FINITE DIMENSIONAL REDUCING AND SMOOTH APPROXIMATING FOR A CLASS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This work provides a finite dimensional reducing and a smooth approximating for a class of stochastic partial differential equations with an additive white noise. Using the invariant random cone to show the asymptotical completion, this stochastic partial differential equation is reduced to a stochastic ordinary differential equation on a random invariant manifold. Furthermore, after deriving the finite dimensional reducing for another stochastic partial differential equation driven by a Wong-Zakai scheme via a smooth colored noise, it is proved that when the smooth colored noise tends to the white noise, the solution and the finite dimensional reducing of the approximate system converge pathwisely to those of the original system.

1. Introduction. The stochastic partial differential equations are the very important mathematical models for various complex phenomena under uncertain influence, such as fluctuating forcing, uncertain parameters, random sources, and random conditions. In many circumstances, the random fluctuations may have delicate or even profound impact in modeling, analyzing, simulating and predicting these complex phenomena. They may drastically affect the qualitative behaviors and results in new properties (see [3, 7, 9, 12, 13, 31] and the references therein).

However, for stochastic partial differential equations, the infinite dimensional state space makes the system more difficult to be visualized geometrically or computed numerically, while the uncertainties make the system too complicated to be analyzed or too expensive to be simulated. The main goal of the present paper is to provide a finite dimensional reducing and a smooth approximating for a class of stochastic partial differential equations with an additive white noise.

We consider a nonlinear stochastic partial differential equation with an additive noise
\[
\frac{du}{dt} = Au + F(u) + \dot{W}(t),
\]
subject to the homogeneous Dirichlet boundary conditions on a bounded domain \(D\) in \(\mathbb{R}^n\). The linear operator \(A\) generates a \(C_0\)-semigroup \(S(t)\), the nonlinearity \(F(u)\) satisfies the Lipschitz condition, and \(\dot{W}(t)\) is a white noise which is the generalized time derivative of a Wiener process \(W(t)\).

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Our first purpose is to derive the finite dimensional reducing of Eq.(1) on a random invariant manifold. It is well known that a random invariant manifold is an invariant set in the state space, carries essential dynamical behavior and characteristics, and provides a geometric structure to describe random dynamics of stochastic partial differential equations. More works about random invariant manifolds of SPDE see [4, 5, 6, 8, 10, 11, 26, 28].

In this paper, borrowing from the invariant random cone to show the asymptotical completion of random invariant manifolds, we reduced Eq.(1) to a stochastic ordinary differential equation on the random invariant manifold.

Our second purpose is to study the approximation of the white noise in Eq.(1). In the pioneer work about approximations, Wong and Zakai [40, 41] studied the approximations for one-dimensional Brownian motion. Their work was later extended to stochastic differential equations of higher dimension ([18, 19, 22, 27, 35, 33, 34, 36]). The Wong-Zakai approximations have also been generalized to stochastic differential equations driven by martingales and semi-martingales ([24, 25, 29, 30]). In addition, Tessitore and Zabczyk [37] investigated Wong-Zakai approximations for a stochastic partial differential equation with the multiplicative noise. Hairer and Pardoux [16] further used a more general Wong-Zakai correction term instead of infinite Ito-Stratonovich correction term to show pathwise convergence. Various works on Wong-Zakai approximations for stochastic partial differential equations see [15, 20, 21, 32, 38, 42] and the references therein.

In the present paper, we use a Wong-Zakai scheme via a smooth colored noise to approximate the white noise in Eq.(1). More precisely, we will investigate the approximate system

\[
\frac{dX^\varepsilon}{dt} = AX^\varepsilon + F(X^\varepsilon) + \Phi^\varepsilon(t), \quad 0 < \varepsilon \ll 1, \tag{2}
\]

where \(\Phi^\varepsilon(t)\) is an approximation of \(\dot{W}(t)\) parameterized by \(\varepsilon\), which will be specified in Section 2. It is known that \(\Phi^\varepsilon(t)\) pathwisely converges to \(\dot{W}(t)\) in the sense of probability as \(\varepsilon\) tends to zero (see [1, 17, 20]). We remark that our approximation \(\Phi^\varepsilon(t)\) of \(\dot{W}(t)\) is smooth, which allows us to demonstrate some pathwise convergence results similar to those in [14, 16] via the convergence argument in [1]. In fact, \(\Phi^\varepsilon(t)\) is everywhere differentiable, which makes that the solutions of the system have higher regularity than the solutions of system driven by the white noise via Brownian motion. We finally can prove that the solution and the finite dimensional reducing of the approximate system (2) converge pathwisely to those of the original system (1) as \(\varepsilon\) tends to zero.

This paper is organized as follows. In the next section, we state some preliminaries including random dynamical system, invariant random cone and the smooth colored noise. Section 3 addresses the existence of random invariant manifolds. In the fourth section, we derive the finite dimensional reducing of the original system (1) and the approximate system (2), respectively. In the last section, we show the convergence between the original system (1) and the approximate system (2).

2. Preliminaries.

2.1. Basic setup. Let \(D\) be a bounded domain with smooth boundary in \(\mathbb{R}^n\), and \(H\) be a separable Hilbert space with the norm \(|\cdot|\) and the inner product \(\langle \cdot, \cdot \rangle\), whose a complete orthonormal basis is \(\{e_1, e_2, \ldots, e_i, \ldots\}\). For Eq.(1), we suppose that the linear operator \(A\) generates a \(C_0\)-semigroup \(S(t) = e^{At}\) on \(H\). In addition, we
denote the spectrum $\sigma(A)$ of this linear operator $A$ in $H$ as $\{\lambda_i, i \in \mathbb{N}\}$, which is assumed to consist of two nonempty parts $\sigma_m$ in $\{z \in \mathbb{C} \mid Rez \geq 0\}$ and $\sigma_\infty$ in $\{z \in \mathbb{C} \mid Rez < 0\}$. Here $\mathbb{C}$ is the complex number set. Further, we assume $\sigma_m$ only has $m$ points (finite points), which just is the reason that we use $m$ as the subscript. The subscript $\infty$ denotes $\sigma_\infty$ has infinite elements.

Decompose $H$ into two spaces $H_m$ and $H_\infty$ corresponding to the spectrum $\sigma_m$ and $\sigma_\infty$. Obviously, $H_m$ is the finite dimensional space, whose dimension is $m$. Denoting $P_m$ as the projection operator from $H$ to $H_m$, then $Q_\infty := I - P_m$ is the operator from $H$ to $H_\infty$, where $I$ is the identity operator. For any $u \in H$, we sometimes use the notation $u_m$ and $u_\infty$ to be as $P_m u \in H_m$ and $Q_\infty u \in H_\infty$, respectively. Also, we denote the operators $A_m$ and $A_\infty$ as the restricting $A$ onto the spaces $H_m$ and $H_\infty$, respectively. Suppose that there are two positive constants $\alpha$ and $\beta$ with the property $0 < \alpha < \beta$ such that (called the dichotomy condition)

$$|e^{tA_m}x| \leq e^{\alpha t} |x|, \quad t \leq 0,$$

$$|e^{tA_\infty}x| \leq e^{-\beta t} |x|, \quad t \geq 0.$$  \(3\)

Furthermore, assume that the nonlinearity $F$ is a continuous nonlinear function with $F(0) = 0$ on $H$ and satisfies (called the Lipschitz condition)

$$|F(u_1) - F(u_2)| \leq L_F |u_1 - u_2|,$$

where the positive constant $L_F$ is called the Lipschitz constant.

2.2. Random dynamical systems. We recall some basic concepts in random dynamical systems (also see [2, 11]). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A flow $\theta$ of mappings $\{\theta_t\}_{t \in \mathbb{R}}$ is defined on the sample space $\Omega$ such that

$$\theta : \mathbb{R} \times \Omega \to \Omega, \quad \theta_0 = id_{\Omega}, \quad \theta_t \circ \theta_s = \theta_{t+s}, \quad (4)$$

for $t, \tau \in \mathbb{R}$. Further, the flow $\{\theta_t\}_{t \in \mathbb{R}}$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F})$-measurable, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra of $\mathbb{R}$. In addition, the measure $\mathbb{P}$ is ergodic with respect to this flow $\{\theta_t\}_{t \in \mathbb{R}}$. Then $\Theta = (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system.

Let $W(t)$ be a two-sided Wiener process taking values in a Hilbert space $H$. Its sample paths are in the space $C_0(\mathbb{R}, H)$ of real continuous functions defined on $\mathbb{R}$, taking zero value at zero time. This set is equipped with the compact open topology. On this set we consider the measurable flow $\theta = \{\theta_t\}_{t \in \mathbb{R}}$, defined by

$$\theta_t \omega = \omega(t + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.$$

The distribution of this process induces a probability measure on $\mathcal{B}(C_0(\mathbb{R}, H))$, which is ergodic with respect to $\theta_t$ (see Arnold [2]). We also consider, instead of the whole $C_0(\mathbb{R}, H)$, a $\{\theta_t\}_{t \in \mathbb{R}}$-invariant subset $\Omega \subset C_0(\mathbb{R}, H)$ of $\mathbb{P}$-measure one and the trace $\sigma$-algebra $\mathcal{F}$ of $\mathcal{B}(C_0(\mathbb{R}, H))$ with respect to $\Omega$. Recall that a set $\Omega$ is called $\{\theta_t\}_{t \in \mathbb{R}}$-invariant if $\theta_t \Omega = \Omega$ for $t \in \mathbb{R}$. On $\mathcal{F}$, we consider the restriction of the probability measure, which still is denoted as $\mathbb{P}$.

A cocycle $\phi$ is a mapping from $\mathbb{R} \times \Omega \times H$ to $H$, which is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{F})$-measurable such that

$$\phi(0, \omega)x = x,$$

$$\phi(t + \tau, \omega)x = \phi(t, \theta_\tau \omega)\phi(\tau, \omega)x$$

for all $t, \tau \in \mathbb{R}^+, \omega \in \Omega$ and $x \in H$. Then $\phi$ together with the metric dynamical system $\Theta$ forms a random dynamical system.
2.3. **Invariant random cone.** In this subsection, we recall some definitions about invariant manifold, asymptotical completion and invariant random cone ([10, 39]).

**Definition 2.1.** A random set $M(\omega)$ is called an invariant set for a random dynamical system $\phi(t, \omega, x)$ if
\[ \phi(t, \omega, M(\omega)) \subset M(\theta_t \omega) \quad \text{for } t \geq 0. \]
Furthermore, if there exists a Lipschitz mapping $h(\cdot, \omega)$ from $H_m$ to $H_\infty$ with $H = H_m \oplus H_\infty$, such that
\[ M(\omega) = \{ \xi + h(\xi, \omega) | \xi \in H_m \}, \]
then $M(\omega)$ is called a Lipschitz invariant manifold.

**Definition 2.2.** An invariant manifold $M(\omega)$ of a random dynamical system $\phi(t, \omega)$ is called almost surely asymptotically complete if for every $x \in H$, there exists $y \in M(\omega)$ such that
\[ |\phi(t, \omega)x - \phi(t, \omega)y| \leq K(\omega)|x - y|e^{-kt}, \quad t \geq 0, \]
for almost all $\omega \in \Omega$, where $k$ is a positive constant and $K(\omega)$ is a positive random variable.

**Definition 2.3.** For a positive random variable $\delta(\omega)$, the random set
\[ C_\delta(\omega) := \{ v | (v, \omega) \in H \times \Omega \text{ and } |Q_\infty v| \leq \delta(\omega)|P_m v| \} \]
is called a random cone.

**Definition 2.4.** Let $C_\delta(\omega)$ be a random cone. For arbitrary $x, y \in H$ and $x - y \in C_\delta(\omega)$, there exists a random variable $\delta(\omega) \in (0, \delta(\omega)]$ almost surely such that
\[ \phi(t, \omega)x - \phi(t, \omega)y \in C_{\delta(\theta_t \omega)}(\theta_t \omega) \quad \text{for almost all } \omega \in \Omega. \]
Then we call that the random dynamical system $\phi(t, \omega)$ has invariant random cone property for the random cone $C_{\delta(\omega)}$.

2.4. **The approximate system.** For the Langevin equation
\[
\begin{aligned}
&dz^\varepsilon = \frac{-1}{\varepsilon} z^\varepsilon dt + \frac{1}{\varepsilon} dW(t), \\
&z^\varepsilon(0) = \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{\frac{s}{\varepsilon}} dW(s),
\end{aligned}
\] (5)
where $\varepsilon$ is a enough small parameter. Then it has the unique solution [10], which is also called an Ornstein-Uhlenbeck process. Take $\varepsilon = \frac{1}{n}$ ($n = 1, 2, \cdots$) to be discrete. Then that $\varepsilon$ tends to 0 actually means that $n$ goes to $\infty$.

Define
\[ \Phi^\varepsilon(t) := \int_0^t z^\varepsilon(\theta_s \omega) ds. \]
Then the following properties hold (see [10, 11]).

(i) There exists a $\{ \theta_t | t \in \mathbb{R} \}$-invariant set $\Omega$ of full measure such that the sample paths $\omega(t)$ satisfy
\[ \lim_{t \to \pm \infty} \frac{\omega(t)}{t} = 0, \quad \omega \in \Omega. \]

(ii) The random variable
\[ z^\varepsilon(\omega) = \int_{-\infty}^{0} \frac{1}{\varepsilon} e^{\frac{s}{\varepsilon}} dW(s) = - \int_{-\infty}^{0} \frac{1}{\varepsilon^2} e^{\frac{s}{\varepsilon^2}} \omega(s) ds, \quad \omega \in \Omega, \]
is well-defined and the unique stationary solution of Eq.(5) is given by
\[ z^\varepsilon(\theta t \omega) = -\int_{-\infty}^{0} \frac{1}{\varepsilon^2} e^{\varepsilon t} \theta \omega(s) ds = \frac{1}{\varepsilon} \omega(t) - \int_{-\infty}^{0} \frac{1}{\varepsilon^2} e^{\varepsilon t} \omega(t + s) ds. \]

Moreover, the mapping \( t \to z^\varepsilon(\theta t \omega) \) is continuous.

(iii) \[ \lim_{t \to \pm \infty} \frac{|z^\varepsilon(\theta t \omega)|}{|t|} = 0 \quad \text{and} \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} z^\varepsilon(\theta t \omega) ds = 0. \]

**Lemma 2.5.** [1] Let \( W(t) \) be a scalar two-sided Brownian motion on \( \mathbb{R} \). For every fixed \( T > 0 \), then \( \Phi^\varepsilon(t) \) almost surely converges \( W(t) \) uniformly in \([0, T]\) as \( \varepsilon \) tends to zero.

Now we investigate the stationary solutions of two linear equations

\[ \begin{align*}
\frac{du}{dt} &= Au + \dot{W}(t), \\
u^*(\omega) &= \int_{-\infty}^{0} e^{-A \tau} dW(\tau) + \int_{0}^{t} e^{-A_m \tau} dW(\tau),
\end{align*} \tag{6} \]

and

\[ \begin{align*}
\frac{dX}{dt} &= AX + \dot{\Phi^\varepsilon}(t), \\
X^*(\omega) &= \int_{-\infty}^{0} e^{-A \tau} d\Phi^\varepsilon(\tau) + \int_{0}^{t} e^{-A_m \tau} d\Phi^\varepsilon(\tau). \tag{7}
\end{align*} \]

**Lemma 2.6.** There exist the stationary solutions \( u^*(\theta t \omega) \) and \( X^*(\theta t \omega) \) of Eq.(6) and Eq.(7), respectively. Furthermore,

\[ \begin{align*}
u^*(\omega) &= \int_{-\infty}^{t} e^{-A \omega(\tau-t)} dW(\tau) + \int_{0}^{t} e^{-A_m \omega(\tau-t)} dW(\tau), \tag{8} \\
X^*(\omega) &= \int_{-\infty}^{t} e^{-A \omega(\tau-t)} d\Phi^\varepsilon(\tau) + \int_{0}^{t} e^{-A_m \omega(\tau-t)} d\Phi^\varepsilon(\tau). \tag{9}
\end{align*} \]

**Proof.** Put \( U(t, \omega; x) \) to be the solution of Eq.(6) with the initial value \( x \). Obviously, \( U(0, \omega; x) = x \). For the linear equation (6), combining the dichotomy condition (3) with the classic stochastic differential equation theory, it implies that

\[ \begin{align*}
U(t, \omega; u^*(\omega)) &= e^{At} u^*(\omega) + \int_{0}^{t} e^{-A(\tau-t)} dW(\tau) \\
&= e^{At} \int_{-\infty}^{0} e^{-A \omega(\tau-t)} dW(\tau) + e^{At} \int_{0}^{t} e^{-A_m \omega(\tau-t)} dW(\tau) + \int_{0}^{t} e^{-A(\tau-t)} dW(\tau) \\
&= \int_{-\infty}^{0} e^{-A \omega(\tau-t)} dW(\tau) + \int_{0}^{t} e^{-A_m \omega(\tau-t)} dW(\tau) + \int_{0}^{t} e^{-A(\tau-t)} dW(\tau) \\
&= \int_{-\infty}^{t} e^{-A \omega(\tau-t)} dW(\tau) + \int_{0}^{t} e^{-A_m \omega(\tau-t)} dW(\tau) \\
&= u^*(\theta t \omega),
\end{align*} \]

which further satisfies \( U(0, \omega; u^*(\omega)) = \int_{-\infty}^{0} e^{-A \omega(\tau-t)} dW(\tau) + \int_{0}^{t} e^{-A_m \omega(\tau-t)} dW(\tau) = u^*(\omega) \) and \( \text{E} u^*(\theta t \omega) = \text{E} u^*(\omega) \). Therefore, the process \( u^*(\theta t \omega) \) is the stationary solution of Eq.(6).

Using the same method, it can be shown that \( X^*(\theta t \omega) \) is also the stationary solution of Eq.(7).
Put \( v = u - u^*(\theta_t \omega) \) and \( Y^\varepsilon = X^\varepsilon - X^*(\theta_t \omega) \). Then Eq.(1) and Eq.(2) can be written, respectively, as

\[
\frac{dv}{dt} = Av + G_1(\theta_t \omega, v),
\]

\[
\frac{dY^\varepsilon}{dt} = AY^\varepsilon + G_2(\theta_t \omega, Y^\varepsilon),
\]

where \( G_1(\theta_t \omega, x) = F(x + u^*(\theta_t \omega)) \) and \( G_2(\theta_t \omega, x) = F(x + X^*(\theta_t \omega)) \) and they have the same Lipschitz constant \( L_F \) as \( F \).

Supplement Eq.(10) and Eq.(11) the initial conditions, respectively, \( v(0) = v_0 \) and \( Y^\varepsilon(0) = Y^\varepsilon_0 \).

Under the basic setup in the above subsection 2.1, by the classical evolutionary equation theory, the system (10) and (11) have the unique global solutions for every \( \omega \in \Omega \). Hence the solution mappings \( (t, \omega, v_0) \rightarrow \phi(t, \omega; v_0) \) and \( (t, \omega, Y^\varepsilon_0) \rightarrow \phi^\varepsilon(t, \omega; Y^\varepsilon_0) \) generate continuous random dynamical systems \( \phi \) and \( \phi^\varepsilon \), which further are \((B(\mathbb{R}) \otimes F \otimes B(H), B(H))\)-measurable.

We introduce the random transformations

\[
T(\omega, x) = x - u^*(\omega), \quad \text{with} \quad T^{-1}(\omega, x) = x + u^*(\omega),
\]

\[
T^\varepsilon(\omega, x) = x - X^*(\omega), \quad \text{with} \quad T^{\varepsilon, -1}(\omega, x) = x + X^*(\omega).
\]

Then we have the following results.

**Lemma 2.7.** Suppose that \( v \) and \( Y^\varepsilon \) are the solutions of Eq.(10) and Eq.(11), respectively. Then

\[
u(t, \omega; x) := T^{-1}(\theta_t \omega, v(t, \omega; T(\omega, x)))
\]

and

\[
X^\varepsilon(t, \omega; x) := T^{\varepsilon, -1}(\theta_t \omega, Y^\varepsilon(t, \omega; T^\varepsilon(\omega, x)))
\]

are solutions for Eq.(1) and Eq.(2), respectively.

3. **Random invariant manifolds.** In this section, we briefly use the Lyapunov-Perron method to derive the random invariant manifolds for Eq.(10) and Eq.(11) (more details see [11]).

Project Eq.(10) onto \( H_m \) and \( H_\infty \), respectively, as

\[
\dot{v}_m = A_m v_m + g_m(\theta_t \omega, v_m + v_\infty),
\]

\[
\dot{v}_\infty = A_\infty v_\infty + g_\infty(\theta_t \omega, v_m + v_\infty),
\]

where

\[
g_m(\theta_t \omega, v_m + v_\infty) = P_m G_1(\theta_t \omega, v_m + v_\infty),
\]

\[
g_\infty(\theta_t \omega, v_m + v_\infty) = Q_\infty G_1(\theta_t \omega, v_m + v_\infty).
\]

Define the Banach space

\[
C_\eta = \{ v \mid v \text{ is a continuous function from } (-\infty, 0] \text{ to } H \}
\]

\[
\sup_{t \in (-\infty, 0]} e^{-\eta t} |v(t)| < \infty,
\]

\[
\sup_{t \in (-\infty, 0]} e^{-\eta t} |v(t)| < \infty,
\]
with norm
\[ |v|_{C^+_{\eta}} = \sup_{t \in (-\infty, 0]} e^{-\eta t} |v(t)|, \]
where the parameter \( \eta \) is in \((-\beta, -\alpha)\).

Define the nonlinear operators \( \mathcal{I} \) and \( \mathcal{J} \) on \( C^+_{\eta} \) as

\[
\mathcal{I}(v, \xi)(t, \omega) = e^{A_m t} \xi + \int_0^t e^{A_m (t-\tau)} g_m(\theta_{\tau t}, v_m + v_{\infty}) d\tau
+ \int_{-\infty}^t e^{A_{\infty} (t-\tau)} g_{\infty}(\theta_{\tau t}, v_m + v_{\infty}) d\tau
\]

and

\[
\mathcal{J}(Y^\varepsilon, \xi)(t, \omega) = e^{A_m t} \xi + \int_0^t e^{A_m (t-\tau)} g_m(\theta_{\tau t}, Y^\varepsilon_m + Y^\varepsilon_{\infty}) d\tau
+ \int_{-\infty}^t e^{A_{\infty} (t-\tau)} g_{\infty}(\theta_{\tau t}, Y^\varepsilon_m + Y^\varepsilon_{\infty}) d\tau,
\]

where \( \xi = P_m v_0 \in H_m, g_m = P_m G_2, g_\infty = Q_\infty G_2. \) Then for any given \( \xi \in H_m \) and each \( v, \bar{v} \in C^+_{\eta} \), it has

\[
|\mathcal{I}(v, \xi) - \mathcal{I}(\bar{v}, \xi)|_{C^+_{\eta}}
\leq \sup_{t \leq 0} e^{-\eta t} \left\{ \int_0^t e^{A_m (t-\tau)} |g_m(v) - g_m(\bar{v})| d\tau \right\}
+ \left\{ \int_{-\infty}^t e^{A_{\infty} (t-\tau)} |g_{\infty}(v) - g_{\infty}(\bar{v})| d\tau \right\}
\]

\[
\leq \sup_{t \leq 0} e^{-\eta t} \left\{ \int_0^t e^{A_m (t-\tau)} |g_m(v) - g_m(\bar{v})| d\tau \right\}
+ \left\{ \int_{-\infty}^t e^{A_{\infty} (t-\tau)} |g_{\infty}(v) - g_{\infty}(\bar{v})| d\tau \right\}
\]

\[
\leq L_F \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) |v(\cdot) - \bar{v}(\cdot)|_{C^+_{\eta}}.
\]

If

\[
L_F \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) < 1,
\]

then by the fixed point argument, \( \mathcal{I}(v) \) has a unique solution \( v(t, \omega; \xi) \in C^+_{\eta}, \) which is the solution of Eq.(10). Let \( h^v(\xi, \omega) = Q_\infty v(0, \omega; \xi), \) that is,

\[
h^v(\xi, \omega) = \int_0^t e^{-A_{\infty} \tau} g(\theta_{\tau \omega}, v(\tau, \omega; \xi)) d\tau.
\]

In a similar way, we have

\[
h^{Y^\varepsilon}(\xi, \omega) = \int_{-\infty}^0 e^{-A_{\infty} \tau} g(\theta_{\tau \omega}, Y^\varepsilon(\tau, \omega; \xi)) d\tau,
\]

where \( Y^\varepsilon(\tau, \omega; \xi) \in C^+_{\eta}. \)

**Lemma 3.1.** Let the assumptions about linear operator \( A \) and nonlinearity \( F \) in subsection 2.1 and the spectrum gap condition (18) hold. Then there exist Lipschitz continuous random invariant manifolds \( M^v(\omega) \) and \( M^{Y^\varepsilon}(\omega) \) for the random systems (10) and (11), represented further as

\[
M^v(\omega) = \{(\xi, h^v(\xi, \omega)) \mid \xi \in H_m\} \quad \text{and} \quad M^{Y^\varepsilon}(\omega) = \{(\xi, h^{Y^\varepsilon}(\xi, \omega)) \mid \xi \in H_m\}.
\]
where the Lipschitz functions \( h^u(\xi, \omega) \) and \( h^{Y^r}(\xi, \omega) \) are defined as (19) and (20).

Then by the transformations \( T \) and \( T^\tau \), defined as (12) and (13), it derives the invariant manifolds of Eq.(1) and Eq.(2) as follows.

**Lemma 3.2.** Let the assumptions about linear operator \( A \) and nonlinearity \( F \) in subsection 2.1 and the spectrum gap condition (18) hold. Then Eq.(1) and Eq.(2) have the Lipschitz invariant manifolds \( M^u(\omega) = T^{-1}(\omega, M^u(\omega)) \) and \( M^{X^r}(\omega) = T^{\tau^-1}(\omega, M^{X^r}(\omega)) \), respectively. Moreover,

\[
M^u(\omega) = \{ (\xi, h^u(\xi, \omega)) \mid \xi \in H_m \}
\]

\[
M^{X^r}(\omega) = \{ (\xi, h^{X^r}(\xi, \omega)) \mid \xi \in H_m \}
\]

\[
:= \{ (\xi, h^u(\xi, \omega) - \int_0^\infty e^{-A_{m,\tau} dW(\tau)} , \omega) + \int_{-\infty}^0 e^{-A_{m,\tau} dW(\tau)} | \xi \in H_m \},
\]

\[
(21)
\]

**4. Finite dimensional reducing.** In this section, we first use the invariant random cone to prove that the invariant random manifold is almost surely asymptotically complete, and then reduce the dynamical behaviors of Eq.(10) and Eq.(11) into random ordinary differential equations on the random invariant manifolds. Further we derive the finite dimensional reducing of Eq.(1) and Eq.(2).

**Lemma 4.1.** Let the assumptions about linear operator \( A \) and nonlinearity \( F \) in subsection 2.1 hold. And suppose the Lipschitz constant \( L_F \) is small enough. Then the random dynamical system \( \phi(t, \omega) \) generated by Eq.(10) has the invariant random cone property for the random cone \( C_\delta(\omega) \) with a deterministic positive parameter \( \delta \). In addition, if there exists \( t_0 > 0 \) such that \( x, y \in H \) and \( \phi(t_0, \omega)x - \phi(t_0, \omega)y \notin C_\delta(\theta_{t_0} \omega) \), then

\[
|\phi(t, \omega)x - \phi(t, \omega)y| \leq K(\omega)|x - y|e^{-kt} \quad \text{for any } \quad t \in [0, t_0],
\]

where \( K(\omega) \) is a positive random variable and \( k = \beta - L_F - \delta^{-5} L_F > 0 \).

**Remark 1.** Notice that the smallness of Lipschitz constant \( L_F \) in Lemma 4.1 is to ensure that the condition (26) holds.

**Proof.** Let \( v \) and \( \bar{v} \) be two solutions of Eq.(10), and put \( v_m := v_m - \bar{v}_m, v_\infty := v_\infty - \bar{v}_\infty \). Then

\[
\dot{v}_m = A_m v_m + g_m(\theta t, \omega, v_m + v_\infty) - g_m(\theta t, \omega, \bar{v}_m + \bar{v}_\infty),
\]

\[
\dot{v}_\infty = A_\infty v_\infty + g_\infty(\theta t, \omega, v_m + v_\infty) - g_\infty(\theta t, \omega, \bar{v}_m + \bar{v}_\infty),
\]

which implies that

\[
\frac{1}{2} \frac{d}{dt} |v_m|^2 \geq -\alpha |v_m|^2 - L_F |v_m|^2 - L_F |v_m| \cdot |v_\infty| \quad (24)
\]

and

\[
\frac{1}{2} \frac{d}{dt} |v_\infty|^2 \leq -\beta |v_\infty|^2 + L_F |v_\infty|^2 + L_F |v_m| \cdot |v_\infty|. \quad (25)
\]
Then it follows from (24) and (25) that
\[
\frac{1}{2} \frac{d}{dt} (|v_\infty|^2 - \delta^2 |v_m|^2) \leq -\beta |v_\infty|^2 + L_F |v_\infty|^2 + L_F |v_m| \cdot |v_\infty| + \alpha \delta^2 |v_m|^2
\]
+ $L_F \delta^2 |v_m|^2 + L_F \delta^2 |v_m| \cdot |v_\infty|$.

Also notice that if $(v_m, v_\infty)$ is in the boundary of the random cone $C_\delta(\omega)$, then $|v_\infty| = \delta |v_m|$. Therefore
\[
\frac{1}{2} \frac{d}{dt} (|v_\infty|^2 - \delta^2 |v_m|^2) \leq (\alpha - \beta + 2L_F + \delta L_F + \delta^{-1} L_F)|v_\infty|^2.
\]

By the assumption $L_F$ is small enough such that
\[
\alpha - \beta + 2L_F + \delta L_F + \delta^{-1} L_F < 0,
\]
then $|v_\infty|^2 - \delta^2 |v_m|^2$ is decreasing on the boundary of the random cone $C_\delta(\omega)$. Consequently, it is concluded that $\phi(t, \omega)x - \phi(t, \omega)y \in C_\delta(\theta_t \omega)$ whenever $x - y \in C_\delta(\omega)$. The first result in Lemma 4.1 holds.

In the next, we prove the second result in Lemma 4.1.

If there exists a positive time $t_0$ such that $\phi(t_0, \omega)x - \phi(t_0, \omega)y \notin C_\delta(\theta_{t_0} \omega)$, the invariant random cone property infers that
\[
\phi(t, \omega)x - \phi(t, \omega)y \notin C_\delta(\theta_t \omega), \quad 0 \leq t \leq t_0,
\]
that is
\[
|v_\infty(t)| > \delta |v_m(t)|, \quad 0 \leq t \leq t_0,
\]
which implies from (25) that
\[
\frac{1}{2} \frac{d}{dt} |v_\infty|^2 \leq -(\beta - L_F - \delta^{-1} L_F)|v_\infty|^2, \quad 0 \leq t \leq t_0.
\]

Then
\[
|v_m(t)|^2 \leq \frac{1}{\delta^2} |v_\infty(t)|^2 \leq \frac{1}{\delta^2} e^{-2kt} |v_\infty(0)|^2, \quad 0 \leq t \leq t_0.
\]

Therefore there exists a positive random variable $K(\omega)$ such that
\[
|\phi(t, \omega)x - \phi(t, \omega)y| \leq |v_m(t)| + |v_\infty(t)| \leq K(\omega)|x - y| e^{-kt}, \quad 0 \leq t \leq t_0,
\]
which completes the proof. $\Box$

For the purpose of the finite dimensional reducing, we will use the backward solvability argument to study Eq. (10) on the invariant manifold $M^v(\omega)$.

For arbitrary given end time $R > 0$ and $t \in [0, R]$, we investigate Eq. (10) in the following form
\[
\dot{v}_m = A_m v_m + g_m(\theta_t \omega, v_m + v_\infty), \quad v_m(R) = \zeta \in H_m,
\]
\[
\dot{v}_\infty = A_\infty v_\infty + g_\infty(\theta_t \omega, v_m + v_\infty), \quad v_\infty(0) = h^v(v_m(0)),
\]
where $h^v$ is defined as (19). Rewrite Eq.(27) and Eq.(28) in the integral form as
\[
v_m(t) = e^{A_m(t-R)} \zeta + \int_R^t e^{A_m(t-\tau)} g_m(\theta_\tau \omega, v(\tau)) d\tau, \quad t \in [0, R],
\]
\[
v_\infty(t) = e^{A_\infty t} h^v(v_m(0)) + \int_0^t e^{A_\infty(t-\tau)} g_\infty(\theta_\tau \omega, v(\tau)) d\tau, \quad t \in [0, R].
\]
Remark 2. Similarly, Eq. (11) can be written as

\[
Y_m^\varepsilon(t) = e^{A_m(t-R)}Y_m^\varepsilon(R) + \int_R^t e^{A_m(t-\tau)}g_m(\theta_\tau, Y_\tau^\varepsilon) d\tau, \quad t \in [0, R],
\]

(31)

\[
Y_\infty^\varepsilon(t) = e^{A_m(t)}h^Y(\varepsilon Y_\infty^\varepsilon(0)) + \int_0^t e^{A_m(\tau)}g_\infty(\theta_\tau, Y_\tau^\varepsilon) d\tau, \quad t \in [0, R],
\]

(32)

where \( h^Y \) is defined as (20).

Lemma 4.2. Let the assumptions about linear operator \( A \) and nonlinearity \( F \) in subsection 2.1 hold. Then for any end time \( R > 0 \), there exists a unique solution \((v_m(\cdot), v_\infty(\cdot)) \in C([0, R]; H_m \times H_\infty)\) for Eq. (29) and Eq. (30). Moreover for any \( t \geq 0 \), \((v_m(t, \theta_{-\tau} \omega), v_\infty(t, \theta_{-\tau} \omega)) \in M^v(\omega)\) for almost all \( \omega \in \Omega \).

Proof. Under the assumptions of \( A \) and \( F \), using the contraction argument, it is easy to derive the existence and uniqueness in small time intervals, which is then extended to the any time interval (also see [10]). \( \square \)

Theorem 4.3. Let the assumptions about linear operator \( A \) and nonlinearity \( F \) in subsection 2.1 hold. Suppose the Lipschitz constant \( L_F \) is small enough. Then Eq. (10) has an \( m \)-dimensional invariant manifold \( M^v(\omega) \). Furthermore, for any solution \( v(t, \theta_{-\tau} \omega) \) of Eq. (10), there exists a positive random variable \( K(\omega) \), a positive constant \( k \), and an orbit \( \mathcal{V}(t, \theta_{-\tau} \omega) \) on the invariant manifold \( M^v(\omega) \) such that

\[
|v(t, \theta_{-\tau} \omega) - \mathcal{V}(t, \theta_{-\tau} \omega)| \leq K(\omega)|v(0) - \mathcal{V}(0)|e^{-kt} \quad \text{for any} \quad t \geq 0.
\]

(33)

Proof. By Lemma 3.1, it only needs to prove (33).

For any solution \( v(t, \theta_{-\tau} \omega) = (v_m(t, \theta_{-\tau} \omega), v_\infty(t, \theta_{-\tau} \omega)) \) of Eq. (10), it follows from Lemma 4.2 that for any end time \( R > 0 \), there exists a solution \( \tilde{v}(t, \theta_{-\tau} \omega) = (\tilde{v}_m(t, \theta_{-\tau} \omega), \tilde{v}_\infty(t, \theta_{-\tau} \omega)) \in C([0, R]; H_m \times H_\infty) \) of Eq. (10) on \( M^v(\omega) \) such that \( \tilde{v}_m(R, \theta_{-\tau} \omega) = v_m(R, \theta_{-\tau} \omega) \). Obviously, \( \tilde{v}(t, \theta_{-\tau} \omega) \) depends on the end time \( R \).

Hence we use the notation \( \tilde{v}(t, \theta_{-\tau} \omega; R) = (\tilde{v}_m(t, \theta_{-\tau} \omega; R), \tilde{v}_\infty(t, \theta_{-\tau} \omega; R)) \) to denote \( \tilde{v}(t, \theta_{-\tau} \omega) = (\tilde{v}_m(t, \theta_{-\tau} \omega), \tilde{v}_\infty(t, \theta_{-\tau} \omega)) \). Taking \( t = 0 \), obviously, \( \tilde{v}_m(0, \omega; R) = \tilde{v}_m(0, \omega) \) and \( \tilde{v}_\infty(0, \omega; R) = \tilde{v}_\infty(0, \omega) \).

Since the solution \( (\tilde{v}_m(t, \theta_{-\tau} \omega; R), \tilde{v}_\infty(t, \theta_{-\tau} \omega; R)) \) is on \( M^v(\omega) \), then it infers

\[
|\tilde{v}_\infty(0, \omega; R)| \leq \int_0^\infty e^{-\beta s}\|g_\infty(\theta_{-s} \omega, v(-s))\| ds
\]

\[
\leq L_F \int_0^\infty e^{-(\beta + \eta)s + \eta s}|v(-s) + u^*(\theta_{-s} \omega)| ds
\]

\[
:= A_{L_F}(\omega)
\]

\[
\leq L_F(|v(\cdot)|_{C^\eta} + |u^*(\theta_{\cdot} \omega)|_{C^\eta}) \int_0^\infty e^{-(\beta + \eta)s} ds,
\]

(34)

where \( A_{L_F}(\omega) \) is a finite tempered random variable due to \( \eta \in (-\beta, -\alpha) \).

Noticing that \( \tilde{v}_m(R, \theta_{-R} \omega) = v_m(R, \theta_{-R} \omega) \), it then deduces that \( \tilde{v}_m(R, \theta_{-R} \omega) - v_m(R, \theta_{-R} \omega) \notin C_\beta(\omega) \), which implies from the invariant random cone property of Eq. (10) (Lemma 4.1) that

\[
\tilde{v}_m(t, \theta_{-t} \omega) - v_m(t, \theta_{-t} \omega) \notin C_\beta(\omega), \quad 0 \leq t \leq R.
\]
Particularly, \( \bar{v}_m(0, \omega) - v_m(0, \omega) \notin C_{\delta}(\omega) \), which immediately implies from (34) that
\[
|\bar{v}_m(0, \omega; R)| \leq |\bar{v}_m(0, \omega; R) - v_m(0, \omega)| + |v_m(0, \omega)| \\
< \frac{1}{\delta} |\bar{v}_\infty(0, \omega; R) - v_\infty(0, \omega)| + |v_m(0, \omega)| \\
\leq \frac{1}{\delta} (L_F(\omega) + |v_\infty(0, \omega)|) + |v_m(0, \omega)|.
\]

Then the random set \( \Gamma(\omega) := \{ \bar{v}_m(0, \omega; R) \mid R \geq 0 \} \) is bounded in \( H_m \). For almost all \( \omega \in \Omega \), choose a sequence \( \{ R_n \}_{n \in \mathbb{N}} \) tending to infinity as \( n \to \infty \), such that \( \lim_{n \to \infty} \bar{v}_m(0, \omega; R_n) = V_m(\omega) \), which is \( F \)-measurable.

Let \( \mathcal{V}(t, \theta_{-t} \omega) = (\mathcal{V}_n(t, \theta_{-t} \omega), \mathcal{V}_\infty(t, \theta_{-t} \omega)) \) be a solution of Eq.(10) with the initial value \( \mathcal{V}(0, \omega) = (\mathcal{V}_n(\omega), h^v(\mathcal{V}_n(\omega), \omega)) \), where \( h^v \) is defined as (19). Then it follows from Lemma 4.2 and Lemma 4.1 that \( \mathcal{V}(t, \theta_{-t} \omega) \in M^v(\omega) \) and \( v(t, \theta_{-t} \omega) - \mathcal{V}(t, \theta_{-t} \omega) \notin C_{\delta}(\omega), 0 \leq t < \infty \), which implies again from Lemma 4.1 that (33) holds. The proof is completed.

**Remark 3.** Theorem 4.3 implies that Eq.(10) is reduced to the following random ordinary differential equation on \( H_m \)
\[
\dot{v}_m = A_m v_m + g_m(\theta_t \omega, v_m + h^v(v_m)), \tag{35}
\]
where \( h^v \) is defined as (19).

**Remark 4.** Eq.(1) is reduced to the following finite dimensional stochastic differential system on \( H_m \)
\[
\dot{u}_m = A_m u_m + F_m(u_m + h^u(u_m)) + \dot{W}(t), \tag{36}
\]
where \( F_m = P_m F \) and \( h^u \) is given by (21).

Similarly, for Eq.(2), we have the following results.

**Theorem 4.4.** Let the assumptions about linear operator \( A \) and nonlinearity \( F \) in subsection 2.1 hold. And Suppose the Lipschitz constant \( L_F \) is small enough. Then Eq.(11) has an \( n \)-dimensional invariant manifold \( M^{Y^\varepsilon}(\omega) \). Furthermore, for any solution \( Y^\varepsilon(t, \theta_{-t} \omega) \) of Eq.(11), there exists a positive random variable \( K(\omega) \), a positive constant \( k \), and an orbit \( \mathcal{Y}(t, \theta_{-t} \omega) \) on the invariant manifold \( M^{Y^\varepsilon}(\omega) \) such that
\[
|Y^\varepsilon(t, \theta_{-t} \omega) - Y(t, \theta_{-t} \omega)| \leq K(\omega)|Y^\varepsilon(0) - Y(0)|e^{-kt} \quad \text{for any} \quad t \geq 0. \tag{37}
\]

**Remark 5.** Theorem 4.4 implies that Eq.(11) is reduced to the following random ordinary differential equation on \( H_m \)
\[
\dot{Y}_m^\varepsilon = A_m Y_m^\varepsilon + \bar{g}_m(\theta_t \omega, Y_m^\varepsilon + h^{Y^\varepsilon}(Y_m^\varepsilon)), \tag{38}
\]
where \( h^{Y^\varepsilon} \) is defined as (20).

**Remark 6.** Eq.(2) is reduced to the following finite dimensional stochastic differential system on \( H_m \)
\[
\dot{X}_m^\varepsilon = A_m X_m^\varepsilon + F_m(X_m^\varepsilon + h^{X^\varepsilon}(X_m^\varepsilon)) + \Phi^\varepsilon(t), \quad 0 < \varepsilon \ll 1, \tag{39}
\]
where \( F_m = P_m F \) and \( h^{X^\varepsilon} \) is given by (21).
5. **Approximations.** In this section, we prove the second main result, that is, when the smooth colored noise tends to the white noise, the solution and the finite dimensional reducing of the approximate system (2) converge pathwisely to those of the original system (1). To the end, we need some preliminaries.

**Lemma 5.1.** [21] Let \( W(t) \) be a scalar two-sided Brownian motion, then for any \( \epsilon > 0, \sigma > 0 \), there exist \( T > 0 \) and \( \varepsilon_1 > 0 \) such that \( e^{-\sigma t}|W(t) - \Phi^\varepsilon(t)| \leq \epsilon \), uniformly for both \( t \geq T \) and \( \varepsilon \leq \varepsilon_1 \).

**Lemma 5.2.** Let \( u^*(\theta_1 \omega) \) and \( X^*(\theta_2 \omega) \) be the stationary solutions of (6) and (7), respectively. Then for any \( \eta \in (-\beta, -\alpha) \), it has \( |u^*(\theta_1 \omega) - X^*(\theta_2 \omega)|_{C^\eta} \to 0 \) as \( \varepsilon \to 0 \), a.s.

**Proof.** It follows from Eq.(8) and Eq.(9) that

\[
|u^*(\theta_1 \omega) - X^*(\theta_2 \omega)|_{C^\eta} = \sup_{t \leq 0} e^{-\eta t} \int_{-\infty}^t e^{-A\omega(\tau-t)}d(W(\tau) - \Phi^\varepsilon(\tau)) + \int_t^{\infty} e^{-A\omega(\tau-t)}d(W(\tau) - \Phi^\varepsilon(\tau))
\]

\[
\leq \sup_{t \leq 0} e^{-\eta t} \int_{-\infty}^t e^{-A\omega(\tau-t)}d(W(\tau) - \Phi^\varepsilon(\tau)) + \int_{-\infty}^{\infty} e^{-A\omega(\tau-t)}d(W(\tau) - \Phi^\varepsilon(\tau))
\]

\[
:= I_1 + I_2.
\]

Firstly consider \( I_1 \). Since that

\[
\int_{-\infty}^t e^{-A\omega(\tau-t)}d(W(\tau) - \Phi^\varepsilon(\tau)) = (W(t) - \Phi^\varepsilon(t))
\]

\[
+ \int_{-\infty}^t (W(\tau) - \Phi^\varepsilon(\tau))e^{-A\omega(\tau-t)}A\omega d\tau,
\]

then

\[
I_1 \leq \sup_{t \leq 0} e^{-\eta t}|W(t) - \Phi^\varepsilon(t)|
\]

\[
+ \sup_{t \leq 0} e^{-\eta t} \int_{-\infty}^t e^{-A\omega(\tau-t)}|A\omega||W(\tau) - \Phi^\varepsilon(\tau)|d\tau
\]

\[
\leq \sup_{t \leq 0} e^{-\eta t}|W(t) - \Phi^\varepsilon(t)|
\]

\[
+ \sup_{t \leq 0} e^{-\eta t} \int_{-\infty}^t e^{\beta(\tau-t)}|A\omega||W(\tau) - \Phi^\varepsilon(\tau)|d\tau
\]

\[
:= I_{11} + I_{12}.
\]

It is easy to see that \( I_{11} \to 0 \) as \( \varepsilon \to 0 \). Now turn to investigate \( I_{12} \).

By Lemma 5.1, for any \( \epsilon > 0 \), there exist \( T > 0 \) and \( \sigma \in (\eta, 0) \) such that \( e^{-\sigma t}|W(t) - \Phi^\varepsilon(t)| \leq \epsilon \) uniformly for both \( t \leq -T \) and \( \varepsilon \leq \varepsilon_1 \). For fixed \( T \), by Lemma 2.5, there exists \( \varepsilon_0 > 0 \) such that \( |W(t) - \Phi^\varepsilon(t)| \leq \epsilon \) for \( \varepsilon \leq \varepsilon_0 \) and \( t \in [-T, 0] \).
If $t \leq -T$, then
\[ e^{-\eta t} \int_{-\infty}^{t} e^{\beta (\tau - t)} |A_{\infty}||W(\tau) - \Phi^{\xi}(\tau)|d\tau \leq |A_{\infty}|e^{-(\eta + \beta) t} \int_{-\infty}^{t} e^{(\beta + \sigma) \tau} d\tau \leq \frac{|A_{\infty}|e^{(\sigma - \eta) t}}{\beta + \sigma}. \]

If $t \geq -T$, then
\[ e^{-\eta t} \int_{-\infty}^{t} e^{\beta (\tau - t)} |A_{\infty}||W(\tau) - \Phi^{\xi}(\tau)|d\tau \leq |A_{\infty}|e^{-(\eta + \beta) t} \int_{-T}^{t} e^{(\beta + \sigma) \tau} d\tau + |A_{\infty}|e^{(\sigma - \eta) t} \int_{-T}^{t} e^{\beta \tau} d\tau \leq \frac{|A_{\infty}|e^{(\sigma - \eta) t}}{\beta + \sigma} + \frac{|A_{\infty}|e^{-\eta t}}{\beta}. \]

Consequently, it has
\[ I_{12} \leq \frac{|A_{\infty}|\epsilon}{\beta + \sigma} + \frac{|A_{\infty}|\epsilon}{\beta}, \]
which means that $I_{12} \to 0$ as $\epsilon \to 0$. Then $I_{1} \to 0$ as $\epsilon \to 0$.

Similarly, we can prove that $I_{2} \to 0$ as $\epsilon \to 0$. This completes the proof of Lemma 5.2.

**Lemma 5.3.** Let the assumptions about linear operator $A$ and nonlinearity $F$ in subsection 2.1 and the spectrum gap condition (18) hold. Then for any $\eta \in (-\beta, -\alpha)$, the solution $u(t)$ of Eq.(1) almost surely converges to the solution $X^{\xi}(t)$ of Eq.(2) in $C_{\bar{\eta}}$ as $\epsilon \to 0$, that is, $|u(\cdot) - X^{\xi}(\cdot)|_{C_{\bar{\eta}}} \to 0$ as $\epsilon \to 0$, a.s.

**Proof.** It follows from Eq.(16) and Eq.(17) that
\[
\begin{align*}
|v(\cdot) - Y^{\xi}(\cdot)|_{C_{\bar{\eta}}} &= \sup_{t \leq 0} e^{-\eta t}|v(t) - Y^{\xi}(t)| \\
&\leq \sup_{t \leq 0} e^{-\eta t} |L_{F} \int_{0}^{t} e^{\alpha(t-\tau)}(v(\tau) - Y^{\xi}(\tau))d\tau| \\
&\quad + \sup_{t \leq 0} e^{-\eta t} |L_{F} \int_{0}^{t} e^{\alpha(t-\tau)}(u(\cdot, \omega) - X^{\xi}(\cdot, \omega))d\tau| \\
&\quad + \sup_{t \leq 0} e^{-\eta t} |L_{F} \int_{-\infty}^{t} e^{\beta(t-\tau)}(v(\tau) - Y^{\xi}(\tau))d\tau| \\
&\quad + \sup_{t \leq 0} e^{-\eta t} |L_{F} \int_{-\infty}^{t} e^{\beta(t-\tau)}(u(\cdot, \omega) - X^{\xi}(\cdot, \omega))d\tau| \\
&\leq L_{F} \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) |v(\cdot) - Y^{\xi}(\cdot)|_{C_{\bar{\eta}}} \\
&\quad + L_{F} \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) |u(\cdot, \omega) - X^{\xi}(\cdot, \omega)|_{C_{\bar{\eta}}},
\end{align*}
\]
which implies that
\[
1 - L_F \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) \leq L_F \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right)\]

Then
\[
|v(\cdot) - Y^\varepsilon(\cdot)|_C^n \leq \frac{L_F \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right)}{1 - L_F \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right)} |u^*(\theta, \omega) - X^*(\theta, \omega)|_C^n.
\]

It immediately follows from Lemma 5.2 that \( |v(\cdot) - Y^\varepsilon(\cdot)|_C^n \to 0 \) as \( \varepsilon \to 0 \). Furthermore, using (12) and (13), Lemma 5.3 holds.

**Theorem 5.4.** (Approximation of the finite dimensional reducing) Let the assumptions about linear operator \( A \) and nonlinearity \( F \) in subsection 2.1 and the spectrum gap condition (18) hold. Then the finite dimensional reducing of Eq. (1) almost surely converges to that of Eq. (2) in \( H \) as \( \varepsilon \to 0 \), that is, for any \( t \geq 0 \),

\[
|u_m(t, \theta_\varepsilon \omega) - X_m^\varepsilon(t, \theta_\varepsilon \omega)| \to 0, \quad \text{as} \quad \varepsilon \to 0, \quad \text{a.s.}
\]

**Proof.** For any end time \( R > 0 \), assume that \( v_m(R) = Y_m^\varepsilon(R) \), it follows from Eq. (29) and Eq. (31) that

\[
|v_m(t, \theta_\varepsilon \omega) - Y_m^\varepsilon(t, \theta_\varepsilon \omega)|
\leq |e^{A_m(t-R)}(v_m(R) - Y_m^\varepsilon(R))| + \left| \int_R^t e^{A_m(t-\tau)}(g_m(\theta_\varepsilon \omega, v(\tau)) - \bar{g}_m(\theta_\varepsilon \omega, Y^\varepsilon(\tau)))d\tau \right|
\leq \int_R^t e^{\alpha(t-\tau)} L_F |v(\tau) - Y^\varepsilon(\tau) + u^*(\theta_\varepsilon \omega) - X^*(\theta_\varepsilon \omega)|d\tau
\leq L_F \int_{-t}^R e^{\alpha(t+s)}|v(-s) - Y^\varepsilon(-s)|d(-s)
+ L_F \int_{-t}^R e^{\alpha(t+s)}|u^*(\theta_\varepsilon \omega) - X^*(\theta_\varepsilon \omega)|d(-s)
\leq L_F |v(\cdot) - Y^\varepsilon(\cdot)|_C^n \int_{-t}^R e^{\alpha(t+s)-\eta s}ds
+ L_F |u^*(\theta, \omega) - X^*(\theta, \omega)|_C^n \int_{-t}^R e^{\alpha(t+s)+\eta s}ds
:= I_{31} + I_{32}.
\]

Notice that for any \( \eta \in (-\beta, -\alpha) \) and \( t \geq 0 \), the integral \( \int_{-R}^t e^{\alpha(t+s)-\eta s}ds \) is finite whenever \( R \in [0, \infty) \). It follows from Lemma 5.2 and (42) that \( I_{31} \to 0 \) and \( I_{32} \to 0 \) as \( \varepsilon \to 0 \).

Consequently,

\[
|v_m(t, \theta_\varepsilon \omega) - Y_m^\varepsilon(t, \theta_\varepsilon \omega)| \to 0, \quad \text{as} \quad \varepsilon \to 0, \quad \text{a.s.}
\]

which implies from (12) and (13) that Theorem 5.4 holds. This completes the proof. □
Theorem 5.5. (Approximation of the solution) Let the assumptions about linear operator $A$ and nonlinearity $F$ in subsection 2.1 and the spectrum gap condition (18) hold. Then the solution $u(t, \theta - \omega)$ of Eq. (1) almost surely converges to the solution $X^\varepsilon(t, \theta - \omega)$ of Eq. (2) in $H$ as $\varepsilon \to 0$, that is, for any $t \geq 0$,

$$|u(t, \theta - \omega) - X^\varepsilon(t, \theta - \omega)| \to 0, \quad \text{as} \quad \varepsilon \to 0, \quad \text{a.s.}$$

Proof. It follows from Lemma 4.2 that for any $t \geq 0$, the solution $(v_m(t, \theta - \omega), v_{\infty}(t, \theta - \omega))$ of Eq. (10) is in $M^\varepsilon(\omega)$ for almost all $\omega \in \Omega$. Similarly, for any $t \geq 0$, the solution $(Y_{m}^\varepsilon(t, \theta - \omega), Y_{\infty}^\varepsilon(t, \theta - \omega))$ of Eq. (11) is in $M^Y(\omega)$ for almost all $\omega \in \Omega$. Furthermore,

$$v(t, \theta - \omega) = (v_m(t, \theta - \omega), v_{\infty}(t, \theta - \omega)) = (v_m(t, \theta - \omega), h^\varepsilon(v_m(t, \theta - \omega), \theta - \omega)), \quad Y^\varepsilon(t, \theta - \omega) = (Y_{m}^\varepsilon(t, \theta - \omega), Y_{\infty}^\varepsilon(t, \theta - \omega)) = (Y_{m}^\varepsilon(t, \theta - \omega), h^\varepsilon(Y_{m}^\varepsilon(t, \theta - \omega), \theta - \omega)).$$

Also it follows from (19) and (20) that

$$|h^\varepsilon(v_m(t, \theta - \omega), \theta - \omega) - h^\varepsilon(Y_{m}^\varepsilon(t, \theta - \omega), \theta - \omega)|$$

$$= |\int_{-\infty}^{0} e^{-A^\varepsilon \tau} g_{\infty}(\tau, \theta - \omega; v_m(t, \theta - \omega))d\tau - \int_{-\infty}^{0} e^{-A^\varepsilon \tau} g_{\infty}(\tau, \theta - \omega; Y_{m}^\varepsilon(t, \theta - \omega))d\tau|$$

$$\leq \int_{-\infty}^{0} e^{\beta \tau} L_F |v(\tau, \theta - \omega; v_m(t, \theta - \omega)) - Y^\varepsilon(\tau, \theta - \omega; Y_{m}^\varepsilon(t, \theta - \omega))|d\tau$$

$$+ \int_{-\infty}^{0} e^{\beta \tau} L_F |u^*(\theta - \omega) - X^*(\theta - \omega)|d\tau$$

$$\leq L_F |v(\cdot) - Y^\varepsilon(\cdot)|_{C_n} + \frac{L_F}{\beta + \eta} |u^*(\theta - \omega) - X^*(\theta - \omega)|_{C_n},$$

which implies from Lemma 5.2 and (42) that

$$|h^\varepsilon(v_m(t, \theta - \omega), \theta - \omega) - h^\varepsilon(Y_{m}^\varepsilon(t, \theta - \omega), \theta - \omega)| \to 0, \quad \text{as} \quad \varepsilon \to 0, \quad \text{a.s.}$$

(45)

Therefore, it follows from (44), (43) and (45) that

$$|v(t, \theta - \omega) - Y^\varepsilon(t, \theta - \omega)| \to 0, \quad \text{as} \quad \varepsilon \to 0, \quad \text{a.s.,}$$

which further implies from (12) and (13) that Theorem 5.5 holds. The proof is completed. \qed

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