NOTE ON THE PAPER OF FU AND WONG ON STRICTLY PSEUDOCONVEX DOMAINS WITH KÄHLER–EINSTEIN BERGMAN METRICS

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Abstract. It is shown that the Ramadanov conjecture implies the Cheng conjecture. In particular it follows that the Cheng conjecture holds in dimension two.

In this brief note we use our uniformization result from [10, 11] to extend the work of Fu and Wong [7] on the relationship between two long-standing conjectures about the behaviour of the Bergman metric of a strictly pseudoconvex domain in \( C^n \), \( n \geq 2 \).

1. Let \( D \in C^n \) be an arbitrary bounded domain. The Bergman kernel function of \( D \) can be defined by the formula

\[
K_D(z) := \sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(z)},
\]

where \( \{ \varphi_j \}_{j=1}^{\infty} \) is any orthonormal basis of the space \( L^2O(D) \) of square-integrable holomorphic functions in \( D \).

It is a standard result that the function \( \log K_D(z) \) is strictly plurisubharmonic and the positive \((1,1)\)-form

\[
k_D := i\partial\overline{\partial}\log K_D(z)
\]

is invariant with respect to biholomorphic mappings between bounded domains. The Bergman metric on \( D \) is the Kähler metric associated with this Kähler form.

2. Fefferman [6] (see also [2]) established the following deep result on the boundary behaviour of the kernel function of a smoothly bounded strictly pseudoconvex domain \( D \). Let \( \rho \in C^\infty(\overline{D}) \) be a defining function for \( D \). Then there is a decomposition

\[
K_D(z) = \varphi(z)\rho(z)^{-(n+1)} + \psi(z) \log |\rho(z)|
\]

where the functions \( \varphi, \psi \in C^\infty(\overline{D}) \), and \( \varphi \neq 0 \) everywhere on \( \partial D \). Note that the latter property implies that the Bergman metric of a strictly pseudoconvex domain is complete.

Although the kernel function is defined globally, its asymptotic behaviour as \( z \to z_0 \in \partial D \) depends only on the local CR geometry of the boundary at the point \( z_0 \) (see [6]). For instance, the kernel function of the unit ball \( B \subset C^n \) can be explicitly decomposed as

\[
K_B(z) = \frac{n!}{\pi^n} (1 - \|z\|^2)^{-(n+1)}
\]

with identically vanishing logarithmic term. Thus the coefficient \( \psi \) in the logarithmic term of (1) vanishes to infinite order at the boundary for any strictly pseudoconvex domain whose boundary is spherical (i.e., locally CR diffeomorphic to the unit sphere \( \partial B \subset C^n \)).

The Ramadanov conjecture [12] asserts that, conversely, the vanishing condition \( \psi = O(\rho^\infty) \) implies that the boundary of \( D \) is spherical. This conjecture has been proved for domains in \( C^2 \) by Graham and Burns [8] and Boutet de Monvel [1].

3. A classical problem, proposed in different forms by Bergman, Hua, and Yau, asks to describe the domain in terms of the differential-geometric properties of its Bergman metric. For example, a well-known theorem of Lu Qi-Keng [9] states that a bounded domain with complete Bergman metric of constant holomorphic sectional curvature is biholomorphic to the ball.
The Cheng conjecture asserts that the hypotheses of Lu’s theorem can be weakened for a smoothly bounded strictly pseudoconvex domain. Namely, such a domain has to be biholomorphic to the ball if and only if its Bergman metric is Kähler–Einstein.

**Theorem.** The Cheng conjecture in \(\mathbb{C}^n\) follows from the Ramadanov conjecture in \(\mathbb{C}^2\).

Since the Ramadanov conjecture is known to be true in dimension 2, we obtain the following result.

**Corollary.** The Bergman metric of a smoothly bounded strictly pseudoconvex domain \(D \subseteq \mathbb{C}^2\) is Kähler–Einstein if and only if this domain is biholomorphic to the ball.

**Remark.** Fu and Wong proved these results for simply connected domains using a weaker uniformization result of Chern and Ji and stated the general case as an open question.

**Proof of the theorem.** Suppose that the Bergman metric of a strictly pseudoconvex domain \(D\) is Kähler–Einstein. Fu and Wong computed (rather ingeniously) that in this case the logarithmic coefficient \(\psi\) in the decomposition vanishes to infinite order at \(\partial D\). Assuming the Ramadanov conjecture, we conclude that the boundary of \(D\) is spherical. Hence, by the uniformization theorem (see Thm. A.2 and Cor. 3.2) the domain \(D\) is covered by the unit ball.

Since \(D\) is the quotient of the ball by the group of holomorphic deck transformations, there is a natural complete metric of constant holomorphic sectional curvature on \(D\) obtained by taking the quotient of the standard invariant metric on the ball. According to Cheng and Yau, the complete Kähler–Einstein metric on \(D\) is unique up to a constant factor. Hence, the Bergman metric of \(D\) is proportional to the quotient metric and so has constant holomorphic sectional curvature. It follows that \(D\) is biholomorphic to the ball by Lu’s theorem.

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1The formulation in [3] is somewhat vague; the precise statement below is taken from [7].