QUASILINEAR PARABOLIC EQUATIONS WITH FIRST ORDER TERMS AND $L^1$-DATA IN MOVING DOMAINS

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ABSTRACT. The global existence of weak solutions to a class of quasilinear parabolic equations with nonlinearities depending on first order terms and integrable data in a moving domain is investigated. The class includes the $p$-Laplace equation as a special case. Weak solutions are shown to be global by obtaining appropriate estimates on the gradient as well as a suitable version of Aubin-Lions lemma in moving domains.

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1. INTRODUCTION

Problems defined on a domain which changes its shape in time have recently attracted a lot of attention from mathematical community since not only they lead to interesting mathematical questions, but also they arise naturally in physics, biology, chemistry and many other fields. Examples include the studies of pattern formation on evolving surfaces [BEM11, GMM+11], of surfactants in two-phase flows [GLS14], of dealloying by surface dissolution of a binary alloy [EE08], of a diffusion interface model for linear surface partial differential equations [ES09], or of modeling and simulation of cell mobility [CMEV12]. We refer the interested reader to the extensive review paper [KK15]. In this paper, we study the global existence of a quasilinear parabolic equation in moving domains, i.e. domains with shapes evolving in time. The equation contains a quasilinear diffusion operator, which includes the $p$-Laplacian as a special case, has a nonlinearity depending on the zero and first order terms, and has external force and initial data which are only integrable.

To precisely state the problem under consideration, we consider a bounded domain $\Omega_0 \subset \mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial \Omega_0$. Let $v : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be a smooth and compactly supported vector field and $\zeta : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be its corresponding flow, i.e. $\zeta$ solves

$$\partial_t \zeta(x, t) = v(\zeta(x, t), t), \quad \zeta(x_0, 0) = x_0$$

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for any $x_0 \in \mathbb{R}^d$. Note that for each fixed $x$, the mapping $t \mapsto \zeta_t(x)$ is an integral curve of $v$ and for fixed $t$, the mapping $x_0 \mapsto \zeta_t(x_0)$ is a diffeomorphism. Assuming that $\Omega_0 \subset \text{supp}(v)$, we define $\Omega_t = \zeta_t(\Omega_0)$ and the non-cylindrical domain as

$$
Q_T := \bigcup_{t \in (0, T)} \Omega_t \times \{t\} = \bigcup_{t \in (0, T)} \zeta_t(\Omega_0) \times \{t\},
$$

$$
\Sigma_T := \bigcup_{t \in (0, T)} \partial \Omega_t \times \{t\} = \bigcup_{t \in (0, T)} \zeta_t(\partial \Omega_0) \times \{t\}.
$$

We choose an open and bounded subset $\hat{\Omega}$ of $\mathbb{R}^d$ such that $\cup_{t \in [0, T]} \Omega_t \subset \hat{\Omega}$ and let $\hat{Q} := \hat{\Omega} \times (0, T)$.

We also need to define time-space spaces in moving domains. Let $\{X(t)\}_{t \in [0, T]}$ be a family of Banach spaces, then we define

$$
L^p(0, T; X(t)) = \{ f : Q_T \to \mathbb{R} : f(t) \in X(t) \text{ for a.e. } t \in (0, T) \}
$$

with the norm

$$
\|f\|_{L^p(0, T; X(t))} = \left( \int_0^T \|f(t)\|^p_{X(t)} dt \right)^{\frac{1}{p}} < +\infty.
$$

Very common in this paper we use $X(t) = L^q(\Omega_t)$ or $X(t) = W^{1,q}_0(\Omega_t)$. In particular, when $p = q$, then we write simply $L^p(\Omega_t)$ instead of $L^p(0, T; L^p(\Omega_t))$.

The main goal of the present paper is to study the global existence of the following quasilinear problem

$$
\begin{cases}
\partial_t u - \text{div}(a(x, t, \nabla u)) + \text{div}(uv) + g(x, t, u, \nabla u) = f, & (x, t) \in Q_T, \\
u(x, t) = 0, & (x, t) \in \Sigma_T, \\
u(x, 0) = u_0(x), & x \in \Omega_0,
\end{cases}
$$

with the external force $f \in L^1(Q_T)$ and initial data $u_0 \in L^1(\Omega_0)$. The nonlinear diffusion $a$ is assumed to satisfy

(A1) $a : \hat{Q} \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function;

(A2) there exists $p > \frac{2d+1}{d+1}$ such that, for $(x, t) \in \hat{Q}$ and $\xi \in \mathbb{R}^d$,

$$
|a(x, t, \xi)| \leq \varphi(x, t) + K|\xi|^{p-1}
$$

where $\varphi \in L^p(\hat{Q}), 1/p + 1/p' = 1$ and $K \geq 0$;

(A3) there exists $\alpha > 0$ such that

$$
a(x, t, \xi)\xi \geq \alpha|\xi|^p,
$$

where $(x, t) \in \hat{Q}$ and $\xi \in \mathbb{R}^d$;

(A4) for almost $(x, t) \in \hat{Q}$ and all $\xi, \xi' \in \mathbb{R}^d$,

$$
(a(x, t, \xi) - a(x, t, \xi'))(\xi - \xi') \geq \frac{1}{\Theta(x, t, \xi', \xi''')}|\xi - \xi''|^\theta 
$$

and $a(x, t, 0) = 0$;

for $\theta > 1$, and $\Theta$ is a nonnegative function which satisfies

$$
|\Theta(x, t, \xi, \xi')| \leq C(1 + |\xi| + |\xi'|)^\vartheta
$$

where

$$
\vartheta < (\theta - 1)\left(p - \frac{d}{d+1}\right).
$$
The nonlinearity $g : \hat{Q} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies: $g$ is continuous with respect to the third and fourth variables, and

(G1) it holds

$$\lambda g(x, t, \lambda, \xi) \geq 0 \quad \text{for all} \quad \lambda \in \mathbb{R}, \xi \in \mathbb{R}^d;$$

(G2) $g$ has a subcritical growth on the gradient, i.e. there exists $0 \leq \sigma < p$ such that

$$|g(x, t, \lambda, \xi)| \leq h(|\lambda|) (\gamma(x, t) + |\xi|^\sigma)$$

where $\gamma \in L^1(\hat{Q})$, and $h$ is an increasing function in $\mathbb{R}_+$. Let us briefly discuss about the above conditions. The conditions (A1)–(A4) of $a$ assure that it contains the important case of $p$-Laplacian, i.e.

$$a(x, t, \xi) \equiv a_p(\xi) = |\nabla \xi|^{p-2} \nabla \xi \quad \text{for} \quad p > \frac{2d+1}{d+1}.$$ 

Moreover, the technical condition (A4) is weaker than the usual strong monotonicity condition

$$(a(x, t, \xi) - a(x, t, \xi'))(\xi - \xi') \geq C|\xi - \xi'|^p,$$

for some $C > 0$, but still stronger than the mere monotone property condition, i.e.

$$(a(x, t, \xi) - a(x, t, \xi'))(\xi - \xi') \geq 0. \quad (4)$$

We also remark that the condition (G2) allows $g$ to have arbitrary growth in the zero order term, as long as it has the suitable sign stated in (G1). A typical example of $g$ is

$$g(x, t, u, \nabla u) = Cu^{2k+1}(\gamma(x, t) + |\nabla u|^\sigma)$$

where $k \in \mathbb{N}$ is arbitrary and $0 \leq \sigma < p$.

Elliptic or parabolic equations with $L^1$-data, such as external force of initial data, appear frequently in applications and are therefore of interest and importance. Concrete examples include elliptic systems modeling electronical devices [GH94], the Fokker-Planck equation arising from populations dynamics [GS98], models of turbulent flows in oceanography and climatology [Lew97], incompressible flows with small Reynolds number [Lio96], or Keller-Segel or Shigesada-Kawasaki-Teramoto type systems [Win19]. Global existence of weak or renormalized solutions to special cases of (1) in fixed domains has been studied extensively in the literature. Let us mention several related works: in [BG89], the authors considered (1) where conditions (A1)–(A4) are imposed, but the function $g$ does not have the first order term; similar results were shown in [Bla93] assuming (4) instead of (A4); the case when $g$ depends on the first order term was considered in [GS01], but the second order term therein is simply the linear elliptic operator, for instance $\text{div}(a(x, t, \nabla u)) = \Delta u$; when $p > 1$ arbitrary, one can only hope for renormalized solutions (see Remark 1.1), its global existence was shown in [BM97]. Related results are also obtained for systems without the first order terms [BS05].

The global existence of solutions to (1) in moving domains is, up to our knowledge, completely open, and that is the main motivation of our paper. We would like also to emphasize that, even in the case of a fixed domain, our results extend that of [BG89] and [GS01].

The main goal of this paper is to prove the global existence of a weak solution to (1) under the conditions (A1)–(A4), (G1)–(G2) and data $f \in L^1(\hat{Q})$ and $u_0 \in L^1(\Omega_0)$. To state the main result, we first give the precise definition of a weak solution.
Definition 1.1 (Weak solutions). Let $T > 0$ be arbitrary. A function $u \in C([0, T]; L^q(\Omega_t)) \cap L^q(0, T; W_0^{1,q}(\Omega_t))$ for all $1 \leq q < p - d/(d + 1)$ is called a weak solution to (1) on $(0, T)$ if $g(x, t, u, \nabla u) \in L^1(Q_T)$ and for all $\psi \in C(0, T; W_0^{1,q}(\Omega_t)) \cap C^1(0, T; L^q(\Omega_t))$, with $q' = q/(q - 1)$ the following weak formulation holds

$$
\int_{\Omega_T} u(T)\psi(T)dx - \int_0^T \int_{\Omega_t} u\psi_t dx dt
$$
$$
+ \int_0^T \int_{\Omega_t} [a(x, t, \nabla u) \cdot \nabla \psi - u \nabla \psi + g(x, t, u, \nabla u)\psi] dx dt
$$
$$
= \int_{\Omega_0} u_0\psi(0)dx + \int_0^T \int_{\Omega_t} f \psi dx dt.
$$

All the terms above are obviously well-defined except for the term containing $g(x, t, u, \nabla u)\psi$. Since $1 < q < p - \frac{d}{d+1}$, the conjugate Hölder exponent $q' = \frac{q}{q-1} > d$ when $q$ is close enough to 1. Therefore, thanks to the embedding $W_0^{1,q}(\Omega_t) \hookrightarrow L^\infty(\Omega_t)$, we have $\psi \in L^\infty(Q_T)$, and therefore, the integration $\int_{Q_T} g(x, t, u, \nabla u)\psi dx dt$ makes sense since $g(x, t, u, \nabla u) \in L^1(Q_T)$.

Remark 1.1. The condition $p > (2d + 1)/(d + 1)$ is needed to define the weak solution. When $p \leq (2d + 1)/(d + 1)$, we can only obtain $\nabla u \in L^q(Q_T)$ for $q \in (0, 1)$. In this case, one expects to show the existence of renormalized solutions instead, which goes beyond the scope of this paper.

The main result of this paper is the following theorem.

Theorem 1.1 (Global existence of weak solutions). Assume the conditions (A1)–(A4) and (G1)–(G2). Then for any $u_0 \in L^1(\Omega_0)$ and any $f \in L^1(Q_T)$, there exists a global weak solution $u$ to (1) on $(0, T)$ as in Definition 1.1.

Let us describe the main ideas in proving Theorem 1.1. To treat moving domains, one can transform the problem into the case of fixed domains and then study the new equation, with the cost of some additional terms. Usually these additional terms depend significantly on the problem itself, and therefore each problem needs to be treated separately. As an attempt to have a more unified mechanism, a different approach is to derive a mechanism to work on the moving domains directly, that is to establish parallel tools for moving domains corresponding to that of fixed domains. This research direction has been investigated by many authors (see e.g. [AES15, AET18, MB08, Vie14]).

In this paper, we adapt the second approach to prove Theorem 1.1, meaning that we treat (1) directly on the non-cylindrical domain $Q_T$. More precisely, first, we consider an approximation of (1) in which the data is approximated by $f_\varepsilon \in L^\infty(Q_T)$ and by $u_0 \in L^\infty(\Omega_0)$. Moreover, we also regularize the nonlinearity $g_\varepsilon = g(1 + \varepsilon|g|)^{-1}$ which is bounded for any fixed $\varepsilon > 0$. Thanks to this regularization, we can use the method from [CNO17] to obtain the global existence of an approximate solution $u_\varepsilon$. The next goal is to derive estimates of this approximate solution uniformly in $\varepsilon$. In order to do that, due to the low regularity of the data, we refine the analysis in [GS01] to adapt to the case of quasilinear problem (1). Once the uniform estimates for $u_\varepsilon$ are obtained, we would like to pass to the limit as $\varepsilon \to 0$, which consequently requires an Aubin-Lions lemma in the case of moving domains. A similar lemma has been shown in different works (see e.g. [Mou16] or [Fuj70]), but they are not applicable to our situation. Therefore, we prove a new Aubin-Lions lemma in moving domains, which allows us to first obtain the almost everywhere convergence $u_\varepsilon \rightarrow u$ and then consequently $\|u_\varepsilon - u\|_{L^1(Q_T)} \rightarrow 0$. Due to the dependence of the
nonlinearity on $\nabla u$, this convergence is not yet enough. By using the ideas from [GS01], we utilize the assumptions (G1) and (G2) to show that the convergence $\nabla u_\varepsilon \to \nabla u$ holds almost everywhere. This in turn helps to get $g_\varepsilon(x, t, u_\varepsilon, \nabla u_\varepsilon) \to g(x, t, u, \nabla u)$ and $a(x, t, \nabla u_\varepsilon) \to a(x, t, \nabla u)$ in appropriate spaces, and eventually to obtain $u$ to be a weak solution to (1).

**The rest of this paper is organized as follows:** In the next Section, we derive uniform a-priori estimates for approximate solutions, which are needed to pass to the limit in Section 3 to obtain the weak solution of (1). The Appendix A and B provide the existence of an approximate solution and a proof of the Aubin-Lions lemma in moving domains respectively.

**Notation.** We will use in this paper the following set of notations.

- Recall that we simply write $L^p(Q_T)$ instead of $L^p(0, T; L^p(\Omega_t))$.
- The double integration $\int_0^T \int_{\Omega_t} dxdt$ is written using the shorthand notation $\int_{Q_T} dxdt$.
- We usually write $C = C(\alpha, \beta, \gamma, \ldots)$ to indicate that the constant $C$ depends on the arguments $\alpha, \beta, \gamma$, etc.

## 2. Uniform estimates

In this section, we consider an approximate problem to (1) and derive uniform *a priori* estimates for the approximate solution. These estimates play a crucial role in passing to the limit to obtain a weak solution to (1). For simplicity we write $g(u, \nabla u)$ instead of $g(x, t, u, \nabla u)$.

Fix an arbitrary time horizon $T > 0$. As usual we regularize the initial data $u_0$ and the external term $f$ by more regular data $u_{0, \varepsilon} \in L^\infty(\Omega_0)$ and $f_\varepsilon \in L^\infty(Q_T)$ for $\varepsilon > 0$, such that

$$\lim_{\varepsilon \to 0} \|u_{0, \varepsilon} - u_0\|_{L^1(\Omega_0)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_{L^1(Q_T)} = 0,$$

and

$$\|u_{0, \varepsilon}\|_{L^1(\Omega_0)} \leq \|u_0\|_{L^1(\Omega_0)} \quad \text{and} \quad \|f_\varepsilon\|_{L^1(Q_T)} \leq \|f\|_{L^1(Q_T)}.$$

Moreover, we also regularize the nonlinear first order term by a bounded nonlinearity, namely, for $\varepsilon > 0$,

$$g_\varepsilon(w, \nabla w) := \frac{g(w, \nabla w)}{1 + \varepsilon |g(w, \nabla w)|}.$$

Note that for any fixed $\varepsilon > 0$, we have

$$|g_\varepsilon(w, \nabla w)| \leq \frac{1}{\varepsilon} \quad \text{for all} \quad (x, t) \in Q_T \quad \text{and all} \quad w.$$

The approximate problem reads as,

$$\begin{cases}
\partial_t u_\varepsilon - \text{div}(a(x, t, \nabla u_\varepsilon)) + \text{div}(u_\varepsilon \nabla v) + g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) = f_\varepsilon, \quad (x, t) \in Q_T, \\
u_\varepsilon(x, t) = 0, \quad (x, t) \in \Sigma_T, \\
u_\varepsilon(x, 0) = u_{0, \varepsilon}(x), \quad x \in \Omega_0.
\end{cases}$$

**Definition 2.1** (Weak solutions to (7)). A weak solution to (7) on $(0, T)$ is a function $u_\varepsilon \in C([-T, T]; L^p(\Omega_t)) \cap L^p(0, T; W^{1, p}_0(\Omega_t))$ with $\partial_t u_\varepsilon \in L^p(0, T; W^{1, -p}(\Omega_t))$, where $W^{1, -p}(\Omega_t) = (W^{1, p}_0(\Omega_t))^*$, such that

$$\begin{align*}
\int_0^T \int_{\Omega_t} \partial_t u_\varepsilon \phi \cdot \nabla \phi dxdt + \int_0^T \int_{\Omega_t} a(x, t, \nabla u_\varepsilon) \cdot \nabla \phi dxdt \\
- \int_0^T \int_{\Omega_t} u_\varepsilon \nabla \phi dxdt + \int_0^T \int_{\Omega_t} g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \phi dxdt = \int_0^T \int_{\Omega_t} f_\varepsilon \phi dxdt.
\end{align*}$$
for all test function $\phi \in L^p(0, T; W_0^{1,p}(\Omega_t))$.

The global existence of a weak solution to (7) can be obtained by the slicing technique in e.g. [CNO17] with suitable, slight modifications. We postpone this proof to the Appendix A in order to not interrupt the train of thought.

**Theorem 2.1** (Existence of a global solution to the approximate problem). Fix $T > 0$. For any $u_{0, \varepsilon} \in L^\infty(\Omega_0)$ and $f_\varepsilon \in L^\infty(Q_T)$, there exists a weak solution to (7) on $(0, T)$.

The focus of this section is therefore to obtain a-priori estimates of solutions to (7) which are uniform in $\varepsilon$. We divide the section further into two subsections, in which the first one shows uniform bounds of approximate solutions in Sobolev spaces, while the second provides uniform bounds of the nonlinearity $g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$.

### 2.1. Uniform bounds of approximate solutions.

The following lemma is the main result of this subsection.

**Lemma 2.1.** There exists a constant $C(T)$ depending on $T$, $v$, $\|u_0\|_{L^1(\Omega_0)}$, and $\|f\|_{L^1(Q_T)}$ but independent of $\varepsilon$ such that

$$\|u_\varepsilon\|_{L^q(0,T;W_0^{1,q}(\Omega_t))} \leq C(T)$$

for all $1 \leq q < p - d/(d+1)$.

The proof of this is long and technical and is therefore divided into several steps. As a preparation, we need a lemma about Sobolev embeddings in moving domains.

**Lemma 2.2** (Sobolev embeddings). Fix $T > 0$. Then there exists a constant $C_{\Omega,T}$ depending on $T$ and $\varepsilon_0$ such that

$$\|u\|_{L^{q^*(\Omega_t)}} \leq C_{\Omega,T}\|\nabla u\|_{L^q(\Omega_t)} \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad u \in W_0^{1,q}(\Omega_t)$$

where $q < d$ and

$$q^* = \frac{dq}{d - q}.$$

**Proof.** The classical Sobolev embedding gives

$$\|u\|_{L^{q^*}(\Omega_t)} \leq C(\Omega_0)\|\nabla u\|_{L^q(\Omega_t)}.$$

Since $\zeta \in C([0, T]; C^1(\mathbb{R}^d))$, there exists $a(T), b(T)$ such that

$$a(T) \leq |\det(D\zeta_t)(x)| \leq b(T) \quad \text{for all} \quad t \in [0, T].$$

Now, for $t \in [0, T]$, we know that $\Omega_t = \zeta_t(\Omega_0)$. Therefore,

$$\left(\int_{\Omega_t} |u(x)|^{q^*} \, dx\right)^{\frac{1}{q^*}} = \left(\int_{\Omega_0} |u(y)|^{q^*} |\det(D\zeta_t)| \, dy\right)^{\frac{1}{q^*}} \leq b(T)^{\frac{1}{q^*}} C(\Omega_0) \left(\int_{\Omega_0} |\nabla u(y)|^q \, dy\right)^{\frac{1}{q}} \leq b(T)^{\frac{1}{q^*}} a(T)^{-\frac{1}{q}} C(\Omega_0) \left(\int_{\Omega_t} |\nabla u(x)|^q \, dx\right)^{\frac{1}{q}},$$

which proves the desired estimate (8). □
Lemma 2.3. Assume that \( u_\varepsilon \in L^p(0,T; W^{1,p}_0(\Omega_t)) \) satisfies
\[
\sup_{t \in (0,T)} \int_{\Omega_t} |u_\varepsilon| dx \leq \beta,
\]
and for each \( n \in \mathbb{N} \),
\[
\int_{B_n} |\nabla u_\varepsilon|^p dxdt \leq C_0 + C_1 \int_{E_n} |\nabla u_\varepsilon| dxdt
\]
for some \( \beta, C_0, C_1 > 0 \) independent of \( \varepsilon \) where
\[
B_n = \{(x,t) \in Q_T : n \leq |u_\varepsilon(x,t)| \leq n+1\} \quad \text{and} \quad E_n = \{(x,t) \in Q_T : |u_\varepsilon(x,t)| > n+1\}.
\]
Then there exists \( C(T,p,q,\beta,C_0,C_1) \) depending on \( T,p,q,\beta,C_0 \) and \( C_1 \), but independent of \( \varepsilon \), such that
\[
\|u_\varepsilon\|_{L^q(0,T;W^{1,q}_0(\Omega_t))} \leq C(T,p,q,\beta,C_0,C_1)
\]
for all \( 1 \leq q < p - \frac{d}{d+1} \).

Remark 2.1. We remark that since \( p > q \), obviously \( u_\varepsilon \in L^q(0,T; W^{1,q}_0(\Omega_t)) \) follows immediately from \( u_\varepsilon \in L^p(0,T; W^{1,p}_0(\Omega_t)) \) and \( \|u_\varepsilon\|_{L^q(0,T;W^{1,q}_0(\Omega_t))} \leq C(T)\|u_\varepsilon\|_{L^p(0,T;W^{1,p}_0(\Omega_t))} \). However, the essential role of (12) is that the constant \( C(T) \) therein is independent of \( \varepsilon \), while the norm \( \|u_\varepsilon\|_{L^p(0,T;W^{1,p}_0(\Omega_t))} \) might blow up as \( \varepsilon \to 0 \).

Proof of Lemma 2.3. Let \( 1 \leq q < p \) be arbitrary. From (10), by using Hölder’s inequality, we have
\[
\int_{B_n} |\nabla u_\varepsilon|^p dxdt \leq C_0 + C_1 \left( \int_{E_n} |\nabla u_\varepsilon|^q dxdt \right)^{1/q} |E_n|^{(q-1)/q}
\]
\[
\leq C_0 + C_1 \|\nabla u_\varepsilon\|_{L^q(Q_T)} |E_n|^{(q-1)/q}.
\]

Since \( q < p \) we can use Hölder’s inequality, inequality (13), and the elementary inequality \( (a+b)^{q/p} \leq a^{q/p} + b^{q/p} \) for \( a, b \geq 0 \), to obtain
\[
\int_{B_n} |\nabla u_\varepsilon|^q dxdt \leq |B_n|^{(p-q)/p} \left( \int_{B_n} |\nabla u_\varepsilon|^p dxdt \right)^{q/p}
\]
\[
\leq |B_n|^{(p-q)/p} \left( C_0^{q/p} + C_1^{q/p} \|\nabla u_\varepsilon\|_{L^p(Q_T)} |E_n|^{(q-1)/p} \right).
\]

Let \( r \geq 0 \) be chosen later. We have, by using the definitions of \( B_n \) and \( E_n \),
\[
\begin{cases}
|B_n| \leq \frac{1}{n^r} \int_{B_n} |u_\varepsilon|^r dxdt, \\
|E_n| \leq \frac{1}{n^r} \int_{E_n} |u_\varepsilon|^r dxdt \leq \frac{1}{n^r} \|u_\varepsilon\|_{L^r(Q_T)}^r.
\end{cases}
\]

Inserting (15) into (14) yields
\[
\int_{B_n} |\nabla u_\varepsilon|^q dxdt \leq C_0^{q/p} \left( \frac{1}{n^r} \right)^{(r-q)/p} \left( \int_{B_n} |u_\varepsilon|^r dxdt \right)^{(p-q)/p}
\]
\[
+ C_1^{q/p} \|\nabla u_\varepsilon\|_{L^p(Q_T)} |E_n|^{(q-1)/p} \left( \frac{1}{n^r} \right)^{(r-1)/p} \left( \int_{B_n} |u_\varepsilon|^r dxdt \right)^{(p-q)/p}.
\]

Let \( K \in \mathbb{N} \) be chosen later. We split \( \|\nabla u_\varepsilon\|_{L^q(Q_T)}^q \) as follows
\[
\|\nabla u_\varepsilon\|_{L^q(Q_T)}^q = \int_{Q_T} |\nabla u_\varepsilon|^q dxdt = \sum_{n=0}^{K} \int_{B_n} |\nabla u_\varepsilon|^q dxdt + \sum_{n=K+1}^{\infty} \int_{B_n} |\nabla u_\varepsilon|^q dxdt.
\]
Since $|B_n| \leq |Q_T|$ and $|E_n| \leq |Q_T|$, we simply evaluate the first term in the right hand side of (17) using (14) as follows
\[
\sum_{n=0}^{K} \int_{B_n} |\nabla u_\varepsilon|^q dx dt \leq (K + 1)C_2 \left( 1 + \|\nabla u_\varepsilon\|_L^q(Q_T) \right),
\] (18)
where $C_2 = \max\{C_0^{q/p} |Q_T|^{(p-q)/p}, C_1^{q/p} |Q_T|^{(q-1)/p}\}$. Using Young’s inequality in (17)-(18), we get
\[
\|\nabla u_\varepsilon\|_{L^q(Q_T)}^q \leq C(K) + 2 \sum_{n=K+1}^{\infty} \int_{B_n} |\nabla u_\varepsilon|^q dx dt,
\] (19)
where
\[
C(K) = 2^{\frac{2}{p} - 1} \cdot ((K + 1)C_2)^{\frac{p}{p-1}} \left( \frac{2}{p} \right)^{\frac{1}{p-1}} + 2(K + 1)C_2.
\]
Note that the constant $C(K)$ tends to infinity as $K \to \infty$. It remains to proceed to the study of the series which appears on the right hand side of (19). Applying Hölder’s inequality on the series with exponents $p/(p-q)$ and $p/q$ and using (16), we have
\[
\sum_{n=K+1}^{\infty} \int_{B_n} |\nabla u_\varepsilon|^q dx dt
\leq C_0^{q/p} \left( \sum_{n=K+1}^{\infty} \frac{1}{n^{(p-q)/q}} \right)^{q/p} \left( \sum_{n=K+1}^{\infty} \int_{B_n} |u_\varepsilon|^r dx dt \right)^{(p-q)/p}
+ C_1^{q/p} \|\nabla u_\varepsilon\|_{L^q(Q_T)}^q \|u_\varepsilon\|^r_{L^q(Q_T)} \left( \sum_{n=K+1}^{\infty} \frac{1}{n^{(p-1)/q}} \right)^{q/p}
\]
\[
\leq C_0^{q/p} \left( \sum_{n=K+1}^{\infty} \frac{1}{n^{(p-q)/q}} \right)^{q/p} \|u_\varepsilon\|^r_{L^q(Q_T)}
+ C_1^{q/p} \|\nabla u_\varepsilon\|_{L^q(Q_T)}^q \|u_\varepsilon\|^r_{L^q(Q_T)} \left( \sum_{n=K+1}^{\infty} \frac{1}{n^{(p-1)/q}} \right)^{q/p}.
\] (20)
We choose $r$ so that the remainder of the series above converges to zero as $K \to \infty$, i.e.
\[
\frac{r(p-q)}{q} > 1.
\] (21)
Note that due to $q \geq 1$, this already implies $r(p-1)/q > 1$. It follows from (20) that
\[
\|\nabla u_\varepsilon\|_{L^q(Q_T)}^q \leq C(K) + \delta(K) \left( \|u_\varepsilon\|^{r(p-q)/p}_{L^q(Q_T)} + \|\nabla u_\varepsilon\|^{q/p}_{L^q(Q_T)} \|u_\varepsilon\|^{r(p-1)/p}_{L^q(Q_T)} \right)
\] (22)
with
\[
\delta(K) = 2 \max \left\{ C_0^{q/p} \left( \sum_{n=K+1}^{\infty} \frac{1}{n^{(p-q)/q}} \right)^{q/p} ; C_1^{q/p} \left( \sum_{n=K+1}^{\infty} \frac{1}{n^{(p-1)/q}} \right)^{q/p} \right\}
\]
with the property $\lim_{K \to \infty} \delta(K) = 0$ thanks to (21). From Young’s inequality, and recalling that $q/p < q$, we have
\[
\|\nabla u_\varepsilon\|_{L^q(Q_T)}^q \leq \frac{1}{p} \|\nabla u_\varepsilon\|_{L^q(Q_T)}^q + \frac{p-1}{p} \|u_\varepsilon\|_{L^p(Q_T)}^p.
\]
Therefore, (22) implies
\[ \| \nabla u_\varepsilon \|_{L^q(T_r)}^q \]
\[ \leq C(K) + \delta(K) \left[ \| u_\varepsilon \|_{L^{r}(Q_r)}^{r(p-q)/p} + \frac{p-1}{p} \| u_\varepsilon \|_{L^{r}(Q_r)}^{r-1} + \frac{1}{p} \| \nabla u_\varepsilon \|_{L^q(Q_r)}^q \right] \]
(23)
\[ \leq C(K) + \delta(K) \left[ \frac{q}{p} + \frac{2p-q-1}{p} \| u_\varepsilon \|_{L^{r}(Q_r)}^{r-1} + \frac{1}{p} \| \nabla u_\varepsilon \|_{L^q(Q_r)}^q \right] \]
where we used \( r(p-q)/p = r - \frac{r}{p} < r \) and the Young inequality \( y^{r(p-q)/q} \leq \frac{p-q}{p} y^r + \frac{q}{p} \) at the last step. We will show now that by choosing a suitable \( r \) (which satisfies (21)) we can estimate
\[ \| u_\varepsilon \|_{L^{r}(Q_r)}^r \leq C(T, \beta) \| \nabla u_\varepsilon \|_{L^q(Q_r)}^q \]
with \( \beta \) is in (9). Indeed, by setting
\[ r = \frac{q(d+1)}{d}, \]
(24)
we have
\[ \frac{r(p-q)}{q} = \frac{(d+1)(p-q)}{d} > 1 \quad \text{since} \quad q < p - \frac{d}{d+1}, \]
thus (21) is satisfied. Note that from (24) we also have \( r < q^* = \frac{dq}{d-q} \). Therefore, we can use the interpolation inequality with \( \frac{1}{r} = \frac{\eta}{1} + \frac{1-\eta}{q^*} \), and \( \sup_{t \in (0,T)} \| u_\varepsilon \|_{L^1(\Omega_t)} \leq \beta \) to estimate
\[ \| u_\varepsilon \|_{L^r(Q_r)} = \int_0^T \| u_\varepsilon \|_{L^{r}(\Omega_t)} dt \leq \int_0^T \| u_\varepsilon \|_{L^{1}(\Omega_t)}^{r(1-\eta)} \| u_\varepsilon \|_{L^{q^*}(\Omega_t)}^{r(\eta)} dt \]
\[ \leq \beta^{r\eta} \int_0^T \| u_\varepsilon \|_{L^{q^*}(\Omega_t)}^{r(1-\eta)} dt. \]
(25)
From (24), we can easily check that \( r(1-\eta) = q \). Therefore, (25) yields
\[ \| u_\varepsilon \|_{L^{r}(Q_r)} \leq \beta^{r\eta} \| u_\varepsilon \|_{L^{q}(0,T;L^{q^*}(\Omega_\ast))}^{q}. \]
(26)
By using Lemma 2.2,
\[ \| u_\varepsilon \|_{L^{q}(0,T;L^{q^*}(\Omega_\ast))}^{q} \leq C_{\Omega,T}^{q} \int_0^T \| \nabla u_\varepsilon \|_{L^{q}(\Omega_t)}^q dt = C_{\Omega,T}^{q} \| \nabla u_\varepsilon \|_{L^{q}(Q_r)}^q. \]
(27)
Combining (23), (26) and (27) leads to
\[ \| \nabla u_\varepsilon \|_{L^{q}(Q_r)}^q \leq C(K) + \delta(K) \left[ \frac{q}{p} + \left( 2p-q-1 \beta^{r\eta} C_{\Omega,T}^{q} + \frac{1}{p} \right) \| \nabla u_\varepsilon \|_{L^{q}(Q_r)}^q \right]. \]
(28)
Recalling that \( \lim_{K \to \infty} \delta(K) = 0 \). We choose \( K \) large enough to have
\[ \delta(K) \left( \frac{2p-q-1}{p} \beta^{r\eta} C_{\Omega,T}^{q} + \frac{1}{p} \right) \leq \frac{1}{2}, \]
which, in combination with (28), implies
\[ \| \nabla u_\varepsilon \|_{L^{q}(Q_r)}^q \leq 2 \left( C(K) + \delta(K) \frac{q}{p} \right), \]
which is the desired estimate (12). \( \square \)

In order to prove Lemma 2.1, thanks to Lemma 2.3, it is sufficient to prove (9) and (10) for solutions to the approximate problem (7). These will be shown in the next consecutive lemmas.
Lemma 2.4. There exists a constant $\beta = \beta \left(T, \|u_0\|_{L^1(\Omega_0)}, \|f\|_{L^1(Q_T)}\right)$ independent of $\varepsilon$ such that for any solution to (7), the following holds

$$\|u_\varepsilon\|_{L^\infty(0,T;L^1(\Omega_t))} \leq \beta.$$  

Proof. Let $k \in \mathbb{R}^+$. We define the truncated function

$$T_k(z) = \begin{cases}  
  z, & \text{if } |z| \leq k, \\
  k, & \text{if } z > k, \\
  -k, & \text{if } z < -k. 
\end{cases}$$

It is clear that $T_k$ is a Lipschitz function, and if $u_\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega_t))$ then $T_k(u_\varepsilon) \in L^p(0,T;W_0^{1,p}(\Omega_t))$ with

$$\nabla T_k(u_\varepsilon) = \chi_{\{|u_\varepsilon| \leq k\}} \nabla u_\varepsilon,$$  

where $\chi_{\{|u_\varepsilon| \leq k\}}$ is the characteristic function of the set $\{(x,t) \in Q_T : |u_\varepsilon(x,t)| \leq k\}$. Choosing $\phi = T_k(u_\varepsilon)$ as test function for (7), we obtain, with $S_k = \int_0^\tau T_k(d\tau)$,

$$d\int_{\Omega_t} S_k(u_\varepsilon) dx + \int_{\Omega_t} a(x,t,\nabla u_\varepsilon) \nabla T_k(u_\varepsilon) dx + \int_{\Omega_t} \text{div}(u_\varepsilon \nabla u_\varepsilon) T_k(u_\varepsilon) dx + \int_{\Omega_t} g_\varepsilon(u_\varepsilon,\nabla u_\varepsilon) T_k(u_\varepsilon) dx = \int_{\Omega_t} f_\varepsilon T_k(u_\varepsilon) dx.$$

From (A3) and (29), we have

$$\int_{\Omega_t} \text{div}(u_\varepsilon \nabla u_\varepsilon) T_k(u_\varepsilon) dx = -\int_{\Omega_t} u_\varepsilon \nabla u_\varepsilon \nabla T_k(u_\varepsilon) dx = \int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq k\}} |\nabla u_\varepsilon|^p dx. \tag{31}$$

Applying integration by parts for penultimate step on the left hand side of (30), we obtain

$$\int_{\Omega_t} \text{div}(u_\varepsilon \nabla u_\varepsilon) T_k(u_\varepsilon) dx = -\int_{\Omega_t} u_\varepsilon \nabla u_\varepsilon \nabla T_k(u_\varepsilon) dx = -\int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq k\}} u_\varepsilon \nabla u_\varepsilon dx. \tag{32}$$

Combining (31)-(32) with (30), we get

$$\frac{d}{dt} \int_{\Omega_t} S_k(u_\varepsilon) dx + \alpha \int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq k\}} |\nabla u_\varepsilon|^p dx + \int_{\Omega_t} g_\varepsilon(u_\varepsilon,\nabla u_\varepsilon) T_k(u_\varepsilon) dx \leq \int_{\Omega_t} |f T_k(u_\varepsilon)| dx + \int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq k\}} u_\varepsilon \nabla u_\varepsilon dx. \tag{33}$$
Applying Young’s inequality for last term in right-hand side above, we have
\[
\int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq \varepsilon\}} u_\varepsilon \nabla u_\varepsilon dx \leq \|v\|_\infty \int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq \varepsilon\}} \|\nabla u_\varepsilon\| dx
\]
\[
\leq \frac{\alpha}{2} \int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq \varepsilon\}} \|\nabla u_\varepsilon\|^p dx + C(\alpha, \|v\|_\infty) \int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq \varepsilon\}} |u_\varepsilon|^p dx
\]
\[
\leq \frac{\alpha}{2} \int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq \varepsilon\}} \|\nabla u_\varepsilon\|^p dx + C(\alpha, \|v\|_\infty) \|u_\varepsilon\|_{L^p(\Omega_t)}^p
\]
\[
= \frac{\alpha}{2} \int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq \varepsilon\}} \|\nabla u_\varepsilon\|^p dx + C(T, \alpha, \|v\|_\infty, k),
\]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Since \( |T_k(u_\varepsilon)| \leq k \),
\[
\int_{\Omega_t} \|f_\varepsilon T_k(u_\varepsilon)\| dx \leq k \|f_\varepsilon\|_{L^1(\Omega_t)}. \tag{35}
\]

We remark that \( u_\varepsilon T_k(u_\varepsilon) \geq 0 \), combining with (G1), we have \( g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) T_k(u_\varepsilon) \geq 0 \). Therefore, inserting (34) and (35) into (33) gives
\[
\frac{d}{dt} \int_{\Omega_t} S_k(u_\varepsilon) dx + \frac{\alpha}{2} \int_{\Omega_t} \chi_{\{|u_\varepsilon| \leq \varepsilon\}} \|\nabla u_\varepsilon\|^p dx \leq k \|f_\varepsilon\|_{L^1(\Omega_t)} + C(T, \alpha, \|v\|_\infty, k). \tag{36}
\]

Integrating (36) on \((0, t)\) for \( t \in (0, T) \) we obtain, in particular
\[
\sup_{t \in (0, T)} \|S_k(u_\varepsilon)(t)\|_{L^1(\Omega_t)} \leq \|S_k(u_\varepsilon,0)\|_{L^1(\Omega_0)} + k \|f_\varepsilon\|_{L^1(Q_T)} + TC(T, \alpha, \|v\|_\infty, k). \tag{37}
\]

We set \( k = 1 \) in (37). Note that \( 0 \leq S_1(z) \leq |z| \) and recall (6), we get
\[
\sup_{t \in (0, T)} \int_{\Omega_t} S_1(u_\varepsilon)(t) dx \leq \|u_0\|_{L^1(\Omega_0)} + k \|f\|_{L^1(Q_T)} + TC(T, \alpha, \|v\|_\infty, 1).
\]

Therefore, by using \( u_\varepsilon = S_1(u_\varepsilon) \) for \(|u_\varepsilon| \geq 1\),
\[
\sup_{t \in (0, T)} \|u_\varepsilon\|_{L^1(\Omega_t)} = \sup_{t \in (0, T)} \int_{\{x \in \Omega_t: |u_\varepsilon| \leq 1\}} |u_\varepsilon| dx + \sup_{t \in (0, T)} \int_{\{x \in \Omega_t: |u_\varepsilon| \geq 1\}} |u_\varepsilon| dx
\]
\[
\leq \sup_{t \in (0, T)} \|u_\varepsilon\|_{L^1(\Omega_t)} + \sup_{t \in (0, T)} \int_{\Omega_t} |S_1(u_\varepsilon)| dx
\]
\[
\leq \sup_{t \in (0, T)} \|u_\varepsilon\|_{L^1(\Omega_t)} + \|u_0\|_{L^1(\Omega_0)} + k \|f\|_{L^1(Q_T)} + TC(T, \alpha, \|v\|_\infty, 1)
\]
\[
=: \beta.
\]

This completes the proof of Lemma 2.4. \( \square \)

**Lemma 2.5.** There exist positive constants \( C_0, C_1 \) independent of \( \varepsilon \) and \( n \in \mathbb{N} \) such that the following estimate holds
\[
\int_{B_n} |\nabla u_\varepsilon|^p dx dt \leq C_0 + C_1 \int_{E_n} |\nabla u_\varepsilon| dx dt \quad \text{for all} \quad \varepsilon > 0 \text{ and all } n \in \mathbb{N},
\]
where \( u_\varepsilon \) is a solution to (7).
Proof. For $n \in \mathbb{N}$, we define the function $\phi_n : \mathbb{R} \to \mathbb{R}$ as

$$
\phi_n(z) = \begin{cases} 
1, & \text{if } z > n + 1, \\
z - n, & \text{if } n \leq z \leq n + 1, \\
0, & \text{if } -n < z < n, \\
-z - n, & \text{if } -n - 1 \leq z \leq -n, \\
1, & \text{if } z \leq -n - 1,
\end{cases}
$$

(38)

and we set $\Psi_n(z) = \int_0^z \phi_n(\tau)d\tau$. We note that $\phi_n$ is a Lipschitz function, and therefore $u_\varepsilon \in L^p(0, T; W^{1, p}_0(\Omega_t))$ implies $\phi_n(u_\varepsilon) \in L^p(0, T; W^{1, p}_0(\Omega_t))$ with

$$
\nabla \phi_n(u_\varepsilon) = \chi_{B_n} \nabla u_\varepsilon,
$$

$\chi_{B_n}$ denoting the characteristic function of the set $B_n = \{(x, t) \in Q_T : n \leq |u_\varepsilon(x, t)| \leq n + 1\}$ defined in (11). We now take $\phi_n(u_\varepsilon) \in L^p(0, T; W^{1, p}_0(\Omega_t))$ as test function for (7) to get

$$
\frac{d}{dt} \int_{\Omega_t} \Psi_n(u_\varepsilon)dx + \int_{\Omega_t} a(x, t, \nabla u_\varepsilon) \nabla \phi_n(u_\varepsilon)dx + \int_{\Omega_t} \text{div}(u_\varepsilon v) \phi_n(u_\varepsilon)dx + \int_{\Omega_t} g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \phi_n(u_\varepsilon)dx \leq \int_{\Omega_t} |f_\varepsilon \phi_n(u_\varepsilon)|dx,
$$

(39)

and consequently, by integrating on $(0, T)$,

$$
\int_{\Omega_T} \Psi_n(u_\varepsilon)(T)dx + \int_{Q_T} a(x, t, \nabla u_\varepsilon) \nabla \phi_n(u_\varepsilon)dxdt + \int_{Q_T} \text{div}(u_\varepsilon v) \phi_n(u_\varepsilon)dxdt + \int_{Q_T} g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \phi_n(u_\varepsilon)dxdt \leq \int_{\Omega_0} \Psi_n(u_{0, \varepsilon})dx + \int_{Q_T} |f_\varepsilon \phi_n(u_\varepsilon)|dxdt.
$$

(40)

From (A3), we obtain

$$
\int_{Q_T} a(x, t, \nabla u_\varepsilon) \nabla \phi_n(u_\varepsilon)dxdt = \int_{Q_T} \chi_{B_n} a(x, t, \nabla u_\varepsilon) \nabla u_\varepsilon dxdt \\
g \leq \alpha \int_{Q_T} \chi_{B_n} |\nabla u_\varepsilon|^p dxdt \\
= \alpha \int_{B_n} |\nabla u_\varepsilon|^p dxdt.
$$

(41)

The penultimate term on the left hand side of (39) can be rewritten as

$$
\int_{Q_T} \text{div}(u_\varepsilon v) \phi_n(u_\varepsilon)dxdt = \int_{Q_T} (\nabla u_\varepsilon \cdot v + u_\varepsilon \text{div}v) \phi_n(u_\varepsilon)dxdt.
$$

(42)
Combining (41)-(42) with (40), and the fact that $\Psi_0$ is nonnegative, we get

$$
\alpha \int_{B_n} |\nabla u_\varepsilon|^p dx dt + \int_{Q_T} g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \phi_n(u_\varepsilon) dx dt
\leq \int_{\Omega_0} \Psi_0(u_{0,\varepsilon}) dx + \int_{Q_T} |f_\varepsilon| \phi_n(u_\varepsilon) dx dt
+ \int_{Q_T} |\nabla u_\varepsilon \cdot v| \phi_n(u_\varepsilon) dx dt + \int_{Q_T} |\text{div} v| \phi_n(u_\varepsilon) dx dt
\leq \int_{\Omega_0} \Psi_0(u_{0,\varepsilon}) dx + \|f\|_{L^1(Q_T)} + \|v\|_\infty \int_{Q_T} |\nabla u_\varepsilon| \phi_n(u_\varepsilon) dx dt + \|\text{div} v\|_\infty T \beta
$$

where we used $\sup_{t \in (0,T)} \|u_\varepsilon\|_{L^1(\Omega_0)} \leq \beta$ at the last step. Using $|\Psi_0(z)| \leq |z|$ and $\text{supp}(\phi_n) \subset (-\infty, -n] \cup [n, \infty)$, we can estimate

$$
\left| \int_{\Omega_0} \Psi_0(u_{0,\varepsilon}) dx \right| \leq \|u_{0,\varepsilon}\|_{L^1(\Omega_0)} \leq \|u_0\|_{L^1(\Omega_0)}
$$

and

$$
\int_{Q_T} |\nabla u_\varepsilon| \phi_n(u_\varepsilon) dx dt \leq \int_{B_n} |\nabla u_\varepsilon| dx dt + \int_{E_n} |\nabla u_\varepsilon| dx dt
\leq \frac{\alpha}{2} \int_{B_n} |\nabla u_\varepsilon|^p dx dt + C(\alpha) Q_T + \int_{E_n} |\nabla u_\varepsilon| dx dt,
$$

where Young’s inequality was applied at the last step. Inserting these estimates into (43) yields

$$
\alpha \int_{B_n} |\nabla u_\varepsilon|^p dx dt + 2 \int_{Q_T} g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \phi_n(u_\varepsilon) dx dt
\leq C \left( \|u_0\|_{L^1(\Omega_0)}, \|f\|_{L^1(\Omega_0)}, \alpha, \beta, v, T \right) + 2 \|v\|_\infty \int_{E_n} |\nabla u_\varepsilon| dx dt,
$$

which implies the desired estimate and therefore completes the proof of Lemma 2.5.

Proof of Lemma 2.1. The proof of Lemma 2.1 is an immediate consequence of Lemmas 2.3, 2.4 and 2.5.

2.2. Uniform bounds of the nonlinearity.

Lemma 2.6. Let $u_\varepsilon$ be a solution of (7). Then the following estimate holds

$$
\|g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\|_{L^1(Q_T)} \leq K
$$

where $K$ is independent of $\varepsilon$.

Proof. To prove (45) we fix $n \in \mathbb{N}$ and write

$$
\|g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\|_{L^1(Q_T)} \leq \int_{\{|u_\varepsilon| \leq n+1\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dx dt + \int_{\{|u_\varepsilon| \geq n+1\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dx dt =: G_1 + G_2.
$$

Recall the function $\phi_n$ defined in (38), we have $\phi_n(z) = 1$ for $z \geq n + 1$. Therefore, by using the fact that $\phi_n(u_\varepsilon) g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \geq 0$ thanks to (G1),

$$
G_2 = \int_{\{|u_\varepsilon| \geq n+1\}} \phi_n(u_\varepsilon) g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) dx dt
$$

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\[ \leq \int_{Q_T} \phi_n(u_\varepsilon) g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) dxdt \]
\[ \leq C + \|v\|_\infty \int_{E_n} |\nabla u_\varepsilon| dxdt \quad \text{(by using (44))} \]
\[ \leq C + \|v\|_\infty \|\nabla u_\varepsilon\|_{L^1(Q_T)} \]
\[ \leq C \quad \text{(by applying Lemma 2.1 for } q = 1). \]

Therefore, \( G_2 \) is bounded uniformly in \( \varepsilon \). We now estimate \( G_1 \) by using the assumption (G2)
\[
G_1 \leq \int_{\{ |u_\varepsilon| \leq n+1 \}} h(u_\varepsilon)(\gamma(x,t) + |\nabla u_\varepsilon|^\sigma) dxdt
\leq h(n+1)\|\gamma\|_{L^1(Q_T)} + h(n+1) \int_{\{ |u_\varepsilon| \leq n+1 \}} |\nabla u_\varepsilon|^\sigma dxdt.
\]

Now, recalling \( B_j = \{ (x,t) : j \leq u_\varepsilon(x,t) \leq j + 1 \} \),
\[
\int_{\{ |u_\varepsilon| \leq n+1 \}} |\nabla u_\varepsilon|^\sigma dxdt = \sum_{j=0}^{n} \int_{B_j} |\nabla u_\varepsilon|^\sigma dxdt
\leq \sum_{j=0}^{n} |B_j|^\frac{p-\sigma}{p} \left( \int_{B_j} |\nabla u_\varepsilon|^p dxdt \right)^\frac{\sigma}{p}
\leq |Q_T|^\frac{p-\sigma}{p} \sum_{j=0}^{n} \left( C + \frac{2\|v\|_\infty}{\alpha} \int_{E_j} |\nabla u_\varepsilon| dxdt \right)^\frac{\sigma}{p} \quad \text{(using (44))}
\leq |Q_T|^\frac{p-\sigma}{p} (n+1) \left( C + \frac{2\|v\|_\infty}{\alpha} \|\nabla u_\varepsilon\|_{L^1(Q_T)} \right)^\frac{\sigma}{p}
\leq C(n,T)
\]

where we used Lemma 2.1 with \( q = 1 \) at the last step. From this and (46), it follows that \( G_1 \) is bounded uniformly in \( \varepsilon > 0 \). Thus (45) is proved. \( \square \)

3. PROOF OF THEOREM 1.1

The uniform bounds in Section 2 imply that there exists a subsequence of \( \{ u_\varepsilon \}_{\varepsilon > 0} \) such that
\[ u_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad L^q(0,T; W^{1,q}_0(\Omega_t)) \quad \text{for all} \quad 1 < q < p - \frac{d}{d+1}. \]

This limit function \( u \) is a candidate for a weak solution to (1), but the weak convergence is far from enough to show that it is the case. We need convergence in stronger topologies, especially to pass to the limit for the linearities. We start with a pointwise and \( L^1 \)-convergence.

**Lemma 3.1.** Let \( \{ u_\varepsilon \}_{\varepsilon > 0} \) be solutions to (7). Then there exists a subsequence of \( \{ u_\varepsilon \}_{\varepsilon > 0} \) (not relabeled) such that
\[ u_\varepsilon \rightarrow u \quad \text{strongly in} \quad L^s(Q_T) \quad \text{for all} \quad 1 \leq s < p - \frac{d}{d+1}. \]

To prove Lemma 3.1, we need an Aubin-Lions lemma for the case of moving domains. A similar lemma was recently shown in [Mou16], but it is not directly applicable to our case. Therefore, a new version is necessary.
Lemma 3.2 (An Aubin-Lions lemma in moving domains). Let \( 1 \leq q < +\infty \) and \( \{u_n\}_{n \geq 1} \) be a sequence which is bounded in \( L^q((0, T); W^{1,q}_0(\Omega_t)) \). Moreover, for any smooth function \( \psi \in \mathcal{D}(Q_T) \) it holds
\[
\left| \int_{Q_T} \psi \partial_t u_n \, dx \, dt \right| \leq C \sup_{t \in (0, T)} \| \psi \|_{H^m(\Omega_t)}
\] (47)
for some \( m \in \mathbb{N} \). Then \( \{u_n\}_{n \geq 1} \) is precompact in \( L^q(Q_T) \), and when \( q > 1 \) then \( \{u_n\}_{n \geq 1} \) is precompact in \( L^q(Q_T) \) for all \( 1 \leq s < q \).

Remark 3.1. In [Mou16], instead of (47), the following stronger condition was imposed
\[
\left| \int_{Q_T} \psi \partial_t u_n \, dx \, dt \right| \leq C \sum_{|\alpha| \leq m} \| \partial_\alpha \psi \|_{L^2(0,T;L^2(\Omega_t))}.
\]

In our case, due to the fact that the right hand side belongs only \( L^1(Q_T) \), it seems that (47) is unavoidable.

Proof of Lemma 3.2. Though Lemma 3.2 is an improved version of that in [Mou16], its proof still follows closely from the ideas therein with some suitable changes. We therefore postpone it and provide the full technical proof in the Appendix B.

We can now apply the Aubin-Lions lemma to prove Lemma 3.1.

Proof of Lemma 3.1. Thanks to Lemma 2.1, we only need to check the condition (47). First, we choose \( m \in \mathbb{N} \) such that \( H^{m-1}(\Omega_t) \subseteq L^\infty(\Omega_t) \) for all \( t \in [0, T] \). Moreover, using similar arguments to Lemma 2.2 we deduce that there exists a constant \( C = C(v, T) \) such that
\[
\| v \|_{L^r(\Omega_t)} \leq C \| v \|_{H^k(\Omega_t)} \quad \text{for all} \quad t \in [0, T]
\]
for \( k \in \{m - 1, m\} \) and for any \( r \in [1, \infty] \). Now, we multiply the approximating problem (7) by \( \psi \in \mathcal{D}(Q_T) \) and then integrate on \( Q_T \) to get
\[
\int_{Q_T} \psi \partial_t u_\varepsilon \, dx \, dt = -\int_{Q_T} a(x, t, \nabla u_\varepsilon) \cdot \nabla \psi \, dx \, dt - \int_{Q_T} [\nabla u_\varepsilon \cdot v + u_\varepsilon \text{div} v] \psi \, dx \, dt
\]
\[
- \int_{Q_T} g_\varepsilon (u_\varepsilon, \nabla u_\varepsilon) \psi \, dx \, dt + \int_{Q_T} f_\varepsilon \psi \, dx \, dt.
\] (48)

From the assumption (A2) of \( a \) we have
\[
\left| \int_{Q_T} a(x, t, \nabla u_\varepsilon) \cdot \nabla \psi \, dx \, dt \right| \leq \int_{Q_T} |\varphi| |\nabla \psi| \, dx \, dt + K \int_{Q_T} |\nabla u_\varepsilon|^{p-1} |\nabla \psi| \, dx \, dt
\]
\[
\leq |\varphi|_{L^p(Q_T)} \| \nabla \psi \|_{L^p(Q_T)} + K \| \nabla u_\varepsilon \|_{L^q(Q_T)}^{p-1} \| \nabla \psi \|_{L^{q'}(Q_T)}
\]
\[
\leq C \sup_{t \in (0, T)} \| \psi \|_{H^m(\Omega_t)}.
\]

Similarly, by using the bounds in Lemmas 2.1 and 2.6 and \( \| f_\varepsilon \|_{L^1(Q_T)} \leq \| f \|_{L^1(Q_T)} \) we get
\[
\left| \int_{Q_T} [\nabla u_\varepsilon \cdot v + u_\varepsilon \text{div} v] \psi \, dx \, dt \right|
\]
\[
\leq \| v \|_{\infty} \| \nabla u_\varepsilon \|_{L^p(Q_T)} \| \psi \|_{L^{p'}(Q_T)} + \| \text{div} v \|_{\infty} \| u_\varepsilon \|_{L^q(Q_T)} \| \nabla \psi \|_{L^{q'}(Q_T)}
\]
\[
\leq C (\| \psi \|_{L^{p'}(Q_T)} + \| \nabla \psi \|_{L^{p'}(Q_T)})
\]
\[
\leq C \sup_{t \in (0, T)} \| \psi \|_{H^m(\Omega_t)}.
\]
and
\[
\left| \int_{Q_T} g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \psi \, dx \, dt \right| \leq \|g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\|_{L^1(Q_T)} \|\psi\|_{L^\infty(Q_T)} \leq C \sup_{t \in (0,T)} \|\psi\|_{H^m(\Omega_t)},
\]
and
\[
\left| \int_{Q_T} f_\varepsilon \psi \, dx \, dt \right| \leq \|f\|_{L^1(Q_T)} \|\psi\|_{L^\infty(Q_T)} \leq C \sup_{t \in (0,T)} \|\psi\|_{H^m(\Omega_t)}.
\]
Putting all these into (48) we get (47), and therefore Lemma 3.2 implies the desired result of Lemma 3.1 since \( q < p - \frac{d}{d+1} \) is arbitrary. 

Due to the nonlinearities in the gradient in \( a(x, t, \nabla u_\varepsilon) \) and in \( g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \), we need also stronger convergence of the gradient.

**Lemma 3.3** (Almost everywhere convergence of the gradient). Let \( \{u_\varepsilon\}_{\varepsilon > 0} \) be solutions of the approximate problem (7). Then the sequence \( \{\nabla u_\varepsilon\}_{\varepsilon > 0} \) converges to \( \nabla u \) almost everywhere as \( \varepsilon \) goes to zero.

**Proof.** We will show that \( \{\nabla u_\varepsilon\}_{\varepsilon} \) is a Cauchy sequence in measure, i.e. for all \( \mu > 0 \)
\[
\mathcal{A} := \text{meas}\{(x, t) \in Q_T : |\nabla u_{\varepsilon'} - \nabla u_\varepsilon| > \mu\} \to 0,
\]
as \( \varepsilon', \varepsilon \to 0 \). From this, after extracting a subsequence, we have the convergence \( \nabla u_\varepsilon \to \nabla u \) almost everywhere.

To prove (49), we let \( k > 0 \) and \( \delta > 0 \) be chosen later and observe that
\[
\mathcal{A} \subset \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4
\]
where
\[
\mathcal{A}_1 = \{(x, t) \in Q_T : |\nabla u_\varepsilon| \geq k\},
\]
\[
\mathcal{A}_2 = \{(x, t) \in Q_T : |\nabla u_{\varepsilon'}| \geq k\},
\]
\[
\mathcal{A}_3 = \{(x, t) \in Q_T : |u_{\varepsilon} - u_{\varepsilon'}| \geq \delta\},
\]
\[
\mathcal{A}_4 = \{(x, t) \in Q_T : |\nabla u_{\varepsilon} - \nabla u_{\varepsilon'}| \geq \mu, |\nabla u_{\varepsilon} - \nabla u_{\varepsilon'}| \leq k, |\nabla u_{\varepsilon'}| \leq k \text{ and } |u_{\varepsilon} - u_{\varepsilon'}| \leq \delta\}.
\]
We will estimate \( \mathcal{A}_i \), \( i = 1, \ldots, 4 \) separately. Firstly, for \( \mathcal{A}_1 \), by applying Lemma 2.1 with \( q = 1 \), we have
\[
|\mathcal{A}_1| = \int_{\mathcal{A}_1} dx \, dt \leq \frac{1}{k} \int_{\mathcal{A}_1} \nabla u_\varepsilon \, dx \, dt \leq \frac{1}{k} \|\nabla u_\varepsilon\|_{L^1(Q_T)} \leq \frac{C}{k},
\]
for \( C \) independent of \( \varepsilon \). Similarly,
\[
|\mathcal{A}_2| \leq \frac{1}{k} \|\nabla u_{\varepsilon'}\|_{L^1(Q_T)} \leq \frac{C}{k}.
\]
For \( \mathcal{A}_3 \),
\[
|\mathcal{A}_3| = \int_{\mathcal{A}_3} dx \, dt \leq \frac{1}{\delta} \|u_{\varepsilon} - u_{\varepsilon'}\|_{L^1(Q_T)}.
\]
It remains to estimate \( \mathcal{A}_4 \). Firstly, by using \( T_\delta(u_{\varepsilon} - u_{\varepsilon'}) = u_{\varepsilon} - u_{\varepsilon'} \) on the set \( \{(x, t) : |u_{\varepsilon} - u_{\varepsilon'}| \leq \delta\} \), we have
\[
|\mathcal{A}_4| \leq \frac{1}{\mu} \int_{\{u_{\varepsilon} - u_{\varepsilon'} \leq \delta\}} |\nabla (u_{\varepsilon} - u_{\varepsilon'})| \, dx \, dt = \frac{1}{\mu} \int_{Q_T} \chi_{\{|u_{\varepsilon} - u_{\varepsilon'}| \leq \delta\}} |\nabla (u_{\varepsilon} - u_{\varepsilon'})| \, dx \, dt.
\]
Subtracting the equation (7) for \( \varepsilon \) and \( \varepsilon' \), then taking \( \phi = T_\delta(u_\varepsilon - u_{\varepsilon'}) \) as a test function, we get

\[
\int_{\Omega_T} S_\delta(u_\varepsilon(x, T) - u_{\varepsilon'}(x, T))dx + \int_{Q_T} (a(x, t, \nabla u_\varepsilon) - a(x, t, \nabla u_{\varepsilon'}))\nabla T_\delta(u_\varepsilon - u_{\varepsilon'})dxdt
\]
\[
= \int_{\Omega_0} S_\delta(u_{0,\varepsilon} - u_{0,\varepsilon'})dx + \int_{Q_T} (f_\varepsilon - f_{\varepsilon'})T_\delta(u_\varepsilon - u_{\varepsilon'})dxdt
\]
\[
- \int_{Q_T} ((u_\varepsilon - u_{\varepsilon'})\text{div}v + v \cdot \nabla (u_\varepsilon - u_{\varepsilon'}))T_\delta(u_\varepsilon - u_{\varepsilon'})dxdt
\]
\[
- \int_{Q_T} (g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) - g_{\varepsilon'}(u_{\varepsilon'}, \nabla u_{\varepsilon'}))T_\delta(u_\varepsilon - u_{\varepsilon'})dxdt.
\]

(54)

Since \( S_\delta \) is nonnegative and thanks to the assumption (A4), the left hand side of (54) is bounded below by

\[
\int_{Q_T} (a(x, t, \nabla u_\varepsilon) - a(x, t, \nabla u_{\varepsilon'})(\nabla u_\varepsilon - \nabla u_{\varepsilon'})\chi\{|u_\varepsilon - u_{\varepsilon'}| \leq \delta \}dxdt
\]
\[
\geq C \int_{Q_T} \chi\{|u_\varepsilon - u_{\varepsilon'}| \leq \delta \} \frac{1}{\Theta(x, t, \nabla u_\varepsilon, \nabla u_{\varepsilon'})}\|
\nabla u_\varepsilon - \nabla u_{\varepsilon'}\|^\theta dxdt.
\]

(55)

For the right hand side of (54), we use \(|T_\delta(z)| \leq \delta\) and \(S_\delta(z) \leq \delta|z|\) to estimate

Right hand side of (54) \leq \delta\|u_{0,\varepsilon} - u_{0,\varepsilon'}\|_{L^1(\Omega_0)} + \delta\|f_\varepsilon - f_{\varepsilon'}\|_{L^1(Q_T)}
\]
\[
+ \delta\|\text{div}v\|_{\infty}\|u_\varepsilon - u_{\varepsilon'}\|_{L^1(Q_T)} + \delta\|v\|_{\infty}\|\nabla (u_\varepsilon - u_{\varepsilon'})\|_{L^1(Q_T)}
\]
\[
+ \delta\|g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\|_{L^1(Q_T)} + \|g_{\varepsilon'}(u_{\varepsilon'}, \nabla u_{\varepsilon'})\|_{L^1(Q_T)}
\]
\[
\leq C\delta.
\]

(56)

with \(C\) independent of \(\varepsilon, \varepsilon'\), and where we used the fact that \(\{u_{0,\varepsilon}\}\) is bounded in \(L^1(\Omega_0)\), and all \(\{f_\varepsilon\}, \{u_\varepsilon\}, \{\nabla u_\varepsilon\}, \{g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\}\) are bounded in \(L^1(Q_T)\). Inserting (55) and (56) into (54) gives

\[
\int_{Q_T} \chi\{|u_\varepsilon - u_{\varepsilon'}| \leq \delta \} \frac{1}{\Theta(x, t, \nabla u_\varepsilon, \nabla u_{\varepsilon'})}\|
\nabla u_\varepsilon - \nabla u_{\varepsilon'}\|^\theta dxdt \leq C\delta.
\]

By using Hölder’s inequality, we have

\[
\int_{Q_T} \chi\{|u_\varepsilon - u_{\varepsilon'}| \leq \delta \}\|
\nabla u_\varepsilon - \nabla u_{\varepsilon'}\|dxdt \leq \left(\int_{Q_T} \chi\{|u_\varepsilon - u_{\varepsilon'}| \leq \delta \} \frac{1}{\Theta(x, t, \nabla u_\varepsilon, \nabla u_{\varepsilon'})}\|
\nabla u_\varepsilon - \nabla u_{\varepsilon'}\|^\theta dxdt\right)^\frac{1}{\theta}
\]
\[
\times \left(\int_{Q_T} \chi\{|u_\varepsilon - u_{\varepsilon'}| \leq \delta \} \Theta(x, t, \nabla u_\varepsilon, \nabla u_{\varepsilon'})^\frac{1}{\theta-1} dxdt\right)^\frac{\theta-1}{\theta}
\]
\[
\leq C\delta^\frac{\theta}{\theta-1} \left(\int_{Q_T} \left[1 + \|
\nabla u_\varepsilon\|_{\theta-1}^\theta + \|
\nabla u_{\varepsilon'}\|_{\theta-1}^\theta\right] dxdt\right)^\frac{\theta-1}{\theta}
\]

where we used (2) at the last step. Thanks to (3), \(\frac{\theta}{\theta-1} < p - \frac{d}{d+1}\). Therefore, Lemma 2.1 implies

\[
\int_{Q_T} \left[\|
\nabla u_\varepsilon\|_{\theta-1}^\theta + \|
\nabla u_{\varepsilon'}\|_{\theta-1}^\theta\right] dxdt \leq C
\]

and thus

\[
\int_{Q_T} \chi\{|u_\varepsilon - u_{\varepsilon'}| \leq \delta \}\|
\nabla u_\varepsilon - \nabla u_{\varepsilon'}\|dxdt \leq C\delta^\frac{\theta}{\theta-1}.
\]
Inserting this into (53) leads to
\[
|\mathcal{A}_4| \leq \frac{C\delta^\frac{1}{q}}{\mu}
\]  
for a constant $C'$ independent of $\varepsilon, \varepsilon'$.

Now let $\kappa > 0$ be arbitrary. We first choose $k$ to be large enough so that (50) and (51) give
\[
|\mathcal{A}_1| + |\mathcal{A}_2| \leq \frac{\kappa}{2}.
\]

We next choose $\delta$ to be small enough ($k$ is now fixed) so that (57) implies
\[
|\mathcal{A}_4| \leq \frac{\kappa}{4}.
\]

With $k$ and $\delta$ are fixed, since $\{u_\varepsilon\}_{\varepsilon > 0}$ is a Cauchy sequence in $L^1(Q_T)$, thanks to Lemma 3.1, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon, \varepsilon' \leq \varepsilon_0$, (52) implies
\[
|\mathcal{A}_3| \leq \frac{1}{\delta} \|u_\varepsilon - u_{\varepsilon'}\|_{L^1(Q_T)} \leq \frac{\kappa}{4}.
\]

Therefore,
\[
|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| + |\mathcal{A}_4| \leq \kappa \quad \text{for all} \quad \varepsilon, \varepsilon' \leq \varepsilon_0.
\]

Thus (49) is proved. \qed

We are now ready to get strong convergence for the nonlinear term $a(x, t, \nabla u)$.

**Lemma 3.4.** Let $\{u_\varepsilon\}_{\varepsilon > 0}$ be solutions to the equation (7). Then, up to a subsequence,
\[
a(x, t, \nabla u_\varepsilon) \to a(x, t, \nabla u) \quad \text{strongly in} \quad L^s(Q_T) \quad \text{for all} \quad 1 \leq s < 1 + \frac{1}{(p-1)(d+1)}.
\]

**Proof.** From Lemma 3.3 and the fact that $a$ is continuous with respect to the third variable, we have
\[
a(x, t, \nabla u_\varepsilon) \to a(x, t, \nabla u) \quad \text{almost everywhere in} \quad Q_T.
\]  
(58)

By using assumption (A2) and Lemma 2.1 we have for any $1 \leq s < 1 + \frac{1}{(p-1)(d+1)}$,
\[
\|a(x, t, \nabla u_\varepsilon)\|_{L^s(Q_T)}^s \leq C \int_{Q_T} |\varphi|^s dxdt + CK^s \int_{Q_T} |\nabla u_\varepsilon|^s dxdt \leq C
\]  
(59)

thanks to $s(p-1) < d - \frac{p}{p+1}$ and $1 + \frac{1}{(p-1)(d+1)} < \frac{p}{p-1} = p'$. From (58) and (59), the Egorov theorem implies that $\{a(x, t, \nabla u_\varepsilon)\}_{\varepsilon > 0}$ is precompact in $L^s(Q_T)$ for all $1 \leq s < 1 + \frac{1}{(p-1)(d+1)}$, which finishes the proof of Lemma 3.4. \qed

Due to the subcritical growth of the nonlinearity $g$ in (G2), its convergence cannot be obtained in the same way as for $a$ in Lemma 3.4. A different approach should be used, for which we need the following lemma.

**Lemma 3.5.** Let $\{u_\varepsilon\}_{\varepsilon > 0}$ be solutions to (7). Then
\[
\lim_{k \to \infty} \sup_{\varepsilon > 0} \int_{\{|u_\varepsilon| \geq k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt = 0.
\]
Proof. Since $T_k(u_\varepsilon) = k$ for $u_\varepsilon \geq k$, we have
\[
\int_{\{|u_\varepsilon| \geq k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dx dt \leq \frac{1}{k} \int_{Q_T} g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) T_k(u_\varepsilon) dx dt.
\] (60)

By integrating (33) on $(0, T)$ and using (32) we obtain, in particular,
\[
\int_{Q_T} g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) T_k(u_\varepsilon) dx dt
\leq \int_{\Omega_0} S_k(u_{0,\varepsilon}) dx + \int_{Q_T} |f_\varepsilon T_k(u_\varepsilon)| dx dt
\qquad + \|\text{div}\, v\|_\infty \int_{Q_T} |u_\varepsilon||T_k(u_\varepsilon)| dx dt.
\] (61)

Let $M > 0$. We then have the following useful estimates
\[
0 \leq S_k(z) \leq M^2 + k|z| \chi_{\{|z| > M\}} \quad \text{and} \quad |T_k(z)| \leq M + k \chi_{\{|z| > M\}}.
\]

Therefore,
\[
\int_{\Omega_0} S_k(u_{0,\varepsilon}) dx \leq M^2 |\Omega_0| + k \int_{\{|u_{0,\varepsilon}| > M\}} |u_{0,\varepsilon}| dx,
\]
\[
\int_{Q_T} |f_\varepsilon T_k(u_\varepsilon)| dx dt \leq M \|f_\varepsilon\|_{L^1(Q_T)} + k \int_{\{|u_\varepsilon| > M\}} |f_\varepsilon| dx dt,
\]
\[
\int_{Q_T} |\nabla u_\varepsilon||T_k(u_\varepsilon)| dx dt \leq M \|\nabla u_\varepsilon\|_{L^1(Q_T)} + k \int_{\{|u_\varepsilon| > M\}} |\nabla u_\varepsilon| dx dt,
\]

and
\[
\int_{Q_T} |u_\varepsilon||T_k(u_\varepsilon)| dx dt \leq M \|u_\varepsilon\|_{L^1(Q_T)} + k \int_{\{|u_\varepsilon| > M\}} |u_\varepsilon| dx dt.
\]

Using these estimates in (60) and (61), we get
\[
\int_{\{|u_\varepsilon| \geq k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dx dt \leq \frac{M^2}{k} |\Omega_0| \frac{CM}{k} \left( \|f_\varepsilon\|_{L^1(Q_T)} + \|u_\varepsilon\|_{L^1(0,T;W^{1,1}_0(\Omega_0))} \right)
\qquad + \int_{\{|u_{0,\varepsilon}| > M\}} |u_{0,\varepsilon}| dx + \int_{\{|u_\varepsilon| > M\}} |f_\varepsilon| dx dt
\qquad + \|\text{div}\, v\|_\infty \int_{\{|u_\varepsilon| > M\}} |\nabla u_\varepsilon| dx dt + \|\text{div}\, v\|_\infty \int_{\{|u_\varepsilon| > M\}} |u_\varepsilon| dx dt.
\] (62)

Due to the uniform bound of $\{\|u_\varepsilon\|_{L^1(Q_T)}\}_{\varepsilon > 0}$ we have
\[
\lim_{M \to \infty} \sup_{\varepsilon > 0} |\{(x, t) \in Q_T : u_\varepsilon(x, t) > M\}| \leq \lim_{M \to \infty} \frac{1}{M} \sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^1(Q_T)} = 0.
\]

Therefore, from the fact that, as $\varepsilon \to 0$, $\|u_{0,\varepsilon} - u_\varepsilon\|_{L^1(\Omega_0)} \to 0$, $\|f_\varepsilon - f\|_{L^1(Q_T)} \to 0$ (by the constructions of $u_{0,\varepsilon}$ and $f_\varepsilon$), and $\|u_\varepsilon - u\|_{L^1(Q_T)} \to 0$ (due to Lemma 3.1), and $\|\nabla u_\varepsilon - \nabla u\|_{L^1(Q_T)} \to 0$ (due the fact that $\nabla u_\varepsilon \to \nabla u$ almost everywhere, and $\|\nabla u_\varepsilon\|_{L^q(Q_T)}$ is bounded uniformly in $\varepsilon$ for some $q > 1$), we imply that the last four terms on the right hand side of (62) become arbitrary small as $M$ tends to infinity.

Let $\kappa > 0$ be arbitrary. We first choose $M$ large enough such that the sum of the last four terms on the right hand side of (62) is smaller than $\kappa/2$. Then using the boundedness of $\|f_\varepsilon\|_{L^1(Q_T)}$
and $\|u_\varepsilon\|_{L^1(0,T;W^{1,1}_0(\Omega_t))}$, there exists $k_0$ large enough, which is independent of $\varepsilon$, such that for all $k \geq k_0$,

$$\frac{M^2}{k} |\Omega_0| + \frac{CM^2}{k} \left( \|f_\varepsilon\|_{L^1(Q_T)} + \|u_\varepsilon\|_{L^1(0,T;W^{1,1}_0(\Omega_t))} \right) \leq \frac{\kappa}{2}.$$  

Therefore,

$$\sup_{\varepsilon > 0} \int_{\{u_\varepsilon \geq k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt \leq \kappa \quad \text{for all} \quad k \geq k_0,$$

which proves the claim \ref{lemma:3.5}.

\textbf{Lemma 3.6} (Strong convergence of the first order terms). \textit{As $\varepsilon \to 0$, there exists a subsequence of $\{g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\}$ that converges to $g(u, \nabla u)$ almost everywhere in $Q_T$ and strongly in $L^1(Q_T)$.}

\textbf{Proof.} From Lemmas 3.1 and 3.3, and the fact that $g$ is continuous with respect to the third and fourth variables, we have

$$g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) = \frac{g(x, t, u_\varepsilon, \nabla u_\varepsilon)}{1 + \varepsilon |g(x, t, u_\varepsilon, \nabla u_\varepsilon)|} \to g(x, t, u, \nabla u) \quad \text{almost everywhere in} \quad Q_T.$$

To show that this convergence is in fact strong in $L^1(Q_T)$-topology, it’s sufficient to show that the set $\{g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\}_{\varepsilon > 0}$ is weakly compact in $L^1(Q_T)$, or equivalently to show that

$$\lim_{A \in \text{meas}(Q_T), |A| \to 0} \sup_{\varepsilon > 0} \int_A |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt = 0 \quad (63)$$

where $A \in \text{meas}(Q_T)$ means that $A \subset Q_T$ is a measurable subset of $Q_T$. Indeed, we have for any $k \in \mathbb{N}$,

$$\int_A |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt = \int_{A \cap \{u_\varepsilon \leq k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt + \int_{A \cap \{u_\varepsilon > k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt. \quad (64)$$

For the second part, we have

$$\int_{A \cap \{u_\varepsilon \geq k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt \leq \int_{\{u_\varepsilon \geq k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt \quad (65)$$

in which the right-hand side tends to 0, as $k \to \infty$, uniformly in $\varepsilon$, thanks to Lemma 3.5. It remains to estimate the first part in (64). From the assumption (G2), we have

$$\int_{A \cap \{u_\varepsilon \leq k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt$$

$$\leq \int_{A \cap \{u_\varepsilon \leq k\}} |g(u_\varepsilon, \nabla u_\varepsilon)| dxdt$$

$$\leq h(k) \int_{A \cap \{u_\varepsilon \leq k\}} (|\gamma(x, t)| + |\nabla u_\varepsilon|^p) dxdt$$

$$\leq h(k) \int_A |\gamma(x, t)| dxdt + h(k) \left( \int_{A \cap \{u_\varepsilon \leq k\}} |\nabla u_\varepsilon|^p dxdt \right)^{\frac{1}{p}} |A|^{\frac{p-1}{p}} \quad (66)$$

where we used Hölder’s inequality and the obvious estimate $|A \cap \{u_\varepsilon \leq k\}| \leq |A|$ at the last step. By using Hölder’s inequality again we find

$$h(k) \int_A |\gamma(x, t)| dxdt \leq h(k) \|\gamma\|_{L^p(Q_T)} |A|^{\frac{p-1}{p}} \quad (67)$$
where we recall that $p' = \frac{p}{p-1}$. From Lemmas 2.5 and 2.1 (with $q = 1$) we can estimate

$$\int_{A \cap \{|u_\varepsilon| \leq k\}} |\nabla u_\varepsilon|^p dx dt \leq \sum_{j=0}^k \int_{B_j} |\nabla u_\varepsilon|^p dx dt$$

$$\leq \sum_{j=0}^k \left( C_0 + C_1 \int_{E_j} |\nabla u_\varepsilon| dx dt \right)$$

$$\leq \sum_{j=0}^k \left( C_0 + C_1 \|\nabla u_\varepsilon\|_{L^1(Q_T)} \right)$$

$$\leq C(k+1).$$

Inserting (67) and (68) into (66) gives us

$$\int_{A \cap \{|u_\varepsilon| \leq k\}} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dx dt \leq Ch(k) |A| \left( \frac{p-1}{p} \right) + C(k+1)h(k) |A| \frac{p-n}{p}.$$  \hspace{1cm} (69)

Using (65) and (69) yields the desired estimate (63) which finishes the proof of Lemma 3.6. \hspace{1cm} \Box

The last lemma is about the continuity in time.

**Lemma 3.7.** The sequence $\{u_\varepsilon\}_{\varepsilon > 0}$ is a Cauchy sequence in $C([0, T]; L^1(\Omega_t))$ as $\varepsilon \to 0$, and therefore $u \in C([0, T]; L^1(\Omega_t))$.

**Proof.** Let $\varepsilon, \varepsilon' > 0$, subtracting the equations for $u_\varepsilon$ and $u_\varepsilon'$ and taking $T_1(u_\varepsilon - u_\varepsilon')$ as the test function, we have

$$\int_{\Omega_t} S_1(u_\varepsilon - u_\varepsilon')(t) dx + \int_0^t \int_{\Omega_s} (a(x, s, \nabla u_\varepsilon) - a(x, s, \nabla u_\varepsilon'))(\nabla u_\varepsilon - \nabla u_\varepsilon') \chi_{\{|u_\varepsilon| \leq 1\}} dx ds$$

$$\leq \int_{\Omega_0} S_1(u_{0, \varepsilon} - u_{0, \varepsilon'}) dx - \int_0^t \int_{\Omega_s} (\nabla \cdot (u_\varepsilon - u_\varepsilon') + (u_\varepsilon - u_\varepsilon') \div v) T_1(u_\varepsilon - u_\varepsilon') dx ds$$

$$\leq \int_0^t \int_{\Omega_s} g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) - g_\varepsilon'(u_\varepsilon', \nabla u_\varepsilon') |T_1(u_\varepsilon - u_\varepsilon')| dx ds + \int_0^t \int_{\Omega_s} (f_\varepsilon - f_\varepsilon') T_1(u_\varepsilon - u_\varepsilon') dx ds.$$

Using the assumption (A4) and $|T_1(z)| \leq 1$ and $S_1(z) \leq |z|$, we obtain

$$\sup_{t \in (0, T)} \int_{\Omega_t} S_1(u_\varepsilon - u_\varepsilon')(t) dx$$

$$\leq m_{\varepsilon, \varepsilon'} := \|u_{0, \varepsilon} - u_{0, \varepsilon'}\|_{L^1(\Omega_0)} + \|\nabla \varepsilon\|_{\infty} \|\nabla u_\varepsilon - \nabla u_\varepsilon'\|_{L^1(Q_T)} + \|\div v\|_{\infty} \|u_\varepsilon - u_\varepsilon'\|_{L^1(Q_T)}$$

$$+ \|g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) - g_\varepsilon'(u_\varepsilon', \nabla u_\varepsilon')\|_{L^1(Q_T)} + \|f_\varepsilon - f_\varepsilon'\|_{L^1(Q_T)}$$
where clearly \( \lim_{\varepsilon, \varepsilon' \to 0} m_{\varepsilon, \varepsilon'} = 0 \). Now by using \( |z| \chi_{\{|z|>1\}}/2 \leq S_1(z) \chi_{\{|z|>1\}} \) and \( |z|^2 \chi_{\{|z|\leq 1\}}/2 \leq S_1(z) \chi_{\{|z|\leq 1\}} \), we can estimate

\[
\|u_\varepsilon(t) - u_{\varepsilon'}(t)\|_{L^1(\Omega_t)} \leq \int_{\{|u_\varepsilon(t) - u_{\varepsilon'}(t)| \leq 1\}} |u_\varepsilon(t) - u_{\varepsilon'}(t)| dx + \int_{\{|u_\varepsilon(t) - u_{\varepsilon'}(t)| > 1\}} |u_\varepsilon(t) - u_{\varepsilon'}(t)| dx
\]

\[
\leq |\Omega_t|^{1/2} \left( \int_{\{|u_\varepsilon(t) - u_{\varepsilon'}(t)| \leq 1\}} |u_\varepsilon(t) - u_{\varepsilon'}(t)|^2 dx \right)^{1/2} + 2 \int_{\Omega_t} S_1(u_\varepsilon - u_{\varepsilon'})(t) dx
\]

\[
\leq |\Omega_t|^{1/2} \left( 2 \int_{\Omega_t} S_1(u_\varepsilon - u_{\varepsilon'})(t) dx \right)^{1/2} + 2 \int_{\Omega_t} S_1(u_\varepsilon - u_{\varepsilon'})(t) dx
\]

\[
\leq \sqrt{2|\Omega_t|^{1/2}} \sqrt{m_{\varepsilon, \varepsilon'}} + 2m_{\varepsilon, \varepsilon'}.
\]

Hence,

\[
\lim_{\varepsilon, \varepsilon' \to 0} \sup_{t \in (0,T)} \|u_\varepsilon(t) - u_{\varepsilon'}(t)\|_{L^1(\Omega_t)} = 0.
\]

Therefore, \( \{u_\varepsilon\}_{\varepsilon>0} \) is a Cauchy sequence in \( C([0,T]; L^1(\Omega_t)) \), and thus \( u \in C([0,T]; L^1(\Omega_t)) \). \( \Box \)

We are now ready to prove the main theorem of this paper.

**Proof of Theorem 1.1.** Let \( \phi \in C(0,T; W_0^{1,q'}(\Omega_t)) \cap C^1(0,T; L^q(\Omega_t)) \) be the test function to the approximate problem. We have

\[
\int_{\Omega_T} u_\varepsilon(T) \phi(T) dx - \int_{Q_T} u_\varepsilon \partial_t \phi dx dt + \int_{Q_T} a(x,t,\nabla u_\varepsilon) \cdot \nabla \phi dx dt
\]

\[
- \int_{Q_T} u_\varepsilon \mathbf{v} \cdot \nabla \phi dx dt + \int_{Q_T} g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \phi dx dt
\]

\[
= \int_{\Omega_0} u_0 \phi(0) dx + \int_{Q_T} f_\varepsilon \phi dx dt.
\]

By applying Lemmas 3.1, 3.4, 3.6, and 3.7, and using (5), we can pass to the limit as \( \varepsilon \to 0 \) in all the terms to obtain that

\[
\int_{\Omega_T} u(T) \phi(T) dx - \int_{Q_T} u_0 \phi dx dt + \int_{Q_T} a(x,t,\nabla u) \cdot \nabla \phi dx dt
\]

\[
- \int_{Q_T} u \mathbf{v} \cdot \nabla \phi dx dt + \int_{Q_T} g(u, \nabla u) \phi dx dt
\]

\[
= \int_{\Omega_0} u_0 \phi(0) dx + \int_{Q_T} f \phi dx dt
\]

or in other words, \( u \) is a weak solution to (1) on \( (0,T) \). The proof of Theorem 1.1 is complete. \( \Box \)

**Appendix A. Existence of approximate solutions**

This section is devoted to a proof of the global existence of a weak solution to the approximate system (7). We follow the ideas in [CNO17].

We divide the time interval \( [0; T] \) into \( N \in \mathbb{N} \) smaller intervals \( (t_j, t_{j+1}) \) for \( j = 0, \ldots, N - 1 \) and define \( \Delta := \max_j |t_j - t_{j+1}| \). The points \( t_j \) are chosen so that

1. \( \bigcup_{j=0,N-1} \Omega_{t_j} \times [t_j, t_{j+1}] \subset \widehat{\Omega} \),
2. \( \Omega_{t_j} \) has smooth boundary for all \( j \in \{0, \ldots, N - 1\} \),
We define the extended function \( f_{\varepsilon} : \hat{Q} \to \mathbb{R} \) as \( f_{\varepsilon}(x, t) = f_{\varepsilon}(x, t) \) if \((x, t) \in \hat{Q}\) and \(f_{\varepsilon}(x, t) = 0\) otherwise. Let us denote by \( I_j = [t_j, t_{j+1}) \). For each \( j \in \{0, \ldots, N-1\} \) we consider the following equation

\[
\begin{cases}
\partial_t w^{(j)} - \text{div}(a(x, t, j, \nabla w^{(j)})) + \text{div}(w^{(j)} v) + g_{\varepsilon}(w^{(j)}, \nabla w^{(j)})) = f_{\varepsilon}, & x \in \Omega_{t_j}, t \in I_j, \\
w^{(j)}(x, t) = \lim_{t \to t_j^-} w^{(j-1)}(\zeta_{t_j-t_{j-1}}^{-1}(x), t), & x \in \Omega_{t_j} \cap \Omega_{t_{j-1}}, \\
0, & x \in \partial \Omega_{t_j}, t \in I_j,
\end{cases}
\]

(70)

If \( t_0 = 0 \) then we let \( w^{(0)}(x, 0) = u_{0,\varepsilon}(x) \). Note that we have the semigroup property \( \zeta_{t+s} = \zeta_t \circ \zeta_s \) and the domains \( \Omega_{t_j} = \zeta_{t_j-t_{j-1}}^{-1}(\Omega_{t_{j-1}}) \) for \( j = 0, N-1 \).

For any fixed \( j \in \{0, \ldots, N-1\} \), by classical results, see e.g. [Lio69], one obtains the existence of a solution \( w^{(j)} \in L^1(\Omega_{t_j} \times I_j) \cap L^p(I_j; W^{1,p}(\Omega_{t_j})) \) with \( \partial_t w^{(j)} \in L^{p'}(I_j; W^{-1,p'}(\Omega_{t_j})) \) of (70). Denote by

\[
\Omega^\Delta := \{(x, t) : x \in \Omega_{t_j}, t \in I_j, j = 0, \ldots, N-1\} = \bigcup_{j=0,\ldots,N-1} \Omega_{t_j} \times I_j.
\]

From [CNO17, Lemma 3.4], we know that as \( \Delta \to 0 \), \( \Omega^\Delta \) converges to \( Q_T \) in Hausdorff sense, and as a consequence \( \chi_{\Omega^\Delta} \) converges strongly to \( \chi_{Q_T} \) in \( L^s(\hat{Q}) \) for all \( s < \infty \). We now glue the solutions \( w^{(j)}(x, t) \) of (70) together and define the approximate solutions

\[
w^\Delta(x, t) = \sum_{j=0}^{N-1} \chi_{\Omega_{t_j}}(x) \chi_{(t_j, t_{j+1})}(t) w^{(j)}(x, t)
\]

for \((x, t) \in \hat{Q}\). The function \( w^{(j)}(x, t) \chi_{\Omega_{t_j}}(x) \) in the formulae above is the function which coincides with \( w^{(j)}(x, t) \) in \( \Omega_{t_j} \) and is equal to zero outside \( \Omega_{t_j} \).

In the sequel, we prove some \( a \text{ priori} \) estimates of \( w^\Delta \) which are independent of \( \Delta \), thus allowing us to pass to the limit \( \Delta \to 0 \). In conclusion we have \( w^\Delta \to v \) where \( v \) is a solution to (7).

**Lemma A.1** (Existence of approximate solutions). *There exists a weak solution to the approximating problem (7).*

**Proof.** The existence of solutions for \( t \in (0, T) \) is classical, so we sketch some main steps here. For simplicity we set

\[
G_{\varepsilon}(u, \nabla u) = \text{div}(uv) + g_{\varepsilon}(u, \nabla u).
\]

**Step 1: Establishing a priori estimates of \( w^\Delta \).**

First, we will prove \( w^\Delta \in L^\infty(\Omega^\Delta) \) for any \( t > 0 \). It is enough to prove the estimate in \( \Omega_0 \times (0, t_1) \).
For $p \in (1, \infty)$, choosing $|w^\Delta|^{k-2}w^\Delta$ as a test function of (7), we have
\[
\frac{d}{dt} \int_{\Omega_0} |w^\Delta|^k dx + k(k-1) \int_{\Omega_0} a(x, 0, \nabla w^\Delta) \cdot \nabla w^\Delta |w^\Delta|^{k-2} dx
= -k \int_{\Omega_0} G_{\varepsilon}(w^\Delta, \nabla w^\Delta)|w^\Delta|^{k-2}w^\Delta dx + k \int_{\Omega_0} f_{\varepsilon}|w^\Delta|^{k-2}w^\Delta dx.
\]
(71)

From (A3), equation (71) becomes
\[
\frac{d}{dt} \int_{\Omega_0} |w^\Delta|^k dx + k(k-1) \alpha \left( \frac{p}{p + k - 2} \right)^p \int_{\Omega_0} |\nabla (w^\Delta)|^{\frac{p+k-2}{p}} |dx
\leq k \int_{\Omega_0} |G_{\varepsilon}(w^\Delta, \nabla w^\Delta)||w^\Delta|^{k-1} dx + k \int_{\Omega_0} |f_{\varepsilon}| |w^\Delta|^{k-1} dx.
\]

By integrating the inequality above from 0 to $t_1$ we have
\[
\int_{\Omega_0} |w^\Delta(t)|^k dx \leq \int_{\Omega_0} |u_{0,\varepsilon}|^k dx + k \int_{0}^{t_1} \int_{\Omega_0} (|G_{\varepsilon}| + |f_{\varepsilon}|) |w^\Delta|^{k-1} dx dt.
\]

Fix $\xi > t_1$. By using Hölder and Young inequalities, we have
\[
(1-t_1\xi^{-\frac{k-1}{k-1}}) \sup_{t \in (0, t_1)} \int_{\Omega_0} |w^\Delta(t)|^k dx + \xi^{-\frac{k-1}{k-1}} \int_{0}^{t_1} \int_{\Omega_0} |w^\Delta(t)|^k dx dt
\leq \sup_{t \in (0, t_1)} \int_{\Omega_0} |w^\Delta(t)|^k dx \leq \int_{\Omega_0} |u_{0,\varepsilon}|^k dx + k \int_{0}^{t_1} \int_{\Omega_0} (|G_{\varepsilon}| + |f_{\varepsilon}|) |w^\Delta|^{k-1} dx dt.
\]
\[
\leq \int_{\Omega_0} |u_{0,\varepsilon}|^k dx + \xi \int_{0}^{t_1} \int_{\Omega_0} (|G_{\varepsilon}| + |f_{\varepsilon}|)^k dx dt + \xi^{-\frac{k-1}{k-1}} \int_{0}^{t_1} \int_{\Omega_0} |w^\Delta(t)|^k dx dt.
\]

Hence
\[
(1-t_1\xi^{-\frac{k-1}{k-1}})^{1/k} \sup_{t \in (0, t_1)} \left( \int_{\Omega_0} |w^\Delta(t)|^k dx \right)^{1/k} \leq \left( \int_{\Omega_0} |u_{0,\varepsilon}|^k dx + \xi \int_{0}^{t_1} \int_{\Omega_0} (|G_{\varepsilon}| + |f_{\varepsilon}|)^k dx dt \right)^{1/k}.
\]

Letting $k \to \infty$, we obtain
\[
\|w^\Delta(t)\|_{L^\infty(\Omega_0)} \leq \|u_{0,\varepsilon}\|_{L^\infty(\Omega_0)} + \xi(\|G_{\varepsilon}\|_{L^\infty(0, t_1; \Omega_0)} + \|f_{\varepsilon}\|_{L^\infty(0, t_1; \Omega_0)}), \forall \xi > t_1.
\]

Second, by using the same arguments in [CNO17, Lemma 3.6 and 3.9], we obtain two results respectively, for precisely there is some constant $C > 0$ depending only on $Q_T$ such that
\[
\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{\Omega_{t_j}} |\nabla w^\Delta|^p dx \leq C,
\]
and let $0 < j \leq N$ be fixed, then
\[
w^{(j)}_{t_j} \chi_{\Omega_{t_j}} \in L^{p'}(t_j, t_{j+1}, W^{-1,p'}(\Omega_{t_j})).
\]

**Step 2: Passing to the limits.** From the above estimates, we can extract a subsequence of $\{w^\Delta\}_N$, also denoted by $\{w^\Delta\}_N$, such that
- $w^\Delta \rightharpoonup v$ in $L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty(\widetilde{Q})$,
- $a(x, t, \nabla w^\Delta) \rightharpoonup \tilde{a}$ in $L^p'(\widetilde{Q})$,
- $G_{\varepsilon}(w^\Delta, \nabla w^\Delta) \rightharpoonup \tilde{g}$ in $L^p'(\widetilde{Q})$.
where \( a(x, t, \nabla w^\Delta) := \sum_{j=0}^{N-1} \chi_{[t_j, t_{j+1})}(t)a(x, t_j, \nabla w^{(j)})\chi_{\Omega_j}(x) \).

In order to pass the second limit, we refer the reader to [CNO17, Lemma 3.10 and 4.7] for more details, we give some main ideas here.

At first, we show that
\[
\int_{t_1}^{t_2-h} \| w^\Delta(t + h) - w^\Delta(t) \|_{W^{-1,p'}(K)} \to 0, \quad \text{as } h \to 0^+
\]
uniformly in \( N \), where \( K \) is compact in \( \hat{\Omega} \). Then by applying a compact theorem in [Sim86, Theorem 1] we deduce that the sequence \( \{ w^\Delta \} \) is relatively compact in \( L^1_{\text{loc}}(Q_T) \). Together with the uniform bound of \( w^\Delta \) in \( L^\infty(\Omega^\Delta) \), we obtain the following lemma.

**Lemma A.2.** There exists a subsequence of \( \{ w^\Delta \} \) which converges strongly in \( L^1(\hat{Q}) \).

Moreover, we have the following result.

**Lemma A.3.** Let \( \phi \) be smooth and such that \( \text{supp} \phi \subset \Omega^\Delta \cap ([0, T] \times \mathbb{R}^d) \). Then
\[
\limsup_{N \to \infty} \int_0^T \int_{\Omega^N} a(x, t, \nabla w^\Delta) \cdot \nabla w^\Delta \phi \, dx \, dt \leq \int_0^T \int_{\Omega_t} \overline{a} \cdot \nabla w \phi \, dx \, dt.
\]

Then, we can now use the same arguments as in [CNO17, Lemma 4.8] to obtain \( \overline{\sigma}(x, t, \nabla v) = a(x, t, \nabla v) \) a.e. in \( Q_T \).

It remains to prove \( \overline{\sigma} = G_\epsilon(v, \nabla v) \) a.e. in \( Q_T \). Since \( G_\epsilon \) is a continuous function with respect to \( w \) and \( \nabla w \), by classical results (see e.g. [Lio69]), the sequence \( G_\epsilon(w^\Delta, \nabla w^\Delta) \to G_\epsilon(v, \nabla v) \) in \( L^1(\hat{Q}) \) if we show that the sequence \( \{ \nabla w^\Delta \} \) converges to \( \nabla v \) a.e. as \( \Delta \to 0 \). This property is obtained as we show that \( \{ \nabla w^\Delta \} \) is a Cauchy sequence in measure, see [Edw65], i.e. for all \( \mu > 0 \)
\[
\text{meas}\{(x, t) \in \hat{Q} : |\nabla w^\Delta - \nabla w'^\Delta| \geq \mu\} \to 0, \quad \text{as } \Delta, \Delta' \to 0. \tag{72}
\]

Let us denote by \( \mathcal{A} \) the subset of \( \hat{Q} \) involved in (72). Let \( k > 0 \) and \( \eta > 0 \), we have
\[
\mathcal{A} \subset \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4,
\]
where
\[
\mathcal{A}_1 = \{(x, t) \in Q_T : |\nabla w^\Delta| \geq k\},
\mathcal{A}_2 = \{(x, t) \in Q_T : |\nabla w'^\Delta| \geq k\},
\mathcal{A}_3 = \{(x, t) \in Q_T : |w^\Delta - w'^\Delta| \geq \eta\},
\mathcal{A}_4 = \{(x, t) \in Q_T : |\nabla w^\Delta - \nabla w'^\Delta| \geq \mu, |\nabla w^\Delta| \leq k, |\nabla w'^\Delta| \leq k, |w^\Delta - w'^\Delta| \leq \eta\}.
\]

By repeating the arguments in Lemma 3.3 we have (72).

**Step 3: Recovery of boundary and initial conditions.**

We refer the reader to [CNO17, Proposition 4.9] to show that \( v \) in **Step 2** is a weak solution of problem (7) and furthermore, \( v(t) \to u_{0, \epsilon} \) a.e. as \( t \to 0 \). \( \square \)
This appendix provides a proof of the Aubin-Lions lemma in Lemma 3.2. We follow the ideas from [Mou16]. For any \( \delta > 0 \), we write \( \Omega^\delta_t = \{ x \in \Omega_t : d(x, \partial \Omega_t) > \delta \} \) and

\[
Q^\delta_T = \bigcup_{t \in (0, T)} \Omega^\delta_t \times \{ t \}.
\]

Let \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be a \( \mathcal{C}^\infty_c \) function such that

- \( \varphi \) is radially symmetric;
- \( \text{supp}(\varphi) \subset B(0, 1) \);
- \( \int_{\mathbb{R}^d} \varphi(x) dx = 1 \).

We define the scaled mollifier as \( \varphi^\varepsilon(x) = \varepsilon^d \varphi(x/\varepsilon) \) and for any distribution \( g \in \mathcal{D}'(Q_T) \) we have the convolution

\[
(g(\cdot, t) * \varphi^\varepsilon)(x) = \int_{\Omega_t} g(t, x - y) \varphi^\varepsilon(y) dy = \int_{\mathbb{R}^d} g(x - y, t) \varphi^\varepsilon(y) dy
\]

defined on \( \Omega^\frac{\varepsilon}{\delta} \), and consequently on \( \Omega^\delta_t \) for all \( \delta < \varepsilon \) by trivial extension.

**Lemma B.1.** Let \( \delta > 0 \). If \( \{ \nabla u_n \} \) is bounded in \( L^p(Q_T) \) for some \( p \geq 1 \), then

\[
\lim_{\varepsilon \to 0} \sup_{n \geq 1} \| u_n * \varphi^\varepsilon - u_n \|_{L^p(Q^\varepsilon_T)} = 0. \tag{73}
\]

**Proof.** By definition and the fact that \( \int_{\mathbb{R}^d} \varphi^\varepsilon(y) dy = 1 \) we have

\[
\|(u_n * \varphi^\varepsilon)(x, t) - u_n(x, t)\| \leq \int_{\mathbb{R}^d} |u_n(x - y, t) - u_n(x, t)| |\varphi^\varepsilon(y)| dy
\]
\[
\leq \int_{\mathbb{R}^d} |y||\nabla u_n(s(x - y) + (1 - s)x, t)||\varphi^\varepsilon(y)| dy
\]
\[
= \int_{|y| \leq \varepsilon} |y||\nabla u_n(s(x - y) + (1 - s)x, t)||\varphi^\varepsilon(y)| dy.
\]

Integrating the above inequality over \( Q^\varepsilon_T \) and using the fact that \( \{ \nabla u_n \} \) is bounded in \( L^p(Q_T) \), we get

\[
\sup_{n \geq 1} \| u_n * \varphi^\varepsilon - u_n \|_{L^p(Q^\varepsilon_T)} \leq C \int_{Q_T} \int_{|y| \leq \varepsilon} |y||\varphi^\varepsilon(y)| dy dx dt,
\]

and consequently (73) as \( \varepsilon \to 0 \). \qed

**Lemma B.2** (A local compactness lemma). Assume all the conditions in Lemma 3.2 are fulfilled. Then there exists \( \delta_0 > 0 \) small enough such that for any \( \delta \leq \delta_0 \), \( \{ u_n \} \) is precompact in \( L^s(Q^\varepsilon_T) \) for all \( 1 \leq s < p \).
Proof. We first prove that for any fixed $\varepsilon < \delta_0$, the sequence $\{u_n * \varphi^\varepsilon\}_n$ is precompact in $L^1(Q_T^\delta)$. Indeed, using the condition (47), and the fact that $\varphi^\varepsilon$ is radially symmetric we have
\[
\left| \int_{Q_T} \psi \partial_t (u_n * \varphi^\varepsilon) \, dx \, dt \right| = \left| \int_{Q_T} (\psi * \varphi^\varepsilon) \partial_t u_n \, dx \, dt \right| \leq C \sup_{t \in (0,T)} \| \psi * \varphi^\varepsilon \|_{H^m(\Omega_t)} \leq C_\varepsilon \sup_{t \in (0,T)} \| \psi \|_{L^\infty(\Omega_t)} \leq C_\varepsilon \| \psi \|_{L^\infty(Q_T^\delta)}.
\]

By duality, we get that $\{\partial_t (u_n * \varphi^\varepsilon)\}_n$ is bounded in $L^1(Q_T^\delta)$. From the assumption of $u_n$, we obtain that $\{u_n * \varphi^\varepsilon\}_n$ and $\{\nabla (u_n * \varphi^\varepsilon)\}_n$ are bounded in $L^1(Q_T^\delta)$. Therefore we have $\{u_n * \varphi^\varepsilon\}_n$ is bounded in $W^{1,1}(Q_T^\delta)$, and thus, by the compact embedding $W^{1,1}(Q_T^\delta) \hookrightarrow L^1(Q_T^\delta)$ we get that $\{u_n * \varphi^\varepsilon\}_n$ is a Cauchy sequence in $L^1(Q_T^\delta)$.

By applying estimate (73) in Lemma B.1 and by writing
\[
\|u_n - u_m\|_{L^1(Q_T^\delta)} \leq \|u_n - u_n * \varphi^\varepsilon\|_{L^1(Q_T^\delta)} + \|u_n * \varphi^\varepsilon - u_m * \varphi^\varepsilon\|_{L^1(Q_T^\delta)} + \|u_m * \varphi^\varepsilon - u_m\|_{L^1(Q_T^\delta)}
\]
we obtain that $\{u_n\}_n$ is precompact in $L^1(Q_T^\delta)$. Using the boundedness of $\{u_n\}_n$ in $L^p(Q_T^\delta)$ and interpolation we obtain the precompactness of $\{u_n\}_n$ in $L^s(Q_T^\delta)$ for all $1 \leq s < p$. \hfill \qed

We will also use the following result from [Mou16].

Lemma B.3. [Mou16, Proposition 8] If $\{u_n\}_n$ and $\{\nabla u_n\}_n$ are bounded in $L^p(Q_T)$ and $\{u_n\}_n$ is precompact in $L^p(Q_T^\delta)$ for all $\delta < \delta_0$, then $\{u_n\}_n$ is precompact in $L^p(Q_T)$.

We have all the ingredients to prove Lemma 3.2. \hfill \qed

Proof of Lemma 3.2. The proof follows directly from Lemmas B.2 and B.3 above. \hfill \qed

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References

[AES15] Amal Alphonse, Charles M Elliott, and Björn Stinner. An abstract framework for parabolic pdes on evolving spaces. Portualiae Mathematica, 1:1–46, 2015.

[AET18] Amal Alphonse, Charles M Elliott, and Joana Terra. A coupled ligand-receptor bulk-surface system on a moving domain: Well posedness, regularity, and convergence to equilibrium. SIAM Journal on Mathematical Analysis, 50(2):1544–1592, 2018.

[BEM11] R. Barreira, C.M. Elliott, and A. Madzvamuse. The surface finite element method for pattern formation on evolving biological surfaces. Journal of Mathematical Biology, 63(6):1095–1119, 2011.

[BG89] Lucio Boccardo and Thierry Gallouët. Non-linear elliptic and parabolic equations involving measure data. Journal of Functional Analysis, 87(1):149–169, 1989.

[Bla93] Dominique Blanchard. Truncations and monotonicity methods for parabolic equations. Nonlinear Analysis: Theory, Methods & Applications, 21(10):725–743, 1993.
[BM97] Dominique Blanchard and François Murat. Renormalised solutions of nonlinear parabolic problems with $L^1$ data: existence and uniqueness. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 127(6):1137–1152, 1997.

[BS05] Mostafa Bendahmane and Mazen Saad. Entropy solution for anisotropic reaction-diffusion-advection systems with $L^1$ data. *Revista Matemática Complutense*, 18(1):49–67, 2005.

[CMEV12] B. Stinner C. M. Elliott and C. Venkataraman. Modelling cell motility and chemotaxis with evolving surface finite elements. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 127(6):1137–1152, 1997.

[CNO17] Juan Calvo, Matteo Novaga, and Giandomenico Orlandi. Parabolic equations in time-dependent domains. *Journal of Evolution Equations*, 17(2):781–804, 2017.

[EE08] C. Eikls and C. M. Elliott. Numerical simulation of dealloying by surface dissolution via the evolving surface finite element method. *Journal of Computational Physics*, 227(23):9727–9741, 2008.

[ES09] C. M. Elliott and B. Stinner. Analysis of a diffuse interface approach to partial differential equations on moving surfaces. *Math. Models Methods Appl. Sci.*, 19:787–802, 2009.

[Fuj70] Hiroshi Fujita. On existence of weak solutions of the Navier-Stokes equations in regions with moving boundaries. *J. Fac. Sci., Univ. Tokyo, Sect. I*, 17:403–420, 1970.

[GH94] Thierry Gallouët and Raphaële Herbin. Existence of a solution to a coupled elliptic system. *Applied Mathematics Letters*, 7(2):49–56, 1994.

[GLS14] H. Garcke, K. F. Lam, and B. Stinner. Diffuse interface modelling of soluble surfactants in two-phase flow. *Communications in Mathematical Sciences*, 12(8):1475–1522, 2014.

[GMM+11] E.A. Gaffney, A. Madzvamuse, P. K. Maini, T. Sekimura, and C. Venkataraman. Modeling parr-mark pattern formation during the early development of Amago trout. *Physics Review E*, 84(4):041923, 2011.

[GS98] Thierry Goudon and Mazen Saad. On a Fokker-Planck equation arising in population dynamics. *Rev. Mat. Complut.*, 11(2):353–372, 1998.

[GS01] Thierry Goudon and Mazen Saad. Parabolic equations involving 0th and 1st order terms with $L^1$ data. *Revista Matemática Iberoamericana*, 17(3):433–469, 2001.

[KK15] E Knobloch and R Krechetnikov. Problems on time-varying domains: Formulation, dynamics, and challenges. *Acta Applicandae Mathematicae*, 137(1):123–157, 2015.

[Lew97] Roger Lewandowski. The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier-Stokes equation with an eddy viscosity. *Nonlinear Analysis: Theory, Methods & Applications*, 28(2):393–417, 1997.

[Lio69] Jacques Louis Lions. Quelques méthodes de résolution des problèmes aux limites non linéaires. 1969.

[Lio96] Pierre-Louis Lions. *Mathematical Topics in Fluid Mechanics: Volume 2: Compressible Models*, volume 2. Oxford University Press on Demand, 1996.

[MB08] SA Meier and Michael Böhm. A note on the construction of function spaces for distributed-microstructure models with spatially varying cell geometry. *Int. J. Numer. Anal. Model.*, 5(5):109–125, 2008.

[Mou16] Ayman Moussa. Some variants of the classical Aubin–Lions lemma. *Journal of Evolution Equations*, 16(1):65–93, 2016.

[Sim86] J. Simon. Compact sets in the space $L^p(0, T; B)$. *Annali Mat. Pura e Appl.*, 146:65–96, 1986.

[Vie14] M. Vierling. Parabolic optimal control problems on evolving surfaces subject to pointwise box constraints on the control—theory and numerical realization. *Interfaces Free Bound.*, 16(2):137–173, 2014.

[Win19] Michael Winkler. The role of superlinear damping in the construction of solutions to drift-diffusion problems with initial data in $L^1$. *Advances in Nonlinear Analysis*, 9(1):526–566, 2019.
QUASILINEAR EQUATIONS IN MOVING DOMAINS AND $L^1$ DATA

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