Twist-2 light-quark distribution functions in a singly heavy baryon in the large $N_c$ limit

Hyeon-Dong Son$^{1,2,∗}$ and Hyun-Chul Kim$^{2,3,†}$

$^1$Center for Extreme Nuclear Matters (CENaM), Korea University, Republic of Korea
$^2$Department of Physics, Inha University, Incheon 22212, Republic of Korea
$^3$School of Physics, Korea Institute for Advanced Study (KIAS), Seoul 02455, Republic of Korea

A singly heavy baryon can be considered as a state consisting of the $N_c - 1$ light valence quarks that create the pion mean field in the large $N_c$ limit. In the limit of the infinitely heavy quark mass, a heavy quark inside a singly heavy baryon is regarded as a mere static color source. It is required only to make the singly heavy baryon a color singlet. Thus, the $N_c - 1$ valence quarks govern quark dynamics inside the singly heavy baryon. Within this pion mean field framework, we investigate the twist-2 unpolarized and longitudinally polarized light-quark distribution functions inside charmed and bottom baryons at a low renormalization point. We observe that the light quarks inside a heavy baryon carry less momentum than those inside a nucleon. This feature is more prominent as the heavy quark mass increases. We discuss the baryon sum rule, momentum sum rule, and the Bjorken spin sum rule for singly heavy baryons. We also discuss the inequality conditions for the quark distribution functions. In addition, we present the results for the light-quark quasi-distribution functions.

I. INTRODUCTION

A singly heavy baryon consists of two light quarks and one heavy quark. If one takes the limit of the infinitely heavy quark ($M_Q → ∞$), the spin of the heavy quark is conserved, which also leads to the spin of the light-quark degrees of freedom is conserved. This is called the heavy-quark spin symmetry [1, 2]. In this limit, the singly heavy baryon is independent of the heavy-quark flavor, which is called the heavy-quark flavor symmetry. Thus, the heavy quark inside a singly heavy baryon remains as a mere static color source. It is only required to make the singly heavy baryon a color singlet. This indicates that the quark dynamics inside it is governed by the light quarks. The two light quarks provide the flavor SU(3) representations of the lowest-lying heavy baryons: the baryon antitriplet with spin $J = 1/2$ and two degenerate the baryon sextet with spin $J = 1/2$ and $J = 3/2$. This degeneracy is removed by the chromomagnetic hyperfine interaction that arises from the $1/M_Q$ corrections.

Witten [3] proposed that, in the large $N_c$ limit, a light baryon emerges as a $N_c$ valence quarks bound by the pion mean field. This picture of the light baryon was realized by the chiral quark-soliton model ($\chi$QSM) [4]. The model was very successful in describing various properties of the nucleon and low-lying hyperons (see a review [5]). The $\chi$QSM was further applied to the parton distribution functions (PDFs) of the nucleon [6–8] and the generalized parton distributions (GPDs) [9] (see also a review [10]). Recently, the $\chi$QSM was extended to singly heavy baryons, motivated by Diakonov [11]. In the limit of the infinitely heavy quark mass ($M_Q → ∞$), a singly heavy baryon can be viewed as a bound state of the $N_c - 1$ valence quarks [12]. A heavy quark inside the singly heavy baryon remains a mere static color source. This implies that quark dynamics of the singly heavy baryon is governed by the light quarks. The presence of the $N_c - 1$ valence quarks create the pion mean field that makes them bound. Combining the bound state with a heavy quark, we can construct a singly heavy baryon state. This extended $\chi$QSM has been used to study the charmed baryon properties such as the mass splitting, isospin mass splitting, magnetic moments, electromagnetic form factors, radiative transition form factors, gravitational form factors, axial-vector transition form factors, and quark spin content [13–21].

In the present work, we investigate the light-quark distribution functions of the singly heavy baryons. Despite the difficulty to measure them from hard scattering processes such as deep inelastic scattering (DIS), the PDFs themselves are fundamental physical quantities that represent the probability densities to find quarks and gluons inside a singly heavy baryon with the momentum fraction $x$ and with a particular

* E-mail: hdsong@korea.ac.kr
† E-mail: hchkim@inha.ac.kr

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polarization configuration considered. Thus, it is of great importance to scrutinize the PDFs so that one can understand how the heavy and light quarks are distributed inside the singly heavy baryon. We anticipate that relevant experimental information on the PDFs inside a singly heavy baryon may be extracted from fragmentation functions for inclusive heavy baryon productions with the Drell-Levy-Yan relation used (see Refs. 22–27).

In Refs. 6, 7, the properties of the twist-2 quark distribution functions in the nucleon were studied within the $\chi$QSM. In the present work, we extend the formalism to the light-quark distribution functions inside a singly heavy baryon. We compute the isoscalar unpolarized and isovector longitudinally-polarized distributions for the light quarks and antiquarks inside the singly heavy baryon. We will discuss the baryon number, momentum and spin sum rules for the light-quark distributions within the current theoretical framework. We also discuss the positivity and inequality conditions for the PDFs. We demonstrate how the $x$ dependence of the light quarks and antiquarks inside the singly heavy baryon is drastically changed, compared with those inside a nucleon. We want to mention that in Ref. 28 the behavior of the heavy-quark distribution was examined in a heavy-quark light-diquark approach. So far as we know, we show for the first time the light quark distributions inside a singly heavy baryon.

This work is organized as follows: In Section II, we present the general formalism of the $\chi$QSM. In the following Section, we show the expressions for the quark and antiquark distribution functions in a singly heavy baryon and discuss the sum rules. Then, the numerical results are shown and the positivity and inequality are discussed. In the penultimate section, the quark quasi-distributions are considered. The final section is devoted to summary and conclusions.

II. BARYONS IN THE CHIRAL QUARK-SOLITON MODEL

In this section, we will briefly explain how the nucleon and singly heavy baryon emerge respectively as the $N_c$ and $N_c - 1$ valence-quark states bound by the pion mean field. We first recapitulate the $\chi$QSM for the nucleon and then extend it to the singly heavy baryon.

A. Nucleon

We start from the low-energy QCD partition function given by in the large $N_c$ limit:

$$Z[\psi, \pi] = \int D\pi D\psi D\psi^\dagger \exp \left[\int d^4x \bar{\psi} \left( i\partial + iM \exp(i\pi^a \tau^a \gamma^5) \right) \psi \right],$$  \hspace{1cm} (1)

where $\pi^a$ and $\psi$ denote the pseudo-Nambu-Goldstone (pNG) fields and quark fields, respectively. $M$ is called the dynamical quark mass arising from the spontaneouse breakdown of chiral symmetry (SB$\chi$S). In the instanton liquid model for the QCD vacuum at low energy [29, 30], which realizes a legitimate mechanism of the SB$\chi$S, the value of $M$ is determined to be $M \approx 350$ MeV. Originally, $M$ is momentum-dependent and plays a role of an regulator, which causes complexity in calculating physical observables. In the current work, we turn off its momentum dependence and introduce a regularization scheme for quark loops with the ultraviolet (UV) cutoff $\Lambda$ introduced. This implies that the quark distribution functions are computed at the normalization point $\mu \simeq \Lambda$, which is approximately given by $\Lambda \simeq 600$ MeV. In fact, the normalization point can not be uniquely determined. $\mu$ is proportional to the inverse of the average instanton size $\bar{\rho} \approx 1/3$ fm and a dimensional parameter that varies with bulk properties of the instanton medium [31, 32]. Since $\mu$ is insensitive to this parameter, we take $\mu = \Lambda \simeq 1/\bar{\rho}$ in the present work.

Having integrated out the quark fields in Eq. (1), we derive the one-loop effective chiral action:

$$S_{\text{eff}} = -N_c \text{Tr} \ln [i\partial + iMU^\gamma],$$  \hspace{1cm} (2)

where $U^\gamma$ in Eq. (2) is defined as

$$U^\gamma(x) := \frac{1 + \gamma_5}{2} U(x) + \frac{1 - \gamma_5}{2} U^\dagger(x), \quad U(x) = \exp(i\pi^a(x)\tau^a).$$  \hspace{1cm} (3)

The pion mean field is defined by a solution of the classical equation of motion that can be derived from the effective chiral action, which is expressed by $U_{\text{cl}}(r)$. The pion mean field minimizes the nucleon mass,
once the symmetries of the pion field are all known. Since the pion field carries isospin indices, one needs to couple them to the spatial components. A minimal way of this coupling can be done by the hedgehog ansatz for the pion mean-field:

$$\pi^a(x) = \hat{n}^a P(r) \tau^a,$$

(4)

where $\tau^a$ denote the SU(2) Pauli matrices and $n^a$ stands for a radial unit vector. $P(r)$ is called the profile function of the classical solution.

The procedure for deriving the classical nucleon mass is performed by the self-consistent Hartree approximation. Given a trial profile function, one diagonalizes the Dirac Hamiltonian in the basis of $n = (K, P)$

$$h(U)\Phi_n(\vec{x}) = E_n\Phi_n(\vec{x})$$

(5)

to find the energy spectrum of the quarks in the trial pion mean field. $K$ and $P$ designate respectively the grand spin defined by $K = J + T$ and parity. The eigenvalues and eigenvectors of the quarks yield a new profile function. We continue this procedure till we obtain the classical nucleon mass with the minimized energy functionals:

$$M_{cl} = N_c E_{level}[U_{cl}(r)] + \sum_{n<0} E_n[U_{cl}(r)] - \sum_{n<0} E_n(U = 1).$$

(6)

The first term arises from the $N_c$ quarks at the distinct level, whereas the second one emerges from the accumulated energy due to the polarized vacuum of the Dirac continuum. The last term is required to subtract the vacuum energy. We need to regularize the second term, since it comes from the vacuum polarization (quark loops), which diverges logarithmically. In the current study, we use the Pauli-Villars regularization scheme with single subtraction to tame the logarithmic divergence of the quark loops. Also, note that we use value of $M = 420$ MeV in the numerical calculation as used in describing properties of the light baryons [5].

The cutoff $\Lambda$ is fixed by computing the pion decay constant $f_\pi = 93$ MeV of which the expression can also be derived by the effective chiral action (2). The classical nucleon mass is numerically obtained as

$$M_{cl} = E_{level} + E_{cont} = 1069 \text{ MeV},$$

(7)

where $E_{level} = 351 \text{ MeV}$ and $E_{cont} = 718 \text{ MeV}$. While we perform the integral over $\pi^a$ by employing the saddle-point approximation in the large $N_c$ limit, we have to treat the zero modes exactly. A rotated field in both three-dimensional ordinary space and isospin space provides the same nucleon mass, because the nucleon mass is minimized in both spaces. Thus, we have to integrate over the rotational and translational zero modes, which is known as the zero-mode collective quantization. Once we quantize the classical solution or the chiral soliton, we restore the quantum numbers of the nucleon, i.e., the nucleon spin, isospin, and momentum. For details, we refer to Ref. [5].

### B. Singly heavy baryon

A singly heavy baryon can be constructed in exactly the same manner as we have shown in the previous subsection. The only difference comes from the fact that the singly heavy baryon consists of $N_c - 1$ valence quarks. In the large $N_c$ limit, we can strip off the heavy quark from the singly heavy baryon. Then the $N_c - 1$ valence quarks will create the pion mean field, which turns out weaker than the nucleon case. The chiral soliton that constitutes the $N_c - 1$ valence quarks carry the color quantum number. Having performed the self-consistent method described previously, we obtain the classical mass for the singly heavy baryon

$$M_{sol} = (N_c - 1)E_{level}(U_{cl}) + \sum_{n<0} E_n(U_{cl}) - \sum_{n<0} E_n(U = 1).$$

(8)

Comparing this with Eq. (6), we find that the prefactor in the first term is different from that in Eq. (6) whereas the expression for the energy of the Dirac continuum is the same. However, the presence of the
\(N_c - 1\) valence quarks yield a weaker pion mean field, so that we get \(E_{\text{level}} = 213\ \text{MeV}\) and \(E_{\text{cont.}} = 475\ \text{MeV}\), thus we have \(M_{\text{sol}} = 901\ \text{MeV}\). Compared with those for the nucleon given in Eq. (7), these results indicate that the pion mean field becomes weaker in the presence of the \(N_c - 1\) valence quarks and produce the continuum energy less than that for the nucleon.

Since the heavy quark remains as the static color source, the total classical mass of the singly heavy baryon should be given as the sum of the soliton mass in Eq. (8) and the heavy quark mass \(M_Q\):

\[
M^Q_{cl} = M_{\text{sol}} + M_Q.
\]  

In Refs. [18, 21], it was found that the self-consistent solution for the pion mean field given in Eq. (8) inside a singly heavy baryon plays a crucial role for the stability of the system. Had one used a parametrized one, the singly heavy baryon would not have satisfied the stability conditions related to the pressure density that comes from its \(D\)-term form factor. Thus, it is essential to construct the pion mean field in this self-consistent approach.

### III. QUARK DISTRIBUTION FUNCTIONS IN THE LARGE \(N_c\) LIMIT

In this section, we provide the expressions for the isoscalar unpolarized and isovector polarized distributions derived from the \(\chi\)QSM.

#### A. Light quark and antiquark distribution functions

Following Ref. [7], we define the quark and antiquark quasi-number densities for the light quarks in a singly heavy baryon as:

\[
D(x, P_h) = \frac{1}{2E_h} \int \frac{d^3k}{(2\pi)^3} \delta \left(x - \frac{k}{P_h}\right) \int d^3x e^{-ikx} \langle h|\bar{\psi}_f(-x/2, t)\Gamma\psi_f(x/2, t)|h\rangle,
\]

\[
\bar{D}(x, P_h) = \frac{1}{2E_h} \int \frac{d^3k}{(2\pi)^3} \delta \left(x - \frac{k}{P_h}\right) \int d^3x e^{-ikx} \langle h|\bar{\psi}_f(-x/2, t)\Gamma\psi_f(x/2, t)|h\rangle,
\]

where the path-ordered exponential is assumed. \(x\) is the longitudinal momentum fraction of the quarks and antiquarks to the heavy baryon momentum. \(E_h\) and \(P_h\) are the energy and momentum of the baryon moving with the velocity \(v\):

\[
E_h = \frac{E_h}{\sqrt{1 - v^2}}, \quad P_h = \frac{M_h v}{\sqrt{1 - v^2}}.
\]

The matrix \(\Gamma\) depends on the choice of the Dirac and isospin structures. For instance, in the case of the singlet quarks polarized parallel or anti-parallel with the nucleon momentum, \(\Gamma\) is given by

\[
\Gamma = \gamma_0 \frac{1 \pm \gamma_5}{2}.
\]

The PDFs on the light one are obtained by taking the Lorentz boost \(P_h \to \infty\) (see Refs. [7, 33]).

The above matrix elements are written in terms of the Dirac spectral representation of the chiral quark-soliton model Eq. (5). We will only recapitulate the formulae for the quark distribution functions, since their derivation was discussed in Ref. [7] in detail. The isoscalar unpolarized quark and antiquark distribution functions are written as follows (\(x \in [0, 1]\))

\[
u(x) + d(x) = (N_c - 1)M_h \int \frac{d^3k}{(2\pi)^3} \Phi_{\text{level}}^\dagger(\vec{k})(1 + \gamma^0\gamma^3)\Phi_{\text{level}}(\vec{k})\delta(k_3 - xM_h + E_{\text{level}}) + N_cM_h \sum_{E_n < 0} \int \frac{d^3k}{(2\pi)^3} \Phi_n^\dagger(\vec{k})(1 + \gamma^0\gamma^3)\Phi_n(\vec{k}) - (U \to 1),
\]

\[
u(x) + d(x) = -u(-x) + d(-x).
\]
Note that the factor $N_c - 1$ for the first term (level contribution) as the singly heavy baryon consists of $N_c - 1$ level quarks. The second term arises from the vacuum polarization (Dirac continuum). For the antiquark distribution function, we can easily obtain by using the property $\bar{q}(x) = -q(-x)$. Similarly, the isovector polarized distributions are expressed as

$$\Delta u(x) - \Delta d(x) = -\frac{1}{6}(2T_3)(N_c - 1)M_h \int \frac{d^3k}{(2\pi)^3} \Phi^\dagger_{\text{level}}(\vec{k})(1 + \gamma^0 \gamma^3)\tau^3 \Phi_{\text{level}}(\vec{k})$$

$$- \frac{1}{6}(2T_3)N_cM_h \sum_{E_n<0} \int \frac{d^3k}{(2\pi)^3} \Phi^\dagger_n(\vec{k})(1 + \gamma^0 \gamma^3)\tau^3 \Phi_n(\vec{k}) - (U \rightarrow 1),$$

$$\Delta \bar{u}(x) - \Delta \bar{d}(x) = \Delta u(-x) - \Delta d(-x). \quad (16)$$

### B. Heavy quark distribution functions

Concerning the valence charm heavy quark, we only consider it as a static color source. So, we obtain the matrix element for the heavy quark PDFs as

$$\int \frac{d^3k}{(2\pi)^3} \delta \left( x - \frac{\vec{k}}{M_h} \right) \int d^3xe^{-i\vec{k} \cdot \vec{x}}|\bar{Q}_f(-\vec{x}/2,t)\Gamma Q_f(\vec{x}/2,t)|h_v). \quad (17)$$

In the heavy quark limit $M_Q \rightarrow \infty$, its four-velocity is fixed. Thus, in this limit, one can eliminate the trivial kinetic part that depends on the momentum for each velocity [2, 34]:

$$\Psi_Q(x) = e^{-iM_Q v \cdot x} \tilde{\Psi}_Q(x). \quad (18)$$

Treating $\tilde{\Psi}$ as the non-interacting field, one obtains the unpolarized and polarized distributions for the heavy quark

$$Q(x) = \delta(x - M_Q/M_h), \quad (19)$$

$$\Delta Q(x) = -\frac{1}{3}\delta(x - M_Q/M_h). \quad (20)$$

As expected, they are given as $\delta$–functions with the fixed point $x = M_Q/M_h$. In Ref. [23], the heavy-quark distribution functions in a singly heavy baryon were studied within a heavy-quark light-diquark picture, where the interactions between the heavy-quark and the diquark in a heavy baryon were considered. It was observed that the heavy-quark distributions have a finite size. In general, one expects a similar behavior once the interaction between the heavy quark and the light soliton is considered.

### C. Sum rules

Let us examine the baryon number and the momentum sum rules as the leading and next-to-leading Mellin moments of the isoscalar unpolarized distributions. Using Eqs. (14) and (19), we obtain the following sum rules:

$$\int_0^1 dx \left( u(x) + d(x) - \bar{u}(x) - \bar{d}(x) \right) = N_c - 1 = 2, \quad (21)$$

$$\int_0^1 dx \ Q(x) = 1, \quad (22)$$

$$\int_0^1 dx \ x(u(x) + d(x) + \bar{u}(x) + \bar{d}(x)) = M_{sol}/M_h, \quad (23)$$

$$\int_0^1 dx \ xQ(x) = M_Q/M_h. \quad (24)$$

Note that the baryon number sum rule Eq. (21) is satisfied by the $N_c - 1$ light quarks occupying the bound level, i.e. no contributions from the polarized vacuum. In the picture of the $\chi$QSM, the momentum of the
singly heavy baryon is carried only by the quarks. Thus, the light-quark momentum sum rule is identified as the ratio of the soliton mass to the baryon mass as \( M_{cl} / (M_{cl} + M_Q) \). These results are also predicted in a study of the energy-momentum tensor (EMT) form factors of the heavy baryons \([21]\), because the momentum sum rule given in Eq. \((23)\) corresponds to the mass form factor \( A_q(0) \) in Ref. \([21]\). For the sake of comparison, we provide the sum rules for the nucleon:

\[
\int_0^1 dx \left( u(x) + d(x) - \bar{u}(x) - \bar{d}(x) \right) = N_c = 3, \\
\int_0^1 dx \left( x u(x) + d(x) + \bar{u}(x) + \bar{d}(x) \right) = M_q = 1.
\]  

(25)  
(26)

Now let us discuss the Bjorken spin sum rule. The leading moment of the isovector polarized quark distribution is proportional to the isovector axial charge \( g^{(3)}_{A,q} \) as follows

\[
\int_0^1 dx \left( \Delta u(x) - \Delta d(x) + \Delta \bar{u}(x) - \Delta \bar{d}(x) \right) = g^{(3)}_{A,q}.
\]  

(27)

We find that the isovector axial charge does not depend on the heavy baryon mass \( M_h \). Also, the singlet heavy quark spin is computed in a similar way and we obtain

\[
\Delta Q \equiv \int_0^1 dx \Delta Q(x) = -1/3,
\]  

(28)

which is identical with that derived from the nonrelativistic quark model \([35]\).

IV. NUMERICAL RESULTS AND DISCUSSIONS

We adopt the numerical method to compute the quark distribution functions, which was already explained in Ref. \([7]\) in detail. Since we focus on the light quark distributions inside \( \Sigma^+ \) and \( \Sigma^0 \), we introduce the heavy quark masses \( M_c = 1300 \text{ MeV} \) and \( M_b = 4200 \text{ MeV} \), which are close to those given by the PDG \([36]\). We want to mention that specific values of the heavy-quark mass are not important. We are interested in how the quark parton distribution functions change as \( M_Q \) increases, so we will see how the difference between the values of \( M_c \) and \( M_b \) come into play. We will show that when \( M_Q \) increases, the PDFs undergo drastic changes. Since we take the limit of \( M_Q \to \infty \), we ignore \( 1/M_Q \) corrections: the pion mean-field for both heavy baryons is identical and the differences of the distributions are originated from the varied \( M_Q \) values at the leading order.

In the current work, we will employ the ansatz for the profile function that was suggested in Ref. \([37]\)

\[
P(r) = 2 \arctan(r_0^2/r^2 \tanh(bMr))
\]  

(29)

instead of using the self-consistent one. The function \([29]\) correctly reproduces the linear behavior of the self-consistent mean-field profile as \( r \approx 0 \). We fit the parameters \( r_0 \) and \( b \) using the self-consistent profile functions for the \( N_c \) and \( N_c - 1 \) valence quarks. It was already shown that the shapes of the distribution functions with the ansatz in the interpolation approximation \([4]\) vary within only a few percents in comparison with those obtained by using the self-consistent one. Thus, the numerical results presented in the following sections are obtained with Eq. \((29)\) utilized The contribution of the Dirac continuum was derived by the interpolation formula \([4]\).

A. Isoscalar Unpolarized Distributions

We first check that the baryon number sum rule Eq. \((21)\) is well satisfied numerically. However, the momentum sum rule Eq. \((23)\) is broken by around 1 \%, because the momentum sum rule is related to the saddle-point equation for the pion mean-field. This small discrepancy arises from the ansatz in Eq. \((29)\). If one uses the self-consistent profile function, the momentum sum rule is exactly satisfied.
FIG. 1. Isoscalar unpolarized quark distribution $u(x) + d(x)$ for quark (black solid curve) and antiquark (blue dashed one) for the proton, $\Sigma_c$, and $\Sigma_b$ drawn from the left to the right panels, respectively.

In Fig. 1, we depict the isoscalar unpolarized quark distributions $u(x) + d(x)$ in the solid curve as well as the antiquark distribution $\bar{u}(x) + \bar{d}(x)$ in the dashed one for the nucleon, $\Sigma_c$, and $\Sigma_b$ from the left to the right panels, respectively. The light quark and antiquark distributions in the $\Sigma_c$, compared with those in the proton, are more concentrated in the smaller $x$ region. We observe that as $M_Q$ grows (from $c$ to $b$), the quark and antiquark distributions are extremely squeezed to the even smaller $x$ region. We will later discuss this behavior in detail.

B. Isovector Polarized Distributions

Before we proceed to present the numerical results for the isovector polarized quark distribution functions, we examine the Bjorken spin sum rule given in Eq. (27). We obtain numerically the axial charge

$$g_A^{(3)} = \int_0^\infty dx (\Delta u(x) - \Delta d(x) - \Delta \bar{u}(x) + \Delta \bar{d}(x)) = 0.7$$

which are identical to the $\Sigma_c$ and $\Sigma_b$ baryons ($T_3 = 1$). We compare this numerical value with that obtained from the study of axial-vector form factors for singly heavy baryons computed within the $SU_f(3)$ chiral quark-soliton model [35], which was given as $g_A^{(3)} = 1.026$. At first glance, this discrepancy seems to be very large and concerns us. However, we consider in the current work only the leading contribution in the large $N_c$ limit, whereas in Ref. [35] the rotational $1/N_c$ corrections are included. Thus, this discrepancy will be removed once the rotational $1/N_c$ corrections are introduced to the quark distribution functions. For comparison, we present the spin sum rule for the proton ($T_3 = 1/2$), which is $g_A^{(3)} = 0.95$. By the same token, we can get $g_A^{(3)} = 1.163$ [35], when we include the rotational $1/N_c$ corrections.

FIG. 2. Isovector polarized quark distribution $\Delta u(x) - \Delta d(x)$ for the quark (black solid curve) and antiquark (blue dashed one) for the proton, $\Sigma_c^{++}$, and $\Sigma_b^{++}$ drawn from the left to the right panels, respectively.
In Fig. 2, the isovector polarized quark (solid curve) and antiquark (dashed one) distributions are displayed for the proton, $\Sigma_c^{++}$, and $\Sigma_b^{++}$ (isospin $T_3 = +1$) consecutively from the left to the right panels. We find the similar $x$-dependence of the quark and antiquark distributions in a singly heavy baryon.

C. Momentum distribution of quarks in a heavy baryon

In the infinitely heavy-quark mass limit, we expect that the heavy quark carries the entire longitudinal-momentum of the system. When one uses a large but finite heavy-quark mass, the light quarks start to share the momentum. As already shown in Fig. 1, the light quarks inside a heavy quark carry a much smaller portion of the momentum of the singly heavy baryon. That is, the light-quark distribution functions exhibit the shapes squeezed to the smaller $x$ region. When $M_Q$ increases from $M_c$ to $M_b$, the shape is further squeezed to the even smaller $x$ region. We observe a similar tendency in the isovector polarized quark distribution functions as shown in Fig. 2.

To analyze this behavior quantitatively, we define the following integral for the isoscalar unpolarized distributions

$$I_h(y) := M_h \int_0^y dx \left( x(u(x) + d(x)) + \bar{u}(x) + \bar{d}(x) \right),$$

where $h$ denotes the baryon of interest: the proton, $\Sigma_c$, or $\Sigma_b$. $y$ stands for the upper limit of the integral, $y \in [0, 1]$. In Fig. 3, the momentum sum given by Eq. (31) is plotted for the proton (solid curve), $\Sigma_c$ (dashed one), and $\Sigma_b$ (dotted one). We observe that the definite momentum shared by the light quarks inside a singly heavy baryon is smaller than that in a nucleon. This is due to the fact that the light-quark soliton mass is smaller in the case of the singly heavy baryons. The light quarks are less energetic in a singly heavy baryon, compared with those in a nucleon. Changing the heavy quark mass does not affect the value of this sum $I_h(y = 1)$, as far as the $1/M_Q$ heavy-light quark interactions are not taken into account. However, varying the heavy quark mass influences the size and the position of the quark and antiquark distributions. For instance, we find that the $y$ value satisfies $I_\Sigma_b(y) = 0.8$ are $y = (0.35, 0.15)$ for the heavy quark masses $M_Q = (1.3, 4.2)$ GeV, as demonstrated in the left panel of Fig. 3. The right panel of Fig. 3 shows the normalized momentum sum: $I_h(y) := I_h(y)/I_h(y = 1)$.

Let us consider the case where the characteristic size $R$ of the pion mean-field shrinks, i.e., $R \to 0$. In this limit, the $\chi$QSM is reduced to the naive quark-model, where the nucleon contains only the non-interacting $N_c$ constituent quarks. For the nucleon, the isoscalar unpolarized distribution function becomes a delta function as follows:

$$u(x) + d(x) = N_c \delta(x - m/M_N),$$

FIG. 3. In the left panel, we draw the light-quark momentum integral defined in Eq. (31) for the proton ($p$, black solid curve), $\Sigma_c$ (blue dashed one), and $\Sigma_b$ (green dotted one). The right panel depicts the same curves but normalized by their maximum values, $I_h(y) := I_h(y)/I_h(y = 1)$. (dashed one), and $\Sigma_b$ (dotted one). We observe that the definite momentum shared by the light quarks inside a singly heavy baryon is smaller than that in a nucleon. This is due to the fact that the light-quark soliton mass is smaller in the case of the singly heavy baryons. The light quarks are less energetic in a singly heavy baryon, compared with those in a nucleon. Changing the heavy quark mass does not affect the value of this sum $I_h(y = 1)$, as far as the $1/M_Q$ heavy-light quark interactions are not taken into account. However, varying the heavy quark mass influences the size and the position of the quark and antiquark distributions. For instance, we find that the $y$ value satisfies $I_\Sigma_b(y) = 0.8$ are $y = (0.35, 0.15)$ for the heavy quark masses $M_Q = (1.3, 4.2)$ GeV, as demonstrated in the left panel of Fig. 3. The right panel of Fig. 3 shows the normalized momentum sum: $I_h(y) := I_h(y)/I_h(y = 1)$.

Let us consider the case where the characteristic size $R$ of the pion mean-field shrinks, i.e., $R \to 0$. In this limit, the $\chi$QSM is reduced to the naive quark-model, where the nucleon contains only the non-interacting $N_c$ constituent quarks. For the nucleon, the isoscalar unpolarized distribution function becomes a delta function as follows:

$$u(x) + d(x) = N_c \delta(x - m/M_N),$$

(32)
where \( m \) designates the constituent quark mass and \( M_N = N_c m \). When it comes to the heavy baryon, on the other hand, we obtain

\[
\begin{align*}
    u(x) + d(x) &= (N_c - 1)\delta(x - m/M_h), \\
    c(x) &= \delta(x - M_Q/M_h).
\end{align*}
\]

The singly heavy baryon mass is identified as \( M_h = (N_c - 1)m + M_Q \). The momentum sum rule then reads

\[
\int_0^1 dx \, x \left[ u(x) + d(x) + c(x) \right] = (N_c - 1)m/M_h + M_Q/M_h = 1.
\]

From Eqs. (32) and (33), we observe clearly that the center of the valence-like light-quark PDFs is shifted to smaller \( x \): from \( 1/N_c \) to \( 1/(N_c + M_Q/m) \). In a more realistic case, where the heavy and light quark interaction of order \( 1/M_Q \) is included, the heavy quark distribution has a non-zero width, as demonstrated in the heavy-quark – diquark picture of the singly heavy baryons in Ref. [28]. In the current work, such an effect is ignored as mentioned earlier.

Last but not least, we want to emphasize that the singly heavy baryon in the current picture is subject to a certain hierarchy arising from the parametrically large \( M_Q \) and \( N_c \). We have constructed the model first by taking the limit of \( M_Q \to \infty \) and then of \( N_c \to \infty \). This results in the following hierarchy for various scales:

\[
1 < M/\Lambda_{\text{QCD}} < N_c < M_Q/\Lambda_{\text{QCD}},
\]

where \( M \) denotes the dynamical quark mass and \( \Lambda_{\text{QCD}} \) is called the QCD scale parameter. Note that the parameters for the instanton vacuum are related to \( \Lambda_{\text{QCD}} \) [41, 51]. In this scheme, the mass of the singly heavy baryon is given by \( M_h = M_{\text{sol}} + M_Q \), where \( M_{\text{sol}} \) is proportional to \( N_c \). The momentum sum rule of the light baryon \( [23] \) vanishes as \( N_c \Lambda_{\text{QCD}}/M_Q \to 0 \). This ordering of the hierarchy between the parameters \( N_c \) and \( M_Q/\Lambda_{\text{QCD}} \) allows one to strip off the heavy quark from a singly heavy baryon, since the heavy quark mainly remains as a static color source (see Refs. [13, 17, 21]). This ordering is strongly supported by the fact that the hyperfine splitting between the heavy baryon states \( \sim O(1/M_Q) \sim 70 \text{ MeV} \) in the case of the charmed baryons is much smaller than the strength of the rotational excitation energy \( \sim O(N_c) \sim 300 \text{ MeV} \). If we reverse the ordering of \( M_Q \to \infty \) and \( N_c \to \infty \) limits, it is very complicated to consider the singly heavy baryons in the pion mean-field approach with the heavy-quark flavor-spin symmetry taken into account. Having constructed the model, we restore physical values of \( N_c = 3 \) and \( M_Q \). The numerical values for \( M \) and \( M_Q \) used in the current study satisfies the inequality (35), as they should be.

### D. Inequalities

The parton distribution functions are expected to satisfy a set of inequalities. The inequality conditions related to the present work are as follow:

\[
\begin{align*}
    f_1^q(x) &\geq 0, \\
    f_1^a(x) &\geq |g_1^a(x)|,
\end{align*}
\]

where \( f_1 \) and \( g_1 \) denote the singlet unpolarized and longitudinally polarized parton distribution functions with \( a = q, \bar{q}, g \). Assuming that the small components \( \sim O(N_c) \) \( (u - d) \) and \( \Delta u + \Delta d \) vanish in the large \( N_c \) limit, we can rewrite Eqs. (36) and (37) in the following form:

\[
\begin{align*}
    u(x) + d(x) &\geq 0, \\
    u(x) + d(x) &\geq |\Delta u(x) - \Delta d(x)|.
\end{align*}
\]

* Strictly speaking, the positivity condition depends on the factorization and regularization schemes. The equations shown in the current work are correct to order of \( \alpha_s \).
The positivity and the inequality conditions stem from the probability interpretation, so that they provide strong constraints on the PDFs. Let us examine whether the present results satisfy the inequalities given by Eqs. (38) and (39) in the large $N_c$ limit. It was already shown in Fig. 1 that the quark and antiquark isoscalar unpolarized distributions satisfy the positivity condition Eq. (38). In Fig. 1, the function $u(x) + d(x) - |\Delta u(x) - \Delta d(x)|$ is illustrated to check numerically the inequality Eq. (39) for the quarks (solid curve) and the antiquarks (dashed one). One observes that the inequality condition is manifestly satisfied for the light and singly heavy baryons within the $\chi$QSM.

![Figure 4](image)

**FIG. 4.** Inequality conditions $u(x) + d(x) - |\Delta u(x) - \Delta d(x)|$ for the quark (black solid curve) and antiquark (blue dashed one) are plotted for the proton (left panel), $\Sigma_c$ (middle one), and $\Sigma_b$ (right one).

V. QUARK QUASI-DISTRIBUTION FUNCTIONS

The PDFs of a singly heavy baryon can also be studied in lattice QCD. In recent years, the large momentum effective theory (LaMET) is widely used. In LaMET, one computes the quasi-distributions on the Euclidean lattice and utilizes the perturbative matching relations to obtain the corresponding light-cone distributions. The properties of the quark quasi-PDFs in the nucleon in the large $N_c$ limit have already been studied within the $\chi$QSM, where the convergence of the quasi-PDFs in the nucleon momentum evolution and the sum rules were discussed. In this section, we briefly mention about the light-quark quasi-distribution functions in a singly heavy baryon. The model expressions for the twist-2 quasi-PDFs are similar as those in Refs. [33, 39]. Only differences lie in the number of bound-level quarks $N_c - 1$ instead of $N_c$ and the corresponding change of the pion mean field. The Mellin moments of the quasi-PDFs are related to the physical quantities pertinent to the symmetries, but they depend on the baryon momentum and the Dirac matrix defining them [33]. In the case of the heavy baryon we have the following expressions for the isoscalar unpolarized light-quark quasi-distributions

\[
\int_{-\infty}^{\infty} dx \left( u(x, P) + d(x, P) \right) = \left\{ \frac{N_c - N_Q}{\sqrt{M^2 - 2P}} (N_c - N_Q), \quad \Gamma = \gamma_0^0, \quad \Gamma = \gamma_3^3 \right\},
\]

(40)

\[
\int_{-\infty}^{\infty} dx \left( u(x, P) + d(x, P) \right) = \left\{ \frac{M_q}{\sqrt{M^2 - 2P}} M_q, \quad \Gamma = \gamma_0^0, \quad \Gamma = \gamma_3^3 \right\},
\]

(41)

and for the isovector polarized ones

\[
\int_{-\infty}^{\infty} dx \left( \Delta u(x, P) - \Delta d(x, P) \right) = \left\{ \frac{P}{\sqrt{M^2 - 2P}} (2T_3) g_{A q}, \quad \Gamma = \gamma_0^0, \quad \Gamma = \gamma_3^3 \right\},
\]

(42)

where $P$ and $M$ denote respectively the heavy baryon momentum and mass. In Fig. 5, the light-quark quasi-distribution functions in $\Sigma_c^{++}$ are depicted where the baryon momentum $P = 3$ GeV is used, which is the typical hadronic momentum used in the LaMET framework. The isoscalar unpolarized distribution $u + d$ is shown in the left panel whereas the isovector polarized distribution $\Delta u - \Delta d$ is drawn in the right panel. Especially, one finds that the isoscalar unpolarized distribution is significantly different from the light-cone distribution. The sum rules given in Eqs. (40) to (42) are satisfied numerically.
Note that the corresponding QCD matrix elements for the quasi-PDFs can be evaluated on the lattice, in principle. However, the numerical results from the current study suggest that the LaMET would require unrealistically high momentum for the singly heavy baryons. A similar observation was found in the case of the heavy quarkonium distribution amplitude, where the required charmonium momentum $P$ is 2-3 times its mass $\Gamma$. Thus, the numerical studies on the Mellin-moment would be more feasible in a lattice QCD study.

![Figure 5](image_url)  
**FIG. 5.** Quark quasi distribution functions in $\Sigma^{++}_c$. Isoscalar unpolarized distribution $u + d$ is shown on the left and the isovector polarized distribution $\Delta u - \Delta d$ is shown on the right panel. Black curves denote the light-cone distribution functions ($P \to \infty$) and the quasi distributions are evaluated with a heavy baryon momentum $P = 3$ GeV along the boost direction. Blue dotted and red dashed plots denote $\Gamma = \gamma^0, \gamma^3$, respectively.

### VI. SUMMARY AND CONCLUSIONS

In this paper, we studied the properties of the twist-2 light-quark distribution functions in the large $N_c$ limit inside a singly heavy baryon within the framework of the chiral quark-soliton model. In the limit of the infinitely heavy-quark mass, the heavy quark remains as a mere static color source, which provides heavy quark spin-flavor symmetry. This indicates that the light quarks govern the quark dynamics inside a singly heavy baryon. Thus, we focused on the isoscalar unpolarized $u + d$ and isovector longitudinally polarized $\Delta u - \Delta d$ quark distributions.

In the limit of $M_Q \to \infty$, the light quark distributions would become a $\delta$-function positioned at $x = 0$, whereas the heavy quark carries the entire momentum of the baryon. In reality, however, the heavy-quark mass is large but finite, so the light-quark distributions have finite size and are centered roughly at $x \equiv M/M_h$. For the numerical calculation, we took heavy-quark masses $M_c = 1.3$ GeV and $M_b = 4.2$ GeV to demonstrate the light-quark distributions inside $\Sigma_c$ and $\Sigma_b$ baryons, respectively. Compared to the nucleon case, the light-quark distributions are squeezed to the small momentum fraction $x$. We numerically confirm that the positivity and the inequalities are well satisfied.

As can be seen from Ref. [28], including the interaction between the heavy quark and the light quarks mostly affects the shape and position of the heavy quark distribution in a singly heavy baryon. On the other hand, we concentrate on the behavior of the light quarks inside a singly heavy baryon, assuming that such effects are only order of $1/M_Q$. The heavy-quark distribution functions can be investigated within the framework of the $\chi$QSM with the heavy-light quark interaction considered. This interaction can be derived from the instanton vacuum and will be discussed in future studies.

The light-quark distributions in a singly heavy baryon can be studied also within lattice QCD. Firstly, one can compute their Mellin moments to study the momentum balance between the light and heavy quark sector. On the other hand, it seems that the current LaMET approach is not suitable to study the $x$-dependence of the PDFs, because high baryon momenta $P_h \gg M_h$ are required. Nevertheless, we briefly discussed the quasi-PDFs and the extended sum rules as their Mellin moments as done in Refs. [33, 39] for the proton, anticipating future results from lattice QCD.
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