A NOTE ON SOLVABLE GRAPHS OF FINITE GROUPS

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Abstract. Let $G$ be a finite non-solvable group with solvable radical $Sol(G)$. The solvable graph $\Gamma_s(G)$ of $G$ is a graph with vertex set $G \setminus Sol(G)$ and two distinct vertices $u$ and $v$ are adjacent if and only if $\langle u, v \rangle$ is solvable. We show that $\Gamma_s(G)$ is not a star graph, a tree, an $n$-partite graph for any positive integer $n \geq 2$ and not a regular graph for any non-solvable finite group $G$. We compute the girth of $\Gamma_s(G)$ and derive a lower bound of the clique number of $\Gamma_s(G)$. We prove the non-existence of finite non-solvable groups whose solvable graphs are planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the paper by obtaining a relation between $\Gamma_s(G)$ and the solvability degree of $G$.

1. Introduction

Let $G$ be a finite group and $u \in G$. The solvabilizer of $u$, denoted by $Sol_G(u)$, is the set given by $\{v \in G : \langle u, v \rangle$ is solvable$\}$. Note that the centralizer $C_G(u) := \{v \in G : uv = vu\}$ is a subset of $Sol_G(u)$ and hence the center $Z(G) \subseteq Sol_G(u)$ for all $u \in G$. By [21 Proposition 2.13], $|C_G(u)|$ divides $|Sol_G(u)|$ for all $u \in G$ though $Sol_G(u)$ is not a subgroup of $G$ in general. A group $G$ is called a S-group if $Sol_G(u)$ is a subgroup of $G$ for all $u \in G$. A finite group $G$ is a S-group if and only if it is solvable (see [21 Proposition 2.22]). Many other properties of $Sol_G(u)$ can be found in [21]. We write $Sol(G) = \{u \in G : \langle u, v \rangle$ is solvable for all $v \in G\}$. It is easy to see that $Sol(G) = \bigcap_{u \in G} Sol_G(u)$. Also, $Sol(G)$ is the solvable radical of $G$ (see [18]). The solvable graph of a finite non-solvable group $G$ is a simple undirected graph whose vertex set is $G \setminus Sol(G)$, and two vertices $u$ and $v$ are adjacent if $\langle u, v \rangle$ is solvable. We write $\Gamma_s(G)$ to denote this graph. It is worth mentioning that $\Gamma_s(G)$ is the complement of the non-solvable graph of $G$ considered in [21] and extension of commuting and nilpotent graphs of finite groups that are studied extensively in [1 2 3 5 6 9 10 11 13 14 15 16 25 26]. It is worth mentioning that the study of commuting graphs of finite groups is originated from a question posed by Erdős [23].

In this paper, we show that $\Gamma_s(G)$ is not a star graph, a tree, an $n$-partite graph for any positive integer $n \geq 2$ and not a regular graph for any non-solvable finite group $G$. In Section 2, we also show that the girth of $\Gamma_s(G)$ is 3 and the clique number of $\Gamma_s(G)$ is greater than or equal to 4. In Section 3, we first show that for a given non-negative integer $k$, there are at the most finitely many finite non-solvable groups whose solvable graph have genus $k$. We also show that there is no finite non-solvable group, whose solvable graph is planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the paper by obtaining a relation between $\Gamma_s(G)$ and $P_s(G)$ in Section 4, where $P_s(G)$ is the probability that a randomly chosen pair of elements of $G$ generate a solvable group (see [20]).

The reader may refer to [24] and [25] for various standard graph theoretic terminologies. For any subset $X$ of the vertex set of a graph $\Gamma$, we write $\Gamma[X]$ to denote the induced subgraph of $\Gamma$ on $X$. The girth of $\Gamma$ is the minimum of the lengths of all cycles in $\Gamma$, and is denoted by $\text{girth}(\Gamma)$. We write
\(\omega(\Gamma)\) to denote the clique number of \(\Gamma\) which is the least upper bound of the sizes of all the cliques of \(\Gamma\). The smallest non-negative integer \(k\) is called the genus of a graph \(\Gamma\) if \(\Gamma\) can be embedded on the surface obtained by attaching \(k\) handles to a sphere. Let \(\gamma(\Gamma)\) be the genus of \(\Gamma\). Then, it is clear that \(\gamma(\Gamma) \geq \gamma(\Gamma_0)\) for any subgraph \(\Gamma_0\) of \(\Gamma\). Let \(K_n\) be the complete graph on \(n\) vertices and \(mK_n\) the disjoint union of \(m\) copies of \(K_n\). It was proved in [21, Corollary 1] that \(\gamma(\Gamma) \leq \gamma(K_m) + \gamma(K_n)\) if \(\Gamma\) has two disjoint subgraphs isomorphic to \(K_m\) and \(K_n\). Also, by [28, Theorem 6-38] we have

\[
\gamma(K_n) = \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor \quad \text{if} \ n \geq 3.
\]

(1)

A graph \(\Gamma\) is called planar, toroidal, double-toroidal and triple-toroidal if \(\gamma(\Gamma) = 0, 1, 2\) and 3 respectively.

Let \(N_k\) be the connected sum of \(k\) projective planes. A simple graph which can be embedded in \(N_k\) but not in \(N_{k-1}\), is called a graph of crosscap \(k\). The notation \(\gamma(\Gamma)\) for any subgraph \(\Gamma\) of \(\Gamma_0\). It was shown in [8] that

\[
\gamma(\Gamma) = \begin{cases} 
\left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor & \text{if } n \geq 3 \text{ and } n \neq 7, \\
3 & \text{if } n = 7.
\end{cases}
\]

(2)

A graph \(\Gamma\) is called a projective graph if \(\gamma(\Gamma) = 1\). It is worth mentioning that \(2K_5\) is not projective graph (see [17]).

2. Graph realization

We begin with the following lemma.

**Lemma 2.1.** For every \(u \in G \setminus \text{Sol}(G)\) we have

\[
\text{deg}(u) = |\text{Sol}_G(u)| - |\text{Sol}(G)| - 1.
\]

**Proof.** Note that \(\text{deg}(u)\) represents the number of vertices from \(G \setminus \text{Sol}(G)\) which are adjacent to \(u\). Since \(u \in \text{Sol}_G(u)\), therefore \(|\text{Sol}_G(u)| - 1\) represents the number of vertices which are adjacent to \(u\). Since we are excluding \(\text{Sol}(G)\) from the vertex set therefore \(\text{deg}(u) = |\text{Sol}_G(u)| - |\text{Sol}(G)| - 1\). □

**Proposition 2.2.** \(\Gamma_s(G)\) is not a star.

**Proof.** Suppose for a contradiction \(\Gamma_s(G)\) is a star. Let \(|G| - |\text{Sol}(G)| = n\). Then there exists \(u \in G \setminus \text{Sol}(G)\) such that \(\text{deg}(u) = n - 1\). Therefore, by Lemma 2.1 \(|\text{Sol}_G(u)| = |G|\). This gives \(u \in \text{Sol}(G)\), a contradiction. Hence, the result follows. □

**Proposition 2.3.** \(\Gamma_s(G)\) is not complete bipartite.

**Proof.** Let \(\Gamma_s(G)\) be complete bipartite. Suppose that \(A_1\) and \(A_2\) are parts of the bi-partition. Then, by Proposition 2.2 \(|A_1| \geq 2\) and \(|A_2| \geq 2\). Let \(u \in A_1, v \in A_2\). If \(|\langle u, v \rangle \text{Sol}(G) \setminus \text{Sol}(G)| > 2\), then there exists \(y \in \langle u, v \rangle \text{Sol}(G) \setminus \text{Sol}(G)\) with \(u \neq y \neq v\) such that \(\langle u, y \rangle\) and \(\langle v, y \rangle\) are both soluble. But then \(y \notin A_1 \text{ and } y \notin A_2\), a contradiction.

It follows that \(|\langle u, v \rangle \text{Sol}(G) \setminus \text{Sol}(G)| = 2\). In particular, \(\text{Sol}(G) = 1 \text{ and } \langle u, v \rangle\) is cyclic of order 3 or \(\text{Sol}(G) = 2\) and \(v = uz\) for \(z\) an involution in \(\text{Sol}(G)\). Now the neighbours of \(u \in A_1\) is just \(u^2 \in A_2\) or \(uz\) in the respective cases. Hence \(|A_2| = |A_1| = 1\), a contradiction. Hence, the result follows. □

Following similar arguments as in the proof of Proposition 2.3 we get the following result.

**Proposition 2.4.** \(\Gamma_s(G)\) is not complete \(n\)-partite.

**Proposition 2.5.** For any finite non-solvable group \(G\), \(\Gamma_s(G)\) has no isolated vertex.
Proof. Suppose \( x \) is an isolated vertex of \( \Gamma_x(G) \). Then \( |\text{Sol}(G)| = 1 \); otherwise \( x \) is adjacent to \( xz \) for any \( z \in \text{Sol}(G) \setminus \{1\} \). Thus it follows that \( o(x) = 2 \); otherwise \( x \) is adjacent to \( x^2 \). Let \( y \in G \). Then \( \langle x, x^y \rangle \) is dihedral and so \( x = x^y \) as \( x \) is isolated. Hence \( x \in Z(G) \) and so \( x \in Z(G) \leq \text{Sol}(G) \), a contradiction. Hence, \( \Gamma_x(G) \) has no isolated vertex.

The following lemma is useful in proving the next two results as well as some results in subsequent sections.

**Lemma 2.6.** Let \( G \) be a finite non-solvable group. Then there exist \( x \in G \) such that \( x, x^2 \not\in \text{Sol}(G) \).

**Proof.** Suppose that for all \( x \in G \), we have \( x^2 \in \text{Sol}(G) \). Therefore, \( G/\text{Sol}(G) \) is elementary abelian and hence solvable. Also, \( \text{Sol}(G) \) is solvable. It follows that \( G \) is solvable, a contradiction. Hence, the result follows.

**Theorem 2.7.** Let \( G \) be a finite non-solvable group. Then \( \text{girth}(\Gamma_x(G)) = 3 \).

**Proof.** Suppose for a contradiction that \( \Gamma_x(G) \) has no 3-cycle. Let \( x \in G \) such that \( x, x^2 \not\in \text{Sol}(G) \) (Lemma 2.6 guarantees the existence of such element). Suppose \( |\text{Sol}(G)| \geq 2 \). Let \( z \in \text{Sol}(G), z \neq 1 \), then \( x, x^2 \) and \( xz \) form a 3-cycle, which is a contradiction. Thus \( |\text{Sol}(G)| = 1 \). In this case, every element of \( G \) has order 2 or 3; otherwise, \( \{x, x^2, x^3\} \) forms a 3-cycle in \( \Gamma_x(G) \) for all \( x \in G \) with \( o(x) > 3 \). Therefore, \( |G| = 2^m3^n \) for some non-negative integers \( m \) and \( n \). By Burnside's Theorem, it follows that \( G \) is solvable; a contradiction. Hence, \( \text{girth}(\Gamma_x(G)) = 3 \).

**Theorem 2.8.** Let \( G \) be a finite non-solvable group. Then \( \omega(\Gamma_x(G)) \geq 4 \).

**Proof.** Suppose for a contradiction that \( G \) is a finite non-solvable group with \( \omega(\Gamma_x(G)) \leq 3 \). Let \( x \in G \setminus \text{Sol}(G) \) such that \( x^2 \not\in G \). Existence of such element is guaranteed by Lemma 2.6. Suppose \( |\text{Sol}(G)| \geq 2 \). Let \( z \in \text{Sol}(G), z \neq 1 \), then \( \{x, x^2, xz, x^2z\} \) is a clique which is a contradiction. Thus \( |\text{Sol}(G)| = 1 \). In this case every element of \( G \setminus \text{Sol}(G) \) has order 2, 3 or 4 otherwise \( \{x, x^2, x^3, x^4\} \) is a clique with \( o(x) > 4 \), which is a contradiction. Therefore \( |G| = 2^m3^n \) where \( m, n \) are non-negative integers. Again, by Burnside’s Theorem, it follows that \( G \) is solvable; a contradiction. This completes the proof.

As a corollary to Theorem 2.7 and Theorem 2.8 we have the following corollary.

**Corollary 2.9.** The solvable graph of a finite non-solvable group is not a tree.

We conclude this section with the following result.

**Proposition 2.10.** \( \Gamma_x(G) \) is not regular.

**Proof.** Follows from [21] Corollary 3.17], noting the fact that a graph is regular if and only if its complement is regular.

3. **Genus and Diameter**

We begin this section with the following useful lemma.

**Lemma 3.1.** Let \( G \) be a finite group and \( H \) a solvable subgroup of \( G \). Then \( \langle H, \text{Sol}(G) \rangle \) is a solvable subgroup of \( G \).

**Proof.** Since \( \text{Sol}(G) \) is normal we have \( \langle H, \text{Sol}(G) \rangle = H \text{Sol}(G) \). Now the proof follows from the fact that solvability is inherent by extension and quotient as

\[
\frac{H \text{Sol}(G)}{\text{Sol}(G)} \cong \frac{H}{H \cap \text{Sol}(G)}.
\]


Proposition 3.2. Let $G$ be a finite non-solvable group such that $\gamma(\Gamma_s(G)) = m$.

(a) If $S$ is a nonempty subset of $G \setminus \text{Sol}(G)$ such that $(x, y)$ is solvable for all $x, y \in S$, then $|S| \leq \left\lceil \frac{7 + \sqrt{1 + 48m}}{2} \right\rceil$.

(b) $|\text{Sol}(G)| \leq \frac{1}{m - 1} \left( \frac{7 + \sqrt{1 + 48m}}{2} \right)$, where $t = \max\{o(x \text{Sol}(G)) \mid x \text{Sol}(G) \in G/\text{Sol}(G)\}$.

(c) If $H$ is a solvable subgroup of $G$, then $|H| \leq \left\lceil \frac{7 + \sqrt{1 + 48m}}{2} \right\rceil + |H \cap \text{Sol}(G)|$.

Proof. We have $\Gamma_s(G)[S] \cong K_{|S|}$ and $\gamma(K_{|S|}) = \gamma(\Gamma_s(G)[S]) \leq \gamma(\Gamma_s(G))$. Therefore, if $m = 0$ then $\gamma(K_{|S|}) = 0$. This gives $|S| \leq 4$, otherwise $K_{|S|}$ will have a subgraph $K_5$ having genus 1. If $m > 0$ then, by Heawood’s formula [27, Theorem 6.3.25], we have

$$|S| = \omega(\Gamma_s(G)[S]) \leq \omega(\Gamma_s(G)) \leq \chi(\Gamma_s(G)) \leq \left\lceil \frac{7 + \sqrt{1 + 48m}}{2} \right\rceil$$

where $\chi(\Gamma_s(G))$ is the chromatic number of $\Gamma_s(G)$. Hence part (a) follows.

Part (b) follows from Lemma 3.1 and part (a) considering $S = \bigcup_{i=1}^{t-1} y_i \text{Sol}(G)$, where $y \in G \setminus \text{Sol}(G)$ such that $o(y \text{Sol}(G)) = t$.

Part (c) follows from part (a) noting that $H = (H \setminus \text{Sol}(G)) \cup (H \cap \text{Sol}(G))$. □

Theorem 3.3. Let $G$ be a finite non-solvable group. Then $|G|$ is bounded above by a function of $\gamma(\Gamma_s(G))$.

Proof. Let $\gamma(\Gamma_s(G)) = m$ and $h_m = \left\lceil \frac{7 + \sqrt{1 + 48m}}{2} \right\rceil$. By Lemma 3.1 we have $\Gamma_s(G)[x \text{Sol}(G)] \cong K_{|\text{Sol}(G)|}$, where $x \in G \setminus \text{Sol}(G)$. Therefore by Proposition 3.2(a), $|\text{Sol}(G)| \leq h_m$.

Let $P$ be a Sylow $p$-subgroup of $G$ for any prime $p$ dividing $|G|$ having order $p^n$ for some positive integer $n$. Then $P$ is a solvable. Therefore, by Proposition 3.2(c), we have $|P| \leq h_m + |\text{Sol}(G)| \leq 2h_m$. Hence, $|G| < (2h_m)^{h_m}$ noting that the number of primes less than $2h_m$ is at most $h_m$. This completes the proof. □

As an immediate consequence of Theorem 3.3 we have the following corollary.

Corollary 3.4. Let $n$ be a non-negative integer. Then there are at the most finitely many finite non-solvable groups $G$ such that $\gamma(\Gamma_s(G)) = n$.

The following two lemmas are essential in proving the main results of this section.

Lemma 3.5. [24, Lemma 3.4] Let $G$ be a finite group.

(a) If $|G| = 7m$ and the Sylow 7-subgroup is normal in $G$, then $G$ has an abelian subgroup of order at least 14 or $|G| \leq 42$.

(b) If $|G| = 9m$, where $3 \nmid m$ and the Sylow 3-subgroup is normal in $G$, then $G$ has an abelian subgroup of order at least 18 or $|G| \leq 72$.

Lemma 3.6. If $G$ is a non-solvable group of order not exceeding 120 then $\Gamma_s(G)$ has a subgraph isomorphic to $K_{11}$ and $\gamma(\Gamma_s(G)) \geq 5$.

Proof. If $G$ is a non-solvable group and $|G| \leq 120$ then $G$ is isomorphic to $A_5$, $A_5 \times Z_2$, $S_5$ or $\text{SL}(2, 5)$. Note that $|\text{Sol}(A_5)| = |\text{Sol}(S_5)| = 1$ and $|\text{Sol}(A_5 \times Z_2)| = |\text{Sol}(\text{SL}(2, 5))| = 2$. Also, $A_5$ has a solvable subgroup of order 12 and $S_5, A_5 \times Z_2, \text{SL}(2, 5)$ have solvable subgroups of order 24. It follows that $\Gamma_s(G)$ has a subgraph isomorphic to $K_{11}$. Therefore, by [4], $\gamma(\Gamma_s(G)) \geq \gamma(K_{11}) = 5$. □

Theorem 3.7. The solvable graph of a finite non-solvable group is neither planar, toroidal, double-toroidal nor triple-toroidal.
Proof. Let $G$ be a finite non-solvable group. Note that it is enough to show $\gamma(\Gamma_s(G)) \geq 4$ to complete the proof. Suppose that $\gamma(\Gamma_s(G)) \leq 3$. Let $x \in G \setminus \operatorname{Sol}(G)$ such that $x^2 \notin \operatorname{Sol}(G)$. Such element exists by Lemma 2.6. Since any two elements of the set $A = x\operatorname{Sol}(G) \cup x^2\operatorname{Sol}(G)$ generate a solvable group, by Proposition 3.2(a), we have $2|\operatorname{Sol}(G)| = |A| \leq \left\lfloor \frac{2 + \sqrt{1 + 4 \cdot 3}}{2} \right\rfloor = 9$. Thus $|\operatorname{Sol}(G)| \leq 4$. Let $p$ be a prime divisor of $|G|$ and $P$ is a Sylow $p$-subgroup of $G$. Since $P$ is solvable, by Proposition 3.2(c), we get $|P| \leq 9 + |P \cap \operatorname{Sol}(G)| \leq 13$. If $|P| = 11$ or $13$ then $|P \cap \operatorname{Sol}(G)| = 1$. Therefore, $\Gamma_s(G)[P \setminus \operatorname{Sol}(G)] \cong K_{10}$ or $K_{12}$. Using (1), we get $\gamma(\Gamma_s(G)[P \setminus \operatorname{Sol}(G)]) = 4$ or $6$. Therefore, $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[P \setminus \operatorname{Sol}(G)]) \geq 4$, a contradiction. Thus $|P| \leq 9$ and hence $p \leq 7$. This shows that $|G|$ divides $2^3.3^2.5.7$.

We consider the following cases.

Case 1. $|\operatorname{Sol}(G)| = 4$.

If $H$ is a Sylow $p$-subgroup of $G$ where $p = 5$ or $7$ then $\langle H, \operatorname{Sol}(G) \rangle$ is solvable since $H$ is solvable (by Lemma 3.1). We have $|H \cap \operatorname{Sol}(G)| = 1$ and $|\langle H, \operatorname{Sol}(G) \rangle| = 20, 28$ according as $p = 5, 7$ respectively. Therefore $\Gamma_s(G)[\langle H, \operatorname{Sol}(G) \rangle \setminus \operatorname{Sol}(G)] \cong K_{16}$ or $K_{24}$. By (1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \operatorname{Sol}(G) \rangle \setminus \operatorname{Sol}(G)]) \geq 13$, which is a contradiction.

Thus $|G|$ is a divisor of $72$. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

Case 2. $|\operatorname{Sol}(G)| = 3$.

If $H$ is a Sylow $p$-subgroup of $G$ where $p = 5$ or $7$ then $\langle H, \operatorname{Sol}(G) \rangle$ is solvable since $H$ is solvable (by Lemma 3.1). We have $|H \cap \operatorname{Sol}(G)| = 1$ and $|\langle H, \operatorname{Sol}(G) \rangle| = 15, 21$ according as $p = 5, 7$ respectively. Therefore $\Gamma_s(G)[\langle H, \operatorname{Sol}(G) \rangle \setminus \operatorname{Sol}(G)] \cong K_{12}$ or $K_{18}$. By (1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \operatorname{Sol}(G) \rangle \setminus \operatorname{Sol}(G)]) \geq 6$, which is a contradiction.

Thus $|G|$ is a divisor of $72$. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

Case 3. $|\operatorname{Sol}(G)| = 2$.

If $H$ is a Sylow $7$-subgroup of $G$ then $\langle H, \operatorname{Sol}(G) \rangle$ is solvable since $H$ is solvable (by Lemma 3.1). We have $|H \cap \operatorname{Sol}(G)| = 1$ and $|\langle H, \operatorname{Sol}(G) \rangle| = 14$. Therefore $\Gamma_s(G)[\langle H, \operatorname{Sol}(G) \rangle \setminus \operatorname{Sol}(G)] \cong K_{12}$. By (1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \operatorname{Sol}(G) \rangle \setminus \operatorname{Sol}(G)]) \geq 6$, which is a contradiction. Let $K$ be a Sylow $3$-subgroup of $G$. If $|K| = 9$ then $\langle K, \operatorname{Sol}(G) \rangle$ is solvable since $K$ is solvable (by Lemma 3.1). We have $|K \cap \operatorname{Sol}(G)| = 1$ and $|\langle K, \operatorname{Sol}(G) \rangle| = 18$. Therefore $\Gamma_s(G)[\langle K, \operatorname{Sol}(G) \rangle \setminus \operatorname{Sol}(G)] \cong K_{16}$. By (1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle K, \operatorname{Sol}(G) \rangle \setminus \operatorname{Sol}(G)]) \geq 13$, which is a contradiction.

Thus $|G|$ is a divisor of $120$. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

Case 4. $|\operatorname{Sol}(G)| = 1$.

In this case, first we shall show that $7 \nmid |G|$. On the contrary, assume that $7 \mid |G|$. Let $n$ be the number of Sylow $7$-subgroup of $G$. Then $n \mid 2^3.3^2.5$ and $n \equiv 1 \pmod{7}$. If $n \neq 1$ then $n \geq 8$. Let $H_1, \ldots, H_8$ be eight distinct Sylow $7$-subgroup of $G$. Then the subgraph induced $\Gamma_s(G)[H_i \setminus \operatorname{Sol}(G)]$ for each $1 \leq i \leq 8$ will contribute $\gamma(\Gamma_s(G)[H_i \setminus \operatorname{Sol}(G)]) = 1$ to the genus of $\Gamma_s(G)$. Thus

$$\gamma(\Gamma_s(G)) \geq \sum_{i=1}^{8} \gamma(\Gamma_s(G)[H_i \setminus \operatorname{Sol}(G)]) = 8,$$

a contradiction. Therefore, Sylow $7$-subgroup of $G$ is unique and hence normal. Since we have started with a non-solvable group, by Lemma 3.5 it follows that $G$ has an abelian subgroup of order at least $14$. Therefore, by (1) we have $\gamma(\Gamma_s(G)) \geq \gamma(K_{13}) = 8$, a contradiction. Hence, $|G|$ is a divisor of $2^3.3^2.5$.

Now, we shall show that $9 \nmid |G|$. Assume that, on the contrary, $9 \mid |G|$. If Sylow $3$-subgroup of $G$ is not normal in $G$, then the number of Sylow $3$-subgroup is greater than or equal to $4$. Let $H_1, H_2, H_3$ be three Sylow $3$-subgroup of $G$. Then the induced subgraph $\Gamma_s(G)[H_i \setminus \operatorname{Sol}(G)] \cong K_8$ and so it contributes $\gamma(\Gamma_s(G)[H_1 \setminus \operatorname{Sol}(G)]) = 2$ to the genus of $\Gamma_s(G)$. If $|H_1 \cap H_2| = 1$, then the
induced subgraph $\Gamma_S(G)[H_2 \setminus \text{Sol}(G)] \cong K_8$ and so it contributes +2 to the genus $\Gamma_S(G)$. Thus
\[
\gamma(\Gamma_S(G)) \geq \gamma(\Gamma_S(G)[(H_1 \cup H_2) \setminus \text{Sol}(G)]) = 4
\]
which is a contradiction. So assume that $|H_1 \cap H_2| = 3$. Similarly $|H_1 \cap H_3| = 3$ and $|H_2 \cap H_3| = 3$.
Let $M = H_2 \setminus H_1$. Then $|M| = 6$. Also note that if $L = H_1 \cup H_2$ and $K = H_3 \setminus L$, then $|K| \geq 4$.
Also $H_1 \cap M = H_2 \cap K = M \cap K = \emptyset$.
If $|K| \geq 5$ then $H_1$ contribute +2 to genus of $\Gamma_S(G)$, $M$ and $K$ each contribute +1 to genus of $\Gamma_S(G)$. Hence genus of $\Gamma_S(G)$ is greater than or equal to 4, a contradiction.
Assume that $|K| = 4$. In this case $|M \cap H_3| = 2$. Let $x \in M \cap H_3$. Then $H_1$ contribute +2 to genus of $\Gamma_S(G)$, $M \setminus \{x\}$ and $K \cup \{x\}$ each contribute +1 to genus of $\Gamma_S(G)$. Hence genus of $\Gamma_S(G)$ is greater than or equal to 4, a contradiction.
These show that the Sylow 3-subgroup of $G$ is unique and hence normal in $G$. Therefore, by Lemma 3.5 and Lemma 3.6, $G$ has an abelian subgroup $A$ of order at least 18. Hence,
\[
\gamma(\Gamma_S(G)) \geq \gamma(\Gamma_S(G)[A \setminus \text{Sol}(G)]) \geq \gamma(K_{17}) = 16
\]
which is a contradiction.
It follows that $9 \nmid |G|$ and $G$ is a divisor of 120. Therefore, by Lemma 3.6, we get $\gamma(\Gamma_S(G)) \geq 5$, a contradiction. Hence, $\gamma(\Gamma_S(G)) \geq 4$ and the result follows. □

The above theorem gives that $\gamma(\Gamma_s(G)) \geq 4$. Usually, genera of solvable graphs of finite non-solvable groups are very large. For example, if $G$ is the smallest non-solvable group $A_5$, then $\Gamma_s(G)$ has 59 vertices and 571 edges. Also $\gamma(\Gamma_s(G)) = 571/6 - 59/2 + 1 = 68$ (follows from [28, Corollary 6–14]). The following theorem shows that the crosscap number of the solvable graph of a finite non-solvable group is greater than 1.

Proposition 3.8. The solvable graph of a finite non-solvable group is not projective.

Proof. Suppose $G$ is a finite non-solvable group whose solvable graph is projective. Note that if $\Gamma_s(G)$ has a subgraph isomorphic to $K_n$, then, by (2), we must have $n \leq 6$. Let $x \in G$, such that $x, x^2 \notin \text{Sol}(G)$. Then $\Gamma_s(G)[x, \text{Sol}(G) \cup x^2 \text{Sol}(G)] \cong K_{2|\text{Sol}(G)|}$. Therefore, $2|\text{Sol}(G)| \leq 6$ and hence $|\text{Sol}(G)| \leq 3$.
Let $p | |G|$ be a prime and $P$ be a Sylow $p$-subgroup of $G$. Then $\Gamma_s(G)[P \setminus \text{Sol}(G)] \cong K_{|P|, \text{Sol}(G)|}$ since $P$ is solvable. Therefore, $|P \setminus \text{Sol}(G)| = |P| - |P \cap \text{Sol}(G)| \leq 6$ and hence $|P| \leq 9$. This shows that $|G|$ is a divisor of $2^3, 3^2, 5.7$.
If $7 | |G|$ then the Sylow 7-subgroup of $G$ is unique and hence normal in $G$; otherwise, let $H$ and $K$ be two Sylow 7-subgroup of $G$. Then $|H \cap K| = |H \cap \text{Sol}(G)| = |K \cap \text{Sol}(G)| = 1$. Therefore, $\Gamma_s(G)[(H \cup K) \setminus \text{Sol}(G)]$ has a subgraph isomorphic to $2K_5$. Hence, $\Gamma_s(G)$ has a subgraph isomorphic to $2K_5$, which is a contradiction. Similarly, if $9 | |G|$, then the Sylow 3-subgroup of $G$ is normal in $G$. Therefore, by Lemma 3.6, it follows that $|G| \leq 72$ or $|G|$ is a divisor of $2^3, 3, 5$. In both the cases, by Lemma 3.6, $\Gamma_s(G)$ has complete subgraphs isomorphic to $K_{11}$, which is a contradiction. This completes the proof. □

We conclude this section, by an observation and a couple of problems regarding the diameter and connectedness of $\Gamma_s(G)$. Using the following programme in GAP [29], we see that the solvable graph of the groups $A_5, S_5, A_5 \times Z_2, SL(2,5), PSL(3,2)$ and $GL(2,4)$ are connected with diameter 2. The solvable graphs of $S_6$ and $A_6$ are connected with diameters greater than 2.

```plaintext
g:=PSL(3,2);
sol:=RadicalGroup(g);
L:=[ ];
gsol:=Difference(g,sol);
```
for x in gsol do
    AddSet(L, [x]);
    for y in Difference(gsol, L) do
        if IsSolvable(Subgroup(g, [x, y])) = true then
            break;
        fi;
        i := 0;
        for z in gsol do
            if IsSolvable(Subgroup(g, [x, z])) = true and IsSolvable(Subgroup(g, [z, y])) = true then
                i := 1;
                break;
            fi;
        od;
        if i = 0 then
            print("Diameter>2");
            print(x, " ", y);
        fi;
    od;
od;

In this connection, we have the following problems.

**Problem 3.1.** Is $\Gamma_s(G)$ connected for any finite non-solvable group $G$?

**Problem 3.2.** Is there any finite bound for the diameter of $\Gamma_s(G)$ when $\Gamma_s(G)$ is connected?

### 4. Relations with solvability degree

The solvability degree of a finite group $G$ is defined by the following ratio

$$P_s(G) := \frac{|\{ (u, v) \in G \times G : \langle u, v \rangle \text{ is solvable} \}|}{|G|^2}.$$ 

Using the solvability criterion (see [12, Section 1]),

“A finite group is solvable if and only if every pair of its elements generates a solvable group” for finite groups we have $G$ is solvable if and only if its solvability degree is 1. It was shown in [20, Theorem A] that $P_s(G) \leq \frac{11}{39}$ for any finite non-solvable group $G$. In this section, we study a few properties of $P_s(G)$ and derive a connection between $P_s(G)$ and $\Gamma_s(G)$ for finite non-solvable groups $G$. We begin with the following lemma.

**Lemma 4.1.** Let $G$ be a finite group. Then $P_s(G) = \frac{1}{|G|^2} \sum_{u \in G} |\text{Sol}_G(u)|$.

*Proof.* Let $S = \{(u, v) \in G \times G : \langle u, v \rangle \text{ is solvable}\}$. Then

$$S = \bigcup_{u \in G} \{u\} \times \{v \in G : \langle u, v \rangle \text{ is solvable}\} = \bigcup_{u \in G} \{u\} \times \text{Sol}_G(u).$$

Therefore, $|S| = \sum_{u \in G} |\text{Sol}_G(u)|$. Hence, the result follows. \qed

**Corollary 4.2.** $|G|P_s(G)$ is an integer for any finite group $G$.

*Proof.* By Proposition 2.16 of [21] we have that $|G|$ divides $\sum_{u \in G} |\text{Sol}_G(u)|$. Hence, the result follows from Lemma 4.1. \qed
We have the following lower bound for $P_s(G)$.

**Theorem 4.3.** For any finite group $G$,

$$P_s(G) \geq \frac{|\text{Sol}(G)|}{|G|} + \frac{2(|G| - |\text{Sol}(G)|)}{|G|^2}.$$  

**Proof.** By Lemma 4.1, we have

$$|G|^2 P_s(G) = \sum_{u \in \text{Sol}(G)} |\text{Sol}_G(u)| + \sum_{u \in G \setminus \text{Sol}(G)} |\text{Sol}_G(u)|$$

$$= |G||\text{Sol}(G)| + \sum_{u \in G \setminus \text{Sol}(G)} |\text{Sol}_G(u)|. \quad (3)$$

By Proposition 2.13 of [21], $|C_G(u)|$ is a divisor of $|\text{Sol}_G(u)|$ for all $u \in G$ where $C_G(u) = \{ v \in G : uv = vu \}$, the centralizer of $u \in G$. Since $|C_G(u)| \geq 2$ for all $u \in G$ we have $|\text{Sol}_G(u)| \geq 2$ for all $u \in G$. Therefore

$$\sum_{u \in G \setminus \text{Sol}(G)} |\text{Sol}_G(u)| \geq 2(|G| - |\text{Sol}(G)|).$$

Hence, the result follows from (3). □

The following theorem shows that $P_s(G) \geq \text{Pr}(G)$ for any finite non-solvable group where $\text{Pr}(G)$ is the commuting probability of $G$ (see [19]).

**Theorem 4.4.** Let $G$ be a finite group. Then $P_s(G) \geq \text{Pr}(G)$ with equality if and only if $G$ is a solvable group.

**Proof.** The result follows from Lemma 4.1 and the fact that $\text{Pr}(G) = \frac{1}{|G|^2} \sum_{u \in G} |C_G(u)|$ noting that $C_G(u) \subseteq \text{Sol}_G(u)$ and so $|\text{Sol}_G(u)| \geq |C_G(u)|$ for all $u \in G$.

The equality holds if and only if $C_G(u) = \text{Sol}_G(u)$ for all $u \in G$, that is $\text{Sol}_G(u)$ is a subgroup of $G$ for all $u \in G$. Hence, by Proposition 2.22 of [21], the equality holds if and only if $G$ is solvable. □

Let $|E(\Gamma_s(G))|$ be the number of edges of the graph the non-solvable graph $\Gamma_s(G)$ of $G$. The following theorem gives a relation between $P_s(G)$ and $|E(\Gamma_s(G))|$.  

**Theorem 4.5.** Let $G$ be a finite non-solvable group. Then

$$2|E(\Gamma_s(G))| = |G|^2 P_s(G) + |\text{Sol}(G)|^2 + |\text{Sol}(G)| - |G|(2|\text{Sol}(G)| + 1).$$

**Proof.** We have

$$2|E(\Gamma_s(G))| = |\{(x, y) \in (G \setminus \text{Sol}(G)) \times (G \setminus \text{Sol}(G)) : (x, y) \text{ is solvable}\}| - |G| + |\text{Sol}(G)|.$$  

Also

$$\mathcal{S} = \{(x, y) \in G \times G : (x, y) \text{ is solvable}\}$$

$$= \text{Sol}(G) \times \text{Sol}(G) \sqcup \text{Sol}(G) \times (G \setminus \text{Sol}(G)) \sqcup (G \setminus \text{Sol}(G)) \times \text{Sol}(G)$$

$$\sqcup \{(x, y) \in (G \setminus \text{Sol}(G)) \times (G \setminus \text{Sol}(G)) : (x, y) \text{ is solvable}\}.$$

Therefore

$$|\mathcal{S}| = |\text{Sol}(G)|^2 + \text{Sol}(G)(|G| - |\text{Sol}(G)|) + 2|E(\Gamma_s(G))| + |G| - |\text{Sol}(G)|$$

$$\implies |G|^2 P_s(G) = |G|(2|\text{Sol}(G)| + 1) - |\text{Sol}(G)|^2 - |\text{Sol}(G)| + 2|E(\Gamma_s(G))|.$$  

Hence, the result follows. □
We conclude this paper noting that lower bounds for $|E(\Gamma_s(G))|$ can be obtained from Theorem 4.5 using the lower bounds given in Theorem 4.3, Theorem 4.4 and the lower bounds for $\Pr(G)$ obtained in [22].

References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, Non-commuting graph of a group, J. Algebra, 298 (2006), 468–492.
[2] A. Abdollahi, M. Zarrin, Non-nilpotent graph of a group, Comm. Algebra 38 (2010), 4390–4403.
[3] M. Akhampi, D. G. M. Farrokhi and K. Khayyarmahesh, Planar, toroidal, and projective commuting and non-commuting graphs, Comm. Algebra, 43 (2015), 2964–2970.
[4] B. Akbari, More on the Non-Solvable Graphs and Solvabilizers, arXiv:1806.01012v1, 2018.
[5] S. Akbari, A. Mohammadian, H. Radjavi and P. Raja, On the diameters of commuting graphs, Linear Algebra Appl., 418 (2006), 161–176.
[6] C. Bates, D. Bundy, S. Hart and P. Rowley, A Note on Commuting Graphs for Symmetric Groups, Electron. J. Combin., 16 (2009), 1–13.
[7] J. Battle, F. Harary, Y. Kodama and J. W. T. Youngs, Additivity of the genus of a graph, Bull. Amer. Math. Soc. 68 (1962), 565–568.
[8] A. Bouchet, Orientable and nonorientable genus of the complete bipartite graph, J. Combin. Theory Ser. B 24 (1978), 24–33.
[9] M. R. Darafsheh, H. Bigdely, A. Bahrami and M. D. Monfared, Some results on non-commuting graph of a finite group, Ital. J. Pure Appl. Math., 268 (2014), 371–387.
[10] A. K. Das, D. Nongsiang, On the genus of the commuting graphs of finite non-abelian groups, Int. Electron. J. Algebra 19 (2016), 91–109.
[11] A. K. Das and D. Nongsiang, On the genus of the nilpotent graphs of finite groups, Comm. Algebra, 43(12) (2015), 5282–5290.
[12] S. Dolfi, R. M. Guralnick, M. Herzog and C. E. Praeger, A new solvability criterion for finite groups, J. London Math. Soc. 85 (2) (2012), 269–281.
[13] P. Dutta, J. Dutta and R. K. Nath, Laplacian spectrum of non-commuting graphs of finite groups, Indian J. Pure Appl. Math. 49 (2018), 205–216.
[14] J. Dutta and R. K. Nath, Spectrum of commuting graphs of some classes of finite groups, Matematika, 33 (2017), 87–95.
[15] J. Dutta and R. K. Nath, Finite groups whose commuting graphs integral, Mat. Vesnik, 69 (2017), 226–230.
[16] J. Dutta and R. K. Nath, Laplacian and signless Laplacian spectrum of commuting graphs of finite groups, Khayyam J. Math., 4 (2018), 77–87.
[17] H. H. Glover, J. P. Huneke and C. S. Wang, 103 graphs that are irreducible for the projective plane, J. Combin. Theory Ser. B 27 (1978), 332–370.
[18] R. Guralnick, B. Kunyavskii, E. Plotkin and A. Shalev, Thompson-like characterizations of the solvable radical, J. Algebra 300 (2006), 363–375.
[19] R. M. Guralnick and G. R. Robinson, On the commuting probability in finite groups, J. Algebra 300(2) (2006), 509–528.
[20] R. Guralnick and J. Wilson, The probability of generating a finite soluble group, Proc. London Math. Soc. 81(3) (2000), 405–427.
[21] D. Hai-Reuven, Non-solvable graph of a finite group and solvabilizers, arXiv:1307.2924v1, 2013.
[22] R. K. Nath and A. K. Das, On a lower bound of commutativity degree, Rend. Circ. Mat. Palermo, b(59(1)) (2010), 137–141.
[23] B. H. Neumann, A problem of Paul Erdős on groups, J. Aust. Math. Soc. (Series A), 21 (1976), 467–472.
[24] D. Nongsisang, Double-Toroidal and Tripe-Toroidal Commuting and Nilpotent Graph, Communicated.
[25] D. Nongsisang and P. K. Saikia, On the non-nilpotent graphs of a group, Int. Electron. J. Algebra, 22 (2017), 78–96.
[26] A. A. Talebi, On the non-commuting graphs of group $D_{2n}$, Int. J. Algebra, 20 (2008), 957–961.
[27] D. B. West, Introduction to Graph Theory (Second Edition), PHI Learning Private Limited, New Delhi, 2009.
[28] Arthur T. White, Graphs, Groups and Surfaces, North-Holland Mathematics Studies, no. 8., American Elsevier Publishing Co., Inc., New York, 1973.
[29] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.6.4, 2013 (http://www.gap-system.org).
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