NONLINEAR DIFFUSION IN THE KELLER-SEGEL MODEL OF PARABOLIC-PARABOLIC TYPE

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Abstract. In this paper we study the initial boundary value problem for the system
\[ u_t - \Delta u^m = -\text{div}(u^q \nabla v), \quad v_t - \Delta v + v = u. \]
This problem is the so-called Keller-Segel model with nonlinear diffusion. Our investigation reveals that nonlinear diffusion can prevent overcrowding. To be precise, we show that solutions are bounded as long as \( m > q > 0 \), thereby substantially generalizing the known results in this area. Furthermore, our result seems to imply that the Keller-Segel model can have bounded solutions and blow-up ones simultaneously.

1. Introduction

Theoretical and mathematical modeling of chemotaxis dates back to the works of Patlak in the 1950s [16] and Keller and Segel in the 1970s [12]. The general form of the model reads:
\begin{align}
\frac{\partial u}{\partial t} &= \text{div} \left( k_1(u,v) \nabla u - k_2(u,v) \nabla v \right) + k_3(u,v) \quad \text{in } \Omega_T \equiv \Omega \times (0,T], \\
\frac{\partial v}{\partial t} &= k_c \Delta v + k_4(u,v) \quad \text{in } \Omega_T, \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{on } \Sigma_T \equiv \partial \Omega \times (0,T], \\
(u,v) \big|_{t=0} &= (u_0(x),v_0(x)) \quad \text{on } \Omega.
\end{align}
Here \( u \) denotes the cell density and \( v \) is the concentration of the chemical signal. The function \( k_1 \) is the diffusivity of the cells, \( k_2 \) is the chemotactic sensitivity, \( k_3 \) describes the cell growth and death. In the signal concentration model, \( k_4 \) describes the net effect of the production and degradation of the chemical signal. As for the remaining terms in the problem, \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with \( C^{1,1} \) boundary \( \partial \Omega \), \( n \) the unit outward normal to \( \partial \Omega \), and \( T \) any positive number.

Motivated by applications, various assumptions on the given data were suggested to further simplify the model [6, 19]. In this paper we focus our attention on the so-called nonlinear-diffusion model. In this case,
\[ k_1 = mu^{m-1}, \quad k_2 = u^q, \quad k_3 = 0, \quad k_c = 1, \quad k_4 = u - v, \]
where \( m, q \in (0, \infty) \).

The resulting problem is:
\begin{align}
\frac{\partial u}{\partial t} - \Delta u^m &= -\text{div}(u^q \nabla v) \quad \text{in } \Omega_T, \\
\frac{\partial v}{\partial t} - \Delta v + v &= u \quad \text{in } \Omega_T, \\
\frac{\partial u^m}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{on } \Sigma_T, \\
(u,v) \big|_{t=0} &= (u_0(x),v_0(x)) \quad \text{on } \Omega.
\end{align}
It is certainly beyond the scope of this paper to give a comprehensive review for the Keller-Segel model. In this regard, we would like to refer the reader to [8, 9]. A problem similar to (1.5)-(1.8) was investigated in [10, 11, 20] under the assumptions that \( N \geq 2, m \geq 1, q \geq 1 \) (note that our \( q \) here is their \( q - 1 \)). The global existence of a weak solution was established if, in addition, \( m > q + 1 - \frac{2}{N} \). When this inequality fails, one obtains local existence and the global existence only holds for small data. Hölder continuity and uniqueness of weak solutions were considered in [13]. Some relevance of nonlinear diffusion in chemotaxis was discussed in [2].

The objective of this paper is to show that the results in the preceding papers can be substantially improved. Before stating our results, let us define our notion of a weak solution.

**Definition 1.1.** We say that \((u, v)\) is a weak solution to (1.5)-(1.8) if
\[
\begin{align*}
u &\in L^\infty(\Omega_T), \ u \geq 0, \ \nu^m \in L^2(0, T; W^{1,2}(\Omega)), \\
v &\in L^\infty(0, T; W^{1,\infty}(\Omega)), \ v \geq 0
\end{align*}
\]
and
\[
\begin{align*}
-\int_{\Omega_T} \frac{\partial u}{\partial t} \xi dx dt + \int_{\Omega_T} \nabla u^m \cdot \nabla \xi dx dt &= \int_{\Omega_T} u_0 \xi(x, 0) dx + \int_{\Omega_T} u^q \nabla v \cdot \nabla \xi dx dt, \\
-\int_{\Omega_T} \frac{\partial v}{\partial t} \eta dx dt + \int_{\Omega_T} \nabla v \cdot \nabla \eta dx dt &= \int_{\Omega_T} v_0 \eta(x, 0) dx + \int_{\Omega_T} (u - v) \eta dx dt
\end{align*}
\]
for each pair of smooth functions \((\xi, \eta)\) with \(\xi(x, T) = \eta(x, T) = 0\).

Our main result is:

**Theorem 1.2 (Main theorem).** Assume:

(H1) \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( N \geq 3 \), with \( C^{1,1} \) boundary \( \partial \Omega \);

(H2) \( u_0 \in L^\infty(\Omega), \ v_0 \in W^{1,\infty}(\Omega) \) with \( u_0 \geq 0, v_0 \geq 0 \);

Then there is a weak solution \((u, v)\) to (1.5)-(1.8), provided that one of the following conditions holds.

(H3) \( m > 0, \ q > 0, \ and \ m > q \);

(H4) \( m > 0, \ 1 \geq q > 0, \ and \ q + \frac{q-1}{N+1} \leq m \leq q \).

Note that (H4) allows the possibility that \( m = q = 1 \). This is the classical Keller-Segel system, which is well known to have blow-up solutions. Thus our theorem actually implies that the Keller-Segel model can have bounded solutions and blow-up ones simultaneously. As far as we know, this is the first result in this direction. Our method seems to suggest that solutions blow up as \( m \to q^+ \), while solutions remain bounded as \( m \to q^- \) with \( q \leq 1 \). All the results are established under the assumption \( N > 2 \). But it is not difficult to see that Theorem 1.2 remains true for \( N = 2 \).

Motivated by numerical and modeling issues, the question of how blow-up of cells can be avoided has received a lot of attention. One way of doing this is to add a cross-diffusion term to the equation for \( v \) [7]. A second way is to alter the cell diffusion [1]. There are other related works. See, e.g., [2] in the context of volume effects. Here we show that nonlinear diffusion can also prevent blow-up.

Throughout this paper the letter \( c \) is always used to represent a positive number whose value is determined by the given data. The norm of a function in \( L^p(\Omega) \) is denoted by \( \|\cdot\|_{p,\Omega} \). The Lebesgue measure of a set \( D \) in \( \mathbb{R}^N \) is represented by \(|D|\). Whenever there is no confusion, we suppress the dependence of a function on its variables, e.g., we write \( u \) for \( u(x, t) \).

2. Preliminaries

In this section we collect a few preparatory results. The first one deals with sequences of non-negative numbers which satisfy certain recursive inequalities.
Proposition 2.1. Let \( \{y_n\}, n = 0, 1, 2, \cdots \), be a sequence of positive numbers satisfying the recursive inequalities
\[
y_{n+1} \leq cb^n y_n^{1+\alpha} \text{ for some } b > 1, c, \alpha \in (0, \infty).
\]
If
\[
y_0 \leq c^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},
\]
then \( \lim_{n \to \infty} y_n = 0 \).

This proposition can be found in ([4], p.12).

The following proposition plays a key role in the proof of our main theorem. It can be viewed as a continuous version of Lemma 3.1 in [15, 17].

Proposition 2.2. Let \( h(\tau) \) be a continuous non-negative function defined on \([0, T_0]\) for some \( T_0 > 0 \). Suppose that there exist three positive numbers \( \varepsilon, \delta, b \) such that
\[
(2.1) \quad h(\tau) \leq \varepsilon h^{1+\delta}(\tau) + b \text{ for each } \tau \in [0, T_0].
\]
Then
\[
(2.2) \quad h(\tau) \leq \frac{1}{|\varepsilon(1+\delta)|^{\frac{1}{\delta}}} \equiv s_0 \text{ for each } \tau \in [0, T_0],
\]
provided that
\[
(2.3) \quad \varepsilon \leq \frac{\delta}{(b+\delta)^\delta(1+\delta)^{1+\delta}} \quad \text{and } h(0) \leq s_0.
\]

Proof. Consider the function \( f(s) = \varepsilon s^{1+\delta} - s + b \) on \([0, \infty)\). Then condition (2.1) simply says
\[
(2.4) \quad f(h(\tau)) \geq 0 \text{ for each } \tau \in [0, T_0].
\]
It is easy to check that the function \( f \) achieves its minimum value at \( s_0 = \frac{1}{|\varepsilon(1+\delta)|^{\frac{1}{\delta}}} \). The minimum value
\[
(2.5) \quad f(s_0) = \frac{\varepsilon}{|\varepsilon(1+\delta)|^{\frac{1}{\delta}}} - \frac{1}{|\varepsilon(1+\delta)|^{\frac{1}{\delta}}} + b = b - \frac{\delta}{\varepsilon^{\frac{1}{\delta}}(1+\delta)^{\frac{1}{\delta}}}.\]

By the first inequality in (2.3), \( f(s_0) \leq -\delta \). Consequently, the equation \( f(s) = 0 \) has exactly two solutions \( 0 < s_1 < s_2 \) with \( s_0 \) lying in between. Evidently, \( f \) is positive on \([0, s_1) \), negative on \((s_1, s_2)\), and positive again on \((s_2, \infty)\). The range of \( h \) is a closed interval because of its continuity, and this interval is either contained in \([0, s_1) \) or \((s_2, \infty)\) due to (2.4). The latter cannot occur due to the second inequality in (2.3). Thus the proposition follows. \( \square \)

Proposition 2.3. Let \( v \) be the solution of the problem
\[
(2.6) \quad v_t - \Delta v + v = u \text{ in } \Omega_T,
\]
\[
(2.7) \quad \frac{\partial v}{\partial n} = 0 \text{ on } \Sigma_T,
\]
\[
(2.8) \quad v(x, 0) = v_0(x) \text{ on } \Omega.
\]
If (H1) holds, then for each \( p > \frac{N+2}{2} \) there is a positive number \( c \) such that
\[
(2.9) \quad \sup_{0 \leq t \leq T} \|\nabla v\|_{W^{1,\infty}(\Omega)} \leq c\|\nabla v_0\|_{W^{1,\infty}(\Omega)} + c\|u\|_{2p, \Omega_T}.
\]
Proof. We do not believe that this result is new. However, we cannot find a good reference to it. So we offer a proof here. First we obtain a local interior estimate. The boundary estimate is achieved by flattening the relevant portion of the boundary.

Now fix a point \( z_0 = (x_0, t_0) \in \Omega_T \). Then pick a number \( R \) from \((0, \min\{\text{dist}(x_0, \partial \Omega), \sqrt{t_0}\})\). Define a sequence of cylinders \( Q_{R_n}(z_0) \) in \( \Omega_T \) as follows:

\[
Q_{R_n}(z_0) = B_{R_n}(x_0) \times (t_0 - R_n^2, t_0],
\]

where

\[
R_n = \frac{R}{2} + \frac{R}{2^{n+1}} \quad n = 0, 1, 2, \ldots.
\]

Choose a sequence of smooth functions \( \theta_n \) so that

\[
\begin{align*}
\theta_n(x, t) &= 1 \quad \text{in } Q_{R_n}(z_0), \\
\theta_n(x, t) &= 0 \quad \text{outside } B_{R_{n-1}}(x_0) \text{ and } t < t_0 - R_{n-1}^2, \\
|\partial_t \theta_n(x, t)| &\leq \frac{c4^n}{R^2} \quad \text{on } Q_{R_{n-1}}(z_0), \\
|\nabla \theta_n(x, t)| &\leq \frac{c2^n}{R} \quad \text{on } Q_{R_{n-1}}(z_0), \quad \text{and}
\end{align*}
\]

\[0 \leq \theta_n(x, t) \leq 1 \quad \text{on } Q_{R_{n-1}}(z_0).
\]

Let \( p \) be given as in the lemma. Select

\[
(2.10) \quad K \geq R^{1 - \frac{N+2}{2p}} \|u\|_{2p, Q_R(z_0)}
\]

as below. Set

\[
K_n = K - \frac{K}{2^{n+1}}, \quad n = 0, 1, 2, \ldots.
\]

Fix an \( i \in \{1, \ldots, N\} \). Define

\[
(2.11) \quad w = v_{x_i}.
\]

Then \( w \) satisfies the equation

\[
(2.12) \quad w_t - \Delta w + w = u_{x_i} \quad \text{in } \Omega_T.
\]

Without loss of generality, assume \( \sup_{\Omega_T} w = \|w\|_{\infty, \Omega_T} \). We use \( \theta_{n+1}^2(w - K_{n+1})^+ \) as a test function in (2.12) to derive

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \theta_{n+1}^2 [(w - K_{n+1})^+]^2 \, dx + \int_{\Omega} \theta_{n+1}^2 \nabla (w - K_{n+1})^+ \cdot \nabla w \, dx + \int_{\Omega} \theta_{n+1}^2 w (w - K_{n+1})^+ \, dx
\]

\[
= \int_{\Omega} \theta_{n+1} \partial_t \theta_{n+1} [(w - K_{n+1})^+]^2 \, dx - 2 \int_{\Omega} \theta_{n+1} \nabla \theta_{n+1} \cdot \nabla w (w - K_{n+1})^+ \, dx
\]

\[
- \int_{\Omega} u \theta_{n+1} \partial_{x_i} (w - K_{n+1})^+ - 2 \int_{\Omega} u \theta_{n+1} \partial_{x_i} \theta_{n+1} (w - K_{n+1})^+ \, dx,
\]

from whence follows

\[
\sup_{0 \leq t \leq t_0} \int_{\Omega} \theta_{n+1}^2 [(w - K_{n+1})^+]^2 \, dx + \int_0^{t_0} \int_{\Omega} \theta_{n+1}^2 |\nabla (w - K_{n+1})^+|^2 \, dx \, dt
\]

\[
\leq \frac{c4^n}{R^2} \int_{Q_{R_n}(z_0)} [(w - K_{n+1})^+]^2 \, dx \, dt + c \int_{A_{n+1}} u \theta_{n+1}^2 \, dx \, dt
\]

\[
(2.14) \quad \leq \frac{c4^n}{R^2} y_n + c \|u^2\|_{p, Q_R(z_0)} |A_{n+1}|^{1 - \frac{1}{p}},
\]
It immediately follows that
\begin{align}
(2.15) \quad y_n &= \int_{Q_{R_n}(z_0)} [(w - K_n)^+]^2 \, dx dt, \\
(2.16) \quad A_{n+1} &= \{(x, t) \in Q_{R_n}(z_0) : w(x, t) \geq K_{n+1}\}.
\end{align}

By Poincaré’s inequality,
\begin{align*}
&\int_0^{t_0} \int_{\Omega} [\theta_{n+1}(w - K_{n+1})^+]^N \frac{\partial}{\partial t} dx dt \\
&\leq \int_0^{t_0} \left( \int_{\Omega} [\theta_{n+1}(w - K_{n+1})^+]^2 dx \right)^{\frac{N}{N+2}} \left( \int_{\Omega} [\theta_{n+1}(w - K_{n+1})^+]^\frac{2N}{N+2} \right)^{\frac{N}{N+2}} dt \\
&\leq \left( \sup_{0 \leq t \leq t_0} \int_{\Omega} [\theta_{n+1}(w - K_{n+1})^+]^2 dx \right)^{\frac{N}{N+2}} \int_0^{t_0} \int_{\Omega} \left| \nabla \theta_{n+1}(w - K_{n+1})^+ \right|^2 dx dt \\
&\leq c (1 + 4^n) \left( \frac{c^n R^2}{R^2} y_n + c \|u^2\|_{p, Q_{R_n}(z_0)} |A_{n+1}|^{1 - \frac{1}{p}} \right)^{\frac{N}{N+2}}.
\end{align*}

Subsequently,
\begin{align}
(2.17) \quad y_{n+1} &= \int_{Q_{R_{n+1}}(z_0)} [(w - K_{n+1})^+]^2 \, dx dt \\
&\leq \int_0^{t_0} \int_{\Omega} [\theta_{n+1}(w - K_{n+1})^+]^2 dx dt \\
&\leq \left( \int_0^{t_0} \int_{\Omega} [\theta_{n+1}(w - K_{n+1})^+]^2 dx \right)^{\frac{N}{N+2}} |A_{n+1}|^{\frac{2}{N+2}} \\
&\leq c A_{n+1} \left( \frac{c^n R^2}{R^2} y_n + c \|u^2\|_{p, Q_{R_n}(z_0)} |A_{n+1}|^{1 - \frac{1}{p}} \right)^{\frac{N}{N+2}} |A_{n+1}|^{\frac{2}{N+2}}. \\
&\leq c A_{n+1} \left( \frac{c^n R^2}{R^2} y_n + c R^{\frac{N+2}{2} - 2} K^2 |A_{n+1}|^{1 - \frac{1}{p}} \right)^{\frac{N}{N+2}} |A_{n+1}|^{\frac{2}{N+2}}. \\
&\leq c A_{n+1} \left( \frac{c^n R^2}{R^2} y_n + c R^{\frac{N+2}{2} - 2} K^2 |A_{n+1}|^{1 - \frac{1}{p}} \right)^{\frac{N}{N+2}} |A_{n+1}|^{\frac{2}{N+2}}.
\end{align}

The last step is due to (2.10). We also have
\begin{align}
(2.18) \quad y_n \geq \int_{A_{n+1}} (K_{n+1} - K_n)^2 \, dx dt = \frac{K^2}{4^n+1} |A_{n+1}|.
\end{align}

It immediately follows that
\begin{align}
(2.19) \quad y_n |A_{n+1}|^{\frac{1}{N+2}} &= y_n |A_{n+1}|^{\frac{1}{p}} |A_{n+1}|^{\frac{2}{N+2} - \frac{1}{p}} \\
&\leq \frac{c R^{\frac{N+2}{p} - 2} (\frac{n+1}{2} - \frac{2}{p} - (N+2))}{p(N+2)} \frac{1 + 2p - (N+2)}{p(N+2)} y_n. \\
(2.20) \quad K^2 |A_{n+1}|^{1 - \frac{1}{p}} |A_{n+1}|^{\frac{2}{N+2}} &= K^2 |A_{n+1}| |A_{n+1}|^{\frac{2}{N+2} - \frac{1}{p}} \\
&\leq \frac{c R^{\frac{N+2}{2} - 2} (\frac{n+1}{2} - \frac{2}{p} - (N+2))}{2p(N+2)} \frac{1 + 2p - (N+2)}{p(N+2)} y_n.
\end{align}

Use these in (2.18) to derive
\begin{align}
(2.21) \quad y_{n+1} \leq \frac{c R^{\frac{N+2}{p} - 2} y_n}{K^{2p - (N+2)}} y_n.
\end{align}
By Proposition 2.1, if we choose $K$ so large that
\begin{align}
y_0 &\leq cK^2R^{N+2}, \tag{2.23} \\
\end{align}
then
\begin{align}
\sup_{Q_{\frac{2}{3}}(z_0)} w &\leq K. \tag{2.24}
\end{align}

In view of (2.10), it is enough for us to take
\begin{align}
K &= c\left(\frac{y_0}{R^{N+2}}\right)^\frac{1}{2} + R^{1-\frac{N+2}{2p}}\|u\|_{2p, Q_{R}(z_0)}. \tag{2.25}
\end{align}

Recall that
\begin{align}
y_0 &= \int_{Q_{\frac{2}{3}}(z_0)} \left[\left(w - \frac{K}{2}\right)^+\right]^2 dxdt \leq \int_{Q_{R}(z_0)} (w^+)^2 dxdt. \tag{2.26}
\end{align}

Hence,
\begin{align}
\sup_{Q_{\frac{2}{3}}(z_0)} w &\leq c\left(\frac{1}{R}\int_{Q_{\frac{2}{3}}(z_0)} (w^+)^2 dxdt\right)^\frac{1}{2} + R^{1-\frac{N+2}{2p}}\|u\|_{2p, Q_{R}(z_0)}. \tag{2.27}
\end{align}

This is the so-called local interior estimate. Now we proceed to derive the boundary estimate.

Suppose $x_0 \in \partial \Omega$. Our assumption on the boundary implies that there exist a neighborhood $U(x_0)$ of $x_0$ and a $C^{1,1}$ diffeomorphism $T$ defined on $U(x_0)$ such that the image of $U(x_0) \cap \Omega$ under $T$ is the half ball $B^+(y_0) = \{y : |y - y_0| < \delta, y_i > 0\}$, where $\delta > 0, y_0 = T(x_0)$, and $i$ is given as in (2.11). This implies that we have flatten $U(x_0) \cap \partial \Omega$ into a region in the plane $y_i = 0$ in the $y$ space [3].

Set
\begin{align}
\tilde{v} &= v \circ T^{-1}, \quad \tilde{w} = \tilde{v}_y.
\end{align}

We can choose $T$ so that $\tilde{w} = \tilde{w}(y, t)$ satisfies the boundary condition
\begin{align}
\tilde{w} |_{y_i=0} = \tilde{v}_y |_{y_i=0} = \frac{\partial \tilde{v}}{\partial n} |_{y_i=0} = 0. \tag{2.28}
\end{align}

One way of doing this is to pick $T = \begin{pmatrix} f_1(x) \\ \vdots \\ f_N(x) \end{pmatrix}$ so that the graph of $f_1(x) = 0$ is $U(x_0) \cap \partial \Omega$ and the set of vectors $\{\nabla f_1, \cdots, \nabla f_N\}$ is orthogonal. By a result in [21], $\tilde{w}$ satisfies the equation
\begin{align}
\partial_t \tilde{w} - \text{div} \left[(J_T^T J_T) \circ T^{-1} \tilde{w}_\nabla\right] + \tilde{w} = (hJ_T) \circ T^{-1} \tilde{w} + (J_T \circ T^{-1} \tilde{v}_\nabla)_i \quad \text{in } B^+(y_0),
\end{align}

where $J_T$ is the Jacobian matrix of $T$, i.e.,
\begin{align}
J_T = \nabla T,
\end{align}

$(J_T \circ T^{-1} \tilde{v}_\nabla)_i$ is the $i$-th component of the vector $J_T \circ T^{-1} \tilde{v}_\nabla$, and the row vector $h$ is roughly $\text{div}(J_T^T J_T)$ and is, therefore, bounded by our assumption on $T$. In view of (2.28), the method employed to prove (2.27) still works here. The only difference is that we use $B^+_{R_n}(y_0)$ instead of $B_{R_n}(y_0)$ in the proof. If $t_0 = 0$, then we just need to change $Q_{R_n}(z_0)$ to $B_{R_n} \times [0, R_n^2)$ and require
\begin{align}
K &\geq 2\|\nabla v_0\|_{\infty, \Omega},
\end{align}
in addition to (2.10) in the proof. Subsequently, (2.27) follows.

Finally, use $v$ as a test function in (3.14) to derive
\begin{align}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 + \int_{\Omega} |\nabla v|^2 dx + \int_{\Omega} v^2 dx = \int_{\Omega} uv dx. \tag{2.29}
\end{align}
It immediately follows that
\[(2.30) \quad \int_{\Omega_T} |\nabla v|^2 dx \,dt \leq c \int_{\Omega_T} u^2 dx \,dt + c \int_{\Omega} v_0^2 dx.\]

Finally, we have
\[(2.31) \quad \| v \|_{\infty, \Omega_T} \leq c \| u \|_{p, \Omega_T} + \| v_0 \|_{\infty, \Omega}.\]

This completes the proof. \(\square\)

3. Proof of Theorem 1.2

A solution to (1.5)-(1.8) is constructed as the limit of a sequence of approximate solutions. Our approximate problems are formulated as follows (also see [20]):
\[(3.1) \quad \partial_t U - m \text{div} ((U^+ + \sigma)^{m-1} \nabla U) = -\text{div} ((U^+) \nabla V) \quad \text{in } \Omega_T,\]
\[(3.2) \quad \partial_t V - \Delta V + V = U \quad \text{in } \Omega_T,\]
\[(3.3) \quad \frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} = 0 \quad \text{on } \Sigma_T,\]
\[(3.4) \quad (U, V) \mid_{t=0} = (u_0, v_0) \quad \text{on } \Omega,\]

where \(\sigma > 0\). The existence of a solution to the above problem can be established via the Leray-Schauder fixed point theorem ([5], p.280). To this end, we define an operator \(T: L^\infty(\Omega_T) \to L^\infty(\Omega_T)\) as follows: Let \(U \in L^\infty(\Omega_T)\). We say \(w = T(U)\) if \(w\) is the unique solution of the problem
\[(3.5) \quad \partial_t w - m \text{div} ((U^+ + \sigma)^{m-1} \nabla w) = -\eta \text{div} ((U^+) \nabla V) \quad \text{in } \Omega_T,\]
\[(3.6) \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma_T,\]
\[(3.7) \quad w \mid_{t=0} = u_0 \quad \text{on } \Omega,\]

where \(V\) solves the problem
\[(3.8) \quad \partial_t V - \Delta V + V = U \quad \text{in } \Omega_T,\]
\[(3.9) \quad \frac{\partial V}{\partial n} = 0 \quad \text{on } \Sigma_T,\]
\[(3.10) \quad V \mid_{t=0} = v_0 \quad \text{on } \Omega.\]

To see that \(T\) is well-defined, we conclude from Proposition 2.3 that \(|\nabla V| \in L^\infty(\Omega_T)\). Moreover, the two initial boundary value problems in the definition of \(T\) are both linear and uniformly parabolic. We can infer from ([14], Chap. III) that \(w\) is Hölder continuous in \(\Omega_T\). It follows that \(T\) is continuous and maps bounded sets into precompact ones. We still need to show that there is a positive number \(c\) such that
\[(3.11) \quad \| U \|_{\infty, \Omega_T} \leq c\]
for all \(U \in L^\infty(\Omega_T)\) and \(\eta \in (0,1)\) satisfying \(U = \eta T(U)\). This equation is equivalent to the following problem
\[(3.12) \quad \partial_t U - m \text{div} ((U^+ + \sigma)^{m-1} \nabla U) = -\eta \text{div} ((U^+) \nabla V) \quad \text{in } \Omega_T,\]
\[(3.13) \quad \partial_t V - \Delta V + V = U \quad \text{in } \Omega_T,\]
\[(3.14) \quad \frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} = 0 \quad \text{on } \Sigma_T,\]
\[(3.15) \quad (U, V) \mid_{t=0} = (\eta u_0, v_0) \quad \text{on } \Omega.\]
Use $U^-$ as a test function in (3.6) to get
\[-\frac{1}{2} \frac{d}{dt} \int_\Omega (U^-)^2 dx - m \int_\Omega (U^+ + \sigma)^{m-1} |\nabla U^-|^2 dx = 0.\]
Integrate to get
(3.10) $U \geq 0$ a.e. on $\Omega_T$.
This implies that
(3.11) $V \geq 0$ a.e. on $\Omega_T$.
We introduce the following change of dependent variables
(3.12) $u = U + \sigma, \quad v = V + \sigma$.
Then $(u, v)$ satisfies the problem
(3.13) $u_t - \Delta u^m = -\eta \text{div}(u - \sigma)^q \nabla v$ in $\Omega_T$,
(3.14) $v_t - \Delta v + v = u$ in $\Omega_T$,
(3.15) $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$ on $\Sigma_T$,
(3.16) $(u, v)_{t=0} = (\eta u_0(x) + \sigma, v_0(x) + \sigma)$ on $\Omega$.

There is no loss of generality for us to assume that $T \leq 1$. Otherwise, we simply consider $(u(x, Tt), v(x, Tt))$ on $[0, 1]$. From here on we will do that, and also let
(3.17) $\sigma \in (0, 1)$.
We already have $\eta \in (0, 1)$. The generic positive number $c$ will be independent of all three of them.

**Lemma 3.1.** Let (H3) hold. Then for each $s$ sufficiently large there is a positive number $c$ such that
(3.18) $\sup_{0 \leq t \leq T} \int_\Omega u^{s+1} dx + \int_\Omega \left| \nabla u^{\frac{m+s}{2}} \right|^2 dx dt \leq c \|\nabla v\|_{\infty, \Omega_T}^{\frac{m+s}{2}} + c.$

**Proof.** First remember that
(3.19) $\sigma \leq u \in L^\infty(\Omega_T)$.
Thus for each $r \in \mathbb{R}$, we have
(3.20) $u^r \in L^2(0, T; W^{1,2}(\Omega))$.
Now pick a number
(3.21) $s > \max\{0, m - 2q\}$.
Use $u^s$ as a test function in (3.13) to derive
\[
\frac{1}{s+1} \frac{d}{dt} \int_\Omega u^{s+1} dx + ms \int_\Omega u^{m+s-2} |\nabla u|^2 dx
= s\eta \int_\Omega (u - \sigma)^q u^{s-1} \nabla v \nabla u dx \leq s \int_\Omega u^{q+s-1} |\nabla v| \nabla u dx
\leq \frac{1}{2} ms \int_\Omega u^{m+s-2} |\nabla u|^2 dx + \frac{s \|\nabla v\|^2_{\infty, \Omega_T}}{2m} \int_\Omega u^{2q-m-s} dx.
\]
To estimate the last integral, first notice that
(3.22) $\int_\Omega u^{m+s-2} |\nabla u|^2 dx = \frac{4}{(m+s)^2} \int_\Omega |\nabla u^{\frac{m+s}{2}}|^2 dx.$
Recall the Sobolev embedding theorem which states that for each \( r \in [1, N) \) there is a positive number \( c \) such that

\[
\|w\|_{N/r, \Omega} \leq c\|\nabla w\|_{r, \Omega} + c\|w\|_{1, \Omega} \quad \text{for each } w \in W^{1,r}(\Omega).
\]

We wish to apply this inequality with \( w = u^{m+s} \) and \( r = 2 \). For this purpose, we further require

\[
\frac{2q - m + s}{m + s} \leq \frac{N}{N - 2}.
\]

Or equivalently,

\[
s \geq \frac{(2q - m)(N - 2) - Nm}{2}.
\]

We derive from Hölder’s inequality and (3.24) that

\[
\int_{\Omega} u^{2q - m + s} \, dx = \int_{\Omega} \left( \int_{\Omega} \left( u^{m+s} \right)^{\frac{2q-m+s}{m+s}} \right)^{\frac{N}{N-2}} \, dx
\]

\[
\leq c \left( \int_{\Omega} \left| \nabla u^{\frac{m+s}{2}} \right|^2 \, dx + \left( \int_{\Omega} u^{\frac{m+s}{2}} \, dx \right)^2 \right)^{\frac{2q-m+s}{m+s}}
\]

\[
\leq c \left( \int_{\Omega} \left| \nabla u^{\frac{m+s}{2}} \right|^2 \, dx \right)^{\frac{2q-m+s}{m+s}} + c \left( \int_{\Omega} u^{\frac{m+s}{2}} \, dx \right)^{\frac{2(2q-m+s)}{m+s}}.
\]

We integrate (3.13) over \( \Omega \) to get

\[
\frac{d}{dt} \int_{\Omega} u \, dx = 0.
\]

Subsequently,

\[
\int_{\Omega} u(x,t) \, dx = \int_{\Omega} (\eta u_0(x) + \sigma) \, dx \leq c \quad \text{for each } t > 0.
\]

If we further assume that

\[
\frac{m + s}{2} < 2q - m + s,
\]

then we can appeal to the interpolation inequality ([5], p.146), thereby deriving

\[
\|u\|_{m+s/2, \Omega} \leq c \|u\|_{2q-m+s, \Omega} + \frac{1}{\varepsilon_h} \|u\|_{1, \Omega} \leq c \|u\|_{2q-m+s, \Omega} + \frac{c}{\varepsilon_h},
\]

where \( \varepsilon > 0, \quad \mu = \left(1 - \frac{2}{m+s}\right) / \left(\frac{2}{m+s} - \frac{1}{2q-m+s}\right) \). Condition (3.29) is equivalent to

\[
s > 3m - 4q.
\]

Use (3.30) in (3.27) and choose \( \varepsilon \) suitably small in the resulting inequality to obtain

\[
\int_{\Omega} u^{2q-m+s} \, dx \leq c \left( \int_{\Omega} \left| \nabla u^{\frac{m+s}{2}} \right|^2 \, dx \right)^{\frac{2q-m+s}{m+s}} + c.
\]
Plug this into (3.22) to get
\[
\frac{1}{s+1} \frac{d}{dt} \int_\Omega u^{s+1} dx + \frac{2ms}{(m+s)^2} \int_\Omega \left| \nabla u^{\frac{m+s}{2}} \right|^2 dx \\
\leq c \left( \int_\Omega \left| \nabla u^{\frac{m+s}{2}} \right|^2 dx \right)^{\frac{2q-m+s}{m+s}} \left\| \nabla v \right\|_{L^2(\Omega_T)}^2 + c\left\| \nabla v \right\|_{L^2(\Omega_T)}^2 \\
\leq \varepsilon \int_\Omega \left| \nabla u^{\frac{m+s}{2}} \right|^2 dx + c(\varepsilon) \left\| \nabla v \right\|_{L^2(\Omega_T)}^{\frac{m+s}{m-q}} + c\left\| \nabla v \right\|_{L^2(\Omega_T)}^2.
\]
(3.33)

The last step is due to the assumption \( m > q \) and Young’s inequality ([5], p. 145). Once again, by taking \( \varepsilon \) suitably small, we arrive at
\[
\sup_{0 \leq t \leq T} \int_\Omega u^{s+1} dx + \int_\Omega \left| \nabla u^{\frac{m+s}{2}} \right|^2 dx dt \leq c\left\| \nabla v \right\|_{L^2(\Omega_T)}^{\frac{m+s}{m-q}} dt + c.
\]
(3.34)

Here we have used the fact \( \frac{m+s}{m-q} > 2 \) due to (3.21). That is to say, the lemma is valid for any \( s \) that satisfies (3.21), (3.26), and (3.31). This completes the proof. \( \square \)

**Lemma 3.2.** Let (H3) hold and \( s \) be given as in Lemma 3.1. Then there is a positive number \( c \) such that
\[
\left\| u \right\|_{L^\infty(\Omega_T)} \leq c\left\| \nabla v \right\|_{L^\infty(\Omega_T)}^\gamma + c,
\]
where
\[
\gamma = \frac{((s+1)(N+2) + N(m-1)^+)(m+s) + (s+1)N(m-q)(N+2)}{(s+1)(m-q)[(N+2)(s+1) + 2N(m-q)]}.
\]
(3.36)

**Proof.** Let
\[
K \geq 2(\left\| u_0 \right\|_{L^\infty(\Omega)} + 1)
\]
be selected as below. Define
\[
K_n = K - \frac{K}{2^{n+1}}, n = 0, 1, \ldots
\]
(3.38)

Obviously,
\[
\frac{K}{2} \leq K_n \leq K.
\]
(3.39)

Set
\[
S_n(t) = \{ x \in \Omega : u(x,t) \geq K_n \},
\]

(3.40)

\[
A_n = \cup_{0 \leq t \leq T} S_n(t) = \{ (x,t) \in \Omega_T : u(x,t) \geq K_n \}.
\]

(3.41)

Subsequently,
\[
\int_0^T |S_{n+1}(t)| dt = |A_{n+1}|.
\]
(3.42)

To simplify our presentation, we also introduce two parameters
\[
m_s = (s+1)\frac{2}{N} + m + s,
\]
(3.43)

\[
q_s = m_s - (2q - m + s).
\]
(3.44)
where $s$ is given as in Lemma 3.1, i.e., $s$ is sufficiently large. Then use $(u^s - K_{n+1}^s)^+$ as a test function in (3.13) to derive

\[
\frac{d}{dt} \int_\Omega \int_0^t (\tau^s - K_{n+1}^s)^+ d\tau dx + ms \int_{S_{n+1}(t)} u^{m+s-2} |\nabla u|^2 dx
\]

\[
= s\eta \int_{S_{n+1}(t)} (u - \sigma)^q u^{s-1} \nabla v \nabla u dx \leq s \int_{S_{n+1}(t)} u^{q+s-1} |\nabla v||\nabla u| dx.
\]

After a suitable application of Cauchy’s inequality ([14], p. 58), we integrate to obtain

\[
\sup_{0 \leq t \leq T} \int_\Omega \int_0^t (\tau^s - K_{n+1}^s)^+ d\tau dx + \int_{A_{n+1}} |\nabla u_m^{m+s}|^2 dx dt
\]

\[
\leq c \int_{A_{n+1}} u^{2q-m-s} |\nabla v|^2 dx dt \leq c ||\nabla v||^2_{\infty, \Omega_T} \int_{A_{n+1}} u^{2q-m-s} dx dt.
\]

Since $s > 1$, we have

\[
\int_{K_{n+1}} (\tau^s - K_{n+1}^s)^+ d\tau \chi_{A_{K_{n+1}}} \geq \int_{K_{n+1}} [(\tau - K_{n+1})^+]^s d\tau
\]

\[
= \frac{1}{s+1} [(u - K_{n+1})^+]^{s+1}.
\]

Recall that $m_s = (s + 1)\frac{2}{N} + m + s$. We estimate, with the aid of Hölder’s inequality and (3.24), that

\[
y_{n+1} = \int_0^T \int_\Omega [(u - K_{n+1})^+]^{m_s} dx dt
\]

\[
\leq c \left( \sup_{0 \leq t \leq T} \int_\Omega [(u - K_{n+1})^+]^{s+1} dx \right)^{\frac{2}{N}} \left( \int_\Omega [(u - K_{n+1})^+]^{(m_s+N-2)} dx \right)^{\frac{N-2}{N}} dt
\]

\[
\leq c \left( \int_\Omega |\nabla [(u - K_{n+1})^+]|^{\frac{m+1}{2}} dx + \left( \int_\Omega [(u - K_{n+1})^+]^{m_s} dx \right)^{\frac{m+1}{2}} \right)^2 dt
\]

We can easily verify that

\[
|\nabla [(u - K_{n+1})^+]|^{\frac{m+1}{2}} = \frac{m+1}{2} [(u - K_{n+1})^+]^{\frac{m+1}{2}-1}|\nabla u|
\]

\[
\leq \frac{m+1}{2} u^{\frac{m+1}{2}-1} |\nabla u| \chi_{S_{n+1}(t)} = |\nabla u^{\frac{m+1}{2}}| \chi_{S_{n+1}(t)}
\]

\[
\int_\Omega [(u - K_{n+1})^+]^{m_s} dx \leq \left( \int_\Omega [(u - K_{n+1})^+]^{m_s} dx \right)^{\frac{m+1}{2m_s}} |S_{n+1}(t)|^{\frac{m+1}{2m_s}}.
\]
The latter yields
\[
\int_0^T \left( \int_\Omega \left[ (u - K_{n+1})^+ \right]^{m+s} \right)^2 \, dx \, dt \\
\leq \int_0^T \left( \int_\Omega \left[ (u - K_{n+1})^+ \right]^{m+s} \right)^{\frac{m+s}{m+s}} |S_{n+1}(t)|^{2-\frac{m+s}{m+s}} \, dt \\
\leq \left( \int_\Omega_T \left[ (u - K_{n+1})^+ \right]^{m+s} \, dx \, dt \right)^{\frac{m+s}{m+s}} \left( \int_0^T |S_{n+1}(t)|^{1+\frac{N ms}{2(s+1)}} \, dt \right)^{\frac{2(s+1)}{N ms}} \\
\leq c |A_{n+1}|^{\frac{2(s+1)}{N ms}} y_n^{m+s}. \tag{3.51}
\]

Here we have used the fact that \( \{y_n\} \) is a decreasing sequence. Use (3.49) and (3.51) in (3.48) and take (3.46) into account to derive
\[
y_{n+1} \leq c \|\nabla v\|_{\infty, \Omega_T} \left( \int_{A_{n+1}} u^{2q-m+s} \, dx \right)^{\frac{N+2}{N}} \\
+ c \|\nabla v\|_{\infty, \Omega_T} \left( \int_{A_{n+1}} u^{2q-m+s} \, dx \right)^{\frac{2}{N}} \left( \int_{A_{n+1}} u^{m+s} \, dx \right)^{\frac{2(s+1)}{N ms}} y_n^{m+s}. \tag{3.52}
\]

The first integral on the right-hand side of (3.52) can be estimated as follows:
\[
\left( \int_{A_{n+1}} u^{2q-m+s} \, dx \right)^{\frac{N+2}{N}} = \frac{(N+2)(2q-m+s)}{K_{n+1}^{N+2}} \left( \int_{A_{n+1}} \left( \frac{u}{K_{n+1}} \right)^{2q-m+s} \, dx \right)^{\frac{N+2}{N}} \\
\leq \frac{1}{K_{n+1}^{N+2}} \left( \int_{A_{n+1}} \left( \frac{u}{K_{n+1}} \right)^{2q-m+s} \, dx \right)^{\frac{N+2}{N}} \\
= \frac{1}{K_{n+1}^{N+2}} \int_{A_{n+1}} u^{m+s} \, dx \tag{3.53}
\]

Similarly,
\[
\left( \int_{A_{n+1}} u^{2q-m+s} \, dx \right)^{\frac{2}{N}} \leq \frac{1}{K_{n+1}^{N+2}} \int_{A_{n+1}} u^{m+s} \, dx \tag{3.54}
\]

Recall that
\[
K_{n+1} - K_n = \frac{K}{2n+2}, \quad \frac{K_{n+1} - K_n}{K_{n+1}} = \frac{1}{2n+2 - 1} > \frac{1}{2n+2}.
\]

With the aid of the preceding two results, we obtain
\[
y_n \geq \int_{A_{n+1}} \left[ (K_{n+1} - K_n)^+ \right]^{m_s} \, dx \, dt = \frac{K_{n+1}^{m_s}}{2(n+2)m_s} |A_{n+1}|, \tag{3.55}
\]
\[
y_n \geq \int_{A_{n+1}} u^{m_s} \left[ \left( \frac{1 - K_n}{u} \right)^+ \right]^{m_s} \, dx \, dt \\
\geq \int_{A_{n+1}} u^{m_s} \left( 1 - \frac{K_n}{K_{n+1}} \right)^{m_s} \, dx \, dt \tag{3.56}
\]
By (3.55),

$$|A_{n+1}| \leq \frac{2(n+2)(s+1)}{N} \frac{2(s+1)}{N} y_n^{\frac{2}{N}}.$$ 

Keeping this, (3.54), (3.39), and (3.56) in mind, we derive from (3.52) that

$$y_{n+1} \leq c \frac{2(n+2)ms(N+2)}{K} \frac{2}{N} y_n^{\frac{2}{N}} + \frac{1}{2} \frac{2(s+1)+2ms}{N} y_n^{\frac{2}{N}}$$

\[ \leq \begin{array}{l}
\frac{cb^n}{K} \left( \frac{2(n+2)}{N} \frac{2}{N} y_n^{\frac{2}{N}} + \frac{\| \nabla v \|^4_{\infty, \Omega_T}}{2(s+1)+2ms} \right) y_n^{\frac{2}{N}},
\end{array} \]

where

$$b = \max \left\{ 2 \frac{ms(N+2)}{N}, 2 \frac{2(s+1)+2ms}{N} \right\}.$$ 

We can easily check from (3.43) and (3.44) that

$$(N+2)qs \geq 2(s+1) + 2qs \quad \text{if and only if} \quad m \geq q.$$ 

Recall that $K_n \geq 1$. Thus if (H3) holds, we can deduce from (3.57) that

$$y_{n+1} \leq \frac{cb^n}{K} \left( \frac{2(n+2)}{N} \frac{2}{N} y_n^{\frac{2}{N}} + \frac{\| \nabla v \|^4_{\infty, \Omega_T}}{2(s+1)+2ms} \right) y_n^{\frac{2}{N}}.$$ 

According to Proposition 2.1, if we choose $K$ so large that

$$y_0 = \int_{\Omega_T} \left[ \left( u - \frac{K}{2} \right)^+ \right]^{ms} dt \leq \int_{\Omega_T} u^{ms} dt$$

$$\leq \frac{cK^{s+1+qs}}{\| \nabla v \|^{N+2}_{\infty, \Omega_T} + \| \nabla v \|^2_{\infty, \Omega_T}},$$

then

$$\sup_{\Omega_T} u \leq K.$$ 

In view of (3.37), it is enough for us to take

$$K = c \left( \int_{\Omega_T} u^{ms} dt \right)^{\frac{1}{s+1+qs}} \left( \frac{\| \nabla v \|^{N+2}_{\infty, \Omega_T} + \| \nabla v \|^2_{\infty, \Omega_T}}{s+1+qs} \right)^{\frac{1}{s+1+qs}}$$

\[ + 2 \| u_0 \|_{\infty, \Omega} + 2. \]
In light of (3.48), (3.51), and (3.18), we have
\[
\int_{\Omega_T} u^{m_s} \, dx \, dt \leq \left( \sup_{0 \leq t \leq T} \int_{\Omega} u^{s+1} \, dx \right) \frac{2}{N} \int_{\Omega_T} \left| \nabla u^{m_s} \right|^2 \, dx \, dt \\
+ c \left( \sup_{0 \leq t \leq T} \int_{\Omega} u^{s+1} \, dx \right)^{\frac{2}{N}} \left( \int_{\Omega_T} u^{m_s} \, dx \, dt \right)^{\frac{m_s}{m_s}} \\
\leq c \left\| \nabla v \right\|_{\infty, \Omega_T}^{\left( \frac{N+2}{m} \right) \left( m-s \right)} + c + \varepsilon \int_{\Omega_T} u^{m_s} \, dx \, dt + c(\varepsilon) \left( \sup_{0 \leq t \leq T} \int_{\Omega} u^{s+1} \, dx \right)^{\frac{m_s}{s+1}}.
\]
Choosing $\varepsilon$ suitably small, we arrive at
\[
\int_{\Omega_T} u^{m_s} \, dx \, dt \leq c \left\| \nabla v \right\|_{\infty, \Omega_T}^{\left( \frac{N+2}{m} \right) \left( m-s \right)} + c + c \left\| \nabla v \right\|_{\infty, \Omega_T}^{\left( \frac{s+1}{m} \right) \left( m-s \right)}.
\]
Substituting this into (3.58) yields
\[
\left\| u \right\|_{\infty, \Omega_T} \leq c \left[ \left( \left\| \nabla v \right\|_{\infty, \Omega_T}^{\left( \frac{N+2}{m} \right) \left( m-s \right)} + \left\| \nabla v \right\|_{\infty, \Omega_T}^{\left( \frac{s+1}{m} \right) \left( m-s \right)} \right) + 1 \right] \left( \left\| \nabla v \right\|_{\infty, \Omega_T}^{N+2} + \left\| \nabla v \right\|_{\infty, \Omega_T}^{2} \right) \frac{1}{s+1+q} + c
\]
\[
\leq c \left[ \left\| \nabla v \right\|_{\infty, \Omega_T}^{\left( \frac{N+2}{m} \right) \left( m-s \right)} + \left\| \nabla v \right\|_{\infty, \Omega_T}^{\left( \frac{s+1}{m} \right) \left( m-s \right)} + 1 \right] \left( \left\| \nabla v \right\|_{\infty, \Omega_T}^{N+2} + 1 \right) \frac{1}{s+1+q} + c
\]
\[
\leq c \left\| \nabla v \right\|_{\infty, \Omega_T}^{\left( \frac{s+1}{m} \right) \left( m-s \right)} \left( \frac{1}{s+1+q} \right) \frac{1}{s+1+q} + c
\]
This together with (3.44) implies the lemma. \hfill \Box

**Proof of Theorem 1.2 under (H3).** We wish to show
\[
\left\| v \right\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \left\| u \right\|_{\infty, \Omega_T} \leq c.
\]
Let $\gamma$ be given as in Lemma 3.2. Note that
\[
\lim_{s \to \infty} \gamma = \lim_{s \to \infty} \frac{(s+1)(N+2) + N(m-1)^+ (m+s) + (s+1)N(m-q)(N+2)}{(s+1)(m-q)(s+1) + 2N(m-q)} = \frac{1}{m-q}.
\]
If $\frac{1}{m-q} > 1$, then there is a $\beta > 0$ such that
\[
\gamma = 1 + \beta \quad \text{for some suitably large $s$.}
\]
Fix this $s$ and let $p$ be given as in Proposition 2.3. We can derive from (2.9) and Lemma 3.2
\[
\left\| v \right\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \left\| u \right\|_{\infty, \Omega_T} \leq c \left\| u \right\|_{2p, \Omega_T} + c \left\| \nabla v \right\|_{\infty, \Omega_T}^{1+\beta} + c
\]
\[
\leq c T^{\frac{1}{2p}} \left\| u \right\|_{\infty, \Omega_T} + c \left\| u \right\|_{2p, \Omega_T}^{1+\beta} + c
\]
\[
\leq c T^{\frac{1}{2p}} \left\| \nabla v \right\|_{\infty, \Omega_T}^{1+\beta} + c T^{\frac{1+\beta}{2p}} \left\| u \right\|_{\infty, \Omega_T}^{1+\beta} + c
\]
(3.61)
\[
\leq c T^{\frac{1}{2p}} \left( \left\| v \right\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \left\| u \right\|_{\infty, \Omega_T} \right)^{1+\beta} + c.
\]
Here we have used the fact that $T \leq 1$. Set
\[
h(\tau) = \left\| v \right\|_{L^\infty(0,\tau;W^{1,\infty}(\Omega))} + \left\| u \right\|_{\infty, \Omega_T}.
\]
Let $T_0 \in (0, T]$ be selected below. It follows from (3.61) that

$$h(\tau) \leq cT_0^{\frac{1}{2p}}h^{1+\beta}(\tau) + c \quad \text{for each } \tau \in [0, T_0].$$

It is not difficult for us to see from the proof of Proposition 2.3 that $\nabla v$ is actually Hölder continuous on $\Omega_T$, so is $u$ for each fixed $\sigma > 0$. Thus $h(\tau)$ is a continuous function of $\tau$. In view of Proposition 2.2, if we choose $T_0$ so that

$$cT_0^{\frac{1}{2p}} \leq \frac{\beta^2}{(c+\beta)^2(1+\beta)^{1+\beta}}$$

and $\|\nabla(v_0 + 1)\|_{W^{1,\infty}(\Omega)} + \|u_0 + 1\|_{\infty,\Omega} \leq \frac{1}{cT_0^{\frac{1}{2p}}(1+\beta)}$, then

$$\|\nabla v\|_{\infty,\Omega \times [0, T_0]} \leq \frac{1}{\left[cT_0^{\frac{1}{2p}}(1+\beta)\right]^{\frac{1}{\beta}}}.\]$$

By setting $T = 0$ in (3.61), we see that $\|\nabla(v_0 + 1)\|_{W^{1,\infty}(\Omega)} + \|u_0 + 1\|_{\infty,\Omega} \leq c$. If we take

$$cT_0^{\frac{1}{2p}} = \frac{\beta^2}{(c+\beta)^2(1+\beta)^{1+\beta}},$$

then the second inequality in (3.64) is automatically satisfied. Upon doing so, we arrive at

$$\|v\|_{L^\infty(0,T_0;W^{1,\infty}(\Omega))} + \|u\|_{\infty,\Omega \times [0, T_0]} \leq \frac{(c+\beta)(1+\beta)}{\beta}.$$

Set $k = \lfloor \frac{T_0}{T_0} \rfloor$, the integer part of the number $\frac{T_0}{T_0}$. If $k \geq 1$, we consider

$$u_{T_0}(x,t) = u(t + T_0, x), \quad v_{T_0}(x,t) = v(t + T_0, x) \quad \text{on } [0, T_0].$$

Obviously, $(u_{T_0}, v_{T_0})$ satisfies the same conditions as $(u, v)$ on $\Omega \times (0, T_0)$. Thus we can repeat the previous arguments to yield (3.67) for $(u_{T_0}, v_{T_0})$. After a finite number of steps, we obtain (3.59). Of course, in the last step, we will have to use $\min\{T_0, T - kT_0\}$ instead of $T_0$.

If $\frac{1}{m-q} < 1$, an application of Young’s inequality is enough to reach (3.59).

If $\frac{1}{m-q} = 1$, this can also be handled easily. We verify that $\frac{dx}{dt}$ changes signs at least three times. Thus either $\gamma$ decreases toward 1 as $s \to \infty$, which can be treated like the first case, or $\gamma$ increases toward 1 as $s \to \infty$, which is essentially the second case.

Clearly, (3.5) is a consequence of (3.59). Thus we can conclude from the Leray-Schauder fixed point theorem that (3.1)-(3.4) has a solution. Denote the solution by $(U_{\sigma}, V_{\sigma})$. In view of (3.10), we can rewrite (3.1)-(3.4) as

$$\partial_t U_{\sigma} - m\text{div}((U_{\sigma} + \sigma)^{m-1}\nabla U_{\sigma}) = -\text{div}(U_{\sigma}^q\nabla V_{\sigma}) \quad \text{in } \Omega_T,$$

$$\partial_t V_{\sigma} - \Delta V_{\sigma} + V_{\sigma} = U_{\sigma} \quad \text{in } \Omega_T,$$

$$\frac{\partial U_{\sigma}}{\partial n} = \frac{\partial V_{\sigma}}{\partial n} = 0 \quad \text{on } \Sigma_T,$$

$$(U_{\sigma}, V_{\sigma}) \big|_{t=0} = (u_0, v_0) \quad \text{on } \Omega.$$

Furthermore,

$$U_{\sigma} \geq 0, \quad V_{\sigma} \geq 0, \quad \text{and } \|V_{\sigma}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \|U_{\sigma}\|_{\infty,\Omega_T} \leq c.$$

We wish to show that we can take $\sigma \to 0$ in (3.69)-(3.72). For this purpose, we use $(U_{\sigma} + \sigma)^m$ as a test function in (3.69) to derive

$$\frac{1}{m+1} \sup_{0 \leq t \leq T} \int_{\Omega} (U_{\sigma} + \sigma)^{m+1} dx + \int_{\Omega_T} |\nabla(U_{\sigma} + \sigma)^{m}|^2 dxdt \leq c.$$
We compute
\[
\partial_t (U_\sigma + \sigma)^{m+1} = (m + 1)(U_\sigma + \sigma)^m \partial_t U_\sigma \\
= (m + 1) \text{div} ((U_\sigma + \sigma)^m \nabla (U_\sigma + \sigma)^m) - (m + 1) |\nabla (U_\sigma + \sigma)^m|^2 \\
- (m + 1) \text{div} ((U_\sigma + \sigma)^m U_\sigma^g \nabla V_\sigma) + (m + 1) U_\sigma^g \nabla V_\sigma \cdot \nabla (U_\sigma + \sigma)^m,
\]
\[
\nabla (U_\sigma + \sigma)^{m+1} = \frac{m + 1}{m} (U_\sigma + \sigma)^m \nabla (U_\sigma + \sigma)^m.
\]

Thus the sequence \( \{\partial_t (U_\sigma + \sigma)^{m+1}\} \) is bounded in \( L^2(0, T; (W^{1,2}(\Omega))^*) + L^1(\Omega_T) \equiv \{\psi_1 + \psi_2 : \psi_1 \in L^2(0, T; (W^{1,2}(\Omega))^*), \psi_2 \in L^1(\Omega_T)\} \) and the sequence \( \{(U_\sigma + \sigma)^{m+1}\} \) is bounded in \( L^2(0, T; W^{1,2}(\Omega)). \)

This puts us in a position to apply the Lions-Aubin lemma [19]. Upon doing, we obtain the pre-compactness of \( \{(U_\sigma + \sigma)^{m+1}\} \) in \( L^2(\Omega_T) \). We can extract a subsequence of \( \{U_\sigma + \sigma\} \), still denoted by \( \{U_\sigma + \sigma\} \), such that \( U_\sigma + \sigma \) converges a.e. on \( \Omega_T \). This is enough to justify passing to the limit in (3.69)-(3.72). The proof is complete.

□

We would like to remark that as \( m \to q^+ \) the upper bound in (3.67) deteriorates. This foretells the possibility that solutions blow up if \( m = q \).

**Proof of Theorem 1.2 under (H4).** We will show that an estimate like (3.35) remains true even without the benefit of Lemma 3.1. Let \( s \) be given as before, i.e., \( s \) is large enough. With the aid of (H4), we can derive from (3.57) that
\[
y_{n+1} \leq \frac{c b^n \left( \|\nabla v\|_{\infty, \Omega_T}^{2(N+2)} + \|\nabla v\|_{\infty, \Omega_T}^{q+2} \right)}{K^{(N+2)q_s}} y_n^{1+\frac{q}{N}}.
\]

In light of Proposition 2.1, if \( K \) is so chosen that
\[
y_0 \leq \frac{c K^{(N+2)q_s}}{2 \|\nabla v\|_{\infty, \Omega_T}^{N+2} + \|\nabla v\|_{\infty, \Omega_T}^{2q+4}},
\]
then
\[
(3.74) \quad \sup_{\Omega_T} u \leq K.
\]

In view of (3.37), it is enough for us to take
\[
K = c \left( \int_{\Omega_T} u_m s \ dx \ dt \right)^{\frac{2}{(N+2)q_s}} \left( \|\nabla v\|_{\infty, \Omega_T}^{N+2} + \|\nabla v\|_{\infty, \Omega_T}^{2q+4} \right)^{\frac{2}{(N+2)q_s}}
+ 2\|u_0\|_{\infty, \Omega_T} + 2.
\]

If
\[
(3.76) \quad \frac{2m_s}{(N + 2)q_s} < 1,
\]
or equivalently,
\[
q < 1 \quad \text{and} \quad m > q + \frac{q - 1}{N + 1},
\]
then Young’s inequality asserts
\[
K \leq \varepsilon \|u\|_{m_s, \Omega_T} + c(\varepsilon) \left( \|\nabla v\|_{\infty, \Omega_T}^{N+2} + \|\nabla v\|_{\infty, \Omega_T}^{2q+4} \right)^{\frac{2}{(N+2)q_s - 2m_s}}
+ 2\|u_0\|_{\infty, \Omega_T} + 2.
\]

Use this in (3.74) to derive
\[
(3.77) \quad \|u\|_{\infty, \Omega_T} \leq c \|\nabla v\|_{\infty, \Omega_T}^{\frac{N+2}{(N+1)m - (N+2)q+1}} + c \|u_0\|_{\infty, \Omega_T} + c.
\]
If
\begin{equation}
\frac{2m_s}{(N + 2)q_s} = 1,
\end{equation}
we can appeal to the interpolation inequality ([5], p. 146) to obtain
\begin{equation}
\|u\|_{m_s, \Omega_T} \leq \varepsilon \|u\|_{\infty, \Omega_T} + \frac{1}{\varepsilon^{m_s-1}} \|u\|_{1, \Omega_T} \leq \varepsilon \|u\|_{\infty, \Omega_T} + \frac{c}{\varepsilon^{m_s-1}}.
\end{equation}
With this in mind, we derive from (3.75) that
\begin{equation}
K \leq c \left( \varepsilon \|u\|_{\infty, \Omega_T} + \frac{c}{\varepsilon^{m_s-1}} \right) \left( \|\nabla v\|_{\infty, \Omega_T}^{N + 2} + \|\nabla v\|_{\infty, \Omega_T}^2 \right)^{\frac{2}{(N + 2)q_s}} + 2\|u_0\|_{\infty, \Omega} + 2
\end{equation}
\begin{equation}
= \alpha \|u\|_{\infty, \Omega_T} + \frac{c}{\alpha^{m_s-1}} \left( \|\nabla v\|_{\infty, \Omega_T}^{N + 2} + \|\nabla v\|_{\infty, \Omega_T}^2 \right)^{\frac{m_s}{(N + 2)q_s} + 2} + 2\|u_0\|_{\infty, \Omega} + 2.
\end{equation}
Plug this into (3.74) and choose \(\alpha\) suitably small in the resulting inequality to derive
\begin{equation}
\|u\|_{\infty, \Omega_T} \leq c\|\nabla v\|_{\infty, \Omega_T}^{N + 2} + c\|u_0\|_{\infty, \Omega} + c.
\end{equation}
The rest of the proof is similar to that under (H3). That is, (3.59) can be inferred from either (3.77) or (3.80).

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