Macroscopic Quantum Self-trapping and Atomic Tunneling in Two-species Bose-Einstein Condensates

Le-Man Kuang¹,², Zhong-Wen Ouyang²
¹CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China
²Department of Physics, Hunan Normal University, Changsha 410081, China

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We present a new theoretical treatment of macroscopic quantum self-trapping (MQST) and quantum coherent atomic tunneling in a zero-temperature two-species Bose-Einstein condensate system in the presence of the nonlinear self-interaction of each species, the interspecies nonlinear interaction, and the Josephson-like tunneling interaction. It is shown that the nonlinear interactions can dramatically affect the MQST and the atomic tunneling, and lead to the collapses and revivals (CR) of population imbalance between the two condensates. The competing effects between the self-interaction of each species and the interspecies interaction can lead to the quenching of the MQST and the suppression of the CR and the Shapiro-like steps of the atomic tunneling current. It is revealed that the interatomic nonlinear interactions can induce the coherent atomic tunneling between two condensates even though there does not exist the interspecies Josephson-like tunneling coupling.

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I. INTRODUCTION

Recently, much attention has been paid to the investigations of systems consisting of two weakly interacting Bose-Einstein condensates [1-17] due to the appearance of quantum interference [18-34] and new macroscopic quantum phenomena [35-37]. In principle, such condensate systems can be produced in a double trap with two condensates coupled by quantum tunneling and ground collisions, or in a system with two different magnetic sublevels of an atom, in which case the two species condensates correspond two electronic states involved. The coupling between two condensates could be realized by the near-resonant dipole-dipole interaction.

The first experiment [6] involving interactions between two condensates was performed with atoms evaporatively cooled in the $|F = 2, M_f = 2 \rangle$ and $|1, -1 \rangle$ spin states of $^{87}$Rb. One of the latest experimental advances [3,7] in this direction is the realization of measurements of relative phase in two-component Bose-Einstein condensates. In experiments at JILA [3], two condensates in two different internal atomic states are produced by using a single two-photon coupling pulse. The two condensates have a well defined relative phase. After a time during which the condensates evolve in the trapping potentials, the two condensates interfere through mixing coherently the two internal atomic states. Then, the relative phase of the two condensates is obtained from the spatial interference pattern. The realization of measurements of the relative phase between two condensates opened the fascinating possibility of experimentally examining the phase-related phenomena in Bose condensates, such as atomic Josephson effect and macroscopic quantum self-trapping (MQST) [37].

Theoretical studies of such systems began in Ho and Shenoy’s work [9] which shown that binary mixtures of condensates of alkali atoms have a great variety of ground state and vortex structures. Then, the stability and collective excitations of two-species condensate systems [1,2,4,5] have been extensively studied. More recently, Smerzi and coworkers [37] have shown that the in a system of two Bose condensates the quantum coherent atomic tunneling between two condensates induces two types of interesting effects. One is an atomic Josephson effect in Bose condensates, which [43] is a generalization of the sinusoidal Josephson effects familiar in superconductors. The other is macroscopic quantum self-trapping (MQST), which is a kind of a self-locked population imbalance between two Bose condensates. It arises because of the interatomic nonlinear self-interaction. The MQST has a quantum nature, involving the coherence of a macroscopic number of atoms in the two condensates. It has been known that the MQST depends upon the trap parameters, the total atoms and initial states of the system and is self-maintained in a closed conserved system without external drives. As pointed out in Ref.[37], it is easier to observe the MQST in Bose condensates than self-trapping phenomena in other systems, such as the single-electron Coulomb Blockade effect [38] arising from the Coulomb interaction between electrons, single polaron trapping in a medium [39] which arises from single electrons, interacting with a polarizable lattice, and external gravitational effects on He II baths [40,41].

In Ref.[37], Smerzi and coworkers only considered the MQST induced by interatomic nonlinear self-interaction in each condensate. However, in a system consisting of two Bose condensates, there are not only nonlinear self-interaction but also interspecies nonlinear interaction. Questions that naturally arise are, what is the effect of the interspecies nonlinear interaction on the MQST and quantum coherent atomic tunneling? Does interspecies nonlinear interaction strengthen or weaken the MQST...
and the atomic tunneling current between them?

In this paper, we present a theoretical treatment of the MQST and the quantum coherent atomic tunneling in a more general two-species Bose condensate system in terms of a two-mode approximate model and the rotating wave approximation. Our treatment involves not only the interatomic nonlinear self-interaction in each species but also the interspecies nonlinear interaction. We find that the presence of the interspecies nonlinear interaction gives rise to new insight to the MQST and the atomic tunneling between the two condensates. This paper is organized as follows. In Sec. II, we establish our model and present an approximate analytic solution. In Sec. III, we discuss the collapse and revival (CR) phenomenon on population imbalance between two condensates. In Sec. IV, we investigate the MQST in the two-condensate system, and discuss the dependence of the MQST upon the initial states and the tunneling interaction and the nonlinear interactions. In Sec. V, we study quantum dynamics of the atomic tunneling current and its dc characteristics. We shall conclude our paper with discussions and remarks in the last section.

II. MODEL AND SOLUTION

We consider a zero-temperature two-species Bose condensate system in which the atoms interact via $aa$ and $bb$ and $ab$ elastic collisions, and there is a Josephson-like coupling term denoted by $a^\dagger b$ and $ab^\dagger$. In the formalism of the second quantization, Hamiltonian of such a system can be written as

$$
\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{\text{int}} + \hat{H}_{\text{jos}},
$$

$$
\hat{H}_i = \int \! dx \hat{\psi}_i^\dagger(x)[-\frac{\hbar^2}{2m} \nabla^2 + V_i(x)] \hat{\psi}_i(x), \quad (i = 1, 2),
$$

$$
\hat{H}_{\text{int}} = U_{12} \int \! dx \hat{\psi}_1^\dagger(x) \hat{\psi}_2^\dagger(x) \hat{\psi}_1(x) \hat{\psi}_2(x),
$$

$$
\hat{H}_{\text{jos}} = \lambda \int \! dx [\hat{\psi}_1^\dagger(x) \hat{\psi}_2(x) + \hat{\psi}_1(x) \hat{\psi}_2^\dagger(x)],
$$

where $\hat{\psi}_i(x)$ and $\hat{\psi}_i^\dagger(x)$ are the atomic field operators which annihilate and create atoms at position $x$, respectively, they satisfy the commutation relation

$$
[\hat{\psi}_i(x), \hat{\psi}_j^\dagger(x')] = \delta_{ij} \delta(x - x').
$$

In Eq.(1), $\hat{H}_1$ and $\hat{H}_2$ describe the evolution of each condensate in the absence of interspecies interaction. $\hat{H}_{\text{int}}$ describes interspecies collisions. $\hat{H}_{\text{jos}}$ is the Josephson-like tunneling coupling term. Atoms are confined in harmonic potentials $V_i(x)$ ($i = 1, 2$) of frequencies $\omega_i$. Interactions between atoms are described by a nonlinear self-interaction term $U_i = 4\pi \hbar^2 a_i^{sc}/m$ and a term that corresponds the nonlinear interaction between different condensates $U_{12} = 4\pi \hbar^2 a_{12}^{sc}/m$, where $a_i^{sc}$ is s-wave scattering length of condensate $i$ and $a_{12}^{sc}$ that between condensate 1 and 2. For simplicity, throughout this paper we let $\hbar = 1$ and assume that $a_i^{sc} = a_{12}^{sc} = a^{sc}$, and $V_i(x) = V_{2}(x)$.

It is well known that the above Hamiltonian can be reduced to two-mode boson Hamiltonian $[30, 32, 35]$ through expanding the atomic field operators over single-particle states $[35]$

$$
\hat{\psi}_i(x) = \hat{a}_i \phi_{iN}(x) + \hat{\psi}_i(x),
$$

where $\hat{a}_i^\dagger = \int \! dx \phi_{iN}(x) \hat{\psi}_i^\dagger(x)$ create particles with distributions $\phi_{iN}(x)$ with $[\hat{a}_i, \hat{a}_i^\dagger] = 1$. The first term in the mode expansion (6) acts only on the condensate state vector, whereas the second term $\hat{\psi}_i(x)$ accounts for non-condensed atoms. Substituting the mode expansions of the atomic field operators into the Hamiltonian (1), retaining only the first term representing the condensates, we arrive at the following two-mode approximate Hamiltonian

$$
\hat{H} = \omega_0(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + q(\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1^\dagger) + 2\chi(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + 2\chi(a_1^\dagger a_2 + a_2^\dagger a_1) + 2\chi a_1^\dagger a_2^\dagger a_2^\dagger a_2 + \hat{H}_{\text{jos}},
$$

where the frequency and the coupling constants are defined by

$$
\omega_0 = \frac{1}{2} \sum_{i=1}^{2} \int \! dx [\frac{1}{2} \nabla V_i(x)]^2 + V(x) |\phi_iN(x)|^2],
$$

$$
q = U_0 \int \! dx |\phi_1N(x)|^2 + |\phi_2N(x)|^2),
$$

$$
\chi = \frac{1}{2} U_{12} \int \! dx |\phi_1N(x)|^2 |\phi_2N(x)|^2.
$$

From Eqs. (6) and (7) we can see that the two-mode approximation essentially consists in neglecting all modes except the condensate modes. At zero temperature, this amounts to ignoring the atoms which have left the condensate mode due to the two-body interactions. In other words, what the two-mode approximation involves is only the first order effects of interactions. The mode expansion of the condensate function over single-particle states (6) makes the condensate shape not to be changed, this limits migration of condensed atoms from one condensate to the other. The constraint on the shapes of condensates implies that the two-mode approximation can be applied only for weak nonlinearity. The valid conditions of the two-mode approximation were demonstrated in Refs. [30, 32, 35], which indicate that this approximation provides a reasonably accurate picture for weak many-body interactions, i.e., for small number of condensed atoms.
For large condensates, the mode functions of condensates are altered due to the collisional interactions, and the two-mode approximation breaks down. As shown in Ref.[30,32], a simple estimate shows that this happens when the number of atoms $Na^{sc} \gg r_0$, where $a^{sc}$ is a typical scattering length and $r_0$ is a measure of the trap size. If we consider a large trap with the size $r_0 = 10 \mu m$ and the typical scattering length $a^{sc} = 5 \text{ nm}$, the two-mode approximation is applicable for $N \leq 200$. This is the case which we consider here. We shall show that the MQST and atomic tunneling between the two condensates are strongly affected by the nonlinear many-body interactions.

We note that the two-mode approximate Hamiltonian has the same form with that of a two-mode nonlinear optical directional coupler [42]. The two-mode Hamiltonian (7) can not be exactly solved, but for weak nonlinear interactions a closed analytical solution can be obtained under the rotating wave approximation suggested by Alodjanc et al. [43].

In order to obtain an approximate analytic solution of the Hamiltonian (7), we introduce a new pair of bosonic operators $\hat{A}_1$ and $\hat{A}_2$ by the following expressions:

$$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{A}_1 e^{igt} - i\hat{A}_2 e^{-igt}), \quad \hat{a}_2 = \frac{1}{\sqrt{2}}(\hat{A}_1 e^{igt} + i\hat{A}_2 e^{-igt}),$$

(12)

where $\hat{A}_1$ and $\hat{A}_2$ are slowly varying operators, they satisfy the usual bosonic commutation relations: $[\hat{A}_i, \hat{A}_j] = 0$, and $[\hat{A}_i, \hat{A}_j^\dagger] = \delta_{ij}$ with $\hat{A}_j^\dagger$ being the hermitian conjugation of $\hat{A}_j$. Then the Hamiltonian (7) reduces to the following form

$$\hat{H} = \omega \hat{N} + \frac{1}{4} q [3\hat{N}^2 - (\hat{A}_1^\dagger \hat{A}_1 - \hat{A}_2^\dagger \hat{A}_2)^2] + g (\hat{A}_1^\dagger \hat{A}_1 - \hat{A}_2^\dagger \hat{A}_2) + \frac{1}{2} \chi \hat{N}^2 - \chi \hat{A}_1^\dagger \hat{A}_1 \hat{A}_2^\dagger \hat{A}_2 + \hat{H}',$$

(13)

where the detuning is given by $\omega = \omega_0 - (\chi + q)/2$, the total number operator $\hat{N}$ is a conserved constant which is given by

$$\hat{N} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{A}_1^\dagger \hat{A}_1 + \hat{A}_2^\dagger \hat{A}_2,$$

(14)

and $\hat{H}'$ is a nonresonant term which is given by

$$\hat{H}' = \frac{1}{2} (\chi - q)(\hat{A}_1^\dagger \hat{A}_2^\dagger e^{-igt} + \hat{A}_2^\dagger \hat{A}_1 e^{igt}),$$

(15)

which oscillates at the frequency $4g$. The account of the fast oscillating term results only in some additional oscillations which play no essential role in the evolution of the measurable quantities specifying the macroscopic quantum phenomena of the two-condensate system, so that it is fully negligible. This means the rotating wave approximation [43]. After neglecting the nonresonant term $\hat{H}'$, we get the following approximate Hamiltonian:

$$\hat{H}_A = \omega \hat{N} + g (\hat{A}_1^\dagger \hat{A}_1 - \hat{A}_2^\dagger \hat{A}_2) + \frac{1}{4} q [3\hat{N}^2 - (\hat{A}_1^\dagger \hat{A}_1 - \hat{A}_2^\dagger \hat{A}_2)^2] + \frac{1}{2} \chi \hat{N}^2 - \chi \hat{A}_1^\dagger \hat{A}_1 \hat{A}_2^\dagger \hat{A}_2.$$  

(16)

In order to solve the Hamiltonian (16) we introduce two Fock spaces of $(\hat{A}_1, \hat{A}_2)$ and $(\hat{a}_1, \hat{a}_2)$ in which the bases are defined by

$$|n,m\rangle = \frac{1}{\sqrt{n!m!}} \hat{A}_1^n \hat{A}_2^m |0,0\rangle,$$

(17)

$$|n,m\rangle = \frac{1}{\sqrt{n!m!}} \hat{a}_1^n \hat{a}_2^m |0,0\rangle,$$

(18)

where $n$ and $m$ take non-negative integers. Obviously, $\hat{H}_A$ is diagonal in the Fock space of $(\hat{A}_1, \hat{A}_2)$, and we find that

$$\hat{H}_A |n,m\rangle = E(n,m)|n,m\rangle,$$

$$E(n,m) = \omega (n + m) + g(n - m) - \frac{1}{2} (3q + \chi)(n + m)^2 - \frac{1}{2} q(n - m)^2 - \chi nm.$$  

(20)

Consider two coherent states defined in Fock spaces of $(\hat{A}_1, \hat{A}_2)$ and $(\hat{a}_1, \hat{a}_2)$, respectively,

$$|\alpha_1, \alpha_2\rangle = D_{\alpha_1}(\alpha_1)D_{\alpha_2}(\alpha_2)|0,0\rangle,$$

(21)

$$|u_1, u_2\rangle = D_{\alpha_1}(u_1)D_{\alpha_2}(u_2)|0,0\rangle,$$

(22)

where $D_{\alpha_i}(\alpha_i)$ and $D_{\alpha_i}(\alpha_i)$ are displacement operators defined by

$$D_{\alpha_i}(\alpha_i) = \exp(\alpha_i \hat{a}_i + \alpha_i^* \hat{a}_i^\dagger),$$

(23)

$$D_{\alpha_i}(u_i) = \exp(u_i \hat{A}_i + u_i^* \hat{A}_i^\dagger).$$

(24)

Note the fact that $|0,0\rangle = |0,0\rangle$, we can find a useful relation to connect $|\alpha_1, \alpha_2\rangle$ and $|u_1, u_2\rangle$ with each other

$$|\alpha_1, \alpha_2\rangle = \frac{|\alpha_1 + \alpha_2, i(\alpha_1 - \alpha_2)|}{\sqrt{2}},$$

(25)

$$|\alpha_1, \alpha_2\rangle = \frac{|\alpha_1 - i\alpha_2, \alpha_1 + i\alpha_2|}{\sqrt{2}}.$$  

(26)

Following the arguments of Bose broken symmetry, we assume that the two condensates are initially in the coherent states $|\alpha_1\rangle$ and $|\alpha_2\rangle$, which are eigenstates of $\hat{a}_1$ and $\hat{a}_2$, respectively. Then the wave function of the two species condensate system at time $t$ can be explicitly expressed as

$$|\Phi(t)\rangle = e^{-\frac{i}{\hbar}Ht} \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n!m!}} u_1^n(u_2^m) \times e^{-iE(n,m)t} |n,m\rangle.$$  

(27)
where
\[ u_1 = \frac{1}{\sqrt{2}}(\alpha_1 + \alpha_2), u_2 = \frac{1}{\sqrt{2}}(\alpha_1 - \alpha_2) \]
\[ N = |\alpha_1|^2 + |\alpha_2|^2 = |u_1|^2 + |u_2|^2, \]
where we have used Eqs.(25), (26) in the derivation of Eq.(27).

III. COLLAPSE AND REVIVALS OF POPULATION IMBALANCE

In this section we show that the two condensate system under our consideration exhibits a collapse and revival phenomenon of population imbalance between two condensates. Denote the number difference of atoms between the two condensates by
\[ D(t) = N_1(t) - N_2(t). \]
Then from Eq.(27) we can find that at time \( t \), the number of atoms in each condensate \( N_i(t) = (a_i^\dagger a_i) \) is given by
\[ N_i(t) = \frac{1}{2}(N - (-1)^i2|u_1||u_2|\cos[4gt + \theta(t)]) \times e^{-2N\sin^2(\frac{1}{4}(q - \chi))t}, (i = 1, 2) \]
where we have used the following symbols:
\[ u_i = |u_i|e^{i\varphi_{u_i}}, \theta(t) = (\varphi_{u_2} - \varphi_{u_1}) + u_{21}\sin(q - \chi)t, \]
with \( u_{21} \) and \( \varphi_{u_i} \) being defined by
\[ u_{21} = |u_2|^2 - |u_1|^2, \]
\[ \varphi_{u_i} = \tan^{-1}\left\{ \frac{|\alpha_1|\sin\varphi_1 \mp (-1)^i|\alpha_2|\sin\varphi_2}{|\alpha_1|\cos\varphi_1 \mp (-1)^i|\alpha_2|\cos\varphi_2} \right\}. \]
Then, the population difference is given by
\[ D(t) = 2|u_1||u_2|\cos[4gt + \theta(t)] \times e^{-2N\sin^2(\frac{1}{4}(q - \chi))t}, \]
where \( N = |\alpha_1|^2 + |\alpha_2|^2 \) is the total number of the atoms in the two condensates. Eq.(35) indicates that the population imbalance periodically oscillates with the time evolution. From Eq.(35) we can see that \( D(t) \) exhibits collapse and revival phenomenon which is a kind of nonclassical effect well known in the Jaynes-Cummings model [44] to describe interaction between a single-mode radiation field and a two-level atom. The CR is also found in a Bose condensate system [35,36]. From Eq.(35) we see that the CR of the population imbalance in the two-condensate system depends on the tunneling interaction \( g \) and interatomic nonlinear interactions \( (q \text{ and } \chi) \). When \( g > |q - \chi|/8 \), since the function \( \cos[4gt + \theta(t)] \) is the rapidly varying part in (35), so that the shape of the CR is determined by the envelope function \( \exp[-2N\sin^2(\frac{1}{4}(q - \chi))t] \). The maximal revivals take place at time \( t = 2n\pi/|q - \chi| \), where \( n \) is an integer. When \( g < |q - \chi|/8 \), the function \( \exp[-2N\sin^2(\frac{1}{4}(q - \chi))t] \) becomes the rapidly varying part in (35), the CR then is determined by the envelope function \( \cos[4gt + \theta(t)] \). In Fig. 1 we plot the evolution of the population difference between two condensates with respect to the time which is in units of \( |q - \chi| \), when the two condensates are in the initial state of \( \alpha_1 = 5 \) and \( \alpha_2 = 4 \), and the tunneling coupling is \( g = 25|q - \chi| \). Fig. 1 clearly indicates the CR phenomenon of the population difference.

It is worthwhile to note that when \( q = \chi \), the population difference (35) becomes
\[ D(t) = 2|u_1||u_2|\cos[4gt + (\varphi_{u_2} - \varphi_{u_1})], \]
which is a simple sinusoidal oscillation, no CR occurs. The suppression of the CR can be explained by looking at the expression (35) which indicates that both the self-interaction \( g \) and the interspecies interaction \( \chi \) can induce the CR, but the CR produced by one can weaken that by another. It is the CR produced by the self-interaction completely counteracts the CR by the interspecies interaction that leads to the suppression of the CR.

When there is no nonlinear interactions, i.e., \( q = \chi = 0 \), from Eq.(35) it is easy to find that the population imbalance has the same form with that of the case \( q = \chi \neq 0 \). This means that the CR vanishes when the nonlinearity vanishes. Hence, the CR of the population imbalance is a consequence of nonlinear interactions in condensates. The CR of the oscillatory transfer of atoms between the two condensates constitutes a novel macroscopic quantum phenomenon induced by interatomic nonlinear interactions for the two species condensate system.

IV. MACROSCOPIC QUANTUM SELF-TRAPPING

In this section we are concerned with the MQST. The MQST effect is characterized by the nonzero time mean value of the fractional population imbalance between the two condensates defined by
\[ p(t) = \frac{N_1(t) - N_2(t)}{N}. \]
From Eqs.(35) and (37) we get that
\[ p(t) = \frac{2|u_1||u_2|\cos[4gt + \theta(t)]}{N} \times e^{-2N\sin^2(\frac{1}{4}(q - \chi))t}. \]
In order to investigate the MQST, we expand the above equation as
where \( J_n(A) \) and \( I_n(A) \) are Bessel function and modified Bessel function. From Eq.(35) it is easy to find that when the tunneling interaction and nonlinear interactions satisfy the condition:

\[
4g = K(\chi - q),
\]

where \( K \) is an integer, we can get a nonzero time-averaged value of population imbalance

\[
\bar{p} = \frac{2|u_1||u_2|}{N} e^{-N} \sum_{n=-\infty}^{+\infty} J_n(u_{21})I_{K-n}(N) \times \cos(\varphi_{u_2} - \varphi_{u_1}),
\]

which indicates the existence of the MQST. Eq.(40) is the condition under which the MQST happens.

From Eq.(39) we can see that when the tunneling interaction vanishes and nonlinear self-interaction equals nonlinear interspecies interaction, i.e., \( g = 0 \) and \( q = \chi \), we arrive at a constant population imbalance

\[
p(t) = \frac{2|u_1||u_2|}{N} \cos(\varphi_{u_2} - \varphi_{u_1}).
\]

This is a time-independent state, called the self-trapping stationary state, which is the consequence of competing between nonlinear self-interaction and nonlinear interspecies interaction.

When there exists the tunneling coupling, i.e., \( g \neq 0 \), from Eq.(40) we can find the critical value of the tunneling coupling at which the MQST happens \( g_c = |q - \chi|/4 \).

This critical value \( g_c \) depends upon only the difference between the nonlinear self-interaction and interspecies nonlinear interaction, not the nonlinear self-interaction and interspecies nonlinear interaction themselves. Therefore, it becomes possible that the MQST occurs only when the tunneling coupling equals or exceeds the critical value \( g_c \).

In Fig.2 we plot the time evolution of the fractional population imbalance when the two condensates are in the initial state of \( \alpha_1 = 10 \) and \( \alpha_2 = 0 \) for (a) \( K = 1 \), (b) \( K = 20 \). Here the time is in units of \( |q - \chi| \). From Fig.2 (a) and (b) we can see that the weaker the tunneling coupling \( (g) \), the more apparent the MQST becomes. This indicates that the MQST is an effect induced by interatomic nonlinear interactions \( q \) and \( \chi \) not the tunneling interaction \( (g) \). In what follows we shall discuss the dependence of the MQST in detail upon the initial states and nonlinear interactions for specific cases.

A. The initial state dependence

We here discuss the dependence of the MQST on the initial states of the two condensates in the following four cases.

Case 1: \( |\alpha_1| = |\alpha_2|, \varphi_1 \neq \varphi_2 \). In this case, the two condensates initially have the same number of atoms but different phases. From Eq.(32) and (33) we get that \( |u_1|^2 = N \cos^2[(\varphi_1 - \varphi_2)/2], |u_2|^2 = N \sin^2[(\varphi_1 - \varphi_2)/2], \) and \( u_{21} = \chi \cos(\varphi_1 - \varphi_2) \). Making use of Eq.(28), from Eqs.(35) and we can see that the fractional population periodically evolves with respect to time, \( p(t) \neq 0 \) except that \( \varphi_1 - \varphi_2 = n\pi, \) where \( n \) is an integer. If \( 4g/(\chi - q) = K \) (an integer), we get the locked population imbalance

\[
\bar{p} = |\sin(\varphi_1 - \varphi_2)| \cos(\varphi_{u_1} - \varphi_{u_2}) e^{-N} \times \sum_{n=-\infty}^{+\infty} J_n(-N \cos(\varphi_1 - \varphi_2))I_{K-n}(N),
\]

which implies that the MQST does exist, even if the two condensates initially have the same number of atoms but different phases. From Eq.(28) we have \( \bar{u}_1 = \bar{u}_2 = \alpha_1/\sqrt{\pi} \), and \( \theta(t) = 0 \). The fractional population evolution is given by

\[
p(t) = e^{-N} \sum_{n=-\infty}^{+\infty} I_n(N) \cos\{n(q - \chi) + 4g)t,\]

So that when the tunneling interaction and the nonlinear interactions satisfy the relation \( 4g/(\chi - q) = K \) (an integer), we can see the appearance of the MQST with the locked population imbalance

\[
\bar{p} = e^{-N} I_K(N).
\]

Case 2: \( |\alpha_1| \neq |\alpha_2|, \varphi_1 = \varphi_2 \). In this case the two condensates initially have the same phases but different number of atoms. From Eq.(38) we see that when \( 4g/(\chi - q) = K \) (an integer), the MQST occurs with the following locked population imbalance:

\[
\bar{p} = |\alpha_1|^2 + |\alpha_2|^2 / 2N e^{-N} \sum_{n=-\infty}^{+\infty} J_n(-2|\alpha_1|\alpha_2))I_{K-n}(N).
\]

Case 3: \( N = |\alpha_1|^2, |\alpha_2| = 0 \). In this case, the system starts with all atoms being in one condensate. Making use of Eq.(28), we get that \( u_1 = u_2 = \alpha_1/\sqrt{\pi} \), and \( \theta(t) = 0 \). The fractional population evolution is given by

\[
p(t) = e^{-N} \sum_{n=-\infty}^{+\infty} I_n(N) \cos\{n(q - \chi) + 4g)t,\]

So that when the tunneling interaction and the nonlinear interactions satisfy the relation \( 4g/(\chi - q) = K \) (an integer), we can see the appearance of the MQST with the locked population imbalance

\[
\bar{p} = e^{-N} I_K(N).
\]

Case 4: \( |\alpha_1| = |\alpha_2| \) and \( \varphi_1 = \varphi_2 \). In this case the two condensates initially have the same number of atoms and the same phases. From Eq.(28) we have \( u_2 = 0 \). Making use of Eq.(38) we can see that the oscillations of the population imbalance vanish, i.e., \( p(t) = 0 \), and no MQST occurs. This is in agreement with the result in Ref.[37].

B. The dependence on nonlinear interactions
Then, we turn to the dependence of the MQST upon the tunneling coupling \((q)\) and the nonlinear interactions between atoms, which are described by the parameters \(q\) and \(\chi\) corresponding to self-interactions and interspecies interactions, respectively.

Case 1: \(q = 0, q \neq 0, \text{and } \chi \neq 0\). In this case there is no tunneling interaction, but there exists interatomic nonlinear interactions. From Eq.(39) we find that the fractional population imbalance becomes

\[
p(t) = \frac{2|u_1||u_2|}{N} e^{-N} \sum_{n,m=-\infty}^{+\infty} J_n(u_{21}) I_m(N)
\times \cos\{(\mu + \nu) N q - \chi)t + (\varphi u_2 - \varphi u_1)\},
\]

which indicates that no MQST occurs if the coupling of interatomic self-interaction does not equals that of interspecies nonlinear interaction, i.e., \(q = \chi\). However, when the self-interaction equals the interspecies interaction, i.e., \(q = \chi\), we can observe the self-trapping stationary state with a constant population imbalance \(p(t) = p(0)\).

Case 2: \(q \neq 0, q = \chi = 0\). In this case, we consider only the effect of the tunneling coupling while the interatomic nonlinear interactions are not involved. Eq.(38) tells us that the MQST does not occur, although there exists oscillations of the population imbalance between the two condensates. This further confirms the validity of Smerzi et al.’s conclusion [37] which the MQST arises from the interatomic nonlinear interaction.

Case 3: \(q \neq 0, q = \chi \neq 0\). In this case there exist both the tunneling interaction and the nonlinear interactions, but self-interaction equals interspecies interaction. From Eq.(38) we can find that the population imbalance exhibits a simple oscillation with

\[
p(t) = \frac{2|u_1||u_2|}{N} e^{-N} \cos[4gt + (\varphi u_2 - \varphi u_1)],
\]

which means that the MQST vanishes.

Case 4: \(g \neq 0, q \neq 0, \chi = 0, \text{or } g \neq 0, q = 0, \chi \neq 0\). In this case, there exist the tunneling interaction and one of the self-interaction and the interspecies interaction. It is easy to see that the fractional population imbalance Eq.(38) reduces to

\[
p(t) = \frac{|u_1||u_2|}{N} e^{-N} \sum_{n,m=-\infty}^{+\infty} J_n(u_{21}) I_m(N)
\times \cos\{(\mu + \nu) N q + 4gt + (\varphi u_2 - \varphi u_1)\},
\]

where \(\kappa = q\) or \(-\chi\). Eq.(49) reflects the fact that when \(4g/\kappa = n + m = K\) (an integer), the MQST happens with the nonzero \(\rho\) given by Eq.(41). This implies that both the nonlinear self-interaction in each condensate and the interspecies nonlinear interactions contribute to the MQST. Since the values of \(K\) to determine \(\rho\) have opposite signs for the self-interaction \((\chi)\) and the interspecies interaction \((q)\), the MQST produced by the self-interaction can weaken that by the interspecies interaction. It is the competition between the MQST induced by the nonlinear self-interactions of each condensate and that the interspecies nonlinear interaction that leads to the quenching of the MQST in the above case 3.

V. THE COHERENT ATOMIC TUNNELING CURRENT

In this section, we study quantum dynamics of the coherent atomic tunneling current between two condensates and its \(dc\) characteristics, and discuss the influence of the initial state of condensates and the tunneling interaction and the nonlinear interactions. The coherent atomic tunneling current between the two condensates is defined by \(I(t) = N_1(t) - N_2(t)\). Making use of Eq.(31), it is straightforward to get that

\[
I(t) = -2|u_1||u_2|\{4g \sin(\theta(t))
+(q - \chi)[|u_1|^2 \sin((q - \chi)t - \theta(t))
+|u_2|^2 \sin((q - \chi)t + \theta(t))]\}.
\]

This indicates that the atomic tunneling current periodically changes, atoms periodically transfer between the two condensates with the time evolution.

In order to see \(dc\) characteristic of the atomic tunneling current, we expand the atomic tunneling current (50) as the following expression

\[
I(t) = -2|u_1||u_2| e^{-N} \sum_{n,m=-\infty}^{+\infty} \{8g J_n(u_{21})
-(q - \chi) [|u_1|^2 J_{n-1}(u_{21}) - |u_2|^2 J_{n+1}(u_{21})]\}
\times I_m(N) \sin\{|(\mu + \nu) N q + 4gt\}
+(\varphi u_2 - \varphi u_1)\},
\]

which implies that when the tunneling coupling, and nonlinear couplings satisfy the condition:

\[
4g = K(\chi - q),
\]

we get the \(dc\) component of the atomic tunneling current with the following form,

\[
I_{dc}(K) = -2|u_1||u_2| \{q - \chi\} \sin(\varphi u_2 - \varphi u_1) e^{-N}
\times \sum_{n=-\infty}^{+\infty} \{2KJ_n(u_{21}) - |u_1|^2 J_{n-1}(u_{21})
-|u_2|^2 J_{n+1}(u_{21})\}\} I_{K-n}(N),
\]

where \(K\) is an integer. This indicates that the \(dc\) component of the atomic tunneling current exhibits a step structure with respect to the integer \(K\). This step structure is a resonant phenomenon among the tunneling interaction and nonlinear interactions with the resonant condition given by Eq.(51). It is the analogue of the Shapiro
steps observed in the superconductor Josephson junction [45], so that we call the steps in the step structure of the dc component of the atomic tunneling current the Shapiro-like steps. In what follows we discuss in detail the dependence of the atomic tunneling current and the Shapiro-like steps upon the initial states and nonlinear interactions for some specific cases.

A. The initial state dependence

In this subsection we discuss the initial-state dependence of the atomic tunneling current and the Shapiro-like steps for the following four cases.

Case 1: $|\alpha_1| = |\alpha_2|, \varphi_1 \neq \varphi_2$. In this case, the two condensates initially have the same number of atoms but different phases. From Eq.(50) we can get the expression of the atomic tunneling current

$$I(t) = -N|\sin(\varphi_1 - \varphi_2)|4g\sin(\theta(t)) + (q - \chi) N \cos \frac{1}{2} (\varphi_1 - \varphi_2) \sin((q - \chi)t - \theta(t)) + \sin \frac{1}{2} (\varphi_1 - \varphi_2) \sin((q - \chi)t + \theta(t)))$$

(54)

where

$$\theta(t) = (\varphi_{u_1} - \varphi_{u_2}) - N \cos(\varphi_1 - \varphi_2) \sin((q - \chi)t)$$. (55)

Form Eq.(54) we can obtain the dc component of the tunneling current with the following result,

$$I_{dc}(K) = -N(q - \chi)|\sin(\varphi_1 - \varphi_2)|\sin(\varphi_{u_2} - \varphi_{u_1})$$

$$\times e^{-N} \sum_{n=-\infty}^{+\infty} \{2KJ_n(-N \cos(\varphi_1 - \varphi_2))$$

$$-N \cos \frac{1}{2} (\varphi_1 - \varphi_2) J_{n-1}(-N \cos(\varphi_1 - \varphi_2))$$

$$- \sin \frac{1}{2} (\varphi_1 - \varphi_2) J_{n+1}(-N \cos(\varphi_1 - \varphi_2))\} \times I_{K-n}(N),$$

(56)

where $n$ is an integer. Eq.(56) gives rise to the Shapiro-like steps of the atomic tunneling current.

It is interesting to note that when the initial phases of two condensates satisfy the condition: $\varphi_1 - \varphi_2 = n\pi$, where $n$ is an integer, we can find zero atomic tunneling current. This means that the blockade of the atomic tunneling happens.

When the initial phases satisfy the relation: $\varphi_1 - \varphi_2 = (2n + 1)\pi/2$, $n$ being an integer, we find that

$$I(t) = -4gN \sin(\varphi_{u_2} - \varphi_{u_1})$$

$$+(q - \chi) N \cos(\varphi_{u_2} - \varphi_{u_1}) \sin(2(q - \chi)t),$$

(57)

which indicates that the atomic tunneling current is a simple superposition of a alternating current with the sinusoidal oscillations and a dc current. The dc component is

$$I_{dc} = -4gN \sin(\varphi_{u_2} - \varphi_{u_1}),$$

(58)

which implies that the dc component of the atomic tunneling current depends only upon the tunneling coupling, it is independent of the nonlinear interactions in two condensates, and increases linearly with the tunneling strength ($g$) and the the total number of the atoms ($N$). In this case no Shapiro-like steps appears.

Case 2: $|\alpha_1| \neq |\alpha_2|, \varphi_1 = \varphi_2$. In this case, the two condensates initially have the same phases but the different number of atoms. The atomic tunneling current is given by

$$I(t) = -|\alpha_1|^2 - |\alpha_2|^2 \{4g \sin(\theta(t)) + \frac{1}{2} (q - \chi) (|\alpha_1| + |\alpha_2|)^2 \sin((q - \chi)t - \theta(t))$$

$$+(|\alpha_1| - |\alpha_2|)^2 \sin((q - \chi)t + \theta(t))\},$$

(59)

where

$$\theta(t) = -2\alpha_1 \alpha_2 \sin(q - \chi)t.$$ (60)

And the dc component of the atomic tunneling current has the following form,

$$I_{dc}(K) = -|\alpha_1|^2 - |\alpha_2|^2 \{q - \chi \sin(\varphi_{u_2} - \varphi_{u_1})$$

$$\times e^{-N} \sum_{n=-\infty}^{+\infty} \{2KJ_n(-|\alpha_1| \alpha_2)$$

$$- \frac{1}{2} (|\alpha_1| - |\alpha_2|)^2 J_{n-1}(-|\alpha_1| \alpha_2)$$

$$- (|\alpha_1| - |\alpha_2|)^2 J_{n+1}(-|\alpha_1| \alpha_2)\} I_{K-n}(N).$$

(61)

In FIG. 3, we plot the time evolution of the atomic tunneling current between the two condensates. Results are shown for the case of $g = 0.25$ and $q - \chi = 0.1$ when the two condensates are in the initial state with $|\alpha_1| = 5$ and $|\alpha_2| = 4$. FIG. 3 indicates that the atomic tunneling current exhibits complicated oscillating behaviors.

Case 3: $N = |\alpha_1|^2$, $|\alpha_2| = 0$. In this case, the system starts with all atoms being in one condensate. Taking into account Eqs.(32) and (33), from Eq.(50) we find the atomic tunneling current to be

$$I(t) = -(q - \chi) N^2 \sin(q - \chi)t,$$

(62)

which is a pure sinusoidal alternating atomic current with the period $T = 2\pi/|q - \chi|$ without dc component. It is worthwhile to note that the atomic tunneling current (62) depends on only the difference between the self-coupling $q$ and the interspecies coupling $\chi$, it is independent of the tunneling coupling $g$ at all. Thus, we can conclude that the nonlinear interactions can induce the atomic tunneling, even if there is no tunneling coupling between two condensates.

Case 4: $|\alpha_1| = |\alpha_2|$ and $\varphi_1 = \varphi_2$. In this case, the two condensates initially have the same number of atoms
and phases, we see that the atomic tunneling current vanishes, and no Shapiro-like step occurs.

The above analyses indicate that the atomic tunneling current and the Shapiro-like steps strongly depend on the initial number of atoms in each condensate and the initial phase difference between the two condensates.

B. The dependence on nonlinear interactions

In what follows we show that the interatomic nonlinear interactions also significantly affect the atomic tunneling current and the Shapiro-like steps.

If we consider only the influence of the tunneling coupling while the interatomic nonlinear interactions are not involved, i.e., $g \neq 0$, $q = \chi = 0$, we find that

$$I(t) = -8g|u_1||u_2| \sin[4gt + (\varphi_{u_2} - \varphi_{u_1})].$$

(63)

This atomic tunneling current is a pure sinusoidally alternating atomic current with the period $T = \pi/2g$, without dc component. Hence no Shapiro-like steps appears.

It is interesting to note that when the interatomic nonlinear interactions are involved, but $q = \chi \neq 0$, we get the same results with those of the case of $q = \chi = 0$ for the atomic tunneling current and the Shapiro-like steps. This indicates that the contributions of the nonlinear self-interaction ($q$) in each condensate to the atomic tunneling can counteract that of the interspecies nonlinear interactions ($\chi$). It is the competition between the nonlinear self-interaction and the interspecies nonlinear interaction that leads to a simple form of the atomic tunneling current and the suppression of the Shapiro-like steps.

In particular, we note that when $g = 0$, i.e., there is no tunneling coupling, and $q \neq \chi$, we find that

$$I(t) = -2|u_1||u_2|(|q - \chi||u_1|^2 \sin((q - \chi)t - \theta(t)) + |u_2|^2 \sin((q - \chi)t + \theta(t)),$$

(64)

where $\theta(t)$ is given by Eq.(32). In FIG. 4, we display the time evolution of the atomic tunneling current (64) when the tunneling coupling vanishes. Results are shown for the case of $q - \chi = 0.1$ when the two condensates are in the initial state with $\alpha_1 = 10$ and $\alpha_2 = 5$. From FIG.4 we see that the evolution of the atomic tunneling current exhibits the CR phenomenon.

Eq.(64) implies that the atomic tunneling current is nonzero, but no the Shapiro-like steps appears. From Eq.(64) we can see that even though there is no the tunneling coupling between the two condensates ($g = 0$), the atomic tunneling between the two condensates may happen. This atomic tunneling is completely induced by the nonlinearity of interatomic interactions which are characterized by interatomic collisions ($q$ and $\chi$). Therefore, we may conclude that the nonlinearity of interatomic interactions in the two condensates can lead to the atomic tunneling between the two condensates.

VI. CONCLUDING REMARKS

We have studied the MQST and the quantum coherent atomic tunneling in a two species Bose condensate system in the presence of nonlinear self-interaction of each species, the interspecies nonlinear interaction, and the Josephson-like tunneling interaction, and have given new insight to the MQST and the atomic tunneling. We have shown that the interatomic nonlinear interactions in the two condensates induce not only the MQST but also the CR of the population difference between two condensates. The CR phenomenon can be considered as a novel macroscopic quantum effect. We have indicated that the nonlinear interactions significantly affect the atomic tunneling, and the Shapiro-like steps of the atomic tunneling current. Comparing with Smerzi and coworkers’ work [37], the present work involves the interspecies nonlinear interaction. The involvement of the interspecies nonlinear interaction gives rise to new characteristics on the MQST and the atomic tunneling. We have shown that when both the nonlinear self-interaction ($g$) and the interspecies nonlinear interaction ($\chi$) present at the same time, the atomic tunneling dynamics and the MQST and the Shapiro-like steps depends upon the difference ($q - \chi$), not $q$ and $\chi$ themselves. We have also found that the interspecies nonlinear interaction generates the MQST at the same level with nonlinear self-interaction. However, contribution from the interspecies interaction to the MQST and that from the self-interaction weaken themselves with each other. It is the competing effects between the nonlinear self-interaction in each species and the interspecies nonlinear interaction that leads to the quenching of the MQST and the suppression of the CR and the Shapiro-like steps of the atomic tunneling current. Especially, we have revealed that the nonlinearity of interatomic interactions in the two condensates can induce the coherent atomic tunneling between two condensates occurs, even though there does not exist the Josephson-like tunneling coupling. It should be mentioned that these results are obtained under the two-mode approximation and the rotating wave approximation, so they are valid for weak nonlinear interactions between atoms. Finally, It should be noted that in order to observe macroscopic quantum phenomena such as MQST and Shapiro-like steps in Bose condensates, one has to control various interactions between atoms. This controlling can be carried out through manipulating interatomic scattering lengths. Several theoretical and experimental approaches [46-51] to alter the scattering length have been proposed. In particular, recent experiments on Feshbach resonances in a Bose condensate [46,47] have indicated that the scattering length of ultracold atoms can be altered through Feshbach resonance. These experimental progresses provide the possibility to observe the MQST and Shapiro-like steps in Bose condensates.
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Figure Captions

FIG. 1. Diagram of the time evolution of the population difference between the two condensates. The time is in units of $|q - \chi|$. Results are shown for the case of $g = 25|q - \chi|$, when the two condensates are in the initial state with $\alpha_1 = 5$ and $\alpha_2 = 4$.

FIG. 2. Diagram of the time evolution of the fractional population imbalance. The time is in units of $|q - \chi|$. Results are shown for (a) $K = 1$, and (b) $K = 20$ when the two condensates are in the initial state with $\alpha_1 = 10$ and $\alpha_2 = 0$.

FIG. 3. The atomic tunneling current between the two condensates as a function of time $t$ (in arbitrary units). Results are shown for the case of $g = 0.25$ and $q - \chi = 0.1$ when the two condensates are in the initial state with $\alpha_1 = 5$ and $\alpha_2 = 4$.

FIG. 4. The atomic tunneling current between the two condensates as a function of time (in arbitrary units) when there does not exist the tunneling coupling. Results are shown for the case of $q - \chi = 0.1$ when the two condensates are in the initial state with $\alpha_1 = 10$ and $\alpha_2 = 5$. 
FIG. 1
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FIG. 2(a)

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Out[1]=

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FIG. 2(b)
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Out[2]=
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FIG. 3
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