A Model with Interacting Composites

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Abstract
We show that we can construct a model in $3 + 1$ dimensions where only composite scalars take place in physical processes as incoming and outgoing particles, whereas constituent spinors only act as intermediary particles. Hence while the spinor-spinor scattering goes to zero, the scattering of composites gives nontrivial results.

Keywords: Composite particles; trivial models; spinor models

1 Introduction

Fermions are an essential ingredient in nature. It is an ever repeating idea to build a model of nature using only fermions, where all the observed bosons are constructed as composites of these entities. In solid state physics electrons, fermions in character, come together to form bosons [1]. Heisenberg spent years to formulate a "theory of everything" using only fermions [2]. Another attempt in this direction came with the work of Gürsey, [3], where a non-polynomial Lagrangian was written to describe self-interacting fermions. Kortel found solutions to this theory [2] which were shown to be instantonic and meronic solutions much later [5].

One of us, with collaborators, tried to make quantum sense of this model a while ago [?], finding that even if these attempts are justified, this model went to a trivial model as the cut-off is removed. We calculated [9] several processes involving incoming and outgoing spinors which gave exactly the naive quark model results, missing the observed logarithmic behaviour predicted by QCD calculations.

During the last twenty years, many papers were written on making sense out of "trivial models", interpreting them as effective theories without taking the cutoff to infinity. One of these models is the Nambu-Jona-Lasinio model [10]. Although this model is shown to be a trivial in four dimensions [?], since the
coupling constant goes to zero with a negative power of the logarithm of the ultraviolet cut-off, as an effective model in low energies it gives us important insight to several processes. In QCD, the studies of hadron mass generation through spontaneous symmetry breaking, important clues to results of the nuclear pairing interaction and the approximate validity of the interacting boson model can be cited as some examples.

There were also attempts to couple the Nambu-Jona-Lasinio model to a gauge field, the so called gauged Nambu-Jona-Lasinio model [14, 15] to be able to get a non-trivial field theory. It was shown that if one has sufficient number of fermion flavors, such a construction is indeed possible [16]. Actually the Nambu-Jona-Lasinio model was constructed based on an analogy with the BCS theory of superconductivity, where fermions come together to form the bosonic interaction necessary to explain the physical phenomena [17].

Here we want to give a new interpretation of our old work. First we see that the polynomial form of the original model really does not correspond to it in the exact sense. The two versions have obey different symmetries. Then we go to higher orders in our calculation in the new version, beyond the one loop for the scattering processes. It is shown that by using the Dyson-Schwinger and Bethe-Salpeter equations some of the fundamental processes can be better understood. We see that while the non-trivial scattering of the fundamental fields is not allowed, bound states can scatter from each other with non-trivial amplitudes. This phenomena is another example of treating the bound states, instead of the principal fields, as physical entities, that go through physical processes such as scattering.

In our model we need an infinite renormalization in one of the diagrams. Further renormalization is necessary at each higher loop, like any other renormalizable model. The difference between our model and other renormalizable models lies in the fact that, although our model is a renormalizable one using naive dimensional counting arguments, we have only one set of diagrams which is divergent. We need to renormalize only one of the coupling constants by an infinite amount. This set of diagrams, corresponding to the scattering of two bound states to two bound states, have the same type of divergence, i.e. $\frac{1}{\varepsilon}$ in the dimensional regularization scheme for all odd number of loops. The contributions from even number of diagrams are finite, hence require no infinite renormalization. The scattering of two scalars to four, or to any higher even number of scalars is finite, as expected to have a renormalizable model, whereas production of spinors from the scattering of scalars go to zero as the cut-off is removed.

We will outline the model as is given in Refs. [6] and [7] in Section I and give our new results in subsequent sections.

2 The Model

We start with the Lagrangian of the model given as

$$L = i\bar{\psi}\gamma^\mu\partial_\mu \psi + g\bar{\psi}\psi\phi + \lambda(\bar{\psi}\psi - a\phi^3).$$

(1)
Here the only terms with kinetic part are the spinors. Here $\lambda$ is a Lagrange multiplier field, $\phi$ is a scalar field with no kinetic part, $g$ and $a$ are coupling constants. This expression contains two constraint equations, obtained from writing the Euler-Lagrange equations for the $\lambda$ and $\phi$ fields.

$$g\bar{\psi}\psi - 3\lambda a\phi^2 = 0,$$

(2)

and

$$g\bar{\psi}\psi - a\phi^3 = 0.$$  

(3)

The Lagrangian given above is just an attempt in writing the original Gursey Lagrangian

$$L = i\bar{\psi}\partial\psi + g'(\bar{\psi}\psi)^{4/3},$$

(4)

in a polynomial form.

We see that the $\gamma^5$ invariance of the original Lagrangian is retained in the form written in Eq. (1). In this form, when $\psi$ is sent to $\gamma^5\psi$, the scalar fields $\phi$ and $\lambda$ are sent to their negatives (minus times the field). This discrete symmetry prevents $\psi$ from acquiring a finite mass in higher orders.

We see that these two models are not equivalent since the latter does not obey one symmetry obeyed by the former one. If we take the original Lagrangian

$$L = \bar{\psi}\partial\psi + (\bar{\psi}\psi)^{4/3} + s(\bar{\omega}(\bar{\psi}\psi + \phi^3)),$$

(5)

and define a symmetry operation $s$ where $s\bar{\omega} = \lambda, s\lambda = 0, s\phi = \omega, s\omega = 0, s\psi = s\bar{\psi} = 0$, so that $L$ is invariant under $s$. If we replace the original Lagrangian by that given in equation (1), by replacing $(\bar{\psi}\psi)^{4/3}$ by a combination of $\phi^4, \bar{\phi}\psi\psi$ we see that this symmetry is not retained. We, therefore, take the second model as a model which only approximates the original one, without claiming equivalence. It is a constrained model which will replace in the original model only in a ”naive” sense.

To quantize the latter system consistently we proceed through the path integral method. In addition to the usual spinor-Dirac primary constraints, fixing the momenta corresponding to the spinor fields $\psi$ and $\bar{\psi}$, we have two new primary constraints, setting the canonical momenta corresponding to the scalar fields $\lambda$ and $\phi$ equal to zero. The primary Hamiltonian is obtained by adding these four constraints multiplied by arbitrary constants to the canonical Hamiltonian, obtained from the Lagrangian given in Eq. (1). The consistency requirement of all the primary constraints, which is setting the Poisson bracket of the constraint equations with the primary Hamiltonian equal to zero, results in two new, secondary constraints, given by our Eqs. (2) and (3). When we calculate the Poisson bracket of these constraints with the primary Hamiltonian to check whether additional constraints are present, we see that the system is closed, determining all the arbitrary constants in the primary Hamiltonian.

Next we compute the determinant of the Poisson brackets of all the second class constraints, the so called Faddeev-Popov determinant. We see that the spinor-Dirac constraints, resulting from the canonical momenta of the spinor
fields has no field dependent contribution to the Faddeev-Popov determinant. This determinant is given as

$$\Delta_F = \left[\det\{\theta_i, \theta_j\}\right]^{1/2} = det\phi^4.$$  \hspace{1cm} (6)

the field dependent contribution coming from the constraints in Eqs. (2) and (3).

We write the generating functional for the Green’s functions of the model as

$$Z = \int D\pi D\chi \delta(\theta_i) \Delta_F \exp\left(-i \int (\chi \pi - H_c)\right).$$  \hspace{1cm} (7)

Here $\chi$ is the generic symbol for all the fields, $\pi$ is the generic symbol for all momenta and $\theta$ is the generic symbol for all the constraints in the model. Performing all the momenta integrals we obtain

$$Z = \int D\psi D\bar{\psi} D\phi D\lambda \Delta_F \left(\frac{\Delta_F}{3 det\phi^2}\right) \exp\left(i \int L' d^4x\right),$$ \hspace{1cm} (8)

where

$$\frac{\Delta_F}{det\phi^2} = det\phi^2,$$ \hspace{1cm} (9)

This contribution is inserted into the Lagrangian using ghost fields $c$ and $c^*$, and the resulting lagrangian reads

$$L'' = \bar{\psi}[\phi + g(\phi + \lambda)]\psi - a\lambda\phi^3 + ic^*\phi^2 c.$$ \hspace{1cm} (10)

We can rewrite this expression by defining

$$\Phi = \phi + \lambda,$$ \hspace{1cm} (11)

$$\Lambda = \phi - \lambda,$$ \hspace{1cm} (12)

as

$$L'' = \bar{\psi}[\phi + g(\phi + \lambda)]\psi - \frac{a}{16}(\Phi^4 + 2\Phi^3\Lambda - 2\Phi\Lambda^3 - \Lambda^4) + i c^*(\Phi^2 + 2\Phi\Lambda + \Lambda^2)c.$$ \hspace{1cm} (13)

By this transformation the $\Lambda$ field is decoupled from the spinor sector of the lagrangian.

The integration over the spinor fields in the functional yields the effective action which is expressed in terms of $\Phi, \Lambda$ and $c, c^*$ fields only.

$$S_{eff} = -Trln(i\theta + g\Phi) + \int d^4x \left[\frac{a}{16}(\Phi^4 + 2\Phi^3\Lambda - 2\Phi\Lambda^3 - \Lambda^4) - \frac{i}{4}c^*(\Phi^2 + 2\Phi\Lambda + \Lambda^2)c\right].$$ \hspace{1cm} (14)

The condition to get rid of the tadpole contribution, which is setting the first derivative of the effective action with respect to the $\Phi$ and $\Lambda$ fields to zero, gives us two equations

$$\frac{-i g}{(2\pi)^2} Tr \int \frac{d^4p}{p - g\nu} - \frac{a}{8}(2\nu^3 + 3\nu^2 s - s^3) = 0,$$ \hspace{1cm} (15)
and
\[ \frac{a}{8}(v^3 - 3vs^2 - 2s^3) = 0. \] (16)

In these expressions, \(-v\) and \(s\) are the vacuum expectation values of the fields \(\Phi\) and \(\Lambda\) respectively and the vacuum expectation value of the ghost fields are set to zero.

A consistent solution satisfying both equations is

\[ s = v = 0, \] (17)

Since the \(\gamma^5\) symmetry is not dynamically broken, no mass is generated for the fermion dynamically. In this respect this model differs from the famous Gross-Neveu model, \([18]\) where this dynamical breaking takes place. It also differs from the Nambu-Jona-Lasinio model. The main reason for this behaviour is the conformal invariance present in the model. Gürsey’s original intention was to construct a conformal invariant model, at least classically. We find that upon quantization of our approximate model at least one phase exists which respects this symmetry.

The fermion propagator is the usual Dirac propagator in lowest order, as can be seen from the Lagrangian. The second derivative of the effective action with respect to the \(\Phi\) field gives us the induced inverse propagator for the \(\Phi\) field, with the infinite part given as

\[ \text{Inf} \left[ \frac{ig^2}{(2\pi)^4} \int \frac{d^4p}{p(p + q)} \right] = \frac{g^2q^2}{4\pi\epsilon}. \] (18)

Here dimensional regularization is used for the momentum integral and \(\epsilon = 4 - n\). We see that the \(\Phi\) field propagates as a massless field.

When we study the propagators for the other fields, we see that no linear or quadratic term in \(\Lambda\) exists, so the one loop contribution to the \(\Lambda\) propagator is absent. Similarly the mixed derivatives of the effective action with respect to \(\Lambda\) and \(\Phi\) is zero at one loop, so no mixing between these two fields occurs. We can also set the propagators of the ghost fields to zero, since they give no contribution in the one loop approximation. The higher loop contributions are absent for these fields.

### 3 Dressed Fermion Propagator

In this section we calculate the above results in higher orders. To justify our result that no mass is generated for the fermion we may study the Bethe-Salpeter equation obeyed for this propagator. The Dyson-Schwinger equation for the spinor propagator is written as

\[ iA\phi + B = i\phi + 4\pi\epsilon \int \frac{d^4q}{(iA\phi + B)(p - q)^2}. \] (19)
Here $iA\not\!\!\!\!\!\not\!p + B$ is the dressed fermion propagator. We use the one loop result for the scalar propagator. After rationalizing the denominator, we can take the trace of this expression over the $\gamma$ matrices to give us

$$B = 4\pi\epsilon \int d^4q \frac{B}{(A^2q^2 + B^2)(p - q)^2}.$$  \hspace{1cm} (20)

The angular integral on the right hand side can be performed to give

$$B = 4\pi\epsilon \left[ \int_0^{p^2} dq^2 \frac{q^2B}{p^2(A^2q^2 + B^2)} + \int_{p^2}^\infty dq^2 \frac{B}{(A^2q^2 + B^2)} \right]. \hspace{1cm} (21)$$

If we differentiate this expression with respect to $p^2$ on both sides, we get

$$\frac{dB}{dp^2} = -4\pi\epsilon \left[ \int_0^{p^2} dq^2 \frac{q^2B}{(p^2)^2(A^2q^2 + B^2)} \right]. \hspace{1cm} (22)$$

This integral is clearly finite. We get zero for the right hand side as $\epsilon$ goes to zero. Since mass is equal to zero in the free case we get this constant equal to zero. This choice satisfies the Eq. (19).

The similar argument can be used to show that $A$ is the dressed spinor propagator is a constant. We multiply Eq. (18) by $\not\!\!\!\!\!\not\!p$ and then take the trace over the spinor indices. We end up with

$$p^2A = p^2 + 4\pi\epsilon \left[ \int_0^{p^2} dq^2 \left( \frac{(q^4)A}{p^2(A^2q^2 + B^2)} \right) + \int_{p^2}^\infty dq^2 \frac{p^2A}{(A^2q^2 + B^2)} \right]. \hspace{1cm} (23)$$

We divide both sides by $p^2$ and differentiate with respect to $p^2$. The end result

$$\frac{dA}{dp^2} = -8\pi\epsilon \int_0^{p^2} dq^2 \left( \frac{(q^4)A}{(p^2)^3(A^2q^2 + B^2)} \right) \hspace{1cm} (24)$$

shows that $A$ is a constant as $\epsilon$ goes to zero. Since the integral is finite, it equals unity for the free case, we take $A = 1$.

### 4 Higher Orders

If our fermion field had a color index $i$ where $i = 1...N$, we could perform an $1/N$ expansion to justify the use of only ladder diagrams for higher orders for the scattering processes. Although in our model the spinor has only one color, we still consider only ladder diagrams anticipating that one can construct a variation of the model with $N$ colors.

We first see that we do not need infinite regularization for the $\langle \bar{\psi}\psi\phi \rangle$ vertex. The one loop correction to the tree vertex involves two fermion and one $\phi$ propagator and one integration. The infinity coming from the momentum integration is cancelled by the $\epsilon$ in the $\phi$ propagator. All the higher order
contributions vanish because the powers of $\epsilon$ exceed the number of infinities coming momentum integrations. Indeed there is only one momentum integration that results in an infinity. We see that there is only a finite renormalization of the spinor-scalar coupling constant $g$.

We come to the same result after we write the Dyson-Schwinger equation for this vertex. We need the result of the four fermion scattering kernel to be able to perform this calculation. There is no four fermion coupling in our Lagrangian; so, this process will use at least one scalar propagator. Since the scalar particle propagator has an $\epsilon$ factor, this process vanishes as $\epsilon$ goes to zero. All higher orders, including the one loop contribution also vanish as $\epsilon$ goes to zero, since they have higher powers of $\epsilon$.

We can justify our claims also by writing the Bethe-Salpeter equations for this process. The Bethe-Salpeter equation for the fermion interaction reads as

$$G^{(2)}(p, q; P) = G^{(2)}_0(p, q; P)$$

$$+ \frac{1}{(2\pi)^8} \int d^4p' d^4q' G^{(2)}_0(p, p'; P)K(p', q'; P)G^{(2)}(q', q; P).$$ (25)

Here $G^{(2)}_0(p, q; P)$ is two non-crossing spinor lines, $G^{(2)}(p, q; P)$ is the proper four point function. $K$ is the Bethe-Salpeter kernel.

We note that this expression involves the four spinor kernel which we found to be of order $\epsilon$. This expression can be written in the quenched ladder approximation [1], where the kernel is separated into a scalar propagator with two spinor legs joining the proper kernel. If the proper kernel is of order $\epsilon$, the loop involving two spinors and a scalar propagator can be at most finite that makes the whole diagram in first order in $\epsilon$. This fact also shows that there is no nontrivial spinor-spinor scattering.

We use this result in calculating the Dyson-Schwinger equation for the spinor-scalar vertex. This vertex involves the finite coupling constant $g$ plus the diagram where the scalar particle goes into two spinors which go to the four spinor Kernel. Here the $\epsilon$ factor coming from the Kernel is cancelled with the loop integration. The loop does not involve any scalar propagators, so it diverges as $\frac{1}{\epsilon}$. The result is finite renormalization of the three point vertex. Hence the spinor-scalar coupling constant does not run.

We see that the only infinite renormalization is needed for the four $\phi$ vertex; hence the coupling constant for this process runs. The first correction to the tree diagram is the box diagram. This diagram has four spinor propagators and give rise to a $\frac{1}{\epsilon^2}$ type divergence. Since we included the four $\phi$ term in our original lagrangian, we can renormalize the coupling constant of this vertex to incorporate this divergence. The finite part of this diagram is just a constant, renormalizing the initial coupling constant by a finite amount. There are no higher infinities for this vertex. The two loop diagram contains a $\phi$ propagator which makes this diagram finite. The three-loop diagram is made out of eight spinor and two scalar lines. At worst we end up with a first order infinity of the form $\frac{1}{\epsilon^3}$ using the dimensional regularization scheme. Higher order ladder
diagrams give at worst the same type of divergence. This divergence for the four scalar vertex can be renormalized using standard means.

5 Conclusion

As a result of this analysis we end up with a model where there is no scattering of the fundamental fields, i.e. the spinors, whereas the composite fields, the scalar field, can take part in a scattering process. Here we do not give the exact expression for this amplitude, but it will be a series in $a$ and even powers of $g$, starting with $g^4$.

We can also have scattering processes where two scalar particles go to an even number of scalar particles. In the one loop approximation all these diagrams give finite results, like the case in the standard Yukawa coupling model. Since going to an odd number of scalars is forbidden by the $\gamma^5$ invariance of the theory, we can also argue that scalar $\phi$ particles can go to an even number of scalar particles only. This assertion is easily checked by diagrammatic analysis.

Any diagram which describes the process of producing spinor particles out of two scalars contains scalar propagators. The lowest of these diagrams where two scalars go to two spinors vanish since it either involves a triangle diagram made out of spinors, or a box diagram, made out of three spinors and one scalar. It vanishes due to fall of the scalar propagator in the latter case, although it is not zero unless the cut-off is removed. The diagram which involves the production of four spinors out of two scalars carries two scalar propagators and the diagram vanishes with the first power of $\epsilon$.

As a result of our calculations we find a model which is trivial for the constituent spinor fields, whereas finite results are obtained for the scattering of the composite scalar particles. The coupling constant for the scattering of the composite particles run, whereas the coupling constant for the spinor-scalar interaction does not run.

In the classical model, described by the Lagrangian given by Eq. (4), we used one coupling constant $g'$, which is divided into two as $g$ and $a$ in Eq. (1). We see that these two behave differently in the quantum case.

Our model is a toy model. We could not yet find a physical system that is effectively described by it.

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