Data-Dependent Coresets for Compressing Neural Networks with Applications to Generalization Bounds

Cenk Baykal†*, Lucas Liebenwein†*, Igor Gilitschenski†, Dan Feldman‡, Daniela Rus†

Abstract

We present an efficient coresets-based neural network compression algorithm that provably sparsifies the parameters of a trained fully-connected neural network in a manner that approximately preserves the network’s output. Our approach is based on an importance sampling scheme that judiciously defines a sampling distribution over the neural network parameters, and as a result, retains parameters of high importance while discarding redundant ones. We leverage a novel, empirical notion of sensitivity and extend traditional coreset constructions to the application of compressing parameters. Our theoretical analysis establishes guarantees on the size and accuracy of the resulting compressed neural network and gives rise to new generalization bounds that may provide novel insights on the generalization properties of neural networks. We demonstrate the practical effectiveness of our algorithm on a variety of neural network configurations and real-world data sets.

1 Introduction

Within the past decade, large-scale neural networks have demonstrated unprecedented empirical success in high-impact applications such as object classification, speech recognition, computer vision, and natural language processing. However, with the ever-increasing size of state-of-the-art neural networks, the resulting storage requirements and performance of these models are becoming increasingly prohibitive in terms of both time and space. Recently proposed architectures for neural networks, such as [29, 42, 35, 6], contain millions of parameters, rendering them prohibitive to deploy on platforms that are resource-constrained, e.g., embedded devices, mobile phones, or small scale robotic platforms.

In this work, we consider the problem of neural network compression with theoretical guarantees. In particular, we address the problem of sparsifying the parameters of a trained fully-connected neural network in a principled way so that the output of the compressed neural network is preserved. We introduce a neural network compression approach based on identifying and removing weighted edges, i.e., weights, with low relative importance via coresets, small weighted subsets of the original set that approximate the pertinent cost function [1, 30, 18, 5]. Our compression algorithm hinges on extensions of the traditional sensitivity-based coresets framework [30, 8], and to the best of our knowledge, is the first to apply coresets to parameter downsizing. In this regard, our work aims

†Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, emails: {baykal, lucasl, igilitschenski, rus}@mit.edu
‡Robotics and Big Data Laboratory, University of Haifa, email: dannyf.post@gmail.com
*These authors contributed equally to this work
to simultaneously introduce a practical algorithm for compressing neural network parameters with provable guarantees and close the research gap in prior coresets work, which has predominantly focused on compressing input data points.

In particular, this paper contributes the following:

1. A novel coreset approach to compressing problem-specific parameters, e.g., neural network weights, that improves upon and extends state-of-the-art coreset constructions.

2. An efficient neural network compression algorithm, CoreNet, based on our extended coreset approach that sparsifies the parameters via importance sampling of weighted edges.

3. Extensions of the CoreNet method, CoreNet+ and CoreNet++, that improve upon the edge sampling approach by additionally performing neuron pruning and amplification.

4. Analytical results establishing guarantees on the approximation accuracy and size of the compressed neural network. In addition, a corollary to our main result that establishes new generalization bounds for neural networks.

5. Evaluations on real-world data sets that demonstrate the practical effectiveness of our algorithm in compressing neural network parameters and validate our theoretical results.

2 Related Work

Our work builds upon the following prior work in coresets and compression approaches.

Coresets

Coreset constructions were originally introduced in the context of computational geometry [1] and subsequently generalized for applications to other problems via an importance sampling-based, sensitivity framework [30, 18, 8]. Coresets have been used successfully to accelerate various machine learning algorithms such as $k$-means clustering [18, 8], graphical model training [36], and logistic regression [21] (see the surveys of [5, 37] for a complete list). In contrast to prior work, we generate coresets for reducing the number of parameters – rather than data points – via a novel construction scheme based on an efficiently-computable notion of sensitivity.

Low-rank Approximations and Weight-sharing

Among others, Denil et al. [13] were among the first to empirically demonstrate the existence of significant parameter redundancy in deep neural networks. A predominant class of compression approaches consists of using low-rank matrix decompositions, such as Singular Value Decomposition (SVD) [14], to approximate the weight matrices with their low-rank counterparts. Similar works entail the use of low-rank tensor decomposition approaches applicable both during and after training [24, 27, 44, 23, 3, 51]. Another class of approaches uses feature hashing and weight sharing [47, 41, 10, 9, 45]. Building upon the idea of weight-sharing, quantization [19, 49, 54] or regular structure of weight matrices can be used for reducing the effective number of parameters [53, 43, 11, 12, 48]. Despite their favorable empirical effectiveness in compressing neural networks, these works generally lack performance guarantees on the quality of their approximations and/or the size of the resulting compressed network.

Weight Pruning

Similar to our proposed method, weight pruning [33] hinges on the idea that only a few dominant weights within a layer are required to approximately preserve the output. Approaches of this flavor have been investigated in [31, 16], e.g., by embedding sparsity as a
constraint [22, 2, 34]. Another related approach is that of [20], which considers a combination of weight pruning and weight sharing methods. To the best of our knowledge, our work is the first to introduce a practical, sampling-based weight pruning algorithm with provable guarantees that imposes minimal assumptions on the neural network weights and input data distribution.

**Generalization** The generalization properties of neural networks have been extensively investigated in various contexts [17, 39, 26, 7]. However, as was pointed out by [52] and [38], current approaches to obtaining non-vacuous generalization bounds do not fully or accurately capture the empirical success of state-of-the-art neural network architectures. Recently, Arora et al. [4] and Zhou et al. [55] highlighted the close connection between compressibility and generalization of neural networks. In [4], the authors presented a compression method based on the Johnson-Lindenstrauss (JL) Lemma [25] and proved generalization bounds based on succinct reparameterizations of the original neural network. Building upon the works of [4] and [55], we extend our theoretical compression results to establish novel generalization bounds for fully-connected neural networks.

In contrast to prior work, this paper addresses the problem of provably compressing a fully-connected neural network while preserving the output for any point from the data distribution up to any user-specified approximation accuracy and failure probability. Unlike the method of [4], which exhibits guarantees of the compressed network’s performance only on the set of training points, our method’s guarantees hold for any point drawn from the distribution.

3 Problem Definition

3.1 Fully-Connected Neural Networks

A feedforward fully-connected neural network with \( L \in \mathbb{N}_+ \) layers and parameters \( \theta \) defines a mapping \( f_\theta : \mathcal{X} \rightarrow \mathcal{Y} \) for a given input \( x \in \mathcal{X} \subseteq \mathbb{R}^d \) to an output \( y \in \mathcal{Y} \subseteq \mathbb{R}^k \) as follows. Let \( \eta^\ell \in \mathbb{N}_+ \) denote the number of neurons in layer \( \ell \in [L] \), where \( [L] = \{1, \ldots, L\} \) denotes the index set, and where \( \eta^1 = d \) and \( \eta^L = k \). Further, let \( \eta = \sum_{\ell=2}^{L} \eta^\ell \) and \( \eta^* = \max_{\ell \in \{2, \ldots, L\}} \eta^\ell \). For layers \( \ell \in \{2, \ldots, L\} \), let \( W^\ell \in \mathbb{R}^{\eta^{\ell-1} \times \eta^\ell} \) be the weight matrix for layer \( \ell \) with entries denoted by \( w_{ij}^\ell \), rows denoted by \( w_i^\ell \in \mathbb{R}^{1 \times \eta^\ell-1} \), and \( \theta = (W^2, \ldots, W^L) \). For notational simplicity, we assume that the bias is embedded in the weight matrices. Then for an input vector \( x \in \mathbb{R}^d \), let \( a^1 = x \) and \( z^\ell = W^\ell a^{\ell-1} \in \mathbb{R}^{\eta^\ell} \), \( \forall \ell \in \{2, \ldots, L\} \), where \( a^{\ell-1} = \phi(z^{\ell-1}) \in \mathbb{R}^{\eta^{\ell-1}} \) denotes the activation. We consider the activation to be the Rectified Linear Unit (ReLU) function, i.e., \( \phi(\cdot) = \max(\cdot, 0) \) (entry-wise, if the input is a vector). The output of the network for an input \( x \) is \( f_\theta(x) = z^L \) and in particular for classification tasks the prediction is \( \arg\max_{i \in [k]} f_\theta(x)_i = \arg\max_{i \in [k]} z_i^L \).

3.2 Neural Network Coreset Problem

Consider the setting where a neural network \( f_\theta(\cdot) \) has been trained on a training set of independent and identically distributed (i.i.d.) samples from a joint distribution on \( \mathcal{X} \times \mathcal{Y} \), yielding parameters \( \theta \). We further denote the input points of a validation data set as \( \mathcal{P} = \{x_i\}_{i=1}^n \subseteq \mathcal{X} \) and the marginal distribution over the input space \( \mathcal{X} \) as \( D \). We define the size of the parameter tuple \( \theta = (W^2, \ldots, W^L), \) \( \text{nnz}(\theta), \) to be the sum of the number of non-zero entries in the weight matrices \( W^2, \ldots, W^L \).
For any given $\varepsilon, \delta \in (0, 1)$, our overarching goal is to generate a reparameterization $\hat{\theta}$, yielding the neural network $f_{\hat{\theta}}(\cdot)$, using a randomized algorithm, such that $\text{nnz}(\hat{\theta}) \ll \text{nnz}(\theta)$, and the neural network output $f_{\theta}(x)$, $x \sim D$ can be approximated up to $1 \pm \varepsilon$ multiplicative error with probability greater than $1 - \delta$. We define the $1 \pm \varepsilon$ multiplicative error between two $k$-dimensional vectors $a, b \in \mathbb{R}^k$ as the following entry-wise bound: $a \in (1 \pm \varepsilon)b \iff a_i \in (1 \pm \varepsilon)b_i \forall i \in [k]$, and formalize the definition of an $(\varepsilon, \delta)$-coreset as follows.

**Definition 1** ($(\varepsilon, \delta)$-coreset). Given user-specified $\varepsilon, \delta \in (0, 1)$, a set of new parameters $\hat{\theta} = (\hat{W}_2, \ldots, \hat{W}_L)$, $\text{nnz}(\hat{\theta}) \ll \text{nnz}(\theta)$, is an $(\varepsilon, \delta)$-coreset of the neural network parameter set $\theta$ if for $x \sim D$ drawn from the data distribution, the following holds

$$P_{\hat{\theta}, x}(f_{\hat{\theta}}(x) \in (1 \pm \varepsilon)f_{\theta}(x)) \geq 1 - \delta,$$

where $P_{\hat{\theta}, x}$ denotes a probability measure with respect to a random data point $x$ (random data point) and the (random) output $\hat{\theta}$ generated by a randomized compression scheme.

## 4 Method

In this section, we introduce our neural network compression algorithm as depicted in Alg. 1. Our method is based on an important sampling-scheme that extends traditional sensitivity-based coreset constructions of [30, 18, 8] to the application of compressing parameters.

### 4.1 CoreNet

Our method (Alg. 1) hinges on the insight that a validation set of data points $P \overset{i.i.d.}{\sim} D^n$ can be leveraged to approximate the relative importance, i.e., sensitivity, of each weighted edge with respect to the input data distribution $D$. For this purpose, we first pick a subsample of the data points $S \subseteq P$ of appropriate size (see Sec. 5 for details) and cache each neuron’s activation (Lines 2-6). We also compute a neuron-specific constant $\Delta_i^\ell(x)$ for each $x \in S$ to be used later in our main procedure to determine the required edge sampling complexity (Lines 2).

Subsequently, we apply our core sparsification procedure to sparsify the set of incoming weighted edges to each neuron in all layers (Lines 9-19). For technical reasons (see Sec. 5 and the corresponding proofs in the Appendix), we perform the sparsification on the positive and negative weighted edges separately and then consolidate the results (Lines 16-19). By repeating this procedure for all neurons in every layer, we obtain a set $\hat{\theta} = (\hat{W}_2, \ldots, \hat{W}_L)$ of sparse weight matrices such that the output of each layer and the entire network is approximately preserved, i.e., $W^\ell a^\ell_{i-1}(x) \approx \hat{W}^\ell a^\ell_{i-1}(x)$ and $f_{\hat{\theta}}(x) \approx f_{\theta}(x)$, respectively\(^1\).

### 4.2 Sparsifying Weights

The crux of our compression scheme lies in Alg. 2 (invoked from Alg. 1, Lines 17-18) and in particular, in the importance sampling scheme used to select a small subset of edges of high importance. The cached incoming edge activations are used to compute the sensitivity, i.e., relative importance, of each considered incoming edge $j \in W$ to neuron $i \in [\eta]^\ell$, $\ell \in \{2, \ldots, L\}$ (Lines 1-2). The relative importance of

\(^1\) $a^\ell_{i-1}(x)$ denotes the approximation from previous layers for an input $x \sim D$; see Sec. 5 for details.
Algorithm 1 CoreNET

Input: $\varepsilon, \delta \in (0, 1)$: error and failure probability, respectively; $P \subseteq X$: a set of $n$ points from the input space $X$ such that $P \subseteq D$: $\theta = (W^2, \ldots, W^L)$: parameters of the original uncompressed neural network.

Output: $\hat{\theta} = (W^2, \ldots, W^L)$: sparsified parameter set such that $f_{\theta}(\cdot) \in (1 + \varepsilon) f_{\theta}(\cdot)$ (see Sec. 5 for details).

1. $\varepsilon' \leftarrow \frac{\varepsilon}{2(1-\varepsilon)}$; $\eta_* \leftarrow \max_{\ell \in \{2, \ldots, L-1\}} \eta^\ell$; $\eta \leftarrow \sum_{\ell=2}^L \eta^\ell$;
2. $S \leftarrow$ Uniform sample (without replacement) of $[\log (8 \eta^* / \delta) \log (\eta^*)]$ points from $P$;
3. $a^1(x) \leftarrow x \quad \forall x \in S$;
4. for $x \in S$ do
5.  for $\ell \in \{2, \ldots, L\}$ do
6.    $a^\ell(x) \leftarrow \phi(W^\ell a^{\ell-1}(x))$; $\Delta^\ell(x) \leftarrow \sum_{k \in [\eta^{\ell-1}]} |w^\ell_{ik} a^{\ell-1}(x)| / \sum_{k \in [\eta^{\ell-1}]} |w^\ell_{ik} a^{\ell-1}(x)|$;
7.  $\kappa \leftarrow \sqrt{2\lambda_* (1 + \sqrt{2\lambda_* \log (8 \eta^*) / \delta})}$, where $\lambda_* = \log (\eta^*) / 2$;
8.  $\hat{\Delta}^\ell \leftarrow \left(\frac{1}{|S|} \max_{i \in [\eta]} \sum_{x \in S} \Delta^\ell(x)\right) + \kappa \quad \forall \ell \in \{2, \ldots, L\}$;
9. for $\ell \in \{2, \ldots, L\}$ do
10. $\hat{W}^\ell \leftarrow (0, \ldots, 0) \in \mathbb{R}^{\eta^\ell \times \eta^{\ell-1}}$;
11. $\hat{\Delta} \leftarrow \prod_{\ell=1}^L \hat{\Delta}^\ell$; $\varepsilon_\ell \leftarrow \varepsilon' / \hat{\Delta}^\ell$;
12. for all $i \in [\eta^\ell]$ do
13.   if $\max_{x \in S} \phi(w^\ell_{ij} a^{\ell-1}(x)) = 0$ then \>
14.     $\hat{w}^\ell_{ij} \leftarrow (0, \ldots, 0) \in \mathbb{R}^{1 \times \eta^{\ell-1}}$; \>
15.   else \>
16.     $\mathcal{W}_+ \leftarrow \{j \in [\eta^{\ell-1}] : w^\ell_{ij} > 0\}$; $\mathcal{W}_- \leftarrow \{j \in [\eta^{\ell-1}] : w^\ell_{ij} < 0\}$;
17.     $\hat{w}^\ell_{ij} \leftarrow \text{SPARSIFY}((\mathcal{W}_+, w^\ell_{ij}, \varepsilon_\ell, \delta, S, a^{\ell-1}(\cdot)))$;
18.     $\hat{w}^\ell_{ij} \leftarrow \text{SPARSIFY}((\mathcal{W}_-, -w^\ell_{ij}, \varepsilon_\ell, \delta, S, a^{\ell-1}(\cdot)))$;
19.     $\hat{w}^\ell_{ij} \leftarrow \hat{w}^\ell_{ij} + \hat{w}^\ell_{ij} - \hat{w}^\ell_{ij}$; $\hat{W}^\ell \leftarrow \hat{W}^\ell$; \>
20. return $\hat{\theta} \leftarrow (\hat{W}^2, \ldots, \hat{W}^L)$.

Algorithm 2 SPARSIFY($W, w, \varepsilon, \delta, S, a(\cdot)$)

Input: $W \subseteq [\eta^{\ell-1}]$: index set; $w \in \mathbb{R}^{1 \times \eta^{\ell-1}}$: row vector corresponding to the weights incoming to node $i \in [\eta^\ell]$ in layer $\ell \in \{2, \ldots, L\}$; $\varepsilon, \delta \in (0, 1)$: error and failure probability, respectively; $S \subseteq P$: subsample of the original point set; $a(\cdot)$: cached activations of previous layer for all $x \in S$.

Output: $\hat{w}$: sparse weight vector.

1. for $j \in W$ do
2.     $s_j \leftarrow \max_{x \in S} \sum_{x \in S} w_{ij} a_j(x)$; \>
3. for $j \in W$ do \>
4.     $q_j \leftarrow 2^{-1}$; \>
5. for $j \in W$ do \>
6.     $m \leftarrow \left[ \frac{8 \log (8 \eta^*) \log (8 \eta^*)}{\varepsilon^2} \right]$; \>
7. for $j \in \mathcal{C}$ do \>
8.     $\hat{w} \leftarrow (0, \ldots, 0) \in \mathbb{R}^{1 \times \eta^{\ell-1}}$; \>
9.     $\hat{w}_{ij} \leftarrow \hat{w}_{ij} + w_{ij} / m q_j$; \>
10. return $\hat{w}$;
each edge \( j \) is computed as the maximum (over \( x \in S \)) ratio of the edge’s contribution to the sum of contributions of all edges. In other words, the sensitivity \( s_j \) of an edge \( j \) captures the highest impact \( j \) had on the output of neuron \( i \in [\eta^\ell] \) in layer \( \ell \) across all \( x \in S \).

The sensitivity of each edge is then used to compute the importance sampling distribution over the incoming weighted edges (Lines 4-5). The intuition behind the importance sampling distribution is that if \( s_j \) is high, then edge \( j \) is more likely to have a high impact on the output of neuron \( i \), therefore we should keep edge \( j \) with a higher probability. \( m \) edges are then sampled with replacement (Lines 6-7) and the sampled weights are then reweighed to ensure unbiasedness of our estimator (Lines 9-10).

### 4.3 Extensions: Neuron Pruning and Amplification

In this subsection we outline two improvements to our algorithm that (i) that do not violate any of our theoretical properties, and (ii) may improve compression rates in practical settings.

**Neuron pruning (CoreNet+)** Similar to removing redundant edges, we can leverage the empirical evaluations to gauge the importance of each neuron. In particular, if the maximum activation (over all evaluations \( x \in S \)) of neuron \( i \) is equal to 0, then the neuron – along with all of the incoming and outgoing edges – can be pruned without significantly affecting the output with reasonable probability (Alg. 1, Lines 13-14). This intuition can be made rigorous under the assumptions outlined in Sec. 5 and it can be shown that the neuron can be pruned away, while approximately preserving the output for any \( x \sim \mathcal{D} \) with high probability.

**Amplification (CoreNet++)** In addition to neuron pruning, it can be shown that coresets exhibiting significantly stronger approximation guarantees can be constructed via amplification, i.e., the procedure of constructing multiple approximations \( (\hat{w}^\ell_i)^1, \ldots, (\hat{w}^\ell_i)^\tau \) over \( \tau \) trials, and picking the best one. To evaluate the quality of each approximation, a different subset \( T \subseteq \mathcal{P} \setminus S \) can be used to infer the performance. In practice, amplification would entail constructing multiple approximations by executing Lines 17-18 of Alg. 1 and picking the one that achieves the lowest relative error on \( T \).

### 5 Analysis

In this section, we establish the theoretical guarantees of our neural network compression algorithm (Alg. 1). In Lemma 1, we analyze how the concept of empirical sensitivity can be used to provably approximate a neuron with only positively-valued incoming edges. This analysis is extended in Lemma 2 to account for negatively-valued edges, which constitutes the core of our importance sampling scheme (Alg. 2) to efficiently construct small-sized coresets. We apply Lemma 2 to establish bounds on the error propagation between layers (Lemma 3) and other intermediate results that culminate in Theorem 4, which establishes the guarantees of our algorithm. In addition, we present a novel generalization bound obtained as a corollary to Theorem 4. The full proofs of all the claims presented in this section can be found in the supplementary material.
5.1 Preliminaries

Fix a neuron $i \in [\eta^\ell]$, $\ell \in \{2, \ldots, L\}$, and note that $z_i^\ell = \sum_{j=1}^{\eta^{\ell-1}} w_{ij} a_j^{\ell-1} = \langle w_i^\ell, a^{\ell-1} \rangle$. We let $\mathcal{D}$ be the input data distribution and let $x \sim \mathcal{D}$ be a randomly drawn input point. Henceforth, we explicitly refer to the pre-activation and activation values at layer $\ell$ with respect to the input $x \in \text{supp}(\mathcal{D})$ as $z^\ell(x)$ and $a^\ell(x)$, respectively. The values of $z^\ell(x)$ and $a^\ell(x)$ at each layer $\ell \in \{2, \ldots, L\}$ will depend on whether or not we compressed the previous layers $\ell' \in \{2, \ldots, \ell\}$. To formalize this interdependency, we let $\hat{z}^\ell(x)$ and $\hat{a}^\ell(x)$ denote the respective quantities of layer $\ell$ when we replace the weight matrices $W^2, \ldots, W^\ell$ in layers $2, \ldots, \ell$ by $\hat{W}^2, \ldots, \hat{W}^\ell$, respectively. Finally, we define $\log(\cdot)$ to be the natural logarithm, i.e., $\log(\cdot) = \log_e(\cdot)$, unless otherwise stated.

For the remainder of this section (Sec. 5) we let $\ell \in \{2, \ldots, L\}$ be an arbitrary layer and let $i \in [\eta^\ell]$ be an arbitrary neuron in layer $\ell$. For purposes of clarity and readability, we will omit the the variable denoting the layer $\ell \in \{2, \ldots, L\}$, the neuron $i \in [\eta^\ell]$, and the incoming edge index $j \in [\eta^{\ell-1}]$, whenever they are clear from the context. For example, when referring to the intermediate value of an arbitrary neuron $i \in [\eta^\ell]$ in layer $\ell \in \{2, \ldots, L\}$, $z_i^\ell(x) = \langle w_i^\ell, a^{\ell-1}(x) \rangle \in \mathbb{R}$ with respect to a point $x$, we will simply write $z(x) = \langle w, a(x) \rangle \in \mathbb{R}$, where $w := w_i \in \mathbb{R}^{1 \times \eta^{\ell-1}}$ and $a(x) := a^{\ell-1}(x) \in \mathbb{R}^{\eta^{\ell-1} \times 1}$. Under this abbreviated notation, the weight of an incoming edge $j$ is denoted by $w_j \in \mathbb{R}$.

5.2 Importance Sampling Bounds for Positive Weights

In this subsection, we establish approximation guarantees under the assumption that the weights are positive. Moreover, we will also assume that the input, i.e., the activation from the previous layer, is non-negative (entry-wise). The subsequent subsection will then relax these assumptions to conclude that a neuron’s value can be approximated well even when the weights and activations are not all positive and non-negative, respectively. Let $\mathcal{W} = \{ j \in [\eta^{\ell-1}] : w_j > 0 \} \subseteq [\eta^{\ell-1}]$ be the set of indices of incoming edges with strictly positive weights, i.e., we neglect edges with weight zero. To sample the incoming edges to a neuron, we will consider and quantify the relative importance of each edge as follows.

**Definition 2** (Relative Importance). The importance of an incoming edge $j \in \mathcal{W}$ with respect to an input $x \in \text{supp}(\mathcal{D})$ is given by the function $g_j(x)$, where

$$g_j(x) = \frac{w_j a_j(x)}{\sum_{k \in \mathcal{W}} w_k a_k(x)} \quad \forall j \in \mathcal{W}.$$

Note that for a randomly drawn $x \sim \mathcal{D}$, $g_j(x)$ is a function of the random variable $x$. We now outline our first assumption that pertains to the Cumulative Distribution Function (CDF) of the relative importance random variable.

**Assumption 1.** For all $j \in \mathcal{W}$, the CDF of the random variable $g_j(x)$, denoted by $F_j(\cdot)$, satisfies

$$F_j\left(\frac{M}{K}\right) \leq \exp(-1/K), \quad \text{where} \quad M = \min\{x \in [0, 1] : F_j(x) = 1\},$$

and $K \in [2, \log(\eta \eta^*)]$ is a universal constant.

Assumption 1 is a technical assumption on the ratio of the weighted activations that will enable us to rule out pathological problem instances where the relative importance of each edge cannot
be well-approximated using a small number of data points \( S \subseteq P \). In particular, we consider a uniformly drawn subsample (without replacement) \( S \subseteq P \) as in Line 2 of Alg. 1, where \(|S| = \lceil \log (8 \eta \eta^* / \delta) \log(\eta \eta^*) \rceil \), and define the empirical sensitivity to be the maximum relative importance over \( S \).

**Definition 3** (Empirical Sensitivity). Let \( S \subseteq P \) be a subset of distinct points from \( P \) i.i.d. \( D^n \). Then, the sensitivity over positive edges \( j \in W \) directed to a neuron is defined as

\[
s_j = \max_{x \in S} g_j(x) = \max_{x \in S} \frac{w_j a_j(x)}{\sum_{k \in W} w_k a_k(x)}.
\]

In Lemma 9 (Appendix A.1), we establish that empirical sensitivity over the set \( S \) is indeed a good proxy for the relative importance of edges. Furthermore, note that \( s_j \geq 0 \) for all \( j \in W \) by definition and the assumption that we only consider positive quantities.

Our first lemma establishes a core result that relates the weighted sum with respect to the sparse row vector \( \hat{w} \), \( \sum_{k \in W} w_k \hat{a}_k(x) \), to the value of the of the weighted sum with respect to the ground-truth row vector \( w \), \( \sum_{k \in W} w_k a_k(x) \). We remark that there is randomness with respect to the randomly generated row vector \( \hat{w}_\ell \), a randomly drawn input \( x \sim D \), and the function \( \hat{a}(\cdot) = \hat{a}_\ell^{-1}(\cdot) \) defined by the randomly generated compressed matrices \( \hat{W}^2, \ldots, \hat{W}^{\ell-1} \) in the previous layers. To clarify the random variables in question over which the probability is measured, we let \( \mathbb{P}_{\hat{w}_\ell, x, \hat{a}_\ell^{-1}(\cdot)} \) denote the probability over the randomly generated row vector \( \hat{w}_\ell \), a randomly drawn input \( x \sim D \), and the function \( \hat{a}_\ell^{-1}(\cdot) \) defined by the randomly generated compressed matrices \( \hat{W}^2, \ldots, \hat{W}^{\ell-1} \) in the previous layers\(^2\). Unless otherwise stated, we will henceforth use the shorthand notation \( \mathbb{P}(\cdot) \) to denote \( \mathbb{P}_{\hat{w}_\ell, x, \hat{a}_\ell^{-1}(\cdot)} \). Moreover, for ease of presentation, we will first condition on the event that \( \hat{a}(x) \in (1 \pm 1/2)a(x) \), i.e., that \( \hat{a}(x) \) is a relatively good approximation for \( a(x) \). Let the above event be denoted by \( \mathcal{E}_{1/2} \). This conditioning will simplify the preliminary analysis and the conditioning will be removed in our subsequent analytical results later in the section.

**Lemma 1** (Positive-Weights Sparsification). Let \( \varepsilon, \delta \in (0, 1) \), and \( x \sim D \). \( \text{SPARSIFY}(W, w, \varepsilon, \delta, S, a(\cdot)) \) generates a row vector \( \hat{w} \) such that, conditioned on the event \( \mathcal{E}_{1/2} \) occurring,

\[
\mathbb{P} \left( \sum_{k \in W} \hat{w}_k \hat{a}_k(x) \notin (1 \pm \varepsilon) \sum_{k \in W} w_k a_k(x) \mid \mathcal{E}_{1/2} \right) \leq \frac{3\delta}{8\eta}
\]

where \( \text{nnz}(\hat{w}) \leq \left\lfloor \frac{8 S \log(\eta \eta^*) \log(8 \eta / \delta)}{\varepsilon^2} \right\rfloor \), and \( S = \sum_{j \in W} s_j \).

### 5.3 Importance Sampling Bounds

In this section, we relax the requirement that the weights are strictly positive and instead consider the following index sets that partition the weighted edges: \( W_+ = \{ j \in [\eta^{\ell-1}] : w_j > 0 \} \) and \( W_- = \{ j \in [\eta^{\ell-1}] : w_j < 0 \} \) as in Alg. 1, Line 16. We still assume that the incoming activations from the previous layers are positive. We remark that we can also relax the assumption on the activations being nonnegative as discussed in Appendix A.2.4. We define \( \Delta_\ell^i(x) \) for a point \( x \sim D \) and neuron \( i \in [\eta^\ell] \) as \( \Delta_\ell^i(x) = \frac{\sum_{k \in [\eta^{\ell-1}]} |w_{ik} a_k^{-1}(x)|}{\sum_{k \in [\eta^{\ell-1}]} w_{ik} a_k^{-1}(x)} \). The following assumption requires that the

\(^2\text{Since } \hat{a}_\ell^{-1}(\cdot) = \phi(2^{-1}(\cdot)) = \phi(\hat{W}^{\ell-1} \hat{a}^{\ell-2}(\cdot)), \text{ and so on.}
random variable $\Delta_\ell^f(x)$ (as a function of the random $x$) is subexponential with a parameter that is bounded by an appropriate constant.

**Assumption 2** (Subexponentiality of $\Delta_\ell^f(x)$). For any layer $\ell \in \{2, \ldots, L\}$ and neuron $i \in [\eta^f]$, the centered random variable $\Delta = \Delta_\ell^f(x) - \mathbb{E}_{x \sim \mathcal{D}}[\Delta_\ell^f(x)]$ is subexponential [46] with parameter $\lambda \leq \log(\eta \eta^*)/2$, i.e., $\mathbb{E}[\exp(s\Delta)] \leq \exp(s^2\lambda^2)$, $\forall |s| \leq 1/\lambda$.

For $\epsilon \in (0, 1)$ and $\ell \in \{2, \ldots, L\}$, we let $\epsilon' = \frac{\epsilon}{2(L-1)}$ and define $\epsilon_\ell = \frac{\epsilon'}{\Delta \rightarrow} = \frac{\epsilon'}{2(L-1) \prod_{k=\ell}^{L} \Delta_k}$, where $\Delta_\ell = \left( \frac{1}{|S_i|} \max_{x \in [\eta^f]} \sum_{x' \in S} \Delta_\ell^f(x') \right) + \kappa$, as in Alg. 1. Moreover, to formalize the interdependency between layers, we let $\mathcal{E}^{\ell-1}$ denote the event that the approximation $\hat{a}^{\ell-1}(-)$ in layer $\ell - 1$ is a good approximation. To this end, for $i \in [\eta^f]$, let $\mathcal{E}_i^\ell$ denote the (desirable) event that $\hat{z}_\ell^f(x) = (1 \pm 2(\ell - 1) \epsilon_{\ell+1}) z_\ell^f(x)$ holds, and let $\mathcal{E}^\ell = \cap_{i \in [\eta^f]} \mathcal{E}_i^\ell$ be the intersection over the events corresponding to each neuron in layer $\ell$. Also note that if event $\mathcal{E}^\ell$ occurs, then the event that $\hat{a}^\ell(x) = (1 \pm 2(\ell - 1) \epsilon_{\ell+1}) a^\ell(x)$ also occurs by definition of the ReLU function.

**Lemma 2** (Conditional Neuron Value Approximation). Let $\epsilon, \delta, \delta' \in (0, 1)$, $\ell \in \{2, \ldots, L\}$, $i \in [\eta^f]$, and $x \sim \mathcal{D}$. CORENET generates a row vector $\hat{w}_i^\ell = \hat{w}_i^\ell + \hat{\Delta}_i^\ell \in \mathbb{R}^{1 \times \eta^f}$ such that, conditioned on the event $\mathcal{E}^{\ell-1}$ occurring, i.e., $\hat{z}_\ell^f(x) = (1 \pm 2(\ell - 1) \epsilon_{\ell+1}) z_\ell^f(x)$, 

$$
P \left( \mathcal{E}_i^\ell \mid \mathcal{E}^{\ell-1} \right) = P \left( \hat{z}_\ell^f(x) = (1 \pm 2(\ell - 1) \epsilon_{\ell+1}) z_\ell^f(x) \mid \mathcal{E}^{\ell-1} \right) \geq 1 - \frac{\delta}{\eta},$$

where $\epsilon_\ell = \frac{\epsilon'}{\Delta \rightarrow}$ and $\text{nnz}(\hat{w}_i^\ell) \leq \left[ \frac{8 S \log(\eta \eta^*) \log(8 \eta / \delta)}{\epsilon'^2} \right] + 1$, where $S = \sum_{j \in W_+} s_j + \sum_{j \in W_-} s_j$ is the sum of sensitivities with respect to neuron $i \in [\eta^f]$.

The following core lemma establishes unconditional layer-wise approximation guarantees, i.e., removes the conditioning on the event $\mathcal{E}^{\ell-1}$, and culminates in our main compression theorem.

**Lemma 3** (Layer-wise Approximation). In the context of the Corollary 11, for any arbitrary layer $\ell \in \{2, \ldots, L\}$ it holds for $\hat{z}_\ell^f(x) = W^\ell \hat{a}^\ell(x)$ that

$$
P \left( \hat{w}_{\ell, i}^\ell \mid \mathcal{E}^{\ell-1} \right) = \frac{P \left( \hat{z}_\ell^f(x) = (1 \pm 2(\ell - 1) \epsilon_{\ell+1}) z_\ell^f(x) \right)}{P \left( \hat{w}_{\ell, i}^\ell \mid \mathcal{E}^{\ell-1} \right)} \geq 1 - \frac{\delta}{\eta} \sum_{\ell'=2}^{L} \eta^{\ell '}.$$

**Theorem 4** (Network Compression). For $\epsilon, \delta \in (0, 1)$, Algorithm 1 generates a set of parameters $\hat{\theta} = (\hat{W}^2, \ldots, \hat{W}^L)$ of size

$$
\text{nnz}(\hat{\theta}) \leq \sum_{\ell=2}^{L} \sum_{i=1}^{\eta^f} \left\lfloor \frac{32 (L - 1)^2 (\Delta \rightarrow)^2 S_{\ell}^f \log(\eta \eta^*) \log(8 \eta / \delta)}{\epsilon^2} \right\rfloor + 1
$$

in $O \left( \eta \eta^* \log(\eta \eta^*/\delta) \right)$ time such that $P_{\hat{\theta}, x \sim \mathcal{D}} \left( \hat{f}_\theta(x) \in (1 \pm \epsilon) f_0(x) \right) \geq 1 - \delta$.

We note that we can obtain a guarantee for a set of $n$ randomly drawn points by invoking Theorem 4 with $\delta' = \delta/n$ and union-bounding over the failure probabilities, while only increasing the sampling complexity logarithmically, as formalized in Corollary 14, Appendix A.2.
5.4 Amplification

In the context of Lemma 2, define the relative error of a (randomly generated) row vector \( \hat{w}^\ell = \hat{w}_R^\ell - \hat{w}_L^\ell \in \mathbb{R}^{1 \times \eta^{\ell-1}} \) with respect to a realization \( x \) of a point \( x \sim \mathcal{D} \) as

\[
\text{err}_{\hat{w}^\ell}(x) = \left| \frac{\langle \hat{w}^\ell, \hat{a}_R^{\ell-1}(x) \rangle}{\langle \hat{w}^\ell, \hat{a}_L^{\ell-1}(x) \rangle} - 1 \right|.
\]

Consider a set \( \mathcal{T} \subseteq \mathcal{P} \setminus \mathcal{S} \) of size \( |\mathcal{T}| \) such that \( \mathcal{T} \overset{i.i.d.}{\sim} \mathcal{D}^{|\mathcal{T}|} \), and let

\[
\text{err}_{\hat{w}^\ell}(\mathcal{T}) = \frac{1}{|\mathcal{T}|} \sum_{x \in \mathcal{T}} \text{err}_{\hat{w}^\ell}(x).
\]

When the layer is clear from the context we will refer to \( \text{err}_{\hat{w}^\ell}(x) \) as simply \( \text{err}_{\hat{w}^\ell} \)

**Lemma 5 (Expected Error).** Let \( \varepsilon, \delta \in (0, 1) \), \( \ell \in \{2, \ldots, L\} \), and \( i \in [\eta^\ell] \). Conditioned on the event \( \mathcal{E}^{\ell-1} \), CORENET generates a row vector \( \hat{w} = \hat{w}_R^\ell - \hat{w}_L^\ell \in \mathbb{R}^{1 \times \eta^{\ell-1}} \) such that

\[
\mathbb{E} \left[ \text{err}_{\hat{w}^\ell}(x) \mid \mathcal{E}^{\ell-1} \right] \leq \varepsilon \ell \hat{\Delta}^\ell \left( k + \frac{5 (1 + k \varepsilon \ell)}{\sqrt{\log(8 \eta / \delta)}} \right) + \frac{\delta (1 + k \varepsilon \ell)}{\eta} \mathbb{E}_{x \sim \mathcal{D}} \left[ \Delta^\ell_i(x) \mid \Delta^\ell_i(x) > \hat{\Delta}^\ell \right],
\]

where \( k = 2(\ell - 1) \).

We now state the advantageous effect of amplification, i.e., constructing multiple approximations for each neuron’s incoming edges and then picking the best one, as formalized below.

**Theorem 6 (Amplification).** Given \( \varepsilon, \delta \in (0, 1) \) such that \( \frac{\delta}{\eta} \) is sufficiently small, let \( \ell \in \{2, \ldots, L\} \) and \( i \in [\eta^\ell] \). Let \( \tau = \left\lceil \frac{\log (4 \eta / \delta)}{\log (10/9)} \right\rceil \) and consider the reparameterized variant of Alg. 1 where we instead have

1. \( \mathcal{S} \subseteq \mathcal{P} \) of size \( |\mathcal{S}| \geq \left\lceil \log \left( 32 |\mathcal{P}| \tau \eta^\tau / \delta \right) \log (\eta \eta^\tau) \right\rceil \),

2. \( \hat{\Delta}^\ell = \left( \frac{1}{|\mathcal{S}|} \max_{i \in [\eta^\ell]} \sum_{x' \in \mathcal{S}} \Delta^\ell_i(x') \right) + \kappa \) as before, but \( \kappa \) is instead defined as
   \[
   \kappa = \sqrt{2 \lambda_\ast \left( 1 + \sqrt{2 \lambda_\ast \log (32 |\mathcal{P}| \tau \eta^\tau / \delta)} \right)},
   \]
   and

3. \( m \geq \left\lceil \frac{8 \log (\eta \eta^\tau) \log (32 |\mathcal{P}| \tau / \delta)}{\varepsilon \ell^2} \right\rceil \) in the sample complexity in SPARSIFYWEIGHTS.

Among \( \tau \) approximations \( (\hat{w}_1^\ell)^1, \ldots, (\hat{w}_1^\ell)^\tau \), generated by Alg. 2, let

\[
\hat{w}_1^\ell = \arg\min_{\hat{w}_1^\ell \in \{\hat{w}_1^\ell)^1, \ldots, (\hat{w}_1^\ell)^\tau \}} \text{err}_{\hat{w}_1^\ell}(\mathcal{T}),
\]

and \( \mathcal{T} \subseteq (\mathcal{P} \setminus \mathcal{S}) \) be a subset of points of size \( |\mathcal{T}| = \left\lceil 8 \log (8 \tau \eta / \delta) \right\rceil \). Then,

\[
\mathbb{P}_{\hat{w}_1^\ell, \hat{a}_1^{\ell-1}()} \left( \mathbb{E}_{x \sim \hat{w}_1^\ell} \left[ \text{err}_{\hat{w}_1^\ell}(x) \mid \hat{w}_1^\ell, \mathcal{E}^{\ell-1} \right] \leq k \varepsilon \ell + 1 \mid \mathcal{E}^{\ell-1} \right) \geq 1 - \frac{\delta}{\eta},
\]

where \( k = 2(\ell - 1) \).
5.5 Generalization Bounds

As a corollary to our main results, we obtain novel generalization bounds for neural networks in terms of empirical sensitivity. To contextualize the following corollary pertaining to the generalization error of the compressed network, we introduce the terminology of [4]. The expected margin loss of a classifier \( f_\theta : \mathbb{R}^d \to \mathbb{R}^k \) parameterized by \( \theta \) with respect to a desired margin \( \gamma > 0 \) and distribution \( \mathcal{D} \) is defined by \( L_\gamma(f_\theta) = \mathbb{E}_{(x,y) \sim \mathcal{D}_{X,Y}}(f_\theta(x)_y \leq \gamma + \max_{i \neq y} f_\theta(x)_i) \). As in [4], we let \( \hat{L}_\gamma \) denote the empirical estimate of the margin loss. The following corollary follows directly from the argument presented in [4] and the bounds on the size and approximation guarantee of the compressed network established in Theorem 13.

**Corollary 7** (Generalization Bounds). For any fully connected network \( f_\theta \) parameterized by \( \theta \), \( \delta \in (0,1) \), and margin \( \gamma \), our compression scheme (Alg. 1) generates weights \( \hat{\theta} \) such that with probability at least \( 1 - \delta \), the expected error \( L_0(f_{\hat{\theta}}) \) with respect to the input points \( \mathcal{P} \subseteq \mathcal{X} \) consisting of \( n \) points is bounded by

\[
L_0(f_{\hat{\theta}}) \leq \hat{L}_\gamma(f_\theta) + \tilde{O}\left(\sqrt{\max_{x \in \mathcal{P}} \|f_\theta(x)\|^2_2 L^2 \sum_{l=2}^{L} \Delta^{(l-1)} S_l \frac{\sum_{i=1}^{L} \eta_i}{\gamma^2 n}}\right).
\]

6 Results

In this section, we evaluate the practical effectiveness of our compression algorithm on popular benchmark data sets (MNIST [32], FashionMNIST [50], and CIFAR-10 [28]) and varying fully-connected trained neural network configurations: 2 to 5 hidden layers, 100 to 1000 hidden units, either fixed hidden sizes or decreasing hidden size denoted by pyramid in the figures. We further compare the effectiveness of our sampling scheme in reducing the number of non-zero parameters of a network, i.e., in sparsifying the weight matrices, to that of uniform sampling and Singular Value Decomposition (SVD). The details of the experimental setup and results of additional evaluations may be found in Appendix B.

**Experiment Setup** Note that we compare against three variations of our compression algorithm: (i) sole edge sampling (CoreNet), (ii) edge sampling with neuron pruning (CoreNet+), and (iii) edge sampling with neuron pruning and amplification (CoreNet++). The two metrics we compare against are average relative error at the output (\( \ell_1 \)-norm) and average loss in classification accuracy compared to the uncompressed network. Both metrics are evaluated on a test set that was not used for training and/or compression.

**Results** A number of results for various architectures and datasets can be found in Figure 1 and Figure 2 for the average drop in classification accuracy and the average relative error (\( \ell_1 \)-norm), respectively. As apparent from Figure 1, we are able to compress networks to about 15% of their original size without significant loss of accuracy for networks trained on MNIST and FashionMNIST, and to about 50% of their original size for CIFAR.

**Discussion** The simulation results presented in this section validate our theoretical results established in Sec. 5. In particular, our empirical results indicate that we are able to outperform...
networks compressed via uniform sampling or SVD across the considered experiments and trials. The results presented in this section further suggest that empirical sensitivity can effectively capture the relative importance of neural network parameters, leading to a more informed importance sampling scheme. Moreover, the relative performance of our algorithm tends to increase as we consider deeper architectures. These findings suggest that our algorithm may also be effective in compressing modern convolutional architectures, which tend to be very deep.

### 7 Conclusion

We presented a coresets-based neural network compression algorithm for compressing the parameters of a trained fully-connected neural network in a manner that approximately preserves the network’s output. Our method and analysis extend traditional coreset constructions to the application of compressing parameters, which may be of independent interest. Our work distinguishes itself from prior approaches in that it establishes theoretical guarantees on the approximation accuracy and size of the generated compressed network. As a corollary to our analysis, we obtain generalization bounds for neural networks, which may provide novel insights on the generalization properties of neural networks. We empirically demonstrated the practical effectiveness our compression algorithm on a variety of neural network configurations and real-world data sets. In future work, we plan to extend our algorithm and analysis to compress Convolutional Neural Networks (CNNs) and other network architectures. We conjecture that our compression algorithm can be
used to reduce storage requirements of neural network models and enable fast inference in practical settings.

**Acknowledgments**

We thank Brandon Araki and Kiran Vodrahalli for valuable discussions and helpful suggestions. We would also like to thank Kasper Green Larsen, Alan Mathiasen, and Allan Gronlund for pointing out an error in an earlier formulation of Lemma 8.

**References**

[1] P. K. Agarwal, S. Har-Peled, and K. R. Varadarajan. Geometric approximation via coresets. *Combinatorial and computational geometry*, 52:1–30, 2005.

[2] A. Aghasi, A. Abdi, N. Nguyen, and J. Romberg. Net-trim: Convex pruning of deep neural networks with performance guarantee. In *Advances in Neural Information Processing Systems*, pages 3180–3189, 2017.

[3] J. M. Alvarez and M. Salzmann. Compression-aware training of deep networks. In *Advances in Neural Information Processing Systems*, pages 856–867, 2017.

[4] S. Arora, R. Ge, B. Neyshabur, and Y. Zhang. Stronger generalization bounds for deep nets via a compression approach. *arXiv preprint arXiv:1802.05296*, 2018.

[5] O. Bachem, M. Lucic, and A. Krause. Practical coreset constructions for machine learning. *arXiv preprint arXiv:1703.06476*, 2017.

[6] V. Badrinarayanan, A. Kendall, and R. Cipolla. Segnet: A deep convolutional encoder-decoder architecture for image segmentation. *CoRR*, abs/1511.00561, 2015.

[7] P. L. Bartlett, D. J. Foster, and M. J. Telgarsky. Spectrally-normalized margin bounds for neural networks. In *Advances in Neural Information Processing Systems*, pages 6241–6250, 2017.

[8] V. Braverman, D. Feldman, and H. Lang. New frameworks for offline and streaming coreset constructions. *arXiv preprint arXiv:1612.00889*, 2016.

[9] W. Chen, J. T. Wilson, S. Tyree, K. Q. Weinberger, and Y. Chen. Compressing convolutional neural networks. *CoRR*, abs/1506.04449, 2015.

[10] W. Chen, J. T. Wilson, S. Tyree, K. Q. Weinberger, and Y. Chen. Compressing neural networks with the hashing trick. *CoRR*, abs/1504.04788, 2015.

[11] Y. Cheng, F. X. Yu, R. S. Feris, S. Kumar, A. Choudhary, and S.-F. Chang. An exploration of parameter redundancy in deep networks with circulant projections. In *Proceedings of the IEEE International Conference on Computer Vision*, pages 2857–2865, 2015.
[12] A. Choromanska, K. Choromanski, M. Bojarski, T. Jebara, S. Kumar, and Y. LeCun. Binary embeddings with structured hashed projections. In *International Conference on Machine Learning*, pages 344–353, 2016.

[13] M. Denil, B. Shakibi, L. Dinh, M. A. Ranzato, and N. de Freitas. Predicting parameters in deep learning. In C. J. C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 26*, pages 2148–2156. Curran Associates, Inc., 2013.

[14] E. Denton, W. Zaremba, J. Bruna, Y. LeCun, and R. Fergus. Exploiting linear structure within convolutional networks for efficient evaluation. *CoRR*, abs/1404.0736, 2014.

[15] B. Doerr. Probabilistic tools for the analysis of randomized optimization heuristics. *arXiv preprint arXiv:1801.06733*, 2018.

[16] X. Dong, S. Chen, and S. Pan. Learning to prune deep neural networks via layer-wise optimal brain surgeon. In *Advances in Neural Information Processing Systems*, pages 4860–4874, 2017.

[17] G. K. Dziugaite and D. M. Roy. Computing nonvacuous generalization bounds for deep (stochastic) neural networks with many more parameters than training data. *arXiv preprint arXiv:1703.11008*, 2017.

[18] D. Feldman and M. Langberg. A unified framework for approximating and clustering data. In *Proceedings of the forty-third annual ACM symposium on Theory of computing*, pages 569–578. ACM, 2011.

[19] Y. Gong, L. Liu, M. Yang, and L. Bourdev. Compressing Deep Convolutional Networks using Vector Quantization. *arXiv preprint arXiv:1412.6115*, 2014.

[20] S. Han, H. Mao, and W. J. Dally. Deep compression: Compressing deep neural network with pruning, trained quantization and huffman coding. *CoRR*, abs/1510.00149, 2015.

[21] J. H. Huggins, T. Campbell, and T. Broderick. Coresets for scalable bayesian logistic regression. *arXiv preprint arXiv:1605.06423*, 2016.

[22] F. N. Iandola, S. Han, M. W. Moskewicz, K. Ashraf, W. J. Dally, and K. Keutzer. SqueezeNet: Alexnet-level accuracy with 50x fewer parameters and 0.5 mb model size. *arXiv preprint arXiv:1602.07360*, 2016.

[23] Y. Ioannou, D. Robertson, J. Shotton, R. Cipolla, and A. Criminisi. Training cnns with low-rank filters for efficient image classification. *arXiv preprint arXiv:1511.06744*, 2015.

[24] M. Jaderberg, A. Vedaldi, and A. Zisserman. Speeding up convolutional neural networks with low rank expansions. *arXiv preprint arXiv:1405.3866*, 2014.

[25] W. B. Johnson and J. Lindenstrauss. Extensions of lipschitz mappings into a hilbert space. *Contemporary mathematics*, 26(189-206):1, 1984.

[26] K. Kawaguchi, L. P. Kaelbling, and Y. Bengio. Generalization in deep learning. *arXiv preprint arXiv:1710.05468*, 2017.
[27] Y.-D. Kim, E. Park, S. Yoo, T. Choi, L. Yang, and D. Shin. Compression of deep convolutional neural networks for fast and low power mobile applications. *arXiv preprint arXiv:1511.06530*, 2015.

[28] A. Krizhevsky and G. Hinton. Learning multiple layers of features from tiny images. 2009.

[29] A. Krizhevsky, I. Sutskever, and G. E. Hinton. Imagenet classification with deep convolutional neural networks. In F. Pereira, C. J. C. Burges, L. Bottou, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 25*, pages 1097–1105. Curran Associates, Inc., 2012.

[30] M. Langberg and L. J. Schulman. Universal $\varepsilon$-approximators for integrals. In *Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms*, pages 598–607. SIAM, 2010.

[31] V. Lebedev and V. Lempitsky. Fast convnets using group-wise brain damage. In *Computer Vision and Pattern Recognition (CVPR), 2016 IEEE Conference on*, pages 2554–2564. IEEE, 2016.

[32] Y. LeCun, L. Bottou, Y. Bengio, and P. Haffner. Gradient-based learning applied to document recognition. *Proceedings of the IEEE*, 86(11):2278–2324, 1998.

[33] Y. LeCun, J. S. Denker, and S. A. Solla. Optimal brain damage. In *Advances in neural information processing systems*, pages 598–605, 1990.

[34] J. Lin, Y. Rao, J. Lu, and J. Zhou. Runtime neural pruning. In *Advances in Neural Information Processing Systems*, pages 2178–2188, 2017.

[35] J. Long, E. Shelhamer, and T. Darrell. Fully convolutional networks for semantic segmentation. In *The IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, June 2015.

[36] A. Molina, A. Munteanu, and K. Kersting. Core dependency networks. In *Proceedings of the 32nd AAAI Conference on Artificial Intelligence (AAAI)*. AAAI Press Google Scholar, 2018.

[37] A. Munteanu and C. Schwiegelshohn. Coresets-methods and history: A theoreticians design pattern for approximation and streaming algorithms. *KI-Künstliche Intelligenz*, 32(1):37–53, 2018.

[38] B. Neyshabur, S. Bhojanapalli, D. McAllester, and N. Srebro. Exploring generalization in deep learning. In *Advances in Neural Information Processing Systems*, pages 5949–5958, 2017.

[39] B. Neyshabur, S. Bhojanapalli, D. McAllester, and N. Srebro. A pac-bayesian approach to spectrally-normalized margin bounds for neural networks. *arXiv preprint arXiv:1707.09564*, 2017.

[40] A. Paszke, S. Gross, S. Chintala, G. Chanan, E. Yang, Z. DeVito, Z. Lin, A. Desmaison, L. Antiga, and A. Lerer. Automatic differentiation in pytorch. In *NIPS-W*, 2017.

[41] Q. Shi, J. Petterson, G. Dror, J. Langford, A. Smola, and S. Vishwanathan. Hash kernels for structured data. *Journal of Machine Learning Research*, 10(Nov):2615–2637, 2009.
[42] K. Simonyan and A. Zisserman. Very deep convolutional networks for large-scale image recognition. *CoRR*, abs/1409.1556, 2014.

[43] V. Sindhwani, T. Sainath, and S. Kumar. Structured transforms for small-footprint deep learning. In *Advances in Neural Information Processing Systems*, pages 3088–3096, 2015.

[44] C. Tai, T. Xiao, Y. Zhang, X. Wang, et al. Convolutional neural networks with low-rank regularization. *arXiv preprint arXiv:1511.06067*, 2015.

[45] K. Ullrich, E. Meeds, and M. Welling. Soft weight-sharing for neural network compression. *arXiv preprint arXiv:1702.04008*, 2017.

[46] R. Vershynin. High-dimensional probability. *An Introduction with Applications*, 2016.

[47] K. Q. Weinberger, A. Dasgupta, J. Attenberg, J. Langford, and A. J. Smola. Feature hashing for large scale multitask learning. *CoRR*, abs/0902.2206, 2009.

[48] W. Wen, C. Wu, Y. Wang, Y. Chen, and H. Li. Learning structured sparsity in deep neural networks. In *Advances in Neural Information Processing Systems*, pages 2074–2082, 2016.

[49] J. Wu, C. Leng, Y. Wang, Q. Hu, and J. Cheng. Quantized Convolutional Neural Networks for Mobile Devices. In *Proceedings of the International Conference on Computer Vision and Pattern Recognition (CVPR)*, 2016.

[50] H. Xiao, K. Rasul, and R. Vollgraf. Fashion-mnist: a novel image dataset for benchmarking machine learning algorithms, 2017.

[51] X. Yu, T. Liu, X. Wang, and D. Tao. On compressing deep models by low rank and sparse decomposition. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 7370–7379, 2017.

[52] C. Zhang, S. Bengio, M. Hardt, B. Recht, and O. Vinyals. Understanding deep learning requires rethinking generalization. *arXiv preprint arXiv:1611.03530*, 2016.

[53] L. Zhao, S. Liao, Y. Wang, J. Tang, and B. Yuan. Theoretical properties for neural networks with weight matrices of low displacement rank. *CoRR*, abs/1703.00144, 2017.

[54] A. Zhou, A. Yao, Y. Guo, L. Xu, and Y. Chen. Incremental Network Quantization: Towards Lossless CNNs with Low-Precision Weights. In *Proceedings of the International Conference on Learning Representations (ICLR) 2017*, feb 2017.

[55] W. Zhou, V. Veitch, M. Austern, R. P. Adams, and P. Orbanz. Compressibility and generalization in large-scale deep learning. *arXiv preprint arXiv:1804.05862*, 2018.
A Proofs of the Analytical Results in Section 5

This section includes the full proofs of the technical results given in Sec. 5.

A.1 Analytical Results for Section 5.2 (Importance Sampling Bounds for Positive Weights)

Recall that the sensitivity $s_j$, of an edge $j \in W$ is defined as the maximum (approximate) relative importance over a subset $S \subseteq \mathcal{P}^{i.i.d.} \sim \mathcal{D}^n$, $|S| = n'$ (Definition 3).

A.1.1 Order Statistic Sampling

We now establish a couple of technical results that will quantify the accuracy of our approximations of edge importance (i.e., sensitivity).

Lemma 8. Let $K > 0$ be a universal constant and let $\mathcal{D}$ be a distribution with CDF $F(\cdot)$ satisfying $F(M/K) \leq \exp(-1/K)$ where $K \in \mathbb{R}_+$ is a universal constant and $M = \min\{x \in [0,1] : F(x) = 1\}$.

Let $\mathcal{P} = \{X_1, \ldots, X_n\}$ be a set of $n = |\mathcal{P}|$ i.i.d. samples each drawn from the distribution $\mathcal{D}$. Let $X_{n+1} \sim \mathcal{D}$ be an i.i.d. sample. Then,

$$\mathbb{P}(K \max_{X \in \mathcal{P}} X < X_{n+1}) \leq \exp(-n/K)$$

Proof. Let $X_{\text{max}} = \max_{X \in \mathcal{P}}$; then,

$$\mathbb{P}(K X_{\text{max}} < X_{n+1}) = \int_0^M \mathbb{P}(X_{\text{max}} < x/K | X_{n+1} = x) \, d\mathbb{P}(x)$$

$$= \int_0^M \mathbb{P}(X < x/K)^n \, d\mathbb{P}(x) \quad \text{since } X_1, \ldots, X_n \text{ are i.i.d.}$$

$$\leq \int_0^M F(x/K)^n \, d\mathbb{P}(x) \quad \text{where } F(\cdot) \text{ is the CDF of } X \sim \mathcal{D}$$

$$\leq F(M/K)^n \int_0^M \, d\mathbb{P}(x)$$

$$= F(M/K)^n \quad \text{by monotonicity of } F$$

$$\leq \exp(-n/K) \quad \text{CDF Assumption,}$$

and this completes the proof. \(\square\)

We now proceed to establish that the notion of empirical sensitivity is a good approximation for the relative importance. For this purpose, let the relative importance $\hat{g}_j(x)$ of an edge $j$ after the previous layers have already been compressed be

$$\hat{g}_j(x) = \frac{w_j \hat{a}_j(x)}{\sum_{k \in \mathcal{W}} w_k \hat{a}_k(x)}.$$
Lemma 9 (Empirical Sensitivity Approximation). Let \( \varepsilon \in (0, 1/2), \delta \in (0, 1), \ell \in \{2, \ldots, L\} \), Consider a set \( S = \{x_1, \ldots, x_n\} \subseteq P \) of size \( |S| \geq \lceil \log(8\eta \eta^* \delta) \log(\eta^*) \rceil \). Then, conditioned on the event \( \mathcal{E}_{i/2} \) occurring, i.e., \( \hat{a}(x) \in (1 \pm 1/2)a(x) \),

\[
\mathbb{P}_{x \sim D}(\exists j \in W : C s_j < \hat{g}_j(x) \mid \mathcal{E}_{i/2}) \leq \frac{\delta}{8\eta},
\]

where \( C = 3 \log(\eta^*) \) and \( W \subseteq [\eta^{\ell-1}] \).

Proof. Consider an arbitrary \( j \in W \) and \( x' \in S \) corresponding to \( g_j(x') \) with CDF \( F_j(\cdot) \) and recall that \( M = \min\{x \in [0, 1] : F_j(x) = 1\} \) as in Assumption 1. Note that by Assumption 1, we have

\[
F(M/K) \leq \exp(-1/K),
\]

and so the random variables \( g_j(x') \) for \( x' \in S \) satisfy the CDF condition required by Lemma 8. Now let \( \mathcal{E} \) be the event that \( Ks_j < g_j(x) \) holds. Applying Lemma 8, we obtain

\[
\mathbb{P}(\mathcal{E}) = \mathbb{P}(Ks_j < g_j(x)) = \mathbb{P}\left(K \max_{x' \in S} g_j(x') < g_j(x)\right) \leq \exp(-|S|/K).
\]

Now let \( \hat{\mathcal{E}} \) denote the event that the inequality \( C s_j < \hat{g}_j(x) = \frac{w_j \hat{a}_j(x)}{\sum_{k \in W} w_k a_k(x)} \) holds and note that the right side of the inequality is defined with respect to \( \hat{g}_j(x) \) and not \( g_j(x) \). Observe that since we conditioned on the event \( \mathcal{E}_{i/2} \), we have that \( \hat{a}(x) \in (1 \pm 1/2)a(x) \).

Now assume that event \( \hat{\mathcal{E}} \) holds and note that by the implication above, we have

\[
C s_j < \hat{g}_j(x) = \frac{w_j \hat{a}_j(x)}{\sum_{k \in W} w_k a_k(x)} \leq \frac{(1 + 1/2)w_j a_j(x)}{(1 - 1/2) \sum_{k \in W} w_k a_k(x)} \leq 3 \cdot \frac{w_j a_j(x)}{\sum_{k \in W} w_k a_k(x)} = 3 g_j(x),
\]

where the second inequality follows from the fact that \( 1+1/2/1-1/2 \leq 3 \). Moreover, since we know that \( C \geq 3K \), we conclude that if event \( \hat{\mathcal{E}} \) occurs, we obtain the inequality

\[
3K s_j \leq 3g_j(x) \iff K s_j \leq g_j(x),
\]

which is precisely the definition of event \( \mathcal{E} \). Thus, we have shown the conditional implication \( (\hat{\mathcal{E}} \mid \mathcal{E}_{i/2}) \Rightarrow \mathcal{E} \), which implies that

\[
\mathbb{P}(\hat{\mathcal{E}} \mid \mathcal{E}_{i/2}) = \mathbb{P}(C s_j < \hat{g}_j(x) \mid \mathcal{E}_{i/2}) \leq \mathbb{P}(\mathcal{E})
\]

18
\[ \leq \exp(-|S|/K). \]

Since our choice of \( j \in W \) was arbitrary, the bound applies for any \( j \in W \). Thus, we have by the union bound

\[
P(\exists j \in W : C s_j < \hat{g}_j(x) \mid E_{1/2}) \leq \sum_{j \in W} P(C s_j < \hat{g}_j(x) \mid E_{1/2}) \\
\leq |W| \exp(-|S|/K) \\
= \left( \frac{|W|}{\eta^*} \right) \frac{\delta}{8\eta} \\
\leq \frac{\delta}{8\eta},
\]

and this concludes the proof. \( \square \)

In practice, the set \( S \) referenced above is chosen to be a subset of the original data points, i.e., \( S \subseteq P \) (see Alg. 1, Line 2). Thus, we henceforth assume that the size of the input points \(|P|\) is large enough (or the specified parameter \( \delta \in (0, 1) \) is sufficiently large) so that \(|P| \geq |S|\).

### A.1.2 Proof of Lemma 1

We now state the proof of Lemma 1. In this subsection, we establish approximation guarantees under the assumption that the weights are strictly positive. The next subsection will then relax this assumption to conclude that a neuron’s value can be approximated well even when the weights are not all positive.

**Lemma 1** (Positive-Weights Sparsification). Let \( \varepsilon, \delta \in (0, 1) \), and \( x \sim \mathcal{D} \). \textsc{Sparsify}(\( W, w, \varepsilon, \delta, S, a(\cdot) \)) generates a row vector \( \hat{w} \) such that, conditioned on the event \( E_{1/2} \) occurring,

\[
P \left( \sum_{k \in W} \hat{w}_k \hat{a}_k(x) \notin (1 \pm \varepsilon) \sum_{k \in W} w_k \hat{a}_k(x) \mid E_{1/2} \right) \leq \frac{3\delta}{8\eta}
\]

where \( \text{nnz}(\hat{w}) \leq \left[ \frac{8S \log(\eta \eta^*) \log(8\eta/\delta)}{\varepsilon^2} \right] \), and \( S = \sum_{j \in W} s_j \).

**Proof.** Let \( \varepsilon, \delta \in (0, 1) \) be arbitrary. Moreover, let \( C \) be the coreset with respect to the weight indices \( W \subseteq [\eta^{\ell-1}] \) used to construct \( \hat{w} \). Note that as in \textsc{Sparsify}, \( C \) is a multiset sampled from \( W \) of size \( m = \left[ \frac{8S \log(\eta \eta^*) \log(8\eta/\delta)}{\varepsilon^2} \right] \), where \( S = \sum_{j \in W} s_j \) and \( C \) is sampled according to the probability distribution \( q \) defined by

\[
q_j = \frac{s_j}{S} \quad \forall j \in W.
\]

Let \( \hat{a}(\cdot) \) be an arbitrary realization of the random variable \( \hat{a}^{\ell-1}(\cdot) \), let \( x \) be a realization of \( x \sim \mathcal{D} \), and let

\[
\hat{z} = \sum_{k \in W} \hat{w}_k \hat{a}_k(x)
\]

be the approximate intermediate value corresponding to the sparsified matrix \( \hat{w} \) and let

\[
\bar{z} = \sum_{k \in W} w_k \hat{a}_k(x).
\]

19
Now define $\mathcal{E}$ to be the (favorable) event that $\hat{z}$ $\varepsilon$-approximates $\tilde{z}$, i.e., $\hat{z} \in (1 \pm \varepsilon)\tilde{z}$. We will now show that the complement of this event, $\mathcal{E}^c$, occurs with sufficiently small probability. Let $Z \subseteq \text{supp}(D)$ be the set of \textit{well-behaved} points (defined implicitly with respect to neuron $i \in [\eta^t]$ and realization $\hat{a}$) and defined as follows:

$$Z = \{x' \in \text{supp}(D) : \hat{g}_j(x') \leq C s_j \quad \forall j \in W\},$$

where $C = 3 \log(\eta \eta^t)$. Let $\mathcal{E}_Z$ denote the event that $x \in Z$ where $x$ is a realization of $x \sim D$.

**Conditioned on $\mathcal{E}_Z$, event $\mathcal{E}^c$ occurs with probability $\leq \frac{\delta}{m}$**. Let $x$ be a realization of $x \sim D$ such that $x \in Z$ and let $C = \{c_1, \ldots, c_m\}$ be $m$ samples from $W$ with respect to distribution $q$ as before. Define $m$ random variables $T_{c_1}, \ldots, T_{c_m}$ such that for all $j \in C$

$$T_j = \frac{w_j \hat{a}_j(x)}{m q_j} = \frac{S w_j \hat{a}_j(x)}{m s_j}. \quad (2)$$

For any $j \in C$, we have for the conditional expectation of $T_j$:

$$\mathbb{E}[T_j \mid \hat{a}(\cdot), x, \mathcal{E}_Z, \mathcal{E}_{1/2}] = \sum_{k \in W} \frac{w_k \hat{a}_k(x)}{m q_k} \cdot q_k = \sum_{k \in W} \frac{w_k \hat{a}_k(x)}{m} = \frac{\tilde{z}}{m},$$

where we use the expectation notation $\mathbb{E}[\cdot]$ with the understanding that it denotes the conditional expectation $\mathbb{E}_{\mathcal{C}}[\hat{a}^{-1}(\cdot), x \mid \cdot]$. Moreover, we also note that conditioning on the event $\mathcal{E}_Z$ (i.e., the event that $x \in Z$) does not affect the expectation of $T_j$. Let $T = \sum_{j \in C} T_j = \hat{z}$ denote our approximation and note that by linearity of expectation,

$$\mathbb{E}[T \mid \hat{a}(\cdot), x, \mathcal{E}_Z, \mathcal{E}_{1/2}] = \sum_{j \in C} \mathbb{E}[T_j \mid \hat{a}(\cdot), x, \mathcal{E}_Z, \mathcal{E}_{1/2}] = \hat{z}$$

Thus, $\hat{z} = T$ is an unbiased estimator of $\tilde{z}$ for any realization $\hat{a}(\cdot)$ and $x$; thus, we will henceforth refer to $\mathbb{E}[T \mid \hat{a}(\cdot), x]$ as simply $\hat{z}$ for brevity.

For the remainder of the proof we will assume that $\tilde{z} > 0$, since otherwise, $\tilde{z} = 0$ if and only if $T_j = 0$ for all $j \in C$ almost surely, which follows by the fact that $T_j \geq 0$ for all $j \in C$ by definition of $W$ and the non-negativity of the ReLU activation. Therefore, in the case that $\tilde{z} = 0$, it follows that

$$\mathbb{P}(|\tilde{z} - \hat{z}| > \varepsilon \tilde{z} \mid \hat{a}(\cdot), x) = \mathbb{P}(\hat{z} > 0 \mid \hat{a}(\cdot), x) = \mathbb{P}(0 > 0) = 0,$$

which trivially yields the statement of the lemma, where in the above expression, $\mathbb{P}(\cdot)$ is short-hand for the conditional probability $\mathbb{P}_{\hat{a}(\cdot)}[\hat{a}^{-1}(\cdot), x(\cdot)]$.

We now proceed with the case where $\tilde{z} > 0$ and leverage the fact that $x \in Z^3$ to establish that for all $j \in W$:

$$Cs_j \geq \hat{g}_j(x) = \frac{w_j \hat{a}_j(x)}{\sum_{k \in W} w_k \hat{a}_k(x)} = \frac{w_j \hat{a}_j(x)}{\hat{z}}$$

\footnote{Since we conditioned on the event $\mathcal{E}_Z$.}
\[ w_j \hat{a}_j(x) \leq C \hat{z}. \tag{3} \]

Utilizing the inequality established above, we bound the conditional variance of each \( T_j, j \in C \) as follows

\[
\begin{align*}
\text{Var}(T_j \mid \hat{a}(\cdot), x, \mathcal{E}_Z, \mathcal{E}_{1/2}) & \leq \mathbb{E} \left[ (T_j)^2 \mid \hat{a}(\cdot), x, \mathcal{E}_Z, \mathcal{E}_{1/2} \right] \\
& = \sum_{k \in W} \frac{(w_k \hat{a}_k(x))^2}{(m q_k)^2} \cdot q_k \\
& = \frac{S}{m^2} \sum_{k \in W} \frac{(w_k \hat{a}_k(x))^2}{s_k} \\
& \leq \frac{S}{m^2} \left( \sum_{k \in W} w_k \hat{a}_k(x) \right) C \hat{z} \\
& = \frac{SC \hat{z}^2}{m^2},
\end{align*}
\]

where \( \text{Var}(\cdot) \) is short-hand for \( \text{Var}_{C \mid \hat{a}^{-1}(\cdot), x}(\cdot) \). Since \( T \) is a sum of (conditionally) independent random variables, we obtain

\[
\begin{align*}
\text{Var}(T \mid \hat{a}(\cdot), x, \mathcal{E}_Z, \mathcal{E}_{1/2}) &= m \text{Var}(T_j \mid \hat{a}(\cdot), x, \mathcal{E}_Z, \mathcal{E}_{1/2}) \\
& \leq \frac{SC \hat{z}^2}{m}.
\end{align*}
\tag{4}
\]

Now, for each \( j \in C \) let

\[ \tilde{T}_j = T_j - \mathbb{E} [T_j \mid \hat{a}(\cdot), x, \mathcal{E}_Z, \mathcal{E}_{1/2}] = T_j - \hat{z}, \]

and let \( \tilde{T} = \sum_{j \in C} \tilde{T}_j \). Note that by the fact that we conditioned on the realization \( x \) of \( x \) such that \( x \in Z \) (event \( \mathcal{E}_Z \)), we obtain by definition of \( T_j \) in (2) and the inequality (3):

\[ T_j = \frac{S w_j \hat{a}_j(x)}{m s_j} \leq \frac{SC \hat{z}}{m}. \tag{5} \]

We also have that \( S \geq 1 \) by definition. More specifically, using the fact that the maximum over a set is greater than the average and rearranging sums, we obtain

\[
S = \sum_{j \in W} s_j = \sum_{j \in W} \max_{x' \in S} g_j(x') \\
\geq \frac{1}{|S|} \sum_{j \in W} \sum_{x' \in S} g_j(x') = \frac{1}{|S|} \sum_{x' \in S} \sum_{j \in W} g_j(x') \\
= \frac{1}{|S|} \sum_{x' \in S} 1 = 1.
\]

Thus, the inequality established in (5) with the fact that \( S \geq 1 \) we obtain an upper bound on the absolute value of the centered random variables:

\[ |\tilde{T}_j| = \left| T_j - \frac{\hat{z}}{m} \right| \leq \frac{SC \hat{z}}{m} = M, \tag{6} \]

which follows from the fact that:
if \( T_j \geq \frac{\tilde{z}}{m} \): Then, by our bound in (5) and the fact that \( \frac{\tilde{z}}{m} \geq 0 \), it follows that
\[
\left| \tilde{T}_j \right| = T_j - \frac{\tilde{z}}{m} \leq \frac{SC \tilde{z}}{m} - \frac{\tilde{z}}{m} \leq \frac{SC \tilde{z}}{m}.
\]

if \( T_j < \frac{\tilde{z}}{m} \): Then, using the fact that \( T_j \geq 0 \) and \( S \geq 1 \), we obtain
\[
\left| \tilde{T}_j \right| = \frac{\tilde{z}}{m} - T_j \leq \frac{\tilde{z}}{m} \leq \frac{SC \tilde{z}}{m}.
\]

Applying Bernstein’s inequality to both \( \tilde{T} \) and \(-\tilde{T}\) we have by symmetry and the union bound,
\[
P(\mathcal{E}^c \mid \hat{a}(\cdot), x, \mathcal{E}_{Z}, \mathcal{E}_{1/2}) = P \left( |T - \tilde{z}| \geq \varepsilon \tilde{z} \mid \hat{a}(\cdot), x, \mathcal{E}_{Z}, \mathcal{E}_{1/2} \right) \]
\[
\leq 2 \exp \left( -\frac{\varepsilon^2 \tilde{z}^2}{2 \text{Var}(T \mid \hat{a}(\cdot), x) + \frac{2 \varepsilon \tilde{z} \hat{M}}{3}} \right) \]
\[
\leq 2 \exp \left( -\frac{2 \varepsilon^2 \tilde{z}^2}{m} + \frac{2SC \tilde{z}^2}{3m} \right) \]
\[
= 2 \exp \left( -\frac{3 \varepsilon^2 m}{8SC} \right) \]
\[
\leq \frac{\delta}{4\eta},
\]
where the second inequality follows by our upper bounds on \( \text{Var}(T \mid \hat{a}(\cdot), x) \) and \( \left| \tilde{T}_j \right| \) and the fact that \( \varepsilon \in (0, 1) \), and the last inequality follows by our choice of \( m = \left\lceil \frac{8S \log(\eta^* \eta^*)}{\varepsilon^2} \log(8\eta/\delta) \right\rceil \). This establishes that for any realization \( \hat{a}(\cdot) \) of \( \hat{a}^{-1}(\cdot) \) and a realization \( x \) of \( x \) satisfying \( x \in \mathcal{Z} \), the event \( \mathcal{E}^c \) occurs with probability at most \( \frac{\delta}{4\eta} \).

**Removing the conditioning on \( \mathcal{E}_{Z} \):** We have by law of total probability
\[
P(\mathcal{E} \mid \hat{a}(\cdot), \mathcal{E}_{1/2}) \geq \int_{x \in \mathcal{Z}} P(\mathcal{E} \mid \hat{a}(\cdot), x, \mathcal{E}_{Z}, \mathcal{E}_{1/2}) P_{x \sim D}(x = x \mid \hat{a}(\cdot), \mathcal{E}_{1/2}) dx \]
\[
\geq \left( 1 - \frac{\delta}{4\eta} \right) \int_{x \in \mathcal{Z}} P_{x \sim D}(x = x \mid \hat{a}(\cdot), \mathcal{E}_{1/2}) dx \]
\[
= \left( 1 - \frac{\delta}{4\eta} \right) \mathbb{P}(\mathcal{E}_{Z} \mid \hat{a}(\cdot), \mathcal{E}_{1/2}) \]
\[
\geq \left( 1 - \frac{\delta}{4\eta} \right) \left( 1 - \frac{\delta}{8\eta} \right) \]

22
\[ \geq 1 - \frac{3\delta}{8\eta} \]

where the second-to-last inequality follows from the fact that \( P(\mathcal{E}^c \mid \hat{a}(\cdot), x, \mathcal{E}_x, \mathcal{E}_{1/2}) \leq \frac{\delta}{4\eta} \) as was established above and the last inequality follows by Lemma 9.

**Putting it all together** Finally, we marginalize out the random variable \( \hat{a}^{\ell-1}(\cdot) \) to establish

\[
P(\mathcal{E} \mid \mathcal{E}_{1/2}) = \int_{\hat{a}(\cdot)} P(\mathcal{E} \mid \hat{a}(\cdot), \mathcal{E}_{1/2}) P(\hat{a}(\cdot) \mid \mathcal{E}_{1/2}) d\hat{a}(\cdot) \\
\geq \left( 1 - \frac{3\delta}{8\eta} \right) \int_{\hat{a}(\cdot)} P(\hat{a}(\cdot) \mid \mathcal{E}_{1/2}) d\hat{a}(\cdot) \\
= 1 - \frac{3\delta}{8\eta}.
\]

Consequently,

\[
P(\mathcal{E}^c \mid \mathcal{E}_{1/2}) \leq 1 - \left( 1 - \frac{3\delta}{8\eta} \right) = \frac{3\delta}{8\eta},
\]

and this concludes the proof. \( \square \)

**A.2 Analytical Results for Section 5.3 (Importance Sampling Bounds)**

We begin by establishing an auxiliary result that we will need for the subsequent lemmas.

**A.2.1 Empirical \( \Delta^\ell_i \) Approximation**

**Lemma 10** (Empirical \( \Delta^\ell_i \) Approximation). Let \( \delta \in (0, 1) \), \( \lambda_* = \log(\eta \eta^*)/2 \), and define

\[
\hat{\Delta}^\ell = \left( \frac{1}{|S|} \max_{i \in [\eta^*]} \sum_{x' \in S} \Delta^\ell_i(x') \right) + \kappa,
\]

where \( \kappa = \sqrt{2\lambda_*} \left( 1 + \sqrt{2\lambda_* \log (8 \eta \eta^*/\delta)} \right) \) and \( S \subseteq \mathcal{P} \) is as in Alg. 1. Then,

\[
P_{x \sim \mathcal{D}} \left( \max_{i \in [\eta^*]} \Delta^\ell_i(x) \leq \hat{\Delta}^\ell \right) \geq 1 - \frac{\delta}{4\eta}.
\]

**Proof.** Define the random variables \( \mathcal{Y}_{x'} = \mathbb{E} [\Delta^\ell_i(x')] - \Delta^\ell_i(x') \) for each \( x' \in S \) and consider the sum

\[
\mathcal{Y} = \sum_{x' \in S} \mathcal{Y}_{x'} = \sum_{x' \in S} \left( \mathbb{E} [\Delta^\ell_i(x)] - \Delta^\ell_i(x') \right).
\]

We know that each random variable \( \mathcal{Y}_{x'} \) satisfies \( \mathbb{E} [\mathcal{Y}_{x'}] = 0 \) and by Assumption 2, is subexponential with parameter \( \lambda \leq \lambda_* \). Thus, \( \mathcal{Y} \) is a sum of \( |S| \) independent, zero-mean \( \lambda_* \)-subexponential random
variables, which implies that $E[\mathcal{Y}] = 0$ and that we can readily apply Bernstein’s inequality for subexponential random variables [46] to obtain for $t \geq 0$

$$
\mathbb{P}\left( \frac{1}{|\mathcal{S}|} \mathcal{Y} \geq t \right) \leq \exp\left( -|\mathcal{S}| \min\left\{ \frac{t^2}{4\lambda^2}, \frac{t}{2\lambda} \right\} \right).
$$

Since $\mathcal{S} = [\log (8 \eta \eta^*/\delta) \log(\eta \eta^*)] \geq \log (8 \eta \eta^*/2 \lambda^*)$, we have for $t = \sqrt{2\lambda^*}$,

$$
\mathbb{P}\left( E[\Delta^\delta_i(x)] - \frac{1}{|\mathcal{S}|} \sum_{x' \in \mathcal{S}} \Delta^\delta_i(x') \geq t \right) = \mathbb{P}\left( \frac{1}{|\mathcal{S}|} \mathcal{Y} \geq t \right) \leq \exp\left( -|\mathcal{S}| \frac{t^2}{4\lambda^2} \right) \leq \exp (-\log (8 \eta \eta^*/\delta)) = \frac{\delta}{8 \eta \eta^*}.
$$

Moreover, for a single $\mathcal{Y}_x$, we have by the equivalent definition of a subexponential random variable [46] that for $u \geq 0$

$$
\mathbb{P}(\Delta^\delta_i(x) - E[\Delta^\delta_i(x)] \geq u) \leq \exp\left( -\min\left\{ \frac{u^2}{4\lambda^2}, \frac{u}{2\lambda} \right\} \right).
$$

Thus, for $u = 2\lambda^* \log (8 \eta \eta^*/\delta)$ we obtain

$$
\mathbb{P}(\Delta^\delta_i(x) - E[\Delta^\delta_i(x)] \geq u) \leq \exp (-\log (8 \eta \eta^*/\delta)) = \frac{\delta}{8 \eta \eta^*}.
$$

Therefore, by the union bound, we have with probability at least $1 - \frac{\delta}{4 \eta \eta^*}$:

$$
\Delta^\delta_i(x) \leq E[\Delta^\delta_i(x)] + u
\leq \left( \frac{1}{|\mathcal{S}|} \sum_{x' \in \mathcal{S}} \Delta^\delta_i(x') + t \right) + u
= \frac{1}{|\mathcal{S}|} \sum_{x' \in \mathcal{S}} \Delta^\delta_i(x') + \left( \sqrt{2\lambda^*} + 2\lambda^* \log (8 \eta \eta^*/\delta) \right)
= \frac{1}{|\mathcal{S}|} \sum_{x' \in \mathcal{S}} \Delta^\delta_i(x') + \kappa
\leq \hat{\Delta}^\delta,
$$

where the last inequality follows by definition of $\hat{\Delta}^\delta$.

Thus, by the union bound, we have

$$
\mathbb{P}_{x \sim \mathcal{D}} \left( \max_{i \in \eta^\ell} \Delta^\delta_i(x) > \hat{\Delta}^\delta \right) = \mathbb{P} \left( \exists i \in \eta^\ell : \Delta^\delta_i(x) > \hat{\Delta}^\delta \right) \leq \sum_{i \in \eta^\ell} \mathbb{P} \left( \Delta^\delta_i(x) > \hat{\Delta}^\delta \right)
$$

24
$$\leq \eta^f \left( \frac{\delta}{4\eta \eta^*} \right) \leq \frac{\delta}{4\eta},$$

where the last line follows by definition of $\eta^* \geq \eta^f$.

## A.2.2 Notation for the Subsequent Analysis

Let $\hat{w}_i^{\ell+}$ and $\hat{w}_i^{\ell-}$ denote the sparsified row vectors generated when \textsc{Sparsify} is invoked with first two arguments corresponding to $(\mathcal{W}_+, w_i^f)$ and $(\mathcal{W}_-, -w_i^f)$, respectively (Alg. 1, Lines 17-18). We will at times omit including the variables for the neuron $i$ and layer $\ell$ in the proofs for clarity of exposition, and for example, refer to $\hat{w}_i^{\ell+}$ and $\hat{w}_i^{\ell-}$ as simply $\hat{w}^+$ and $\hat{w}^-$, respectively.

Let $x \sim \mathcal{D}$ and define

$$\hat{z}^+(x) = \sum_{k \in \mathcal{W}_+} \hat{w}_k^+ \hat{a}_k(x) \geq 0 \quad \text{and} \quad \hat{z}^-(x) = \sum_{k \in \mathcal{W}_-} (-\hat{w}_k^-) \hat{a}_k(x) \geq 0$$

be the approximate intermediate values corresponding to the sparsified matrices $\hat{w}^+$ and $\hat{w}^-; let

$$\hat{z}^+(x) = \sum_{k \in \mathcal{W}_+} w_k \hat{a}_k(x) \geq 0 \quad \text{and} \quad \hat{z}^-(x) = \sum_{k \in \mathcal{W}_-} (-w_k) \hat{a}_k(x) \geq 0$$

be the corresponding intermediate values with respect to the original row vector $w$; and finally, let

$$z^+(x) = \sum_{k \in \mathcal{W}_+} w_k a_k(x) \geq 0 \quad \text{and} \quad z^-(x) = \sum_{k \in \mathcal{W}_-} (-w_k) a_k(x) \geq 0$$

be the true intermediate values corresponding to the positive and negative valued weights.

Note that in this context, we have by definition

$$\hat{z}_i^f(x) = \langle \hat{w}, \hat{a}(x) \rangle = \hat{z}^+(x) - \hat{z}^-(x),$$

$$z_i^f(x) = \langle w, \hat{a}(x) \rangle = \hat{z}^+(x) - \hat{z}^-(x), \quad \text{and}$$

$$z_i^f(x) = \langle w, a(x) \rangle = z^+(x) - z^-(x),$$

where we used the fact that $\hat{w} = \hat{w}^+ - \hat{w}^- \in \mathbb{R}^{1 \times \eta^{f-1}}$.

## A.2.3 Proof of Lemma 2

**Lemma 2** (Conditional Neuron Value Approximation). Let $\varepsilon, \delta \in (0, 1)$, $\ell \in \{2, \ldots, L\}$, $i \in [\eta^f]$, and $x \sim \mathcal{D}$. \textsc{CoreNet} generates a row vector $\hat{w}_i^f = \hat{w}_i^{\ell+} - \hat{w}_i^{\ell-} \in \mathbb{R}^{1 \times \eta^{f-1}}$ such that, conditioned on the event $\mathcal{E}^{\ell-1}$ occurring, i.e., $\hat{z}_i^{\ell-1}(x) \in (1 \pm 2(\ell - 2) \varepsilon_\ell) \hat{z}_i^{\ell-1}(x)$, $\mathbb{P} \left( \mathcal{E}_i^\ell \mid \mathcal{E}^{\ell-1} \right) = \mathbb{P} \left( \hat{z}_i^f(x) \in (1 \pm 2(\ell - 1) \varepsilon_{\ell+1}) z_i^f(x) \mid \mathcal{E}^{\ell-1} \right) \geq 1 - \frac{\delta}{\eta}, \quad (1)$

where $\varepsilon_\ell = \frac{e^2}{\Delta \varepsilon} \eta^{f-1}$ and $\text{nnz}(w_i^f) \leq \left\lfloor \frac{8 S \log(q \eta^*) \log(8 \eta/\delta)}{\varepsilon_\ell^2} \right\rfloor + 1$, where $S = \sum_{j \in \mathcal{W}_+} s_j + \sum_{j \in \mathcal{W}_-} s_j$ is the sum of sensitivities with respect to neuron $i \in [\eta^f]$. 25
Proof. Let \( \varepsilon, \delta \in (0, 1) \) be arbitrary and let \( \mathcal{W}_+ = \{ j \in [\eta^{L-1}] : w_j > 0 \} \) and \( \mathcal{W}_- = \{ j \in [\eta^{L-1}] : w_j < 0 \} \) as in Alg. 1. Let \( \varepsilon_\ell \) be defined as before,

\[
\varepsilon_\ell = \frac{\varepsilon'}{\Delta^\ell \rightarrow},
\]

where \( \Delta^\ell \rightarrow = \prod_{k=\ell}^{L} \hat{\Delta}^k \) and \( \hat{\Delta}^\ell = \left( \frac{1}{|S|} \max_{i \in [\eta^\ell]} \sum_{x' \in S} \Delta^\ell_i(x') \right) + \kappa. \)

Observe that \( w_j > 0 \forall j \in \mathcal{W}_+ \) and similarly, for all \( (-w_j) > 0 \forall j \in \mathcal{W}_- \). That is, each of index sets \( \mathcal{W}_+ \) and \( \mathcal{W}_- \) corresponds to strictly positive entries in the arguments \( w^\ell_i \) and \( -w^\ell_i \), respectively passed into \textsc{Sparsify}. Observe that since we conditioned on the event \( \mathcal{E}^{\ell-1} \), we have

\[
2 (\ell - 2) \varepsilon_\ell \leq 2 (\ell - 2) \frac{\varepsilon}{2(L - 1) \prod_{k=\ell}^{L} \hat{\Delta}^k} \leq \frac{\varepsilon}{2L - \ell + 1} \leq \frac{\varepsilon}{2},
\]

where the inequality \( \hat{\Delta}^k \geq 2 \) follows from the fact that

\[
\hat{\Delta}^k = \left( \frac{1}{|S|} \max_{i \in [\eta^\ell]} \sum_{x' \in S} \Delta^\ell_i(x') \right) + \kappa \geq 1 + \kappa \quad \text{Since } \Delta^\ell_i(x') \geq 1 \quad \forall x' \in \text{supp}(D) \text{ by definition} \geq 2.
\]

we obtain that \( \hat{a}(x) \in (1 \pm \varepsilon/2)a(x) \), where, as before, \( \hat{a} \) and \( a \) are shorthand notations for \( \hat{a}^{\ell-1} \in \mathbb{R}^{\eta^{L-1} \times 1} \) and \( a^{\ell-1} \in \mathbb{R}^{\eta^{L-1} \times 1} \), respectively. This implies that \( \mathcal{E}^{\ell-1} \Rightarrow \mathcal{E}_{1/2} \) and since \( m = \left\lceil 8 \log(\eta \eta^*) \log(8 \eta/\delta) \right\rceil \) in Alg. 2 we can invoke Lemma 1 with \( \varepsilon = \varepsilon_\ell \) on each of the \textsc{Sparsify} invocations to conclude that

\[
\mathbb{P} \left( \hat{z}^+(x) \notin (1 \pm \varepsilon_\ell) \hat{z}^+(x) \mid \mathcal{E}^{\ell-1} \right) \leq \mathbb{P} \left( \hat{z}^+(x) \notin (1 \pm \varepsilon_\ell) \hat{z}^+(x) \mid \mathcal{E}_{1/2} \right) \leq \frac{3 \delta}{8 \eta},
\]

and

\[
\mathbb{P} \left( \hat{z}^-(x) \notin (1 \pm \varepsilon_\ell) \hat{z}^-(x) \mid \mathcal{E}^{\ell-1} \right) \leq \frac{3 \delta}{8 \eta}.
\]

Therefore, by the union bound, we have

\[
\mathbb{P} \left( \hat{z}^+(x) \notin (1 \pm \varepsilon_\ell) \hat{z}^+(x) \text{ or } \hat{z}^-(x) \notin (1 \pm \varepsilon_\ell) \hat{z}^-(x) \mid \mathcal{E}^{\ell-1} \right) \leq \frac{3 \delta}{8 \eta} + \frac{3 \delta}{8 \eta} = \frac{3 \delta}{4 \eta}.
\]

Moreover, by Lemma 10, we have with probability at most \( \frac{\delta}{4 \eta} \) that

\[
\Delta^\ell_i(x) > \hat{\Delta}^\ell.
\]
Thus, by the union bound over the failure events, we have that with probability at least \(1 - (3\delta/4n + \delta/4n) = 1 - \delta/n\) that both of the following events occur

1. \(\hat{z}^+(x) \in (1 \pm \varepsilon_\ell)\hat{z}^+(x)\) and \(\hat{z}^-(x) \in (1 \pm \varepsilon_\ell)\hat{z}^-(x)\)  \(\tag{7}\)

2. \(\Delta^i_\ell(x) \leq \hat{\Delta}^\ell\)  \(\tag{8}\)

Recall that \(\varepsilon' = \frac{\varepsilon}{2(L-1)}\), \(\varepsilon_\ell = \frac{\varepsilon'}{\Delta^\ell}\), and that event \(\mathcal{E}^\ell_i\) denotes the (desirable) event that

\[
\hat{z}^\ell_\ell(x) (1 + 2 (\ell - 1) \varepsilon_{\ell+1}) z^\ell_\ell(x)
\]

holds, and similarly, \(\mathcal{E} = \cap_{i \in [\ell]} \mathcal{E}^\ell_i\) denotes the vector-wise analogue where

\[
\hat{z}^\ell(x) (1 + 2 (\ell - 1) \varepsilon_{\ell+1}) z^\ell(x).
\]

Let \(k = 2 (\ell - 1)\) and note that by conditioning on the event \(\mathcal{E}^{\ell-1}\), i.e., we have by definition

\[
\hat{a}^{\ell-1}(x) \in (1 + 2 (\ell - 2) \varepsilon_\ell) a^{\ell-1}(x)
\]

\[
= (1 + k \varepsilon_\ell) a^{\ell-1}(x),
\]

which follows by definition of the ReLU function. Recall that our overarching goal is to establish that

\[
\hat{z}^\ell_i(x) \in (1 + 2 (\ell - 1) \varepsilon_{\ell+1}) z^\ell_i(x),
\]

which immediately imply by definition of the ReLU function that

\[
\hat{a}^\ell_i(x) \in (1 + 2 (\ell - 1) \varepsilon_{\ell+1}) a^\ell_i(x).
\]

Having clarified the conditioning and our objective, we will once again drop the index \(i\) from the expressions moving forward.

Proceeding from above, we have with probability at least \(1 - \delta/n\)

\[
\hat{z}(x) = \hat{z}^+(x) - \hat{z}^-(x)
\]

By Event (7) above

\[
\leq (1 + \varepsilon_\ell) \hat{z}^+(x) - (1 - \varepsilon_\ell) \hat{z}^-(x)
\]

Conditioning on event \(\mathcal{E}^{\ell-1}\)

\[
\leq (1 + \varepsilon_\ell)(1 + k \varepsilon_\ell) \hat{z}^+(x) - (1 - \varepsilon_\ell)(1 - k \varepsilon_\ell) \hat{z}^-(x)
\]

\[
= (1 + \varepsilon_\ell(k + 1) + k \varepsilon^2_\ell) \hat{z}^+(x) + (-1 + (k + 1) \varepsilon_\ell - k \varepsilon^2_\ell) \hat{z}^-(x)
\]

\[
= (1 + k \varepsilon^2_\ell) \hat{z}(x) + (k + 1) \varepsilon_\ell \left( \hat{z}^+(x) + \hat{z}^-(x) \right)
\]

\[
= (1 + k \varepsilon^2_\ell) \hat{z}(x) + \frac{(k + 1) \varepsilon'}{\prod_{k=1}^{\ell} \Delta^k} \left( \hat{z}^+(x) + \hat{z}^-(x) \right)
\]

By Event (8) above

\[
\leq (1 + k \varepsilon^2_\ell) \hat{z}(x) + \frac{(k + 1) \varepsilon'}{\prod_{k=1}^{\ell} \Delta^k} \left( \hat{z}^+(x) + \hat{z}^-(x) \right)
\]

By \(\Delta^\ell_i(x) = \frac{\hat{z}^+(x) + \hat{z}^-(x)}{|\hat{z}(x)|}\)

\[
= (1 + k \varepsilon^2_\ell) \hat{z}(x) + \frac{(k + 1) \varepsilon'}{\prod_{k=1}^{\ell+1} \Delta^k} |\hat{z}(x)|
\]

\[
= (1 + k \varepsilon^2_\ell) \hat{z}(x) + (k + 1) \varepsilon_{\ell+1} |\hat{z}(x)|.
\]

To upper bound the last expression above, we begin by observing that \(k \varepsilon^2_\ell \leq \varepsilon_\ell\), which follows from the fact that \(\varepsilon_\ell \leq \frac{1}{\prod(L-1)} \leq \frac{1}{k}\) by definition. Moreover, we also note that \(\varepsilon_\ell \leq \varepsilon_{\ell+1}\) by definition of \(\hat{\Delta}^\ell \geq 1\).

Now, we consider two cases:
Case of \( z(x) \geq 0 \): In this case, we have
\[
\hat{z}(x) \leq (1 + k \varepsilon^2_\ell) z(x) + (k + 1) \varepsilon_{\ell+1} |z(x)| \\
\leq (1 + \varepsilon_\ell) z(x) + (k + 1) \varepsilon_{\ell+1} z(x) \\
\leq (1 + \varepsilon_{\ell+1}) z(x) + (k + 1) \varepsilon_{\ell+1} z(x) \\
= (1 + (k + 2) \varepsilon_{\ell+1}) z(x) \\
= (1 + 2(\ell - 1) \varepsilon_{\ell+1}) z(x),
\]
where the last line follows by definition of \( k = 2(\ell - 2) \), which implies that \( k + 2 = 2(\ell - 1) \). Thus, this establishes the desired upper bound in the case that \( z(x) \geq 0 \).

Case of \( z(x) < 0 \): Since \( z(x) \) is negative, we have that \((1 + k \varepsilon^2_\ell) z(x) \leq z(x)\) and \( |z(x)| = -z(x)\) and thus
\[
\hat{z}(x) \leq (1 + k \varepsilon^2_\ell) z(x) + (k + 1) \varepsilon_{\ell+1} |z(x)| \\
\leq z(x) - (k + 1) \varepsilon_{\ell+1} z(x) \\
\leq (1 - (k + 1) \varepsilon_{\ell+1}) z(x) \\
\leq (1 - (k + 2) \varepsilon_{\ell+1}) z(x) \\
= (1 - 2(\ell - 1) \varepsilon_{\ell+1}) z(x),
\]
and this establishes the upper bound for the case of \( z(x) \) being negative.

Putting the results of the case by case analysis together, we have the upper bound of \( \hat{z}(x) \leq z(x) + 2(\ell - 1) \varepsilon_{\ell+1} |z(x)| \). The proof for establishing the lower bound for \( z(x) \) is analogous to that given above, and yields \( \hat{z}(x) \geq z(x) - 2(\ell - 1) \varepsilon_{\ell+1} |z(x)| \). Putting both the upper and lower bound together, we have that with probability at least \( 1 - \frac{\delta}{n} \):
\[
\hat{z}(x) \in (1 \pm 2(\ell - 1) \varepsilon_{\ell+1}) z(x),
\]
and this completes the proof.

A.2.4 Remarks on Negative Activations

We note that up to now we assumed that the input \( a(x) \), i.e., the activations from the previous layer, are strictly nonnegative. For layers \( \ell \in \{3, \ldots, L\} \), this is indeed true due to the nonnegativity of the ReLU activation function. For layer 2, the input is \( a(x) = x \), which can be decomposed into \( a(x) = a_{pos}(x) - a_{neg}(x) \), where \( a_{pos}(x) \geq 0 \in \mathbb{R}^{\ell-1} \) and \( a_{neg}(x) \geq 0 \in \mathbb{R}^{\ell-1} \). Furthermore, we can define the sensitivity over the set of points \( \{a_{pos}(x), a_{neg}(x) \mid x \in \mathcal{S}\} \) (instead of \( \{a(x) \mid x \in \mathcal{S}\} \)), and thus maintain the required nonnegativity of the sensitivities. Then, in the terminology of Lemma 2, we let
\[
z_{pos}^+(x) = \sum_{k \in \mathcal{W}_+} w_k a_{pos,k}(x) \geq 0 \quad \text{and} \quad z_{neg}^-(x) = \sum_{k \in \mathcal{W}_-} (-w_k) a_{neg,k}(x) \geq 0
\]
be the corresponding positive parts, and
\[
z_{neg}^+(x) = \sum_{k \in \mathcal{W}_+} w_k a_{neg,k}(x) \geq 0 \quad \text{and} \quad z_{pos}^-(x) = \sum_{k \in \mathcal{W}_-} (-w_k) a_{pos,k}(x) \geq 0
\]
be the corresponding negative parts of the preactivation of the considered layer, such that
\[ z^+(x) = z^+_{\text{pos}}(x) + z^+_{\text{neg}}(x) \quad \text{and} \quad z^-(x) = z^+_{\text{neg}}(x) + z^+_{\text{pos}}(x). \]

We also let
\[ \Delta^i_\ell(x) = \frac{z^+(x) + z^-(x)}{|z(x)|} \]
be as before, with \( z^+(x) \) and \( z^-(x) \) defined as above. Equipped with above definitions, we can rederive Lemma 2 analogously in the more general setting, i.e., with potentially negative activations. We also note that we require a slightly larger sample size now since we have to take a union bound over the failure probabilities of all four approximations (i.e. \( \hat{\mathbf{z}}^+_{\text{pos}}(x), \hat{\mathbf{z}}^-_{\text{neg}}(x), \hat{\mathbf{z}}^+_{\text{neg}}(x), \) and \( \hat{\mathbf{z}}^-_{\text{pos}}(x) \)) to obtain the desired overall failure probability of \( \delta/\eta \).

### A.2.5 Proof of Theorem 4

The following corollary immediately follows from Lemma 2 and establishes a layer-wise approximation guarantee.

**Corollary 11** (Conditional Layer-wise Approximation). Let \( \varepsilon, \delta \in (0, 1), \ell \in \{2, \ldots, L\}, \) and \( x \sim \mathcal{D}. \) Conditioned on the event \( \mathcal{E}^{\ell-1} \) occurring, CORENET generates a sparse weight matrix \( \hat{W}^\ell = (\hat{w}^\ell_1, \ldots, \hat{w}^\ell_{\eta^\ell})\top \in \mathbb{R}^{\eta^\ell \times \eta^{\ell-1}} \) such that
\[
\mathbb{P}(\mathcal{E}^\ell \mid \mathcal{E}^{\ell-1}) = \mathbb{P}(\hat{\mathbf{z}}^\ell(x) \in (1 \pm 2 (\ell - 1) \varepsilon_{\ell+1}) z^\ell(x) \mid \mathcal{E}^{\ell-1}) \geq 1 - \frac{\delta \eta^\ell}{\eta}, \tag{9}
\]
where \( \varepsilon_\ell = \frac{\varepsilon}{\Delta_{\varepsilon-}}, \hat{\mathbf{z}}^\ell(x) = \hat{W}^\ell \hat{a}^\ell(x), \) and \( z^\ell(x) = W^\ell a^\ell(x) \).

**Proof.** Since (1) established by Lemma 2 holds for any neuron \( i \in [\eta^\ell] \) in layer \( \ell \) and since \( (\mathcal{E}^\ell)^c = \cup_{i \in [\eta^\ell]} (\mathcal{E}^\ell_i)^c \), it follows by the union bound over the failure events \( (\mathcal{E}^\ell_i)^c \) for all \( i \in [\eta^\ell] \) that with probability at least \( 1 - \frac{\eta^\ell}{\eta} \)
\[
\hat{\mathbf{z}}^\ell(x) = \hat{W}^\ell \hat{a}^{\ell-1}(x)
\in (1 \pm 2 (\ell - 1) \varepsilon_{\ell+1}) W^\ell a^{\ell-1}(x)
= (1 \pm 2 (\ell - 1) \varepsilon_{\ell+1}) z^\ell(x).
\]

The following lemma removes the conditioning on \( \mathcal{E}^{\ell-1} \) and explicitly considers the (compounding) error incurred by generating coresets \( \hat{W}^2, \ldots, \hat{W}^\ell \) for multiple layers.

**Lemma 3** (Layer-wise Approximation). In the context of the Corollary 11, for any arbitrary layer \( \ell \in \{2, \ldots, L\} \) it holds for \( \hat{\mathbf{z}}^\ell(x) = \hat{W}^\ell \hat{a}^\ell(x) \) that
\[
\mathbb{P}_{(\hat{W}^2, \ldots, \hat{W}^\ell), x} (\mathcal{E}^\ell) = \mathbb{P}_{(\hat{W}^2, \ldots, \hat{W}^\ell), x} \left( \hat{\mathbf{z}}^\ell(x) \in (1 \pm 2 (\ell - 1) \varepsilon_{\ell+1}) z^\ell(x) \right) \geq 1 - \frac{\delta}{\eta} \sum_{\ell=2}^{L} \eta^\ell. \tag{10}
\]
Proof. Invoking Corollary 11, we know that for any layer $\ell' \in \{2, \ldots, L\}$,
\[
\mathbb{P}_{\hat{W}^{\ell'}, x, \hat{a}^{\ell'-1}(\cdot)} (\mathcal{E}^{\ell'} \mid \mathcal{E}^{\ell'-1}) \geq 1 - \frac{\delta}{\eta} \eta^{\ell'}.
\tag{10}
\]
We also have by the law of total probability that
\[
\mathbb{P}(\mathcal{E}^{\ell'}) = \mathbb{P}(\mathcal{E}^{\ell'} \mid \mathcal{E}^{\ell'-1}) \mathbb{P}(\mathcal{E}^{\ell'-1}) + \mathbb{P}(\mathcal{E}^{\ell'} \mid (\mathcal{E}^{\ell'-1})^c) \mathbb{P}((\mathcal{E}^{\ell'-1})^c)
\geq \mathbb{P}(\mathcal{E}^{\ell'} \mid \mathcal{E}^{\ell'-1}) \mathbb{P}(\mathcal{E}^{\ell'-1})
\tag{11}
\]
Repeated applications of (10) and (11) in conjunction with the observation that $\mathbb{P}(\mathcal{E}^1) = 1^4$ yield
\[
\mathbb{P}(\mathcal{E}^\ell) \geq \mathbb{P}(\mathcal{E}^{\ell'} \mid \mathcal{E}^{\ell'-1}) \mathbb{P}(\mathcal{E}^{\ell'-1}) \geq \prod_{\ell'=2}^{\ell} \mathbb{P}(\mathcal{E}^{\ell'} \mid \mathcal{E}^{\ell'-1})
\geq \prod_{\ell'=2}^{\ell} \left( 1 - \frac{\delta}{\eta} \eta^{\ell'} \right)
\geq 1 - \frac{\delta}{\eta} \sum_{\ell'=2}^{\ell} \eta^{\ell'}
\]
By the Weierstrass Product Inequality, where the last inequality follows by the Weierstrass Product Inequality\(^5\) and this establishes the lemma. \qed

 Appropriately invoking Lemma 3, we can now establish the approximation guarantee for the entire neural network. This is stated in Theorem 4 and the proof can be found below.

**Theorem 4** (Network Compression). For $\epsilon, \delta \in (0, 1)$, Algorithm 1 generates a set of parameters $\hat{\theta} = (\hat{W}^2, \ldots, \hat{W}^L)$ of size
\[
\text{nnz}(\hat{\theta}) \leq \sum_{\ell=2}^{L} \sum_{i=1}^{\eta^\ell} \left( \left\lfloor \frac{32 (L-1)^2 (\hat{\Delta}^{\ell-1})^2 S^\ell_i \log(\eta \eta^*) \log(8 \eta/\delta)}{\epsilon^2} \right\rfloor + 1 \right)
\]
in $O \left( \eta \eta^* \log \left( \eta \eta^*/\delta \right) \right)$ time such that $\mathbb{P}_{\hat{\theta}, x \sim \mathcal{D}} (f_{\hat{\theta}}(x) \in (1 \pm \epsilon) f_{\theta}(x)) \geq 1 - \delta$.

**Proof.** Invoking Lemma 3 with $\ell = L$, we have that for $\hat{\theta} = (\hat{W}^2, \ldots, \hat{W}^L)$,
\[
\mathbb{P}_{\hat{\theta}, x} (f_{\hat{\theta}}(x) \in 2 (L-1) \varepsilon_{L+1} f_{\theta}(x)) = \mathbb{P}_{\hat{\theta}, x} (z^L(x) \in 2 (L-1) \varepsilon_{L+1} z^L(x))
\]
\(^4\)Since we do not compress the input layer.
\(^5\)The Weierstrass Product Inequality [15] states that for $p_1, \ldots, p_n \in [0, 1],
\[
\prod_{i=1}^{n} (1 - p_i) \geq 1 - \sum_{i=1}^{n} p_i.
\]

30
\[ P(\mathcal{E}^L) \]

\[ \geq 1 - \frac{\delta}{\eta} \sum_{\ell'=2}^{L} \eta_{\ell'} \]

\[ = 1 - \delta, \]

where the last equality follows by definition of \( \eta = \sum_{\ell=2}^{L} \eta_{\ell} \). Note that by definition,

\[ \varepsilon_{L+1} = \frac{\varepsilon}{2(L-1) \prod_{k=L+1}^{L} \Delta^k} \]

\[ = \frac{\varepsilon}{2(L-1)}, \]

where the last equality follows by the fact that the empty product \( \prod_{k=L+1}^{L} \Delta^k \) is equal to 1.

Thus, we have

\[ 2(L-1)\varepsilon_{L+1} = \varepsilon, \]

and so we conclude

\[ P_{\theta, x} (f_{\theta}(x) \in (1 \pm \varepsilon) f_{\theta}(x)) \geq 1 - \delta, \]

which, along with the sampling complexity of Alg. 2 (Line 6), establishes the approximation guarantee provided by the theorem.

For the computational time complexity, we observe that the most time consuming operation per iteration of the loop on Lines 9-19 is the weight sparsification procedure. The asymptotic time complexity of each SPARSIFY invocation for each neuron \( i \in [\eta^{\ell}] \) in layers \( \ell \in \{2, \ldots, L\} \) (Alg. 1, Lines 17-18) is dominated by the relative importance computation for incoming edges (Alg. 2, Lines 1-2). This can be done by evaluating \( w_{\ell}^{i,k} a_{\ell-1}^{k}(x) \) for all \( k \in \mathcal{W} \) and \( x \in \mathcal{S} \), for a total computation time that is bounded above by \( \mathcal{O}(|\mathcal{S}| \eta^{\ell-1}) \) since \( |\mathcal{W}| \leq \eta^{\ell-1} \) for each \( i \in [\eta^{\ell}] \). Thus, SPARSIFY takes \( \mathcal{O}(|\mathcal{S}| \eta^{\ell-1}) \) time. Summing the computation time over all layers and neurons in each layer, we obtain an asymptotic time complexity of \( \mathcal{O}(\sum_{\ell=2}^{L} \eta^{\ell} \eta^* \eta) \leq \mathcal{O}(\log(\eta \eta^* / \delta)). \)

Since \( |\mathcal{S}| \leq \mathcal{O}(\log(\eta \eta^* / \delta)) \), we conclude that the computational complexity our neural network compression algorithm is

\[ \mathcal{O}(\eta * \log(\eta \eta^* / \delta)). \] \hfill (12)

\section*{A.2.6 Proof of Theorem 13}

In order to ensure that the established sampling bounds are non-vacuous in terms of the sensitivity, i.e., not linear in the number of incoming edges, we show that the sum of sensitivities per neuron \( S \) is small. The following lemma establishes that the sum of sensitivities can be bounded \emph{instance-independent} by a term that is logarithmic in roughly the total number of edges \((\eta \cdot \eta^*)\).

\textbf{Lemma 12 (Sensitivity Bound).} \textit{For any} \( \ell \in \{2, \ldots, L\} \) \textit{and} \( i \in [\eta^{\ell}] \), \textit{the sum of sensitivities} \( S = S_+ + S_- \) \textit{is bounded by}

\[ S \leq 2|\mathcal{S}| = 2 \left[ \log \left( \frac{8 \eta \eta^*}{\delta} \right) \log(\eta \eta^*) \right]. \]
Proof. Consider $S_{\ell}$ for an arbitrary $\ell \in \{2, \ldots, L\}$ and $i \in [\eta^\ell]$. For all $j \in W$ we have the following bound on the sensitivity of a single $j \in W$,

$$s_j = \max_{x \in S} g_j(x) \leq \sum_{x \in S} g_j(x) = \sum_{x \in S} w_j a_j(x) \sum_{k \in W} w_k a_k(x),$$

where the inequality follows from the fact that we can upper bound the max by a summation over $x \in S$ since $g_j(x) \geq 0, \forall j \in W$. Thus,

$$S_+ = \sum_{j \in W} s_j \leq \sum_{j \in W} \sum_{x \in S} g_j(x) = \sum_{x \in S} \sum_{j \in W} w_j a_j(x) \sum_{k \in W} w_k a_k(x) = |S|,$$

where we used the fact that the sum of sensitivities is finite to swap the order of summation.

Using the same argument as above, we obtain

$$S_- = \sum_{j \in W_-} s_j \leq |S|,$$

which establishes the lemma. \qed

Note that the sampling complexities established above have a linear dependence on the sum of sensitivities, $\sum_{\ell=2}^{L} \sum_{i=1}^{\eta^\ell} S_{\ell}^i$, which is instance-dependent, i.e., depends on the sampled $S \subseteq P$ and the actual weights of the trained neural network. By applying Lemma 12, we obtain a bound on the size of the compressed network that is independent of the sensitivity.

**Theorem 13 (Sensitivity-Independent Network Compression).** For any given $\varepsilon, \delta \in (0, 1)$ our sampling scheme (Alg. 1) generates a set of parameters $\hat{\theta}$ of size $\text{nnz}(\hat{\theta}) \in \mathcal{O}\left(\frac{\log(\eta/\delta) \log(\eta \eta^*/\delta) \log^2(\eta \eta^*) \eta L^2}{\varepsilon^2} \sum_{\ell=2}^{L} (\hat{\Delta} \ell^-)^2\right)$, in $\mathcal{O}\left(\eta \eta^* \log(\eta \eta^*/\delta)\right)$ time, such that $\mathbb{P}_{\hat{\theta}, x \sim D}(f_{\hat{\theta}}(x) \in (1 \pm \varepsilon)f_\theta(x)) \geq 1 - \delta$.

Proof. Combining Lemma 12 and Theorem 4 establishes the theorem. \qed
A.2.7 Generalized Network Compression

Theorem 4 gives us an approximation guarantee with respect to one randomly drawn point \( x \sim \mathcal{D} \). The following corollary extends this approximation guarantee to any set of \( n \) randomly drawn points using a union bound argument, which enables approximation guarantees for, e.g., a test data set composed of \( n \) i.i.d. points drawn from the distribution. We note that the sampling complexity only increases by roughly a logarithmic term in \( n \).

**Corollary 14** (Generalized Network Compression). For any \( \varepsilon, \delta \in (0,1) \) and a set of i.i.d. input points \( \mathcal{P}' \) of cardinality \( |\mathcal{P}'| \in \mathbb{N}_+ \), i.e., \( \mathcal{P}' \overset{i.i.d.}{\sim} \mathcal{D}^{|\mathcal{P}'|} \), consider the reparameterized version of Alg. 1 with

1. \( S \subseteq \mathcal{P} \) of size \( |S| \geq \lceil \log (16 |\mathcal{P}'| \eta \eta^* / \delta) \log (\eta \eta^*) \rceil \),
2. \( \hat{\Delta}^\ell = \left( \frac{1}{|S|} \max_{i \in \eta^\ell} \sum_{x' \in S} \Delta_i^\ell(x') \right) + \kappa \) as before, but \( \kappa \) is instead defined as
   \[ \kappa = \sqrt{2 \lambda} \left( 1 + \sqrt{2 \lambda} \log (16 |\mathcal{P}'| \eta \eta^*/\delta) \right), \]
   and
3. \( m \geq \left\lceil \frac{8 S \log(\eta \eta^*) \log(16 |\mathcal{P}'| \eta \delta)}{\varepsilon^2} \right\rceil \) in the sample complexity in SPARSIFYWEIGHTS.

Then, Alg. 1 generates a set of neural network parameters \( \hat{\theta} \) of size at most

\[
\text{nnz}(\hat{\theta}) \leq \sum_{\ell=2}^{L} \sum_{i=1}^{\eta^\ell} \left( \frac{32 (L - 1)^2 (\hat{\Delta}^\rightarrow)^2 S_i^\ell \log(\eta \eta^*) \log(16 |\mathcal{P}'| \eta \delta)}{\varepsilon^2} \right) + 1 \in \mathcal{O} \left( \frac{\log(\eta \eta^*)}{\varepsilon^2} \log(\eta |\mathcal{P}'| / \delta) \right) \sum_{\ell=2}^{L} (\hat{\Delta}^\rightarrow)^2 \sum_{i=1}^{\eta^\ell} S_i^\ell \),
\]

in \( \mathcal{O} \left( \eta \eta^* \log \left( \eta \eta^* |\mathcal{P}'| / \delta \right) \right) \) time such that

\[ \mathbb{P}_{\hat{\theta}, x} (\forall x \in \mathcal{P}' : f_{\hat{\theta}}(x) \in (1 \pm \varepsilon) f_{\theta}(x)) \geq 1 - \frac{\delta}{2}. \]

**Proof.** The reparameterization enables us to invoke Theorem 4 with \( \delta' = \delta/2 |\mathcal{P}'| \); applying the union bound over all \( |\mathcal{P}'| \) i.i.d. samples in \( \mathcal{P}' \) establishes the corollary.

A.3 Analytical Results for Section 5.4 (Theorem 6, Amplification)

In the context of the notation introduced in Section 5.4 recall that the relative error of a (randomly generated) row vector \( \hat{w} : = \hat{w}^\ell = \hat{w}^+ - \hat{w}^- \in \mathbb{R}^{1 \times \eta_{\ell-1}} \) with respect to a point \( x \in \text{supp}(\mathcal{D}) \) is defined as

\[ \text{err}_{\hat{w}}(x) = \frac{\hat{\ell}(x)}{\hat{z}(x)} - 1 = \frac{\langle \hat{w}^\ell, \hat{a}^{\ell-1}(x) \rangle}{\langle \hat{w}^\ell, \hat{a}^{\ell-1}(x) \rangle} - 1, \]

where \( \hat{w} \) is shorthand for \( \hat{w}^\ell_i \) as before. Similarly define

\[ \text{err}_{\hat{w}^+}(x) = \frac{\hat{\ell}^+(x)}{\hat{z}^+(x)} - 1 = \frac{\sum_{k \in W_+} w_k^\ell \hat{a}_k^{\ell-1}(x)}{\sum_{k \in W_+} w_k^\ell \hat{a}_k^{\ell-1}(x)} - 1 \]

and
The following lemma establishes the expected performance of a randomly constructed coreset with respect to the distribution of points $\mathcal{D}$ conditioned on coreset constructions for previous layers $(\hat{W}^2, \ldots, \hat{W}^{\ell-1})$ that define the realization $\hat{a}^{\ell-1}(-)$.

**Lemma 5** (Expected Error). Let $\varepsilon, \delta \in (0, 1)$, $\ell \in \{2, \ldots, L\}$, and $i \in [\eta^\ell]$. Conditioned on the event $\mathcal{E}^{\ell-1}$, CORENET generates a row vector $\hat{w} = \hat{w}^+ - \hat{w}^- \in \mathbb{R}^{1 \times \eta^{\ell-1}}$ such that

$$
\mathbb{E} [\text{err}_{\hat{w}}(x) \mid \mathcal{E}^{\ell-1}] \leq \varepsilon_\ell \hat{\Delta}^{\ell} \left( k + \frac{5 (1 + k \varepsilon_\ell)}{\sqrt{\log(8 \eta/\delta)}} \right) + \frac{\delta (1 + k \varepsilon_\ell)}{\eta} \mathbb{E} \left[ \Delta_\ell^i(x) \mid \Delta_\ell^i(x) > \hat{\Delta} \right],
$$

where $k = 2(\ell - 1)$.

**Proof.** For clarity of exposition we will omit explicit references to the layer $\ell$ and neuron $i$ as they are assumed to be arbitrary. In the context of this definition, let $\hat{w} = \hat{w}^+ - \hat{w}^- \in \mathbb{R}^{1 \times \eta^{\ell-1}}$ and let $x \sim \mathcal{D}$. The proof outline is to bound the overall error term $\text{err}_{\hat{w}}(x)$ by bounding $\text{err}_{\hat{w}^+}(x)$ and $\text{err}_{\hat{w}^-}(x)$.

Let $\mathcal{E}_\Delta(x)$ denote the event that the inequality

$$
\Delta_\ell^i(x) \leq \hat{\Delta}
$$

holds and recall that we condition on the event $\mathcal{E}^{\ell-1}$ occurring as in the premise of the lemma.

Let $k = 2(\ell - 2)$ and let $\xi = k \varepsilon_\ell$. We begin by observing that conditioned on the events $\mathcal{E}_\Delta$ and $\mathcal{E}^{\ell-1}$, for any constant $u \geq 0$, the following inequality

$$
\max \{ \text{err}_{\hat{w}^+}(x), \text{err}_{\hat{w}^-}(x) \} \leq \frac{u \xi}{1 + \xi} := \varepsilon_*$
$$

implies that

$$
\text{err}_{\hat{w}}(x) \leq k \varepsilon_{\ell+1} (u + 1).
$$

Henceforth we will at times omit the variable $x$ when referring to the point-specific variables for clarity of exposition with the understanding that the results hold for any arbitrary $x$.

To see the previous implication explicitly, observe that conditioning on $\mathcal{E}^{\ell-1}$ implies that we have $\hat{a}^{\ell-1}(x) \in (1 \pm 2 (\ell - 2) \varepsilon_\ell) a^{\ell-1}(x) = (1 \pm \xi) a^{\ell-1}(x)$, which yields by the triangle inequality

$$
|\hat{z} - z| \leq |\hat{z}^+ - z^+| + |\hat{z}^- - z^-| = \left| \sum_{k \in \mathcal{W}_+} w_k (\hat{a}_k - a_k) \right| + \left| \sum_{k \in \mathcal{W}_-} (-w_k) (\hat{a}_k - a_k) \right|
$$

$$
\leq \sum_{k \in \mathcal{W}_+} w_k |\hat{a}_k - a_k| + \sum_{k \in \mathcal{W}_-} (-w_k) |\hat{a}_k - a_k|
$$

$$
\leq \sum_{k \in \mathcal{W}_+} w_k \xi a_k + \sum_{k \in \mathcal{W}_-} (-w_k) \xi a_k
$$

$$
= \xi (z^+ + z^-).
$$

(13)
Moreover, via a similar triangle-inequality type argument and the premise \(\max\{\hat{\text{err}}_{\hat{\omega}}^+ (x), \hat{\text{err}}_{\hat{\omega}}^- (x)\} \leq \varepsilon_*\) we obtain

\[
|\hat{z} - \bar{z}| \leq |\hat{z}^+ - \bar{z}^+| + |\hat{z}^- - \bar{z}^-|
\]

\[
\leq \varepsilon_* (\hat{z}^+ + \bar{z}^-)
\]

\[
\leq \varepsilon_* ((1 + \xi) \hat{z}^+ + (1 + \xi) \bar{z}^-)
\]

\[
= \varepsilon_* (1 + \xi)(\hat{z}^+ + \bar{z}^-).
\]

By event \(\mathcal{E}^{t-1}\)

Combining the inequalities (13) and (14), we obtain

\[
|\hat{z} - z| \leq |\hat{z} - \bar{z}| + |\hat{z} - z|
\]

\[
\leq \varepsilon_* (1 + \xi) (\hat{z}^+ + \bar{z}^-) + \xi (\hat{z}^+ + \bar{z}^-)
\]

\[
= |z| \Delta_t^v (x) (\varepsilon_* (1 + \xi) + \xi)
\]

\[
= |z| \Delta_t^v (x) (u \xi + \xi)
\]

\[
\leq |z| \Delta^\xi (u + 1)
\]

\[
= |z| \Delta^\xi \kappa \varepsilon_\ell (u + 1)
\]

\[
= |z| \kappa \varepsilon_{\ell+1} (u + 1),
\]

By event \(\mathcal{E}_{\Delta}\)

By definition of \(\xi = k \varepsilon_\ell\)

By definition of \(\varepsilon_{\ell+1} = \Delta^\xi \varepsilon_\ell\)

and dividing both sides by \(|z|\) yields the bound on \(\text{err}_{\omega}(x)\).

Let \(\mathcal{Z} \subseteq \text{supp}(\mathcal{D})\) denote the set of well-behaved points, i.e., the set of points that satisfy the sensitivity inequality with respect to edges in both \(\mathcal{W}_+\) and \(\mathcal{W}_-\), i.e.,

\[
\mathcal{Z} = \{ x' \in \text{supp}(\mathcal{D}) : \hat{g}_j (x') \leq C s_j \quad \forall j \in \mathcal{W}_+ \cup \mathcal{W}_- \},
\]

and let \(\mathcal{E}_{\mathcal{Z}(x)}\) denote the event \(x \in \mathcal{Z}\). Let \(\mathcal{E}_{\mathcal{G}(x)} = \mathcal{E}_{\mathcal{Z}(x)} \cap \mathcal{E}_{\Delta}(x)\) denote the good event that both of the events \(\mathcal{E}_{\mathcal{Z}(x)}\) and \(\mathcal{E}_{\Delta}(x)\) occur.

Note that since \(\text{err}_{\omega}(x) \geq 0\) for all \(x\) and \(\hat{\omega}\), we obtain by the equivalent formulation of expectation of non-negative random variables

\[
\mathbb{E} [\text{err}_{\omega}(x) \mid \mathcal{E}_{\mathcal{G}(x)} \cap \mathcal{E}^{t-1}] = \int_0^\infty \mathbb{P} (\text{err}_{\omega}(x) \geq v \mid \mathcal{E}_{\mathcal{G}(x)} \cap \mathcal{E}^{t-1}) \, dv
\]

\[
\leq k \varepsilon_{\ell+1} + \int_0^\infty \mathbb{P} (\text{err}_{\omega}(x) \geq v \mid \mathcal{E}_{\mathcal{G}(x)} \cap \mathcal{E}^{t-1}) \, dv
\]

\[
= k \varepsilon_{\ell+1} \left( 1 + \int_0^\infty \mathbb{P} (\text{err}_{\omega}(x) \geq k \varepsilon_{\ell+1} (u + 1) \mid \mathcal{E}_{\mathcal{G}(x)} \cap \mathcal{E}^{t-1}) \, dv \right)
\]

(15)

where the last equality follows by the change of variable \(v = k \varepsilon_{\ell+1} (u + 1)\).

Recall that by the argument presented in the proof of Lemma 1, we have by Bernstein’s inequality for \(t \geq 0\)

\[
\mathbb{P} (\text{err}_{\omega}^+ (x) \geq t \mid \mathcal{E}_{\mathcal{G}(x)} \cap \mathcal{E}^{t-1}) \leq 2 \exp \left( -\frac{3t^2 m}{SC (6 + 2t)} \right)
\]

\[
\leq 2 \exp \left( -\frac{8 \log(8 \eta/\delta)}{6 + 2t} \cdot \left( \frac{t}{\varepsilon_\ell} \right)^2 \right)
\]

35
\[
\begin{align*}
\text{where } \\
\quad &a := \frac{8 \log(8\eta/\delta)}{\varepsilon_t^2},
\end{align*}
\]

in the last equality and the second inequality follows by definition of \(m = \left\lceil \frac{8 S \log(\eta \eta^*) \log(8\eta/\delta)}{\varepsilon_t^2} \right\rceil \) and the fact that \(C = 3 \log(\eta \eta^*)\). Via the same reasoning, we have for \(\tilde{\text{err}}_{\tilde{w}^*}^{-}(x)\):

\[
\begin{align*}
\mathbb{P}\left( \tilde{\text{err}}_{\tilde{w}^*}^{-}(x) \geq t \mid \mathcal{E}_G(x) \cap \mathcal{E}^{\ell-1} \right) &\leq 2 \exp\left( -\frac{a t^2}{6 + 2t} \right).
\end{align*}
\]

Hence, combining the implication established in the beginning of the proof with the bounds established above, we invoke the union bound to obtain

\[
\begin{align*}
\mathbb{P}\left( \text{err}_{\tilde{w}}(x) \geq k\varepsilon_{\ell+1}(u + 1) \mid \mathcal{E}_G(x) \cap \mathcal{E}^{\ell-1} \right) &\leq \mathbb{P}\left( \max\{\tilde{\text{err}}_{\tilde{w}^*}^{-}(x), \tilde{\text{err}}_{\tilde{w}^*}^{-}(x)\} > \frac{u \xi}{1 + \xi} \mid \mathcal{E}_G(x) \cap \mathcal{E}^{\ell-1} \right) \leq \min\left\{ 4 \exp\left( -\frac{a t^2}{6 + 2t} \right), 1 \right\},
\end{align*}
\]

where \(t = \frac{u \xi}{1 + \xi}\) and as before, \(a = \frac{8 \log(8\eta/\delta)}{\varepsilon_t^2}\). From the expression above, we see that for a value of \(t\) satisfying

\[
\begin{align*}
t &\geq \frac{2\sqrt{a \log 8 + \log^2 2 + \log 4}}{a},
\end{align*}
\]

we have \(4 \exp\left( -\frac{a t^2}{6 + 2t} \right) \leq 1\). Bounding the expression above via elementary computations, we have

\[
\begin{align*}
\frac{2\sqrt{a \log 8 + \log^2 2 + \log 4}}{a} &\leq \frac{2\sqrt{2a \log 8 + \log 4}}{a} \\
&\leq \frac{3\sqrt{2a \log 8}}{a} \\
&\leq \frac{7}{\sqrt{a}} := t^*.
\end{align*}
\]

Now note that for \(t \geq t^*\), we have

\[
\begin{align*}
\exp\left( -\frac{a t^2}{6 + 2t} \right) &\leq \exp\left( -\frac{a t^2}{6 (t/t^*) + 2t} \right) = \exp\left( -\frac{a t t^*}{6 + 2t^*} \right).
\end{align*}
\]

Let

\[
\begin{align*}
b &= \frac{\xi}{1 + \xi},
\end{align*}
\]

and recall that \(t = \frac{u \xi}{1 + \xi} = ub\). Letting

\[
\begin{align*}
u^* &= \frac{t^*}{b} = \frac{7}{b \sqrt{a}},
\end{align*}
\]

36
we reformulate the bound above in terms of $u$ and $u^*$,

$$\exp \left( - \frac{at^*}{6 + 2t^*} \right) = \exp \left( - \frac{ab^2 u^* u}{6 + 2u^* b} \right) = \exp \left( - \left( \frac{ab^2 u^*}{6 + 2u^* b} \right) u \right) = \exp(-c u),$$

where

$$c = \frac{ab^2 u^*}{6 + 2u^* b}.$$  

This implies that for $u \geq u^*$, we have

$$\mathbb{P} \left( \text{err}_{\hat{w}}(x) \geq k\varepsilon_{\ell+1}(u + 1) \mid \mathcal{E}_{\mathcal{G}(x)} \cap \mathcal{E}^{\ell-1} \right) \leq 4 \exp(-cu),$$

and for $u \in [0, u^*]$, we trivially have

$$\mathbb{P} \left( \text{err}_{\hat{w}}(x) \geq k\varepsilon_{\ell+1}(u + 1) \mid \mathcal{E}_{\mathcal{G}(x)} \cap \mathcal{E}^{\ell-1} \right) \leq 1.$$

Putting it all together, we bound the integral from (15) as follows

$$\int_0^\infty \mathbb{P} \left( \text{err}_{\hat{w}}(x) \geq k\varepsilon_{\ell+1}(u + 1) \mid \mathcal{E}_{\mathcal{G}(x)} \cap \mathcal{E}^{\ell-1} \right) du \leq \int_0^{u^*} 1 du + 4 \int_{u^*}^\infty \exp(-cu) du$$

$$= u^* + \frac{4 \exp(-cu^*)}{c} \leq u^* + \frac{4 \exp(-2)}{c} \leq u^* + \frac{80 \exp(-2)}{49} u^* \leq 2u^*,$$

where the first inequality follows from the definitions of $u^*$ and $c$, which imply that $u^* b = 7/\sqrt{a}$ and so by straightforward simplification,

$$cu^* = \left( \frac{ab(u^* b)}{6 + 2(u^* b)} \right) u^* = \left( \frac{7ab}{6\sqrt{a} + 14} \right) u^* \geq \frac{49\sqrt{a}}{6\sqrt{a} + 14 \sqrt{a}} = \frac{49}{20} > 2,$$

where we used the inequality $a = \frac{8 \log(8\eta/\delta)}{\epsilon_1^2} \geq 1$. This implies that $\exp(-cu^*) \leq \exp(-2)$. Similarly, the second inequality follows from the calculations above and the definition of $u^*$

$$\frac{1}{c} \leq \frac{20}{7b \sqrt{a}} = \frac{20}{49} u^*.$$
Plugging this inequality on the integral back to our bound on the conditional expectation (15), we establish
\[ \mathbb{E} [\operatorname{err}_{\hat{w}}(x) \mid \mathcal{E}_{G(x)} \cap \mathcal{E}^{\ell-1}] \leq k \varepsilon_{\ell+1} (1 + 2 u^*). \]

To bound the conditional expectation given the event \((\mathcal{E}_{G(x)})^c\) we first observe that since \(\hat{w}^+\) and \(\hat{w}^-\) are unbiased estimators, we have
\[ \mathbb{E} [(\hat{w}^+, \cdot) \mid \mathcal{E}_{G(x)}^c \cap \mathcal{E}^{\ell-1}, x] = (w^+, \cdot) \quad \text{and} \quad \mathbb{E} [(\hat{w}^-, \cdot) \mid \mathcal{E}_{G(x)}^c \cap \mathcal{E}^{\ell-1}, x] = (w^-, \cdot). \]

Moreover, note that conditioning on event \(\mathcal{E}^{\ell-1}\) implies that for any \(x \in \text{supp}(D)\)
\[ |\hat{z}(x)| \leq |\tilde{z}(x)| + \xi (\tilde{z}^+(x) + \tilde{z}^-(x)), \]
where \(\xi = k \varepsilon_\ell\) as before. Thus, invoking the triangle inequality and applying the definition of \(\Delta_\ell^f(x)\), we bound \(\operatorname{err}_{\hat{w}}(x)\) as
\[ \operatorname{err}_{\hat{w}}(x) = \left| \frac{\hat{z}(x)}{z(x)} - 1 \right| \leq \frac{\hat{z}(x)}{z(x)} + 1 \]
\[ \leq \Delta_\ell^f \frac{\hat{z}(x)}{z^+(x) + z^-(x)} + \xi (\hat{z}^+(x) + \hat{z}^-(x)) + 1 \]
\[ \leq \Delta_\ell^f \frac{\hat{z}^+(x) + \hat{z}^-(x) + \xi (\hat{z}^+(x) + \hat{z}^-(x))}{z^+(x) + z^-(x)} + 1 \]
\[ = (1 + \xi) \Delta_\ell^f \frac{\hat{z}^+(x) + \hat{z}^-(x)}{z^+(x) + z^-(x)} + 1. \]

Since the above bound holds for any arbitrary \(x\), we obtain by monotonicity of expectation, law of iterated expectation, and the unbiasedness of our estimators
\[ \mathbb{E} [\operatorname{err}_{\hat{w}}(x) \mid \mathcal{E}_{G(x)}^c \cap \mathcal{E}^{\ell-1}] \]
\[ \leq (1 + \xi) \mathbb{E} \left[ \Delta_\ell^f \frac{\hat{z}^+(x) + \hat{z}^-(x)}{z^+(x) + z^-(x)} \mid \mathcal{E}_{G(x)}^c \cap \mathcal{E}^{\ell-1} \right] + 1 \]
\[ = (1 + \xi) \mathbb{E} \left[ \Delta_\ell^f \frac{\hat{z}^+(x) + \hat{z}^-(x)}{z^+(x) + z^-(x)} \mid \mathcal{E}_{G(x)}^c \cap \mathcal{E}^{\ell-1} \right] + 1 \]
\[ = (1 + \xi) \mathbb{E} \left[ \Delta_\ell^f \mid \mathcal{E}_{G(x)}^c \cap \mathcal{E}^{\ell-1} \right] + 1 \]
\[ = (1 + \xi) \mathbb{E} \left[ \Delta_\ell^f \mid \mathcal{E}_{G(x)}^c \right] + 1 \]
\[ = (1 + \xi) \mathbb{E} \left[ \Delta_\ell^f \mid \Delta_\ell^f > \hat{\Delta}_\ell \right] + 1 \]
\[ \leq 2 (1 + \xi) \mathbb{E} \left[ \Delta_\ell^f \mid \Delta_\ell^f > \hat{\Delta}_\ell \right]. \]

By the law of total expectation, we obtain
\[ \mathbb{E} [\operatorname{err}_{\hat{w}}(x) \mid \mathcal{E}^{\ell-1}] = \underbrace{\mathbb{E} [\operatorname{err}_{\hat{w}}(x) \mid \mathcal{E}_{G(x)}^c \cap \mathcal{E}^{\ell-1}] \mathbb{P}(\mathcal{E}_{G(x)})}_{A} + \underbrace{\mathbb{E} [\operatorname{err}_{\hat{w}}(x) \mid \mathcal{E}_{G(x)}^c \cap \mathcal{E}^{\ell-1}] \mathbb{P}(\mathcal{E}_{G(x)})}_{B} \].

38
\[ A (1 - \mathbb{P}(E_{\text{G}(x)}^c)) + B \mathbb{P}(E_{\text{G}(x)}^c) \leq A_{\max} \left(1 - \mathbb{P}(E_x^c)\right) + B_{\max} \mathbb{P}(E_x^c), \]

where \( A_{\max} \) and \( B_{\max} \) are the upper bounds on the conditional expectations as established above:

\[ A_{\max} = k \varepsilon_{\ell+1} (1 + 2 u^*) \quad \text{and} \quad B_{\max} = 2 (1 + \xi) \mathbb{E}_x \left[ \Delta_\ell^\ell(x) \mid \Delta_i^\ell(x) > \hat{\Delta}_\ell^\ell \right]. \]

We now bound \( \mathbb{P}(E_x^c) \) by the union bound and applications of Lemma 9 (twice, one for positive and the other for negative weights) and Lemma 10

\[ \mathbb{P}(E_x^c) \leq \left( \frac{\delta}{8\eta} + \frac{\delta}{8\eta} \right) + \frac{\delta}{4\eta} = \frac{\delta}{2\eta}. \]

Moreover, by definition of \( a, \xi, u^* \) we have

\[ A_{\max} = \xi \hat{\Delta}_\ell^\ell (1 + 2 u^*) \]
\[ = \hat{\Delta}_\ell^\ell \left( \xi + \frac{14(1 + \xi)}{\sqrt{a}} \right) \]
\[ \leq \hat{\Delta}_\ell^\ell \left( \xi + \frac{5 \varepsilon_\ell (1 + \xi)}{\sqrt{\log(8\eta/\delta)}} \right) \]
\[ = \varepsilon_\ell \hat{\Delta}_\ell^\ell \left( k + \frac{5 (1 + \xi)}{\sqrt{\log(8\eta/\delta)}} \right). \]

Putting it all together, we establish

\[ \mathbb{E} [\text{err}_{\hat{w}^\ell}(x) \mid \mathcal{E}_{\ell-1}] \leq \left(1 - \frac{\delta}{2\eta}\right) A_{\max} + \frac{\delta}{2\eta} B_{\max} \]
\[ \leq A_{\max} + \frac{\delta}{2\eta} B_{\max} \]
\[ \leq \varepsilon_\ell \hat{\Delta}_\ell^\ell \left( k + \frac{5 (1 + \xi)}{\sqrt{\log(8\eta/\delta)}} \right) + \frac{\delta (1 + \xi)}{\eta} \mathbb{E}_x \left[ \Delta_i^\ell(x) \mid \Delta_i^\ell(x) > \hat{\Delta}_\ell^\ell \right] \]
\[ = \varepsilon_\ell \hat{\Delta}_\ell^\ell \left( k + \frac{5 (1 + k \varepsilon_\ell)}{\sqrt{\log(8\eta/\delta)}} \right) + \frac{\delta (1 + k \varepsilon_\ell)}{\eta} \mathbb{E}_x \left[ \Delta_i^\ell(x) \mid \Delta_i^\ell(x) > \hat{\Delta}_\ell^\ell \right] \]

and this concludes the proof.

Next, consider \( \tau \in \mathbb{N}_+ \) coreset constructions corresponding to the approximations \( \{(\hat{w}_\ell^\ell)_1, \ldots, (\hat{w}_\ell^\ell)_\tau\} \) generated as in Alg. 2 for layer \( \ell \in \{2, \ldots, L\} \) and neuron \( i \in [\eta^\ell] \). We overload the \( \text{err}_C(\cdot) \) function so that the error with respect to the set \( T \) is defined as the mean error, i.e.,

\[ \text{err}_{\hat{w}^\ell}(T) = \frac{1}{|T|} \sum_{x \in T} \text{err}_{\hat{w}^\ell}(x). \] (16)

Equipped with this definition, we proceed to prove Theorem 6.
Theorem 6 (Amplification). Given \( \varepsilon, \delta \in (0,1) \) such that \( \frac{\varepsilon}{\delta} \) is sufficiently small, let \( \tau \in \{2, \ldots, L\} \) and \( i \in [\eta^\ell] \). Let \( \tau = \left\lceil \frac{\log(4 \eta/\delta)}{\log(10/9)} \right\rceil \) and consider the reparameterized variant of Alg. 1 where we instead have

1. \( S \subseteq \mathcal{P} \) of size \( |S| \geq \left\lceil \frac{\log (32 |\mathcal{P}| \tau \eta^* / \delta)}{\log (32 |\mathcal{P}| \tau \eta^*/\delta)} \right\rceil \),

2. \( \hat{\Delta}^\ell = \left( \frac{1}{|S|} \max_{i \in [\eta^\ell]} \sum_{x' \in S} \Delta_i^\ell (x') \right) + \kappa \) as before, but \( \kappa \) is instead defined as
   \[
   \kappa = \sqrt{2 \lambda^* \left( 1 + \sqrt{2 \lambda^* \log (32 |\mathcal{P}| \tau \eta^*/\delta)} \right)},
   \]
   and

3. \( m \geq \left\lceil \frac{8 \log (\eta^*) \log (32 |\mathcal{P}| \tau \eta^*/\delta)}{\varepsilon^2} \right\rceil \) in the sample complexity in \textsc{SparsifyWeights}.

Among \( \tau \) approximations \( \hat{w}^\ell_1, \ldots, \hat{w}^\ell_\tau \) generated by Alg. 2, let

\[
\hat{w}^\ast_i = \operatorname{argmin}_{\hat{w}^\ell_i \in \{\hat{w}^\ell_1, \ldots, \hat{w}^\ell_\tau\}} \operatorname{err}_{\hat{w}^\ell_i}(T),
\]
and \( T \subseteq (\mathcal{P} \setminus S) \) be a subset of points of size \( |T| = \left\lceil 8 \log (8 \tau \eta / \delta) \right\rceil \). Then,

\[
\mathbb{P}(\hat{w}^\ast_i, \operatorname{err}_{\hat{w}^\ast_i}(T) \geq \varepsilon_{\ell+1} | \mathcal{E}^{\ell-1}) \geq \frac{1 - \delta}{\eta},
\]
where \( k = 2(\ell - 1) \).

Proof. Let

\[
\xi = k \varepsilon_{\ell+1}.
\]
We observe that the reparameterization above enables us to invoke Lemma 2 with \( \delta' = \frac{\delta}{\delta |\mathcal{P}|} \) to obtain

\[
\mathbb{P}(\operatorname{err}_{\hat{w}}(x) \geq \xi \mid \mathcal{E}^{\ell-1}) \leq \frac{\delta}{4 |\mathcal{P}| \tau \eta}.
\]

Now let \( \mathcal{B} \) denote the event that the inequality

\[
\max_{\hat{w} \in \{\hat{w}_1, \ldots, \hat{w}_\tau\}} \max_{x \in T} \operatorname{err}_{\hat{w}}(x) < \xi
\]
holds, where \( T \subseteq (\mathcal{P} \setminus S) \) is a set of size \( \left\lceil 8 \log (8 \tau \eta / \delta) \right\rceil \).

By the probabilistic inequality established above, we have by the union bound

\[
\mathbb{P}(\mathcal{B}^c \mid \mathcal{E}^{\ell-1}) = \mathbb{P} \left( \max_{\hat{w} \in \{\hat{w}_1, \ldots, \hat{w}_\tau\}} \max_{x \in T} \operatorname{err}_{\hat{w}}(x) \geq \xi \mid \mathcal{E}^{\ell-1} \right)
\leq \sum_{\hat{w} \in \{\hat{w}_1, \ldots, \hat{w}_\tau\}} \sum_{x \in T} \mathbb{P}(\operatorname{err}_{\hat{w}}(x) \geq \xi \mid \mathcal{E}^{\ell-1})
\leq \frac{\tau |T| \delta}{4 |\mathcal{P}| \tau \eta}
\leq \frac{\delta}{4 \eta},
\]

40
where the last inequality follows from the fact that $|\mathcal{T}| \leq |\mathcal{P}|$.

Conditioning on $\mathcal{B}$ enables us to reason about the bounded random variables $\text{err}_{\hat{w},\ell}(x)$ for each $x \in \mathcal{T}$ via Hoeffding’s inequality to establish that for any $\hat{w}^\ell \in \{(\hat{w})_1, \ldots, (\hat{w})_\tau\}$

$$\mathbb{P}\left( |\text{err}_{\hat{w},\ell}(\mathcal{T}) - \mathbb{E}_{x|\hat{w}^\ell}[\text{err}_{\hat{w},\ell}(x) | \hat{w}^\ell, \mathcal{B} \cap \mathcal{E}^{\ell-1}]| \geq \frac{\xi}{4} | \mathcal{B} \cap \mathcal{E}^{\ell-1} \right) \\ \leq 2 \exp\left(-\frac{(\xi |\mathcal{T}|)^2}{8 (\xi)^2 |\mathcal{T}|} \right) \\ = 2 \exp\left(-\frac{|\mathcal{T}|}{8} \right)$$

where, as stated earlier, we implicitly condition on the realization $\hat{a}(\cdot)$ of $\hat{a}^\ell(\cdot)$ in the expression above and in the subsequent parts of the proof since it can be marginalized out and does not affect our bounds. Applying the union bound, we further obtain

$$\mathbb{P}\left( \max_{\hat{w}^\ell \in \{(\hat{w})_1, \ldots, (\hat{w})_\tau\}} |\text{err}_{\hat{w},\ell}(\mathcal{T}) - \mathbb{E}_{x|\hat{w}^\ell}[\text{err}_{\hat{w},\ell}(x) | \hat{w}^\ell, \mathcal{B} \cap \mathcal{E}^{\ell-1}]| \leq \frac{\xi}{4} | \mathcal{B} \cap \mathcal{E}^{\ell-1} \right) \\ \leq 2 \tau \exp\left(-\frac{|\mathcal{T}|}{8} \right) \tag{17}$$

$$\leq \frac{\delta}{4 \eta}, \tag{18}$$

where the last inequality follows by our choice of $|\mathcal{T}|$:

$$|\mathcal{T}| = \lceil 8 \log (8 \tau \eta / \delta) \rceil.$$

Let $\mathcal{L}$ denote the event that

$$\max_{\hat{w}^\ell \in \{(\hat{w})_1, \ldots, (\hat{w})_\tau\}} |\text{err}_{\hat{w},\ell}(\mathcal{T}) - \mathbb{E}_{x|\hat{w}^\ell}[\text{err}_{\hat{w},\ell}(x) | \hat{w}^\ell, \mathcal{B} \cap \mathcal{E}^{\ell-1}]| \leq \frac{\xi}{4}$$

By law of total probability, we have

$$\mathbb{P}(\mathcal{L}^c \mid \mathcal{E}^{\ell-1}) = \mathbb{P}(\mathcal{L}^c \mid \mathcal{B}, \mathcal{E}^{\ell-1}) \mathbb{P}(\mathcal{B} \mid \mathcal{E}^{\ell-1}) + \mathbb{P}(\mathcal{L}^c \mid \mathcal{B}^c, \mathcal{E}^{\ell-1}) \mathbb{P}(\mathcal{B}^c \mid \mathcal{E}^{\ell-1})$$

$$\leq \frac{\delta}{4 \eta} \left( 1 - \mathbb{P}(\mathcal{B}^c \mid \mathcal{E}^{\ell-1}) \right) + \mathbb{P}(\mathcal{B}^c \mid \mathcal{E}^{\ell-1})$$

$$= \frac{\delta}{4 \eta} + \mathbb{P}(\mathcal{B}^c \mid \mathcal{E}^{\ell-1}) \left( 1 - \frac{\delta}{4 \eta} \right)$$

$$\leq \frac{\delta}{4 \eta} + \frac{\delta}{4 \eta}$$

$$= \frac{\delta}{2 \eta}.$$ 

Now let $\hat{w}^\dagger$ denote the true minimizer of $\mathbb{E}_{x|\hat{w}^\ell}[\text{err}_{\hat{w},\ell}(x) | \hat{w}^\ell, \mathcal{E}^{\ell-1}]$ among $\hat{w}^\ell \in \{(\hat{w})_1, \ldots, (\hat{w})_\tau\}$ (note that it is not necessarily the case that $\hat{w}^\dagger = \hat{w}^\star$), i.e.,

$$\hat{w}^\dagger = \arg\min_{\hat{w} \in \{(\hat{w})_1, \ldots, (\hat{w})_\tau\}} \mathbb{E}_{x|\hat{w}^\ell}[\text{err}_{\hat{w},\ell}(x) | \hat{w}, \mathcal{E}^{\ell-1}].$$
For each constructed $(\hat{w})_t$, $t \in [\tau]$, invoking Markov’s inequality and the result of Lemma 5 corresponding to the adjusted size of $S$ and sample complexity $m$ yields

\[
\mathbb{P}\left( \mathbb{E}_{x|\hat{w}^t} [\text{err}_{(\hat{w})_t}(x) | (\hat{w})_t, \mathcal{E}^t] \geq \frac{\xi}{4} | \mathcal{E}^t \right) \leq \frac{4 \mathbb{E}[\text{err}_{(\hat{w})_t}(x) | \mathcal{E}^t]}{\xi} \\
\leq \frac{4\varepsilon_t \hat{\Delta}^t}{\xi} \left( k + \frac{5(1+k\varepsilon_t)}{\sqrt{\log(8\eta/\delta)}} \right) + \frac{4\delta (1+k\varepsilon_t)}{\xi \eta} \mathbb{E}_{x \sim \mathcal{D}} \left[ \Delta^t(x) | \Delta^t(x) > \hat{\Delta}^t \right] \\
\leq \frac{9}{10},
\]

where the last inequality follows for $\frac{\delta}{\eta}$ small enough. Thus, the event $\mathbb{E}[\text{err}_{\hat{w}^t}(x) | \hat{w}^t, \mathcal{E}^t] \geq \frac{\xi}{4}$ occurs if and only if we fail (i.e., exceed $\xi/4$ expected error) in all $\tau$ trials. This implies that

\[
\mathbb{P}\left( \mathbb{E}_{x|\hat{w}^t} [\text{err}_{\hat{w}^t}(x) | \hat{w}^t, \mathcal{E}^t] \geq \frac{\xi}{4} | \mathcal{E}^t \right) = \mathbb{P}\left( \forall t \in [\tau] : \mathbb{E}_{x|\hat{w}^t} [\text{err}_{(\hat{w})_t}(x) | (\hat{w})_t, \mathcal{E}^t] \geq \frac{\xi}{4} | \mathcal{E}^t \right) \\
\leq \left( \frac{9}{10} \right)^\tau \\
\leq \frac{\delta}{4 \eta},
\]

where the last inequality follows by our choice of $\tau$:

\[
\tau = \left\lceil \log(4 \eta/\delta) \right\rceil / \log(10/9).
\]

Let $\mathcal{G}$ denote the event that $\mathbb{E}_{x|\hat{w}^t} [\text{err}_{\hat{w}^t}(x) | \hat{w}^t, \mathcal{E}^t] \leq \frac{\xi}{4}$ and recall that

\[
\hat{w}^* = \arg\min_{\hat{w}^t \in \{\hat{w}^t\}_1^{\tau}} \text{err}_{\hat{w}^t}(\mathcal{T}).
\]

If events $\mathcal{B}$, $\mathcal{L}$, and $\mathcal{G}$ all occur, i.e., $\mathcal{B} \cap \mathcal{L} \cap \mathcal{G} \neq \emptyset$, then we obtain

\[
\mathbb{E}_{x|\hat{w}^*} [\text{err}_{\hat{w}^*}(x) | \hat{w}^*, \mathcal{E}^t] \\
= \mathbb{E}_{\mathcal{T}|\hat{w}^*} [\text{err}_{\hat{w}^*}(\mathcal{T}) | \hat{w}^*, \mathcal{E}^t] \\
= \mathbb{E} [\text{err}_{\hat{w}^*}(\mathcal{T}) | \hat{w}^*, \mathcal{B}, \mathcal{E}^t] \mathbb{P}(\mathcal{B} | \mathcal{E}^t) + \mathbb{E} [\text{err}_{\hat{w}^*}(\mathcal{T}) | \hat{w}^*, \mathcal{B}^c, \mathcal{E}^t] \mathbb{P}(\mathcal{B}^c | \mathcal{E}^t) \\
\leq \mathbb{E} [\text{err}_{\hat{w}^*}(\mathcal{T}) | \hat{w}^*, \mathcal{B}, \mathcal{E}^t] + \mathbb{E} [\text{err}_{\hat{w}^*}(x) | \hat{w}^*, \mathcal{B}^c, \mathcal{E}^t] \left( \frac{\delta}{4 \eta} \right) \\
\leq \mathbb{E} [\text{err}_{\hat{w}^*}(\mathcal{T}) | \hat{w}^*, \mathcal{B}, \mathcal{E}^t] + \frac{\xi}{4} \\
\leq \text{err}_{\hat{w}^*}(\mathcal{T}) + \frac{\xi}{2},
\]

for $\frac{\delta}{\eta}$ small enough and $\mathcal{B} \cap \mathcal{L} \neq \emptyset$. 

42
\[
\leq \text{err}_{\hat{w}^\dagger} (\mathcal{T}) + \frac{\xi}{2} \quad \text{By definition of } \hat{w}^* \\
\leq \mathbb{E}_{\mathcal{T} | \hat{w}^\dagger} [\text{err}_{\hat{w}^\dagger} (\mathcal{T}) | \hat{w}^\dagger, \mathcal{B}, \mathcal{E}^{\ell-1}] + \frac{3\xi}{4} \quad \text{By } \mathcal{B} \cap \mathcal{L} \neq \emptyset \\
\leq \mathbb{E}_{x | \hat{w}^\dagger} [\text{err}_{\hat{w}^\dagger} (x) | \hat{w}^\dagger, \mathcal{E}^{\ell-1}] + \frac{3\xi}{4} \quad \text{By } \mathcal{G} \neq \emptyset, \\
\leq \xi
\]

where in the second to last inequality, we used the fact that conditioning on \( \mathcal{B} \) leads to a decrease in the expected value relative to the unconditional expectation.

By the union bound over the failure events, we have that the sequence of inequalities above holds with probability at least \( 1 - \frac{\delta}{\eta} \):

\[
P \left( \mathbb{E}_{x | \hat{w}^*} [\text{err}_{\hat{w}^*} (x) | \hat{w}^*, \mathcal{E}^{\ell-1}] \leq \xi | \mathcal{E}^{\ell-1} \right) \geq P (\mathcal{B} \cap \mathcal{L} \cap \mathcal{G} | \mathcal{E}^{\ell-1}) \\
= 1 - P \left( (\mathcal{B} \cap \mathcal{L} \cap \mathcal{G})^c | \mathcal{E}^{\ell-1} \right) \\
\geq 1 - \left( P(\mathcal{B}^c | \mathcal{E}^{\ell-1}) + P(\mathcal{L}^c | \mathcal{E}^{\ell-1}) + P(\mathcal{G}^c | \mathcal{E}^{\ell-1}) \right) \\
\geq 1 - \left( \frac{\delta}{4\eta} + \frac{\delta}{2\eta} + \frac{\delta}{4\eta} \right) \\
= 1 - \frac{\delta}{\eta},
\]

and this establishes the theorem. \( \square \)
B Additional Results

In this section, we give more details on the evaluation of our compression algorithm on popular benchmark data sets and varying fully-connected neural network configurations. In the experiments, we compare the effectiveness of our sampling scheme in reducing the number of non-zero parameters of a network to that of uniform sampling and the singular value decomposition (SVD). All algorithms were implemented in Python using the PyTorch library [40] and simulations were conducted on a computer with a 2.60 GHz Intel i9-7980XE processor (18 cores total) and 128 GB RAM.

For training and evaluating the algorithms considered in this section, we used the following off-the-shelf data sets:

- **MNIST** [32] — 70,000 images of handwritten digits between 0 and 9 in the form of $28 \times 28$ pixels per image.
- **CIFAR-10** [28] — 60,000 $32 \times 32$ color images, a subset of the larger CIFAR-100 dataset, each depicting an object from one of 10 classes, e.g., airplanes.
- **FashionMNIST** [50] — A recently proposed drop-in replacement for the MNIST data set that, like MNIST, contains 60,000, $28 \times 28$ grayscale images, each associated with a label from 10 different categories.

We considered a diverse set of network configurations for each of the data sets. We varied the number of hidden layers between 2 and 5 and used either a constant width across all hidden layers between 200 and 1000 or a linearly decreasing width (denoted by "Pyramid" in the figures). Training was performed for 30 epochs on the normalized data sets using an Adam optimizer with a learning rate of 0.001 and a batch size of 300. The test accuracies were roughly 98% (MNIST), 45% (CIFAR10), and 96% (FashionMNIST), depending on the network architecture. To account for the randomness in the training procedure, for each data set and neural network configuration, we averaged our results across 4 trained neural networks.

B.1 Details on the Compression Algorithms

We evaluated and compared the performance of the following algorithms on the aforementioned data sets.

1. **Uniform (Edge) Sampling** — A uniform distribution is used, rather than our sensitivity-based importance sampling distribution, to sample the incoming edges to each neuron in the network. Note that like our sampling scheme, uniform sampling edges generates an unbiased estimator of the neuron value. However, unlike our approach which explicitly seeks to minimize estimator variance using the bounds provided by empirical sensitivity, uniform sampling is prone to exhibiting large estimator variance.

2. **Singular Value Decomposition** (SVD) — The (truncated) SVD decomposition is used to generate a low-rank (rank-$r$) approximation for each of the weight matrices $(\hat{W}_2, \ldots, \hat{W}_L)$ to obtain the corresponding parameters $\hat{\theta} = (\hat{W}_2^r, \ldots, \hat{W}_L^r)$ for various values of $r \in \mathbb{N}_+$. Unlike the compared sampling-based methods, SVD does not sparsify the weight matrices. Thus, to achieve fair comparisons of compression rates, we compute the size of the rank-$r$ matrices
constituting $\hat{\theta}$ as,

$$\text{nnz}(\hat{\theta}) = \sum_{\ell=2}^{L} \sum_{i=1}^{r} \left( \text{nnz}(u^\ell_i) + \text{nnz}(v^\ell_i) \right),$$

where $W^\ell = U^\ell \Sigma^\ell (V^\ell)^\top$ for each $\ell \in \{2, \ldots, L\}$, with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\eta^\ell-1}$ and $u^\ell_i$ and $v^\ell_i$ denote the $i$th columns of $U^\ell$ and $V^\ell$ respectively.

3. **CoreNet (Edge Sampling)** — Our core algorithm for edge sampling shown in Alg. 2, but without the neuron pruning procedure.

4. **CoreNet+ (CoreNet & Neuron Pruning)** — Our algorithm shown in Alg. 1 that includes the neuron pruning step.

5. **CoreNet++ (CoreNet+ & Amplification)** — In addition to the features of Corenet+, multiple coresets $C_1, \ldots, C_\tau$ are constructed over $\tau \in \mathbb{N}_+$ trials, and the best one is picked by evaluating the empirical error on a subset $T \subseteq P \setminus S$ (see Sec. 4 for details).

**B.2 Preserving the Output of a Neural Network**

We evaluated the accuracy of our approximation by comparing the output of the compressed network with that of the original one and compute the $\ell_1$-norm of the relative error vector. We computed the error metric for both the uniform sampling scheme as well as our compression algorithm (Alg. 1). Our results were averaged over 50 trials, where for each trial, the relative approximation error was averaged over the entire test set. In particular, for a test set $P_{\text{test}} \subseteq \mathbb{R}^d$ consisting of $d$ dimensional points, the average relative error of with respect to the $f_{\hat{\theta}}$ generated by each compression algorithm was computed as

$$\text{error}_{P_{\text{test}}}(f_{\tilde{\theta}}) = \frac{1}{|P_{\text{test}}|} \sum_{x \in P_{\text{test}}} \left\| f_{\hat{\theta}}(x) - f_{\tilde{\theta}}(x) \right\|_1.$$

Figures 3, 4, and 5 depict the average performance of the compared algorithms for various network architectures trained on MNIST, CIFAR-10, and FashionMNIST, respectively. Our algorithm is able to compress networks trained on MNIST and FashionMNIST to about 10% of their original size without significant loss of accuracy. On CIFAR-10, a compression rate of 50% yields classification results comparable to that of uncompressed networks. The shaded region corresponding to each curve represents the values within one standard deviation of the mean.

**B.3 Preserving the Classification Performance**

We also evaluated the accuracy of our approximation by computing the loss of prediction accuracy on a test data set, $P_{\text{test}}$. In particular, let $\text{acc}_{P_{\text{test}}}(f_{\tilde{\theta}})$ be the average accuracy of the neural network $f_{\tilde{\theta}}$, i.e.,

$$\text{acc}_{P_{\text{test}}}(f_{\tilde{\theta}}) = \frac{1}{|P_{\text{test}}|} \sum_{x \in P_{\text{test}}} \mathbb{1} \left( \arg\max_{i \in [\eta^L]} f_{\tilde{\theta}}(x) \neq y(x) \right),$$

where $y(x)$ denotes the (true) label associated with $x$. Then the drop in accuracy is computed as

$$\text{acc}_{P_{\text{test}}}(f_{\hat{\theta}}) - \text{acc}_{P_{\text{test}}}(f_{\tilde{\theta}}).$$
Figures 6, 7, and 8 depict the average performance of the compared algorithms for various network architectures trained on MNIST, CIFAR-10, and FashionMNIST respectively. The shaded region corresponding to each curve represents the values within one standard deviation of the mean.

B.4 Discussion

As indicated in Sec. 6, the simulation results presented here validate our theoretical results and suggest that empirical sensitivity can lead to effective, more informed sampling compared to other methods. Moreover, we are able to outperform networks that are compressed via uniform sampling or SVD. We also note that there is a notable difference in the performance of our algorithm between different datasets. In particular, the difference in performance of our algorithm compared to the other method for networks trained on FashionMNIST and MNIST is much more significant than for networks trained on CIFAR. We conjecture that this is partially due to considering only fully-connected networks as these network perform fairly poorly on CIFAR (around 45% classification accuracy) and thus edges have more uniformly distributed sensitivity as the information content in the network is limited. We envision that extending our guarantees to convolutional neural networks may enable us to further reason about the performance on data sets such as CIFAR.
Figure 3: Evaluations against the MNIST dataset with varying number of hidden layers ($L$) and number of neurons per hidden layer ($\eta^*$). Shaded region corresponds to values within one standard deviation of the mean. The figures show that our algorithm’s relative performance increases as the number of layers (and hence the number of redundant parameters) increases.
Figure 4: Evaluations against the CIFAR-10 dataset with varying number of hidden layers ($L$) and number of neurons per hidden layer ($\eta^*$). The trend of our algorithm’s improved relative performance as the number of parameters increases (previously depicted in Fig. 3) also holds for the CIFAR-10 data set.
Figure 5: Evaluations against the FashionMNIST dataset with varying number of hidden layers ($L$) and number of neurons per hidden layer ($\eta^*$).
Figure 6: Evaluations against the MNIST dataset with varying number of hidden layers ($L$) and number of neurons per hidden layer ($\eta^*$). Shaded region corresponds to values within one standard deviation of the mean. The figures show that our algorithm’s relative performance increases as the number of layers (and hence the number of redundant parameters) increases.
Figure 7: Evaluations against the CIFAR-10 dataset with varying number of hidden layers ($L$) and number of neurons per hidden layer ($\eta^*$). The trend of our algorithm’s improved relative performance as the number of parameters increases (previously depicted in Fig. 6) also holds for the CIFAR-10 data set.
Figure 8: Evaluations against the FashionMNIST dataset with varying number of hidden layers ($L$) and number of neurons per hidden layer ($\eta^*$).