Abstract. In order to compute with \( l \)-adic sheaves or crystals on a line over \( \mathbb{F}_q \) a low-technology alternative to the traditional computation with the Hecke operators on the automorphic side could be helpful. A program which has evolved over the years in our discussions with M. Kontsevich centers around the concept that, in the geometric case, there must exist certain multiplication laws on the Galois–representation side that could be thought of as precursors of the automorphic lifts: non-abelian Abel’s theorems, and their restrictions to diagonal, Clausen identities. To a varying extent, they can determine the trace functions of \( l \)-adic sheaves or crystals with prescribed ramification without directly appealing to the Hecke–eigen property on the automorphic side.

1. Motivation. The central calculation of this paper says that the convolution of the Markov local system on the punctured genus 1 curve with the quadratic character sheaf is a quaternionic rank 3 local system whose monodromy is given by the conjugation action by units in an order in \( A_{(2,3)} \). We use it, and related facts, as a proof of concept, or illustration, for a few ideas about Galois representations that are specific to the function-field or geometric (= curve over \( \mathbb{C} \) situation — namely, that:

1. the classical ‘per prime’ formulation of lifting by Langlands might be too restrictive, and formulations seeking to ‘link’ the Euler factors at different places should be sought for [22];
2. Taylor’s formula, rather than being an isolated fact, is merely the simplest member of the family of ‘master formulas’ that relate Hecke–type operators to hamiltonians in various quantizations of differential equations;
3. a proper formulation of Abel’s theorem should take on the multiplicative form, and in such form it is capable of surviving in the non-abelian situation, potentially furnishing an answer to the problem of computing Galois representations;
4. Clausen–type formulas [21] should be viewed, depending on the optic, as (precursors of) Langlands’s lifts or as restrictions to diagonal of the multiplication kernels of non-abelian Abel’s theorems; nevertheless, even these restrictive Clausen formulas impose strong conditions on the trace functions, and can be used instead of the Hecke–eigen property on the automorphic side for practically computing with them;
5. in the Betti rendering [3], the presence of a generalized lift can be signaled by the existence of a map between character varieties.
The conceptual difference from the geometric Langlands correspondence is that we are somewhat agnostic as to the nature of the quantization model; non-abelian Abel’s theorems arise when the external powers of the DE are considered.

We also say a few words about our original motivation to classify D4 congruence sheaves in the context of mirror symmetry. We work up from the

**Fact.** Classification of Picard rank 1 Fano threefolds [19] is mirrored by the classification of D3 Kugo-Sato 3-folds [14]. Namely, Fano 3-folds of index \(d\) and the anticanonical degree \((-K)^3 = 2d^2N\) correspond 1 : 1 to \(KS_{N,d}\)’s as in the top right corner of the diagram

\[
\begin{array}{c}
\mathcal{E} \times \mathcal{E}^N \\
\downarrow \\
\mathcal{E} \times \mathcal{E}^N \\
\downarrow \\
X_0(N)^N \\
\longleftarrow \mathbb{P}^1
\end{array}
\]

such that the Picard–Fuchs DE arising from the right arrow is of D3 type.

### 2. DN equations

A DN equation [13] is obtained from an \((N+1) \times (N+1)\) matrix \(A = (a_{ij})_{i,j=0}^N\) that satisfies

\[
\begin{align*}
    a_{ij} &= 0, \quad i - j > 1 \\
    a_{ij} &= 1, \quad i - j = 1 \\
    a_{ij} &= a_{N-j,N-i}, \quad i - j < 1
\end{align*}
\]

The respective differential operator is then defined as

\[
L_A(t) = D^{-1} \det_{\text{right}} \left( \delta_{ij}D - a_{ij}(Dt)^{j-i+1} \right)
\]

where \(\delta_{ij}\) is the Kronecker symbol and \(\det_{\text{right}}\) means the right determinant. For a sufficiently generic \(A\), the respective differential equation has maximal unipotent monodromy at \(t = 0\). The other singularities are the inverse eigenvalues of \(A\); the respective monodromies are orthogonal \((N\ \text{odd})\) or symplectic \((N\ \text{even})\) reflections.

Thus, the mirror duals of Picard rank 1 Fano threefolds are 1-parametric families of motives whose specificity is so strong that it entails modularity. A literal analogue won’t work in the non-Shimura situation: for D4’s, the moduli space of Hodge structures of type

\[
h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1
\]

is 4-dimensional complex, and the universal Hodge structure over it is not a variation of Hodge structures. We need pencils to play the role of ‘4-dimensional Kugo-Sato’s’, but not in any traditional sense. Another drawback is that we know essentially nothing about the Calabi-Yau geometries that will appear in the fibers. A way out is to construct congruence sheaves: representations of \(\mathbb{Q}(t)\) with prescribed (small) geometric ramification and congruence properties.

We want to construct them locally over each \(\mathbb{F}_p\), an glue them together across Spec \(\mathbb{Z}\) using congruences as a rigidity constraint.

### 3. Galois representations and spectral problems

The prevailing dogma is that in order to compute Galois representations, we must be solving certain spectral problems. This is not immediately visible in the apparently symmetric Gauss’s quadratic reciprocity

\[
\left( \frac{\alpha}{\beta} \right) = \left( \frac{\beta}{\alpha} \right) (-1)^{(\alpha-1)(\beta-1)/4},
\]
however, the modern interpretation is that the meaning of, say, the LHS of \((\frac{\gamma}{\alpha}) = (\frac{\eta}{\beta})\) is Galois-representational (the action of the \(q\)-Frobenius in \(\mathbb{Q}(\sqrt{5})\)), whereas the meaning of the RHS is spectral:

\[
\begin{array}{c}
\text{\[\frac{\gamma}{\alpha}\]} \\
\text{\[\frac{\eta}{\beta}\]}
\end{array}
\]

define \(\tilde{T}_q(f)(x) = \sum_{k=0}^{q-1} f(\frac{ax+k}{q})\), then the function on the circle shown in the picture is \(\tilde{T}_q\)-eigen with the eigenvalue \(\left(\frac{\alpha}{\beta}\right)\). This is clearer in the case of cubic equations. Put, for instance, \(f(x) = x^3 - 4x - 1\). There are three ways in which \(f\) can split mod prime \(p \neq 229 = \text{disc } f\): with \(r_p = 0, 1\) or 3 roots in \(\mathbb{F}_p\). Put \(a_p = r_p - 1, a_{229} = 1\). Extend to \(a_n\) by multiplicativity in the usual way: if \(\text{ord}_p n = 1, a_n = a_p a_{n/p}\), if \(\text{ord}_p n > 1, a_n = a_p a_{n/p} - \left(\frac{229}{q}\right) a_{n/p}^2\). Let \(K_0(y)\) be the \(K\)-Bessel function so that \(K_0(y) = \int_0^\infty \exp(-y \cosh t) \, dt\). Finally, define \(M\) as the function on the upper half-plane given by \(M(x+iy) = y^{1/2} \sum_{n=1}^{\infty} a_n \cos(nx) K_0(ny)\). Then ‘reciprocity’ means, in particular, that \(M\left(\frac{1}{229}\right) = M(iy)\), which translates into the functional equation for the respective \(L\)-function.

What is the spectral problem in question? Let \(\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\) denote the laplacian on the upper half-plane. By separation of variables, a series such as above with any choice of \(a_n\) is \(\Delta\)-eigen with the eigenvalue \(1/4\). The deep fact is that \(M\) is cuspidal and is \(\pm\)-invariant under the action of \(\Gamma_0(229)\) and, in fact, its normalizer: \(M|_\gamma = \pm M\) for \(\gamma \in \Gamma_0(229)\).

The established pattern therefore seems to be: in order to compute a Galois representation one tries to study a spectral problem and recover the Frobenius eigenvalues from the spectral parameters. This is true without any stretch even historically, e.g.,

- Ramanujan’s \(\Delta\) had been found long before the motive whose \(L\)-function is \(\Delta\)’s Mellin transform [5].
- The Poor–Yuen [26] paramodular form of conductor 61 over \(\mathbb{Q}\) had been found before a Calabi–Yau threefold whose \(H^3\) gives the respective Galois rep.

The spectral problem in question is typically similar to the one depicted above: given a lattice \(L\) and a prime number \(p\) one considers the collection of lattices ‘mutated at \(p\)’. For \(gl(2)\) and rank 2–lattices these are the lattices that are of index \(p\) in \(L\); for a general reductive group the mutations \(m\) correspond to the shape of the theorem on elementary divisors for that group which is in turn dictated by the structure of its Cartan algebra. Averaging over the mutations \(m_{sp}\) of a given shape \(s_p\), we obtain the action of the Hecke operator \(T_{sp}\) on the space of functions on lattices. It turns out that the information contained in the collection \(\{T_{sp}\}\) can be encoded by a semisimple conjugacy class in a group \(G\) with dual root data (the Satake transform), which one can try to interpret as a collection of the eigenvalues of the action of \(p\)-Frobenius in a Galois representation.

4. Lifts. This by now standard worldview suggests that with any group morphism \(G \to K\) should be associated some transform (lifting) of the respective spectral problems for the dual groups. This was properly codified by Langlands as the ‘functoriality principle’ in 1966-7.
considering the $L$–function

$$L(p, s) = \prod_p \frac{1}{\prod_i (1 - \alpha(p)_i p^{-s})},$$

where \{\alpha(p)_i\} is the collection of the eigenvalues described above. The $L$–functions (as Euler products) may undergo simple formal transformations, such as, e.g.

$$\prod_p \frac{1}{(1 - \alpha_p p^{-s})(1 - p/\alpha_p p^{-s})} \rightsquigarrow \prod_p \frac{1}{(1 - \alpha_p^2 p^{-s})(1 - (p/\alpha_p)^2 p^{-s})(1 - pp^{-s})}.$$

for the Shimura $\text{Sym}^2$ lift. It can be expected that these automorphic $L$–functions should be again automorphic, which can in certain situations be proved with the help of the converse theorems. At the level of $L$–functions, such transforms can be represented, roughly, by the convolution with theta and Eisenstein kernels. There also exist transforms at the level of automorphic functions themselves.

It should be noted that $L$–functions lift ‘per individual prime’, at the level of each individual Euler factor. In other words, one considers an essentially non-linear operation at each place, and seeks to express it in terms of some integral transform/convolution with a kernel. There are far fewer examples where one tensors, or multiplies, the material at different places. One notable example, not of this, but having such flavor, is the formula of Gross–Kohnen–Zagier [16] that gives integral representation for the height pairing of two Heegner points corresponding to discriminants $D_1, D_2$.

A different theta–type transform, the Eichler (–Shimizu/Jacquet–Langlands) correspondence [29], establishes a link between Hecke operators acting on 2–lattices of $GL(2)$ and 4–lattices acted upon by units in orders of quaternionic algebras. This is done implicitly by using the trace formula; the mechanism in the background is the integral transform kernel arising from the Howe duality [18] between $Gl(2)$ and the quaternions commuting in the metaplectic representation of $GSp(4)$. Since the metaplectic representation is essentially infinite-dimensional and non-geometric, it implies that the Eichler correspondence should not expected to be given by any geometric kernel.

5. Lifts as systems of equations on Frobenius traces. The geometric Langlands correspondence is a ‘double metaphor’ of this, via the standard chain of abstractions, whereby one first passes to the case of function fields over the finite field, then to curves over $C$. The role of $X_0(229)$ in our example with the Maass form is played by the space of principal bundles $\text{Bun}G^\vee$; the role of Galois representations is played by differential equations. However, the relationship between the differential equation on a curve and the differential system on $\text{Bun}G^\vee$ is now more direct and has geometric nature (is really a correspondence). It follows that there should exist certains precursors of the automorphic lifts at the level of the differential equations themselves, which would relate non–linear operations, such as symmetric/wedge/Schur powers to linear operations given by convolutions with kernels. For $\text{Sym}^2$ we will call these ‘duplication formulas’, in contrast to ‘multiplication formulas’ where integral representations of the result of multiplying solutions at different arguments are sought for. (Thus, duplication is similar in flavor to Langlands’s original formulation, while multiplication feels more like the Gross-Kohnen-Zagier formula.) The fact that the differential equation, or a crystal in the $p$–adic setup, has a multiplication kernel (= satisfies a non-abelian Abel’s theorem), imposes relations on its solutions or the trace functions; even duplication alone can be a very strong condition.
The simplest instance of a duplication formula on $\mathbb{A}^1$ reads $(a + x) + x = a + (2x)$; it can be obtained by restricting to diagonal the multiplication law $(a + x) + y = a + (x + y)$, which only looks uninspiring if we forget the machinery of the 'master formula' behind it.

6. Multiplication laws, Abël’s theorems and master formulas. Indeed, let $L$ be the differential operator $\frac{d}{dx}$ on $\mathbb{A}^1$ and consider the spectral problem $(L - \lambda)f(z) = 0$. Consider the ‘normalised’ ($f(0) = 1$) solution $\Phi(z, \lambda)$ of this spectral problem, $f(z) = \exp(\lambda z)$ and substitute $L$ for $\lambda$. Then $\exp(\lambda x) \exp(\lambda y) = \exp(\lambda(x + y))$ implies $\exp(xL)\exp(yL) = \exp(\lambda(x + y)L)$, which in turn implies an identity between the Hecke/shift operators $T_xT_y = T_{x+y}$.

In a higher genus situation, multiplication theorems in the abelian setup are known as addition theorems, but addition is a no-go beyond geometric class field theory. Abël’s discovery was that for each algebraic differential $\omega = R(x, y)dx$ there exists a number $P$, such that every sum of $N \geq P$ integrals can be reduced to a sum of $P$ integrals.

\[
\int_a^{b_1} \omega + \int_a^{b_2} \omega + \cdots + \int_a^{b_N} \omega = \int_a^{b_1} \omega + \int_a^{b_2} \omega + \cdots + \int_a^{b_P} \omega + E
\]

where the $y_1, \ldots, y_P$ depend algebraically on $x_1, \ldots, x_N$ and $E$ denotes an elementary function, i.e. rational+$\log$(rational). An obvious multiplicative reformulation of Abël’s theorem as we know it is that there exists a simple kernel $K(x|y)$ such that

\[\Phi(x_0)\Phi(x_1)\ldots\Phi(x_g) = \int K(x\mid y) \Phi(y_1)\ldots\Phi(y_g) dy_1\ldots dy_g\]

for any differential equation $d\Phi(x) = \Phi(x)\omega$ on a compact Riemann surface $C$ of genus $g$. One can fill in all the details; $K$, obviously, is a delta-kernel supported on the graph of $\sum(x - o) = \sum(y - o)$ in Jac$^0(C)$; $\Phi(z) = \exp \int_0^z \omega$. It is in the multiplicative formulation that Abël’s theorem is capable of surviving in the non-abelian situation.

The ‘master formula’ works similarly in the general situation. Once a quantization of the DE in question is chosen, the accessory parameters are turned into a commuting system of hamiltonians on the quantization model. The Hecke/shift-type integral operator $T_x$ is now obtained by evaluating $\Phi(x)$ with every accessory parameter replaced by the respective hamiltonian.

7. Clausen’s formula as a duplication formula for Bessel’s equation. It runs

\[
\left( \sum_{n=0}^{\infty} \frac{z^n}{n!} \right)^2 = \sum_{n=0}^{\infty} \frac{z^n}{n!} \binom{2n}{n} = 1 + 2z + \frac{3}{2}z^2 + \frac{5}{9}z^3 + \frac{35}{288}z^4 + \frac{7}{400}z^5 + \cdots
\]

and should be interpreted as a Sym$^2$ lift: the LHS is a non-linear transform, whereas the RHS is an integral representation, namely, by convolution on $\mathbb{O}_m$ with a ‘quadratic character’ DE whose solution is $\frac{1}{\sqrt{1 - 4z}} = \sum_{n=0}^{\infty} \binom{2n}{n} z^n$.

The relevance of this observation is that having a multiplication formula imposes relations on trace functions and enables one to recover them. To put it even simpler, at the level of the differential equation itself, the functional equation

\[
\left( \sum_{n=0}^{\infty} a_n z^n \right)^2 = \sum_{n=0}^{\infty} \binom{2n}{n} a_n z^n
\]

together with the initial condition $a_0 = a_1 = 1$ enables one to recover the series inductively. Bessel’s equation is, of course, rigid hypergeometric; the observation acquires real significance in examples with accessory parameters.
8. Bessel multiplication law. The way to pass from Bessel duplication to Bessel multiplication is given by the Sonine–Gegenbauer formula [15], [24]. Define

\[ J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!^2} \left( \frac{z}{2} \right)^{2m}; \]

it then reads, for \(0 < x < y < \infty, \ y - x < z < y + x,\)

\[ J_0(x)J_0(y) = \int_{y-x}^{y+x} K(x,y,z) J_0(z) \, dz, \]

with

\[ K(x,y,z) = \frac{1}{2\pi} S(x,y,z)^{-1/2} \]

where \(S(x,y,z) = 16^{-1}(2x^2y^2 + 2y^2z^2 + 2x^2z^2 - x^4 - y^4 - z^4)\) is the square of the area of a triangle of sides \(x, y, z.\)

9. Bessel’s equation as a degenerate case of D2. A natural idea would be to try tobettify various differential equations and automate the search for lifts for them. One expects the lifts for a DE to be expressible as identities between the traces in the monodromy of the local system of its solutions. Since Bessel’s equation has an irregular singularity at infinity, it would not be convenient to bettify the Clausen identity or the Sonine–Gegenbauer formula directly. Instead, it is useful to think of Bessel’s equation as a degenerate case of the generically regular D2 equation. Let \(f(t) = t^x + A t^y + B t,\) and put \(\mathcal{L} = f(t) \partial^2 + f'(t) \partial + t, \ \partial = \frac{d}{dt}.\) D2 equations are, up to trivial transformation, the equations of the form

\[ \mathcal{L}\varphi_\lambda(t) = \lambda \varphi_\lambda(t), \]

\(\lambda\) being the (only) accessory parameter. Define a sequence of polynomials \(b_n(\lambda) \in \mathbb{C}[\lambda]\) by

\[ b_0 = 1, \ b_{n+1}(\lambda) = \frac{1}{B(n+1)^2} \left( (\lambda - An(n+1)) b_n(\lambda) - n^2 b_{n-1}(\lambda) \right). \]

so that \(\varphi_\lambda(t) = \sum_{n=0}^{\infty} b_n(\lambda)t^n\) satisfies the D2 equation. Since the degree of \(b_n\) is \(n,\) it is natural to define the numbers \(c_{klm}\) by the formula \(b_k(\lambda)b_l(\lambda) = \sum_{m=0}^{\infty} c_{klm} b_m(\lambda).\) Expanding, one has

\[ \sum_{k,l,m=0}^{\infty} c_{klm} x^k y^l z^m = B \cdot P(x,y,Bz)^{-\frac{1}{2}}, \]

where

\[ P(x,y,z) = (Bxy - yz - xz)^2 - 4xyz (x + y + z + A) = \text{Discrim}_t \left( f(t) - (t-x)(t-y)(t-z) \right). \]

Put \(K(x,y,z) = Bz^{-1}P(x,y,Bz^{-1})^{-\frac{1}{2}} \in z^{-1}\mathbb{C}[z^{-1}][[x,y]],\) cf. [22]. One checks that \(\mathcal{L}_xK = \mathcal{L}_yK = \mathcal{L}_zK.\) For any \(\lambda \in \mathbb{C}\) define

\[ \psi_\lambda(x,y) = \int K(x,y,z) \varphi_\lambda(z) \, dz. \]

We have \(\mathcal{L}_x\psi_\lambda = \mathcal{L}_y\psi_\lambda = \lambda \psi_\lambda.\) Therefore

\[ \int K(x,y,z) \varphi_\lambda(z) \, dz = r(\lambda) \varphi_\lambda(x) \varphi_\lambda(y), \]

and by substituting \(x = 0,\) from \(\varphi_\lambda(0) = 1, K(0,0,y,z) = (z-y)^{-1}\) we obtain \(r(\lambda) = 1.\) Finally, the degenerate case \(A = B = 0\) can be transformed into Bessel’s equation:
\[(L - \lambda) r^{-1} J_0 \left( 2 \sqrt{-\lambda/t} \right) = 0.\] Returning to the Clausen duplication kernel, we see that the one arising in the D2 case is

\[(16) \quad K(x, x, z) = B \left( z^{-1} \left( x^2 - B \right)^2 - 4B f(x) z^{-1} \right)^{-1/2},\]

defining again the convolution with a quadratic character sheaf.

Remark. In our language, the differential operator \( L - \lambda \) is ‘quantized’ tautologically: the model is an instance of \( \mathbb{P}^1 \), the hamiltonian is \( L \) itself, and the Hecke–type shift operator at \( t_0 \) is \( \varphi_L(t_0) \); the Bun\(_G\)–quantization coincides with the ‘separation of variables’, or Beauville–Mukai, quantization introduced by Enriquez and Rubtsov. We refer the reader to [7] for the separation-of-variables quantization of second–order DEs with more than one accessory parameter.

10. Bettification and quaternionic rotation. Katz’s theorem [20] says that any two rigid differential equations (with r.s.) can be connected by a chain of elementary transformations, namely, twists by character sheaves or convolutions with Kummer sheaves w.r.t. to the additive group. One wants to expand this type of thinking to non–rigid situations. One might attempt to make the following principle precise (to begin with, on a case-by-case basis):

- Two differential equations with r.s. are in the same class if their respective character varieties are biregular and there is an algebraic transformation formula between the traces;
- If two differential equations are in the same class, they must be related by a chain of convolutions with ‘elementary’ kernels.

All rigid DEs are in the same class from this perspective: the spaces of accessible parameters are all one-pointed, and the DEs are indeed related by chains of elementary transforms. Clausen identities, or ‘functoriality’, would become instances of this principle, the integral representation for, say, the \( \text{Sym}^2 \) being one convolution step away from the original DE. However, there will also be more complicated formulas, possibly going beyond Langlands’s functoriality.

The main experimental results of this paper are as follows. Markov’s cubic

\[(17) \quad m_1^2 + m_2^2 + m_3^2 = m_1 m_2 m_3\]

is a character variety for the free group \( F_2 < A, B > \) realized as the fundamental group of a punctured torus \( E \setminus \{ O \} \); we consider those representations \( \varphi : F_2 \rightarrow SL_2 \) for which the loop around the puncture is anti-unipotent:

\[(18) \quad \varphi([A, B]) \sim \begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}\]

and set \( m_1 = \text{Tr} \varphi(A), m_2 = \text{Tr} \varphi(B), m_3 = \text{Tr} \varphi(AB). \) Let \( M \) be the Markov local system on \( E \) corresponding to a Markov triple of natural numbers \( (m_1, m_2, m_3) \). Consider its ‘convolution with the quadratic character’: the fiber at \( x \) of the local system \( M \ast \mathcal{L} \) is, by definition, \( H^1(E, j_!(M \otimes \mathcal{L})) \) where \( \mathcal{L} \) is the quadratic character sheaf ramified at \( 2x \) and the puncture \( O \) and uniquely determined by \( x \). The local system \( M \ast \mathcal{L} \) is defined on the 4:1 cover of \( E \setminus \{ O \} \) that corresponds to the homomorphism \( F_2 \rightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) given by \( w \mapsto (\deg_A w, \deg_B w) \) mod 2. Computing, we find experimentally:
The traces of the elements $A^2, B^2, (AB)^2$ in the monodromy representation of $M \ast L$ are respectively $m_1^2 - 1, m_2^2 - 1, m_3^2 - 1$.

One might therefore think that the local system $M \ast L$ is, up to a simple geometric pullback, the $\text{Sym}^2$ of $M$. However,

(2) there are no unipotents in the image of the monodromy representation in $M \ast L$.

In fact,

(3) the local system $M \ast L$ is a rank 3 local system arising from the action by a group of units in an order in $A\{2,3\}$ (the quaternion algebra over $\mathbb{Q}$ ramified at 2 and 3) in the conjugation representation on traceless quaternions.

Yet, according to our principle stated above, the identity in (1) must not happen without there being some longer chain of relations to $\text{Sym}^2$, or, a ‘twisted’ Clausen formula of some sort. Moreover, as integer points are Zariski–dense on Markov’s surface, one would expect such an identity to hold for any Markov–type local system on $E\{O\}$, i.e. for an arbitrary not necessarily integral point $(m_1, m_2, m_3)$. Indeed,

(4) let $M$ be any Markov–type local system on $E \setminus \{O\}$, and let $M_{A^1}$ be the Markov-type sheaf on $A^1$ whose pullback under the usual double cover $\sigma : E \setminus \{O\} \to A^1$ is $M$; its monodromies around the three finite critical values of $\sigma$ are reflections, and the conjugacy class of the local monodromy around $\infty$ is a size 2 Jordan block with the eigenvalue $\exp(2\pi i/4)$.

Let $L_\chi$ be the Kummer sheaf on $A^1$ corresponding to a fourth-order character so that the local monodromies at 0 and $\infty$ are $\exp(2\pi i/4)$ and $\exp(-2\pi i/4)$, and let $L_\rho$ be the local system on $A^1 \setminus \{\text{critical values of } \sigma\}$ corresponding to an eighth-order character so that the local monodromies at the three critical values are all equal to $\exp(2\pi i/8)$.

One has the following twisted Clausen formula:

$$\sigma^*\left(\text{Sym}^2(L_\rho \otimes (M_{A^1} \ast_{\text{Gal}} L_\chi))\right) = \sigma^*(M_{A^1}) \ast L$$

For the actual Markov system $M$, the term $M_{A^1} \ast_{\text{Gal}} L_\chi$ in the LHS can be identified as a subgroup of the triangle group $(3,4,4)$, which is known to be arithmetic, from which fact statement (3) follows immediately, as the RHS is $M \ast L$.

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