A quaternionic fractional Borel-Pompeiu type formula

José Oscar González-Cervantes(1) and Juan Bory-Reyes(2)∗

(1) Departamento de Matemáticas, ESFM-Instituto Politécnico Nacional. 07338, Ciudad México, México
Email: jogc200678@gmail.com
(2) SEPI, ESIME-Zacatenco-Instituto Politécnico Nacional. 07338, Ciudad México, México
Email: juanboryreyes@yahoo.com

Abstract

In theoretical setting, associated with a fractional $\psi$–Fueter operator that depends on an additional vector of complex parameters with fractional real parts, this paper establishes a fractional analogue of Borel-Pompeiu formula as a first step to develop a fractional $\psi$–hyperholomorphic function theory and the related operator calculus.

Keywords. Quaternionic analysis; Stokes and Borel-Pompeiu formulas; fractional $\psi$–Fueter operator.
AMS Subject Classification (2020): 30G30; 30G35; 35A08; 35R11; 45P05.

1 Introduction

Fractional calculus is a theory allowing integrals and derivatives of arbitrary real or complex order. The interest in the subject has been growing continuously during the last few decades because of numerous applications in diverse fields of science and engineering, see [1, 12, 19, 20, 24, 25, 28, 31, 32, 38]. Fractional integrals and derivatives can be traced back to the genesis of differential calculus when in 1695 G. Leibnitz mentioned a derivative of order $\frac{1}{2}$. However a rigorous investigation was first carried out by J. Liouville in a series of papers from 1832-1837, where he defined the first outcast of an operator of fractional integration, see

∗corresponding author
for instance [19]. Later investigations and further developments by among others B. Riemann [29] led to the construction of the integral-based Riemann-Liouville fractional integral operator (see [7]), which has been a valuable cornerstone in fractional calculus ever since. For a brief history and exposition of the fundamental theory of fractional calculus we refer the reader to [30].

A framework for a fractional Euclidean Clifford analysis represents a very recent topic of research, see [6, 8, 11, 15, 27, 40] and the references given there. In particular the interest for considering fractional Dirac Operator is devoted in [2, 3, 9, 10].

Nowadays, quaternionic analysis is regarded as a broadly accepted branch of classical analysis offering a successful generalization of complex holomorphic function theory, the most renowned examples are Sudbery’s paper [37] and the books [13, 14, 17, 18]. It relies heavily on results on functions defined on domains in $\mathbb{R}^4$ with values in the skew field of real quaternions $\mathbb{H}$ associated to a generalized Cauchy-Riemann operator (the so-called $\psi$–Fueter operator) by using a general orthonormal basis in $\mathbb{R}^4$ (to be named structural set) $\psi$ of $\mathbb{H}^4$, see, e.g., [21–24]. This theory is centered around the concept of $\psi$–hyperholomorphic functions, see [21, 33–36].

The original contribution of this paper is to give a re-contextualization of the $\psi$–hyperholomorphic function theory and the related operator calculus in a fractional setting, based on the modification of the $\psi$–Fueter operator in the Riemann-Liouville sense that depends on an additional vector of complex parameters with fractional real parts. In addition to the general extent of our proposed research, the important fact that the introduced fractional $\psi$–Fueter operator factorizes a fractional $\psi$–Laplacian is also shown.

The structure of the papers is as follows. After this brief introduction, in the preliminary section we recall some basic facts about a quaternionic analysis associated to a structural set $\psi$, such as the $\psi$–Fueter operator, Stokes and the Borel-Pompeiu formulas, as well as the notions of fractional Riemann-Liouville integral and derivative. Sec. 3 deals with a quaternionic version of the fractional derivative of the Riemann-Liouville associated to a structural set. Stokes and Borel-Pompeiu type formulas are proved.

2 Preliminaries

Below we give basic definitions and facts on the fractional calculus and quaternionic analysis. These notions will be used throughout the whole paper.

2.1 Standard definition of and results on Riemann-Liouville fractional integro-differential operators

There are different definitions of fractional derivatives. One of the most popular (even though it has disadvantages for applications to real world problems) is the

2
Riemann–Liouville derivative (see, e.g., [20]). For completeness, we recall the key definitions and results on Riemann-Liouville fractional integro-differential operators.

Given $\alpha \in \mathbb{C}$ with $\Re \alpha > 0$, let us recall that the Riemann-Liouville integrals of order $\alpha$ of a $f \in L^1([a,b], \mathbb{R})$, with $-\infty < a < b < \infty$, on the left and on the right, are defined by

$$(I_\alpha^a f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(\tau)}{(x-\tau)^{1-\alpha}} d\tau, \quad \text{with } x > a$$

and

$$(I_\alpha^b f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(\tau)}{(\tau-x)^{1-\alpha}} d\tau, \quad \text{with } x < b,$$

respectively.

What is more, let $n = \lceil \Re \alpha \rceil + 1$, where $\lceil \cdot \rceil$ means the integer part of $\cdot$ and $f \in AC^n([a,b], \mathbb{R})$: i.e., the class of functions $f$ which are continuously differentiable on the segment $[a,b]$ up to the order $n-1$ and $f^{(n-1)}$ is supposed to be absolutely continuous on $[a,b]$. The fractional derivatives in the Riemann-Liouville sense, on the left and on the right, are defined by

$$(D_\alpha^a f)(x) := \frac{d}{dx^n} [(I_{n-\alpha}^a f)(x)]$$

and

$$(D_\alpha^b f)(x) := (-1)^n \frac{d}{dx^n} [(I_{n-\alpha}^b f)(x)]$$

respectively. It is worth noting that the derivatives in (1), (2) exist for $f \in AC^n([a,b], \mathbb{R})$. Fractional Riemann-Liouville integral and derivative are linear operators.

Fundamental theorem for Riemann-Liouville fractional calculus [7] shows that

$$(D_\alpha^a I_{n-\alpha}^a f)(x) = f(x) \quad \text{and} \quad (D_\alpha^b I_{n-\alpha}^b f)(x) = f(x). \quad (3)$$

Let us mention an important property of the fractional Riemann-Liouville integral and derivative, see [4, pag. 23], [39, pag. 1835].

**Proposition 2.1.**

$$(D_\alpha^a 1)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad \forall x \in [a,b]. \quad (4)$$

### 2.2 Rudiments of quaternionic analysis

Consider the skew field of real quaternions $\mathbb{H}$ with its basic elements $1, i, j, k$. Thus any element $x$ from $\mathbb{H}$ is of the form $x = x_0 + x_1 i + x_2 j + x_3 k$, $x_k \in \mathbb{R}, k = 0, 1, 2, 3$.  

3
The basic elements define arithmetic rules in $\mathbb{H}$: by definition $i^2 = j^2 = k^2 = -1$, $ij = -ji = k; jk = -jk = i$ and $ki = -ik = j$. For $x \in \mathbb{H}$ we define the mapping of quaternionic conjugation: $x \mapsto \overline{x} := x_0 - x_1i - x_2j - x_3k$. In this way it is easy seen that $x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2$. Note that $\overline{q\overline{x}} = \overline{q}\overline{x}$ for $q, x \in \mathbb{H}$.

The quaternionic scalar product of $q, x \in \mathbb{H}$ is given by $\langle q, x \rangle := \frac{1}{2} (\overline{qx} + \overline{xq}) = \frac{1}{2} (q\overline{x} + x\overline{q})$.

A set of quaternions $\psi = \{\psi_0, \psi_1, \psi_2, \psi_3\}$ is called structural set if $\langle \psi_k, \psi_s \rangle = \delta_{k,s}$, for $k, s = 0, 1, 2, 3$ and any quaternion $x$ can be rewritten as $x_\psi := \sum_{k=0}^3 x_k \psi_k$, where $x_k \in \mathbb{R}$ for all $k$. Given $q, x \in \mathbb{H}$ we follow the notation used in [35] to write

$$\langle q, x \rangle_\psi = \sum_{k=0}^3 q_k x_k,$$

where $q_k, x_k \in \mathbb{R}$ for all $k$.

Let $\psi$ an structural set. From now on, we will use the mapping

$$\sum_{k=0}^3 x_k \psi_k \rightarrow (x_0, x_1, x_2, x_3). \quad (5)$$

in essential way.

We have to say something about the set of complex quaternions, which are given by

$$\mathbb{H}(\mathbb{C}) = \{ q = q_1 + i q_2 \mid q_1, q_2 \in \mathbb{H} \},$$

where $i$ is the imaginary unit of $\mathbb{C}$. The main difference to the real quaternions is that not all non-zero elements are invertible. There are so-called zero-divisors.

Let us recall that $\mathbb{H}$ is embedded in $\mathbb{H}(\mathbb{C})$ as follows:

$$\mathbb{H} = \{ q = q_1 + i q_2 \in \mathbb{H}(\mathbb{C}) \mid q_1, q_2 \in \mathbb{H} \text{ and } q_2 = 0 \}.$$ 

The elements of $\mathbb{H}$ are written in terms of the structural set $\psi$ hence those of $\mathbb{H}(\mathbb{C})$ can be written as $q = \sum_{k=0}^3 \psi_k q_k$, where $q_k \in \mathbb{C}$.

Functions $f$ defined in a bounded domain $\Omega \subset \mathbb{H} \cong \mathbb{R}^4$ with value in $\mathbb{H}$ are considered. They may be written as: $f = \sum_{k=0}^3 f_k \psi_k$, where $f_k, k = 0, 1, 2, 3$, are $\mathbb{R}$-valued functions in $\Omega$. Properties as continuity, differentiability, integrability and so on, which ascribed to $f$ have to be posed by all components $f_k$. We will follow standard notation, for example $C^1(\Omega, \mathbb{H})$ denotes the set of continuously differentiable $\mathbb{H}$-valued functions defined in $\Omega$.

The left- and the right-$\psi$-Fueter operators are defined by $\psi D[f] := \sum_{k=0}^3 \psi_k \partial_k f$ and $\psi D_r[f] := \sum_{k=0}^3 \partial_k f \psi_k$, for all $f \in C^1(\Omega, \mathbb{H})$, respectively, where $\partial_k f = \frac{\partial f}{\partial x_k}$ for all $k$, see [35][36].
Particularly, if $\partial \Omega$ is a 3-dimensional smooth surface then the Borel-Pompieu formula shows that

\[
\int_{\partial \Omega} \left( K_\psi (\tau - x) \sigma_x^\psi f(\tau) + g(\tau) \sigma_{x}^\psi K_\psi (\tau - x) \right)
- \int_{\Omega} \left( K_\psi (y - x) \psi D[f](y) + \psi D_r[g](y) K_\psi (y - x) \right) dy
= \begin{cases}
  f(x) + g(x), & x \in \Omega, \\
  0, & x \in \mathbb{H} \setminus \overline{\Omega}.
\end{cases}
\]  

(6)

Differential and integral versions of Stokes’ formulas for the $\psi$-hyperholomorphic functions theory are given by

\[
d(g \sigma_x^\psi f) = \left( g \psi D[f] + \psi D_r[g] f \right) dx,
\]  

(7)

\[
\int_{\partial \Omega} g \sigma_x^\psi f = \int_{\Omega} \left( g \psi D[f] + \psi D_r[g] f \right) dx,
\]  

(8)

for all $f, g \in C^1(\mathbb{R}^4)$, see [35–37]. Here, $d$ stands for the exterior differentiation operator, $dx$ denotes the differential form of the 4-dimensional volume in $\mathbb{R}^4$ and

\[
\sigma_x^\psi := -sgn\psi \left( \sum_{k=0}^{3} (-1)^k \psi_k d\hat{x}_k \right)
\]

is the quaternionic differential form of the 3-dimensional volume in $\mathbb{R}^4$ according to $\psi$, where $d\hat{x}_k = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ omitting factor $dx_k$. In addition, $sgn\psi$ is 1, or $-1$, if $\psi$ and $\psi_{std} := \{1, i, j, k\}$ have the same orientation, or not, respectively. Note that, $|\sigma_x^\psi| = dS_3$ is the differential form of the 3-dimensional volume in $\mathbb{R}^4$ and write $\sigma_x = \sigma_x^{\psi, std}$. Let us recall that the $\psi$-hyperholomorphic Cauchy Kernel is given by

\[
K_\psi (\tau - x) = \frac{1}{2\pi^2} \frac{\overline{\tau_\psi - x_\psi}}{|\tau_\psi - x_\psi|^4},
\]

and the integral operator

\[
\psi T[f](x) = \int_{\Omega} K_\psi (y - x) f(y) dy
\]

defined for all $f \in L^2(\Omega, \mathbb{H}) \cup C(\Omega, \mathbb{H})$ satisfies

\[
\psi D \circ \psi T[f] = f, \quad \forall f \in L^2(\Omega, \mathbb{H}) \cup C(\Omega, \mathbb{H}).
\]  

(9)

This can be found in [21][33][36].
3 Main results

We shall consider vector parameters $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^4$, requiring the values of $\Re \alpha_\ell$, for $\ell = 0, 1, 2, 3$, are varied between any two consecutive integer values, that is, $n = |\Re \alpha_\ell| + 1$, for $n \in \mathbb{N}$.

Firstly, let us start with the case $n = 1$, i.e., $0 < \Re \alpha_\ell < 1$ for $\ell = 0, 1, 2, 3$ and finally the general case will be presented.

3.1 Fractional $\psi$-Fueter operator of order $\vec{\alpha}$

**Definition 3.1.** Let $a = \sum_{k=0}^{3} \psi_k a_k, b = \sum_{k=0}^{3} \psi_k b_k \in \mathbb{H}$ such that $a_k < b_k$ for all $k$. Write

$$J_a^b := \{ \sum_{k=0}^{3} \psi_k x_k \in \mathbb{H} \mid a_k < x_k < b_k, \ k = 0, 1, 2, 3 \}$$

$$= (a_0, b_0) \times (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3),$$

and define $m(J_a^b) := (b_0 - a_0)(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$.

In what follows, with notation $J_a^b$ we assume $a_k < b_k$ for all $k$.

**Remark 1.** Set $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^4$ and $f = \sum_{i=0}^{3} \psi_i f_i \in AC^1(J_a^b, \mathbb{H})$; i.e., $f_0, f_1, f_2, f_3$, the real components of $f$, belongs to $AC^1(\Omega, \mathbb{R})$. The mapping $x_j \mapsto f_i(q_0, \ldots, x_j, \ldots, q_3)$ belongs to $AC^1((a_i, b_i), \mathbb{R})$ for each $q \in J_a^b$ and all $i, j = 0, 1, 2, 3$.

Now, given $q, x \in J_a^b$ and $i, j = 0, \ldots, 3$, the fractional integral of Riemann-Liouville of order $\alpha_j$ of the mapping $x_j \mapsto f_i(q_0, \ldots, x_j, \ldots, q_3)$ is defined by:

$$\big(\Gamma_{\alpha_j}^{\psi} f_i\big)(q_0, \ldots, x_j, \ldots, q_3) = \frac{1}{\Gamma(\alpha_j)} \int_{a_j}^{x_j} \frac{f_i(q_0, \ldots, \tau_j, \ldots, q_3)}{(x_j - \tau_j)^{1-\alpha_j}} d\tau_j.$$

By the above, as $f = \sum_{i=0}^{3} \psi_i f_i$ it follows that

$$\big(\Gamma_{\alpha_j}^{\psi} f\big)(q_0, \ldots, x_j, \ldots, q_3) := \frac{1}{\Gamma(\alpha_j)} \int_{a_j}^{x_j} \frac{f(q_0, \ldots, \tau_j, \ldots, q_3)}{(x_j - \tau_j)^{1-\alpha_j}} d\tau_j$$

$$= \sum_{i=0}^{3} \psi_i \frac{1}{\Gamma(\alpha_j)} \int_{a_j}^{x_j} \frac{f_i(q_0, \ldots, \tau_j, \ldots, q_3)}{(x_j - \tau_j)^{1-\alpha_j}} d\tau_j$$

$$= \sum_{i=0}^{3} \psi_i \big(\Gamma_{\alpha_j}^{\psi} f_i\big)(q_0, \ldots, x_j, \ldots, q_3).$$

for every $f \in AC^1(J_a^b, \mathbb{H})$ and $q, x \in J_a^b$. 

6
What is more, the fractional derivative in the Riemann-Liouville sense of the mapping \( x_j \mapsto f(q_0, \ldots, x_j, \ldots, q_3) \) of order \( \alpha_j \) is given by

\[
D_{a_j}^{\alpha_j} f(q_0, \ldots, x_j, \ldots, q_3) = \frac{\partial}{\partial x_j} \sum_{i=0}^{3} \psi_i (I_{a_j}^{\alpha_j} f_i)(q_0, \ldots, x_j, \ldots, q_3)
= \frac{\partial}{\partial x_j} \frac{1}{\Gamma(\alpha_j)} \int_{a_j}^{x_j} f(q_0, \ldots, \tau_j, \ldots, q_3) (x_j - \tau_j)^{1-\alpha_j} d\tau_j.
\]

Note that \((I_{a_j}^{\alpha_j} f)\) and \(D_{a_j}^{\alpha_j} f\) are \(\mathbb{H}(\mathbb{C})\)-valued functions for every \(j\). In a similar way we can introduce \((I_{b_j}^{\alpha_j} f)\) and \(D_{b_j}^{\alpha_j} f\).

**Definition 3.2.** Let \( f \in AC^1(J_{b_j}^b, \mathbb{H}) \), and \( \vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^4 \) such that \( 0 < \Re \alpha_j < 1 \) for \( j = 0, 1, 2, 3 \). The fractional psi-Fueter operator of order \( \vec{\alpha} \) is defined to be

\[
\psi D_{a_j}^\vec{\alpha} [f](q, x) := \sum_{j=0}^{3} \psi_j (D_{a_j}^{\alpha_j} f)(q_0, \ldots, x_j, \ldots, q_3)
= \sum_{j=0}^{3} \psi_j \frac{\partial}{\partial x_j} \frac{1}{\Gamma(\alpha_j)} \int_{a_j}^{x_j} f(q_0, \ldots, \tau_j, \ldots, q_3) (x_j - \tau_j)^{1-\alpha_j} d\tau_j,
\]

for \( q, x \in J_{b_j}^b \). Note that \( q \) is considered a fixed point since the integration and derivation variables are the real components of \( x \). Moreover, \( \psi D_{a_j}^\vec{\alpha} [f] \) is a \(\mathbb{H}(\mathbb{C})\)-valued function.

Particularly, \( \psi D_{a_j}^\vec{\alpha} [f](q, x) |_{x=q} \) can be considered as \( \psi D_{a_j}^\vec{\alpha} [f] \) at point \( q \). Then denote \( \psi D_{a_j}^\vec{\alpha} [f](q, x) |_{x=q} = \psi D_{a_j}^\vec{\alpha} [f](q) \).

On the other hand, given \( f \in AC^1(J_{b_j}^b, \mathbb{H}) \) define

\[
\psi \mathcal{T}_{a_j}^\vec{\alpha} [f](q, x) = \sum_{j=0}^{3} \frac{1}{2 \Gamma(\alpha_j)} \int_{a_j}^{x_j} \psi_j f(q_0, \ldots, \tau_j, \ldots, q_3) + \frac{f(q_0, \ldots, \tau_j, \ldots, q_3)}{(x_j - \tau_j)^{1-\alpha_j}} d\tau_j
\]
and

\[
\psi \mathcal{I}_{a_j}^\vec{\alpha} [f](q, x, \vec{\alpha}) = \int_{J_{a_j}^b} \frac{f(\tau_0, q_1, \ldots, q_3) (x_0 - \tau_0)^{\alpha_0}}{1 \cdot \Gamma(\alpha_0)} + \cdots + \frac{f(q_0, q_2, \ldots, \tau_3) (x_3 - \tau_3)^{\alpha_3}}{1 \cdot \Gamma(\alpha_3)} d\mu_{\tau}.
\]

where \( \tau = \sum_{k=0}^{3} \psi_k \tau_k \) and \( d\mu_{\tau} \) is the differential of volume.

**Remark 2.** Throughout the paper we shall be mainly interested in the study of null-solutions of the operator \( \psi D_{a_j}^\vec{\alpha} \) and Riemann-Liouville integrals \( \psi \mathcal{T}_{a_j}^\vec{\alpha} \) and \( \psi \mathcal{I}_{a_j}^\vec{\alpha} \).

But similar analyzes have arisen for operators \( \psi D_{b_j}^\vec{\alpha} \), \( \psi \mathcal{T}_{b_j}^\vec{\alpha} \) and \( \psi \mathcal{I}_{b_j}^\vec{\alpha} \) introduced with \((I_{b_j}^{\alpha_j})\) and \( D_{b_j}^{\alpha_j} \) replaced by \((I_{a_j}^{\alpha_j})\) and \( D_{a_j}^{\alpha_j} \) and the corresponding right chance of integration in \( \psi \mathcal{T}_{a_j}^\vec{\alpha} \) and \( \psi \mathcal{I}_{a_j}^\vec{\alpha} \).
Proposition 3.3. Consider \( f \in AC^1(J_{a}^b, \mathbb{H}) \), and \( \vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^4 \) with \( 0 < \Re \alpha_\ell < 1 \) for \( \ell = 0, 1, 2, 3 \). Then

1. \( \psi D_\vec{\alpha}^a [f](q, x) = \psi D_x \circ \psi I_\vec{\alpha}^a [f](q, x, \vec{\alpha}) \).

2. \( \psi D_\vec{\alpha}^a \circ \psi I_\vec{\alpha}^a [f](q, x) = \sum_{j=0}^{3} \psi_j f_j(q_0, \ldots, x_j, \ldots, q_3) \) for \( f \in AC^1(J_{a}^b, \mathbb{H}) \), where \( x = \sum_{j=0}^{3} \psi_j x_j \) and \( q = \sum_{j=0}^{3} \psi_j q_j \in J_{a}^b \).
   Particularly, \( \psi D_\vec{\alpha}^a \circ \psi I_\vec{\alpha}^a [f](q, x) |_{x=q} = f(q) \).

3. \( \psi D_x \circ \psi I_\vec{\alpha}^a [f](q, x) = \Delta R^4 \circ \psi I_\vec{\alpha}^a [f](q, x, \vec{\alpha}) \), where \( \Delta R^4 \) denotes the Laplacian in \( \mathbb{R}^4 \) according to the real components of \( x \).

4. If the mapping \( x \rightarrow I_\vec{\alpha}^a [f](q, x, \vec{\alpha}) \) belongs to \( C^2(J_{a}^b, \mathbb{H}) \) for all \( q \) and set \( \vec{\beta} = (\beta_0, \beta_1, \beta_2, \beta_3) \in \mathbb{C}^4 \) with \( 0 < \Re \beta_\ell < 1 \) for \( \ell = 0, 1, 2, 3 \) then we have

\[
\psi D_\vec{\alpha}^a \circ \psi D_\vec{\beta}^a [f](q, x) = \sum_{j=0}^{3} \psi_j^2 D_{a_j}^{\alpha_j+\beta_j} f(q_0, \ldots, x_j, \ldots, q_3)
\]

and

\[
\tilde{\psi} D_\vec{\alpha}^a \circ \psi D_\vec{\beta}^a [f](q, x) = \sum_{j=0}^{3} D_{a_j}^{\alpha_j+\beta_j} f(q_0, \ldots, x_j, \ldots, q_3).
\]

Note that, for \( \vec{\alpha} = \vec{\beta} \), the above formula drawn the fact that the fractional \( \psi \)-Fueter operator of order \( \frac{1+\vec{\alpha}}{2} \) factorizes a fractional \( \psi \)-Laplace operator defined by \( \psi \Delta_\vec{\alpha}^a := \sum_{j=0}^{3} D_{a_j}^{1+\alpha_j} \).
Proof. 1. The first identity is a consequence from the following:

\[
\sum_{j=0}^{3} \left( I_{a_j}^{x_j} f(q_0, \ldots, x_j, \ldots, q_3) \right)
= \sum_{j=0}^{3} \frac{1}{\Gamma(\alpha_j)} \int_{a_j}^{x_j} \frac{f(q_0, \ldots, \tau_j, \ldots, q_3)}{(x_j - \tau_j)^{1-\alpha_j}} d\tau_j
= \frac{1}{\Gamma(\alpha_0)} \int_{a_0}^{x_0} f(\tau_0, q_1, \ldots, q_3) d\tau_0 + \ldots + \frac{1}{\Gamma(\alpha_3)} \int_{a_3}^{x_3} f(q_0, \ldots, q_2, \tau_3) d\tau_3
= \frac{1}{\Gamma(\alpha_0)} \int_{a_0}^{x_0} \frac{f(\tau_0, q_1, \ldots, q_3)(x_0 - \tau_0)^{1-\alpha_0}}{\Gamma(\alpha_0)} d\tau_0 + \ldots + \frac{1}{\Gamma(\alpha_3)} \int_{a_3}^{x_3} \frac{f(q_0, \ldots, q_2, \tau_3)\Gamma(\alpha_0)}{(x_3 - \tau_3)^{1-\alpha_3}} d\tau_3
= \int_{a_0}^{x_0} \int_{a_1}^{x_1} \int_{a_2}^{x_2} \int_{a_3}^{x_3} f(\tau_0, q_1, \ldots, q_3) \frac{(x_0 - \tau_0)^{1-\alpha_0}}{\Gamma(\alpha_0)} d\tau_0 + \ldots + f(q_0, \ldots, q_2, \tau_3) \frac{\Gamma(\alpha_0)}{(x_3 - \tau_3)^{1-\alpha_3}} d\tau_3
\]

From Fubini’s Theorem one has that

\[
\sum_{j=0}^{3} \frac{1}{\Gamma(\alpha_j)} \int_{a_j}^{x_j} \frac{f(q_0, \ldots, \tau_j, \ldots, q_3)}{(x_j - \tau_j)^{1-\alpha_j}} d\tau_j
= \int_{J_a^x} \frac{f(\tau_0, q_1, \ldots, q_3) \frac{(x_0 - \tau_0)^{1-\alpha_0}}{\Gamma(\alpha_0)}}{m(J_a^x)} \frac{\Gamma(\alpha_0)}{(x_3 - \tau_3)^{1-\alpha_3}} d\mu_\tau,
\]

where \( \tau = (\tau_0, \tau_1, \tau_2, \tau_3) \).

Due to

\[
\frac{\partial}{\partial x_j} \frac{1}{\Gamma(\alpha_j)} \int_{a_j}^{x_j} \frac{f(q_0, \ldots, \tau_j, \ldots, q_3)}{(x_j - \tau_j)^{1-\alpha_j}} d\tau_j = \frac{\partial}{\partial x_j} \sum_{n=0}^{3} \frac{1}{\Gamma(\alpha_n)} \int_{a_n}^{x_n} \frac{f(q_0, \ldots, \tau_n, \ldots, q_3)}{(x_n - \tau_n)^{1-\alpha_n}} d\tau_n,
\]

one concludes that

\[
\psi D_a^\alpha[f](q, x)
= \sum_{j=0}^{3} \psi \frac{\partial}{\partial x_j} \frac{1}{\Gamma(\alpha_j)} \int_{a_j}^{x_j} \frac{f(q_0, \ldots, \tau_j, \ldots, q_3)}{(x_j - \tau_j)^{1-\alpha_j}} d\tau_j
= \sum_{j=0}^{3} \psi \frac{\partial}{\partial x_j} \sum_{n=0}^{3} \frac{1}{\Gamma(\alpha_n)} \int_{a_n}^{x_n} \frac{f(q_0, \ldots, \tau_n, \ldots, q_3)}{(x_n - \tau_n)^{1-\alpha_n}} d\tau_n
= \psi \tilde{D}_x \psi T_a^\alpha[f](q, x, \bar{\alpha}).
\]
2. The proof of this identity is a consequence of the Fundamental Theorem in the Riemann-Liouville fractional derivative calculus:

\[ \psi \mathcal{D}_{\alpha}^a \circ \psi \mathcal{I}_{\alpha}^a [f](q, x) = \sum_{j=0}^{3} \psi_j D_{a_j}^{\alpha_j} \int_{a_j}^{x_j} \frac{f_j(q_0, \ldots, \tau_j, \ldots, q_3)}{(x_j - \tau_j)^{1+\alpha_j}} d\tau_j \]

\[ = \sum_{j=0}^{3} \psi_j f_j(q_0, \ldots, x_j, \ldots, q_3). \]

3. \[ \tilde{\psi} D_x \circ \psi \mathcal{D}_{\alpha}^a [f](q, x) = \tilde{\psi} D_x \circ \psi \mathcal{I}_{\alpha}^a [f](q, x, \bar{\alpha}) = \Delta_{\mathbb{R}^4} \circ \psi \mathcal{I}_{\alpha}^a [f](q, x, \bar{\alpha}). \]

4. \[ \psi \mathcal{D}_{\alpha}^a \circ \psi \mathcal{D}_{\beta}^a [f](q, x) = \sum_{j=0}^{3} \psi_j D_{a_j}^{\alpha_j} \sum_{k=0}^{3} \psi_k (D_{a_k}^{\beta_k} f)(q_0, \ldots, x_k, \ldots, q_3) \]

\[ = \sum_{j,k=0}^{3} \psi_j \psi_k (D_{a_j}^{\alpha_j} D_{a_k}^{\beta_k} f)(q_0, \ldots, x_k, \ldots, q_3) \]

\[ = \sum_{j=0}^{3} \psi_j^2 D_{a_j}^{\alpha_j+\beta_j} f(q_0, \ldots, x_j, \ldots, q_3). \]

\[ \psi \mathcal{D}_{\alpha}^a \circ \psi \mathcal{D}_{\bar{\alpha}}^a [f](q, x) = \sum_{j=0}^{3} \psi_j D_{a_j}^{\alpha_j} \sum_{k=0}^{3} \psi_k (D_{a_k}^{\beta_k} f)(q_0, \ldots, x_k, \ldots, q_3) \]

\[ = \sum_{j,k=0}^{3} \psi_j \psi_k (D_{a_j}^{\alpha_j} D_{a_k}^{\beta_k} f)(q_0, \ldots, x_k, \ldots, q_3) \]

\[ = \sum_{j=0}^{3} D_{a_j}^{\alpha_j+\beta_j} f(q_0, \ldots, x_j, \ldots, q_3). \]

**Remark 3.** Properties exhibited by Proposition 3.3 gives an extension of basic formulas related to the standard fractional Riemann-Liouville derivative to the context of a fractional quaternionic analysis. This, essentially with \( \frac{d}{dx} \) and \( (\mathcal{I}_{a+}^x f)(x) \) replaced by \( \psi \mathcal{D} \) and \( \psi \mathcal{I}_{a+}^x [f](q, x, \bar{\alpha}) \) respectively.

In particular, 2. establish a quaternionic analogous of the Fundamental Theorem, comparing with 3.

An additional observation is that operator \( \psi \mathcal{D}_{r,a}^x \) (right action of \( \psi \mathcal{D}_{a}^x \)) meets similar properties given in Proposition 3.3. For example

\[ \psi \mathcal{D}_{r,a}^x [f](q, x) = \psi \mathcal{D}_{r,x} \psi \mathcal{I}_{a+}^x [f](q, x, \bar{\alpha}). \]
Proposition 3.4. (Stokes type integral formula induced by $\psi\mathcal{D}_a^\alpha$) If $\bar{\alpha}, \bar{\beta} \in \mathbb{C}^4$ with $0 < \Re\alpha_\ell, \Re\beta_\ell < 1$ for $\ell = 0, 1, 2, 3$ and let $f, g \in AC^1(J_a^b, \mathbb{H})$ consider $q \in J_a^b$ such that the mappings $x \mapsto \psi\mathcal{I}_{a}^\alpha[f](q, x, \bar{\alpha})$ and $x \mapsto \psi\mathcal{I}_{a}^\alpha[g](q, x, \bar{\beta})$ belong to $C^1(J_a^b, \mathbb{H}(\mathbb{C}))$. Then

$$
\int_{\partial J_a^b} \psi\mathcal{I}_{a}^\alpha[g](q, x, \bar{\beta})\sigma_x^\alpha \psi\mathcal{I}_{a}^\alpha[f](q, x, \bar{\alpha}) \, dx.
$$

Proof. Applies [1] to $\psi\mathcal{I}_{a}^\alpha[g](q, x, \bar{\beta})$ and $\psi\mathcal{I}_{a}^\alpha[f](q, x, \bar{\alpha})$ to obtain

$$
\int_{\partial J_a^b} \left( \psi\mathcal{I}_{a}^\alpha[g](q, x, \bar{\beta}) \psi\mathcal{D}_a^\alpha[f](q, x) + \psi\mathcal{D}_{r,a}[g](q, x)\psi\mathcal{I}_{a}^\alpha[f](q, x) \right) \, dx.
$$

\[ \square \]

Theorem 3.5. (Borel-Pompieu type formula induced by $\psi\mathcal{D}_a^\alpha$ and $\psi\mathcal{D}_{r,a}$) Let $\bar{\alpha}, \bar{\beta} \in \mathbb{C}^4$ with $0 < \Re\alpha_\ell, \Re\beta_\ell < 1$ for $\ell = 0, 1, 2, 3$ and $f, g \in AC^1(J_a^b, \mathbb{H})$. Consider $q \in J_a^b$ such that the mappings $x \mapsto \psi\mathcal{I}_{a}^\alpha[f](q, x, \bar{\alpha})$ and $x \mapsto \psi\mathcal{I}_{a}^\alpha[g](q, x, \bar{\beta})$, for $x \in J_a^b$, belong to $C^1(J_a^b, \mathbb{H}(\mathbb{C}))$ then

$$
\begin{align*}
\int_{\partial J_a^b} & \left( \mathcal{R}_a^\alpha(\tau, x)\sigma_x^\alpha \psi\mathcal{I}_{a}^\alpha[f](q, \tau, \bar{\alpha}) + \psi\mathcal{I}_{a}^\alpha[g](q, \tau, \bar{\beta})\sigma_x^\alpha \mathcal{R}_a^\alpha(\tau, x) \right) \, d\tau \\
& - \int_{\partial J_a^b} \left( \mathcal{R}_{r,a}^\alpha(y, x)\psi\mathcal{D}_a^\alpha[f](q, y) + \psi\mathcal{D}_{r,a}[g](q, y)\mathcal{R}_{r,a}^\alpha(y, x) \right) \, dy \\
& = \sum_{i=0}^{3} (f + g)(q_0, \ldots, x_i, \ldots, q_3) + N[f](q, x, \bar{\alpha}) + N[g](q, x, \bar{\beta}), \quad x \in J_a^b,
\end{align*}
$$

where

$$
\mathcal{R}_a^\alpha(y, x) := \sum_{i=0}^{3} \left[ D_{a_i^+}^\alpha K(y - x) \right]
$$

and the partial derivative $D_{a_i^+}^\alpha$ is in terms of real the component $x_i$ of $x$ and

$$
N[f](q, x, \bar{\alpha}) = \sum_{i, j = 0}^{3} \frac{(\Gamma_{a_j^+}^\alpha f)(q_0, \ldots, x_j, \ldots, q_3)}{\Gamma(a_i)(x - a_i)^{\alpha_i}}.
$$
Proof. Borel-Pompieiu formula associated to the ψ-Fueter operator, see [10], applied in $\mathcal{I}_a^\tau[f](q, x, \alpha)$ and $\mathcal{I}_a^\tau[g](q, x, \beta)$ gives us

$$
\int_{\partial J_a^b} (K_\psi(\tau - x)\sigma^\tau \psi \mathcal{I}_a^\tau[f](q, \tau, \alpha) + \psi \mathcal{I}_a^\tau[g](q, \tau, \beta)\sigma^\tau K_\psi(\tau - x)) \\
- \int_{J_a^b} (K_\psi(y - x)\psi \mathcal{D}_y^\tau \mathcal{I}_a^\tau[f](q, y, \alpha) + \psi \mathcal{D}_y \mathcal{I}_a^\tau[g](q, y, \beta)K_\psi(y - x))dy
$$

$$
= \begin{cases} \\
\psi \mathcal{D}_a^\alpha[f](q, x, \alpha) + \psi \mathcal{I}_a^\tau[g](q, x, \beta), & x \in J_a^b, \\
0, & x \in \mathbb{H} \setminus J_a^b,
\end{cases}
$$

As

$$
\psi \mathcal{D}_a^\alpha[f](q, y) = \psi \mathcal{D}_y \mathcal{I}_a^\tau[f](q, y, \alpha),
\psi \mathcal{D}_r \mathcal{I}_a^\tau[g](q, y) = \psi \mathcal{D}_r \mathcal{I}_a^\tau[g](q, y, \beta),
$$

then

$$
\int_{\partial J_a^b} (K_\psi(\tau - x)\sigma^\tau \psi \mathcal{I}_a^\tau[f](q, \tau, \alpha) + \psi \mathcal{I}_a^\tau[g](q, \tau, \beta)\sigma^\tau K_\psi(\tau - x)) \\
- \int_{J_a^b} (K_\psi(y - x)\psi \mathcal{D}_a^\beta \mathcal{I}_a^\tau[f](q, y, \beta)K_\psi(y - x))dy
$$

$$
= \begin{cases} \\
\psi \mathcal{I}_a^\tau[f](q, x, \alpha) + \psi \mathcal{I}_a^\tau[g](q, x, \beta), & x \in J_a^b, \\
0, & x \in \mathbb{H} \setminus J_a^b,
\end{cases}
$$

Particularly, for $g = 0$ acting $\sum_{i=0}^3 D_{a_i}^{\alpha_i}[f](q, y) = \psi \mathcal{D}_r \mathcal{I}_a^\tau[f](q, y)$, where $D_{a_i}^{\alpha_i}$ is given in terms of the real component $x_i$ of $x$, on both sides, we see that

$$
\sum_{i=0}^3 D_{a_i}^{\alpha_i} \left[ \int_{\partial J_a^b} (K_\psi(\tau - x)\sigma^\tau \psi \mathcal{I}_a^\tau[f](q, \tau, \alpha)) \\
- \int_{J_a^b} K_\psi(y - x)\psi \mathcal{D}_{a_i}^{\alpha_i} \mathcal{I}_a^\tau[f](q, y)dy \right]
$$

$$
= \begin{cases} \\
\sum_{i=0}^3 D_{a_i}^{\alpha_i} \psi \mathcal{I}_a^\tau[f](q, x, \alpha), & x \in J_a^b, \\
0, & x \in \mathbb{H} \setminus J_a^b,
\end{cases}
$$

Combining fundamental theorem for Riemann-Liouville fractional calculus and [14].
we obtain the following:

\[
\sum_{i=0}^{3} D_{a_i}^{\alpha_i} \psi \mathcal{T}_a(x, a_i, a) = \sum_{i=0}^{3} \sum_{j=0}^{3} D_{a_i}^{\alpha_i} [ (\Gamma_{a_j}^{\alpha_i}) f(q_0, \ldots, x_j, \ldots, q_3)]
\]

\[
= \sum_{i=0}^{3} D_{a_i}^{\alpha_i} [ (\Gamma_{a_i}^{\alpha_i}) f(q_0, \ldots, x_i, \ldots, q_3)]
\]

\[
+ \sum_{i, j = 0}^{3} D_{a_i}^{\alpha_i} [ (\Gamma_{a_j}^{\alpha_i}) f(q_0, \ldots, x_j, \ldots, q_3)]
\]

\[
= \sum_{i=0}^{3} f(q_0, \ldots, x_i, \ldots, q_3)
\]

\[
+ \sum_{i, j = 0}^{3} \frac{(\Gamma_{a_j}^{\alpha_i}) f(q_0, \ldots, x_j, \ldots, q_3)}{\Gamma[\alpha_i](x - a_i)^{\alpha_i}}
\]

for all \(x \in J_b\).

Finally, Fubini’s Theorem, Leibniz formula and the previous computations give us that

\[
\int_{\partial J_a^b} \left[ \sum_{i=0}^{3} D_{a_i}^{\alpha_i} K_\psi(x - x) \right] \sigma_x \psi \mathcal{T}_a(x, \sigma_x, \alpha)
\]

\[
- \int_{J_a^b} \left[ \sum_{i=0}^{3} D_{a_i}^{\alpha_i} K_\psi(y - x) \right] \psi \mathcal{D}_a^{\alpha_i}[f(q, y)] dy
\]

\[
= \begin{cases} 
\sum_{i=0}^{3} f(q_0, \ldots, x_i, \ldots, q_3) \\
+ \sum_{i, j = 0}^{3} \frac{(\Gamma_{a_j}^{\alpha_i}) f(q_0, \ldots, x_j, \ldots, q_3)}{\Gamma[\alpha_i](x - a_i)^{\alpha_i}}, & x \in J_a^b, \\
0, & x \in \mathbb{H} \setminus J_a^b.
\end{cases}
\]
For $f = 0$, we can repeat the argument to conclude that

$$\int_{\partial J_a^b} \psi^{\mathcal{I}_a}(q, \tau, \beta) \sigma^\psi_{\tau} \mathcal{R}_{\psi,a}(\tau, x) = \int_{J_a^b} \psi^{\mathcal{D}_{r,a}}(q, y) \mathcal{R}_{\psi,a}(y, x) dy$$

$$= \left\{ \begin{array}{ll}
3 \sum_{i=0}^3 g(q_0, \ldots, x_i, \ldots, q_3) + N[g](q, x, \beta), & x \in J_a^b, \\
0, & x \in \mathbb{H} \setminus J_a^b.
\end{array} \right.$$ 

Remark 4. According to decomposition of the hiperholomorphic Cauchy kernel in terms of Gegenbauer polynomials given in page 93 of [14] we see that

$$R_{\psi,a}^\beta(y, x) := \frac{1}{2\pi^2} \sum_{k=0}^\infty \frac{1}{|y|^{k+3}} \sum_{i=0}^3 D_{\alpha_i}^\alpha [\omega^k A_4,k(x, y)],$$

with

$$2A_4,k(x, y) := [(k+1)C_{k+1}(s) + (2-n)C_k(s) \omega_y \wedge \omega_x] \omega_x,$$

where $C_{k+1}$ and $C_k$ are the Gegenbauer polynomials, $x = |x|\omega_x$, $y = |y|\omega_y$ and $s = (\omega_x, \omega_y)$, see [14].

Corollary 3.6. Under the same the hypothesis of Proposition 3.5 we have:

$$\int_{\partial J_a^b} \left( R_{\psi,a}^\beta(\tau, q) \sigma^\psi_{\tau} \mathcal{I}_a^\tau[f](q, \tau, \alpha) + \psi^{\mathcal{I}_a^\tau}[g](q, \tau, \beta) \sigma^\psi_{\tau} \mathcal{R}_{\psi,a}(\tau, q) \right)$$

$$- \int_{J_a^b} \left( R_{\psi,a}^\beta(y, q) \psi^{\mathcal{D}_{r,a}}[f](q, y) + \psi^{\mathcal{D}_{r,a}}[g](q, y) \mathcal{R}_{\psi,a}(y, q) \right) dy$$

$$= 4(f + g)(q) + N[f](q, q, \alpha) + N[g](q, q, \beta).$$

If $\psi^{\mathcal{D}_{r,a}}[f](q, \cdot) = \psi^{\mathcal{D}_{r,a}}[g](q, \cdot) = 0$ on $J_a^b$, then

$$\int_{\partial J_a^b} \left( R_{\psi,a}^\beta(\tau, x) \sigma^\psi_{\tau} \mathcal{I}_a^\tau[f](q, \tau, \alpha) + \psi^{\mathcal{I}_a^\tau}[g](q, \tau, \beta) \sigma^\psi_{\tau} \mathcal{R}_{\psi,a}(\tau, x) \right)$$

$$= \left\{ \begin{array}{ll}
3 \sum_{i=0}^3 (f + g)(q_0, \ldots, x_i, \ldots, q_3) + N[f](q, x, \alpha) + N[g](q, x, \beta), & x \in J_a^b, \\
0, & x \in \mathbb{H} \setminus J_a^b.
\end{array} \right.$$ 

and for $x = q$ we have that

$$\int_{\partial J_a^b} \left( R_{\psi,a}^\beta(\tau, q) \sigma^\psi_{\tau} \mathcal{I}_a^\tau[f](q, \tau, \alpha) + \psi^{\mathcal{I}_a^\tau}[g](q, \tau, \beta) \sigma^\psi_{\tau} \mathcal{R}_{\psi,a}(\tau, q) \right)$$

$$= 4(f + g)(q) + N[f](q, q, \alpha) + N[g](q, q, \beta).$$
3.2 An iterated fractional $\Psi$-Fueter operator

Now we shall study properties of an iterated fractional $\Psi$-Fueter operator.

Let $n \in \mathbb{N}$ and let $\Psi := \{\psi_1, \ldots, \psi_n\}$ be the collection of $n$ structural sets of $\mathbb{H}$. The left and right iterated $\Psi$-Fueter operators are denoted by

$$\Psi D = \psi_1 D \circ \cdots \circ \psi_n D,$$

and

$$\Psi D_r = \psi_1 D_r \circ \cdots \circ \psi_n D_r,$$

respectively. We will restrict our attention to the case $\Psi := \{\psi, \ldots, \psi\}$. Motivated by [5], it is possible to consider the general situation, but we will not develop this point here.

**Definition 3.7.** Let $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^4$ with $n = \lceil \Re(\alpha_\ell) \rceil + 1$ for $\ell = 0, 1, 2, 3$. We define the iterated fractional $\Psi$-Fueter operator of order $\vec{\alpha}$ on $\text{AC}^n(J_b^R, \mathbb{H})$ by

$$\Psi D_{\vec{\alpha}}^n[f](q, x) = \Psi D_x \circ \psi_I^n[f](q, x, n - \vec{\alpha}),$$

for all $q, x \in J_b^R$ and similarly

$$\Psi D_{r,\vec{\alpha}}^n[f](q, x) = \Psi D_{r,x} \circ \psi_I^n[f](q, x, n - \vec{\alpha}),$$

Here and subsequently by $\vec{1}$ we mean the vector $(1, 1, 1, 1)$. Hence $n\vec{1} - \vec{\alpha} = (n - \alpha_0, n - \alpha_1, n - \alpha_2, n - \alpha_3)$.

**Proposition 3.8.** Consider $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^4$ with $n = \lceil \Re(\alpha_\ell) \rceil + 1$ for $\ell = 0, 1, 2, 3$. Then

1. If $f \in C(J_b^R, \mathbb{H}) \cup L_2(J_b^R, \mathbb{H})$ then

$$\frac{1}{2} \sum_{j=0}^3 f(q_0, \ldots, x_j, \ldots, q_3) + \frac{1}{2} \sum_{j=0}^3 \psi D^{n-1} \circ \psi T^{(n-1)}[f](q_0, \ldots, x_j, \ldots, q_3) \psi_j,$$

where $\psi T^{(n-1)} = \psi T \circ \cdots \circ \psi T$, $(n - 1)$-times.

2. If $f \in \text{AC}^n(J_b^R, \mathbb{H})$ then

$$\psi D^{(n)} \circ \psi D_{\vec{\alpha}}^n[f](q, x) = \Delta_{\mathbb{R}^4}^n \circ \psi I_{\mathbb{R}^4}^n[f](q, x, \vec{\alpha}),$$

where $\psi D^{(n)} = \psi D \circ \cdots \circ \psi D$ and $\Delta_{\mathbb{R}^4}^n = \Delta_{\mathbb{R}^4} \circ \cdots \circ \Delta_{\mathbb{R}^4}$ $n$-times.
3. Let \( \vec{\beta} = (\beta_0, \ldots, \beta_3) \in C^4 \) with \( m = [\Re(\beta_\ell)] + 1 \) for \( \ell = 0, 1, 2, 3 \). Let \( f \in AC^{n+m}(J_{a, H}) \) be such that the mapping \( x \to f(q, x, \vec{\alpha}) \) belongs to \( C^{n+m}(J_{a, H}) \) then

\[
\psi D_{a}^{\vec{\alpha}} \circ \psi D_{a}^{\vec{\beta}}[f](q, x) = \sum_{j=0}^{3} \psi_{j}^{m+n} D_{a}^{\alpha_j + \beta_j} f(q_0, \ldots, x_j, \ldots, q_3)
\]

and

\[
\bar{\psi} D_{a}^{\vec{\alpha}} \circ \bar{\psi} D_{a}^{\vec{\beta}}[f](q, x) = \sum_{j,k=0}^{3} \psi_{j}^{-n+m} D_{a}^{\alpha_j + \beta_j} f(q_0, \ldots, x_j, \ldots, q_3).
\]

**Proof.** 1. Note that

\[
\psi D_{a}^{\vec{\alpha}} \circ \psi J_{a}^{\vec{\alpha}} \circ \psi T^{(n-1)}[f](q, x) = \psi D^{(n-1)} \circ (\psi D_{a}^{n-\vec{\alpha}} \circ \psi J_{a}^{n-\vec{\alpha}})[\psi T^{(n-1)}(f)](q, x, n \vec{\alpha} - \vec{\alpha})
\]

\[
= \psi D^{(n-1)} \circ (\sum_{j=0}^{3} \psi_{j}(\psi T^{(n-1)}(f)\psi_{j})(q_0, \ldots, x_j, \ldots, q_3))
\]

\[
= \frac{1}{2} \psi D^{(n-1)} \sum_{j=0}^{3} \psi_{j} \psi T^{(n-1)}(f)(q_0, \ldots, x_j, \ldots, q_3) + \sum_{j=0}^{3} \psi T^{(n-1)}(f)(q_0, \ldots, x_j, \ldots, q_3) \psi_{j}
\]

\[
= \frac{1}{2} \sum_{j=0}^{3} \psi D^{(n-1)} \circ \psi T^{(n-1)}(f)(q_0, \ldots, x_j, \ldots, q_3) + \sum_{j=0}^{3} \psi D^{(n-1)} \circ \psi T^{(n-1)}(f)(q_0, \ldots, x_j, \ldots, q_3) \psi_{j}
\]

\[
= \frac{1}{2} \sum_{j=0}^{3} f(q_0, \ldots, x_j, \ldots, q_3) + \sum_{j=0}^{3} \psi D^{(n-1)} \circ \psi T^{(n-1)}(f)(q_0, \ldots, x_j, \ldots, q_3) \psi_{j}.
\]

The proof is completed by using Proposition 3.3 and (9).

2. The verification of 2. and 3. are reduced to direct computations.
Proposition 3.9. (The Borel-Pompieu formula of higher order) Let $J_1, \ldots, J_n \subset \mathbb{H}$ be a sequence of open bounded rectangles such that $J_k \supset J_{k+1}$ for $k = 1, \ldots, n-1$. Let $f : \mathcal{J}_1 \to \mathbb{H}$ such that $\psi \mathcal{D}^{(n-\ell)}[f] \in C^1(J_\ell, \mathbb{H}) \cap C(\mathcal{J}_\ell, \mathbb{H})$ for $\ell = 1, \ldots, n$ then

$$
\int_{\partial J_n} K_\psi(y_n - x) \sigma_{y_n}^\psi f(y_n)
- \int_{J_n \times \partial J_{n-1}} K_\psi(y_n - x) K_\psi(y_{n-1} - y_n) \sigma_{y_{n-1}}^\psi \psi \mathcal{D}^{(1)}[f](y_{n-1})d\mu_{y_n}
+ \int_{J_n \times J_{n-1} \times \partial J_{n-2}} K_\psi(y_n - x) K_\psi(y_{n-1} - y_n) K_\psi(y_{n-2} - y_{n-1}) \sigma_{y_{n-2}}^\psi \psi \mathcal{D}^{(2)}[f](y_{n-2})d\mu_{y_{n-1}}d\mu_{y_n}
+ \\
\vdots \\
+ (-1)^{n-1} \int_{J_n \times J_{n-1} \times \cdots \times J_2 \times \partial J_1} K_\psi(y_n - x) K_\psi(y_{n-1} - y_n) \cdots K_\psi(y_2 - y_3) \sigma_{y_1}^\psi \psi \mathcal{D}^{(n-1)}[f](y_1)d\mu_{y_2}d\mu_{y_3} \cdots d\mu_{y_{n-1}}d\mu_{y_n}
+ (-1)^n \int_{J_n \times J_{n-1} \times \cdots \times J_2 \times J_1} K_\psi(y_n - x) K_\psi(y_{n-1} - y_n) \cdots K_\psi(y_2 - y_3) K_\psi(y_1 - y_2) \psi \mathcal{D}^{(n)}[f](y_1)d\mu_{y_2}d\mu_{y_3} \cdots d\mu_{y_{n-1}}d\mu_{y_n}
= \begin{cases} 
 f(x), & x \in J_n, \\
 0, & x \in \mathbb{H} \setminus \mathcal{J}_n,
\end{cases}
$$

Proof. Fixing each $\ell = 1, \ldots, n$, we can assert that

$$
\int_{\partial J_\ell} K_\psi(y_\ell - x) \sigma_{y_\ell}^\psi \psi \mathcal{D}^{(n-\ell)}[f](y_\ell) - \int_{J_\ell} K_\psi(y_\ell - x) \psi \mathcal{D}^{(n-1-\ell)}[f](y_\ell)d\mu_{y_\ell}
= \begin{cases} 
 \psi \mathcal{D}^{(n-\ell)}f(x), & x \in J_\ell, \\
 0, & x \in \mathbb{H} \setminus \mathcal{J}_\ell,
\end{cases}
$$

where $\psi \mathcal{D}^0 = \psi \mathcal{D}^0$ is the identity operator.

Particularly, for $\ell = 1$ we obtain

$$
\int_{\partial J_1} K_\psi(y_1 - x) \sigma_{y_1}^\psi \psi \mathcal{D}^{(n-1)}[f](y_1) - \int_{J_1} K_\psi(y_1 - x) \psi \mathcal{D}^{(n)}[f](y_1)d\mu_{y_1}
= \begin{cases} 
 \psi \mathcal{D}^{(n-1)}f(x), & x \in J_1, \\
 0, & x \in \mathbb{H} \setminus \mathcal{J}_1,
\end{cases}
$$

for $\ell = 2$:

$$
\int_{\partial J_2} K_\psi(y_2 - x) \sigma_{y_2}^\psi \psi \mathcal{D}^{(n-2)}[f](y_2) - \int_{J_2} K_\psi(y_2 - x) \psi \mathcal{D}^{(n-1)}[f](y_2)d\mu_{y_2}
= \begin{cases} 
 \psi \mathcal{D}^{(n-2)}f(x), & x \in J_2, \\
 0, & x \in \mathbb{H} \setminus \mathcal{J}_2,
\end{cases}
$$

17
and for $\ell = 3$

$$
\int_{\partial J_3} K_\psi(y_3 - x)\sigma_{y_3}^\psi \psi D^{(n-3)}[f](y_3) - \int_{J_3} K_\psi(y_3 - x)\psi D^{(n-2)}[f](y_3)d\mu_{y_3} = \left\{
\begin{array}{ll}
\psi D^{(n-3)}f(x), & x \in J_3, \\
0, & x \in \mathbb{H} \setminus J_3.
\end{array}
\right.
$$

Combining the previous representation formulas yields

$$
\int_{\partial J_3} K_\psi(y_3 - x)\sigma_{y_3}^\psi \psi D^{(n-3)}[f](y_3)
- \int_{J_3 \times \partial J_3} K_\psi(y_3 - x)K_\psi(y_2 - y_3)\sigma_{y_2}^\psi \psi D^{(n-2)}[f](y_2)d\mu_{y_3}
+ \int_{J_3 \times J_2 \times \partial J_1} K_\psi(y_3 - x)K_\psi(y_2 - y_3)K_\psi(y_1 - y_2)\sigma_{y_1}^\psi \psi D^{(n-1)}[f](y_1)d\mu_{y_2}d\mu_{y_3}
- \int_{J_3 \times J_2 \times J_1} K_\psi(y_3 - x)K_\psi(y_2 - y_3)K_\psi(y_1 - y_2)\psi D^{(n)}[f](y_1)d\mu_{y_1}d\mu_{y_2}d\mu_{y_3}
= \left\{
\begin{array}{ll}
\psi D^{(n-3)}f(x), & x \in J_3, \\
0, & x \in \mathbb{H} \setminus J_3.
\end{array}
\right.
$$

We next proceed by induction to obtain the result. \hfill \Box

Remark 5. In the same manner we can see that, given $g : J_\ell \to \mathbb{H}$ such that

$$
\psi D_r^{(n-\ell)}[g] \in C^1(J_\ell, \mathbb{H}) \cap C(\overline{J_\ell}, \mathbb{H}),
$$
for \( \ell = 0, \ldots, n \), we get

\[
\int_{\partial J_n} g(y_n) \sigma_{y_n} K_\psi(y_n - x) \\
- \int_{J_n} \psi \mathcal{D}_r^{(1)}[g](y_{n-1}) \sigma_{y_{n-1}} K_\psi(y_{n-1} - y_n) K_\psi(y_n - x) d\mu_{y_n} \\
+ \int_{J_n \times J_{n-1} \times \partial J_{n-2}} \psi \mathcal{D}_r^{(2)}[g](y_{n-2}) \sigma_{y_{n-2}} K_\psi(y_{n-2} - y_{n-1}) K_\psi(y_{n-1} - y_n) K_\psi(y_n - x) d\mu_{y_{n-1}} d\mu_{y_n} \\
+ \ldots \\
+ (-1)^{n-1} \int_{J_n \times \cdots \times J_2 \times \partial J_1} \psi \mathcal{D}_r^{(n-1)}[g](y_1) K_\psi(y_1 - y_2) \sigma_{y_1} \\
K_\psi(y_2 - y_3) \cdots K_\psi(y_{n-1} - y_n) K_\psi(y_n - x) d\mu_{y_2} d\mu_{y_3} \cdots d\mu_{y_{n-1}} d\mu_{y_n} \\
+ (-1)^n \int_{J_n \times \cdots \times J_2 \times J_1} \psi \mathcal{D}_r^{(n)}[g](y_1) K_\psi(y_1 - y_2) \\
K_\psi(y_2 - y_3) \cdots K_\psi(y_{n-1} - y_n) K_\psi(y_n - x) d\mu_{y_1} d\mu_{y_2} \cdots d\mu_{y_{n-1}} d\mu_{y_n} \\
= \left\{ \begin{array}{ll}
g(x), & x \in J_n, \\
0, & x \in \mathbb{H} \setminus J_n, \end{array} \right.
\]

**Corollary 3.10.** Let \( J_1, \ldots, J_n \subset \mathbb{H} \) be a sequence of open bounded rectangles such that \( J_k \supset J_{k+1} \) for \( k = 0, \ldots, n \) and set \( f : J_1 \to \mathbb{H} \) such that such \( g(x) = \psi \mathcal{T}_a^\ell[f](q, x, \alpha) \), for all \( x \in J_1 \), satisfies the hypothesis of Proposition \( \mathcal{X}_\mathcal{X} \) i.e., \( \psi \mathcal{D}_r^{(n-\ell)}[g] \in C^1(J_\ell, \mathbb{H}) \cap C(J_{\ell+1}, \mathbb{H}) \) for \( \ell = 1, \ldots, n \) and also \( f \big|_{J_n} \in AC^1(J_n, \mathbb{H}) \) satisfies that the mapping \( x \to \psi \mathcal{T}_a^\ell[f](q, x, \alpha) \), for all \( x \in J_n^a = J_n \), belongs to
$C^1(J_n, \mathbb{H}(\mathbb{C}))$. Then
\[
\int_{\partial J_n} \mathbf{R}_{\psi, a}^\alpha (y_n - x) \sigma_{y_n}^\psi \mathbf{T}_{a}^{y_n} [f](q, y_n, \alpha) \\
- \int_{J_n \times \partial J_{n-1}} \mathbf{R}_{\psi, a}^\alpha (y_n - x) K_{\psi} (y_{n-1} - y_n) \sigma_{y_{n-1}}^\psi \Psi \mathbf{D}_{a}^{\alpha_{n-1}} [f](q, y_{n-1}) d\mu_{y_n} \\
+ \int_{J_n \times J_{n-1} \times \partial J_{n-2}} \mathbf{R}_{\psi, a}^\alpha (y_n - x) K_{\psi} (y_{n-1} - y_n) K_{\psi} (y_{n-2} - y_{n-1}) \sigma_{y_{n-2}}^\psi \Psi \mathbf{D}_{a}^{\alpha_{n-2}} [f](q, y_{n-2}) d\mu_{y_{n-1}} d\mu_{y_n} \\
+ \cdots \\
+ (-1)^{n-1} \int_{J_n \times J_{n-1} \times \cdots \times J_2 \times \partial J_1} \mathbf{R}_{\psi, a}^\alpha (y_n - x) K_{\psi} (y_{n-1} - y_n) \cdots K_{\psi} (y_2 - y_3) \\
K_{\psi} (y_1 - y_2) \sigma_{y_1}^\psi \Psi \mathbf{D}_{a}^{\alpha_1} [f](q, y_1) d\mu_{y_2} d\mu_{y_3} \cdots d\mu_{y_{n-1}} d\mu_{y_n} \\
+ (-1)^n \int_{J_n \times J_{n-1} \times \cdots \times J_2 \times J_1} \mathbf{R}_{\psi, a}^\alpha (y_n - x) K_{\psi} (y_{n-1} - y_n) \cdots K_{\psi} (y_2 - y_3) \\
K_{\psi} (y_1 - y_2) \Psi \mathbf{D}_{a}^{\alpha_2} [f](q, y_1) d\mu_{y_2} d\mu_{y_3} \cdots d\mu_{y_{n-1}} d\mu_{y_n} \\
= \begin{cases} 
\sum_{i=0}^{3} f(q_0, \ldots, x_i, \ldots, q_3) + N[f](q, x, \alpha), & x \in J_n, \\
0, & x \in \mathbb{H} \setminus \mathcal{T}_n.
\end{cases}
\]

Proof. Applying Proposition 3.9 to the mapping $x \rightarrow \psi \mathbf{T}_{a}^\alpha [f](q, x, \alpha)$ and acting on both sides the operator $\sum_{i=0}^{3} \mathbf{D}_{a_i}^{\alpha_i}$ the proof is complete. \[\square\]

Corollary 3.11. The following assertions may be proved in much the same way as before.
1. If \( q \in J_n \) doing \( x = q \) we obtain

\[
\int_{\partial J_n} \tilde{R}_{\Psi,a}^\alpha(y_n - q) \sigma_{y_n}^\Psi \mathcal{J}^y_a[f](q, y_n, \tilde{\alpha}) \\
- \int_{J_n \times \partial J_{n-1}} \tilde{R}_{\Psi,a}^\alpha(y_n - q) K_\psi(y_{n-1} - y_n) \sigma_{y_n}^\psi \Psi \mathcal{D}_a^{\alpha-(n-1)}[f](q, y_{n-1})d\mu_{y_n} \\
+ \int_{J_n \times J_{n-1} \times \partial J_{n-2}} \tilde{R}_{\Psi,a}^\alpha(y_n - q) K_\psi(y_{n-1} - y_n) K_\psi(y_{n-2} - y_{n-1}) \sigma_{y_n}^\psi \\
\Psi \mathcal{D}_a^{\alpha-(n-2)}[f](q, y_{n-2})d\mu_{y_{n-1}}d\mu_{y_n}
\]

\[
+ \cdots
\]

\[
+ (-1)^{n-1} \int_{J_n \times J_{n-1} \times \cdots \times J_2 \times \partial J_1} \tilde{R}_{\Psi,a}^\alpha(y_n - q) K_\psi(y_{n-1} - y_n) \cdots K_\psi(y_2 - y_3) \\
K_\psi(y_1 - y_2) \sigma_{y_1}^\psi \Psi \mathcal{D}_a^{\alpha-1}[f](q, y_1) d\mu_{y_2} d\mu_{y_3} \cdots d\mu_{y_{n-1}} d\mu_{y_n}
\]

\[
= \begin{cases} 
4f(q) + N[f](q, q, \tilde{\alpha}), & x \in J_n, \\
0, & x \in \mathbb{H} \setminus \overline{J_n},
\end{cases}
\]

2. If \( \Psi \mathcal{D}_a^\alpha[f](q, x) = 0 \) for all \( x \in J_1 \) then

\[
\int_{\partial J_n} \tilde{R}_{\Psi,a}^\alpha(y_n - x) \sigma_{y_n}^\Psi \mathcal{J}^y_a[f](q, y_n, \tilde{\alpha}) \\
- \int_{J_n \times \partial J_{n-1}} \tilde{R}_{\Psi,a}^\alpha(y_n - x) K_\psi(y_{n-1} - y_n) \sigma_{y_n}^\psi \Psi \mathcal{D}_a^{\alpha-(n-1)}[f](q, y_{n-1})d\mu_{y_n} \\
+ \int_{J_n \times J_{n-1} \times \partial J_{n-2}} \tilde{R}_{\Psi,a}^\alpha(y_n - x) K_\psi(y_{n-1} - y_n) K_\psi(y_{n-2} - y_{n-1}) \sigma_{y_n}^\psi \\
\Psi \mathcal{D}_a^{\alpha-(n-2)}[f](q, y_{n-2})d\mu_{y_{n-1}}d\mu_{y_n}
\]

\[
+ \cdots
\]

\[
+ (-1)^{n-1} \int_{J_n \times J_{n-1} \times \cdots \times J_2 \times \partial J_1} \tilde{R}_{\Psi,a}^\alpha(y_n - x) K_\psi(y_{n-1} - y_n) \cdots K_\psi(y_2 - y_3) \\
K_\psi(y_1 - y_2) \sigma_{y_1}^\psi \Psi \mathcal{D}_a^{\alpha-1}[f](q, y_1) d\mu_{y_2} d\mu_{y_3} \cdots d\mu_{y_{n-1}} d\mu_{y_n}
\]

\[
= \begin{cases} 
\sum_{i=0}^3 f(q_0, \ldots, x_i, \ldots, q_3) + N[f](q, x, \tilde{\alpha}), & x \in J_n, \\
0, & x \in \mathbb{H} \setminus \overline{J_n},
\end{cases}
\]
3. If $\Psi D_\vec{\alpha} \circ \psi_n[f](q, x) = 0$ for all $x \in \partial J_k$, for $k = 1, \ldots, n-1$, and $\Psi D_\vec{\alpha}[f](q, x) = 0$ for all $x \in J_1$ we obtain

$$\int_{\partial J_n} R^{\vec{\alpha}}_{\psi, a}(y_n - x) \sigma_{y_n}^{\psi} T_\vec{\alpha}^{y_n}[f](q, y_n, \vec{\alpha}) \left\{ \begin{array}{ll}
\sum_{i=0}^3 f(q_0, \ldots, x_i, \ldots, q_3) + N[f](q, x, \vec{\alpha}), & x \in J_n, \\
0, & x \in \mathbb{H} \setminus J_n. \end{array} \right.$$ 

If $q \in J_n$ doing $x = q$ we obtain

$$\int_{\partial J_n} R^{\vec{\alpha}}_{\psi, a}(y_n - q) \sigma_{y_n}^{\psi} T_\vec{\alpha}^{y_n}[f](q, y_n, \vec{\alpha}) = 4f(q) + N[f](q, q, \vec{\alpha}).$$

Remark 6. Much of the formulas in previous proposition and corollaries can be extended to the right-hand versions of the iterated fractional $\Psi$-Fueter operator.

**Conclusion and Future development**

In this work, we extend basic results from $\psi$–hyperholomorphic function theory and the related operator calculus to the fractional setting. We consider fractional derivative in the sense of Riemann–Liouville to introduce a fractional form of the $\psi$–Fueter operator that depends on an additional vector of complex parameters with fractional real parts. More precisely, we present a fractional version of the classical quaternionic Borel-Pompeiu type formula associated to this fractional $\psi$–Fueter operator.

The results of Subsection 3.2 have been encouraging enough to merit further investigation considering the general case $\Psi := \{\psi_1, \ldots, \psi_n\}$ for different structural sets, dealing with a generalized iterated fractional $\Psi$-Fueter operator of order $\vec{\alpha}$ on $AC^n(J_{a,d}, \mathbb{H})$ given by

$$\Psi, \psi_{n+1} D_\vec{\alpha}^\vec{\alpha}[f](q, x) = \Psi D_x \circ \psi_{n+1} T_\vec{\alpha}[f](q, x, n\vec{1} - \vec{\alpha}),$$

for all $q, x \in J_{a,d}$ and similarly

$$\Psi, \psi_{n+1} D_{r,a}^\vec{\alpha}[f](q, x) = \Psi D_{r,x} \circ \psi_{n+1} T_{\vec{\alpha}}[f](q, x, n\vec{1} - \vec{\alpha}),$$

where $\psi_{n+1}$ is another structural set. Work in this direction is currently under progress, some aspects are still challenging and require further research.

**Declarations**

**Funding**

Instituto Politécnico Nacional (grant number SIP20211188) and CONACYT.
Conflict of interest
The authors declare that they have no conflict of interest regarding the publication of this paper.

Author contributions
Both authors contributed equally to the manuscript and typed, read, and approved the final form of the manuscript, which is the result of an intensive collaboration.

References
[1] Baleanu, D., Magin, R. L., Bhalekar, S., Daftardar-Gejji, V. Chaos in the fractional order nonlinear Bloch equation with delay. Commun. Nonlinear Sci. Numer. Simul. 25 (2015), no. 1-3, 41–49.

[2] Baleanu, D., Restrepo, J. E., Suragan, D. A class of time-fractional Dirac type operators. Chaos Solitons Fractals 143 (2021) 110590.

[3] Bernstein, S. A fractional Dirac operator. In: Noncommutative analysis, operator theory and applications Operator Theory: Advances and Applications, vol 252.. eds Alpay D., Cipriani F., Colombo F., Guido D., Sabadini I., Sauvageot J. L. (Birkhäuser/Springer, Cham, 2016) 27 - 41.

[4] Bonilla B., Kilbas A. A. and Trujillo J. J. Cálculo Fraccionario y Ecuaciones Diferenciales Fraccionarias, Servicio de Publicaciones de la UNED, Madrid, (2003).

[5] Bory Reyes, J. De Schepper, H., Guzmán Adán, A., Sommen, F. Higher order Borel-Pompeiu representations in Clifford analysis. Math. Methods Appl. Sci. 39(16) (2016) 4787-4796.

[6] Coloma, N.; Di Teodoro, A.; Ochoa-Tocachi, D.; Ponce, F. Fractional Elementary Bicomplex Functions in the Riemann–Liouville Sense. Adv. Appl. Clifford Algebr. 31 (2021), no. 4, Paper No. 63.

[7] Contharteze Grigoletto, E., Capelas de Oliveira, E. Fractional Versions of the Fundamental Theorem of Calculus, Appl. Math. 4 (2013) 23–33.

[8] Delgado, B.B.; Macías-Díaz, J.E. On the General Solutions of Some Non-Homogeneous Div-Curl Systems with Riemann–Liouville and Caputo Fractional Derivatives. Fractal Fract. 2021, 5, 117.

[9] Ferreira, M., Vieira, N. Eigenfunctions and fundamental solutions of the fractional Laplace and Dirac operators: the Riemann–Liouville case. Complex Anal. Oper. Theory 10 (5), 1081-1100 (2016).
[10] Ferreira, M., Vieira, N. Eigenfunctions and fundamental solutions of the fractional Laplace and Dirac operators using Caputo derivatives. Complex Var. Elliptic Equ. 62 (9), 1237-1253, (2017).

[11] Ferreira, M, Kraußhar, R. S., Rodrigues, M. M., Vieira, N. A higher dimensional fractional Borel-Pompeiu formula and a related hypercomplex fractional operator calculus. Math. Methods Appl. Sci. 42(10) (2019) 3633–3653.

[12] Gorenflo, R., Mainardi, F. Fractional calculus: integral and differential equations of fractional order. In: Fractals and fractional calculus in continuum mechanics (Udine, 1996), CISM Courses and Lect., 378, (Springer, Vienna, 1997), pp 223–276.

[13] Gürlebeck, K., Sprössig, W. Quaternionic analysis and elliptic boundary value problems. (Birkhäuser, Verlag, 1990).

[14] Gürlebeck, K., Sprössig, W. Quaternionic and Clifford calculus for physicists and engineers. (John Wiley and Sons, New York, 1997).

[15] Kähler, U., Vieira, N. Fractional Clifford analysis. In: Bernstein, S., Kähler, U., Sabadini, I., Sommen, F. (eds.) Hypercomplex analysis: new perspectives and applications. Trends in mathematics, 191-201. Birkhäuser, Basel, 2014.

[16] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204. 1st ed. (Elsevier Science., Amsterdam, 2006).

[17] Kravchenko, V. V., Shapiro, M. V. Integral Representations for Spatial Models of Mathematical Physics. (Addison Wesley Longman Inc, Essex, 1996).

[18] Kravchenko, V.V. Applied Quaternionic Analysis. Research and Exposition in Mathematics. (Heldermann Verlag, Lemgo, 2003).

[19] Liouville, J. Mémoire sur le calcul des différentielles a indices quelconques. J. Ecole Polytech., 13 (1832) 71-162.

[20] Miller, K. S., Ross, B. An Introduction to the Fractional Calculus and Fractional Differential Equations. (John Wiley & Sons, Inc., New York, 1993).

[21] Mitelman, I., Shapiro, M. Differentiation of the Martinelli-Bochner integrals and the notion of the hyperderivability. Math. Nachr., 1995, 172, 211-238.

[22] Naser, M. Hyperholomorphic functions. Siberian Math. J. 12 (1971) 959-968.

[23] Nôno, K. On the quaternion linearization of Laplacian $\triangle$. Fukuoka Univ. 35 (1986) 5-10.

[24] Nôno, K. Hyperholomorphic functions of a quaternion variable. Fukuoka Univ. 32 (1983) 21-37.
[25] Oldham, K. B., Spanier, J. The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. (Dover Publ. Inc., New York, 2006).

[26] Ortigueira, M. D. Fractional calculus for scientists and engineers. Lecture Notes in Electrical Engineering, 84. (Springer, Dordrecht, 2011).

[27] Peña Pérez, Y. Abreu Blaya, R. Árciga Alejandre, M. P., Bory Reyes, J. Biquaternionic reformulation of a fractional monochromatic Maxwell system. Adv. High Energy Phys. 2020, Art. ID 6894580, 9 pp.

[28] Podlubny, I. Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering, 198. (Academic Press, Inc., San Diego, CA, 1999).

[29] Riemann, B. Versuch einer allgemeinen Auffassung der Integration und Differentiation (An attempt to a general understanding of integration and differentiation) (1847). In: H. Weber (ed.), Bernhard Riemanns gesammelte mathematische Werke und wissenschaftlicher Nachlass. 2nd ed. (Dover Publications, New York, 1953).

[30] Ross B. A brief history and exposition of the fundamental theory of fractional calculus. In: Ross B. (eds) Fractional Calculus and Its Applications. Lecture Notes in Mathematics, vol 457. (Springer, Berlin, Heidelberg, 1975)

[31] Samko, S.G., Kilbas, A.A., Marichev, O.I. Fractional Integrals and Derivatives. Theory and Applications. (Gordon and Breach Sci. Publ. London, New York, 1993 (Nauka, Minsk, 1987)).

[32] Srivastava, H. M., Dubey, V. P., Kumar, R.; Singh, J., Kumar, D., Baleanu, D. An efficient computational approach for a fractional-order biological population model with carrying capacity. Chaos Solitons Fractals 138 (2020), 109880, 13 pp.

[33] Shapiro, M. Quaternionic analysis and some conventional theories. In: Operator Theory ed. Alpay, D. (Springer, Basel, 2015) pp. 1423-1446.

[34] Shapiro, M. Some remarks on generalizations of the one dimensional complex analysis: hypercomplex approach (Trieste, 1993). In: Functional Analytic Methods in Complex Analysis and Applications to Partial Differential Equations. Ed. Tutschke W. (World Scienti. Publ., River Edge, NJ, 1995) pp. 379-401.

[35] Shapiro, M, Vasilevski, N. L. Quaternionic ψ-monogenic functions, singular operators and boundary value problems. I. ψ-Hyperholomorphy function theory. Compl. Var. Theory Appl. 27 (1995) 17–46.
[36] Shapiro, M., Vasilevski, N. L. Quaternionic $\psi$-hyperholomorphic functions, singular operators and boundary value problems II. Algebras of singular integral operators and Riemann type boundary value problems. Compl. Var. Theory Appl. 27 (1995) 67–96.

[37] Sudbery, A. Quaternionic analysis, Math. Proc. Phil. Soc. 85 (1979) 199–225.

[38] Tarasov, V. E. Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media. (Springer, New York, 2011).

[39] Valério, D., Trujillo, J.J., Rivero, M. et al. Fractional calculus: A survey of useful formulas. Eur. Phys. J. Spec. Top. 222 (2013) 1827-1846.

[40] Vieira, N. Fischer decomposition and Cauchy-Kovalevskaya extension in fractional Clifford analysis: the Riemann-Liouville case. Proc. Edinb. Math. Soc. II. 60 (1), 251-272, 2017.