Sub-classical fields and polarization in electrodynamics

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Expectation values of the electromagnetic field and the electric current are introduced at space-time resolution which belongs to the quantum domain. These allow us to approach some key features of classical electrodynamics from the underlying QED. One is the emergence of the radiation field in the retarded solution of the Maxwell equation, derived from an action principle. Another question discussed is the systematic derivation of the polarizability of a charge system. Furthermore, the decoherence and the consistency of the photon field are established by a perturbative calculation of the reduced density matrix for the electromagnetic field within the Closed Time Path formalism.

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I. INTRODUCTION

Classical electrodynamics consists of a set of equations of motion for the electromagnetic (EM) field and the electric current, the Maxwell and the mechanical equations, respectively. But the classical theory should be derived from QED rather than used as a starting point to quantize the EM field. When approached in this manner, these equations become approximate, govern the expectation values of the EM field and the electric current, and are the subject of an infinite hierarchy of quantum corrections.

The degrees of freedom common in quantum and classical physics are the expectation values of local fields. But we have to keep in mind that such expectation values are not necessarily classical. The crucial quantity which controls the classicality of the field expectation value is the number of degrees of freedom $N = n\xi^3$ within the internal distance scale of the field, the correlation length $\xi$ of a many-body system at particle density $n$. The UV cutoff $\Lambda$, the maximal energy scale of QED, will be kept large but finite in this work: therefore, the internal scale of the field expectation value $\xi > 1/\Lambda$ may fall into the quantum or classical domain, depending on initial conditions and external sources. For $N \sim 1$ the local expectation value does not follow the laws of classical physics and will be called a subclassical field. Such fields are used for instance in the density functional method [5]. These fields are useful to monitor the quantum-classical transition because they bridge the gap between observables and expectation values, used on the microscopic and macroscopic domain, respectively. This can be seen simplest within the framework of the renormalization group where the classical action governing the dynamics of the subclassical field can be set as an initial condition for the integration of the renormalization group equations in the macroscopic domain. This point of view motivated this work, the study of some necessary ingredients of the quantum-classical transition by means of the dynamics of subclassical fields. It includes the study of the dynamical origin of decoherence [1, 2] and consistency [3], the breakdown of the time reversal invariance [4] and the way the expectation values are formed at the microscopic scale.

A reassuring feature of subclassical fields is that their dynamics is stable and well defined in the UV. In fact, the particles can not be localized with a precision better than their Compton wavelength therefore the subclassical fields are smeared out at shorter distances. This scale is independent of the underlying quantum field theory. The subclassical field is classical in the sense that quantum fluctuations are averaged out. But it is still in the quantum domain due to its fine spatial resolution because the decoherence within this space region is not strong enough to restore classical probabilities, in particular, the additivity for exclusive events.

The present work is the continuation of the program outlined in Ref. [20] where the equations of motion for the expectation value of the EM field and the electric current were discussed in QED. We go beyond that work and render that approach more realistic by including a reservoir of charges at finite density and by calculating explicitly and by interpreting the vacuum polarization effects in the equation of motion for the EM field and in decoherence/consistency. The loop expansion will be employed to include the vacuum polarization effects and we shall be satisfied by calculating the leading order, one-loop approximation to the linearized variational equations of motion. The current terminology of the traditional construction of the averaged fields in classical electrodynamics can be borrowed from the renormalization group method, it is a special blocking. In the present work we are satisfied by a simpler realization of this idea, the coarse graining generated by the elimination of the charged degrees of freedom from the dynamics without lowering the space resolution. The decoherence and consistency become simple perturbative effects in this setup, they are generated by the collective modes of the many-body vacuum. The dynamical breakdown of the time reversal invariance can be traced back to the collective modes as well, it is driven by the finiteness of the life-time of the quasiparticles. Both the decoherence/consistency and dynamical breakdown of the time reversal...
invariance are actually generated by the same piece of the photon polarization tensor. This is how the dynamical breakdown of the time reversal invariance and decoherence appear coupled in the classical limit. Note that there is no need of a particular definition of pointer states in this procedure, quasiparticles arising form our coarse graining provide a natural, dynamically generated robust basis in the Fock space for the study of the reduced density matrix.

The breakdown of the time reversal invariance brings the boundary conditions in time into the foreground. The other goal of this work is the establishment of the dynamical symmetry breaking and its clear separation from the explicit breakdown by the boundary conditions in time. The not entirely trivial issue of the latter is the way the external sources rearrange the radiation field in the variation principle to comply with the given boundary conditions. The simplest way to regulate the corresponding formal, $0/0$ terms in the equations of motion is the introduction of the usual $\imath\epsilon$ term in the quadratic part of the action in classical field theory, as well as in its quantum counterpart.

The quantum corrections to the space-time dependence of the averages of local observables are usually calculated by means of the linear response formalism. We need a more powerful method, capable of dealing with the eventual nonperturbative issues of the quantum-classical crossover regime to arrive at a reliable description of quantum effects, turning classical electrodynamics into QED at short distances. The Closed Time Path (CTP) method developed by J. Schwinger [6] provides us with a nonperturbative setting not only for the determination of the space-time dependence of local observables but for other key ingredients of the quantum-classical transition, to the reduced density matrix and the possible dynamical breakdown of the time reversal invariance, too.

The CTP method has been extended a number of times [7–15]. Despite being the only way of obtaining expectation values as opposed to transition amplitudes reproduced by conventional quantum field theory its inherent algebraic complications somehow limited its applications. The generator functional for connected Green functions, our starting point given by Eq. (7) below is closely related to the influence functional introduced in Ref. [16]. One may simple say that all that is presented below is the application of the influence functional scheme to derive the effective action and the equations of motion for the expectation value of local operators. The solution of the latter provides us with a partial resummation of the perturbation series with rich physical content. Furthermore, certain details of the influence functional contain decoherence/consistency and irreversibility, the key conditions for the classical limit. The reduced density matrix and dissipative forces have been studied in the framework of nonrelativistic quantum mechanics in a manner reminiscent of the CTP formalism [17]. The unusual feature of quantum transition amplitudes, the possibility of fixing the initial and the final states of the motion has already been addressed in a manner, equivalent to the CTP formalism in Ref. [18]. The Schwinger-Dyson equation, the removal of the UV divergences, the equation of motion for the EM field derived in the CTP formalism and the pair creation generated by a homogeneous external electric field have already been thoroughly studied for a large number of charged fermions in Ref. [21]. Our results are complementary since they retain the adiabatic, IR tail of the vacuum polarization only. It is worth mentioning that the problem of establishing variational equations for the expectation values [19] can be resolved by a suitable parametrization of the degrees of freedom [20]. The role of collective modes in the quantum-classical transition can conveniently be studied within the CTP scheme [22], as well.

We start in Section II with a short introduction of the CTP formalism, including the systematic construction of the Heisenberg representation in terms of Green functions. Instead of following the historical route, the motivation of this scheme by the calculation of the expectation values rather than transition amplitudes, an alternative argument is presented. The CTP formalism is motivated by the generalization of the classical action principle for initial condition problems which produce the retarded solutions. Beside the CTP case we need the Open Time Path (OTP) version, as well, to determine the reduced density matrix. Some functionals, used subsequently are introduced here, too, namely those which generates the connected Green functions and their Legendre transform, the effective action.

A rather general issue is addressed in Section III where a nonrenormalization theorem is shown for the Maxwell equations when all charges are included. One finds dressing only when charges are ignored. This result is in agreement with the usual derivation of the dielectric constant and the magnetic permeability in classical electrodynamics.

The double-time formalism is introduced for QED in Section IV, followed by the calculation of the quadratic part of the generator functional for the connected Green functions and the effective action for the electric current and the EM field, as well as a short discussion of the ways the mass-shell modes of the photon and the zero modes of the free EM field action satisfy the desired boundary conditions in time by formal, $0/0$-type terms in the equations of motion. The one-loop result for this functional is summarized in Section V.

The explicit calculation of the reduced density matrix for the EM field is carried out in Section VI. The real part of its logarithm controls the dielectric and the magnetic permeability functions and determines the quasiparticles, the imaginary part is responsible for the finite life-time of the quasiparticles, and the decoherence and the consistency of the EM field. Finally, Section VII summarizes our conclusions.

The Appendixes contain some technical details. A short summary of the CTP propagators in the vacuum is presented in Appendix A, followed by the extension to thermodynamical equilibrium in Appendix B.
II. TIME ARROW AND THE DOUBLE-TIME FORMALISM

The dynamical breakdown of the time reversal invariance is a provocative, not yet understood problem [4]. Weak interactions set apart, the fundamental forces and dynamical equations of Nature are invariant under a reversion of the direction of time. Nevertheless, daily experiences show that external perturbations influence the systems at later times only, and therefore the retarded solutions of the time reversal invariant equations of motion should be used. We set up our quadrature to solve the differential equations accordingly, by specifying the initial conditions. The resulting scheme obviously does not obey time reversal invariance. Such an explicit breakdown of the symmetry is a trivial phenomenon, the real, dynamical issue is the simplicity of the initial conditions, and the absence of the radiation field in electrodynamics or excited states, in general, in quantum mechanics.

Even this trivial issue poses an interesting question. How can such an initial condition problem be dealt with in physics? The Newton equation, being a second order differential equation, requires two input data per degrees of freedom to fix a solution uniquely. For a differential equation we can choose freely among parameters of the motion, either at the initial or at the final time. This allows us to treat problems where the initial or the final condition is simple. A typical example is the use of the retarded or advanced Liénard-Wiechert potentials in classical electrodynamics. But the fundamental, most flexible layer of classical mechanics is the variational principle where boundary problems can be defined only because the variation performed at the end point of the trajectory in an initial problem cancels the generalized momentum. Such a shortcoming of the variation method is not a serious problem in classical dynamics since we can switch back to the canonical equations of motion anytime and set up the desired Cauchy problem. The situation is more involved in quantum mechanics, where the action principle remains important. In fact, after having set up a problem in the operator formalism, the actual calculation of the transition amplitudes or expectation values is usually done in the path integral formalism. Hence the goal of generalizing the variation principle to cope with retarded or advanced forces remains an important question.

How can we construct a variational method which automatically produces the retarded solution of the equations of motion? One should modify the procedure in such a manner that it is enough to provide the coordinates and the velocities at \( t = t_f \), the beginning of the time evolution. But what does one do with the variation of the trajectory at the final time \( t = t_f \)? A simple way to render the variation at the final time consistent with the classical trajectory is to cancel altogether its contribution to the equation of motion. This can be achieved by extending the dynamical problem for twice as long as \( 0 \leq t_{\text{CTP}} \leq 2T = 2(t_f - t_i) \). We follow the usual trajectory \( x^+(t) \) for the time interval \( T \) then we flip the time arrow and follow the time backward along another trajectory \( x^-(t) \), which returns to the initial conditions. The closed trajectory obtained in this manner is

\[
x^\text{CTP}(t_{\text{CTP}}) = \begin{cases} 
  x^+(t_i + t_{\text{CTP}}) & 0 \leq t_{\text{CTP}} \leq T, \\
  x^-(2t_f - t_i - t_{\text{CTP}}) & T \leq t_{\text{CTP}} \leq 2T.
\end{cases}
\]

The variation problem for such a time reversed trajectory is well defined, and it yields a trivial equation of motion \( 0=0 \) for \( t = t_f \) and produces the retarded solution. The quantum analogy of this scheme is well known, it is the CTP formalism of Schwinger [6]. We shall need a slight generalization of this scheme, the OTP formalism [23] to access the reduced density matrix of subsystems to cope with a genuine quantum effect, the decoherence.

The CTP formalism was originally developed as the standard perturbation expansion for expectation values in the Heisenberg representation of Quantum Field Theory [6]. When Feynman’s insight to QED, based on a simpler scheme of transition amplitudes became widely accepted, then the perturbation expansion for the expectation values was gradually abandoned. This shift of attention seemed to be further justified by the simplicity and generality of the path integral representation of the transition amplitudes. The interest in the CTP formalism was kept in condensed matter physics, where expectation values corresponding to given, time-dependent external perturbation are important.

The expectation value

\[
\langle \psi(t)|A|\psi(t) \rangle = \langle \psi_i|A_H(t)|\psi_i \rangle,
\]

obtained in the Heisenberg representation, where where \( A_H(t) = e^{i(t-t_i)H}Ae^{-i(t-t_i)H} \) and \( |\psi(t_i)\rangle = |\psi_i\rangle \) can be written as

\[
\langle \psi(t)|A|\psi(t) \rangle = \text{Tr}[Ae^{-i(t-t_i)H}\rho_i e^{i(t-t_i)H}].
\]

in terms of the initial density matrix \( \rho_i = |\psi_i\rangle \langle \psi_i |. \) This expectation value can be related in a simple manner to the transition amplitude

\[
A = \langle \psi_i|e^{-i(t_f-t_i)H}Ae^{-i(t-t_i)H}|\psi_i \rangle
\]
only when the initial state \( |\psi_i\rangle \) is an eigenstate of the Hamiltonian. This condition is not met in a realistic situation where the initial state contains complicated, collective excitations. Expectation values like (3) will be obtained in the CTP formalism, used in this work to derive the effective action whose Euler-Lagrange equation is satisfied by the expectation values of the EM field and the electric current.

### A. Generator functional for connected Green functions

We recall first some general features of the CTP formalism and next we introduce this scheme for QED. The expectation value (3) motivates the introduction of the generator functional

\[
e^{iW[j^+,j^-]} = \Tr T\left[ e^{-i\int_{t_i}^{t_f} dt \oint d^3x \rho_i(T[H(x) + j^-(x)O(x) - j^+(x)O(x)]^\dagger) \rho_i(T[H(x) - j^+(x)O(x) + j^-(x)O(x)])} \right]
\]

where \( j^\pm \) are two sources coupled to a local operator \( O \) and we use the units \( \hbar = c = 1 \). This generator functional can easily be given in the path integral formalism. Indeed, the time evolution operator to the left of \( \rho_i \) has the standard path integral representation. The same holds for the time evolution operator on the right except that the replacement \( j^+ \rightarrow -j^- \) is carried out, as well. Note that the Hermitian conjugation of the operator standing to the right of \( \rho_i \) leads to anti-time ordering. Consequently, the number of degrees of freedom is doubled, \( \phi^+ \) and \( \phi^- \) will be used to denote the CTP doublets, field trajectories for the left and right time evolution operator, respectively. We therefore have pairs of trajectories in the full path integral

\[
e^{iW[j^+,j^-]} = \int D[\phi^+]D[\phi^-]D[\phi^+\phi^-] \Psi^*_0[\phi^+] \Psi_0[\phi^-] \int_{\phi_i^+}^{\phi_f^+} D[\phi^+] \int_{\phi_i^-}^{\phi_f^-} D[\phi^-] e^{iS[\phi^+] + iS_{CT}[\phi^+] - iS[\phi^-] - iS_{CT}[\phi^-] + ij^+ \sigma^+ + ij^- \sigma^-}
\]

where \( O^\pm \) is constructed by means of the field \( \phi^\pm \), the integral measure \( D[\phi^\pm] \) includes the field variables for \( t_i < t < t_f \) and the space coordinate of \( \phi \) is suppressed. The initial condition \( \rho_i = |0\rangle\langle 0| \) is used where \( |0\rangle \) stands for the noninteracting vacuum state with wave functional \( \Psi_0[\phi] \) and the trajectories \( \phi^+ \) and \( \phi^- \) are joined at the final time due to the trace operation in Eq. (5). The boundary conditions in time do not influence UV divergences therefore the counterterms contained in \( S_{CT}[\phi] \) are imposed separately for each time axes. It is not difficult to check that the boundary conditions in time, used in Eq. (6) can be incorporated by the replacement \( S[\phi^+] - S[\phi^-] \rightarrow \tilde{S}[\phi] = S[\phi^+] - S[\phi^-] + S_{BC}[\phi^+, \phi^-] \) where \( S_{BC} \) is a quadratic expression of the fields taken at \( t = t_i \) and \( t = t_f \). The scalar product \( fg \) of space-time functions stands for space-time integration, \( f g = \int d^4x f(x)g(x) \) for the minimal coupling. Note that the expectations values, calculated for unitary time evolution \( j^+ = -j^- \) are independent of the value of this final time \( t_f \) as long as they are considered before \( t_f \). The bonus of these apparent complications in the CTP formalism is a way to handle the dynamical breakdown of the time reversal invariance, decoherence and dissipative forces [20, 22].

We consider in this work elementary fermions of electric charge \( e_n \), \( n \) being a flavor index, interacting with the EM field. For this kind of equations (3) can be obtained by the use of the generator functional

\[
e^{iW[a^-,a^+,j^+,j^-]} = \Tr T\left[ e^{-i\int_{t_i}^{t_f} dt \oint d^3x [H(x) + \sum_n a_n^+ (x) J_n(x) + j^-(x)A(x) + j^+(x)A(x)]} \rho_i T^*\left[ e^{i\int_{t_i}^{t_f} dt \oint d^3x [H(x) - \sum a_n^- (x) J_n(x) + j^-(x)A(x) + j^+(x)A(x)]} \right]\right]
\]

where \( T^* \) denotes anti-time ordering, \( j^\pm \) are two sources used to generate the EM field \( A \) and \( a^\pm \) are two vectorial sources coupled to the EM current \( \psi^\mu \psi = J^\mu \). Eq. (7) can be written as

\[
e^{iW[a,j]} = \int D[A]\left( \prod_n D[a_n^+]D[a_n^-] \right) e^{\frac{i}{2} \sum_n \sum_{\sigma \sigma'} \langle \hat{\psi}^n(\hat{G}_{\sigma n})^{-1}_{\sigma' \sigma} + \delta_{\sigma' \sigma} (n^\sigma - e_n^\sigma A^\sigma) \rangle \hat{\psi}^{\sigma'} + \frac{i}{2} \hat{A} D_{\sigma n}^{-1} \hat{A} + ij \hat{A} + iS_{CT}\right)
\]

using the notation

\[
\hat{\psi}^n = \left( \begin{array}{c} \psi_n^+ \\ \psi_n^- \end{array} \right), \quad \hat{A} = \left( \begin{array}{c} A^+ \\ -A^- \end{array} \right), \quad \hat{a}_n = \left( \begin{array}{c} a_n^+ \\ a_n^- \end{array} \right), \quad \hat{j} = \left( \begin{array}{c} j^+ \\ -j^- \end{array} \right).
\]

In Eq. (8), the CTP inverse-propagators

\[
\hat{G}^{-1}_{\sigma n} = \begin{pmatrix} i\hat{\gamma}^\sigma - m_n + i\epsilon & 0 \\ 0 & -\gamma^\sigma (i\hat{\gamma} - m_n + i\epsilon)^\dagger \gamma^0 \end{pmatrix} + \hat{G}^{-1}_{BCn}, \quad \hat{D}^{-1}_{\sigma n} = \begin{pmatrix} \square T + \xi \square L + i\epsilon & 0 \\ 0 & -\square T - \xi \square L + i\epsilon \end{pmatrix} + \hat{D}^{-1}_{BC},
\]

are used.
already take into account boundary conditions in time, represented by the full CTP matrices \( \hat{G}_{BC}^{-1} \) and \( \hat{D}_{BC}^{-1} \). The notation

\[
T^{ab} = g^{ab} - L^{ab}, \quad L^{ab} = \frac{\partial^a \partial^b}{\Box}
\]

(11)
will be used for the transverse and longitudinal projection operators. The counterterm \( S_{CT} \) will be suppressed below because it is given by the usual transition amplitude formalism.

The CTP propagators are introduced in Appendix A for free fields. The bosonic two-point functions, such as the photon propagator, possess the structure

\[
\hat{D} = \left( \begin{array}{cc} D & D^+ \vspace{2mm} \\ D^- & D \end{array} \right) = \left( \begin{array}{cc} D^n & -D^f \vspace{2mm} \\ D^f & -D^n \end{array} \right) + iD_i \left( \begin{array}{cc} 1 & 1 \\
1 & 1 \end{array} \right)
\]

(12)

involving three real space-time dependent functions. It is an important property of radiative corrections that the self energies appearing in the quadratic generator functionals display the same structure and allows us to define the interactive near and far field propagators. The actual calculation of the propagator for photons reveals that \( D^n \) and \( D^f \) are the near and far field Green functions of classical electrodynamics, respectively. These propagators are usually introduced by splitting the Liénard-Wiechert potential of classical electrodynamics into an \( \mathcal{O}(r^{-1}) \) part and an \( \mathcal{O}(r^{-2}) \) part and the retarded and advanced propagators are defined as \( \hat{D}^\pm = D^n \pm D^f \). The propagator \( \hat{D} \) is symmetric with respect to the exchange of its indices, cf. Eq. (A1). Therefore, the near and far field propagators can be identified by the symmetric and antisymmetric part of the retarded Green function, \( D^n_{ab} = D^n_{ba} \), \( D^f_{ab} = -D^f_{ba} \). Another, more microscopic separation of these two Green functions, based on the quantum level, is mentioned at the end of this section. Finally, we shall find in Section IV C an additional, classical way to define the near and far field propagators.

### B. Expectation values of the EM field and electric current

Notice that the unitarity of the time evolution assures that expectation values can be calculated in two different ways, e.g.

\[
\langle A \rangle = \frac{\delta W}{\delta j^+} |_{j = \hat{a} = 0} = \frac{\delta W}{\delta j^-} |_{j = \hat{a} = 0}
\]

(13)

and the final time \( t_f \) plays no role as long as the expectation value is considered before it, \( x^0 < t_f \), in the equations above. Such a noncompact \( R \) symmetry, called the CTP symmetry, suggests the re-parametrization [20]

\[
a^\pm = \frac{a}{2} (1 \pm \kappa) \pm \hat{a}, \quad j^\pm = \frac{j}{2} (1 \pm \kappa) \pm \hat{j}
\]

(14)
of the external sources because we have

\[
\langle A \rangle = \frac{\delta W}{\delta j} |_{j = \hat{a} = 0}
\]

(15)

for an arbitrary real value of \( \kappa \). The book-keeping sources \( a \) and \( j \) must be canceled after calculating the functional derivatives in order to recover the physical expectation values. On the contrary, the sources \( \hat{a} \) and \( \hat{j} \) can be nonvanishing for unitary time evolution and represent the devices which are supposed to drive the system adiabatically from the vacuum at \( t = -\infty \) to the desired initial state at \( t = t_i \). Eq. (15) shows that the fields coupled to \( a \) and \( j \) can be identified by the physical fields because their averages reproduce the field expectation values. We shall see later that the field coupled to the sources \( \hat{a} \) and \( \hat{j} \) will be responsible for decoherence.

The obvious difficulty of the construction of variational equations for the expectation values is the fact that we have twice as many degrees of freedom in the CTP formalism as in the classical counterpart. This problem can be overcome by considering the effective dynamics obtained by the elimination of a degree of freedom by its equation of motion from each CTP doublet. But the preceding argument reveals an apparent problem in this plan. On the one hand, the reduplication of the degrees of freedom in the CTP formalism, \( A \to A^\pm \), produces two identical looking variables as far as expectation values are concerned and it is not clear which one to eliminate. On the other hand, decoherence, a mechanism not directly encoded by expectation values is related to the difference \( A^d = A^+ - A^- \) only. There are different dynamical issues mixed in these variables. Therefore, it appears to be appropriate to go over the
Keldysh basis, \((A^+, A^-) \rightarrow (A, A^d) = ((A^+ + A^-)/2, A^+ - A^-)\) and eliminate \(A^d\). But there is a new problem which arises in this manner. In fact, it is not difficult to see that the dynamics of the field variable \(A\) and its canonically conjugated momentum \(\Pi = -i\frac{\delta}{\delta A}\) is contained in the dependence of \(A\) or \(A^d\), respectively of the density matrix \(\rho[A^+, A^-]\). The complete elimination of \(A^d\) removes all information about the momentum in the effective theory. But the time dependence of the expectation value \(\langle A \rangle\) can be used to reconstruct the expectation value \(\langle \Pi \rangle = \partial_0 \langle A \rangle\). Thus \(A^d\) should appear in the combination of \(A^+\) and \(A^-\) to be retained in the effective theory. This condition is realized by the restriction \(\kappa \neq 0\) on the otherwise free parameter \([20]\).

The effective action, a device used to derive the equation of motion for the expectation values, is usually the functional Legendre transform of \(W\). This step must be slightly modified because the latter object is complex in the double-time formalism. Since \(W = 0\) for the physical case with unitary time evolution we define the effective action as the Legendre transform of the real part of \(W\),

\[
\Re W[\hat{a}, \hat{j}] = \Gamma[\hat{J}, \hat{A}] + \hat{a}\hat{J} + \hat{j}\hat{A},
\]

and

\[
\hat{J} = \frac{\delta W}{\delta \hat{a}}, \quad \hat{A} = \frac{\delta W}{\delta \hat{j}},
\]

by using the external sources \(a^\pm\) and \(j^\pm\). The inverse transformation, Eq. (16) and

\[
\hat{a} = -\frac{\delta \Gamma}{\delta \hat{J}}, \quad \hat{j} = -\frac{\delta \Gamma}{\delta \hat{A}},
\]

appear as variational equations of motion satisfied by the fields \(\hat{J}\) and \(\hat{A}\). It is advantageous to find the effective action for the expectation values. To this end we write

\[
\Gamma[J, \bar{J}, A, \bar{A}] = \Re W[\hat{a}, \hat{j}] - aJ - \bar{a}\bar{J} - jA - \bar{j}\bar{A}
\]

by means of the parametrization (14) and

\[
\hat{J} = \frac{\delta \Re W}{\delta \hat{a}}, \quad \hat{A} = \frac{\delta \Re W}{\delta \hat{j}}, \quad \bar{A} = \frac{\delta \Re W}{\delta \hat{j}},
\]

The inverse transformation,

\[
a = -\frac{\delta \Gamma}{\delta \hat{J}}, \quad \bar{a} = -\frac{\delta \Gamma}{\delta \hat{J}}, \quad j = \frac{\delta \Gamma}{\delta \hat{A}}, \quad \bar{j} = \frac{\delta \Gamma}{\delta \hat{A}},
\]

represents the Euler-Lagrange equations. The fields \(\bar{J}\) and \(\bar{A}\) stand for the expectation values and the auxiliary fields \(\hat{J}\) and \(\hat{A}\) incorporate the effects of quantum fluctuations.

The imaginary part \(\Im W\), left out from the construction of the our effective action is important in establishing decoherence. In fact, decoherence stands for the suppression of the off diagonal elements of the density matrix in the pointer representation. By assuming that the field variables \(\bar{J}\) and \(\bar{A}\) are good pointer variables decoherence corresponds to the suppression of the absolute magnitude of \(e^{iW[\hat{a}, \hat{j}]}\) as \(\bar{J}\) and \(\bar{A}\) are increased. Such a suppression comes from \(\Re W[\hat{a}, \hat{j}]\).

Finally, one can construct an effective action for the physical expectation values only,

\[
\Gamma[J, A] = \Re W[a, \hat{a}, j, \hat{j}] - aJ - jA
\]

where

\[
J = \frac{\delta \Re W}{\delta a}, \quad A = \frac{\delta \Re W}{\delta j}.
\]

The inverse transformation,

\[
a = -\frac{\delta \Gamma}{\delta \hat{J}}, \quad j = -\frac{\delta \Gamma}{\delta \hat{A}},
\]

give the Euler-Lagrange equations for the expectation values only.
C. Reduced density matrix

We have surveyed so far the means to find equations of motion for expectation values. The right-hand side of Eq. (3) is not sufficient for our other goal, for the calculation of the reduced density matrix. The reduced density matrix for the EM field is obtained by eliminating the charged degrees of freedom and can be written as

$$\rho[A^{(1)}, A^{(2)}] = \langle A^{(1)} | \text{Tr}_{\text{ch}} [A e^{-i(t-t_1)H} \rho e^{i(t-t_1)H}] | A^{(2)} \rangle$$

(25)

where the trace operation is over the Fock space of the charged particles and the state $|A\rangle$ is the eigenstate of the photon field operator with field configurations $A_\mu(x)$ in the functional Schrödinger representation. The generator functional of Eq. (8) for $\rho[A^{(1)}, A^{(2)}]$ will be given in the OTP formalism. It consists of a path integral where the charged fields have a closed time path and the integration over the photon field is restricted to trajectories with open end points $A^+_\mu(t_f, x) = A^{(1)}_\mu(x)$ and $A^-_\mu(t_f, x) = A^{(2)}_\mu(x)$.

The OTP formalism displays entanglement of subsystems in a specially clear manner. Let us consider for instance QED as a closed system, described by a factorizable, pure state at the initial time. The interactions generate entanglement between the charges and the EM field; the charges considered alone will be found in a mixed state and their reduced density matrix can be obtained by eliminating the EM field. The contributions of the perturbative integration over the EM field configurations are labeled by graphs. Those which contain photon lines connecting the two time axes represent mixed state contributions. In fact, the integration over the momentum of these lines produces the sum of pure state contributions to the reduced density matrix. The structure (12) of the photon propagator reveals that the near field is responsible for the interactions within a time axis. These elementary processes contribute to the self energy and preserve the purity of the charge states. The far field connects the two time axes and represents the mixing resulting from the entanglement of the charge-EM field subsystems.

The perturbative evaluation of the generator functional of Eq. (7) implies a strong limitation. The naive perturbation expansion relies on the translation invariance of the vacuum, allowing weak inhomogeneous components in the external sources. This is enough to support sufficiently inhomogeneous field expectation value profiles when there is no gap in the excitation spectrum above the vacuum, such as in the case for a system of charges at finite density. But once the localized states arise through a gap in the spectrum then the weak external inhomogeneities are not sufficient to produce localized states on the mass-shell. In fact, all we achieve at finite temperature and vanishing charge density is a weak inhomogeneous vacuum polarization by virtual pairs. The creation of states with few on-shell localized charges remains a nonperturbative issue in this formalism [20].

III. FLUCTUATIONS AND DRESSING

Both the external, classical source $j$ and the dynamical current $\langle 0 | j(0) | 0 \rangle = \hat{J}$ generate vacuum polarization. The simple structure of the QED action allows us to compare these two polarizations. To this end we consider the expectation value of the EM field induced by weak external and dynamical, quantum currents. We shall seek the linearized expression for the EM field. The result will be equally valid for the transition amplitude formalism or for any double-time functionals. The integration over the fermion fields in the generator functional (8) yields

$$e^{iW[\hat{a}, \hat{j}]} = \int \mathcal{D}[\hat{A}] e^{iW^c[\hat{a} - e\hat{\sigma} \hat{A}] + \frac{e^2}{2} \hat{A} \mathcal{D}^{-1}_{\sigma} \hat{A} + \hat{j} \hat{A}}$$

(26)

where $\hat{\sigma}$ denotes flipping the sign of the $-\sigma$ component of CTP doublets

$$\hat{\sigma} \begin{pmatrix} A^+ \\ A^- \end{pmatrix} = \begin{pmatrix} A^+ \\ -A^- \end{pmatrix},$$

(27)

the generator functional of non-interacting fermions is denoted by

$$W^c[\hat{a}] = \sum_n W_n[\hat{a}_n],$$

(28)

with

$$W_n[\hat{a}] = -i \text{Tr} \ln \left[ \mathcal{G}^{-1}_{0n} + \begin{pmatrix} \hat{\mu}^+ & 0 \\ 0 & \hat{\mu}^- \end{pmatrix} \right]$$

(29)

being the generator functional for the connected Green functions of the current for the flavor $n$ in the absence of electromagnetic interactions. Note that there are connected Green functions of arbitrarily high order for the
noninteracting Dirac-see because the current is a composite operator. We perform the shift of the integral variable, $A \rightarrow A - D_0 j$, in other words, we use the equation of motion for the EM field to write

$$e^{iW[\hat{a}, \hat{j}]} = e^{-\frac{i}{2} A D_0 j} \int D[A] e^{iW'[\hat{a} - e\hat{\sigma}(A - D_0 j)]} + \frac{i}{2} \hat{A} D_0^{-1} \hat{A}$$

with

$$W[\hat{a}] = W[\hat{a}, \hat{j} = 0]$$

being the generator functional for the current in full QED. Therefore, we obtain the relation

$$W[\hat{a}, \hat{j}] = W[\hat{a} + e\hat{\sigma} D_0 j] - \frac{1}{2} \hat{\sigma} D_0 j,$$

the reduction of the two-variable generator functional into a single variable functional. The simple form of the initial Lagrangian allows us to separate the tree-level dependence on the source $j$ and to place the loop-induced dependence into the dressed fermion one-loop graphs. In doing so, we rely on the fact that the only part of the QED action which contains higher-than-quadratic terms in the fields is the minimal coupling.

The natural use of such a reduction is the simplification of the equations of motion for the currents

$$\hat{J}_n = \frac{\delta W[\hat{a} + e\hat{\sigma} D_0 j]}{\delta \hat{a}_n}$$

and the EM field

$$\hat{A} = \sum_n e_n \hat{D}_0 \hat{\sigma} \frac{\delta W[\hat{a} + e\hat{\sigma} D_0 j]}{\delta \hat{a}_n} - \hat{D}_0 j$$

which can be written as

$$\hat{A} = \hat{D}_0 (\hat{\sigma} e^{tr} \hat{j} - j)$$

where $e$ denotes a flavor column vector made up of the electric charges, $e^{tr} = (e_0, e_1, \cdots)$.

The external sources $\hat{a}_n$ and $\hat{j}$ are the independent variables in these equations. When the functional Legendre transformation is performed then $\hat{J}_n$ and $\hat{A}$ become independent variables but the equations remain valid by considering $\hat{j}$ and $\hat{a}_n$ as dependent variables. Comparing Eq. (35) with the second equation in Eqs. (18) the exact result [24]

$$\Gamma[\hat{J}, \hat{A}] = \Gamma_{mech}[\hat{J}] + \frac{1}{2} \hat{A} D_0^{-1} \hat{A} - \hat{A} \hat{\sigma} \sum_n e_n \hat{J}_n$$

follows. Here the first term represents the mechanical contribution of the Dirac-see and plays the role of an integration "constant". There are two sources of complexities in the dynamics, the implications of the Pauli exclusion principle for the electric current as a composite operator and the electromagnetic interactions. The latter has a small parameter, $\hbar$ or $e^2$ to organize a systematical approximation scheme but the former type of dynamical correlations admit no small parameter and induce arbitrary high order connected Green functions for the non-interacting Dirac-see in Eq. (29) and arbitrary high order vertices in the effective action (36).

The lesson of Eq. (35) is rather surprising, it is the absence of renormalization in the nonmechanical part of the theory. We shall see below, when a similar result is recovered in the loop-expansion, that this triviality results from a cancellation between the two channels the source $j$ can induce EM field. What is observed here is that the relation among the expectation values and the external current $j$ is not renormalized because the former already includes all the dressing. The nontrivial source of dressing according to Eq. (32) arises from the Green functions for the currents, the structure of the functional $W[\hat{a}]$ of Eq. (31). The $j$ dependence comes in a trivial manner, dictated by the minimal coupling, gauge invariance.

It is instructive to follow what happens with the vacuum polarization when we select the flavor $n = 0$ as our valence charge, $e_v = e_0$, and cancel the external source $a_{n_b}$, $n_b \neq 0$ for the remaining flavors, called background charges. Let us introduce the notation $\hat{a}_v = \hat{a}_0$,

$$\hat{J}_v = \frac{\delta W[\hat{a} + e\hat{\sigma} D_0 j]}{\delta \hat{a}_v},$$

$$\Gamma_{mech}[\hat{J}] = \Gamma_{mech}[\hat{J}_v] + \frac{1}{2} \hat{A} D_0^{-1} \hat{A} - \hat{A} \hat{\sigma} \sum_n e_n \hat{J}_n$$
and \( e_b \) will denote the background charge vector, \( e_b^i = (0, e_1, \cdots) \). The polarization effects now arise from the dependence of the electric current of any flavor on the external sources \( \hat{a}_v \) and \( \hat{j} \) and one can find these contributions by expanding Eq. (33) in \( \hat{a}_v \) and in \( \hat{j} \). We have to keep in mind that the current of the valence charge, given by Eq. (37), includes all external source dependence and it is an independent variable of the effective action therefore, its dependence on \( j \) has already been accounted for. Thus the expansion can be restricted to the background charges which yields, after inserting it into Eq. (34),

\[
\hat{A} = \hat{D}_0 \left[ \hat{\sigma} e_v \hat{J}_v - \hat{j} + \hat{\sigma} e_b^i \frac{\delta^2 W[\hat{a}]}{\delta \hat{a} \delta \hat{a}_v} \left| \hat{a}_v = 0 \right. \hat{a}_v + \hat{\sigma} e_b^i \frac{\delta^2 W[\hat{a}]}{\delta \hat{a} \delta \hat{a}} \left| \hat{a} = 0 \right. e_b^j \hat{D}_0 \right] + \cdots
\]

(38)

up to quadratic terms in the external sources for charge conjugation invariant vacuum. The first two terms on the right-hand side stand for the direct, tree-level part of Eq. (35) and the polarization effects are represented by the third and fourth terms in the linearized equation of motion. Recall that the external field \( \hat{a}_v \) is used only to generate the valence current \( \hat{J}_v \). Therefore, \( \hat{a}_v \) in the third term should be expressed in terms of \( \hat{J}_v \) when the relation between the valence current and the EM field is sought. Thus this term represents the polarization effects due to the dependence of the background currents on the valence current. The last term shows the polarization due to the dependence of the background currents on the external current \( \hat{j} \). These two terms are not directly incorporated in the valence current dynamics and they represent the dressing in the equation of motion when \( \hat{a}_v \) is expressed in terms of \( \hat{J}_v \). In other words, the dressing in the Maxwell equation (38) is due to the fact that our experimentally monitored valence current does not cover all charges in the system. It is completely natural that the uncontrolled charges generate the dressing, which gives rise to the proportionality between the averaged and the local EM field.

IV. EFFECTIVE ACTION

After some general remarks we now turn to explicit loop-expansion expressions for the quadratic part of the functionals mentioned in the previous Section.

A. Connected Green functions

The generator functionals of Eq. (26) for a single valence charge \( n = 0 \) can be written after carrying out the integration over the fermion fields as

\[
e^{iW[\hat{a}, \hat{j}]} = \int D[\hat{A}] e^{\sum_n W_n[\delta_0, a] + i\hat{J} + \frac{1}{2} \hat{A} \hat{D}^{-1} \hat{A} + \mathcal{O}(\hat{a}^2) + \mathcal{O}(\hat{A}^3)}
\]

(39)

where \( W_n \) is defined by Eq. (29) and a homogeneous, classical background charge is assumed to cancel the \( \mathcal{O}(\hat{A}) \) tadpole terms when the vacuum is charged. The improved photon propagator

\[
\hat{D} = (\hat{D}_0^{-1} - \hat{\Pi})^{-1}
\]

(40)

contains the one-loop polarization from all charges,

\[
\hat{\Pi} = \hat{\sigma} e^{tr} \hat{G} e \hat{\sigma},
\]

(41)

where

\[
\hat{G}_{(n\sigma x\mu), (n'\sigma' y\nu)} = -i \delta_{n'n'} \text{tr}(G_{0 n y x} G_{0 n y x}^\dagger G_{0 n y x}^\dagger)
\]

(42)

The background charge improved photon propagator, used later is given by

\[
\hat{D}_b = \frac{1}{\hat{D}_0^{-1} - \hat{\Pi}_b}
\]

(43)

including the polarization

\[
\hat{\Pi}_b = \hat{\sigma} e_b^{i r} \hat{G} e_b \hat{\sigma}.
\]

(44)
The CTP expression for the functional $W[\hat{a}, \hat{j}]$ is obtained when the photon field trajectories are closed in the functional integral of Eq. (39). We shall need the OTP result which is obtained by carrying out this functional integral for uncorrelated $A^\pm$ trajectories, having different end points. The functional integration yields

$$W[\hat{a}, \hat{j}] = -\frac{1}{2} \langle \hat{a}, \hat{j} \rangle \left( \begin{array}{c} \hat{G}_v^0 \hat{G}_e \frac{\hat{G}_0}{\hat{D}_0} \\ \hat{G}_e \hat{D}_0 \hat{G}_v^0 \hat{F}_v \end{array} \right) \left( \begin{array}{c} \hat{a} \\ \hat{j} \end{array} \right) + \mathcal{O}(\hbar^2) + \mathcal{O}(\text{source}^3)$$

(45)

where $\hat{G}_v = \hat{G}_{00}, \hat{G}_e = (\hat{G}e)_{0}$. The dressed propagators are the sum of the products $\hat{D}_1 \delta \hat{D}_2 \sigma \cdots \delta \hat{D}_n$ where $\hat{D}_j$ has the form shown in Eq. (12). The product displays the same structure as the factors $\hat{D}_j$, namely the retarded or advanced part of the product is the product of the retarded or advanced parts,

$$\langle \hat{D}_1 \delta \hat{D}_2 \sigma \cdots \delta \hat{D}_n \delta \rangle = \hat{D}_1^r \hat{D}_2^r \cdots \hat{D}_n^r,$$

(46)

yielding

$$\hat{D}^r = \frac{1}{\hat{D}_0^r - \Pi},$$

(47)

We need the real part of $W$, considered in real space as opposed to momentum space, for the effective action in Eq. (19). To simplify matters we exclude pair creation processes by restricting ourselves external sources with modes $\omega, k < m$ and the heat and particle baths are chosen to be nonrelativistic, $T, k_F \ll m$. The one-loop expressions for both $\Re \hat{G}$ and $\Im \hat{G}$ are vanishing on the photon mass-shell, the support of $\Im \hat{D}_0$, according to the calculation reported in Section V. These properties allows us to simplify the real part of the products in the matrix elements in Eq. (45) and one finds

$$\mathcal{R}W[\hat{a}, \hat{j}] = -\frac{1}{2} \langle \hat{a}, \hat{j} \rangle \left( \begin{array}{c} \mathcal{R} \hat{G}_v \\ \mathcal{R} \hat{G}_e \end{array} \right) \left( \begin{array}{c} \hat{a} \\ \hat{j} \end{array} \right),$$

(48)

The form (12) of $\hat{G}$ and $\hat{D}$ gives after a lengthy but straightforward calculation

$$\mathcal{R}W[\hat{a}, \hat{j}] = -\frac{1}{2} \langle \hat{a}, \hat{j} \rangle \left( \begin{array}{cc} -\kappa \hat{G}^a & \kappa e_v \hat{G}^a D_0^a - \hat{G}^r D_0^r \\ \kappa e_v D_0^a \hat{G}^a - \kappa D_0^a & e_v \hat{G}^r D_0^r - \hat{D}^r \end{array} \right) \left( \begin{array}{c} \hat{a} \\ \hat{j} \end{array} \right),$$

(49)

where $\hat{G}^a$ denotes $(\mathcal{R}G)^a$. This result leads to the linearized expressions

$$J = \kappa \hat{G}^a a - \kappa e_v \hat{G}^a D_0^a j + \hat{G}^r \hat{a} - e_v \hat{G}^r D_0^r \hat{j}$$

$$A = -\kappa e_v D_0^a \hat{G}^a a + \kappa D_0^a j - e_v \hat{D}_0^a \hat{G}^r \hat{a} + \hat{D}^r \hat{j}$$

$$J^a = \hat{G}^a a - e_v \hat{G}^a D_0^a j$$

$$A^a = -e_v \hat{D}_0^a \hat{G}^a a + D_0^a j$$

(50)

for the expectation values in terms of the external sources for all charges or for a single valence charge, respectively.

Notice that the physical expectation values $J$ and $A$ contain the retarded field of the physical sources $\hat{a}$ and $\hat{j}$. This is not a dynamical breakdown of the time reversal invariance, but rather a trivial result of the boundary conditions. In fact, the open ended, null boundary condition at the final time leads to a destructive interference between the two time axes which cancels the advanced field generated by the physical sources [20]. The book-keeping variables $a$ and $j$ generate an $O(\kappa)$ time reversal invariant near field, as a result of the equal coupling of these sources to the dynamical variables of the two time axes by the $\kappa$-dependent term in Eq. (14).

After the formal manipulations we set $a = j = 0$ to regain the physical case with unitary time evolution. The auxiliary fields are vanishing in this case, cf. Eq. (13), and the Maxwell equation, the expression of the EM field in terms of the dressed, retarded analogy of the Liénard-Wiechert potential, reads

$$A = \hat{D}_0^r \hat{j} - e_v \hat{D}_0^r J$$

(51)

where $\hat{D}_0^{r-1} = \hat{D}_0^{-1} - \Pi^r_0.$
TABLE I: Properties of the near and far propagator on and off the mass-shell

| Operator          | Mass-shell | Off mass-shell |
|-------------------|------------|----------------|
| $\Box D^n$        | 0          | 1              |
| $D_0^n$           | 0          | $-\Box^{-1}$   |
| $D_1^{\bar{a}-1}$| $\neq 0$   | 0              |
| $D_2^{n-1}D_0^{\bar{a}-1}$ | 0          | 1              |
| $D_2^{n-1}D_0^{\bar{a}}D_0^{\bar{a}-1}$ | 0          | $D_0^{\bar{a}-1}$ |

B. Legendre transform

The variation principle governing the expectation values is based on the effective action obtained by a functional Legendre transformation of $\Re W[a, j]$. We mention first a peculiar feature of the loop-expansion. The independent variables of the generator functional for the connected Green functions $W[a, j]$ are the external sources $\bar{a}$ and $\bar{j}$ which are classical, $\mathcal{O}(\hbar^0)$ quantities. The result is the well-known equivalence of the expansion in $\hbar$ and in the number of loops in the Feynman graphs in the connected Green functions. But in the present case the electric current $\bar{j}$ is $\mathcal{O}(\hbar)$ and the expansion in $\hbar$ mixes different loop-orders in the effective action.

The Legendre transform of a quadratic functional remains quadratic, and the two kernels are the inverse of each other up to a sign. The inverse of the block matrices of the quadratic functionals of Eq. (45) can be obtained by performing the change of variable of the Legendre transformation explicitly with the result

$$
\Gamma = -\frac{1}{2}(J, A, J^a, A^a) \begin{pmatrix}
0 & 0 & \tilde{G}^{n-1} & e_v \\
0 & 0 & e_v & D_0^b \\
e_v & D_0^b & -\kappa e_v(P_{on} \tilde{G}^{n-1} + P_{off}) & -\kappa(P_{off} + D_0^b \tilde{G}^{n} P_{on})e_v \\
e_v & D_0^b & -\kappa e_v(P_{on} \tilde{G}^{n-1} + P_{off}) & -\kappa(P_{off} + D_0^b \tilde{G}^{n} P_{on})e_v
\end{pmatrix} \begin{pmatrix}
J \\
A \\
J^a \\
A^a
\end{pmatrix},
$$

(52)

The operator $P_{on}$ projects on to the photon mass-shell and $P_{off} = \mathbb{1} - P_{on}$. Some important properties of $D^n$ and $D^j$ related to the mass-shell and used in the derivation are listed in Table I. For instance, the replacements $D_0^{n-1} D_0^{\bar{a}} \Pi_{0}^{\alpha}(J - D_0^n D_0^{n-1}) \rightarrow 0$, $D_0^{n-1} D_0^{\bar{a}} D_0^{n-1} \rightarrow D_0^{n-1} P_{off}$ and $\Pi_{0}^{\alpha}(J - D_0^n D_0^{n-1}) \rightarrow \Pi_{0}^{\alpha} P_{on}$ have been carried out in obtaining the $\mathcal{O}(A^n)$ term in the effective action. The corresponding linearized, one-loop equations of motion are

$$
a = e_v A^a + \tilde{G}^{n-1} J^a \\
j = D_0^b A^n + e_v J^a \\
\bar{a} = \tilde{G}^{n-1} J - \kappa \tilde{G}^{n-1} \tilde{G}^{n} \tilde{G}^{n-1} J^a + e_v A - \kappa (P_{off} + \tilde{G}^{n} P_{on})e_v A^a \\
\bar{j} = D_0^b A - \kappa (\Pi_{0}^{\alpha} P_{on} + D_0^{n-1} P_{off}) A^a - \kappa e_v (P_{on} \tilde{G}^{n} \tilde{G}^{n-1} + P_{off}) J^a + e_v J
$$

(53)

Notice that the auxiliary fields couple in a different manner to the physical external sources on and off the mass shell. The last expression is the vacuum polarization corrected Maxwell equation. It corresponds to the equation of motion Eq. (4.18) in Ref. [21] when the effects of the physical sources which drive the system adiabatically to the desired initial state are incorporated implicitly in the propagators and the self-energy.

The auxiliary fields are not needed for the dynamics of the observable expectation values therefore, it is natural to seek the effective action involving the physical fields alone. To this end we consider $\bar{a}$ and $\bar{j}$ as parameters in the generator functional of Eq. (49) and perform the functional Legendre transformation on $a$ and $j$ only, with the result

$$
\kappa \Gamma [J, A] = \frac{1}{2} (J, A) \begin{pmatrix}
\tilde{G}^{n-1} & -e_v \\
-e_v & D_0^{n-1}
\end{pmatrix} \begin{pmatrix}
J \\
A
\end{pmatrix} + J a'' + A j'',
$$

(54)

where the inverse of the near field propagators is defined to be zero on the mass-shell. One may even keep only one field performing the Legendre transformation for only a single source with the result

$$
\kappa \Gamma [A] = \frac{1}{2} A D_0^{n-1} A + A j',
$$

(55)

and

$$
\kappa \Gamma [J] = \frac{1}{2} J \tilde{G}^{n-1} J + J a'.
$$

(56)
The source terms \( a'' \), \( j'' \), \( a' \) and \( j' \) are the sum of the action of different retarded propagators on \( \tilde{a} \) and \( \tilde{j} \), their detailed form is such that Eqs. (50) are satisfied. Notice the need for \( \kappa \neq 0 \) to arrive at any variational scheme for the physical variables only [20].

What is the scale regime where the renormalization group treatment, performed in its complexity is supposed to preserve the form of these equations? The UV cutoff of the equations is the Compton wavelength because the vacuum polarization effects suppress the variation in the space-time within this distance scale. The infrared limitation is provided by the typical length scale of the state generated by the external sources, such as the de Broglie wavelength. In fact, the smearing of the configuration beyond this length scale modifies the space-time dependence and the equations of motion. Once the blocking scale passes the quantum-classical crossover one arrives at the true, classical equations supported by decoherence.

C. Radiation field

The external sources or boundary conditions in time determine the radiation field in classical electrodynamics. The non-trivial issue here is the realization of this well-understood circumstance within the variational principle. The point is that the radiation field enters, in a singular manner, into the variational scheme of the actions (54)-(55). The first sign of this complication is the vanishing of the near field propagator on the photon mass-shell due to the principal value prescription in its Fourier integral representation. Another way to see that the radiation field does not enter into the quadratic part of the action is to recall the remark made after Eq. (12), namely that the far field propagator is antisymmetric thus the radiation field drops out entirely from the \( \mathcal{O}(A^2) \) part of the action. This state of affairs is in agreement with the formal time reversal symmetry of the elementary processes because the field operators act forward and backward in time by annihilating and creating elementary excitations.

The retarded propagators may appear in the \( \mathcal{O}(A jj') \) terms of the action and are attached to the external sources. The \( \mathcal{O}(A^2) \) part of the action controls the restoring force acting on the oscillation around the stable vacuum thus the external source induces a singular response on the mass shell, rendering the mass-shell radiation field nondynamical, to be settled by the boundary conditions in time. In fact, a simple way to arrive at the desired equations of motion without separating explicitly the on- and off-shell components of the equation as in Eqs. (53) is to write the source in Eq. (55) as \( j' = Dn^{-1}A_{on} \) and note that the equation of motion \( A = A_{on} \) contains the mass-shell component with a formal 0/0 coefficient. This scheme is actually realized by the usual \( m^2 \rightarrow m^2 - i\epsilon \) prescription for the free action when applied in classical field theory.

When all fields arising from the reduplication of the degrees of freedom inherent in the double-time formalism are retained then the physical fields can be coupled to nonsymmetrical kernels. The result is the simple and natural preservation of the retarded and advanced solutions in the variational scheme. Such a treatment of the boundary conditions in time opens up possibilities to address the genuine, dynamical breakdown of time reversal invariance. For instance, the friction forces signal in classical physics that the coupling of our system to its environment breaks the time reversal invariance in a dynamical manner. Such kind of forces can easily be encoded in the variation principle of the double-time formalism. Another example is the dynamical building up of decoherence in a system due to a coupling to its environment with a gapless spectrum, to be seen explicitly below.

V. ONE-LOOP POLARIZATION TENSOR

We return in this Section to the loop expansion and present the photon-polarization tensor in the one-loop approximation. We shall work with the Fourier transform

\[
G^{\sigma\sigma'}_{(x\mu)(y\nu)} = \int_q e^{-ixq} \hat{G}^{\sigma\sigma'}_{q\mu\nu},
\]

where the integration in the Fourier space is denoted by

\[
\int_p = \int d^4p \frac{1}{(2\pi)^4}.
\]

The real and imaginary parts of the two-point function are defined in the space-time. Their Fourier transform is

\[
(\Re \hat{G})_q = \frac{1}{2} (\hat{G}_q + \hat{G}^*_{-q}), \quad (\Im \hat{G})_q = \frac{1}{2i} (\hat{G}_q - \hat{G}^*_{-q}).
\]
Our goal is to obtain
\[ \hat{G}_{\mu}(\tau\nu) = -\frac{i\hbar}{4} \sum_{\eta,\eta'} \int_p \text{tr}[\gamma^\mu G_{\eta+p}^{\sigma\nu} \gamma^\nu G_{\eta}^{\sigma\nu}] \] (60)
for a single charge in terms of the CTP propagator (B18). Because of the structure (12) it is sufficient to find \( \hat{G}^{++} \)
and \( \hat{G}^{+-} \). It will be useful to recall that this satisfies the Ward identity
\[ k^\mu \hat{G}_{\mu\nu} = 0. \] (61)
For the sake of simplicity we seek the detailed expressions below the pair creation threshold, \( q^2 < 4m^2 \), and keep the density and the temperature low enough to keep the charges nonrelativistic.

The CTP propagators are the sum of the contributions from the vacuum and from the environment. The latter is linear in the Fermi-Dirac distribution function \( n_p = 1/\{\exp(\beta(\epsilon_p - \text{sign}(p^0)\mu)) + 1\} \). As a result we have the similar separation in the polarization tensor (60), which will be the sum of three kinds of terms. The vacuum contributions are independent of \( n \). The \( O(n) \) terms contain a particle from the environment on the mass-shell. Finally, there will be \( O(n^2) \) terms where both particles come from the environment and are on the mass-shell.

As far as the environment is concerned, we shall have either finite temperature at vanishing density or finite density at vanishing temperature. We write the two-point function as \( \hat{G}^{++} \) and \( \hat{G}^{+-} \).

\[ \hat{G}^{++}_{\mathrm{vac},q} = \frac{\alpha}{3\pi} (g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}) \left\{ \frac{1}{3} + 2 \left( 1 + \frac{2m^2}{q^2} \right) \left[ \sqrt{\frac{4m^2}{q^2} - 1} \arccot \frac{4m^2}{q^2} - 1 - 1 \right] \right\}. \] (62)
with \( \alpha = e^2/4\pi \). The matrix element \( +\nu \),
\[ \hat{G}^{+-}_{\mathrm{vac},q} = -128\pi^3 i\alpha m^2 (g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}) \left( 1 + \frac{q^2}{2m^2} \right) \int_p \delta(q^2 + 2pq)\delta(p^2 - m^2)\Theta(-p^0 - q^0)\Theta(p^0) \] (63)
is vanishing below the pair creation threshold because it consists of on-shell amplitudes only.

The environment breaks the formal Lorentz covariance of \( \hat{G}^{++}_{\mathrm{env}} \). Let us denote the temporal unit vector by \( u^\mu = (1, 0) \) in the rest frame of the environment where the expressions for the distribution functions of Eqs. (B19) are valid. We use two invariants \( q^2 = -|q - u(q)|^2 \) and \( \nu = uq/|q| = \omega/|q| \) with \( q^\mu = (\omega, q) \) to parameterize the momentum dependence. For each CTP index, covariance with respect to three-dimensional rotations suggests that we use three three-dimensional scalars whose number is further reduced to two by gauge invariance.

These two sets of scalar are the easiest to obtain in terms of \( \hat{A} = g_{\mu\nu}\hat{G}^{\nu\mu} \) and \( \hat{B} = u\hat{G}u \). The final useful form is
\[ \hat{G}^{\mu\nu}_{\mu}(\tau\nu) = \hat{G}_{\mu}(\tau\nu) \left( \begin{array}{cc} \frac{1}{\nu} & \frac{1}{\nu} \\ \nu & \nu \end{array} \right) + \frac{1}{2} \left( \hat{G}_{\mu}(1 - \nu^2) - \hat{A}_{\mu} \right) \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \] (64)
where the space-time tensor structure is explicitly shown for the environment contributions with \( n = q/|q|, \ L = n \otimes n, \ T = 1 - \ L \).

The expression (64) makes the space-time tensor structure explicit. The CTP index structure is contained in the Lorentz-scalar CTP matrices \( \hat{A} \) and \( \hat{B} \) which will be detailed now. These matrices are the sum of the vacuum and environment contributions,
\[ \hat{X}_q = \hat{X}_{\mathrm{vac},q} + \hat{X}_{\mathrm{env},q} \] (65)
where \( X \) stands for \( \hat{A} \) or \( \hat{B} \). As far as the vacuum contribution is concerned in the framework of the gradient expansion it contains \( \hat{G}_{\mathrm{vac},q} \) only which gives
\[ (\Re\hat{G}_{\mathrm{vac}})_q = \frac{\alpha}{15\pi m^2} q^2 \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) = \frac{(\Re\hat{A}_{\mathrm{vac}})_q}{3(1 - \nu^2)} \] (66)
and \( \delta \mathcal{A}_{\text{vac}} = \delta \mathcal{B}_{\text{vac}} = 0 \) with the help of Eq. (62). The structure of Eq. (12) is preserved by the environmental contribution, and we find the real and imaginary parts

\[
(\Re X_{\text{env}}^{++})_q = \Re \int_p 2\pi \delta(p^2 - m^2)P \frac{X_{p,q}(n_p + n_{-p})}{q^2 + 2pq + i\epsilon}
\]

\[
(\Re X_{\text{env}}^{+-})_q = \frac{i}{2} \int_p 2\pi \delta(q^2 + 2pq)2\pi \delta(q^2 - m^2)\text{sign}(p^0 + q^0)(n_p + n_{-p})X_{p,q}
\]

\[
(\Im X_{\text{env}}^{++})_q = \frac{1}{2} \int_p 2\pi \delta(q^2 + 2pq)2\pi \delta(p^2 - m^2)[n_p(n_{q+p} - 1) + n_{-p}(n_{-p-q} - 1)]X_{p,q},
\]

where \( P \) denotes the principal value integral and the kinematical factors are

\[
\mathcal{A}_{p,q} = 8(m^2 - pq),
\]

\[
\mathcal{B}_{p,q} = 8\left[p'^2 + p^0 q^0 - \frac{1}{2}pq\right].
\]

We shall consider the cases of finite temperature and vanishing density and finite density and vanishing temperature when the one-particle distribution function \( n_p \) generates the scales \( k_{ch} = T \) and \( k_{ch} = k_F \), respectively and keep the environment in the nonrelativistic regime, \( T, k_F \ll m \) for simplicity. The frequency integrals are carried out in Eqs. (67) by means of the residuum theorem, resulting in three-dimensional integration over the momentum \( p \). By splitting these integrals into two parts, corresponding to \( |p| < m \) and \( |p| > m \) and by finding an upper bound for the latter contributions one can verify that the simple nonrelativistic expressions give reliable leading order approximations.

What are the kinematical regions contributing to the various pieces of the two-point function for a nonrelativistic environment? At low temperature and vanishing charge density it is enough to retain the \( \mathcal{O}(n) \) terms which decreases rapidly when \( |q| \) is increased. The situation changes drastically at finite density because the gap disappears in the excitation spectrum. \( \Re X^{++} \), given in the first equation of Eq. (67) describes a particle-hole pair where one member of the pair belongs to the environment and is on the mass shell. It is advantageous to use the dimensionless ratio

\[
r = \frac{q^2 + 2\omega q^0}{2q|k_{ch}|}
\]

which gives

\[
\omega \approx \frac{(q + rk_{ch} n)^2}{2m} - \frac{(rk_{ch})^2}{2m}
\]

indicating that the Pauli blocking allows particle-hole excitations at small \( q \) for \( |r| \approx 1 \). Thus \( \Re X^{++} \) is expected to take larger values for \( q \ll m \) at \( |r| \approx 1 \) only. Both the particle and the hole are on the mass-shell in the remaining expressions in Eqs. (67) because the time ordering is the only mechanism to produce off-shell amplitudes in the CTP propagators. The distribution function \( n_p \) restricts the integration approximately to \( |p| < k_{cr} \). The conditions \( p^2 = (p - q)^2 = m^2 \) for nonrelativistic four momentum \( p \) with \( p^0 \approx m \) give \( k_{cr} = pn \), the integration is over a plane, orthogonal to \( q \). The largest area corresponds to \( r = 0 \) and the remaining functions in Eqs. (67) tend to be large for \( r \approx 0 \).

The actual calculation of the integrals leads to the expressions

\[
(\Re X_{q}^{++}) = X_{q^0,q}^+ + X_{-q^0,q}^-,
\]

\[
(\Re X_{q}^{+-}) = i(X_{-q^0,q}^- - X_{q^0,q}^+),
\]

\[
(\Im X_{q}^{++}) = X_{q^0,q}^+ + X_{-q^0,q}^-.
\]
where the case of finite temperature and vanishing density gives

\[ A_q^+ = \frac{4\alpha m^2 T}{\pi |q|} \left( 1 + \frac{q^2}{2m^2} \right) L_q - \frac{4\alpha T^2}{\pi} \tilde{I}_q, \]

\[ A_q^- = -\frac{4\alpha m^2 T}{|q|} \left( 1 + \frac{q^2}{2m^2} \right) J_q, \]

\[ A_q^i = -\frac{4\alpha m^2 T}{|q|} \left( 1 + \frac{q^2}{2m^2} \right) K_q, \]

\[ B_q^+ = -\frac{4\alpha m^2 T}{\pi |q|} \left( 1 + \frac{q^2}{4m^2} + \frac{q^0}{m} \right) L_q - \frac{2\alpha T^2}{\pi} \tilde{I}_q, \]

\[ B_q^- = -\frac{4\alpha m^2 T}{|q|} \left( 1 + \frac{q^2}{4m^2} + \frac{q^0}{m} \right) J_q, \]

\[ B_q^i = -\frac{4\alpha m^2 T}{|q|} \left( 1 + \frac{q^2}{4m^2} + \frac{q^0}{m} \right) K_q, \]

(72)

where

\[ I_q = \int_0^\infty dz z \frac{n_z}{\tilde{\omega}_z} \ln \left| \frac{r_z + z}{r_z - z} \right|, \]

\[ \tilde{I}_q = \int dz z^2 \frac{n_z}{\tilde{\omega}_z}, \]

\[ J_q = \int_0^\infty dz z \frac{n_z}{\tilde{\omega}_z} \Theta(1 - r_z) \text{sign}(\text{sign}(q^0) \tilde{\omega}_z + \beta q^0), \]

\[ K_q = -\int_0^\infty dz z \frac{n_z}{\tilde{\omega}_z} \Theta(1 - r) \frac{1 + e^{-\sqrt{(5m)^2 + z^2} + 1}}{1 + e^{-\sqrt{(5m)^2 + z^2} + 1}}, \]

(73)

with \( r_z = \frac{q^2}{2q^m T} + \frac{q^0}{q^m} \tilde{\omega}_z, \tilde{\omega}_z = \sqrt{(5m)^2 + z^2} \) and \( n_z = 1/(e\sqrt{(5m)^2 + z^2} + 1) \). The results for finite density and vanishing temperature are

\[ A_q^+ = \frac{2\alpha k_F^2 m}{\pi |q|} \left( 1 + \frac{q^2}{2m^2} \right) L_q, \]

\[ A_q^- = -\frac{2\alpha k_F^2 m}{|q|} \left( 1 + \frac{q^2}{2m^2} \right) M_q, \]

\[ B_q^+ = \frac{2\alpha k_F^2 m}{\pi |q|} \left( 1 + \frac{q^2}{4m^2} + \frac{q^0}{m} \right) L_q, \]

\[ B_q^- = -\frac{2\alpha k_F^2 m}{|q|} \left( 1 + \frac{q^2}{4m^2} + \frac{q^0}{m} \right) M_q, \]

\[ A_q^i = -\frac{2\alpha k_F^2 m}{|q|} \left( 1 + \frac{q^2}{2m^2} \right) N_q, \]

\[ B_q^i = -\frac{2\alpha k_F^2 m}{|q|} \left( 1 + \frac{q^2}{4m^2} + \frac{q^0}{m} \right) N_q, \]

(74)

with

\[ L_q = r + \frac{1}{2}(1 - r^2) \ln \left| \frac{r + 1}{r - 1} \right|, \]

\[ M_q = \Theta(1 - |r|)(1 - r^2), \]

\[ N_q = \begin{cases} 1 - r^2 & |q| > 2k_F, -1 < r < 1 \\ 1 - \frac{q^2}{2m^2} & |q| < 2k_F, -1 - \frac{|q|}{2k_F} < r < 1 - \frac{|q|}{2k_F} \end{cases} \]

(75)

The variable \( r \) in these equations is given by Eq. (69) after the replacement \( k_{ch} = k_F \) and \( \omega_p = m \).

\( L \) is the relativistic generalization of the Lindhard function. The last two expressions in Eqs. (74) give the relativistic version of the decoherence factor of the Coulomb field, considered in Ref. [22]. The expressions (72)-(75) are valid for nonrelativistic electron gas but naturally comprise the relativistic kinematics for photons.
VI. REDUCED DENSITY MATRIX

We use now the functional in Eq. (8) to obtain the reduced density matrix for the EM field. A single charged field is considered for simplicity and it will be treated by CTP boundary conditions and the OTP boundary conditions are used for the EM field. The generator functional in Eq. (8) with $\hat{a} = \hat{j} = 0$, considered as a functional of the final field configurations gives the reduced density matrix

$$\rho[A^+_f \mathbf{x}, A^-_f \mathbf{x}] = \int D[\hat{A}] \left( \prod_n D[\hat{\psi}_n] D[\hat{\bar{\psi}}_n] \right) e^{i \sum_n [\hat{\bar{\psi}}_n (\hat{\mathcal{D}} \dot{\hat{\psi}}_n) - \hat{\psi}^\dagger_n (\hat{\mathcal{D}} \dot{\hat{\bar{\psi}}}_n) + \frac{1}{2} \hat{\mathcal{D}} \dot{\hat{\psi}}_n \dot{\hat{\bar{\psi}}}_n]}.$$  

(76)

The integration over the charged field can be carried out with the result

$$\rho[A^+_f \mathbf{x}, A^-_f \mathbf{x}] = \int D[\hat{A}] e^{i S_{eff}[^{\hat{A}}]}$$

(77)

where the effective bare action is

$$S_{eff}[^{\hat{A}}] = \frac{1}{2} \hat{\mathcal{A}} (\hat{\mathcal{D}}^{-1} - \Pi) \hat{\mathcal{A}} + O(\hat{\mathcal{A}}^3),$$

(78)

$\hat{\Pi}$ being given by Eq. (41). This functional integral contains information about the dynamics described by the density matrix in a manner similar the path integral describes the transition amplitude between pure states in the single time formalism. The real part $\Re S[^{\hat{A}}]$ is identical to $\Gamma[^{\hat{A}}]$, obtained as the Legendre transform of $\Re W[^{\hat{\bar{\psi}}}]$ for $\hat{a} = 0$, it determines the expectation value of the EM field and will be used to extract the polarization induced by the charges. The imaginary part controls the width of the peak in the reduced density matrix, and the decoherence and consistency of the EM field.

The normal modes of the quadratic part of the action in the path integral expression for the transition amplitudes are plane waves. It is easy to see that $\Pi_0$ is not diagonalizable in the time axis index only. The normal modes of the CTP action (78) couple the wave vector $\mathbf{q} = (\omega, \mathbf{q})$ with the CTP index and are labeled by the wave vector, space-time and CTP indices. The dynamical role of each normal mode is characterized by three numbers,

$$\Pi^{nm}_{q}, \Pi^{mf}_{q}, \Pi^{mf}_{q}$$

by applying the parametrization (12) for the current-current two-point function $\hat{G}$. It is easy to check that the Fourier transforms $\Pi^n_{q}$ and $\Pi^f_{q}$ are real and $\Pi^f_{q}$ is purely imaginary. In the one-loop approximation the real quantities arise from the interactions within a single time axis and the purely imaginary quantity corresponds to correlation between the time axes. This correlation, realized by Green functions connecting the two time axes, arises because the charged particles which are exchanged in the process represented by the Green functions are entangled with the EM field.

We shall consider these quantities in the $(q, \omega)$ space. The zeros of $\Pi^{nm}_{q,\omega}$ locate the mass-shell of the quasiparticles, represented by the normal modes of the diagonal, $++$ or $--$ blocks of the quadratic bare action. The imaginary part of the two-point function in the space-time, which is given by $i \Pi^{nf}_{q,\omega}$ in Fourier-space, controls two, superficially different dynamical processes. On the one hand, the decay of the quasiparticles, the inverse life-time of a quasiparticle can be identified by $\Pi^{nf}_{q,\omega}$ evaluated on the mass-shell. On the other hand, according to the structure displayed in Eq. (12), this parameterizes the suppression of the off-diagonal matrix elements of the reduced density matrix, realized by the mode $\omega, q$ of the path integral. Thus the consistency and decoherence of the quasiparticle modes on the mass-shell have the same dynamical origin as the finite life-time of the quasiparticles. This is the expected relation between the dynamical breakdown of the time reversal invariance and the classical limit.

We now turn to the discussion of the qualitative features of $\Pi^n_{q}$ and $\Pi^f_{q}$ in describing the polarizability of the charges and the impact of $\Pi^f_{q}$ on the classical limit.

A. Electric and magnetic susceptibilities

To make contact with the usual three-dimensional notation we introduce the parametrization $A^\mu = (\phi, \mathbf{A})$ of the vector potential, giving rise to $\mathbf{E} = -\nabla \phi - \partial_0 \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. We consider quadratic action in the EM field in a
formal manner, as a book-keeping device for the equation of motion (51), namely we assume that the actions contain the inverse of the retarded Green function. On the one hand, we can introduce the retarded electric and magnetic susceptibilities χ and ˜χ, respectively with the phenomenological parametrization

\[ S_0[A] = \frac{1}{2} \int_{x,y} (E_x \epsilon_{xy} E_y - B_x \mu_{xy} B_y) \]

of the Maxwell action. On the other hand, the effective action for the EM field expectation value yielding the equation of motion is

\[ \Gamma = \frac{1}{2} AD^{r-1}A \]

where \( D^{r-1} = D_{0}^{r-1} - e(\tilde{G}^{++} - \tilde{G}^{+-})e \) according to Eq. (47). Therefore the equivalence of the two quadratic form, expressed by the equation

\[ q^2 \begin{pmatrix} 1 & -n \nu \\ -n \nu & \nu^2 I - T \end{pmatrix} - B^r \begin{pmatrix} 1 & n \nu \\ n \nu & \nu^2 L \end{pmatrix} + \frac{1}{2} q^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = q^2 \begin{pmatrix} 1 & 1 \\ -n \nu & \nu^2 I - T \end{pmatrix} - (\tilde{\mu} - \epsilon) \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \]

follows in Fourier space with \( X^r = X^{++} - X^{+-} \). The left hand side contains the free inverse propagator and the retarded self energy as given by Eqs. (64) and (66)-(67). The right-hand side is \( \delta^2 S_0[A] / \delta A_{\nu \nu} \delta A_{\nu \nu} \) with \( \nu = \omega / |q| \), \( n = q / |q| \), \( L = n \otimes n \), and \( T = I - L \). We insert into the right-hand side the expressions \( \epsilon = 1 + \chi \) and \( \tilde{\mu} = 1 + \tilde{\chi} \) which gives, after canceling the free inverse propagator on the left-hand side by the contributions of the one on the right-hand side

\[ \chi_q = -\frac{B'_q}{q^2} \]

\[ \tilde{\chi}_q = \frac{B'^r_q (1 - 3 \nu^2) - A'^r_q}{2q^2} \]

Let us now see the more detailed, numerical features of the polarization functions at finite density, obtained by using Eqs. (71), (74)-(75) to compute \( X_q^r = X^{++}_q - X^{+-}_q \) in Eqs. (83). The susceptibilities display a divergent structure which is well known for nonrelativistic Coulomb gas. The magnetic sector and the relativistic effects for the EM field are retained in this calculation which slightly modifies the picture. The susceptibilities, \( q^2 \epsilon_{\omega,q} \) and \( q^2 \tilde{\mu}_{\omega,q} \) are shown in Figs. 1 for \( m = 1 \) and \( \mu = 0.1 \). They all take appreciable values for \( |r| \approx 1 \), cf. Eqs. (74)-(75) and (70). The quasiparticle conditions \( \Re \epsilon_{\omega,q} = 0 \), \( \Re \tilde{\mu}_{\omega,q} = 0 \) have solution along the valley of the line \( r = 1 \), shown on the \( (|q|, \omega) \) plane in Fig. 2. The approximately linear parts in the electric and magnetic sectors belong to the zero sound, the back turning sections correspond to the plasmonlike excitations. The EM field may have relativistic energy and momenta and the relativistic correction factors \( O(q^2 / 2m^2) \) in Eqs. (74) modify the plasmon lines in an essential manner. They prevent the plasmons from becoming long range and push the frequency up sharply as the wave vector tends to zero in the electric sector, as seen in Fig. 2 (a). For \( \omega \geq 2 \) we naturally run into the pair creation singularities. The normal modes of the magnetic sector behave in a rather peculiar manner, the plasmon line, shown in Fig. 2 (c) follows approximately a nonrelativistic dispersion relation curve of mass \( m_{eff} = 10^{-3} \) even in the relativistic domain.

The screening of the electric sector manifests itself in the nonvanishing value of the product \( q^2 \Re \epsilon \) in the limit \( q \to 0 \) and the resulting infrared divergence in \( |\epsilon| \) as shown in Fig. 3 (a). The valley in \( |\epsilon| \) identifies the strongly coupled modes, they are along the plasmon line of Fig. 2 (a) and on the continuation of the zero-sound line. The zero-sound line shown in Fig. 2 (a) is weakly coupled due to the short life-time. The magnetic sector is qualitatively similar, \( |\tilde{\mu}| \) plotted in Fig. 3 (b) indicates strongly coupled magnetic plasmons and weakly coupled magnetic zero sound modes. The magnetic field is screened as well as by the nonvanishing value of \( q^2 \Re \tilde{\mu} \) in Fig. 1 (c) in the infrared, \( q \to 0 \) limit.

The physics is simpler at low temperature and vanishing density due to the gap in the excitation spectrum for the charges. One finds qualitatively similar frequency and wave vector dependence as displayed in the figures above except that the overall scale is suppressed by \( \exp(-m/T) \).

**B. Consistency and decoherence**

The reduced density matrix, given by Eq. (76) can always be diagonalized and the resulting basis consists of completely decohered states. But one can gain no more insight into the classical limit by using this basis. Instead, we
FIG. 1: The susceptibilities, (a): \( q^2 \Re \chi = q^2 (\Re \epsilon - 1) \), (b): \( q^2 \Im \epsilon \), (c): \( q^2 \Re \tilde{\chi} = q^2 (\Re \tilde{\mu} - 1) \), (d): \( q^2 \Im \tilde{\mu} \) as functions of \( \omega \) and \( |q| \).

FIG. 2: Quasi-particles on the \(|q|, \omega\) plane. (a): In the electric sector they are the roots of the equations \( \Re \epsilon = 0 \). The zero sound and the plasmon modes correspond to the straight line starting at the origin and after the turning point, respectively. The magnetic quasiparticles are at \( \Re \tilde{\mu} = 0 \), (b) is around the origin and (c) is on a larger part of the \(|q|, \omega\) plane.

are interested in the role of physically motivated states which persist at both side of the quantum-classical transition. These correspond to quasiparticles, defined by the \( \mathcal{O}(A^2) \) part of the bare action (78). Therefore, we seek the contribution of the quasiparticles to decoherence in what follows. It is easy to see by means of the structure (12) of \( \hat{G} \) and the expression (41) for the self energy that the imaginary part of the effective bare action \( \Im S[A] \) of Eq. (78)
depends on the combination $A^+ - A^-$ of the EM field only. Eq. (82) can be used again to parameterize it as

$$\Im S[\hat{A}] = \frac{1}{2}\int_x (E^d\epsilon^d - B^d\tilde{\mu}^d B^d)$$

(84)

with $E^d = E^+ - E^-$, $B^d = B^+ - B^-$ where the decoherence parameters

$$\epsilon^d_q = \frac{B^i_q}{q^2},$$

$$\tilde{\mu}^d_q = \frac{E^i_q(1 - 3\nu^2) - A^i_q}{q^2},$$

(85)

are expressed in terms of the imaginary parts introduced in Eq. (71) and are plotted in Fig. 4. $\Im S[\hat{A}]$ governs the suppression of each plane wave mode along the off-diagonal direction in the reduced density matrix. It is easy to see that this functional is semidefinite because it measures the phase space available for particle-antiparticle or particle-hole excitations which incorporate decoherence. Its value, taken on the quasiparticle line of Fig. 2, yields the inverse life-time. We see that the zero-sound quasiparticles are rendered unimportant by their short life-time. Naturally, higher loop contributions to the effective action generate imaginary part and finite life-time beyond the zero-sound line by the multi particle-antiparticle or particle-hole excitations, but this remains a weak effect.

When considered from the point of view of the quantum-classical transition, the effective bare action monitors the impact of the entanglement between the EM field and the charges. One measure of this entanglement is decoherence which corresponds to an instantaneous state. But we find more information in this action. The suppression of the contributions in the path integral representation of the density matrix as the functional of the difference of the two field trajectories carries information about the consistency of histories of the EM field [3]. The contribution of a plane wave, a normal mode of the one-loop dynamics of the EM field, to the conditional probability distributions is closer to the one expected in the standard, classical probability if $\Im S[\hat{A}]$ increases with $A^+ - A^-$ because the quantum interference term which violates the additivity of probabilities is suppressed by $e^{-\Im S}$ [23].

The lesson is that the classical probabilities are recovered by plane wave modes whose dispersion relation is that of the short life-time quasiparticles. Their short decay time assures the dynamical breakdown of the time reversal invariance, needed for the classical limit. These modes which maximize $\Im S$ are located around the long wavelength part of the zero sound curve at finite density and vanishing temperature. The suppression of the off-diagonal elements of the reduced density matrix for these plane wave EM fields shows that these field configurations become good pointer states in the classical limit.

VII. CONCLUSIONS

The problem of deriving classical electrodynamics from QED is pursued further in this work. It is based on the monitoring of subclassical fields, and the expectation values of local operators with space-time resolution limited...
formally by the UV cutoff of QED. The final goal is to weaken the resolution and thereby recover the macroscopic, classical dynamics for the space-time dependence of the expectation values. As a preparatory step, we make a coarse graining by eliminating the charged degrees of freedom with a fixed, microscopic space-time resolution. Polarization effects, arising in this manner, reflect competing coherent and decoherent phenomena, both at the elementary and at the collective level.

The double-time axes formalism used in this work was motivated by the implementation of initial conditions, as opposed to the boundary value problems within the variational method. This step leads to the double-time axis formulation in a natural manner. The double-time axes formalism is well suited to follow the eventual breakdown of time reversal invariance at the quantum level, driven dynamically by degeneracies or externally, by the boundary conditions in time. The breakdown of the time reversal invariance appears at the level of the expectation value of the EM field as the radiation field. A simple microscopic way of separating the EM field into near and far components emerges as well for they represent pure state self-interactions for the charges and generate entanglement between charges and the EM field, respectively.

Some well-known issues appear in a new point of view, such as the origin of the dynamics for the free-field components in the variational equations. The free-field component belongs to the null space of the kernel of the free action and drops out from the linearized equation of motion. This problem is avoided in the double-time formalism, and Feynman’s iε prescription seems to be the analogous treatment in the single time formalisms of both quantum and classical field theory.

Another question which can easily be addressed is the relation between the classical limit and the dynamical breakdown of the time reversal invariance. The block structure of the CTP propagators shows that a shorter lifetime of the quasiparticles means more consistent EM field trajectories and a narrower peak in the absolute magnitude of the reduced density matrix $|\rho(x, y)|$ around the diagonal, $x = y$. Another way to see this relation by means of the EM field expectation value is to note that the purely imaginary far field component of the CTP polarization tensor, $\Pi^f_{\omega, q}$, plays a double role in the dynamics. On the one hand, it characterizes the mixed states arising from the charge-EM field entanglement, and on the other hand, it leads to a phase delay in the propagation of the EM field and thereby breaks the time reversal invariance.

These results open the way for further developments, and we close by mentioning some of them. The subclassical fields are expectation values but they are not necessarily classical quantities because they may display the microscopic structure in space-time where decoherence is not efficient. In order to identify initial conditions which lead to classical fields, we need to control either the decoherence itself or the characteristic scale of the field expectation value. The latter is determined by the two-point functions and its handling requires the extension of the present scheme to external sources coupled to the bi-local product of the elementary fields. Such an explicit treatment of the correlation functions would, in addition, allows us to test different collapse scenarios of the wave function.

Another extension of the present study is the detailed analysis of the effective action for the EM field and the electric current to derive a variational scheme for magnetohydrodynamics. The more accurate determination of the susceptibilities from first principles may lead to new experimental devices to identify some signatures of the underlying microscopic quantum theory by precision measurements in classical electrodynamics.
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Appendix A: CTP propagators in the vacuum

The CTP vacuum propagators for scalar, vector, and spin-half particles are summarized in this Appendix. The relation between expectation values and the parametrization of the propagator is achieved by path integration, and the explicit expression, for the different expectation values are computed in the operator formalism.

1. Neutral bosons

We introduce the propagator for a boson field $\phi$ with or without vector indices by means of the free generator functional,

$$e^{iW[j]} = \int D[\hat{\phi}] e^{\frac{i}{2}\hat{\phi}^\dagger D^{-1} \hat{\phi} + ij\cdot \hat{\phi}} = e^{-\frac{1}{2}j^\dagger D j}$$

which gives, in the Heisenberg representation,

$$i \frac{\delta^2 W[j]}{\delta j_a^+ \delta j_b^+} = \sum_n (0|n)\langle n|T[\phi_a \phi_b]|0\rangle = \langle 0|T[\phi_a \phi_b]|0\rangle = iD_{ab}^{++},$$

$$i \frac{\delta^2 W[j]}{\delta j_a^+ \delta j_b^-} = \sum_n (0|\bar{T}[\phi_a \phi_b]|n\rangle\langle n|0\rangle = \langle 0|T[\phi_b \phi_a]|0\rangle^* = iD_{ab}^{--},$$

$$i \frac{\delta^2 W[j]}{\delta j_a^- \delta j_b^-} = \sum_n (0|\phi_b|n\rangle\langle n|\phi_a|0\rangle = \langle 0|\phi_b \phi_a|0\rangle = iD_{ab}^{-+},$$

$$i \frac{\delta^2 W[j]}{\delta j_a^- \delta j_b^+} = \sum_n (0|\phi_a|n\rangle\langle n|\phi_b|0\rangle = \langle 0|\phi_a \phi_b|0\rangle = iD_{ab}^{+-},$$

written as

$$i \begin{pmatrix} D & D^{++} \\ D^{+-} & D^{--} \end{pmatrix}_{ab} = i \begin{pmatrix} D & -D^{++} \\ D^{+-} & D^{--} \end{pmatrix}_{ab} = \begin{pmatrix} \langle 0|T[\phi_a \phi_b]|0\rangle & \langle 0|\phi_b \phi_a|0\rangle \\ \langle 0|\phi_a \phi_b|0\rangle & \langle 0|T[\phi_b \phi_a]|0\rangle^* \end{pmatrix}$$

by means of the notation $D^{++} = D$. The basic CTP identity,

$$T[\phi_a \phi_b] + \bar{T}[\phi_a \phi_b] = \phi_a \phi_b + \phi_b \phi_a,$$

written as

$$D - D^\dagger = D^{--} - D^{++}$$

allows us to parameterize the propagator in terms of three real functions, the near and far field Green functions, $D^n$ and $D^f$, respectively, and the common imaginary part, $D_i = 3D$, as in Eq. (12). The detailed form of the real and imaginary parts,

$$\Re D_{aa'} = -\frac{1}{2}\text{sign}(t-t')i\langle 0|\{\phi_a, \phi_{a'}\}|0\rangle = D_{aa'}^n,$$

$$D_{aa'} = -\frac{1}{2}\langle 0|\{\phi_a, \phi_{a'}\}|0\rangle,$$

$$\Re D_{aa'}^{-} = -\frac{i}{2}\langle 0|\{\phi_a, \phi_{a'}\}|0\rangle = D_{aa'}^f,$$

(A6)
justifies the introduction of retarded and advanced propagators
\[
D^\alpha_{a\alpha'} = D^a_{a\alpha'} \pm D_a^\alpha, \\
= \mp \Theta(\pm(t-t'))i(0)[\phi_a, \phi_{a'}]0).
\] (A7)

The expectation values of the propagator (A3) are easiest to compute in the operator formalism by means of the quantum field
\[
\phi(x) = \int_k [a(k)e^{-ikx} + a^\dagger(k)e^{ikx}],
\] (A8)
where the notation
\[
\int_k = \int \frac{d^3k}{(2\pi)^3 2\omega_k}
\] (A9)
\[\omega_k = \sqrt{m^2 + k^2}\] has been introduced. The nonvanishing canonical commutation relation for the creation and annihilation operators,
\[
[a(p), a^\dagger(p')] = (2\pi)^3 2\omega_p \delta(p - p'),
\] (A10)
gives, in a trivial manner,
\[
iD^{\pm\mp}(x, x') = \int_k e^{\pm ik(x-x')}
\] (A11)
for a scalar particle. The multiplication by the Heaviside function,
\[
\Theta(t-t')iD^{\mp+}(x, x') = -\frac{1}{i} \int_\omega \frac{e^{-i\omega(t-t')}}{\omega + i\epsilon} \int_k \frac{e^{-i\omega_k(t-t')+i k(x-x')}}{\omega - \omega_k + i\epsilon},
\]
\[
\Theta(t'-t)iD^{+-}(x, x') = -\frac{1}{i} \int_\omega \frac{e^{-i\omega(t'-t')}}{\omega + i\epsilon} \int_k \frac{e^{-i\omega_k(t'-t)+i k(x'-x)}}{\omega + \omega_k - i\epsilon},
\] (A12)
gives the usual causal propagator and, we finally find
\[
\begin{pmatrix}
D & D^{+-} \\
D^{+-} & D^{--}
\end{pmatrix}_k
= \begin{pmatrix}
\frac{1}{k^2 - m^2 + i\epsilon} & -2\pi i\delta(k^2 - m^2)\Theta(-k^0) \\
-2\pi i\delta(k^2 - m^2)\Theta(k^0) & \frac{1}{k^2 - m^2 - i\epsilon}
\end{pmatrix}
\] (A13)
at vanishing temperature.

The Fourier transforms of the real and imaginary parts of the causal propagator are
\[
D^0_k = P\frac{1}{k^2 - m^2}, \\
D_{ik} = -\pi\delta(k^2 - m^2).
\] (A14)

In computing the real and imaginary parts of \(D^--\), one has to keep in mind that they are defined in Eqs. (A6) in the real space-time as opposed to Fourier space, and the expression
\[
D^{+-} = -\pi i \int_k e^{-ik(x-x')}\delta(k^2 - m^2)[\Theta(k^0) - \Theta(-k^0)] - \pi i \int_k e^{-ik(x-x')}\delta(k^2 - m^2)[\Theta(k^0) + \Theta(-k^0)]
\] (A15)
yields
\[
D^I_k = -\pi i\delta(k^2 - m^2)e(k^0)
\] (A16)
These operators transform under the charge conjugation transformation values computed in the particle and the antiparticle sectors, and it originates from the charge conjugation invariance of the commutation relations (A23) and of the vacuum. When the propagator is expressed in terms of the same as in the case of neutral particles. The non-Hermitian quantum field

\[ D^{-1}D^f = 0, \]  

(cf. Eq. (D11) in [20] for the regulated expression.

2. Charged Bosons

In the case of a non-Hermitian boson field the free generator functional is

\[ e^{iW[j^, j]} = \int D[\hat{\phi}^\dagger]D[\hat{\phi}] e^{i\phi^\dagger \cdot D^{-1} \cdot \phi + i\phi^\dagger \cdot j} = e^{-i\phi^\dagger \cdot D \cdot j}. \]  

and the CTP propagator block-matrix elements, identified in a manner analogous to Eqs. (A2), are

\[ i \begin{pmatrix} D & D^{+-} \\ D^{+} & D^{-} \end{pmatrix} = \begin{pmatrix} \langle 0|T[\phi_a \phi_a]|0 \rangle & \langle 0|\phi_a \phi_a^\dagger|0 \rangle \\ \langle 0|\phi_a \phi_a^\dagger|0 \rangle^* & \langle 0|T[\phi_0 \phi_0^\dagger]|0 \rangle^* \end{pmatrix} \]  

and they can be written as in Eq. (12) due to the CTP identity

\[ T[\phi_x \phi_x^\dagger] + T^*[\phi_x \phi_x^\dagger] = \phi_x^\dagger \phi_x + \phi_x \phi_x^\dagger, \]

\[ D - D^\dagger = D^{+ -} + D^{+ -}. \]  

It is the charge conjugation invariance which guarantees that the detailed form of the functions \( D_i, D^a \) and \( D^f \) is the same as in the case of neutral particles. The non-Hermitian quantum field

\[ \phi(x) = \int_k [a(p)e^{-ikx} + b^\dagger(p)e^{ikx}] \]  

involves the creation and annihilation operators whose nonvanishing canonical commutators are

\[ [a(p), a^\dagger(p')] = [b(p), b^\dagger(p')] = (2\pi)^3 2\omega_p. \]  

These operators transform under the charge conjugation transformation \( C, C^2 = 1 \), as \( C^\dagger a(p)C = b(p), C^\dagger b(p)C = a(p) \) and \( C^\dagger \psi_x C = \psi_x^\dagger \). Now, the relation \( D^{+ -} = D^{- +} \) actually establishes the equivalence of vacuum expectation values computed in the particle and the antiparticle sectors, and it originates from the charge conjugation invariance of the commutation relations (A23) and of the vacuum. When the propagator is expressed in terms of \( D \) and \( D^{- +} \), the complex conjugation like in Eq. (A5) reverses the order of \( \psi \) and \( \psi^\dagger \), and one needs another complex conjugation, this time embedded into the charge conjugation operator, to place the dagger at the appropriate operator in \( D^{+ -} \),

\[ iD^{+ -} = \langle 0|\phi^\dagger \phi_a|0 \rangle = \langle 0|C^\dagger \phi_a \phi|^0 \rangle^* = iD^{- +} C^*. \]  

In the presence of a noncharge conjugate invariant vacuum, \( D^{+ -} \neq D^{- +} \).

3. Fermions

The generator functional for the propagator,

\[ e^{iW[j^, j]} = \int D[\hat{\psi}]D[\hat{\bar{\psi}}] e^{i\bar{\psi} \cdot \hat{G}^{-1} \cdot \psi + i\phi^\dagger \cdot j} = e^{-i\phi^\dagger \cdot \hat{G} \cdot j}, \]  

and

\[ D^a_k = \frac{1}{k^2 - m^2 \pm i\varepsilon \text{sign}(k^0)}. \]  

Note the remarkable feature of the near field propagator \( D^n \) in the first expression of Eqs. (A14). It is the only part of the CTP propagator which is nonvanishing off the mass shell, the support of the other functions \( D_i \) and \( D^f \) is just the mass shell. Furthermore, \( D^a_k \) diverges in momentum space as we approach the mass shell, \( k^2 \to m^2 \); therefore

\[ D^{n-1}D^f = 0, \]  

(A18)
leads to
\[
i \begin{pmatrix} G & G^{+-} \\ G^{+} & G^{-} \end{pmatrix}_{xy} = \begin{pmatrix} (0|T[\bar{\psi}_x \gamma_x \psi_y]|0) & (0|T[\bar{\psi}_y \gamma_x \psi_x]|0) \\ 0|T[\bar{\psi}_x \gamma_x \psi_y]|0 & 0|T[\bar{\psi}_y \gamma_x \psi_x]|0 \end{pmatrix}
\] (A26)

by following the strategy of Eqs. (A2). The CTP identity,
\[
T[\bar{\psi} \psi] + \bar{T}[\bar{\psi} \psi] = -\bar{\psi} \psi + \psi \bar{\psi}
\] (A27)

now leads to the parametrization
\[
\begin{pmatrix} G & G^{+-} \\ G^{+} & G^{-} \end{pmatrix}_{xy} = \begin{pmatrix} G^n + iG_i & G^f - iG_i \\ -G^f - iG_i & -G^n + iG_i \end{pmatrix}.
\] (A28)

The field operator is written as
\[
\psi(x) = \int \frac{m}{\omega_k} \sum \alpha \left[ b_\alpha(k) u^{(\alpha)}(k) e^{-ikx} + d_\alpha^\dagger(k) v^{(\alpha)}(k) e^{ikx} \right]
\] (A29)

where \((k - m)u(k) = (k + m)v(k) = 0\) and the creation and annihilation operators are defined by their nonvanishing anti-commutation relations
\[
\{b_\alpha(p), b_\beta^\dagger(p')\} = \{d_\alpha(p), d_\beta^\dagger(p')\} = (2\pi)^3 \frac{\omega_p}{m} \delta_{\alpha,\beta} \delta(p - p').
\] (A30)

The charge conjugation is represented by \(C^{-1} \psi C = U \psi^*\) with \(U = i\gamma^2\) and the identity (A27) allows us to write
\[
i(G^{+-})_{xy} = (U(0|C^\dagger \bar{\psi}_x \gamma_x \psi_y C|0)^* U)_{\alpha \beta} = (U iG^{--} C U)_{\alpha \beta}
\]
\[
iG^{--}_{xy} = (\gamma_0 iG_{yx} \gamma^0)_{\alpha \beta}.
\] (A31)

The Fourier decomposition (A29) gives the explicit expression
\[
\begin{pmatrix} G & G^{+-} \\ G^{+} & G^{-} \end{pmatrix} = (i\partial_x + m) \begin{pmatrix} D & -D^{+-} \\ -D^{+} & D^{--} \end{pmatrix},
\] (A32)

where the scalar propagator in the right hand side is given by Eq. (A13), in a standard manner.

Appendix B: CTP propagators at finite temperature and density

We extend the propagators recorded in the previous appendix for the presence of heat and particle reservoirs.

1. Neutral bosons

It is advantageous to use a finite quantization box, where the commutation relation (A10) reads
\[
[a(p), a^\dagger(p')] = 2V \omega_p \delta_{p,p'}
\] (B1)

and the quantum field becomes
\[
\phi(x) = \frac{1}{2V \omega_k} \sum_k [a(k) e^{-ikx} + a^\dagger(k) e^{ikx}].
\] (B2)
The relation

\[
\text{Tr}[e^{-\beta H_0} \phi(x)\phi(x')] = \sum_{\{n\}} \sum_k \frac{e^{-ik(x-x')}}{2V\omega_k} \langle\{n\}|e^{-\beta H}|\{n\}\rangle \\
+ \sum_{\{n\}} \sum_k \frac{1}{4V^2\omega_k} (e^{ik(x-x')} + e^{-ik(x-x')}) \langle\{n\}|e^{-\beta H} a^\dagger(k)a(k)|\{n\}\rangle
\]

\[
= \int_k e^{-ik(x-x')} \prod_n e^{-\beta n\epsilon_k} \sum \frac{1}{2V\omega_k} (e^{ik(x-x')} + e^{-ik(x-x')}) \langle\{n\}|e^{-\beta H} n(k)|\{n\}\rangle
\]

\[
= \prod_k \frac{1}{1 - e^{-\beta \epsilon_k}} \left[ iD_{T=0}^+(x,x') + \int_k (e^{ik(x-x')} + e^{-ik(x-x')}) \sum_n ne^{-\beta \epsilon_k} \right], \quad (B3)
\]

where \(H_0\) is the free Hamiltonian gives

\[
\frac{iD^+(x,x')} = \frac{\text{Tr}[e^{-\beta H} \phi(x)\phi(x')]}{\text{Tr}[e^{-\beta H}]} = iD_{T=0}^+(x,x') + \Delta D(x,x'), \quad (B4)
\]

with \(D_{T=0}^+\) being taken from Eq. (A11) and

\[
\Delta D = \int_k e^{-ik(x-x')} \frac{2\pi \delta(k^2 - m^2)}{e^{\beta \epsilon_k} - 1}. \quad (B5)
\]

Similar steps produce

\[
\frac{iD^-(x,x')} = \frac{\text{Tr}[e^{-\beta H} \phi(x)\phi(x')]}{\text{Tr}[e^{-\beta H}]} = iD_{T=0}^-(x,x') + i\Delta D(x,x'). \quad (B6)
\]

The lesson is that finite temperature modifies the imaginary part of the propagator only, where

\[
\Delta D_{ik} = -\frac{2\pi \delta(k^2 - m^2)}{e^{\beta \epsilon_k} - 1} \quad (B7)
\]

and \(\Delta D^R = \Delta D^A = 0\). The independence of the retarded and advanced propagators from the temperature can be understood by recalling that these propagators correspond to fixed initial or final conditions.

2. Charged bosons

The non-Hermitian field

\[
\phi(x) = \frac{1}{2V\omega_k} \sum_k [a(k)e^{-ikx} + b^\dagger(k)e^{ikx}] \quad (B8)
\]

involving the operators which satisfy the nonvanishing commutation relations

\[
[a(p), a^\dagger(p')] = [b(p), b^\dagger(p')] = 2V\omega_p \delta_{p,p'} \quad (B9)
\]

give rise to

\[
\text{Tr}[e^{-\beta(H-\mu Q)} \phi(x)\phi(x')] = \int_k e^{-ik(x-x')} \prod_k e^{-\beta n(\epsilon_k-\mu)} \sum_n \frac{1}{2V\omega_k} \langle\{n\}|e^{-\beta(H-\mu Q)} (b^\dagger(k)b(k)e^{ik(x-x')} + a^\dagger(k)a(k)e^{-ik(x-x')})|\{n\}\rangle
\]
which in turn gives

\[ iD^+(x, x') = \frac{\text{Tr}[e^{-\beta(H - \mu Q)}\psi(x)\psi^\dagger(x')]}{\text{Tr}[e^{-\beta(H - \mu Q)}]} \]

\[ = iD^+_{T=0}(x, x') + \int_k e^{ik(x-x')} \sum_n \frac{n e^{-\beta(\epsilon_k + \mu)}}{e^{-\beta(\epsilon_k + \mu)}} + \int_k e^{-ik(x-x')} \sum_n \frac{n e^{-\beta(\epsilon_k - \mu)}}{e^{-\beta(\epsilon_k - \mu)}} \]

\[ = iD^+_{T=0}(x, x') + \int_k e^{-ik(x-x')} \left( \frac{2\pi\delta(k^2 - m^2)\Theta(-k^0)}{e^{\beta(\epsilon_k + \mu)} - 1} + \frac{2\pi\delta(k^2 - m^2)\Theta(k^0)}{e^{\beta(\epsilon_k - \mu)} - 1} \right). \quad (B11) \]

The modification due to the environment is the same in \( \text{Tr}[e^{-\beta(H - \mu Q)}\phi^\dagger(x')\phi(x)] \) as in Eq. (B10), leading to the same conclusion as in the case of a neutral boson, namely that the only modification of the environment is \( D^\pm_{T=0} = D^\pm_{T=0} + \Delta D, \) and

\[ i\Delta D_{ik} = -\frac{2\pi\delta(k^2 - m^2)\Theta(-k^0)}{e^{\beta(\epsilon_k + \mu)} - 1} - \frac{2\pi\delta(k^2 - m^2)\Theta(k^0)}{e^{\beta(\epsilon_k - \mu)} - 1}. \quad (B12) \]

### 3. Fermions

The field operator

\[ \psi(x) = \frac{1}{V} \sum_{k,\alpha} \frac{m}{\omega_k} b_\alpha(k) u^{(\alpha)}(k) e^{-ikx} + \bar{d}_\alpha(k) \bar{u}^{(\alpha)}(k) e^{ikx} \]  

(B13)

and the anti-commutation relations

\[ \{b_\alpha(p), \bar{b}_\beta(p')\} = \{d_\alpha(p), \bar{d}_\beta(p')\} = V \frac{\delta_{\alpha,\beta}}{m} \delta_{p, p'} \]  

(B14)

give

\[ \text{Tr}[e^{-\beta(H - \mu Q)}\psi(x)\psi^\dagger(x')] = \sum_{\{n\}} \sum_{k,k'} \frac{m^2}{2V^2\omega_k^2} \langle \{n\} | e^{-\beta(H - \mu Q)} \]

\[ (d^\dagger(k)d(k')\psi^{(\alpha)}(k')\psi^\dagger(k)e^{ikx-ik'x'} + b(k)b^\dagger(k')\psi^{(\alpha)}(k')\psi^\dagger(k)e^{ik'x-ikx})|\{n\} \rangle \]

\[ = \sum_{\{n\}} \sum_{k} \frac{m}{2V^2\omega_k^2} \langle \{n\} | e^{-\beta(H - \mu Q)}(d^\dagger(k)d(k)(k - m)e^{ik(x - x')} + b(k)b^\dagger(k)(k + m)e^{-ik(x - x')})|\{n\} \rangle \]

\[ = \sum_{\{n\}} \sum_{k} \frac{1}{2V^2\omega_k^2} (k + m)e^{-ik(x - x')} \langle \{n\} | e^{-\beta(H - \mu Q)}|\{n\} \rangle_{E > 0} \]

\[ + \sum_{\{n\}} \sum_{k} \frac{m}{2V^2\omega_k^2} \langle \{n\} | e^{-\beta(H - \mu Q)}(d^\dagger(k)d(k)(k - m)e^{ik(x - x')} - b^\dagger(k)b(k)(k + m)e^{-ik(x - x')})|\{n\} \rangle \]

\[ = \int_k (k + m)e^{-ik(x - x')} \prod_k (1 + e^{-\beta(\epsilon_k - \mu)}) \]

\[ + \sum_{k} \frac{1}{2V^2\omega_k} (e^{-\beta(\epsilon_k + \mu)}(k - m)e^{ik(x - x') - e^{-\beta(\epsilon_k - \mu)}(k + m)e^{-ik(x - x')}). \quad (B15) \]

This result, together with the analogous one for \( \text{Tr}[e^{-\beta(H - \mu Q)}\psi(x')\psi(x)] \) yield

\[ G^\pm_{T=0} = -(i\partial + m)(D^\pm_{T=0} - \Delta D), \quad (B16) \]

where the last term is given by Eq. (B12).
4. Summary

The different propagators encountered in the discussion presented above can be summarized as

\[
D_k = \begin{pmatrix}
\frac{1}{k^2 - m^2 + i\epsilon} & -2\pi i\delta(k^2 - m^2)\Theta(k^0)

\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

and

\[
G_k = (k + m)^2 \begin{pmatrix}
\frac{1}{k^2 - m^2 + i\epsilon} & 2\pi i\delta(k^2 - m^2)\Theta(k^0)

\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

where the occupation numbers

\[
n^\pm_b = \frac{1}{e^{\beta (\epsilon_k + \mu)} - 1}
\]

\[
n^\pm_f = \frac{1}{e^{\beta (\epsilon_k + \mu)} + 1}
\]

are used.

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