The Moduli Space of Many BPS Monopoles for Arbitrary Gauge Groups

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ABSTRACT

We study the moduli space for an arbitrary number of BPS monopoles in a gauge theory with an arbitrary gauge group that is maximally broken to $U(1)^k$. From the low energy dynamics of well-separated dyons we infer the asymptotic form of the metric for the moduli space. For a pair of distinct fundamental monopoles, the space thus obtained is $R^3 \times (R^1 \times \mathcal{M}_0)/Z$ where $\mathcal{M}_0$ is the Euclidean Taub-NUT manifold. Following the methods of Atiyah and Hitchin, we demonstrate that this is actually the exact moduli space for this case. For any number of such objects, we show that the asymptotic form remains nonsingular for all values of the intermonopole distances and that it has the symmetries and other characteristics required of the exact metric. We therefore conjecture that the asymptotic form is exact for these cases also.
1 Introduction

Among the many remarkable features of the Bogomol’nyi-Prasad-Sommerfield (BPS) limit [1] is the existence of families of degenerate static multimonopole solutions. For any given topological charge, the space of such solutions, with gauge equivalent configurations identified, forms a finite dimensional moduli space. A metric for this space is defined in a natural way by the kinetic energy terms of the Yang-Mills-Higgs Lagrangian. As was first pointed out by Manton [2], a knowledge of this metric is sufficient for determining the low energy dynamics of a set of monopoles and dyons. More recently [3-5], such knowledge has been used in theories with extended supersymmetry to show the existence of some of the dyonic states required by the electromagnetic duality conjecture of Montonen and Olive [6].

For the case of an $SU(2)$ gauge symmetry spontaneously broken to $U(1)$, the moduli space $M$ of solutions carrying $n$ units of magnetic charge has $4n$ dimensions. This naturally suggests that these solutions should be interpreted as configurations of $n$ monopoles, each of which is specified by three position coordinates and a $U(1)$ phase angle. (Recall that time variation of this phase angle gives an electrically charged dyon.) The metric for the two-monopole moduli space was determined by Atiyah and Hitchin [7]. For three or more monopoles, the moduli space metric is still unknown. An asymptotic form, valid in the regions of $M$ corresponding to widely separated monopoles, has been found by Gibbons and Manton [8], but this develops singularities if any of the intermonopole distances becomes too small, and hence cannot be exact.

In this paper, we consider an arbitrary gauge group $G$ with rank $k \geq 2$, with the adjoint Higgs field assumed to be such as to give the maximal symmetry breaking, to the Cartan subgroup $U(1)^k$. In this case, there are $k$ independent topological charges $n_a$ and, correspondingly, $k$ “fundamental monopoles”, each of which carries a single unit of one of these charges [9]. The moduli spaces corresponding to any combination of $n$ fundamental monopoles are $4n$-dimensional, just as in $SU(2)$. We determine the moduli space metric for all two-monopole solutions; this generalizes the $SU(3)$ result outlined by us in [10] and also found by Gauntlett and Lowe [11] and by Connell [12]. In addition, we find the asymptotic form of the metric for any number of monopoles. We then argue that in a number of cases with three or more monopoles this asymptotic form is in fact likely to be the exact metric over the entire moduli space.

It might seem odd to try to obtain the moduli space metric for three or more monopoles with a larger gauge group when this cannot even be done for $SU(2)$. The reason that it might actually be easier to work with a larger group is that in a number of such cases the moduli
space possesses greater symmetry. Specifically, consider a multimonopole solution with \( n \leq k \) fundamental monopoles, each corresponding to a different topological charge. The corresponding \( U(1) \) factors of the unbroken gauge group act nontrivially on this solution, and this action generates an isometry on the moduli space. Furthermore, for any such collection of topological charges there is a solution that, although it is composite, is spherically symmetric. On the moduli space, such a solution corresponds to a fixed point under the isometries that correspond to overall rotation of the monopole configuration. In \( SU(2) \) there are no spherically symmetric solutions with multiple magnetic charge \([13]\), and hence no such fixed points.

We recall some properties of BPS monopoles in an \( SU(2) \) theory in Sec. 2, and then discuss the solution for higher groups in Sec. 3. We then begin our analysis of the moduli spaces in Sec. 4. Here, following the approach that Gibbons and Manton used for \( SU(2) \), we use our knowledge of the interactions between widely separated dyons to infer the asymptotic form of the moduli space metric. In Sec. 5, we specialize to the case of two monopoles. After separating out the center-of-mass motion, we are left with a relative moduli space \( \mathcal{M}_0 \) that is simply a Taub-NUT manifold. Although our methods thus far only establish that this is the correct manifold asymptotically, we note that it remains smooth as the monopole separation is taken to zero and, further, that it possesses the appropriate symmetries. This suggests that this asymptotic form might in fact be correct. In Sec. 6 we use the methods of Atiyah and Hitchin to verify this. The argument here rests on the fact that the moduli space must be hyperkähler. The Taub-NUT space is the only such four-dimensional manifold with both the proper symmetry and the correct asymptotic behavior. In Sec. 7 we return to the case of an arbitrary number of distinct fundamental monopoles. We show that the asymptotic metric can be smoothly continued to all values of the intermonopole distances, and that it has all of the symmetry properties required of the exact metric. Hence, we conjecture that this form is, in fact, exact. Section 8 contains some concluding remarks.

### 2 BPS Monopoles in \( SU(2) \) Gauge Theory

We begin by recalling some properties of the BPS solutions \([1]\) in an \( SU(2) \) theory spontaneously broken to \( U(1) \). We fix the normalization of the electric and magnetic charges \( g \) and \( q \) are defined by writing the large distance behavior of the electromagnetic field strength (in radial gauge) as

\[
B^a_i = \frac{\hat{r} \hat{r}_a g}{4\pi r^2}, \quad E^a_i = \frac{\hat{r} \hat{r}_a q}{4\pi r^2}.
\]  

(2.1)
The dyon solution carrying one unit of magnetic charge (i.e., \( g = 4\pi/e \), with \( e \) being the gauge coupling) may be written as

\[
\Phi^a(r) = \hat{r}^a K(r; v),
\]
\[
A_\alpha^a(r) = \epsilon_{\alpha k} \hat{r}^k \left( A(r; v) - \frac{1}{er} \right),
\]
\[
A_0^a(r) = \frac{q}{\sqrt{g^2 + q^2}} \Phi^a,
\]

where \( v \) is the asymptotic magnitude of the Higgs field and

\[
K(r; v) = v \coth(erv\eta) - \frac{1}{erv\eta},
\]
\[
A(r; v) = \frac{v\eta}{\sinh(erv\eta)},
\]

where

\[
\eta = \frac{g}{\sqrt{g^2 + q^2}}.
\]

The mass of this dyon is

\[
\tilde{m} = v\sqrt{g^2 + q^2}.
\]

This radial gauge form of the solution makes the spherical symmetry of the dyon manifest, in that the effects of a spatial rotation can be entirely compensated by a global \( SU(2) \) gauge transformation. However, for understanding the nature of the solutions with higher magnetic charges, it is useful to apply a gauge transformation that brings the Higgs field to a constant direction in internal space. This introduces a Dirac string singularity that can be chosen to lie along the \( z \)-axis. In cylindrical coordinates the purely magnetic solution then takes the form

\[
\Phi^a = \delta^{a3} K(r; v),
\]
\[
A^3 = -\frac{1}{e} \left( \frac{z}{r} \pm 1 \right) d\phi,
\]
\[
W \equiv \frac{1}{\sqrt{2}} (A^1 + iA^2) = ie^{\pm i\phi + i\gamma} \frac{1}{\sqrt{2}} A(r; v) \left( \frac{z}{r} d\rho - i\rho d\phi - \frac{\rho}{r} dz \right),
\]

where the upper and lower signs correspond to solutions with Dirac strings lying along the positive and negative \( z \)-axes, respectively. Although the spherical symmetry of the solution is obscured in this gauge, the action of the unbroken \( U(1) \) is made clearer. In particular, a global \( U(1) \) transformation simply shifts the arbitrary phase \( \gamma \), while the dyon solutions can be obtained by allowing \( \gamma \) to vary uniformly with time. The fact that \( \gamma \) has a period of \( 2\pi \) implies that in the quantized theory \( q \) must be an integer multiple of \( e \).

The solutions with higher magnetic charge can all be understood as multimonopole solutions. To begin, consider a doubly charged solution corresponding to two monopoles separated by a
distance \( R \). If we work in a gauge with uniform Higgs field orientation, a first approximation to the solution can be obtained by superimposing two single monopole solutions, provided that the two Dirac strings do not overlap. Hence, let us combine a solution centered about the point \((0,0,R/2)\) with its Dirac string running upward with one centered at \((0,0,-R/2)\) with a Dirac string running downward. Using an obvious notation, we can write the gauge fields of the solution as

\[
A_j = A_j^{(1)} + A_j^{(2)} + \delta A_j, \quad W_j = W_j^{(1)} + W_j^{(2)} + \delta W_j. \tag{2.7}
\]

Before superimposing the Higgs fields, we must first separate out the asymptotic value \( \Phi_0 \). Defining \( \Delta \Phi^{(i)}(r) = \Phi^{(i)}(r) - \Phi_0 \) for the Higgs field associated with each monopole, we have

\[
\Phi = \Phi_0 + \Delta \Phi^{(1)} + \Delta \Phi^{(2)} + \delta \Phi. \tag{2.8}
\]

For \( R \gg 1/ev \), we expect that \( \delta A_j, \delta W_j \), and \( \delta \Phi \) will be small corrections that can be determined perturbatively by substituting these expressions into the BPS field equations.

Contrary to what one might have expected, this solution is not symmetric under rotations about the line joining the centers of the two monopoles. To see this, note that whereas the \( A_j^{(i)} \) and \( \Delta \Phi^{(i)} \) are explicitly independent of the azimuthal angle \( \phi \), the \( W_j^{(i)} \) are of the form

\[
W_j^{(1)} = f_j^{(1)}(\rho,z)e^{i(\phi+\gamma_1)}, \quad W_j^{(2)} = f_j^{(2)}(\rho,z)e^{i(-\phi+\gamma_2)}. \tag{2.9}
\]

The effect on \( W_j^{(1)} \) of a rotation about the \( z \)-axis can be compensated by a global \( U(1) \) rotation by an equal and opposite amount, but this just doubles the phase change of \( W_j^{(2)} \); alternatively, one can leave \( W_j^{(2)} \) unchanged and shift the phase of \( W_j^{(1)} \). The best one can do in compensating the effects of the rotation is to apply a \( z \)-dependent \( U(1) \) transformation that shifts the phase in one sense for \( z > 0 \) and in the other for \( z < 0 \); because the magnitudes of the \( W_j \) fall exponentially at large distances from the center of the monopoles, the violations of axial symmetry that remain after this gauge transformation are exponentially small. An explicit gauge-invariant measure of the asymmetry is

\[
(W_j^{(2)}(r))^* W_j^{(1)}(r) \sim e^{-2i\delta}; \quad \text{the magnitude of this quantity is everywhere of order } e^{-evR} \text{ or smaller.}
\]

When the two monopoles are not widely separated, superposition does not even give a good first approximation to the solution and more sophisticated techniques are needed to obtain the solutions. Using such methods, one finds that the rotational asymmetry about the axis joining the

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1To completely determine these fields, one must both impose a gauge condition and require that they be orthogonal to the one-monopole zero modes about each of the component monopoles.
two monopoles persists as long as the centers of the two monopoles do not coincide. If the centers do coincide, the solution has a single axis of rotational symmetry; general arguments show that spherically symmetric solutions are impossible [13].

3 BPS Monopoles for Larger Gauge Groups

Recall that a basis for the Cartan subalgebra of a rank \( k \) gauge group can be chosen to be \( k \) generators \( H_i \) normalized so that

\[
\text{tr} H_i H_j = \delta_{ij}.
\]

(3.1)

One can then define raising and lowering operators \( E_\alpha \) obeying

\[
[H, E_\alpha] = \alpha E_\alpha,
\]

(3.2)

and normalized so that

\[
[E_\alpha, E_{-\alpha}] = \alpha \cdot H.
\]

(3.3)

The roots \( \alpha \) may be viewed as vectors forming a lattice in a \( k \)-dimensional Euclidean space. It is always possible to choose a basis of \( k \) simple roots for this lattice in such a way that all other roots are linear combinations of the simple roots with integer coefficients all of the same sign; a root is called positive or negative according to this sign. A set of simple roots of particular importance is defined as follows. Let \( \Phi_0 \) be the asymptotic value of the Higgs field in some fixed direction (say, the positive \( z \)-axis). We may choose \( \Phi_0 \) to lie in the Cartan subalgebra and then define a vector \( h \) by

\[
\Phi_0 = h \cdot H.
\]

(3.4)

In this paper we are concerned with the case of maximal symmetry breaking, with the gauge group spontaneously broken to \( U(1)^k \). This is achieved if and only if \( h \) has a nonzero inner product with all of the roots. When this is so, there is a unique set of simple roots \( \beta_a \) that satisfy the requirement that \( h \cdot \beta_a \) be positive for all \( a \); we shall use this basis for the remainder of this paper.

Asymptotically, the magnetic field must commute with the Higgs field. Hence, in the direction chosen to define \( \Phi_0 \), it must be of the form

\[
B_k = \frac{\hat{r}_k}{4 \pi r^2} g \cdot H.
\]

(3.5)

Topological arguments lead to the quantization condition [4]

\[
g = \frac{4 \pi}{e} \sum_{a=1}^{k} n_a \beta_a^*,
\]

(3.6)
where

$$\beta_a^* = \frac{\beta_a}{\beta_a^2} \quad (3.7)$$

are the duals of the simple roots and the integers $n_a$ are the topologically conserved charges corresponding to the homotopy class of the scalar field at spatial infinity.

Monopole solutions carrying a single unit of topological charge can be obtained by simple embeddings of the $SU(2)$ solution. Each simple root $\beta_a$ defines an $SU(2)$ subgroup with generators

$$t^1 = \frac{1}{\sqrt{2\beta_a^2}} (E_{\beta_a} + E_{-\beta_a}),$$

$$t^2 = -\frac{i}{\sqrt{2\beta_a^2}} (E_{\beta_a} - E_{-\beta_a}),$$

$$t^3 = \beta_a^* \cdot H. \quad (3.8)$$

If $\Phi^s(r; v)$ and $A^s_i(r; v) \ (s = 1, 2, 3)$ is the $SU(2)$ solution corresponding to a Higgs expectation value $v$, then

$$A_i(r) = \sum_{s=1}^{3} A^s_i(r; h \cdot \beta_a) t^s,$$

$$\Phi(r) = \sum_{s=1}^{3} \Phi^s(r; h \cdot \beta_a) t^s + (h - h \cdot \beta_a^* \beta_a) \cdot H, \quad (3.9)$$

is a solution with topological charges

$$n_b = \delta_{ab}, \quad (3.10)$$

and mass

$$m_a = \frac{4\pi}{e} h \cdot \beta_a^*. \quad (3.11)$$

We will refer to such solutions as fundamental monopoles \cite{footnote}. As in the $SU(2)$ case, there are dyon solutions corresponding to these fundamental monopoles, with their electric charges quantized so that asymptotically

$$E_k = \frac{en \vec{v}_k}{4\pi r^2} \beta_a \cdot H \quad (3.12)$$

for some integer $n$.

Solutions corresponding to several widely separated fundamental monopoles can be constructed in a manner similar to that described above for the $SU(2)$ case. A notable difference, which will be of importance later, concerns the symmetry of the two-monopole solutions. In the $SU(2)$ case these failed to be axially symmetric because any gauge transformation that compensated the effects of a spatial rotation on one of the monopoles shifted the phase of the other monopole in the wrong
direction. By contrast, if the two are different fundamental monopoles, it is always possible to find two gauge transformations such that each gives the required phase shift on one of the monopoles and leaves the other invariant. As a result, the superposition construction yields solutions with exact axial symmetry. (Note that $\text{Tr} \left( W_j^{(2)} \right)^* W_j^{(1)}$, the analogue of the gauge-invariant measure of the asymmetry in the $SU(2)$ case, vanishes identically.)

A second difference from the $SU(2)$ case is the existence of relatively simple, spherically symmetric solutions that can be interpreted as superpositions of several fundamental monopoles at the same point. These are given by Eqs. (3.8) and (3.9), but with a composite root $\alpha$, rather than a simple root $\beta_a$, defining the $SU(2)$ subgroup. The coefficients $n_a$ in the expansion

$$\alpha^* = \sum_{a=1}^{k} n_a \beta^*_a$$

are the topological charges of the solution, while the mass is

$$m = \sum_{a} n_a m_a.$$  

Although the mass and topological charge of these solutions are consistent with their interpretation as superpositions of several noninteracting monopoles, one might still ask why these spherically symmetric solutions should be viewed on such a different basis than those obtained from the simple roots. An answer is obtained by counting the normalizable zero modes about these solutions. After gauge fixing, the number of zero modes about an arbitrary BPS solution is

$$N = 4 \sum_{a=1}^{k} n_a.$$  

Thus, each of the fundamental monopoles has four zero modes, three corresponding to spatial translations and the fourth to a global $U(1)$ gauge rotation. By contrast, the solutions based on composite roots all have additional zero modes, with the number precisely that expected if these are in fact superpositions of several fundamental monopoles.

The zero modes about the spherically symmetric solutions clearly must arrange themselves in angular momentum multiplets. For each of the fundamental monopoles the four zero modes divide into a triplet (the spatial translation modes) and a singlet (the global gauge rotation mode). Consider next an embedding solution corresponding to a composite root $\alpha$ for which the expansion Eq. (3.13) has only two nonvanishing $n_a$, both equal to unity. Of the eight zero modes about such a solution, four keep the solution within the embedding subgroup; these are just the overall translation and global gauge rotation modes. The remaining four separate into two doublets that transform in opposite fashion under the “hypercharge” $U(1)$ generated by $(\mathbf{h} - \mathbf{h} \cdot \beta_a \beta^*_a) \cdot \mathbf{H}$. (This
implies, by the way, that none of these modes correspond in a simple way to separation of the centers of the constituent monopoles; such separation only occurs at second order in the deviations from the spherically symmetric solution.) More generally, one can show that, for any embedded solution based on a root for which the nonzero $n_a$ are all unity, the $N - 4$ zero modes that lie outside the embedding subgroup are all arranged in pairs of doublets carrying opposite hypercharges. This is not the case if any of the $n_a$ are greater than unity.

4 Widely Separated Fundamental Monopoles

The metric on the moduli space determines the motions of slowly moving dyons. Conversely, the form of the moduli space metric can be inferred from a knowledge the interactions between the dyons. These become quite complicated when several dyons approach one another. However, for finding the metric in the regions of the moduli space corresponding to large intermonopole distances it is sufficient to examine the long-range pairwise interactions between widely separated dyons. This analysis has been carried out previously for $SU(2)$ [8][16]; in this section we will extend it to larger gauge groups.

We begin by considering the interactions between a single pair of $SU(2)$ dyons, with positions $x_j$ and velocities $v_j$ and carrying $U(1)$ magnetic charges $g_j$ and electric charges $q_j$ ($j = 1, 2$). If the separation between the dyons is much larger than the radius of a monopole core, the electromagnetic interactions between them can be well approximated by the standard results for a pair of moving point charges. There is also a long-range scalar force that is manifested as a position-dependent shift in the dyon mass. Recall that the mass of an isolated dyon is $v\sqrt{g^2 + q^2}$, where $v$ is the magnitude of the vacuum expectation value of the Higgs field (in $SU(2)$ theory). When there is a second dyon present, its scalar field must be added to $v$ in this formula. To be more precise, Lorentz transformation of the dyon solution of Eq.(2.2) reveals that the magnitude of the Higgs field about a moving dyon is

$$|\Phi_0 + \Delta \Phi^{(j)}(x)| = v - \frac{1}{4\pi |x - x_j|} \sqrt{1 - v_j^2} \sqrt{g_j^2 + q_j^2} + O(r^{-2}).$$

The effective mass of dyon 1 in the presence of dyon 2 can then be written as

$$\sqrt{g_1^2 + q_1^2} |\Phi_0 + \Delta \Phi^{(2)}(x_1)|.$$  (4.2)

These effects of these interactions on dyon 1 are described by the Lagrangian

$$L_{SU(2)}^{(1)} = \sqrt{g_1^2 + q_1^2} |\Phi_0 + \Delta \Phi^{(2)}(x_1)| \sqrt{1 - v_1^2}$$

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\[ + q_1 \left[ v_1 \cdot A^{(2)}(x_1) - A_0^{(2)}(x_1) \right] + g_1 \left[ v_1 \cdot \tilde{A}^{(2)}(x_1) - \tilde{A}_0^{(2)}(x_1) \right]. \]  

(4.3)

Here \( A^{(2)} \) and \( A_0^{(2)} \) are the ordinary vector and scalar electromagnetic potentials due to charge 2, while \( \tilde{A}^{(2)} \) and \( \tilde{A}_0^{(2)} \) are dual potentials defined so that \( E = -\nabla \times \tilde{A} \) and \( B = -\nabla \tilde{A}_0 - \partial \tilde{A}/\partial t \).

These potentials can be obtained by standard methods [16]. Substituting the results, together with Eq. (4.1) for the Higgs field, into Eq. (4.3) and keeping only terms of up to second order in \( q_j \) or \( v_j \), we obtain

\begin{align*}
L_{SU(2)}^{(1)} &= -m_1 \left( 1 - \frac{1}{2} v_1^2 + \frac{q_1^2}{2g_1^2} \right) \\
&\quad - \frac{g_1 g_2}{8 \pi r_{12}} \left[ (v_1 - v_2)^2 - \left( \frac{q_1}{g_1} - \frac{q_2}{g_2} \right)^2 \right] \\
&\quad - \frac{1}{4\pi} (g_1 q_2 - g_2 q_1) (v_2 - v_1) \cdot w_{12},
\end{align*}

(4.4)

where \( m_1 = v g_1 \), \( r_{12} = |x_1 - x_2| \) and \( w_{12} \) is a Dirac monopole potential, defined so that

\[ \nabla_1 \times w_{12}(x_1 - x_2) = -\frac{x_1 - x_2}{r_{12}^3}. \]

(4.5)

We now want to generalize this to the case of a larger group, with dyon \( j \) being obtained by embedding the \( SU(2) \) solution via the subgroup defined by the simple root \( \alpha_j \). Working in a gauge where the Higgs field is everywhere in the Cartan subalgebra, we use the long-range behavior of the electromagnetic field strengths

\begin{align*}
B^{(j)} &= g_j (\alpha_j^* \cdot H) \frac{(x - x_j)}{4\pi |x - x_j|^3}, \\
E^{(j)} &= q_j (\alpha_j^* \cdot H) \frac{(x - x_j)}{4\pi |x - x_j|^3},
\end{align*}

(4.6)

to define the magnetic and electric charges \( g_j \) and \( q_j \). (Note that \( q_j \) multiplies \( \alpha_j^* \cdot H \) rather than \( \alpha_j \cdot H \). This turns out to be convenient for writing the Lagrangian, but leads to the somewhat unusual quantization condition that \( q_j \) be an integer multiple of \( e\alpha_j^2 \). For \( SU(2) \) this agrees with our previous convention if the single root \( \alpha \) is take to have unit length.) In the absence of other dyons, the long-range Higgs field of this dyon would be

\[ \Phi^{(j)} = h \cdot H - \frac{(\alpha_j^* \cdot H)}{4\pi |x - x_j|} \sqrt{1 - v_j^2} \sqrt{g_j^2 + q_j^2}, \]

(4.7)

and its mass would be

\[ m_j = (\alpha_j^* \cdot h) \sqrt{g_j^2 + q_j^2}. \]

(4.8)

The electromagnetic interactions between the two dyons can all be traced back to terms in the fundamental field theory Lagrangian of the form \( \text{tr} F_{\mu\nu}^{(j)} F^{(j)\mu\nu} \) and hence acquire multiplicative
factors of
\[ \text{tr} \left( (\alpha_1^* \cdot H)(\alpha_2^* \cdot H) \right) = \alpha_1^* \cdot \alpha_2^*. \eqno(4.9) \]

The shift in the mass of dyon 1 due to the Higgs field of dyon 2 is found by replacing the factor of \( \alpha_1^* \cdot h \) in Eq. (4.8) by
\[ \alpha_1^* \cdot h - \frac{\alpha_1^* \cdot \alpha_2^*}{4\pi r_{12}} \sqrt{1 - v_2^2 g_2^2 + g_2^2}. \eqno(4.10) \]

From this we see that the scalar interaction terms acquire the same factor of \( \alpha_1^* \cdot \alpha_2^* \) as the electromagnetic terms. Aside from this factor, the long-range interactions are exactly as in the \( SU(2) \) case, and so \( I_{SU(2)}^{(1)} \) can be generalized to a higher rank group simply by replacing \( 1/r_{12} \) by \( \alpha_1^* \cdot \alpha_2^* / r_{12} \) and \( w_{12} \) by \( \alpha_1^* \cdot \alpha_2^* w_{12} \).

The extension to an arbitrary number of dyons is straightforward, provided that their mutual separations are all large. Since we are considering fundamental dyons that all carry unit magnetic charges, we can set all of \( g_j \) equal to \( g = 4\pi/e \). The Lagrangian obtained by adding all the pairwise interactions can be written as
\[ L = \frac{1}{2} M_{ij} \left( \mathbf{v}_i \cdot \mathbf{v}_j - \frac{q_i q_j}{g^2} \right) + \frac{g}{4\pi} q_i W_{ij} \cdot \mathbf{v}_j, \eqno(4.11) \]

where
\[
M_{ii} = m_i - \sum_{k \neq i} \frac{g^2 \alpha_i^* \cdot \alpha_k^*}{4\pi r_{ik}}, \\
M_{ij} = \frac{g^2 \alpha_i^* \cdot \alpha_j^*}{4\pi r_{ij}} \quad \text{if} \ i \neq j, \eqno(4.12)
\]

with \( m_i = g \alpha_i^* \cdot h \), and
\[
W_{ii} = -\sum_{k \neq i} \alpha_i^* \cdot \alpha_k^* w_{ik}, \\
W_{ij} = \alpha_i^* \cdot \alpha_j^* w_{ij} \quad \text{if} \ i \neq j. \eqno(4.13)
\]

with \( w_{ij} \) being value at \( x_i \) of the Dirac potential due to the \( j \)th monopole. The \( q \)-independent part of the monopole rest energies has been omitted.

To obtain the moduli space metric, we need a Lagrangian that is purely kinetic; i.e., one in which all terms are quadratic in velocities. This can be done by interpreting the \( q_j/e \) as conserved momenta conjugate to cyclic angular variables \( \xi_j \); because \( q_j/e \) is quantized in integer multiples of \( \alpha_j^2 \), the period of \( \xi_j \) must be \( 2\pi/\alpha_j^2 \). Thus, if we make the identification
\[ q_i/e = \frac{g^4}{(4\pi)^2} (M^{-1})_{ij} (\dot{\xi}_j + W_{jk} \cdot \mathbf{v}_k), \eqno(4.14) \]
the desired Lagrangian $\mathcal{L}$ is the Legendre transform

$$\mathcal{L} = L + \sum_j \dot{\xi}_j q_j / e$$

$$= \frac{1}{2} M_{ij} v_i \cdot v_j + \frac{g^4}{2 (4\pi)^2} (M^{-1})_{ij} \left( \dot{\xi}_i + W_{ik} \cdot v_k \right) \left( \dot{\xi}_j + W_{jl} \cdot v_l \right).$$  \hspace{1cm} (4.15)

From this we immediately obtain the large separation approximation to the moduli space metric,

$$\mathcal{G} = \frac{1}{2} M_{ij} dx_i \cdot dx_j + \frac{g^4}{2 (4\pi)^2} (M^{-1})_{ij} (d\xi_i + W_{ik} \cdot dx_k) (d\xi_j + W_{jl} \cdot dx_l).$$  \hspace{1cm} (4.16)

Note that this metric is equipped with a number of $U(1)$ isometries, each of which is generated by the constant shift of one of the $\xi_j$'s.

For the case of $SU(2)$, one can easily see that this approximation to the moduli space metric cannot be exact [17]. First of all, it develops singularities if any of the intermonopole distances becomes too small, whereas the moduli space metric should be nonsingular. Second, for the case of two monopoles the approximate metric is independent of the relative phase angle $\xi_1 - \xi_2$. If this isometry were exact, the two-monopole solutions would be axially symmetric, which we know is not the case.

Neither of these objections arises for the moduli space corresponding to a collection of several distinct fundamental monopoles in a larger group, provided that each corresponds to a different simple root. In Secs. 5 and 7 we will show that the metric is nonsingular for all values of the $r_{ij}$ in such cases. Further, we argued in the previous section that solutions with two different fundamental monopoles are axially symmetric, and so the $U(1)$ isometries of the metric $\mathcal{G}$ are just what we want.

In the next two sections we will concentrate on the two-monopole case and show that the metric obtained in this section must be the exact moduli space metric. We will then return to the case of three or more distinct fundamental monopoles and argue that $\mathcal{G}$ is likely to be exact for this case also.

## 5 A Pair of Distinct Fundamental Monopoles

We now specialize the results of the previous section to the case of two fundamental monopoles. There are three possibilities, distinguished by the sign of $\lambda = -2\alpha_1^* \cdot \alpha_2^*$. This is negative only if both monopoles are based on the same simple root. In this case the moduli space is the same as that for two $SU(2)$ monopoles [7], where we know that the asymptotic metric develops a singularity at small monopole separations. For two distinct monopoles, $\lambda = 0$ if their roots are not connected in the Dynkin diagram of the original gauge algebra and $\lambda > 0$ if the roots are so connected. In
the former case there are clearly no interactions between the monopoles, and the moduli space is simply the product of two one-monopole moduli spaces. This leaves only the case $\lambda > 0$, to which we now turn. We then have two distinct roots $\alpha_1$ and $\alpha_2$. Because these are both simple, the combinations $\lambda \alpha_1^2 = -2\alpha_1 \cdot \alpha_2 / \alpha_2^2$ and $\lambda \alpha_2^2 = -2\alpha_1 \cdot \alpha_2 / \alpha_1^2$ are positive integers no larger than 3. On the other hand, $(\lambda \alpha_1^2)(\lambda \alpha_2^2)/4 = (\alpha_1 \cdot \alpha_2)^2/\alpha_2^2 \alpha_2^2$ must be less than one. Thus, without loss of generality, we can set $\lambda \alpha_1^2 = p$ and $\lambda \alpha_2^2 = 1$, where $p$ is either 1, 2, or 3. In particular, the rank-two gauge groups $SU(3)$, $SO(5)$, and $G_2$ yield $p = 1, 2, \text{and } 3$, respectively.

We begin by separating center-of-mass from relative variables. For the spatial coordinates we introduce the usual quantities

$$R = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad r = x_1 - x_2. \quad (5.1)$$

To separate the phase variables, we first introduce the total charge $q_\chi$ and the relative charge $q_\psi$ as linear combinations of the electric charges $q_1$ and $q_2$,

$$q_\chi = \frac{(m_1 q_1 + m_2 q_2)}{e (m_1 + m_2)}, \quad q_\psi = \frac{\lambda (q_1 - q_2)}{2e}. \quad (5.2)$$

(We noted in the previous section that $q_j$ is quantized in integer units of $e\alpha_j^2$. It follows that the relative charge $q_\psi$ has eigenvalues $0, \pm 1/2, \pm 1, \ldots$. In contrast, the total charge $q_\chi$ is not quantized unless the ratio $m_1/m_2$ is a rational number.) The conjugate variables to these charges are

$$\chi = (\xi_1 + \xi_2), \quad \psi = \frac{2(m_2 \xi_1 - m_1 \xi_2)}{\lambda (m_1 + m_2)}. \quad (5.3)$$

When expressed in these variables, the Lagrangian separates into a sum of two terms,

$$L^{(2)} = L^{(2)}_{cm} + L^{(2)}_{rel}, \quad (5.4)$$

where

$$L^{(2)}_{cm} = \frac{1}{2} (m_1 + m_2) \dot{R}^2 + \frac{g^4}{32\pi^2 (m_1 + m_2)} \dot{\chi}^2; \quad (5.5)$$

and $L^{(2)}_{rel}$ is the rest:

$$L^{(2)}_{rel} = \frac{1}{2} \left( \mu + \frac{g^2 \lambda}{8\pi r} \right) \dot{r}^2 + \frac{1}{2} \left( \frac{g^2 \lambda}{8\pi r} \right)^2 \left( \mu + \frac{g^2 \lambda}{8\pi r} \right)^{-1} \left( \dot{\psi} + w(r) \cdot \dot{r} \right)^2. \quad (5.6)$$

Here $\mu$ is the reduced mass and $w(r) = w_{12}(r)$.

The center-of-mass part of the moduli space is parameterized by $R$ and $\chi$ with the metric,

$$G^{(2)}_{cm} = \frac{m_1 + m_2}{2} \left( dR^2 + \frac{g^4}{16\pi^2 (m_1 + m_2)^2} d\chi^2 \right). \quad (5.7)$$
and is a flat four-dimensional manifold. Defining $r_0 \equiv g^2 \lambda / 8\pi \mu$, we find that the dynamics of $L^{(2)}_{rel}$ is reproduced by the geodesic motion on a four-dimensional manifold with metric

$$G^{(2)}_{rel} = \frac{\mu}{2} \left( (1 + r_0/r) \, dr^2 + r_0^2 (1 + r_0/r)^{-1} \left( d\psi + w(r) \cdot dr \right)^2 \right).$$

Apart from an overall rescaling, this is the metric of a Taub-NUT manifold with length parameter $l = r_0/2$. This is smooth everywhere but at $r = 0$, where there is a singularity unless $\psi$ has a period of $4\pi$.

Locally, the full moduli space is a product of these two manifolds. However, the periodicity of the $\xi_i$'s imposes certain identifications on $\psi$ and $\chi$, so that the total moduli space is obtained only after a division by a discrete group, as we now show. In the previous section we noted that the quantization of the charge operators $q_i$ implied that the $\xi_i$ had periods $2\pi/\alpha_i^2$. A shift of $\xi_1$ by $2\pi/\alpha_1^2$ then implies the identification

$$(\chi, \psi) = (\chi + \frac{2\pi}{\alpha_1^2}, \psi + \frac{4\pi m_2}{\lambda \alpha_1^2 (m_1 + m_2)}),$$

while a $-2\pi/\alpha_2^2$ shift of $\xi_2$ gives

$$(\chi, \psi) = (\chi - \frac{2\pi}{\alpha_2^2}, \psi + \frac{4\pi m_1}{\lambda \alpha_2^2 (m_1 + m_2)}).$$

Then, after $p = \lambda \alpha_1^2$ steps of the first shift and one step of the second, we discover that

$$(\chi, \psi) = (\chi, \psi + 4\pi).$$

This shows that the $\psi$ has a period of $4\pi$, in agreement with the half-integer quantization of $q_\psi$ noted above, and hence that the relative coordinate metric $G^{(2)}_{rel}$ is smooth everywhere.

Without these identifications, the surface spanned by the coordinates $(\chi, \psi)$ (at any $r \neq 0$) would be a cylinder. However, the identification (5.9) tells us to cut out a section of this cylinder of length $\delta \chi = 2\pi \lambda / p$ and to identify the two rims with a twist $\delta \psi = 4\pi m_2 / (p(m_1 + m_2))$. The result is that the moduli space is of the form

$$\mathcal{M} = R^3 \times \frac{R^1 \times \mathcal{M}_0}{Z},$$

where $\mathcal{M}_0$ is the Taub-NUT manifold with metric (5.8). If the ratio of the two monopole masses is rational, $\chi$ is in fact periodic. For instance, when the masses are equal, the generator of the identification map becomes $(\chi, \psi) = (\chi + 2\pi \lambda / p, \psi + 2\pi / p)$, which collapses the $R^1$ to a circle of length $4\pi \lambda$. Accordingly, the moduli space can be written as

$$\mathcal{M} = R^3 \times \frac{S^1 \times \mathcal{M}_0}{Z_{2p}}.$$
Such compactification of the $R^1$ to $S^1$ corresponds to the fact that $\rho$ is quantized when the mass ratio is rational.

The Lagrangian $L_{\text{rel}}^{(2)}$ possess a rotational invariance. Although the term involving $\dot{\psi}$ is not manifestly invariant under rotation of the vector $r$, the situation is similar to that of a charged particle in the presence of a point-like monopole and the effect of this last term is to simply modify the conserved angular momentum in a familiar way:

$$\mathbf{J} = \left( \mu + \frac{g^2 \lambda}{8\pi r} \right) \mathbf{r} \times \dot{\mathbf{r}} + q_\psi \hat{r}.$$

Note that for $q_\psi = \pm 1/2, \pm 3/2, \ldots$, this angular momentum is quantized at half-integer values, so that the rotation group is actually $SU(2)$ instead of the naive $SO(3)$.

From a more geometrical point of view, the conserved angular momentum $\mathbf{J}$ implies a set of Killing vector fields on $M_0$ that generate an $SU(2)$ translational symmetry. In other words, the spatial rotations of the monopoles induce an $SU(2)$ isometry of the moduli space itself. Since a rotation leaves the relative distance $r$ between the two monopoles invariant, the orbit is spanned by angular coordinates only. Furthermore, because of the extra term proportional to $q_\psi$, the angular momentum $\mathbf{J}$ shifts $\psi$ in addition to rotating the two-sphere $|\mathbf{r}| = r$, so the orbit is generically three-dimensional. In fact, the three-dimensional orbit at $r > 0$ is simply an $S^3$ parameterized by the three Euler angles $\psi$, $\theta$, and $\phi$, with the latter two spanning the two-sphere $|\mathbf{r}| = r$. The only exception is at origin, $r = 0$, where the orbit collapses to a point.

We have shown in this section that for a pair of distinct fundamental monopoles the asymptotic metric found in Sec. 4 remains nonsingular for all values of the monopole separation. While this means that the asymptotic form could, in fact, be exact, it certainly not prove that it is. In the next section we will use the methods that Atiyah and Hitchin used for the $SU(2)$ case to demonstrate that Eq. (5.12) does indeed give the exact two-monopole moduli space metric.

6 The Two-Monopole Moduli Space

Symmetry considerations tell us a good deal about the form of the two-monopole moduli space $\mathcal{M}$. First of all, there must be three flat directions corresponding to overall spatial translations of the two-monopole system. Next, note that the dyon solutions obtained from a time-dependent phase rotation generated by $\Phi_0$ are BPS solutions and therefore do not interact when at rest. This implies the existence of a fourth flat direction, corresponding to the coordinate $\chi$ introduced in the previous section. Allowing for the possibility of identifications of the sort that we found in Sec. 5,
we conclude that the space must be of the form

$$\mathcal{M} = R^3 \times \frac{R^1 \times \mathcal{M}_0}{\mathcal{D}}.$$  \hfill (6.1)

where $\mathcal{D}$ is a discrete normal subgroup of the isometry group of $R^1 \times \mathcal{M}_0$.

The isometry group of $\mathcal{M}_0$ is also easily determined. Since a spatial rotation of a BPS solution about any fixed point is again a solution, $\mathcal{M}_0$ must possess a three-dimensional rotational isometry. One linear combination of the two unbroken $U(1)$ gauge degrees of freedom generates the translational symmetry, alluded to above, along the overall $R^1$. The remaining generator must then induce a $U(1)$ isometry acting on $\mathcal{M}_0$. Hence, $\mathcal{M}_0$ must be a four-dimensional manifold with at least four Killing vector fields that span an $su(2) \times u(1)$ algebra. Furthermore, we saw in the previous section that the orbits of the rotational isometry on the asymptotic metric were three-dimensional; clearly the exact metric must also possess this property at large $r$.

In addition, the moduli space must be hyperkähler. In four dimensions this implies that the manifold must be a self-dual Einstein manifold. From this, together with the rotational symmetry properties of the manifold, it follows that the metric can be written as

$$ds^2 = f(r)^2 dr^2 + a(r)^2 \sigma_1^2 + b(r)^2 \sigma_2^2 + c(r)^2 \sigma_3^2,$$  \hfill (6.2)

where the metric functions obey

$$\frac{2bc da}{f} \frac{da}{dr} = b^2 - c^2 - a^2 - 2\lambda bc, \quad \text{and cyclic permutations thereof},$$  \hfill (6.3)

with $\lambda$ either 0 or 1, while the three one-forms $\sigma_k$ satisfy

$$d\sigma_i = \frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k.$$  \hfill (6.4)

An explicit representation for these one-forms is

$$\begin{align*}
\sigma_1 &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \\
\sigma_2 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi, \\
\sigma_3 &= d\psi + \cos \theta d\phi.
\end{align*}$$  \hfill (6.5)

The ranges of $\theta$ and $\phi$ are $[0, \pi]$ and $[0, 2\pi]$, respectively. In order that the metric tend to the $G_{rel}$ found in Sec. 5, the range of $\psi$ must be $[0, 4\pi]$. A 4n-dimensional manifold with a metric is hyperkähler if it possesses three covariantly constant complex structures $\mathcal{J}(k)$ that also form a quaternionic structure, and if the metric is pointwise Hermitian with respect to each $\mathcal{J}(k)$. If we recast the zero modes $(\delta \Phi, \delta A_i)$ into a spinor $\Psi = \Phi + i\tau_i \delta A_i$ where $\tau_i$’s are the Pauli matrices, the zero mode equation is manifestly invariant under right multiplications by the $i\tau_i$’s, and this induces the almost quaternionic structure on the moduli space. Detailed arguments that the moduli space is hyperkähler can be found in Refs. [7] and [18].
Following Atiyah and Hitchin [7], we may reduce Eq. (6.3) to a two-dimensional flow equation for \( x = a/b \) and \( y = c/b \) in terms of \( \eta = \int b/ac \, dr \).

\[
\frac{dx}{d\eta} = x(1 - x)(1 + x - \lambda y), \quad \frac{dy}{d\eta} = y(1 - y)(1 + y - \lambda x).
\]

(6.6)

The case of \( \lambda = 1 \) was studied in detail by Atiyah and Hitchin, where they found only three possibilities (up to irrelevant permutations of \( \sigma_k \)'s) that corresponded to complete manifolds:

1) \( a = b = c \) case: flat \( R^4 \).
2) \( a = b \neq c \) case: the Taub-NUT geometry with an \( SU(2) \) rotational isometry.
3) \( a \neq b \neq c \) case: the Atiyah-Hitchin geometry with an \( SO(3) \) rotational isometry.

The case of \( \lambda = 0 \) leads to only one new possibility:

4) the Eguchi-Hanson gravitational instanton [20].

The moduli space is clearly curved, since otherwise there would be no interaction, and so we can exclude \( R^4 \). Of the remaining three, the only one that asymptotes to the geometry of \( G^{(2)}_{rel} \) is the Taub-NUT geometry, and thus \( M_0 \) must be a Taub-NUT manifold [10-12]. With a suitable choice of \( r \), its metric can be written as

\[
ds^2 = \left( 1 + \frac{2l}{r} \right) \left( dr^2 + r^2 \sigma_1^2 + r^2 \sigma_2^2 \right) + \left( \frac{4l^2}{1 + 2l/r} \right) \sigma_3^2
\]
\[
= \left( 1 + \frac{2l}{r} \right) dr^2 + \left( \frac{4l^2}{1 + 2l/r} \right) (d\psi + \cos \theta \, d\phi)^2.
\]

(6.7)

(Note that \( \psi \) dependence drops out of the metric coefficients so that we have an additional Killing vector field, \( \xi_3 = \partial/\partial \psi \), that generates the extra \( U(1) \) isometry anticipated above.) A comparison of this metric with the asymptotic form \( G_{rel}^{(2)} \) in Eq. (3.8) reveals that they are identical, up to a trivial rescaling and a gauge choice for the vector potential \( w \), if we identify the length parameter \( l \) with \( r_0/2 = -g^2 \alpha_1^* \cdot \alpha_2^*/8\pi\mu \).

A further check that \( M_0 \) must be Taub-NUT follows from the existence of the spherically symmetric BPS solution obtained, as described below Eq. (3.13) by embedding the \( SU(2) \) monopole using the subgroup defined by the composite root whose dual is the sum of the duals of the two relevant simple roots. This solution obviously corresponds to a point of \( M_0 \) that is invariant under

\[\text{footnote}{\text{In fact, as pointed out by Atiyah and Hitchin, the case of \( \lambda = 0 \) need not be considered. When \( \lambda = 0 \), the rotational isometry leaves the three complex structures of the moduli space individually invariant, but this cannot be the case for the real moduli space.}}\]
the rotational isometry. While there are no such points in the Atiyah-Hitchin and Eguchi-Hanson geometries, the point \( r = 0 \) of the Taub-NUT metric is just what we want. The apparent singularity in the metric at this point turns out to be a simple coordinate singularity, also known as a “NUT” singularity, that is easily removed by introducing a new radial coordinate with \( \rho^2 \simeq 8lr \) so that

\[
\begin{align*}
 ds^2 &= \left[ 1 + O(\rho^2) \right] d\rho^2 + \frac{\rho^2}{4} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^3 \right) + O(\rho^4). \\
&= \left( 1 + O(\rho^2) \right) d\rho^2 + \frac{\rho^2}{4} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^3 \right) + O(\rho^4).
\end{align*}
\] (6.8)

To the leading order, this is the metric of the flat \( R^4 \) Euclidean space and thus obviously smooth at the origin. (The unfamiliar factor of \( 1/4 \) is due to the normalization of the \( \sigma_k \).) From this expression it is clear that the infinitesimal motions away from the origin (i.e., the zero modes about the spherically symmetric solution) transform under the vector representation of the \( SO(4) \) symmetry of \( R^4 \). In terms of the three-dimensional rotations of physical space, this vector decomposes into a pair of doublets, in agreement with the zero mode multiplet structure noted at the end of Sec. 3.

### 7 The Smooth Moduli Space of \( n \) Fundamental Monopoles

We saw in Sec. 5 that for a pair of distinct fundamental monopoles the asymptotic form of the moduli space metric remained nonsingular for all values of the monopole separation, suggesting that this asymptotic form might be the exact metric, a conjecture that was verified by the analysis of Sec. 6. We now return to the more general case of an arbitrary number of distinct fundamental monopoles. We will argue the asymptotic form is likely to be exact in this case also.

Thus, suppose we have \( n \) distinct fundamental monopoles of charges \( \alpha_i^\star \). These \( n \) \( \alpha_i \)'s form a subset of the simple roots of the Lie algebra and define a subdiagram of its Dynkin diagram. In general, this subdiagram would be composed of several connected components. However, monopoles that belong to one connected component have no interaction with those belonging to others, so that the total moduli space would be a product of moduli spaces for each connected component. Hence, it is sufficient to consider the case where the diagram spanned by the \( \alpha_i \)'s is connected, and is therefore the full Dynkin diagram of some (possibly smaller) simple gauge group.

The moduli space metric for this \( n \)-monopole system must be both hyperkähler and nonsingular. The demonstration that the asymptotic metric is hyperkähler is essentially the same as for the \( SU(2) \) case treated by Gibbons and Manton \[8\]. The key ingredient is that the fact that the gradient of the scalar potential \( 1/r_{ij} \) is equal to the curl of the vector potential \( w_{ij} \) for each pair of monopoles; since the only modification we have introduced is to multiply both potentials by \( \alpha_i^\star \cdot \alpha_j^\star \), the hyperkähler property is preserved.
We next turn to the issue of singularities. The most obvious set of potential singularities are the places where one or more of the \( r_{ij} \) vanish; we will leave these for last. A second possibility is that a metric component might diverge, even though all of the \( r_{ij} \) are nonzero, because the \( k \times k \) matrix \( M_{ij} \) failed to be invertible. The final possibility is that the metric might become degenerate, which would be seen as a zero of its determinant

\[
\text{Det } G = \left( \frac{g^2}{16 \pi} \right)^{2n} (\text{Det } M)^2. \tag{7.1}
\]

To rule out these last two possibilities it suffices to show that \( \text{Det } M \) is nonzero whenever the \( r_{ij} \) are nonzero.

To show this, all we need is the fact that \( M_{ij} \) can be written in the form

\[
M_{ii} = m_i + \sum_{j \neq i} c_{ij}, \quad M_{ij} = -c_{ij} \quad \text{if } i \neq j, \tag{7.2}
\]

where the \( c_{ij} \) are all nonnegative and the \( m_i \) are all positive definite. For \( n = 2 \) it is trivial to see that \( \text{Det } M > 0 \). For \( n > 2 \) we proceed inductively. First, we note that the determinant is equal to zero if all of the \( m_i \) vanish. Second, the the partial derivative of \( \text{Det } M \) with respect to any one of the masses, say \( m_1 \), is the determinant of the symmetric \( (n - 1) \times (n - 1) \) matrix obtained by eliminating the first row and column. This new matrix is of the same type as \( M \) (except that now we have \( m_j \to m_j + c_{1j} > 0 \)), and hence has a positive determinant by the induction hypothesis. It then immediately follows that \( \text{Det } M > 0 \) and that the metric is nonsingular if none of the \( r_{ij} \) vanish.

Before proceeding further, it is convenient to separate out the center-of-mass coordinates. A crucial point to observe here is that there are exactly \( n - 1 \) number of links among the \( \alpha_i \)'s in the connected Dynkin subdiagram spanned by the \( \alpha_i \)'s. Because of this, there are exactly \( n - 1 \) number of unordered pair of distinct indices \{\( i, j \)\} such that \( \alpha_i \cdot \alpha_j \neq 0 \). It is easy to see that the metric coefficients are then functions only of these \( x_i - x_j \). Thus introducing a label \( A \) for these unordered pairs \{\( i, j \)\}, we can introduce coordinates

\[
R = \frac{\sum m_i x_i}{\sum m_i}, \quad r_A = x_i - x_j. \tag{7.3}
\]

Similarly, we can split the electric charges \( q_i \) of the \( n \) monopoles into a total charge \( q_\chi \) and relative charges \( q_A \), defined as

\[
q_\chi = \frac{\sum m_i q_i}{e \sum m_i}, \quad q_A = \frac{\lambda_A}{2e} (q_i - q_j). \tag{7.4}
\]
with $\lambda_A \equiv -2\alpha^*_i \cdot \alpha^*_j$. As in the two-monopole case the relative charges $q_A$'s are quantized at half-integer values, and so their conjugate variables $\psi_A$ all have period $4\pi$. The center-of-mass degrees of freedom $\mathbf{R}$ and $\chi$ now decouple from the rest, and we arrive at the relative moduli space metric,

$$G_{\text{rel}} = \frac{1}{2} C_{AB} \, d\mathbf{r}_A \cdot d\mathbf{r}_B + \frac{g^4 \lambda_A \lambda_B}{2(8\pi)^2} (C^{-1})_{AB} (d\psi_A + w(r_A) \cdot d\mathbf{r}_A) (d\psi_B + w(r_B) \cdot d\mathbf{r}_B). \quad (7.5)$$

Here the $(n-1) \times (n-1)$ matrix $C_{AB}$ is

$$C_{AB} = \mu_{AB} + \delta_{AB} \frac{g^2 \lambda_A}{8\pi r_A}, \quad (7.6)$$

where $r_A = |\mathbf{r}_A|$ and $\mu_{AB}$ is a constant matrix that can be found by using the redefinition (7.3) and the free part of the Lagrangian $\mathcal{L}$. (Note that $\mu_{AB}$ is diagonal only if $n = 2$.)

This expression for $G_{\text{rel}}$ is manifestly invariant under independent constant shifts of the periodic coordinate $\psi_A$. The $\partial/\partial \psi_A$’s are thus Killing vectors that generate $n-1$ $U(1)$ isometries of the relative moduli space. These, together with the isometry under uniform translation of the global phase $\chi$, correspond to the action of the $k$ independent global $U(1)$ gauge rotations generated by the $\alpha_i \cdot \mathbf{H}$.

We have assumed that the $\alpha_i$'s are connected and distinct. It is easy to see that the sum $\sum \alpha^*_i$ is then equal to $\gamma^*$ where $\gamma$ is some positive root of the group $G$. Embedding of the $SU(2)$ BPS monopole using the subgroup generated by $\gamma$ gives a spherically symmetric solution on which $n-1$ of the $U(1)$ gauge rotations (i.e., those orthogonal to $\gamma \cdot \mathbf{H}$) act trivially. The existence of this solution implies that there must be a maximally symmetric point on the relative moduli space that is a fixed point both under overall rotation of the $n$-monopoles and under the $n-1$ $U(1)$ translations. This fixed point is just the origin, $\mathbf{r}_A = 0$ for all $A$.

In the neighborhood of the origin, the factors of $1/r_A$ are all sufficiently large that the matrix $C_{AB}$ is effectively diagonal. The behavior of the metric $G_{\text{rel}}$ is then

$$G_{\text{rel}} \simeq \frac{g^2}{16\pi} \sum_A \lambda_A \left( \frac{1}{r_A^2} dr_A^2 + r_A (d\psi_A + w(r_A) \cdot d\mathbf{r}_A)^2 \right), \quad (7.7)$$

with the leading corrections linear in the $r_A$’s. Comparing this with the results of the two previous sections, we see that the manifold is nonsingular at the origin, and that the four-dimensional component corresponding to each $A$ gives the metric of $R^4$, with the invariant distance from the origin measured, up to a common multiplicative constant, by $\lambda_{AT_A}$. As with the two-monopole case, this is consistent with the fact that the zero modes about the spherically symmetric solution
include $n-1$ pairs of rotational doublets in addition to the four modes corresponding to overall translation or gauge rotation.

Finally, we consider the points where only some of the $r_A$'s vanish; we distinguish those that vanish by replacing the subscript $A$ by $V$. In inverting $C_{AB}$ to leading order, it suffices to remove all components of $\mu_{AB}$ in the rows or the columns labeled by the $V$'s. The matrix $C$ is then effectively block-diagonal, and consists of the diagonal entries $g^2\lambda_V/8\pi r_V$'s as well as a chunk of smaller square matrices. Looking for the part of metric along the $r_V$ and $\psi_V$ directions, we find

$$G_{rel} \simeq \frac{g^2}{16\pi} \sum_V \lambda_V \left( \frac{1}{r_V} d\psi_V^2 + r_V (d\psi_V + w(r_V) \cdot d r_V)^2 \right) + \cdots. \quad (7.8)$$

The terms shown explicitly indicated that the four-dimensional part corresponding to each of the $V$'s is again a smooth $R^4$. The remaining terms, indicated by the ellipsis, consist of harmless finite parts quadratic the other $dr_A$'s and $d\psi_A$'s as well as mixed terms that involve one of the $dr_V$'s or $d\psi_V$'s multiplied by a $d r_A$ or $d\psi_A$. The off-diagonal metric coefficients corresponding to the latter vanish linearly in terms of local Cartesian coordinates at $r_V = 0$, and hence cannot introduce any singular behavior at $r_V = 0$. We conclude that the metric $G_{rel}$, and thus $G$, remain smooth as any number of monopoles come close together.

8 Discussion

In this paper we have shown that the metric obtained from the dynamics of well-separated fundamental monopoles is in fact smooth for all values of the intermonopole distances, provided that the monopoles are all based on different simple roots of the maximally broken gauge group. Furthermore, this metric is hyperkähler, and displays the isometries and the behavior near its rotationally symmetric fixed point that the exact metric must have. For the case of any two distinct monopoles we have shown that this asymptotic metric is indeed exact, and it seems likely that this is the case for any number of such objects.

The asymptotic form for the metric can be exact only if the fundamental monopoles are all distinct, since it develops a curvature singularity if a pair of identical monopoles are brought too close together. However, there are some cases involving two or more identical monopoles where the existence of spherically symmetric solutions suggests that there may be some simplifications compared to the general case. While these can arise for any group other than $SU(n)$, the simplest example occurs in maximally broken $SO(5)$. If $\alpha$ and $\beta$ are the long and the short simple roots for this case, then $2\alpha^* + \beta^* = (\alpha + \beta)^*$, which means that there is a spherically symmetric embedded solution with triple magnetic charge $2\alpha^* + \beta^*$. This solution must correspond to a rotationally
invariant fixed point of the eight-dimensional relative moduli space. If one of the \( \alpha^* \) monopoles is removed to a large distance while the others are kept close together, the four-dimensional subspace spanned by the relative coordinates of the latter pair should resemble the smooth Taub-NUT space, but if the two \( \alpha^* \) monopoles are kept together while the \( \beta^* \) monopole is taken far away, an approximation to the Atiyah-Hitchin geometry should emerge.

A second open issue concerns theories for which the unbroken gauge group has a non-Abelian factor. The zero mode counting has been worked out in detail for the cases where the magnetic charge has only Abelian components. As one might expect, the dimension of the moduli space is typically greater than \( 4 \sum \tilde{n}_a \), where the \( \tilde{n}_a \) are the topologically conserved magnetic charges [21]. At the same time, the relative moduli space must have an isometry group larger than \( SU(2) \times U(1)^{(k-1)} \), since at least some of the extra zero modes are induces by the unbroken non-Abelian symmetry. One might hope that this larger isometry group would act to simplify the moduli space, just as the Taub-NUT manifold is much simpler than the Atiyah-Hitchin. Some partial progress in this direction has been reported [22], and we are currently pursuing this direction.

Finally, one of the more compelling motivations for studying the low energy dynamics of monopoles and dyons arises from the Montonen-Olive duality conjecture [6]. In certain supersymmetric Yang-Mills theories, duality maps strongly coupled electric theories to weakly coupled magnetic ones, thus enabling one to probe the nonperturbative nature of strongly coupled Yang-Mills theories. A notable example of this is the \( N = 2 \) supersymmetric gauge theories softly broken to \( N = 1 \), where confinement is explicitly realized through magnetic monopole condensation [23].

In order for such a duality to make sense, however, the spectrum of magnetically charged particles must be consistent with that predicted by the duality mapping of the electrically charged ones. In \( N = 4 \) supersymmetric Yang-Mills theories, duality relates each elementary massive vector meson of electric charge \( \gamma \) to a tower of dyons of magnetic charge \( \gamma^* \) among others, where \( \gamma \) is any root of the gauge algebra. The analysis of the classical solutions tells us, however, that a monopole based on a composite root is not a fundamental entity, but rather corresponds to a mere coincidence point on a multi-monopole moduli space. Thus, it is imperative to see if the quantization of the multi-monopole dynamics leads to a bound state that could conceivably be dual to the vector meson whenever \( \gamma \) is composite.

This program has recently been carried out in the simplest case, that of a maximally broken \( SU(3) \) gauge theory, by the authors [10], and also independently by Gauntlett and Lowe [11]. The most massive of the charged vector mesons in this theory is dual to a threshold bound state with composite magnetic charge, which is realized as a unique normalizable harmonic form on the Taub-
NUT manifold. Since, as we have seen above, the Taub-NUT manifold is the universal relative moduli space for a pair of interacting distinct fundamental monopoles, the same two-monopole bound state immediately carries over to arbitrary gauge groups [10][11].

It remains an open problem to find the bound states of more than two fundamental monopoles. Now that there is a strong candidate for the exact moduli space metric, however, it should be a matter of differential calculus to search for harmonic forms and check if they are suitably normalizable. If this approach were to find the dyon spectrum predicted by duality, it would provide further evidence in support of proposed moduli space metric.

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References

[1] E.B. Bogomol'nyi, Sov. J. Nucl. Phys. 24, 449 (1976); M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975); S. Coleman, S. Parke, A. Neveu and C.M. Sommerfield, Phys. Rev. D15, 544 (1977).

[2] N.S. Manton, Phys. Lett. 110B, 54 (1982).

[3] A. Sen, Phys. Lett. B329, 217 (1994).

[4] S. Sethi, M. Stern and E. Zaslow, Nucl. Phys. B457, 484 (1995); J.P. Gauntlett and J. Harvey, S-Duality and the Dyon Spectrum in N=2 Super Yang-Mills Theory, hep-th/9508156.

[5] M. Porrati, On the existence of states saturating the Bogomol'nyi bound in N=4 supersymmetry, hep-th/9505187.

[6] C. Montonen and D. Olive, Phys. Lett. 72B, 117 (1977); H. Osborn, Phys. Lett. 83B, 321 (1979).

[7] M.F. Atiyah and N.J. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton Univ. Press, Princeton (1988); Phys. Lett. 107A, 21 (1985); Phil. Trans. R. Soc. Lon. A315, 459 (1985).
[8] G.W. Gibbons and N.S. Manton, Phys. Lett. B356, 32 (1995).

[9] E.J. Weinberg, Nucl. Phys. B167, 500 (1980).

[10] K. Lee, E.J. Weinberg and P. Yi, *Electromagnetic duality and SU(3) monopoles*, CU-TP-734, hep-th/9601097.

[11] J.P. Gauntlett and D.A. Lowe, *Dyons and S-duality in N=4 supersymmetric gauge theory*, hep-th/9601085.

[12] S.A. Connell, *The dynamics of the SU(3) charge (1,1) magnetic monopoles*, University of South Australia preprint.

[13] A.H. Guth and E.J. Weinberg, Phys. Rev. D14, 1660 (1976).

[14] P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. B125, 1 (1977); F. Engert and P. Windey, Phys. Rev. D14, 2728 (1976).

[15] C. Athorne, Commun. Math. Phys. 88, 43 (1983).

[16] N.S. Manton, Phys. Lett. 154B, 397 (1985); (E) 157B, 475 (1985).

[17] G.W. Gibbons and N.S. Manton, Nucl. Phys. B274, 183 (1986).

[18] J. Gauntlett, Nucl. Phys. B411, 443 (1994).

[19] G.W. Gibbons and C.N. Pope, Commun. Math. Phys. 66, 267 (1979).

[20] T. Eguchi and A.J. Hanson, Ann. Phys. 120, 82 (1979).

[21] E.J. Weinberg, Nucl. Phys. B203, 445 (1982).

[22] N. Dorey, C. Fraser, T.J. Hollowood and M.A.C. Kneipp, *Non-Abelian duality in N = 4 supersymmetric gauge theories*, SWAT/96, CBPF-NF-001/96, hep-th/9512116.

[23] N. Seiberg and E. Witten, Nucl. Phys. B426, 19 (1994); (E) B430, 485 (1994); ibid. B431, 484 (1994).