Rogue Waves and Their Patterns in the Vector Nonlinear Schrödinger Equation

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Abstract
In this paper, we study the general rogue wave solutions and their patterns in the vector (or $M$-component) nonlinear Schrödinger (NLS) equation. By applying the Kadomtsev–Petviashvili reduction method, we derive an explicit solution for the rogue wave expressed by \( \tau \) functions that are determinants of $K \times K$ block matrices ($1 \leq K \leq M$) with an index jump of $M + 1$. Patterns of the rogue waves for $M = 3, 4$ and $K = 1$ are thoroughly investigated. It is found that when one of the internal parameters is large enough, the wave pattern is linked to the root structure of a generalized Wronskian–Hermite polynomial hierarchy in contrast with rogue wave patterns of the scalar NLS equation, the Manakov system, and many others. Moreover, the generalized Wronskian–Hermite polynomial hierarchy includes the Yablonskii–Vorob’ev polynomial and Okamoto polynomial hierarchies as special cases, which have been used to describe the rogue wave patterns of the scalar NLS equation and the Manakov system, respectively. As a result, we extend the most recent results by Yang et al. for the scalar NLS equation and the Manakov system. It is noted that the case $M = 3$ displays a new feature different from the previous results. The predicted

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rogue wave patterns are compared with the ones of the true solutions for both cases of $M = 3, 4$. An excellent agreement is achieved.

**Keywords** Kadomtsev–Petviashvili reduction method · Vector nonlinear Schrödinger equation · Rogue wave pattern · Wronskian–Hermite polynomials

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1 Introduction

Rogue waves have been known in the maritime community as part of folklore for centuries. Notable features of such waves include sudden emergence, abnormally large amplitude, and disappearance without any trace. These characteristics indicate that rogue waves may result in tremendous impacts on their surrounding environment and have been associated with many maritime disasters (Dysthe et al. 2008). Systematic studies on rogue waves started only after the first verified measurement of an extreme water wave in 1995 (Haver 2004). Remarkably, research on rogue waves has developed considerably since 2007, following the discovery of rogue waves in optical fibers (Solli et al. 2007), which has attracted much interest in both optics and hydrodynamics. Since then, there has been an explosion of studies to explore rogue waves extended to other physical systems, such as superfluid helium (Ganshin et al. 2008), Bose–Einstein condensates (Bludov et al. 2009), capillary waves (Shats et al. 2010), and plasmas (Bailung et al. 2011).

In optics and hydrodynamics, the mathematical models governing wave propagation can be derived from Maxwell’s equations and the Euler equations, respectively (Dudley et al. 2019). Under further assumptions, the nonlinear Schrödinger (NLS) equation, which describes the evolution of slowly varying wave packets in nonlinear wave systems, can be reduced from both of these two models (Ablowitz et al. 1991). Owing to its integrability, the NLS equation has been widely studied (Besse et al. 2002; Eliasson and Kuksin 2010; Kato 1987; Pelinovsky and Yang 2004; Weinstein 1982; Yang 2016) and shown to admit a number of analytic solutions. In particular, one of its rational solutions, namely the Peregrine soliton (Peregrine 1983), is widely regarded as the prototype of rogue waves. In the past two decades, mathematical study on rogue waves has attracted much attention, and various higher-order rogue wave solutions of the NLS equation have been constructed (Akhmediev et al. 2009; Dubard et al. 2010; Kedziora et al. 2011; Guo et al. 2012; Ohta and Yang 2012). It is worth noting that these solutions in turn have facilitated the experimental studies of rogue waves. On the other hand, explicit rogue wave solutions have been derived in various integrable equations, such as the derivative NLS equation (Yang et al. 2020), the Yajima–Oikawa equation (Chen et al. 2015, 2018), the three-wave equation (Yang and Yang 2021b), the Manakov system (Baronio et al. 2014; Chen and Mihalache 2015), and the Sasa–Satsuma equation (Feng et al. 2022; Wu et al. 2022). Besides, rogue waves of infinite order have been uncovered (Bilman et al. 2020; Bilman and Miller 2022) by making use of the Riemann–Hilbert approach (Yang 2010), while rogue waves on the periodic background (Chen and Pelinovsky 2018; Chen et al. 2019; Feng et al. 2020) have also
been explored. Large-order asymptotics for solitons (Bilman and Buckingham 2019) and rogue waves (Bilman and Miller 2019) of the NLS equation were analyzed by using the inverse-scattering transform method.

In addition to their physical significance, rogue waves may exhibit extremely regular and symmetric patterns, which are intriguing and can provide critical information for predicting subsequent rogue waves from previous ones. For instance, circular rogue wave clusters of the NLS equation were reported in Kedziora et al. (2011) by using Darboux transformation and numerical simulations. Soon after this phenomenon was confirmed analytically in He et al. (2013), a systematic classification of the NLS rogue wave patterns was obtained in Kedziora et al. (2013) according to the order of rogue waves and the parameter shifts involved in Akhmediev breathers in the rogue wave limit. Moreover, this study reveals various highly symmetric geometric structures of rogue waves under certain choices of parameters, including triangles, pentagons, heptagons, and nonagons. A even more remarkable observation is that, which was first shown in Yang and Yang (2021c), the distribution of rogue waves for specific choices of parameters looks very similar to another independent object, that is, the root structure of the Yablonskii–Vorob’ev polynomial hierarchy, which is closely related to rational solutions of the Painlevé II hierarchy (Clarkson and Mansfield 2003). When specific internal parameter is large enough, the deep connection between these two objects has been established analytically in Yang and Yang (2021c), which is a remarkable progress in the study of rogue waves. Following this work, it is found that such patterns are universal, as rogue waves of many other integrable equations demonstrate similar patterns, such as the Boussinesq equation and the Manakov system, as long as the Schur polynomials involved in the $\tau$ functions have index jumps of two (Yang and Yang 2021a). Beyond that, Yang and Yang (2023) very recently discovered that other rogue wave patterns exist when the index jumps are three, and these patterns are characterized by root structures of Okamoto polynomial hierarchies.

Inspired by the works in Yang and Yang (2021c, a, 2023), some natural problems arise.

- Can we construct the rogue wave solution in $M$-component NLS equation?
- What about the patterns of rogue waves for $M = 3, 4$ or even the general case? Are these patterns related to the root structure of some other polynomial hierarchy?

The main objective of this paper is to solve the problems listed above by considering the vector NLS equation

$$i u_{j,t} + u_{j,xx} + \left( \sum_{k=1}^{M} \sigma_k |u_k|^2 \right) u_j = 0, \quad j = 1, 2, \ldots, M,$$

where $M$ is a positive integer and $\sigma_k = \pm 1$. For $M = 2$, it is known as the Manakov system (Manakov 1974), which is a model that governs soliton propagation through optical fiber arrays (Akhmediev and Ankiewicz 1997; Kang et al. 1996; Kanna and Lakshmanan 2001). Our results consist of two ingredients. First, we will apply the Kadomtsev–Petviashvili reduction technique to derive rogue wave solutions of the vector NLS equation whose $\tau$ functions are represented by determinants.
of $K \times K$ block matrices ($1 \leq K \leq M$) with index jumps of $M + 1$. The crucial point of this part is to solve a system of algebraic equations (see Lemma 2.1 and its proof). Then, we will study the rogue wave patterns for $M = 3, 4$ and $K = 1$. We find that when a specific internal parameter is large enough, these patterns are connected to the root structure of a new polynomial hierarchy, which is called generalized Wronskian–Hermite polynomials (see Sect. 2.2). Moreover, we notice that the Yablonskii–Vorob’ev polynomial and Okamoto polynomial hierarchies are special cases of the generalized Wronskian–Hermite polynomial hierarchy. Accordingly, our results have unified rogue wave patterns of the scalar NLS equation and the vector NLS equation (1) for $M = 2, 3, 4$. In addition, we find that the proof in the inner region for $M = 3$ is different from other cases. In the proofs of the inner region of the scalar NLS equation and the Manakov system, one can perform row and column operations to reduce the $\tau$ functions into determinants of block matrices with lower triangular matrices at the (1,1) entry whose elements on the diagonal are all 1. Then, the sizes of the determinants can be decreased; hence, these waves can be approximated by possible lower-order rogue waves in the inner region. However, although the waves for $M = 3$ can also be approximated by possible lower-order rogue waves in the inner region, it turns out that in certain cases the sizes of the determinants remain unchanged after row and column operations, as the (1,1) entries of the block matrices are no longer triangular. The predicted rogue wave patterns are compared with actual ones, and excellent agreement is achieved.

The structure of this paper can now be explained. Section 2 presents some preliminary results that will be used in the subsequent discussions. We first provide explicit rogue wave solutions with index jumps of $M + 1$ of the vector NLS equation, and it is shown that these solutions are expressed by $K \times K$ block matrices ($1 \leq K \leq M$). This is followed by an introduction to the generalized Wronskian–Hermite polynomials and the study of their root structures. Then, rogue wave patterns for the three- and four-component NLS equations under the condition that one of the internal parameters is large enough are stated in Sect. 3, which form the main results of this paper. Section 4 is devoted to comparing predicted and actual rogue wave patterns, while the proofs of the main results are provided in Sect. 5. We summarize the main results of this paper in Sect. 6. Finally, the proof of Lemma 2.1, which involves the study of multiple roots of some rational function and plays a pivotal role in this paper, and derivations of rogue wave solutions in the vector NLS equation are given in Appendices A and B, respectively, while the results on root structures of the generalized Wronskian–Hermite polynomials of jump $k = 4, 5$ are proved in Appendix C.

## 2 Preliminaries

### 2.1 Rogue Wave Solutions of the Vector Nonlinear Schrödinger Equation

This section presents rogue wave solutions of the vector NLS equation (1), which possesses an infinite dimensional algebra of non-commutative symmetries (Kodama and Mikhailov 2001). We note that these solutions have been studied previously (Baronio et al. 2012; Chen and Mihalache 2015; Ling et al. 2014; Mu et al. 2015; Rao et al. 2013).
In particular, vector Peregrine solitons were found by applying the loop group method in Zhang et al. (2021), in which the authors proposed the problem of whether patterns of these rogue waves are related to Yablonskii–Vorob’ev polynomial hierarchy. This problem was later confirmed in the Manakov system (Yang and Yang 2021a), which has been taken as an example to show that universal rogue wave patterns associated with the Yablonskii–Vorob’ev polynomial hierarchy exist in integrable systems. Very recently, new patterns of another class of (degenerate) rogue waves of the Manakov system have been obtained by Yang and Yang (2023) through establishing the connection between these waves and the Okamoto polynomial hierarchies. A remarkable feature of these new patterns is that, unlike previous patterns, the transformations between the locations of fundamental rogue waves and zeros of the Okamoto polynomial hierarchies are nonlinear, thereby leading to deformations of rogue patterns. Inspired by these studies, we will extend the results in Yang and Yang (2023) and solve the problem mentioned above in Zhang et al. (2021) for $M = 3, 4$ by studying patterns of degenerate rogue waves of the vector NLS equation (1). To this end, we introduce some notations and lemma that will be needed.

The Schur polynomials $S_n(x)$ are defined by

$$\sum_{n=0}^{\infty} S_n(x)\lambda^n = \exp \left( \sum_{k=1}^{\infty} x_k \lambda^k \right),$$

where $x = (x_1, x_2, \ldots)$. To be more specific, we have

$$S_0(x) = 1, \quad S_1(x) = x_1, \quad S_2(x) = \frac{1}{2} x_1^2 + x_2, \ldots,$$

$$S_j(x) = \sum_{l_1+2l_2+\ldots+ml_m=j} \left( \prod_{i=1}^{m} \frac{x_i^{l_i}}{l_i!} \right). \quad (2)$$

Further, we define $S_j(x) \equiv 0$ for $j < 0$.

**Lemma 2.1** Let $M$ be a positive integer and $\lambda_1 > 0, r_j \neq 0, k_j$ be distinct real constants, $j = 1, 2, \ldots, M$. Let $\mathcal{R}_M(z)$ be a rational function defined by

$$\mathcal{R}_M(z) = \sum_{j=1}^{M} \frac{r_j}{(z+k_j)^2} + 2. \quad (3)$$

Then, $\mathcal{R}_M(z) = 0$ has a pair of complex conjugate roots with nonzero imaginary parts of multiplicity $M$

$$\lambda_1 \cos[\pi/(M + 1)] - k_1 \pm i\lambda_1 \sin[\pi/(M + 1)], \quad (4)$$

if the parameters $r_j, k_j, j = 2, \ldots, M$, satisfy the conditions

$$k_j = k_1 + \lambda_1 \left( \sin[\pi/(M + 1)] \cot[j\pi/(M + 1)] - \cos[\pi/(M + 1)] \right), \quad (5)$$
and

\[ r_j = 2(-1)^{j+1} \prod_{i=1, i \neq j}^M (k_j - k_i)^{-1} \left( \lambda_1 \frac{\sin[\pi/(M + 1)]}{\sin[j\pi/(M + 1)]} \right)^{M+1}. \]  \hspace{1cm} (6)

**Remark 1** The equation \( \mathcal{R}_M(z) = 0 \) may have real roots of multiplicity \( M \) as well. For instance, the equation

\[
\frac{162}{(z + 2)^2} - \frac{256}{(z + 3)^2} - \frac{16}{(z + 1)^2} + 2 = 0
\]

has a real root 1 of multiplicity 3, while the equation

\[
-\frac{1024}{(z + 2)^2} + \frac{3125}{(z + 3)^2} - \frac{2592}{(z + 4)^2} + \frac{81}{(z + 1)^2} + 2 = 0
\]

has a real root 2 of multiplicity 4. Nevertheless, this case will not occur in our subsequent discussions.

We provide the proof of Lemma 2.1 in Appendix A. Next, we define the function \( G_M(p) \) by

\[ G_M(p) = \sum_{j=1}^M \frac{\sigma_j \rho_j^2}{p - ik_j} + 2p, \] \hspace{1cm} (7)

where \( \sigma_j = \pm 1 \) and \( \rho_j > 0, k_j \) are real constants, \( j = 1, 2, \ldots, M \). We may deduce from Lemma 2.1 that if \( \sigma_j, \rho_j \) and \( k_j, j = 1, 2, \ldots, M \), satisfy the constraints

\[
k_j = k_1 + \lambda_1 (\sin[\pi/(M + 1)] \cot[j\pi/(M + 1)] - \cos[\pi/(M + 1)]) ,
\]

\[
\sigma_j \rho_j^2 = 2(-1)^{j+1} \prod_{i=1, i \neq j}^M (k_j - k_i)^{-1} \left( \lambda_1 \frac{\sin[\pi/(M + 1)]}{\sin[j\pi/(M + 1)]} \right)^{M+1}, \] \hspace{1cm} (8)

then the algebraic equation

\[ G_M'(p) = 0 \] \hspace{1cm} (9)

has a pair of non-imaginary roots of multiplicity \( M \) given by

\[ \pm \lambda_1 \sin[\pi/(M + 1)] - i\lambda_1 \cos[\pi/(M + 1)] + ik_1. \] \hspace{1cm} (10)

This is the case that we will encounter later. Additionally, we will need the function \( p(\kappa) \) defined by (see (146) in Appendix B)

\[ G_M(p(\kappa)) = \frac{G_M(p(0))}{M + 1} \sum_{n=1}^{M+1} \exp \left( \exp \left( \frac{2n\pi i}{M + 1} \kappa \right) \right). \]
\[ G_M(p(0)) = \frac{M+1}{M+1} \sum_{n=1}^{M+1} \exp \left( \cos \left( \frac{2n\pi}{M+1} \right) \kappa \right) \cos \left( \sin \left( \frac{2n\pi}{M+1} \right) \kappa \right). \]

(11)

**Theorem 2.2** Let \( M \) be a positive integer, \( \rho_j > 0, k_j \) be real constants, and \( \sigma_j = 1, \) where \( j = 1, 2, \ldots, M. \) Assume \( \rho_j \) and \( k_j \) are given by (8). Let \( G_M(p), p(\kappa) \) be functions defined by (7) and (11), respectively, and \( p_0 \) be a zero of \( G'_M(p) \) of multiplicity \( M. \)

Let \( x^\pm_I = (x^\pm_{1,I}, x^\pm_{2,I}, \ldots), I = 1, 2, \ldots, M, \) and \( s = (s_1, s_2, \ldots) \) be the vectors defined by

\[ x^+_i,I = \alpha_i x + \beta_i t + \sum_{j=1}^{M} n_j \theta_{ij} + a_{i,I}, \]

(12)

\[ x^-_i,I = \alpha_i^* x - \beta_i^* t - \sum_{j=1}^{M} n_j \theta^*_{ij} + a^*_{i,I}, \]

(13)

\[ \ln \left[ \frac{1}{\kappa} \left( \frac{p_0 + p_1}{p_1} \right) \left( \frac{p(\kappa) - p_0}{p(\kappa) + p_1} \right) \right] = \sum_{r=1}^{\infty} s_r \kappa^r, \]

(14)

where the asterisk “*” represents complex conjugation, \( p_0 = p(0), \ p_1 = p'(0), \) the \( a_{i,I} \)'s are arbitrary constants, and \( \alpha_i, \ \beta_i, \ \theta_{ij}, \ j = 1, 2, \ldots, M, \) are defined by the expansions

\[ p(\kappa) - p_0 = \sum_{r=1}^{\infty} \alpha_r \kappa^r, \quad p^2(\kappa) - p_0^2 = \sum_{r=1}^{\infty} \beta_r \kappa^r, \quad \ln \frac{p(\kappa) - i k_j}{p_0 - i k_j} = \sum_{r=1}^{\infty} \theta_{r,j} \kappa^r. \]

In this case, the \( M \)-component NLS equation (1) admits \( \mathcal{N} \)-th-order rogue wave solutions

\[ u_{j,\mathcal{N}} = \rho_j g_{j,\mathcal{N}} \frac{e^{i(k_j x + w_j t)}}{f_{\mathcal{N}}}, \quad j = 1, 2, \ldots, M, \]

(15)

where

\[ w_j = \sum_{i=1}^{M} \sigma_i \rho_i^2 - k_j^2, \quad \mathcal{N} = (N_1, N_2, \ldots, N_M), \]

(16)

with \( N_j (j = 1, 2, \ldots, M) \) being nonnegative integers, and \( f \) and \( g_j \) are given by

\[ f_{\mathcal{N}} = \tau_{n_0}, \quad g_{j,\mathcal{N}} = \tau_{n_j} \]

(17)

with

\[ n_0 = (0, 0, \ldots, 0) \in \mathbb{R}^M, \quad n_j = e_j, \]
Remark 2 The rogue wave solutions to the vector NLS equation (1) in Theorem 2.2 are represented by \( \tau \) functions which have matrix elements expressed by Schur polynomials with an index jump of \( M + 1 \). These rogue waves exist only when \( G'_M(p) = 0 \) has non-imaginary roots of order \( M \). This indicates that the scalar NLS equation and the Manakov system have no such kind of rogue waves with index jumps of \( J \geq 4 \). In addition, when \( G'_M(p) = 0 \) has simple or double roots, the vector NLS equations would possess similar types of rogue waves as the scalar NLS equation or the Manakov system, where the index jumps of the corresponding Schur polynomials are 2 or 3. In such cases, the rogue wave patterns are similar to those of the scalar NLS equation (Yang and Yang 2021c) or the Manakov system (Yang and Yang 2023) and hence we will not further discuss these rogue waves.

Remark 3 We restrict the study of rogue wave patterns to the cases of \( M = 3, 4 \). Then, the \( p(\kappa) \) introduced in (11) can be expressed by

\[
G_M(p(\kappa)) = \begin{cases} 
G_3(p(0)) & M = 3, \\
\frac{4}{5} G_4(p(0)) & M = 4.
\end{cases}
\]

(23)
It is also clear that there are other parameter choices for \( G'_M(p) = 0 \) \((M = 3, 4)\) to have a pair of non-imaginary roots of order \( M \), on account of the symmetry of the vector NLS equation \((1)\). Let \((i, j, l)\) be any permutation of the set \( \{1, 2, 3\} \), then the condition \((8)\) can be replaced by

\[
\rho_i = \rho_j = \sqrt{2}\rho_l = 2|k_l - k_i| = 2|k_l - k_j| \neq 0, \quad k_i \neq k_j. \tag{24}
\]

Similarly, assume \((i, j, l, m)\) is any permutation of the set \( \{1, 2, 3, 4\} \), then the condition \((8)\) can be replaced by

\[
2\rho_i^2 = (3 - \sqrt{5})\rho_j^2 = (3 - \sqrt{5})\rho_l^2 = 2\rho_m^2 = (6 - 2\sqrt{5})(k_j - k_l)^2 = 4(k_l - k_i)^2 = (6 + 2\sqrt{5})(k_m - k_l)^2 \neq 0. \tag{25}
\]

Under these conditions, the roots are

\[
p_0 = \begin{cases} 
\pm \frac{\rho_i}{2} + ik_l, & \text{for } M = 3, \\
\pm \frac{1}{4} \sqrt{5 + \sqrt{5}}\rho_i + i \left( k_i - \frac{1}{4} \sqrt{3 - \sqrt{5}}\rho_i \right), & \text{for } M = 4.
\end{cases} \tag{26}
\]

**Remark 4** For a fixed value of \( p(0) \), there exist \( M + 1 \) functions of \( p(\kappa) \) that satisfy \((11)\). However, these multiple functions of \( p(\kappa) \) are connected by symmetry and give rise to the same rogue wave solutions \( u_{j,N} \), allowing us to select any of these functions. As the right-hand side of \((11)\) can be expressed as

\[
\frac{G_M(p(0))}{M+1} \left[ \exp \left( \exp \left( \frac{2\pi i}{M+1} \kappa \right) \right) + \exp \left( \exp \left( \frac{4\pi i}{M+1} \kappa \right) \right) + \ldots + \exp \left( \exp \left( 2\pi i \kappa \right) \right) \right],
\]

which is invariant when \( \kappa \) changes to \( \exp(2\pi i/(M+1))\kappa \), we find that if \( p(\kappa) \) is a solution to this equation, so are \( p(\exp(2n\pi i/(M+1))\kappa) \), \( n = 1, 2, 3, \ldots, M \). As a result, these \( p(\kappa) \) functions are related as \( p(\exp(2n\pi i/(M+1))\kappa) \). By applying these symmetries and employing similar arguments as Remark 3 in Yang and Yang (2021b), it can be demonstrated that the \( u_{j,N} \) solutions with a branch of the function \( p(\kappa) \) and complex parameters \( a_{k,i} \) coincide with those \( u_{j,N} \) solutions corresponding to the branches \( p(\exp(2n\pi i/(M+1))\kappa) \) and complex parameters \( a_{k,i} \exp(2kn\pi i/(M+1)) \), where \( i = 1, 2, \ldots, M \). Therefore, without loss of generality, we can choose any of these \( p(\kappa) \) functions and keep the complex parameters \( a_{k,i} \) free.

**Remark 5** When \( K = 1 \), the \( \tau \) functions are comprised of determinants of single block matrices, i.e.,

\[
\tau_n = \det_{1 \leq i, j \leq N} \left( m_{(M+1)i-I_1,(M+1)j-I_1}^{(n, I_1, I_1)} \right), \quad 1 \leq I_1 \leq M,
\]
where \( m_{i,j}^{(n,l)} \) is given by (22). In this case, we define the rogue wave solutions in Theorem 2.2 to be the \( I_1 \)-th type, \( 1 \leq I_1 \leq M \), and simply denote \( x_1^\pm \) by \( x^\pm \) by ignoring the dependence on \( I \).

**Remark 6** By rewriting \( \tau_n \) into a larger determinant similar to Ohta and Yang (2012), we can show that the degrees of the polynomials \( \tau_n \) for \( M = 3, 4 \) with respect to \( x \) and \( t \) in Theorem 2.2 are

\[
\text{deg}(\tau_n) = \begin{cases} 
3 \left( N_1^2 + N_2^2 + N_3^2 \right) - 2(N_1 N_2 + N_1 N_3 + N_2 N_3) + (3N_1 + N_2 - N_3), & M = 3, \\
5 \left( N_1^2 + N_2^2 + N_3^2 + N_4^2 \right) - (N_1 + N_2 + N_3 + N_4)^2 + 4N_1 + 2N_2 - 2N_4, & M = 4, 
\end{cases}
\]

(27)

where \( N_j (j = 1, 2, \ldots, M) \) are nonnegative integers such that \( N_1 + N_2 + \ldots + N_M = N \). We note that when \( N_I = 0 \), it means that the block matrices \( \tau_n^{[l_1,l_2]} \) and \( \tau_n^{[l_1,l_2]} (l = 1, 2, \ldots, M) \) do not appear in (18).

**Remark 7** It can be calculated that

\[
s_1 = s_2 = s_3 = s_5 = s_6 = s_7 = s_9 = s_{10} = s_{11} = 0, \quad \text{when } M = 3, \quad (28)
\]

and

\[
s_1 = s_2 = s_3 = s_4 = s_6 = s_7 = s_8 = s_9 = s_{11} = 0, \quad \text{when } M = 4, \quad (29)
\]

in (14), but we do not know whether or not \( s_i = 0 \) holds for all \( i \in \mathbb{N} \) such that \( i \not\equiv 0 \mod (M + 1) \) when \( M = 3, 4 \). We also note that \( x_1^\pm \) can be removed from the solution when \( r \equiv 0 \mod (M + 1) \), by using the technique developed in Yang and Yang (2021c).

### 2.2 Generalized Wronskian–Hermite Polynomials

Hermite polynomials are a sequence of classical orthogonal polynomials, and they arise in many areas of mathematics, such as probability, combinatorics, and random matrix theory. Like other orthogonal polynomials, Hermite polynomials can be defined from various viewpoints. It is also worth noting that there are two different standardizations in common use. However, it turns out that neither of them is convenient for the analysis of wave patterns. Instead, we will introduce a slightly different definition (Yang and Yang 2022). Let \( p_j(z) \) be Schur polynomials defined by

\[
\sum_{j=0}^{\infty} p_j(z) \epsilon^j = \exp \left( z \epsilon + \epsilon^2 \right),
\]

(30)

with \( p_j(z) \equiv 0 \) for \( j < 0 \). Then, it can be shown that the polynomials \( p_j(z) \) are related to Hermite polynomials via certain rescaling.
Next, we introduce Wronskian–Hermite polynomials, which have appeared in the study of certain monodromy-free Schrödinger operators (Oblomkov 1999). Let $N$ be a positive integer and $\Lambda_1 = (n_1, n_2, \ldots, n_N)$, where $\{n_i\}$ are distinct positive integers such that $n_1 < n_2 < \ldots < n_N$, then the Wronskian–Hermite polynomial $W_{\Lambda_1}(z)$ is defined as

\[
W_{\Lambda_1}(z) = \begin{vmatrix}
  p_{n_1}(z) & p_{n_1-1}(z) & \cdots & p_{n_1-N+1}(z) \\
  p_{n_2}(z) & p_{n_2-1}(z) & \cdots & p_{n_2-N+1}(z) \\
  \vdots & \vdots & \ddots & \vdots \\
  p_{n_N}(z) & p_{n_N-1}(z) & \cdots & p_{n_N-N+1}(z)
\end{vmatrix} .
\]  

(31)

Note from (30) that $p'_{k+1}(z) = p_k(z)$. This implies that the Wronskian–Hermite polynomial $W_{\Lambda_1}(z)$ can be rewritten as

\[
W_{\Lambda_1}(z) = \text{Wronskian} \left[ p_{n_1}(z), p_{n_2}(z), \ldots, p_{n_N}(z) \right].
\]  

(32)

In particular, when the indices $(n_1, n_2, \ldots, n_N)$ are consecutive, these polynomials are called generalized Hermite polynomials, which are closely related to rational solutions of the fourth Painlevé equation (Clarkson 2003).

The Yablonskii–Vorob’ev polynomials (Yablonskii 1959; Vorob’ev 1965) and Okamoto polynomials (Okamoto 1986) are another two important classes of special polynomials. As shown in Clarkson and Mansfield (2003), Yang and Yang (2023), they can be generalized to hierarchies that have close connections with rogue wave patterns of certain integrable systems (Yang and Yang 2021a, 2023). It turns out that the Wronskian–Hermite polynomials can be generalized in a similar way. Let $p^{[m]}_j(z)$, where $m$ is a positive integer, be Schur polynomials defined by

\[
\sum_{j=0}^{\infty} p^{[m]}_j(z) \epsilon^j = \exp{(z \epsilon + \epsilon^m)},
\]  

(33)

with $p^{[m]}_j(z) \equiv 0$ for $j < 0$. Then, the generalized Wronskian–Hermite polynomials are defined by

\[
W^{[m]}_{\Lambda_1}(z) = \begin{vmatrix}
  p^{[m]}_{n_1}(z) & p^{[m]}_{n_1-1}(z) & \cdots & p^{[m]}_{n_1-N+1}(z) \\
  p^{[m]}_{n_2}(z) & p^{[m]}_{n_2-1}(z) & \cdots & p^{[m]}_{n_2-N+1}(z) \\
  \vdots & \vdots & \ddots & \vdots \\
  p^{[m]}_{n_N}(z) & p^{[m]}_{n_N-1}(z) & \cdots & p^{[m]}_{n_N-N+1}(z)
\end{vmatrix} .
\]  

(34)

In particular, these polynomials are called generalized Wronskian–Hermite polynomials of jump $k > 0$ if $n_{j+1} - n_j = k, j = 1, 2, \ldots, N - 1$. Further, when $n_1 = l$, where $1 \leq l < k$, we denote the generalized Wronskian–Hermite polynomial of jump $k > 0$ by $W^{[m,k,l]}_N(z)$, i.e.,
We remark that when \( \mathbf{W} \) makes \( \mathbf{W} \) polynomial hierarchies of \( \mathbf{W} \) are special cases of the generalized Wronskian–Hermite polynomials. For the convenience of later use, we have multiplied a constant \( c^N \) in (35), which makes \( W_N^{[m,k,l]}(z) \) a monic polynomial.

Since we can deduce from (33) that \( (p_j^m)'(z) = p_j^m(z) \), \( W_N^{[m,k,l]}(z) \) can be rewritten as

\[
W_N^{[m,k,l]}(z) = \text{Wronskian} \left[ p_l(z), p_{l+k}(z), \ldots, p_{l+k(N-1)}(z) \right].
\]

If we take \( m = 2, k = 4 \), then the first few \( W_N^{[2,4,l]}(z) \) \((N, l = 1, 2, 3)\) are

\[
\begin{align*}
W_1^{[2,4,1]}(z) &= z, \\
W_2^{[2,4,1]}(z) &= z^3 (z^2 + 10), \\
W_3^{[2,4,1]}(z) &= z^6 (z^6 + 42z^4 + 540z^2 + 2520), \\
W_1^{[2,4,2]}(z) &= z^2 + 2, \\
W_2^{[2,4,2]}(z) &= z (z^6 + 18z^4 + 60z^2 + 120), \\
W_3^{[2,4,2]}(z) &= z^3 (z^{12} + 60z^{10} + 1260z^8 + 12000z^6 + 54000z^4 + 181440z^2 + 302400), \\
W_1^{[2,4,3]}(z) &= z^2 + 6, \\
W_2^{[2,4,3]}(z) &= z^3 (z^6 + 30z^4 + 252z^2 + 840), \\
W_3^{[2,4,3]}(z) &= z^6 (z^{12} + 84z^{10} + 2700z^8 + 43680z^6 + 388080z^4 + 1995840z^2 + 4656960).
\end{align*}
\]

We remark that when \( k = 2 \), the generalized Wronskian–Hermite polynomials \( W_N^{[2m+1,2,1]}(z) \) are related to the Yablonskii–Vorob’ev polynomial hierarchy through some rescaling. In addition, \( W_N^{[m,3,1]}(z) \) and \( W_N^{[m,3,2]}(z) \) are multiples of the Okamoto polynomial hierarchies of \( Q_N^{[m]}(z) \) and \( R_N^{[m]}(z) \), respectively (Yang and Yang 2023). In other words, the Yablonskii–Vorob’ev polynomial hierarchy and the Okamoto polynomial hierarchies are special cases of the generalized Wronskian–Hermite polynomials.

As seen in subsequent sections, rogue wave patterns of the vector NLS equation (1) are asymptotically determined by the distribution of zeros of the generalized
Wronskian–Hermite polynomials. Root structures of certain special cases of the generalized Wronskian–Hermite polynomials have been obtained in previous studies, such as the Yablonskii–Vorob’ev polynomial hierarchy (Clarkson and Mansfield 2003; Balogh et al. 2016; Fukutani et al. 2000; Taneda 2000; Buckingham and Miller 2014) and the Okamoto polynomials hierarchies (Clarkson 2003; Kametaka 1983; Fukutani et al. 2000). For instance, it has been shown that all nonzero roots of the Yablonskii–Vorob’ev polynomials and the Okamoto polynomials \( Q_N^{[1]}(z) \) and \( R_N^{[1]}(z) \) are simple (Fukutani et al. 2000; Kametaka 1983). Despite that, as far as we know, root structures for higher members of generalized Wronskian–Hermite polynomials have not been studied yet.

Now, we discuss root structures of the generalized Wronskian–Hermite polynomials of jump 4 and 5, which will be used in later studies on rogue wave patterns. Let \( N_0 \) be the remainder of \( N \) divided by \( m \), i.e.,

\[
N_0 \equiv N \mod m \quad \text{or} \quad N = km + N_0,
\]

where \( k \) is a nonnegative integer, and we denote \([a]\) by the largest integer less than or equal to a real number \( a \). Then, our results can be summarized as follows.

**Theorem 2.3** The generalized Wronskian–Hermite polynomials \( W_N^{[m,4,l]} \) of jump 4 are monic with degree \( N(3N - 3 + 2l)/2 \) and have the form

\[
W_N^{[m,4,l]} = z^\Gamma w_N^{[m,4,l]}(\zeta), \quad \zeta = z^m, \tag{39}
\]

where \( w_N^{[m,4,l]}(\zeta) \) is a monic polynomial with real coefficients, \( w_N^{[m,4,l]}(0) \neq 0 \), and \( \Gamma \) is the multiplicity of the zero root given by

\[
\Gamma = \frac{3}{2} \left( N_1^2 + N_2^2 + N_3^2 \right) - (N_1N_2 + N_1N_3 + N_2N_3) + \frac{1}{2} (3N_1 + N_2 - N_3) \tag{40}
\]

with the values of \( N_1, N_2, \) and \( N_3 \) characterized as follows.

- **When** \( m \equiv 1 \mod 4, \) we have
  
  \[
  l = 3 : (N_1, N_2, N_3) = \begin{cases} 
  (0, 0, 0), & 0 \leq N_0 \leq \lfloor \frac{m}{4} \rfloor \\
  (\lfloor \frac{m}{4} \rfloor, N_0 - \lfloor \frac{m}{4} \rfloor, 0), & \lfloor \frac{m}{4} \rfloor + 1 \leq N_0 \leq 2 \lfloor \frac{m}{4} \rfloor \\
  (\lfloor \frac{m}{4} \rfloor, \lfloor \frac{m}{4} \rfloor, N_0 - 2 \lfloor \frac{m}{4} \rfloor), & 2 \lfloor \frac{m}{4} \rfloor + 1 \leq N_0 \leq 3 \lfloor \frac{m}{4} \rfloor \\
  (m - 1 - N_0, m - 1 - N_0, m - 1 - N_0), & 3 \lfloor \frac{m}{4} \rfloor + 1 \leq N_0 \leq m - 1
  \end{cases}
  \]

  \[
  l = 2 : (N_1, N_2, N_3) = \begin{cases} 
  (0, 0, 0), & 0 \leq N_0 \leq \lfloor \frac{m}{4} \rfloor \\
  (0, \lfloor \frac{m}{4} \rfloor, N_0 - \lfloor \frac{m}{4} \rfloor), & \lfloor \frac{m}{4} \rfloor + 1 \leq N_0 \leq 2 \lfloor \frac{m}{4} \rfloor \\
  (\lfloor \frac{m}{4} \rfloor - 1, \lfloor \frac{m}{4} \rfloor - 1, N_0 - 2 \lfloor \frac{m}{4} \rfloor - 1), & 2 \lfloor \frac{m}{4} \rfloor + 1 \leq N_0 \leq 3 \lfloor \frac{m}{4} \rfloor \\
  (m - 1 - N_0, m - 1 - N_0, m - 1 - N_0), & 3 \lfloor \frac{m}{4} \rfloor + 1 \leq N_0 \leq m - 1
  \end{cases}
  \]
\( l = 1 : (N_1, N_2, N_3) \)
\[
\begin{align*}
&= \begin{cases}
(0, 0, N_0), & 0 \leq N_0 \leq \left\lceil \frac{m}{4} \right\rceil \\
\left( \left\lceil \frac{m}{4} \right\rceil - 1, N_0 - \left\lceil \frac{m}{4} \right\rceil - 1, 0 \right), & \left\lceil \frac{m}{4} \right\rceil + 1 \leq N_0 \leq 2 \left\lceil \frac{m}{4} \right\rceil + 1 \\
\left( \left\lceil \frac{m}{4} \right\rceil - 1, \left\lceil \frac{m}{4} \right\rceil, N_0 - 2 \left\lceil \frac{m}{4} \right\rceil - 1 \right), & 2 \left\lceil \frac{m}{4} \right\rceil + 2 \leq N_0 \leq 3 \left\lceil \frac{m}{4} \right\rceil + 1 \\
(m - 1 - N_0, m - N_0, m - N_0), & 3 \left\lceil \frac{m}{4} \right\rceil + 2 \leq N_0 \leq m - 1.
\end{cases}
\end{align*}
\]

- When \( m \equiv 2 \pmod{4} \), we have

\( l = 3 : (N_1, N_2, N_3) \)
\[
\begin{align*}
&= \begin{cases}
\left( \frac{km}{2} + N_0, 0, \frac{km}{2} \right), & 0 \leq N_0 \leq \left\lceil \frac{m}{4} \right\rceil \\
\left( \frac{km}{2} + \left\lceil \frac{m}{4} \right\rceil, 0, \frac{km}{2} + N_0 - \left\lceil \frac{m}{4} \right\rceil \right), & \left\lceil \frac{m}{4} \right\rceil + 1 \leq N_0 \leq 3 \left\lceil \frac{m}{4} \right\rceil + 1 \\
\left( \frac{km}{2} + N_0 - 2 \left\lceil \frac{m}{4} \right\rceil - 1, 0, \frac{km}{2} + 2 \left\lceil \frac{m}{4} \right\rceil + 1 \right), & 3 \left\lceil \frac{m}{4} \right\rceil + 2 \leq N_0 \leq m - 1
\end{cases}
\end{align*}
\]

\( l = 2 : (N_1, N_2, N_3) \)
\[
\begin{align*}
&= \begin{cases}
\left( \frac{km}{2} - 1, 0, \frac{km}{2} + N_0 \right), & 0 \leq N_0 \leq \left\lceil \frac{m}{4} \right\rceil \\
\left( \frac{km}{2} + N_0 - \left\lceil \frac{m}{4} \right\rceil - 1, 0, \frac{km}{2} + \left\lceil \frac{m}{4} \right\rceil \right), & \left\lceil \frac{m}{4} \right\rceil + 1 \leq N_0 \leq 3 \left\lceil \frac{m}{4} \right\rceil + 1 \\
\left( \frac{km}{2} + 2 \left\lceil \frac{m}{4} \right\rceil, 0, \frac{km}{2} + N_0 - 2 \left\lceil \frac{m}{4} \right\rceil - 1 \right), & 3 \left\lceil \frac{m}{4} \right\rceil + 2 \leq N_0 \leq m - 1
\end{cases}
\end{align*}
\]

- When \( m \equiv 3 \pmod{4} \), we have

\( l = 3 : (N_1, N_2, N_3) \)
\[
\begin{align*}
&= \begin{cases}
(N_0, 0, 0), & 0 \leq N_0 \leq \left\lceil \frac{m}{4} \right\rceil \\
(N_0 - 1, \left\lceil \frac{m}{4} \right\rceil, 0), & \left\lceil \frac{m}{4} \right\rceil + 1 \leq N_0 \leq 2 \left\lceil \frac{m}{4} \right\rceil + 1 \\
(N_0 - 2 \left\lceil \frac{m}{4} \right\rceil - 2, \left\lceil \frac{m}{4} \right\rceil, \left\lceil \frac{m}{4} \right\rceil), & 2 \left\lceil \frac{m}{4} \right\rceil + 2 \leq N_0 \leq 3 \left\lceil \frac{m}{4} \right\rceil + 2 \\
(m - 1 - N_0, m - 1 - N_0, m - 1 - N_0), & 3 \left\lceil \frac{m}{4} \right\rceil + 3 \leq N_0 \leq m - 1
\end{cases}
\end{align*}
\]

\( l = 2 : (N_1, N_2, N_3) \)
\[
\begin{align*}
&= \begin{cases}
(0, N_0, 0), & 0 \leq N_0 \leq \left\lceil \frac{m}{4} \right\rceil + 1 \\
(N_0 - \left\lceil \frac{m}{4} \right\rceil - 1, \left\lceil \frac{m}{4} \right\rceil + 1, 0), & \left\lceil \frac{m}{4} \right\rceil + 2 \leq N_0 \leq 2 \left\lceil \frac{m}{4} \right\rceil + 1 \\
(N_0 - 2 \left\lceil \frac{m}{4} \right\rceil - 2, \left\lceil \frac{m}{4} \right\rceil + 1), & 2 \left\lceil \frac{m}{4} \right\rceil + 2 \leq N_0 \leq 3 \left\lceil \frac{m}{4} \right\rceil + 2 \\
(m - 1 - N_0, m - 1 - N_0, m - N_0), & 3 \left\lceil \frac{m}{4} \right\rceil + 3 \leq N_0 \leq m - 1
\end{cases}
\end{align*}
\]

\( l = 1 : (N_1, N_2, N_3) \)
\[
\begin{align*}
&= \begin{cases}
(0, 0, N_0), & 0 \leq N_0 \leq \left\lceil \frac{m}{4} \right\rceil + 1 \\
(0, N_0 - \left\lceil \frac{m}{4} \right\rceil - 1, \left\lceil \frac{m}{4} \right\rceil + 1), & \left\lceil \frac{m}{4} \right\rceil + 2 \leq N_0 \leq 2 \left\lceil \frac{m}{4} \right\rceil + 2 \\
(N_0 - 2 \left\lceil \frac{m}{4} \right\rceil - 2, \left\lceil \frac{m}{4} \right\rceil + 1, \left\lceil \frac{m}{4} \right\rceil + 1), & 2 \left\lceil \frac{m}{4} \right\rceil + 3 \leq N_0 \leq 3 \left\lceil \frac{m}{4} \right\rceil + 2 \\
(m - 1 - N_0, m - 1 - N_0, m - N_0), & 3 \left\lceil \frac{m}{4} \right\rceil + 3 \leq N_0 \leq m - 1
\end{cases}
\end{align*}
\]
Theorem 2.4 The generalized Wronskian–Hermite polynomials $W_{N}^{[m,5,l]}$ of jump 5 are monic with degree $N(2N - 2 + l)$ and have the form

$$ W_{N}^{[m,5,l]} = z^{\Gamma} w_{N}^{[m,5,l]}(\xi), \quad \xi = z^{m}, \quad (41) $$

where $w_{N}^{[m,5,l]}(\xi)$ is a monic polynomial with real coefficients, $w_{N}^{[m,5,l]}(0) \neq 0$, and $\Gamma$ is the multiplicity of the zero root given by

$$ \Gamma = \frac{5}{2} \left( N_{1}^{2} + N_{2}^{2} + N_{3}^{2} + N_{4}^{2} \right) - \frac{1}{2} \left( N_{1} + N_{2} + N_{3} + N_{4} \right)^{2} + 2N_{1} + N_{2} - N_{4}. \quad (42) $$

The values of $N_{1}, N_{2}, N_{3}, N_{4}$ can be characterized in a similar way as Theorem 2.3. (The details are provided in Lemma 6.1 of Appendix C.)

We provide the proofs of Theorems 2.3 and 2.4 in Appendix C.

Remark 8 We note that Theorems 2.3 and 2.4 provide the multiplicities of the zero root of the generalized Wronskian–Hermite polynomials $W_{N}^{[m,4,l]}$ and $W_{N}^{[m,5,l]}$, respectively, and as we will see subsequently, these multiplicities are essential in the analysis of rogue wave patterns in the inner region when specific parameters are very large (see Theorems 3.1 and 3.2). It is also clear that the roots of $W_{N}^{[m,4,l]}$ are distributed symmetrically on some circles in the sense that if $z_{0}$ is a root of $W_{N}^{[m,4,l]}$, then so is $z_{0} \exp(2k\pi i/m)$, where $k = 0, 1, \ldots, m - 1$.

In the analytical study of rogue wave patterns, a crucial assumption is that all the nonzero roots of the corresponding generalized Wronskian–Hermite polynomials $W_{N}^{[m,k,l]}$ are simple (Yang and Yang 2021c, a, 2023). This assumption has been proved for Yablonskii–Vorob’ev polynomials $W_{N}^{[3,2,1]}(z)$ and Okamoto polynomials $W_{N}^{[2,3,1]}(z)$ (Fukutani et al. 2000) and $W_{N}^{[2,3,2]}(z)$ (Fukutani et al. 2000; Kametaka 1983). Nevertheless, it has not been proved for the general case. Since our results will also rely on this assumption, we propose a conjecture similar to those in Yang and Yang (2021c, a, 2023).

Conjecture All nonzero roots of the generalized Wronskian–Hermite polynomials $W_{N}^{[m,k,l]}$ are simple for any integers $N \geq 1, m \geq 1, k \geq 2, 1 \leq l \leq k - 1$.

Although we are not able to prove this conjecture, we have verified it numerically for a variety of special cases which include all the particular generalized Wronskian–Hermite polynomials $W_{N}^{[m,k,l]}$ that will be involved in this paper.

The distributions of roots of the Yablonskii–Vorob’ev polynomial and Okamoto polynomial hierarchies demonstrate highly regular and symmetric structures (Clarkson and Mansfield 2003; Clarkson 2003). The original Yablonskii–Vorob’ev polynomials form approximately equilateral triangles, while the higher members of the Yablonskii–Vorob’ev polynomial hierarchy form various shapes, such as pentagons, septagons, nonagons, and undecagons, depending on the values of $m$ (Yang and Yang 2023). The roots of Okamoto polynomials exhibit completely different structures compared with Yablonskii–Vorob’ev polynomials. Both of the $W_{N}^{[2,3,1]}(z)$ and $W_{N}^{[2,3,2]}(z)$ have similar
Fig. 1 Plots of the roots of the polynomials $W_N^{[m,4,3]}(z)$ for $2 \leq N \leq 5$ and $m = 2, 3, 5, 6, 7$

root structures as these roots are located on two “triangles” except that $W_N^{[2,3,1]}(z)$ has an extra row of roots on a straight line between these two triangles. Here, we use “triangles” because the edges of these triangles are curved rather than straight lines. A natural question is what characteristics the root structures of the generalized Wronskian–Hermite polynomials would exhibit. To this end, we plot the roots of $W_N^{[m,4,1]}(z)$ and $W_N^{[m,5,1]}(z)$ in Figs. 1-2 and 3, respectively.

3 Rogue Wave Patterns of the Three- and Four-Component Nonlinear Schrödinger equation

**Theorem 3.1** Let $p_0, p_1, \theta_{1n}, \rho_n, k_n, w_n (n = 1, 2, 3)$ be the same as in Theorem 2.2. Assume that $|a_m| \gg 1$ and all other parameters are $O(1)$ in the $i$-th-type $N_i$-th-order rogue waves ($i = 1, 2, 3$)

$$u_{1,N_i}(x, t), \quad u_{2,N_i}(x, t), \quad u_{3,N_i}(x, t),$$

of the three-component nonlinear Schrödinger equation, where $N_i = N e_i$, $N$ is a positive integer, and $e_j$ is the standard unit vector in $\mathbb{R}^3$. We also assume that all nonzero roots of the generalized Wronskian–Hermite polynomials $W_N^{[m,4,4-i]}$ of jump
Fig. 2 Plots of the roots of the polynomials $W_{N}^{[m, A, 1]}(z)$ for $2 \leq N \leq 5$ and $m = 2, 3, 5, 6, 7$

4 are simple. Then, we have the following results concerning the asymptotics of the rogue waves (43).

(1) In the outer region on the $(x, t)$ plane, when $\sqrt{x^2 + t^2} = O(|a_m|^{1/m})$, the $N_i$-th-order rogue waves separate into $N(3N + 5 - 2i)/2 - \Gamma$ fundamental rogue waves, where $\Gamma$ is given in (39). These fundamental rogue waves are

\[
\hat{u}_1(x, t) = \rho_1 e^{i(k_1 x + \omega_1 t)} \frac{[p_1 x + 2p_0 p_1(\dot{u}t) + \theta_{11}] [p_1^* x - 2p_0^* p_1^*(\dot{u}t) - \theta_{11}^*]}{|p_1 x + 2p_0 p_1(\dot{u}t)|^2 + |h_0|^2},
\]

\[
\hat{u}_2(x, t) = \rho_2 e^{i(k_2 x + \omega_2 t)} \frac{[p_1 x + 2p_0 p_1(\dot{u}t) + \theta_{12}] [p_1^* x - 2p_0^* p_1^*(\dot{u}t) - \theta_{12}^*]}{|p_1 x + 2p_0 p_1(\dot{u}t)|^2 + |h_0|^2},
\]

\[
\hat{u}_3(x, t) = \rho_3 e^{i(k_3 x + \omega_3 t)} \frac{[p_1 x + 2p_0 p_1(\dot{u}t) + \theta_{13}] [p_1^* x - 2p_0^* p_1^*(\dot{u}t) - \theta_{13}^*]}{|p_1 x + 2p_0 p_1(\dot{u}t)|^2 + |h_0|^2},
\]
where \(|h_0|^2 = |p_1|^2 / (p_0 + p_0^*)^2\), and their positions \((\hat{x}_0, \hat{t}_0)\) are given by

\[
\hat{x}_0 = \frac{1}{\Re (p_0)} \Re \left[ \frac{p_0^*}{p_1} \left( z_0 a_m^{1/m} - \Delta_i \right) \right], \quad \hat{t}_0 = \frac{1}{2\Re (p_0)} \Im \left[ \frac{1}{p_1} \left( z_0 a_m^{1/m} - \Delta_i \right) \right],
\]

(47)

where \(z_0\) is any one of the nonzero simple roots of \(W_N^{[m,4,4-i]}(z)\). \(\Delta_i\) is a \(z_0\)-dependent \(O(1)\) quantity (the expression of \(\Delta_1\) is provided in (79)), and \((\Re, \Im)\) refer to the real and imaginary parts of a complex number, respectively. The approximation error here is \(O\left(|a_m|^{-1/m}\right)\). In other words, when \(|a_m| \gg 1\) and \((x - \hat{x}_0)^2 + (t - \hat{t}_0)^2 = O(1)\), we have the following asymptotics

\[
u_{n,N;i}(x, t) = \hat{u}_n \left( x - \hat{x}_0, t - \hat{t}_0 \right) + O \left( |a_m|^{-1/m} \right), \quad n = 1, 2, 3. \quad (48)
\]

(2) In the inner region, where \(x^2 + t^2 = O(1)\), if zero is a root of the generalized Wronskian–Hermite polynomials \(W_N^{[m,4,4-i]}(z)\), then \([u_{1,N;i}(x, t), u_{2,N;i}(x, t), u_{3,N;i}(x, t)]\) is approximately a lower \(\hat{N}_i\)-th-order rogue wave

\[
u_{1,N;i}(x, t), \quad \nu_{2,N;i}(x, t), \quad \nu_{3,N;i}(x, t)
\]
where \( \hat{N}_i \) refers to the value of \( N_j \) against \( l \in \{1, 2, 3\} \) given in Theorem 2.3. Moreover, the internal parameters

\[
(\hat{a}_{1,n}, \hat{a}_{2,n}, \hat{a}_{3,n}, \hat{a}_{5,n}, \hat{a}_{6,n} \ldots, \hat{a}_{4N_n,4-i-n,n}) , \quad n = 1, 2, 3, \]

in this lower-order rogue wave are related to those in the original rogue wave as follows.

- For \( m \equiv 1 \text{ or } 3 \mod 4 \), we have

\[
\hat{a}_{j,1} = \hat{a}_{j,2} = \hat{a}_{j,3} = a_j + \left( N - \sum_{n=1}^{3} N_{n,4-i} \right) s_j, \quad j = 1, 2, 3, 5, 6, 7 \ldots
\]

- For \( m \equiv 2 \mod 4 \), we have

\[
\hat{a}_{j,1} = \hat{a}_{j,3} = a_j + \left( N - \sum_{n=1}^{3} N_{n,4-i} \right) s_j, \quad \text{if } j = 1, 2, 3, 5, \ldots, m - 1, m + 1, \ldots,
\]
\[
\left( N - \sum_{n=1}^{3} N_{n,4-i} \right) s_j, \quad \text{if } j = m.
\]

Here, \( s_j \) is defined in Theorem 2.2. The approximation error of this lower-order rogue wave is \( O \left( |a_m|^{-1} \right) \). In other words, when \( |a_m| \gg 1 \) and \( x^2 + t^2 = O(1) \), we have

\[
u_{n,\hat{N}_i} (x, t; a_2, a_3, a_5, a_6, \ldots) = u_{n,\hat{N}_i} (x, t; \hat{a}_{j,1}, \hat{a}_{j,2}, \hat{a}_{j,3}, j = 1, 2, 3, 5, 6 \ldots) + O \left( |a_m|^{-1} \right) , \quad n = 1, 2, 3.
\]

If zero is not a root of \( W_N^{[m,4,4-i]}(z) \), the solution

\[
[u_{1,\hat{N}_i}(x, t), \quad u_{2,\hat{N}_i}(x, t), \quad u_{3,\hat{N}_i}(x, t)]
\]

is approximately the constant background

\[
\begin{bmatrix}
\rho_1 e^{i(k_1 x + \omega_1 t)}, & \rho_2 e^{i(k_2 x + \omega_2 t)}, & \rho_3 e^{i(k_3 x + \omega_3 t)}
\end{bmatrix}.
\]

**Theorem 3.2** Let \( p_0, p_1, \theta_{1n}, \rho_n, k_n, w_n (n = 1, 2, 3, 4) \) be the same as in Theorem 2.2. Assume that \( |a_m| \gg 1 \) and all other parameters are \( O(1) \) in the \( i \)-th-type \( \hat{N}_i \)-th-order rogue waves \( i = 1, 2, 3, 4 \)

\[
[u_{1,\hat{N}_i}(x, t), \quad u_{2,\hat{N}_i}(x, t), \quad u_{3,\hat{N}_i}(x, t), \quad u_{4,\hat{N}_i}(x, t)]
\]

\[ (49) \]
of the four-component nonlinear Schrödinger equation, where $N_1 = N e_i$, $N$ is a positive integer and $e_j$ is the standard unit vector in $\mathbb{R}^4$. We also assume that all nonzero roots of the generalized Wronskian–Hermite polynomials $W_N^{[m,5.5-i]}$ of jump 5 are simple. Then, we have the following results concerning the asymptotics of the rogue waves (49).

(1) In the outer region on the $(x, t)$ plane, when $\sqrt{x^2 + t^2} = O \left( |a_m|^{-1/m} \right)$, the $N$-th-order rogue waves separate into $N(2N + 3 - i) - \Gamma$ fundamental rogue waves, where $\Gamma$ is given in (41). These fundamental rogue waves are

$$\tilde{u}_1(x, t) = \rho_1 e^{i(k_1 x + \omega_1 t)} \left[ p_1 x + 2 p_0 p_1 (it) + \theta_{11} \right] \frac{\left| p_1 x - 2 p_0^* p_1^*(it) - \theta_{11}^* \right| + |h_0|^2}{|p_1 x + 2 p_0 p_1 (it)|^2 + |h_0|^2},$$

$$\tilde{u}_2(x, t) = \rho_2 e^{i(k_2 x + \omega_2 t)} \left[ p_1 x + 2 p_0 p_1 (it) + \theta_{12} \right] \frac{\left| p_1 x - 2 p_0^* p_1^*(it) - \theta_{12}^* \right| + |h_0|^2}{|p_1 x + 2 p_0 p_1 (it)|^2 + |h_0|^2},$$

$$\tilde{u}_3(x, t) = \rho_3 e^{i(k_3 x + \omega_3 t)} \left[ p_1 x + 2 p_0 p_1 (it) + \theta_{13} \right] \frac{\left| p_1 x - 2 p_0^* p_1^*(it) - \theta_{13}^* \right| + |h_0|^2}{|p_1 x + 2 p_0 p_1 (it)|^2 + |h_0|^2},$$

$$\tilde{u}_4(x, t) = \rho_4 e^{i(k_4 x + \omega_4 t)} \left[ p_1 x + 2 p_0 p_1 (it) + \theta_{14} \right] \frac{\left| p_1 x - 2 p_0^* p_1^*(it) - \theta_{14}^* \right| + |h_0|^2}{|p_1 x + 2 p_0 p_1 (it)|^2 + |h_0|^2},$$

where $|h_0|^2 = |p_1|^2 / \left( p_0 + p_0^* \right)^2$, and their positions $(\bar{x}_0, \bar{t}_0)$ are given by

$$\bar{x}_0 = \frac{1}{\Re \left( p_0 \right)} \Re \left( \frac{p_0^*}{p_1} \left( z_0 d_m^{1/m} - \bar{\Delta}_i \right) \right), \quad \bar{t}_0 = \frac{1}{2 \Re \left( p_0 \right)} \Im \left[ \frac{1}{p_1} \left( z_0 d_m^{1/m} - \bar{\Delta}_i \right) \right],$$

where $z_0$ is any one of the nonzero simple roots of $W_N^{[m,5.5-i]}(z)$, and $\bar{\Delta}_i$ is a $z_0$-dependent $O(1)$ quantity. (The expression of $\bar{\Delta}_1$ is provided in (109).) The approximation error here is $O \left( |a_m|^{-1/m} \right)$. In other words, when $|a_m| \gg 1$ and $(x - \bar{x}_0)^2 + (t - \bar{t}_0)^2 = O(1)$, we have the following asymptotics

$$u_{n,N_1}(x, t) = \tilde{u}_n \left( x - \bar{x}_0, t - \bar{t}_0 \right) + O \left( |a_m|^{-1/m} \right), \quad n = 1, 2, 3, 4. \quad (55)$$

(2) In the inner region, where $x^2 + t^2 = O(1)$, if zero is a root of the generalized Wronskian–Hermite polynomials $W_N^{[m,5.5-i]}(z)$, then $\left[ u_{1,N_1}(x, t), u_{2,N_1}(x, t), u_{3,N_1}(x, t), u_{4,N_1}(x, t) \right]$ is approximately a lower $N_1$-th-order rogue wave

$$u_{1,N_1}(x, t), u_{2,N_1}(x, t), u_{3,N_1}(x, t), u_{4,N_1}(x, t)$$

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where \( \overline{N}_i = \sum_{j=1}^{4} N_{j, 5-i} e_j \) and \( N_{j, 1} \) refers to the value of \( N_j \) against \( l \in \{1, 2, 3, 4\} \) which are given in Theorem 2.4. Moreover, the internal parameters

\[
(\tilde{a}_{1,n}, \tilde{a}_{2,n}, \tilde{a}_{3,n}, \tilde{a}_{4,n}, \tilde{a}_{6,n}, \ldots, \tilde{a}_{5 N_{n, 5-i - n}, n}) , \quad n = 1, 2, 3, 4 ,
\]

in this lower-order rogue waves are related to those in the original rogue wave by

\[
\tilde{a}_{j, 1} = \tilde{a}_{j, 2} = \tilde{a}_{j, 3} = \tilde{a}_{j, 4} = a_j + (N - \sum_{n=1}^{4} N_{n, 5-i}) s_j , \quad j = 1, 2, 3, 4, 6, 7 \ldots,
\]

where \( s_j \) is defined in Theorem 2.2. The approximation error of this lower-order rogue wave is \( O(|a_m|^{-1}) \). In other words, when \( |a_m| \gg 1 \) and \( x^2 + t^2 = O(1) \), we have

\[
\begin{align*}
&u_{n, N_i}(x, t; a_2, a_3, a_4, a_6, \ldots) \\
&= u_{n, \overline{N}_i}(x, t; a_{j, 1}, a_{j, 2}, a_{j, 3}, a_{j, 4}, j = 1, 2, 3, 4, 6, 7 \ldots) \\
&+ O\left(|a_m|^{-1}\right), \quad n = 1, 2, 3, 4.
\end{align*}
\]

If zero is not a root of \( W_{N}^{[m, 5, 5-i]}(z) \), the solution

\[
[u_{1, N_i}(x, t), \ u_{2, N_i}(x, t), \ u_{3, N_i}(x, t), \ u_{4, N_i}(x, t)]
\]

is approximately the constant background

\[
\begin{bmatrix}
\rho_1 e^{i(k_1 x + \omega_1 t)}, \rho_2 e^{i(k_2 x + \omega_2 t)}, \rho_3 e^{i(k_3 x + \omega_3 t)}, \rho_4 e^{i(k_4 x + \omega_4 t)}
\end{bmatrix}.
\]

4 Comparison Between Predicted and True Rogue Wave Patterns

4.1 Comparison in the Three-Component NLS Equation

In this subsection, we compare our predictions of rogue wave patterns in Theorem 3.1 with true rogue waves of the three-component NLS equation. It is noted that the predicted \( i \)-th-type \( |u_{n, N_i}(x, t)|, n = 1, 2, 3 \), from Theorem 3.1 can be divided into a simple form

\[
|u_{n, N_i}(x, t)| = |u_{n, \overline{N}_i}(x, t)| + \sum_{j=1}^{N_p} \left(|\hat{u}_n(x - x_0^{(j)}, t - t_0^{(j)})| - \rho_n\right), \quad n = 1, 2, 3,
\]

where \( u_{n, \overline{N}_i}(x, t) \) is a lower-order rogue wave of the three-component NLS equation, \( \overline{N}_i = (N_1, N_2, N_3) \) is given by Theorems 3.1 and 2.3, \( \hat{u}_n(x, t), n = 1, 2, 3, \) is
the fundamental rogue wave of the three-component NLS equation, whose predicted location \( (\hat{x}_0^{(j)}, \hat{z}_0^{(j)}) \) can be obtained from (47), and \( N_p \) is the number of fundamental rogue waves given in Theorem 3.1. To analyze the triple root case, we choose the background wavenumbers \( k_2 = -k_1 = 1 \) and \( k_3 = 0 \), which gives \( \rho_1 = \rho_2 = \rho_3^2 = 2 \) by (24). Further, we select \( p_0 = 1, p_1 = -1/\sqrt{3} \) and \( p_2 = 1/\sqrt{3} \) for the subsequent analysis.

4.1.1 First-Type Rogue Waves of the Three-Component NLS Equation

We start with \((2, 0, 0)\)-th-order rogue wave solutions. Moreover, we let one of the internal parameters \((a_2, a_3, a_5, a_6, a_7)\) be large and set others to 0. We note that \(a_1\) can be set to 0 by normalization, and \(a_4\) is a parameter that can be removed. Then, the very large parameter is one of

\[
\begin{align*}
  a_2 &= 30, & a_3 &= 100, & a_5 &= 1200, & a_6 &= 3000, & a_7 &= 7000. \\
\end{align*}
\]

According to Theorem 3.1, the position \((\hat{x}_0, \hat{z}_0)\) of each fundamental rogue wave corresponding to

\[
\begin{align*}
  u_{1,N_1}^{(x,t)}, & \quad u_{2,N_1}^{(x,t)}, & \quad u_{3,N_1}^{(x,t)}
\end{align*}
\]

can be predicted by equation (47). The lower \((N_1, N_2, N_3)\)-th-order rogue wave would appear in the inner region, and the value of \((N_1, N_2, N_3)\) can be obtained from Theorems 3.1 and 2.3. In our prediction, the \((N_1, N_2, N_3)\) values for these five rogue wave solutions are

\[
\begin{align*}
  (N_1, N_2, N_3) &= (1, 0, 1), & (0, 0, 0), & (1, 1, 0), & (1, 0, 1), & (0, 1, 0),
\end{align*}
\]

respectively. Note that \((0, 0, 0)\) means no lower-order rogue wave exists in the inner region. Because of our choice of parameters \(a_m\) and the value of \(s_j\) shown in Remark 7, the internal parameters in these predicted lower \((N_1, N_2, N_3)\)-th-order rogue waves of the inner region are all zero.

For \([u_{1,N_1}^{(x,t)}, u_{2,N_1}^{(x,t)}, u_{3,N_1}^{(x,t)}]\), their corresponding predicted rogue wave patterns are illustrated in the last three rows of Fig. 4, with the first row being the locations of predicted rogue waves. These predicted rogue waves are generated in the following way. We first replace each non-center dot, which is the nonzero root, in the first row of Fig. 4 by a fundamental rogue wave according to (44)-(46). Then, the center dot is replaced by a lower \((N_1, N_2, N_3)\)-th-order rogue wave with all internal parameters set to zero.

It can be seen from Fig. 4 that the large-\(a_2\) solution displays a skewed double-triangle, corresponding to the double-triangle root structure of \(W_2^{[2,4,3]}(z)\). The large-\(a_3\) solution exhibits a skewed triple-triangle, corresponding to the triple-triangle root structure of \(W_2^{[3,4,3]}(z)\). The large-\(a_5\) solution displays a deformed pentagon, corresponding to the pentagon-shaped root structure of \(W_2^{[5,4,3]}(z)\). The large-\(a_6\) solution exhibits a deformed hexagon, corresponding to the hexagon-shaped root structure of \(W_2^{[6,4,3]}(z)\). The large-\(a_7\) solution displays a deformed heptagon, corresponding to the heptagon-shaped root structure of \(W_2^{[7,4,3]}(z)\). It seems that triple-triangle is a new
Predicted 1st-type $(2, 0, 0)$-th-order rogue waves of the three-component NLS from Theorem 3.1. Each column depicts rogue waves with a single large parameter $\alpha_m$, whose value is indicated on top, and all other internal parameters are set to zero. Top row: predicted $(\tilde{x}_0, \tilde{t}_0)$ locations by formulae (47). Second row: predicted $|u_1(x, t)|$. Third row: predicted $|u_2(x, t)|$. Bottom row: predicted $|u_3(x, t)|$. First column: the $(x, t)$ intervals are $-21 \leq x \leq 21, -25 \leq t \leq 25$. Second column: the $(x, t)$ intervals are $-30 \leq x \leq 30, -25 \leq t \leq 25$. Third column: the $(x, t)$ intervals are $-30 \leq x \leq 20, -16 \leq t \leq 16$. Fourth column: the $(x, t)$ intervals are $-30 \leq x \leq 25, -15 \leq t \leq 15$. Fifth column: the $(x, t)$ intervals are $-30 \leq x \leq 25, -15 \leq t \leq 15$

Fig. 4 Predicted 1st-type $(2, 0, 0)$-th-order rogue waves of the three-component NLS from Theorem 3.1.

By comparison of the true rogue waves to the predicted ones (see Figs. 4 and 5), we can observe that each of the rogue waves matches perfectly in terms of position and rogue wave shape. Notice that the predicted pattern looks very different from the root structure of $W_{2}^{[m,4,3]}(z)$. This is due to the term $\Delta_1$ leading to a nonlinear transformation from the root structure. When $|\alpha_m|$ is set to be very large, the term $\Delta_1$ can be neglected, and the patterns become much closer to certain linear transformations of the root structure of $W_{2}^{[m,4,3]}(z)$.

Apart from above observations, we can also quantitatively compare the differences between predicted and true rogue waves. To illustrate this, we choose the 1st-type $(2, 0, 0)$-th-order rogue waves and then select various large real values of $\alpha_3$ to analyze errors in the outer region and various large real values of $\alpha_5$ to analyze errors in the inner region. Referring to the work of Yang and Yang (2021c), we define...
Fig. 5 True 1st-type \((2, 0, 0)\)-th-order rogue waves of the three-component NLS with the same parameters as Fig. 4. The \((x, t)\) interval for each column is the same as the corresponding column in Fig. 4.

The error of Peregrine location

\[
\text{error of Peregrine location} = \sqrt{\left(\hat{x}_0 - x_0\right)^2 + \left(\hat{t}_0 - t_0\right)^2},
\]

(58)

and

\[
\text{error of inner region} = \left| u_{j,N_1}(x, t) - u_{j,\hat{N}_1}(x, t) \right|_{(x,t)\in \{(x,t)|x^2+t^2=O(1)\}},
\]

(59)

where \(j = 1, 2, \ldots, M\), \((x_0, t_0)\) is the location at which each rogue wave reaches maximum modulus value and \((\hat{x}_0, \hat{t}_0)\) is the predicted location of each fundamental rogue wave. The errors and decay rates associated with large values of \(a_3\) and \(a_5\) are depicted in Fig. 6. As can be seen, the results of the numerical analysis satisfy the corresponding error estimates stated in Theorem 3.1. A notable observation is that the numerical decay of solution error with respect to \(a_5\) at \(x = t = 0\) is \(O(|a_5|^{-2})\), surpassing the predicted estimate \(O(|a_5|^{-1})\) in Theorem 3.1. This phenomenon can possibly be attributed to the vanishing of certain components of the solutions at \(x = t = 0\), leading to a smaller error than initially expected.

4.1.2 Second-Type Rogue Waves of the Three-Component NLS Equation

In this case, we mainly carry out the second-type rogue waves in detail by taking \(N_2 = 3\), i.e., \((N_1, N_2, N_3) = (0, 3, 0)\). For brevity, we only let one of the internal parameters \((a_2, a_3, a_5, a_6, a_7)\) be large, and the others are set to 0. The very large parameter is one of

\[
a_2 = 30, \quad a_3 = 400, \quad a_5 = 4800, \quad a_6 = 3000, \quad a_7 = 7000.
\]

(60)
Fig. 6 Decay of errors in our predictions of Theorem 3.1 for the outer and inner regions of the 1st-type \((2, 0, 0)\)-th-order rogue waves in the three-component NLS with various large real values of \(a_3\) or \(a_5\), while other internal parameters are set to zero. (a) \(|u_3(x, t)|\) of the true rogue wave with \(a_3 = 10000\). (b) Decay of error versus \(a_3\) for the outer fundamental rogue wave marked by the left red box, together with the \(|a_3|^{-1/3}\) decay for comparison. (c) Decay of error versus \(a_3\) for the outer fundamental rogue wave marked by the right red box, together with the \(|a_3|^{-1/3}\) decay for comparison. (d) \(|u_3(x, t)|\) of the true rogue wave with \(a_5 = 1000\). (e) Decay of error versus \(a_5\) for the lower-order rogue wave at point \((x, t) = (0, 0)\) marked by the red box, together with the \(|a_5|^{-1}\) decay and \(|a_5|^{-2}\) decay for comparison. (f) Decay of error versus \(a_5\) for the lower-order rogue wave at point \((x, t) = (0.1, 0.2)\) marked by the red box, together with the \(|a_5|^{-1}\) decay for comparison.

According to Theorem 3.1, the position \((\hat{x}_0, \hat{t}_0)\) of each fundamental rogue wave

\[
u_{1, N_2}^2(x, t), \quad u_{2, N_2}^2(x, t), \quad u_{3, N_2}^2(x, t)
\]

can be predicted by equation (47). The lower \((N_1, N_2, N_3)\)-th-order rogue wave would appear in the inner region. The values of \((N_1, N_2, N_3)\) can be deduced by Theorems 3.1 and 2.3. In our prediction, the \((N_1, N_2, N_3)\) values for these five rogue wave solutions are

\[
(N_1, N_2, N_3) = (1, 0, 1), \quad (0, 0, 0), \quad (0, 0, 0), \quad (1, 0, 1), \quad (1, 2, 0),
\]

respectively. Note that \((0, 0, 0)\) means no lower-order rogue wave exists in the inner region. On account of our choice of parameters \(a_m\) and the value of \(s_j\) shown in Remark 7, the internal parameters in these predicted lower \((N_1, N_2, N_3)\)-th-order rogue waves of the inner region are all chosen to be zero.

For \([u_{1, N_2}^2(x, t), u_{2, N_2}^2(x, t)]\), their corresponding lower-order rogue wave patterns are shown in the last two rows of Fig. 7, with the first row being the predicted locations.
Fig. 7 Predicted 2nd-type \((0, 3, 0)\)-th-order rogue waves of the three-component NLS from Theorem 3.1. Each column depicts rogue waves with a single large parameter \(a_{0j}\), whose value is indicated on top, and all other internal parameters are set to zero. Top row: predicted \((\hat{x}_0, \hat{t}_0)\) locations by formulae (47). Middle row: predicted \(|u_1(x, t)|\). Bottom row: predicted \(|u_2(x, t)|\). First column: the \((x, t)\) intervals are \(-25 \leq x \leq 25, -26 \leq t \leq 26\). Second column: the \((x, t)\) intervals are \(-45 \leq x \leq 51, -29 \leq t \leq 27\). Third column: the \((x, t)\) intervals are \(-47 \leq x \leq 31, -22 \leq t \leq 22\). Fourth column: the \((x, t)\) intervals are \(-40 \leq x \leq 35, -20 \leq t \leq 20\). Fifth column: the \((x, t)\) intervals are \(-35 \leq x \leq 30, -20 \leq t \leq 20\).

of the rogue waves. As shown in Fig. 7, solutions in the first column are skewed double-triangles, while solutions from the second to the fifth columns are skewed triple-triangles, pentagons, hexagons, and heptagons, respectively.

Comparing the true rogue waves with predicted ones (see Figs. 7 and 8), we can observe that each of the rogue waves strikingly matches in position and rogue wave shape. Not only that, but it is also numerically demonstrated that the actual and predicted results match very well. Since they are very similar to the previous error analysis, we omit the details.

4.1.3 Third-Type Rogue Waves of the Three-Component NLS Equation

In this case, we choose \((0, 0, 4)\)-th-order rogue wave solutions. We only set one of the internal parameters \((a_2, a_3, a_5, a_6, a_7)\) to be large, and the remaining parameters are set to 0. The very large parameter is one of

\[
a_2 = 100, \quad a_3 = 200, \quad a_5 = 1500, \quad a_6 = 5000, \quad a_7 = 10000.
\]  

According to Theorem 3.1, the position \((\hat{x}_0, \hat{t}_0)\) of each fundamental rogue wave

\[
\begin{align*}
&u_{1,N_5}(x, t), \quad u_{2,N_5}(x, t), \quad u_{3,N_5}(x, t)
\end{align*}
\]
can be predicted by equation (47). The lower \((N_1, N_2, N_3)\)-th-order rogue wave would appear in the inner region. The value of \((N_1, N_2, N_3)\) can be obtained from Theorems 3.1 and 2.3. In our prediction, the \((N_1, N_2, N_3)\) values for these five rogue wave solutions are

\[
(N_1, N_2, N_3) = (2, 0, 2), \quad (0, 0, 1), \quad (0, 1, 1), \quad (2, 0, 2), \quad (0, 2, 2),
\]

respectively. We remark that \((0, 0, 1)\) means there is only a fundamental rogue wave in the inner region. The internal parameters in these predicted lower \((N_1, N_2, N_3)\)-th-order rogue waves of the inner region are all zero, due to our choice of parameters \(a_m\) and the value of \(s_j\) shown in Remark 7.

For \([u_{1,N_1}(x, t), u_{3,N_3}(x, t)]\), their corresponding predicted rogue wave patterns are shown in the last two rows of Fig. 9, with the first row being the locations of the rogue waves. It can be seen from Fig. 9 that the large-\(a_3\) solution exhibits a skewed triple-triangle, corresponding to the triple-triangle root structure of \(W_4^{[3,4,1]}(z)\). The large-\(a_5\) solution displays a deformed pentagon, corresponding to the pentagon-shaped root structure of \(W_4^{[5,4,1]}(z)\). The large-\(a_6\) solution exhibits a deformed hexagon, corresponding to the hexagon-shaped root structure of \(W_4^{[6,4,1]}(z)\). The large-\(a_7\) solution exhibits a deformed heptagon, corresponding to the heptagon-shaped root structure of \(W_4^{[7,4,1]}(z)\).

Comparing the actual rogue waves with the predicted ones (see Figs. 9 and 10), we can observe that each of the rogue waves matches perfectly in position and rogue wave shape. Moreover, one can further compare them numerically. The results also support our prediction, and since they are very similar to previous analysis, the details are omitted.
Fig. 9 Predicted 3rd-type $(0, 0, 4)$-th-order rogue waves of the three-component NLS from Theorem 3.1. Each column corresponds to rogue waves with a single large parameter $a_m$, whose value is indicated on top, and all other internal parameters are set to zero. Top row: predicted $(\hat{x}, \hat{t})$ locations by formulae (47). Middle row: predicted $|u_1(x, t)|$. Bottom row: predicted $|u_3(x, t)|$. First column: the $(x, t)$ intervals are $-30 \leq x \leq 20$, $-50 \leq t \leq 50$. Second column: the $(x, t)$ intervals are $-45 \leq x \leq 50$, $-32 \leq t \leq 32$. Third column: the $(x, t)$ intervals are $-55 \leq x \leq 40$, $-25 \leq t \leq 25$. Fourth column: the $(x, t)$ intervals are $-55 \leq x \leq 40$, $-25 \leq t \leq 25$. Fifth column: the $(x, t)$ intervals are $-45 \leq x \leq 35$, $-20 \leq t \leq 20$.

Fig. 10 True 3rd-type $(0, 0, 4)$-th-order rogue waves of the three-component NLS with the same parameters as Fig. 9. The $(x, t)$ interval for each column is the same as the corresponding column in Fig. 9.

4.1.4 Effect of Parameters on the Rogue Wave Shapes

We first represent the complex parameter $a_m$ as $a_m = |a_m| \exp(i\vartheta_m)$. In what follows, we will discuss the effects of the modulus $|a_m|$ and the argument $\vartheta_m$ on the shapes of rogue waves.

To illustrate the effect of the changes of $\vartheta_m$, we consider the 2nd-type $(0, 3, 0)$-th-order rogue waves of the three-component NLS equation and set $|a_3| = 300$, while
Fig. 11 Effect of the argument in $a_3$ on orientations of the 2nd-type $(0, 3, 0)$-th-order rogue waves of the three-component NLS equation. Each column represents a rogue wave with a different value of $a_3$ with the same modulus $300$ but a different argument, and all other internal parameters are set to zero. The $(x, t)$ intervals are $-55 \leq x \leq 55$ and $-30 \leq t \leq 30$.

The remaining parameters $a_m$ are set to $0$. Here, we simply choose four values of $a_3$, namely

$$(300 \exp(-\pi i/3), \ 300, \ 300 \exp(\pi i/3), \ 300 \exp(\pi i)), $$

and then, the corresponding predicted and true rogue wave patterns are shown in Fig. 11. It can be seen that the orientation of the rogue wave pattern is changed when we vary the values of $\vartheta_3$. In fact, this can be seen from Theorem 3.1 as well. In addition, we find that the orientation of the rogue wave pattern is obtained by rotating angle $\text{arg}(-a_m)/m$ of the root structure of $W_{3}^{[3,4,2]}(z)$, where “arg” represents the argument of a complex number.

To study the effect of changes in $|a_m|$ on rogue wave patterns, we choose 2nd-type $(0, 3, 0)$-th-order rogue waves. Set the argument of $a_2$ to $0$, while the remaining parameters $a_m$ are chosen to be $0$. The corresponding predicted and true rogue wave patterns are depicted in Fig. 12. It can be observed that the shape of the rogue wave pattern becomes closer and closer to the linear transformation of the roots of $W_{3}^{[2,4,2]}(z)$ when the modulus of $a_2$ gets larger. Specifically, when $a_2 = 20$, both predicted and true rogue wave patterns look very irregular, especially those located on the negative $x$-axis. However, further increasing the value of $a_2$, the distortion will gradually be weakened. For $a_2 = 500$, the shape of rogue waves is very close to a linear transformation of the roots of $W_{3}^{[2,4,2]}(z)$. The reason is that $\Delta_2$ is a $z_0$-dependent $O(1)$ quantity. Specifically, when $a_2$ takes a very large value, the value of $\Delta_2$ in (47) can be ignored and the predicted rogue wave can be obtained approximately by some linear transformation from the root structure of $W_{3}^{[2,4,2]}(z)$. 
Fig. 12 Effect of the size $a_2$ on shapes of the 2nd-type $(0, 3, 0)$-th-order rogue waves of the three-component NLS equation. Each column represents a rogue wave with different values of $a_2$ indicated on top, and all other internal parameters are set to zero. Top row: prediction location of $|u_3(x, t)|$. Bottom row: true $|u_3(x, t)|$. First column: the $(x, t)$ intervals are $-30 \leq x, t \leq 30$. Second column: the $(x, t)$ intervals are $-45 \leq x, t \leq 45$. Third column: the $(x, t)$ intervals are $-85 \leq x, t \leq 85$.

4.2 Comparison in the Four-Component NLS Equation

In this subsection, we compare our predicted rogue wave patterns in Theorem 3.2 with true rogue waves of the four-component NLS equation. For the quadruple root case, we choose background wavenumbers $k_1 = -k_4 = \left(\sqrt{5} - 1\right)/4$ and $k_2 = -k_3 = -\left(\sqrt{5} + 1\right)/8$, which implies $\rho_1 = \rho_4 = \sqrt{2}$ and $\rho_2 = \rho_3 = \sqrt{\sqrt{5} + 3}$ according to (25). In this circumstance, we select $p_0 = \sqrt{\left(\sqrt{5} + 5\right)/2}/2$ and $p_1 = \sqrt{10(\sqrt{5} + 25)/4608}$.

4.2.1 Third-Type Rogue Waves of the Four-Component NLS Equation

In this case, we focus on $(0, 0, 2, 0)$-th-order rogue wave solution. For brevity, we only consider the first four irreducible parameters $(a_2, a_3, a_4, a_6)$ and set the rest of the parameters to 0. Then, for the internal parameters $(a_2, a_3, a_4, a_6)$, we let one of them be large and set the others to 0. The very large parameter is one of

$$a_2 = 50, \quad a_3 = 200, \quad a_4 = 500, \quad a_6 = 5000.$$
Fig. 13 Predicted 3rd-type (0, 0, 2, 0)-th-order rogue waves of the four-component NLS equation from Theorem 3.2. Each column corresponds to a rogue wave with a single large parameter $a_m$, whose value is indicated on top, and all other internal parameters are set to zero. Top row: predicted $(\bar{x}_0, \bar{t}_0)$ locations by formulae (54). Middle row: predicted $|u_1(x, t)|$. Bottom row: predicted $|u_2(x, t)|$. First column: the $(x, t)$ intervals are $-25 \leq x \leq 25, -40 \leq t \leq 40$. Second column: the $(x, t)$ intervals are $-50 \leq x \leq 40, -35 \leq t \leq 35$. Third column: the $(x, t)$ intervals are $-30 \leq x \leq 30, -20 \leq t \leq 20$. Fourth column: the $(x, t)$ intervals are $-30 \leq x \leq 30, -15 \leq t \leq 15$

According to Theorem 3.2, the position $(\bar{x}_0, \bar{t}_0)$ of each fundamental rogue wave

$$u_{1,N_1}(x, t),\ u_{2,N_1}(x, t),\ u_{3,N_1}(x, t),\ u_{4,N_1}(x, t)$$

can be predicted by (54). The $(N_1, N_2, N_3, N_4)$-th-order rogue wave would appear in the inner region, and the $(N_1, N_2, N_3, N_4)$ values for these five rogue wave solutions are obtained from Theorems 3.2 and 2.4 as

$$(N_1, N_2, N_3, N_4) = (0, 0, 0, 0), (0, 0, 1, 1), (0, 1, 1, 0), (0, 0, 1, 1),$$

Note that $(0, 0, 0, 0)$ means no lower-order rogue wave exists in the center region. Owing to our choice of parameters $a_m$ and the value of $s_j$ shown in Remark 7, the internal parameters in these predicted lower $(N_1, N_2, N_3, N_4)$-th-order rogue waves of the center region are all chosen to be zero.

For $[u_{1,N_1}(x, t), u_{2,N_1}(x, t)]$, their corresponding predicted rogue wave patterns are shown in the last two rows of Fig. 13, with the first row being the locations of the rogue waves. Each column is separated when one of the parameters $(a_2, a_3, a_4, a_6)$ is large.
As depicted in Fig. 13, the large-$a_2$ solution exhibits a skewed double-triangle, corresponding to the double-triangle root structure of $W_2^{[2,5,2]}(z)$. The large-$a_3$ solution exhibits a skewed triple-triangle, corresponding to the triple-triangle root structure of $W_2^{[3,5,2]}(z)$. The large-$a_4$ solution exhibits a deformed rectangle, corresponding to the rectangle-shaped root structure of $W_2^{[4,5,2]}(z)$. The large-$a_6$ solution exhibits a deformed hexagon, corresponding to the hexagon-shaped root structure of $W_2^{[6,5,2]}(z)$.

Comparing the actual rogue waves with the predicted rogue waves (see Figs. 13 and 14), we can observe that each of the rogue waves matches perfectly in terms of position and rogue wave shape. Notice that the predicted pattern looks very different from the root structure of $W_2^{[m,5,2]}(z)$. This is caused by $\tilde{\Delta}_3$, which leads to a nonlinear transformation from the root structure. When we take $|a_m|$ very large, the term $\tilde{\Delta}_3$ can be neglected, and our pattern becomes more similar to a certain linear transformation of the root structure of $W_2^{[m,5,2]}(z)$. The numerical results also match very well, and as they are very similar to the previous error analysis, we omit the details.

4.2.2 Fourth-Type Rogue Waves of the Four-Component NLS Equation

In this circumstance, we consider $(0, 0, 0, 2)$-th-order rogue waves. For brevity, we only let one of the internal parameters $(a_2, a_3, a_4, a_6)$ be large and the others are set to 0. The very large parameter is one of

$$a_2 = 50, \quad a_3 = 200, \quad a_4 = 500, \quad a_5 = 5000.$$  \hspace{1cm} (63)

According to Theorem 3.2, the position $(\bar{x}_0, \bar{t}_0)$ of each fundamental rogue wave

$$u_{1, N_4}(x,t), \quad u_{2, N_4}(x,t), \quad u_{3, N_4}(x,t), \quad u_{4, N_4}(x,t)$$
Fig. 15 Predicted 4th-type \((0, 0, 0, 2)\)-th-order rogue waves of the four-component NLS equation from Theorem 3.2. Each column corresponds to rogue waves with a single large parameter \(a_m\), whose value is indicated on top, and all other internal parameters are set to zero. Top row: predicted \((\tilde{x}_0, \tilde{t}_0)\) locations by formulae (54). Middle row: predicted \(|u_3(x, t)|\). Bottom row: predicted \(|u_4(x, t)|\). First column: the \((x, t)\) intervals are \(-25 \leq x \leq 25, -35 \leq t \leq 35\). Second column: the \((x, t)\) intervals are \(-55 \leq x \leq 35, \mathbf{25} \leq t \leq 25\). Third column: the \((x, t)\) intervals are \(-25 \leq x \leq 25, -25 \leq t \leq 25\). Fourth column: the \((x, t)\) intervals are \(-30 \leq x \leq 30, -20 \leq t \leq 20\).

can be predicted by (54). The possible \((N_1, N_2, N_3, N_4)\)-th-order rogue wave would appear in the inner region, and the \((N_1, N_2, N_3, N_4)\) values for these five rogue wave solutions are obtained from Theorems 3.2 and 2.4 as

\[(N_1, N_2, N_3, N_4) = (0, 0, 0, 0), \quad (0, 0, 0, 0), \quad (0, 0, 1, 1), \quad (0, 0, 0, 0), \]

respectively. Note that \((0, 0, 0, 0)\) means that there are no lower-order rogue waves in the center region. For the same reason as previous cases, the internal parameters in these predicted lower \((N_1, N_2, N_3, N_4)\)-th-order rogue waves of the center region are all taken to be zero.

For \([u_{3,N_2}(x, t), u_{4,N_2}(x, t)]\), their corresponding predicted rogue wave patterns are shown in the last two rows of Fig. 15, with the first row being the predicted locations of the rogue waves. It can be seen from Fig. 15 that the first to fourth columns are skewed double-triangles, skewed triple-triangles, rectangles, and hexagons, respectively.

Comparing the true rogue waves with predicted ones (see Figs. 15 and 16), we can observe that each of the rogue waves matches perfectly in terms of position and rogue
wave shape. The results of the numerical analysis also match very well, but we omit the details because they are very similar to the previous error analysis.

5 Proof of the Main Results

**Proof of Theorem 3.1** We will only provide the proof for $i = 1$ as the proofs are similar in other cases. Assume $|a_m|$ is large and the rest parameters are $O(1)$ in the 1st-type rogue wave solutions of the three-component NLS equation. We first consider the case when $(x, t)$ is far away from the origin and $(x^2 + t^2)^{1/2} = O(|a_m|^{-1/m})$. In this circumstance, we have

\[
S_j \left( x^+ (n) + vs \right) = S_j \left( x_1^+, x_2^+, x_3^+, v s_4, x_5^+, x_6^+, x_7^+, v s_8, \ldots, x_m^+ + v s_m, \ldots \right) = S_j (v) \left[ 1 + O \left( a_m^{-1/m} \right) \right], \tag{64}
\]

where

\[
v = (p_0 x + 2 p_0 p_1 i t, 0, \ldots, 0, a_m, 0, \ldots).\]

According to Remark 7, we have $s_1 = s_2 = s_3 = s_5 = s_6 = s_7 = 0$. By the definition of Schur polynomials, we have the relation

\[
S_j (v) = a_m^{j/m} p_j^{|m|} (z), \tag{65}
\]

where

\[
z = a_m^{-1/m} (\alpha_1 x + \beta_1 i t) = a_m^{-1/m} (p_0 x + 2 p_0 p_1 i t). \tag{66}
\]
Then, it follows that
\[
\det_{1 \leq i, j \leq N} \left[ S_{4i-j} \left( x^+ (n) + v_j s^* \right) \right] = \left( c_N^{[m,4,3]} \right)^{-1} a_m^{-3N(N+1)/2m} W_N^{[m,4,3]}(z) \left[ 1 + O \left( a_m^{-1/m} \right) \right] \tag{67}
\]
and
\[
\det_{1 \leq i, j \leq N} \left[ S_{4i-j} \left( x^- (n) + v_j s^* \right) \right] = \left( c_N^{[m,4,3]} \right)^{-1} \left( a_m^* \right)^{-3N(N+1)/2m} W_N^{[m,4,3]}(z^*) \left[ 1 + O \left( a_m^{-1/m} \right) \right] \tag{68}
\]
Here, \( S_j \equiv 0 \) when \( j < 0 \).

Next, we rewrite the function \( \tau_n \) into the following form by Laplace expansion
\[
\tau_n = \sum_{0 \leq v_1 < v_2 < \ldots < v_N \leq 4N-1} \det_{1 \leq i, j \leq N} \left[ (h_0)^{v_j} S_{4i-1-v_j} \left( x^+ (n) + v_j s^* \right) \right] \\
\times \det_{1 \leq i, j \leq N} \left[ (h_0)^{v_j} S_{4i-1-v_j} \left( x^- (n) + v_j s^* \right) \right], \tag{69}
\]
where \( h_0 = p_1 / (p_0 + p_0^*) \).

It is clear that the highest-order term in \( a_m \) in this \( \tau_n \) comes from the index choices of \( v_j = j - 1 \). Therefore, we have
\[
\tau_n = |\alpha|^2 |a_m|^{3N(N+1)/m} \left| W_N^{[m,4,3]}(z) \right|^2 \left[ 1 + O \left( a_m^{-1/m} \right) \right], \tag{70}
\]
where
\[
\alpha = h_0^{N(N-1)/2} \left( c_N^{[m,4,3]} \right)^{-1}. \tag{71}
\]
From the asymptotic analysis above, we conclude that the leading-order term of \( \tau_n \) is independent of \( n \). Consequently, when \((x, t)\) is not close to \((\tilde{x}_0, \tilde{t}_0)\), which is related to the roots of \( W_N^{[m,4,3]}(z) \) by
\[
z_0 = a_m^{-1/m} \left( p_0 \tilde{x}_0 + 2 p_0 p_1 \tilde{t}_0 \right),
\]
we have
\[
\frac{\tau_{n_1}}{\tau_{n_0}} = 1 + O \left( a_m^{-1/m} \right), \quad \frac{\tau_{n_2}}{\tau_{n_0}} = 1 + O \left( a_m^{-1/m} \right), \quad \frac{\tau_{n_3}}{\tau_{n_0}} = 1 + O \left( a_m^{-1/m} \right), \quad |a_m| \gg 1. \tag{72}
\]

However, when \((x, t)\) is close to \((\tilde{x}_0, \tilde{t}_0)\), the coefficient of the term with highest order in \( a_m \) vanishes. To deal with this case, we have to consider lower-order terms in \( a_m \), which require more precise asymptotics. In this circumstance, i.e., \((x, t)\) is near \((\tilde{x}_0, \tilde{t}_0)\), we find that
\[
S_j \left( x^+ (n) + v s^* \right) = S_j \left( x_1^+, x_2^+, x_3^+, v s_4^+, x_5^+, x_6^+, x_7^+, v s_8^+, \ldots, x_m^+ + v s_m, \ldots \right) \\
= \left[ S_j (\tilde{v}) + \tilde{x}_2^+ (\tilde{x}_0, \tilde{t}_0) S_{j-2} (\tilde{v}) \right] \left[ 1 + O \left( a_m^{-2/m} \right) \right], \quad |a_m| \gg 1,
\]
where
\[
\dot{\mathbf{v}} = \left(p_0x + 2p_0p_1i + n_1\theta_{11} + n_2\theta_{12} + n_3\theta_{13}, 0, \ldots, 0, a_m, 0, \ldots \right),
\]
\[
\dot{x}_2^+(x, t) = p_2x + \left(2p_0p_2 + p_1^2\right)(it).
\]

Then, we expand
\[
W = \left(p_0x + 2p_0p_1i + n_1\theta_{11} + n_2\theta_{12} + n_3\theta_{13}, 0, \ldots, 0, a_m, 0, \ldots \right).
\]

As a result, the corresponding leading-order term in \(a_m\) results from the determinants containing
\[
\hat{\mathbf{v}} = \left(\theta_{11}, \theta_{12}, \theta_{13}\right),
\]
where
\[
\hat{x}_2^+(x, t) = p_2x + \left(2p_0p_2 + p_1^2\right)(it).
\]

\(p_2 = p_1^2\) and \(a_1\) in \(x_1^+\) is set to 0. Similar to (65), we can get
\[
S_j(\hat{\mathbf{v}}) = a_m^{j/m} p_j^{[m]}(\hat{\mathbf{v}}),
\]
where
\[
\hat{\mathbf{v}} = a_1^{-1/m} \left(p_0x + 2p_0p_1i + n_1\theta_{11} + n_2\theta_{12} + n_3\theta_{13}\right).
\]

In this case, there are two index choices of \(\nu_j\) that will produce leading-order terms in \(a_m\) for \(\tau_n\). One of them is \(\nu = (0, 1, \ldots, N-1)\), while the other is \(\nu = (0, 1, \ldots, N-2, N)\).

(I) For the first choice of index, i.e., \(\nu_j = \nu - 1\), there are two parts that will provide leading-order terms. The first part stems from \(S_j(\hat{\mathbf{v}})\), and we find that the dominant term involving \(x^+(\mathbf{n})\) is expressed as
\[
\alpha a_m^{\frac{3N(N+1)}{2m}} W_N^{[m, 4, 3]}(\hat{\mathbf{v}}) \left[1 + O\left(a_m^{-2/m}\right)\right].
\]

Then, we expand \(W_N^{[m, 4, 3]}(\hat{\mathbf{v}})\) around \(z_0\), and noting \(W_N^{[m, 4, 3]}(z_0) = 0\), we obtain
\[
W_N^{[m, 4, 3]}(\hat{\mathbf{v}}) = a_m^{-1/m} \left[p_0(x - \hat{x}_0) + 2p_0p_1i(t - \hat{t}_0) + n_1\theta_{11} + n_2\theta_{12} + n_3\theta_{13}\right]
\times \left[W_N^{[m, 4, 3]}(z_0) \left[1 + O\left(a_m^{-1/m}\right)\right]\right].
\]

As a result, the corresponding leading-order term in \(a_m\) is
\[
\alpha a_m^{\frac{3N(N+1)-2}{2m}} \left[p_0(x - \hat{x}_0) + 2p_0p_1i(t - \hat{t}_0) + n_1\theta_{11} + n_2\theta_{12} + n_3\theta_{13}\right]
\times \left[W_N^{[m, 4, 3]}(z_0) \left[1 + O\left(a_m^{-1/m}\right)\right]\right].
\]

The other leading-order term results from the determinants containing \(\dot{x}_2^+(\hat{x}_0, \hat{t}_0) S_{j-2}\), that is,
\[
\dot{x}_2^+(\hat{x}_0, \hat{t}_0) h_0^{N(N-1)/2} \sum_{j=1}^{N} \det_{1 \leq i \leq N}
\left[S_{4i-1}(\hat{\mathbf{v}}), \ldots, S_{4i-(j-1)}(\hat{\mathbf{v}}), S_{4i-j-2}(\hat{\mathbf{v}}), S_{4i-(j+1)}(\hat{\mathbf{v}}), \ldots, S_{4i-N}(\hat{\mathbf{v}})\right]
\times \left[1 + O\left(a_m^{-1/m}\right)\right].
\]
Combining (77) and (78) yields the leading-order term in \( a_m \) (Yang and Yang 2023) of the first determinant in (69) containing \( x^\dagger(\mathbf{n}) \) corresponding to the index choice \( \nu_j = j - 1 \), that is,

\[
\alpha a_m^{3N(N+1)-2/2m} \left[ p_0 (x - \hat{x}_0) + 2 p_0 p_1 i (t - \hat{t}_0) + n_1 \theta_{11} + n_2 \theta_{12} + n_3 \theta_{13} + \Delta_1 \right]
\times \left[ W_N^{[m,4,3]}(z_0) \right] \left[ 1 + O \left( a_m^{-1/m} \right) \right]
\]

where

\[
\Delta_1 = \frac{\hat{k}_2^+ (\hat{x}_0, \hat{t}_0)}{a_m^{1/m}} \sum_{j=1}^N \det_{1 \leq i \leq N} \left[ p_{4i-1}^{[m]} (z_0), \ldots, p_{4i-j+1}^{[m]} (z_0), p_{4i-j-2}^{[m]} (z_0), p_{4i-j-1}^{[m]} (z_0), \ldots, p_{4i-N}^{[m]} (z_0) \right].
\]

and \( \Delta_1 = O(1) \) as

\[
\hat{x}_2^+ (\hat{x}_0, \hat{t}_0) = p_2 \hat{x}_0 + \left( 2 p_0 p_2 + p_1^2 \right) (i\hat{t}_0) = O(|a_m^{-1/m}|).
\]

Further, we can absorb the \( \Delta_1 \) into \((\hat{x}_0, \hat{t}_0)\) (see Yang and Yang (2023, p. 38)) and obtain

\[
\alpha a_m^{3N(N+1)-2/2m} \left[ p_0 (x - \hat{x}_0) + 2 p_0 p_1 i (t - \hat{t}_0) + n_1 \theta_{11} + n_2 \theta_{12} + n_3 \theta_{13} \right]
\times \left[ W_N^{[m,4,3]}(z_0) \right] \left[ 1 + O \left( a_m^{-1/m} \right) \right],
\]

where \( \hat{x}_0 \) and \( \hat{t}_0 \) are given in (47).

Similarly, the second determinant in (69) containing \( x^- (\mathbf{n}) \) corresponding to the index choice \( \nu_j = j - 1 \) contributes the term

\[
\alpha^* (a_m^*)^{3N(N+1)-2/2m} \left[ p_0^* (x - \hat{x}_0) - 2 p_0^* p_1^* i (t - \hat{t}_0) - n_1 \theta_{11}^* - n_2 \theta_{12}^* - n_3 \theta_{13}^* \right]
\times \left[ W_N^{[m,4,3]}(z_0^*) \right] \left[ 1 + O \left( a_m^{-1/m} \right) \right].
\]

(II) For the second choice of index, i.e., \( \nu = (0, 1, \ldots, N - 2, N) \), the dominant term in \( a_m \) can be calculated in a similar way as (67), that is,

\[
\hat{k}_0^2 a_m^{3N(N+1)-2/2m} \det_{1 \leq i \leq N} \left[ p_{4i-1}^{[m]} (z_0), p_{4i-2}^{[m]} (z_0), \ldots, p_{4i-(N-1)}^{[m]} (z_0), p_{4i-N-1}^{[m]} (z_0) \right]
\times \left[ 1 + O \left( a_m^{-1/m} \right) \right].
\]
Since \( p_{j-1}^{[m]}(z) = \left[ p_j^{[m]} \right]'(z) \), the above term can be expressed as
\[
h_0\alpha a_m^{2N(N+1)-2} \left[ W_N^{[m,4,3]} \right]'(z_0) \left[ 1 + O \left( a_m^{-1/m} \right) \right].
\]
Similarly, its conjugate counterpart reads
\[
h_0^*\alpha^* (a_m)^{2N(N+1)-2} \left[ W_N^{[m,4,3]} \right]'(z_0^*) \left[ 1 + O \left( a_m^{-1/m} \right) \right].
\]
Summarizing the above two contributions, we conclude that
\[
\tau_n(x, t) = |\alpha|^2 \left| \left[ W_N^{[m,4,3]} \right]'(z_0) \right|^2 |a_m|^{[2N(N+1)-2]/m}
\times \left( \left[ p_1 (x - \hat{x}_0) + 2i p_0 p_1 (t - \hat{t}_0) + n_1 \theta_1 + n_2 \theta_2 + n_3 \theta_3 \right] \right.
\times \left[ p_1^* (x - \hat{x}_0^*) - 2i p_0^* p_1^* (t - \hat{t}_0) - n_1 \theta_1^* + n_2 \theta_2^* - n_3 \theta_3^* \right] + |h_0|^2 \right)
\times \left[ 1 + O \left( a_m^{-1/m} \right) \right]. \tag{83}
\]
Finally, under the assumption that all nonzero roots of the generalized Wronskian–Hermite polynomials \( W_N^{[m,k,l]} \) are simple, the above leading-order term in \( a_m \) for \( \tau_n(x, t) \) is nonzero. Hence, using (83), we conclude that, near \((\hat{x}_0, \hat{t}_0)\), the \( N \)-th-order rogue wave is approximated by a fundamental rogue wave of the three-component NLS equation given in Theorem 3.1 with error \( O(|a_m|^{-1/m}) \).

In order to study the patterns of the 1st-type rogue waves of the three-component NLS rogue waves under the condition \(|a_m| \gg 1\) in the inner region with \(x^2 + t^2 = O(1)\), we first use similar method as that in Ohta and Yang (2012) to rewrite the determinant \( \tau_n \) as a \( 5N \times 5N \) determinant
\[
\tau_n = \begin{vmatrix} O_{N \times N} & \Phi_{N \times 4N} \\ -\Psi_{4N \times N} & I_{4N \times 4N} \end{vmatrix}, \tag{84}
\]
where
\[
\Phi_{i,j} = \left( \frac{p_1}{p_0 + p_0^*} \right)^{i-1} \left( S_{4i-j} \left[ x^+(n) + (j-1)s \right] \right),
\]
\[
\Psi_{i,j} = \left( \frac{p_1^*}{p_0 + p_0^*} \right)^{i-1} \left( S_{4j-i} \left[ x^-(n) + (i-1)s^* \right] \right).
\]
It is clear that each element in (84) is a polynomial in \( a_m \). To express these polynomials explicitly, we define \( y^\pm \) to be the vector \( x^\pm \) without the \( a_m \) term, i.e.,
\[
x^+ = y^+ + (0, \cdots, 0, a_m, 0, \cdots), \quad x^- = y^- + (0, \cdots, 0, a_m^*, 0, \cdots). \tag{85}
\]
Then, we can expand the Schur polynomials $S_j \left( x^\pm + \nu s \right)$ by

\[
S_j \left( x^+ + \nu s \right) = \sum_{l=0}^{[j/m]} \frac{a^m_l}{l!} S_{j-lm} \left( y^+ + \nu s \right),
\]

\[
S_j \left( x^- + \nu s^* \right) = \sum_{l=0}^{[j/m]} \left( a^m_l \right)^* \frac{1}{l!} S_{j-lm} \left( y^- + \nu s^* \right), \tag{86}
\]

where $[a]$ refers to the largest integer less than or equal to $a$. To determine the highest-order term in $a_m$ of $\tau_\Phi$, a straightforward way is to keep only the highest order of $a_m$ in each element. However, it turns out the resulting determinant will vanish. To tackle this issue, we can use a similar argument as that in Yang and Yang (2021c) to perform row and column operations. Notice that we have three cases in total to consider, i.e., $m \equiv j \mod 4, j = 1, 2, 3$. Since the proof for $j = 2$ is different from those in the NLS equation (Yang and Yang 2021c) and the Manakov system (Yang and Yang 2023), we first focus on the proof of this case. As the proofs for the cases $j = 1$ and $3$ are similar to Yang and Yang (2023), we only provide a brief proof for $j = 1$.

For $m \equiv 2 \mod 4$, i.e., $m = 4r + 2 \ (r \geq 0)$, according to the block structure of the determinant $\tau_\Phi$, we can perform row operations on the matrix $\Phi_{N \times N}$. For convenience, we define $\hat{S}_j = S_j \left( y^+ + \nu s \right)$ and omit $\left( p_1/(p_0 + p_0^*) \right)^{j-1}$ in the following representation because they are the same in each column and have no effect on the row operations. Then, we can substitute (86) into $\Phi_{N \times N}$ and rewrite it into the form

\[
\Phi_{N \times N} \sim \begin{bmatrix}
\hat{S}_3 & \hat{S}_2 & \hat{S}_{m-4} & \hat{S}_{2m-2} & \cdots \\
\hat{S}_2 & \hat{S}_0 & \hat{S}_{m-5} & \hat{S}_{2m-1} & \cdots \\
a_m \hat{S}_1 + \hat{S}_{m+1} & a_m \hat{S}_4 + \hat{S}_{m+4} & \cdots \\
a_m \hat{S}_5 + \hat{S}_{m+5} & \cdots \\
a_m \hat{S}_{m-1} + \hat{S}_{2m-1} & \cdots \\
a^m_{2l} \hat{S}_3 + a_m \hat{S}_{m+3} + \hat{S}_{2m+3} & \cdots \\
a^m_{2l} \hat{S}_2 + a_m \hat{S}_{m+2} + \hat{S}_{2m+2} & \cdots \\
a^m_{2l} \hat{S}_{m-3} + a^m_s \hat{S}_{m-3} + \hat{S}_{3m-3} & \cdots \\
a^m_{2l} \hat{S}_{m-4} + a_m \hat{S}_{2m-4} + \hat{S}_{3m-4} & \cdots \\
a^m_{3l} \hat{S}_1 + a^m_{2l} \hat{S}_{m+1} + O(a_m) & \cdots \\
a^m_{3l} \hat{S}_0 + a^m_{2l} \hat{S}_m + O(a_m) & \cdots \\
a^m_{4l} \hat{S}_{m-1} + a^m_{3l} \hat{S}_{m+1} + O(a_m) & \cdots \\
a^m_{4l} \hat{S}_2 + a^m_{3l} \hat{S}_{m+2} + a^m_{2l} \hat{S}_{2m+2} + O(a_m) & \cdots \\
a^m_{4l} \hat{S}_{m-3} + a^m_{3l} \hat{S}_{2m-3} + a^m_{2l} \hat{S}_{3m-3} + O(a_m) & \cdots \\
a^m_{4l} \hat{S}_{m-4} + a^m_{3l} \hat{S}_{2m-4} + a^m_{2l} \hat{S}_{3m-4} + O(a_m) & \cdots
\end{bmatrix}
\]
In this case, we can notice that the coefficients of the highest $a_m$ power terms in the first column are proportional to

$$\hat{S}_3, \hat{S}_7, \ldots, \hat{S}_{m-3}, \hat{S}_1, \hat{S}_5, \ldots, \hat{S}_{m-1}$$ (87)

and repeating. To be more precise, the first $r$ rows are a sequence starting with $\hat{S}_3$, and the subscripts of $\hat{S}$ increase by 4. The next $r + 1$ rows, i.e., rows $r + 1$ to $2r + 1$, are a sequence starting with $\hat{S}_1$, and the subscripts increase by 4 as well. After that, the subsequent $r$ rows are the sequence starting with a multiple of $\hat{S}_3$, followed by $r + 1$ rows starting with a multiple of $\hat{S}_1$, and so on and so forth. Each element in the second and higher columns maintains the same form as the elements in the first column, except that the subscripts decrease by 1, where $\hat{S}_j \equiv 0$ for $j < 0$.

Notice that each $2r + 1 (= m/2)$ row circulates a multiple of the sequence (87) and $N_0 \equiv N \mod m$, i.e., $N = km + N_0$. Hence, we define the first $m/2$ rows of $\Phi_{N \times 4N}$ as the first block matrix, the next $m/2$ rows as the second block matrix, and so on. The first $km$ rows consist of $2k$ blocks, each containing two parts. The last remaining $N_0$ rows are called the remaining block matrix, i.e., the $(2k + 1)$-th block matrix.

The first round of the row operation uses the first block to eliminate the highest power term of $a_m$ in each subsequent block, leaving the lower power terms of $a_m$. This can be achieved by multiplying each row of the first part of the first block matrix by $-a_m^2/(2n - 2)!$, multiplying each row of the second part of the first block matrix by $-a_m^2/(2n - 1)!$ and adding them to the corresponding row of the $n$-th block matrix. The resulting $\Phi_{N \times 4N}$ is

$$\Phi_{N \times 4N} \sim \begin{bmatrix}
\hat{S}_3 & \hat{S}_2 & \ldots \\
\hat{S}_7 & \hat{S}_6 & \ldots \\
\vdots & \vdots & \ddots \\
\hat{S}_{m-3} & \hat{S}_{m-4} & \ldots \\
a_m \hat{S}_1 + \hat{S}_{m+1} & a_m \hat{S}_0 + \hat{S}_m & \ldots \\
a_m \hat{S}_5 + \hat{S}_{m+5} & a_m \hat{S}_4 + \hat{S}_{m+4} & \ldots \\
\vdots & \vdots & \ddots \\
a_m \hat{S}_{m-1} + \hat{S}_{2m-1} & a_m \hat{S}_{m-2} + \hat{S}_{2m-2} & \ldots \\
a_m \hat{S}_{m+3} + \hat{S}_{2m+3} & a_m \hat{S}_{m+2} + \hat{S}_{2m+2} & \ldots \\
\vdots & \vdots & \ddots \\
(a_m^3 \hat{S}_{2m-3} + \hat{S}_{3m-3}) & (a_m^3 \hat{S}_{2m-4} + \hat{S}_{3m-4}) & \ldots \\
\left(\frac{1}{2!} - \frac{1}{3!}\right) a_m^2 \hat{S}_{m+1} + O(a_m) & \left(\frac{1}{2!} - \frac{1}{3!}\right) a_m^2 \hat{S}_m + O(a_m) & \ldots \\
\left(\frac{1}{2!} - \frac{1}{3!}\right) a_m^2 \hat{S}_{m-1} + O(a_m) & \left(\frac{1}{2!} - \frac{1}{3!}\right) a_m^2 \hat{S}_{2m-2} + O(a_m) & \ldots \\
\frac{1}{3!} a_m^3 \hat{S}_{m+3} + \frac{2}{2!} a_m^2 \hat{S}_{2m+3} + O(a_m) & \frac{1}{3!} a_m^3 \hat{S}_{m+2} + \frac{2}{2!} a_m^2 \hat{S}_{2m+2} + O(a_m) & \ldots \\
\frac{1}{3!} a_m^3 \hat{S}_{2m-3} + \frac{2}{2!} a_m^2 \hat{S}_{3m-3} + O(a_m) & \frac{1}{3!} a_m^3 \hat{S}_{2m-4} + \frac{2}{2!} a_m^2 \hat{S}_{3m-4} + O(a_m) & \ldots \\
\vdots & \vdots & \ddots \\
\end{bmatrix}$$ (88)
The second round of the row operation uses the second block to eliminate the highest power term of $a_m$ in each subsequent block, leaving the lower power terms of $a_m$. This results in

$$
\Phi_{N \times 4N} \sim \begin{bmatrix}
\hat{S}_1 & \hat{S}_2 & \cdots \\
\hat{S}_7 & \hat{S}_6 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
a_m \hat{S}_1 + \hat{S}_{m+1} & a_m \hat{S}_0 + \hat{S}_m & \cdots \\
a_m \hat{S}_5 + \hat{S}_{m+5} & a_m \hat{S}_4 + \hat{S}_{m+4} & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
a_m \hat{S}_{m-1} + \hat{S}_{2m-1} & a_m \hat{S}_{m-2} + \hat{S}_{2m-2} & \cdots \\
a_m \hat{S}_{m+3} + \hat{S}_{2m+3} & a_m \hat{S}_{m+2} + \hat{S}_{2m+2} & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{2!} - \frac{1}{3!} a_m \hat{S}_{m+1} + O(a_m) & \left( \frac{1}{2!} - \frac{1}{3!} \right) a_m^2 \hat{S}_m + O(a_m) & \cdots \\
\frac{1}{2!} - \frac{1}{3!} a_m^2 \hat{S}_{m-1} + O(a_m) & \left( \frac{1}{2!} - \frac{1}{3!} \right) a_m^2 \hat{S}_{2m-2} + O(a_m) & \cdots \\
\frac{1}{2!} - \frac{1}{3!} a_m^2 \hat{S}_{m+3} + O(a_m) & \left( \frac{1}{2!} - \frac{1}{3!} \right) a_m^2 \hat{S}_{2m+2} + O(a_m) & \cdots \\
\frac{1}{2!} - \frac{1}{3!} a_m^2 \hat{S}_{m-3} + O(a_m) & \left( \frac{1}{2!} - \frac{1}{3!} \right) a_m^2 \hat{S}_{3m-4} + O(a_m) & \cdots 
\end{bmatrix}.
$$

(89)

We can continue to perform these row operations to $\Phi_{N \times 4N}$, which have $2k$ rounds in total. Similar column operations can be applied to the matrix $\Psi_{4N \times N}$. At the end of these operations, we arrive at the situation where the determinant (84) does not vanish when we keep only the highest-order term in $a_m$ for each element. The difference with the previous work in Yang and Yang (2021c) is that we cannot generate the lower triangular block matrix or upper triangular block matrix after keeping the highest-order terms and row switchings. This indicates that the size of determinant $\tau_n$ is unchanged. Moreover, it can be shown that $\tau_n$ reduces to the form

$$
\tau_n = \beta_1 |a_m|^{\gamma_1} \left| \begin{array}{c}
\text{O}_{N \times N} \\
\hat{\Psi}_{4N \times N} \\
\text{I}_{4N \times 4N} \\
\end{array} \right| \left[ 1 + O\left( a_m^{-1} \right) \right],
$$

(90)

where $\beta_1 \neq 0$, $\gamma_1 > 0$ are constants, and

$$
\hat{\Phi} = \left( \hat{\Phi}_{N_1 \times 4N}^{(1)}, \hat{\Phi}_{N_3 \times 4N}^{(3)} \right), \quad \hat{\Psi} = \left( \hat{\Psi}_{4N \times N_1}^{(1)}, \hat{\Psi}_{4N \times N_3}^{(3)} \right),
$$

$$
\hat{\Phi}_{i,j}^{(1)} = (h_0)^{-j-1} S_{4i+1-j} \left[ y^+ (n) + (j - 1) s \right],
$$

$$
\hat{\Psi}_{i,j}^{(3)} = (h_0^*)^{-i-1} S_{4j+1-j} \left[ y^- (n) + (i - 1) s^* \right].
$$

(91)
Since the rogue wave solutions are independent of the constants $\beta_1$ and $\gamma_1$, we can rewrite (90) into a $2 \times 2$ block determinant (Yang and Yang 2023)

$$\tau_n = \det \left( \begin{array}{cc} \tau_n^{[1,1]} & \tau_n^{[1,3]} \\ \tau_n^{[3,1]} & \tau_n^{[3,3]} \end{array} \right) \left[ 1 + O \left( a_m^{-1} \right) \right]$$

(92)

where

$$\tau_n^{[I,J]} = \left( m_{4i-I,4j-J}^{(n,I,J)} \right)_{1 \leq i \leq N_I, 1 \leq j \leq N_J}, \quad 1 \leq I, J \leq 3,$$

and

$$m_{i,j}^{(n,I,J)} = \sum_{v=0}^{\min(i,j)} \left[ \frac{|p_1|^2}{(p_0 + p_0^*)^2} \right]^v S_i-v \left( y^+(n) + vs \right) S_j-v \left( y^-(n) + vs^* \right).$$

(94)

Note that the determinant $\tau_n$ is still of order $N$, but the degree of $\tau_n$ with respect to $x$ or $t$ is reduced, so this is still a lower-order rogue wave. Moreover, in this case, $\tau_n$ in the inner region is always approximately a $2 \times 2$ block matrix regardless of the values of $N$ and $m$, i.e., $N_2 = 0$ in (16). As a result of this, when $x^2 + t^2 = O(1)$ and $|a_m| \gg 1$, the determinant in (84) is approximately a $(N_1, 0, N_3)$-th-order rogue wave of the three-component NLS equation

$$[u_{1,\tilde{N}_1}(x,t), \quad u_{2,\tilde{N}_1}(x,t), \quad u_{3,\tilde{N}_1}(x,t)]$$

where $\tilde{N}_1 = (N_1, 0, N_3), u_{j,\tilde{N}_1}$ ($j = 1, 2, 3$) is given in Theorem 2.2 with $N_j = N_{j,3}$, and the internal parameters

$$\left( \hat{a}_{1,n}, \hat{a}_{2,n}, \hat{a}_{3,n}, \hat{a}_{5,n}, \hat{a}_{6,n}, \ldots, \hat{a}_{4N_{n,3-n,n}} \right), \quad n = 1, 3,$$

are related to those in the original rogue wave as

$$\hat{a}_{j,1} = \hat{a}_{j,3} = a_j, \quad j = 1, 2, 3, 5, 6, 7 \ldots, m - 1, m + 1, \ldots$$

and

$$\hat{a}_{m,1} = \hat{a}_{m,3} = 0.$$

From (92), we deduce that the approximation error of this lower-order rogue wave is $O \left( |a_m|^{-1} \right)$.

Next, we consider the case $m \equiv 1 \mod 4$, i.e., $m = 4r + 1$ ($r > 1$). Notice that the coefficients of the highest $a_m$ power terms in the first column of $\Psi_{N \times 4N}$ are proportional to

$$\hat{S}_3, \hat{S}_7, \ldots, \hat{S}_{4r-1}, \hat{S}_2, \hat{S}_6, \ldots, \hat{S}_{4r-2}, \hat{S}_1, \hat{S}_5, \ldots, \hat{S}_{4r-3}, \hat{S}_0, \hat{S}_4, \ldots, \hat{S}_4r$$

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and repeating. Similar to the previous case, we can think of the first \( m \) rows as the first block matrix of \( \Phi_{N \times 4N} \), the next \( m \) rows as the second block matrix, and so on. On account of \( N = km + N_0 \), the remaining \( N_0 \) rows are called the remaining block matrix, i.e., the \( (k + 1) \)-th block matrix. Each block matrix can be divided into four parts; for example, the first column of these four parts are sequences starting with \( \hat{S}_3 \), \( \hat{S}_2 \), \( \hat{S}_1 \), and \( \hat{S}_0 \), respectively.

Then, using a similar argument as in Yang and Yang \( \text{(2021c)} \), we arrive at the situation where the determinant \( (84) \) does not vanish when we keep only the highest-order term in \( a_m \) for each element. In this case, after row and column swapping, upper and lower triangular block matrices will be generated. After expanding these block matrices, \( \tau_n \) reduces to the form

\[
\tau_n = \beta_2 |a_m|^\gamma_2 \left[ O_{\hat{N}_3 \times \hat{N}_3} \Phi_{\hat{N}_3 \times \hat{N}} \begin{pmatrix} I_{\hat{N} \times \hat{N}} \end{pmatrix} \right] \left[ 1 + O\left( a_{m}^{-1} \right) \right], \tag{95} \]

where \( \beta_2 \neq 0, \gamma_2 > 0 \) are constants, \( \hat{N}_3 = \sum_{n=1}^{3} N_{n,3} \), \( \hat{N} = \max_{1 \leq i \leq 3} (4N_{i,3} - i + 1) \),

\[
\Phi = \begin{pmatrix} \Phi^{(1)}_{N_1 \times \hat{N}} \\ \Phi^{(2)}_{N_2 \times \hat{N}} \\ \Phi^{(3)}_{N_3 \times \hat{N}} \end{pmatrix}, \quad \Psi = \begin{pmatrix} \Psi^{(1)}_{\hat{N} \times N_1} \\ \Psi^{(2)}_{\hat{N} \times N_2} \\ \Psi^{(3)}_{\hat{N} \times N_3} \end{pmatrix} \]

\[
\Phi^{(l)}_{i,j} = (h_0)^{-(j-1)} S_{4i+1-j} \left[ y^+(n) + (j - 1 + \nu_0) s \right] \]

\[
\Psi^{(l)}_{i,j} = (h_0)^{-(i-1)} S_{4j+1-j} \left[ y^-(n) + (i - 1 + \nu_0) s^* \right] \tag{96} \]

and \( \nu_0 = N - \hat{N}_1 \). Finally, using a similar argument as in Yang and Yang \( \text{(2023)} \), we find that \( \tau_n \) can be asymptotically reduced to a \( (N_{1,3}, N_{2,3}, N_{3,3}) \)-th-order rogue wave of the three-component NLS equation in the inner region. Notice that the internal parameters

\[
(\hat{a}_{1,n}, \hat{a}_{2,n}, \hat{a}_{3,n}, \hat{a}_{5,n}, \hat{a}_{6,n}, \ldots, \hat{a}_{4N_{n,3}-n,n}), \quad n = 1, 2, 3, \]

are related to those in the original rogue wave as

\[
\hat{a}_{j,1} = \hat{a}_{j,2} = \hat{a}_{j,3} = a_j + (N - \hat{N}_3)s_j, \quad j = 1, 2, 3, 5, 6, 7 \ldots . \]

As pointed before, the proofs of our 1st-type and 3rd-type rogue waves of the three-component NLS are very similar, so we omit the proof of 3rd type. However, there are some differences in the proof of 2nd-type rogue waves in the inner region. This is explained as follows.
We rewrite the determinant \( \tau_n \) as a \( 5N \times 5N \) determinant as in (84). Note that \( m \equiv j \mod 4 \), \( j = 1, 2, 3 \), and the different case is still \( j = 2 \), i.e., \( m = 4r + 2 \) \( (r \geq 0) \). In this case, we can substitute (86) into (84) to expand each element into a polynomial in \( a_m \). Similar to the proof of 1st type, we can rewrite \( \Phi_{N \times 4N} \) as follows

\[
\Phi_{N \times 4N} \sim \begin{bmatrix}
\hat{S}_2 & \hat{S}_1 & \cdots \\
\hat{S}_6 & \hat{S}_5 & \\
\vdots & \vdots & \ddots \\
\hat{S}_{m-4} & \hat{S}_{m-5} & \\
a_m \hat{S}_0 + \hat{S}_m & \hat{S}_{m-1} & \\
a_m \hat{S}_4 + \hat{S}_{m+4} & a_m \hat{S}_3 + \hat{S}_{m+3} & \\
\vdots & \vdots & \ddots \\
a_m \hat{S}_{m-2} + \hat{S}_{2m-2} & a_m \hat{S}_{m-3} + \hat{S}_{2m-3} & \\
\frac{a_m^2}{2!} \hat{S}_2 + a_m \hat{S}_{m+2} + \hat{S}_{2m+2} & \frac{a_m^2}{2!} \hat{S}_1 + a_m \hat{S}_{m+1} + \hat{S}_{2m+1} & \\
\vdots & \vdots & \\
\frac{a_m^3}{3!} \hat{S}_0 + \frac{a_m^2}{2!} \hat{S}_m + O(a_m) & \frac{a_m^2}{2!} \hat{S}_{m-1} + O(a_m) & \\
\frac{a_m^3}{3!} \hat{S}_{m-2} + \frac{a_m^2}{2!} \hat{S}_{2m-2} + O(a_m) & \frac{a_m^3}{3!} \hat{S}_{m-3} + \frac{a_m^2}{2!} \hat{S}_{2m-3} + O(a_m) & \\
\vdots & \vdots & \\
\end{bmatrix}
\]

(97)

It can be seen that the matrix \( \Phi_{N \times 4N} \) can be divided into a number of blocks. We use the same method as before; that is, we use the preceding blocks to eliminate the highest-order terms in \( a_m \) of the subsequent blocks in turn. After the above operations, we find that only the coefficient of the highest-power term in \((r + 1)\)th row is \( \hat{S}_0 \). This inspires us to eliminate one row and one column through some operations.

We first keep only the highest remaining power of \( a_m \) in the \((r + 1)\)th row of \( \Phi_{N \times 4N} \). Then, from the original determinant \( \tau_n \), we can expand it according to the \((r + 1)\)th row and obtain
\[
\Phi_{N \times 4N} \sim \begin{bmatrix}
\hat{S}_1 & \hat{S}_0 & \cdots \\
\hat{S}_5 & \hat{S}_4 & \\
\vdots & \vdots & \\
\hat{S}_{m-5} & \hat{S}_{m-6} & \\
a_m \hat{S}_3 + \hat{S}_{m+3} & a_m \hat{S}_2 + \hat{S}_{m+2} & \\
\vdots & \vdots & \\
a_m \hat{S}_{m-3} + \hat{S}_{2m-2} & a_m \hat{S}_{m-4} + \hat{S}_{2m-3} & \\
a_m \hat{S}_{m+1} + \hat{S}_{2m+1} & a_m \hat{S}_m + \hat{S}_{2m} & \\
\vdots & \vdots & \\
a_m \hat{S}_{2m-5} + \hat{S}_{3m-5} & a_m \hat{S}_{2m-6} + \hat{S}_{3m-6} & \\
\left(\frac{1}{2!} - \frac{1}{3!}\right) a_m^2 \hat{S}_{m-1} + O(a_m) & \left(\frac{1}{2!} - \frac{1}{3!}\right) a_m^2 \hat{S}_{m-2} + O(a_m) & \\
\vdots & \vdots & \\
\left(\frac{1}{2!} - \frac{1}{3!}\right) a_m^2 \hat{S}_{2m-3} + O(a_m) & \left(\frac{1}{2!} - \frac{1}{3!}\right) a_m^2 \hat{S}_{2m-4} + O(a_m) & \\
\vdots & \vdots & \\
\end{bmatrix}
\]

(98)

Similar treatment can be applied to the matrix \(\Psi_{4N \times N}\). It can be observed that we have a similar situation to the inner region of 1st-type rogue wave with \(m = 4r + 2\). Finally, we can rewrite (90) into a \(2 \times 2\) block determinant

\[
\tau_n = \det \left( \begin{bmatrix} 
\tau_n^{[1,1]} & \tau_n^{[1,3]} \\
\tau_n^{[3,1]} & \tau_n^{[3,3]} 
\end{bmatrix} \right) \left[ 1 + O\left(a_m^{-1}\right) \right]
\]

(99)

where

\[
\tau_n^{[I,J]} = \left( m_{4I-I,4J-J}^{(n,I,J)} \right)_{1 \leq i \leq N_I, 1 \leq j \leq N_J}, \quad 1 \leq I, J \leq 3,
\]

(100)

and

\[
m_{i,j}^{(n,I,J)} = \sum_{v=0}^{\min(i,j)} \left[ \frac{|p_1|^2}{(p_0 + p_0^*)^2} \right]^v S_{i-v} \left( x_i^{+}(n) + vs \right) S_{j-v} \left( x_j^{-}(n) + vs^* \right).
\]

Note that the determinant \(\tau_n\) is always \((N-1) \times (N-1)\) and \(\tau_n\) in the inner region is always approximately a \(2 \times 2\) block matrix regardless of the values of \(N\) and \(m\), i.e., \(N_2 = 0\) in (16). Moreover, we remark that the internal parameters

\[
(\hat{a}_{1,n}, \hat{a}_{2,n}, \hat{a}_{3,n}, \hat{a}_{4,n}, \hat{a}_{5,n}, \hat{a}_{6,n} \ldots, \hat{a}_{4N_2-n,n}) , \quad n = 1, 3,
\]

are related to those in the original rogue wave as

\[
\hat{a}_{j,1} = \hat{a}_{j,3} = a_j + s_j, \quad j = 1, 2, 3, 5, 6, 7 \ldots, m - 1, m + 1, \ldots
\]
and
\[ \hat{a}_{m,1} = \hat{a}_{m,3} = s_m. \]

This completes the proof of Theorem 3.1 for the inner region. \qed

**Proof of Theorem 3.2** Since the proofs are similar for different \( i \in \{1, 2, 3, 4\} \), it suffices to present the proof for \( i = 1 \).

Assume \( |a_m| \) is large and other parameters are \( O(1) \). We first consider the situation when \( (x, t) \) is located in the outer region, i.e., \( \sqrt{x^2 + t^2} = O(1) \). Since the proof is very similar to Theorem 3.1, we only show the differences.

To begin with, we have
\[
S_j \left( x^+(n) + vs \right) = S_j \left( x_1^+, x_2^+, x_3^+, x_4^+, vs_5, x_6^+, x_7^+, x_8^+, x_9^+, vs_{10}, \ldots, x_m^+ + vs_m, \cdots \right)
\]
\[ = S_j(\nu) \left[ 1 + O \left( a_m^{-1/m} \right) \right], \tag{101} \]

where
\[ \nu = (p_0 x + 2p_0 p_1 it, 0, \ldots, 0, a_m, 0, \ldots). \tag{103} \]

This relation is the same as (64), but the values of \( p_0 \) and \( p_1 \) are different from the three-component NLS equation. Then, after some calculations similar to the proof of Theorem 3.1, we find that the highest-order term in \( a_m \) for \( \tau_n \) is
\[
\tau_n = |\alpha|^2 |a_m|^{4N(N+1)/m} \left| W_N^{[m,5,4]}(z) \right|^2 \left[ 1 + O \left( a_m^{-1/m} \right) \right], \tag{104} \]

where
\[ \alpha = h_0^{N(N-1)/2}(c_N^{[m,5,4]})^{-1}, \quad h_0 = p_1 / (p_0 + p_0^*), \quad z = a_m^{-1/m}(p_0 x + 2p_0 p_1 it). \]

Note that the order of \( a_m \) is changed from \( 3N(N+1)/m \) in the three-component case to \( 4N(N+1)/m \). Thus, the solutions
\[ u_{1,N_1}(x, t), \quad u_{2,N_1}(x, t), \quad u_{3,N_1}(x, t), \quad u_{4,N_1}(x, t) \]

are the plane-wave backgrounds, except at or near \((\tilde{x}_0, \tilde{t}_0)\), where
\[
z_0 = a_m^{-1/m} \left( p_0 \tilde{x}_0 + 2p_0 p_1 i\tilde{t}_0 \right) \tag{105} \]
is a root of \( W_N^{[m,5,4]}(z) \).

In what follows, we show that when \( (x, t) \) is contained in a small neighborhood of \((\tilde{x}_0, \tilde{t}_0)\) given by (105), the underlying rogue wave is approximately a fundamental rogue wave. Denote by
\[ \hat{x}_2^+(x, t) = p_2 x + \left( 2p_0 p_2 + p_1^2 \right)(it). \]

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which contains the dominant terms of \( x_2^+ (x, t) \) in (12) with the index “\( I \)” removed. Then, for \((x, t)\) in the neighborhood of \((\bar{x}_0, \bar{t}_0)\), we have a more refined asymptotics for \( S_j (x^+ (n) + v s) \)

\[
S_j (x^+ (n) + v s) = [S_j (\tilde{\nu}) + \tilde{x}_2^+ (\bar{x}_0, \bar{t}_0) S_{j-2} (\tilde{\nu})] \left[ 1 + O \left( a_m^{-2/m} \right) \right], \quad |a_m| \gg 1,
\]

where

\[
\tilde{\nu} = (p_0 x + 2 p_0 p_1 t + n_1 \theta_{11} + n_2 \theta_{12} + n_3 \theta_{13} + n_4 \theta_{14}, 0, \cdots, 0, a_m, 0, \cdots).
\]

Here, the normalization of \( a_1 = 0 \) has been utilized. Next, we rewrite \( \tau_n \) in a similar form as (69) by means of Laplace expansion. Further, the contribution from the first index choice of \( \nu_j = j - 1 \) can be expressed as

\[
\alpha a_m \frac{2N(N+1)-1}{m} \left[ p_0 (x - \bar{x}_0) + 2 p_0 p_1 (t - \bar{t}_0) + \sum_{k=1}^{4} n_k \theta_{1k} + \tilde{\Delta}_1 \right] \left[ W_N^{[m, 5, 4]} \right]' (z_0) \left[ 1 + O \left( a_m^{-1/m} \right) \right]
\]

where

\[
\tilde{\Delta}_1 = \frac{\tilde{x}_2^+ (\bar{x}_0, \bar{t}_0)}{a_m^{1/m}} \sum_{j=1}^{N} \det_{1 \leq i \leq N} \left[ p_{si-1}^{[m]} (z_0), \cdots, p_{si-j-2}^{[m]} (z_0), \cdots, p_{si-N}^{[m]} (z_0) \right] \left[ W_N^{[m, 5, 4]} \right]' (z_0)
\]

and \( \tilde{\Delta}_1 = O(1) \) as \( \tilde{x}_2^+ (\bar{x}_0, \bar{t}_0) = O \left( \left| a_m^{1/m} \right| \right) \). By absorbing \( \tilde{\Delta}_1 \) into \((\bar{x}_0, \bar{t}_0)\) (Yang and Yang 2023), we obtain

\[
\alpha a_m \frac{2N(N+1)-1}{m} \left[ p_0 (x - \bar{x}_0) + 2 p_0 p_1 (t - \bar{t}_0) + \sum_{k=1}^{4} n_k \theta_{1k} \right] \left[ W_N^{[m, 5, 4]} \right]' (z_0) \left[ 1 + O \left( a_m^{-1/m} \right) \right].
\]

where \( \bar{x}_0 \) and \( \bar{t}_0 \) are given in Theorem 3.2.

For the second index choice, i.e., \( \nu = (0, 1, \cdots, N - 2, N) \), the dominant terms in \( a_m \) can be calculated in a similar way as (82), that is,

\[
\frac{N(N-1)+2}{h_0^2} \frac{2N(N+1)-1}{m} \det_{1 \leq i \leq N} \left[ p_{si-1}^{[m]} (z_0), \cdots, p_{si-N}^{[m]} (z_0) \right] \left[ 1 + O \left( a_m^{-1/m} \right) \right].
\]
Since \( p_j^{[m]}(z) = \left[p_j^{[m]}\right]'(z) \), the above term can be expressed as

\[
h_0 \alpha a_m^{\frac{2N(N+1)-1}{m}} \left[ W_N^{[m,5,4]} \right]'(z_0) \left[ 1 + O \left( a_m^{-1/m} \right) \right].
\] (112)

Summarizing the above two contributions, we conclude that

\[
\tau_n(x, t) = |\alpha|^2 \left[ W_N^{[m,5,4]} \right]'(z_0) \left| a_m \right|^{\frac{2N(N+1)-1}{m}} \times \left( \begin{array}{l}
[p_1 (x - \tilde{x}_0) + 2i p_0 p_1 (t - \tilde{t}_0) + n_1 \theta_{11} + n_2 \theta_{12} + n_3 \theta_{13} + n_4 \theta_{14}] \\
[p_1^* (x - \tilde{x}_0) - 2i p_0^* p_1^* (t - \tilde{t}_0) - n_1 \theta_{11}^* - n_2 \theta_{12}^* - n_3 \theta_{13}^* - n_4 \theta_{14}^*] + |h_0|^2
\end{array} \right)
\]
\[
\left[ 1 + O \left( a_m^{-1/m} \right) \right].
\] (113)

Thus, the proof for outer region is completed. \(\square\)

In order to study the patterns of the 1st-type rogue waves of the four-component NLS rogue waves under the condition \(|a_m| \gg 1\) in the inner region with \(x^2 + t^2 = O(1)\), we first rewrite the determinant \(\tau_n\) as a \(6N \times 6N\) determinant

\[
\tau_n = \begin{vmatrix}
O_{N \times N} & \Phi_{N \times 5N} \\
-\Psi_{5N \times N} & I_{5N \times 5N}
\end{vmatrix},
\] (115)

where

\[
\Phi_{i,j} = \left( \frac{p_1}{p_0 + p_0^*} \right)^{j-1} S_{5i-j} \left[ x^+(n) + (j-1)s \right],
\]

\[
\Psi_{i,j} = \left( \frac{p_1^*}{p_0 + p_0^*} \right)^{i-1} S_{5j-i} \left[ x^-(n) + (i-1)s^* \right].
\]

Then, we can apply (85) and (86) to express each element in (115) into a polynomial in \(a_m\) explicitly. Notice there are in total four cases to consider, i.e., \(m \equiv j \mod 5\), \(j = 1, 2, 3, 4\). Since the proofs for all cases are similar, it suffices to provide the proof for \(j = 1\). To determine the highest-order term in \(a_m\) of \(\tau_n\), we can use similar argument as that in Yang and Yang (2021c, 2023) to perform row and column operations. After these operations, \(\tau_n\) can be reduced to the form

\[
\tau_n = \beta |a_m|^\gamma \begin{vmatrix}
O_{\tilde{N} \times \tilde{N}} & \tilde{\Phi}_{\tilde{N} \times 5\tilde{N}} \\
-\tilde{\Psi}_{\tilde{N} \times \tilde{N}} & I_{\tilde{N} \times \tilde{N}}
\end{vmatrix} \left[ 1 + O \left( a_m^{-1} \right) \right],
\] (116)
where $\beta \neq 0$, $\gamma > 0$ are constants, $\overline{N}_4 = \sum_{n=1}^{4} N_{n,4}$, $\overline{N} = \max_{1 \leq i \leq 4} (5N_{1,4} - i + 1)$, and $\nu = N - \overline{N}_4$. Since the rogue wave solutions are independent of the constants $\beta$ and $\gamma$, we can rewrite (116) into a $4 \times 4$ block determinant

$$\tau_n = \det \left( \begin{array}{cccc} \tau_n^{[1,1]} & \tau_n^{[1,2]} & \tau_n^{[1,3]} & \tau_n^{[1,4]} \\ \tau_n^{[2,1]} & \tau_n^{[2,2]} & \tau_n^{[2,3]} & \tau_n^{[2,4]} \\ \tau_n^{[3,1]} & \tau_n^{[3,2]} & \tau_n^{[3,3]} & \tau_n^{[3,4]} \\ \tau_n^{[4,1]} & \tau_n^{[4,2]} & \tau_n^{[4,3]} & \tau_n^{[4,4]} \end{array} \right) \left[ 1 + O \left( a_m^{-1} \right) \right]$$

(118)

where

$$\tau_n^{[i,j]} = \left( m_{S_{i-j}}^{(n,1,J)} \right)_{1 \leq i \leq N_{1,4}, 1 \leq j \leq N_{J,4}}$$

(119)

and

$$m_{i,j}^{(n,1,J)} = \sum_{v=0}^{\min(i,j)} \left[ \frac{|p_1|^2}{(p_0 + p_v^0)^2} \right]^v S_{i-v} \left( y^+(n) + \nu_0 s + \nu s \right) S_{j-v} \left( y^-(n) + \nu_0 s^* + \nu s^* \right).$$

(120)

Finally, the determinant in (118) becomes a $(N_{1,4}, N_{2,4}, N_{3,4}, N_{4,4})$-th-order rogue wave of the four-component NLS equation, and the internal parameters

$$\left( \tilde{a}_{1,n}, \tilde{a}_{2,n}, \tilde{a}_{3,n}, \tilde{a}_{4,n}, \tilde{a}_{6,n} \ldots, \tilde{a}_{5N_{1,4},-n,n} \right), \quad n = 1, 2, 3, 4,$$

are related to those in the original rogue wave as

$$\tilde{a}_{j,1} = \tilde{a}_{j,2} = \tilde{a}_{j,3} = \tilde{a}_{j,4} = a_j + \left( N - \overline{N}_4 \right) s_j, \quad j = 1, 2, 3, 4, 6, 7 \ldots.$$

From (118), we deduce that the approximation error of this lower-order rogue wave is $O \left( |a_m|^{-1} \right)$. This completes the proof of Theorem 3.2 for the inner region.
6 Conclusion

In summary, we have constructed rogue waves of the vector (or $M$-component) NLS equation (1) and analyzed their patterns for $M = 3, 4$. These solutions are expressed in terms of Gram-type determinants of $K \times K$ block matrices ($1 \leq K \leq M$) with index jumps of $M + 1$ via Kadomtsev–Petviashvili reduction technique. One crucial step in this process is solving a system of algebraic equations (see Lemma 2.1 and its proof). The rogue wave patterns corresponding to $M = 3, 4$ and $K = 1$ have been investigated comprehensively. We find that when specific internal parameters are large enough, these patterns are described by new polynomial hierarchies, i.e., the generalized Wronskian–Hermite polynomials, in contrast with the scalar NLS equation and the Manakov system. Since the Yablonskii–Vorob’ev polynomial hierarchy and Okamoto polynomial hierarchies are special cases of the generalized Wronskian–Hermite polynomials, our results have unified rogue wave patterns of the scalar NLS equation and the vector NLS equation for $M = 2, 3, 4$. It is worth noting that the case $M = 3$ presents a unique feature as, in certain cases, the sizes of the Gram-type determinants cannot be reduced in the approximation of inner regions.

The rogue wave patterns for $M = 3, 4$ exhibit very rich structures similar to the Manakov system (Yang and Yang 2023), and they are, in general, distorted from root structures of the generalized Wronskian–Hermite polynomials. The predicted rogue wave patterns have been compared with true solutions, and excellent agreement is achieved. As pointed out in Yang and Yang (2021a, 2023), universal rogue wave patterns, which depend on the index jumps, exist in integrable systems. We expect that the patterns uncovered in the present paper will appear in many other systems and thus are universal, as long as the corresponding Schur polynomials have index jumps of 4 or 5.

Appendix A

In this appendix, we provide the proof of Lemma 2.1. Assume $\xi$ is a root of $\mathcal{R}_M(z) = 0$ of multiplicity $M$ with $\Im(\xi) \neq 0$, then we have

$$\mathcal{R}^{(n)}_M(\xi) = 0, \quad n = 0, 1, 2, \ldots, M - 1, \quad (121)$$

where

$$\mathcal{R}^{(m)}_M(\xi) = (-1)^m (m + 1)! \sum_{j=1}^{M} \frac{r_j}{(\xi + k_j)^{m+2}}, \quad m \geq 1.$$ 

The system of equations (121) is linear in $r_j$, $j = 1, 2, \ldots, M$, so we can solve for them and obtain

$$(\xi + k_j)^{M+1} = -\frac{1}{2} \prod_{i=1}^{M} (k_j - k_i) r_j. \quad (122)$$
Denote by
\[
\xi = x + iy, \quad -\frac{1}{2} \prod_{i=1, i\neq j}^{M} (k_j - k_i)r_j = \lambda_j^{M+1} \exp(i\theta_j \pi), \quad j = 1, 2, \ldots, M, \tag{123}
\]
where \(\lambda_j > 0\), \(x, y\) are real, \(y \neq 0\) and
\[
\theta_j = \begin{cases} 
0, & \text{if } -\frac{1}{2} \prod_{i=1, i\neq j}^{M} (k_j - k_i)r_j > 0, \\
1, & \text{if } -\frac{1}{2} \prod_{i=1, i\neq j}^{M} (k_j - k_i)r_j < 0,
\end{cases} \tag{124}
\]
then we deduce from \(122\) that, for each \(k_j\), there exits \(l_j \in \{0, 1, \ldots, M\}\) such that
\[
x + k_j + iy = \begin{cases} 
\lambda_j \exp[2l_j \pi i/(M + 1)], & \text{if } \theta_j = 0, \\
\lambda_j \exp[(2l_j + 1) \pi i/(M + 1)], & \text{if } \theta_j = 1,
\end{cases} \tag{125}
\]
Comparing both sides of \(125\) gives
\[
y = \begin{cases} 
\lambda_j \sin[2l_j \pi /(M + 1)], & \text{if } \theta_j = 0, \\
\lambda_j \sin[(2l_j + 1) \pi /(M + 1)], & \text{if } \theta_j = 1.
\end{cases} \tag{126}
\]
This implies that all the corresponding \(\sin[2l_j \pi /(M + 1)]\) or \(\sin[(2l_j + 1) \pi /(M + 1)]\), \(j = 1, 2, \ldots, M\), should have the same sign. Without loss of generality, we may assume \(y > 0\). Note that the set
\[
\{1, \exp[\pi i/(M + 1)], \exp[2\pi i/(M + 1)], \ldots, \exp[2M\pi i/(M + 1)], \exp[(2M + 1)\pi i/(M + 1)]\}
\]
contains exactly \(M\) elements with positive imaginary parts, which are
\[
\exp[\pi i/(M + 1)], \quad \exp[2\pi i/(M + 1)], \ldots, \quad \exp[M\pi i/(M + 1)]. \tag{127}
\]
Since the \(k_j\)’s are distinct, it then follows that
\[
x + k_j + iy = \lambda_j \exp[\sigma_j \pi i/(M + 1)], \tag{129}
\]
where \((\sigma_1, \sigma_2, \ldots, \sigma_M)\) can be any permutation of the set \(\{1, 2, \ldots, M\}\). Without loss of generality, we may take
\[
\sigma_j = j, \tag{130}
\]
where \(j = 1, 2, \ldots, M\). In this circumstance, we have \(\theta_j = [1 + (-1)^{j+1}]/2\) and
\[
x = \lambda_j \cos[j \pi /(M + 1)] - k_j, \tag{131}
\]
\[
y = \lambda_j \sin[j \pi /(M + 1)], \tag{132}
\]
and hence
\[ \lambda_j = \lambda_1 \frac{\sin[j \pi/(M+1)]}{\sin[j \pi/(M+1)]}, \quad (133) \]
\[ r_j = 2(-1)^{j+1} \prod_{i=1, i \neq j}^{M} (k_j - k_i)^{-1} \left( \frac{\lambda_1 \sin[j \pi/(M+1)]}{\sin[j \pi/(M+1)]} \right)^{M+1}. \quad (136) \]

As $\mathcal{R}_M(z)$ is a rational function with real coefficients, it is clear that $\xi^*$ is a root of $\mathcal{R}_M(z) = 0$ of multiplicity $M$ as well. This completes the proof.

**Appendix B**

In this appendix, we apply Hirota’s bilinear method to derive rogue wave solutions of the vector NLS equation (1) presented in Theorem 2.2 based on the KP reduction technique. For convenience, we only consider the case when $t_n$ given in (18) consists of $M \times M$ block matrices, i.e., $K = M$, as other cases can be treated in a similar manner. In such case, we have $I_j = j (j = 1, 2, \ldots, M)$ in (18).

We first transform the vector NLS equation (1) into a set of bilinear equations
\[
\left( D_x^2 + \sum_{j=1}^{M} \sigma_j \rho_j^2 \right) f \cdot f = \sum_{j=1}^{M} \sigma_j \rho_j^2 g_j g_j^*,
\]
\[
(iD_t + D_x^2 + 2ik_j D_x) g_j \cdot f = 0, \quad j = 1, 2, \ldots, M,
\]
under the nonzero boundary condition at $\pm \infty$ by the variable transformation
\[
u_j = \rho_j g_j f e^{i(k_j x + w_j t)}, \quad j = 1, 2, \ldots, M,
\]
where $w_j = \sum_{j=1}^{M} \sigma_j \rho_j^2 - k_j^2$, $f$ is a real-valued function, $g_j$ is a complex-valued function, and $D$ is the Hirota’s bilinear operator (Hirota 2004) defined by
\[
D_x^n D_t^m f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n \left[ f(x, t) g(x', t') \right]_{x' = x, t' = t}.
\]
Next we define
\[
m^n = \frac{1}{p + q} \left[ \prod_{j=1}^{M} \left( -\frac{p - ik_j}{q + ik_j} \right)^{n_j} \right] e^{\xi + \eta},
\]
\[
\xi = px + p^2 y + \sum_{j=1}^{M} \frac{1}{p - ik_j} v_j + \xi_0(p),
\]
\[
\eta = qx - q^2 y + \sum_{j=1}^{M} \frac{1}{q + ik_j} v_j + \eta_0(q),
\]
where \(n = (n_1, n_2, \ldots, n_M)\) with \(n_j\) being integers, \(p, q, v_j\) are arbitrary complex constants, \(j = 1, 2, \ldots, M\), and \(\xi_0(p), \eta_0(q)\) are arbitrary functions of \(p\) and \(q\), respectively. Let \(A_i\) and \(B_j\) be differential operators of order \(i\) and \(j\), respectively, defined by
\[
A_i(p) = \frac{1}{i!} \left[ f_1(p) \partial_p \right]^i, \quad B_j(q) = \frac{1}{j!} \left[ f_2(q) \partial_q \right]^j,
\]
where \(f_1(p), f_2(q)\) are arbitrary functions of \(p\) and \(q\), respectively. Then, it can be calculated that (Ohta and Yang 2012) the determinant
\[
\tau_n = \det_{1 \leq i, \mu \leq N} \left( m^n_{i, \nu, \mu} \right)
\]
where \((i_1, i_2, \ldots, i_N)\) and \((j_1, j_2, \ldots, j_N)\) are arbitrary sequences of indices, and the matrix element \(m^n_{ij}\) is defined as
\[
m^n_{ij} = A_i B_j m^n,
\]
would satisfy the bilinear equations
\[
\left( \frac{1}{2} D_x D_{v_j} - 1 \right) \tau_n \cdot \tau_n = -\tau_{n,1} \tau_{n,-1}, \quad j = 1, 2, \ldots, M,
\]
\[
\left( D_x^2 - D_y + 2ik_j D_x \right) \tau_{n,1} \cdot \tau_n = 0, \quad j = 1, 2, \ldots, M,
\]
where \(n_{j,i} = n + i \times e_j\), and \(e_j\) is the standard unit vector in \(\mathbb{R}^M\).

In what follows, we will establish the reductions from the bilinear equations (140) in the KP hierarchy to the bilinear equations (137), thereby obtaining rogue wave solutions of the vector NLS equation (1). This procedure consists of several steps.

i) Dimension reduction
Note that
\[
\left( 2\partial_x + \sum_{k=1}^{M} \sigma_k \rho_k^2 \partial_{v_k} \right) m^n_{ij} = A_i B_j \left[ G_M(p) + H_M(q) \right] m^n,
\]
where
\[ G_M(p) = \sum_{j=1}^{M} \frac{\sigma_j \rho_j^2}{p - i k_j} + 2p, \quad H_M(q) = \sum_{j=1}^{M} \frac{\sigma_j \rho_j^2}{q + i k_j} + 2q. \] (142)

This implies that
\[
\left( 2\partial_x + \sum_{k=1}^{M} \sigma_k \rho_k^2 \partial v_k \right) m_{ij}^{n} = \sum_{\mu=0}^{i} \frac{1}{\mu!} \left[ (f_1 \partial_p)^\mu G_M(p) \right] m_{i-\mu,j}^{n} + \sum_{l=0}^{j} \frac{1}{l!} \left[ (f_2 \partial_q)^l H_M(q) \right] m_{i,j-l}^{n}. \] (143)

Then, we can use the method introduced in Yang and Yang (2021b) to find \( f_1(p) \) and \( f_2(q) \) such that
\[
(f_1 \partial_p)^{M+1} G_M(p) = G_M(p), \quad (f_2 \partial_q)^{M+1} H_M(q) = H_M(q). \] (144)

Specifically, by expressing \( f_1 \) in the form
\[ f_1(p) = \frac{U(p)}{U'(p)}, \] (145)

we can convert (144) into
\[ \partial_{\ln U}^{M+1} G_M(p) = G_M(p). \]

Normalize \( U(p_0) = 1 \), then, under the condition that \( p_0 \) is a root of \( G_M \) of multiplicity \( M \), this equation admits the unique solution
\[
G_M(p) = \frac{G_M(p(0))}{M+1} \sum_{n=1}^{M+1} \exp \left( \exp \left( \frac{2n \pi i}{M+1} \right) \ln U(p) \right). \] (146)

Then, \( f_1(p) \) can be obtained by using the relation (145) after solving this equation for \( U \). In a similar way, \( f_2(q) \) can be found.

Choosing \( q_0 = p_0^* \) and using (144) and the assumption that \( p_0 \) is a root of \( G'_M(p) = 0 \) of multiplicity \( M \), the equation (143) reduces to
\[
\left( 2\partial_x + \sum_{k=1}^{M} \sigma_k \rho_k^2 \partial v_k \right) m_{ij}^{n} \bigg|_{p=p_0,q=q_0} = G_M(p_0) \sum_{\mu=0}^{i} \frac{1}{\mu!} m_{i-\mu,j}^{n} + H_M(q_0) \sum_{l=0}^{j} \frac{1}{l!} m_{i,j-l}^{n} \bigg|_{p=p_0,q=q_0}. \] (147)
Let $N = N_1 + N_2 + \cdots + N_M$, where $N_j$, $j = 1, 2, \cdots, M$, are positive integers, and define the determinant $\tau_n$ by

$$\tau_n = \det \begin{pmatrix} \tau_{[1,1]} & \tau_{[1,2]} & \cdots & \tau_{[1,M]} \\ \tau_{[2,1]} & \tau_{[2,2]} & \cdots & \tau_{[2,M]} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{[M,1]} & \tau_{[M,2]} & \cdots & \tau_{[M,M]} \end{pmatrix},$$

(148)

where

$$\tau_{[I,J]} = \text{mat}_{1 \leq i \leq N_I, 1 \leq j \leq N_J} \left( m_{(M+1)i-I,(M+1)j-J}^{n} \right) \bigg|_{p=p_0, q=q_0, \xi_0=\xi_0, I_0=\eta_0, J_0}, \quad 1 \leq I, J \leq M,$$

(149)

and $m_{i,j}^n$ is given by (139).

With (147), we can use similar argument as in Ohta and Yang (2012) to show that the determinant $\tau_n$ satisfies the dimensional reduction condition

$$\left( 2\partial_x + \sum_{k=1}^{M} \sigma_k \rho_k^2 \partial v_k \right) \tau_n = N \left[ G_M(p_0) + H_M(q_0) \right] \tau_n.$$

(150)

Therefore, we can use (150) to eliminate the variables $v_j$, $j = 1, 2, \cdots, M$, from the higher-dimensional bilinear system (140). As a result of this, we have

$$\left( D_x^2 + \sum_{j=1}^{M} \sigma_j \rho_j^2 \right) \tau_n \cdot \tau_n = \sum_{j=1}^{M} \sigma_j \rho_j^2 \tau_{n,j,1} \tau_{n,j,-1},$$

$$\left( iD_t + D_x^2 + 2ik_j D_x \right) \tau_{n,j,1} \cdot \tau_n = 0, \quad j = 1, 2, \cdots, M,$$

(151)

where $t = -iy$.

ii) Complex conjugate reduction

Impose the parameter constraint

$$\xi_{0,I} = \eta_{0,I}^*,$$

and in view of $p_0 = q_0^*$, we have $[f_1(p_0)]^* = f_2(q_0)$. It then follows that

$$\tau_n = \tau_{-n}^*.$$

(152)

Define

$$f = \tau_{n_0}, \quad g_j = \tau_{n_j}, \quad j = 1, 2, \cdots, M,$$

(153)
then the complex conjugacy condition (152) implies that $f$ is real. Therefore, from (151) and (152), we conclude that the functions $f$ and $g_j$ satisfy the bilinear system (137), thereby yielding rational solutions to the vector NLS equation (1) via the transformation (138).

iii) **Introduction of free parameters**

We apply the method proposed in Yang and Yang (2021b) to introduce free parameters in the following form

$$\xi_{0,l} = \sum_{n=1}^{\infty} a_{n,l} \ln^n U(p),$$

(154)

where $U(p)$ is defined by (145) and the $a_{n,l}$’s are free complex constants.

iv) **Simplification of solutions**

With the aid of the generator $D$ of the differential operators $(p\partial_p)^k (q\partial_q)^l$ given as

$$D = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{k^l}{k! l!} (p\partial_p)^k (q\partial_q)^l = \exp (\kappa p\partial_p + \lambda q\partial_q) = \exp (\kappa \partial_{\ln p} + \lambda \partial_{\ln q}),$$

(155)

we are able to simplify the solutions expressed by (153) using differential operators into the form of Schur polynomials as presented in Theorem 2.2. Since the computations are very similar to those in the three-wave system by Yang and Yang (2021b), we omit the details.

Thus, the proof of Theorem 2.2 is completed.

**Appendix C**

In the first part of this appendix, we provide the values of $N_1, N_2, N_3, N_4$ that appear in Theorem 2.4 in the following lemma.

**Lemma 6.1** The values of $N = (N_1, N_2, N_3, N_4)$ involved in Theorem 2.4 are characterized as follows.

- When $m \equiv 1 \mod 5$, we have

  \[ l = 4 : N \]

  $$\begin{align*}
  &= \{(N_0, 0, 0, 0), \quad 0 \leq N_0 \leq \left\lceil \frac{m}{5} \right\rceil \\
  &\quad (\left\lceil \frac{m}{5} \right\rceil, N_0 - \left\lceil \frac{m}{5} \right\rceil, 0, 0), \quad \left\lceil \frac{m}{5} \right\rceil + 1 \leq N_0 \leq 2 \left\lceil \frac{m}{5} \right\rceil \\
  &\quad (\left\lceil \frac{m}{5} \right\rceil, \left\lceil \frac{m}{5} \right\rceil, N_0 - 2 \left\lceil \frac{m}{5} \right\rceil), \quad 2 \left\lceil \frac{m}{5} \right\rceil + 1 \leq N_0 \leq 3 \left\lceil \frac{m}{5} \right\rceil \\
  &\quad (\left\lceil \frac{m}{5} \right\rceil, \left\lceil \frac{m}{5} \right\rceil, \left\lceil \frac{m}{5} \right\rceil, N_0 - 3 \left\lceil \frac{m}{5} \right\rceil), \quad 3 \left\lceil \frac{m}{5} \right\rceil + 1 \leq N_0 \leq 4 \left\lceil \frac{m}{5} \right\rceil \\
  &\quad (m - 1 - N_0, m - 1 - N_0, m - 1 - N_0, m - 1 - N_0), \quad 4 \left\lceil \frac{m}{5} \right\rceil + 1 \leq N_0 \leq m - 1 \end{align*} \]

- When $l = 3$, we have

  $$\begin{align*}
  &= \{(0, N_0, 0, 0), \quad 0 \leq N_0 \leq \left\lceil \frac{m}{5} \right\rceil \\
  &\quad (0, \left\lceil \frac{m}{5} \right\rceil, N_0 - \left\lceil \frac{m}{5} \right\rceil, 0), \quad \left\lceil \frac{m}{5} \right\rceil + 1 \leq N_0 \leq 2 \left\lceil \frac{m}{5} \right\rceil \\
  &\quad (0, \left\lceil \frac{m}{5} \right\rceil, \left\lceil \frac{m}{5} \right\rceil, N_0 - 2 \left\lceil \frac{m}{5} \right\rceil), \quad 2 \left\lceil \frac{m}{5} \right\rceil + 1 \leq N_0 \leq 3 \left\lceil \frac{m}{5} \right\rceil \\
  &\quad (\left\lceil \frac{m}{5} \right\rceil - 1, \left\lceil \frac{m}{5} \right\rceil - 1, \left\lceil \frac{m}{5} \right\rceil - 1, N_0 - 3 \left\lceil \frac{m}{5} \right\rceil - 1), \quad 3 \left\lceil \frac{m}{5} \right\rceil + 1 \leq N_0 \leq 4 \left\lceil \frac{m}{5} \right\rceil + 1 \\
  &\quad (m - 1 - N_0, m - 1 - N_0, m - 1 - N_0, m - 1 - N_0), \quad 4 \left\lceil \frac{m}{5} \right\rceil + 2 \leq N_0 \leq m - 1\end{align*} \]
When \( m \equiv 2 \mod 5 \), we have

\[
\begin{align*}
&l = 4 : \mathbb{N} \\
&\left\{
\begin{array}{l}
(N_0, 0, 0, 0), \\
\left(\frac{m}{5}, 0, N_0 - \left\lfloor \frac{m}{5} \right\rfloor, 0\right), \\
(N_0 - 2 \left\lfloor \frac{m}{5} \right\rfloor - 1, \left\lfloor \frac{m}{5} \right\rfloor, 0, \left\lfloor \frac{m}{5} \right\rfloor + 1)
\end{array}
\right. \\
&\left(\frac{m}{5}, 0, N_0 - 3 \left\lfloor \frac{m}{5} \right\rfloor - 1, \left\lfloor \frac{m}{5} \right\rfloor + 1
\right)
\end{align*}
\]

\[
\begin{align*}
&(m - 1 - N_0, m - 1 - N_0, m - 1 - N_0, m - 1 - N_0), \\
&4 \left\lfloor \frac{m}{5} \right\rfloor + 2 \leq N_0 \leq m - 1
\end{align*}
\]

When \( m \equiv 3 \mod 5 \), we have

\[
\begin{align*}
&l = 4 : \mathbb{N} \\
&\left\{
\begin{array}{l}
(N_0, 0, 0, 0), \\
\left(\frac{m}{5}, 0, N_0 - \left\lfloor \frac{m}{5} \right\rfloor\right), \\
(N_0 - 2 \left\lfloor \frac{m}{5} \right\rfloor - 1, \left\lfloor \frac{m}{5} \right\rfloor + 1
\end{array}
\right. \\
\left(\frac{m}{5}, 0, N_0 - 3 \left\lfloor \frac{m}{5} \right\rfloor - 2, \left\lfloor \frac{m}{5} \right\rfloor + 1
\right)
\end{align*}
\]

\[
\begin{align*}
&(m - 1 - N_0, m - 1 - N_0, m - 1 - N_0, m - 1 - N_0), \\
&4 \left\lfloor \frac{m}{5} \right\rfloor + 3 \leq N_0 \leq m - 1
\end{align*}
\]
l = 3 : \mathbb{N}
\begin{align*}
&\text{l, } N_0, 0, 0, \quad 0 \leq N_0 \leq [\frac{m}{5}] \\
&= (N_0 - [\frac{m}{5}] - 1, 0, [\frac{m}{5}], 0), \quad [\frac{m}{5}] + 1 \leq N_0 \leq 2[\frac{m}{5}] + 1 \\
&= (\lceil \frac{m}{5} \rceil, N_0 - 2[\frac{m}{5}] - 2, [\frac{m}{5}], [\frac{m}{5}] + 1), \quad 3[\frac{m}{5}] + 3 \leq N_0 \leq 4[\frac{m}{5}] + 2 \\
&= (m - 1 - N_0, m - 1 - N_0, m - 1 - N_0, m - N_0), \quad 4[\frac{m}{5}] + 3 \leq N_0 \leq m - 1
\end{align*}

l = 2 : \mathbb{N}
\begin{align*}
&\text{l, } N_0, 0, 0, \quad 0 \leq N_0 \leq [\frac{m}{5}] + 1 \\
&= (N_0 - [\frac{m}{5}] - 1, 0, [\frac{m}{5}], 0), \quad [\frac{m}{5}] + 2 \leq N_0 \leq 2[\frac{m}{5}] + 1 \\
&= (\lceil \frac{m}{5} \rceil, N_0 - 2[\frac{m}{5}] - 2, [\frac{m}{5}], [\frac{m}{5}] + 1), \quad 3[\frac{m}{5}] + 3 \leq N_0 \leq 4[\frac{m}{5}] + 2 \\
&= (m - 1 - N_0, m - N_0, m - N_0, m - N_0), \quad 4[\frac{m}{5}] + 4 \leq N_0 \leq m - 1
\end{align*}

l = 1 : \mathbb{N}
\begin{align*}
&\text{l, } N_0, 0, 0, \quad 0 \leq N_0 \leq [\frac{m}{5}] + 1 \\
&= (N_0 - [\frac{m}{5}] - 1, [\frac{m}{5}], 0, 0), \quad [\frac{m}{5}] + 2 \leq N_0 \leq 2[\frac{m}{5}] + 1 \\
&= (N_0 - 2[\frac{m}{5}] - 2, [\frac{m}{5}], [\frac{m}{5}], 0), \quad 2[\frac{m}{5}] + 2 \leq N_0 \leq 3[\frac{m}{5}] + 2 \\
&= (N_0 - 3[\frac{m}{5}] - 3, [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], 0), \quad 3[\frac{m}{5}] + 3 \leq N_0 \leq 4[\frac{m}{5}] + 3 \\
&= (m - 1 - N_0, m - N_0, m - N_0, m - N_0, m - 1 - N_0), \quad 4[\frac{m}{5}] + 4 \leq N_0 \leq m - 1
\end{align*}

\textbf{\textbullet} When \( m \equiv 4 \mod 5 \), we have
\begin{align*}
l = 4 : \mathbb{N}
&\text{l, } N_0, 0, 0, \quad 0 \leq N_0 \leq [\frac{m}{5}] + 1 \\
&= (N_0 - [\frac{m}{5}] - 1, [\frac{m}{5}], 1, 0, 0), \quad [\frac{m}{5}] + 2 \leq N_0 \leq 2[\frac{m}{5}] + 1 \\
&= (N_0 - 2[\frac{m}{5}] - 2, [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], 0), \quad 2[\frac{m}{5}] + 2 \leq N_0 \leq 3[\frac{m}{5}] + 2 \\
&= (N_0 - 3[\frac{m}{5}] - 3, [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], 0), \quad 3[\frac{m}{5}] + 3 \leq N_0 \leq 4[\frac{m}{5}] + 3 \\
&= (m - 1 - N_0, m - N_0, m - N_0, m - 1 - N_0, m - 1 - N_0), \quad 4[\frac{m}{5}] + 4 \leq N_0 \leq m - 1
\end{align*}

l = 3 : \mathbb{N}
\begin{align*}
&\text{l, } N_0, 0, 0, \quad 0 \leq N_0 \leq [\frac{m}{5}] + 1 \\
&= (N_0 - [\frac{m}{5}] - 1, [\frac{m}{5}], 1, 0, 0), \quad [\frac{m}{5}] + 2 \leq N_0 \leq 2[\frac{m}{5}] + 1 \\
&= (N_0 - 2[\frac{m}{5}] - 2, [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], 0), \quad 2[\frac{m}{5}] + 2 \leq N_0 \leq 3[\frac{m}{5}] + 2 \\
&= (N_0 - 3[\frac{m}{5}] - 3, [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], 0), \quad 3[\frac{m}{5}] + 3 \leq N_0 \leq 4[\frac{m}{5}] + 3 \\
&= (m - 1 - N_0, m - 1 - N_0, m - 1 - N_0, m - 1 - N_0, 0), \quad 4[\frac{m}{5}] + 4 \leq N_0 \leq m - 1
\end{align*}

l = 2 : \mathbb{N}
\begin{align*}
&\text{l, } N_0, 0, 0, \quad 0 \leq N_0 \leq [\frac{m}{5}] + 1 \\
&= (N_0 - [\frac{m}{5}] - 1, [\frac{m}{5}], 1, 0, 0), \quad [\frac{m}{5}] + 2 \leq N_0 \leq 2[\frac{m}{5}] + 2 \\
&= (N_0 - 2[\frac{m}{5}] - 2, [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], 0), \quad 2[\frac{m}{5}] + 3 \leq N_0 \leq 3[\frac{m}{5}] + 2 \\
&= (N_0 - 3[\frac{m}{5}] - 3, [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], 0), \quad 3[\frac{m}{5}] + 3 \leq N_0 \leq 4[\frac{m}{5}] + 3 \\
&= (m - 1 - N_0, m - 1 - N_0, m - 1 - N_0, m - N_0, m - N_0), \quad 4[\frac{m}{5}] + 4 \leq N_0 \leq m - 1
\end{align*}

l = 1 : \mathbb{N}
\begin{align*}
&\text{l, } N_0, 0, 0, \quad 0 \leq N_0 \leq [\frac{m}{5}] + 1 \\
&= (N_0 - [\frac{m}{5}] - 1, [\frac{m}{5}], 1, 0, 0), \quad [\frac{m}{5}] + 2 \leq N_0 \leq 2[\frac{m}{5}] + 2 \\
&= (N_0 - 2[\frac{m}{5}] - 2, [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], 0), \quad 2[\frac{m}{5}] + 3 \leq N_0 \leq 3[\frac{m}{5}] + 2 \\
&= (N_0 - 3[\frac{m}{5}] - 3, [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], [\frac{m}{5}], 0), \quad 3[\frac{m}{5}] + 3 \leq N_0 \leq 4[\frac{m}{5}] + 3 \\
&= (m - 1 - N_0, m - N_0, m - N_0, m - N_0, m - N_0), \quad 4[\frac{m}{5}] + 4 \leq N_0 \leq m - 1.
\end{align*}

In the second part of this appendix, we will prove Theorems 2.3 and 2.4 for root structures of the generalized Wronskian–Hermite polynomials \( W_N^{[m,k,l]} \). We only pro-
vide the proof for \( k = 5, l = 4 \) and \( m \equiv 1, 2, 3, 4 \) mod 5, as other cases can be proved in a similar manner.

Firstly, we define a new class of special Schur polynomials \( S_j^m(z; a) \) and the polynomials \( \hat{W}_N^{[m,5,4]}(z; a) \) as

\[
\sum_{j=0}^{\infty} S_j^m(z; a) e^j = \exp \left[ z e + a e^m \right],
\]

(156)

\[
\hat{W}_N^{[m,5,4]}(z; a) = c_N^{[m,5,4]},
\]

(157)

where \( a \) is a parameter, \( c_N^{[m,5,4]} \) is a constant defined in (36), and \( S_j^m(z; a) \equiv 0 \) when \( j < 0 \). Compared with the generalized Wronskian–Hermite polynomials \( W_N^{[m,k,l]} \), the polynomials \( \hat{W}_N^{[m,5,4]}(z; a) \) have a new parameter \( a \). Note that the polynomials \( S_j^m(z; a) \) are related to \( p_j^m(z) \) by

\[
S_j^m(z; a) = a^{j/m} p_j^m(\hat{z}), \quad \hat{z} = a^{-1/m} z.
\]

(158)

In addition, we have

\[
\hat{W}_N^{[m,5,4]}(z; a) = a^{2N(N+1)/m} W_N^{[m,5,4]}(\hat{z}).
\]

(159)

From (158) and (159), we find that each term in the polynomial \( \hat{W}_N^{[m]}(z; a) \) is a constant multiple of \( a^s z^k \) and \( k + ms = 2N(N + 1) \). This indicates that when the power of \( a \) is larger, the power of \( z \) is lower. Thus, to find the lowest order of \( z \), it suffices to find the highest order of \( a \). To this end, we rewrite the polynomials \( S_j^m(z; a) \) as

\[
S_j^m(z; a) = \sum_{n=0}^{[j/m]} \frac{a^n}{n!(j - nm)} z^{j-nm}
\]

(160)

and substitute (160) into the determinant (156). Note that coefficients of each \( a^s z^k \) in each row are proportional to each other, thus we can ignore them. In particular, for
\( m = 5r + 1 \), where \( r \) is positive integer, we get

\[
\hat{W}_N^{m, 5, 4}(z; \alpha) \sim c_N
\]

\[
\begin{pmatrix}
  \cdots \\
  \cdots \\
  z^{5r-1} & \cdots \\
  az^3 & \cdots \\
  \vdots & \ddots \\
  az^{5r-2} + z^{5r-2+m} & \cdots & az^{5r-3} + z^{5r-3+m} & \cdots \\
  a^2z^2 + az^{2+m} & \cdots & a^2z^1 + az^{1+m} & \cdots \\
  a^3z^1 + a^2z^{1+m} & \cdots & a^3z^0 + a^2z^m & \cdots \\
  a^4z^0 + a^3z^m & \cdots & a^4z^{1} & \cdots \\
  a^4z^{5r} + a^3z^{5r+m} & \cdots & a^4z^{5r-1} + a^3z^{5r-1+m} & \cdots
\end{pmatrix}
\]

(161)

Next, we perform row operations to (161), which consist of several steps.

1. Note that the coefficients of the highest-order terms in \( a \) in the first column of (161) are periodic and one period is given by

\[
\begin{align*}
  &z^4, \ldots, z^{5r-1}, \\
  &z^3, \ldots, z^{5r-2}, \\
  &z^2, \ldots, z^{5r-3}, \\
  &z^1, \ldots, z^{5r-4}, \\
  &z^0, \ldots, z^{5r}.
\end{align*}
\]

(162)

According to this periodicity, we divide the determinants (161) into \( \lfloor N/m \rfloor \) block matrices of size \( m \times N \) and one \( N_0 \times N \) block matrix, where \( N_0 \equiv N \mod m \). In addition, we divide the first column in each block into five parts, which have distinct initial powers in \( z \), and the difference of the powers in \( z \) of consecutive terms in each part is 5. We denote the number of parts starting with power \( j \) by \( N_{5-j}, j = 1, \ldots, 4 \).

2. We are only concerned with the first column of (161), since other columns have similar structures. According to the above discussions, we may use each part of the first block to cancel the highest-order terms in \( a \) of the corresponding parts for the subsequent blocks. After the first round row operations, the coefficients of the highest-order terms in \( a \) in the first column of the second block become

\[
\begin{align*}
  &z^{4+m}, \ldots, z^{5r-1+m}, \\
  &z^{3+m}, \ldots, z^{5r-2+m}, \\
  &z^{2+m}, \ldots, z^{5r-3+m}, \\
  &z^{1+m}, \ldots, z^{5r-4+m}, \\
  &z^{m}, \ldots, z^{5r+m}.
\end{align*}
\]

(163)
and from the third to the last blocks, the corresponding coefficients change to $z^{i+m}$ from $z^i$. In the second round, we can use the second block to cancel the highest-order terms in $a$ of the blocks below. Then, we continue this process until the last block. At the end of these operations, we can exchange the rows in each block such that the highest-order terms in $a$ of the first column are

$$z^0, z^1, z^2, \ldots, z^{5r}, z^{m}, z^{m+1}, z^{m+2}, \ldots, z^{km}, z^{km+1}, z^{km+2}, \ldots, z^{km+5r}, \ldots.$$  

and the determinant (161) becomes

$$\hat{W}^{[m,5,4]}_{N \times N} \sim \begin{pmatrix} L_{(N-N_0) \times (N-N_0)} & 0_{(N-N_0) \times N_0} \\ M_{N_0 \times (N-N_0)} & \hat{W}_{N_0 \times N_0} \end{pmatrix}$$

where $L_{(N-N_0) \times (N-N_0)}$ is a lower triangular matrix whose diagonal entries are all 1.

3. Therefore, to calculate the lowest power of $z$ of $\hat{W}^{[m,5,4]}_{N \times N}(z; a)$, it suffices to compute the power of the reduced $N_0 \times N_0$ determinant $\hat{W}_{N_0 \times N_0}$ and the final result is $\Gamma$ as given in (42).

Next, we derive the factorization of $W^{[m,5,4]}_N(z)$ provided in Theorem 2.4. Since the multiplicity of the zero root of $W^{[m,5,4]}_N(z)$ is $\Gamma$, we can write

$$W^{[m,5,4]}_N(z) = z^\Gamma q^{[m]}_N(z).$$ (164)

Note that

$$p^{[m]}_j(\omega z) = \omega^j p^{[m]}_j(z),$$ (165)

where $\omega$ is any of the $m$-th root of 1, i.e., $\omega^m = 1$. From (164) and (165), we immediately have

$$W^{[m,5,4]}_N(\omega z) = \omega^{2N(N+1)} W^{[m,5,4]}_N(z)$$

and

$$q^{[m]}_N(\omega z) = \omega^{2N(N+1)-\Gamma} q^{[m]}_N(z).$$

Since $2N(N+1)-\Gamma$ is a multiple of $m$, we have $\omega^{2N(N+1)-\Gamma} = 1$, and hence,

$$q^{[m]}_N(\omega z) = q^{[m]}_N(z).$$

This completes the proof.

**Remark 9** We note that the row operations performed above are similar to those in the proof of rogue patterns in the inner region. For some special cases, such as $m = 4r + 2$ in the three-component NLS equation, the proof needs some modifications similar to the proof of Theorem 2.3.
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Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

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