Abstract. We prove the dynamical large deviations for a particle system in which particles may have different velocities. We assume that we have two infinite reservoirs of particles at the boundary: this is the so-called boundary driven process. The dynamics we considered consists of a weakly asymmetric simple exclusion process with collision among particles having different velocities.

1. Introduction

In the last years there has been considerable progress in understanding stationary nonequilibrium states: reversible systems in contact with different reservoirs at the boundary imposing a gradient on the conserved quantities of the system. In these systems there is a flow of matter through the system and the dynamics is not reversible. The main difference with respect to equilibrium (reversible) states is the following. In equilibrium, the invariant measure, which determines the thermodynamic properties, is given for free by the Gibbs distribution specified by the Hamiltonian. On the contrary, in nonequilibrium states the construction of the stationary state requires the solution of a dynamical problem. One of the most striking typical property of these systems is the presence of long-range correlations. For the symmetric simple exclusion this was already shown in a pioneering paper by Spohn [14]. We refer to [5, 7] for two recent reviews on this topic.

We discuss this issue in the context of stochastic lattice gases in a box of linear size \( N \) with birth and death process at the boundary modeling the reservoirs. We consider the case when there are many thermodynamic variables: the local density denoted by \( \rho \), and the local momentum denoted by \( p_k, k = 1, \ldots, d \), \( d \) being the dimension of the box.

The model which we will study can be informally described as follows: fix a velocity \( v \), an integer \( N \geq 1 \), and boundary densities \( 0 < \alpha_v(\cdot) < 1 \) and \( 0 < \beta_v(\cdot) < 1 \); at any given time, each site of the set \( \{1, \ldots, N - 1\} \times \{0, \ldots, N - 1\}^{d-1} \) is either empty or occupied by one particle at velocity \( v \). In the bulk, each particle attempts to jump at any of its neighbors at the same velocity, with a weakly asymmetric rate. To respect the exclusion rule, the particle jumps only if the target site at the same velocity \( v \) is empty; otherwise nothing happens. At the boundary, sites with first coordinates given by 1 or \( N - 1 \) have particles being created or removed in such a way that the local densities are \( \alpha_v(\tilde{x}) \) and \( \beta_v(\tilde{x}) \): at rate \( \alpha_v(\tilde{x}/N) \) a particle is created at \( \{1\} \times \{\tilde{x}\} \) if the site is empty, and at rate \( 1 - \alpha_v(\tilde{x}) \) the particle at \( \{1\} \times \{\tilde{x}\} \) is removed if the site is occupied, and at rate \( \beta_v(\tilde{x}) \) a particle is created at \( \{N - 1\} \times \{\tilde{x}\} \) if the site is empty, and at rate \( 1 - \beta_v(\tilde{x}) \) the particle at \( \{N - 1\} \times \{\tilde{x}\} \) is removed if the site is occupied. Superposed to this dynamics, there is a collision process which exchange velocities of particles in the same site in a way that momentum is conserved.

Similar models have been studied by [1, 8, 11]. In fact, the model we consider here is based on the model of Esposito et al. [8] which was used to derive the Navier-Stokes equation. It is also noteworthy that the derivation of hydrodynamic limits and macroscopic fluctuation theory for a system with two conserved quantities have been studied in [14].

The hydrodynamic limit for the above model has been proved in [12]. The hydrodynamic equation derives from the underlying stochastic dynamics through an appropriate scaling limit in which the microscopic time and space coordinates are rescaled diffusively. The hydrodynamic equation thus represents the law of large numbers for the empirical density of the stochastic lattice gas. The convergence has to be understood in

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\]

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probability with respect to the law of the stochastic lattice gas. Once it is established a natural question is to consider large deviations.

This article thus provides a derivation of the dynamical large deviations for this model, and the proof follows the method introduced in [9]. The main difference is that their proof of $I_T(\cdot|\gamma)$-density relied on some energy estimates that we were not able to achieve due to the presence of velocities. Therefore, we had to overcome problem by taking a different approach at that part.

The article is organized as follows: in Section 2 we establish the notation and state the main results of the article; in Section 3 we review the hydrodynamics for this model, that was obtained in [12]; in Section 4 several properties of the rate function are derived; Section 5 proves the $I_T(\cdot|\gamma)$-density, which is a key result for proving the lower bound; finally, in Section 6 the proofs of the upper and lower bounds of the dynamical large deviations are given.

2. Notation and Results

Fix a positive integer $d \geq 1$. Denote by $D^d$ the open set $(0,1) \times \mathbb{T}^{d-1}$, where $\mathbb{T}^k$ is the $k$-dimensional torus $[0,1)^k$, and by $\Gamma$ the boundary of $D^d$, $\Gamma = \{(u_1, \ldots, u_d) \in [0,1] \times \mathbb{T}^{d-1} : u_1 = \pm 1\}$.

For an open subset $\Lambda$ of $\mathbb{R} \times \mathbb{T}^{d-1}$, $C^m(\Lambda)$, $1 \leq m \leq +\infty$, stands for the space of $m$-continuously differentiable real functions defined on $\Lambda$. Let $C^0(\Lambda)$ (resp. $C^\infty(\Lambda)$), $1 \leq m \leq +\infty$, be the subset of functions in $C^m(\Lambda)$ which vanish at the boundary of $\Lambda$ (resp. with compact support in $\Lambda$).

For an integer $N \geq 1$, denote by $\mathbb{T}^{d-1}_N = \{0, \ldots, N-1\}^{d-1}$, the discrete $(d-1)$-dimensional torus of length $N$. Let $\mathbb{T}^d_N = \{1, \ldots, N - 1\} \times \mathbb{T}^{d-1}_N$ be the cylinder in $\mathbb{Z}^d$ of length $N - 1$ and basis $\mathbb{T}^{d-1}_N$, and let $\Gamma_N = \{(x_1, \ldots, x_d) \in \mathbb{Z} \times \mathbb{T}^{d-1}_N \mid x_1 = 1 \text{ or } x_1 = (N-1)\}$ be the boundary of $\mathbb{T}^d_N$.

Let $\mathcal{V} \subset \mathbb{R}^d$ be a finite set of velocities $v = (v_1, \ldots, v_d)$. Assume that $\mathcal{V}$ is invariant under reflexions and permutations of the coordinates: 

$$(v_1, \ldots, v_{i-1}, -v_i, v_{i+1}, \ldots, v_d) \text{ and } (v_{\sigma(1)}, \ldots, v_{\sigma(d)})$$

belong to $\mathcal{V}$ for all $1 \leq i \leq d$, and all permutations $\sigma$ of $\{1, \ldots, d\}$, provided $(v_1, \ldots, v_d)$ belongs to $\mathcal{V}$.

On each site of $\mathbb{T}^d_N$, at most one particle for each velocity is allowed. We denote: the number of particles with velocity $v$ at $x$, $v \in \mathcal{V}$, $x \in \mathbb{T}^d_N$, by $\eta(x,v) \in \{0,1\}$; the number of particles in each velocity $v$ at a site $x$ by $\eta_x = \{\eta(x,v) \mid v \in \mathcal{V}\}$; and a configuration by $\eta = \{\eta_x; x \in \mathbb{T}^d_N\}$. The set of particle configurations is $X_N = \{\{0,1\}^\mathcal{V}\}^{\mathbb{T}^d_N}$.

On the interior of the domain, the dynamics consists of two parts: (i) each particle of the system evolves according to a nearest neighbor weakly asymmetric random walk with exclusion among particles of the same velocity, and (ii) binary collision between particles of different velocities. Let $p(x,v)$ be an irreducible probability transition function of finite range, and mean velocity $v$:

$$\sum_x xp(x,v) = v.$$ 

The jump law and the waiting times are chosen so that the jump rate from site $x$ to site $x+y$ for a particle with velocity $v$ is

$$P_N(y,v) = \frac{1}{2} \sum_{j=1}^d (\delta_{y,e_j} + \delta_{y,-e_j}) + \frac{1}{N} p(y,v),$$

where $\delta_{x,y}$ stands for the Kronecker delta, which equals one if $x = y$ and 0 otherwise, and $\{e_1, \ldots, e_d\}$ is the canonical basis in $\mathbb{R}^d$.

2.1. The boundary driven exclusion process. Our main interest is to examine the stochastic lattice gas model given by the generator $\mathcal{L}_N$ which is the superposition of the boundary dynamics with the collision and exclusion:

$$\mathcal{L}_N = N^2 \{ \mathcal{L}_N^b + \mathcal{L}_N^\tau + \mathcal{L}_N^c \},$$

(2.1)

where $\mathcal{L}_N^b$ stands for the generator which models the part of the dynamics at which a particle at the boundary can enter or leave the system, $\mathcal{L}_N^\tau$ stands for the generator which models the collision part of the dynamics
and lastly, $\mathcal{L}^{ex}_N$ models the exclusion part of the dynamics. Let $f$ be a local function on $X_N$. The generator of the exclusion part of the dynamics, $\mathcal{L}^{ex}_N$, is given by
\[
(\mathcal{L}^{ex}_N f)(\eta) = \sum_{x,v,z} \eta(x,v)[1 - \eta(z,v)]P_N(z - x,v)[f(\eta^{x,z,v}) - f(\eta)],
\]
where
\[
\eta^{x,y,v}(z,w) = \begin{cases} 
\eta(y,v) & \text{if } w = v \text{ and } z = x, \\
\eta(x,v) & \text{if } w = v \text{ and } z = y, \\
\eta(z,w) & \text{otherwise}.
\end{cases}
\]

The generator of the collision part of the dynamics, $\mathcal{L}^c_N$, is given by
\[
(\mathcal{L}^c_N f)(\eta) = \sum_{y \in D^i_N} \sum_{q \in Q} p(y,q,\eta)[f(\eta^{y,q}) - f(\eta)],
\]
where $Q$ is the set of all collisions which preserve momentum:
\[
Q = \{ q = (v,w,v',w') \in V^4 : v + w = v' + w' \},
\]
the rate $p(y,q,\eta)$ is given by
\[
p(y,q,\eta) = \eta(y,v)\eta(y,w)[1 - \eta(y,v')][1 - \eta(y,w')],
\]
and for $q = (v_0,v_1,v_2,v_3)$, the configuration $\eta^{y,q}$ after the collision is defined as
\[
\eta^{y,q}(z,u) = \begin{cases} 
\eta(y,v_{j+2}) & \text{if } z = y \text{ and } u = v_j \text{ for some } 0 \leq j \leq 3, \\
\eta(z,u) & \text{otherwise}.
\end{cases}
\]
where the index of $v_{j+2}$ should be taken modulo 4.

Particles of velocities $v$ and $w$ at the same site collide at rate one and produce two particles of velocities $v'$ and $w'$ at that site.

Finally, the generator of the boundary part of the dynamics is given by
\[
(\mathcal{L}^b_N f)(\eta) = \sum_{x \in D^i_N} \sum_{v \in V} \left[ \alpha_v(\hat{x}/N)[1 - \eta(x,v)] + (1 - \alpha_v(\hat{x}/N))\eta(x,v) \right][f(\sigma^{x,v}\eta) - f(\eta)]
\]
\[
+ \sum_{x \in D^i_N} \sum_{v \in V} \left[ \beta_v(\hat{x}/N)[1 - \eta(x,v)] + (1 - \beta_v(\hat{x}/N))\eta(x,v) \right][f(\sigma^{x,v}\eta) - f(\eta)],
\]
where $\hat{x} = (x_2,\ldots,x_d)$,
\[
\sigma^{x,v}\eta(y,w) = \begin{cases} 
1 - \eta(x,v), & \text{if } w = v \text{ and } y = x, \\
\eta(y,w), & \text{otherwise}.
\end{cases}
\]
and for every $v \in V$, $\alpha_v, \beta_v \in C^2(T^{d-1})$. Note that time has been speeded up diffusively in $\mathcal{L}^b_N$. We also assume that, for every $v \in V$, $\alpha_v$ and $\beta_v$ have images belonging to some compact subset of $(0,1)$. The functions $\alpha_v$ and $\beta_v$, which affect the birth and death rates at the two boundaries, represent the densities of the reservoirs.

Let $D(\mathbb{R}_+,X_N)$ be the set of right continuous functions with left limits taking values on $X_N$. For a probability measure $\mu$ on $X_N$, denote by $P_\mu$ the measure on the path space $D(\mathbb{R}_+,X_N)$ induced by $\{\eta(t) : t \geq 0\}$ and the initial measure $\mu$. Expectation with respect to $P_\mu$ is denoted by $E_\mu$.

2.2. Mass and momentum. For each configuration $\xi \in \{0,1\}^V$, denote by $I_0(\xi)$ the mass of $\xi$ and by $I_k(\xi)$, $k = 1,\ldots,d$, the momentum of $\xi$:
\[
I_0(\xi) = \sum_{v \in V} \xi(v), \quad I_k(\xi) = \sum_{v \in V} v_k\xi(v).
\]
Set $I(\xi) := (I_0(\xi),\ldots,I_d(\xi))$. Assume that the set of velocities is chosen in such a way that the unique quantities conserved by the random walk dynamics described above are mass and momentum: $\sum_{x \in D^i_N} I(\eta_x)$. Two examples of sets of velocities satisfying these conditions can be found at [S].
For each chemical potential $\lambda = (\lambda_0, \ldots, \lambda_d) \in \mathbb{R}^{d+1}$, denote by $m_\lambda$ the measure on $\{0, 1\}^V$ given by
\[
m_\lambda(\xi) = \frac{1}{Z(\lambda)} \exp \{\lambda \cdot I(\xi)\},
\]
where $Z(\lambda)$ is a normalizing constant. Note that $m_\lambda$ is a product measure on $\{0, 1\}^V$, i.e., that the variables $\{\xi(v) : v \in V\}$ are independent under $m_\lambda$.

Denote by $\mu^N_\lambda$ the product measure on $X_N$, with marginals given by
\[
\mu^N_\lambda(\eta : \eta(x, \cdot) = \xi) = m_\lambda(\xi),
\]
for each $\xi$ in $\{0, 1\}^V$ and $x \in D^d_N$. Note that $\{\eta(x, v) : x \in D^d_N, v \in V\}$ are independent variables under $\mu^N_\lambda$, and that the measure $\mu^N_\lambda$ is invariant for the exclusion process with periodic boundary condition.

The expectation under $\mu^N_\lambda$ of the mass and momentum are given by
\[
\rho(\lambda) := E_{\mu^N_\lambda} [I_0(\eta_u)] = \sum_{v \in V} \theta_v(\lambda),
\]
\[
p_k(\lambda) := E_{\mu^N_\lambda} [I_k(\eta_u)] = \sum_{v \in V} v_k \theta_v(\lambda).
\]

In this formula $\theta_v(\lambda)$ denotes the expected value of the density of particles with velocity $v$ under $m_\lambda$:
\[
\theta_v(\lambda) := E_{m_\lambda} [\xi(v)] = \frac{\exp \left\{ \lambda_0 + \sum_{k=1}^d \lambda_k v_k \right\}}{1 + \exp \left\{ \lambda_0 + \sum_{k=1}^d \lambda_k v_k \right\}}.
\]

Denote by $(\rho, p)(\lambda) := (\rho(\lambda), p_1(\lambda), \ldots, p_d(\lambda))$ the map that associates the chemical potential to the vector of density and momentum. It is possible to prove that $(\rho, p)$ is a diffeomorphism onto $U \subset \mathbb{R}^{d+1}$, the interior of the convex envelope of $\{I(\xi), \xi \in \{0, 1\}^V\}$. Denote by $\Lambda = (\Lambda_0, \ldots, \Lambda_d) : U \rightarrow \mathbb{R}^{d+1}$ the inverse of $(\rho, p)$. This correspondence allows one to parameterize the invariant states by the density and momentum: for each $(\rho, p)$ in $U$ we have a product measure $\nu_{\rho, p} = \mu^N_\lambda(\rho, p)$ on $X_N$.

2.3. Dynamical large deviations. Fix $T > 0$, let $M_+$ be the space of finite positive measures on $D^d$ endowed with the weak topology, and let $M$ be the space of bounded variation signed measures on $D^d$ endowed with the weak topology. Let $M_+ \times M^d$ be the cartesian product of these spaces endowed with the product topology, which is metrizable. Let also $M^0$ be the subset of $M_+ \times M^d$ of all absolutely continuous measures with respect to the Lebesgue measure satisfying:

\[
M^0 = \{ \pi \in M_+ \times M^d : \pi(du) = (\rho, p)(u)du \text{ and } 0 \leq \rho(u) \leq |V|, |p_k(u)| \leq \bar{v}|V|, k = 1, \ldots, d, \text{ a.e.} \},
\]

where $\bar{v} = \max_{v \in V} v_1$. Let $D([0, T], M_+ \times M^d)$ be the set of right continuous functions with left limits taking values on $M_+ \times M^d$ endowed with the Skorohod topology. $M^0$ is a closed subset of $M_+ \times M^d$ and $D([0, T], M^0)$ is a closed subset of $D([0, T], M_+ \times M^d)$.

For a measure $\pi \in M$, denote by $\langle \pi, G \rangle$ the integral of $G$ with respect to $\pi$.

Let $\Omega_T = (0, T) \times D^d$ and $\Omega_T^\tau = [0, T] \times D^d$. For $1 \leq m, n \leq +\infty$, denote by $C^{m,n}(\Omega_T)$ the space of functions $G = G_t(u) : \Omega_T \rightarrow \mathbb{R}$ with $m$ continuous derivatives in time and $n$ continuous derivatives in space. We also denote by $C^{m,n}_0(\Omega_T)$ (resp. $C^{\infty, \infty}(\Omega_T)$) the set of functions in $C^{m,n}(\Omega_T)$ (resp. $C^{\infty, \infty}(\Omega_T)$) which vanish at $[0, T] \times \Gamma$ (resp. with compact support in $\Omega_T$).

Let the energy $Q : D([0, T], M^0) \rightarrow [0, \infty]$ be given by
\[
Q(\pi) = \sum_{k=0}^d \sum_{i=1}^d \sup_{G \in C^{m,n}_0(\Omega_T)} \left\{ 2 \int_0^T dt \langle p_{k,t}, \partial_u G_t \rangle - \int_0^T dt \int_{D^d} G(t, u)^2 du \right\}.
\]

where $p_{k,t}(u) = p_k(t, u)$ and $p_{0,t}(u) = \rho(t, u)$. 

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For each $G \in C^{1,2}_0([0,T]) \times [C_0^2(D)]^d$ and each measurable function $\gamma : \overline{D^d} \to [0,|\mathcal{V}|] \times [-\bar{v}|\mathcal{V}|,\bar{v}|\mathcal{V}|]^d$, $\gamma = (\rho_0,p_0)$, let $\hat{J}_G = \hat{J}_{G,\gamma,T} : D([0,T],\mathcal{M}^0) \to \mathbb{R}$ be the functional given by

$$
\hat{J}_G(\pi) = \int_{D^d} G(T,u) \cdot (\rho,p)(T,u) du - \int_{D^d} G(0,u) \cdot (\rho_0,p_0)(u) du \\
- \int_0^T dt \int_{D^d} du \left\{ (\rho,p)(t,u) \cdot \partial_t G(t,u) + \frac{1}{2} (\rho,p)(t,u) \cdot \sum_{1 \leq i \leq d} \partial_{\nu_i}^2 G(t,u) \right\} \\
+ \int_0^T dt \int_{\{1\} \times \mathbb{T}^{d-1}} dS b(\tilde{u}) \cdot \partial_{\nu_1} G(t,u) - \int_0^T dt \int_{\{0\} \times \mathbb{T}^{d-1}} dS a(\tilde{u}) \cdot \partial_{\nu_1} G(t,u) \\
+ \int_0^T dt \int_{D^d} du \sum_{i \in \mathcal{V}} \tilde{v} \cdot \chi(\theta_v(\Lambda(\rho,p))) \sum_{1 \leq i \leq d} v_i \partial_{\nu_i} G(t,u) \\
- \int_0^T dt \int_{D^d} du \sum_{i \in \mathcal{V}} \sum_{k=0}^d v_k \partial_{x_i} G_k^i(u) \chi(\theta_v(\Lambda(\rho,p))),
$$

where $\chi(r) = r(1-r)$ is the static compressibility and $\pi_t(du) = (\rho,p)(t,u) du$. Define $J_G = J_{G,\gamma,T} : D([0,T],\mathcal{M}_+ \times \mathcal{M}^d) \to \mathbb{R}$ by

$$
J_G(\pi) = \begin{cases} 
\hat{J}_G(\pi) & \text{if } \pi \in D([0,T],\mathcal{M}^0), \\
+\infty & \text{otherwise}.
\end{cases}
$$

We define the rate functional $I_T(\cdot|\gamma) : D([0,T],\mathcal{M}_+ \times \mathcal{M}^d) \to [0,\infty]$ as

$$
I_T(\pi|\gamma) = \begin{cases} 
\sup \left\{ J_G(\pi) \middle| G \in C^{1,2}_0([0,T]) \times [C_0^2(D)]^d, \pi \in \mathcal{M}_+ \times \mathcal{M}^d \right\} & \text{if } Q(\pi) < \infty, \\
+\infty & \text{otherwise}.
\end{cases}
$$

We now present the main result of this article, whose proof is given in Section 6, which is the dynamical large deviations for this boundary driven exclusion process with many conserved quantities.

**Theorem 2.1.** Fix $T > 0$ and a measurable function $(\rho_0,p_0) : D^d \to [0,|\mathcal{V}|] \times [-\bar{v}|\mathcal{V}|,\bar{v}|\mathcal{V}|]^d$. Consider a sequence $\eta^N$ of configurations in $X_N$ associated to $\gamma = (\rho_0,p_0)$ in the sense that:

$$
\lim_{N \to \infty} \langle \pi_0^N(\eta^N), G \rangle = \int_{D^d} G(u) \rho_0(u) du,
$$

and

$$
\lim_{N \to \infty} \langle \pi_k^N(\eta^N), G \rangle = \int_{D^d} G(u) p_k(u) du, \quad k = 1, \ldots, d,
$$

for every continuous function $G : \overline{D^d} \to \mathbb{R}$. Then, the measure $Q_{\eta^N} = \mathbb{P}_{\eta^N}(\pi^N)^{-1}$ on $D([0,T],\mathcal{M}_+ \times \mathcal{M}^d)$ satisfies a large deviation principle with speed $N^d$ and rate function $I_T(\cdot|\gamma)$. Namely, for each closed set $C \subset D([0,T],\mathcal{M}_+ \times \mathcal{M}^d)$,

$$
\lim_{N \to \infty} \frac{1}{N^d} \log Q_{\eta^N}(C) \leq -\inf_{\pi \in C} I_T(\pi|\gamma)
$$

and for each open set $O \subset D([0,T],\mathcal{M}_+ \times \mathcal{M}^d)$,

$$
\lim_{N \to \infty} \frac{1}{N^d} \log Q_{\eta^N}(O) \geq -\inf_{\pi \in O} I_T(\pi|\gamma).
$$

Moreover, the rate function $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.
3. HYDRODYNAMICS

Fix $T > 0$ and let $(B, \| \cdot \|_B)$ be a Banach space. We denote by $L^2([0,T], B)$ the Banach space of measurable functions $U : [0,T] \to B$ for which

$$\|U\|_{L^2([0,T], B)}^2 = \int_0^T \|U_t\|_B^2 dt < \infty.$$ 

Moreover, we denote by $H^1(D^d)$ the Sobolev space of measurable functions in $L^2(D^d)$ that have generalized derivatives in $L^2(D^d)$.

For $x = (x_1, \hat{x}) \in \{0, 1\} \times \mathbb{T}^{d-1}$, let

$$d(x) = \begin{cases} a(\hat{x}) = \sum_{\nu \in \mathcal{V}} (a_{\nu}(\hat{x}), v_1 a_\nu(\hat{x}), \ldots, v_d a_\nu(\hat{x})), & \text{if } x_1 = 0, \\ b(\hat{x}) = \sum_{\nu \in \mathcal{V}} (b_{\nu}(\hat{x}), v_1 b_\nu(\hat{x}), \ldots, v_d b_\nu(\hat{x})), & \text{if } x_1 = 1. \end{cases} \quad (3.1)$$

Fix a bounded density profile $\rho_0 : D^d \to \mathbb{R}^+$, and a bounded momentum profile $p_0 : D^d \to \mathbb{R}^d$. A bounded function $(\rho, p) : [0,T] \times D^d \to \mathbb{R}_+ \times \mathbb{R}^d$ is a weak solution of the system of parabolic partial differential equations

$$\begin{cases} \partial_t (\rho, p) + \sum_{\nu \in \mathcal{V}} \D \cdot \nabla \chi(\theta_\nu(\Lambda(\rho, p))) = \frac{1}{2} \Delta (\rho, p), \\ (\rho, p)(0, \cdot) = (\rho_0, p_0)(\cdot) \text{ and } (\rho, p)(t, x) = d(x), x \in \{0, 1\} \times \mathbb{T}^{d-1}, \end{cases} \quad (3.2)$$

if for every vector valued function $H : [0,T] \times D^d \to \mathbb{R}^{d+1}$ of class $C^{1,2}([0,T] \times D^d)$ vanishing at the boundary, we have

$$\int_{D^d} H(T, u) \cdot (\rho, p)(T, u) du - \int_{D^d} H(0, u) \cdot (\rho, p_0)(u) du$$

$$= \int_0^T dt \int_{D^d} d\hat{x} \left\{ (\rho, p)(t, u) \cdot \partial_t H(t, u) + \frac{1}{2} \frac{\partial^2_{u_i} H(t, u)}{\sum_{1 \leq i \leq d} |\partial_{u_i} H(t, u)|^2} \right\}$$

$$- \int_0^T dt \int_{\{1\} \times \mathbb{T}^{d-1}} dS \cdot \partial_{u_i} H(t, u) + \int_0^T dt \int_{\{0\} \times \mathbb{T}^{d-1}} dS \cdot \frac{\partial_{u_i} H(t, u)}{\sum_{1 \leq i \leq d} |\partial_{u_i} H(t, u)|^2} \sum_{1 \leq i \leq d} v_i \partial_{u_i} H(t, u),$$

$dS$ being the Lebesgue measure on $\mathbb{T}^{d-1}$.

We say that the solution $(\rho, p)$ has finite energy if its components belong to $L^2([0,T], H^1(D^d))$:

$$\int_0^T ds \left( \int_{D^d} \|\nabla \rho(s, u)\|^2 du \right) < \infty,$$

and

$$\int_0^T ds \left( \int_{D^d} \|\nabla p_k(s, u)\|^2 du \right) < \infty,$$

for $k = 1, \ldots, d$, where $\nabla f$ represents the generalized gradient of the function $f$.

In [12] the following theorem was proved:

**Theorem 3.1.** Let $(\mu^N)_{N}$ be a sequence of probability measures on $X_N$ associated to the profile $(\rho_0, p_0)$. Then, for every $t \geq 0$, for every continuous function $H : D^d \to \mathbb{R}$ vanishing at the boundary, and for every $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}_{\mu^N} \left[ \frac{1}{N^d} \sum_{x \in D^d_N} H \left( \frac{x}{N} \right) I_0(\eta_x(t)) - \int_{D^d} H(u) \rho(t, u) du \right] > \delta = 0,$$

and for $1 \leq k \leq d$

$$\lim_{N \to \infty} \mathbb{P}_{\mu^N} \left[ \frac{1}{N^d} \sum_{x \in D^d_N} H \left( \frac{x}{N} \right) I_k(\eta_x(t)) - \int_{D^d} H(u) p_k(t, u) du \right] > \delta = 0.$$
where \((\rho, p)\) has finite energy and is the unique weak solution of equation \((3.2)\).

4. The rate function \(I_T(\cdot|\gamma)\)

We examine in this section the rate function \(I_T(\cdot|\gamma)\). The main result, presented in Theorem 4.6 below, states that \(I_T(\cdot|\gamma)\) has compact level sets. The proof relies on two ingredients. The first one, stated in Lemma 4.2, is an estimate of the energy and of the \(H^{-1}\) norm of the time derivative of a trajectory in terms of the rate function. The second one, stated in Lemma 4.3, establishes that sequences of trajectories, with rate function uniformly bounded, which converges weakly in \(L^2\) converge in fact strongly. We follow the strategy introduced in [9].

Recall that \(V\) is an open neighborhood of \(D^d\), and consider, for each \(v \in V\), smooth functions \(\kappa^v_k : V \to (0, 1)\) in \(C^2(V)\), for \(k = 0, \ldots, d\). We assume that each \(\kappa^v_k\) has its image contained in some compact subset of \((0, 1)\), that the restriction of \(\kappa = \sum_{v \in V}(\kappa^v_0, v_1\kappa^v_1, \ldots, v_d\kappa^v_d)\) to \(\{0\} \times \mathbb{T}^{d-1}\) equals the vector valued function \(a(\cdot)\) defined in (3.1), and that the restriction of \(\kappa\) to \(\{1\} \times \mathbb{T}^{d-1}\) equals the vector valued function \(b(\cdot)\), also defined in (3.1), in the sense that \(\kappa(x) = d(x_1, \tilde{x})\) if \(x \in [0, 1) \times \mathbb{T}^{d-1}\).

Let \(L^2(D^d)\) be the Hilbert space of functions \(G : D^d \to \mathbb{R}\) such that \(\int_{D^d} |G(u)|^2 \, du < \infty\) equipped with the inner product

\[
\langle G, F \rangle_2 = \int_\Omega G(u) F(u) \, du,
\]

and the norm of \(L^2(D^d)\) is denoted by \(\| \cdot \|_2\).

Recall that \(H^1(D^d)\) is the Sobolev space of functions \(G\) with generalized derivatives \(\partial_{u_1} G, \ldots, \partial_{u_d} G\) in \(L^2(D^d)\). \(H^1(D^d)\) endowed with the scalar product \(\langle \cdot, \cdot \rangle_{1,2}\), defined by

\[
\langle G, F \rangle_{1,2} = \langle G, F \rangle_2 + \sum_{j=1}^d \langle \partial_{u_j} G, \partial_{u_j} F \rangle_2,
\]

is a Hilbert space. The corresponding norm is denoted by \(\| \cdot \|_{1,2}\).

Recall that we denote by \(C_\infty_c(D^d)\) the set of infinitely differentiable functions \(G : D^d \to \mathbb{R}\), with compact support in \(D^d\). Denote by \(H^1_0(D^d)\) the closure of \(C_\infty_c(D^d)\) in \(H^1(D^d)\). Since \(D^d\) is bounded, by Poincaré’s inequality, there exists a finite constant \(C\) such that for all \(G \in H^1_0(D^d)\)

\[
\|G\|_2^2 \leq C \sum_{j=1}^d \langle \partial_{u_j} G, \partial_{u_j} G \rangle_2.
\]

This implies that, in \(H^1_0(D^d)\)

\[
\|G\|_{1,2,0} = \left\{ \sum_{j=1}^d \langle \partial_{u_j} G, \partial_{u_j} G \rangle_2 \right\}^{1/2}
\]

is a norm equivalent to the norm \(\| \cdot \|_{1,2}\). Moreover, \(H^1_0(D^d)\) is a Hilbert space with inner product given by

\[
\langle G, J \rangle_{1,2,0} = \sum_{j=1}^d \langle \partial_{u_j} G, \partial_{u_j} J \rangle_2.
\]

To assign boundary values along the boundary \(\Gamma\) of \(D^d\) to any function \(G\) in \(H^1(D^d)\), recall, from the trace Theorem (\[15\], Theorem 21.A.(c)), that there exists a continuous linear operator \(\text{Tr} : H^1(D^d) \to L^2(\Gamma)\), called trace, such that \(\text{Tr}(G) = G|_{\Gamma}\) if \(G \in H^1(D^d) \cap C(\overline{D^d})\). Moreover, the space \(H^1_0(D^d)\) is the space of functions \(G\) in \(H^1(D^d)\) with zero trace (\[16\], Appendix (48b)):

\[
H^1_0(D^d) = \left\{ G \in H^1(D^d) : \text{Tr}(G) = 0 \right\}.
\]

Finally, denote by \(H^{-1}(D^d)\) the dual of \(H^1_0(D^d)\). \(H^{-1}(D^d)\) is a Banach space with norm \(\| \cdot \|_{-1}\) given by

\[
\|v\|_{-1}^2 = \sup_{G \in C_\infty_c(D^d)} \left\{ 2\langle v, G \rangle_{-1,1} - \int_{D^d} \|\nabla G(u)\|^2 \, du \right\},
\]

where \(\langle v, G \rangle_{-1,1}\) stands for the values of the linear form \(v\) at \(G\).
For each $G \in C_c^{\infty}(\Omega_T)$ and each integer $1 \leq i \leq d$, let $Q^{G}_{i,k} : D([0,T], \mathcal{M}) \to \mathbb{R}$ be the functional given by
\[
Q^{G}_{i,k}(\pi) = 2 \int_0^T dt \langle \pi^i, \partial_u G_i \rangle - \int_0^T dt \int_{D^d} du \ G(t,u)^2,
\]
where $\pi = (\pi^0, \pi^1, \ldots, \pi^d)$. Recall, from subsection 2.2, that the energy $\mathcal{Q}(\pi)$ is given by
\[
\mathcal{Q}(\pi) = \sum_{i=0}^d \sum_{k=1}^d Q_{i,k}(\pi), \quad \text{with} \quad Q_{i,k}(\pi) = \sup_{G \in C_c^{\infty}(\Omega_T)} \mathcal{Q}^{G}_{i,k}(\pi).
\]

The functional $Q^{G}_{i,k}$ is convex and continuous in the Skorohod topology. Therefore $Q_{i,k}$ and $\mathcal{Q}$ are convex and lower semicontinuous. Furthermore, it is well known that a measure $\delta > 0$.

To conclude the proof, we observe that the left-hand side is bounded by $(r-s)^2 \mathcal{I}_T(\pi|\gamma)$, follows from Lemma 6.4.

Fix a path $\pi \in D([0,T], \mathcal{M})$ and let $\gamma$ be the subset of $[0,T]$ for almost all $\pi$. Let $\gamma$ be the subset of $[0,T]$ such that
\[
\mathcal{Q}(\pi) = \sup_{G \in C_c^{\infty}(\Omega_T)} \mathcal{Q}^{G}_{i,k}(\pi).
\]

Lemma 4.1. Let $\pi$ be a trajectory in $D([0,T], \mathcal{M})$ such that $I_T(\pi|\gamma) < \infty$. Then $\pi$ belongs to $D_\gamma$.

Proof. Fix a path $\pi$ in $D([0,T], \mathcal{M})$ with finite rate function, $I_T(\pi|\gamma) < \infty$. By definition of $I_T$, $\pi$ belongs to $D([0,T], \mathcal{M})$. Denote its density and momentum by $(\rho, p)$:
\[
\mathcal{Q}(\pi) = \mathcal{Q}(\rho, p),
\]

The proof that $(\rho, p)(0, \cdot) = \gamma(\cdot)$ is similar to the one of Lemma 3.5 in [9], and the proof that $\mathcal{Q}(\rho) = d_0$, $\mathcal{Q}(p) = d_k$, $k = 1, \ldots, d$, is similar to the one found in Lemma 4.1 in [9]. The fact that $\pi$ has finite energy follows from Lemma 3.4.

We deal now with the continuity of $\pi$. We claim that there exists a positive constant $C_0$ such that, for any $g \in C_c^{\infty}(\Omega)$, and any $0 \leq s < r < T$,
\[
|\langle \pi_r, g \rangle - \langle \pi_s, g \rangle| \leq C_0(r-s)^{1/2} \left\{ C_1 + I_T(\pi|\gamma) + \|g\|_{1,2,0} + (r-s)^{1/2}\|\Delta g\|_1 \right\}.
\]

Indeed, for each $\delta > 0$, let $\psi^\delta : [0, T] \to \mathbb{R}$ be the function given by
\[
(r-s)^{1/2}\psi^\delta(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq s \text{ or } r + \delta \leq t \leq T, \\ \frac{t-s}{\delta} & \text{if } s \leq t \leq s + \delta, \\ 1 & \text{if } s + \delta \leq t \leq r, \\ 1 - \frac{r-t}{\delta} & \text{if } r \leq t \leq r + \delta, \end{cases}
\]

and let $G^\delta(t, u) = \psi^\delta(t)g(u)$. Of course, $G^\delta$ can be approximated by functions in $C_c^{1,2}(\Omega_T)$ and then
\[
(r-s)^{1/2} \lim_{\delta \to 0} J_{G^\delta}(\pi) = |\langle \pi_r, g \rangle - \langle \pi_s, g \rangle| - \int_s^r dt \langle \pi_t, \Delta g \rangle + \int_s^r dt \int_{D^d} du \sum_{v \in \mathcal{V}} \tilde{v} \cdot \chi(\theta_v(\Lambda(\rho, p))) \sum_{i=1}^d v_i \partial_u_i g(u) - \frac{1}{(r-s)^{1/2}} \int_s^r dt \int_{D^d} du \sum_{v \in \mathcal{V}} \left( \sum_{k=0}^d \partial_v v_k g^k(u) \right)^2 \chi(\theta_v(\Lambda(\rho, p)))
\]

To conclude the proof, we observe that the left-hand side is bounded by $(r-s)^{1/2}I_T(\pi|\gamma)$, that $\chi$ is positive and bounded above on $[0, 1]$ by $1/4$, and finally, we use the elementary inequality $2ab \leq a^2 + b^2$. \qed
Denote by \(L^2([0,T],H^1_0(D^d))^*\) the dual of \(L^2([0,T],H^1_0(D^d))\). By Proposition 23.7 in [13], \(L^2([0,T],H^1_0(D^d))^*\) corresponds to \(L^2([0,T],H^{-1}(D^d))\) and for \(v\) in \(L^2([0,T],H^1_0(D^d))^*\), \(G\) in \(L^2([0,T],H^1_0(D^d))\),

\[
\langle v, G \rangle_{-1,1} = \int_0^T \langle v_t, G_t \rangle_{-1,1} dt ,
\]

where the left hand side stands for the value of the linear functional \(v\) at \(G\). Moreover, if we denote by \(\|v\|_{-1}\) the norm of \(v\),

\[
\|v\|_{-1}^2 = \int_0^T \|v_t\|_{-1}^2 dt .
\]

Fix a path \(\pi(t, du) = (\rho, p)(t, u)du\) in \(D\), and suppose that for \(k = 0, \ldots, d\)

\[
\sup_{H \in C_0^\infty(\Omega_T)} \left\{ 2 \int_0^T dt \langle p_{k,t}, \partial_t H_t \rangle_{2} - \int_0^T dt \int_{D^d} du \|\nabla H_t\|^2 \right\} < \infty .
\]

In this case, for each \(k\), \(\partial_k p_k : C_0^\infty(\Omega_T) \to \mathbb{R}\) defined by

\[
\partial_k p_k(H) = -\int_0^T \langle p_{k,t}, \partial_t H_t \rangle_{2} dt
\]

can be extended to a bounded linear operator \(\partial_k p_k : L^2([0,T],H^1_0(D^d)) \to \mathbb{R}\). It belongs therefore to \(L^2([0,T],H^1_0(D^d))^* = L^2([0,T],H^{-1}(D^d))\). In particular, there exists \(v^k = \{v^k_t : 0 \leq t \leq T\}\) in \(L^2([0,T],H^{-1}(D^d))\), which we denote by \(v^k_t = \partial_k p_k, t\), such that for any \(H\) in \(L^2([0,T],H^1_0(D^d))\),

\[
\langle \partial_k p_k, H \rangle_{-1,1} = \int_0^T \langle \partial_k p_k, H_t \rangle_{-1,1} dt .
\]

Moreover,

\[
\|\partial_k p_k\|_{-1}^2 = \int_0^T \|\partial_k p_k, t\|_{-1}^2 dt
\]

\[
= \sup_{H \in C_0^\infty(\Omega_T)} \left\{ 2 \int_0^T dt \langle p_{k,t}, \partial_t H_t \rangle_{2} - \int_0^T dt \int_{D^d} du \|\nabla H_t\|^2 \right\} .
\]

Denote by \(\langle \partial_k (\rho, p), G \rangle_{-1,1}\) the linear functional given by

\[
\langle \partial_k (\rho, p), G \rangle_{-1,1} = \sum_{k=0}^d \langle \partial_k p_k, H \rangle_{-1,1},
\]

with

\[
\|\partial_k (\rho, p)\|_{-1}^2 = \sum_{k=0}^d \|\partial_k p_k\|_{-1}^2.
\]

Let \(W\) be the set of paths \(\pi(t, du) = (\rho, p)(t, u)du\) in \(D\), such that [13] holds, i.e., such that \(\partial_k p_k\) belongs to \(L^2((0,T],H^{-1}(D^d))\). For \(G\) in \(L^2((0,T],H^1_0(D^d)^{d+1})\), let \(J_G : W \to \mathbb{R}\) be the functional given by

\[
J_G(\pi) = \langle \partial_k (\rho, p), G \rangle_{-1,1} + \frac{1}{2} \int_0^T dt \int_{D^d} du \nabla (\rho, p)(t, u) \cdot \nabla G(t, u)
\]

\[
+ \int_0^T dt \int_{D^d} du \sum_{\nu \in \mathcal{V}} \tilde{v} \cdot \chi(\theta_\nu(\Lambda(\rho, p))) \sum_{1 \leq i \leq d} v_i \partial_{\nu_i} G(t, u)
\]

\[
- \int_0^T dt \int_{D^d} du \sum_{\nu \in \mathcal{V}} \sum_{k=0}^d \left( \sum_{i=1}^d v_k \partial_{\nu_i} G^k_i(u) \right)^2 \chi(\theta_\nu(\Lambda(\rho, p))).
\]

Note that \(J_G(\pi) = J_G(\pi)\) for every \(G\) in \(C_0^\infty(\Omega_T) \times [C_0^\infty(D^d)]^d\). Moreover, since \(J(\pi)\) is continuous in \(L^2((0,T],H^1_0(D^d)^{d+1})\), and since \(C_0^\infty(\Omega_T)\) is dense in \(C_0^{\infty,2}(\Omega_T)\) and in \(L^2((0,T],H^1_0(D^d))\), for any \(\pi\) in \(W\),

\[
I_T(\pi|\gamma) = \sup_{G \in C_0^\infty(\Omega_T) \times [C_0^\infty(D^d)]^d} J_G(\pi) = \sup_{G \in L^2(0,T,H^1_0(D^d)^{d+1})} J_G(\pi).
\]

(4.4)
Lemma 4.2. There exists a constant $C_0 > 0$ such that if the density and momentum $(\rho, p)$ of some path $\pi(t, du) = (\rho, p)(t, u)du$ in $D([0, T], \mathcal{M}^0)$ has generalized gradients, $\nabla \rho$ and $\nabla p$, $k = 1, \ldots, d$. Then
\[
\|\partial_t(\rho, p)\|_{-1}^2 \leq C_0 \{ I_T(\pi|\gamma) + \mathcal{Q}(\pi) \},
\]
and in the last inequality we used the Cauchy-Schwartz inequality and the elementary inequalities $2\chi(\gamma) \leq 1$.\hfill (4.5)

Proof. Fix a path $\pi(t, du) = (\rho, p)(t, u)du$ in $D([0, T], \mathcal{M}^0)$. In view of the discussion presented before the lemma, we need to show that the left hand side of (4.3) is bounded by the right hand side of (4.5). Such an estimate follows from the definition of the rate function $I_T(\cdot|\gamma)$ and from the elementary inequality $2ab \leq a^2 + b^2$.

To prove (4.6), observe that
\[
I(\pi) \geq J_G(\pi) - \frac{1}{2} \int_0^T dt \int_{D^d} du \sum_{i=1}^d (\partial_{x_i}(\rho, p), \partial_{x_i}G)^2
\]
\[
+ \int_0^T dt \int_{D^d} du \sum_{\nu \in \mathcal{V}} (\chi(\theta_v(\Lambda(\rho, p)))) \sum_{i=1}^d \tilde{v}_i \partial_{x_i}G
\]
\[
\geq \partial_t(\rho, p) + \frac{1}{2} \int_0^T dt \int_{D^d} du \sum_{i=1}^d (\partial_{x_i}(\rho, p), \partial_{x_i}G)^2 - C \int_0^T dt \sum_{k=0}^d \|\nabla G_k\|_2^2,
\]
where $C$ is constant obtained from the elementary inequality $2ab \leq a^2 + b^2$, the fact that $\mathcal{V}$ is finite, and that $\chi$ is bounded above by $1/4$ in $[0, 1]$.

Now, consider $G = K(\pi - \kappa)$, and note that $\pi - \kappa$ belong to $L^2([0, T], H^1_0(D^d))$, which implies that it may be approximated by $C_c^\infty$ functions. Therefore $\partial_t(\rho, p) = (\gamma_T, \gamma_T - \kappa) - (\gamma_0, \gamma_0 - \kappa)$, which is bounded by some constant $C_1$. We, then, obtain that
\[
I(\pi) \geq \int_0^T dt \Big\{ -C_1 + \int_{\kappa=0}^K \sum_{k=0}^d \|\nabla p_k\|_2^2 - CK \sum_{k=0}^d \|\nabla (p_k - \kappa_k)\|_2^2 \Big\}
\]
\[
\geq \int_0^T dt \Big\{ (K/4 - 2CK) \sum_{k=0}^d \|\nabla p_k\|_2^2 \Big\} - \frac{K}{4} \sum_{k=0}^d \|\nabla \kappa_k\|_2^2 - 2CK \sum_{k=0}^d \|\nabla \kappa_k\|_2^2 - C_1
\]
where in the last inequality we used the Cauchy-Schwartz inequality and the elementary inequalities $2ab \leq a^2 + b^2$. The proof thus follows from choosing a suitable $K$, the estimate given in (4.5), and the fact we have a fixed smooth function $\kappa$. \hfill \Box

Corollary 4.3. The density $(\rho, p)$ of a path $\pi(t, du) = (\rho, p)(t, u)du$ in $D([0, T], \mathcal{M}^0)$ is the weak solution of the equation (3.2) and initial profile $\gamma$ if and only if the rate function $I_T(\pi|\gamma)$ vanishes. Moreover, if any of the above conditions hold, $\pi$ has finite energy ($\mathcal{Q}(\pi) < \infty$).

Proof. On the one hand, if the density $(\rho, p)$ of a path $\pi(t, du) = (\rho, p)(t, u)du$ in $D([0, T], \mathcal{M}^0)$ is the weak solution of equation (3.2) with initial condition is $\gamma$, in the formula of $J_G(\pi)$, the linear part in $G$ vanishes which proves that the rate functional $I_T(\pi|\gamma)$ vanishes. On the other hand, if the rate functional vanishes, the path $(\rho, p)$ belongs to $L^2([0, T], [H^1(D^d)]^{d+1})$ and the linear part in $G$ of $J_G(\pi)$ has to vanish for all functions $G$. In particular, $(\rho, p)$ is a weak solution of (3.2). Moreover, if the rate function is finite, by the previous lemma, $\pi$ has finite energy. Accordingly, if $\pi$ is a weak solution, we have from Theorem 3.1 that it has finite energy. \hfill \Box

For each $q > 0$, let $E_q$ be the level set of $I_T(\pi|\gamma)$ defined by
\[
E_q = \{ \pi \in D([0, T], \mathcal{M}) : I_T(\pi|\gamma) \leq q \}.
\]
By Lemma 4.4, $E_q$ is a subset of $C([0, T], M^0)$. Thus, from the previous lemma, it is easy to deduce the next result.

**Corollary 4.4.** For every $q \geq 0$, there exists a finite constant $C(q)$ such that

$$\sup_{\pi \in E_q} \left\{ \left\| \partial_t (\rho, p) \right\|_{L^1} + \sum_{k=0}^d \int_0^T dt \int_{D^d} du \left\| \nabla p_k(t, u) \right\|_2 \right\} \leq C(q).$$

Next result together with the previous estimates provide the compactness needed in the proof of the lower semicontinuity of the rate function.

**Lemma 4.5.** Let $\{\rho^n : n \geq 1\}$ be a sequence of functions in $L^2(\Omega_T)$ such that uniformly on $n$,

$$\int_0^T dt \left\| \rho^n \right\|_{L^2}^2 + \int_0^T dt \left\| \partial_t \rho^n \right\|_{L^1}^2 < C$$

for some positive constant $C$. Suppose that $\rho \in L^2(\Omega_T)$ and that $\rho^n \to \rho$ weakly in $L^2(\Omega_T)$. Then $\rho_n \to \rho$ strongly in $L^2(\Omega_T)$.

**Proof.** Since $H^1(D^d) \subseteq L^2(D^d)$, from Corollary 8.4, the sequence $\{\rho_n\}$ is relatively compact in $L^2([0, T], L^2(D^d))$. Therefore the weak convergence implies the strong convergence in $L^2([0, T], L^2(D^d))$.

**Theorem 4.6.** The functional $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.

**Proof.** We have to show that, for all $q \geq 0$, $E_q$ is compact in $D([0, T], M)$. Since $E_q \subset C([0, T], M^0)$ and $C([0, T], M^0)$ is a closed subset of $D([0, T], M)$, we just need to show that $E_q$ is compact in $C([0, T], M^0)$.

We will show first that $E_q$ is closed in $C([0, T], M^0)$. Fix $q \in \mathbb{R}$ and let $\{\pi^n : n \geq 1\}$ be a sequence in $E_q$ converging to some $\pi \in C([0, T], M^0)$. Then, for all $G \in C(\Omega_T) \times \left[C(\Omega_T)\right]^d$,

$$\lim_{n \to \infty} \int_0^T dt \left( \pi^n_t : G \right) = \int_0^T dt \left( \pi_t : G \right).$$

Notice that this means that $\pi^{n,k} \to \pi^k$ weakly in $L^2(\Omega_T)$, for each $k = 0, \ldots, d$, which together with Corollary 4.4 and Lemma 4.5 imply that $\pi^{n,k} \to \pi^k$ strongly in $L^2(\Omega_T)$. From this fact and the definition of $J_G$ it is easy to see that, for all $G$ in $C_0^{1,2}(\Omega_T) \times \left[C_0^2(\Omega_T)\right]^d$,

$$\lim_{n \to \infty} J_G(\pi^n) = J_G(\pi).$$

This limit, Corollary 4.4 and the lower semicontinuity of $\mathcal{Q}$ permit us to conclude that $\mathcal{Q}(\pi) \leq C(q)$ and that $I_T(\pi|\gamma) \leq q$.

We prove now that $E_q$ is relatively compact. To this end, it is enough to prove that for every continuous function $G : \mathbb{R}^d \to \mathbb{R}$, and every $k = 0, \ldots, d$,

$$\lim_{\delta \to 0} \sup_{\pi \in E_q} \sup_{0 \leq s, \tau \leq T} \left\| \pi^k_{s, \tau} - \pi^k_{s, \tau} \right\| = 0. \quad (4.7)$$

Since $E_q \subset C([0, T], M^0)$, we may assume by approximations of $G$ in $L^1(D^d)$ that $G \in C^\infty_r(D^d)$. In which case, (4.7) follows from (4.4).

We conclude this section with an explicit formula for the rate function $I_T(\cdot|\gamma)$. For each $\pi(t, du) = (\rho, p)(t, u)du$ in $D([0, T], M^0)$, denote by $H_0^\pi$ the Hilbert space induced by $C_0^{1,2}(\Omega_T)$ endowed with the inner product $\langle \cdot, \cdot \rangle_\pi$ defined by

$$\langle H, G \rangle_\pi = \sum_{v \in V} \int_0^T dt \int_{D^d} du \chi(\theta_v(A(\rho, p)))(\hat{v} \cdot \nabla H)(\hat{v} \cdot \nabla G). \quad (4.8)$$

Induced means that we first declare two functions $F, G$ in $C_0^{1,2}(\Omega_T)$ to be equivalent if $\langle F - G, F - G \rangle_\pi = 0$ and then we complete the quotient space with respect to the inner product $\langle \cdot, \cdot \rangle_\pi$. The norm of $H_0^\pi$ is denoted by $\| \cdot \|_\pi$. 

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Fix a path $\pi$ in $D([0, T], M^0)$ and a function $H$ in $H^1_0(\pi)$. A measurable function $\lambda : [0, T] \times D^d \to \mathbb{R}_+ \times \mathbb{R}^d$ is said to be a weak solution of the nonlinear boundary value parabolic equation

$$
\begin{align*}
\frac{\partial I}{\partial t} + \sum_{i=1}^d \int_{\mathbb{R}} \partial_x [\chi(\theta_v(\Lambda(\lambda)))(v_i - \lambda \cdot \partial_x H)] = \frac{1}{2} \Delta I,
\end{align*}
$$

if it satisfies the following two conditions.
(i) For $k = 0, \ldots, d$, $\lambda_k$ belongs to $L^2([0, T], H^1(D^d))$:

$$
\int_0^T ds \left( \int_{D^d} \| \nabla \lambda_k(s, u) \|^2 du \right) < \infty,
$$

(ii) For every function $G(t, u) = G_1(u)$ in $c_0^1(D^d)$,

$$
\begin{align*}
&\int_{D^d} G(T, u) \cdot \lambda(T, u) du - \int_{D^d} G(0, u) \cdot \gamma(u) du \\
&= \int_0^T dt \int_{D^d} du \left\{ \lambda(t, u) \cdot \partial_t G(t, u) + \frac{1}{2} \lambda(t, u) \cdot \sum_{1 \leq i \leq d} \partial_{u_i}^2 G(t, u) \right\} \\
&\quad - \int_0^T dt \int_{\{1\} \times T^{d-1}} dS b(\bar{u}) \cdot \partial_{u_1} G(t, u) + \int_0^T dt \int_{\{0\} \times T^{d-1}} dS a(\bar{u}) \cdot \partial_{u_1} G(t, u)
\end{align*}
$$

Uniqueness of solutions of equation (4.9) follows from the same arguments of the uniqueness proved in [12].

**Lemma 4.7.** Assume that $\pi(t, du) = (\rho, p)(t, u) du$ in $D([0, T], M^0)$ has finite rate function: $I_T(\pi|\gamma) < \infty$. Then, there exists a function $H$ in $H^1_0(\pi)$ such that $(\rho, p)$ is a weak solution to (4.10). Moreover,

$$
I_T(\pi|\gamma) = \frac{1}{4} \| H \|^2_{\pi}.
$$

The proof of this lemma is similar to the one of Lemma 10.5.3 in [10] and is therefore omitted.

5. $I_T(\cdot|\gamma)$-Density

The main result of this section, stated in Theorem 5.5, asserts that any trajectory $\lambda_t$, $0 \leq t \leq T$, with finite rate function, $I_T(\lambda|\gamma) < \infty$, can be approximated by a sequence of smooth trajectories $\{\lambda^n : n \geq 1\}$ such that

$$
\lambda^n \to \lambda \quad \text{and} \quad I_T(\lambda^n|\gamma) \to I_T(\lambda|\gamma).
$$

This is one of the main steps in the proof of the lower bound of the large deviations principle for the empirical measure. The proof is mainly based on the regularizing effects of the hydrodynamic equation. This strategy was introduced by [9].

A subset $A$ of $D([0, T], M_+ \times M^d)$ is said to be $I_T(\cdot|\gamma)$-dense if for every $\pi$ in $D([0, T], M_+ \times M^d)$ such that $I_T(\pi|\gamma) < \infty$, there exists a sequence $\{\pi^n : n \geq 1\}$ in $A$ such that $\pi^n$ converges to $\pi$ and $I_T(\pi^n|\gamma)$ converges to $I_T(\pi|\gamma)$.

Let $\Pi_1$ be the subset of $D([0, T], M^0)$ consisting of paths $\pi(t, du) = (\rho, p)(t, u) du$ whose density $(\rho, p)$ is a weak solution of the hydrodynamic equation (5.2) in the time interval $[0, \delta]$ for some $\delta > 0$.

**Lemma 5.1.** The set $\Pi_1$ is $I_T(\cdot|\gamma)$-dense.
Proof. Fix \( \pi(t, du) = (\rho, p)(t, u)du \) in \( D([0, T], \mathcal{M}_+ \times \mathcal{M}^0) \) such that \( I_T(\pi|\gamma) < \infty \). By Lemma 4.1 \( \pi \) belongs to \( C([0, T], \mathcal{M}^0) \). For each \( \delta > 0 \), let \( (\rho^\delta, p^\delta) \) be the path defined as

\[
(\rho^\delta, p^\delta)(t, u) = \begin{cases} 
\tau(t, u) & \text{if } 0 \leq t \leq \delta, \\
\tau(2\delta - t, u) & \text{if } \delta \leq t \leq 2\delta, \\
(\rho, p)(t - 2\delta, u) & \text{if } 2\delta \leq t \leq T,
\end{cases}
\]

where \( \tau \) is the weak solution of the hydrodynamic equation (3.2) starting at \( \gamma \). It is clear that \( \pi^\delta(t, du) = (\rho^\delta, p^\delta)(t, u)du \) belongs to \( D_\gamma \), because so do \( \pi \) and \( \tau \) and that \( Q(\pi^\delta) \leq Q(\pi) + 2Q(\tau) < \infty \). Moreover, \( \pi^\delta \) converges to \( \pi \) as \( \delta \downarrow 0 \) because \( \pi \) belongs to \( C([0, T], \mathcal{M}^0) \). By the lower semicontinuity of \( I_T(\cdot|\gamma) \), \( I_T(\pi|\gamma) \leq \liminf_{\delta \to 0} I_T(\pi^\delta|\gamma) \). Then, in order to prove the lemma, it is enough to prove that \( I_T(\pi|\gamma) \geq \limsup_{\delta \to 0} I_T(\pi^\delta|\gamma) \).

To this end, decompose the rate function \( I_T(\pi^\delta|\gamma) \) as the sum of the contributions on each time interval \([0, \delta], [\delta, 2\delta] \) and \([2\delta, T]\). The first contribution vanishes because \( \pi^\delta \) solves the hydrodynamic equation in this interval. On the time interval \([\delta, 2\delta] \), \( \partial_t \rho^\delta = -\partial_t \tau_{2\delta-t} = \frac{\partial}{\delta} \int_D \int \partial_t \chi(\theta_v(\Lambda(p, p))) = -\Delta(\rho^\delta, p^\delta) + \sum_{v \in V} \tilde{v}[v \cdot \nabla \chi(\theta_v(\Lambda(p, p)))]. \) In particular, the second contribution is equal to

\[
\sup_{G \in \mathcal{G}_0, |\delta| < |c| < \delta} \left\{ \sum_{i=1}^d \int_0^\delta ds \int_D du \partial_{x_i}(\rho, p) \cdot \partial_{x_i} G - \sum_{v \in V} \sum_{i=1}^d \int_0^\delta du \chi(\theta_v(\Lambda(p, p)))[\tilde{v} \cdot \nabla G]^2 \right\}
\]

which, by Lemma 6.3, is bounded from above, and therefore this last expression converges to zero as \( \delta \downarrow 0 \). Finally, the third contribution is bounded by \( I_T(\pi|\gamma) \) because \( \pi^\delta \) in this interval is just a time translation of the path \( \pi \).

Let \( \Pi_2 \) be the set of all paths \( \pi \) in \( \Pi_1 \) with the property that for every \( \delta > 0 \) there exists \( \epsilon > 0 \) such that, for \( k = 0, \ldots, d \), \( d(\pi^\delta, \cdot, \delta | \Omega) \geq \epsilon \) for all \( t \in [\delta, T] \), where \( \partial \Omega \) stands for the boundary of \( \Omega \).

We begin by proving an auxiliary lemma.

Lemma 5.2. Let \( \pi, \lambda \in \mathcal{U} \), and let \( \pi^\epsilon = (1-\epsilon)\pi + \epsilon \lambda, 0 \leq \epsilon \leq 1 \). Then, for all \( v \in \mathcal{V} \), we have

\[
\theta_v(\Lambda(\pi^\epsilon)) = (1-\epsilon)\theta_v(\Lambda(\pi)) + \epsilon \theta_v(\Lambda(\lambda)).
\]

Proof. Fix some \( \lambda \in \mathcal{U} \). Observe that

\[
\left( \sum_{v \in \mathcal{V}} \theta_v(\Lambda(\lambda)), \sum_{v \in \mathcal{V}} v_1 \theta_v(\Lambda(\lambda)), \ldots, \sum_{v \in \mathcal{V}} v_d \theta_v(\Lambda(\lambda)) \right) = (\lambda_0, \lambda_1, \ldots, \lambda_d)
\]

is a linear system with \( d+1 \) equations and \( |\mathcal{V}| \) unknowns (given by \( \theta_v(\Lambda(\lambda)), \) for \( v \in \mathcal{V} \)). Therefore, any solution of this linear system can be expressed as a linear combination of \( \lambda_i, i = 0, 1, \ldots, d \). The proof follows from this fact.

Remark 5.3. In the particular case when \( d = 1 \) and the set of velocities is \( \mathcal{V} = \{v, -v\} \subset \mathbb{R} \), a simple computation gives the unique solution

\[
\theta_v(\Lambda(\lambda_0, \lambda_1)) = \frac{\lambda_0}{2} + \frac{\lambda_1}{2v} \quad \text{and} \quad \theta_{-v}(\Lambda(\lambda_0, \lambda_1)) = \frac{\lambda_0}{2} - \frac{\lambda_1}{2v}.
\]

Lemma 5.4. The set \( \Pi_2 \) is \( I_T(\cdot|\gamma) \)-dense.

Proof. By Lemma 6.1 it is enough to show that each path \( \pi(t, du) = (\rho, p)(t, u)du \) in \( \Pi_1 \) can be approximated by paths in \( \Pi_2 \). Fix \( \pi \) in \( \Pi_1 \) and let \( \tau \) be as in the proof of the previous lemma. For each \( 0 < \epsilon < 1 \), let \( (\rho^\epsilon, p^\epsilon) = (1-\epsilon)(\rho, p) + \epsilon \tau \), \( \pi^\epsilon(t, du) = (\rho^\epsilon, p^\epsilon)(t, u)du \). Note that \( Q(\pi^\epsilon) \leq \frac{\partial}{\delta} \int_D \int \partial_t \chi(\theta_w(\Lambda(p, p))) \leq \infty \) because \( Q \) is convex and both \( Q(\pi) \) and \( Q(\tau) \) are finite. Hence, \( \pi^\epsilon \) belongs to \( D_\gamma \) since both \( \rho \) and \( \tau \) satisfy the boundary conditions. Moreover, it is clear that \( \pi^\epsilon \) converges to \( \pi \) as \( \epsilon \downarrow 0 \). By the lower semicontinuity of \( I_T(\cdot|\gamma) \), in order to conclude the proof, it is enough to show that

\[
\lim_{N \to \infty} I_T(\pi^\epsilon|\gamma) \leq I_T(\pi|\gamma).
\]

By Lemma 4.1 there exists \( H \in \mathcal{H}_0^{d}(\pi) \) such that \( (\rho, p) \) solves the equation (4.4). Let us denote \( \chi(\theta_v(\Lambda(p, p))) \) simply by \( \chi_v(\pi) \), and define \( P_{\tau,v}(\pi) = \chi_v(\pi)[\tilde{v} \cdot \partial_{x_i} H - v_i] \), and note that \( P_{\tau,v}(\tau) = -v_i \chi(\theta_v(\Lambda(\tau))). \) Let also
Observe that, by Lemma 4.7

$$I(\pi) = \frac{1}{4}\|H\|_\pi^2,$$

and that, using the definition of \( \| \cdot \|_\pi \) in (4.8),

$$\frac{1}{4}\|H\|_\pi^2 = \frac{1}{4} \sum_{i,v} \int_0^T dt \int_{D^d} du \chi_v(\pi)(\hat{v} \cdot \partial_x, H)^2 = \frac{1}{4} \sum_{i,v} \int_0^T dt \int_{D^d} du \frac{(P^v_{i,v} + v_i \chi_v(\pi))^2}{\chi_v(\pi)}.$$

A simple computation shows that

$$J_G(\pi^\epsilon) = \sum_{i,v} \int_0^T \int_{D^d} [P^v_{i,v} + \chi_v(\pi^\epsilon) v_i](\hat{v} \cdot \partial_x, G) - \chi_v(\pi^\epsilon)(\hat{v} \cdot \partial_x, G)^2$$

$$= \frac{1}{4} \sum_{i,v} \int_0^T dt \int_{D^d} du \frac{|P^v_{i,v} + \chi_v(\pi^\epsilon) v_i|^2}{\chi_v(\pi^\epsilon)} - \left( \frac{1}{2} \frac{P^v_{i,v} + \chi_v(\pi^\epsilon)}{\sqrt{\chi_v(\pi^\epsilon)}} - \sqrt{\chi_v(\pi)(\hat{v} \cdot \partial_x, G)} \right)^2.$$

Let

$$A_\epsilon = \frac{1}{4} \sum_{i,v} \int_0^T dt \int_{D^d} du \frac{|P^v_{i,v} + \chi_v(\pi^\epsilon) v_i|^2}{\chi_v(\pi^\epsilon)},$$

and

$$B_\epsilon(G) = \int_0^T dt \int_{D^d} du \left( \frac{1}{2} \frac{P^v_{i,v} + \chi_v(\pi^\epsilon)}{\sqrt{\chi_v(\pi^\epsilon)}} - \sqrt{\chi_v(\pi)(\hat{v} \cdot \partial_x, G)} \right).$$

This implies that

$$I(\pi^\epsilon) = \sup_G J_G(\pi^\epsilon) = \sup_G \{ A_\epsilon - B_\epsilon(G)^2 \} = A_\epsilon - \inf_G B_\epsilon(G)^2 \leq A_\epsilon,$$

where the supremum and infimum are taken over in \( G \) in \( C^\infty_c(\Omega_T) \times [C^\infty_c(D^d)]^d \).

It remains to be shown that \( A_\epsilon \) is uniformly integrable in \( \epsilon \). However, this is a simple consequence of Lemma 5.2.

Let \( \Pi \) be the subset of \( \Pi_2 \) consisting of all those paths \( \pi \) which are solutions of the equation (4.9) for some \( H \in C^{1,2}_0(\Omega_T) \times [C^\infty_c(D^d)]^d \).

**Theorem 5.5.** The set \( \Pi \) is \( I_T(\cdot|\gamma) \)-dense.

**Proof.** By the previous lemma, it is enough to show that each path \( \pi \in \Pi_2 \) can be approximated by paths in \( \Pi \). Fix \( \pi(t, du) = (\rho, p)(t, u)du \in \Pi_2 \). By Lemma 4.7, there exists \( H \in H^1_0(\pi) \) such that \( (\rho, p) \) is the weak solution of (4.2) in some time interval \([0, 2\delta]\) for some \( \delta > 0 \). In particular, \( \nabla H^k = 0 \) a.e in \([0, 2\delta] \times D^d \). On the other hand, since \( \pi \) belongs to \( \Pi_1 \), there exists \( \epsilon > 0 \) such that, for \( k = 0, \ldots, d, d(\pi^\epsilon_t(\cdot), \partial_t) \geq \epsilon \) for \( \delta \leq t \leq T \). Therefore,

$$\int_0^T dt \int_{D^d} \| \nabla H_t(\pi) \|^2 < \infty. \quad (5.2)$$

Since \( H \) belongs to \( H^1_0(\pi) \), there exists a sequence of functions \( \{ H^n : n \geq 1 \} \) in \( C^{1,2}_0(\Omega_T) \) converging to \( H \) in \( H^1_0(\pi) \). We may assume of course that \( \nabla H^n_t \equiv 0 \) in the time interval \([0, \delta]\). In particular,

$$\lim_{n \to \infty} \int_0^T dt \int_{D^d} \| \nabla H^n_t(\pi) - \nabla H_t(\pi) \|^2 = 0. \quad (5.3)$$

For each integer \( n > 0 \), let \((\rho^n, p^n)\) be the weak solution of (4.9) with \( H^n \) in place of \( H \) and set \( \pi^n(t, du) = (\rho^n, p^n)(t, u)du \). By (4.10) and since \( \chi \) is bounded above in \([0, 1]\) by \( 1/4 \), we have that

$$I_T(\pi^n|\gamma) = \frac{1}{2} \sum_{\gamma \in \tilde{V}} \int_0^T dt \langle \chi(\theta_v(\Lambda(\rho^n_t, p^n_t))), \nabla H_t^n \rangle \leq C_0 \int_0^T dt \int_{D^d} \| \nabla H^n_t(\pi) \|^2.$$
In particular, by (5.2) and (5.3), \( I_T(\pi^n|\gamma) \) is uniformly bounded on \( n \). Thus, by Theorem 1.6 the sequence \( \pi^n \) is relatively compact in \( D([0,T],\mathcal{M}_+ \times \mathcal{M}^0) \).

Let \( \{\pi^{n_k}: k \geq 1\} \) be a subsequence of \( \pi^n \) converging to some \( \pi^0 \) in \( D([0,T],\mathcal{M}^0) \). For every \( G \) in \( C^1_0(\Omega_T) \),

\[
\int_{D^d} G(T,u) \cdot (\rho^n_{t_k},\mathbf{p}^{n_k}_{t_k})(T,u) du - \int_{D^d} G(0,u) \cdot \gamma(u) du
\]

\[
= \int_0^T dt \int_{D^d} \left\{ (\rho^n_{t_k},\mathbf{p}^{n_k}_{t_k})(t,u) \cdot \partial_t G(t,u) + \frac{1}{2} (\rho^n_{t_k},\mathbf{p}^{n_k}_{t_k})(t,u) \cdot \sum_{1 \leq i \leq d} \partial^2 u_i G(t,u) \right\}
\]

\[-\int_0^T dt \int_{D^d} dS b(\tilde{u}) \cdot \partial_u G(t,u) + \int_0^T dt \int_{\{0\} \times T_{d-1}} dS a(\tilde{u}) \cdot \partial_u G(t,u)
\]

\[-\int_0^T dt \int_{D^d} du \sum_{v \in V} \tilde{v} \cdot \chi(\theta_v(\Lambda(\rho^{n_k}_{t_k},\mathbf{p}^{n_k}_{t_k}))) \sum_{1 \leq i \leq d} v_i \partial_{u_i} G(t,u),
\]

\[+ \sum_{v \in V} \int_0^T dt \int_{D^d} du \chi(\theta_v(\Lambda(\rho^{n_k}_{t_k},\mathbf{p}^{n_k}_{t_k}))) [\tilde{v} \cdot \nabla H^{n_k}][\tilde{u} \cdot \nabla G].
\]

Letting \( k \to \infty \) in this equation, we obtain the same equation with \( \pi^0 \) and \( H \) in place of \( \pi^{n_k} \) and \( H^{n_k} \), respectively, if

\[
\lim_{k \to \infty} \int_0^T dt \int_{D^d} du \sum_{v \in V} \tilde{v} \cdot \chi(\theta_v(\Lambda(\rho^{n_k}_{t_k},\mathbf{p}^{n_k}_{t_k}))) \sum_{1 \leq i \leq d} v_i \partial_{u_i} G(t,u)
\]

\[= \int_0^T dt \int_{D^d} du \sum_{v \in V} \tilde{v} \cdot \chi(\theta_v(\Lambda(\rho^0_{t_k},\mathbf{p}^0_{t_k}))) \sum_{1 \leq i \leq d} v_i \partial_{u_i} G(t,u),
\]

\[
\lim_{k \to \infty} \sum_{v \in V} \int_0^T dt \int_{D^d} du \chi(\theta_v(\Lambda(\rho^{n_k}_{t_k},\mathbf{p}^{n_k}_{t_k}))) [\tilde{v} \cdot \nabla H^{n_k}][\tilde{u} \cdot \nabla G]
\]

\[= \sum_{v \in V} \int_0^T dt \int_{D^d} du \chi(\theta_v(\Lambda(\rho^0_{t_k},\mathbf{p}^0_{t_k}))) [\tilde{v} \cdot \nabla H][\tilde{u} \cdot \nabla G].
\]

We prove the second claim, the first one being simpler. Note first that we can replace \( H^{n_k} \) by \( H \) in the previous limit, because \( \chi \) is bounded in \([0,1]\) by \( 1/4 \), and (5.3) holds. Now, \( (\rho^{n_k},\mathbf{p}^{n_k}) \) converges to \( (\rho^0,\mathbf{p}^0) \) weakly in \( L^2(\Omega_T) \) because \( \pi^{n_k} \) converges to \( \pi^0 \) in \( D([0,T],\mathcal{M}^0) \). Since \( I_T(\pi^n|\gamma) \) is uniformly bounded, by Corollary 4.3 and Lemma 4.5 \( (\rho^{n_k},\mathbf{p}^{n_k}) \) converges to \( (\rho^0,\mathbf{p}^0) \) strongly in \( L^2(\Omega_T) \) which implies (5.4). In particular, since (5.2) holds, by uniqueness of weak solutions of equation (4.9), \( \pi^n = \pi \) and we are done. □

6. Large deviations

We prove in this section Theorem 2.1 which is the dynamical large deviations principle for the empirical measure of boundary driven stochastic lattice gas model with many conserved quantities. The proof uses some of the ideas introduced in [9].

6.1. Superexponential estimates. It is well known that one of the main steps in the derivation of the upper bound is a super-exponential estimate which allows the replacement of local functions by functionals of the empirical density in the large deviations regime.

Let \( \kappa \) be as in the beginning of Section 4. Note that since \( \nu^N_\kappa \) is not the invariant state, there are no reasons for \( -N^2 \mathcal{L}_N f, f \rangle_{\nu^N_\kappa} \) to be positive. The next statement shows that this expression is almost positive.

For each function \( f: X_N \to \mathbb{R} \), let \( D_{\nu^N_\kappa}(f) \) be

\[
D_{\nu^N_\kappa}(f) = D^c_{\nu^N_\kappa}(f) + D^b_{\nu^N_\kappa}(f) + D^b_{\nu^N_\kappa} f, (f),
\]

where

\[
D^c_{\nu^N_\kappa}(f) = \sum_{v \in V} \sum_{x \in D_n^c} \sum_{x+z \in D_n^c} P_N(z-x,v) \int \left[ \sqrt{f(\eta_1,\ldots,\eta_N)} - \sqrt{f(\eta)} \right]^2 \nu^N_\kappa(\eta),
\]
\[D^c_{\nu_N}(f) = \sum_{q \in Q} \sum_{x \in D_N^d} \int p(x, q, \eta) \left[ \sqrt{f(\eta^x \eta)} - \sqrt{f(\eta)} \right]^2 \nu_N^\kappa(\eta),\]

and
\[D^h_{\nu_N}(f) = \sum_{v \in V} \sum_{x \in (N-1) \times \mathbb{T}_x^{d-1}} \int [\alpha_v(\tilde{x}/N)(1 - \eta(x, v)) + (1 - \alpha_v(\tilde{x}/N))\eta(x, v)] \times \left[ \sqrt{f(\sigma^x \eta)} - \sqrt{f(\eta)} \right]^2 \nu_N^\kappa(\eta) + \sum_{v \in V} \sum_{x \in (N-1) \times \mathbb{T}_x^{d-1}} \int [\beta_v(\tilde{x}/N)(1 - \eta(x, v)) + (1 - \beta_v(\tilde{x}/N))\eta(x, v)] \times \left[ \sqrt{f(\sigma^x \eta)} - \sqrt{f(\eta)} \right]^2 \nu_N^\kappa(\eta).

**Proposition 6.1.** There exist constants \(C_1 > 0\) and \(C_2 = C_2(\alpha, \beta) > 0\) such that for every density \(f\) with respect to \(\nu_N^\kappa\), then
\[< \mathcal{L}_N \sqrt{f}, \sqrt{f} >_{\nu_N^\kappa} \leq -C_1 D_{\nu_N^\kappa}(f) + C_2 N^{d-2}.\]

The proof of this proposition is elementary and is thus omitted.

Further, we may choose \(\kappa\) for which there exists a constant \(\theta > 0\) such that:

\[\kappa(u_1, \tilde{u}) = d(-1, \tilde{u}) \quad \text{if} \quad 0 \leq u_1 \leq \theta,\]
\[\kappa(u_1, \tilde{u}) = d(1, \tilde{u}) \quad \text{if} \quad 1 - \theta \leq u_1 \leq 1,\]

for all \(\tilde{u} \in \mathbb{T}_x^{d-1}\). In that case, for every \(N\) large enough, \(\nu_N^\kappa\) is reversible for the process with generator \(\mathcal{L}_N^h\) and then \((-N^2 \mathcal{L}_N^h f, f)_{\nu_N^\kappa}\) is positive.

Fix \(L \geq 1\) and a configuration \(\eta\), let \(I^L(x, \eta) := I^L(x) = (I_0^L(x), \ldots, I_L^k(x))\) be the average of the conserved quantities in a cube of the length \(L\) centered at \(x\):
\[I^L(x) = \frac{1}{|A_L|} \sum_{z \in x + A_L} I(\eta_z),\]

where, \(A_L = \{-L, \ldots, L\}^d\) and \(|A_L| = (2L + 1)^d\) is the discrete volume of box \(A_L\).

For each \(G \in C(\mathbb{T}_x^d) \times C(\mathbb{T}_x^d)^d\), and each \(\varepsilon > 0\), let
\[V_{N, \varepsilon}^{G, 1}(s, \eta) = \frac{1}{N^d} \sum_{k=0}^d \sum_{i, j=1}^d \sum_{x \in D_N^d} \partial_{u_i} G^k(s, x/N) \left[ \tau_z \tilde{V}_{N, \varepsilon}^{j, k}(\eta) \right] ,\]

where \(\tilde{V}_{N, \varepsilon}^{j, k}(\eta) = \frac{1}{(2L + 1)^d} \sum_{y \in A_N \times v \in V} \sum_{z \in \mathbb{Z}^d} p(z, v) \sum_{x \in D_N^d} v_j v_k \chi(\theta_v(G(\Lambda(I^0(0))))),\]

and let
\[V_{N, \varepsilon}^{G, 2}(s, \eta) = \frac{1}{2N^d} \sum_{v \in V} \sum_{x \in D_N^d} \sum_{i=1}^d \sum_{j, k=0}^d v_j v_k \partial_{u_i} G^k(x/N) \partial_{u_i} G^j(x/N) \times \left\{ \eta(x, v)[1 - \eta(x + e_i, v)] + \eta(x, v)[1 - \eta(x - e_i, v)] - 2\chi(\theta_v(\Lambda(I^0(0)))) \right\} .\]

Let, again, \(G : [0, T] \times \mathbb{T}_x^{d-1} \rightarrow \mathbb{R}^{d+1}\) be a continuous function, and consider the quantities
\[V_{N, \varepsilon}(s, \eta, G) = \frac{1}{N^{d-1}} \sum_{k=0}^d \sum_{x \in \mathbb{T}_x^{d-1}} G_k(s, \tilde{x}/N) \left( I_k(\eta(1, \tilde{x})(s)) - \sum_{v \in V} v_k \alpha_v(\tilde{x}/N) \right) ,\]
\[ V^+_N(s, \eta, G) = \frac{1}{N^{d-1}} \sum_{k=0}^{d} \sum_{\tilde{x} \in \mathbb{T}^{d-1}} G_k(s, \tilde{x}/N) \left( I_k(\eta_{N-1, \tilde{x}}(s)) - \sum_{v \in \mathcal{V}} \nu_k \beta_v(\tilde{x}/N) \right), \]

**Proposition 6.2.** Fix \( G \in C(\Omega_T) \times C(\mathbb{D}^d) \), \( H \) in \( C([0, T] \times \Gamma) \times C(\mathcal{G}) \), a cylinder function \( \Psi \) and a sequence \( \{\eta^N : N \geq 1\} \) of configurations with \( \eta^N \) in \( X_N \). For every \( \delta > 0 \),

\[
\lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N} \left[ \left| \int_0^T V_{N^j}(s, \eta) \, ds \right| > \delta \right] = -\infty,
\]

\[
\lim_{N \to \infty} \frac{1}{N^d} \mathbb{P}_{\eta^N} \left[ \left| \int_0^T V_{N^\pm}(s, \eta, G) \right| > \delta \right] = -\infty,
\]

for \( j = 1, 2 \).

The proof of the above proposition follows from Proposition 6.1, the replacement lemmas proved in [12], and the computation presented in [3], p. 78, for non reversible processes.

For each \( \varepsilon > 0 \) and \( \pi \in \mathcal{M}_+ \times \mathcal{M}_d \), for \( k = 0, \ldots, d \), denote by \( \Xi_\varepsilon(\pi^k) = \pi^k_\varepsilon \) the absolutely continuous measure obtained by smoothing the measure \( \pi_k \):

\[
\Xi_\varepsilon(\pi^k)(dx) = \pi^k_\varepsilon(dx) = \frac{1}{U_\varepsilon} \frac{\pi_k(\Lambda_\varepsilon(x))}{|\Lambda_\varepsilon'(x)|} \, dx,
\]

where \( \Lambda_\varepsilon(x) = \{y \in D^d : |y-x| \leq \varepsilon\}, |A| \) stands for the Lebesgue measure of the set \( A \), and \( \{U_\varepsilon : \varepsilon > 0\} \) is a strictly decreasing sequence converging to 1: \( U_\varepsilon > 1, U_\varepsilon > U_{\varepsilon'} \) for \( \varepsilon > \varepsilon' \), \( \lim_{\varepsilon \to 0} U_\varepsilon = 1 \). Let

\[
\pi^{N,\varepsilon} = \left( \Xi_\varepsilon(\pi^0_1), \Xi_\varepsilon(\pi^1_1), \ldots, \Xi_\varepsilon(\pi^d_1) \right).
\]

A simple computation shows that \( \pi^{N,\varepsilon} \) belongs to \( \mathcal{M}_d \) for \( N \) sufficiently large because \( U_\varepsilon > 1 \), and that for each continuous function \( H : D^d \to \mathbb{R}^{d+1} \),

\[
\langle \pi^{N,\varepsilon}, H \rangle = \frac{1}{N^d} \sum_{x \in D^d_N} H(x/N) \cdot \mathbf{1}^N(x) + O(N, \varepsilon),
\]

where \( O(N, \varepsilon) \) is absolutely bounded by \( C_0 \{N^{-1} + \varepsilon\} \) for some finite constant \( C_0 \) depending only on \( H \).

For each \( H \) in \( C_0^{1,2}(\Omega_T) \times C_0^2(\mathbb{D}^d) \) consider the exponential martingale \( M^H_t \) defined by

\[
M^H_t = \exp \left\{ N^d \left[ \langle \pi^0_t, H_t \rangle - \langle \pi^0_0, H_0 \rangle \right. \right.
\]

\[
- \frac{1}{N^d} \int_0^t e^{-N^d(\pi^0_t, H_t)} \left( \partial_s + N^2 \mathcal{L}_N \right) e^{N^d(\pi^0_t, H_t)} \, ds \left. \right\}.
\]

Recall from subsection 2.2 the definition of the functional \( \tilde{J}_H \). An elementary computation shows that

\[
M^H_T = \exp \left\{ N^d \left[ \tilde{J}_H(\pi^{N,\varepsilon}) + \mathcal{V}^H_{N,\varepsilon} + c^1_H(\varepsilon) + c^2_H(\gamma, H) \right] \right\}.
\]

(6.1)

In this formula,

\[
\mathcal{V}^H_{N,\varepsilon} = - \int_0^T V_{N^1}(s, \eta) \, ds - \sum_{i=1}^d \int_0^T V_{N^2}(s, \eta) \, ds
\]

\[
+ V^+_N(s, \eta, \partial_u H) - V^-_N(s, \eta, \partial_u H) + \langle \pi^0_0, H_0 \rangle - \langle \gamma, H_0 \rangle;
\]

and \( c^j_H : \mathbb{R}_+ \to \mathbb{R}, j = 1, 2 \), are functions depending only on \( H \) such that \( c^j_H(\varepsilon) \) converges to 0 as \( \varepsilon \downarrow 0 \). In particular, the martingale \( M^H_T \) is bounded by \( \exp \{ C(H, T) N^d \} \) for some finite constant \( C(H, T) \) depending only on \( H \) and \( T \). Therefore, Proposition 6.2 holds for \( \mathbb{P}_{\eta^N} = \mathbb{P}_{\eta^N} M^H_T \) in place of \( \mathbb{P}_{\eta^N} \).
6.2. **Energy estimates.** To exclude paths with infinite energy in the large deviations regime, we need an energy estimate. We state first the following technical result.

**Lemma 6.3.** There exists a finite constant $C_0$, depending on $T$, such that for every $G$ in $C^\infty_c(\Omega_T)$, every integer $1 \leq i \leq d$, $0 \leq k \leq d$, and every sequence $\{\eta^N : N \geq 1\}$ of configurations with $\eta^N$ in $X_N$,

$$
\lim_{N \to \infty} \frac{1}{N^d} \log \mathbb{E}_{\eta^N} \left[ \exp \left\{ N^d \int_0^T \langle \pi_t^{N,k}, \partial_u G \rangle \right\} \right] \leq C_0 \left\{ 1 + \int_0^T \|G_t\|_2^2 \, dt \right\}.
$$

The proof of this proposition follows from Lemma 3.8 in [12], and the fact that $d\eta_t^N/d\nu_t^N \leq CN^d$, for some positive constant $C = C(\kappa)$.

For each $G$ in $C^\infty_c(\Omega_T)$ and each integer $1 \leq i \leq d$, let $\bar{Q}^{G}_{i,k} : D([0,T], \mathcal{M}_+ \times \mathcal{M}^d) \to \mathbb{R}$ be the function given by

$$
\bar{Q}^{G}_{i,k}(\pi) = \int_0^T dt \langle \pi_t^k, \partial_u G_t \rangle - C_0 \int_0^T dt \int_{D^d} du G(t,u)^2.
$$

Notice that

$$
sup_{G \in C^\infty_c(\Omega_T)} \left\{ \bar{Q}^{G}_{i,k}(\pi) \right\} = \frac{\bar{Q}^{G}_{i,k}(\pi)}{4C_0}.
$$

Fix a sequence $\{G_r : r \geq 1\}$ of smooth functions dense in $L^2([0,T], H^1(D^d))$. For any positive integers $m,l$, let

$$
B^k_{m,l} = \left\{ \pi \in D([0,T], \mathcal{M}_+ \times \mathcal{M}^d) : \max_{1 \leq j \leq m, 1 \leq i \leq d} \bar{Q}^{G}_{i,j} \leq l \right\}.
$$

Since, for fixed $G$ in $C^\infty_c(\Omega_T)$ and $1 \leq i \leq d$ integer, the function $\bar{Q}^{G}_{i,k}$ is continuous, $B_{m,l}$ is a closed subset of $D([0,T], \mathcal{M})$.

**Lemma 6.4.** There exists a finite constant $C_0$, depending on $T$, such that for any positive integers $r,l$ and any sequence $\{\eta^N : N \geq 1\}$ of configurations with $\eta^N$ in $X_N$,

$$
\lim_{N \to \infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N} \left[ (B^k_{m,l})^c \right] \leq -l + C_0,
$$

where $k = 0, \ldots, d$.

**Proof.** For integers $1 \leq k \leq r$ and $1 \leq i \leq d$, by Chebychev inequality and by Lemma 6.3,

$$
\lim_{N \to \infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N} \left[ \bar{Q}^{G}_{i,k} > l \right] \leq -l + C_0.
$$

Hence, from

$$
\lim_{N \to \infty} \frac{1}{N^d} \log (a_N + b_N) \leq \max \left\{ \lim_{N \to \infty} \frac{1}{N^d} \log a_N, \lim_{N \to \infty} \frac{1}{N^d} \log b_N \right\},
$$

we obtain the desired inequality. \(\square\)

**Lemma 6.5.** There exists a finite constant $C_0$, depending on $T$, such that for every $G$ in $C^\infty_c(\Omega_T) \times [C^\infty_c(D^d)]^d$, and every sequence $\{\eta^N : N \geq 1\}$ of configurations with $\eta^N$ in $X_N$,

$$
\lim_{N \to \infty} \frac{1}{N^d} \log \mathbb{E}_{\eta^N} \left[ \exp \left\{ N^d \sum_{i=1}^d \sum_{k=0}^d dt \langle \pi_t^N, \partial_u G^k \rangle \right\} \right] \leq C_0 \left\{ 1 + \int_0^T \|G_t\|_2^2 \, dt \right\}.
$$

In particular, we have that if $(\rho, \phi)$ is the solution of (3.2), then

$$
sup_{G \in C^\infty_c(\Omega_T)} \left\{ \sum_{i=1}^d \int_0^T ds \int_{D^d} du \partial_x (\rho, \phi) \cdot \partial_x G - \int_0^T dt \int_{D^d} du \partial_x (\phi_\tau(L(\rho, \phi))) \partial \cdot \nabla G \right\}_2,
$$

is finite, and vanishes if $T \to 0$.  

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Proof. Applying Feynman-Kac’s formula and using the same arguments of Lemma 3.3 in [12], we have that
\[
\frac{1}{N^d} \log E_{\nu_N^d} \left[ \exp \left\{ N \int_0^T ds \sum_{i=1}^{d} \sum_{k=0}^{d} (I_k(\eta^d(s)) - I_k(\eta^d-e_i(s))) \partial_u G^k(s, x/N) \right\} \right]
\]
is bounded above by
\[
\frac{1}{N^d} \int_0^T \lambda_N^d ds,
\]
where \( \lambda_N^d \) is equal to
\[
\sup \left\{ \left( N \sum_{i,k} \sum_{x \in D_N^d} (I_k(\eta^d(x)) - I_k(\eta^d(x-e_i))) \partial_u G^k(s, x/N), f \right)_{\nu_N^d} \right\} + N^2 < \mathcal{L}_N \sqrt{f}, \sqrt{f} > \nu_N^d \}
\]
where the supremum is taken over all densities \( f \) with respect to \( \nu_N^d \). By Proposition 6.1, the expression inside brackets is bounded above by
\[
CN^d - \frac{N^2}{2} D_{nu_N^d}(f) + \sum_{i,k} \sum_{x \in D_N^d} \left\{ N \partial_u G^k(s, x/N) \int \left[ I_k(\eta^d) - I_k(\eta^d-e_i) \right] f(\eta) \nu_N^d(d\eta) \right\}.
\]
We now rewrite the term inside the brackets as
\[
\sum_{v \in V} \sum_{i=1}^{d} \sum_{x \in D_N^d} \left\{ \int N(\tilde{v} \cdot \partial_u G(s, x/N)) [\eta(x, v) - \eta(x - e_i, v)] f(\eta) \nu_N^d(d\eta) \right\}.
\]
Writing \( \eta(x, v) - \eta(x - e_i, v) = \eta(x, v)[1 - \eta(x - e_i, v)] - \eta(x - e_i, v)[1 - \eta(x, v)] \), and applying the same arguments in Lemma 3.8 of [12], we obtain that
\[
N(\tilde{v} \cdot \partial_u G(s, x/N)) \int [\eta(x, v) - \eta(x - e_i, v)] f(\eta) \nu_N^d(d\eta)
\]
\[
\leq (\tilde{v} \cdot \partial_u G(s, x/N))^2 \int [\eta(x, v)[1 - \eta(x - e_i, v)] f(\eta^d-e_i,x,v) d\nu_N^d
\]
\[
+ \frac{1}{4} \int f(\eta^d-e_i,x,v) \left[ \left( 1 - \frac{\gamma_{x-e_i,v}}{\gamma_{x,v}} \right)^2 \right] \nu_N^d(d\eta)
\]
\[
+ N^2 \left\{ \frac{1}{2} [\sqrt{f(\eta^d-e_i,x,v)} - \sqrt{f(\eta)}]^2 \nu_N^d(d\eta) \right\}
\]
\[
+ 2(\tilde{v} \cdot \partial_u G(s, x/N))^2 \int [\eta(x, v)[1 - \eta(x - e_i, v)] f(\eta) + f(\eta^d-e_i,x,v)]^2 \nu_N^d(d\eta),
\]
we have that \((\sqrt{f(\eta)} + \sqrt{f(\eta^d-e_i,x,v)})^2 \leq 2(f(\eta) + f(\eta^d-e_i,x,v))\). An application of the replacement lemma (Lemma 3.7 in [12]) concludes the proof. \( \square \)

6.3. Upper Bound. Fix a sequence \( \{F_j : j \geq 1\} \) of smooth functions dense in \( C(\overline{D^d}) \) for the uniform topology, with positive coordinates. For \( j \geq 1 \) and \( \delta > 0 \), let
\[
D_{j,\delta} = \left\{ \pi \in D([0, T], \mathcal{M}_+ \times \mathcal{M}^d) : |\langle \pi^k_t, F_j \rangle| \leq \tilde{v}^k|\pi| \int_{D^d} F_j(x) dx + C_j \delta, k = 0, \ldots, d, 0 \leq t \leq T \right\},
\]
where \( \tilde{v}^0 = 0 \) and \( \tilde{v}^k = \tilde{v}, C_j = ||\nabla F_j||_\infty \) and \( \nabla F \) is the gradient of \( F \). Clearly, the set \( D_{j,\delta}, j \geq 1, \delta > 0 \), is a closed subset of \( D([0, T], \mathcal{M}_+ \times \mathcal{M}^d) \). Moreover, if
\[
E_{m,\delta} = \bigcap_{j=1}^{m} D_{j,\delta},
\]
we have that \( D([0, T], \mathcal{M}^d) = \cap_{n \geq 1} \cap_{m \geq 1} E_{m,1/n} \). Note, finally, that for all \( m \geq 1, \delta > 0, \)
\[
\pi_N^{s,\delta} \text{ belongs to } E_{m,\delta} \text{ for } N \text{ sufficiently large.} \tag{6.4}
\]
Fix a sequence of configurations \( \{ \eta^n : N \geq 1 \} \) with \( \eta^n \) in \( X_N \) and such that \( \pi^n(\eta^n) \) converges to \( \gamma(u)du \) in \( M \). Let \( A \) be a subset of \( D([0,T], \mathcal{M}_+ \times \mathcal{M}^d) \),

\[
\frac{1}{N^d} \log \mathbb{P}_{\eta^n}[\pi^n \in A] = \frac{1}{N^d} \log \mathbb{E}_{\eta^n}[M_H^T (M_H^T)^{-1} \mathbf{1}\{\pi^n \in A\}] .
\]

Maximizing over \( \pi^n \) in \( A \), we get from (6.1) that the last term is bounded above by

\[
- \inf_{\pi \in A} \hat{J}_H(\pi^+) + \frac{1}{N^d} \log \mathbb{E}_{\eta^n}[M_H^T e^{-N^d \psi_{N,\varepsilon}}] - c_H^j(\varepsilon) - c_H^2(N^{-1}) .
\]

Since \( \pi^n(\eta^n) \) converges to \( \gamma(u)du \) in \( M \) and since Proposition 6.2 holds for \( \mathbb{P}_{\eta^n} = \mathbb{P}_{\eta^n} M_H^T \) in place of \( \mathbb{P}_{\eta^n} \), the second term of the previous expression is bounded above by some \( C_H(\varepsilon, N) \) such that

\[
\lim_{\varepsilon \to 0} \lim_{N \to \infty} C_H(\varepsilon, N) = 0 .
\]

Hence, for every \( \varepsilon > 0 \), and every \( H \) in \( C^1_0(\overline{D_T}) \times [C^2_0(\overline{D_T})]^d \),

\[
\lim_{N \to \infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^n}[A] \leq - \inf_{\pi \in A} \hat{J}_H(\pi^+) + C_H^j(\varepsilon) ,
\]

where \( \lim_{\varepsilon \to 0} C_H^j(\varepsilon) = 0 \). Let

\[
B_{r,l} = \left\{ \pi \in D([0,T], \mathcal{M}_+ \times \mathcal{M}^d) : \max_{1 \leq j \leq r} \sum_{1 \leq i \leq d} \hat{Q}_{i,j}^1(\pi) \leq l \right\} ,
\]

and, for each \( H \in C^1_0(\overline{D_T}) \times [C^2_0(\overline{D_T})]^d \), each \( \varepsilon > 0 \) and any \( r, l, m, n \in \mathbb{Z}_+ \), let \( J_{H,\varepsilon}^{r,l,m,n} : D([0,T], \mathcal{M}_+ \times \mathcal{M}^d) \to \mathbb{R} \cup \{ \infty \} \) be the functional given by

\[
J_{H,\varepsilon}^{r,l,m,n}(\pi) = \begin{cases} J_H(\pi^+) & \text{if } \pi \in B_{r,l} \cap E_{m,1/n} , \\ +\infty & \text{otherwise} . \end{cases}
\]

This functional is lower semicontinuous because so is \( J_H \circ \Xi_\varepsilon \) and because \( B_{r,l} \), \( E_{m,1/n} \) are closed subsets of \( D([0,T], \mathcal{M}_+ \times \mathcal{M}^d) \).

Let \( O \) be an open subset of \( D([0,T], \mathcal{M}_+ \times \mathcal{M}^d) \). By Lemma 6.3, 6.3, and 6.5,

\[
\lim_{N \to \infty} \frac{1}{N^d} \log Q_{\eta^n}[O] \leq \max \left\{ \lim_{N \to \infty} \frac{1}{N^d} \log Q_{\eta^n}[O \cap B_{r,l} \cap E_{m,1/n}] , \right. \\
\left. \lim_{N \to \infty} \frac{1}{N^d} \log Q_{\eta^n}[(B_{r,l})^c] \right\} \\
\leq \max \left\{ - \inf_{\pi \in O \cap B_{r,l} \cap E_{m,1/n}} J_H(\pi^+) + C_H^j(\varepsilon) , -l + C_0 \right\} \\
= - \inf_{\pi \in O} L_{H,\varepsilon}^{r,l,m,n}(\pi) ,
\]

where

\[
L_{H,\varepsilon}^{r,l,m,n}(\pi) = \min \left\{ J_{H,\varepsilon}^{r,l,m,n}(\pi) - C_H^j(\varepsilon) , l - C_0 \right\} .
\]

In particular,

\[
\lim_{N \to \infty} \frac{1}{N^d} \log Q_{\eta^n}[O] \leq - \sup_{H,\varepsilon,r,l,m,n} \inf_{\pi \in O} L_{H,\varepsilon}^{r,l,m,n}(\pi) .
\]

Note that, for each \( H \in C^1_0(\overline{D_T}) \times [C^2_0(\overline{D_T})]^d \), each \( \varepsilon > 0 \) and \( r, l, m, n \in \mathbb{Z}_+ \), the functional \( L_{H,\varepsilon}^{r,l,m,n} \) is lower semicontinuous. Then, by Lemma A2.3.3 in [10], for each compact subset \( K \) of \( D([0,T], \mathcal{M}) \),

\[
\lim_{N \to \infty} \frac{1}{N^d} \log Q_{\eta^n}[K] \leq - \inf_{\pi \in K} \sup_{H,\varepsilon,r,l,m,n} L_{H,\varepsilon}^{r,l,m,n}(\pi) .
\]
By (6.2) and since $D([0,T],\mathcal{M}^0) = \cap_{n \geq 1} \cap_{m \geq 1} E_{m,1/n}$,
\[
\lim_{\varepsilon \to 0} \lim_{l \to \infty} \lim_{r \to \infty} \lim_{m \to \infty} \lim_{n \to \infty} I_{H,\varepsilon}^{r,l,m,n}(\pi) = \begin{cases} 
\hat{J}_H(\pi) & \text{if } Q(\pi) < \infty \text{ and } \pi \in D([0,T],\mathcal{M}^0), \\
+\infty & \text{otherwise}.
\end{cases}
\]

This result and the last inequality imply the upper bound for compact sets because $\hat{J}_H$ and $J_H$ coincide on $D([0,T],\mathcal{M}^0)$. To pass from compact sets to closed sets, we have to obtain exponential tightness for the sequence $\{Q_{\eta^n}\}$. This means that there exists a sequence of compact sets $\{K_n : n \geq 1\}$ in $D([0,T],\mathcal{M})$ such that
\[
\lim_{N \to \infty} \frac{1}{N^d} \log Q_{\eta^n}(K_n^c) \leq -n.
\]
The proof presented in [2] for the non-interacting zero range process is easily adapted to our context.

6.4. **Lower Bound.** The proof of the lower bound is similar to the one in the convex periodic case. We just sketch it and refer to [10], Section 10.5. Fix a path $\pi$ in $\Pi$ and let $H \in C_0^{\infty}(\Omega_T)$ be such that $\pi$ is the weak solution of equation (4.9). Recall from the previous section the definition of the martingale $M^H_\eta$ and denote by $P^H_\eta$ the probability measure on $D([0,T],X_N)$ given by $P^H_\eta[A] = P_{\eta^n}[M^H_\eta A]$. Under $P^H_\eta$ and for each $0 \leq t \leq T$, the empirical measure $\pi_t^n$ converges in probability to $\pi_t$. Further, we have
\[
\lim_{N \to \infty} \frac{1}{N^d} H \left( P^H_\eta \left[ \frac{P^H_\eta}{P^H_\eta} \right] \right) = I_T(\pi|\gamma),
\]
where $H(\mu|\nu)$ stands for the relative entropy of $\mu$ with respect to $\nu$. From these two results we can obtain that for every open set $O \subset D([0,T],\mathcal{M}_+ \times \mathcal{M}^d)$ which contains $\pi$,
\[
\lim_{N \to \infty} \frac{1}{N^d} \log P_{\eta^n}[O] \geq -I_T(\pi|\gamma).
\]
The lower bound follows from this and the $I_T(\cdot|\gamma)$-density of $\Pi$ established in Theorem 5.5.

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References

[1] Beltrán, J. and Landim, C. A lattice gas model for the incompressible Navier-Stokes equation, *Ann. Inst. H. Poincaré, Probab. Statist.*, 44, 886-914.
[2] Benois, O. : *Large deviations for the occupation times of independent particle systems*, Ann. Appl. Probab. 6, 269–296 (1996).
[3] Benois, O., Kipnis, C. and Landim, C. : *Large deviations from the hydrodynamical limit of mean zero asymmetric zero range processes*, Stochastic Process. Appl. 55, 65–89 (1995).
[4] Bernardin, C. Stationary nonequilibrium properties for a heat conduction model. *Physical Review E*, 78, 021134, 2008.
[5] Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C. Large deviation approach to non equilibrium processes in stochastic lattice gases. *Bull. Braz. Math. Soc.*, 37, 611-643, 2006.
[6] Bertini, L., De Sole, A., Gabrielli, G., Jona-Lasinio, G. and Landim, C. Stochastic interacting particle systems out of equilibrium. *J. Stat. Mech. Theory Exp.*, (7):P07014, 35pp (electronic), 2007.
[7] Derrida, B., Non equilibrium steady states: fluctuations and large deviations of the density and of the current. *J. Stat. Mech. Theory Exp.*, P07023, 2007.
[8] Esposito, R., Marra, R., Yau, H. T. *Navier-Stokes Equations for Stochastic Particle Systems on the Lattice*. *Comm. Math. Phys.*, 182, 395-456, 1996.
[9] Farfan, J., Landim, C. and Mourragui, M. Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion. Preprint. Available at [http://arxiv.org/abs/0903.5526](http://arxiv.org/abs/0903.5526).
[10] Kipnis, C. and Landim, C. *Scaling limits of interacting particle systems*, Grundlehren Math. Wiss. 320, Springer-Verlag, Berlin, 1999.
[11] Quastel, J. and Yau, H. T. *Lattice Gases, Large Deviations, and the Incompressible Navier-Stokes Equations*. *Annals of Mathematics*, 148, 51-108, 1998.
[12] Simas, A.B. Hydrodynamic limit for a boundary driven stochastic lattice gas model with many conserved quantities. Preprint. (2009)
[13] Simon, J. *Compact Sets in the Space $L^p(0,T;B)$*. Annali di Matematica pura ed applicata, (IV), Vol. CXLVI, 65-96 (1987).
[14] Spohn, H. Long range correlations for stochastic lattice gases in a non-equilibrium steady state, J. Stat. Phys. A:Math. Gen., 16, 4275-4291, 1983.

[15] E. Zeidler, Nonlinear Functional Analysis and its Applications. II/A, Linear Monotone Operators and II/B Nonlinear Monotone Operators. Springer-Verlag, Berlin. (1980).

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