872 | 227 | 10 | 6 | 12 | 9 | 1 |
| homog.           |          | 22 432 | 3508 | 18 672 | 227 | 1 | 6 | 8 | 9 | 1 |
| transitive       |          | 18 836 | 0 | 18 599 | 227 | 0 | 0 | 5 | 4 | 1 |
| st. homog.       |          | 41 63 | 99 | 3935 | 121 | 0 | 0 | 5 | 2 | 1 |
| st. trans.       |          | 3923 | 0 | 3799 | 118 | 0 | 0 | 5 | 0 | 1 |

| \( 6 \times 7 \) | \( \Sigma \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 10 | 12 | 42 | 84 |
|------------------|----------|---|---|---|---|---|---|---|----|----|----|----|
| non-ISO          |          | 11 582 | 10 912 | 492 | 20 | 24 | 102 | 11 | 1 | 12 | 4 | 3 | 1 |
| homog.           |          | 409  | 2 | 345 | 0 | 24 | 8 | 9 | 1 | 12 | 4 | 3 | 1 |
| transitive       |          | 392  | 0 | 342 | 0 | 24 | 0 | 8 | 0 | 12 | 4 | 1 | 1 |
| st. homog.       |          | 91   | 0 | 61 | 0 | 4 | 2 | 7 | 0 | 12 | 2 | 2 | 1 |
| st. trans.       |          | 84   | 0 | 60 | 0 | 3 | 0 | 7 | 0 | 12 | 1 | 0 | 1 |

| \( 7 \times 7 \) | \( \Sigma \) | 2 | 3 | 6 | 12 | 21 | 42 | 294 | 588 |
|------------------|----------|---|---|---|----|----|----|------|-----|
| non-ISO          |          | 20  | 5 | 5 | 3 | 1 | 1 | 2 | 2 | 1 |
| homog.           |          | 9   | 1 | 0 | 2 | 1 | 0 | 2 | 2 | 1 |
| transitive       |          | 6   | 0 | 0 | 2 | 1 | 0 | 2 | 0 | 1 |
| st. homog.       |          | 6   | 0 | 0 | 1 | 1 | 0 | 1 | 2 | 1 |
| st. trans.       |          | 4   | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 |

Table 25: 2-MOLR for \( n = 7 \).
| 2×7 | Σ    | 1  | 2  | 3  | 4  | 6  | 12 | 14 | 42 |
|-----|------|----|----|----|----|----|-----|----|----|
| non-ISO | 2858 | 2300 | 512 | 3  | 28 | 9  | 2  | 3  | 1  |
| homog. | 318  | 194 | 100 | 3  | 6  | 9  | 2  | 3  | 1  |
| transitive | 15  | 0  | 0  | 3  | 0  | 9  | 2  | 0  | 1  |
| st. homog. | 318  | 194 | 100 | 3  | 6  | 9  | 2  | 3  | 1  |
| st. trans. | 15  | 0  | 0  | 3  | 0  | 9  | 2  | 0  | 1  |
| 3×7 | Σ    | 1  | 2  | 3  | 6  | 7  | 9  | 14 | 18 | 21 | 42 | 63 |
| non-ISO | 65 883 453 | 65 822 447 | 60 195 | 635 | 143 | 17 | 3  | 3  | 4  | 4  | 4  | 1  | 1  |
| homog. | 49 370 | 48 126 | 566 | 542 | 116 | 4  | 3  | 3  | 4  | 4  | 1  | 1  |
| transitive | 647  | 0  | 0  | 524 | 113 | 0  | 3  | 0  | 4  | 1  | 1  | 1  |
| st. homog. | 13 975 | 13 397 | 305 | 189 | 64  | 4  | 3  | 3  | 4  | 4  | 1  | 1  |
| st. trans. | 185  | 0  | 0  | 125 | 51  | 0  | 3  | 0  | 4  | 0  | 1  | 1  |
| 4×7 | Σ    | 1  | 2  | 3  | 4  | 6  | 7  | 9  | 12 | 14 | 18 | 21 | 42 | 63 |
| non-ISO | 323 112 477 | 323 002 195 | 107 997 | 1975 | 120 | 116 | 43 | 10 | 3  | 8  | 2  | 6  | 1  | 1  |
| homog. | 2985 | 1232 | 147 | 1474 | 1  | 91 | 9  | 10 | 3  | 8  | 2  | 6  | 1  | 1  |
| transitive | 1578 | 0  | 0  | 1468 | 0  | 90 | 0  | 10 | 3  | 0  | 2  | 3  | 1  | 1  |
| st. homog. | 283  | 59  | 27  | 136 | 1  | 34 | 2  | 10 | 3  | 4  | 2  | 3  | 1  | 1  |
| st. trans. | 160  | 0  | 0  | 113 | 0  | 30 | 0  | 10 | 3  | 0  | 2  | 0  | 1  | 1  |
| 5×7 | Σ    | 1  | 2  | 3  | 4  | 6  | 7  | 14 | 21 | 42 |
| non-ISO | 55 545 | 52 981 | 2500 | 32 5  | 2  | 15 | 8  | 1  | 1  |
| homog. | 111  | 31  | 21  | 32  | 0  | 2  | 15 | 8  | 1  | 1  |
| transitive | 36  | 0  | 0  | 32  | 0  | 2  | 0  | 0  | 1  | 1  |
| st. homog. | 8  | 0  | 0  | 2  | 0  | 2  | 0  | 3  | 0  | 1  |
| st. trans. | 5  | 0  | 0  | 2  | 0  | 2  | 0  | 0  | 0  | 1  |
| 6×7 | Σ    | 1  | 2  | 3  | 4  | 6  | 12 | 42 | 126 |
| non-ISO | 16  | 1  | 4  | 1  | 1  | 3  | 2  | 3  | 1  |
| homog. | 11  | 0  | 1  | 0  | 1  | 3  | 2  | 3  | 1  |
| transitive | 6  | 0  | 0  | 0  | 0  | 3  | 2  | 0  | 1  |
| st. homog. | 4  | 0  | 0  | 0  | 0  | 0  | 0  | 3  | 1  |
| st. trans. | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  |
| 7×7 | Σ    | 294 | 882 |
| non-ISO | 4  | 3  | 1  |
| homog. | 4  | 3  | 1  |
| transitive | 1  | 0  | 1  |
| st. homog. | 4  | 3  | 1  |
| st. trans. | 1  | 0  | 1  |

Table 26: 3-MOLR for \( n = 7 \).
| $4 \times 7$ | $\Sigma$ | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 14 | 16 | 28 |
|-------------|----------|---|---|---|---|---|---|---|----|----|----|
| non-ISO     | 204      | 96| 41| 32| 6 | 18| 6 | 4 | 1  |    |    |
| homog.      | 67       | 1 | 0 | 31| 6 | 18| 6 | 4 | 1  |    |    |
| transitive  | 53       | 0 | 0 | 31| 0 | 18| 0 | 3 | 1  |    |    |
| st. homog.  | 3        | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 0  |    |    |
| st. trans.  | 1        | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0  | 0  | 0  |

Table 27: 4-MOLR for $n = 7$. 
| $2\times 7$ | $\Sigma$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 10 | 14 |
|-------------|---------|---|---|---|---|---|---|---|-----|-----|
| non-ISO     | 10 626  | 9590 | 957 | 48 | 4  | 24 | 2  | 1  |     |     |
| homog.      | 176     | 122  | 37  | 4  | 4  | 6  | 2  | 1  |     |     |
| transitive  | 6       | 0    | 0   | 0  | 4  | 0  | 2  | 0  |     |     |
| st. homog.  | 176     | 122  | 37  | 4  | 4  | 6  | 2  | 1  |     |     |
| st. trans.  | 6       | 0    | 0   | 0  | 4  | 0  | 2  | 0  |     |     |

| $3\times 7$ | $\Sigma$ | 1 | 2 | 3 | 5 | 6 | 7 | 10 | 14 | 21 |
|-------------|---------|---|---|---|---|---|---|-----|-----|-----|
| non-ISO     | 22 982  | 21 848 | 1039 | 39 | 30 | 7 | 1  | 12  | 1  | 5  |
| homog.      | 49      | 3    | 1   | 0  | 30 | 0 | 0  | 12  | 1  | 2  |
| transitive  | 42      | 0    | 0   | 0  | 30 | 0 | 0  | 12  | 0  | 0  |
| st. homog.  | 48      | 2    | 1   | 0  | 30 | 0 | 0  | 12  | 1  | 2  |
| st. trans.  | 42      | 0    | 0   | 0  | 30 | 0 | 0  | 12  | 0  | 0  |

| $4\times 7$ | $\Sigma$ | 1 | 2 | 4 | 8 | 14 | 21 |
|-------------|---------|---|---|---|---|-----|-----|
| non-ISO     | 19      | 2  | 4  | 8 | 3 | 1   | 1   |
| homog.      | 2       | 0  | 0  | 0 | 0 | 1   | 1   |
| st. homog.  | 2       | 0  | 0  | 0 | 0 | 1   | 1   |

| $5\times 7$ | $\Sigma$ | 4 | 14 | 20 |
|-------------|---------|---|-----|-----|
| non-ISO     | 5       | 2  | 1   | 2   |
| homog.      | 4       | 1  | 1   | 2   |
| transitive  | 2       | 0  | 0   | 2   |
| st. homog.  | 1       | 0  | 1   | 0   |

| $6\times 7$ | $\Sigma$ | 20 | 24 | 42 | 120 |
|-------------|---------|----|----|----|-----|
| non-ISO     | 5       | 1  | 1  | 1  | 2   |
| homog.      | 5       | 1  | 1  | 1  | 2   |
| transitive  | 3       | 1  | 0  | 0  | 2   |
| st. homog.  | 1       | 0  | 0  | 1  | 0   |

| $7\times 7$ | $\Sigma$ | 294 |
|-------------|---------|-----|
| non-ISO     | 1       | 1   |
| homog.      | 1       | 1   |
| st. homog.  | 1       | 1   |

Table 28: 5-MOLR for $n = 7$. 
| $2 \times 7$ | $\Sigma$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 10 | 12 | 16 | 48 | 84 |
|----------------|---------|---|---|---|---|---|---|---|---|---|---|---|---|
| non-ISO | 1895 | 1505 | 328 | 2 | 29 | 4 | 14 | 2 | 1 | 2 | 2 | 1 |
| homog. | 26 | 7 | 7 | 1 | 1 | 1 | 2 | 0 | 1 | 1 | 2 | 2 | 1 |
| transitive | 5 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 2 | 1 |
| st. homog. | 26 | 7 | 7 | 1 | 1 | 1 | 2 | 0 | 1 | 1 | 2 | 2 | 1 |
| st. trans. | 5 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 2 | 1 |

| $3 \times 7$ | $\Sigma$ | 1 | 2 | 3 | 6 | 12 | 21 | 84 | 126 |
|----------------|---------|---|---|---|---|---|---|---|---|
| non-ISO | 23 | 8 | 4 | 1 | 5 | 1 | 1 | 1 | 2 |
| homog. | 7 | 0 | 0 | 0 | 3 | 1 | 0 | 1 | 2 |
| transitive | 7 | 0 | 0 | 0 | 3 | 1 | 0 | 1 | 2 |
| st. homog. | 6 | 0 | 0 | 0 | 2 | 1 | 0 | 1 | 2 |
| st. trans. | 4 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 2 |

| $4 \times 7$ | $\Sigma$ | 84 | 126 |
|----------------|---------|---|---|
| non-ISO | 2 | 1 | 1 |
| homog. | 2 | 1 | 1 |
| transitive | 2 | 1 | 1 |
| st. homog. | 2 | 1 | 1 |
| st. trans. | 2 | 1 | 1 |

| $5 \times 7$ | $\Sigma$ | 84 |
|----------------|---------|---|
| non-ISO | 1 | 1 |
| homog. | 1 | 1 |
| transitive | 1 | 1 |
| st. homog. | 1 | 1 |
| st. trans. | 1 | 1 |

| $6 \times 7$ | $\Sigma$ | 252 |
|----------------|---------|---|
| non-ISO | 1 | 1 |
| homog. | 1 | 1 |
| transitive | 1 | 1 |
| st. homog. | 1 | 1 |
| st. trans. | 1 | 1 |

| $7 \times 7$ | $\Sigma$ | 1764 |
|----------------|---------|---|
| non-ISO | 1 | 1 |
| homog. | 1 | 1 |
| transitive | 1 | 1 |
| st. homog. | 1 | 1 |
| st. trans. | 1 | 1 |

Table 29: 6-MOLR for $n = 7$. 

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Sizes of Autotopism Groups of MOLR for $n = 8$
| $2 \times 8$ | $\Sigma$ | 1 | 2 | 4 | 6 | 8 | 12 | 16 | 30 | 32 | 60 | 64 | 128 |
|----------|-----------|---|---|---|---|---|----|----|----|----|----|----|-----|
| st. homog. | 186 | 52 | 65 | 43 | 0 | 10 | 2 | 2 | 1 | 6 | 1 | 2 | 2 |
| st. trans. | 99 | 0 | 35 | 39 | 0 | 10 | 2 | 2 | 0 | 6 | 1 | 2 | 2 |

| $3 \times 8$ | $\Sigma$ | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 30 | 32 | 48 | 60 | 64 | 192 |
|----------|-----------|---|---|---|---|---|----|----|----|----|----|----|----|-----|
| st. homog. | 446 443 | 394 387 | 50 556 | 11 | 1311 | 28 | 101 | 22 | 12 | 2 | 8 | 1 | 1 | 1 | 2 |
| st. trans. | 45 429 | 0 | 43 990 | 0 | 1265 | 27 | 99 | 22 | 12 | 1 | 8 | 1 | 1 | 1 | 2 |

| $4 \times 8$ | $\Sigma$ | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 24 | 32 | 48 | 64 | 192 | 768 |
|----------|-----------|---|---|---|---|---|----|----|----|----|----|----|----|-----|-----|
| st. homog. | 4 432 284 | 3 196 674 | 1 222 057 | 168 | 11 703 | 343 | 1002 | 50 | 194 | 11 | 54 | 11 | 14 | 1 | 2 |
| st. trans. | 1 097 655 | 0 | 1 086 038 | 0 | 10 123 | 317 | 857 | 50 | 181 | 9 | 52 | 11 | 14 | 1 | 2 |

| $5 \times 8$ | $\Sigma$ | 1 | 2 | 4 | 8 | 16 | 32 |
|----------|-----------|---|---|---|----|----|-----|
| st. homog. | 3 826 527 | 884 803 | 2 929 324 | 11 629 | 702 | 65 | 4 |
| st. trans. | 2 569 679 | 0 | 2 558 291 | 10 693 | 629 | 62 | 4 |

| $6 \times 8$ | $\Sigma$ | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 32 | 96 |
|----------|-----------|---|---|---|---|----|----|----|----|----|-----|
| st. homog. | 242 732 | 7954 | 227 619 | 2 | 6289 | 46 | 611 | 23 | 181 | 6 | 1 |
| st. trans. | 206 612 | 0 | 200 153 | 0 | 5745 | 35 | 503 | 22 | 147 | 6 | 1 |

| $7 \times 8$ | $\Sigma$ | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 24 | 32 | 48 | 56 | 96 |
|----------|-----------|---|---|---|---|----|----|----|----|----|----|----|----|-----|
| st. homog. | 484 | 28 | 175 | 5 | 109 | 72 | 48 | 29 | 9 | 2 | 2 | 2 | 1 | 2 |
| st. trans. | 305 | 0 | 92 | 0 | 82 | 64 | 24 | 29 | 7 | 2 | 2 | 1 | 0 | 2 |

| $8 \times 8$ | $\Sigma$ | 4 | 8 | 16 | 32 | 64 | 448 |
|----------|-----------|---|---|----|----|----|-----|
| st. homog. | 70 | 9 | 19 | 22 | 8 | 11 | 1 |
| st. trans. | 13 | 0 | 2 | 4 | 4 | 3 | 0 |

Table 30: 2-MOLR for $n = 8$. 

|   | 2×8 | Σ   | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 24 | 32 | 48 | 64 | 96 | 128 | 384 |
|---|-----|-----|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|
| homog. | 11 565 | 10 583 | 803 | 24 | 83 | 22 | 20 | 14 | 6 | 2 | 4 | 2 | 0 | 1 | 0 | 1 |
| transitive | 66 | 0 | 0 | 24 | 0 | 22 | 0 | 14 | 0 | 2 | 0 | 2 | 0 | 1 | 0 | 1 |
| st. homog. | 11 565 | 10 583 | 803 | 24 | 83 | 22 | 20 | 14 | 6 | 2 | 4 | 2 | 0 | 1 | 0 | 1 |
| st. trans. | 66 | 0 | 0 | 24 | 0 | 22 | 0 | 14 | 0 | 2 | 0 | 2 | 0 | 1 | 0 | 1 |
| 3×8 | Σ   | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 18 | 24 | 36 | 48 | 144 | 576 |
| st. homog. | 9 144 025 | 9 121 524 | 13 878 | 7463 | 479 | 564 | 27 | 64 | 8 | 4 | 3 | 3 | 5 | 1 | 2 |
| st. trans. | 7627 | 0 | 0 | 7067 | 0 | 485 | 0 | 57 | 0 | 4 | 3 | 3 | 5 | 1 | 2 |
| 4×8 | Σ   | 1 | 2 | 3 | 4 | 6 | 8 | 9 | 12 | 16 | 18 | 24 | 32 | 36 | 48 | 64 | 72 | 96 | 144 | 192 | 2304 |
| st. homog. | 178 502 | 91 562 | 40 127 | 41 876 | 2464 | 1665 | 397 | 38 | 177 | 94 | 26 | 39 | 8 | 3 | 10 | 5 | 2 | 3 | 1 | 3 | 2 |
| st. trans. | 41 505 | 0 | 0 | 39 650 | 0 | 1566 | 0 | 38 | 168 | 0 | 26 | 36 | 0 | 3 | 7 | 0 | 2 | 3 | 1 | 3 | 2 |
| 5×8 | Σ   | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 24 | 48 |
| st. homog. | 628 | 34 | 142 | 311 | 59 | 34 | 26 | 5 | 5 | 10 | 2 |
| st. trans. | 75 | 0 | 0 | 38 | 0 | 25 | 0 | 3 | 0 | 7 | 2 |
| 6×8 | Σ   | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 24 | 48 |
| st. homog. | 111 | 18 | 8 | 42 | 8 | 12 | 8 | 2 | 5 |
| st. trans. | 32 | 0 | 8 | 0 | 8 | 0 | 12 | 0 | 2 | 2 |
| 7×8 | Σ   | 8 | 9 | 24 | 56 | 168 |
| st. homog. | 10 | 2 | 3 | 2 | 1 | 2 |
| st. trans. | 6 | 0 | 3 | 1 | 0 | 2 |
| 8×8 | Σ   | 48 | 64 | 192 | 448 | 1344 |
| st. homog. | 7 | 1 | 2 | 1 | 1 | 2 |
| st. trans. | 3 | 1 | 0 | 0 | 0 | 2 |

Table 31: 3-MOLR for \( n = 8 \).
| 2×8  | Σ    | 1  | 2  | 3  | 4  | 6  | 8  | 12 | 16 | 24 | 32 | 48 | 64 | 96 | 128 | 384 |
|------|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| st. homog. | 216 950 | 212 259 | 4241 | 20 | 313 | 8  | 52 | 2  | 31 | 1  | 9  | 6  | 3  | 1  | 3  | 1  |
| st. trans. | 152 | 0  | 0  | 0  | 98 | 0  | 17 | 0  | 20 | 1  | 4  | 5  | 3  | 0  | 3  | 1  |
| 3×8  | Σ    | 1  | 2  | 3  | 4  | 6  | 8  | 12 | 16 | 24 | 32 | 48 |
| st. homog. | 1 648 723 | 1 596 362 | 45 732 | 46 | 6203 | 20 | 279 | 5  | 58 | 6  | 8  | 4  |
| st. trans. | 4284 | 0  | 0  | 0  | 4028 | 0  | 196 | 2  | 46 | 2  | 8  | 2  |
| 4×8  | Σ    | 1  | 2  | 3  | 4  | 6  | 8  | 12 | 16 | 24 | 32 | 48 | 64 | 96 |
| st. homog. | 3 547 | 296 | 1420 | 14 | 1273 | 5  | 340 | 5  | 106 | 12 | 45 | 1  | 26 | 4  |
| st. trans. | 712 | 0  | 0  | 0  | 510 | 0  | 124 | 4  | 32 | 6  | 15 | 0  | 21 | 0  |
| 5×8  | Σ    | 2  | 4  | 8  | 16 | 24 | 32 | 48 | 64 |
| st. homog. | 58  | 2  | 10 | 25 | 9  | 4  | 4  | 2  | 2  |
| st. trans. | 20  | 0  | 3  | 5  | 3  | 1  | 4  | 2  | 2  |
| 6×8  | Σ    | 16 | 24 | 48 |
| st. homog. | 4  | 1  | 1  | 2  |
| 7×8  | Σ    | 56 | 168 |
| st. homog. | 3  | 1  | 2  |
| 8×8  | Σ    | 448 | 1344 |
| st. homog. | 3  | 1  | 2  |

Table 32: 4-MOLR for \( n = 8 \).
| $2 \times 8$ | $\Sigma$ | 1 | 2 | 4 | 5 | 8 | 10 | 16 | 20 | 32 | 64 | 128 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| st. homog. | 509 622 | 505 439 | 3896 | 185 | 9 | 52 | 8 | 21 | 2 | 5 | 3 | 2 |
| st. trans. | 19 | 0 | 0 | 0 | 9 | 0 | 8 | 0 | 2 | 0 | 0 | 0 |
| $3 \times 8$ | $\Sigma$ | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 24 | 48 |
| st. homog. | 2652 | 2112 | 419 | 2 | 75 | 13 | 15 | 2 | 9 | 2 | 3 |
| $4 \times 8$ | $\Sigma$ | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 24 | 32 | 48 | 64 | 96 | 192 |
| st. homog. | 267 | 129 | 30 | 12 | 30 | 1 | 23 | 3 | 17 | 2 | 11 | 1 | 4 | 2 | 2 |
| $5 \times 8$ | $\Sigma$ | 8 | 16 |
| st. homog. | 2 | 1 | 1 |
| $6 \times 8$ | $\Sigma$ | 16 |
| st. homog. | 1 | 1 |
| $7 \times 8$ | $\Sigma$ | 56 |
| st. homog. | 1 | 1 |
| $8 \times 8$ | $\Sigma$ | 448 |
| st. homog. | 1 | 1 |

Table 33: 5-MOLR for $n = 8$. 
| Size | Column 1 | Column 2 | Column 3 | Column 4 | Column 5 | Column 6 | Column 7 | Column 8 | Column 9 | Column 10 | Column 11 | Column 12 | Column 13 | Column 14 |
|------|----------|----------|----------|----------|----------|----------|----------|----------|----------|-----------|-----------|-----------|-----------|-----------|
| 2×8  | Σ        | 1        | 2        | 3        | 4        | 5        | 6        | 8        | 10       | 12        | 16        | 24        | 32        | 48        | 64        | 96        | 128       | 384       |
|      | st. homog. | 91 013   | 89 017   | 1597     | 59 138   | 4 71     | 35 8     | 37 22    | 3 7 7    | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     |
|      | st. trans. | 109      | 0        | 0        | 0        | 0        | 60 0     | 0 36 0   | 3 0 6    | 0 2 0     | 0 2 0     | 0 2 0     | 0 2 0     | 0 2 0     | 0 2 0     | 0 2 0     | 0 2 0     |
| 3×8  | Σ        | 1        | 2        | 3        | 4        | 6        | 8        | 12       | 18       | 24        | 36        | 48        | 96        | 288       |
|      | st. homog. | 975      | 3        | 10       | 18       | 7        | 797      | 3 120    | 2 5      | 5 1 3     | 1 3 1     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     |
|      | st. trans. | 908      | 0        | 0        | 0        | 0        | 775      | 0 119    | 0 4      | 5 1 3     | 1 3 1     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     |
| 4×8  | Σ        | 3        | 6        | 9        | 12       | 18       | 24       | 36       | 48       | 96        | 144       |
|      | st. homog. | 155      | 2        | 122      | 2        | 14       | 8        | 1 1      | 1 1      | 2 1       | 1 1       | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     |
|      | st. trans. | 146      | 0        | 122      | 0        | 13       | 8        | 0 1      | 1 1      | 0 1       | 1 1       | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     | 2 2 2     |
| 5×8  | Σ        | 24       |
|      | st. homog. | 1        | 1        |
| 6×8  | Σ        | 48       |
|      | st. homog. | 1        | 1        |
| 7×8  | Σ        | 168      |
|      | st. homog. | 1        | 1        |
| 8×8  | Σ        | 1344     |
|      | st. homog. | 1        | 1        |

Table 34: 6-MOLR for \( n = 8 \).
| $2\times 8$ | $\sum$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 10 | 32 | 42 | 48 | 84 | 128 | 192 | 2688 |
|------------|-------|---|---|---|---|---|---|---|---|----|----|----|----|----|-----|-----|-------|
| st. homog. | 4538  | 4439 | 67 | 9  | 3  | 2  | 3  | 1  | 2  | 4   | 1   | 2   | 1   | 1   | 1    | 1    | 1     |
| st. trans. | 5     | 0   | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 0   | 2   | 0   | 1   | 0   | 0    | 1    |
| $3\times 8$ | $\sum$ | 21 | 168 |
| st. homog. | 2     | 1   | 1   |
| st. trans. | 2     | 1   | 1   |
| $4\times 8$ | $\sum$ | 168 | 672 |
| st. homog. | 2     | 1   | 1   |
| st. trans. | 2     | 1   | 1   |
| $5\times 8$ | $\sum$ | 168 |
| st. homog. | 1     | 1   |
| st. trans. | 1     | 1   |
| $6\times 8$ | $\sum$ | 336 |
| st. homog. | 1     | 1   |
| st. trans. | 1     | 1   |
| $7\times 8$ | $\sum$ | 1176 |
| st. homog. | 1     | 1   |
| st. trans. | 1     | 1   |
| $8\times 8$ | $\sum$ | 9408 |
| st. homog. | 1     | 1   |
| st. trans. | 1     | 1   |

Table 35: 7-MOLR for $n = 8$. 
Sizes of Autotopism Groups of MOLR for $n = 9$
| n×9 | \(\Sigma\) | 4 | 6 | 12 | 16 | 24 | 36 | 72 | 80 | 648 |
|-----|---------|---|---|----|----|----|----|----|----|-----|
| st. trans. | | 126 | 95 | 3 | 13 | 4 | 3 | 3 | 2 | 2 | 1 |
| 3×9 | \(\Sigma\) | 2 | 4 | 6 | 8 | 12 | 18 | 24 | 36 | 40 | 54 | 72 | 108 | 1944 |
| st. trans. | | 1 418 577 | 1 413 987 | 3905 | 511 | 8 | 115 | 19 | 8 | 13 | 3 | 2 | 3 | 2 | 1 |
| 4×9 | \(\Sigma\) | 2 | 4 | 6 | 8 | 12 | 16 | 18 | 24 | 36 | 54 | 72 | 80 | 108 | 144 |
| st. trans. | | 560 524 587 | 560 437 428 | 82 404 | 3520 | 958 | 220 | 15 | 19 | 12 | 5 | 2 | 1 | 1 | 1 |
| 5×9 | \(\Sigma\) | 2 | 4 | 6 | 8 | 12 | 16 | 18 | 24 | 36 | 54 | 72 | 108 | 144 |
| st. trans. | | 20019 499 500 | 2001 908 825 | 400 369 | 9228 | 1223 | 370 | 6 | 29 | 9 | 11 | 2 | 1 | 1 | 1 |
| 6×9 | \(\Sigma\) | 2 | 4 | 6 | 8 | 12 | 18 | 24 | 36 | 54 | 72 | 108 | 324 | 648 |
| st. trans. | | 674 803 643 7 | 674 792 712 7 | 423 557 | 12 685 | 521 | 577 | 89 | 32 | 28 | 3 | 6 | 9 | 2 | 1 |
| 7×9 | \(\Sigma\) | 2 | 4 | 6 | 8 | 12 | 18 | 24 | 36 | 54 | 108 | 216 |
| st. trans. | | 1 577 270 689 | 1 577 056 397 | 208 362 | 4900 | 384 | 595 | 13 | 18 | 2 | 4 | 1 |
| 8×9 | \(\Sigma\) | 2 | 4 | 6 | 8 | 12 | 16 | 18 | 24 | 36 | 54 | 72 | 96 | 108 | 864 |
| st. trans. | | 14 940 988 | 14 917 761 | 20 900 | 1879 | 114 | 268 | 8 | 22 | 13 | 13 | 2 | 1 | 2 | 4 | 1 |
| 9×9 | \(\Sigma\) | 2 | 4 | 6 | 8 | 12 | 16 | 18 | 24 | 36 | 54 | 72 | 96 | 108 | 144 | 162 | 324 | 972 | 7776 |
| st. trans. | | 28 955 | 25 570 | 2802 | 307 | 110 | 99 | 3 | 13 | 11 | 17 | 2 | 3 | 2 | 6 | 1 | 1 | 5 | 2 | 1 |

Table 36: 2-MOLR for \(n = 9\).
| 2×9 | Σ | 3  | 6  | 12 | 18 | 24 | 36 | 54 | 108 |
|-----|----|----|----|----|----|----|----|----|-----|
| st. trans. | 202 | 86 | 102 | 4  | 3  | 2  | 3  | 1  | 1   |
| 3×9 | Σ | 3  | 6  | 9  | 12 | 18 | 36 | 54 | 162 | 324 |
| st. trans. | 72836 | 71109 | 1618 | 40 | 6  | 56 | 1  | 4  | 1   |
| 4×9 | Σ | 3  | 6  | 9  | 12 | 18 | 36 | 54 | 162 | 324 |
| st. trans. | 1746912 | 1742486 | 4209 | 149 | 37 | 21 | 4  | 6  |     |

Table 37: 3-MOLR for $n = 9$. 
| $2 \times 9$ | $\Sigma$ | 4 | 8 | 12 | 16 | 24 | 144 |
|-------------|----------|----|----|-----|-----|-----|------|
| st. trans.  | 1017     | 881| 121| 5   | 2   | 7   | 1    |

| $3 \times 9$ | $\Sigma$ | 4 | 8 | 12 | 16 | 24 | 36 | 48 | 72 | 108 | 432 |
|-------------|----------|----|----|-----|-----|-----|-----|-----|-----|------|------|
| st. trans.  | 356680   | 355023| 1511| 112| 6   | 18 | 4   | 1   | 3   | 1    | 1    |

| $4 \times 9$ | $\Sigma$ | 4 | 8 | 12 | 16 | 24 | 32 | 36 | 72 | 144 | 288 |
|-------------|----------|----|----|-----|-----|-----|-----|-----|-----|------|------|
| st. trans.  | 2640163  | 2635762| 4131| 95 | 147 | 1 | 17 | 2   | 5   | 2    | 1    |

| $5 \times 9$ | $\Sigma$ | 4 | 8 | 12 | 16 | 32 | 36 | 72 | 144 | 288 |
|-------------|----------|----|----|-----|-----|-----|-----|-----|------|------|
| st. trans.  | 645453   | 641633| 3467| 30 | 305 | 1 | 11 | 1   | 3   | 2    | 1    |

| $6 \times 9$ | $\Sigma$ | 4 | 8 | 12 | 16 | 24 | 48 | 72 | 216 | 432 |
|-------------|----------|----|----|-----|-----|-----|-----|-----|------|------|
| st. trans.  | 1816     | 1662| 124| 15 | 9   | 1   | 1   | 1   | 1    | 1    |

| $7 \times 9$ | $\Sigma$ | 4 | 8 | 16 | 72 | 144 |
|-------------|----------|----|----|-----|-----|-----|
| st. trans.  | 31       | 3  | 12 | 14 | 1   | 1   |

| $8 \times 9$ | $\Sigma$ | 8 | 64 | 288 | 576 |
|-------------|----------|----|-----|-----|-----|
| st. trans.  | 7        | 2  | 3   | 1   | 1   |

| $9 \times 9$ | $\Sigma$ | 64 | 576 | 2592 | 5184 |
|-------------|----------|-----|------|-------|-------|
| st. trans.  | 5        | 2   | 1    | 1     | 1     |

Table 38: 4-MOLR for $n = 9$.

| $2 \times 9$ | $\Sigma$ | 5 | 10 |
|-------------|----------|----|----|
| st. trans.  | 17       | 7  | 10 |

Table 39: 5-MOLR for $n = 9$. 
| $2 \times 9$ | $\Sigma$ | 6 | 12 | 24 | 36 | 54 | 72 | 108 | 216 |
|---|---|---|---|---|---|---|---|---|---|
| st. trans. | 543 | 422 | 104 | 7 | 4 | 1 | 2 | 2 | 1 |
| $3 \times 9$ | $\Sigma$ | 6 | 12 | 18 | 24 | 36 | 54 | 72 | 108 | 324 | 648 |
| st. trans. | 21 620 | 20 528 | 946 | 69 | 12 | 48 | 5 | 3 | 3 | 5 | 1 |
| $4 \times 9$ | $\Sigma$ | 6 | 12 | 18 | 24 | 36 | 54 | 72 | 108 |
| st. trans. | 244 | 157 | 41 | 24 | 2 | 8 | 3 | 3 | 6 |
| $5 \times 9$ | $\Sigma$ | 6 | 12 | 18 | 36 | 54 | 108 |
| st. trans. | 33 | 10 | 11 | 1 | 4 | 3 | 4 |
| $6 \times 9$ | $\Sigma$ | 6 | 12 | 24 | 36 | 72 | 108 | 216 | 324 | 648 |
| st. trans. | 16 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 2 |
| $7 \times 9$ | $\Sigma$ | 12 |
| st. trans. | 1 | 1 |
| $8 \times 9$ | $\Sigma$ | 48 |
| st. trans. | 1 | 1 |
| $9 \times 9$ | $\Sigma$ | 432 |
| st. trans. | 1 | 1 |

Table 40: 6-MOLR for $n = 9$. 
Table 41: 7-MOLR for $n = 9$. 

|     | $\Sigma$ | 7  | 14 |
|-----|----------|----|----|
| 2×9 | 39       | 37 | 2  |
| st. trans. | 39       | 37 | 2  |
| 3×9 | 1532     | 1522 | 10 |
| st. trans. | 1532     | 1522 | 10 |
| 4×9 | 300      | 298 | 2  |
| st. trans. | 300      | 298 | 2  |
| n×9 | Σ | 8 | 16 | 32 | 48 | 56 | 96 | 864 |
|-----|---|----|----|----|----|----|----|-----|
| st. trans. | 54 | 39 | 9 | 2 | 1 | 1 | 1 | 1 |

| n×9 | Σ | 8 | 16 | 48 | 96 | 144 | 288 | 432 | 2592 |
|-----|---|----|----|----|----|-----|-----|-----|------|
| st. trans. | 48 | 22 | 17 | 3 | 2 | 1 | 1 | 1 | 1 |

| n×9 | Σ | 16 | 24 | 32 | 48 | 64 | 144 | 288 | 432 | 576 |
|-----|---|----|----|----|----|----|-----|-----|-----|-----|
| st. trans. | 27 | 11 | 2 | 2 | 3 | 4 | 1 | 1 | 1 | 2 |

| n×9 | Σ | 16 | 24 | 48 | 64 | 144 | 288 | 432 | 576 |
|-----|---|----|----|----|----|-----|-----|-----|-----|
| st. trans. | 22 | 8 | 2 | 3 | 4 | 1 | 1 | 1 | 2 |

| n×9 | Σ | 16 | 48 | 96 | 144 | 288 | 432 | 864 | 2592 |
|-----|---|----|----|----|-----|-----|-----|-----|------|
| st. trans. | 16 | 6 | 3 | 2 | 1 | 1 | 1 | 1 | 1 |

| n×9 | Σ | 16 | 48 | 96 | 288 | 864 |
|-----|---|----|----|-----|-----|-----|
| st. trans. | 9 | 2 | 2 | 3 | 1 | 1 |

| n×9 | Σ | 48 | 384 | 1152 | 3456 |
|-----|---|-----|--------|--------|-----|
| st. trans. | 7 | 2 | 3 | 1 | 1 |

| n×9 | Σ | 384 | 3456 | 10368 | 31104 |
|-----|---|-----|----------|--------|------|
| st. trans. | 5 | 2 | 1 | 1 | 1 |

Table 42: 8-MOLR for \( n = 9 \).
The Stepwise Transitive 8-MOLR for $n = 9$
Figure 4: The 8-MOLS of size $9 \times 9$ with $|\text{Aut}| = 10368$, corresponding to the Galois plane.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |   |
| 8 | 7 | 6 | 2 | 1 | 0 | 3 | 4 | 5 |   |
| 7 | 6 | 8 | 0 | 2 | 1 | 5 | 3 | 4 |   |
| 6 | 8 | 7 | 1 | 0 | 2 | 4 | 5 | 3 |   |
| 5 | 4 | 3 | 6 | 7 | 8 | 2 | 1 | 0 |   |
| 4 | 3 | 5 | 8 | 6 | 7 | 0 | 2 | 1 |   |
| 3 | 5 | 4 | 7 | 8 | 6 | 1 | 0 | 2 |   |
| 2 | 0 | 1 | 4 | 5 | 3 | 7 | 8 | 6 |   |
| 1 | 2 | 0 | 5 | 3 | 4 | 8 | 6 | 7 |   |

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |   |
| 8 | 7 | 6 | 0 | 2 | 1 | 5 | 3 | 4 |   |
| 7 | 6 | 8 | 0 | 2 | 1 | 5 | 3 | 4 |   |
| 6 | 8 | 7 | 1 | 0 | 2 | 4 | 5 | 3 |   |
| 5 | 4 | 3 | 6 | 7 | 8 | 2 | 1 | 0 |   |
| 4 | 3 | 5 | 8 | 6 | 7 | 0 | 2 | 1 |   |
| 3 | 5 | 4 | 7 | 8 | 6 | 1 | 0 | 2 |   |
| 2 | 0 | 1 | 4 | 5 | 3 | 7 | 8 | 6 |   |
| 1 | 2 | 0 | 5 | 3 | 4 | 8 | 6 | 7 |   |

Figure 4: The 8-MOLS of size $9 \times 9$ with $|\text{Aut}| = 10368$, corresponding to the Galois plane.
Figure 5: The 8-MOLS of size $9 \times 9$ with $|\text{Aut}| = 31\,104$, corresponding to the dual of the Hall plane.
|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 8 | 7 | 6 | 5 | 3 | 4 | 2 | 1 | 0 |
| 7 | 8 | 1 | 6 | 2 | 0 | 3 | 5 | 4 |
| 6 | 0 | 8 | 7 | 1 | 2 | 4 | 3 | 5 |
| 5 | 4 | 3 | 2 | 6 | 7 | 8 | 0 | 1 |
| 4 | 3 | 5 | 1 | 7 | 8 | 0 | 6 | 2 |
| 3 | 5 | 4 | 0 | 8 | 6 | 1 | 2 | 7 |
| 2 | 6 | 7 | 8 | 0 | 1 | 5 | 4 | 3 |
| 1 | 2 | 0 | 4 | 5 | 3 | 7 | 8 | 6 |

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 7 | 6 | 8 | 4 | 5 | 3 | 1 | 0 | 2 |
| 3 | 5 | 4 | 0 | 6 | 7 | 2 | 8 | 1 |
| 5 | 4 | 3 | 2 | 7 | 8 | 0 | 1 | 6 |
| 1 | 8 | 6 | 7 | 2 | 0 | 4 | 3 | 5 |
| 6 | 7 | 0 | 8 | 1 | 2 | 5 | 4 | 3 |
| 8 | 2 | 7 | 6 | 0 | 1 | 3 | 5 | 4 |
| 4 | 3 | 5 | 1 | 8 | 6 | 7 | 2 | 0 |
| 2 | 0 | 1 | 5 | 3 | 4 | 8 | 6 | 7 |

Figure 6: The 8-MOLS of size $9 \times 9$ with $|\text{Aut}| = 384$, corresponding to the Hall plane.
Figure 7: The 8-MOLS of size $9 \times 9$ with $|\text{Aut}| = 384$, corresponding to the dual of the Hall plane.
Figure 8: The 8-MOLS of size $9 \times 9$ with $|\text{Aut}| = 3456$, corresponding to the Hall plane.