OPTIMAL EIGENVALUES ESTIMATE FOR THE DIRAC OPERATOR ON DOMAINS WITH BOUNDARY

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Abstract. We give a lower bound for the eigenvalues of the Dirac operator on a compact domain of a Riemannian spin manifold under the MIT bag boundary condition. The limiting case is characterized by the existence of an imaginary Killing spinor.

1. Introduction

Let Ω be a compact domain in an $n$-dimensional Riemannian spin manifold $(N^n, g)$ whose boundary is denoted by $\partial \Omega$. In [HMR02], the authors studied four elliptic boundary conditions for the Dirac operator $D$ of the domain $\Omega$. More precisely, they prove a Friedrich-type inequality [Fri80] which relates the spectrum of the Dirac operator and the scalar curvature of the domain $\Omega$. These boundary conditions are the following: the Atiyah-Patodi-Singer (APS) condition based on the spectral resolution of the boundary Dirac operator; a modified version of the APS condition, the mAPS condition; the boundary condition CHI associated with a chirality operator; and a Riemannian version of the MIT bag boundary condition. In fact, they show that, if the boundary $\partial \Omega$ of $\Omega$ has non-negative mean curvature, then under the APS, CHI or mAPS boundary conditions, the spectrum of the classical Dirac operator of the domain $\Omega$ is a sequence of unbounded real numbers $\{\lambda_k : k \in \mathbb{Z}\}$ satisfying

$$\lambda_k^2 \geq \frac{n}{4(n-1)} R_0,$$

where $R_0$ is the infimum of the scalar curvature of the domain $\Omega$. Moreover, equality holds only for the CHI and the mAPS conditions and in these cases, $\Omega$ is respectively isometric to a half-sphere or it carries a non-trivial real Killing spinor and has minimal boundary. In the case of the MIT boundary condition, they show that the spectrum of the Dirac operator on $\Omega$ is an unbounded discrete set of complex numbers $\lambda_{\text{MIT}}$ with positive imaginary part satisfying

$$|\lambda_{\text{MIT}}|^2 > \frac{n}{4(n-1)} R_0,$$

if the mean curvature of the boundary is non-negative. This result leads to the following question: can one improve this inequality in order to obtain some boundary geometric
invariants on the right hand side of (2)? We show in this paper that such a result can be obtained. More precisely, we prove the following theorem:

**Theorem 1.** Let $\Omega$ be a compact domain of an $n$-dimensional Riemannian spin manifold $(N^n, g)$ whose boundary $\partial \Omega$ satisfies $H > 0$. Under the MIT boundary condition $\mathbb{B}^\text{MIT}_-$, the spectrum of the classical Dirac operator $D$ on $\Omega$ is an unbounded discrete set of complex numbers with positive imaginary part. Any eigenvalue $\lambda^\text{MIT}$ satisfies

$$|\lambda^\text{MIT}|^2 \geq \frac{n}{4(n-1)} R_0 + n \text{Im}(\lambda^\text{MIT}) H_0,$$

(3)

where $H_0$ is the infimum of the mean curvature of the boundary. Moreover, equality holds if and only if the associated eigenspinor is an imaginary Killing spinor on $\Omega$ and if the boundary $\partial \Omega$ is a totally umbilical hypersurface with constant mean curvature.

The proof of this theorem is based on a modification of the spinorial Levi-Civita connection which leads to a spinorial Reilly-type formula. This formula can be seen as a hyperbolic version of the Reilly inequality used in [HMR02].

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2. Geometric preliminaries

In this section, we give some standard facts about Riemannian spin manifolds with boundary. For more details, we refer to [BBW93] or [HMR02].

On a compact domain $\Omega$ with smooth boundary $\partial \Omega$ in a $n$-dimensional Riemannian spin manifold $(N^n, g)$, denote by $\Sigma \Omega$ the complex spinor bundle corresponding to the metric $g$ and by $\nabla$ its Levi-Civita connection acting on $T\Omega$ as well as its lift to $\Sigma \Omega$. The map $\gamma : \mathbb{C}l(\Omega) \to \text{End}(\Sigma \Omega)$ is the Clifford multiplication where $\mathbb{C}l(\Omega)$ is the Clifford bundle over $\Omega$. The spinor bundle is endowed with a natural Hermitian scalar product, denoted by $\langle \cdot , \cdot \rangle$, compatible with $\nabla$ and $\gamma$. The Dirac operator is then the first order elliptic operator acting on sections of $\Sigma \Omega$ locally given by

$$D : \Gamma(\Sigma \Omega) \to \Gamma(\Sigma \Omega) \quad \psi \mapsto -\sum_{i=1}^n \gamma(e_i) \nabla e_i \psi,$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame of $T\Omega$.

Consider now the boundary $\partial \Omega$ which is an oriented hypersurface of the domain $\Omega$ with induced orientation and Riemannian structure. Since the normal bundle of $\partial \Omega$ is trivial, the boundary itself is a spin manifold. This spin structure on the boundary allows to construct an intrinsic spinor bundle $\Sigma(\partial \Omega)$ over $\partial \Omega$ naturally endowed with a Hermitian metric, a Clifford multiplication $\gamma^{\partial \Omega}$ and a spinorial Levi-Civita connection $\nabla^{\partial \Omega}$. Moreover the restriction $S(\partial \Omega) := \Sigma \Omega|_{\partial \Omega}$ to the boundary of the spinor bundle $\Sigma \Omega$ is a Dirac bundle, i.e. there exist on $S(\partial \Omega)$ a Hermitian metric denoted by $\langle \cdot , \cdot \rangle$ compatible with the Levi-Civita connection $\nabla^S$ and the Clifford multiplication $\gamma^S$. The Clifford multiplication $\gamma^S : \mathbb{C}l(\partial \Omega) \to \text{End}(S(\partial \Omega))$ is given by $\gamma^S(X)\psi = \gamma(X)\gamma(\nu)\psi$ for all $X \in \Gamma(T\Omega)$ and $\psi \in \Gamma(S(\partial \Omega))$. Similarly we can relate the Levi-Civita connection acting on $\Sigma \Omega$ with that acting on $S(\partial \Omega)$ by the spinorial Gauss formula (see [BBW93]):

$$(\nabla_X \psi)_{\partial \Omega} = \nabla_X^S \psi_{\partial \Omega} + \frac{1}{2} \gamma^S(AX)\psi_{\partial \Omega},$$
for all $X \in \Gamma(T(\partial\Omega))$, $\psi \in \Gamma(\Sigma\Omega)$ and where $AX = -\nabla_X \nu$ is the shape operator of the boundary $\partial\Omega$ with respect to the inner normal vector field $\nu$. We can then define the boundary Dirac operator acting on $S(\partial\Omega)$ which is an elliptic first order differential operator locally given by

$$D^S = \sum_{j=1}^{n-1} \gamma_S^j(e_j) \nabla^S_{e_j}.$$  \hfill (4)

Recall that there is a standard identification

$$S(\partial\Omega) \equiv \begin{cases} \Sigma(\partial\Omega) & \text{if } n \text{ is odd} \\ \Sigma(\partial\Omega) \oplus \Sigma(\partial\Omega) & \text{if } n \text{ is even} \end{cases}$$

Taking into account the relation between the Hermitian bundle $S(\partial\Omega)$ and $\Sigma(\partial\Omega)$, one can see that

$$\nabla^S \equiv \begin{cases} \nabla^{\partial\Omega} & \text{if } n \text{ is odd} \\ \nabla^{\partial\Omega} \oplus \nabla^{\partial\Omega} & \text{if } n \text{ is even} \end{cases}$$

and

$$\gamma^S \equiv \begin{cases} \gamma^{\partial\Omega} & \text{if } n \text{ is odd} \\ \gamma^{\partial\Omega} \oplus -\gamma^{\partial\Omega} & \text{if } n \text{ is even} \end{cases}$$

3. THE MIT BOUNDARY CONDITION

First, note that on a closed compact Riemannian spin manifold, the classical Dirac operator has exactly one self-adjoint $L^2$ extension, so it has real discrete spectrum. In the setting of manifolds with boundary, a defect of self-adjointness appears. It is given by the Green formula

$$\int_\Omega \langle D\varphi, \psi \rangle dv(g) - \int_\Omega \langle \varphi, D\psi \rangle dv(g) = - \int_{\partial\Omega} \langle \gamma(\nu)\varphi, \psi \rangle ds(g),$$

for all $\varphi, \psi \in \Gamma(\Sigma\Omega)$. Furthermore, in this case, the Dirac operator has a closed range of finite codimension, but an infinite-dimensional kernel, which varies depending on the choice of the Sobolev space. We refer to [BBW93], [Lop53] or [HMR02] for a careful treatment of boundary conditions for elliptic operators.

The MIT bag boundary condition has first been introduced by physicists of the Massachusetts Institute of Technology in a Lorentzian setting (see [CJJ74], [CJJT74] or [Joh75]). The Riemannian version of this condition has been studied in [HMR02] in order to get Friedrich estimates and in [HMZ02] because of its conformal covariance to give a conformal lower bound for the first eigenvalue of the intrinsic Dirac operator of hypersurfaces bounding a compact domain in a Riemannian spin manifold. Consider the pointwise endomorphism

$$i\gamma(\nu) : \Gamma(S(\partial\Omega)) \rightarrow \Gamma(S(\partial\Omega))$$

acting on the restriction to the boundary $\partial\Omega$ of the spinor bundle over $\Omega$ and where $i$ is the fundamental imaginary number. This map is an involution, and so the bundle $S(\partial\Omega)$
splits into two eigensubbundles $V^\pm$ associated with the eigenvalues $\pm 1$. We then have two associated orthogonal projections given by

$$B^\pm_{\text{MIT}} : L^2(S(\partial \Omega)) \to L^2(V^\pm) \quad \varphi \mapsto \frac{1}{2}(\text{Id} \pm i\gamma(\nu))\varphi,$$

which define local elliptic boundary conditions for the Dirac operator $D$ on the domain $\Omega$. So under this boundary condition, the eigenvalue problem

$$\begin{cases}
D\varphi = \lambda^{\text{MIT}}\varphi & \text{on } \Omega \\
B^\pm_{\text{MIT}}\varphi = 0 & \text{along } \partial \Omega
\end{cases}$$

has a discrete spectrum with finite dimensional eigenspaces consisting of smooth spinor fields.

**Remark 1.** Under the MIT boundary condition $B^-_{\text{MIT}}$, the spectrum of the Dirac operator $D$ is contained in the upper half complex plane $\{z \in \mathbb{C} / \text{Im}(z) > 0\}$. Indeed, let $\lambda^{\text{MIT}}$ be an eigenvalue of $D$ under the MIT boundary condition and $\varphi \in \Gamma(\Sigma \Omega)$ the associated spinor field, then taking $\psi = i\varphi$ in the Formula (5) leads to

$$2\text{Im}(\lambda^{\text{MIT}}) \int_{\Omega} |\varphi|^2 dv(g) = \int_{\partial \Omega} |\varphi|^2 ds(g)$$

Two possibilities can occur: we have either $\text{Im}(\lambda^{\text{MIT}}) > 0$ or $\text{Im}(\lambda^{\text{MIT}}) = 0$. If $\text{Im}(\lambda^{\text{MIT}}) = 0$, then the spinor field $\varphi$ should vanish along the boundary $\partial \Omega$ and by the unique continuation principle (see [BBW93]), it should be identically zero on the manifold $\Omega$. This is impossible because the spinor $\varphi$ is supposed to be an eigenspinor, so a non trivial field. The first case is the only possibility, i.e. $\text{Im}(\lambda^{\text{MIT}}) > 0$. For the boundary condition $B^+_{\text{MIT}}$, we can show that the imaginary part of all eigenvalues of the Dirac operator is negative.

4. **THE HYPERBOLIC REILLY FORMULA**

In this section, we give a spinorial Reilly formula based on a modification of the spinorial Levi-Civita connection. Let $\alpha \in \mathbb{R}$, then we define the connection $\nabla^\alpha$ acting on $\Sigma \Omega$ by

$$\nabla^\alpha_X \varphi := \nabla_X \varphi + i\alpha \gamma(X)\varphi,$$

for all $\varphi \in \Gamma(\Sigma \Omega)$ and $X \in \Gamma(T\Omega)$. We can now derive an integral version of the Schrödinger-Lichnerowicz formula using the modified connection $\nabla^\alpha$. Indeed, we have:

**Proposition 2.** For all spinor fields $\varphi \in \Gamma(\Sigma \Omega)$, we have:

$$\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{L^2} = \langle D^2 \varphi, \varphi \rangle_{L^2} - \frac{R}{4} \varphi, \varphi \rangle_{L^2} + n\alpha^2 ||\varphi||^2_{L^2} - \int_{\partial \Omega} \langle \nabla^\alpha \varphi, \varphi \rangle ds(g),$$

where $R$ is the scalar curvature of the domain $\Omega$.

**Proof:** First note that the $L^2$-formal adjoint of the connection $\nabla^\alpha$ is, by definition, given by

$$\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{L^2} = ||\nabla^\alpha \varphi||^2_{L^2} = \sum_{j=1}^n \int_{\Omega} \langle \nabla^\alpha_{e_j} \varphi, \nabla^\alpha_{e_j} \varphi \rangle dv(g),$$
for all \( \varphi \in \Gamma(\Sigma\Omega) \) and where \( \{e_1, \ldots, e_n\} \) is a local orthonormal frame of \( T\Omega \). An easy calculation using the compatibility properties of the Hermitian metric with the spinorial connection and the Clifford multiplication gives

\[
\sum_{j=1}^{n} \langle \nabla_{e_j}^\alpha \varphi, \nabla_{e_j}^\alpha \varphi \rangle = \sum_{j=1}^{n} \left( e_j \langle \nabla_{e_j}^\alpha \varphi, \varphi \rangle - \langle \nabla_{e_j}^{-\alpha} \nabla_{e_j}^\alpha \varphi, \varphi \rangle \right),
\]

and Stokes theorem leads to

\[
\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{L^2} = \langle -\sum_{j=1}^{n} \nabla^{-\alpha}_e \nabla^\alpha_{e_j} \varphi, \varphi \rangle_{L^2} - \int_{\partial \Omega} \langle \nabla^\alpha_\nu \varphi, \varphi \rangle ds(g).
\]

We can now easily compute

\[
\langle -\sum_{j=1}^{n} \nabla^{-\alpha}_e \nabla^\alpha_{e_j} \varphi, \varphi \rangle_{L^2} = \langle -\sum_{j=1}^{n} \nabla_{e_j} \nabla_{e_j} \varphi, \varphi \rangle_{L^2} + n \alpha^2 ||\varphi||^2_{L^2}
\]

\[
= \langle \nabla^\ast \nabla \varphi, \varphi \rangle_{L^2} + n \alpha^2 ||\varphi||^2_{L^2},
\]

and then the classical Schrödinger-Lichnerowicz formula (see [LM 89]) leads to Identity (9).

This formula is a first step to obtain Inequality (3). However, we have now to introduce the Dirac operator and the twistor operator associated with the connection \( \nabla^\alpha \). The modified Dirac operator is locally defined by

\[
D^\alpha \varphi = \sum_{j=1}^{n} \gamma(e_j) \nabla_{e_j}^\alpha \varphi,
\]

and the associated twistor operator by

\[
P^\alpha_X \varphi = \nabla_X^\alpha \varphi + \frac{1}{n} \gamma(X) D^\alpha \varphi,
\]

for all \( X \in \Gamma(T\Omega) \) and \( \varphi \in \Gamma(\Sigma\Omega) \). Note that for \( \alpha = 0 \), the operators \( D^0 \) and \( P^0 \) are respectively the classical Dirac operator and the classical twistor operator which satisfy the relation (see [BHMM] or [Fri00] for example)

\[
||\nabla \varphi||^2 = ||P \varphi||^2 + \frac{1}{n} ||D \varphi||^2
\]

We can then check that the modified operators satisfy the same relation, i.e.

\[
||\nabla^\alpha \varphi||^2 = ||P^\alpha \varphi||^2 + \frac{1}{n} ||D^\alpha \varphi||^2.
\]
Indeed, if \(\{e_1, ..., e_n\}\) is a local orthonormal frame of \(T\Omega\), we have
\[
|P^\alpha \varphi|^2 = \sum_{j=1}^n \langle \nabla_{e_j} \varphi + \frac{1}{n} \gamma(e_j) D^\alpha \varphi, \nabla_{e_j} \varphi + \frac{1}{n} \gamma(e_j) D^\alpha \varphi \rangle
\]
\[
= |\nabla^\alpha \varphi|^2 - \frac{2}{n} |D^\alpha \varphi|^2 + \frac{1}{n} |D^\alpha \varphi|^2
\]
\[
= |\nabla^\alpha \varphi|^2 - \frac{1}{n} |D^\alpha \varphi|^2,
\]
and so Identity (12) follows directly. We are now ready to establish the hyperbolic version of the spinorial Reilly formula given in [HMR02]. This formula can be seen as an analogous of the one used in [HMR03] to give a lower bound of the first eigenvalue of the intrinsic Dirac operator for hypersurfaces bounding a compact domain of a manifold with negative scalar curvature. More precisely, we prove:

**Proposition 3.** For all \(\varphi \in \Gamma(\Sigma\Omega)\), we have:
\[
||P^\alpha \varphi||^2_{L^2} = \frac{n-1}{n} ||D^\alpha \varphi||_{L^2} - \frac{R}{4} \varphi, \varphi\rangle_{L^2} - n(n-1)\alpha^2 ||\varphi||^2_{L^2}
\]
\[
+ \int_{\partial\Omega} \langle D^\alpha \varphi + \frac{n-1}{2}(2\alpha \gamma(\nu) \varphi - H \varphi), \varphi \rangle ds(g),
\]
where \(H\) is the mean curvature of the boundary \(\partial\Omega\) of \(\Omega\).

**Proof:** Observe first that the modified Dirac operator \(D^\alpha\) is not formally self-adjoint. Indeed an easy calculation using (5) gives
\[
\int_{\Omega} \langle D^\alpha \varphi, \psi \rangle dv(g) = \int_{\Omega} \langle \varphi, D^{-\alpha} \psi \rangle dv(g) - \int_{\partial\Omega} \langle \gamma(\nu) \varphi, \psi \rangle ds(g),
\]
for all \(\varphi, \psi \in \Gamma(\Sigma\Omega)\). However, we have:
\[
D^2 \varphi = D^{-\alpha} D^\alpha \varphi - n^2 \alpha^2 \varphi,
\]
and so substituting in Formula (12) gives
\[
\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{L^2} = \langle D^{-\alpha} D^\alpha \varphi, \varphi \rangle_{L^2} - \frac{R}{4} \varphi, \varphi\rangle_{L^2} - n(n-1)\alpha^2 ||\varphi||^2_{L^2} - \int_{\partial\Omega} \langle \nabla^\alpha \varphi, \varphi \rangle ds(g).
\]
The integration by parts formula (14) leads to
\[
\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{L^2} = ||D^\alpha \varphi||^2_{L^2} - \frac{R}{4} \varphi, \varphi\rangle_{L^2} - n(n-1)\alpha^2 ||\varphi||^2_{L^2} - \int_{\partial\Omega} \langle \gamma(\nu) D^\alpha \varphi + \nabla^\alpha \varphi, \varphi \rangle ds(g).
\]
With the help of Identity (12), we have
\[
||P^\alpha \varphi||^2_{L^2} = \frac{n-1}{n} ||D^\alpha \varphi||_{L^2} - \frac{R}{4} \varphi, \varphi\rangle_{L^2} - n(n-1)\alpha^2 ||\varphi||^2_{L^2} - \int_{\partial\Omega} \langle \gamma(\nu) D^\alpha \varphi + \nabla^\alpha \varphi, \varphi \rangle ds(g).
\]
However the boundary term can be written
\[-\gamma(\nu)D^{\alpha}\varphi - \nabla^{\alpha}_{\nu}\varphi = -\gamma(\nu)D\varphi - \nabla_{\nu}\varphi + (n-1)\alpha i\gamma(\nu)\varphi,\]
and using the identity
\[-\gamma(\nu)D\varphi - \nabla_{\nu}\varphi = D^{S}\varphi - \frac{n-1}{2}H\varphi,\]
Formula (13) follows directly.

We are now ready to prove Theorem 1.

5. The estimate

Proof of Theorem 1 Consider now a compact domain \(\Omega\) of a Riemannian spin manifold such that the mean curvature \(H\) of the boundary satisfies \(H \geq 2\alpha\), for \(\alpha > 0\). By ellipticity of the MIT boundary condition \(B_{\text{MIT}}\), consider a smooth spinor field \(\varphi \in \Gamma(\Sigma\Omega)\) solution of the eigenvalue boundary problem (6), i.e. \(\varphi\) satisfies
\[
\begin{align*}
D\varphi &= \lambda_{\text{MIT}} \varphi \quad \text{on} \ \Omega \\
B_{\text{MIT}}\varphi &= 0 \quad \text{along} \ \partial\Omega
\end{align*}
\]
with \(\text{Im}(\lambda_{\text{MIT}}) > 0\) by Remark 1. We now apply the hyperbolic Reilly formula (13) to the spinor field \(\varphi\) to get
\[
||P^{\alpha}\varphi||_{L^{2}}^{2} = \left(\frac{n-1}{n}|\lambda_{\text{MIT}} - n\alpha i|^{2} - n(n-1)\alpha^{2}\right)||\varphi||_{L^{2}}^{2} - \frac{R}{4}\varphi,\varphi_{L^{2}}
+ \int_{\partial\Omega} \langle D^{S}\varphi + \frac{n-1}{2}(2\alpha i\gamma(\nu)\varphi - H\varphi),\varphi\rangle ds(g).
\]
Note that since \(i\gamma(\nu)\varphi = \varphi\) along the boundary, we can compute
\[
\langle D^{S}\varphi,\varphi\rangle = \langle D^{S}\varphi, i\gamma(\nu)\varphi\rangle = \langle i\gamma(\nu)D^{S}\varphi,\varphi\rangle = -\langle D^{S} (i\gamma(\nu)\varphi),\varphi\rangle = -\langle D^{S}\varphi,\varphi\rangle,
\]
and so the preceding formula gives
\[
||P^{\alpha}\varphi||_{L^{2}}^{2} + \frac{n-1}{2} \int_{\partial\Omega} (H - 2\alpha)|\varphi|^{2} ds(g) = \frac{n-1}{n} \left(\frac{1}{n}\lambda_{\text{MIT}}^{2} - 2n\alpha \text{Im}(\lambda_{\text{MIT}})\right)||\varphi||_{L^{2}}^{2} - \frac{R}{4}\varphi,\varphi_{L^{2}}.
\]
(16)
The assumption on the mean curvature gives:
\[
|\lambda_{\text{MIT}}^{2} - 2n\alpha \text{Im}(\lambda_{\text{MIT}})| \geq \frac{n}{4(n-1)}R_{0}.
\]
For \(\alpha_{0} = \frac{1}{2}H_{0}\), where \(H_{0} = \inf_{\partial\Omega}(H)\), we get Inequality (16). Suppose now that equality is achieved, thus
\[
||P^{\alpha_{0}}\varphi||_{L^{2}}^{2} = 0 \quad \text{and} \quad \frac{n-1}{2} \int_{\partial\Omega} (H - 2\alpha_{0})|\varphi|^{2} ds(g) = 0.
\]
Moreover the spinor field \(\varphi\) is a solution of (15), so it satisfies the Killing equation
\[
\nabla_{X}\varphi = -\frac{\lambda_{\text{MIT}}}{n}\gamma(X)\varphi, \quad \text{for all} \ X \in \Gamma(T\Omega).
\]
Since such a spinor field has no zeroes (see [Fri00]), the mean curvature of the boundary is constant with $H = 2\alpha_0$. Furthermore, it is a well-known result [BFGK90] that, in this case, the eigenvalue $\lambda^{MIT}$ has to be either real or purely imaginary. Here we have $\text{Im}(\lambda^{MIT}) > 0$, then $\lambda^{MIT} \in i\mathbb{R}^+$. The domain $\Omega$ is in particular an Einstein manifold. We now show that the boundary has to be totally umbilical. Indeed, note that we have for all $X \in \Gamma(T(\partial\Omega))$:

$$
\nabla_X (i\gamma(\nu)\varphi) = i\gamma(\nabla_X \nu)\varphi + i\gamma(\nu)\nabla_X \varphi
= i\gamma(\nabla_X \nu)\varphi + \alpha_0 \gamma(\nu)\gamma(X)\varphi
= i\gamma(\nabla_X \nu)\varphi + i\alpha_0 \gamma(X)\varphi.
$$

However along the boundary we have $i\gamma(\nu)\varphi = \varphi$, so we obtain

$$
\gamma(\nabla_X \nu)\varphi = -2\alpha_0 \gamma(X)\varphi.
$$

Since the spinor field $\varphi$ has no zeros, we have $\Lambda(X) = -\nabla_X \nu = 2\alpha X$ and the boundary is totally umbilical. We can again show that in the equality case, we have $\text{Im}(\lambda^{MIT}) = n\alpha_0$. In fact, just note that the boundary term can be rewritten as

$$
\int_{\partial\Omega} \langle D^S \varphi - \frac{n-1}{2} H \varphi + (n-1)\alpha_0 \varphi, \varphi \rangle ds(g) = -\int_{\partial\Omega} \langle \nabla_\nu \varphi + \gamma(\nu)D \varphi - (n-1)\alpha_0 \varphi, \varphi \rangle ds(g).
$$

This term is zero since we have equality in (16). Now using that the spinor field $\varphi$ is an imaginary Killing spinor satisfying (6) gives

$$
\nabla_\nu \varphi + \gamma(\nu)D \varphi = \frac{n-1}{n} \text{Im}(\lambda^{MIT})\varphi.
$$

Substituting in the preceding identity gives

$$
(n-1) \int_{\partial\Omega} (\alpha_0 - \frac{\text{Im}(\lambda^{MIT})}{n}) |\varphi|^2 ds(g) = 0,
$$

and since $\varphi$ has no zeroes, $\text{Im}(\lambda^{MIT}) = n\alpha_0 = \frac{nH_0}{2}$. $\square$

**Remark 2.**

1. The orthogonal projection $B^{\perp}_{MIT}$ defines a local elliptic boundary condition for the Dirac operator $D$ of $\Omega$. We can easily check that in this case, the imaginary part of an eigenvalue $\lambda^{MIT}$ of $D$ satisfies $\text{Im}(\lambda^{MIT}) < 0$. Inequality (3) is then given by

$$
|\lambda^{MIT}|^2 \geq \frac{n}{4(n-1)} R_0 - n \text{Im}(\lambda^{MIT}) H_0.
$$

2. For $H_0 = 0$, we obtain Inequality (2). In fact, if we suppose that equality is achieved, Theorem [Bau89a] implies $\text{Im}(\lambda^{MIT}) = \frac{nH_0}{2} = 0$ which is impossible by Remark [Bau89b].

3. Note that the Riemannian spin manifolds with an imaginary Killing spinor with Killing number $i\alpha$ have been classified by H. Baum in [Bau89a] and [Bau89b]. Such manifolds are called pseudo-hyperbolic and they are given by

$$
(\mathbb{R} \times \exp M_0, g) = (\mathbb{R} \times M_0, dt^2 \oplus e^{-4\alpha t} g_{M_0}),
$$
where \((M_0, g_{M_0})\) is a complete Riemannian spin manifold carrying a non-trivial parallel spinor. After suitable rescaling of the metric, we can assume that the Killing number is either \(i/2\) or \(-i/2\), i.e. we have
\[
\nabla_X \phi = \pm \frac{i}{2} \gamma(X) \phi.
\]

Moreover, constant mean curvature hypersurfaces in pseudo-hyperbolic manifolds are classified by the Hyperbolic Alexandrov Theorem proved in [Mon99] (see also [HMR03] for a proof using spinors). Indeed, such a hypersurface is either a round geodesic hypersphere (and, in this case, \(M_0\) is flat and \(H > 1\)) or a slice \(\{s\} \times M_0\) (and, in this case, \(M_0\) is compact and \(H = 1\)).

We can then prove the following corollary:

**Corollary 4.** If the boundary of the compact domain \(\Omega\) is connected, there is no manifold satisfying the equality case in Inequality (3).

**Proof:** If \(\Omega\) is a compact domain with connected boundary achieving equality in (3), then there exists an imaginary Killing spinor on \(\Omega\) and the boundary \(\partial\Omega\) is a totally umbilical constant mean curvature hypersurface with \(H = 2\alpha\). However, using Remark (2.3), \(\Omega\) is a domain in a pseudo-hyperbolic space whose connected boundary is a slice \(\{s\} \times M_0\) and then \(\Omega\) is non-compact. \(\square\)

**Remark 3.** With a slight modification of the boundary condition, we give a domain \(\Omega\) whose boundary has two connected components carrying an imaginary Killing spinor field \(\varphi \in \Gamma(\Sigma \Omega)\) which satisfy
\[
i \gamma(\nu_1) \varphi|_{\partial \Omega_1} = \varphi|_{\partial \Omega_1} \quad \text{and} \quad i \gamma(\nu_2) \varphi|_{\partial \Omega_2} = -\varphi|_{\partial \Omega_2},
\]
where \(\nu_1\) (resp. \(\nu_2\)) is an inner unit vector field normal to \(\partial \Omega_1\) (resp. \(\partial \Omega_2\)). First recall that one distinguishes two types of imaginary Killing spinors (see [Bau89a] and [Bau89b]). Indeed, if \(\varphi \in \Gamma(\Sigma \Omega)\) is an imaginary Killing spinor, denote by \(f\) its length function, then the function
\[
q_{\varphi}(x) := f(x)^2 - \frac{1}{4\alpha^2} ||\nabla f||^2
\]
satisfies \(q_{\varphi}\) is constant and \(q_{\varphi} \geq 0\). If \(q_{\varphi} = 0\), \(\varphi\) is a Killing spinor of type I whereas if \(q_{\varphi} > 0\), \(\varphi\) is a Killing spinor of type II. If \((N^n, g)\) is a complete connected Riemannian spin manifold with an imaginary Killing spinor of type II associated with the Killing number \(i\alpha\), then \((N^n, g)\) is isometric to the hyperbolic space \(\mathbb{H}^{n-4\alpha^2}_0\). If \((N^n, g)\) admits an imaginary Killing spinor of type I, then \((N^n, g)\) is isometric to the warped product \((\mathbb{R} \times M_0, dt^2 \oplus e^{-4\alpha t} g_{M_0})\), where \(M_0\) is a complete Riemannian spin manifold with a non-trivial parallel spinor field. Moreover, \(q_{\varphi} = 0\) if and only if there exists a unit vector field \(\xi\) on \(N\) such that \(\gamma(\xi) \varphi = i\varphi\). In fact, we can easily prove that the vector field \(\xi\) is the normal field of \(\{t\} \times M_0\) for all \(t \in \mathbb{R}\). So consider the domain given by the warped product \(\Omega := ([a, b] \times M_0, dt^2 \oplus e^{-4\alpha t} g_{M_0})\), where \(M_0\) is a compact spin manifold carrying a non-trivial parallel spinor field and with \(-\infty < a < b < +\infty\). The domain \(\Omega\) carries an imaginary Killing spinor \(\varphi\) of type I, so there exists \(\xi\) normal to \(\{t\} \times M_0\) for all \(t \in [a, b]\)
such that $\gamma(\xi)\varphi = i\varphi$. The boundary of $\Omega$ has two connected components which are slices $\{a\} \times M_0$ and $\{b\} \times M_0$ of $\Omega$ and with mean curvature $H_a = H_b = 2\alpha$, where $H_t$ is the mean curvature of a slice $\{t\} \times M_0$. The spinor field $\varphi$ clearly satisfies the boundary conditions (17).

References

[Bär98] C. Bär, *Extrinsic bounds of the Dirac operator*, Ann. Glob. Anal. Geom. **16** (1998), 573–596.
[Bau89a] H. Baum, *Complete Riemannian manifolds with imaginary Killing spinors*, Ann. Glob. Anal. Geom. **7** (1989), 205–226.
[Bau89b] ———, *Odd-dimensional Riemannian manifolds admitting imaginary Killing spinors*, Ann. Glob. Anal. Geom. **7** (1989), 141–153.
[BBW93] B. Booß-Bavnbek and K.P. Wojciechowski, *Elliptic boundary problems for the Dirac operator*, Birkhäuser, Basel, 1993.
[BFGK90] H. Baum, T. Friedrich, R. Grünewald, and I. Kath, *Twistor and Killing spinors on Riemannian manifolds*, vol. 108, Seminarbericht, 1990, Humboldt-Universität zu Berlin.
[BHMM] J.P. Bourguignon, O. Hijazi, J.L. Milhorat, and A. Moroainu, *A spinorial approach to Riemannian and conformal geometry*, Monograph (In Preparation).
[CJJ+74] A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, and V.F. Weisskopf, *New extended model of hadrons*, Phys. Rev. D **9** (1974), 3471–3495.
[CJJT74] A. Chodos, R.L. Jaffe, K. Johnson, and C.B. Thorn, *Baryon structure in the bag theory*, Phys. Rev. D **10** (1974), 2599–2604.
[Fri80] T. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nicht negativer Skalarkrümmung*, Math. Nach. **97** (1980), 117–146.
[Fri00] ———, *Dirac operators in Riemannian geometry*, vol. 25, Amer. Math. Soc. Graduate Studies in Math., 2000.
[HMR02] O. Hijazi, S. Montiel, and S. Roldán, *Eigenvalue boundary problems for the Dirac operator*, Commun. Math. Phys. **231** (2002), 375–390.
[HMR03] ———, *Dirac operators on hypersurfaces of manifolds with negative scalar curvature*, Ann. Global Anal. Geom. **23** (2003), 247–264.
[HMZ02] O. Hijazi, S. Montiel, and X. Zhang, *Conformal lower bounds for the Dirac operator on embedded hypersurfaces*, Asian J. Math. **6** (2002), 23–36.
[Joh75] K. Johnson, *The M.I.T bag model*, Acta Phys. Pol. **B6** (1975), 865–892.
[LM89] H.B. Lawson and M.L. Michelsohn, *Spin Geometry*, Princeton University Press ed., vol. 38, Princeton Math. Series, 1989.
[Lop53] Ya.B. Lopatinski˘ı, *On a method for reducing boundary problems for a system of differential equations of elliptic type to regular integral equations*, Ukrain. Math. Ž. **5** (1953), 123–151, (Russian).
[Mon99] S. Montiel, *Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds*, Indiana Univ. Math. J. **48** (1999), 711–748.

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