The rank of a divisor on a finite graph: geometry and computation

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Abstract: We study the problem of computing the rank of a divisor on a finite graph, a quantity that arises in the Riemann-Roch theory on a finite graph developed by Baker and Norine (Advances in Mathematics, 215(2): 766-788, 2007). Our work consists of two parts: the first part is an algorithm whose running time is polynomial for a multigraph with a fixed number of vertices. More precisely, our algorithm has running time $O(2^n \log^n n \poly(size)(G))$, where $n + 1$ is the number of vertices of the graph $G$. The second part consists of a new proof of the fact that testing if rank of a divisor is non-negative or not is in the complexity class $NP \cap co-NP$ and motivated by this proof and its generalisations, we construct a new graph invariant that we call the critical automorphism group of the graph.

1 Introduction

The study of chip firing games on a graph with their several variants is a classic topic in discrete mathematics, in particular in algebraic graph theory and this is reflected in the fact that standard text books on algebraic graph theory, such as the book by Godsil and Royle [7] have chapters devoted to chip firing games. See the article of Merino [14] for a broad survey of the topic.

Despite their simplicity, chip firing games have connections to various other areas of mathematics and physics. Some examples include chip firing games studied in dynamical systems under the name “sandpile models” and in fact, early algorithmic work on chip firing games was done independently by physicists. Chip firing games of graphs also play a role in counting the number of points on elliptic curves over finite fields and is the topic of the Phd thesis of Greg Musikar. Recently, chip firing games have played a key role in developing analogies between graphs and Riemann surfaces. This line of research was initiated by Baker and Norine in their pioneering work “Riemann-Roch and Abel-Jacobi theory on a finite graph” [2] where they show analogues of the Riemann-Roch theorem on a finite graph and the theorem is best explained in the language of chip-firing games.

A chip-firing game is a solitary game played on an undirected connected multigraph and is defined as follows: Each vertex of the graph is assigned an integer, refereed to as “chips” and this assignment is called the initial configuration. At each move of the game, an arbitrary vertex $v$ is allowed to either lend or borrow one chip along each edge incident with it and we obtain a new configuration. We define two configurations $C_1$ and $C_2$ to be equivalent if $C_1$ can be reached from $C_2$ by a sequence of chip firings. The Laplacian matrix $Q$ of the graph naturally comes into the picture as follows:

**Lemma 1.** Configurations $C_1$ and $C_2$ are equivalent if and only if $C_1 - C_2$ can be expressed as $Q \cdot w$ for some vector $w$ with integer coordinates.

We can ask some natural questions on such a game:

1. Is a given configuration equivalent to an effective configuration i.e., a configuration where each vertex has a non-negative number of chips?

2. More generally, given a configuration what is the minimum number of chips that must be removed from the system so that the resulting configuration is not equivalent to an effective configuration?
The Riemann-Roch theorem of Baker and Norine provides insights into answering these questions. We need the following definitions before we can state the theorem.

**Definition 1. (Divisor)** A configuration on a finite graph is called a divisor and is represented as an integer vector on \( n + 1 \) coordinates.

**Definition 2. (Degree of a divisor)** For a divisor \( D \), the total number of chips, i.e., the sum of chips over all the vertices of the graph, is called the degree of the divisor \( D \), denoted by \( \deg(D) \).

**Definition 3. (Rank of a divisor)** For a divisor \( D \), one less than the minimum number of chips that must be removed from the divisor \( D \) so that the resulting configuration is not equivalent to an effective divisor is called the rank of the divisor \( D \), denoted by \( r(D) \).

**Theorem 1. (Riemann-Roch theorem for graphs)** For any undirected connected graph \( G \), there exists a divisor \( K \) called the canonical divisor such that for any divisor \( D \) we have the following formula:

\[
r(D) - r(K - D) = \deg(D) - (g - 1)
\]

where \( g \) is the cyclotomic number of \( G \) and is equal to \( m - n + 1 \) where \( m \) is the number of edges, \( n \) is the number of vertices of \( G \).

1.1 Related work and a brief description of our work

The central quantity in the Riemann-Roch theorem is the rank of a divisor and the efficient computation of rank is a natural problem, attributed to Hendrik Lenstra (see [4]). In fact this work, gave an algorithm, i.e., a procedure that terminates in a finite number of steps to compute the rank of a divisor on a tropical curve. But the algorithm does not run in polynomial time in the size of the multigraph even when the number of vertices are fixed since the algorithm involves iterating over all the spanning trees in the graph (see Proof of Theorem 23 in [4]) and the number of spanning trees in indeed not polynomially bounded in the size of the mutligraph even if the number of vertices are fixed. On the other hand, there are polynomial time algorithms for deciding if the rank of a divisor on a finite multigraph is non-negative, see [6], [17] and [15].

Our paper is centered around the problem of computing the rank. More precisely, (Section 5) we obtain an algorithm whose running time is polynomial for a multigraph with a fixed number of vertices. More precisely, our algorithm has running time \( 2^{O(n \log n)} \text{poly(size)}(G) \), where \( n + 1 \) is the number of vertices of the multigraph \( G \). Recall that we are working with arbitrary undirected connected multigraphs or equivalently graphs with positive integer weights on the edges and indeed, the original Riemann-Roch theory was also developed in this setting. The main tools involved are the Riemann-Roch formula and a formula for rank (Theorem 3) that is used in the proof of the Riemann-Roch formula, these results were first obtained in the work of Baker and Norine [2]. We obtain a geometric interpretation of rank (Theorem 5) and combine this geometric interpretation along with algorithms from the geometry of numbers to obtain the algorithm for computing the rank (Algorithm 3.5). We find it satisfying that geometric tools seem to be essential in obtaining the algorithm, though the definition of rank of a divisor can be stated in purely combinatorial terms.

The second part (Section 4) starts with a new proof of the fact that testing if \( r(D) \geq 0 \) is in \( NP \cap co-NP \). A duality theorem characterising divisors with \( r(D) \geq 0 \) plays a key role in our proof. Motivated by the observation that generalisations of the duality theorem lead to more general complexity results on computing the rank, we generalise the duality theorem and this generalisation leads to the construction of a new graph invariant that we call the **critical automorphism group** of the graph.

2 Preliminaries

We will now describe mainly geometric notions and results that we frequently use in the rest of our paper.
2.1 Lattices

A lattice $L$ is a discrete subgroup of the Euclidean vector space $\mathbb{R}^n$. More concretely, a lattice is the Abelian group obtained by taking all the integral combinations of a set of linearly independent vectors $b_1, \ldots, b_k$ in $\mathbb{R}^n$. More precisely,

$$L = \left\{ \sum_{i=1}^{k} \alpha_i b_i \mid \alpha_i \in \mathbb{Z} \right\} \quad (2)$$

The set $B = \{b_1, \ldots, b_k\}$ is called a basis of $L$ and the integer $k$ is called the dimension of $L$, denoted by $\text{dim}(L)$, is independent of the choice of the basis. We denote the subspace spanned by the elements of $B$ by $\text{Span}(L)$.

Let us now look at the important geometric invariants of a lattice. The volume of the lattice $\text{Vol}(L)$, also known as the discriminant or determinant, is defined as

$$\sqrt{\det(BB^t)}$$

where the basis $B$ is represented as a matrix with its elements row-wise and hence, $BB^t$ is the Gram matrix of the basis elements. Another important invariant is the norm of a shortest vector of a lattice.

Definition 4. A shortest vector of $L$ in the Euclidean norm is an element $q$ of $L$ such that $q \cdot q \leq q' \cdot q'$ for all non-zero elements $q'$ in $L$ and we denote $||q||_2$ as $\nu_E(L)$.

2.2 The Laplacian Lattice of a Graph

For an undirected connected graph $G$, the Laplacian matrix $Q(G)$ is defined as $D(G) - A(G)$ where $D(G)$ is the diagonal matrix with the degree of every vertex in its diagonal and $A(G)$ is the vertex-adjacency matrix of the graph. We assume the following standard form of the Laplacian matrix:

$$Q = \begin{bmatrix}
\delta_0 & -b_{01} & -b_{02} & \cdots & -b_{0n} \\
-b_{10} & \delta_1 & -b_{12} & \cdots & -b_{1n} \\
& \ddots & \ddots & \ddots & \ddots \\
-b_{n0} & -b_{n1} & -b_{n2} & \cdots & \delta_n \\
\end{bmatrix} \quad (3)$$

has the following properties:

(C1) $b_{ij}$’s are integers, $b_{ij} \geq 0$ for all $0 \leq i \neq j \leq n$ and $b_{ij} = b_{ji}$, $\forall i \neq j$.

(C2) $\delta_i = \sum_{j=1,j\neq i}^{n} b_{ij} = \sum_{j=1,j\neq i}^{n} b_{ji}$ (and is the degree of the $i$-th vertex).

Though the Laplacian matrix of a graph contains essentially the same information as the adjacency matrix of a graph, it enjoys other nice properties. See, for example, the chapter “The Laplacian matrix of a Graph” in the algebraic graph theory book of Godsil and Royle [7] for a more complete discussion.

Lemma 2. The Laplacian matrix $Q(G)$ is a symmetric positive semi-definite matrix.

Another remarkable property of the Laplacian matrix is described in the Matrix-Tree theorem:

Theorem 2. (Kirchoff’s Matrix-Tree Theorem) The absolute of value of any cofactor is equal to the number of spanning trees of the graph.

A remarkable aspect of the matrix-tree theorem is that it reduces counting the number of spanning trees of a graph into a determinant computation and hence provides a polynomial time algorithm for it.

Definition 5. (The Laplacian lattice of a graph) Given the Laplacian matrix $Q(G)$, the lattice generated by the rows (or equivalently the columns) of $Q(G)$ is called the Laplacian lattice $L_G$ of the graph.

In the language of chip firing games, the Laplacian lattice is the set of all divisors that are equivalent to the divisor $(0, \ldots, 0)$.
Definition 6. (The Hyperplanes $H_k$) For a fixed real number $k$, we denote $n$-dimensional hyperplane \( \{(x_0, \ldots, x_n) | \sum_{i=0}^{n} x_i = k\} \) by $H_k$.

Definition 7. (The Root Lattice $A_n$) The root lattice $A_n$ is the lattice of integer points in the hyperplane $H_0 = \{(x_0, \ldots, x_n) | \sum_{i=0}^{n+1} x_i = 0, x_i \in \mathbb{R} = (1, \ldots, 1)^\perp \}$. More precisely,
\[
A_n = \{(x_0, \ldots, x_n) | \sum_{i=0}^{n+1} x_i = 0, x_i \in \mathbb{Z}\}.
\]

Remark 1. The name “root lattice” is derived from the fact that the lattice $A_n$ is generated by a root system i.e. a set of vectors that satisfy reflection symmetries. See pages 96–98 of Conway and Sloane [5] for a precise definition of a root system and for a discussion on root lattices. In the case of $A_n$, the corresponding root system is $e_i - e_j$ where $e_i$ and $e_j$ run over the standard basis of $\mathbb{R}^{n+1}$.

We now make a few simple observations on the Laplacian lattice of a graph.

Lemma 3. The Laplacian lattice of a graph on $n+1$-vertices is a sublattice of the root lattice $A_n$.

Definition 8. (The covolume of a sublattice) A full-dimensional sublattice $L_s$ of a lattice $L$ is a subgroup of the Abelian group $L$ the cardinality of the quotient group $L/L_s$ is called the covolume of $L_s$ with respect to $L$.

Lemma 4. [13] The covolume of the Laplacian lattice of $G$ with respect to $A_n$ is equal to the number of spanning trees of $G$.

The elements of $A_n/L_G$ naturally possess an Abelian group structure and this group is known as the Picard group of $G$, also known as the Jacobian of $G$. A number of works have been devoted to the study of the structure of this group and the information that it contains about the underlying graph, see for example the works of Biggs [3], Kotani and Sunada [11] and Lorenzini [12]. As a straightforward corollary to Lemma 4 we obtain:

Corollary 1. The cardinality of the Picard group of $G$ is equal to the number of spanning trees of $G$.

2.3 Algorithmic Geometry of Numbers

We will now briefly discuss some important algorithmic problems related to lattices and geometry of numbers, in particular those that we employ in our algorithm. Two important and closely related problems in the algorithmic geometry of numbers are:

- **Closest vector problem (CVP)**: Given a lattice $L$ by an arbitrary basis and a target vector $v$, find a lattice that is closer to $v$ than to any other lattice point in the $\ell_2$-norm.

- **Shortest vector problem (SVP)**: Given a lattice $L$, by an arbitrary basis find a non-zero lattice point that has the smallest $\ell_2$-norm.

Indeed, CVP and SVP have versions with respect to the other $\ell_p$-norms. For the $\ell_2$-norm both CVP and SVP are known to be NP-hard and in fact SVP has a polynomial time reduction to CVP in any given norm but the converse is not known. A problem that is closely related to CVP is the integer programming problem:

**Integer programming problem**: Given a polyhedron $P$ in the form $A \cdot x \leq b$ where $A$ is an $m \times n$ matrix and $b$ is a vector in $\mathbb{R}^n$. Decide if $P$ has an integer point or not i.e., $P \cap \mathbb{Z}^n = \emptyset$ or not.

Indeed, the integer programming problem is also known to be NP-hard. In the 1980’s there was a great amount of progress in algorithmic geometry of numbers, triggered by the algorithm of Lenstra that solves the integer programming problem in polynomial time for a fixed dimension [10]. Lenstra’s algorithm has running time $2^{O(n^3)} \cdot poly(|I|)$, where $|I|$ is the size of the input. The factor $2^{O(n^3)}$ was subsequently improved and the current best factor being $2^{O(n \log n)}$ by Kannan [8]. Note that the Kannan’s algorithm also works when the polytope is presented as a separation oracle and in fact, the algorithm works also for general convex bodies presented in terms of a separation oracle, see the remark “General convex bodies and mixed integer programs” in [8]. We will crucially use Kannan’s algorithm and the polytope will be presented to Kannan’s algorithm in terms of a separation oracle.
2.4 Polyhedral Distance Functions

Let $\mathcal{P}$ be a convex polytope in $\mathbb{R}^n$ with the reference point $O = (0, \ldots, 0)$ that we call the “center” in its interior. By $\mathcal{P}(p, \lambda)$ we denote a dilation of $\mathcal{P}$ by a factor $\lambda$ and its center translated to the point $p$ i.e. $\mathcal{P}(p, \lambda) = p + \lambda \cdot \mathcal{P}$ and $\lambda \cdot \mathcal{P} = \{ \lambda x \mid x \in \mathcal{P} \}$. We define the $\mathcal{P}$-midpoint of two points $p$ and $q$ in $\mathbb{R}^n$ as $\inf \{R \mid \mathcal{P}(p, R) \cap \mathcal{P}(q, R) \neq \emptyset \}$. The polyhedral distance function $d_\mathcal{P}(\cdot, \cdot)$ between the points of $\mathbb{R}^n$ is defined as follows:

$$\forall p, q \in \mathbb{R}^n, \quad d_\mathcal{P}(p, q) := \inf \{ \lambda \geq 0 \mid q \in \mathcal{P}(p, \lambda) \}.$$ 

$d_\mathcal{P}$ is not generally symmetric, indeed it is easy to check that $d_\mathcal{P}(\cdot, \cdot)$ is symmetric if and only if the polyhedron $\mathcal{P}$ is centrally symmetric i.e. $\mathcal{P} = -\mathcal{P}$. Nevertheless $d_\mathcal{P}(\cdot, \cdot)$ satisfies the triangle inequality.

**Lemma 5.** For every three points $p, q, r \in \mathbb{R}^n$, we have $d_\mathcal{P}(p, q) + d_\mathcal{P}(q, r) \geq d_\mathcal{P}(p, r)$. In addition, if $q$ is a convex combination of $p$ and $r$, then $d_\mathcal{P}(p, q) + d_\mathcal{P}(q, r) = d_\mathcal{P}(p, r)$.

**Proof.** To prove the triangle inequality, it will be sufficient to show that if $q \in p + \lambda \cdot \mathcal{P}$ and $r \in q + \mu \cdot \mathcal{P}$, then $r \in p + (\lambda + \mu) \cdot \mathcal{P}$. We write $q = p + \lambda \cdot q'$ and $r = q + \mu \cdot r'$ for two points $q'$ and $r'$ in $\mathcal{P}$. We can then write $r = p + \lambda \cdot q' + \mu \cdot r' = p + (\lambda + \mu) \cdot (\frac{\lambda}{\lambda + \mu} q' + \frac{\mu}{\lambda + \mu} r')$. $\mathcal{P}$ being convex and $\lambda, \mu \geq 0$, we infer that $\frac{\lambda}{\lambda + \mu} q' + \frac{\mu}{\lambda + \mu} r' \in \mathcal{P}$, and so $r \in p + (\lambda + \mu) \cdot \mathcal{P}$. The triangle inequality follows.

To prove the second part of the lemma, let $t \in [0, 1]$ be such that $q = t \cdot p + (1 - t) \cdot r$. By the triangle inequality, it will be enough to prove that $d_\mathcal{P}(p, q) + d_\mathcal{P}(q, r) \leq d_\mathcal{P}(p, r)$. Let $d_\mathcal{P}(p, r) = \lambda$ so that $r = p + \lambda \cdot r'$ for some point $r'$ in $\mathcal{P}$. We infer first that $q = t \cdot p + (1 - t) \cdot r = t \cdot p + (1 - t) \cdot (p + \lambda \cdot r') = p + (1 - t) \lambda \cdot r'$, which implies that $d_\mathcal{P}(p, q) \leq (1 - t) \lambda$. Similarly we have $t \cdot r = t \cdot p + t \lambda \cdot r' = q - (1 - t) \cdot r + t \lambda \cdot r'$. It follows that $r = q + t \lambda \cdot r'$ and so $d_\mathcal{P}(q, r) \leq t \lambda$. We conclude that $d_\mathcal{P}(p, q) + d_\mathcal{P}(q, r) \leq d_\mathcal{P}(p, r)$, and the lemma follows. \hfill $\square$

We also observe that the polyhedral metric $d_\mathcal{P}(\cdot, \cdot)$ is translation invariant, i.e.

**Lemma 6.** For any two points $p, q$ in $\mathbb{R}^n$, and for any vector $v \in \mathbb{R}^n$, we have $d_\mathcal{P}(p, q) = d_\mathcal{P}(p - v, q - v)$. In particular, $d_\mathcal{P}(p, q) = d_\mathcal{P}(p - q, O) = d_\mathcal{P}(O, q - p)$.

**Proof.** The proof is easy: if $q \in p + \lambda \cdot \mathcal{P}$, then $q - v \in p - v + \lambda \cdot \mathcal{P}$, and vice versa. \hfill $\square$

**Remark 2.** The notion of a polyhedral distance function is essentially the concept of a gauge function of a convex body that has been studied in [16]. Lemmas 5 and 6 can be derived in a straightforward way from the results in [16].

Recall that the n-dimensional hyperplane $H_k$ is defined as $H_k = \{ (x_0, \ldots, x_n) \mid \sum_{i=0}^n x_i = k \}$. We will be mainly be using distance functions defined by regular simplices $\triangle$ and $\bar{\triangle}$ where the simplices $\triangle$ and $\bar{\triangle}$ are defined as follows:

**Definition 9.** The regular simplex $\triangle$ is the convex hull of $t_0, \ldots, t_n$ in $H_0$ where

$$t_{ij} = \begin{cases} n, & \text{if } i = j, \\ -1, & \text{otherwise} \end{cases}$$

for $i$ from 0, $\ldots$, $n$ and $t_{ij}$ is the $j$-th coordinate of $t_i$. We define $\triangle$ as $-\triangle$.

We note that for points in $H_0$, the distance functions $d_\triangle$ and $d_{\bar{\triangle}}$ have a simple formula:

**Lemma 7.** (Lemma 4.7, [16]) For any pair of points $p, q$ in $H_0$, we have:

$$d_\triangle(p, q) = | \min_i (q_i - p_i) |,$$

$$d_{\bar{\triangle}}(p, q) = | \min_i (p_i - q_i) |.$$
Given a permutation $\pi$ on the $n+1$ vertices of $G$, define the ordering $\pi(v_0) <_\pi \pi(v_1) <_\pi \cdots <_\pi \pi(v_n)$ and orient the edges of graph $G$ according to the ordering defined by $\pi$ i.e., there is an oriented edge from $v_i$ to $v_j$ if $(v_i, v_j) \in E$ and if $v_i <_\pi v_j$ in the ordering defined by $\pi$. Consider the acyclic orientation induced by a permutation $\pi$ on the set of vertices of $G$ and define $\nu_\pi = (\text{indeg}_\pi(v_0) - 1, \ldots, \text{indeg}_\pi(v_n) - 1)$, where $\text{indeg}_\pi(v)$ is the indegree of the vertex $v$ in the directed graph oriented according to $\pi$. Define

$$\text{Ext}(L_G) = \{-\nu_\pi + q \mid \pi \in S_{n+1}, q \in L_G\}. \tag{4}$$

### 2.5 Distance function induced by a Discrete Point Set

Given a polyhedral distance function $P$ and a discrete point set $S$, we define a function $h_{P,S} : \mathbb{R}^n \to \mathbb{R}$ as:

$$h_{P,S}(p) = \min_{q \in S} d_P(p, q) \tag{5}$$

In particular, the notion of local minima and local maxima of the distance function $h_{P,S}$ turns out to be useful:

**Definition 10.** (Local Maxima and Local Minima of $h_{P,S}$) Let $B(p, \epsilon)$ be the Euclidean ball of radius $\epsilon$ centered at $p$. A point $c$ in $\mathbb{R}^n$ is called a local minimum of $h_{P,S}$ if there exists an $\epsilon > 0$ such that $h_{P,S}(c) \leq h_{P,S}(q)$ for all $q \in B(c, \epsilon)$. A point $c$ in $\mathbb{R}^n$ is called a local maximum of $h_{P,S}$ if there exists an $\epsilon > 0$ such that $h_{P,S}(c) \geq h_{P,S}(q)$ for all $q \in B(c, \epsilon)$.

We denote the set of local maxima of $h_{P,S}$ by $\text{Crit}_P(S)$.

### 3 Algorithms for computing the rank

In this section, we will construct algorithms for computing the rank with the main result being an algorithm for computing the rank that runs in polynomial time when the number of vertices of the multigraph is fixed.

#### 3.1 A simplification

We shall first observe that by using the Riemann-Roch theorem we can restrict our attention to divisors of degree between zero and $g - 1$. Firstly, a divisor of negative degree must have rank minus one. Furthermore, by the Riemann-Roch formula we have:

**Lemma 8.** If the degree of $D$ is strictly greater than $2g - 2$, then $r(D) = \text{deg}(D) - g$.

**Proof.** Observe that if the degree of $D$ is strictly greater than $2g - 2$ then the rank of $K - D$ is $-1$ and apply the Riemann-Roch theorem. \hfill $\Box$

Furthermore, we can compute the rank of divisors of degree between $g$ and $2g - 2$ by computing the rank of $K - D$, a divisor that has degree between zero and $g - 1$ and then applying the Riemann-Roch theorem. Hence, we consider the problem of computing the rank of a divisor of degree between zero and $g - 1$. In fact, we consider the decision version of the problem i.e., we want to decide (efficiently) if $r(D) \leq k$ for every $k$ between zero and $g - 1$; observe that such a procedure combined with a binary search over the parameter $k$ will compute the rank in time $O(\ln(g))$ times the running time of the procedure.

#### 3.2 A first attempt at computing the rank

Let us discuss a first attempt at computing the rank. We will compute rank directly from its definition (Definition 3). We will use the fact that there is a polynomial time algorithm for testing if $r(D) \geq 0$ due to the independent work of Dhar [6] and Tardos [17].
Algorithm 1. 1. Enumerate all effective divisors of degree at most the degree of the divisor \( D \).

2. Find an effective divisor \( E \) of smallest degree such that \( r(D - E) = -1 \) by using Dhar’s algorithm.

Theorem 3. The running time of Algorithm 1 is \( O(2^{n \ln g}) \).

The running time of the Algorithm 1 is not polynomial in the size of the input even for a fixed number of vertices since the quantity \( 2^{n \ln g} \) is not polynomially bounded in the size of the input. There is general interest in obtaining an algorithm that runs in polynomial time for a fixed number of vertices and furthermore, in obtaining a singly exponential time algorithm i.e., an algorithm with running time \( 2^{O(n^{\omega})} \text{poly}(\text{size}(G)) \). We will now undertake a deeper study of rank to obtain an algorithm that runs in time polynomial in the size of the input provided that the number of vertices is fixed. More precisely, our algorithm has running time \( 2^{O(n^{\log n})} \text{poly}(\text{size}(G)) \). An important ingredient is a geometric interpretation of rank that we shall obtain in the following section.

3.3 A geometric interpretation of rank

We start with the following formula for rank first shown in Baker and Norine [2] and later reproven in Amini and Manjunath [1].

Theorem 4. For any divisor \( D \), we have:

\[
 r(D) = \min_{\nu \in \text{Ext}(L_D)} \deg^+(D - \nu) - 1
\]

where \( \deg^+(D) = \sum_{i: D > 0} D_i \).

Remark 3. Some remarks on the proof(s) of Theorem 4 are in order: as we mentioned earlier, Theorem 4 has two proofs, the original proof due to Baker and Norine [2] was based on combinatorial tools. In particular, the main component of the proof was to establish the existence and uniqueness of a certain special type of divisors called “\( v \)-reduced” divisors in each linear equivalence class of divisors, while the approach of Amini and Manjunath [1] involved studying the Laplacian lattice under the simplicial distance function \( d_\Delta \).

3.3.1 A sketch of the approach

Let us now briefly sketch our approach to computing the rank: We start with the formula to compute the rank and proceed as follows: we run over all the permutations \( \pi \in S_{n+1} \) and for each permutation \( \pi \) suppose that we could compute \( \min_{q \in L_G} \deg^+(D - \nu_\pi + q) \) in time that is possibly exponential but only in \( n \) and then we would obtain an algorithm with running time \( O(f(n) \text{poly}(\text{size}(G))) \) for some function \( f \). But, how do we compute \( \min_{q \in L_G} \deg^+(D - \nu_\pi + q) \)? One hope would be to reduce the problem to a closest vector problem on lattices or more generally to integer programming. Fortunately, the integer programming problem has an algorithm that runs in time that exponential only in \( n \) (the dimension of the lattice). Such an algorithm would run in time \( O(2^{n^{\log n}} \text{poly}(\text{size}(G))) \). This approach requires a better understanding of the \( \deg^+ \) function that we now obtain.

Definition 11. (Orthogonal projections onto \( H_k \)) For a point \( P \in \mathbb{R}^{n+1} \) we denote by \( \pi_k(P) \) the orthogonal projection of \( P \) onto the hyperplane \( H_k \).

For the sake of presentation, we first consider the case where the divisor has degree \( g - 1 \). In this case, we observe that \( \deg^+(D - \nu) = \frac{\ell_1(D - \nu)}{2} \) and that \( \ell_1(D - \nu) = \ell_1(\pi_0(D) - \pi_0(\nu)) \). We denote the set \( \pi_0(\text{Ext}(L_G)) \) the orthogonal projection of \( \text{Ext}(L_G) \) onto the hyperplane \( H_0 \) by \( \text{Crit}_\Delta(L_G) \), as defined in Subsection 2.5, and indeed, the orthogonal projections of \( \text{Ext}(L_G) \) are the local maxima of the distance function \( h_{\Delta,L_G} \) we refer to [1] for more details.

Corollary 2. For any divisor \( D \) with \( \deg(D) = g - 1 \), we have:

\[
 r(D) = \min_{c \in \text{Crit}_\Delta(L_G)} \frac{\ell_1(\pi_0(D) - c)}{2} - 1.
\]
Lemma 10. \(\inf_{\deg} \deg\) and scaled by a factor of \(\pi\) on \(H_0\) we define the generalised degree-plus distance between \(P\) and \(Q\) as

\[
d^+_k(P, Q) = \sup \left\{ r \mid \Delta(P, r) \cap \Delta(Q, r + k) = \emptyset \right\}.
\]

Note that though \(d^+_k\) does not appear to be a distance function at first glance, we will actually show that it can be realised by a sequence of distance functions (See Section 3.4 for a definition).

We will now note some basic properties of the function \(d^+_k\):

Lemma 9. \((\text{Translation Invariance})\) For any points \(P, Q, T\) in \(H_0\) and for any positive real numbers \(r_1\) and \(r_2\) we have: \(\Delta(P, r_1) \cap \Delta(Q, r_2) = \emptyset\) if and only if \(\Delta(P + T, r_1) \cap \Delta(Q + T, r_2) = \emptyset\).

Proof. Assume that \(\Delta(P, r_1) \cap \Delta(Q, r_2) = \emptyset\) and consider a point \(S \in \Delta(P, r_1) \cap \Delta(Q, r_2)\). Now, \(S = r_1 \sum_{i=1}^{n+1} \alpha_i v_i + P = r_2 \sum_{i=1}^{n+1} \beta_i v_i + Q\) for some \(\alpha_i \geq 0, \beta_i \geq 0\) and \(\sum_i \alpha_i = \sum_i \beta_i = 1\). Now this implies that \(S + T = r_1 \sum_{i=1}^{n+1} \alpha_i v_i + P + T = r_2 \sum_{i=1}^{n+1} \beta_i v_i + Q + T\). Hence, \(S + T \in \Delta(P + T, r_1) \cap \Delta(Q + T, r_2)\). The converse follows by symmetry.

Lemma 10. \((\text{Projection Lemma})\) Let \(R\) be a point in \(\mathbb{R}^{n+1}\) with \(\deg(R) \geq 0\), let \(O\) be the origin and let \(\pi_0(R)\) be the orthogonal projection of \(R\) onto \(H_0\). We have:

\[
\inf_{Z \in H^+(R) \cap H^+(O)} \deg(Z) = (n + 1) \sup \left\{ r \mid \Delta(\pi_0(R), r) \cap \Delta(O, r + \frac{\deg(R)}{n + 1}) = \emptyset \right\} + \deg(R).
\]

Proof. Consider a point \(X\), say, in the intersection of \(H^+(O)\) and \(H^+(R)\) and consider the intersection of the hyperplane \(H_{\deg(X)}\) with \(H^+(O)\) and \(H^+(R)\). Observe that the intersection of \(H_{\deg(X)}\) and \(H^+(O)\) is a simplex that is a scaled and translated copy of \(\Delta\), call it \(\Delta_1\), centered at \(\frac{\deg(X)}{n+1}(1, \ldots, 1)\) and scaled by a factor of \(\frac{\deg(X)}{n+1}\). Similarly, the intersection of \(H_{\deg(X)}\) and \(H^+(R)\) is also a simplex that is a scaled and translated copy of \(\Delta\), call it \(\Delta_2\) centered at \(R + \frac{\deg(X-R)}{n+1}(1, \ldots, 1)\) scaled by a factor of \(\frac{\deg(X-R)}{n+1}\). Observe that \(\deg(X) \geq \deg(R) \geq \deg(O)\). Indeed simplices \(\Delta_1\) and \(\Delta_2\) intersect at \(X\) and \(\deg(X)\) is equal to \(n + 1\) times the radius of \(\Delta_2\) plus \(\deg(R)\). We now project the simplices \(\Delta_1\) and \(\Delta_2\) onto \(H_0\) and obtain inf \{\(\deg(Z)\) \(Z \in H^+(R) \cap H^+(O)\}\} \(\geq (n + 1) \sup \left\{ r \mid \Delta(\pi_0(R), r) \cap \Delta(O, r + \frac{\deg(R)}{n + 1}) = \emptyset \right\} + \deg(R)\). Now, consider a point \(P\) in \(\Delta(\pi_0(R), r) \cap \Delta(O, r + \frac{\deg(R)}{n + 1})\) and observe that the point \(X = P + (r + \frac{\deg(R)}{n + 1})(1, \ldots, 1)\) is a point in the intersection of \(H^+(O)\) and \(H^+(R)\). This shows that inf \{\(\deg(Z)\) \(Z \in H^+(R) \cap H^+(O)\}\} \(\leq (n + 1) \sup \left\{ r \mid \Delta(\pi_0(R), r) \cap \Delta(O, r + \frac{\deg(R)}{n + 1}) = \emptyset \right\} + \deg(R)\). This completes the proof.

We are now ready to establish the connection between the \(\deg^+\) function and the function \(d^+_k\).
Lemma 11. For any pair of points $P$ and $Q$ in $\mathbb{R}^{n+1}$ with $\deg(P) \geq \deg(Q)$, we have

$$\deg^+(P - Q) = (n + 1) d_k^+ (\pi_0(P), \pi_0(Q)) + \deg(P - Q)$$

for $k = \frac{\deg(P - Q)}{n + 1}$.

Proof. First consider a point $R$ in $\mathbb{R}^{n+1}$. We have $\deg^+(R) = \sum_{r \geq 0} R_i = \deg(R \oplus O)$, where $R \oplus O = (\max(R_0, 0), \ldots, \max(R_n, 0))$. Now we have:

$$\deg(R \oplus O) = \inf_{z \in H^+(R) \cap H^+(O)} \deg(Z).$$

By Lemma 10 we have

$$\inf_{z \in H^+(R) \cap H^+(O)} \deg(Z) = (n + 1) \sup \left\{ r \mid \triangle(\pi_0(R), r) \cap \triangle(O, r + \frac{\deg(R)}{n + 1}) = \emptyset \right\} + \deg(R).$$

Now, for two points $P$ and $Q$ in $\mathbb{R}^{n+1}$, letting $R = P - Q$ in the above formula and applying Lemma 9 we obtain the relation given in the proposition.

The function $d_k^+$ is motivated naturally by the definition of the $\deg^+$ function but is not very handy for geometric as well as computational reasons. In the following, we will obtain a more convenient representation of $d_k^+$. In fact, $d_k^+$ is closely related to the following family of polytopes: For a point $P \in H_0$ and $m, n > 0$, let $\mathcal{P}_{m,n}(P) = (\triangle(O, m) \oplus_{\text{Mink}} \triangle(O, n)) + P$, where $\oplus_{\text{Mink}}$ denotes the Minkowski sum. Note that we use the notation $\oplus$ for the tropical maximum sum.

Lemma 12. For any positive real numbers $m, n$, $\mathcal{P}_{m,n}(P)$ is a convex polytope.

Proof. Using the fact that Minkowski sum of two convex polytopes is a convex polytope and hence, $\mathcal{P}_{m,n}(O)$ is a convex polytope. Indeed translates of a convex polytope is also a convex polytope and hence, $\mathcal{P}_{m,n}(P)$ is also a convex polytope.

Lemma 13. For points $P$ and $Q$ in $H_0$ and for $k \geq 0$, $d_k^+(P, Q) = \inf \{ r \mid Q \in (\triangle(O, r) \oplus_{\text{Mink}} \triangle(O, r + k)) + P \}$, where $O$ is the origin.

Proof. By definition $d_k^+(P, Q) = \sup \{ r \mid \triangle(P, r) \cap \triangle(Q, r + k) \}$. Let $r_0 = d_k^+(P, Q)$ and consider a point $R$ in the intersection of $\triangle(P, r_0)$ and $\triangle(Q, r_0 + k)$. Rephrasing $d_k^+(P, R) = r_0$ and $d_k^+(Q, R) = d_k^+(R, Q) = r_0 + k$. This implies that $R - P \in \triangle(O, r_0)$ and $Q - R \in \triangle(O, r_0 + k)$. This shows that $Q - P \in \triangle(O, r_0) \oplus_{\text{Mink}} \triangle(O, r_0 + k)$ and we obtain $Q \in (\triangle(O, r_0) \oplus_{\text{Mink}} \triangle(O, r_0 + k)) + P$. Hence, $d_k^+(P, Q) \geq \inf \{ r \mid Q \in (\triangle(O, r) \oplus_{\text{Mink}} \triangle(O, r + k)) + P \}$.

Furthermore, if $Q$ is contained in $\triangle(O, r) \oplus_{\text{Mink}} \triangle(O, r + k) + P$ then there exists a point $R = R_1 + R_2$ such that $R_1 \in \triangle(O, r)$ and $R_2 \in \triangle(O, r + k)$ with $Q = R_1 + R_2 + P$ and we take $R_3$ such that $R_3 = Q - R_2 = R_1 + P$. Therefore, the point $R_3$ is contained in both $\triangle(Q, r + k)$ and $\triangle(P, r)$ and we obtain $\inf \{ r \mid Q \in (\triangle(O, r) \oplus_{\text{Mink}} \triangle(O, r + k)) + P \} \geq d_k^+(P, Q)$. This concludes the proof.

As a corollary we obtain a handy characterisation of the polytope $\mathcal{P}_{m,n}$.

Corollary 3. Let $P$, $Q$ be points in $H_0$, a point $Q$ belongs to the polytope $\mathcal{P}_{r,r+d}(P)$ if and only if $\deg^+(P + \frac{d(1, \ldots, 1)}{n+1} - Q) \leq r(n+1)$.

Proof. Let a point $Q$ be contained in $\mathcal{P}_{r,r+d}(P)$, then $\inf \{ r' \mid Q \in (\triangle(O, r') \oplus_{\text{Mink}} \triangle(O, r' + d) + P \} \leq r$ and hence by Lemma 13 we know that $r \geq \inf \{ r' \mid Q \in (\triangle(O, r') \oplus_{\text{Mink}} \triangle(O, r' + d) + P \} = d_k^+(P, Q)$. By Lemma 11 we know that $\deg^+(P + \frac{d(1, \ldots, 1)}{n+1} - Q) = (n+1)d_k^+(P, Q)$. Hence, $\deg^+(P + \frac{d(1, \ldots, 1)}{n+1} - Q) \leq r(n+1)$. Conversely, if a point $Q$ satisfies $\deg^+(P + \frac{d(1, \ldots, 1)}{n+1} - Q) \leq r(n+1)$ then, $d_k^+(P, Q) \leq r$ and hence, $r \geq \inf \{ r' \mid Q \in (\triangle(O, r') \oplus_{\text{Mink}} \triangle(O, r' + d) + P \}$. This completes the proof.
Now putting together, the formula for rank in Theorem 4, Lemma 11 and Lemma 13 we obtain the following geometric interpretation of rank:

**Theorem 5. (A Geometric Interpretation of rank)** Consider a divisor $D$ of degree $d$ between zero and $g - 1$, then $D$ has rank $r_0 - 1$ if and only if $\pi_0(D)$ is contained in the boundary of the arrangement $\cup_{c \in \text{Crit}_{\ell_1}(L_G)} \mathcal{P}_{r_1,r_2}(c)$ where $r_1 = r_0/(n + 1)$ and $r_2 = (r_0 + g - 1 - d)/(n + 1)$.

**Remark 4.** The fact that $\cup \mathcal{P}_{r_1,r_1+k}(c) \subseteq \mathcal{P}_{r_2,r_2+k}(c)$ if $r_1 \leq r_2$ is implicit in the statement of Theorem 5.

### 3.4 Computing the rank for divisors of degree between zero and $g - 1$

We now give an algorithm for computing the rank that runs in polynomial time for a fixed number of vertices. The algorithm uses two main ingredients:

1. The geometric interpretation of rank (Theorem 5).
2. Reduction to the algorithm for integer programming by Kannan [8]:

For the sake of exposition, we first consider the slightly easier case of divisors with degree exactly $g - 1$. We employ Theorem 2 to obtain the following algorithm:

**Algorithm 2.** 1. For each permutation $\pi \in S_{n+1}$, we compute $\min_{q \in L_G} \ell_1(\pi_0(D) - q)/2 - 1$ using Kannan’s algorithm. The $\ell_1$ unit ball is given to Kannan’s algorithm as a separation oracle (we will provide an efficient implementation of the separation oracle in Lemma 14).
2. We minimise over all permutations $\pi$.

We now turn to the general case: we start with the geometric interpretation for rank and we would like to reduce the problem to the integer programming algorithm. We construct a preliminary algorithm as follows:

**Algorithm 3.** 1. Find the smallest integer $r$ such that $\pi_0(D)$ is contained in $\cup_{c \in \text{Crit}_{\ell_1}(L_G)} \mathcal{P}_{r_1,r_2}(c)$ where $r_1 = \frac{r}{n+1}$, $r_2 = \frac{r+g-1+d}{n+1}$ by testing for all values of $r$ from zero to $g - 1$.

Using the fact that the degree of the divisor is between zero and $g - 1$, the algorithm would run in time $O(g \cdot 2^{O(n \log n)} \cdot \text{poly(size(G)))}$. Since $g$ is not polynomially bounded in the size of the input, the algorithm does not run in polynomial time for fixed values of $n$. We resolve this problem by performing a binary search over the parameter $r$ in the polytope $\mathcal{P}_{r_1,r_2}(c)$ and apply Kannan’s algorithm at each step of the binary search. Since we know from Subsection 3.1 that the rank of the divisor is at most $g - 1$ the algorithm terminates in $O(2^{n \log n} \text{poly(size(G))})$. Here is a formal description of the algorithm:

**Algorithm 4.** 1. For each permutation $\pi \in S_{n+1}$, use binary search on the parameter $r$ along with Kannan’s algorithm to test if $\pi_0(D)$ is contained in $\cup_{q \in L_G} \mathcal{P}_{r,r+g-1-d}(c_n + q)$, the polytope is presented to Kannan’s algorithm as a separation oracle.
2. Repeat over all permutations $\pi$.

The straightforward way of presenting the polytope $\mathcal{P}_{r,r+k}$ to Kannan’s algorithm is in terms of its facets. But since the number of facets of $\mathcal{P}_{r,r+k}$ is $2^{n+1}$, the factor depending on $n$ in the time complexity of the algorithm becomes larger than $2^{n \log n}$. Hence, we present the polytope $\mathcal{P}_{r,r+k}$ by a separation oracle to Kannan’s algorithm and the following efficient implementation of the separation oracle ensures that the algorithm runs in time $2^{O(n \log n)} \text{poly(size(G))}$.

**Lemma 14.** (A separation oracle for the polytope $\mathcal{P}_{m,n}$) There is a polynomial time separation oracle for the polytope $\mathcal{P}_{r,r+d}$ i.e., given any point $p$ there is a polynomial time (in the bit length of $p$ and the vertex description of $\mathcal{P}_{r,r+d}$) algorithm that either decides that $p$ is contained in $\mathcal{P}_{r,r+d}$ or outputs a hyperplane separating the point $p$ and the polytope $\mathcal{P}_{r,r+d}$.
Lemma 15. Observe that

\[ \text{Proof.} \]

the vertices of \( \triangle \) dimension is fixed. \( \) we do not have an explicit description of the extremal points as we have in the case of Laplacian lattices. \( \) polynomial time even when the dimension of the lattice is fixed. In our algorithm we crucially exploit \( \) sublattice of the root lattice \( \) As described in \( \) the notion of rank of a divisor can also be defined for an arbitrary definition it can written as

\[ R \]

1. If \( \) We now summarise the results that we obtained in the previous section to obtain an algorithm for

3.5 The Algorithm

We now summarise the results that we obtained in the previous section to obtain an algorithm for computing the rank of a divisor.

Algorithm 5. 1. If \( \) output \( \) if \( \) then, set \( \)

2. If \( \) set \( \) and compute \( \) Output \( \)\[
\]
3. If \( \) then \( \)

4. If \( \) then we invoke Algorithm \( \) to compute \( \).

Remark 5. As described in \( \) the notion of rank of a divisor can also be defined for an arbitrary sublattice of the root lattice \( \) In such a general setting, we do not know if rank can computed in polynomial time even when the dimension of the lattice is fixed. In our algorithm we crucially exploit our knowledge of the extremal points and the problem with handling the general case is that we do not have an explicit description of the extremal points as we have in the case of Laplacian lattices. As a consequence, we do not know how to find the extremal points in polynomial time even when the dimension is fixed.

The correctness of the algorithm is clear from Theorem 5.

Theorem 6. For any divisor \( D \) with degree between zero to \( g-1 \), Algorithm \( \) computes the rank of the divisor \( D \).

Theorem 7. Algorithm \( \) runs in time \( 2^{O(n \log n)} \text{poly(size}(G)) \) and hence, runs in polynomial time for a fixed number of vertices.

Proof. The first step in the algorithm takes time \( O(\ln(g)2^{O(n \log n)} \text{poly(size}(G)) \) since a separation oracle for \( \mathcal{P}_{m,n} \) can be constructed in polynomial time in the size of \( G \) and Kannan’s algorithm takes \( 2^{O(n \log n)} \text{poly(size}(G)) \) and we iterate \( 2^n \log n \) times. Since \( \ln(g) \) is polynomially bounded in the size of the input, the time complexity of the algorithm is \( O(2^{O(n \log n)} \text{poly(size}(G)) \).

We will end this section by determining the vertices of the polytope \( \mathcal{P}_{m,n} \).

Lemma 15. The vertices of \( \mathcal{P}_{R_1,R_2} \) are of the form \( w_{i,j} = R_1 t_i - R_2 t_j \) for \( i \neq j \) where \( t_0, \ldots, t_n \) are the vertices of \( \triangle \).

Proof. Observe that \( w_{i,j} \) is contained in \( \mathcal{P}_{R_1,R_2} \) for all pairs \( i, j \) from 0 to \( n \). We now show that \( \mathcal{P}_{R_1,R_2} \) is contained in the convex hull of \( w_{i,j} \) where \( i, j \) vary from 0 to \( n \). Let \( p \) be a point in \( \mathcal{P}_{R_1,R_2} \) by definition it can written as \( R_1 \sum_{i=0}^{n} \lambda_i t_i - R_2 \sum_{i=0}^{n} \sigma_i v_i \) with \( \lambda_i \geq 0 \), \( \sigma_i \geq 0 \) for \( i \) from 0 to \( n \) and \( \sum_{i=0}^{n} \lambda_i = \sum_{i=0}^{n} \sigma_i = 1 \). We let \( \ell_{ij} = \lambda_i \sigma_j \) and write \( p = \sum_{i=0}^{n} \sum_{j=0}^{n} \ell_{ij} w_{ij} \). We now verify that \( \sum_{i,j} \ell_{ij} = 1 \) and \( \ell_{ij} \geq 0 \). This shows that \( \mathcal{P}_{R_1,R_2} \) is contained in the convex hull of \( \{ w_{i,j} \}_{i,j} \).

We now show that \( w_{i,j} \) are not vertices of \( \mathcal{P}_{R_1,R_2} \) since \( w_{i,j} \) is contained in \( \triangle(O,R_1 - R_2) \) and \( \triangle(O,R_1 - R_2) \) is contained in \( \triangle(O,R_1) \) if \( R_1 \geq R_2 \) and is contained in \( \triangle(O,R_2) \) otherwise. To
conclude the proof of the lemma, it suffices to show that \( w_{i,j} \) is a vertex if \( i \neq j \). To this end, we consider the linear function \( f_{i,j} = x_i - x_j \) and note that \( w_{i,j} \) is the unique maximum of \( f_{i,j} \) among all points in the set \( \{ w_{i,j} \}_{i,j} \).

\[ \square \]

4 A duality theorem and its generalisations

We obtain another proof of the fact that testing if a divisor \( D \) is effective or not i.e., testing if \( r(D) \geq 0 \) is contained in \( NP \cap co - NP \). Although, this fact is already implicit in the algorithm of Tardos, our proof motivates a generalisation of the duality theorem which will see in the next subsection.

An interesting characterisation of divisors of negative rank (i.e., rank equal to minus one) is the following theorem first established in [2]:

**Theorem 8.** A point \( D \) has rank minus one if and only if it dominates a point in \( Ext(L_G) \).

Theorem 8 was further generalised to arbitrary full dimensional sublattices of \( H_0 \) in [1]. We will state a “projected” version of the theorem here. Let \( L \) be a full-dimensional sublattice of \( H_0 \). For any real number \( t > 0 \), define the arrangements \( A_t = \cup_{q \in L} \Delta(q,t) \) and \( B_t = \cup_{c,t} \Delta(c,t) \) (see Subsection 2.5 for the definition). We have the following duality theorem for the arrangements \( A_t \) and \( B_t \).

**Theorem 9.** (A Duality Theorem for the arrangement of Simplices) Let \( L \) be a full dimensional lattice in \( H_0 \) for any real number \( 0 \leq t \leq Cov_{\Delta}(L) \), then the arrangements \( A_t \) and \( B_{Cov_{\Delta}(L) - t} \) tile \( H_0 \) i.e., \( int(A_t) \cup B_{Cov_{\Delta}(L) - t} = H_0 \) and \( int(A_t) \cap B_{Cov_{\Delta}(L) - t} = \emptyset \).

We now use Theorem 12 and the fact that in the case of Laplacian lattices, the set of Crit\( _{\Delta}(L_G) \) have a combinatorial interpretation help to obtain an efficient verification procedure for \( r(D) \geq 0 \).

**Theorem 10.** The problem of deciding if \( r(D) \geq 0 \) is contained in \( NP \cap co - NP \).

*Proof.* Suppose that \( r(D) \geq 0 \) then by definition there is a point \( q \in L_G \) such that \( D \geq q \). We can assume that \( q \) is contained in \( \Delta(\pi_0(D), Cov_{\Delta}(L_G)) \) and hence, \( q \) has a description that is polynomial in the description of \( D \) and \( Cov_{\Delta}(L_G) \). We know that \( Cov_{\Delta}(L_G) \) is the density of the graph (see [1]). The verifier then tests if \( D \geq q \) and whether \( q \in L_G \) i.e., and hence, if \( q \) can be written as an integral combination of the basis elements of \( L_G \). This test is the lattice membership problem and can be performed in polynomial time, see [9] for more details.

This shows that the problem of deciding if \( r(D) \geq 0 \) is in \( NP \). Conversely, to show that the problem of testing if \( r(D) < 0 \) is also contained in \( NP \), we use the structural theorem of Theorem 8. By Theorem 8 we know that if \( r(D) < 0 \) then there is a point \( c' \) in \( Crit_{\Delta}(L_G) \) of the form \( c' = c + q \) where \( q \in L_G \) such that \( c \in \Delta(\pi_0(D), Cov_{\Delta}(L_G)) \) and \( -deg(D) + 1 \). Since, \( c' \) is contained in \( \Delta(\pi_0(D), Cov_{\Delta}(L_G)) \) we know that \( c' \) has a description that is polynomial in the description of \( D \) and the description of \( G \). In order to verify that \( c' \) is of the form \( c' = c + q \) for some \( c \in S_{n+1} \) and \( q \in L_G \), the verifier also produces \( c' \) and \( q \). The verifier then tests if \( q \) is in \( L_G \) by testing if \( q \) can be written as an integral combination of a basis of \( L_G \), this test is known as the membership testing problem and can be performed in polynomial time and then test if \( c' - q \) is equal to the projection of \( (\text{indeg}_\pi(v_1) - 1, \text{indeg}_\pi(v_2) - 1, \ldots, \text{indeg}_\pi(v_{n+1}) - 1) \) onto \( H_0 \). Hence, deciding if \( r(D) < 0 \) is also contained in \( NP \).

\[ \square \]

4.1 Generalised Duality

Reformulating Corollary 2 in a more geometric language, the rank of the divisor is essentially the smallest possible radius of the arrangement of polytopes that are up to scaling the unit ball of the \( \ell_1 \)-norm and centered at points in \( Crit_{\Delta}(L_G) \) that contains \( \pi_0(D) \). A “duality theorem” similar to Theorem 9 for the
\[ \ell_1 \text{-unit ball instead of the simplex } \Delta \text{ will, under some mild assumptions, have similar implications as Theorem } 10 \text{ for the problem of testing if a divisor of degree } g - 1 \text{ has rank at most } c \text{ for any } c \text{ between } -1 \text{ and } g - 1. \text{ This observation also raises the problem of extending the notion of duality from the regular simplex } \Delta \text{ to a general polytope.} \]

**Definition 13. (Duality for polytopes)** The graph \( G \) is said to have duality with respect to the polytope \( P \) if there exists a star-shaped body \( P^* \) such that for \( R > 0 \), there exists an \( R' > 0 \) such that a point \( p \) is contained in the interior of the arrangement \( \cup_{c \in Crit(\Delta)} P(c, R) \) if and only if it is not contained in \( \cup_{q \in L_G} P^*(q, R') \).

**Remark 6.** Our imposition that \( P^* \) has to be star-shaped in the definition of duality is somewhat arbitrary and is motivated from the fact that in practice \( P^* \) happens to be star-shaped.

While we do not know of tools to obtain duality theorems with respect to polytopes in general we will now sketch a general strategy to prove duality theorems for simplices. Here are the main ingredients:

(I1). From the results in [1], we know that the elements of \( Crit(\Delta) \) have a geometric interpretation namely as the local maximum of the simplicial distance function \( \Delta \) on the lattice \( L_G \). We use the fact that the set of local maxima of the simplicial distance function behave “well” under non-singular linear maps i.e., if \( M \) is a non-singular linear map then \( M(Crit(\Delta(L_G))) \) is the set of local maxima of \( M(L_G) \) under the simplicial distance function \( d_{M(\Delta)}. \)

(I2). Suppose that the linear map \( M \) fixes \( L_G \) and also \( Crit(\Delta(L_G)) \) as sets then we know that \( Crit(\Delta(L_G)) \) must be the set of local maxima of the simplicial distance function induced by the simplex \( M(\Delta) \) on \( L_G \). This motivates a new notion of automorphism of a graph i.e., the group of linear transformations that fix both \( L_G \) and \( Crit(\Delta(L_G)) \). Furthermore, the graph \( G \) has duality with respect to the simplex \( M(\Delta) \) with \( (M(\Delta))^* \) being \( -M(\Delta). \)

Let us now discuss each item of the general strategy in more detail.

**Lemma 16.** Let \( M \) be a non-singular linear transformation on \( H_0 \) and let \( L \) be a full dimensional lattice in \( H_0 \), then for every point \( p \in H_0 \), \( h_{\Delta,L}(p) = h_{M(\Delta),M(L)}(M(p)) \).

**Proof.** We know that there exists a lattice point \( q \in L \) such that \( h_{\Delta,L}(p) = d_{\Delta}(p,q) \). Hence, \( p \in \Delta(O,h_{\Delta,L}(p)) + q \). We have: \( M(p) \in \Delta(O,h_{\Delta,L}(p)) + M(q) \) and we have: \( d_{M(\Delta)}(M(p),M(q)) = h_{\Delta,L}(p) \) and hence, \( h_{M(\Delta),M(L)}(M(p)) \leq h_{\Delta,L}(p) \). Since, \( M \) is non-singular, \( M^{-1} \) is well defined and we apply the argument for a point in \( M(L) \) and the distance function \( d_{M(\Delta)} \) to deduce that \( h_{M(\Delta),M(L)}(M(p)) \geq h_{\Delta,L}(p) \).

Another result we use is the behaviour of arrangements of simplices under non-singular linear transformations. Let \( D \) be a discrete point set of \( H_0 \) and let \( A_{S,D,t} \) be the arrangement of simplices of type \( S \) on \( D \), more precisely \( A_{S,D,t} = \cup_{q \in D} S(q,t) \).

**Lemma 17.** For any non-singular linear transformation \( M \) on \( H_0 \), \( M(A_{D,S,t}) = A_{M(D),M(S),t} \). Furthermore, the map \( M \) preserves the interior of the arrangements i.e., \( M(\text{int}(A_{D,S,t})) = \text{int}(A_{M(D),M(S),t}) \).

**Proof.** By definition: \( M(A_{D,S,t}) = M(\cup_{q \in D} S(O,t) + q) = (\cup_{q \in D} M(S(O,t)) + q) = (\cup_{q \in D} M(S(O,t))) + M(q) = A_{M(D),M(S),t} \). Now, suppose \( p \) is a point in the interior of the arrangement \( A_{D,S,t} \) then we know that there exists \( \epsilon > 0 \) such that \( B(p, \epsilon) \) in the arrangement \( A_{D,S,t} \). Applying the linear transformation \( M \), we know that \( M(B(p, \epsilon)) \) is contained in the arrangement \( M(A_{D,S,t}) = A_{M(D),M(S),t} \). Now, we know that \( M(B(p, \epsilon)) \) contains the ball \( B(M(p), \sqrt{\lambda(M)} \epsilon) \) where \( \lambda(M) = \inf(||M(x)||^2 ||x||^2 = 1, x \in H_0 \) since \( M \) is non-singular we know that \( \lambda(M) > 0 \) and hence, \( M(p) \) is contained in the interior of \( A_{M(D),M(S),t} \) and similarly, if we apply the same argument for a point \( p' \) in the interior of \( A_{M(D),M(S),t} \) then \( M^{-1}(p) \) is a point in the interior of \( A_{D,S,t} \). Hence, \( M(\text{int}(A_{D,S,t})) = \text{int}(A_{M(D),M(S),t}) \).

**Corollary 4.** Let \( M \) be a non-singular linear transformation on \( H_0 \) and let \( L \) be a full dimensional lattice in \( H_0 \), then \( Cov_{\Delta,L} = Cov_{M(\Delta)}(M(L)) \).

**Remark 7.** The second part of Lemma 17 is essentially a topological fact and in topological terms we use the fact a non-singular linear transformation from \( H_0 \) to itself is a homeomorphism from \( H_0 \) to itself where \( H_0 \) is equipped with the Euclidean topology.
Remark 8. Note that we did not use our assumption that the set is a discrete point set anywhere in the proof of Lemma 17 and can be generalised to non-discrete sets but since we mainly deal with discrete sets we restrict to them.

Theorem 11. Let $M$ be a non-singular linear transformation on $H_0$ and let $L$ be a full dimensional lattice in $H_0$, the set of local maxima of the function $h_{M(\Delta),M(L)}$ is precisely the set $M(Crit_{\Delta}(L))$.

Proof. Suppose $c$ is an element of $Crit_{\Delta}(L)$, we know that there exists an $\epsilon > 0$ such that $h_{\Delta,L}(c) \geq h_{\Delta,L}(c')$ for all $c' \in B(c,\epsilon)$. Now by Lemma 16, we know that $h_{M(\Delta),M(L)}(c) \geq h_{M(\Delta),M(L)}(c'')$ for all $c'' \in M(B(c,\epsilon))$. The set $M(B(c,\epsilon))$ is an ellipsoid that contains the ball $B(M(c),\epsilon \sqrt{\lambda(M)})$ where $\lambda(M)$ is inf$\{ ||x||_2^2 \ | \ |x|^2 = 1, x \in H_0 \}$ and $M$ is non-singular we have $\lambda(M) > 0$. Hence, $M(Crit_{\Delta}(L)) \subseteq Crit_{M(\Delta)}(M(L))$. Since, $M$ is a non-singular linear map, we consider the map $M^{-1}$ and apply the above argument for a point in $Crit_{M(\Delta)}(M(L))$ to deduce that $Crit_{M(\Delta)}(M(L)) \subseteq M(Crit_{\Delta}(L))$. \qed

We will now generalise the duality theorem with respect to any full dimensional simplex on $H_0$ with centroid at the origin.

Theorem 12. Let $S$ be a full dimensional simplex on $H_0$ with centroid at the origin and let $A_{L,S,t}$ be the arrangement $\cup_{q \leq L} - S(q,t)$ and let $B_{L,S,t}$ be the arrangement $\cup_{c \in Crit\_S(L)} S(c,t)$. Fix a real number $0 \leq t \leq Cov_{\Delta}(L)$, the arrangements $A_{L,S,t}$ and $B_{L,S,Cov\_S(L)-t}$ tile $H_0$ i.e., $A_{L,S,t} \cup B_{L,S,Cov\_S(L)-t} = H_0$ and $int(A_{L,S,t}) \cap B_{L,S,Cov\_S(L)-t} = \emptyset$.

Proof. Since $S$ is a full-dimensional simplex with centroid at the origin we know that there is a non-singular map $M$ that takes $S$ to the simplex $\Delta$. Furthermore, by Corollary 14 we know that $Cov_{-S}(L) = Cov_{\Delta}(M(L))$ and hence $0 \leq t \leq Cov_{\Delta}(M(L))$ and we can apply Theorem 9 to the lattice $M(L)$, a full dimensional lattice in $H_0$ and with the simplex $\Delta$, that a point $p \in H_0$ is contained exclusively in either exclusively the interior of the arrangement $A_{M(L),\Delta,t}$ or the arrangement $B_{M(L),\Delta,Cov_{\Delta}(M(L))-t}$. Now note that $M(L)$ and $Crit_\Delta(M(L))$ are discrete sets and apply Lemma 17 with the linear transformation $M^{-1}$. Hence, we know that $int(A_{L,S,t}) \cup B_{L,S,Cov\_S(L)-t} = H_0$ and $int(A_{L,S,t}) \cap B_{L,S,Cov\_S(L)-t} = \emptyset$ since, if there is a point $p \in H_0$ in $int(A_{L,S,t}) \cap B_{L,S,Cov\_S(L)-t}$ then $M^{-1}(p)$ is contained in $int(A_{M(L),\Delta,t}) \cap B_{M(L),\Delta,Cov_{\Delta}(M(L))-t}$ which is a contradiction. \qed

4.2 The Automorphism Group of $Crit_\Delta(L)$

We now discuss the second item of the general strategy, the notion of automorphisms:

Definition 14. (Automorphism of a lattice) For a lattice $L$, a non-singular linear map $M$ on the linear space spanned by the elements of $L$ is called an automorphism of $L$ if it induces a bijection on $L$.

Here is a concrete characterisation of the automorphism group of a lattice:

Lemma 18. Let $B$ be an arbitrary basis of a $n$-dimensional lattice $L$ written as a matrix with the basis elements row-wise. A linear transformation $M$ is an automorphism of the lattice $L$ if and only if it $M(B) = T \cdot B$ where $T \in SL(n,\mathbb{Z})$.

The set of automorphisms of a lattice has a natural group structure under composition (if you think of the linear maps as matrices then the group operation is matrix multiplication) and this group is called the automorphism group of $L$.

Definition 15. (Automorphism of $Crit_\Delta(L)$) Let $L$ be a sublattice of $H_0$, a non-singular linear map $M$ on the linear space spanned by the elements of $L$ is called an automorphism of $Crit_\Delta(L)$ if it induces a bijection on $Crit_\Delta(L)$.

The set of automorphisms of $Crit_\Delta(L)$ has a natural group structure under composition (if you think of the linear maps as matrices then the group operation is matrix multiplication). The group gives rise to a graph invariant that we call the critical automorphism group of the graph.
Remark 9. Note that the notion of a critical automorphism group can be developed more generally, for any distance function induced by a convex body \( P \) on a lattice \( L \), we can define the group of transformations that fix both \( L \) and \( \text{Crit}_P(L) \) as sets.

A convenient way to think of the automorphisms of \( \text{Crit}_\triangle(L_G) \) is to consider \( \text{Crit}_\triangle(L_G)/L_G \) as a subset of the torus \( T_G = H_0/L_G \) and observe the automorphisms of \( L_G \) induce a bijection of the torus \( T_G \) to itself; among these bijections the automorphisms of \( \text{Crit}_\triangle(L_G) \) are those that take the set \( \text{Crit}_\triangle(L_G)/L_G \) to itself. Let us now give some concrete examples of \( \text{Crit}_\triangle(L_G) \):

1. For a regular graph \( G \), we have \( \text{Crit}_\triangle(L_G) = -\text{Crit}_\triangle(L_G) \). Hence, the linear transformation \( M(x) = -x \) is an example of an automorphism of \( \text{Crit}_\triangle(L_G) \).

2. The permutation map induced by every automorphism of \( H_0 \) also is indeed an automorphism of \( \text{Crit}_\triangle(L_G) \); but since permutation maps fix the simplex \( \triangle \), these are not useful to obtain simplices other than \( \triangle \) for which duality holds. Note that these automorphism take one class in \( \text{Crit}_\triangle(L_G) \) to another.

3. Every \( c \in \text{Crit}_\triangle(L_G) \) where \( c = c_\pi + q \) and \( q \in L_G \). We can write each \( c_\pi \) as a rational linear combination of \( \{b_1, \ldots, b_n\} \). Define the height of \( c_\pi \) as the least common multiple of the denominators of these rational coefficients in their reduced form and define the height of \( \text{Crit}_\triangle(L_G) \) as least common multiple of the ratio of the denominators of the height of each \( c_\pi \). A linear transformation \( M(b_i) = b_i + H(\sum_{j<i} \alpha_j b_j) \) is an automorphism of \( L_G \) where \( H \) is a multiple of the height of \( \text{Crit}_\triangle(L_G) \) and \( \alpha_j \)'s are arbitrary integers is an automorphism of \( \text{Crit}_\triangle(L_G)/L_G \). These automorphisms takes an equivalence class \( \text{Crit}_\triangle(L_G)/L_G \) to itself.

The usefulness of the critical automorphism group arises from the following partial characterization of simplices (with centroid at the origin) for which duality holds:

Theorem 13. Let \( G \) be an undirected connected graph, duality holds with respect to a simplex \( S \) with centroid at the origin if there is a linear map \( M \) taking \( \triangle \) to \( S \) that belongs to the automorphism group of \( \text{Crit}_\triangle(L_G) \).

Remark 10. We do not know if the converse of Theorem 13 also holds, the main difficulty in proving the converse is to show that if \( S \) has duality then \( \text{Crit}_\triangle(L_G) \) is the precisely the set of local maxima under the distance function \( d_S \).

The fact that an automorphism \( M \) of \( \text{Crit}_\triangle(L_G) \) fixes \( \text{Crit}_\triangle(L_G) \) combined with the Theorem 12 means that the graph \( G \) has a duality with respect to the simplex \( M(\triangle) \) with the dual \( (M(\triangle))^* \) being \(-M(\triangle)\).

### 4.3 The case of complete graphs

We will study the critical automorphism group of the complete graph. Let \( b_0, \ldots, b_n \) be the rows of the Laplacian matrix of the complete graph \( K_{n+1} \). The extremal points are permutations of the point \((-1, 0, 1, \ldots, n)\) of the coordinates translated by a lattice point i.e., a point in the Laplacian lattice of \( K_{n+1} \). Since, the points in \( \text{Crit}_\triangle(L_G) \) are orthogonal projections of the extremal points onto the hyperplane \( H_0 = (1, \ldots, 1)^\perp \), they are points of the form \( c_\pi + q \) where \( c_\pi = \pi((n/2, n/2 - 1, n/2 - 2, \ldots, -n/2)) \), where \( \pi \) is a permutation. Now, since we now that up to equivalence modulo \( L_G \) there are only \( n! \) class of \( \text{Crit}_\triangle(L_G) \) and that permutations such that \( \pi(n) = n \) form representatives of the set \( \text{Crit}_\triangle(L_G)/L_G \). We now consider permutations such that \( \pi(n) = n \) and for such a permutation, we can rewrite \( c_\pi \) in the basis of \( \{b_0, \ldots, b_n\} \) as

\[
c_\pi = \frac{\left(\sum_{j=0}^{n-1} (j+1) \cdot b_\pi(j)\right)}{n}. \tag{8}
\]

Hence, the height of \( \text{Crit}_\triangle(K_{n+1}) \) is \( n \). We can use the exploit Equation 8 to obtain a better formula for rank. We will now construct critical automorphisms of \( K_{n+1} \):
Lemma 19. Every map of the form \( b_i = b_{\pi(i)} \) where \( \pi \) is a permutation is an element of the critical automorphism group.

More generally, we have the following result:

Theorem 14. A map \( M \) that takes \( b_i \) to \( b_{\pi(i)} + H \cdot q_i \) for \( i \) from 0 to \( n - 1 \) where \( \pi \) is a permutation and \( H \) is an integer that divides \( n \), the height of \( K_{n+1} \) and \( q_i \) is contained in the sublattice spanned by \( b_{\pi(0)}, \ldots, b_{\pi(i-1)} \) i.e., \( q_i = \sum_{j=0}^{i-1} \alpha_{ij} b_{\pi(j)} \) where \( \alpha_{ij}s \) are integers is an element of the critical automorphism group of \( K_{n+1} \).

Proof. First, we show that the map takes \( L_{K_{n+1}} \) to itself as follows: we write \( b_{\pi(i)} \) for \( i \) from 0 to \( n \) as an integral combination of \( M(b_{\pi(0)}), \ldots, M(b_{\pi(n)}) \) as follows: \( b_{\pi(0)} = M(b_i) \) and for \( i \) from 1 to \( n - 1 \) we have:

\[
b_{\pi(i)} = M(b_i) - H \sum_{j=0}^{i-1} \sum_{k=0}^{i-1} \alpha_{jk} M(b_j)
\]

Hence, \( M \) takes a basis of \( L_{K_{n+1}} \) to another basis of \( L_{K_{n+1}} \) and hence takes \( L_{K_{n+1}} \) to itself.

Now, to show that \( M \) takes \( \text{Crit}_\Delta(L_{K_{n+1}}) \) to itself we consider \( M(c_\pi) \) and form Equation [8] we have:

\[
M(c_\pi) = \left( \sum_{i=0}^{n-1} (j+1) \cdot M(b_{\pi(i)}) \frac{n}{n} \right) = \left( \sum_{i=0}^{n-1} (j+1) \frac{n}{n} M(b_{\pi(i)}) + H \cdot q_{\pi(i)} \right).
\]

Now since \( H \) divides \( n \) we know that \( M(c_\pi) \) is mapped to \( c_\sigma \pi + q' \) for some lattice point \( q' \) and since \( M \) takes \( K_{n+1} \) to itself it takes \( M(c_\pi + L_{K_{n+1}}) \) to \( c_{\sigma \pi} + L_{K_{n+1}} \). Furthermore, since \( \pi \) and \( \sigma \) are permutations, \( M \) induces a permutation on equivalence classes in \( \text{Crit}_\Delta(L_{K_{n+1}})/L_{K_{n+1}} \), represented by \( c_\pi \) with \( \pi(n) = n \). These arguments show \( M \) takes \( \text{Crit}_\Delta(L_{K_{n+1}}) \) to itself and hence \( M \) is an element of the critical automorphism group of \( K_{n+1} \).

\[ \square \]

Remark 11. Note that Theorem [4] provides a construction of elements of the critical automorphism group of \( K_{n+1} \), but the construction is not exhaustive in the sense, there are other elements of the critical automorphism group of \( K_{n+1} \) that are not constructed by Theorem [4].

4.4 Concluding remarks

A natural outgrowth of our work would be to obtain a singly exponential algorithm for computing the rank and furthermore, to resolve the complexity of computing the rank i.e., to determine whether it is NP-hard or it is computable in polynomial time. The bottleneck in our algorithm is the enumeration of all permutations. We believe that another interesting direction would be obtain duality theorems for polytopes generalising the results that we obtained on simplices.

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