FORCING ALGEBRAS, SYZYGY BUNDLES, AND TIGHT CLOSURE

HOLGER BRENNER

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Abstract. We give a survey about some recent work on tight closure and Hilbert-Kunz theory from the viewpoint of vector bundles. This work is based on understanding tight closure in terms of forcing algebras and the cohomological dimension of torsors of syzygy bundles. These geometric methods allowed to answer some fundamental questions of tight closure, in particular the equality between tight closure and plus closure in graded dimension two over a finite field and the rationality of the Hilbert-Kunz multiplicity in graded dimension two. Moreover, this approach showed that tight closure may behave weirdly under arithmetic and geometric deformations, and provided a negative answer to the localization problem.

Introduction

In this survey article we describe some developments which led to a detailed geometric understanding of tight closure in dimension two in terms of vector bundles and torsors. Tight closure is a technique in positive characteristic introduced by M. Hochster and C. Huneke 20 years ago ([21], [22]). We recall its definition. Let $R$ be a commutative ring of positive characteristic $p$ with $e$th Frobenius homomorphism $F^e : R \to R, f \mapsto f^q$, $q = p^e$. For an ideal $I$ let $I^{[q]} := F^e(I)$ be the extended ideal under the $e$th Frobenius. Then the tight closure of $I$ is given by

$$I^* = \{ f \in R : \text{there exists } t, \text{not in any minimal prime},$$

such that $tf^q \in I^{[q]}$ for $q \gg 0 \}$$

(in the domain case this means just $t \neq 0$, and for all $q$). In this paper we will not deal with the applications of tight closure in commutative algebra, homological algebra and algebraic geometry, but with some of its intrinsic problems. One of them is whether tight closure commutes with localization, that is, whether for a multiplicative system $S \subseteq R$ the equality

$$(I^*)_S = (I_S)^*$$

holds (the inclusion $\subseteq$ is always true). A directly related question is whether tight closure is the same as plus closure. The plus closure of an ideal $I$ in a
domain $R$ is defined to be

$$I^+ = \{ f \in R : \text{there exists } R \subseteq S \text{ finite domain extension such that } f \in IS \}.$$ 

This question is known as the *tantalizing question* of tight closure theory. The inclusion $I^+ \subseteq I^*$ always holds. Since the plus closure commutes with localization, a positive answer to the tantalizing question for a ring and all its localizations implies a positive answer for the localization problem for this ring. The tantalizing question is a problem already in dimension two, the localization problem starts to get interesting in dimension three.

What makes these problems difficult is that there are no exact criteria for tight closure. There exist many important inclusion criteria for tight closure, and in all these cases the criteria also hold for plus closure (in general, with much more difficult proofs). The situation is that the heartlands of “tight closure country” and of “non tight closure country” have been well exploited, but not much is known about the thin line which separates them. This paper is about approaching this thin line geometrically.

The original definition of tight closure, where one has to check infinitely many conditions, is difficult to apply. The starting point of the work we are going to present here is another description of tight closure due to Hochster as *solid closure* based on the concept of *forcing algebras*. Forcing algebras were introduced by Hochster in [19] in an attempt to build up a characteristic-free closure operation with similar properties as tight closure. This approach rests on the fact that $f \in (f_1, \ldots, f_n)^*$ holds in $R$ if and only if $H^\dim R_m(A) \neq 0$, where $A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$ is the forcing algebra for these data (see Theorem 4.1 for the exact statement). This gives a new interpretation for tight closure, where, at least at first glance, not infinitely many conditions are involved. This cohomological interpretation can be refined geometrically, and the goal of this paper is to describe how this is done and where it leads to. We give an overview.

We will describe the basic properties of forcing algebras in Section 1. A special feature of the cohomological condition for tight closure is that it depends only on the open subset $D(mA) \subseteq \text{Spec } A$. This open subset is a “torsor” over $D(\mathfrak{m}) \subseteq \text{Spec } R$, on which the *syzygy bundle* Syz($f_1, \ldots, f_n$) acts. This allows a more geometric view of the situation (Section 2). In general, closure operations for ideals can be expressed with suitable properties of forcing algebras. We mention some examples of this correspondence in Section 3 and come back to tight closure and solid closure in Section 4.

To obtain a detailed understanding, we restrict in Section 5 to the situation of a two-dimensional standard-graded normal domain $R$ over an algebraically closed field and homogeneous $R_+$-primary ideals. In this setting, the question about the cohomological dimension is the question whether a torsor coming from forcing data is an affine scheme. Moreover, to answer this question we can look at the corresponding torsor over the smooth projective curve Proj $R$. 

This translates the question into a projective situation. In particular, we can then use concepts from algebraic geometry like semistable bundles and the strong Harder-Narasimhan filtration to prove results. We obtain an exact numerical criterion for tight closure in this setting (Theorems 5.2 and 5.3). The containment in the plus closure translates to a geometric condition for the torsors on the curve, and in the case where the base field is the algebraic closure of a finite field we obtain the same criterion. This implies that under all these assumptions, tight closure and plus closure coincide (Theorem 5.4).

With this geometric approach also some problems in Hilbert-Kunz theory could be solved, in particular it was shown that the Hilbert-Kunz multiplicity is a rational number in graded dimension two (Theorem 6.3). In fact, there is a formula for it in terms of the strong Harder-Narasimhan filtration of the syzygy bundle. In Section 7, we change the setting and look at families of two-dimensional rings parametrized by a one-dimensional base. Typical bases are \text{Spec} \mathbb{Z} (arithmetic deformations) or \mathbb{A}^1_K (geometric deformations). Natural questions are how tight closure, Hilbert-Kunz multiplicity and strong semistability of bundles vary in such a family. Examples of P. Monsky already showed that the Hilbert-Kunz multiplicity behaves “weirdly” in the sense that it is not almost constant. It follows from the geometric interpretation that also strong semistability behaves wildly. Further extra work is needed to show that tight closure also behaves wildly under such a deformation. We present the example of Brenner-Katzman in the arithmetic case and of Brenner-Monsky in the geometric case (Examples 7.4 and 7.7). The latter example shows also that tight closure does not commute with localization and that even in the two-dimensional situation, the tantalizing question has a negative answer, if the base field contains a transcendental element. We close the paper with some open problems (Section 8).

As this is a survey article, we usually omit the proofs and refer to the original research papers and to [9]. I thank Helena Fischbacher-Weitz, Almar Kaid and Axel Stäbler for useful comments.

1. Forcing algebras

Let $R$ be a commutative ring, let $M$ be a finitely generated $R$-module and $N \subseteq M$ a finitely generated $R$-submodule. Let $s \in M$ be an element. The forcing algebra for these data is constructed as follows: choose generators $y_1, \ldots, y_m$ for $M$ and generators $x_1, \ldots, x_n$ for $N$. This gives rise to a surjective homomorphism $\varphi : R^n \to M$, a submodule $N' = \varphi^{-1}(N)$ and a morphism $R^m \to R^n$ which sends $e_i$ to a preimage $x'_i$ of $x_i$. Altogether we get the commutative diagram with exact rows
The element $s$ is represented by $(s_1, \ldots, s_m) \in \mathbb{R}^m$, and $s$ belongs to $N$ if and only if the linear equation

$$
\begin{pmatrix}
  t_1 \\
  \vdots \\
  t_n
\end{pmatrix}
= 
\begin{pmatrix}
  s_1 \\
  \vdots \\
  s_m
\end{pmatrix}
$$

has a solution. An important insight due to Hochster is that this equation can be formulated with new variables $T_1, \ldots, T_n$, and then the “distance of $s$ to $N$” - in particular, whether $s$ belongs to a certain closure of $N$ - is reflected by properties of the resulting (generic) forcing algebra. Explicitly, if $\alpha$ is the matrix describing the submodule $N$ as above and if $(s_1, \ldots, s_m)$ represents $s$, then the forcing algebra is defined by

$$
\begin{aligned}
A &= \mathbb{R}[T_1, \ldots, T_n]/(\alpha T - s),
\end{aligned}
$$

where $\alpha T - s$ stands for

$$
\begin{pmatrix}
  T_1 \\
  \vdots \\
  T_n
\end{pmatrix}
= 
\begin{pmatrix}
  s_1 \\
  \vdots \\
  s_m
\end{pmatrix}
$$

or, in other words, for the system of inhomogeneous linear equations

$$
\begin{align*}
  a_{11}T_1 + \ldots + a_{1n}T_n &= s_1 \\
  a_{21}T_1 + \ldots + a_{2n}T_n &= s_2 \\
  \vdots &
\end{align*}
$$

In the case of an ideal $I = (f_1, \ldots, f_n)$ and $f \in R$ the forcing algebra is just $A = \mathbb{R}[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$. Forcing algebras are given by the easiest algebraic equations at all, namely linear equations. Yet we will see that forcing algebras already have a rich geometry. Of course, starting from the data $(M, N, s)$ we had to make some choices in order to write down a forcing algebra, hence only properties which are independent of these choices are interesting.

The following lemma expresses the universal property of a forcing algebra.
Lemma 1.1. Let the situation be as above, and let $R \to R'$ be a ring homomorphism. Then there exists an $R$-algebra homomorphism $A \to R'$ if and only if $s \otimes 1 \in \text{im}(N \otimes R' \to M \otimes R')$.

Proof. This follows from the right exactness of tensor products applied to the sequence (*) above. □

The lemma implies in particular that for two forcing algebras $A$ and $A'$ we have (not uniquely determined) $R$-algebra homomorphisms $A \to A'$ and $A' \to A$. It also implies that $s \in N$ if and only if there exists an $R$-algebra homomorphism $A \to R$ (equivalently, Spec $A \to$ Spec $R$ has a section).

We continue with some easy geometric properties of the mapping Spec $A \to$ Spec $R$. The formation of forcing algebras commutes with arbitrary base change $R \to R'$. Therefore for every point $p \in$ Spec $R$ the fiber ring $A \otimes_R \kappa(p)$ is the forcing algebra given by

$$\alpha(p)T = s(p),$$

which is an inhomogeneous linear equation over the field $\kappa(p)$. Hence the fiber of Spec $A \to$ Spec $R$ over $p$ is the solution set to a system of linear inhomogeneous equations.

We know from linear algebra that the solution set to such a system might be empty, or it is an affine space (in the sense of linear algebra) of dimension $\geq n - m$. Hence one should think of Spec $A \to$ Spec $R$ as a family of affine-linear spaces varying with the base. Also, from linear algebra we know that such a solution set is given by adding to one particular solution a solution of the corresponding system of homogeneous of linear equations. The solution set to $\alpha(p)T = 0$ is a vector space over $\kappa(p)$, and this solution set is the fiber over $p$ of the forcing algebra of the zero element, namely

$$B = R[T_1, \ldots, T_n]/(\alpha T) = R[T_1, \ldots, T_n]/(\sum_{i=1}^n a_{ij}T_i, j = 1, \ldots, m).$$

As just remarked, the fibers of $V = $ Spec $B$ over a point $p$ are vector spaces of possibly varying dimensions. Therefore $V$ is in general not a vector bundle. It is, however, a commutative group scheme over Spec $R$, where the addition is given by

$$V \times V \longrightarrow V, (s_1, \ldots, s_n), (s'_1, \ldots, s'_n) \longmapsto (s_1 + s'_1, \ldots, s_n + s'_n)$$

(written on the level of sections) and the coaddition by

$$R[T_1, \ldots, T_n]/(\alpha T) \to R[T_1, \ldots, T_n]/(\alpha T) \otimes R[\tilde{T}_1, \ldots, \tilde{T}_n]/(\alpha \tilde{T}), T_i \mapsto T_i + \tilde{T}_i.$$

This group scheme is the kernel group scheme of the group scheme homomorphism

$$\alpha : \mathbb{A}^n_R \longrightarrow \mathbb{A}^m_R$$

between the trivial additive group schemes of rank $n$ and $m$. We call it the syzygy group scheme for the given generators of $N$. 
The syzygy group scheme acts on the spectrum of a forcing algebra \( \text{Spec} A, A = R[T]/(\alpha T - s) \) for every \( s \in M \). The action is exactly as in linear algebra, by adding a solution of the system of homogeneous equations to a solution of the system of inhomogeneous equations. An understanding of the syzygy group scheme is necessary before we can understand the forcing algebras.

Although \( V \) is not a vector bundle in general, it is not too far away. Let \( U \subseteq \text{Spec} R \) be the open subset of points \( p \) where the mapping \( \alpha(p) \) is surjective. Then the restricted group scheme \( V|_U \) is a vector bundle of rank \( n - m \). If \( M/N \) has its support in a maximal ideal \( m \), then the syzygy group scheme induces a vector bundle on the punctured spectrum \( \text{Spec} R - \{m\} \), which we call the syzygy bundle. Hence on \( U \) we have a short exact sequence

\[
0 \rightarrow \text{Syz} \rightarrow \mathcal{O}^m_U \rightarrow \mathcal{O}^m_U \rightarrow 0
\]

of vector bundles on \( U \).

We will mostly be interested in the situation where the submodule is an ideal \( I \subseteq R \) in the ring. We usually fix ideal generators \( I = (f_1, \ldots, f_n) \) and (\*) becomes

\[
R^n \xrightarrow{f_1, \ldots, f_n} R \rightarrow R/I \rightarrow 0.
\]

The ideal generators and an element \( f \in R \) defines then the forcing equation \( f_1T_1 + \ldots + f_nT_n - f = 0 \). Moreover, if the ideal is primary to a maximal ideal \( m \), then we have a syzygy bundle \( \text{Syz} = \text{Syz}(f_1, \ldots, f_n) \) defined on \( D(m) \).

2. Forcing algebras and torsors

Let \( Z \subseteq \text{Spec} R \) be the support of \( M/N \) and let \( U = \text{Spec} R - Z \) be the open complement where \( \alpha \) is surjective. Let \( s \in M \) with forcing algebra \( A \). We set \( T = \text{Spec} A|_U \) and we assume that the fibers are non-empty (in the ideal case this means that \( f \) is not a unit). Then the action of the group scheme \( V \) on \( \text{Spec} A \) restricts to an action of the syzygy bundle \( \text{Syz} = V|_U \) on \( T \), and this action is simply transitive. This means that locally the actions looks like the action of \( \text{Syz} \) on itself by addition.

In general, if a vector bundle \( \mathcal{S} \) on a separated scheme \( U \) acts simply transitively on a scheme \( T \rightarrow U \) – such a scheme is called a geometric \( \mathcal{S} \)-torsor or an affine-linear bundle –, then this corresponds to a cohomology class \( c \in H^1(U, S) \) (where \( S \) is now also the sheaf of sections in the vector bundle \( S \)). This follows from the Čech cohomology by taking an open covering where the action can be trivialized. Hence forcing data define, by restricting the forcing algebra, a torsor \( T \) over \( U \).

On the other hand, the forcing data define the short exact sequence \( 0 \rightarrow \text{Syz} \rightarrow \mathcal{O}^m_U \rightarrow \mathcal{O}^0_U \rightarrow 0 \) and \( s \) is represented by an element \( s' \in R^n \xrightarrow{\delta} \Gamma(U, \mathcal{O}^n_U) \). By the connecting homomorphism \( s' \) defines a cohomology class \( c = \delta(s') \in H^1(U, \text{Syz}) \).
An explicit computation of Čech cohomology shows that this class corresponds to the torsor given by the forcing algebra.

Starting from a cohomology class \( c \in H^1(U, S) \), one may construct a geometric model for the torsor classified by \( c \) in the following way: because of \( H^1(U, S) \cong \text{Ext}^1(O_U, S) \) we have an extension

\[
0 \to S \to S' \to O_U \to 0.
\]

This sequence induces projective bundles \( \mathbb{P}(S') \hookrightarrow \mathbb{P}(S'') \) and \( T(c) \cong \mathbb{P}(S'') - \mathbb{P}(S') \). If \( S = \text{Syz}(f_1, \ldots, f_n) \) is the syzygy bundle for ideal generators, then the extension given by the cohomology class \( \delta(f) \) coming from another element \( f \) is easy to describe: it is just

\[
0 \to \text{Syz}(f_1, \ldots, f_n) \to \text{Syz}(f_1, \ldots, f_n, f) \to O_U \to 0
\]

with the natural embedding (extending a syzygy by zero in the last component). This follows again from an explicit computation in Čech cohomology.

If the base \( U \) is projective, a situation in which we will work starting with Section 5, then \( \mathbb{P}(S') \) is also a projective variety and \( \mathbb{P}(S'') \) is a subvariety of codimension one, a divisor. Then properties of the torsor are reflected by properties of the divisor and vice versa.

3. Forcing algebras and closure operations

A closure operation for ideals or for submodules is an assignment

\[
N \mapsto N^c
\]

for submodules \( N \subseteq M \) of \( R \)-modules \( M \) such that \( N \subseteq N^c = (N^c)^c \) holds. One often requires further nice properties of a closure operation, like monotony or the independence of representation (meaning that \( s \in N^c \) if and only if \( \bar{s} \in O' \) in \( M/N \)). Forcing algebras are very natural objects to study such closure operations. The underlying philosophy is that \( s \in N^c \) holds if and only if the forcing morphism \( \text{Spec } A \to \text{Spec } R \) fulfills a certain property (depending on and characterizing the closure operation). The property is in general not uniquely determined; for the identical closure operation one can take the properties to be faithfully flat, to be (cyclic) pure, or to have a (module- or ring-) section.

Let us consider some easy closure operations to get a feeling for this philosophy. In Section 4 we will see how tight closure can be characterized with forcing algebras.

Example 3.1. For the radical \( \text{rad}(I) \) the corresponding property is that \( \text{Spec } A \to \text{Spec } R \) is surjective. It is not surprising that a rough closure operation corresponds to a rough property of a morphism. An immediate consequence of this viewpoint is that we get at once a hint of what the radical of a submodule should be: namely \( s \in \sqrt{N} \) if and only if the forcing algebra is Spec-surjective. This is equivalent to the property that
such that $g$ complex topology in the
bers there is another interesting closure operation, calle d
s
An element $C$ is such that the morphism
Example
forcing algebra respectively .

$s \otimes 1 \in \text{im}(N \otimes_R K \to M \otimes_R K)$ for all homomorphism $R \to K$ to fields (or
just for all $k(p), p \in \text{Spec } R$).

**Example 3.2.** We now look at the integral closure of an ideal, which is defined by

$$\bar{I} = \{ f \in R : \text{ there exists } f^k + a_1 f^{k-1} + \ldots + a_{k-1} f + a_k = 0, a_i \in I \}.$$  

The integral closure was first used by Zariski as it describes the normalization of blow-ups. What is the corresponding property of a morphism?

We look at an example. For $R = K[X, Y]$ we have $X^2 Y \in (X^3, Y^3)$ and $XY \not\in (X^3, Y^3)$. The inclusion follows from $(X^2 Y)^3 = X^6 Y^3 \in (X^3, Y^3)^3$.

The non-inclusion follows from the valuation criterion for integral closure: This says for a noetherian domain $R$ that $f \in \bar{I}$ if and only if for all mappings to discrete valuation rings $\varphi : R \to V$ we have $\varphi(f) \in IV$. In the example the mapping $K[X, Y] \to K[X], Y \mapsto X$, yields $X^2 \not\in (X^3)$, so it can not belong to the integral closure. In both cases the mapping $\text{Spec } A \to \text{Spec } R$ is surjective. In the second case, the forcing algebra over the line $V(Y - X)$ is given by the equation $T_1 X^3 + T_2 X^3 + X^2 = X^2((T_1 + T_2)X + 1)$. The fiber over the zero point is a plane and is an affine line over a hyperbola for
every other point of the line. The topologies above and below are not much related: The inverse image of the non-closed punctured line is closed, hence the topology downstairs does not carry the image topology from upstairs. In
fact, the relationship in general is

$$f \in \bar{I} \text{ if and only if } \text{Spec } A \to \text{Spec } R \text{ is universally a submersion}$$

(a submersion in the topological sense). This relies on the fact that both properties can be checked with (in the noetherian case discrete) valuations
(for this criterion for submersions, see [15] and [1]).

Let us consider the forcing algebras for $(X, Y)$ and 1 and for $(X^2, Y^2)$ and $XY$ in $K[X, Y]$. The restricted spectra of the forcing algebras over the punctured plane for these two forcing data are isomorphic, because both represent
the torsor given by the cohomology class $\frac{1}{XY} = \frac{XY}{X^2 Y} \in H^1(D(X, Y), \mathcal{O})$.

However, $XY \in (X^2, Y^2)$, but $1 \not\in (X, Y)$ (not even in the radical). Hence integral closure can be characterized by the forcing algebra, but not by the
induced torsor. An interesting feature of tight closure is that it only depends
on the cohomology class in the syzygy bundle and the torsor induced by the
forcing algebra respectively.

**Example 3.3.** In the case of finitely generated algebras over the complex numbers there is another interesting closure operation, called continuous closure. An element $s$ belongs to the continuous closure of $N$ if the forcing algebra $A$ is such that the morphism $C - \text{Spec } A \to C - \text{Spec } R$ has a continuous section in the complex topology. For an ideal $I = (f_1, \ldots, f_n)$ this is equivalent to
the existence of complex-continuous functions $g_1, \ldots, g_n : C - \text{Spec } R \to C$
such that $\sum_{i=1}^n g_i f_i = f$ (as an identity on $C - \text{Spec } R$).
Remark 3.4. One can go one step further with the understanding of closure operations in terms of forcing algebras. For this we take the forcing algebras which are allowed by the closure operation (i.e., forcing algebras for \( s, N, M, s \in N^c \)) and declare them to be the defining covers of a (non-flat) Grothendieck topology. This works basically for all closure operations fulfilling certain natural conditions. This embeds closure operations into the much broader context of Grothendieck topologies, see [6].

4. Tight closure as solid closure

We come back to tight closure, and its interpretation in terms of forcing algebras and solid closure.

Theorem 4.1. Let \((R, \mathfrak{m})\) be a local excellent normal domain of positive characteristic and let \(I\) denote an \(\mathfrak{m}\)-primary ideal. Then \(f \in I^*\) if and only if \(H_{\mathfrak{m}}^{\dim R}(A) \neq 0\), where \(A\) denotes the forcing algebra.

Proof. We indicate the proof of the direction that the cohomological property implies the tight closure inclusion. By the assumptions we may assume that \(R\) is complete. Because of \(H_{\mathfrak{m}}^{\dim R}(A) \neq 0\) there exists by Matlis-duality a non-trivial \(R\)-module homomorphism \(\psi : A \to R\) and we may assume \(\psi(1) =: c \neq 0\). In \(A\) we have the equality \(f = \sum_{i=1}^{n} f_i T_i\) and hence

\[
f^q = \sum_{i=1}^{n} f_i^q T_i^q \quad \text{for all } q = p^e.
\]

Applying the \(R\)-homomorphism \(\psi\) to these equations gives

\[
c f^q = \sum_{i=1}^{n} f_i^q \psi(T_i^q),
\]

which is exactly the tight closure condition (the \(\psi(T_i^q)\) are the coefficients in \(R\)). For the other direction see [19].

This theorem provides the bridge between tight closure and cohomological properties of forcing algebras. The first observation is that the property about local cohomology on the right hand side does not refer to positive characteristic. The closure operation defined by this property is called solid closure, and the theorem says that in positive characteristic and under the given further assumptions solid closure and tight closure coincide. The hope was that this would provide a closure operation in all (even mixed) characteristics with similar properties as tight closure. This hope was however destroyed by the following example of Paul Roberts (see [34]).

Example 4.2. (Roberts) Let \(K\) be a field of characteristic zero and consider

\[
A = K[X, Y, Z]/(X^3T_1 + Y^3T_2 + Z^3T_3 - X^2Y^2Z^2).
\]

Then \(H_3^{(X,Y,Z)}(A) \neq 0\). Therefore \(X^2Y^2Z^2 \in (X^3, Y^3, Z^3)_{\text{sc}}\) in the regular ring \(K[X, Y, Z]\). This means that in a three-dimensional regular ring an
ideal needs not be solidly-closed. It is however an important property of tight closure that every regular ring is $F$-regular, namely that every ideal is tightly closed. Hence solid closure is not a good replacement for tight closure (for a variant called parasolid closure with all good properties in equal characteristic zero, see [2]).

Despite this drawback, solid closure provides an important interpretation of tight closure. First of all we have for $d = \dim(R) \geq 2$ (the one-dimensional case is trivial) the identities

$$H^d_m(A) \cong H^d_{mA}(A) \cong H^{d-1}(D(mA), O).$$

This easy observation is quite important. The open subset $D(mA) \subseteq \text{Spec } A$ is exactly the torsor induced by the forcing algebra over the punctured spectrum $D(m) \subseteq \text{Spec } R$. Hence we derive at an important particularity of tight closure: tight closure of primary ideals in a normal excellent local domain depends only on the torsor (or, what is the same, only on the cohomology class of the syzygy bundle). We recall from the last section that this property does not hold for integral closure.

By Theorem 4.1, tight closure can be understood by studying the global sheaf cohomology of the torsor given by a first cohomology class of the syzygy bundle. The forcing algebra provides a geometric model for this torsor. An element $f$ belongs to the tight closure if and only if the cohomological dimension of the torsor $T$ is $d - 1$ (which is the cohomological dimension of $D(m)$), and $f \not\in I^*$ if and only if the cohomological dimension drops. Recall that the cohomological dimension of a scheme $U$ is the largest number $i$ such that $H^i(U, F) \neq 0$ for some (quasi-)coherent sheaf $F$ on $U$. In the quasiaffine case, where $U \subseteq \text{Spec } B$ (as in the case of torsors inside the spectrum of the forcing algebra), one only has to look at the structure sheaf $F = \mathcal{O}$.

In dimension two this means that $f \in I^*$ if and only if the cohomological dimension of the torsor is one, and $f \not\in I^*$ if and only if this is zero. By a theorem of Serre ([13, Theorem III.3.7]) cohomological dimension zero means that $U$ is an affine scheme, i.e., isomorphic as a scheme to the spectrum of a ring (do not confuse the “affine” in affine scheme with the “affine” in affine-linear bundle).

It is in general a difficult question to decide whether a quasiaffine scheme is an affine scheme. Even in the special case of torsors there is no general machinery to answer it. A necessary condition is that the complement has pure codimension one (which is fulfilled in the case of torsors). So far we have not gained any criterion from our geometric interpretation.
5. Tight closure in graded dimension two

From now on we deal with the following situation: \( R \) is a two-dimensional normal standard-graded domain over an algebraically closed field of any characteristic, \( I = (f_1, \ldots, f_n) \) is a homogeneous \( R_+ \)-primary ideal with homogeneous generators of degree \( d_i = \deg(f_i) \). Let \( C = \Proj R \) be the corresponding smooth projective curve. The ideal generators define the homogeneous resolution

\[
0 \longrightarrow \Syz(f_1, \ldots, f_n) \longrightarrow \bigoplus_{i=1}^n R(-d_i) \xrightarrow{f_1, \ldots, f_n} R \longrightarrow R/I \longrightarrow 0,
\]

and the short exact sequence of vector bundles on \( C \)

\[
0 \longrightarrow \Syz(f_1, \ldots, f_n) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_C(-d_i) \xrightarrow{f_1, \ldots, f_n} \mathcal{O}_C \longrightarrow 0.
\]

We also need the \( m \)-twists of this sequence for every \( m \in \mathbb{Z} \),

\[
0 \longrightarrow \Syz(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_C(m-d_i) \xrightarrow{f_1, \ldots, f_n} \mathcal{O}_C(m) \longrightarrow 0.
\]

It follows from this \textit{presenting sequence} by the additivity of rank and degree that the vector bundle \( \Syz(f_1, \ldots, f_n)(m) \) has rank \( n-1 \) and degree

\[
(m(n-1) - \sum_{i=1}^n d_i) \deg C
\]

(where \( \deg C = \deg \mathcal{O}_C(1) \) is the degree of the curve).

A homogeneous element \( f \in R_m = \Gamma(C, \mathcal{O}_C(m)) \) defines again a cohomology class \( c \in H^1(C, \mathcal{O}_C(m)) \) as well as a torsor \( T(c) \to C \). This torsor is a homogeneous version of the torsor induced by the forcing algebra on \( D(m) \subset \Spec R \). This can be made more precise by endowing the forcing algebra \( A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f) \) with a (not necessarily positive) \( \mathbb{Z} \)-grading and taking \( T = D_+(R_+) \subseteq \Proj A \). From this it follows that the affineness of this torsor on \( C \) is decisive for tight closure. The translation of the tight closure problem via forcing algebras into torsors over projective curves has the following advantages:

1. We can work over a smooth projective curve, i.e., we have reduced the dimension of the base and we have removed the singularity.
2. We can work in a projective setting and use intersection theory.
3. We can use the theory of vector bundles, in particular the notion of semistable bundles and their moduli spaces.

We will give a criterion when such a torsor is affine and hence when a homogeneous element belongs to the tight closure of a graded \( R_+ \)-primary ideal. For this we need the following definition.

\textbf{Definition 5.1.} Let \( S \) be a locally free sheaf on a smooth projective curve \( C \). Then \( S \) is called \textit{semistable}, if \( \deg(T)/\rk(T) \leq \deg(S)/\rk(S) \) holds for all subbundles \( T \neq 0 \). In positive characteristic, \( S \) is called \textit{strongly semistable},
if all Frobenius pull-backs $F^e(\mathcal{S})$, $e \geq 0$, are semistable (here $F : C \to C$ denotes the absolute Frobenius morphism).

Note that for the syzygy bundle we have the natural isomorphism (by pulling back the presenting sequence)

$$F^e(\text{Syz}(f_1, \ldots, f_n)) \cong \text{Syz}(f_1^q, \ldots, f_n^q).$$

Therefore the Frobenius pull-back of the cohomology class $\delta(f) \in H^1(C, \text{Syz}(f_1, \ldots, f_n)(m))$ is

$$F^e(\delta(f)) = \delta(f^q) \in H^1(C, \text{Syz}(f_1^q, \ldots, f_n^q)(qm)).$$

The following two results establish an exact numerical degree bound for tight closure under the condition that the syzygy bundle is strongly semistable.

**Theorem 5.2.** Suppose that $\text{Syz}(f_1, \ldots, f_n)$ is strongly semistable. Then we have $R_m \subseteq I^*$ for $m \geq \sum_{i=1}^n d_i/(n-1)$.

**Proof.** Note that the degree condition implies that $\mathcal{S} := \text{Syz}(f_1, \ldots, f_n)(m)$ has non-negative degree. Let $c \in H^1(C, \mathcal{S})$ be any cohomology class (it might be $\delta(f)$ for some $f$ of degree $m$). The pull-back $F^e(c)$ lives in $H^1(C, F^e(\mathcal{S}))$. Let now $k$ be such that $\mathcal{O}_{C}(-k) \otimes \omega_C$ has negative degree, where $\omega_C$ is the canonical sheaf on the curve. Let $z \in \Gamma(C, \mathcal{O}_C(k)) = R_k$, $z \neq 0$. Then $zF^e(c) \in H^1(C, F^e(\mathcal{S}) \otimes \mathcal{O}_C(k))$. However, by degree considerations, these cohomology groups are zero: by Serre duality they are dual to $H^0(C, F^e(\mathcal{S}^\vee) \otimes \mathcal{O}_C(-k) \otimes \omega_C)$, and this bundle is semistable of negative degree, hence it can not have global sections. Because of $zF^e(c) = 0$ it follows that $zf^q$ is in the image of the mapping given by $f_1^q, \ldots, f_n^q$, so $zf^q \in I^q$ and $f \in I^*$. \hfill \Box

**Theorem 5.3.** Suppose that $\text{Syz}(f_1, \ldots, f_n)$ is strongly semistable. Let $m < \sum_{i=1}^n d_i/(n-1)$ and let $f$ be a homogeneous element of degree $m$. Suppose that $f^{p^a} \notin I^{[p^a]}$ for a such that $p^a > gn(n-1)$ (where $g$ is the genus of $C$). Then $f \notin I^*.$

**Proof.** Here the proof works with the torsor $T$ defined by $c = \delta(f)$. The syzygy bundle $\mathcal{S} = \text{Syz}(f_1, \ldots, f_n)(m)$ has now negative degree, hence its dual bundle $\mathcal{F} = \mathcal{S}^\vee$ is an ample vector bundle (as it is strongly semistable of positive degree). The class defines a non-trivial dual extension $0 \to \mathcal{O}_C \to \mathcal{F}' \to \mathcal{F} \to 0$. By the assumption also a certain Frobenius pull-back of this extension is still non-trivial. Hence $\mathcal{F}'$ is also ample and therefore $\mathbb{P}(\mathcal{F}) \subset \mathbb{P}(\mathcal{F}')$ is an ample divisor and its complement $T = \mathbb{P}(\mathcal{F}') - \mathbb{P}(\mathcal{F})$ is affine. Hence $f \notin I^*$. \hfill \Box

It is in general not easy to establish whether a bundle is strongly semistable or not. However, even if we do not know whether the syzygy bundle is strongly semistable, we can work with its strong Harder-Narasimhan filtration. The
Harder Narasimhan filtration of a vector bundle $\mathcal{S}$ on a smooth projective curve is a filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

with $\mathcal{S}_i/\mathcal{S}_{i-1}$ semistable and descending slopes

$$\mu(\mathcal{S}_1) > \mu(\mathcal{S}_2/\mathcal{S}_1) > \ldots > \mu(\mathcal{S}/\mathcal{S}_{t-1}).$$

Since the Frobenius pull-back of a semistable bundle need not be semistable anymore, the Harder-Narasimhan filtration of $F^\ast(\mathcal{S})$ is quite unrelated to the Harder-Narasimhan filtration of $\mathcal{S}$. However, by a result of A. Langer [27, Theorem 2.7], there exists a certain number $e$ such that the quotients in the Harder-Narasimhan filtration of $F^e\ast(\mathcal{S})$ are strongly semistable. Such a filtration is called strong. With a strong Harder-Narasimhan filtration one can now formulate an exact numerical criterion for tight closure inclusion building on Theorems 5.2 and 5.3.

The criterion basically says that a torsor is affine (equivalently, $f \notin I^+$), if and only if the cohomology class is non-zero in some strongly semistable quotient of negative degree of the strong Harder-Narasimhan filtration. One should remark here that even if we start with a syzygy bundle, the bundles in the filtration are no syzygy bundles, hence it is important to develop the theory of torsors of vector bundles in full generality. From this numerical criterion one can deduce an answer to the tantalizing question.

**Theorem 5.4.** Let $K = \mathbb{F}_p$ be the algebraic closure of a finite field and let $R$ be a normal standard-graded $K$-algebra of dimension two. Then $I^+ = I^+$ for every $R_+$-primary homogeneous ideal.

**Proof.** This follows from the numerical criterion for the affineness of torsors mentioned above. The point is that the same criterion holds for the non-existence of projective curves inside the torsor. One reduces to the situation of a strongly semistable bundle $\mathcal{S}$ of degree 0. Every cohomology class of such a bundle defines a non-affine torsor and hence we have to show that there exists a projective curve inside, or equivalently, that the cohomology class can be annihilated by a finite cover of the curve. Here is where the finiteness assumption about the field enters. $\mathcal{S}$ is defined over a finite subfield $\mathbb{F}_q \subseteq K$, and it is represented (or rather, its $\mathcal{S}$-equivalence class) by a point in the moduli space of semistable bundles of that rank and degree 0. The Frobenius pull-backs $F^{e\ast}(\mathcal{S})$ are again semistable (by strong semistability) and they are defined over the same finite field. Because semistable bundles form a bounded family (itself the reason for the existence of the moduli space), there exist only finitely many semistable bundles defined over $\mathbb{F}_q$ of degree zero. Hence there exists a repetition, i.e., there exists $e' > e$ such that we have an isomorphism $F^{e\ast}(\mathcal{S}) \cong F^{e'}(\mathcal{S})$. By a result of H. Lange and U. Stuhler [26] there exists a finite mapping $C' \xrightarrow{\psi} C \xleftarrow{F^e} C$ (with $\psi$ étale) such that the pull-back of the bundle is trivial. Then one is left with the
problem of annihilating a cohomology class $c \in H^1(C, \mathcal{O}_C)$, which is possible using Artin-Schreier theory (or the graded version of K. Smith’s parameter theorem, [36]). □

Remark 5.5. This theorem was extended by G. Dietz for $R_+$-primary ideals which are not necessarily homogeneous [14]. The above proof shows how important the assumption is that the base field is finite or the algebraic closure of a finite field. Indeed, we will see in the last section that the statement is not true when the base field contains transcendental elements. Also some results on Hilbert-Kunz functions depend on the property that the base field is finite.

6. Applications to Hilbert-Kunz theory

The geometric approach to tight closure was also successful in Hilbert-Kunz theory. This theory originates in the work of E. Kunz ([24], [25]) and was largely extended by P. Monsky ([30], [17]).

Let $R$ be a commutative ring of positive characteristic and let $I$ be an ideal which is primary to a maximal ideal. Then all $R/I^{[q]}$, $q = p^e$, have finite length, and the Hilbert-Kunz function of the ideal is defined to be

$$e \mapsto \varphi(e) = \log(R/I^{[p^e]}).$$

Monsky proved the following fundamental theorem of Hilbert-Kunz theory ([30], [22, Theorem 6.7]).

**Theorem 6.1.** The limit

$$\lim_{e \to \infty} \frac{\varphi(e)}{p^e \dim(R)}$$

exists (as a positive real number) and is called the Hilbert-Kunz multiplicity of $I$, denoted by $e_{HK}(I)$.

The Hilbert-Kunz multiplicity of the maximal ideal in a local ring is usually denoted by $e_{HK}(R)$ and is called the Hilbert-Kunz multiplicity of $R$. It is an open question whether this number is always rational. Strong numerical evidence suggests that this is probably not true in dimension $\geq 4$, see [33]. We will deal with the two-dimensional situation in a minute, but first we relate Hilbert-Kunz theory to tight closure (see [22, Theorem 5.4]).

**Theorem 6.2.** Let $R$ be an analytically unramified and formally equidimensional local ring of positive characteristic and let $I$ be an $m$-primary ideal. Let $f \in R$. Then $f \in I^*$ if and only if

$$e_{HK}(I) = e_{HK}((I, f)).$$

This theorem means that the Hilbert-Kunz multiplicity is related to tight closure in the same way as the Hilbert-Samuel multiplicity is related to integral closure.
We restrict now again to the case of an $R_{+}$-primary homogeneous ideal in a standard-graded normal domain $R$ of dimension two over an algebraically closed field $K$ of positive characteristic $p$. In this situation Hilbert-Kunz theory is directly related to global sections of the Frobenius pull-backs of the syzygy bundle on Proj $R$ (see Section 5). We shall see that it is possible to express the Hilbert-Kunz multiplicity in terms of the strong Harder-Narasimhan filtration of this bundle.

For homogeneous ideal generators $f_1, \ldots, f_n$ of degrees $d_1, \ldots, d_n$ we write down again the presenting sequence on $C = \text{Proj} R$,

$$0 \longrightarrow \text{Syz}(f_1, \ldots, f_n) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_C(-d_i) \overset{f_1, \ldots, f_n}{\longrightarrow} \mathcal{O}_C \longrightarrow 0.$$ 

The $m$-twists of the Frobenius pull-backs of this sequence are

$$0 \longrightarrow \text{Syz}(f_1^q, \ldots, f_n^q)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_C(m - qd_i) \overset{f_1^q, \ldots, f_n^q}{\longrightarrow} \mathcal{O}_C(m) \longrightarrow 0.$$ 

The global evaluation of the last short exact sequence is

$$0 \longrightarrow \Gamma(C, \text{Syz}(f_1^q, \ldots, f_n^q)(m)) \longrightarrow \bigoplus_{i=1}^{n} R_{m-qd_i} \overset{f_1^q, \ldots, f_n^q}{\longrightarrow} R_m,$$ 

and the cokernel of the map on the right is

$$R_m / (f_1^q, \ldots, f_n^q) = (R/I^{[q]})_m.$$ 

Because of $R/I^{[q]} = \bigoplus_{m} (R/I^{[q]})_m$, the length of $R/I^{[q]}$ is the sum over the degrees $m$ of the $K$-dimensions of these cokernels. The sum is in fact finite because the ideals $I^{[q]}$ are primary (or because $H^1(C, \text{Syz}(f_1^q, \ldots, f_n^q)(m)) = 0$ for $m \gg 0$), but the bound for the summation grows with $q$. Anyway, we have

$$\dim(R/I^{[q]})_m = \dim(\Gamma(C, \mathcal{O}_C(m))) - \sum_{i=1}^{n} \dim(\Gamma(C, \mathcal{O}_C(m - qd_i))) + \dim(\Gamma(C, \text{Syz}(f_1^q, \ldots, f_n^q)(m))).$$ 

The computation of the dimensions $\dim(\Gamma(C, \mathcal{O}_C(\ell)))$ is easy, hence the problem is to control the global sections of $\text{Syz}(f_1^q, \ldots, f_n^q)(m)$, more precisely, its behavior for large $q$, and its sum over a suitable range of $m$. This behavior is encoded in the strong Harder-Narasimhan filtration of the syzygy bundle. Let $e$ be fixed and large enough such that the Harder-Narasimhan filtration of the pull-back $\mathcal{H} = F^{e*}(\text{Syz}(f_1, \ldots, f_n)) = \text{Syz}(f_1^q, \ldots, f_n^q)$ is strong. Let $\mathcal{H}_j \subseteq \mathcal{H}$, $j = 1, \ldots, t$, be the subsheaves occurring in the Harder-Narasimhan filtration and set

$$\nu_j := \frac{-\mu(\mathcal{H}_j/\mathcal{H}_{j-1})}{p^e \deg(C)} \quad \text{and} \quad r_j = \text{rk}(\mathcal{H}_j/\mathcal{H}_{j-1}).$$ 

Because the Harder-Narasimhan filtration of $\mathcal{H}$ and of all its pull-backs is strong, these numbers are independent of $e$. The following theorem was shown by Brenner and Trivedi independently ([7], [37]).
Theorem 6.3. Let $R$ be a normal two-dimensional standard-graded domain over an algebraically closed field and let $I = (f_1, \ldots, f_n)$ be a homogeneous $R_+\text{-primary}$ ideal, $d_i = \deg(f_i)$. Then the Hilbert-Kunz multiplicity of $I$ is given by the formula

$$e_{HK}(I) = \frac{\deg(C)}{2} \left(\sum_{j=1}^t r_j \nu_j^2 - \sum_{i=1}^n d_i^2\right).$$

In particular, it is a rational number.

We can also say something about the behavior of the Hilbert-Kunz function under the additional condition that the base field is the algebraic closure of a finite field (see [8]).

Theorem 6.4. Let $R$ and $I$ be as before and suppose that the base field is the algebraic closure of a finite field. Then the Hilbert-Kunz function has the form

$$\varphi(e) = e_{HK}(I)p^{2e} + \gamma(e),$$

where $\gamma$ is eventually periodic.

This theorem also shows that here the “linear term” in the Hilbert-Kunz function exists and that it is zero. It was proved in [23] that for normal excellent $R$ the Hilbert-Kunz function looks like

$$e_{HK}q^{\dim(R)} + \beta q^{\dim(R)-1} + \text{smaller terms}.$$  

For possible behavior of the second term in the non-normal case in dimension two see [32]. See also Remark 7.2.

7. Arithmetic and geometric deformations of tight closure

The geometric interpretation of tight closure theory led to a fairly detailed understanding of tight closure in graded dimension two. The next easiest case is to study how tight closure behaves in families of two-dimensional rings, parametrized by a one dimensional ring. Depending on whether the base has mixed characteristic (like $\text{Spec } \mathbb{Z}$) or equal positive characteristic $p$ (like $\text{Spec } K[T] = \mathbb{A}_K^1$) we talk about arithmetic or geometric deformations.

More precisely, let $D$ be a one-dimensional domain and let $S$ be a $D$-standard-graded domain of dimension three, such that for every point $p \in \text{Spec } D$ the fiber rings $S_{\kappa(p)} = S \otimes_D \kappa(p)$ are normal standard-graded domains over $\kappa(p)$ of dimension two. The data $I = (f_1, \ldots, f_n)$ in $S$ and $f \in S$ determine corresponding data in these fiber rings, and the syzygy bundle $\text{Syz}(f_1, \ldots, f_n)$ on $\text{Proj } S \to \text{Spec } D$ determines syzygy bundles on the curves $C_{\kappa(p)} = \text{Proj } S_{\kappa(p)}$. The natural questions here are: how does the property $f \in I^*$ (in $S_{\kappa(p)}$) depend on $p$, how does $e_{HK}(I)$ depend on $p$, how does strong semistability depend on $p$, how does the affineness of torsors depend on $p$?
Semistability itself is an open property and behaves nicely in a family in the sense that if the syzygy bundle is semistable on the curve over the generic point, then it is semistable over almost all closed points. D. Gieseker gave in [16] an example of a collection of bundles such that, depending on the parameter, the $e$th pull-back is semistable, but the $(e+1)$th is not semistable anymore (for every $e$). The problem how strong semistability behaves under arithmetic deformations was explicitly formulated by Y. Miyaoka and by N. Shepherd-Barron ([29], [35]).

In the context of Hilbert-Kunz theory, this question has been studied by P. Monsky ([17], [31]), both in the arithmetic and in the geometric case. Monsky (and Han) gave examples that the Hilbert-Kunz multiplicity may vary in a family.

**Example 7.1.** Let $R_p = \mathbb{Z}/(p)[X,Y,Z]/(X^4 + Y^4 + Z^4)$. Then the Hilbert-Kunz multiplicity of the maximal ideal is

$$e_{HK}(R_p) = \begin{cases} 3 & \text{for } p = \pm 1 \mod 8 \\ 3 + 1/p^2 & \text{for } p = \pm 3 \mod 8 \end{cases}.$$ 

Note that by the theorem on prime numbers in arithmetic progressions there exist infinitely many prime numbers of all these congruence types.

**Remark 7.2.** In the previous example there occur infinitely many different values for $e_{HK}(R_p)$ depending on the characteristic, the limit as $p \to \infty$ is however well defined. Trivedi showed [38] that in the graded two-dimensional situation this limit always exists, and that this limit can be computed by the Harder-Narasimhan filtration of the syzygy bundle in characteristic zero. Brenner showed that one can define, using this Harder-Narasimhan filtration, a Hilbert-Kunz multiplicity directly in characteristic zero, and that this Hilbert-Kunz multiplicity characterizes solid closure [3] in the same way as Hilbert-Kunz multiplicity characterizes tight closure in positive characteristic (Theorem 6.2 above). Combining these results one can say that “solid closure is the limit of tight closure” in graded dimension two, in the sense that $f \in I^e$ in characteristic zero if and only if the Hilbert-Kunz difference $e_{HK}((I,f)) - e_{HK}(I)$ tends to 0 for $p \to \infty$.

It is an open question whether in all dimensions the Hilbert-Kunz multiplicity always has a limit as $p$ goes to infinity, whether this limit, if it exists, has an interpretation in characteristic zero alone (independent of reduction to positive characteristic) and what closure operation it would correspond to. See also [12].

In the geometric case, Monsky gave the following example ([31]).

**Example 7.3.** Let $K = \mathbb{Z}/(2)$ and let

$$S = \mathbb{Z}/(2)[T][X,Y,Z]/(Z^4 + Z^2XY + Z(X^3 + Y^3) + (T + T^2)X^2Y^2).$$
We consider $S$ as an algebra over $\mathbb{Z}/(2)[T]$ ($T$ has degree 0). Then the Hilbert-Kunz multiplicity of the maximal ideal is

$$e_{\text{HK}}(S_{\kappa(p)}) = \begin{cases} 
3 & \text{if } \kappa(p) = K(T) \text{ (generic case)} \\
3 + 1/4^m & \text{if } \kappa(p) = \mathbb{Z}/(2)(t) \text{ is finite over } \mathbb{Z}/(2) \text{ of degree } m.
\end{cases}$$

By the work of Brenner and Trivedi (see Section 6) these examples can be translated immediately into examples where strong semistability behaves weirdly. From the first example we get an example of a vector bundle of rank two over a projective relative curve over $\text{Spec} \mathbb{Z}$ such that the bundle is semistable on the generic curve (in characteristic zero), and is strongly semistable for infinitely many prime reductions, but also not strongly semistable for infinitely many prime reductions.

From the second example we get an example of a vector bundle of rank two over a projective relative curve over the affine line $\mathbb{A}^1_{\mathbb{Z}/(2)}$, such that the bundle is strongly semistable on the generic curve (over the function field $\mathbb{Z}/(2)(T)$), but not strongly semistable for the curve over any finite field (and the degree of the field extension determines which Frobenius pull-back destabilizes).

To get examples where tight closure varies with the base one has to go one step further (in short, weird behavior of Hilbert-Kunz multiplicity is a necessary condition for weird behavior of tight closure). Interesting behavior can only happen for elements of degree $(\sum d_i)/(n-1)$ (the degree bound, see Theorems 5.2 and 5.3).

In [11], Brenner and M. Katzman showed that tight closure does not behave uniformly under an arithmetic deformation, thus answering negatively a question in [19].

**Example 7.4.** Let

$$R = \mathbb{Z}/(p)[X,Y,Z]/(X^7 + Y^7 + Z^7)$$

and $I = (X^4, Y^4, Z^4)$, $f = X^3Y^3$. Then $f \in I^*$ for $p = 3 \mod 7$ and $f \notin I^*$ for $p = 2 \mod 7$ (see [11] Proposition 2.4 and Proposition 3.1). Hence we have infinitely many prime reductions where the element belongs to the tight closure and infinitely many prime reductions where it does not.

**Remark 7.5.** Arithmetic deformations are closely related to the question “what is tight closure over a field of characteristic zero”. The general philosophy is that characteristic zero behavior of tight closure should reflect the behavior of tight closure for almost all primes, after expressing the relevant data over an arithmetic base. By declaring $f \in I^*$, if this holds for almost all primes, one obtains a satisfactory theory of tight closure in characteristic zero with the same impact as in positive characteristic. This is a systematic way to do reduction to positive characteristic (see [22] Appendix 1 and [20]). However, the example above shows that there is not always a uniform behavior in positive characteristic. A consequence is also that
solid closure in characteristic zero is not the same as tight closure (but see Remark 7.2). From the example we can deduce that $f \in I^s$, but $f \notin I^*$ in $\mathbb{Q}[X,Y,Z]/(X^7 + Y^7 + Z^7)$. Hence, the search for a good tight closure operation in characteristic zero remains.

We now look at geometric deformations. They are directly related to the localization problem and to the tantalizing problem which we have mentioned in the introduction.

**Lemma 7.6.** Let $D$ be a one-dimensional domain of finite type over $\mathbb{Z}/(p)$ and let $S$ be a $D$-domain of finite type. Let $f \in S$ and $I \subseteq S$ be an ideal. Suppose that localization holds for $S$. If then $f \in I^*$ in the generic fiber ring $S_{Q(D)}$, then also $f \in I^*$ in $S_{\kappa(p)} = S \otimes_D \kappa(p)$ for almost all closed points $p \in \text{Spec } D$.

**Proof.** The generic fiber ring is the localization of $S$ at the multiplicative system $M = D - \{0\}$ (considered in $S$). So if $f \in I^*$ holds in $S_{Q(D)} = S_M$, and if localization holds, then there exists $h \in M$ such that $hf \in I^*$ in $S$ (the global ring of the deformation). By the persistence of tight closure ([22, Theorem 2.3] applied to $S \rightarrow S_{\kappa(p)}$) it follows that $hf \in I^*$ in $S_{\kappa(p)}$ for all closed points $p \in \text{Spec } D$. But $h$ is a unit in almost all residue class fields $\kappa(p)$, so the result follows. \qed

**Example 7.7.** Let $S = \mathbb{Z}/(2)[T][X,Y,Z]/(Z^4 + Z^2XY + Z(X^3 + Y^3) + (T + T^2)X^2Y^2)$ as in Example 7.3 and let $I = (X^4, Y^4, Z^4)$, $f = Y^3Z^3$ ($X^3Y^3$ would not work). Then $f \in I^*$, as is shown in [13], in the generic fiber ring $S_{\mathbb{Z}/(2)[T]}$, but $f \notin I^*$ in $S_{\kappa(p)}$ for all closed points $p \in \text{Spec } D$. Hence tight closure does not commute with localization.

**Example 7.8.** Let $K = \mathbb{Z}/(2)(T)$ and $R = K[X,Y,Z]/(Z^4 + Z^2XY + Z(X^3 + Y^3) + (T + T^2)X^2Y^2)$. This is the generic fiber ring of the previous example. It is a normal, standard-graded domain of dimension two and it is defined over the function field. In this ring we have $Y^3Z^3 \in (X^4, Y^4, Z^4)^*$, but $Y^3Z^3 \notin (X^4, Y^4, Z^4)^+$. Hence tight closure is not the same as plus closure, not even in dimension two.

### 8. Some open problems

We collect some open questions and problems, together with some comments of what is known and some guesses. We first list problems which are weaker variants of the localization problem.

**Problem 8.1.** Is $F$-regular the same as weakly $F$-regular?

Recall that a ring is called **weakly $F$-regular** if every ideal is tightly closed, and **$F$-regular** if this is true for all localizations. A positive answer would...
have followed from a positive answer to the localization problem. This path is not possible anymore, but there are many positive results on this: it is true in the Gorenstein case, in the graded case ([28]), it is true over an uncountable field (proved by Murthy, see [22, Theorem 12.2]). All this shows that a positive result is likely, at least under some additional assumptions.

Problem 8.2. Does tight closure commute with the localization at one element?

There is no evidence why this should be true. It would be nice to see a counterexample, and it would also be nice to have examples of bad behavior of tight closure under geometric deformations in all characteristics.

Problem 8.3. Suppose $R$ is of finite type over a finite field. Is tight closure the same as plus closure?

This is known in graded dimension two for $R_+$-primary ideals by Theorem 5.4 and the extension for non-homogeneous ideals (but still graded ring) by Dietz (see [14]). To attack this problem one probably needs first to establish new exact criteria of what tight closure is. Even in dimension two, but not graded, the best way to establish results is probably to develop a theory of strongly semistable modules on a local ring.

Can one characterize the rings where tight closure is plus closure? Are rings, where every ideal coincides with its plus closure, $F$-regular?

For a two-dimensional standard-graded domain and the corresponding projective curve, the following problems remain.

Problem 8.4. Let $C$ be a smooth projective curve over a field of positive characteristic, and let $\mathcal{L}$ be an invertible sheaf of degree zero. Let $c \in H^1(C, \mathcal{L})$ be a cohomology class. Does there exist a finite mapping $C' \to C$, $C'$ another projective curve, such that the pull-back annihilates $c$.

This is known for the structure sheaf $\mathcal{O}_C$ and holds in general over (the algebraic closure of) a finite field. It is probably not true over a field with transcendental elements, the heuristic being that otherwise there would be a uniform way to annihilate the class over every finite field (an analogue is that every invertible sheaf of degree zero over a finite field has finite order in $\text{Pic}^0(C)$, but the orders do not have much in common as the field varies, and the order over larger fields might be infinite).

Problem 8.5. Let $R$ be a two-dimensional normal standard-graded domain and let $I$ be an $R_+$-primary homogeneous ideal. Write $\varphi(e) = e_{HK}p^2 + \gamma(e)$. Is $\gamma(e)$ eventually periodic?

By Theorem 6.4 this is true if the base field is finite, but this question is open if the base field contains transcendental elements. How does (the lowest term of) the Hilbert-Kunz function behave under a geometric deformation?
Problem 8.6. Let $C \to \text{Spec } D$ be a relative projective curve over an arithmetic base like $\text{Spec } \mathbb{Z}$, and let $\mathcal{S}$ be a vector bundle over $C$. Suppose that the generic bundle $\mathcal{S}_0$ over the generic curve of characteristic zero is semistable. Is then $\mathcal{S}_p$ over $C_p$ strongly semistable for infinitely many prime numbers $p$?

This question was first asked by Y. Miyaoka (29). Corresponding questions for an arithmetic family of two-dimensional rings are: Does there exist always infinitely many prime numbers where the Hilbert-Kunz multiplicity coincides with the characteristic zero limit? If an element belongs to the solid closure in characteristic zero, does it belong to the tight closure for infinitely many prime reductions? In [3], there is a series of examples where the number of primes with not strongly semistable reduction has an arbitrary small density under the assumption that there exist infinitely many Sophie Germain prime numbers (a prime number $p$ such that also $2p + 1$ is prime).

We come back to arbitrary dimension.

Problem 8.7. Understand tight closure geometrically, say for standard-graded normal domains with an isolated singularity. The same for Hilbert-Kunz theory.

Some progress in this direction has been made in [4] and in [10], but much more has to be done. What is apparent from this work is that positivity properties of the top-dimensional syzygy bundle coming from a resolution are important. A problem is that strong semistability controls global sections and by Serre duality also top-dimensional cohomology, but one problem is to control the intermediate cohomology.

Problem 8.8. Find a good closure operation in equal characteristic zero, with tight closure like properties, with no reduction to positive characteristic.

The notion of parasolid closure gives a first answer to this [2]. However, not much is known about it beside that it fulfills the basic properties one expects from tight closure, and many proofs depend on positive characteristic (though the notion itself does not). Is there a more workable notion?

One should definitely try to understand here several candidates with the help of forcing algebras and the corresponding Grothendieck topologies. A promising approach is to allow the forcing algebras as coverings which do not annihilate (top-dimensional) local cohomology unless it is annihilated by a resolution of singularities.

Is there a closure operation which commutes with localization (this is also not known for characteristic zero tight closure, but probably false)?

Problem 8.9. Find a good closure operation in mixed characteristic and prove the remaining homological conjectures.

In Hilbert-Kunz theory, the following questions are still open.
Problem 8.10. Is the Hilbert-Kunz multiplicity always a rational number? Is it at least an algebraic number?

The answer to the first question is probably no, as the numerical material in [33] suggests. However, this still has to be established.

Problem 8.11. Prove or disprove that the Hilbert-Kunz multiplicity has always a limit as the characteristic tends to $\infty$.

If it has, or in the cases where it has, one should also find a direct interpretation in characteristic zero and study the corresponding closure operation.

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FACHBEREICH FÜR MATHEMATIK UND INFORMATIK, UNIVERSITÄT ONSBRÜCK, ONSBRÜCK, GERMANY

E-mail address: hbrenner@uni-osnabrueck.de