ON STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH RAPIDLY OSCILLATING COEFFICIENTS

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Abstract. The paper deals with the functional differential equation

\[ y'(t) + \int_0^\infty \mu_0(ds) y(t-s) + \sum_{k=1}^{\infty} e^{i\omega_k t} \int_0^\infty \mu_k(ds) y(t-s) = f(t), \]

where the functions \( y \) and \( f \) take their values in a Hilbert space, \( \omega_k \in \mathbb{R} \), \( \mu_k \) are bounded operator-valued measures concentrated on \([0, +\infty)\), and \( \sum_{k=1}^{\infty} \|\mu_k\| < \infty \). It is shown that the equation is stable provided the unperturbed equation \( y'(t) + \int_0^\infty \mu_0(ds) y(t-s) = f(t) \) is at least strictly passive (and consequently stable) and a special estimate holds; this estimate is certainly true if \( |\omega_k| \) are sufficiently large.

1. Introduction. The paper describes (Theorem 3.1) conditions that guarantee the input-output stability of a linear functional differential equation with almost periodic coefficients and infinite delay. The first condition is close (Proposition 2.2) to the stability of an unperturbed equation with constant coefficients; in the case of the equation without delay, this condition is equivalent (Corollary 3.4) to stability. The second condition consists in the special estimate (6). This estimate is certainly satisfied (Remark 3) if the frequencies \( \omega_k \) of oscillation of coefficients are sufficiently large in absolute value which is in agreement with the idea of the averaging principle.

The averaging principle \([16, 28, 32]\) says that rapidly oscillating terms of a differential equation may be omitted when the stability problem is investigated. Most of assertions connected with the averaging principle are qualitative, i.e., they do not explain how rapidly must oscillate a term of an equation in order to become unessential. In this connection we note that in some cases estimate (6) can be effectively verified, see Examples 3 and 4.

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Earlier versions of our result were presented in [18, 19]: paper [18] is devoted to ordinary differential equations, in paper [19] the case of feedback systems is considered (without proofs). In this paper we discuss the case of a functional differential equation for a function $y$ taking its values in an arbitrary Hilbert space.

The consideration of a functional differential equation is more complicated, because the stability of ordinary differential equation (8) implies that the spectrum of the values of the characteristic function $\omega \mapsto \mathrm{i}\omega 1 + a$ is contained in the right half-plane, while for the characteristic function $\omega \mapsto \mathrm{i}\omega 1 + \hat{\mu}_0(\omega)$ of functional differential equation (4) this is not the case, see Examples 1 and 2. Therefore we are forced to impose an additional assumption (the spectrum of the operator $\mathrm{i}\omega 1 + \hat{\mu}_0(\omega)$, $\omega \in \mathbb{R}$, does not intersect the semi-axis $(-\infty, 0]$, see Theorem 3.1) which we need in order to define the square root of operator (9). For other results connected with averaging in functional differential equations, see [9, 22, 23] and references therein.

Roughly speaking, the main ideas of the proof are as follows. The stability of the equation $L y = f$ is equivalent to the causal invertibility of the operator $L$ (Proposition 4.1). If the operator $M$ is small in the sense that $\|ML - 1\| < 1$, the causal invertibility of $L$ implies the causal invertibility of $L + M$ (cf. Proposition 4.2) and, consequently, implies the stability of the equation $(L + M)y = f$. It is important that the number $\|ML^{-1}\|$ depends on the spaces in which the operators $L$ and $M$ are considered (nevertheless, the causal invertibility and stability usually do not depend on the underlying spaces). The number $\|ML^{-1}\|$ is smallest if one considers the differential operator $L$ as acting from $H^{1/2}$ to $H^{-1/2}$, where $H^s$ are the Sobolev spaces [24]. In order to further reduce the number $\|ML^{-1}\|$ we use the number $\|L^{-1/2}ML^{-1/2}\|$ instead of it: thus, the proof implicitly uses square roots of differential operators, which are fractional differential operators [14, 27]. Since our equation contains factors $e^{\mathrm{i}\omega t}$ and convolutions, it is convenient to use the Fourier transform. Since we want the Fourier transform to possess good properties, we require that the unknown function $y$ takes its values in a Hilbert (not Banach) space. Close methods (utilising other functional spaces and other estimates) were published in [6, 7, 36].

In Section 2 we describe the equation under investigation and some facts connected with the notion of stability. In Section 3 the main result (Theorem 3.1) is formulated and discussed. Section 4 is devoted to its proof.

2. Preliminaries. In this Section we recall the definition of an operator-valued measure and describe the main equation. We also discuss some properties of stability.

Let $H$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ induced by it, and let $B(H)$ be the algebra of all bounded linear operators acting in $H$. We denote by $a^*$ the adjoint of $a \in B(H)$, and we denote by $1$ the identity operator.

A bounded operator-valued measure on $\mathbb{R}$ is the sum

$$
\mu = \sum_{m=1}^{\infty} a_m \mu_m, \tag{1}
$$

where $a_m \in B(H)$ and $\mu_m$ are bounded complex measures on $\mathbb{R}$ satisfying the estimate $\sum_{m=1}^{\infty} \|a_m\| \cdot \|\mu_m\| < \infty$; we recall that a complex measure $\mu$ is called [2, ch. III, § 1.8] bounded if the supremum of $|\int y(t)\mu(dt)|$ over all continuous functions $y : \mathbb{R} \to \mathbb{C}$ with compact support such that $|y(t)| \leq 1$ for all $t$ is finite. For a function
$y : [\alpha, \beta] \to \mathbb{H}$ or $y : [\alpha, \beta] \to \mathbb{C}$, the integral with respect to $\mu$ is defined as

$$\int_{\alpha}^{\beta} \mu(dt)y(t) = \sum_{m=1}^{\infty} a_m \int_{\alpha}^{\beta} y(t)\mu_m(dt).$$

We stress that, if $y$ takes its values in $\mathbb{H}$, the result of the integration is a vector from $\mathbb{H}$, but if $y$ is a scalar function, then the result of the integration is an operator acting in $\mathbb{H}$.

The representation (1) is not unique: we do not distinguish measures that give the same integrals $\int_{\alpha}^{\beta} \mu(dt)y(t)$ for all $y$. We define $\|\mu\|$ to be the infimum of $\sum_{m=1}^{\infty} \|a_m\| \cdot \|\mu_m\|$ over all representations (1). Clearly,

$$\left\| \int_{\alpha}^{\beta} \mu(dt)y(t) \right\| \leq \|\mu\| \sup_{t \in [\alpha, \beta]} \|y(t)\|.$$

We denote the set of all bounded operator-valued measures by $\mathcal{M} = \mathcal{M}(\mathbb{R}, \mathcal{B}(\mathbb{H}))$.

We denote by $\mathcal{M}^+ = \mathcal{M}^+(\mathbb{R}, \mathcal{B}(\mathbb{H}))$ the set of all bounded operator-valued measures $\mu \in \mathcal{M}$ that possess at least one representation (1) with $\mu_m$ concentrated on $[0, +\infty)$.

For more about operator-valued measures, see [17, § 4.4].

We set $\mathbb{C}^+ = \{ \lambda \in \mathbb{C} : \text{Re} \lambda > 0 \}$, $\overline{\mathbb{C}^+} = \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq 0 \}$.

We define the Laplace and Fourier transforms of $\mu \in \mathcal{M}^+$ by the formulas

$$\tilde{\mu}(\lambda) = \int_{0}^{+\infty} e^{-\lambda t} \mu(dt), \quad \lambda \in \overline{\mathbb{C}^+},$$

$$\hat{\mu}(\omega) = \int_{0}^{+\infty} e^{-i\omega t} \mu(dt), \quad \omega \in \mathbb{R}.$$

Clearly,

$$\tilde{\mu}(i\omega) = \hat{\mu}(\omega), \quad \omega \in \mathbb{R},$$

$$\|\tilde{\mu}(\lambda)\| \leq \|\mu\|, \quad \lambda \in \overline{\mathbb{C}^+}. \quad (2)$$

We consider the functional differential equation [10, 13, 15, 17]

$$y'(t) + (\mu_0 * y)(t) + \sum_{k=1}^{\infty} e^{i\omega_k t} (\mu_k * y)(t) = f(t), \quad (3)$$

where $\mu_k \in \mathcal{M}^+$, $\sum_{k=1}^{\infty} \|\mu_k\| < \infty$, $\omega_k \in \mathbb{R}$, and the symbol $*$ means the convolution, i.e.,

$$(\mu_k * y)(t) = \int_{0}^{\infty} \mu_k(ds) y(t-s).$$

We interpret equation (3) as a perturbed one with respect to the equation

$$y'(t) + (\mu_0 * y)(t) = f(t). \quad (4)$$

The initial value problem for equation (3) on $[a, b]$, $-\infty < a < b < +\infty$, has the form

$$y'(t) + (\mu_0 * y)(t) + \sum_{k=1}^{\infty} e^{i\omega_k t} (\mu_k * y)(t) = f(t), \quad a < t \leq b,$$

$$y(t) = \varphi(t), \quad t \leq a.$$
For definiteness, we assume that $f$ and $\varphi$ are bounded continuous functions taking their values in $H$, and the solution $y$ must be continuous on $(-\infty, b]$ and differentiable on $[a, b)$. It is well known [10, 13, 15, 17] that for any bounded continuous functions $f$ and $\varphi$ and any $-\infty < a < b < +\infty$ the solution of the initial value problem exists and is unique.

We say that equation (3) is input-output stable if there exists $K < \infty$ such that for all bounded continuous functions $f$ and $\varphi$ and any $-\infty < a < b < +\infty$ the solution of the initial value problem satisfies the estimate

$$\sup_{a < t \leq b} \left(\|y'(t)\| + \|y(t)\|\right) \leq K \left(\sup_{t \leq a} \|\varphi(t)\| + \sup_{a < t \leq b} \|f(t)\|\right).$$

Remark 1. It can be shown [17, theorem 3.5.9] that input-output stability of equation (3) is equivalent to exponential stability provided measures $\mu_k$ exponentially decrease at infinity, namely, $\sum_{k=0}^{\infty} \|\mu_k\| < \infty$, where $\int_{0}^{\infty} e^{\nu s} \mu_k(ds) y(s) = \int_{0}^{\infty} e^{\nu s} \mu_k(ds) y(s)$ for some $\nu > 0$.

The following assertion is well known.

**Proposition 2.1** (see, e.g., [17, theorem 4.5.9(c)]). Equation (4) is input-output stable if and only if the operator

$$\lambda I + \tilde{\mu}_0(\lambda)$$

is invertible for all $\lambda \in \mathbb{C}^+$.  

In the following assertion we show that assumption (i) of our main Theorem 3.1 (see below) implies the input-output stability.

**Proposition 2.2.** Let for all $\omega \in \mathbb{R}$, the spectrum of the operator

$$i \omega I + \tilde{\mu}_0(\omega)$$

do not intersect the semi-axis $(-\infty, 0]$. Then for all $\lambda \in \mathbb{C}^+$, the spectrum of the operator

$$\lambda I + \tilde{\mu}_0(\lambda)$$

does not intersect the semi-axis $(-\infty, 0]$ as well and, consequently (by Proposition 2.1), equation (4) is input-output stable.

**Proof.** By estimate (2), the spectrum of the operator $\tilde{\mu}_0(\lambda)$ is contained in the circle $\{z : |z| \leq \|\mu_0\|\}$. Hence the spectrum of the operator $\lambda I + \tilde{\mu}_0(\lambda)$ is contained in the circle $\{z : |z - \lambda| \leq \|\mu_0\|\}$. Consequently, the spectra of the operators $\lambda I + \tilde{\mu}_0(\lambda)$, where $\lambda \in \mathbb{C}^+$ and $|\lambda| \geq \|\mu_0\| + 1$, are contained in the set

$$\bigcup_{\lambda \in \mathbb{C}^+, |\lambda| \geq \|\mu_0\| + 1} \{z : |z - \lambda| \leq \|\mu_0\|\},$$

which does not intersect the semi-axis $(-\infty, 0]$. Thus, it remains to show that the set

$$\Sigma = \bigcup_{\lambda \in K} \sigma(\lambda I + \tilde{\mu}_0(\lambda)),$$

where $K$ is the semicircle

$$K = \{\lambda : \lambda \in \mathbb{C}^+, |\lambda| \leq \|\mu_0\| + 1\},$$

does not intersect the semi-axis $(-\infty, 0]$.  

We consider the set
\[ \hat{\Sigma} = \bigcup_{|\omega| \leq \|\mu_0\| + 1} \sigma(i\omega \mathbf{1} + \hat{\mu}_0(\omega)). \]

We observe that, by the assumption of the Proposition, the set \( \hat{\Sigma} \) does not intersect the semi-axis \((-\infty, 0] \).

We denote by \( C = C(K, \mathcal{B}(\mathbb{H})) \) the Banach algebra \([1, 12, 26]\) of all continuous functions \( \psi : K \to \mathcal{B}(\mathbb{H}) \) with the pointwise operations and the norm \( \|\psi\| = \max_{\lambda \in K} \|\psi(\lambda)\| \). The spectrum \( \sigma_C(\psi) \) of \( \psi \in C \) in the Banach algebra \( C \) is clearly the set
\[ \sigma_C(\psi) = \bigcup_{\lambda \in K} \sigma(\psi(\lambda)). \]

We note that the boundary \( \Gamma \) of \( K \) is the union of the sets
\[ \Gamma_1 = [-i(\|\mu_0\| + 1), i(\|\mu_0\| + 1)], \]
\[ \Gamma_2 = \{ \lambda : |\lambda| = \|\mu_0\| + 1 \} \cap \mathbb{C}^+. \]

We also note that
\[ \hat{\Sigma} = \sigma_C(\psi_0), \]
\[ \tilde{\Sigma} = \bigcup_{\lambda \in \Gamma_1} \sigma(\psi_0(\lambda)), \]
where
\[ \psi_0(\lambda) = \lambda \mathbf{1} + \hat{\mu}_0(\lambda). \]

With this notation, it suffices to show that \( \sigma_C(\psi_0) \) does not intersect the semi-axis \((-\infty, 0] \).

We denote by \( A = A(K, \mathcal{B}(\mathbb{H})) \) the subset of \( C \) consisting of all functions \( \psi \in C \) that are holomorphic in the interior of \( K \). Clearly, \( A \) is a closed subalgebra of \( C \).

We make use of the following Theorem from [20]. *Let a closed subset \( \Gamma \subseteq K \) possess the Property: for any \( \psi \in A \) the function \( \lambda \mapsto \|\psi(\lambda)\| \) attains its maximal value on \( \Gamma \). Then for any \( \psi \in A \) we have*
\[ \sigma_C(\psi) \subseteq \text{po}\left( \bigcup \{ \sigma(\psi(\lambda)) : \lambda \in \Gamma \} \right), \]

*where, for a compact set \( E \subseteq \mathbb{C} \), \( \text{po}(E) \) means the union of \( E \) and all the bounded connected components of the complement \( \mathbb{C} \setminus E \).*

Let us verify that the Property holds for the boundary \( \Gamma \) of \( K \). For any \( \psi \in A \) and bounded linear functional \( l \) on \( \mathcal{B}(\mathbb{H}) \), the maximum value of the function \( \lambda \mapsto |l(\psi(\lambda))|, \lambda \in K \), occurs on \( \Gamma \) by the maximum modulus theorem [8, Corollary 5.4.3]. Now, the Property follows from the Hanh–Banach theorem [12, theorem 2.7.4].

Applying [20] to the function \( \psi_0 \) we obtain
\[ \sigma_C(\psi_0) \subseteq \text{po}\left( \bigcup \{ \sigma(\psi_0(\lambda)) : \lambda \in \Gamma \} \right). \]

Therefore, if \( \bigcup \{ \sigma(\psi_0(\lambda)) : \lambda \in \Gamma \} \) does not intersect \((-\infty, 0] \), then \( \sigma_C(\psi_0) \) does not intersect \((-\infty, 0] \) as well.

Clearly, the set
\[ \bigcup_{\lambda \in \Gamma} \sigma(\psi_0(\lambda)) \]
is the union of the sets
\[
\bigcup_{\lambda \in \Gamma_1} \sigma(\psi_0(\lambda)), \quad \bigcup_{\lambda \in \Gamma_2} \sigma(\psi_0(\lambda)).
\]
The former set is the set \(\hat{\Sigma}\), which does not intersect \((-\infty, 0]\). The latter one does not intersect \((-\infty, 0]\) as it was shown in the beginning of the proof.

An operator \(a \in B(H)\) is called strictly positive definite if it is self-adjoint and
\[
(a x, x) \geq \kappa \|x\|^2, \quad x \in H,
\]
for some \(\kappa > 0\). It is well known that the spectrum of a strictly positive definite operator \(a\) is contained in the half-line \(\mathbb{R}^+ = \{ \lambda \in \mathbb{R} : \lambda \geq \kappa \}\), where \(\kappa > 0\) satisfies (5).

An operator-valued measure \(\mu_0 \in M^+\) is called strictly passive if the operator \((\tilde{\mu}_0(\lambda) + \tilde{\mu}_0(\lambda)^*)/2\) is strictly positive definite for all \(\lambda \in \mathbb{C}^+\). From the physical point of view, the equation \(y'(t) + (\mu_0 * y)(t) = f(t)\) with a passive measure \(\mu_0\) describes a system without energy sources. Passivity leads to stability, see Proposition 2.3.

For more information on passivity, including its applications, see [4, 25, 33] and references therein.

**Proposition 2.3.** If the operator-valued measure \(\mu_0\) in (4) is strictly passive, then the spectrum of the operator \(\lambda 1 + \tilde{\mu}_0(\lambda)\) does not intersect the semi-plane \(\mathbb{C}^- = \{ z \in \mathbb{C} : \text{Re} z \leq 0 \}\) for all \(\lambda \in \mathbb{C}^+\). In particular, the assumption of Proposition 2.2 is satisfied and equation (4) is input-output stable.

**Proof.** Let \(\lambda \in \mathbb{C}^+\). We set \(a = \tilde{\mu}_0(\lambda)\) for brevity. We have
\[
\text{Re}(a \psi, \psi) = \frac{(a \psi, \psi) + (a \psi, \psi)}{2} = \frac{(a \psi, \psi) + (a^* \psi, \psi)}{2} = \frac{\langle a + a^* \psi, \psi \rangle}{2} \geq \kappa \|\psi\|^2, \quad \psi \in H,
\]
for some \(\kappa > 0\).

Consequently, if \(\text{Re} z \leq 0\) and \(\text{Re} \lambda \geq 0\), then \(-\text{Re} \langle z 1 - \lambda 1 - a \psi, \psi \rangle \geq \kappa \|\psi\|^2\) for some \(\kappa > 0\). Therefore the kernel of \(z 1 - \lambda 1 - a\) consists of zero and the image of \(z 1 - \lambda 1 - a\) is closed.

The same reasoning shows that the kernel of \((z 1 - \lambda 1 - a)^*\) consists of zero. Hence [26, theorem 12.10] the closure of the image of \(z 1 - \lambda 1 - a\) coincides with \(H\).

Thus, the spectrum of the operator \(\lambda 1 + \tilde{\mu}_0(\lambda)\) does not intersect the semi-plane \(\mathbb{C}^-\) for all \(\lambda \in \mathbb{C}^+\).

**Example 1.** We consider the functional differential equation
\[
y'(t) + y(t - 1) = f(t).
\]

The values of the characteristic function
\[
\varphi_2(\omega) = i \omega + e^{-i \omega}, \quad \omega \in \mathbb{R},
\]
do not intersect the semi-axis \((-\infty, 0]\), though some values of \(\varphi_2(\omega)\) with \(\omega \in \mathbb{R}\) have negative real parts (for example, \(\varphi_2(\omega) = -1 + i \pi\) for \(\omega = \pi\)); see fig. 1(a). Thus, the assumption of Proposition 2.2 is satisfied, and, in particular, the equation is input-output stable.
Example 2. We consider the functional differential equation
\[ y'(t) - 0.82 y(t) + y(t - 1) + 0.1 y(t - 9.2) = f(t). \]
The values of the characteristic function
\[ \varphi_3(\omega) = i \omega - 0.82 + e^{-i \omega} + 0.1 e^{-9.2 i \omega}, \quad \omega \in \mathbb{R}, \]
intersect the semi-axis \((-\infty, 0];\) see fig. 1(b). Nevertheless,
\[ \lambda - 0.82 + e^{-\lambda} + 0.1 e^{-9.2 \lambda} \neq 0, \quad \lambda \in \mathbb{C}^+. \]
Thus, the assumption of Proposition 2.2 is not satisfied, though the equation is input-output stable by Proposition 2.1.

3. The main result. In this Section we formulate our main result (Theorem 3.1). We recall that we discuss the stability of the equation
\[ y'(t) + (\mu_0 * y)(t) + \sum_{k=1}^{\infty} e^{i \omega_k t} (\mu_k * y)(t) = f(t), \]
which we interpret as a perturbed one with respect to the equation
\[ y'(t) + (\mu_0 * y)(t) = f(t). \]
Theorem 3.1 assumes that the unperturbed equation satisfies assumption (i) that is slightly stronger than stability (see Proposition 2.2) and the perturbation is small in the sense of estimate (6). Estimate (6) implies (see Remark 3) that the greater the frequency \(\omega_k\) the large \(\mu_k\) is allowed; this conclusion is analogous to the idea of the averaging principle.

**Theorem 3.1.** Assume that the following assumptions are fulfilled.
(i) For all \(\omega \in \mathbb{R}\) the spectrum of the operator
\[ i \omega \mathbf{1} + \hat{\mu}_0(\omega) \]
does not intersect the semi-axis \((-\infty, 0];\)
(ii) The estimate
\[ \sum_{k=1}^{\infty} \sup_{\omega \in \mathbb{R}} \| \sqrt{\Phi(\omega + \omega_k)} \hat{\mu}_k(\omega) \sqrt{\Phi(\omega)} \| < 1 \] (6)
holds, where
\[ \Phi(\omega) = \left( i\omega \mathbf{1} + \hat{\mu}_0(\omega) \right)^{-1}, \]
and \( \sqrt{\cdot} \) means a holomorphic branch of the square root function defined in \( \mathbb{C} \) with the cut along the semi-axis \( (-\infty, 0] \).

Then equation (3) is input-output stable.

We prove Theorem 3.1 in the next Section.

**Remark 2.** We recall [1, 12, 26] that the square root of an operator \( a \in \mathcal{B}(\mathbb{E}) \), where \( \mathbb{E} \) is a Banach space, is defined by the formula
\[ \sqrt{a} = \frac{1}{2\pi i} \int_{\Gamma} \sqrt{\lambda} \left( \lambda \mathbf{1} - a \right)^{-1} d\lambda, \]
where \( \Gamma \) surrounds the spectrum \( \sigma(a) \) of \( a \). The operator \( \sqrt{a} \) exists if \( \sigma(a) \) does not intersect \( (-\infty, 0] \). In particular, in our case
\[ \Phi(\omega) = \frac{1}{2\pi i} \int_{\Gamma_1} \sqrt{\lambda} \left[ (\lambda - i\omega) \mathbf{1} - \hat{\mu}_0(\omega) \right]^{-1} d\lambda, \]
where \( \Gamma_1 \) surrounds the spectrum of \( (i\omega \mathbf{1} + \hat{\mu}_0(\omega))^{-1} \), and \( \Gamma \) surrounds the spectrum of \( i\omega \mathbf{1} + \hat{\mu}_0(\omega) \). These formulas will not be explicitly used in our exposition.

**Corollary 3.2.** Let for all \( \omega \in \mathbb{R} \) the spectrum of the operator
\[ i\omega \mathbf{1} + \hat{\mu}_0(\omega) \]
do not intersect the semi-axis \( (-\infty, 0] \) and
\[ \sum_{k=1}^{\infty} \sup_{\omega \in \mathbb{R}} \left\| \sqrt{\Phi(\omega + \omega_k)} \right\| \cdot \left\| \Phi(\omega) \right\| \cdot \| \mu_k \| < 1. \tag{7} \]

Then equation (3) is input-output stable.

**Proof.** Evidently, estimate (7) implies estimate (6).

**Remark 3.** It is easy to see that \( \left\| \Phi(\omega) - \frac{1}{\omega} \mathbf{1} \right\| \to 0 \) as \( \omega \to \infty \). Therefore, \( \left\| \sqrt{\Phi(\omega)} \right\| \to 0 \) as \( \omega \to \infty \). Consequently, \( \sup_{\omega \in \mathbb{R}} \left\| \sqrt{\Phi(\omega + \omega_k)} \right\| \cdot \left\| \Phi(\omega) \right\| \to 0 \) as \( \omega_k \to \infty \). Thus, estimate (7) certainly holds provided \( |\omega_k| \) are sufficiently large.

**Corollary 3.3.** Let the measure \( \mu_0 \) from (3) be strictly passive, and let estimate (6) or (7) hold. Then equation (3) is input-output stable.

**Proof.** The proof follows from Proposition 2.3 and Theorem 3.1.

In Example 2 we have seen that assumption (i) of Theorem 3.1 is stronger than input-output stability. The following corollary shows that in the case of ordinary differential equation this assumption can be (formally) weakened to the input-output stability.

**Corollary 3.4.** Let equation (4) have the form
\[ y'(t) + ay(t) = f(t), \tag{8} \]
where \( a \in \mathcal{B}(\mathbb{E}) \), and be input-output stable, and let inequality (6) or (7) hold, where
\[ \Phi(\omega) = (i\omega \mathbf{1} + a)^{-1}. \]

Then equation (3) is input-output stable.
Proof. We observe that equation (8) can be written in the form (4) if we take
\( \mu_0 = a \delta \), where \( \delta \) is the Dirac measure, i.e., \( \int_0^\infty y(t) \, \delta(t) = y(0) \). In this case
\( \hat{\mu}_0(\omega) = a \) for all \( \omega \in \mathbb{R} \). We recall that equation (8) is input-output stable if and only if the spectrum \( \sigma(a) \) of \( a \) is contained in \( \mathbb{C}^+ \). Clearly, in such a case the spectrum of the operator \( i\omega 1 + \hat{\mu}_0(\omega) = i\omega 1 + a \) does not intersect the semi-axis \( (-\infty, 0] \). Thus, the assumptions of Theorem 3.1 are satisfied. \( \square \)

Example 3. Let us consider the equation
\[
y'(t) + ay(t) + b \cos(\omega_1 t) \, y(t - h) = f(t)
\]
with \( a > 0, b \in \mathbb{C}, \omega_1 \in \mathbb{R}, \) and \( h \geq 0 \). The unperturbed equation \( y'(t) + ay(t) = f(t) \) is obviously input-output stable. We rewrite the equation in the form
\[
y'(t) + a(\delta * y)(t) + \frac{b}{2} e^{i\omega t} (\delta_h * y)(t) + \frac{b}{2} e^{-i\omega t} (\delta_h * y)(t) = f(t),
\]
where \( \delta_h \) is the shifted Dirac measure, i.e., \( \int_0^\infty y(s) \, \delta_h(ds) = y(h) \). Clearly, \( \|\delta_h\| = 1 \).

We note that
\[
\delta_h(\omega) = e^{-i\omega h}.
\]
In our case,
\[
\Phi(\omega) = (i\omega + a)^{-1}.
\]
Therefore, estimate (7) takes the form
\[
sup_{\omega \in \mathbb{R}} \left| \sqrt{(i(\omega + \omega_1) + a)^{-1}} \right| \cdot \frac{b}{2} \cdot \left| \sqrt{(i\omega + a)^{-1}} \right| + sup_{\omega \in \mathbb{R}} \left| \sqrt{(i(\omega - \omega_1) + a)^{-1}} \right| \cdot \frac{b}{2} \cdot \left| \sqrt{(i\omega + a)^{-1}} \right| < 1.
\]
Or equivalently (after shifting by \( \pm \omega_1/2 \)),
\[
sup_{\omega \in \mathbb{R}} \frac{|b|}{\sqrt{a^2 + (\omega - \omega_1/2)^2} \sqrt{a^2 + (\omega + \omega_1/2)^2}} < 1.
\]
Or equivalently (after finding the points of maximum),
\[
\max \left\{ \frac{|b|}{\sqrt{a^2 + \omega_1^2/4}}, \frac{|b|}{\sqrt{a\omega_1}} \right\} < 1.
\]
Or equivalently (after expressing \( |b| \) from the above inequality),
\[
|b| < \min \left\{ \sqrt{a^2 + \omega_1^2/4}, \sqrt{a\omega_1} \right\}.
\]
Thus, the equation is input-output stable provided the last estimate is satisfied.

Example 4. Let us consider the equation
\[
y'(t) + ay(t) + b \sin(\omega_1 t) \, (y(t - h_1) - y(t - h_2)) = f(t)
\]
with \( a = 1, b = 3, \omega_1 = 15, h_1 = 1, \) and \( h_2 = 2 \). The unperturbed equation \( y'(t) + ay(t) = f(t) \) is input-output stable. We rewrite the equation in the form
\[
y'(t) + a(\delta * y)(t) + \frac{b}{2} e^{i\omega t} ((\delta_{h_1} - \delta_{h_2}) * y)(t) - \frac{b}{2} e^{-i\omega t} ((\delta_{h_1} - \delta_{h_2}) * y)(t) = f(t).
\]
In our case, estimate (6) takes the form
\[
\sup_{\omega \in \mathbb{R}} \left| \sqrt{i(\omega + \omega_1) + a}^{-1} \cdot \frac{b}{2} (e^{-i\omega h_1} - e^{-i\omega h_2}) \right| \cdot \left| \sqrt{i\omega + a}^{-1} \right| + \sup_{\omega \in \mathbb{R}} \left| \sqrt{i(\omega - \omega_1) + a}^{-1} \cdot \frac{b}{2} (e^{-i\omega h_1} - e^{-i\omega h_2}) \right| \cdot \left| \sqrt{i\omega + a}^{-1} \right| < 1.
\]
Numerical calculations show that the sum of these suprema approximately equals 0.982376. Thus, the equation is input-output stable.

For more complicated applications, it may be required a numerical calculation of the square root of an operator. We refer to [11] and references therein for this topic.

4. The proof of Theorem 3.1. This Section is devoted to the proof of Theorem 3.1.

We denote by \( C = C(\mathbb{R}, \mathbb{H}) \) the Banach space of all bounded continuous functions \( y : \mathbb{R} \to \mathbb{H} \) equipped with the norm \( \| y \| = \| y \|_C = \sup_{t \in \mathbb{R}} \| y(t) \| \). We denote by \( C^1 = C^1(\mathbb{R}, \mathbb{H}) \) the Banach space of all continuously differentiable functions \( y \in C \) such that \( y' \in C \) with the norm \( \| y \| = \| y \|_{C^1} = \| y' \|_C + \| y \|_C \).

We associate with equation (3) the operator
\[
(\mathcal{L}y)(t) = y'(t) + (\mu_0 * y)(t) + \sum_{k=1}^{\infty} e^{i\omega_k t} (\mu_k * y)(t).
\] (9)
It is easy to see that \( \mathcal{L} \) continuously acts from \( C^1 \) to \( C \). We represent \( \mathcal{L} \) as the sum
\[
\mathcal{L} = \mathcal{L}_0 + \sum_{k=1}^{\infty} \mathcal{M}_k,
\]
where
\[
(\mathcal{L}_0 y)(t) = y'(t) + (\mu_0 * y)(t),
\]
\[
(\mathcal{M}_k y)(t) = e^{i\omega_k t} (\mu_k * y)(t).
\]
We note that \( \mathcal{L}_0 \) corresponds to equation (4).

A linear operator \( \mathcal{A} \) acting from one space of functions (or distributions) defined on \( \mathbb{R} \) to another one is called causal [4, 17, 33, 35] if for all functions \( y \) from the domain of \( \mathcal{A} \) and for all \( t \in \mathbb{R} \) we have
\[
y(s) = 0 \text{ for all } s < t \quad \Rightarrow \quad (\mathcal{A}y)(s) = 0 \text{ for all } s < t.
\]
Unformally, causality means that the values of the output signal \( (\mathcal{A}y)(t) \) at any instant of time \( t \) may depend on the past values of the input signal \( y \) (i.e. the values \( y(s) \) for \( s \leq t \)), but can not depend on the future ones.

A causal operator \( \mathcal{A} \) is called causally invertible if the inverse operator exists and is also causal. The following proposition shows that the causal invertibility is closely connected with stability, see also Example 5 below.

**Proposition 4.1** ([17, Theorem 3.3.6 and Corollary 6.3.2]). Equation (3) is input-output stable if and only if the operator \( \mathcal{L} : C^1 \to C \) is causally invertible.
Example 5. We consider the causal operator \( Uy = y' + y \) as acting from \( C^1 \) to \( C \). The inverse operator

\[
(U^{-1}f)(t) = \int_{-\infty}^{t} e^{-(t-s)} f(s) \, ds
\]

is causal, see [17, Proposition 2.3.4] for more detailed explanation. According to Proposition 4.1 this correlates with the stability of the equation \( y' + y = f \). Next, we consider the causal operator \( Vy = y' - y \). The inverse operator is

\[
(V^{-1}f)(t) = -\int_{t}^{\infty} e^{t-s} f(s) \, ds.
\]

The operator \( V^{-1} \) is not causal; thus, \( V \) is invertible, but not causally invertible. In view of Proposition 4.1 this corresponds to the unstability of the equation \( y' - y = f \).

The following assertion (together with Proposition 4.1) can be used for the investigation of stability of perturbed equations.

**Proposition 4.2** ([17, Proposition 2.2.12]). Let \( A, B : C^1 \to C \) be bounded linear causal operators. If the operator \( A \) is causally invertible and \( \|B\| \cdot \|A^{-1}\| < 1 \), then the operator \( A - B \) is also causally invertible.

**Remark 4.** We show that the direct application of Proposition 4.2 does not lead to results similar to the averaging principle. We consider the scalar equation \( y'(t) + y(t) + e^{\omega t}y(t) = f(t) \). We represent it in the form

\[ Uy + My = f, \]

where the operators

\[
(Uy)(t) = y'(t) + y(t), \quad (My)(t) = e^{\omega t}y(t)
\]

are considered as acting from \( C^1 \) to \( C \). We have

\[
(U^{-1}f)(t) = \int_{-\infty}^{t} e^{-(t-s)} f(s) \, ds.
\]

Therefore

\[
(MU^{-1}f)(t) = e^{\omega t} \int_{-\infty}^{t} e^{-(t-s)} f(s) \, ds = \int_{-\infty}^{t} e^{\omega t} e^{-(t-s)} f(s) \, ds.
\]

It is well known [17, Theorem 1.5.12] that the norm of an integral operator (with \( k : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \))

\[
(Kf)(t) = \int_{-\infty}^{\infty} k(t,s) f(s) \, ds
\]

considered as acting from \( C \) to \( C \) is

\[
\|K\|_{C \to C} = \sup_{t} \int_{-\infty}^{\infty} |k(t,s)| \, ds.
\]

Consequently,

\[
\|MU^{-1}\|_{C \to C} = 1,
\]

which does not tend to zero as \( \omega \to \infty \). One may prefer to consider the operator \( U^{-1}M : C^1 \to C^1 \). But the norm of \( U^{-1}M \) does not tend to zero as well. Indeed, if \( \|U^{-1}M\| \) tends to zero, it would follow from the representation

\[
MU^{-1} = U(U^{-1}M)U^{-1},
\]
that $\mathcal{M}^{-1}$ tends to zero.

Nevertheless, below we will see that, if one replaces the spaces $C^1$ and $C$ by $H^{1/2}$ and $H^{-1/2}$ respectively, the situation will be qualitatively different.

We denote by $L^2_s = L^2_s(\mathbb{R}, \mathbb{H})$, $s \in \mathbb{R}$, the space of all equivalence classes of measurable functions $y : \mathbb{R} \to \mathbb{H}$ bounded in the norm
\[
\|y\| = \|y\|_{L^2_s} = \sqrt{\int_{-\infty}^{+\infty} (1 + \omega^2)^s \|y(\omega)\|^2 d\omega}.
\]
Evidently, $\|y\|_{L^2_s} \leq \|y\|_{L^2_{s'}}$ for $s \leq s'$.

We denote by $H^s = H^s(\mathbb{R}, \mathbb{H})$, $s \in \mathbb{R}$, the space of all distributions. It is known that the Fourier transform $\hat{y}$ of $y$ belongs to $L^2_s$ with the norm $\|\hat{y}\|_{H^s} = \|y\|_{L^2_s}$. Thus, the Fourier transform $\mathcal{F}$ is an isometric isomorphism from $H^s$ onto $L^2_s$. It is well known that since $\mathbb{H}$ is a Hilbert space, $H^0 = L^2$. In [21] it is explained that the equality $H^0 = L^2$ may not be true if $\mathbb{H}$ is not a Hilbert space, but is only a Banach one.) The spaces $H^s$ are called Sobolev spaces or Bessel potential spaces. For definiteness, we assume that the Fourier transform of an integrable function $y$ is defined by the formula (with the normalizing coefficient 1)
\[
(\mathcal{F}y)(\omega) = \hat{y}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} y(t) dt.
\]
It is easy to see that $H^1$ consists of functions $y \in L^2$ such that $y' \in L^2$ also, and the norm on $H^1$ is equivalent to the norm $\|y\| = \|y\|_{L^2} + \|y'\|_{L^2}$. Evidently, the operator $\mathcal{L}$ acts from $H^1$ to $L^2$. We show that the invertibility of $\mathcal{L} : H^1 \to L^2$ implies the invertibility of $\mathcal{L} : C^1 \to C$.

We denote by $L^2_{2\infty} = L^2_{2\infty}(\mathbb{R}, \mathbb{H})$ the space of all equivalence classes of measurable functions $y : \mathbb{R} \to \mathbb{H}$ bounded in the norm [5]
\[
\|y\| = \|y\|_{L^2_{2\infty}} = \sup_{k \in \mathbb{Z}} \sqrt{\int_{k}^{k+1} \|y(t)\|^2 dt}.
\]
We denote by $W^1_{2\infty} = W^1_{2\infty}(\mathbb{R}, \mathbb{H})$ the space of all distributions $y \in L^2_{2\infty}$ such that the distribution derivative $y'$ of $y$ also belongs to $L^2_{2\infty}$ with the norm
\[
\|y\| = \|y\|_{W^1_{2\infty}} = \|y\|_{L^2_{2\infty}} + \|y'\|_{L^2_{2\infty}}.
\]
Clearly, $W^1_{2\infty} \subset C \subset L^2_{2\infty}$. It is also evident that operator (9) acts from $W^1_{2\infty}$ to $L^2_{2\infty}$. In [17, Theorem 6.3.1] it is shown that the invertibility of $\mathcal{L} : H^1 \to L^2$ implies the invertibility of $\mathcal{L} : W^1_{2\infty} \to L^2_{2\infty}$.

Further, we show that the invertibility of $\mathcal{L} : W^1_{2\infty} \to L^2_{2\infty}$ implies the invertibility of $\mathcal{L} : C^1 \to C$. Indeed, let the operator $\mathcal{L} : W^1_{2\infty} \to L^2_{2\infty}$ be invertible. Then for any $f \in C \subset L^2_{2\infty}$ the equation $\mathcal{L} y = f$ has a unique solution $y \in W^1_{2\infty} \subset C^1$. From the representation
\[
y'(t) = f(t) - (\mu_0 * y)(t) - \sum_{k=1}^{\infty} e^{i\omega_k t}(\mu_k * y)(t)
\]
it is seen that $y' \in C$, i.e., $y \in C^1$. Thus the operator $\mathcal{L} : C^1 \to C$ is invertible.

We consider the operators
\[
\mathcal{\Lambda}_0 = \mathcal{F} \mathcal{L}_0 \mathcal{F}^{-1}, \quad \mathcal{\Lambda}_k = \mathcal{F} \mathcal{M}_k \mathcal{F}^{-1}.
\]
Obviously, the invertibility of the operator \( \mathcal{L} : H^1 \to L_2 \) is equivalent to the invertibility of the operator \( \hat{\mathcal{L}} : L^2_2 \to L_2 \).

Clearly,

\[
\hat{\mathcal{L}} = \hat{\mathcal{L}}_0 + \sum_{k=1}^{\infty} \hat{\mathcal{M}}_k.
\]

Straightforward calculations show that

\[
(\hat{\mathcal{L}}_0 \varphi)(\omega) = (i\omega 1 + \hat{\mu}_0(\omega)) \varphi(\omega),
\]

\[
(\hat{\mathcal{M}}_k \varphi)(\omega) = \hat{\mu}_k (\omega - \omega_k) \varphi(\omega - \omega_k).
\]

We recall from (2) that \( \| \hat{\mu}_k(\omega) \| \leq \| \mu_k \|, \omega \in \mathbb{R} \).

We show that the operator \( \hat{\mathcal{L}}_0 \) establishes an isomorphism from \( L^2_2 \) onto \( L^{2-1}_2 \) for all \( s \in \mathbb{R} \); more precisely, we show that the inverse is the operator (by assumption (i) of Theorem 3.1 the inverse of \( i\omega 1 + \hat{\mu}_0(\omega) \) exists for all \( \omega \in \mathbb{R} \))

\[
(\hat{\mathcal{K}} \varphi)(\omega) = (i\omega 1 + \hat{\mu}_0(\omega))^{-1} \varphi(\omega).
\]

First, we verify that \( \hat{\mathcal{L}}_0 \) acts from \( L^2_2 \) to \( L^{2-1}_2 \), and \( \hat{\mathcal{K}} \) acts from \( L^{2-1}_2 \) to \( L^2_2 \) for all \( s \in \mathbb{R} \). Indeed, since the function \( \hat{\mu}_0(\cdot) \) is bounded, the function \( \varphi_0(\omega) = i\omega 1 + \hat{\mu}_0(\omega) \) is equivalent to the function \( \omega \mapsto i\omega 1 \) as \( \omega \to \infty \) in the sense that

\[
\lim_{\omega \to \infty} \frac{\| i\omega 1 + \hat{\mu}_0(\omega) \|}{\| i\omega 1 \|} = 1. \tag{10}
\]

Consequently, the function \( \omega \mapsto (i\omega 1 + \hat{\mu}_0(\omega))^{-1} \) is equivalent to the function \( \omega \mapsto (i\omega 1)^{-1} \) as \( \omega \to \infty \) in the sense that

\[
\lim_{\omega \to \infty} \frac{\| (i\omega 1 + \hat{\mu}_0(\omega))^{-1} \|}{\| (i\omega 1)^{-1} \|} = 1. \tag{11}
\]

Next, we observe that

\[
\| \hat{\mathcal{L}}_0 y \|_{L^{2-1}_2} \leq \sqrt{\int_{-\infty}^{+\infty} (1 + \omega^2)^{s-1} \| i\omega 1 + \hat{\mu}_0(\omega) \|^2 d\omega}
\]

\[
= \sqrt{\int_{-\infty}^{+\infty} \frac{\| i\omega 1 + \hat{\mu}_0(\omega) \|^2}{1 + \omega^2} (1 + \omega^2)^s \| y(\omega) \|^2 d\omega}
\]

\[
\leq \sup_{\omega \in \mathbb{R}} \frac{\| i\omega 1 + \hat{\mu}_0(\omega) \|}{\sqrt{1 + \omega^2}} \| y \|_{L^2_2},
\]

\[
\| \hat{\mathcal{K}} y \|_{L^2_2} \leq \sqrt{\int_{-\infty}^{+\infty} (1 + \omega^2)^{s-1} \| (i\omega 1 + \hat{\mu}_0(\omega))^{-1} \|^2 \| y(\omega) \|^2 d\omega}
\]

\[
= \sqrt{\int_{-\infty}^{+\infty} \frac{\| (i\omega 1 + \hat{\mu}_0(\omega))^{-1} \|^2}{1/(1 + \omega^2)} (1 + \omega^2)^s \| y(\omega) \|^2 d\omega}
\]

\[
\leq \sup_{\omega \in \mathbb{R}} \frac{\| (i\omega 1 + \hat{\mu}_0(\omega))^{-1} \|}{\sqrt{1 + \omega^2}} \| y \|_{L^{2-1}_2}.\]
By (10) and (11), these suprema are finite. Therefore, \( \hat{L}_0 \) continuously acts from \( L_{2}^{s} \) to \( L_{2}^{s-1} \), and \( \hat{K} \) continuously acts from \( L_{2}^{s-1} \) to \( L_{2}^{s} \). Evidently, both \( \hat{L}_0 \hat{K} \) and \( \hat{K} \hat{L}_0 \) are the identity operators. So, \( \hat{K} \) is the inverse of \( \hat{L}_0 \).

We consider the operator

\[
(\hat{L}_0^{1/2} \varphi)(\omega) = \sqrt{i \omega 1 + \mu_0(\omega)} \varphi(\omega),
\]

where \( \sqrt{\cdot} \) is a holomorphic branch of the square root function defined in \( \mathbb{C} \setminus (-\infty, 0] \). By assumption (i) of Theorem 3.1, the spectrum of \( i \omega 1 + \mu(\omega), \omega \in \mathbb{R} \), does not intersect \( (-\infty, 0] \). Therefore, the holomorphic function \( \sqrt{\cdot} \) is applicable to the operator \( i \omega 1 + \mu_0(\omega) \); moreover, the function \( \omega \mapsto \sqrt{i \omega 1 + \mu_0(\omega)} \) is continuous. Clearly, the function \( \omega \mapsto \sqrt{i \omega 1 + \mu_0(\omega)} \) is equivalent to the function \( \omega \mapsto \sqrt{i \omega 1} \) as \( \omega \to \infty \). Consequently, the operator \( \hat{L}_0^{1/2} \) establishes an isomorphism from \( L_{2}^{s} \) onto \( L_{2}^{s-1/2} \) for all \( s \in \mathbb{R} \). We denote by \( \hat{L}_0^{-1/2} \) the inverse of the operator \( \hat{L}_0^{1/2} \).

Evidently,

\[
(\hat{L}_0^{-1/2} \varphi)(\omega) = (i \omega 1 + \mu_0(\omega))^{-1/2} \varphi(\omega).
\]

Further, we consider the operator

\[
\hat{\mathcal{L}} = \hat{L}_0 + \sum_{k=1}^{\infty} \hat{M}_k
\]

as acting from \( L_{2}^{1/2} \) to \( L_{2}^{-1/2} \). Here we interpret the operator \( \hat{M}_k : L_{2}^{1/2} \to L_{2}^{-1/2} \) as the composition of the embedding of \( L_{2}^{1/2} \) into \( L_{2} \), the operator \( \hat{M}_k : L_{2} \to L_{2} \), and the embedding of \( L_{2} \) into \( L_{2}^{-1/2} \). Clearly, the norms of the embeddings are less than or equal to 1 and \( \| \hat{M}_k : L_{2} \to L_{2} \| \leq \| \mu_k \| \). Therefore, \( \| \hat{M}_k : L_{2}^{1/2} \to L_{2}^{-1/2} \| \leq \| \mu_k \| \), and thus the series \( \sum_{k=1}^{\infty} \hat{M}_k : L_{2}^{1/2} \to L_{2}^{-1/2} \) converges. Evidently, \( \hat{\mathcal{L}} : L_{2}^{1/2} \to L_{2}^{-1/2} \) is an extension of \( \hat{\mathcal{L}} : L_{2}^{1/2} \to L_{2} \).

We show that the invertibility of \( \hat{\mathcal{L}} : L_{2}^{1/2} \to L_{2}^{-1/2} \) implies the invertibility of \( \hat{\mathcal{L}} : L_{2}^{1/2} \to L_{2} \). Indeed, let the operator \( \hat{\mathcal{L}} : L_{2}^{1/2} \to L_{2}^{-1/2} \) be invertible. Then for any \( f \in L_{2} \subset L_{2}^{-1/2} \) the equation \( \tilde{\mathcal{L}} y = f \) has a unique solution \( y \in L_{2}^{1/2} \). From the representation \( \hat{\mathcal{L}}_0 y + \sum_{k=1}^{\infty} \hat{M}_k y = f \) or, equivalently, \( y = \hat{\mathcal{L}}(f - \sum_{k=1}^{\infty} \hat{M}_k y) \) it is seen that \( y \in L_{2}^{1/2} \). Thus the operator \( \hat{\mathcal{L}} : L_{2}^{1/2} \to L_{2} \) is invertible.

Next we consider the composition

\[
\hat{L}_0^{-1/2} \hat{\mathcal{L}} \hat{L}_0^{-1/2},
\]

where \( \hat{\mathcal{L}} \) acts from \( L_{2}^{1/2} \) to \( L_{2}^{-1/2} \), the left operator \( \hat{L}_0^{-1/2} \) acts from \( L_{2} \) onto \( L_{2}^{1/2} \), and the right operator \( \hat{L}_0^{-1/2} \) acts from \( L_{2}^{-1/2} \) onto \( L_{2} \); thus the whole composition acts from \( L_{2} \) to itself. Clearly,

\[
\hat{L}_0^{-1/2} \hat{\mathcal{L}} \hat{L}_0^{-1/2} = \hat{L}_0^{-1/2} (\hat{L}_0 - \sum_{k=1}^{\infty} \hat{M}_k) \hat{L}_0^{-1/2} = 1 - \sum_{k=1}^{\infty} \hat{L}_0^{-1/2} \hat{M}_k \hat{L}_0^{-1/2}.
\]
Simple calculations show that
\[
(\hat{L}_0^{-1/2}\hat{M}_k\hat{L}_0^{-1/2}\varphi)(\omega) = (i\omega\mathbf{1} + \hat{\mu}_0(\omega))^{-1/2}\hat{\mu}_k(\omega - \omega_k)
\]
\[
\times (i(\omega - \omega_k)\mathbf{1} + \hat{\mu}_0(\omega - \omega_k))^{-1/2}\varphi(\omega - \omega_k).
\]
The operator \(\hat{\mathcal{L}}_0^{-1/2}\hat{\mathcal{M}}_k\hat{\mathcal{L}}_0^{-1/2}\) is considered in \(L_2\). Therefore, its norm is equal to the norm of the operator (the operator \(\hat{\mathcal{N}}_k\) is only shifted with respect to the operator \(\hat{\mathcal{L}}_0^{-1/2}\hat{\mathcal{M}}_k\hat{\mathcal{L}}_0^{-1/2}\))
\[
(\hat{\mathcal{N}}_k\varphi)(\omega) = (i(\omega + \omega_k)\mathbf{1} + \hat{\mu}_0(\omega + \omega_k))^{-1/2}\hat{\mu}_k(\omega)(i\omega\mathbf{1} + \hat{\mu}_0(\omega))^{-1/2}\varphi(\omega)
\]
\[
= \sqrt{\Phi(\omega + \omega_k)\hat{\mu}_k(\omega)}\sqrt{\Phi(\omega)}\varphi(\omega).
\]
 Clearly,
\[
\|\hat{\mathcal{L}}_0^{-1/2}\hat{\mathcal{M}}_k\hat{\mathcal{L}}_0^{-1/2}\| = \|\hat{\mathcal{N}}_k\| \leq \sup_{\omega \in \mathbb{R}}\|\sqrt{\Phi(\omega + \omega_k)\hat{\mu}_k(\omega)}\sqrt{\Phi(\omega)}\|.
\]
Therefore, assumption (6) guarantees the estimate
\[
\left\|\sum_{k=1}^{\infty}\hat{\mathcal{L}}_0^{-1/2}\hat{\mathcal{M}}_k\hat{\mathcal{L}}_0^{-1/2}\right\| = \|\hat{\mathcal{L}}_0^{-1/2}\hat{\mathcal{M}}_k\hat{\mathcal{L}}_0^{-1/2}\| < 1.
\]
The last estimate implies the invertibility of operator (12) which in turn implies the invertibility of \(\hat{\mathcal{L}}\) and \(\mathcal{L}\) in all spaces under consideration.

But, because of Proposition 4.1, in order to finish the proof of Theorem 3.1 we need to establish the causal invertibility of the operator \(\hat{\mathcal{L}}\). We make use of the following auxiliary assertion.

**Proposition 4.3.** Let a family \(\mathcal{L}[s] : C^1 \to C\), \(s \in [0, 1]\), of bounded linear causal operators depend on the parameter \(s\) continuously in norm. Let all operators \(\mathcal{L}[s]\), \(s \in [0, 1]\), be invertible, and the operator \(\mathcal{L}[0]\) be causally invertible. Then all operators \(\mathcal{L}[s]\), \(s \in [0, 1]\), are causally invertible.

**Proof.** It is clear that the inverse operators \(\mathcal{L}[s]^{-1} : C \to C^1\) depend on \(s \in [0, 1]\) continuously in norm. Hence,
\[
\eta = \sup_{s \in [0, 1]} \|\mathcal{L}[s]^{-1}\| < \infty.
\]
Since the function \(s \mapsto \mathcal{L}[s]\) is uniformly continuous on \([0, 1]\), we can choose a partition \(0 = s_0 < s_1 < \cdots < s_n = 1\) such that
\[
\|\mathcal{L}[s_{i+1}] - \mathcal{L}[s_i]\| < 1/\eta, \quad i = 1, 2, \ldots, n.
\]
By Proposition 4.2, the causal invertibility of \(\mathcal{L}[s_i]\) implies the causal invertibility of \(\mathcal{L}[s_{i+1}]\) for all \(i = 1, 2, \ldots, n\). As a consequence, we obtain that the causal invertibility of \(\mathcal{L}[0]\) implies that of \(\mathcal{L}[1]\). \(\square\)

We consider the family of operators
\[
(\mathcal{L}[s]y)(t) = y'(t) + (\mu_0 * y)(t) + s \sum_{k=1}^{\infty} e^{i\omega_k t}(\mu_k * y)(t)
\]
depending on the parameter \(s \in [0, 1]\). All above reasonings are clearly valid for all operators \(\mathcal{L}[s]\), \(s \in [0, 1]\), and therefore these operators are invertible. It is easy to see that the operators \(\mathcal{L}[s] : C^1 \to C\) depend on \(s \in [0, 1]\) continuously in norm.
We observe that, by Propositions 2.2 and 4.1, the operator $L_0 = L$ is causally invertible. Therefore, by Proposition 4.3, the operator $L_1 = L$ is causally invertible. The proof of Theorem 3.1 is complete.

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