Entanglement entropy of a simple quantum system

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Abstract

We propose a simple approach to the calculation of the entanglement entropy of a spherically symmetric quantum system composed of two separate regions. We consider bound states of the system described by a wave function that is scale invariant and vanishes exponentially at infinity. Our result is in accordance with the holographic bound on entropy and shows that entanglement entropy scales with the area of the boundary.

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I. INTRODUCTION

Entanglement is one of the fundamental features of quantum mechanics and has led to the development of new areas of research, such as quantum information and quantum computing. Some reviews and recent results on entanglement entropy in many-body systems, conformal field theory and black hole physics can be found in [1, 2, 3, 4, 5].

Einstein, Podolsky and Rosen (EPR) proposed a thought experiment [6] to prove that quantum mechanics predicts the existence of “spooky” nonlocal correlations between spatially separate parts of a quantum system, a phenomenon that Schrödinger [7] called entanglement. Afterward, Bell [8] derived some inequalities that can be violated in quantum mechanics but must be satisfied by any local hidden variable model. It was Aspect [9] who first verified in laboratory that the EPR experiment, in the version proposed by Bohm [10], violates Bell inequalities, showing therefore that quantum entanglement and nonlocality are correct predictions of quantum mechanics.

A renewed interest in entanglement came from black hole physics: as suggested in [11, 12], black hole entropy can be interpreted in terms of quantum entanglement, since the horizon of a black hole divides spacetime into two subsystems, such that observers outside cannot communicate the results of their measurements to observers inside, and vice versa. Black hole entanglement entropy turns out to scale with the area $A$ of the event horizon, in accordance with the renowned Bekenstein-Hawking formula [13, 14, 15, 16]:

$$S_{BH} = \frac{A}{4\ell_P},$$

where $S_{BH}$ is the black hole entropy and $\ell_P$ is the Planck length.

Let us consider a spherically symmetric quantum system, composed of two regions $A$ and $B$ separated by a spherical surface of radius $R$ (Fig. 1). The entanglement entropy of each part obeys an “area law”, as discussed e.g. in [1, 2] for many-body systems. This result can be justified by means of a simple argument proposed by Srednicki in [12]. If we trace over the field degrees of freedom located in region $B$, the resulting density matrix $\rho_A$ depends only on the degrees of freedom inside the sphere, and the associated von Neumann entropy is $S_A = -\text{Tr}(\rho_A \ln \rho_A)$. If then we trace over the degrees of freedom in region $A$, we obtain an entropy $S_B$ which depends only on the degrees of freedom outside the sphere. It is straightforward to show that $S_A = S_B = S$, therefore the entropy $S$ should depend only on properties shared by the two regions inside and outside the sphere. The only feature they have in common is their boundary, so it is reasonable to expect that $S$ depends only on the
area of the boundary, $A = 4\pi R^2$.

The area scaling of entanglement entropy has been investigated much more in the context of quantum field theory than in quantum mechanics. In order to bridge the gap, in this paper we study the entanglement entropy of a quantum system composed of two separate parts (Fig. 1), described by a wave function $\psi$, which we assume invariant under scale transformations and vanishing exponentially at infinity. In Section III we will show that the entropy $S$ of both parts of our system is bounded by $S \lesssim \eta \frac{A}{4\pi r_p}$, where $\eta$ is a numerical constant related to the dimensionless parameter $\lambda$ appearing in the wave function $\psi$. This result, obtained at the leading order in $\lambda$, is in accordance with the so-called holographic bound on the entropy $S$ of an arbitrary system [17, 18, 19]: $S \leq \frac{A}{4\pi r_p}$, where $A$ is the area of any surface enclosing the system.

In Section II we present the main features of our approach, focusing in particular on the properties of entanglement entropy and on the form of the wave function describing the system. In Section III we calculate analytically the bound on entanglement entropy. In Section IV we summarize both the limits and the goals of our approach.

II. MAIN FEATURES OF THE MODEL

Let us suppose that a quantum system consists of two parts, $A$ and $B$, which have previously been in contact but are no longer interacting. The variables $\varrho$, $r$ describing the

FIG. 1: A quantum system composed of two parts $A$ and $B$, separated by a spherical surface of radius $R$. 

$\varrho$, $r$
system are subjected to the following constraints (see Fig. 1):

\[
\begin{cases}
0 \leq \varrho \leq R & \text{region A} \\
r \geq R & \text{region B},
\end{cases}
\]

where \( R \) is the radius of the spherical surface separating the two regions. It is convenient to introduce two dimensionless variables

\[ x = \frac{\varrho}{R}, \quad y = \frac{r}{R}, \]

subjected to the constraints \( 0 \leq x \leq 1 \) and \( y \geq 1 \).

In the following we will assume that the system is spherically symmetric, in order to treat the problem as one-dimensional in each region \( A \) and \( B \), with all physical properties depending only on the radial distance from the origin.

A. Entanglement entropy

Let \( \psi(x, y) \) be a generic wave function describing the system in Fig. 1 composed of two parts \( A \) and \( B \).

As discussed e.g. in \([20, 21]\), we can provide a description of all measurements in region \( A \) through the so-called density matrix \( \rho_A(x, x') \):

\[
\rho_A(x, x') = \int_B d^3y \, \psi^*(x, y) \psi(x', y),
\]

where \( d^3y \) is related to the volume element \( d^3r \) in \( B \) through the relation \( d^3r = R^3 d^3y \), with \( d^3y = y^2 \sin \theta d\theta d\phi dy \) in spherical coordinates.

Similarly, experiments performed in \( B \) are described by the density matrix \( \rho_B(y, y') \):

\[
\rho_B(y, y') = \int_A d^3x \, \psi^*(x, y) \psi(x, y'),
\]

where \( d^3x \) is related to the volume element \( d^3\varrho \) in \( A \) through the relation \( d^3\varrho = R^3 d^3x \), with \( d^3x = x^2 \sin \theta d\theta d\phi dx \) in spherical coordinates.

Notice that \( \rho_A \) is calculated tracing out the exterior variables \( y \), whereas \( \rho_B \) is evaluated tracing out the interior variables \( x \).

Density matrices have the following properties:

1. \( \text{Tr} \rho = 1 \) (total probability equal to 1)
2. $\rho = \rho^\dagger$ (hermiticity)

3. $\rho_j \geq 0$ (all eigenvalues are positive or zero).

When only one eigenvalue of $\rho$ is nonzero, the nonvanishing eigenvalue is equal to 1 by virtue of the trace condition on $\rho$. This case occurs only if the wave function can be factorized into an uncorrelated product

$$\psi(x, y) = \psi_A(x) \cdot \psi_B(y).$$

(4)

States that admit a wave function are called “pure” states, as distinct from “mixed” states, which are described by a density matrix. A quantitative measure of the departure from a pure state is provided by the so-called von Neumann entropy

$$S = -\text{Tr}(\rho \log \rho).$$

(5)

$S$ is zero if and only if the wave function is an uncorrelated product.

The von Neumann entropy is a measure of the degree of entanglement between the two parts $A$ and $B$ of the system, therefore it is called entanglement entropy.

When the two subsystems $A$ and $B$ are each the complement of the other, entanglement entropy can be calculated by tracing out the variables associated to region $A$ or equivalently to region $B$, since it turns out to be $S_A = S_B$.

B. Wave function

As already said, the spherical region $A$ in Fig. 1 is part of a larger closed system $A \cup B$, described by a wave function $\psi(x, y)$, where $x$ denotes the set of coordinates in $A$ and $y$ the coordinates in $B$.

We will exploit for $\psi$ the following analytic form:

$$\psi(x, y) = Ce^{-\lambda y/x},$$

(6)

where $C$ is the normalization constant and $\lambda$ is a dimensionless parameter, that we assume much greater than unity ($\lambda \gg 1$).

If the system is in a bound state due to a central potential, the complete wave function $\hat{\psi}$ should contain an angular part expressed by spherical harmonics $Y_{lm}(\theta, \phi)$:

$$\hat{\psi}(x, y; \theta, \phi) = \hat{C} Y_{lm}(\theta, \phi) \psi(x, y),$$

where $\hat{C}$ is the normalization constant for the angular part.
where \( \hat{C} \) is the normalization constant. If the angular momentum is zero, the angular component of the wave function reduces to a constant \( Y_{00}(\theta, \phi) = 1/\sqrt{4\pi} \), which can be included in the overall normalization constant \( C = \hat{C}/\sqrt{4\pi} \) appearing in the expression (6) of the wave function \( \psi \).

In order to justify the form (6) of the wave function \( \psi \), let us list the main properties it satisfies.

1) \( \psi \) depends on both sets of variables \( x, y \) defined in the two separate regions \( A \) and \( B \), but it is not factorizable in the product (4) of two distinct parts depending only on one variable:

\[
\psi(x, y) \neq \psi_A(x) \cdot \psi_B(y).
\]

This assumption guarantees that the entanglement entropy of the system is not identically zero.

2) \( \psi \) depends on the variables \( x, y \) through their ratio \( y/x \), hence it is invariant under scale transformations:

\[
x \rightarrow \mu x \quad \text{and} \quad y \rightarrow \mu y, \quad \text{with} \quad \mu \ \text{constant}.
\]

3) \( \psi \) has the asymptotic form of an exponential decay.

The last ansatz on \( \psi \) is equivalent to consider the quantum state of a central potential vanishing at infinity, with negative energy eigenvalues. The asymptotic form of the radial part \( f(r) \) of the wave function describing this state is

\[
f(r) = C_B e^{-\kappa r} = C_B e^{-\kappa Ry},
\]

where \( r \rightarrow \infty \) is the radial distance from the origin, \( C_B \) is the normalization constant and \( y = r/R \). In particular, we could consider a particle with mass \( m \) and negative energy \( E \), in a bound state due to a central potential going to zero as \( r \rightarrow \infty \). In this case, the asymptotic form of the Schrödinger equation, in spherical coordinates, is given by

\[
\frac{d^2 f(r)}{dr^2} = \kappa^2 f(r), \quad \text{with} \quad \kappa = \sqrt{\frac{2m|E|}{\hbar}}.
\]

Apart from the normalization constant, \( f(r) \) coincides with the restriction \( \psi_B(y) \) of the wave function \( \psi(x, y) \) to the exterior region \( B \), as seen by an inner observer localized for instance at \( x = 1 \), i.e. on the boundary between the two regions:

\[
\psi_B(y) = \psi(x, y)|_{x=1} = Ce^{-\lambda y}.
\]
By comparing the asymptotic behaviour of the wave functions (7) and (9) as \( y \to \infty \), we find:

\[
\lambda = \gamma \frac{R}{\ell_p}, \quad \text{with} \quad \gamma = \kappa \ell_p, \quad (10)
\]

where the Planck length \( \ell_p = (\hbar G/c^3)^{1/2} \) has been introduced to make the parameter \( \gamma \) dimensionless, without any further physical meaning in this context. In Section III we will assume \( \lambda \gg 1 \), which is always true in a system with \( R \) sufficiently larger than \( \ell_p \), as it easily follows from Eq. (10).

If the inner observer is not localized on the boundary but in a fixed point \( x_0 \) inside the spherical region (with \( 0 < x_0 < 1 \)), then the expression (10) of \( \lambda \) has to be multiplied by \( x_0 \).

Notice that the dependence of \( \lambda \) on the radius \( R \) of the boundary has been derived by imposing an asymptotic form on the wave function \( \psi(x, y) \) as \( y \to \infty \), with respect to a fixed point \( x \sim 1 \) inside the spherical region of the system in Fig. I.

In Section III we will show that the entropy of both parts of our system depends on \( \lambda^2 \sim R^2/\ell_p^2 \), i.e. on the area of the spherical boundary. The area scaling of the entanglement entropy is, essentially, a consequence of the nonlocality of the wavefunction \( \psi(x, y) \), which establishes a correlation between points inside \( (x \sim 1) \) and outside \( (y \to \infty) \) the boundary.

### III. ANALYTIC RESULTS

We normalize the wave function (6) by means of the condition

\[
\int_A d^3x \int_B d^3y \psi^*(x, y) \psi(x, y) = 1,
\]

with \( d^3x = x^2 \sin \theta d\theta d\phi dx \) and \( d^3y = y^2 \sin \theta d\theta d\phi dy \) in spherical coordinates. Under the assumption \( \lambda \gg 1 \), the normalization constant \( C \) turns out to be

\[
C \approx 2\lambda e^{\lambda} \frac{\lambda}{4\pi}. \quad (11)
\]

Let us focus on the interior region \( A \) represented in Fig. I. We calculate the density matrix \( \rho_A(x, x') \) by tracing out the variables \( y \) related to the subsystem \( B \), as expressed in Eq. (2):

\[
\rho_A(x, x') = \int_B d^3y \psi^*(x, y) \psi(x', y)
\]

\[
\approx 4\pi C^2 \frac{xx'}{\lambda} e^{-\lambda \frac{x+x'}{xx'}}. \quad (12)
\]
where we have assumed $\lambda \gg 1$.

It is easy to verify that the density matrix $\rho_A(x, x')$ satisfies all properties listed in Section II A:

1. Total probability equal to 1:

$$\int_A d^3x \rho_A(x, x) = 1 \iff \text{Tr} (\rho_A) = 1.$$  

2. Hermiticity:

$$\rho_A(x, x') = \rho^*_{A}(x', x) \iff \rho_A = \rho_A^\dagger.$$  

3. All eigenvalues are positive or zero:

$$\rho_A(x, x') \geq 0 \quad \forall \quad x, x' \in (0, 1) \implies (\rho_A)_j \geq 0.$$  

The entanglement entropy (15) can be expressed in the form

$$S_A = - \int_A d^3x \rho_A(x, x) \log[\rho_A(x, x)]. \quad (13)$$

Substituting the expression (12) of $\rho_A(x, x')$, with $x' = x$, we find:

$$S_A \approx 4\pi \frac{C^2}{\lambda} \int_A d^3x e^{-2\lambda/x} \left[ \lambda \frac{x}{2} \log \left( \frac{4\pi C^2 x}{\lambda} \right) \right].$$

We can maximize the previous integral by means of the condition

$$e^{-2\lambda/x} \leq e^{-2\lambda} \quad \forall \quad x \in (0, 1).$$

The entanglement entropy $S_A$ turns out to be bounded by

$$S_A \lesssim (4\pi)^2 C^2 e^{-2\lambda} \left\{ \frac{1}{3} \lambda^2 - \frac{1}{\lambda} \left[ a \log \left( \frac{4\pi C^2}{\lambda} \right) + b \right] \right\},$$

with $a = \frac{1}{8}$ and $b = -\frac{1}{32} (1 + 4 \log 2)$.

By inserting the expression (11) of the normalization constant $C$, we obtain

$$S_A \lesssim \frac{1}{3} \lambda^2 \left\{ 1 - \frac{12}{\lambda} \left[ a \log(\lambda/\pi) + b \right] \right\}.$$  

If we neglect the subleading terms in $\lambda \gg 1$ and substitute $\lambda = \gamma R/\ell_P$, as given in Eq. (10), we finally find:

$$S_A \lesssim \eta A \frac{\ell}{4\ell_P}, \quad \text{with} \quad \eta = \frac{1}{3\pi} \gamma^2,$$  

(14)
where $A = 4\pi R^2$ is the area of the spherical boundary in Fig. 1. The result (14) is in accordance with the holographic bound on entropy $S \leq \frac{A}{4\ell^2}$, introduced in [17, 18], and shows that the entanglement entropy of both parts of our composite system obeys an “area law”, as discussed e.g. in [1, 2] for many-body systems.

For a particle with energy $E$ and mass $m$, satisfying the asymptotic form (8) of the Schrödinger equation, we can express the parameter $\eta$ in the form

$$\eta = \frac{2}{3\pi} \frac{m|E|}{m^2 c^2},$$

where we have combined Eqs. (8), (10), (14) and have introduced, for dimensional reasons, the Planck mass $m_P = (\hbar c/G)^{1/2}$. Under the assumptions $|E| \ll m_P c^2$ and $m \lesssim m_P$, we obtain $\eta \ll 1$, therefore in this case the bound (14) on entropy is much stronger than the holographic bound $S \leq \frac{A}{4\ell^2}$.

All calculations performed in this Section can be repeated focusing on the exterior region $B$ represented in Fig. 1. By tracing out the interior variables $x$, as in Eq. (3), the density matrix $\rho_B(y, y')$ turns out to be

$$\rho_B(y, y') = \int_A d^3x \psi^*(x, y)\psi(x, y')$$

$$\approx \frac{4\pi C^2 e^{-\lambda(y+y')}}{\lambda y + y'},$$

where we have substituted the expression (6) of the wave function $\psi$ and have applied the usual assumption $\lambda \gg 1$.

Analogously to Eq. (13), the entanglement entropy is given by the integral

$$S_B = -\int_B d^3y \rho_B(y, y) \log[\rho_B(y, y)].$$

The numerical evaluation of the previous integral confirms that the entropy bound calculated tracing out the interior variables $x$ coincides, under the assumption $\lambda \gg 1$, with the entropy bound evaluated tracing out the exterior variables $y$, i.e. $S_A = S_B$.

**IV. CONCLUSIONS**

We have proposed a simple approach to the calculation of the entanglement entropy of a spherically symmetric quantum system. The result obtained in Eq. (14), $S_A \lesssim \eta \frac{A}{4\ell^2}$, is in
accordance with the holographic bound on entropy introduced in [17, 18] and with the “area law” discussed e.g. in [1, 2]. Our result, in fact, shows that the entanglement entropy of both parts of the system in Fig. 1 depends only on the area of the boundary that separates the two regions.

The area scaling of the entanglement entropy is a consequence of the nonlocality of the wave function, which relates the points inside the boundary with those outside. In particular, we have derived the area law for entropy by imposing an asymptotic behaviour on the wave function $\psi(x, y)$ as $y \to \infty$, with respect to a fixed point $x \sim 1$ inside the interior region.

The main limit of our model is that we have considered only one particular form of the wave function $\psi$. However, more general forms of $\psi$ might be considered, hopefully, in future developments of the model. Let us finally stress that our results are valid at the leading order in the dimensionless parameter $\lambda \gg 1$ appearing in the wave function $\psi$ of the system.

The treatment presented in this paper for the entanglement entropy of a quantum system is an extremely simplified model, but the accordance of our result with the holographic bound and with the area scaling of the entanglement entropy indicates that we have isolated the essential physics of the problem in spite of all simplifications.

**Acknowledgments**

The author is very greatful to Prof. M. Cadoni for helpful discussions and criticism.

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