A MINIMAL KUREPA TREE WITH RESPECT TO CLUB EMBEDDINGS

HOSSEIN LAMEI RAMANDI

Abstract. We will show it is consistent with GCH that there is a minimal Kurepa tree with respect to club embeddings. That is, there is a Kurepa tree $T$ which club embeds in all of its Kurepa subtrees in the sense of [1]. Moreover the Kurepa tree we introduce, has no Aronszajn subtree.

1. Introduction

In this paper we study some specific $\omega_1$-trees with respect to isomorphisms restricted to a closed unbounded subset of $\omega_1$. Similarity of $\omega_1$-trees with respect to clubs of $\omega_1$ was first considered by Abraham and Shelah.

Theorem 1.1. [1] $PFA$ implies that every two Aronszajn trees are club isomorphic.

Here two $\omega_1$-trees $S, T$ are club isomorphic, if there is a club $C \subset \omega_1$ such that $T \upharpoonright C$ is isomorphic to $S \upharpoonright C$. This theorem may be regarded as an evidence that under some reasonable forcing axioms, like $PFA$ or some strengthening of that, Aronszajn trees behave like non-atomic countable trees. For instance, considering the fact that $2^{<\omega}$ is a minimal countable non-atomic tree, one might ask whether or not there are minimal Aronszajn trees. However the notion of Lipschitz trees, introduced by Todorcevic, made it clear that the class of Aronszajn trees is a lot more complicated if they are considered with actual embeddings, rather than club embeddings.

Theorem 1.2. [6] $PFA$ implies that there is no minimal Aronszajn tree.

Although there is a powerful structural theorem regarding the club isomorphisms of Aronszajn trees, similar questions regarding Kurepa trees do not seem to be addressed. In this paper we will prove:

\textbf{2010 Mathematics Subject Classification.} 03E05.

\textbf{Key words and phrases.} Kurepa trees, club embedding.
Theorem 1.3. It is consistent with GCH that there is a Kurepa tree $T$ which is club isomorphic to all of its downward closed everywhere Kurepa subtrees. Moreover $T$ has no Aronszajn subtrees.

An $\omega_1$-tree $T$ is said to be everywhere Kurepa if for all $x \in T$, the tree of all $y \in T$ that are compatible with $x$, is Kurepa. Since every Kurepa subtree of an everywhere Kurepa tree contains an everywhere Kurepa subtree, this theorem implies that the tree in the theorem is actually club minimal with respect to being Kurepa, i.e. for every downward closed Kurepa subtree $U \subset T$ there is a club $C \subset \omega_1$ and a one to one, level and order preserving function $f : T \upharpoonright C \rightarrow U \upharpoonright C$.

The club minimality of an everywhere Kurepa tree clarifies the behavior of the invariant $\Omega$ introduced in [2]. This invariant was originally defined for linear orders, but it can be translated for the class of $\omega_1$-trees as follows. Here $B(T)$ is the collection of all branches in $T$, for $b, b' \in B(T)$, $b(\alpha)$ is the element in $b$ which has height $\alpha$, and $b \Delta b'$ is the minimum $\alpha \in \omega_1$ such that $b(\alpha) \neq b'(\alpha)$.

Definition 1.4. $\Omega(T)$ is the set of all countable $Z \subset B(T)$ with the property that for all $t \in T_{\alpha_Z}$ there is a $b \in Z$ with $t \in b$, where $\alpha_Z = \sup \{ b \Delta b' : b, b' \in Z \}$.

The relation between the $\Omega$ defined here and the one in [2] can be described as follows. Assume $T$ is an $\omega_1$-tree which is equipped with a lexicographic order. Let $L$ be the linear order consisting of the elements of $T$ with the lexicographic order. $\Omega(T)$ defined above is equivalent to $\Omega(L)$ defined in [2], in the sense that their symmetric difference in non-stationary in $[B(T)]^{\omega}$. The invariant $\Omega$ played an essential role in the proof of the following results.

Theorem 1.5. [2] Assume PFA+ . If $L$ is a minimal non $\sigma$-scattered linear order, then it is either a real or Countryman type.

Theorem 1.6. [4] If there is a supercompact cardinal then there is a forcing extension which satisfies CH in which there is no minimal non $\sigma$-scattered linear order.

In order to see the role of $\Omega$, first recall from [2], $\Omega(L)$ contains a club iff $L$ is $\sigma$-scattered. Also for linear orders $L_0 \subset L$, $L$ does not embed in $L_0$ if $\Omega(L_0) \prec \Omega(L)$ is stationary. Part of the work in [2] and [4] was to deduce, from appropriate hypothesis, that if $L$ is a non $\sigma$-scattered linear order that does not contain any real or Aronszajn type then there is $L_0 \subset L$ such that $\Omega(L_0)$ does not contain a club and $\Omega(L_0) \prec \Omega(L)$ is stationary. So one might ask, aside from linear orders which contain real types or Aronszajn types, whose $\Omega$ is non
stationary, are there non $\sigma$-scattered linear orders $L$, such that for all $L_0 \subset L$ either $\Omega(L_0) \equiv \Omega(L)$ or else $\Omega(L_0)$ contains a club. A consistent negative answer is given in [2] and [4]. The existence of a club minimal Kurepa tree gives a consistent affirmative answer to this question.

The forcings we use to add embeddings are not proved to be proper, but their behavior towards suitable models $M$ are similar to proper posets often enough. This property of posets is called $E$-completeness, and shown to be sufficient criteria for preserving $\omega_1$ in [5]. The notion $S$-completeness here seems to coincide with $E$-completeness.

In section 2, based on the work in [5], and the notion of proper isomorphism condition for proper posets we will prove the lemmas needed for certain chain conditions which are not included in [5]. We have also included the proof of the fact that $S$-complete forcings are closed under countable support iterations although it is proved in [5]. This makes the proof of the lemmas needed for chain condition properties more clear. Section 3 is devoted to the proof of Theorem 1.3.

To avoid ambiguity we fix some notation and terminology. An $\omega_1$-tree $T$ is a tree which has countable levels and does not branch at limit heights, i.e. there are no distinct pair $s, t \in T$ which have the same height and predecessors. A chain $b \subset T$ is called a branch of $T$ if it intersects all levels of $T$. An $\omega_1$-tree $T$ is called Aronszajn if it has no branches. It is called Kurepa if it has at least $\omega_2$ many branches. For $C \subset \omega_1$, $T \upharpoonright C = \{t \in T: \text{height of } t \text{ is in } C\}$. If $S, T$ are trees, $f: T \rightarrow S$ is called a tree embedding if for all $t, s \in T$, $t \triangleleft_T s$ iff $f(t) \triangleleft_S f(s)$.

2. $S$-Completeness, Iteration and Chain Condition

We will work with forcings which may not be proper but up to a fixed stationary set they behave very much like $\sigma$-complete forcings. In this section we provide the machinery to iterate these posets and sufficient criteria for verifying the chain conditions of the forcings we will use. Everything in this section is built on the material in [5].

Definition 2.1. Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. A poset $\mathcal{P}$ is said to be $S$-complete, if every descending $(M, \mathcal{P})$-generic sequence, $\langle p_n : n \in \omega \rangle$ has a lower bound, for all $M$ with $M \cap X \in S$ and $M$ suitable for $X, \mathcal{P}$.

First note that $S$-complete forcings preserve the stationarity of all stationary subsets of $S$. Although it is clear from the definition we emphasize that $S$-complete is not stronger than properness unless $S$ is a club. In that case $S$-complete is very close to being $\sigma$-complete. The following fact vacuously follows from the definition.
Fact 2.2. Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. If $\mathcal{P}$ is an $S$-complete forcing then it preserves $\omega_1$ and adds no new countable sequences of ordinals.

Now we prove that for a given stationary $S \subset [X]^\omega$ where $X$ is uncountable, the class of all $S$-complete forcings is closed under countable support iterations. We follow the same strategy as in the proof of the similar lemma for proper posets in [5].

Fact 2.3. Assume $S, X$ are as above, $\mathcal{P}$ is $S$-complete, and $\models \mathcal{P} \ " \hat{\mathcal{Q}} \ is \ S$-complete". Then $\mathcal{P} \ast \hat{\mathcal{Q}}$ is $S$-complete.

Proof. Assume $M$ is suitable for $\mathcal{P} \ast \hat{\mathcal{Q}}$ and $M \cap X \in S$. Let $\langle p_n \ast \hat{q}_n : n \in \omega \rangle$ be a descending $(M, \mathcal{P} \ast \hat{\mathcal{Q}})$-generic sequence. Since $\langle p_n : n \in \omega \rangle$ is an $(M, \mathcal{P})$-generic sequence, it has a lower bound $p \in \mathcal{P}$. Moreover $p \models \ " \langle \hat{q}_n : n \in \omega \rangle \ is \ an \ (M[\hat{G}_\mathcal{P}], \hat{\mathcal{Q}})$-generic."$

On the other hand, the $(M, \mathcal{P})$-generic condition $p$ forces that $M[\hat{G}_\mathcal{P}] \cap V = M$ and consequently $M[\hat{G}_\mathcal{P}] \cap X \in \hat{S}$. So it forces that the sequence $\langle \hat{q}_n : n \in \omega \rangle$ has a lower bound as well. Let $\hat{q}$ be a $\mathcal{P}$-name for such a condition, then $p \ast \hat{q}$ is a lower bound for $\langle p_n \ast \hat{q}_n : n \in \omega \rangle$. □

Lemma 2.4. Assume $X$ is uncountable, $S \subset [X]^\omega$ is stationary, $\langle \mathcal{P}_i, \hat{\mathcal{Q}}_j : i \leq \delta, j < \delta \rangle$ is a countable support iteration of $S$-complete forcings, $N$ is suitable for $\mathcal{P}_\delta$, $N \cap X \in S$, $\langle p_n : n \in \omega \rangle$ is an $(N, \mathcal{P}_\delta)$-generic descending sequence of conditions, $\alpha < \delta$ is in $N$ and $q \in \mathcal{P}_\alpha$ is a lower bound for $\langle p_n \upharpoonright \alpha : n \in \omega \rangle$. Then there is a lower bound $q' \in \mathcal{P}_\delta$ for $\langle p_n : n \in \omega \rangle$, such that $q' \upharpoonright \alpha = q$.

Proof. We use induction on $\delta$. If $\delta$ is a successor ordinal the lemma follows from the induction hypothesis and the argument in the proof of the previous fact. if $\delta$ is limit, let $\langle \alpha_n : n \in \omega \rangle$ be a cofinal sequence in $N \cap \delta$ such that $\alpha_0 = \alpha$, and for all $i$, $\alpha_i \in N$. Note that for all $i$, $\langle p_n \upharpoonright \alpha_i : n \in \omega \rangle$ is a descending $(N, \mathcal{P}_{\alpha_i})$-generic sequence. So by the induction hypothesis there is a sequence $q_i, i \in \omega$, such that

- $q_0 = q$
- $q_i \in \mathcal{P}_{\alpha_i}$ is a lower bound for $\langle p_n \upharpoonright \alpha_i : n \in \omega \rangle$
- $i < j \rightarrow q_j \upharpoonright \alpha_i = q_i$

Now $q' = \bigcup_{i \in \omega} q_i$ works. □

Corollary 2.5. Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. Then the class of $S$-complete forcings are closed under countable support iterations.
We will use the following fact in the next section which follows vacuously from the last definition.

**Fact 2.6.** Assume $T$ is an $\omega_1$-tree which has no Aronszajn subtree in the ground model $V$, $\Omega(T) \subset [B(T)]^\omega$ is stationary, and $P$ is an $\Omega(T)$-complete forcing. Then $T$ has no Aronszajn subtree in $V^P$.

**Proof.** Assume $U$ is a $\mathcal{P}$-name for a downward closed Aronszajn subtree of $T$. Let $p \in \mathcal{P}$, $M$ be suitable with $M \cap B(T) \in \Omega(T)$ and $p, \dot{U} \in M$. Also let $\delta = M \cap \omega_1$. For all $b \in M \cap B(T)$ the set $D_b$ consisting of all conditions $q \in \mathcal{P}$ which forces that $b(\alpha) \notin \dot{U}$ for some $\alpha \in \omega_1$ is dense and in $M$. Note that if $q \in D_b$, it decides the minimum $\alpha \in \omega_1$, which witnesses that $q \in D_b$. Now let $\langle p_n : n \in \omega \rangle$ be a decreasing $(M, \mathcal{P})$-generic sequence, with $p_0 = p$, and $\bar{p}$ be a lower bound for this sequence. Then $\bar{p}$ forces that $\dot{U}$ has no element in $\{ b(\delta) : b \in M \cap B(T) \} = T_\delta$. This implies that $U$ is a countable set which is a contradiction. $\square$

Now we deal with the chain condition issue for $S$-complete forcings. The following definition is a modification of the $\kappa$-properness isomorphism condition.

**Definition 2.7.** Assume $S, X$ are as above. We say that $\mathcal{P}$ satisfies the $S$-closedness isomorphism condition for $\kappa$, or $\mathcal{P}$ has the $S$-cic for $\kappa$, if whenever

- $M, N$ are suitable models for $\mathcal{P}$,
- both $M \cap X, N \cap X$ are in $S$,
- $h : M \to N$ is an isomorphism such that $h \upharpoonright (M \cap N) = id$,
- there are $\alpha_M, \alpha_N$ in $M \cap \kappa$ and $N \cap \kappa$ respectively with $h(\alpha_M) = \alpha_N$, $sup(M \cap \kappa) < \alpha_N$, $M \cap \alpha_M = N \cap \alpha_N$, and
- $\langle p_n : n \in \omega \rangle$ is an $(M, \mathcal{P})$-generic sequence,

then there is a common lower bound $q \in \mathcal{P}$ for $\langle p_n : n \in \omega \rangle$ and $\langle h(p_n) : n \in \omega \rangle$.

**Lemma 2.8.** Assume $2^{\aleph_0} < \kappa$, $\kappa$ is a regular cardinal and that $S, X$ are as above. If $\mathcal{P}$ satisfies the $S$-cic for $\kappa$ then it has the $\kappa$-c.c.

**Proof.** Let $\langle p_\xi : \xi \in \kappa \rangle$ be a collection of conditions in $\mathcal{P}$, and for each $\xi \in \kappa$, $M_\xi$ be a suitable model for $\mathcal{P}$ such that $M \cap X \in S$, $\kappa, \xi$, and $\langle p_\xi : \xi \in \kappa \rangle$ are in $M$. Consider the function $f : \kappa \to \kappa$ defined by $\xi \mapsto sup(M_\xi \cap \xi)$. Obviously for all $\xi$ with $cf(\xi) > \omega$, $f(\xi) < \xi$. So there is a stationary $W \subset \kappa$ such that the function $f \upharpoonright W$ is a constant. Now find $U \subset W$ of size $\kappa$ such that for all $\xi < \eta$ in $U$, $sup(M_\xi \cap \kappa) < \eta$ and $M_\xi \cap \xi = M_\eta \cap \eta$. 

Now let for each $\xi \in U \langle p_\xi^n : n \in \omega \rangle$ be descending and $(M_\xi, P)$-generic with $p_\xi^0 = p_\xi$. Since $2^{\aleph_0} < \kappa$ we can thin down $U$ if necessary to get

for all $\xi, \eta$ in $U$, $M_\xi$ is isomorphic to $M_\eta$ via the map, $h_{\xi\eta} : M_\xi \to M_\eta$, induced by the transitive collapse maps.

Now consider models $M_\xi$ together with $\langle p_\xi^n : n \in \omega \rangle$ as constants. There are at most continuum many of the isomorphism types of these models and by extensionality the isomorphism between $M_\xi$ and $M_\eta$ is unique if it exists. So we can thin down the collection $\langle p_\xi : \xi \in \omega_2 \rangle$ again, to have

for all $\xi, \eta$ and $n \in \omega$, $h_{\xi\eta}(p_\xi^n) = p_\eta^n$

in addition to what we had so far.

Now since $P$ satisfies $S$-cic, for every pair of distinct $\xi, \eta$ in $U$, there is a condition $q \in P$ which is a common lower bound for sequences $\langle p_\xi^n : n \in \omega \rangle$ and $\langle p_\eta^n : n \in \omega \rangle$, meaning in particular that $p_\xi$ is compatible with $p_\eta$ which was desired. \hfill $\square$

We are now ready to state and prove the lemma we need for the chain condition issues.

**Lemma 2.9.** Suppose $\langle P_i, \dot{Q}_j : i \leq \delta, j < \delta \rangle$ is a countable support iteration of $S$-complete forcings, where $S \subset [X]^{\omega}$ is stationary and $X$ is uncountable. Assume in addition that

$\models_{P_i} \text{"} \dot{Q} \text{ has the } S \text{-cic for } \kappa \text{"},$

for all $i \in \delta$. Then $P_\delta$ has the $S$-cic for $\kappa$.

**Proof.** First note that if $P$ is any forcing, $M, N$ are suitable for $P$, $h : M \to N$ is an isomorphism, $p$ is both $(M, P)$-generic and $(N, P)$-generic, and $G \subset P$ is $V$-generic with $p \in G$, then $h[G] : M[G] \to N[G]$ defined by $\tau_G \mapsto (h(\tau))_G$ is an isomorphism as well.

Before we deal with the general case, we prove the lemma for $P \ast \dot{Q}$. Let $M, N, h$ be as in definition 2.7 for $P \ast \dot{Q}$, and let $\langle p_n, \dot{q}_n : n \in \omega \rangle$ be a descending $(M, P \ast \dot{Q})$-generic such that the sequences $\langle p_n : n \in \omega \rangle$ and $\langle h(p_n) : n \in \omega \rangle$ have a common lower bound $p \in P$. Since $p$ is both $(M, P)$-generic and $(N, P)$-generic, it forces the hypotheses of the definition 2.7 for $M[G_P], N[G_P], \dot{Q}, h[G_P]$, and $\langle \dot{q}_n : n \in \omega \rangle$. By the assumption on $\dot{Q}$, there is a $P$-name $\dot{q}$ which is forced by $p$ to be a common lower bound for $\langle \dot{q}_n : n \in \omega \rangle$ and $\langle h[G_P](\dot{q}_n) : n \in \omega \rangle$. So $p \ast \dot{q}$ is a common lower bound for $\langle p_n \ast \dot{q}_n : n \in \omega \rangle$ and its image under $h$.

Now let $(\ast)$ be the assertion that
"suppose $\alpha < \delta < \kappa$, $M, N, h$, and $\langle p_n : n \in \omega \rangle$, are as in definition 2.7 for $\mathcal{P} = \mathcal{P}_\delta$, with $\alpha \in M$. Moreover $r \in \mathcal{P}_\alpha$ and $r_h \in \mathcal{P}_{h(\alpha)}$ are lower bounds for $\langle p_n \upharpoonright \alpha : n \in \omega \rangle$ and $\langle h(p_n \upharpoonright \alpha) : n \in \omega \rangle$, respectively such that

- $\text{supp}(r) \subset M$, and $\text{supp}(r_h) \subset N$, and
- $r(\xi) = r_h(\xi)$ for all $\xi$ in $M \cap N$.

Then there are lower bounds $r \in \mathcal{P}_\delta$, $\bar{r} \in \mathcal{P}_\delta$ for $\langle p_n : n \in \omega \rangle$ and $\langle h(p_n) : n \in \omega \rangle$ respectively such that

- $\text{supp}(\bar{r}) \subset M$, and $\text{supp}(\bar{r}_h) \subset N$,
- $\bar{r}(\xi) = \bar{r}_h(\xi)$ for all $\xi$ in $M \cap N$,
- $\bar{r} \upharpoonright \alpha = r$ and $\bar{r}_h \upharpoonright h(\alpha) = r_h$.

First note that (*) is stronger than the lemma by letting $\alpha = 0$ and gluing $\bar{r}$ and $\bar{r}_h$ together, in order to get the desired condition.

We use induction to show (*). The successor step is trivial by what we just proved, and if $\delta$ is limit the proof is exactly the same as the Lemma 2.4. This is by considering sequences $\langle \alpha_i : i \in \omega \rangle$ which is cofinal in $M \cap \delta$ with $\alpha_0 = \alpha$ as well as its image under $h$ which is cofinal in $N \cap \delta$, because $h$ fixes the intersection. □

We finish this section with a few remarks.

**Remark 2.10.**

- Unlike Lemma 2.4 of chapter VIII of [5], in the last lemma there is no hypothesis on the length of the iteration. In other words by the lemmas in this section, as long as $\kappa$ is regular and greater than the continuum, any countable support iteration of posets that have the $S$-cic for $\kappa$ has the $\kappa$-cc.
- It is possible to define $S$-proper posets to be the ones which have $M$-generic condition $q$ below $p$, whenever $M$ comes from a stationary set $S$, and $p$ is a condition inside $M$. These posets inherit many nice properties of proper posets. For instance, they preserve stationarity of all stationary subsets of $S$, and their countable support iterations do not add new branches to $\omega_1$-trees provided that the iterands have this property.
- $S$-properness is obviously weaker than both $S$-completeness and properness. The behavior of $S$-proper posets might be of interest if they are useful tools for interesting problems.

3. Minimal Kurepa Trees

In this section I will prove Theorem 1.3. A fastness notion for closed unbounded subsets of $\omega_2$ is used in the definition of the forcings which add embeddings. A club $C_U \subset \omega_2$ is fast enough for $U, T$ if it is the
set of all \( \sup(M_{\xi} \cap \omega_2) \) where \( \langle M_{\xi} : \xi \in \omega_2 \rangle \) is a continuous \( \in \)-chain of \( \aleph_1 \)-sized elementary submodels of \( H_\theta \) such that \( \xi \cup \omega_1 \subset M_{\xi} \), \( U, T \) are in \( M_0 \), and \( \langle M_\eta : \eta \leq \xi \rangle \) is in \( M_{\xi+1} \).

**Definition 3.1.** Suppose \( T \) is an everywhere Kurepa tree with \( B(T) = \langle b_\xi : \xi \in \omega_2 \rangle \), \( U \) a downward closed everywhere Kurepa subtree of \( T \), and \( C_U \subset \omega_2 \) a club that is fast enough. \( Q(= Q_{T,U}) \) is the set of all conditions \( p = (f_p, \phi_p) \) such that,

1. \( f_p : T \upharpoonright A_p \rightarrow U \upharpoonright A_p \) is a level preserving tree isomorphism, where \( A_p \subset \omega_1 \) is countable and closed with \( \max A_p = \alpha_p \),
2. \( \phi_p \) is a countable partial injection from \( \omega_2 \) to \( \omega_2 \) such that
   1.a. for all \( \xi \in \text{dom}(\phi_p) \), \( b_{\phi_p(\xi)} \in B(U) \).
   1.b. \( \phi_p \) respects \( C_U \), i.e. for all \( \alpha \in C_U, \xi \in \text{dom}(\phi_p), \xi < \alpha \leftrightarrow \phi_p(\xi) < \alpha \),
3. for each \( t \in T_{\alpha_p} \) there are at most finitely many \( \xi \in \text{dom}(\phi_p) \) with \( t \in b_\xi \), and
4. if \( \xi \in \text{dom}(\phi_p) \) then \( f_p(b_\xi(\alpha_p)) = b_{\phi_p(\xi)}(\alpha_p) \).

We let \( p \leq q \) if \( f_q \subset f_p \) and \( \phi_q \subset \phi_p \).

Note that the set of all conditions \( q \) with \( \alpha_q \geq \alpha \) is dense for all \( \alpha \in \omega_1 \). Also it is easy to see that for all \( b \in B(T) \) the set of all conditions \( q \) with \( b \in \text{dom}(\phi_q) \) is dense, as well as the set of all conditions \( q \) with \( b \in \text{ran}(\phi_p) \) when \( b \in B(U) \).

**Lemma 3.2.** Suppose \( T \) is an everywhere Kurepa tree with \( \Omega(T) = S \) stationary, \( \mathcal{P} \) is an \( S \)-complete forcing, and \( \check{U} \) is a \( \mathcal{P} \)-name for a downward closed everywhere Kurepa subtree of \( T \). Then

1) \( \Vdash_{\mathcal{P}} " \check{Q}_{T,U} \text{ is } \check{S} \text{-c.c.}" \), and
2) \( \Vdash_{\mathcal{P}} " \check{Q}_{T,U} \text{ has the } \check{S} \text{-c.c. for } \omega_2 \" \).

**Proof.** Let \( G \subset \mathcal{P} \) be a \( V \)-generic filter. Note that \( \mathcal{P} \) does not add new branches to \( \omega_1 \)-trees and \( S \subset [B(T)]^\omega \) is stationary in \( V[G] \).

Now we work in \( V[G] \). To see (1) assume \( \check{M} \) is suitable for \( \check{Q}_G = \check{Q} \), and \( M \cap B(T) \in S \). Also let \( \langle p_n = (f_n, \phi_n) : n \in \omega \rangle \) be a descending \((M, \check{Q})\)-generic sequence, and \( \delta = M \cap \omega_1 \). By elementarity and density argument,

- \( \bigcup_{n \in \omega} \text{dom}(\phi_n) = M \cap B(T) \), and
- \( \bigcup_{n \in \omega} \text{dom}(f_n) = T \upharpoonright A \), for some \( A \) which is cofinal in \( \delta \).

Let \( \phi_p = \bigcup_{n \in \omega} \phi_n \) and for each \( \xi \in M \cap \omega_2 \) define \( f_p(b_\xi(\delta)) = b_{\phi_p(\xi)}(\delta) \). This makes \( p \) a condition in the poset and a lower bound for the sequence \( \langle p_n : n \in \omega \rangle \), since \( T_\delta \subset \bigcup (B(T) \cap M) \).

For (2), still in \( V[G] \), let \( M, N, \langle p_n = (f_n, \phi_n) : n \in \omega \rangle \) and \( h \) be as in definition 2. Then with \( M \cap \omega_1 = N \cap \omega_1 = \delta \). We let \( h(\phi_n) = \psi_n \) and since
$h$ fixes the intersection, $h(f_n) = f_n$. Also note that $b(\delta) = [h(b)](\delta)$, for all $b \in B(T) \cap M$. Let $\phi = \bigcup_{n \in \omega} (\phi_n \cup \psi_n)$ , and $f(b(\xi)) = b(\phi(\xi))$. To see $\phi$ is one to one, note that

$$h(\xi) \neq \xi \iff \alpha_M \leq \xi < sup(M \cap \omega_2) < \alpha_N \leq h(\xi).$$

But $sup(M \cap \omega_2)$ is in $C_U$. Therefore $\phi(\xi) < sup(M \cap \omega_2) < h(\phi(\xi)) = \psi(\xi)$ This makes $(\bigcup_n f_n \cup f, \phi)$ a condition in the poset which is a lower bound for both $\langle p_n : n \in \omega \rangle$ and its image under $h$. \qed

Now by Lemmas 3.2, 2.4, 2.8, and 2.9, the following proposition is obvious.

**Proposition 3.3.** Assume GCH. If $T$ is an everywhere Kurepa tree with $\omega_2$ many branches such that $\Omega(T) \subset [B(T)]^\omega$ is stationary , then there is a forcing extension in which GCH is still true and $T$ is a club isomorphic to all of its downward closed everywhere Kurepa subtrees.

In order to prove Theorem 1.3 it suffices to show that there is a Kurepa tree that satisfies the hypothesis of the proposition. Let $K$ be the poset consisting of conditions of the form $p = (T_p, b_p)$ where

- $T_p$ is a countable tree of height $\alpha_p + 1$ such that for all $t \in T_p$ there exists $s \in (T_p)_{\alpha_p}$ with $t < s$,
- $b_p$ is countable partial function from $\omega_2$ to the last level of $T_p$.

$p \leq q$ in $K$ if

- $(T_p)_{\leq \alpha_q} = T_q$
- $dom(b_p) \supset dom(b_q)$
- for all $\xi \in dom(b_q)$, $b_q(\xi) \leq b_p(\xi)$

It is well known that $K$ is countably closed and under $CH$, has the $\omega_2$-chain condition. Let $T$ be the $K$-generic tree, then $\Omega(T)$ is stationary in $[B(T)]^\omega$. To see that let $p$ be a condition that forces the contrary and $\dot{E}$ be a $K$-name for a club in $[B(T)]^\omega$ which is forced by $p$ to be disjoint from $\Omega(T)$. Let $M$ be suitable for $K$ with $p, \dot{E}$, etc in $M$. Then for any sequence $\langle p_n : n \in \omega \rangle$ which is $(M, K)$-generic and $p_0 \leq p$ we can form a lower bound $\bar{p}$ for the sequence such that $dom(\bar{p}) = M \cap \omega_2$. Note that such a condition forces that $M \cap \omega_2 = M[\dot{G}] \cap \omega_2$, where $\dot{G}$ is the canonical name for the $K$-generic filter. On the other hand $\bar{p}$ forces that $M[\dot{G}] \cap B(T) \in \dot{E}$, because it is $M$-generic. So $\bar{p}$ forces that $M[\dot{G}] \cap B(T) \in \dot{E} \cap \Omega(T)$ which is a contradiction.

In order to show that $T$ does not have any Aronszajn subtree in the final model, after embeddings added, we will show that

$$\models_K \hat{T} \text{ has no Aronszajn subtrees.}$$
Note that this suffices by Fact 2.6. Let $\dot{U}$ be a $\mathcal{K}$-name for an uncountable downward closed subtree of $\dot{T}$, where $\dot{T}$ is a $\mathcal{K}$-name for the tree $T$. Let $M$ be a suitable model for $\mathcal{K}$ with $\dot{U} \in M$. By the assumptions, for all $\xi \in \omega_2 \cap M$, and $p \in M \cap P$ there is an extension $q \in M \cap P$ of $p$ such that for some $\alpha \in \omega_1$, $q$ forces that $b_\xi(\alpha) \notin \dot{U}$, where $b_\xi = \bigcup_{p \in \dot{G}} b_p(\xi)$ and $\dot{G}$ is the canonical name for the generic filter of the forcing $\mathcal{K}$. Note that by elementarity $\alpha$ is in $M \cap \omega_1$. Now let $\langle p_n : n \in \omega \rangle$ be an $(M, \mathcal{K})$-generic sequence such that for all $\xi \in \omega_2 \cap M$ there is an $n \in \omega$ such that $p_n \vDash b_\xi(M \cap \omega_1) \notin \dot{U}$. Let $q$ be a lower bound for this sequence such that $\text{dom}(b_q) = M \cap \omega_2$. Then $q$ forces that $\dot{U}$ is countable, which is a contradiction. We showed that every downward closed subtree of $T$ contains $b_\xi$ for some $\xi \in \omega_2$. This shows that $T$ has no Aronszajn subtree and $\langle b_\xi : \xi \in \omega_2 \rangle$ is the collection of all branches of $T$.

Remark 3.4. The Kurepa tree constructed in [3] also satisfies the hypothesis of the last proposition. So it can be made minimal in the same way as above. It is shown in [3] that this tree has no Aronszajn subtree, so by Fact 2.6 this tree has no Aronszajn subtree after embeddings are added either.

Acknowledgments

For continual support and encouragement, the author would like to thank Justin Tatch Moore.

The research presented in this paper was supported in part by NSF grants DMS-1262019 and DMS-1600635.

References

[1] U. Abraham and S. Shelah. Isomorphism types of Aronszajn trees. Israel J. Math., 50(1-2):75–113, 1985.
[2] T. Ishiu and J. T. Moore. Minimality of non $\sigma$-scattered orders. Fund. Math., 205(1):29–44, 2009.
[3] B. Kuzeljevic and S. Todorcevic. Forcing with matrices of countable elementary submodels. Proc. Amer. Math. Soc., 145(5):2211–2222, 2017.
[4] H. Lamei Ramandi and J. Tatch Moore. There may be no minimal non $\sigma$-scattered linear order. Math. Res. Lett. accepted.
[5] S. Shelah. Proper and Improper Forcing. Springer-Verlag, Berlin, second edition, 1998.
[6] S. Todorcevic. Lipschitz maps on trees. J. Inst. Math. Jussieu, 6(3):527–556, 2007.

Department of Mathematics, Cornell University, Ithaca, NY 14853–4201, USA
E-mail address: hossein@math.cornell.edu