Long-time asymptotic behavior of a mixed schrödinger equation with weighted Sobolev initial data

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Abstract

We apply the steepest descent method to obtain sharp asymptotics for a mixed schrödinger equation

\[ q_t + iq_{xx} - i\alpha(|q|^2 q)_x - 2b^2|q|^2 q = 0, \]

\[ q(x, t = 0) = q_0(x), \]

under essentially minimal regularity assumptions on initial data in a weighted Sobolev space \( q_0(x) \in H^{2,2}(\mathbb{R}) \). In the asymptotic expression, the leading order term \( \mathcal{O}(t^{-1/2}) \) comes from dispersive part \( q_t + iq_{xx} \) and the error order \( \mathcal{O}(t^{-3/4}) \) from a \( \bar{\partial} \) equation.

Keywords: Mixed schrödinger equation; Lax pair; Riemann-Hilbert problem; \( \bar{\partial} \) steepest descent method; long-time asymptotic.
1 Introduction

In this paper, we consider a mixed nonlinear Schrödinger (NLS) equation with a usual cubic nonlinear term and a derivative cubic nonlinear term

\[ q_t + iq_{xx} - ia(|q|^2q)_x - 2b^2|q|^2q = 0, \tag{1.1} \]
\[ q(x, t = 0) = q_0(x) \in H^{2,2}(\mathbb{R}), \tag{1.2} \]

where

\[ H^{2,2}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : x^2 f, f'' \in L^2(\mathbb{R}) \}. \]

The equation (1.1) was presented and further solved by the inverse scattering method [1, 2] and can be used to describe Alfvén waves propagating along the magnetic field in cold plasmas and the deep-water gravity waves [3, 4]. The term \( i(|q|^2q)_x \) in the equation (1.1) is called the self-steepening term, which causes an optical pulse to become asymmetric and steepen upward at the trailing edge [5, 6]. The equation (1.1) also describes the short pulses propagate in a long optical fiber characterized by a nonlinear refractive index [7, 8]. Brizhik et al showed that the modified NLS equation (1.1), unlike the classical NLS equation, possesses static localized solutions when the effective nonlinearity parameter is larger than a certain critical value [9].

For \( a = 0 \), the equation (1.1) reduces the classical defocusing Schrödinger equation

\[ iu_t + u_{xx} - 2b^2|u|^2u = 0, \tag{1.3} \]

which has important applications in a wide variety of fields such as nonlinear optics, deep water waves, plasma physics, etc [10, 11].

For \( b = 0 \), the equation (1.1) reduces to the Kaup-Newell equation [12]

\[ iu_t + u_{xx} - ia(|u|^2u)_x = 0. \tag{1.4} \]
For $a, b \neq 0$, the equation (1.1) is equivalent to the modified NLS equation

$$q_t + q_{xx} + ia(|q|^2q)_x - 2|q|^2q = 0,$$

which was also called the perturbation NLS equation [13].

There are various works on mixed NLS equation (1.1) and the modified NLS equation (1.5). For example, Date analyzed the periodic soliton solutions [14]. In 1991, Guo and Tian studied the unique existence and the decay behaviours of the smooth solutions to the initial value problem [15]. In 1994, Tian and Zhang proved that the Cauchy problem of initial value in Sobolev space has a unique weak solution [16]. The existence and nonexistence of global solution and blow-up solution for this equation were also investigated [17]. The bright, dark envelope solutions and Soliton behavior with N-fold Darboux transformation for mixed nonlinear Schrödinger equation were also discussed [18–20].

The local well-posedness of smooth solutions in the Sobolev spaces $H^s(\mathbb{R})$, $s > 3/2$ was established by Tsutsumi and Fukuda [21] and later extended to solutions with low regularity in the Sobolev space $H^{1/2}(\mathbb{R})$ by Takaoka [22]. For initial conditions in the energy space $H^1(\mathbb{R})$, Hayashi-Ozawa proved that solutions exist in $H^1(\mathbb{R})$ [23]. Colliander et al extended this result to $u_0 \in H^{1/2+\epsilon}(\mathbb{R})$. More recently, global well-posedness of derivative NLS equation with initial conditions $u_0 \in H^1$ and $H^{1/2}(\mathbb{R})$, respectively has been discussed by Guo [25] and Wu [26]. According to the existing conclusion, the global well-posedness of DNLS in $H^{2,2}(\mathbb{R})$ has been rigorously proved by using the inverse scattering transformation [27, 28].

In recent years, McLaughlin and Miller developed a $\bar{\partial}$-steepest decedent method for obtaining asymptotic of RH problems based on $\bar{\partial}$-problems [29]. This method has been successfully adapted to study the NLS equation and derivative NLS equation [30–32]. We recently have applied this method to ob-
tain asymptotic for the Kundu-Eckhaus equation and modified NLS equation with weighted Sobolev initial data [33, 34].

Recently, for the mixed defocusing nonlinear Schrödinger equation (1.1) with Schwartz initial data \( q_0(x) \in \mathcal{S}(\mathbb{R}) \), by defining a general analytical domain and two reflection coefficients, we found an unified long time asymptotic formula via the Deift-Zhou nonlinear steepest descent method [35]

\[
q(x, t) = t^{-1/2} \alpha(z_0) e^{i(4tz_0^2 - \nu(z_0) \log 8t)} e^{-\frac{2ia}{\nu(z_0)} \int_{z_0}^{+\infty} \log(1 - |r(\lambda)|^2) (a\lambda - b) d\lambda} + O(t^{-1} \log t),
\]

which covers results on classical defocusing Schrödinger equation, derivative Schrödinger equation and modified Schrödinger equation as special cases. In this paper, for much weaker weighted Sobolev initial data \( q_0(x) \in H^{2,2}(\mathbb{R}) \), we apply \( \bar{\partial} \) steepest deccedent method to give long time asymptotic for the initial value problem (1.1)-(1.2)

\[
q(x, t) = t^{-1/2} \alpha(z_0) e^{i(4tz_0^2 - \nu(z_0) \log 8t)} e^{-\frac{2ia}{\nu(z_0)} \int_{z_0}^{+\infty} \log(1 - |r(\lambda)|^2) (a\lambda - b) d\lambda} + O(t^{-3/4}).
\]

The advantages of this method are that it avoids delicate estimates involving \( L^p \) estimates of Cauchy projection operators. Here we consider the mixed NLS equation (1.1) in a weighted Sobolev space \( H^{2,2}(\mathbb{R}) \) to global existence of solutions and provide a long time asymptotic result in classical sense by using inverse scattering theory.

Our article is arranged as follows. In section 2 and section 3, based on the Lax pair of the mixed NLS equation (1.1), we set up a RH problem \( \mathcal{N}(z) \) associated with the initial value problem (1.1)-(1.2), which will be used to analyze long-time asymptotics of the mixed NLS equation (1.1) in our paper. In section 4, by defining a transformation \( \mathcal{N}^{(1)}(z) = \mathcal{N}(z) \delta^{-\sigma_3} \), we get a RH problem whose jump matrices admits two type of up and down triangular decompositions. In section 5, we establish a hybrid \( \bar{\partial} \) problem by the transformation \( \mathcal{N}^{(2)}(z) = \mathcal{N}^{(1)}(z) \mathcal{R}^{(2)} \). The \( \mathcal{N}^{(2)}(z) \) is further decomposed into a pure
RH problem for $N^{rhp}(z)$ and a pure $\overline{\partial}$-problem for $E(x,t,z)$, where the pure RH problem $N^{rhp}(z)$ is a solvable model associated with a Weber equation; the error estimates on the pure $\overline{\partial}$-problem will be given in section 6. Finally, in section 7, according to a series of transformations made above, we establish a relation $N(z) = E(z)N^{rhp}(z)\mathcal{R}^{(2)}(z)^{-1}\delta\sigma_3$, from which the asymptotic behavior of the mixed NLS equation can be obtained.

2 Spectral analysis

The defocusing mixed NLS equation (1.1) admits the following Lax pair

\begin{align}
\psi_x + i\lambda(a\lambda - 2b)\sigma_3\psi &= P_1\psi, \\
\psi_t + 2i\lambda^2(a\lambda - 2b)^2\sigma_3\psi &= P_2\psi,
\end{align}

where

\begin{align}
P_1 &= (\lambda a - b)Q, \\
P_2 &= -i(a^2\lambda^2 - 2ab\lambda + b^2)Q^2\sigma_3 + (2a^2\lambda^3 - 6ab\lambda^2 + 4b^2\lambda)Q \\
&\quad + i(\lambda a - b)Q_x\sigma_3 + (\lambda a^2 - ab)Q^3.
\end{align}

2.1 Asymptotic

Due to $g_0(x) \in H^{2,2}(\mathbb{R})$, so as $x \to \pm\infty$, the Lax pair (2.1)-(2.2) becomes

\begin{align}
\psi_x + i\lambda(a\lambda - 2b)\sigma_3\psi &\sim 0, \\
\psi_t + 2i\lambda^2(a\lambda - 2b)^2\sigma_3\psi &\sim 0,
\end{align}

which implies that the Jost solutions of the Lax pair (2.1)-(2.2) admit asymptotic

\begin{align}
\psi &\sim e^{-it\theta(x,t,\lambda)\sigma_3}, \quad x \to \pm\infty,
\end{align}
where \( \theta(x, t, \lambda) = \lambda (a\lambda - 2b)x/t + 2\lambda^2(a\lambda - 2b)^2 \). Therefore, making transformation
\[
\Psi = \psi e^{i\theta(x,t,\lambda)\sigma_3},
\]
we have
\[
\Psi \sim I, \ x \to \pm \infty,
\]
and \( \Psi \) satisfies a new Lax pair
\[
\begin{align*}
\Psi_x + i\lambda(a\lambda - 2b)[\sigma_3, \Psi] &= P_1 \Psi, \\
\Psi_t + 2i\lambda^2(a\lambda - 2b)^2[\sigma_3, \Psi] &= P_2 \Psi.
\end{align*}
\]

We consider asymptotic expansion
\[
\Psi = \Psi_0 + \frac{\Psi_1}{\lambda} + \frac{\Psi_2}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right), \quad \lambda \to \infty,
\]
where \( \Psi_0, \Psi_1, \Psi_2 \) are independent of \( \lambda \). Substituting (2.9) into (2.7) and comparing the coefficients of \( \lambda \), we obtain that \( \Psi_0 \) is a diagonal matrix and
\[
\begin{align*}
\Psi_1 &= \frac{i}{2} Q \Psi_0 \sigma_3, \\
\Psi_{0x} + i a[\sigma_3, \Psi_2] &= a Q \Psi_1 + b Q \Psi_0.
\end{align*}
\]

In the same way, substituting (2.9) into (2.8) and comparing the coefficients of \( \lambda \) in the same order leads to
\[
\begin{align*}
\Psi_{0x} &= \frac{i}{2} a|q|^2 \sigma_3 \Psi_0, \\
\Psi_{0t} &= \left[ \frac{3}{4} ia^2|q|^4 + \frac{1}{2} a(\bar{q}q_x - \bar{q}_x q) \right] \sigma_3 \Psi_0.
\end{align*}
\]
Noting that the mixed NLS equation (1.1) admits the conservation law
\[
\left( \frac{i}{2} a|q|^2 \right)_t = \left[ \frac{3}{4} ia^2|q|^4 + \frac{1}{2} a(\bar{q}q_x - \bar{q}_x q) \right]_x,
\]

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so two equations (2.12) and (2.13) are compatible if we define

$$\Psi_0(x, t) = e^{ia \int_{-\infty}^{x} |q(x', t)|^2 dx' \sigma_3}. \quad (2.15)$$

We introduce a new function $\mu = \mu(x, t, \lambda)$ by

$$\Psi(x, t, \lambda) = \Psi_0(x, t) \mu(x, t, \lambda), \quad (2.16)$$

then $\mu$ admits the asymptotic

$$\mu = I + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty, \quad (2.17)$$

and satisfies the Lax pair

$$\mu_x + i\lambda(a\lambda - 2b)\sigma_3, \mu = e^{-\frac{ia}{2} \int_{-\infty}^{x} |q|^2(x', t) dx' \sigma_3} H_1 \mu, \quad (2.18a)$$

$$\mu_t + 2i\lambda^2(a\lambda - 2b)^2 \sigma_3, \mu = e^{-\frac{ia}{2} \int_{-\infty}^{x} |q|^2(x', t) dx' \sigma_3} H_2 \mu, \quad (2.18b)$$

where

$$H_1 = P_1 - \frac{ia}{2} |q|^2(x, t) \sigma_3, \quad H_2 = P_2 - \int_{-\infty}^{x} (q \bar{q}' + q' \bar{q}) dx' \sigma_3.$$

### 2.2 Analyticity and symmetry

According to (2.10) and (2.16), we can establish the relationship between the solution $q(x, t)$ of the mixed NLS and $\Psi$ as follows

$$q = 2ie^{ia x} \int_{-\infty}^{x} |q(x', t)|^2 dx' \lim_{\lambda \to \infty} (\lambda \Psi)_{12},$$

which combines with (2.16) gives

$$q(x, t) = 2ie^{ia x} \int_{-\infty}^{x} |q(x', t)|^2 dx' \lim_{\lambda \to \infty} (\lambda \mu)_{12}. \quad (2.19)$$

The Lax pair (2.18a)-(2.18b) can be written into a full derivative form

$$d(e^{i \theta(x, t, \lambda) \tilde{\sigma}_3} \mu) = e^{i \theta(x, t, \lambda) \tilde{\sigma}_3} e^{-\frac{ia}{2} \int_{-\infty}^{x} |q|^2(x', t) dx' \tilde{\sigma}_3} (H_1 dx + H_2 dt) \mu. \quad (2.21)$$
The two solutions of equation (2.21) are

\[
\mu_- = I + \int_{-\infty}^{x} e^{-i\lambda(a\lambda-2b)(x-y)\hat{\sigma}_3} e^{-\frac{ia}{2} \int_{-\infty}^{y} |q|^2(x',t)dx'} \hat{\sigma}_3 H_1(y,t,\lambda) \mu_-(y,t)dy,
\]

(2.22)

\[
\mu_+ = I + \int_{\infty}^{x} e^{-i\lambda(a\lambda-2b)(x-y)\hat{\sigma}_3} e^{-\frac{ia}{2} \int_{-\infty}^{y} |q|^2(x',t)dx'} \hat{\sigma}_3 H_1(y,t,\lambda) \mu_+(y,t)dy.
\]

(2.23)

We define two domains by

\[
D^+ = \{\lambda|(a\text{Re}\lambda - b) \text{Im}\lambda > 0\}, \quad D^- = \{\lambda|(a\text{Re}\lambda - b) \text{Im}\lambda < 0\},
\]

(2.24)

then boundary of the domains \(D^+\) and \(D^-\) is given by

\[
\Sigma = \{\lambda|(a\text{Re}\lambda - b) \text{Im}\lambda = 0\}.
\]

(2.25)

![Figure 1: Analytical domains \(D^+, D^-\) and boundary \(\Sigma\) corresponding to the mixed NLS equation.](image)

We denote \(\mu_\pm = (\mu_{\pm,1}, \mu_{\pm,2})\). Starting from the integrated equation (2.22)-(2.23), by constructing the Neumann series, it follows that

**Proposition 1.** For \(q(x) \in L^1(\mathbb{R})\) and \(t \in \mathbb{R}^+\), the eigenfunctions defined by (2.22) exist and are unique. Moreover, \(\mu_{-,1}\) and \(\mu_{+,2}\) are analytic in \(D^+\); \(\mu_{-,2}\) and \(\mu_{+,1}\) are analytic in \(D^-\).
Again by using transformations (2.5) and (2.16), we have

\[
\mu_-(x, t, \lambda) = \mu_+(x, t, \lambda)e^{-i\theta(x, t, \lambda)\hat{\sigma}_3}S(\lambda).
\]

where

\[
S(\lambda) = \begin{pmatrix}
s_{11}(\lambda) & s_{12}(\lambda) \\
s_{21}(\lambda) & s_{22}(\lambda)
\end{pmatrix}.
\]

Proposition 2. The eigenfunctions \(\mu_{\pm}\) and scattering matrix \(S(\lambda)\) satisfy the symmetry relations

\[
\mu_{\pm}(x, t, \lambda) = \sigma_1 \mu_{\pm}(x, t, \overline{\lambda})\sigma_1, \quad S(\lambda) = \sigma_1 S(\overline{\lambda})\sigma_1, \quad (2.27)
\]

\[
\mu_{\pm}(x, t, \lambda) = -\sigma_* \mu_{\pm}(x, t, 2b/a - \overline{\lambda})\sigma_* , \quad S(\lambda) = -\sigma_* S(2b/a - \overline{\lambda})\sigma_*, \quad (2.28)
\]

with

\[
\sigma_* = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad \sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

3 The construction of a RH problem

We further use eigenfunction \(\mu(x, t, z)\) to set up a RH problem. Define

\[
M(x, t, \lambda) = \begin{cases}
\frac{\mu_{-1}(\lambda)}{s_{11}(\lambda)}; & \lambda \in D^+, \\
\frac{\mu_{+2}}{s_{22}(\lambda)}; & \lambda \in D^-,
\end{cases}
\]

then equation (2.26) leads to the following Riemann-Hilbert problem:

**RH problem 1.** Find a matrix function \(M(x, t, \lambda)\) satisfying

(i) \(M(x, t, \lambda)\) is analytic in \(\mathbb{C}\setminus\Sigma\);

(ii) The boundary value \(M_{\pm}(x, t, \lambda)\) at \(\Sigma\) satisfies the jump condition

\[
M_+(x, t, \lambda) = M_-(x, t, \lambda)V(x, t, \lambda), \quad \lambda \in \Sigma,
\]

and the jump matrix \(V(\lambda)\) is given by

\[
V(x, t, \lambda) = \begin{pmatrix}
1 - r(\lambda)r(\overline{\lambda}) & -r(\lambda)e^{2i\theta(\lambda)} \\
r(\lambda)e^{-2i\theta(\lambda)} & 1
\end{pmatrix}, \quad r(\lambda) = \frac{s_{21}(\lambda)}{s_{11}(\lambda)},
\]

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where
\[ \theta(x, t, \lambda) = 2\lambda^2(a\lambda - 2b)^2 + \frac{x}{t}\lambda(a\lambda - 2b); \] (3.4)

(iii) Asymptotic behavior
\[ M(x, t, \lambda) = I + O(\lambda^{-1}), \quad \text{as} \quad \lambda \to \infty. \] (3.5)

The solution for the initial-value problem (1.1) can be expressed in terms of the RH problem
\[ q(x, t) = 2ie^{i\alpha \int_{-\infty}^{x} |q(x', t)|^2 dx'} \lim_{\lambda \to \infty} [\lambda M(x, t, \lambda)]_{12}. \] (3.6)

We set
\[ m(x, t) = \lim_{\lambda \to \infty} \lambda [\lambda M(x, t, \lambda)]_{12}; \] (3.7)
then from (3.6) and its complex conjugate we obtain
\[ |q|^2 = 4|m|^2. \] (3.8)

Thus (3.6) becomes
\[ q(x, t) = 2ie^{4\alpha \int_{-\infty}^{x} |m|^2 dx'} m(x, t). \] (3.9)

Let \( z = \lambda(a\lambda - 2b) \) and define
\[ N(x, t, z) = (a\lambda - 2b)^{-\frac{i\alpha}{2}} M(x, t, \lambda), \] (3.10)
then we translate the RH problem 1 to a new RH problem:

**RH problem 2.** Find a matrix function \( N(x, t, z) \) satisfying

(i) \( N(x, t, z) \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \),

(ii) The boundary value \( N_{\pm}(x, t, z) \) at \( \Sigma \) satisfies the jump condition
\[ N_{\pm}(x, t, z) = N_{\pm}(x, t, z)V_N, \quad z \in \mathbb{R}, \] (3.11)
where
\[ V_N = e^{-i\theta(z)} \begin{pmatrix} 1 - z\rho_1(z)\rho_2(z) & -\rho_1(z) \\ z\rho_2(z) & 1 \end{pmatrix} \]

\[ \theta(z) = \frac{x}{t} z + 2z^2, \quad (3.12) \]

and two reflection coefficients are given by
\[ \rho_1(z) = \frac{r(\lambda)}{a\lambda - 2b}, \quad \rho_2(z) = \frac{r(\lambda)}{\lambda}. \quad (3.13) \]

(iii) Asymptotic behavior
\[ N(x,t,z) = I + O\left(\frac{1}{z}\right), \quad as \quad z \to \infty. \quad (3.14) \]

From (3.12), we get the stationary point
\[ z_0 = -\frac{x}{4t}, \quad (3.15) \]

and two steepest descent lines
\[ L = \{ z = z_0 + ue^{i\pi/4}, \quad u \geq 0 \} \cup \{ z = z_0 + ue^{5i\pi/4}, \quad u \geq 0 \}, \]
\[ \bar{L} = \{ z = z_0 + ue^{-i\pi/4}, \quad u \geq 0 \} \cup \{ z = z_0 + ue^{3i\pi/4}, \quad u \geq 0 \}. \]

**Remark 4.1.** Comparing with the RH problem 1, the RH problem 2 possesses three special properties: 1) It is a RH problem on a real axis; 2) it possesses two reflection coefficients \( \rho_1 \) and \( \rho_2 \); 3) Its phase factor \( \theta(z) \) has only one stationary point \( z_0 \).

Next based on the RH problem 2, we analyze the asymptotic behavior of the solution of the defocusing mixed NLS equation by using the \( \bar{\partial} \) steepest descent method.
4 Triangular factorizations of jump matrix

We decompose the jump matrix $V_N$ into appropriate upper/lower triangular factorizations which can help us to make continuous extension of the RH problem. It can be shown that the matrix $V_N$ admits the following two triangular factorizations:

$$V_N(z) = \begin{cases} \begin{pmatrix} 1 - \rho_1(z)e^{-2it\theta} & 0 \\ z\rho_2(z)e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv W_LW_R, & z > z_0 \\ \begin{pmatrix} 1 - z\rho_1(z)\rho_2(z) & 0 \\ 1 - z\rho_1(z)\rho_2(z) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv U_LU_0U_R, & z < z_0 \end{cases}$$

which is shown in figure 2.

To remove the diagonal matrix $U_0$ across $(-\infty, z_0]$ in the second factorization, we introduce a scalar RH problem

$$\begin{cases} \delta_+(z) = \delta_-(z)(1 - z\rho_1(z)\rho_2(z)), & z < z_0, \\ \delta_+(z) = \delta_-(z), & z > z_0, \\ \delta(z) \to 1, & \text{as } z \to \infty, \end{cases} \quad (4.1)$$

which has a solution

$$\delta(z) = \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{z_0} \frac{\ln(1 - \xi\rho_1(\xi)\rho_2(\xi))}{\xi - z} d\xi \right] = \exp \left( i \int_{-\infty}^{z_0} \frac{\nu(\xi)}{\xi - z} d\xi \right), \quad (4.2)$$

Figure 2: The jump matrices of $N$. 

To remove the diagonal matrix $U_0$ across $(-\infty, z_0]$ in the second factorization, we introduce a scalar RH problem
where
\[ \nu(z) = -\frac{1}{2\pi} \ln[1 - z\rho_1(z)\rho_2(z)]. \]

We write (4.2) in the form
\[
\delta(z) = \exp \left( i \int_{z_0}^{z} \frac{\nu(z_0)}{\xi - z} d\xi \right) \exp \left( i \int_{-\infty}^{z_0} \frac{\nu(\xi) - \chi(\xi)\nu(z_0)}{\xi - z} d\xi \right)
\]
\[ = (z - z_0)^{i\nu(z_0)} e^{i\beta(z, z_0)}, \tag{4.3} \]

with
\[
\beta(z, z_0) = -\nu(z_0) \log(z - z_0 + 1) + \int_{-\infty}^{z_0} \frac{\nu(\xi) - \chi(\xi)\nu(z_0)}{\xi - z} d\xi, \tag{4.4} \]

where \(\chi\) is the characteristic function of the interval \((z_0 - 1, z_0)\). We choose the branch of the logarithm with \(-\pi < \arg(z) < \pi\) as \(z \to z_0\) for \(z - z_0 = re^{i\phi}\) with \(-\pi < \phi < \pi\) and implied constants independent of \(z_0 \in \mathbb{R}\).

By using (2.26) and (2.27), we get
\[
|r(\lambda)|^2 + 1/|s_{11}(\lambda)|^2 = 1,
\]

which implies that \(|z\rho_1(z)\rho_2(z)| = |r(\lambda)| < 1. In a similar way [30], we can show that

**Theorem 4.1.** Suppose that \(r(\lambda) = z\rho_1(z)\rho_2(z) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})\) and \(\|r\|_{L^\infty} \leq c < 1\), the function \(\delta(z)\) has properties:

(i) \(\delta(z)\) is uniformly bounded in \(z\), namely
\[
(1 - \|r\|_{L^\infty})^{1/2} \leq |\delta(z)| \leq (1 - \|r\|_{L^\infty})^{-1/2}, \tag{4.5}
\]

(ii) \(\delta(z)\overline{\delta(z)} = 1\).

(iii) \(\delta(z)\) admits asymptotic expansion
\[
\delta(z) = 1 + \frac{i}{z} \int_{-\infty}^{z_0} \nu(\xi) d\xi + \mathcal{O}\left(\frac{1}{z^2}\right), \quad z \to \infty. \tag{4.6}
\]
Asymptotics along any ray of the form $z = z_0 + e^{i\phi}R^+$ with $-\pi < \phi < \pi$ as $z \to z_0$

$$|\delta(z) - e^{i\beta(z_0,z_0)}(z_0 - z)^{-i\nu(z_0)}| \lesssim -|z - z_0| \log |z - z_0|. \quad (4.7)$$

Making a transformation

$$N^{(1)} = N^{\delta-\sigma_3}, \quad (4.8)$$

we get the following RH problem.

**RH problem 3.** Find an analytic function $N^{(1)}$ with the following properties:

(i) $N^{(1)}(x, t, z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$;

(ii) The boundary value $N^{(1)}(x, t, z)$ at $\mathbb{R}$ satisfies the jump condition

$$N^{(1)}_+(x, t, z) = N^{(1)}_-(x, t, z)V^{(1)}_N(x, t, z), \quad z \in \mathbb{R}, \quad (4.9)$$

where the jump matrix $V^{(1)}_N(z)$ is given by

$$V^{(1)}_N = \begin{cases} 
\left( \begin{array}{cc} 1 - \rho_1(z)\delta^2 e^{-2it\theta} & 0 \\
0 & 1 
\end{array} \right) \begin{array}{c} \delta^2 e^{2it\theta} \\
1 
\end{array}, & z > z_0, \\
\left( \begin{array}{cc}
z\rho_2(z) & 0 \\
1 & 0 
\end{array} \right) \begin{array}{c} 1 - \rho_1(z) \delta e^{2it\theta} \\
1 
\end{array}, & z < z_0, 
\end{cases} \quad \triangleq \begin{array}{c} GL, \\
GR 
\end{array} \quad (4.10)$$

see Figure 3.

(iii) Asymptotic condition

$$N^{(1)}(x, t, z) = I + O(z^{-1}), \quad as \quad z \to \infty. \quad (4.11)$$

$$\frac{V^{(1)}_N}{z_0} = H_LH_R, \quad \frac{V^{(1)}_N}{z_0} = GLGR \quad \to \mathbb{R}$$

Figure 3: The jump matrix of $N^{(1)}(z)$. 

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5 A hybrid $\bar{\partial}$ problem

We continuously extend the scattering data in the jump matrix $V_N^{(1)}$ by the following way:

1. $z\rho_2(z)$ is extended to the sector $\Lambda_1 : \{ z : \arg z \in (0, \pi/4) \}$,
2. $-\frac{\rho_1(z)}{1-z\rho_1(z)\rho_2(z)}$ is extended to the sector $\Lambda_3 : \{ z : \arg z \in (3\pi/4, \pi) \}$,
3. $\frac{z\rho_2(z)}{1-z\rho_1(z)\rho_2(z)}$ is extended to the sector $\Lambda_4 : \{ z : \arg z \in (\pi, 5\pi/4) \}$,
4. $-\rho_1(z)$ is extended to the sector $\Lambda_6 : \{ z : \arg z \in (7\pi/4, 2\pi) \}$.

**Lemma 5.1.** There exist function $R_j : \Lambda_j \to \mathbb{C}$, $j = 1, 3, 4, 6$ with boundary values

\[
R_1(z) = \begin{cases} 
  z\rho_2(z)\delta(z)^{-2}, & z \in (z_0, \infty), \\
  z_0\rho_2(z_0)\delta_0(z_0)^{-2}(z - z_0)^{-2i\nu}, & z \in \Sigma_1,
\end{cases} \tag{5.1}
\]

\[
R_3(z) = \begin{cases} 
  -\delta^2 \frac{\rho_1(z)}{1-z\rho_1(z)\rho_2(z)}, & z \in (-\infty, z_0), \\
  -\frac{\rho_1(z_0)}{1-z_0\rho_1(z_0)\rho_2(z_0)}\delta_0(z_0)^2(z - z_0)^{2i\nu}, & z \in \Sigma_2,
\end{cases} \tag{5.2}
\]

\[
R_4(z) = \begin{cases} 
  \delta^{-2} \frac{z\rho_2(z)}{1-z\rho_1(z)\rho_2(z)}, & z \in (-\infty, z_0), \\
  \frac{z_0\rho_2(z_0)}{1-z_0\rho_1(z_0)\rho_2(z_0)}\delta_0(z_0)^{-2}(z - z_0)^{-2i\nu}, & z \in \Sigma_3,
\end{cases} \tag{5.3}
\]

\[
R_6(z) = \begin{cases} 
  -\rho_1(z)\delta(z)^2, & z \in (z_0, \infty), \\
  -\rho_1(z_0)\delta_0(z_0)^2(z - z_0)^{2i\nu}, & z \in \Sigma_4,
\end{cases} \tag{5.4}
\]

moreover, $R_j$ admit the following estimates

\[|\bar{\partial}R_j| \leq c_1|z - z_0|^{-\frac{1}{2}} + c_2|p_j'(Rez)|, \quad j = 1, 3, 4, 6,\]

\[\bar{\partial}R_j = 0, \quad j = 2, 5.\]

where

\[
p_1(z) = z\rho_2(z), \quad p_3(z) = -\frac{\rho_1(z)}{1-z\rho_1(z)\rho_2(z)}, \]

\[
p_4(z) = \frac{z\rho_2(z)}{1-z\rho_1(z)\rho_2(z)}, \quad p_6(z) = -\rho_1(z).
\]
Proof. We only give the proof for $R_1$, the others can be proved in a similar way. Define $f_1(z)$ on $\Lambda_1$ by

$$f_1(z) = p_1(z_0)\delta_0(z_0)^{-2}(z - z_0)^{-2i\nu(z_0)}\delta(z)^2,$$

(5.5)

Denote $z = u + iv = z_0 + \varrho e^{i\varphi}$, then we have

$$\varrho = |z - z_0|, \quad \overline{\partial} = \frac{1}{2}e^{i\varphi}(\partial_{\varrho} + i\varrho^{-1}\partial_{\varphi}).$$

(5.6)

Define

$$R_1(u, v) = p_1(u)\cos 2\varphi + (1 - \cos 2\varphi)f_1(u + iv),$$

(5.7)

it follows from (5.1) that

$$|\overline{\partial}R_1| \leq c_1 ||p_1 - p_1(z_0)|| + |p_1(z_0) - f_1| + c_2|p_1'(u)|.$$  

(5.9)

While

$$|p_1(z) - p_1(z_0)| = \left| \int_{z_0}^{z} p_1'(s)ds \right| \leq ||p_1||_{L^2((z,z_0))} \cdot |z - z_0|^{1/2}.$$  

(5.10)

From (5.5), we have

$$p_1(z_0) - f_1 = p_1(z_0) - p_1(z_0) \exp[2i\nu((z - z_0) \ln(z - z_0) - (z - z_0 + 1) \ln(z - z_0 + 1))]$$

$$\times \exp[2(\beta(z, z_0) - \beta(z_0, z_0))].$$

Noticing that in $\Lambda_1$,

$$|\beta(z, z_0) - \beta(z_0, z_0)| = O(\sqrt{z - z_0}),$$

and

$$|\nu(z_0) \ln(z - z_0)| \leq O(\sqrt{z - z_0}).$$
Therefore
\[ |p_1(z_0) - f_1| = p_1(z_0)\{1 - \exp[O(\sqrt{z - z_0})]\} \]
\[ = p_1(z_0)\{O(\sqrt{z - z_0})\}. \]  \hspace{1cm} (5.11)

Combining these estimates yields
\[ |\bar{\partial}R_1| \leq c_1|z - z_0|^{-\frac{1}{2}} + c_2|p'_1|. \]  \hspace{1cm} (5.12)

We use \( R_j \) obtained above to define a new unknown function
\[ N^{(2)} = N^{(1)}(z)R^{(2)}(z) \]  \hspace{1cm} (5.13)

where
\[ R^{(2)} = \left\{ \begin{array}{ll}
\begin{pmatrix}
1 & 0 \\
R_1e^{2it\theta} & 1
\end{pmatrix}, & z \in \Lambda_1, \\
\begin{pmatrix}
1 & -R_3e^{-2it\theta} \\
0 & 1
\end{pmatrix}, & z \in \Lambda_3, \\
\begin{pmatrix}
1 & 0 \\
R_4e^{2it\theta} & 1
\end{pmatrix}, & z \in \Lambda_4, \\
\begin{pmatrix}
1 & -R_6e^{-2it\theta} \\
0 & 1
\end{pmatrix}, & z \in \Lambda_6, \\
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, & z \in \Lambda_2 \cup \Lambda_5,
\end{array} \right. \]  \hspace{1cm} (5.14)

which is shown in Figure 4.
\[ \mathcal{R}^{(2)} = \begin{pmatrix} 1 & -2 \delta_0(z_0)^2 (z - z_0) - 2i \nu \\ 1 & 0 \end{pmatrix}, \quad z \in \Sigma_1^{(2)}, \]

\[ e^{-2i \theta} \mathcal{R}^{(2)} = \begin{pmatrix} 1 & \frac{1}{1 - \rho_1(z_0) \rho_2(z_0)} \delta_0^2 (z - z_0)^2 + 2i \nu \\ 0 & 1 \end{pmatrix}, \quad z \in \Sigma_2^{(2)}, \]

\[ e^{-2i \theta} \mathcal{R}^{(2)} = \begin{pmatrix} 1 & \frac{1}{1 - \rho_1(z_0) \rho_2(z_0)} \delta_0 (z - z_0)^2 + 2i \nu \\ 0 & 1 \end{pmatrix}, \quad z \in \Sigma_3^{(2)}, \]

\[ e^{-2i \theta} \mathcal{R}^{(2)} = \begin{pmatrix} 1 & -\rho_1(z_0) \delta_0^2 (z - z_0)^2 + 2i \nu \\ 0 & 1 \end{pmatrix}, \quad z \in \Sigma_4^{(2)}, \]

Figure 4: \( \mathcal{R}^{(2)} \) in each \( \Lambda_j \).

Making \( \bar{\partial} \)-differential on the equation (5.13) and noting that \( N^{(2)} \) is analytic in \( \Lambda_j, \ j = 1, 3, 4, 6 \), we have

\[ \bar{\partial} N^{(2)} = N^{(2)} \mathcal{R}^{(2)} - 1 \mathcal{R}^{(2)} = N^{(2)} \mathcal{R}^{(2)}. \]  

(5.15)

Let \( \Sigma^{(2)} = \bigcup_{j=1}^{4} \Sigma_j^{(2)} \). It can be shown that \( N^{(2)} \) satisfies the following \( \bar{\partial} \)-RH problem.

**RH problem 4.** Find a function \( N^{(2)} \) with the following properties.

(i) \( N^{(2)}(x, t, z) \) is continuous in \( \mathbb{C} \setminus \Sigma^{(2)} \);

(ii) The boundary value \( N^{(2)}(x, t, z) \) at \( \Sigma^{(2)} \) satisfies the jump condition

\[ N^{(2)}_+(x, t, z) = N^{(2)}_-(x, t, z) V^{(2)}_N(x, t, z), \quad z \in \Sigma^{(2)}, \]  

(5.16)

where the jump matrix \( V^{(2)}_N(z) \) is defined by (see Figure 5)

\[ V^{(2)}_N = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
-z_0 \rho_2(z_0)^2 (z - z_0)^2 + 2i \nu & 1 
\end{pmatrix}, & z \in \Sigma_1^{(2)}, \\
\begin{pmatrix} 1 & \frac{1}{1 - \rho_1(z_0) \rho_2(z_0)^2} \delta_0^2 (z - z_0)^2 + 2i \nu \\
0 & 1 
\end{pmatrix}, & z \in \Sigma_2^{(2)}, \\
\begin{pmatrix} 1 & \frac{1}{1 - \rho_1(z_0) \rho_2(z_0)^2} \delta_0 (z - z_0)^2 + 2i \nu \\
0 & 1 
\end{pmatrix}, & z \in \Sigma_3^{(2)}, \\
\begin{pmatrix} 1 & -\rho_1(z_0) \delta_0^2 (z - z_0)^2 + 2i \nu \\
0 & 1 
\end{pmatrix}, & z \in \Sigma_4^{(2)}, 
\end{cases} \]  

(5.17)
Figure 5: The jump matrices $V_N^{(2)}$ for $N^{(2)}$, $\partial R^{(2)} \neq 0$ in pink domain; and $\partial R^{(2)} = 0$ in white domain

(iii) Asymptotic condition

$$N^{(2)}(x, t, z) = I + O(z^{-1}), \quad \text{as} \quad z \to \infty. \quad (5.18)$$

(iv) Away from $\Sigma^{(2)}$, we have

$$\overline{\partial N^{(2)}} = N^{(2)}\overline{\partial R^{(2)}}, \quad z \in \mathbb{C} \setminus \Sigma^{(2)}, \quad (5.19)$$

where

$$\overline{\partial R^{(2)}} = \begin{cases} \left( \begin{array}{cc} 0 & 0 \\ -\partial R_1 e^{2it\theta} & 1 \end{array} \right), & z \in \Lambda_1, \\ \left( \begin{array}{cc} 0 & -\partial R_3 e^{-2it\theta} \\ 0 & 0 \end{array} \right), & z \in \Lambda_3, \\ \left( \begin{array}{cc} 0 & 0 \\ -\partial R_4 e^{2it\theta} & 0 \end{array} \right), & z \in \Lambda_4, \\ \left( \begin{array}{cc} 0 & -\partial R_6 e^{-2it\theta} \\ 0 & 0 \end{array} \right), & z \in \Lambda_6, \\ \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), & \text{otherwise}. \end{cases} \quad (5.20)$$

Diagrammatically, the jump matrices are as in Figure 5.
In order to solve the RH problem 4, we decompose it into a solvable RH Problem for $N^{\text{rhp}}(x, t, z)$ with $\overline{\partial}R^{(2)} = 0$ and a pure $\overline{\partial}$-Problem $E(x, t, z)$ with $\overline{\partial}R^{(2)} \neq 0$. The pure RH problem for $N^{\text{rhp}}(z)$ is a solvable model associated with a Weber equation, which will be analyzed in next section 6; The error estimates on the pure $\overline{\partial}$-problem for $E(x, t, z)$ will given in In section 7.

6 Analysis on a solvable model

The hybrid RH problem $N^{(2)}(x, t, z)$ with $\overline{\partial}R^{(2)} = 0$ leads to the following pure RH problem for the $M^{\text{rhp}}(x, t, z)$.

**RH problem 5.** Find a matrix-valued function $N^{\text{rhp}}(x, t, z)$ with following properties:

(i) Analyticity: $N^{\text{rhp}}(z)$ is analytical in $\mathbb{C} \setminus \Sigma^{(2)}$,

(ii) Jump condition:

$$N^{\text{rhp}}_+(z) = N^{\text{rhp}}_-(z)V^{(2)}_N(z), \quad z \in \Sigma^{(2)},$$

where $V^{(2)}_N(z)$ is given by (5.17).

(iii) Asymptotic condition

$$N^{\text{rhp}}(z) = I + O(z^{-1}), \quad \text{as} \quad z \to \infty.$$ 

Making a scaling transformation

$$N^{\text{sol}}(\zeta) = N^{\text{rhp}}(z) \bigg|_{z = \zeta / \sqrt{8t + z_0}},$$

$$\rho_2(z_0) = \rho_{20} \delta_0^2 e^{-2i\nu(z_0) \ln \sqrt{8t} e^{4iz_0^2}},$$

$$\rho_1(z_0) = \rho_{10} \delta_0^{-2} e^{2i\nu(z_0) \ln \sqrt{8t} e^{-4iz_0^2}},$$

then the RH problem 5 is changed into the following RH problem.
**RH problem 6.** Find a matrix-valued function $N^{sol}(\zeta)$ with the following properties:

(i) $N^{sol}(\zeta)$ is analytic on $\zeta \in \mathbb{C}\setminus\Sigma^{(2)}$,

(ii) The boundary value $N^{sol}(\zeta)$ satisfies the jump condition

$$N^+_\zeta(\zeta) = N^-_{\zeta}(\zeta)V^{(2)}_N(\zeta), \quad \zeta \in \Sigma^{(2)},$$

(6.6)

where

$$V^{(2)}_N(\zeta) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
\frac{1}{z_0 \rho_20} \zeta^{-2i\nu} e^{i\frac{\zeta^2}{2}} & 1 
\end{pmatrix}, & \zeta \in \Sigma_1, \\
\begin{pmatrix} 1 & 0 \\
-\frac{\rho_{10}}{1-z_0 \rho_{10} \rho_20} \zeta^{-2i\nu} e^{-i\frac{\zeta^2}{2}} & 1 
\end{pmatrix}, & \zeta \in \Sigma_2, \\
\begin{pmatrix} 1 & 0 \\
\frac{1}{1-z_0 \rho_{10} \rho_20} \zeta^{-2i\nu} e^{i\frac{\zeta^2}{2}} & 1 
\end{pmatrix}, & \zeta \in \Sigma_3, \\
\begin{pmatrix} 1 & 0 \\
-\rho_{10} \zeta^{-2i\nu} e^{-i\frac{\zeta^2}{2}} & 1 
\end{pmatrix}, & \zeta \in \Sigma_4.
\end{cases}$$

(6.7)

(iii) Asymptotic condition

$$N^{sol}(\zeta) = I + \frac{N^1_{\zeta^{sol}(\zeta)}}{\zeta} + O(\zeta^{-1}) \quad \text{as } \zeta \to \infty,$$

(6.8)
The contour $\Sigma^{(2)}$ and real axis $\mathbb{R}$ divide complex plane $\mathbb{C}$ into six different domains $\Lambda_j$, $j = 1, \ldots, 6$ which are shown in Figure 6. Let

$$N^{sol}(\zeta) = \vartheta(\zeta) P_0 e^{i \frac{2}{4 \sigma_3} \zeta^{-i \nu(z_0)} \sigma_3},$$

(6.9)

where

$$P_0 = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\rho_{10} & 1 \end{pmatrix}, & \zeta \in \Lambda_1 \\ \begin{pmatrix} 1 & -z_0 \rho_{20} \\ -z_0 \rho_{10} \rho_{20} & 1 \end{pmatrix}, & \zeta \in \Lambda_3 \\ \begin{pmatrix} 1 & 0 \\ -\rho_{10} \rho_{20} & 1 \end{pmatrix}, & \zeta \in \Lambda_4 \\ \begin{pmatrix} 1 & -\rho_{10} \rho_{20} \\ 0 & 1 \end{pmatrix}, & \zeta \in \Lambda_6 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \zeta \in \Lambda_2 \cup \Lambda_5. \end{cases}$$

(6.10)

It is easy to check that $\vartheta$ satisfies the following RH problem, see [35].
**RH problem 7.** Find a matrix-valued function \( \vartheta(\zeta) \) with the following properties

(i) \( \vartheta \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \),

(ii) The boundary value \( \vartheta(\zeta) \) at \( \mathbb{R} \) satisfies the jump condition

\[
\vartheta_{+}(\zeta) = \vartheta_{-}(\zeta)V(0), \quad \zeta \in \mathbb{R}, \quad (6.11)
\]

where

\[
V(0) = \begin{pmatrix}
1 - z_0 \rho_1(z_0) \rho_2(z_0) & -\rho_1(z_0) \\
\rho_2(z_0) & 1
\end{pmatrix}.
\]

(iii) Asymptotic condition

\[
\vartheta e^{\frac{i\kappa^2}{\pi} \sigma_3 \zeta^{-i\nu(z_0)} \sigma_3} \to I, \quad \text{as} \quad \zeta \to \infty. \quad (6.12)
\]

This kind of RH problem can be changed into a Werber equation to get the solution in terms of parabolic cylinder functions.

**Proposition 3.** The solution to the RH problem 6 is given by \( \vartheta_{+}(\zeta) \) and \( \vartheta_{-}(\zeta) \), defined in \( \mathbb{C}_+ \) and \( \mathbb{C}_- \) respectively. For \( \zeta \in \mathbb{C}_+ \),

\[
(\vartheta_{+})_{11}(\zeta) = e^{-\frac{3\pi \nu}{4}} D_{i\nu}(e^{-\frac{3\pi}{4} \zeta}), \\
(\vartheta_{+})_{12}(\zeta) = e^{\frac{\pi}{4}} (\beta_{21})^{-1}[\partial_{\zeta} D_{-i\nu}(e^{-\frac{3\pi}{4} \zeta}) - \frac{i\zeta}{2} D_{-i\nu}(e^{-\frac{3\pi}{4} \zeta})], \\
(\vartheta_{+})_{21}(\zeta) = e^{-\frac{3\pi \nu}{4}} (\beta_{12})^{-1}[\partial_{\zeta} D_{i\nu}(e^{-\frac{3\pi}{4} \zeta}) + \frac{i\zeta}{2} D_{i\nu}(e^{-\frac{3\pi}{4} \zeta})], \\
(\vartheta_{+})_{22}(\zeta) = e^{\frac{\pi}{4}} D_{-i\nu}(e^{-\frac{3\pi}{4} \zeta}).
\]

For \( \zeta \in \mathbb{C}_- \),

\[
(\vartheta_{-})_{11}(\zeta) = e^{\frac{\pi \nu}{4}} D_{i\nu}(e^{\frac{\pi}{4} \zeta}), \\
(\vartheta_{-})_{12}(\zeta) = e^{-\frac{3\pi \nu}{4}} (\beta_{21})^{-1}[\partial_{\zeta} D_{-i\nu}(e^{-\frac{3\pi}{4} \zeta}) - \frac{i\zeta}{2} D_{-i\nu}(e^{-\frac{3\pi}{4} \zeta})], \\
(\vartheta_{-})_{21}(\zeta) = e^{\frac{\pi}{4}} (\beta_{12})^{-1}[\partial_{\zeta} D_{i\nu}(e^{\frac{3\pi}{4} \zeta}) + \frac{i\zeta}{2} D_{i\nu}(e^{\frac{3\pi}{4} \zeta})], \\
(\vartheta_{-})_{22}(\zeta) = e^{-\frac{3\pi \nu}{4}} D_{-i\nu}(e^{\frac{3\pi}{4} \zeta}).
\]
where
\[ \beta_{12} = \frac{(2\pi)^{\frac{1}{2}} e^{\frac{i\pi}{4}} e^{-\frac{3i\pi}{4}}}{z_0 \rho_2 \Gamma(-i\nu)}, \quad \beta_{21} = \frac{\nu}{\beta_{12}}, \]
(6.15)
and \( D_a(\xi) = D_a(e^{-\frac{3i\pi}{4}} \zeta) \) is a solution of the Weber equation
\[ \partial_{\xi}^2 D_a(\xi) + \left[ \frac{1}{2} - \frac{\xi^2}{4} + a \right] D_a(\xi) = 0. \]

Finally, by using the proposition and (6.9), we then get
\[ N_{\text{sol}} = I + \frac{N_{1\text{sol}}}{\zeta} + \mathcal{O}(\zeta^{-2}), \]
where
\[ N_{1\text{sol}} = \begin{pmatrix} 0 & -i\beta_{12} \\ i\beta_{21} & 0 \end{pmatrix}. \]
(6.16)

7 Analysis on a pure \( \bar{\partial} \) problem

Suppose that \( N^{(2)} \) is a solution of the RH problem 4, then the ratio
\[ E(z) = N^{(2)}(z) N^{rhp}(z)^{-1}, \]
will have no jumps in the plane, but is a solution of the following problem.

**RH problem 8.** Find a function \( E(z) \) with the following properties:

(i) \( E(z) \) is continuous in \( \mathbb{C} \),

(ii) Asymptotic condition
\[ E(z) = I + O(z^{-1}), \quad \text{as} \quad z \to \infty. \]
(7.2)

(iii) \( E(x, t, z) \) satisfies the \( \bar{\partial} \)-equation
\[ \bar{\partial} E = EW(z), \]
(7.3)
where

\[ W(z) = \begin{cases} 
N_{rhp}(z) \begin{pmatrix} 0 & -iR_1 e^{2i\theta} \delta^{-2} & 0 \\
n \partial R_1 e^{2i\theta} \delta^{-2} & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix} N_{rhp}(z)^{-1}, & z \in \Lambda_1, \\
N_{rhp}(z) \begin{pmatrix} 0 & -iR_3 e^{-2i\theta} \delta^2 \\
n \partial R_3 e^{-2i\theta} \delta^2 & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix} N_{rhp}(z)^{-1}, & z \in \Lambda_3, \\
N_{rhp}(z) \begin{pmatrix} 0 & -iR_4 e^{2i\theta} \delta^{-2} & 0 \\
n \partial R_4 e^{2i\theta} \delta^{-2} & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix} N_{rhp}(z)^{-1}, & z \in \Lambda_4, \\
N_{rhp}(z) \begin{pmatrix} 0 & -iR_6 e^{-2i\theta} \delta^2 \\
n \partial R_6 e^{-2i\theta} \delta^2 & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix} N_{rhp}(z)^{-1}, & z \in \Lambda_6, \\
\begin{pmatrix} 0 & 0 \\
0 & 0 
\end{pmatrix}, & \text{otherwise.} 
\end{cases} \] (7.4)

**Proof.** Noticing that \( N_{rhp}(z) \) is holomorphic in \( \mathbb{C} \setminus \Sigma^{(2)} \), by using (5.19), direct calculation shows that

\[ \overline{\partial} E = \overline{\partial} N^{(2)}(N^{PC})^{-1} = N^{(2)} \overline{\partial} R^{(2)}(N^{PC})^{-1} = \{ N^{(2)}(N^{PC})^{-1} \}{N^{PC} \overline{\partial} R^{(2)}(N^{PC})^{-1}} = EW(x, t, z), \]

where \( W(x, t, z) \) is given by (7.4). \( \square \)

The solution of the \( \overline{\partial} \)-equation (7.3) can be expressed by the following integral

\[ E = I - \frac{1}{\pi} \int \int \frac{EW}{s - z} \ dA(s), \] (7.5)

which can be written in operator equation

\[ (I - J)E = I, \] (7.6)

where

\[ J(E) = -\frac{1}{\pi} \int \int \frac{EW}{s - z} \ dA(s). \] (7.7)
In order to show the solvability of (7.7), we prove that $J$ is small norm as $t \to \infty$.

**Proposition 4.** As $t \to \infty$, we have the following estimate

$$||J||_{L^\infty \rightarrow L^\infty} \leq ct^{-1/4}. \quad (7.8)$$

**Proof.** We give the details for sector $\Lambda_1$ only as the corresponding arguments for other sectors which are identical with appropriate modifications. Let $f \in L^\infty(\Lambda_1)$, then

$$|J(f)| \leq \int \int_{\Lambda_1} \frac{|f\delta R_1 \delta^{-2} e^{2it\theta}|}{|s-z|} dA(s)$$

$$\leq ||f||_{L^\infty(\Lambda_1)} ||\delta^{-2}||_{L^\infty(\Lambda_1)} \int \int_{\Lambda_1} \frac{|\delta R_1 e^{2it\theta}|}{|s-z|} dA(s)$$

$$\leq I_1 + I_2, \quad (7.9)$$

where

$$I_1 = \int \int_{\Lambda_1} \frac{|p_1'| e^{-4tv(u-z_0)}}{|s-z|} dA(s), \quad (7.10a)$$

$$I_2 = \int \int_{\Lambda_1} \frac{|s-z_0|^{-1/2} e^{-4tv(u-z_0)}}{|s-z|} dA(s). \quad (7.10b)$$

Since $\rho_1(z), \rho_2(z) \in H^{2,2}(\mathbb{R})$ and we set $s = u + iv, z = \alpha + i\tau$, then

$$I_1 \leq \int_0^{\infty} \int_v^{\infty} \frac{|p_1'| e^{-4tv(u-z_0)}}{|s-z|} dudv$$

$$\leq \int_0^{\infty} e^{-4tv^2} \int_v^{\infty} \frac{|p_1'|}{|s-z|} dudv$$

$$\leq ||p_1'(u)||_{L^2(\mathbb{R})} \int_0^{\infty} e^{-4tv^2} ||\frac{1}{s-z}||_{L^2((v,\infty))} dv. \quad (7.11)$$

Moreover,

$$||\frac{1}{s-z}||_{L^2((v,\infty))} \leq \left(\int_0^{\infty} \frac{1}{|s-z|^2} du\right)^{1/2} \leq \left(\frac{\pi}{|v-\tau|}\right)^{1/2}, \quad (7.12)$$
Thus we have

\[ |I_1| \leq C_1 \int_0^\infty \frac{e^{-4tv^2}}{\sqrt{\tau - v}} \, dv = C_1 \left[ \int_0^\tau \frac{e^{-4tv^2}}{\sqrt{\tau - v}} \, dv + \int_\tau^\infty \frac{e^{-4tv^2}}{\sqrt{\tau - v}} \, dv \right]. \tag{7.13} \]

Using the fact \( \sqrt{\tau} e^{-4t\tau^2} \leq ct^{-1/4}p^{-1/2} \), we obtain

\[ \int_0^\tau \frac{e^{-4tv^2}}{\sqrt{\tau - v}} \, dv \leq \int_0^1 \sqrt{\tau} \frac{e^{-4t\tau^2}p^2}{\sqrt{1 - p}} \, dp \leq ct^{-1/4} \int_0^1 \frac{1}{\sqrt{p(1 - p)}} \, dp \leq c_1 t^{-1/4}, \tag{7.14} \]

whereas using the variable substitution \( w = v - \tau \),

\[ \int_\tau^\infty \frac{e^{-4tv^2}}{\sqrt{v - \tau}} \, dv \leq \int_0^\infty \frac{e^{-4tw^2}}{\sqrt{w}} \, dw. \tag{7.15} \]

According to \( e^{-tw^2}w^{5/2} \leq ct^{-1/4} \), it becomes

\[ \int_\tau^\infty \frac{e^{-4tw^2}}{\sqrt{v - \tau}} \, dv \leq c_2 t^{-1/4}. \tag{7.16} \]

Hence the final estimation is

\[ |I_1| \leq C_2 t^{-1/4}. \tag{7.17} \]

To arrive at a similar estimate for \( I_2 \), we start with the following \( L^p \)-estimate for \( p > 2 \).

\[ \| |s - z_0|^{-1/2} \|_{L^p(du)} = \left( \int_{z_0 + v}^\infty \frac{1}{|u + iv - z_0|^{p/2}} \, du \right)^{1/p} \]

\[ = \left( \int_v^\infty \frac{1}{|u + iv|^{p/2}} \, du \right)^{1/p} \]

\[ = \left( \int_v^\infty \frac{1}{(u^2 + v^2)^{p/4}} \, du \right)^{1/p} \]

\[ = v^{(1/p - 1/2)} \left( \int_{v/4}^{\pi/4} (\cos x)^{p/2 - 2} \, dx \right)^{1/p} \]

\[ \leq c v^{(1/p - 1/2)} \]

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Similarly to the $L^2$-estimate above, we obtain for $L^q$ with $1/p + 1/q = 1$.

$$|| \frac{1}{s-z} ||_{L^p(v, \infty)} \leq c | v - \tau |^{1/q-1}. \quad (7.19)$$

It follows that

$$|I_2| \leq c \left[ \int_0^T e^{-4tv^2} v^{(1/p-1/2)} |v - \tau|^{1/q-1} dv + \int_\tau^\infty e^{-tv^2} v^{(1/p-1/2)} |v - \tau|^{1/q-1} dv \right]. \quad (7.20)$$

By using the fact $\sqrt{\tau} e^{-4t\tau^2 w^2} \leq ct^{-1/4} w^{-1/2}$, the first integral yields

$$\int_0^T e^{-4tv^2} v^{(1/p-1/2)} |v - \tau|^{1/q-1} dv \leq ct^{-1/4}. \quad (7.21)$$

Let $v = \tau + w$, the estimate for the second integral $I_2$ leads to

$$\int_0^\infty e^{-4t(\tau+w)^2} (\tau + w)^{(1/p-1/2)} w^{1/q-1} dw \leq \int_0^\infty e^{-tw^2} w^{-1/2} dw. \quad (7.22)$$

Then making use of the variable substitution $y = \sqrt{t}w$ yields

$$\int_\tau^\infty e^{-tv^2} v^{(1/p-1/2)} |v - \tau|^{1/q-1} dv \leq ct^{-1/4}. \quad (7.23)$$

Combining the previous estimates of $I_1$ in (7.21), the final result is described below

$$|I_2| \leq ct^{-1/4}. \quad (7.24)$$

Finally, combing (7.21) and (7.24) gives

$$|J| \leq ct^{-1/4}. \quad (7.25)$$

□

**Proposition 5.** For sufficiently large $t$, the integral equation (7.7) may be inverted by Neumann series. Its asymptotic expression is

$$E(z) = I + O(t^{-1/4}). \quad (7.26)$$

After proving the existence and analyzing the asymptotic result of $E(z)$, our goal is to establish the relation between $E(z)$ and $N(z)$, and to obtain the asymptotic behavior of $N(z)$ by using the estimation (7.26).
8 Long time asymptotics of the mixed NLS

Recall transformations (4.8), (5.13) and (7.1), we get

\[ N(z) = E(z) N^{rhp}(z) R^{(2)}(z)^{-1} \delta^{\sigma_3}. \] (8.1)

Making the Laurent expansions

\[
N(z) = I + \frac{N_1}{z} + o(z^{-1}), \\
E(z) = I + \frac{E_1}{z} + o(z^{-1}), \\
N^{rhp}(z) = I + \frac{N_{sol}^{rhp}}{\sqrt{8t}(z-z_0)} + o(z^{-1}), \\
\delta(z)^{\sigma_3} = I + \frac{\delta_1 \sigma_3}{z} + o(z^{-1}),
\]

and taking \( z \to \infty \) in vertical direction in \( \Lambda_2, \Lambda_5 \), the formula (8.1) gives

\[ N_1 = \frac{1}{\sqrt{8t}} N_{sol}^{rhp} + E_1 + \delta_1 \sigma_3. \] (8.3)

Next we evaluate the decay rate of the matrix \( E_1 \). From the integral equation (7.26) satisfied by \( E \), we have

\[ E_1 = \frac{1}{\pi} \int \int EWdA(s), \] (8.4)

which can be divided into two parts for calculation

\[
|E_1| \leq \int_0^\infty \int_{v+z_0}^{\infty} e^{-4tv(u-z_0)}|p'|dudv + \int_0^\infty \int_{v+z_0}^{\infty} e^{-4tv(u-z_0)}|s-z_0|^{-1/2}dudv
\]

\[ = I_3 + I_4. \] (8.5)

By using the Cauchy-Schwarz inequality, we have

\[ I_3 \leq c \int_0^\infty \left( \int_v^\infty e^{-4tv} du \right)^{1/2} dv \leq ct^{-3/4}. \] (8.6)
Similarly, by the Hölder inequality for $1/p + 1/q = 1$, $2 < p < 4$, we let $w = \sqrt{tv}$ and get

$$I_4 \leq \frac{c}{t^{1/q}} \int_0^\infty v^{2/p-3/2} e^{-4tv^2} dv \leq ct^{-3/4} \int_0^\infty w^{2/p-3/2} e^{-4w^2} dw \leq ct^{-3/4} \leq ct^{-3/4}. \tag{8.7}$$

Substituting (8.6) and (8.7) into (8.5) yields

$$|E_1| \leq ct^{-3/4}. \tag{8.8}$$

Recall the formula (3.9), we know

$$q(x, t) = 2ie^{4ia \int_{-\infty}^x |m(x', t)|^2 dx'} m(x, t), \tag{8.9}$$

which implies that we just make an estimate on the $m(x, t)$ to obtain the asymptotic of the potential $q(x, t)$.

From (3.7), (3.10), (8.3) and (8.8), we have

$$m(x, t) = \lim_{\lambda \to \infty} (\lambda M)_{12} = \lim_{z \to \infty} (z N)_{12} = (N_1)_{12}
= \frac{\beta_{12}}{i\sqrt{8t}} + O(t^{-3/4}) = \frac{1}{\sqrt{2t}} \alpha(z_0) e^{i(4tz_0^2 - \nu(z_0) \log 8t)} + O(t^{-3/4}), \tag{8.10}$$

which leads to

$$|m(x, t)|^2 \sim \frac{\nu}{8t}, \quad t \to \infty.$$  

Further calculating gives

$$\int_{-\infty}^x |m(x', t)|^2 dx' \sim \int_{-\infty}^x \frac{\nu}{8t} dx' = -\frac{1}{2\pi} \int_{\lambda_0}^{+\infty} \log(1 - |r(\lambda)|^2) (a\lambda - b) d\lambda. \tag{8.11}$$

with $z_0 = \lambda_0 (a\lambda_0 - 2b)$. Finally, substituting (8.10) and (8.11) into (8.9) yields

$$q(x, t) = \frac{1}{\sqrt{t}} \alpha(z_0) e^{i(4tz_0^2 - \nu(z_0) \log 8t)} e^{-\frac{2ia}{\lambda_0} \int_{\lambda_0}^{+\infty} \log(1 - |r(\lambda)|^2) (a\lambda - b) d\lambda} + O(t^{-\frac{3}{4}}), \tag{8.12}$$

which leads to our main results.

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**Theorem 8.1.** For initial data $q_0(x) \in H^{2,2}(\mathbb{R})$ in the Sobolev space with reflection coefficient $\rho_1(z), \rho_2(z) \in H^{2,2}(\mathbb{R})$, then long-time asymptotic of the solution for the mixed NLS equation (1.1) is given by

$$q(x,t) = \frac{1}{\sqrt{t}} \alpha(z_0) e^{i(\sqrt{2}t - \nu(z_0) \log 8t)} e^{i\pi \frac{1}{2} \int_{z_0}^{\infty} \log(1-|r(\lambda)|^2)(a\lambda - b) d\lambda} + O(t^{-3/4}),$$

where

$$\alpha(z_0) = \frac{1}{(a\lambda_0 - 2b) r(\lambda_0) \Gamma(-a)} e^{\frac{i\pi}{4}},$$

whose modulus is

$$|\alpha(z_0)|^2 = \frac{1}{4\pi} \ln(1 - |r(\lambda_0)|^2),$$

and angle is

$$\arg \alpha(z_0) = \frac{1}{\pi} \int_{-\infty}^{z_0} \log |z_0 - \lambda(a\lambda - 2b)| d\log(1 - |r(\lambda)|^2) + \frac{\pi}{4} - \arg[(a\lambda_0 - 2b) r(\lambda_0)] + \arg \Gamma(i\nu).$$

**Acknowledgements**

This work is supported by the National Natural Science Foundation of China (Grant No. 11671095, 51879045).

**References**

[1] M. Wadati, K. Konno and Y. H. Ichikawa, A Generalization of Inverse Scattering Method, *J. Phys. A*, 36(1979), 1965-1966.

[2] T. Kawata, J. Sakai and N. Kobayashi, Inverse method for the mixed nonlinear schrödinger-equation and solution-solutions, *J. Phys. Soc. Jpn.*, 48(1980), 1371-1379.

[3] M. Stiassnie, Note on the modified nonlinear schrödinger equation for deep water waves, *Wave Motion*, 6(1984), 431-433.
[4] K. Mio, T. Ogino, K. Minami and S. Takeda, Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas, *J. Phys. Soc. Jpn.*, 41(1976), 265-271.

[5] G. P. Agrawal, Nonlinear Fiber Optics, 4th ed., Academic Press, Boston, 2007.

[6] J. K. Yang, Nonlinear Waves in Integrable and Nonintegrable Systems, SIAM, Philadelphia, 2010.

[7] H. Nakatsuka, D. Grischkowsky and A. C. Balant, Nonlinear picosecond-Pulse propagation through optical fibers with positive group velocity dispersion, *Phys. Rev. Lett.*, 47(1981), 910-913.

[8] N. Tzoar and M. Jain, Self-phase modulation in long-geometry optical waveguides, *Phys. Rev. A*, 23(1981), 1266-1270.

[9] L. Brizhik, A. Eremko and B. Piette, Solutions of a D-dimensional modified nonlinear Schrödinger equation, *Nonlinearity*, 13(2003), 1481-1497.

[10] G. Biondini, G. Kovacic, Inverse scattering transform for the focusing nonlinear Schrodinger equation with nonzero boundary conditions, *Journal of Mathematical Physics*, 55(2014), 339-351.

[11] F. Demontis, B. Prinari, C. Van Der Mee, The inverse scattering transform for the defocusing nonlinear Schrodinger equations with nonzero boundary conditions, *Studies in Applied Mathematics*, 131(2013), 1-40.

[12] D. J. Kaup, A. C. Newell, An exact solution for a derivative nonlinear Schrodinger equation, *Journal of Mathematical Physics*, 19(1978), 798-801.
[13] A. I. Maimistov, Evolution of solitary waves which are approximately solitons of a nonlinear Schrödinger equation, *J. Exp. Theor. Phys.*, 77(1993), 727-731.

[14] A. Roy Chowdhury, S. Paul, and S. Sen, Periodic solutions of the mixed nonlinear Schrödinger equation, *Phys. Rev. D*, 32(1985), 3233-3237.

[15] B. L. Guo, S. B. Tan, On smooth solutions to the initial value problem for the mixed nonlinear Schrödinger equations, *Proc. Royal Soc.*, 119A(1991), 31-45.

[16] S. B. Tian, L.H. Zhang, On a weak solution of the Mixed Nonlinear Schrödinger Equations, *J. Math. Anal. Appl.*, 182(1994), 409-421.

[17] S. B. Tian, Blow-up Solutions for Mixed Nonlinear Schrödinger Equations, *Acta Mathematica Sinica*, 20(2004), 115-124.

[18] X. Lü, Madelung fluid description on a generalized mixed nonlinear Schrödinger equation, *Nonlinear Dyn*, 81(2015), 239-247.

[19] X. Lü, Soliton behavior for a generalized mixed nonlinear Schrödinger model with N-fold Darboux transformation, *Chaos*, 23(2013), 033137.

[20] J. S. He, S. W. Xu, and Y. Cheng, The rational solutions of the mixed nonlinear Schrödinger equation, *AIP Advances*, 5(2015), 017105.

[21] M. Tsutsumi, I. Fukuda, On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem, *Funkcial. Ekvac.*, 23 (1980), 259-277.

[22] H. Takaoka, Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity, *Adv. Differential Equations*, 4 (1999), 561-580.
[23] N. Hayashi, T. Ozawa, On the derivative nonlinear Schrödinger equation, *Phys. D*, 55 (1992), 14-36.

[24] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, A refined global well-posedness result for Schrödinger equations with derivative, *SIAM J. Math. Anal.*, 34 (2002), 64-86.

[25] Z. Guo, Y. Wu, Global well-posedness for the derivative nonlinear Schrödinger equation in $H^{1/2}(\mathbb{R})$, *Discrete Contin. Dyn. Syst.*, 37 (2017), 257-264.

[26] Y. Wu, Global well-posedness on the derivative nonlinear Schrödinger equation, *Anal. PDE*, 8 (2015), 1101-1112.

[27] R. Jenkins, J. Liu, P. Perry, C. Sulem, The derivative nonlinear Schrödinger equation: global well-posedness and soliton resolution, *Quarterly of applied mathematicas*, 78(2020), 33-73.

[28] R. Jenkins, J. Liu, P. Perry, C. Sulem, Global well-posedness for the derivative nonlinear Schrödinger equation, *Math. Phys.*, 363(2018), 1003-1049.

[29] K. D. T-R McLaughlin, P. D. Miller, $\bar{\partial}$-steepest descent method and the asymptotic behavior of polynomials orthogonal and exponentially varying nonanalytic weights, *Int. Math. Res. Not.*, IMRN (ISSN1687-3017) (2006) 48673.

[30] M. Dieng, K.D.T-R McLaughlin, Long-time asymptotics for the NLS equation via $\bar{\partial}$ methods, arXiv:0805.2807v1.
[31] M. Borghese, R. Jenkins, K. K.D.T-R McLaughlin, Long time asymptotics behavior of the focusing nonlinear Schrödinger equation, *Ann. I. H. Poincaré*, AN 35(2018), 887-920.

[32] J. Liu, P.A. Perry, C. Sulem, Long-time behavior of solutions to the derivative nonlinear Schröinger equation for soliton-free initial data, *Ann. I. H. Poincaré*, AN 35(2018), 217-265.

[33] R. H. Ma, E.G. Fan, Long time asymptotics behavior of the focusing nonlinear Kundu-Eckhaus equation, arXiv:1912.01425v1, 2019

[34] Y. L. Yang, E.G. Fan, Long-time asymptotic behavior of the modified Schrodinger equation via Dbar-steepest descent method, arXiv:2980918, 2019

[35] Q. Y. Cheng, E. G. Fan, Long-time asymptotics for a mixed nonlinear Schrödinger equation with the Schwartz initial data, *J. Math. Anal. Appl*, 489(2020), 124188.1-24.

[36] P. Deift, X. Zhou, Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space, *Comm. Pure and Appl. Math.*, LVI(2003), 1029-1077.