Nonlinear Spinor field in isotropic space-time and dark energy models

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Within the scope of isotropic FRW cosmological model the role of nonlinear spinor field in the evolution of the Universe is studied. It is found that unlike in anisotropic cosmological models in the present case the spinor field does not possess nontrivial non-diagonal components of energy-momentum tensor, consequently does not impose any additional restrictions on the components of the spinor field or metric function. The spinor description of different matter was given and evolution of the Universe corresponding to these sources is illustrated. In the framework of a three fluid system the utility of spinor description of matter is established.

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I. INTRODUCTION

Nonlinear self-couplings of the spinor fields may arise as a consequence of the geometrical structure of the space-time and, more precisely, because of the existence of torsion. As early as 1938, Iwanenko [1] showed that a relativistic theory imposes in some cases a fourth-order self-coupling. This theory was further developed in [2–4]. The influence of nonlinear (fourth-order) terms in the Lagrangian of some classical relativistic field theories was investigated in [5]. In case of spinor field, stable localized configurations with a lowest energy state are shown to exist always for positive values of the coupling constant. As the self-action is of spin-spin type, it allows the assignment of a dynamical role to the spin and offers a clue about the origin of the nonlinearities. This question was further clarified in some important papers by Utiyama, Kibble, and Sciama [6–8]. Particle-like solutions of classical spinor field equations were obtained in [9–11]. Stability of optical gap solitons, i.e. localized solutions of spinor-like system, is analyzed in [12]. A nonlinear spinor field, suggested by the symmetric coupling between nucleons, muons, and leptons, was investigated in [9] in the classical approximation. A classical spinor field defined by a variational principle on a Lagrangian with quadratic Dirac and quartic Fermi terms was investigated in [10]. In the simplest scheme, the self-action is of pseudovector type, but it can be shown that one can also get a scalar coupling [13]. An excellent review of the problem may be found in [14].

Nonlinear quantum Dirac fields were used by Heisenberg [11, 15] in his ambitious unified theory of elementary particles. They are presently the object of renewed interest since the widely known paper by Gross and Neveu [16] where the two-dimensional massless fermion field theories with quartic interaction were studied. Nonlinear spinor field within the scope of static plane-symmetric model of gravitational field was studied in [17–20]. Recently a variational method for studying the evolution of solitary wave solution of nonlinear Dirac equation was developed in [21].

But thanks to its ability to describe different stages of evolution [22–36] as well as simulate different characteristic of matter from perfect fluid to phantom matter nonlinear spinor field is now extensively exploited in cosmology [37–41].
But some recent studies showed that the presence of non-trivial non-diagonal components of the energy-momentum tensor of the spinor field plays very crucial role in the evolution of both spinor field and the metric functions [42–52]. Unlike in anisotropic cosmological models the non-diagonal components of the energy-momentum tensor of the spinor field in the isotropic FRW space-time are trivial. Moreover, the FRW model gives a surprisingly accurate picture of the present day Universe. Hence in this paper we study the role of nonlinear spinor field in the evolution of an isotropic and homogeneous FRW Universe. Cosmological models with nonlinear spinor field within scope of FRW space-time was studied in [46, 47, 53]. The purpose of this paper is to study the role of spinor field nonlinearity in the evolution of the isotropic space-time. Beside this we give spinor descriptions of fluid and dark energy and show why this method is convenient to exploit to study the evolution of the Universe.

II. BASIC EQUATION

Let us consider the case when the isotropic and homogeneous space-time is filled with nonlinear spinor field. The corresponding action can be given by

\[ S(g; \psi, \bar{\psi}) = \int L\sqrt{-g}d\Omega \]  

(2.1)

with

\[ L = L_g + L_{sp} \]  

(2.2)

Here \( L_g \) corresponds to the gravitational field

\[ L_g = \frac{R}{2\kappa} \]  

(2.3)

where \( R \) is the scalar curvature, \( \kappa = 8\pi G \) is the Einstein’s gravitational constant and \( L_{sp} \) is the spinor field Lagrangian.

Let us consider the isotropic FRW space-time given by

\[ ds^2 = dt^2 - a^2 [dx^2 + dy^2 + dz^2] \]  

(2.4)

with \( a \) being the functions of time only.

The spinor field Lagrangian is given by

\[ L_{sp} = \frac{i}{2} [\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi] - m_{sp} \bar{\psi} \psi - F. \]  

(2.5)

We choose the nonlinear term \( F \) to be the function of \( K \) only, i.e., \( F = F(K) \), with \( K \) taking one of the following expressions \{I, J, I+J, I-J\}, where \( I \) is the scalar bilinear invariant: \( I = S^2 = (\bar{\psi}\psi)^2 \) and \( J \) is the pseudoscalar bilinear invariant: \( J = P^2 = (i\bar{\psi}\gamma^5 \psi)^2 \). In (2.5) \( \nabla_\mu \) is the covariant derivative of spinor field:

\[ \nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \quad \nabla_\mu \bar{\psi} = \frac{\partial \bar{\psi}}{\partial x^\mu} + \bar{\psi} \Gamma_\mu, \]  

(2.6)

with \( \Gamma_\mu \) being the spinor affine connection.

Variation of (2.5) with respect to \( \bar{\psi} \) and \( \psi \) yields spinor field equations

\[ i\gamma^\mu \nabla_\mu \psi - m_{sp} \bar{\psi} - \mathcal{D} \psi - i\mathcal{D} \gamma^5 \psi = 0, \]  

(2.7a)

\[ i\nabla_\mu \bar{\psi} \gamma^\mu + m_{sp} \bar{\psi} + \mathcal{D} \bar{\psi} + i\mathcal{D} \bar{\psi} \gamma^5 = 0, \]  

(2.7b)
where we denote $\mathcal{D} = 2SF_K K_I$ and $\mathcal{D} = 2PF_K K_J$, with $F_K = dF / dK$, $K_I = dK / dI$ and $K_J = dK / dJ$. In view of (2.7) the spinor field Lagrangian can be rewritten as

$$L_{sp} = \frac{1}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - m_{sp} \bar{\psi} \psi - F(I, J)$$

$$= \frac{1}{2} \bar{\psi} [\gamma^\mu \nabla_\mu \psi - m_{sp} \psi] - \frac{1}{2} \nabla_\mu [\bar{\psi} \gamma^\mu + m_{sp} \bar{\psi}] \psi - F(I, J),$$

$$= 2F_K (IK_I + JK_J) - F = 2KF_K - F(K). \quad (2.8)$$

Choosing the tetrad for the metric (2.4) in the following way

$$e^{(0)}_0 = 1, \quad e^{(1)}_1 = a, \quad e^{(2)}_2 = a, \quad e^{(3)}_3 = a. \quad (2.9)$$

from

$$\Gamma_\mu = \frac{1}{4} \bar{\gamma}_a \gamma^\nu \partial_\mu e^{(a)}_\nu - \frac{1}{4} \gamma_\rho \bar{\gamma}^\nu \Gamma^\rho_{\mu \nu}. \quad (2.10)$$

one finds the expressions for spinor affine connections as:

$$\Gamma_0 = 0, \quad \Gamma_1 = \frac{1}{2} \varphi^1 \varphi^0, \quad \Gamma_2 = \frac{1}{2} \varphi^2 \varphi^0, \quad \Gamma_3 = \frac{1}{2} \varphi^3 \varphi^0. \quad (2.11)$$

From the definition [26]

$$T^\rho_\mu = \frac{1}{4} g^{\rho \nu} \left( \bar{\psi} \gamma_\mu \nabla_\nu \psi + \psi \gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma_\nu \psi - \nabla_\nu \bar{\psi} \gamma_\mu \psi \right) - \delta^\rho_\mu L_{sp}. \quad (2.12)$$

in this case one finds the the following nontrivial components of energy-momentum tensor of the spinor field

$$T^{00} = m_{sp} S + F(K), \quad (2.13a)$$

$$T^{11} = T^{22} = T^{33} = F(K) - 2KF_K, \quad (2.13b)$$

Further, taking into account that the Einstein tensor corresponding to the metric (2.4) has only nontrivial diagonal components:

$$G_1 = G_2 = G_3 = - \left( \frac{2 \ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right), \quad (2.14a)$$

$$G^{00} = -3 \frac{\dot{a}^2}{a^2}. \quad (2.14b)$$

we write the complete set of Einstein equation for FRW metric

$$2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \kappa (F(K) - 2KF_K), \quad (2.15a)$$

$$3 \frac{\dot{a}^2}{a^2} = \kappa (m_{sp} S + F(K)). \quad (2.15b)$$

Note that one can solve (2.15b) to find a, but to obtain the solution that satisfies both (2.15a) and (2.15b) it is worthy to combine them as follows:

$$\ddot{a} = \frac{\kappa}{6} \left( 3T^1_1 - T^0_0 \right) a = \frac{\kappa}{6} \left( 2F(K) - 6KF_K - m_{sp} S \right) a. \quad (2.16)$$
To solve the equation \((2.16)\) one should know the explicit form of spinor field nonlinearity and also the relation between the invariant \(K\) and metric function \(a\). To find this relation, let us go back to spinor field equations. In this case in view of \((2.6)\) and \((2.11)\) the spinor field equations \((2.7)\) take the form

\[
\begin{align*}
\dot{\gamma}^0 \left( \psi + \frac{3}{2} \dot{a} \psi \right) - m_{sp} \psi - \mathcal{D} \psi - \mathcal{I} \gamma^5 \psi &= 0, \quad (2.17a) \\
\dot{\psi} + \frac{3}{2} \dot{a} \psi &= 0.
\end{align*}
\]

From \((2.17)\) it can be shown that \([48]\)

\[
K = V_0^2, \quad V_0 = \text{Const.} \quad (2.18)
\]

The relation \((2.18)\) holds for \(K = I\) both for massive and massless spinor, whereas, in case of \(K\) being one of the expressions \(\{J, I + J, I - J\}\) it is true for the massless spinor field only.

From \((2.17)\) for the invariants of bilinear spinor form in this case we have

\[
\begin{align*}
\dot{S}_0 + \mathcal{I}A^0_0 &= 0, \quad (2.19a) \\
\dot{P}_0 - \Phi A^0_0 &= 0, \quad (2.19b) \\
\dot{A}^0_0 + \Phi P_0 - \mathcal{I}S_0 &= 0, \quad (2.19c) \\
\dot{A}^3_0 &= 0, \quad (2.19d) \\
\dot{v}^3_0 &= 0, \quad (2.19e) \\

\dot{Q}_{30}^3 + \mathcal{I}Q_{30}^{31} &= 0, \quad (2.19f) \\
\dot{Q}_{30}^3 - \Phi v^3_0 &= 0, \quad (2.19g) \\
\dot{Q}_{31}^{30} - \mathcal{I}v^3_0 &= 0. \quad (2.19h)
\end{align*}
\]

where we denote \(S_0 = Sa^3, P_0 = Pa^3, A^\mu_0 = A^\mu a^3, v^\mu_0 = v^\mu a^3, Q_{0}^{\mu \nu} = Q_{0}^{\mu \nu} a^3\). Here \(v^\mu = \bar{\psi} \gamma^\mu \psi, A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi\) and \(Q_{\mu \nu} = \bar{\psi} \sigma_{\mu \nu} \psi\) are the vector, pseudovector and tensor bilinear spinor forms, respectively.

From \((2.19a)\) - \((2.19h)\) one finds the following relations:

\[
\begin{align*}
(S_0)^2 + (P_0)^2 + (A^0_0)^2 &= C_1 = \text{Const}, \quad (2.20a) \\
A^3_0 &= C_2 = \text{Const}, \quad (2.20b) \\
v^3_0 &= C_3 = \text{Const}, \quad (2.20c) \\
(Q_{0}^{30})^2 + (Q_{0}^{31})^2 + (v^3_0)^2 &= C_4 = \text{Const}. \quad (2.20d)
\end{align*}
\]

It should be noted that in case of a Bianchi type-I space-time for bilinear spinor forms one obtains the system that is exactly the same as \((2.19)\) with the solution \((2.20)\). But the presence of non-trivial non-diagonal components of energy-momentum tensor of the spinor field in that case leads to \(A^1 = A^2 = A^3 = 0\) if the restrictions are imposed on the spinor fields only. And in that specific case thanks to Fierz identity the relation \(A^\mu v^\mu = 0\) leads to \(S = 0\) and \(P = 0\), hence vanishing mass and nonlinear terms in the spinor field Lagrangian \([47]\). But the absence of non-trivial non-diagonal components of the energy-momentum tensor of spinor field in FRW model does not lead to such a severe situation.

### III. SOLUTION TO THE FIELD EQUATIONS

In this section we will solve the field equations derived in the previous section.
A. Solution to the spinor field equations

Let us now solve the spinor field equation. We consider the 4-component spinor field given by

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \quad (3.1)$$

Denoting $\phi_i = a^{3/2} \psi_i$ the spinor field equation (2.17a) can be written as

$$\dot{\phi}_1 + i (m_{sp} + D) \phi_1 + \mathcal{D} \phi_3 = 0, \quad (3.2a)$$
$$\dot{\phi}_2 + i (m_{sp} + D) \phi_2 + \mathcal{D} \phi_4 = 0, \quad (3.2b)$$
$$\dot{\phi}_3 - i (m_{sp} + D) \phi_3 - \mathcal{D} \phi_1 = 0, \quad (3.2c)$$
$$\dot{\phi}_4 - i (m_{sp} + D) \phi_4 - \mathcal{D} \phi_2 = 0. \quad (3.2d)$$

We study the cases for different $K$ in details.

In case of $K = I$ for the components of spinor field from (3.2) one finds [26]

$$\psi_1(t) = C_1 a^{3/2} \exp \left( -i \int m_{sp} + \mathcal{D} \right) dt, \quad (3.3a)$$
$$\psi_2(t) = C_2 a^{3/2} \exp \left( -i \int m_{sp} + \mathcal{D} \right) dt, \quad (3.3b)$$
$$\psi_3(t) = C_3 a^{3/2} \exp \left( i \int m_{sp} + \mathcal{D} \right) dt, \quad (3.3c)$$
$$\psi_4(t) = C_4 a^{3/2} \exp \left( i \int m_{sp} + \mathcal{D} \right) dt, \quad (3.3d)$$

with $C_1, C_2, C_3, C_4$ being the integration constants and related to $V_0$ as

$$C_1^* C_1 + C_2^* C_2 - C_3^* C_3 - C_4^* C_4 = V_0,$$

where $V_0$ such that $I = V_0^2 / a^6$.

As it was mentioned earlier for $K = J$ only a massless spinor field provides exact solution [24].

For the massless spinor field in this case we find [26]

$$\psi_1 = \frac{1}{a^{3/2}} \left( D_1 e^{i \sigma} + i D_3 e^{-i \sigma} \right), \quad (3.4a)$$
$$\psi_2 = \frac{1}{a^{3/2}} \left( D_2 e^{i \sigma} + i D_4 e^{-i \sigma} \right), \quad (3.4b)$$
$$\psi_3 = \frac{1}{a^{3/2}} \left( i D_1 e^{i \sigma} + D_3 e^{-i \sigma} \right), \quad (3.4c)$$
$$\psi_4 = \frac{1}{a^{3/2}} \left( i D_2 e^{i \sigma} + D_4 e^{-i \sigma} \right), \quad (3.4d)$$

with the constants $D_i$'s obeying $D_1^* D_1 + D_2^* D_2 - D_3^* D_3 - D_4^* D_4 = V_0$, where $J = V_0^2 / a^6$.

As far as cases with $K = I + J$ or $K = I - J$ are concerned, the solution to the equation (3.2) can be written in the form [52]

$$\phi(t) = T \exp \left( - \int_{t_1}^t A(\tau) d\tau \right) \phi(t_1), \quad (3.5)$$
expanding Universe when stage of evolution, in (3.8) we should set

Thus the spinor field equations are totally solved and it was found that the components of spinor field are inverse to $a^{3/2}$.

B. Solutions to the gravitational field equations

Let us now go back to the equation (2.16). We choose the polynomial nonlinearity in the from

$$F = \sum_k \lambda_k K^{n_k},$$

where $\lambda_k$ are the coupling constants. For simplicity we consider the case with $K = I$. Taking into account that $S = V_0/a^3$ ($I = V_0^2/a^6$) we now have

$$\ddot{a} = \frac{\kappa}{6} \left( 2 \sum_k \lambda_k (1 - 3n_k) \frac{V_0^{2n_k}}{a^{6n_k}} - m_{sp} \frac{V_0}{a^3} \right) a. \tag{3.8}$$

We solve this equation numerically. Since at any space-time point where $a = 0$ the invariants of spinor and gravitational fields become singular, it is a singular point [26]. So, we consider the initial value of $a(0)$ is small but non-zero. As a result for the nonlinear term to prevail at the early stage of evolution, in (3.8) we should set $n_k = n_1 : 1 - 6n_1 < -2$, i.e., $n_1 > 1/2$, whereas for an expanding Universe when $a \to \infty$ as $t \to \infty$ one should have $n_k = n_2 : 1 - 6n_2 > -2$, i.e., $n_2 < 1/2$. On account of this we can write the foregoing equation as

$$\dot{a} = \Phi(a), \tag{3.9}$$

$$\Phi(a) = \frac{\kappa}{6} \left( 2\lambda_1 (1 - 3n_1) \frac{V_0^{2n_1}}{a^{6n_1}} + 2\lambda_2 (1 - 3n_2) \frac{V_0^{2n_2}}{a^{6n_2}} - m_{sp} \frac{V_0}{a^3} \right) a,$$

with the first integral

$$\dot{a} = \Phi_1(a), \tag{3.10}$$

$$\Phi_1(a) = \sqrt{\frac{\kappa}{3} \left( V_0^{2n_1} \lambda_1 a^{2(1-3n_1)} + V_0^{2n_2} \lambda_2 a^{2(1-3n_2)} + m_{sp} \frac{V_0}{a} \right) + \tilde{C}},$$

with $\tilde{C}$ being an integration constant.

In what follows we study the equation (3.9) numerically. As it was mentioned earlier, we set a small but non-zero initial value for $a(t)$, precisely, $a(0) = 0.5$, while $\dot{a}(t)$ is calculated from (3.10). We also set $V_0 = 1$, $m_{sp} = 0.01$, $\tilde{C} = 1$, $\kappa = 1$, $n_1 = 2/3$, and $n_2 = -1/6$. We have considered both positive and negative values for $\lambda_1$ ($\lambda_1 = \pm 0.0001$) and $\lambda_2$ ($\lambda_2 = \pm 0.03$). It is found that the sign of $\lambda_1$ has almost no effect, while that of $\lambda_2$ is crucial. A positive $\lambda_2$ generates an expanding Universe [cf. Fig. 1], while a negative $\lambda_2$ gives rise to an Universe that after attaining some maximum begins to contract and finally ends in Big Crunch [cf. Fig. 2]. In Figs. 3 and 4 the corresponding pictures of evolution of deceleration parameter are given.
IV. SPINOR DESCRIPTION OF MATTER

One of the principal advantages of using spinor description of source fields lies in the fact that in this case the Bianchi identity is fulfilled automatically. To solve a self-consistent system beside Einstein field equations one has to consider the Bianchi identity that gives relation between material field and geometry. The spinor description of source fields gives a straightforward exit when a multiple sources are considered. To show that let us write the Bianchi identity $G^{\nu}_{\mu;\nu} = 0$, that leads to

$$T^{\nu}_{\mu;\nu} = T^{\nu}_{\mu,\nu} + \Gamma^{\nu}_{\rho\nu} T^{\rho}_{\mu} - \Gamma^{\rho}_{\mu\nu} T^{\nu}_{\rho} = 0,$$

(4.1)
FIG. 3: Plot of deceleration parameter $q$ for a positive $\lambda_2$

FIG. 4: Plot of deceleration parameter $q$ for a negative $\lambda_2$

which for the metric (2.4) on account of the components of the energy-momentum tensor takes the form

$$\dot{\varepsilon} + 3\frac{\ddot{a}}{a} (\varepsilon + p) = 0.$$  \hspace{1cm} (4.2)

Inserting $\varepsilon$ and $p$ from (2.13a) and (2.13b) from (4.2) one finds

$$\frac{m_{sp}}{S} \frac{d}{dt} \left( Sa^3 \right) + \frac{F_K}{a^6} \frac{d}{dt} \left( Ka^6 \right) = 0.$$ \hspace{1cm} (4.3)

In case of $K = I = S^2$ (4.3) fulfills identically as $Sa^3 = const.$ and $Ka^6 = const.$, whereas in the case when $K$ takes one of the following expressions $\{J, I + J, I - J\}$, for a massless spinor field
Equation (4.3) fulfills identically as $K \delta^5 = \text{const}$. Hence if we use spinor description of different fluid and dark energy simulated from corresponding equation of state, the Bianchi identity will be fulfilled identically without invoking any additional condition.

Let us now consider the case with massless spinor field when it can successfully simulate different types of fluid and dark energy. Here we simply give the expressions corresponding to different matters. A detailed description of how to obtain these expressions can be found in [46, 47].

### A. perfect fluid and quintessence

Let us first consider the case when the spinor filed generates a fluid with barotropic equation of state

$$p = W \varepsilon, \quad W = \text{const}. \quad (4.4)$$

In this case from (4.2) we have

$$F(S) = \lambda S^{1+W}, \quad \lambda = \text{const}. \quad (4.5)$$

Depending on the value of $W$ the nonlinearity (4.5) gives rise to dust ($W = 0$), radiation ($W = 1/3$), hard Universe ($W \in (1/3, 1)$), stiff matter ($W = 1$), quintessence ($W \in (-1/3, -1)$), cosmological constant ($W = -1$), phantom matter ($W < -1$) and ekpyrotic matter ($W > 1$). In Fig. 5 we have given the evolution of $a$ when the Universe is filled with quintessence setting $W = -1/2$. It should be mentioned that if one considers a massive spinor field, the nonlinear term obtained from (4.2) in that case contains a term that cancels the mass term in the spinor field Lagrangian [46, 47].

![Plot of metric function $a$ in case of a quintessence.](image)

### B. Chaplygin gas

A Chaplygin gas can be given by

$$p_{\text{ch}} = -\frac{A}{\varepsilon_{\text{ch}}}, \quad (4.6)$$
which corresponds to the nonlinearity

\[ F = \left( A + \lambda K^{(1+\alpha)/2} \right)^{1/(1+\alpha)}. \]  

(4.7)

with \( A \) being a positive constant and \( 0 < \alpha \leq 1 \). In this case evolution of \( a \) is given in Fig. 6.

\[ p = W(\epsilon - \epsilon_{cr}), \quad W \in (-1, 0), \]

(4.8)

can be simulated by

\[ F = \lambda K^{(1+W)/2} + W \frac{\epsilon_{cr}}{1+W}, \]

(4.9)

with \( \epsilon_{cr} \) being some constant. In this case we have a oscillatory mode of expansion given in Fig. 7.

FIG. 6: Plot of metric function \( a \) in case of a Chaplygin gas.

C. Oscillating dark energy

An oscillating dark energy (modified quintessence) with EoS [54]

\[ p = W(\epsilon - \epsilon_{cr}), \quad W \in (-1, 0), \]

(4.8)

can be simulated by

\[ F = \lambda K^{(1+W)/2} + W \frac{\epsilon_{cr}}{1+W}, \]

(4.9)

with \( \epsilon_{cr} \) being some constant. In this case we have a oscillatory mode of expansion given in Fig. 7.

Corresponding energy density and pressure is given in Fig. 8. As one sees, as the energy density becomes less than \( \epsilon_{cr} \), pressure becomes positive. It results in contraction of space-time, hence the increase of energy density. This leads to pressure becoming negative and hence the Universe expanding very fast. Thus we see that in this case the Universe begins from initial singularity (Big Bang), expands to some maximum, then begin to contract and ends in Big Crunch only to begin from a Big Bang.

D. Modified Chaplygin gas

In case of a modified Chaplygin gas with equation of state

\[ p = W\epsilon - A/\epsilon^\alpha, \]

(4.10)
we have

\[ F = \left( \frac{A}{1+W} + \lambda S^{(1+\alpha)(1+W)} \right)^{1/(1+\alpha)}. \]  

(4.11)

The corresponding evolution is given in Fig. 9.
E. Quintom

In order to understand the behavior of the equation of state of dark energy (4.4) with \( w_{\text{quint}} > -1 \) in the past and with \( w_{\text{quint}} < -1 \) at present, quintom model of dark energy was proposed [55]. Quintom model is a dynamic model of dark energy and compare to the other models of dark energy it defines the cosmic evolution in a different way. One of the characteristics of Quintom model is the fact that its equation of state can smoothly pass the value of \( w_{\text{quint}} = -1 \) [56]. In contrast to (4.4), where \( w \) is a constant, in Quintom model it depends on time and can be given by the EoS

\[
 w_{\text{quint}}(t) = -r - \frac{s}{t^2},
\]

(4.12)

where \( r \) and \( s \) are some parameters. Many authors have used Quintom model in order to generate a bouncing Universe. Spinor description of Quintom model was given in [57]. Following [57] let us construct corresponding spinor field nonlinearity. In doing so let us first write the EoS parameter. Taking into account that \( \varepsilon = T_0^0 \) and \( p = -T_1^1 \) from (2.13a) and (2.13b) for the massless spinor field we have

\[
 w_{\text{quint}} = \frac{p}{\varepsilon} = \frac{2KF_K - F(K)}{F(K)} = -1 + \frac{2KF_K}{F}.
\]

(4.13)

Since energy density \( \varepsilon \) should be positive, \( F(K) = \varepsilon \) should be positive as well. For \( w_{\text{quint}} > -1 \) then we should have \( F_K > 0 \), while for \( w_{\text{quint}} < -1 \) we should have \( F_K < 0 \). Moreover from (2.18) we see that for an expanding Universe \( K \) is a decreasing function of time. Taking all this into account we like in [57] construct 3 scenario:

1. quintom-A scenario that describes the Universe evolving from quintessence phase with \( w_{\text{quint}} > -1 \) to a phantom phase with \( w_{\text{quint}} < -1 \). In this case we have

\[
 F_K > 0 \quad \rightarrow \quad F_K < 0;
\]

As a study case we can consider

\[
 F(K) = \lambda \left[ (K - b)^2 + c \right], \quad F_K = 2\lambda(K - b),
\]

(4.14)
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where \( b \) and \( c \) are some positive constants. In this case \( F(K) > 0 \). As far as \( F_K \) is concerned, at the initial stage when the Universe is small enough, i.e. \( V << 1 \), \( K = V_0^2/V^2 \) is quite large, hence \( F_K > 0 \). This leads to \( w_{\text{quint}} > -1 \), i.e. we have quintessence like phase. At \( K = b \) we have \( F_K = 0 \), with EoS \( w_{\text{quint}} = -1 \). After that as the Universe expands \( K \) becomes small, hence \( F_K < 0 \). As a result we have phantom-like phase with \( w_{\text{quint}} < -1 \). Under this choice we have

\[
\begin{align*}
    w_{\text{quint}} &= -1 + \frac{4K(K-b)}{[(K-b)^2 + c]}.

d(4.15)
\end{align*}
\]

Corresponding behavior of metric function \( a \) and EoS parameter are given in Figs. 10 and 11

FIG. 10: Plot of metric function \( a \) in case of a type 1 quintom scenario.

(2) quintom-B scenario that describes the Universe evolving from phantom phase with \( w_{\text{quint}} < -1 \) to a quintessence phase with \( w_{\text{quint}} > -1 \). In this case we have

\[
F_K < 0 \quad \rightarrow \quad F_K > 0;
\]

In this case Cai and Wang [57] proposed \( F = \lambda [- (K-b)K + c] \). Though in this case the condition \( F_K < 0 \quad \rightarrow \quad F_K > 0 \) fulfills, the function \( F \) hence the energy density becomes negative at the early stage of evolution. So we propose

\[
F(K) = \frac{\lambda}{[(K-b)^2 + c]}, \quad F_K = -\frac{2\lambda(K-b)}{[(K-b)^2 + c]^2}.
\]

(4.16)

As one sees, \( F(K) \) is always positive. From (4.16) it can be easily verified that at the initial stage when \( K \) in large, \( F_K < 0 \). It is a phantom-like phase with \( w_{\text{quint}} < -1 \). With the expansion of the Universe \( K \) becomes small. At \( K = b \) we have \( F_K = 0 \), that is \( w_{\text{quint}} = -1 \). Further as \( K \) decreases, \( F_K \) becomes positive, thus giving rise to quintessence-like phase with \( w_{\text{quint}} > -1 \). In this case we have

\[
\begin{align*}
    w_{\text{quint}} &= -1 - \frac{4K(K-b)}{[(K-b)^2 + c]}.
\end{align*}
\]

(4.17)

Corresponding behavior of metric function \( a \) and EoS parameter are given in Figs. 12 and 13.
(3) quintom-C scenario when $F_K$ changes its sign more than one time. In this case one can obtain a new quintom scenario with EoS crossing $-1$ many times. In this case one can set

$$F(K) = \lambda \left[ K(K-b)^2 + c \right], \quad F_K = \lambda (K-b)(3K-b).$$

From (4.18) we again have $F(K) > 0$ and $F_K$ changes its sign more than once. In this case

$$w_{\text{quint}} = -1 - \frac{2K(K-b)(3K-b)}{K(K-b)^2 + c}.$$  

Note that this case can be simulated with $F(K) = c + \sin(K)$, where $c > 1$, hence $F(K) > 0$. In this case $F_K = \cos(K)$ is the alternating quantity.

FIG. 11: Plot of EoS parameter in case of a type 1 quintom scenario.

FIG. 12: Plot of metric function $a$ in case of a type 2 quintom scenario.
Corresponding behavior of metric function $a$ and EoS parameter are given in Figs. 14 and 15.

**FIG. 13:** Plot of EoS parameter in case of a type 2 quintom scenario.

**FIG. 14:** Plot of metric function $a$ in case of a type 3 quintom scenario.

**F. Why spinor description is convenient?**

Question may arise “Why does one need spinor description of matter?” To answer this question, let us recall that the Universe was filled with different matter in the course of evolution. While in the past the evolution was matter dominated, now the dominant component is the dark energy. In case when there is only two components, say radiation and dark energy, the continuity equation
can be easily separated thanks to the fact that the dark energy does not interact with usual matter. But in case of three or more components the situation becomes complicated. Exploiting the spinor description one may avoid this situation, as the continuity equation for spinor field is fulfilled identically [cf. (4.3) and discussion thereafter]. To demonstrate it, let us consider the case with a Van-der-Waals gas, radiation and modified Chaplygin gas. In this case we have

\[
\epsilon = \epsilon_{\text{vdw}} + \epsilon_{\text{rad}} + \epsilon_{\text{mchap}},
\]

\[
p = p_{\text{vdw}} + p_{\text{rad}} + p_{\text{mchap}} = \frac{8W_{\text{vdw}}\epsilon_{\text{vdw}}}{3 - \epsilon_{\text{vdw}}} - 3\epsilon_{\text{vdw}}^2 + W_{\text{rad}}\epsilon_{\text{rad}} + W_{\text{mchap}}\epsilon_{\text{mchap}} - A/\epsilon_{\text{mchap}}^{\alpha}.
\]

Acting straightforward, from (4.2) in this case we obtain

\[
\dot{\epsilon}_{\text{vdw}} + \dot{\epsilon}_{\text{rad}} + \dot{\epsilon}_{\text{mchap}} + \frac{3}{a}\left[(\epsilon_{\text{vdw}} + p_{\text{vdw}}) + (\epsilon_{\text{rad}} + p_{\text{rad}}) + (\epsilon_{\text{mchap}} + p_{\text{mchap}})\right] = 0
\]

(4.21)

Now assuming that the dark energy does not interact with normal matter, the foregoing equation can be decomposed to the following system:

\[
\dot{\epsilon}_{\text{vdw}} + \dot{\epsilon}_{\text{rad}} + \frac{3}{a}\left[(\epsilon_{\text{vdw}} + p_{\text{vdw}}) + (\epsilon_{\text{rad}} + p_{\text{rad}})\right] = 0,
\]

(4.22a)

\[
\dot{\epsilon}_{\text{mchap}} + \frac{3}{a}\left(\epsilon_{\text{mchap}} + p_{\text{mchap}}\right) = 0.
\]

(4.22b)

Thus we have only two equations for three unknowns, i.e., the system is underdetermined. Whereas, expressing only two of the three components (say, radiation and modified Chaplygin gas) in terms of spinor field nonlinearity we can easily overcome this situation.

Further taking into account that radiation and modified Chaplygin gas we can simulate by means of spinor field nonlinearity as

\[
\epsilon_{\text{rad}} = \lambda_{\text{rad}}S^{4/3}, \quad \epsilon_{\text{mchap}} = \left(\frac{A}{1+W} + \lambda_{\text{mchap}}S^{(1+W)(1+\alpha)}\right)^{1/(1+\alpha)}.
\]

(4.23)
the equation (2.16) together with the continuity equations can be written as

\[
\begin{align*}
\dot{H} &= -H^2 - \frac{\kappa}{6} f(a, \varepsilon_{\text{vdw}}), \\
\dot{a} &= aH, \\
\dot{\varepsilon}_{\text{vdw}} &= -3H \left( \varepsilon_{\text{vdw}} + \frac{8W_{\text{vdw}}\varepsilon_{\text{vdw}}}{3 - \varepsilon_{\text{vdw}}} - 3\varepsilon_{\text{vdw}}^2 \right),
\end{align*}
\]

with

\[
f(a, \varepsilon_{\text{vdw}}) = \frac{24W_{\text{vdw}}\varepsilon_{\text{vdw}}}{3 - \varepsilon_{\text{vdw}}} - 9\varepsilon_{\text{vdw}}^2 + \varepsilon_{\text{vdw}} + 2\lambda_{\text{rad}}V_0^4/3a^4
+ (3W + 1) \left( \frac{A}{1 + W} + \lambda_{\text{mchap}}V_0^{(1+W)(1+\alpha)}/a^{3(1+W)(1+\alpha)} \right)^{1/(1+\alpha)} \right)
- 3A \left( \frac{A}{1 + W} + \lambda_{\text{mchap}}V_0^{(1+W)(1+\alpha)}/a^{3(1+W)(1+\alpha)} \right)^{\alpha/(1+\alpha)}.
\]

In what follows, we solve the foregoing system numerically. Our goal is here pure pedagogical, so we use possible simple values for problem parameters setting \(\kappa = 1, V_0 = 1, \lambda_{\text{rad}} = 1, \lambda_{\text{mchap}} = 1, W_{\text{vdw}} = 1/2, \alpha = 1/2, W = -1/2,\) and \(A = 1.\) For initial values we use \(H(0) = 0.1, a(0) = 0.9\) and \(\varepsilon_{\text{vdw}}(0) = 0.9.\) In Fig. 16 we have illustrated the evolution of metric function when the Universe is filled with Van-der-Waals gas, radiation and modified Chaplygin gas. Evolution of corresponding Hubble parameter is given in Fig. 17. The Figs. 18, 19 and 20 show the evolution of energy density, pressure and EoS parameter of the Van-der-Waals gas, respectively. As one sees, the Van-der-Waals gas possesses negative pressure at the early stage which gives rise to initial inflation.

![FIG. 16: Plot of metric function a in case of the FRW Universe filled with Van-der-Waals gas, radiation and modified Chaplygin gas.](image)
V. CONCLUSION

Within the scope of isotropic FRW cosmological model the role of nonlinear spinor field in the evolution of the Universe is studied. It is found that unlike in anisotropic cosmological models in the present case the spinor field does not possess nontrivial non-diagonal components of energy-momentum tensor. As a result, there is no need to impose any additional restriction on either the spinor field or on the metric function or on both of them, as it takes place in case of anisotropic models. The spinor description of different matter was given and evolutions of the Universe corre-
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FIG. 19: Evolution of pressure of a Van-der-Waals gas

FIG. 20: Evolution of EoS parameter of a Van-der-Waals Gas

sponding to these sources are illustrated. In the framework of a three fluid system we have shown why the spinor description of matter is more convenient to study the evolution of the Universe filled with multiple sources.

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