On the Ergodic Theorem for Non-Linear Wave Propagations

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Abstract: We present a complete study of the Ergodic Theorem for the difficult problem of Non-Linear Wave Propagations through Cylindrical Measures/Path Integrals and the famous Ruelle-Amrein-Geogerscu-Enss (R.A.G.E.) theorem on the characterization of continuous spectrum of self-adjoint operators

1 Introduction

One of the most important phenomenon in numerical studies of the non-linear wave motion especially in the two-dimensional case - is the existence of overwhelming majority of wave motions that wandering over all possible system’s phase space and, given enough time, coming as close as desired (but not entirely coinciding) to any given initial condition ([1]).

This phenomenon is signaling certainly that the famous recurrence theorem of Poincaré is true for infinite dimensional continuum mechanical systems like that one represented by a bounded domain when subject to non-linear vibrations.

A fundamental question appears in the context of this recurrence phenomenon and concerned to the existence of the time average for the associated non-linear wave motion and naturally leading to the concept of a infinite - dimensional invariant measure for the non-linear wave equation ([1]), the mathematical phenomenon subjacente to the Poincaré recurrence theorem.
In this paper, we intend to give applied mathematics arguments for the validity of the famous Ergodic theorem in a class of polynomial non-linear and Lipschitz wave motions. Our approach is fully based on Hilbert Space methods previously used to study Dynamical System’s in Classical Mechanics which by its turn simplifies enormously the task of constructing explicitly the associated invariant measure on certain Sobolev Spaces: the infinite - dimensional space of wave’s motion initial conditions. This study is presented on the main section of this paper, namely section 4.

In section 2, we give a very detailed proof of the RAGE’S theorem, a basic functional analysis rigorous method used in our proposed Hilbert Space generalization of the usual finite - dimensional Ergodic theorem and fully presented in section 3.

In section 5, we complement our studies by analyzing the important case of wave - diffusion under random stirring.

In appendices A and B, we additionally present important technical details related to the section 4 and they contain material as important as those presented in the previous sections of our work.

2 On the detailed mathematical proof of the R.A.G.E. theorem.

In this purely mathematical first section 2 of our study, we intend to present a detailed mathematical proof of the R.A.G.E theorem ([2]) used on the analytical proof of ours of the Ergodic theorem on section 3 for Hamiltonian systems of N-particles.

Let us, thus, start our analysis by considering a self - adjoint operator $\mathcal{L}$ on a Hilbert Space $(H,(,))$ where $H_c(\mathcal{L})$ denotes the associated continuity sub-space obtained from the spectral theorem applied to $\mathcal{L}$. We have the following result:

**Proposition (RAGE theorem)** - Let $\psi \in H_c(\mathcal{L})$ and $\tilde{\psi} \in H$. We have the Ergodic limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \left| \langle \tilde{\psi}, \exp(it\mathcal{L})\psi \rangle \right|^2 dt = 0$$

(1)

**Proof** - In order to show the validity of the above ergodic limit, let us re-write eq.(1) in
terms of the spectral resolution of $\mathcal{L}$, namely.

$$ I = \frac{1}{T} \int_0^T |(\tilde{\psi}, \exp(it\mathcal{L})\psi)|^2 dt = \frac{1}{T} \int_0^T \left\{ \int_{-\infty}^{+\infty} e^{+it\lambda} d\lambda(\tilde{\psi}, E_j(\lambda)\psi) \right\} \times \left\{ \int_{-\infty}^{+\infty} e^{-it\mu} d\mu(\tilde{\psi}, E_j(\mu)\psi) \right\} $$

(2)

Here we have used the usual spectral representation of $\mathcal{L}$

$$ (g, \mathcal{L}h) = \int_{-\infty}^{+\infty} \lambda d\lambda (g, E_j(\lambda)h) $$

(3)

with $g \in H$ and $h \in \text{Dom}(\mathcal{L})$.

Let us remark that the function $\exp(i(\lambda - \mu)t)$ is majorized by the function 1 which, by its turn, is an integrable function on the domain $[0, T] \times \mathbb{R} \times \mathbb{R}$ with the product measure

$$ \frac{dt}{T} \otimes d\lambda(\tilde{\psi}, E_j(\lambda)\psi) \otimes d\mu(\tilde{\psi}, E_j(\mu)\psi) $$

(4)

since

$$ \int_0^T \frac{dt}{T} \otimes d\lambda(\tilde{\psi}, E_j(\lambda)\psi) \otimes d\mu(\tilde{\psi}, E_j(\mu)\psi) = (\tilde{\psi}, \psi)^2 < \infty $$

(5)

At this point, we can safely apply the Fubini theorem for interchange the order of integration in relation to the $t$-variable which leads to the partial result below

$$ I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( \frac{e^{i(\lambda - \mu)t} - 1}{i(\lambda - \mu)t} \right) d\lambda(\tilde{\psi}, E_j(\lambda)\psi) \otimes d\mu(\tilde{\psi}, E_j(\mu)\psi) $$

(6)

Let us consider two cases. Firstly we take $\tilde{\psi} = \psi$. In this case, we have the bound

$$ \left| \frac{e^{i(\lambda - \mu)t} - 1}{i(\lambda - \mu)t} \right| \leq \left| \frac{2 \text{sen}((\lambda - \mu)t)}{(\lambda - \mu)t} \right| \leq 1 $$

(7)

If we restrict ourselves to the case of $\lambda \neq \mu$, a direct application of the Lebesgue convergence theorem on the limit $T \to \infty$ yields that $I = 0$.

On the other hand, in the case of $\lambda = \mu$ we intend now to show that the set $\mu = \lambda$ is a set of zero measure in $\mathbb{R}^2$ in respect with the measure $d\lambda(\psi, E_j(\lambda)\psi) \otimes d\mu(\psi, E_j(\mu)\psi)$.

This can be seen by considering the real function on $\mathbb{R}$

$$ f(\lambda) = (\psi, E_j(\lambda)\psi) $$

(8)
We observe that this function is continuous and non-decreasing since $\psi \in H_c(L)$ and range $E_j(\lambda_1)C$ range $E_j(\lambda_2)$ for $\lambda_1 \leq \lambda_2$. Now, $f(\lambda)$ is an uniform continuous function on the whole real line $R$, since for a given $\varepsilon > 0$, there is a constant $\tilde{\lambda}$ such that
\begin{align*}
(\psi, E_j(-\infty, -\tilde{\lambda})\psi) < \frac{\varepsilon}{2} 
\end{align*}
(9)
\begin{align*}
\|\psi\|^2 - (\psi, E_j(-\infty, \tilde{\lambda})\psi) < \frac{\varepsilon}{2} 
\end{align*}
(10)

Additionally, $f(\lambda)$ is uniform continuous on the closed interval $[-\tilde{\lambda}, \tilde{\lambda}]$ and $f(\lambda)$ can not make variations greater than $\frac{\varepsilon}{2}$ on $(-\infty, -\tilde{\lambda}]$ and $[+\tilde{\lambda}, \infty)$, besides of being a monotonic function on $R$. These arguments show the uniform continuity of $f(\lambda)$ on whole line $R$. Hence we have that for a given $\varepsilon > 0$, exists a $\delta > 0$ such that
\begin{align*}
\left|(\psi, E_j(\lambda')\psi) - (\psi, E_j(\lambda'')\psi)\right| \leq \frac{\varepsilon}{\|\psi\|^2} 
\end{align*}
(11-A)
for
\begin{align*}
|\lambda' - \lambda''| \leq \delta 
\end{align*}
(12)

In particular for $\lambda' = \lambda + \delta$ e $\lambda'' = \lambda - \delta$
\begin{align*}
(\psi, E_j(\lambda + \delta)\psi) - (\psi, E_j(\lambda - \delta)\psi) \leq \frac{\varepsilon}{\|\psi\|^2} 
\end{align*}
(13)

As a consequence, we have the estimate
\begin{align*}
\int_{-\infty}^{+\infty} d\lambda(E_j(\lambda)\psi, \psi) \int_{\lambda-\delta}^{\lambda+\delta} d\mu(\psi, E_j(\mu)\psi) \\
&\leq \|\psi\|^2 \times ((\psi, E_j(\lambda + \delta)\psi) - (\psi E_j(\lambda - \delta)\psi)) \\
&\leq \|\psi\|^2 \times \frac{\varepsilon}{\|\psi\|^2} \leq \varepsilon 
\end{align*}
(14)

Let us note that for each $\varepsilon > 0$, there is a set $D_\delta = \{(\lambda, \mu) \in R^2; |\lambda - \mu| < \delta\}$ wich contains the line $\lambda = \mu$ and has measure less than $\varepsilon$ in relation to the measure $d\lambda(E_j(\lambda)\psi, \psi) \otimes d\mu(E_j(\mu)\psi, \psi)$ as a result of eq.(14). This shows our claim that $I = 0$ in our special case.

In the general case of $\tilde{\psi} \neq \psi$, we remark that solely the orthogonal component on the continuity sub-space $H_c(L)$ has a non - vanishing inner product with $\exp(itL)\psi$. By using now the polarization formulae, we reduce this case to the first analyzed result of $I = 0$. 

4
At this point we arrive at the complete R.A.G.E theorem ([2])

**Theorem** - Let $K$ be a compact operator on $(H, (\cdot, \cdot))$. We have thus the validity of the Ergodic limit

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T \|K \exp(i t \mathcal{L}) \psi\|_H^2 \, dt = 0
$$
with $\psi \in H_c(\mathcal{L})$.

We leave the details of the proof of this result for the reader, since any compact operator is the norm operator limit of finite - dimension operators and one only needs to show that

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T \| \sum_{n=0}^N c_n (e^{i t \mathcal{L}} \psi, e_n) g_n \|_H \, dt = 0
$$
for $c_n$ constants and $\{e_n\}, \{g_n\}$ a finite set of vector of $(H; (\cdot, \cdot))$.

### 3 On the Boltzmann Ergodic Theorem in Classical Mechanics as a result of the R.A.G.E theorem

One of the most important statement in Physics is the famous zeroth law of thermodynamics: “any system approaches an equilibrium state”. In the classical mechanics frameworks, one begins with the formal elements of the theory. Namely, the phase-space $R^{6N}$ associated to a system of $N$-classical particles and the set of Hamilton equations

$$
\dot{p}_i = -\frac{\partial H}{\partial q_i}; \quad \dot{q}_i = \frac{\partial H}{\partial p_i}
$$
where $H(q,p)$ is the energy function.

The above cited thermodynamical equilibrium principle becomes the mathematical statement that for each compact support continuous functions $C_c(R^{6N})$, the famous ergodic limit should holds true ([3]).

$$
\int_{R^{6N}} d^{3N} q(0) d^{3N} p(0) \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T f(q(t); p(t)) \, dt \right\} = \eta(f)
$$
where $\eta(f)$ is a linear functional on $C_c(R^{6N})$ given exactly by the Boltzmann statistical weight and $\{q(t), p(t)\}$ denotes the (global) solutions of the Hamilton equations (12).
We aim at this section point out a simple new mathematical argument of the fundamental eq.(18) by means of Hilbert Space Techniques and the R.A.G.E’S theorem.

Let us begin by introducing for each initial condition \((q(0), p(0))\), a function \(\omega_{q_0, p_0}(t) \equiv (q(t), p(t))\), here \((q(t), p(t))\) is the assumed global unique solution of eq.(17) with prescribed initials conditions. Let \(U_t : L^2(\mathbb{R}^{6N}) \to L^2(\mathbb{R}^{6N})\) be the unitary operator defined by

\[
(U_t f)(q, p) = f(\omega_{q_0, p_0}(t)) \quad (19)
\]

We have the following theorem (the Liouville’s theorem) ([5]).

**Theorem 1.** \(U_t\) is a unitary one-parameters group whose infinitesimal generator is \(-i\bar{L}\), where \(-iL\) is the essential self-adjoint operator acting on \(C_0^\infty(\mathbb{R}^{6N})\) defined by the Poisson bracket.

\[
(Lf)(p, q) = \{f, H\}(q, p) = \sum_{i=1}^{3N} \left( \frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) (q, p) \quad (20)
\]

The basic result we are using to show the validity of the ergodic limit eq. (18) is the famous R.A.G.E’s theorem exposed on section 1.

**Theorem 2.** Let \(\phi \in H_c(-i\bar{L})(L^2(\mathbb{R}^{6N}))\), here \(H_c(-i\bar{L})\) is the continuity sub-space associated to self-adjoint operator \(-i\bar{L}\). For every vector \(\beta \in L^2(\mathbb{R}^{6N})\), we have the result

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left| \langle \beta, U_t \psi \rangle \right|^2 dt = 0, \quad (21)
\]

or equivalently for every \(\psi \in L^2(\mathbb{R}^{6N})\)

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \beta, U_t \psi \rangle dt = \langle \beta, P_{ker(-i\bar{L})} \psi \rangle \quad (22)
\]

where \(P_{ker(-i\bar{L})}\) is the projection operator on the (closed) sub-space \(ker(-i\bar{L})\).

That eq.(22) is equivalent to eq.(21), is a simple consequence of the Schwartz inequality below written

\[
\left| \int_0^T \langle \beta, U_t \psi \rangle dt \right| \leq \left( \int_0^T \left| \langle \beta, U_t \psi \rangle \right|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T 1 dt \right)^{\frac{1}{2}} \quad (23)
\]

or

\[
\left| \frac{1}{T} \int_0^T \langle \beta, U_t \psi \rangle dt \right| \leq \left( \frac{1}{T} \int_0^T \left| \langle \beta, U_t \psi \rangle \right|^2 dt \right)^{\frac{1}{2}} \quad (24)
\]
As a consequence of eq.(23), we can see that the linear functional \( \eta(f) \) of the Ergodic theorem is exactly given by (just consider \( \beta(p,q) \equiv 1 \) on supp of \( \mathbb{P}_{\text{ker}(-i\bar{L})}(\psi) \))

\[
\eta(f) = \int_{\mathbb{R}^{3N}} dq dp \mathbb{P}_{\text{ker}(-i\bar{L})}(f)(q,p).
\]

(25)

By the Riesz’ s theorem applied to \( \eta(f) \), we can re-write (represent) eq.(25) by means of a (kernel)-function \( h_{\eta(H)}(q,p) \), namely

\[
\eta(f) = \int_{\mathbb{R}^{3N}} dq^3N dp^3N p(f(q,p))h_{\eta(H)}(p,q)
\]

(26)

where the function \( h_{\eta(H)}(q,p) \) satisfies the relationship

\[
\{h_{\eta(H)}, L\} = \sum_{i=1}^{3N} \left( \frac{\partial h_{\eta}}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i} \frac{\partial h_{\eta}}{\partial p_i} \right) = 0
\]

(27)

or equivalently \( h_{\eta(H)}(q,p) \) is a “smooth” function of the Hamiltonian function \( H(q,p) \), by imposing the additive Boltzman behavior for \( h_{\eta(H)}(q,p) \) namely, \( h_{\eta(H_1+H_2)}(q,p) = h_{\eta(H_1)}h_{\eta(H_2)} \), one obtains the famous Boltzman weight as the (unique) mathematical output associated to the Ergodic Theorem on Classical Statistical Mechanics in the presence of a thermal reservoir ([4]).

\[
h_{\eta(H)}(q,p) = \exp\{-\beta H(q,p)\} \int d^3N q^3N p \exp\{-\beta H(q,p)\}
\]

(28)

with \( \beta \) a (positive) constant which is identified with the inverse macroscopic temperature of the combined system after evaluating the system internal energy in the equilibrium state. Note that \( \|h_{\eta(H)}\|_{L^2} = 1 \) since \( \|\eta(f)\| = 1 \).

A last remark should be made related to eq.(28). In order to obtain this result one should consider the non-zero value in ergodic limit

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T dt (U_t^2)(q,p)h_{\eta(H)}(q,p) = \langle h_{\eta(H)}, \mathbb{P}_{\text{ker}(-i\bar{L})}(\psi) \rangle
\]

(29)

or by a pointwise argument (for every \( t \))

\[
(U_t^2 h_{\eta(H)}) \in \mathbb{P}_{\text{ker}(-i\bar{L})},
\]

(30)

that is

\[
h_{\eta(H)} \in \mathbb{P}_{\text{ker}(-i\bar{L})} \Leftrightarrow Lh = 0.
\]

(31)
4 On the invariant Ergodic functional measure for some non-linear wave equations

Let us start this section by considering the discretized \((N\text{-particle})\) wave motion Hamiltonian associated to a non-linear polynomial wave equation related to the vibration of a one-dimensional string of length \(2a\)

\[
H(p_i, q_i) = \sum_{i=1}^{N} \left( \frac{2a}{N} \right) \left( \frac{p_i^2}{2} + \frac{1}{2} (q_{i+1} - q_i)^2 + g(q_i)^{2k} \right)
\]  

with the initial one boundary Dirichlet conditions

\[
q_i(0) = q_i \\
p_i(0) = p_i \\
q_i(-a) = q_i(a) = 0
\]  

Here \(k\) denotes a positive integer associated to the non-linearity power and \(g > 0\) the non-linearity positive coupling constant.

Since one can argue that the above Hamiltonian dynamical system possess unique global solutions on the time interval \(t \in [0, \infty)\) one can, thus, straightforwardly write the associated invariant (Ergodic) measure on basis of the Ergodic theorem exposed in section 2 restricted to the configuration space after integrating out the term involving the canonical momenta

\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} dt \ F(q_1(t), \ldots, q_N(t)) = \int_{R^N} F(q_1, \ldots, q_N) d^{(\text{inv})} \mu(q_1, \ldots, q_N)
\]  

here the explicitly expression for the Hamiltonian system invariant measure in configuration space is given by [with \(\beta = 1\)]

\[
d^{(\text{inv})}_i \mu(q_1, \ldots, q_N) = \frac{1}{Z_n^{(0)}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{N} \left( \frac{2a}{N} \right) \left[ (q_{i+1} - q_i)^2/a^2 \right] + \frac{g}{2k} (q_i)^{2k} \right\} \times (dq_1 \cdots dq_N)
\]

where

\[
Z_n^{(0)} = \int_{R^N} d^{(\text{inv})} \mu(q_1, \ldots, q_N)
\]

Now it is a non-trivial result on the theory of integration on infinite dimensional spaces that the cylindrical measures (normalized to the unity !) converges in a weak
(star) topology sense to the well-defined non-linear polynomial Wiener measure at the continuum limit \( N \to \infty \), or \( \sum_{l=1}^{\infty} i \to \int_{-L}^{L} dx \) and \( q_{l}(t) \to U(x,t) \) (see appendix B for the relevant mathematical arguments supporting the mathematical existence of such limit).

As a consequence of eqs. (35) at the continuum limit, we have the infinite-dimensional analogous of Ergodic result for the non-linear polynomial wave equation (with the Boltzman constant equal to unity — see eq. (28) — section 3 of this study)

\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} F(U(x,t)) \, dt = \int_{C^\alpha(-a,a)} d\text{Wiener} \mu[U^{(0)}(x)] \times F(U^{(0)}(x)) \exp \left\{ -\frac{g}{2k} \int_{-a}^{a} dx (U^{(0)}(x))^{2k} \right\}
\]

where the wave field \( U(x,t) \in L^\infty([0,\infty),L^{2k-1}((-a,a))) \) satisfies in a generalized weak sense the non-linear wave equation below (see appendix A).

\[
\frac{\partial^2 U(x,t)}{\partial t^2} = \frac{\partial^2 U(x,t)}{\partial x^2} + g(U(x,t))^{2k-1}
\]

\[
U(x,0) = U^{(0)}(x) \in H^1((-a,a))
\]

\[
U_t(x,0) = 0
\]

\[
U(-a,t) = U(a,t) = 0
\]

\( d\text{Wiener} \mu[U^{(0)}(x)] \) denotes the Wiener measure over the uni-dimensional Brownian field end-points trajectories \( \{U^{(0)}(x)| \text{ with } U^{(0)}(-a) = U^{(0)}(a) = 0 \} \) and \( F(U^{(0)}(x)) \) denotes any Wiener-integrable functional.

Note that \( C^\alpha(-a,a) \) denotes the Banach space of the \( \alpha \)-Holder continuous function (for \( \alpha \leq \frac{1}{2} \)) which is obviously the support of the Wiener measure \( d\text{Wiener} \mu[U^{(0)}(x)] \).

It is worth remark the normalization factor \( Z \) of the non-linear Wiener measure implicitly used on eq. (36).

\[
\frac{1}{Z} \int_{C^\alpha(-a,a)} d\text{Wiener} \mu[U^{(0)}(x)] \exp \left\{ -\frac{g}{2k} \int_{-a}^{a} dx (U^{(0)}(x))^{2k} \right\} = 1
\]

Let us pass to the important case of existence of a dissipation term on the wave equation problem eq. (37) - eq. (40) with \( \nu \) the positive viscosity parameter and leading straightforwardly to solutions on the whole time propagation interval \([0,\infty)\)

\[
\frac{\partial^2 U(x,t)}{\partial t^2} = \frac{\partial^2 U(x,t)}{\partial x^2} - \nu \frac{\partial U(x,t)}{\partial t} + g(U(x,t))^{2k-1}
\]
It is a simple observation that there is a bijective correspondence between the solution of eq.(42) with the Klein-Gordon like wave equation for the re-scaled wave field \( \beta(x,t) = e^{\frac{\nu^2}{4}t}U(x,t) \), namely

\[
\frac{\partial^2 \beta(x,t)}{\partial t^2} - \frac{\partial^2 \beta(x,t)}{\partial x^2} = \left( \frac{\nu^2}{4} \right) \beta(x,t) + g \ e^{-\frac{(2k-1)\nu t}{2}}(\beta(x,t))^{2k-1}
\]

(45)

\[
\beta(x,0) = U^{(0)}(x) = H^1((-a,a))
\]

(46)

\[
\beta_t(x,0) = U^{(1)}(x) - \frac{\nu}{2}U^{(0)}(x) = 0
\]

(47)

\[
\beta(-a,t) = \beta(a,t) = 0
\]

(48)

As a result we have the analogous of the Ergodic theorem applied to eq.(45) and a sort of Caldirola-Kanai action is obtained ([5]), a new result of ours in this non-linear dissipative case (with the path-integral identification \( x \rightarrow \sigma; U^{(0)}(x) = X(\sigma) \))

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \ F(e^{\frac{\nu^2}{4}t}U(x,t))
\]

\[
= \int_{C^2_{\gamma}(-a,a)} d^{\text{Wiener}} \mu[X(\sigma)] \ F(X(\sigma))
\]

\[
\times \exp \left\{ -\frac{g}{2k} \int_{-a}^a d\sigma \ e^{-\frac{(2k-1)\nu \sigma}{2}} \times (X(\sigma))^{2k} \right\}
\]

\[
\times \exp \left\{ + \left( \frac{\nu^2}{4} \right) \int_{-a}^a d\sigma \ (X(\sigma))^2 \right\}
\]

(49)

Concerning the higher-dimensional case, let us consider \( A \) a strongly positive elliptic operator of order \( 2m \) associated to the free vibration of a domain \( \Omega \) in a general space \( R^n \), and satisfying the Garding coerciviness condition

\[
A = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha_x (a_{\alpha \beta}(x)) D^\beta_x
\]

(50)

with the Sobolev spaces operator’s domain

\[
D(A) = H^{2m}(\Omega) \cap H^m_0(\Omega).
\]

(51)
Note that there exists a constant $C_0(\Omega) > 0$ ($C_0(\Omega) \to 0$ if $\text{vol}(\Omega) \to \infty$) such that the Garding coerciviness condition holds true on $D(A)$

$$\text{Real } (AU, U)_{L^2(\Omega)} \geq C_0(\Omega) \|f\|_{H^{2m}(\Omega)}$$  \hspace{1cm} (52)

We can thus associate a Lipschitz non-linear external vibration on the domain $\Omega$ as governed by the wave equation below.

$$\frac{\partial^2 U(x, t)}{\partial^2 t} = (AU)(x, t) + G'(U(x, t))$$  \hspace{1cm} (53)

$$U(x, 0) = U^{(0)}(x) \in H^{2m}(\Omega) \cap H_0^{m}(\Omega)$$  \hspace{1cm} (54)

$$U_t(x, 0) = g(x) \in L^2(\Omega)$$  \hspace{1cm} (55)

Here $G(z)$ denotes an differentiable Lipschitzian function, like $G(z) = A \sin(Bz), e^{-\gamma z}$, etc... and thus, leading to the uniqueness and existence of global weak solutions on $C([0, \infty), H^{2m}(\Omega) \cap H_0^{m}(\Omega))$ by the Fixed point theorem ([6]).

Analogously to the discretization / $N$-particle technique of the 1+1 case, one obtains as a mathematical candidate for the Ergodic invarainte measure, the following rigorously measure on the Hilbert Space of functions $H^{2m}(\Omega)$ (with $\beta = 1$)

$$d^{(\text{inv})}\mu(\phi(x)) = d_{A\mu}(\phi(x)) \times e^{-\frac{1}{2} \int_{\Omega} d^\nu x G(\phi(x))}$$  \hspace{1cm} (56)

Here $d_{A\mu}(\phi(x))$ denotes the Gaussian measure generated by the quadratic form associated to the operator $A$ on its domain $D(A)$ for real initial conditions functions $\phi(x)$.

$$\int_{H^{2m}(\Omega)} d_{A\mu}(\phi(x))(\phi(x_1)\phi(x_2)) = (A^{-1})(x_1, x_2)$$  \hspace{1cm} (57)

Note that the Green function of the operator $A$ belongs to the trace class operators on $H^{2m}(\Omega)$, which results on eq.(57) through the application of the Minlo’s theorem ([7]).

At this point, we remark that $d_{\text{inv}}\mu(\phi(x))$ defines a mathematical rigorous Radon measure in $H^{2m}(\Omega)$ since the function (functional)

$$F(\phi(x)) = \exp \left\{-\frac{1}{2} \int_{\Omega} d^\nu x G(\phi(x)) \right\}$$  \hspace{1cm} (58)

belongs to $L^1(H^{2m}(\Omega), d_{A\mu}(\phi(x)))$ due to the Lipschitz property of the non-linearity $G(z)$, i.e. there is constants $c^+$ and $C^-$, such that

$$c^- \phi(x) \leq G(\phi(x)) \leq c^+ \phi(x)$$  \hspace{1cm} (59)
As a consequence, we have the obvious estimate below
\[ e^{-c^{+}} \int_{\Omega} d^{n}x \phi(x) \leq e^{-\frac{1}{2}c^{+}} \int_{\Omega} d^{n}x G(\phi(x)) \leq e^{-c^{-}} \int_{\Omega} d^{n}x \phi(x) \]  
(60)
so by the Lebesgue comparasion theorem, we get the functional integrability of the non-linearity term of the ergodic invariant measure eq.(56):
\[ \left| \int_{H^{2m}(\Omega)} dA_{\mu}(\phi(x)) e^{-\frac{1}{2}c^{-}} \int_{\Omega} d^{n}x G(\phi(x)) \right| \leq e^{+\frac{1}{2}(c^{-}+c^{+})^{2} Tr_{H^{2m}(\Omega)}(A^{-1})} < \infty \]  
(61-a)
and leading thus to the mathematical validity of eq.(56).

At this point it is worth remark that one can easily generalize the type of our allowed Lipschitz non-linearity for those of the following form
\[ \left( \int_{\Omega} d^{n}x G(\phi(x)) \right) \geq \left\{ + \int_{\Omega} d^{n}x \frac{\gamma(x)}{2} (\phi)^{2}(x) + \int_{\Omega} d^{n}x C(x) \phi(x) \right\} \]  
(61-b)
with \( \gamma(x) \) and \( C(x) \) real positive \( L^{\infty}(\Omega) \) functions.

This claim is a result of the straightforward Gaussian - infinite - dimensional integration below
\[ \left| \int_{H^{2m}(\Omega)} dA_{\mu}(\phi) \exp \left\{ - \int_{\Omega} d^{n}x \frac{\gamma(x)}{2} (\phi \phi)(x) + \int_{\Omega} d^{n}x C(x) \bar{\phi}(x) \right\} \right| \leq \left| \det^{-\frac{1}{2}} \left[ 1 + \int_{\Omega} d^{n}y A^{-1}(x-y) \cdot \frac{\gamma(y)}{2} \right] \right| \exp \left\{ + \frac{1}{2} \int_{\Omega} d^{n}x \int_{\Omega} d^{n}y C(x) \cdot A^{-1}(x,y) C(y) \right\} \]  
(61-c)

In the usual (Euclidean) Feynman path-integral formal notation, the Ergodic theorem takes the physicist’s form after re-introducing the reservoir temperature parameter \( \beta = 1/kT \).
\[ \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} F(U(x,t)) = \frac{1}{Z} \int_{H^{2m}(\Omega)} D^{F} [ \varphi(x) ] e^{-\frac{1}{2}c^{+} \int_{\Omega} d^{n}x (\varphi(x) \cdot (A\varphi(x)))} \times F(\varphi(x)) \times e^{-\beta \int_{\Omega} d^{n}x G(\varphi(x))} \]  
(62)
with the normalization factor
\[ Z = \int_{H^{2m}(\Omega)} D^{F} [ \varphi(x) ] e^{\frac{1}{2}c^{-} \int_{\Omega} \varphi(x)(A\varphi(x)) d^{n}x} e^{-\beta \int_{\Omega} d^{n}x G(\varphi(x))} \]  
(63)
The functionals $F(\varphi(x))$ are objects belonging to the space $L^1(H^{2m}(\Omega), d_A(\varphi(x))) \subset C(H^{2m}(\Omega), R)$.

At this point let us remark that by a direct application of the mean value theorem there is a sequence of growing times $\{t_n\}_{n \in \mathbb{Z}}$ (with $t_n \to \infty$) such that

$$\lim_{n \to \infty} F(\varphi(x, t_n)) = \frac{1}{2} \int_{H^{2m}(\Omega)} DF[\varphi(x)] e^{-\frac{1}{2} \beta \int_{\Omega} d^x x (\varphi \cdot A \varphi)(x)} \times F(\varphi(x)) e^{-\beta \int_{\Omega} d^x x G(\varphi(x))}$$

It is worth recall that the (positive) parameter $\beta$ is not determined directly from the parameters of the mechanical underlyng system and can be thought as a remnant of the initial conditions “averaged out” by the Ergodic integral (see section 3) and sometimes related to the presence of a intrinsic dissipation mechanism involved on the Ergodic - time average process responsible for the extensivity of the system’s thermodynamics (see eq.(28) - section 3). For instance, at the limit of vanishing temperature (absolute Newtonian-Boltzman-Maxwell zero temperatire $\beta \to \infty$ and somewhat related to the so called ergodic mixing microcomonomial - Ensemble ), on can see the appearance of an “attractor” on the space $H^{2m}(\Omega) \cap H^0_{\Omega}$ and exactly given by the stationary strong unique solution of the wave equation eq.(53) since, we always have an soluble Elliptic problem on $H^{2m}(\Omega) \cap H^0_{\Omega}$ by means of the Lax-Milgram-Gelfand theorem ([6]).

$$(A^* \varphi)(x) = C'(\varphi^*(x))$$

$$\varphi^*(x) \big|_{\partial \Omega} = 0$$ (65)

(66)

note that the stationary strong solution $\varphi^*(x)$ is not a real attractor in the sense of the theory of Dynamical systems, since we can only say that there is a specific sequence of growing $\{t_n\}_{n \in \mathbb{Z}}$, with $t_n \to \infty$, such that at the leading asymptotic limit $\beta \to \infty$ we have the time-infinite limit for every solution of eqs.(65) - (66).

$$\lim_{t_n \to \infty} F(\varphi(x, t_n)) = F(\varphi^*(x))$$ (67)

A (somewhat) formal path-integral calculation of the next term on the result eq.(41) for $F(z)$ being smooth real-valued functions can be implemented by means of the usual path-integrals Saddle-point techniques applied to the (normalized) functional integral.
\[
\lim_{t_n \to \infty} F(\varphi(x, t_n)) = F(\varphi^*(x))
\]
\[
+ \left\{ \frac{1}{\beta} \frac{\delta^2 F}{\delta^2 \varphi} (\varphi^*(x)) \times \frac{d}{d\alpha} \left[ \ln \det \left[ 1 + A^{-1} \left( \frac{\delta^2 G}{\delta \varphi} (\varphi^*) + \alpha \right) \right] \right] \bigg|_{\alpha=0} \right\}
+ O(\beta^{-2})
\]
\[
= F(\varphi^*(x)) + \frac{1}{\beta} \left\{ \frac{\delta^2 F}{\delta^2 \varphi} (\varphi^*(x)) \times \text{Tr}_{H^{2m}(\Omega)} \left( \left[ 1 + A^{-1} \left( \frac{\delta^2 G}{\delta \varphi} (\varphi^*) \right) \right]^{-1} \right) \right\} + O(\beta^{-2})
\]
\[
= F(\varphi^*(x)) + \frac{1}{\beta} \left\{ \frac{\delta^2 F}{\delta^2 \varphi} (\varphi^*(x)) \times \sum_{n=0}^{\infty} \left[ (-1)^n \text{Tr}_{H^{2m}(\Omega)} \left( A^{-1} \left( \frac{\delta^2 G}{\delta \varphi} (\varphi^*) \right) \right) \right]^n \right\} + O(\beta^{-2})
\]
\[
(68)
\]
Another important remark to be made is related to the existence of time-independent simple constraints on the discretized Hamiltonian eq.(32), namely
\[
H^{(a)} (q_1, \ldots, q_N) = 0 \quad ; \quad a = 1, \ldots, n
\]
\[
(70)
\]
In this case, one should consider that the continuum version of eq.(70) leads to (strongly) continuous functionals on \(H^{2m}(\Omega) \cap H^m_0(\Omega)\)
\[
H^{(a)} (\phi(x, t)) = 0 \quad ; \quad a = 1, \ldots, n
\]
\[
(71)
\]
They have a direct consequence of restricting the wave motion to a closed non-linear manifold of the original vector Hilbert Space \(H^{2m}(\Omega) \cap H^m_0(\Omega)\).

Now it is straightforward to apply the Ergodic theorem on the effective mechanical configuration space and obtain, thus, as the Ergodic invariant measure the constraint path-integral.
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \ F(\phi(x, t))
\]
\[
= \frac{1}{Z} \int_{H^{2m}(\Omega)} D^F [\varphi(x)] e^{-\frac{1}{2} \beta \int_\Omega d^x \left( \varphi(x) \cdot A \varphi(x) \right)} e^{-\beta \int_\Omega d^x G(\varphi(x))}
\]
\[
\times F(\varphi(x)) \times \left( \prod_{a=1}^n \delta^{(F)}(H^{(a)}(\varphi(x))) \right)
\]
\[
(72)
\]
where \(\delta^{(F)}(\cdot)\) denotes the usual formal delta - functionals.
5 An Ergodic theorem in Banach Spaces and Applications to Stochastic-Langevin Dynamical Systems

In this complementary section 5, we intend to present our approach to study long-time / ergodic behavior of infinite-dimensional dynamical systems by analyzing the somewhat formal diffusion equation with polynomial terms and driven by a white noise stirring ([4]).

In order to implement such studies, let us present the author’s generalization of the R.A.G.E theorem for a contraction self-adjoint semi-group $T(t)$ on a Banach space $X$. We have, thus, the following theorem.

**Theorem 3.** Let $f \in X$. We have the ergodic generalized theorem

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt (e^{-tA} f) = \mathbb{P}_{ker(A)}(f)$$

where $A$ is the infinitesimal generator of $T(t)$.

Let us sketch the proof of the above claimed theorem of ours.

As a first step, one should consider eq.(73) re-written in terms of the “resolvent operator of $A$” by means of a Laplace Transform (The Hile-Yosida - Dunford Spectral Calculus)

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \left\{ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dz e^{zt}((z + A)^{-1}f) \right\}.$$  \hspace{1cm} (74)

Now it is straightforward to apply the Fubbini theorem to exchange the order of integrations $(dt, dz)$ in eq.(74) and get, thus, the result

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt (e^{-tA} f) = \lim_{z \to 0^-} +((z + A)^{-1} f).$$ \hspace{1cm} (75)

The $z \to 0^-$ limit of the integral of eq.(75) (since Real $(z) \subset \rho(A) \subset (-\infty, 0)$) can be evaluated by means of saddle point techniques applied to Laplace’s transforms. We have the following result

$$\lim_{z \to 0^-} ((z + A)^{-1} f)$$

$$= \lim_{z \to 0^-} \int_0^\infty e^{-zt} (e^{-tA} f) \ dt$$

$$= \lim_{t \to \infty} (e^{-tA} f) = \mathbb{P}_{ker(A)}(f).$$  \hspace{1cm} (76)
Let us apply the above theorem to the Langevin Equation. Let us consider the Fokker-Planck equation associated to the following Langevin Equation

\[
\frac{dq^i}{dt}(t) = -\frac{\partial V}{\partial q_i}(q^i(t)) + \eta^i(t)
\]

(77)

where \( \{\eta^i(t)\} \) denotes a white-noise stochastic time process representing the thermal coupling of our mechanical system with a thermal reservoir at temperature \( T \). Its two point function is given by the “Fluctuation-Dissipation” theorem

\[
\langle \eta^i(t)\eta^j(t') \rangle = kT \delta(t - t').
\]

(78)

The associated Fokker-Planck equation associated do eq.(74) has the following explicitly form

\[
\frac{\partial P}{\partial t}(q^i, \bar{q}^i; t) = +kT \Delta_{q_i} P(q^i, \bar{q}^i; t) + \nabla_{q_i} \nabla_{q_i} V \cdot P(q^i, \bar{q}^i; t)
\]

(79)

\[
\lim_{t \to 0^+} P(q^i, \bar{q}^i; t) = \delta^{(N)}(q^i - \bar{q}^i).
\]

(80)

By noting that we can associate a contractive semi-group to the initial value problem eq.(80) in the Banach Space \( L^1(R^{3N}) \), namely:

\[
\int P(q^i, \bar{q}^i; t) f(\bar{q}^i) d\bar{q}^i = (e^{-tA} f).
\]

(81)

Here the closed positive accretive operator \( A \) is given explicitly by

\[
-A(\cdot) = kT \Delta_{q_i}(\cdot) + \nabla_{q_i}[(\nabla_{q_i} V)(\cdot)]
\]

(82)

and acts firstly on \( X = C_\infty(R^{3N}) \). It is instructive to point out that the perturbation accretive operator \( B = \nabla_{q_i} \cdot [(\nabla_{q_i} V)(\cdot)] \) on \( C_\infty(R^{3N}) \) with \( V(q^i) \in C_\infty(R^{3N}) \) is such that it satisfies the estimate on \( S(R^{3N}) \) : \( \|Bf\|_{S(R^{3N})} \leq a\|f\|_{S(R^{3N})} + b\|f\|_{S(R^{3N})} \) with \( a > 1 \) and for some \( b \). As a consequence \( A(\cdot) = kT \Delta_{q_i}(\cdot) + (B(\cdot)) \) generates a contractive semi-group on \( C_\infty(R^{3N}) \) or by an extension argument on the whole \( L^1(R^{3N}) \) since the \( L^1 \)-closure of \( C_\infty(R^{3N}) \) is the Banach space \( L^1(R^{3N}) \).

At this point we may apply our Theorem 3 to obtain the Langevin-Brownian Ergodic theorem applied to our Fokker-Planck equation

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \left( \int_{R^{3N}} dq(e^{-tA} f)(q) \right) = \int_{R^{3N}} dq_{\text{ker}(A)} f(q) = \int_{R^{N}} d^N q^i \int d^N \bar{q}^i f(\bar{q}^i) \mu_{eq}(\bar{q}^i)
\]

(83)
where the equilibrium probability distribution is given explicitly by the \textbf{unique normalizable} element of the closed sub-space \( \ker(A) \).

\begin{equation}
O = +(kT)\Delta_{q_i}\mathbb{P}^{eq}(\bar{q}^i) + \nabla_{\bar{q}^i}[(\nabla_{\bar{q}^i}V)\mathbb{P}(\bar{q}^i)]
\end{equation}

or exactly, we have the Boltzmann’s weight for our equilibrium Langevin-Brownian probability distribution

\begin{equation}
\mathbb{P}^{eq}(\bar{q}^i) = \exp\left\{-\frac{1}{kT}(V(\bar{q}^i))\right\}.
\end{equation}

For the general Langevin equation in the complete phase space \( \{q_i, p_i\} \) as in the bulk of this note, one should re-obtains the complete Boltzman statistical weight as the equilibrium ergodic probability distribution.

\begin{equation}
\mathbb{P}^{eq}(\bar{q}^i, \bar{p}^i) = \frac{1}{Z} \exp\left\{-\frac{1}{kT}\left[\sum_{l=1}^{N} \frac{1}{2}(\bar{p}^i)^2 + V(\bar{q}^-)\right]\right\}
\end{equation}

with the normalization factor

\begin{equation}
Z = \int_{\mathbb{R}^n} d\bar{p}^i \int_{\mathbb{R}^n} d\bar{q}^i \mathbb{P}^{eq}(\bar{q}^i, \bar{p}^i)
\end{equation}

Let us apply the above exposed result for the following non-linear (polinomial) stochastic diffusion equation on a one-dimensional domain \( \Omega = (-a, a) \) with all positive coefficients \( \{\lambda_j\} \) an the non-linear polinomial term

\begin{equation}
\frac{\partial U(x,t)}{\partial t} = + \frac{1}{2} \frac{d^2 U(x,t)}{d^2 x} - \left( \sum_{j=0, j=\text{odd}}^{2k-1} \lambda_j U^j(x,t) \right) + \eta(x,t)
\end{equation}

\begin{align}
U(x,t)\big|_{\partial \Omega} &= 0 \\
U(x,0) &= f(x) \in L^2(\Omega).
\end{align}

Here \( \{\eta(x,t)\} \) are samplings of a white-noise stirring.

In this case, one can show by standard techniques that the global solution \( U(x,t) \in C([0, \infty), L^2(-a, a)) \) and by using the discreticized approach of section 4, the invariant Ergodic measure associated to the non-linear diffusion equation is given by the mathematically functional equilibrium - Langevin Brownian probability distribution

\begin{equation}
\mathbb{P}^{eq}(U(x)) = \exp\left\{-\int_{-a}^{a} dx \left( \sum_{j=0}^{2k} \frac{\lambda_j}{j} U^j(x) \right) \right\} \times d_{\frac{1}{2}} \frac{d^2}{d^2 x} \mu(U(x))
\end{equation}
Here $d \frac{1}{2} \frac{d^2}{dx^2} \mu(U(x))$ denotes the Gaussian functional measure (normalized to unity) on the Sobolev Space $H^1(-a,a)$ space. In the Feynman path-integral notation (with $\beta = 1$)

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \ F(U(x,t))$$

$$= \int_{H^1(-a,a)} D^F[U(x)] e^{-\frac{1}{2} \int_a^a \frac{d^2}{dx^2}(x)} dU$$

$$\times F[U(x)] \quad (92)$$

Note that eq.(91) defines a rigorous mathematical object since it does not need to be renormalized from a calculational point of view on the Sobolev space $H_1(-a,a)$ ([10]) as its non-linear exponential term is bounded by the unity (Lebesgue theorem). It is worth point out that for non-linearities of Lipschitz type on general domains $\Omega CR^\nu$, one can see that the support of the Gaussian measure is readily $H^2(\Omega)$ and the equilibrium Langevin - Brownian probability distribution

$$P^{eq}(U(x)) = \exp \left\{ - \int_\Omega d^Dx \ G(U(x)) \right\} \times d(-\frac{1}{2}\Delta)\mu(U(x)) \quad (93)$$

is a perfect well-defined Randon measure on $H^2(\Omega)$ by the same Lebesgue convergence theorem.

6 The existence and uniqueness results for some non-linear wave motions in 2D

In this technical section 6, we give an argument for the global existence and uniqueness solution of the Hamiltonian motion equations associated firstly to eq.(32) - section 3 and by secondly to eq.(53) - section 3. Related to the two-dimensional case, let us equivalently show the weak existence and uniqueness of the associated continuum non-linear polynomial wave equation in the domain $(-a,a) \times R^+.$

$$\frac{\partial^2 U(x,t)}{\partial t^2} - \frac{\partial^2 U(x,t)}{\partial x^2} + g(U(x,t))^{2k-1} = 0 \quad (94)$$
\[ U(-a, t) = U(a, t) = 0 \]
\[ U(x, 0) = U_0(x) \in H^1([-a,a]) \] (95)

\[ U_t(x, 0) = U_1(x) \in L^2([-a,a]) \] (96)

Let us consider the Galerkin approximants functions to eq.(95) - eq.(96) as given below

\[ \bar{U}_n(t) \equiv \sum_{\ell=1}^{n} U_\ell(t) \text{ sen} \left( \frac{\ell \pi}{a} x \right) \] (97)

Since there is a \( \gamma_0(a) \) positive such that

\[ \left( -\frac{d^2}{dx^2} \bar{U}_n, \bar{U}_n \right)_{L^2([-a,a])} \geq \gamma_0(a) \left( \bar{U}_n, \bar{U}_n \right)_{H^1([-a,a])} \] (98)

we have the a priori estimate for any \( t \)

\[ 0 \leq \varphi(t) \leq \varphi(0) \] (99)

with

\[ \varphi(t) = \frac{1}{2} \| \bar{U}_n(t) \|_{L^2}^2 + \gamma_0(a) \| \bar{U}_n \|_{H^1}^2 \]
\[ + \frac{1}{2k} \| \bar{U}_n \|_{L^2}^{2k} \] (100)

As a consequence of the bound eq.(100), we get the bounds for any given \( T \) (with \( A_i \) constants)

\[ \sup_{0 \leq t \leq T} \text{ess} \| \bar{U}_n \|_{H^1(-a,a)}^2 \leq A_1 \] (101)
\[ \sup_{0 \leq t \leq T} \text{ess} \| \bar{U}_n \|_{L^2(-a,a)}^2 \leq A_2 \] (102)
\[ \sup_{0 \leq t \leq T} \text{ess} \| \bar{U}_n \|_{L^{2k}(-a,a)}^{2k} \leq A_3 \] (103)

By usual functional - analytical theorems on weak-compactness on Banach-Hilbert Spaces, one obtains that there is a sub-sequence \( \bar{U}_n(t) \) such that for any finite \( T \)

\[ \bar{U}_n(t) \overset{\text{weak} - \text{star}}{\longrightarrow} \bar{U}(t) \quad \text{in} \quad L^\infty([0,T], H^1(-a,a)) \] (104)
\[
\begin{align*}
U_n(t)(\text{weak} - \text{star}) \overline{\nu}(t) & \quad \text{in} \quad L^\infty([0, T], L^2(-a, a)) \quad (105) \\
U_n(t)(\text{weak} - \text{star}) \overline{\rho}(t) & \quad \text{in} \quad L^\infty([0, T], L^{2k}(-a, a)) \quad (106)
\end{align*}
\]

At this point we observe that for any \( p > 1 \) (with \( \tilde{A}_i \) constants) and \( T < \infty \) we have the relationship below:

\[
\int_0^T \| \dot{U}_n \|^p_{L^2(-a, a)} dt \leq T (A_1)^{\frac{p}{2}} \quad (107)
\]

\[
\int_0^T \| \ddot{U}_n \|^p_{L^{2k}(-a, a)} dt \leq T (A_2)^{\frac{p}{2}} \quad (108)
\]

since we have the continuous injection below:

\[
H^1(-a, a) \hookrightarrow L^2(-a, a) \quad (109)
\]

\[
L^{2k}(-a, a) \hookrightarrow L^2(-a, a) \quad (110)
\]

As a consequence of the Aubin-Lion theorem ([6]), one obtains straightforwardly the strong convergence on \( L^p((0, T), L^2(-a, a)) \) together with the almost everywhere pointwise equality among the solutions candidate

\[
\dot{U}_n \rightarrow \dot{U}(t) = \int_0^t \nu(s) ds \quad (111)
\]

By the H"older inequality applied to the pair \((q, k)\):

\[
\| U_n - \overline{U} \|_{L^q(-a, a)} \leq \| U_n - \overline{U} \|_{L^{2k}(-a, a)}^{1-\theta} \| U_n - \overline{U} \|_{L^{2k}(-a, a)}^\theta \quad (112)
\]

with \( 0 \leq \theta \leq 1 \)

\[
\frac{1}{q} = \frac{1}{2} - \frac{\theta}{2k} \quad (113)
\]

in particular with \( q = 2k - 1 \), one obtains the strong convergence of \( U_n(t) \) in the general Banach space \( L^\infty((0, T), L^{2k-1}(-a, a)) \), with \( \overline{\theta} = \frac{k}{1-k} \left( \frac{3-2k}{2k-1} \right) \)

As a consequence of the above obtained results, one can pass safely the weak limit on

\[
C^\infty((0, T), L^2(-a, a))
\]

\[
\lim_{n \to \infty} \left\{ \frac{d^2}{dt^2} \langle \overline{U}_n, v \rangle_{L^2(-a, a)} + \left( -\frac{d^2}{dx^2} \overline{U}_n, v \right)_{L^2(-a, a)} + g(\overline{U}_n^{2k-1}, v)_{L^2(-a, a)} \right\}
\]

\[
= \frac{d^2}{dt^2} \langle \overline{U}, v \rangle_{L^2(-a, a)} + \left( -\frac{d^2}{dx^2} \overline{U}, v \right)_{L^2(-a, a)} + g(\overline{U}^{2k-1}, v)_{L^2(-a, a)} = 0 \quad (114)
\]

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for any $v \in C^\infty((0,T),L^2(-a,a))$

At this point we sketchy a somewhat rigorous argument to prove the problem’s uniqueness.

Let us consider the hypothesis that the finite function

$$a(x,t) = \frac{(U)^{2k+1}(x,t) - (\overline{v})^{2k+1}(x,t)}{(U(x,t) - \overline{v}(x,t))} \quad (115)$$

is essentially bounded on the domain $[0,\infty) \times (-a,a)$ where $U(x,t)$ and $\overline{v}(x,t)$ denotes, two hypothesized different solutions for the 2D-polinomial wave equation eq.(94) - eq.(96).

It is straightforward to see that its difference $W(x,t) = (U - \overline{v})(x,t)$ satisfies the “Linear” wave equation problem

$$\frac{\partial^2 W}{\partial t^2} - \frac{\partial^2 W}{\partial x} + (aW)(x,t) = 0 \quad (116)$$

$$W(0) = 0 \quad (117)$$

$$W_t(0) = 0 \quad (118)$$

At this point we observe the estimate (where $H^1 \hookrightarrow L^2$)

$$\frac{1}{2} \frac{d}{dt} \left( \| \frac{d}{dt} W \|^2_{L^2(-a,a)} + \| W \|^2_{H^1(-a,a)} \right) \leq \| a \|^2_{L^\infty((0,T) \times (-a,a))} \times \left( \| W \|_{L^2(-a,a)} \times \| \frac{dW}{dt} \|^2_{L^2(-a,a)} \right) \leq M \left( \| \frac{dW}{dt} \|^2_{L^2(-a,a)} + \| W \|^2_{H^1(-a,a)} \right) \leq M \left( \| \frac{dW}{dt} \|^2_{L^2(-a,a)} + \| W \|^2_{H^1(-a,a)} \right) \quad (119)$$

which after an application of the Gronwall’s inequality give us that

$$\left( \| \frac{d}{dt} W \|^2_{L^2(-a,a)} + \| W \|^2_{H^1(-a,a)} \right) (0) \leq \left( \| \frac{dW}{dt} \|^2_{L^2(-a,a)}(0) + \| W \|^2_{H^1(-a,a)}(0) \right) = 0 \quad (120)$$

which proves the problem’s uniqueness under the not proved yet hypothesis that in the two-dimensional case (at least for compact support infinite differentiable initial conditions)

$$\sup_{x \in (-a,a), t \in [0,\infty)} \| a(x,t) \| \leq M \quad (121)$$
As a last comment on the 1 + 1-polynomial non-linear wave motion, we call the reader attention that one can easily extend the technical results analyzed here to the complete polynomial non-linearity

$$\frac{\partial^2 U}{\partial^2 t} - \frac{\partial^2 U}{\partial^2 x} = \sum_{j=1}^{j<2k, j, \text{odd}} c_j U^{2k-j} = C_1 U^{2k-1} + C_3 U^{2k-3} + \ldots$$

(122)

with the set of couplings \( \{C_j\} \) belonging to a positive set of real number. In this case we have that the general solution belongs a priori to the space of functions \( \bigcap_{j=1}^{j<2k} L^\infty((0,T), L^{2k-j}(-a,a)) \).
Appendix A

On sequences of Random measures on Functional Spaces.

Let us consider a completely regular topological space $X$ and the Banach space of continuous function space there $C(X) = \{ f : X \to R, f \text{ continuous on the } X\text{-topology with } \|f\|_\infty = \sup_{x \in X} |f(x)| < \infty \}$

(A-1)

Associated to the Banach Space $C(X)$, there is its dual $\mathcal{M}(X)$ which is the set of Random measures on $X$ with norm given by its total variation $<<\mu>> = |\mu|(X)$ (where $\mu \in \mathcal{M}(X)$).

It is easy to see that if $\beta(X)$ is the Stone-chech compactification of $X$, we have the isometric immersion $\mathcal{M}(X) \hookrightarrow \mathcal{M}(\beta(X))$

We have, thus, the important theorem of Prokhorov on the accumulation points of sets of Random measures ([7]).

**Theorem (Prokhorov) -** If a family of measures $\{\mu_\alpha\}$ on $\mathcal{M}(X)$ is such that it satisfies the following conditions

a) Bounded uniformly

$$<<\mu_\alpha>> = |\mu_\alpha|(X) \leq M$$

(A-2)

c) Uniformly regular - For any given $\varepsilon > 0$, there is a compact set $K(\varepsilon) \subset X$ such that

$$\left| \int_X f(x) \, d\mu_\alpha(x) \right| \leq \varepsilon \|f\|_\infty$$

for any $f(x) \in C(X)$ satisfying the condition

$$\text{(supp } f(x)) \cap K(\varepsilon) = \{\phi\}$$

(A-3)

We have that the set of Random measures is a relatively compact set on the weak-star topology on $\mathcal{M}(X)$. Namely, for any $f(x) \in C(X)$, we have that there is a measure $\nu \in \mathcal{M}(X)$ such that for a given sequence $\{\alpha_k\}$
\[
\lim \sup_{\{\alpha_K\}} \int_X f(x) \, d\mu_{\alpha_K}(x) = \int_X f(x) \, d\nu(x) \quad (A-4)
\]

In order to apply such theorem to the set of Random measures as given by eq.(35), we introduce the completely Regular space formed by the one-point compactification of the Real line

\[
X = \prod_{i \in \mathbb{Z}} (\mathcal{R}_i) \quad \text{(or } X = \prod_{i \in \mathbb{Z}} \beta(R) \text{)} \quad (A-5)
\]

and note the strict positivity of the integrand (the path-integral weight) on eq.(35) and its boundedness by the Gaussian free term since \( \exp\left\{ -\frac{1}{2} \frac{g(i \cdot \cdot \cdot)}{2} \sum_{l=1}^{N} (q_l)^{2k} \right\} < 1 \) for \( \{q_i\} \in \prod_{l=1}^{N} (\mathcal{R}_i) \).

We note either the chain property of the invariant measure set ensuring us the measure uniqueness limit on eq.(B-4) when applied to eq.(35) - section 4, namely:

\[
\int_{\mathbb{R}^N} \left| f(q_1, \cdots, q_{N-1}) \right| d^{(inv)}\mu(q_1, \cdots, q_N) \geq \int_{\mathbb{R}^{N-1}} \left| f(q_1, \cdots, q_{N-1}) \right| d^{(inv)}\mu(q_1, \cdots, q_{N-1}) \quad (A-6)
\]

Finally, we remark that the well-defined non-linear Wiener functional measure coincides with those finite-dimensional as given by eq.(35-a) - eq.(35-b) when it is restricts to the set of continuous polygonal paths and, thus, ensuring the convergence of the measure set eq.(35-a) - eq.(35-b) to the full Wiener path measure eq.(36), since its functional support is contained on the set of continuous functions on the interval \([-a, a]\).
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