Santalo’s formula and stability of trapping sets of positive measure

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Abstract
Billiard trajectories (broken generalised geodesics) are considered in the exterior of an obstacle $K$ with smooth boundary on an arbitrary Riemannian manifold. We prove a generalisation of the well-known Santalo’s formula. As a consequence, it is established that if the set of trapped points has positive measure, then for all sufficiently small smooth perturbations of the boundary of $K$ the set of trapped points for the new obstacle $K'$ obtained in this way also has positive measure. More generally the measure of the set of trapped points depends continuously on perturbations of the obstacle $K$. Some consequences are derived in the case of scattering by an obstacle $K$ in $\mathbb{R}^n$. For example, it is shown that, for a large class of obstacles $K$, the volume of $K$ is uniquely determined by the average travelling times of scattering rays in the exterior of $K$.

MSC: 37D20, 37D40, 53D25, 58J50

Keywords: geodesic, billiard trajectory, Santalo’s formula, trapped point, Liouville measure, travelling time, scattering by obstacles

1 Introduction

Let $g_t$ be the geodesic flow on the unit sphere bundle $S\widetilde{M}$ of a $C^k$ ($k \geq 2$) Riemannian manifold $\widetilde{M}$ without boundary, and let $n = \dim(\widetilde{M}) = n \geq 2$. Consider an arbitrary compact subset $M$ of $\widetilde{M}$ with non-empty smooth ($C^k$) boundary $\partial M$ and non-empty interior $M \setminus \partial M$.

Next, suppose that $K$ is a compact subset of the interior $M \setminus \partial M$ of $M$ with smooth boundary $\partial K$ and non-empty interior $K \setminus \partial K$. Consider the compact subset

$$\Omega = M \setminus K$$

of $M$. Then the boundary of $\Omega$ in $\widetilde{M}$ is $\partial \Omega = \partial M \cup \partial K$. Define the billiard flow $\phi_t$ in $S(\Omega)$ in the usual way: it coincides with the geodesic flow $g_t$ in the interior of $S(\Omega)$, and when a geodesic hits the boundary $\partial \Omega$ it reflect following the law of geometrical optics. Let $dq$ be the measure on $\widetilde{M}$ determined by the Riemannian metric and let $dv$ be the Lebesgue measure on the unit sphere $S_q\widetilde{M}$. It is well-known (see Sect. 2.4 in [CFS]) that the geodesic flow $g_t$ preserves the Liouville measure $dq dv$ on $S\widetilde{M}$, and so the billiard flow $\phi_t$ preserves the restriction $\lambda$ of $dq dv$ to $S(\Omega)$. Finally, denote by $\mu$ the Liouville measure on $S^+(\partial \Omega)$ defined by

$$d\mu = d\rho(q)d\omega_q|\langle v(q), v\rangle|,$$

where $\rho$ is the measure on $\partial \Omega$ determined by the Riemannian metric on $\partial \Omega$ and $\omega_q$ is the Lebesgue measure on the $(n-1)$-dimensional unit sphere $S_q(M)$ (see e.g. Sect. 6.1 in [CFS]).

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For any \( q \in \partial \Omega \), denote by \( \nu(q) \in S\widetilde{M} \) the unit normal vector to \( \partial \Omega \) pointing into the interior of \( \Omega \). Set
\[
S^+(\partial \Omega) = \{ x = (q, v) : q \in \partial \Omega, \langle v, \nu(q) \rangle \geq 0 \},
\]
and in a similar way define \( S^+(\partial M) \subset S^+(\partial \Omega) \). For any \( x = S^+(\partial \Omega) \) denote by \( \tau(x) \geq 0 \) the maximal number (or \( \infty \)) such that \( \phi_t(x) = g_t(x) \) is in the interior of \( \Omega \) for all \( 0 < t < \tau(x) \). Set \( \tau(x) = 0 \) if \( \langle \nu(q), v \rangle = 0 \). The function \( \tau \) thus defined is called the first return time function for the flow \( \phi_t \) on \( \Omega \).

In the present paper we will assume that the geodesic flow \( g_t \) has no trapped trajectories in \( M \), i.e.
\[
(A): \text{ For every } x = (q, v) \in S^+(\partial M) \text{ with } \langle v, \nu(q) \rangle > 0 \exists t > 0 \text{ with } \text{pr}_1(g_t(x)) \in \partial M,
\]
where we set \( \text{pr}_1(q, v) = q \). In particular, it follows from this assumption that \( \tau(x) < \infty \) for every \( x \in S^+(\partial \Omega) \).

Then Santalo’s formula (see e.g. pp. 336-338 in [Sa] or [Cha]) gives
\[
\int_{S(\Omega)} f(x) \, d\lambda(x) = \int_{S^+(\partial \Omega)} \left( \int_0^{\tau(x)} f(g_t(x)) \, dt \right) \, d\mu(x)
\]
for every \( \lambda \)-integrable function \( f \) on \( S(\Omega) \).

Santalo’s formula has been widely used in geometry and dynamical systems (including billiards) – see e.g. [CULV], [Ch], [CaG], [C] and the references there.

By assumption \((A)\), the geodesic flow \( g_t \) has no trapped trajectories in \( M \). However, the billiard flow \( \phi_t \) may have trapped trajectories in \( M \) with respect to the obstacle \( K \). More precisely, we will now consider the billiard trajectories in \( \Omega \) as scattering trajectories in \( M \) reflecting at the boundary \( \partial K \) when they hit \( K \). Given \( x = S(\Omega) \), let \( t(x) \geq 0 \) be the maximal number (or \( \infty \)) such that \( \phi_t(x) \) is in the interior of \( M \) for all \( 0 < t < t(x) \). For \( x = (q, v) \in S^+(\partial \Omega) \) with \( \langle \nu(q), v \rangle = 0 \) set \( t(x) = 0 \). Then \( t(x) \) is called the travelling time function on \( \Omega \). Given \( x \in S^+(\partial M) \), consider the billiard trajectory
\[
\gamma^+(x) = \{ \phi_t(x) : 0 \leq t \leq t(x) \}
\]
which starts at a point \( x \in S^+(\partial M) \) and, if \( t(x) < \infty \), it ends at another point \( y \in S^+(\partial M) \). Between \( x \) and \( y \) the trajectory may hit \( \partial K \) several times.

Problems related to recovery of information about the manifold \( M \) from travelling times \( \tau(x) \) of geodesics in the interior of \( M \) have been considered in Riemannian geometry for a rather long time – see [SUV], [SU], [CULV] and the references there for some general information. Similar problems have been studied when an obstacle \( K \) is present; then one deals with travelling times \( t(x) \) of billiard trajectories (generalised geodesics) in the exterior of \( K \) – see [NS1], [NS2], [St3] and the references there (see also [LP] and [M] for some general information on inverse scattering problems). For some classes of obstacles \( K \) the travelling time function \( t(x) = t_K(x) \) completely determines \( K \) ([NS1], [NS2]). For example, it was recently established ([NS1]) that this is so in the class of obstacles \( K \) in Euclidean spaces that are finite disjoint unions of strictly convex domains with smooth boundaries.
Let \( \text{Trap}_Q^+(\partial M) \) be the set of the trapped points \( x = (q, v) \in S^+ (\partial M) \) of the billiard flow \( \phi_t \) in \( \Omega \), i.e. the points for which \( t(x) = \infty \). In general, it may happen that \( \text{Trap}_Q^+(\partial M) \neq \emptyset \), however it follows from Theorem 1.6.2 in [LP] (see also Proposition 2.4 in [St1]) that

\[
\mu(\text{Trap}_Q^+(\partial M)) = 0. \tag{1.2}
\]

Next, let \( \text{Trap}(\Omega) \) be the set of all trapped points \( x = (q, v) \in S(\Omega) \) of the billiard flow \( \phi_t \) in \( \Omega \), i.e. the points for which \( t(q, v) = \infty \) or \( t(q, -v) = \infty \). As Livschits’ example shows (see Figure 1), in general \( \text{Trap}(\Omega) \) may have positive Liouville measure and even a non-empty interior.

Generalising Santalo’s formula (1.1), we prove the following.

**Theorem 1.1.** Assume that \( M \) and the geodesic flow \( g_t \) in \( M \) satisfy the condition (A). Let \( f : S(\Omega) \setminus \text{Trap}(\Omega) \to \mathbb{C} \) be a \( \lambda \)-measurable function. Assume that either:

(i) \( \text{Trap}(\Omega) = \emptyset \) and \( f \) is integrable on \( S(\Omega) \), or

(ii) \( |f| \) is integrable.

Then we have

\[
\int_{S(\Omega) \setminus \text{Trap}(\Omega)} f(x) \, d\lambda(x) = \int_{S^+(\partial M) \setminus \text{Trap}_Q^+(\partial M)} \left( \int_0^{t(x)} f(\phi_t(x)) \, dt \right) \, d\mu(x). \tag{1.3}
\]

Clearly, in the case (ii) above \( \text{Trap}(\Omega) \) is allowed to have positive measure.

We will now describe the main consequence of Theorem 1.1 derived in this paper.

Let \( k \geq 3 \) and let \( C^k(\partial K, M) \) be the space of all smooth embedding \( F : \partial K \to M \) endowed with the Whitney \( C^k \) topology (see [Hi]). Given \( F \in C^k(\partial K, M) \), let \( K_F \) be the obstacle in \( M \) with boundary \( \partial K_F = F(\partial K) \) so that \( K_F \cap \partial M = \emptyset \), and let \( \Omega_F = M \setminus K_F \).

Our second main result in this paper is the following.

**Theorem 1.2.** Assume that \( M \) and the geodesic flow \( g_t \) in \( M \) satisfy the condition (A).

(a) If \( \lambda(\text{Trap}(\Omega)) > 0 \), then there exists an open neighbourhood \( U \) of \( \text{id} \) in \( C^k(\partial K, M) \) such that for every \( F \in U \) we have \( \lambda(\text{Trap}(\Omega_F)) > 0 \).

(b) More generally, for every \( \epsilon > 0 \) there exists an open neighbourhood \( U \) of \( \text{id} \) in \( C^k(\partial K, M) \) such that for every \( F \in U \) we have \( |\lambda(\text{Trap}(\Omega_F)) - \lambda(\text{Trap}(\Omega))| < \epsilon \).

We prove Theorem 1.1 in Sect. 2 below, and then use it in Sect.3 to derive Theorem 1.2. Let us note that in the case \( \text{Trap}(\Omega) = \emptyset \) the formula (1.3) with \( f = 1 \) was mentioned without proof and used by Plakhov and Roshchina in [PlaR] (see also [Pla]).

Formula (1.3) with \( f = 1 \) implies the following.

**Corollary 1.3.** Under the assumptions of Theorem 1.1 we have

\[
\lambda(\text{Trap}(\Omega)) = \lambda(S(\Omega)) - \int_{S^+(\partial M)} t(x) \, d\mu(x).
\]

So, if we know the travelling time function \( t(x) \) and have enough information about \( \Omega \) to determine its volume, then we can determine the measure of the set of trapped points in \( \Omega \), as well.
Some other consequences of Theorem 1.1 concerning scattering by obstacles are discussed in Sects. 4 and 5 below.

![Figure 1: Livshits' Example (Ch. 5 in [M])](image)

$E$ is a half-ellipse with end points $P$ and $Q$ and foci $F_1$ and $F_2$. Any trajectory entering the interior of the ellipse between the foci must exit between the foci after reflection. So, no trajectory ‘coming from infinity’ has a common point with the parts $U$ and $V$ of the boundary and the nearby regions.

Similar examples in higher dimensions are described in [NS3].

## 2 A generalised Santalo’s formula

Given $x \in S(\Omega)$, we will say that $\gamma^+(x)$ contains a gliding segment on the boundary $\partial \Omega$ if there exist $0 \leq t_1 < t_2 \leq t(x)$ such that $\phi_t(x) = g_t(x) \in S(\partial \Omega)$ for all $t \in [t_1, t_2]$ (i.e. $\gamma^+(x)$ contains a non-trivial geodesic segment lying entirely in $\partial \Omega$).

It follows from results\(^2\) in [MS2] that for $\lambda$-almost all $x \in S(\Omega)$, the billiard trajectory $\gamma^+(x)$ does not contain any gliding segments on the boundary $\partial \Omega$, and $\gamma^+(x)$ has only finitely many reflections.

First, we prove a special case of Theorem 1.1.

**Lemma 2.1.** Assume that $M$ and the geodesic flow $g_t$ in $M$ satisfy the condition (A). Let $V$ be an open subset of $S(\Omega)$ containing $\text{Trap}(\Omega)$ and such that $\phi_t(x) \in V$ for any $x \in V$ and any $t \in [0, t(x)]$. Assume that $f : S(\Omega) \to \mathbb{C}$ is an integrable function such that $f = 0$ on $V$. Then (1.3) holds.

\(^2\)See Sect. 3 in [MS2]; see also [MS1], [H] or [PS] for general information about generalised geodesics.
Proof of Lemma 2.1. Let $V$ and $f$ satisfy the assumptions of the lemma. Then we have
\[ \int_{S(\Omega)} f(x) \, d\lambda(x) = \int_{S(\Omega) \setminus V} f(x) \, d\lambda(x) \]  
(2.1)
and
\[ \int_{S^+(\partial M)} \left( \int_0^{t(x)} f(\phi_t(x)) \, dt \right) \, d \mu(x) = \int_{S^+(\partial M) \setminus V} \left( \int_0^{t(x)} f(\phi_t(x)) \, dt \right) \, d \mu(x). \]  
(2.2)

We will prove that the right-hand-sides of (2.1) and (2.2) are equal.

Given an integer $k \geq 0$ denote by $\Gamma_k$ the set of those $x \in S^+(\partial M) \setminus V$ such that $\gamma^+(x)$ contains no gliding segments on the boundary $\partial \Omega$, and $\gamma^+(x)$ has exactly $k$ reflections at $\partial K$. Clearly $\Gamma_k$ are disjoint, measurable subsets of $S^+(\partial M)$ and, as remarked earlier, it follows from [MS2] that
\[ \mu \left( S^+(\partial M) \setminus (V \cup \cup_{k=0}^{\infty} \Gamma_k) \right) = 0. \]  
(2.3)

The billiard ball map $B$ is defined in the usual way: given $x = (q, v) \in S^+(\partial \Omega)$ with $g_t(x) \in S(\partial \Omega)$ for some $t \in (0, \infty)$, take the smallest $t > 0$ with this property, and let $g_t(x) = (p, w)$ for some $p \in \partial \Omega$, $w \in S^{n-1}$. Set $B(x) = (p, \sigma_p(w)) \in S^+(\partial \Omega)$, where
\[ \sigma_p : T_p(\widetilde{M}) \to T_p(\widetilde{M}) \]
is the symmetry through the tangent plane $T_p(\partial \Omega)$, i.e.
\[ \sigma_p(\xi) = \xi - 2 \langle \nu(p), \xi \rangle \nu(p). \]

It follows from the condition (A) that $B$ is well-defined on a set $\Lambda$ of full $\mu$-measure in $S^+(\partial \Omega)$ and the Liouville measure $\mu$ is invariant with respect to $B$ (see e.g. Lemma 6.6.1 in [CFS]).

Notice that each of the sets $\Gamma_k$ is contained in $\Lambda$ and moreover
\[ \Gamma_{k,j} = B^j(\Gamma_k) \subset \Lambda \]
for all $j = 0, 1, \ldots, k$. Also, by the definition of the sets $\Gamma_k$ we have
\[ B^j(\Gamma_k) \subset S^+(\partial K) , \quad 1 \leq j \leq k. \]  
(2.4)
The sets $\Gamma_{k,j}$ are clearly measurable, and moreover
\[ \Gamma_{k,j} \cap \Gamma_{m,i} = \emptyset \quad \text{whenever } k \neq m \text{ or } j \neq i. \]  
(2.5)

Indeed, assume that $y \in \Gamma_{k,j} \cap \Gamma_{m,i}$ for some non-negative numbers $k, j, m, i$ with $0 \leq j \leq k$ and $0 \leq i \leq m$. Then $y = B^j(x)$ for some $x \in \Gamma_k$ and $y = B^i(z)$ for some $z \in \Gamma_m$. Assume e.g. $j > i$. Then $B^i(z) = y = B^j(x) = B^i(B^{j-i}(x))$ implies $z = B^{j-i}(x)$. Now (2.4) gives $z \in S^+(\partial K)$ which is a contradiction with $z \in \Gamma_m \subset S^+(\partial M)$. Thus, we must have $j = i$ and therefore $B^i(z) = y = B^i(x)$, so $z = x \in \Gamma_k \cap \Gamma_m$. The latter is non-empty only when $k = m$. This proves (2.5).
Finally, notice that

\[ \mu \left( S^+ (\partial \Omega) \setminus ( V \cup \bigcup_{k=0}^{\infty} \bigcup_{j=0}^{k} \Gamma_{k,j}) \right) = 0. \]  \hspace{1cm} \text{(2.6)}

Indeed, as mentioned earlier, it follows from results in [MS2] that for \( \lambda \)-almost all \( x \in S(\Omega) \), the billiard trajectory \( \gamma^+(x) \) does not contain any gliding segments on the boundary \( \partial \Omega \), and \( \gamma^+(x) \) has only finitely many reflections. Since \( \text{Trap}(\Omega) \subset V \), for almost all \( y \in S^+(\partial K) \setminus V \) there exist \( t \geq 0 \) and \( x \in S^+(\partial M) \) such that \( y = \phi_t(x) \), \( t(x) < \infty \), the billiard trajectory \( \gamma^+(x) \) does not contain any gliding segments on \( \partial \Omega \) and has only finitely many reflections. Let \( k \) be the number of those reflections; then for some \( j \in \mathbb{N} \) we have \( y = B^j(x) \). Thus, \( x \in \Gamma_k \) and \( y \in \Gamma_{k,j} \). This proves (2.6).

Next, given \( k > 0 \) and \( x \in \Gamma_k \), clearly we have

\[ t(x) = \tau(x) + \tau(B(x)) + \tau(B^2(x)) + \ldots + \tau(B^k(x)). \]

Set

\[ T_j(x) = \tau(x) + \tau(B(x)) + \ldots + \tau(B^j(x)) \]

for \( j = 0, 1, \ldots, k \). Clearly \( \phi_{T_j(x)}(x) = B^{j+1}(x) \) for \( 0 \leq j \leq k - 1 \). Hence for any \( j = 1, \ldots, k \) we have

\[ \phi_{s+T_{j-1}(x)}(x) = \phi_s(\phi_{T_{j-1}(x)}(x)) = g_s(B^j(x)), \]

and therefore, using the substitution \( t = s + T_{j-1}(x) \) below we get

\[ \int_{T_{j-1}(x)}^{T_j(x)} f(\phi_t(x)) \, dt = \int_0^{\tau(B^j(x))} f(\phi_s(B^j(x))) \, ds. \]

Thus,

\[
\int_0^{t(x)} f(\phi_t(x)) \, dt = \int_0^{\tau(x)} f(\phi_t(x)) \, dt + \sum_{j=1}^{k} \int_{T_{j-1}(x)}^{T_j(x)} f(\phi_t(x)) \, dt \\
= \int_0^{\tau(x)} f(\phi_t(x)) \, dt + \sum_{j=1}^{k} \int_0^{\tau(B^j(x))} f(\phi_s(B^j(x))) \, ds \\
= \sum_{j=0}^{k} \int_0^{\tau(B^j(x))} f(\phi_s(B^j(x))) \, dt.
\]

Now, using the above and the fact that \( B^j : \Gamma_{k,j} \rightarrow B^j(\Gamma_{k,j}) \) is a measure preserving bijection, we get

\[
\int_{\Gamma_k} \left( \int_0^{\tau(B^j(x))} f(\phi_s(B^j(x))) \, ds \right) \, d\mu(x) = \int_{\Gamma_{k,j}} \left( \int_0^{\tau(y)} f(\phi_s(y)) \, ds \right) \, d\mu(y),
\]
which implies
\[
\int_{\Gamma_k} \left( \int_0^{t(x)} f(\phi_t(x)) \, dt \right) \, d\mu(x) = \int_{\Gamma_k} \left( \sum_{j=0}^k \int_0^{\tau(B^j(x))} f(\phi_t(B^j(x))) \, dt \right) \, d\mu(x) = \sum_{j=0}^k \int_{\Gamma_k} \left( \int_0^{\tau(B^j(x))} f(\phi_t(B^j(x))) \, dt \right) \, d\mu(x) = \sum_{j=0}^k \int_{\Gamma_k,j} \left( \int_0^{\tau(y)} f(\phi_t(y)) \, dt \right) \, d\mu(y).
\]

Using (2.2), (2.3), (2.4) and (2.6), we derive
\[
\int_{S^+(\partial M)} \left( \int_0^{t(x)} f(\phi_t(x)) \, dt \right) \, d\mu(x) = \int_{S^+(\partial M) \setminus V} \left( \int_0^{t(x)} f(\phi_t(x)) \, dt \right) \, d\mu(x) = \sum_{k=0}^{\infty} \int_{\Gamma_k} \left( \int_0^{\tau(x)} f(\phi_t(x)) \, dt \right) \, d\mu(x) = \sum_{k=0}^{\infty} \sum_{j=0}^k \int_{\Gamma_k,j} \left( \int_0^{\tau(y)} f(\phi_t(y)) \, dt \right) \, d\mu(y) = \int_{S^+(\partial \Omega) \setminus V} \left( \int_0^{\tau(y)} f(\phi_t(y)) \, dt \right) \, d\mu(y) = \int_{S^+(\partial \Omega)} \left( \int_0^{\tau(y)} f(g_t(y)) \, dt \right) \, d\mu(y).
\]

By Santalo’s formula (1.1), the latter is equal to \( \int_{S(\Omega)} f(x) \, d\lambda(x). \)

**Proof of Theorem 1.1.** If \( \text{Trap}(\Omega) = \emptyset \), taking \( V = \emptyset \) in Lemma 2.1, proves the case (i).

To prove the case (ii), assume that \( \text{Trap}(\Omega) \neq \emptyset \). Let \( f \) be a measurable function of \( S(\Omega) \setminus \text{Trap}(\Omega) \) such that \( |f| \) is integrable. Extend \( f \) as 0 on \( \text{Trap}(\Omega) \). Since \( \text{Trap}_+^{\Omega}(\partial M) \) is closed in \( S^+(\partial M) \), there exists a sequence of compact sets
\[
F_1 \subset F_2 \subset \ldots \subset F_m \subset \ldots \subset S^+(\partial M) \setminus \text{Trap}_+^{\Omega}(\partial M)
\]
such that
\[
\bigcup_{m=1}^{\infty} F_m = S^+(\partial M) \setminus \text{Trap}_+^{\Omega}(\partial M).
\]  
(2.7)
Choose such a sequence, and for every \( m \) set
\[
G_m = \{ \phi_t(x) : x \in F_m, \ 0 \leq t \leq t(x) \}.
\]

\(^3\)This is already assumed in the right-hand-side of (1.3).
Then we have
\[ \bigcup_{m=1}^{\infty} G_m = S(\Omega) \setminus \text{Trap}(\Omega). \] (2.8)
Indeed, if \( x \in G_m \) for some \( m \), then \( x = \phi_t(y) \) for some \( y \in F_m \subset S^+(\partial M) \setminus \text{Trap}_t^+(\partial M) \) and some \( t \in [0, t(x)] \). Then \( y \notin \text{Trap}_m^+(\partial M) \) gives \( t(y) < \infty \) and so \( t(x) < \infty \), too. Also, if \( x = (q,v) \), then \( t(q,-v) = t < \infty \). Thus, \( x \notin \text{Trap}(\Omega) \). This proves that \( G_m \subset S(\Omega) \setminus \text{Trap}(\Omega) \) for all \( m \). Next, let \( x = (q,v) \in S(\Omega) \setminus \text{Trap}(\Omega) \). Then \( t = t(q,-v) < \infty \), so \( x = \phi_t(y) \) for some \( y \in S^+(\partial M) \). Moreover, \( t(y) = t(x) + t < \infty \), so \( y \in S^+(\partial M) \setminus \text{Trap}_t^+(\partial M) \), and by (2.7) we have \( y \in F_m \) for some \( m \geq 1 \). Moreover \( 0 \leq t \leq t(y) \). Thus, \( x \in G_m \). This proves (2.8).

Notice that all sets \( G_m \) are closed in \( S(\Omega) \). Indeed, given \( m \), a standard argument (see e.g. the proof of Lemma 3.1 below) shows that the function \( t(x) \) is uniformly bounded on the compact set \( F_m \). Let \( \{x_p\}_{p=1}^{\infty} \) be a sequence in \( G_m \) with \( x_p \to x \) as \( p \to \infty \). Then \( x_p = \phi_{t_p}(y_p) \) for some \( t_p \in [0,T] \) and some \( y_p \in F_m \), where \( T = \sup_{y \in F_m} t(y) \). Taking an appropriate sub-sequence, we may assume \( y_p \to y \) and \( t_p \to t \) as \( p \to \infty \). Since \( F_m \) is compact, \( y \in F_m \). Now a simple continuity argument gives \( x = \phi_t(y) \), so \( x \in G_m \).

Thus, \( G_m \) is closed in \( S(\Omega) \) and so \( V_m = S(\Omega) \setminus G_m \) is open in \( S(\Omega) \). Moreover, it is clear that \( \phi_t(x) \in V_m \) for all \( x \in V_m \) and all \( t \in [0, t(x)] \). Let \( \chi_{G_m} \) be the characteristic function of \( G_m \) in \( S(\Omega) \). Applying Lemma 2.1 to \( V = V_m \) and \( f \) replaced by \( f_m = f \cdot \chi_{G_m} \), we get
\[
\int_{S(\Omega) \setminus \text{Trap}(\Omega)} f_m(x) \, d\lambda(x) = \int_{S^+(\partial M)} \left( \int_0^{t(x)} f_m(\phi_t(x)) \, dt \right) \, d\mu(x).
\]
Extending \( f_m \) as 0 on \( V_m \), we get a sequence of measurable functions on \( S(\Omega) \) with \( |f_m| \leq |f| \) for all \( m \) and \( f_m(x) \to f(x) \) for all \( x \in S(\Omega) \). Since \( |f| \) is integrable, Lebesgue’s Theorem now implies
\[
\int_{S(\Omega) \setminus \text{Trap}(\Omega)} f(x) \, d\lambda(x) = \int_{S^+(\partial M)} \left( \int_0^{t(x)} f(\phi_t(x)) \, dt \right) \, d\mu(x),
\]
which proves the theorem. ■

3 Proof of Theorem 1.2

It is enough to prove the more general part (b).

Let \( M \) and \( \Omega \) satisfy the assumptions of Theorem 1.2, so (A) holds. Let \( \epsilon > 0 \). Assume that there is no open neighbourhood \( U \) of \( \text{id} \) in \( C^k(\partial M, M) \) such that for every \( F \in U \) we have \( |\lambda(\text{Trap}(\Omega_m)) - \lambda(\text{Trap}(\Omega))| < 2\epsilon \). Then there exists a sequence \( \{F_m\}_{m=1}^{\infty} \subset C^k(\partial M, M) \) with \( F_m \to \text{id} \) as \( m \to \infty \) in the \( C^k \) Whitney topology such that
\[ |\lambda(\text{Trap}(\Omega_m)) - \lambda(\text{Trap}(\Omega))| \geq 2\epsilon \] (3.1)
for all \( m \), where we set \( \Omega_m = \Omega \setminus \Omega_m \) for brevity. Set \( K_m = M \setminus \Omega_m \); then \( \partial K_m = F_m(\partial K) \). Since all obstacles \( K_m \) are in the interior of \( M \), we have \( \Omega_m \subset M \), so the Liouville measure
\[ \lambda = dqdv \] is well-defined on \( S(\Omega_m) \). The function \( t(x) \) and the billiard trajectory \( \gamma^+(x) \) for \( \Omega_m \) will be denoted by \( t_m(x) \) and \( \gamma^+_m(x) \), respectively. Set

\[ \mathcal{T}' = \text{Trap}^+_\Omega(\partial M) \cup \bigcup_{m=1}^\infty \text{Trap}^+_\Omega_m(\partial M). \]

Then \( \mu(\mathcal{T}') = 0 \) by (1.2). Theorem 1.1 with \( f = 1 \) implies

\[
\lambda(S(\Omega) \setminus \text{Trap}(\Omega)) = \int_{S^+(\partial M) \setminus \mathcal{T}'} t(x) \, d\mu(x) = \int_{S^+(\partial M) \setminus \mathcal{T}'} t(x) \, d\mu(x),
\]

and

\[
\lambda(S(\Omega_m) \setminus \text{Trap}(\Omega_m)) = \int_{S^+(\partial M)} t_m(x) \, d\mu(x) = \int_{S^+(\partial M) \setminus \mathcal{T}'} t_m(x) \, d\mu(x).
\]

Since \( \lambda(S(\Omega_m)) \to \lambda(S(\Omega)) \) as \( m \to \infty \), we may assume that \( |\lambda(S(\Omega_m)) - \lambda(S(\Omega))| < \epsilon \) for all \( m \geq 1 \). Combining this with (3.1) and the above equalities, we get

\[
|I_m - I| \geq \epsilon
\]

for all \( m \geq 1 \), where we set for brevity

\[
I_m = \int_{S^+(\partial M) \setminus \mathcal{T}'} t_m(x) \, d\mu(x) \quad \text{and} \quad I = \int_{S^+(\partial M) \setminus \mathcal{T}'} t(x) \, d\mu(x).
\]

For every \( m \geq 1 \) denote by \( \mathcal{T}''_m \) the set of those \( x \in S^+(\partial M) \) such that \( \gamma^+_m(x) \) has a tangent point to \( \partial \Omega_m \), and let \( \mathcal{T}''_0 \) be the set of those \( x \in S^+(\partial M) \) such that \( \gamma^+(x) \) has a tangent point to \( \partial \Omega \). As remarked earlier, we have \( \mu(\mathcal{T}''_m) = 0 \) for all \( m \geq 0 \). Thus, for the set

\[ \mathcal{T}'' = \bigcup_{m=0}^\infty \mathcal{T}''_m \]

we have \( \mu(\mathcal{T}'') = 0 \), and therefore \( \mu(\mathcal{T}' \cup \mathcal{T}'') = 0 \). Let

\[ D_1 \subset D_2 \subset \ldots \subset D_r \subset \ldots \subset S^+(\partial M) \setminus (\mathcal{T}' \cup \mathcal{T}'') \]

be compact sets so that \( \mu(D_r) \not\subset \mu(S^+(\partial M)) \) as \( r \to \infty \). Set

\[
I_m^{(r)} = \int_{D_r} t_m(x) \, d\mu(x) \quad \text{and} \quad I^{(r)} = \int_{D_r} t(x) \, d\mu(x)
\]

for brevity. Then

\[
I_m^{(r)} \not\subset I_m \quad \text{and} \quad I^{(r)} \not\subset I
\]

as \( r \to \infty \), for any fixed \( m \) in the first limit. It follows from this that there exists an integer \( r_0 \) so that

\[
I - \frac{\epsilon}{3} < I^{(r)} \leq I
\]

for all \( r \geq r_0 \).

Next, consider an arbitrary point \( x = (q, v) \in S^+(\partial M) \setminus (\mathcal{T}' \cup \mathcal{T}'') \). Then \( x \notin \text{Trap}^+_\Omega(\partial M) \) gives \( t(x) < \infty \), while \( x \notin \mathcal{T}'_0'' \) shows that \( \gamma^+(x) \) is a simply (transversally)
reflected trajectory in \( \Omega \) with no tangencies to \( \partial \Omega \). Thus, \( \text{the number } j_0(x) \text{ of reflection points of } \gamma^+(x) \text{ is finite, and } \gamma^+(x) \text{ has no other common points with } \partial \Omega. \)

**Lemma 3.1.** Let \( r \geq 1 \). There exists an integer \( m_0 = m_0(r) \geq 1 \) such that for every \( x \in D_r \) and every integer \( m \geq m_0 \) the trajectory \( \gamma^+_m(x) \) has at most \( j_0(x) \) common points\(^4\) with \( \partial \Omega_m \).

**Proof of Lemma 3.1.** Fix \( r \geq 1 \) and assume there exist arbitrarily large \( m \) such that \( \gamma^+_m(x) \) has at least \( j_0(x) + 1 \) reflection points for some \( x \in D_r \). Choosing a subsequence, we may assume that the latter is true for all \( m \geq 1 \) and that \( x_m \to x \in D_r \) as \( m \to \infty \). Since both \( \gamma^+_m(x) \) and \( \gamma^+(x) \) are simply reflecting, it follows that \( j_0(x_m) = j_0(x) \) for all sufficiently large \( m \); we will assume this is true for all \( m \). Set \( j_0 = j_0(x) \). Thus, for every \( m \) the trajectory \( \gamma^+_m(x_m) \) has at least \( j_0 + 1 \) reflection points. Let \( q_1, \ldots, q_{j_0} \in \partial K \) be the successive reflection points of \( \gamma^+(x) \). Set \( q_{j_0+1} = g_1 \) for convenience. Notice that some of the points \( q_j \) may coincide; however, \( q_{j+1} \neq q_j \) for all \( j \). For each \( j = 1, \ldots, j_0 \) choose a small open neighbourhood \( V_j \) of \( q_j \) on \( \partial K \) so that \( V_j \cap V_{j+1} = \emptyset \) for all \( j = 1, \ldots, j_0 \). Since \( F_m \to \text{id} \) in the \( C^k \) Whitney topology, a simple continuity argument shows that for sufficiently large \( m \) we have \( F_m(V_j) \cap F_m(V_{j+1}) = \emptyset \) for all \( j = 1, \ldots, j_0 \). Assuming that the neighbourhoods \( V_j \) are chosen sufficiently small and \( m \) is sufficiently large, it follows that for each \( j = 1, \ldots, j_0 \), \( \gamma^+_m(x_m) \) has an unique reflection point \( q_j^{(m)} \) in \( F_m(V_j) \). Let \( q_{j_0+1}^{(m)} \) be a reflection point of \( \gamma^+_m(x_m) \) which is not in \( \bigcup_{j=1}^{j_0} F_m(V_j) \); such a point exists, since by assumption the trajectory \( \gamma^+_m(x_m) \) has at least \( j_0 + 1 \) reflection points. Choosing an appropriate sub-sequence again, we may assume that \( q_{j_0+1}^{(m)} \to q_{j_0+1} \in \partial K \) as \( m \to \infty \). It is now clear that \( q_{j_0+1} \) is a common point of \( \gamma^+(x) \) and \( \partial K \) which does not belong to \( \bigcup_{j=1}^{j_0} V_j \), so \( q_{j_0+1} \neq q_j \) for all \( j = 1, \ldots, j_0 \). This is a contradiction which proves the lemma. \( \square \)

We now continue with the proof of Theorem 1.2.

According to (3.2), we either have \( I_m \leq I - \epsilon \) for infinitely many \( m \), or \( I_m \geq I + \epsilon \) for infinitely many \( m \).

**Case 1.** \( I_m \geq I + \epsilon \) for infinitely many \( m \). Considering an appropriate subsequence, we may assume \( I_m \geq I + \epsilon \) for all \( m \geq 1 \).

Let \( r \geq r_0 \) so that (3.4) holds for \( r \), and let \( m_0 = m_0(r) \) be as in Lemma 3.1. Then the compactness of \( D_r \) and a simple continuity argument show that

\[
i_0 = \sup_{x \in D_r} j_0(x) < \infty.
\]

This and Lemma 3.1 now imply

\[
t_m(x) \leq D i_0 , \quad m \geq m_0 , \quad x \in D_r.
\]

Another simple continuity argument shows that for \( x \in D_r \) the only possible limit point of the sequence \( \{t_m(x)\}_{m=1}^{\infty} \) is \( t(x) \), so we must have \( \lim_{m \to \infty} t_m(x) = t(x) \) for all \( x \in D_r \). This is true for all \( r \), so \( \lim_{m \to \infty} t_m(x) = t(x) \) for all \( x \in S^+(\partial M) \setminus (T' \cup T''). \) Setting

\[
\tilde{t}_m(x) = \sup_{m \geq m} t_m(x) , \quad x \in S^+(\partial M) \setminus (T' \cup T''),
\]

\(^4\)Clearly all of them must be simple reflection points, since \( x \notin T'_m \cup T''_m \).
we get a sequence of measurable functions with \( \tilde{t}_m(x) \) \( \subseteq \) \( t(x) \) on \( S^+(\partial M) \setminus (T' \cup T'') \). By Lebesgue’s Theorem (or Fatou’s Lemma),

\[
\lim_{m \to \infty} \int_{S^+(\partial M)} \tilde{t}_m(x) \, d\mu(x) = \lim_{m \to \infty} \int_{S^+(\partial M) \setminus (T' \cup T'')} \tilde{t}_m(x) \, d\mu(x) = \int_{S^+(\partial M) \setminus (T' \cup T'')} t(x) \, d\mu(x) = \int_{S^+(\partial M)} t(x) \, d\mu(x).
\]

Thus, there exists \( m_1 \geq m_0 \) such that

\[
\int_{S^+(\partial M)} \tilde{t}_m(x) \, d\mu(x) < \int_{S^+(\partial M)} t(x) \, d\mu(x) + \frac{\epsilon}{3} = I + \frac{\epsilon}{3}
\]

for all \( m \geq m_1 \). Since \( t_m(x) \leq \tilde{t}_m(x) \), it follows that

\[
I_m = \int_{S^+(\partial M)} t_m(x) \, d\mu(x) \leq \int_{S^+(\partial M)} \tilde{t}_m(x) \, d\mu(x) < I + \frac{\epsilon}{3}
\]

for \( m \geq m_1 \), which is a contradiction with our assumption that \( I_m \geq I + \epsilon \) for all \( m \geq 1 \).

**Case 2.** \( I_m \leq I - \epsilon \) for infinitely many \( m \). Considering an appropriate subsequence, we may assume \( I_m \leq I - \epsilon \) for all \( m \geq 1 \). Combining this with (3.3) and (3.4) gives

\[
I_m^{(r)} \leq I_m \leq I - \epsilon < I^{(r)} + \frac{\epsilon}{3} - \epsilon = I^{(r)} - \frac{2\epsilon}{3}
\]

for all \( r \geq r_0 \) and all \( m \geq 1 \).

Next, fix an arbitrary integer \( r \geq r_0 \) so that (3.4) holds for \( r \), and let \( m_0 = m_0(r) \) be as in Lemma 3.1. As in Case 1, the compactness of \( D_r \) and a simple continuity argument show that

\[
i_0 = \sup_{x \in D_r} j_0(x) < \infty,
\]

while Lemma 3.1 implies

\[
t_m(x) \leq D i_0, \quad m \geq m_0, \quad x \in D_r.
\]

As in Case 1, for any \( x \in D_r \) we must have \( \lim_{m \to \infty} t_m(x) = t(x) \) for all \( x \in D_r \).

Moreover, for the fixed \( r \), this convergence is uniform. Indeed, if this is not true, then there exist \( \delta > 0 \) and infinite sequences \( \{x_s\} \subseteq D_r \) and \( 1 \leq m_1 < m_2 < \ldots < m_s < \ldots \) such that \( |t_m(x_s) - t(x_s)| \geq \delta \) for all \( s \). Using the compactness of \( D_r \), we may assume that \( x_s \to x \in D_r \) as \( s \to \infty \). Also, since \( \{t_m(x_s)\} \) is a bounded sequence, we may assume that \( t_m(x_s) \to t \in [0, D i_0] \) as \( s \to \infty \). Since the trajectory \( \gamma_{m_s}(x_s) \) has at most \( i_0 \) reflection points, choosing an appropriate subsequence again, we may assume that the billiard trajectory \( \gamma_{m_s}(x_s) \) has the same number \( p \) of reflections points for all \( s \geq 1 \).

Let \( y_1(s), y_2(s), \ldots, y_p(s) \) be the successive reflection points of \( \gamma_{m_s}(x_s) \). By compactness, choosing appropriate subsequences again, we may assume that \( y_i(s) \to y_i \) as \( s \to \infty \) for all \( i = 1, \ldots, p \). It is now clear that \( y_1, \ldots, y_p \in \partial K \) and these are the successive reflection points of a billiard trajectory in \( \Omega \). Since \( x_s \to x \) as \( x \to \infty \), we must have that \( y_1, \ldots, y_p \).
are the successive reflection points of $\gamma^+(x)$. In particular, we must have $t_{m_s}(x_s) \to t(x)$ as $s \to \infty$. However, $t(x_s) \to t(x)$ as well, so it follows that $|t_{m_s}(x_s) - t(x_s)| < \delta$ for all sufficiently large $s$; a contradiction with our assumption.

Thus, $t_m(x) \to t(x)$ as $m \to \infty$ uniformly for $x \in D_r$. This implies that there exists $m_1 \geq m_0$ such that

$$|I_m^{(r)} - I| = \left| \int_{D_r} t_m(x) \, d\mu(x) - \int_{D_r} t(x) \, d\mu(x) \right| < \frac{\epsilon}{3}$$

for all $m \geq m_1$. In particular we have $I_{m_1}^{(r)} > I - \frac{\epsilon}{3}$ for all $m \geq m_1$, which is a contradiction with (3.6).

In this way we have show that (3.2) cannot hold for (infinitely many) $m \geq 1$. This completes the proof of the theorem.

4 Scattering by obstacles in $\mathbb{R}^n$

Here we consider the case when $\tilde{M} = \mathbb{R}^n$ with the standard Riemannian metric for some $n \geq 2$, and $K$ is a compact subset of $\mathbb{R}^n$ whose boundary $\partial K$ is a $C^k$ manifold of dimension $n - 1$ for some $k \geq 2$. We assume that $\mathbb{R}^n \setminus K$ is connected. Let $M$ be a large closed ball in $\mathbb{R}^n$ containing $K$ in its interior. As in Sect. 1,

$$\Omega = \tilde{M} \setminus K$$

has a smooth boundary $\partial \Omega = S_0 \cup \partial K$, where $S_0$ is the boundary sphere of $M$.

In the present case the scattering rays from Sect. 1 are simply billiard trajectories in the exterior of $K$ that come from infinity, enter $M$ at some point $q \in S_0$ with a certain direction $v \in S^{n-1}$ and after a time $t(q,v)$ spent in $\Omega$, leave $M$ and go to infinity. Then $t(q,v)$ is what we called the travelling time of $x = (q,v)$ in Sect. 1.

It is a natural problem to try to recover information about the obstacle $K$ from measurements along billiard trajectories (generalised geodesics) in the exterior of $K$. As we mentioned in the Introduction, problems of this kind have been considered for a rather long time in Riemannian geometry and more recently in scattering by obstacles, as well. Reconstructing $K$ in practice from the travelling times data appears to be a rather difficult problem, although in relatively simple cases there is enough scope to achieve this – see for example Sect. 4 in [NS2] which describes how to recover a planar obstacle $K$ which is a disjoint union of two strictly convex domains.

In the case considered in this section, condition (A) from Sect. 1 is always satisfied. A point $x = (q,v) \in S(\Omega)$ is trapped if either its forward billiard trajectory $\gamma^+(x)$ or its backward trajectory $\gamma^+(q,-v)$ is infinitely long. That is, either the billiard trajectory in the exterior of $K$ issued from $q$ in the direction of $v$ is bounded (contained entirely in the ball $M$) or the one issued from $q$ in the direction of $-v$ is bounded.

What concerns the problem of obtaining information about $K$ from travelling times $t(x)$, Theorem 1.1 provides some general information. In particular when $f = 1$ we get the following consequence.
Theorem 4.1. Assume that the set Trap(\(\Omega_K\)) of trapped points in \(\Omega_K\) has Lebesgue measure zero. Then

\[
Vol_n(K) = Vol_n(M) - \frac{1}{Vol_{n-1}(S^{n-1})} \int_{S^+(S_0)} t(x) \, d\mu(x),
\]

(4.1)

where \(Vol_n(K)\) is the standard volume of \(K\) in \(\mathbb{R}^n\) and \(Vol_{n-1}(S^{n-1})\) is the standard \((n - 1)\)-dimensional volume (surface area) of \(S^{n-1}\).

Proof. It follows from Theorem 1.1 with \(f = 1\) that

\[
Vol(S(\Omega)) = \int_{S^+(S_0)} t(x) \, d\mu(x).
\]

Combining this with \(Vol(S(\Omega)) = Vol_n(\Omega) Vol_{n-1}(S^{n-1})\) and \(Vol_n(K) = Vol_n(M) - Vol_n(\Omega)\) proves (4.1).

Remarks. (a) Formula (4.1) shows that when Trap(\(\Omega_K\)) has measure zero, from travelling times data we can recover the volume of \(K\). That is, without seeing \(K\) and without any preliminary information about \(K\), just measuring travelling times of a certain kind of signals incoming through points on the sphere \(S_0\) and outgoing through points on \(S_0\), we can compute the amount of mass in \(K\), i.e. the volume of \(K\). Apart from that, it appears that (4.1) could be useful in numerical approximations of the volume of \(K\).

(b) In Theorem 4.1 we only used the trivial function \(f = 1\). Naturally, one would expect that using Theorem 1.1 for a large family of functions \(f\) would bring much more significant information about the obstacle \(K\). It is already known (see e.g. [St3] and [NS2]) that a solid amount of information about \(K\) is recoverable from travelling times. However by means of formula (1.3) it might be possible to get such information in a more explicit way.

In simple cases when \(K\) is a disjoint union of connected pieces of roughly the same size and shape, we can estimate the number \(k\) of these pieces. Naturally, \(k\) can be a very large number\(^5\).

Example. Assume that \(K\) is a disjoint union of \(k\) balls of the same radius \(a > 0\), where \(k \geq 1\) is arbitrary (possibly a large number). Suppose that we know \(a\) from some preliminary information. Then measuring travelling times \(t(x)\) for a relatively large number of points \(x = (q, v) \in S(S_0)\) we get an approximation of the integral \(\int_{S^+(S_0)} t(x) \, d\mu(x)\), and therefore by means of (4.1), we obtain an approximate value for the number \(k\) of connected components of \(K\). The precise formula (assuming we can measure almost all travelling times) follows from (4.1):

\[
k = \frac{Vol_n(K)}{\pi^{n/2} a^n/\Gamma(n/2 + 1)} = \frac{R^n}{a^n} - \frac{\Gamma(n/2) \Gamma(n/2 + 1)}{2\pi^n a^n} \int_{S^+(S_0)} t(x) \, d\mu(x),
\]

where \(R\) is the radius of the large ball \(M\) containing \(K\), and \(\Gamma\) is Euler’s Gamma function.

\(^5\)E.g. consider a model that resembles the molecules in a gas container.
5 A Corollary

Given an integer $k \geq 0$, recall from Sect. 2 that (in the case $V = \emptyset$ in the proof of Lemma 2.1) $\Gamma_k$ is the set of all $x \in S^+(\partial M)$ such that $t(x) < \infty$, $\gamma^+(x)$ contains no gliding segments on the boundary $\partial \Omega$ and has exactly $k$ reflections at $\partial K$. As another consequence of Theorem 1.1 we get the following, the first part of which concerns the general situation considered in Sects. 1 and 2, while the second deals with a special kind of obstacles in $\mathbb{R}^n$.

**Corollary 5.1.** Let $D = \text{diam}(M)$.

(a) Under the assumptions of Theorem 1.1 we have
\[
\frac{1}{D} \text{Vol}(S(\Omega)) \leq \sum_{k=0}^{\infty} (k+1) \mu(\Gamma_k).
\]

(b) Let $K$ be a finite disjoint union of strictly convex domains in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundaries, and let $d > 0$ be a constant such that the minimal distance between distinct connected components of $K$ is not less than $d$ and $d \leq \text{dist}(K, \partial M)$, where $M$ is a ball of diameter $D$ containing $K$ in its interior. Then
\[
\frac{1}{D} \text{Vol}(S(\Omega)) \leq \sum_{k=0}^{\infty} (k+1) \mu(\Gamma_k) \leq \frac{1}{d} \text{Vol}(S(\Omega)).
\]

In particular, there exists a constant $C > 0$ such that
\[
\mu(\Gamma_k) \leq \frac{C}{k+1}
\]
for all $k \geq 0$.

**Proofs.** (a) By Theorem 1.1 with $f = 1$ we get
\[
\text{Vol}(S(\Omega)) = \int_{S^+(S_0)} t(x) \, d\mu(x) = \sum_{k=0}^{\infty} \int_{\Gamma_k} t(x) \, d\mu(x) \leq \sum_{k=0}^{\infty} \int_{\Gamma_k} (k+1)D \, d\mu(x)
\]
\[
= D \sum_{k=0}^{\infty} (k+1) \mu(\Gamma_k),
\]
which proves part (a).

(b) First, notice that $\widehat{M} = \mathbb{R}^n$, $M$, $K$ and $\Omega = \overline{M \setminus K}$ satisfy the condition (A) from Sect. 1. Moreover, it follows from Proposition 2.3 in [St1] and Proposition 5.1 in [St2] that the set $\text{Trap}(K)$ of trapped points has Lebesgue measure zero. By the nature of $K$ and $M$, no billiard trajectory in $S(\Omega)$ contains non-trivial gliding segments on $\partial \Omega$. Thus, for any non-trapped point $x \in S(\Omega)$ the trajectory $\gamma^+(x)$ has only finitely many reflection points.

Hence Theorem 1.1 is applicable. Using it again with $f = 1$ as in the proof of part (a), this times estimating from below $t(x) \geq (k+1)d$, proves the assertion. 

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