Recursion relations and scattering amplitudes in the light-front formalism

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Abstract

The fragmentation functions and scattering amplitudes are investigated in the framework of light-front perturbation theory. It is demonstrated that, the factorization property of the fragmentation functions implies the recursion relations for the off-shell scattering amplitudes which are light-front analogs of the Berends-Giele relations. These recursion relations on the light-front can be solved exactly by induction and it is shown that the expressions for the off-shell light-front amplitudes are represented as a linear combinations of the on-shell amplitudes. By putting external particles on-shell we recover the scattering amplitudes previously derived in the literature.

1. Introduction

Quantization procedure on the the light-front (or the null-plane) was first proposed a long time ago by Dirac [1] as an alternative approach to the more standard instant-time quantization. One of the interesting features of the light-front quantization is the presence of only three dynamical Poincaré generators which describe the evolution of a system in light-front time, see for example [2]. Thus, one may hope that the light-front formalism may lead to a simpler solution of problems in relativistic quantum mechanics than other quantization schemes which typically possess larger number of dynamical operators. It can be also shown that, there exists a subgroup on the light-front which exhibits algebraic structure isomorphic to the Galilean symmetry group of non-relativistic quantum mechanics in two dimensions [3, 4]. It has been also argued, see for example [5], that the vacuum on the light-front is essentially structureless (the arguments about simplicity of the vacuum have been provided earlier by analysis of graphs in the infinite momentum frame [6]). This stems from the fact that the lines in the diagrams for amplitudes only have positive $p^+$ momenta which dramatically reduces the number of diagrams which are needed to be considered and eliminates vacuum graphs. To be precise, the vacuum on the light-front is structureless up to zero modes, for which special treatment may be necessary, like discrete quantization which isolates these modes, see for example [7]. It was also shown that zero modes contribute to the Higgs VEV in the standard model [8]. Due to the apparent simplicity of vacuum, light-front methods have been also used to study the chiral symmetry breakdown, for a recent nice review see [9]. In any case this property of the light-front vacuum allows to define unambiguously the partonic content of hadrons and of hadronic wave functions and has been used to argue about the presence of in-hadron quark condensates [10]. The light-front framework has been used to investigate the hadron dynamics from AdS/CFT correspondence [11] and in the high energy approximation to compute the soft gluon component of the heavy onium wave function and to obtain a correspondence with the hard Pomeron in QCD [12].

In the previous works [13, 14] we have investigated in some detail the gluon wave functions, fragmentation functions and scattering amplitudes within the framework of the light-front perturbation theory (LFPT). The wave functions differ from fragmentation functions in a way the light-front energy denominators are treated. In the case of the wave functions the energy denominators are kept for the last state (which is kept off-shell). For the fragmentation functions, the last state is on-shell whereas the first incoming particle is off-shell and the corresponding energy denominator is non-zero. Otherwise there exists close relations between these two objects as they both describe transitions...
for 1 → n particles. It has been shown in [13, 14] that one can construct the recursion relations for each of these objects (related factorization properties or cluster decomposition of the light front wave functions were discussed earlier in [15] and the ladder relations between different Fock state components were constructed in [16]). In simpler cases, some of these recursion relations have been solved exactly and the solution for arbitrary number of gluons in the wave function and fragmentation functions has been found. This was done for the case of gluon wave functions and fragmentation functions in the case when the helicities of the outgoing gluons are the same. Interestingly, the compact recursion relations for the gluon wave functions derived are related to the vanishing property of the on-shell helicity amplitudes for these selected configurations of the helicities. It turns out that the property of vanishing of the amplitudes for special cases of the helicities, which was proven [17, 18] using supersymmetry relations, in the light-front formalism originates from the angular momentum [19, 20, 21] and energy conservation laws. The recursion relations were then generalized to include a different configuration of helicities, and in the case of the fragmentation functions they are light-front analogs of the Berends-Giele recursion relations [22]. It was also shown that there are general relations between gluon wave functions and scattering amplitudes. More precisely, it was demonstrated that the amplitude $M$ for 2 → n can be obtained through analytical continuation from the light-front wave functions $\Psi_{n=1}$, which contain, in general, a smaller number of graphs. This property was used to reproduce some lowest order results for the scattering amplitudes which were previously available in the literature, [23].

Despite the fact that there has been an enormous progress in the computation of the multi-particle helicity amplitudes in QCD and many results are now implemented into numerical automated algorithms, (see for example [24, 25, 26, 27, 28, 29]), it is still an interesting question to ask whether the obtained results for the scattering amplitudes can be derived in a simpler way in the light-front theory, thus giving a better insight into the structure of the theory. For example, one of the interesting aspects of using the LFPT is the fact that the variables used to express the helicity amplitudes naturally arise in this framework.

In this paper we make significant progress towards answering this question by showing how to derive the maximally helicity violating (MHV) amplitudes in the framework of light-front perturbation theory. We will use the previously constructed recursion relations for the fragmentation functions, which are then solved exactly. The solution for the (off-shell) scattering amplitude is proved by mathematical induction. The method, of course, can be used to find the amplitudes for arbitrary configuration of helicities of outgoing particles. The obtained result is more general than the on-shell amplitude, since it is the solution for an off-shell amplitude with the non-vanishing energy denominator in the first state. Interestingly, it is expressed as a linear combination of the on-shell amplitudes with different number of external legs, with the first term being proportional to the on-shell amplitude with maximum number of external legs and the subsequent terms containing energy denominator terms. These terms vanish when evaluating the on-shell amplitude, thus reproducing the exact MHV result.

The structure of this paper is the following. In the next section we shall set up a useful notation, and discuss some preliminaries about light-front calculations. We shall recall previously derived factorized recursion relations for the fragmentation functions, which are the light-front analogs of Berends-Giele recursion relations. In Sec. 3 recursion relations for the off-shell amplitudes are constructed, which are directly related to the recursion relations for the light-front fragmentation functions. In Sec. 4 we present the solution to the recursion relations and show that it reduces to the MHV amplitude. Finally in Sec. 5 we state the conclusions.

2. Light-front perturbation theory and fragmentation functions

In this section we briefly recall the notation and variables used in the previous works [13, 14] which will be used in the current paper. We will mainly consider transitions from one to $n + 1$ final state particles or transitions from 2 to $n$ final state particles. We shall specialize in this paper to amplitudes with gluons as external particles. In this paper we will use as the building blocks the 'fragmentation functions', which describe the transition of one particle (ex. a gluon) which is off shell into $n$ on-shell final state particles. This object, introduced in [13], will be denoted by $T_n$ and is depicted in Fig. 1. The off-shell initial gluon is labeled by $1 \ldots n$ and the final state on-shell gluons are labeled as $1, \ldots, n$. The initial gluon has transverse momentum $k_{(1...n)} = \sum_{i=1}^{n} k_i$ and longitudinal momentum fraction $z_{(1...n)} = \sum_{j=1}^{n} z_j$.

The momenta of the last $n$ gluons are labeled $k_1, \ldots, k_n$, as in Fig 1. Each of these momenta can be represented as $k_i = (z_i P^+, k_i^- \mathcal{L})$, with $z_i$ being the fraction of the initial $P^+$ momentum which is carried by the gluon labeled by $i$ and $k_i^-$ being the transverse component of the gluon momentum. $P^+$ is the total longitudinal momentum in the initial
Figure 1: Pictorial representation of the fragmentation amplitude $T_n[(1 \ldots n)^{k_0} \rightarrow 1^{k_1}, 2^{k_2}, \ldots, n^{k_n}]$ for a single off-shell initial gluon. Variables $\lambda_0, \ldots, \lambda_n$ denote the polarization of the gluons. The initial gluon $(1 \ldots n)$ fragments into $n$ final state gluons $1, \ldots, n$. The vertical dashed line indicates that for this part of the diagram one needs to take an energy denominator, i.e. the leftmost gluon is in an intermediate state. The other energy denominators which are taken for the intermediate states inside the blob are implicit and are not shown in the picture.

state, for the example depicted in Fig. 1 it could be the initial state of the total graph to which the subgraph in Fig. 1 is attached. In the LFPT [6, 30, 31, 32, 33] one has to evaluate the energy denominators for each of the intermediate states for the process. The energy denominator for say $j$ intermediate gluons is defined as the difference between the light-front energies of the final and intermediate state in question

$$D_j = \sum_{\text{out}} E_i - \sum_{i=1}^{j} E_i,$$

where

$$E_i(0) \equiv k_i(0) = \frac{k_i^2(0)}{k_i^2(0)},$$

are the light-front energies and the first sum represents a sum over the energies of all final state gluons present in the fragmentation function. Furthermore, one has to sum over all possible vertex orderings. The fragmentation function shown in example in Fig. 1 would thus be given schematically by the expression

$$T_n \sim \sum_{\text{vertex orderings}} g^{n-1} \Pi_{j=1}^{n-1} \frac{V_j}{z_j D_j},$$

where $V_j$ are the vertices and $z_j$ and $D_j$ are the corresponding fractional momenta and denominators for all the intermediate states. Note the important fact that for the fragmentation function depicted in Fig. 1 the first gluon is not really an initial state. As mentioned above, it is understood that the fragmentation function is only a subgraph, attached via this gluon to a bigger graph. Therefore, the leftmost gluon is in fact an intermediate state for which the energy denominator, denoted by the dashed line, has to be taken into account. The rightmost gluons are the final on-shell particles, and the energy denominator is not included there. Finally, one needs to sum over all the vertex orderings in the light-front time. The results derived in [13] and in the following sections are for the color ordered multi-gluon amplitudes. Hence, we focus only on the kinematical parts of the subamplitudes.

The fragmentation function for a special choice of the helicities was evaluated exactly in [13]. The explicit results for the transition $+ \rightarrow + \cdot \cdot \cdot +$ reads

$$T_n[(12\ldots n)^{+} \rightarrow 1^{+}, 2^{+}, \ldots, n^{+}] = (-ig)^{n-1} \left( \frac{z_{1\ldots n}}{z_1 \ldots z_n} \right)^{3/2} \frac{1}{v_{n-1} v_{n-2} \ldots v_2},$$

where the variables $v_{ij}$ were defined as

$$v_{ij} := \left( \frac{k_j}{z_j} - \frac{k_i}{z_i} \right), \quad \Sigma_j := \epsilon^{(-)} \cdot v_j,$$

and $\epsilon^{(-)}$ will be defined shortly. It is well known [3, 30, 31] that on the light-front the Poincaré group can be decomposed onto a subgroup which contains the Galilean-like nonrelativistic dynamics in 2-dimensions. The `$+$' components of the momenta can be interpreted as the 'masses'. In this case the variable $\Sigma_j$ can be interpreted as a relative
transverse light-front velocity of the two gluons. The same variable is present when evaluating the energy denominators of different intermediate states. The above variable is closely related to the variables used in the framework of helicity amplitudes, see [34].

For a given pair of momenta $k_i$ and $k_j$ we have the result

$$
\langle ij \rangle = \sqrt{\varepsilon_i \varepsilon_j} \langle \psi \langle k_i \rangle \cdot \psi \langle k_j \rangle \rangle = \sqrt{\varepsilon_i \varepsilon_j} \langle \psi \langle k_i \rangle \cdot \psi \langle k_j \rangle \rangle,
$$

where the variables $\langle ij \rangle$ and $[ij]$ are defined by

$$
\langle ij \rangle = (i - |j^+|), \quad [ij] = (i + |j^-|),
$$

and where chiral projections of the spinors for massless particles are defined as

$$
|\pm\rangle = \psi_{\pm}(k_i) = \frac{1}{2}(1 \pm \gamma_5)\psi_{\pm}(k_i), \quad \langle \pm | = \overline{\psi}_{\pm}(k_i),
$$

for a given momentum $k_i$. Above, we have also introduced the polarization four-vector of the gluon with four-momentum $k$

$$
\varepsilon^{(\pm)} = \varepsilon^{(\pm)} + \frac{2\varepsilon^{(\pm)} \cdot k}{\eta \cdot k} \eta,
$$

where $\varepsilon^{(\pm)} = (0, 0, \varepsilon^{(\pm)})$, and the transverse vector is defined by $\varepsilon^{(\perp)} = \frac{1}{\sqrt{2}}(1, \pm i)$. Vector $\eta$ is related to the choice of the light-cone gauge, $\eta \cdot A = 0$, where $\eta^\mu = (0, 2, 0)$ in the light-front coordinates. It is interesting that in the light-front formalism the variables $\langle ij \rangle$ appear naturally in the vertices and in the energy denominators.

The fragmentation functions introduced above possess an important property which will be widely utilized in this paper. Namely, it was demonstrated in [13] that the fragmentation functions factorize after the summation over all the light-front time orderings. This property can then be used to write down the explicit recursion formula for the fragmentation functions. That is to say, the fragmentation into $n+1$ gluons which is denoted by $T_{n+1}[(1, 2, \ldots, n+1) \rightarrow 1, 2, \ldots, n+1]$ can be represented as the product of two lower fragmentation functions $T_i[(1 \ldots i) \rightarrow 1, \ldots, i]$ and $T_{n+1-i}[(i+1 \ldots n+1) \rightarrow i+1, \ldots, n+1]$. Finally, one needs to sum over the splitting combinations. This procedure is schematically expressed in Fig. 2 and, to be precise, the expression which reflects the factorization reads

$$
T_{n+1}[(12 \ldots n+1) \rightarrow 1, 2, \ldots, n+1] = \frac{2ig}{D_{n+1}} \sum_{i=1}^{n} \left\{ \frac{\nu_i^{(12 \ldots n+1)}}{\varepsilon^{(12 \ldots n+1)}} \times T_i[(1 \ldots i) \rightarrow 1, \ldots, i] T_{n+1-i}[(i+1 \ldots n+1) \rightarrow i+1, \ldots, n+1] \right\}. \quad (10)
$$

Figure 2: Pictorial representation of the factorization property represented in Eq. (10), a light-front analog of the Berends-Giele recursion relations [22]. The helicities of the outgoing gluons are chosen to be the same in this particular case. The dashed vertical line indicates the energy denominator $D_{n+1}$.

The energy denominator $D_{n+1}$ in the above equation has been defined as

$$
D_{n+1} = \frac{k_1^2}{z_1} + \frac{k_2^2}{z_2} + \ldots + \frac{k_{n+1}^2}{z_{n+1}} - \frac{k_{n+1}^2}{z_{n+1}}, \quad (11)
$$
i.e. the overall $P^+$ momentum cancels with the normalization of the internal lines (note the difference with respect to the definition of $D$ previously).

The following complex representation of the transverse vectors has been introduced $v_{ij} = \xi^{(i)} \cdot \mathbf{v}_{j}$, $v'_{ij} = \xi^{(i)} \cdot \mathbf{v}_{j}$, and along with a useful notation,

$$
\mathcal{M}_{h_1 h_2 \ldots h_p j_1 j_2 \ldots j_q} = \frac{k_{i_1} + k_{i_2} + \ldots + k_{i_p}}{z_i + z_2 + \ldots + z_p} - \frac{k_{j_1} + k_{j_2} + \ldots + k_{j_q}}{z_{j_1} + z_{j_2} + \ldots + z_{j_q}},
$$

$$
\xi_{h_1 h_2 \ldots h_p j_1 j_2 \ldots j_q} = \frac{(z_{i_1} + z_{i_2} + \ldots + z_{i_p})(z_{j_1} + z_{j_2} + \ldots + z_{j_q})}{z_{i_1} + z_{i_2} + \ldots + z_{i_p} + z_{j_1} + z_{j_2} + \ldots + z_{j_q}}.
$$

(12)

(13)

We also introduced notation for the partial sums $z_{(1\ldots)} = z_1 + z_2 + \ldots + z_t$ and $k_{(1\ldots)} = k_1 + k_2 + \ldots + k_t$.

It turns out that the above defined fragmentation functions $T_n$ are related to the gluonic currents which are building blocks in the Berends-Giele recursion relations [22]. These recursive relations utilize the (gauge-dependent) current $J^\mu$, which is obtained from the scattering amplitudes by putting one of the particles off-shell. The dual subamplitudes can be obtained by contraction with the polarization vector and setting the gluon back on-shell, [23]

$$
M(0, 1, 2, \ldots, n) = i P^\mu J_\mu(1, 2, \ldots, n)_{P_{n-1} P} ,
$$

(14)

where we have defined $P = \sum_{i=1}^n p_i$, and in this formula $p_i$ denote the four-vectors for the momenta of the outgoing particles. In the light-front perturbation theory the current can also be defined and is related to the fragmentation function as

$$
T_n((12 \ldots n) \rightarrow 1, 2, \ldots, n) \equiv e^\mu(12 \ldots n) J_\mu(1, 2, \ldots, n),
$$

(15)

where by $e^\mu(12 \ldots n)$ we denote the polarization vector of the incoming (off-shell) gluon in the fragmentation function. With such a definition the factorization property for the fragmentation function [10] is a light-front analog of the Berends-Giele [22] recursion formula (see also [23] for the recursion relations in the light-cone gauge)

$$
J^\mu(1, 2, \ldots, n) = -\frac{i}{P^2} \sum_{i=1}^{n-1} V_3^{\nu_{ij}}(p_{1, i}, \ldots, p_{i, n}) J_\nu(1, \ldots, i) J_\mu(i + 1, \ldots, n)
$$

$$
- \frac{i}{P^2} \sum_{i=j+1}^{n-1} \sum_{j=1}^{n-2} V_4^{\nu_{ijkl}} J_\nu(1, \ldots, j) J_\nu(i + 1, \ldots, i) J_\mu(i, \ldots, n).
$$

(16)

The simpler form of [10] (as compared to [16]), which only includes 3-gluon vertex, stems from the fact that it has been written for a particular configuration of helicities, namely for identical helicities of outgoing particles. It is possible to write down a general factorization (recursion) relation for the fragmentation function which will include the 4-gluon vertex as well as the Coulomb term. We will investigate and use this more complex case in the next section. We should also remark that the Berends-Giele relations are written on the level of individual diagrams, whereas for the derivation of the analogous recursion relations on the light-front [10], the summation over the different orderings in light-front time is necessary to decouple the fragmentation trees.

3. Recursion relations for off-shell amplitudes on the light-front

The main goal of this section is to reproduce, within the light-front perturbation theory, the Parke-Taylor [23] amplitudes by solving the appropriate recursion relations. In the following, we will mostly deal with the light-front matrix elements which describe transitions from 1 to $n + 1$ gluons. It has been demonstrated that one can obtain easily the amplitudes for 2 to $n$ transitions from the 1 to $n + 1$ transitions [14], and on the light-front the latter typically involve a smaller number of graphs to evaluate. The reason for dealing with the matrix elements for 1 to $n + 1$ transitions is that we can directly utilize the factorization property for the fragmentation functions mentioned above. Thus, following [14], let us introduce the following notation

$$
\mathcal{A}(k_1, z_1; \ldots; k_n, z_n) = -N M_{1+n}(k_1, z_1; k_2, z_2; \ldots; k_n, z_n),
$$

(17)
where $\mathcal{A}_{2 \rightarrow n}$ is the helicity amplitude for 2 particles going to $n$ particles, for which one needs to set the initial and final state particles as on-shell, and $N$ is a normalization factor which will be specified later.

The above relation is illustrated in Fig. 3 and was discussed at length in previous work [14] and used to compute the on-shell amplitudes. This is obtained by assuming the incoming gluons are off-shell and it is similar to the Berends-Giele recursion relations which involve the currents $J^*$ necessary for evaluation of the on-shell amplitudes. $M_{1 \rightarrow n+1}$ is more general than $M_{1 \rightarrow n+1}$ in the sense that

$$M_{1 \rightarrow n+1} = \frac{1}{N} M_{1 \rightarrow n+1}|_{D_{n+1} \rightarrow 0},$$

where $D_{n+1}$ is the energy denominator for the first state. In the following, we will concentrate on a particular configuration of helicities, $M_{1 \rightarrow n+1}(+ \rightarrow + \cdots +)$, which corresponds to transition of $2 \rightarrow n$ particles for $(++ \rightarrow + \cdots +)$ when the two particles are incoming and the rest is outgoing or to scattering amplitude ($- - \cdots -$) when all the particles are outgoing. A key component in this derivation will be the fragmentation functions as defined in [13], since it allows us to write $M_{1 \rightarrow n+1}$ as the sum of all the graphs in Fig. 4. The convenience comes from the use of the factorization property of fragmentation functions. One can write,

$$M_{1 \rightarrow n+1} = \sum_{j=2}^{n} V_+ \sqrt{\frac{z_{1j} z_{2j} \cdots z_{n+1}}{z_{1j} z_{2j} \cdots z_{n+1}}} T_j[(1 \ldots j)^+ \rightarrow 1^{-}, 2^+, \ldots, j^+] T_{n+1-j}[(j+1 \ldots n+1)^+ \rightarrow (j+1)^+, \ldots, (n+1)^+]$$

$$+ \sum_{j=1}^{n} V_- \sqrt{\frac{z_{1j} z_{2j} \cdots z_{n+1}}{z_{1j} z_{2j} \cdots z_{n+1}}} T_j[(1 \ldots j)^- \rightarrow 1^{-}, 2^+, \ldots, j^+] T_{n+1-j}[(j+1 \ldots n+1)^+ \rightarrow (j+1)^+, \ldots, (n+1)^+]$$

$$+ \sum_{j=2}^{n} \sum_{j=1}^{j-1} (V_+ + V_{\text{Coul}}) \sqrt{\frac{z_{1j} z_{2j} \cdots z_{n+1}}{z_{1j} z_{2j} \cdots z_{n+1}}} T_{j-1}[(1 \ldots i)^- \rightarrow 1^{-}, 2^+, \ldots, i^+] T_{n+1-j}[(j+1 \ldots n+1)^+ \rightarrow (j+1)^+, \ldots, (n+1)^+].$$

The first, second and third line come from Fig. 4a, Fig. 4b and Figs. 4c - 4d respectively. The $V$’s are the vertex factors and these are given by

$$V_+ = 2 g z_{1n+1} V_{(j+1 \ldots n+1)(1 \ldots j)}',$$

$$V_- = 2 g z_{j1 \ldots n} V_{(1 \ldots n+1)(1 \ldots j)}'.$$
\[ V_\delta = ig^2, \]
\[ V_{\text{Coul}} = ig^2 \frac{(z_{i1,n+1} + z_{i+j+1,n+1})(z_{i1,j} - z_{i1,i})}{(z_{i1,n+1} - z_{i+j+1,n+1})^3}. \]

Inspecting formula (19) we see that the fragmentation functions involved in the process correspond to three different helicity configurations. One of them \( T_n[(12\ldots n)^+ \to 1^+, 2^\ldots,n^+] \) was found in (13) and its explicit expression was given in Eq. 4. The second one can be easily derived using similar methods (see Appendix A) with the result

\[ T_n[(12\ldots n)^- \to 1^-, 2^+, \ldots,n^+] = (-ig)^{n-1} \left( \frac{z_{i1,n}}{z_{i1,n}} \right)^{3/2} \frac{1}{V_{n-1}V_{n-1-n}V_{21}}. \]

The third fragmentation function, \( T_n[(12\ldots n)^+ \to 1^-, 2^+, \ldots,n^+] \), however, remains unknown. To find it we would, once again, need the graphs depicted in Fig. 4. This implies a relationship between \( M_{1-n} \) and \( T_n[(12\ldots n)^+ \to 1^-, 2^+, \ldots,n^+] \) which one can express as

\[ T_n[(12\ldots n)^+ \to 1^-, 2^+, \ldots,n^+] = \frac{1}{\sqrt{V_{1-n} \cdots V_n}} i M_{1-n}. \]

Therefore this fragmentation function is directly proportional to \( M_{1-n} \), but it includes the denominator for the first (leftmost state) and different normalization of the external particles. Thus, Eq. (19) which is depicted in Fig. 4 turns out to be a recursion relation for \( M_{1-n+1} \). In the next section we will find a solution to this equation and prove it via the method of mathematical induction.

Figure 4: Graphs involved in the fragmentation of a single off-shell gluon into \( n+1 \) on-shell gluons. The initial and final helicities are specified in the figures. We denote the 3-gluon vertex in Figs. (a) and (b) as \( V_\delta \) and \( V_- \) respectively, the 4-gluon vertex in Fig. (c) as \( V_\delta \), and the Coulomb term in Fig. (d) as \( V_{\text{Coul}} \). Vertical lines denote the energy denominators that need to be taken, they are implicit in all intermediate states denoted by blobs. There are no energy denominators in the final state.
4. Explicit solution to the recursion formula

In this section we will solve the recursion formula given in Eq. (19) supplemented by relation (25) and find the general expression for the off-shell amplitude \( \overline{M}_{1-n+1} \). We shall see that it can be expressed as a linear combination of the on-shell amplitudes with different number of external legs. By putting on-shell constraint on the external gluons, that is by putting \( D_{n+1} = 0 \), the on-shell MHV amplitude will be reproduced from the solution.

Let us start with the initial conditions for the recursion formula. The normalization is such that the initial fragmentation functions are set to \( T_1[1^+ \rightarrow 1^-] = 0 \), \( T_1[1^+ \rightarrow 1^+] = T_1[1^- \rightarrow 1^-] = 1 \). Finding \( \overline{M}_{1-2} (n = 1) \) is trivial,

\[
\overline{M}_{1-2} = 2g z_2 v_{1(12)} | \frac{z_1 z_2}{z_2} | \frac{v_{1(12)}}{v_{1(12)}} = 2g \left( \frac{z_1 z_2}{z_2} \right) \frac{v_{1(12)}}{v_{1(12)}} = 2g \frac{z_1 z_2}{z_2} \frac{v_{1(12)}}{v_{1(12)}} = 2 g 1 M_{1-2} , \tag{26}
\]

where we defined

\[
M_{1-2} = \frac{z_1 z_2}{z_2} v_{1(12)} . \tag{27}
\]

Finding \( \overline{M}_{1-3} (n = 2) \) is much more complicated and we should remark that the order in which terms will be added is the same as when we perform the proof via induction. The order is important because it makes the structure of the solution easier to see. To begin, it will be convenient for our calculations to add \( V_4 \) and \( V_{\text{Coul}} \) to get

\[
V_4 + V_{\text{Coul}} = V_{\text{comb}, a} + V_{\text{comb}, b} , \tag{28}
\]

where

\[
V_{\text{comb}, a} = 2i g^2 \frac{z_{1+n+1} z_{j+1-j}}{z_{1-j}} , \tag{29}
\]

\[
V_{\text{comb}, b} = -2i g^2 \frac{z_{j-i} z_{j+1-n+1}}{z_{1-j}} . \tag{30}
\]

Thus, we can replace Figs. 4c and 4d with Figs. 5a and 5b. The white and black blobs represent the contributions from the vertices \( V_{\text{comb}, a} \) and \( V_{\text{comb}, b} \) respectively. From recursion (19) we see that there are five different terms which contribute to \( \overline{M}_{1-3} \):

\[
I = 2g z_2 v_{1(12)} | \frac{z_1 z_2 z_3}{z_1 z_2} | \frac{1}{v_{121} z_2} | \frac{i}{D_2} | \overline{M}_{1-2} | , \tag{31}
\]

\[
II = 2g z_2 v_{1(12)} | \frac{z_1 z_2 z_3}{z_1 z_2} | \frac{(-ig)^{1/2}}{v_{32}} | \frac{1}{v_{32}} = -2i g^2 \frac{z_{123}}{z_1 z_2 z_3} v_{1(21)} v_{121} , \tag{32}
\]

\[
III = 2g z_2 v_{1(12)} | \frac{z_1 z_2 z_3}{z_1 z_2 z_3} | \frac{(-ig)^{1/2}}{v_{32}} | \frac{1}{v_{32}} = -2i g^2 \frac{z_{123}}{z_1 z_2 z_3} v_{1(21)} v_{121} , \tag{33}
\]

\[
IV = 2i g^2 \frac{z_{123}}{z_1 z_2 z_3} | \frac{1}{v_{121}} = -2i g^2 \frac{z_{123}}{z_1 z_2 z_3} M_{1-2} , \tag{34}
\]

\[
V = -2i g^2 \frac{z_{123}}{z_1 z_2 z_3} | \frac{1}{v_{121}} . \tag{35}
\]

Here we have already written IV in terms of \( M_{1-2} \) as a simple example of how the term coming from Fig. 5b will be simplified later on. Next, III is added to \( V \) to get

\[
VI = III + V = -2i g^2 \frac{z_{123}}{z_1 z_2 z_3} | \frac{1}{v_{121}} + \frac{1}{v_{121}} = -2i g^2 \frac{z_{123}}{z_1 z_2 z_3} v_{1(21)} v_{121} , \tag{36}
\]
which added to II gives
\[
\text{VI + II = VII} = -2ig^2v_{1(23)i} \left[ \frac{z_{12}^2}{z_{23}} \frac{1}{z_{23}v_{12} + \frac{1}{z_{12}v_{21}}} \right].
\] (37)

However, using the following relation
\[
\frac{z_{123}z_2}{z_{12}} + \frac{z_1z_{12}^2}{z_{122}} v_{3(23)} + \frac{z_1^2}{z_{12}} v_{3(23)} = \frac{1}{z_{122}} v_{12v23} \left[ z_{12}z_2^2v_{12}v_{23} + z_1z_{12}^2v_{3(23)}v_{23} + z_1^2z_{23}v_{3(23)}v_{12} \right]
\]
\[
= \frac{1}{z_{122}} v_{12v23} \left[ z_{12}z_2v_{12} (z_2v_{23} + z_2v_{3(23)})) + z_1z_2v_{3(23)} (z_3v_{23} + z_2v_{32}) \right]
\]
\[
= \frac{z_{123}z_1}{z_{122}} v_{12v23} \left[ -z_{12}v_{2(13)i} \right],
\] (38)
\[
\text{we see that term VII can be written as}
\]
\[
\text{VII} = -2ig^2v_{1(23)i} \left[ M_{1\rightarrow 3} \right]_{12} \left[ \frac{1}{z_{23}v_{12(23)i}} - \frac{1}{z_{23}v_{12(23)i}} + 2v_{3(2)}} \right].
\] (42)

Now that \( \overline{M}_{1\rightarrow 3} = I + IV + VII \) is written completely in terms of on-shell amplitudes we can collect terms proportional to \( M_{1\rightarrow 3} \) to get,
\[
\overline{M}_{1\rightarrow 3} = -2ig^3M_{1\rightarrow 3} + 2ig^2 \frac{z_{123}v_{23}}{z_{12}} M_{1\rightarrow 3} - 2ig^2 \frac{z_{123}}{z_{122}z_{12}} v_{3(23)} \left[ M_{1\rightarrow 2} \right]_{123} \left[ \frac{1}{z_{3}v_{3(23)i}} + \frac{1}{z_{3}v_{3(23)i}} - \frac{2v_{3(2)}}{D_2} \right],
\] (43)
\[
= -2ig^2M_{1\rightarrow 3} - 2ig^2 \frac{z_{123}}{z_{122}z_{12}} v_{3(23)} \left[ M_{1\rightarrow 2} \right]_{123} \left[ \frac{1}{z_{3}v_{3(23)i}} + \frac{1}{z_{3}v_{3(23)i}} - \frac{2v_{3(2)}}{D_2} \right]
\]
\[
\times \left[ D_2 \left( -z_{123}z_2v_{12(23)i} + z_3z_{23}v_{3(23)} \right) + z_2v_{12(23)i} (2z_{123}v_{3(2)}v_{3(2)}(23)) \right].
\] (45)

Using
\[
2z_{123}v_{3(2)}v_{3(23)} = -z_2z_3 \left( \frac{k_2}{z_3} - \frac{k_2}{z_2} \right) = z_{123}(D_2 - D_3),
\]
and
\[
- z_{123}z_2v_{12(23)i} + z_3z_{23}v_{3(23)} = -z_{123}z_2v_{12(23)i} - z_2z_{123}v_{12(23)i} = -z_{123}z_2v_{12(23)i},
\] (47)
our final result for \( \overline{M}_{1\rightarrow 3} \) is then
\[
\overline{M}_{1\rightarrow 3} = 2\beta^3g^2M_{1\rightarrow 3} - 2\beta^2g^2 \frac{z_{123}}{z_{23}z_{12}} v_{3(23)} \left[ M_{1\rightarrow 2} \right]_{123} \left[ \frac{1}{z_{3}v_{3(23)i}} + \frac{1}{z_{3}v_{3(23)i}} - \frac{2v_{3(2)}}{D_2} \right],
\] (48)
where \( M_{1\rightarrow 3} \) is a special case of the general definition for arbitrary number \( n \) of particles
\[
M_{1\rightarrow n} = \frac{z_{1\rightarrow n}z_{1\rightarrow n} \cdots z_{n} \cdots v_{12v23} \cdots v_{n-1}v_{n(1\rightarrow n)}}{z_{23} \cdots z_{n} v_{3(23)} \cdots v_{3(n(1\rightarrow n))}},
\] (49)
which (up to some factors) is the on-shell scattering amplitude for \( 2 \rightarrow n - 1 \) transition, \( {\mathcal{A}}_{2-n-1} \), see Eq. (17).

The next step in the iteration, \( \overline{M}_{1\rightarrow 4} \), can be found following the same procedure, yet it is a much more tedious process. The result ends up being
\[
\overline{M}_{1\rightarrow 4} = 2\beta^4g^3 \left( M_{1\rightarrow 4} - D_4 \frac{z_{123}v_{34}}{z_{23}z_{12}z_{123}v_{34}} \left[ M_{1\rightarrow 3} \right]_{123} \left[ \frac{1}{z_{3}v_{3(23)i}} + \frac{1}{z_{3}v_{3(23)i}} - \frac{2v_{3(2)}}{D_2} \right] \right).
\] (50)
Interestingly, the off-shell amplitude is expressed as a linear combination of the on-shell objects with the pre factors which are proportional to the energy denominators. In particular we see that by putting the on-shell constraint $D_4 = 0$ we recover the on-shell amplitude.

Following the pattern found, one would then expect, for a general integer $n \geq 2$,

$$\overline{M}_{1-n} = 2g^n \mathcal{M}_{1-n} = 2g^n \sum_{i=2}^{n-1} \frac{1}{z_i} \frac{1}{z_{i+1}} \cdots \frac{1}{z_n} \frac{M_{1-i}}{v_{i+1} v_{i+2} \cdots v_{n+1} v_{n} v_{i+1} v_{i+2}}. \quad (51)$$

We shall now present the proof of this result using the method of mathematical induction. Before we begin, the following are two relationships which we will use many times in the rest of this paper,

$$z_{2j+1} v_{i+1} = z_{2j+1} v_{i+2} = \sum_{i=1}^{j} z_{2j+1} v_{i+1} \quad (52)$$

and

$$z_{2j+1} v_{i+1} = z_{2j+1} v_{i+2} = \sum_{i=j}^{n} z_{2j+1} v_{i+1} \quad (53)$$

To perform the proof we assume (51) is true and then use it in (19) to find $M_{n+1}$. At the end, $M_{n+1}$ should be of the

$$\overline{M}_{n+1} = 2g^{n+1} \mathcal{M}_{n+1} = 2g^{n+1} \sum_{i=2}^{n-1} \frac{1}{z_i} \frac{1}{z_{i+1}} \cdots \frac{1}{z_n} \frac{M_{1-i}}{v_{i+1} v_{i+2} \cdots v_{n+1} v_{n} v_{i+1} v_{i+2}}.$$

for form given by expression (51) for $n \to n+1$. Let us remind that for the result one needs to add all the contributions from Figs. 4a, 4b, 5a and 5b. We begin with Fig. 5a. For fixed $j$, the expression for this graph reads

$$E_j = -\sum_{i=1}^{j-1} 2(-i)^{n-1} g^n \frac{z_{i+1} z_{i+2} \cdots z_{n+1}}{z_{i+1} z_{i+2} \cdots z_{n+1} v_{n+1} v_{i+1} v_{i+2} v_{i+3} \cdots v_{21}}.$$

Next, we add the expressions for the graphs presented in Figs. 4b and 5b for fixed $j$ ($j \neq 1$),

$$A_j = 2(-i)^{n-1} g^n \frac{z_{i+1} z_{i+2} \cdots z_{n+1}}{z_{i+1} z_{i+2} \cdots z_{n+1} v_{n+1} v_{i+1} v_{i+2} v_{i+3} \cdots v_{21}}.$$

where we have used Eq. (52). For $j = 1$ in Fig. 4b

$$B = 2(-i)^{n-1} g^n \frac{z_{i+1} z_{i+2} \cdots z_{n+1}}{z_{i+1} z_{i+2} \cdots z_{n+1} v_{n+1} v_{i+1} v_{i+2} v_{i+3} \cdots v_{21}}.$$

Figure 5: Graphs representing the contribution from the equivalent 4-gluon vertices $V_{comb.a}$ and $V_{comb.b}$ to the fragmentation function.
The overall contribution from Figs. 4b and 5b would then be given by

\[ E_2 = \sum_{j=2}^{n} A_j + B = 2(-i)^{n-1} g^n \sum_{j=1}^{n} \frac{1}{z_{2} \cdots z_{n+1}} \frac{z_{j+1,n+1}^{2}}{v_{n+1,n \cdots v_{j+1}} v_{j+1} \cdots v_{2}}. \]  

(57)

For Fig. 4a for fixed \( j \), we get

\[ E_3 = 2(-1)^{n-j}(ig)^{n-j} \frac{z_{j+n+1}z_{j+1,n+1}^{2}}{z_{j+1,j+1} \cdots z_{n+1} v_{n+1,n \cdots v_{j+1}} D_{j+1} M_{1-j} M_{1+j}}. \]  

(58)

Now, we need to add the contributions from (54), (57) and (58) to get the final expression for \( M_{1-n} \). To simplify the calculations it is useful to rewrite the expressions entirely in terms of \( M_{1-j} \). We can use the following identity (which is proven in Appendix B)

\[ \frac{v_{k+1}(1 \cdots k+1)}{z_{k+1} \cdots z_{k}} \sum_{j=1}^{k} \frac{z_{j+1,k+1}^{2}}{z_{j+1,j} v_{1+i,1+i} v_{j+i+1} v_{j+i} \cdots v_{2}} = (-1)^{k} z_{k+1}v_{k+1}(1 \cdots k+1) \left\{ M_{1-k+1} \frac{1}{v_{1+i,1+i}} - \sum_{j=2}^{k} \frac{1}{z_{j+1,j} z_{1,i+1} z_{1,j+1} v_{j+i+1} v_{j+i} v_{j+i} \cdots v_{2}} \right\}, \]  

(59)

where \( k \) is the overall contribution from Figs. 4b and 5b. To get the final expression for \( M_{1-n} \), we get

\[ E_1 = 2(-i)^{n-1} g^n \frac{z_{1,n+1} z_{j+1,n+1}^{2}}{z_{1,j} v_{j+1} \cdots v_{n+1} v_{n+1,n \cdots v_{j+1}} w_{1-i,1-i} \left\{ M_{1-j} \frac{1}{v_{1+i,1+i}} - \sum_{j=2}^{n} \frac{1}{z_{j+1,j} z_{1,i+1} z_{1,j+1} v_{j+i+1} v_{j+i} v_{j+i} \cdots v_{2}} \right\}}, \]

(60)

\[ E_2 = -2(i)^{n-1} g^n \frac{z_{1,n+1} z_{1,n+1}^{2}}{z_{1,j} v_{j+1} \cdots v_{n+1} v_{n+1,n \cdots v_{j+1}} w_{1-i,1-i} \left\{ M_{1-j} \frac{1}{v_{1+i,1+i}} - \sum_{j=2}^{n} \frac{1}{z_{j+1,j} z_{1,i+1} z_{1,j+1} v_{j+i+1} v_{j+i} v_{j+i} \cdots v_{2}} \right\}}, \]

(62)

We can now find \( M_{1-n+1} \) from the contributions of (58), (61) and (62), where we must remember to sum over \( j \) from \( j = 2 \) to \( j = n \) and collect terms proportional to \( M_{1-l} \), where \( 2 \leq l \leq n+1 \). For \( l = n+1 \) we get only one term, which comes from the first term in (62).

\[ 2(i)^{n+1} g^n M_{1-n+1}. \]  

(63)

We should note that this term is, in fact, the MHV amplitude, \( A_{2-m} \), which we wished to obtain. For any other \( l \), after simplifying and remembering to use (51) we get

\[ 2(i)^{n+1} g^n \frac{1}{z_{1,j} v_{j+1} \cdots v_{n+1} v_{n+1,n \cdots v_{j+1}} w_{1-i,1-i}} \frac{1}{v_{1+i,1+i}} \frac{M_{1-l} D_{i} C + z_{1,i+1} v_{1+i}}{1}, \]

(64)

where

\[ C = -z_{1,n+1} z_{1,j} v_{1+i,1+i} v_{j+i+1} v_{j+i+1} - \sum_{j=1}^{n} z_{j+1,n+1} z_{j+1} v_{j+i+1} v_{j+i+1}, \]  

(65)

\[ F = 2z_{1,n+1} z_{1,j} v_{1+i,1+i} v_{1+i,1+i} v_{j+i+1} v_{j+i+1} - 2 \sum_{j=1}^{n} z_{j+1,n+1} z_{j+1} v_{1+i,1+i} v_{j+i+1} v_{j+i+1}. \]  

(66)
However, we can write $C$ as

$$C = -z_{1...n}z_{1...1} \left( z_{1...1} \sum_{j=l+1}^{n+1} z_j v_{j1} + z_{1...1} \left( \sum_{j=l+1}^{n+1} z_j \right) v_{1(l...1)} - z_{1...1} \sum_{j=l+1}^{n} \sum_{j'=j+1}^{n+1} z_{m} v_{j'j+1} \right). \quad (67)$$

After some little algebra and changing the order of summation in the last term, i.e. replacing $\sum_{j=l+1}^{n} \sum_{j'=j+1}^{n+1} z_{m} v_{j'j+1}$ with $\sum_{m=l+2}^{n+1} \sum_{j=l+1}^{m-1}$, we arrive at

$$C = -z_{1...n}z_{1...1} v_{1(1...1)}. \quad (68)$$

Furthermore, we can write $F$ as

$$F = 2 \sum_{j=l+1}^{n+1} z_j z_{1...1} v_{j1} v_{1(1...1)} + 2 \sum_{j=l+1}^{n+1} z_j v_{j1} v_{1(1...1)} + \sum_{j=l+1}^{n+1} z_j v_{j1} v_{1(1...1)}.$$

Since $j$ will always be greater than $l$ we can write $z_{1...j} v_{m(l...1)} = z_{1...j} v_{m(l...1)} + \sum_{j'=l+1}^{j} z_{m} v_{m l}$. We use this to expand the second term in $F$. Changing the order of summation in these two new terms we get

$$F = 2 \sum_{j=l+1}^{n+1} z_j z_{1...1} v_{j1} v_{1(1...1)} + 2 \sum_{j=l+1}^{n+1} z_j v_{j1} v_{1(1...1)} + \sum_{j=l+1}^{n+1} z_j v_{j1} v_{1(1...1)}.$$

We can rewrite this in terms of $k$'s by using

$$v_{ab} v_{ab} = \frac{1}{2} \left( \frac{k}{z_a} - \frac{k}{z_b} \right)^2.$$

Manipulating the sums and simplifying the expression we end up with

$$F = -z_{1...n+1} \left( \sum_{j=l+1}^{n+1} z_j \right) + \left( \sum_{j=l+1}^{n+1} z_j \right) + 2 \sum_{j=l+1}^{n+1} z_j \cdot \sum_{j=l+1}^{n+1} z_j \cdot \sum_{j=l+1}^{n+1} z_j$$

$$= z_{1...n+1} \left( \sum_{j=l+1}^{n+1} z_j \right) - \sum_{j=l+1}^{n+1} z_j \cdot \sum_{j=l+1}^{n+1} z_j$$

$$= z_{1...n+1}(D_l - D_{n+1}). \quad (73)$$

Thus, (64) reduces to

$$-2g^{n+1} g^{n+1} g^n \frac{1}{z_{1...n+1}} \sum_{j=l+1}^{n+1} \frac{z_j^2}{z_{1...n+1}} v_{1(1...1)} v_{1(1...1)} \frac{1}{D_{n+1}} M_{1-l}. \quad (76)$$

The final result is the sum of (63) and (70)

$$2g^{n+1} g^n M_{1-n+1} = 2g^{n+1} g^n \sum_{j=l+1}^{n} \frac{1}{z_{j+1} \ldots z_{n+1}} \sum_{j=1}^{n+1} \frac{z_{j}^2}{z_{j+1} \ldots z_{n+1}} v_{1(1...1)} v_{1(1...1)} \frac{1}{D_{n+1}} M_{1-l},$$

which gives us $M_{1-n+1}$ written as a linear combination of on-shell amplitudes. Comparing these terms to (51) we see that, indeed, $M_{1-n+1}$ is of the same form, which completes our proof. It is important to note that if we now apply the condition that the initial state be on-shell, i.e. $D_{n+1} \to 0$, $M_{1-n+1}$ does reduce to the known expression for the MHV amplitudes.
Conclusions

In this paper we have investigated the fragmentation functions and scattering amplitudes within the framework of the light front perturbation theory. We have shown that the recursion relations for the fragmentation functions actually imply the recursion relations for the off-shell light-front scattering amplitudes. These recursion relations are light-front analogs of the previously derived Berends-Giele recursion relations. Using these methods we have been able to reproduce the lowest order scattering amplitudes. Finally, it was shown that the recursion relations can be solved exactly to all orders in the number of external legs and thus compact expressions for the off-shell amplitudes have been derived. Interestingly, the expression for the off-shell amplitude can be expressed as a sum of terms proportional to the on-shell amplitudes multiplied by the appropriate light-front energy denominators. When the external gluons are put on-shell, the energy denominators vanish, and the first term in the sum reproduces the previously known results for the MHV amplitudes. The light-front methods presented in this paper can be readily generalized to compute the scattering amplitudes for different helicity configurations.

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Appendix A

In this appendix we will show how the fragmentation function \( T_n[(12 \ldots n)^- \rightarrow 1^-, 2^+, \ldots, n^+] \) is derived. We will follow the procedure used in [13] to calculate \( T_n[(12 \ldots n)^- \rightarrow 1^+, 2^+, \ldots, n^+] \).

We begin with \( n = 2 \). It can easily be seen that the fragmentation function for this is

\[
T_2[(12)^- \rightarrow 1^-, 2^+] = 2i g \frac{z_1}{\sqrt{z_2(1-z_2)}} \frac{v_2(1)}{2D_2} = 2i g \frac{z_1}{\sqrt{z_2(1-z_2)}} \frac{v_2(1)}{2 - 2z_1 v_2 v_1} = -i g \left( \frac{z_1}{z_2} \right)^2 \frac{1}{v_1}. \tag{78}
\]

One should note that this is the same as that for \( T_2[(12)^+ \rightarrow 1^+, 2^+] \) multiplied by an extra factor of \((z_1/z_2)^2\). For \( n = 3 \) we then get

\[
T_3[(123)^- \rightarrow 1^-, 2^+, 3^+] = \frac{-2(i g)^2}{D_3} \left[ \frac{z_1}{\sqrt{z_2(1-z_2)}} \right] \frac{v_2(1)}{v_2} + \frac{z_1}{\sqrt{z_2(1-z_2)}} \frac{v_2(1)}{v_2} = \frac{z_1}{\sqrt{z_2(1-z_2)}} \frac{v_2(1)}{v_2}. \tag{79}
\]

\[
T_3[(123)^+ \rightarrow 1^+, 2^+, 3^+] = \frac{-2(i g)^2}{D_3} \left[ \frac{z_1}{\sqrt{z_2(1-z_2)}} \right] \frac{v_2(1)}{v_2} + \frac{z_1}{\sqrt{z_2(1-z_2)}} \frac{v_2(1)}{v_2} = \frac{z_1}{\sqrt{z_2(1-z_2)}} \frac{v_2(1)}{v_2}. \tag{80}
\]

We recognize that this is the expression given in [13] for \( T_3[(13)^+ \rightarrow 1^+, 2^+, 3^+] \). Thus, the final result for \( n = 3 \) is

\[
T_3[(123)^- \rightarrow 1^-, 2^+, 3^+] = (i g)^2 \left( \frac{z_1}{z_2} \right)^2 \left( \frac{z_2}{z_1 z_3} \right)^{3/2} \left( \frac{1}{v_1 v_2 v_3} \right). \tag{81}
\]

This pattern implies that for general \( n \) the fragmentation function should be of the form

\[
T_n[(12 \ldots n)^- \rightarrow 1^-, 2^+, \ldots, n^+] = \left( \frac{z_1}{z_{1 \ldots n}} \right)^2 T_n[(12 \ldots n)^+ \rightarrow 1^+, 2^+, \ldots, n^+]. \tag{82}
\]
We show that this expression is indeed correct by substituting (83) into the recursion relation for $T_{n+1}[(12\ldots n+1)^-\rightarrow 1^-, 2^+,\ldots, n+1^+]$ given below.

$$T_{n+1}[(12\ldots n+1)^-\rightarrow 1^-, 2^+,\ldots, n+1^+] = -\frac{2i}{D_{n+1}} \sum_{j=1}^{n} \left\{ \left( \frac{z_{l,j}}{z_{1\ldots n+1}} \right)^2 \frac{v_{j+1}(\xi(j+1\ldots n+1))}{\sqrt{6}^{(j-1)(j+1\ldots n+1)}} \right\} x T_{l}[(1\ldots i)^-\rightarrow 1^-, 2^+,\ldots, i^+] T_{n+1-}[((i+1\ldots n+1)^+\rightarrow i+1^+,\ldots, n+1^+)] \right\}. \quad (84)$$

We see that $T_{l}[(1\ldots i)^-\rightarrow 1^-, 2^+,\ldots, i^+]$ includes a factor of $(z_l/z_{1\ldots n+1})^2$ which can be combined with the $(z_l/z_{1\ldots n+1})^2$ in (84) to give

$$T_{n+1}[(1\ldots n+1)^-\rightarrow 1^-, 2^+,\ldots, n+1^+] = -\frac{2i}{D_{n+1}} \left( \frac{z_l}{z_{1\ldots n+1}} \right)^2 \sum_{j=1}^{n} \left\{ \frac{v_{j+1}(\xi(j+1\ldots n+1))}{\sqrt{6}^{(j-1)(j+1\ldots n+1)}} \right\} x T_{l}[(1\ldots i)^+\rightarrow 1^+, 2^+,\ldots, i^+] T_{n+1-}[(i+1\ldots n+1)^+\rightarrow i+1^+,\ldots, n+1^+] \right\}. \quad (85)$$

This is simply the recursion relation for $T_{n+1}[(1\ldots n+1)^+\rightarrow 1^+, 2^+,\ldots, n+1^+]$ given in (10) multiplied by $(z_l/z_{1\ldots n+1})^2$. Thus, combining (10), (83) and (84) we get

$$T_{n+1}[(1\ldots n+1)^-\rightarrow 1^-, 2^+,\ldots, n+1^+] = \left( \frac{z_l}{z_{1\ldots n+1}} \right)^2 T_{n+1}[(1\ldots n+1)^+\rightarrow 1^+, 2^+,\ldots, n+1^+], \quad (86)$$

which proves (83) to be correct.

**Appendix B**

In this appendix we provide the proof for (59) which is done using induction. We begin by rewriting the (59) as

$$\sum_{j=1}^{k} \frac{z_{j+1\ldots k+1}}{z_{1\ldots j}} \frac{v_{j+1\ldots k+1}}{v_{k+1}(k+1\ldots k+1)} - \sum_{j=2}^{k} \frac{z_{j+1\ldots k+1}}{z_{1\ldots j}} \frac{v_{j+1\ldots k+1}}{v_{j+1\ldots k+1}^{2}(j+1\ldots k+1)} = 0. \quad (87)$$

To get this we have substituted (59) into (59) and taken out some overall factors. We will label the left hand side of this equation as $f_k$ and assume that $f_k = 0$ for $k < k'$, where $k'$ is an arbitrary upper limit. If, under our assumption, $f_k = 0$ for $k' = k + 1$ then it will be true that $f_k = 0$ for all $k$ subject to $f_k = 0 = f_2$.

Our current expression for $f_k$ is, however, too cumbersome to work with. Thus, we will derive a simpler form. We can combine the first and third terms by using (52). Then using $v_{j+1\ldots k+1} = -v_{k+1\ldots j+1} + v_{j+1\ldots k+1}$ and (53) we get, after some manipulation,

$$\sum_{j=1}^{k} \frac{z_{j+1\ldots k+1}}{z_{1\ldots j}} v_{j+1\ldots k+1} = \sum_{j=1}^{k} \frac{z_{j+1\ldots k+1} v_{j+1\ldots k+1}}{v_{k+1}(k+1\ldots k+1)} = \sum_{j=1}^{k} \frac{\left( \frac{z_{j+1\ldots k+1}}{z_{1\ldots j}} \right) \left( \frac{v_{j+1\ldots k+1}}{v_{k+1}(k+1\ldots k+1)} \right)}{v_{j+1\ldots k+1}} = \sum_{j=1}^{k} \frac{z_{j+1\ldots k+1}}{z_{1\ldots j}} \frac{v_{j+1\ldots k+1}}{v_{k+1}(k+1\ldots k+1)} = \sum_{j=1}^{k} \frac{z_{j+1\ldots k+1}}{z_{1\ldots j}} \frac{v_{j+1\ldots k+1}}{v_{k+1}(k+1\ldots k+1)} \sum_{j=1}^{k} \frac{z_{j+1\ldots k+1}}{z_{1\ldots j}} \frac{v_{j+1\ldots k+1}}{v_{k+1}(k+1\ldots k+1)} \sum_{j=1}^{k} \frac{z_{j+1\ldots k+1}}{z_{1\ldots j}} \frac{v_{j+1\ldots k+1}}{v_{k+1}(k+1\ldots k+1)}. \quad (88)$$

Here we have changed the lower and upper limits of $j$ appropriately. We now choose to change the order in which the sums are performed in the second term of (88). I.e., we replace $\sum_{j=1}^{k} \sum_{j=1}^{k} \frac{z_{j+1\ldots k+1}}{z_{1\ldots j}} \frac{v_{j+1\ldots k+1}}{v_{k+1}(k+1\ldots k+1)}$ by $\sum_{j=1}^{k} \sum_{j=1}^{k} \frac{z_{j+1\ldots k+1}}{z_{1\ldots j}} \frac{v_{j+1\ldots k+1}}{v_{k+1}(k+1\ldots k+1)}$. Furthermore, since $j$ and
\[ l \] are dummy indices we can exchange them so that we can now combine the two terms in (88) into a single double sum. Thus, for the sum of the first and third terms of (87) we end up with
\[
- \sum_{j=2}^{k+1} \sum_{i=1}^{k+1} \frac{z_{j+1}}{z_{1...j}} v_{j+1} \sum_{j=2}^{k+1} \sum_{i=1}^{k+1} \frac{z_{j+1}}{z_{1...j}} v_{j+1} l = \sum_{j=2}^{k+1} \sum_{i=1}^{k+1} \frac{z_{j+1}}{z_{1...j}} v_{j+1} l
\]
(89)

We can also rewrite the second term in (87) in the following way.
\[
- \sum_{j=2}^{k+1} \sum_{i=1}^{k+1} \frac{z_{j+1}}{z_{1...j}} v_{j+1} \sum_{j=2}^{k+1} \sum_{i=1}^{k+1} \frac{z_{j+1}}{z_{1...j}} v_{j+1} l = - \sum_{j=2}^{k+1} \sum_{i=1}^{k+1} \frac{z_{j+1}}{z_{1...j}} v_{j+1} l
\]
(90)

Finally, \( f_k \) will be given by the sum of (89) and (90). Let us now define
\[
g_k = \frac{v_{k+1(1...k+1)}}{z_{1...k+1}} f_k.
\]
(91)

For \( k' = k + 1 \) we would then have
\[
g_k' = \frac{v_{k+2(1...k+2)}}{z_{1...k+2}} f_k + g_k
\]
(92)

To get the third line we have used (52) and \( z_{1...k+2} v_{k+2(1...k+2)} = z_{1...k+1} v_{k+2(1...k+1)} = z_{1...k} (v_{k+2(1...k+1)} + v_{k+1(1...k+1)}) \). Our assumption \( f_k = 0 \) implies \( g_k = 0 \), since \( v_{k+1(1...k+1)} \) is in general non-zero. Hence, this shows that \( f_k + g_k = 0 \). It is fairly easy to show that (87) is true for \( k = 1, 2 \). Thus, we have shown that (87) is true for all \( k \geq 1 \).

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