General static spherically symmetric solutions in Hořava gravity

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Abstract

We derive general static spherically symmetric solutions in the Hořava theory of gravity with nonzero shift field. These represent “hedgehog” versions of black holes with radial “hair” arising from the shift field. For the case of the standard de Witt kinetic term (λ = 1) there is an infinity of solutions that exhibit a deformed version of reparametrization invariance away from the general relativistic limit. Special solutions also arise in the anisotropic conformal point λ = 1/3.

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1 Introduction

The Hořava theory of gravity, introduced in [1, 2], is intended to be a power-counting
renormalizable UV completion of the standard Hilbert-Einstein Gravity. It is a
theory with higher spatial derivatives of the intrinsic curvature of a spatial slice that
treats space and time differently. Indeed, the basic property that makes this theory
power-counting renormalizable is its invariance under the anisotropic rescaling

\[ x \rightarrow b x \quad t \rightarrow b^z t, \]

which makes the conformal dimensions (\([\ ]_s\)) of space and time to be different:

\[ [x]_s = -1 \quad [t]_s = -z. \]

For a (3 + 1)-dimensional space-time \( z = 3 \).

In this theory space-time must be of the form \( M = \mathbb{R} \times \Sigma \) where \( \Sigma \) is a space-like
3-dimensional surface. Because of the anisotropy, the theory is invariant only under
diffeomorphisms that leave unchanged the foliation structure (\([3, 4]\)) \( \mathcal{F} \)

\[ x^i \rightarrow \tilde{x}^i = \tilde{x}^i(x, t) \quad t \rightarrow \tilde{t} = \tilde{t}(t). \]

The space-time metric \( g_{\mu \nu} \), because of the foliation structure, can be globally
decomposed in terms of the ADM parametrization:

\[ g_{\mu \nu} = \begin{pmatrix} -N^2 + N_i N^i & N_j \\ N_i & h_{ij} \end{pmatrix} \quad g^{\mu \nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N_j}{N^2} \\ \frac{N_i}{N^2} & h^{ij} - \frac{N_i N^i}{N^2} \end{pmatrix} \quad (1) \]

where \( h_{ij}(x, t) \) is the metric on \( \Sigma \) and \( N(x, t) \) and \( N_i(x, t) \) are, respectively, lapse
and shift functions.

The Hořava-Lifshitz action

\[ S_{HL} = S_K - S_V \quad (2) \]

contains a kinetic term \( S_K \), involving time derivatives, and a potential term \( S_V \),
involving only space derivatives. In the potential term there are also higher space
derivatives, as well as higher powers of the 3-dimensional curvature on \( \Sigma \). The
potential term was first introduced using the detailed balance condition \cite{2}; more
general expressions were considered in \cite{5, 6, 7}. In particular, Kehagias and Sfetsos
considered in \cite{8} an action obtained by softly breaking the detailed balance condition
with a curvature term $\mu^4 R$. The full modified Horava-Lifshitz action, in order of
descending dimensions, is

$$S = \int d^4x \sqrt{h} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4} C_{ij} C^{ij} + \frac{\kappa^2 \mu^2}{2w^2} \varepsilon^{ijk} R_{il} \nabla_j R^l \right\}$$

$$- \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} + \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \left( \frac{1 - 4\lambda}{4} R^2 + \Lambda_W R - 3\Lambda_W^2 \right) + \mu^4 R \right\}$$

where the kinetic term corresponds to the first bracket, in which

$$K_{ij} = \frac{1}{2N} (\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i),$$

$R_{ij}$ are the spatial components of the Ricci tensor on $\Sigma$, $R$ is its trace and $C_{ij}$ are
the spatial components of the Cotton tensor. This new term, as observed in \cite{8},
makes the action have a well-behaved limit

$$\Lambda_W \rightarrow 0$$

and admits a Minkowski vacuum solution.

The above theory has a UV critical point $z = 3$ and an IR critical point $z = 1$, for which
$w \rightarrow \infty$, which corresponds to the relativistic case. Indeed, in the IR limit
the quadratic terms in the curvature vanish, obtaining

$$S = \int d^4x \sqrt{h} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \Lambda_W (R - 3\Lambda_W) + \mu^4 R \right\},$$

which is isotropic under rescalings of space and time. Comparing the IR limit of the
modified Horava-Lifshitz action to the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G} \int \sqrt{g} d^4x \left[ R^{(4)} - 2\Lambda_E \right] = \frac{c}{16\pi G} \int d^4x dt \sqrt{h} N \left[ \frac{1}{c^2} (K_{ij} K^{ij} - K^2) + \frac{R}{2} - 2\Lambda_E \right]$$

we obtain, respectively, the emergent velocity of light, the emergent Newton constant
and the cosmological constant

$$c = \frac{\kappa \mu}{4} \sqrt{8\mu^2 + \frac{\kappa^2 \Lambda_W}{1 - 3\lambda}}, \quad G_N = \frac{\kappa^2}{32\pi c}, \quad \Lambda = \frac{3\kappa^2 \Lambda_W^2}{16(1 - 3\lambda)\mu^2 + 2\kappa^2 \Lambda_W^2}.$$

The IR limit of the Horava-Lifshitz action will recover Einstein gravity only if the
running constant $\lambda$ becomes 1 in the $z = 1$ fixed point.
Several aspects of the Kehagias-Sfetsos action were analyzed: cosmological solutions [9], possible tests [10, 11, 12, 13], fundamental aspects of the theory [14, 15, 16, 17, 18] and black hole solutions (with vanishing shift variables) [19, 20, 21, 22, 23, 24, 25, 26, 32, 28, 29, 30] and special cases such as $\lambda = 1/3$ [31]. In particular, Kiritsis and Kofinas in [32] studied more general solutions considering the Hořava-Lifshitz action with generic (independent) coefficient as coupling constants, that is, for an action not derived from a detailed balance condition.

In the present work we study and derive spherically symmetric solutions for the Kehagias-Sfetsos action with general $\lambda$ and nonzero shift variables. We call these “hedgehog” solutions, in analogy with the field theoretic soliton configurations of the same name, as they possess radially-pointing “hair” due to the shift field. In the process we uncover conserved quantities for the system, and a special “deformed” gauge invariance for the case $\lambda = 1$. The conformal value $\lambda = 1/3$ is also shown to have special properties. Our solutions recover previously known solutions in the appropriate limits.

2 The spherically symmetric ansatz

We shall work with the action (3) in the ADM parametrization, but with somewhat redefined coefficients. Specifically, we shall rescale

$$\mu^2 \rightarrow (3 \lambda - 1) \mu^2$$

which will allow us to recover a nontrivial conformal limit when $\lambda = 1/3$. We will also denote the total coefficient of the linear Ricci term $\mathcal{R}$ (which receives contributions both from $\Lambda_W$ in the action and from the added extra term) as $\omega \kappa^2 \mu^2/8$. Finally, we will use the freedom to rescale time and $N_i$ (which amounts to a choice of time units) in order to make the coefficient of the kinetic term equal to the coefficient of the Ricci term. This will ensure that at the IR limit the speed of light comes out 1. With these choices, the action becomes

$$S = \frac{\kappa^2 \mu^2}{8} \int dt \sqrt{h} N \left\{ \omega \left( K_{ij} R^{ij} - \lambda K^2 \right) - \frac{4}{\mu^2 w^4} C_{ij} C^{ij} + \frac{4}{\mu w^2} \sqrt{3 \lambda - 1} \epsilon^{ijk} \mathcal{R}_{il} \nabla_j \mathcal{R}^l_k ight. - (3 \lambda - 1) \mathcal{R}_{ij} \mathcal{R}^{ij} + \frac{4 \lambda - 1}{4} \mathcal{R}^2 - 3 \Lambda_W^2 + \omega \mathcal{R} \right\}$$

(Note that our $\omega$ corresponds to $\omega - \Lambda_W$ in [9].) The standard Einstein gravity is recovered in the limit $\lambda \rightarrow 1$, $\omega \rightarrow \infty$, and the cosmological constant $\Lambda$ in this limit.
is identified as 
\[ \Lambda = \frac{3\Lambda_W^2}{2\omega} \]
We shall keep \( \lambda \) arbitrary, as there may be measurable deviations from its general relativistic value (\( \lambda = 1 \)).

The most general static spherically symmetric ansatz involves a spherically symmetric 3-dimensional metric in terms of a radial coordinate \( r \) with metric \( f^{-1}(r) \) and spherical angles \( \theta, \phi \), a lapse function \( N(r) \) depending only on \( r \) and a “hedgehog” configuration for the shift vector \( N_i \) of the form \( N_r = N_r(r) \), \( N_\theta = N_\phi = 0 \). In this parametrization the metric is
\[
ds^2 = (-N^2 + N^2_r f)dt^2 + 2N_r dt dr + \frac{1}{f} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi ) \quad (6)
\]
In the general relativistic case the term \( dt dr \) can be eliminated by an appropriate redefinition \( t \to t + F(r) \). In the present case, however, such a transformation is not an invariance of the action, and the variable \( N_r \) remains a relevant degree of freedom (this point was also stressed in [32]).

For nonvanishing \( N_i \) the kinetic term for \( h_{ij} \) (involving the extrinsic curvature) is nonvanishing and must be included in the action. Using the expressions derived in the appendix for the extrinsic and intrinsic curvatures for the spherical metric (6), the action (3) after integration over the angular part, and omitting the trivial integration over \( t \), becomes
\[
S = 4\pi \frac{\kappa^2 \mu^2}{8} \int dr \left[ L_K - L_V \right] \quad (7)
\]
where
\[
L_V = \frac{N}{\sqrt{f}} \left[ (2\lambda - 1) \frac{(f-1)^2}{r^2} - 2\lambda \frac{f-1}{r} f' + \lambda - 1 \frac{1}{2} f'^2 - 2\omega(1 - f - rf') - 3\Lambda_W^2 r^2 \right]
\]
\[
L_K = \omega \frac{\sqrt{f}}{N} \left[ (1 - \lambda) \frac{r^2}{f} \left( f N' + \frac{1}{2} f' N_r \right)^2 + 2(1 - 2\lambda) f N_r^2 - 4\lambda r \left( f N' + \frac{1}{2} f' N_r \right) N_r \right]
\]
in which prime denotes differentiation with respect to \( r \).

To facilitate the treatment of the problem and identify its essential mathematical structure, we define new variables as follows:
\[
p = \frac{1 + \omega r^2 - f}{\sqrt{\omega^2 - \Lambda_W^2 r^2}}, \quad q = \sqrt{\frac{2\omega f}{\omega^2 - \Lambda_W^2 r^2}} r^2 N_r, \quad M = \frac{N r^3}{\sqrt{f}} \quad (8)
\]
assuming $\omega > |\Lambda_W|$. We further define a new logarithmic radial coordinate
\[ s = \ln r \] (9)

In terms of the new variables and coordinate, the action becomes
\[ S = 2\pi\kappa^2\mu^2(\omega^2 - \Lambda_W^2) \int ds \mathcal{L} \]
\[ \mathcal{L} = M \left( \frac{\lambda - 1}{2} p^2 - 2p\dot{p} - 3\dot{p}^2 + 3 \right) + \frac{1}{M} \left( \frac{\lambda - 1}{2} \dot{q}^2 + 2q\dot{q} - 3q^2 \right) \] (10)

where overdot denotes derivative with respect to $s$. For the classical theory the overall coefficient in the action is irrelevant and will be omitted from now on.

In the above form, some features are immediately obvious: the explicit appearance of the radial variable has dropped. Further, the only relevant parameter is $\lambda$, all other parameters (such as $\omega$ and $\Lambda_W$) having been absorbed in field redefinitions. Also note that the spatial metric ($p$) and shift ($q$) variables enter the action in a remarkably similar way.

The equations of motion obtain as
\[ \frac{\lambda - 1}{2} \dot{p}^2 - 2p\dot{p} - 3\dot{p}^2 + 3 = \frac{1}{M^2} \left( \frac{\lambda - 1}{2} \dot{q}^2 + 2q\dot{q} - 3q^2 \right) \] (11)
\[ -M(2\dot{p} + 6p) = \frac{d}{ds} \left\{ M[(\lambda - 1)\dot{p} - 2p] \right\} \] (12)
\[ \frac{1}{M}(2\dot{q} - 6q) = \frac{d}{ds} \left\{ \frac{1}{M}[(\lambda - 1)\dot{q} + 2q] \right\} \] (13)

Upon elimination of $M$ using its (algebraic) equation of motion the above reduce to two coupled second-order differential equations for $p$ and $q$. The general solution will contain 4 integration constants. The equations of motion, however, are invariant under a simultaneous rescaling of $N$ and $N_r$, or
\[ M \rightarrow cM, \quad q \rightarrow cq \] (14)

for any constant $c$, corresponding to a rescaling of time in the metric. This can be used to set their scale (usually by requiring $N \rightarrow 1$ as $r \rightarrow \infty$) thus eliminating one integration constant. The solutions will therefore contain 3 relevant constants, corresponding to the mass of the black hole plus two additional “hair” parameters.

The above equations are invariant under independent changes of sign for $M$, $p$ and $q$, so the solution manifold will exhibit this symmetry. The flip $M \rightarrow -M$ is inconsequential, since only $N^2$ appears in the spacetime structure. The flip $q \rightarrow
$-q$ is essentially time reversal and corresponds to inverting the hedgehog direction $N_r \rightarrow -N_r$, while the flip $p \rightarrow -p$ corresponds to changing the radial metric as $f \rightarrow 2 + \omega r^2 - f$.

In addition to the above, the action (10) has two radial “invariants”, that is, two first integrals of the equations of motion. The first one is obvious: since $L$ does not depend explicitly on the parameter $s$, the action is invariant under shifts $s \rightarrow s + \epsilon$, that is, under the infinitesimal variations

$$\delta M = \dot{M}, \quad \delta p = \dot{p}, \quad \delta q = \dot{q}.$$  

The lagrangian changes by a total derivative,

$$\delta L = \dot{\mathcal{L}}$$  

and so the conserved quantity, analogous to energy for the radial coordinate $s$, is:

$$E = \frac{\partial \mathcal{L}}{\partial \dot{M}} \delta M + \frac{\partial \mathcal{L}}{\partial \dot{p}} \delta p + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q - \mathcal{L}$$

$$= M \left( \frac{\lambda - 1}{2} \dot{p}^2 + 3p^2 - 3 \right) + \frac{1}{M} \left( \frac{\lambda - 1}{2} \dot{q}^2 + 3q^2 \right)$$  

(15)

$E$ is essentially the mass parameter, reducing to $E = 12m$ in the case of an ordinary (de Sitter) black hole.

The other invariance is more nontrivial. The fact that $p$ and $q$ enter the action in a similar form suggests a possible new invariance under a variation involving just these two fields. Indeed, it can be checked that the variation

$$\delta p = \frac{1}{M}[(\lambda - 1)\dot{q} + 2q], \quad \delta q = M[(\lambda - 1)\dot{p} - 2p]$$  

(16)

will make $\delta \mathcal{L} = \dot{K}$ a total derivative. Therefore the conserved quantity is

$$G = \left( \frac{\partial \mathcal{L}}{\partial \dot{p}} \delta p + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q - \mathcal{L} \right)$$

$$= 2(\lambda - 1) \left( \frac{\lambda - 1}{2} \dot{p} \dot{q} + q \dot{p} - p \dot{q} + 3pq \right)$$  

(17)

This is one of the nontrivial “hair” parameters of the black hole.

The above two constants of motion allow in principle for the reduction of the system into one ordinary differential equation. Indeed, $E$, $G$, and the equation of motion for $M$ (11) are algebraic expressions in $M$, $\dot{p}$ and $\dot{q}$, and therefore can be used to express $M$, $\dot{p}$ and $\dot{q}$ in terms of $p$ and $q$:

$$\dot{p} = P(p, q), \quad \dot{q} = Q(p, q)$$
Considering $p$ as the new independent variable, $q$ can be obtained by solving the equation

\[
\frac{dq}{dp} = \frac{Q(p, q)}{P(p, q)}
\]

after which $M$ and the variable $s$ can be determined.

Due to the rather complicated form of $P(p, q)$ and $Q(p, q)$, the above procedure is quite involved. We postpone a full treatment of the general case for a future publication. There are, however, special values of $\lambda$ with interesting features, for which the problem can be readily solved, and we expose them in the next sections. Further, a more explicit solution for general $\lambda$ can be found in the “bald” configuration $N_r = 0$. This case will be analyzed in section 5.

### 3 The case $\lambda = 1$

The value $\lambda = 1$ is special, as it is required for recovering general relativity (together with $\omega \to \infty$). The equations of motion (11,12,13) for $\lambda = 1$ become first-order and simplify dramatically:

\[
\begin{align*}
-2p\dot{p} - 3p^2 + 3 &= \frac{1}{M^2} (2q\dot{q} - 3q^2) \quad (18) \\
(\dot{M} - 3M)p &= 0 \quad (19) \\
(\dot{M} - 3M)\frac{q}{M} &= 0 \quad (20)
\end{align*}
\]

We see that the last two equations become essentially identical and imply

\[
\dot{M} = 3M \quad \Rightarrow \quad M = c e^{3s} = c r^3
\]

(The other solution $p = q = 0$ implies also $M = 0$ and is trivial.) Using the time scale invariance (14) to set $c = 1$, we obtain

\[
N = \sqrt{f}
\]

as in the standard general relativistic case. The remaining equation can then be written as

\[
\frac{d}{ds} \left[ r^3 (1 - p^2) \right] = \frac{d}{ds} \left( \frac{q^2}{r^3} \right)
\]

which determines $p$ in terms of $q$ or vice-versa:

\[
p^2 = 1 + \frac{k}{r^3} - \frac{q^2}{r^6} \quad (21)
\]
It is evident that the case $\lambda = 1$ has an infinity of solutions. The corresponding solutions for the metric function $f(r)$ and the shift variable $N_r(r)$ in terms of an arbitrary function $g(r)$ read

$$f = 1 + \omega r^2 \pm \sqrt{(\omega^2 - \Lambda W^2)r^4 + 4\omega m r - 2\omega g^2(r)} \quad (22)$$

$$N_r = \frac{g(r)}{r\sqrt{f}} \quad (23)$$

The expressions for $f$ obtained in [8], [9] are recovered for $g(r) = 0$ and the negative choice of sign for $p$, after identifying our $\omega$ with their $\omega - \Lambda W$.

The above suggests that the theory for $\lambda = 1$ has a gauge invariance. A further indication for this is that the integral of motion $G$ [17] is identically zero for $\lambda = 1$. Indeed, the $\lambda = 1$ action is invariant under the variation

$$\delta(q^2) = -M^2 \delta(p^2), \quad \delta M = 0 \quad (24)$$

with $\delta(p^2)$ an arbitrary function of $r$. Clearly the symmetry transformation (16) is a special case of the above gauge transformation, justifying the vanishing of its charge.

The above symmetry (24) reduces to the usual reparametrization invariance under $t \rightarrow t + F(r)$ in the IR limit $\omega \rightarrow \infty$, as can be checked by using the expressions (22, 23). For finite $\omega$, however, it corresponds to a “deformed” transformation. The full meaning of this symmetry is under investigation.

### 4 The case $\lambda = 1/3$

As observed in [2] the value $\lambda = \frac{1}{3}$ corresponds to the action being invariant under an anisotropic conformal (Weyl) symmetry. That this value is special also manifests in the fact that the action in this case becomes a sum of perfect squares:

$$\mathcal{L} = -\frac{M}{3}(\dot{p} + 3p)^2 + 3M - \frac{1}{3M}(\dot{q} - 3q)^2 = -\frac{\bar{M}}{3} \bar{p}^2 + 3\bar{M}r^6 - \frac{1}{3\bar{M}}\bar{p}^2$$

where we redefined

$$\bar{p} = r^3 p, \quad \bar{q} = \frac{q}{r^3}, \quad \bar{M} = \frac{M}{r^6}$$
The equations of motion (11,12,13) simplify accordingly

\[
\frac{\ddot{q}^2}{M^2} + 9r^6 = \ddot{p}^2 \quad (25)
\]

\[
d\left(\frac{\dot{M}\ddot{p}}{s}\right) = 0 \quad (26)
\]

\[
d\left(\frac{\dot{q}}{M}\right) = 0 \quad (27)
\]

The above equations integrate readily giving

\[
\dot{p} = \pm 3\sqrt{A^2 + r^6}, \quad \dot{q} = 3\ddot{M} = \frac{3AB}{\sqrt{A^2 + r^6}}
\]

with \(A, B\) integration constants. From these, the fields \(p, q, M\) are obtained as

\[
p = \pm \frac{3}{r^3} \int \frac{dr}{r} \sqrt{A^2 + r^6} = \pm \frac{1}{r^3} \left(\sqrt{A^2 + r^6} + \frac{A}{2} \ln \frac{\sqrt{A^2 + r^6} - A}{\sqrt{A^2 + r^6} + A}\right) + K_1
\]

\[
q = 3ABr^3 \int \frac{dr}{r} \sqrt{A^2 + r^6} = \frac{B_1r^3}{2} \ln \frac{\sqrt{A^2 + 9r^6} - A}{\sqrt{A^2 + 9r^6} + A} + K_2r^3
\]

\[
M = \frac{Br^6}{\sqrt{A^2 + 9r^6}}
\]

with \(K_1, K_2\) new integration constants. Fixing the scale of \(M\) by choosing \(B = 3\), the corresponding solutions for \(f, N_r\) are

\[
f = 1 + \omega r^2 \pm \frac{\sqrt{\omega^2 - \Lambda^2_{\text{fw}}}}{r} \left(\sqrt{A^2 + r^6} + \frac{A}{2} \ln \frac{\sqrt{A^2 + r^6} - A}{\sqrt{A^2 + r^6} + A}\right) + K_1
\]

\[
N_r = r \sqrt{\frac{\omega^2 - \Lambda^2_{\text{fw}}}{2\omega f}} \left(\frac{3}{2} \ln \frac{\sqrt{A^2 + 9r^6} - A}{\sqrt{A^2 + 9r^6} + A} + K_2\right).
\]

5 \(N_r = 0\) solutions

For \(q = 0\) the equation of motion (12) is satisfied and we can determine \(p\) from (11):

\[
\frac{\lambda - 1}{2} p^2 - 2p\dot{p} - 3(p^2 - 1) = 0. \quad (28)
\]

Solving for \(\dot{p}\) we have

\[
\dot{p} = \frac{2p - \sqrt{4p^2 + 6(\lambda - 1)(p^2 - 1)}}{\lambda - 1}
\]
where $\epsilon = \pm 1$. Note that only the case $\epsilon = 1$ has a finite limit for $\lambda \to 1$. This is trivially separable, giving

$$\frac{dr}{r} = \frac{\lambda - 1}{2} \frac{dp}{p - \epsilon \sqrt{\frac{3\lambda - 1}{2} p^2 - \frac{3}{2} (\lambda - 1)}}$$

and upon doing the integral we obtain

$$\ln[C r] = -\frac{1}{6} \left\{ \ln \left[ \frac{\sqrt{a p^2 + b - \epsilon p}}{\sqrt{a p^2 + b + \epsilon p}} \right] + \ln \left[ b + (a - 1) p^2 \right] + 2 \epsilon \sqrt{a} \ln[a p + \sqrt{a \sqrt{b + a p^2}}] \right\}$$

(29)

where

$$a = \frac{3\lambda - 1}{2}, \quad b = -\frac{3}{2} (\lambda - 1)$$

and $C$ is an integration constant.

Although such an expression is not explicit, it becomes explicit by considering $p$ as the independent variable and expressing $r$ in terms of $p$ in the spacetime structure. Moreover, it allows for a qualitative investigation of the behavior of the solution under a variation of $\lambda$. It also reproduces the explicit solutions found earlier in the limit $\lambda \to 1$ and $\lambda \to 1/3$.

6 Conclusions

We have examined the full static spherically symmetric configuration in Hořava gravity, which, as shown, admits hedgehog solutions. The solutions for $\lambda = 1$ present a gauge invariance corresponding to a “deformed” coordinate transformation, not previously observed because the usually chosen condition $g_{rt} = 0$ fixes the gauge. This invariance does not survive for $\lambda \neq 1$. A specific gauge can thus be fixed by continuity as $\lambda \to 1$ (we must take into account that in the theory $\lambda$ is a running constant), for example by matching the value of

$$\lim_{\lambda \to 1} \frac{G}{\lambda - 1},$$

which remains finite and nonzero, or, alternatively, by coupling to matter. Both possibilities are under investigations.

The case $\lambda = 1/3$, corresponding to an anisotropic Weyl-invariant theory, is another relevant value for which the equations of motion simplify and hence admit quite explicit solutions.
Although it is hoped (and required) that $\lambda$ goes to 1 in the IR limit of the theory, this has not been proved yet. Therefore, it is interesting to study the behavior of the solutions of the theory for generic $\lambda$. The solutions we find, as well as the solutions found in [32] for zero shift variables, are not explicit. Our solutions for $\lambda \neq 1, \frac{1}{3}$, $N_r \neq 0$ may be integrable but the expressions for

$$
\dot{p} = P(p, q) \quad \dot{q} = Q(p, q),
$$

although algebraic, are quite complicated. On the other hand, our solutions for $N_r = 0$ based on the softly-broken detailed balance condition have a simpler form, and can be rendered explicit by changing variable to $p$ from the original $r$. The qualitative behavior of this and other solutions is under investigation.

### A Static Spherical Case

The most general static spherically symmetric ansatz for a metric is

$$
g_{\mu\nu} = \begin{pmatrix}
-N^2 + N_r^2 f & N_r & 0 & 0 \\
N_r & \frac{1}{f} & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
$$

for which

$$
h_{\mu\nu} = \begin{pmatrix}
N_r^2 f & N_r & 0 & 0 \\
N_r & \frac{1}{f} & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
\quad N_\alpha = (fN_r^2, N_r, 0, 0) \quad n_\alpha = (-N, 0, 0, 0)
$$

where $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the metric on the space-like surface $\Sigma$ orthogonal to the direction $n_\alpha$, $N_\alpha$ the shift function and $N$ the lapse function.

The kinetic term in the action (3) is constructed from the extrinsic curvature defined as

$$
K_{\alpha\beta} \equiv \frac{1}{2} \mathcal{L}_n h_{\alpha\beta} = \frac{1}{2N} [\partial_\mu h_{\alpha\beta} - (\nabla_\alpha N_\beta)_h - (\nabla_\beta N_\alpha)_h]
$$

where with $(\nabla_\alpha N_\beta)_h$ we mean that the covariant derivative is respect to the metric $h_{\mu\nu}$. In our case the metric is static then

$$
K_{\alpha\beta} = -\frac{1}{2N} [(\nabla_\alpha N_\beta)_h + (\nabla_\beta N_\alpha)_h].
$$
In particular the spacial components of the extrinsic curvature are

\[ K_{rr} = -\frac{1}{N} \left( N'_r + \frac{1}{2} f' N_r \right) \quad K_{\theta\theta} = -\frac{1}{N} f N_r r \sin^2 \theta, \]

and, using the relation \( K_{\alpha\beta} = h_{\alpha}^i h_{\beta}^j K_{ij} \), the remaining non-zero components are

\[ K_{tt} = f^2 N_r^2 K_{rr} \quad K_{tr} = f N_r K_{rr}. \]

Raising one index with \( g^{\mu\nu} \) we find

\[ K_{rr} = K_{rt} g^{tr} + K_{rt} g^{rt} = -\frac{1}{N} \left( N'_r + \frac{1}{2} f' N_r \right) \left( f N'_r + \frac{1}{2} f' N_r \right) = -\frac{1}{N} \left( f N'_r + \frac{1}{2} f' N_r \right)^2 \]

\[ K_{tt} = g^{tt} K_{tt}^r + g^{tr} K_{tr}^r = 0 \quad K^{tr} = g^{tt} K_{tt}^r + g^{tr} K_{tr}^r = 0 \]

\[ K_{tt} = g^{tt} K_{tt}^t + g^{tr} K_{tr}^t = 0 \quad K^{tt} = -\frac{1}{N} f N_r \quad K^{\phi\phi} = -\frac{1}{N r^3 \sin^2 \theta} \]

Therefore the kinetic term of the action (3) is given by

\[ K_{ij} K^{ij} - \lambda K^2 = \frac{1}{N^2} \left( f N'_r + \frac{1}{2} f' N_r \right)^2 + \frac{2 f^2 N_r^2}{r^2} - \lambda \left[ \frac{1}{N} \left( f N'_r + \frac{1}{2} f' N_r \right) + \frac{2 f N_r}{r} \right]^2 = \]

\[ = 1 - \frac{1}{N^2} \left( f N'_r + \frac{1}{2} f' N_r \right)^2 \quad \frac{2(1 - \lambda) f N_r^2}{r^2} - \frac{4 \lambda}{N^2} \left( f N'_r + \frac{1}{2} f' N_r \right) \left( f N_r \right) \quad (30) \]

In the potential term the intrinsic curvature \( R_{\alpha\beta\gamma\delta} \) in different contractions is related to the 3 + 1-dimensional curvature as follows

\[ R_{\alpha\beta\gamma\delta} = h_{\alpha}^\mu h_{\beta}^\nu h_{\gamma}^\rho h_{\delta}^\lambda R_{\mu\nu\rho\lambda}^{(4)} - 2 K_{[\alpha} | K_{\beta]} |. \]

In our case the intrinsic Ricci tensor for \( \Sigma \) has the following spatial non-zero components:

\[ R_{rr} = -\frac{1}{r} f' \quad R_{\theta\theta} = -\frac{1}{2} f' r - (f - 1) \quad R_{\phi\phi} = R_{\theta\theta} \sin^2 \theta \quad (31) \]
giving

\[
R_{\mu\nu}R^{\mu\nu} = R_{\rho\sigma}h^{\rho\alpha}h^{\sigma\beta} = R_{ij}R_{kl}h^{ik}h^{jl} = \sum_{i=1}^{3}(R_{ii})^2 = (R_{rr})^2 + \frac{2}{r^4}(R_{\theta\theta})^2 = \frac{1}{r^6}f'^2 + \frac{2}{r^4}
\left(-\frac{1}{2}f'r-(f-1)\right)^2 = \frac{3}{2} \frac{1}{r^2}f'^2 + \frac{2}{r^4}(f-1)^2 + \frac{2}{r^6}(f-1)f' \tag{32}
\]

and

\[
R = g^{\mu\nu}R_{\mu\nu} = h^{ij}R_{ij} = -\frac{1}{r}f' + \frac{2}{r^2}
\left(-\frac{1}{2}f'r-(f-1)\right) = -\frac{2}{r^2}[f'r+(f-1)] \tag{33}
\]

The Cotton tensor, because of the symmetry, is still null:

\[C_{ij} = 0. \tag{34}\]

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