Revisiting the Darmois and Lichnerowicz junction conditions

Kayll Lake

Abstract What have become known as the “Darmois” and “Lichnerowicz” junction conditions are often stated to be equivalent, “essentially” equivalent, in a “sense” equivalent, and so on. One even sees not infrequent reference to the “Darmois-Lichnerowicz” conditions. Whereas the equivalence of these conditions is manifest in Gaussian-normal coordinates, a fact that has been known for close to a century, this equivalence does not extend to a loose definition of “admissible” coordinates (coordinates in which the metric and its first order derivatives are continuous). We show this here by way of a simple, but physically relevant, example. In general, a loose definition of the “Lichnerowicz” conditions gives additional restrictions, some of which simply amount to a convenient choice of gauge, and some of which amount to real physical restrictions, away from strict “admissible” coordinates. The situation was totally confused by a very influential, and now frequently misquoted, paper by Bonnor and Vickers, that erroneously claimed a proof of the equivalence of the “Darmois” and “Lichnerowicz” conditions within this loose definition of “admissible” coordinates. A correct proof, based on a strict definition of “admissible” coordinates, was given years previous by Israel. It is that proof, generally unrecognized, that we must refer to. Attention here is given to a clarification of the subject, and to the history of the subject, which, it turns out, is rather fascinating in itself.

Keywords Junction Conditions
1 Introduction

1.1 The Conditions

At a non-null boundary surface $\Sigma$ in spacetime (we do not consider surface layers) consider the following junction conditions:\[1\]

**Darmois 1927** (D [1]): The continuity of the first and second fundamental forms (the intrinsic metric and extrinsic curvature) [2] across $\Sigma$ [2].

**Lichnerowicz 1955** (L [3]): The continuity of the metric and all first order partial derivatives of the metric across $\Sigma$ in coordinates that traverse $\Sigma$ (such coordinates being loosely referred to as “admissible”) [3].

1.2 Gaussian Normal Coordinates

In Gaussian-normal coordinates [4] the spacetime is given by

$$ds^2 = \pm dn^2 + g_{ij}(n,x^k)dx^i dx^j$$

(1)

where $\Sigma$ is defined by $n = n_0$ where $n_0$ is a constant. $\Sigma$ is timelike with “+” and spacelike with “-”. Trajectories of constant $x^i$ are geodesics affinely parameterized by $n$. We suppose that the coordinates traverse $\Sigma$. As is well known, the second fundamental forms of $\Sigma$ are simply

$$K_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial n}$$

(2)

and since the other partial derivatives of $g_{ij}$ lie in $\Sigma$, it immediately follows that

$$D \iff L$$

(3)

and it is important to note, in these coordinates [5]. At this point it is, perhaps, important to point out that the junction conditions are background geometrical smoothness criteria and as such they can give rise to physically unreasonable situations even when satisfied [6].

---

1. We include an absolute minimum of technical complexity so as not to detract from the main points. Further, it is important to note that references through this communication that were once obscure are now (for the most part) immediately available through the internet, though not always in translation.

2. It is to be emphasized that two different spacetimes are involved and their respective coordinates are unrelated prior to matching.

3. This is not exactly what Lichnerowicz said, but it is what many references think he said. We start with this loose definition of “admissible coordinates” and then rectify the definition in the section Admissible Coordinates below.
1.3 Review of the Original Literature

In [1] page 29 Darmois says

"The conditions that we will meet, and whose importance is very high if we want to understand the interdependence of the masses and the field, were introduced by Schwarzschild. We will put as a condition that there is a system of coordinates in which the $g_{\alpha\beta}$ and their first derivatives are continuous functions even at the crossing of the border."

What is interesting is that not only did Darmois state a weak form of the L conditions, he attributes them to Schwarzschild. Whereas Darmois quotes both of Schwarzschild’s papers (the exterior and interior solutions) it is really the interior solution paper [7] that makes the point clear. In any event, whereas it might be argued that Schwarzschild was considering a special case, Darmois goes on to say

"These conditions are not invariant under an arbitrary change of variables and in reality can present difficulties. It is not always obvious whether the variables employed for solving the two problems, interior and exterior, allow the implementation of these conditions."

Darmois then points out that it is always possible to introduce Gaussian normal coordinates [1], states that the first fundamental form is $g_{ij}$ and that the second fundamental form is (2). He then says

"the [continuity of the] first and second fundamental forms of the border represent the common boundary."

and

"This condition is invariant, and can now be expressed, without otherwise effecting the change of coordinates, in any system."

And so, whereas Darmois used Gaussian normal coordinates to introduce the idea of the continuity of the first and second fundamental forms, he clearly realized that these conditions do not depend on the coordinates of the enveloping spacetimes.

In [3] Lichnerowicz devotes a chapter (III) to junction conditions. (The idea of “admissible” coordinates can be found early in chapter I.) However, essentially the same material can be found in his earlier work [5]. There, on page 31, in a section called “The junction conditions of Schwarzschild” one can find essentially everything except precise details concerning “admissible”
Darmois was Lichnerowicz’s supervisor, and one can find a reference to [1] in the Introduction of [8] and in [3] one can find a long introduction by Darmois. It remains a fact of history that Lichnerowicz, fully aware of Darmois’ suggestion to use the first and second fundamental forms, did not pursue this approach. The reason, most certainly, is the fact that Lichnerowicz was interested primarily in *global* problems, even in these early days, and “admissible” coordinates (when properly defined) suited his needs. Of course, global problems have evolved into an important area of study [12].

1.4 The Early Impact

The work of Lichnerowicz [3] was soon well known. In the influential book by Synge [13], one can find reference to [3] on page 1. Further, on page 40, Synge says

“Much of the work done on junction conditions prior to the introduction of admissible coordinates by Lichnerowicz is mathematically obscure.”

The work of Darmois [1] was not well known, at least outside of France until the middle 1960’s. As late as 1962, Beckedorff [14] solved the Oppenheimer - Snyder problem using the continuity of the first and second fundamental forms without reference to Darmois. The D conditions, with reference to Beckedorff, but not to Darmois, are stated explicitly by Misner and Sharp in 1964 [15]. In the fall of 1965 Israel submitted his now famous work [17] in which he developed the D conditions and extended the analysis to the study of surface layers based on discontinuities in the second fundamental forms. Israel gives an extensive review of previous work but with no mention of Darmois. The first reference to Darmois, outside of France, would appear to be the paper by Cocke [18]. The fact that Cocke was at the Institut Henri Poincaré might help explain this. After the paper by Cocke, there was a growing list of references to Darmois. Because of the importance of surface layers, one now finds frequent reference to the “Darmois-Israel” conditions.

This is explained by Lichnerowicz in [9].

The idea that “admissible” coordinates could be constructed by way of the introduction of coordinate changes fixed by the matching procedure was, of course, not new even at that time. This is the procedure used by Oppenheimer and Snyder [10], and on a mathematically equivalent problem by Einstein and Straus [11].

Within France, the equivalence of the Darmois and (correctly formulated) Lichnerowicz conditions were well known (José Senovilla, private communication).

For a related work that cites [3] and [15] but not Darmois see [16].

Israel was simply not aware of the paper by Darmois (Werner Israel, private communication).

It must be noted that Darmois never considered discontinuities in the second fundamental forms.
2 The Paper by Bonnor and Vickers

In 1981 a very influential paper by Bonnor and Vickers [19] appeared [20]. Whereas this paper was primarily interested in the junction conditions of O’Brien and Synge, which we do not discuss here, it is known primarily for the claimed proof of the equivalence of D and L. Their argument can be summarized as follows:

Given D one can introduce Gaussian-normal coordinates and so L is satisfied. Let us write this here as $D \Rightarrow L$. To establish $L \Rightarrow D$ they argued that if L is satisfied then so is D.

This brief “proof” is not adequate to establish the equivalence of D and L. As regards $L \Rightarrow D$, one must address the issue as to whether L is simply assumed in some set of coordinates, and whether or not the set of conditions L are in fact the same as the conditions D. For example, the conditions D could be a subset of the conditions L in which case the conditions can not be said to be equivalent. As regards $D \Rightarrow L$, aside from the fact that the introduction of Gaussian-normal coordinates that traverse $\Sigma$ may be impractical, one needs precise conditions under which, given D, coordinates that traverse $\Sigma$ can be constructed in which the metric is differentiable at $\Sigma$. This is discussed below.

2.1 The Impact

An inexhaustive search of the literature reveals the following: (i) An outright claim that D and L are equivalent (citing [19] for a proof) is given in, for example, [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], and [31]. (ii) A claim of equivalence (not citing [19]) is in [32]. (iii) A claim that D and L are “essentially” equivalent is in [33]. (iv) Sometimes D and L have been discussed without the alleged equivalence given in [19], for example in [34], [35] and [36]. (v) More cautionary statements that D are more “reliable” than L can be found in [37]. (vi) Also cautionary statements that D, “according to [19]”, are equivalent to L can be found in [38], [39] and [40]. (vii) There are statements in which I think that the author is saying D is equivalent to L according to [19], but I am not sure, as for example in [41]. (viii) There are references that paraphrase [19], for example [42] page 47, but do not state that the L conditions, away from Gaussian-normal coordinates, are not necessarily equivalent to the D conditions. (ix) There are references that attribute the L conditions to [19], for example [43], [44] and [45]. (x) There are references that refer to the Darmois-Lichnerowicz conditions [46]. (xi) There is even a reference that claims that the book by Lichnerowicz [3] was coauthored by Darmois [47]!
3 An Example

Here we consider the classic problem of the junction of a spherically symmetric static (not necessarily perfect) fluid onto the Schwarzschild vacuum. To compare the D and L conditions, we work out the junction problem in Gaussian normal coordinates and then in curvature coordinates.

3.1 Gaussian Normal Coordinates

We start with the line element

$$ds^2 = dr^2 + A(r)d\Omega_2^2 + B(r)dt^2$$  \hspace{1cm} (4)

where $d\Omega_2^2$ represents the line element of a two-sphere $(d\theta^2 + \sin(\theta)^2d\phi^2)$. We take the coordinates to be comoving and take an energy-momentum tensor to be of the form

$$T_\alpha^\beta = \text{diag}[p(r), P(r), P(r), -\rho(r)].$$  \hspace{1cm} (5)

By writing out the Einstein tensor $G_\alpha^\beta$ we see that $P$ and $\rho$ involve second order derivatives of the functions $A$ and $B$ but $p$ does not, it involves only first order derivatives. We conclude that $p$ is necessarily 0 at $\Sigma$ which can be taken to be $r = \text{constant} = r_\Sigma > 0$. This allows us to write $B$ in terms of $A$ for $r \geq r_\Sigma$. A fundamental property of the metric (4) is the effective gravitational mass, the invariant properties of which were first explored by Hernandez and Misner [48] who wrote the function in the form

$$m(r) = \frac{A^{3/2}}{2} R_{\theta\phi}^{\theta\phi},$$  \hspace{1cm} (6)

where $R$ is the Riemann tensor. See also [49]. From (4) and (6) we find

$$m(r) = \frac{4A - A'^2}{8\sqrt{A}}$$  \hspace{1cm} (7)

and so $m(r)$ is also continuous across $\Sigma$ with $m = m(r_\Sigma)$ for $r \geq r_\Sigma$ [50]. This completes the junction problem in Gaussian normal coordinates.

3.2 Curvature Coordinates

Let us start with the D conditions. The exterior Schwarzschild vacuum, in terms of exterior curvature coordinates $(r, \theta, \phi, T)$, is of course given by

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2d\Omega_2^2 - (1 - \frac{2m}{r})dT^2.$$  \hspace{1cm} (8)
The interior line element, in terms of interior comoving curvature coordinates \((r, \theta, \phi, t)\), can be taken to be

\[
ds^2 = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2 - e^{2\Phi(r)} dt^2.
\] (9)

The coordinates in (4) and in (9) are entirely distinct. Without loss in generality we can take

\[
ds^2_\Sigma = R(\tau)^2 d\Omega^2_2 - d\tau^2
\] (10)

where \(\tau\) is the proper time on \(\Sigma\).

The continuity of the intrinsic metric associated with \(\Sigma\) ensures that the continuity of \(\theta\) and \(\phi\) in metrics (8) and (9) is allowed and that the history of the timelike boundary surface \(\Sigma\) is given by

\[
R(\tau) = r_\Sigma(> 2m(r_\Sigma)) = r_\Sigma(> 2m).
\] (11)

The continuity of the extrinsic curvature component \(K_{\tau \tau}\) gives us

\[
\Phi' |_\Sigma = \frac{m}{r(2m - r)} |_\Sigma
\] (12)

so that along with the source equation for \(\Phi\) (the generalized Tolman - Oppenheimer - Volkoff equation) we have

\[
p(r_\Sigma) = 0
\] (13)

where \(p\) is, again, the isotropic pressure. We can take (13) (with (11)) as the definition of the boundary \(\Sigma\). The continuity of the extrinsic curvature components \(K_{\theta \theta}\) and \(K_{\phi \phi}\), along with (11), gives

\[
m(r_\Sigma) = m.
\] (14)

Finally, the correct rigging of the 4-normals to \(\Sigma\) (interior to exterior) is verified by the continuity of the trace of the extrinsic curvature across \(\Sigma\). Equations (11), (13) and (14) then constitute the solution to the D junction problem for the case of a static spherically symmetric fluid joined onto vacuum. We see that the D conditions are invariant to the change in coordinates, as they must be. Even in this simple case, it is to be noted that the D conditions are not entirely trivial to execute.

To explore the L conditions let us start with the assumption that the coordinates used in (8) and (9) are admissible in the weak sense (i.e. there is no distinction between \((t, T)\) and \((r, r)\) and the 4-metric components are \(C^1\) at \(\Sigma\)). Then, in addition to the previous conditions, when applied to \((t, T)\) the L conditions also give

\[
e^{2\Phi(r_\Sigma)} = 1 - \frac{2m}{r_\Sigma}
\] (15)
(the derivative requirement already appearing in (12)) and when applied to \((r, r)\) they give the further condition

\[ m'(r \Sigma) = \rho(r \Sigma) = 0, \]  

(16)

where \(\rho\) is, again, the comoving energy density. Condition (15) is a forced, but convenient, choice of gauge that we discuss below. The condition (16) is a physical restriction. There are solutions that satisfy (16) \([51]\) and so for these, with (15), the L conditions are satisfied. We conclude that D does not imply L for these solutions.

3.3 An aside on Gauge Conditions

It is well-known that the evolution of null geodesics in the spacetime (9) is governed by

\[ \frac{\dot{r}^2}{l^2} = \frac{1}{r^2} \left( 1 - \frac{2m(r)}{r} \right) \left( \frac{B(r)^2}{b^2} - 1 \right) \]  

(17)

where \(\dot{\cdot} = d/d\lambda\), \(\lambda\) being an affine parameter, \(b^2 = \gamma^2/l^2\) where \(\gamma\) (the “energy”) is associated with the Killing vector \(\delta_{\alpha}^\alpha\) and \(l\) (the “angular momentum”) is associated with the Killing vector \(\delta_{\alpha}^\phi\), and the “potential impact parameter” \(B(r)^2\) is given by

\[ B(r)^2 = \frac{r^2}{e^{2\Phi(r)}}. \]  

(18)

With the aide of condition (15) we see that \(B^2\) is continuous through \(\Sigma\) and with the addition of (12) we have \((B^2)'/\Sigma\) continuous through \(\Sigma\). The gauge condition (15) therefore provides a convenient choice when studying, for example, transparent spherical gravitational lenses.

For the complete Schwarzschild solution, the interior is given by

\[ ds^2 = \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\Omega^2 - (a - b\sqrt{1 - \frac{r^2}{R^2}})^2 dt^2 \]  

(19)

where \(a\) and \(b\) are constants as is \(\rho\) where \(8\pi \rho/3 = 1/R^2\). The lensing properties of the complete Schwarzschild solution have been studied \([52]\). If \(r_{\Sigma}\) exists, it is given by

\[ 3\sqrt{1 - \frac{r_{\Sigma}^2}{R^2}} = \frac{a}{b}. \]  

(20)

In the first edition of the Exact Solutions book \([53]\), the gauge condition (15), which amounts to \(b = 1/2\) in this case, is applied. This is also true of the book by Griffiths and Podolský \([54]\). In the second edition of the Exact Solutions book \([42]\), there is no mention of \(r_{\Sigma}\) and no gauge condition is applied. The point is that the gauge condition (15) is convenient, but not required.
4 Admissible Coordinates

In the foregoing, an essential element to the Lichnerowicz approach has been lost. In [3], on page 5, Lichnerowicz says

“We demand that in the intersection between the domains of two admissible coordinate systems, the second derivatives of the coordinate transformation must be piecewise \( C^2 \) functions.”

At first sight, this might appear a little strong. The fact that Lichnerowicz frequently uses the requirement that derivatives of the coordinate transformation must be piecewise \( C^2 \) functions perhaps confuses the issue.\(^{10}\) The original statement is correct. If \( D \) has been established, then one can prove (see [56], [17] and more recently [36]) that there always exists coordinates, by way of \( C^1 \) (piecewise \( C^3 \)) transformations, that traverse \( \Sigma \) such that the metric is differentiable at \( \Sigma \). Call these coordinates \( \mathcal{C} \), Israel’s transparent notation for \( C \) is\(^{11}\)

\[
g_{\alpha\beta} = (C^1_{\Sigma}, C^3) \quad (21)
\]

Further (see the same references above), all coordinates obtained from \( \mathcal{C} \) via \( C^2 \) (piecewise \( C^4 \)) transformations, in Israel’s notation

\[
(C^2_{\Sigma}, C^4), \quad (22)
\]

form a set of coordinates, say \( S \), that we call “admissible” or “natural” coordinates exactly as stated by Lichnerowicz in \( [9] \). If the conditions \( L \) are carried out in \( S \) then clearly \( D \) and \( L \) are equivalent. The flaw in the proof by Bonnor and Vickers \( [19] \) is the lack of a requirement of smoothness which is the essential feature of \( L \). The flaw in the example given above is that curvature coordinates are simply not “admissible”, they were simply assumed to be.

5 Conclusions

As should be well known, the \( D \) and \( L \) conditions are equivalent in Gaussian normal coordinates and in coordinates related to Gaussian normal coordinates by \( (C^2_{\Sigma}, C^4) \) transformations (such coordinates being referred to as “natural” or “admissible” coordinates \( S \)). Whereas the \( D \) conditions can be examined in

---

\(^{10}\) This confusion persists. To quote from a recent paper [55] “the term “admissible” designates a coordinate system of a \( C^2 \) class (atlas) manifold structure describing the space-time”.

\(^{11}\) Curiously, Synge [13], on page 2, refers to the \( C^3 \) condition as “not the important thing”, but it is.

\(^{12}\) An explanation of these conditions, in English, was given by Lichnerowicz at the 1957 Chapel Hill Conference. Unfortunately, his lecture remained rather difficult to get until relatively recently. See [57].
any set of enveloping coordinates, albeit with some effort, the L conditions are restricted to “natural” or “admissible” coordinates in the strong sense of \( S \). It is fitting to end with a quotation from [17]. When reflecting on section 7 of his paper, Israel says

“The formulas of this Section, despite their simple appearance, are actually of limited utility, since natural [admissible] coordinates are seldom the most convenient for handling a problem in practice.”

Acknowledgments. This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada. It is a pleasure to thank Bill Ballik, Werner Israel, Dmitri Lebedev, Peter Musgrave, Eric Poisson and José Senovilla for discussions.

References

1. G. Darmois (1927) *Mémoire de Sciences Mathématiques, Fascicule XXV*, “Les équations de la gravitation einsteinienne”

2. Readers unfamiliar with these terms can find a very clear and detailed explanation in the classic text by Eisenhart. See L. P. Eisenhart (1947) *An Introduction to Differential Geometry*. Princeton University Press.

3. A. Lichnerowicz (1955) *Théories Relativistes de la Gravitation et de l’Electromagnétisme*. Masson, Paris.

4. These have been in use in spacetime for a century. For a convenient resource see, for example, C. Misner, K. Thorne and J. Wheeler (1973) *Gravitation*. W.H. Freeman and Company, Section 21.13

5. This has to be considered the most trivial case of a junction problem possible and many specific examples of this type of junction can be found. See, for example, A. Lightman, W. Press, R. Price and S. Teukolsky (1975) *Problem Book in Relativity and Gravitation*. Princeton University Press, Problem 9.29. They consider the special case \( g_{ij} = a(n)^2 \gamma^{ij}(x^k) \).

6. Following on [11] and considering the case of Robertson-Walker geometries, we see that for any (spacelike) surface of comoving proper-time, we only need \( a \) and its derivative to be continuous. This allows instantaneous “phase changes” which are unjustifiable without further physics describing the situation.

7. K. Schwarzschild—“Über das Gravitationsfeld einer Kugel aus incompressibler Flüssigkeit nach der Einsteinschen Theorie”, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, 1916, S. 424. In English translation “On the Gravitational Field of a Sphere of Incompressible Liquid, According to Einstein’s Theory” The Abraham Zelmanov Journal, The journal for General Relativity, gravitation and cosmology, Vol. 1, 2008, ISSN 1654-9163, translation by Larissa Borissova and Dmitri Rabounski.

8. A. Lichnerowicz (1939) *Problèmes Globaux en Mécanique Relativiste*. Hermann, Paris.

9. A. Lichnerowicz (1988) “Mathematics and General Relativity: A Recollection” in *Studies in the History of General Relativity* Edited by J. Eisenstaedt and A. Knox Birkhäuser, Boston.

In an attempt to alleviate the complications of following the Darmois-Israel approach, the package *GRJunction* [58], [59] was developed by Peter Musgrave to run with *GRTensorII*. Unfortunately, due to my own negligence, *GRJunction* was never in wide distribution, despite the very wide spread use of *GRTensorII*. This has now changed. *GRTensorII* has been updated to *GRTensorIII*, thanks to Peter’s efforts, and an update to *GRJunction* has been completed and is now available [60].
10. J. Oppenheimer and H. Snyder, Phys. Rev. 56, 455 (1939).
11. A. Einstein and E. Straus, Rev. Mod. Phys. 17, 120 (1945) and 18, 148 (1946).
12. Y. Choquet-Bruhat (2009) General Relativity and the Einstein Equations Oxford University Press, Oxford.
13. J. Synge (1960) Relativity: The General Theory North-Holland, Amsterdam.
14. D. Beckedoff (1962) Terminal Configurations of Stellar Evolution, Undergraduate Thesis Princeton, supervised by C. Misner.
15. C. Misner and D. Sharp, Phys. Rev. 136, B571 (1964).
16. L. Bel and A. Hamoui, Ann. Inst. Henri Poincaré VII, 229 (1967).
17. W. Israel, Nuovo Cimento 44B, 1 (1966) (see 48B, 463 for corrections).
18. W. Cocke, J. Math. Phys. 7, 1171 (1966).
19. W. Bonnor and P. Vickers, General Relativity and Gravitation 13, 29 (1981).
20. At the time of writing, this paper has 96 citations via the Web of Science.
21. M. Sharif and A. Siddiqua, General Relativity and Gravitation 43, 73 (2011).
22. D. Giang and C. C. Dyer, Int. J. Modern Physics D 18, 1913 (2009).
23. R. Wiltshire, General Relativity and Gravitation 35, 175 (2003).
24. C. Oliwa, Some Mathematical Problems in Inhomogeneous Cosmology, Ph. D. thesis, University of Toronto (2001).
25. U. von der Gönna and D. Kramer, General Relativity and Gravitation 31, 349 (1999).
26. M. MacCallum and N. Santos, Classical and Quantum Gravity 15, 1627 (1998).
27. A. Bernui and E. Portocarrero, Astrophysical Journal 427, 947 (1994).
28. F. Fayos, X. Jaen, E. Llanta and J. M. M. Senovilla, Classical and Quantum Gravity 8, 2057 (1991).
29. J. Griffiths, (1991) Colliding plane waves in general relativity, Oxford University Press.
30. C. Clarke and T. Dray, Classical and Quantum Gravity 4, 265 (1987).
31. L. Herrera and J. Jiménez, Physical Review D 28, 2987 (1983).
32. C. Hellaby and T. Dray, Physical Review D 52, 7333 (1995).
33. P. Pereira and A. Wang, General Relativity and Gravitation 32, 2189 (2000).
34. F. Fayos, J. Senovilla and R. Torres, Physical Review D 54, 4862 (1996).
35. M. Mars and J. Senovilla, Classical and Quantum Gravity 10, 1865 (1993).
36. C. Hellaby and T. Dray, Physical Review D 49, 5096 (1994).
37. P. Bhar, Astrophys. Space Sci. 356, 309 (2015).
38. P. Bhar, Astrophys. Space Sci. 357, 46 (2015).
39. C. Hellaby, A. Sumeruk and G. F. R. Ellis, International Journal of Modern Physics D 6, 211 (1997).
40. M. MacCallum, General Relativity and Gravitation 30, 131 (1998).
41. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt (2003) Exact Solutions of Einstein's Field Equations Cambridge University Press, Cambridge.
42. N. Pant, M. Ahmad and N. Pradhan J. Astrophys. Astr. 37, 6 (2016).
43. N. Pant, N. Pradham and M. H. Murad, Astrophys. Space Sci. 355, 137 (2015).
44. M. Murad and N. Pant, Astrophys. Space Sci. 350, 349 (2014).
45. P. Jacewicz and A. Krasinski, General Relativity and Gravitation 44, 81 (2012).
46. J. Cuchí, A. Gil-Rivero and E. Ruiz, General Relativity and Gravitation 45, 1433 (2013).
47. W. C. Hernandez and C. W. Misner, Astrophys. J. 143, 452 (1966).
48. M. Cahill and G. McVittie, J. Math. Phys. 11, 1360 (1970), E. Poisson and W. Israel, Physical Review D 41, 1796 (1990), T. Zannias, Physical Review D 41, 3252 (1990) and S. Hayward, Physical Review D 53, 1938 (1996).
49. Setting $m(r) = m$ in (48A) it does not follow that $A$ in fact be expressed in terms of elementary functions. This in no way affects the argument given here.
50. Consider, for example, $m(r) = -r^4 + ar^3$ with $0 < a < (32/9)^{1/3}$. We have $4\pi\rho(0) = 3a$, $\rho(r):=0$ for $r>3a/4$ and $\Phi$ is a monotone increasing function of $r$ with a regular minimum at $r=0$ and subject to the boundary condition $\Phi(r):=6a^2/(32-9a^3)$.
51. J. Jaffe, Mon. Not. R. Astron. Soc. 149, 395 (1970); J. Lawrence, Astrophys. J. 230, 249 (1979), T. Klings and E. Newman, Physical Review D 59, 124002 (1999).
52. D. Kramer, H. Stephani, E. Herlt and M. MacCallum (1980) Exact Solutions of Einstein’s Field Equations Cambridge University Press, Cambridge.
53. J. Griffiths and J. Podolský (2009) Exact Space-Times in Einstein’s General Relativity Cambridge University Press, Cambridge.
55. R. Lapiedra and J. Morales-Lladosa, Physical Review D 95, 064025 (2017).
56. W. Israel, Proc. Roy. Soc. London A 208, 404 (1958).
57. A. Lichnerowicz, “On the Integration of the Einstein Equations”, Chapter 5 in The Role of Gravitation in Physics Report from the 1957 Chapel Hill Conference Edited by C. DeWitt and D. Rickles, Edition Open Access, 2011.
58. P. Musgrave and K. Lake, Class. Quantum Grav. 13, 1885 (1996).
59. P. Musgrave and K. Lake, Class. Quantum Grav. 14, 1285 (1997).
60. GRTensorIII 2.0: Hypersurfaces and Junctions is available free of charge. Release information is at: http://hyperspace.uni-frankfurt.de/2017/02/20/grtensoriii-2-0-hypersurfaces-and-junctions/