Tietze Equivalences as Weak Equivalences

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Abstract
A given monoid usually admits many presentations by generators and relations and the notion of Tietze equivalence characterizes when two presentations describe the same monoid: it is the case when one can transform one presentation into the other using the two families of so-called Tietze transformations. The goal of this article is to provide an abstract and geometrical understanding of this well-known fact, by constructing a model structure on the category of presentations, in which two presentations are weakly equivalent when they present the same monoid. We show that Tietze transformations form a pseudo-generating family of trivial cofibrations and give a proof of the completeness of these transformations by an abstract argument in this setting.

Keywords Monoid · Presentation · Tietze transformation · Model category

1 Introduction

In order to navigate between the various presentations of a monoid, a very convenient tool is provided by Tietze transformations, originally investigated for groups [12] (see also [9, chapter II]): these are two families of elementary transformations one can perform on a monoid while preserving the presented monoid. Typically, the Knuth-Bendix completion procedure for string rewriting systems uses such transformations in order to turn a presentation of a monoid into another presentation of the same monoid which has the property of being convergent [6,8], and thus for which the word problem is easily decidable. The Tietze transformations moreover enjoy a completeness property: given any two presentations of a given monoid, there is a way of transforming the first into the second by performing a series of such transformations.

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In this article, we provide a conceptual and geometrical point of view on Tietze transformations, by showing that they can be abstractly thought of as “continuously deforming” the presentations. In order to make this formal, we consider the category of presentations of monoids with suitably chosen morphisms (it turns out that we need to allow some sort of degeneracies) and construct a model structure on it, where weakly equivalent presentations are presentations of a same monoid. We then show that the Tietze transformations can then be interpreted in this setting as a pseudo-generating family of trivial cofibrations: they generate trivial cofibrations with fibrant codomain. Finally, the classical proof of completeness for Tietze transformations proceeds by constructing some kind of cospan of Tietze transformations between two presentations of the same monoid: we explain here how to reconstruct this proof by purely abstract arguments based on our model structure.

The main goal of this article is thus to shed new light on theses well-known concepts and proofs, and advocate the relevance of homotopical methods to people working with presentations of monoids, which is why we have done our best to have a self-contained exposition. We see this work as a first step in order to tackle generalizations of Tietze transformations to higher dimension (e.g. coherent presentations of categories [5, Sect. 2.1]) or more involved structures (Lawvere theories, operads, etc.).

We recall the notion of Tietze transformation between presentations of monoids in Sect. 2, and of model category in Sect. 3. We construct our model structure on the category of presentations in Sect. 4, show that Tietze transformations form a pseudo-generating family of trivial cofibrations in Sect. 5 and use this to abstractly study Tietze equivalences in Sect. 6.

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2 Tietze Equivalences of Presentations of Monoids

2.1 Monoid

A monoid \((M, \cdot, 1)\) consists of a set \(M\) equipped with a binary multiplication operation \(\cdot\) and a unit element \(1\) such that multiplication is associative and the unit acts as a neutral element. A morphism \(f : M \to N\) between two monoids is a function which preserves multiplication and unit. We write \(\text{Mon}\) for the resulting category.

2.2 Free and Quotient Monoids

Given a set \(X\), a word over \(X\) is a finite sequence \(u = a_1 \ldots a_n\) of elements of \(X\), and its length is \(|u| = n\). we write \(X^{*}\) for the free monoid generated by \(X\): its elements are words over \(X\), multiplication \(uv\) of two words is their concatenation, and the unit is the empty word, noted 1.

Given a binary relation \(\sim\) on a monoid \(M\), we write \(M/\sim\) for the quotient monoid whose elements and equivalence classes of elements of \(M\) by the congruence generated by \(\sim\), and multiplication and unit are induced by those of \(M\).
2.3 Presentation

A presentation $P = \langle P_1 | P_2 \rangle$ consists of
- a set $P_1$ of generators,
- a set $P_2 \subseteq P_1^* \times P_1^*$ of relations.

Such a presentation is finite when both the sets $P_1$ and $P_2$ are. A relation $(u, v) \in P_1^*$ is generally denoted by “$u \Rightarrow v$” and we write $\equiv$ for the smallest congruence generated by $P_2$. A morphism $f : P \to Q$ between presentations is a function $f : P_1 \to Q_1$ such that, for every $u \Rightarrow v \in P_2$, we have $f^*(u) \Rightarrow f^*(v) \in Q_2$. A subpresentation $P'$ of $P$ is a presentation equipped with a morphism $P' \to P$ whose underlying function is an inclusion. We write $\text{Pres}$ for the category of presentations and their morphisms. Note that, by definition, there is a forgetful functor $\text{Pres} \to \text{Set}$ sending a presentation $P$ to its set $P_1$ of generators.

2.4 Presented Monoid

The monoid $P$ presented by a presentation $P = \langle P_1 | P_2 \rangle$ is the quotient monoid $P = P_1^*/P_2$ i.e., the quotient of the free monoid $P_1^*$ by the congruence $\equiv$ generated by $P_2$. We often write $q^P : P_1^* \to \bar{P}$ for the quotient morphism and, given $u \in P_1^*$, we write $\bar{u}$ for its equivalence class $q^P(u)$. More generally, we say that a monoid $M$ is presented by $P$ when $M$ is isomorphic to $P^*$. This construction extends as a functor $\text{Pres} \to \text{Mon}$.

Example 1 We have the following presentations:

| $\mathbb{N}$ | $\mathbb{N} \times \mathbb{N}$ | $\mathbb{N}/2\mathbb{N}$ | $\mathbb{Z}$ |
| --- | --- | --- | --- |
| $\simeq \langle a \mid \rangle$ | $\simeq \langle a, b \mid ab \Rightarrow ba \rangle$ | $\simeq \langle a \mid aa \Rightarrow 1 \rangle$ | $\simeq \langle a, b \mid ab \Rightarrow 1, ba \Rightarrow 1 \rangle$. |

2.5 Standard Presentation

To any monoid $M$, one can associate a presentation $\langle M \rangle$, called the standard presentation of $M$, defined by

\[
P_1 = \{a \mid a \in M\} \\
P_2 = \{a_1 \ldots a_n \Rightarrow b_1 \ldots b_m \mid a_1 \ldots a_n = b_1 \ldots b_m\}.
\]

i.e., it contains the elements of the monoids as generators and there is a relation between two words of generators when the product of their elements are equal. This construction extends as a functor $\text{Mon} \to \text{Pres}$. It can be used to show that any monoid admits at least one presentation:

Lemma 1 Given a monoid $M$, its standard presentation is a presentation of $M$: $\langle M \rangle \simeq M$.

Lemma 2 The presentation functor is left adjoint to the standard presentation functor

\[
\begin{array}{ccc}
\text{Pres} & \xrightarrow{\simeq} & \text{Mon} \\
\downarrow & & \downarrow \\
\langle - \rangle & & \\
\end{array}
\]

the counit of the adjunction being an isomorphism.
2.6 Reflexive Presentations

A presentation $P$ is reflexive when for every word $u \in P_1^*$ there is a relation $u \Rightarrow u \in P_2$. We write $r\text{Pres}$ for the full subcategory of $\text{Pres}$ on reflexive presentations.

**Lemma 3** The expected forgetful functor admits a left adjoint

$$
\text{Pres} \overset{\bot}{\longrightarrow} r\text{Pres}
$$

sending a presentation $P$ to the presentation $Q$ with

$$
Q_1 = P_1 \quad Q_2 = P_2 \cup \{ u \Rightarrow u \mid u \in P_1^* \}
$$

and $r\text{Pres}$ is equivalent to the Kleisli category of the monad on $\text{Pres}$ induced by the adjunction.

**Lemma 4** The category $r\text{Pres}$ is equivalent to the category whose objects are presentations (not necessarily reflexive) and a morphism $f : P \to Q$ is a function $f : P_1 \to Q_1$ such that for every relation $u \Rightarrow v \in P_2$ we have either $f(u) \Rightarrow f(v) \in Q_2$ or $f(u) = f(v)$.

In the following, when describing concrete examples of reflexive presentations, we generally omit mentioning reflexivity relations (or, alternatively, the description of morphisms given by previous lemma could be considered).

**Remark 1** The standard presentation is clearly reflexive and thus the adjunction of Lemma 2 restricts to an adjunction between reflexive presentations and monoids.

2.7 Equivalence Between Presentations

There is a very natural notion of equivalence of presentations: two presentations can be considered as equivalent when they present isomorphic monoids. In order to provide a concrete and amenable description of this relation, Tietze has introduced a family of transformations on presentations which characterize the equivalence. Those were originally formulated in the context of presentations of groups [12].

We begin with a simpler but useful characterization of the equivalence:

**Lemma 5** Two presentations $P$ and $Q$ are such that $P \simeq Q$ if and only if there is a cospan of presentations

$$
P \xrightarrow{f} R \xleftarrow{g} Q
$$

such that the induced monoid morphisms $\overline{f} : \overline{P} \to \overline{R}$ and $\overline{g} : \overline{Q} \to \overline{R}$ are isomorphisms.

**Proof** If there is a cospan as above then we have $\overline{P} \simeq \overline{R} \simeq \overline{Q}$ and $P$ and $Q$ are thus equivalent. Conversely, suppose that $P$ presents the monoid $M$, i.e., there is an isomorphism $\overline{P} \to M$. Under the adjunction of Lemma 2, this induces a map $f : P \to \langle M \rangle$ such that $\overline{f} : \overline{P} \to \overline{\langle M \rangle} = M$. Similarly, we can construct a map $g : P \to \langle M \rangle$.

2.8 Tietze Transformation

The elementary Tietze transformations are the following transformations producing a new presentation $Q$ from a presentation $P$:
Adding a derivable generator: given a new generator \( a \notin P_1 \) and word \( u \in P_1^* \), we define the presentation \( Q \) by
\[
Q_1 = P_1 \cup \{ a \} \\
Q_2 = P_2 \cup \{ u \Rightarrow a \} ,
\]

Adding a derivable relation: given two words \( u, v \in P_1^* \) such that \( u \overset{\mathcal{P}}{=} v \), we define the presentation \( Q \) by
\[
Q_1 = P_1 \\
Q_2 = P_2 \cup \{ u \Rightarrow v \} .
\]

It is easy to see that those transformations preserve the presented monoids:

**Lemma 6** Given an elementary Tietze transformation from \( P \) to \( Q \), we have an isomorphism \( P \simeq Q \).

A Tietze transformation from \( P \) to \( Q \) consists in a finite sequence of presentations
\[
P = P^0, P^1, P^2, \ldots, P^n = Q
\]
such that for every \( i \) with \( 0 \leq i < n \) there is an elementary Tietze transformation from \( P^i \) to \( P^{i+1} \). In this situation, we sometimes write
\[
P \leadsto Q
\]

Note that contrarily to the usual convention, we do not allow here removing generators or relations.

The transformation (T2) can be replaced by the following four transformations:

(T2r) reflexivity: given \( u \in P_1^* \), we define \( Q \) by
\[
Q_1 = P_1 \\
Q_2 = P_2 \cup \{ u \Rightarrow u \} ,
\]

(T2s) symmetry: given \( u, v \in P_1^* \) such that \( u \Rightarrow v \in P_2 \), we define \( Q \) by
\[
Q_1 = P_1 \\
Q_2 = P_2 \cup \{ v \Rightarrow u \} ,
\]

(T2t) transitivity: given \( u, v, w \in P_1^* \) such that \( u \Rightarrow v, v \Rightarrow w \in P_2 \), we define \( Q \) by
\[
Q_1 = P_1 \\
Q_2 = P_2 \cup \{ u \Rightarrow w \} .
\]

(T2c) context: given \( u, v, v', w \in P_1^* \) such that \( v \Rightarrow v' \in P_2 \), we define \( Q \) by
\[
Q_1 = P_1 \\
Q_2 = P_2 \cup \{ uvw \Rightarrow uv'w \} .
\]

The resulting systems are the same in the following sense:

**Lemma 7** The following assertions are equivalent: there is a Tietze transformation from \( P \) to \( Q \)

(i) using (T1) and (T2),
(ii) using (T1), (T2r), (T2s), (T2t) and (T2c).

In the following, unless otherwise mentioned, we use the second set of Tietze transformations which are easier to work with because they are more “atomic”. 
2.9 Tietze Equivalence

A Tietze equivalence from \( P \) to \( Q \) is a finite sequence of presentations \( P = P^0, P^1, P^2, \ldots, P^n = Q \) such that for every \( i \) with \( 0 \leq i < n \) there is a Tietze transformation from \( P^i \) to \( P^{i+1} \) or from \( P^{i+1} \) to \( P^i \). Two presentations are Tietze equivalent when there is a Tietze equivalence between them. Otherwise, the Tietze equivalence is the smallest equivalence relation relating any two presentations between which there is an (elementary) Tietze transformation. By Lemma 6 above, Tietze equivalences preserve the presented monoids. It well known that, for finite presentations, the converse holds [9, chapter II]:

**Proof** The right-to-left implication follows from Lemma 6. For the left-to-right implication, suppose given an isomorphism \( P \cong Q \). For the sake of simplicity we suppose that we actually have \( P = Q \) and more generally that Tietze equivalent presentations give rise to identical presented monoids (the proof without this assumption can be constructed from the one below by inserting isomorphisms at required places). Given a generator \( a \in P_1 \), there exists an element \( u \in Q_1 \) such that \( q^P(a) = q^Q(u) \). We write \( a^Q \) for a choice of such an element. Dually, given \( b \in Q_1 \), we write \( b^P \in P_1^* \) for a word such that \( q^P(b^P) = q^Q(b) \). We generalize this notation to words \( u = a_1 \ldots a_n \in P_1^* \), by setting \( u^Q = a_1^Q \ldots b_n^Q \) (and we define \( v^Q \) for \( v \in Q_1^* \) similarly). Note that, for \( u \in P_1^* \), we have

\[
(u^Q)^P = u
\]

(and dually). We construct a presentation \( R \) by

\[
R_1 = P_1 \cup Q_1 \quad R_2 = P_2 \cup Q_2 \cup R_1^P \cup R_1^Q
\]

where

\[
R_1^P = \{ a^Q \Rightarrow a \mid a \in P_1 \} \quad R_1^Q = \{ b^P \Rightarrow b \mid b \in Q_1 \}
\]

We now construct a Tietze transformation from \( P \) to \( R \). Dually, we will be able to construct a Tietze transformation from \( Q \) to \( R \) and we will be able to conclude that \( P \) and \( Q \) are Tietze equivalent:

\[
P \leadsto R \leadsto Q.
\]

By using Tietze transformations (T1), starting from \( P \), we can add each generator \( b \in Q_1 \) along with the relation \( b^P \Rightarrow b \), thus obtaining a transformation

\[
P = \langle P_1 \mid P_2 \rangle \leadsto P' = \langle P_1, Q_1 \mid P_2, R_2^Q \rangle.
\]

Note that, given a word \( u \in Q_1^* \), we have \( u^P = u \). Therefore, given \( a \in P_1 \), we have \( a^Q = (a^Q)^P \Rightarrow a \) by (1). By using Tietze transformations (T2) we can add each derivable relation \( a^Q \Rightarrow a \) to \( P' \) thus reaching the presentation \( R \):

\[
P \leadsto P' \leadsto R.
\]

**Remark 2** The proof above uses Tietze transformations (T1) and (T2). The proof can be performed by using the other set of transformations given by Lemma 7, at the cost of having to take a slightly bigger \( R \).

The proof of theorem 1 constructs a “cospan” of Tietze transformations. We will see that it can be constructed by using tools coming from model categories.
2.10 An Example

Consider the presentations 
\[ \langle a \mid \rangle \text{ and } \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb \rangle. \]
Both present the additive monoid \( \mathbb{N} \), and indeed there is a Tietze equivalence between them:

\[
\begin{align*}
\langle a \mid \rangle &\rightarrow \langle a, b \mid 1 \Rightarrow b \rangle \quad \text{(T1)} \\
&\rightarrow \langle a, b \mid 1 \Rightarrow b, b \Rightarrow bb \rangle \quad \text{(T2c)} \\
&\rightarrow \langle a, b \mid 1 \Rightarrow b, b \Rightarrow bb, 1 \Rightarrow bb \rangle \quad \text{(T2t)} \\
&\rightarrow \langle a, b \mid 1 \Rightarrow b, b \Rightarrow bb, 1 \Rightarrow bb \Rightarrow b \rangle \quad \text{(T2s)}
\end{align*}
\]

\[
\begin{align*}
\langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb, bb \Rightarrow b \rangle &\rightarrow \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb \rangle \quad \text{(T2t)} \\
&\rightarrow \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb \rangle \quad \text{(T2s)}
\end{align*}
\]

Also note that both presentations are “minimal”: there is no way to remove a derivable generator or a relation without changing the presented monoid. In particular, starting from the second presentation, we have to add relations first in order to be able remove the generator \( b \) and all the relations.

2.11 Generalization to Infinite Presentations

The above theorem 1 holds only for finite presentations, which is the way it is usually stated. It can easily be generalized to presentations of arbitrary cardinality by allowing the Tietze transformations to add sets of derivable generators and sets of derivable relations (instead of only one), what we call generalized Tietze transformations. The right way to think of those is as being obtained as cellular extensions of elementary Tietze transformations and we will prove in theorem 5 the following generalization of theorem 1, which was already known, see for instance \([10, \text{Sect. } 1.5]\):

**Theorem 1** Given two presentations \( P \) and \( Q \), we have \( P \simeq Q \) if and only if they are related by a zig-zag of generalized Tietze transformations, i.e., there exists a finite sequence of presentations 
\[
P = P^0, P^1, \ldots, P^{2n} = Q
\]
such that for every index \( i \), there is a generalized Tietze transformation from \( P^{2i} \) to \( P^{2i+1} \) and from \( P^{2i+2} \) to \( P^{2i+1} \).

**Remark 3** The naive generalization of Theorem 1, which states that two presentations have the same presented monoid if and only if they are related by a “possibly infinite zig-zag” of elementary Tietze transformations, is plain wrong (and this is not what the above theorem states). For instance, consider the following presentation of the monoid \( \mathbb{N} \):
\[
P = \langle a, b_i \mid a \Rightarrow b_i, b_i \Rightarrow b_{i+1} \rangle_{i \in \mathbb{N}}
\]
and write \( P^i \) for \( P \) with the relations \( a \Rightarrow b_i \) removed for \( i < k \). We have \( P^0 = P \) and the relation \( a \Rightarrow b_k \) is derivable in \( P^k \), so that there is an elementary Tietze transformation from \( P^{k+1} \) to \( P^k \). However, writing
\[
P^\infty = \langle a, b_i \mid b_i \Rightarrow b_{i+1} \rangle_{i \in \mathbb{N}}
\]
we have that $P^0$ does not present the same monoid as $P^\infty$ even though there is an “infinite sequence of elementary Tietze transformations” between them. Namely, $P$ presents $\mathbb{N}$ whereas $P^\infty$ presents $\mathbb{N} \# \mathbb{N}$, the free product of two copies of $\mathbb{N}$, and two are not isomorphic (the former is commutative whereas the later is not).

### 3 Model Categories

In this section, we recall elementary definitions and facts about model categories which we will use in the following and refer the reader to classical textbooks for details [7].

#### 3.1 Lifting Properties

Suppose fixed a category. A morphism $p : X \to Y$ has the right lifting property, or rlp, with respect to a morphism $i : A \to B$ when for every pair of morphisms $f : A \to X$ and $g : B \to Y$ such that $p \circ f = g \circ i$ there exists a morphism $h : B \to X$ making the following diagram commute:

$$
\begin{array}{c}
A & \xrightarrow{f} & X \\
\downarrow{i} & & \downarrow{p} \\
B & \xrightarrow{h} & X
\end{array}
$$

In this situation, we also say that $i$ has the left lifting property, or llp, with respect to $f$, and write $i \boxslash p$. Given two classes $\mathcal{L}$ and $\mathcal{R}$, we write $\mathcal{L} \boxslash \mathcal{R}$ whenever $i \boxslash p$ for every $i \in \mathcal{L}$ and $p \in \mathcal{R}$. We also write $\mathcal{L} \boxslash \mathcal{R}$ (resp. $\mathcal{R} \boxslash \mathcal{L}$) for the class of morphism with the rlp (resp. llp) with respect to $\mathcal{L}$ (resp. $\mathcal{R}$).

**Lemma 8** Given classes of morphisms $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{R}$ and $\mathcal{R}'$,

- $\mathcal{L} \subseteq (\boxslash \mathcal{L})$ implies $\mathcal{L} \boxslash \mathcal{R} \subseteq \mathcal{L}' \boxslash \mathcal{R}$,
- $\mathcal{R} \subseteq (\boxslash \mathcal{R})$ implies $\mathcal{L} \boxslash \mathcal{R} \subseteq \mathcal{L} \boxslash \mathcal{R}'$.

**Lemma 9** We suppose the ambient category cocomplete. A class of the form $\mathcal{L} = \boxslash \mathcal{R}$ contains isomorphisms and is closed under

- coproducts: for any family $(i_k : A_k \to B_k)_{k \in K}$ of morphisms in $\mathcal{L}$, the morphism

$$
\bigsqcup_{k \in K} i_k : \bigsqcup_{k \in K} A_k \to \bigsqcup_{k \in K} B_k
$$

is also in the class,

- pushouts: for any morphism $i : A \to B$ in $\mathcal{L}$ and morphism $f : A \to A'$, for any pushout diagram

$$
\begin{array}{c}
A & \xrightarrow{f} & A' \\
\downarrow{i} & & \downarrow{j} \\
B & \xrightarrow{g} & B'
\end{array}
$$

the morphism $j$ also belongs to $\mathcal{L}$,
– countable compositions: for any diagram

\[
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots
\]

consisting of morphisms \( f_k : A_k \to A_{k+1} \) in \( \mathcal{L} \) for \( k \in \mathbb{N} \), the canonical morphism

\[
A_0 \to \text{colim}_k A_k
\]

also belongs to \( \mathcal{L} \).

– retracts: given a morphism \( i : A \to B \) and two retracts \( r \circ s = \text{id}_{A'} \) and \( r' \circ s' = \text{id}_{B'} \), any morphism \( j : A' \to B' \) for which there is a commutative diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{j} & A' \\
\downarrow r & & \downarrow r \\
B' & \xrightarrow{s'} & B'
\end{array}
\]

also belongs to \( \mathcal{L} \).

Dually, any class for the form \( \mathcal{L} \mathbb{L} \) contains isomorphisms and is closed under products, pullbacks, countable compositions and retracts.

Given a class \( \mathcal{I} \) of morphisms, the class \( \mathcal{I} \)-cell of \( \mathcal{I} \)-cellular extensions is defined as the smallest class of morphisms closed under sums, pushouts and countable compositions (note that we do not require closure under retracts).

**Lemma 10** A morphism is an \( \mathcal{I} \)-cellular extension if and only if it is a composite of pushouts of sums of elements of \( \mathcal{I} \).

**Lemma 11** Given a class \( \mathcal{I} \) of morphisms, the class of \( \mathcal{I} \)-cellular extensions is included in \( \mathbb{L}(\mathcal{I} \mathbb{L}) \).

**Proof** By Lemma 8, we have \( \mathcal{I} \) included in \( \mathbb{L}(\mathcal{I} \mathbb{L}) \) and, by Lemma 9, this class is closed under sums, pushouts and countable compositions.

**Lemma 12** (Retract lemma) Given a factorization \( f = p \circ i \) such that \( f \mathbb{L} p \), \( f \) is a retract of \( i \).

Dually, given a factorization \( f = p \circ i \) such that \( i \mathbb{L} f \), \( f \) is a retract of \( p \).

**Proof** Since \( f \mathbb{L} p \), we have a map \( h \) such that

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow f & & \downarrow p \\
Z & \xleftarrow{h} & Z
\end{array}
\]

and the map \( f \) is thus a retract of \( i \):

\[
\begin{array}{ccc}
X & \xrightarrow{i} & X \\
\downarrow f & & \downarrow f \\
Z & \xrightarrow{h} & Y & \xrightarrow{p} & Z
\end{array}
\]

as claimed.
3.2 Weak Factorization System

A weak factorization system on a category is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that

- every morphism $f$ factors as $f = p \circ i$ with $i \in \mathcal{L}$ and $p \in \mathcal{R}$,
- $\mathcal{L} = \mathcal{R} \cap \mathcal{R}$ and $\mathcal{R} = \mathcal{L} \cap \mathcal{L}$.

**Remark 4** From Lemmas 9 and 12, the second condition can be equivalently be replaced by the two following conditions

- $\mathcal{L} \not\subseteq \mathcal{R}$,
- the classes $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

One of the main techniques in order to construct weak factorization systems is due to the following proposition [7, Sect. 2.1.2]. The notion of locally finitely presentable category is recalled in Sect. 4.5.

**Proposition 1** (Small object argument) Suppose that the category is cocomplete and locally finitely presentable. For any class $\mathcal{I}$ of morphisms, $(\mathcal{L} \cap \mathcal{I}, \mathcal{I})$ is a weak factorization system. Moreover, every morphism $f$ factors as $f = p \circ i$ where $i \in \mathcal{L} \cap \mathcal{I}$ is an $\mathcal{I}$-cellular extension and $p \in \mathcal{I}$. Moreover, every element of $\mathcal{L} \cap \mathcal{I}$ is a retract of an $\mathcal{I}$-cellular extension.

3.3 Model Category

A model category is a category equipped with three classes of morphisms

- $\mathcal{C}$: cofibrations,
- $\mathcal{W}$: weak equivalences,
- $\mathcal{F}$: fibrations

such that

- the category is complete and cocomplete,
- weak equivalences satisfy the 2-out-of-3 property: given a diagram

\[
\begin{array}{ccc}
\mathcal{C} & \mathcal{W} \cap \mathcal{F} & \\
\mathcal{C} \cap \mathcal{W}, \mathcal{F} & \end{array}
\]

if two morphisms belong to $\mathcal{W}$ then so does the third,

- $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ forms a weak factorization system,
- $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ forms a weak factorization system.

An object $X$ is cofibrant when the initial morphism $\emptyset \to X$ is a cofibration, and fibrant when the terminal morphism $X \to 1$ is a fibration.

From the previous section, we can expect that the weak factorization system can be generated as lifting completions of some classes. Indeed, many model categories are cofibrantly generated (also sometimes called combinatorial since we work here with locally presentable categories) [7, Theorem 2.1.19]:

**Proposition 2** In a locally presentable, complete and cocomplete category, suppose given a subcategory $\mathcal{W}$ closed under retracts and satisfying the 2-out-of-3 property, and two sets $\mathcal{I}$ and $\mathcal{J}$ of morphisms such that the inclusions

$\mathcal{I} \subseteq \mathcal{J} \cap \mathcal{W}$

$\mathcal{I} \subseteq \mathcal{J} \cap \mathcal{W}$
hold, one of them being an equality. Then we have a model category with \( \mathcal{W} \) as weak equivalences, \( \mathcal{I}(\mathcal{J}) \) as cofibrations and \( \mathcal{J}(\mathcal{I}) \) as fibrations. In this case, the elements of \( \mathcal{I} \) as \( \mathcal{J} \) are respectively called generating cofibrations and generating trivial cofibrations.

4 A Model Structure on Reflexive Presentations

Our aim is to construct a model structure on the category of reflexive presentations where weak equivalences correspond to presenting isomorphic categories and trivial cofibrations are Tietze transformations. The general strategy here is to use Proposition 2 and thus to satisfy all the required hypotheses: in particular, we want to show the equality \( \mathcal{I}(\mathcal{J}) = \mathcal{J}(\mathcal{I}) \cap \mathcal{W} \).

Unless otherwise mentioned, all the presentations considered in this section are supposed to be reflexive; the reason for this shall be discussed in Sect. 7.1. We first study some of the properties of the category of reflexive presentations.

4.1 Colimits

The category \( \mathbf{rPres} \) has coproducts. Namely, given two presentations \( P \) and \( Q \), their coproduct \( P \sqcup Q \) is given by

\[
(P \sqcup Q)_1 = P_1 \sqcup Q_1 \quad \quad (P \sqcup Q)_2 = P_2 \sqcup Q_2
\]

and the argument generalizes to show that the category has small coproducts. In particular, the initial object \( \emptyset \) is the empty presentation, with \( \emptyset_1 = \emptyset \) and \( \emptyset_2 = \emptyset \). Suppose given two morphisms of presentations

\[
P \xrightarrow{f} Q \xleftarrow{g}
\]

Their coequalizer is the presentation \( R \) whose set of generators is the coequalizer

\[
P_1 \xrightarrow{f} Q_1 \xrightarrow{g} R_1
\]

i.e., the quotient set \( R_1 = Q_1 / \sim \) under the smallest equivalence relation such that \( f(a) \sim f(b) \) for \( a \in P_1 \), the function \( h \) being the quotient map, and the set of relations is

\[
R_2 = \{ h^*(u) \Rightarrow h^*(v) \mid u \Rightarrow v \in Q_1 \}.
\]

The category is thus cocomplete. In particular, the pushout of a diagram

\[
Q^1 \xleftarrow{f^1} P \xrightarrow{f^2} Q^2
\]

is the presentation \( R \) whose set \( R_1 \) of generators is the pushout of the underlying sets of generators, with cococone maps \( h^1 : Q^1 \to R \) and \( h^2 : Q^2 \to R \), and relations

\[
R_2 = \{ h^1(u) \Rightarrow h^1(v) \mid u \Rightarrow v \in Q^1_1 \} \cup \{ h^2(u) \Rightarrow h^2(v) \mid u \Rightarrow v \in Q^2_2 \}.
\]

Note that the forgetful functor \( \mathbf{rPres} \to \mathbf{Set} \), sending a presentation to its underlying set of generators, preserves colimits.
4.2 Limits

The product \( P \times Q \) of two reflexive presentations \( P \) and \( Q \) has generators \((P \times Q)_1 = P_1 \times Q_1\) and the set \((P \times Q)_2\) of relations is

\[
\left\{ (a_1, a'_1) \ldots (a_m, a'_m) \Rightarrow (b_1, b'_1) \ldots (b_n, b'_n) \mid a_1 \ldots a_m \Rightarrow b_1 \ldots b_n \in P_2 \right\}.
\]

This generalizes to small products. In particular, the terminal presentation \(1\) has one generator \(a\) and all relations of the form \(a^m \Rightarrow a^n\) for \(m, n \in \mathbb{N}\). Given two morphisms of presentations

\[
P \xrightarrow{f} Q \xleftarrow{g} Q
\]

their equalizer \(R\) is given by

\[
R_1 = \{ a \in P_1 \mid f(a) = g(a) \}
\]

i.e., this is the equalizer of the underlying sets, and relations are

\[
R_2 = \{ u \Rightarrow v \in P_2 \mid f^*(u) = g^*(u) \text{ and } f^*(v) = g^*(v) \}.
\]

The category is thus complete and the forgetful functor \(rPres \rightarrow \text{Set}\) preserves limits.

4.3 Monomorphisms

A monomorphism \(f : P \rightarrow Q\) is a morphism whose underlying function \(f : P_1 \rightarrow Q_1\) is injective, i.e., the forgetful functor \(rPres \rightarrow \text{Set}\) reflects monomorphisms. In this sense, the monomorphisms of presentations inherit the properties of those of the categories of sets. For instance,

Lemma 13 \text{In } rPres, monomorphisms are stable under coproducts, pushouts and countable compositions.

\textbf{Proof} \ The forgetful functor to sets preserves coproducts, pushouts and countable compositions, and reflects monomorphisms.

Remark 5 \ These stability conditions are not generally true in a category. As a counterexample, in the category of commutative rings, the inclusion \(i : \mathbb{Z} \rightarrow \mathbb{Q}\) is a mono, but the sum (which is here the tensor product, and corresponds to the usual tensor product of \(\mathbb{Z}\)-modules)

\[
\text{id}_{\mathbb{Z}/2} \otimes i : \mathbb{Z}/2 = \mathbb{Z}/2 \otimes \mathbb{Z} \rightarrow \mathbb{Z}/2 \otimes \mathbb{Q} = 1
\]

is not a mono. It is however the case that monomorphisms are stable under pushout in a topos (and, more generally, an adhesive category).

4.4 Epimorphisms

Similarly, an epimorphism \(f : P \rightarrow Q\) is a morphism whose underlying function \(P_1 \rightarrow Q_1\) is a surjection.
4.5 Local Finite Presentability

We refer to [1] for a detailed presentation of the notions introduced here. An object $X$ of a category $C$ is *finitely presentable* when the representable functor

$$\text{Hom}(X, -) : C \to \text{Set}$$

preserves filtered limits: this means that for a diagram $(Y_i)_{i \in I}$ indexed by a filtered category $I$, the canonical morphism

$$\text{colim}_i \text{Hom}(X, Y_i) \to \text{Hom}(X, \text{colim}_i Y_i)$$

is an isomorphism. In particular, finitely presentable presentations objects are precisely the finite presentations.

A locally small category $C$ is *locally finitely presentable* when it is cocomplete and there is a set of finitely presentable objects such that every object of $C$ is a filtered colimit of objects in this set. In the case of the category of presentations, every presentation is the filtered colimit of its finite subpresentations, and the category $\text{rPres}$ is thus locally finitely presentable. The category $\text{Mon}$ is also locally finitely presentable, as the category of models of a Lawvere theory.

4.6 Weak Equivalences

We write $\mathcal{W}$ for the class of morphisms $f : P \to Q$ such that the induced morphism $\overline{f} : \overline{P} \to \overline{Q}$ between presented monoids is an isomorphism. Many of the properties of isomorphisms are thus reflected on weak equivalences:

**Proof** The class of isomorphisms in any category satisfies the 2-out-of-3 property. Isomorphisms are closed under sums. Namely, given two isomorphisms $i : A \to B$ and $i' : A \to B'$, the two following diagrams commute:

By universal property of coproducts, we deduced that $(i^{-1} + i'^{-1}) \circ (i + i') = \text{id}_{A+A'}$. Similarly, we can show $(i + i') \circ (i^{-1} + i'^{-1}) = \text{id}_{B+B'}$ and $i + i'$ is thus an isomorphism. Isomorphisms are also closed under pushouts. Namely, consider $j : A' \to B'$ the pushout of an isomorphism $i : A \to B$ along a morphism $f : A \to A'$. The two following diagrams commute, where $j' : B' \to A'$ is defined by universal property of the pushout:
This shows that $j \circ j' = \text{id}_{B'}$. Similarly, we have $j' \circ j = \text{id}_{A'}$ and $j$ is thus an isomorphism. Consider a countable composition of isomorphisms $f_i : A_i \to A_{i+1}$ as in (2). There is a cocone on $A_0$ consisting of the morphisms

$$f_0^{-1} \circ f_1^{-1} \circ \ldots \circ f_{i-1}^{-1} : A_i \to A_0$$

which is easily seen to be universal and the composite is thus (isomorphic to) $\text{id}_{A_0}$.

Consider a retract $j$ of an isomorphism $i$ as in (3). We claim that the morphism $j' = s \circ i^{-1} \circ r'$ is the inverse of $j$. Namely, one has

$$j' \circ j = s \circ i^{-1} \circ r' \circ j \quad j \circ j' = j \circ s \circ i^{-1} \circ r'$$

$$= s \circ i^{-1} \circ i \circ r \quad = s' \circ i \circ i^{-1} \circ r'$$

$$= s \circ r \quad = s' \circ r'$$

$$= \text{id}_{A'} \quad = \text{id}_{B'}$$

\[\square\]

### 4.7 Generating Cofibrations

Consider the presentation with one generator and no relation:

$$G = \langle a \mid \rangle.$$

Given $m, n \in \mathbb{N}$, we introduce notations for the following presentations, respectively with $n$ generators and no relation, and with one relation between words of respective lengths $m$ and $n$:

$$G^n = \langle a_1, \ldots, a_n \mid \rangle \quad R^{m,n} = \langle a_1, \ldots, a_{m+n} \mid a_1 \ldots a_m \Rightarrow a_{m+1} \ldots a_{m+n} \rangle.$$

We write $I$ for the class of morphisms, called generating cofibrations, consisting of the obvious inclusions of presentations

$$g : \emptyset \hookrightarrow G \quad r^{m,n} : G^{m+n} \hookrightarrow R^{m,n}$$

for some $m, n \in \mathbb{N}$.  

\[\square\] Springer
4.8 Cofibrations

We write $C = \square(I^{\square})$ for the class of morphisms whose elements are called *cofibrations*. Note that, given a presentation $P$, the pushouts

$$
\begin{array}{c}
\emptyset \xrightarrow{g} G \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{array}
$$

are respectively the presentation obtained from $P$ by adding a generator and the presentation obtained from $P$ by adding a relation (between the two words of $P_1^*$ specified by $f$).

**Lemma 14** Every presentation $P$ is cofibrant, in the sense that the initial morphism $\emptyset \hookrightarrow P$ is a cofibration.

**Proof** By Proposition 1, it is enough to show that the initial morphism $\emptyset \hookrightarrow P$ can be obtained as a composite of pushouts of generating cofibrations, which amounts to show that every presentation can be obtained from the empty one by adding generators and relations, which we will do in this order (generators first, and then relations). Given a relation $u \Rightarrow v \in P_2$, we have a canonical inclusion

$$
G^{[u]+[v]} \xrightarrow{r^{[u]+[v]}} R^{[u],[v]}
$$

and a canonical inclusion

$$
G^{[u]+[v]} \longrightarrow \bigsqcup_{a \in P_1} G.
$$

By summing those morphisms over relations $(u, v) \in P_2$, and post-composing with the codiagonal

$$
\bigsqcup_{(u,v) \in P_2} \bigsqcup_{a \in P_1} G \longrightarrow \bigsqcup_{a \in P_1} G.
$$

we obtain a diagram

$$
\bigsqcup_{(u,v) \in P_2} G^{[u]+[v]} \longrightarrow \bigsqcup_{(u,v) \in P_2} R^{[u],[v]}
$$

whose pushout is precisely $P$. Finally, we consider the composite of morphisms

$$
\emptyset \longrightarrow \bigsqcup_{a \in P_1} G \longrightarrow P
$$

where the second morphism is constructed in the cocone of the pushout. Again, this composite expresses the fact that any presentation can be constructed from the empty one by first adding all the generators, and then adding all the relations. \hfill \Box

The construction given in the above proof easily generalizes to show:
Lemma 15 Any monomorphism $f : P \to Q$ is a cofibration (and, in fact, an $I$-cellular extension).

Conversely, one has:

Lemma 16 Cofibrations are monomorphisms.

Proof The generating cofibrations are monomorphisms. Moreover, monomorphisms are closed under coproducts, under pushouts and countable compositions by Lemma 13. By Proposition 1, cofibrations are thus retracts of monomorphisms. We conclude using the fact that monomorphisms are closed under retracts.

Namely, suppose given a retract $j$ of a monomorphism $i$, as in (3), and two morphisms $h_1, h_2$ such that $j \circ h_1 = j \circ h_2$, we have

\[
\begin{align*}
  j \circ h_1 &= j \circ h_2 \\
  s' \circ j \circ h_1 &= s' \circ j \circ h_2 \\
  i \circ s \circ h_1 &= i \circ s \circ h_2 \\
  s \circ h_1 &= s \circ h_2 \\
  r \circ s \circ h_1 &= r \circ s \circ h_2 \\
  h_1 &= h_2
\end{align*}
\]

and we conclude. □

Corollary 1 The class $C$ of cofibrations is the class of monomorphisms in $r\text{Pres}$.

4.9 Trivial Fibrations

The morphisms in the class $I^{\sqsubset}$ are called trivial fibrations. From the lifting property with respect to the generators we immediately deduce,

Lemma 17 The morphisms $f : P \to Q$ in $I^{\sqsubset}$ are those

- whose underlying function $f : P_1 \to Q_1$ is surjective, and
- such that for every $u, v \in P^*_1$, $f^*(u) \Rightarrow f^*(v) \in Q_2$ implies $u \Rightarrow v \in P_2$.

Lemma 18 Trivial fibrations are weak equivalences: $I^{\sqsubset} \subseteq W$.

Proof Since $f : P_1 \to Q_1$ is surjective, we have that $\overline{f} : P \to Q$ is also surjective. We have to show that it is also injective in order to conclude. Suppose given $u, v \in P_1$ such that $f^*(u) \equiv f^*(v)$: we have a sequence

\[
f^*(u) = w_0 \Leftrightarrow w_1 \Leftrightarrow \ldots \Leftrightarrow w_n = f^*(v)
\]

where the arrows “$\Leftrightarrow$” mean that, for $0 \leq i < n$, there is a decomposition of $w_i$ and $w_{i+1}$ as

\[
w_i = t_i u_i v_i \quad \text{and} \quad w_{i+1} = t'_i u'_i v'_i \quad \text{with} \quad u_i \Rightarrow u'_i \in Q_2 \quad \text{or} \quad u_i \Leftarrow u'_i \in Q_2.
\]

Moreover, since $Q$ is reflexive, we can always suppose that this sequence is non-empty, i.e., $n > 0$: we can replace the empty sequence by the reflexivity relation $f^*(u) \Rightarrow f^*(u)$. By surjectivity, for $0 \leq i \leq n$, there are words $t_i^P, u_i^P, v_i^P, t'_i^P, u'_i^P, v'_i^P$ in $P_1^*$ whose image under $f$ is respectively $t_i, u_i, v_i, t'_i, u'_i, v'_i$, and we may moreover assume $t_0^P u_0^P v_0^P = u$ and $t_{n-1}^P u_{n-1}^P v_{n-1}^P = v$. Finally, since $f$ is a trivial fibration, we have $u_i \Rightarrow u'_i$ or $u_i \Leftarrow u'_i$ and we conclude that $u \equiv v$. □
From the results of Sect. 4.4, one has:

**Lemma 19** Every trivial fibration is an epimorphism.

### 4.10 Trivial Cofibrations

The class of **trivial cofibrations** is $\mathcal{C} \cap \mathcal{W}$ and consists of monomorphisms $f : P \to Q$ such that the induced morphism of monoids $\overline{f} : \overline{P} \to \overline{Q}$ is an isomorphism.

**Lemma 20** A morphism $f : P \to Q$ is a trivial cofibration when

- $f$ is a monomorphism,
- for every $a \in Q_1$, there exists $u \in P_1^*$ such that $f(\overline{u}) = \overline{a}$,
- for $u, v \in P_1^*$ such that $f(\overline{u}) = f(\overline{v})$, we have $\overline{u} = \overline{v}$.

**Proof** Suppose that $f$ is a trivial cofibration. Since $f$ is a cofibration, it is a monomorphism by Lemma 16. Given $a \in Q_1$, since $\overline{f}$ is surjective there is $u \in P_1^*$ such that $f(\overline{u}) = \overline{a}$, and we have $f(\overline{u}) = \overline{a}$. Given $u, v \in P_1^*$ such that $f(\overline{u}) = f(\overline{v})$, we have $\overline{f}(\overline{u}) = \overline{f}(\overline{v})$, thus $\overline{u} = \overline{v}$ in $\overline{P}$ since $\overline{f}$ injective, and finally $u = v$. Conversely, suppose given a monomorphism $f : P \to Q$. Given $a \in Q_1$, by hypothesis, there exists $u_a \in P_1^*$ such that $\overline{f}(\overline{u}_a) = \overline{a}$. Therefore, given $\overline{v} \in Q$, for some $v = a_1 \ldots a_n \in Q_1$, we have

$$\overline{f}(\overline{u}_{a_1} \ldots \overline{u}_{a_n}) = \overline{f}(\overline{u}_{a_1}) \ldots \overline{f}(\overline{u}_{a_n}) = \overline{a}_1 \ldots \overline{a}_n = \overline{v}$$

and $\overline{f}$ is thus surjective. Suppose given $u, v \in P_1^*$ such that $\overline{f}(\overline{u}) = \overline{f}(\overline{v})$: we have $f(\overline{u}) = f(\overline{v})$, thus $u = \overline{v}$ and finally $\overline{u} = \overline{v}$. □

**Lemma 21** The class of trivial cofibrations satisfies $\sqcap((\mathcal{C} \cap \mathcal{W}) \sqcap) = \mathcal{C} \cap \mathcal{W}$.

**Proof** By Lemma 8, we have $\mathcal{C} \cap \mathcal{W} \subseteq \sqcap((\mathcal{C} \cap \mathcal{W}) \sqcap)$. Conversely, by Proposition 1, every element of $\sqcap((\mathcal{C} \cap \mathcal{W}) \sqcap)$ is a retract of a $(\mathcal{C} \cap \mathcal{W})$-cellular extension, and thus belongs to $\mathcal{C} \cap \mathcal{W}$, because this last class is closed under sums, pushouts, countable compositions and retracts by Lemmas 9 and 14. □

### 4.11 Fibrations

The class $\mathcal{F}$ of **fibrations** is determined by the two other classes: should there be a model structure, it is necessarily $\mathcal{F} = (\mathcal{C} \cap \mathcal{W}) \sqcap$. An explicit description of fibrant objects is given by Lemmas 32 and 36.

### 4.12 Toward a Model Structure

We now have almost all the ingredients required to construct a model structure on the category $r\text{Pres}$ of reflexive presentations with $\mathcal{W}$ as weak equivalences, $\mathcal{C} = \sqcap(\mathcal{I} \sqcap)$ as cofibrations and $\mathcal{F} = (\mathcal{C} \cap \mathcal{W}) \sqcap$ as fibrations. Namely, assuming that $\mathcal{J} = \mathcal{C} \cap \mathcal{W}$ is a set, we can apply Proposition 2, whose hypothesis can be shows as follows. Closure under retracts and the 2-out-of-3 property for $\mathcal{W}$ were shown in Lemma 14. We first show $\mathcal{I} \sqcap \subseteq \mathcal{J} \sqcap \cap \mathcal{W}$. We have $\mathcal{J} = \mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}$ thus, by Lemma 8, $\mathcal{I} \sqcap = (\sqcap(\mathcal{I} \sqcap)) \sqcap = \mathcal{C} \sqcap \subseteq \mathcal{J} \sqcap$; and we have...
\[ T^\varnothing \subseteq W \text{ by Lemma 18.} \] Finally, the remaining inclusion is shown by Lemma 21, since we have
\[ \varnothing(J^\varnothing) = \varnothing((C \cap W)^\varnothing) = C \cap W = \varnothing(I^\varnothing) \cap W \]
which concludes the proof.

However, it is not clear at all that the class \( J \) should be a set. This is why our actual construction of a model category uses Smith’s theorem, which provides sufficient conditions in order to ensure that there exists a set \( J \) which can be used as a replacement for \( J \), in the sense that we have \( \varnothing(J^\varnothing) = J \).

### 4.13 Recognizing Weak Equivalences

In preparation for the use of Smith’s theorem, we show that morphisms of presentations which are weak equivalences can be characterized by factorization properties as follows. First note that the words in a given presentation can be represented in the following way.

**Lemma 22** Given a natural number \( n \) and a presentation \( P \), there is a bijection between morphisms \( G^n \to P \) and words in \( P_1^n \) of length \( n \).

**Proof** To a word \( u = b_1 \ldots b_n \) of length \( n \), we associate the morphism \( f : G^n \to P \) such that \( f(a_i) = b_i \) for \( 1 \leq i \leq n \). Conversely, a morphism \( f : G^n \to P \), we associate the word \( u = f(a_1) \ldots f(a_n) \). The two operations are easily seen to be inverse of each other. \( \square \)

**Lemma 23** Suppose given a morphism \( w : P \to Q \) and \( u \in P_1^n \). Writing \( f : G^{\lvert u \rvert} \to P \) for the morphism associated to \( u \) by Lemma 22, the morphism associated to \( w \circ f \) by Lemma 22 is \( w^*(u) \).

**Proof** Direct by inspection of the bijection constructed in the proof of Lemma 22. \( \square \)

Similarly, we can represent pairs of words as follows, where we write \( G^{m,n} \) instead of \( G^{m+n} \):

**Lemma 24** Given natural numbers \( m, n \) and a presentation \( P \), there is a bijection between morphisms \( G^{m,n} \to P \) and pairs of words \( u, v \in P_1^m \) with \( \lvert u \rvert = m \) and \( \lvert v \rvert = n \).

**Proof** A word \( u = b_1 \ldots b_m b_{m+1} \ldots b_{m+n} \) of length \( m + n \) in \( P_1^n \) can be seen as the pair of words \( b_1 \ldots b_m \) and \( b_{m+1} \ldots b_{m+n} \) in \( P_1^m \) of respective lengths \( m \) and \( n \). Thus the result by Lemma 22. \( \square \)

We now construct a family of presentations to represent pairs of equivalent words. Suppose fixed natural numbers \( m \) and \( n \). Given a natural number \( o \), consider the set
\[ G = \{a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_o\} \]
Suppose moreover given a non-zero natural number \( k \) and words \( u_i, v_i, v'_i, w_i \in G^* \) with \( 1 \leq i \leq k \), such that
\[ u_1 v_1 w_1 = a_1 \ldots a_n \quad u_i v'_i w_i = u_{i+1} v_{i+1} w_{i+1} \quad u_k v_k w_k = b_1 \ldots b_n \]
for every index \( i \) with \( 1 \leq i < n \). We write \( E^{m,n} \) for a presentation of the form
\[ E^{m,n} = \langle G \mid v_1 \leftrightarrow v'_1, \ldots, v_k \leftrightarrow v'_k \rangle \]
where \( v_i \leftrightarrow v'_i \) is either \( v_i \Rightarrow v'_i \) or \( v'_i \Rightarrow v_i \), called an equivalence presentation. Note that there is a canonical inclusion \( e : G^{m,n} \to E^{m,n} \) such that \( e(a_i) = a_i \) for \( 1 \leq i \leq m \)

\( \square \) Springer
and $e(a_{m+i}) = b_i$ for $1 \leq i \leq n$. Also note that $R^{n,n}$ is a particular case of $E^{m,n}$ (where $k = 1$). We could further reduce the number of equivalence presentations that we consider by imposing conditions such as the fact that there is no repeated generator within each $v_i$ or $v'_i$, that $v_i$ and $v'_i$ do not share any common generator for every index $i$, that each generator $c_j$ occurs within $v_i$ or $v'_i$ for some index $i$, and so on, but this will play no significant role in the following.

**Lemma 25** Suppose given a presentation $P$ and a pair of words $u, v \in P^*_1$ with $|u| = m$ and $|v| = n$, corresponding to a morphism $f : G^{m,n} \to P$ via Lemma 24. We have $u \equiv v$ if and only if there exists a morphism $g : E^{m,n} \to P$ making the following diagram commute, for some equivalence presentation $E^{m,n}$,

\[
\begin{array}{ccc}
G^{m,n} & \xrightarrow{f} & P \\
\downarrow & & \downarrow \\
E^{m,n} & \xrightarrow{g} & P
\end{array}
\]

where the vertical arrow is the canonical inclusion.

**Proposition 3** Suppose given a morphism of presentations $w : P \to Q$. The induced morphism $\overline{w} : \overline{P} \to \overline{Q}$ is surjective if and only if every square as on the left factors as on the right

\[
\begin{array}{ccc}
\varnothing & \xrightarrow{} & P \\
\downarrow & & \downarrow \\
G & \xrightarrow{g} & Q
\end{array}
\quad
\begin{array}{ccc}
\varnothing & \xrightarrow{} & G^n \\
\downarrow & & \downarrow \\
G & \xrightarrow{g'} & E^{1,n}
\end{array}
\]

where $g' : G \to E^{1,n}$ sends $a$ to $a_1$ and $h : G^n \to E^{1,n}$ sends $a_i$ to $b_i$.

**Proof** The presentation $\varnothing$ being initial, the square on the left is uniquely determined by $g$ which, by Lemma 22, corresponds to a generator $a$ in $Q_1$. The diagram on the right corresponds to the existence of a word $v$ of length $n$ in $P^*_1$ (given by $f''$ via Lemma 22) such that $a \equiv w^*(v)$ (this is given by $g''$ via Lemma 23 and 25). The above factorization property thus amounts to requiring that every generator $a$ in $Q_1$ is equivalent to some word of the form $w^*(v)$ for some $v \in P^*_1$, and thus that every word $u \in Q^*_1$ is equivalent to some word of the form $w^*(v)$ for some $v \in P^*_1$, i.e., that $\overline{w}$ is surjective.

**Proposition 4** Suppose given a morphism of presentations $w : P \to Q$. The induced morphism $\overline{w} : \overline{P} \to \overline{Q}$ is injective if and only if every square as on the left (where the vertical morphism $i : G^{m,n} \to E^{m,n}$ is the canonical inclusion into some equivalence presentation) factors as on the right, where $E^{1,n}_1$ is an equivalence presentation (the index stresses the fact that it might be different from $E^{m,n}$) and the square on the left is a pushout:
Proof The square on the left amounts to giving two words \( u \) and \( v \) in \( P^*_1 \) of respective lengths \( m \) and \( n \) (by \( f \) via Lemma 24) such that \( w^*(u) \cong w^*(v) \) (by \( g \) via Lemma 23 and 25). The diagram on the right corresponds to supposing that we have \( u \cong v \) (by \( f'' \) via Lemma 25) (note that \( g'' \) does not bring any information by the universal property of the pushout). The above factorization property thus amounts to requiring that for every words \( u, v \in P^*_1 \) such that \( w^*(u) \cong w^*(v) \), we have \( u \cong v \), i.e., that \( w : P \to Q \) is injective. \( \square \)

A presentation \( P \) is countable when the set \( P_1 \) is countable. We write \( \mathcal{W} \leq \omega \) for the class of weak equivalences \( w : P \to Q \) such that both \( P \) and \( Q \) are countable. Up to isomorphism, we can always suppose that we have \( P_1 \subseteq \mathbb{N} \) and \( Q_1 \subseteq \mathbb{N} \), so that \( \mathcal{W} \leq \omega \) is essentially a set (as opposed to a class).

**Proposition 5** Every commutative square as on the left, where \( R \) and \( S \) are finite and \( w \) is a weak equivalence, factors as a square as on the right with \( w^\omega \in \mathcal{W} \leq \omega : \)

\[
\begin{array}{ccc}
R & \xrightarrow{f} & P \\
\downarrow i & & \downarrow w \\
S & \xrightarrow{g} & Q
\end{array}
\]

\[
\begin{array}{ccc}
R & \xrightarrow{f^\omega} & P^\omega \\
\downarrow i^\omega & & \downarrow w^\omega \\
S & \xrightarrow{g^\omega} & Q^\omega
\end{array}
\]

Proof We are going to construct a sequence \( w^k : P^k \to Q^k \) of morphisms, indexed by \( k \in \mathbb{N} \), such that both \( P^k \) and \( Q^k \) are finite, the left square above factors through each \( w^k \), i.e.,

\[
\begin{array}{ccc}
R & \xrightarrow{f_k} & P^k \\
\downarrow i & & \downarrow w^k \\
S & \xrightarrow{g_k} & Q^k
\end{array}
\]

\[
\begin{array}{ccc}
R & \xrightarrow{f'_k} & P \\
\downarrow f''_k & & \downarrow w \\
S & \xrightarrow{g'_k} & Q
\end{array}
\]

we have \( P^k \subseteq P^{k+1} \) and \( Q^k \subseteq Q^{k+1} \), and the morphisms \( w^k \) as well as the factorizations respect those inclusions. Since \( P^k \) and \( Q^k \) are finite, it will be the case that we can moreover suppose that \( P^0_1 \) and \( Q^0_1 \) are initial segments of \( \mathbb{N} \), so that it makes sense to consider the smallest generator or relation satisfying a property. We first define \( w^0 \) to be a morphism isomorphic to \( i \), such that both \( P^0_1 \) and \( Q^0_1 \) are initial segments of \( \mathbb{N} \). Suppose \( w^k \) defined, we alternate between the two operations below in order to construct \( w^{k+1} \).

(A) Consider a generator \( a \in Q^k \) which has no antecedent under \( w^k : P^k \to Q^k \). By Lemma 22, it corresponds to a morphism \( G \to Q^k \). By Proposition 3, since \( w \) is surjective, the outer square of the diagram on the left factors as on the right, for some equivalence presentation \( E^{1,n} : \)

\[
\begin{array}{ccc}
\varnothing & \xrightarrow{u^k} & P^k \\
\downarrow \quad & & \downarrow \quad \\
G & \xrightarrow{s^k} & Q^k
\end{array}
\]

\[
\begin{array}{ccc}
\varnothing & \xrightarrow{C^k} & P \\
\downarrow \quad & & \downarrow \quad \\
G & \xrightarrow{E^{1,n}} & Q
\end{array}
\]
We define the pushouts \( P^{k+1} = P^{k} \sqcup G^{n} \) and \( Q^{k+1} = Q^{k} \sqcup E^{1,n} \), and the morphism \\
\( w^{k+1} : P^{k+1} \to Q^{k+1} \) is defined by the expected universal property of \( P^{k+1} \), see the diagram on the left below:

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & G^{n} \\
\downarrow & & \downarrow \\
P^{k} & \longrightarrow & P^{k+1} \\
\downarrow & & \downarrow \quad \ldots \quad \downarrow \quad j \\
G & \longrightarrow & E^{1,n} \\
\downarrow & & \downarrow \\
Q^{k} & \longrightarrow & Q^{k+1} \\
\end{array}
\]

We can construct the pushouts so that \( P^{k} \subseteq P^{k+1} \), \( Q^{k} \subseteq Q^{k+1} \) and both \( P^{k+1} \) and \( Q^{k+1} \) are initial segments of \( \mathbb{N} \). Finally, the horizontal morphisms of the diagram (c) are defined from (a) and (b) by universal property of the pushouts, and the outer square on the right above is a factorization of (a).

(B) Consider a pair of words \( u, v \in (P_{1}^{k})^{*} \) such that there is an equivalence \( (w^{k})^{*}(u) \equiv (w^{k})^{*}(v) \) and \( u \neq \bar{v} \) in \( P^{k} \). By Lemma 24, the pair of words \( (u, v) \) in \( P^{k} \) corresponds to a morphism \( G^{m,n} \to P^{k} \) and, by Lemma 25, the equivalence \( (w^{k})^{*}(u) \equiv (w^{k})^{*}(v) \) to a morphism \( E^{m,n} \to Q \), where \( m \) and \( n \) are the respective lengths of \( u \) and \( v \). By Proposition 4, the outer square on the left (where the morphism \( G^{m,n} \to E^{m,n} \) is the canonical inclusion) factors as on the right,

\[
\begin{array}{ccc}
G^{m,n} & \longrightarrow & P^{k} \\
\downarrow & & \downarrow w^{k} \\
E^{m,n} & \longrightarrow & Q^{k} \\
\end{array} \quad \leftrightarrow \quad \begin{array}{ccc}
G^{m,n} & \longrightarrow & P^{k} \\
\downarrow & & \downarrow w \\
E^{m,n} & \longrightarrow & Q \\
\end{array}
\]

We define the pushouts \( P^{k+1} = P^{k} \sqcup G^{m,n} E^{1,n} \) and \( Q^{k+1} = Q^{k} \sqcup E^{m,n} E \), and the morphism \\
\( w^{k+1} : P^{k+1} \to Q^{k+1} \) is defined by the expected universal property of \( P^{k+1} \), see the diagram on the left below:

\[
\begin{array}{ccc}
G^{m,n} & \longrightarrow & E^{m,n} \\
\downarrow & & \downarrow \\
P^{k} & \longrightarrow & P^{k+1} \\
\downarrow & & \downarrow \quad \ldots \quad \downarrow \quad j \\
E^{m,n} & \longrightarrow & E \\
\downarrow & & \downarrow \\
Q^{k} & \longrightarrow & Q^{k+1} \\
\end{array}
\]

We can construct the pushouts so that \( P^{k} \subseteq P^{k+1} \), \( Q^{k} \subseteq Q^{k+1} \) and both \( P^{k+1} \) and \( Q^{k+1} \) are initial segments of \( \mathbb{N} \). Finally, the horizontal morphisms of the diagram (f) are defined from (d) and (e) by universal property of the pushouts, and the outer square on the right above is a factorization of (d).

By alternatively using (A) and (B), we construct a sequence of \( w^{k} : P^{k} \to Q^{k} \) of morphisms of presentations, such that every generator of \( Q^{k} \) is eventually handled by (A) and every relation in \( Q^{k} \) between words in the image is eventually handled by (B). More explicitly, this
can be performed as follows. We say that a generator \( a \in P^k \) has \emph{appeared} at step \( i \) when \((u, v) \in P^i\) and \( a \notin P^{i-1} \); more generally, we say that a word \( u \in (P^k)^* \) has appeared at step \( i \) if it contains a generator which has appeared at step \( i \) and all the generators it contains have appeared at step \( j \geq i \); we say that a pair \((u, v)\) of words have appeared at step \( i \) if \( u \) has appeared at step \( i \) and \( v \) has appeared at step \( j \geq i \), or the converse. We then iteratively perform the following steps in order to define the terms of the sequence \((w^k)_{k \in \mathbb{N}}\): supposing that \( w^k \) is defined, we defined \( w^{k+1} \) to \( w^{2k+2} \) as follows.

1. We construct \( w^{k+1} \) by using (A) on the smallest generator \( a \) which is not in the image of \( w^k \) (we take \( w^{k+1} = w^k \) if no such generator exists).
2. We construct \( w^{k+1+i+1} \), for \( 0 \leq i \leq k \), where \( w^{k+1+i+1} \) is defined from \( w^{k+1+i} \) by using (B) on the smallest pair of words \( u, v \in (P^k)^* \) which has appeared at step \( i \), such that there is an equivalence \( (w^k)^* \overset{\text{Q}}{\Rightarrow} (w^k) \) and \( u \neq v \in P^k \) (we take \( w^{k+1+i+1} = w^{k+1+i} \) if no such pair exists).

Finally, we define the colimits \( P^\omega = \bigcup_{k \in \mathbb{N}} P^k \) and \( Q^\omega = \bigcup_{k \in \mathbb{N}} Q^k \), which are countable presentations as unions of finite ones, and \( w^\omega : P^\omega \to Q^\omega \) as the morphism induced by the cocone consisting of the morphisms \( w^k : P^k \to Q^k \subseteq Q^\omega \). Given a square

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & P^\omega \\
\downarrow & & \downarrow w^\omega \\
G & \overset{g}{\longrightarrow} & Q^\omega
\end{array}
\]

the generator corresponding to the morphism \( g : G \to Q \) was handled by (A) at some step \( k \) (or had a lifting from the beginning, this case being simple). There is therefore a factorization of the square on the left as on the middle

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & P^k+1 \\
\downarrow & & \downarrow w^k+1 \\
G & \overset{g}{\longrightarrow} & Q^k+1
\end{array}
\]

for some equivalence presentation \( E^{1,n} \), from which we deduce the factorization

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & G^n \\
\downarrow & & \downarrow w^n \\
G & \overset{g'}{\longrightarrow} & E^{1,n} \\
\downarrow & & \downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \\
G & \overset{g''}{\longrightarrow} & E^{1,n} & \overset{g''}{\longrightarrow} & Q^{k+1} & \overset{g}{\longrightarrow} & Q^\omega
\end{array}
\]

of the original square by post-composing with the canonical inclusions into the colimit.

By Proposition 3, the morphism \( \overline{w}^\omega \) is thus surjective. Similarly, by considering steps (B) and Proposition 4, one can show that \( \overline{w}^\omega \) is surjective. The morphism \( w^\omega \) is thus a weak equivalence as desired. \( \square \)
4.14 A Model Structure

Given a class $\mathcal{W}$ of morphisms and a morphism $i$, we say that $\mathcal{W}$ satisfies the solution set condition at $i$ if there is a set $\mathcal{W}_i \subseteq \mathcal{W}$ of morphisms such that any commutative square as on the left, with $w \in \mathcal{W}$, factors as a square as on the right, for some $w' \in \mathcal{W}_i$:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow^i & & \downarrow^w \\
B & \xrightarrow{g} & Y \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
A & \xrightarrow{f'} & X' \\
\downarrow & & \downarrow^w \\
B & \xrightarrow{g'} & Y' \\
\end{array}
$$

(4)

By extension, given a set $\mathcal{I}$ of morphisms, we say that $\mathcal{W}$ satisfies the solution set condition at $\mathcal{I}$, if it satisfies the solution set condition at any $i \in \mathcal{I}$. The following theorem is due to Smith, see [3, Theorem 1.7]:

**Theorem 2** In a locally finitely presentable category, suppose given a subcategory $\mathcal{W}$ and a set $\mathcal{I}$ of morphisms such that

1. $\mathcal{W}$ is closed under retracts and has the 2-out-of-3 property,
2. $\mathcal{I} \subseteq \mathcal{W}$,
3. $\mathcal{I} \cap \mathcal{W}$ is closed under pushouts and countable compositions,
4. $\mathcal{W}$ satisfies the solution set condition at $\mathcal{I}$.

Then there is a cofibrantly generated model structure with $\mathcal{I}$ as cofibrations, $\mathcal{W}$ as weak equivalences and $(\mathcal{I} \cap \mathcal{W})$ as fibrations.

**Theorem 3** There is a model structure on the category $\text{rPres}$ of reflexive presentations with $\mathcal{W}$ (as defined in Sect. 4.6) as weak equivalences, $\mathcal{C} = \mathcal{I}$ as cofibrations (with $\mathcal{I}$ as defined in Sect. 4.7) and $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})$ as fibrations.

**Proof** We apply theorem 2. First point was shown in Lemma 14, second point in Lemma 18. For third point, the closure of $\mathcal{I}$ under pushouts and countable compositions was shown in Lemma 8 and the one of $\mathcal{W}$ in Lemma 14, from which we deduce the one of their intersection. The last point is the object of Proposition 5.

4.15 A Quillen Functor

The category $\text{Mon}$ can canonically be equipped with the trivial model structure where weak equivalences are isomorphisms and every morphism is both fibrant and cofibrant. The presentation functor $\text{rPres} \to \text{Mon}$ described in Sect. 2.4 is a left adjoint (Lemma 2 and remark 1) which trivially preserves cofibrations and trivial cofibrations, and is thus a Quillen functor. Moreover, this functor reflects weak equivalences and, given a presentation $P$, the counit $P \to \langle \bar{P} \rangle$ of the adjunction is a weak equivalence: by [7, Corollary 1.3.16], the presentation functor is thus a Quillen equivalence. By [7, Proposition 1.3.13], this means that the derived functor induces, as expected, an equivalence of categories between the localization of $\text{rPres}$ under weak equivalences and the one of $\text{Mon}$ (which is $\text{Mon}$ itself):

$$
\text{Ho}(\text{rPres}) \cong \text{Ho}(\text{Mon}) \simeq \text{Mon}.
$$
5 Tietze Transformations as Trivial Cofibrations

In Sect. 5.1 below, we introduce a class $J$ of morphisms of reflexive presentations such that pushouts of morphisms in this class correspond to elementary Tietze transformations. Contrarily to what one could expect, this family does not generate all trivial cofibrations: we have a strict inclusion $/\boxslash(J/\boxslash) \subsetneq C \cap W$. However, we show that the two classes coincide for morphisms with fibrant codomain: we thus say that the class $J$ is pseudo-generating, following the terminology of Simpson [11, Sect. 8.7].

5.1 Pseudo-Generating Trivial Cofibrations

We write $J$ for the class of morphisms of $r\text{Pres}$, called pseudo-generating trivial cofibrations

$$\langle a_1, \ldots, a_m | \rangle \hookrightarrow \langle a_1, \ldots, a_m, a_{m+1} | u \Rightarrow a_{m+1} \rangle$$

$$\langle a_1, \ldots, a_m | \rangle \hookrightarrow \langle a_1, \ldots, a_m | u \Rightarrow u \rangle$$

$$\langle a_1, \ldots, a_{m+n} | u \Rightarrow v \rangle \hookrightarrow \langle a_1, \ldots, a_{n+m} | u \Rightarrow v, v \Rightarrow u \rangle$$

$$\langle a_1, \ldots, a_{m+n+p} | u \Rightarrow v, v \Rightarrow w, u \Rightarrow w \rangle$$

for some $m, n, p \in \mathbb{N}$ with

$$u = a_1 \ldots a_m$$

$$w = a_{m+1} \ldots a_{m+n}$$

$$v = a_{m+1} \ldots a_{m+n+p}$$

$$w' = a_{m+n+p+1} \ldots a_{m+n+p+q}$$

Lemma 26 Given a pseudo-generating cofibrations $j : P \rightarrow Q$ and a morphism of presentations $f : P \rightarrow P'$, consider the pushout $j' : P' \rightarrow Q'$ of $j$ along $f$:

$$\begin{array}{ccc}
P & \xrightarrow{f} & P' \\
\downarrow j & & \downarrow j' \\
Q & \rightarrow & Q'
\end{array}$$

then there is an elementary Tietze transformation from $P'$ to $Q'$, and conversely every elementary Tietze transformation arises in this way.

Proof Pushout of the five kinds of morphisms in $J$ precisely give rise to the five kinds of Tietze transformations (T1), (T2r), (T2s), (T2t) and (T2c). \qed

We are thus tempted to call generalized Tietze transformation a morphism in $J$-cell. In particular, every element of $J$ is itself a Tietze transformation and thus, by theorem 1,

Lemma 27 Generating trivial cofibrations are weak equivalences: $J \subseteq W$.

Moreover, those morphisms are monomorphisms and thus, by Lemma 15,

Lemma 28 The pseudo-generating trivial cofibrations are cofibrations: $J \subseteq /\boxslash(I/\boxslash)$.

Remark 6 By general properties [7, Proposition 2.1.18], we have that morphisms in $/\boxslash(I/\boxslash)$ are retracts of Tietze transformations. We do not know whether the morphisms in $/\boxslash(J/\boxslash)$ are precisely Tietze transformations or not.
5.2 Morphisms in \([\mathcal{J}]\)

The following lemmas show that the morphisms in the class \([\mathcal{J}]\) are trivial cofibrations. We will however see in Sect. 5.4 that not every trivial cofibration is in this class, i.e., the inclusion is strict.

**Proof** By Lemma 28, we have that \(\mathcal{J} \subseteq [\mathcal{I}]\). Thus, by Lemma 8, we have \([\mathcal{J}] \subseteq ([\mathcal{I}] \subseteq [\mathcal{I}]\). \(\square\)

**Lemma 29** We have \([\mathcal{J}] \subseteq \mathcal{W}\).

**Proof** By Lemma 26, a pushout of an element in \(\mathcal{J}\) is an elementary Tietze transformation and thus a weak equivalence by Lemma 6. By Proposition 1, any element of \([\mathcal{J}]\) is a countable composition of elementary Tietze transformations, and thus a weak equivalence by lemma 14. \(\square\)

**Lemma 30** We have \([\mathcal{J}] \subseteq \mathcal{C} \cap \mathcal{W}\).

**Proof** By Lemmas 30 and 29. \(\square\)

5.3 Pseudo-fibrations

The morphisms in \(\mathcal{J}\) are called pseudo-fibrations. A pseudo-fibrant object \(P\) is one such that the terminal morphism \(P \to 1\) is a pseudo-fibration.

**Lemma 31** A presentation \(\mathcal{P}\) is pseudo-fibrant when

- for every word \(u \in \mathcal{P}_1\), there is a generator \(a \in \mathcal{P}_1\) such that \(u \Rightarrow a \in \mathcal{P}_2\),
- the relation \(\mathcal{P}_2\) on \(\mathcal{P}_1\) is a congruence.

*In particular, we have \(u \Rightarrow v\) if and only if \(u \Rightarrow v \in \mathcal{P}_2\).*

More generally, pseudo-fibrations can be described as follows:

**Lemma 32** A morphism \(f : \mathcal{P} \to \mathcal{Q}\) is a pseudo-fibration when

- for every \(u \in \mathcal{P}_1^*\) and \(b \in \mathcal{Q}_1\) such that \(f(u) \Rightarrow b \in \mathcal{Q}_2\), there exists \(a \in \mathcal{P}_1\) with \(f^*(a) = b\) and \(u \Rightarrow a \in \mathcal{P}_2\),
- for every \(u \in \mathcal{P}_1^*\),
  \[f^*(u) \Rightarrow f^*(v) \in \mathcal{Q}_2 \text{ implies } u \Rightarrow v \in \mathcal{P}_2,\]
- for every \(u, v \in \mathcal{P}_1^*\) with \(u \Rightarrow v \in \mathcal{P}_2\),
  \[f^*(u) \Rightarrow f^*(v) \in \mathcal{Q}_2 \text{ implies } v \Rightarrow u \in \mathcal{P}_2,\]
- for every \(u, v \in \mathcal{P}_1^*\) with \(u \Rightarrow v \in \mathcal{P}_2\) and \(v \Rightarrow w \in \mathcal{P}_2\),
  \[f^*(u) \Rightarrow f^*(w) \in \mathcal{Q}_2 \text{ implies } u \Rightarrow w \in \mathcal{P}_2,\]
- for every \(u, v, w, w' \in \mathcal{P}_1^*\) with \(u \Rightarrow v \in \mathcal{P}_2\),
  \[f^*(wvw') \Rightarrow f^*(ww') \in \mathcal{Q}_2 \text{ implies } wvw' \Rightarrow ww' \in \mathcal{P}_2.\]
Proof. By Lemma 30, we have \( \square(\mathcal{J} \square) \subseteq \mathcal{C} \cap \mathcal{W} \). Therefore, by Lemma 8,
\[
\mathcal{F} = (\mathcal{C} \cap \mathcal{W})\square \subseteq (\square(\mathcal{J} \square))\square = \mathcal{J} \square.
\]

Lemma 33  For any object \( P \), there exists a pseudo-fibrant object \( \tilde{P} \), called a pseudo-fibrant replacement of \( P \), together with a map \( P \to \tilde{P} \) in \( \square(\mathcal{J} \square) \).

Proof  Use the small object argument (Proposition 1) to factor the terminal morphism \( P \to 1 \) as a morphism in \( \square(\mathcal{J} \square) \) followed by a morphism in \( \mathcal{J} \square \).

5.4 \( \mathcal{J} \) Is Not Generating

Contrarily to what one might expect, the class \( \mathcal{J} \) is not a generating class for the trivial cofibrations. This can be seen by observing that the following inclusion does not hold:
\[
\mathcal{J} \square \cap \mathcal{W} \not\subseteq \mathcal{I} \square
\]
For instance, consider the inclusion
\[
\langle a \mid \rangle \to \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb \rangle
\]
which corresponds to the example developed Sect. 2.10. This morphism is both a pseudo-fibration since the only relations to lift are the reflexivity relations (which are not noted here, see Sect. 2.6) and a weak equivalence since both presented monoids are \( \mathbb{N} \). However, it is not a trivial fibration since it is not surjective on generators. The same example can be used to show that the inclusion
\[
\square(\mathcal{I} \square) \cap \mathcal{W} \not\subseteq \square(\mathcal{J} \square)
\]
do not hold either: the map above is a trivial cofibration since it is both a monomorphism and a weak equivalence, but it cannot be obtained as a retract of a composite of pushouts of sums of elements of \( \mathcal{J} \). Namely, the generator \( b \) has to be added using a Tietze transformation (T1), but the relations are not of the right form. Intuitively, the relation \( 1 \Rightarrow b \) has to be added first, see Sect. 2.10.

Remark 7  As a simpler (but less illuminating) example, consider the inclusion
\[
\langle a \mid \rangle \to \langle a, b \mid b \Rightarrow aa \rangle
\]
which is not an elementary Tietze transformation, because of the chosen orientation for the relation (T1).

Similarly, the inclusion
\[
\langle a, b, c, d \mid aa \Rightarrow bb, bb \Rightarrow cc, cc \Rightarrow dd \rangle \to \langle a, b, c, d \mid aa \Rightarrow bb, bb \Rightarrow cc, cc \Rightarrow dd, aa \Rightarrow dd \rangle
\]
is a pseudo-fibration and a weak equivalence, but not a trivial fibration one since the relation \( aa \Rightarrow dd \) cannot be lifted.

5.5 \( \mathcal{J} \) Is Pseudo-Generating

It is interesting to note that the inclusions of previous section are satisfied if we restrict to fibrations whose codomain is fibrant. We begin by a reciprocal to Lemma 30:
Lemma 34 Any trivial cofibration \( i : P \to Q \) with pseudo-fibrant codomain \( Q \) belongs to \( \mathcal{J} \text{-cell} \), and thus to \( \mathcal{J} \text{-}(\mathcal{J} \text{-}) \).

Proof Since \( i \) is a trivial cofibration, it is an injection and we have \( \overline{P} = \overline{Q} \). For simplicity, we suppose that \( i \) is an inclusion. For every generator in \( a \in Q_1 \setminus P_1 \), there is a word \( u_a \in P_1^i \) such that \( u_a \in a \) and therefore \( u_a \Rightarrow a \in Q_2 \) since \( Q \) is pseudo-fibrant (\( Q_2 \) is a congruence). Writing \( P^0 \) for \( P \) with the generator \( a \) and a relation \( u_a \Rightarrow a \) added, for every \( a \in Q_1 \setminus P_1 \), we have a morphism \( P \to P^0 \) in \( \mathcal{J} \text{-cell} \) factoring \( f \) (the inclusion \( P \to P^0 \) can be expressed as a pushout of a coproduct of pseudo-generating trivial cofibrations of the first form). We write \( P^{k+1} \) for the presentation obtained from \( P^k \) by adding

- a relation \( u \Rightarrow u \) for every word \( u \) over \( P_1^k \),
- a relation \( v \Rightarrow u \) for every relation \( u \Rightarrow v \in P_2^k \),
- a relation \( u \Rightarrow w \) for every relations \( u \Rightarrow v, v \Rightarrow w \in P_2^k \),
- a relation \( wuw' \Rightarrow wv \) for every relation \( u \Rightarrow v \in P_2^k \) and words \( w, w' \) over \( P_1^k \).

There is a morphism \( P^k \to P^{k+1} \) in \( \mathcal{J} \text{-cell} \). Every generator of \( Q \) gets added at the first step and every relation of \( Q \) gets added at some step. Therefore \( Q = \text{colim}_k P^k \) and \( i \) belongs to \( \mathcal{J} \text{-cell} \).

Remark 8 The above proof essentially consists in using the small object argument to construct a factorization \( i = g \circ f \) with \( f \in \mathcal{J} \text{-cell} \) and \( g \in \mathcal{J} \text{-}(\mathcal{J} \text{-}) \), and observing that \( g \) can be chosen to be an identity when \( Q \) is pseudo-fibrant.

Lemma 35 Any pseudo-fibration \( p : P \to Q \in \mathcal{J} \text{-}(\mathcal{J} \text{-}) \) with pseudo-fibrant target \( Q \) is a fibration, i.e., \( p \in (\mathcal{C} \cap \mathcal{W}) \text{-}(\mathcal{J} \text{-}) \).

Proof Suppose given a trivial cofibration \( i : P' \to Q' \in \mathcal{C} \cap \mathcal{W} \) and two morphisms \( f : P' \to P \) and \( g : Q' \to Q \) such that \( p \circ f = g \circ i \). By Lemma 33, we can consider a pseudo-fibrant replacement \( \tilde{Q}' \) of \( Q' \) together with the associated morphism \( j : Q' \to \tilde{Q}' \) in \( \mathcal{J} \text{-}(\mathcal{J} \text{-}) \), and thus in \( \mathcal{C} \cap \mathcal{W} \) by Lemma 30. By orthogonality, there is a map \( k : \tilde{Q}' \to Q \) such that \( k \circ j = g \).

Finally, by Lemma 34 \( (j \circ i) \mathcal{J} \text{-} p \), from which we deduce the existence of \( h : \tilde{Q}' \to P \) such that \( h \circ j \circ i = f \) and \( p \circ h = k \).

\[
\begin{array}{ccc}
P' & \xrightarrow{f} & P \\
\downarrow{C \cap \mathcal{W} \ni \exists h} & & \downarrow{p \in \mathcal{J} \text{-}} \\
Q' & \xrightarrow{g} & Q \\
\downarrow{C \cap \mathcal{W} \ni (C \cap \mathcal{W}) \ni j} & \downarrow{k} & \downarrow{\in \mathcal{J} \text{-}} \\
\tilde{Q}' & \xrightarrow{\in \mathcal{J} \text{-}} & 1
\end{array}
\]

Therefore the morphism \( h \circ j : Q' \to P \) is a filler and thus \( i \mathcal{J} \text{-} p \).

Lemma 36 Pseudo-fibrant and fibrant objects coincide.

Proof By Lemma 32, any fibrant object is pseudo-fibrant. Conversely, by Lemma 35, it suffices to check that the terminal object is pseudo-fibrant, which can be verified directly.

Lemma 37 Given a monoid \( M \), its standard presentation \( \langle M \rangle \) is fibrant.

Proof The presentation \( \langle M \rangle \) satisfies the conditions of Lemma 31 and is thus pseudo-fibrant and thus fibrant by Lemma 36.
6 Tietze Equivalences as Cospans

In this section we reconstruct the proof of the Tietze theorem by showing that any two presentations of the same monoid can be related by a cospan of generalized Tietze transformations.

6.1 Coproduct

We begin by showing that, under suitable hypothesis, the canonical injections into coproducts are cofibrations.

Lemma 38 In a model category, when X is cofibrant, the canonical injections \( \iota_0 : Y \to Y \sqcup X \) and \( \iota_1 : Y \to X \sqcup Y \) are cofibrations.

**Proof** We have a pushout diagram

\[
\begin{array}{ccc}
\emptyset & \rightarrow & Y \\
\downarrow & \searrow \iota_1 & \\
X & \rightarrow & X \sqcup Y \\
\end{array}
\]

When \( X \) is cofibrant, the initial map into \( X \) is a cofibration, and the map \( \iota_1 \) is thus also a cofibration, as a pushout of a cofibration. The other case is similar. \( \square \)

6.2 Weak Equivalences as Cospans

We now recall the contents of the proof of the celebrated Ken Brown lemma, which shows that every weak equivalence between cofibrant objects factors as a cospan of trivial cofibrations.

Lemma 39 (Ken Brown’s lemma) In a model category, every weak equivalence \( w : X \to Y \) between cofibrant objects \( X \) and \( Y \) factors as \( w = p \circ i \) where \( i \) is a trivial cofibration and \( p \) a trivial fibration which admits a section by a trivial cofibration \( j : Z \to Y \).

**Proof** We can factor the map \( (w, \text{id}_Y) : X \sqcup Y \to Y \) as a cofibration \( k : X \sqcup Y \to Z \) followed by a trivial fibration \( p : Z \to Y \). Since \( X \) and \( Y \) are cofibrant, by Lemma 38, the injections into \( X \sqcup Y \) are cofibrations. We define \( i = k \circ \iota_0 \) and \( j = k \circ \iota_1 \):

\[
\begin{array}{ccc}
X & \xrightarrow{\iota_0} & X \sqcup Y \\
\downarrow i & \searrow \iota_1 & \downarrow j \\
Y & \xrightarrow{\iota_0} & Y \\
\end{array}
\]

\[
\begin{array}{ccc}
X \sqcup Y & \xrightarrow{k} & Z \\
\downarrow & \searrow \iota_1 & \downarrow \text{id}_Y \\
Y & \xrightarrow{p} & Y \\
\end{array}
\]

\[
\begin{array}{ccc}
Z & \xrightarrow{p} & Y \\
\downarrow & \searrow j \\
X \sqcup Y & \xrightarrow{k} & Z \\
\downarrow i & \searrow \iota_1 \\
X & \xrightarrow{\iota_0} & X \sqcup Y \\
\end{array}
\]

The maps \( i \) and \( j \) are cofibrations as composites of cofibrations and are weak equivalences by the 2-out-of-3 property. \( \square \)

Remark 9 In the previous lemma, the cospan \((i, j)\) can be considered as a factorization of \( w \), in the sense that we have \( j \circ w = j \circ p \circ i = i \).
Remark 10 In a model category where monomorphisms are cofibrations (such as the case of interest here, see Lemma 16), a simpler argument can be given: since $Y$ is cofibrant and $p$ is a trivial fibration, the diagram

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & Z \\
\downarrow & \nearrow & \\
Y & \longrightarrow & Y
\end{array}
\]

admits a filler $j : Y \rightarrow Z$, which is a section of $p$; moreover, since $j$ is a monomorphism, it is a cofibration, and it is a weak equivalence by the 2-out-of-3 property.

Theorem 4 In a model category $\mathcal{M}$ in which every object is cofibrant, every isomorphism in $\text{Ho}(\mathcal{M})$ is the localization of a cospan of trivial cofibrations.

Proof Consider an isomorphism $f : X \rightarrow Y$ in $\text{Ho}(\mathcal{M})$. We write $\mathcal{M}'$ for the full subcategory of $\mathcal{M}$ whose objects are fibrant. The fibrant replacement functor $F : \mathcal{M} \rightarrow \mathcal{M}'$ induces an equivalence between the homotopy categories [7, Proposition 1.2.3]. Moreover, $\text{Ho}(\mathcal{M}')$ is a quotient of $\mathcal{M}'$ by homotopy equivalences [7, Theorem 1.2.10], the map $Ff$ is thus a homotopy equivalence and thus a weak equivalence [7, Proposition 1.2.8]. The map $f$ is thus the localization of a span of weak equivalences

\[
\begin{array}{ccc}
X & \xleftarrow{i_X} & FX \\
\downarrow & \nearrow & \downarrow \overline{f} \\
Y & \longrightarrow & FY
\end{array}
\]

where $i_X : X \rightarrow FX$ is the trivial cofibration associated to the fibrant replacement. By Lemma 39, we thus have two cospans of trivial cofibrations

\[
\begin{array}{ccc}
X' & \leftarrow & FY \\
\uparrow & \nearrow & \downarrow \overline{f} \\
X & \longrightarrow & Y
\end{array}
\]

and we conclude to the existence of one cospan of trivial cofibrations using the fact that trivial cofibrations are closed under pushouts. \qed

6.3 Tietze Equivalences

We can now conclude with the abstract proof of the Tietze theorem.

Theorem 5 In the category $r\text{Pres}$, two presentations $P$ and $Q$ are such that $P \simeq Q$ if and only if there is a cospan of generalized Tietze transformations (of morphisms in $\mathcal{J}$-cell) from $P$ to $Q$.

Proof Suppose given two presentations $P, Q \in r\text{Pres}$ such that $P \simeq Q$. With the model structure introduced in Sect. 4, this can be rewritten as $\text{Ho}(P) \simeq \text{Ho}(Q)$, and therefore we deduce that there is a cospan of trivial cofibrations

\[
\begin{array}{ccc}
& \longrightarrow & R \\
\downarrow & \nearrow & \\
P & \longrightarrow & Q
\end{array}
\]
Up to taking a fibrant replacement of $R$ and suppose that $R$ is fibrant and thus pseudo-fibrant by Lemma 36. We deduce that this is a span of Tietze transformations by Lemma 34. Conversely, Tietze transformations are weak equivalences by Lemma 29 and thus $P$ and $Q$ become isomorphic after localizing under weak equivalences.

7 Variants and Extensions

Many variants of the situation considered here could be thought of and are left for future work.

7.1 Non-reflexive Presentations

If we consider the category $\text{Pres}$ of (non-necessarily reflexive) presentations, many of the constructions performed in previous section can still be carried over. However, Lemma 18 does not hold anymore, preventing the construction of a model category: the elements of $T^\otimes$ are not necessarily weak equivalences. As a counter-example consider the morphism $\langle a, b \mid \rangle \to \langle c \mid \rangle$.

It belongs to $T^\otimes$ since it satisfies the conditions of lemma 17 (which still holds): it is surjective on generators and lifts every required relation since there are none. It is however not a weak equivalence since the monoids presented by the source and the target are respectively $\mathbb{N} * \mathbb{N}$ and $\mathbb{N}$ which are not isomorphic (the first one is not commutative for instance). We expect that there is however a right semi-model structure in the sense of [2], whose cofibrations are generated by $T$.

7.2 Multisets of Relations

The notion of presentation can be modified in order to allow multiple relations with the same source and the same target: such a presentation $P$ consists of a set $P_1$ of generators together with a set $P_2$ of relations equipped with source and target maps $s, t : P_2 \to P_1$. Here, an element $\alpha \in P_2$ with $s(\alpha) = u$ and $t(\alpha) = v$ encodes a relation $u \Rightarrow v$. We expect that this modification does not significantly changes the situation studied here.

7.3 Presentations of Categories

As a further generalization, one can consider presentations of categories. Such a presentation $P$ of a category consists of a set $P_0$ of objects, a set $P_1$ of generators for morphisms equipped with source and target maps $s_0, t_0 : P_1 \to P_0$, and a set $P_2$ of relations equipped with source and target maps $s_1, t_1 : P_2 \to P_1^*$ such that $s_0^* \circ s_1 = s_0^* \circ t_1$ and $t_0^* \circ s_1 = t_0^* \circ t_1$. Here, $P_1^*$ denotes the morphisms of the free category over the graph $(P_0, P_1)$ and the category presented by $P$ is obtained by quotienting the morphisms of this free category under the congruence generated by $P_2$. The notion of presentation of monoid of Sect. 7.2, is the particular case where $P_0 = \{ \star \}$ is reduced to one element. We expect the proofs of this paper to generalize to this setting.
7.4 Presentations of \( n \)-categories

This notion of presentation sketched in the previous section, is a particular case of the notion of polygraph, see [4], which generalizes to give presentations of \( n \)-categories. It would be interesting to see whether the model structure extends to this case.

7.5 Presentations of Groupoids

The notion of Tietze transformation was originally developed for presentations of groups. It would be interesting to generalize the model structure to this case, as well as generalizations of presentations of groupoids.

7.6 Coherent Presentations

A notion of Tietze transformation for coherent presentations of categories is introduced in [5]. We would like to investigate this case, as well as, more generally, developing a notion of Tietze transformation for resolutions of categories by \((\infty, 1)\)-polygraphs.

Data Availability  Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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