SOME ARITHMETIC PROPERTIES OF WEIL POLYNOMIALS OF THE FORM $t^{2g} + at^g + q^g$

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Abstract. An isogeny class $\mathcal{A}$ of abelian varieties defined over a finite field is said to be cyclic if every variety in $\mathcal{A}$ has a cyclic group of rational points. In this paper we study the local cyclicity of Weil-central isogeny classes of abelian varieties, i.e. those with Weil polynomials of the form $f_{\mathcal{A}}(t) = t^{2g} + at^g + q^g$, as well as the local growth of the groups of rational points of the varieties in $\mathcal{A}$ after finite field extensions. We exploit the criterion: an isogeny class $\mathcal{A}$ with Weil polynomial $f$ is cyclic if and only if $f'(1)$ is prime with $f(1)$ divided by its radical.

1. Introduction

In this paper we study abelian varieties defined over finite fields with a cyclic group of rational points. This subject is motivated by both applications and theory:

- Finite subgroups of abelian varieties over finite fields are suitable for multiple applications. Cyclic subgroups of the group of rational points are used for example in cryptography, where the discrete logarithm problem is exploited. Abelian varieties can be very abstract objects. Jacobians of algebraic curves are abelian varieties and they are more tractable for application purposes.
- Statistics on cyclic varieties is related to Cohen-Lenstra heuristics ([1]), which asserts, roughly speaking, that random abelian groups tend to be cyclic. Historically, the cyclicity question arose in the context of conjectures of Lang and Trotter ([6]): given an elliptic curve defined over the rational numbers, we are interested in the set of primes such that the reduction is a cyclic elliptic curve. This question was studied also by Serre, Gupta and Murty. Generalizations to higher dimensions was also done.

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This leads to give the following:

**Definition 1.** Given an abelian variety $A$ defined over a finite field $k$, an isogeny class $\mathcal{A}$ of abelian varieties defined over $k$ and a rational prime $\ell$, we say that

1. $A$ is **cyclic** if its group $A(k)$ of rational points is cyclic;
2. $A$ is **$\ell$-cyclic** if the $\ell$-primary component $A(k)_\ell$ of its group $A(k)$ of rational points is cyclic;
3. $\mathcal{A}$ is **cyclic** if the abelian variety $A$ is cyclic for all $A \in \mathcal{A}$; and,
4. $\mathcal{A}$ is **$\ell$-cyclic** if the abelian variety $A$ is $\ell$-cyclic for all $A \in \mathcal{A}$.

This paper concerns cyclicity of isogeny classes. The Honda-Tate theory simplify the study of isogeny classes by studying their Weil polynomials. In addition, it is easy to verify the cyclicity of an isogeny class given its Weil polynomial:

**Theorem 1** (A. Giangreco, 2019, [2]). Let $\mathcal{A}$ be a $g$-dimensional $\mathbb{F}_q$-isogeny class of abelian varieties corresponding to the Weil polynomial $f_{\mathcal{A}}(t)$. Then $\mathcal{A}$ is cyclic if and only if $f_{\mathcal{A}}'(1)$ is coprime with $\widehat{f_{\mathcal{A}}}(1)$.

Here $\widehat{n}$ denotes the ratio of an integer $n$ to its radical. This is in fact a local criterion that can be easily deduced from the proof of Theorem 1.

**Definition 2.** An isogeny class $\mathcal{A}$ of $g$-dimensional abelian varieties defined over the finite field $\mathbb{F}_q$ is said to be **Weil-central** if its Weil polynomial has the form

$$f_{\mathcal{A}}(t) = t^{2g} + at^g + q^g.$$

In this paper we study the local cyclicity of Weil-central isogeny classes after base field extension as well as the local growth of their group of rational points. Given an abelian variety $A$ defined over the finite field $\mathbb{F}_q$ with $q$ elements, and belonging to an isogeny class $\mathcal{A}$, we denote by $\mathcal{A}_n$ the $\mathbb{F}_{q^n}$-isogeny class of $A$. For any prime number $\ell$, let $v_\ell(\cdot)$ be the usual $\ell$-adic valuation over the rational numbers. Thus, for an $\ell$-cyclic isogeny class $\mathcal{A}$ we are interested in the following sets:

$$g_\ell(\mathcal{A}) := \{ n \in \mathbb{N} : v_\ell(f_{\mathcal{A}_n}(1)) > v_\ell(f_{\mathcal{A}}(1)) \} \cup \{1\}$$

and

$$c_\ell(\mathcal{A}) := \{ n \in \mathbb{N} : \mathcal{A}_n \text{ is } \ell\text{-cyclic and } v_\ell(f_{\mathcal{A}_n}(1)) > v_\ell(f_{\mathcal{A}}(1)) \} \cup \{1\}.$$

The first set gives the extensions for which the $\ell$-primary component of the group of rational points is strictly bigger than the one over the base field. The second set gives the cyclic behavior of the $\ell$-component after such finite field extensions. Observe that we have

$$c_\ell(\mathcal{A}) = g_\ell(\mathcal{A}) \setminus \{ n \in \mathbb{N} : \mathcal{A}_n \text{ is not } \ell\text{-cyclic} \}.$$
For an integer \( z \) we denote by \( \omega(\ell) \) the order of \( z \) in the multiplicative group \((\mathbb{Z}/\ell\mathbb{Z})^*\), i.e. the smallest integer \( m \) such that \( z^m \equiv 1 \pmod{\ell} \). Then our main result:

**Theorem 2.** Let \( \ell \) be a prime and \( A \) be an \( \ell \)-cyclic Weil-central isogeny class of ordinary abelian varieties of dimension \( g \) defined over \( \mathbb{F}_q \), such that \( \ell \) does not divide \( g \) and \( \ell \) does not divide \( q^g - 1 \). Then we have

\[
g_\ell(A) \supset \ell N - 2N - \cup_{1<d|g} dN,
\]

\[
c_\ell(A) \supset \ell N - 2N - \cup_{1<d|g} dN - \omega(\ell(q^g))N,
\]

provided that \( v_\ell(f_A(1)) \geq 1 \).

**Remark 1.** From Lemma 6 about the cyclicity, it follows that \( v_\ell(f_A(1)) \geq 2 \) implies that \( \ell \) does not divide \( q^g - 1 \), provided that the other hypothesis of the Theorem 2 holds.

**Organization of the paper.** In Section 2 we recall some generalities about abelian varieties. Section 3 is devoted to the proof of Theorem 2. It will be proved in different lemmas that can be useful by themselves. We first study (Lemma 3) for which extensions a Weil-central isogeny class “remains” Weil-central. Then we study in Lemma 5 the growth behavior and we prove the assertion of Theorem 2 about the set \( g_\ell(A) \). Finally, we study the cyclicity and we prove, as a consequence of Corollary 7, the assertion of Theorem 2 about the set \( c_\ell(A) \).

## 2. Generalities on abelian varieties

We refer the reader to [8] for the general theory of abelian varieties, and to [12] for abelian varieties over finite fields.

Let \( q = p^r \) be a power of a prime, and let \( k = \mathbb{F}_q \) be a finite field with \( q \) elements. Let \( A \) be an abelian variety of dimension \( g \) over \( k \). The set \( A(k) \) of rational points of \( A \) is a finite abelian group. It is the kernel of the endomorphism \( 1 - F \), where \( F \) is the well known Frobenius endomorphism of \( A \). Multiplication by an integer \( n \) is a group homomorphism whose kernel \( A_n \) is a finite group scheme of rank \( n^{2g} \). It is known the group structure of the groups of points over \( \overline{k} \):

\[
A_n(\overline{k}) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}, \quad p \nmid n
\]

\[
A_p(\overline{k}) \cong (\mathbb{Z}/p\mathbb{Z})^i, \quad 0 \leq i \leq g.
\]

For a fixed prime \( \ell \) (\( \neq p \)), the \( A_{\ell n} \) form an inverse system under \( \ell \to A_{\ell n} \), and we can define the Tate module \( T_\ell(A) \) by its inverse limit \( \varprojlim A_{\ell n}(\overline{k}) \). This is a free \( \mathbb{Z}_\ell \)-module of rank \( 2g \) and the absolute Galois group of \( \overline{k} \) over \( k \) operates on it by \( \mathbb{Z}_\ell \)-linear maps.
The Frobenius endomorphism $F$ of $A$ acts on $T_\ell(A)$ by a semisimple linear operator, and its characteristic polynomial $f_A(t)$ is called *Weil polynomial of $A$* (also called *characteristic polynomial of $A$*). The Weil polynomial is independent of the choice of the prime $\ell$. Tate proved in [9] that a $k$-isogeny class $A$ is determined by the Weil polynomial $f_A$ of any $A \in A$, i.e. two abelian varieties $A$ and $B$ defined over $k$ are isogenous (over $k$) if and only if $f_A = f_B$. Thus the notation $f_A$ is justified. If $A$ is simple, $f_A(t) = h_A(t)^e$ for some irreducible polynomial $h_A$.

Weil proved that all of the roots of a Weil polynomial have absolute value $\sqrt{q}$ (they are called $q$-Weil numbers). Thus, the Weil polynomial of an isogeny class $A$ has the general form

$$f_A(t) = t^{2g} + a_1 t^{2g-1} + \cdots + a_g t^g + a_{g-1} qt^{g-1} + \cdots + a_1 q^{g-1} t + q^g.$$ 

The cardinality of the group $A(k)$ of rational points of $A$ equals $f_A(1)$, and thus it is an invariant of the isogeny class. An abelian variety $A$ is said to be *ordinary* if it has a maximal $p$-rank. This is equivalent of having the central term $a_g$ of its Weil polynomial coprime with $p$. Thus, ordinarity it is also an invariant of the isogeny class.

Let $\{\alpha_i\}_{i=1}^{2g}$ be the set of roots of the Weil polynomial of the abelian variety $A \in A$ defined over $\mathbb{F}_q$. For a positive integer $n$, we denote by $A_n$ the isogeny class defined over $\mathbb{F}_{q^n}$ corresponding to the Weil polynomial with $\{\alpha_i^n\}_{i=1}^{2g}$ as its set of roots. It is the $\mathbb{F}_{q^n}$-isogeny class of the variety $A$.

### 3. Weil-central isogeny classes

As we stated, we are interested in Weil-central isogeny classes of abelian varieties. They have a Weil polynomial as follows

$$f_A(t) = t^{2g} + at^g + q^g.$$ 

Among these isogeny classes are those of elliptic curves and zero-trace abelian surfaces, with Weil polynomials $f_E(t) = t^2 + at + q$ and $f_S(t) = t^4 + at^2 + q^2$, respectively. Cyclicity of elliptic curves and their extensions was studied by Vlăduț in [10] and [11].

The following facts motivate the study of such isogeny classes. We know from [3] that among Weil-central isogeny classes of abelian surfaces, only such with Weil polynomial $t^4 - qt^2 + q^2$, and $p \equiv 1 \pmod{3}$ do not contain a principally polarizable variety. Also, from [4], very few do not contain the Jacobian of a 2-genus curve.

Note that in order that a polynomial of the form (2) to be a Weil polynomial, we need that $0 \leq |a| \leq 2\sqrt{q^g}$. Indeed, if not, the polynomial $x^2 + ax + q^g$ would have two different real roots, which would
imply that \(2\) has complex roots of absolute value different than \(\sqrt{q}\). This also implies that the real part of the roots of \(x^2 + ax + q^g\) is exactly \(-a/2\).

**Notations.** We denote simply by \((a, q)_g\) the central isogeny class \(A\) with Weil polynomial given by the expression \(2\) above. We denote by \(N_{g,n}(a)\) the cardinalities of the groups of rational points of the varieties in \(A_n\), where \(A\) is defined by \((a, q)_g\). If \(A\) is clear from the context, we write \(N_{g,n}\). We write \(N\) instead of \(N_{g,1}\) and \(N_n\) instead of \(N_{g,n}\) if the dimension \(g\) is clear from the context. We recall that \(N_{g,n}(A) = f_{A_n}(1)\).

**Weil polynomial after field extension.** Our results on cyclicity and growth are for Weil-central isogeny classes. Thus, in this section we study which extensions \(A_n\) of a Weil-central isogeny class \(A\) are Weil-central as well. Lemma 3 below gives the answer and is from where the “\(-\bigcup_{1 < dg} dN\)” of Theorem 2 comes from.

In the case of such a surface \(S\), from [5, Theorem 6], we know that \(S_n\) splits for \(n\) even. However, our criterion is independent of the simplicity or not of the isogeny class, so we do not worry about that.

**Lemma 3.** Suppose the isogeny class \(A\) has Weil polynomial \(f_A(t) = t^{2g} + a_1 t^g + q^g\), with \(\gcd(a_1, p) = 1\), i.e. it is an ordinary isogeny class. Then, its extensions \(A_n\) have Weil polynomials \(f_{A_n}(t) = t^{2g} + a_n t^g + q^{ng}\) for every \(n\) such that \(\gcd(n, g) = 1\), where \(a_n\) is obtained recursively

\[
a_n = (-1)^n a_1^n - \sum_{i=1}^{[n/2]} \binom{n}{i} a_{n-2i} q^{gi}.
\]

**Proof.** We first prove that if \(R = \{\alpha_1, \ldots, \alpha_g, q/\alpha_1, \ldots, q/\alpha_g\}\) is the set of roots of \(f_A\), then

\[
\{\alpha_1^n, \ldots, \alpha_g^n, (q/\alpha_1)^n, \ldots, (q/\alpha_g)^n\}
\]

is the set of roots of \(f_n(t) := t^{2g} + a_n t^g + q^{ng}\). Then we prove that \(f_n\) is irreducible if and only if \(\gcd(n, g) = 1\). Thus, for \(\beta \in R\), we will show that \(\beta^n\) is a root of

\[
t^{2g} + a_n t^g + q^{ng}.
\]

It is clear that \(\beta^n\) is a root of

\[
t^{2g} - (\beta^{ng} + (q/\beta)^{ng}) t^g + q^{ng} \in \mathbb{C}[t].
\]

Thus we have to show that

\[
a_n = -(\beta^{ng} + (q/\beta)^{ng}) \in \mathbb{Z}.
\]
In general, if we define $c_n = -x^n - (z/x)^n$ for $n > 0$ and $c_0 = -1$, we have that

\[-1^n c_1^n = (x + z/x)^n = x^n + (z/x)^n + \binom{n}{1} x^{n-1}(z/x) + x(z/x)^{n-1} + \ldots \]

\[\ldots + \binom{n}{i} x^{n-i}(z/x)^i + x^i(z/x)^{n-i} + \ldots \]

\[\ldots + \left( \frac{n}{\lfloor n/2 \rfloor} \right) A, \]

where (observe that for $n$ odd we have that $\lfloor n/2 \rfloor = (n - 1)/2$)

\[A = x^{n/2}(z/x)^{n/2} \text{ or } A = x^{(n+1)/2}(z/x)^{(n-1)/2} + x^{(n-1)/2}(z/x)^{(n+1)/2}\]

for $n$ even or odd, respectively. Equivalently

\[A = z^{\lfloor n/2 \rfloor} (x + z/x)^{2(n/2 - \lfloor n/2 \rfloor)}.\]

Then

\[-1^n c_1^n = -c_n - \binom{n}{1} z c_{n-2} - \ldots - \binom{n}{i} z^i c_{n-2i} - \ldots - \left( \frac{n}{\lfloor n/2 \rfloor} \right) z^{\lfloor n/2 \rfloor} c_\epsilon,\]

where $\epsilon = 0, 1$ for $n$ even or odd, respectively. Finally

\[c_n = (-1)^{n+1} c_1^n - \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{i} c_{n-2i} z^i.\]

By taking $x = \alpha^g$ and $z = q^g$, we finish the first part of the proof. Now we prove that $f_n(t)$ is irreducible if and only if $\gcd(n, g) = 1$.

For $k = 0, 1, \ldots, g - 1$, let

\[\beta/g + k2\pi/g, \quad \text{and} \quad -\beta/g - k2\pi/g,\]

be the arguments of the complex roots in $R$ of $f_\alpha$, respectively, where $\beta$ and $-\beta$ are the arguments of the roots of $t^2 + a_1 t + q^g$, and $0 < \beta < \pi$. The polynomial $f_n$ is reducible if and only if two of the roots in $R$ to the $n$-th power are equal. We have

1. either $\alpha_i^n = \alpha_j^n$ for some $i \neq j \in \{0, \ldots, g - 1\}$. Then

\[\frac{2\pi n}{g} (i - j) \equiv 0 \pmod{2\pi}\]

which holds if and only if $n(i - j)/g \in \mathbb{Z}$ if and only if $\gcd(n, g) > 1$.
(2) either \( \alpha^n_i = (q^i / \alpha_j)^n \) for some \( i, j \in \{0, \ldots, g - 1\} \). Then
\[
\frac{n}{g} [2 \beta + 2 \pi (i + j)] \equiv 0 \pmod{2 \pi}
\]
A necessary condition is that \( \beta \in \pi \mathbb{Q} \). Note that \( \cos(\beta) = -a_1/(2 \sqrt{q^g}) \). From one side, it can be shown (from [7, Thm. 1], for example) that \( 2 \cos(\beta) \) is an algebraic integer if \( \beta \in \pi \mathbb{Q} \).

On the other side, \( a_1 / \sqrt{q^g} \) is an algebraic number of degree \( \leq 2 \) which is not an algebraic integer since \( \gcd(a_1, p) = 1 \). Thus, this would be a contradiction.

The Honda-Tate theory for ordinary abelian varieties ensures that the polynomial \( f_n(t) = t^{2g} + a_n t^g + q^g \) obtained is indeed the Weil polynomial \( f_{A_n} \) of the extension \( A_n \).

**Local growth.** Given an \( \ell \)-cyclic isogeny class \( A \) (with \( \ell \mid f_A(1) \), i.e. with non-trivial \( \ell \)-primary component) it is clear that for all \( n, \ell \mid f_{A_n}(1) \) since \( A(\mathbb{F}_q) \subset A(\mathbb{F}_{q^n}) \). Thus, it is more interesting to know for which of these values of \( n \) the \( \ell \)-part increases (relatively to the base field). Lemma 5 below gives an answer.

We first fix a polynomial that will be useful. For every positive integer \( n \), we set
\[
P_n(x) := \sum_{i=0}^{n-1} x^i.
\]
Note that \((x - 1)P_n(x) = x^n - 1 \). Notice that Lemma 5 below is only valid for \( n \) odd (so the “\(-2N\)” in Theorem 2). We write first the polynomial \( P_n \) in a convenient way:

**Lemma 4.** For \( n \) odd, the polynomial \( P_n(x) \) can be obtained recursively:
\[
P_n(x) = (x + 1)^{n-1} - \sum_{i=1}^{(n-1)/2} \left( \binom{n}{i} - 2 \binom{n - 1}{i - 1} \right) x^i P_{n-2i}(x),
\]
with \( P_1(x) = 1 \).

**Proof.** The proof is straightforward by using induction on \( n \) and showing directly that the equality \((x - 1)P_n(x) = (x - 1) \) “right-hand-side” holds. \( \square \)

**Lemma 5.** For every positive odd integer \( n \) and any prime integer \( \ell \), we have
\[
v_\ell(N_n) \geq v_\ell(N_1) + v_\ell(nP_n(q^g)),
\]
provided that \( \ell \mid N_1 \).
Proof. We suppose $n$ odd. Recall that for Weil-central isogeny classes
\[ N_{g,1} = q^g + a_1 + 1, \]
and,
\[ N_{g,n} = q^{gn} + a_n + 1, \]
where $a_n$ can be computed by using Lemma 3. From the hypothesis $v_\ell(N_1) := m > 0$ then
\[ N_{g,1} \equiv q^g + a_1 + 1 \equiv z\ell^m \pmod{\ell^{2m}}, \quad 0 < z < \ell^m, \ell \nmid z. \]
From now, all congruences are modulo $\ell^{2m}$. First, we show by induction on $n$ that:
\[ a_n \equiv -q^{gn} - 1 + z\ell^m nP_n(q^g). \]
For $n = 1$,
\[ a_1 \equiv -q^g - 1 + z\ell^m P_1(q^g), \]
with $P_1 = 1$.
Using the induction hypothesis for $i = 1, \ldots, (n - 1)/2$ (so that $n - 2i < n$), we have that
\[ a_{n-2i}q^{gi} \equiv \left[ -q^{g(n-2i)} - 1 + z\ell^m (n - 2i)P_{n-2i}(q^g) \right] q^{gi} \equiv -q^{gn-gi} - q^{gi} + z\ell^m q^{gi}(n - 2i)P_{n-2i}(q^g), \]
then taking the sum over $i = 1, \ldots, (n - 1)/2$
\[ \sum \binom{n}{i} a_{n-2i}q^{gi} \equiv \sum_{i=1}^{(n-1)/2} \binom{n}{i} \left[ -q^{g(n-2i)} - q^{gi} + z\ell^m q^{gi}(n - 2i)P_{n-2i}(q^g) \right] \equiv -(q^g + 1)^n + q^{gn} + 1 + z\ell^m \sum_{i=1}^{(n-1)/2} \binom{n}{i} (n - 2i) [q^{gi}P_{n-2i}(q^g)]. \]
From Lemma 3, $a_n \equiv a_1^n - \sum \binom{n}{i} a_{n-2i}q^{gi} \equiv \left[ -(q^g + 1) + z\ell^m \right] n - \left[ -(q^g + 1)^n + q^{gn} + 1 + z\ell^m \sum_{i=1}^{(n-1)/2} \binom{n}{i} (n - 2i) [q^{gi}P_{n-2i}(q^g)] \right] \quad \text{(Here we used the fact that } m > 0: (x + y\ell^m)^n \equiv x^n + nx^{n-1}y\ell^m \pmod{\ell^{2m}}). \]
\[ \equiv -(q^g + 1)^n + n(q^g + 1)^{n-1}z\ell^m - \left[ -(q^g + 1)^n + q^{gn} + 1 + z\ell^m \sum_{i=1}^{(n-1)/2} \binom{n}{i} (n - 2i) [q^{gi}P_{n-2i}(q^g)] \right] \equiv -q^{gn} - 1 + z\ell^m \left[ (q^g + 1)^{n-1} - \sum_{i=1}^{(n-1)/2} \binom{n}{i} (n - 2i) [q^{gi}P_{n-2i}(q^g)] \right] \equiv -q^{gn} - 1 + z\ell^m \left[ (q^g + 1)^{n-1} - \sum_{i=1}^{(n-1)/2} \left[ \binom{n}{i} (n - 2i) - 2\binom{n-1}{i-1} \right] [q^{gi}P_{n-2i}(q^g)] \right]. \]
which completes the induction part. Then we compute \( N_n \):

\[
N_n \equiv q^{gn} + a_n + 1
\]

\[
\equiv q^{gn} + [-q^{gn} - 1 + z\ell^m nP_n(q^g)] + 1
\]

\[
\equiv z\ell^m nP_n(q^g).
\]

This completes the proof. \( \Box \)

Since \( v_\ell(nP_n(q^g)) \geq 1 \) if \( \ell \mid n \), we have then proved that \( g_\ell(A) \supset \ell N - 2N - \bigcup_{1<d|g} dN \).

**Remark 2.** From Lemma 5 above, we also have that \( g_\ell(A) \supset \{ n \in \mathbb{N} : \ell \mid P_n(q^g) \} \). However, from Lemma 6 (see Section below), \( \ell \mid P_n(q^g) \) implies \( \ell \mid (q^g)^n - 1 \), then \( A_n \) is not necessarily \( \ell \)-cyclic.

**Local cyclicity.** In this section we will give the characterization of the local cyclicity of Weil-central isogeny classes and their extensions. Following the definition of cyclicity for isogeny classes, a local version of [2, Theorem 2.2] gives the cyclicity criterion: for any prime \( \ell \) and for any isogeny class \( A \) we have that

\[
\begin{align*}
(3) & \quad A \text{ is } \ell \text{-cyclic if and only if } \ell \text{ does not divide } \gcd(f_A(1), f'_A(1)). \\
\end{align*}
\]

This has a meaning only if \( A(k)_\ell \) is not trivial for some \( A \in A \) (and thus for all \( A \in A \)), equivalently if \( \ell \) divides \( f_A(1) \). Sometimes we use a weaker version of the cyclicity criterion, namely \( \ell \nmid \gcd(f_A(1), f'_A(1)) \) implies that the isogeny class is \( \ell \)-cyclic. We give a complete description of the local cyclicity:

**Lemma 6.** Given a Weil-central isogeny class \((a, q)_g\) and a rational prime \( \ell \):

\[
\begin{align*}
(1) & \quad \text{if } \ell \nmid g \text{ and } \ell \nmid q^g - 1 \text{ then } (a, q)_g \text{ is } \ell \text{-cyclic}; \\
(2) & \quad \text{if } \ell \nmid g, \ell \mid q^g - 1 \text{ and } \ell \mid f(1) \text{ then } (a, q)_g \text{ is } \ell \text{-cyclic if and only if } \ell^2 \nmid f(1); \\
(3) & \quad \text{if } \ell \mid g, \text{ then } (a, q)_g \text{ is } \ell \text{-cyclic if and only if } \ell^2 \nmid f(1). \\
\end{align*}
\]

**Proof.** Recall that \((a, q)_g\) corresponds to the isogeny class with Weil polynomial \( f(t) = t^{2g} + at^g + q^g \). Then we have

\[
\begin{align*}
f(1) & = 1 + a + q^g = (q^g - 1) + (a + 2), \\
f'(1) & = g(2 + a).
\end{align*}
\]

Then we prove the lemma point by point using the cyclicity criterion [3]:

\[
\begin{align*}
(1) & \quad \text{If we suppose that } (a, q)_g \text{ is not } \ell \text{-cyclic (equivalently } \ell \mid \gcd(f(1), f'(1))) \text{, then } \ell \mid q^g - 1 \text{, contradiction.}
\end{align*}
\]
(2) If \( \ell \nmid g \) and \( \ell | q^g - 1 \), then \( \ell | \gcd(f(1), f'(1)) \) if and only if \( \ell^2 | f(1) \).

(3) In the case \( \ell \mid g \), we have that \( \ell | f'(1) \), then the result follows.

\[
{\text{□}}
\]

From Lemma 6, the local cyclicity at a prime \( \ell \) is possible only if \( \ell \nmid g \) and \( \ell \nmid q^g - 1 \), when we consider only extensions such that the \( \ell \)-part grows.

**Corollary 7.** Suppose \( \ell \nmid g \) and \( \ell \nmid q^g - 1 \) (in particular \( A_1 \) is \( \ell \)-cyclic), then

\[
\{ n \in \mathbb{N} : A_n \text{ is } \ell \text{-cyclic} \} \supset \mathbb{N} - \cup_{1<d|g} d\mathbb{N} - \omega_\ell(q^g)\mathbb{N}.
\]

**Proof.** In order to have that \( A_n \) is Weil-central, we have to dismiss \( \cup_{1<d|g} d\mathbb{N} \). For the \( \ell \)-cyclicity of \( (a_n, q^n)_g \) we use item (1) of Lemma 6. Observe that in this case, the cyclicity is independent of the value \( a_n \). We set \( \delta := \omega_\ell(q^g) \) and we consider the Euclidean division

\[
n = c\delta + r, \; 0 \leq r < \delta.
\]

Then we have

\[
q^{gn} \equiv q^{g(c\delta + r)} \equiv q^{gr\delta}q^{gr} \equiv q^{gr} \pmod{\ell},
\]

which is congruent to 1 if and only if \( r \) is zero, if and only if \( n \) is a multiple of \( \delta \).

\[
{\text{□}}
\]

Observe that here we do not consider the growth of the \( \ell \)-part, only the cyclicity. Moreover, the isogeny classes \( A_n \) can be “\( \ell \)-trivial”. We have then proved that

\[
c_\ell(A) \supset \ell\mathbb{N} - 2\mathbb{N} - \cup_{1<d|g} d\mathbb{N} - \omega_\ell(q^g)\mathbb{N}.
\]

This completes the proof of Theorem 2.

**Examples.** Let consider the ordinary elliptic curve \( (g = 1) \) defined by the Weil polynomial

\[
f_\ell(t) = t^2 + t + 73.
\]

We have \( f_\ell(1) = 75 = 3 \times 5^2 \) and \( q^g - 1 = 72 \) is not divisible by 5. Then the isogeny class \( \mathcal{E} = (1, 73)_1 \) is 5-cyclic and so the conditions of Theorem 2 are verified. We also have \( \omega_5(73) = 4 \). Consequently

\[
g_5(\mathcal{E}) \supset 5\mathbb{N} - 2\mathbb{N},
\]

\[
c_5(\mathcal{E}) \supset 5\mathbb{N} - 2\mathbb{N} - 4\mathbb{N} = 5\mathbb{N} - 2\mathbb{N}.
\]

Note that the elliptic curve \( \mathcal{E} \) is 3-cyclic since \( v_3(f_\ell(1)) = 1 \). However, we cannot apply Theorem 2 for \( \ell = 3 \) since \( q^g - 1 = 72 \) is divisible by 3. This and another few examples are listed in Table 1.
Table 1. Examples for Theorem 2

| \((a, q)\) | \(N = f_A(1)\) | Prime \(\ell\) | \(\omega_\ell(q^a)\) | \(g_\ell(A)\) | \(c_\ell(A)\) |
|---------|----------------|---------------|-----------------|-------------|-------------|
| \((1, 73)_1\) | \(3 \times 5^2\) | 5 | 4 | \(5N - 2N\) | \(5N - 2N\) |
| \((11, 17)_3\) | \(5^2 \times 197\) | 5 | 4 | \(5N - 2N - 3N\) | \(5N - 2N - 3N\) |
| \((17, 19)_3\) | \(13 \times 23^2\) | 23 | 22 | \(23N - 2N - 3N\) | \(23N - 2N - 3N - 22N\) |
| \((20, 7)_6\) | \(70 \times 41^2\) | 41 | 20 | \(41N - 2N - 3N\) | \(41N - 2N - 3N - 20N\) |

References

[1] H. Cohen and H. W. Lenstra, *Heuristics on class groups of number fields*, in Number Theory Noordwijkerhout 1983, H. Jager, ed., Berlin, Heidelberg, 1984, Springer Berlin Heidelberg, pp. 33–62.

[2] A. J. Giangreco-Maidana, *On the cyclicity of the rational points group of abelian varieties over finite fields*, Finite Fields and Their Applications, 57 (2019), pp. 139 – 155.

[3] E. W. Howe, D. Maisner, E. Nart, and C. Ritzenthaler, *Principally polarizable isogeny classes of abelian surfaces over finite fields*, Mathematical Research Letters, 15 (2008), pp. 121–127.

[4] E. W. Howe, E. Nart, and C. Ritzenthaler, *Jacobians in isogeny classes of abelian surfaces over finite fields*, Annales de l’Institut Fourier, 59 (2009), pp. 239–289.

[5] E. W. Howe and H. J. Zhu, *On the existence of absolutely simple abelian varieties of a given dimension over an arbitrary field*, Journal of Number Theory, 92 (2002), pp. 139 – 163.

[6] S. Lang and H. Trotter, *Primitive points on elliptic curves*, Bull. Amer. Math. Soc., 83 (1977), pp. 289–292.

[7] D. H. Lehmer, *A note on trigonometric algebraic numbers*, American Mathematical Monthly, 40 (1933), p. 165.

[8] D. Mumford, *Abelian varieties*, vol. 5 of Tata Institute of fundamental research studies in mathematics, Oxford University Press, London, 1970.

[9] J. Tate, *Endomorphisms of abelian varieties over finite fields*, Inventiones mathematicae, 2 (1966), pp. 134–144.

[10] S. G. Vlăduț, *Cyclicity statistics for elliptic curves over finite fields*, Finite Fields and Their Applications, 5 (1999), pp. 13 – 25.

[11] ———, *On the cyclicity of elliptic curves over finite field extensions*, Finite Fields and Their Applications, 5 (1999), pp. 354 – 363.

[12] W. C. Waterhouse, *Abelian varieties over finite fields*, Annales scientifiques de l’École Normale Supérieure, 2 (1969), pp. 521–560.

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