Boundary Correlators in 2D Quantum Gravity:
Liouville versus Discrete Approach

Ivan K. Kostov∗†
Service de Physique Théorique, CNRS – URA 2306,
C.E.A. - Saclay, F-91191 Gif-Sur-Yvette, France#
and
Jefferson Physical Laboratory, Harvard University
Cambridge, MA 02138, USA

We calculate a class of two-point boundary correlators in 2D quantum gravity using its microscopic realization as loop gas on a random surface. We find a perfect agreement with the two-point boundary correlation function in Liouville theory, obtained by V. Fateev, A. Zamolodchikov and Al. Zamolodchikov. We also give a geometrical meaning of the functional equation satisfied by this two-point function.

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∗ kostov@spht.saclay.cea.fr
† Associate member of the INRNE, Bulgarian Academy of Sciences
# Permanent address
1. introduction

The Liouville \([1]\) and the matrix-model \([2]\) descriptions of 2D quantum gravity\footnote{A good review of both approaches is given in \([3]\).}, known also as 2D string theory, are complementary to each other in the sense that the each of the two approaches has its advantages for certain class of problems. The correspondence between these two approaches is historically the first and technically the most elementary example of the matrix-string duality.

In the theory of closed strings (world sheet without a boundary), the predictions of the Liouville theory are restricted to the exponents characterizing the scaling behavior of the correlation functions. These have been formulated long ago in \([4]\) and \([5]\) and are known as the KPZ rule, relating the “flat” and the “gravitational” dimensions of the matter fields. The KPZ rule was checked on various microscopic realizations of the 2D quantum gravity in terms of statistical models on random lattices and in the last years have been applied successfully to evaluate some difficult “flat” critical exponents \([6]\).

The situation is much more interesting in the case of a world sheet with boundaries, which describes open string amplitudes. The simplest such manifold is the disc. In this case one can compare not only critical exponents, but also functions of several dimensionless quantities. Indeed, an \(n\)-point boundary correlator depends on \(n\) dimensionless parameters, which can be made of the area of the world sheet and the \(n\) distances between the points along the boundary.

A well known example is the bulk one-point function in presence of a boundary, which plays an important role in 2D quantum gravity \([7,8]\). This function depends on the dimensionless ration \(\mu_B/\sqrt{\mu}\) and correctly reproduces the results obtained using the statistical mechanics of randomly triangulated surfaces.

Another, much less trivial example is the boundary 2-point function for two boundary operators intertwining between the free (Neumann) and fixed (Dirichlet) boundary conditions. This function depends on two dimensionless parameters associated with the two boundary cosmological constants along the Dirichlet and Neumann pieces of the boundary and have been first calculated in \([9]\) using the representation of the matter fields as a gas of non-intersecting loops (domain walls) on a randomly triangulated surface.

Other solvable examples of boundary correlation functions are given, again in the loop gas formulation, by various configurations of domain walls connecting points at the boundary. Such correlators have found application in some statistical problems associated with percolating clusters and polymers \([6]\).

Recently, due to the impressive results in \([10]\) and \([11]\), it became possible to compare the results of the matrix model for the boundary correlators with the predictions of the Liouville theory. The authors of \([10]\) calculated the general boundary two-point correlation function, and the boundary three-point function was calculated in \([11]\).

In this paper we will show that the expression for the boundary Liouville two-point correlator obtained in \([10]\) is in perfect agreement with the results obtained using the microscopic definition of 2D quantum gravity as a loop gas on a random planar graph. Moreover, we will give a simple geometrical meaning of the remarkable functional equation for the two-point correlator, obtained in \([10]\).
2. The boundary 2D gravity as a boundary Liouville theory

2.1. World-sheet description

The world-sheet formulation of 2D quantum gravity (see, for example the review \[3\]) involves a Liouville field \(\varphi\) and a matter field \(\chi\), defined on the world-sheet manifold. From the point of view of the 2D string theory these two fields represent the two components of the position field of the string. It is most convenient to choose the world sheet with the disc topology as the upper half-plane \(H_+\), in which case all the curvature is concentrated at the infinite point. Then the action is given by the integral

\[
\mathcal{A}[\varphi, \chi] = \int_{H_+} d^2 z \left( \frac{g}{4\pi} (\nabla \varphi^2 + \nabla \chi^2) + \mu e^{2\varphi} \right) + \int_{-\infty}^{\infty} dx \mu B e^\varphi + \text{ghosts} \tag{2.1}
\]

and the background charges are defined by the asymptotic of the fields at \(|x| \to \infty\)

\[
\varphi \sim -(g + 1) \log z\bar{z}, \quad \chi \sim -(g - 1) \log z\bar{z} \tag{2.2}
\]

The central charges of the matter and Liouville fields are

\[
c_\varphi = 1 + 6 \frac{(g + 1)^2}{g}, \quad c_\chi = 1 - 6 \frac{(g - 1)^2}{g} \tag{2.3}
\]

so that \(c_\varphi + c_\chi = 26\).

2.2. Bulk and boundary states

The bulk and boundary\(^3\) KPZ states are marginal operators \(V_p\) and \(B_p\), representing products of Liouville and matter vertex operators

\[
V_p(z, \bar{z}) = e^{(g+1-|p|)\varphi(z, \bar{z})} e^{-i(p-g+1)\chi(z, \bar{z})}, \tag{2.4}
\]

\[
B_q(x) = e^{(\frac{g+1}{2}-|q|)\varphi(x)} e^{-i(q-\frac{g-1}{2})\chi(x)} \tag{2.5}
\]

Here we use notations, which are more natural from the point of view of string theory, where \(p\) make sense of target-space momentum. The dimensions of the matter and the Liouville components of the bulk KPZ state \(V_p\) are

\[
\Delta_\chi[V_p] = \frac{p^2 - (g - 1)^2}{4g}, \quad \Delta_\varphi[V_p] = \frac{-p^2 + (g + 1)^2}{4g} \tag{2.6}
\]

\(^2\) Here we adopt the notations commonly used in the Coulomb gas picture, while in the notations used in \[10\] the duality transformation takes simpler form. The relation between our notations and these of \[10\] is given in Appendix A.

\(^3\) The boundary operators in 2D quantum gravity have been first discussed in \[12\].
so that its sum is equal to 1. For the boundary KPZ state $\mathbf{B}_q$ these two dimensions read

$$
\Delta \chi[\mathbf{B}_q] = \frac{4q^2 - (g - 1)^2}{4g}, \quad \Delta \varphi[\mathbf{B}_q] = \frac{-4q^2 + (g + 1)^2}{4g}.
$$

(2.7)

In the 1+1 dimensional string theory, the operators $\mathbf{V}_p$ and $\mathbf{V}_{-p}$ are interpreted as left- and right moving closed string states. Similarly, the operators $\mathbf{B}_q$ and $\mathbf{B}_{-q}$ describe left- and right moving modes of open string states with target-space momentum $p$. These states are related to each other by hermitian conjugation.

There is another, less trivial symmetry, associated with the Liouville direction of the target space. This symmetry exchanges the operators (2.4) and the operators with the same dimension (2.6), but with the “wrong” sign of the Liouville dressing

$$
\tilde{\mathbf{V}}_p(z, \bar{z}) = e^{(g+1+|p|)\varphi} e^{-i(-p-g+1)\chi},
$$

(2.8)

$$
\tilde{\mathbf{B}}_q(x) = e^{(g+1+|q|)\varphi} e^{-i(-q-g-1)\chi}.
$$

(2.9)

The correlation functions containing the dual operators (2.8) are obtained by analytic continuation to negative $|p|$ of the correlations of the “physical” states (2.7).

There is a one-parameter family of conformally invariant boundary conditions characterized by different (generically complex) values of the boundary cosmological constant $\mu^B$. In general, the boundary operators, intertwine between different boundary conditions. In the presence of boundary operators it is possible to impose different boundary conditions at different pieces of the boundary, each being characterised by its own value of $\mu^B$. Therefore a boundary operator is not characterized only by its dimension and its position on the boundary, but also by the two boundary conditions it joins, characterized by the boundary cosmological constants $\mu_1^B$ and $\mu_2^B$. Following [10], we will denote such a boundary operator as $\mathbf{B}^{\mu_1^B \mu_2^B}$.

If the theory is defined by the boundary term in (2.1), then the observables depend on the scale invariant ratios $\mu^B/\sqrt{\mu}$. For example, a disc correlation function with the bulk operators $\mathbf{V}_{p_1}, \ldots, \mathbf{V}_{p_m}$ and the boundary operators $\mathbf{B}^{\mu_1^B \mu_2^B}_{q_1}, \mathbf{B}^{\mu_2^B \mu_3^B}_{q_2}, \ldots, \mathbf{B}^{\mu_n^B \mu_1^B}_{q_n}$ scales as

$$
\left\langle \prod_a \mathbf{V}_{p_a} \prod_b \mathbf{B}^{\mu_b^B \mu_{b+1}^B}_{q_b} \right\rangle = \mu^\gamma F \left( \frac{\mu_1^B}{\sqrt{\mu}}, \ldots, \frac{\mu_n^B}{\sqrt{\mu}} \right)
$$

(2.10)

$$
\gamma = \frac{1}{2} (2 - \gamma_{\text{str}}) - \sum_a (1 - \delta_{pa}) - \frac{1}{2} \sum_b (1 - \delta_{2qb})
$$

(2.11)

where we introduced the gravitational dimensions

$$
\delta_p = \frac{|p| + \gamma_{\text{str}}}{2}, \quad \gamma_{\text{str}} = 1 - g
$$

(2.12)
and $F$ is some scaling function.

The gravitational dimension $\delta = \delta_p$ is related to flat dimension $\Delta = \Delta^\chi_p$ (eq. (2.6)) by the KPZ formula [4]

$$\Delta = \frac{\delta (\delta - \gamma_{\text{str}})}{1 - \gamma_{\text{str}}}.$$  \hfill (2.13)

The dimensions of the boundary KPZ states, or the open string states, are given by the same formulas, but with twice larger momentum $p = 2q$.

This setting of the boundary Liouville problem is not the most general one. As we shall discuss below, the discrete models of quantum gravity can exhibit a critical behavior with anomalous scaling of the boundary cosmological constant. In this case the boundary interaction is $e^{\sqrt{g} \phi}$.

2.3. Duality property of the Liouville theory

Introduce the parametrization of the bulk and boundary cosmological constants through the positive constant $M$ and the (in general complex) variable $\tau$ as

$$\mu = \frac{1}{\pi} \frac{\Gamma(1 - 1/g)}{\Gamma(1/g)} M^2$$  \hfill (2.14)

and

$$\mu_B(\tau) = \frac{\Gamma(1 - 1/g)}{\pi} M \cosh \tau.$$  \hfill (2.15)

Then the duality states (see [10], eqs. (1.19) and (2.26)) that the observables are invariant under $g, p, M, \tau \to \tilde{g}, \tilde{p}, \tilde{M}, \tilde{\tau}$ with

$$\tilde{g} = 1/g, \quad \tilde{p} = p/g, \quad \tilde{M} = M^g, \quad \tilde{\tau} = g\tau.$$  \hfill (2.16)

We will assume this parametrization of $\mu$ and $\mu_B$ in the rest of the text.

The meaning of the parameter $\tau$ is the following. The partition function and all correlation functions on the disc are meromorphic functions of $\mu_B$ with a cut along the semi-infinite interval $-\infty < \mu_B < -\mu_B(0)$. The singularity associated with the branch point is a square-root one and describes the critical behavior of 2D quantum gravity with $c = -2$, in which the boundary is critical, but the world sheet has finite area. The map $\mu_B \to \tau$ represents the uniformization map that resolves the branch point. The two sides of the cut are parametrized by $\tau = t \pm i\pi, 0 < t < \infty$. Note that the the boundary cosmological constant is an even function of the parameter $\tau$, which means that all correlation functions should have the symmetry $\tau \to -\tau$.  

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2.4. Degenerate boundary KPZ operators

The degenerate bulk KPZ states, or the closed string states, are labeled by the target-space momenta
\[ p_{rs} = gr - s, \quad r, s = 1, 2, \ldots \]  
(2.17)
They satisfy the same fusion rule as the degenerate matter fields in the conformal field theory
\[(r, s) \ast (r', s') = \{(r + r' - 1 - 2m, s + s' - 1 - 2n)\}_{m, n \geq 0}. \]

The degenerate boundary KPZ states, or the open string states, are labeled by the target-space momenta
\[ q_{rs} = \frac{gr - s}{2}. \]  
(2.18)
The spacing of the degenerate target-space momenta of the open string states is thus twice less than that of the closed string states.

2.5. Functional equation for the two-point boundary correlator

The central subject of our discussion will be the boundary two-point correlator, considered as a function of the complex variables \( \tau_1 \) and \( \tau_2 \) defined in (2.15) and the target-space momentum \( q \)
\[ \langle B^\mu_1^B B^\mu_2^B B_{\mu_1^B}^B \rangle = D_q(\tau_1, \tau_2). \]  
(2.19)
It satisfies the evident symmetries
\[ D_q(\tau_1, \tau_2) = D_q(\tau_2, \tau_1) = D_q(-\tau_1, \tau_2) = D_{-q}(\tau_1, \tau_2). \]  
(2.20)

This correlation function coincides with the coordinate-independent part of the two-point boundary correlator in the Liouville theory (eq. (3.18) of [10])
\[ D_q(\tau_1, \tau_2) = d(\beta|s_1, s_2), \quad \beta = Q/2 - b|q|, \]  
(2.21)
where \( b = g^{-1/2}, \quad Q = b + 1/b \) and \( s = \tau / \pi b \) (see Appendix A).

It is also convenient to introduce the two-point correlator of the “wrongly dressed” states (2.8)
\[ \tilde{D}_q(\tau_1, \tau_2) = d(\beta|s_1, s_2), \quad \beta = Q/2 + b|q|. \]  
(2.22)
The “unitarity” condition satisfied by the Liouville two-point function implies a relation between the correlator of two “physical” KPZ states to the correlator of the “wrongly dressed” states (see Appendix A)
\[ D_q(\tau_1, \tau_2)\tilde{D}_q(\tau_1, \tau_2) = 1, \]  
(2.23)
where \( \tilde{D}_q(\tau_1, \tau_2) \) is given by the analytic continuation of \( D_{-q}(\tau_1, \tau_2) \) from positive values of \(-q\).

The authors of [10] showed that the boundary two-point correlator, considered as a function of the complex variables \( \tau_1 \) and \( \tau_2 \) defined in (2.15), the target-space momentum \( q \) and the dimensionfull constant \( M \) defined in (2.14), satisfies a remarkable pair of functional equations:

\[
D_q(\tau_1 + i\pi/g, \tau_2) - D_q(\tau_1 - i\pi/g, \tau_2) = i \frac{M}{c_g(q)} \sinh \tau_1 D_{q-\frac{1}{2}}(\tau_1, \tau_2), \tag{2.24}
\]

\[
D_q(\tau_1 + i\pi, \tau_2) - D_q(\tau_1 - i\pi, \tau_2) = i \frac{M g}{c_g(q)} \sinh \tau_1 D_{q-\frac{1}{2}q}(\tau_1, \tau_2), \tag{2.25}
\]

where the multiplicative factors are given by

\[
c_g(q) = \frac{2}{g} \frac{\Gamma(1-2q) \Gamma(2q)}{\Gamma(2q-g) \Gamma(-2q)}, \quad \tilde{c}_g(q) = \frac{C_1}{g(2q/g)}. \tag{2.26}
\]

These two equations are related by the duality transformation (2.16).

The equations (2.24)-(2.25) hold only if all the momenta are positive. When the moment on the right-hand side becomes negative, eqs. (2.24) and (2.25) make sense as analytic continuation from positive values of \( q - \frac{1}{2}g \) or \( q - \frac{1}{2} \). Using the “unitarity” relation (2.23), we obtain another pair of functional equations

\[
D_q(\tau_1 + i\pi/g, \tau_2) - D_q(\tau_1 - i\pi/g, \tau_2) = i \frac{M}{\tilde{c}_g(q)} \sinh \tau_1 D_{q-\frac{1}{2}}(\tau_1, \tau_2), \tag{2.27}
\]

\[
D_q(\tau_1 + i\pi, \tau_2) - D_q(\tau_1 - i\pi, \tau_2) = i \frac{M g}{\tilde{c}_g(q)} \sinh \tau_1 D_{q-\frac{1}{2}q}(\tau_1, \tau_2). \tag{2.28}
\]

The solution of these equations is unique and reads explicitly [10]

\[
D_q(\tau_1, \tau_2) = d_g(|q|) \tilde{D}_{|q|}(\tau_1 + \tau_2) \tilde{D}_{|q|}(\tau_1 - \tau_2), \tag{2.29}
\]

where \( \tilde{D}_q(\tau) \) is given by the integral

\[
\log \tilde{D}_q(\tau) = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dy}{y} \left[ \frac{\sinh(2\pi q y/g)}{\sinh(\pi y)\sinh(\pi y/g)} e^{i\gamma \tau} - \frac{2q}{\pi y} \right], \tag{2.30}
\]

and the normalization factor \( d_g(q) \) can be found in Appendix A.

In the rest of this paper we will study the boundary two-point correlation function using the discrete formulation of quantum gravity. We will calculate explicitly the boundary correlators for for

\[
q = \frac{(L + 1)g - 1}{2} \quad \text{and} \quad q = \frac{(L + \frac{1}{2})g - 1}{2}, \quad L = 1, 2, ...
\]

and will show that they satisfy (with appropriate normalization) either (2.24) or (2.28). In both cases the answer is given by the same formula (2.29)-(2.30).

\footnote{We corrected a misprint in the final formula (4.7) of [10].}
3. Loop gas formulation of 2D quantum gravity

3.1. The loop gas model

The 2D quantum gravity can be constructed microscopically as strings with discrete target spaces. The target spaces of the rational CFT coupled to gravity are Dynkin graphs of ADE type \[13,8\]. These simple string theories represent ADE statistical models \[14\] defined on random graphs and can be realized as models of coupled random matrices \[13\]. A non-unitary theory with continuous spectrum of central charge \(c \leq 1\) is the \(O(n)\) model on a random lattice, which is realized as a one-matrix model \[16\]. All these theories are described in terms of nonintersecting loops of the world sheet.

The simplest statistical model allowing interpretation in terms of the gas of loops is the \(O(n)\) model coupled to 2D gravity \[16\]. In this model, the matter field is defined as a system of self- and mutually avoiding loops on a random planar trivalent graph. Each loop is taken with a length factor \(ne^{-M_0\ell}\), where \(\ell\) is the length of the loop (= the number of links visited by the loop). The fugacity of the loops \(n\), whose original meaning is the number of flavors of the \(O(n)\) vector field, is in fact any real number in the interval \(-2 \leq n \leq 2\). The continuous parameter \(n\) is related to the Coulomb gas coupling constant \(g\) by

\[
n = -2 \cos \pi g.
\] (3.1)

We are concerned only by planar trivalent graphs \(G\) with the topology of a disc. We associate with such a graph the area \(A_G\) and the boundary length \(L_G\). The area is defined as the number of vertices and the boundary length as the number of external lines of the graph. Then the loop gas partition function is

\[
Z = \sum_{\text{graphs } G} e^{-\mu_0 A_G - \mu_0^B L_G} \sum_{\text{loops on } G} n^{\#\text{loops}} e^{-M_0 L_{\text{loops}}}. \tag{3.2}
\]

where by \(L_{\text{loops}}\) we denoted the total length of the loops (= the number of links occupied by loops).

The critical phases of the the loop gas on a random graph \[17\] are qualitatively the same as these on a flat hexagonal lattice \[18\]. The dense phase is described by the vicinity of the critical point of \(M_0\). In this phase the world sheet is densely covered by loops, whose characteristic length diverges at the critical point. The critical phase is obtained when both \(M_0\) and \(\mu_0\) become critical. In this phase the length of the loops diverges, but the area that is not covered by loops also diverges. This is why the critical phase is also called dilute phase of the loop gas. Further, one can construct multicritical phases by adding more coupling constants associated with the planar graph \[17\]. As the loop gas model on a random surface can be considered as a \(n \neq 0\) deformation of the ensemble of empty (without loops) planar graphs, its multi-critical regimes are related to the multi-critical regimes of the ensemble of planar graphs \[19\].
The critical regimes of the loop gas on a random graph are described by conformal field theories of matter fields with central charge \( c = c_\chi \) as given by eq. (2.3), where the branch of the inverse function \( g(n) \) is determined by the way the criticality is achieved. In the dense phase \( 0 < g < 1 \), in the dilute phase \( 1 < g < 2 \), and in the \( k \)-critical phase \( k + 1 < g < k + 2 \).

Each value of the central charge between \(-\infty\) and 1 has two realizations: one in the dense phase \( (g < 1) \) and the other in the dilute (or a multicritical) phase \( (g > 1) \). These two realizations are not equivalent, but related by duality. For example the pure gravity \( (g = 3/2 \) or \( n = 0 \)) is related by duality to the critical percolation problem on a random graph \( (g = 2/3 \) or dense loops with \( n = 1 \)). The exact solution out of criticality allows to determine the string susceptibility in the different phases

\[
\gamma_{\text{str}} = \begin{cases} 
1 - g, & g > 1; \\
1 - \frac{1}{g}, & 1 < g < 1.
\end{cases}
\] (3.3)

This means that the partition function on a surface with genus \( h \) scales as \( \mu^{(1-h)(2-\gamma_{\text{str}})} \), which in the case of the disc (genus 1/2) gives \( \mu^{1+\frac{2\chi}{g}} \). This scaling behavior matches with the scaling law (2.10) only if \( g > 1 \). From here we conclude that the dense phase is described by the world-sheet action (2.1) with Liouville interaction \( \tilde{\mu} e^{2\phi} \) instead of \( \mu e^{2\phi} \), which is related to the original one by duality transformation.

3.2. Critical behavior in presence of boundary

In the case of a world sheet with boundary, one has to fix the boundary conditions both for the matter and the gravity. The boundary condition for the gravitational field is defined by the boundary cosmological constant, which we will denote by the letter \( z \) instead of \( \mu_B \). This notation is inherited from the matrix model description of the loop gas model [16], where the boundary cosmological constant has the meaning of a spectral parameter for the random matrix.

The boundary condition for the matter field is defined by the behavior imposed on the vacuum loops near the boundary. The most natural boundary condition for loop gas model is obtained by requiring that the vacuum loops avoid the boundary. We call this boundary condition fixed or \textit{Dirichlet type}, because in the SOS model interpretation, in which the loops are the domain walls separating two neighboring heights, it means that all points at the boundary have the same height. The partition function with this boundary condition satisfies a simple loop equation [13].

Another boundary condition can be prepared by allowing not only closed loops, but also open lines ending at the points of the boundary. The free or \textit{Neumann type} boundary condition is defined by requiring that all links along the boundary are endpoints of such lines (Fig.1). This boundary condition is studied in [9]. From the point of view of the SOS model it means that there the SOS field changes freely along the boundary. In
the continuum limit and after the Coulomb gas mapping, these two boundary conditions become the usual Dirichlet and Neumann boundary conditions for the gaussian field $\chi$.

![Fig.1: Neumann and Dirichlet type boundary conditions for the loop gas.](image)

The explicit solution of the loop equations [17,9] shows that the boundary cosmological constant $z \sim \mu^B$ scales as $\mu^{1/2}$ only in the case of Dirichlet boundary in the dilute phase or a Neumann boundary in the dense phase. In the two other cases it scales as $\mu^{\frac{1}{2g}}$ and corresponds to taking the “wrong” branch in the KPZ dressing of the identity operator on the boundary⁵. Therefore we can identify the bulk and boundary interactions in these four cases as follows

| Bulk | Dirichlet boundary | Neumann boundary |
|------|--------------------|------------------|
| $g > 1$ : | $\mu e^{2\varphi}$ | $\mu^B e^{\varphi}$ | $\tilde{\mu}^B e^g \varphi$ |
| $g < 1$ : | $\mu e^{2\varphi/g}$ | $\tilde{\mu}^B e^{\varphi}$ | $\mu^B e^{\varphi/g}$ |

Thus the duality transformation (2.16) also exchanges the Dirichlet and Neumann boundary conditions for the matter field, which is natural from the point of view of the Coulomb gas mapping. The fact that the boundary can have anomalous dimension is not very surprising. Take for example the topological point $g = 1/2$ in which $n = 0$ and the world sheet has zero area. In this case the cosmological constant measures the length of the boundary and thus $\mu \sim \mu^B$.

⁵ This means that the Seiberg rule for choosing these branches does not necessarily hold for boundary operators.
3.3. Star polymers in the bulk and at the boundary

The order parameters (the magnetic operators) in the $O(n)$ model have a simple description in terms of the loop gas. The $m$-th magnetic operator $S_L$ is represented as the source of $L$ nonintersecting lines meeting at a point \[20,21\]. Following Duplantier \[21\], we call these operators *star polymers* (Fig. 2a). The correlation function of two such operators can be evaluated as the partition function of a network consisting of $L$ nonintersecting lines tied at their extremities, moving in the sea of vacuum loops of the $O(n)$ model.

In the Coulomb gas picture \[18\] the star polymers describe point-like topological defects. The flat conformal dimensions of the corresponding local operators are \[21\]

$$\Delta_{L/2,0} = \frac{g L^2}{16} - \frac{(g - 1)^2}{4g}. \quad (3.4)$$

The smallest dimension $\Delta_1$ is positive only in the interval $-1/2 \leq g \leq 2$. Outside this interval the propagator of the nonintersecting random walk grows with the distance between its two extremities. The fact that two points are connected with a line leads to an effective repulsion between them. Such a phenomenon is typical for non-unitary theories.

By the KPZ formula \[2.13\], the gravitational scaling dimensions of the star operators are

$$\delta_{L}^{\text{bulk}} = \left(\frac{Lg}{2\nu} + \gamma_{\text{str}}\right)/2 = \frac{Lg}{4} - \frac{g - 1}{2}. \quad (3.5)$$

These values have been confirmed by direct calculations in the matrix model \[22\].

![Fig. 2: Star polymers in the bulk (a) and at the boundary (b).](image)

The correlation function of two *boundary star polymers* with $L$ lines is described geometrically as the partition function of the loop gas in presence of $L$ non-intersecting lines whose ends meet at two boundary points (Fig.2b).
The operator identification of the star polymers is different in the bulk and on the boundary, which is a general phenomenon in the boundary CFT \[23\]. In the loop gas coupled to gravity, the boundary operators \( S_L \) have gravitational dimensions

\[
\delta_L^{\text{bound}} = L \frac{g}{2} = \delta_{L+1,1},
\]  

(3.6)

(see, for example, sect. 5 of \[9\].) Therefore, when considered as a boundary operators, the star operators \( S_n \) can be identified with the degenerate boundary fields \( B_{g(L+1)-1} \). The flat dimensions of these fields are

\[
\Delta_L^{\text{bound}} = \frac{L}{2} \left( \frac{L + 2}{2} g - 1 \right) = \Delta_{L+1,1}.
\]

(3.7)

4. Loop equation for the disc amplitude with Dirichlet-type boundary

In the next two sections we will derive loop equations for the boundary correlation functions of two star operators separating two Dirichlet-Dirichlet or Dirichlet-Neumann boundary conditions. These loop equations include as an ingredient the disc amplitude with Dirichlet-type boundary \( W(z) \). For the case of self-consistency we will derive first the loop equation for this amplitude.

Let \( \hat{W}(\ell) \) be the partition function of the loop gas on a random surface with the topology of a disc with fixed boundary length \( \ell \). We define the disc partition function \( \Phi(z) \) with boundary cosmological constant \( z \) as

\[
\Phi(z) = \int_0^\infty \frac{d\ell}{\ell} e^{-z\ell} \hat{W}(\ell).
\]

(4.1)

Then \( W(z) = \frac{\partial \Phi}{\partial z} \) is the partition function with one marked point at the boundary

\[
W(z) = \int_0^\infty d\ell \ e^{-z\ell} \hat{W}(\ell).
\]

(4.2)

The disc amplitude with two marked points on the boundary is given by the derivative

\[
\frac{\partial W(z)}{\partial z} = \int_0^\infty \ell d\ell \ e^{-z\ell} \hat{W}(\ell)
\]

(4.3)

and so on. One can consider the the amplitude with \( k \) marked points and \( k \) different boundary cosmological constants between them. For example, the disc amplitudes with two marked points and two different boundary cosmological constants, \( z_1 \) and \( z_2 \) between them, is given by

\[
W(z_1, z_2) = \int_0^\infty d\ell \ e^{-z_1\ell_1 - z_2\ell_2} \hat{W}(\ell_1 + \ell_2) = \frac{W(z_1) - W(z_2)}{z_1 - z_2}.
\]

(4.4)
Considered as a function of the complex variable $z$, $W(z)$ is analytic in the complex plane cut along the interval $-\infty < z < -M$ along the negative real axis. The singularity at $z = -M$ comes from the dominance of surfaces with divergent boundary length. The constant $M = M(\mu)$ is a function of the cosmological constant $\mu$ and can be calculated using the microscopic definition of the model [8]. It has the meaning of the effective boundary interaction induced by the fluctuations in the bulk.

It is convenient to introduce, as in (2.15), the uniformization parameter $\tau$, which resolves the branch point singularity

$$ z = M \cosh \tau. \tag{4.5} $$

Then all disc amplitudes are even analytic functions of $\tau$.

The disc partition function $W(z)$ satisfies a quadratic loop equation (see, for example [8], sect 3.4)

$$ V'(z)W(z) = \int \frac{dz'}{2\pi i} \frac{1}{z - z'} \left[ W(z')^2 - 2 \cos \pi g \ W(z')W(-z') \right] \tag{4.6} $$

where $V(z)$ is a polynomial (the matrix model potential) tuned such that the ensemble of empty planar graphs is near the $m$-critical point. The contour of integration circles the cut in the $z$-plane along the interval $-\infty < z < -M$ on the negative half of the real axis. The partition function of the empty planar graphs satisfies the same equation, but without the last term. Therefore by subtracting this trivial piece, we can get rid of the potential $V(z)$. Then equation (4.6) implies the following boundary condition along the real axis

$$ \text{Im} W(z)[W(z + i0) + W(z - i0) - 2 \cos \pi g \ W(-z)] = 0. \tag{4.7} $$

In the parametrization (4.5), the upper and the lower sides of the cut correspond to the lines $\tau + i\pi$ and $\tau - i\pi$, $0 < \tau < \infty$. Therefore (4.7) can be written as the following finite-difference equation for the analytic function $W(\tau) \equiv W(z(\tau))$

$$ W(\tau + i\pi) + W(\tau - i\pi) - 2 \cos \pi g \ W(\tau) = 0. \tag{4.8} $$

Its solution is

$$ W(\tau) = -\frac{M^g}{2\sin \pi g} \cosh(g\tau) \tag{4.9} $$

or in terms of $z$

$$ W(z) = -\frac{(z + \sqrt{z^2 - M^2})^g + (z + \sqrt{z^2 - M^2})^g}{4\sin \pi g}, \tag{4.10} $$

where we chose the normalization for later convenience. The solution depends on the cosmological constant through the parameter $M$. If the potential $V(z)$ is tuned near the $m$-th multicritical point of the ensemble of planar graphs, then the Coulomb coupling constant $g$ should be taken in the interval $m - 1 < g < m$. 

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5. Boundary correlation functions of star operators

From the operator identification (3.7), we expect that this partition function coincides with the two-point function (2.29) of two boundary KPZ states with target space momenta

\[ q = \frac{1}{2}((L + 1)g - 1) \]

\[ W_L(\tau_1, \tau_2) \sim D_{(L+1)g-1}(\tau_1, \tau_2). \]  \hspace{1cm} (5.1)

Let us check that this is indeed the case.

As we already pointed out, the correlation function \( W_L(\tau_1, \tau_2) \) of two boundary star operators with \( L \) lines is described geometrically as the partition function of the loop gas in presence of \( L \) non-intersecting lines whose ends meet at two boundary points (Fig.2b). The inverse Laplace transform of \( W_L \), which is the partition function for fixed lengths \( \ell_1 \) and \( \ell_2 \) of the two boundaries

\[ W_L(\tau_1, \tau_2) = \int_{0}^{\infty} d\ell_1 d\ell_2 \, e^{-z_1\ell_1 - z_2\ell_2} \hat{W}_L(\ell_1, \ell_2) \] \hspace{1cm} (5.2)

is obtained from the disc partition function \( W(\ell) \) as follows. Consider the sum over all loop configurations with fixed lengths \( \ell'_1, ..., \ell'_L \) of the branches of the star polymers, which is given by the product of \( L + 1 \) disc partition functions, and then integrate with respect to the lengths

\[ \hat{W}_L(\ell_1, \ell_2) = \int_{0}^{\infty} d\ell'_1 ... d\ell'_L \, W(\ell_1 + \ell'_1) \hat{W}(\ell'_1 + \ell'_2) ... \hat{W}(\ell'_L + \ell_2). \] \hspace{1cm} (5.3)

Fig. 3: The loop equation for the boundary correlation function of two star polymers.

The integral representation (5.3) is equivalent to the recurrence relation between \( \hat{W}_L \) and \( \hat{W}_{L-1} \) depicted in Fig.3

\[ \hat{W}_L(\ell_1, \ell_2) = \int_{0}^{\infty} d\ell \, \hat{W}(\ell_1 + \ell) \hat{W}_{L-1}(\ell, \ell_2), \quad \hat{W}_0(\ell_1, \ell_2) = \hat{W}(\ell_1 + \ell_2) \] \hspace{1cm} (5.4)
or, after a Laplace transformation,

\[ W_L(z_1, z_2) = \oint \frac{dz}{2\pi i} \frac{W(z_1) - W(z)}{z_1 - z} W_{L-1}(-z, z_2) \]  

(5.5)

where the contour of integration circles the cut \(-\infty < z < -M\). This integral equation is equivalent to the following condition on the discontinuity of \(W_L(z)\) along the cut

\[ \text{Im} W_L(-z, z_2) = \text{Im} W(-z) \cdot W_{L-1}(z, z_2), \quad z > M. \]  

(5.6)

In terms of the uniformization variable \(\tau\) this gives the finite difference equation

\[ W_L(\tau_1 + i\pi, \tau_2) - W_L(\tau_1 - i\pi, \tau_2) = [W(\tau_1 + i\pi) - W(\tau_1 - i\pi)]W_{L-1}(\tau_1, \tau_2) \]  

(5.7)

or, taking into account the explicit expression for \(W(\tau)\)

\[ W_L(\tau_1 + i\pi, \tau_2) - W_L(\tau_1 - i\pi, \tau_2) = -iM^g \sinh g\tau_1 W_{L-1}(\tau_1, \tau_2). \]  

(5.8)

This equation coincides (up to a normalization factor) with the functional equation (2.25) obtained in [10] if we take \(q = \frac{Lg}{2} + \text{constant}\). The constant is fixed by identifying

\[ W_{L=0}(z_1, z_2) = \frac{W(z_1) - W(z_2)}{z_1 - z_2} \]  

(5.9)

with the correlation function \(D_{g-1}(z_1, z_2)\) of two identity operators (4.4). This gives

\[ q = \frac{(L + 1)g - 1}{2}, \quad L = 1, 2, 3, ... \]  

(5.10)

which is in accord with the identification (3.6).

The explicit form of the solution is

\[ W_L(\tau_1, \tau_2) = -\frac{M^g(L+1)-1}{2\cos \pi g} \hat{W}_L(\tau_1 + \tau_2)\hat{W}_L(\tau_1 - \tau_2) \]  

(5.11)

with

\[ \hat{W}_L(\tau) = \frac{\sinh \left( g\frac{\tau}{2} + i\pi \frac{Lg}{2} \right) \sinh \left( g\frac{\tau}{2} + i\pi \frac{(L-2)g}{2} \right) \cdots \sinh \left( g\frac{\tau}{2} - i\pi \frac{Lg}{2} \right)}{\sinh \left( \frac{\tau}{2} - i\pi \frac{Lg}{2} \right)} \]  

(5.12)

6. The intertwiner between Dirichlet and Neumann boundary conditions

Now let us consider another, less trivial, example of a boundary correlator, which have been calculated in [3]. This is the correlation function \(\Omega(z, \tilde{z})\) of two boundary changing
operators that intertwine between the Dirichlet and Neumann boundary conditions (see Fig. 10 of [9]). We denote by
\[ z = M \cosh \tau, \quad \tilde{z} = M^g \cosh g\tilde{\tau} \]  
(6.1)
the cosmological constants along the Dirichlet and Neumann type boundaries, correspondingly (see Fig.4).

\[ \text{Fig. 4: Disc amplitude with Dirichlet/Neumann boundary condition.} \]

Consider first the partition function \( \hat{\Omega}(\ell, \tilde{z}) \) with fixed length \( \tilde{\ell} \) of the Dirichlet boundary. It satisfies the following integral equation depicted in Fig.5
\[ \hat{\Omega}(\ell, \tilde{z}) = 1 + \sum_{n=0}^{\infty} \int_0^\infty d\ell_1...d\ell_n \hat{W}(\ell + \ell_1 + ... + \ell_n, \tilde{z}) \hat{\Omega}(\ell_1\tilde{z})...\hat{\Omega}(\ell_n\tilde{z}), \]  
(6.2)

\[ \text{Fig. 5: Loop equation for the amplitude with Dirichlet/Neumann boundary conditions} \]

or in terms of the Laplace transform \( \Omega(z, \tilde{z}) = 1 - \int_0^\infty d\ell e^{-\ell z} \hat{\Omega}(\ell, \tilde{z}) \)
\[ \Omega(z, \tilde{z}) = -\oint \frac{dz'}{2\pi i} \frac{1}{z - z'} \frac{W(z')}{\hat{\Omega}(-z', \tilde{z})}. \]  
(6.3)
Taking the imaginary part along the cut, we get the functional equation

\[ \text{Im}\Omega(z, \bar{z}) = -\frac{\text{Im}W(z)}{\Omega(-z, \bar{z})} \]  

(6.4)

or, in terms of \( \tau \),

\[ \Omega(\tau+i\pi, \bar{\tau}) - \Omega(\tau-i\pi, \bar{\tau}) = -\frac{W(\tau+i\pi) - W(\tau-i\pi)}{\Omega(\tau, \bar{\tau})}. \]  

(6.5)

This equation coincides, up to a normalization factor, with the functional equation (2.28) for the special value \( q = g/4 \) of the target space momentum. Its solution, found in [9], is given by the integral (2.30) with \( q = g/4 \).

The Dirichlet/Neumann intertwiner \( B_{g/4} \) is nothing but the gravitationally dressed twist operator, which should be inserted because the curvatures \( \pi/2 \) associated with the points where the Dirichlet and Neumann boundaries join. The flat dimension of this operator is

\[ \Delta^\chi = \frac{(3g-2)(2g-1)}{16g^2} = \frac{\Delta_{21}\Delta_{01}}{g^2}. \]  

(6.6)

For \( c = 1 \), this is the well known \( \Delta = 1/16 \). Note also that the dimension of the Dirichlet/Neumann intertwiner vanishes in the case of percolation \( g = 2/3 \) and trees \( g = 2 \).

Note that in the dual theory \( (\tilde{g} = 1/g) \), in which the Dirichlet and Neumann boundary conditions exchange their places, this operator should be identified as \( B_{1/4} \). This is consistent, since the Dirichlet-type and Neumann-type boundaries have different dimensions,

\[ \mu^B_{\text{Dirichlet}} \sim \mu^{1/2}, \quad \mu^B_{\text{Neumann}} \sim \mu^{1/2g}. \]

Now it is easy to apply the method of the previous section to calculate a class of operators obtained by fusing the twist operator with a star polymer. Let us denote by \( \Omega_L \) the partition function of the loop gas on the disc, with two \( L \)-star polymers at the boundary, with Dirichlet type boundary condition on one side and Neumann type boundary condition on the other side. We obtain the same recurrence equation (5.5)

\[ \Omega_L(\tau+i\pi, \bar{\tau}) - \Omega_L(\tau-i\pi, \bar{\tau}) = i \sinh g\tau_1 \Omega_{L-1}(\tau_1, \tau_2), \]  

(6.7)

but with different initial condition \( \Omega_0(\tau, \bar{\tau}) = \Omega(\tau, \bar{\tau}) \). This reproduces the FZZ functional equation for the values of the target-space momentum

\[ q = \frac{g(L + \frac{1}{2}) - 1}{2}, \quad L = 1, 2, 3, ... \]
7. Boundary correlator for generic boundary operators.

Although obtained in quite different way, equations (5.8) and (6.5) reduce to the functional equation of [10] for particular values of the momentum $q$. These values correspond to the boundary changing operators that are simple to construct in the loop gas model, and it is plausible that the functional equation holds for any value of the charge $q$.

A heuristic argument in favor of that is the following. As we saw in the previous section, the operator with momentum $q_{21} = g - \frac{1}{2}$ can be inserted by adding a line starting at some point of the boundary. Thus the operator boundary KPZ operator $B_q$ with momentum $q$ can be considered as the result of the fusion of the boundary KPZ operators with momenta $g - \frac{1}{2}$ and $q - \frac{1}{2}g$. In the 2D gravity the fusion rules of KPZ operators are determined by the the fusion rules of their matter components. The fusion rules for the Liouville components follow from the requirement that the result of the fusion is again a marginal operator. Proceeding as in the previous section, we write the integral equation

$$W_q(z_1, z_2) = \oint \frac{dz}{2\pi i} \frac{W(z) - W(z)}{z_1 - z} W_{q-\frac{1}{2}g}(-z, z_2)$$ \hspace{1cm} (7.1)

which leads to the functional equation

$$W_q(\tau_1 + i\pi, \tau_2) - W_q(\tau_1 - i\pi, \tau_2) = i M^g \sinh g \tau_1 W_{q-\frac{1}{2}g}(\tau_1, \tau_2).$$ \hspace{1cm} (7.2)

Up to a normalization factor, this is the functional equation (2.24) obtained from the Liouville theory. The above “derivation” however applies only for operators that can be constructed in the loop gas model. A rigorous derivation of the functional equation in the general case will be presented elsewhere [24].

8. Concluding remarks

We derived a functional recurrence equation for the two-point function of two discrete series of boundary operators in 2D quantum gravity. The recurrence equation is analogous to the fusion procedure with the lowest degenerate operator in the matter CFT. It is identical to the functional equation for the Liouville boundary two-point function obtained in [10]. We used a geometrical derivation (world-sheet surgery) of the recurrence equation, but it is also possible to write it as Ward identity in the corresponding matrix model.

The method explained in this paper can be applied to arbitrary multi-point amplitudes and thus give an alternative procedure to derive the boundary correlation functions in Liouville theory [24].

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6 This is the boundary “leg factor”, which reflects the different normalization of the wave functions in the world sheet and target space approach.
Since the boundary correlators in a CFT coupled to quantum gravity depend on the same number of variables as the boundary correlators in the corresponding CFT on the semi-plane, one is tempted to try to generalize the KPZ formula (2.13) relating the flat and gravitational scaling dimensions to a one-to-one correspondence between the boundary correlation functions on flat and random surfaces.

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Appendix A. The boundary two-point function in Liouville theory

The authors of [10] considered pure boundary Liouville theory with action

\[ A_\phi = \int_{H_+} d^2z \left( \frac{1}{4\pi} (\nabla \phi)^2 + \mu e^{2b\phi} \right) + \int_{-\infty}^{\infty} dx \mu_B e^{b\phi} \]  

(A.1)

where the background charge is introduced by the asymptotics

\[ \phi(z, \bar{z}) \sim -Q \log(z\bar{z}), \quad Q = b + 1/b. \]  

(A.2)

The Liouville bulk and boundary vertex operators are defined as

\[ V_\alpha = e^{2\alpha \phi(z, \bar{z})}, \quad B_\beta = e^{\beta \phi(x)}. \]  

(A.3)

Considered as a function of the boundary cosmological constant \( \mu^B \), the disc partition function is analytic in the complex plane cut along the interval

\[ -\infty < \mu^B < -\sqrt{\mu \sin \pi b^2}. \]

The branch point singularity is resolved by the uniformization map \( \mu^B \rightarrow s \) defined as

\[ \mu^B = \sqrt{\mu \sin \pi b^2} \cosh(\pi bs). \]  

(A.4)

The boundary 2-point function

\[ \langle B_{\beta_1}^\mu_{\mu_1} B_{\beta_2}^\mu_{\mu_2} (x) B_{\beta_1}^\mu_{\mu_1} (0) \rangle = d(\beta|\mu_1^B, \mu_2^B) |x|^{-2\Delta_\beta} \]  

(A.5)
is given by eq. (3.18) in [10]. It is related to the correlation function of two boundary KPZ states in 2D quantum gravity as

\[
d(\beta|\mu_1^B, \mu_2^B) = \begin{cases} 
D_q(\tau_1, \tau_2) & \text{if } \beta < \frac{Q}{2} \\
D_q(\tau_1, \tau_2) & \text{if } \beta > \frac{Q}{2}.
\end{cases}
\] (A.6)

In the notations used in this paper

\[
\varphi = b\phi, \quad g = \frac{1}{b^2}, \quad \tau = \pi bs, \\
p = \frac{Q - 2\alpha}{b}, \quad q = \frac{Q - 2\beta}{2b} \\
M = \pi \sqrt{\mu \sin \frac{\pi b^2}{2}} \Gamma(1 - \frac{b^2}{2})
\] (A.7)

the coordinate-independent part of the boundary two-point function reads

\[
d(\beta|(s_1, s_2)) = d_g(q) \hat{G}_q(\tau_1 + \tau_2) \hat{G}_q(\tau_1 - \tau_2),
\] (A.8)

where

\[
\log \hat{G}_q(\tau) = -\frac{1}{2} \int_{-\infty}^{\infty} dy \left[ \frac{\sinh(2\pi q y/g)}{\sinh(\pi y) \sinh(\pi y/g)} e^{iy\tau} - \frac{2q}{\pi y} \right],
\] (A.9)

\[
d_g(q) = \left( g^{\frac{q-1}{g}} M \right)^q \frac{G\left(\frac{2q}{\sqrt{g}}\right)}{G\left(\frac{-2q}{\sqrt{g}}\right)},
\] (A.10)

and the function \(G\) is defined by (\(b = 1/\sqrt{g}\))

\[
G(x + b) = \frac{b^{\frac{1}{\sqrt{g}} - bx}}{\sqrt{2\pi}} \Gamma(bx)G(x), \quad G(x + 1/b) = \frac{b^{-\frac{1}{\sqrt{g}} + x/b}}{\sqrt{2\pi}} \Gamma(x/b)G(x).
\] (A.11)

The function \(G_q(\tau)\) satisfies the unitarity condition

\[
\hat{G}_q(\tau) \hat{G}_{-q}(\tau) = 1.
\] (A.12)

and the recurrence equations

\[
\hat{G}_q(\tau) \hat{G}_{-q}(\tau) = 1
\] (A.13)

\[
\hat{G}_{q+\frac{1}{2}g}(\tau) = \cosh \left( \frac{g\tau}{2} \pm i\pi q \right) \hat{G}_q(\tau \mp i\pi)
\] (A.14)

\[
\hat{G}_{q+\frac{1}{2}g}(\tau) = \cosh \left( \frac{\tau}{2} \pm i\pi \frac{q}{g} \right) \hat{G}_q(\tau \mp \frac{1}{g})
\] (A.15)

which determine it up to a multiplicative constant. For boundary operators corresponding to degenerate KPZ states \(q = \frac{1}{2}(rg - s)\) the function \(\hat{G}_q\) takes a simple form

\[
\hat{G}_{r,r-s}(\tau) = \prod_{k=0}^{r-1} \cosh \frac{g\tau + i\pi [(r-2k-1)g-s]}{2} \prod_{j=0}^{s-1} \cosh \frac{\tau + i\pi [(s-2j-1)g-r]}{2}.
\] (A.16)
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