Light deflection due to a charged, rotating body

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Abstract
According to the general theory of relativity and subsequent developments in the field, it is known that there are three factors, namely mass, rotation and charge, that can influence the space–time geometry. Accordingly, we discuss the effect of the space–time geometry of a charged, rotating body on the motion of the light ray. We obtained the expression for the equatorial deflection of light due to such a body up to the fourth-order contribution of both the mass and charge. We used the null geodesic approach of the light ray for our calculation. If we set the charge equal to zero, our expression of the bending angle is reduced to the Kerr equatorial bending angle. If we set the rotation to zero, our expression reduces to the Reissner–Nordström deflection angle, and if we set both charge and rotation to zero, our expression reduces to the Schwarzschild bending angle.

Keywords: light deflection, rotation, Kerr–Newman

1. Introduction

General relativity, a theory of gravitation, was developed by Einstein as early as 1907, and the final form was given in 1915. One of the very important consequences of general relativity is the bending of a light ray in the presence of a gravitational field. Three factors, likely mass distribution, charge and rotation, can influence the path of the light ray. The exact solution of Einstein’s field equation for a static, spherical, uncharged body was found by Schwarzschild in 1915 [1], that for an uncharged rotating body by Kerr in 1963 [2], and that for a rotating charged body by Newman, which is known as the Kerr–Newman metric [3, 4]. The solution for Einstein’s field equation for a static, charged, spherically symmetric body was obtained by Reissner and Nordström independently, and is known as the Reissner–Nordström solution [5, 6]. Einstein himself calculated the bending angle ($\alpha = 4GM/Rc^2$) of light up to the first-order term of $k(\equiv GM/Rc^2)$, assuming a null geodesic path of light in the space–time...
manifold, a basic postulate for light in general relativity. In the above expression, $G$ is the gravitational constant, $M$ is the mass of the gravitating body, $c$ the speed of light in free space, and $R$ the distance of the closest approach. The observational verification of his prediction was made in 1919 during the total solar eclipse [7].

After Einstein, many authors calculated the light deflection angle up to the second or higher order terms of $\hbar$. Virbhadra and Ellis obtained the lens equation for the Schwarzschild black hole [8] and naked singularity [9], and calculated the light deflection angle. Keeton and Petters [10] calculated the higher order terms of the light deflection angle for a Schwarzschild mass. Iyer and Petters [11] calculated it for a strong field, and found that under weak field approximation their expression matches with that of Keeton and Petters [10].

The bending angle for the Kerr mass in the equatorial plane was calculated by Iyer and Hansen [12, 13] using the null geodesic of the photon. According to their results, the deflection produced in the presence of a rotating black hole explicitly depends on the direction of motion of the light. If the light ray is moving in the direction of spin, then the deflection angle is higher than the zero rotation Schwarzschild field, and if the light ray is moving in the opposite direction to the rotation, then the bending angle is smaller than the Schwarzschild field. Bozza [14] obtained the lensing formula, and calculated the relativistic image position for a light-ray trajectory close to the equatorial plane of a Kerr black hole. Aazami et al [15, 16] calculated the two components of the light bending angle, along the direction of the equatorial plane and perpendicular to the equatorial plane of a Kerr black hole in the quasi-equatorial regime. All of the above mentioned calculations were done using the null geodesic of the photon.

On the other hand, some authors have used the material medium approach, where the gravitational effect on the light ray was calculated by assuming some effective refractive index assigned to the medium through which light is propagating. With a similar approach, Atkinson [17] studied the trajectory of a light ray near a very massive, static and spherically symmetric star. Fischback and Freeman [18] calculated the second-order contribution to gravitational deflection by a static mass using the same method. Sen [19] used this method to calculate the gravitational deflection of light without any weak field approximation. A similar method was used by Balaz [20] to calculate the change in the direction of the polarization vector of an electromagnetic wave passing close to a rotating body.

All of the above calculations were done for the Schwarzschild and Kerr mass. Virbhadra et al [21] worked on the Janis–Newman–Winicour (JNW) mass, which is a charged, static mass, and calculated the light deflection angle up to second order. Eiroa et al [22] worked on the Reissner–Nordström (RN) mass and calculated the light deflection angle in both the strong and weak deflection limit. Hasse and Pelrick [23] worked on the lensing by the Kerr–Newman mass (charged, rotating) using Morse theory, and showed that an infinite number of images were formed by such a body. Kranotitis [24] derived the analytic solutions of the lens equations in terms of the Appell and Lauricella hypergeometric functions and the Weierstrass modular form for a Kerr–Newman mass. He used this formula to calculate the light deflection angle. However, he did not apply any limiting condition (zero spin, zero charge, both spin and charge zero) to the expression of the deflection angle [24, equation 87] to verify his result. In this paper, we calculate the exact expression for the light deflection angle by a Kerr–Newman mass in the equatorial plane, which is a function of mass, spin and charge. We use the null geodesic of the photon approach to calculate the light deflection angle, and verify our results through the above-mentioned limiting conditions.
2. General geodesic equations in Kerr–Newman space–time

In the general theory of relativity, the Kerr–Newman line element represents the most generalised form of the space–time curvature, where all three factors (mass distribution, rotation, charge) have their contribution.

The Kerr–Newman line element given by Newman et al. [3, 4],

\[
\begin{align*}
\text{ds}^2 = & \left(1 - \frac{2mr - Q^2}{\rho^2}\right)c^2 dr^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \\
& - \left(r^2 + a^2 + \frac{a^2 \sin^2 \theta (2mr - Q^2)}{\rho^2}\right) \sin^2 \theta d\phi^2 \\
& + \frac{2a(2mr - Q^2) \sin^2 \theta}{\rho^2} c dt d\phi
\end{align*}
\]

(1)

where, \(\Delta = r^2 - 2mr + a^2 + Q^2\), \(\rho^2 = r^2 + a^2 \cos^2 \theta\), \(m = \frac{GM}{c^2}\), \(a = \frac{J}{cM}\) and \(Q^2 = \frac{Ge^2}{4\pi \epsilon_0 c^4}\), further \(c\), \(G\), \(M\), \(J\) and \(e\) are the velocity of light in free space, the gravitational constant, mass, angular momentum of the gravitating body and the static charge. Here, \(\frac{1}{4\pi \epsilon_0}\) is Coulomb’s force constant, with \(\epsilon_0\) the free-space permittivity. Both \(m\) and \(Q\) have the dimension of length. If we set \(a = 0\), then equation (1) will reduce to the Reissner–Nordström solution in curved space–time. If we set \(Q = 0\), then equation (1) will reduce to the Kerr solution, and if we set \(a = \frac{Q}{m}\), then the equation will reduce to the Schwarzschild solution in curved space–time.

The two horizons are located at the roots of \(r^2 - 2mr + a^2 + Q^2 = 0\), i.e.

\[
r_{\pm} = m \pm \sqrt{m^2 - a^2 - Q^2}
\]

(2)

Let us now consider that the light ray is moving in Kerr–Newman space–time. Carter [25] showed that the Lagrangian \((\mathcal{L})\) and the Hamiltonian (H) of the particle will be

\[
\mathcal{L} = \frac{1}{2} g_{ij} \frac{dx^i}{dr} \frac{dx^j}{dr}
\]

(3)
[25, equation 30] and

\[
H = \frac{1}{2} g^{ij} p_i p_j
\]

(4)
[25, equation 35].

In the above equation, \(p_i = g_{ij} \frac{dx^j}{dr}\) is the momenta obtained from the Lagrangian \((\mathcal{L})\) [25, equation 34], \(\tau\) is the affine parameter and \(A\) is the vector potential [25].

From the Hamiltonian (H), the generalised form of the Hamilton–Jacobi equation is

\[
\frac{dS}{d\tau} = \frac{1}{2} g^{ij} \frac{dx}{d\tau} \frac{dx}{d\tau}
\]

(5)

where \(S\) is the Jacobi action [25, equation 46]. The solution of the above equation takes the form [25, equation 47]
\[
S = -E \epsilon t + L \phi + \int^r \frac{\sqrt{R(r)}}{\Delta} \, dr + \int^\theta \frac{\sqrt{\Theta(\theta)}}{d\theta} \, d\theta \tag{6}
\]

where,
\[
R(r) = \left[ E \left( r^2 + a^2 \right) - aL \right]^2 - \Delta \left[ K + (aE - L)^2 \right] \tag{7}
\]
and
\[
\Theta(\theta) = K - \cos^2 \theta \left[ -a^2 E^2 + \frac{L^2}{\sin^2 \theta} \right] \tag{8}
\]

[25, equations (53)–(55)].

Here, \( E, L, K \) are the constants of motion. \( E \) and \( L \) represent the energy and angular momentum of the particle along the direction of the black hole spin axis and \( K \) is the Carter constant. For \( K = 0 \), particle motion remains entirely in the equatorial plane.

Carter [25, equations (62)–(65)] and Heisnang et al [26, equations (17)–(20)] showed that it is possible to obtain the four geodesic equations governing the motion of the particle in Kerr–Newman space–time using the above action \( S \). The null geodesic equations for the light ray are,

\[
\rho^2 \frac{d\tau}{\partial r} = \sqrt{R(r)} \tag{9}
\]

\[
\rho^2 \frac{d\theta}{\partial \tau} = \sqrt{\Theta(\theta)} \tag{10}
\]

\[
\rho^2 \frac{d\phi}{\partial \tau} = -\left[ aE \sin^2 \theta - L \right] + \frac{a}{\Delta} \left[ E \left( r^2 + a^2 \right) - aL \right] \tag{11}
\]

\[
\rho^2 \frac{d(ct)}{\partial \tau} = -a \left( aE \sin^2 \theta - L \right) + \frac{(r^2 + a^2)}{\Delta} \left[ E \left( r^2 + a^2 \right) - aL \right]. \tag{12}
\]

### 3. Equatorial geodesic equations

We want to calculate the equatorial light deflection angle, and for that we need equatorial geodesic equations. The conditions for the equatorial plane are \( \theta = \frac{\pi}{2} \) and \( K = 0 \). In this section, we aim to calculate the geodesic equations of a light ray whose trajectory remains at the equatorial plane throughout the motion. We will obtain modified geodesic equations with the above conditions.

From equation (9) we have,

\[
r^4 \dot{r}^2 = \left[ E \left( r^2 + a^2 \right) - aL \right]^2 - \Delta (aE - L)^2 \tag{13}
\]

here \( (\cdot) \) means the derivative with respect to the affine parameter \( \tau \). Putting the value \( \Delta = r^2 - 2mr + a^2 + Q^2 \), we have,

\[
r^4 \dot{r}^2 = \left[ E \left( r^2 + a^2 \right) - aL \right]^2 - \left( r^2 - 2mr + a^2 + Q^2 \right)(aE - L)^2. \tag{14}
\]
With some simplification, we have

\[ \dot{r}^2 = E^2 + \frac{1}{r^4} (aE - L)^2 (2mr - Q^2) - \frac{1}{r^2} (L^2 - a^2 E^2). \]  

(15)

It is possible to show that the impact parameter is the ratio of \( L \) and \( E \) [27, page 328]. Following [12, equation 7], we can write the impact parameter \( b_s = \frac{L}{E} \), where \( s = +1 \) for the prograde and \( s = -1 \) for the retrograde orbit of the light ray, and \( b \) is the positive magnitude of the impact parameter. However, \( b_s^2 = (sb)^2 = (\pm b)^2 = b^2 \), so we can drop \( s \) for the even power of the impact parameter. So the new form of the above equation is,

\[ \dot{r}^2 = L^2 \left[ \frac{1}{b^2} + \frac{1}{r^4} \left( \frac{a}{b_s} - 1 \right)^2 (2mr - Q^2) - \frac{1}{r^2} \left( 1 - \frac{a^2}{b^2} \right) \right] \]  

(16)

or,

\[ \dot{r} = L \left[ \frac{1}{b^2} + \frac{1}{r^4} \left( \frac{a}{b_s} - 1 \right)^2 (2mr - Q^2) - \frac{1}{r^2} \left( 1 - \frac{a^2}{b^2} \right) \right]^{1/2}. \]  

(17)

\( r \) obtains a local extremum for the closest approach \( r_0 \), and we can write:

\[ \dot{r} \bigg|_{r=r_0} = 0 \]

Thus, from equation (17),

\[ \frac{1}{b^2} + \frac{1}{r_0^2} \left( \frac{a}{b_s} - 1 \right)^2 (2mr_0 - Q^2) - \frac{1}{r_0^2} \left( 1 - \frac{a^2}{b^2} \right) = 0 \]  

(18)

or,

\[ \frac{r_0^2}{b^2} = \left( 1 - \frac{a^2}{b^2} \right) - \left( \frac{a}{b_s} - 1 \right)^2 \left( \frac{2m}{r_0} - \frac{Q^2}{r_0^2} \right) \]  

(19)

From equation (10), it is clear that,

\[ \dot{\theta} \bigg|_{\theta=\frac{\pi}{2}} = 0 \]  

(20)

From equation (11)

\[ r^2 \dot{\phi} = -(aE - L) + \frac{a}{\Delta} \left[ E \left( r^2 + a^2 \right) - aL \right]. \]  

(21)

With some simplification, we have,

\[ \dot{\phi} = \frac{L}{\Delta} \left[ \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) + \frac{1}{b_s} \left( \frac{2ma}{r} - \frac{aQ^2}{r^2} \right) \right]. \]  

(22)

From equation (12), we have,

\[ r^2 \left( c \dot{t} \right) = -a(aE - L) + \frac{r^2 + a^2}{\Delta} \left[ E \left( r^2 + a^2 \right) - aL \right]. \]  

(23)
Simplifying we get,

$$
(ct) = \frac{1}{\Delta} \left[ \left( r^2 + a^2 + \frac{2ma}{r} - \frac{Q^2a^2}{r^2} \right) E - \left( \frac{2ma}{r} - \frac{aQ^2}{r^2} \right) L \right],
$$

(24)

Here, equations (17), (22), (24) represent the equatorial geodesic equations at the Kerr–Newman space–time.

4. Radius of the photon sphere

Claudel et al [28] studied the photon sphere in Schwarzschild geometry and obtained the generalized definition of the photon surface for spherically symmetric space–time.

Vries [29] had studied photon orbits in Kerr–Newman space–time, and obtained the necessary and sufficient conditions for the existence and uniqueness of such photon orbits. According to Vries [29], circular photon orbits mark the limit of the innermost light trajectories coming from infinity. One of the very important orbits is the circular photon orbit at equatorial plane i.e \( \theta = \frac{\pi}{2} \) and \( K = 0 \). For the circular orbit, Dadhich and Kale [30] showed that,

$$
\frac{E}{\mu} = \frac{r^2 - 2mr + Q^2 \pm a\left( mr - Q^2 \right)^{\frac{1}{2}}}{r \left[ r^2 - 3mr + 2Q^2 \pm 2a\left( mr - Q^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}}
$$

and

$$
\frac{L}{\mu} = \frac{\sqrt{mr - Q^2} \left[ r^2 + a^2 \pm 2a\sqrt{\left( mr - Q^2 \right)} \right] \pm aQ^2}{r \left[ r^2 - 3mr + 2Q^2 \pm 2a\left( mr - Q^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}}
$$

where the upper sign is for direct orbit and the lower sign for retrograde orbit, with \( \mu \) the mass of the orbiting particle. The above two equations will be real if [30, equation (16)]

$$
r^2 - 3mr + 2Q^2 \pm 2a\left( mr - Q^2 \right)^{\frac{1}{2}} \geq 0
$$

the limiting case of equality corresponds to the orbit of infinite energy, i.e. the photon orbit [31]. The smallest root of the above equation \( r = r_{\text{ph}} \) represents the radius of the photon sphere [30]. In the above relation, if we set \( Q = 0 \), this relation is an exact match to the relation given in [31, equation (2.17)].

As a general procedure in the present work, we can also derive the expression for the radius of the equatorial circular photon orbit. For the circular orbit we must have \( R(r) = \frac{dR}{dr} = 0 \), where \( R \) is defined by equation (7). Following [27, page 329] and using equation (15), we can have equations (25) and (26) to determine the radius of the circular photon orbit.
\begin{equation}
E^2 + \frac{(aE - L)}{r_{ph}} \left( \frac{2m}{r_{ph}} - \frac{Q^2}{r_{ph}} \right) - \frac{L^2 - a^2E^2}{r_{ph}} = 0 \tag{25}
\end{equation}

and

\begin{equation}
\frac{d}{dr} \left[ E^2 + \frac{1}{r^4} (aE - L)^2 \left( 2mr - Q^2 \right) - \frac{1}{r^2} \left( L^2 - a^2E^2 \right) \right] \bigg|_{r=r_{ph}} = 0 \tag{26}
\end{equation}

or,

\[ r_{ph}^2 \frac{L + aE}{L - aE} = 3mr_{ph} + 2Q^2 = 0. \tag{27} \]

Solving the above equation for \( r_{ph} \), we have,

\begin{equation}
r_{ph} = \frac{3m \pm \sqrt{9m^2 - 8Q^2} \left( \frac{L + aE}{L - aE} \right)}{2} \tag{28}
\end{equation}

here \( a > 0 \) and \( a < 0 \) correspond to a direct and retrograde orbit. If we set \( a = 0 \), i.e. for the Reissner–Nordström space–time, then the above expression will reduce to

\begin{equation}
r_{ph} = \frac{3}{2}m \pm \frac{1}{2} \sqrt{9m^2 - 8Q^2} \tag{29}
\end{equation}

which is exactly the same as obtained by Vries [29, equation (38)].

5. Equatorial light deflection angle

The light deflection angle can be expressed as [32, page 188],\n
\begin{equation}
\alpha = 2 \int_n^\infty \left( \frac{d\varphi}{dr} \right) \, dr - \pi \tag{30}
\end{equation}

Now, using equations (17) and (22),

\begin{equation}
\frac{d\varphi}{dr} = \frac{\left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) + \frac{1}{b_2} \left( \frac{2ma}{r} - \frac{aQ^2}{r^2} \right)}{\Delta \sqrt{\frac{1}{b_1^2} + \frac{1}{r^4} \left( \frac{a}{b_1} - 1 \right)^2 \left( 2mr - Q^2 \right) - \frac{1}{r^2} \left( 1 - \frac{a^2}{b_1^2} \right)}} \tag{31}
\end{equation}

Using equation (31) in equation (30), we will get

\begin{equation}
\alpha = 2 \int_n^\infty \frac{\left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) + \frac{1}{b_2} \left( \frac{2ma}{r} - \frac{aQ^2}{r^2} \right)}{\Delta \sqrt{\frac{1}{b_1^2} + \frac{1}{r^4} \left( \frac{a}{b_1} - 1 \right)^2 \left( 2mr - Q^2 \right) - \frac{1}{r^2} \left( 1 - \frac{a^2}{b_1^2} \right)}} \, dr - \pi \tag{32}
\end{equation}

After some algebraic calculations (shown in appendix A), we can have the following expression from equation (32)
\[
\alpha = 2 \int_0^1 \frac{dx}{\sqrt{G} \sqrt{1 - x^2}} f f_1^{-1} (1 - 2 \delta_2)^{-\frac{1}{2}} \left( 1 + \delta_1 \left[ 1 - 2 \delta_2 \right]^{-1} \right)^{-\frac{1}{2}} - \pi
\]

where,
\[
\begin{align*}
  f_1 &= 1 - 2Fhx + Fn^2x^2 \\
  f_2 &= 1 - 2hx + x^2n^2 + \hat{a}^2h^2x^2 \\
  \delta_1 &= \frac{F^2n^2 (1 + x^2)}{G} \\
  \delta_2 &= \frac{F^2h (1 - x^3)}{G (1 - x^2)}
\end{align*}
\]

and \( h = \frac{m}{n_0} \) and \( n^2 = \frac{Q^2}{r_0^2} \), \( \hat{a} = \frac{a}{m} \) and \( x = \frac{r_0}{r} \). And following [15, 16], we substituted
\[
G = 1 - \left( \frac{a}{b} \right)^2 = 1 - \hat{a}^2 \left( \frac{n}{b} \right)^2 \text{ and } F = 1 - \left( \frac{a}{b} \right)^2 = 1 - s \hat{a} \frac{m}{b}.
\]

The above equation (33) is the general expression for the deflection of light due to the rotating charged sphere in the equatorial plane.

For the weak deflection limit, following [15, 16] one can assume, \( Q, m \ll n_0 \), in other words, \( h, n \ll 1 \). So the above equation can be expanded in the Taylor series in terms of both \( h \) and \( n \) by following a procedure as detailed in appendix B, and finally one can write:
\[
\alpha = c_0 \pi + 4h \left( c_1 + \frac{7}{2} - \frac{3\pi}{8} \right) \frac{F^4n^2}{G^2} - \left( 1 - F \right) \frac{4n^2}{3} + \left( \frac{13}{6} - \frac{\pi}{4} \right) \frac{F^2n^2}{G^2}
\]
\[
+ h^2 \left( -4c_2 + \frac{15\pi}{4} - 16 \right) - \frac{825\pi}{32} - 50 \left( \frac{F^6n^2}{G^2} \right) - \left( 1 - F \right) \frac{3\pi}{8} \left( 8 - \hat{a}^2 \right)
\]
\[
+ \left( \frac{15\pi}{2} - 16 \right) \frac{F^2}{G^2} + \frac{F^4}{G^2} \left( \frac{105\pi}{16} - 16 \right)
\]
\[
+ \left( \frac{81\pi}{8} - 18 \right) \frac{F^4}{G^2} + \frac{n^2s_0}{8\sqrt{G}} \left( \frac{7F^2}{2G} + 3 \right)
\]
\[
+ h^3 \left( \frac{122}{3}b_3 - \frac{15\pi}{2} - d_3 \right) + h^3 \left( -130c_4 + \frac{3465\pi}{64}d_4 \right)
\]
\[
+ \frac{n^2\pi}{8\sqrt{G}} \left( \frac{7F^2}{2G} + \frac{57F^4}{8G^2} \right)
\]

(34)

Where (following [15, 16]),
\[
c_0 = \frac{1}{\sqrt{G}} - 1 \quad (35a)
\]
\[ c_1 = \frac{F^2 + G - FG}{G^3} \]  
\[ c_2 = \frac{F^2}{G}c_1 \]  
\[ d_2 = \frac{1}{15G^5} \left( 15F^4 - 4G[F - 1] \left[ 3F^2 + 2G \right] - 2G^2\dot{a}^2 \right) \]  
\[ c_3 = \frac{1}{61G^7} \left( 61F^6 - G[F - 1] \left[ 45F^4 + 32F^2G + 16G^2 \right] 
- 4G^2\dot{a}^2 \left[ 2F^2 + 2G - FG \right] \right) \]  
\[ d_3 = \frac{F^2}{G}d_2 \]  
\[ c_4 = \frac{F^2}{65G^9} \left( 65F^6 - 49[F - 1]F^4G + 8F^2G^2s_0 + 2s_1G^3 \right) \]  
\[ d_4 = \frac{1}{1155G^9} \left( 1155F^8 - 840[F - 1]F^6G + 140F^4s_0G^2 
+ 40s_1F^2G^3 + 8s_2G^4 \right) \]

and

\[ s_0 = -\dot{a}^2 + 4 - 4F \]  
\[ s_1 = -4\dot{a}^2 + 8 + 2F\dot{a}^2 - 8F \]  
\[ s_2 = \dot{a}^4 - 12\dot{a}^2 + 16 + 8\dot{a}^2F - 16F \]

The above equation (34) represents the equatorial deflection of light due to the rotating charged sphere in the weak field limit. As mentioned earlier in text after equation (33), the assumption used for derivation here is that both \( Q \) and \( m \ll \rho_0 \). Thus this solution can also be applied for the Kerr metric by setting \( Q = 0 \). It is also implicit that, \( a^2 + Q^2 < m^2 \) as stated in equation (2). This expression is a function of mass, rotation and charge. We verify our result with some limiting conditions in the following section of the paper.

If we set charge equal to zero in equation (34), we will get the expression of deflection for light by the Kerr mass obtained by Aazami \textit{et al} [16, equation (B17)].

If we set rotation equal to zero with non-zero charge in equation (34), we get,

\[ a = 4h + \left( -4 + \frac{15\pi}{4} \right)h^2 + \left( \frac{122}{3} - \frac{15\pi}{2} \right)h^3 + \left( -130 + \frac{3465\pi}{64} \right)h^4 - \frac{3\pi}{4}n^2 + \frac{57\pi}{64}n^4 
- \left( 14 - \frac{3\pi}{2} \right)n^2h - \left( -50 + \frac{825\pi}{32} \right)n^2h^2 \]  
\[ \text{(37)} \]
This is the expression for deflection by a charged, non-rotating body. This expression, up to the second order contribution of charge, i.e. \( n^2 \), was obtained by Eiroa et al. [22, equation (55)].

If we set both charge and rotation equal to zero in equation (34), we get,

\[
\alpha = 4h + \left( -4 + \frac{15\pi}{4} \right) h^3 + \left( \frac{122}{3} - \frac{15\pi}{2} \right) h^3 + \left( -130 + \frac{3465\pi}{64} \right) h^4.
\]

(38)

This is the well known expression of the light deflection angle due to the Schwarzschild mass given by Keeton and Petters [10, equation (23)].

If we set mass and rotation equal to zero, i.e. \( h = 0 \) and \( F = G = 1 \) in equation (34), we will get,

\[
\alpha = n^2 \left[ -\frac{3\pi}{4} + n^4 \left( \frac{57\pi}{64} \right) \right].
\]

(39)

This is the deflection amount of the light ray occurring only due to charge. A hypothetical massless, static body can influence the space–time curvature.

6. Discussion of results

We have the expression of the event horizon from equation (2), which is

\[ r = m \pm \sqrt{m^2 - a^2 - Q^2}. \]

The above expression represents two event horizons, and for \( Q^2 + a^2 > m^2 \), naked singularity occurs. This clearly gives a limit to the value of \( (Q^2 + a^2) \), i.e. it must be less than \( m^2 \). We choose the values of \( a \) and \( Q \) in such a way that it holds the condition,

\[ m^2 \geq Q^2 + a^2. \]

However, \( m^2 = Q^2 + a^2 \) is a very special case, as two event horizons coincide with each other.

In order to understand the physical significance of the calculations done in this paper, we plot bending angle \( (\alpha) \) against various physical parameters \( (\hat{a}, n, b) \) in figures 1–3, respectively, by taking the Sun as a test case. We consider that the Sun has some charge \( (n = 8.483105679 \times 10^7) \), and the closest approach is the radius of the Sun \((6.955 \times 10^8m)\). The value of \( n = Q/n_0 \) is chosen in such a way that \( (Q^2 + a^2) \) is always less than \( m^2 \).

Figure 1 clearly shows the difference between prograde and retrograde motion with respect to the zero rotation (RN) case. The pattern of the graph is similar to the result given by Iyer and Hansen [12] for Kerr equatorial bending.

From figure 2, we can say that the deflection of the light ray reduces with the increase in the charge of the body. When charge is zero, the light ray gets the maximum deflection.

Figure 3 shows the change in bending angle with impact parameter. Though the pattern is similar, as given by Iyer and Hansen [12], the prograde, retrograde and zero spin (RN) plot overlap with each other as the difference between them is small for the Sun. So we consider pulsar PSR J 1748-2446 [33] as a test case and plot the bending angle versus impact parameter graph in figure 4. We consider that the pulsar has some charge \( (n = 8.483105679 \times 10^{-7}) \) and the closest approach is the radius of the pulsar, i.e. 20 km. The pattern of the graph is similar, as given by Iyer and Hansen [12]. For this pulsar we take \( m = 1.99 \) km and calculate \( a \) as 0.96 km from the input values of the time period \( T = 1.393 \) ms and \( n_0 = 20 \) km, as listed by Nuñez and Nowakowski [33]. We note that under
Figure 1. The bending angle (arcsec) as a function of rotation parameter ($a/m$) with constant charge $n = 8.483105679 \times 10^{-7}$ and impact parameter one solar radius. Here, three different curves represent the prograde (when the light ray moves in the direction of the rotation of the body), corresponding RN (zero rotation) and retrograde (when the light ray moves in the opposite direction of the rotation of the body) motion of the light ray.

Figure 2. The bending angle (arcsec) as a function of charge $n = Q/r_0$ with constant rotation parameter $\tilde{a} = 0.5$ and impact parameter one solar radius. Geometries are explained in the caption for figure 1.
Figure 3. The bending angle (arcsec) as a function of impact parameter in the unit of solar radius with constant rotation ($\dot{a} = 0.5$) and charge $n = 1.413850947 \times 10^6$. The geometries are explained in the caption for figure 1. All the plots for prograde, RN and retrograde merge together for the Sun.

Figure 4. The bending angle (arcsec) as a function of impact parameter $b/r_g$ with constant rotation ($\dot{a} = 0.5$) and charge $n = 8.483105679 \times 10^7$. The geometries are explained in the caption for figure 1.
this condition $\sqrt{m^2 - a^2} = 1.73$ km and we remain away from naked singularity. Further, the weak field condition approximation also remains just valid as $\frac{m}{a} \sim 2$. However, we choose the value of $Q^2$ carefully such that $m^2 > a^2 + Q^2$.

7. Conclusions

From the above study, we may conclude the following.

1. The expression for the equatorial deflection of light due to a charged, rotating body has been calculated considering contributions from mass and charge up to fourth order terms.

2. The charge has a noticeable effect on the path of the light ray. When compared with the Kerr expression for bending, we find that there are some extra terms in the expression for deflection which occur due to the presence of charge. If we set the charge equal to zero, the deflection angle will reduce to that of the Kerr deflection angle. If we set both charge and rotation equal to zero, the deflection angle will reduce to that of the Schwarzschild deflection angle.

3. As we know, with the Kerr–Newman metric, even with a hypothetical body of zero mass and non-zero charge, we can introduce curvature in space–time geometry—this has only been reconfirmed up until the final step of our calculation.

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Appendix A

We rewrite equation (32) as follows,

$$\alpha = 2 \int_{r_0}^{\infty} \frac{\left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) + \frac{1}{b_s} \left( \frac{2ma}{r} - \frac{aQ^2}{r^2} \right)}{\Delta \left( \frac{1}{b^2} + \frac{1}{b^4} \left( \frac{a}{b_s} - 1 \right)^2 \left( 2mr - Q^2 \right) - \frac{1}{r^2} \left( 1 - \frac{a^2}{b^2} \right) \right)} \, dr - \pi \quad (A.1)$$

Let us introduce a new variable $x = \frac{r_0}{r}$. So,

$$dx = -\frac{r_0 \, dr}{r^2}$$
or,
\[ \frac{dx}{n_0} = -\frac{dr}{r^2} \]
the limits will change as and when \( r \to \infty \), then \( x \to 0 \) and when \( r \to n_0 \), then \( x \to l \).
Using this in the above equation, we have,
\[ \alpha = 2 \int_0^1 \frac{f_1}{f_2 \sqrt{f_3}} \, dx - \pi \quad (A.2) \]
Where,
\[ f_1 = \left( 1 - \frac{2m}{n_0} + \frac{Q^2x^2}{n_0^2} \right) + \frac{1}{b_y} \left( \frac{2ma}{n_0} - \frac{aQ^2x^2}{n_0^2} \right) \]
\[ f_2 = 1 - \frac{2m}{n_0} + \frac{Q^2x^2}{n_0^2} + \frac{a^2x^2}{n_0^2} \]
and
\[ f_3 = r_0^2 \left( \frac{1}{b^2} + x^2 \left( \frac{a}{b} \right) - 1 \right)^2 \left( \frac{2mr - Q^2}{n_0^2} \right) - \frac{x^2}{n_0^2} \left( 1 - \frac{a^2}{b^2} \right) \]
Let us substitute \( h = \frac{m}{n_0} \) and \( n^2 = \frac{Q^2}{n_0^2} \) and \( \hat{a} = \frac{a}{m} \). So the new forms of \( f_1 \), \( f_2 \) and \( f_3 \) are,
\[ f_1 = \left( 1 - 2hx + x^2n^2 \right) + \frac{1}{b_y} \left( 2hax - anx^2 \right) \]
\[ f_2 = 1 - 2hx + x^2n^2 + \hat{a}^2h^2x^2 \]
and
\[ f_3 = r_0^2 \left( \frac{1}{b^2} + x^2 \left( \frac{a}{b} \right) - 1 \right)^2 \left( 2hx - x^2n^2 \right) - x^2 \left( 1 - \frac{a^2}{b^2} \right) \]
Let us put the expression of \( \frac{r_0^2}{b^2} = \left( 1 - \frac{a^2}{b^2} \right) - \left( \frac{a}{b} \right) \left( 2h - n^2 \right) \) from equation (19) in the expression of \( f_3 \), and rearranging we get,
\[ f_3 = \left( 1 - \frac{a^2}{b^2} \right) \left( 1 - x^2 \right) - \left( \frac{a}{b} \right)^2 \left( 2h \left( 1 - x^3 \right) + n^2 \left( 1 - \frac{a^2}{b^2} \right) \left( 1 - x^4 \right) \right. \]
Now following [15, 16], we have substituted \( G = 1 - \left( \frac{a}{b} \right)^2 = 1 - \hat{a}^2 \left( \frac{m}{b} \right)^2 \) and \( F = 1 - \left( \frac{a}{b} \right) = 1 - s\hat{a} \frac{m}{b} \). Thus for zero rotation (\( \hat{a} = 0 \)), \( F = G = 1 \). Now using \( F \) and \( G \) we can write,
\[ f_1 = 1 - 2Fh + Fn^2x^2 \]
\[ f_2 = 1 - 2hx + x^2n^2 + \delta h^2x^2 \]

and
\[ f_3 = G\left(1 - x^2\right) - 2F^2h\left(1 - x^3\right) + F^2n^2\left(1 - x^4\right) \]
or,
\[ f_3 = G\left(1 - x^2\right)\left(1 - \frac{2F^2h\left(1 - x^3\right)}{G\left(1 - x^2\right)}\right)\left(1 + \frac{F^2n^2\left[1 + x^2\right]}{G\left[1 - x^2\right]}\left[1 - \frac{2F^2h\left(1 - x^3\right)}{G\left(1 - x^2\right)}\right]^{-1}\right) \]

Let us substitute \( \frac{F^2h\left(1 - x^3\right)}{G\left(1 - x^2\right)} = \delta_2 \) and \( \frac{F^2n^2\left(1 + x^2\right)}{G} = \delta_1 \). So the new form of \( f_3 \) using \( \delta_2 \) and \( \delta_1 \) is,
\[ f_3 = G\left(1 - x^2\right)\left(1 - 2\delta_2\right)\left(1 + \delta_1\left[1 - 2\delta_2\right]^{-1}\right) \]

Putting these values of \( f_1, f_2 \) and \( f_3 \) in equation (A.2), we get,
\[
\alpha = 2 \int_0^1 \frac{f_1 f_2^{-1}}{\sqrt{G\sqrt{1 - x^2}}\sqrt{1 - 2\delta_2} \sqrt{1 + \delta_1\left[1 - 2\delta_2\right]^{-1}}} \, dx - \pi \quad \text{(A.3)}
\]

Now, rearranging the above equation we can write
\[
\alpha = 2 \int_0^1 \frac{dx}{\sqrt{G\sqrt{1 - x^2}}} f_2^{-1}\left(1 - 2\delta_2\right)^{-\frac{1}{2}}\left(1 + \delta_1\left[1 - 2\delta_2\right]^{-1}\right)^{-\frac{1}{2}} - \pi. \quad \text{(A.4)}
\]

**Appendix B**

For the weak deflection limit, following [15, 16] one can assume, \( Q, m \ll \eta_0 \), in other words, \( h, n \ll 1 \). So equation (33) can be expanded in the Taylor series in terms of both \( h \) and \( n \). Here we calculate the deflection angle, considering contribution up to fourth order terms in mass (represented by \( h \)) and charge (represented by \( n \)) only, and write
\[
\alpha = 2 \int_0^1 \frac{dx}{\sqrt{G\sqrt{1 - x^2}}} f_1 \left(1 + 2hx + x^2h^2\left[4 - \delta^2\right]\right.
+ x^3h^3\left[8 - 4\delta^2\right] + x^4h^4\left[\delta^4 - 12\delta^2 + 16\right]
\left(1 - x^2n^2\left[1 + 2hx + x^2h^2\left[4 - \delta^2\right]\right]\right. + x^4n^4\right)
\times \left(1 + \delta_2 + \frac{3}{2}\delta_2^2 + \frac{5}{2}\delta_2^3 + \frac{35}{8}\delta_2^4\right)
\left(1 - \delta_1\left[1 + 2\delta_2 + 4\delta_2^2\right]\right. + \frac{3}{8}\delta_2^3\left) - \pi. \right.
\]
Multiplying term-by-term and retaining only up to fourth order in both $h$ and $n$ we get,

\[
\alpha = 2 \int_{0}^{1} \frac{dx}{\sqrt{1 - x^2}} \left( 1 + h \left[ 2x \{1 - F\} \left\{ 1 - 2n^2 x^2 \right\} \right] + \frac{\delta_2}{h} - \frac{\delta_2}{h} \delta x [1 - F] - \frac{3 \delta_1 \delta_2}{2} \right) + h^2 \left[ -n^2 x^2 \{1 - F\} \delta_2 \right] + \frac{3 \delta_2^2}{2} - \frac{3}{2} x^2 n^2 \{1 - F\} \frac{\delta_2 G}{h^2} \\
- \frac{x^2 s_0 \delta_1}{2} - \frac{3 x \{1 - F\} \delta_1 \delta_2}{4} - \frac{15 \delta_1 \delta_2^2}{h^2} \\
+ h^3 \left[ \frac{x^3 s_1}{h} + \frac{x^2 s_0 \delta_2}{h} + \frac{3 x \{1 - F\} \delta_2^2}{h^2} + \frac{5 \delta_2^3}{2h^3} \right] \\
+ h^4 \left[ \frac{x^4 s_2}{h} + \frac{x^3 s_0 \delta_2}{h} + \frac{3 x^2 s_0 \delta_2^3}{2h^2} + 5 x \{1 - F\} \frac{\delta_2^3}{h^3} + 35 \frac{\delta_2^4}{8 h^4} \right] \\
+ n^2 \left[ -x^2 \{1 - F\} - \frac{\delta_1}{2n^2} \right] + n^4 \left[ x^4 \{1 - F\} + \frac{x^2 \delta_1}{2n^2} \{1 - F\} + \frac{3 \delta_2^2}{8n^4} \right] - \pi \]

where, $s_0$, $s_1$ and $s_2$ have been substituted from equations (36a), (35b), (35c). Now integrating the above equation term-by-term we get,

\[
\alpha = c_0 \pi + 4h \left[ c_1 - \left( \frac{7}{2} - \frac{3\pi}{8} \right) \frac{F^4 n^2}{G^2} - \frac{[1 - F]}{\sqrt{G}} \left( \frac{4n^2}{3} + \frac{13}{6} - \pi \right) \frac{F^2 n^2}{G} \right] \\
+ h^2 \left( -4c_2 + \frac{15\pi}{4} d_2 - \left( \frac{825\pi}{32} - 50 \right) \frac{F^6 n^2}{G^2} - \frac{[1 - F]}{\sqrt{G}} \left( \frac{3\pi}{8} \right) \{8 - a^2\} \right) \\
+ \left\{ \frac{15\pi}{2} - 16 \right\} \frac{F^2}{G} + \frac{F^4}{G^2} \left( \frac{105\pi}{16} - 16 \right) \\
+ \left\{ \frac{81\pi}{8} - 18 \right\} \frac{F^4}{G^2} - \frac{n^2 s_0}{8 \sqrt{G}} \left[ 7F^2 \frac{2G}{3} \right] \\
+ h^3 \left( \frac{122}{3} c_3 - \frac{15\pi}{2} d_3 \right) + h^4 \left( -130c_4 + \frac{3465\pi}{64} d_4 \right) + \frac{n^2 s_0}{2 \sqrt{G}} \left[ -1 - F \right] \frac{3F^2}{2G} \\
+ \frac{n^2 \pi^2}{8 \sqrt{G}} \left( 3 \{1 - F\} + \frac{7 F^2 \{1 - F\}}{2G} + \frac{57F^4}{8G^2} \right) \right)
\]

(B.1)

where $c_0$, $c_1$, $c_2$, $c_3$, $c_4$, $d_2$, $d_3$ and $d_4$ have been substituted from equations (35a) to (35h). The above equation represents the light deflection due to charged, rotating mass up to the fourth order term of both $h$ and $n$. 


References

[1] Schwarzschild K 1916 *Berliner Sitzungsbesichte (Phys. Math Klasse)* Proceedings of the Prussian Royal Academy of Sciences 7 189–96
[2] Kerr R P 1963 *Phys. Rev. Lett.* 11 237
[3] Newman E T and Janis A I 1965 *J. Math. Phys.* 6 915
[4] Newman E T, Couch E, Chinnapared R, Exton A, Prakash A and Torrence R 1965 *J. Math. Phys.* 6 918
[5] Reissner H 1916 *Ann. d. Physik* 50 106–20
[6] Nordström G 1918 *Proc. Kon. Ned. Akad. Wet* 20 1238–45
[7] Dyson F W, Eddington A S and Davidson C 1920 *Phil. Trans. R. Soc. Lond. A* 220 291–333
[8] Virbhadra K S and Ellis G F R 2000 *Phys. Rev.* D 62 084003
[9] Virbhadra K S and Ellis G F R 2002 *Phys. Rev.* D 65 103004
[10] Keeton C R and Petters A O 2005 *Phys. Rev.* D 72 104006
[11] Iyer S V and Petters A O 2007 *Gen. Relativ. Gravit.* 39 1563–82
[12] Iyer S V and Hansen E C 2009 *Phys. Rev.* D 80 124023
[13] Iyer S V and Hansen E C 2009 arXiv:gr-qc/0908.0085
[14] Bozza V 2003 *Phys. Rev.* D 67 103006
[15] Aazami A B, Keeton C R and Petters A O 2011 *J. Math. Phys.* 52 092502
[16] Aazami A B, Keeton C R and Petters A O 2011 *J. Math. Phys.* 52 102501
[17] Atkinson R D E 1965 *Astron. J.* 70 8
[18] Fishback E and Freeman B S 1982 *Phys. Rev.* D 22 12
[19] Sen A K 2010 *Astrophysika* 53 4
[20] Balazs N L 1958 *Phys. Rev.* 110 1
[21] Virbhadra K S, Narasimha D and Chitre S M 1998 *Astron. Astrophys.* 337 1–8
[22] Eiroa E F, Romero G E and Torres D F 2002 *Phys. Rev.* D 66 024010
[23] Hasse W and Perlick V 2006 *J. Math. Phys.* 47 042503
[24] Kraniotis G V 2014 *Gen. Relativ. Gravit.* 46 11
[25] Carter B 1968 *Phys. Rev.* 175 5
[26] Heisnam S, Meitei I A and Singh K Y 2014 *Int. J. Astron. Astrophys.* 4 365–73
[27] Chandrasekhar S 1983 *The Mathematical Theory of Blackholes* ed R J Elliot, J A Krumhansl and D H Wilkinson (New York: Oxford University Press)
[28] Claudela C M, Virbhadra K S and Ellis G F R 2001 *J. Math. Phys.* 42 2
[29] Vries A D 2000 *Class. Quantum Grav.* 17 123–44
[30] Dadhich N and Kale P P 1977 *J. Math. Phys.* 18 1727
[31] Bardeen J M, Press W H and Teukolsky S A 1972 *Astrophys. J.* 178 347–69
[32] Weinberg S 1972 *Gravitation and Cosmology: Principle and Application of General Theory of Relativity* (New York, London, Sydney, Toronto: J. Wiley and sons Inc.)
[33] Nuñez P D and Nowakowski M 2010 *J. Astrophys. Astr.* 31 105–19