Asymptotic representation for spectrum of linear problem describing flows of viscoelastic polymeric fluid

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Abstract. We study a new rheological model (a modification of a known Pokrovski – Vinogradov model). As was shown by numerical simulations, it takes into account nonlinear effects arising in flows of melts and solutions of polymers in domains with a complex boundary geometry. In the case when the main solution is a Poiseuille-type flow in an infinite plane channel (one considers a viscoelastic polymeric fluid) we obtain an asymptotic formula for the distribution of spectrum points of the linear problem. Small perturbations have such the additional property that they are periodic with respect to the variable going along the side of the channel.

1. Introduction
A new rheological model in which nonlinear effects arising in a polymer medium are taken into account with a reasonable accuracy is studied in the paper. From the physical point of view, the media under consideration is a suspension of noninteracting elastic dumbbells [1]. Each dumbbell is a couple of Brownian particles connected by an elastic force and moves in an anisotropic liquid formed by a solvent and other dumbbells.

This model is a modification of the known model by Pokrovsky and Vinogradov [2], [3]; its main element is a new rheological relation which describes the connection between kinematic characteristics of the flow and internal thermodynamical parameters. Carried out numerical experiments show that results match experimental data well [4], [5].

As the test regime we study the experimentally observed Poiseuille-type flow in an infinite plane channel for the system of the Navier–Stokes equations. The asymptotics of eigenvalues is studied for the problem linearized about a given stationary solution and possible small perturbations are harmonic functions with respect to the variable along the infinite channel with a fixed frequency.

The main result of this work continues a number of facts obtained earlier: in [6] a formal asymptotic representation of the spectrum points for the linear problem in an infinite plane channel was, in particular, found when as the class of perturbations we choose the class of functions decreasing at infinity; it follows from this formula that for the chosen model the Poiseuille-type flow is linearly unstable by Lyapunov; in [7] it is proved that there is no asymptotic linear stability by Lyapunov for Poiseuille-type flows in the class of periodic...
perturbations provided that the Fourier series contain only harmonics with a chosen limited number of frequencies.

2. Problem statement. Auxiliary information. Formulation of the main result
In [1] a new mathematical model describing flows of an incompressible viscoelastic polymeric fluid was presented. In the 2D case nonstationary flows of polymeric media are described by the following rheological model (written already in a dimensionless form):

\[ u_x + v_y = 0, \]  
\[ \frac{du}{dt} + p_x = \frac{1}{Re}\{(a_{11})_x + (a_{12})_y\}, \]  
\[ \frac{dv}{dt} + p_y = \frac{1}{Re}\{(a_{12})_x + (a_{22})_y\}, \]  
\[ \frac{da_{11}}{dt} - 2A_1 u_x - 2a_{12} u_y + K_I a_{11} = -\beta(a_{11}^2 + a_{22}^2), \]  
\[ \frac{da_{12}}{dt} - A_1 v_x - A_2 u_y + \tilde{K}_I a_{12} = 0, \]  
\[ \frac{da_{22}}{dt} - 2A_2 v_y - 2a_{12} v_x + \tilde{K}_I a_{22} = -\beta(a_{12}^2 + a_{22}^2). \]

Here \( t \) is the time, \( u \) and \( v \) are the components of the velocity vector \( \mathbf{u} \) in the Cartesian coordinate system \( x, y \), \( p \) is the hydrostatic pressure, \( a_{ij} \) is the symmetrical anisotropy tensor of the second rank; \( \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u}, \nabla) \) is the substantial derivative.

The system of linear equations resulting from the linearization of system (1)–(6) about a chosen stationary solution (further its components are marked with "\(^\wedge\)") in the case when the fluid is moving in an infinite plane channel was obtained in [9]. In the vector form it is formulated as follows. In the domain (see Figure 1)

\[ G = \{(t, x, y) | t > 0, (x, y) \in \Pi = \{(x, y) | |x| < \infty, 0 < y < 1\}\} \]
we need to find a solution to the following system of equations:

\[ U_t + \hat{B} U_x + \hat{C} U_y + \hat{R} U + F = 0, \]  
\[ \Delta \Omega = \frac{1}{Re} \{\sigma_{xx} + 2(a_{12})_{xy}\} - 2\hat{\omega} v_x. \]  

Here \( U = \begin{pmatrix} u \\ v \\ a_{11} \\ a_{12} \\ a_{22} \end{pmatrix} \) is the unknown vector–function, \( \sigma = a_{11} - a_{22}, \Omega = p - \frac{1}{Re} a_{22}, \)

the matrices \( \hat{B} = B(\hat{U}), \hat{C} = C(\hat{U}), \hat{R} = R(\hat{U}) \) are written with the help of the components of the stationary solution \( \hat{U}(y) \):

\[ \hat{U}(y) = \begin{pmatrix} \hat{u}(y) \\ 0 \\ \hat{a}_{11}(y) \\ \hat{a}_{12}(y) \\ \hat{a}_{22}(y) \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \hat{u} & 0 & -\frac{1}{Re} & 0 & 0 \\ 0 & \hat{u} & 0 & -\frac{1}{Re} & 0 \\ -2\hat{A}_1 & 0 & \hat{u} & 0 & 0 \\ 0 & -\hat{A}_1 & 0 & \hat{u} & 0 \\ 0 & 0 & -2\hat{a}_{12} & 0 & 0 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{Re} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{Re} \\ -2\hat{a}_{12} & 0 & 0 & 0 & 0 \\ -\hat{A}_2 & 0 & 0 & 0 & 0 \\ 0 & -2\hat{A}_2 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} 0 & \hat{\omega} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{a}_{11} & R_{33} & R_{34} & R_{35} \\ 0 & \hat{a}_{12} & R_{43} & R_{44} & R_{45} \\ 0 & \hat{a}_{22} & R_{53} & R_{54} & R_{55} \end{pmatrix}, \]  

where \( \hat{A}_1 = \hat{a}_{11} + \frac{1}{W}, \hat{A}_2 = \hat{a}_{22} + \frac{1}{W}, \)

\[ R_{33} = \frac{1}{W} + \frac{k}{3} \hat{\ell} + \frac{k + 5\beta}{3} \hat{a}_{11}, \quad R_{34} = -2(\hat{\omega} - \beta \hat{a}_{12}), \quad \hat{\omega} = \hat{u}_y, \quad R_{35} = \frac{k}{3} \hat{a}_{11}, \]
\[ R_{43} = \frac{k}{3} \hat{a}_{12}, \quad R_{44} = \frac{1}{W} + \frac{k}{3} \hat{I}, \quad R_{45} = -\hat{\omega} + \frac{k}{3} \hat{a}_{12}, \]
\[ R_{53} = \frac{k}{3} \hat{a}_{22}, \quad R_{54} = 2\beta \hat{a}_{12}, \quad R_{55} = \frac{1}{W} + \frac{k}{3} + \frac{k + 5\beta}{3} \hat{a}_{22}, \]

\[ F = \begin{pmatrix} p_x \\ p_y \\ 0 \\ 0 \end{pmatrix}, \quad \"\Delta\" \text{ is the symbol of the Laplace operator.} \]

We will assume that on the boundary of the domain \( G \) the conditions
\[ u\big|_{y=0} = v\big|_{y=0} = u\big|_{y=1} = v\big|_{y=1} = 0; \quad (10) \]
\[ \Omega_y = \frac{1}{Re} (a_{12})_x \quad \text{for } y = 0, 1 \quad (11) \]
hold the initial data
\[ U\big|_{t=0} = U_0(x,y) \quad (12) \]
are prescribed. Moreover, the initial data satisfy the boundary conditions (10).

Remark 2.2 As the basic solution we can consider, for example, a solution similar to the Poiseuille solution of the Navier-Stokes equations (see [4], [10], [11]) which is symmetric with respect to the channel’s axis \( y = \frac{1}{2} \) (and \( \hat{p}(x,y) = \frac{1}{Re} \hat{a}_{22}(y) + \hat{p}_0 - \hat{A}x, \hat{p}_0 \) is the pressure on the channel’s axis, \( \hat{A} \) is the parameter connected with the dimensionless pressure difference on the segment \( h \)).

In [9] for \( k = \beta \) the components \( \hat{u}(y), \hat{a}_{11}(y), \hat{a}_{12}(y), \hat{a}_{22}(y) \) of this solution are found explicitly, and for \( k \neq \beta \) they are obtained numerically with the help of the proposed iterative process. Figure 2 shows profiles of the function \( \hat{u}(y) \) for the following values of the parameters \( k, \beta, \hat{D} (= Re \hat{A}) \) and \( W: \hat{k} = 0, \hat{D} = 2, W = 1 \). The graphs numbered as 1,2,3,4 correspond to the values \( \beta = 0.1, 0.2, 0.3, 0.4 \).

Remark 2.3 As was noted in [4], [10] (see also [9]), the profile of the function \( \hat{u}(y) \) is different from the parabolic Poiseuille profile for the viscous fluid. The maximum value of the velocity \( \hat{u}_{\text{max}} \) is found by using the following formula:

\[ \hat{u}_{\text{max}} = \hat{u}\left(\frac{1}{2}\right) = \frac{\hat{D}}{8} + \int_0^{\frac{1}{2}} \frac{k - \hat{a}_{22}}{A_2}(\xi)\hat{a}_{12}(\xi)d\xi. \]

Remark 2.4 In [9] it is proved that system (7) with the given pressure \( p(t,x,y) \) \( t \) is hyperbolic (see [12]) if \( \hat{A}_1 > 0, \hat{A}_2 > 0 \) and \( \hat{A}_1 \hat{A}_2 - \hat{a}_{12}^2 > 0 \) (see the representation of the matrices \( \hat{B} \) and \( \hat{C} \)). These inequalities are valid when as the basic solution we chose, for example, the Poiseuille solution (this is checked explicitly for \( k = \beta \) numerically for \( k \neq \beta \)). The information about the roots of the characteristic equation plays a significant role in the formulation of initial boundary value problems for \( t \)-hyperbolic systems.

We will search a solution of the linear problem (7)–(12) in the class of vector-functions which have the following representation:

\[ M(t,x,y) = \begin{pmatrix} U(t,x,y) \\ \Omega(t,x,y) \\ \frac{\partial \Omega}{\partial y} (= L) \end{pmatrix} = e^{-i\xi x} \begin{pmatrix} \hat{U}(t,y) \\ \hat{\Omega}(t,y) \\ \hat{L}(t,y) \end{pmatrix}, \quad \xi \in \mathbb{R} \text{ is fixed parameter.} \quad (13) \]
Applying representation (13) and using formally the Laplace transform in time, we obtain the model problem with the parameters $\omega$ and $\lambda$:

\begin{equation}
\dot{C}U_y^L + (\lambda I - i\xi \tilde{B} + \tilde{R})U^L + \tilde{F} = U^*_0(y),
\end{equation}

\begin{equation}
\Omega^L_y = L^L,
\end{equation}

\begin{equation}
L_y^L - \xi^2 \Omega^L = -\frac{2i\xi}{Re}(a_{12}^L)_y - \frac{1}{Re}(a_{11}^L - a_{22}^L)\xi^2 + 2i\xi\hat{\omega}v^L,
\end{equation}

where $L$ denotes the Laplace transform of a function or a component of a vector-function, $\lambda$ is the dual variable for $t$ with respect to the Laplace transform:

\[
\hat{F} = \begin{pmatrix}
-i\xi(\Omega^L + \frac{1}{Re}a_{22}^L)
L + \frac{1}{Re}(a_{22}^L)_y
0
0
0
\end{pmatrix},
U_0(x,y) = e^{ixy}U^*_0(y).
\]

The boundary conditions (10), (11) are transformed as

\begin{equation}
u^L|_{y=0} = v^L|_{y=0} = u^L|_{y=1} = v^L|_{y=1} = 0,
\end{equation}

\begin{equation}L^L = -i\xi a_{12}^L \text{ for } y = 0, 1.
\end{equation}

Let us assume that the following inequality holds:

\begin{equation}\hat{A}_2(y) \neq Re, \quad y \in [0,1].
\end{equation}

We are now in a position to formulate the main result of the paper.
Theorem 2.1 If condition (20) is satisfied, then the eigenvalues of the boundary value problem (14)–(19) have the following asymptotic representation:

\[ \lambda_k = -\frac{1}{2} \int_0^1 \sqrt{Re A_2} d\eta \left[ \int_0^1 (d_{11} - d_{22}) d\xi + (2k + 1)i\pi \right] + O\left(\frac{1}{|k|}\right), \]

where \( k \) is an integer number, \( |k| \to \infty \), the functions \( d_{ii} \), \( i = 1, 2 \) are defined as follows

\[
d_{11} = \sqrt{\frac{Re A_2}{A_2}} \left( -i\xi \hat{u} + R_{44} + i\xi \frac{\hat{a}_{12}}{\sqrt{Re A_2}} \hat{R}_{43} + \frac{1}{2} \frac{\hat{a}_{12} R_{43}}{\sqrt{Re A_2}} \right),
\]

\[
d_{22} = -\sqrt{\frac{Re A_2}{A_2}} \left( \frac{\sqrt{Re A_2}}{2} \left( \frac{-i\xi \hat{u}}{\sqrt{Re A_2}} \hat{a}_{12} i\xi \hat{R}_{43} - \frac{1}{2} \frac{\hat{a}_{12}}{A_2} R_{43} - i\xi \hat{u} \hat{R}_{44} \right) + \frac{1}{2} \frac{\sqrt{Re A_2}}{\sqrt{Re A_2}} \right),
\]

and \( \xi \) is a fixed parameter.

Remark 2.5 The initial boundary value problem (7)–(12) thus allows the Laplace transform with respect to the time.

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