Grassmannian $\sigma$-models and Topological-Antitopological Fusion

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Abstract

We study the topological-antitopological fusion equations for supersymmetric $\sigma$-models on Grassmannian manifolds $G(k, N)$. We find a basis in which the metric becomes diagonal and the $tt^*$ equations become tractable. The solution for the metric of $G(k, N)$ can then be described in terms of the metric for the $CP^{N-1}$ models. The IR expansion helps clarify the picture of the vacua and gives the soliton numbers and masses. We also show that the $tt^*$ equation for $G(k, N)$ in the large $N$ limit is solvable, for any $k$. 

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1 Introduction

We investigate two-dimensional supersymmetric sigma models on Grassmannian target spaces using techniques developed in the study of $N = 2$ supersymmetric theories in two dimensions. These techniques follow from the possibility of describing many models in terms of Landau-Ginsburg type actions (for a review, see [1, 2]). These actions are characterized by a superpotential which obeys non-renormalization theorems and can be used to study both conformal and massive theories. The superpotential encodes the chiral ring of the model under consideration and many of the properties of the model can be determined through a metric and a new index computed from the chiral ring. This metric, defined on the space of Ramond supersymmetric ground-states (which, by spectral flow, are in one-to-one correspondence with the chiral superfields), is the ordinary inner product in the Hilbert space of states and is determined by the topological-antitopological differential fusion equations obtained from the chiral ring. The metric can be thought of as a generalization of the Zamolodchikov metric away from the conformal point. This metric and the new index derived from it are helpful for understanding various properties of the model, like the scale and coupling dependence and the soliton spectrum. The new index can be obtained from a set of integral equations by means of the thermodynamical Bethe ansatz, given the exact S-matrix. The index being related to the metric, these integral equations are equivalent to the $tt^*$ differential equations. However both sets of equations have proven difficult to solve, and their equivalence has only been shown numerically for some simple cases.

The differential equations simplify when one considers a model with an infinite number of chiral superfields, as was done in [18] for the $\mathbb{CP}^{N-1}$ model in the large $N$ limit. In this case the $tt^*$ equations determining the metric become an equation first studied in the context of self-dual gravity which is related to a symmetry reduction of Plebański’s “heavenly” equation for a self-dual Kähler potential in D=4. The model is solved using finite temperature results and methods inspired from self-dual gravity. This example shows how the $tt^*$ formalism contains a lot of information about a particular quantum field theory without having to solve the theory completely.

Here we consider supersymmetric $\sigma$-models on Grassmannian target spaces $G(k, N)$, the $\mathbb{CP}^{N-1}$ model being the simplest Grassmannian model $G(1, N)$. These $(1 + 1)$ dimensional field theories have many analogies with $(3 + 1)$ dimensional non-abelian gauge theories: both have instantons, are conformally invariant at the classical level, and have dynamical mass generation and asymptotic freedom at the quantum level.

We derive the $tt^*$ equations for the ground-state metric of the Grassmannian $\sigma$-models $G(k, N)$ for any $k$ and $N$. The equations become solvable in terms of the metric for $\mathbb{CP}^{N-1}$ when written in a basis for which the metric becomes diagonal. The boundary conditions in the infra-red and ultra-violet are then easily obtained. In the IR limit, a clear picture emerges for the numbers and masses of solitons interpolating between various vacua and completes the description in [15].

The $tt^*$ equations for the $G(k, N)$ models are then studied in the large $N$ limit. We
show that the equation is solvable in this limit for any \( k \geq 2 \), once the solution for \( k = 1 \) (the large \( N \mathbb{C}P^{N-1} \) model) is known.

The Grassmannian models are interesting in other ways, since they have not been solved completely as quantum field theories, in particular their exact S-matrices and spectrum have not been fully determined, as well as their finite temperature properties. The results reported here should provide further insight into solving them.

In section 2, we review the Grassmannian \( \sigma \)-models. In section 3., we review the fusion equations for the \( \mathbb{C}P^{N-1} \) model and derive the \( tt^* \) equations and their asymptotics for \( \mathbb{G}(k, N) \). In section 4., we give some examples, and in section 5. we derive the \( tt^* \) equation for large \( N \) and fixed \( k \), and give its solution. The concluding remarks are in section 6.

2 Grassmannian \( \sigma \)-models

Supersymmetric non-linear \( \sigma \)-models define maps from spacetime into a riemannian target manifold \( M \). If the target manifold is Kähler, the model will have \( N = 2 \) supersymmetry. (For a review, see refs [1, 6, 7, 8]). We study here supersymmetric \( \sigma \)-models defined on the complex Grassmannian manifolds \( \mathbb{G}(k, N) \). These spaces have (complex) dimension \( k(N-k) \) and consist of all \( k \) dimensional subspaces of the complex vector space \( \mathbb{V} \cong \mathbb{C}^N \).

The supersymmetric Ramond ground states of an \( N = 2 \) \( \sigma \)-model (and thus, by spectral flow, the chiral superfields) are in one-to-one correspondence with the complex cohomology classes of the target space. The chiral ring will be a deformation of the classical cohomology ring of the manifold, due to instanton corrections. For the \( \mathbb{C}P^{N-1} \) models, the classical cohomology ring is generated by the Kähler form \( X \), with relations \( X^N = 0 \). Instanton corrections modify the ring relations to \( X^N = \beta \).

We now describe the classical cohomology of the Grassmannian manifold \( \mathbb{G}(k, N) \). Over the Grassmannian \( \mathbb{G}(k, N) \) there is a “tautological” \( k \)-plane bundle \( E \), whose fiber at each point \( x \) in \( \mathbb{G}(k, N) \) is the \( k \)-plane labelled by \( x \), and a complementary bundle \( F \) of rank \( (N-k) \), producing the exact sequence

\[
0 \to E \to V \cong \mathbb{C}^N \to F \to 0.
\]  

(2.1)

The classical cohomology of \( \mathbb{G}(k, N) \) is generated from the Chern classes \( X_i \equiv c_i(E^*) \), with certain relations, where \( X_i \) is a \((i,i)\) form and \( E^* \) is the dual of \( E \) (see [3, 10, 11, 12]).

Define the following generating function

\[
c_t = \sum_{i=0}^{k} c_t(E^*) t^i = \sum_{i=0}^{k} X_i t^i
\]

(2.2)

In the same way, let \( Y_j = c_j(F^*) \), where \( F^* \) is dual to \( F \).

It follows from (2.1) that

\[
c_t(E^*) \cdot c_t(F^*) = \sum_{i \geq 0} X_i t^i \cdot \sum_{j \geq 0} Y_j t^j = 1
\]

(2.3)
and the classical cohomology is generated by the $X_i$, $Y_j$ with conditions (2.3).

In particular they imply

$$Y_j = 0, \quad \text{for} \quad N - k + 1 \leq j \leq N. \tag{2.4}$$

The quantum cohomology ring results from a modification to relations (2.3) of the form:

$$c_t(E^*) \cdot c_t(F^*) = 1 + (-1)^{N-k} t^N \tag{2.5}$$

Then,

$$\left( \sum_{i=0}^{k} X_i t^i \right) \cdot \left( \sum_{j=0}^{N-k} Y_j t^j \right) = 1 + (-1)^{N-k} t^N \tag{2.6}$$

and (2.4) gets modified to

$$Y_{N+1-i} + (-1)^{N-k} \delta_{i,1} = 0, \quad 1 \leq i \leq k \tag{2.7}$$

The quantum cohomology ring is generated by polynomials in the $X_i$’s (if one eliminates for example the $Y_j$’s) subject to the constraints (2.6) and its dimension is $N!/k!(N-k)!$.

We are now interested in considering the Grassmannian $\sigma$-models as quantum field theories and finding the chiral superfields which are in one-to-one correspondence with the cohomology classes of the target space, and thus generate the quantum cohomology ring.

A convenient way to find these fields and determine the chiral ring is through a Landau-Ginsburg type description for the effective action of the model under consideration.

Many $N = 2$ supersymmetric theories in two dimensions admit a Landau-Ginsburg description if their superspace Lagrangian is of the form:

$$L = \int d^4 \theta \sum_i \phi_i \bar{\phi}_i + \int d^2 \theta W(\phi_i) + h.c. \tag{2.8}$$

where $\phi_i, \bar{\phi}_i$ are the chiral and antichiral $(a,c)$ superfields and the superpotential $W$ is an analytic function of the complex superfields which obeys non-renormalization theorems. The ground states of the theory are $dW(\phi) = 0$. The chiral ring is the ring of polynomials generated by the $\phi_i$ modulo the relations $dW(\phi)/d\phi_i = D \bar{D} \phi_i \sim 0$. (For a review, see [4, 5].)

It turns out that the effective action for both the $\mathbb{C}P^{N-1}$ and $\mathbb{G}(k, N)$ $\sigma$-models have the form of a Landau-Ginsburg theory:

$$L = \int d^4 \theta \left[ \sum_{i=1}^{N} \bar{S}_i e^{-V} S_i + \frac{A}{2\pi} V \right] \tag{2.9}$$

where $S_i$ are $N$ chiral superfields which become the homogeneous coordinates on $\mathbb{C}P^{N-1}$. Their complex components are the $N$ complex scalar fields $n_i$ and fermion fields $\psi_i$. $V$ is
a real vector superfield which contains the auxiliary fields of the theory. (For a review, see [18] and references therein).

As quantum field theories, the Grassmannian $\sigma$-models (for $k \geq 2$) can be thought of as generalizations of the $\mathbb{C}P^{N-1}$ models. The Lagrangian has a similar form

$$L = \int d^4\theta \left[ \sum_a \bar{S}_a e^{-V} S_a + \alpha \text{Tr} V \right].$$

(2.10)

where now the chiral fields $S_{ia}$ carry two indices, a ‘gauge’ $U(k)$ index $i$, and a ‘flavour’ $SU(N)$ index $a$, since there are now $(N \times k)$-matrix scalar fields $n = (n^a_i)$ and $(N \times k)$-matrix Dirac spinor fields $\psi = (\psi^a_i)$. $V$ is a $k \times k$ matrix of superfields with gauge group $U(k)$. (For a detailed description of Grassmannian $\sigma$-models, see [5].)

Integrating out the superfields $S_i$ in (2.9), one obtains an effective action for the $\mathbb{C}P^{N-1}$ models which has the form of a Landau-Ginsburg model. By gauge invariance only the field-strength superfields $X$ and $\bar{X}$ remain since $V$ is not gauge invariant:

$$S_{\text{eff}} = \frac{N}{2\pi} \int d^2x \left\{ \int d^2\theta W(X) + \int d^2\theta \bar{W}(\bar{X}) + \int d^4\theta [Z(X, \bar{X}, \Delta, \bar{\Delta})] \right\}$$

(2.11)

with

$$W(X) = \frac{1}{2\pi} X (\log X^N - N + A(\mu) - i\theta).$$

(2.12)

$A$ is a renormalized coupling and $\theta$ the instanton angle, and

$$X = D_L \bar{D}_R V, \quad \bar{X} = D_R \bar{D}_L V.$$  

(2.13)

The chiral ring is the powers of $X$ mod $dW = 0$, or

$$X^N = e^{-A+i\theta} \equiv \beta.$$  

(2.14)

For the Grassmannian $\sigma$-models, the generalization is the following. The field-strength superfields (which we now call $\lambda$, $\bar{\lambda}$) belong to the adjoint representation of $U(k)$ and are now gauge covariant. The gauge-invariant objects of (2.10) are now the Ad-invariant polynomials in the field-strengths $\lambda$, with a ring generated by the $(a, c)$ superfields $X_i \ (i = 1, 2, \ldots, k)$ defined by

$$\det[t - \lambda] = t^k + \sum_{j=1}^{k} (-1)^j t^{k-j} X_j.$$  

(2.15)

However, Cecotti and Vafa observe [15] that at the topological field theory level, the $\lambda$’s and $\bar{\lambda}$’s are independent of each other and therefore, as matrices, are diagonalizable and all their eigenvalues $\lambda_m$ are distinct. Then, without loss of generality, the functional determinants in the path integral (with all background fields constant and fermions vanishing) are the same as for the $\mathbb{C}P^{N-1}$ case, since the full background is now abelian, and the superpotential takes the form

$$W_f(\lambda_1, \lambda_2, \ldots, \lambda_k) = \frac{1}{2\pi} \sum_{j=1}^{k} \lambda_j (\log \lambda_j^N - N + A(\mu) - i\theta),$$  

(2.16)
which fixes the theory completely.

The gauge-invariant fields are now polynomials in the eigenvalues \( \lambda_m \) of the field-strengths \( \lambda \) and are generated by the elementary symmetric functions

\[
X_i(\lambda) \equiv \sum_{1 \leq l_1 < l_2 < \ldots < l_i \leq k} \lambda_{l_1} \lambda_{l_2} \ldots \lambda_{l_i} \quad (i = 1, \ldots, k) \tag{2.17}
\]

The ring relations are \( \lambda_j^N = \text{const.} \) and the quantum cohomology of the Grassmannian \( \sigma \)-models will be generated by the elementary symmetric functions \( X_i \)'s. As quantum field theories, the Grassmannian \( \sigma \)-models \( G(k, N) \) can thus be identified as the tensor product of \( k \) copies of the \( \mathbb{C}P^{N-1} \sigma \)-model with certain redundant states eliminated.

This form of the superpotential is also derived in [12]. There Witten shows, by relating the quantum cohomology of the Grassmannians to a cubic quantum form in a quantum gauge theory, that the Grassmannian \( \sigma \)-model reduces at very large distances (where massless particles dominate) to a topological field theory which is a gauged WZW model. He shows that it is possible to work in a region of space where the auxiliary fields and the field-strength superfields which couple to them in the effective action belong now to a diagonal subgroup \( U(1)^k \) of \( U(k) \) and \( (2.16) \) is then the gauged WZW model of \( U(1)^k \).

### 3 Topological-antitopological fusion equations

Here we derive the \( tt^* \) equations for the Grassmannian models \( G(k, N) \) for any \( k \) and \( N \), starting with \( G(2, N) \). We first review the \( tt^* \) equations for the \( \mathbb{C}P^{N-1} \) model and their ground-state metric.

#### \( tt^* \) equations for \( \mathbb{C}P^{N-1} \) model

The chiral ring for the \( \mathbb{C}P^{N-1} \) model on a Kähler manifold is generated by a single element \( X \), the Kähler class, with relation

\[
X^N = \beta \tag{3.1}
\]

where \( - \ln \beta \) is the action for a holomorphic instanton.

As a basis for the chiral ring, we take

\[
1, X, X^2, X^3, \ldots, X^{N-1}. \tag{3.2}
\]

These fields are in one-to-one correspondence with the Ramond ground-states \( |i\rangle \) of the supersymmetric theory. The inner product of these ground-states \( g_{ij} \equiv \langle \bar{i}|j \rangle \) is the metric we are interested in.

The \( tt^* \) equations describe the way in which the metric changes along the renormalization group flow and are determined by the following differential equation:

\[
\bar{\partial}_j (g \partial_i g^{-1}) = [C_i, gC^j g^{-1}]. \tag{3.3}
\]
The $C_i$ represent the action on the chiral ring of the operators corresponding to a perturbation by the couplings. In the present case, we have one coupling $\beta$, and the equations are characterized by a single matrix $C_\beta$

$$
C_\beta = -\frac{1}{\beta} \left( \begin{array}{cccc}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
\beta & 0 & 0 & \ldots & 0 & 0
\end{array} \right)
$$

The model has a $\mathbb{Z}_N$ symmetry, which implies that the metric $g_{ji} \equiv \langle \bar{X}^j | X^i \rangle$ is diagonal. The metric depends only on $|\beta|^2$ due to chiral charge conservation, and the $tt^*$ equations reduce to the affine $A_{N-1}$ Toda equations

$$
\partial_z \partial_{z^*} q_i + e^{(q_{i+1} - q_i)} - e^{(q_i - q_{i-1})} = 0 \quad (3.4)
$$

where we have

$$
q_i = \log \langle \bar{i} | i \rangle - \frac{2i - N + 1}{2N} \log |\beta|^2 \\
z = N\beta^{\frac{1}{N}} \\
q_i \equiv q_{i+N} \quad (3.5)
$$

One can also define the usual topological metric, which is the two-point function $\eta_{ij}$ on the space of states and has non-vanishing elements

$$
\eta_{i,N-1-i} = 1 \quad (3.6)
$$

The ‘reality’ constraints result from a relation between $\eta$ and $g$. They are

$$
\eta^{-1} g (\eta^{-1} g)^* = 1 \quad (3.7)
$$

and imply

$$
\langle \bar{i} | i \rangle \langle N-1-i | N-1-i \rangle = 1 \quad (3.8)
$$

or

$$
0 = q_i + q_{N-i-1} \quad (3.9)
$$

A solution to these equations should be determined by the boundary conditions near $\beta \sim 0$ and $\beta \sim \infty$.

A semiclassical calculation of the metric in the UV (small $\beta$) limit shows that

$$
\langle \bar{i} | i \rangle = \frac{i!}{(N-1-i)!} [z (N \ln |\beta| - N\gamma)]^{N-1-2i} \quad (3.10)
$$

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In the IR (large $\beta$), the $tt^*$ equations give information about the solitons of the theory. In \cite{13}, semiclassical considerations are used to derive the leading order IR expansion for the metric
\begin{equation}
q_i \sim \frac{-2\sin[2\pi(i + \frac{1}{2})]\exp(-4|z|\sin \frac{\pi}{N})}{\sqrt{8\pi |z| \sin \frac{\pi}{N}}}
\end{equation}

For the $\mathbb{C}P^1$ and $\mathbb{C}P^2$ cases, the $tt^*$ equations become special cases of the Painlevé III equation, for which the connection formula between small- and large-$\beta$ asymptotics is known\cite{17}, although the exact solution is not known. For $N > 4$, the equations have not been studied explicitly.

There exists another basis useful in describing the $tt^*$ asymptotics. This is the canonical basis.\cite{13, 19, 15}

This basis diagonalizes the ring $\mathcal{R}$
\begin{equation}
(C_k)_i^j = \delta_{ki}\delta_{ji}
\end{equation}
in the sense that one can choose representatives of the chiral ring $\phi_j$, such that
\begin{equation}
\phi_j|i\rangle = \delta_j^i|i\rangle \sqrt{W''(X_i)}.
\end{equation}
The canonical vacua are thus eigenstates under multiplication by the chiral ring.

The large-$\beta$ behavior for the $\mathbb{C}P^{N-1}$ metric in the canonical basis can be derived semiclassically.\cite{13, 15}

Call $|l_r\rangle$ the canonical vacua at the $l_r$ critical point,
\begin{equation}
X(l_r) = t^\frac{1}{N} \exp[2\pi il_r/N] \quad (l_r = 0, 1, \ldots N).
\end{equation}
(where $X$ is the chiral primary of $\mathbb{C}P^{N-1}$).

Then
\begin{equation}
\langle l_s|l_r\rangle \cong \delta_{sr} - i \text{sign}(r - s) \left( \frac{N}{|r - s|} \right)^{1/2} K_0(m_{rs}\beta) + \ldots,
\end{equation}
where
\begin{equation}
m_{rs} = 4N|\beta|^{1/N} \sin \left( \frac{\pi|r - s|}{N} \right)
\end{equation}
is the mass of the soliton connecting the two vacua.\cite{17, 20}

For the $\mathbb{C}P^{N-1}$ model, it is easy to show that the relation between the chiral basis and the canonical basis is just a Fourier transform
\begin{equation}
|X^s\rangle = \frac{1}{\sqrt{N}} e^{2\pi i \frac{1}{N} \sum_{r=0}^{N-1} e^{2\pi i s(r + \frac{1}{2})}} |l_r\rangle
\end{equation}
**The ground-state metric for the Grassmannian σ-model**

We are interested in finding a suitable basis for the metric of the Grassmannian σ-model. In view of (2.16, 2.17), a basis for the chiral ring will consist of polynomials \( P_r(X_i) \) (for \( r = 1, \ldots, N!/(N-k)! \)) in the \( X_i \)'s. Then one can determine \( C_p \) from the ring relations and derive the \( tt^* \) equations from (3.3). However the equations are difficult to handle in this form. A more enlightening way to obtain and study the equations is to use the map (2.17). The metric can then be defined in terms of the variables \( \lambda_m \), in the following way

\[
\langle P_r(X_i)|P_s(X_j)\rangle = \frac{1}{k!}(\Delta(\lambda)P_r(X_i(\lambda))|\Delta(\lambda)P_s(X_j(\lambda)))_f
\]  

(3.18)

where \( \Delta(\lambda) \) is the Vandermonde determinant and is the Jacobian \( J = \det(\partial X_i/\partial \lambda_j) \) of the transformation in (2.17).

Each basis element \( P_s(X_j) \) can then be written as a polynomial \( \Delta(\lambda)P_s(X_j(\lambda)) \) in the different \( \lambda_m \)'s. The metric on the left side of (3.18) is then a sum over the products of the metrics for the \( CP^{N-1} \)-models, which are diagonal \( \langle \lambda^m_m|\lambda^s_s \rangle = \delta_{ks}(\lambda^m_m|\lambda^s_s) \).

For example, take the Grassmannian model \( G(2, N) \). The Chern classes are \( X_1 \) and \( X_2 \), where \( X_1 \) has (complex) dimension 1 and \( X_2 \) has (complex) dimension 2. The dimension of the ring is \( N(N-1)/2 \) and the ring is generated by the \( \{X_1, X_2\} \). The ring relations will be determined by constraints on the \( \{X_1, X_2\} \) from (2.16, 2.17).

A possible choice of basis is

\[
\mathcal{R} = \{1, X_1, X_2, X_1^2, X_1X_2, X_1^3, \ldots, X_1^iX_2^j\}
\]  

(3.19)

where \( (i + 2j) \leq \dim_{c} G(2, N) = 2(N-2) \). The number of elements with \( \dim_{c}|m| = \) number of elements with \( \dim_{c}|2(N-2) - m| \).

Then using (3.18), with

\[
X_1 = \lambda_1 + \lambda_2 \quad X_2 = \lambda_1\lambda_2
\]  

(3.20)

and where the Jacobian of the transformation is \( \Delta(\lambda) = \lambda_1 - \lambda_2 \), we have

\[
\langle X_1^2|X_2 \rangle = \frac{1}{2}[(\lambda_1|\lambda_1)(\lambda_2^2|\lambda_2^2) + (\lambda_2|\lambda_2)(\lambda_1^2|\lambda_1^2)] \equiv (\lambda|\lambda)(\lambda^2|\lambda^2)
\]

\[
\langle X_1^2|X_1^2 \rangle = (1|1)(\lambda^3|\lambda^3) + (\lambda|\lambda)(\lambda^2|\lambda^2)
\]

\[
\langle X_1X_2|X_1X_2 \rangle = (\lambda|\lambda)(\lambda^3|\lambda^3)
\]  

(3.21)

and so forth, where \( (\lambda^i|\lambda^j) \) are solutions to the affine \( \hat{A}_{N-1} \) Toda equations.

The ground-state metric is still non-diagonal and complicated. However, the form of the metric suggests that one might try to eliminate off-diagonal terms by taking linear combinations of the above basis elements and finding an orthogonal basis in which the metric becomes diagonal and simple (i.e. with each component consisting of a single term). We now show that this is possible.
In [13], the authors mention that the $tt^*$ equations will have the simplest form in a particular basis, the ‘flat coordinates’ basis, characterized by a two-point function metric $\eta$ which is independent of the perturbing parameters of the model and squares to 1 ($\eta^* = \eta^{-1} = \eta$). (These coordinates are the ones supplied by conformal perturbation theory and their chiral algebra, defined through an effective LG potential, give the structure constants $C_{ij}^k$ obtained by conformal perturbation theory [22, 23, 24, 10, 2].) We now show that we can find a basis with such an $\eta$, and this choice makes the $tt^*$ equations tractable.

Let’s first consider $G(2,N)$. We need a basis with $N(N - 1)/2$ elements. Consider basis elements of the form $|mn\rangle \equiv |\lambda^m\lambda^n\rangle$ with $n > m$ such that

\[
|mn\rangle = -|nm\rangle = 0 \quad \text{if} \quad m = n \tag{3.22}
\]

We can write

\[
\langle m'n'|mn \rangle = \langle m'|m \rangle \langle n'|n \rangle - \langle m'|n \rangle \langle n'|m \rangle = \delta_{mm'}\delta_{nn'}\langle m'|m \rangle \langle n'|n \rangle - \delta_{mn'}\delta_{m'n} \langle m'|n \rangle \langle n'|m \rangle \tag{3.23}
\]

since we know that the $\mathbb{C}P^{N-1}$ metric is diagonal.

Then the metric is defined as

\[
\langle mn|mn \rangle = \langle m|m \rangle \langle n|n \rangle = -\langle mn|nm \rangle = \langle nm|nm \rangle = 0 \quad \text{if} \quad m = n \tag{3.24}
\]

Our basis elements for the chiral ring are now the $N(N - 1)/2$ elements $|mn\rangle$ with $n > m$.

The reality constraints for $\mathbb{C}P^{N-1}$

\[
\langle m|m \rangle \langle N - 1 - m|N - 1 - m \rangle = 1 \tag{3.25}
\]

become for $G(2,N)$

\[
\langle mn|mn \rangle \langle N - 1 - m, N - 1 - n|N - 1 - m, N - 1 - n \rangle = 1. \tag{3.26}
\]

The topological metric for the $\mathbb{C}P^{N-1}$ model is

\[
\eta_{i,N-1-i} = \langle X^i X^{N-1-i} \rangle = 1. \tag{3.27}
\]

For the $G(2,N)$ models, the non-zero elements of $\eta$ are now

\[
\eta_{mn,N-1-m,N-1-n} = \langle mn, N - 1 - m, N - 1 - n \rangle = 1 \tag{3.28}
\]
or, since our basis is defined with \( m < n \),

\[
\langle mn, N - 1 - n, N - 1 - m \rangle = -1. \tag{3.29}
\]

This shows that \( \eta \) has the form

\[
\eta = \begin{pmatrix}
0 & 0 & \ldots & \ldots & 0 & 0 & -1 \\
0 & 0 & \ldots & \ldots & 0 & -1 & 0 \\
0 & 0 & -1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & -1 & 0 & 0 \\
0 & -1 & \ldots & \ldots & 0 & 0 & 0 \\
-1 & 0 & \ldots & \ldots & 0 & 0 & 0
\end{pmatrix}
\]

where the block in the middle consists of a diagonal matrix with elements \(-1\), the size of the block being determined by the condition that \( m + n = N - 1 \).

We have \( \eta^* = \eta^{-1} = \eta \) and the reality constraints (3.7) are now

\[
g \eta g^T = \eta. \tag{3.30}
\]

The metric \( g \) is orthogonal with respect to \( \eta \).

The matrix \( C_\beta \) characterizing the relations in the chiral ring is easily determined in this basis. The perturbing operator corresponds to the chiral superfield \( X_1 \equiv \lambda_1 + \lambda_2 \) (with a factor of \( \frac{1}{\beta} \) coming from the action of \( \mathbb{C}P^{N-1} \)).

The algebra of the perturbing operator is then represented by

\[
X_1 |mn\rangle = |m + 1, n\rangle + |m, n + 1\rangle \tag{3.31}
\]

with

\[
|r, N\rangle = \beta|r, 0\rangle = -\beta|0, r\rangle = -|N, r\rangle \tag{3.32}
\]

since one can think of it as being

\[
(|\lambda_1 1\rangle + |1\lambda_2\rangle) \otimes |\lambda_1^m \lambda_2^n\rangle = |\lambda_1^{m+1} \lambda_2^n\rangle + |\lambda_1^m \lambda_2^{n+1}\rangle \tag{3.33}
\]

The chiral algebra for the other operators can be determined in the same way, take for example \( X_2 = \lambda_1 \lambda_2 \), then

\[
|\lambda_1 \lambda_2\rangle \otimes |\lambda_1^m \lambda_2^n\rangle = |\lambda_1^{m+1} \lambda_2^{n+1}\rangle \tag{3.34}
\]

or

\[
X_2 |mn\rangle = |m + 1, n + 1\rangle \tag{3.35}
\]

and in general, we have combinations of the following possibilities

\[
|\lambda_1^i + \lambda_2^i\rangle \otimes |\lambda_1^m \lambda_2^n\rangle = |\lambda_1^{m+i} \lambda_2^n\rangle + |\lambda_1^m \lambda_2^{n+i}\rangle \\
|\lambda_1^k \lambda_2^l\rangle \otimes |\lambda_1^m \lambda_2^n\rangle = |\lambda_1^{m+k} \lambda_2^{n+l}\rangle \tag{3.36}
\]

with

\[
|r, N + s\rangle = \beta|r, s\rangle = -\beta|s, r\rangle = -|N + s, r\rangle \tag{3.37}
\]
The $tt^*$ equations for $G(2, N)$

From the form of the metric, we can either use (3.3) with $C_\beta$ as defined in (3.31), or (3.4) to find the $tt^*$ equations. We have

$$g_{ij,ij} = \langle i|j|ij \rangle = \langle i|i\rangle \langle j|j \rangle$$

where $j > i$, $i = 0, \ldots, N - 2$

$$j = 1, \ldots, N - 1$$  \hspace{1cm} (3.38)

Defining

$$q_{ij} = \ln g_{ij,ij} - \frac{(i + j) - N + 1}{N} \log |\beta|^2$$

$$z = N\beta^\frac{1}{N}$$

$$q_{ij} \equiv q_{i+Nj} \text{ and } q_{ij} \equiv q_{i+Nj}$$  \hspace{1cm} (3.39)

the $tt^*$ equations become for $G(2, N)$

$$\partial_q \partial_z q_{ij} + e^{q_{i+1,j} - q_{ij}} - e^{q_{i-1,j} - q_{ij}} + e^{q_{i+1,j} - q_{ij}} - e^{q_{i,j} - q_{i-1,j}} = 0$$  \hspace{1cm} (3.40)

where an exponential term is ignored if it contains any $q_{ij}$ with $i = j$ (this will happen if $j = i + 1$, then it is easy to see using (3.39) that the 3rd and 4th terms on the LHS cancel each other out).

The reality constraints can be expressed as

$$q_{ij} + q_{N-i-1N-j-1} = 0,$$  \hspace{1cm} (3.41)

which imply

$$q_{i,N-i-1} = 0, \quad i = 0, \ldots, N - 2.$$  \hspace{1cm} (3.42)

We still need the boundary conditions. For the small-$\beta$ limit, we use (3.10) applied to each factor in the metric. For the large-$\beta$ limit, we first determine all one-soliton contributions to $q_i$ for the $CP^{N-1}$ models using (3.15) and (3.17). This will give us the numerical coefficient in front of the RHS of (3.11) (which is related to the soliton numbers) and all the next order contributions to $q_i$ coming from fundamental solitons. Then we can get the one-soliton contributions to the asymptotic behavior of $q_{ij}$.

In \[13\], the leading order one-soliton contribution in the chiral basis (3.11) is derived from taking the Fourier transform of the leading correction (calculated semiclassically) to $\langle l_{r+1}|l_r \rangle$, the one-soliton sector of smallest mass. Extending the calculation to include all one-soliton sectors, we find all fundamental soliton contributions (coming from the canonical basis) to the IR asymptotics of the metric for $CP^{N-1}$

$$q_i = \sum_{r=1}^{\frac{N-1}{2}} \binom{N}{r} \sin \left[ \frac{2\pi r}{N} \left( i + \frac{1}{2} \right) \right] \frac{1}{\pi} K_0(m_r \beta) + \ldots$$  \hspace{1cm} (3.43)
for $i = 0, \ldots, N - 1$ and the masses are

$$m_r = 4N|\beta| \sin \left(\frac{\pi r}{N}\right).$$  (3.44)

The IR boundary conditions for the $G(2, N)$ models then become

$$q_{ij} = q_i + q_j = \sum_{r=1}^{N-1} \binom{N}{r} \left\{ \sin \left[ \frac{2\pi r}{N} (i + \frac{1}{2}) \right] + \sin \left[ \frac{2\pi r}{N} (j + \frac{1}{2}) \right] \right\} \frac{1}{\pi} K_0(m_r \beta) + \ldots$$

$$= \sum_{r=1}^{N-1} \binom{N}{r} 2 \sin \left[ \frac{\pi r}{N} (i + j + 1) \right] \cos \left[ \frac{\pi r}{N} (i - j) \right] \frac{1}{\pi} K_0(m_r \beta) + \ldots$$  (3.45)

and the leading contribution is given by the sum of the $r = 1$ and $r = N - 1$ terms since the corresponding masses are smallest.

**The $tt^*$ equations for $G(k, N)$**

The $tt^*$ equations easily generalize to $G(k, N)$. The symmetries of the metric are such that

$$\langle m_1 m_2 \ldots m_k | m_{i_1} m_{i_2} \ldots m_{i_k} \rangle = \epsilon_{i_1 i_2 \ldots i_k} \langle m_1 m_2 \ldots m_k | m_1 m_2 \ldots m_k \rangle$$  (3.46)

where $\epsilon_{i_1 i_2 \ldots i_k}$ is the totally antisymmetric tensor. Defining

$$q_{l_1 \ldots l_k} = \ln g_{l_1 \ldots l_k, l_1 \ldots l_k} - \frac{2}{2N} \sum_i l_i - k(N - 1) \log |\beta|^2$$

$$= q_{l_1} + q_{l_2} + \ldots + q_{l_k},$$  (3.47)

the $tt^*$ equations become

$$\partial_x \partial_{\bar{x}} q_{l_1 \ldots l_k} + \sum_{i=1}^k \left\{ \exp[q_{l_1 \ldots l_1+1 \ldots l_k} - q_{l_1 \ldots l_k}] - \exp[q_{l_1 \ldots l_k} - q_{l_1 \ldots l_i}] \right\} = 0$$  (3.48)

with appropriate boundary conditions and where

$$0 \leq l_1 < l_2 < \ldots < l_i < \ldots < l_k \leq N - 1,$$  (3.49)

and again any exponential containing any $q$ with 2 indices the same is ignored.

**The canonical basis for $G(k, N)$**

We now show that our basis, when Fourier transformed to the canonical basis, agrees with the expression derived in [15]. There, the authors derive a formal expression for the metric of the $G(k, N)$ $\sigma$-models in the canonical basis in terms of the canonical metric for $\mathbb{C}P^{N-1}$. In view of (2.16), the Grassmannian $\sigma$-models $G(k, N)$ can be identified as the
quotient of a tensor product of $k$ copies of the $\mathbb{C}P^{N-1}$ $\sigma$-model reduced by the action of the replica symmetry $S_N$, which eliminates certain states, as explained in [13]. They find then

$$\langle \{h_1, h_2, \ldots, h_k\} | \{l_1, l_2, \ldots, l_k\} \rangle = \frac{1}{k!} \sum_{s, t \in S_k} \sigma(s) \sigma(t) \prod_{\alpha=1}^{k} \langle h_t(\alpha) | l_s(\alpha) \rangle_{\alpha}$$

$$= \det_{\{h_\alpha \}, \{l_\beta \}} \left[ \langle \bar{h}_\alpha | l_\beta \rangle \right] (3.50)$$

where $\sigma(s)$ is the signature of the permutation $s$, $\langle \bar{h} | l \rangle$ is the $N \times N$ matrix giving the ground state metric for the $\mathbb{C}P^{N-1}$ $\sigma$-model in a canonical basis, and $\det_{\{h_\alpha \}, \{l_\beta \}}$ is the determinant of the $k \times k$ minor obtained by selecting the rows $(h_1, h_2, \ldots, h_k)$ and the columns $(l_1, l_2, \ldots, l_k)$.

Consider $\mathbb{G}(2, N)$, we then see that

$$\langle m' n' | m n \rangle = | \beta |^{(m+n-N+1)} \sum_{l_1, l_2, h_1, h_2 = 0}^{N-1} e^{2\pi i (m+\frac{1}{2}) l_1 + (n+\frac{1}{2}) l_2} e^{-2\pi i (m'+\frac{1}{2}) h_1 + (n'+\frac{1}{2}) h_2} \langle \bar{h}_1 | \bar{h}_2 | l_1, l_2 \rangle$$

when we Fourier transform each component of the metric separately

$$\langle \bar{h}_1, \bar{h}_2 | l_1, l_2 \rangle = \langle \bar{h}_1 | l_1 \rangle \langle \bar{h}_2 | l_2 \rangle - \langle \bar{h}_2 | l_1 \rangle \langle \bar{h}_1 | l_2 \rangle. (3.52)$$

This generalizes to $\mathbb{G}(k, N)$.

### 4 Examples

$\mathbb{G}(2, 4)$

We look first at the simplest non-trivial Grassmannian $\sigma$-model $\mathbb{G}(2, 4)$. By using (2.8), the relations that determine the chiral ring are the following

$$X_1^3 = 2X_1X_2 \quad \beta = X_2^2 - X_1^2X_2$$

(4.1)

There are 6 elements in the ring and since cohomology elements are of even degree, with highest degree the (real) dimension of $\mathbb{G}(2, 4)$, or 8, the ring is

$$\mathcal{R} = \{1, X_1, X_2, X_1^2 - X_2, X_1X_2, X_2^2\} \equiv \{|01\}, |02\}, |12\}, |03\}, |13\}, |23\} (4.2)$$

The matrix $C_\beta$ is defined by the relations in (3.31) and takes the form

$$C_\beta = \frac{1}{\beta} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & 0 & 0 & 1 \\ 0 & -\beta & 0 & 0 & 0 & 0 \end{pmatrix}$$
The reality constraints are
\[ g_{01,01} \cdot g_{23,23} = 1 \]
\[ g_{03,03} \cdot g_{13,13} = 1 \]
\[ g_{21,21} = 1 \]
\[ g_{03,03} = 1 \] (4.3)

where
\[ g_{ij,ij} = \langle \lambda^i | \lambda^i \rangle \langle \lambda^j | \lambda^j \rangle \] (4.4)

The \(tt^*\) equations are, with
\[ q_{01} = \log g_{01,01} |\beta| = -q_{32} \]
\[ q_{02} = \log g_{02,02} |\beta|^{\frac{1}{2}} = -q_{31} \]
\[ q_{03} = q_{12} = 0 \] (4.5)

\[ \partial \bar{z} \partial z q_{01} + e^{q_{02} - q_{01}} - e^{q_{01} + q_{02}} = 0 \]
\[ \partial \bar{z} \partial z q_{02} + 2e^{-q_{02}} - e^{q_{02} - q_{01}} - e^{q_{02} + q_{01}} = 0 \] (4.6)

There are only two unknown functions, since this system has the same number of unknown functions as \( \mathbb{C}P^3 \).

Let’s now consider the behavior of the metric in the IR. From the metric of the \( \mathbb{C}P^3 \) model, the leading term in the one-soliton sector of lowest mass is of the form
\[ e^{q_{01}} = \langle 01 | 01 \rangle \cong |\beta|^{-1} \{ 1 + \left( \frac{4}{1} \right) \frac{2\sqrt{2}}{\pi} K_0(m_1 \beta) \} \]
\[ e^{q_{02}} = \langle 02 | 02 \rangle \cong |\beta|^{-\frac{1}{2}} \{ 1 + \left( \frac{4}{2} \right) \frac{2}{\pi} K_0(m_2 \beta) \} \] (4.7)

and \( \langle 03 | 03 \rangle = \langle 12 | 12 \rangle = 1, \quad q_{23} = -q_{01}, \quad q_{13} = -q_{02} \).

By then going to the canonical basis, one can determine the soliton numbers between various vacua.

For example, calling \( \langle h_1, h_2 | k_1, k_2 \rangle = \langle l_{h_1 h_2} | l_{k_1 k_2} \rangle \),
\[ \langle l_{02} | l_{01} \rangle = i \frac{4}{\pi} K_0(m_1 \beta) - \frac{24}{\pi^2} K_0(m_1 \beta) K_0(m_2 \beta) + \ldots \]
\[ \langle l_{03} | l_{01} \rangle = i \frac{6}{\pi} K_0(m_2 \beta) - \frac{16}{\pi^2} K_0(m_1 \beta) K_0(m_1 \beta) + \ldots \] (4.8)

and so forth.

The soliton structure that emerges is the following. The Grassmannian \( \sigma \)-models having the same ring structures as the Kazama-Suzuki LG models \( SU(N)/SU(N - k) \times SU(k) \times U(1) \) at level 1, perturbed by the most relevant operator, which have been
discussed in [1, 27, 28, 29, 30], the geometry of their vacuum images in the $W$-plane (the complex plane with the values of $W(t_{ij})$ plotted) is similar.

For $\mathbb{G}(2,4)$, the representation in the $W$-plane is the same as for the perturbed $SU(4)/SU(2) \times SU(2) \times U(1)$ model discussed in [27]. The states $|02\rangle$ and $|13\rangle$ sit at the top of 2 tetrahedra joined together by the 4 other vacua. Between these 2 states and each of the 4 other states there are solitons interpolating, of mass $m_1$ and multiplicity 4. There are also solitons of mass $m_2$ and multiplicity 6 between each adjacent vacua of the square formed by the 4 states. There are no fundamental solitons linking the 2 vertices ($|02\rangle, |13\rangle$).

$\mathbb{G}(3,6)$

The ring is

$$\mathcal{R} = \{ 1, X_1, X_2, X_1^2 - X_2, X_3, X_1X_2 - X_3, -X_1^3 + 2X_1X_2 - X_3, X_1^2X_2 - X_2^2 - X_1X_3, X_2^2 - X_1X_3, X_1X_2, X_1X_2^2 - X_2X_3 - X_2^2X_3, X_3X_2^2 - X_2X_3, X_2X_3, X_3^2 + X_2^2 - 2X_1X_2X_3, X_1X_2X_3 - X_3, X_1X_2^2, X_3X_2 - X_1X_3^2, X_2X_3^2, X_3^3 \}$$

$$= \{ |012\rangle, |013\rangle, |023\rangle, |014\rangle, |024\rangle, |015\rangle, |035\rangle, |123\rangle, |025\rangle, |034\rangle, |124\rangle, |035\rangle, |125\rangle, |134\rangle, |045\rangle, |135\rangle, |234\rangle, |235\rangle, |145\rangle, |245\rangle, |345\rangle \}$$

The $tt^*$ equations are easily derived from (3.48).

In the IR, we have for example the following expansions

$$q_{012} = -q_{345} = \left(6 \right) \frac{4}{\pi} K_0(m_1\beta) + \left(6 \right) \frac{1}{\pi} K_0(m_3\beta)$$

$$q_{013} = -q_{245} = \left(6 \right) \frac{2}{\pi} K_0(m_1\beta) + \left(6 \right) \frac{\sqrt{3}}{\pi} K_0(m_2\beta) - \left(6 \right) \frac{1}{\pi} K_0(m_3\beta)$$

$$q_{024} = -q_{135} = \left(6 \right) \frac{3}{\pi} K_0(m_3\beta)$$

(4.10)

with $m_1 = 12|\beta^{\frac{7}{3}}\rangle$, $m_2 = 12\sqrt{3}|\beta^{\frac{1}{3}}\rangle$, $m_3 = 24|\beta^{\frac{7}{3}}\rangle$.

We can determine which vacua are joined by fundamental solitons. The metric in the canonical basis $\langle l_{ijk} | l_{ijk} \rangle$ can be obtained by inverse Fourier transform of (3.51) or by using (3.50) where the metric is given by a three by three determinant.

For example

$$\langle l_{012} | l_{013} \rangle = -i \frac{6}{\pi} K_0(m_1\beta) + \ldots$$

$$\langle l_{013} | l_{035} \rangle = i \frac{15}{\pi} K_0(m_2\beta) + \ldots$$

$$\langle l_{012} | l_{015} \rangle = -i \frac{20}{\pi} K_0(m_3\beta) + \ldots$$

(4.11)
The $W$-plane picture is the following (see also [28]). It consists of two concentric circles with the smaller one having half the radii of the larger one (in the ratios $m_3 : m_1$), plus two vacua in the middle. Six vacua are at the vertices of a regular hexagon on the outer circle and consist of the states $\{|012\rangle|015\rangle|045\rangle|345\rangle|234\rangle|123\rangle\}$ and there are 12 vacuum images on the inner circle (since images come in pairs and are mapped to the same point here) with the two sets of states $\{|013\rangle|014\rangle|145\rangle|245\rangle|235\rangle|023\rangle\}$ and $\{|125\rangle|025\rangle|035\rangle|034\rangle|134\rangle|124\rangle\}$. The hexagons have the same orientation. The two degenerate vacuum images in the middle are $\{|024\rangle|135\rangle\}$.

We now look for the soliton polytope. From our (4.10), we find a fundamental soliton interpolating between 2 vacua if any two indices characterizing the two vacua are equal. For example take the vacua $|012\rangle$. It is on the outer hexagon and will be connected only to the two adjacent vacua $|015\rangle, |123\rangle$ on the outer hexagon (with solitons of mass $m_3$ and multiplicity 20). It is also connected to the 6 nearest neighbors on the inner circle, (the vacuum images being degenerate 2 by 2 on the inner circle) $|014\rangle, |025\rangle$ and $|023\rangle, |124\rangle$ with masses $m_2$ and multiplicity 15 and $|013\rangle, |125\rangle$ with masses $m_1$ and multiplicity 6. Finally $|012\rangle$ is connected with the vacuum in the middle $|024\rangle$ with mass $m_3$ and multiplicity 20. In the inner circle, each of the vacua in the first set $\{|013\rangle|014\rangle|145\rangle|245\rangle|235\rangle|023\rangle\}$ are connected to all the other vacua in the first set with masses $m_1, m_2, m_3$ and multiplicities 6, 15, 20 and the other set is connected among itself in the same way. Finally each vacua in the first set (the same is true for the second set) is connected alternatively to either one or the other vacuum in the middle, depending on whether there are any two indices the same.

As a summary then, solitons fall into multiplets of completely antisymmetric representations (one column Young tableaux) of $SU(6)$, and the representations and masses are determined by how far apart the respective vacua are.

5 The large $N$ limit

We are now interested in checking whether the $tt^*$ equations become solvable in the large $N$ and $k$ limit, in the same way as they were for the large $N \mathbb{C}P^{N-1}$model.[8]

We consider first the equations for $G(2, N)$ in the large-$N$ limit. There is an immediate generalization to $G(k, N)$ for any $k$. We proceed as in [18].

We assume that the metric becomes a continuous function of the two variables $s_1 \equiv i/N$, and $s_2 \equiv j/N$.

Redefining (see [18] for notation)

$$q_{ij} = \frac{1}{N} \ln \langle \bar{i}|i\rangle \langle \bar{j}|j\rangle + 2\frac{(i + j)}{N} \log |\beta|^2,$$

the $tt^*$ equations become

$$\frac{4}{N} \partial_{\beta} \partial_{\bar{\beta}} q_{ij} + e^{N(q_{i,j+1} - q_{ij})} - e^{N(q_{i,j} - q_{i,j-1})} + e^{N(q_{i+1,j} - q_{ij})} - e^{N(q_{i,j} - q_{i-1,j})} = 0$$

(5.2)
with $q_{i+N,j} = q_{ij}$ and $q_{i,j+N} = q_{ij}$.

Then, with $q(s_1, s_2) = \ln g(s_1, s_2)$, we have

$$4\partial_\beta \partial_\tilde{\beta} q(s_1, s_2) = \exp \left[ N \left\{ q(s_1, s_2 + \frac{1}{N}) - q(s_1, s_2) \right\} \right] - \exp \left[ N \left\{ q(s_1, s_2) - q(s_1, s_2 - \frac{1}{N}) \right\} \right]$$

$$+ \exp \left[ N \left\{ q(s_1 + \frac{1}{N}, s_2) - q(s_1, s_2) \right\} \right] - \exp \left[ N \left\{ q(s_1, s_2) - q(s_1 - \frac{1}{N}, s_2) \right\} \right]$$

and the equation reduces to

$$4\partial_\beta \partial_\tilde{\beta} q(s_1, s_2) = \frac{\partial}{\partial s_1} \exp \left[ \frac{\partial q}{\partial s_1} \right] + \frac{\partial}{\partial s_2} \exp \left[ \frac{\partial q}{\partial s_2} \right]$$

with general solution

$$q(s_1, s_2) = \ln g(s_1, s_2) = \ln [g(s_1)g(s_2)] = q(s_1) + q(s_2)$$

where $s_2 > s_1$, and where $q(s_i)$ is the solution found in [18] for the $\mathbb{C}P^{N-1}$ model in the large $N$ limit. The solution is again a $k$-th order product of the metric for the $\mathbb{C}P^{N-1}$ model with appropriate configurations eliminated.

The generalization for arbitrary (finite or infinite) $k$ is, for $\mathbb{G}(k, N)$

$$\partial_2 \partial_\tilde{2} q(s_1, s_2, \ldots, s_k) = \sum_{i=1}^{k} \frac{\partial}{\partial s_i} \exp \left[ \frac{\partial q}{\partial s_i} \right]$$

with solution

$$q(s_1, s_2, \ldots, s_k) = \ln g(s_1, s_2, \ldots, s_k) = \ln [g(s_1)g(s_2) \ldots g(s_k)]$$

with

$$s_1 < s_2 < s_3 < \ldots < s_k.$$  

We note that there is no particular difficulty in going beyond $k = 1$. This is consistent with an analysis in [12] where the author shows that the Grassmannian $\sigma$-model simplifies in the limit of $k$ fixed, $N \to \infty$, and therefore everything can be calculated in an asymptotic expansion in powers of $\frac{1}{N}$, by expansion around a certain extremum.

## 6 Concluding Remarks

We have shown that the $tt^*$ fusion equations for the ground-state metric of supersymmetric $\sigma$-models defined on Grassmannian manifolds $\mathbb{G}(k, N)$ can be determined for any $k$ and $N$. The result follows from writing the $tt^*$ equations in a particular orthogonal basis which makes them tractable in terms of the equations for $\mathbb{C}P^{N-1}$. The solution is a $k$-th order product of the ground-state metric for the $\mathbb{C}P^{N-1}$ model, where appropriate anti-symmetrization and normalization procedures are carried out. We have computed all fundamental soliton contributions to the $tt^*$ asymptotics in the IR and, by going to the
canonical basis, have shown that our result is consistent with that proposed by Cecotti and Vafa\footnote{15} from path-integral considerations on the full quantum field theory.

As pointed out by the authors in \cite{15}, the Grassmannian models do not fit in the classification scheme of two-dimensional $N=2$ supersymmetric field theories which determines the number of vacua and solitons between them. These models have various vacua aligned in the $W$ plane, yielding monodromy matrices $H$ which do not satisfy the classification equations, and there is an ambiguity in asking what the soliton numbers are (see \cite{15, 25}). By showing how the $tt^*$ equations for the Grassmannians are related to the ones for the $\mathbb{C}P^{N-1}$ models, we were able to study their behavior in the IR and make the vacua picture clearer. The multiplicity and masses of fundamental solitons interpolating between any two given vacua are easily determined.

We are also able to solve the $tt^*$ equation for $\mathbb{G}(k, N)$ in the large $N$ limit for any $k$. The solution can be written in terms of the metric for $\mathbb{C}P^{N-1}$ in the large $N$ limit, which was found in \cite{18}, using finite temperature results and large $N$ techniques, and methods inspired from self-dual gravity. The Grassmannian $\sigma$-models have not been solved completely as quantum field theories and our results may offer further insights into them.

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