VALIRON AND ABEL EQUATIONS FOR HOLOMORPHIC
SELF-MAPS OF THE POLYDISC

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Abstract. We introduce a notion of hyperbolicity and parabolicity for a holomorphic
self-map $f: \Delta^N \to \Delta^N$ of the polydisc which does not admit fixed points in $\Delta^N$. We
generalize to the polydisc two classical one-variable results: we solve the Valiron equation
for a hyperbolic $f$ and the Abel equation for a parabolic nonzero-step $f$. This is done by
studying the canonical Kobayashi hyperbolic semi-model of $f$ and by obtaining a normal
form for the automorphisms of the polydisc. In the case of the Valiron equation we also
describe the space of all solutions.

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1. Introduction

A holomorphic self-map $f: \Delta \to \Delta$ of the unit disc can be classified in three types
according to its dynamical behavior. The self-map $f$ is called elliptic if it admits a fixed
point $p \in \Delta$. If $f$ is not an automorphism of $\Delta$, i.e. if $|f'(p)| < 1$, then $(f^n)$ converges to
the constant map $p$ uniformly on compact subsets. If $f$ admits no fixed points in $\Delta$, then
the classical Denjoy–Wolff theorem states that there exists a point $p \in \partial \Delta$ such that $(f^n)$
converges to the constant map $p$ uniformly on compact subsets, and such that

$$\angle \lim_{z \to p} f(z) = p \quad \text{and} \quad \lambda_f := \liminf_{z \to p} \frac{1 - |f(z)|}{1 - |z|} \leq 1,$$

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where \(\angle \lim\) stands for the non-tangential limits. The point \(p\) is called the Denjoy–Wolff point of \(f\) and the number \(\lambda_f\) is called the dilation of \(f\) at \(p\). The self-map \(f\) is called parabolic if \(\lambda_f = 1\) and hyperbolic if \(\lambda_f < 1\).

By \(k_\Delta\) denote the Poincaré distance on \(\Delta\). The class of all parabolic self-maps splits into two subclasses: parabolic self-maps \(f\) of non-zero step, for which the limit of the non-increasing sequence \((k_\Delta(f^n(z), f^{n+1}(z)))_{n \geq 0}\) is positive for some (and hence for any) \(z \in \Delta\), and those of zero step, for which this limit is identically zero.

Each type in this classification is associated with a functional equation that gives a “model” for the dynamics of the self-map. In the following theorem we summarize the results obtained by Königs [19], Valiron [23], Pommerenke [20], and by Baker and Pommerenke [7]. It is also worth to mention the important work of Cowen [13], who unified these equations in a common framework. We follow Cowen’s presentation of these results. We denote by \(\mathbb{H}\) the upper-half plane.

**Theorem 1.1.** Let \(f: \Delta \to \Delta\) be a holomorphic self-map. If \(f\) is elliptic, we suppose that \(0 < |f'(p)| < 1\), where \(p \in \Delta\) is the (unique) fixed point of \(f\). Then there exists an \(f\)-invariant domain \(U \subset \Delta\) such that for all \(z \in \Delta\) the orbit \((f^n(z))\) eventually lies in \(U\) and such that \(f|_U\) is univalent. Moreover:

1. [Königs] If \(f\) is an elliptic map, then there exists a holomorphic function \(\Theta: \Delta \to \mathbb{C}\) which is univalent on \(U\), solves the Schröder equation
   \[
   \Theta(f(z)) = f'(p)\Theta(z) \quad \text{for all } z \in \Delta,
   \]
   and satisfies \(\bigcup_{n \geq 0} f'(p)^{-n}\Theta(\Delta) = \mathbb{C}\).

2. [Valiron] If \(f\) is a hyperbolic map with dilation \(\lambda_f\) at its Denjoy–Wolff point, then there exists a holomorphic function \(\Theta: \Delta \to \mathbb{H}\) which is univalent on \(U\), solves the Valiron equation
   \[
   \Theta(f(z)) = \frac{1}{\lambda_f}\Theta(z) \quad \text{for all } z \in \Delta,
   \]
   and satisfies \(\bigcup_{n \geq 0} \lambda_f^n\Theta(\Delta) = \mathbb{H}\).

3. [Pommerenke] If \(f\) is a parabolic map of non-zero step, then there exists a holomorphic function \(\Theta: \Delta \to \mathbb{H}\) which is univalent on \(U\), solves the Abel equation
   \[
   \Theta(f(z)) = \Theta(z) \pm 1 \quad \text{for all } z \in \Delta,
   \]
   and satisfies \(\bigcup_{n \geq 0} (\Theta(\Delta) \mp n) = \mathbb{H}\).

4. [Baker–Pommerenke] If \(f\) is a parabolic map of zero step, then there exists a holomorphic function \(\Theta: \Delta \to \mathbb{C}\) which is univalent on \(U\), solves the Abel equation
   \[
   \Theta(f(z)) = \Theta(z) + 1 \quad \text{for all } z \in \Delta,
   \]
   and satisfies \(\bigcup_{n \geq 0} (\Theta(\Delta) - n) = \mathbb{C}\).
Remark 1.2. It is easy to see that solutions to the Schröder equation (1.1) differ just by a constant complex factor. Analogous (but much deeper) uniqueness result for the Valiron equation (1.2) is due to Bracci and Poggi-Corradini [8]: every holomorphic solution $\eta: \mathbb{D} \to \mathbb{H}$ to the Valiron equation is a positive multiple of $\Theta$ and satisfies $\bigcup_{n \geq 0} \lambda^n \eta(\Delta) = \mathbb{H}$.

Some (weaker) uniqueness results for the Abel equation are known, see, e.g., [12] for a detailed discussion. The reason for the $\pm$ sign in the Abel equation (1.3) is that we only consider solutions with values in the upper half-plane $\mathbb{H}$. This is not the case for the Abel equation (1.4), for which solutions with values in the whole $\mathbb{C}$ are allowed.

The Denjoy–Wolff theorem was generalized to the unit ball $\mathbb{B}^N$ by Hervé [17], to any $C^2$-smooth bounded strongly convex domain by Abate [1], and to arbitrary (not necessarily smooth) bounded strongly convex domains by Budzyńska [10], see also [3, 11]. Using the dilation at the Denjoy–Wolff point, one can extend the dynamical classification to holomorphic self-maps on the ball. This made possible to generalize Theorem 1.1 in several ways, see, e.g., [6] for a brief history.

For the polydisc $\Delta^N$, holomorphic dynamics has been studied by a number of specialists, e.g., by Hervé [18], Frosini [14], Abate and Raissy [3]. The main issue in this case is that the Denjoy–Wolff theorem fails, as the following example from [2] shows. Let $f: \Delta^2 \to \Delta^2$ be defined by $f(z, w) := (\lambda z, \frac{1+w}{3-w})$, where $|\lambda| = 1, \lambda \neq 1$. Then $f$ has no fixed points in $\Delta^2$, but $(f^n)$ does not converge.

Hence, in the polydisc case, a different approach is needed in order to obtain a dynamical classification. In [6] it is proved that given a holomorphic self-map $f: \mathbb{B}^N \to \mathbb{B}^N$ with no fixed points in $\mathbb{B}^N$, one can determine whether $f$ is parabolic or hyperbolic by looking at how fast the orbits diverge to infinity w.r.t. the Kobayashi distance in the ball. The divergence rate is defined in [6] not only in $\mathbb{B}^N$ but on any complex manifold. This allows us to use the divergence rate in order to define dynamical types of holomorphic self-maps of the polydisc, see Definition 4.1.

The main results of this note are the generalizations to the polydisc of assertions (2) and (3) of Theorem 1.1.

**Theorem 1.3.** Let $f: \Delta^N \to \Delta^N$ be a hyperbolic holomorphic self-map with dilation $\lambda_f$. Then there exists a holomorphic function $\Theta : \Delta^N \to \mathbb{H}$ that solves the Valiron equation

$$\Theta(f(z)) = \frac{1}{\lambda_f} \Theta(z) \quad \text{for all } z \in \Delta,$$

and satisfies $\bigcup_{n \geq 0} \lambda^n \Theta(\Delta) = \mathbb{H}$.

**Theorem 1.4.** Let $f: \Delta^N \to \Delta^N$ be a parabolic holomorphic self-map, with nonzero-step. Then there exists a holomorphic function $\Theta : \Delta^N \to \mathbb{H}$ that solves the Abel equation

$$\Theta(f(z)) = \Theta(z) \pm 1 \quad \text{for all } z \in \Delta,$$

and satisfies $\bigcup_{n \geq 0} (\Theta(\Delta) \mp n) = \mathbb{H}$. 


The proof of both theorems consists of two main steps. First we apply the theory of canonical semi-models developed by Bracci and the first-named author in [6, 5] and a result of Heath and Suffridge [15] to reduce the problem to the case of an automorphism \( \tau \) of the polydisc. Then we solve the problem for \( \tau \) by conjugating it to a suitable normal form.

Theorem 1.4 is a part of Theorem 6.2, which we prove in Section 6. Theorem 1.3 follows, in view of Remark 5.5, from Theorem 5.6, which we prove in Section 5. Moreover, Theorem 5.6 gives a complete description of all solutions to the Valiron equation with values in \( \mathbb{H} \), generalizing the uniqueness result by Bracci and Poggi-Corradini [8] for the unit disc.

2. Preliminaries

For a complex manifold \( X \), we denote by \( k_X \) its Kobayashi pseudo-distance.

**Definition 2.1.** Let \( f: X \to X \) be a holomorphic self-map of a complex manifold \( X \). The step \( s(x) \) of \( f \) at \( x \in X \) is the limit
\[
s(x) := \lim_{n \to \infty} k_X(f^n(x), f^{n+1}(x)).
\]

This limit exists because the sequence \((k_X(f^n(x), f^{n+1}(x)))_{n \geq 0}\) is non-increasing.

The divergence rate \( c(f) \) of \( f \) is the limit
\[
c(f) := \lim_{m \to \infty} \frac{k_X(f^m(x), x)}{m},
\]

which, according to [6], exists and does not depend on the choice of \( x \in X \).

**Remark 2.2.** In the context of non-expansive mapping theory in Banach spaces and in the Hilbert ball, \( s(x) \) and \( c(f) \) were considered in [9, 21, 22].

The following result is proved in [6].

**Theorem 2.3.** Let \( f: \mathbb{B}^q \to \mathbb{B}^q \) be a holomorphic self-map with no fixed points in \( \mathbb{B}^q \), and let \( \lambda_f \in (0, 1] \) be the dilation of \( f \) at its Denjoy–Wolff point. Then \( \lambda_f = e^{-c(f)} \).

**Definition 2.4.** Let \( X \) be a complex manifold and let \( f: X \to X \) be a holomorphic self-map. A semi-model for \( f \) is a triple \((\Lambda, h, \varphi)\), where \( \Lambda \) is a complex manifold, \( h: X \to \Lambda \) is a holomorphic mapping, and \( \varphi: \Omega \to \Omega \) is an automorphism such that \( h \circ f = \varphi \circ h \) and \( \bigcup_{n \geq 0} \varphi^{-n}(h(X)) = \Lambda \). We call the manifold \( \Lambda \) the base space and the mapping \( h \) the intertwining mapping.
Let \((Z, \ell, \tau)\) and \((\Lambda, h, \varphi)\) be two semi-models for \(f\). A morphism of semi-models \(\hat{\eta}: (Z, \ell, \tau) \to (\Lambda, h, \varphi)\) is given by a holomorphic map \(\eta: Z \to \Lambda\) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{h} & \Lambda \\
\downarrow f & & \downarrow \eta \\
Z & \xrightarrow{\varphi} & \Lambda \\
\downarrow \tau & & \downarrow \eta \\
X & \xrightarrow{h} & \Lambda
\end{array}
\]

**Remark 2.5.** It is shown in [6, Lemmas 3.6 and 3.7] that if \((Z, \ell, \tau), (\Lambda, h, \varphi)\) are semi-models for \(f\), then there exists at most one morphism \(\hat{\eta}: (Z, \ell, \tau) \to (\Lambda, h, \varphi)\), and that the holomorphic map \(\eta: Z \to \Lambda\) is surjective.

**Definition 2.6.** Let \(X\) be a complex manifold and let \(f: X \to X\) be a holomorphic self-map. Let \((Z, \ell, \tau)\) be a semi-model for \(f\) whose base space \(Z\) is Kobayashi hyperbolic. We say that \((Z, \ell, \tau)\) is a canonical Kobayashi hyperbolic semi-model for \(f\) if for any semi-model \((\Lambda, h, \varphi)\) for \(f\) such that the base space \(\Lambda\) is Kobayashi hyperbolic, there exists a morphism of semi-models \(\hat{\eta}: (Z, \ell, \tau) \to (\Lambda, h, \varphi)\).

**Remark 2.7.** If \((Z, \ell, \tau)\) and \((\Lambda, h, \varphi)\) are two canonical Kobayashi hyperbolic semi-model for \(f\), then they are isomorphic.

In what follows we will need the following result from [5] (see also [6]).

**Theorem 2.8.** Let \(X\) be a cocompact Kobayashi hyperbolic complex manifold, and let \(f: X \to X\) be a holomorphic self-map. Then there exists a canonical Kobayashi hyperbolic semi-model \((Z, \ell, \tau)\) for \(f\), where \(Z\) is a holomorphic retract of \(X\). Moreover, the following holds:

1. For any \(n \geq 0\), \(\lim_{m \to \infty} (f^m)^* k_X = (\tau^{-n} \circ \ell)^* k_Z\);
2. The divergence rate of \(\tau\) satisfies \(c(\tau) = c(f)\).

**Remark 2.9.** Notice that \(Z\) could reduce to a point. In such a case, \(c(f)\) is necessarily zero.
3. A NORMAL FORM FOR THE AUTOMORPHISMS OF THE POLYDISC

We start by introducing some notation. For \( k \in \mathbb{N} \) and \( j = 1, \ldots, k \) we introduce notation for the following maps:

\[
\sigma_k : \mathbb{C}^k \to \mathbb{C}^k; \quad (z_1, z_2, \ldots, z_k) \mapsto (z_2, \ldots, z_{k-1}, z_1),
\]

\[
\pi_{j,k} : \mathbb{C}^k \to \mathbb{C}; \quad z \mapsto \langle z, e_j \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the Hermitian inner product and \( (e_j) \) is the standard basis in \( \mathbb{C}^k \).

The following classical theorem due to Poincaré describes all holomorphic automorphisms of \( \Delta^q \) in terms of disc automorphisms. We formulate this theorem for the "poly-halfplane" \( \mathbb{H}^q \), which is, on the one hand, biholomorphic to the polydisc \( \Delta^q \), but, on the other hand, seems to be more convenient in the study of fixed-point free holomorphic self-maps.

**Theorem 3.1 (Poincaré).** Every holomorphic automorphism \( \tau \) of \( \mathbb{H}^q \) is a map of the form

\[
\mathbb{H}^q \ni (z_1, z_2, \ldots, z_q) \mapsto \left( \gamma_1(z_{p(1)}), \gamma_2(z_{p(2)}), \ldots, \gamma_q(z_{p(q)}) \right),
\]

where \( p \) is a permutation of \( \{1, \ldots, q\} \) and \( \gamma_j \in \text{Aut}(\mathbb{H}) \) for all \( j = 1, \ldots, q \).

Unfortunately, the study of dynamics of polydisc automorphisms does not seem to benefit much from the direct application of the representation given in Theorem 3.1. In this section we show that Theorem 3.1 leads to another representation, a normal form for polydisc automorphisms, which turns out to be much more informative from the dynamical point of view. We start by introducing what we call cycle automorphisms.

**Definition 3.2.** A cycle automorphism of \( \mathbb{H}^k \) is an automorphism \( \tau \) of the form

\[
\mathbb{H}^k \ni z = (z_1, z_2, \ldots, z_k) \mapsto \left( \gamma_1(z_2), \gamma_2(z_3), \ldots, \gamma_k(z_1) \right),
\]

where \( \gamma_j \in \text{Aut}(\mathbb{H}) \) for all \( j = 1, \ldots, k \).

In this definition we allow \( k = 1 \), in which case the cycle automorphisms are just automorphisms of \( \mathbb{H} \). The reason to consider cycle automorphisms is explained in the following remark.

**Remark 3.3.** Using the decomposition of the permutation \( \sigma \) into cycles, one can rephrase Poincare’s Theorem 3.1 as follows: every holomorphic automorphism \( \tau : \mathbb{H}^q \to \mathbb{H}^q \) can be represented, up to a permutation of the coordinates, as a direct sum of cycle automorphisms. More precisely, for any \( \tau \in \text{Aut}(\mathbb{H}^q) \) there exists a partition \( \prod_{\nu=1}^n J_\nu = \{1, \ldots, q\} \), bijective maps \( p_\nu : \{1, \ldots, k_\nu\} \to J_\nu \) and cycle automorphisms \( \tau_\nu : \mathbb{H}^{k_\nu} \to \mathbb{H}^{k_\nu}, \nu = 1, \ldots, n, \) such that

\[
\bar{\pi}_\nu \circ \tau = \tau_\nu \circ \bar{\pi}_\nu \quad \text{for all } \nu = 1, \ldots, n,
\]

where \( \bar{\pi}_\nu : \mathbb{H}^q \ni (z_1, \ldots, z_q) \mapsto (z_{p_\nu(1)}, \ldots, z_{p_\nu(k_\nu)}) \in \mathbb{H}^{k_\nu} \).
In what follows, representation (3.2) in the above remark will be referred to as a cycle decomposition of \( \tau \). Obviously, it is unique up to a change of order of \( J_\nu \)'s and pre-compositions of \( p_\nu \)'s with cyclic shifts of the sets \( \{1, \ldots, k_\nu\} \).

**Remark 3.4.** Note that for a cycle automorphism \( \tau \) defined by (3.1),

\[
\tau^k(z_1, z_2, \ldots, z_k) = (\Gamma_1(z_1), \Gamma_2(z_2), \ldots, \Gamma_k(z_k)),
\]

where \( \Gamma_j, j = 1, \ldots, k \), are automorphisms of \( \mathbb{H} \) determined recurrently by

\[
\Gamma_1 := \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_k \quad \text{and} \quad \Gamma_{j+1} = \gamma_j^{-1} \circ \Gamma_j \circ \gamma_j \quad \text{for} \ j = 1, \ldots, k - 1.
\]

In particular, all \( \Gamma_j \) are conjugate to each other.

As a consequence, it is particularly easy to extend the dynamical classification of the disc automorphisms to the cycle automorphisms.

**Definition 3.5.** Let \( \tau : \mathbb{H}^k \to \mathbb{H}^k \) be a cycle automorphism. We say that \( \tau \) is elliptic, parabolic, or hyperbolic, depending on whether the automorphism \( \Gamma_1 \) (and hence any of \( \Gamma_j \)'s) is an elliptic, parabolic or hyperbolic self-map of \( \mathbb{H} \), respectively.

In the elliptic case, we define the multipliers \( \lambda_\tau \) of \( \tau \) to be the \( k \)-th roots of \( \Gamma'_1(p) \), where \( p \in \mathbb{H} \) is the (unique) fixed point of \( \Gamma_1 \). In the parabolic and hyperbolic cases, we define the dilation of \( \tau \) to be \( \lambda_\tau := \sqrt[k]{\lambda_{\Gamma_1}} > 0 \), where \( \lambda_{\Gamma_1} \) stands for the dilation of \( \Gamma_1 \) at its Denjoy–Wolff point.

The following result gives a normal form for cycle automorphisms (and, thanks to Remark 3.3, for general automorphisms). For completeness we will treat also the elliptic case, although we will not need it in what follows.

**Theorem 3.6.** Let \( \tau : \mathbb{H}^k \to \mathbb{H}^k \) be a cycle automorphism. Then following statements hold.

(i) If \( \tau \) is hyperbolic or parabolic, then there exists \( g \in \text{Aut}(\mathbb{H}^k) \) of the form

\[
g(z_1, z_2, \ldots, z_k) = (g_1(z_1), g_2(z_2), \ldots, g_k(z_k)), \quad \text{where} \ g_j \in \text{Aut}(\mathbb{H}), \ j = 1, \ldots, k,
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{H}^k & \xrightarrow{\tau} & \mathbb{H}^k \\
g \downarrow & & \downarrow \ g \\
\mathbb{H}^k & \xrightarrow{L} & \mathbb{H}^k
\end{array}
\]

where \( L := \frac{1}{\lambda_\tau} \sigma_k \) in the hyperbolic case and \( L := \sigma_k \pm (1, 1, \ldots, 1) \) in the parabolic case.

(ii) If \( \tau \) is elliptic, then for each of the \( k \) values of the multiplier \( \lambda_\tau \) there exists a biholomorphism \( g : \mathbb{H}^k \to \Delta^k \) of the form
where $T$ and define the functions $V$ and $g$

3.7 Remark

In the hyperbolic case, the intertwining map is a translation of the form $z \mapsto z + (a, \ldots, a)$, where $a \in \mathbb{R}$. The canonical form $L$ is unique in this case.

Proof. Let us first assume first that $\tau$ is hyperbolic. Adopting notation of Remark 3.4, we see that there exists $g_1 \in \text{Aut}(\mathbb{H})$ such that $g_1 \circ \Gamma_1 = g_1/\lambda_{\Gamma_1}$. Then the function $V_1 := g_1 \circ \pi_{1,k}$ satisfies

$$V_1 \circ \tau^k = \frac{1}{\lambda_{\Gamma_1}} V_1. \quad (3.3)$$

For $j = 2, \ldots, k + 1$, define the functions $V_j$ recurrently by $V_j := \lambda_{\tau}(V_{j-1} \circ \tau)$. Thanks to (3.3) we see that $V_{k+1} = V_1$. It follows that

$$(V_1, V_2, \ldots, V_k) \circ \tau = \frac{1}{\lambda_{\tau}} (V_2, \ldots, V_k, V_1) \quad (3.4)$$

Using the very definition of cycle automorphism (Definition 3.2), it is easy to see that $V_j = g_j \circ \pi_{j,k}$ for all $j = 1, \ldots, k$, where the $g_j$’s are defined recurrently by $g_j := \lambda_{\tau}(g_{j-1} \circ \gamma_{j-1})$, $j = 2, \ldots, k$. This proves the theorem for a hyperbolic cycle automorphisms $\tau$.

The case of a parabolic automorphism $\tau$ is treated essentially in the same way, except that $g_1 \in \text{Aut}(\mathbb{H})$ is defined as a solution to the functional equation $g_1 \circ \Gamma_1 = g_1 \pm k$, the equation (3.3) is replaced by

$$V_1 \circ \tau^k = V_1 \pm k,$$

and the functions $V_j, g_j$ for $j = 2, \ldots, k + 1$ are defined by $V_j := (V_{j-1} \circ \tau) \mp 1$ and $g_j := (g_{j-1} \circ \gamma_{j-1}) \mp 1$. In all the formulas, the choice of the sign “+” or “−” is determined uniquely by the canonical form of $\Gamma_1$.

Finally, the proof for the elliptic case can be also carried out using essentially the same argument as above if we define the biholomorphism $g_1 : \mathbb{H} \to \Delta$ to be a solution to $g_1 \circ \Gamma_1 = \lambda_{\Gamma_1} g_1$, replace equation (3.3) with the Schröder equation

$$V_1 \circ \tau^k = \lambda_{\Gamma_1} V_1,$$

and define the functions $V_j, g_j$ for $j = 2, \ldots, k + 1$ recurrently by $V_j := \lambda_{\tau}^{-1}(V_{j-1} \circ \tau)$ and $g_j := \lambda_{\tau}^{-1}(g_{j-1} \circ \gamma_{j-1})$. \hfill \Box

Remark 3.7. Using the fact that $L^k$ is of the form $(z_1, \ldots, z_k) \mapsto (T(z_1), \ldots, T(z_k))$, where $T : \mathbb{C} \to \mathbb{C}$ is an affine map, it is not difficult to see that for the hyperbolic case, the intertwining map $g$ in the above theorem is unique up to multiplication by a positive number. Similarly, for the parabolic case, $g$ is unique up to post-composing with a translation of the form $z \mapsto z + (a, \ldots, a)$, where $a \in \mathbb{R}$. The canonical form $L$ is unique in
both cases, in particular, the choice of the sign for \( L = \sigma_k \pm (1, \ldots, 1) \) in the parabolic case is determined uniquely by the automorphism \( \tau \). At the same time, for the elliptic case the canonical form \( L \) is determined by \( \tau \) and by the choice of the multiplier \( \lambda_\tau \). Independently of the choice of the multiplier, the intertwining map \( g \) is unique up to multiplication by a number of absolute value one, unless \( \tau_k = \text{id}_{\mathbb{H}^k} \). The canonical forms corresponding to different choices of the multiplier are conjugate by the linear map \((z_1, z_2, \ldots, z_k) \mapsto (z_1, \mu z_2, \ldots, \mu^{k-1} z_k)\), where \( \mu^k = 1 \). Finally, in the “degenerate” case \( \tau_k = \text{id}_{\mathbb{H}^k} \), the intertwining map \( g \) is defined up to an arbitrary holomorphic automorphism of \( \mathbb{H} \).

Remark 3.8. From Theorem 3.6 it follows that a cycle automorphism \( \tau : \mathbb{H}^k \to \mathbb{H}^k \) is elliptic if and only if it has a fixed point in \( \mathbb{H}^k \).

4. DYNAMICAL CLASSIFICATION FOR HOLOMORPHIC SELF-MAPS OF THE POLYDISC

We now introduce a classification for holomorphic self-maps of the polydisc using the divergence rate.

**Definition 4.1.** Let \( f : \mathbb{H}^q \to \mathbb{H}^q \) be a holomorphic self-map. Then we say that

i) \( f \) is elliptic if it admits a fixed point \( z \in \mathbb{H}^q \),

ii) \( f \) is parabolic if it admits no fixed point in \( \mathbb{H}^q \) and \( c(f) = 0 \),

iii) \( f \) is hyperbolic if \( c(f) > 0 \) (which implies that \( f \) admits no fixed point in \( \mathbb{H}^q \)).

If \( f \) is not elliptic, then we define its dilation to be \( \lambda_f := e^{-c(f)} \in (0, 1] \). We say that \( f \) is of non-zero step if \( \lim_{n \to \infty} k_{\mathbb{H}^q}(f^n(z), f^{n+1}(z)) \neq 0 \) for all \( z \in \mathbb{H}^q \).

It is easy to see that, if \( \tau \) is a cycle automorphism, then this definition is coherent with Definition 3.5. We now study the case of an arbitrary automorphism \( \tau \) of \( \mathbb{H}^q \).

**Proposition 4.2.** Let \( \tau \in \text{Aut}(\mathbb{H}^q) \). Then \( c(\tau) = \max_{\nu} c(\tau_\nu) \), where the maximum is taken over all cycle automorphisms \( \tau_\nu \) in the cycle decomposition of \( \tau \).

**Proof.** We adopt here the notation of Remark 3.3. Using that remark together with Remark 3.4, we see that if \( Q := \prod_{\nu=1}^n k_\nu \), then

\[
\tau^Q(z_1, z_2, \ldots, z_q) = (T_1(z_1), T_2(z_2), \ldots, T_q(z_q)) \quad \text{for all } (z_1, \ldots, z_q) \in \mathbb{H}^q,
\]

where \( T_j \in \text{Aut}(\mathbb{H}) \), \( j = 1, \ldots, q \). Moreover, \( c(T_j) = Qc(\tau_\nu) \) for all \( j \in J_\nu \) and all \( \nu = 1, \ldots, n \).

Recall that \( k_{\mathbb{H}^q}(z, w) = \max_{j=1,\ldots,q} k_{\mathbb{H}}(z_j, w_j) \) for any pair of points \( z = (z_1, \ldots, z_q) \) and \( w = (w_1, \ldots, w_q) \) in \( \mathbb{H}^q \). Then \( c(\tau^Q) = \max_{\nu} Qc(\tau_\nu) \), which implies the result, see [6, Remark 2.5].

This immediately yields the following

**Corollary 4.3.** Let \( \tau \in \text{Aut}(\mathbb{H}^q) \). Then:

1. \( \tau \) is elliptic if and only if its cycles are all elliptic,
(2) \( \tau \) is parabolic if and only if it admits a parabolic cycle and does not admit hyperbolic ones.
(3) \( \tau \) is hyperbolic if and only if it admits a hyperbolic cycle, and in this case its dilation \( \lambda_{\tau} \) is the minimum of the dilations of its hyperbolic cycles.

5. The Valiron equation

First we show that hyperbolicity of \( f \) is a necessary condition for the Valiron equation to have solutions with values in \( \mathbb{H} \).

**Proposition 5.1.** Let \( f : \mathbb{H}^N \to \mathbb{H}^N \) be a holomorphic self-map. Suppose that there exists a holomorphic function \( V : \mathbb{H}^N \to \mathbb{H} \) and a constant \( \mu \in (0, 1) \) such that

\[
V \circ f = \frac{1}{\mu} V.
\]

Then \( f \) is hyperbolic with dilation \( \lambda_f \leq \mu \).

**Proof.** Since the divergence rate of the automorphism \( z \mapsto \frac{1}{\mu} z \) equals \(-\log \mu\), by [6, Lemma 2.9] we have that \(-\log \lambda_f := c(f) \geq -\log \mu > 0\). \( \square \)

**Definition 5.2.** Let \( f : \mathbb{H}^N \to \mathbb{H}^N \) be a hyperbolic holomorphic self-map with dilation \( \lambda_f \). By a Valiron function \( V \) of \( f \) we mean a holomorphic function \( V : \mathbb{H}^N \to \mathbb{H} \) that solves the Valiron equation

\[
V \circ f = \frac{1}{\lambda_f} V. \tag{5.1}
\]

For hyperbolic cycle automorphisms we can give a complete characterization of the family of all Valiron functions. According to Theorem 3.6, without loss of generality we may assume that \( \tau \) is in its canonical form, i.e. \( \tau := \frac{1}{\lambda} \sigma_k \) for some constant \( \lambda \in (0, 1) \).

**Proposition 5.3.** Let \( \tau := \frac{1}{\lambda} \sigma_k \), where \( \lambda \in (0, 1) \). Then \( V \) is a Valiron function of \( \tau \) if and only if

(i) \( V(rw) = rV(w) \) for all \( r > 0 \) and all \( w \in \mathbb{H}^k \);
(ii) \( V \circ \sigma_k = V \).

Moreover, \( V(\mathbb{H}^k) = \mathbb{H} \) for any Valiron function \( V \) of \( \tau \).

**Proof.** It is an easy exercise to check that if \( V : \mathbb{H}^k \to \mathbb{H} \) is holomorphic and satisfies (i) and (ii), then it is a Valiron function of \( \tau \).

Let us prove the converse. Suppose that \( V \) is a Valiron function of \( \tau \). Then

\[
V(w/\lambda^k) = V(w)/\lambda^k \quad \text{for all } w \in \mathbb{H}^k. \tag{5.2}
\]

We have to show that it satisfies (i) and (ii). For \( a = (a_1, \ldots, a_k) \in (\mathbb{R}^+)^k \) consider the map \( V_a(\zeta) := V(\zeta a), \zeta \in \mathbb{H} \). Then

\[
V_a(\zeta/\lambda^k) = V_a(\zeta)/\lambda^k \quad \text{for all } \zeta \in \mathbb{H}.
\]
By a result of Heins [16], it follows that $V_a(\zeta) = -iV(ia)\zeta$ for all $\zeta \in \mathbb{H}$.

In particular,

$$V(ir) = V_a(ir) = V(ia)r \quad \text{for all } r > 0 \text{ and all } a \in (\mathbb{R}^+)^k.$$

Applying now the Uniqueness Principle for holomorphic functions, we obtain (i). In its turn (i) yields

$$V \circ \sigma_k = V \circ (\lambda \tau) = \lambda(V \circ \tau) = V.$$

Finally, the chain of inclusions

$$\mathbb{H} = V_a(\mathbb{H}) \subset V(\mathbb{H}^k) \subset \mathbb{H},$$

shows that $V(\mathbb{H}^k) = \mathbb{H}$ whenever $V$ is a Valiron function of $\tau$. \hfill \Box

Now we consider the case of an arbitrary hyperbolic automorphism $\tau : \mathbb{H}^q \to \mathbb{H}^q$ with dilation $\lambda$. According to Remark 3.3 and Proposition 4.2, up to a reordering of variables, $\tau$ can be written in the following form

$$\mathbb{H}^q \cong \mathbb{H}^m \times \mathbb{H}^{q-m} \ni (z, w) \mapsto (\hat{\tau}(z), \tilde{\tau}(w)) \in \mathbb{H}^m \times \mathbb{H}^{q-m}, \quad (5.3)$$

where $0 < m \leq q$ and the cycle decomposition for $\hat{\tau}$ contains only hyperbolic cycle automorphism with dilation $\lambda$ while the cycle decomposition for $\tilde{\tau}$ may contain only non-hyperbolic cycle automorphism or hyperbolic cycle automorphisms with strictly greater dilation.

Applying Theorem 3.6 to each of the cycle automorphisms in the decomposition of $\hat{\tau}$, we see that we can assume without loss of generality that $\hat{\tau} := \frac{1}{\lambda} \hat{\sigma}$, where $\hat{\sigma} : \mathbb{C}^m \to \mathbb{C}^m$ is a linear map of the form

$$\hat{\sigma}(z_1, z_2, \ldots, z_m) := (z_{\hat{\sigma}(1)}, z_{\hat{\sigma}(2)}, \ldots, z_{\hat{\sigma}(m)}),$$

defined by a permutation $\hat{\sigma}$ of $\{1, \ldots, m\}$.

**Theorem 5.4.** In the above notation, a holomorphic function $V : \mathbb{H}^m \times \mathbb{H}^{q-m} \to \mathbb{H}$ is a Valiron function of $\tau$ if and only if $V(z, w)$ does not depend on $w \in \mathbb{H}^{q-m}$ and

1. $V(rz) = rV(z)$ for all $r > 0$ and all $z \in \mathbb{H}^m$;
2. $V \circ \hat{\sigma} = V$.

Moreover, $V(\mathbb{H}^q) = \mathbb{H}$ for any Valiron function $V$ of the automorphism $\tau$.

**Remark 5.5.** It follows immediately from the above theorem that a hyperbolic automorphism of a polydisc has infinitely many Valiron functions.

**Proof of Theorem 5.4.** As in Proposition 5.3, it is easy to check that if $V(z, w)$ does not depend on $w \in \mathbb{H}^{q-m}$ and satisfies (i) and (ii), then it is a Valiron function of $\tau$.

Now we show that if $V$ is a Valiron function, then $V(z, w)$ does not depend on $w \in \mathbb{H}^{q-m}$. Once this is proved, the rest of the proof of the theorem repeats almost literally that of Proposition 5.3 except that the shift map $\sigma_k$ is replaced by the map $\hat{\sigma}$.

We assume $m < q$, otherwise there is nothing to prove. Let $Q \in \mathbb{N}$ be defined as in the proof of Proposition 4.2. Then we have:
is also a canonical Kobayashi hyperbolic semi-model for $f$.

(b) in particular, $k_{H^m}(z, \hat{\tau}^Q(z)) = jk_H(i, i/\lambda^Q) = jQc(\tau)$ for all $j \in \mathbb{N}$ and any $z \in (i\mathbb{R}^+)^m$;

(c) $k_{H^{q-m}}(w, \tilde{\tau}^Q(w))/j \to Qc(\tilde{\tau}) < Qc(\tau)$ as $j \to +\infty$ uniformly on compact subsets with respect to $w \in H^{q-m}$.

Fix some $z_0 \in (i\mathbb{R}^+)^m$ and some closed ball $B \subset H^{q-m}$. According to (b) and (c) there exists $j_0 \in \mathbb{N}$ such that if $M := j_0Q$, then

$$k_H((z_0, w), \hat{\tau}^M(z_0, w)) = \max \{k_{H^m}(z_0, \hat{\tau}^M(z_0)), k_{H^{q-m}}(w, \tilde{\tau}^M(w))\} = Mc(\tau)$$

for all $w \in B$. Therefore, on the one hand,

$$k_H(V(z_0, w), V(\tau^M(z_0, w))) \leq k_{H^q}((z_0, w), \tau^M(z_0, w)) = Mc(\tau).$$

On the other hand, $V(\tau^M(z_0, w)) = V(z_0, w)/\lambda^M$ because $V$ is a Valiron function of $\tau$. Thus

$$k_H(V(z_0, w), V(\tau^M(z_0, w))) \geq k_H(i, i/\lambda^M) = Mc(\tau),$$

and the equality is attained only if $V(z_0, w) \in i\mathbb{R}^+$.

This shows that $V(z_0, w) \in i\mathbb{R}^+$ for all $w \in B$ and hence, by the Open Mapping Theorem, $V(z_0, \cdot) : H^{q-m} \to H$ is a constant function for any $z_0 \in (i\mathbb{R}^+)^m$. By the Uniqueness Principle for holomorphic functions, this means that $V(z, w)$ depends only on $z$ in the whole $H^m \times H^{q-m}$.

Now we prove the main result of this section.

**Theorem 5.6.** Let $f : H^N \to H^N$ be a hyperbolic holomorphic self-map with dilation $\lambda_f$. Then $f$ admits a canonical Kobayashi hyperbolic semi-model $(H^q, \ell, \tau)$, where $1 \leq q \leq N$ and $\tau$ is a hyperbolic automorphism of $H^N$ with dilation $\lambda_\tau = \lambda_f$. Moreover, a holomorphic function $V : H^N \to H$ is a Valiron function of $f$ if and only if $V = \hat{V} \circ \ell$, where $\hat{V} : H^q \to H$ is a Valiron function of the automorphism $\tau$. In particular, every Valiron function $V$ of $f$ satisfies

$$\bigcup_{n \geq 0} \lambda_f^n V(H^N) = H. \quad (5.4)$$

**Proof.** By Theorem 2.8, there exists a canonical Kobayashi hyperbolic semi-model $(Z, \hat{\ell}, \hat{\tau})$ for $f$. Since $Z$ is a holomorphic retract of $H^N$, by [15] there exists a biholomorphism $\psi : Z \to H^q$, where $0 \leq q \leq N$. Clearly

$$(H^q, \ell := \psi \circ \hat{\ell}, \tau := \psi \circ \hat{\tau} \circ \psi^{-1})$$

is also a canonical Kobayashi hyperbolic semi-model for $f$.

By assertion (2) of Theorem 2.8 we have that $c(\tau) = c(f) > 0$. Hence $q \geq 1$ and $\tau$ is a hyperbolic automorphism with $\lambda_\tau = \lambda_f$.

If $\hat{V} : H^q \to H$ is a Valiron function of the automorphism $\tau$, then clearly $V := \hat{V} \circ \ell$ is a Valiron function of $f$. Conversely, assume that $V : H^N \to H$ is a Valiron function
of \( f \), and set \( \Omega := \cup_{n \geq 0} \lambda^n V(\mathbb{H}^N) \). Then \( (\Omega, V, z \mapsto \frac{1}{\lambda} z) \) is a semi-model for \( f \) with Kobayashi hyperbolic base space. Since \( (\mathbb{H}^q, \ell, \tau) \) is a canonical Kobayashi hyperbolic semi-model for \( f \), there exists a semi-model morphism \( \hat{\eta} : (\mathbb{H}^q, \ell, \tau) \to (\Omega, V, z \mapsto \frac{1}{\lambda} z) \).

The holomorphic function \( \hat{V} := \eta \) is a Valiron function of \( \tau \) and it satisfies \( V = \hat{V} \circ \ell \).

Finally, equality (5.4) holds because, by Theorem 5.4, \( H = \hat{V}(\mathbb{H}^q) \subset \Omega \subset \mathbb{H} \).

**Remark 5.7.** At the end of preparation of this paper, the authors got to know that Wang and Deng [24] recently proved existence of a Valiron function under quite restrictive additional assumptions: in particular, it is supposed that there exists an orbit \( (f^n(z_0))_{n \in \mathbb{N}} \) converging to a point on the boundary of the polydisc within a \( K \)-region. Consider, e.g., the self-map \( \mathbb{H} \times \mathbb{D} \ni (z, w) \mapsto (e^{\alpha \pi z}, \frac{2 + w}{3} \exp(i \log z)) \in \mathbb{H} \times \mathbb{D} \), where \( \alpha > 0 \) is an irrational number and \( \log z \) stands for the single-valued branch that takes values in \( \{ \zeta : \text{Im} \zeta \in (0, \pi) \} \) for \( z \in \mathbb{H} \). It is easy to see that according to Definition 4.1, this self-map is hyperbolic and hence, by Theorem 5.6, it admits a Valiron function, although neither the map itself nor any of its iterates has an orbit convergent to a boundary point.

### 6. The Abel equation

First we show that \( f \) must be a self-map of non-zero step for the Abel equation to have a solution with values in \( \mathbb{H} \).

**Proposition 6.1.** Let \( f : \mathbb{H}^N \to \mathbb{H}^N \) be a holomorphic self-map, and assume that there exists a holomorphic function \( \Theta : \mathbb{H}^N \to \mathbb{H} \) and a constant \( \alpha \in \{-1, 1\} \) such that

\[
\Theta \circ f = \Theta + \alpha.
\]

Then \( f \) is of non-zero step.

**Proof.** For all \( x \in \mathbb{H}^N \) and all \( m \geq 0 \), we have

\[
k_{\mathbb{H}^N}(f^m(x), f^{m+1}(x)) \geq k_{\mathbb{H}}(\Theta(f^m(x)), \Theta(f^{m+1}(x)))
= k_{\mathbb{H}}(\Theta(x) + \alpha m, \Theta(x) + \alpha(m + 1)) = k_{\mathbb{H}}(\Theta(x), \Theta(x) + \alpha).
\]

Therefore,

\[
s(x) = \lim_{m \to \infty} k_{\mathbb{H}^N}(f^m(x), f^{m+1}(x)) \geq k_{\mathbb{H}}(\Theta(x), \Theta(x) + \alpha) > 0
\]

for all \( x \in \mathbb{H}^N \), what was to be shown. \( \square \)

**Theorem 6.2.** Let \( f : \mathbb{H}^N \to \mathbb{H}^N \) be a parabolic holomorphic self-map of non-zero step. Then it admits a canonical Kobayashi hyperbolic semi-model \( (\mathbb{H}^q, \ell, \tau) \), where \( 1 \leq q \leq N \) and \( \tau \) is a parabolic automorphism of \( \mathbb{H}^N \). Moreover, there exists \( \alpha \in \{-1, 1\} \) and a holomorphic solution \( \Theta : \mathbb{H}^N \to \mathbb{H} \) to the Abel equation

\[
\Theta \circ f = \Theta + \alpha
\]
that satisfies
\[
\bigcup_{n \geq 0} \{ \Theta(H^N) - n\alpha \} = \mathbb{H}.
\]

**Proof.** By Theorem 2.8 there exists a canonical Kobayashi hyperbolic semi-model \((Z, \hat{\ell}, \hat{\tau})\) for \(f\). Since \(Z\) is a holomorphic retract of \(H^N\), by [15] there exists a biholomorphism \(\psi: Z \to H^q\), where \(0 \leq q \leq N\). Clearly
\[
(H^q, \ell := \psi \circ \hat{\ell}, \tau := \psi \circ \hat{\tau} \circ \psi^{-1})
\]
is also a canonical Kobayashi hyperbolic semi-model for \(f\).

Fix any \(z \in H^q\). By the definition of a semi-model, there exists \(x \in H^N\) and \(n \geq 0\) such that \(\tau^{-n}(\ell(x)) = z\). Then, by assertion (1) of Theorem 2.8, \(k_{H^q}(z, \tau(z)) = s(x) > 0\).

Hence \(q \geq 1\) and \(\tau\) is not elliptic. By (2) of Theorem 2.8, we have \(c(\tau) = c(f) = 0\). Hence \(\tau\) is parabolic. By Corollary 4.3, the cycle decomposition of \(\tau\) contains at least one parabolic cycle automorphism \(\tau_\nu\). By Theorem 3.6 (i), there exists \(g \in \text{Aut}(H^{k_\nu})\) such that
\[
g^{-1} \circ \tau_\nu \circ g = \sigma_{k_\nu} + \alpha(1,1,\ldots,1)
\]
for \(\alpha \equiv 1\) or \(\alpha \equiv -1\).

With notations of Remark 3.3, set \(\mu(z_1, \ldots, z_{k_\nu}) := \frac{1}{k_\nu} \sum_{j=1}^{k_\nu} z_j\) and \(A := \mu \circ g \circ \pi_\nu\). Using (3.2) and (6.2), it easy to see that \(A \circ \tau = A + \alpha\). Therefore, \(\Theta \circ f = \Theta + \alpha\), where \(\Theta := A \circ \ell\). This proves the existence of solutions to the Abel equation.

Finally, note that \(A(H^q) = \mu(H^{k_\nu}) = H^q\). By definition of a semi-model, we have \(\bigcup_{n \geq 0} \tau^{-n}(\ell(H^N)) = H^q\). Therefore, \(H = A(H^q) = \bigcup_{n \geq 0} A(\tau^{-n}(\ell(H^N))) = \bigcup_{n \geq 0} [A(\ell(H^N)) - \alpha n]\). This implies (6.1). The proof is complete.

**Remark 6.3.** There are parabolic univalent self-maps of the polydisc whose canonical Kobayashi hyperbolic semi-models are elliptic\(^1\). An elementary example is the self-map of \(\Delta \times \mathbb{H}\) defined by \(f(z, w) = (\lambda z, w + i)\) with \(|\lambda| = 1\). Its canonical Kobayashi hyperbolic semi-model is \((\Delta, (z, w) \mapsto z, z \mapsto \lambda z)\). Clearly, by Theorem 6.2 such self-maps are not of non-zero step. Whether or not such a phenomenon can appear in the unit ball is an open question [6, Section 5.5].

\(^1\)A semi-model \((Z, \ell, \psi)\) is said to be elliptic, if the automorphism \(\psi\) has a fixed point in \(Z\).


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