Reconstruction of Bandlimited Functions from Space–Time Samples

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Abstract

For a wide family of even kernels \( \{ \varphi_u, u \in I \} \), we describe discrete sets \( \Lambda \) such that every bandlimited signal \( f \) can be reconstructed from the space-time samples \( \{(f \ast \varphi_u)(\lambda), \lambda \in \Lambda, u \in I \} \).

Keywords: Dynamical sampling, Paley–Wiener spaces, Bernstein spaces

1 Introduction

The classical sampling problem asks when a continuous signal (function) \( f \) can be reconstructed from its discrete samples \( f(\lambda), \lambda \in \Lambda \). In the dynamical sampling problem, the set of space samples is replaced by a set of space-time samples (see e.g. \([1],[2],[3],[5]\) and references therein). An interesting case is the problem of reconstruction of a bandlimited signal \( f \) from the space-time samples of its states \( f \ast \varphi_u \) resulting from the convolution with a kernel \( \varphi_u \). An important example (see \([3],[4]\)) is the Gaussian kernel \( \varphi_u(x) = \exp(-ux^2) \), which arises from the diffusion process. More generally, the kernel

\[
\varphi_u(x) = \exp(-u|x|^{\alpha}), \quad \alpha > 0,
\]

arises from the fractional diffusion equation.

Denote by \( PW_\sigma \) the Paley–Wiener space

\[
PW_\sigma := \{ f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subseteq [-\sigma, \sigma] \},
\]

where \( \hat{f} \) denotes the Fourier transform

\[
\hat{f}(t) = \int_{\mathbb{R}} e^{-itx} f(x) \, dx.
\]

A set \( \Lambda \subset \mathbb{R} \) is called uniformly discrete (u.d.) if

\[
\delta(\Lambda) := \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0.
\]

The following problem is considered in \([3]\): Given a u.d. set \( \Lambda \subset \mathbb{R} \) and a kernel \( \{ \varphi_u, u \in I \} \), where \( I \) is an interval. What are the conditions that allow one to recover a function \( f \in PW_\sigma \) in a stable way from the data set

\[
\{(f \ast \varphi_u)(\lambda) : \lambda \in \Lambda, u \in I \}.
\]
In what follows, we denote by $\Phi_u$ the Fourier transform of $\varphi_u$ and assume that the functions $\varphi_u(x)$ and $\Phi_u(t)$ are continuous functions of $(x, u)$ and $(t, u)$, respectively.

It is remarked in [3], that the property of stable recovery formulated above is equivalent to the existence of two constants $A, B$ such that

$$A\|f\|_2^2 \leq \int \sum_{\lambda \in \Lambda} \left| (f * \varphi_u)(\lambda) \right|^2 du \leq B\|f\|_2^2, \quad \forall f \in PW_\sigma. \quad (4)$$

It often happens in the sampling theory that inequalities similar to the one in the right hand-side of (4) are not difficult to check. It is also the case here, it suffices to assume the uniform boundedness of the $L^1(\mathbb{R})$-norms $\|\varphi_u\|_1$:

Proposition 1

Assume

$$\sup_{u \in I} \|\varphi_u\|_1 < \infty. \quad (5)$$

Then for every $\sigma > 0$ and every u.d. set $\Lambda$ there is a constant $B$ such that

$$\int \sum_{\lambda \in \Lambda} \left| (f * \varphi_u)(\lambda) \right|^2 du \leq B\|f\|_2^2, \quad \forall f \in PW_\sigma.$$

We present a simple proof in Section 3.

Hence, the main difficulty lies in proving the left hand-side inequality.

Recall that the classical Shannon sampling theorem states that every $f \in PW_\sigma$ admits a stable recovery from the uniform space samples $f(k/a), k \in \mathbb{Z}$, if and only if $a \geq \sigma/\pi$. The critical value $a = \sigma/\pi$ is called the Nyquist rate. Since the space-time samples [3] produce “more information” compared to the space samples, one may expect that every $f \in PW_\sigma$ can be recovered from the space-time uniform samples at sub-Nyquist spatial density. However, it is not the case, as shown in [4] for the convolution with the Gaussian kernel. On the other hand, it is proved in [3] that uniform dynamical samples at sub-Nyquist spatial rate allow one to stably reconstruct the Fourier transform $\hat{f}$ away from certain, explicitly described blind spots.

It is well-known that the nonuniform sampling is sometimes more efficient than the uniform one. For example, this is so for the universal sampling, see e.g. [6], Lecture 6. It is also the case for the problem above: For a wide class of even kernels, we show that data [3] always allows stable reconstruction, provided $\Lambda$ is any relatively dense set “different” from an arithmetic progression.

To state precisely our main result, we need the following definition: Given a u.d. set $\Lambda$, the collection of sets $W(\Lambda)$ is defined as all weak limits of the translates $\Lambda - x_k$, where $x_k$ is any bounded or unbounded sequence of real numbers (for the definition of weak limit see e.g. Lecture 3.4.1 in [6]).

Consider the following condition:

$(\alpha)$ $W(\Lambda)$ does not contain the empty set, and no element $\Lambda^* \in W(\Lambda)$ lies in an arithmetic progression.

The first property in $(\alpha)$ means that $\Lambda$ is relatively dense, i.e. there exists $r > 0$ such that every interval $(x, x + r)$ contains at least one point of $\Lambda$. It follows that every element $\Lambda^* \in W(\Lambda)$ is also a relatively dense set.

The second condition in $(\alpha)$ means that no $\Lambda^* \in W(\Lambda)$ is a subset of $b + (1/a)\mathbb{Z}$, for some $a > 0$ and $b \in \mathbb{R}$.

Let us now define a collection of kernels $\mathcal{C}$: A kernel $\{\varphi_u, u \in I\}$, where $I$ is an interval, belongs to $\mathcal{C}$ if it satisfies the following five conditions:
(β) There is a constant $C$ such that
\[ \sup_{u \in I} |\varphi_u(x)| \leq \frac{C}{1 + x^4}, \quad x \in \mathbb{R}; \]  
(6)

(γ) There is a constant $C$ such that
\[ \|\varphi_{u'} - \varphi_u\|_1 \leq C|u - u'|, \quad u, u' \in I; \]  
(7)

(ζ) Every $\varphi_u$ is real and even: $\varphi_u(x) \in \mathbb{R}$, $\varphi_u(-x) = \varphi_u(x), x \in \mathbb{R}, u \in I$;
(η) $\sup_{u \in I} |\Phi_u(t)| > 0$ for every $t \in \mathbb{R}$;
(θ) For every $w \in \mathbb{C}$ and every $\sigma > 0$, the family $\{\Phi_u''(t) + w\Phi_u(t), u \in I\}$ forms a complete set in $L^2(0, \sigma)$.

Clearly, condition (β) implies that the derivatives $\Phi_u''(t), u \in I$, are continuous and uniformly bounded. Condition (ζ) implies that the functions $\Phi_u$ are real and even.

One may easily check that $C$ contains the kernels defined in (1), where $I = (a, b)$ is any interval such that $0 < a < b < \infty$.

Our main result is as follows:

**Theorem 1** Given a u.d. set $\Lambda \subset \mathbb{R}$ and a kernel $\{\varphi_u, u \in I\} \in \mathcal{C}$. The following conditions are equivalent:

(a) The left inequality in (4) is true for every $\sigma > 0$ and some $A = A(\sigma)$;
(b) $\Lambda$ satisfies condition (α).

2 Space–Time Sampling in Bernstein Spaces

The aim of this section is to prove a variant of Theorem 1 for the Bernstein space $B_\sigma$.

It is well-known that every function $f \in PW_\sigma$ admits an analytic continuation to the complex plane and satisfies
\[ |f(x + iy)| \leq Ce^{\sigma|y|}, \quad x, y \in \mathbb{R}, \]  
(8)

where $C$ depends only on $f$.

The Bernstein space $B_\sigma$ is defined as the set of entire functions $f$ satisfying (8) with some $C$ depending only on $f$. An equivalent definition is that $B_\sigma$ consists of the bounded continuous functions that are the inverse Fourier transforms of tempered distributions supported by $[-\sigma, \sigma]$.

Denote by $\mathcal{C}_0$ the collection of kernels $\{\varphi_u, u \in I\}$ satisfying the properties (β)-(η) in the definition of $C$ above. However, we do not require $I$ to be an interval. In particular, it can be a countable set.

**Theorem 2** Given a u.d. set $\Lambda \subset \mathbb{R}$ and a kernel $\{\varphi_u, u \in I\} \in \mathcal{C}_0$. The following conditions are equivalent:

(a) For every $\sigma > 0$ there is a constant $K = K(\sigma)$ such that
\[ \|f\|_\infty \leq K \sup_{\lambda \in \Lambda, u \in I} |(f * \varphi_u)(\lambda)|, \quad \forall f \in B_\sigma; \]  
(9)

(b) $\Lambda$ satisfies condition (α).

To prove this theorem we need a lemma:
Lemma 1 Assume $f \in B_{\sigma}$ and $\{\varphi_u, u \in I\} \in \mathcal{C}_0$. If $(f * \varphi_u)(0) = 0, u \in I$, then $f$ is odd, $f(-x) = -f(x), x \in \mathbb{R}$.

Proof. 1. Given a function $f \in B_{\sigma}$, set

$$f_r(z) := \frac{f(z) + f(z)}{2}, \quad f_i(z) := \frac{f(z) - f(z)}{2i}.$$ 

Then $f_r, f_i$ are real (on $\mathbb{R}$) entire functions satisfying $f = f_r + if_i$. It is clear that both $f_r$ and $f_i$ satisfy $\mathcal{S}$, so that they both lie in $B_{\sigma}$. Hence, since every $\varphi_u$ is real, it suffices to prove the lemma for the real functions $f \in B_{\sigma}$.

2. Let us assume that $f \in B_{\sigma}$ is real. Write

$$f_e(x) := \frac{f(z) + f(-z)}{2}, \quad f_o(x) := \frac{f(z) - f(-z)}{2}.$$ 

Clearly, $f_e \in B_{\sigma}$ is even, $f_o \in B_{\sigma}$ is odd and $f = f_e + f_o$. Since $\varphi_u$ is even, we have $(f_o * \varphi_u)(0) = 0, u \in I$. Hence, to prove Lemma 1 it suffices to check that if a real even function $f \in B_{\sigma}$ satisfies $(f * \varphi_u)(0) = 0, u \in I$, then $f = 0$.

3. Let us assume that $f \in B_{\sigma}$ is real, even and satisfies $(f * \varphi_u)(0) = 0, u \in I$. If $f$ does not vanish in $\mathbb{C}$ then $f(z) = e^{iaz}$ for some $-\sigma \leq a \leq \sigma$, which implies $a = 0, f(z) = 1$. Then $(f * \varphi_u)(0) = \Phi_u(0) = 0, u \in I$, which contradicts condition $(\eta)$. Hence, $f(w) = 0$ for some $w \in \mathbb{C}$. It follows that $f(-w) = 0$. Set

$$g(z) := \frac{f(z)}{s^2 + u^2}.$$ 

Denote by $G$ the Fourier transform of $g$. Then $G$ is continuous, even and vanishes outside $(-\sigma, \sigma)$. Now, condition $(f * \varphi_u) = 0, u \in I$, implies:

$$0 = \int_{\mathbb{R}} \varphi_u(s)f(s)\,ds = \int_{\mathbb{R}}(s^2 - u^2)\varphi_u(s)g(s)\,ds =$$

$$- \int_{\sigma}^{\infty} (\Phi_u'(t) + w^2\Phi_u(t))G(t)\,dt = -2 \int_{0}^{\infty} (\Phi_u'(t) + w^2\Phi_u(t))G(t)\,dt.$$ 

Using property $(\theta)$, we conclude that $G = 0$ and so $f = 0$.

2.1 Proof of Theorem 2

We denote by $C$ different positive constants.

1. Suppose $W(\Lambda)$ contains an empty set. It means that $\Lambda$ contains arbitrarily long gaps: For every $\rho > 0$ there exists $x_\rho$ such that $\Lambda \cap (x_\rho - 2\rho, x_\rho + 2\rho) = \emptyset$. Set

$$f_\rho(x) := \frac{\sin(\sigma(x - x_\rho))}{\sigma(x - x_\rho)} \in B_{\sigma}. \quad (10)$$ 

Then $\|f_\rho\|_\infty = 1$. Using $\mathcal{S}$, for all $x$ such that $|x - x_\rho| \geq 2\rho$, we have

$$|(f_\rho * \varphi_u)(x)| \leq \int_{|s| > \frac{|x - x_\rho|}{2}} \frac{2}{\sigma|x - x_\rho|} |\varphi_u(s)|\,ds +$$

$$\int_{|s| \leq \frac{|x - x_\rho|}{2}} |\varphi_u(s)|\,ds \leq \frac{C}{|x - x_\rho|}. \quad (11)$$
It readily follows that \([\ref{3}]\) is not true.

2. Suppose \(\Lambda^* \subset b + (1/a)\mathbb{Z}\) for some \(\Lambda^* \in W(\Lambda), b \in \mathbb{R}\) and \(a > 0\). Since 
\(\Lambda^* - b \in W(\Lambda)\), we may assume that \(b = 0\).

Consider two cases: First, let us assume that \(\Lambda \subset (1/a)\mathbb{Z}\). Clearly, 
the function \(f(z) := \sin(\pi az) \in B_{\sigma}\). Since every function \(\varphi_u\) is even 
while \(f\) is odd, one may easily check that \((f \ast \varphi_u)(k/a) = 0, k \in \mathbb{Z}\), so that \([\ref{3}]\) is not true.

Now, assume that \(\Lambda^* \subset (1/a)\mathbb{Z}\), for some \(\Lambda^* \in W(\Lambda)\). This means that for every 
small \(\varepsilon > 0\) and large \(R > 0\) there is a point \(v = v(\varepsilon, R) \in \mathbb{R}\) such that \((\Lambda - v) \cap (-R, R)\) 
is close to a subset of \((1/a)\mathbb{Z}\) in the sense that for every \(\lambda \in \Lambda \cap (v - R, v + R)\) there exists 
k(\lambda) \in \mathbb{Z} with 
\[
|\lambda - v - k(\lambda)/a| \leq \varepsilon, \quad \lambda \in \Lambda \cap (v - R, v + R).
\]

For simplicity of presentation, we assume that \(v = 0, a = 1\), and that 
\[
\Lambda \cap (-R, R) = \{\lambda_k : |k| \leq m\}, \quad |\lambda_k - k| \leq \varepsilon, \quad m = [R], \quad |k| \leq m. \tag{12}
\]
The proof of the general case is similar.

Fix \(\varepsilon := 1/\sqrt{R}\). Set 
\[
f(x) := \sin(\pi x) \frac{\sin(\varepsilon x)}{\varepsilon x} \in B_{\varepsilon x + \varepsilon} \tag{13}
\]
and 
\[
f_k(x) := \sin(\pi x) \frac{\sin(\varepsilon \lambda_k)}{\varepsilon \lambda_k},
\]
Then 
\[
|f(\lambda_k - s) - (-1)^{k+1} f_k(s)| \leq \left| \sin(\pi(\lambda_k - s)) - \sin(\pi(k - s)) \right| \left| \frac{\sin(\varepsilon(\lambda_k - s))}{\varepsilon \lambda_k} \right|
+ \left| \frac{\sin(\varepsilon s)}{\varepsilon s} \frac{\sin(\varepsilon \lambda_k)}{\varepsilon \lambda_k} \right|.
\tag{14}
\]
By \([\ref{12}]\), 
\[
|\sin(\pi(\lambda_k - s)) - \sin(\pi(k - s))| \leq \pi \varepsilon, \quad s \in \mathbb{R},
\]
and so the first term in the right-hand-side of (14) is less than \(\pi \varepsilon\) for every \(s \in \mathbb{R}\). To 
estimate the second term in (14), we use the classical Bernstein’s inequality (see e.g. 
[6], Lecture 2.10):
\[
\left| \frac{\sin(\varepsilon(\lambda_k - s))}{\varepsilon \lambda_k} - \frac{\sin(\varepsilon \lambda_k)}{\varepsilon \lambda_k} \right| = \left| \int_0^s \left( \frac{\sin(\varepsilon(\lambda_k - u))}{\varepsilon(\lambda_k - u)} \right)' \, du \right| \leq \|s\| \left| \left( \frac{\sin(u)}{u} \right)' \right|_\infty \leq \varepsilon |s|.
\]
Therefore, 
\[
|f(\lambda_k - s) - (-1)^{k+1} f_k(s)| \leq \pi \varepsilon (1 + |s|), \quad s \in \mathbb{R}.
\]
Observe that 
\[
(f \ast \varphi_u)(\lambda_k) = \int_{\mathbb{R}} (f(\lambda_k - s) - (-1)^{k+1} f_k(s)) \varphi_u(s) \, ds + (-1)^{k+1} \int_{\mathbb{R}} f_k(s) \varphi_u(s) \, ds.
\]
Since \(f_k\) is odd, the last integral is equal to zero. It follows that for every \(|k| \leq m\) we have 
\[
|(f \ast \varphi_u)(\lambda_k)| \leq \pi \varepsilon \int_{\mathbb{R}} (1 + |s|) |\varphi_u(s)| \, ds, \quad u \in I.
\]
Hence, using (6) we conclude that
\[ |(f \ast \varphi_u)(\lambda)| \leq C\epsilon, \quad \lambda \in \Lambda \cap (-R, R), \quad u \in I. \]

On the other hand, for all \( \lambda \in \Lambda, |\lambda| \geq R \) and \( |s| < 1/\epsilon = \sqrt{R} \), we get
\[ |f(\lambda - s)| \leq \frac{1}{|\lambda - s|} \leq \frac{\sqrt{R}}{R - \sqrt{R}} < 2\epsilon, \]
provided \( R \) is sufficiently large. This and (6) imply
\[ |(f \ast \varphi_u)(\lambda)| \leq 2\epsilon \int_{|s| < \sqrt{R}} |\varphi_u(s)| \, ds + \int_{|s| > \sqrt{R}} |\varphi_u(s)| \, ds \leq C\epsilon, \quad \lambda \in \Lambda, |\lambda| \geq R, u \in I. \]

Since \( \epsilon \) can be chosen arbitrarily small, we conclude that (9) is not true.

3. Assume condition (a) holds. We have to show that for every \( \sigma > 0 \) there is a constant \( K = K(\sigma) \) such that (9) is true.

Assume this is not so. It means that there exists \( \sigma > 0 \) and a sequence of functions \( f_n \in B_\sigma \) satisfying
\[ \|f_n\|_\infty = 1, \quad \sup_{u \in I, \lambda \in \Lambda} |(f_n \ast \varphi_u)(\lambda)| \leq 1/n. \]

Choose points \( x_n \in \mathbb{R} \) such that \( |f_n(x_n)| > 1 - 1/n \), and set \( g_n(x) := f_n(x + x_n) \).
It follows from the compactness property of Bernstein spaces (see e.g. [6], Lecture 2.8.3), that there is a subsequence \( n_k \) such that \( g_{n_k} \) converge (uniformly on compacts in \( \mathbb{C} \)) to some non-zero function \( g \in B_\sigma \). We may also assume (by taking if necessary a subsequence of \( n_k \)) that the translates \( \Lambda - x_{n_k} \) converge weakly to some \( \Gamma \in W(\Lambda) \). By property (a), \( \Gamma \) is an infinite set which is not a subset of any arithmetic progression.

Clearly, we have
\[ (g \ast \varphi_u)(\gamma) = 0, \quad u \in I, \quad \gamma \in \Gamma. \]

By Lemma 4 we see that every function \( g(x - \gamma), \gamma \in \Gamma \), is odd. Clearly, this implies that \( g \) is a periodic function and \( \Gamma \) is a subset of an arithmetic progression whose difference is a half-integer multiple of the period of \( g \). Contradiction.

3 Space–Time Sampling in Paley-Wiener Spaces

Throughout this section we denote by \( C \) different positive constants.

In what follows we assume that \( I \) is an interval. We denote by \( C \) different positive constants.

The following statement easily follows from (6) and (7):

**Corollary 1** Assume condition (a) holds for some kernel \( \{\varphi_u\} \) satisfying (7), a u.d. set \( \Lambda \) and \( \sigma > 0 \). Then there is a constant \( K' = K'(\sigma) \) such that
\[ \|f\|_\infty^2 \leq K' \int I \sup_{\lambda \in \Lambda} |(f \ast \varphi_u)(\lambda)|^2 \, du, \quad \forall f \in B_\sigma. \]

We skip the simple proof.
3.1 Proof of Proposition

Take any function \( f \in PW_\sigma \) and denote by \( F \) its Fourier transform. It follows from (5) that \( \| \Phi_u \|_\infty \leq C, \ u \in I \). Hence, the functions \( F \cdot \Phi_u \in L^2(-\sigma, \sigma) \) and

\[
\| f \ast \varphi_u \|_2 = \| F \cdot \Phi_u \|_2 \leq \| \Phi_u \|_\infty \| F \|_2 \leq C \| f \|_2.
\]

Clearly, \( f \ast \varphi_u \in PW_\sigma \), for every \( u \). Using Bessel’s inequality (see e.g. Proposition 2.7 in [6]), we get

\[
\sum_{\lambda \in \Lambda} |(f \ast \varphi_u)(\lambda)|^2 \leq C \| f \ast \varphi_u \|_2 \leq C \| f \|_2^2, \ u \in I,
\]

which proves Proposition.

3.2 Connection between space–time sampling in \( B_\sigma \) and \( PW_\sigma \)

Observe that if \( \Lambda \) is a sampling set (in the ‘classical sense’) for the Paley-Wiener space \( PW_\sigma' \), then it is a sampling set for the Bernstein spaces \( B_\sigma \) with a ‘smaller’ spectrum \( \sigma < \sigma' \), and vice versa (see Theorem 3.32 in [6]). We provide a corresponding statement for the space-time sampling problem.

For the reader’s convenience, we recall the main inequalities:

\[
\| f \|_2^2 \leq D \int \sum_{\lambda \in \Lambda} |(f \ast \varphi_u)(\lambda)|^2 \, du,
\]

(15)

\[
\| f \|_\infty \leq K \sup_{\lambda \in \Lambda, u \in I} |(f \ast \varphi_u)(\lambda)|.
\]

(16)

**Theorem 3** Let \( \Lambda \) be a u.d. set, a kernel \( \{ \varphi_u \} \) satisfy (9) and (10) and \( \sigma' > \sigma > 0 \).

(i) Assume that (10) holds with some constant \( K \) for all \( f \in B_{\sigma'} \). Then there is a constant \( D \) such that (15) is true for every \( f \in PW_{\sigma'} \).

(ii) Assume that (15) holds with some constant \( D \) for all \( f \in PW_{\sigma'} \). Then there is a constant \( K \) such that (10) is true for every \( f \in B_{\sigma} \).

**Proof.** The proof is somewhat similar to the proof of Theorem 3.32 in [6], but is more technical.

(i) Assume that (10) holds for every \( f \in B_{\sigma'} \). Fix any positive number \( \varepsilon \) satisfying

\[
\sigma + \varepsilon \leq \sigma'.
\]

(17)

Set

\[
h_\varepsilon(x) := \frac{\sin \varepsilon x}{\varepsilon x}, \quad \varepsilon > 0.
\]

(18)

It is easy to check that

\[
h_\varepsilon(0) = 1, \quad \| h_\varepsilon \|_2^2 = \frac{C}{\varepsilon}, \quad \| h_\varepsilon' \|_2^2 = C \varepsilon.
\]

(19)

For every \( f \in PW_{\sigma'} \), we have

\[
\| f \|_2^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx \leq \int_{\mathbb{R}} \sup_{s \in \mathbb{R}} |h_\varepsilon(x-s) f(s)|^2 \, dx.
\]
Note that $h_ε(x - s)f(s) ∈ PW_{σ+ε} ⊂ B_{σ'}$. By Corollary 1, for every $x$ and $s$,

$$|h_ε(x - s)f(s)|^2 ≤ C \int \sup_{\lambda \in \Lambda} \left| \int \varphi_u(\lambda - s)h_ε(x - s)f(s) \, ds \right|^2 \, d u ≤ C \int \sum_{\lambda \in \Lambda} \left| \int \varphi_u(\lambda - s)h_ε(x - s)f(s) \, ds \right|^2 \, d u.$$ 

Write

$$J = J_u(x, \lambda) := \left| \int \varphi_u(\lambda - s)h_ε(x - s)f(s) \, ds \right|^2. \tag{20}$$

Then

$$\|f\|^2 \leq C \int \sum_{\lambda \in \Lambda} \int J \, d u d x.$$

Clearly,

$$J ≤ 2(J_1 + J_2),$$

where

$$J_1 := \left| \int \varphi_u(\lambda - s)h_ε(x - \lambda)f(s) \, ds \right|^2 = |h_ε(x - \lambda)|^2 |(f * \varphi_u)(\lambda)|^2,$$

and using property (11) and the Cauchy–Schwarz inequality, we have

$$J_2 := \left| \int \varphi_u(\lambda - s)(h_ε(x - s) - h_ε(x - \lambda))f(s) \, ds \right|^2 ≤ \int |\varphi_u(s - \lambda)| \, d s \int |\varphi_u(\lambda - s)||h_ε(x - s) - h_ε(x - \lambda)|^2 |f(s)|^2 \, d s \leq C \int |\varphi_u(\lambda - s)||h_ε(x - s) - h_ε(x - \lambda)|^2 |f(s)|^2 \, d s.$$

Observe that

$$|h_ε(x - s) - h_ε(x - \lambda)|^2 = \left| \int_s^\lambda h_ε'(x - v) \, d v \right|^2 ≤ |s - \lambda| \int_s^\lambda |h_ε'(x - v)|^2 \, d v.$$

Hence,

$$J_2 ≤ C \int |\varphi_u(\lambda - s)||s - \lambda| \left( \int_s^\lambda |h_ε'(x - v)|^2 \, d v \right) |f(s)|^2 \, d s.$$

Using (10), we have

$$\int \sum_{\lambda \in \Lambda} \int J_1 \, d u d x = \int |h_ε(\lambda - x)|^2 \, d x \sum_{\lambda \in \Lambda} \int |(f * \varphi_u)(\lambda)|^2 \, d u ≤$$

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\[ \frac{C}{\varepsilon} \sum_{\lambda \in \Lambda} \int (f * \varphi_u)(\lambda)^2 du. \]

To estimate the second sum we switch the order of integration and apply (19):

\[ \int \sum_{\lambda \in \Lambda} \int J_2 du dx \leq \int \sum_{\lambda \in \Lambda} \int \left| \varphi_u(\lambda - s)\right| |s - \lambda| f(s)^2 \left( \int_{s - \lambda}^{s + \lambda} |h'_{\varepsilon}(s - v)|^2 dv dx \right) du ds. \]

Now, by (6) we get

\[ \sum_{\lambda \in \Lambda} \left| \varphi_u(\lambda - s)\right| |s - \lambda| f(s)^2 \leq C \sum_{\lambda \in \Lambda} (\lambda - s)^2 \frac{1 + (\lambda - s)^2}{1 + (\lambda - s)^2} < C, \quad u \in I, s \in \mathbb{R}. \]

where the second inequality holds since \( \Lambda \) is a u.d. set (see definition in (2)). Hence,

\[ \int \sum_{\lambda \in \Lambda} \int J_2 du dx \leq C \varepsilon |I| \| f \|_2^2, \]

where \( |I| \) is the length of \( I \).

Combining this with the estimate for \( J_1 \) and using (20), we conclude that

\[ \| f \|_2^2 \leq \frac{C}{\varepsilon} \sum_{\lambda \in \Lambda} \int (f * \varphi_u)(\lambda)^2 du + C \varepsilon |I| \| f \|_2^2. \]

Choosing \( \varepsilon \) small enough, we obtain (15).

(ii) Assume (15) holds with some constant \( D \) for all \( f \in PW_{\sigma'} \).

We will argue by contradiction. Assume that there is no constant \( K \) such that (16) holds for every \( f \in B_{\sigma} \). This means that there exist \( g_j \in B_{\sigma} \) such that \( \| g_j \|_{\infty} = 1 \),

\[ \sup_{u \in I, \lambda \in \Lambda} \left| (g_j * \varphi_u)(\lambda) \right| < \frac{1}{j}, \quad \tag{21} \]

and for some points \( x_j \) we have \( |g_j(x_j)| \geq 1/2 \).

Assume \( \varepsilon > 0 \) satisfies (17) and let \( h_{\varepsilon} \) be defined by formula (18). Set

\[ f_j(x) := g_j(x) h_{\varepsilon}(x - x_j). \]

It is clear that for every \( j \) we have \( f_j \in PW_{\sigma'} \), \( \| f_j \|_{\infty} \leq 1 \), and that \( |f_j(x_j)| \geq 1/2 \). The last two inequalities and the Bernstein’s inequality imply that there is a constant \( K' > 0 \) such that

\[ \| f_j \|_2 \geq K', \quad j \in \mathbb{N}. \quad \tag{22} \]

By (16), we get

\[ \| f_j \|_2^2 \leq C \int \sum_{\lambda \in \Lambda} |(f_j * \varphi_u)(\lambda)|^2 du = C \int \sum_{\lambda \in \Lambda} \left| \int_{x_j}^{x_j + \varepsilon} g_j(x) \varphi_u(\lambda - x) h_{\varepsilon}(x - x_j) dx \right|^2 du. \]
This gives
\[ \|f_j\|_2^2 \leq C(\tilde{J}_1 + \tilde{J}_2), \] (23)
where \( \tilde{J}_1 \) and \( \tilde{J}_2 \) are defined as follows:
\[
\tilde{J}_1 := \left( \sum_{\lambda \in \Lambda} \left| \int_\mathbb{R} g_j(x) \varphi_u(\lambda - x)(h_\varepsilon(x - x_j) - h_\varepsilon(\lambda - x_j)) \, dx \right| \right)^2 du,
\]
\[
\tilde{J}_2 := \left( \sum_{\lambda \in \Lambda} \int_\mathbb{R} g_j(x) \varphi_u(\lambda - x) h_\varepsilon(\lambda - x_j) \, dx \right)^2 du.
\]

By Bessel’s inequality (see, e.g. [6], Proposition 2.7) and (19),
\[
\sum_{\lambda \in \Lambda} |h_\varepsilon(\lambda - s)|^2 \leq C \|h_\varepsilon\|_2^2 \leq \frac{C}{\varepsilon^2}, \quad \forall s \in \mathbb{R}.
\]

Therefore, using (21) we arrive at
\[
\tilde{J}_2 \leq \frac{C}{\varepsilon^2} |I|.
\]

Let us now estimate \( \tilde{J}_1 \). Recall that \( \|g_j\|_\infty = 1 \). Using the change of variables \( x = t + \lambda \), we get
\[
\tilde{J}_1 \leq \left( \int \varphi_u(-t) \left( \int_0^t h_\varepsilon'(s + \lambda - x_j) \, ds \right) \, dt \right)^2 du.
\]

Now, use the Cauchy–Schwarz inequality:
\[
\tilde{J}_1 \leq \int \left( \int_\mathbb{R} |\varphi_u(-t)(1 + t^2)^2 \int_\mathbb{R} \frac{1}{(1 + t^2)^2} \left| h_\varepsilon'(s + \lambda - x_j) \right| \, ds \right)^2 dt \, du.
\]

Using again the Cauchy–Schwarz inequality and condition (21), we arrive at
\[
\tilde{J}_1 \leq C \int \left( \int_\mathbb{R} \frac{|I|}{(1 + t^2)^2} \int_0^t \left| h_\varepsilon'(s + \lambda - x_j) \right|^2 \, ds \right) \, dt \, du.
\]

Finally, Bessel’s inequality yields
\[
\sum_{\lambda \in \Lambda} |h_\varepsilon'(s + \lambda - x_j)|^2 \leq C \|h_\varepsilon\|_2^2 \leq C \varepsilon,
\]
and we conclude that
\[
\tilde{J}_1 \leq C |I| \varepsilon.
\]

We now insert the estimate for \( \tilde{J}_1, \tilde{J}_2 \) in (23) and use (22) to get the estimate
\[
(K')^2 \leq \frac{C}{\varepsilon^2} + C |I| \varepsilon.
\]
Choosing \( \varepsilon \) sufficiently small, we arrive at contradiction for all large enough \( j \).
3.3 Proof of Theorem 1

The proof easily follows from Theorems 2 and 3. Assume that the assumptions of Theorem 1 hold.

(i) Assume that Λ satisfies condition (α). Then by Theorem 2, for every σ > 0 there exists K = K(σ) such that inequality (9) is true. Applying Theorem 3, we see that there exists A = A(σ) > 0 the left hand-side inequality in (4) is also true for every σ > 0.

(ii) Assume that Λ does not satisfy condition (α). Then by Theorem 2 there exists σ > 0 such that there is no constant K for which condition (9) is true. Applying Theorem 3 we see that for every positive σ′ > σ there is no constant D such that inequality (15) holds for every f ∈ PWσ′.

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