Quasi-time reversal invariance of Brownian motion and a cycle symmetry for diffusion processes on the circle

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Abstract

In this paper, we find a new symmetry of Brownian motion. We prove that Brownian motion is invariant under a transformation named as quasi-time reversal. Moreover, using the quasi-time reversal invariance of Brownian motion, we prove an equality which characterizes the symmetry of the forming times of the forward and backward cycles for general diffusion processes on the circle.

Keywords: Bessel processes, path decomposition, Haldane equality

1 Introduction

Let $W$ be a standard one-dimensional Brownian motion and let $b > 0$ be a fixed real number. Let $\beta = \inf \{ t \geq 0 : W_t = b \}$ be the hitting time of $b$ by $W$ and let $\alpha = \sup \{ 0 \leq t < \beta : W_t = 0 \}$ be the last zero of $W$ before $\beta$. Williams [1, 2] proved a striking result that the Brownian motion $W$ taken between $\alpha$ and $\beta$ is a three-dimensional Bessel process. Specifically, let $Y$ be a three-dimensional Bessel process starting from 0 and let $\tau_b = \inf \{ t \geq 0 : Y_t = b \}$ be the hitting time of $b$ by $Y$. Then the processes $\{W_{\alpha+t} : 0 \leq t \leq \beta - \alpha\}$ and $\{Y_t : 0 \leq t \leq \tau_b\}$ have the same distribution. This result is part of the brilliant Williams’ Brownian path decomposition theorem. In addition, some other interesting relations between one-dimensional Brownian motion and the three-dimensional Bessel process were found by Williams [1, 2] and Pitman [3].

In this paper, we do not consider Brownian motion taken between $\alpha$ and $\beta$, but consider that taken between the other two random times. Let $\tau = \inf \{ t \geq 0 : \|W_t\| = b \}$ be the hitting time of $\{-b, b\}$ by $W$ and let $\sigma = \sup \{ 0 \leq t < \tau : W_t = 0 \}$ be the last zero of $W$ before $\tau$. Using a deep result about the time reversal of Markov processes established by Nagasawa [4] and some deep properties of Bessel processes, we prove that the Brownian motion $W$ taken between $\sigma$ and $\tau$ is a “coin-flipping” three-dimensional Bessel process. Specifically, let $Y$ and $Z$ be two independent three-dimensional Bessel processes starting from 0 and let $\xi$ be a random variable independent of $Y$ and $Z$ whose distribution is $P(\xi = b) = P(\xi = -b) = 1/2$. Let $L$ be a process defined by

$$L_t = \begin{cases} Y_t, & \text{if } \xi = b, \\ -Z_t, & \text{if } \xi = -b, \end{cases}$$

(1)
and let $\gamma = \inf\{t \geq 0 : |R_t| = b\}$ be the hitting time of $\{-b, b\}$ by $R$. We prove that the processes $\{W_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ and $\{R_t : 0 \leq t \leq \gamma\}$ have the same distribution. Note that the process $L$ is nothing but a Bessel process $Y$ if we get heads in a coin toss and a minus Bessel process $-Z$ if we get tails in a coin toss. That is why we say that the Brownian motion $W$ taken between $\sigma$ and $\tau$ is a “coin-flipping” three-dimensional Bessel process.

Brownian motion is probably the stochastic process with most symmetries. In classical textbooks, Brownian motion is described to have four major symmetries: reflection invariance, translation invariance, scaling invariance, and time inversion invariance. In this paper, we find a new symmetry of Brownian motion. Using the structure of the Brownian motion $W$ between $\sigma$ and $\tau$ mentioned above, we prove that Brownian motion is invariant under a transformation $\phi$ named as quasi-time reversal. Specifically, let $w$ be a continuous path in $\mathbb{R}$ starting from 0, let $\tau_0(w) = \inf\{t \geq 0 : |w_t| = 1\}$ be the hitting time of $\{-b, b\}$ by $w$, and let $\sigma_0(w) = \sup\{0 \leq t < \tau_0(w) : w_t = 0\}$ the last zero of $w$ before $\tau_0(w)$. Under the quasi-time reversal $\phi$, the continuous path $w$ is taken time reversal between $\sigma_0(w)$ and $\tau_0(w)$ and is continuously spliced when needed (see Definition 4). We prove that the process $\phi(W)$ is also a standard one-dimensional Brownian motion.

The quasi-time reversal invariance of Brownian motion established in this paper has wide applications, both in probability theory and statistical physics. In this paper, we exhibit one of its application in probability theory by proving an interesting symmetry for general diffusion process on the circle. Let $X$ be a one-dimensional diffusion process with diffusion coefficient $a > 0$ and drift coefficient $b$, where $a$ and $b$ are continuous functions with period 1. Since the diffusion and drift coefficients are periodic functions, the process $X$ can be viewed as a diffusion process on the circle. A continuous stochastic process on the circle constantly forms forward (counterclockwise) and backward (clockwise) cycles. Let $T^+$ be the forming time of the forward cycle by $X$, which is defined as the time needed for $X$ to form a forward cycle for the first time, and let $T^-$ be the forming time of the backward cycle by $X$. In this paper, we find that although the distributions of $T^+$ and $T^-$ may be different, their distributions, conditional on the corresponding cycle is formed earlier than its reversed cycle, are the same:

$$P(T^+ \leq u | T^+ < T^-) = P(T^- \leq u | T^- < T^+).$$

This relation, which characterizes the symmetry of the forming times of the forward and backward cycles for general diffusion processes on the circle, is named as the cycle symmetry. Let $T = T^+ \wedge T^-$. Then the cycle symmetry can be rewritten as

$$P(T \leq u | T^+ < T^-) = P(T \leq u | T^- < T^+),$$

which implies that the time needed for general diffusion processes on the circle to form a forward or backward cycle is independent of which one of these two cycles is formed (see Remark 6). The cycle symmetry also suggests that the hitting time of $\{-1, 1\}$ by general diffusion processes with periodic diffusion and drift coefficients is independent of which one of $-1$ and 1 is hit (see
Remark 7). For Brownian motion, this independence is obvious. In this paper, we generalize this independence to general diffusions with periodic diffusion and drift coefficients.

We remark that such kind of cycle symmetry not only holds for diffusion processes on the circle, but also holds for other stochastic processes. Similar cycle symmetries, which are named as the generalized Haldane equality, were first found in three-state Markov chains by biophysicists [5–7] and were then proved in general Markov chains by the authors of this paper [8] using the cycle representation theory of Markov chains. The finding of the cycle symmetry for general diffusion processes on the circle is motivated by these works. Finally, we point out that the cycle symmetry established in this paper is closed related to the fluctuation theorems in nonequilibrium statistical physics. However, this topic will not be included in this paper.

2 Some structural properties of Brownian motion

Let \( W = \{W_t : t \geq 0\} \) be a standard one-dimensional Brownian motion defined on some filtered space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) satisfying the usual conditions. In this section, we fix a constant \( b > 0 \) and define two random times \( \tau \) and \( \sigma \) as follows. Let

\[
\tau = \inf\{t \geq 0 : |W_t| = b\}
\]

be the hitting time of \( \{-b, b\} \) by \( W \) and let

\[
\sigma = \sup\{0 \leq t < \tau : W_t = 0\}
\]

be the last zero of \( W \) before \( \tau \).

Let \( x \geq 0 \). Recall that a process \( Y = \{Y_t : t \geq 0\} \) is called a three-dimensional Bessel process starting from \( x \) if \( Y \) has the same distribution with \( \{|B_t| : t \geq 0\} \), where \( B_t \) is a three-dimensional Brownian motion starting from \( (x, 0, 0) \).

The next theorem, which is the main result of this section, gives the structure of the Brownian motion \( W \) between \( \sigma \) and \( \tau \).

**Theorem 1.** Let \( Y = \{Y_t : t \geq 0\} \) and \( Z = \{Z_t : t \geq 0\} \) be two independent three-dimensional Bessel processes starting from 0. Let \( \xi \) be a random variable independent of \( Y \) and \( Z \) whose distribution is \( P(\xi = b) = P(\xi = -b) = 1/2 \). Let \( L = \{L_t : t \geq 0\} \) be a process defined as

\[
L_t = \begin{cases} 
Y_t, & \text{if } \xi = b, \\
-Z_t, & \text{if } \xi = -b.
\end{cases}
\]

Let \( \gamma = \inf\{t \geq 0 : |L_t| = b\} \). Then the processes \( \{W_{\sigma+t} : 0 \leq t \leq \tau-\sigma\} \) and \( \{L_t : 0 \leq t \leq \gamma\} \) have the same distribution.

**Remark 1.** Intuitively, the process \( L \) can be constructed as follows. We first toss a coin. If we get heads, then the process \( L \) takes the value of \( Y \) and if we get tails, then the process \( L \) takes the value of \( -Z \). This shows that the Brownian motion \( W \) taken between \( \sigma \) and \( \tau \) is nothing but a “coin-flipping” three-dimensional Bessel process.
In order to prove the above theorem, we need some results about the time reversal of Markov processes. For further references, we introduce some notations. Let $E$ be a topological space. Let $C(E)$ denote the space of continuous functions on $E$. Let $C_c(E)$ denote the space of continuous functions on $E$ with compact support. Let $C_0(E)$ denote the space of continuous functions on $E$ which vanish at infinity.

**Definition 1.** Let $E$ be a locally compact Hausdorff space with countable base. Let $X = \{X_t : t \geq 0\}$ be a Markov process with state space $E$. Let $\zeta = \inf\{t \geq 0 : X_t = \partial\}$ be the lifetime of $X$. Assume that $\zeta$ is finite and positive and assume that $X$ has continuous paths on $[0, \zeta)$. Then the time-reversed process of $X$ with respect to its lifetime $\zeta$ is a process $\tilde{X} = \{\tilde{X}_t : t > 0\}$ defined as

$$
\tilde{X}_t = \begin{cases} 
X_{\zeta-t}, & \text{if } 0 < t < \zeta, \\
\partial, & \text{if } t \geq \zeta.
\end{cases}
$$

(7)

If $X_{\zeta-}$ exists almost surely, then we define $\tilde{X}_0 = X_{\zeta-}$.

The following lemma, which is due to Nagasawa [4], will play a key role in the proof of Theorem 1. Let $P_t(x, dy)$ be the transition function of the Markov process $X$. Recall that the potential kernel $U(x, dy)$ of $X$ is defined as

$$
U(x, dy) = \int_0^\infty P_t(x, dy) dt.
$$

(8)

**Lemma 1.** The conditions are the same as in Definition 1. Let $P_t(x, dy)$ be the transition function of $X$ and let $U(x, dy)$ be the potential kernel of $X$. Let $\tilde{X} = \{\tilde{X}_t : t > 0\}$ be the time-reversed process of $X$ with respect to its lifetime $\zeta$. Assume that there exist a probability measure $\mu$ on $E$ and another transition function $Q_t(x, dy)$ on $E$, such that

(i) the potential $\nu(dy) = \int_E U(x, dy) \mu(dx)$ is a Radon measure on $E$;

(ii) for any $f \in C_c(E)$ and $x \in E$, the map $t \mapsto Q_t f(x)$ is right-continuous on $(0, \infty)$;

(iii) for any nonnegative Borel functions $f$ and $g$ on $E$,

$$
\int_E P_t f(x) g(x) \nu(dx) = \int_E f(x) Q_t g(x) \nu(dx).
$$

(9)

Then under $P_\mu$, $\tilde{X}$ is a strong Markov process with respect to some filtration whose transition function is $Q_t(x, dy)$.

**Proof.** The proof of this lemma can be found in Revuz and Yor [9].

The above theorem does not give the explicit expression of the transition function $Q_t(x, dy)$. For practical use, we need the following theorem. Recall that if the Markov process $X$ has a transition density $p(t, x, y)$ with respect to some $\sigma$-finite measure $m$, then the Green’s function of $X$ with respect to $m$ is defined as

$$
u(x, y) = \int_0^\infty p(t, x, y) dt.
$$

(10)
Theorem 2. The conditions are the same as in Lemma 1. Assume that $X$ has a transition density $p(t, x, y)$ with respect to a $\sigma$-finite measure $m$. Let $u(x, y)$ be the Green’s function of $X$ with respect to $m$. Assume that there exists $x_0 \in E$, such that

(i) $u(x_0, x)$ is locally integrable in $x$;
(ii) $u(x_0, x) > 0$ for each $x$;
(iii) the map $(t, x) \mapsto p(t, x, y)$ is continuous on $(0, \infty) \times E$.

Then under $P_{x_0}$, $\tilde{X}$ is a strong Markov process with respect to some filtration whose transition function is

$$Q_t(x, dy) = \frac{p(t, y, x)u(x_0, y)}{u(x_0, x)}m(dy). \quad (11)$$

Proof. We first prove that $Q_t(x, dy)$ is a transition function. To this end, we must prove that $Q_t(x, E) \leq 1$ for each $x \in E$ and $Q_t(x, dy)$ satisfies the Chapman-Kolmogorov equation. Since $p(t, x, y)$ is a transition density with respect to $m$, for each $x \in E$,

$$Q_t(x, E) = \int_E p(t, y, x)u(x_0, y) \frac{m(dy)}{u(x_0, x)} = \frac{1}{u(x_0, x)} \int_E p(t, y, x)m(dy) \int_0^\infty p(s, x_0, y)ds$$

$$= \frac{1}{u(x_0, x)} \int_0^\infty p(t + s, x_0, x)ds \leq \frac{1}{u(x_0, x)} \int_0^\infty p(s, x_0, x)ds = 1. \quad (12)$$

Let

$$q(t, x, y) = \frac{p(t, y, x)u(x_0, y)}{u(x_0, x)}. \quad (13)$$

In order to prove that $Q_t(x, dy)$ satisfies the Chapman-Kolmogorov equation, we only need to prove that $q(t, x, y)$ is a transition density with respect to $m$. For any $s, t \geq 0$ and $x, y \in E$,

$$\int_E q(t, x, z)q(s, z, y)m(dz) = \int_E p(t, z, x)u(x_0, z) \frac{p(s, y, z)u(x_0, y)}{u(x_0, z)}m(dz) \frac{m(dy)}{u(x_0, x)}$$

$$= \int_E p(t + s, y, x)u(x_0, y)u(x_0, x) = q(t + s, x, y). \quad (14)$$

This shows that $Q_t(x, dy)$ is a transition function.

In order to prove this theorem, we only need to check that the conditions (i), (ii), and (iii) of Lemma 1 are satisfied. If we take $\mu$ to be the unit mass at $x_0$, then the potential $\nu(dy) = \int_E U(x, dy)\mu(dx) = U(x_0, dy)$ is a Radon measure since $u(x_0, y)$ is locally integrable in $y$. For any $f \in C_c(E)$ and $x \in E$,

$$Q_t f(x) = \frac{1}{u(x_0, x)} \int_E f(y)p(t, y, x)U(x_0, dy). \quad (15)$$

Note that $p(t, y, x)$ is continuous in $t$ and $y$ and $U(x_0, dy)$ is a Radon measure. Using the dominated convergence theorem, we see that $Q_t f(x)$ is continuous in $t$. For any nonnegative Borel functions $f$ and $g$ on $E$,

$$\int_E P_t f(y)g(y)U(x_0, dy) = \int_{E \times E} f(x)g(y)p(t, y, x)u(x_0, y)m(dx)m(dy)$$

$$= \int_{E \times E} f(x)g(y)u(x_0, x)m(dx)Q_t(x, dy) = \int_E f(x)Q_t g(x)U(x_0, dx). \quad (16)$$
This completes the proof of this theorem.

Remark 2. Under the conditions in Lemma 1 and Theorem 2, we can only prove that the time-reversed process \( \{\tilde{H}_t : t > 0\} \) is a Markov process and we do not know whether \( \{\tilde{H}_t : t \geq 0\} \) is a time-homogeneous Markov process even if \( \tilde{H}_0 \) is well-defined.

Lemma 2. Let \( H = \{H_t : t \geq 0\} \) be a process defined as

\[
H_t = \begin{cases} \ \ W_t, & \text{if } t < \tau, \\ \ \ \partial, & \text{if } t \geq \tau. \end{cases}
\]

(17)

Then \( H \) is a Feller process whose potential kernel is

\[
U_H(x, dy) = \frac{1}{b} (b - x \lor y)(b + x \land y)I_{\{\partial < y < b\}} dy.
\]

(18)

Proof of Lemma 2. Since \( H \) is a Brownian motion killed outside \((-b, b)\), it is a Feller process.

For any bounded continuous function \( f \) on \((-b, b)\),

\[
U_H f(x) = E_x \int_0^\infty f(H_t) dt = E_x \int_0^\tau f(W_t) dt.
\]

(19)

Let

\[
g(x) = \int_{-b}^x dy \int_{-b}^y f(z) dz.
\]

(20)

Then \( g'' = f \). By Ito’s formula,

\[
M_t = g(W_t) - g(W_0) - \frac{1}{2} \int_0^t f(W_s) ds
\]

(21)

is a local martingale. It is easy to see that

\[
M_{t\wedge \tau} = g(W_{t\wedge \tau}) - g(W_0) - \frac{1}{2} \int_0^{t\wedge \tau} f(W_s) ds
\]

(22)

is a martingale. This suggests that

\[
E_x \int_0^\tau f(W_s) ds = 2(E_x g(W_\tau) - g(x))
\]

\[
= 2g(b)P_x(W_\tau = b) + 2g(-b)P_x(W_\tau = -1) - g(x)
\]

\[
= (g(b) - g(x)) \frac{b + x}{b} + (g(-b) - g(x)) \frac{b - x}{b}
\]

\[
= \frac{b + x}{b} \int_x^b dy \int_{-b}^y f(z) dz - \frac{b - x}{b} \int_{-b}^x dy \int_{-b}^y f(z) dz.
\]

(23)

Tedious but elementary calculations show that

\[
U_H f(x) = E_x \int_0^\tau f(W_s) ds = \frac{1}{b} \int_{-b}^b f(y)(b - x \lor y)(b + x \land y) dy.
\]

(24)

This completes the proof of this lemma.
Lemma 3. Let \( \tilde{H} = \{ \tilde{H}_t : t \geq 0 \} \) be a process defined as

\[
\tilde{H}_t = \begin{cases} W_{\tau-t}, & \text{if } 0 \leq t < \tau, \\ \partial, & \text{if } t \geq \tau. \end{cases}
\] (25)

Then \( \{ \tilde{H}_t : t > 0 \} \) is a strong Markov process with respect to some filtration whose potential kernel is

\[
U_{\tilde{H}}(x, dy) = \frac{(b - |y|(b - x \lor y))(b + x \land y)}{b(b - |x|)} I_{\{ -b < y < b \}} dy.
\] (26)

Proof. Let \( H \) be the process defined in Lemma 2. Then \( \tilde{H} \) is just the time-reversal process of \( H \) with respect to its lifetime \( \tau \). Since \( H \) is a Brownian motion killed outside \((-b, b)\), it has a transition density \( p_H(t, x, y) \) with respect to the Lebesgue measure. Moreover, it follows from a theorem of Hunt that the map \((t, x, y) \mapsto p_H(t, x, y)\) is continuous on \((0, \infty) \times (-b, b) \times (-b, b)\).

By Lemma 2, the Green’s function of \( H \) with respect to the Lebesgue measure is

\[
u_H(x, y) = \frac{1}{b}(b - x \lor y)(b + x \land y).
\] (27)

This shows that

\[
u_H(0, y) = (b - |y|).
\] (28)

It is easily to see that \( U_H(0, dy) = u_H(0, y)dy \) is a Radon measure on \((-b, b)\) and \( u_H(0, y) > 0 \) for each \( y \in (-b, b) \). Thus all conditions in Theorem 2 are satisfied. By Theorem 2, \( \tilde{H} \) is a strong Markov process with respect to some filtration whose potential kernel is

\[
U_{\tilde{H}}(x, dy) = \int_0^\infty \frac{p_H(t, y, x)u_H(0, y)}{u_H(0, x)} dy dt = \frac{u_H(y, x)u_H(0, y)}{u_H(0, x)} dy
\] (29)

This completes the proof of this lemma. \(\square\)

Remark 3. The proof of the above theorem is based on Theorem 2. According to Theorem 2 we can only prove that \( \{ \tilde{H}_t : t > 0 \} \) is a Markov process and we do not know whether \( \{ \tilde{H}_t : t \geq 0 \} \) is a time-homogeneous Markov process when \( t = 0 \) is added.

Lemma 4. Let \( S = \{ S_t : t \geq 0 \} \) be a process defined as

\[
S_t = \begin{cases} W_{\tau-t}, & \text{if } 0 \leq t < \tau - \sigma, \\ \partial, & \text{if } t \geq \tau - \sigma. \end{cases}
\] (30)

Then \( \{ S_t : t > 0 \} \) is a Markov process whose potential kernel is

\[
U_S(x, dy) = 2(b - |y|)^2 \left( \frac{1}{b - |x|} \land \frac{1}{b - |y|} - \frac{1}{b} \right) I_{\{ x, y > 0, -b < y < b \}} dy.
\] (31)
Proof. Let \( \tilde{H} \) be the process defined in Lemma\[3\] It is easy to see that \( \tau - \sigma = \inf \{ t \geq 0 : \tilde{H}_t = 0 \} \). Thus \( S \) is just the Markov process \( \tilde{H} \) killed at 0. This shows that \( S \) is a Markov process. For any any bounded measurable function \( f \) on \((-b, b)\),

\[
U_S f(x) = E_x \int_0^\infty f(S_t)dt = E_x \int_0^{\tau - \sigma} f(\tilde{H}_t)dt
\]

\[
= E_x \int_0^\infty f(\tilde{H}_t)dt - E_x \int_0^{\sigma} f(\tilde{H}_t)dt
\]

\[
= E_x \int_0^\infty f(\tilde{H}_t)dt - E_x \int_0^{\tau - \sigma + t} f(\tilde{H}_t)dt. \tag{32}
\]

Note that \( \tilde{H}_{\tau - \sigma} = W_\sigma = 0 \). By the strong Markov property, \( \tilde{H}_{\tau - \sigma + t} \) is a Markov process starting from 0 which has the same transition function as \( \tilde{H} \). This shows that

\[
U_S f(x) = E_x \int_0^\infty f(\tilde{H}_t)dt - E_x \int_0^{\tau - \sigma} f(\tilde{H}_t)dt = U_{\tilde{H}} f(x) - U_{\tilde{H}} f(0). \tag{33}
\]

By Lemma\[3\]

\[
U_{\tilde{H}} f(x, dy) = \frac{(b - |y|)(b - x \vee y)(b + x \wedge y)}{b(b - |x|)} I_{(-b < y < b)} dy. \tag{34}
\]

Tedious but elementary calculations show that

\[
U_S f(x) = 2 \int_{-b}^b f(y)(b - |y|)^2 \left( \frac{1}{b - |x|} \wedge \frac{1}{b - |y|} - \frac{1}{b} \right) I_{(xy > 0)} dy. \tag{35}
\]

This completes the proof of this lemma. \( \square \)

We are now in a position to determine the distribution of the process \( S \). To this end, we need some deep results about three-dimensional Bessel processes. For clarity, we make the following definition.

**Definition 2.** Let \( Y = \{ Y_t : t \geq 0 \} \) be a three-dimensional Bessel process starting from \( x < b \) and let \( \tau_b = \inf \{ t \geq 0 : Y_t = b \} \). Let \( Y^b = \{ Y^b_t : t \geq 0 \} \) be a process defined as

\[
Y^b_t = \begin{cases} 
Y_t, & \text{if } 0 \leq t < \tau_b, \\
\partial, & \text{if } t \geq \tau_b.
\end{cases} \tag{36}
\]

Then \( Y^b \) is called a three-dimensional Bessel process starting from \( x \) and killed at \( b \).

**Lemma 5.** Let \( Y^b = \{ Y^b_t : t \geq 0 \} \) be a three-dimensional Bessel processes starting from \( x < b \) and killed at \( b \). Then \( Y^b \) is a Feller process.

**Proof.** Let \( B \) be a three-dimensional Brownian motion starting from \((x, 0, 0)\). Let \( B^b \) be the Brwonian motion \( B \) killed outside the ball \( B(0, b) = \{ x \in \mathbb{R}^3 : |x| < b \} \). It is easy to see that \(|B^b|\) and \( Y^b \) have the same distribution. Let \( P_t(x, dy) \) be the transition function of \( B^b \) and let \( Q_t(x, dy) \) be the transition function of \( Y^b \). In order to prove that \( Y^b \) be a Feller process, we only need to prove that \( Q_t \) maps \( C_0[0, b] \) into \( C_0[0, b] \). For any \( f \in C_0[0, b] \),

\[
Q_t f(x) = E_x f(Y^b_t) = E_{(x,0,0)} f(|B^b_t|) = P_t g(x, 0, 0), \tag{37}
\]

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where \( g \) is a function on \( B(0, b) \) defined as \( g(x) = f(|x|) \). Since \( f \in C_0[0, b) \), it is easy to see that \( g \in C_0(B(0, b)) \). Since \( B^b \) is the Brwonian motion \( B \) killed outside \( B(0, b) \), it is a Feller process, which shows that \( P_t g \in C_0(B(0, b)) \). This further implies that \( Q_t f \in C_0[0, b) \), which shows that \( Y^b \) is a Feller process.

**Lemma 6.** Let \( Y^b = \{Y^b_t : t \geq 0\} \) be a three-dimensional Bessel processes starting from 0 and killed at \( b \). Let \( J_t = b - Y^b_t \). Then \( J = \{J_t : t \geq 0\} \) is a Feller process whose potential kernel is

\[
U_J(x, dy) = 2(b - y)^2 \left( \frac{1}{b-x} \wedge \frac{1}{b-y} - \frac{1}{b} \right) I_{\{0 < y \leq b\}} dy.
\]

(*Proof.\) By Lemma 5, \( Y^b \) is a Feller process, which implies that \( J \) is a Feller process. Let \( Y \) be a three-dimensional Bessel process starting from 0. Let \( \tau_b = \inf\{t \geq 0 : Y_t = b\} \). For any bounded measurable function \( f \) on \([0, b)\), using the strong Markov property, we obtain that

\[
U_J f(x) = E_x \int_0^\infty f(J_t) dt = E_{b-x} \int_0^\infty f(b - Y^b_t) dt = E_{b-x} \int_0^{\tau_b} f(b - Y_t) dt \\
= E_{b-x} \int_0^\infty f(b - Y_t) dt - E_{b-x} \int_0^{\tau_b} f(b - Y_t) dt \\
= E_{b-x} \int_0^\infty f(b - Y_t) dt - E_{b-x} \int_0^{\tau_{b+t}} f(b - Y_{b+t}) dt \\
= E_{b-x} \int_0^\infty f(b - Y_t) dt - E_b \int_0^{\infty} f(b - Y_t) dt \\
= U_Y g(b-x) - U_Y g(b),
\]

where \( g \) is a function on \([0, b)\) defined as \( g(x) = f(b - x) \). Recall that the potential kernel of \( Y \) is

\[
U_Y(x, dy) = 2 \left( \frac{1}{x} \wedge \frac{1}{y} \right) y^2 I_{\{y \geq 0\}} dy.
\]

Tedious but elementary calculations show that

\[
U_J f(x) = 2 \int_0^b f(y)(b - y)^2 \left( \frac{1}{b-x} \wedge \frac{1}{b-y} - \frac{1}{b} \right) y^2 d\mu_y.
\]

This completes the proof of this lemma.\)

**Lemma 7.** Let \( Y^b = \{Y^b_t : t \geq 0\} \) and \( Z^b = \{Z^b_t : t \geq 0\} \) be two independent three-dimensional Bessel processes starting from 0 and killed at \( b \). Let \( \xi \) be a random variable independent of \( Y^b \) and \( Z^b \) whose distribution is \( P(\xi = b) = P(\xi = -b) = 1/2 \). Let \( \tilde{S} = \{\tilde{S}_t : t \geq 0\} \) be a process defined as

\[
\tilde{S}_t = \begin{cases} 
  b - Y^b_t, & \text{if } \xi = b, \\
  Z^b_t - b, & \text{if } \xi = -b.
\end{cases}
\]

Then \( \tilde{S} \) is a Feller process whose potential kernel is

\[
U_{\tilde{S}}(x, dy) = 2(b - |y|)^2 \left( \frac{1}{b - |x|} \wedge \frac{1}{b - |y|} - \frac{1}{b} \right) I_{\{|xy| = b, y < b\}} dy.
\]
Proof. Let \( E = [-b, 0) \cup (0, b] \) be the state space of \( \tilde{S} \). Let \( \mathcal{F}_t \) denote the \( \sigma \)-algebra
\[
\mathcal{F}_t = \sigma\{\xi, Y^b_s, Z^b_s : 0 \leq s \leq t\}.
\] (44)

We first prove that \( \tilde{S} \) is a Markov process with respect to the filtration \( \{\mathcal{F}_t\} \). Let \( J_t = b - Y^b_t \) and let \( K_t = b - Z^b_t \). Let \( P_t(x, dy) \) be the common transition function of \( J \) and \( K \). For any bounded measurable function \( f \) on \( E \),
\[
E[f(\tilde{S}_{s+t})|\mathcal{F}_s] = E[f(J_{s+t})I_{\xi=b} + f(-K_{s+t})I_{\xi=-b}|\mathcal{F}_s]
= E[f(J_{s+t})I_{\xi=b}] + E[f(-K_{s+t})|\mathcal{F}_s]I_{\xi=-b}
= P_t f(J_s)I_{\xi=b} + P_t g(K_s)I_{\xi=-b}
= P_t f(\tilde{S}_s)I_{\xi=b} + P_t g(-\tilde{S}_s)I_{\xi=-b}
= P_t f(\tilde{S}_s)I_{\xi=b, \tilde{S}_s \neq 0} + P_t g(-\tilde{S}_s)I_{\xi=\tilde{S}_s \neq 0}
\] (45)

where \( g \) is a function on \( E \) defined as \( g(x) = f(-x) \). This shows that \( \tilde{S} \) is a Markov process with respect to the filtration \( \{\mathcal{F}_t\} \). Let \( Q_t(x, dy) \) be the transition function of \( \tilde{S} \). Then
\[
Q_t f(x) = P_t f(x)I_{\xi=b} + P_t g(-x)I_{\xi<-b}.
\] (46)

We next prove that \( \tilde{S} \) is a Feller process. To this end, we only need to prove that \( Q_t \) maps \( C_0(E) \) into \( C_0(E) \). For any \( f \in C_0(E) \), we have \( g \in C_0(E) \). It then follows that \( f|_{(0,b]} \in C_0(0,b] \) and \( g|_{(0,b]} \in C_0(0,b] \). By Lemma \( 6 \), \( J \) is a Feller process, which shows that \( P_t f|_{(0,b]} \in C_0(0,b] \) and \( P_t g|_{(0,b]} \in C_0(0,b] \). This fact, together with (46), implies that \( Q_t f \in C_0(E) \). Thus \( \tilde{S} \) is a Feller process.

We finally calculate the potential kernel of \( \tilde{S} \). Let \( U_J(x, dy) \) be the potential kernel of \( J \) and let \( U_S(x, dy) \) be the potential kernel of \( \tilde{S} \). It follows from (46) that
\[
U_{\tilde{S}} f(x) = U_J f(x)I_{\xi>b} + U_J g(-x)I_{\xi<b}.
\] (47)

By Lemma \( 6 \)
\[
U_J(x, dy) = 2(b - y)^2 \left( \frac{1}{b - x} - \frac{1}{b} \right) dy.
\] (48)

Tedious but elementary calculations show that
\[
U_{\tilde{S}} f(x) = 2 \int_{-b}^{b} f(y)(b - |y|)^2 \left( \frac{1}{b - |x|} - \frac{1}{b} \right) I_{|xy|>0} dy.
\] (49)

This completes the result of this lemma. \( \Box \)

Lemma 8. Let \( X \) be a Markov process with state space \( E \). Let \( B(E) \) be the space of bounded measurable function on \( E \). Let \( U(x, dy) \) be the potential kernel of \( X \) and let \( \{R_\lambda : \lambda > 0\} \) be the resolvent of \( X \). Assume that \( U \) is a bounded linear operator on \( B(E) \). Then when \( \lambda \) is sufficiently small, we have
\[
R_\lambda = \sum_{n=0}^{\infty} (-\lambda)^n U^{n+1}.
\] (50)
Proof. For any \( f \in B(E) \) and \( x \in E \), by the dominated convergence theorem, we have
\[
\lim_{\mu \to 0} \int_0^\infty e^{-\mu t} P_t f(x) dt \to \int_0^\infty P_t f(x) dt.
\]
This shows that
\[
\lim_{\mu \to 0} R_\mu f(x) dt \to U f(x).
\]
Letting \( \mu \to 0 \) in the resolvent equation
\[
R_\lambda - R_\mu + (\lambda - \mu) R_\mu R_\lambda = 0,
\]
we obtain that
\[
(I + \lambda U) R_\lambda = U.
\]
Since \( U \) is a bounded linear operator on \( B(E) \), it follows that \( I + \lambda U \) is invertible in \( B(E) \) when \( \lambda \) is sufficiently small. This shows that \( R_\lambda = (I + \lambda U)^{-1} U = \sum_{n=0}^{\infty} (-\lambda)^n U^{n+1} \),
which gives the desired result.

The following theorem gives the distribution of the process \( S \).

**Theorem 3.** Let \( S = \{ S_t : t \geq 0 \} \) be the process defined in Lemma 4 and let \( \bar{S} = \{ \bar{S}_t : t \geq 0 \} \) be the process defined in Lemma 7. Then the processes \( S \) and \( \bar{S} \) have the same distribution.

**Proof.** By Lemma 4 \( \{ S_t : t > 0 \} \) is a Markov process. We shall now prove that \( S = \{ S_t : t \geq 0 \} \) is a Markov process when \( t = 0 \) is included. To distinguish the difference between \( t > 0 \) and \( t \geq 0 \), we denote the process \( \{ S_t : t > 0 \} \) by \( S^* \).

By Lemma 4 and Lemma 7 we see that \( S^* \) and \( \bar{S} \) are Markov processes with the same potential kernel. Let \( E = [-b, 0) \cup (0, b] \) and let \( U(x, dy) \) be the common potential kernel of \( S^* \) and \( \bar{S} \). It is easy to see that \( U \) is a bounded linear operator on \( B(E) \). Let \( \{ R_\lambda : \lambda > 0 \} \) be the resolvent of \( S \). By Lemma 8 when \( \lambda \) is sufficiently small, we have
\[
R_\lambda = (I + \lambda U)^{-1} U = \sum_{n=0}^{\infty} (-\lambda)^n U^{n+1}.
\]
This shows that \( S^* \) and \( \bar{S} \) have the same resolvent when \( \lambda \) is small. The resolvent equation
\[
R_\lambda - R_\mu + (\lambda - \mu) R_\mu R_\lambda = 0,
\]
进一步 implies that \( R_\lambda \) for \( \lambda \) sufficiently small determines \( R_\lambda \) for each \( \lambda \). This shows that \( S^* \) and \( \bar{S} \) have the same resolvent. By the uniqueness of the Laplace transformation, we see that \( S^* \) and \( \bar{S} \) have the same transition function, denoted by \( P_t(x, dy) \). By Lemma 7 \( \bar{S} \) is a Feller process, which shows \( \{ P_t : t \geq 0 \} \) is a Feller semigroup. For any \( f \in C_0(E) \), by the Markov property, we have
\[
E\{ f(S_{s+t}) | \mathcal{F}_s \} = P_t f(S_s).
\]
It then follows from the dominated convergence theorem and $P_t f \in C_0(E)$ that

$$E\{f(S_t) | \mathcal{F}_{0+}\} = P_t f(S_0). \quad (59)$$

This shows that $S$ is a Markov process with transition function $P_t(x, dy)$ when $t = 0$ is included.

Finally, note that the initial distributions of $S$ and $\tilde{S}$ are both the distribution of $\xi$ and $S$ and $\tilde{S}$ have the same transition function. It follows that $S$ and $\tilde{S}$ have the same distribution. \qed

We still need a result about three-dimensional Bessel processes.

**Lemma 9.** Let $Y^b = \{Y^b_t : t \geq 0\}$ be a three-dimensional Bessel process starting from 0 and killed at $b$. Let $\tilde{Y}^b = \{\tilde{Y}^b_t : t \geq 0\}$ be the time-reversed process of $Y^b$ with respect to its lifetime. Then the processes $Y^b$ and $b - \tilde{Y}^b$ have the same distribution.

**Proof.** The proof of this lemma can be found in Revuz and Yor \cite{RevuzYor} \qed

**Theorem 4.** Let $R = \{R_t : t \geq 0\}$ be a process defined as

$$R_t = \begin{cases} W_{\sigma+t}, & \text{if } 0 \leq t < \tau - \sigma \\ \partial, & \text{if } t \geq \tau - \sigma. \end{cases} \quad (60)$$

Let $Y^b$ and $Z^b$ be two independent three-dimensional Bessel processes starting from 0 and killed at $b$. Let $\xi$ be a random variable independent of $Y^b$ and $Z^b$ whose distribution is $P(\xi = b) = P(\xi = -b) = 1/2$. Let $R = \{R_t : t \geq 0\}$ be a process defined as

$$R_t = \begin{cases} Y^b_t, & \text{if } \xi = b \\ -Z^b_t, & \text{if } \xi = -b. \end{cases} \quad (61)$$

Then the processes $R$ and $\tilde{R}$ have the same distribution.

**Proof.** Let $S$ be the process defined in Lemma \cite{lemma} and let $\tilde{S}$ be the process defined in Lemma \cite{lemma}. Then $R$ is just the time-reversed process of $S$ with respect to its lifetime $\tau - \sigma$. Note that the time-reversal process of $\tilde{S}$ with respect to its lifetime is $(b - \tilde{Y}^b)I_{\{\xi = b\}} + (\tilde{Z}^b - b)I_{\{\xi = -b\}}$, where $\tilde{Y}^b$ and $\tilde{Z}^b$ are the time-reversed processes of $Y^b$ and $Z^b$ with respect to their lifetimes, respectively. By Lemma \cite{lemma} $(b - \tilde{Y}^b)I_{\{\xi = b\}} + (\tilde{Z}^b - b)I_{\{\xi = -b\}}$ has the same distribution as $\tilde{R}$. This shows that $\tilde{R}$ and the time-reversed process of $\tilde{S}$ have the same distribution. By Theorem \cite{theorem} $S$ and $\tilde{S}$ have the same distribution. This suggests that $R$ and $\tilde{R}$ have the same distribution. \qed

We are now in a position to to prove Theorem \cite{theorem}.

**Proof of Theorem \cite{theorem}** Note that the process $\{W_{\sigma+t} : 0 \leq t < \tau - \sigma\}$ is just the process $R$ taken before its lifetime and the process $\{L_t : 0 \leq t < \gamma\}$ is just the process $\tilde{R}$ taken before its lifetime. By Theorem \cite{theorem} $R$ and $\tilde{R}$ have the same distribution. This shows that the processes $\{W_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ and $\{L_t : 0 \leq t \leq \gamma\}$ have the same distribution. \qed
3 Quasi-time reversal invariance of Brownian motion

Let $\mathbb{W}$ denote the canonical continuous path space

$$\mathbb{W} = \{ w : [0, \infty) \to \mathbb{R} : w \text{ is continuous and } w(0) = 0 \}. \quad (62)$$

Let $w$ be a continuous path in $\mathbb{W}$. Let

$$\tau_0(w) = \inf \{ t \geq 0 : |w_t| = 1 \} \quad (63)$$

be the hitting time of $\{-b, b\}$ by $w$ and let

$$\sigma_0(w) = \sup \{ 0 \leq t < \tau_0(w) : w_t = 0 \} \quad (64)$$

be the last zero of $w$ before $\tau_0(w)$.

Note that $\tau_0(w)$ and $\sigma_0(w)$ defined above are closed related to the random times $\tau$ and $\sigma$ defined in (4) and (5). If we take $b = 1$ in (4) and (5), then it is easy to see that $\tau = \tau_0(W)$ and $\sigma = \sigma_0(W)$.

Remark 4. In the following discussion, we shall denote $\tau_0(w)$ by $\tau(w)$ and denote $\sigma_0(w)$ by $\sigma(w)$ unless this convention will cause confusion.

We next define two transformations on the canonical continuous path space $\mathbb{W}$.

**Definition 3.** Let $\psi$ be a transformation on $\mathbb{W}$ defined by

$$\psi(w)_t = \begin{cases} w_t, & \text{if } 0 \leq t < \sigma(w), \\ -w_t, & \text{if } \sigma(w) \leq t \leq \tau(w), \\ w_t - 2, & \text{if } w_\tau = 1, t > \tau(w), \\ w_t + 2, & \text{if } w_\tau = -1, t > \tau(w). \end{cases} \quad (65)$$

The transformation $\psi$ is called the quasi-reflection.

**Definition 4.** Let $\phi$ be a transformation on $\mathbb{W}$ defined by

$$\phi(w)_t = \begin{cases} w_t, & \text{if } 0 \leq t < \sigma(w), \\ w_{\sigma + \tau - t} - 1, & \text{if } w_\tau = 1, \sigma(w) \leq t \leq \tau(w), \\ w_{\sigma + \tau - t} + 1, & \text{if } w_\tau = -1, \sigma(w) \leq t \leq \tau(w), \\ w_t - 2, & \text{if } w_\tau = 1, t > \tau(w), \\ w_t + 2, & \text{if } w_\tau = -1, t > \tau(w). \end{cases} \quad (66)$$

The transformation $\psi$ is called the quasi-time reversal.

The expressions of these two transformations are somewhat complicated at the first sight. However, the intuitive meanings of these two transformations are extremely clear. Under quasi-reflection, a continuous path $w$ is reflected between $\sigma(w)$ and $\tau(w)$ and is continuously spliced
Then the map following discussion, we do not distinguish a continuous path in standard Brownian motion quasi-reflection can be viewed as a transformation canonical path space in formation to form a cycle for the first time and the processes independent. We make a crucial observation that the above two processes are the time reversal of processes \( \bar{w} \). Let \( \tilde{S} \) denote the unit circle \( \{ z \in \mathbb{C} : |z| = 1 \} \) in the complex plane. Let \( \mathbb{W}(S^1) \) denote the canonical path space in \( S^1 \) defined as

\[
\mathbb{W} = \{ w : [0, \infty) \to S^1 : w \text{ is continuous and } w(0) = 1 \}.
\] (67)

Then the map \( w \mapsto e^{2\pi iw} \) gives a one-to-one correspondence between \( \mathbb{W} \) and \( \mathbb{W}(S^1) \). In the following discussion, we do not distinguish a continuous path \( w \) in \( \mathbb{W} \) and its counterpart \( e^{2\pi iw} \) in \( \mathbb{W}(S^1) \). As a result, the standard Brownian motion \( W \) on the real line is equated with the standard Brownian motion \( e^{2\pi i W} \) on the circle.

Let \( w \) be a continuous path in \( \mathbb{W}(S^1) \). Under the above convention, \( \tau(w) \) is time need for \( w \) to form a cycle for the first time and \( \sigma(w) \) is last exit time of 1 by \( w \) before \( \tau(w) \). Moreover, the quasi-reflection can be viewed as a transformation \( \psi \) on \( \mathbb{W}(S^1) \) defined as

\[
\psi(w)_t = \begin{cases} 
    w_t, & \text{if } 0 \leq t < \sigma(w), \\
    \bar{w}_t, & \text{if } \sigma(w) \leq t \leq \tau(w), \\
    w_t, & \text{if } t > \tau(w), 
\end{cases}
\] (68)

where \( \bar{w}_t \) is the complex conjugate of \( w_t \), and the quasi-time reversal can be viewed as a transformation \( \phi \) on \( \mathbb{W}(S^1) \) defined as

\[
\phi(w)_t = \begin{cases} 
    w_t, & \text{if } 0 \leq t < \sigma(w), \\
    w_{\sigma + \tau - t}, & \text{if } \sigma(w) \leq t \leq \tau(w), \\
    w_t, & \text{if } t > \tau(w). 
\end{cases}
\] (69)

We shall now prove that Brownian motion is invariant under quasi-reflection and quasi-time reversal. In the rest of this section, \( \tau \) and \( \sigma \) are understood to be \( \tau(W) \) and \( \sigma(W) \).

**Lemma 10.** Let \( W \) be a standard one-dimensional Brownian motion. Then the processes \( \{ W_t : 0 \leq t \leq \sigma \} \), \( \{ W_{\sigma+t} : 0 \leq t \leq \tau - \sigma \} \), and \( \{ W_{\tau+t} - W_{\tau} : t \geq 0 \} \) are independent.

**Proof.** By the strong regenerate property of Brownian motion, the third process is independent of the first two processes. Thus we only need to prove that the first two processes are independent. Let \( \tilde{H} \) be the process defined in Lemma 3. By Lemma 3 \( \{ \tilde{H}_t : t > 0 \} \) is a strong Markov process with respect to some filtration. Note that \( \tau - \sigma \) is the hitting time of 0 by \( \tilde{H} \). Using the strong Markov property, we see that the processes \( \{ \tilde{H}_t : 0 \leq t \leq \tau - \sigma \} \) and \( \{ \tilde{H}_{\tau - \sigma + t} : 0 \leq t \leq \sigma \} \) are independent. We make a crucial observation that the above two processes are the time reversal of the processes \( \{ W_{\sigma+t} : 0 \leq t \leq \tau - \sigma \} \) and \( \{ W_t : 0 \leq t \leq \tau \} \), respectively. This shows that the processes \( \{ W_t : 0 \leq t \leq \tau \} \) and \( \{ W_{\sigma+t} : 0 \leq t \leq \tau - \sigma \} \) are independent. \( \square \)
Theorem 5. Let $W$ be a standard one-dimensional Brownian motion. Let $\psi$ be the quasi-reflection on $\mathcal{W}$. Then $\psi(W)$ is also a standard one-dimensional Brownian motion.

Proof. By the definition of the quasi-reflection $\psi$, we see that the processes $\{W_t : 0 \leq t \leq \sigma\}$ and $\{\psi(W)_t : 0 \leq t \leq \sigma\}$ are the same and the processes $\{W_{\tau+t} - W_{\tau} : t \geq 0\}$ and $\{\psi(W)_{\tau+t} - \psi(W)_{\tau} : t \geq 0\}$ are also the same. By Lemma 3 in order to prove that $\psi(W)$ is a Brownian motion, we only need to prove that the processes $\{W_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ and $\{\psi(W)_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ have the same distribution.

By Theorem 1 the processes $\{W_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ and $\{Y_t I_{\xi=1} - Z_t I_{\xi=-1} : 0 \leq t \leq \gamma\}$ have the same distribution. This shows that the processes $\{\psi(W)_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ and $\{-Y_t I_{\xi=1} + Z_t I_{\xi=-1} : 0 \leq t \leq \gamma\}$ have the same distribution. Since $Y$ and $Z$ are independent three-dimensional processes starting from 0 and since $\xi$ is independent of $Y$ and $Z$, it is easy to see that the processes $\{W_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ and $\{\psi(W)_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ have the same distribution.

\[\square\]

Corollary 1. Let $W$ be a standard Brownian motion on $S^1$. Let $\psi$ be the quasi-reflection on $\mathcal{W}(S^1)$. Then $\psi(W)$ is also a standard Brownian motion on $S^1$.

Theorem 6. Let $W$ be a standard one-dimensional Brownian motion. Let $\phi$ be the quasi-time reversal on $\mathcal{W}$. Then $\phi(W)$ is also a standard one-dimensional Brownian motion.

Proof. Similar to the proof of Theorem 3 in order to prove that $\phi(W)$ is a Brownian motion, we only need to prove that the processes $\{W_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ and $\{\phi(W)_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ have the same distribution. By Corollary 1 the processes $\{W_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ and $\{Y_t I_{\xi=1} - Z_t I_{\xi=-1} : 0 \leq t \leq \gamma\}$ have the same distribution. For any $0 \leq t \leq \tau - \sigma$,

$$\phi(W)_{\sigma+t} = \begin{cases} W_{\tau-t} - 1 & \text{if } W_{\tau} = 1, \\ W_{\tau-t} + 1 & \text{if } W_{\tau} = -1. \end{cases}$$

Thus the processes $\{\phi(W)_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ and $\{(Y_{\tau-t} - 1)I_{\xi=1} + (1 - Z_{\tau-t})I_{\xi=-1} : 0 \leq t \leq \gamma\}$ have the same distribution. By Lemma 2 the processes $\{(Y_{\tau-t} - 1)I_{\xi=1} + (1 - Z_{\tau-t})I_{\xi=-1} : 0 \leq t \leq \gamma\}$ and $\{-Y_t I_{\xi=1} + Z_t I_{\xi=-1} : 0 \leq t \leq \gamma\}$ have the same distribution. Since $Y$ and $Z$ are independent three-dimensional processes starting from 0 and since $\xi$ is independent of $Y$ and $Z$, it is easy to see that the processes $\{W_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ and $\{\phi(W)_{\sigma+t} : 0 \leq t \leq \tau - \sigma\}$ have the same distribution.

\[\square\]

Corollary 2. Let $W$ be a standard Brownian motion on $S^1$. Let $\phi$ be the quasi-time reversal on $\mathcal{W}(S^1)$. Then $\phi(W)$ is also a standard Brownian motion on $S^1$.

4 A cycle symmetry for diffusion processes on the circle

The quasi-time reversal invariance of Brownian motion established in this paper has many applications. In this section, we shall use the quasi-time reversal invariance of Brownian motion
to prove an interesting symmetry for general diffusion processes on the circle named as the cycle symmetry.

In this section, we consider a one-dimensional time-homogeneous diffusion process $X$ with diffusion coefficient $a : \mathbb{R} \to (0, \infty)$ and drift coefficient $b : \mathbb{R} \to \mathbb{R}$. We assume that $a$ and $b$ are continuous functions satisfying

$$a(x + 1) = a(x), \quad b(x + 1) = b(x). \quad (71)$$

Since $a$ and $b$ are all periodic functions, the diffusion process $X$ can be viewed as defined on the circle. In the following discussion, we shall always regard $X$ as a diffusion process on $S^1$.

We shall construct the diffusion process $X$ as the weak solution to the following stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad (72)$$

where $\sigma$ is a function on $\mathbb{R}$ defined as $\sigma(x) = \sqrt{a(x)}$ and $W$ is a standard one-dimensional Brownian motion defined on some filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfying the usual conditions. Since $a$ and $b$ are continuous periodic functions and $a(x) > 0$ for each $x$, by the Stroock-Varadhan uniqueness theorem [10], the martingale problem for $(a,b)$ is well-posed. By the equivalence between the martingale-formulation formulation and the weak-solution formulation [11], the weak solution to the stochastic differential equation (72) always exists and is unique in law.

We shall now study the trajectory properties of diffusion processes on the circle. The path of a diffusion process on the circle constantly forms the forward (counterclockwise) and backward (clockwise) cycles. To rigorously define the forming times of the forward and backward cycles, we need the following definitions.

**Definition 5.** The cycle forming time $T$ of $X$ is defined as

$$T = \inf\{t \geq 0 : |X_t - X_0| = 1\}. \quad (73)$$

If $X_T - X_0 = 1$, then we say that $X$ forms the forward cycle at time $T$. If $X_T - X_0 = -1$, then we say that $X$ forms the backward cycle at time $T$.

**Definition 6.** For each $n \geq 1$, the $n$-th cycle forming time $T_n$ of $X$ is defined as

$$T_n = \inf\{t \geq T_{n-1} : |X_t - X_{T_{n-1}}| = 1\}, \quad (74)$$

where we assume that $T_0 = 0$. If $X_{T_n} - X_{T_{n-1}} = 1$, then we say that $X$ forms the forward cycle at time $T_n$. If $X_{T_n} - X_{T_{n-1}} = -1$, then we say that $X$ forms the backward cycle at time $T_n$.

Intuitively, $T$ is the time needed for $X$ to form a forward or backward cycle for the first time and $T_n$ is the time needed for $X$ to form a forward or backward cycle for the $n$-th time.

**Definition 7.** Let $\tau^+ = \inf\{n \geq 1 : X \text{ forms the forward cycle at time } T_n\}$ and let $\tau^- = \inf\{n \geq 1 : X \text{ forms the backward cycle at time } T_n\}$. Then the forming time $T^+$ of the forward cycle by $X$ is defined as

$$T^+ = T_{\tau^+}. \quad (75)$$
and the forming time $T^-$ of the backward cycle by $X$ is defined as

$$T^- = T_{r-}. \quad (76)$$

Intuitively, $T^+$ is the time needed for $X$ to form a forward cycle for the first time and $T^-$ is the time needed for $X$ to form a backward cycle for the first time. It is easy to check that the following three relations hold.

**Lemma 11.** The following three relations hold:

(i) $T = T^+ \wedge T^-$;

(ii) $T^+ < T^-$ is equivalent to $X_T - X_0 = 1$;

(iii) $T^- < T^+$ is equivalent to $X_T - X_0 = -1$.

We are now in a position to state the main result of this section.

**Theorem 7.** Let $X$ be a diffusion process solving the stochastic differential equation (72), where $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ are continuous functions with period 1. Then

(i) for any $u \geq 0$,

$$P(T^+ \leq u, T^+ < T^-) = P(T^- < T^+) = \exp \left( 2 \int_0^1 \frac{b(y)}{\sigma^2(y)} dy \right); \quad (77)$$

(ii) for any $u \geq 0$,

$$P(T^+ \leq u | T^+ < T^-) = P(T^- \leq u | T^- < T^+). \quad (78)$$

**Remark 5.** The above theorem, which seems a bit counter-intuitive at first sight, shows that although the distributions of $T^+$ and $T^-$ may not be the same, their distributions, conditional on the corresponding cycle is formed earlier than its reversed cycle, are the same. The relation (78), which characterizes the symmetry of the forming times of the forward and backward cycles for general diffusion processes on the circle, will be named as the cycle symmetry in this paper.

**Corollary 3.** The notations are the same as in Theorem 7. Then $T$ and $X_T - X_0$ are independent.

**Proof.** By Lemma 11 and Theorem 7 we obtain that

$$P(T \leq u | X_T - X_0 = 1) = P(T \leq u | X_T - X_0 = -1). \quad (79)$$

This implies that $T$ and $X_T - X_0$ are independent.

**Remark 6.** Note that $T$ is the time needed for $X$ to form a forward or backward cycle for the first time and $X_T - X_0$ characterizes which one of these two cycles is formed. Thus the above corollary shows that the forming time of a forward or backward cycle for general diffusion processes on the circle is independent of which one of these two cycles is formed.

**Remark 7.** The above corollary also suggests that the hitting time of $\{-1, 1\}$ by general diffusion processes with periodic diffusion and drift coefficients is independent of which one of $-1$ and 1 is hit. It is easy to see that this independence holds for Brownian motion. In fact, let $W$ be a...
standard one-dimensional Brownian motion and let \( \tau = \inf\{t \geq 0 : |W_t| = 1\} \) be the hitting time of \( \{-1, 1\} \) by \( W \). It is a classical result \[12\] that

\[
P(\tau \in dt | W_\tau = 1) = P(\tau \in dt | W_\tau = -1) = \frac{\sqrt{2}}{\pi t^3} \sum_{n=-\infty}^{\infty} \left[ (4n + 1)e^{-\frac{(4n+1)^2}{2t}} \right],
\]

which shows that \( \tau \) and \( W_\tau \) are independent. The above corollary generalizes this independence to general diffusion processes with periodic diffusion and drift coefficients.

5 Proof of the cycle symmetry

The proof of the cycle symmetry is divided into three steps. First, using the quasi-time reversal invariance of Brownian motion, we shall prove the cycle symmetry for diffusion processes with \( \sigma \equiv 1 \) and smooth \( b \). Second, using some transformation techniques, we shall prove the cycle symmetry for diffusion processes with smooth \( \sigma \) and \( b \). Third, using some approximation techniques, we shall prove the cycle symmetry for diffusion processes with continuous \( \sigma \) and \( b \).

For further reference, we introduce some notations. Let \( X \) be a diffusion process solving the stochastic differential equation (72) with \( X_0 = 0 \). Recall that the potential function \( U \) of \( X \) is defined as

\[
U(x) = -2 \int_0^x \frac{b(y)}{\sigma^2(y)} dy
\]

and the scale function \( s \) of \( X \) is defined as

\[
s(x) = \int_0^x e^{U(y)} dy.
\]

Let \( \mathbb{W} \) be the canonical continuous path space defined in (62). Let \( \mathcal{G}_t \) and \( \mathcal{G} \) be \( \sigma \)-algebras on \( \mathbb{W} \) defined as

\[
\mathcal{G}_t = \sigma\{w_s : 0 \leq s \leq t\}, \quad \mathcal{G}_t = \sigma\{w_s : s \geq 0\}.
\]

Recall that the distribution \( \mu_X \) of \( X \) is a probability measure on \( \mathbb{W} \) defined as

\[
\mu_X(A) = P(X \in A), \quad A \in \mathcal{G}.
\]

The following lemma is a direct application of Girsanov’s theorem.

Lemma 12. Let \( X \) be a diffusion process solving the stochastic differential equation

\[
dX_t = b(X_t)dt + dW_t, \quad X_0 = 0,
\]

where \( b : \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function with period 1. Let \( \mu_X \) and \( \mu_W \) be the distributions of \( X \) and \( W \), respectively. Then for any \( u \geq 0 \) and \( A \in \mathcal{G}_u \),

\[
\mu_X(A) = \int_A \exp \left( -\frac{1}{2} \left[ U(w_u) + \int_0^u (b(w_t)^2 + b'(w_t))dt \right] \right) d\mu_W(w),
\]

where \( U \) is the potential function of \( X \).
Proof. Since $b$ is a continuous periodic function, it follows that $b(X_t)$ is a bounded continuous adapted process. By Novikov’s condition, the process
\[ M_t = \exp \left( - \int_0^t b(X_t) dW_t - \frac{1}{2} \int_0^t b(X_t)^2 dt \right) \]  
(87)
is a martingale. Thus we can define a probability measure $Q$ by
\[ dQ = \exp \left( - \int_0^u b(X_t) dW_t - \frac{1}{2} \int_0^u b(X_t)^2 dt \right) dP. \]  
(88)
Note that $X_t = W_t + \int_0^t b(X_s) ds$. By Girsanov’s theorem, $\{X_t : 0 \leq t \leq u\}$ is a standard Brownian motion under $Q$. Thus the distribution of $\{X_t : 0 \leq t \leq u\}$ under $Q$ is the same as that of $\{W_t : 0 \leq t \leq u\}$ under $P$. For any bounded measurable function $f$ on $C[0, u]$,
\[ E^P f(W) = E^Q f(X) = E^P f(X) \frac{dQ}{dP} \]
\[ = E^P f(X) \exp \left( - \int_0^u b(X_t) dW_t - \frac{1}{2} \int_0^u b^2(X_t) dt \right) \]
\[ = E^P f(X) \exp \left( - \int_0^u b(X_t) dX_t + \frac{1}{2} \int_0^u b^2(X_t) dt \right) \]
\[ = E^P f(X) \exp \left( \frac{1}{2} \int_0^u U'(X_t) dX_t + \frac{1}{2} \int_0^u b^2(X_t) dt \right). \]
(89)
Since $b$ is a $C^1$ function, it follows that $U$ is a $C^2$ function. Using Ito’s formula, we obtain that
\[ dU(X_t) = U'(X_t) dX_t + \frac{1}{2} U''(X_t) dt = U'(X_t) dX_t - b'(X_t) dt. \]
(90)
This shows that
\[ E^P f(W) = E^P f(X) \exp \left( \frac{1}{2} \left[ U(X_u) + \int_0^u (b^2(X_t) + b'(X_t)) dt \right] \right). \]
(91)
Thus on the measurable space $(\mathcal{M}, \mathcal{G}_u)$,
\[ d\mu_W(w) = \exp \left( \frac{1}{2} \left[ U(w_u) + \int_0^u (b^2(w_t) + b'(w_t)) dt \right] \right) d\mu_X(w). \]
(92)
This completes the proof of this lemma. \square

We are now in a position to prove the cycle symmetry for diffusion process with $\sigma \equiv 1$ and smooth $b$. The proof of the following result strongly depends on the quasi-time reversal invariance of Brownian motion established in this paper.

**Lemma 13.** Let $X$ be a diffusion process solving the stochastic differential equation (85), where $b : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function with period 1. Then for any $u \geq 0$,
\[ \frac{P(T \leq u, X_T = 1)}{P(T \leq u, X_T = -1)} = \frac{P(X_T = 1)}{P(X_T = -1)} = \exp \left( 2 \int_0^1 b(y) dy \right). \]
(93)
Proof. Let $\tau$ and $\sigma$ be two functions on $W$ defined in (63) and (64). Let $A$ and $B$ be two sets in $\mathbb{W}$ defined as

$$A = \{ w \in \mathbb{W} : \tau(w) \leq u, w_\tau = 1 \}, \quad B = \{ w \in \mathbb{W} : \tau(w) \leq u, w_\tau = -1 \}. \quad (94)$$

Let $\mu_X$ and $\mu_W$ be the distributions of $X$ and $W$, respectively. Then we have

$$P(T \leq u, X_T = 1) = \mu_X(A), \quad P(T \leq u, X_T = -1) = \mu_X(B). \quad (95)$$

It is easy to see that $A, B \in \mathcal{F}_u$. By Lemma[12] we have

$$\mu_X(A) = \int_A \exp \left( -\frac{1}{2} \left[ U(w_u) + \int_0^u p(w_t)dt \right] \right) d\mu_W(w), \quad (96)$$

where $p = b^2 + b'$ is a continuous function with period 1.

Let $\phi$ be the quasi-time reversal on $W$. By Lemma[5] we have

$$\phi_* \mu_W = \mu_W, \quad (97)$$

where $\phi_* \mu_W$ is the push-forward measure defined by $\phi_* \mu_W(A) = \mu_W(\phi^{-1}(A))$. Moreover, we make a crucial observation that $\phi$ is a one-to-one map from $A$ onto $B$. This shows that

$$\mu_X(A) = \int_B \exp \left( -\frac{1}{2} \left[ U(\phi(w)_u) + \int_0^u p(\phi(w)_t)dt \right] \right) d\mu_W(w) \quad (98)$$

Since $b$ is a $C^1$ function with period 1, for any $w \in B$,

$$U(\phi(w)_u) = U(w_u + 2) = -2 \int_0^{w_u} b(y)dy - 2 \int_{w_u}^{w_u+1} b(y)dy - 2 \int_{w_u+1}^{w_u+2} b(y)dy$$

$$= -2 \int_0^{w_u} b(y)dy - 4 \int_0^1 b(y)dy = U(w_u) + 2U(1). \quad (99)$$

Since $p$ is a continuous function with period 1, for any $w \in B$,

$$\int_0^u p(\phi(w)_t)dt = \int_0^\sigma p(w_t)dt + \int_\sigma^\tau p(w_{\sigma+t-\tau} + 1)dt + \int_\tau^u p(w_t + 2)dt$$

$$= \int_0^\sigma p(w_t)dt + \int_\sigma^\tau p(w_t)dt + \int_\tau^u p(w_t)dt = \int_0^u p(w_t)dt. \quad (100)$$

The above calculations show that

$$\mu_X(A) = \int_B \exp \left( -\frac{1}{2} \left[ U(w_u) + 2U(1) + \int_0^u p(w_t)dt \right] \right) d\mu_W(w)$$

$$= e^{-U(1)} \int_B \exp \left( -\frac{1}{2} \left[ U(w_u) + \int_0^u p(w_t)dt \right] \right) d\mu_W(w) \quad (101)$$

$$= e^{-U(1)} \mu_X(B).$$

This shows that

$$\frac{P(T \leq u, X_T = 1)}{P(T \leq u, X_T = -1)} = \frac{\mu_X(A)}{\mu_X(B)} = e^{-U(1)}, \quad (102)$$

which gives the desired result.

\[ \square \]
We are now in a position to prove the cycle symmetry for diffusion process $X$ with smooth $\sigma$ and $b$.

**Lemma 14.** Let $X$ be a diffusion process solving the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = 0,$$

where $b: \mathbb{R} \to \mathbb{R}$ is a $C^1$ function with period 1 and $\sigma: \mathbb{R} \to (0, \infty)$ is a $C^2$ function with period 1. Then for any $u \geq 0$,

$$\frac{P(T \leq u, X_T = 1)}{P(T \leq u, X_T = -1)} = \frac{P(X_T = 1)}{P(X_T = -1)} = \exp \left( 2 \int_0^1 \frac{b(x)}{\sigma^2(x)} dx \right).$$

**Proof.** Let $f$ be a function on $\mathbb{R}$ defined as

$$f(x) = \int_0^x \frac{1}{\sigma(y)} dy.$$

It is easy to see that $f$ is a $C^3$ diffeomorphism on $\mathbb{R}$. Using the Ito’s formula, we obtain that

$$df(X_t) = \left( f'(X_t)b(X_t) + \frac{1}{2} f''(X_t)\sigma^2(X_t) \right) dt + f'(X_t)\sigma(X_t)dW_t.$$ 

Since $f'\sigma \equiv 1$, we obtain that

$$df(X_t) = \left( f'b + \frac{1}{2} f''\sigma^2 \right) f^{-1}(f(X_t)) dt + dW_t.$$ 

This shows that $f(X)$ is a diffusion process solving the stochastic differential equation

$$dY_t = c(Y_t)dt + dW_t, \quad Y_0 = 0.$$ 

where $c = (f'b + \frac{1}{2} f''\sigma^2) f^{-1}$. Since $f'b + \frac{1}{2} f''\sigma^2$ is a function with period 1, it follows that $c(f(x+1)) = c(f(x))$ for each $x$. Since $\sigma$ is a $C^2$ periodic function with period 1, we have

$$f(x+1) - f(x) = \int_x^{x+1} \frac{1}{\sigma(y)} dy = \int_0^1 \frac{1}{\sigma(y)} dy = f(1).$$

This shows that $c(x + f(1)) = c(x)$ for each $x$. Thus $c$ is a $C^1$ periodic function with period $f(1)$.

Since $f$ is a $C^3$ diffeomorphism, it is easy to see that the cycle forming time $T$ of the diffusion process $X$ is exactly the cycle forming time of the diffusion process $f(X)$, that is,

$$T = \inf\{t \geq 0 : |f(X_t)| = f(1)\}.$$ 

By Lemma [13] we have

$$\frac{P(T \leq u, f(X_T) = f(1))}{P(T \leq u, f(X_T) = -f(1))} = \frac{P(f(X_T) = f(1))}{P(f(X_T) = -f(1))} = \exp \left( 2 \int_0^{f(1)} c(y)dy \right).$$

Since $\sigma$ is a $C^2$ function with period 1,

$$-f(1) = -\int_0^1 \frac{1}{\sigma(y)} dy = -\int_{-1}^0 \frac{1}{\sigma(y)} dy = f(-1).$$
Moreover, by the periodicity of $\sigma$, we have

$$
\int_0^{f^{(1)}} c(y)dy = \int_0^{f^{(1)}} \left( f' b + \frac{1}{2} f'' \sigma^2 \right) \circ f^{-1}(y)dy
$$

$$
= \int_0^1 \left( f'(x) b(x) + \frac{1}{2} f''(x) \sigma^2(x) \right) f'(x)dx = \int_0^1 b(x) - \frac{\sigma'(x)}{\sigma(x)} dx
$$

(113)

$$
= \int_0^1 \frac{b(x)}{\sigma^2(x)}dx - \log(\sigma(1)) + \log(\sigma(0)) = \int_0^1 \frac{b(x)}{\sigma^2(x)}dx.
$$

Thus we obtain that

$$
P(T \leq u, f(X_T) = f(1)) = P(f(X_T) = f(1)) = \exp \left( 2 \int_0^1 \frac{b(x)}{\sigma^2(x)}dx \right). \quad (114)
$$

Since $f$ is a $C^3$ diffeomorphism, we obtain the desired result. \hfill \square

**Lemma 15.** Let $X$ be a diffusion process solving the stochastic differential equation (103), where $b : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function and $\sigma : \mathbb{R} \to (0, \infty)$ is a bounded continuous function. Let $P_t(x, dy)$ be the transition function of $X$. Then for any $t > 0$ and $x \in \mathbb{R}$, $P_t(x, dy)$ has a density $p(t, x, y)$ with respect to the Lebesgue measure.

**Proof.** The proof of this lemma can be found in Stroock and Varadhan [10]. \hfill \square

**Lemma 16.** Let $X$ be a diffusion process solving the stochastic differential equation (103), where $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ are continuous functions with period 1. Let $\mu_X$ be the distribution of $X$. Let $A$ and $B$ be two subsets of $\mathbb{W}$ defined in (94). Then $\mu_X(\partial A) = \mu_X(\partial B) = 0$.

**Proof.** Let $\bar{A}$ be the closure of $A$ and let $A^o$ be the interior of $A$. Note that

$$
\bar{A} \subset A \cup \{ \tau < u, w(\tau) = -1, \exists h > 0, \forall \tau \leq t \leq \tau + h, \text{ we have } w(t) \geq -1 \}. \quad (115)
$$

Moreover,

$$
A^o \supset \{ \tau < u, w(\tau) = 1, \forall h > 0, \exists \tau \leq t \leq \tau + h, \text{ such that } w(t) > 1 \}. \quad (116)
$$

The above two equations show that

$$
\partial A \subset \{ \tau = u, w(\tau) = -1 \}
$$

$$
\cup \{ \tau < u, w(\tau) = 1, \exists h > 0, \forall \tau \leq t \leq \tau + h, \text{ we have } w(t) \leq 1 \}
$$

(117)

$$
\cup \{ \tau < u, w(\tau) = -1, \exists h > 0, \forall \tau \leq t \leq \tau + h, \text{ we have } w(t) \geq -1 \}.
$$

Let $M$ be the diffusion process solving to the stochastic differential equation

$$
dM_t = \sigma(M_t)dW_t, \quad M_0 = 0. \quad (118)
$$

Let $\mu_M$ be the distribution of $M$. By Girsanov’s theorem, $\mu_X$ and $\mu_M$ are mutually absolutely continuous on the measurable space $(\mathbb{W}, \mathcal{G}_u)$. In order to prove that $\mu_X(\partial A) = 0$, we only need
to prove that $\mu_M(\partial A) = 0$. Note that $M$ is a strong Markov process. According to (117), we have

$$\mu_M(\partial A) \leq P_0(M_u = -1) + P_1(\exists h > 0, \forall 0 \leq t \leq h, \text{ we have } M_t \leq 1)$$

$$+ P_{-1}(\exists h > 0, \forall 0 \leq t \leq h, \text{ we have } M_t \geq -1).$$

By Lemma [15] the transition function $P_{n}(x, dy)$ of $M$ has a density with respect to the Lebesgue measure for each $u > 0$ and $x \in \mathbb{R}$. This shows that $P_0(M_u = -1) = 0$ for each $u \geq 0$. Since $M$ is a continuous local martingale, there exists a Brownian motion $B$, such that $M_t = B_{[M,M]_t}$. Note that $[M,M]_t = \int_0^t \sigma^2(M_s)ds$ is a strictly increasing process. Thus

$$P_1(\exists h > 0, \forall 0 \leq t \leq h, \text{ we have } M_t \leq 1)$$

$$= P_1(\exists h > 0, \forall 0 \leq t \leq h, \text{ we have } B_t \leq 1) = 0.$$ (120)

Similarly, we can prove that $P_{-1}(\exists h > 0, \forall 0 \leq t \leq h, \text{ we have } M_t \geq -1) = 0$. This shows that $\mu_X(\partial A) = \mu_M(\partial A) = 0$. Similarly, we can prove that $\mu_X(\partial B) = 0$.

**Lemma 17.** Let $X$ be a diffusion process solving the stochastic differential equation (103) and for each $n \geq 1$, let $X_n$ be a diffusion process solving the stochastic differential equation

$$dX_t = b_n(X_t)dt + \sigma_n(X_t)dW_t, \quad X_t = 0,$$

where $b, b_n : \mathbb{R} \to \mathbb{R}$ and $\sigma, \sigma_n : \mathbb{R} \to (0, \infty)$ are bounded continuous functions. Let $\mu_X$ and $\mu_{X_n}$ be the distributions of $X$ and $X_n$, respectively. Assume that for any $M > 0$, the following two conditions hold:

$$\sup_{n \geq 1} \sup_{|x| \leq M} \left( |\sigma_n^2(x)| + |b_n(x)| \right) < \infty,$$

$$\lim_{n \to \infty} \sup_{|x| \leq M} \left( |\sigma_n^2(x) - \sigma^2(x)| + |b_n(x) - b(x)| \right) = 0.$$ (122)

Then $\mu_{X_n} \Rightarrow \mu_X$.

**Proof.** The proof of this lemma can be found in Stroock and Varadhan [10].

We are now in a position to prove the cycle symmetry for diffusion process $X$ with continuous $\sigma$ and $b$.

**Proof of Theorem 2.** It is easy to see that (ii) is a direct corollary of (i). Thus we only need to prove (i). By Lemma [11], we only need to prove that

$$\frac{P(T \leq u, X_T = 1)}{P(T \leq u, X_T = -1)} = \frac{P(X_T = 1)}{P(X_T = -1)} = \exp \left( 2 \int_0^1 \frac{b(x)}{\sigma^2(x)}dx \right).$$ (123)

In order to prove the result under general initial distributions, we only need to prove the result for diffusion process $X$ starting from 0. Since the space of smooth periodic functions are dense in the space of continuous periodic functions, we can find $C^1$ functions $b_n$ with period 1 such that $b_n$ converges to $b$ uniformly. Similarly, we can find $C^2$ functions $\sigma_n$ with period 1 such that
$\sigma_n$ converges to $\sigma$ uniformly. Let $X^n$ be the diffusion process solving the stochastic differential equation
\[ dX_t = b_n(X_t)dt + \sigma_n(X_t)dW_t, \quad X_0 = 0. \tag{124} \]
Let $\mu_X$ and $\mu_{X^n}$ be the distributions of $X$ and $X^n$, respectively. By Lemma 14 we have
\[ \frac{\mu_{X^n}(A)}{\mu_{X^n}(B)} = \exp \left( 2 \int_0^1 \frac{b_n(x)}{\sigma_n^2(x)} dx \right). \tag{125} \]
Since $b, \sigma, b_n, \sigma_n$ are all continuous periodic functions, it is easy to see that the conditions in Lemma 17 are satisfied. It then follows from Lemma 17 that $\mu_{X^n} \Rightarrow \mu_X$. By Lemma 16 we have $\mu_X(\partial A) = 0$. This shows that $\mu_{X^n}(A) \rightarrow \mu(A)$. Similarly, we have $\mu_{X^n}(B) \rightarrow \mu(B)$. Moreover, it is easy to see that $b_n/\sigma_n^2$ converges to $b/\sigma^2$ uniformly. Letting $n \rightarrow \infty$ in (125), we obtain that
\[ \frac{\mu_X(A)}{\mu_X(B)} = \frac{P(T \leq u, X_T = 1)}{P(T \leq u, X_T = -1)} = \exp \left( 2 \int_0^1 \frac{b(x)}{\sigma^2(x)} dx \right). \tag{126} \]
Since the above equation holds for each $u \geq 0$, we obtain the desired result.

\[ \square \]

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