Electromagnetic Modeling of Superconductors With Commercial Software: Possibilities With Two Vector Potential-Based Formulations

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Abstract—In recent years, the $H$ formulation of Maxwell’s equations has become the de facto standard for simulating the time-dependent electromagnetic behavior of superconducting applications with commercial software. However, there are cases where other formulations are desirable, for example for modeling superconducting turns in electrical machines or situations where the superconductor is better described by the critical state than by a power-law resistivity. In order to accurately and efficiently handle these situations, here we consider two approaches based on the magnetic vector potential: the $T$-$A$ formulation of Maxwell’s equations (with power-law resistivity) and the implementation of a quasi critical state model (QCSM) with a steep $E$-$J$ relationship limited at $J_c$. In this article, we extend the $T$-$A$ formulation to thick conductors so that large coils with different coupling scenarios between the turns can be considered. We also discuss the QCSM in terms of its ability to calculate ac losses; in particular, we investigate the dependence of the calculated ac losses on the frequency of the ac excitation and the possibility of using quick one-step (instead of full-cycle) simulations to calculate the ac losses.

Index Terms—AC losses, critical state model, finite-element method (FEM), high-temperature superconductor (HTS) machines, $T$-$A$-formulation.

I. INTRODUCTION

NUMERICAL models have become popular tools for understanding the behavior of superconductors and for designing applications. Among the models used for investigating the electromagnetic behavior of superconductors, the finite-element method (FEM) based on the $H$ formulation of Maxwell’s equations combined with the power-law model of the superconductor is by far the most widely adopted approach, used by tens of research groups around the world [1], [2]. The reason of such popularity mainly resides in the easiness of implementation in the FEM program COMSOL Multiphysics [3], [4], although implementations in other commercial software packages like FlexPDE [5] and MATLAB [6], open-source environments like GetDP [7], [8], and home-made FEM codes like Daryl Maxwell [9] also exist.

In this article, we discuss two approaches based on the magnetic vector potential, which—for different reasons—can be considered as an alternative to the $H$ formulation for some application contexts.

The $T$-$A$ formulation, proposed by Zhang et al. in [10], is becoming a popular tool for solving electromagnetic problems involving high-temperature superconductor (HTS) coated conductors which can be treated as infinitely thin objects [11]–[14]. This model too uses the power law as constitutive relation of the superconductor. Here, we extend the formulation to thick superconductors: not only does this allow simulating other types of wires for which the superconductor cannot be approximated as an infinitely thin object (like Bi-2223 or MgB$_2$ flat rectangular tapes), but—perhaps more importantly—it also allows simulating stacks of electromagnetically coupled coated conductors, which are often used in high-current HTS cables [15]–[17]. In stacks of coated conductors, the superconducting layers of the various tapes are electromagnetically coupled and the whole stack can be assimilated to a thick superconductor. The main advantage of this formulation is that it can be directly used to simulate HTS in electrical machines, if these are modeled with a formulation based on the magnetic vector potential $A$ [18].

Numerical formulations using the vector potential $A$ can also be combined with different (from the power law) constitutive models of the superconductor. In this article, a smooth $E$-$J$ characteristics, with $J$ limited to $\pm J_c$, is used for obtaining a fast solution in terms of vector potential $A$ by solving a backward sequence of nonlinear magnetostatic problems. We show that, at least in certain cases, a one-step calculation of the field distribution corresponding to the peak of the ac excitation...
The formulation is passed to the \( A \) formulation as an external current density source in (1). In this way, the electromagnetic interaction between multiple tapes can be calculated.

For the 2-D problems considered here (Fig. 1), (2) becomes

\[
J_z = \frac{\partial T_y}{\partial x} - \frac{\partial T_x}{\partial y}.
\]

In the original article on the \( T-A \) formulation [10], the superconductors were considered as infinitely thin objects, which led to a simplification of the governing equations, because \( T \) had only one component.

In the \( T-A \) formulation, the current \( I \) flowing in a conductor of cross section \( S \) is given by

\[
I = \iint_{\Omega} J \, d\Omega = \iiint_{\Omega} \nabla \times T \, d\Omega = \oint_{\partial \Omega} T \, ds
\]

where \( \partial \Omega \) represents the boundary edges of the cross section \( \Omega \). In thin and thick superconductors, there are different ways to impose such condition. In thin conductors, the current is imposed by setting appropriate (0D) boundary conditions at the extremities of each tape, as explained in [10]. In thick conductors of rectangular cross section, the current vector potential \( T \) has two components, \( T_z \) and \( T_y \), and the desired transport current can be obtained by using different sets of conditions on the superconductor’s boundary for those two components. Four examples are represented in Fig. 1. One can easily verify that they are all consistent with (5) for imposing a current \( I \) of desired amplitude. This way of implementing (5) takes advantage of the rectangular geometry: for example the projection of \( T_y \) onto the top and bottom boundaries of the rectangle (which are parallel to the \( x \)-direction) is automatically zero. For more general shapes of the cross section, the implementation of (5) is less straightforward, and the details are given in the appendix. The \( T-A \) formulation is implemented in COMSOL Multiphysics, by using the PDE-coefficient form module for the \( T \) part and the magnetic fields module for the \( A \) part, respectively. The \( T \) and \( A \) parts use Lagrange first and second order elements, respectively. A discussion on the use of elements of different order can be found in the appendix of [14].

The superconductor is modeled as a material with power-law resistivity

\[
\rho(J) = \frac{E_c}{J_c} \left( \frac{|J|}{J_c} \right)^{n-1}
\]

where \( E_c \) is the critical electric field, \( J_c \) is the critical current density, and \( n \) the power-law exponent defining the steepness of the \( E-J \) curve.

B. \( A \)-Formulation With QCSM

The quasi critical state model (QCSM) solves the equation

\[
\nabla^2 A = -\mu_0 J
\]

where \( J \) takes values approximating the transition between \( +J_c \) and \( -J_c \) or zero. In the version discussed here (implemented in
COMSOL Multiphysics), we model this transition as

\[ J = J_c \text{erf} \left( \frac{E}{E_0} \right) \]  

(8)

where \( \text{erf} \) is the error function [19] and \( E_0 \) is a parameter defining the steepness of the switch between \( +J_c \) and \( -J_c \) (or from 0 to \( \pm J_c \) for steep points). In this work, unless specified otherwise, we used \( E_0 = 1 \times 10^9 \) V m\(^{-1}\). Instead of \( \text{erf} \), other functions based on exponentials [20] or hyperbolic tangent [21] can be used to smooth the transition between \( \pm J_c \). The model is defined here as a QCSM because it uses a smooth \( E-J \) characteristic—(8)—, which results in a different behavior than that of a “pure” critical state model (prescribing a sharp shift to \( \pm J_c \) produced by a nonzero electric field regardless of its magnitude and rate of variation), as it will be illustrated later.

In addition, since we consider 2-D problems, the magnetic vector potential has only one component (along \( z \) in Fig. 1), and from now on it will be treated as a scalar. The electric field driving the current originates from the time-variation of the magnetic vector potential \( A \) plus a voltage gradient [20], [21]. In the present article, we consider a single isolated tape and the voltage gradient term can be mostly ignored although it will still influence the boundary conditions. In this way, we are assuming Weyl’s gauge, where \( A = A_c + \nabla \int d\phi \), being \( A_c \) and \( \phi \) the vector potential in Coulomb’s gauge and the electrostatic scalar potential, respectively. With this gauge

\[ E = -\frac{\partial A}{\partial t} \approx \frac{A_{r+\Delta t} - A_t}{\Delta t} . \]  

(9)

By substituting (9) and (8) in (7), we finally obtain

\[ \nabla^2 A(t + \Delta t) = -\mu_0 J_c \text{erf} \left( \frac{A_{r+\Delta t} - A_t}{E_0 \Delta t} \right) . \]  

(10)

This equation, which corresponds to the backward Euler solution of nonlinear and time-dependent problem, allows solving the time evolution of \( A \) by simulating a series of static problems (one for each time step). An external magnetic field or a transport current is imposed by setting the appropriate conditions for the magnetic vector potential on the boundary of the air domain surrounding the superconductor. For example, in 2-D cartesian coordinates, a boundary condition

\[ A = B_0 (-x \cos \theta + y \sin \theta) \sin(\omega t) \]  

(11)

generates a magnetic field of amplitude \( B_0 \), angle \( \theta \) with respect to the \( y \) axis, and sinusoidal time dependence with angular frequency \( \omega \). A boundary condition

\[ A = A_0 \sin(\omega t) \]  

(12)

where \( A_0 \) is a constant, generates a periodic transport current in the superconductor. The value of the current can be calculated \( a \) \( \text{posteriori} \) in the postprocessing, by integrating \( J \) over the superconductor’s cross section at the peak of the current.

In [20], it was shown that, in the case of an ac excitation, the superconductor’s cyclic losses could be simply computed by knowing the current density \( J_p \) and the magnetic vector potential \( A_p \) at the peak of the excitation as

\[ Q = -4 \int_{\Omega} J_p A_0 d\Omega \]  

(13)

where \( \Omega \) is the superconductor’s domain.

This expression was also mentioned in [22]–[24]. However, as pointed out in Section 2.5 of [24] and in Section II.C.2 of [25], its applicability for computing the cyclic ac losses is limited to certain conditions. First, we use Weyl’s gauge, where \( A = A_c + \nabla \int d\phi \). Using this relation, we can see that (13) is equivalent to (20) in [25] for Coulomb’s (or any other) gauge. Second, this equation assumes that at each half cycle the current density fronts penetrate monotonically from all external surfaces inwards, and hence the region with \( J = +J_c \) grows toward that of \( J = -J_c \), and vice versa. It is also necessary that at the initial stage the current fronts penetrate only toward the current-free kernel, where \( A \) vanishes in Weyl’s gauge. This gauge is satisfied because, first, \( J = 0 \) causes \( E = 0 \), and hence \( \partial A/\partial t = 0 \) and, second, \( A \) is zero initially and \( \partial A/\partial t = 0 \) follows from the beginning of the curve, so that \( A \) remains null. As the field increases from zero to the peak, the current density of the points of the superconductor for which \( J = \pm J_c \) never changes (until when the field or the current is reversed). Examples of scenarios when this is not the case are combinations of simultaneous alternating transport current and magnetic field [24] or the magnetization of a superconductor of elliptical cross section with inclined field [26]. In the latter case, the problem stems from the fact that, while the increase of the field from zero to the peak is monotonic, the evolution of the current density in the superconductor is not: in other words, due to the deformation of the field lines inside the superconductor as the field is increased, some points inside the superconductor may switch between \( +J_c \) and \( -J_c \) (or vice versa) during the field ramp from zero to the peak value. In Section III-B, we will verify this and try to assess the magnitude of the error committed by the one-step calculation and (13) for calculating the cyclic ac losses.

C. Other Models Used for Comparison

The \( T-A \) formulation with the power law and the \( A \) formulation with the QCSM are validated with a comparison with other models: the minimum electro-magnetic entropy production (MEMEP) model and two “pure” critical state models, respectively. This subsection quickly summarizes these models used for comparison.

The MEMEP model uses the current density as state variable, avoiding meshing the air. Differently from integral methods, it solves \( J \) by minimizing a certain functional [27], [28]. This method can take any \( E(J) \) relation into account, including the multivalued relation of the critical state model (CSM) [28]. However, in this article we use the power law \( E(J) \) relation defined by the resistivity in (6).

As for the “pure” CSM, a sharp shift to \( \pm J_c \) is produced by a nonzero electric field regardless of its magnitude and rate of variation. This means that only the sign of the electric field rather than its magnitude determines the electrodynamics of the
ratios are obtained by changing $E$.

The minimum magnetic energy variation (MMEV) method proposed in [24] is based on the variational principle in [30] for infinitely long geometries and assuming the general critical state model, where $|J| < J_c$ if $E = 0$ and $|J| = J_c$ if $|E| > 0$. This multivalued relation corresponds to the power law $E(J)$ characteristics with the resistivity in (6) with the limit of $n \to \infty$. The current density is found by minimizing a certain functional, which also contains $A$ in Coulomb’s gauge. As shown in [28] for any $n$ (including $n \to \infty$), finding $J(x, y)$ that minimizes this functional is the same as solving $E = -\partial A/\partial t - \nabla \phi$, being $\phi$ the electrostatic potential. As well, this minimum is unique and always exists [28].

We emphasize that, in order to obey the pure critical state model, the VIEM imposes the constraint $|J| = \{0, J_c\}$. The MMEV method allows having subcritical current density values, but—differently from the QCSM—their are corresponding to an electric field equal to zero. We also emphasize that (8) represents a good mathematical representation of the CSM as far as the problem is dominated by a sufficiently high electric field, arising from an intense time derivative of excitation [related to boundary condition (11)] due to high frequency and/or high magnitude. This means that in these operating conditions the results of the QCSM and the pure CSM coincide. However, in the low electric field regime (which can occur when a low frequency or a small ripple current is considered), the two models differ, as it will become clear from the results shown in Section III-B. (including

III. RESULTS

A. T-A Formulation: Validation and Application to Electrical Machines

As mentioned in Section II-A, the T-A formulation for thick conductors can be used for simulating not only individual conductors with rectangular cross section, but also stacks of coupled coated conductors. Here, we present the validation of the model for the latter case, in particular for the stand-alone racetrack coil considered in [18] and [31]. The coil is made of four cable turns, each made of 13 tape turns, which can be considered as electrically insulated or in electrical contact. For brevity, the two situations are referred to as “uncoupled” and “coupled,” respectively. Fig. 2 shows the transport ac loss (at 500 Hz) of such coils as a function of the normalized critical current. The transport current of each cable (each made of 13 turns) is 2248.6 A. Different $I/I_c$ ratios are obtained by changing $I_c$. The figure presents a comparison between different models for the coupled and uncoupled case: the T-A formulation, the $H$ formulation and the MEMEP method. With the T-A formulation, the uncoupled case is given by the original 1-D model developed in [10], [11], whereas the coupled case is given by the approach presented in Section II-A, with each cable meshed with 60 × 30 elements. With the $H$ formulation and the MEMEP method, the tapes are simulated as individual objects (mesh $60 \times 1$), with different constraints on the current for the two coupling scenarios.

The results are in very good agreement with each other: the difference between the models is in the range of only a few %, with the exception of a few points at very low current ratios for the coupled case. The difference at low currents is due to the fact that the T-A formulation computes coupled tapes as large rectangular bulks (see Fig. 1), whereas the $H$ formulation and MEMEP simulate individual tapes, and that at low currents ratios the current penetrates very little inside superconductors.1

Due to the nonuniform current distribution among the tapes, the ac losses of the coupled case are about twice as high as those of the uncoupled case.

The T-A formulation can be used to study the difference between the two coupling scenarios in electrical machines. As an example, Fig. 3 shows the current density distribution in two half coils of the stator of the superconducting motor considered in [18] and [31], whose design is based on that of the SUTOR motor [32]. The average power loss dissipation of the stator coils in the coupled case is higher than in the uncoupled case, increasing from 52.31 W to 111.5 W at 65 K, and from 90.79 W to 173.22 W at 77 K.

B. QCSM: One-Step Versus Full-Cycle Simulations, Influence of Frequency

We compared the one-step and full-cycle simulations of the QCSM for the magnetization losses of a superconductor of

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1 Simulations of large rectangular bulks with the $H$ formulation—not shown here—confirmed the same results of the T-A, even at low current ratios.
elliptical cross section caused by an inclined magnetic field—a case for which one-step simulations and (13) should not give the correct results for the reasons explained in Section II-B. We started with the simulation of an ellipse representative of the superconducting cross section of a Bi-2223 tape, with semi-axes $a = 2$ mm and $b = 0.1$ mm (black ellipse in Fig. 4), self-field critical current $I_c = 160$ A and constant $J_c$. Then, we considered two progressively narrower and thicker ellipses, while leaving $I_c$ and the total area unchanged: in particular, the semi-axes were divided and multiplied by the same factor $\eta$, which was set equal first to 2 (red ellipse in Fig. 4) and then to 2.83 (blue ellipse in Fig. 4).

The ac applied field was set equal to 20 and 120 mT. These two values correspond to cases of partial and total penetration in the superconductor for $\theta = 0$° (see insets of Figs. 5 and 6 for the definition of the angle). In both cases, the orientation of the field was varied from 0° to 90°. The ac losses were computed as follows: for the full-cycle model, by integrating the product $J \cdot E$ over the superconductor’s cross section and averaging over the second cycle; for the one-step model, by using (13). Figs. 5 and 6 show that the losses calculated with the full-cycle and one-step models are very similar, for both field amplitudes and all angles: the difference does not exceed 5%. These simulations seem to indicate that the differences in ac losses calculated with the full-cycle and one-step models are rather small, probably because the differences in current distribution are not great and only occur over a small part of the cycle; once the loss is averaged over a cycle, the effect is even smaller. We also report that, due to similar reasons, the one-step calculation is also used in power law based finite modeling for the fast calculation of trapped magnetization of HTS bulks for sufficiently large values of $n$ [33].
As a second comparison between the one-step and full-cycle simulations, we calculated the losses of a thin ellipse ($\eta = 1$, see Fig. 4) under the simultaneous action of an ac transport current and an in-phase ac magnetic field with orientation $\theta = 0^\circ$ (see insets of Figs. 5 and 6 for the definition of the angle). The current and the field are applied by using both (11) and (12) as contributing terms to the boundary condition. The results are given in Fig. 7, which shows that the predictions of the two models are in excellent agreement, except for the cases of high current and high field (top-right part of the figure). This is to be expected, because in those cases the sample is fully penetrated by the magnetic field resulting from the transport current and external magnetic field; in such situations, certain assumptions for the validity of (13)—such as the existence of a kernel inside the sample where $A = 0$—are no longer valid [24].

In the full-cycle simulations with high currents and fields, the contribution (12) of the boundary condition does not produce a sinusoidal transport current (see inset of Fig. 8). This is due to the nonlinear penetration of the magnetic flux into the sample. Since the external field is sinusoidal, this means that the transport current and the external field are not synchronous, and the obtained ac losses are not representative of a true in-phase ac–ac scenario. In order to obtain the desired sinusoidal transport current, additional constraints need to be added, for example as proposed in [34]. The results are reported in Fig. 8. For confirmation, the results obtained with the MMEV method—which correspond to a pure CSM with in-phase current and field—are also shown. This demonstrates that the simple boundary condition (12) is not sufficient for applying a transport current in the presence of external magnetic field, especially when the field is sufficiently large so that the induced currents severely limit the transport capability of the superconductor.

Finally, with the full-cycle model, we varied the frequency of the external ac field (20 mT, $\theta = 0^\circ$) applied to the thin elliptical conductor of Fig. 4. As shown by the red lines in Fig. 9, if the frequency is sufficiently low, the results depart from those obtained with a pure critical state model (black crosses) and the calculated losses strongly depend on the frequency. This is due to the fact that rate-independent results are obtained only when the generated electric field is sufficiently large [compared to the chosen $E_0$ in (8)]. This is not the case when the frequency of the applied magnetic field is too low: in that case, the results depart from that of a “pure” critical state model.
The reason of this dependence lies in the fact that the constitutive law of the superconductor is an $E-J$ relationship—(8), and not an $A-J$ relationship, as in the model proposed by Prof. A. M. Campbell [20]. Because of this dependence on the rate of the excitation, we called this model a QCSM. Care should therefore be taken, if the model is used to simulate situations with slowly changing fields and/or transport currents.

The rate-independence of the QCSM can be reestablished by adapting the parameter $E_0$ to the rate of change of the fields and currents, more specifically in such a way that the product $E_0\Delta t$ that appears at the denominator of (10) remains constant and much smaller than the numerator. In this way, the product $E_0\Delta t$ plays the same role of the parameter $A_c$ in (8) of [20].

As a confirmation, Fig. 10 shows the magnetization cycles for the same geometry and field amplitude and direction as in Fig. 9, at a fixed (low) frequency of 0.5 mHz for various values of $E_0$.

An advantage of the QCSM proposed here is that a sufficiently fine time-stepping in (10) makes the process automatic when considering nontrivial driving forces. Furthermore, it is not necessary to track the maximum and minimum local values of the field to avoid overlooking local field reversals inside the superconducting domain.

IV. CONCLUSION

With this article, we extended the $T$-$A$ formulation to the case of thick superconductors. This extension allows simulating stacks (or windings) of electromagnetically coupled HTS coated conductors. The formulation was applied to calculate the losses of the stator coils of a superconducting motor and to compare the case of uncoupled and coupled turns. This approach is particularly appealing for 2-D problems, where the geometry is assumed infinitely long or has cylindrical symmetry. For 3-D problems, the computational burden of solving for both $A$ and $T$ in the superconducting domains is probably excessive compared to methods using only one formulation in each domain [35], [36].

Full-cycle and one-step simulations of the QCSM were compared in terms of ac loss calculation. In the case of individual superconducting tapes subjected to external ac magnetic fields, the difference of the ac loss results is rather marginal. In that case, the static model can be used to rapidly evaluate the ac losses of superconducting tapes. In the case of simultaneous ac current and ac field, however, the losses calculated with the one-step model can be significantly different from those calculated with full-cycle simulations, particularly in the case of high current and high field. Even with full-cycle simulations, however, attention should be paid to the shape of the transport current: the boundary condition (12) might not be sufficient to produce a sinusoidal transport current with a background field. In order to have a perfectly synchronous ac–ac scenario, one therefore needs to apply current constraints.

In the low-current and low-field range, the agreement between one-step and full-cycle simulations is excellent and one-step simulations can be used for a very rapid estimation of the losses. The simplicity of implementation and the speed of the one-step calculation can be attractive for didactical purposes, e.g., for introducing students to the topic of ac losses in superconductors [37].
Finally, the full-cycle simulations with the QCSM revealed a dependence of the ac loss results on the frequency of the ac excitation, if the frequency range is sufficiently large. This dependence should be taken into account, especially for the simulation of slow varying fields. A CSM-like behavior at low frequencies can be reestablished by reducing the value of the parameter $E_0$ in (8).

**APPENDIX**

In 2-D superconductors of arbitrary shape, an effective way of obtaining the desired transport current $I(t)$ by means of (5) is by using a weak form [38]:

$$
\int_{\partial \Omega} \left( \frac{I(t)}{p} - T_t \right) \hat{n} \, ds = 0
$$

(14)

where $\hat{n}$ is a test function, $\partial \Omega$ is the superconductor’s boundary, $p$ is the superconductor’s perimeter, $T_t$ is the tangential projection of the current potential $T$ on the superconductor’s boundary, and $s$ is the coordinate along that boundary. In COMSOL Multiphysics, this can be done by adding a weak contribution on the superconductor’s boundary, as follows:

$$
(I(t)/p - T_t) \ast \text{test}(T_t).
$$

(15)

This procedure was successfully tested on several shapes, and an example is reported in Fig. 11. The figure shows the distribution of the normalized current density in a superconductor of arbitrary shape at the end of an ac cycle of transport current. The results obtained with the $T$–$A$ formulation (top) are compared with those obtained with the $H$ formulation. The agreement is very good and the calculated ac losses (not shown here) are identical.

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