Phase space interpretation of exponential Fermi acceleration

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Abstract. Recently, the occurrence of exponential Fermi acceleration (FA) has been reported in a rectangular billiard with an oscillating bar inside (Shah et al 2010 Phys. Rev. E 81 056205). In this paper, we analyze the underlying physical mechanism and show that the phenomenon can be understood as a sequence of highly correlated motions, consisting of alternating phases of free propagation and motion along the invariant spanning curves of the well-known one-dimensional Fermi–Ulam model. The key mechanism for the occurrence of exponential FA can be captured in a random walk model in velocity space with step width proportional to the velocity itself. The model reproduces the occurrence of exponential FA and provides a good \textit{ab initio} prediction of the value of the growth rate, including its full parameter dependence. Our analysis clearly points out the requirements for exponential FA, thereby opening the prospect of finding other systems exhibiting this unusual behavior.

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1. Introduction

In recent years, the study of Fermi acceleration (FA) in two-dimensional (2D) time-dependent billiards has attracted a great deal of attention [1–7]. FA is the unbounded energy gain of particles exposed to driving forces and was first proposed in 1949 by Enrico Fermi [8] to explain the high energies of cosmic radiation (for a review of FA, see [9]). He suggested that charged particles repeatedly interact with time-dependent magnetic fields (originating either from shock waves of supernovae or from magnetized interstellar clouds) in such a way that on average they gain energy. Nowadays, FA is investigated in a variety of systems belonging to different areas of physics, such as astrophysics [10–12], plasma physics [13, 14] and atom optics [15, 16] and has even been used in the interpretation of experimental results in atomic physics [17].

The 1D prototype system allowing the investigation of FA is the so-called Fermi–Ulam model (FUM) [18], which consists of non-interacting particles moving between one fixed and one oscillating wall. The FUM and its variants have been the subject of extensive theoretical (see [18, 19] and references therein) and experimental [20, 21] studies. In the FUM, the existence of FA depends exclusively on the driving law of the oscillating wall: as long as the driving law is sufficiently smooth, there is no unlimited energy growth due to the existence of invariant spanning curves [18]. In particular, this means that harmonic driving laws do not lead to FA in the FUM.

In 2D time-dependent billiards, a smooth driving law may already lead to FA. For example, the existence of FA was shown for a harmonically oscillating stadium-like billiard [3, 4], in the driven eccentric annular billiard [5], in an oval billiard [6] and in the time-dependent elliptical billiard [7, 22, 23]. On the other hand, the breathing concentric annular billiard [5] and the circular billiard [24] do not exhibit FA. The ensemble-averaged energy \( E(t) \) in all the 2D time-dependent billiards that do show FA grows as a function of time according to a power law, i.e. \( E(t) \sim t^d \), with some exponent \( d \). For such power laws, it is known that FA is not structurally stable in the sense that a finite amount of dissipation will destroy it [1, 25–27], independently of whether dissipation is introduced via inelastic collisions or via drag forces (e.g. Stokes’ friction).

Thus, a natural question that arises is whether there are certain time-dependent billiards that show a somewhat ‘faster’ acceleration of energy, in particular whether there is e.g. an exponential acceleration. The first hint that such a fast acceleration process is possible is given in [28], where the authors prove the existence of single orbits with exponential energy growth under certain conditions. In [29], it is shown that the energy of a whole ensemble of particles grows exponentially in a rectangular billiard with an oscillating bar inside. This result is generalized in [2], where the authors show by means of an analysis of the
Figure 1. Setup: rectangular billiard of height $H$ and length $L$ with an oscillating bar of length $l$ placed in the center, parallel to the $x$-axis (here $H = 2$, $L = 4$ and $l = 2$). The two areas above and below the bar are the interaction areas (with the moving bar) A and B, respectively.

Anosov–Kasuga invariant that in special classes of billiard systems the ensemble-averaged energy accelerates exponentially.

Whereas in [2, 28, 29], a mathematical analysis of exponential FA is provided, the aim of this paper is to investigate the physical mechanism leading to exponential FA in the setup proposed in [29]. This setup consists of an oscillating bar inside a rectangular billiard, where the bar is aligned parallel to the long side of the rectangle. This can be interpreted as particles moving alternately in an FUM and in a static rectangular billiard. Since neither the static rectangular billiard nor the FUM alone even shows FA, we want to clarify from a physical point of view how the combination of both leads to exponential FA, i.e. what are the microscopic processes that cause the astonishingly fast acceleration. To this end, we will show that in the high velocity regime, the temporal movement of the particles on invariant curves of the FUM can be modeled by a suitable random walk with step sizes being proportional to the velocity itself. This random walk model shows exponential acceleration. Furthermore, the corresponding parameters of the random walk can be extracted from the underlying FUM, even enabling an alternative (compared with that given in [29]) prediction of the exponential acceleration rate without any free parameters.

The paper is structured as follows. In section 2, we introduce the setup and present the results of our numerical simulations. How the microscopic dynamics of single trajectories can be interpreted as piecewise motion along invariant spanning curves of an appropriate FUM is shown in section 3. With this picture in mind, we construct a random walk model in section 4 that generalizes the considered model, incorporating the basic characteristics of the underlying physical mechanism. Finally, a short summary and outlook is given in section 5.

2. The setup and results

The investigated setup is shown in figure 1. It consists of a rectangular billiard of length $L$ and height $H$ with an oscillating bar inside. The rectangular billiard without the bar is integrable, since upon collisions with the billiard boundary, only the sign of the corresponding component of the velocity $v$ is reversed; that is, $|v_x|$ and $|v_y|$ in particular are preserved. Now, a bar of
Figure 2. Semi-logarithmic plot of the time evolution of the ensemble averaged modulus of the velocity $\langle |v| \rangle(t)$. The velocity grows exponentially, $\langle |v| \rangle(t) \sim e^{Rt}$, with a growth rate of $R = 1.1 \times 10^{-5}$ ($a = 0.1$, $\omega = 0.02$).

length $L$ is placed in the middle of the billiard, parallel to the longer side ($x$-direction) of the billiard. Here, we assume a harmonic oscillation law: $y_b = a \cos(\omega t)$, with $a$ and $\omega$ being the driving amplitude and frequency, respectively. Since the oscillating bar transfers momentum in the $y$-direction only, $|v_x|$ is preserved. As an initial ensemble, we take $N = 10^4$ classical, non-interacting particles with a fixed velocity $v_x = 0.16$; and $v_y$ is randomly chosen in the interval $[0, 40v_x]$ (we use $H = 2$, $L = 4$ and $l = 2$ for the simulations). We iterate these particles by numerically solving the corresponding discrete mapping, i.e. by calculating the successive collisions with the billiard boundary, which consists of the rectangle and the bar. The main computational effort is to determine the time of the next collision with the oscillating bar, where the smallest root of an implicit equation (with possibly many roots) has to be found; see e.g. [30, 31]. The main quantity of interest is the time evolution of the ensemble-averaged modulus of the velocity in the $y$-direction ($v_x = $ const; $v_y \equiv v$), which is given by

$$\langle |v| \rangle(t) = \frac{1}{N} \sum_{i=1}^{N} |v_i(t)|,$$

where $v_i(t)$ is velocity of the $i$th particle at time $t$. The results of the simulations are shown in figure 2 for 3000 oscillations of the bar on a semi-logarithmic scale. The ensemble averaged velocity clearly grows exponentially $\langle |v| \rangle(t) \sim \exp(Rt)$, as reported in [29], with a growth rate of $R \approx 1.1 \times 10^{-5}$. Let us now develop a physical picture of this acceleration process, answering the question of how we can link the microscopic dynamics of single trajectories to the appearance of exponential FA.

3. Connection with the Fermi–Ulam model

Since the $v_x$ component of a particle’s velocity remains constant (see figure 1) as long as the particle is in one of the interaction areas, the dynamics in the $y$-direction corresponds exactly to
Figure 3. Phase space of the 1D FUM. For low $v$ there is a chaotic sea, whereas for high velocities there are invariant spanning curves. The difference between the maximum and the minimum of such a curve grows with increasing $v$. The red lines show the analytic results for the invariant spanning curves based on static wall approximation (SWA). In the inset, a single invariant spanning curve is shown for different driving amplitudes $a$. For large $a$, the SWA deviates significantly from the exact result.

that of a particle moving in a 1D FUM, where the distance between the equilibrium positions of the moving wall and the static wall is given by $h = H/2$. The time $t_I$ that the particle spends inside the interaction area is simply given by $t_I = l/v_x$. We define the time $t_F$ that the particle spends in the FUM by each passing of the interaction area (A or B, see figure 1) as the time difference between the first and the last collision with the oscillating bar while the particle is in the interaction area. The motivation for this definition is that only collisions with the oscillating bar change $v_y$; that is, we want to keep track of the time during which a certain change in the velocity takes place. The times $t_I$ and $t_F$ are not identical, since once a particle enters the interaction area, a certain amount of time will elapse before it collides with the bar. However, for high velocities $v$, which means that $v_x$ is large since $v_x = \text{const}$, $t_F$ converges toward $t_I$. By ‘high’ we mean that $v$ is large compared to the maximal velocity of the bar; that is, for $|v| \gg \omega a_0$ (which implies that $|v| \approx |v_y|$) we obtain $t_F \approx t_I = l/v_x$.

Since the dynamics of particles can be described for some time spans $t_F$ as a 1D FUM, it is convenient to summarize some of the properties of the FUM (for a more detailed description, see [18] and references therein). The phase space of the FUM is shown in figure 3. For low velocities there is a large chaotic sea containing many regular islands (this regime is shrunk to a narrow band $0 < v \lesssim 0.3$ in figure 3). Above the first invariant spanning curve (FISC) with velocity $v_c$, the motion becomes more and more regular, until for $v \gg v_c$ there are exclusively invariant spanning curves corresponding to a synchronized motion between the oscillating wall and the particles. Due to these invariant curves, there is no diffusion in momentum space and the FUM with a harmonic oscillation of the wall does not show FA.

The invariant spanning curves $v_{isc}(\phi)$ are not just straight lines, but show a characteristic shape (see figure 3). There are infinitely many of them, which can be labeled by the velocity $a_a$.
$v^{\text{isc}}(\phi = 0)$ and are parameterized as $v^{\text{isc}} = v^{\text{isc}}(\phi, \tilde{v})$. The minimum of these curves is always at $\phi = 0$ and the maximum at $\phi = \pi / \omega$. The difference $\Delta v^{\text{isc}} = v^{\text{isc}}(\pi, \tilde{v}) - v^{\text{isc}}(0, \tilde{v})$ grows linearly with increasing $\tilde{v}$, i.e. $\Delta v^{\text{isc}} \sim \tilde{v}$. For high velocities ($v \gg v_c$) this can be rigorously shown within the so-called SWA [18, 19], which assumes that the oscillating bar is fixed in coordinate space but transfers momentum as if it were moving. The distance between two collisions is then simply $2h$ and the time between two collisions is $\Delta t = 2h / v$, where $v$ is the velocity after the preceding wall collision. After a collision with the (only in momentum space moving) wall, the velocity change is

$$v(t) + \Delta t / \omega \sim \Delta v^{\text{isc}}.$$  

This means that the velocity is also much larger than the velocity $v_c$ of the FISC. A particle enters the interaction area, let us say above the bar (interaction area $A$), at time $t_i$ with a certain velocity $v_i = |v_i|$. If it is fast, it collides many times during the time $t_f$ with the bar, thus moving along the corresponding invariant spanning curve of the FUM and leaving the interaction area at time $t_2 = t_i + t_c$ with velocity $v_2$. The particle propagates in the free part of the rectangular billiard (i.e. the part where no collisions with the bar take place) for a time $t_f = 2d / v_c$. During this time, the modulus of the velocity does not change and thus the particle re-enters the interaction area at time $t_3 = t_2 + t_c$ with velocity $v_2$. The particle can enter either the interaction area $A$ or $B$, i.e. above or below the bar depending on the exact dynamics. However, for high velocities, to a good approximation this is a random process, as argued in [29]; that is, the particle will be injected with probability one-half above and with probability one-half below the bar. If the particle is injected in part $A$, it re-enters the same FUM as described above, now at the phase $\phi = \omega t_f \mod 2\pi$. If it is injected in part $B$ we have to add a phase shift of $\pi$, i.e. $\phi = (\omega t_f + \pi) \mod 2\pi$. Since the two FUMs (above and below the bar) can be transformed into each other simply by shifting the phase by $\pi$. Again, the particle

\[ v_1 = v^{\text{isc}}(t, \tilde{v}) - 2v_w(t), \]

where $v_w(t) = \tilde{y}_w(t) = -a\omega \sin(\omega t)$ is the velocity of the wall. Since for high velocity the particle moves on an invariant curve, we set $v_1 = v^{\text{isc}}(t + \Delta t, \tilde{v})$ and get

\[ v^{\text{isc}}(t + \Delta t, \tilde{v}) = v^{\text{isc}}(t, \tilde{v}) - 2v_w(t). \]

Expanding this equation into a Taylor series up to first order and applying the continuous limit $\Delta t \to 0$, we obtain $2h\tilde{v}^{\text{isc}}(t, \tilde{v}) / v^{\text{isc}}(t, \tilde{v}) = -2v_w(t)$. Integration yields

\[ v^{\text{isc}}(\phi, \tilde{v}) = \tilde{v} e^{(\phi / h)[1 - \cos \phi]}, \quad \phi \in [0, 2\pi). \]

Here, $\phi$ is the phase of the wall oscillation and is given by $\phi = \omega t \mod 2\pi$. Obviously, the difference $\Delta v^{\text{isc}}$ between the velocity maximum and minimum of an invariant spanning curve is

\[ \Delta v^{\text{isc}} = v^{\text{isc}}(\pi, \tilde{v}) - v^{\text{isc}}(0, \tilde{v}) = \tilde{v} \cdot (e^{2\pi / h} - 1) \sim \tilde{v} \]

and thus proportional to $\tilde{v}$. The $v^{\text{isc}}(\phi, \tilde{v})$ of equation (4) are shown in figure 3 as red lines. We obtained very good agreement with the exact results of the numerical simulations. However, the inset shows a single invariant spanning curve for different driving amplitudes $a$, and for large $a$, the SWA (equation (4)) deviates significantly from the exact result; that is, the SWA is valid for small driving amplitudes only.

Based on the above phase space description of the FUM, we can investigate the microscopic dynamics of a typical trajectory moving inside the oscillating bar billiard. Since we are ultimately interested in the acceleration process, we can assume that the particles are fast compared to the motion of the bar, i.e. again $v \gg a\omega$.

This means that the velocity is also much larger than the velocity $v_c$ of the FISC. A particle enters the interaction area, let us say above the bar (interaction area $A$), at time $t_i$ with a certain velocity $v_i = |v_i|$ and spends time $t_1 \approx t_f$ in it. Since it is fast, it collides many times during the time $t_f$ with the bar, thus moving along the corresponding invariant spanning curve of the FUM and leaving the interaction area at time $t_2 = t_i + t_c$ with velocity $v_2$. Now the particle propagates in the free part of the rectangular billiard (i.e. the part where no collisions with the bar take place) for a time $t_f = 2d / v_c$. During this time, the modulus of the velocity does not change and thus the particle re-enters at time $t_3 = t_2 + t_c$ the interaction area with velocity $v_2$. The particle can enter either the interaction area $A$ or $B$, i.e. above or below the bar depending on the exact dynamics. However, for high velocities, to a good approximation this is a random process, as argued in [29]; that is, the particle will be injected with probability one-half above and with probability one-half below the bar. If the particle is injected in part $A$, it re-enters the same FUM as described above, now at the phase $\phi = \omega t_f \mod 2\pi$. If it is injected in part $B$ we have to add a phase shift of $\pi$, i.e. $\phi = (\omega t_f + \pi) \mod 2\pi$. Since the two FUMs (above and below the bar) can be transformed into each other simply by shifting the phase by $\pi$. Again, the particle
Figure 4. A typical trajectory in the investigated setup (see figure 1). The particle enters the interaction area four times (curves 1, 3 and 4: area B; curve 2: area A). For a fixed time \( t_F \) (or phase \( \Delta \phi = \omega t_F \mod 2\pi \)) it moves along the invariant spanning curves of the corresponding FUM (thin red lines); see also figure 3. It then leaves the interaction area with a certain velocity before it re-enters area A or B with the same velocity but now with a different phase.

Let us describe the above process in a more quantitative way. The particle enters the interaction area at \( t_1 \) with a high velocity \( v_1 \gg a_0 \omega \) and \( v_1 \gg v_x \). The corresponding invariant spanning curve on which the particle will move for the time \( t_F \) can be calculated as follows: the entry phase \( \phi_1 \) is given by \( \phi_1 = \omega t_1 \mod 2\pi \); by setting \( v_1 = v^{isc}(\phi_1, \tilde{v}) = \tilde{v} e^{(a_0/\hbar)(1-\cos \phi_1)} \) we obtain \( \tilde{v} = v_1 e^{(-a_0/\hbar)(1-\cos \phi_1)} \). The exit velocity is then (remember \( t_F = l/v_x \))

\[
v_2 = v^{isc}(\phi_1 + \omega l/v_x, \tilde{v}) = v_1 e^{(a_0/\hbar)(1-\cos (\phi_1 + \omega l/v_x))}. \tag{6}
\]

This procedure can be repeated again and again, yielding \( v_n = \sum_{i=1}^{n} \Delta v_i + v_1 \), where \( \Delta v_i \) are obtained in the above described manner by exploiting the piecewise motion on the invariant spanning curves. However, this sum is not suitable for obtaining a closed expression for \( v(t) \) (or \( \langle |v| \rangle(t) \)), especially since it contains the random phase shifts of \( \pi \). We therefore employ, in the next section, a random walk model based on statistical properties of the above described procedure, and this will enable us to calculate explicitly the exponential growth rate.

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4. Random walk model

The velocity in the random walk model is written as

\[ v_{n+1} = v_n + \Delta v_n, \quad (7) \]

where \( \Delta v_n \) can be positive or negative. Since the \( \Delta v_n \) are determined by moving along parts of the invariant spanning curves of the FUM and for the latter we know that \( \Delta v_n \sim \tilde{v} \), we conclude that the \( \Delta v_n \) are proportional to \( v_n \), so

\[ \Delta v_n = \pm c v_n, \quad (8) \]

yielding

\[ v_{n+1} = v_n \pm c \cdot v_n = (1 \pm c)v_n. \]

The constant \( c \) is an effective constant; in addition to the geometry of the billiard, \( c \) depends in particular on the entry phase \( \phi \) of a particle into the FUM and of course on the time \( t_F = l/v_x \) (or phase \( \Delta \phi = \omega l/v_x \mod 2\pi \)) that the particle spends in the FUM. We consider an ensemble of particles all starting with the same \( v_x \); thus the phase shift \( \Delta \phi \) is the same for all particles and \( \Delta \phi \) is a constant. Under these assumptions, we proceed as follows.

- We show that the random walk model of equation \((8)\) leads to exponential FA.
- We determine the effective \( c \) using the phase space properties of the FUM.

To this end, we consider a particle after \( N \) steps, i.e. \( N \) cycles through the FUM. The probability \( p(k) \) to have completed \( k \) more positive than negative steps \( \Delta v \) is then

\[ p(k) = \frac{1}{2^N} \binom{N}{(N+k)/2}. \quad (9) \]

We define \( v_{(k)} \) as the velocity that is reached after \( N \) steps with \( k \) steps more in the positive than in the negative direction and \( v_{(-k)} \) as the velocity that is reached after \( N \) steps with \( k \) steps more in the negative than in the positive direction. There are of course many different paths, leading to the same \( v_{(k)} \); however, the order of the steps is irrelevant. Let us assume (without loss of generality) \( N \) to be even; then \( k \) can be any even number between \(-N\) and \( N \). The expectation value of the modulus of the velocity after \( N \) steps is given by summing over all possible \( k \)s and weighting them with the corresponding probability \( p(k) \):

\[ \langle |v| \rangle_N = \sum_{k=-N, \text{ even}}^{N} p(k) |v|_{(k)} = \frac{1}{2^N} \sum_{k=-N/2}^{N/2} \binom{N}{(N+k)/2} |v|_{(2k)}. \quad (10) \]

By setting \( \gamma := 1 + c \) and \( v_0 \) as the initial velocity, using equation \((8)\) for the positive sign, we obtain

\[ \langle |v| \rangle_N = v_0 \left( \frac{1 + \gamma^2}{2\gamma} \right)^N. \quad (11) \]

To switch from the number of cycles \( N \) to the actual time \( t \), we assume that the time between two collisions with the vertical walls is given by \( L/v_x \). Substituting \( N = v_x t/L \) yields the exponential time law

\[ \langle |v| \rangle (t) = v_0 e^{R t}, \quad (12) \]
where the growth rate $R$ is given by

$$R = \frac{v_c}{L} \ln \left( \frac{\gamma}{2} + 1/2\gamma \right).$$

(13)

The random walk model reproduces an exponential dependence of the ensemble averaged velocity on time. Nevertheless, in order to determine the system-specific value of the growth rate $R$, we still have to determine the effective constant $c$. To this end, we rewrite the part of equation (8) with the ‘+’ as $c = (v_{n+1} - v_n)/v_n$. The velocity $v_{n+1}$ is, according to equation (6), given by

$$v_{n+1} = v^{isc}(\phi_n + \Delta \phi, \tilde{v}(v_n, \phi_n)),$$

(14)

where we write $\tilde{v} = \tilde{v}(v_n, \phi_n)$, since $\tilde{v}$ depends on the entry phase $\phi_n$ and the entry velocity $v_n$. For a fixed $\Delta \phi$, we thus have $c = c(v_n, \phi_n) = v^{isc}(v_n, \phi_n + \Delta \phi)/v^{isc}(v_n, \phi_n) - 1$ with $v^{isc} \propto v_n$ leading to $c(v_n, \phi_n) = c(\phi_n)$. Thus, the effective $c$ is given by averaging over all entry phases $\phi_n$ that lead to a positive $\Delta v_n$ (this is sufficient, since the random walk model of equations (8) and (10) has intrinsically included the ‘−’ part, allowing the $\Delta v$ to be negative):

$$c_{\text{eff}} = \frac{1}{\phi_{n.2} - \phi_{n.1}} \int_{\phi_{n.1}}^{\phi_{n.2}} c(\phi_n) \, d\phi_n,$$

(15)

where

$$c(\phi_n) = \frac{\exp[1 - \cos(\phi_n + \Delta \phi) \Delta \phi / \omega]}{\exp[1 - \cos(\phi_n) \Delta \phi / \omega]} - 1.$$

(16)

The integral over $\phi_n$ has to be evaluated such that $\Delta v_n$ is positive in order to account for all accelerating trajectories, i.e. $\phi_{n.1} = -\Delta \phi/(2\omega)$ and $\phi_{n.2} = \pi/\omega - \Delta \phi/(2\omega)$ yielding for the normalization $N_\phi = 1/(\phi_{n.2} - \phi_{n.1}) = \omega/\pi$. Since the growth rate $R$ depends on $\gamma$ (see equation (13)) and $\gamma = 1 + c$, we finally obtain

$$\gamma_{\text{eff}} = 1 + c_{\text{eff}} = \frac{\omega}{\pi} \int_{-\Delta \phi/2\omega}^{\pi/\omega - \Delta \phi/2\omega} \frac{\exp[1 - \cos(\phi_n + \Delta \phi) \Delta \phi / \omega]}{\exp[1 - \cos(\phi_n) \Delta \phi / \omega]} \, d\phi_n$$

$$\equiv 1 + \langle |v| \rangle_{\phi}.$$

(17)

Here we assumed that the entry phases $\phi_n$ are uniformly distributed, which is valid on a sufficiently long time scale if the ratio $L/v_n$ is incommensurate with the driving period $T = 2\pi/\omega$. Note that the growth rate $R$ of equation (13), together with the corresponding $\gamma$ of equation (17), has been obtained ab initio without any fit parameters. However, the corresponding values are too small. From the results of the simulation shown in figure 2, for $a = 0.1$ and $\omega = 0.02$ we obtain for the growth rate $R \approx 1.1 \times 10^{-5}$. Inserting equation (17) into (13) leads to $R = 0.51 \times 10^{-5}$, which provides the order of magnitude of the numerical result, but is too small by about a factor 2. The reason for this is as follows: the sum of two sequences of the dynamics each consisting of e.g. three steps, one with a $c(\phi)$ close to the maximally possible value $c_{\text{max}}$ in each step and one with steps close to the minimal value of $c(\phi)$ (i.e. $c_{\text{min}} = 0$), contributes more significantly to the ensemble average $\langle |v| \rangle$ than the sum of two corresponding sequences both with step width $c = (c_{\text{max}} + c_{\text{min}})/2$. However, according to equation (17) we calculated a mean of the latter type and therefore obtained a lower bound of the correct growth rate. The deviation of the result obtained from the simulation and that from the model can thus be understood as a consequence of the negligence of correlations. One way of effectively

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including correlations to the definition of \(c_{\text{eff}}\) is to define \(c_{\text{eff}} = \langle c^m \rangle_\phi / \langle c^{m-1} \rangle_\phi\) with \(m > 1\). Here, \(\langle c^m \rangle_\phi\) denotes the average of \(c^m(\phi)\) over all phases in the interval \([-\Delta \phi/(2\omega), (\pi - \Delta \phi/2)/\omega]\) with \(c\) given by equation (16). An upper bound for the growth rate can be obtained within the assumption that all steps but those with maximal \(c(\phi)\) are suppressed, i.e. by calculating the effective \(c\) with the assumption that \(c(\phi)\) is equal to the maximally possible value \(c_{\text{max}}\) at each step:

\[
\gamma_{\text{max}} = 1 + c_{\text{max}} = \frac{e^{\frac{a}{\pi}[1-\cos(\pi/2+\Delta \phi/2)]}}{e^{\frac{a}{\pi}[1-\cos(\pi/2-\Delta \phi/2)]}} = 1 + \lim_{N \to \infty} \frac{\langle c^N \rangle_\phi}{\langle c^{N-1} \rangle_\phi}. \tag{18}
\]

This upper bound leads to \(R = 1.24 \times 10^{-5}\), which is quite close to the result obtained from the simulation. In order to test these estimations for a whole range of parameters and also to show that our random walk model correctly describes the whole dependence of the growth rate of the parameters of the system, we extract the growth rate \(R\) for different driving frequencies \(\omega\) at a fixed amplitude \(a = 0.025\) by performing a numerical simulation for each value of the frequency and compare \(R\) with the corresponding result obtained from our random walk model. The growth rate \(R(\omega)\) (see figure 5) shows characteristic (decaying) oscillations, as already theoretically predicted in [29]. The minima where \(R(\omega)\) is exactly zero can be easily understood. At these values of \(\omega\), the driving period \(T = 2\pi/\omega\) and the time between two collisions with the same vertical wall \(2L/v_x\) are commensurable, which leads to a \(\phi\)-periodic entering and leaving of the FUM for \(v_y \to \infty\) (when the first and the last collision with the oscillating bar converge to its edges). Note that the occurrence of the minima in the growth rate \(R(\omega)\) is based on the fact that all the particles of the ensemble possess the same, constant velocity in the \(x\)-direction. Apparently, these characteristic oscillations are fully reproduced by our model.

The inset of figure 5 shows the analogous comparison between the model and the simulation for a fixed driving frequency \(\omega = 0.1\) and different but small values of the amplitude.
(the regime where the invariant spanning curves of the FUM can be well approximated within
the SWA). From this, we observe firstly that the growth rate increases strongly with the
amplitude of the oscillating bar, and secondly that this dependence can be explained well by the
random walk model. According to the good agreement between the simulation and the model,
we may conclude from equations (17) and (13) that the amplitude dependence of the growth
rate is given approximately by \( R(a) \propto \ln (\cosh(a/h)) \). As figure 3 reveals, we may not expect
that this is also true for large values of the amplitude, since then the expressions for the invariant
spanning curves obtained within the SWA strongly differ from the numerical results. Obviously,
the result of the simulation is between the estimations for the lower and the upper bound for all
values of the system parameters.

These results indicate that all the details of the specific system under consideration that are
not accounted for in the random walk model, including the existence of a chaotic sea, do not
contribute crucially to the growth rate. Even more, all the details of the specific system are only
needed to calculate the effective \( c \). The requirements for the occurrence of exponential FA are
comparatively weak: a temporally periodic entering and leaving of invariant spanning curves
with \( v_{\text{isc}}^\text{max} - v_{\text{isc}}^\text{min} \propto v_{\text{isc}}^\text{min} \) at different phases is sufficient.

5. Conclusion

In this paper, we have investigated the physical mechanism leading to exponential FA in the
rectangular billiard with an oscillating bar inside. In particular, we showed that the dynamics
of individual trajectories can be understood as alternating phases of motion in an appropriate
1D FUM and free propagation. During the temporal FUM phases, the particles move (in the high velocity regime that is of interest here) on invariant spanning curves of the FUM,
which can be—at least for small driving amplitudes—obtained analytically within the SWA.
Using the intrinsic property of the invariant spanning curves of the FUM that the difference
between the maximal and the minimal velocity grows linearly with the minimal velocity of
the invariant curve and the fact that acceleration and deceleration have equal probability, the
process can be modeled as a random walk with step width proportional to the velocity itself, i.e.
\( v_{n+1} = v_n \pm c v_n \). This model explains the occurrence of the exponential acceleration. Calculating
an effective step width \( c_{\text{eff}} \), we obtain a good \textit{ab initio} estimation of the growth rate and
reproduce the whole qualitative dependence of the system parameters. We emphasize that our
random walk model reflects that a temporally periodic entering and leaving of equally shaped
invariant spanning curves that have the property that the difference between the maximal and
the minimal velocity grows linearly with the minimal velocity of the latter is the key ingredient
for the occurrence of exponential FA (full details of the specific system are contained in the
factor \( c \)). This opens the prospect of searching for other systems exhibiting the phenomenon of
exponential FA.

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