EXTREMAL UNIPOTENT REPRESENTATIONS FOR THE HOWE CORRESPONDENCE OVER FINITE FIELDS

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ABSTRACT. We study the Howe correspondence for unipotent representations of irreducible dual pairs \((G', G) = (U_m(\mathbb{F}_q), U_n(\mathbb{F}_q))\) and \((G', G) = (\text{Sp}_{2m}(\mathbb{F}_q), \text{O}_{2n}^\epsilon(\mathbb{F}_q))\), where \(\mathbb{F}_q\) denotes the finite field with \(q\) elements (\(q\) odd) and \(\epsilon = \pm 1\). We show how to extract extremal (i.e., minimal and maximal) irreducible subrepresentations from the image of \(\pi\) under the correspondence of a unipotent representation \(\pi\) of \(G\).

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1. Introduction

Let \(\mathbb{F}_q\) is a finite field with \(q\) elements and odd characteristic. A pair \((G', G)\) of mutually centralized reductive subgroups of \(\text{Sp}_{2n}(q) := \text{Sp}_{2n}(\mathbb{F}_q)\) is called a reductive dual pair. Roger Howe has introduced (cf. [17]) a correspondence \(\Theta : \mathcal{R}(G) \to \mathcal{R}(G')\) between the category of complex representations of these subgroups. It is obtained by restricting the Weil representation of \(\text{Sp}_{2n}(q)\) to the product \(G' \cdot G\).

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Irreducible dual pairs \((G', G)\) in \(\text{Sp}_{2n}(q)\) can be either symplectic-orthogonal \((\text{Sp}_{2m}(q), \text{O}_{m'}(q))\), unitary \((\text{U}_m(q), \text{U}_{m'}(q))\) or linear pairs \((\text{GL}_m(q), \text{GL}_{m'}(q))\) with \(n = mm'\) in all cases.

The Howe correspondence for finite fields sends irreducible representations of \(G\) to (in general) reducible representations of \(G'\). So for an irreducible representation \(\pi\) of \(G\), the representation \(\Theta(\pi)\) of \(G'\) decomposes as a sum of irreducible subrepresentations. The main goal of the present work is to find certain extremal (i.e. minimal and maximal) representations in this set of irreducible representations for unitary pairs and symplectic-orthogonal pairs (we deal only with even dimensional orthogonal groups). This generalizes the case \((\text{Sp}_4(q), \text{O}_{2m}^+(q))\) studied by Aubert, Kraskiewicz, and Przebinda in [4].

One of the main facts we use in finding this extremal representations is the compatibility between the Howe correspondence and both the Harish-Chandra and Lusztig decompositions.

If we call \(\mathcal{R}(G)_\lambda\) the representations spanned by the Harish-Chandra series \(\text{Irr}(G)_\lambda\) corresponding to \((L, \lambda)\), then one can find [5, Theorem 3.7] a cuspidal representation \(\lambda'\) of the group of rational points of a certain Levi of \(G'\) such that the theta correspondence sends representations in \(\mathcal{R}(G)_\lambda\) to representations in \(\mathcal{R}(G')_{\lambda'}\).

Considering the groups \(G\) and \(G'\) whose groups of rational points are \(G\) and \(G'\), if \(\mathcal{E}(G, (s))\) is the Lusztig series corresponding to the semisimple rational element \(s\) of \(G^*\) then there is a semisimple \(G'^*\)-conjugacy class \((s')\) such that the Howe correspondence sends representations in \(\mathcal{E}(G, (s))\) to representations in the subcategory \(\mathcal{R}(G', (s'))\) of \(\mathcal{R}(G')\) spanned by \(\mathcal{E}(G', (s'))\) [5, Proposition 2.3]. Moreover, the application between semisimple classes if induced by the natural inclusion of one of the groups \(G'^*F'^*\) or \(G'^*F'^*\) in the other. In particular the image of an unipotent representation (that is the member of \(\mathcal{E}(G, (1))\)) decomposes as a sum of unipotent representations.

Not many classical groups have representations both cuspidal and unipotent, between those appearing in type I dual pairs we find the groups \(\text{Sp}_{2k(k+1)}(q), \text{O}_{2k^2}(q)\) and \(\text{U}_{k(k+1)/2}(q)\) for a positive integer \(k\) and \(\epsilon = \text{sgn}(-1)^k\); these groups have only one cuspidal unipotent representation except for the orthogonal groups, which has two. The Howe correspondence for unipotent representations can be seen as a correspondence between Harish-Chandra series corresponding to cuspidal unipotent representations. It is known [12] that these series are in bijection with the set of irreducible representations of a certain Weyl group, so the Howe correspondence can be expressed as a correspondence between Weyl groups. For dual pairs \((\text{Sp}_{2m}(q), \text{O}_{2m'}^+(q))\) and \((\text{U}_m(q), \text{U}_{m'}(q))\) it
becomes a correspondence between the pairs $(B_{m-k(k+1)}, B_{m'-k'(k'+1)})$ and $(B_{(1/2)(m-k(k+1)/2)}, B_{(1/2)(m'-k'(k'+1)/2)})$, $k'$ depending on $k$.

Call $(W_l, W_{l'})$ one of these pairs. In [3] Aubert, Michel and Rouquier showed that the Howe correspondence is given by the characters below, all the sums are over $0 \leq r \leq \min(l, l')$ and $\chi \in \text{Irr}(W_r)$.

\[ \sum_r \sum_{\chi} (\text{Ind}_{W_r \times W_{l-r}}^{W_l} \chi \otimes 1) \otimes (\text{Ind}_{W_r \times W_{l'-r}}^{W_{l'}} \text{sgn} \chi \otimes 1), \]

for the pair $(U_m(q), U_{m'}(q))$ if $k$ is odd or $k = k' = 0$;

\[ \sum_r \sum_{\chi} (\text{Ind}_{W_r \times W_{l-r}}^{W_l} \chi \otimes \text{sgn}) \otimes (\text{Ind}_{W_r \times W_{l'-r}}^{W_{l'}} \chi \otimes \text{sgn}), \]

for the pair $(U_m(q), U_{m'}(q))$ otherwise. They also conjectured that in the symplectic-orthogonal case the correspondence is given by the following characters, again the sums are over $0 \leq r \leq \min(l, l')$ and $\chi \in \text{Irr}(W_r)$:

\[ \sum_r \sum_{\chi} (\text{Ind}_{W_r \times W_{l-r}}^{W_l} \chi \otimes \text{sgn}) \otimes (\text{Ind}_{W_r \times W_{l'-r}}^{W_{l'}} \chi \otimes \text{sgn}), \]

for $(\text{Sp}_{2m}(q), O_{2m}^+(q))$ if $k$ is even and $(\text{Sp}_{2m}(q), O_{2m}^-(q))$ if $k$ is odd;

\[ \sum_r \sum_{\chi} (\text{Ind}_{W_r \times W_{l-r}}^{W_l} \chi \otimes 1) \otimes (\text{Ind}_{W_r \times W_{l'-r}}^{W_{l'}} \chi \otimes \text{sgn}), \]

for $(\text{Sp}_{2m}(q), O_{2m'}^+(q))$ if $k$ is odd and $(\text{Sp}_{2m}(q), O_{2m'}^-(q))$ if $k$ is even.

For an irreducible representation $\chi$ of $W_l$ we call $\tau(\chi)$ the set of those irreducible representations $\chi'$ of $W_{l'}$ such that $\chi \otimes \chi'$ is an irreducible component of one of the above sums.

The Springer correspondence relates an irreducible representation of the Weyl group $W_l$ of $\text{Sp}_{2l}(\mathbb{F}_q)$ to a pair $(\mathcal{O}, \rho)$ consisting of a unipotent conjugacy orbit $\mathcal{O}$ in $\text{Sp}_{2l}(\mathbb{F}_q)$ and an irreducible representation $\rho$ of the group of connected components of the centralizer of any $u \in \mathcal{O}$. For symplectic-orthogonal pairs we explicit a representation in $\tau(\chi)$ which provides the smallest (resp. largest) unipotent orbit (for the closure order) via the Springer correspondence (cf. Theorems [9] and [10]). We will consider it to be the minimal (resp. maximal) representation in the symplectic-orthogonal setting. For an irreducible representation $\chi_{\xi', \eta'}$ of $W_{l'}$ this minimal (resp. maximal) representation is indexed by the bipartition $(\xi', (l-l') \cup \eta')$ (resp. $(\xi', (l-l' + \eta_1', \eta_2', \ldots, \eta_{l'}'))$) in the cases covered by (1), and by the bipartition $((l-l') \cup \xi', \eta')$ (resp. $((l-l' + \eta_1', \xi_2', \ldots, \xi_r'), (\eta_2', \ldots, \eta_{l'}'))$) in the cases covered by (2).
Call $\lambda_k$ the unique cuspidal unipotent representation of $U_{k(k+1)/2}(q)$. The representations in the Harish-Chandra series $\text{Irr}(U_m(q))_{\lambda_k}$ are the virtual characters $R^{U_m}_{\mu}$ [11] Appendix, proposition p.224] up to a sign. The partitions $\mu$ parametrizing these characters are those having $(k k-1 \cdots 1)$ as 2-core and having as 2-quotients (of parameter 1) the irreducible representations of $B_{(1/2)(m-k(k+1)/2)}$ obtained by the bijection $\text{Irr}(U_m(q))_{\lambda_k} \simeq \text{Irr}(B_{(1/2)(m-k(k+1)/2)})$ mentioned above (cf. [12]). It is natural to order the representations in $\tau(\chi)$ by the order between the corresponding partitions $\mu$ which parametrize the virtual characters $R^{U_m}_{\mu}$. We show that there is a smallest (resp. largest) representation in $\tau(\chi)$ for this order (cf. Theorems 11 and 12), this is the minimal (resp. maximal) representation in the unitary setting. For an irreducible representation $\chi$ of $W_l$ this minimal (resp. maximal) representation is indexed by the bipartition $(\{l-l\} \cup \mu, \lambda)$ (resp. $(\{l-l\} \cup \mu, \lambda)$) in the cases covered by (3), and by the bipartition $(\{l-l\} \cup \mu, \lambda)$ (resp. $(\{l-l\} \cup \mu, \lambda)$) in the cases covered by (4).

2. Dual pairs

In this section we will present the reductive dual pairs over finite fields $\mathbb{F}_q$. We will suppose the characteristic $p$ of the field to be odd. All forms are supposed to be nondegenerate.

Let $W$ be a symplectic vector space over $\mathbb{F}_q$. The group of isometries for the symplectic form over $W$ is denoted by $\text{Sp}(W)$. By choosing a suitable base we can consider the symplectic group as a group of matrices, in this situation we will also denote it by $\text{Sp}_{2n}$, where $\dim W = 2n$.

For a group $G$ and a subgroup $H$, we let

$$C_G(H) = \{x \in G \mid xz = zx \text{ for all } z \in H\}$$

**Definition 1.** A reductive dual pair $(G, G')$ in $\text{Sp}(W)$ is a pair of reductive subgroups $G$ and $G'$ of $\text{Sp}(W)$ such that

$$C_{\text{Sp}(W)}(G) = G' \quad \text{and} \quad C_{\text{Sp}(W)}(G') = G$$

We will usually omit the word reductive and call $(G, G')$ a dual pair.

If $W = W_1 + W_2$ is an orthogonal decomposition, and if $(G_1, G'_1)$ and $(G_2, G'_2)$ are dual pairs in $\text{Sp}(W_1)$ and $\text{Sp}(W_2)$ respectively, then $(G, G') = (G_1 \times G_2, G'_1 \times G'_2)$ is a dual pair in $\text{Sp}(W)$. Such a pair is said to be reducible. A dual pair $(G, G')$ which does not arise in this way is said to be irreducible. For example, if $W$ is irreducible for the action of $G \times G'$, then the dual pair $(G, G')$ is irreducible. Every dual
pair can be written as a product of irreducible dual pairs. We present
now the classification of irreducible dual pairs over finite fields. All the
inner forms are supposed to be non degenerate.

(1) Let $V_1$ and $V_2$ be vector spaces over $\mathbb{F}_q$. Suppose $V_1$ has a sym-
plectic form $\langle \cdot, \cdot \rangle_1$ and $V_2$ has a quadratic form $\langle \cdot, \cdot \rangle_2$. The group of
isometries of the latter is called \textit{orthogonal} and denoted by $O(V_2)$.
The $\mathbb{F}_q$ vector space $W = V_1 \otimes_{\mathbb{F}_q} V_2$ has a symplectic form defined by
\[ \langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle = \langle u_1, v_1 \rangle_1 \langle u_2, v_2 \rangle_2. \]
We can see $Sp(V_1)$ and $O(V_2)$ as subgroups of $Sp(W)$ via the natural
map $Sp(V_1) \times O(V_2) \rightarrow Sp(W)$. The pair $(Sp(V_1), O(V_2))$ so obtained
is an irreducible dual pair, it is called \textit{symplectic-orthogonal}.

(2) Consider the quadratic extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$ and let $F$ denote
its Frobenius morphism. Let $V_1$ be a vector space over $\mathbb{F}_{q^2}$ with a
nondegenerate skew Hermitian form $\langle \cdot, \cdot \rangle_1$, so
$\langle \alpha u, \beta v \rangle_1 = \alpha \langle u, v \rangle_1 \beta^*$ and $\langle u, v \rangle_1^t = -\langle v, u \rangle_1$,
and let $U(V_1)$ be the isometry group of this form. Similarly, let $V_2$ be
a $\mathbb{F}_{q^2}$-vector space with a Hermitian form $\langle \cdot, \cdot \rangle_2$ so
$\langle \alpha u, \beta v \rangle_2 = \alpha \langle u, v \rangle_2 \beta^*$ and $\langle u, v \rangle_2^t = \langle v, u \rangle_2$,
and let $U(V_2)$ be its isometry group of this form.
The $\mathbb{F}_q$ vector space $W = V_1 \otimes_{\mathbb{F}_{q^2}} V_2$ has a symplectic form defined by
\[ \langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle = tr_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\langle u_1, v_1 \rangle_1 \langle u_2, v_2 \rangle_2^F). \]
Again, via the natural map $U(V_1) \times U(V_2) \rightarrow Sp(W)$, we can see
$U(V_1)$ and $U(V_2)$ as subgroups of $Sp(W)$. The irreducible dual pair
$(U(V_1), U(V_2))$ obtained this way is called \textit{unitary}.

Unitary and symplectic-orthogonal dual pairs are said to be of \textit{type I}.

(3) Let $V_1$ and $V_2$ be vector spaces over $\mathbb{F}_q$. As for type I dual pairs
we have a natural action of $GL(V_1) \times GL(V_2)$ on $V = V_1 \otimes V_2$ and an
induced action on its dual $V^*$. By considering the diagonal action we
get a map
\[ GL(V_1) \times GL(V_2) \rightarrow GL(W) \]
where $W = V \oplus V^*$. This last vector space can be given a symplectic form
\[ \langle x + x^*, y + y^* \rangle = y^*(x) - x^*(y), \]
that makes $GL(V_1)$ and $GL(V_2)$ subgroups of $Sp(W)$. Irreducible dual
pairs $(GL(V_1), GL(V_2))$ arising this way are called \textit{linear}.

Linear dual pairs are also said to be of \textit{type II}. 

3. Howe Correspondence

The Howe correspondence relates representations of the members of a dual pair. In order to introduce it we need to study the representation theory of the Heisenberg group.

**Definition 2.** The Heisenberg group is the group with underlying set \( H = \{ (w, t) : w \in W, t \in \mathbb{F}_q \} \) and product

\[
(w, t) \cdot (w', t') = (w + w', t + t' + \frac{1}{2} \langle w, w' \rangle)
\]

The representation theory of the Heisenberg group is simple. Let us take an irreducible representation \( \rho \) of \( H \), Schur’s lemma implies that its restriction to the center \( Z = \mathbb{F}_q \) of \( H \) has form \( \chi_\rho \cdot 1 \) where \( \chi_\rho \) is a character or \( \mathbb{F}_q \).

If \( \chi_\rho = 1 \) then \( \rho \) factors to \( H/Z \cong W \) which is abelian, so \( \rho \) is itself a character (has dimension one). The case \( \chi_\rho \) non trivial is given by the following theorem.

**Theorem 1** (Stone-von-Neumann). For any non trivial character \( \chi \) of \( Z \) there exists (up to equivalence) a unique irreducible representation \( \rho \) of \( H \) such that \( \chi_\rho = \chi \).

The action of \( \text{Sp}(W) \) on \( W \) lifts to an action on \( H \) and hence to one on the set \( \text{Irr}(H) \) of equivalence classes of irreducible representations. The action on \( H \) fixes the element of the center so \( \rho \) and \( x \cdot \rho \) agree on \( Z \), for any \( x \in \text{Sp}(W) \) and any irreducible representation \( \rho \) of \( H \).

Fix a character \( \chi \) of the center, and let \( \rho \) be an irreducible representation corresponding of \( H \) corresponding to it by the theorem. As we saw in the previous paragraph \( \rho \) and \( x \cdot \rho \) have same restriction to the center. Hence, the unicity part implies that there is an operator \( \omega(x) \) verifying

\[
\rho(x \cdot w, t) = \omega(x)\rho(w, t)\omega(x)^{-1}.
\]

Schur’s lemma shows that \( \omega \) is a projective representation of \( \text{Sp}(W) \), that is

\[
\omega(xy) = \alpha(x, y)\omega(x)\omega(y),
\]

for a complex two cocycle \( \alpha(x, y) \). As \( H^2(\text{Sp}(W), \mathbb{C}^\times) = 0 \), this cocycle is a coboundary, so that \( \alpha(x, y) = f(x)f(y)f(xy)^{-1} \) for a complex function \( f \) on \( \text{Sp}(W) \). Scaling \( \omega \) by \( f \) gives us a true representation of \( \text{Sp}(W) \). We call \( \omega \) the Weil representation of \( \text{Sp}(W) \).

The Weil representation depends on the character \( \chi \) first defined, but this dependance is weak. In fact using Schur’s lemma we can show that if \( \chi \) and \( \chi' \) differ by a square (i.e. there is \( s \in \mathbb{F}_q \) so that \( \chi(t) = \chi'(t + s) \))
For an irreducible dual pair \((G, G')\), there is a natural map from \(G \times G'\) to \(\text{Sp}(W)\). Pulling back the Weil representation by this map we get a representation \(\omega_{G, G'}\) of \(G \times G'\). This representation decomposes as a sum:

\[
\omega_{G, G'} = \sum_{\pi, \pi'} m_{\pi, \pi'} \pi \otimes \pi',
\]

where the sum is over the set of irreducible representations \(\pi\) and \(\pi'\) of \(G\) and \(G'\) respectively. We can rearrange this sum in order to get

\[
\omega_{G, G'} = \sum_{\pi' \in \text{Irr}(G')} \Theta(\pi') \otimes \pi',
\]

where \(\Theta(\pi') = \sum_{\pi, \pi'} m_{\pi, \pi'}\pi\) is a (not necessarily irreducible) representation of \(G\). The correspondence \(\pi' \mapsto \Theta(\pi')\) is called the Howe correspondence.

4. Harish-Chandra theory

In this section \(G\) will denote a reductive algebraic group defined over \(\mathbb{F}_q\) with Frobenius morphism \(F\). We will treat both connected and non-connected groups. As the theta correspondance deals with representations of \(G^F\) for such \(G\), we are interested in deepening our understanding of \(\text{Irr}(G^F)\). Harish-Chandra theory provides a way to do so.

All the groups concerned will be rational so we’ll omit this assumption.

Parabolic and Levi subgroups for connected groups are well known. For non connected groups we have the following definition.

**Definition 3.** A parabolic subgroup \(P\) of \(G\) is defined as the normalizer in \(G\) of a parabolic subgroup \(P^o\) of the connected component \(G^o\) of \(G\). A Levi subgroup of \(P\) is the normalizer in \(G\) of the couple \(L^o \subset P^o\) where \(L^o\) is a Levi subgroup of \(P^o\).

The following lines apply to connected and non-connected groups.

**Proposition 1.** 1. Let \(Q \subset P\) be two parabolic groups. If \(M\) is a Levi subgroup of \(Q\), there exists a unique Levi subgroup \(L\) of \(P\) containing it.

2. For \(L\) and \(P\) before, the following are equivalent:
   a) \(M\) is a Levi subgroup of a parabolic subgroup of \(L\).
   b) \(M\) is a Levi subgroup of a parabolic subgroup of \(G\), and \(M \subset L\).
If \( L \) is a Levi subgroup of a parabolic subgroup of \( G \) and we do not need to specify what the parabolic is, then we will say that \( L \) is a Levi of \( G \).

Let \( L \) be a Levi subgroup and \( U \) the unipotent radical of the parabolic \( P \) of \( G \). Let \( \delta \) be a representation of the group \( L^F \) of rational points of \( L \). The canonical isomorphism \( P/U = L \) allows us to lift \( \delta \) to a character of \( P^F \), that we denote by the same letter. Setting:

\[
R^G_L(\delta) := \text{Ind}^{G^F}_{P^F} \delta.
\]

where \( \text{Ind} \) is the classical induction, we obtain a representation of \( G^F \) which is known as the parabolic induction of \( \delta \).

Let now \((\pi, V)\) be a representation of \( G^F \) and \( V(U) \) the subspace generated by \( \pi(u)v - v \), with \( u \in U^F \) and \( v \in V \). The fact that \( L^F \) normalizes \( U^F \) implies that \( V(U) \) is stable by the former. We obtain in this way a representation of \( L^F \) in \( V/V(U) \), known as the parabolic restriction of \( \pi \), and denoted by \( ^*R^G_L(\pi) \).

The relation between the two functors just defined is similar to that between classic induction and restriction. For example we have the Frobenius reciprocity:

\[
\text{Hom}_{L^F}(^*R^G_L(\pi), \delta) = \text{Hom}_{G^F}(\pi, R^G_L(\delta)).
\]

The parabolic subgroup used in the constructions above does not appear in the notation since these functors do not depend upon the choice of a parabolic containing our Levi. This is a consequence of the Mackey formula whose proof can be found in [9, Proposition 6.1].

We next talk about transitivity. This is a crucial property of parabolic induction and restriction.

**Proposition 2.** Let \( Q \subset P \) two parabolic subgroups of \( G \) and \( M \subset L \) two corresponding Levi subgroups. Then

\[
R^G_L \circ R^L_M = R^G_M
\]

Using Frobenius reciprocity we conclude that a similar result holds for parabolic restriction.

**Proof.** [9, Proposition 4.7]
Definition 4. Let $L$ be a Levi subgroup and $\delta$ an irreducible representation of $L^F$. We say that $\delta$ is cuspidal (or that the pair $(L, \delta)$ is cuspidal) if $^*R^L_M(\delta) = 0$ for all proper Levi subgroups $M$ of $L$.

Frobenius reciprocity implies that $\delta$ is cuspidal if and only if for all Levi $M \subset L$ and $\rho \in \text{Irr}(M^F)$,

$$\langle \delta, R^L_M(\rho) \rangle = 0.$$ 

It is not hard to see that if two pairs $(L, \delta)$ and $(L', \delta')$ are $G^F$-conjugate (that is, if it exist $x \in G^F$ such that $L' = L^x$ and $\delta' = \delta^x$) then $R^G_L(\delta)$ and $R^G_{L'}(\delta')$ are equivalent.

Theorem 2. Let $\chi \in \text{Irr}(G^F)$; then, up to $G^F$-conjugacy, there exists a unique cuspidal pair $(L, \delta)$ such that $\langle \chi, R^G_L(\delta) \rangle \neq 0$.

This theorem gives us a partition of $\text{Irr}(G^F)$ in series parametrized by cuspidal pairs $(L, \delta)$. The set of irreducible representation of $G^F$ appearing in $R^G_L(\delta)$ is called Harish-Chandra series of $(L, \delta)$.

Harish-Chandra series are in turn parametrized by irreducible representations of certain Hecke algebras.

For a cuspidal representation $\delta$ of $L^F$ we put

$$W_G(\delta) = \{x \in N_{G^F}(L)/L^F : \delta^x = \delta \}.$$ 

Theorem 3. There is an isomorphism

$$\text{End}_{G^F}(R^G_L(\delta)) \simeq \mathbb{C}[W_G(\delta)].$$

In particular, the set of irreducible components of $R^G_L(\delta)$ is in bijective correspondence with the irreducible representations of $W_G(\delta)$.

Proof. [12]}

In other words, the irreducible representations in the Harish-Chandra series of $(L, \delta)$ are parametrized by characters of $W_G(\delta)$.

We have seen that the definition of parabolic induction and restriction is the same for connected and non connected groups. They relate in the following way.

Lemma 1. Let $G$ be a non connected group, $G^\circ$ its identity component, a Levi $L^\circ \subset P^\circ$ of $G^\circ$ and $L \subset P$ the corresponding Levi of $G$. Then

$$^*R^G_{L^\circ} \circ \text{Res}^G_{G^\circ} = \text{Res}^L_{L^\circ} \circ ^*R^G_L$$

$$^*R^G_L \circ \text{Ind}^G_{G^\circ} = \text{Ind}^L_{L^\circ} \circ ^*R^G_{L^\circ}$$

Proof. [6] Proposition 10.10]
We end this section showing the relation between the cuspidal representations of a non connected group and those of its identity component. It’s an easy corollary of the previous Lemma.

**Proposition 3.** An irreducible representation $\psi$ of $G$ is cuspidal if and only if it is an irreducible component of $\text{Ind}^{G}_{G^0} \psi^0$ for a certain cuspidal irreducible representation $\psi^0$ of $G^0$.

5. DELIGNE-LUSZTIG THEORY

The definition of parabolic induction involves a rational Levi contained in a rational parabolic. Deligne and Lusztig extended this construction to the case where the rational Levi is not contained in any rational parabolic. This construction, when specialized to maximal tori will give us a decomposition of the category of irreducible representations of $G^F$, similar to that obtained from parabolic induction.

**Proposition 4.** Let $G$ be a connected group defined over $\mathbb{F}_q$, and let $F$ be its Frobenius morphism. The map $\mathcal{L} : x \in G \mapsto x^{-1}Fx \in G$ is surjective.

**Proof.** [18]

This map is known as the Lang map and the result actually holds if we replace the Frobenius by any surjective endomorphism of $G$ with a finite number of fixed points.

Let $L$ be a rational Levi of a parabolic $P$, and let $U$ be its unipotent radical. The group $G^F$ acts on the left on $\mathcal{L}^{-1}(U)$ while (as $L$ normalizes $U$) $L^F$ acts on the right. This induces, for all integers $i$, a $G^F$-module-$L^F$ structure on the vector spaces $H^i_c(\mathcal{L}^{-1}(U))$ of $l$-adic cohomology with compact support. We can in this way see the virtual vector space $H^*_c(\mathcal{L}^{-1}(U)) = \sum_i (-1)^i H^i_c(\mathcal{L}^{-1}(U))$ as a $G^F$-module-$L^F$.

**Definition 5.** The functor $R^G_L : E \rightarrow H^*_c(\mathcal{L}^{-1}(U)) \otimes_{\mathbb{C}[L]} E$ going from the category of $L^F$-modules to that of $G^F$-modules is known as the Lusztig induction.

It is important to stress that, for a representation $\rho$ of $L^F$, the Lusztig induction $R^G_L(\rho)$ does not provide necessarily a representation of $G^F$. In fact, its associated character decomposes as sum of irreducible characters with coefficients in $\mathbb{Z}$, not necessarily positive. They are called virtual representations.

If the group $L$ is contained in a rational parabolic $P$ then Deligne-Lusztig induction becomes parabolic induction, that is why we use the same notation for both inductions. Indeed, in this case $U$ is also rational and the fact that $x$ and $Fx$ have same class modulo $U$ for
any $x \in \mathcal{L}^{-1}(U)$ implies that $\mathcal{L}^{-1}(U) \to G/U$ has image $(G/U)^F \simeq G^F/U^F$. This morphism has fibers isomorphic to $U$. As $G^F/U^F$ is finite its cohomology groups are trivial except in degree zero where $H^*(G^F/U^F) \simeq \mathbb{Q}_l[G^F/U^F]$. This implies [Digne-Michel, pag. 81] that $\mathbb{Q}_l[G^F/U^F]$.

In order to get a partition of the set of irreducible representations of $G^F$ we need the Mackey formula to hold. It does if we restrict ourselves to maximal rational tori [9, Theorem 11.13]. It follows from this that, for $\theta \in \hat{T}^F$ and $\theta' \in \hat{T}'^F$:

$$\langle R_{T^F}^G(\theta), R_{T'^F}^G(\theta') \rangle = 0$$

whenever $(T, \theta)$ and $(T', \theta')$ are not $G^F$-conjugate. So $R_{T^F}^G(\theta)$ and $R_{T'^F}^G(\theta')$ are orthogonal to each other, but they may have a common constituent as they are virtual characters.

This tells us that the partition of Irr($G^F$) we want to obtain cannot be indexed by the set of $G^F$-conjugacy classes of pairs $(T, \theta)$. In order to get a disjoint union we need the weaker notion of geometric conjugacy classes, it basically tells us that two pairs are conjugate up to scalar extension.

**Definition 6.** Let $T$ and $T'$ be two rational maximal tori, and let $\theta$ and $\theta'$ be characters respectively of $T^F$ and $T'^F$. We say that the pairs $(T, \theta)$ and $(T', \theta')$ are geometrically conjugate if there exists a positive integer $n$ and $g \in G^{F^n}$ such that $T' = gT$ and $\theta' \circ N_{F^n/F} = \theta \circ N_{F^n/F} \circ \text{ad}(g)$

In the previous definition $\text{ad}(g) : T^{F^n} \to T^{F^n}$ is the conjugation by $g$ and $N_{F^n/F} : T^{F^n} \to T^F$ is the map given by $t \mapsto t^F t \cdots F^{n-1} t$. This last generalizes the Frobenius morphism.

**Proposition 5.** Let $T$ and $T'$ be two rational maximal tori, and let $\theta$ and $\theta'$ be characters respectively of $T^F$ and $T'^F$. If $R_{T^F}^G(\theta)$ and $R_{T'^F}^G(\theta')$ share an irreducible constituent then the pairs $(T, \theta)$ and $(T', \theta')$ are geometrically conjugate.

**Proof.** [9, Proposition 13.3]}

We still need to show that any irreducible character of $G^F$ appears in the induced $R_{T^F}^G(\theta)$ of some pair $(T, \theta)$. We first give a definition.

**Definition 7.** We call uniform functions the class functions of $G^F$ that are linear combinations of Deligne-Lusztig characters.

The character $reg_G$ of the regular representation of $G^F$ is a uniform function [9, Corollary 12.14]. As every irreducible representation $\chi$ appears on the regular representation, there exists a pair $(T, \theta)$ such that $\chi$ is an irreducible component of $R_{T^F}^G(\theta)$. 


Let \([T, \theta]\) be the geometric conjugacy class of \((T, \theta)\). Consider the set of irreducible representations of \(G^F\) appearing in \(R_{T'}(\theta')\) for a certain \((T', \theta') \in [T, \theta]\). The previous observations and Proposition 5 tell us that these sets partition the category of irreducible representations of \(G^F\).

The following proposition gives a parametrization of the set of geometric conjugacy classes.

**Proposition 6.** Let \((G, F)\) and \((G^*, F^*)\) be dual to each other. Geometric conjugacy classes of pairs \((T, \theta)\) in \(G\) are in bijection with \(F^*\)-stable conjugacy classes of semi-simple elements of \(G^*\).

**Proof.** [9, Proposition 13.12] □

**Definition 8.** A Lusztig series \(\mathcal{E}(G^F, (s))\) corresponding to the geometric conjugacy class \((s)\) of semi-simple \(s \in G^{sF^*}\) is the set of irreducible representations of \(G^F\) appearing in \(R_T(\theta)\) for \((T, \theta)\) belonging to the geometric conjugacy class associated to \((s)\) by Proposition 6.

The paragraph preceding proposition 6 can be rewritten in terms of the last definition.

**Proposition 7.** Lusztig series associated to different geometric conjugacy classes of rational semi-simple \((s) \in G^{sF^*}\) form a partition of the set of irreducible representations of \(G^F\).

The representations corresponding to the series \(\mathcal{E}(G^F, 1)\) of the trivial element in \(G^{sF^*}\) are called unipotent representations.

### 6. A correspondence between Weyl groups

In this section we will replace the Weil representation \(\omega\) by a representation \(\omega^b\) introduced by Gérardin (cf. [14]). The correspondence obtained from the latter behaves well with respect to both Harish-Chandra and Lusztig decompositions. Moreover, it preserves cuspidal and unipotent representations. For symplectic orthogonal pairs \(\omega\) and \(\omega^b\) have the same restriction to \(G_m, G'_{m'}\), for unitary or linear pairs these restrictions differ by multiplication by a character \(\sigma \otimes \sigma'\) of \(G_m : G'_{m'}\) with values in \(\{1, -1\}\) (cf. [5, Section 1C]). We can therefore obtain the Howe correspondence from the correspondence induced by \(\omega^b\).

Let \(T\) and \(T'\) be two Witt towers such that pairs \((G_m, G'_{m'})\) with \(G_m \in T\) and \(G'_{m'} \in T'\) form a dual pair of type I.

- For unitary groups there are two Witt towers, one whose groups are \(U_{2n}(q)\) for \(n \in \mathbb{N}\) and the other for groups \(U_{2n+1}(q)\) for \(n \in \mathbb{N}\). The first one will be denoted by \(U^+\) and the second one by \(U^-\).
• In the symplectic case there is only one Witt tower, formed by groups $\text{Sp}_{2n}(q)$ for $n \in \mathbb{N}$. It will be denoted by $\text{Sp}$.
• Even orthogonal groups provide two Witt towers whose groups are respectively $\text{O}^+_{2n}(q)$, $\text{O}^-_{2n}(q)$, for positive integers $n$. These will be denoted by $\text{O}^+$ and $\text{O}^-$ respectively.

**Theorem 4.** Suppose $\pi'$ is a cuspidal representation of $G'_{m'}$ and

1. $\Theta_{m,m'}(\pi') = \pi$ in the correspondence for $(G_m, G'_{m'})$.
2. $\pi'$ does not occur in the correspondence for $(G_k, G'_{m'})$ for $k < m$, that is $\Theta_{k,m'}(\pi) = 0$.

Then $\pi$ is a cuspidal representation of $G_m$.

**Proof.** [2, Theorem 2.2]

The occurrence of a representation $\pi'$ of $G'_{m'}$ in the Howe correspondence for $(G_m, G'_{m'})$ with $m$ minimal (that is $\Theta_{k,m'}(\pi) = 0$ for $k < m$) is referred to as the *first occurrence*. The integer $m$ will be called the *first occurrence index* for $\pi'$.

**Theorem 5.** Under the notation of theorem 4, let $\pi$ and $\pi'$ be irreducible representations of $G_m$ and $G'_{m'}$ respectively. Suppose the representation $\pi \otimes \pi'$ appears in the Weil representation $\omega_{m,m'}$ of the pair $(G_m, G'_{m'})$. Then $\pi$ is unipotent if and only if $\pi'$ is unipotent.

**Proof.** [5, Proposition 2.3]

Theorems 4 and 5 together imply that, in the case of first occurrence, for dual pairs of type I, the Howe correspondence takes cuspidal unipotent representations to cuspidal unipotent representations.

The only type II group having a cuspidal unipotent representation is $\text{GL}_1(q)$, moreover this representation is the trivial one. Between type I groups there are not many having cuspidal unipotent representations either.

**Theorem 6.** The following groups:

1. $\text{Sp}_{2n}(q)$, $n = k(k + 1)$,
2. $\text{SO}^+_{2n}(q)$, $n = k^2$, $\epsilon = \text{sgn}(-1)^k$,
3. $\text{U}_n(q)$, $n = (k^2 + k)/2$

are the only groups in their respective Witt towers having a cuspidal unipotent representation. Moreover, in each case the group possesses a unique cuspidal unipotent representation denoted by $\lambda_k$.

**Proof.** [2, Theorem 5.1]
Irreducible cuspidal (resp. unipotent) representations of \(O_n^\pm(q)\) appear as constituents of \(\text{Ind}_{O_n^\pm(q)}^{SO_n^\pm(q)}(\pi)\) where \(\pi\) is a cuspidal (resp. unipotent) representation of \(SO_n^\pm(q)\).

Using Frobenius reciprocity and Clifford theory we can show that the representation \(\text{Ind}_{SO_{2k^2}(q)}^{O_{2k^2}(q)}(\lambda_k)\) is not irreducible, so it decomposes as
\[
\text{Ind}_{SO_{2k^2}(q)}^{O_{2k^2}(q)}(\lambda_k) = \lambda_k^+ \oplus \lambda_k^-
\]
where \(\lambda_k^+\) and \(\lambda_k^-\) are the irreducible cuspidal unipotent representations of \(O_{2k^2}(q)\). Furthermore, Frobenius reciprocity implies that these representations differ by tensoring with the sgn character of \(O_{2k^2}(q)\).

Theorem 7 gives us restrictions on the dimensions of type I groups in order to have cuspidal unipotent representations. The following theorem gives us first occurrence indices for these representations.

**Theorem 7.** The Howe correspondence for dual pairs \((U_n(q), U_m(q))\) and \((Sp_{2m}(q), O_{2n}^s(q))\) takes cuspidal unipotent representations to cuspidal unipotent representations as follows:

- For towers \((Sp, O')\), \(\lambda_k\) correspond to \(\lambda_k^+\) if \(\epsilon\) is the sign of \((-1)^k\) and to \(\lambda_{k+1}^-\) otherwise.
- For towers \((U', U')\), \(\lambda_k\) correspond to \(\lambda_{k'}\) where \(k' = k + 1\) or \(k' = k - 1\). We take \(k\) so that \(\epsilon\) is the sign of \((-1)^{(k+1)/2}\) and we choose \(k'\) such that \(\epsilon' = (-1)^{(k'+1)/2}\).

Moreover these cases give the first occurrence of \(\lambda_k\).

**Proof.** [2, Theorems 4.1 and 5.2] \(\square\)

This theorem allows us to write the Howe correspondence between cuspidal unipotent representations as a function \(\theta\) on integers. For instance, for towers \((Sp, O^+)\) we get \(\theta(k) = k = \theta(k-1)\) for \(k\) even. For unitary towers this function depends on the modulo 4 class of \(k\).

Given two groups \(G_l\) and \(G_m\) of the same Witt tower such that \(l < m\), we can include \(G_l\) inside the Levi subgroup \(G_l \times T_{m-l}\) of \(G_m\). Let \(\lambda\) be a cuspidal representation of \(G_l\), we will denote by \(R(G_m)_\lambda\) the subset of \(R(G_m)\) whose elements are spanned by
\[
\text{Irr}(G_m)_\lambda = \{\gamma \in \text{Irr}(G_m) : \gamma \text{ is a constituent of } R_{G_l \times T_{m-l}}^{G_m}(\lambda \otimes 1)\}.
\]

**Theorem 8.** Let \(T\) and \(T'\) be two Witt towers such that pairs \((G_m, G'_{m'})\) with \(G_m \in T\) and \(G'_{m'} \in T'\) form a dual pairs of type I. Let \(\lambda'\) be a cuspidal representation of \(G'_{m'}\), let \(l\) be its first occurrence index and \(\lambda = \Theta_{l', l}(\lambda')\) the corresponding cuspidal representation of \(G_l\). For \(\gamma \in \text{Irr}(G'_{m'})_\lambda, \Theta_{m, m'}(\gamma) = 0\) whenever \(m < l\) and \(\Theta_{m, m'}(\gamma) \in R(G_m)_\lambda\).
otherwise. Moreover, the character $\lambda'$ of $G'_1$, is unipotent if and only if $\lambda$ is a unipotent character of $G_1$.

Proof. [5, Théorème 3.7]

The last theorem tells us that the Howe correspondence is compatible with the Harish-Chandra decomposition and that it preserves cuspidal unipotent representations.

The standard Levi subgroups $L$ of $\text{Sp}_{2n}(q)$ are $L = \text{GL}_{n_1}(q) \times \cdots \times \text{GL}_{n_l}(q) \times \text{Sp}_{2l}(q)$ such that $m = n_1 + \cdots + n_r + l$. A unipotent (resp. cuspidal) representation $\rho$ of $L$ is then given by $\rho = \rho_1 \otimes \cdots \otimes \rho_r \otimes \sigma$ where $\sigma$ and $\rho_i$ are unipotent (resp. cuspidal) representations of $\text{Sp}_{2l}(q)$ and $\text{GL}_{n_i}(q)$ respectively. Therefore, theorem 6 and the remark preceding it, imply that the only Levi having a cuspidal unipotent representation are $L_k = T_r \times \text{Sp}_{2k(k+1)}(q)$, this representation is unique and given by $1 \otimes \lambda_k$ where $\lambda_k$ is the only cuspidal unipotent representation of $\text{Sp}_{2k(k+1)}(q)$. The Harish-Chandra series corresponding to the pair $(L_k, 1 \otimes \lambda_k)$ will be denoted by $\text{Irr}(\text{Sp}_{2n}(q))_k$ and the set $\mathcal{R}(\text{Sp}_{2n}(q))_{\lambda_k}$ by $\mathcal{R}(\text{Sp}_{2n}(q))_k$.

Similar reasoning apply to orthogonal groups $\text{O}_{2n}^+(q)$, in this case, for $k$ verifying $\epsilon = (-1)^k$, we’ll have two unipotent cuspidal representations $\lambda_k^+$ and $\lambda_k^-$ giving place to two Harish-Chandra series $\text{Irr}(\text{O}_{2n}^+(q))_{k+}$, $\text{Irr}(\text{O}_{2n}^+(q))_{k-}$ and two sets $\mathcal{R}(\text{O}_{2n}^+(q))_{k+}$, $\mathcal{R}(\text{O}_{2n}^+(q))_{k-}$ respectively.

We know that representations in the Harish-Chandra series corresponding to a cuspidal unipotent representation, are parametrized by irreducible representations of a Weyl group. For the series $\text{Irr}(\text{Sp}_{2n}(q))_k$ the corresponding group is

$$W_{\text{Sp}_{2n}(q)}(\lambda_k) = \{x \in N_{\text{Sp}_{2n}(q)}(L_k)/L_k : \lambda_k^x = \lambda_k\}.$$  

The unicity of $\lambda_k$ implies that the condition on the elements of the previous group is trivial so $W_{\text{Sp}_{2n}(q)}(\lambda_k)$ reduces to $N_{\text{Sp}_{2n}(q)}(L_k)/L_k$ which is a Weyl group of type $B_{m-k(k+1)}$. The same reasoning allow us to state that the series $\text{Irr}(\text{O}_{2n}^+(q))_{k\pm}$ are in bijection with the irreducible representations of a Weyl group of type $B_{n-k^2}$.

This together with Theorem 6 imply that for type I dual pairs $(\text{Sp}_{2m}(q), \text{O}_{2n}^\pm(q))$ and $(\text{U}_m(q), \text{U}_n(q))$ the Howe correspondence between Harish-Chandra series of cuspidal unipotent representations is given by a correspondence between pairs $(B_{m-k(k+1)}, B_{n-\theta(k^2)})$ for symplectic-orthogonal pairs, and $(B_{m-k(k+1)/2}, B_{n-\theta(k)(\theta(k)+1)/2})$ for unitary pairs.

Let $(W_l, W_r')$ be such a pair. For symplectic-orthogonal dual pairs it is conjectured [5, Conjecture 3.11] that the Howe correspondence is given by the characters below, all the sums are over $0 \leq r \leq \min(l, l')$ and $\chi \in \text{Irr}(W_r')$:
\[ I_1 = \sum_r \sum_\chi (\text{Ind}_{W_r \times W_{l-r}} W_r \chi \otimes \text{sgn}) \otimes (\text{Ind}_{W_r \times W_{l'-r}} W_r' \chi \otimes \text{sgn}), \]

for \((\text{Sp}_{2m}(q), O^+_{2n}(q))\) if \(k\) is even and \((\text{Sp}_{2m}(q), O^-_{2n}(q))\) if \(k\) is odd;

\[ I_2 = \sum_r \sum_\chi (\text{Ind}_{W_r \times W_{l-r}} W_r \chi \otimes 1) \otimes (\text{Ind}_{W_r \times W_{l'-r}} W_r' \chi \otimes \text{sgn}), \]

for \((\text{Sp}_{2m}(q), O^+_{2n}(q))\) if \(k\) is odd and \((\text{Sp}_{2m}(q), O^-_{2n}(q))\) if \(k\) is even.

For unitary dual pair it is proven \([5, \text{Theorem 3.10}]\) that the Howe correspondence is given by the characters below, again the sums are over \(0 \leq r \leq \min(l, l')\) and \(\chi \in \text{Irr}(W_r)\) :

\[ J_1 = \sum_r \sum_\chi (\text{Ind}_{W_r \times W_{l-r}} W_r \chi \otimes 1) \otimes (\text{Ind}_{W_r \times W_{l'-r}} W_r' \text{sgn} \chi \otimes 1), \]

for the pair \((U_m(q), U_n(q))\) if \(k\) is odd or \(k = \theta(k) = 0;\)

\[ J_2 = \sum_r \sum_\chi (\text{Ind}_{W_r \times W_{l-r}} W_r \chi \otimes \text{sgn}) \otimes (\text{Ind}_{W_r \times W_{l'-r}} W_r' \text{sgn} \chi \otimes 1), \]

for the pair \((U_m(q), U_n(q))\) otherwise.

7. Extremal representations

We will deal with the symplectic-orthogonal and unitary pairs separately because the definition of "minimal" and "maximal" representation changes from one to the other.

Following \([5]\) we introduce the following order between partitions.

**Definition 9.** Let \(\mu\) and \(\nu\) be two partitions (of possibly different integers). We say that \(\nu\) is contained in \(\mu\) if its Young diagram is contained in that of \(\mu\). They are said to be close if \(|\nu_i - \mu_i| \leq 1\) for all \(i\). Finally we say that \(\nu\) precedes \(\mu\) and we denote it by \(\nu \preceq \mu\) if \(\nu\) is contained in \(\mu\) and they are close, this defines an order relation.

It is important to stress that the order just introduced is stronger the classical order between partitions, i.e. \(\nu \preceq \mu\) implies \(\nu \leq \mu\).

Irreducible characters of a Weyl group \(W_l\) of type \(B\) or \(C\) are known to be parametrised by bipartitions of \(l\) \([13, \text{Theorem 5.5.6}]\). We will denote by \(\mathcal{P}_2(l)\) the set of bipartitions \((\lambda, \mu)\) of \(l\) and by \(\chi_{\lambda,\mu}\) the irreducible representation of \(W_l\) corresponding to this bipartition.

**Proposition 8.** Let \((\lambda, \mu)\) be a bipartition of the integer \(r\), then
1. \( \text{Ind}^{W_i}_{W_{i-1}} \chi_{\lambda, \mu} \otimes \text{sgn} = \sum_{\mu \leq \mu'} \chi_{\lambda, \mu} \).
2. \( \text{Ind}^{W_i}_{W_{i-1}} \chi_{\lambda, \mu} \otimes 1 = \sum_{\lambda \leq \lambda'} \chi_{\lambda', \mu} \).
3. \( \text{sgn} \cdot \chi_{\lambda, \mu} = \chi_{\mu, \lambda} \).

Proof. \cite{13} Chapter 5

Achar and Henderson \cite{1} introduced the following order between bipartitions

**Definition 10.** For \((\rho, \sigma), (\mu, \nu) \in \mathcal{P}_2(n)\) we say that \((\rho, \sigma) \leq (\mu, \nu)\) if and only if the following inequalities hold for all \(k \geq 0\):

\[
\rho_1 + \sigma_1 + \cdots + \rho_k + \sigma_k \leq \mu_1 + \nu_1 + \cdots + \mu_k + \nu_k, \quad \text{and}
\rho_1 + \sigma_1 + \cdots + \rho_k + \rho_{k+1} + \sigma_k \leq \mu_1 + \nu_1 + \cdots + \mu_k + \nu_k + \mu_{k+1}.
\]

We will refer to this as the Achar-Henderson order.

Let \(V\) be a vector space of dimension \(n\) over an algebraically closed field. The \(\text{GL}(V)\)-orbits on the enhanced nilpotent cone \(V \times \mathcal{N}\) (where \(\mathcal{N}\) is the variety of nilpotent endomorphisms of \(V\)) are parametrized by bipartitions of \(n\) (cf. \cite{1} Proposition 2.3). If we denote by \(O_{\mu, \nu}\) the orbit corresponding to the bipartition \((\mu, \nu)\) of \(n\), then \(O_{\rho, \sigma} \subset O_{\mu, \nu}\) if and only if \((\rho, \sigma) \leq (\mu, \nu)\) (cf. \cite{1} Theorem 3.9), the former defines an order called the closure order. The Achar-Henderson order on bipartitions is then compatible with this closure order.

### 7.1. Symplectic-orthogonal pairs

Let \(W_n = \text{W}(C_n)\) be the Weyl group of \(\text{Sp}_{2n}(\mathbb{F}_q)\). In \cite{16} Lusztig generalized the Springer correspondence introduced by the Springer in \cite{21} for finite fields of large characteristic. This correspondence is an injective map from the set of irreducible representations of \(W_n\) into the set of pairs \((O, \psi)\) where \(O\) is a unipotent conjugacy class of \(\text{Sp}_{2n}(\mathbb{F}_q)\) and \(\psi\) is an irreducible character of the group \(A(u)\) of connected components of the centraliser \(C(u)\) of any \(u \in O\).

Recall that a partition is called symplectic if each odd part appears with even multiplicity. There is a bijection between symplectic partitions of \(2n\) and unipotent conjugacy classes of \(\text{Sp}_{2n}(\mathbb{F}_q)\). We denote by \(O_\lambda\) the unipotent orbit associated to the symplectic partition \(\lambda\).

Consider a symplectic partition of \(2n\) by adding a zero if necessary we can suppose \(\lambda\) has an even number \(2k\) of parts. We now define \(\lambda^* = \lambda_{2k-j+1} + j - 1\) for \(j = 1, \ldots, 2k\). We divide \(\lambda^*\) into its odd and even parts. Is has the same number of odd parts as even parts. Let the odd parts be

\[
2\xi_1^* + 1 < 2\xi_2^* + 1 < \ldots < 2\xi_k^* + 1
\]
and the even parts be
\[ 2\eta_1^* < 2\eta_2^* < \ldots < 2\eta_k^*. \]

Then we have
\[ \xi_1^* < \xi_2^* < \ldots < \xi_k^* \]
\[ \eta_1^* < \eta_2^* < \ldots < \eta_k^*. \]

Next we define \( \xi_i = \xi_{k-i+1}^* - (k - i) \) and \( \eta_i = \eta_{k-i+1}^* - (k - i) \) for each \( i \). We obtain in this way a bipartition \((\xi, \eta) \in \mathcal{P}_2(n)\), the injective map \( \lambda \to (\xi, \eta) \) so obtained is closely related to the Springer correspondence.

Given a bipartition \((\xi, \eta)\) of \( n \), we ensure that \( \xi \) has one more part than \( \eta \) by adding zeroes to \( \xi \) if necessary, call \( k \) the number of parts of \( \eta \). We associate to \((\xi, \eta)\) the following \( u \)-symbol
\[
\begin{pmatrix}
\xi_{k+1} & \xi_k + 2 & \cdots & \xi_1 + 2k \\
\eta_k + 1 & \cdots & \eta_1 + 2k - 1
\end{pmatrix}
\]

The bipartition \((\xi, \eta)\) is in the image of the above map if and only if its associated \( u \)-symbol is distinguished, that is

\[ \xi_{k+1} \leq \eta_k + 1 \leq \xi_k + 2 \leq \eta_{k-1} + 3 \leq \cdots \]

In this situation the Springer map sends the representation \( \chi_{\xi, \eta} \) of \( W_n \) to the pair \((O_\lambda, 1)\) where \( \lambda \) is the symplectic partition with \( \lambda \to (\xi, \eta) \) and 1 is the trivial representation of \( A(u) \).

The set of all \( u \)-symbols which contain the same entries with the same multiplicities as a given \( u \)-symbol is called the similarity class of the latter. Each similarity class contains exactly one distinguished \( u \)-symbol.

Suppose the bipartition \((\xi, \eta)\) is not in the image of the above map. If we call \((\xi', \eta')\) the distinguished \( u \)-symbol similar to \((\xi, \eta)\) and we let \( \lambda \to (\xi', \eta') \), then the Springer correspondence maps \( \chi_{\xi, \eta} \) into the pair \((O_\lambda, \psi)\) for some character \( \psi \) of \( A(u) \).

The closure order between unipotent conjugacy classes is defined by \( O \leq O' \) if and only if \( O \subset \overline{O'} \). We saw above that unipotent classes are in indexed by symplectic partitions. This bijection is so that the closure order between unipotent classes corresponds to the classical order between the corresponding symplectic partitions.

Recall that for dual pairs \((\text{Sp}_{2m}(q), O_{2n}^\pm(q))\) the Howe correspondence between Harish-Chandra series of cuspidal unipotent representations is given by a correspondence between the pair \((W_{m-k(k+1)}, W_{n-\theta(k)^2})\) of type B Weyl groups. Let \( l = m - k(k+1), l' = n - \theta(k)^2 \) and \((\xi', \eta')\) be a bipartition of \( l' \).
Remark 1. The first step in getting the extremal representations is to describe the set \( \tau(\chi_{\xi',\eta'}) \) of all the representations \( \chi_{\xi,\eta} \) of \( \text{Irr}(W_l) \) such that the outer tensor product \( \chi_{\xi,\eta} \boxtimes \chi_{\xi',\eta'} \) appears in \( I_1 \) or \( I_2 \). We will denote this set by \( \tau(\xi',\eta') \) for short. We must then find the representation in this set which gives the smallest symplectic partition for the classical order (or equivalently the smallest unipotent or bit of the group \( \text{Sp}_{2l}(\mathbb{F}_q) \)) via the Springer correspondence, this will be our minimal representation, the one providing the largest symplectic partition is our maximal representation.

We start by considering the character \( I_1 \). We suppose from now on that \( l > 2l' \).

7.1.1. First character. Proposition 8 allows us to index the second sum in \( I_1 \) by bipartitions \( (\xi,\zeta) \in \mathcal{P}_2(r) \), the representation then becomes.

\[
I_1 = \sum_{0 \leq r \leq \min(l,l')} \sum_{(\xi,\zeta) \in \mathcal{P}_2(r)} \sum_{\eta,\eta'} \chi_{\xi,\eta} \boxtimes \chi_{\xi',\eta'},
\]

where the third sum is over partitions \( \eta \) and \( \eta' \) of \( l - |\xi| \) and \( l' - |\xi| \) such that \({}^t\zeta \preceq {}^t\eta \) and \({}^t\zeta \preceq {}^t\eta' \).

Lemma 2. Let \( (\xi',\eta') \) be a bipartition of \( l' \) and \( \tau(\xi',\eta') \) be the set of representations \( \chi_{\xi,\eta} \) of \( \text{Irr}(W_l) \) such that \( \chi_{\xi,\eta} \boxtimes \chi_{\xi',\eta'} \) appears in \( I_1 \).

1. The bipartition \( (\xi,\eta) \) belongs to \( \tau(\xi',\eta') \) if and only if \({}^t\eta \) and \({}^t\eta' \) have a common predecessor for the \( \preceq \) order, and \( \xi = \xi' \).
2. The smallest element of \( \tau(\xi',\eta') \) for the Achar-Henderson corresponds to \( (\xi',(l-l') \cup \eta') \).
3. The largest element of \( \tau(\xi',\eta') \) for the Achar-Henderson corresponds to \( (\xi',(l-l' + \eta_1 + \eta_2,\eta_3,\ldots,\eta_r)) \).

Proof. Item 1 is an easy consequence of equality (5).

The representations belonging to \( \tau(\xi',\eta') \) correspond to bipartitions have \( \xi' \) as first component, so we just need to prove that the smallest partition having a common predecessor with \( \eta' \) is \( (l-l') \cup \eta' \).

So let \( \eta = (\eta_1,\ldots,\eta_r) \) and \( \eta' = (\eta'_1,\ldots,\eta'_{r+1}) \) have a common predecessor \( \zeta = (\zeta_1,\ldots,\zeta_r) \) for the \( \preceq \) order (we can suppose that \( l(\zeta) = l(\eta') = l(\eta) - 1 \) by adding zeros), this implies

\[
0 \leq \zeta_r \leq \eta'_r \leq \cdots \leq \eta'_2 \leq \zeta_1 \leq \eta'_1
\]

\[
0 \leq \eta_{r+1} \leq \zeta_r \leq \eta_r \leq \cdots \leq \eta_2 \leq \zeta_1 \leq \eta_1,
\]

which in turn imply

\[
\eta'_k \geq \eta_{k+1} \text{ for } k = 1,\ldots,r.
\]
As \((\xi, \eta)\) and \((\xi', \eta')\) are bipartitions of \(l\) and \(l'\) respectively, \(|\eta| - |\eta'| = l - l'\), i.e.

\[
\sum_{i=1}^{r+1} \eta_i = l - l' + \sum_{i=1}^r \eta'_i.
\]

(7)

This equality and the inequalities in (6) provide

\[
\sum_{i=1}^{k+1} \eta_i \geq l - l' + \sum_{i=1}^k \eta'_i,
\]

for \(k = 0, \ldots, r\), i.e. \(\eta' \geq (l - l') \cup \eta\). This proves item 2.

The proof of item 3 is analogous to that of the previous. Indeed, we have

\[
\eta_k \geq \eta'_{k+1} \text{ for } k = 1, \ldots, r - 1.
\]

These together with equality (7) imply

\[
\sum_{i=1}^{k} \eta_i \leq l - l' + \eta'_1 + \sum_{i=1}^{k} \eta'_i,
\]

for \(k = 1, \ldots, r + 1\), where we set \(\eta'_i = 0\) for \(i > r\). This implies the assertion. \(\Box\)

**Theorem 9.** Let \((\xi', \zeta) \leq (\xi', \zeta')\) belong to \(\tau(\xi', \eta')\) and \(\lambda, \lambda'\) denote the symplectic partitions related to them by the Springer correspondence, then \(\lambda \leq \lambda'\). In particular the minimal and maximal representations (cf. Remark 4) in \(\tau(\xi', \eta')\) correspond to the bipartitions \((\xi', (l - l') \cup \eta')\) and \((\xi', (l - l' + \eta_1 + \eta_2 + \eta_3, \ldots, \eta'_r))\) respectively.

**Proof.** By adding zeros we can suppose that the \(\zeta\) and \(\zeta'\) have same number \(k\) of parts and that \(\xi'\) has one more part than both. Let

\[
\left(\xi_{k+1}' \xi_k' + 2 \cdots \xi_1' + 2k \right) \quad \text{and} \quad \left(\xi_{k+1}' \xi_k' + 2 \cdots \xi_1' + 2k \right)
\]

be the \(u\)-symbols corresponding to \((\xi', \zeta)\) and \((\xi', \zeta')\) respectively. Let

\[
\begin{pmatrix}
\gamma_{2k+1} & \gamma_{2k-1} & \cdots & \gamma_1 \\
\gamma_{2k} & \gamma_2 & & \\
\gamma_{2k} & & & \\
\end{pmatrix}
\quad \text{and} \quad 
\begin{pmatrix}
\gamma'_{2k+1} & \gamma'_{2k-1} & \cdots & \gamma'_1 \\
\gamma'_{2k} & \gamma'_2 & & \\
\gamma'_{2k} & & & \\
\end{pmatrix}
\]

their associated distinguished \(u\)-symbols. The bipartitions \((\alpha, \beta)\) and \((\alpha', \beta')\) corresponding to these (by the algorithm described at the beginning of this section) verify \((\alpha, \beta) \leq (\alpha', \beta')\) if and only if \(\gamma \leq \gamma'\), where \(\gamma = (\gamma_1, \ldots, \gamma_{2k+1})\) and \(\gamma = (\gamma_1, \ldots, \gamma_{2k+1})\). This is in turn equivalent to the symplectic partitions \(\lambda\) and \(\lambda'\) verifying \(\lambda \leq \lambda'\).
In order to verify $\gamma \leq \gamma'$, take $r \in \{1, \ldots, 2k+1\}$. We need to show that

$$\gamma_1 + \cdots + \gamma_r \leq \gamma'_1 + \cdots + \gamma'_r.$$ 

Let

$$\gamma_1 + \cdots + \gamma_r = \sum_{i=1}^{t} \zeta_i + 2(k-i) + 1 + \sum_{i=1}^{s} \zeta'_i + 2(k-i+1),$$

where $t + s = r$, and

$$\gamma'_1 + \cdots + \gamma'_r = \sum_{i=1}^{t'} \zeta'_i + 2(k-i) + 1 + \sum_{i=1}^{s'} \zeta'_i + 2(k-i+1),$$

where $t' + s' = r$. Suppose that $s' \geq s$ (the case $s \geq s'$ has a similar proof). Distinguishedness implies

$$\xi'_{s+i} + 2(k+1-s-i) \geq \xi'_{s'+i} + 2(k-t'-i) + 1$$

for all $i = 1, \ldots, s' - s$, so

$$\sum_{i=1}^{t'} \zeta'_i + 2(k-i) + 1 + \sum_{i=1}^{s'} \zeta'_i + 2(k-i+1) \geq \sum_{i=1}^{t} \zeta'_i + 2(k-i) + 1 + \sum_{i=1}^{s} \xi_i + 2(k-i+1) \geq \sum_{i=1}^{t} \zeta_i + 2(k-i) + 1 + \sum_{i=1}^{s} \xi_i + 2(k-i+1),$$

the last inequality coming from $\zeta' \geq \zeta$.

The final statement is a consequence of Lemma 2. □

7.1.2. Second character. We now analyse the case given by character $I_2$. Proposition 8 allows us to rewrite this character as:

$$(8) \quad I_2 = \sum_{0 \leq r \leq \min(l, l')} \sum_{(\xi, \eta) \in \mathcal{P}_2(r)} \sum_{\epsilon' \epsilon''} \chi_{\xi', \eta} \boxtimes \chi_{\xi', \eta},$$

Where the sum is over partitions $\xi'$ and $\eta'$ of $l - |\eta|$ and $l' - |\epsilon|$ such that $^t\xi \leq ^t\epsilon'$ and $^t\eta \leq ^t\epsilon''$.

Lemma 3. Let $(\xi', \eta')$ be a bipartition of $l'$ and $\tau(\xi', \eta')$ be the set of the representations $\chi_{\xi, \eta}$ of $\text{Irr}(W_l)$ such that $\chi_{\xi, \eta} \boxtimes \chi_{\xi', \eta'}$ appears in $I_2$.

1. The representation $\chi_{\xi, \eta}$ belongs to $\tau(\xi', \eta')$ if and only if $^t\xi' \leq ^t\xi$ and $^t\eta \leq ^t\eta'$. 

2. Suppose the number of parts of \( \eta, \eta' \) and \( \xi' \) are the same and equal to an integer \( r \), and that \( \xi \) has one more part than these three. Then for \( P, Q \subset \{1, \ldots, r\} \) arbitrary,

\[
\xi_1 + \sum_{P} \xi_{i+1} + \sum_{Q} \eta_i \geq (l - l') + \sum_{P} \xi'_i + \sum_{Q} \eta'_i.
\]

In particular, the smallest element of \( \tau(\xi', \eta') \) for Achar Henderson order is \((l - l') \cup \xi', \eta')\).

3. Under the same assumptions of the previous item, for \( P \subset \{1, \ldots, r + 1\} \) and \( Q \subset \{1, \ldots, r\} \)

\[
\sum_{P} \xi_i + \sum_{Q} \eta_i \leq l - l' + \eta'_1 + \sum_{P} \xi'_i + \sum_{Q} \eta'_{i+1}.
\]

In particular, the largest representation of \( \tau(\xi', \eta') \) for the Achar Henderson order corresponds to the bipartition \((l - l' + \eta'_1 + \xi'_1, \xi'_2, \ldots, \xi'_r), (\eta'_2, \ldots, \eta'_r)\).

Proof. Item 1 is a straightforward consequence of equality (8).

Let \( \chi_{\xi, \eta} \) belong to \( \tau(\xi', \eta') \) so that \( ^t\xi' \preceq ^t\xi \) and \( ^t\eta \preceq ^t\eta' \). These imply

\[
(9) \quad \xi_{r+1} \leq \xi_k \leq \xi_k \quad \text{and} \quad \eta_k \leq \eta'_k
\]

for all \( k = 1, \ldots, r \). Rewriting

\[
|\xi| + |\eta| - |\xi'| - |\eta'| = l - l'
\]

as

\[
\xi_1 - (l - l') + \sum_{i=1}^{r} \xi_{i+1} - \xi'_i + \sum_{i=1}^{r} \eta_i - \eta'_i = 0,
\]

inequalities (9) imply

\[
\xi_1 - (l - l') + \sum_{P} \xi_{i+1} - \xi'_i + \sum_{Q} \eta_i - \eta'_i \geq 0,
\]

for \( P, Q \subset \{1, \ldots, r\} \) arbitrary. It is simple to see this implies \( (\xi, \eta) \succeq ((l - l') \cup \xi', \eta') \).

The proof of item 3 is similar to that of 2 and will be omitted. \( \square \)

**Theorem 10.** Let \( (\xi, \eta) \) be a bipartition in \( \tau(\xi', \eta') \). If \( \lambda, \lambda_m \) and \( \lambda_M \) denote the symplectic partitions related to \( (\xi, \eta), ((l-l') \cup \xi', \eta') \) and \(((l-l' + \eta'_1 + \xi'_1, \xi'_2, \ldots, \xi'_r), (\eta'_2, \ldots, \eta'_r)) \) by the Springer correspondence, then \( \lambda_m \leq \lambda \leq \lambda_M \). In particular, the minimal and maximal representations in \( \tau(\xi, \eta) \) (cf. Remark 7) correspond to the bipartitions \(((l-l') \cup \xi', \eta') \) and \(((l-l' + \eta'_1 + \xi'_1, \xi'_2, \ldots, \xi'_r), (\eta'_2, \ldots, \eta'_r)) \).
Proof. As in the proof of Theorem 9 we can suppose that \(\eta, \eta'\) and \(\xi'\) have the same number \(k\) of parts and that \(\xi\) has \(k+1\) parts. Let

\[
\begin{pmatrix}
\xi_{k+1} & \xi_k + 2 & \cdots & \xi_1 + 2k \\
\eta_k + 1 & \cdots & \eta_1 + 2k - 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\xi'_{k+1} & \xi'_{k-1} + 2 & \cdots & l - l' + 2k \\
\eta'_k + 1 & \cdots & \eta'_1 + 2k - 1
\end{pmatrix}
\]

be the \(u\)-symbols corresponding to \((\xi, \eta)\) and \(((l - l') \cup \xi', \eta')\) respectively. Let

\[
\begin{pmatrix}
\gamma_{2k+1} & \gamma_{2k-1} & \cdots & \gamma_1 \\
\gamma_{2k} & \cdots & \gamma_2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\gamma'_{2k+1} & \gamma'_{2k-1} & \cdots & \gamma'_1 \\
\gamma'_{2k} & \cdots & \gamma'_2
\end{pmatrix}
\]

be their associated distinguished \(u\)-symbols. As the last two \(u\)-symbols are distinguished, \(l > 2l'\) imply that \(\gamma'_1 = l - l' + 2k\) and \(\gamma_1 = \xi_1 + 2k\).

For \(r \in \{1, \ldots, 2k + 1\}\), there are \(P, Q \subset \{1, \ldots, r\}\) such that

\[
\gamma_1' + \cdots + \gamma_r' = (l - l' + 2k) + \sum_P \xi'_i + 2(k - i) + \sum_Q \eta'_i + 2(k - i) + 1.
\]

The right side in the above inequality is smaller than

\[
(\xi_1 + 2k) + \sum_P \xi_{i+1} + 2(k - i - 1) + \sum_Q \eta_i + 2(k - i) + 1
\]

by item 2 of Lemma 3. The last sum is smaller than \(\gamma_1 + \cdots + \gamma_r\) (because we deal with a distinguished symbol). This means that \(\gamma' \leq \gamma\) and, as in proof of Theorem 9, this is equivalent to \(\lambda_m \leq \lambda\).

The assertion concerning the maximal representation has a similar proof. \(\square\)

7.2. Unitary pairs. As before, let \(G\) be a connected reductive group over \(\overline{\mathbb{F}}_q\) defined over \(\mathbb{F}_q\) with Frobenius map \(F\) and \(T\) a fixed rational maximal torus. It is known that \(G^F\)-conjugacy classes of rational tori are in bijection with \(F\)-conjugacy classes of the Weyl group \(W\) of \(G\). This bijection relates the rational torus \(T^f\) to its type relative to \(T\), this is just the \(F\)-class in \(W\) of \(g^{-1}Fg \in N_G(T)\). We will denote by \(T_w\) a torus of type \(w \in W\) and by \(R_w\) the virtual character \(R^G_{T_w}(1)\).

Let \(\phi\) be a central function of \(W\) we define

\[
R^G_\phi = \frac{1}{|W|} \sum_{w \in W} \phi(w)R_w.
\]

Characters of the symmetric group are parametrized by partitions of \(n\), we denote by \([\mu]\) the character corresponding to the partition \(\mu\). Let
$R^{\text{U}_m(q)}_\mu$ be the character of $\text{U}_m(q)$ corresponding to the central function $[\mu]$ and let
\[
\varepsilon_\mu = (-1)^{\sum_{i=1}^k \binom{\ell_i}{2} + \binom{m(m-1)}{2}},
\]
so that $\varepsilon_\mu R^{\text{U}_m(q)}_\mu$ is a true character of $\text{U}_m(q)$.

From Theorem 3 we have a bijection between the category $\mathcal{R}(\text{U}_m(q))_k$ (of complex representations spanned by $\text{Irr}(\text{U}_m(q))_{\lambda_k}$) and the set of irreducible representations of $W_{m-k(k+1)/2}$. This bijection allows us to describe explicitly the characters in this Harish-Chandra series \[\text{Appendice, proposition p. 224}].\]

**Proposition 9.** 1. The unique cuspidal unipotent representation $\lambda_k$ of $\text{U}_{1/2(k^2+k)}(q)$ is $\varepsilon_{\tau_k} R^{\text{U}_m(q)}_{\tau_k}$ where $\tau_k$ is the $k$-th 2-core $\tau_k = (k, \ldots, 1)$.

2. For $m \geq 1/2(k^2 + k)$ the irreducible characters of $\mathcal{R}(\text{U}_m(q))_k$ are $\varepsilon_\mu R^{\text{U}_m(q)}_\mu$ where $\mu$ is a partition of $m$ of 2-core $\tau_k$. This character is related to the bipartition $(\mu(0), \mu(1))$ (where $\mu(0)$ and $\mu(1)$ are the 2-quotients of parameter 1 of $\mu$) under the bijection given in Theorem 3.

This theorem tells us that for a fixed $k$ there’s a bijection between the bipartitions of $m - k(k+1)/2$, the representations in the Harish-Chandra series $\mathcal{R}(\text{U}_m(q))_k$ and the partitions of $m$ with 2-core $\tau_k$.

The definition of 2-core is not useful to do computations. We need to express it otherwise. Let’s first recall the following [5, Lemma 5.8]

**Proposition 10.** If $\mu'$ is a partition obtained from $\mu$ by removing a 2-rimhook, then the $\beta$-set of $\mu'$ is $\{\beta_1, \ldots, \beta_j, \beta_j-2, \beta_j+1, \ldots, \beta_i\}$. In particular, the $\beta$-sets of a partition and its 2-core have the same number of even (resp. odd) elements.

For a triangular partition $\tau_k = (k k - 1 \ldots 1)$, an easy calculation shows that its $\beta$-set is $\beta_k = \{0, 2, \ldots, 2t_0 - 2, 1, 3, \ldots, 2t_1 - 1\}$ where $t_0 = |\beta_k(0)|$ and $t_1 = |\beta_k(1)|$. These two last numbers depend on the parity of $k$: if $k$ is even then $t_0 = t + k + 1/2$, $t_1 = t - k - 1/2$ and if $k$ is odd then $t_0 = t - k/2$, $t_1 = t + k/2$.

**Proposition 11.** Let $\mu$ and $\mu'$ be two bipartitions of the same integer $l$ and let $\beta$ and $\beta'$ be their $\beta$-sets respectively. Then $\mu$ and $\mu'$ have the same 2-core if and only if $|\beta(0)| = |\beta'(0)|$ and $|\beta(1)| = |\beta'(1)|$.

**Proof.** Suppose that $\mu$ and $\mu'$ have the same 2-core. Corollary 11 says that the number of even (resp. odd) elements in the $\beta$-sets of $\mu$ and $\mu'$ equal the number of even (resp. odd) elements in the $\beta$-set of the common 2-core, so we have $|\beta(0)| = |\beta'(0)|$ and $|\beta(1)| = |\beta'(1)|$. 

If \( \mu \) and \( \mu' \) have different 2-cores \( \tau_k \) and \( \tau_{k'} \), assuming that \( k < k' \) we have 4 cases depending on the parity of \( k \) and \( k' \). For instance if they’re both odd then \( |\beta_k(1)| = t + k/2 < t + k'/2 = |\beta_{k'}(1)| \) and \( |\beta_{k'}(0)| = t - k'/2 < t - k/2 = |\beta_k(0)| \) so by Corollary 10 \( |\beta'(0)| < |\beta(0)| \) and \( |\beta(1)| < |\beta'(1)| \). The other 3 cases are analogous. \( \square \)

Recall that for dual pairs \((U_m(q), U_n(q))\) the Howe correspondence between Harish-Chandra series of cuspidal unipotent representations is given by a correspondence between pairs of type \( B \) Weyl groups \((W_l, W_{l'})\) with \( l = m - k(k + 1)/2 \) and \( l' = n - \theta(k)(\theta(k) + 1)/2 \). As for symplectic-orthogonal dual pairs we will fix an irreducible representation \( \chi_{\lambda,\mu} \) of \( W_l \) and we will describe explicitly the set \( \tau(\lambda, \mu) \) of representations of \( W_{l'} \) such that the outer tensor product \( \chi_{\lambda,\mu} \boxtimes \chi_{\lambda',\mu'} \) appears in \( J_1 \) or \( J_2 \).

**Remark 2.** From proposition 8 we have a bijection

\[
\text{Irr}(W_{l'}) \simeq \mathcal{B}(U_n(q))_{k'}
\]

\[
\chi_{\lambda,\mu} \mapsto \epsilon_\nu R_{\nu}^{U(m)}
\]

where \( k' = \theta(k) \) and \( \nu \) is a partition of \( n \) with \( \tau_{k'} \) as 2-core and \((\lambda, \mu)\) as 2-quotient of parameter 1. A representation in \( \tau(\lambda, \mu) \subset \text{Irr}(W_{l'}) \) is said to be minimal (resp. maximal) if it provides the smallest (resp. largest) representation \( \epsilon_\nu R_{\nu}^{U(m)} \) in \( \mathcal{B}(U(m))_{k'} \) (where the order in this Harish-Chandra series is given by the classical order between the indexing partitions \( \nu \)).

We show in the following sections that such a representation exists and that it is unique. We start with character \( J_1 \). As for symplectic-orthogonal pairs we suppose that \( l' > 2l \).

**7.2.1. First character.** Proposition 8 allows us to write \( J_1 \) as

\[
J_1 = \sum_{0 \leq r \leq \min(l, l')} \sum_{(\lambda, \mu) \in \mathcal{P}_2(r)} \sum_{\lambda', \mu'} \chi_{\lambda', \mu} \boxtimes \chi_{\mu', \lambda},
\]

where the sum is over partitions \( \lambda' \) and \( \mu' \) of \( l - |\mu| \) and \( l' - |\lambda| \) such that \( ^t \lambda \leq ^t \lambda' \) and \( ^t \mu \leq ^t \mu' \).

**Lemma 4.** Let \( (\lambda, \mu) \) be a bipartition of \( l \) and \( \tau(\lambda, \mu) \) the set of representations \( \chi_{\mu', \lambda'} \) such that \( \chi_{\lambda,\mu} \boxtimes \chi_{\mu', \lambda'} \) appears in \( J_1 \).

1. The representations in \( \tau(\lambda, \mu) \) correspond to bipartitions \((\mu', \lambda')\) such that \( ^t \lambda' \leq ^t \lambda \) and \( ^t \mu \leq ^t \mu' \).

2. The smallest element of \( \tau(\lambda, \mu) \) for the Achar-Henderson order is \((l' - l) \cup \mu, \lambda)\).
3. The largest representation in $\tau(\lambda, \mu)$ for the Achar-Henderson order is $((l' - l + \lambda_1 + \mu_1, \mu_2, \ldots, \mu_r), (\lambda_2, \ldots, \lambda_r))$

Proof. Analogous to the proof of Lemma 2 $\square$

Theorem 11. There exists a unique minimal (resp. maximal) representation in $\tau(\lambda, \mu)$ (cf. Remark 3) it corresponds to the bipartition $((l' - l) \cup \mu, \lambda)$ (resp. $((l' - l + \lambda_1 + \mu_1, \mu_2, \ldots, \mu_r), (\lambda_2, \ldots, \lambda_r))$ of $l'$.

Proof. Consider the elements of $\tau(\lambda, \mu)$ having the same second component, say $\lambda'$. As in Lemma 2 we can prove that the smallest (resp. largest) of these bipartitions for the Achar-Henderson order is $((a) \cup \mu, \lambda')$ (resp. $((a + \mu_1, \mu_2, \ldots, \mu_r), \lambda')$ with $a = l' - l + |\lambda| - |\lambda'|$.

Recall that representations $e^\nu R_{U_n(q)^0}$ in $R(U_n(q))_{k'}$ have $\tau_{k'}$ as common 2-core and that this fixes the length of the partitions in the 2-quotient (see Proposition 11). We can use Theorem 12 to assert that the smallest (resp. largest) partition $\nu$ (for the classical order on partitions) having a 2-quotient with $\lambda'$ as second component corresponds to the 2-quotient $((a) \cup \mu, \lambda')$ (resp. $((a + \mu_1, \mu_2, \ldots, \mu_r), \lambda')$).

We need still to compare partitions $\nu$ having 2-quotients of the form $((a') \cup \kappa, \lambda')$ for fixed $\kappa$. Indeed, for $\lambda'$ fixed, both the minimal and maximal 2-quotients are of this form (with $\kappa = \mu$ for the minimal and $\kappa = (\mu_2, \ldots, \mu_r)$ for the maximal). Let’s consider two 2-quotients $((a') \cup \kappa, \lambda')$ and $((\tilde{a}) \cup \kappa, \tilde{\lambda})$ such that

$$((a') \cup \kappa, \lambda') \leq ((\tilde{a}) \cup \kappa, \tilde{\lambda}).$$

This amounts to the following inequalities

$$a' + \sum_{i=1}^{k} \lambda'_i \leq \tilde{a} + \sum_{i=1}^{k} \tilde{\lambda}_i$$

for $k = 0, 1, \ldots, t_1$ where $t_1$ is the number of parts of $\lambda'$ and $\tilde{\lambda}$ and $t_0 = l(\nu) + 1$ ((recall that by Proposition 11 these lengths are fixed by the 2-core). The beta sets $\beta'$ and $\tilde{\beta}$ corresponding to $((a') \cup \kappa, \lambda')$ and $((\tilde{a}) \cup \kappa, \tilde{\lambda})$ respectively,

$$\beta' = \{2(a' + t_0 - 1), 2(\kappa_{i-1} + t_0 - i), 2(\lambda'_j + t_1 - j) + 1\}$$

$$\tilde{\beta} = \{2(\tilde{a} + t_0 - 1), 2(\kappa_{i-1} + t_0 - i), 2(\tilde{\lambda}_j + t_1 - j) + 1\},$$

where $2 \leq i \leq t_0, 1 \leq j \leq t_1$ in both sets.
Let $t = t_0 + t_1$ and $\beta'_1 > \cdots > \beta'_t$ (resp. $\tilde{\beta}_1 > \cdots > \tilde{\beta}_t$) denote the elements of $\beta'$ (resp. $\tilde{\beta}$) after ordering. For $l = 1, \ldots, t$

$$ \sum_{i=1}^{l} \beta'_i = 2(a' + t_0 - 1) + \sum_{n=2}^{N} 2(\kappa_{n-1} + t_0 - n) + \sum_{m=1}^{M} 2(\lambda'_m + t_1 - 1) + 1 $$

$$ = 2(a + t_0 - 1) + \sum_{n=2}^{N} 2(\kappa_{n-1} + t_0 - n) + \sum_{m=1}^{M} 2(\tilde{\lambda}_m + t_1 - 1) + 1 $$

$$ \leq \sum_{i=1}^{l} \tilde{\beta}_i $$

the second equality coming from (10) and the last from the fact that the terms on the right are the $l$ biggest in $\tilde{\beta}$. The result holds thanks to Lemma 4.

\[ \square \]

7.2.2. Second character. Once again, we use Proposition 8 to rewrite $J_2$ as

\[ J_2 = \sum_{0 \leq r \leq \min(l, l')} \sum_{(\lambda, \mu) \in P_2(r)} \sum_{\mu, \mu'} \chi_{\lambda, \mu} \boxtimes \chi_{\mu', \lambda}, \]

where the third sum is over partitions $\mu$ and $\mu'$ of $l - |\lambda|$ and $l' - |\lambda|$ such that $^t\nu \leq ^t\mu$ and $^t\nu \leq ^t\mu'$.

Lemma 5. Let $(\lambda, \mu)$ be a bipartition of $l$ and $\tau(\lambda, \mu)$ be the set of representations $\chi_{\mu', \lambda'}$ of $W_\nu$ such that $\chi_{\lambda, \mu} \boxtimes \chi_{\mu', \lambda'}$ appears in $J_2$.

1. An irreducible representation $\chi_{\mu', \lambda'}$ of $W_\nu$ belongs to $\tau(\lambda, \mu)$ if and only if $^t\mu$ and $^t\mu'$ have a common predecessor for the order $\preceq$ and that $\lambda' = \lambda$.

2. The smallest element of $\tau(\lambda, \mu)$ for the Achar-Henderson order is $((l' - l) \cup \mu, \lambda)$.

3. The largest element of $\tau(\lambda, \mu)$ for the Achar-Henderson order is $((l' - l + \mu_1 + \mu_2, \mu_3, \ldots, \mu_r), \lambda)$.

Proof. Similar to that of Lemma 2. \[ \square \]

Theorem 12. Let $\epsilon_\nu R_\nu$ and $\epsilon_{\nu'} R_{\nu'}$ denote two irreducible characters of $\mathcal{R}(U_m(q))$. If the 2-quotients $(\mu, \lambda)$ and $(\mu', \lambda)$ of $\nu$ and $\nu'$ of parameter 1 verify $(\mu, \lambda) \preceq (\mu', \lambda)$, then $\nu \preceq \nu'$. In particular, the minimal (resp. maximal) representations in $\tau(\lambda, \mu)$ (cf. Remark 3) corresponds to the bipartition $((l' - l) \cup \mu, \lambda)$ (resp. $((l' - l + \mu_1 + \mu_2, \mu_3, \ldots, \mu_r), \lambda)$).

Proof. Call $t_1$ the number of parts of $\lambda$. As $\nu$ and $\nu'$ have the same 2-core we can suppose $\mu$ and $\mu'$ to have the same number of parts $t_0$ so
that the $\beta$-sets of $\nu$ and $\nu'$ are
\[ \beta = \{2(\mu_i + t - i), 2(\lambda_j + t - j) + 1 | 1 \leq i \leq t_0, 1 \leq j \leq t_1 \} \]
and
\[ \beta' = \{2(\mu'_i + t - i), 2(\lambda'_j + t - j) + 1 | 1 \leq i \leq t_0, 1 \leq j \leq t_1 \} \]
respectively.

Suppose after ordering, the elements of $\beta$ are $\beta_1 > \cdots > \beta_t$ and those of $\beta'$ are $\beta'_1 > \cdots > \beta'_t$ for $t = t_0 + t_1$. The hypothesis $(\mu, \lambda) \leq (\mu', \lambda)$ is equivalent to $\mu \leq \mu'$. For all $k$ we can find non negatives integers $r$ and $s$ verifying $r + s = k$ such that
\[
\sum_{i=1}^{k} \beta_i = \sum_{i=1}^{r} 2(\mu_i + t - i) + \sum_{i=1}^{s} 2(\lambda_i + t - i) + 1 \\
\leq \sum_{i=1}^{r} 2(\mu'_i + t - i) + \sum_{i=1}^{s} 2(\lambda'_i + t - i) + 1 \leq \sum_{i=1}^{k} \beta'_i.
\]
The last inequality is true because the elements to its right are the $k$ biggest elements of $\beta'$.

It is easy to see that the set of inequalities
\[
\sum_{i=1}^{k} \beta_i \leq \sum_{i=1}^{k} \beta'_i
\]
for $k = 1, \ldots, t$ is equivalent to $\nu \leq \nu'$. The last assertion is a consequence of Lemma 5. \(\square\)

8. Perspectives

The study conducted in this paper has been done for all type I dual pairs but for pairs $(\text{Sp}_{2m}(q), O_{2n+1}(q))$. Indeed, we used the characters giving the Howe correspondence between Harish-Chandra series (see end of section 6) found by Aubert, Michel and Rouquier in [5]. They, in turn found these characters from the results in [22]. In the latter paper Srinivasan studied how the Weil representations decompose in terms of the Deligne-Lusztig virtual representations for all dual pairs (including linear pairs) but for pairs with odd orthogonal groups. It could be possible to extend our study to all dual pairs by using the results in a recently published paper by Pan (cf. [20]).

Pan also proved in [19, Theorem 3.10] that the description of the theta correspondence of cuspidal representations for a finite reductive dual pair of unitary groups can be reduced to the case of cuspidal unipotent representations. This should allow us to extend our results
from unipotent representations to any irreducible representation for unitary groups.

Recently, Gurevich and Howe [15, Theorem 3.3.3] have found a way to extract an irreducible subrepresentation $\eta(\pi)$ of $\Theta(\pi)$ for symplectic-orthogonal dual pairs. It would be interesting to compare this representation with the extremal representations we have obtained. This could lead to solve Conjecture 3.3.4 in their paper.

Let $K$ be a field of characteristic 0, and $G$ be either a symplectic or orthogonal (resp. unitary) group with coefficients in $K$. The local Langlands correspondence classifies irreducible representations $\pi$ of $G$ in terms of their $L$-parameters $(\phi, \varepsilon)$, where $\phi$ is a conjugate self-dual representation of the Weil-Deligne group of $K$ and $\varepsilon$ is an irreducible character of the component group associated to $\phi$. In [3], Atobe and Gan express the local Howe correspondence as a correspondence between $L$-parameters for discrete (in fact tempered) representations.

Let $\chi_{\xi, \eta'}$ is an irreducible representation of $W_\nu$ and $\chi_{\xi, \eta}$ denote one of the extremal representations obtained in Theorems 9 - 12. It should be possible to obtain an analogue over finite fields to the results in [3] can be obtained by studying the relation between the pairs $(O', \psi')$ and $(O, \psi)$ corresponding to $\chi_{\xi, \eta'}$ and $\chi_{\xi, \eta}$ via the Springer correspondence.

References

[1] Achar, P. N., and Henderson, A. Orbit closures in the enhanced nilpotent cone. Advances in Mathematics 219, 1 (2008), 27 – 62.
[2] Adams, J., and Moy, A. Unipotent representations and reductive dual pairs over finite fields. Transactions of the American Mathematical Society 340, 1 (1993), 309–321.
[3] Atobe, H., and Gan, W. T. Local Theta correspondence of Tempered Representations and Langlands parameters. ArXiv e-prints (Feb. 2016).
[4] Aubert, A.-M., Kräskiewicz, W., and Przebinda, T. Howe correspondence andspringer correspondence for dual pairs over a finite field. Proceedings of the Symposia in Pure Mathematics 92 (2016), 17–44.
[5] Aubert, A.-M., Michel, J., and Rouquier, R. Correspondance de Howe pour les groupes réductifs sur les corps finis. Duke Math. J. 83, 2 (1996), 353–397.
[6] Bonnafé, C. Sur les caractères des groupes réductifs finis : applications aux groupes spéciaux linéaires et unitaires. Astérisque 306 (2006), 165 pages. 97 pages, in french.
[7] Carter, R. W. Finite groups of Lie type. Wiley Classics Library. John Wiley & Sons, Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.
[8] Digne, F., and Michel, J. Foncteurs de Lusztig et caractères des groupes linéaires et unitaires sur un corps fini. *Journal of Algebra* 107, 1 (1987), 217–255.

[9] Digne, F., and Michel, J. *Representations of finite groups of Lie type*. London Mathematical Society Student Texts (Book 21). Cambridge University Press, 1991.

[10] Digne, F., and Michel, J. Groupes réductifs non connexes. *Ann. Sci. École Norm. Sup. (4)* 27, 3 (1994), 345–406.

[11] Fong, P., and Srinivasan, B. Brauer trees in classical groups. *Journal of Algebra* 131, 1 (1990), 179–225.

[12] Geck, M. A note on Harish-Chandra induction. *Manuscripta mathematica* 80, 4 (1993), 393–402.

[13] Geck, M., and Pfeiffer, G. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*. London Mathematical Society Monographs (Numero 21). Clarendon Press, 2000.

[14] Gérardin, P. Weil representations associated to finite fields. *J. Algebra* 46, 1 (1977), 54–101.

[15] Gurevich, S., and Howe, R. Small Representations of Finite Classical Groups. *ArXiv e-prints* (Sept. 2016).

[16] Lusztig, G. Green polynomials and singularities of unipotent classes. *Adv. in Math.* 42, 2 (1981), 169–178.

[17] Mœglin, C., Vignéras, M.-F., and Waldspurger, J.-L. *Correspondances de Howe sur un corps p-adique*, vol. 1291 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1987.

[18] Müller, P. Algebraic groups over finite fields, a quick proof of Lang’s theorem. *Proceedings of the american mathematical society* 131, 2 (2002), 369–370.

[19] Pan, S.-Y. Supercuspidal representations and preservation principle of theta correspondence. *Journal fr die reine und angewandte Mathematik (Crelles Journal)* (2016).

[20] Pan, S.-Y. Weil representations of finite symplectic groups and finite odd-dimensional orthogonal groups. *Journal of Algebra* 453 (2016), 291–324.

[21] Springer, T. A. Trigonometric sums, Green functions of finite groups and representations of Weyl groups. *Invent. Math.* 36 (1976), 173–207.

[22] Srinivasan, B. Weil representations of finite classical groups. *Invent. Math.* 51, 2 (1979), 143–153.