Moving intervals for packing and covering

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Abstract

We study several problems on geometric packing and covering with movement. Given a family $\mathcal{I}$ of $n$ intervals of $\kappa$ distinct lengths, and another interval $B$, can we pack the intervals in $\mathcal{I}$ inside $B$ (respectively, cover $B$ by the intervals in $\mathcal{I}$) by moving $\tau$ intervals and keeping the other $\sigma = n - \tau$ intervals unmoved? We show that both packing and covering are W[1]-hard with any one of $\kappa$, $\tau$, and $\sigma$ as single parameter, but are FPT with combined parameters $\kappa$ and $\tau$. We also obtain improved polynomial-time algorithms for packing and covering, including an $O(n \log^2 n)$ time algorithm for covering, when all intervals in $\mathcal{I}$ have the same length.

1 Introduction

Let $\mathcal{I}$ be a family of $n$ intervals of $\kappa$ distinct lengths and of total length $\ell_{\mathcal{I}}$. Let $B$ be another interval of length $\ell_{B}$, in the same line as the intervals in $\mathcal{I}$. We study the following problems of deciding, for two parameters $\sigma$ and $\tau$ where $\sigma + \tau = n$, whether we can move $\tau$ intervals in $\mathcal{I}$, and keep the other $\sigma$ intervals in $\mathcal{I}$ unmoved, to achieve certain geometric configurations:

- **PACK**: the $n$ intervals in $\mathcal{I}$ are pairwise-disjoint and are contained in $B$,
- **COVER**: the union of the $n$ intervals in $\mathcal{I}$ contains $B$,
- **JOIN**: the $n$ intervals in $\mathcal{I}$ are joined into a contiguous interval of length $\ell_{\mathcal{I}}$,
- **J-PACK**: the $n$ intervals in $\mathcal{I}$ are joined into a contiguous interval of length $\ell_{\mathcal{I}}$ that is contained in $B$,
- **J-COVER**: the $n$ intervals in $\mathcal{I}$ are joined into a contiguous interval of length $\ell_{\mathcal{I}}$ that contains $B$,
- **TILE**: the $n$ intervals in $\mathcal{I}$ are joined into a contiguous interval of length $\ell_{\mathcal{I}}$ that coincides with $B$.

Without loss of generality, we assume that $\ell_{\mathcal{I}} \leq \ell_{B}$ for PACK and J-PACK, $\ell_{\mathcal{I}} \geq \ell_{B}$ for COVER and J-COVER, and $\ell_{\mathcal{I}} = \ell_{B}$ for TILE. The problem TILE is a special case of the four problems PACK, COVER, J-PACK, and J-COVER, which are equivalent when $\ell_{\mathcal{I}} = \ell_{B}$. The problem JOIN is similar to the problem TILE, but does not have the constraint of the interval $B$. Indeed JOIN can be viewed as a special case of J-PACK where $B$ contains all intervals in $\mathcal{I}$ and has extra spaces of $\ell_{\mathcal{I}}$ at both ends.

We represent all intervals in the form $[x, y)$, which is a set closed at left endpoint $x$ and open at right endpoint $y$. To avoid unnecessary technicality, we allow zero-distance moves. That is, an interval may be “moved” without changing its position. This ensures natural monotonicities of $\tau$ and $\sigma$ for these problems. For example, denote by PACK$(\tau, \sigma)$ the predicate whether the $n$ intervals in $\mathcal{I}$ can be packed inside the interval $B$ by moving $\tau$ intervals and keeping $\sigma = n - \tau$ intervals unmoved. If PACK$(\tau, \sigma)$ is true and $\sigma > 0$, then PACK$(\tau + 1, \sigma - 1)$ is also true. Let PACK*, COVER*, JOIN*, J-PACK*, J-COVER*, and TILE*
be the optimization versions of these decision problems, for minimizing $\tau$ and maximizing $\sigma$, and let $\tau^*$ and $\sigma^*$ denote the optimal values, where $\tau^* + \sigma^* = n$.

In slightly different formulations, our problems have been studied previously by Mehrandish, Narayanan, and Opatrny [5], for the application of intrusion detection and barrier coverage by sensors. When the number $\kappa$ of distinct lengths of intervals in $I$ is unrestricted, Mehrandish et al. proved that these problems are all weakly NP-hard by reductions from PARTITION [3, Problem SP12]. In particular, JOIN is hard already for $\sigma = 2$ [5, Theorem 2], and TILE, hence PACK, COVER, J-PACK, and J-COVER, are hard already for $\sigma = 1$ [5, Theorem 3]. The complexities of these problems for fixed $\kappa$ or $\tau$, however, have not been examined.

Using standard techniques of dynamic programming, we show that these problems admit polynomial-time algorithms when either $\kappa$ or $\tau$ is constant:

**Proposition 1.** PACK, JOIN, J-PACK, J-COVER, and TILE admit algorithms running in $O(n \log n + n \cdot f(\kappa, \tau))$ time, and COVER admits an algorithm running in $O(n^2 \cdot f(\kappa, \tau))$ time, for some function $f$ that is bounded by a polynomial in $n$ when either $\kappa$ or $\tau$ is constant.

In terms of parameterized complexity, Proposition 1 implies that these problems are fixed-parameter tractable (FPT) with combined parameters $\kappa$ and $\tau$. Extending the previous hardness results, we prove the following theorem:

**Theorem 1.** PACK, COVER, JOIN, J-PACK, J-COVER, and TILE are NP-hard, and are W[1]-hard with any one of $\kappa$, $\tau$, and $\sigma$ as single parameter, even when all interval coordinates are integers encoded in unary.

When all intervals in $I$ have the same length, that is, when $\kappa = 1$, Mehrandish et al. [5, Theorems 4, 7, 10, 11] presented an $O(n^2)$ time algorithm for $\text{JOIN}^*$ and $\text{J-PACK}^*$, and an $O(n^3)$ time algorithm for $\text{PACK}^*$ and $\text{COVER}^*$. We obtain improved algorithms for these problems:

**Proposition 2.** When $\kappa = 1$, $\text{JOIN}^*$, $\text{J-PACK}^*$, $\text{J-COVER}^*$, and $\text{TILE}^*$ admit algorithms running in $O(n \log n)$ time.

**Theorem 2.** When $\kappa = 1$, $\text{PACK}^*$ and $\text{COVER}^*$ admit algorithms running in $O(n \log n + n \min\{\sigma^*, \tau^*\})$ time, and $\text{COVER}^*$ admits an algorithm running in $O(n \log^2 n)$ time.

We leave two open questions:

- Is there an exact algorithm running in $O(n \log^2 n)$ time for $\text{PACK}^*$ on intervals of the same length?
- Are there FPT algorithms with combined parameters $\kappa$ and $\sigma$ for PACK and COVER?

## 2 Algorithms on intervals of the same length for J-PACK* and J-COVER*

In this section we prove Proposition 2 on the four problems $\text{JOIN}^*$, $\text{J-PACK}^*$, $\text{J-COVER}^*$, and $\text{TILE}^*$. Recall that $\text{JOIN}^*$ is a special case of $\text{J-PACK}^*$, and $\text{TILE}^*$ is a special case of $\text{J-PACK}^*$ and $\text{J-COVER}^*$. Thus it suffices to present algorithms for $\text{J-PACK}^*$ and $\text{J-COVER}^*$.

For any real number $x$, denote by $\{x\} = x - \lfloor x \rfloor$ the fractional part of $x$. For any interval $[x, x + \ell)$ of integer length $\ell$, define its fractional coordinate as $\{x\}$.

Let $B = [0, \ell_B)$, and $I = \{I_1, \ldots, I_n\}$, where $I_i = [x_i, x_i + 1)$. Then $\ell_I = n$. For $\bar{x} \in [0, 1)$, let $I(\bar{x})$ be the subfamily of intervals in $I$ with fractional coordinate $\bar{x}$. As in [5, Theorem 4 and Theorem 7], the problem $\text{J-PACK}^*$ (respectively, $\text{J-COVER}^*$) reduces to computing, for every distinct fractional coordinate $\bar{x} \in [0, 1)$ of the intervals in $I$, the maximum number of pairwise-disjoint intervals in $I(\bar{x})$ that are inside
some interval $B_{\tilde{x}}$ of fractional coordinate $\tilde{x}$ and of length $\ell_{\tilde{x}} = n$, such that $B_{\tilde{x}} \subseteq B$ (respectively, $B_{\tilde{x}} \supseteq B$). Then the intervals in $I$ can be joined into a contiguous interval that coincides with $B_{\tilde{x}}$.

The algorithm for J-PACK* (respectively, J-COVER*) works as follows. First sort the intervals in $I$ lexicographically, by considering each interval $I_i$ as a pair of numbers $(\{x_i\}, \ell_i)$. Then the intervals in each subfamily $I(\tilde{x})$ appear consecutively in the sorted list. Next scan each subfamily $I(\tilde{x})$ independently.

For each $\tilde{x}$, regardless of the intervals in $I(\tilde{x})$, there exists an interval $B_{\tilde{x}} \subseteq B$ (respectively, $B_{\tilde{x}} \supseteq B$) of fractional coordinate $\tilde{x}$ and length $n$ if and only if either $\tilde{x} = 0$ or $[\tilde{x}, \tilde{x} + n] \subseteq [0, \ell_B)$ (respectively, $[\tilde{x} - 1, \tilde{x} - 1 + n] \supseteq [0, \ell_B)$), which can be checked in constant time. This condition is satisfied, then proceed to find the maximum number of pairwise-disjoint intervals in $I(\tilde{x})$ that are contained in $[0, \ell_B)$ (respectively, in $[\ell_B - n, n]$) such that the span $x_j - x_i + 1$ of the first interval $I_i$ and the last interval $I_j$ is at most $n$. This can be done in linear time using the standard “two pointers” technique. These intervals in $I(\tilde{x})$, with span at most $n$, are inside some interval $B_{\tilde{x}}$ of length exactly $n$ such that $B_{\tilde{x}} \subseteq [0, \ell_B)$ (respectively, $B_{\tilde{x}} \supseteq [\ell_B - n, n]$) and hence $B_{\tilde{x}} \supseteq [0, \ell_B)$.

The overall running time of the algorithm is $O(n \log n)$, which is dominated by the sorting. This completes the proof of Proposition 2.

**Remark.** We have used fractional numbers for the interval coordinates so that the intervals in $I$ have a convenient unit length of 1. Alternatively, we could let $B$ and the intervals in $I$ have integer coordinates, and let the uniform length $\ell$ of the intervals in $I$ be an integer greater than 1. In this formulation, we would define $\{x\} = x \mod \ell$. Then the same idea yields an algorithm of the same $O(n \log n)$ running time.

## 3 Algorithms on intervals of the same length for PACK* and COVER*

In this section we prove Theorem 2 on the two problems PACK* and COVER*.

Let $B = [0, \ell_B)$, and $I = \{I_1, \ldots, I_n\}$, where $I_i = [x_i, x_i + 1)$. Then $\ell_I = n$. Add two dummy intervals $I_0 = [x_0, x_0 + 1) = [-1, 0)$ and $I_{n+1} = [x_{n+1}, x_{n+1} + 1) = [\ell_B, \ell_B + 1)$). Any interval in $I$ that does not intersect $B$ can be relocated to either $(-1, 0)$ or $[\ell_B, \ell_B + 1)$ without affecting the answer to either problem. After relocating outside intervals and sorting, we can assume that

$$-1 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq x_{n+1} = \ell_B.$$

When comparing pairs, we use notations such as $<, \leq, >, \geq, \min, \max$, and use terms such as largest and smallest, in terms of lexicographic order. For example, $(1, 2) < (2, 1)$, and $\min\{(1, 2), (2, 1)\} = (1, 2)$.

### 3.1 Algorithms for PACK*

For $0 \leq i < j \leq n + 1$, write $i \prec j$ if $x_j - x_i \geq 1$. For $0 \leq j \leq n + 1$, let $l_j$ be the largest index $i$ such that $0 \leq i \prec j$, or $-1$ if no such $i$ exists. With increasing $x_i$ for $0 \leq i \leq n + 1$, $l_i$ for all $i$ can be computed in $O(n)$ time using the standard “two pointers” technique.

Recall that $\{x\} = x - \lfloor x \rfloor$. For $0 \leq i < j \leq n + 1$, define

$$w(i, j) = \begin{cases} 1 & \text{if } \{x_i\} > \{x_j\}, \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to check that the $n$ intervals in $I$ can be packed inside $B$ by moving $n - \sigma$ intervals and keeping $\sigma$ intervals unmoved if and only if there is a sequence $s$ of $\sigma + 2$ indices $s_h$, $0 \leq h \leq \sigma + 1$, where

$$0 = s_0 < s_1 < \ldots < s_{\sigma + 1} = n + 1,$$
such that the following condition holds:

$$\sum_{h=0}^{\sigma} \left( [x_{s_{h+1}}] - [x_{s_h}] - w(s_h, s_{h+1}) \right) \geq n + 1,$$

which simplifies to

$$[x_{n+1}] - [x_0] - \sum_{h=0}^{\sigma} w(s_h, s_{h+1}) \geq n + 1$$

$$[\ell_B] + 1 - \sum_{h=0}^{\sigma} w(s_h, s_{h+1}) \geq n + 1$$

$$\sum_{h=0}^{\sigma} w(s_h, s_{h+1}) \leq [\ell_B] - n.$$

Let $\pi$ be a permutation of the $n + 2$ indices $0, 1, \ldots, n, n + 1$ such that for $0 \leq i < j \leq n + 1$, $\pi_i > \pi_j$ if and only if $\{x_i\} > \{x_j\}$, which can be found by sorting in $O(n \log n)$ time. For $0 \leq i < j \leq n + 1$, denote by $\text{inv}(i, j)$ the indicator variable which is 1 when $\pi_i > \pi_j$ and is 0 otherwise. Then $w(i, j) = \text{inv}(i, j)$.

From this new perspective, the problem of determining the maximum value $\sigma^*$, such that the $n$ intervals in $I$ can be packed inside $B$ by moving $n - \sigma^*$ intervals and keeping $\sigma^*$ intervals unmoved, reduces to the problem of finding the maximum length $\sigma^* + 2$ of a sequence $s$ of indices $0 = s_0 < s_1 < \ldots < s_{\sigma^*+1} = n + 1$, such that the number of inversions $\pi_i > \pi_j$ between consecutive indices $i < j$ in $s$ (which we will call drops) is at most $d^* = [\ell_B] - n$.

**Dynamic programming** For $0 \leq h \leq i \leq n$, define $dp(h, i)$ as the pair $(d(h, i), p(h, i))$, where $d(h, i)$ is the minimum number of drops in any sequence $s$ of $h + 1$ indices $0 = s_0 < \ldots < s_h \leq i$, and $p(h, i)$ is the minimum value of $p = \pi_{s_h}$ over all such sequences $s$ with exactly $d = d(h, i)$ drops. If there are no such sequences $s$, define $dp(h, i) = (\infty, 0)$. For convenience, also define $dp(h, i) = (\infty, 0)$ for $h > i$.

The table $dp$ can be computed by dynamic programming. For the base case when $h = 0$ or $h = i$, let $dp(0, i) = (0, \pi_0)$ for $0 \leq i \leq n$, and, for $1 \leq i \leq n$, let

$$dp(i, i) = \begin{cases} dp(i-1, i-1) \rightarrow i & \text{if } i-1 < i, \\ (\infty, 0) & \text{otherwise.} \end{cases}$$

where

$$(d, p) \rightarrow i = \begin{cases} (\infty, 0) & \text{if } (d, p) = (\infty, 0), \\ (d, \pi_i) & \text{else if } p < \pi_i, \\ (d+1, \pi_i) & \text{otherwise.} \end{cases}$$

Then for $0 < h < i \leq n$, in particular, for $h = 1, \ldots, n - 1$ and $i = h + 1, \ldots, n$, we can compute $dp(h, i)$ using the recurrence

$$dp(h, i) = \min\{ dp(h, i-1), dp(h-1, i) \rightarrow i \}.$$

Note that $\sigma^*$ is the maximum $h$ such that the $d$-part of $dp(h, l_{n+1})$ $\rightarrow$ $n$ + 1 is at most $d^*$, which can be found in $O(n\sigma^*)$ time by a sequential search for increasing values of $h$.

We can also find $\tau^* = n - \sigma^*$ in $O(n\tau^*)$ time by another sequential search, for increasing values of $t = i - h$ instead of $h$. First compute $dp(h, i)$ for the base case when $h = 0$ or $h = i$ as before, then compute $dp(h, h+t)$ for $t = 1, \ldots, n - 1$ and $h = 1, \ldots, n - t$. Note that $\tau^*$ is the minimum $t$ such that the $d$-part of $dp(n-t, l_{n+1})$ $\rightarrow$ $n + 1$ is at most $d^*$.

Thus we have an algorithm for PACK$^*$ running in $O(n \log n + n \min\{\sigma^*, \tau^*\})$ time.
3.2 Algorithms for COVER

Recall that \( \{ x \} = x - \lfloor x \rfloor \). For \( 0 \leq i < j \leq n + 1 \), define

\[
 w(i, j) = \begin{cases} 
 1 & \text{if } \{ x_i \} < \{ x_j \} \text{ or } x_i = x_j, \\
 0 & \text{otherwise}. 
\end{cases}
\]

It is easy to check that \( B \) can be covered by the \( n \) intervals in \( I \) by moving \( n - \sigma \) intervals and keeping \( \sigma \) intervals unmoved if and only if there is a sequence \( s \) of \( \sigma + 2 \) indices \( s_h, 0 \leq h \leq \sigma + 1 \), where

\[
 0 = s_0 < s_1 < \ldots < s_{\sigma + 1} = n + 1,
\]

such that the following condition holds:

\[
 \sum_{h=0}^{\sigma} \left( \lfloor x_{s_{h+1}} \rfloor - \lfloor x_{s_h} \rfloor + w(s_h, s_{h+1}) \right) \leq n + 1,
\]

which simplifies to

\[
 \lfloor x_{n+1} \rfloor - \lfloor x_0 \rfloor + \sum_{h=0}^{\sigma} w(s_h, s_{h+1}) \leq n + 1
\]

\[
 \lfloor \ell_B \rfloor + 1 + \sum_{h=0}^{\sigma} w(s_h, s_{h+1}) \leq n + 1
\]

\[
 \sum_{h=0}^{\sigma} w(s_h, s_{h+1}) \leq n - \lfloor \ell_B \rfloor.
\]

We prove a technical lemma in the following:

**Lemma 1.** For \( 0 \leq i < j \leq n + 1 \), \((\{ x_i \}, -x_i, i) < (\{ x_j \}, -x_j, j)\) if and only if \( \{ x_i \} < \{ x_j \} \) or \( x_i = x_j \).

**Proof.** We first prove the only if implication. Suppose that \((\{ x_i \}, -x_i, i) < (\{ x_j \}, -x_j, j)\). Then by the lexicographic order, there are three cases: either \( \{ x_i \} < \{ x_j \} \), or \( \{ x_i \} = \{ x_j \} \) and \( x_i > x_j \), or \( \{ x_i \} = \{ x_j \} \) and \( x_i = x_j \) and \( i < j \). Recall that \( x_i \leq x_j \) for \( i < j \). Thus the second case does not hold. The third case simplifies to \( x_i = x_j \). In summary, we have either \( \{ x_i \} < \{ x_j \} \) or \( x_i = x_j \).

We next prove the if implication. Suppose that \( \{ x_i \} < \{ x_j \} \) or \( x_i = x_j \). If \( \{ x_i \} < \{ x_j \} \), then clearly \( (\{ x_i \}, -x_i, i) < (\{ x_j \}, -x_j, j) \). If \( x_i = x_j \), then \( \{ x_i \} = \{ x_j \} \) and \( -x_i = -x_j \), but since \( i < j \), we again have \( (\{ x_i \}, -x_i, i) < (\{ x_j \}, -x_j, j) \). \( \square \)

Let \( \pi \) be a permutation of the \( n + 2 \) indices \( 0, 1, \ldots, n, n + 1 \) such that for \( 0 \leq i < j \leq n + 1 \), \( \pi_i > \pi_j \) if and only if \( (\{ x_i \}, -x_i, i) < (\{ x_j \}, -x_j, j) \), which can be found by sorting in \( O(n \log n) \) time. For \( 0 \leq i < j \leq n + 1 \), denote by \( \text{inv}(i, j) \) the indicator variable which is \( 1 \) when \( \pi_i > \pi_j \) and \( 0 \) otherwise. Then, by Lemma 1, \( w(i, j) = \text{inv}(i, j) \).

From this new perspective, the problem of determining the maximum value \( \sigma^* \), such that \( B \) can be covered by the \( n \) intervals in \( I \) by moving \( n - \sigma^* \) intervals and keeping \( \sigma^* \) intervals unmoved, reduces to the problem of finding the maximum length \( \sigma^* + 2 \) of a sequence \( s \) of indices increasing from \( 0 \) to \( n + 1 \), such that the number of inversions \( \pi_i > \pi_j \) between consecutive indices \( i < j \) in \( s \) (which we will call drops) is at most \( d^* = n - \lfloor \ell_B \rfloor \).
**Dynamic programming** For $0 \leq h \leq i \leq n$, define $dp(h, i)$ as the pair $(d(h, i), p(h, i))$, where $d(h, i)$ is the minimum number of drops in any sequence $s$ of $h + 1$ indices $0 = s_0 < \ldots < s_h \leq i$, and $p(h, i)$ is the minimum value of $p = \pi_{s_h}$ over all such sequences $s$ with exactly $d = d(h, i)$ drops.

The table $dp$ can be computed by dynamic programming. For the base case when $h = 0$ or $h = i$, let $dp(0, i) = (0, \pi_0)$ for $0 \leq i \leq n$, and let $dp(i, i) = dp(i - 1, i - 1) \rightarrow i$ for $1 \leq i \leq n$, where

$$(d, p) \rightarrow i = \begin{cases} (d, \pi_i) & \text{if } p < \pi_i, \\ (d + 1, \pi_i) & \text{otherwise.} \end{cases}$$

Then for $0 < h < i \leq n$, in particular, for $h = 1, \ldots, n - 1$ and $i = h + 1, \ldots, n$, we can compute $dp(h, i)$ using the recurrence

$$dp(h, i) = \min \{ dp(h, i - 1), dp(h - 1, i - 1) \rightarrow i \}.$$

Note that $\sigma^*$ is the maximum $h$ such that the $d$-part of $dp(h, n) \rightarrow n + 1$ is at most $d^*$, which can be found in $O(n\sigma^*)$ time by a sequential search for increasing values of $h$.

We can also find $\tau^* = n - \sigma^*$ in $O(n\tau^*)$ time by another sequential search, for increasing values of $t = i - h$ instead of $h$. First compute $dp(h, i)$ for the base case when $h = 0$ or $h = i$ as before, then compute $dp(h, h + t)$ for $t = 1, \ldots, n - 1$ and $h = 1, \ldots, n - t$. Note that $\tau^*$ is the minimum $t$ such that the $d$-part of $dp(n - t, n) \rightarrow n + 1$ is at most $d^*$.

Thus we have an algorithm for COVER* running in $O(n \log n + n \min\{\sigma^*, \tau^*\})$ time.

**Lagrangian relaxation** We next present an algorithm for COVER* running in $O(n \log^2 n)$ time. To do this, we first redesign the dynamic programming algorithm, then speed it up using Lagrangian relaxation. The technique of Lagrangian relaxation is also known as **Aliens trick** in the competitive programming community. Our use of this trick here is inspired by a related problem created by Compton and Qi [1].

For $0 \leq d \leq i \leq n + 1$, let $\sigma(d, i)$ be the maximum $h$ such that there exists a sequence $s$ of $h + 2$ indices increasing from 0 to $i$, $0 = s_0 < \ldots < s_{h+1} = i$, with at most $d$ drops. Let $\sigma(d, i) = -\infty$ if such a sequence does not exist. For convenience, also define $\sigma(d, i) = -\infty$ for $d > i$. Our goal is to get $\sigma^* = \sigma(d^*, n + 1)$, where $d^* = n - \lfloor E_B \rfloor \leq n$.

We can compute $\sigma^*$ by dynamic programming. For $d = 0$, $\sigma(0, i)$ is 2 less than the length of a longest increasing subsequence of $\pi_0 \ldots \pi_i$ starting at $\pi_0$ and ending at $\pi_i$, or $-\infty$ if it does not exist. In particular, $\sigma(0, 0) = -1$. With the help of some data structures such as balanced search trees, $\{\sigma(0, i) | 0 \leq i \leq n + 1\}$ can be computed in $O(n \log n)$ time. If $d^* = 0$, then we already have $\sigma^* = \sigma(0, n + 1)$. In the following, we assume that $d^* \geq 1$.

For $d = 1, \ldots, d^*$, we can compute $\{\sigma(d, i) | d \leq i \leq n + 1\}$ from $\{\sigma(d - 1, i) | d - 1 \leq i \leq n + 1\}$.

For the base case, $\sigma(d, 0) = \sigma(d - 1, 0)$. For $1 \leq j \leq n + 1$,

$$\sigma(d, j) = \max\{ \sigma(d - 1, j), \max\{ \sigma(d - \text{inv}(i, j), i) + 1 | 0 \leq i < j \} \}.$$ 

With balanced search trees, this takes $O(n \log n)$ time for each $d$. Thus we can get $\sigma^* = \sigma(d^*, n + 1)$ in $O(d^* n \log n)$ time.

We next speed up the algorithm using Aliens trick, which depends on a property of $\sigma$ stated in the following lemma. We say that a function $f : \mathbb{Z} \to \mathbb{Z}$ is concave for $a \leq x \leq b$ if $f(x + 1) - f(x) \leq f(x) - f(x - 1)$ for $a < x < b$.

**Lemma 2.** $\sigma(d, n + 1)$ is concave for $1 \leq d \leq n + 1$.

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1The problem can be translated to the following: Given a permutation $\pi$ of $[1, n]$, compute for each $0 \leq d < n$ the maximum length of a sequence $s$ of indices such that the number of inversions $\pi_i > \pi_j$ between consecutive indices $i < j$ in $s$ is at most $d$. One of the solutions by the problem setters uses Aliens trick and runs in $O(n \sqrt{n} \log n)$ time.
The trick is to assign a penalty $\lambda$ for each drop. Instead of computing $\sigma(d, i)$ sequentially for increasing $d$ from 0 to $d^*$ as in the dynamic programming algorithm, we will compute $\sigma(d, i) - d\lambda$, and get $\sigma^* = \sigma(d^*, n + 1)$ indirectly as $\sigma(d^*, n + 1) - d^*\lambda + d^*\lambda$, by a binary search for a suitable $\lambda$.

For any $\lambda \geq 0$, and for $1 \leq i \leq n + 1$, let 

$$\sigma_\lambda(i) = \max\{\sigma(d, i) - d\lambda \mid 1 \leq d \leq i\},$$

and correspondingly, let $d_\lambda(i)$ be the minimum $d$, $1 \leq d \leq i$, such that $\sigma(d, i) - d\lambda = \sigma_\lambda(i)$.

Consider $\sigma(d, n + 1) - d\lambda$ as a function of $d$ and $\lambda$. Geometrically, it can be viewed as the dot product of two vectors $(d, \sigma(d, n + 1))$ and $(-\lambda, 1)$. As $\lambda$ decreases from $\infty$ to 0, the vector $(-\lambda, 1)$ rotates from $-\vec{x}$ direction to $+\vec{y}$ direction. For any fixed $\lambda$, the dot products for different $d$ are proportional to the projection lengths of different vectors $(d, \sigma(d, n + 1))$ onto the same vector $(-\lambda, 1)$.

For each $d$, $1 \leq d \leq n$, equating the two dot products for $d$ and $d + 1$ yields an equation of $\lambda$, 

$$\sigma(d, n + 1) - d\lambda = \sigma(d + 1, n + 1) - (d + 1)\lambda,$$

which has an integer solution,

$$\lambda_d = \sigma(d + 1, n + 1) - \sigma(d, n + 1).$$

By Lemma 2, $\lambda_d$ is decreasing for $1 \leq d \leq n$. Since any sequence with at most $d$ drops is also a sequence with at most $d + 1$ drops, we have $\sigma(d, n + 1) \leq \sigma(d + 1, n + 1)$ and hence $\lambda_d \geq 0$. Also, since $\sigma(d, n + 1) + 2 \geq 2$ for $d \geq 1$, and $\sigma(d + 1, n + 1) + 2 \leq n + 2$, we have $\lambda_d \leq n$. Thus

$$0 \leq \lambda_n \leq \ldots \leq \lambda_1 \leq n.$$

Thus for each $d$, $1 \leq d \leq n + 1$, there is a non-empty range of integer values for $\lambda$, between 0 and $n$, such that 

$$\sigma_\lambda(n + 1) = \sigma(d, n + 1) - d\lambda.$$

Specifically, these ranges are $\lambda_1 \leq \lambda \leq n$ for $d = 1$, $\lambda_d \leq \lambda \leq \lambda_{d-1}$ for $2 \leq d \leq n$, and $0 \leq \lambda \leq \lambda_n$ for $d = n + 1$.

Correspondingly, $d_\lambda(n + 1)$ is decreasing for $0 \leq \lambda \leq n$: it is equal to $n + 1$ for $0 \leq \lambda < \lambda_n$, to $d = n, \ldots, 2$ for $\lambda_d \leq \lambda < \lambda_{d-1}$, and to 1 for $\lambda_1 \leq \lambda \leq n$. Some of these ranges may be empty, so $d_\lambda(n + 1)$ may not assume every integer value from $n + 1$ down to 1, as $\lambda$ increases from 0 to $n$. But we always have $d_\lambda(n + 1) = 1$ for $\lambda = n$. Recall our assumption that $d^* \geq 1$. Thus $d_\lambda(n + 1) \leq d^*$ for $\lambda = n$.

Let $\lambda^*$ be the smallest integer $\lambda$, $0 \leq \lambda \leq n$, such that $d_\lambda(n + 1) \leq d^*$. Then $\lambda^* = \lambda_{d^*}$, and hence $\sigma_{\lambda^*}(n + 1) = \sigma(d^*, n + 1) - d^*\lambda^*$. Then $\sigma^* = \sigma(d^*, n + 1) = \sigma_{\lambda^*}(n + 1) + d^*\lambda^*$.

It remains to find $\lambda^*$. Recall that $\{\sigma(0, i) \mid 0 \leq i \leq n + 1\}$ can be computed in $O(n \log n)$ time, and subsequently $\{\sigma(1, i) \mid 1 \leq i \leq n + 1\}$ can be computed from $\{\sigma(0, i) \mid 0 \leq i \leq n + 1\}$ in $O(n \log n)$ time. After these two preliminary steps, we can find $\lambda^*$ by a binary search, which amounts to first computing $\sigma_\lambda(n + 1)$ and $d_\lambda(n + 1)$, and then checking whether $d_\lambda(n + 1) \leq d^*$, for $O(\log n)$ different $\lambda$ between 0 and $n$.

For each such $\lambda$, we can compute the $n + 1$ pairs $(\sigma_\lambda(i), d_\lambda(i))$, $1 \leq i \leq n + 1$, sequentially, by dynamic programming. For the base case, let $(\sigma_\lambda(1), -d_\lambda(1)) = (\sigma(1, 1) - \lambda, -1)$. Then, for $j = 2, \ldots, n + 1$, use the recurrence

$$(\sigma_\lambda(j), -d_\lambda(j)) = \max\{ (\sigma(1, j) - \lambda, -1), \max\{ (\sigma_\lambda(i) + 1 - \text{inv}(i, j)\lambda, -d_\lambda(i) - \text{inv}(i, j)) \mid 1 \leq i < j \} \}.$$

Again, with balanced search trees, this can be done in $O(n \log^* n)$ time for each $\lambda$. Thus the overall running time is $O(n \log^2 n)$. 7
3.2.1 Proof of Lemma 2

Recall $dp(h, i)$ and $(d, p) \rightarrow i$ defined earlier. For $0 \leq i \leq n$, let

$$DP_i = \{ dp(h, i) \mid 0 \leq h \leq i \}.$$ 

We first prove some easy lemmas. The following lemma is about the lexicographic order of pairs:

**Lemma 3.** For all integers $a$, $b$, $c$, and $d$, $(a, b) < (c, d) \iff (a - 1, b) < (c - 1, d)$.

**Proof.** In lexicographic order, $(a, b) < (c, d)$ is equivalent to $a < c$ or $a = c$ and $b < d$, which is equivalent to $a - 1 < c - 1$ or $a - 1 = c - 1$ and $b < d$, and hence $(a - 1, b) < (c - 1, d)$. □

The next two lemmas are on basic properties of $(d, p) \rightarrow i$:

**Lemma 4.** For $1 \leq i \leq n$, the $p$-part of $(d, p) \rightarrow i$ is $\pi_i$, and $(d, p) \rightarrow i$ is the smallest pair greater than $(d, p)$ with $p$-part equal to $\pi_i$.

**Proof.** Recall that $(d, p) \rightarrow i$ yields either $(d, \pi_i)$, if $p < \pi_i$, or $(d + 1, \pi_i)$, otherwise. The greedy choice of $d$ or $d + 1$ ensures that it is the smallest among all pairs $(a, \pi_i)$ that are greater than $(d, p)$. □

**Lemma 5.** For $1 \leq i \leq n$, if $(d, p) < (d', p') \rightarrow i$, then $(d - 1, p) < (d' - 1, p') \rightarrow i$.

**Proof.** If $p' < \pi_i$, then $(d', p') \rightarrow i = (d', \pi_i)$, and $(d' - 1, p') \rightarrow i = (d' - 1, \pi_i)$. If $p' \geq \pi_i$, then $(d', p') \rightarrow i = (d' + 1, \pi_i)$, and $(d' - 1, p') \rightarrow i = (d', \pi_i)$. Write $(d', p') \rightarrow i$ as $(a, \pi_i)$. Then $(d' - 1, p') \rightarrow i = (a - 1, \pi_i)$. By Lemma 3, if $(d, p) < (a, \pi_i)$, then $(d - 1, p) < (a - 1, \pi_i)$. □

The next few lemmas are on basic properties of $dp(h, i)$ and $DP_i$:

**Lemma 6.** For $0 \leq i \leq n$, $dp(h, i)$ is strictly increasing for $0 \leq h \leq i$.

**Proof.** It suffices to show that $dp(h, i) < dp(h + 1, i)$ for $0 \leq h < i$. For any sequence $s$ of $h + 2$ indices, $0 = s_0 < \ldots < s_h < s_{h+1} \leq i$, with exactly $d(h + 1, i)$ drops and with $\pi_{s_{h+1}} = p(h + 1, i)$, let $s'$ be the subsequence of the first $h + 1$ indices in $s$, $0 = s_0 < \ldots < s_h \leq i$. Then either $\pi_{s_h} < \pi_{s_{h+1}}$, and $s$ has the same number of drops as $s'$, or $\pi_{s_h} > \pi_{s_{h+1}}$, and $s$ has one more drop than $s'$. □

**Lemma 7.** For $0 \leq i \leq n$, $dp(0, i) = (0, \pi_0)$ is the smallest pair in $DP_i$, and is the only pair in $DP_i$ with $p$-part equal to $\pi_0$.

**Proof.** By Lemma 6, $dp(0, i)$ is the smallest pair in $DP_i$. In the definition of $dp(h, i)$, the sequence $s$ of $h + 1$ distinct indices $s_0, \ldots, s_h$ always starts with $s_0 = 0$, so the last index $s_h$ can be $0$ only if $h = 0$. This implies that the $p$ part of $dp(h, i)$ is equal to $\pi_0$ for $h = 0$, and is not equal to $\pi_0$ for $0 < h \leq i$. Thus $dp(0, i) = (0, \pi_0)$ is the only pair in $DP_i$ with $p$-part equal to $\pi_0$. □

**Lemma 8.** For $0 \leq i \leq n$, $dp(i, i)$ is the largest pair in $DP_i$, and its $p$-part is equal to $\pi_i$. For $1 \leq i \leq n$, $dp(i, i)$ is the smallest pair greater than $dp(i - 1, i - 1)$ with $p$-part equal to $\pi_i$.

**Proof.** By Lemma 6, $dp(i, i)$ is the largest pair in $DP_i$, for $0 \leq i \leq n$. Recall that $dp(0, 0) = (0, \pi_0)$, and $dp(i, i) = dp(i - 1, i - 1) \rightarrow i$ for $1 \leq i \leq n$. By Lemma 4, the $p$-part of $dp(i, i)$ is $\pi_i$, and $dp(i, i)$ is the smallest among all pairs $(a, \pi_i)$ that are greater than $dp(i - 1, i - 1)$, for $1 \leq i \leq n$. □

**Lemma 9.** For $0 < h \leq i \leq n$, if $dp(h, i) > (d, p) \in DP_i$, then $dp(h - 1, i) \geq (d, p)$. 

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Proof. Suppose that \( dp(h, i) > (d, p) \in DP_i \). Since \((d, p) \in DP_i \), we have \((d, p) = dp(h', i)\) for some \( h' \), \( 0 \leq h' \leq i \). Then \( dp(h', i) = (d, p) < dp(h, i) \). By Lemma 6, it follows that \( h' < h \), and hence \( h - 1 \geq h' \).

By Lemma 6 again, we have \( dp(h - 1, i) \geq dp(h', i) = (d, p) \).

We are now ready to prove Lemma 2 that \( \sigma(d, n + 1) \) is concave for \( 1 \leq d \leq n + 1 \), that is, \( \sigma(d + 1, n + 1) - \sigma(d, n + 1) \) is decreasing for \( 1 \leq d \leq n \).

Fix any \( d \), where \( 1 \leq d \leq n + 1 \). From the definitions of \( dp(h, i) \) for \( 0 \leq h \leq i \leq n \) and \( \sigma(d, i) \) for \( 0 \leq d \leq i \leq n + 1 \), we can see that \( \sigma(d, n + 1) \) is the maximum \( h \), \( 0 \leq h \leq n \), such that the \( d \)-part of \( dp(h, n) \rightarrow n + 1 \) is at most \( d \). By Lemma 6, \( \sigma(d, n + 1) + 1 \) is equal to the number of distinct values for \( h \), \( 0 \leq h \leq n \), such that the \( d \)-part of \( dp(h, n) \rightarrow n + 1 \) is at most \( d \). Note that the \( d \)-part of \( dp(h, n) \rightarrow n + 1 \) is at most \( d \) if and only if \( dp(h, n) < (d, \pi_{n+1}) \). Then \( \sigma(d, n + 1) + 1 \) is equal to the number of pairs in \( DP_n \) that are less than \( (d, \pi_{n+1}) \).

Therefore, for \( 1 \leq d \leq n \), \( \sigma(d + 1, n + 1) - \sigma(d, n + 1) \) is equal to the number of pairs \((a, p) \in DP_n \) such that \((d, \pi_{n+1}) \leq (a, p) < (d + 1, \pi_{n+1}) \). Note that no pair \((a, p) \in DP_n \) can have \( p = \pi_{n+1} \). For \( 1 \leq d \leq n + 1 \), define \( k_<(d) \) (respectively, \( k_>(d) \)) as the number of pairs \((d, p) \in DP_n \) with \( p < \pi_{n+1} \) (respectively, \( p > \pi_{n+1} \)). Then \( \sigma(d + 1, n + 1) - \sigma(d, n + 1) = k_<(d) + k_>(d + 1) \) for \( 1 \leq d \leq n \).

To show that \( \sigma(d + 1, n + 1) - \sigma(d, n + 1) \) is decreasing for \( 1 \leq d \leq n \), it suffices to show that both \( k_<(d) \) and \( k_>(d) \) are decreasing for \( 1 \leq d \leq n + 1 \). We need the following lemma:

**Lemma 10.** For \( 0 \leq i \leq n \), if \((d + 1, p) \in DP_i \) and \((d, p) \geq (0, \pi_0) \), then \((d, p) \in DP_i \).

Now fix any \( d \), where \( 1 \leq d \leq n \). Then \((d, p) > (0, \pi_0) \) for all \( p \). By Lemma 10 with \( i = n \), any pair \((d + 1, p) \in DP_n \), either \( p < \pi_{n+1} \) or \( p > \pi_{n+1} \), has a unique corresponding pair \((d, p) \in DP_n \). Thus \( k_<(d + 1) \leq k_<(d) \) and \( k_>(d + 1) \leq k_>(d) \).

To complete the proof of Lemma 2, it remains to prove Lemma 10. Our proof is by induction on \( i \). For the base case when \( i = 0 \), \( dp(h, i) \) is defined only for \( h = 0 \). Since \( dp(0, 0) = (0, \pi_0) \) is included in \( DP_0 \), the lemma clearly holds.

We now proceed to the inductive step, and fix \( i > 0 \). By Lemma 7, \( dp(0, i) = (0, \pi_0) \) is the only pair in \( DP_1 \) with \( p \)-part equal to \( \pi_0 \). Thus the lemma holds for \( p = \pi_0 \). There are two cases remaining: either \( p = \pi_i \) or \( p \neq \pi_i \) and \( p \neq \pi_0 \).

**The first case.** We first consider the case that \( p = \pi_i \). By Lemma 8, the \( p \)-part of \( dp(i, i) \) is \( \pi_i \).

Write \( dp(i, i) \) as \((d, \pi_i) \). By Lemma 6, \( dp(i, i) \) is the largest pair in \( DP_i \). It suffices to prove that \((c, \pi_i) \in DP_i \) for all \( c < d \) such that \((c, \pi_i) \geq (0, \pi_0) \).

Fix any \( c < d \) such that \((c, \pi_i) \geq (0, \pi_0) \). Then \((c, \pi_i) > (0, \pi_0) \) since \( \pi_i \neq \pi_0 \) for \( i > 0 \). By Lemma 8, \( dp(i, i) = (d, \pi_i) \) is the smallest pair greater than \( dp(i - 1, i - 1) \) with \( p \)-part equal to \( \pi_i \). Since \( c < d \), we must have \((c, \pi_i) \leq dp(i - 1, i - 1) \).

Let \( h \geq 0 \) be the smallest integer such that \((c, \pi_i) \leq dp(h, i - 1) \). Then \( h \leq i - 1 \). Since \((c, \pi_i) > (0, \pi_0) \), we also have \( h > 0 \).

Write \( dp(h, i - 1) \) as \((d', p') \). Then \((d', p') \in DP_{i-1} \). From \((d', p') \geq (c, \pi_i) \), it follows by Lemma 3 that \((d' - 1, p') \geq (c - 1, \pi_i) \).

We have \((d', p') \in DP_{i-1} \), \( dp(h, i - 1) = dp(h, i - 1) = (d', p') > (d' - 1, p') \), and \( h > 0 \).

We next show that \( dp(h - 1, i - 1) \geq dp(h - 1, i - 1) \). Consider two cases:

- \((d' - 1, p') < (0, \pi_0) \). Since \( h > 0 \), it follows by Lemma 6 that \( dp(h - 1, i - 1) \geq dp(0, i - 1) = (0, \pi_0) > (d' - 1, p') \).

- \((d' - 1, p') \geq (0, \pi_0) \). Since \((d', p') \in DP_{i-1} \) and \((d' - 1, p') \geq (0, \pi_0) \), it follows by induction that \((d' - 1, p') \in DP_{i-1} \). Since \( dp(h, i - 1) > (d' - 1, p') \), it follows by Lemma 9 that \( dp(h - 1, i - 1) \geq (d' - 1, p') \).
Thus \( dp(h - 1, i - 1) \geq (d' - 1, p') \geq (c - 1, \pi_i) \). On the other hand, our choice of \( h \) implies that \( dp(h - 1, i - 1) < (c, \pi_i) \leq dp(h, i - 1) \). Therefore, \( dp(h - 1, i - 1) \rightarrow i = (c, \pi_i) \).

Recall the following recurrence for \( 0 < h < i \leq n \):

\[
dp(h, i) = \min\{dp(h, i - 1), dp(h - 1, i - 1) \rightarrow i\}.
\]

Thus \( dp(h, i) = dp(h - 1, i - 1) \rightarrow i = (c, \pi_i) \). Thus \((c, \pi_i) \in DP_i\).

The second case We next consider the case that \( p 
\neq \pi_i \) and \( p 
\neq \pi_0 \). Suppose that \((d, p) \in DP_i \) and \((d - 1, p) \geq (0, \pi_0) \). We will show that \((d - 1, p) \in DP_i \).

Since \((d, p) \in DP_i \), it follows that \((d, p) = dp(h', i) \) for some \( h' \), where \( 0 \leq h' \leq i \). By Lemma 8, the \( p \)-part of \( dp(i, i) \) is equal to \( \pi_i \). By Lemma 7, the \( p \)-part of \( dp(0, i) \) is equal to \( \pi_0 \). Since \( p 
\neq \pi_i \) and \( p 
\neq \pi_0 \), we must have \( 0 < h' < i \).

By (1), we have \((d, p) = dp(h', i) = \min\{dp(h', i - 1), dp(h' - 1, i - 1) \rightarrow i\} \). By Lemma 4, the \( p \)-part of \( dp(h' - 1, i - 1) \rightarrow i \) is \( \pi_i \). Since \( p 
\neq \pi_i \), it follows that \((d, p) \neq dp(h' - 1, i - 1) \rightarrow i \). Thus \((d, p) = dp(h', i - 1) \in DP_{i-1} \), and \((d, p) < dp(h' - 1, i - 1) \rightarrow i \).

Write \( dp(h' - 1, i - 1) \) as \( (d', p') \). Then \((d', p') \in DP_{i-1} \). Also, \((d, p) < (d', p') \rightarrow i \). By Lemma 5, it follows that \((d - 1, p) < (d' - 1, p') \rightarrow i \). By Lemma 6, we have \( dp(h', i - 1) > dp(h' - 1, i - 1) \), that is, \((d, p) > (d', p') \). By Lemma 3, it follows that \((d - 1, p) > (d' - 1, p') \).

Since \((d, p) \in DP_{i-1} \) and \((d - 1, p) \geq (0, \pi_0) \), it follows by induction that \((d - 1, p) \in DP_{i-1} \). Thus \((d - 1, p) = dp(h, i - 1) \) for some \( h \), where \( 0 \leq h \leq i - 1 \). By Lemma 7, the \( p \)-part of \( dp(0, i - 1) \) is \( \pi_0 \). Since \( p 
\neq \pi_0 \), we must have \( h > 0 \). Thus \( 0 < h < i \).

We have \((d', p') \in DP_{i-1} \), \( dp(h, i - 1) = (d - 1, p) > (d' - 1, p') \), and \( h > 0 \).

We next show that \( dp(h - 1, i - 1) \geq (d' - 1, p') \). Consider two cases:

- \((d' - 1, p') < (0, \pi_0) \). Since \( h > 0 \), it follows by Lemma 6 that \( dp(h - 1, i - 1) \geq dp(0, i - 1) = (0, \pi_0) \). \( (d' - 1, p') \).

- \((d' - 1, p') \geq (0, \pi_0) \). Since \((d', p') \in DP_{i-1} \) and \((d' - 1, p') \geq (0, \pi_0) \), it follows by induction that \((d' - 1, p') \in DP_{i-1} \). Since \( dp(h, i - 1) > (d' - 1, p') \), it follows by Lemma 9 that \( dp(h - 1, i - 1) \geq (d' - 1, p') \).

Thus \( dp(h - 1, i - 1) \geq (d' - 1, p') \). By Lemma 4, it follows that \( dp(h - 1, i - 1) \rightarrow i \geq (d' - 1, p') \rightarrow i \).

Recall that \( dp(h, i - 1) = (d - 1, p) < (d' - 1, p') \rightarrow i \).

By (1), we have \( dp(h, i) = dp(h, i - 1) = (d - 1, p) \). Thus \((d - 1, p) \in DP_i \), as desired.

This completes the proof of Lemma 10, Lemma 2, and Theorem 2.

4 Algorithms on intervals of different lengths

In this section we prove Proposition 1. Recall that JOIN is a special case of J-PACK, and TILE is a special case of the four problems PACK, COVER, J-PACK, and J-COVER. It suffices to present algorithms for these four problems.

Suppose that \( \kappa > 1 \). Let \( \ell = (\ell_1, \ldots, \ell_\kappa) \) be the \( \kappa \) distinct lengths of the \( n \) intervals in \( \mathcal{I} \). In the following, we use the notation \( u \) as a shorthand for a \( \kappa \)-tuple \((u_1, \ldots, u_\kappa) \). In particular, \( 0 = (0, \ldots, 0) \) and \( \tau = (\tau, \ldots, \tau) \). For \( u = (u_1, \ldots, u_\kappa) \), denote by \( -u \) the \( \kappa \)-tuple \((-u_1, \ldots, -u_\kappa) \). For \( u = (u_1, \ldots, u_\kappa) \) and \( v = (v_1, \ldots, v_\kappa) \), denote by \( u - v \) the \( \kappa \)-tuple \((u_1 - v_1, \ldots, u_\kappa - v_\kappa) \). Denote by \( u \cdot v \) the dot product \( \sum_{h=1}^{\kappa} u_h v_h \), and write \( u \leq v \) (respectively, \( u \geq v \), \( u = v \)) if \( u_h \leq v_h \) (respectively, \( u_h \geq v_h \), \( u_h = v_h \)) for all \( 1 \leq h \leq \kappa \). Denote by \( |u| \) the sum \( \sum_{h=1}^{\kappa} |u_h| \). For \( u \geq 0 \), we use the phrase “\( u \) intervals” to refer to \( |u| \) intervals, including \( u_h \) intervals of each length \( \ell_h, 1 \leq h \leq \kappa \).
Let $B = [0, \ell_B)$. Let $I_1, \ldots, I_n$, where $I_i = [x_i, y_i)$, be the $n$ intervals in $\mathcal{I}$ sorted in increasing $x_i + y_i$. Add two dummy intervals $I_0$ and $I_{n+1}$, where $I_0 = [x_0, y_0)$, with $y_0 = 0$ and $x_0 + y_0 < x_1 + y_1$, and $I_{n+1} = [x_{n+1}, y_{n+1})$, with $x_{n+1} = \ell_B$ and $x_n + y_n < x_{n+1} + y_{n+1}$.

For $0 \leq i \leq n+1$, denote by $\mathcal{I}_i$ the subfamily of intervals $I_0, I_1, \ldots, I_i$. For $0 \leq i \leq n+1$, denote by $m_i$ the $\kappa$-tuple $(m_{i,1}, \ldots, m_{i,\kappa})$ where $m_{i,h}$ for $1 \leq h \leq \kappa$ is the multiplicity of $\ell_h$ (that is, the number of intervals of length $\ell_h$) in $\mathcal{I}_i \setminus \{I_0, I_{n+1}\}$.

**Algorithm for PACK** Let $A(i, u, v)$, where $0 \leq i \leq n+1$, $0 \leq u \leq \tau$, and $0 \leq v \leq \tau$, be the predicate whether there exists a subfamily $J \subseteq \mathcal{I}_i$ of pairwise-disjoint intervals including $I_0$ and $I_i$, such that

- $\mathcal{I}_i \setminus J$ includes exactly $u$ intervals,
- the positive gaps between consecutive intervals in $J$ can accommodate $v$ intervals (that is, the $v$ intervals can be partitioned into subfamilies, one subfamily for each gap, such that each gap has space for the corresponding intervals).

Then the $n$ intervals in $\mathcal{I}$ can be packed inside $B$ by moving $\tau$ intervals if and only if $A(n+1, v, v)$ is true for some $v \geq 0$ with $|v| \leq \tau$.

The table $A$ can be computed by dynamic programming. For the base case when $i = 0$, $A(0, u, v)$ is true if and only if $u = v = 0$. For $1 \leq i \leq n+1$, $A(i, u, v)$ is true if and only if $A(i', u', v')$ is true for some $0 \leq i' < i$, $0 \leq u' \leq u$, and $0 \leq v' \leq v$ such that

$$i' \geq i - 1 - \tau, \quad u - u' = m_{i-1} - m_{i'}, \quad (v - v') \cdot \ell \leq x_i - y_i.$$

**Algorithm for J-PACK** Let $A(i, u, v)$, where $0 \leq i \leq n+1$, $0 \leq u \leq \tau$, and $0 \leq v \leq \tau$, be the predicate whether there exists a subfamily $J \subseteq \mathcal{I}_i$ of pairwise-disjoint intervals including $I_0$ and $I_i$, such that

- $\mathcal{I}_i \setminus J$ includes exactly $u$ intervals,
- the positive gaps between consecutive intervals in $J$ can accommodate $v$ intervals (that is, the $v$ intervals can be partitioned into subfamilies, one subfamily for each gap, such that the two boundary gaps, the one bounded by $I_0$ on the left, and the one bounded by $I_{n+1}$ on the right, if any, have space for, while the other gaps fit exactly, the corresponding intervals).

Then the $n$ intervals in $\mathcal{I}$ can be joined into a contiguous interval contained in $B$ by moving $\tau$ intervals if and only if $A(n+1, v, v)$ is true for some $v \geq 0$ with $|v| \leq \tau$.

The table $A$ can be computed by dynamic programming. For the base case when $i = 0$, $A(0, u, v)$ is true if and only if $u = v = 0$. For $1 \leq i \leq n+1$, $A(i, u, v)$ is true if and only if $A(i', u', v')$ is true for some $0 \leq i' < i$, $0 \leq u' \leq u$, and $0 \leq v' \leq v$ such that

$$i' \geq i - 1 - \tau, \quad u - u' = m_{i-1} - m_{i'}, \quad \begin{cases} (v - v') \cdot \ell = x_i - y_i, & \text{if } 0 < i < i' < n + 1 \\ (v - v') \cdot \ell \leq x_i - y_i, & \text{otherwise}. \end{cases}$$

**Algorithm for J-COVER** Let $A(i, u, v)$, where $0 \leq i \leq n+1$, $0 \leq u \leq \tau$, and $0 \leq v \leq \tau$, be the predicate whether there exists a subfamily $J \subseteq \mathcal{I}_i$ of intervals including $I_0$ and $I_i$, such that the intervals in $J \setminus \{I_0, I_{n+1}\}$ are pairwise-disjoint, and moreover,
\begin{itemize}
  \item \(\mathcal{I}_i \setminus \mathcal{J}\) includes exactly \(u\) intervals,
  \item the positive gaps between consecutive intervals in \(\mathcal{J}\) can accommodate \(v\) intervals (that is, the \(v\) intervals can be partitioned into subfamilies, one subfamily for each gap, such that the two boundary gaps, the one bounded by \(I_0\) on the left, and the one bounded by \(I_{n+1}\) on the right, if any, can be covered by, while the other gaps fit exactly, the corresponding intervals).
\end{itemize}

Then the \(n\) intervals in \(\mathcal{I}\) can be joined into a contiguous interval containing \(B\) by moving \(\tau\) intervals if and only if \(A(n+1, v, v)\) is true for some \(v \geq 0\) with \(|v| \leq \tau\).

The table \(A\) can be computed by dynamic programming. For the base case when \(i = 0\), \(A(0, u, v)\) is true if and only if \(u = v = 0\). For \(1 \leq i \leq n+1\), \(A(i, u, v)\) is true if and only if \(A(i', u', v')\) is true for some \(0 \leq i' < i\), \(0 \leq u' \leq u\), and \(0 \leq v' \leq v\) such that

\[
i' \geq i - 1 - \tau, \quad u - u' = m_{i-1} - m_{i'},
\]

\[
\begin{cases}
(v - v') \cdot \ell = x_i - y_i & \text{if } 0 < i < i' < n + 1 \\
(v - v') \cdot \ell \geq x_i - y_i & \text{otherwise}.
\end{cases}
\]

**Algorithm for COVER** Let \(A(i, u, v)\), where \(0 \leq i \leq n + 1\), \(0 \leq u \leq \tau\), and \(0 \leq v \leq \tau\), be the predicate whether there exists a subfamily \(\mathcal{J} \subseteq \mathcal{I}_i\) of intervals including \(I_0\) and \(I_i\), with no interval properly contained in another, such that

\begin{itemize}
  \item \(\mathcal{I}_i \setminus \mathcal{J}\) includes at least \(u\) intervals,
  \item the positive gaps between consecutive intervals in \(\mathcal{J}\) can accommodate \(v\) intervals (that is, the \(v\) intervals can be partitioned into subfamilies, one subfamily for each gap, such that each gap can be covered by the corresponding intervals).
\end{itemize}

Then \(B\) can be covered by the \(n\) intervals in \(\mathcal{I}\) by moving \(\tau\) intervals if and only if \(A(n+1, v, v)\) is true for some \(v \geq 0\) with \(|v| \leq \tau\).

The table \(A\) can be computed by dynamic programming. For the base case when \(i = 0\), \(A(0, u, v)\) is true if and only if \(u = v = 0\). For \(1 \leq i \leq n + 1\), \(A(i, u, v)\) is true if and only if \(A(i', u', v')\) is true for some \(0 \leq i' < i\), \(0 \leq u' \leq u\), and \(0 \leq v' \leq v\) such that

\[
x_{i'} \leq x_i \text{ and } y_{i'} \leq y_i, \quad (v - v') \cdot \ell \geq x_i - y_i.
\]

**Running time analysis** The number of entries in the table \(A\) is \((n + 2)(\tau + 1)^{2k}\). But at the end of the algorithm, we need to check only entries \(A(n+1, v, v)\) for \(v \geq 0\) with \(|v| \leq \tau\). Thus we can restrict the computation of \(A(i, u, v)\) to \(u \geq 0\) with \(|u| \leq \tau\) and \(v \geq 0\) with \(|v| \leq \tau\).

Consider the directed graph \(G_{\kappa, \tau}\) with a vertex for each \(\kappa\)-tuple \(w \geq 0\) with \(|w| \leq \tau\), and an edge from \(w\) to \(w'\) if and only if \(w' \leq w\). Then \(G_{\kappa, \tau}\) has \((\tau + \kappa)^2\) vertices, \(O((\tau + \kappa)^2)^2\) edges, and can be built in \(O((\tau + \kappa)^2)^2\) time. Thus with some pre-processing, we can reduce the number of relevant table entries to \((n + 2)(\tau + \kappa)^2\). For the recurrence, each entry is computed by looking up at most \((\tau + 1)(\tau + \kappa)^2\) other entries for PACK / J-PACK / J-COVER, and at most \((n + 1)(\tau + \kappa)^2\) other entries for COVER, with \(O(\kappa)\) time on each look-up. So the overall running time, including the \(O(n \log n)\) time on sorting, is \(O(n \log n + n \tau \kappa (\tau + \kappa)^4)\) for PACK / J-PACK / J-COVER, and is \(O(n^2 \kappa (\tau + \kappa)^4)\) for COVER.

Since \(\tau \leq n\) and \(\kappa \leq n\), \((\tau + \kappa)^2\) is bounded by a polynomial in \(n\) when either \(\kappa\) or \(\tau\) is constant. Thus the overall running time is bounded by a polynomial in \(n\) when either \(\kappa\) or \(\tau\) is constant.
A more careful implementation  The running time of the algorithms for PACK, J-PACK, and J-COVER can be improved by a more careful implementation. Let \( A'(i, t, d) \), where \( 0 \leq i \leq n + 1, 0 \leq t \leq \tau, \) and \(-\tau \leq d \leq \tau\), be the predicate whether there exist \( u \geq 0 \) with \(|u| \leq \tau\) and \( v \geq 0 \) with \(|v| \leq \tau\), such that \(|u| = t, u - v = d, \) and \( A(i, u, v) \) is true. Then \( A(n + 1, v, v) \) is true for some \( v \geq 0 \) with \(|v| \leq \tau\) if and only if \( A'(n + 1, t, 0) \) is true for some \( t \leq \tau\). We can compute \( A' \) by dynamic programming in a similar way as \( A \).

Since \( d = u - v \), we have \( d_h = u_h - v_h \) for each \( h, 1 \leq h \leq \kappa \). If \( d_h \) is negative, then \(|d_h|\) is the number of intervals of length \( \ell_h \) in \( I \setminus I \) that need to be moved to the gaps between consecutive intervals in \( J \). If \( d_h \) is positive, then \(|d_h|\) is the number of intervals of length \( \ell_h \) in \( I \setminus J \) that remain to be moved to later gaps. In both cases, a nonzero component \( d_h \) of \( d \) signifies a commitment to move \(|d_h|\) intervals in \( I \setminus J \). The \(|d_h|\) moves for different values of \( h \) are independent because they correspond to intervals of different lengths. Thus we only need to consider \( d \) with \(|d| \leq \tau\).

There are exactly \( \binom{\tau + \kappa}{\kappa} \) nonnegative tuples \( d \geq 0 \). For each such tuple, the number of nonzero components is at most \( \min\{\kappa, \tau\} \), and there are at most \( 2^{\min\{\kappa, \tau\}} \) ways to add positive or negative signs to them. Thus the number of \( \kappa \)-tuples \( d \) with \(|d| \leq \tau\) is at most \( \binom{\tau + \kappa}{\kappa} 2^{\min\{\kappa, \tau\}} \). With some pre-processing, we can reduce the number of entries of \( A' \) to \( O(n \tau) \binom{\tau + \kappa}{\kappa} 2^{\min\{\kappa, \tau\}} \), and correspondingly reduce the number of table look-ups for each entry to \( O(\tau) \binom{\tau + \kappa}{\kappa} 2^{\min\{\kappa, \tau\}} \). Then the overall running time becomes \( O(n \log n + n \cdot \kappa \tau^2 \binom{\tau + \kappa}{\kappa} 2^{\min\{\kappa, \tau\}}) \) for PACK, J-PACK, and J-COVER.

This completes the proof of Proposition 1.

5  Intractability

In this section we prove Theorem 1. Recall that TILE is a special case of the four problems PACK, COVER, J-PACK, and J-COVER, which are equivalent when \( \ell_I = \ell_B \). Thus it suffices to prove the hardness of the two problems TILE and JOIN.

5.1  Strong-NP-hardness and W[1]-hardness with parameter \( \sigma \)

As a warm-up exercise, we first present a simple proof of the strong-NP-hardness, and W[1]-hardness with parameter \( \sigma \), of the two problems TILE and JOIN.

Our proof is by a reduction from the strongly NP-hard problem BIN PACKING [3, Problem SR1]. Given \( \hat{n} \) items of integer lengths \( a_i, 1 \leq i \leq \hat{n} \), and \( \hat{k} \) bins each of integer length \( b \), the problem BIN PACKING asks whether the \( \hat{n} \) items can be packed inside the \( \hat{k} \) bins, that is, whether the \( \hat{n} \) items can be partitioned into \( \hat{k} \) subsets, such that the total length of items in each subset is at most \( b \). Our reduction is from a restricted version of BIN PACKING where all integers \( a_i \) and \( b \) are encoded in unary and moreover \( \sum_{i=1}^{\hat{n}} a_i = \hat{k} b \). BIN PACKING is W[1]-hard with parameter \( \hat{k} \) even for this restricted version [4]. Without loss of generality, we assume that \( a_i \leq b \) for \( 1 \leq i \leq \hat{n} \).

Our reduction works as follows. Let \( \ell_B = \hat{k}(b + 1) \). Put the interval \( B \) at \([0, \ell_B]\), then partition it into \( 2\hat{k} \) intervals of alternating lengths \( b \) and \( 1 \), where the \( \hat{k} \) intervals of length \( b \) are called bin intervals, and the \( \hat{k} \) intervals of length \( 1 \) are called separator intervals. Let \( I \) be a family of \( n = \hat{n} + \hat{k} \) intervals of total length \( \ell_I = \sum_{i=1}^{\hat{n}} a_i + \hat{k} = \ell_B \), including \( \hat{n} \) item intervals of lengths \( a_i \), all sharing the same left endpoint as \( B \), and the \( \hat{k} \) separator intervals from \( B \). Let \( \tau = \hat{n} - 1 \) and \( \sigma = \hat{k} + 1 \).

This completes the construction. Refer to Figure 1 for an example. We claim that the \( \hat{n} \) items can be partitioned into \( \hat{k} \) subsets each of total length \( b \) if and only if \( B \) can be tiled with \( I \) by moving \( \tau \) intervals and keeping \( \sigma \) intervals unmoved, if and only if \( I \) can be joined into a contiguous interval by moving \( \tau \) intervals and keeping \( \sigma \) intervals unmoved.
Figure 1: An interval $B$ and a family $I$ of $n = 13$ intervals constructed from a Bin Packing instance consisting of $\hat{n} = 9$ items of lengths $1, 2, 2, 2, 3, 4, 5$ and $\hat{\kappa} = 4$ bins of length $6$. $B$ can be tiled by moving $\tau = 8$ intervals and keeping $\sigma = 5$ intervals unmoved. Initial positions of intervals in $I$ are shown in black; final positions of moved intervals are shown in red.

It is easy to see that if the $\hat{n}$ items can be partitioned into $\hat{\kappa}$ subsets each of total length $b$, then $B$ can be tiled with $I$ by moving $\tau$ intervals and keeping $\sigma$ intervals unmoved. Also, any tiling of $B$ with $I$ necessarily joins $I$ into a contiguous interval. Thus we have the two direct, only if, implications of the claim. To complete the proof, it suffices to show that if $I$ can be joined into a contiguous interval by moving $\tau$ intervals and keeping $\sigma$ intervals unmoved, then the $\hat{n}$ items can be partitioned into $\hat{\kappa}$ subsets each of total length $b$.

Note that the $\hat{n}$ item intervals pairwise intersect, but have to become pairwise disjoint when joined into a contiguous interval. Thus by our choice of $\tau$ and $\sigma$, all but one of the item intervals must move, and all separator intervals must not move. Since there is an unmoved interval in $I$ at either end of the interval $B$, and since $\ell_I = \ell_B$, the intervals in $I$ must be joined into a contiguous interval that coincides with $B$. It follows that each bin interval of $B$ must be covered by a subset of item intervals of total length $b$. Correspondingly, we get a partition of the $\hat{n}$ items in $\hat{\kappa}$ bins.

The reduction is clearly polynomial, and is FPT with parameter $\sigma = \hat{\kappa} + 1$. Thus Tile and Join are NP-hard, and W[1]-hard with parameter $\sigma$, even when the input is encoded in unary.

5.2 W[1]-hardness with parameter $\kappa$ and with parameter $\tau$

To prove the W[1]-hardness, with parameter $\kappa$ and with parameter $\tau$, of the two problems Tile and Join, we use two reductions from the same W[1]-hard problem Colored Clique [2].

Let $G$ be a graph with $\hat{n}$ vertices and $\hat{m}$ edges, where each vertex has one of $\hat{\kappa}$ colors. The problem Colored Clique asks whether there exists in $G$ a colored clique of $\hat{\kappa}$ pairwise-adjacent vertices including exactly one vertex of each color. Denote the $\hat{n}$ vertices by $0, \ldots, \hat{n} - 1$, and denote the $\hat{\kappa}$ colors by $0, \ldots, \hat{\kappa} - 1$. Without loss of generality, we assume that every edge in $G$ is incident to two vertices of different colors, every vertex in $G$ is adjacent to at least one vertex of each of the other $\hat{\kappa} - 1$ colors, and $\hat{\kappa} \geq 3$.

5.2.1 W[1]-hardness with parameter $\kappa$

We first prove the W[1]-hardness with parameter $\kappa$ of Tile and Join. Let $\tau = 3\hat{\kappa} + 3(\hat{\kappa}/2) + 8(\hat{n} - 1)(\hat{\kappa}/2)$, and $\ell = 6\hat{\kappa}^2$. We will construct an interval $B$ and a family $I$ of intervals. The lengths of the intervals in $I$ include:

- a fractional length $\ell_f = \frac{1}{\tau + 1}$,
- a vertex length $\ell_v = (2 + 2(\hat{n} - 1)(\hat{\kappa} - 1)) \cdot \ell$,
- an edge length $\ell_e = (2 + 4(\hat{n} - 1)) \cdot \ell$,
- a padding length $\ell_p = \ell$. 


• two incidence lengths $\ell_{ij}^+ = \ell + \ell_{ij}$ and $\ell_{ij}^- = \ell - \ell_{ij}$, where $\ell_{ij} = \hat{\kappa} \cdot i + j$, for each ordered pair of colors $i, j$ with $i \neq j$.

• two color lengths $\ell_{\{i\}}^+ = \ell + \ell_{\{i\}}$ and $\ell_{\{i\}}^- = \ell - \ell_{\{i\}}$, where $\ell_{\{i\}} = \hat{\kappa}^2 + \hat{\kappa} \cdot i + \hat{\kappa}$, for each color $i$.

• two color-pair lengths $\ell_{\{ij\}}^+ = \ell + \ell_{\{ij\}}$ and $\ell_{\{ij\}}^- = \ell - \ell_{\{ij\}}$, where $\ell_{\{ij\}} = \hat{\kappa}^2 + \hat{\kappa} \cdot i + j$, for each unordered pair of colors $\{i, j\}$ with $i < j$.

In total, there are $k = 4 + 4\left(\hat{\kappa}^2\right) + 2\hat{\kappa} + 2\left(\hat{\kappa}\right)$ lengths listed above. These $k$ lengths are all distinct except that $\ell_v = \ell_e$ when $\hat{\kappa} = 3$. Thus the number $\kappa$ of distinct lengths is either $k$ or $k - 1$, which is $O(\hat{\kappa}^2)$. Since $\ell_p = \ell = 6\hat{\kappa}^2$, $0 < \ell_{ij} < \hat{\kappa}^2$, $\hat{\kappa}^2 \leq \ell_{\{i\}} < 2\hat{\kappa}^2$, and $\hat{\kappa}^2 < \ell_{\{ij\}} < 2\hat{\kappa}^2$, the lengths $\ell_p, \ell_{\{i\}}, \ell_{\{ij\}}$, and $\ell_{\{ij\}}^+$ are all greater than $4\hat{\kappa}^2$ and less than $8\hat{\kappa}^2$, and hence differ from $\ell$ and from each other by factors less than two.

The interval $B$ and its partition Let $g_1 = 2\hat{\kappa} + 2\left(\hat{\kappa}\right)$, $g_2 = 4(\hat{\kappa} - 1)$, $n_B = (\hat{n} + \hat{m} + g_1 + g_2 + 1) + \hat{n} + \hat{m} + g_1 + g_2$, and $B_B = (\hat{n} + \hat{m} + g_1 + g_2 + 1) \cdot 1 + \hat{n} \cdot \ell_v + \hat{m} \cdot \ell_e + (g_1 + g_2) \cdot \ell$. Put the interval $B$ at $[0, \ell_B)$, then partition it into $n_B$ subintervals, including

• $\hat{n} + \hat{m} + g_1 + g_2 + 1$ separator intervals of length 1,

• $\hat{n}$ vertex intervals of length $\ell_v$,

• $\hat{m}$ edge intervals of length $\ell_e$,

• $g_1$ type-1 gap intervals of various lengths with average $\ell$, including
  
  – one interval of each length $\ell_{\{i\}}^+$ and $\ell_{\{i\}}^-$, for each color $i$,
  
  – one interval of each length $\ell_{\{ij\}}^+$ and $\ell_{\{ij\}}^-$, for each unordered pair of colors $\{i, j\}$ with $i < j$,

• $g_2$ type-2 gap intervals of various lengths with average $\ell$, including
  
  – $\hat{n} - 1$ intervals of each length $\ell_{ij}^+$ and $\ell_{ij}^-$, for each ordered pair of colors $ij$ with $i \neq j$,
  
  – $4(\hat{n} - 1)$ of length $\ell_p = \ell$,

where the $\hat{n} + \hat{m} + g_1 + g_2$ vertex/edge/gap intervals are interspersed between the $\hat{n} + \hat{m} + g_1 + g_2 + 1$ separator intervals.

The family $\mathcal{I}$ of $n$ intervals Let $n = (\hat{n} + \hat{m} + g_1 + g_2 + 1) \cdot (\tau + 1) + \hat{n} \cdot (2 + 2(\hat{n} - 1)(\hat{\kappa} - 1)) + \hat{m} \cdot (2 + 4(\hat{\kappa} - 1)) + \hat{\kappa} + 6\hat{\kappa}$. Construct the family $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$ of $n$ intervals in four parts as follows.

First, for each of the $\hat{n} + \hat{m} + g_1 + g_2 + 1$ separator intervals of length 1, partition it further into $\tau + 1$ fractional intervals of length $\ell_f$, and put them in $\mathcal{I}_0$.

Next, for each vertex $i$ of color $i$, take a distinct vertex interval of length $\ell_v$ in the partition of $B$, partition it further into $2 + 2(\hat{n} - 1)(\hat{\kappa} - 1)$ intervals, which encode the color $i$ and the vertex $i$, and put them in $\mathcal{I}_1$:

• one interval of each length $\ell_{\{i\}}^+$ and $\ell_{\{i\}}^-$,

• $2(\hat{n} - 1)$ intervals for each color $j \neq i$, including
  
  – $\hat{n} - i$ intervals of each length $\ell_{ij}^+$ and $\ell_{ij}^-$,
  
  – $\hat{n} - i$ pairs of intervals of length $\ell_p = \ell$.

Next, for each edge $\{i, j\}$ of color pair $\{i, j\}$ with $i < j$, take a distinct edge interval of length $\ell_e$ in the partition of $B$, partition it further into $2 + 4(\hat{n} - 1)$ intervals, which encode the color pair $\{i, j\}$ and the vertices $\{i, j\}$, and put them in $\mathcal{I}_2$:
• one interval of each length $\ell^+_i$ and $\ell^-_i$,

• $4(\tilde{n} - 1)$ intervals including
  
  – $\tilde{n} - 1 - i$ intervals of each length $\ell^+_i$ and $\ell^-_i$,
  
  – $i$ pairs of intervals of length $\ell_p = \ell$,
  
  – $\tilde{n} - 1 - j$ intervals of each length $\ell^+_j$ and $\ell^-_j$,
  
  – $j$ pairs of intervals of length $\ell_p = \ell$.

Finally, construct $\hat{\kappa}$ intervals of length $\ell_v$, and $\left(\frac{\hat{\kappa}}{2}\right)$ intervals of length $\ell_e$, all sharing the same left endpoint as $B$, and put them in $\mathcal{I}_3$.

![Graph](image)

Figure 2: Top: A graph of $\hat{n} = 6$ vertices and $\hat{m} = 7$ edges with $\hat{\kappa} = 3$ colors, where vertex $i$ has color $i \mod 3$. The three vertices $2, 3, 4$ form a colored clique. Center: The $\hat{n} = 6$ vertex intervals and $\hat{m} = 7$ edge intervals are partitioned into the listed lengths (in particular, $\ell_p = 54$ is the padding length), where superscripts denote multiplicities, and the incidence lengths for the colored clique $2, 3, 4$ are highlighted in red. Bottom: The multiplicities of the highlighted lengths from the vertex intervals are complementary with those from the edge intervals, and add up to $\hat{n} - 1 = 5$.

This completes the construction of $B$ and $\mathcal{I}$. Refer to Figure 2 for an example. Among the intervals in the partition of $B$, the separator, vertex, and edge intervals are further partitioned to construct intervals in
\(\mathcal{I}_0, \mathcal{I}_1, \text{ and } \mathcal{I}_2\), respectively. Only the gap intervals are not used in the construction of \(\mathcal{I}\); their total length is \((g_1 + g_2) \cdot \ell\). On the other hand, the total length of the intervals in \(\mathcal{I}_3\) is

\[
\hat{k} \cdot \ell_v + \left(\frac{\hat{k}}{2}\right) \cdot \ell_e = \hat{k} \cdot (2 + 2(\hat{n} - 1)(\hat{k} - 1)) \cdot \ell + \left(\frac{\hat{k}}{2}\right) \cdot (2 + 4(\hat{n} - 1)) \cdot \ell \\
= \left(2\hat{k} + 2\left(\frac{\hat{k}}{2}\right)ight) + 4(\hat{n} - 1)\left(\frac{\hat{k}}{2}\right) + 4(\hat{n} - 1)\left(\frac{\hat{k}}{2}\right) \cdot \ell = (g_1 + g_2) \cdot \ell.
\]

Thus \(\ell_\mathcal{I} = \ell_B\), as expected.

For convenience we have used fractional intervals of length \(\ell_f = \frac{1}{\tau + 1}\). By scaling with factor \(\tau + 1\), the coordinates of all intervals can be converted to integers polynomial in \(\hat{k}\) and \(\hat{n}\). Thus the reduction is strongly polynomial. Since \(\kappa = O(\hat{k}^2)\), the reduction is also FPT. We claim that \(G\) has a colored clique of \(\kappa\) vertices if and only if \(B\) can be tiled with \(\mathcal{I}\) by moving \(\tau\) intervals, if and only if \(\mathcal{I}\) can be joined into a contiguous interval in \(\tau\) moves.

**Colored Clique \(\implies\) Tile** Suppose that \(G\) has a colored clique \(K\) of \(\hat{k}\) vertices. We will tile \(B\) with \(\mathcal{I}\) by moving \(\tau\) intervals as follows. First move the \(\hat{k} + \left(\frac{\hat{k}}{2}\right)\) intervals in \(\mathcal{I}_3\) to cover the \(\hat{k}\) vertex intervals and the \(\left(\frac{\hat{k}}{2}\right)\) edge intervals inside \(B\) corresponding to the vertices and edges in \(K\). Next move the corresponding intervals in \(\mathcal{I}_1\) and \(\mathcal{I}_2\), including the \(2 + 2(\hat{n} - 1)(\hat{k} - 1)\) intervals composing each of these vertex intervals, and the \(2 + 4(\hat{n} - 1)\) intervals composing each of these edge intervals, to cover the \(g_1 + g_2\) gap intervals. The number of these intervals in \(\mathcal{I}_1\) and \(\mathcal{I}_2\) is

\[
\hat{k} \cdot (2 + 2(\hat{n} - 1)(\hat{k} - 1)) + \left(\frac{\hat{k}}{2}\right) \cdot (2 + 4(\hat{n} - 1)) = 2\hat{k} + 2\left(\frac{\hat{k}}{2}\right) + 8(\hat{n} - 1)\left(\frac{\hat{k}}{2}\right) = g_1 + g_2.
\]

The total number of moves is

\[
\hat{k} + \left(\frac{\hat{k}}{2}\right) + g_1 + g_2 = 3\hat{k} + 3\left(\frac{\hat{k}}{2}\right) + 8(\hat{n} - 1)\left(\frac{\hat{k}}{2}\right) = \tau.
\]

Since \(K\) is a colored clique, the intervals in \(\mathcal{I}_1\) composing the \(\hat{k}\) vertex intervals include exactly one pair of intervals of lengths \(\ell^+_i\) for each color \(i\), and the intervals in \(\mathcal{I}_2\) composing the \(\left(\frac{\hat{k}}{2}\right)\) edge intervals include exactly one pair of intervals of lengths \(\ell^+_i\) for each unordered pair of colors \(\{i, j\}\) with \(i \neq j\), to cover the \(g_1 = 2\hat{k} + 2\left(\frac{\hat{k}}{2}\right)\) type-1 gap intervals of these lengths.

Moreover, by design of the complementary multiplicities of incidence and standard lengths in partitioning vertex and edge intervals inside \(B\) into intervals in \(\mathcal{I}_1\) and \(\mathcal{I}_2\), respectively, the intervals composing the \(\hat{k}\) vertex intervals and the \(\left(\frac{\hat{k}}{2}\right)\) edge intervals also include, for each ordered pair of colors \(i, j\) with \(i \neq j\), exactly \(\hat{n} - 1\) pairs of intervals of lengths \(\ell^+_i\), and exactly \(\hat{n} - 1\) pairs of intervals of length \(\ell\), to cover the \(g_2 = 8(\hat{n} - 1)\left(\frac{\hat{k}}{2}\right)\) type-2 gap intervals of these lengths. Thus \(B\) is tiled with \(\mathcal{I}\) by moving \(\tau\) intervals.

**Tile \(\implies\) Join** Any tiling of \(B\) with \(\mathcal{I}\) necessarily joins \(\mathcal{I}\) into a contiguous interval. Thus if \(B\) can be tiled with \(\mathcal{I}\) by moving \(\tau\) intervals, then \(\mathcal{I}\) can be joined into a contiguous interval in \(\tau\) moves.

**Join \(\implies\) Colored Clique** Suppose that \(\mathcal{I}\) can be joined into a contiguous interval in \(\tau\) moves. We will find a colored clique of \(\hat{k}\) vertices in \(G\).

First note that each separator interval inside \(B\) is composed of \(\tau + 1\) fractional intervals in \(\mathcal{I}_0\), which are so numerous and so short that moving \(\tau\) of them is not enough to change their neighboring spaces by 1. Since all other intervals in \(\mathcal{I}\) and all vertex/edge/gap intervals inside \(B\) have integer coordinates, we can
assume without loss of generality that no fractional interval in \( \cal{I}_0 \) is moved. Then, to join the intervals in \( \cal{I} \) into a contiguous interval, we must cover all gap intervals inside \( B \).

All \( \hat{\kappa} + \binom{\hat{\kappa}}{2} \) intervals in \( \cal{I}_3 \) contain the \( \tau + 1 \) fractional intervals in \( \cal{I}_0 \) composing the leftmost separator interval inside \( B \), so they must be moved. Since their lengths, \( \ell_v \) and \( \ell_e \), are greater than the various lengths of the gap intervals inside \( B \), they cannot be moved to fill the gaps directly. To fill the gaps, we have to use intervals in \( \cal{I}_1 \) and \( \cal{I}_2 \).

Recall that the lengths of all intervals in \( \cal{I}_1 \) and \( \cal{I}_2 \), and the lengths of all gap intervals inside \( B \), are around \( \ell \), and differ from each other by factors less than two. Thus each gap interval must be covered by one interval of the same length in \( \cal{I}_1 \) or \( \cal{I}_2 \). In total, we need \( g_1 + g_2 \) moves to fill the gaps. Since \( \tau = \hat{\kappa} + \binom{\hat{\kappa}}{2} + g_1 + g_2 \), there is not a single move to waste. In summary, we must first move the \( \hat{\kappa} + \binom{\hat{\kappa}}{2} \) intervals in \( \cal{I}_3 \) to cover some portions of vertex and edge intervals inside \( B \), and then move exactly \( g_1 + g_2 \) intervals in \( \cal{I}_1 \) and \( \cal{I}_2 \) composing these covered portions to cover the \( g_1 + g_2 \) gap intervals inside \( B \).

First consider the \( g_1 = 2\hat{\kappa} + 2\binom{\hat{\kappa}}{2} \) type-1 gap intervals. Each type-1 gap interval of a color (respectively, color-pair) length must be covered by one interval of the same length in \( \cal{I}_1 \) (respectively, \( \cal{I}_2 \)), which was constructed from some vertex (respectively, edge) interval inside \( B \). Since there is a type-1 gap interval of every color and color-pair length, the \( \hat{\kappa} + \binom{\hat{\kappa}}{2} \) intervals in \( \cal{I}_3 \) must cover exactly \( \hat{\kappa} \) vertex intervals and \( \binom{\hat{\kappa}}{2} \) edge intervals inside \( B \), and the corresponding \( \hat{\kappa} \) vertices and \( \binom{\hat{\kappa}}{2} \) edges in \( G \) must span all \( \hat{\kappa} \) colors and all \( \binom{\hat{\kappa}}{2} \) color pairs.

Next consider the \( g_2 = 8(\hat{n} - 1)\binom{\hat{\kappa}}{2} \) type-2 gap intervals. Among the intervals in \( \cal{I}_1 \) and \( \cal{I}_2 \) composing the \( \hat{\kappa} \) vertex intervals and \( \binom{\hat{\kappa}}{2} \) edge intervals inside \( B \), the intervals of color and color-pair lengths are used to cover the type-1 gap intervals, so the intervals of incidence and padding lengths are left to cover the type-2 gap intervals. The \( g_2 \) type-2 gap intervals include exactly \( \hat{n} - 1 \) type-2 gap intervals of each incidence length, and \( 4(\hat{n} - 1)\binom{\hat{\kappa}}{2} \) intervals of the padding length. In particular, for each unordered pair of colors \( \{i, j\} \) with \( i < j \), the multiplicities of the four incidence lengths \( \ell_{ij}^+ \) and \( \ell_{ij}^- \) are all \( \hat{n} - 1 \). By design of the complementary multiplicities of incidence lengths in \( \cal{I}_1 \) and \( \cal{I}_2 \), the target of \( \hat{n} - 1 \) for \( \ell_{ij}^+ \) and \( \ell_{ij}^- \) can be reached only if the multiplicities of \( \hat{n} - 1 - i \) for \( \ell_{ij}^- \) and \( \hat{n} - 1 - j \) for \( \ell_{ij}^- \) in \( \cal{I}_2 \) are paired with the multiplicities of \( i \) for \( \ell_{ij}^+ \) and \( j \) for \( \ell_{ij}^- \) in \( \cal{I}_1 \). Such pairings require the edge \( \{i, j\} \) of color pair \( \{i, j\} \) be consistent with the vertices \( i' \) and \( j' \) of the corresponding colors \( i \) and \( j \), that is, \( i = i' \) and \( j = j' \). Thus the \( \hat{\kappa} \) vertices and the \( \binom{\hat{\kappa}}{2} \) edges form a colored clique.

### 5.2.2 W[1]-hardness with parameter \( \tau \)

We next prove the W[1]-hardness with parameter \( \tau \) of TILE and JOIN. Let \( \tau = 3\hat{\kappa} + 3\binom{\hat{\kappa}}{2} + 8\binom{\hat{\kappa}}{2} \), \( \kappa = 4 + 4\hat{n}(\hat{\kappa} - 1) + 2\hat{\kappa} + 2\binom{\hat{\kappa}}{2} \), and \( \ell = \hat{n}\kappa + \hat{\kappa}^2 \). We will construct an interval \( B \), and a family \( \cal{I} \) of intervals of \( \kappa \) distinct lengths. The \( \kappa \) lengths include

- a fractional length \( \ell_f = \frac{1}{\tau + 1} \),
- a vertex length \( \ell_v = 2 \cdot 3\ell + 2(\hat{\kappa} - 1) \cdot 7\ell \),
- an edge length \( \ell_e = 2 \cdot 3\ell + 4 \cdot 5\ell \),
- a pairing length \( \ell_p = 12\ell \),
- four incidence lengths \( \ell_{ij}^+ = 7\ell + \ell_{ij}^- \), \( \ell_{ij}^- = 7\ell - \ell_{ij}^+ \), \( \ell_{ij}^+ = 5\ell + \ell_{ij}^- \), \( \ell_{ij}^- = 5\ell - \ell_{ij}^+ \), where \( \ell_{ij} = \hat{\kappa} \cdot i + j + 1 \), for each vertex \( i \) of color \( \tau \) and for each color \( j \neq \tau \),
- two color lengths \( \ell_{(i)}^+ = 3\ell + \ell_{(i)}^- \) and \( \ell_{(i)}^- = 3\ell - \ell_{(i)}^+ \), where \( \ell_{(i)} = \hat{\kappa} \cdot i + 1 \), for each color \( i \),
- two color-pair lengths \( \ell_{(ij)}^+ = 3\ell + \ell_{(ij)}^- \) and \( \ell_{(ij)}^- = 3\ell - \ell_{(ij)}^+ \), where \( \ell_{(ij)} = \hat{\kappa} \cdot i + j + 1 \), for each unordered pair of colors \( \{i, j\} \) with \( i < j \).
These \( \kappa \) lengths are all distinct. From \( \ell = \hat{n}\kappa + \hat{\kappa}^2 \), it follows that \( \hat{n}\kappa < \ell \) and \( \hat{\kappa}^2 < \ell \). Since \( 0 < \ell_{ij} \leq \hat{n}\kappa \), we have \( 4\ell < \hat{\ell}_{ij}^+ < 6\ell < \hat{\ell}_{ij}^+ < 8\ell \). Since \( 0 < \ell_{\{i\}} \leq \hat{\kappa}^2 \) and \( 0 < \ell_{\{ij\}} \leq \hat{\kappa}^2 \), the lengths \( \ell_{\{i\}}^\pm \) and \( \ell_{\{ij\}}^\pm \) are all greater than \( 2\ell \) and less than \( 4\ell \), and hence differ from each other by factors less than two.

**The interval \( B \) and its partition** Let \( g_1 = 2\hat{\kappa} + 2\binom{\hat{\kappa}}{2} \), \( g_2 = 4\binom{\hat{\kappa}}{2} \), \( n_B = (\hat{n} + \hat{m} + g_1 + g_2 + 1) + \hat{n} + \hat{m} + g_1 + g_2 \), and \( \ell_B = (\hat{n} + \hat{m} + g_1 + g_2 + 1) \cdot 1 + \hat{n} \cdot \ell_v + \hat{m} \cdot \ell_e + g_1 \cdot 3\ell + g_2 \cdot 12\ell \). Put the interval \( B \) at \([0, \ell_B)\), then partition it into \( n_B \) subintervals, including:

- \( \hat{n} + \hat{m} + g_1 + g_2 + 1 \) separator intervals of length 1,
- \( \hat{n} \) vertex intervals of length \( \ell_v \),
- \( \hat{m} \) edge intervals of length \( \ell_e \),
- \( g_1 \) type-1 gap intervals of various lengths with average \( 3\ell \), including
  - one interval of each length \( \ell^+_{\{i\}} \) and \( \ell^-_{\{i\}} \), for each color \( i \),
  - one interval of each length \( \ell^+_{\{ij\}} \) and \( \ell^-_{\{ij\}} \), for each unordered pair of colors \( \{i, j\} \) with \( i < j \),
- \( g_2 \) type-2 gap intervals of length \( \ell_p = 12\ell \),

where the \( \hat{n} + \hat{m} + g_1 + g_2 \) vertex/edge/gap intervals are interspersed between the \( \hat{n} + \hat{m} + g_1 + g_2 + 1 \) separator intervals.

**The family \( \mathcal{I} \) of \( n \) intervals** Let \( n = (\hat{n} + \hat{m} + g_1 + g_2 + 1) \cdot (\tau + 1) + \hat{n} \cdot (2 + 2(\hat{\kappa} - 1)) + \hat{m} \cdot (2 + 4) + \hat{\kappa} + \binom{\hat{\kappa}}{2} \).

Construct the family \( \mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3 \) of \( n \) intervals in four parts as follows.

First, for each of the \( \hat{n} + \hat{m} + g_1 + g_2 + 1 \) separator intervals of length 1, partition it further into \( \tau + 1 \) fractional separator intervals of length \( \ell_f \), and put them in \( \mathcal{I}_0 \).

Next, for each vertex \( i \) of color \( \iota \), take a distinct vertex interval of length \( \ell_v \) in the partition of \( B \), partition it further into \( 2 + 2(\hat{\kappa} - 1) \) intervals, which encode the color \( \iota \) and the vertex \( i \), and put them in \( \mathcal{I}_1 \):

- one interval of each length \( \ell^+_{\{i\}} \) and \( \ell^-_{\{i\}} \),
- two intervals for each color \( j \neq \iota \), including
  - one interval of each length \( \ell^+_{\{ij\}} \) and \( \ell^-_{\{ij\}} \).

Next, for each edge \( \{i, j\} \) of color pair \( \{\iota, \jmath\} \), take a distinct edge interval of length \( \ell_e \) in the partition of \( B \), partition it further into \( 2 + 4 \) intervals, and put them in \( \mathcal{I}_2 \):

- one interval of each length \( \ell^+_{\{ij\}} \) and \( \ell^-_{\{ij\}} \),
- four intervals including
  - one interval of each length \( \ell^+_{\{ij\}} \) and \( \ell^-_{\{ij\}} \),
  - one interval of each length \( \ell^+_{\{ij\}} \) and \( \ell^-_{\{ij\}} \),
  - one interval of each length \( \ell^+_{\{ij\}} \) and \( \ell^-_{\{ij\}} \).

Finally, construct \( \hat{\kappa} \) intervals of length \( \ell_v \), and \( \binom{\hat{\kappa}}{2} \) intervals of length \( \ell_e \), all sharing the same left endpoint as \( B \), and put them in \( \mathcal{I}_3 \).

This completes the construction of \( B \) and \( \mathcal{I} \). Refer to Figure 3 for an example. Among the intervals in the partition of \( B \), the separator, vertex, and edge intervals are further partitioned to construct intervals in
Figure 3: Top: A graph of $\hat{n} = 6$ vertices and $\hat{m} = 7$ edges with $\hat{\kappa} = 3$ colors, where vertex $i$ has color $i \mod 3$. The three vertices 2, 3, 4 form a colored clique. Center: The $\hat{n} = 6$ vertex intervals and $\hat{m} = 7$ edge intervals are partitioned into the listed lengths, where the incidence lengths for the colored clique 2, 3, 4 are highlighted in red. Bottom: The highlighted lengths are complementary, and add up to the pairing length $\ell_p = 324$.

$I_0$, $I_1$, and $I_2$, respectively. Only the gap intervals are not used in the construction of $I$; their total length is $g_1 \cdot 3\ell + g_2 \cdot 12\ell$. On the other hand, the total length of the intervals in $I_3$ is

$$\kappa \cdot \ell_v + \binom{\kappa}{2} \cdot \ell_e = \hat{\kappa} \cdot (2 \cdot 3\ell + 2(\hat{\kappa} - 1) \cdot 5\ell) + \binom{\hat{\kappa}}{2} \cdot (2 \cdot 3\ell + 4 \cdot 7\ell) = \left(2\hat{\kappa} + 2 \cdot \binom{\hat{\kappa}}{2}\right) \cdot 3\ell + 4 \cdot \binom{\hat{\kappa}}{2} \cdot 12\ell = g_1 \cdot 3\ell + g_2 \cdot 12\ell.$$

Thus $\ell_I = \ell_B$, as expected.

For convenience we have used fractional intervals of length $\ell_f = \frac{1}{\tau + 1}$. By scaling with factor $\tau + 1$, the coordinates of all intervals can be converted to integers polynomial in $\hat{\kappa}$ and $\hat{n}$. Thus the reduction is strongly polynomial. Since $\tau = O(\kappa^2)$, the reduction is also FPT. We claim that $G$ has a colored clique of $\kappa$ vertices if and only if $B$ can be tiled with $I$ by moving $\tau$ intervals, if and only if $I$ can be joined into a contiguous interval in $\tau$ moves.

$$324 = 196 + 128 = 197 + 127 = 200 + 124 = 201 + 123 = 202 + 122 = 204 + 120 = 182 + 142 = 181 + 143 = 178 + 146 = 177 + 147 = 176 + 148 = 174 + 150$$
COLORED CLIQUE $\implies$ TILE Suppose that $G$ has a colored clique $K$ of $\kappa$ vertices. We will tile $B$ with $\mathcal{I}$ by moving $\tau$ intervals as follows. First move the $\kappa + \binom{\kappa}{2}$ intervals in $\mathcal{I}_3$ to cover the $\kappa$ vertex intervals and the $\binom{\kappa}{2}$ edge intervals inside $B$ corresponding to the vertices and edges in $K$. Next move the corresponding intervals in $\mathcal{I}_1$ and $\mathcal{I}_2$, including the $2 + 2(\kappa - 1)$ intervals composing each of these vertex intervals, and the $2 + 4$ intervals composing each of these edge intervals, to cover the $g_1 + g_2$ gap intervals: one interval of length $\ell^+_{\{i\}}$ or $\ell^+_{\{ij\}}$ for each type-1 gap interval; two intervals of complementary lengths $\ell^+_{ij} + \ell^-_{ij} = \ell_p$ or $\ell^-_{ij} + \ell^+_{ij} = \ell_p$, for each type-2 gap interval. The number of these intervals in $\mathcal{I}_1$ and $\mathcal{I}_2$ is

$$\kappa \cdot (2 + 2(\kappa - 1)) + \binom{\kappa}{2} \cdot (2 + 4) = 2\kappa + 2 \binom{\kappa}{2} + 2 \cdot 4 \binom{\kappa}{2} = g_1 + 2g_2.$$ 

The total number of moves is

$$\kappa + \binom{\kappa}{2} + g_1 + 2g_2 = 3\kappa + 3 \binom{\kappa}{2} + 8 \binom{\kappa}{2} = \tau.$$ 

Since $K$ is a colored clique, the intervals in $\mathcal{I}_1$ composing the $\kappa$ vertex intervals include one pair of intervals of lengths $\ell^+_{\{i\}}$ for each color $i$, and the intervals in $\mathcal{I}_2$ composing the $\binom{\kappa}{2}$ edge intervals include exactly one pair of intervals of lengths $\ell^+_{\{ij\}}$ for each unordered pair of colors $\{i, j\}$ with $i < j$, to cover the $g_1 = 2\kappa + 2 \binom{\kappa}{2}$ type-1 gap intervals of these lengths.

Moreover, by design of the complementary incidence lengths in partitioning vertex and edge intervals inside $B$ into intervals in $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively, the intervals composing the $\kappa$ vertex intervals and the $\binom{\kappa}{2}$ edge intervals also include, for each edge $\{i, j\}$ of color pair $\{i, j\}$ in $K$, one interval of each length $\ell^+_{ij}, \ell^-_{ij}, \ell^+_{ij}, \ell^-_{ij}$, forming four complementary pairs $\ell^+_{ij} + \ell^-_{ij} = \ell_{ij} + \ell^+_{ij} = \ell_{ij} + \ell^-_{ij} = \ell_{ij}$, to cover the $g_2 = 4 \binom{\kappa}{2}$ type-2 gap intervals of length $\ell_p$. Thus $B$ is tiled with $\mathcal{I}$ by moving $\tau$ intervals.

TILE $\implies$ JOIN Any tiling of $B$ with $\mathcal{I}$ necessarily joins $\mathcal{I}$ into a contiguous interval. Thus if $B$ can be tiled with $\mathcal{I}$ by moving $\tau$ intervals, then $\mathcal{I}$ can be joined into a contiguous interval in $\tau$ moves.

JOIN $\implies$ COLORED CLIQUE Suppose that $\mathcal{I}$ can be joined into a contiguous interval in $\tau$ moves. We will find a colored clique of $\kappa$ vertices in $G$.

By the same argument as in the preceding proof for parameter $\kappa$, we can assume without loss of generality that no fractional interval in $\mathcal{I}_0$ is moved. Then, to join the intervals in $\mathcal{I}$ into a contiguous interval, we must cover all gap intervals inside $B$.

All $\kappa + \binom{\kappa}{2}$ intervals in $\mathcal{I}_3$ contain the $\tau + 1$ fractional intervals in $\mathcal{I}_0$ composing the leftmost separator interval inside $B$, so they must be moved. Since their lengths, $\ell_e$ and $\ell_e$, are greater than the various lengths of the gap intervals inside $B$, they cannot be moved to fill the gaps directly. To fill the gaps, we have to use intervals in $\mathcal{I}_1$ and $\mathcal{I}_2$.

Recall that the color lengths $\ell^+_{\{i\}}$ and the color-pair lengths $\ell^+_{\{ij\}}$ differ from each other by factors less than two, and are smaller than the incidence lengths $\ell^+_{ij}$ and $\ell^-_{ij}$. Thus each type-1 gap interval must be covered by one interval of the same length in $\mathcal{I}_1$ or $\mathcal{I}_2$. Also recall that the pairing length $\ell_p$ of the type-2 gap intervals is greater than the color lengths, the color-pair lengths, and the incidence lengths of the intervals in $\mathcal{I}_1$ and $\mathcal{I}_2$. Thus each type-2 gap interval requires two moves to cover. In total, we need $g_1 + 2g_2$ moves to fill the gaps. Since $\tau = \kappa + \binom{\kappa}{2} + g_1 + 2g_2$, there is not a single move to waste. In summary, we must first move the $\kappa + \binom{\kappa}{2}$ intervals in $\mathcal{I}_3$ to cover some portions of vertex and edge intervals inside $B$, and then move exactly $g_1 + 2g_2$ intervals in $\mathcal{I}_1$ and $\mathcal{I}_2$ composing these covered portions to cover the $g_1 + g_2$ gap intervals inside $B$. 

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First consider the $g_1 = 2\hat{\kappa} + 2\binom{\hat{\kappa}}{2}$ type-1 gap intervals. Each type-1 gap interval of a color (respectively, color-pair) length must be covered by one interval of the same length in $I_1$ (respectively, $I_2$), which was constructed from some vertex (respectively, edge) interval inside $B$. Since there is a type-1 gap interval of every color and color-pair length, the $\hat{\kappa} + \binom{\hat{\kappa}}{2}$ intervals in $I_3$ must cover exactly $\hat{\kappa}$ vertex intervals and $\binom{\hat{\kappa}}{2}$ edge intervals inside $B$, and the corresponding $\hat{\kappa}$ vertices and $\binom{\hat{\kappa}}{2}$ edges in $G$ must span all $\hat{\kappa}$ colors and all $\binom{\hat{\kappa}}{2}$ color pairs.

Next consider the $g_2 = 4\binom{\hat{\kappa}}{2}$ type-2 gap intervals. Among the intervals in $I_1$ and $I_2$ composing the $\hat{\kappa}$ vertex intervals and $\binom{\hat{\kappa}}{2}$ edge intervals inside $B$, the intervals of color and color-pair lengths are used to cover the type-1 gap intervals, so the intervals of incidence lengths are left to cover the type-2 gap intervals. They include exactly $\hat{\kappa} \cdot 2(\hat{\kappa} - 1) = g_2$ intervals of incidence lengths $\hat{\ell}_{ij}^\pm$ in $I_1$, and exactly $\binom{\hat{\kappa}}{2} \cdot 4 = g_2$ intervals of incidence lengths $\hat{\ell}_{ij}^\pm$ in $I_2$. Recall that $4\ell < \hat{\ell}_{ij}^+ < 6\ell < \hat{\ell}_{ij}^- < 8\ell$. Thus each type-2 gap interval, of length $\ell_p = 12\ell$, must be covered by two intervals of complementary lengths, either $\hat{\ell}_{ij}^+ + \hat{\ell}_{ij}^- = \ell_p$ or $\hat{\ell}_{ij}^- + \hat{\ell}_{ij}^+ = \ell_p$, with matching subscript $ij$. Such pairings require the edge $\{i, j\}$ of color pair $\{\hat{i}, \hat{j}\}$ be consistent with the vertices $i'$ and $j'$ of the corresponding colors $i$ and $j$, that is, $i = i'$ and $j = j'$. Thus the $\hat{\kappa}$ vertices and the $\binom{\hat{\kappa}}{2}$ edges form a colored clique.

This completes the proof of Theorem 1.

Remark. The four problems PACK, COVER, J-PACK, and TILE have a variant where $B$ is a simple closed curve, and $I$ is a family of intervals on the curve $B$. Since the family $I$ of intervals are inside the interval $B$ in our reductions for the proof of Theorem 1, these reductions can be easily adapted to the closed-curve variant and yield similar hardness results.

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A  Torty and Shields

Torty the sea turtle is fending off invading jellyfish! Torty has an army of \( n \) turtles standing in a line, each holding a shield of length \( \ell \). Initially, the shield of the \( i \)-th turtle is at the interval \([x_i, x_i + \ell]\). Torty wants the turtles to arrange their defense positions so that their shields concatenate into a contiguous interval of length \( n \cdot \ell \) inside the battle interval \([0, b]\).

Turtles are serene creatures. They would rather bask in the sun than move around. But once moving, they can go any distance. What is the minimum number of turtles that have to move to form the target configuration?

Torty entrusts this important problem to you. Solve it quickly!

Input
The first line contains three integers \( n, \ell, \) and \( b \), where \( 1 \leq n \leq 2 \cdot 10^5 \), \( 1 \leq \ell, b \leq 10^9 \), and \( n \cdot \ell \leq b \). The second line contains \( n \) integers \( x_1, x_2, \ldots, x_n \), where \(-10^9 \leq x_i \leq 10^9 \). The shields may overlap at their initial positions.

Output
Print the minimum number of turtles that have to move.

Sample input

\[
6 \ 2 \ 13 \\
-1 \ 3 \ 4 \ 5 \ 12 \ 11
\]

Sample output

\[
3
\]

Note
The best way is to move the turtle with shield at \([-1, 1)\) to \([1, 3)\), the turtle with shield at \([4, 6)\) to \([7, 9)\), and the turtle with shield at \([12, 14)\) to \([9, 11)\). Then all shields are joined into a contiguous interval \([1, 13)\) inside the battle interval \([0, 13]\).
B Blue Puppy and UFOs

Blue Puppy is looking for his little brother Torty. It’s time to go home, but Torty is still playing among sea turtles and jellyfish on the beach, which is an interval $[0, b)$.

Blue Puppy has $n$ UFOs hovering above the beach, where the $i$-th UFO covers an interval $[x_i, x_i + \ell)$ with surveillance cameras.

What is the minimum number of UFOs that have to move so that the $n$ UFOs together cover the whole length of the beach?

Input

The first line contains three integers $n$, $\ell$, and $b$, where $1 \leq n \leq 10^5$, $1 \leq \ell, b \leq 10^9$, and $n \cdot \ell \geq b$. The second line contains $n$ integers $x_1, x_2, \ldots, x_n$, where $-10^9 \leq x_i \leq 10^9$. The intervals covered by the UFOs may overlap at their initial positions.

Output

Print the minimum number of UFOs that have to move.

Sample input

8 2 10
-1 -2 3 4 5 8 9 10

Sample output

2

Note

One of the best ways is to move the UFO at $[4, 6)$ to $[1, 3)$, and move the UFO at $[10, 12)$ to $[6, 8)$. Then the 8 UFOs together cover the interval $[-2, 11)$, which contains the beach $[0, 10)$. 