Quark bilinear step scaling functions and their continuum limit extrapolation

M. Guagnelli\textsuperscript{a}, J. Heitger\textsuperscript{b}, C. Pena\textsuperscript{c} and A. Vladikas\textsuperscript{a} (ALPHA Collaboration)

\textsuperscript{a}INFN Sezione di Roma II, c/o Dipartimento di Fisica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica 1, I-00133 Rome, Italy
\textsuperscript{b}Westfälische Wilhelms-Universität Münster, Institut für Theoretische Physik, Wilhelm-Klemm-Strasse 9, D-48149 Münster, Germany
\textsuperscript{c}DESY, Theory Group, Notkestrasse 85, D-22603 Hamburg, Germany

Some new results on nonperturbative renormalisation of quark bilinears in quenched QCD with Schrödinger Functional techniques are presented. Special emphasis is put on a study of the universality of the continuum limit for step scaling functions computed with different levels of $O(a)$ improvement.

1. Step scaling functions of quark bilinears

Nonperturbative renormalisation is a key ingredient in precision lattice QCD computations. Schrödinger Functional (SF) techniques enable us to determine nonperturbatively the renormalisation group (RG) running of fundamental parameters (coupling and quark masses) and composite operators in QCD over a vast range of scales, thus providing a good control of systematics in the renormalisation procedure. One important issue in this context is the extrapolation to the continuum limit (CL) required to remove the cutoff dependence in the RG running. Whenever this dependence is steep and cannot be tamed by appropriate $O(a)$ improvement of the action and/or the relevant operators, the extrapolation may turn into a major source of uncertainty. Good control of the latter is also interesting in order to test universality of the CL for different lattice regularisations, along the lines of e.g. [2]. Here we report on an ongoing detailed investigation of the CL extrapolation in the determination of the RG running of quark masses and quark bilinears in quenched QCD with Wilson fermions.

To describe the nonperturbative RG running of a bilinear $O_T = \bar{\psi}_i \Gamma \psi_j$, where $i,j$ are different quark flavours, we use a step scaling function (SSF) $\sigma_T$, defined as

$$\sigma_T(u) = \exp \left\{ - \int_{\gamma(\mu)}^{\gamma(\mu/2)} \frac{dg}{\beta(g)} \right\} ,$$

where $\gamma$ is the anomalous dimension of the operator, $\beta(g(\mu)) = \partial g(\mu) / \partial \ln \mu$, and $\gamma$ is the renormalised coupling. In order to compute a SSF in a SF scheme, the theory is put to live in a finite box of size $L^3 \times T$ with SF boundary conditions. The numerical determination of the SSF at a given value of $u = g^2$ starts with the computation of the renormalisation constant on lattices of sizes $L/a$ and $2L/a$ at fixed bare coupling; this allows to construct the finite-cutoff SSF $\Sigma_T(u,a/L) = Z_T(g_0^2,a/L) / Z_T(g^2,a/L)$. Then $\sigma_T$ is obtained

$$Z_T \left( g_0^2, a/T \right) \frac{F(T/2)}{\Theta} = \frac{F(T/2)}{\Theta} \bigg|_{\text{tree level}} ,$$

where $F$ is a suitable SF correlation function, and $\Theta$ is a factor (see below for its precise form) which cancels the multiplicative renormalisation of SF boundary fields.

The numerical determination of the SSF at a given value of $u = g^2$ starts with the computation of the renormalisation constant on lattices of sizes $L/a$ and $2L/a$ at fixed bare coupling; this allows to construct the finite-cutoff SSF $\Sigma_T(u,a/L) = Z_T(g_0^2,a/L) / Z_T(g^2,a/L)$. Then $\sigma_T$ is obtained

\[\text{We will always take } T = L, \text{ vanishing boundary gauge fields and a phase } \theta = 0.5 \text{ in spatial boundary conditions.}\]
by repeating the procedure at various values of $a$ for fixed $L$, and extrapolating to the CL:

$$\sigma_T(u) = \lim_{a/L \to 0} \frac{\Sigma_T(u, a/L)}{g^2(L^{-1})} = \cdots . \tag{3}$$

We will concentrate on two bilinear operators, namely $P = O_{\gamma_5}$ and $T_{0k} = i O_{\sigma_{0k}}$, with $\sigma_{0k} = \frac{i}{2} [\gamma_0, \gamma_k]$. To compute renormalisation constants, Eq. 2 is imposed with $F$ and $\Theta$ set to

$$f_P(x_0) = -\frac{1}{2} \langle \bar{\psi}(x) \gamma_5 \psi(x) O_{ji}[\gamma_5] \rangle , \tag{4}$$

$$f_1 = -\frac{1}{2L^6} \langle O'_{ij}[\gamma_5] O_{ji}[\gamma_5] \rangle \tag{5}$$

for the pseudoscalar density and

$$k_T(x_0) = -\frac{1}{6} \sum_k \langle \bar{\psi}(x) \sigma_{0k} \psi(x) O_{ji}[\gamma_k] \rangle , \tag{6}$$

$$k_1 = -\frac{1}{6L^6} \sum_k \langle O'_{ij}[\gamma_k] O_{ji}[\gamma_k] \rangle \tag{7}$$

for the tensor bilinear, where $O_{ij}[\Gamma] = a^6 \sum_{y, z} \bar{\psi}(y) \Gamma \psi(z)$ is a SF boundary source. The resulting renormalisation constants and SSFs will bear the labels $P$ and $T$, respectively.

We stress that the computation of $\sigma_P$ and $\sigma_T$ suffices to determine the RG running of all quark bilinears in the CL. Recall also that by using axial Ward-Takahashi identities it is easy to show that $\sigma_P = \sigma_m^1$, where $\sigma_m$ is the SSF for quark masses.

2. Cutoff effects in step scaling functions

Cutoff effects in $\Sigma_P$ and $\Sigma_T$ can be analysed by considering the Symanzik expansion of the two-point functions entering their definitions. In the absence of $O(a)$ improvement both quantities are hence predicted to exhibit leading cutoff effects linear in $a$. To implement complete on-shell $O(a)$ improvement at the action level, it is enough to include the clover term proportional to $c_{sw}$ and the boundary counterterms proportional to $c_t$ and $\tilde{c}_t$. $O(a)$ improvement for the operators in the chiral limit can be written schematically as

$$P^I = P ; \quad T^I_{0k} = T_{0k} + a c_T [\partial_k V_k - \partial_k V_0] , \tag{8}$$

with $V_{\mu} = \bar{\psi}_{i} \gamma_{\mu} \psi_{j}$. The coefficient $c_{sw}$ is known nonperturbatively in the whole range of values of $g_0$ needed, while $c_t$ and $\tilde{c}_t$ are known in perturbation theory to $g_0^4$ and $g_0^6$ orders, respectively. $c_T$, on the other hand, has been computed nonperturbatively only at a few values of $g_0$.

We have computed the SSFs $\Sigma_P$ and $\Sigma_T$ in quenched QCD at 14 different values of the renormalised coupling, ranging from $\bar{g}^2(L^{-1}) = 3.48$ to $\bar{g}^2(L^{-1}) = 0.8873$, and four different values of the bare coupling constant in each case (corresponding to $L/a = 6, 8, 12, 16$). The details are essentially the same as in [3]. The main novelty is that we have used two different regularisations, involving different levels of $O(a)$ improvement, the results of which will be referred to as $[I]$ and $[U]$, respectively:

[I] Clover action with nonperturbative $c_{sw}$, one-loop values for $c_t$ and $\tilde{c}_t$.

[U] Wilson action ($c_{sw} = 0$), $c_t = 1$ (tree level value), one-loop value for $c_t$.

$\Sigma_T$ has been always computed with an unimproved $T_{0k}$.

As the effect of neglecting $O(a^4)$ terms in $c_t$ and $\tilde{c}_t$ can be expected to result in very small $O(a)$ effects in the SSFs $\Sigma_P$ and $\Sigma_T$, the results for $\Sigma_P^{[I]}$ should approach the continuum limit as $a^2$; $\Sigma_T^{[U]}$, on the other hand, is expected to exhibit a linear behaviour in $a$. This is indeed observed in numerical results, as exemplified by Fig. 1 where the CL extrapolation for $\sigma_T$ at two different values of the renormalised coupling is shown. By fitting $[I]$ and $[U]$ results independently with various Ansätze, universality of the CL has been checked, up to deviations which never exceed 1.5 $\sigma$ and can be regarded as statistical. Eventually, $[I]$ and $[U]$ results can be combined into a constrained fit to obtain a more precise determination of $\sigma_T$, as advocated in [7] for computations where $O(a)$ improvement of operators is difficult to implement.

$\Sigma_T$ is expected, on the other hand, to exhibit linear cutoff effects even when it is computed with an $O(a)$ improved action, due to the lack of operator improvement. Furthermore, $\Sigma_T^{[U]}$ may show large $O(a)$ effects. This is indeed observed in preliminary results for this SSF (Fig. 2). In this case, the two-action strategy reveals its usefulness, as assuming universality of the CL and performing a
3. Outlook

The results obtained so far demonstrate universality of the CL result for the RG running of quark bilinears computed with actions with different levels of $O(a)$ improvement. A better control of the CL extrapolation can be achieved in cases where nonperturbatively $O(a)$ improved operators are not available by combining results from different actions. This technique has been used in our preliminary computation of the non-perturbative RG running of tensor bilinears in quenched QCD. Pending tasks include comparison with cutoff effects in perturbation theory, and the computation of the NLO anomalous dimension of $T_{0k}$, required to match our results for $\sigma_T$ to conventional renormalisation schemes.

Acknowledgements

Work supported in part by the European Union’s Human Potential Programme under contract HPRN-CT-2000-00145, Hadrons/Lattice QCD.

REFERENCES

1. For a recent review see R. Sommer, Nucl. Phys. Proc. Suppl. 119 (2003) 185.
2. ALPHA, G. de Divitiis et al., Nucl. Phys. B437 (1995) 447.
3. ALPHA, S. Capitani et al., Nucl. Phys. B544 (1999) 669.
4. M. Lüscher et al., Nucl. Phys. B478 (1996) 365.
5. M. Lüscher et al., Nucl. Phys. B491 (1997) 323; ALPHA, A. Bode, U. Wolff and P. Weisz, Nucl. Phys. B540 (1999) 491; M. Lüscher and P. Weisz, Nucl. Phys. B479 (1996) 429.
6. T. Bhattacharya et al., Phys. Rev. D63 (2001) 074505.
7. M. Guagnelli, K. Jansen and R. Petronzio, Phys. Lett. B457 (1999) 153; ALPHA, M. Guagnelli et al., Nucl. Phys. Proc. Suppl. 119 (2003) 436.