Results on para-Sasakian manifold admitting a quarter symmetric metric connection

VISHNUVARDHANA. S.V.¹ AND VENKATESHA²

¹ Department of Mathematics, GITAM School of Science, GITAM (Deemed to be University)
Bengaluru, Karnataka-561 203, INDIA.

² Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga,
Karnataka, INDIA.

svvishnuvardhana@gmail.com, vensmath@gmail.com

ABSTRACT

In this paper we have studied pseudosymmetric, Ricci-pseudosymmetric and projectively pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection and constructed examples of 3-dimensional and 5-dimensional para-Sasakian manifold admitting a quarter-symmetric metric connection to verify our results.

RESUMEN

En este artículo hemos estudiado variedades para-Sasakiáneas seudosimétricas, Ricci-seudosimétricas y proyectivamente seudosimétricas que admiten una conexión métrica cuarto-simétrica, y construimos ejemplos de variedades para-Sasakianas 3-dimensional y 5-dimensional que admiten una conexión métrica cuarto-simétrica para verificar nuestros resultados.

Keywords and Phrases: Para-Sasakian manifold, pseudosymmetric, Ricci-pseudosymmetric, projectively pseudosymmetric, quarter-symmetric metric connection.

2020 AMS Mathematics Subject Classification: 53C35, 53D40.
1 Introduction

One of the most important geometric property of a space is symmetry. Spaces admitting some sense of symmetry play an important role in differential geometry and general relativity. Cartan [5] introduced locally symmetric spaces, i.e., the Riemannian manifold \((M, g)\) for which \(\nabla R = 0\), where \(\nabla\) denotes the Levi-Civita connection of the metric. The integrability condition of \(\nabla R = 0\) is \(R \cdot R = 0\). Thus, every locally symmetric space satisfies \(R \cdot R = 0\), whereby the first \(R\) stands for the curvature operator of \((M, g)\), i.e., for tangent vector fields \(X\) and \(Y\) one has \(R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\), which acts as a derivation on the second \(R\) which stands for the Riemann-Christoffel curvature tensor. The converse however does not hold in general. The spaces for which \(R \cdot R = 0\) holds at every point were called semi-symmetric spaces and which were classified by Szabo [19].

Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. In some spaces \(R \cdot R\) is not identically zero, these turn out to be the pseudo-symmetric spaces of Deszcz [9, 10, 11], which were characterized by the condition \(R \cdot R = LQ(g, R)\), where \(L\) is a real function on \(M\) and \(Q(g, R)\) is the Tachibana tensor of \(M\).

If at every point of \(M\) the curvature tensor satisfies the condition
\[
R(X,Y) \cdot J = L_J[(X \wedge_g Y) \cdot J],
\]
then a Riemannian manifold \(M\) is called pseudosymmetric (resp., Ricci-pseudosymmetric, projectively pseudosymmetric) when \(J = R(\text{resp.}, S, P)\). Here \((X \wedge_g Y)\) is an endomorphism and is defined by \((X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y\) and \(L_J\) is some function on \(U_J = \{x \in M : J \neq 0\}\) at \(x\). A geometric interpretation of the notion of pseudosymmetry is given in [13]. It is also easy to see that every pseudosymmetric manifold is Ricci-pseudosymmetric, but the converse is not true.

An analogue to the almost contact structure, the notion of almost paracontact structure was introduced by Sato [18]. An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be of even dimension as well. Kaneyuki and Williams [14] studied the almost paracontact structure on a pseudo-Riemannian manifold. Recently, almost paracontact geometry in particular, para-Sasakian geometry has taking interest, because of its interplay with the theory of para-Kahler manifolds and its role in pseudo-Riemannian geometry and mathematical physics ([4, 7, 8], etc.).

As a generalization of semi-symmetric connection, quarter-symmetric connection was introduced. Quarter-symmetric connection on a differentiable manifold with affine connection was defined and studied by Golab [12]. From thereafter many geometers studied this connection on different manifolds.

Para-Sasakian manifold with respect to quarter-symmetric metric connection was studied by
De et al., [16, 1], Pradeep Kumar et al., [17] and Bisht and Shanker [15].

Motivated by the above studies in this article we study properties of projective curvature tensor on para-Sasakian manifold admitting a quarter-symmetric metric connection. The organization of the paper is as follows: In Section 2, we present some basic notions of para-Sasakian manifold and quarter-symmetric metric connection on it. Section 3 and 4 are respectively devoted to study the pseudosymmetric and Ricci-pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection. Here we prove that if a para-Sasakian manifold $M^n$ admitting a quarter-symmetric metric connection is Pseudosymmetric (resp., Ricci pseudosymmetric) then $M^n$ is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_R = -2$ (resp., $L_S = -2$). Section 5 and 6 are concerned with projectively flat and projectively pseudosymmetric para-Sasakian manifold $M^n$ admitting a quarter-symmetric metric connection. Finally, we construct examples of 3-dimensional and 5-dimensional para-Sasakian manifold admitting a quarter-symmetric metric connection and we find some of its geometric characteristics.

2 Preliminaries

A differential manifold $M^n$ is said to admit an almost paracontact Riemannian structure $(\phi, \xi, \eta, g)$, where $\phi$ is a tensor field of type $(1, 1)$, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a Riemannian metric on $M^n$ such that

\begin{align}
\phi^2 X &= X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad (2.1) \\
g(X, \xi) &= \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)
\end{align}

for all vector fields $X, Y \in \chi(M^n)$. If $(\phi, \xi, \eta, g)$ on $M^n$ satisfies the following equations

\begin{align}
(\nabla_X \phi) Y &= -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (2.3) \\
d\eta &= 0 \quad \text{and} \quad \nabla_X \xi = \phi X, \quad (2.4)
\end{align}

then $M^n$ is called para-Sasakian manifold [3].

In a para-Sasakian manifold, the following relations hold [6]:

\begin{align}
(\nabla_X \eta) Y &= -g(X, Y) + \eta(X)\eta(Y), \quad (2.5) \\
\eta(R(X, Y)Z) &= g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (2.6) \\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \quad (2.7) \\
S(X, \xi) &= -(n-1)\eta(X), \quad (2.8) \\
S(\phi X, \phi Y) &= S(X, Y) + (n-1)\eta(X)\eta(Y), \quad (2.9)
\end{align}

for every vector fields $X, Y, Z$ on $M^n$. Here $\nabla$ denotes the Levi-Civita connection, $R$ denotes the Riemannian curvature tensor and $S$ denotes the Ricci curvature tensor.
Here we consider a quarter-symmetric metric connection $\nabla$ on a para-Sasakian manifold $[16]$ given by
\[
\nabla_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi.
\] (2.10)

The relation between curvature tensor $\tilde{R}(X, Y)Z$ of $M^n$ with respect to quarter-symmetric metric connection $\hat{\nabla}$ and the curvature tensor $R(X, Y)Z$ with respect to the Levi-Civita connection $\nabla$ is given by
\[
\tilde{R}(X, Y)Z = R(X, Y)Z + 3g(\phi X, Z)\phi Y - 3g(\phi Y, Z)\phi X
+ \{\eta(X)Y - \eta(Y)X\} \eta(Z) - [g(Y, Z)\eta(X) - \eta(Y)g(X, Z)]\xi.
\] (2.11)

Also from (2.11) we obtain
\[
\tilde{S}(Y, Z) = S(Y, Z) + 2g(Y, Z) - (n + 1)\eta(Y)\eta(Z) - 3\text{trace}\phi g(\phi Y, Z),
\] (2.12)

where $\tilde{S}$ and $S$ are Ricci tensors of connections $\hat{\nabla}$ and $\nabla$ respectively.

3 Pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold $M^n$ admitting a quarter-symmetric metric connection is said to be pseudosymmetric if
\[
\tilde{R}(X, Y) \cdot \tilde{R} = L_{\tilde{R}}(X \wedge g Y) \cdot \tilde{R},
\] (3.1)

holds on the set $U_{\tilde{R}} = \{x \in M^n : \tilde{R} \neq 0 \text{ at } x\}$, where $L_{\tilde{R}}$ is some function on $U_{\tilde{R}}$.

Suppose that $M^n$ be pseudosymmetric, then in view of (3.1) we have
\[
\tilde{R}(\xi, Y)\tilde{R}(U, V)W - \tilde{R}(\tilde{R}(\xi, Y)U, V)W - \tilde{R}(U, \tilde{R}(\xi, Y)V)W
- \tilde{R}(U, V)\tilde{R}(\xi, Y)W = L_{\tilde{R}}[(\xi \wedge g Y)\tilde{R}(U, V)W - \tilde{R}((\xi \wedge g Y)U, V)W
- \tilde{R}(U, (\xi \wedge g Y)V)W - \tilde{R}(U, V)((\xi \wedge g Y)W)].
\] (3.2)

By virtue of (2.7) and (2.11), (3.2) takes the form
\[
(L_{\tilde{R}} + 2)[\eta(\tilde{R}(U, V)W)Y - g(Y, \tilde{R}(U, V)W)\xi - \eta(U)\tilde{R}(Y, V)W + g(Y, U)\tilde{R}(\xi, V)W
- \eta(V)\tilde{R}(U, Y)W + g(Y, V)\tilde{R}(U, \xi)W - \eta(W)\tilde{R}(U, V)Y + g(Y, W)\tilde{R}(U, V)\xi]
= 0.
\] (3.3)

Taking inner product of (3.3) with $\xi$ and using (2.6) and (2.11), we get
\[
(L_{\tilde{R}} + 2)[g(Y, R(U, V)W) + 3g(\phi U, W)g(\phi V, Y) - 3g(\phi V, W)g(\phi U, Y)
+ \eta(W)\eta(U)g(V, Y) - \eta(V)g(Y, U)] - \{g(V, W)\eta(U) - \eta(V)g(U, W)\}\eta(Y)
+ 2\{g(V, W)g(Y, U) - g(V, Y)g(U, W)\} = 0.
\] (3.4)
Assuming that $L_R + 2 \neq 0$, the above equation becomes
\[
g(Y, R(U, V)W) + 3g(\phi U, W)g(\phi V, Y) - 3g(\phi V, W)g(\phi U, Y) \\
+ \eta(W)\{\eta(U)g(V, Y) - \eta(V)g(U, Y)\} - [g(V, W)\eta(U) - \eta(V)g(U, W)]\eta(Y) \\
+ 2[g(V, W)g(Y, U) - g((V, Y)g(U, W)] = 0. \tag{3.5}
\]

Putting $V = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i = 1, 2, 3, \ldots, n$, we get
\[
\tilde{S}(Y, U) = -2(n - 1)g(Y, U). \tag{3.6}
\]

Hence, we can state the following:

**Theorem 1.** If a para-Sasakian manifold $M^n$ admitting a quarter-symmetric metric connection is pseudosymmetric then $M^n$ is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\tilde{R}} = -2$.

### 4 Ricci-pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold $M^n$ admitting a quarter-symmetric metric connection is said to be Ricci-pseudosymmetric if the following condition is satisfied
\[
\tilde{R}(X, Y) \cdot \tilde{S} = L_{\tilde{S}}[(X \wedge g Y) \cdot \tilde{S}], \tag{4.1}
\]
on $U_{\tilde{S}}$.

Let para-Sasakian manifold $M^n$ admitting a quarter-symmetric metric connection be Ricci-pseudosymmetric. Then we have
\[
\tilde{S}(\tilde{R}(X, Y)Z, W) + \tilde{S}(Z, \tilde{R}(X, Y)W) = L_{\tilde{S}}[\tilde{S}((X \wedge g Y)Z, W) + \tilde{S}(Z, (X \wedge g Y)W)]. \tag{4.2}
\]

By taking $Y = W = \xi$ and making use of (2.7), (2.8) and (2.11), the above equation turns into
\[
(L_{\tilde{S}} + 2)[\tilde{S}(X, Z) + 2(n - 1)g(X, Z)] = 0 \tag{4.3}
\]

Thus, we have the following assertion:

**Theorem 2.** If a para-Sasakian manifold $M^n$ admitting a quarter-symmetric metric connection is Ricci-pseudosymmetric then $M^n$ is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\tilde{S}} = -2$. 

5 Projectively flat para-Sasakian manifold admitting a quarter-symmetric metric connection

The projective curvature tensor on a Riemannian manifold is defined by [2]

\[ P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[S(Y, Z)X - S(X, Z)Y]. \] (5.1)

For an \( n \)-dimensional para-Sasakian manifold \( M^n \) admitting a quarter-symmetric metric connection, the projective curvature tensor is given by

\[ \tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{(n-1)}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y]. \] (5.2)

**Theorem 3.** A projectively flat para-Sasakian manifold \( M^n \) admitting a quarter-symmetric metric connection is an Einstein manifold with respect to quarter-symmetric metric connection.

**Proof.** Consider a projectively flat para-Sasakian manifold admitting a quarter-symmetric metric connection. Then from (5.2) we have

\[ g(\tilde{R}(X, Y)Z, W) = \frac{1}{(n-1)}[\tilde{S}(Y, Z)g(X, W) - \tilde{S}(X, Z)g(Y, W)]. \] (5.3)

Setting \( X = W = \xi \) in (5.3) and using (2.7), (2.8), (2.11) and (2.12), we get

\[ \tilde{S}(X, Z) = -2(n-1)g(X, Z). \] (5.4)

Hence, the proof is completed. \( \square \)

6 Projectively pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold admitting a quarter-symmetric metric connection is said to be projectively pseudosymmetric if

\[ \tilde{R}(X, Y) \cdot \tilde{P} = L_{\tilde{P}}[(X \wedge g Y) \cdot \tilde{P}], \] (6.1)

holds on the set \( U_{\tilde{P}} = \{ x \in M^n : \tilde{P} \neq 0 \text{ at } x \} \), where \( L_{\tilde{P}} \) is some function on \( U_{\tilde{P}} \).

Let \( M^n \) be projectively pseudosymmetric, then we have

\[ \tilde{R}(X, \xi) \tilde{P}(U, V)\xi = \tilde{P}(\tilde{R}(X, \xi)U, V)\xi - \tilde{P}(U, \tilde{R}(X, \xi)V)\xi \]

\[ -\tilde{P}(U, V)\tilde{R}(X, \xi)\xi = L_{\tilde{P}}[(X \wedge g \xi) \tilde{P}(U, V)\xi - \tilde{P}((X \wedge g \xi)U, V)\xi \]

\[ -\tilde{P}(U, (X \wedge g \xi)V)\xi - \tilde{P}(U, V)(X \wedge g \xi)\xi]. \] (6.2)
By virtue of (2.11), (2.12) and (5.2), (6.2) becomes

\[(L_P + 2)\tilde{P}(U, V)X = 0.\]  \hspace{1cm} (6.3)

So, one can state that:

**Theorem 4.** If a para-Sasakian manifold $M^n$ admitting a quarter-symmetric metric connection is projectively pseudosymmetric then $M^n$ is projectively flat with respect to quarter-symmetric metric connection or $L_P = -2$.

In view of theorem 3, one can state the above theorem as

**Theorem 5.** If a para-Sasakian manifold $M^n$ admitting a quarter-symmetric metric connection is projectively pseudosymmetric then $M^n$ is an Einstein manifold with respect to quarter-symmetric metric connection or $L_P = -2$.

7 Examples

7.1 Example

We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where $(x, y, z)$ are standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame field on $M$ given by

\[E_1 = e^z \frac{\partial}{\partial y}, \quad E_2 = e^z (\frac{\partial}{\partial y} - \frac{\partial}{\partial x}), \quad E_3 = \frac{\partial}{\partial z}.\]

If $g$ is a Riemannian metric defined by

\[g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}\]

for $1 \leq i, j \leq 3$, and if $\eta$ is the 1-form defined by $\eta(Z) = g(Z, E_3)$ for any vector field $Z \in \chi(M)$. We define the $(1, 1)$-tensor field $\phi$ as

$\phi(E_1) = E_1, \quad \phi(E_2) = -E_2, \quad \phi(E_3) = 0.$

The linearity property of $\phi$ and $g$ yields that

\[\eta(E_3) = 1,\]
\[\phi^2 U = U - \eta(U)E_3,\]
\[g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V),\]
for any $U, V \in \chi(M)$.

Now we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = E_2.$$ 

The Riemannian connection $\nabla$ of the metric $g$ known as Koszul’s formula and is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul’s formula we get the followings in matrix form

$$\begin{pmatrix}
\nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 \\
\nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 \\
\nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3
\end{pmatrix} =
\begin{pmatrix}
-E_3 & 0 & E_1 \\
0 & -E_3 & E_2 \\
0 & 0 & 0
\end{pmatrix}.$$ 

Clearly $(\phi, \xi, \eta, g)$ is a para-Sasakian structure on $M$. Thus $M(\phi, \xi, \eta, g)$ is a 3-dimensional para-Sasakian manifold.

Using (2.10) and the above equation, one can easily obtain the following:

$$\begin{pmatrix}
\hat{\nabla}_{E_1}E_1 & \hat{\nabla}_{E_1}E_2 & \hat{\nabla}_{E_1}E_3 \\
\hat{\nabla}_{E_2}E_1 & \hat{\nabla}_{E_2}E_2 & \hat{\nabla}_{E_2}E_3 \\
\hat{\nabla}_{E_3}E_1 & \hat{\nabla}_{E_3}E_2 & \hat{\nabla}_{E_3}E_3
\end{pmatrix} =
\begin{pmatrix}
-2E_3 & 0 & 2E_1 \\
0 & -2E_3 & 2E_2 \\
0 & 0 & 0
\end{pmatrix}.$$ 

With the help of the above results it can be easily verified that

\begin{align*}
R(E_1, E_2)E_3 &= 0, & R(E_2, E_3)E_3 &= -E_2, & R(E_1, E_3)E_3 &= -E_1, \\
R(E_1, E_2)E_2 &= -E_1, & R(E_2, E_3)E_2 &= E_3, & R(E_1, E_3)E_2 &= 0, \\
R(E_1, E_2)E_1 &= E_2, & R(E_2, E_3)E_1 &= 0, & R(E_1, E_3)E_1 &= E_3.
\end{align*}

and

\begin{align*}
\hat{R}(E_1, E_2)E_3 &= 0, & \hat{R}(E_2, E_3)E_3 &= -2E_2, & \hat{R}(E_1, E_3)E_3 &= -2E_1, \\
\hat{R}(E_1, E_2)E_2 &= -4E_1, & \hat{R}(E_2, E_3)E_2 &= 2E_3, & \hat{R}(E_1, E_3)E_2 &= 0, \\
\hat{R}(E_1, E_2)E_1 &= 4E_2, & \hat{R}(E_2, E_3)E_1 &= 0, & \hat{R}(E_1, E_3)E_1 &= 2E_3. & (7.1)
\end{align*}

Since $E_1, E_2, E_3$ forms a basis, any vector field $X, Y, Z \in \chi(M)$ can be written as $X = a_1E_1 + b_1E_2 + c_1E_3$, $Y = a_2E_1 + b_2E_2 + c_2E_3$, $Z = a_3E_1 + b_3E_2 + c_3E_3$, where $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, 2, 3$. Using the expressions of the curvature tensors, we find values of Riemannian curvature
and Ricci curvature with respect to quarter-symmetric metric connection as:

\[
\tilde{R}(X, Y)Z = [-4(a_1b_2 - b_1a_2)b_3 + 2(c_1a_2 - a_1c_2)c_3]E_1 + [-4(b_1a_2 - a_1b_2)a_3 + 2(c_1b_2 - b_1c_2)c_3]E_2 + [-2(c_1a_2 - a_1c_2)a_3 - 2(c_1b_2 - b_1c_2)b_3]E_3,
\]

(7.2)

\[
\tilde{S}(E_1, E_1) = \tilde{S}(E_2, E_2) = -6, \quad \tilde{S}(E_3, E_3) = -4.
\]

(7.3)

Using (7.1), (7.3) and the expression of the endomorphism \((X \wedge g Y)Z = g(Y, Z)X - g(X, Z)Y\), one can easily verify that

\[
\tilde{S}(\tilde{R}(X, E_3)Y, E_3) + \tilde{S}(Y, \tilde{R}(X, E_3)E_3) = -2[\tilde{S}((X \wedge g E_3)Y, E_3) + \tilde{S}(Y, (X \wedge g E_3)E_3)],
\]

(7.4)

here \(L_{\tilde{g}} = -2\). Thus, the above equation verify one part of the Theorem 2 of section 4.

Moreover, the manifold under consideration satisfies

\[
\tilde{R}(X, Y)Z = -\tilde{R}(Y, X)Z,
\]

\[
\tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y = 0.
\]

Hence, from the above equations one can say that this example verifies the condition (c) of Theorem 3.1 in [1] and first Bianchi identity.

### 7.2 Example

We consider a 5-dimensional manifold \(M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5\}\), where \((x_1, x_2, x_3, x_4, x_5)\) are standard coordinates in \(\mathbb{R}^5\). We choose the vector fields

\[
E_1 = \frac{\partial}{\partial x_1}, \quad E_2 = \frac{\partial}{\partial x_2}, \quad E_3 = \frac{\partial}{\partial x_3}, \quad E_4 = \frac{\partial}{\partial x_4}, \quad E_5 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5},
\]

which are linearly independent at each point of \(M\).

Let \(g\) be a Riemannian metric defined by

\[
g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}
\]

for \(1 \leq i, j \leq 5\), and if \(\eta\) is the 1-form defined by \(\eta(Z) = g(Z, E_5)\) for any vector field \(Z \in \chi(M)\). Let \(\phi\) be the \((1, 1)\)-tensor field defined by

\[
\phi(E_1) = E_1, \quad \phi(E_2) = E_2, \quad \phi(E_3) = E_3, \quad \phi(E_4) = E_4, \quad \phi(E_5) = 0.
\]
The linearity property of \( \phi \) and \( g \) yields that
\[
\eta(E_3) = 1,
\]
\[
\phi^2 U = U - \eta(U)E_3,
\]
\[
g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V),
\]
for any \( U, V \in \chi(M) \).

Now we have
\[
[E_1, E_2] = 0, \quad [E_1, E_3] = 0, \quad [E_1, E_4] = 0, \quad [E_1, E_5] = E_1,
\]
\[
[E_2, E_3] = 0, \quad [E_2, E_4] = 0, \quad [E_2, E_5] = E_2,
\]
\[
[E_3, E_4] = 0, \quad [E_3, E_5] = E_3, \quad [E_4, E_5] = E_4.
\]

By virtue of Koszul's formula we get the followings in matrix form
\[
\begin{pmatrix}
\nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 & \nabla_{E_1}E_4 & \nabla_{E_1}E_5 \\
\nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 & \nabla_{E_2}E_4 & \nabla_{E_2}E_5 \\
\nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3 & \nabla_{E_3}E_4 & \nabla_{E_3}E_5 \\
\nabla_{E_4}E_1 & \nabla_{E_4}E_2 & \nabla_{E_4}E_3 & \nabla_{E_4}E_4 & \nabla_{E_4}E_5 \\
\nabla_{E_5}E_1 & \nabla_{E_5}E_2 & \nabla_{E_5}E_3 & \nabla_{E_5}E_4 & \nabla_{E_5}E_5
\end{pmatrix}
\begin{pmatrix}
-E_5 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2 \\
E_3 \\
E_4 \\
E_5
\end{pmatrix}
\]
\[
= \begin{pmatrix}
-E_5 & 0 & 0 & 0 & E_1 \\
0 & -E_5 & 0 & 0 & E_2 \\
0 & 0 & -E_5 & 0 & E_3 \\
0 & 0 & 0 & -E_5 & E_4 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Above expressions satisfies all the properties of para-Sasakian manifold. Thus \( M(\phi, \xi, \eta, g) \) is a 5-dimensional para-Sasakian manifold.

From the above expressions and the relation of quarter symmetric metric connection and Riemannian connection, one can easily obtain the following:
\[
\begin{pmatrix}
\nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 & \nabla_{E_1}E_4 & \nabla_{E_1}E_5 \\
\nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 & \nabla_{E_2}E_4 & \nabla_{E_2}E_5 \\
\nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3 & \nabla_{E_3}E_4 & \nabla_{E_3}E_5 \\
\nabla_{E_4}E_1 & \nabla_{E_4}E_2 & \nabla_{E_4}E_3 & \nabla_{E_4}E_4 & \nabla_{E_4}E_5 \\
\nabla_{E_5}E_1 & \nabla_{E_5}E_2 & \nabla_{E_5}E_3 & \nabla_{E_5}E_4 & \nabla_{E_5}E_5
\end{pmatrix}
\begin{pmatrix}
-2E_5 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
2E_1 \\
2E_2 \\
2E_3 \\
2E_4 \\
2E_5
\end{pmatrix}
\]
\[
= \begin{pmatrix}
-2E_5 & 0 & 0 & 0 & 2E_1 \\
0 & -2E_5 & 0 & 0 & 2E_2 \\
0 & 0 & -2E_5 & 0 & 2E_3 \\
0 & 0 & 0 & -2E_5 & 2E_4 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

With the help of the above results it can be easily obtain the non-zero components of curvature tensors as
\[
R(E_1, E_2)E_1 = E_2, \quad R(E_1, E_2)E_2 = -E_1, \quad R(E_1, E_3)E_1 = E_3, \quad R(E_1, E_3)E_3 = -E_1,
\]
\[
R(E_1, E_4)E_1 = E_4, \quad R(E_1, E_4)E_4 = -E_1, \quad R(E_1, E_5)E_1 = E_5, \quad R(E_1, E_5)E_5 = -E_1,
\]
\[
R(E_2, E_3)E_2 = E_3, \quad R(E_2, E_3)E_3 = -E_2, \quad R(E_2, E_4)E_2 = E_4, \quad R(E_2, E_4)E_4 = -E_2,
\]
\[
R(E_2, E_5)E_2 = E_5, \quad R(E_2, E_5)E_5 = -E_2, \quad R(E_3, E_4)E_3 = E_4, \quad R(E_3, E_4)E_4 = -E_3,
\]
\[
R(E_3, E_5)E_3 = E_5, \quad R(E_3, E_5)E_5 = -E_3, \quad R(E_4, E_5)E_4 = E_5, \quad R(E_4, E_5)E_5 = -E_4,
\]
\[
R(E_5, E_5)E_5 = E_5.
\]
and

\[ \tilde{\mathbf{R}}(E_1, E_2)E_1 = 4E_2, \quad \tilde{\mathbf{R}}(E_1, E_2)E_2 = -4E_1, \quad \tilde{\mathbf{R}}(E_1, E_3)E_1 = 4E_3, \quad \tilde{\mathbf{R}}(E_1, E_3)E_3 = -4E_1, \]
\[ \tilde{\mathbf{R}}(E_1, E_4)E_1 = 4E_4, \quad \tilde{\mathbf{R}}(E_1, E_4)E_4 = -4E_1, \quad \tilde{\mathbf{R}}(E_1, E_5)E_1 = 2E_5, \quad \tilde{\mathbf{R}}(E_1, E_5)E_5 = -2E_1, \]
\[ \tilde{\mathbf{R}}(E_2, E_3)E_2 = 4E_3, \quad \tilde{\mathbf{R}}(E_2, E_3)E_3 = -4E_2, \quad \tilde{\mathbf{R}}(E_2, E_4)E_2 = 4E_4, \quad \tilde{\mathbf{R}}(E_2, E_4)E_4 = -4E_2, \]
\[ \tilde{\mathbf{R}}(E_2, E_5)E_2 = 2E_5, \quad \tilde{\mathbf{R}}(E_2, E_5)E_5 = -2E_2, \quad \tilde{\mathbf{R}}(E_3, E_4)E_3 = 4E_4, \quad \tilde{\mathbf{R}}(E_3, E_4)E_4 = -4E_3, \]
\[ \tilde{\mathbf{R}}(E_3, E_5)E_3 = 2E_5, \quad \tilde{\mathbf{R}}(E_3, E_5)E_5 = -2E_3, \quad \tilde{\mathbf{R}}(E_4, E_5)E_4 = 2E_5, \quad \tilde{\mathbf{R}}(E_4, E_5)E_5 = -2E_4. \] (7.5)

Since \( E_1, E_2, E_3, E_4, E_5 \) forms a basis, any vector field \( X, Y, Z \in \chi(M) \) can be written as
\[ X = a_1E_1 + b_1E_2 + c_1E_3 + d_1E_4 + f_1E_5, \quad Y = a_2E_1 + b_2E_2 + c_2E_3 + d_2E_4 + f_2E_5, \quad Z = a_3E_1 + b_3E_2 + c_3E_3 + d_3E_4 + f_3E_5, \]
where \( a_i, b_i, c_i, d_i, f_i \in \mathbb{R}, \ i = 1, 2, 3, 4, 5 \). Using the expressions of the curvature tensors, we find values of Riemannian curvature and Ricci curvature with respect to quarter-symmetric metric connection as;

\[ \tilde{\mathbf{R}}(X, Y)Z = [-4\{a_1(b_2b_3 + c_2c_3 + d_2d_3) - a_2(b_1b_3 + c_1c_3 + d_1d_3)\} - 2(a_1f_2 - f_1a_2)f_3]E_1 + \]
\[ + [-4\{b_1(a_2a_3 + c_2c_3 + d_2d_3) - b_2(a_1a_3 + c_1c_3 + d_1d_3)\} - 2(b_1f_2 - f_1b_2)f_3]E_2 + \]
\[ + [-4\{c_1(a_2a_3 + b_2b_3 + d_2d_3) - c_2(a_1a_3 + b_1b_3 + d_1d_3)\} - 2(c_1f_2 - f_1c_2)f_3]E_3 + \]
\[ + [-4\{d_1(a_2a_3 + b_2b_3 + c_2c_3) - d_2(a_1a_3 + b_1b_3 + c_1c_3)\} - 2(d_1f_2 - f_1d_2)f_3]E_4 + \]
\[ + [2\{(a_1f_2 - f_1a_2)a_3 + (b_1f_2 - f_1b_2)b_3 + (c_1f_2 - f_1c_2)c_3 + (d_1f_2 - f_1d_2)d_3\}]E_5, \]
\[ \tilde{\mathbf{S}}(E_1, E_1) = \tilde{\mathbf{S}}(E_2, E_2) = \tilde{\mathbf{S}}(E_3, E_3) = \tilde{\mathbf{S}}(E_4, E_4) = -14, \quad \tilde{\mathbf{S}}(E_5, E_5) = -8. \] (7.6)

In view of (7.5), (7.6) and the expression of the endomorphism one can easily verify the equation (7.4) and hence the Theorem 2 of section 4 is verified. This example also verifies the condition (c) of Theorem 3.1 in [1] and first Bianchi identity.

Above two examples verifies the one part of the Theorem 2, that is, if a para-Sasakian manifold \( M^n \) admitting a quarter-symmetric metric connection is Ricci pseudosymmetric then \( M^n \) satisfies \( L_S = -2 \). \( M^n \) is not Einstein manifold with respect to quarter-symmetric metric connection.

Another part of the theorem is that, if a para-Sasakian manifold \( M^n \) admitting a quarter-symmetric metric connection is Ricci pseudosymmetric then \( M^n \) is an Einstein manifold with respect to quarter-symmetric metric connection \( (L_S \neq -2) \). Now, the second part of the Theorem 2 can be verified by using the proper example.

### 7.3 Example

We consider a 5-dimensional manifold \( M = \{(x, y, z, u, v) \in \mathbb{R}^5\} \), where \( (x, y, z, u, v) \) are standard coordinates in \( \mathbb{R}^5 \). Let \( \{E_1, E_2, E_3, E_4, E_5\} \) be a linearly independent global frame field on \( M \) given
by

\[ E_1 = \frac{\partial}{\partial x}, \quad E_2 = e^{-x} \frac{\partial}{\partial y}, \quad E_3 = e^{-x} \frac{\partial}{\partial z}, \quad E_4 = e^{-x} \frac{\partial}{\partial u}, \quad E_5 = e^{-x} \frac{\partial}{\partial v}. \]

Let \( g \) be a Riemannian metric defined by

\[ g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \]

for \( 1 \leq i, j \leq 5 \), and if \( \eta \) is the 1-form defined by \( \eta(Z) = g(Z, E_1) \) for any vector field \( Z \in \chi(M) \).

Let the \((1, 1)\)-tensor field \( \phi \) be defined by

\[ \phi(E_1) = 0, \quad \phi(E_2) = E_2, \quad \phi(E_3) = E_3, \quad \phi(E_4) = E_4, \quad \phi(E_5) = E_5. \]

With the help of linearity property of \( \phi \) and \( g \), we have

\[ \eta(E_1) = 1, \]

\[ \phi^2 V = V - \eta(V)E_1, \]

\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \]

for any \( X, Y \in \chi(M) \).

Now we have

\[ [E_1, E_2] = -E_2, \quad [E_1, E_3] = -E_3, \quad [E_1, E_4] = -E_4, \quad [E_1, E_5] = -E_5, \]

\[ [E_2, E_3] = [E_2, E_4] = [E_2, E_5] = [E_3, E_4] = [E_3, E_5] = [E_4, E_5] = 0. \]

With the help of Koszul’s formula we get the followings in matrix form

\[
\begin{pmatrix}
\nabla_{E_1} E_1 & \nabla_{E_1} E_2 & \nabla_{E_1} E_3 & \nabla_{E_1} E_4 & \nabla_{E_1} E_5 \\
\nabla_{E_2} E_1 & \nabla_{E_2} E_2 & \nabla_{E_2} E_3 & \nabla_{E_2} E_4 & \nabla_{E_2} E_5 \\
\nabla_{E_3} E_1 & \nabla_{E_3} E_2 & \nabla_{E_3} E_3 & \nabla_{E_3} E_4 & \nabla_{E_3} E_5 \\
\nabla_{E_4} E_1 & \nabla_{E_4} E_2 & \nabla_{E_4} E_3 & \nabla_{E_4} E_4 & \nabla_{E_4} E_5 \\
\nabla_{E_5} E_1 & \nabla_{E_5} E_2 & \nabla_{E_5} E_3 & \nabla_{E_5} E_4 & \nabla_{E_5} E_5 \\
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
E_2 & -E_1 & 0 & 0 & 0 \\
E_3 & 0 & -E_1 & 0 & 0 \\
E_4 & 0 & 0 & -E_1 & 0 \\
E_5 & 0 & 0 & 0 & -E_1 \\
\end{pmatrix}.
\]

In this case, \((\phi, \xi, \eta, g)\) is a para-Sasakian structure on \( M \) and hence \( M(\phi, \xi, \eta, g) \) is a 5-dimensional para-Sasakian manifold.

Using (2.10) and the above equation, one can easily obtain the following:

\[
\begin{pmatrix}
\nabla_{E_1} E_1 & \nabla_{E_1} E_2 & \nabla_{E_1} E_3 & \nabla_{E_1} E_4 & \nabla_{E_1} E_5 \\
\nabla_{E_2} E_1 & \nabla_{E_2} E_2 & \nabla_{E_2} E_3 & \nabla_{E_2} E_4 & \nabla_{E_2} E_5 \\
\nabla_{E_3} E_1 & \nabla_{E_3} E_2 & \nabla_{E_3} E_3 & \nabla_{E_3} E_4 & \nabla_{E_3} E_5 \\
\nabla_{E_4} E_1 & \nabla_{E_4} E_2 & \nabla_{E_4} E_3 & \nabla_{E_4} E_4 & \nabla_{E_4} E_5 \\
\nabla_{E_5} E_1 & \nabla_{E_5} E_2 & \nabla_{E_5} E_3 & \nabla_{E_5} E_4 & \nabla_{E_5} E_5 \\
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
2E_2 & -2E_1 & 0 & 0 & 0 \\
2E_3 & 0 & -2E_1 & 0 & 0 \\
2E_4 & 0 & 0 & -2E_1 & 0 \\
2E_5 & 0 & 0 & 0 & -2E_1 \\
\end{pmatrix}.
\]
From above results it can be easily obtain the non-zero components of Riemannian curvature and Ricci curvature tensors as

\[
R(E_1, E_2)E_1 = E_2, \quad R(E_1, E_2)E_2 = -E_1, \quad R(E_1, E_3)E_1 = E_3, \quad R(E_1, E_3)E_3 = -E_1, \\
R(E_1, E_4)E_1 = E_4, \quad R(E_1, E_4)E_4 = -E_1, \quad R(E_1, E_5)E_1 = E_5, \quad R(E_1, E_5)E_5 = -E_1, \\
R(E_2, E_3)E_2 = E_3, \quad R(E_2, E_3)E_3 = -E_2, \quad R(E_2, E_4)E_2 = E_4, \quad R(E_2, E_4)E_4 = -E_2, \\
R(E_2, E_5)E_2 = E_5, \quad R(E_2, E_5)E_5 = -E_2, \quad R(E_3, E_4)E_3 = E_4, \quad R(E_3, E_4)E_4 = -E_3, \\
R(E_3, E_5)E_3 = E_5, \quad R(E_3, E_5)E_5 = -E_3, \quad R(E_4, E_5)E_4 = E_5, \quad R(E_4, E_5)E_5 = -E_4,
\]

and

\[
\tilde{R}(E_1, E_2)E_1 = 2E_2, \quad \tilde{R}(E_1, E_2)E_2 = -2E_1, \quad \tilde{R}(E_1, E_3)E_1 = 2E_3, \quad \tilde{R}(E_1, E_3)E_3 = -2E_1, \\
\tilde{R}(E_1, E_4)E_1 = 2E_4, \quad \tilde{R}(E_1, E_4)E_4 = -2E_1, \quad \tilde{R}(E_1, E_5)E_1 = 2E_5, \quad \tilde{R}(E_1, E_5)E_5 = -2E_1, \\
\tilde{R}(E_2, E_3)E_2 = 2E_3, \quad \tilde{R}(E_2, E_3)E_3 = -2E_2, \quad \tilde{R}(E_2, E_4)E_2 = 2E_4, \quad \tilde{R}(E_2, E_4)E_4 = -2E_2, \\
\tilde{R}(E_2, E_5)E_2 = 2E_5, \quad \tilde{R}(E_2, E_5)E_5 = -2E_2, \quad \tilde{R}(E_3, E_4)E_3 = 2E_4, \quad \tilde{R}(E_3, E_4)E_4 = -2E_3, \\
\tilde{R}(E_3, E_5)E_3 = 2E_5, \quad \tilde{R}(E_3, E_5)E_5 = -2E_3, \quad \tilde{R}(E_4, E_5)E_4 = 2E_5, \quad \tilde{R}(E_4, E_5)E_5 = -2E_4, \quad (7.7) \\
\tilde{S}(E_1, E_1) = \tilde{S}(E_2, E_2) = \tilde{S}(E_3, E_3) = \tilde{S}(E_4, E_4) = \tilde{S}(E_5, E_5) = -8, \quad (7.8) \\
\tilde{S}(X, Y) = -2(5 - 1)g(X, Y) = -8g(X, Y),
\]

where \( X = a_1E_1 + b_1E_2 + c_1E_3 + d_1E_4 + f_1E_5 \) and \( Y = a_2E_1 + b_2E_2 + c_2E_3 + d_2E_4 + f_2E_5 \).

From (7.7), (7.8) and the expression of the endomorphism one can easily substantiate, the equation (7.4) and hence second part of the Theorem 2 (for \( L_\mathcal{\mathcal{F}} \neq -2 \)).
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