Total Unimodularity and Strongly Polynomial Solvability of Constrained Minimum Input Selections for Structural Controllability: An LP-Based Method

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Abstract—This article investigates several cost-sparsity-induced optimal input selection problems for structured systems. Given an autonomous system and a prescribed set of input links, where each input link has a nonnegative cost, the problems include selecting the minimum cost of input links and selecting the input links with the smallest possible cost while bounding their cardinality to achieve system structural controllability. Current studies show that in the dedicated input case, the former problem is polynomially solvable by some graph-theoretic algorithms, while the general nontrivial constrained case is largely unexplored. We show that these problems can be formulated as equivalent integer linear programming (ILP) problems. Subject to the “source strongly connected component groupwise input constraint,” which contains the dedicated input one as a special case, we demonstrate that the constraint matrices of these ILPs are totally unimodular. This property allows us to solve these ILPs efficiently via their linear programming (LP) relaxations, leading to a unifying algebraic method for these problems with polynomial time complexity. We further show that these problems can be solved in strongly polynomial time, independent of the size of the costs and cardinality bounds. Finally, we provide an example to illustrate the power of the proposed method.

Index Terms—Input selection, integer programming, linear programming, structural controllability, total unimodularity.

I. INTRODUCTION

Input/output (I/O) selection, or actuator/sensor placement for a control system to possess certain performances, is of great importance for system design. Most of the I/O selection problems are challenging due to their combinatorial nature [1]. Over the past decades, significant achievements have been achieved in understanding the structure and computational complexity of various I/O selection problems concerning a wide variety of system properties; see, e.g., [2], [3], [4], and [5].

The past decade has also witnessed a growing research interest in I/O selections for structured systems [6], [7], [8], [9]. Structured systems can often be represented by graphs, and have the potential to describe the interaction structure of large-scale networked systems. In particular, some properties defined on structured systems are generic in the sense that almost all realizations of a structured system share the same said properties. Controllability is such a property, and the corresponding notion is structural controllability [10].

Among the related problems, the problems of optimally selecting inputs, actuators, or interconnection links for achieving structural controllability have been extensively explored [6], [7], [11], [12], [13], [14], [15], [16], [17]. Depending on the objective, these problems could be roughly classified into two categories: selecting the minimum number/cost of inputs, and selecting the minimum number/cost of input links (e.g., the sparsest input matrices), both to ensure structural controllability. The differences between them lie in that, in the former problems, selecting an input indicates all input links incident to this input will be selected, while all input links can be selected independently in the latter problems. The former problems are NP-hard in the general constrained case [16], [17]. This article will focus on the latter class of problems. For these problems, the authors [6] give the first polynomial time algorithm when all state variables have dedicated inputs. Olshevsky [11] extends the previous result by restricting that some state variables are forbidden to be actuated and providing a faster algorithm. Later, Pequito et al. [12] show the polynomial solvability of the same problem with cost consideration. The essence of their methods is to convert these problems to some weighted matching problems [18]. So far, all the polynomially solvable cases in [6], [11], and [12] belong to the dedicated input case. Only very little was known about the complexity status for the nondedicated constrained input case, except for some cases reported in [13], [14], and [15] where the corresponding problems could be trivially reduced to the minimum (cost) spanning arborescence problems or the minimum cost maximum matching problems.

In this article, we make an attempt toward the nondedicated constrained input case and provide an alternatively algebraic method for it. More precisely, given an autonomous system and a constrained input configuration, where whether an input can directly actuate a state variable, as well as the corresponding (heterogeneous) cost, is prescribed, we consider three related cost-sparsity-induced optimal input selection problems: selecting the minimum number of input links, selecting the minimum (total) cost of input links, and selecting the input links with the smallest possible cost while their cardinality does not exceed a prescribed number, all for achieving structural controllability. To the best of our knowledge, the third problem has not been considered before, and no polynomial time algorithms have been reported for all the three problems in the nondedicated input case, except for some trivial cases (cf., [13], [14], and [15]).

In this article, as our first contribution, we show these problems can be formulated as equivalent integer linear programming (ILP) problems by suitably choosing the decision variables. Although ILPs are

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usually NP-hard, we reveal that the corresponding constraint matrices of these ILPs are totally unimodular, under a weak constraint on the input configuration, namely, the proposed source strongly connected component grouped input constraint, which contains the dedicated input one as a special case. Total unimodularity (TU) is an important property for combinatorial problems [19], [20], but has not yet been revealed for the input selection problems as far as we know. This inherent structure allows us to solve those ILPs efficiently by solving the linear programming (LP) relaxations. Hence, as our second contribution, we provide a unifying LP-based method with polynomial time complexity toward the three problems, which also gives an algebraic, rather than graph-theoretic proof for the polynomial solvability of these problems for a wide variety of nondedicated input constraints. Furthermore, thanks to the TU structure, it is revealed that the considered problems are strongly polynomially solvable under the addressed condition, meaning there are algorithms that can solve them in polynomial time that is independent of the size of the costs and cardinality bounds.

The rest of this article is organized as follows. Section II gives the problem formulations, and Section III provides some preliminaries in graph theory and structured systems. Section IV presents ILP formulations of the addressed problems, while Section V deals with their TU properties and efficient solvability. Section VI provides an illustrative example. Finally, Section VII concludes this article.

Notations: For two vectors $a$ and $b$, $a \leq b$ means $a_i \leq b_i$ entry wisely. A vector $a$ is integral if every element of it is an integer. For an optimization problem $\min \{ \varphi(x) : x \in \Lambda \}$, $\Lambda$ is the feasible region, $x \in \Lambda$ is a feasible solution, the minimum of the objective $\varphi(x)$ on $x \in \Lambda$ is called the optimal (objective) value, or optimum, while the $x$ for which the optimum is attained is called an optimal solution (if $x$ is integral, it is called the integral optimal solution). $I_{n \times m}$ denotes the $n \times m$ matrix with all entries 1. $0_{n \times m}$ is defined similarly.

II. PROBLEM FORMULATIONS

Consider a linear-time invariant system

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state variables and inputs, and $\hat{A} \in \mathbb{R}^{n \times n}$, $\hat{B} \in \mathbb{R}^{n \times m}$ are the state matrix and input matrix, respectively.

A structured matrix is a matrix with entries from $\{0, *, \}$ where 0 denotes the fixed zero entries, and $*$ the entries that can take value freely. We may use $\{0, *\}^{n_1 \times n_2}$ to denote the set of all structured matrices with the dimension $n_1 \times n_2$. For $M \in \{0, *\}^{n_1 \times n_2}$, $S(M)$ denotes the set of its realizations, i.e., $S(M) = \{ M \in \mathbb{R}^{n_1 \times n_2} : M_{ij} = 0 \text{ if } M_{ij} (= 0) \}$. Let $\hat{A}$ and $\hat{B}$ be structured matrices that characterize the sparsity patterns of $\hat{A}$ and $\hat{B}$, that is, $A_{ij} = 0$ (resp., $B_{ij} = 0$) implies $\hat{A}_{ij} = 0$ (resp., $\hat{B}_{ij} = 0$). $(\hat{A}, \hat{B})$ is said to be structurally controllable, if there exists $A \in S(\hat{A})$ and $B \in S(\hat{B})$ so that $(A, B)$ is controllable. Controllability is a generic property, in the sense that if $(A, B)$ is structurally controllable, then almost all of its realizations are controllable [10].

Given $B \in \{0, *\}^{n \times m}$, let $N(B)$ be the set of $*$ entries in $B$, i.e., $N(B) = \{(i, j) : B_{ij} = *\}$. Define the set of $n \times m$ structured matrix as $K(B) = \{B' : N(B') \subseteq N(B)\}$. We say $B$ is dedicated, if each column of $B$ has at most one nonzero entry. Assign a nonnegativity rational cost $w_{ij}$ to each nonzero entry $B_{ij}$ of $B$, representing the cost of actuating the $i$th state variable using the $j$th input. Let $||B||_w$ be the sum of costs of all nonzero entries (corresponding to input links) in $B$, i.e., $||B||_w = \sum_{(i,j) \in N(B)} w_{ij}$. Let $||B||_0$ be the number of nonzero entries (i.e., sparsity) of $B$.

We consider the following problems:

- **Problem $P_1$: Constrained sparsest input selection**

  $$\min_{B' \in K(B)} ||B'||_0$$

  s.t. $(A, B')$ structurally controllable.

- **Problem $P_2$: Constrained minimum cost input selection**

  $$\min_{B' \in K(B)} ||B'||_w$$

  s.t. $(A, B')$ structurally controllable.

- **Problem $P_3$: Minimum cost $k$-sparsity input selection ($k$ is given)**

  $$\min_{B' \in K(B)} ||B'||_w$$

  s.t. $(A, B')$ structurally controllable, and $||B||_0 \leq k$.

The aforementioned problems all assume that each available input link can be selected independently. As their names suggested, the problem $P_1$ aims to select the sparsest input matrix from $K(B)$, $P_2$ to select the input matrix from $K(B)$ with the smallest total cost of its input links, and $P_3$ to find the input matrix from $K(B)$ with the total cost of input links as small as possible while upper bounding its sparsity, all to achieve structural controllability. It is remarkable that $P_3$ is more general than the problem (2) discussed in [12], which aims to find the sparsest input matrix for achieving structural controllability with the total link cost as small as possible. This problem can be formulated as $P_3$ by setting $k$ to be the optimal value of $P_1$ with $B$ being an $n \times m$ matrix full of nonzero entries, and indeed is a special case of $P_2$ by reassigning the link costs, as we shall explain in Remark 3 in Section IV. Problem $P_3$ may be desirable, for example, in a network where the activation of new links may be expensive, while different links may have different budgets (characterized by the cost $[w_{ij}]$), the goal is to select the input configuration with the size of active input links not exceeding a prescribed number to ensure system controllability, while making the total budgets as small as possible.

Throughout, the following assumptions are adopted.

**Assumption 1**: $(A, B)$ is structurally controllable.

**Assumption 2**: Let $w_{\min} = \min_{(i, j) \in N(A)} w_{ij}$ and $w_{\max} = \max_{(i, j) \in N(A)} w_{ij}$. Assume that $0 \leq w_{\min} \leq w_{\max} < \infty$.

Assumption 1 is necessary for the feasibility of problems $P_1$, $P_2$, and $P_3$. To ensure problem $P_3$ is feasible, one additional requirement is that $k$ should be no less than the optimal value of problem $P_1$ (denoted by $N_{\min}$). Assumption 2 is general enough to meet all practical designs. Say, if all input links are of equal cost, then $w_{\max} = w_{\min} > 0$; if some input links already exist, then $w_{\min} = 0$. When each input link has a uniform cost, the problem $P_2$ is equivalent to problem $P_1$. If $k \geq n$, the problem $P_3$ is equivalent to problem $P_2$ (cf., Lemma 1).

III. PRELIMINARIES

This section introduces some preliminaries in graph theory and structured systems [9], [19].

A directed graph (digraph for short) is denoted by $G = (V, E)$, with $V$ the vertex set and $E$ the edge set. A path in a digraph is a set of ordered edges, in which the terminal vertex of the preceding edge is the starting vertex of the successive edge. A digraph is strongly connected, if for any pair of its vertices, there is a path from each of them to the other. A strongly connected component (SCC) of a digraph is its subgraph that is strongly connected, and no edges or vertices can be included in this subgraph without breaking its property of being strongly connected. A bipartite graph, which often reads $G = (V_L, V_R, E_{RL})$, is a graph whose vertices can be partitioned into two disjoint parts $V_L$ and $V_R$. 

**Example**: Consider the following digraph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$. This digraph is strongly connected, as there is a path from each vertex to any other vertex. The subgraph $G' = (V' = \{v_2, v_3\}, E' = \{(v_2, v_3)\})$ is a strongly connected component of $G$.
such that all its edges $E_{RL}$ have end vertices in both parts. The incidence matrix of the bipartite graph $G$ is a $\{V_L \cup V_R\} \times |E_{RL}|$ matrix $C$, such that $C_{ij} = 1$ if the $i$th vertex $v_i \in V_L \cup V_R$ is an end vertex of the $j$th edge $e_j \in E_{RL}$ and 0 otherwise. A matching of a bipartite graph is a set of edges among which any two do not share a common end vertex. A vertex is matched with respect to a matching if it is the end vertex of an edge in this matching. A maximum matching is the set of edges with the maximum number among all matchings.

Given $A \in \{0, \ast\}^{n \times n}$, $B \in \{0, \ast\}^{n \times m}$, the state digraph is $G(A) = (X, E_A)$, in which $X = \{x_1, \ldots, x_n\}$ is the set of state vertices and $E_A = \{(x_i, x_j) : A_{ij} \neq 0\}$ is the set of state edges. The system digraph is $G(A, B) = (X \cup U, E_A \cup E_U)$, where the input vertices $U = \{u_1, \ldots, u_n\}$, the input links (edges) $E_U = \{(u_i, x_j) : B_{ij} \neq 0\}$, and the decision variables for our ILP formulations. $
 \sum_{E_{RL} \in E_{RL}}$ is matched. \n \r For $G(A, B)$, the bipartite graph associated with $(A, B)$ is defined as $B(A, B) = (X_L \cup X_R, E_{XX} \cup E_{UX})$, in which $X_L = \{x_1', \ldots, x_n'\}$ and $X_R = \{x_1'', \ldots, x_m''\}$ are copies of $X$. $U = \{u_1, \ldots, u_n\}$, $E_{XX} = \{(x_i', x_j') : A_{ij} \neq 0\}$, and $E_{UX} = \{(u_i, x_j') : B_{ij} \neq 0\}$.

Suppose $G(A)$ can be decomposed into $n$ SCCs, $1 \leq n_c \leq n$, and the $i$th SCC has a vertex set $X_i \subseteq X$ ($1 \leq i \leq n_c$). An SCC is called a source SCC, if there is no incoming edge to vertices in this SCC from other SCCs in $G(A)$; otherwise, it is called a nonsource SCC. Suppose there are $r$ source SCCs in $G(A)$, with their indices being $I = \{1, \ldots, r\}$, $1 \leq r \leq n_c$. For each $i \in I$, let $X_i' = \{x_j \in X_i : x_j \in X_{i+1}\}$ be a copy of $X_i$ in $X_i'$, and define $E_i = \{(u, x) \in E_{UX} : x \in X_i' \cup U\}$ as the set of input links between $U$ and $X_i'$ in $B(A, B)$.

A state vertex $x_i \in X$ is said to be input-reachable, if there is a path starting from an input vertex $u \in U$ to $x_i$ in $G(A, B)$. We say $X_i'$ is input reachable in $B(A, B)$ if each vertex of $X_i$ is input reachable in $G(A, B)$.

**Lemma 1 (9):** $(A, B)$ is structurally controllable, if and only if: 1) every state vertex $x_i \in X$ is input reachable, and 2) there is a maximum matching in $B(A, B)$ such that every $x_i' \in X_i$ is matched.

By the definition of input reachability, it is obvious that condition 1) of Lemma 1 is equivalent to $E_i \neq 0$ for each $i \in I$.

**IV. ILP FORMULATIONS OF $\mathcal{P}_1, \mathcal{P}_2$, AND $\mathcal{P}_3$**

In this section, we formulate problems $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ as some equivalent ILPs. Before introducing our ILP formulations, we discuss the essential difficulty in extending the graph-theoretic algorithms for problems $\mathcal{P}_1$ and $\mathcal{P}_2$ in [6] and [12] from the dedicated input case to the nondedicated one. From Lemma 1, any subset of $E_{UX}$ making $(A, B')$ ($B' \in \mathcal{K}(B)$) structurally controllable contains two (possibly overlapping) parts $E_{mat}$ and $E_{rex}$ so that the addition of $E_{mat}$ to $(X_L, U \cup X_R, E_{XX})$ makes the resulting $(X_L, U \cup X_R, E_{XX} \cup E_{mat})$ a maximum matching that matches $X_L$, and the addition of $E_{rex}$ makes the obtained $(X_L, U \cup X_R, E_{XX} \cup E_{rex})$ have a nonempty $E_{i}$ for each $i \in I$. Although both parts with the minimum cardinality/cost can be polynomially determined via the respective graph-theoretic algorithms (see [13], [14], and [15] for details), the challenge is that $E_{mat}$ and $E_{rex}$ might overlap, thus making their union not necessarily optimal. The essential idea of the graph-theoretic methods in [6] and [12] is to find the maximal “intersection” between these two sets. To this end, by introducing some slack variables to $(X_L, X_R, E_{XX})$, they construct a weighted bipartite graph, and for a weighted maximum matching $E_i$ of it, an optimal input solution is obtained by selecting dedicated inputs to those state vertices that are not matched by $E_i \cap E_{XX}$, and adding additional inputs to make every state vertex input reachable. In the dedicated input case, selecting an input link will not affect the ability of the subsequent input links w.r.t. the matching function (in a feasible solution $E_{mat} \cup E_{rex}$ to problem $\mathcal{P}_1$ or $\mathcal{P}_2$, we say an input link $e$ serves the matching function if $e \in E_{mat}$). However, in the nondedicated input case, things are different since multiple input links may be incident to the same input vertex. This makes extending the graph-theoretic algorithms in [6] and [12] to the nondedicated input case nontrivial.

In our ILP formulations for the general input case, however, we do not intend to figure out such an intersection; instead, we directly adopt the corresponding cost functions in problems $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ as our objectives. The key to the ILP formulations is the introduction of binary variables $y = \{y_{uv} : (u, v) \in E_{XX} \cup E_{UX}\}$ and $z = \{z_i : i \in I\}$, where $y_{uv} = 1$ indicates edge $(u, v)$ is in a specific maximum matching $E_i$ of $B(A, B)$, and $y_{uv} = 0$ means the contrary; $z_i = 1$ means $X_i'$ is input reachable after adding the edges $E_i \cap E_{rex}$ to $(X_L, X_R \cup U, E_{XX})$, and $z_i = 0$ means the contrary. The variables $y$ and $z$ will be the decision variables for our ILP formulations.

**Proposition 1:** Under Assumptions 1 and 2, the problem $\mathcal{P}_1$ is equivalent to the following ILP $\mathcal{P}_1^{ILP}$ (i.e., their optimal objective values are equal), for $i = 1, 2, 3$, respectively:

\[ \min_{y,z} \sum_{(u,v) \in E_{UX}} y_{uv} + |I| - \sum_{i \in I} z_i \] (P1 ILP)

\[ \text{s.t.} \quad y_{uv} = 1, \forall v \in X_L \] (2)

\[ \sum_{(u,v) \in E_{UX} \cup E_{UX}} y_{uv} \leq 1, \forall u \in X_R \cup U \] (3)

\[ z_i \leq \sum_{(u,v) \in E_i} y_{uv}, \forall i \in I \] (4)

\[ y_{uv} \in \{0, 1\}, \forall (u, v) \in E_{XX} \cup E_{UX} \] (5)

\[ z_i \in \{0, 1\}, \forall i \in I \] (6)

\[ \min_{y,z} \sum_{(u,v) \in E_{UX}} w_{uv}y_{uv} + \sum_{i \in I} (1 - z_i)w_i^{min} \] (P2 ILP)

\[ \text{s.t.} (2), (3), (4), (5), \text{ and (6)} \] (7)

\[ \min_{y,z} \sum_{(u,v) \in E_{UX}} w_{uv}y_{uv} + \sum_{i \in I} (1 - z_i)w_i^{min} \] (P3 ILP)

\[ \text{s.t.} (2), (3), (4), (5), \text{ and (6)} \] (9)

where $w_{uv} = 1$ for $(u,v) \in E_{XX}$, and $w_i^{min} = \min_{(u,v) \in E_{UX}} w_{uv}$, i.e., the minimum cost of input links incident to $X_i'$, for each $i \in I$. Besides, with Assumptions 1 and 2, for any $k \in \mathbb{N}$, the problem $\mathcal{P}_2$ is feasible, if and only if the ILP $\mathcal{P}_1^{ILP}$ is.

**Proof:** We first focus on $\mathcal{P}_1^{ILP}$. Let $E_i = \{(u, v) \in E_{XX} \cup E_{UX} : y_{uv} = 1, (y, z) \text{ subject to (2) - (6)}\}$. Constraint (2) means every vertex of $X_L$ should be an end vertex of exactly one edge in $E_i$, and constraint (3) means each vertex of $X_R \cup U$ can be the end vertex of at most one edge in $E_i$. Therefore, constraints (2) and (3) make sure $E_i$ is a matching of $B(A, B)$ that matches $X_L$, Moreover, to minimize the objective function, $z_i$ subject to (4) and (6) should take the value $z_i = \min \{\sum_{(u,v) \in E_i} y_{uv}, 1\}$, for each $i \in I$; otherwise, by changing the corresponding $z_i$ from 0 to 1, the constraints are fulfilled while the objective value can decrease. That is, if $X_i'$ is input-reachable in $(X_L, X_R \cup U, E_i \cup E_{XX})$, then $z_i = 1$; otherwise, $z_i = 0$. Hence, to make $X_i'$ input-reachable for all $i \in I$, the minimum number of input links that need to be added to $(X_L, X_R \cup U, E_i \cup E_{XX})$ is $|I| - \sum_{i \in I} z_i$, and adding one element arbitrarily from $E_i$ for each $i$ with $z_i = 0$ is feasible. Therefore, optimizing the objective of $\mathcal{P}_1^{ILP}$...
will obtain the optimal solution to $P_1$. On the other hand, according to Lemma 1 and following a reversed analysis similarly, with Assumption 1, it turns out that any optimal solution to problem $P_1$ should correspond to a $(y^*, z^*)$ that optimizes $P_{\text{ILP}}^5$.

We now consider $P_{\text{ILP}}^2$. With the notations defined previously, after adding $E_2$ to $(X_L, X_R \cup U, E_{XX})$, to make every $X'_k$ with $z_i = 0$ input-reachable while incurring the minimal cost $w_{\min}$. On the other hand, to minimize the objective function, for an $i \in I$ with $w_{\min} > 0$, $z_i$ subject to (8) and (9) must take $z_i = \min\{\sum_{(u,v) \in E_i} w_{uv}, 1\}$, as otherwise one can always change the respective $z_i$ from 0 to 1 so that constraints (8) and (9) are fulfilled, while the objective value decreases (note if $w_{\min} = 0$, the objective value remains unchanged). Therefore, the optimal value of $P_{\text{ILP}}^2$ is equal to that of $P_2$.

Consider $P_{\text{ILP}}^3$. Based on the analysis for $P_{\text{ILP}}^1$, for a fixed matching $E_i$ defined therein, the left-hand side of (8) is the minimal possible number of input links associated with $E_i$ that can make the original system structurally controllable. In other words, constraint (8) ensures the sparsity of the feasible input matrix for the problem $P_3$ does not exceed $k$. Hence, $P_3$ is feasible, if and only if $P_{\text{ILP}}^3$ is. Then, following the analysis for $P_{\text{ILP}}^3$, it turns out the optimal value of $P_3$ is equal to that of problem $P_3$.

The following lemma states how to recover an optimal solution to the problem $P_i$ from solutions to the corresponding $P_{\text{ILP}}^i$.

**Lemma 2:** Suppose Assumptions 1 and 2 hold. Let $(u_{i_{\min}}, x_{i_{\min}}) = \arg \min\{u_{i_{\max}} \mid E_i, w_{ux}\}$ for each $i \in I$. Let $(y^*, z^*)$ be an optimal solution to $P_{\text{ILP}}^i (i = 1, 2, 3)$. Define $E_{\text{mat}}^i = \{(u,v) \in E_{UX} : y_{uv} = 1\}, E_{\text{rea}}^i = \{(u_{i_{\min}}, x_{i_{\min}}) : z_i = 0, i \in I\}$. Then, $E_{\text{mat}}^i \cup E_{\text{rea}}^i$ is the set of input links of an optimal solution to the problem $P_i (i = 1, 2, 3)$.

**Proof:** The statement follows directly from Lemma 1 and the proof of Proposition 1.

**Remark 1:** Problems $P_{\text{ILP}}^4(3)_{i=1}^N$ essentially find the optimal solutions to $P_{\text{ILP}}^1(3)_{i=1}^N$ by optimizing the objective functions over all maximum matchings of $B(A,B)$. Note in these ILPs, the relation $z_i = \min\{\sum_{(u,v) \in E_i} w_{uv}, 1\}$ (for $w_{\min} > 0$) results from optimizing the objective functions, rather than the feasible regions.

**Remark 2:** Finding the sparsest input matrix from $K(B)$ while incurring the total cost as small as possible for structural controllability (cf., [12, Problem (2)], [15], referred to as $P_4$) can be alternatively formulated as the following ILP, given $w_{\max} > 0$.

\[
\min_{y^*, z^*} \sum_{(u,v) \in E_{UX}} w_{uv}y_{uv} + \sum_{i \in I} (1 - z_i)w_{\min}^{\text{ILP}} (P_4^{\text{ILP}}) + \gamma \left( \sum_{(u,v) \in E_{UX}} y_{uv} + |I| - \sum_{i \in I} z_i \right)
\]

s.t. \( (2), (3), (4), (5), \) and (6)

where $\gamma \geq nw_{\max}$ is to penalize the sparsity of the solution. It turns out that for any feasible solution to $P_4^{\text{ILP}}$ with a sparsity larger than $N_{P_4}$ (the optimum of $P_4$), compared to any solution with a sparsity being $N_{P_4}$, the increase in the objective value of $P_4^{\text{ILP}}$ caused by the sparsity penalty term, which is at least $\gamma$, is bigger than the decrease in the total link cost term, which is at most $N_{P_4}w_{\max}^2 - (1 + N_{P_4})w_{\min} \leq nw_{\max}$. Consequently, any optimal solution to $P_4^{\text{ILP}}$ must have a sparsity of $N_{P_4}$. Note $P_{\text{ILP}}^5$ can be formulated as $P_2$ by redefining the cost $w_{uv} \equiv w_{uv} + \gamma, \forall (u,v) \in E_{UX}$, implying $P_4$ is a special case of $P_2$.

**V. MAIN RESULTS: LP RELAXATIONS AND TU**

Although we have formulated problems $P_{\text{ILP}}^{4,5}_{i=1}^N$ as ILPs, it may not be favorable unless those ILPs can be solved efficiently. Typically, ILPs are known to be NP-hard (e.g., the set cover problem can be formulated as an ILP but is NP-hard) [19]. However, by proving the TU of the constraint matrices of the ILPs, we shall show that $P_{\text{ILP}}^{4,5}_{i=1}^N$ can be solved in polynomial time simply by using their LP relaxations, and even in strongly polynomial time, under a wide range of conditions on $B$ that include almost all the known nontrivial conditions with which problems $P_1$ and $P_2$ are reportedly polynomially solvable.

**A. Total Unimodularity (TU)**

**Definition 1 (Total unimodularity [19], TU):** A matrix $M$ is TU if every square submatrix of it has determinant 0, +1, or −1.

**Lemma 3 ([20]):** Let $M$ be a $p \times q$ TU matrix. Then, every vertex of the polyhedron $\{x \in \mathbb{R}^q : Mx \leq b, x \geq 0, x \in \mathbb{Z}^q\}$ has a $p \times q$ TU constraint matrix $M$, then its LP relaxation by removing the integral constraint on $x$, i.e., $\min\{c^T x : Mx \leq b, x \geq 0\}$ yields an integral optimal solution [19, Ch. 4.12] whenever the optimum exists and is finite, for any integral vector $b$ and all rational $c$. It also follows that the original ILP is polynomial-time solvable since the respective LP is [21, Th. 16.2].

We introduce the following constraint on $B$, termed the source SCC grouped input constraint. As shown in the sequel, this constraint defines a large class of systems whose corresponding constraint matrices of $P_{\text{ILP}}^{4,5}_{i=1}^N$ are TU. We shall briefly discuss the case without this constraint in Section V-C.

**Assumption 3:** (Source SCC grouped input constraint) Given $(A,B)$, assume in $G(A,B)$ that no input vertex can simultaneously actuate two state vertices that come from different source SCCs, or one is from a source SCC and the other a nonsource SCC.

Note Assumption 3 does not impose constraints on how each input vertex connects with vertices within the same source SCC, or how each input vertex connects with vertices not belonging to the source SCCs (thus, one input vertex can connect with multiple nonsource SCCs). Particularly, the dedicated input case falls into Assumption 3. The case where $G(A)$ is strongly connected automatically satisfies Assumption 3. It seems Assumption 3 characterizes a two-layer control structure, in which the first layer consists of all source SCCs, and the second one all nonsource SCCs. One input signal cannot directly actuate two state variables that are in different layers. In the first layer, each source SCC somehow has a high priority of autonomy such that it has its own inputs. Typical practical systems that could exhibit such a layered control structure include a class of cyber-physical systems and the emerging Internet of Things (IoT) with a cloud-edge computing paradigm. For

4For notions of polyhedral and vertex, we refer readers to [19] and [21]. A polyhedron $P = \{x \in \mathbb{R}^q : Mx \leq b\}$ is integral (i.e., every vertex of $P$ is integral), if and only if $\min\{c^T x : x \in P\}$ is attained by an integral vector, for any vector $c$ whenever the minimum exists [21, p. 232].

5It is remarkable that some LP solvers may return a nonintegral optimal solution (this happens when there exist an integral and a nonintegral solution, which are both optimal). In this case, an integral optimal solution can always be found from the nonintegral one via some standard manipulations in polynomial time, for example, via computing the Hermite normal form in time $O(d^5)$, with $w \leq 2.373$ being the exponent of matrix multiplication, and if the number of decision variables; see [21, Corollary 5.3b, Th. 16.2] for details.
example, the authors in [22] proposed a middleware named “Etherware” for networked control (which is the early stage of cyber-physical systems [23]). Etherware acts as an integration platform for complex control systems, with higher level supervisory control implemented for soft real-time operation, connected to lower level critical control loops implemented on native platforms for hard real-time operation. In the corresponding control hierarchy, the higher level supervisor could send goals to the controller in the cyber-layer, and the cyber-layer sends the commands to the lower level actuator in the physical layer consisting of physical plants that implement hardware-oriented control locally. In industrial systems (or IoT) with a cloud-edge computing framework [24], the first layer consists of (multiple) cloud servers that conduct complex computational tasks and make decisions for the second layer constituted by physical/industrial systems. Similar layered (hierarchical) structures may also emerge in social networks, political networks, and transportation networks that are layered by SCCs, or different types of subnetworks [25]. In addition, some networked systems with geographically distributed subsystems may also satisfy Assumption 3. For example, consider the species competition-migration model arising in ecology systems [26]. This model describes the spatial heterogeneity of populations due to species competition and migration [26]. In case the coexistence/exclusion relations among the species are dense, the interactions within each patch are highly coupled, corresponding to an SCC. Since the patches are geographically distributed, they usually cannot share the same input (the environment/human effect in each patch, etc.). Hence, such a system might also satisfy Assumption 3.

To show the TU property of $P_i^{ILP}$ in [1], in what follows, we characterize the TU of two augmented incidence matrices of a bipartite graph satisfying a condition resembling Assumption 3, which might be of independent interest. The proofs are postponed to Section V-D.

Proposition 2: Suppose in a bipartite graph $G = (X_L, X_R \cup U, E)$ (not necessarily corresponding to a structured system), $X_L$ and $U$ are partitioned into $r + 1 (r \geq 0)$ disjoint subsets $X_L^0, \ldots, X_L^{r+1}$ and $U^{C_1}, \ldots, U^{C_{r+1}}$, such that $E = E_{XX} \cup (\bigcup_{i=0}^{r} E_i)$, in which $E_i$ (resp., $E_{XX}$) are the edges between $X_L^i$ and $U^{C_i}$ (resp., between $X_L$ and $X_R$). Rewrite $X_L \cup X_R \cup U \equiv \{x_1, \ldots, x_n\}$ and $E \equiv \{e_1, \ldots, e_{m}\}$, where $n_v \equiv |X_L \cup X_R \cup U|$ and $n_E \equiv |E|$. Define the $(n_v + r + 1) \times (n_E + r)$ augmented incidence matrix $M$ as follows:

$$M_{ij} = \begin{cases} 1, & \text{if } v_i \in \partial(e_j), 1 \leq i \leq n_v, 1 \leq j \leq n_E \in E_i, n_{i,v} + 1 \leq i \leq n_v + r, e_j \in E_{i,v} \\ 1, & \text{if } n_v + 1 \leq i \leq n_v + r + 1, j = n_E + i - n_v \\ 0, & \text{otherwise}, \end{cases}$$

where $\partial(e_j)$ is the set of end vertices of $e_j$. Then, matrix $M$ is TU.

Proposition 3: Consider the bipartite graph $G$ and matrix $M$ in Proposition 2. Let the $(n_v + r + 1) \times (n_E + r)$ matrix $\tilde{M}$ be

$$\tilde{M}_{ij} \equiv \begin{cases} M_{ij}, & \text{if } 1 \leq i \leq n_v + r, 1 \leq j \leq n_E + r \\ 1, & \text{if } i = n_v + r + 1, e_j \in E_{i,v} \\ -1, & \text{if } i = n_v + r + 1, n_E + 1 \leq j \leq n_v + r + 1 \\ 0, & \text{otherwise}. \end{cases}$$

Then, the matrix $\tilde{M}$ is TU.

An illustration of $M$ and $\tilde{M}$ is given in Fig. 1. From the construction, $M$ contains two blocks, the first of which is the incidence matrix of $G$, and the second one corresponds to $E_1, \ldots, E_r$. $M$ contains an additional block corresponding to the set $\bigcup_{i=0}^{r} E_i$. The relations between $M$ ($\tilde{M}$) and the constraints of $P_i^{ILP}$ in (1) are explained as follows. Given $(A, B)$ satisfying Assumption 3, let $M_{ILP}$ and $\tilde{b}_{ILP}$ be defined in (14). Define $N \equiv |E_{XX} \cup E_{XU}|$. Then, from (14), the feasible region of $P_i^{ILP}$ can be written as $\{x \in \mathbb{R}^{N+r} : M_{ILP}x \leq \tilde{b}_{ILP}, x \geq 0\}$, where

$$M_{ILP} \equiv \begin{bmatrix} \tilde{M}_{ILP} \\ I_{N+r} \end{bmatrix}, \tilde{b}_{ILP} \equiv \begin{bmatrix} \tilde{b}_{ILP} \\ 1_{N+r} \end{bmatrix}$$

for $i = 1, 2$. Since $M$ is TU from Proposition 2, it follows easily that every square submatrix of $M_{ILP}$ still has determinant 0 or ±1 (via a
similar manner to the proof of Proposition 2 in Section V-D). Hence, $M_{rLP}$ is TU. With Lemma 3, this yields the required assertion.

Theorem 2: With Assumptions 1–3, the minimum cost $k$-sparsity input selection problem $(P_3)$ can be solved in polynomial time via solving the following LP relaxation $P_{3LP}$:

$$\min_{y,z} \sum_{(u,v) \in E_{UX}} w_{uv} y_{uv} + \sum_{i \in I} (1 - z_i) w^\text{min}_i$$

s.t. (2), (3), (4), (8), (17), and (18).  

Proof: From (15) and similar to the proof of Theorem 1, the constraint matrix of $P_{3LP}$ is obtained from $M$ after duplicating its rows corresponding to $X_L$ multiplied by $-1$, then adding a unit matrix $I_{N+r}$, and thus, is TU since $M$ is. The results then follow immediately.

In light of the aforementioned theories, problems $P_{3LP}$ can be solved efficiently simply via solving the respective LP relaxations whenever $(A, B)$ satisfies Assumption 3. This means, some off-the-shelf LP solvers could be directly used, including the interior point method, the ellipsoid algorithm, and their state-of-the-art improvements (cf. [27]; see footnote 5 for the case when a nonintegral optimal solution is returned). Particularly, it is shown in [27] and [28] that an LP with $d$ variables can be solved in time $O(d^{5/2}L)$, where $L$ is the number of input bits. For $P_{3LP}$ in its standard form ($i = 1, 2, 3$), upon defining $N \equiv |E_{UX} \cup E_{XX}|$, it has $N + r$ variables, and $L \approx \log_2 w_{mx} + \log_2 k + \log_2 n$ (for $[w_{ij}]$ integral). Therefore, under Assumption 3, the problem $P_3$ can be solved in time $O((r + N)^{2.5}L) \rightarrow O(N^{2.5}L)$, $i = 1, 2, 3$, noting that determining the SCCs and finding $E_{inut}$ and $E_{inut}$ both have linear time complexity in $N + r$.

When restricted to the dedicated input case for problems $P_3$ and $P_2$, the aforementioned result is not a running time improvement compared with the graph-theoretic methods in [6], [11], and [12]. Nevertheless, the power of our LP-based method lies in it that can handle more complicated cases than the dedicated input one in a unifying manner, remarkably, without using any combinatorial structure of these problems, except for the SCC decompositions. Therefore, it is also conceptually simple and easy for programmatic implementation. By contrast, it seems unclear how to extend the graph-theoretic methods to the $P_3$ even in the dedicated input case, or $P_1$ and $P_2$ beyond the dedicated input case.

Another significance of Theorems 1 and 2 lies in that, it gives an algebraic, rather than algorithmic proof, for the polynomial solvability of the addressed input selection problems under a wide variety of input constraints. So far, it appears that all the identified nontrivial cases that enable polynomial solvability are related to TU. Furthermore, the TU structure allows us to characterize the complexity status of these problems more precisely.

Definition 2 (Strongly polynomial time, [21]): A strongly polynomial (time) algorithm is an algorithm that runs in time that is polynomial in the number of input items (i.e., dimension of the input variables). Importantly, the running time of a strongly polynomial algorithm is independent of the size of the input values, which refers to the length of the unary representation of the input numerical value.

Corollary 1: With Assumptions 1–3, problems $P_1$, $P_2$, and $P_3$ can be solved in strongly polynomial time.

Proof: From [29, Th. 1], ILPs with full column rank constraint matrices that are TU are solvable in strongly polynomial time. The results follow from this fact and the proven assertions in Theorems 1 and 2 that the constraint matrices of $P_{3LP}$ are all TU.

The aforementioned corollary reveals that, with Assumptions 1–3, there exist polynomial algorithms for problems $P_1$, $P_2$, and $P_3$, whose running time depends only on $n$, $m$, $N$, and $r$ (for example, the algorithm given in [29]), but is independent of the size of the sparsity bound $k$ and the input costs $[w_{ij}]$. Especially, for a fixed $(A, B)$, the value of $k$ or $[w_{ij}]$ will not affect the running time. This is a stronger conclusion than Theorems 1 and 2. Note the interior point methods and the ellipsoid algorithms for LPs are usually not strongly polynomial, since their running time might scale with the numerical values of inputs [30].

C. Beyond Assumption 3

For an ILP in its general form $\min \{c^T x : Mx \leq b, x \in \mathbb{Z}^n\}$, $M$ is called totally $\Delta$-modular ($\Delta$-TM), if all subdeterminants of $M$ are at most $\Delta$ in absolute value. It is known for $\Delta$ large enough, ILPs with $\Delta$-TM constraint matrices will become NP-hard in general [29]. Currently, very little is known about the complexity status for general ILPs with $\Delta$-TM constraint matrices, except for $\Delta \leq 2$ (cf. [29]). In the following, we provide an example showing that without Assumption 3, the ILPs $P_{3LP}$ could have $\Delta$-TM constraint matrices with $\Delta \geq 3$, which indicates analyzing and solving those ILPs might be essentially harder than the case with Assumption 3.

Example 1: Consider a system $(A, B)$ with its digraph $\mathcal{G}(A, B)$ given in Fig. 2(a) (denoted by system $\Sigma$), from which we know it has one source SCC with the vertex set $X_1 = \{x_1, x_2, x_3, x_4\}$. This system does not satisfy Assumption 3. For problem $P_1$, $P_3$ with this system, the corresponding matrix $M$ of dimension $17 \times 20$, defined in a similar way to Proposition 2, is given in Fig. 2(b) (not unique w.r.t the order of edges). It turns out that the submatrix of $M$ with rows indexed by $\{3, 4, 7, 8, 11, 13, 15, 16, 17\}$ and columns by $\{3, 7, 9, 10, 12, 14, 15, 16, 19\}$ has determinant $-3$. This means $M$ is $\Delta$-TM for some $\Delta \geq 3$, and so is the constraint matrix of the associated $P_{3LP}$ or $P_{3ILP}$. Furthermore, to show what difference the non-TU has brought to the related LPs, consider the polyhedron defined by $P_{rel} = \{x \in \mathbb{R}^{|\Sigma|} : Mx \leq \{1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0\}, 0_{20-1} \leq x \leq 1_{20-1}\}$, which is the feasible region of problem $P_{3LP}$ associated with the system $\Sigma$ after relaxing the equality in constraint (2) with “$\leq$”. Obviously, $P_{rel}$ surrounds the feasible region of $P_{3LP}$. From [21, Th. 22.2], the integrality of $P_{rel}$ suffices to show the feasible region of $P_{3LP}$ is integral. We know from Proposition 2 that for systems satisfying Assumption 3, the similarly defined $P_{rel}$ is integral. By contrast, for the considered system $\Sigma$, it turns out $P_{rel}$ is not integral. Indeed, by choosing $c^T = \{-0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 0, 0\}$, $\min\{c^T x \in P_{rel}\}$ obtains its minimum $-4.5$ with a fractional $x$, indicating that $P_{rel}$ is not integral.  

Remark 4: It is remarkable that TU is a sufficient condition for an LP to have integral optimal solutions (indeed, it can be verified the LP

$k$ and the input costs $[w_{ij}]$. Especially, for a fixed $(A, B)$, the value of $k$ or $[w_{ij}]$ will not affect the running time. This is a stronger conclusion than Theorems 1 and 2. Note the interior point methods and the ellipsoid algorithms for LPs are usually not strongly polynomial, since their running time might scale with the numerical values of inputs [30].



Fig. 2. System digraph $\mathcal{G}(A, B)$ and the associated matrix $M$ of system $\Sigma$ in Example 1. Dotted red edges represent the input links.

6The code for Example 1, as well as the example in Section VI, is available at https://github.com/Yuanzhang42014/LP-CMIS-SC.
Proof of TU

The following proposition provides the condition necessary for the transitive reduction of a directed graph. It is stated as a lemma.

Lemma 4: (Ghoula–Houri Characterization, [19]) A $p \times q$ integral matrix $A = [a_{ij}]$ is TU if and only if for each column $j$, the sum of the elements in each row is non-negative, and for each row $i$, the sum of the elements in each column is non-negative.

Proof of Proposition 2: Let $M$ be the matrix consisting of the first $n_E$ columns of $M$. We first prove by induction that $M$ is TU. Recall that by definition, each column of $M$ indexed by $e \in E_{r+1} \cup E_{XX}$ has exactly two nonzero entries $1$'s, corresponding to the two end vertices of $e$, and each column indexed by $e \in \bigcup_{i=1}^m E_i$ has exactly three nonzero entries, among which two 1's correspond to the end vertices of $e$ and the third one is less than or equal to $E_i$ (see Fig. 1 for illustration). For the beginning of the induction, the claim is clearly true for any $1 \times 1$ submatrix of $M$.

Assume the claim holds true for all $(k - 1) \times (k - 1)$ submatrices of $M$ ($k \geq 2$). Let $M'$ be a $k \times k$ submatrix of $M$. If there is a column of $M'$, which contains no nonzero entry, then certainly $det M' = 0$. If there is a column of $M'$, which contains exactly one nonzero entry, then $det M' = \pm det M''$, where $M''$ is obtained from $M'$ after deleting the respective row and column containing the aforementioned entry. Hence, $det M' \in \{0, \pm 1\}$, since $det M'' \in \{0, \pm 1\}$ by induction.

Otherwise, each column of $M'$ contains at least two nonzero entries. We consider two cases. In the first case, $M'$ does not contain $-1$'s, which means each column contains exactly two 1's. As $G$ is bipartite, the rows of $M'$ can be partitioned into two sets $R_1$ and $R_2$, such that for each of its columns, there is exactly one 1 in each set. Then, for each column $j \in \{1, \ldots, k\}$ of $M'$, $\sum_{i \in R_1} M'_{ij} - \sum_{i \in R_2} M'_{ij} = 0$, leading to $det M' = 0$. In the second case, $M'$ has some rows which contain $-1$'s. We consider two subcases.

Subcase i): Each column of $M'$ contains $-1$'s. Since each column of $M'$ contains at most two $1$'s and at most one $-1$, summing up all elements in each column will yield $1$ or $0$.

Subcase ii): $M'$ contains some columns which do not have $-1$'s. Suppose the columns of $M'$ that contain $-1$'s are indexed by $E_{12}$, and the columns not containing $-1$'s but corresponding to the subset of $E_{12}$ are indexed by $E_{22}$, and the remaining columns are indexed by $E_{32}$ (we may alternatively use the corresponding edge or vertex to denote the respective column or row of a submatrix). Without losing any generality, assume that the $E_{12}$ is a subset of $\bigcup_{i=1}^m E_i$, and $E_{22}$ is a subset of $\bigcup_{i=r+1}^m E_i$, $1 \leq r \leq r$. Note $G$ is bipartite, and for each $i \in \{1, \ldots, r+1\}$, the rows of $M'$ corresponding to $U^G_i$ do not have $1$'s except in its columns indexed by $E_i$. Consequently, we can partition the rows of $M'$ into $R_1, R_2, \ldots, R_r, R_{r+1}, R_{R}, R_T$, and $R_1$ so that $R_i$ contains all rows of $M'$ indexed by $U^G_i$, $i = 1, \ldots, r$, (some of $R_i$'s may be empty). $R_L$ contains all rows of $M'$ indexed by $X_L$, $R_R$ contains all rows of $M'$ indexed by $U^{G+} \cup X_R$, and all rows containing $-1$'s are in $R_1$. Observe that, for each column of $M'$ that are not indexed by $\bigcup_{i=1}^m E_i$, there is exactly one 1 in $R_L$ and in $R_R$. For each column of $M'$ indexed by $\bigcup_{i=1}^m E_i$, there are at most two 1's and at least one 1 in $\bigcup_{i=1}^m R_i$ or $R_L \cup R_R$, as well as at most one $-1$ in $R_1$. Therefore, for each column $j \in E_{12}$, of $M'$, it holds

\[
\sum_{i \in R_L} M'_{ij} + \sum_{i \in R_R} M'_{ij} = -1, 0, 1, \text{ and for each column } j \in E_{32} \text{ of } M'
\]

\[
\sum_{i \in R_L} M'_{ij} + \sum_{i \in R_R} M'_{ij} = 0, 1, \text{ and for each column } j \in E_{23} \text{ of } M'
\]

By Lemma 4, in all these subcases, we get $det M' \in \{0, \pm 1\}$. This proves that $M$ is TU.

Note $M$ is obtained after adding the matrix $I_r$ next to the lower right corner of $M$. Any square submatrix $M'$ of $M$ that contains some nonzero entries of $I_r$ has a structure as follows:

\[
M' = \begin{bmatrix} M'' & 0 \\ \vdots & I_r \\ M'' & 0 \end{bmatrix}
\]

where $M''$ is obtained by deleting the rows and columns belonging to $I_r$, $r' \leq r$. Therefore, $det M' = \pm det M'' \in \{0, \pm 1\}$ by the TU of $M$. By definition, $M$ is TU. $\square$

Proof of Proposition 3: As $M$ is obtained by adding a row to $M$ and $M$ is TU, it suffices to show that every square submatrix $M'$ of $M$ that contains nonzero entries from the last row of $M$ (denoting this row by $\alpha$; see Fig. 1) is TU. First, consider the case where $M'$ does not contain any nonzero elements in the last row of $M$ (denoting these rows by $\beta_1, \ldots, \beta_t$; from the top down, respectively). Notice $\alpha$ has 1's in its columns indexed by $\bigcup_{i=1}^m E_i$ and $-1$ in its $n_E + 1$ to $n_E + r$ columns, and 0's elsewhere. Also observe that for each row of $M$ in the set indexed by $\bigcup_{i=r+1}^m U^G_i$, there are no nonzero entries except in the columns indexed by $\bigcup_{i=1}^m E_i$. As a result, after changing the sign of the last row of $M'$, by regarding $\bigcup_{i=1}^m E_i$ of $M'$ as a single row $E_i$ of $M$ and adopting a way stated in the proof of Proposition 2, we can find an assignment of signs for each row of $M'$ so that the sum of their signed rows equals a row vector with entries in $\{0, \pm 1\}$.

Now, consider the case where $M'$ contains nonzero elements from $\alpha$ and some $\beta_i$'s simultaneously. Without sacrificing any generality, suppose $M'$ contains nonzero elements from $\beta_1, \ldots, \beta_t$, $1 \leq r_1 \leq \alpha$. Observe that $(\alpha + \sum_{i=1}^t \beta_i)_{ij} = 1$ when $j$ is indexed by $\bigcup_{i=r+1}^m E_i$, $(\alpha + \sum_{i=1}^t \beta_i)_{ij} = -1$ when $j = n_E + r_1 + 1, \ldots, n_E + r$, and $(\alpha + \sum_{i=1}^t \beta_i)_{ij} = 0$ elsewhere. Introduce a $(n_{V+1} + 1) \times (n_{E+1} + r)$ matrix $M'$, consisting of the first $n_{V+1}$ rows of $M$ and its last row being $\alpha + \sum_{i=1}^t \beta_i$. Notice that for each row of $M$ in the set indexed by $\bigcup_{i=r+1}^m U^G_i$, there are no nonzero elements except in the columns indexed by $\bigcup_{i=r+1}^m E_i$. By regarding $\bigcup_{i=r+1}^m E_i$ as a single row $E_i$ of $M$ and following the similar reasoning to the proof of Proposition 2, it turns out every submatrix of $M$ is TU. Consequently, there is a sign assignment for each row of $M'$, such that the sum of their signed rows yields a row vector with entries in $\{0, \pm 1\}$, in which the rows containing nonzero elements from $\alpha$ and $\beta_1, \ldots, \beta_t$ always have the same sign. This means $M$ is TU from Lemma 4. $\square$

VI. ILLUSTRATIVE EXAMPLE

This section provides an example to illustrate the LP-based methods. Consider a system $(A, B)$ with its system digraph $G(A, B)$ given in Fig. 3. The costs of available input links are given near each link.

It is easy to see this system satisfies Assumptions 1-3. This system consists of six SCCs, among which $X_1 = \{x_1, x_2, x_3\}$ and $X_2 = \{x_4, x_5, x_6\}$ are the vertex sets of two source SCCs (thus $r = 2$). Moreover, $w^{\min} = \min_{i \in [m]} w_i = 1$. For problems $P_1$, $P_2$, and $P_3$ associated with this system, introducing variables $\{y_e : e \in E_{U,X} \cup E_{X_X}\}$ and
\{z_1, z_2\}, we can obtain the corresponding LPs \(P_1^{LP}, P_2^{LP}, \) and \(P_3^{LP}\). Particularly, for \(P_3\), we set \(k = 3\).

With the help of the MATLAB LP solver \texttt{linprog}, for the LP \(P_1^{LP}\), it is found the optimal solution is \(y_e = 1\) for \(c = (u_4, x_7)\), \((u_4, x_5)\), \((x_7, x_5)\), \((x_7, x_6)\), \((x_7, x_4)\), \((x_7, x_2)\), \((x_7, x_1)\), and \((x_7, x_8)\), and \(y_e = 0\) otherwise; in addition, \(z_1 = z_2 = 1\). This means the optimum to problem \(P_3\) is 2, and the corresponding optimal input selection is \(\{(u_1, x_1), (u_4, x_4)\}\). The corresponding cost is 140, which is the minimum cost that can be achieved with 2 input links. For \(P_2^{LP}\), it turns out the optimal solution corresponds to \(E_{\text{max}} = \{(u_1, x_1), (u_3, x_3), (u_5, x_9), (u_6, x_7)\}\) and \(E_{\text{soc}} = \emptyset\), with the optimum 13. This indicates the optimum to problem \(P_2\) is 13, and the input selection is \(\{(u_1, x_1), (u_3, x_3), (u_5, x_9), (u_6, x_7)\}\).

Similarly, for \(P_1^{LP}\) with \(k = 3\), it is attained that \(P_1^{LP}\) achieves its optimum 52 with \(E_{\text{max}} = \{(u_1, x_1), (u_4, x_4), (u_5, x_9)\}\) and \(E_{\text{soc}} = \emptyset\). This means, the optimum to \(P_1\) is 52, and the optimal input selection is \(\{(u_1, x_1), (u_4, x_4), (u_9, x_9)\}\).

The aforementioned three solutions also indicate a natural tradeoff between the sparsity and the cost of the optimal input selections for achieving structural controllability, which highlights the significance of the problem \(P_3\). Say, with a smaller sparsity bound, the cost of the corresponding optimal input selection tends to be bigger.

VII. CONCLUSION

This article investigates three related cost-sparsity-induced optimal input selection problems for achieving structural controllability in the nondedicated constrained input case. We first formulate these problems as equivalent ILPs, and then, show that, under the source SCC grouped input constraint, those ILPs could be solved efficiently by their LP relaxations using the off-the-shelf LP solvers. We further show those problems are strongly polynomially solvable. We do this by proving that the corresponding constraint matrices of the ILPs are TU. In this way, we provide an alternative algebraic approach, conceptually different from the graph-theoretic ones, for solving these problems under the addressed condition. It contains all existing known polynomially solvable (nontrivial) cases as special instances. To explore the complexity of problems \(P_{1,2,3}^{LP}\) without Assumption 3, it may be necessary to explore further properties of the ILPs, which could be considered in future work.

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