Bounds on the number of time steps for simulating arbitrary interaction graphs

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September 26, 2002

Abstract

In previous papers we have considered mutual simulation of $n$-partite pair-interaction Hamiltonians. We have focussed on the running time overhead of general simulations, while considering the required number of time steps only for special cases (decoupling and time-reversal). These two complexity measures differ significantly. Here we derive lower bounds on the number of time steps for general simulations. In particular, the simulation of interaction graphs with irrational spectrum requires at least $n$ steps. We discuss as examples graphs that correspond to graph codes and nearest neighbor interactions in 1- and 2-dimensional lattices. In the latter case the lower bounds are almost tight.

1 Introduction

Simulating Hamiltonian evolutions of arbitrary quantum systems on a quantum computer is an idea that goes back already to Feynman [1]. This would be an attractive application of future quantum computers since there are no known efficient classical algorithms for simulating generic dynamics of many particle systems. Here a quantum computer can be any quantum system provided that its time evolution can be controlled in a universal way. In particular, the problem of controllability includes the question which Hamiltonian evolutions can be simulated efficiently by the considered Hamiltonian system [2–13] Assume the natural Hamiltonian $H$ to act on an $n$-fold tensor product Hilbert space

$$\mathcal{H}_n := \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$$

where each $\mathcal{H}$ denotes the Hilbert space of a qudit, i.e. a $d$-dimensional system. The Lie algebra $su(d)$ of traceless Hermitian operators on $\mathcal{H}$ is a $m := (d^2 - 1)$-dimensional real vector space. Let $B := \{\sigma_\alpha \mid \alpha = 1, \ldots, m\}$ be an orthogonal basis of $su(d)$ with respect to the trace inner product $\langle A | B \rangle := tr(A^\dagger B)/d$ for $A, B \in su(d)$.

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Then the most general qudit-qudit interaction on \( n \) coupled qudits is given by

\[
H_J := \sum_{k<l} \sum_{\alpha\beta} J_{k\alpha l\beta} \sigma^{(k)}_{\alpha} \sigma^{(l)}_{\beta},
\]

where \( J \) is chosen to be a real symmetric \( mn \times mn \)-matrix with zeros for \( k = l \). Note that the symmetry of the coupling matrix \( J \) does not imply any physical symmetry of the interaction. It is a consequence of our redundant representation that turns out to be very useful. The coupling matrix \( J \) consists of \( m \times m \)-blocks. The \( m \times m \)-matrix \( J_{kl} \) given by the block at position \((k, l)\) describes the coupling between the qudits \( k \) and \( l \). We have \( J_{lk} = J^T_{kl} \), i.e. the matrix describing the coupling between the qudits \( l \) and \( k \) is just the transpose of the matrix describing the coupling between \( k \) and \( l \). The blocks on the diagonal are zero matrices.

In the setting discussed here and in most other articles on simulation of Hamiltonians the only possibilities of external control are given by local unitaries on each qudit. We assume that one is able to implement them independently. Formally, all control operations are elements of the group \( C := SU(d) \otimes SU(d) \otimes \cdots \otimes SU(d) \). A common approximation is to assume that all operations in \( C \) can be implemented arbitrarily fast ("fast control limit"). The simulation of Hamiltonians is based on the following "average Hamiltonian" approach that has successfully been used for describing Nuclear Magnetic Resonance experiments since many years (e.g. \cite{14, 15}).

Let \( t_1, t_2, \ldots, t_n \) be positive numbers and \( u_1, u_2, \ldots, u_N \in C \) be control operations. Then the algorithm

\[
\text{perform } u_1, \text{ wait } t_1, \text{ perform } u^\dagger_1, \text{ perform } u_2, \text{ wait } t_2, \text{ perform } u^\dagger_2, \ldots, \text{ perform } u^\dagger_N, \text{ wait } t_N, \text{ perform } u_N
\]

implements approximatively the unitary evolution

\[
\prod_{j=1}^{N} \exp(u_j^\dagger H u_j t_j).
\]

If the times \( t_j \) are small compared to the time scale of the natural evolution according to the natural Hamiltonian \( H \) this is approximatively the evolution according to the average Hamiltonian

\[
\sum_j t_j u_j H u_j^\dagger / \tau ,
\]

where \( \tau := \sum_j t_j \) is the slow down factor of the evolution, i.e., the time overhead of the simulation. For investigating the time overhead and the number of time steps of a simulation it turns out to be useful to work with the coupling matrices instead of considering the Hamiltonians themselves. To express the effect of the control operations on the coupling matrix \( J \) note that any unitary operation \( u \in SU(d) \) corresponds to a rotation on the \( m \)-dimensional sphere via the relation

\[
u^\dagger \left( \sum_{\alpha} c_\alpha \sigma_\alpha \right) u = \sum_{\alpha} \tilde{c}_\alpha \sigma_\alpha ,
\]

where the vector \( \tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_m) \) is obtained by applying a rotation \( U \in SO(m) \) on the vector \( c = (c_1, c_2, \ldots, c_m) \). It is straightforward to verify that conjugation of \( H_J \)
by \( v := u^{(1)} \otimes u^{(2)} \otimes \cdots \otimes u^{(n)} \) corresponds to conjugation of \( J \) by a block diagonal matrix of the form

\[
V := U^{(1)} \oplus U^{(2)} \oplus \cdots \oplus U^{(n)} \in \bigoplus_{k=1}^{n} SO(m) .
\]

The condition for correct simulation is hence given by

\[
\tilde{J} = \sum_{j} t_{j} V_{j} J V_{j}^{T} ,
\]

where the orthogonal matrix \( V_{j} \) corresponds to the unitary \( v_{j} \) for \( j = 1, \ldots, N \).

The question of optimal simulation has been completely solved so far only for the case of two-qubit Hamiltonians \([7, 16]\). Optimal simulation protocols are constructed in \([16]\). The number of time steps is between 1 and 3.

For \( n \) qudits we have shown in \([6, 17, 11]\) that the eigenvalues of the coupling matrix \( J \) and \( \tilde{J} \) provide lower bounds on the simulation time overhead. Graph theoretical notions can also provide upper bounds. In certain cases the bounds are known to be tight \([6, 17]\).

Here we do not focus on the time overhead but on the number of time steps in the \( n \) partite case. Upper bounds for decoupling (switching off the Hamiltonian) and time-reversal (simulating of \(-H\) by \( H\)) are directly given by the parameters of known schemes \([3, 4, 8, 11]\). Lower bounds for time-reversal have been derived in \([17]\). These results shown that time overhead and the number of time steps are not connected in any obvious way. Here we address the general problem of mutual simulation of pair-interactions on \( n \) qudits.

In Section 2 we derive lower bounds on the number of time steps for the general problem of mutual simulation of pair-interactions. The type of coupling is assumed to be the same between all nodes, but the strengths and the signs may vary. Like the bounds on the time overhead \([3, 4, 11]\), the bound derived here make us of the spectrum of the corresponding coupling matrices. In the special case that one wants to cancel some interactions while keeping others (“the simulation of a certain interaction graph”) the bounds refer directly to the spectra of the corresponding adjacency matrices. Surprisingly, the greatest lower bound can be given if the smallest eigenvalue of the adjacency matrix is irrational. In Section 3 the bounds shall be applied to three different cases where long range interactions are present between all spins. The first two cases are the simulation of nearest neighborhood coupling in a square lattice and a cyclic chain. The third case is a certain interaction graph that has been proposed to prepare the states of a graph code \([18, 19, 20]\).

## 2 Lower bounds on the number of time steps

We restrict our attention to interactions with an additional symmetry, namely Hamiltonians of the following form

\[
H := \sum_{k<l} w_{kl} \sum_{\alpha \beta} c_{\alpha \beta} \sigma^{(k)}_{\alpha} \sigma^{(l)}_{\beta} .
\]
The matrix \( W := (w_{kl}) \) is a real symmetric \( n \times n \)-matrix with zeros on the diagonal. It describes the coupling strengths and the signs of the interactions between all qudits. The matrix \( C = (c_{\alpha \beta}) \) is a real symmetric \( m \times m \)-matrix characterizing the type of the coupling. This means that all qudits interact with each other via the same interaction and that only the coupling strengths and the signs vary. It is important that in this special case the coupling matrix \( J \) can be expressed as a tensor product of \( W \) and \( C \), i.e., \( J = W \otimes C \).

To derive a general lower bound for simulating arbitrary interactions \( \tilde{J} \) by a tensor product interaction \( J = W \otimes C \) it is useful to observe that the decisive condition in eq. (2) is invariant with respect to the following rescaling of interactions: Multiply each \( m \times m \)-block \( k, l \) of \( J \) and \( \tilde{J} \) with the same factor \( r_{kl} \). In the case \( W \otimes C \) we can therefore assume w.l.o.g. that \( W \) is a matrix with only 1 as non-diagonal entries as long as we restrict our attention to Hamiltonians with complete interaction graphs, i.e., \( w_{kl} \neq 0 \) for \( k \neq l \). Formally, rescaling is denoted as follows. Let \( A \otimes B \) be the entry-wise quotient of the matrices \( A \) and \( B \) provided that \( A \) has only zero entries at those positions where \( A \) has also a zero. Then we consider the simulation of \( \tilde{J}/(W \otimes I) \) by \( K \otimes C \) where \( K \) is the matrix with only one as non-diagonal entries and 0 in the diagonal and \( I \) is the matrix with only 1 as entries.

In order to emphasize that the assumption \( w_{kl} = 1 \) is not an assumption on the real physical coupling strength we formulate the following theorem for general \( W \) and use rescaling only in the proof.

**Theorem 1 (Lower bound)** Let \( J := W \otimes C \) be the coupling matrix of the system Hamiltonian, \( \tilde{J} \) an arbitrary coupling matrix of the interaction that is simulated, and \( \mu \) the time overhead. Denote the minimal and maximal eigenvalues of \( C \) by \( \lambda_{\min} \) and \( \lambda_{\max} \), respectively and its rank by \( r(C) \). Let \( I \) be the \( m \times m \)-matrix whose all entries are 1. Let \( s \) be the number of eigenvalues of \( \tilde{J}/(W \otimes I) \) that are not contained in the interval

\[
I := [-\mu \lambda_{\max}, -\mu \lambda_{\min}].
\]

Then the number of time steps required to simulate \( H_{\tilde{J}} \) by \( H_J \) is at least \( s/r(C) \).

**Proof:** The condition for a scheme \( t_1, V_1, t_2, V_2, \ldots, t_N, V_N \) to be a simulation of \( \tilde{J} \) reads

\[
\sum_{j=1}^{N} t_j V_j (W \otimes C) V_j^T = \tilde{J}.
\]  

(4)

Since the matrices \( V_j \) are block-diagonal we can rescale each \( m \times m \)-block such that we obtain

\[
\sum_{j=1}^{N} t_j V_j (K \otimes C) V_j^T = \tilde{J}/(W \otimes I).
\]

In the following we denote the rescaled coupling matrix \( \tilde{J}/(W \otimes I) \) by \( J' \).

Set \( R := \sum_{j=1}^{N} t_j V_j (1 \otimes C) V_j^T \). By adding \( R \) on both sides we obtain

\[
\sum_{j=1}^{N} t_j V_j ((K + 1) \otimes C) V_j^T = J' + R.
\]  

(5)
The rank of the matrix \((K + 1)\) is 1 since all its entries are 1. Consequently, the rank of the left hand side of eq. \((5)\) is at most \(Nr(C)\).

Set \(a := \mu \lambda_{\min}\) and \(b := \mu \lambda_{\max}\). Note that the eigenvalues of \(R\) are contained in the interval \([a, b]\). The rank of the right hand side is at least the number of eigenvalues of \(J'\) outside the interval \(I = [-b, -a]\). This is seen as follows.

Let \(P\) and \(Q\) be the projections onto the sums of all eigenspaces of \(J'\) with eigenvalues smaller than \(-b\) and greater than \(-a\), respectively. Clearly, \(s\) is equal to the dimension of the image of \(P \oplus Q\). Denote by \(\tilde{J}\) and \(\tilde{R}\) the \(s \times s\)-submatrices of \(J'\) and \(R\) defined by \((P \oplus Q)J'(P \oplus Q)\) and \((P \oplus Q)R(P \oplus Q)\), respectively.

Due to the choice of \(P\) and \(Q\) the spectrum of \(\tilde{J}\) is contained in the interval \((-\infty, -b) \cup (-a, \infty)\). The spectrum of \(\tilde{R}\) is contained in the interval \([a, b]\) since the minimal (maximal) eigenvalues of a matrix cannot decrease (increase) when projecting the matrix.

We prove the theorem by showing that \(\tilde{J} + \tilde{R}\) has full rank. Set \(f := \frac{1}{2}(a + b)\). From the triangle inequality it follows that for every unit vector \(\langle\Psi\rangle \in \mathbb{R}^s\) one has
\[
\|\langle\tilde{J} + \tilde{R}\rangle\| = \|\langle\tilde{J} + f1 + \tilde{R} - f1\rangle\| \\
\geq \|\langle\tilde{J} + f1\rangle\| - \|\langle\tilde{R} - f1\rangle\|.
\]
The eigenvalues of the shifted operators \(\tilde{J} + f1\) and \(\tilde{R} - f1\) are contained in the intervals \((-\infty, -\frac{1}{2}(b - a)) \cup (\frac{1}{2}(b - a), \infty)\) and \([-\frac{1}{2}(b - a), \frac{1}{2}(b - a)]\), respectively. This implies that
\[
\|\langle\tilde{J} + f1\rangle\| - \|\langle\tilde{R} - f1\rangle\| > 0
\]
because the norms can be bounded by the absolute values of the eigenvalues: \(\|\langle\tilde{J} + f1\rangle\| > \frac{1}{2}(b - a)\) and \(\|\langle\tilde{R} - f1\rangle\| \leq \frac{1}{2}(b - a)\). This completes the proof.

The following upper bound can be easily derived from Carathéodory’s theorem \([21]\).

**Theorem 2 (Upper bound)** Every simulation that is possible can be achieved within
\[
\frac{n(n - 1)}{2}m^2 + 1
\]
time steps.

**Proof** If \(\tilde{J}\) can be simulated by \(J\) with time overhead \(\tau\) then \(\tilde{J}/\tau\) is in the convex span of the matrices \(V_jJV_j^T\) with notation as above. The dimension of this convex set is at most \(m^2n(n - 1)/2\) since the diagonal blocks are empty and each matrix \(V_jJV_j^T\) is symmetric. Carathéodory’s theorem states that each point in an \(M\) dimensional convex set can be written as a convex sum of at most \(M + 1\) extreme points.

There are interesting cases where the bound of Theorem \([1]\) can be tightened. Assume \(\tilde{J} = \tilde{W} \otimes C\) with the same matrix \(C\) as the interaction that is used for the simulation. In other words, only the strengths and the signs of some interactions should be changed. By the same rescaling trick as above we can consider the problem to simulate \((\tilde{W}/W) \otimes C\) by \(K \otimes C\) where \(K\) has only 1 as non-diagonal entries.

**Theorem 3 (Lower bound)** Let \(W \otimes C\) be the coupling matrix of the natural Hamiltonian and \(\tilde{W} \otimes C\) the coupling that we want to simulate. Assume all non-diagonal entries of \(W\) to be non-zero.

5
1. Let $C$ be a positive semidefinite matrix. Then the number of time steps is at least the number of positive eigenvalues of $\tilde{W}/W$.

2. Let $C = \text{diag}(1, 1, \ldots, 1)$ be the $m \times m$-identity matrix. Then the number of time steps is at least $n - k$, where $k$ is the multiplicity of the smallest eigenvalue $\mu_{\text{min}}$ of $\tilde{W}/W$.

3. Let the natural coupling be $C := \text{diag}(0, 0, 1)$, i.e., we have $\sigma_z \otimes \sigma_z$ interactions between all spin-1/2-particles. Let the set of local control operations be restricted to $i\sigma_x$-transformations. Then one requires at least $n - k$ time steps with $k$ as in Case 2. If $\mu_{\text{min}}$ is irrational then at least $n$ steps are necessary. In any case, $n(n - 1)/2 + 1$ time steps are always sufficient.

Note that Case (3) is of special interest since it deals with simulation procedures that do not rely on any first order approximation. In this case all summands in eq. (5) commute and the unitary transformation implemented by the simulation scheme coincides exactly with the exponent of the average Hamiltonian.

**Proof (of Theorem 3)**

**Case 1** This statement is a corollary of Theorem 1 since $\tilde{J}/(W \otimes I) = (\tilde{W}/W) \otimes C$. The number of positive eigenvalues of this tensor product matrix is $r(C)$ times the number of positive eigenvalues of $\tilde{W}/W$ if $r(C)$ is the rank of $C$.

**Case 2** Consider the rescaled problem to simulate $A \otimes C$ by $K \otimes C$. In this case the right hand side of eq. (5) reduces to

$$A \otimes C + \tau 1 ,$$

where $\tau := \sum_j t_j$ is the time overhead and $A := \tilde{W}/W$. In [11] we have shown that $\tau$ is at least $-\mu_{\text{min}}$. The reason is that the spectrum of $\tau K$ has to majorize the spectrum of $A$ since $A$ is a convex sum of conjugates of $\tau K$. Therefore the rank of the matrix in expression (5) is at least $m(n - k)$. Since the left hand side of eq. (5) has at most the rank $m N$ the number of time steps is at least $n - k$.

**Case 3** In this case one can use a more convenient formalism. We drop the matrix $C$ and characterize the interactions by $W$ and $\tilde{W}$. If a $zz$-interaction is conjugated by $i\sigma_x$-transformations only two possibilities occur. The interaction term between spin $k$ and spin $l$ acquires a minus sign if exactly one spin of both is subjected to a conjugation. The interaction is unchanged if either no spin or both spins are conjugated. Instead of representing the time step $j$ by the $3n \times 3n$ block diagonal matrix $V_j$ we can represent it by an $n \times n$ diagonal matrix $X_j$. The diagonal entries are $\pm 1$ and indicate which spins are subjected to conjugation. Then eq. (3) reduces to

$$\sum_{j=1}^{N} t_j X_j K X_j = A .$$

We add the identity matrix on both sides. Due to $X_j 1X_j = 1$ we obtain

$$\sum_j t_j X_j (K + 1) X_j = A + \sum_j t_j 1 .$$
The rank of the left hand side is at most the number $N$ of time steps. To estimate the rank of the right hand side note that the time overhead $\tau := \sum j t_j$ is at least $-\mu_{\min}$. Hence the dimension of the kernel of $A + \tau \mathbf{1}$ can at most be the multiplicity of the eigenvalue $\mu_{\min}$. This shows that the number of time steps is at least $n - k$.

Assume $\mu_{\min}$ to be irrational. Then the time overhead $\tau$ is necessarily greater than $-\mu_{\min}$. This can be seen as follows. The optimization with respect to the time overhead reduces to the following convex problem. Consider the matrix $X_j K X_j$ for an arbitrary time step $j$. Its non-diagonal entries are $\pm 1$ and indicate which interactions acquire a minus sign in $j$th step. In graph-theoretical language, the set of matrices that can occur as $X_j K X_j$ are exactly the Seidel matrices of complete bipartite graphs (see (3)). Then the optimal $\tau$ is the minimal positive number such that $A/\tau$ is in the convex span of the set of Seidel matrices of complete bipartite graphs. Geometrically, the convex span is a polytope having the Seidel matrices as its extreme points. It is embedded in the $n(n - 1)/2$ dimensional vector space of real symmetric matrices with zeros on the diagonal. Let $O$ be the origin. Consider the semi-line $\nu A$ for $\nu \geq 0$. Then the optimal simulation is the unique intersection point $P$ of the semi-line with the boundary of the polytope. The quotient of the distance between 0 and $A$ and between 0 and $P$ is the optimal time overhead. This quotient can never be irrational. The reason is that $P$ has rational entries since it is the solution of a linear equation over the field $\mathbb{Q}$ of rational numbers. Hence $\tau$ is greater than $-\mu_{\min}$ and $A + \tau \mathbf{1}$ has necessarily full rank. This proves that we need at least $n$ time steps.

To see that $n(n - 1)/2 + 1$ time steps are always sufficient we can argue as in the proof of Theorem 3 with Caratheodory’s theorem. The dimension of the convex span of the matrices $X_j K X_j$ is at most $n(n - 1)/2$. \hfill $\square$

In the following section we will consider the task to cancel some interactions and keep others. Then $A := \tilde{W}/W$ has only 1 and 0 as entries. In graph-theoretical language, it is the adjacency matrix of the desired interaction graph.

It is surprising that it is relevant for our lower bound whether the smallest eigenvalue of the adjacency matrix is irrational (see Case 3.). It is not clear whether this is only a feature of our proof or whether there is a true connection to the irrationality of graph spectra.

In the following section we will use graph spectra for deriving lower bounds based on the above theorems. We show that some of them are almost tight by sketching simulation schemes based on well-known results on selective decoupling.

### 3 Applications

In this section we consider the simulation of special interaction graphs that appeared in the literature in various applications. Some interesting models in quantum information theory refer to quite idealized types of Hamiltonians like nearest neighbor interactions. If the natural Hamiltonian contains long range interactions between all nodes one may try to simulate the idealized interaction. Then the problem is to cancel the unwanted terms without destroying the desired interactions. The examples below show that it may cause a large number of time steps to cancel unwanted long-range interactions no matter how fast they are decreasing with the distance. As long as they are not
neglected, the control sequences that cancel them may be rather long. Here we restrict our attention to $\sigma_z \otimes \sigma_z$ interactions between $n$ qubits.

### 3.1 Cyclic and open spin chains

We examine first a quantum system of $n$ spins equally spaced on a one-dimensional lattice. The interaction between the spins may decrease with the distance between the spins. We consider the problem to simulate a chain with only nearest neighbor interactions, i.e., we have to cancel the interactions between all non-adjacent pairs. The desired interaction graphs can be seen in Fig. 3.1.

The adjacency matrix of the cyclic spin chain is $S + S^T$, where $S$ denotes the cyclic shift in $n$ dimensions. The eigenvalues of $S$ and $S^T$ are the $n$th roots of unity. The eigenvalues of $S + S^T$ are given by

$$2\Re\left(\exp(i2\pi l/n)\right) \quad \text{with} \quad 0 \leq l < n.$$  

For odd $n$ the lower bound on the number of time steps is $n$ since the least eigenvalue is irrational for all $n > 3$. If $n$ is even it is rational and has multiplicity 1. Hence the bound is $n - 1$ in this case. There are numbers $n$ where this bound is almost tight. This is shown by the following example. Let $n$ be an even number with the property that a Hadamard matrix of dimension $n/2$ exists. This is for instance the case for each power of 2 (see [22]). We construct a simulation scheme that consists of 2 subroutines.

The first subroutine simulates the interaction between the pairs $\{1, 2\}, \{3, 4\}, \ldots, \{n - 1, n\}$ and the second simulates $\{n, 1\}, \{2, 3\}, \ldots, \{n - 2, n - 1\}$. Note that all the pairs in the same subroutine are disjoint. The problem to simulate the interactions between disjoint pairs is a special instance of well-known “cluster decoupling” [11] where the interaction between independent cliques are cancelled and the interactions within the same clique remains. It can be achieved using Hadamard matrices having at least the number of cliques as dimension (compare [4, 3]). The entries $\pm 1$ in column $j$ determine which spins are conjugated in the step $j$. The dimension is the number of time steps of the decoupling subroutine.

Using this method, we need $n/2$ steps in each subroutine. Therefore we have given a simulation with $n$ steps, whereas our lower bound is $n - 1$. In general the number of steps for simulating the circle grows only linearly in $n$. This shows that the lower bound is quite good even for general $n$.

The adjacency matrix corresponding to the open spin chain has the eigenvalues [23]

$$2\cos\left(\frac{\pi}{n+1}i\right), \quad i = 1, \ldots, n.$$  

\[ (7) \]
The smallest eigenvalue is irrational for all \( n > 2 \). By Theorem 3 (Case 3) we conclude that the number of time steps is at least \( n \).

### 3.2 Square lattice

We consider a quantum system of \( n = l^2 \) spins located on a two-dimensional square lattice. For simplicity assume \( l \) to be even. We want to simulate a lattice with only nearest neighbor interactions.

The desired interaction graph is shown on the left of Fig. 3.2. This kind of interaction can for instance be used for preparing the initial state in the ‘One-Way Quantum Computer’ proposed in [24]. The eigenvalues of the corresponding adjacency matrix \( A \) are known in graph theory [23]:

\[
2 \cos \left( \frac{\pi}{l+1} i \right) + 2 \cos \left( \frac{\pi}{l+1} j \right), \quad i, j = 1, \ldots, l.
\]  

We first consider the time overhead. An upper bound is given by 4 since this is the chromatic index of the graph [1]. It is easy to see that the minimal eigenvalue of \( A \) is given by

\[
\lambda_{\text{min}} = 2 \cos \left( \frac{\pi}{l+1} i \right) + 2 \cos \left( \frac{\pi}{l+1} j \right).
\]  

By Theorem 3 (Case 3) the lower bound on the number of time steps is \( n \) since the smallest eigenvalue is irrational. Note that this example shows that the complexity measures time overhead and number of time steps may differ significantly.

An upper bound on the number of time steps can be obtained as follows. The graph has \( 2(l - 1)l \) edges. We can partition the edges into 4 sets of edges such that each set contains only disjoint interacting pairs. These 4 partitions are shown in Fig. 3.2. The simulation consists of 4 subroutines simulating one the interactions in one of the 4 classes. For each subroutine we choose Hadamard matrices with a dimension that is at least the number of cliques. The numbers of cliques are \( l^2/2 \) or \( l^2/2 + l \) in each subroutine. Since there exist Hadamard matrices for every power of 2 the square lattice graph can always be simulated in \( O(l^2) = O(n) \) time steps.

### 3.3 Graph codes

The computational power of different \( n \) spin interactions is not well understood yet. It would be interesting to know which \( n \) qubit transformations can easily be implemented when a certain Hamiltonian is given. However, one of the few examples where the power of specific Hamiltonians is directly used (without using them to implement 2-qubit gates) is the preparation of states of graph codes proposed in [18, 19]. Here the
codes states are obtained by the free time evolution according to a Hamiltonian with $\sigma_z \otimes \sigma_z$ interactions. The graph representing a code is the interaction graph that can be used for preparing the states. We assume the natural interaction to be equal $zz$ interaction between all 6 spins. and would like to simulate the interaction graph in Fig 3.3.

The eigenvalues of the ‘wheel’ in Fig. 3.3 can easily be computed by any computer algebra system. They are

$$1 + \sqrt{6}, \ \frac{1}{2}\sqrt{5} - \frac{1}{2}, \ \frac{1}{2}\sqrt{5} - \frac{1}{2}, \ 1 - \sqrt{6}, \ -\frac{1}{2}\sqrt{5} - \frac{1}{2}, \ -\frac{1}{2}\sqrt{5} - \frac{1}{2}.$$ 

The minimal eigenvalue is $-1/2 - \sqrt{5}/2$. By Theorem 3 (Case 3) the minimal number of time steps is 6. An implementation with 12 time steps is given as follows. The scheme consists of 3 subroutines each consisting of 4 time steps. As above each subroutine simulates the interaction between disjoint cliques and cancels the interaction between different cliques. Subroutine 1 has the cliques $\{1,2,6\}, \{4,5\}, \{3\}$. The clique partitions in subroutine 2 and 3 are $\{3,4,6\}, \{1,5\}, \{2\} \text{ and } \{1\}, \{4\}, \{2,3\}, \{5,6\}$, respectively. In each subroutine decoupling the different cliques can be achieved by Hadamard matrices of dimension 4 since no subroutine has more than 4 cliques. Hence we have 4 time steps in each subroutine.

### 4 Conclusions

We have derived lower bounds on the number of time steps for simulating arbitrary pair-interactions between $n$ qudits. Like the lower bounds on the time overhead, they make use of the spectrum of the coupling matrices. However, there is no direct connection between both complexity measures since the time overhead refers to spectral majorization while the bounds on the number of time steps refer to the number of eigenvalues not contained in a certain interval. We have shown an example where the number of time steps is of the order $n$ but the time overhead is independent of $n$.

### Acknowledgements

Thanks to Debbie Leung for helpful corrections. This work has been founded by the BMBF project “Informatische Prinzipien und Methoden bei der Steuerung komplexer Quantensysteme”.

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Figure 3: Interaction required for preparing states of a graph code of length 5.
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