Highly Sensitive Coupled Oscillator Based on an Exceptional Point of Degeneracy and Nonlinearity

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Abstract—We propose a scheme for obtaining highly-sensitive oscillators in a coupled-resonator system with an exceptional point of degeneracy (EPD) and a small instability. The oscillator with the exceptional degeneracy is realized by using two coupled resonators with an almost balanced small-signal gain and loss, that saturates due to nonlinear effects of the active component, resulting in an oscillation frequency that is very sensitive to a perturbation of the circuit. Two cases are investigated, with two parallel LC resonators with balanced small-signal gain and loss that are either coupled wirelessly by mutual inductance or coupled-wired by a capacitor. This paper demonstrates theoretically and experimentally the conditions to obtain a second-order EPD oscillator and analyzes the ultrasensitivity of the oscillation frequency to components’ perturbation, including the case of asymmetric perturbation that breaks PT-symmetry. We discuss the effects of nonlinearity on the performance of the oscillator and how the proposed scheme improves the sensing’s sensitivity of perturbations. In contrast to previous methods, our proposed degenerate oscillator can sense positive or negative changes of a circuit component. The degenerate oscillator circuit may find applications in various areas such as ultrasensitive sensors, tunable oscillators and modulators.

Index Terms—Exceptional points, Degeneracy, Oscillator, Resonator, Sensor, Nonlinearity

I. INTRODUCTION

Oscillators are fundamental components of radio frequency (RF) electronics. Traditionally, an oscillator is viewed as a positive feedback mechanism utilizing a gain device with a selective reactive circuit. An oscillator generates a continuous, periodic single-frequency output when the Barkhausen’s criteria are satisfied. The oscillator circuit should have a self-sustaining mechanism such that noise gets filtered, quickly grows and becomes a periodic signal. Most RF oscillators are implemented by only one active device for noise and cost considerations, such as Van der Pol and voltage-controlled oscillators [1]. Oscillators can be realized by a simple LC resonator with positive feedback using a negative resistance. Pierce, Colpitts, and tunnel diode oscillators play a role of negative resistance in a circuit, as well as a cross-coupled transistor pair [2], [3], [4]. All oscillators are based on a single-pole operation, i.e., a single pole is rendered unstable when the system is brought above the threshold. While oscillators based on an LC resonator are the most common type of oscillator, other designs may feature distributed [5], [6], ring [7], [8], coupled [9], or multi-mode [10] oscillators, which come with their own challenges and advantages.

In this paper, we study the concept of an oscillator based on a double pole, i.e., an oscillator designed to utilize an exceptional point of degeneracy (EPD) in two coupled resonators. A system reaches the EPD when at least two eigenfrequencies coalesce into a single degenerate one, in their eigenfrequencies (eigenvalues) and polarization states (eigenvectors) [11], [12], [13], [14], [15], [16], [17], [18]. The letter “D” in EPD refers to the key concept of “degeneracy” where the relevant eigenmodes, including the associated eigenvectors are fully degenerate [19]. The degeneracy order refers to the number of coalescing eigenfrequencies. The concept of EPD has been implemented traditionally in systems that evolve in time, like in coupled resonators [20], [21], [22], [23], [24], periodic and uniform multimode waveguides [25], [26], [27], [28], [29], [30], and also in waveguides using Parity-Time (PT) symmetry [31], [32], [29]. EPDs have been recently demonstrated also in temporally-periodic single resonator without a gain element [33], [34], [35], inspired by the finding that EPD exists in spatially periodic lossless waveguides [36], [37], [38], using the non-diagonalizability of the transfer matrix associated to the periodic system.

A very significant feature of a system with EPD is the ultra-sensitivity of its eigenvectors and eigenvalues to a perturbation of a system’s parameter. This property paves the way to conceive a scheme to measure a small change in either physical, chemical, or biological parameters that causes a perturbation in the system. Typically, a sensor’s sensitivity is related to the amount of spectral shift of a resonance mechanism in response to a perturbation in environmental parameters, for example, a glucose concentration or other physical variations like changing pressure, etc. Sensors with EPD can be wired or wirelessly connected to the measuring part of the sensor. In this paper, we show the extreme sensitivity of an oscillator operating at an EPD to external perturbations.

Previous parity-time (PT)-symmetric circuits have been conceived as two coupled resonators where changes happen at one resonator, and the data is detected on the other side [39]. When the circuit is perturbed away from its EPD, PT-symmetry must be maintained in order to obtain two real-valued frequencies. For example, in Ref. [39], when one side’s capacitance is perturbed, the authors tuned the other side’s capacitance using a varactor to keep the PT-symmetry in the circuit, so they can still observe two real-valued shifted frequencies perturbed away from the degenerate EPD frequency. Thus, in previously published schemes (implementing the demonstration of sensitive measurement of a perturbation) the exact value of such perturbation should be exactly known to tune the other side of the system in order to keep the circuit PT-symmetric.

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This seems to contradict the idea that the circuit is used as a sensor of an unknown measurable quantity. That scheme could be saved if combined with an iterative method performing an automatic scan to reconstruct the PT-symmetry. Anyway, this rebalancing procedure (to keep the system PT-symmetric) makes it more complicated to use of such a scheme when designing a sensor.

The limitation of PT-symmetry schemes is that they can detect only perturbations that lead to the same-sign change in a system’s component, such as a capacitor’s value. This is because a PT-symmetric system provides two real-valued frequencies only when the system is perturbed away from its EPD in one direction (for example for $G$ values smaller than the $G_c$ related to the EPD, when looking at the eigenfrequencies in Fig. 1). If the perturbation makes the system move in the other direction, the shift of the frequencies is in the imaginary parts [40], [21], [39], [23], leading to two complex-valued frequencies and hence to instability. One must also consider that any mismatch between the sensor side (typically the part with losses) and the reader side (typically the part with gain), even involuntary, leads to an asymmetric system. Thus, a PT-symmetric system in practice always shows two complex-valued eigenfrequencies and increase the risks of self-sustained oscillations (unless an EPD is designed having a large enough damping factor, larger than the eigenfrequency perturbation due to circuit tolerances). Noise and nonlinearities play a critical role in the robustness of these kinds of applications and affect the possibility of instability [41]. Some error-correction techniques are studied in [42] to overcome some of these drawbacks using a nonlinear PT-symmetry scheme to enhance the robustness of sensing.

In this paper, we provide a scheme that starts by using a quasi PT-symmetric condition, working near an EPD, that makes the double-pole system slightly unstable even before having any perturbation. In other words, we turn the above-mentioned practical problems that occur in PT-symmetric systems to our advantage when the circuit has to be used in a highly sensitive sensor. We set the gain value slightly higher than the loss counterpart to make the system slightly unstable. As a result of instability and nonlinear gain, the signal grows until the active gain component reaches saturation, and the working operation will be close to the EPD.

We first show the behavior of wirelessly coupled LC resonators through the dispersion relation of the resonance frequency versus perturbation and we discuss the occurrence of EPDs in such a system. In section III, we use the nonlinear model for the gain to achieve the oscillator’s characteristics. We show that the oscillation frequency is very close to the EPD frequency. The EPD-based oscillator has an oscillation frequency that is very sensitive to perturbation, exhibiting the typical square root-like behavior of EPD systems, where the change in frequency of the oscillator is proportional to the square root of the perturbation. In section IV, we demonstrate the highly sensitive behavior of the circuit by breaking PT-symmetry, i.e., by perturbing the capacitance on the lossy side (the sensing capacitance). In this case, the circuit oscillates at a shifted frequency compared to the EPD one. Notably, both positive and negative perturbations in the capacitance are shown to lead to opposite shifted frequencies, i.e., the proposed scheme detects positive and negative changes in the capacitance, in contrast to conventional PT-symmetry systems [21], [39], [20] that generate frequency shifts associated to only one sign of the perturbation. The EPD is demonstrated also by analyzing the bifurcation of the dispersion diagram at the EPD frequency by using the Puiseux fractional power series expansion [43], [15].

II. OSCILLATOR BASED ON COUPLED RESONATORS WITH EPD

We investigate the coupled resonators shown in Fig. 1(a), where one parallel LC resonator is connected to gain (left side,
or \( n = 1 \) and the other is connected to loss (right side, or \( n = 2 \)). In this ideal circuit, negative conductance (gain) has the same magnitude as the loss to exactly satisfy PT-symmetry. When a system satisfies PT-symmetry, it means that the system is invariant to the application of the two operators "P" and "T". The "P" stands for parity transformation (making a spatial reflection (e.g., \( x \to -x \)), and "T" stands for time-reversal transformation (\( t \to -t \)), where \( x \) is the coordinate and \( t \) is the time.

By writing Kirchhoff’s current laws, we obtain the equations

\[
\begin{align*}
\frac{d^2Q_1}{dt^2} &= -\frac{1}{LC_1(1-k^2)}Q_1 + \frac{k}{LC_2(1-k^2)}Q_2 + \frac{G_1}{C_1} \frac{dQ_1}{dt} \\
\frac{d^2Q_2}{dt^2} &= +\frac{1}{LC_1(1-k^2)}Q_1 - \frac{k}{LC_2(1-k^2)}Q_2 - \frac{G_2}{C_2} \frac{dQ_2}{dt}
\end{align*}
\]

(1)

where \( Q_n \) is the capacitors charge on the gain side (\( n = 1 \)) and the lossy side (\( n = 2 \)), and \( \dot{Q}_n = \frac{dQ_n}{dt} \) is the current flowing into the capacitor. We define the state vector as \( \Psi(t) = [Q_1, Q_2, Q_1, Q_2]^T \), consisting of a combination of stored charges and currents on both sides, and the superscript \( T \) denotes the transpose operation. Thus, we describe the system in a Liouvillian formalism as

\[
\frac{d\Psi}{dt} = M\Psi.
\]

(2)

We are interested in finding the eigenfrequencies and eigenvectors of the system matrix \( M \) describing the circuit. Assuming signals of the form \( Q_n \propto e^{i\omega t} \), we write the eigenvalues problem associated with the circuit equations, \( (M - j\omega I)\Psi = 0 \), where \( I \) is a 4 by 4 identity matrix. Then, by solving \( P(\omega) \triangleq \det(M - j\omega I) = 0 \), the four eigenfrequencies are found. By assuming \( C_1 = C_2 = C_0 \) and linear \( G_1 = G_2 = G \), a symmetry condition that has been described as PT symmetric [23], the characteristic equation takes the simplified form

\[
P(\omega) = (1 - k^2) \left( \frac{\omega}{\omega_0} \right)^4 + \left( G^2 Z^2 (1 - k^2) - 2 \right) \left( \frac{\omega}{\omega_0} \right)^2 + 1 = 0,
\]

(3)

where \( Z = \sqrt{L/C_0} \) is a convenient normalizing impedance, and \( \omega_0 = 1/(LC_0) \). The characteristic equation is quadratic in \( \omega^2 \); therefore, \( \omega \) and \( -\omega \) are both solutions. Moreover, the \( \omega \)'s coefficients in the characteristic equation are real, hence \( \omega \) and \( \omega^* \) are both solutions, where * represents the complex conjugate operation. The 4 by 4 matrix \( M \) results in 4 angular eigenfrequencies which are found analytically as,

\[
\omega_{1,3} = \pm \omega_0 \sqrt{\frac{1}{1 - k^2} - \frac{G^2 Z^2}{2} - \sqrt{b}},
\]

(4)

\[
\omega_{2,4} = \pm \omega_0 \sqrt{\frac{1}{1 - k^2} - \frac{G^2 Z^2}{2} + \sqrt{b}},
\]

(5)

The EPD frequency is found when the component values obey the condition

\[
b = -\frac{1}{1 - k^2} + \left( \frac{G^2 Z^2}{2} - \frac{1}{1 - k^2} \right)^2.
\]

(6)

The real and imaginary parts of the eigenfrequencies are shown in Fig. (1)(b) and (c) varying \( G \). It is seen from this plot that the system’s eigenfrequencies are coalescing at a specific balanced linear gain/loss value \( G = G_c \), where \( b = 0 \). Note that in this scenario, the EPD-enabling value \( G_c \) is derived from Eq. (7) as

\[
G_c = \frac{1}{Z} \left( \frac{1}{\sqrt{1 - k}} - \frac{1}{\sqrt{1 + k}} \right).
\]

(7)

For clarification, when \( G = 0 \) (lossless and gainless circuit), we have two pairs of resonance frequencies \( \omega_{1,3} = \pm \omega_0/\sqrt{1+k} \) and \( \omega_{2,4} = \pm \omega_0/\sqrt{1-k} \), and \( \omega_1 \neq \omega_2 \) always, except for the trivial case with \( k = 0 \), when these eigenfrequencies are equal to those of the isolated circuits, but since the two circuits are isolated this is not an important degeneracy. With the given values of \( L \) and \( C \) in the caption of Fig. 1, a second-order EPD occurs when \( G = G_c = 20.52 \) mS. In this case, the circuit’s currents and charges grow linearly with increasing time as \( Q_n \propto t \cos(\omega_c t) \), and they oscillate at the degenerate frequency \( \omega_c \). Also, near the EPD point, the eigenfrequencies, when perturbing \( G \), have a square root-like behavior as \( |\omega - \omega_c| \propto \pm \sqrt{(GZ)^2 - (G_cZ)^2} \). A second coalescence (i.e., degeneracy) happens for larger values of \( G \), i.e., at \( G_c' = \frac{1}{2} \left( \frac{1}{\sqrt{1-k}} + \frac{1}{\sqrt{1+k}} \right) \). When \( G > G_c' \), all frequencies are imaginary, so we only study cases of \( G < G_c' \). In the strong coupling regime, \( 0 < G < G_c \), the eigenfrequencies are purely real, and the oscillation wave has two fundamental frequencies. In the weak coupling regime, \( G_c < G < G_c' \), the frequencies are complex conjugate and the imaginary part of the angular eigenfrequencies is non-zero, and it causes two system solutions \( (Q_1 \text{ and } Q_2) \) with damping and exponentially growing signals in the system. Since the solution of the circuit is \( Q_n \propto e^{i\omega t} \), the eigenfrequency with a negative imaginary part is associated to an exponentially growing signal and the oscillation frequency is associated to the real part of the eigenfrequency.

At each positive (real part) angular eigenfrequency \( \omega_1 \) and \( \omega_2 \), calculated by Eqs. (4) and (5), we find the two associated eigenvectors \( \Psi_1 \) and \( \Psi_2 \) by using Eq. (2). A sufficient condition for an EPD to occur is that at least
two eigenvectors coalesce, and that is what we check in the following. Various choices could be made to measure the state vectors’ coalescence at an EPD, and here, the Hermitian angle between the state amplitude vectors $\Psi_1$ and $\Psi_2$ is defined as

$$\theta = \arccos \left( \frac{\langle \Psi_1, \Psi_2 \rangle}{||\Psi_1|| \cdot ||\Psi_2||} \right).$$

(10)

Here the inner product is defined as $\langle \Psi_1, \Psi_2 \rangle = \Psi_1^\dagger \Psi_2$, where the dagger symbol $\dagger$ denotes the complex conjugate transpose operation, $|| \cdot ||$ represents the absolute value, and $|| \cdot ||$ represents the norm of a vector. According to this definition, the state vectors $\Psi_1$ and $\Psi_2$ correspond to resonance frequencies $\omega_1$ and $\omega_2$, respectively. When some system’s parameter is varied, eigenfrequencies and associated eigenvectors are calculated using Eq. (2). In the case when $G$ varies, Fig. (1)(d) shows that the sine of the angle $\theta$ between the two eigenvectors vanishes when the eigenfrequencies coalesce, which indicates the coalescence of the two eigenmodes in their eigenvalues and eigenvectors and hence the occurrence of a second-order EPD.

III. OSCILLATOR CHARACTERISTICS

This section describes the important features of an oscillator made of two coupled resonators with discrete (lumped) elements with balanced gain and loss, coupled wirelessly by a mutual inductance as in Fig. 1. The transient time-domain, frequency spectrum, and double pole (or zero, depending on what we look at) features are discussed. A cubic model (nonlinear) of the active component providing gain is considered. The parameters used here are the same as those used in the previous section, where $G_e = 20.52 \, \text{mS}$ leads to an EPD of order two at a frequency of 16.1 MHz, except that $-G_1$ accounts also for the nonlinear part responsible for the saturation effect.

A. Transient and frequency behavior

Time and frequency-domain responses of the coupled resonators circuit are obtained by using the Keysight Advanced Design System (ADS) circuit time-domain simulator, as shown in Fig. 2(b)-(d). The cubic model for gain, in Fig. 2(a), represented as

$$i = -G_1v + \alpha v^3$$

(11)

is a simplified description of the gain obtained from a cross-coupled transistor or an operational amplifier (opamp) based circuit. Here, $-G_1$ is the small-signal gain provided by the negative slope of the $i - v$ curve, i.e., is the negative conductance in the small-signal region and $\alpha = G_1/(3V_b^3)$ is a third-order nonlinearity that describes saturation, where $V_b$ is a turning point voltage determined by the biasing direct current (DC) voltage. We assume $V_b = 1 \, \text{V}$, and to start self-sustained oscillation we assume that gain $-G_1$ is not a perfect balance of the loss $G_2$. Indeed, we assume that $G_1$ is 0.1% larger than $G_2$. Therefore, the system is slightly perturbed away from the PT-symmetry condition to start with. We also assume white noise (at the temperature of 298 K) is present in the loss resistor and it is indeed the initial condition for starting oscillations.

Using $G_1$ to be 0.1% larger than $G_2$, the circuit is unstable and it starts to oscillate, and after a transient, the circuit saturates, yielding a stable oscillation, as shown in Fig. 2(b)-(d). As it was shown in Figs. 1(b) and (c) assuming linear gain, for values of $G_1 = G_2 < G_e$, the system has two distinct eigenfrequencies $\omega_1$ and $\omega_2$ with zero imaginary part. However, when using the cubic nonlinear model with $G_1 = 1.001 G_2$, with $G_2 \lesssim G_1 < G_e$, the imaginary part is not zero anymore because of the nonlinearity and slightly broken PT-symmetry. Thus, when using the cubic model, after an initial transient, the oscillation signal associated to the eigenfrequency with a negative imaginary part makes the system saturates. Considering again the initial result in Figs. 1(b) and (c) assuming linear gain, it is noted that when $G_1 = G_2 > G_e$, we have two complex conjugate eigenfrequencies, and the one associated to the negative imaginary part makes the circuit oscillate. However, when using the cubic gain model
with \( G_1 = 1.001G_2 \), with \( G_1 \geq G_2 > G_e \), eigenfrequencies approximately follow the linear gain eigenfrequency trend. It means that for the values \( G_1 \geq G_2 > G_e \), we have a larger negative imaginary part of the eigenfrequency than when \( G_2 \leq G_1 \leq G_e \). The rising time is related to the magnitude of the negative imaginary part of the eigenfrequency; indeed, as shown in Fig. 2(b)-(d), the rising time is different in the three cases. By going further from the EPD point, the signal saturates in a shorter time. In all cases, the frequency spectrum of the time-domain signal is found by taking the Fourier transform of the voltage on the gain side after reaching saturation, for a time window of \( 10^3 \) periods.

B. Root locus of zeros of the total admittance

This subsection discusses the frequency (phasor) approach to better understand the degenerate resonance frequencies of the coupled resonators circuit. We use the admittance resonance method and we demonstrate the occurrence of double zeros at the EPD. The resonance condition based on the vanishing of the total admittance implies that

\[
Y_{\text{in}}(\omega) - G_1 = \frac{P(\omega)}{jG_2 (1 - k^2) \omega^3 + L^2 G_1 (1 - k^2) \omega^2 - j L \omega} = 0, \tag{12}
\]

where the \( Y_{\text{in}} \) is the input admittance of the linear circuit, including the capacitor \( C_1 \), looking right as shown in Fig. 2(a). Here, we assume linear gain with \( G_1 = G_2 = G \), i.e., satisfying PT symmetry.

The polynomial \( P(\omega) \) is given in Eq. (3). We calculate the eigenfrequencies by finding the zeros of \( Y_{\text{in}}(\omega) - G \), and this leads to the same \( \omega \)-zeros of \( P(\omega) = \det(\mathbf{M} - j \omega \mathbf{I}) = 0 \). Note that both \( \omega(G) \) and \(-\omega(G)\) are both solutions of Eq. (12), as well as both \( \omega(G)\) and \( \omega^*(G)\). The trajectories of the zeros of this equation, i.e., the resonance frequencies \( \omega(G)\), are shown in Fig. 3 by varying linear \( G \) from 18 mS to 22 mS (we recall that in this case \( G_1 = G_2 = G_2 \)), in the complex frequency plane. We show only the roots with \( \text{Re}(\omega) > 0 \) for simplicity. At the EPD, \( G = G_e = 20.52 \) mS, and the above equation reduces to \( Y_{\text{in}}(\omega) - G \propto (\omega - \omega_e)^2\), i.e., the admittance exhibits a double zero at the EPD angular frequency \( \omega_e \). This unique property is also responsible for the square root-like behavior of resonance frequency variation due to the perturbation in a system, as discussed next, which is the key to high sensitivity. Moreover, for values \( G < G_e \), the two resonance frequencies are purely real, and for \( G > G_e \), the two resonance frequencies are a complex conjugate pair.

IV. SENSOR POINT OF VIEW

A. High sensitivity and the Puiseux fractional power expansion

As mentioned in the Introduction, when the system is operating at an EPD, the eigenfrequencies are extremely sensitive to system perturbations, and this property is intrinsically related to the Puiseux series [43] that provides a fractional power series expansion of the eigenvalues in the vicinity of the EPD point. We consider a small perturbation \( \Delta_X \) of a system parameter \( X \) as

\[
\Delta_X = \frac{X - X_e}{X_e}, \tag{13}
\]

where \( X \) is the perturbed value of a component, and \( X_e \) is the unperturbed value that provides the EPD of second order. A perturbation \( \Delta_X \) leads to a perturbed matrix \( \mathbf{M}(\Delta_X) \) and, as a consequence, it leads to two distinct perturbed eigenfrequencies \( \omega_p(\Delta_X) \), with \( p = 1, 2 \), near the EPD eigenfrequency \( \omega_e \) as predicted by the Puiseux series containing power terms of \( \Delta_X^\frac{1}{2} \). A good approximation of the two \( \omega_p(\Delta_X) \), with \( p = 1, 2 \), is given by the first order expansion

\[
\omega_p(\Delta_X) \approx \omega_e + (-1)^p \alpha_1 \sqrt{\Delta_X}. \tag{14}
\]

Following [43], [15], we calculate \( \alpha_1 \) as

\[
\alpha_1 = \sqrt{-\frac{\partial^2 H(\Delta, \omega)}{\partial \Delta^2} \frac{\partial H(\Delta, \omega)}{\partial \Delta}}, \tag{15}
\]

where \( H(\Delta, \omega) = \det[\mathbf{M}(\Delta) - j \omega \mathbf{I}] \) and its derivatives are evaluated at the EPD, i.e., at \( \Delta_X = 0 \) and \( \omega = \omega_e \).

Consider a coupled LC resonator, as described in Fig. 2(a), assume the capacitor \( C_2 \) on the loss side is perturbed from the initial value as \((1 + \Delta C_2)e \), where \( e \) is unperturbed value for both \( C_1 \) and \( C_2 \): the coefficient \( \alpha_1 \) is found analytically as

\[
\alpha_1 = \sqrt{\frac{L^2 \omega_e^2 G_2^2 \left(1 + \frac{C_2 \omega_e}{G_2} \right) (1 - k^2) + (1 - C_e L \omega_e^2)}{L^2 \left(6C_2 \omega_e^2 + G_2^2 \right)(1 - k^2) - 2C_e L \omega_e^2}}. \tag{16}
\]

The Puiseux fractional power series expansion Eq. (14) indicates that for a small perturbation such that \( |\Delta_X| \ll 1 \), the eigenfrequencies change dramatically from their original degenerate value due to the square root function. The Puiseux series first-order coefficient is evaluated by Eq. (16) as \( \alpha_1 = 10^{-7}(1.693 + j 1.530) \) rad/s. The coefficient \( \alpha_1 \) is a complex number implying that the system always has two complex eigenfrequencies, for any \( C_2 \) value. In Fig. 4(a) and (b), the estimate of \( \omega_p \), with \( p = 1, 2 \), using the Puiseux series is shown.
perturbation of PT symmetry leads to instability. \( \Delta \) resulting in a very large shift in the oscillation frequency. Note like the frequency of oscillation in the circuit. For a small value sensing capacitance to detect possible variations in chemical directly solving the characteristic equation Eq. (3) are shown 

The highest sensitivity of the EPD circuit is achieved by only 

from its loss balanced value \( G_e \) for the single LC resonator could variations, i.e., around \( |\Delta C_2| \approx 5\% \). For larger \( C_2 \) variations, i.e., around \( |\Delta C_2| \approx 5\% \), the slope of the flattened square root-like curve is similar to the slope of the curve relative to the perturbed LC resonator.

To show how a telemetric sensor with nonlinearity works, we now consider that the gain element is nonlinear, following the cubic model in Eq. (11) where the small-signal negative conductance \( -G_1 \) is considered. The lossy element \( \delta = 0.001 \) from its loss balanced value \( G_e \), i.e., increased by 0.1% from its loss balanced value \( G_e \) (we recall that \( G_2 = G_e \)). The frequencies of oscillation are obtained by applying a Fourier transform of the capacitor \( C_1 \) voltage after the system reaches saturation, for each considered value of \( C_2 \). (c) Sensitivity comparison with single linear LC resonator, when varying \( \Delta C_2 \). The much higher sensitivity of the EPD oscillator with double pole is clear. Note that the whole frequency variation relative to the full perturbation range of capacitance \( -5\% < \Delta C_2 < 5\% \) for the single LC resonator could be achieved by only 1/10 of the perturbation \( -0.5\% < \Delta C_2 < 0.5\% \) when the EPD based circuit is used. The highest sensitivity of the EPD circuit is shown for very small perturbations \( \Delta C_2 \).

by a dashed black line. The calculated eigenfrequencies by directly solving the characteristic equation Eq. (3) are shown by solid blue lines. In this example, we can consider \( C_2 \) as a sensing capacitance to detect possible variations in chemical or physical parameters, transformed into electrical parameters, like the frequency of oscillation in the circuit. For a small value of \( \Delta C_2 \), around the EPD value \( \Delta C_2 = 0 \), the imaginary and real parts of the eigenfrequencies experience a sharp change, resulting in a very large shift in the oscillation frequency. Note that this rapid change in the oscillation frequency is valid for both positive and negative changes of \( \Delta C_2 \), which can be useful for various sensing applications. Note also that a perturbation of PT symmetry leads to instability.

To show how the sensitivity is improved when using the second-order EPD (double-pole) oscillator, we compare its sensitivity to an analogous scheme made of one single LC resonator, with an inductance of \( L = 0.1 \mu \text{H} \) and capacitance of \( C_2 = 1 \text{nF} \) without adding gain or loss. The resonance frequency of the LC resonator is \( f_0 = 1/(2\pi\sqrt{LC_2}) \) and by perturbing the capacitance \( C_2 \), the resonance frequency changes as \( f \approx f_0(1 - \Delta C_2/2) \). Figure (c) shows the comparison between two cases: (i) oscillation frequency of the EPD based oscillator with nonlinear gain (red dots) using the time-domain circuit simulator Keysight ADS, and (ii) the resonance frequency of the single LC resonator (dashed green). The results demonstrate that the EPD-based circuit with nonlinearity has higher sensitivity (square root-like behavior due to the perturbation) than a single LC resonator without EPD (linear behavior). The whole frequency variation, relative to the full perturbation range of capacitance \( -5\% < \Delta C_2 < 5\% \) for the single LC resonator, could be achieved by only 1/10 of the perturbation \( -0.5\% < \Delta C_2 < 0.5\% \) when the EPD based circuit is used. The highest sensitivity of the EPD circuit is shown for very small perturbations \( \Delta C_2 \).

Noise is assumed in the lossy element \( G_e \). The capacitor \( C_2 \) on the lossy side is perturbed by \pm 0.5\% steps and we perform time-domain simulations using the circuit simulator implemented in the Keysight ADS circuit simulator. Noise is assumed in the lossy element \( G_2 \) to start oscillations. The time-domain voltage signal at the capacitor \( C_1 \) on the gain side is read, and then, we take the Fourier transform of such signal, after reaching saturation, for a time window of \( 10^3 \) periods. The oscillation frequency evolution by changing \( \Delta C_2 \) is shown in Fig. 4 by red dots. There is no imaginary part associated to such a signal since it is saturated and steady.
and it has the shape of an almost pure sinusoid after reaching saturation (phase noise is discussed later on in this paper). The oscillation frequency curve dispersion (red dots) still has a square root-like shape of the perturbation. Figure 5 shows also another important aspect, the flexibility in choosing the gain value in the nonlinear circuit, i.e., different levels of mismatch between gain and loss, using different values for the small-signal negative conductance \( G_1 = G_c (1 + \delta) \) where \( \delta = 0 \), 0.001 and 0.01, represents the mismatch between the loss and gain side (we recall that \( G_2 = G_c \)). As shown in Fig. 5, even with 1% mismatch between gain and loss, the nonlinear circuit shows the same behavior in the perturbation of the oscillation frequency, that is even matched to the case with \( \delta = 0 \). Thus, working in the unstable oscillation configuration using nonlinearity in the coupled circuit gives us freedom to tune the gain component’s value and it works well even with some mismatch between gain and loss. Note that the oscillation frequency is highly sensitive to the capacitance perturbation on either side of the circuit, either on the loss or gain side. Though not shown explicitly, we have observed this feature theoretically, by calculating the eigenfrequencies from \( \text{det}(\mathbf{M} - j \omega \mathbf{L}) = 0 \) when varying \( C_1 \), and also verified the shifted resonance frequencies using the prediction provided by the Puiseux series. Also, we have observed in time-domain analyses with Keysight ADS circuit simulators using nonlinear gain, that the shift of the oscillation frequency is more sensitive to perturbation of \( C_1 \) than \( C_2 \). In this paper, however, we only show the result from perturbing \( C_2 \) because we want to investigate how a telemetric sensor works (i.e., the sensing capacitance is on the passive part of the coupled resonators circuit).

![Diagram](image)

Fig. 6. (a) Coupled resonators terminated with gain \(-G_1\) and loss \(G_2\), with \(G_1 = G_2 = G_c = 9 \text{ mS}\), and \(L = 10 \mu\text{H}\), coupling capacitance \(C_c = 1.5 \text{nF}\), capacitances \(C_1 = C_2 = C_c = 1.5 \text{nF}\). These parameters lead to an EPD. The isolated (i.e., without coupling) resonance frequency of each LC resonator is \(\omega_0 = 1/\sqrt{LC} \approx 25.8 \times 10^6 \text{ s}^{-1}\). The eigenfrequencies of the coupled circuit are calculated by solving \(\text{det}(\mathbf{M} - j \omega \mathbf{L}) = 0\). (b) Real and (c) imaginary parts of the angular eigenfrequencies normalized by \(\omega_0\), varying \(C_2\) around the EPD value \(C_c\). (d) At the EPD, the coalescence parameter \(\sin(\theta)\) vanishes, indicating that the two state vectors coalesce.

\[
\frac{d\Psi}{dt} = \mathbf{M} \Psi
\]

\[
\mathbf{M} = \mathbf{A}^{-1} = \begin{pmatrix}
0 & 0 & A & 0 \\
0 & 0 & 0 & A \\
-\frac{B_1}{C_1} & -\frac{B_1}{C_1} & -\frac{B_1}{C_1} & -\frac{B_1}{C_1} \\
-\frac{B_2}{C_2} & -\frac{B_2}{C_2} & -\frac{B_2}{C_2} & -\frac{B_2}{C_2}
\end{pmatrix}
\]

\[
A = 1 + \frac{C_1}{C_1}, \quad B_1 = 1 + \frac{C_1}{C_1}, \quad B_2 = 1 + \frac{C_2}{C_2}.
\]

In this configuration, EPD occurs at \(C_1 = C_2 = C_c = 1.5 \text{nF}\), linear gain and loss \(G_1 = G_2 = G_c = 9 \text{ mS}\), \(L = 10 \mu\text{H}\). Figures (6)(b) and (c) show the real and imaginary parts of the eigenfrequencies when perturbing \(C_2\), and Fig. (6)(d) demonstrates the convergence of eigenvectors when \(C_2 = C_c\), calculated by solving the dispersion equation \(\text{det}(\mathbf{M} - j \omega \mathbf{L}) = 0\) for \(\omega\). The coalescence of two eigenvectors is observed by defining the angle between them as in (10),

![Graph](image)

Fig. 7. Experimental proof of exceptional sensitivity. (a) Experimental and theoretical changes in the real part of the resonance frequencies \(f\) due to a positive and negative relative perturbation \(\Delta C_2\) applied to the capacitance \(C_2\) as \((1 + \Delta C_2)C_c\). Solid blue lines: eigenfrequencies calculated by finding the zeros of the dispersion equation \(\text{det}(\mathbf{M} - j \omega \mathbf{L}) = 0\) using linear gain \(G_1 = G_2 = G_c = 9 \text{ mS}\); dashed-black: an estimate using the Puiseux fractional power expansion truncated to its first order, using linear gain. Green triangles: oscillation frequency measured experimentally (using nonlinear gain) after reaching saturation for different values of \(C_2\). The measured oscillation frequency significantly departs from the EPD frequency \(f_e = 988.6 \text{ kHz}\) even for a very small variation of the capacitance, approximately following the fractional power expansion \((\Delta C_2) - f_e \propto \text{sgn}(\Delta C_2)\sqrt{|\Delta C_2|}\). Note that both positive and negative capacitance perturbations are measured.

V. EXPERIMENTAL DEMONSTRATION OF HIGH SENSITIVITY: CASE WITH COUPLING CAPACITANCE

An analogous system with the properties highlighted in the previous sections is made by the two resonators with balanced gain and loss (PT-symmetry) coupled with a capacitor \(C_c\) as shown in Fig. 6. We discuss the condition to have an EPD and show the high sensitivity theoretically and experimentally. First, we find the EPD condition by writing down Kirchhoff’s laws and using the Liouvillian formalism using the system vector \(\Psi = [Q_1, Q_2, \dot{Q}_1, \dot{Q}_2]^T\), where \(Q_n\) is the capacitor charge on the gain side \((n = 1)\) and the lossy side \((n = 2)\), and \(\dot{Q}_n = dQ_n/dt\), leading to

\[
\frac{d\Psi}{dt} = \mathbf{M} \Psi
\]
and this indicates the coalescence of the two eigenmodes in their eigenvalues and eigenvectors, and hence the occurrence of a second-order EP. It is seen from this plot that the system eigenfrequencies are coalescing at a specific capacitance $C_2 = C_e$. The system is unstable for any $C_2 \neq C_e$ because of broken PT symmetry, since there is always an eigenfrequency with $\text{Im} (\omega)<0$. Moreover, the bifurcation of the dispersion diagram at the EPD is in agreement with the one provided by the Puiseux fractional power series expansion truncated to its first order, represented by a dashed black line in Fig. (7). The Puiseux series coefficient is calculated as $\alpha_1 = 1.084 \times 10^6 + j \times 1.43 \times 10^6 \text{ rad/s}$ by using Eq. (15), assuming negative linear gain. The coefficient $\alpha_1$ is a complex number that implies that the system always has two complex eigenfrequencies, for any $C_2$ value; that results in an unstable circuit, since one eigenfrequency has $\text{Im} (\omega) < 0$, for any $C_2$ value.

In order to confirm the high sensitivity to a perturbation in the proposed oscillator scheme based on nonlinear negative conductance (nonlinear gain), the gain is now realized using an opamp (Analog Devices Inc., model ADA4817) whose gain is tuned with a resistance trimmer (Bourns Inc., model 3252W-1-501LF) to reach the proper small-signal gain value of $-G_1 = -9 \text{ mS}$. Note that we assume that the nonlinear gain is a bit larger (around 0.1 %) than the loss on the other side of the circuit to make the system slightly unstable. All the other parameters are as in the previous example: a linear conductance of $G_2 = 9 \text{ mS}$, capacitors of $C_1 = C_2 = C_e = 1.5 \text{ nF}$, and inductors of $L = 10 \mu \text{H}$ (Coilcraft, model MSS7348-103MEC). This nonlinear circuit oscillates at the EPD frequency. The actual experimental circuit differs from the ideal one using nonlinear gain in a couple of points: First, extra losses are present in the reactive components associated with their quality factor. The inductor has the lowest quality factor in this circuit with an internal DC resistance of 45 m\Omega, from its datasheet, which is however small. Second, electronic components have tolerances. To overcome some of the imperfections in the experiment process, we use a capacitance trimmer (Sprague-Goodman, model GMC40300) and a resistance trimer in our printed circuit board (PCB) to tune the circuit to operate at the EPD. Also, to have more tunability, a series of pin headers are connected parallel to the loss side, where extra capacitors and resistors could be connected in parallel, as mentioned in Appendix B. The circuit is designed to work at the EPD frequency of $f_e = 988.6 \text{ kHz}$, and indeed after tuning the circuit, we experimentally obtain an experimental EPD frequency at $f = 989.6 \text{ kHz}$ as shown in Fig. 7 with a green triangle at $C_4 = C_e$, very close to the designed one. The oscillation frequency is obtained by taking the FFT of the experimentally obtained time-domain voltage signal of the capacitor $C_1$ using an oscilloscope (Agnilet Technologies DSO-X 2024A) after the signal reaches saturation for a time window of $10^3$ periods with $10^6$ points. The oscillation frequency is in agreement with the result read by the spectrum analyzer (Rigol, model DSA832E).

We then perturb $C_2$ as $(1 + \Delta C_2)C_e$, where $C_e$ satisfies the EPD condition, with small steps $\Delta C_2$, as explained in Appendix B. As shown in Fig. 7, the measured oscillation frequency dramatically shifts away from the EPD frequency, following the square root of $\Delta C_2$ as theoretically predicted by Eq. (14) for the linear case. The experimental results (green triangles) in Fig. 7 demonstrate that even for a small positive and negative perturbation $C_2 - C_e = \pm 20 \text{ mF}$, corresponding to a $\Delta C_2 = \pm 0.013$, the oscillation frequency significantly changes, which can be easily detected even in practical noisy electronic systems. Figure 8(a) shows the experimental time-domain voltage signal of the capacitor $C_1$ with respect to the ground, when a relative perturbation $\Delta C_2 = 0.013$ is applied to $C_2$, measured by an oscilloscope. The spectrum’s frequency is now measured with a spectrum analyzer, and shown in 8(b) as an inset. The frequency of the spectrum matches the perturbed ($\Delta C_2 = 0.013$) oscillation frequency, green triangle in Fig. 7, obtained from the Fourier transform of the time domain experimental data. These results confirm that the structure is oscillating at the predicted perturbed resonance condition after saturation.

An essential feature of any oscillator is its ability to produce a near-perfect periodic time-domain signal (pure sinusoidal
wave), and this feature is quantified in terms of phase noise, determined here based on the measured power spectrum up to 10 kHz frequency offset. The phase noise and power spectrum in Fig. 8(b) demonstrate that electronic noise (which is significant in opamp) and thermal noise in the proposed highly sensitive oscillator scheme does not discredit the potential of this circuit to exhibit measurable high sensitivity to perturbations. Indeed, the low phase noise of $-80.8$ dB/Hz at 1 kHz offset from the oscillation frequency shows that the frequency shifts observed in Fig. 7 are well measurable. Note that this result is intrinsic in the nonlinear saturation regime proper of an oscillator. The resonance oscillation peaks have a very narrow bandwidth (linewidth), which makes the oscillation frequency shifts very distinguishable and easily readable. In this oscillator-sensor system, we also have some freedom in choosing the small-signal gain value because the dynamics are also determined by the saturation arising from the nonlinear gain behavior. For example, in the experiment, we have verified that circuit has the same oscillation frequency when using an unbalanced small-signal gain 1% larger than the balanced loss value. Figure 8(c) shows two measured frequency spectra corresponding to a relative perturbation $\Delta C_2 = 0.013$ applied to $C_2$, using two different gain values. The spectrum has been measured using a resolution bandwidth of 300 Hz, while the video bandwidth is set to 30 Hz to fully capture the spectrum. The red curve is for case with gain around 1% bigger than the balanced loss whereas the blue curve is for the case where gain and loss are balanced. These two frequency response shows the same oscillation frequency but the power spectrum peak has a very small difference, 0.2 dBm higher for the case with 1% larger gain shown in Fig.8(c). This important feature that helps us design the circuit without a very accurate balance between gain and loss, i.e., oscillator-sensors can be realized without satisfying exactly PT symmetry also when the sensing perturbation is not applied. As mentioned earlier, the nonlinear oscillator with broken PT symmetry exhibits the very important feature that the oscillation frequency shifts are both positive and negative, depending on the sign of the perturbations $\Delta C_2$, hence allowing sensing positive and negative values of $\Delta C_2$.

VI. CONCLUSIONS

We demonstrated that two coupled LC resonators terminated with nonlinear gain, with almost balanced loss and small-signal gain, working near an EPD, make an oscillator whose oscillation frequency is very sensitive to perturbations. The nonlinear behavior of the active component is essential for the three important features observed by simulations and experimentally: (i) the oscillation frequency is very sensitive to perturbations, and both positive and negative perturbations of a capacitor are measured leading to very high sensitivity based on shifted oscillation frequency that approximately follows the square-root law, proper of EPD systems; (ii) the measured spectrum has very low phase noise allowing clean measurements of the shifted oscillation frequencies. (iii) It is not necessary to have a perfect gain/loss balance, i.e., we have shown that slightly broken gain/loss balance leads to the same results as for the case of perfectly balanced gain and loss.

Note that none of the features above are available in current PT-symmetry circuits in the literature [21], [39]: Indeed, only one sign of the perturbation is measurable with the PT-symmetry circuits published so far, since the other sign leads to the circuit instability. Furthermore, to make a single sign perturbation measurement, in the literature, e.g., [39], the capacitor $C_1$ on the gain side has been tuned using a varactor to reach the value of the perturbed capacitor ($C_2$) on the reading side in order to rebuild the PT symmetry (but in a sensor operation it is not possible to know a priori the value that has to be measured); furthermore, to work at or very close to an EPD, using linear gain, the gain has to be set equal to the loss (balanced gain/loss condition).

The oscillation frequency shift follows the square root-like behavior predicted by the Puiseux series expansion, as expected for EPD-based systems. We show the performance of the oscillator-sensor scheme based on two configurations: wireless coupling with a mutual inductor, and wired coupling by a capacitor. The latter oscillator scheme has been fabricated and tested. We have analyzed how the nonlinearity in the gain element makes the circuit unstable and oscillate after reaching saturation. The oscillator’s characteristics have been determined in terms of transient behavior and sensitivity to perturbations due to either capacitance or resistance change in the system. The experimental verification provided results in very good agreement with theoretical expectations. The measured high sensitivity of the oscillator sensor to perturbations can be used as a practical solution for enhancing sensitivity, also because the measured shifted frequencies are well visible with respect to underlying noise. The proposed EPD-based oscillator-sensor can be used in many automotive, medical, and industrial applications where detections of small variations of physical, chemical, or biological variations need to be detected.

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APPENDIX

NEGATIVE RESISTANCE

Several different approaches provide negative nonlinear conductance needed for proposed circuits. In this subsection, we
In the experiment, we setup the nonlinear gain to be a bit larger (around 0.1%) than the balanced gain by tuning the RV1 to make the system slightly unstable. To tune and find the exact value of the capacitance that leads to an EPD \((C_2 = C_e)\), a variable capacitor (Sprague-Goodman, model GMC40300) and a series of pin headers, where extra capacitors could be connected in parallel to \(C_2\), are provided. By adding small and known capacitors values on the load side, we tuned the capacitance \(C_2\) to bring the circuit very close to the EPD and observe the EPD oscillation frequency \(f = f_e\).

To show the square root-like behavior of the oscillator’s frequency due to perturbations in Fig. 7 and 6, we perturbed the capacitor \(C_2\) with pairs of extra 10 pF capacitors to make 20 pF steps, connected in parallel to \(C_2\), using the pin headers shown in Fig. 10. After each perturbation, the oscillation frequency is measured with an oscilloscope and also with a spectrum analyzer (for comparison and verification purposes), as discussed in Section B, and shown in Fig. 7 with green triangles. Moreover, for the perturbed circuit, considering \(\Delta C_2 = 0.013\) applied to \(C_2\) (any perturbed point can be chosen), we changed the variable resistor RV1 to study oscillation frequency variation for different unbalanced nonlinear gains. The goal was to show that the circuit using a bit unbalanced nonlinear gain still has the same oscillation frequency. Indeed, by trimming the RV1, we verified the same oscillation frequency for roughly 1% unbalanced gain and loss, as shown in Fig. 8(c). Note that on the PCB, the ground plane (on the bottom layer) is designed to connect all the ground of the measurement equipments and DC supply to the circuit’s ground.

show the circuit in Fig. 9 that utilizes opamp to achieve negative impedance. The converter circuit converts the impedance as \(Z_{in} = -R_1\) while we design the circuit to work at the EPD point by choosing \(R_1 = 1/G_e\). In the experiment, we used \(R_1 = 100\ \Omega\), and \(R_2 = 2\ \mathrm{k}\Omega\) to achieve the EPD value. We tuned the negative resistance with resistor trimmer \(R_x\) to reach the EPD value \(G_e = 9\ \mathrm{mS}\).

**Appendix B**

**Implementation of the Nonlinear Coupled Oscillator**

We investigate resonances and their degeneracy in the two LC resonators coupled by a capacitor as in Fig. 10(a), whereas Figs. 10(b) and (c) illustrate the PCB layout and assembled circuit. In the fabricated circuit, the sensing capacitance is shown in the red dashed box, the nonlinear gain is in the orange dashed box, and the DC supply is in the yellow dashed box. Inductors have values \(L_1 = L_2 = 10\ \mu\mathrm{H}\), the loss value is set to \(G_2 = 9\ \mathrm{mS}\) with a linear resistor, the capacitor on the gain side and the coupling capacitor are \(C_1 = C_e = 1.5\ \mathrm{nF}\). The gain element is designed with an opamp (Analog Devices Inc., model ADA4817), where the desired value of gain is achieved with a variable resistor RV1.

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