A NOTE ON “A GENERALIZATION OF ROBERTS’ COUNTEREXAMPLE TO THE FOURTEENTH PROBLEM OF HILBERT BY S. KURODA”

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Abstract. In [4], Kuroda generalized Roberts’ counterexample [5] to the fourteenth problem of Hilbert. The counterexample is given as the kernel of a locally nilpotent derivation on a polynomial ring. We replace his construction of the invariant elements by a more straightforward construction and give a more precise form of invariant elements.

1. Introduction

Let $k$ be a field of characteristic zero and let $B$ be a $k$-algebra. We denote by $\text{LND}_k(B)$ the set of $k$-derivations of $B$. In [4], Kuroda proved the following result.

**Theorem 1.1.** Let $B = k[x_1, \ldots, x_n, y_1, \ldots, y_n, y_{n+1}]$ be a polynomial $k$-algebra and define $\delta \in \text{LND}_k(B)$ by $\delta(x_i) = 0$ and $\delta(y_i) = x_i^2$ for all $1 \leq i \leq n$, and $\delta(y_{n+1}) = x_1 \cdots x_n$. Suppose that $n \geq 4$. Then $A := \text{Ker} \delta$ is not finitely generated over $k$.

In order to prove this theorem, he made use of the following lemma.

**Lemma 1.2.** With the notations and assumptions in the above theorem, there exists a positive integer $\alpha$ such that the $k$-subalgebra $A$ contains elements of the form

$$x_1^\alpha y_{n+1}^\ell + (\text{terms of lower degree in } y_{n+1})$$

for each $\ell \geq 1$.

In this paper, we prove that we can take $\alpha = 1$. Namely, we prove the following.

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Theorem 1.3. With the notations and assumptions in the above theorem, the $k$-subalgebra $A$ contains elements of the form
\[ x_1 y_{n+1}^\ell + \text{(terms of lower degree in } y_{n+1}) \]
for each $\ell \geq 1$.

2. Proof of Theorem 1.3

In a subsequent proof, we use the following result.

Lemma 2.1. Let $B = k[x_2, \ldots, x_n, y_2, \ldots, y_n]$ and define $\delta \in \text{LND}_k(B)$ by $\delta(x_i) = 0$ and $\delta(y_i) = x_i^2$ for each $i$. Then $\text{Ker} \, \delta$ is a $k$-algebra generated by $x_2, \ldots, x_n$ and $x_i^2 y_j - x_j^2 y_i$ (2 \leq i, j \leq n, i \neq j).

We can prove this lemma by the same argument in [3, Theorem 1.2].

Now, for each monomial $m = x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_{n+1}^{b_{n+1}}$, define
\[ \tau(m) = \left( \frac{a_2}{2} \right) + \cdots + \left( \frac{a_n}{2} \right) - (b_1 + \cdots + b_n), \]
where we write $[a] = \max\{ n \in \mathbb{Z} \mid n \leq a \}$ for any $a \in \mathbb{R}$. Let $f_{1,n+1} = x_1 y_{n+1} - x_2 \cdots x_n y_1$ and let $f_{i,j} = x_i^2 y_j - x_j^2 y_i$ for each pair $(i,j)$ with $1 \leq i, j \leq n$ and $i \neq j$. It is easy to see that all of $f_{1,n+1}, x_i$ and $f_{i,j}$ belong to $A$. Let $A'$ be the $k$-subalgebra generated by $f_{1,n+1}, x_i$ (1 \leq i \leq n) and $f_{i,j}$ (1 \leq i,j \leq n, i \neq j). Since $A$ is factorially closed in $B$, i.e., $a = b_1 b_2$ with $b_1, b_2 \in B$ implies $b_1, b_2 \in A$, it suffices to show that there exists $f \in A'$ such that $f_{1,n+1}^\ell - f$ is of the form
\[ x_1^\ell y_{n+1}^\ell + x_1^{\ell-1} \text{(terms of lower degree in } y_{n+1}) \]
\[ = x_1^{\ell-1} \left( x_1 y_{n+1}^\ell + \text{(terms of lower degree in } y_{n+1}) \right). \]

We have
\[ f_{1,n+1}^\ell = x_1^\ell y_{n+1}^\ell - \ell x_1^{\ell-1} y_{n+1}^\ell x_2 \cdots x_n y_1 \]
\[ + \binom{\ell}{2} x_1^{\ell-2} y_{n+1}^\ell x_2^2 \cdots x_n^2 y_1 + \cdots + (-1)^\ell x_2^\ell \cdots x_n^\ell y_1 \]
and we construct $f \in A'$ which, when subtracted from $f_{1,n+1}^\ell$, cancels the terms in $f_{1,n+1}^\ell$ of degree < $\ell - 1$ in $x_1$ and produces only the terms of degree $\geq \ell - 1$ in $x_1$. Namely, as the element $f_{1,n+1}^\ell - f$, we construct an element in $A'$ of the form
\[ x_1^\ell y_{n+1}^\ell + g_{\ell-1} y_{n+1}^{\ell-1} + g_{\ell-2} y_{n+1}^{\ell-2} + \cdots + g_0, \]
where $g_i \in k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and $x_1^{\ell-1}$ divides every $g_i$.

By the descending induction on $r$, we suppose that we obtain an element in $A'$ of the form
\[ G_r = x_1^\ell y_{n+1}^\ell + g_{\ell-1} y_{n+1}^{\ell-1} + g_{\ell-2} y_{n+1}^{\ell-2} + \cdots + g_r y_{n+1}^r + \cdots + g_0 \]
with \( g_i \in k[x_1, \ldots, x_n, y_1, \ldots, y_n] \) and \( g_{\ell-1}, \ldots, g_{\ell+1} \) divisible by \( x_1^{\ell-1} \). We show that \( G_r \) is modified by an element of \( A' \) so that a new \( g_r \) is divisible by \( x_1^{\ell-1} \) without changing the terms \( g_{\ell-1}, \ldots, g_{\ell+1} \). Furthermore, we suppose the following conditions are satisfied.

1. For \( 0 \leq i \leq \ell - 1 \), if we write \( g_i = \sum_j x_1^j y_1^{q_{i,j}} h_{i,j} \) with \( h_{i,j} \in k[x_2, \ldots, x_n, y_2, \ldots, y_n] \), then \( i + j + 2q_{i,j} = 2\ell \).
2. We have \( h_{i,0} = \cdots = h_{i,i-1} = 0 \) for \( 0 \leq i \leq \ell - 1 \), i.e., for each \( x_1^j y_1^{q_{i,j}} h_{i,j} \) appearing in \( g_i \), we have \( j \geq i \).
3. For each monomial \( m = x_1^i x_2^{a_2} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} y_{n+1}^i \) in \( y_{n+1} x_1^i y_1^{q_{i,j}} h_{i,j} \), we have
   
   (i) \( 2\tau(m) \geq \ell - j - 3 \) and \( a_2, \ldots, a_n \) are all odd integers if \( j \equiv \ell - 1 \pmod{2} \),
   
   (ii) \( 2\tau(m) \geq \ell - j \) and \( a_2, \ldots, a_n \) are all even integers if \( j \equiv \ell \pmod{2} \).

In order to improve the term \( g_r \) in such a way that \( h_{r,0} = \cdots = h_{r,\ell-2} = 0 \), we suppose by a double induction that \( h_{r,0} = \cdots = h_{r,p-1} = 0 \) and \( h_{r,p} \neq 0 \) with \( r \leq p \leq \ell - 2 \). With this hypothesis taken into account, we denote the polynomial \( G_r \) by \( G_{r,p} \). The beginning polynomial for induction is \( G_{\ell-2,\ell-2} = f_{1,n+1}^\ell \), for which \( g_i = (-1)^i \binom{\ell}{i} x_1^i (x_2 \cdots x_n y_1)^{\ell-i}, h_{i,i} = (-1)^i \binom{\ell}{i} (x_2 \cdots x_n)^{\ell-i} \), \( h_{i,j} = 0 \) \( (i \neq j) \) and \( q_{i,j} = \ell - i \) for \( 0 \leq i, j \leq \ell - 1 \). One can check easily that the above conditions are satisfied for \( G_{\ell-2,\ell-2} = f_{1,n+1}^\ell \).

We explain the process of improving \( g_r \). Since \( g_{r+1} \) is divisible by \( x_1^{\ell-1} \) and

\[
\delta(x_1^p y_1^{q_{r,p}} h_{r,p} y_{n+1}^r) = x_1^p y_1^{q_{r,p}} y_{n+1}^r \delta(h_{r,p}) + \text{(terms of degree } > p \text{ in } x_1) + \text{(terms of degree } < r \text{ in } y_{n+1}),
\]

we have

\[
0 = \delta(G_{r,p}) = x_1^p y_1^{q_{r,p}} y_{n+1}^r \delta(h_{r,p}) + \text{(terms of degree } > p \text{ in } x_1) + \text{(terms of degree } \neq r \text{ in } y_{n+1})
\]

and hence \( \delta(h_{r,p}) = 0 \). Lemma 2.1 implies that \( h_{r,p} \) is a sum of polynomials of the form

\[ cx_2^{d_2} \cdots x_n^{d_n} \prod_{i,j \in \{2, \ldots, n\}} f_{i,j}^{t_{i,j}} \]

with \( c \in k \) and non-negative integers \( d_i, t_{i,j} \). Note that all of \( d_2, \ldots, d_n \) are odd integers (resp. even integers) if \( p \equiv \ell - 1 \pmod{2} \) (resp. if \( p \equiv \ell \pmod{2} \)). In fact, since the contributions of the \( f_{i,j} \) to the exponent \( d_2, \ldots, d_n \) are even, the remark follows from the conditions.
(i) and (ii) of (3). Now we choose any one of the above polynomi-
als and let $H = \prod_{i,j \in \{2,\ldots,n\}} f_{i,j}^{k_{i,j}}$. Then, for each monomial $m$ in $y_{n+1}^r x_1^p y_1^{q_{r,p}x_2^{d_2}} \cdots x_n^{d_n} H$, we have in view of (i) and (ii) of (3),
\begin{align*}
2\tau(y_1^{q_{r,p}x_2^{d_2}} \cdots x_n^{d_n}) &= 2\tau(m) \\
&\geq \begin{cases} \\
\ell - p - 3 & \text{if } p \equiv \ell - 1 \pmod{2} \\
\ell - p & \text{if } p \equiv \ell \pmod{2} \end{cases},
\end{align*}
where multiplying $y_1^{q_{r,p}x_2^{d_2}} \cdots x_n^{d_n}$ by any $x_i^2 y_j (i \neq 1)$, $y_{n+1}$ or $x_1$ does not change the value of $\tau$. Note that $r \leq p \leq \ell - 2$ and that if $p \equiv \ell - 1 \pmod{2}$, then $p \leq \ell - 3$ and hence $\ell - p - 3 \geq 0$. Thus we have $\tau(y_1^{q_{r,p}x_2^{d_2}} \cdots x_n^{d_n}) \geq 0$ and there exists an element $F \in A'$ of the form
\begin{align*}
F &= c x_1^{p-r} f_{1,n+1}^r f_{2,1}^{q_2} \cdots f_{n,1}^{q_n} x_2^{d_2-2q_2} \cdots x_n^{d_n-2q_n} \\
&= c x_1^{p-r} y_1^{q_{r,p}y_{n+1}^r} x_2^{d_2} \cdots x_n^{d_n} \\
&\quad + \text{(terms of degree } > p \text{ in } x_1) + \text{(terms of degree } < r \text{ in } y_{n+1}),
\end{align*}
where $q_2 + \cdots + q_n = q_{r,p}$. We can prove that $G_{r,p} - FH$ satisfies the same conditions as $G_{r,p}$ does except for the condition $h_{r,p} \neq 0$ but the number of nonzero terms in $h_{r,p}$ gets smaller. We prove this below. By repeating this process finitely many times, we obtain a new $G_r$ satisfying the condition $h_{r,p} = 0$. Further, continuing this process finitely many times, we obtain a modified $G_r$ satisfying the condition $h_{r,0} = \cdots = h_{r,\ell-2} = 0$, i.e., $g_r$ is divisible by $x_1^{\ell-1}$. Hence by induction on $r$, we completes a proof.

Now we show that $G_{r,p} - FH$ satisfies the same conditions as $G_{r,p}$ does but the number of nonzero monomial terms in $h_{r,p}$ becomes less. We have only to show that each monomial in $F$ satisfies the conditions (1)-(3) since none of $y_{n+1}$, $x_1$ and $y_1$ appears in $H$ and multiplication of any monomial in $H$ to a monomial does not change the value of $\tau$. Each nonzero monomial $m_F$ in $F$ is of the form
\begin{align*}
x_1^{p-r}(x_1 y_{n+1})^{r_1}(x_2 \cdots x_n y_1)^{r_2} x_2^{d_2-2q_2} \cdots x_n^{d_n-2q_n} \prod_{i=2}^n (x_i^2 y_1)^{\alpha_i} (x_1^2 y_1)^{\beta_i}
\end{align*}
with $r_1 + r_2 = r$ and $\alpha_i + \beta_i = q_i$ for $i = 2, \ldots, n$. We choose one $m_F$ and let $w$, $z_1$, and $z_{n+1}$ be the exponents of $x_1$, $y_1$, and $y_{n+1}$ in $m_F$ respectively. Then we have
\begin{align*}
w &= p - r + r_1 + 2\beta_2 + \cdots + 2\beta_n = p - r_2 + 2\beta_2 + \cdots + 2\beta_n, \\
z_1 &= r_2 + \alpha_2 + \cdots + \alpha_n \quad \text{and} \quad z_{n+1} = r_1.
\end{align*}
First we prove \( m_F \) satisfies the conditions (1) and (2). Indeed, we have
\[
z_{n+1} + w + 2z_1 = p + r_1 + r_2 + 2(\alpha_2 + \beta_2) + \cdots + 2(\alpha_n + \beta_n)
\]
\[
= p + r + 2q_2 + \cdots + 2q_n = p + r + 2q_{r,p} = 2\ell
\]
and
\[
w - z_{n+1} = p - (r_1 + r_2) + 2(\beta_2 + \cdots + \beta_n)
\]
\[
= p - r + 2(\beta_2 + \cdots + \beta_n) \geq p - r \geq 0.
\]
In order to prove that \( m_F \) satisfies the condition (3), we consider four cases
(a) \( p \equiv \ell - 1 \pmod{2} \) and \( r_2 = 2u + 1 \)
(b) \( p \equiv \ell - 1 \pmod{2} \) and \( r_2 = 2u \)
(c) \( p \equiv \ell \pmod{2} \) and \( r_2 = 2u + 1 \)
(d) \( p \equiv \ell \pmod{2} \) and \( r_2 = 2u, \)
where \( u \) is an integer. We only consider the case (a). The remaining cases can be treated in a similar fashion. Then we have
\[
w \equiv \ell - 1 - 2u - 1 + 2\beta_2 + \cdots + 2\beta_n \equiv \ell \pmod{2}.
\]
The exponent of each \( x_i \) \( (i \neq 1) \) in \( m_F \) is equal to \( 2u + 1 + d_i - 2q_i + 2\alpha_i \). Since each \( d_i \) is an odd integer by the condition (i) of (3), it is an even integer. In addition, we have
\[
2\tau(m_F) = 2((n - 1)u + (n - 1) + \tau(x_2^{d_2-2q_2} \cdots x_n^{d_n-2q_n})
\]
\[
- (2u + 1) - (\beta_2 + \cdots + \beta_n)) = 2u(n - 3) + 2(n - 2) + 2\tau(x_2^{d_2-2q_2} \cdots x_n^{d_n-2q_n})
\]
\[
- 2(\beta_2 + \cdots + \beta_n)
\]
\[
=(r_2 - 1)(n - 3) + 2(n - 2) + 2\tau(y_1^{q_{r,p}}x_2^{d_2} \cdots x_n^{d_n})
\]
\[
- 2(\beta_2 + \cdots + \beta_n)
\]
\[
\geq r_2 - 1 + 2 \cdot 2 + \ell - p - 3 - 2(\beta_2 + \cdots + \beta_n)
\]
\[
= \ell - (p - r_2 + 2\beta_2 + \cdots + 2\beta_n) = \ell - w,
\]
where the term \( (n - 1) \) in the first equality is due to the condition that all the \( d_i \) and \( r_2 \) are odd integers and we use the condition \( n \geq 4 \) to show the inequality. Thus the condition (3) holds for \( m_F \). This induction completes a proof of Theorem 1.3.

3. Application to module derivations

In this section, we give application of Theorem 1.3 to locally nilpotent module derivations. First, we recall the following definition (see [6]).
**Definition 3.1.** Let $\delta \in \text{LND}_k(B)$ and let $M$ be a $B$-module with a $k$-linear endomorphism $\delta_M : M \to M$. A pair $(M, \delta_M)$ is called a $(B, \delta)$-module (a $\delta$-module, for short) if the following two conditions are satisfied.

1. For any $b \in B$ and $m \in M$, $\delta_M(bm) = \delta(b)m + b\delta_M(m)$.
2. For each $m \in M$, there exists a positive integer $N$ such that $\delta^n_M(m) = 0$ if $n \geq N$.

Let $A = \text{Ker} \delta$. Then $\delta_M$ is an $A$-module endomorphism. Whenever we consider $\delta$-modules, the derivation $\delta$ on $B$ is fixed once for all. We call $\delta_M$ a module derivation (resp. locally nilpotent module derivation) on $M$ if it satisfies the condition (1) (resp. both conditions (1) and (2)).

If there is no fear of confusion, we simply say that $M$ is a $\delta$-module instead of saying that $(M, \delta_M)$ is a $\delta$-module.

We consider the following problem.

**Problem 3.2.** Let $B$ be an affine $k$-domain with a locally nilpotent derivation $\delta$ and let $M$ be a finitely generated $B$-module with $\delta$-module structure. Is $M_0 := \text{Ker} \delta_M = \{m \in M \mid \delta_M(m) = 0\}$ a finitely generated $A$-module?

We have positive answers to Problem 3.2 if one of the following conditions is satisfied (see [7]).

1. $M$ is torsion-free as a $B$-module and $A$ is a noetherian domain.
2. $M$ is torsion-free as a $B$-module and $\dim B \leq 3$.
3. $M_0$ is a free $A$-module.
4. The $B$-module $BM_0$ generated by $M_0$ is a free $B$-module with a basis $\{e_1, \ldots, e_n\}$ such that $e_i \in M_0$.
5. $B = A[y]$ is a polynomial ring over a noetherian domain $A$, $a := \delta(y)$ is a nonzero element of $A$ and $a$ has no torsion in $M$.

In [6], there is an easy counterexample to Problem 3.2 in the case where $M$ has torsion as a $B$-module. In addition, there are counterexamples in the free case by making use of the counterexamples to the fourteenth problem of Hilbert given by Roberts [5], Kojima-Miyamishi [3], Freudenburg [2] and Daigle-Freudenburg [1]. In such examples, we take $B$ to be a polynomial ring and $M$ to be the differential module $\Omega_{B/k}$. We can give $\Omega_{B/k}$ a natural module derivation as follows.

**Lemma 3.3.** Let $B$ be a $C$-algebra and let $\delta$ be a locally nilpotent $C$-derivation of $B$. Then the differential module $M := \Omega_{B/k}$ is a $\delta$-module if we define $\delta_M$ by $\delta_M(db) = d\delta(b)$ for $b \in B$. 

We can prove this lemma easily (see [7]). Theorem 1.3 gives a new counterexample to Problem 3.2. Namely, we have the following assertions.

**Theorem 3.4.** With the notations and assumptions in Theorem 1.1, let $M = \Omega_{B/k}$ be the differential module with natural $\delta$-module structure. Namely, $M$ is a free $B$-module

$$M = \bigoplus_{i=1}^{n} Bdx_i \oplus \bigoplus_{i=1}^{n} Bdy_i$$

with a free basis $\{dx_1, \ldots, dx_n, dy_1, \ldots, dy_{n+1}\}$ and a module derivation defined by

$$\delta(dx_i) = 0, \quad \delta(dy_i) = 2x_i dx_i \quad (1 \leq i \leq n) \quad \text{and}$$

$$\delta(dy_{n+1}) = \sum_{i=1}^{n} x_1 \cdots \hat{x_i} \cdots x_n dx_i.$$

Then $M_0$ is not a finitely generated $A$-module.

We can prove this in a fashion similar to [7, Theorem 6.2] by making use of Theorem 1.3. The fact that we can take $\alpha = 1$ in Lemma 1.2 plays an important role in the proof of Theorem 3.4.

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