On well-definability of the $L_\infty$/$L_2$ Hankel operator and detection of all the critical instants in sampled-data systems

Tomomichi Hagiwara | Akira Inai | Jung Hoon Kim

Abstract
Because sampled-data systems have $b$-periodic nature with the sampling period $b$, an arbitrary $\Theta \in [0, b)$ is taken and the quasi $L_\infty$/$L_2$ Hankel operator at $\Theta$ is defined as the mapping from $L_2(-\infty, \Theta)$ to $L_\infty(\Theta, \infty)$. Its norm called the quasi $L_\infty$/$L_2$ Hankel norm at $\Theta$ is used to define the $L_\infty$/$L_2$ Hankel norm as the supremum of their values over $\Theta \in [0, b)$. If the supremum is actually attained as the maximum, then a maximum-attaining $\Theta$ is called a critical instant and the $L_\infty$/$L_2$ Hankel operator is said to be well-definable. An earlier study establishes a computation method of the $L_\infty$/$L_2$ Hankel norm, which is called a sophisticated method if our interest lies only in its computation. However, the feature of the method that it is free from considering the quasi $L_\infty$/$L_2$ Hankel norm for any $\Theta \in [0, b)$ prevents the earlier study to give any arguments as to whether the obtained $L_\infty$/$L_2$ Hankel norm is actually attained as the maximum, as well as detecting all the critical instants when the $L_\infty$/$L_2$ Hankel operator is well-definable. This paper establishes further arguments to tackle these relevant questions and provides numerical examples to validate the arguments in different aspects of authors’ theoretical interests.

1 | INTRODUCTION

As real control systems have encountered various practical problems relevant to the increasing size and complexity of the corresponding mathematical models, the model reduction problem has been regarded as one of the most important issues in large-scale systems. For example, recent discussions on the model reduction problems for Markovian jump, descriptor and linear continuous-time systems are, respectively, found in [1–3].

On the other hand, by noting the fact that most systems in engineering and science are described by dynamic models, whose output depends not only on the present input but also on the past input, the Hankel operator defined as the mapping from the past input to the future output has been considered in a number of constraint/model reduction problems [4–7]. More precisely, the associated norm of the Hankel operator is called the Hankel norm, and this norm has been used as a measure for the constraint/model reduction problems.

Consequently, the Hankel operator/norm for continuous-time linear time-invariant (LTI) systems has been deeply studied in [6–9]; the $L_2$ norm is employed to evaluate both the past input and the future output when the Hankel operator is defined, and we call the corresponding Hankel operator the $L_2$/$L_2$ Hankel operator for clarity. There is also a study defining the Hankel operator as a mapping from $L_2(-\infty, 0)$ to $L_\infty(0, \infty)$ [10] (and relevant studies [11, 12]), which we call the $L_\infty$/$L_2$ Hankel operator.

Aiming at extending existing studies on the Hankel operator/norm to sampled-data systems, on the other hand, the corresponding $L_2$/$L_2$ Hankel operator/norm was studied in [13] for the first time. Due to the periodic action of the LTI discrete-time controller together with the hold and sampling devices, the continuous-time input-output behavior of a sampled-data system (consisting of an LTI continuous-time generalised plant together with an LTI discrete-time controller) is $b$-periodic, where $b$ denotes the sampling period. Thus, it deeply matters when to take the time instant that separates the past and the future in defining the Hankel operator/norm. In the pioneering study [13], however, this issue was completely neglected and the past and the future were simply separated at an instant at which the sampler takes its action, that is, a sampling instant.

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This inadequate treatment was amended in our recent study [14] by taking an arbitrary $\Theta \in [0, b]$ as the instant separating the past and the future (under such treatment that 0 is a sampling instant). More precisely, as opposed to the study in [13] corresponding to the $L_2/L_2$ Hankel operator (or a quasi $L_2/L_2$ Hankel operator in the term of the present paper), we considered the mapping from $L_2(-\infty, \Theta)$ to $L_\infty(\Theta, \infty)$, and called it the quasi $L_\infty/L_2$ Hankel operator at $\Theta$; we took this modified treatment because it seemed easier than that with the $L_2/L_2$ Hankel operator in [13].

Then, the quasi $L_\infty/L_2$ Hankel norm at $\Theta$ was defined as the norm of this quasi $L_\infty/L_2$ Hankel operator. Consequently, the $L_\infty/L_2$ Hankel norm was defined as the supremum of this norm over $\Theta \in [0, b]$, and the $L_\infty/L_2$ Hankel operator was defined, if the supremum is attained as the maximum, as the quasi $L_\infty/L_2$ Hankel operator at $\Theta$ (which we call a critical instant) at which the maximum is attained. To put it another way, if a critical instant does not exist, then the $L_\infty/L_2$ Hankel operator fails to be well-definable. In connection with this, what has been shown in our recent study [14] on sampled-data systems can be summarised, very briefly, as follows.

(i) If our interest lies only in the $L_\infty/L_2$ Hankel norm and no other relevant issues at all, it can actually be characterised in such an alternative approach that completely avoids the reference to the quasi $L_\infty/L_2$ Hankel operators/norms.

(ii) The alternative approach, under some circumstances, can actually ensure for a given sampled-data system that the $L_\infty/L_2$ Hankel operator is well-definable. Furthermore, when the approach does ensure its well-definability, we can actually detect a critical instant (but not necessarily all the critical instants).

(iii) The alternative approach is essentially the same as applying the computation method of the $L_\infty/L_2$-induced norm [15–17], which implies that the $L_\infty/L_2$ Hankel norm coincides with the $L_\infty/L_2$-induced norm, as in the continuous-time LTI case.

Similarly for the cases of continuous-time and discrete-time systems, it is very important to consider model reduction problems for sampled-data systems, and the $L_\infty/L_2$ Hankel norm could be taken as an effective performance measure. Because we can naturally conclude from the finding (iii) that the synthesis problem of an optimal controller minimising the $L_\infty/L_2$ Hankel norm is the same as that minimising the $L_\infty/L_2$-induced norm even for sampled-data systems, the problem of minimising the error corresponding to the model reduction problems can be tackled through a more tractable problem of minimising the $L_\infty/L_2$-induced norm of sampled-data systems (see [18] for the treatment of the latter problem). In this sense, it can be expected that the $L_\infty/L_2$ Hankel norm contributes to wider applications to model reduction problems of sampled-data systems.

On the other hand, the finding (ii) above is also attractive particularly because it might suggest not only a chance of successfully computing the $L_\infty/L_2$ Hankel norm but also detecting all the associated critical instants while avoiding the computations of the quasi $L_\infty/L_2$ Hankel norms for all $\Theta \in [0, b]$ (or completely avoiding its computation even for a single $\Theta$). Unfortunately, however, the finding in our previous study [14] was so partial that it was not strong enough to definitely conclude that what might be suggested is indeed the case. This situation motivates us to continue much further studies on the relevant issues in the present paper.

Toward such further studies, this paper follows the structure shown in Figure 1. After providing in Section 2 a more technical review of the aforementioned notions and a quantitative summary of the earlier study [14], this paper first provides in Section 3 a computation method of the quasi $L_\infty/L_2$ Hankel norm at each $\Theta \in [0, b]$ as a key preliminary to the overall arguments. This precisely corresponds to what the earlier study [14] has successfully avoided to deal with numerically as well as theoretically. However, explicitly providing theoretical treatment of the quasi $L_\infty/L_2$ Hankel norm eventually turns out to be quite helpful in developing the intended further studies relevant to the earlier study [14]; at an early stage of the paper, it first leads to explicitly stating in Section 4 the seeming deficiency of [14] (at the very present stage of the paper) and clarifying the intended extension in a fully explicit manner. Very simply put, this paper aims at clarifying how much information our earlier sophisticated computation method of the $L_\infty/L_2$ Hankel norm alone in [14] (recall (i) above) can actually offer on well-definability of the $L_\infty/L_2$ Hankel operator and (if its well-definability is ensured as a by-product of the sophisticated method) further on the locations of all critical instants.

To tackle such a technical issue, Section 5 first provides a key result for a special case on the basis of the theoretical treatment of the quasi $L_\infty/L_2$ Hankel norm in Section 3, which states that $\Theta = 0$ is a critical instant whenever $P_1 = 0$ (where $P_1$ denotes the subsystem of the continuous-time generalised plant

![FIGURE 1](image_url) Flow chart for the overall arguments in this paper
from the disturbance to the controlled output). Furthermore, Section 5 introduces a classification of critical instants, which again follows from the technical arguments on the quasi $L_\infty / L_2$ Hankel norm in Section 3, which is thus a key preliminary section. Section 6 deals with the respective analysis of each case of the classification for the single-output case, and combining the results together with the aforementioned key result for a special case leads to the main results on deciding well-definability and detecting all critical instants. Section 7 extends the main results to the multi-output case. Section 8 gives numerical examples to demonstrate the theoretical results developed in this paper and investigates whether they could possibly be further strengthened. In connection with the relationship between the sections, the structure of the present paper can be depicted as Figure 1.

The notation in this paper is as follows. $\mathbb{R}^p$ and $\mathbb{N}$ denote the set of $p$-dimensional real vectors, and the set of positive integers, respectively. The 2-norm and $\infty$-norm of $x \in \mathbb{R}^p$ are denoted by $\|x\|^2$ and $\|x\|_\infty = \max_{i=1,\ldots,p} |x_i|$, respectively. $\mu_p(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the maximum diagonal entry and maximum eigenvector of a positive semi-definite real symmetric matrix, respectively. The shorthand notation $\mu_p(\cdot)$ means $\lambda_{\max}(\cdot)$ for $p = 2$ and $\mu_p(\cdot)$ for $p = \infty$.

2 THE $L_\infty / L_2$ HANKEL OPERATOR FOR SAMPLED-DATA SYSTEMS

Consider the stable linear time-invariant (LTI) sampled-data system $\Sigma_{\text{SD}}$ shown in Figure 2, by which we mean that the continuous-time generalised plant $P$ is LTI and the controller $\Psi$ is also LTI in the discrete-time sense. The symbols $\Psi$ and $P$ denote the zero-order hold and the ideal sampler, respectively, operating with sampling period $h$ in a synchronous fashion. Solid lines and dashed lines in this figure represent continuous-time and discrete-time signals, respectively. Suppose that $P$ and $\Psi$ are described by

\[
P z = C_1 x + D_{12} u ,
\]

\[
\Psi y = C_2 x ,
\]

respectively, where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$, and $\psi_k \in \mathbb{R}^{m \times n}$. Regarding $\Psi$, we use the notation $y_k := y(kh)$, and we have $u(t) = u_k$ for $k \leq t < (k+1)h$ due to the action of $\H$. We say that $\Sigma_{\text{SD}}$ is a single-output sampled-data system if $n_z = 1$; otherwise, we call it a multi-output sampled-data system.

The sampled-data system $\Sigma_{\text{SD}}$ viewed as a continuous-time mapping from the input $w$ to the output $z$ is periodically time-varying because of the $h$-periodic actions of $S, \Psi$ and $\H$. Hence, it matters quite essentially when to take the time instant separating the past and the future when we are to develop arguments on Hankel operators/norms for $\Sigma_{\text{SD}}$. In this regard, we first review the relevant viewpoint and notions in the earlier study [14], in which the input is taken in the $L_2$ space and the output is regarded as an element in $L_\infty$.

Let $\Theta \in [0, b)$ be the time instant separating the past about the input $w$ and the future about the output $z$. For each $\Theta \in [0, b)$, we consider the input $w$ defined on $(-\infty, \Theta)$ such that its $L_2(-\infty, \Theta)$ norm

\[
\|w(\cdot)\|_{L_2(\Theta)} := \left(\int_{-\infty}^{\Theta} |w(t)|^2 dt \right)^{1/2}
\]

is well-defined. The function space of such $w$ is denoted by $L_2(-\infty, \Theta)$. For simplicity, we also say $w \in L_2(-\infty, \Theta)$ for $w$ defined on $(-\infty, \infty)$, when $w(t) = 0$ for $t \geq \Theta$ and $w$ restricted to $(-\infty, \Theta)$ belongs to $L_2(-\infty, \Theta)$. On the other hand, the output $z$ is handled only on $(\Theta, \infty)$ (assuming that $w(t) = 0$ for $t \geq \Theta$) and regarded as an element of the function space $L_\infty(\Theta, \infty)$, that is, the set of $z$ such that its $L_\infty(\Theta, \infty)$ norm

\[
\|z(\cdot)\|_{L_\infty(\Theta, \infty)} := \text{ess sup}_{t \geq \Theta} |z(t)|
\]

is well-defined, where $\text{ess sup}_{t \geq \Theta} |z(t)|$ denotes the $p$-norm of finite-dimensional vectors. In all the subsequent arguments, we assume that either $p = 2$ or $p = \infty$ is fixed in each context, and the shorthand notation $L_\infty(\Theta, \infty)$ is occasionally used to mean either $L_{\infty, 2}(\Theta, \infty)$ and/or $L_\infty(\Theta, \infty)$.

The mapping from $L_2(-\infty, \Theta)$ to $L_\infty(\Theta, \infty)$ associated with $\Sigma_{\text{SD}}$ is called its quasi $L_\infty / L_2$ (or simply $\Sigma_{\text{SD}}$) Hankel operator at $\Theta$, which we denote by $\H_p^{(\Theta)}$. Its norm defined as

\[
\|\H_p^{(\Theta)}\|_{L_\infty / L_2} := \sup_{w \in L_2(-\infty, \Theta)} \frac{\|z(\cdot)\|_{L_\infty(\Theta, \infty)}}{\|w(\cdot)\|_{L_2(\Theta)}}
\]

is called the quasi $L_\infty / L_2$ (or $L_\infty / L_2$) Hankel norm at $\Theta$.

We further remark that the $L_\infty / L_2$ (or $L_\infty / L_2$) Hankel norm of the sampled-data system $\Sigma_{\text{SD}}$, which we denote by $\|\Sigma_{\text{SD}}\|_{L_\infty / L_2}$, is defined in [14] as

\[
\|\Sigma_{\text{SD}}\|_{L_\infty / L_2} := \sup_{\Theta \in [0, b)} \|\H_p^{(\Theta)}\|
\]

with respect to which we recall the following definition.

**Definition 1** (Definition 3.2 in [14]). If there exists a $\Theta^* \in [0, b)$ such that $\sup_{\Theta \in [0, b]} \|\H_p^{(\Theta)}\| = \|\H_p^{(\Theta^*)}\|$ (= max$\Theta \in [0, b] \|\H_p^{(\Theta)}\|$), we say that $\Theta^*$ is a critical instant and the $L_\infty / L_2$ Hankel operator is well-definable for $\Sigma_{\text{SD}}$. 

**FIGURE 2** Sampled-data system $\Sigma_{\text{SD}}$
Furthermore, $H^{(\Theta^{*})}_\infty$ is called the $L_\infty/L_2$ Hankel operator of $\Sigma_{SD}$ (even though $\Theta^*$ and thus the $L_\infty/L_2$ Hankel operator may not always be unique).

It is obvious from the above definition that the $L_\infty/L_2$ Hankel operator is well-definable if and only if there exists a critical instant. In addition, the $L_\infty/L_2$ Hankel norm equals the norm of the $L_\infty/L_2$ Hankel operator when the $L_\infty/L_2$ Hankel operator is well-definable, while the $L_\infty/L_2$ Hankel norm is always definable through (5) even if the $L_\infty/L_2$ Hankel operator is not well-definable. Regarding such subtleties, what has been clarified in our preceding study [14] is as follows (whose details will be given in Theorems 2 and 3 in Section 4).

(i) The $L_\infty/L_2$ Hankel norm can actually be obtained without computing the quasi $L_\infty/L_2$ Hankel norms at all.

(ii) Such a computation method of the $L_\infty/L_2$ Hankel norm could offer, under some circumstances for a given sampled-data system, some information on a critical instant (if one exists) and thus well-definability of the $L_\infty/L_2$ Hankel operator.

Because of (i) and (ii), the preceding study [14] did not refer to a problem of characterising the quasi $L_\infty/L_2$ Hankel norm $\|H^{(\Theta)}_\infty\|$ at each $\Theta \in [0, b)$. One of the topics dealt with in this paper is precisely the characterisation of $\|H^{(\Theta)}_\infty\|$ in such a way that its numerical computation becomes possible. Furthermore, this paper is ultimately interested in making use of the arguments for the above specific topic to tackle another fundamental topic on well-definability of the $L_\infty/L_2$ Hankel operator.

3 CHARACTERISING THE QUASI $L_\infty/L_2$ HANKEL NORM

This section discusses a numerical computation of the quasi $L_\infty/L_2$ Hankel norm for each $\Theta \in [0, b)$.

3.1 Lifting treatment of sampled-data systems

In the treatment of $\Sigma_{SD}$ with a periodically time-varying input-output mapping, it is convenient to apply the lifting technique [19–21], which converts the continuous-time function $f(\cdot)$ to the sequence $\{f_k(\Theta)\}$ of functions on $[0, b)$ given by

$$\hat{f}_k(\Theta) = f(kb + \Theta) \quad (0 \leq \Theta < b).$$

(6)

The lifted representation of $\Sigma_{SD}$ is given by

$$\begin{align*}
\hat{x}_{k+1} &= A\hat{x}_k + B\hat{w}_k \\
\hat{z}_k &= C\hat{x}_k + D\hat{w}_k,
\end{align*}$$

(7)

with $\xi_k := [x^T_k, \psi^T_k]^T$ (where $x_k := x(kb)$) by introducing the matrix $A: \mathbb{R}^{n+np} \to \mathbb{R}^{n+np}$ and the operators $B: L_2[0, b) \to \mathbb{R}^{n+np}$, $C: \mathbb{R}^{n+np} \to L_\infty[0, b)$ and $D: L_2[0, b) \to L_\infty[0, b)$ defined in an appropriate fashion. Their precise definitions are not necessary in the following arguments and thus are omitted; see, for example, [14, 16] for details.

Let us take an arbitrary $t \in \{\Theta, \Theta + b\}$, then we readily see from the $b$-periodicity of the input-output mapping of $\Sigma_{SD}$ that the output $\hat{z}(kb + t)$ for each input $w \in L_2(-\infty, \Theta)$ is equal to the output $\hat{z}$ at $t$ for another input obtained by shifting the original input $w$ to the left (i.e., toward the past) by $kb$. Note that such a shifted version of $w$ also belongs to $L_2(-\infty, \Theta)$, which implies that we may restrict our attention, without loss of generality, to $t \in \{\Theta, \Theta + b\}$ and evaluate only $\hat{z}(t)$ in the treatment of the quasi $L_\infty/L_2$ Hankel norm at $\Theta$. In addition, it is also obvious from (7) that if $t \in \{\Theta, \Theta + b\} \subset [0, 2b)$ actually belongs to $t \in [0, b)$, then $\hat{z}(t)$ is relevant to $\hat{F}_1\hat{w}$, while if $t \in [b, 2b)$, then $\hat{z}(t)$ is relevant to $\hat{F}_2\hat{w}$ under the assumption that $\hat{v}_0(\Theta) = 0 (\Theta \geq \Theta)$, where

$$\hat{F}_1 := \left[D \begin{array}{c} CB \end{array} \begin{array}{c} C A \end{array} \begin{array}{c} C A \end{array} \begin{array}{c} B \end{array} \cdots \right],$$

(8)

$$\hat{F}_2 := \left[D \begin{array}{c} CB \end{array} \begin{array}{c} C A \end{array} \begin{array}{c} C A \end{array} \begin{array}{c} B \end{array} \cdots \right],$$

(9)

and $\hat{w} := [\hat{w}_0^T, \hat{w}_1^T, \ldots]^T$. Introducing the notation $\|\hat{w}\|^{(\Theta)}_{\infty, L_\infty} := (\sum_{k=0}^{\infty} \|\hat{w}_k\|_{L_\infty}^2)^{1/2}$ under the principle that referring to $\|\hat{w}\|^{(\Theta)}_{L_\infty}$ always implies $\hat{w}_0(\Theta) = 0 (\Theta \geq \Theta)$ (which in turn implies $\|\hat{w}\|^{(\Theta)}_{L_\infty, L_\infty} = \|\hat{w}\|^{(\Theta)}_{L_\infty}$), we readily see the following relation about the quasi $L_\infty/L_2$ Hankel norm at $\Theta$:

$$\|H^{(\Theta)}_\infty\| = \sup_{\|w\|^{(\Theta)}_{L_\infty, L_\infty} \leq 1} \sup_{\|w\|^{(\Theta)}_{L_\infty} \leq \Theta \leq \Theta + b} \|\hat{z}(t)\|_{F_1} = \max \left\{ \sup_{\|\hat{w}\|^{(\Theta)}_{L_\infty, L_\infty} \leq 1} \sup_{\|\hat{w}\|^{(\Theta)}_{L_\infty} \leq \Theta} \|\hat{F}_1\hat{w}\|_{\Theta}\right\},$$

(10)

The following subsection characterises the quasi $L_\infty/L_2$ Hankel norms $\|H^{(\Theta)}_\infty\|$ more explicitly through this representation.

3.2 Explicit characterisation of $\|H^{(\Theta)}_\infty\|$

To give closed-form representations for $\|\hat{F}_1\hat{w}\|_{\Theta}$ with $\Theta \in \{\Theta, b\}$ as well as $\|\hat{F}_2\hat{w}\|_{\Theta}$ with $\Theta \in [0, \Theta)$ under $\hat{v}_0(\tau) = 0$, $\tau \in \{\Theta, b\}$, let us introduce the matrix $\hat{\delta}_t(\cdot) := \hat{\delta}(t - \tau)I_n$, defined on $t \in [0, b)$, where $\hat{\delta}(t)$ is a unit impulse occurring at $t = 0$. We first consider $\hat{z}(t)$ with $t \in \{\Theta, b\}$ and thus $\hat{F}_1\hat{w}$ with $\Theta = t \in \{\Theta, b\}$ in (10). Since $\hat{v}_0(\tau) = 0$ for $\tau \in \{\Theta, b\}$, it is given by
On the other hand, \( \zeta(t) \) with \( t \in [a, \Theta + b] \), that is, \( \{\mathbf{F}_2 \hat{v}(\Theta)\} \) with \( \Theta = t - b \in [0, \Theta] \) in (10), is given by

\[
\{\mathbf{F}_2 \hat{v}(\Theta)\} = \int_0^\Theta (CB\delta_r(\Theta))\hat{v}_0(t)\,dt + \sum_{k=1}^b \int_0^{\Theta} (CA^k B\delta_r(\Theta))\hat{v}_{(k+1)}(\Theta)\,dt,
\]

again by \( \hat{v}_0(t) = 0, \quad t \in [\Theta, b) \). Let us further introduce

\[
I_{1}^{\Theta} (\Theta) := \int_0^\Theta (D\delta_r(\Theta))[(D\delta_r(\Theta))]^T \,dt
\]

\[
I_{2}^{\Theta} (\Theta) := \int_0^\Theta (C\delta_r(\Theta))[(C\delta_r(\Theta))]^T \,dt
\]

(12)

for each \( \Theta \in [0, b) \). Then, essentially the same arguments as those in [16] (which simply use the continuous-time and discrete-time Cauchy-Schwarz inequalities applied to (11) and (12), with the associated necessary and sufficient conditions for yielding equality taken into account) lead to

\[
\sup_{\|\hat{v}\|_{L_2} \leq 1} |\{\mathbf{F}_1 \hat{v}(\Theta)\}|_p = \max_{\|\hat{v}\|_{L_2} \leq 1} |\{\mathbf{F}_2 \hat{v}(\Theta)\}|_p = \max_{\|\hat{v}\|_{L_2} \leq 1} \mu_{1/2}^{p} (F_1^{(\Theta)} (\Theta)), \quad \Theta \in [0, b)
\]

(15)

(16)

where \( \mu_p(\cdot) \) denotes \( \lambda_{\max}(\cdot) \) for \( p = 2 \) and \( d_{\max}(\cdot) \) for \( p = \infty \). Hence, by (10), we are led to the following result.

**Theorem 1.** For each \( \Theta \in [0, b) \), the quasi \( L_{\infty} / L_{2} \) Hankel norm \( \|\mathbf{H}_{\Theta}^{(\Theta)}\| \) is given by

\[
\|\mathbf{H}_{\Theta}^{(\Theta)}\| = \max \{\sup_{\Theta < \Theta} \mu_{1/2}^{p} (F_1^{(\Theta)} (\Theta)) \}
\]

(17)

Computational methods for the matrices \( F_1^{(\Theta)} (\Theta) \) and \( F_2^{(\Theta)} (\Theta) \) are given in [22], from which it is obvious that \( F_1^{(\Theta)} (\Theta) \) is continuous in \( \Theta \in [0, b] \) (where \( F_1^{(\Theta)} (\Theta) := \lim_{\Theta \rightarrow b-0} F_1^{(\Theta)} (\Theta) \)) and \( F_2^{(\Theta)} (\Theta) \) is continuous in \( \Theta \in [0, \Theta] \) for each \( \Theta \in [0, b) \). Hence, \( \sup_{\Theta < \Theta} \mu_{1/2}^{p} (F_1^{(\Theta)} (\Theta)) \) may actually be replaced by \( \max_{\Theta < \Theta} \sup_{\Theta < \Theta} \mu_{1/2}^{p} (F_1^{(\Theta)} (\Theta)) \), respectively, which, for our later purposes, motivates us to define a vital timing \( \theta_{\max}^{\star} (\Theta) \) for each \( \Theta \in [0, b) \) as follows.

**Definition 2.** For each \( p = 2, \infty \) and every \( \Theta \in [0, b) \), we define \( \theta_{\max}^{\star} (\Theta) \) by

\[
\theta_{\max}^{\star} (\Theta) = \arg \max_{\Theta \in [0, \Theta]} \mu_{1/2}^{p} (F_1^{(\Theta)} (\Theta)) \quad (18)
\]

when \( \|\mathbf{H}_{\Theta}^{(\Theta)}\| = \max_{\Theta \in [0, \Theta]} \mu_{1/2}^{p} (F_1^{(\Theta)} (\Theta)) \), and by

\[
\theta_{\max}^{\star} (\Theta) = b + \arg \max_{\Theta \in [0, \Theta]} \mu_{1/2}^{p} (F_2^{(\Theta)} (\Theta))
\]

(19)

when \( \|\mathbf{H}_{\Theta}^{(\Theta)}\| = \max_{\Theta \in [0, \Theta]} \mu_{1/2}^{p} (F_2^{(\Theta)} (\Theta)) \).

**Remark 1.** Definition 2 should be interpreted as uniquely defining \( \theta_{\max}^{\star} (\Theta) \) for each \( \Theta \in [0, b) \) through the following:

1) If more than one \( \theta_{\max}^{\star} \) satisfy

\[
\mu_{1/2}^{p} (F_1^{(\Theta)} (\Theta)) = \max_{\Theta \in [0, \Theta]} \mu_{1/2}^{p} (F_1^{(\Theta)} (\Theta)), \quad \Theta \in [0, \Theta)
\]

then \( \theta_{\max}^{\star} (\Theta) \) in (18) is defined as the minimum of such \( \theta_{\max}^{\star} \).

2) If and only if no such \( \theta_{\max}^{\star} \) exists in 1), \( \theta_{\max}^{\star} (\Theta) \) is defined through (19), similarly with the minimum possible value.

3) When we are led to \( \theta_{\max}^{\star} (\Theta) = b \) through (18), we actually use the notation \( \theta_{\max}^{\star} (\Theta) = b - 0 \) instead, to discriminate the situation from that arising from (19).

The existence of the minimum is easy to see both in 1) and 2) by the continuity arguments.

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1 For each \( \Theta \in [0, b) \) and every \( \Theta \in [0, b] \), the corresponding worst input achieving the maximum in (15) can also be determined, which is unique up to a scalar factor. Similarly for (16). This, together with (10), implies that the worst input to \( \mathbf{H}_{\Theta}^{(\Theta)} \), which does exist because \( \mathbf{H}_{\Theta}^{(\Theta)} \) is a finite-rank operator [14], is also unique up to a scalar factor for each \( \Theta \in [0, b) \).
Remark 2. Roughly speaking, \( \Theta^*(\Theta) \) can be interpreted as the (earliest) instant \( \theta \geq \Theta \) at which the output \( z \) to the worst input
\[
w^* \in W_\Theta := \{w \in L_2(-\infty, \Theta) : \|w\|_{L_2}^\Theta \leq 1\},
\]
“attains” its \( L_{\infty,p}^0(\Theta, \infty) \) norm (i.e. \( \Theta \) such that \( \|z\|_{\infty}^\Theta = \|z(\tau)\|_p \) is satisfied) for the given \( \Theta \in [0, b) \). For notational case, \( \Theta^*(\Theta) \) is sometimes denoted simply by \( \Theta^*(\Theta) \) when the underlying \( p \) is clear from the context or whether \( p = 2 \) or \( p = \infty \) is not significant in the relevant arguments.

Remark 3. In connection with 3) in Remark 1 (which is associated with the definition of \( b - 0 \)), we regard in the following arguments that \( i \) \( b - 0 \neq b \), \( h - 0 \neq h \), \( b - 0 \neq b \), and \( h - 0 \neq h \), and \( \tau < b - 0 \) if and only if \( \tau < b \). Only if \( \Theta^*(\Theta) = b - 0 \), the output \( z \) to the worst input \( w^* \) for \( \Theta \) may possibly fail to attain its \( L_{\infty,p}^0(\Theta, \infty) \) norm (there might not exist \( t \) such that \( \|z\|_{\infty}^\Theta = \|z(t)\|_p \)). This is because the output \( z \) is possibly discontinuous only at sampling instants.

4 | MOTIVATING QUESTION ON WELL-DEFINABILITY OF THE \( L_\infty / L_2 \) HANKEL OPERATOR

Theorem 1 obviously provides us with a computation method of the \( L_\infty / L_2 \) Hankel norm \( \|\Sigma_{SD}\|_{H_2} \) through its definition (5) by computing the quasi \( L_\infty / L_2 \) Hankel norm \( \|H_p^\Theta\| \) for each \( \Theta \in [0, b) \). We call such a computation method a naive computation method of \( \|\Sigma_{SD}\|_{H_2} \). Such a method is capable of detecting any critical instant \( \Theta^* \), if one exists, and can determine whether or not the \( L_\infty / L_2 \) Hankel operator is well-definable, in principle. In the preceding study [14] on the \( L_\infty / L_2 \) Hankel norm \( \|\Sigma_{SD}\|_{H_2} \), on the other hand, the following result was established through entirely different arguments characterising \( \|H_p^\Theta\| \) for each \( \Theta \in [0, b) \). We call such a computation method a sophisticated computation method of \( \|\Sigma_{SD}\|_{H_2} \) alone.

Theorem 2 ([14]). The \( L_\infty / L_2 \) Hankel norm is given by
\[
\|\Sigma_{SD}\|_{H_2} = \sup_{\Theta \in [0, b)} \mu_p^{1/2}(F(\Theta)), \quad p = 2, \infty,
\]
where
\[
F(\Theta) := \int_0^\Theta (D\delta_\tau(\Theta))(D\delta_\tau(\Theta))^T d\tau
+ \sum_{k=0}^\infty \int_0^b (C\delta_\tau^k B\delta_\tau(\Theta))(C\delta_\tau^k B\delta_\tau(\Theta))^T d\tau.
\]

Even though the sophisticated computation method is convenient and efficient as a method for merely computing \( \|\Sigma_{SD}\|_{H_2} \), it is not obvious whether it can offer some information about a critical instant \( \Theta^* \) (i.e. well-definability of the \( L_\infty / L_2 \) Hankel operator, and if a critical instant does exist, how to determine its value, and so on). The remaining part of this paper is mostly devoted to elaborating on this issue, that is, studying how much information the sophisticated computation method of \( \|\Sigma_{SD}\|_{H_2} \) can actually offer on well-definability of the \( L_\infty / L_2 \) Hankel operator (and when it is well-definable, how much information it can offer on the values of the critical instants \( \Theta^* \)).

We begin by recalling that only the following very partial answer has been obtained on the sophisticated computation method about its implication on the relevant issues on well-definability of the \( L_\infty / L_2 \) Hankel operator and (when it is well-definable) the values of the critical instants \( \Theta^* \).

Theorem 3 ([14]). For the underlying \( p = 2 \) or \( p = \infty \), let us suppose that \( \mu_p^{1/2}(F(\Theta)), \Theta \in [0, b) \) is maximum-attaining in the sense that there exists \( \Theta^* \in [0, b) \) such that \( \sup_{\Theta \in [0, b)} \mu_p^{1/2}(F(\Theta)) = \mu_p^{1/2}(F(\Theta^*)) \). For each such \( \Theta^* \), let \( \Theta = \Theta^* \). Then, \( \Theta \) is a critical instant relevant to \( \|\Sigma_{SD}\|_{H_2} \).

This theorem can be interpreted as giving a sort of sufficient condition under which the sophisticated computation method of \( \|\Sigma_{SD}\|_{H_2} \) alone through \( F(\Theta) \) can conclude the well-definability of the \( L_\infty / L_2 \) Hankel operator and also detect (a subset of) the associated critical instants. However, the following issues are not clear from Theorem 3.

(A) Whether \( \mu_p^{1/2}(F(\Theta)), \Theta \in [0, b) \) is maximum-attaining whenever a critical instant does exist (in other words, whether a critical instant never exists and the \( L_\infty / L_2 \) Hankel operator is not well-definable when it fails to be maximum-attaining).

(B) Whether all the critical instants can be detected through \( \mu_p^{1/2}(F(\Theta)), \Theta \in [0, b) \) as the maximum-attaining points.

Our arguments will lead to including a somewhat negative answer to the above two questions, but yet and more importantly, we will eventually show that the sophisticated computation method of \( \|\Sigma_{SD}\|_{H_2} \) alone in [14] can rather often actually provide somewhat more information on a critical instant \( \Theta^* \) and well-definability of the \( L_\infty / L_2 \) Hankel operator than what we can simply see as an immediate consequence of Theorem 3.

5 | PRELIMINARIES TO DECIDING WHETHER A CRITICAL INSTANT EXISTS

Regarding the questions (A) and (B) in the last section motivating our further studies, we will eventually establish the
following facts; they correspond to a brief but quite informative summary of all the results that the present paper clarifies as the hidden abilities of the sophisticated computation method of the $L_\infty / L_2$ Hankel norm $\|\Sigma_{SD}\|_{H_\infty}$ in [14].

For single-output sampled-data systems, if $F (\Theta)^{1/2}$ is not maximum-attaining over $\Theta \in [0, b)$, then

(a) no critical instant exists if $P_{11} \neq 0$;
(b) every $\Theta \in [0, b)$ is a critical instant if $P_{11} = 0$

where $P_{11}$ denotes the subsystem from $w$ to $z$ in the continuous-time generalised plant $P$. In addition,

(c) if $P_{11} \neq 0$ and $P_{11} + D_{12}D_{22}P_{22} \neq 0$ and if $F (\Theta)^{1/2}$, $\Theta \in [0, b)$ is maximum-attaining, then every critical instant $\Theta$ is a maximum-attaining point of $F (\Theta)^{1/2}$ (and vice versa);
(d) Suppose that there exists a critical instant. If $\Theta = 0$ is not a critical instant, then $F (\Theta)^{1/2}$, $\Theta \in [0, b)$ is maximum-attaining and every critical instant is its maximum-attaining point.

In particular, the above claims (a) and (c) imply that a positive answer can be obtained for both questions (A) and (B) if $P_{11} \neq 0$ and $P_{11} + D_{12}D_{22}P_{22} \neq 0$, meaning that the sophisticated computation method fully suffices in this special case in finding all critical instants or deciding their absence. However, let us consider, for example, the case $P_{11} = 0$, then it remains unclear whether that method fully suffices, suggesting a limit of the method; this is because $P_{11} = 0$ implies that $\Theta = 0$ is a critical instant as shown shortly in Theorem 4 and thus if $F (\Theta)^{1/2}$, $\Theta \in [0, b)$ happens to be maximum-attaining, then both claims (b) and (d) become helpless. These arguments will further be generalised to cover the multi-output case.

5.1 $\Theta = 0$ is a critical instant if $P_{11} = 0$

The following result plays an important role in our later arguments, which in particular implies that the $L_\infty / L_2$ Hankel operator is well-definable if $P_{11} = 0$.

**Theorem 4.** If $P_{11} = 0$, then $\Theta = 0$ is a critical instant of $\Sigma_{SD}$ both for $p = 2$ and $p = \infty$.

**Proof.** Since $F_2$ is a submatrix of the operator matrix $F_1$ by (8) and (9), it immediately follows that

$$\sup_{\Theta \in [0, b)} \|F_1(\Theta)^{1/2}\|_p \leq \sup_{\Theta \in [0, b)} \|F_2(\Theta)^{1/2}\|_p, \quad (p = 2, \infty), \quad \Theta \in [0, b).$$

Hence, it follows from (10) and (15) that

$$\|H_p(\Theta)^{1/2}\| \leq \sup_{\Theta \in [0, b)} \mu_p^{1/2}(F_1(\Theta)^{1/2}) \quad (p = 2, \infty) \quad \text{for every } \Theta \in [0, b),$$

once the definition of $F_1(\Theta)$ is extended as it is so that $\Theta \in [0, \Theta]$ is also covered in (13). Since $P_{11} = 0$, it follows that the first term on the right-hand side of (13) vanishes and thus $F_1(\Theta)$ is independent of $\Theta$. Hence, for each $\Theta \in [0, b)$, we have

$$\|H_p(\Theta)^{1/2}\| \leq \sup_{\Theta \in [0, \Theta]} \mu_p^{1/2}(F_1(\Theta)^{1/2}) = \sup_{0 \leq \Theta < b} \mu_p^{1/2}(F_1(\Theta)^{1/2}) = \|H_p^{\infty}\| \quad (p = 2, \infty) \quad \text{by (17)}.$$

This completes the proof.

5.2 Classification of $\vartheta_p^* (\Theta)$ for critical instant $\Theta$ and necessary and sufficient condition for the existence of a critical $\Theta$ satisfying $\vartheta_p^* (\Theta) = \Theta$

We are mostly concerned about a possibility that a critical instant may be overlooked, despite a possible hope (somewhat “suggested” by Theorem 3) that it can always be detected by the sophisticated computation method of $\|\Sigma_{SD}\|_{H_\infty}$ through the curve of $\mu_p^{1/2}(F (\Theta))$, $\Theta \in [0, b)$. Hence, in the following arguments, we virtually confine ourselves to the case when a critical instant $\Theta$ does exist (from now on, a critical instant is denoted simply by $\Theta$ rather than $\Theta^*$ to make the notation concise). Furthermore, for each critical instant $\Theta \in [0, b)$, we begin by classifying the associated $\vartheta_p^* (\Theta)$ given in Definition 2 into the following five cases (recall Remarks 1 and 3 for the notation $\vartheta_p^* (\Theta) = b - 0)$:

(I) $\vartheta_p^* (\Theta)$ is a critical instant satisfying (I) and only if $\mu_p^{1/2}(F (\Theta)) = \max_{\Theta \in [0, b)} \|F_1(\Theta)^{1/2}\|_p$

by $F_1(\Theta)$ and (15) (this is true even if $\Theta$ is not a critical instant). Then, since $\Theta(\Theta)$ is right-continuous at every $\Theta \in [0, b)$ regardless of $\Theta$, we readily have the following result from (21).

**Proposition 1.** $\Theta \in [0, b)$ is a critical instant satisfying (I) if and only if $\mu_p^{1/2}(F (\Theta)) = \max_{\Theta \in [0, b)} \|F_1(\Theta)^{1/2}\|_p$.

Hence, we readily have the following slightly improved statement of Theorem 3 by restricting our attention to case (I).

**Theorem 5.** There exists a critical instant $\Theta$ such that $\vartheta_p^* (\Theta) = \Theta$ if and only if $\mu_p^{1/2}(F (\Theta))$, $\Theta \in [0, b)$ for the underlying $p = 2$ or $p = \infty$ is maximum-attaining. For each $\Theta = \vartheta_p^*$ that attains the maximum, $\Theta = \vartheta_p^*$ is a critical instant such that $\vartheta_p^* (\Theta) = \Theta$.

6 DECIDING WHETHER A CRITICAL INSTANT EXISTS FOR SAMPLED-DATA SYSTEMS: SINGLE-OUTPUT CASE

This section together with the following section considers the remaining cases other than (I) (which is covered by Theorem 5). This section confines itself to single-output sampled-data systems, and deals with the cases (II) (including (II)), (III) and (IV) in separate subsections. Furthermore, we combine the results for all the cases to show the claims (a)–(d) in the beginning of Section 5. These claims will be generalised
in the following section to the multi-output sampled-data case.

6.1 Necessary condition for the existence of a critical \( \Theta \) satisfying (II) or (II')

Suppose that there exists a critical instant \( \Theta \in [0, b) \) satisfying (II) or (II'), which we denote by \( \Theta_1 \). Let

\[
\|w\|_2^2 = \{ w \in L_2(-\infty, \Theta_1) : \|w\|_2^2 = 1 \},
\]

be the worst input to \( H^{\Theta_1} \) and let \( \theta^*_1 := \theta^*(\Theta_1) \) (since we are dealing with single-output sampled-data systems, whether \( p = 2 \) or \( p = \infty \) does not matter at all, and thus \( H^{\Theta_1} \) is simply denoted by \( H^{\Theta_1} \); similarly, \( \|\Sigma_{\text{SD}}\|_{\Delta_1} \) will simply be denoted by \( \|\Sigma_{\text{SD}}\|_{\Delta_1} \)). Then, because of the underlying assumption (II) or (II'), it follows from (13) and (24) that

\[
\|H^{\Theta_1}\| = \|F_1^{\Theta_1}\|_1 \bigg( \frac{w^{\ast}_1}{\|w^{\ast}_1\|_2} \bigg) = \bigg( F_1^{\Theta_1} \bigg) \bigg( \frac{w^{\ast}_1}{\|w^{\ast}_1\|_2} \bigg) \bigg)^{1/2}.
\]

Take an arbitrary \( \Theta_2 \) such that \( \Theta_1 < \Theta_2 < \theta^*_1 \) (note that \( \Theta_2 < b \) then), and take \( w_0 \) such that \( w_0(t) = \theta^*_1(t) \) for \( t < \Theta_1 \) and \( w_0(t) = 0 \) for \( \Theta_1 < t < \Theta_2 \) (\( w^{\ast}_1 \) and \( w_0 \) are thus essentially identical). We readily see that \( w_0 \in \partial W_{\Theta_2} \) and \( (H^{\Theta_1}w_0)(\theta^*_1) = (H^{\Theta_1}w_0)(\theta^*_1) \). Hence, it is obvious that \( \|H^{\Theta_1}\| \geq \|H^{\Theta_1}\| \). However, since \( \Theta_1 \) is a critical instant, this must imply \( \|H^{\Theta_1}\| = \|H^{\Theta_1}\| \), with \( w_0 \) being the worst input to \( H^{\Theta_1} \). In particular, \( \Theta_1 \) is a critical instant and \( \theta^*(\Theta_2) = \theta^*_1 \), which is either less than \( b \) or given by \( b - 0 \). Hence, in a way similar to (24), we have

\[
\|H^{\Theta_2}\| = \left( F_1^{\Theta_1} \right)^{1/2}.
\]

This, together with (13) and (24), leads to

\[
\int_{\Theta_1}^{\Theta_2} (D\Delta_1)(\theta^*_1)(D\Delta_1)(\theta^*_1) \bigg( D\Delta_1 \bigg)^T \bigg( D\Delta_1 \bigg)^T d\tau = 0,
\]

since \( \|H^{\Theta_2}\|^2 - \|H^{\Theta_1}\|^2 = 0 \). Hence, by \( \Theta_1 < \Theta_2 < \theta^*_1 \) and \( (D\Delta_1)(\theta^*_1) = C_1 \exp(-A(t_\ast - \tau))B_1 \), it follows that \( C_1 \exp(-A(t_\ast - \tau))B_1 \equiv 0 \) over the interval \( (\theta^*_1 - \Theta_2, \theta^*_1 - \Theta_1) \) and thus on the whole real axis since \( C_1 \exp(-A(t)B_1) \) is real analytic in \( t \). This implies that \( P_{11} = 0 \).

Summarising the above arguments with some simple additional considerations leads to the following claims.

1. A critical \( \Theta \) \((= \Theta_1)\) satisfying (II) exists only if another critical instant \( \Theta' \) \((= \theta^*_1) \) satisfying (I) exists.
2. A critical \( \Theta \) satisfying (II) exists only if \( P_{11} = 0 \). A critical \( \Theta \) satisfying (II') exists only if \( P_{11} = 0 \).
3. If \( \Theta \in [0, b) \) is a critical instant satisfying (II) (or (II'), respectively), then every \( \Theta' \in [\Theta, \theta^* (\Theta)] \) (or \( \Theta' \in [\Theta, b) \), respectively) is a critical instant satisfying (II) (or (II'), respectively).

6.2 Necessary condition for the existence of a critical \( \Theta \) satisfying (III)

Suppose that there exists a critical instant \( \Theta \in [0, b) \) satisfying (III), which we denote by \( \Theta_1 \), and take an arbitrary \( \Theta_2 \) such that \( \Theta_1 < \Theta_2 < b \). Then, it is not hard to see that essentially the same arguments as those in the preceding subsection lead to \( \|H^{\Theta_2}\| = \|H^{\Theta_1}\| \) (which implies that \( \Theta_2 \) is also a critical instant) and

\[
\|H^{\Theta_2}\| = \left( F_2^{\Theta_1} \right)^{1/2} (i = 1, 2).
\]

Hence, it follows from (14) and (27) that

\[
\int_{\Theta_1}^{\Theta_2} (C_1 + D_{12}D_{1p}C_2) \exp(-A(b - \tau))B_1 \times B_1^T \exp(-A(b - \tau))C_1 + D_{12}D_{1p}C_2^T \bigg( D\Delta_1 \bigg)^T \bigg( D\Delta_1 \bigg)^T d\tau = 0,
\]

or equivalently,

\[
P_{11} + D_{12}D_{1p}P_{21} = 0,
\]

where \( P_{21} \) denotes the subsystem from \( w \) to \( y \) in \( P \).

To summarise, we are led to the following claims, where the first claim can be derived readily if we note the \( b \)-periodicity of \( \Sigma_{\text{SD}} \).

1. A critical \( \Theta \) \((= \Theta_1)\) satisfying (III) exists only if \( \Theta = 0 \) is a critical instant satisfying (I).
2. A critical \( \Theta \) satisfying (III) exists only if \( \Theta = 0 \) is a critical instant satisfying (I).
3. If \( \Theta \in [0, b) \) is a critical instant satisfying (III), then every \( \Theta' \in [\Theta, b) \) is a critical instant satisfying (III).

6.3 Necessary condition for the existence of a critical \( \Theta \) satisfying (IV)

Suppose that there exists a critical instant \( \Theta \in [0, b) \) satisfying (IV). Then, similar arguments with the \( b \)-periodicity of \( \Sigma_{\text{SD}} \) taken into account readily lead to the following claims.

1. A critical \( \Theta \) satisfying (IV) exists only if \( \Theta = \theta^* (\Theta) - b \in (0, b) \) is a critical instant satisfying (I).
2. A critical \( \Theta \) satisfying (IV) exists only if \( P_{11} = 0 \).
3. If \( \Theta \) is a critical instant satisfying (IV), then every \( \Theta' \) such that \( 0 \leq \Theta' < \theta^* (\Theta) - b \in (0, b) \) is a critical instant satisfying (II).
Here, the second claim can be obtained by combining the second claim in Section 6.1 with the last claim above.

6.4 Summary for single-output sampled-data systems

Summarising the above arguments for the single-output sampled-data systems, we readily lead to the following results.

**Proposition 2** (Claim 1 in Sections 6.1–6.3). If there exists a critical instant \( \Theta \) satisfying \( \Theta < \Theta^* (\Theta) < h \) or \( \Theta^* (\Theta) \geq h \), then there also exists a critical instant \( \Theta^* \) such that \( \Theta^* (\Theta) = \Theta \). In particular, if there exists a critical instant \( \Theta^* \) such that \( \Theta^* (\Theta) = h \), then \( \Theta = 0 \) is a critical instant such that \( \Theta^* (\Theta) = \Theta \).

**Proposition 3** (Claim 2 in Sections 6.1–6.3). There exists a critical instant \( \Theta \) satisfying \( \Theta < \Theta^* (\Theta) < h \) only if \( P_1 = 0 \). There exists a critical instant \( \Theta^* \) satisfying \( \Theta^* (\Theta) = h \) only if \( P_1 + D_{12} D_2 P_{21} = 0 \).

**Proposition 4** (Claim 3 in Sections 6.1–6.3). \( \Theta \) is a critical instant satisfying \( \Theta < \Theta^* (\Theta) < h \) only if every \( \Theta^* \in [0, \Theta^* (\Theta)] \) is a critical instant. \( \Theta \) is a critical instant satisfying \( \Theta^* (\Theta) = h \) only if every \( \Theta^* \in (0, \Theta^* (\Theta)) \) is a critical instant. \( \Theta \) is a critical instant satisfying \( \Theta^* (\Theta) = h \) if and only if every \( \Theta^* \in (0, \Theta^* (\Theta) - h) \) is a critical instant.

On the basis of the above results, we give a series of results that are relevant to what information the sophisticated computational method of \( \| \Sigma_{SD} \|_1 \) alone can actually offer on the associated issue of critical instants and well-definability of the Hankel operator as a by-product. First, we can see that combining Proposition 2 with Proposition 1 immediately leads to the following result.

**Theorem 6** (single-output case). There exists a critical instant \( \Theta \) satisfying \( \Theta \leq \Theta^* (\Theta) < h \) or \( \Theta^* (\Theta) \geq h \), if and only if \( F(\Theta)^{1/2}, \Theta \in [0, h] \) is maximum-attaining. In particular, there exists a critical instant \( \Theta \) satisfying \( \Theta^* (\Theta) = h \) only if it satisfies \( F(\Theta)^{1/2} \) is attained at \( \Theta = 0 \).

**Proof.** Sufficiency of the first assertion follows from Proposition 1. From Proposition 2, on the other hand, if there exists a critical \( \Theta \) satisfying \( \Theta < \Theta^* (\Theta) < h \) or \( \Theta^* (\Theta) \geq h \), then there exists a critical \( \Theta \) satisfying \( \Theta^* (\Theta) = h \). This together with Proposition 1 establishes necessity of the first assertion. The second assertion immediately follows by combining the second assertion of Proposition 2 with Proposition 1. This completes the proof.

We can interpret the above theorem as showing the following: even if there exists a critical instant, \( F(\Theta)^{1/2} \) is not maximum-attaining if (and only if) \( \Theta^* (\Theta) = h \) for every critical instant \( \Theta \). Roughly speaking, this implies that the “converse” of Theorem 3 (derived in the earlier study [14]) is not true, in the sense that \( F(\Theta)^{1/2} \) failing to be maximum-attaining over \( \Theta \in [0, h] \) does not necessarily mean that no critical instant exists. This is precisely a weakness of the earlier study on the sophisticated computation method of \( \| \Sigma_{SD} \|_1 \) alone if we are actually further interested in detecting (some or all of) the associated critical instants.

This paper is indeed motivated by our interest in whether we could somehow alleviate such a weakness. In this respect, the above theorem is helpful because it reveals that such a weakness manifests itself only when every critical instant \( \Theta \) satisfies \( \Theta^* (\Theta) = b - 0 \). In fact, we could further apply Propositions 3 and 4 relevant to critical instants \( \Theta \) satisfying \( \Theta^* (\Theta) = b - 0 \) to establish the claims (a) and (b) stated at the beginning of Section 5, which are supplementary to the assertion of Theorem 6 above. The following first result corresponds to (a).

**Theorem 7** (single-output case). Suppose that \( P_{11} \neq 0 \). If \( F(\Theta)^{1/2}, \Theta \in [0, b] \) is not maximum-attaining, then no critical instant exists.

**Proof.** Suppose that \( F(\Theta)^{1/2}, \Theta \in [0, b] \) is not maximum-attaining. It follows from Theorem 6 that if there exists a critical instant \( \Theta \), it should satisfy \( \Theta^* (\Theta) = b - 0 \). It then follows from Proposition 3 that \( P_{11} = 0 \). This contradicts the assumption \( P_{11} \neq 0 \), and thus no critical instant can exist.

On the other hand, the following result corresponds to (b).

**Theorem 8** (single-output case). Suppose that \( P_{11} = 0 \). If \( F(\Theta)^{1/2}, \Theta \in [0, b] \) is not maximum-attaining, then every \( \Theta \in [0, b] \) is a critical instant.

**Proof.** If \( F(\Theta)^{1/2}, \Theta \in [0, b] \) is not maximum-attaining, it follows from Theorem 6 that either (i) there exists no critical instant or (ii) every critical instant \( \Theta \in [0, b] \) satisfies \( \Theta^* (\Theta) = b - 0 \). Since \( P_{11} = 0 \), however, it follows from Theorem 4 that \( \Theta = 0 \) should be a critical instant, so that (i) cannot be the case. Hence, (ii) should be true with a critical instant \( \Theta = 0 \). This implies by Proposition 4 that every \( \Theta \in [0, b] \) is a critical instant.

We can further obtain the following result.

**Theorem 9** (single-output case). There exists a critical instant if and only if either of the following conditions is true:

- \( F(\Theta)^{1/2}, \Theta \in [0, b] \) is maximum-attaining.
- \( P_{11} = 0 \).

**Proof.** Sufficiency follows from Theorems 3 and 4. Necessity follows from Theorem 7.

Furthermore, we have the following result corresponding to the claim (c) stated at the beginning of Section 5.

**Theorem 10** (single-output case). If \( P_{11} \neq 0, P_{11} + D_{12} D_2 P_{21} \neq 0 \) and if \( F(\Theta)^{1/2}, \Theta \in [0, b] \) is maximum-attaining, then every critical instant \( \Theta \in [0, b] \) is a maximum-attaining point of \( F(\Theta)^{1/2} \) (and vice versa).
Proof. It is obvious from Theorem 3 that every maximum attain-
ing point of $F(\Theta)^{1/2}$ is a critical instant, which does exist. By Proposition 3, every critical instant satisfies (I), and thus the assertion follows from Proposition 1. □

As long as single-output sampled-data systems are con-
cerned, Theorems 7–10 imply that, the sophisticated com-
putation method of $\|\Sigma_{SD}\|_1$ alone can actually offer some addi-
tional information (beyond what is directly implied by The-
orem 3 derived in [14]) on a critical instant $\Theta$ and thus well-
definability of the $I_\infty /L_2$ Hankel operator. This consequence was not shown in the earlier study [14] and is shown here for the first time.

We can actually give the following corollary, which gen-
eralises Theorem 10 (and thus Claim (c)) in the follow-
ing respects: it manifests a particular situation where the sophis-
ticated computation method of the $I_\infty /L_2$ Hankel norm alone actually suffices fully in completely answering the relevant questions of (i) well-definability of the $I_\infty /L_2$ Han-
kel operator and (ii) the locations of all the critical instants (if it is well-definable). Such a situation is entirely nontrivial at all in the earlier study [14], and thus the corollary clearly proves the significance of the preceding arguments in this paper.

**Corollary 1** (single-output case). Consider the single-output sampled-data systems with $P_1 \neq 0$ and $P_1 + D_{12}L_0P_2 \neq 0$. If $F(\Theta)^{1/2}, \Theta \in [0, b)$ is maximum-attaining, then every critical instant $\Theta \in [0, b)$ is a maximum-attaining point of $F(\Theta)^{1/2}$ (and vice versa) and satisfies (I), and there exists no critical instant otherwise.

Proof. The first assertion follows readily from Theorem 10 together with Proposition 1 (and also Theorem 3). The second assertion follows readily from Theorem 7. □

In spite of the above positive side of the sophisticated com-
putation method of $\|\Sigma_{SD}\|_1$ alone (see also Examples 1–3), the arguments in this section also clarify its negative side. To see this, we first note that combining Propositions 2 and 3 together with Theorem 4 readily leads to the following result.

**Proposition 5.** If there exists a critical instant $\Theta$ satisfying $\Theta < \theta^* (\Theta) < b$, $\theta^* (\Theta) = b - 0$ or $\theta^* (\Theta) \geq b$, then $\Theta = 0$ is a criti-
cal instant.

What this proposition means is that when there does exist a critical instant, it is “likely” that $\Theta = 0$ is one such instant and, more precisely, it can fail to be true only when every criti-
cal instant $\Theta$ satisfies $\theta^* (\Theta) = \Theta$, or equivalently (by Proposition 1), only when $(F(\Theta)^{1/2}, \Theta \in [0, b)$ is maximum-attaining and) every critical instant is a maximum-attaining point of $F(\Theta)^{1/2}, \Theta \in [0, b)$. The following theorem is nothing but a restatement of this observation; more precisely, it is nothing but the claim (d) stated at the beginning of Section 5 and clearly suggests a limit of the sophisticated computation method in its ability in detecting all the critical instants (see Examples 4–6).

**Theorem 11** (single-output case). Suppose that there exists a criti-
cal instant. If $\Theta = 0$ is not a critical instant, then $F(\Theta)^{1/2}, \Theta \in [0, b)$ is maximum-attaining and every critical instant is its maximum-
attaining point.

### 7 | EXTENSION TO MULTI-OUTPUT SAMPLED-DATA SYSTEMS

This section is devoted to extending the results in the preced-
ing section to those for multi-output sampled-data systems. We omit the details but provide some fundamental key arguments for obtaining some results parallel to the Claims (a)–(d) in Sec-
tion 5. This is because the full process to derive the results is quite technical but can be followed by using similar arguments to those in the preceding section. We separately discuss the cases for $\rho = \infty$ and $\rho = 2$.

#### 7.1 | The case of $\rho = \infty$

We begin with the case of $\rho = \infty$. Suppose that there exists a critical instant $\Theta$ satisfying either of (II), (II)', (III) and (IV), which we denote by $\Theta_1$. Let $w^*_1 \in \partial \Sigma_{SD}$ be the worst input to $H_{\infty}^{[1]}$, and let $\theta^*_1 = \theta^*_\infty (\Theta_1)$. Then, it readily follows from the definition of the $I_{\infty, \infty} (\Theta, \infty)$ norm that, without loss of generality, we may assume $\|H_{\infty} (\Theta_1) \|_{\infty} = \| (H_{\infty} (\Theta_1) w_1^*) \|_{\infty}$ (by reordering the entries of $\zeta_1$ if necessary), where $\| \cdot \|_{\infty}$ is a shorthand notation for $\| \cdot \|_{\infty}$ and $(H_{\infty} (\Theta_1) w_1^*)$, denotes the $r$th entry of $H_{\infty} (\Theta_1) w_1^*$. In this connection, let us consider the single-output sampled-data system $\Sigma_{SD}$ obtained by removing all the entries of $\zeta_1$ but the $r$th one in $\Sigma_{SD}$. Then, we may further assume that $\theta^*(\Theta_1)$ for $\Sigma_{SD}$ does not exceed $\theta^*_1 (\Theta_1)$ for any $\Sigma_{SD}$, such that $\|H_{\infty} (\Theta_1) w_1^* \|_{\infty} = \| (H_{\infty} (\Theta_1) w_1^*) \|_{\infty}$ (if such $i \neq 1$ exists). Here, note that the left-hand side of $\|H_{\infty} (\Theta_1) w_1^* \|_{\infty} = \| (H_{\infty} (\Theta_1) w_1^*) \|_{\infty}$ equals $\| \Sigma_{SD} \|_{H, \infty}$. On the other hand, the right-hand side equals $\| \Sigma_{SD} \|_{H, \infty}$ (which in turn implies $\| \Sigma_{SD} \|_{H, \infty} = \| \Sigma_{SD} \|_{H, \infty}$). It is thus obvious that $\Theta_1$ remains to be a critical instant of $\Sigma_{SD, 1}$, whose $\theta^*(\Theta_1)$ is nothing but $\theta^*_1$ defined above. This implies that $\Theta_1$ is a critical instant of the single-output sampled-data system $\Sigma_{SD, 1}$ satisfying (II), (II)', (III) or (IV). Hence, we can apply Proposi-
tions 2–4 to the single-output sampled-data system $\Sigma_{SD, 1}$ by noting the following result under $\rho = \infty$ (whose proof is omitted).

**Lemma 1.** If $\Theta_2$ is another critical instant of $\Sigma_{SD, 1}$, then it is a critical instant of $\Sigma_{SD}$. Furthermore, $\theta^*_\infty (\Theta_2)$ for $\Sigma_{SD}$ does not exceed $\theta^*_1 (\Theta_2)$ for $\Sigma_{SD, 1}$.

We are thus led to the multi-output ($\rho = 2, \infty$) counterpart of Propositions 2–4; Propositions 2 and 4 remain the same except that $\theta^*(\Theta)$ should read $\theta^*_1 (\Theta)$, while Proposition 3 reads as follows when $\rho = \infty$. 


Proposition 6. There exists a critical instant \( \Theta \) satisfying \( \Theta < \Theta^*_\infty(\Theta) < b \), \( \Theta^*_\infty(\Theta) = b - 0 \) or \( \Theta^*_\infty(\Theta) > b \) only if \( (P_{11})_i = 0 \) for some i, where (\^i) denotes the subsystem corresponding to the i\textsuperscript{th} output. There exists a critical instant \( \Theta \) satisfying \( \Theta^*_\infty(\Theta) = b \) only if \( (P_{11} + D_{12}D\Psi P_{21})_i = 0 \) for some i.

With these propositions, we are led to the following results for \( p = \infty \) corresponding to the multi-output version of the Claims (a)–(d) in Section 5.

If \( \mu^1\overline{2}(F(\Theta)), \Theta \in [0, b) \) is not maximum-attaining, then

(a) no critical instant exists if \( (P_{11})_i \neq 0 \) for every i;
(b-1) every \( \Theta \in [0, b) \) is a critical instant if \( P_{11} = 0 \);
(b-2) if \( (P_{11})_i = 0 \) whenever \( \sup_{\Theta \in [0, b)} \mu^1\overline{2}(F(\Theta)) = \sup_{\Theta \in [0, b)} F_{ii}(\Theta)^{1/2} \), where \( F_{ii}(\Theta) \) denotes the i\textsuperscript{th} diagonal entry of \( F(\Theta) \), then no critical instant exists, and every \( \Theta \in [0, b) \) is a critical instant otherwise.

In addition,

(c) if \( (P_{11})_i \neq 0 \) and \( (P_{11} + D_{12}D\Psi P_{21})_i = 0 \) for each i and \( \mu^1\overline{2}(F(\Theta)), \Theta \in [0, b) \) is maximum-attaining, then every critical instant \( \Theta \in [0, b) \) is a maximum-attaining point of \( \mu^1\overline{2}(F(\Theta)) \) (and vice versa);
(d) Suppose that there exists a critical instant. If \( \Theta = 0 \) is not a critical instant, then \( \mu^1\overline{2}(F(\Theta))^{1/2}, \Theta \in [0, b) \) is maximum-attaining and every critical instant is its maximum-attaining point.

In addition, a counterpart of Corollary 1 reads as follows for \( p = \infty \) (which is slightly more general than Claim (c)).

Corollary 2. Consider the sampled-data systems such that \( (P_{11})_i \neq 0 \) and \( (P_{11} + D_{12}D\Psi P_{21})_i \neq 0 \) for each i. If \( \mu^1\overline{2}(F(\Theta)) \) is maximum-attaining over \( \Theta \in [0, b) \), then every critical instant \( \Theta \in [0, b) \) is a maximum-attaining point of \( \mu^1\overline{2}(F(\Theta)) \) (and vice versa) and satisfies (I), while if \( \mu^1\overline{2}(F(\Theta)), \Theta \in [0, b) \) is not maximum-attaining, then there exists no critical instant.

7.2 The case of \( p = 2 \)

We next consider the case of \( p = 2 \). Suppose that there exists a critical instant \( \Theta \in [0, b) \) satisfying either of (II), (II'), (III) and (IV), which we denote by \( \Theta_1 \). Let \( u^*_1 \in \partial W_{\Theta_1} \) be the worst input to \( H^1_{\Theta_1} \), and let \( \Theta^*_1 = \Theta^*_1(\Theta_1) \). It readily follows from the definition of the \( I_{\infty,2}(\Theta, \infty) \) norm that, whenever \( V \) is an orthogonal matrix, we have \( \| \Sigma_{\Theta_1} \|_{1,2} = \| V \Sigma_{\Theta_1} V^T \|_{1,2} \), \( \| H^1_{\Theta_1} \|_2 = \| V \Sigma_{\Theta_1} V^T \|_2 \) and \( \| H^1_{\Theta_1} u^*_1 \|_{\infty} = \| V \Sigma_{\Theta_1} u^*_1 \|_{\infty} \), where \( \| \cdot \|_2 \) and \( \| \cdot \|_{\infty,2} \) are shorthand notations for \( \| \cdot \|_2 \) and \( \| \cdot \|_{\infty,2} \). Without loss of generality, we can take an orthogonal \( V \) such that all the entries of \( V \Sigma_{\Theta_1} V^T \) for the input \( u^*_1 \) are zero but the first one,

in which case we have \( \| \Sigma_{\Theta_1} \|_{1,2} = \| V \Sigma_{\Theta_1} V^T \|_{1,2} \), \( \| H^1_{\Theta_1} \|_2 = \| V \Sigma_{\Theta_1} V^T \|_2 \) and \( \| H^1_{\Theta_1} u^*_1 \|_{\infty} = \| V \Sigma_{\Theta_1} u^*_1 \|_{\infty} \).

Proposition 7. There exists a critical instant \( \Theta \) satisfying \( \Theta < \Theta^*_\infty(\Theta) < b \), \( \Theta^*_\infty(\Theta) = b - 0 \) or \( \Theta^*_\infty(\Theta) > b \) only if there exists a nonzero vector \( v \) such that \( v^TF(\Theta) = 0 \). There exists a critical instant \( \Theta \) satisfying \( \Theta^*_\infty(\Theta) = b \) only if there exists a nonzero vector \( v \) such that \( v^TF(\Theta) = 0 \).

We are finally led to the following results for \( p = 2 \) as a generalised version of the Claims (a)–(d) in Section 5.

If \( \mu^1\overline{2}(F(\Theta)), \Theta \in [0, b) \) is not maximum-attaining, then

(a) no critical instant exists if \( v^TF(\Theta) = 0 \) for each nonzero vector \( v \);
(b-1) every \( \Theta \in [0, b) \) is a critical instant if \( P_{11} = 0 \);
(b-2) if \( v^TF(\Theta) = 0 \) whenever \( v \) is such that \( |v|^2 = 1 \) and \( \sup_{\Theta \in [0, b)} \mu^1\overline{2}(F(\Theta)) = \sup_{\Theta \in [0, b)} (v^TF(\Theta)) \), then no critical instant exists, and every \( \Theta \in [0, b) \) is a critical instant otherwise.

In addition,

(c) if \( v^TF(\Theta) = 0 \) and \( v^TF(\Theta) = 0 \) for each nonzero \( v \) and if \( \mu^1\overline{2}(F(\Theta)), \Theta \in [0, b) \) is maximum-attaining, then every critical instant \( \Theta \in [0, b) \) is a maximum-attaining point of \( \mu^1\overline{2}(F(\Theta)) \) (and vice versa);
(d) Suppose that there exists a critical instant. If \( \Theta = 0 \) is not a critical instant, then \( \mu^1\overline{2}(F(\Theta))^{1/2}, \Theta \in [0, b) \) is maximum-attaining and every critical instant is its maximum-attaining point.

In addition, a counterpart of Corollary 1 reads as follows for \( p = 2 \) (which is slightly more general than Claim (c)).

Corollary 3. Consider the sampled-data systems such that \( v^TF(\Theta) = 0 \) and \( v^TF(\Theta) = 0 \) for each nonzero \( v \). If \( \mu^1\overline{2}(F(\Theta)), \Theta \in [0, b) \) is maximum-attaining, then every critical instant \( \Theta \in [0, b) \) is a maximum-attaining point of \( \mu^1\overline{2}(F(\Theta)) \) (and vice versa) and satisfies (I), while if \( \mu^1\overline{2}(F(\Theta)), \Theta \in [0, b) \) is not maximum-attaining, then there exists no critical instant.
8 NUMERICAL EXAMPLES

This section is devoted to giving examples to confirm that the numerical computation results of the quasi $L_\infty/L_2$ Hankel norms $\|H^{[\Theta]}\|$ for $\Theta \in [0, b]$ through (17) (and, in particular, the associated conclusions as to whether or not the $L_\infty/L_2$ Hankel operator is well-definable) are consistent with other theoretical and most central arguments in the present paper (i.e. the clarified ability of the sophisticated computation method of the $L_\infty/L_2$ Hankel norm alone). We only consider the single-output case, and with the above purpose of this section in mind, $F(\Theta)$ relevant to the sophisticated computation method is also computed for $\Theta \in [0, b)$.

We study six examples to deal with different aspects in our theoretical interests in the preceding arguments, where the sampling period is $b = 2$ except for Example 5, in which $b = 0.2$. The computation results for the quasi Hankel norms and $F(\Theta)$ for Examples 1–6 are shown in Figures 3–8, respectively. For Example 1, the associated timing $\dot{\Theta}^*(\Theta)$ is also computed for each $\Theta \in [0, b)$ as shown in Figure 3(b), which is obtained readily in the course of the computation of the quasi Hankel norm shown in Figure 3(a) (recall Definition 2).

The first three examples are relevant to the positive side of the sophisticated computation method of the $L_\infty/L_2$ Hankel norm alone, or more precisely, a by-product that the present paper has clarified as its possible hidden ability in deciding well-definability of the $L_\infty/L_2$ Hankel operator and, when it is well-definable, finding all the critical instants. Demonstrating the computation method of the quasi $L_\infty/L_2$ Hankel norm $\|H^{[\Theta]}\|$ for each $\Theta \in [0, b)$ is also aimed at in the first example.

Example 1.

\[
\mathcal{A} = \begin{bmatrix} -3 & 2 \\ -4 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D_{12} = 0,
\]

(30)

FIGURE 3 Results of Example 1

FIGURE 4 Results of Example 2

FIGURE 5 Results of Example 3

FIGURE 6 Results of Example 4
The computation results of Example 6

FIGURE 7 Results of Example 5

FIGURE 8 Results of Example 6

\[
A_p = \begin{bmatrix}
-2.1857 & 2.3763 \\
-1.1134 & 1.2104
\end{bmatrix}, \quad B_p = \begin{bmatrix}
0.0176 \\
0.0090
\end{bmatrix},
\]
\[
C_p = \begin{bmatrix}
0.7610 \\
-0.8274
\end{bmatrix}, \quad D_p = 0.2367
\] (31)

Our observations for this example are as follows.

- Just for reference, this is an example with \( P_1 \) (\( = P_1 + D_{12}D_pP_{21} \)) \( \neq 0 \).
- \( F(\Theta)^{1/2} \), \( \Theta \in [0, b) \) is maximum-attaining (Figure 3).
- The computation results of \( ||H(\Theta)|| \) and \( F(\Theta)^{1/2} \) are consistent with the assertion of Theorem 5 relevant to well-definability of the quasi \( L_{\infty}/L_2 \) Hankel operator.
- More strongly, all the critical instants have been successfully detected as a unique maximum-attaining point of \( F(\Theta)^{1/2} \), \( \Theta \in [0, b) \), which is consistent with Theorem 10 or the claim (c) at the beginning of Section 5, and more importantly, with Corollary 1 (or as an alternative interpretation, because \( \Theta = 0 \) is not a critical instant by Figure 3(a); this is contradictory to the claim (d) there and thus Theorem 11. In this example, the only critical instant is 1.0480.
- The \( L_{\infty}/L_2 \) Hankel norm is 2.1867.

Example 2. Consider Example 1 with \( D_{12} = 0 \) replaced by \( D_{12} = -3 \), for which we can observe the following:

- This is an example with \( P_1 \) \( \neq 0 \).
- \( F(\Theta)^{1/2} \), \( \Theta \in [0, b) \) is not maximum-attaining (Figure 4).
- The claim (a) at the beginning of Section 5 (i.e. no critical instant exists and thus the \( L_{\infty}/L_2 \) Hankel operator is not well-definable) can be confirmed to be true in this example. This claim more formally corresponds to Theorem 7.
- The \( L_{\infty}/L_2 \) Hankel norm is 2.4854.

Example 3.

\[
A = \begin{bmatrix}
-1 & -2 \\
0 & -2
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
1 \\
0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
-1 \\
-1
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
0 & 1
\end{bmatrix},
\]
\[
C_2 = \begin{bmatrix}
1 & 0
\end{bmatrix}, \quad D_{12} = -1,
\] (32)

\[
A_p = \begin{bmatrix}
2.5656 & -2.3760 \\
3.8256 & -3.5417
\end{bmatrix}, \quad B_p = \begin{bmatrix}
0.0247 \\
0.0368
\end{bmatrix},
\]
\[
C_p = \begin{bmatrix}
-0.6388 & 0.5914
\end{bmatrix}, \quad D_p = 0.2367,
\] (33)

We can observe the following for this example.

- This is an example with \( P_1 = 0 \).
- \( F(\Theta)^{1/2} \), \( \Theta \in [0, b) \) is not maximum-attaining (Figure 5).
- The results are consistent with the assertions of Theorem 4 for a sufficient condition that ensures \( \Theta = 0 \) to become a critical instant (which is also consistent with Theorem 11).
- The claim (b) at the beginning of Section 5 (which implies that every \( \Theta \in [0, b) \) is a critical instant) can be confirmed to be true in this example. This claim more formally corresponds to Theorem 8.
- The \( L_{\infty}/L_2 \) Hankel norm is 0.2968.

In contrast to the above examples demonstrating the positive side of the new findings in the present paper on the sophisticated computation method, the following examples are more or less relevant to its negative side; from what has been claimed at the beginning of Section 5, these examples naturally satisfy either \( P_1 = 0 \) or \( P_1 + D_{12}D_pP_{21} = 0 \). Furthermore, they are all contradictory to Theorem 11.

Example 4.

\[
A = \begin{bmatrix}
1 & 2 \\
0 & -2
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
1 & -1 \\
0 & 0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 \\
-1
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
0 & 1
\end{bmatrix},
\]
\[
C_2 = \begin{bmatrix}
1 & 0
\end{bmatrix}, \quad D_{12} = 0,
\] (34)

\[
A_p = \begin{bmatrix}
-0.0057 & 0.3452 \\
0.0095 & -0.5720
\end{bmatrix}, \quad B_p = \begin{bmatrix}
0.3632 \\
-0.6018
\end{bmatrix},
\]
\[
C_p = \begin{bmatrix}
-0.0298 & 1.7975
\end{bmatrix}, \quad D_p = 1.8334,
\] (35)
We can observe the following in this example.

- This is an example with $P_{11} = P_{11} + D_{12}D_{21}$.
- $F(\Theta)^{1/2}, \Theta \in [0, b]$ is maximum-attaining and $\lim \Theta \in [0, b]$ is a critical instant (Figure 6), where $\Theta^*(\Theta) = \Theta$ for $\Theta = 0$ while $\Theta^*(\Theta) = b$ for $\Theta \in (0, b)$ (the corresponding figure is omitted).
- The result is consistent with Theorem 6.
- The sophisticated computation method of the $L_{\infty}/L_2$ Hankel norm alone cannot detect all the critical instants in this example (as well as the following examples); since $F(\Theta)^{1/2}, \Theta \in [0, b]$ is maximum-attaining (unlike the last example), the claims (a) and (b) at the beginning of Section 5 can offer no clue about the locations of all critical instants, while the claims (c) and (d) are also helpless.
- No theoretical results obtained in this paper for the sophisticated computation method can directly indicate that every $\Theta \in [0, b]$ is a critical instant. This suggests that it is hard for us to completely avoid the computation of the quasi $L_{\infty}/L_2$ Hankel norm, in general, when we are interested only in the computation of the $L_{\infty}/L_2$ Hankel norm but also in locating all the associated critical instants.
- The $L_{\infty}/L_2$ Hankel norm is 7.1548.

Example 5.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$C_1 = [14.6388, 0], \quad C_2 = [1, 0], \quad D_{12} = 1.5, \quad (36)$$

$$A_P = \begin{bmatrix} -0.2622 & -0.1254 \\ -0.3679 & -0.3954 \end{bmatrix}, \quad B_P = \begin{bmatrix} 1.9354 \\ 2.5326 \end{bmatrix},$$

$$C_P = \begin{bmatrix} 2.7668 & 3.8898 \end{bmatrix}, \quad D_P = -9.7592, \quad (37)$$

$$b = 0.2, \quad (38)$$

We can observe the following in this example.

- This is an example with $P_{11} = P_{11} + D_{12}D_{21}$.
- Every $\Theta \in [0, b]$ is a critical instant (Figure 7), where $\Theta^*(\Theta) = \Theta$ for $\Theta = 0$ while $\Theta^*(\Theta) = b$ for $\Theta \in (0, b)$. In particular, we can see the assertion is false that if every $\Theta \in [0, b]$ is a critical instant, then $P_{11} = 0$, even though the preceding two examples might have suggested so (its converse is not true, either; see the third observation for the next example).
- The result is consistent with the second assertion of Theorem 6.
- We can see that the converse assertion of Theorem 4 is not true.
- The last two observations in Example 4 apply also to this example, regardless of whether or not $F(\Theta)^{1/2}$ tends to $F(0)^{1/2}$ as $\Theta \to b$.
- The $L_{\infty}/L_2$ Hankel norm is 18.2509.

Example 6. Consider Example 3 with $D_{12} = -1$ replaced by $D_{12} = 1$, for which our observations are as follows.

- This is an example with $P_{11} = 0$.
- $F(\Theta)^{1/2}, \Theta \in [0, b]$ is maximum-attaining and $\Theta = 0$ is a critical instant (Figure 6). This corresponds to a situation where none of the claims (a)–(d) in the beginning of Section 5 can offer any clues for detecting all the critical instants. More precisely, it eventually turns out that the only critical instant $\Theta = 0$ detected by the sophisticated computation method is actually a unique one, but its uniqueness cannot be concluded within the framework of this method itself.
- $P_{11} = 0$ does not always mean that every $\Theta \in [0, b]$ is a critical instant (even though all the preceding examples with $P_{11} = 0$ might seem to suggest that it could be the case). The numerical results can be seen to be contradictory to the contrapositive statement of Theorem 8.
- We can see that the converse assertion of Theorem 11 is not true.
- The $L_{\infty}/L_2$ Hankel norm is 0.1688.

9 | CONCLUSION

This paper was motivated by the success of the sophisticated computation method of the $L_{\infty}/L_2$ Hankel norm ([14]) in possibly providing some information on well-definability of the $L_{\infty}/L_2$ Hankel operator and a subset of critical instants for a given sampled-data system (as briefly summarised in Section 2). More specifically, this paper aimed at leading to more thorough results on well-definability of the $L_{\infty}/L_2$ Hankel operator and detection of critical instants beyond what has already been shown in [14]. To this end, we first tackled in Section 3 for the first time the problem of characterising the quasi $L_{\infty}/L_2$ Hankel norm $\|H(\Theta)\|$ at $\Theta \in [0, b)$. This result immediately leads to a naive computation method of the $L_{\infty}/L_2$ Hankel norm, with which the feature of the sophisticated computation method in [14] was contrasted in Section 4; in particular, the motivation of the present paper in clarifying much further ability of the computation method in [14] was stated in detail. To proceed with such a direction, the characterisation of $\|H(\Theta)\|$ developed in Section 3 was successfully exploited as a preliminary step in Section 5 to show a key fact that $\Theta = 0$ is a critical instant if $P_{11} = 0$. Section 6 dealt with the single-output case and established that $F(\Theta)^{1/2}$ (used in the sophisticated computation method) being not maximum-attaining over $\Theta \in [0, b)$ implies that no critical instant exists if $P_{11} \neq 0$ and that every $\Theta \in [0, b)$ is a critical instant otherwise. A sufficient condition was also given as to when the sophisticated computation method can detect all critical instants. Section 7 extended the results in Section 6 to the multi-output case; these two sections further clarified in much more details on how much information the sophisticated computation method can offer, for a given sampled-data system, on well-definability of the $L_{\infty}/L_2$ Hankel operator as well as all the critical instants. Numerical examples were given in Section 8 to demonstrate the theoretical validity.
of the arguments derived in this paper and investigate whether they could possibly be further strengthened.

Finally, it would be also worthwhile to discuss the effectiveness of detecting critical instants from an application point of view. As cyber issues on real systems have been reported in recent years [23–25], the security problems of cyber-physical systems (CPSs) have attracted interests in the control community. Even though there have been a number of studies such as [26, 27] to characterise various types of cyber attacks to make CPSs unstable on the framework of sampled-data systems, it is still unclear what could be a definite measure for the security problems of CPSs. In this sense, the arguments on the existence of critical instants of sampled-data systems discussed in this paper might be used for encrypting CPSs against external hackers, and the classification of critical instants might be also used for establishing a security measure for CPSs.

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