GENERALIZED EUCLIDEAN OPERATOR RADIUS

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Abstract. In this paper, we introduce the $f$–operator radius of Hilbert space operators as a generalization of the Euclidean operator radius and the $q$–operator radius. Properties of the newly defined radius are discussed, emphasizing how it extends some known results in the literature.

1. Introduction

Let $\mathbb{B}(\mathcal{H})$ denote the $C^*$–algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For $T \in \mathbb{B}(\mathcal{H})$, the operator norm and the numerical radius of $T$ are defined, respectively, by

$$
\|T\| = \sup_{\|x\|=1} \|Tx\| \quad \text{and} \quad \omega(T) = \sup_{\|x\|=1} |\langle Tx, x \rangle|.
$$

It is well known that $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$, that is equivalent to the operator norm via the relation

$$
\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|, \quad T \in \mathbb{B}(\mathcal{H}).
$$

It is interesting to find possible bounds of $\omega(\cdot)$ in terms of $\|\cdot\|$ since the calculations of $\|\cdot\|$ are much easier than those of $\omega(\cdot)$. We refer the reader to [1, 7, 15, 16, 19, 20, 21, 22, 23, 24, 25, 27] as a recent list of references treating numerical radius and operator norm inequalities.

Among the most well-established interesting results in this direction are the following inequalities due to Kittaneh [13, 14]

$$
\omega(T) \leq \frac{1}{2}\|T\| + \|T^*\|,
$$

and

$$
\omega^2(T) \leq \frac{1}{2}\|T\|^2 + \|T^*\|^2.
$$

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and

\[
\omega(T) \leq \frac{1}{2} \left( \|T\| + \|T^2\|^{\frac{1}{2}} \right),
\]

where \( T^* \) is the adjoint operator of \( T \) and \(|T| = (T^*T)^{1/2}\).

Extending the numerical radius, the Euclidean operator radius of the operators \( T_1, \ldots, T_n \in B(\mathcal{H}) \) was defined in [18] as

\[
\omega_e(T_1, \ldots, T_n) = \sup_{\|x\|=1} \left( \sum_{j=1}^{n} |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}}.
\]

This was also generalized in [8] to

\[
\omega_q(T_1, \ldots, T_n) = \sup_{\|x\|=1} \left( \sum_{j=1}^{n} |\langle T_j x, x \rangle|^q \right)^{\frac{1}{q}}; \quad q \geq 1.
\]

We refer the reader to [2, 4, 9, 10, 23, 26] as a list of references treating properties and significance of \( \omega_e \) and \( \omega_q \).

In the literature, it is interesting to introduce and define new related numerical radii or operator radii in a way that extends some well-known concepts. For this particular concern, we refer the reader to [1, 5, 25], where a discussion of other types of numerical radii has been presented.

This paper introduces a generalized form of \( \omega_e \) and \( \omega_q \) that depends on a certain function \( f \). It turns out that both \( \omega_e \) and \( \omega_q \) are special cases of this new concept, which we define as follows.

**Definition 1.1.** Let \( T_1, \ldots, T_n \in B(\mathcal{H}) \) and let \( f : [0, \infty) \to [0, \infty) \) be a continuous increasing function with \( f(0) = 0 \). We define the \( f \)-operator radius of the operators \( T_1, \ldots, T_n \) by

\[
\omega_f(T_1, \ldots, T_n) = \sup_{\|x\|=1} \left( \sum_{j=1}^{n} f \left( |\langle T_j x, x \rangle| \right) \right)
\]

Thus, when \( f(t) = t^2 \), \( \omega_f = \omega_e \), and when \( f(t) = t^q \), \( \omega_f = \omega_q \), for \( q \geq 1 \).

The quantities \( \omega_e, \omega_q \) were defined in [8, 18] as norms on \( B(\mathcal{H}) \times \cdots \times B(\mathcal{H}) \). In what follows, we show norm properties of \( \omega_f \).

It is implicitly understood that \( f : [0, \infty) \to [0, \infty) \) is a continuous increasing function with \( f(0) = 0 \), whenever we write \( \omega_f \).

The Davis-Wielandt radius of \( T \in B(\mathcal{H}) \) is defined as

\[
d\omega(T) = \sup_{\|x\|=1} \left\{ \sqrt{\langle Tx, x \rangle^2 + \|Tx\|^4} \right\}.
\]
It is not hard to see that $d\omega(T)$ is unitarily invariant, but it does not define a norm on $B(H)$. It is well-known that

$$\max\left\{ \omega(T), \|T\|^2 \right\} \leq d\omega(T) \leq \sqrt{\omega^2(T) + \|T\|^4}. $$

Putting $n = 2$, $T_1 = T$, and $T_2 = T^*T$, in Definition 1.1, we deliver

$$
\omega_f(T, T^*T) = \sup_{\|x\|=1} f^{-1}(f(|\langle T x, x \rangle|) + f(|\langle T^*T x, x \rangle|)) \\
= \sup_{\|x\|=1} f^{-1}(f(|\langle T x, x \rangle|) + f(|\langle T x, T x \rangle|)) \\
= \sup_{\|x\|=1} f^{-1}(f(|\langle T x, x \rangle|) + f(\|T x\|^2))
$$

which provides an extension of the Davis-Wielandt radius of $T$. Notice that when $f(t) = t^2$, $\omega_f(T, T^*T) = d\omega(T)$.

We need the following lemmas throughout the subsequent sections. The first lemma has been a helpful tool in studying operator inequalities in the literature.

Lemma 1.1. [6, (4.24)] Let $f$ be a convex function defined on a real interval $I$ and let $T \in B(H)$ be a self-adjoint operator with spectrum in $I$. Then $f(|\langle Tx, x \rangle|) \leq \langle f(T) x, x \rangle$ for all unit vectors $x \in H$.

The second lemma is a useful characterization of numerical radii.

Lemma 1.2. [27] Let $T \in B(H)$. Then

$$\omega(T) = \sup_{\theta \in \mathbb{R}} \|\mathfrak{R}(e^{i\theta} T)\|,$$

where $\mathfrak{R}(T)$ is the real part of the operator $T$, defined by $\mathfrak{R}T = \frac{T + T^*}{2}$.

We also need the following lemma, which holds for convex functions with $f(0) \leq 0$.

Lemma 1.3. If $f : [0, \infty) \to [0, \infty)$ is a convex function with $f(0) = 0$, then $f$ is superadditive. That is

$$f(a + b) \geq f(a) + f(b),$$

for $a, b \geq 0$. The inequality is reversed when $f : [0, \infty) \to [0, \infty)$ is concave, without having $f(0) = 0$.

Recall that the Aluthge transform $\tilde{T}$ of $T \in B(H)$ is defined by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $U$ is the partial isometry appearing in the polar decomposition $T = U|T|$ of $T$, [3]. Yamazaki showed the following better estimates of (1.3) than [27]

$$
\omega(T) \leq \frac{1}{2} \left( \|T\| + \omega(\tilde{T}) \right). 
$$
2. FURTHER DISCUSSION OF $\omega_f$

In this section, we discuss the quantity $\omega_f$. This includes basic properties and possible relations with the numerical radius $\omega$ and the operator norm $\| \cdot \|$. More applications to numerical radius bounds will be discussed too.

We begin with the following basic properties of $\omega_f$.

**Proposition 2.1.** Let $T_1, \ldots, T_n \in \mathfrak{B}(\mathcal{H})$ and let $f : [0, \infty) \to [0, \infty)$ be a continuous increasing function with $f(0) = 0$. Then

(i) $\omega_f(T_1, \ldots, T_n) = 0$ if and only if $T_1 = \cdots = T_n = 0$.

(ii) $\omega_f(\alpha T_1, \ldots, \alpha T_n) = |\alpha| \omega_f(T_1, \ldots, T_n)$ for all $\alpha \in \mathbb{C}$, provided that $f$ is multiplicative.

(iii) $\omega_f(T_1 + T'_1, \ldots, T_n + T'_n) \leq \omega_f(T_1, \ldots, T_n) + \omega_f(T'_1, \ldots, T'_n)$, provided that $f$ is geometrically convex. That is, $f(\sqrt{ab}) \leq \sqrt{f(a)f(b)}$.

(iv) $\omega_f(T_1, \ldots, T_n) = \omega_f(T_{1\ast}, \ldots, T_{n\ast})$.

(v) If $U_1, \ldots, U_n$ are unitary, then

$$\omega_f(U_1^\ast T_1 U_1, \ldots, U_n^\ast T_n U_n) = \omega_f(T_1, \ldots, T_n).$$

(vi) If $g : [0, \infty) \to [0, \infty)$ is an injective function such that $g(0) = 0$, and $f \circ g^{-1}$ is convex, then

$$\omega_f(T_1, \ldots, T_n) \leq g(\omega_f(T_1, \ldots, T_n)).$$

**Proof.** The first, second, and fourth assertions immediately follow the definition of $\omega_f$. For (iii), assume that $f$ is an increasing geometrically convex function. Then

$$\sum_{j=1}^n f\left(\left|\langle T_j + T'_j, x, x \rangle\right|\right) = \sum_{j=1}^n f\left(\left|\langle T_j x, x \rangle + \langle T'_j x, x \rangle\right|\right)$$

$$\leq \sum_{j=1}^n f\left(\left|\langle T_j x, x \rangle\right| + \left|\langle T'_j x, x \rangle\right|\right),$$

where we obtain the last inequality by the triangle inequality and the fact that $f$ is increasing. On the other hand, since $f$ is geometrically convex, it follows that [17, Corollary 1.1]

$$f^{-1}\left(\sum_{j=1}^n f\left(\left|\langle T_j x, x \rangle\right| + \left|\langle T'_j x, x \rangle\right|\right)\right) \leq f^{-1}\left(\sum_{j=1}^n f\left(\left|\langle T_j x, x \rangle\right|\right)\right) + f^{-1}\left(\sum_{j=1}^n f\left(\left|\langle T'_j x, x \rangle\right|\right)\right),$$

which implies,

$$f^{-1}\left(\sum_{j=1}^n f\left(\left|\langle T_j + T'_j, x, x \rangle\right|\right)\right) \leq f^{-1}\left(\sum_{j=1}^n f\left(\left|\langle T_j x, x \rangle\right|\right)\right) + f^{-1}\left(\sum_{j=1}^n f\left(\left|\langle T'_j x, x \rangle\right|\right)\right).$$

Consequently,

$$\omega_f(T_1 + T'_1, \ldots, T_n + T'_n) \leq \omega_f(T_1, \ldots, T_n) + \omega_f(T'_1, \ldots, T'_n).$$
To prove (v), we have

\[ \omega_f (U^*_1 T_1 U_1, \ldots, U^*_n T_n U_n) = \sup_{\|x\|=1} f^{-1} \left( \sum_{j=1}^{n} f \left( |\langle U^*_j T_j U_j x, x \rangle| \right) \right) \]

\[ = \sup_{\|x\|=1} f^{-1} \left( \sum_{j=1}^{n} f \left( |\langle T_j U_j x, U_j x \rangle| \right) \right) \]

\[ = \sup_{\|y\|=1} f^{-1} \left( \sum_{j=1}^{n} f \left( |\langle T_j y, y \rangle| \right) \right) \]

\[ = \omega_f (T_1, \ldots, T_n). \]

Finally, for (vi), we note first that convexity of \( f \circ g^{-1} \), together with the facts that \( f(0) = g(0) = 0 \), implies

\[ f \circ g^{-1}(a) + f \circ g^{-1}(b) \leq f \circ g^{-1}(a + b); \quad a, b \geq 0 \]

thanks to Lemma 1.3. Since \( f^{-1} \) is an increasing function, then

\[ f^{-1} \left( f \circ g^{-1}(a) + f \circ g^{-1}(b) \right) \leq g^{-1}(a + b). \]

Now, replacing \( a \) and \( b \) by \( g(a) \) and \( g(b) \), we get

\[ f^{-1} \left( f(a) + f(b) \right) \leq g^{-1} \left( g(a) + g(b) \right). \]

The last inequality can be extended to \( n \)-tuple as follows

\[ f^{-1} \left( \sum_{j=1}^{n} f(a_j) \right) \leq g^{-1} \left( \sum_{j=1}^{n} g(a_j) \right); \quad a_j \geq 0. \]

Now, let \( x \in \mathcal{H} \) be a unit vector. Replacing \( a_j \) in the above inequality by \( |\langle T_j x, x \rangle| \), then taking the supremum implies

\[ \omega_f (T_1, \ldots, T_n) \leq \omega_g (T_1, \ldots, T_n). \]

This completes the proof. \( \square \)

Next, we attempt to find a relation between \( \omega_f \) and \( \omega \).

Theorem 2.1. Let \( T_1, \ldots, T_n \in \mathbb{B}(\mathcal{H}) \) and let \( f : [0, \infty) \to [0, \infty) \) be a continuous increasing convex function with \( f(0) = 0 \). Then

\[ \omega_f (T_1, \ldots, T_n) \leq \sum_{j=1}^{n} \omega(T_j). \]
Proof. Since \( f \) is convex increasing, it follows that \( f^{-1} \) is increasing and concave. By Lemma 1.3, we have

\[
f^{-1}(a + b) \leq f^{-1}(a) + f^{-1}(b); \quad a, b \geq 0.
\]

Further, since \( f \) is convex, superadditivity of \( f \) implies

\[
\sum_{j=1}^{n} f(a_j) \leq f\left(\sum_{j=1}^{n} a_j\right)
\]

for any \( a_j \in J \). Monotony of \( f^{-1} \) then implies

\[
f^{-1}\left(\sum_{j=1}^{n} f(a_j)\right) \leq \sum_{j=1}^{n} a_j.
\]

By replacing \( a_i \) by \(|\langle T_jx, x \rangle|\) in (2.2), we obtain

\[
f^{-1}\left(\sum_{j=1}^{n} f(|\langle T_jx, x \rangle|)\right) \leq \sum_{j=1}^{n} |\langle T_jx, x \rangle|,
\]

for all unit vectors \( x \in H \). Now, by taking supremum over unit vectors \( x \in H \), we get

\[
\omega_f(T_1, \ldots, T_n) \leq \sum_{j=1}^{n} \omega(T_j),
\]

as desired. \( \square \)

Remark 2.1. For any \( x \in H \) with \( \|x\| = 1 \), it holds

\[|\langle T_jx, x \rangle| \leq \omega(T_j).\]

If \( f : [0, \infty) \to [0, \infty) \) is increasing, we get

\[
\sum_{j=1}^{n} f(|\langle T_jx, x \rangle|) \leq \sum_{j=1}^{n} f(\omega(T_j)).
\]

This implies

\[
\omega_f(T_1, \ldots, T_n) = \sup_{\|x\|=1} f^{-1}\left(\sum_{j=1}^{n} f(|\langle T_jx, x \rangle|)\right) \leq f^{-1}\left(\sum_{j=1}^{n} f(\omega(T_j))\right).
\]

Now, if \( f \) is convex (and increasing of course), \( f^{-1} \) is concave (and increasing), hence \( f^{-1} \) is subadditive. That is

\[
f^{-1}\left(\sum_{j=1}^{n} f(\omega(T_j))\right) \leq \sum_{j=1}^{n} f^{-1}(f(\omega(T_j))) = \sum_{j=1}^{n} \omega(T_j).
\]
Thus, we have shown that if $T_j \in \mathcal{B}(\mathcal{H})$ and $f : [0, \infty) \to [0, \infty)$ is a continuous increasing convex function then

$$
\omega_f(T_1, \ldots, T_n) \leq f^{-1}\left(\sum_{j=1}^{n} f(\omega(T_j))\right) \leq \sum_{j=1}^{n} f^{-1}(f(\omega(T_j))) = \sum_{j=1}^{n} \omega(T_j).
$$

This indeed provides a considerable refinement of (2.3). We notice that the condition $f(0)$ is unnecessary here.

In the following theorem, we present the $\omega_f$ version of the first inequality in (1.1). We notice that (2.3) provides the $\omega_f$ version of the second inequality in (1.1) because $\omega(T_j) \leq \|T_j\|$. In fact, (2.4) provides further details than (2.3). However, we need to be cautious here as (2.4) is valid for convex functions, while the next is for concave functions.

**Theorem 2.2.** Let $T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ and let $f : [0, \infty) \to [0, \infty)$ be a continuous increasing concave function with $f(0) = 0$. Then

$$
\frac{1}{2} \left\| \sum_{j=1}^{n} T_j \right\| \leq \omega \left( \sum_{j=1}^{n} T_j \right) \leq \omega_f(T_1, \ldots, T_n).
$$

**Proof.** Let $x \in \mathcal{H}$ be a unit vector. Since $f$ is concave with $f(0) = 0$, $f^{-1}$ is convex with $f^{-1}(0) = 0$. Applying Lemma 1.3, we have

$$
\omega_f(T_1, \ldots, T_n) \geq f^{-1}\left(\sum_{j=1}^{n} f(|\langle T_j x, x \rangle|)\right)
$$

$$
\geq \sum_{j=1}^{n} |\langle T_j x, x \rangle|
$$

$$
\geq \left| \sum_{j=1}^{n} \langle T_j x, x \rangle \right|
$$

$$
= \left| \left\langle \left( \sum_{j=1}^{n} T_j \right) x, x \right\rangle \right|.
$$

Taking the supremum over unit vectors $x \in \mathcal{H}$, we obtain $\omega_f(T_1, \ldots, T_n) \geq \omega \left( \sum_{j=1}^{n} T_j \right)$. The result follows immediately from (1.1). \qed

The following result is concerned with some lower bounds for $\omega_f(\cdot)$.

**Proposition 2.2.** Let $T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing convex function. Then

$$
\omega_f(T_1, \ldots, T_n) \geq \sup_{|\lambda_j| \leq 1} \omega \left( \sum_{j=1}^{n} \frac{\lambda_j}{n} T_j \right) \geq \frac{1}{2} \sup_{|\lambda_j| \leq 1} \left\| \sum_{j=1}^{n} \frac{\lambda_j}{n} T_j \right\|.
$$
Proof. By convexity of $f$ we have, for any $\lambda_j \in \mathbb{C}$ with $|\lambda_j| \leq 1$ and any unit vector $x \in \mathcal{H}$,

$$f^{-1}\left(\sum_{j=1}^{n} f\left(|\langle T_j x, x \rangle|\right)\right) \geq \sum_{j=1}^{n} \frac{1}{n} |\langle T_j x, x \rangle|$$

$$\geq \left| \sum_{j=1}^{n} \frac{\lambda_j}{n} T_j x, x \right|$$

$$= \left| \left\langle \sum_{j=1}^{n} \frac{\lambda_j}{n} T_j x, x \right\rangle \right|.$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ yields

$$\omega_f(T_1, \ldots, T_n) \geq \omega\left(\sum_{j=1}^{n} \frac{\lambda_j}{n} T_j\right),$$

for any $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $|\lambda_j| \leq 1$. Therefore,

$$\omega_f(T_1, \ldots, T_n) \geq \sup_{|\lambda_j| \leq 1} \omega\left(\sum_{j=1}^{n} \frac{\lambda_i}{n} T_j\right).$$

The second inequality follows quickly from (1.1). \qed

On making use of inequality (2.5), we find different lower bounds for $\omega_f$.

Corollary 2.1. Let $T_1, \ldots, T_n \in \mathbb{B}(\mathcal{H})$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing convex function. Then

$$\omega_f(T_1, \ldots, T_n) \geq \frac{1}{n} \max\{\omega(T_1), \ldots, \omega(T_n)\} \geq \frac{1}{2n} \max\{\|T_1\|, \ldots, \|T_n\|\}.$$  

Proof. For any $j \in \{1, \ldots, n\}$, we consider $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that $\lambda_i = 1$ and $\lambda_j = 0$ if $j \neq i$. Then, by (2.5), we have

$$\omega_f(T_1, \ldots, T_n) \geq \frac{1}{n} \omega(T_j) \geq \frac{1}{2n} \|T_j\|,$$

for any $1 \leq j \leq n$, and this completes the proof. \qed

Corollary 2.2. Let $T_1, \ldots, T_n \in \mathbb{B}(\mathcal{H})$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing convex function. Then

$$\omega_f(T_1, \ldots, T_n) \geq \frac{1}{n} \max\left\{ \omega\left(\sum_{j=1}^{n} \pm T_j\right)\right\} \geq \frac{1}{2n} \max\left\{\left\|\sum_{j=1}^{n} \pm T_j\right\|\right\}.$$  

Proof. It is a simple consequence of (2.5) where we consider $\lambda_j = e^{i\theta}$ with $\theta \in [0, 2\pi]$. \qed

In the previous statement we can consider $\lambda_j = e^{i\theta}$ with $\theta \in [0, 2\pi]$. 

Remark 2.2. From Corollary 2.2, we get
\[ \omega_f (T_1, T_2) \geq \frac{1}{2} \omega (T_1 + T_2). \]

Let \( T = B + iC \) be the Cartesian decomposition of the operator \( T \in \mathbb{B} (\mathcal{H}) \). Setting \( T_1 = B \) and \( T_2 = iC \), we infer that
\[ \omega_f (B, C) = \omega_f (B, iC) \geq \frac{1}{2} \omega (B + iC) = \frac{1}{2} \omega (T). \]

Remark 2.3. Letting \( T_1 = T_2 = \cdots = T_n = T \). From Theorem 2.1, we get
\[ (2.6) \quad \omega_f (T, \ldots, T) \leq n \omega (T). \]

On the other hand, by Corollary 2.2, we infer that
\[ (2.7) \quad \omega_f (T, \ldots, T) \geq \omega (T). \]

Combining two inequalities (2.6) and (2.7), we reach to
\[ \omega (T) \leq \omega_f (T, \ldots, T) \leq n \omega (T). \]

In the following, we present a lower bound for the generalized Davis-Wielandt radius introduced in the introduction.

Corollary 2.3. Let \( T \in \mathbb{B} (\mathcal{H}) \) and let \( f : [0, \infty) \to [0, \infty) \) be a continuous increasing concave function with \( f (0) = 0 \). Then
\[ \| \mathfrak{R} T + T^* T \| + \frac{|\omega (T + T^* T) - \omega (T^* + T^* T)|}{2} \leq \omega_f (T, T^* T). \]

Proof. From Theorem 2.2, we have
\[ \omega (T + T^* T) \leq \omega_f (T, T^* T). \]

Since \( \omega (X) = \omega (X^*) \) for any \( X \in \mathbb{B} (\mathcal{H}) \), we get
\[ \omega (T^* + T^* T) \leq \omega_f (T, T^* T). \]

Thus,
\[ \| \mathfrak{R} T + T^* T \| + \frac{|\omega (T + T^* T) - \omega (T^* + T^* T)|}{2} \]
\[ = \omega (\mathfrak{R} T + T^* T) + \frac{|\omega (T + T^* T) - \omega (T^* + T^* T)|}{2} \]
\[ \leq \frac{\omega (T + T^* T) \omega (T^* + T^* T)}{2} + \frac{|\omega (T + T^* T) - \omega (T^* + T^* T)|}{2} \]
\[ \leq \max \{ \omega (T + T^* T), \omega (T^* + T^* T) \} \]
\[ \leq \omega_f (T, T^* T), \]

as desired. \qed
We notice that Corollary 2.3 provides some possible relation between $\omega_f(T, T^*T)$ and $\|\mathcal{R}T + T^*T\|$ when $f$ is a concave function. In contrast, the following corollary presents a possible relation between these quantities when $f$ is convex.

**Corollary 2.4.** Let $T \in \mathcal{B}(\mathcal{H})$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing convex function. Then

$$
\frac{1}{2} \max \{\omega(T), \|T\|^2\} \leq \omega_f(T, T^*T),
$$

and

$$
\frac{1}{2} \|\mathcal{R}T + T^*T\| + \frac{1}{4} \left| \omega(T + T^*T) - \omega(T^* + T^*T) \right| \leq \omega_f(T, T^*T).
$$

**Proof.** Employing Corollary 2.1, gives

$$
\omega_f(T, T^*T) \geq \frac{1}{2} \max \{\omega(T), \omega(T^*T)\} = \frac{1}{2} \max \{\omega(T), \|T^*T\|\} = \frac{1}{2} \max \{\omega(T), \|T\|^2\}.
$$

This proves the first inequality. To establish the second inequality, by Corollary 2.2, we have

$$
\omega_f(T, T^*T) \geq \frac{1}{2} \omega(T + T^*T).
$$

Applying the same arguments as in the proof of Corollary 2.3 indicates the expected result. □

## 3. More elaborated relations with the numerical radius

In 1994, Furuta [11] proved an attractive generalization of Kato’s (Cauchy–Schwarz) inequality, for an arbitrary $T \in \mathcal{B}(\mathcal{H})$, as follows

$$
\left| \left\langle T \left| T \right|^{\alpha+\beta-1} x, y \right\rangle \right|^2 \leq \left\langle \left| T \right|^{2\alpha} x, x \right\rangle \left\langle \left| T^* \right|^{2\beta} y, y \right\rangle
$$

for any $x, y \in \mathcal{H}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$.

In the following result, we present an upper bound of $\omega_f$ for operators of the form $T|T|^{\alpha+\beta-1}$ appearing in (3.1).

**Theorem 3.1.** Let $T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing continuous geometrically convex function. If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\omega_f \left( T_1|T_1|^{\alpha+\beta-1}, \ldots, T_n|T_n|^{\alpha+\beta-1} \right) \leq \left\| f^{-1} \left( \sum_{j=1}^{n} \left( \frac{1}{p} f^{\frac{2}{p}} \left( \left| T_j \right|^{2\alpha} \right) + \frac{1}{q} f^{\frac{2}{q}} \left( \left| T_j^* \right|^{2\beta} \right) \right) \right) \right\|,
$$

for any $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$. 

Proof. Employing (3.1) for the \( n \)-tuple operators \( (T_1, \ldots, T_n) \), by setting \( y = x \), we have
\[
\sum_{i=1}^{n} f \left( \langle T_j \mid T_j \mid^{\alpha+\beta-1} x, x \rangle \right) \leq \sum_{j=1}^{n} f \left( \langle |T_j|^{2\alpha} x, x \rangle \right)^{\frac{1}{2}} \left( \langle |T_j^*|^{2\beta} x, x \rangle \right)^{\frac{1}{2}} \\
\leq \sum_{j=1}^{n} f^{\frac{1}{2}} \left( \langle |T_j|^{2\alpha} x, x \rangle \right) f^{\frac{1}{2}} \left( \langle |T_j^*|^{2\beta} x, x \rangle \right) \\
\leq \left( \sum_{j=1}^{n} f^{\frac{p}{2}} \left( \langle |T_j|^{2\alpha} x, x \rangle \right) \right)^{\frac{1}{p}} \left( \sum_{j=1}^{n} f^{\frac{q}{2}} \left( \langle |T_j^*|^{2\beta} x, x \rangle \right) \right)^{\frac{1}{q}} \\
\leq \frac{1}{p} \sum_{j=1}^{n} f^{\frac{p}{2}} \left( \langle |T_j|^{2\alpha} x, x \rangle \right) + \frac{1}{q} \sum_{j=1}^{n} f^{\frac{q}{2}} \left( \langle |T_j^*|^{2\beta} x, x \rangle \right) .
\]
Thus,
\[
f^{-1} \left( \sum_{j=1}^{n} f \left( \langle T_j \mid T_j \mid^{\alpha+\beta-1} x, x \rangle \right) \right) \leq f^{-1} \left( \frac{1}{p} \sum_{j=1}^{n} f^{\frac{p}{2}} \left( \langle |T_j|^{2\alpha} x, x \rangle \right) + \frac{1}{q} \sum_{j=1}^{n} f^{\frac{q}{2}} \left( \langle |T_j^*|^{2\beta} x, x \rangle \right) \right) .
\]
We get the required result by taking the supremum over all unit vector \( x \in \mathcal{H} \). \( \square \)

A more straightforward upper bound of \( \omega_f \) can be stated as follows.

**Theorem 3.2.** Let \( T_1, \ldots, T_n \in B(\mathcal{H}) \) and let \( f : [0, \infty) \to [0, \infty) \) be an increasing convex function. Then
\[
\omega_f (T_1, \ldots, T_n) \leq \left\| f^{-1} \left( \sum_{j=1}^{n} \left( \frac{f \left( \langle |T_j|^{2\alpha} x, x \rangle \right) + f \left( \langle |T_j^*|^{2(1-\alpha)} x, x \rangle \right)}{2} \right) \right) \right\| ,
\]
for any \( 0 \leq \alpha \leq 1 \).

**Proof.** For \( 0 \leq \alpha \leq 1 \), the Cauchy-Schwarz inequality, together with the arithmetic-geometric mean inequality, implies
\[
|\langle T_j x, x \rangle| \leq \langle |T_j|^{2\alpha} x, x \rangle^{\frac{1}{2}} \langle |T_j^*|^{2(1-\alpha)} x, x \rangle^{\frac{1}{2}} \\
\leq \left\langle \frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} x, x \right\rangle ,
\]
for the unit vector \( x \in \mathcal{H} \).
Noting that $f$ is increasing, then applying Lemma 1.1 we have
\[
\sum_{j=1}^{n} f(|\langle T_j x, x \rangle|) \leq \sum_{j=1}^{n} f\left(\frac{\left(|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}\right)}{2} x, x\right)
\]
\[
\leq \sum_{j=1}^{n} \left\langle f\left(\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right) x, x\right\rangle
\]
\[
\leq \sum_{j=1}^{n} \left\langle f\left(\frac{|T_j|^{2\alpha}}{2}\right) + f\left(\frac{|T_j^*|^{2(1-\alpha)}}{2}\right) x, x\right\rangle
\]
\[
= \left\langle \sum_{j=1}^{n} \left( f\left(\frac{|T_j|^{2\alpha}}{2}\right) + f\left(\frac{|T_j^*|^{2(1-\alpha)}}{2}\right) \right) x, x \right\rangle,
\]
which implies
\[
f^{-1}\left(\sum_{j=1}^{n} f(|\langle T_j x, x \rangle|)\right) \leq f^{-1}\left(\sum_{j=1}^{n} \left( f\left(\frac{|T_j|^{2\alpha}}{2}\right) + f\left(\frac{|T_j^*|^{2(1-\alpha)}}{2}\right) \right) x, x \right)\).
\]
We get the required result by taking the supremum over all unit vectors $x \in \mathcal{H}$, noting that $f^{-1}$ is also increasing. \qed

Another bound, similar to that in Theorem 3.3, can be stated as follows. The proof is very similar to that of Theorem 3.3, so we do not include it here.

**Theorem 3.3.** Let $T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing convex function. If $p_j > 0$ so that $\sum_{j=1}^{n} p_j = 1$, then
\[
\omega_f(p_1 T_1, \ldots, p_n T_n) \leq \left\| f^{-1}\left(\sum_{j=1}^{n} p_j \left( f\left(\frac{|T_j|^{2\alpha}}{2}\right) + f\left(\frac{|T_j^*|^{2(1-\alpha)}}{2}\right) \right) \right) \right\|,
\]
for any $0 \leq \alpha \leq 1$.

In the following result, a super-multiplicative function refers to a function $f : [0, \infty) \to [0, \infty)$ such that $f(a)f(b) \leq f(ab)$ for all $a, b \in [0, \infty)$. We notice that all power functions $f(t) = t^r, r > 0$ are such functions.

**Theorem 3.4.** Let $T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing, convex and super-multiplicative function. Then
\[
\omega_f(T_1, \ldots, T_n) \leq \left\| f^{-1}\left(\sqrt[n]{\sum_{j=1}^{n} f\left(\frac{T_j^* T_j + T_j T_j^*}{2}\right)} \right) \right\|.
\]
Proof. Let $B_j + iC_j$ be the Cartesian decomposition of the Hilbert space operators $T_j$, for $j = 1, \cdots, n$. We have

$$|\langle T_j x, x \rangle|^2 = \langle B_j x, x \rangle^2 + \langle C_j x, x \rangle^2$$

$$\leq \langle B_j^2 x, x \rangle + \langle C_j^2 x, x \rangle = \langle (B_j^2 + C_j^2) x, x \rangle,$$

where we have used Lemma 1.1 to obtain the last inequality, noting that both $B_j$ and $C_j$ are self-adjoint and that $f(t) = t^2$ is convex. But since $f$ is increasing, super-multiplicative and convex, we have

$$f^2 (|\langle T_j x, x \rangle|) \leq f (|\langle T_j x, x \rangle|^2) \leq f \left( \langle (B_j^2 + C_j^2) x, x \rangle \right) \leq \langle (B_j^2 + C_j^2) x, x \rangle$$

which implies that

$$\sum_{j=1}^{n} f (|\langle T_j x, x \rangle|)^2 \leq \sum_{j=1}^{n} f \left( \langle (B_j^2 + C_j^2) x, x \rangle \right) \leq \sum_{j=1}^{n} \langle (B_j^2 + C_j^2) x, x \rangle.$$

Applying Jensen’s inequality to the function $g(t) = t^2$ implies

$$\frac{1}{n^2} \left( \sum_{j=1}^{n} f (|\langle T_j x, x \rangle|) \right)^2 \leq \frac{1}{n} \sum_{j=1}^{n} f \left( |\langle T_j x, x \rangle| \right)^2$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} \langle (B_j^2 + C_j^2) x, x \rangle,$$

and this is equivalent to

$$\sum_{j=1}^{n} f \left( |\langle T_j x, x \rangle| \right) \leq \left( \sum_{j=1}^{n} \langle (B_j^2 + C_j^2) x, x \rangle \right)^{\frac{1}{2}}.$$

Also, since $f$ is increasing, we get

$$f^{-1} \left( \sum_{j=1}^{n} f \left( |\langle T_j x, x \rangle|^2 \right) \right) \leq f^{-1} \left( \left( \sum_{j=1}^{n} \langle (B_j^2 + C_j^2) x, x \rangle \right)^{\frac{1}{2}} \right)$$

$$= f^{-1} \left( \sqrt{n} \left( \sum_{j=1}^{n} \langle (B_j^2 + C_j^2) x, x \rangle \right)^{\frac{1}{2}} \right)$$

$$= f^{-1} \left( \sqrt{n} \left( \sum_{j=1}^{n} \langle \frac{T_j^* T_j + T_j T_j^*}{2} x, x \rangle \right)^{\frac{1}{2}} \right).$$

We get the required result by taking the supremum over all unit vector $x \in \mathcal{H}$. □

In the following remark, we explain the significance of Theorem 3.4.
Remark 3.1. Taking \( f(t) = t^2, \ t \geq 0 \), Theorem 3.4 implies

\[
\omega_e(T_1, \ldots, T_n) \leq \sqrt{\frac{n}{2}} \left\| \sum_{j=1}^{n} (T_j^*T_j + T_jT_j^*)^2 \right\|^{\frac{1}{2}}.
\]

(3.2)

In particular, choosing \( n = 1 \) and \( T_1 = T \), we get

\[
\omega(T) \leq \sqrt{\frac{1}{2}} \|T^*T + TT^*\|,
\]

or

\[
\omega^2(T) \leq \frac{1}{2} \|T^*T + TT^*\|,
\]

which is an outstanding result of Kittaneh (1.2). A more general form of the inequality (3.2) could be stated by taking \( f(t) = t^p, \ t \geq 0 (p \geq 1) \), in Theorem 3.4

\[
\omega_p(T_1, \ldots, T_n) \leq \sqrt{\frac{n}{2^p}} \left\| \sum_{j=1}^{n} (T_j^*T_j + T_jT_j^*)^p \right\|^{\frac{1}{p}}
\]

holds for all \( p \geq 1 \).

Theorem 3.5. Let \( B_j + iC_j \) be the Cartesian decomposition of the Hilbert space operators \( T_j \in B(H) \) \( (j = 1, \ldots, n) \). Let \( f : [0, \infty) \to [0, \infty) \) be an increasing convex function that satisfies \( f(0) = 0 \). Then

\[
\omega_f(T_1, \ldots, T_n) \leq \left\| f^{-1} \left( \sum_{j=1}^{n} f(|B_j| + |C_j|) \right) \right\|.
\]

Proof. Let \( B_j + iC_j \) be the Cartesian decomposition of the Hilbert space operators \( T_j \) for all \( j = 1, \ldots, n \). If \( x \in H \) is a unit vector, we have

\[
\begin{align*}
\sum_{j=1}^{n} f(|\langle T_jx, x \rangle|) &= \sum_{j=1}^{n} f \left( \sqrt{\langle B_jx, x \rangle^2 + \langle C_jx, x \rangle^2} \right) \\
&\leq \sum_{j=1}^{n} f (|\langle B_jx, x \rangle| + |\langle C_jx, x \rangle|) \\
&\leq \sum_{j=1}^{n} f (f(|B_j| + |C_j|) x, x) \\
&\leq \sum_{j=1}^{n} f (f(|B_j| + |C_j|) x, x)
\end{align*}
\]
where we have used Lemma 1.1 twice to obtain the last two inequalities. Thus, since $f$ is increasing,

$$f^{-1} \left( \sum_{j=1}^{n} f \left( |\langle T_j x, x \rangle| \right) \right) \leq f^{-1} \left( \sum_{j=1}^{n} \langle f(|B_j| + |C_j|) x, x \rangle \right)$$

$$= f^{-1} \left( \left\langle \left( \sum_{j=1}^{n} f \left( |B_j| + |C_j| \right) \right) x, x \right\rangle \right)$$

$$\leq f^{-1} \left( \left\| \sum_{j=1}^{n} f \left( |B_j| + |C_j| \right) \right\| \right)$$

$$= \left\| f^{-1} \left( \sum_{j=1}^{n} f \left( |B_j| + |C_j| \right) \right) \right\| ,$$

where we obtain the last equality because $f$ is increasing. 

Now, extending (1.4) to $\omega_f$, we have the following.

**Theorem 3.6.** Let $T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing convex function. Then

$$\omega_f (T_1, \ldots, T_n) \leq f^{-1} \left( \sum_{j=1}^{n} \left( f \left( \|T_j\| \right) + f \left( \omega \left( T_j^* \right) \right) \right) \right).$$

**Proof.** For each $T_j$, let $T_j = U_j |T_j|$ be the polar decomposition of $T_j$. By Lemma 1.2, if $x \in \mathcal{H}$ is a unit vector, it follows that $|\langle T_j x, x \rangle| \leq \Re \{ e^{i\theta} \langle T_j x, x \rangle \}$, for all $\theta \in \mathbb{R}$. Then, for all $\theta$, we have

$$|\langle T_j x, x \rangle|$$

$$\leq \Re \{ e^{i\theta} \langle T_j x, x \rangle \}$$

$$= \frac{1}{4} \langle (e^{-i\theta} + U_j) |T_j| (e^{i\theta} + U_j^*) x, x \rangle - \frac{1}{4} \langle (e^{-i\theta} - U_j) |T_j| (e^{i\theta} - U_j^*) x, x \rangle$$

$$\leq \frac{1}{4} \langle (e^{-i\theta} + U_j) |T_j| (e^{i\theta} + U_j^*) x, x \rangle.$$
Thus,
\[
\sum_{j=1}^{n} f \left( \langle T_j x, x \rangle \right) \leq \sum_{j=1}^{n} f \left( \frac{1}{4} \langle (e^{-i\theta} + U_j) \| T_j \| (e^{i\theta} + U_j^*) x, x \rangle \right)
\]
\[
\leq \sum_{j=1}^{n} f \left( \frac{1}{4} \langle (e^{-i\theta} + U_j) \| T_j \| (e^{i\theta} + U_j^*) x, x \rangle \right)
\]
\[
\leq \sum_{j=1}^{n} \left< f \left( \frac{1}{4} \langle (e^{-i\theta} + U_j) \| T_j \| (e^{i\theta} + U_j^*) x, x \rangle \right) \right>
\]
\[
\leq \sum_{j=1}^{n} \left< f \left( \frac{1}{4} \langle (e^{-i\theta} + U_j) \| T_j \| (e^{i\theta} + U_j^*) x, x \rangle \right) \right>
\]
\[
= \sum_{i=1}^{n} f \left( \frac{1}{4} \| T_j \| \left( e^{i\theta} + U_j^* \right) \left( e^{-i\theta} + U_j \right) \| T_j \| \right)
\]
\[
= \sum_{i=1}^{n} f \left( \| T_j \| + \Re e^{i\theta} \tilde{T}_j \right)
\]
\[
= \sum_{i=1}^{n} f \left( \| T_j \| + \Re e^{i\theta} \tilde{T}_j \right).
\]

On the other hand,
\[
\sum_{j=1}^{n} \left\| f \left( \frac{T_j + \Re e^{i\theta} \tilde{T}_j}{2} \right) \right\| \leq \frac{1}{2} \sum_{i=1}^{n} \left\| f \left( |T_j| \right) + f \left( \Re \tilde{T}_j \right) \right\|
\]
\[
\leq \frac{1}{2} \sum_{i=1}^{n} \left( \left\| f \left( |T_j| \right) \right\| + \left\| f \left( \Re \tilde{T}_j \right) \right\| \right)
\]
\[
= \frac{1}{2} \sum_{i=1}^{n} \left( f \left( \| T_j \| \right) + f \left( \| \Re \tilde{T}_j \| \right) \right)
\]
\[
= \frac{1}{2} \sum_{i=1}^{n} \left( f \left( \| T_j \| \right) + f \left( \| \tilde{T}_j \| \right) \right).
\]

So,
\[
f^{-1} \left( \sum_{j=1}^{n} f \left( \langle T_j x, x \rangle \right) \right) \leq f^{-1} \left( \sum_{j=1}^{n} \left( f \left( \| T_j \| \right) + f \left( \| \tilde{T}_j \| \right) \right) \right),
\]

which completes the proof.

We close this paper by introducing an upper bound for the generalized Davis-Wielandt radius.
Corollary 3.1. Let $T \in B(\mathcal{H})$ with the polar decomposition $T = U|T|$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing convex function. Then

$$\omega_f(T, T^*T) \leq f^{-1}\left(\frac{f(\|T\|) + f\left(\omega\left(\tilde{T}\right)\right) + f\left(\|T\|^2\right) + f\left(\omega(|T|U|T|)\right)}{2}\right).$$

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