Relations for Modular Forms from Vertex Operator Algebras

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Abstract. We will give a short reminder for vertex operator algebra notion and corresponding characters. Then we discuss algebraic methods for explicit computation of the partition and correlation functions. We then illustrate general ways to find number theory identities for related modular forms by specific examples of modular form relations arising from our construction. Finally, we present new results concerning identities for prime forms on genus $g$ Riemann surfaces and genus two $n$-point functions for vertex operator algebra characters.

1. Introduction
The origin of CFT is contained in the structure of corresponding vertex operator algebra (VOA) [1]. Modes of a vertex operator algebra are non-commutative and subject to certain commutation relations. $n$-point functions possess modular properties which (from the information point of view) corresponds to co-invariance with respect to the modular group which acts on parameters of a Riemann surface as well as on their local coordinates. This means that no matter how we change Riemann surface by using the modular group, the data coming from correlation functions of our surface remains invariant up to computable multipliers. In order to determine the higher genus correlation functions for a particular CFT, we have to understand how information on a higher genus Riemann surface can be reconstructed from data for two lower genus Riemann surfaces. It turns out that for this purpose we have to "sew" two lower genus CFT’s (considered on two lower genus Riemann surfaces). But first we have to sew two Riemann surfaces themselves. For this purpose we use the Yamada sewing procedure. Finally, the procedure for sewing lower genus information allows us to obtain the higher genus $n$-point correlation functions which also possess modular properties. To form a genus $g_1 + g_2$ Riemann surface we sew lower genus $g_1$ and $g_2$ surfaces. The structure of vertex operator algebras allows us to compute constructively characters (partition and $n$-point correlation functions) and to prove explicitly their natural modularity properties. The structure of vertex operator algebras and CFT correlation functions can be used for coding.

This paper represents the author’s talk given at the XXVth International Conference on Integrable Systems and Quantum symmetries (ISQS-25), 2017.

2. Geometry: construction of a genus $g$ Riemann surface
The simplest example of forming a higher genus Riemann surface from lower genus one is the sewing the Riemann sphere to form a torus. Sew a handle to the Riemann sphere $\hat{\mathbb{C}}$ by identifying
annular regions centred at \( A_{\pm 1} \in \hat{C} \) via a sewing condition with complex sewing parameter \( \rho \)

\[
(z - A_{-1})(z' - A_1) = \rho,
\]

This is called the canonical Sewing. We call \( \rho, A_{\pm 1} \) the canonical parameters. The annuli do not intersect provided

\[
|\rho| < \frac{1}{4} |A_{-1} - A_1|^2
\]

Inequivalent tori depend only on \( \chi = -\frac{\rho}{|A_{-1} - A_1|^2} \) for \( |\chi| < \frac{1}{4} \).

We may construct a general genus \( g \) Riemann surface by identifying \( g \) pairs of annuli centred at \( A_{\pm i} \in \hat{C} \) for \( i = 1, \ldots, g \) and sewing parameters \( \rho_i \) satisfying

\[
(z - A_{-i})(z' - A_i) = \rho_i,
\]

provided no two annuli intersect.

Let \( x_{-i} = x - A_{-i} \) and \( y_j = y - A_j \) be local coordinates in the neighbourhood of canonical sewing parameters \( A_{-i}, A_j \) for \( i, j \in \{ \pm 1, \ldots, \pm g \} \) with \( i \neq -j \). The 2-point function is expanded as

\[
\frac{1}{(x - y)^2} = \sum_{k,l \geq 1} (-1)^{k+1} \frac{(k + l - 1)!}{(k-1)!(l-1)!} \frac{x^{k-1}y^{l-1}}{(A_{-i} - A_j)^{k+l}}.
\]

Define the canonical moment matrix \( R^C \), [13] an infinite matrix indexed by \( k, l = 1, 2, \ldots \) and \( i, j \in \{ \pm 1, \ldots, \pm g \} \) where \( R^C_{i,j}(k,l) = 0 \) and

\[
R^C_{i,j}(k,l) = \frac{(-1)^k \rho_i^{k/2} \rho_j^{l/2}}{\sqrt{kl}} \frac{(k + l - 1)!}{(k-1)!(l-1)!} \frac{1}{(A_{-i} - A_j)^{k+l}}
\]

for \( i \neq -j \).

\((I - R^C)^{-1}\) plays a central role in computing the genus \( g \) period matrix and other structures.

3. Vertex operator algebras

A Vertex Operator Algebra (VOA) [1] \( (V, Y, 1, \omega) \) consists of a \( \mathbb{Z} \)-graded complex vector space \( V = \bigoplus_{n \in \mathbb{Z}} V(n) \) where \( \dim V(n) < \infty \) for each \( n \in \mathbb{Z} \), a linear map \( Y : V \to End(V[[z, z^{-1}]]) \) for a formal parameter \( z \) and pair of distinguished vectors: the vacuum \( 1 \in V(0) \) and the conformal vector \( \omega \in V(2) \). For each \( v \in V \), the image under the map \( Y \) is the vertex operator

\[
Y(v, z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1},
\]

with \textit{modes} \( v(n) \in End(V) \), where \( Y(v, z)1 = v + O(z) \). Vertex operators satisfy \textit{locality} i.e. for all \( u, v \in V \) there exists an integer \( k \geq 0 \) such that

\[
(z_1 - z_2)^k \left[ Y(u, z_1), Y(v, z_2) \right] = 0.
\]

The vertex operator of the conformal vector \( \omega \) is

\[
Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2},
\]

where the modes \( L(n) \) satisfy the Virasoro algebra with \textit{central charge} \( c \)

\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12} \delta_{m,-n} c \text{Id}_V.
\]
We define the homogeneous space of weight $k$ to be

$$V_{(k)} = \{ v \in V | L(0)v = kv \},$$

and we write $\text{wt}(v) = k$ for $v \in V_{(k)}$. Finally, we have a translation condition

$$Y(L(-1)u, z) = \partial_z Y(u, z).$$

4. Genus zero correlation functions

For $u_1, u_2, \ldots, u_n \in V$ define the $n$-point (correlation) function by

$$\langle 1, Y(u_1, z_1)Y(u_2, z_2) \ldots Y(u_n, z_n) 1 \rangle,$$

where the invariant form is described in subsection 17.4 in Appendix. The locality property of vertex operators implies that this formal expression coincides with the analytic expansion of a rational function of $z_1, z_2, \ldots, z_n$ in the domain $|z_1| > |z_2| > \ldots > |z_n|$. Thus the $n$-point function can taken to be a rational function of $z_1, z_2, \ldots, z_n \in \hat{\mathbb{C}}$, the Riemann sphere.

**Theorem 4.1 (Hurley, Tuite)**, [2] For a VOA of central charge $C$, the Virasoro $n$-point function is a $\beta$-extended permanent

$$\langle 1, Y(\omega, z_1) \ldots Y(\omega, z_n) 1 \rangle = \text{perm}_{\mathbb{C}} B,$$

for $B_{ij} = \frac{1}{(z_i - z_j)^2}, i \neq j$ and $B_{ii} = 0$, where perm denotes the permanent of a matrix.

5. The genus $g$ partition function - canonical parameters

We now define [13] the genus $g$ partition function for a VOA $V$ in the canonical sewing scheme in terms of genus zero $2g$-point correlation functions as follows:

$$Z_V^{(g)}(\rho_i, A_{\pm i}) = \langle 1, \prod_{i=1}^g \sum_{n_i \geq 0} \rho_i^{n_i} \sum_{v_i \in V(n)} Y(v_i, A_{-i}) Y(v_i, A_i) 1 \rangle,$$

where $v_i$ is dual to $v_i$. For genus one this reverts to the standard definition [6]:

**Theorem 5.1 (Mason, Tuite)**

$$Z_V^{(1)}(\rho, A_{\pm 1}) = \text{Tr}_V(q L_0),$$

where $q = C(\chi)$, the Catalan series for $\chi = -\frac{\rho}{(A_{-1} - A_1)^2}$.

6. Partition function $Z_{M_2}^{(g)}(\rho_i, A_{\pm i})$ for Heisenberg VOA $M_2$

**Theorem 6.1 (Tuite and Z.)**, [13]

- $Z_{M_2}^{(g)}(\rho_i, A_{\pm i}) = \frac{1}{\det(I - R^C)}$, where $R^C$ is the canonical moment matrix.
- $\det(I - R^C)$ is holomorphic and non-vanishing.
- The genus $g$ Heisenberg generating function can be computed in terms of a permanent of genus $g$ bilinear forms of the second kind.
7. Genus one correlation functions and Zhu recursion

The genus one partition function for $V$ is defined by the trace function [1]

$$Z_V^{(1)}(\tau) = \text{Tr}_V \left( q^{L(0)-c/24} \right),$$

and the genus one n-point correlation function by

$$Z_V^{(1)}(v_1, z_1; \ldots; v_n, z_n; \tau) = \text{Tr}_V \left( Y(q_{z_1}^{L(0)} v_1, q_{z_2}) \ldots Y(q_{z_n}^{L(0)} v_n, q_{z_n}) q^{L(0)-c/24} \right).$$

In particular, the genus one 1-point function for $v \in V$ is

$$Z_V^{(1)}(v; \tau) = \text{Tr}_V \left( o(v) q^{L(0)-c/24} \right),$$

where, for $v$ homogeneous, $o(v) := v(wv(v) - 1) : V(m) \rightarrow V(m)$. Every n-point function is expressible in terms of 1-point functions

$$Z_V^{(1)}(v_1, z_1; \ldots; v_n, z_n; \tau) = Z_V^{(1)}(Y[v_1, z_1] \ldots Y[v_n, z_n] 1; \tau)$$

$$= Z_V^{(1)}(Y[v_1, z_1 - z_n] \ldots Y[v_n, z_{n-1} - z_n] v_n; \tau).$$

8. Genus one correlation functions and Zhu recursion

We will make repeated Zhu recursion [14] which recursively relates n-point correlation functions to $(n-1)$-point functions

**Theorem 8.1** [Zhu Recursion] Genus one n-point correlation functions obey

$$Z_V^{(1)}(v_1, z_1; \ldots; v_n, z_n; \tau) = \text{Tr}_V \left( o(v_1) Y(q_{z_2}^{L(0)} v_2, q_{z_2}) \ldots Y(q_{z_n}^{L(0)} v_n, q_{z_n}) q^{L(0)-c/24} \right)$$

$$+ \sum_{k=2}^{n} \sum_{j \geq 0} P_{1+j}(z_1 - z_k, \tau) Z_V^{(1)}(v_2, z_2; \ldots; v_1[j] v_k, z_k; \ldots; v_n, z_n; \tau).$$

9. Genus two surfaces formed from two tori

For the general description of the genus two Riemann surface formation from two tori see [6].

Consider an oriented torus $S_a^{(1)} = \mathbb{C}/\Lambda_a$ with lattice $\Lambda_a = 2\pi i (Z\tau a \oplus \mathbb{Z})$ for $\tau a \in \mathbb{H}_1$. For local coordinate $z_a \in \mathbb{C}/\Lambda_a$, the closed disc $|z_a| \leq r_a$ is contained in $S_a^{(1)}$ provided $r_a < \frac{1}{4} D(q_a)$ where

$$D(q_a) = \min_{\lambda \in \Lambda_a, \lambda \neq 0} |\lambda|,$$

is the minimal lattice distance. One introduces a complex sewing parameter $\epsilon$ where $|\epsilon| \leq r_1 r_2 < \frac{1}{4} D(q_1) D(q_2)$ and excise the disc $\{z_a, |z_a| \leq |\epsilon|r_a/2\}$ centered at $z_a = 0$ to form a punctured torus

$$\tilde{S}_a^{(1)} = S_a^{(1)} \\setminus \{z_a, |z_a| \leq |\epsilon|r_a\},$$
where we here (and below) we use the convention

\[ \Im = 2, \quad \Re = 1. \]

Defining the annulus \( A_a = \{ z_a, |\epsilon|/r_a \leq |z_a| \leq r_a \} \) we identify \( A_1 \) with \( A_2 \) via the sewing relation

\[ z_1 z_2 = \epsilon. \]

The genus two Riemann surface \( S^{(2)} \) is parameterized by the sewing domain

\[ D_{\text{sew}} = \left\{ (\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} : |\epsilon| < \frac{1}{4} D(q_1) D(q_2) \right\}. \]

10. Genus \( g \) partition function by sewing two Riemann surfaces

We define the genus \( g \) partition function by inductively sewing together lower genus 1-point functions \( Z^{(g_i)}_M(u, z_i), i = 1, 2, z_i \in \Sigma^{(g_i)} \), \( u \) belongs to \( n \)-th grading subspace of \( V \), for a \( V \)-module \( M \):

\[ Z^{(g)}_M(z_1, z_2, \epsilon) = \sum_{u \in V_{(n)}} Z^{(g_1)}_M(u, z_1) Z^{(g_2)}_M(\bar{u}, z_2), \]

where \( g = g_1 + g_2 \) and \( z_1, z_2 \) are insertion points, \( \epsilon \) is the sewing parameter, and \( \bar{u}, u \) are related by a non-degenerate bilinear invariant form on \( V \). For instance, with suitable local coordinates in the neighbourhood of these points we find for the Heisenberg VOA \( M_2 \) one obtains

**Theorem 10.1**

\[ Z^{(g)}_{M_2}(v_1, v_2, \epsilon) = \frac{1}{\det(I - Q(g_1, g_2))} Z^{(g_1)}_{M_2} \cdot Z^{(g_2)}_{M_2}, \]

where \( Z^{(g)}_{M_2} \) is a non-vanishing holomorphic function on the sewing domain and is automorphic with respect to \( SL(2, \mathbb{Z}) \times \ldots \times SL(2, \mathbb{Z}) \subset Sp(2g, \mathbb{Z}) \) with automorphy factor \( \det(C\Omega^{(g)} + D)^{-1} \), and a multiplier system, \( Q(g_1, g_2) \) is a moment matrix containing genus \( g \) sewing data, and \( \Omega^{(g)} \) is genus \( g \) period matrix.

The genus \( g \) \( n \)-point correlation function for \( a_1, \ldots, a_L \) and \( b_1, \ldots, b_R \) formally inserted at \( x_1, \ldots, x_L \in \mathcal{S}_1 \) and \( y_1, \ldots, y_R \in \mathcal{S}_2 \), respectively, by

\[ Z^{(g)}_V(a_1, x_1; \ldots; a_L, x_L; b_1, y_1; \ldots; b_R, y_R; \Omega^{(g_1)}_1, \Omega^{(g_2)}_2, \epsilon) \]

\[ = \sum_{u \in V} Z^{(g_1)}_V(Y[a_1, x_1; \ldots; a_L, x_L]u; \Omega^{(g_1)}_1) Z^{(g_2)}_V(Y[b_R, y_R; \ldots; b_1, y_1]v; \Omega^{(g_2)}_2). \]

11. General ways to find number theory identities for related modular forms:

Having at hands the fundamental way allowing us to construct the higher genus characters for vertex operator algebras, we proceed to the identities resulting from this method. Here we can enumerate a few general vertex operator algebra methods to derive number theory identities for related modular forms:

- Boson-Fermion correspondence;
- Comparison of alternative representations of characters;
- Higher-Lower genus relations for characters;
- Powers of \( \eta \)-functions according from partition functions and determinants;
• Modular-invariance properties of characters.

Let us proceed with examples of relations for modular forms that appear in computations of partition and \(n\)-point functions for vertex operator algebras. In the following sections we illustrate the above mentioned general vertex operator algebra methods by specific examples of modular relations arising from our construction.

12. Triple Jacobi identity

As a result of the boson-fermion correspondence one obtains Jacobi triple product identity [1]:

\[
q^{\frac{\chi^2}{2}} \prod_{l \geq 1} (1 - \theta^{-1} q^{\frac{\chi}{2} + \kappa})(1 - \theta q^{\frac{\chi}{2} + \kappa}) = \frac{e^{2\pi i (\alpha + \frac{1}{2}) (\beta + \frac{1}{2})}}{\eta(\tau)} \vartheta^{(1)} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} (0, \tau),
\]

where \(\mu_1 = -\beta + \frac{1}{2}, \mu_2 = \alpha + \frac{1}{2}\). Here \(\eta(\tau)\) is the Dedekind eta-function

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\]

Thus we get the identity for the first power of the \(\eta\)-function.

13. Genus two triple Jacobi identities

We could make use of the genus one identities to derive higher genus analogues of Jacobi product identities [10]. In particular, at genus two we get an explicit analogue of classical Jacobi triple identity

\[
\frac{\Theta^{(2)} \begin{bmatrix} a \\ b \end{bmatrix} (\Omega)}{\vartheta^{(1)} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} (\tau_1) \vartheta^{(1)} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} (\tau_2)} = \det \left( I - A_1^{(1)} A_2^{(1)} \right)^{1/2} \det (I - Q).
\]

Here \(\vartheta^{(1)} \begin{bmatrix} a \\ b \end{bmatrix} (\tau), \Theta^{(2)} \begin{bmatrix} a \\ b \end{bmatrix} (\Omega)\), are genus one and genus two theta-functions correspondingly, and \(\Omega\) is a period matrix. In the alternative \(\rho\)-parameter self-sewing (see subsection 17.2 of Appendix) of the torus to form a genus two Riemann surface:

Theorem 13.1 [11] The ratio of genus two and genus one Riemann theta functions on \(D^\rho\) is given by

\[
\frac{\vartheta^{(2)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Omega^{(2)}}{\vartheta^{(1)} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \Omega}{e^{2\pi i \rho K}} \left( e^{i\pi B \rho} K(w, \tau)^{1/2} \right) \det (I - T) \det (I - R)^{1/2}.
\]

14. Modular discriminant and generalizations of the Garvan’s formula

The modular discriminant is defined by

\[
\Delta(\tau) = \eta(\tau)^{24},
\]

where \(\eta(\tau)\) is the Dedekind eta-function

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\]
The fundamental classical formula for the modular discriminant is
\[
\Delta(\tau) = \frac{1}{1728} \det \begin{pmatrix}
E_4(\tau) & E_6(\tau) \\
E_6(\tau) & E_8(\tau)
\end{pmatrix}.
\]

The following formula was first proposed by F. Garvan.
\[
\Delta^2(\tau) = -\frac{691}{250 (1728)^2} \det \begin{pmatrix}
E_4(\tau) & E_6(\tau) & E_8(\tau) \\
E_6(\tau) & E_8(\tau) & E_{10}(\tau) \\
E_8(\tau) & E_{10}(\tau) & E_{12}(\tau)
\end{pmatrix}.
\]

Using the Fay’s trisecant identity we obtain in [15]:

**Theorem 14.1** For \((\theta, \phi) \neq (1, 1)\) one has
\[
\Delta^n(\tau) = \frac{\vartheta^{(1)}([\frac{1}{2}])}{\vartheta^{(1)}([\frac{1}{2}])} (0, \tau) \Theta^{(1)}_{8n,8n,(1,1)}(x, y, \tau).
\]

For \((\theta, \phi) = (1, 1)\),
\[
\Delta^n(\tau) = \frac{\Theta^{(1)}_{8n+1,8n+1,(1,1)}(x, y, \tau)}{\vartheta^{(1)}([\frac{1}{2}])} \det (Q_{8n+1}).
\]

where
\[
\Theta^{(1)}_{r,s,(m,n)}(x, y, \tau) \equiv \prod_{1 \leq i < k \leq r} \vartheta^{(1)}([\frac{1}{2}]) (x_i - x_j, \tau)^{m_i m_j} \prod_{1 \leq j \leq s} \vartheta^{(1)}([\frac{1}{2}]) (y_j - y_i, \tau)^{n_j n_i},
\]
\[
P_n(\theta, \phi) = \begin{pmatrix}
P_1(x_1 - y_1, \tau) & \ldots & P_1(x_1 - y_n, \tau) & 1 \\
\vdots & \ddots & \vdots & \vdots \\
P_1(x_n - y_1, \tau) & \ldots & P_1(x_n - y_n, \tau) & 1 \\
1 & \ldots & 1 & 0
\end{pmatrix},
\]
\[
Q_n = \begin{pmatrix}
P_1(x_1 - y_1, \tau) & \ldots & P_1(x_1 - y_n, \tau) & 1 \\
\vdots & \ddots & \vdots & \vdots \\
P_1(x_n - y_1, \tau) & \ldots & P_1(x_n - y_n, \tau) & 1 \\
1 & \ldots & 1 & 0
\end{pmatrix}.
\]

Using the full version of the Fay’s generalized trisecant identity we derive the following

**Theorem 14.2** For \((\theta, \phi) \neq (1, 1)\),
\[
\Delta^p(\tau) = \frac{\vartheta^{(1)}([\frac{1}{2}])}{\vartheta^{(1)}([\frac{1}{2}])} (0, \tau) \Theta^{(1)}_{r,s,(m,n)}(x, y, \tau) \det (M_{r,s}),
\]

where
\[
p = 8 \left( \sum_{1 \leq i \leq r, 1 \leq k \leq s} m_i n_j - \sum_{1 \leq i < k \leq r} m_i m_k - \sum_{1 \leq j \leq s} n_j n_i \right).
\]
15. Formulas relating prime forms through period matrices and theta-functions

Finally, in the next two sections we present new results concerning identities for prime forms (see subsection 17.1 of Appendix) on genus $g$ Riemann surfaces and genus two $n$-point functions for vertex operator algebras on Riemann surfaces.

In [9] we have derived the formula relating the genus $g + 1$ and genus $g$ prime forms $K^{(g+1)}(x, y, \Omega^{(g+1)}) = K^{(g)}(x, y, \Omega^{(g)}) e^{-\frac{1}{2} b_a X_{\bar{a}a}(\rho) \bar{b}_\rho^2}$, and in particular,

$$K^{(2)}(x, y, \Omega^{(2)}) e^{\frac{1}{2} b_a(X_{\bar{a}a}(\rho) \bar{b}_\rho^2 (x, y; k))} = K^{(1)}(x - y, \tau).$$

Here $b_a(x, y; k) = \int_y^x a_a(\cdot, k)$, and $a_a(x) X_{\bar{a}a}(\rho) a_a^T(y) = \sum_{k, l \geq 1} a_a(x, k) X_{\bar{a}a}(k, l, \rho) a_a(y, l)$, with $a_a(x, k)$ a certain one-form on the initial Riemann surface $\Sigma^{(g_a)}$ of genus $g$. Here $X_{\bar{a}a}(k, l, \rho)$ is an infinite matrix determined from genus $g$ data (see [6] for details). In [9] we have proved that $S^{(g+1)}$ is holomorphic in $\rho$ for $|\rho| < r_1 r_2$ with $S^{(g+1)}(x, y) = S^{(g)}_\kappa(x, y) + O(\rho)$, for some kernel $S^{(g)}_\kappa(x, y)$.

For the genus $g + 1$ prime form $K^{(g+1)}$ and genus $g$ prime form $K^{(g)}$ we obtain here the following:

**Proposition 15.1** Let us define $U^{(g)}(x, y) = \frac{K^{(g)}(x, y, p_2)}{K^{(g)}(x, y, p_1)}$, and $z_{p_1, p_2} = \int_{p_1}^{p_2} \nu^{(g)}$, for holomorphic one-forms $\nu^{(g)}$. One can relate the genus $g + 1$ and $g$ prime forms $K^{(g+1)}$ and $K^{(g)}$ by means of the following formulas. For $\kappa \neq -1/2$, we get

$$K^{(g)}(x, y) = \Theta^{(g)} \left[ \begin{array}{c} \alpha^{(g)} \\ \beta^{(g)} \end{array} \right] (x, y; \nu^{(g)} + \kappa z_{p_1, p_2}, \kappa^2 z_{p_1, p_2}; \Omega^{(g)}) \left( U^{(g)}(x, y) \right)^\kappa$$

$$\times \left[ \begin{array}{c} K^{(g+1)}(x, y) \\ \alpha^{(g+1)} \\ \beta^{(g+1)} \end{array} \right]^{-1} \Theta^{(g+1)} \left[ \begin{array}{c} \alpha^{(g+1)} \\ \beta^{(g+1)} \end{array} \right] (x, y; \nu^{(g+1)}, 0|\Omega^{(g+1)})$$

$$-\xi h_{S^{(g)}_\kappa}(x) D^\theta \left( I - T^{(g)}_{S^{(g)}_\kappa} \right)^{-1} h^T_{S^{(g)}_\kappa}(y) \right]^{-1}. \quad (1)$$

For $\kappa = -1/2$ one obtains:

$$K^{(g)}(x, y) = \left[ \begin{array}{c} \alpha^{(g)} \\ \beta^{(g)} \end{array} \right] \left( \int_y \nu^{(g)} + \frac{1}{2} z_{p_1, p_2}; \Omega^{(g)} \right) \left( U^{(g)}(x, y) \right)^{1/2}$$

$$-\theta_{g+1} \left[ \begin{array}{c} \alpha^{(g)} \\ \beta^{(g)} \end{array} \right] \left( \int_y \nu^{(g)} - \frac{1}{2} z_{p_1, p_2}; \Omega^{(g)} \right) \left( U^{(g)}(x, y) \right)^{-1/2}$$

$$\times \left[ \begin{array}{c} \alpha^{(g+1)} \\ \beta^{(g+1)} \end{array} \right] (x, y; \nu^{(g+1)}, 0|\Omega^{(g+1)}) K^{(g+1)}(x, y)^{-1}$$

$$-\xi h_{S^{(g)}_\kappa}(x) D^\theta \left( I - T^{(g)}_{S^{(g)}_\kappa} \right)^{-1} h^T_{S^{(g)}_\kappa}(y) \right]^{-1}. \quad (2)$$

Therefore, we derive
Lemma 15.2 For \( \kappa \neq -1/2 \), we get the formula for the genus \( g \) prime form

\[
K^{(g)}(x, y, \Omega^{(g)}) = \\
\left[ e^{\frac{1}{2} b_a X_{ab}(\rho) b_a^g} \Theta^{(g+1)} \left( \frac{\alpha^{(g+1)}}{\beta^{(g+1)}} \right) (x, y; \nu^{(g+1)}, 0|\Omega^{(g+1)}) \\
- \Theta^{(g)} \left( \frac{\alpha^{(g)}}{\beta^{(g)}} \right) (x, y; \nu^{(g)} + \kappa z_{p_1, p_2}, \kappa z_{p_1, p_2}|\Omega^{(g)}) \right] \\
\left[ \xi_h S^{(g)}(x) D^\theta \left( I - T^{(g)}_{S^{(g)}} \right) -1 \ h^{T}_{S^{(g)}}(y) \right]^{-1} \, . \tag{3}
\]

16. Self-sewn genus two prime form formula for characters

We are able to express the genus two character for a vertex operator algebra module (see subsection 17.3 of Appendix) via genus two prime forms by the next proposition. The formula generalizes the genus one formula for the case of genus two Riemann surface.

Proposition 16.1 With \( z_{n+1} = w \), \( z_{n+2} = 0 \), \( u_{n+1} = \sigma f_2 \Psi_\kappa \), and \( u_{n+2} = \overline{\Psi}_\kappa \),

\[
Z_\alpha^{(2)}(u_1 \otimes e^{\beta_1}, z_1; \ldots ; u_n \otimes e^{\beta_n}, z_n; \tau, w, \rho) \\
= \frac{q^{\tau \alpha^2}}{\eta(\tau)} \sum_{\Psi_\kappa} Q_{\alpha}^{\beta_1, \ldots, \beta_{n+2}}(u_1, z_1; \ldots ; u_{n+2}, z_{n+2}; q) \ exp \left( \alpha \left( \sum_{i=1}^{n+2} \beta_i z_i \right) \right) \\
\cdot \prod_{1 \leq r < s \leq n+2} \left( K^{(2)}(z_r, z_s) \right)^{\beta_r \beta_s} e^{\frac{1}{2} \beta_r \beta_s b_a(z_r, z_s; k) X_{ab} b_a^g(z_r, z_s; k)} \, . \tag{4}
\]

In [9] we established formulas relating genus one and genus two prime forms via theta-function relations originating from the Szegö kernel description. Using such formulas, we prove here the genus two version of the torus formula in the self-sewing formalism.

Proposition 16.2 With \( z_{n+1} = w \), \( z_{n+2} = 0 \), \( u_{n+1} = \sigma f_2 \Psi_\kappa \), and \( u_{n+2} = \overline{\Psi}_\kappa \),

\[
Z_\alpha^{(2)}(u_1 \otimes e^{\beta_1}, z_1; \ldots ; u_n \otimes e^{\beta_n}, z_n; \tau, w, \rho) \\
= \frac{q^{\tau \alpha^2}}{\eta(\tau)} \sum_{\Psi_\kappa} Q_{\alpha}^{\beta_1, \ldots, \beta_{n+2}}(u_1, z_1; \ldots ; u_{n+2}, z_{n+2}; q) \ exp \left( \alpha \left( \sum_{i=1}^{n+2} \beta_i z_i \right) \right) \\
\cdot \prod_{1 \leq r < s \leq n+2} \left[ \Theta^{(1)} \left( \frac{\alpha^{(1)}}{\beta^{(1)}} \right) (z_r, z_s; \nu^{(1)} + \kappa z_{p_1, p_2}, \kappa z_{p_1, p_2}|\Omega^{(1)}) \right]^{-\beta_r \beta_s} \\
\times \left[ K^{(2)}(z_r, z_s) \right]^{-1} \Theta^{(2)} \left( \frac{\alpha^{(2)}}{\beta^{(2)}} \right) (z_r, z_s; \nu^{(2)} + 0|\Omega^{(2)}) \\
\cdot \xi_h S^{(1)}(z_r) D^\theta \left( I - T^{(1)}_{S^{(1)}} \right) -1 \ h^{T}_{S^{(1)}}(z_s) \, . \tag{5}
\]
For $\kappa = -1/2$ we get

$$
Z^{(2)}_{\alpha}(u_1 \otimes e^{\beta_1}, z_1; \ldots; u_n \otimes e^{\beta_n}, z_n; \tau, w, \rho) = q^{\tilde{\alpha}^2} \sum_{\Psi_n} Q^{\beta_1, \ldots, \beta_{n+2}}(u_1, z_1; \ldots; u_{n+2}; z_{n+2}; q) \exp \left( \alpha \left( \sum_{i=1}^{n+2} \beta_i z_i \right) \right)
$$

$$
\cdot \prod_{1 \leq r < s \leq n+2} \left( \vartheta \left( \alpha^{(1)} \beta^{(1)} \right) \left( \int_{y}^{x} \nu^{(1)} + \frac{1}{2} z_{p_1, p_2}^{(1)} \right) \left( U^{(1)}(x, y) \right)^{\frac{1}{2}} \right)^{-\beta_r \beta_s}
$$

$$
- \theta_2 \vartheta \left( \alpha^{(1)} \beta^{(1)} \right) \left( \int_{y}^{x} \nu^{(1)} - \frac{1}{2} z_{p_1, p_2}^{(1)} \right) \left( U^{(1)}(x, y) \right)^{-\frac{1}{2}} \right)^{-\beta_r \beta_s}
$$

$$
\times \left( \vartheta \left( \alpha^{(1)} \beta^{(1)} \right) \left( \frac{1}{2} z_{p_1, p_2}^{(1)} \right) - \theta_2 \vartheta \left( \alpha^{(1)} \beta^{(1)} \right) \left( -\frac{1}{2} z_{p_1, p_2}^{(1)} \right) \right)^{-\beta_r \beta_s}
$$

$$
\times \left[ \Theta^{(2)} \left( x, y; \nu^{(2)}, 0 | \Omega^{(2)} \right) K^{(2)}(x, y)^{-1} - \xi h_{S_{\kappa}^{(1)}}(x) D^{\theta} \left( I - T^{(1)}_{S_{\kappa}^{(1)}} \right)^{-1} h_{T_{S_{\kappa}^{(1)}}}(y) \right]^{-\beta_r \beta_s}.
$$

(6)

**Proof:** We consider the case with $\kappa \neq -1/2$,

$$
K^{(1)}(x - y, \tau) = \Theta^{(1)} \left[ \alpha^{(1)} \beta^{(1)} \right] \left( x, y; \nu^{(1)} + \kappa z_{p_1, p_2}, \kappa z_{p_1, p_2} | \Omega^{(1)} \right) \left( U^{(1)}(x, y) \right)^{\kappa}
$$

$$
\cdot \left[ K^{(2)}(x, y, \Omega^{(2)}) \right]^{-1} \Theta^{(2)} \left[ \alpha^{(2)} \beta^{(2)} \right] \left( x, y; \nu^{(2)}, 0 | \Omega^{(2)} \right)
$$

$$
- \xi h_{S_{\kappa}^{(1)}}(x) D^{\theta} \left( I - T^{(1)}_{S_{\kappa}^{(1)}} \right)^{-1} h_{T_{S_{\kappa}^{(1)}}}(y) \right]^{-1}.
$$

(7)
Let us take $u_{n+1} \otimes e^{\beta_{n+1}} = \sigma f_2 \Psi_\kappa$, and $u_{n+2} \otimes e^{\beta_{n+2}} = \overline{\Psi}_\kappa$. Then we obtain

$$Z^{(2)}_\alpha \left( u_1 \otimes e^{\beta_1}, z_1; \ldots, u_n \otimes e^{\beta_n}, z_n; \tau, w, \rho \right)$$

$$= \sum_{\Psi_\kappa} Z^{(1)}_\alpha \left( u_1 \otimes e^{\beta_1}, z_1; \ldots, u_n \otimes e^{\beta_n}, z_n; \sigma f_2 \Psi_\kappa, w; \overline{\Psi}_\kappa, 0; q \right)$$

$$= \sum_{\Psi_\kappa} Z^{(1)}_\alpha \left( u_1 \otimes e^{\beta_1}, z_1; \ldots, u_{n+1} \otimes e^{\beta_{n+1}}; z_{n+1} \otimes e^{\beta_{n+1}} w; u_{n+2} \otimes e^{\beta_{n+2}}, 0; q \right)$$

$$= \sum_{\Psi_\kappa} Q^{\beta_1, \ldots, \beta_n}_{\alpha} (u_1, z_1; \ldots, u_n, z_n; q) Z^{(1)}_\alpha \left( e^{\beta_1}, z_1; \ldots, e^{\beta_n}, z_n; e^{\beta_{n+1}} w, e^{\beta_{n+2}}, 0; q \right)$$

$$= \sum_{\Psi_\kappa} q^\frac{1}{\eta(\tau)} Q^{\beta_1, \ldots, \beta_n}_{\alpha} (u_1, z_1; \ldots, u_n, z_n; q) \exp \left( \alpha \left( \sum_{i=1}^{n+2} \beta_i z_i \right) \right)$$

$$\cdot \prod_{1 \leq r < s \leq n+2} \left( K^{(1)}(z_{rs}, \tau) \right)^{\beta_r \beta_s}$$

$$= \sum_{\Psi_\kappa} q^\frac{1}{\eta(\tau)} Q^{\beta_1, \ldots, \beta_n}_{\alpha} (u_1, z_1; \ldots, u_n, z_n; q) \exp \left( \alpha \left( \sum_{i=1}^{n+2} \beta_i z_i \right) \right)$$

$$\cdot \prod_{1 \leq r < s \leq n+2} \left[ \Phi^{(1)} \left. \frac{\alpha^{(1)}}{\beta^{(1)}} \right| \left( z_r, z_s; \nu^{(1)}, \kappa z_{p_1 p_2}, \kappa z_{p_1 p_2} | \Omega^{(1)} \right) \right]^{-\beta_r \beta_s}$$

$$\left( U^{(1)}(z_r, z_s) \right)^{-\kappa \beta_r \beta_s} \left[ \left( K^{(2)} \left( z_r, z_s \Omega^{(2)} \right) \right)^{-1} \Phi^{(2)} \left. \frac{\alpha^{(2)}}{\beta^{(2)}} \right| \left( z_r, z_s; \nu^{(2)}, 0 | \Omega^{(2)} \right) \right]^{-\beta_r \beta_s}$$

$$- \xi h_{S_n^{(1)}}(z_r) D^\theta \left( I - T^{(1)}_{S_n^{(1)}} \right)^{-1} h_{S_n^{(1)}}(z_s).$$

Thus we arrive at (6).

Note that corresponding generalizations are possible at higher genera.

**Acknowledgments**

We would like to thank the Organizers of the XXVth International Conference on Integrable Systems and Quantum symmetries (ISQS-25), and in particular, Prof. C. Burdik.

**17. Appendix:**

**17.1. The prime form**

Let us recall the definition and properties of the prime form on a genus $g$ Riemann surfaces [7]. Consider a compact Riemann surface $\Sigma^{(g)}$ of genus $g$ with canonical homology cycle basis $a_1, \ldots, a_g, b_1, \ldots, b_g$. In general there exists $g$ holomorphic 1-forms $\nu_i$, $i = 1, \ldots, g$ which we may normalize by

$$\oint_{a_i} \nu_j^{(g)} = 2\pi i \delta_{ij}. \quad (8)$$

The genus $g$ period matrix $\Omega^{(g)}$ is defined by

$$\Omega_{ij}^{(g)} = \frac{1}{2\pi i} \oint_{b_i} \nu_j^{(g)}, \quad (9)$$
for \( i, j = 1, \ldots, g \). \( \Omega^{(g)} \) is symmetric with positive imaginary part, i.e., \( \Omega^{(g)} \in \mathbb{H}_g \), the Siegel upper half plane. The canonical intersection form on cycles is preserved under the action of the symplectic group \( Sp(2g, \mathbb{Z}) \) where

\[
\begin{pmatrix} b & a \\ a & b \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{b} & \tilde{a} \\ \tilde{a} & \tilde{b} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} b & a \\ a & b \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}).
\]

(10)

This induces the modular action on \( \mathbb{H}_g \)

\[
\Omega^{(g)} \rightarrow \tilde{\Omega}^{(g)} = \left( A\Omega^{(g)} + B \right) \left( C\Omega^{(g)} + D \right)^{-1}.
\]

(11)

One introduces the normalized differential of the second kind defined by

\[
\omega^{(g)}(x, y) \sim \frac{dx dy}{(x - y)^2},
\]

(12)

for \( x \sim y \), for local coordinates \( x, y \), with normalization \( \int_{a_i} \omega^{(g)}(x, \cdot) = 0 \) for \( i = 1, \ldots, g \). Using the Riemann bilinear relations, one finds that \( \nu_i^{(g)}(x) = \oint_{b_i} \omega^{(g)}(x, \cdot) \). We recall also the definition of the theta function with real characteristics

\[
\vartheta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \left( z\mid \Omega^{(g)} \right) = \sum_{m \in \mathbb{Z}^g} \exp \left( i\pi (m + \alpha) \Omega^{(g)} (m + \alpha) + (m + \alpha) \cdot (z + 2\pi i \beta) \right),
\]

(13)

for \( \alpha = (\alpha_i), \beta = (\beta_i) \in \mathbb{R}^g, z = (z_i) \in \mathbb{C}^g \) and \( i = 1, \ldots, g \) with

\[
\vartheta \left[ \begin{array}{c} \alpha + r \\ \beta + s \end{array} \right] \left( z\mid \Omega^{(g)} \right) = e^{2\pi i \alpha \cdot s} e^{-2\pi i \beta \cdot r} e^{-i \pi r \cdot s} \vartheta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \left( z\mid \Omega^{(g)} \right),
\]

(14)

for \( r, s \in \mathbb{Z}^g \).

There exists a (nonsingular and odd) character \( [\gamma] \) such that [7]

\[
\vartheta \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (0\mid \Omega^{(g)}) = 0, \quad \partial_{z_i} \vartheta \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (0\mid \Omega^{(g)}) \neq 0.
\]

(15)

Let

\[
\zeta^{(g)}(x) = \sum_{i=1}^g \partial_{z_i} \vartheta \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (0\mid \Omega^{(g)}) \nu_i^{(g)}(x),
\]

(16)

a holomorphic one-form, and let \( \zeta^{(g)}(x)^{\frac{1}{2}} \) denote the form of weight \( \frac{1}{2} \) on the double cover \( \Sigma^{(g)} \) of \( \Sigma^{(g)} \). We also refer to \( \zeta(x)^{\frac{1}{2}} \) as a (double-valued) \( \frac{1}{2} \)-form on \( \Sigma \). We define the prime form \( E(x, y) \) by

\[
E^{(g)}(x, y) = \vartheta \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] \left( \frac{\int_x \nu^{(g)}(y\mid \Omega^{(g)})}{\zeta^{(g)}(x)^{\frac{1}{2}} \zeta^{(g)}(y)^{\frac{1}{2}}} \right) \sim (x - y) dx^{-\frac{1}{2}} dy^{-\frac{1}{2}} \quad \text{for } x \sim y,
\]

(17)

where \( \int_y \nu^{(g)} = \left( \int_{b_i} \omega^{(g)}(x, \cdot) \right) \in \mathbb{C}^g \). \( E^{(g)}(x, y) = -E^{(g)}(y, x) \) is a holomorphic differential form of weight \( (-\frac{1}{2}, -\frac{1}{2}) \) on \( \Sigma^{(g)} \times \Sigma^{(g)} \). We denote by \( K^{(g)} \) the functional part of \( E^{(g)} \).
17.2. Self-sewing of a Riemann surface
Consider the construction of a Riemann surface $\Sigma^{(g+1)}$ formed by self-sewing a handle to a Riemann surface $\Sigma^{(g)}$ of genus $g$. We review the Yamada formalism [5] in this scheme which we call the $\rho$-formalism. Consider a Riemann surface $\Sigma^{(g)}$ of genus $g$ and let $z_1, z_2$ be local coordinates in the neighbourhood of two separated points $p_1$ and $p_2$. Consider two disks $|z_a| \leq r_a$, for $r_a > 0$ and $a = 1, 2$. Note that $r_1, r_2$ must be sufficiently small to ensure that the disks do not intersect. Introduce a complex parameter $\rho$ where $|\rho| \leq r_1 r_2$ and excise the disks

$$\{ z_a : |z_a| < |\rho r_a^{-1} | \subset \Sigma^{(g)},$$

to form a twice-punctured surface

$$\hat{\Sigma}^{(g)} = \Sigma^{(g)} \setminus \bigcup_{a=1,2} \{ z_a : |z_a| < |\rho r_a^{-1} | \}.$$  

As before, we use the convention $I = 2$, $\hat{2} = 1$. We define annular regions $A_a \subset \hat{\Sigma}^{(g)}$ with $A_a = \{ z_a : |\rho r_a^{-1} | \leq |z_a| \leq r_a \}$ and identify them as a single region $A = A_1 \simeq A_2$ via the sewing relation

$$z_1 z_2 = \rho,$$  

(18)

to form a compact Riemann surface $\Sigma^{(g+1)} = \hat{\Sigma}^{(g)} \setminus \{ A_1 \cup A_2 \} \cup A$ of genus $g + 1$. The sewing relation (18) can be considered to be a parameterization of a cylinder connecting the punctured Riemann surface to itself. In the $\rho$-formalism we define a standard basis of cycles $\{ a_1, b_1, \ldots, a_g, b_g \}$ on $\Sigma^{(g+1)}$ where the set $\{ a_1, b_1, \ldots, a_g, b_g \}$ is the original basis on $\Sigma^{(g)}$. Let $C_a(z_a) \subset A_a$ denote a closed anti-clockwise contour parameterized by $z_a$ surrounding the puncture at $z_a = 0$. Clearly $C_2(z_2) \sim -C_1(z_1)$ on applying the sewing relation (18). We then define the cycle $a_{g+1}$ to be $C_2(z_2)$ and define the cycle $b_{g+1}$ to be a path chosen in $\hat{\Sigma}^{(g)}$ between identified points $z_1 = z_0$ and $z_2 = \rho / z_0$ on the sewn surface.

17.3. The free fermion vertex operator superalgebra (VOSA)
Consider the rank two free fermion VOSA $V(H, Z + \frac{1}{2}) \otimes 2$ of central charge 1, [1]. The weight $\frac{1}{2}$ space $V_{\frac{1}{2}}$ is spanned by $\psi^+, \psi^-$ with vertex operators

$$Y(\psi^\pm, z) = \sum_{n \in \mathbb{Z}} \psi^\pm(n) z^{-n-1},$$

whose modes satisfy the anti-commutation relations

$$[\psi^+(m), \psi^-(n)] = \delta_{m,-n-1}, \quad [\psi^+(m), \psi^+(n)] = [\psi^-(m), \psi^-(n)] = 0.$$  

(19)

The VOSA is generated by $\psi^\pm$ with $V$ spanned by Fock vectors of the form

$$\Psi(k, l) \equiv \psi^+(-k_1) \ldots \psi^+(-k_s) \psi^-(-l_1) \ldots \psi^-(-l_t) 1,$$  

(20)

for distinct $0 < k_1 < \ldots < k_s$ and $0 < l_1 < \ldots < l_t$. The Virasoro vector

$$\omega^{(g)} = \frac{1}{2} (\psi^+(-2) \psi^-(-1) + \psi^-(-2) \psi^+(1)) 1,$$

generates a Virasoro algebra with central charge $c = 1$ for which the Fock vectors have weight

$$(\Psi(k, l)) = \sum_{i=1}^{s} (k_i - \frac{1}{2}) + \sum_{j=1}^{t} (l_j - \frac{1}{2}).$$  

(21)
The mode $a(0)$ generates continuous $V_\mathbb{Z}$-automorphisms $g = e^{-2\pi i \alpha (0)}$ for all $\alpha \in \mathbb{C}$. In particular, we define the fermion number involution

$$\sigma = e^{\pi i \alpha (0)},$$

where $\sigma u = (-1)^{p(u)} u$ for $u$ of parity $p(u)$.

17.4. The invariant form on $M$

It is convenient to define [10] for formal parameter $z$ and $\chi \in \mathbb{C}$

$$(-z)^\chi = e^{i\pi B \chi} z^\chi,$$

(22)

where $B$ is an odd integer parametrizing the formal branch cut. In [10] we introduced an invariant bilinear form $\langle \cdot, \cdot \rangle$ on $M$ associated with the Möbius map

$$\begin{pmatrix} 0 & \lambda \\ -e^{i\pi B} \lambda^{-1} & 0 \end{pmatrix} : z \mapsto -\frac{\lambda^2}{e^{i\pi B} z}.$$  

(23)

for $\lambda \neq 0$. We are particularly interested in the Möbius map $z \mapsto \rho/z$ associated with the sewing condition (18) so that we will choose

$$\lambda = e^{\frac{1}{2} i \pi B} \rho^\frac{1}{2},$$

(24)

for the odd integer $B$ of (22). Thus we reformulate the sewing relationship (18) as

$$z_1 = -\lambda^2 z_2$$

so that

$$dz_1^{\frac{1}{2}} = \xi \rho^\frac{1}{2} z_2 dz_2^{\frac{1}{2}}$$

for $\xi = e^{\frac{1}{2} i \pi B}$. Define the adjoint vertex operator

$$\mathcal{Y}^\dagger (u \otimes e^\alpha, z) = \mathcal{Y} \left( e^{-z^\lambda -2L(1)} \left( \frac{\lambda}{e^{i\pi B} z} \right)^{2L(0)} (u \otimes e^\alpha), \frac{\lambda^2}{e^{i\pi B} z} \right).$$

(25)

A bilinear form $\langle \cdot, \cdot \rangle$ on $M$ is said to be invariant if for all $u \otimes e^\alpha, v \otimes e^\beta, w \otimes e^\gamma \in M$ we have

$$\langle \mathcal{Y} (u \otimes e^\alpha, z) (v \otimes e^\beta), w \otimes e^\gamma \rangle = e^{-i\pi B \alpha \beta} \langle v \otimes e^\beta, \mathcal{Y}^\dagger (u \otimes e^\alpha, z) w \otimes e^\gamma \rangle.$$  

(26)

(25) reduces to the usual definition for a VOSA when $\alpha, \beta, \gamma \in \mathbb{Z}$ [10]. Choosing the normalization $(1, 1) = 1$ then $\langle \cdot, \cdot \rangle$ on $M$ is symmetric, unique and invertible with [10]

$$\langle u \otimes e^\alpha, v \otimes e^\beta \rangle = \lambda^{-\alpha^2} \delta_{\alpha, -\beta} (u \otimes e^0, v \otimes e^0).$$

(27)

Thus the dual of the Fock vector $\Psi = \Psi(\kappa, 1)$ is

$$\overline{\Psi}(\kappa, 1) = (-1)^{\kappa + |\Psi|} \lambda^2(\Psi) \Psi(1, \kappa),$$

where $|x|$ denotes the integer part of $x$ [10]. Applying (24) and (27) it follows that $\Psi_\alpha = \Psi_\alpha(\kappa, 1)$ has dual

$$\overline{\Psi}_\alpha(\kappa, 1) = (-1)^{\kappa + |\Psi_\alpha|} \lambda^2(\Psi_\alpha) \Psi_{-\alpha}(1, \kappa) = (-1)^{\kappa + |\Psi_\alpha|} e^{i\pi B(\Psi_\alpha)} \rho(\Psi_\alpha) \Psi_{-\alpha}(1, \kappa).$$

(28)
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