Para-Sasaki-like Riemannian manifolds and new Einstein metrics

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Abstract
We determine a new class of paracontact paracomplex Riemannian manifolds derived from certain cone construction, called para-Sasaki-like Riemannian manifolds, and give explicit examples. We define a hyperbolic extension of a paraholomorphic paracomplex Riemannian manifold, which is a local product of two Riemannian spaces of equal dimension, and show that it is a para-Sasaki-like Riemannian manifold. If the original paraholomorphic paracomplex Riemannian manifold is a complete Einstein space of negative scalar curvature, then its hyperbolic extension is a complete Einstein para-Sasaki-like Riemannian manifold of negative scalar curvature. Thus, we present new examples of complete Einstein Riemannian manifolds of negative scalar curvature.

Keywords  Almost paracontact Riemannian manifolds · Holomorphic paracomplex manifold · Para-Sasaki-like manifolds · Einstein manifolds

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1 Introduction

In 1976, Sato [16] introduced the concept of almost paracontact Riemannian manifolds as an analogue of almost contact Riemannian manifolds [1,14]. Later, in 1980, Sasaki [15] defined the notion of an almost paracontact Riemannian manifold of type \((p, q)\), where \(p\) and \(q\) are the numbers of the multiplicity of the structure eigenvalues 1 and \(-1\), respectively. In addition, there is a simple eigenvalue 0.

In the present paper, we consider a \((2n + 1)\)-dimensional almost paracontact Riemannian manifolds of type \((n, n)\), i.e., \(p = q = n\) such that the paracontact distribution can be considered as a \(2n\)-dimensional almost paracomplex Riemannian distribution with an almost paracomplex structure and a structure group \(O(n) \times O(n)\). Paracomplex geometry has been studied since the first papers by Rashevskij [12], Libermann [6] and Patterson [11] from several different points of view. In particular, the almost paracomplex Riemannian manifolds are classified by Staikova and Gribachev in [19].

We call these \((2n + 1)\)-dimensional manifolds almost paracontact paracomplex Riemannian manifolds (or briefly apcpcR manifolds). A natural example is the direct product of an almost paracomplex Riemannian manifold with the real line. Accordingly, any real hypersurface of an almost paracomplex Riemannian manifold admits an almost paracontact paracomplex Riemannian structure.

A \((2n+1)\)-dimensional manifold \((M, \phi, \xi, \eta)\) is said to be an almost paracontact manifold if \(\phi\) is a \((1, 1)\)-tensor field, \(\xi\) is a vector field, and \(\eta\) is a 1-form satisfying the compatibility conditions

\[
\phi^2 = \text{id} - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0. \tag{1}
\]

Let \(H = \ker(\eta)\) be the paracontact distribution. Then the endomorphism \(\phi\) induces an almost product structure (in particular, an almost paracomplex structure) on \(H\) [4], so that \((H, \phi)\) is a \(2n\)-dimensional almost product distribution (in particular, an almost paracomplex distribution). Let us note that an almost paracomplex structure is an almost product structure \(P\), i.e., \(P^2 = \text{id}\) and \(P \neq \pm \text{id}\), so that the eigenvalues +1 and −1 of \(P\) have the same multiplicity \(n\) [3], i.e., \(\text{tr} P = 0\) follows.

In the present work we consider the case of almost paracontact paracomplex manifolds, i.e., its paracontact distribution is equipped with an almost paracomplex structure. According to Sasaki [15], these manifolds are called almost paracomplex manifolds of type \((n, n)\). Obviously, the property \(\text{tr} \phi = 0\) holds.

Let \(g\) be an associated Riemannian metric such that

\[
g(x, \xi) = \eta(x), \quad g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y).
\]

Then \((M, \phi, \xi, \eta, g)\) is called an almost paracontact paracomplex Riemannian manifold [16]. An almost paracontact paracomplex Riemannian manifold \((M, \phi, \xi, \eta)\) is said to be a paracontact paracomplex Riemannian manifold if the following condition is also fulfilled [17]

\[
2g(x, \phi y) = (\mathcal{L}_\xi g)(x, y) = (\nabla_x \eta)(y) + (\nabla_y \eta)(x), \tag{2}
\]

where \(\mathcal{L}\) denotes the Lie derivative, and \(\nabla\) is the Levi-Civita connection of the Riemannian metric \(g\).

The aim of the paper is to define a new class of almost paracontact paracomplex Riemannian manifolds, which arise under the condition that a certain Riemannian cone over it has a paraholomorphic paracomplex Riemannian (briefly, phpCR) structure. We call it a para-Sasaki-like Riemannian manifold and give explicit examples. We find out that the class of
para-Sasaki-like Riemannian manifolds is a subclass of para-Sasakian spaces introduced by Sato in [17] (see Remark 3.4 below). Studying the structure of para-Sasaki-like Riemannian spaces, we show that the paracontact form $\eta$ is closed and a para-Sasaki-like Riemannian manifold can be considered locally as a specific product of the real line with a phpcR manifold. Recall that each phpcR manifold is locally the Riemannian product of two Riemannian spaces of equal dimension. We also get that the curvature of para-Sasaki-like Riemannian manifolds is completely determined by the curvature of the underlying local phpcR manifold and the Ricci curvature in the direction of $\xi$ is equal to $-2n$, while in the Sasaki case it is $2n$. In this sense, para-Sasaki-like manifolds can be considered as the counterpart of the Sasaki manifolds; the skew-symmetric part of $\nabla \eta$ vanishes, while in the Sasaki case the symmetric (Killing) part of $\nabla \eta$ vanishes.

We define a hyperbolic extension of a (complete) phpcR manifold, which looks like a some kind of warped product, showing that it is a (complete) para-Sasaki-like Riemannian manifold. Moreover, we show that if the starting phpcR manifold is a complete Einstein manifold of negative scalar curvature, then its hyperbolic extension is a complete Einstein para-Sasaki-like Riemannian manifold of negative scalar curvature. Thus we obtain new examples of a complete Einstein Riemannian manifold of negative scalar curvature (see Theorem 4.3 and Example 3).

In the last section, we define and study paracontact conformal/homothetic deformations and determine a subclass of these deformations preserving the para-Sasaki-like condition. We also show that the Ricci tensor of a para-Sasaki-like Riemannian space is invariant under the paracontact homothetic deformation preserving the para-Sasaki-like condition.

**Convention 1** Let $(M, \phi, \xi, \eta, g)$ be an apcpcR manifold.

(a) We shall denote the smooth vector fields on $M$ by $x, y, z, w$, i.e., $x, y, z, w \in \mathfrak{X}(M)$.

(b) We shall use $X, Y, Z, W$ to denote smooth horizontal vector fields on $M$, i.e., $X, Y, Z, W \in H = \ker(\eta)$.

## 2 Almost paracontact paracomplex Riemannian manifolds

Let $(M, \phi, \xi, \eta)$ be a $(2n + 1)$-dimensional almost paracontact paracomplex manifold, i.e., the eigenspaces of $\phi$ on the paracomplex distribution $H = \ker(\eta)$ have the same dimension $n$.

An almost paracontact paracomplex manifold is a normal almost paracontact paracomplex manifold if the corresponding almost paracomplex structure $\tilde{\phi}$ on $\tilde{M} = M \times \mathbb{R}$, defined by

$$\tilde{\phi} X = \phi X, \quad \tilde{\phi} \xi = r \frac{d}{d\tau}, \quad \tilde{\phi} \frac{d}{d\tau} = \frac{1}{r} \xi, \quad (3)$$

is integrable (i.e., $(\tilde{M}, \tilde{\phi})$ is a paracomplex manifold) [3]. The almost paracontact paracomplex structure is normal if and only if the Nijenhuis tensor $N$ of $(\phi, \xi, \eta)$ vanishes, where $N$ is defined by

$$N = [\phi, \phi] - d\eta \otimes \xi, \quad [\phi, \phi](x, y) = [\phi x, \phi y] + \phi^2 [x, y] - \phi \phi [x, y] - \phi [x, \phi y],$$

and $[\phi, \phi]$ is the Nijenhuis torsion of $\phi$ [16].

The associated metric $\tilde{g}$ of $g$ on an almost paracontact paracomplex Riemannian manifold $(M, \phi, \xi, \eta, g)$ is defined by

$$\tilde{g}(x, y) = g(x, \phi y) + \eta(x)\eta(y).$$
It is a pseudo-Riemannian metric of signature \((n + 1, n)\) (see e.g., [8]).

The almost paracontact paracomplex Riemannian manifold is also known as an almost paracontact Riemannian manifold of type \((n, n)\) [7]. The structure group of these manifolds is \(O(n) \times O(n) \times I(1)\), where \(O(n)\) and \(I(1)\) are respectively the orthogonal matrix of size \(n\) and the unit matrix of size 1.

The covariant derivatives of \(\phi, \xi, \eta\) with respect to the Levi-Civita connection \(\nabla\) of \(g\) play a fundamental role in the differential geometry of the almost paracontact Riemannian manifolds. The structure tensor \(F\) of type \((0, 3)\) on \((M, \phi, \xi, \eta, g)\) is defined by

\[
F(x, y, z) = g\left(\nabla_x \phi, y, z\right)
\]

and has the properties [7]

\[
F(x, y, z) = F(x, z, y) = -F(x, \phi y, \phi z) + \eta(y) F(x, \xi, z) + \eta(z) F(x, y, \xi),
\]

\[
(\nabla_x \eta)(y) = g(\nabla_x \xi, y) = -F(x, \phi y, \xi).
\]

The 1-forms associated with \(F\) are given by

\[
\theta(z) = \sum_{i=1}^{2n} F(e_i, e_i, z), \quad \theta^*(z) = \sum_{i=1}^{2n} F(e_i, \phi e_i, z), \quad \omega(z) = F(\xi, \xi, z)
\]

and satisfy the relations \(\theta^* \circ \phi = -\theta \circ \phi^2\), \(\omega(\xi) = 0\).

The symmetric \((1, 2)\)-tensor \(\hat{N}\) on an almost paracontact Riemannian manifold is defined by [8]

\[
\hat{N}(x, y) = \{\phi, \phi\}(x, y) - \left((\nabla_x \eta)(y) + (\nabla_x \eta)(y)\right) \xi
\]

\[
= \{\phi, \phi\}(x, y) - (L_{\xi} g)(x, y) \otimes \xi,
\]

where

\[
\{\phi, \phi\}(x, y) = \phi(x, \phi y) + \phi^2(x, y) - \phi\{\phi x, y\} - \phi\{x, \phi y\}
\]

and the symmetric bracket \(\{x, y\}\) is determined by

\[
g([x, y], z) = g(\nabla_x y + \nabla_y x, z)
\]

\[
= x g(y, z) + y g(x, z) - z g(x, y) + g([x, z], y) + g([z, y], x).
\]

The tensor \(\hat{N}\) is also called the associated Nijenhuis tensor of the almost paracontact Riemannian structure.

We denote the corresponding tensors of type \((0, 3)\) by the same letters, \(N(x, y, z) = g(N(x, y, z), \hat{N}(x, y, z) = g(\hat{N}(x, y, z), z)\). Both tensors \(N\) and \(\hat{N}\) can be expressed in terms of the fundamental tensor \(F\) as follows [8]

\[
N(x, y, z) = F(\phi x, y, z) - F(\phi y, x, z) - F(x, y, \phi z) + F(x, y, \phi z) + \eta(z)\left[F(\phi y, x, \xi) - F(y, \phi x, \xi)\right],
\]

\[
\hat{N}(x, y, z) = F(\phi x, y, z) + F(\phi y, x, z) - F(x, y, \phi z) - F(x, y, \phi z) + \eta(z)\left[F(x, \phi y, \xi) + F(y, \phi x, \xi)\right].
\]

### 2.1 Relation with paraholomorphic paracomplex Riemannian manifolds

Note that the \(2n\)-dimensional paracontact distribution \(H = \ker(\eta)\) of a para-Sasaki-like Riemannian manifold is equipped with an almost paracomplex structure \(P = \phi|_H\), a metric
\( h = g|_H \), where \( \phi|_H, g|_H \) are the restrictions of \( \phi, g \) on \( H \), respectively. The metric \( h \) is compatible with \( P \) as follows

\[
    h(PX, PY) = h(X, Y), \quad \tilde{h}(X, Y) = h(X, PY),
\]

where \( \tilde{h} \) is the associated neutral metric.

Recall that a \( 2n \)-dimensional almost paracomplex manifold \((N, P, h)\) equipped with a Riemannian metric \( h \) satisfying (9) is known as an almost paracomplex Riemannian manifold \([3,6]\) or almost product Riemannian manifold with \( \text{tr} P = 0 \) \([19–21]\). When the almost product structure \( P \) is parallel with respect to the Levi-Civita connection \( \nabla' \) of the metric \( h \), \( \nabla' P = 0 \), then the manifold is known as a Riemannian \( P \)-manifold \([20]\), a locally product Riemannian manifold or a paraholomorphic paracomplex Riemannian manifold \([10]\). In this case the almost product structure \( P \) is integrable.

Let us denote the structure \((0,3)\)-tensor of \((N, P, h)\) as follows

\[
    F'(X, Y, Z) = h((\nabla'_X P) Y, Z).
\]

The equalities \( P^2 = \text{id} \) and (10) imply the properties:

\[
    F'(X, Y, Z) = F'(X, Z, Y) = -F'(X, PY, PZ),
    F'(X, PY, Z) = -F'(X, Y, PZ).
\]

The 1-forms \( \theta' \) and \( \theta'^* \) are given by

\[
    \theta'(Z) = \sum_{i=1}^{2n} F'(e_i, e_i, Z), \quad \theta'^*(Z) = \sum_{i=1}^{2n} F'(e_i, Pe_i, Z).
\]

### 2.2 The case of parallel structures

The simplest case of almost paracontact Riemannian manifolds is when the structures are \( \nabla \)-parallel, \( \nabla\phi = \nabla\xi = \nabla\eta = \nabla g = \nabla\tilde{g} = 0 \), and it is determined by the condition \( F(x, y, z) = 0 \). In this case the distribution \( H \) is involutive. The corresponding integral submanifold is totally geodesic, it inherits a \( \text{phpcR} \) structure and the almost paracontact Riemannian manifold is locally a Riemannian product of a \( \text{phpcR} \) manifold with a real interval.

### 3 Para Sasaki-like Riemannian manifolds

In this section, we consider the Riemannian cone over an \( \text{apcpcR} \) manifold and determine a para-Sasaki-like paracontact paracomplex Riemannian manifold with the condition that its Riemannian cone is a Riemannian manifold with a paraholomorphic paracomplex structure.

#### 3.1 Paraholomorphic Riemannian cone

Let \((M, \phi, \xi, \eta, g)\) be a \((2n+1)\)-dimensional \( \text{apcpcR} \) manifold. We consider the Riemannian cone \( C(M) = M \times \mathbb{R}^+ \) over \( M \), equipped with the almost paracomplex structure \( \tilde{P} \) determined
in (3), and the Riemannian metric defined by
\[ \tilde{g}((x, a \frac{d}{dx}), (y, b \frac{d}{dy})) = r^2 g(x, y)|_H + \eta(x) \eta(y) + ab \]
\[ = r^2 g(x, y) + (1 - r^2) \eta(x) \eta(y) + ab, \] (11)
where \( r \) is the coordinate on \( \mathbb{R}^+ \) and \( a, b \) are \( C^\infty \) functions on \( M \times \mathbb{R}^+ \).

Using the general Koszul formula
\[ 2g(\nabla_x y, z) = xg(y, z) + yg(z, x) - zg(x, y) + g([x, y], z) + g([z, x], y) + g([z, y], x), \] (12)
we calculate from (11) that the non-zero components of the Levi-Civita connection \( \tilde{\nabla} \) of the Riemannian metric \( \tilde{g} \) on \( C(M) \) are given by
\[ \tilde{g}\left(\tilde{\nabla}_X Y, Z\right) = r^2 g \left(\nabla_X Y, Z\right), \quad \tilde{g}\left(\tilde{\nabla}_X \frac{d}{dx}, \right) = -rg \left(\nabla X, Y\right), \]
\[ \tilde{g}\left(\tilde{\nabla}_X Y, \xi\right) = r^2 g \left(\nabla_X Y, \xi\right) + \frac{1}{2} \left( r^2 - 1 \right) d\eta(X, Y), \]
\[ \tilde{g}\left(\tilde{\nabla}_X \xi, Z\right) = r^2 g \left(\nabla_X \xi, Z\right) - \frac{1}{2} \left( r^2 - 1 \right) d\eta(X, Z), \]
\[ \tilde{g}\left(\tilde{\nabla}_Y \xi, Y\right) = r^2 g \left(\nabla_Y \xi, Y\right) - \frac{1}{2} \left( r^2 - 1 \right) d\eta(Y, Z), \]
\[ \tilde{g}\left(\tilde{\nabla}_Y \xi, Z\right) = g \left(\nabla_X \xi, Z\right), \]
\[ \tilde{g}\left(\tilde{\nabla}_X \frac{d}{dx}, Z\right) = rg \left(\nabla X, Z\right), \quad \tilde{g}\left(\tilde{\nabla}_Y \frac{d}{dy}, Z\right) = rg \left(\nabla Y, Z\right). \]

Applying (3), we get that the non-zero components of \( \tilde{\nabla} \tilde{P} \) are given by
\[ \tilde{g}\left(\tilde{\nabla}_X \tilde{P}, Y, Z\right) = r^2 g \left(\nabla_X \phi Y, Z\right), \]
\[ \tilde{g}\left(\tilde{\nabla}_X \tilde{P}, Y, \xi\right) = r^2 \left\{ g \left(\nabla_X \phi Y, \xi\right) + g(X, Y) \right\} + \frac{1}{2} \left( r^2 - 1 \right) d\eta(X, \phi Y), \]
\[ \tilde{g}\left(\tilde{\nabla}_X \tilde{P}, Y, \frac{d}{dx}\right) = r \left\{ g \left(\nabla_X \xi, Y\right) - g(X, \phi Y) \right\} - \frac{1}{2} \left( r^2 - 1 \right) d\eta(Y, X), \]
\[ \tilde{g}\left(\tilde{\nabla}_X \tilde{P}, \xi, Z\right) = -r^2 \left\{ g \left(\nabla_X \xi, \phi Z\right) - g(X, Z) \right\} + \frac{1}{2} \left( r^2 - 1 \right) d\eta(X, \phi Z), \]
\[ \tilde{g}\left(\tilde{\nabla}_Y \tilde{P}, \frac{d}{dy}, Z\right) = r \left\{ g \left(\nabla_Y \xi, Z\right) - g(X, \phi Z) \right\} - \frac{1}{2} \left( r^2 - 1 \right) d\eta(Y, X), \]
\[ \tilde{g}\left(\tilde{\nabla}_X \tilde{P}, Y, \xi\right) = r^2 g \left(\nabla_X \phi Y, Z\right) - \frac{1}{2} \left( r^2 - 1 \right) \left\{ d\eta(\phi Y, Z) - d\eta(Y, \phi Z) \right\}, \]
\[ \tilde{g}\left(\tilde{\nabla}_X \tilde{P}, Y, \xi\right) = -g(\nabla_X \xi, \phi Y), \quad \tilde{g}\left(\tilde{\nabla}_Y \tilde{P}, Y, \xi\right) = -g(\nabla_Y \xi, \phi Y), \]
\[ \tilde{g}\left(\tilde{\nabla}_X \tilde{P}, \frac{d}{dx}, Z\right) = \frac{1}{r} g(\nabla_X \xi, Y), \quad \tilde{g}\left(\tilde{\nabla}_Y \tilde{P}, \frac{d}{dy}, Z\right) = \frac{1}{r} g(\nabla_Y \xi, Z). \] (13)

**Proposition 3.1** The Riemannian cone \( C(M) \) over an apcpcR manifold \((M, \phi, \xi, \eta, g)\) is a Riemannian manifold with a paraholomorphic paracomplex structure if and only if the following conditions hold
\[ F(X, Y, Z) = F(\xi, Y, Z) \neq 0, \] (14)
\[ F(X, Y, \xi) = -g(X, Y). \] (15)
Expressions (13) imply that \( \nabla \tilde{\nabla} = 0 \) on the Riemannian cone \((C(M), \tilde{P}, \tilde{g})\) if and only if the apcpcR manifold \((M, \phi, \xi, \eta, g)\) satisfies the conditions

\[
F(X, Y, Z) = 0, \quad \omega(Z) = 0, \quad \nabla_{\xi} \xi = 0 \tag{16}
\]

\[
F(X, Y, \xi) = -g(X, Y) - \frac{1}{2\sqrt{2}} (r^2 - 1) \, d\eta(X, \phi Y), \tag{17}
\]

\[
F(\xi, Y, Z) = \frac{1}{2\sqrt{2}} (r^2 - 1) \{d\eta(\phi Y, Z) - d\eta(Y, \phi Z)\}. \tag{18}
\]

According to (17), we get

\[
(\nabla_X \eta)(Y) = g(X, \phi Y) + \frac{1}{2\sqrt{2}} (r^2 - 1) \, d\eta(X, Y).
\]

Hence, \( d\eta(X, Y) = \frac{1}{2\sqrt{2}} (r^2 - 1) \, d\eta(X, Y) \) since \( \tilde{g} \) is symmetric. The latter equality shows \( d\eta(X, Y) = 0 \) which yields

\[
(\nabla_X \eta)(Y) = g(X, \phi Y). \tag{19}
\]

Therefore (2) holds and \((M, \phi, \xi, \eta, g)\) is a paracontact Riemannian manifold.

From (16) we get \( d\eta(\xi, X) = (\nabla_\xi \eta)(X) - (\nabla_X \eta)(\xi) = 0 \). Hence, we have \( d\eta = 0 \).

Substitute \( d\eta = 0 \) into (17) and (18) to complete the proof of the proposition.

**Definition 3.2** A manifold \((M, \phi, \xi, \eta, g)\) is said to be a para-Sasaki-like paracontact para-Riemannian manifold (for short, a para-Sasaki-like Riemannian manifold) if the structure tensors \( \phi, \xi, \eta, g \) satisfy equalities (14) and (15).

To characterize para-Sasaki-like Riemannian manifolds, we need the general formula for an apcpcR manifold \((M, \phi, \xi, \eta, g)\) from [8]

\[
g(\nabla_X \phi)_{y, z} = F(x, y, z)
\]

\[
= \frac{1}{4}\left[N(\phi x, y, z) + N(\phi y, z, y) + \tilde{N}(\phi x, y, z) + \tilde{N}(\phi y, z, y)\right] \tag{20}
\]

\[
- \frac{1}{2}\eta(x)\left[N(\xi, y, \phi z) + \tilde{N}(\xi, y, \phi z) + \eta(z)\tilde{N}(\xi, \xi, \phi y)\right].
\]

The next result determines para-Sasaki-like Riemannian manifolds by the structure tensors.

**Theorem 3.3** Let \((M, \phi, \xi, \eta, g)\) be an apcpcR manifold. The following conditions are equivalent:

(a) The manifold \((M, \phi, \xi, \eta, g)\) is para-Sasaki-like Riemannian;
(b) The covariant derivative \( \nabla \phi \) satisfies the equality

\[
(\nabla_X \phi)_y = -g(x, y)\xi - \eta(y)x + 2\eta(x)\eta(y)\xi = -g(\phi x, \phi y)\xi - \eta(y)\phi^2 x; \tag{21}
\]

(c) The Nijenhuis tensors \( N \) and \( \tilde{N} \) satisfy the conditions:

\[
N = 0, \quad \tilde{N} = -4(\tilde{g} - \eta \otimes \eta) \otimes \xi. \tag{22}
\]

**Proof** It is easy to verify, using (5), that (21) is equivalent to the system of equations (14) and (15), which establishes the equivalence between a) and b) in view of Proposition 3.1.

Substitute (21) successively in (7) and (8) to get (22), which gives the implication b) \(\Rightarrow\) c).

Suppose (22) holds. Then (21) follows from (22) and (20). This completes the proof.
Remark 3.4 Equality (21) shows that a para-Sasaki-like Riemannian manifold is a special case of the para-Sasakian manifold considered by Sato in [17] (see also [13]), namely, a para-Sasaki-like Riemannian manifold is para-Sasakian such that the endomorphism $\phi$ restricted on the paracontact distribution $H = \ker(\eta)$ defines a paracomplex structure on $H$.

Corollary 3.5 Let $(M, \phi, \xi, \eta, g)$ be a para-Sasaki-like Riemannian manifold. Then we have:

(a) the manifold $(M, \phi, \xi, \eta, g)$ is a normal paracontact Riemannian manifold, $N = 0$, $2\tilde{g}\vert_H = \mathcal{L}_\xi g$, the fundamental 1-form $\eta$ is closed, $d\eta = 0$, and the integral curves of $\xi$ are geodesics, $\nabla_\xi \eta = 0$;

(b) the 1-forms $\theta$ and $\theta^*$ satisfy the equalities $\theta = -2n\eta$ and $\theta^* = 0$, respectively.

3.2 Example 1: Solvable Lie group as a para-Sasaki-like Riemannian manifold

Consider the solvable Lie group $G$ of dimension $2n + 1$ with a basis of left-invariant vector fields $\{e_0, \ldots, e_{2n}\}$ defined by the commutators

\[ [e_0, e_1] = -e_{n+1}, \ldots, [e_0, e_n] = -e_{2n}, \]
\[ [e_0, e_{n+1}] = -e_1, \ldots, [e_0, e_{2n}] = -e_n. \]  (23)

Define an invariant apcpcR structure on $G$ by

\[ g(e_i, e_i) = 1, \quad g(e_i, e_j) = 0, \quad i, j \in \{0, 1, \ldots, 2n\}, \quad i \neq j, \]
\[ \xi = e_0, \quad \phi e_1 = e_{n+1}, \ldots, \phi e_n = e_{2n}. \]  (24)

Using the Koszul formula (12), we check that (14) and (15) are fulfilled, i.e., $(G, \phi, \xi, \eta, g)$ is a para-Sasaki-like Riemannian manifold.

Let $e^0 = \eta$, $e^1, \ldots, e^{2n}$ be the corresponding dual 1-forms, $\epsilon^i(e_j) = \delta^i_j$. It follows from (23) that the structure equations of the group are

\[ de^0 = d\eta = 0, \quad de^1 = e^0 \wedge e^{n+1}, \ldots, de^n = e^0 \wedge e^{2n}, \]
\[ de^{n+1} = e^0 \wedge e^1, \ldots, de^{2n} = e^0 \wedge e^n. \]  (25)

and the para-Sasaki-like Riemannian structure has the form

\[ g = \sum_{i=0}^{2n} (e_i)^2, \quad \phi e^0 = 0, \quad \phi e^1 = e^{n+1}, \ldots, \phi e^n = e^{2n}. \]  (26)

The basis of dual 1-forms can be the following

\[ e^0 = dt, \quad e^i = \cosh(t)dx^i + \sinh(t)dx^{n+i}, \]
\[ i \in \{1, 2, \ldots, n\}, \quad e^{n+i} = \sinh(t)dx^i + \cosh(t)dx^{n+i}. \]  (27)

The 1-forms defined in (27) satisfy (25) and the para-Sasaki-like Riemannian metric has the form

\[ g = dt^2 + \cosh(2t) \sum_{i=1}^{2n} (dx^i)^2 + \sinh(2t) \sum_{i=1}^{n} dx^i dx^{n+i}. \]  (28)

Equalities (23), (26)–(28) imply that the distribution $H = \text{span}\{e_1, \ldots, e_{2n}\}$ is integrable and the corresponding integral submanifold can be considered as the flat space.
\[ \mathbb{R}^{2n} = \text{span}\{dx^1, \ldots, dx^{2n}\} \]\n
with the following \text{phpcR} structure

\[ Pdx^1 = dx^{n+1}, \ldots, Pdx^n = dx^{2n}; \quad h = \sum_{i=1}^{2n} (dx^i)^2, \quad \tilde{h} = 2 \sum_{i=1}^{n} dx^i dx^{n+i}. \]

### 3.3 Hyperbolic extension of a paraholomorphic paracomplex Riemannian manifold

Inspired by Example 1, we propose a more general construction. Let \((N^{2n}, J, h, \tilde{h})\) be a 2n-dimensional \text{phpcR} manifold, i.e., the almost product structure \(P\) has \(\text{tr } P = 0\), acts as an isometry on the metric \(h\), \(h(PX, PY) = h(X, Y)\) and it is parallel with respect to the Levi-Civita connection of \(h\). In particular, the almost paracomplex structure \(P\) is integrable. The associated neutral pseudo-Riemannian metric \(\tilde{h}\) is defined by \(\tilde{h}(X, Y) = h(PX, Y)\) and it is also parallel with respect to the Levi-Civita connection of \(h\).

Consider the product manifold \(M^{2n+1} = \mathbb{R} \times N^{2n}\). Let \(dr\) be the coordinate 1-form on \(\mathbb{R}\) and define an \text{apcpcR} structure on \(M^{2n+1}\) as follows

\[ \eta = dr, \quad \phi|_H = P, \quad \eta \circ \phi = 0, \quad g = dr^2 + \cosh(2t)h + \sinh(2t)\tilde{h}. \quad (29) \]

**Theorem 3.6** Let \((N^{2n}, P, h, \tilde{h})\) be a 2n-dimensional \text{phpcR} manifold. Then the product manifold \(M^{2n+1} = \mathbb{R} \times N^{2n}\), equipped with the \text{apcpcR} Riemannian structure defined in (29), is a para-Sasaki-like Riemannian manifold. If the Riemannian manifold \((N^{2n}, h)\) is complete then the para-Sasaki-like Riemannian manifold \((M^{2n+1}, g) = (\mathbb{R} \times N^{2n}, g)\) is complete.

**Proof** To show that the metric \(g\) is Riemannian, we consider an orthonormal basis for \(h\) of the form \(\{e_1, Pe_1, \ldots, e_n, Pe_n\}\). Then the matrix of \(g\) with respect to the basis \(\{\xi = \partial_t, e_1, Pe_1, \ldots, e_n, Pe_n\}\) has the form

\[
\begin{pmatrix}
1 & o & o & \cdots & o \\
o^\top & A & O & \cdots & O \\
o^\top & O & A & \cdots & O \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
o^\top & O & O & \cdots & A
\end{pmatrix}
\]

where we have denoted

\[ A = \begin{pmatrix} \cosh(2t) & \sinh(2t) \\ \sinh(2t) & \cosh(2t) \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad o = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad o^\top = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

The matrix of \(g\) is positive-definite since all its principal minors are positive due to the identity \(\cosh^2(2t) - \sinh^2(2t) = 1\).

It is easy to check, using (12), (29) and the fact that the paracomplex structure \(P\) is parallel with respect to the Levi-Civita connection of \(h\), that the structure defined in (29) satisfies (14) and (15) and thus \((M, \phi, \xi, \eta, g)\) is a para-Sasaki-like Riemannian manifold.

To show that the metric \(g\) on \(M^{2n+1} = \mathbb{R} \times N^{2n}\) is complete, we observe that the metric \(dr^2\) on \(\mathbb{R}\) is complete. If the Riemannian metric \(h\) on \(N^{2n}\) is complete, then the Riemannian metrics \(g|_N(t)\) on \(N^{2n}\) belonging to the one-parameter family

\[ g|_N(t) = \cosh(2t)h + \sinh(2t)\tilde{h} \]

are complete since their Levi-Civita connections coincide with the Levi-Civita connection of \(h\) (see (43) below). Now apply [2, Lemma 2] to conclude that \(g\) is complete.
We call the para-Sasaki-like Riemannian manifold constructed in Theorem 3.6 a hyperbolic extension of a paraholomorphic paracomplex Riemannian manifold.

### 3.4 Example 2: Lie group of dimension 5 as a hyperbolic extension of a pphpcR manifold

Let us consider the Lie group $G^5$ of dimension 5 having a basis of left-invariant vector fields $\{e_0, \ldots, e_4\}$ with the commutators

$$
[e_0, e_1] = \lambda e_2 - e_3 + \mu e_4, \quad [e_0, e_2] = -\lambda e_1 - \mu e_3 - e_4, \\
[e_0, e_3] = -e_1 + \mu e_2 + \lambda e_4, \quad [e_0, e_4] = -\mu e_1 - e_2 - \lambda e_3, \quad \lambda, \mu \in \mathbb{R}.
$$

We equip $G^5$ with an invariant pphpcR structure as in (24) for $n = 2$. Then, using (12), we calculate that the non-zero components of the Levi-Civita connection are:

$$
\nabla_{e_0} e_1 = \lambda e_4 + \mu e_4, \quad \nabla_{e_1} e_0 = e_3, \quad \nabla_{e_2} e_0 = -\lambda e_1 - \mu e_3, \quad \nabla_{e_4} e_0 = e_4, \\
\nabla_{e_0} e_3 = \mu e_4 + \lambda e_4, \quad \nabla_{e_3} e_0 = e_1, \quad \nabla_{e_2} e_4 = -\mu e_1 - \lambda e_3, \quad \nabla_{e_4} e_2 = -e_0.
$$

Similarly, as in Example 1, we verify that the constructed manifold $(G^5, \phi, \xi, \eta, g)$ is a para-Sasaki-like Riemannian manifold.

We consider the case for $\mu = 0$ and $\lambda \neq 0$. By virtue of (30), the structure equations of the group become

$$
de^0 = dt, \quad de^1 = \lambda e^0 \wedge e^2 + e^0 \wedge e^3, \quad de^2 = -\lambda e^0 \wedge e^1 + e^0 \wedge e^4, \quad de^3 = e^0 \wedge e^1 + \lambda e^0 \wedge e^4, \quad de^4 = e^0 \wedge e^2 - \lambda e^0 \wedge e^3.
$$

A basis of 1-forms satisfying (31) is given by $e^0 = dt$ and

$$
e^1 = f_1 \ dx^1 + f_2 \ dx^2 + f_3 \ dx^3 + f_4 \ dx^4, \quad e^2 = -f_3 \ dx^1 - f_4 \ dx^2 + f_1 \ dx^3 + f_2 \ dx^4, \\
e^3 = f_1 \ dx^1 - f_2 \ dx^2 + f_3 \ dx^3 - f_4 \ dx^4, \quad e^4 = -f_3 \ dx^1 + f_4 \ dx^2 + f_1 \ dx^3 - f_2 \ dx^4,
$$

where

$$
f_1 = \exp(t) \cos(\lambda t), \quad f_2 = \exp(-t) \cos(\lambda t), \quad f_3 = \exp(t) \sin(\lambda t), \quad f_4 = \exp(-t) \sin(\lambda t).
$$

Then the para-Sasaki-like Riemannian metric has the form

$$
g = dt^2 + 2 \exp(2t) \left( dx^1 \right)^2 + 2 \exp(-2t) \left( dx^2 \right)^2 + 2 \exp(2t) \left( dx^3 \right)^2 + 2 \exp(-2t) \left( dx^4 \right)^2,
$$

which can be written as follows

$$
g = dt^2 + 2 \cosh(2t) \left\{ \left( dx^1 \right)^2 + \left( dx^2 \right)^2 + \left( dx^3 \right)^2 + \left( dx^4 \right)^2 \right\} + 2 \sinh(2t) \left\{ \left( dx^1 \right)^2 - \left( dx^2 \right)^2 + \left( dx^3 \right)^2 - \left( dx^4 \right)^2 \right\}.
$$
Therefore, the associated metric \( \tilde{h}(X, Y) = h(X, PY) \) is given by
\[
\tilde{h} = (dx^1)^2 - (dx^2)^2 + (dx^3)^2 - (dx^4)^2
\]
and the para-Sasaki-like Riemannian metric (32) takes the form as in (29).

\section{4 Curvature properties of para-Sasaki-like Riemannian manifolds: Einstein condition}

We consider an apcpcR manifold \((M, \phi, \xi, \eta, g)\) of dimension \(2n + 1\). Its curvature tensor of type \((1, 3)\) is defined by \( R = [\nabla, \nabla] - \nabla[\cdot, \cdot] \). The corresponding curvature tensor of type \((0, 4)\) is denoted by the same letter and is determined by \( R(x, y, z, w) = g(R(x, y)z, w) \). The Ricci tensor \( Ric \), the scalar curvature \( Scal \) and the \( * \)-scalar curvature \( Scal^* \) are the usual traces of the curvature
\[
Ric(x, y) = \sum_{i=0}^{2n} R(e_i, x, y, e_i), \quad Scal = \sum_{i=0}^{2n} Ric(e_i, e_i), \quad Scal^* = \sum_{i=0}^{2n} Ric(e_i, \phi e_i)
\]
with respect to an arbitrary orthonormal basis \( \{e_0, \ldots, e_{2n}\} \) of its tangent space.

\textbf{Proposition 4.1} \textit{On a para-Sasaki-like Riemannian manifold \((M, \phi, \xi, \eta, g)\) the following formula holds}

\[
R(x, y, \phi z, w) - R(x, y, z, \phi w)
= -[g(y, z) - 2\eta(y)\eta(z)] g(x, \phi w) - [g(y, w) - 2\eta(y)\eta(w)] g(x, \phi z)
+ [g(x, z) - 2\eta(x)\eta(z)] g(y, \phi w) + [g(x, w) - 2\eta(x)\eta(w)] g(y, \phi z).
\]

\textit{In particular, we have}
\[
R(x, y)\xi = -\eta(y)x + \eta(x)y, \quad [X, \xi] \in H, \quad \nabla_\xi X = \phi X - [X, \xi] \in H, \\
R(\xi, X)\xi = X, \quad Ric(x, \xi) = -2n \eta(y), \quad Ric(\xi, \xi) = -2n.
\]

\textbf{Proof} Applying (21) to the Ricci identity for \( \phi \), i.e.,
\[
R(x, y, \phi z, w) - R(x, y, z, \phi w) = g\left((\nabla_x \nabla_y \phi) z, w\right) - g\left((\nabla_y \nabla_x \phi) z, w\right),
\]
and using (19), we obtain (33) by straightforward calculations. Equality (33) for \( z = \xi \) implies (34) due to (1). The assertions in (35) follow from (19) and \( d\eta = 0 \). Equalities (36) are direct consequences of (34).

Note that (34) and (36) are proved for para-Sasakian manifold in [13].
4.1 The horizontal curvature and the Einstein condition

From $\mathrm{d}\eta = 0$ it follows that locally $\eta = \mathrm{d}t$, where $t$ is the coordinate of $\mathbb{R}$. Then, $H = \ker \eta$ is integrable and we get locally the product $M^{2n+1} = \mathbb{R} \times N^{2n}$ with $T N^{2n} = H$. As a result, the submanifold $(N^{2n}, P = \phi|_H, h = g|_H)$ is a phpcR manifold. In fact, by (14) we get that $h \left( (\nabla_X^h) P, Y, Z \right) = F(X, Y, Z) = 0$, where $\nabla^h$ is the Levi-Civita connection of $h$.

The submanifold $N^{2n}$ can be considered as a hypersurface of $M^{2n+1}$ with unit normal $\xi = \frac{\mathrm{d}}{\mathrm{d}t}$. Equality (19) yields

$$g(\nabla_X \xi, Y) = -g(\nabla_X Y, \xi) = g(X, \phi Y) = \tilde{g}|_H(X, Y), \quad \nabla_\xi \xi = 0.$$  

The second fundamental form is therefore equal to $-\tilde{g}|_H = -\tilde{h}$. Then, the Gauss equation (see e.g. [5, Chapter VII, Proposition 4.1]) has the form

$$R(X, Y, Z, W) = R^h(X, Y, Z, W) + g(X, \phi Z)g(Y, \phi W)$$

$$-g(Y, \phi Z)g(X, \phi W),$$

where $R^h$ is the curvature tensor of the phpcR manifold $(N^{2n}, P, h)$.

For the horizontal Ricci tensor, we obtain from (36) and (37) that

$$\text{Ric}(Y, Z) = \sum_{i=1}^{2n} R(e_i, Y, Z, e_i) + R(\xi, Y, Z, \xi)$$

$$= \text{Ric}^h(Y, Z) + g(\phi Y, \phi Z) - g(Y, Z) = \text{Ric}^h(Y, Z),$$

where $\text{Ric}^h$ is the Ricci tensor of $h = g|_H$.

On a para-Sasaki-like Riemannian manifold, the curvature tensor in the direction of $\xi$ is completely determined by $\eta, \phi, g, \tilde{g}$ due to Proposition 4.1. In fact, the properties of the Riemannian curvature and (34) yield

$$R(x, y, z, \xi) = R(\xi, z, y, x) = -\eta(x)g(y, z) + \eta(y)g(x, z).$$

Then (37) and (39) imply that the Riemannian curvature of a para-Sasaki-like Riemannian manifold is completely determined by the curvature of the underlying phpcR manifold $(N^{2n}, T N^{2n} = H, P, h)$ as follows

$$R(x, y, z, w) = R^h(x|_H, y|_H, z|_H, w|_H) - g(y, \phi z)g(x, \phi w) + g(x, \phi z)g(y, \phi w)$$

$$- (g(y, z)\eta(x) - g(x, z)\eta(y))\eta(w)$$

$$- (g(x, w)\eta(y) - g(y, w)\eta(x))\eta(z).$$

For the Ricci tensor and the scalar curvatures, we get:

$$\text{Ric}(y, z) = \text{Ric}^h(y, z) - 2n \eta(y)\eta(z),$$

$$\text{Scal} = \text{Scal}^h - 2n, \quad \text{Scal}^* = \text{Scal}^{h*}.$$  

We obtain from (40), or comparing (38) with (36), the next

**Proposition 4.2** A para-Sasaki-like Riemannian manifold $(M, \phi, \xi, g)$ is Einstein if and only if the underlying local phpcR manifold $(N^{2n}, P, h)$ is Einstein with negative scalar curvature $-4n^2$, i.e.,

$$\text{Ric}^h = -2n h.$$  

Proposition 4.2 allows the construction of a new Einstein manifold (see Example 3 below). We have

___
Theorem 4.3 Let \((N^{2n}, P, h^N)\) be a 2n-dimensional Einstein \(phpcR\) manifold with negative scalar curvature \(-4n^2\), i.e., its Ricci tensor satisfies (41). Then its hyperbolic extension, the \((2n+1)\)-dimensional space \((M^{2n+1} = \mathbb{R} \times N^{2n}, g, \phi, \eta)\) with the \(appcR\) structure \((g, \phi, \eta)\) on \(M^{2n+1}\), defined by

\[
\eta = dt, \quad \phi|_H = P, \quad \eta \circ \phi = 0, \quad g = dt^2 + \cosh(2t) h^N + \sinh(2t) \tilde{h}^N,
\]
is an Einstein para-Sasaki-like Riemannian manifold of negative scalar curvature.

If the Einstein Riemannian manifold \((N^{2n}, h)\) is complete, then the para-Sasaki-like Riemannian manifold \((M^{2n+1}, g) = (\mathbb{R} \times N^{2n}, g)\) is a complete Einstein Riemannian manifold of negative scalar curvature.

Proof According to Theorem 3.6, it remains to show that the Einstein condition on the Riemannian manifold \((M^{2n+1}, g)\) holds.

The horizontal metrics, i.e., the Riemannian metric \(h\) and the pseudo-Riemannian metric \(\tilde{h}\) of signature \((n, n)\) on \(N^{2n}\), are

\[
h = g|_H = \cosh(2t) h^N + \sinh(2t) \tilde{h}^N, \\
\tilde{h} = \tilde{g}|_H = \sinh(2t) h^N + \cosh(2t) \tilde{h}^N. \tag{42}
\]

The Levi-Civita connection \(\nabla_{h^N}\) of the metric \(h^N\) coincides with the Levi-Civita connection of \(\tilde{h}^N\) since \(\nabla_{h^N} P = 0\). Using this fact, the Koszul formula gives

\[
2g \left( \nabla^g_X Y, Z \right) = \cosh(2t) h^N \left( \nabla^h_X Y, Z \right) + \sinh(2t) \tilde{h}^N \left( \nabla^h_X Y, Z \right) \\
= 2h \left( \nabla^h_X Y, Z \right), \tag{43}
\]

\[
2g \left( \nabla^g_X \xi, Y \right) = \xi \ g(X, Y) = 2 \sinh(2t) h^N (X, Y) + 2 \cosh(2t) \tilde{h}^N (X, Y) \\
= 2 \tilde{g}(X, Y)
\]

for \(X, Y, Z \in TN^{2n}\).

The first equality in (43) shows that the Levi-Civita connection \(\nabla^h\) of the horizontal metric \(h\) coincides with the Levi-Civita connection \(\nabla_{h^N}\), \(\nabla^h = \nabla_{h^N}\). Now, (42) yields the next formula for the curvature of \(h\)

\[
R^h = \cosh(2t) \ R^{h^N} + \sinh(2t) \ \tilde{R}^{h^N}, \quad \tilde{R} := PR. \tag{44}
\]

Taking the trace in (44), we get that the Ricci tensor is given by

\[
Ric^h (X, Y) = \cosh(2t) \ Ric^{h^N} (X, Y) + \sinh(2t) \ Ric^{h^N} (X, PY). \tag{45}
\]

Now, (41), (42) and (45) imply

\[
Ric^h (X, Y) = -2n \left\{ \cosh(2t) h^N (X, Y) + \sinh(2t) \tilde{h}^N (X, Y) \right\} \\
= -2n \ h(X, Y). \tag{46}
\]

The second equality in (43) tells us that the manifold \(N^{2n}\) can be considered as a hypersurface of \(M^{2n+1}\) with second fundamental form equal to \(-\tilde{g}\). This fact combined with (46) and Proposition 4.2 gives that the para-Sasaki-like Riemannian manifold \((M^{2n+1}, \phi, \xi = \frac{d}{dt}, \eta = dt, g)\) is an Einstein Riemannian manifold of negative scalar curvature equal to \(-2n(2n+1)\).
4.2 Example 3: Complete para-Sasaki-like Einstein space as a hyperbolic extension

Consider the product of two complete $n$-dimensional Einstein Riemannian manifolds of negative scalar curvature equal to $-2n^2$. For example, take the product of two discs $N^{2n} = D^n \times D^n$ with the Poincaré metric $h^N = g_D \times g_D$. The usual product structure $P$ on $N^{2n}$ is defined by $PA = A, PB = B$ for $(A, B) \in TD^n \times TD^n$. Thus we obtain a complete Einstein phpcR manifold $(N^{2n}, P, h^N)$, whose Ricci tensor satisfies (41). The product manifold $M^{2n+1} = \mathbb{R} \times N^{2n}$ with the metric $g = dr^2 + \cosh(2t)h^N + \sinh(2t)\tilde{h}^N$ is a complete Einstein para-Sasaki-like Riemannian manifold according to Theorem 4.3.

4.3 Example 4: Hyperbolic extension of a $P$-invariant sphere in a flat space

The example below illustrates Theorem 3.6. Let us consider the real space $\mathbb{R}^{2n+2} = \{(x^1, \ldots, x^{2n+2})\}$ for $n \geq 2$ as a flat phpcR manifold. The canonical paracomplex structure $P'$ and the canonical $P'$-compatible Riemannian metrics $h'$ and $\tilde{h}'$ are given by

$$P'x' = (x^{n+2}, \ldots, x^{2n+2}, x^1, \ldots, x^{n+1}),$$

$$h'(x', y') = \sum_{i=1}^{2n+2} (x^i y^i), \quad \tilde{h}'(x', y') = \sum_{i=1}^{n+1} (x^i y^{n+i+1} + x^{n+i+1} y^i)$$

for arbitrary vectors $x' = (x^1, \ldots, x^{2n+2})$ and $y' = (y^1, \ldots, y^{2n+2})$ in $\mathbb{R}^{2n+2}$.

It is clear that $P', h', \tilde{h}'$ satisfy (9), the Levi-Civita connection $\nabla'$ of the Riemannian metric $h'$ preserves the paracomplex structure $P'$, $\nabla' P' = 0$, and therefore we have a phpcR manifold.

The so-called invariant hypersurface $S^{2n}_h(z_0; a, b)$ in the considered phpcR manifold $(\mathbb{R}^{2n+2}, h', P')$ is studied in [20,21]. We outline the construction below.

Identifying the point $z' = (z^1, \ldots, z^{2n+2})$ in $\mathbb{R}^{2n+2}$ with its position vector $z'$, we consider the $P'$-invariant hypersurface $S^{2n}_h(z_0; a, b)$ defined by the equations

$$h'(z' - z_0, z' - z_0) = a, \quad \tilde{h}'(z' - z_0, z' - z_0) = b,$$

where $(0, 0) \neq (a, b) \in \mathbb{R}^2, a > |b|$. The codimension-two submanifold $S^{2n}_h(z_0; a, b)$ is the intersection of the standard $(2n + 1)$-dimensional sphere with the standard hyperboloid in $\mathbb{R}^{2n+2}$. Obviously, this submanifold is $P'$-invariant. The restriction of $h'$ on $S^{2n}_h(z_0; a, b)$ has rank $2n$ due to the condition $(0, 0) \neq (a, b)$. The phpcR structure $(P', h')$ on $\mathbb{R}^{2n+2}$ inherits a phpcR structure $(P' = P'|_{S^{2n}_h}, h = h'|_{S^{2n}_h})$ on $S^{2n}_h(z_0; a, b)$ for $n \geq 2$ which is sometimes called a $P$-invariant sphere with center $z_0$ and a pair of parameters $(a, b)$ [20].

The curvature tensor of $S^{2n}_h(z_0; a, b)$ is given by the formula [18] (see also [21])

$$R'|_{S^{2n}_h} = \frac{1}{a^2 - b^2} \left\{ a \left( \pi_1^{h'} + \pi_2^{h'} \right) - b \pi_3^{h'} \right\},$$

where $2\pi_1^{h'} = h'|_{S^{2n}_h} \otimes h'|_{S^{2n}_h}, 2\pi_2^{h'} = \tilde{h}'|_{S^{2n}_h} \otimes \tilde{h}'|_{S^{2n}_h}, \pi_3^{h'} = h'|_{S^{2n}_h} \otimes \tilde{h}'|_{S^{2n}_h}$ and $\otimes$ stands for the Kulkarni–Nomizu product of two $(0, 2)$-tensors; for example,

$$(h \otimes \tilde{h})(X, Y, Z, W) = h(Y, Z)\tilde{h}(X, W) - h'(X, Z)\tilde{h}(Y, W) + \tilde{h}(Y, Z)h(X, W) - \tilde{h}(X, Z)h(Y, W).$$
Consequently, we derive
\[
\begin{align*}
Ric'\big|_{S_h^{2n}} &= \frac{2(n - 1)}{a^2 - b^2} \left( a h'\big|_{S_h^{2n}} - b \tilde{h}'\big|_{S_h^{2n}} \right), \\
Scal'\big|_{S_h^{2n}} &= \frac{4n(n - 1)a}{a^2 - b^2}, \quad Scal''\big|_{S_h^{2n}} = -\frac{4n(n - 1)b}{a^2 - b^2}.
\end{align*}
\] (48)

The product manifold \( M^{2n+1} = \mathbb{R} \times S_h^{2n}(\zeta_0'; a, b) \) equipped with the apcpcR structure \((\phi, \xi, \eta, g)\) given in (29) is a para-Sasaki-like Riemannian manifold according to Theorem 3.6.

Following the proof of Theorem 4.3, we get from (44) and (47) the next formula for the horizontal curvature
\[
R^h = \frac{1}{a^2 - b^2} \left[ \cosh(2t) \left[ a \left( \pi_1^{h'} + \pi_2^{h'} \right) - b \pi_3^{h'} \right] + \sinh(2t) \left[ a \pi_3^{h'} - b \left( \pi_1^{h'} + \pi_2^{h'} \right) \right] \right]
\]
\[
= \frac{1}{a^2 - b^2} \left[ \left[ a \cosh(2t) - b \sinh(2t) \right] \left( \pi_1^{h'} + \pi_2^{h'} \right) \right.
\]
\[
- \left. \left[ b \cosh(2t) - a \sinh(2t) \right] \pi_3^{h'} \right].
\] (49)

Taking into account (37), (42) and (49), we obtain that the horizontal curvature \( R\big|_H \) of the para-Sasaki-like Riemannian manifold \( M^{2n+1} = \mathbb{R}^+ \times S_h^{2n}(\zeta_0'; a, b) \) is given by
\[
R\big|_H = R^h + \sinh^2(2t) \pi_1^h + \cosh^2(2t) \pi_2^h - \sinh(2t) \cosh(2t) \pi_3^h
\]
\[
= \frac{1}{a^2 - b^2} \left[ \left[ a \cosh(2t) + b \sinh(2t) \right] \left( \pi_1^h + \pi_2^h \right) \right.
\]
\[
- \left. \left[ b \cosh(2t) + a \sinh(2t) \right] \pi_3^h \right].
\]

Then (38), (42) and (48) imply
\[
Ric\big|_H = Ric^h
\]
\[
= \frac{2(n - 1)}{a^2 - b^2} \left[ \left[ a \cosh(2t) + b \sinh(2t) \right] g - \eta \otimes \eta \right] - \left[ b \cosh(2t) + a \sinh(2t) \right] \tilde{g}
\]
\[
- 2n \eta \otimes \eta.
\]

The latter equality, (40), (42) and (49) give
\[
Ric = \frac{2(n - 1)}{a^2 - b^2} \left[ \left[ a \cosh(2t) + b \sinh(2t) \right] (g - \eta \otimes \eta) \right.
\]
\[
- \left[ b \cosh(2t) + a \sinh(2t) \right] g \]
\[
- 2n \eta \otimes \eta.
\]

Therefore, the para-Sasaki-like Riemannian manifold \( M^{2n+1} = \mathbb{R} \times S_h^{2n}(\zeta_0'; a, b) \) has a Ricci tensor of the form \( Ric = \alpha(t)g + \beta(t)\tilde{g} + \gamma(t)\eta \otimes \eta \) for some smooth functions \( \alpha(t) \), \( \beta(t) \) and \( \gamma(t) \). Such a manifold is sometimes called an \textit{almost Einstein-like space} [9].
5 Paraconformal conformal transformations

Let \((M, \phi, \xi, \eta, g)\) be an apcpcR manifold. The transformation

\[
\bar{\phi} = \phi, \quad \bar{\xi} = \exp(-w)\xi, \quad \bar{\eta} = \exp(w)\eta,
\]

\[
\bar{g}(x, y) = \exp(2u) \cosh(2v)g(x, y) + \exp(2u)\sinh(2v)g(x, \phi y) + \{\exp(2w) - \exp(2u) \cosh(2v)\}\eta(x)\eta(y),
\]

where \(u, v, w\) are smooth functions on \(M\), is called a paraconformal conformal transformation of \((\phi, \xi, \eta, g)\). It is easy to verify that \((M, \phi, \bar{\xi}, \bar{\eta}, \bar{g})\) is again an apcpcR manifold and the paraconformal conformal transformations on an apcpcR manifold form a group. When \(u, v, w\) are constants, we have a paracontact homothetic transformation.

In this section we study the behavior of the para-Sasaki-like condition under paraconformal conformal transformations.

**Lemma 5.1** Let \((M, \phi, \xi, \eta, g)\) and \((M, \phi, \bar{\xi}, \bar{\eta}, \bar{g})\) be related by a paraconformal conformal transformation. Then we have

\[
2\bar{F}(x, y, z) = \exp(2u)\left\{\cosh(2v) [2F(x, y, z) - F_2(x, y, z)] + \sinh(2v)F_1(x, y, z) + 2\{\chi_1(z)g(\phi x, \phi y) + \chi_1(y)g(\phi x, \phi z) + \chi_2(z)g(x, \phi y) + \chi_2(y)g(x, \phi z)\}\right\}
\]

where

\[
F_1(x, y, z) = F(x, y, z) + F(\phi y, x, z) - F(z, x, \phi y)
\]

\[
F_2(x, y, z) = [F(x, y, \xi) - F(\phi y, \phi x, \xi)]\eta(z) + [F(x, z, \xi) - F(\phi z, \phi x, \xi)]\eta(y) + [F(y, z, \xi) - F(\phi z, \phi y, \xi)]\eta(x),
\]

\[
\chi_1(z) = \cosh(2v) [du(\phi z) - dv(z)] + \sinh(2v) [dv(\phi z) - du(z)],
\]

\[
\chi_2(z) = \cosh(2v) [dv(\phi z) - du(z)] + \sinh(2v) [du(\phi z) - dv(z)].
\]

**Proof** The Koszul equality (12) for the Levi-Civita connection \(\nabla\) of \(\bar{g}\), (4), (5), (6), (42) and (50) yield

\[
2\bar{g}(\nabla_x y, z) = 2\exp(2u)\left\{\cosh(2v) g(\nabla_x y, z) + \sinh(2v) [g(\nabla_x y, \phi z) + F_3(x, y, z)]
\right\}
\]

\[
+ \psi_1(x)g(\phi x, \phi y) + \psi_1(y)g(\phi x, \phi z) - \psi_1(z)g(\phi x, \phi y) + \psi_2(x)g(\phi y, \phi z) - \psi_2(z)g(x, \phi y)
\]

\[
+ \{\exp(2w) - \exp(2u) \cosh(2v)\} \left\{2\eta(\nabla_x y)\eta(z) + F_4(x, y, z)\right\}
\]

\[
+ 2\exp(2w) \{\eta(y)\eta(z)dw(x) + \eta(x)\eta(z)dw(y) - \eta(x)\eta(y)dw(z)\}
\]

where \(\psi_1 = \cosh(2v)du + \sinh(2v)dv\), \(\psi_2 = \cosh(2v)dv + \sinh(2v)du\),

\[
F_3(x, y, z) = \frac{1}{2} \{F(x, y, z) + F(y, x, z) - F(z, x, y)\},
\]

\[
F_4(x, y, z) = [F(x, \phi y, \xi) - F(\phi y, \phi x, \xi)]\eta(x) + [F(x, \phi x, \xi) - F(x, \phi z, \xi)]\eta(y)
\]

\[
- [F(x, \phi y, \xi) + F(y, \phi x, \xi)]\eta(z).
\]
The form of (51) follows from (4) and (52).

Substitute (50) into (21) to get that the para-Sasaki-like condition for the metric \( \bar{g} \) reads

\[
\bar{F}(x, y, z) = -\exp(w + 2u) \left\{ \cosh(2v) \left[ \eta(z)g(\phi x, \phi y) + \eta(y)g(\phi x, \phi z) \right] \\
+ \sinh(2v) \left[ \eta(z)g(x, \phi y) + \eta(y)g(x, \phi z) \right] \right\}. \tag{53}
\]

Substitute (21) into (51) to get

\[
\bar{F}(x, y, z) = \exp(2w)\eta(x) \left\{ \eta(y)d\varphi(\phi z) + \eta(z)d\varphi(\phi y) \right\} \\
- \exp(2u) \left\{ \cosh(2v)\eta(z)g(\phi x, \phi y) \\
+ \cosh(2v)\eta(y)g(x, \phi y) + \cosh(2v)\eta(z)g(\phi x, \phi z) \\
+ \cosh(2v)\eta(y)g(x, \phi z) \right\}.
\]

Then, (53) and (54) imply

\[
\begin{align*}
&\exp(w) - 1 \exp(2u) \left\{ \cosh(2v) \left[ \eta(z)g(\phi x, \phi y) + \eta(y)g(\phi x, \phi z) \right] \\
&+ \sinh(2v) \left[ \eta(z)g(x, \phi y) + \eta(y)g(x, \phi z) \right] \right\} \\
&+ \exp(2u) \left\{ \chi_1(z)g(\phi x, \phi y) + \chi_1(y)g(\phi x, \phi z) \\
&+ \chi_2(z)g(x, \phi y) + \chi_2(y)g(x, \phi z) \right\} \\
&+ \exp(2w)\eta(x) \left\{ \eta(y)d\varphi(\phi z) + \eta(z)d\varphi(\phi y) \right\} = 0.
\end{align*}
\]

Set \( x = y = z \) into (55) to get

\[
d\varphi(\phi z) = 0. \tag{56}
\]

Now, applying (56), we rewrite (55) in the form

\[
\vartheta_1(z)g(\phi x, \phi y) + \vartheta_2(z)g(x, \phi y) + \vartheta_1(y)g(\phi x, \phi z) + \vartheta_2(y)g(x, \phi z) = 0, \tag{57}
\]

where the 1-forms \( \vartheta_1 \) and \( \vartheta_2 \) are defined by

\[
\begin{align*}
\vartheta_1(z) &= [\exp(w) - 1] \cosh(2v)\eta(z) + \chi_1(z), \\
\vartheta_2(z) &= [\exp(w) - 1] \sinh(2v)\eta(z) + \chi_2(z). \tag{58}
\end{align*}
\]

Taking the trace of (57) first with respect to \( x = e_i, z = e_i \) and then with respect to \( y = e_i, z = e_i \), we get the following system of equations

\[
2(n + 1)\vartheta_1(z) - \eta(z)\vartheta_1(\xi) + \vartheta_1(\xi)\vartheta_2(\phi z) = 0,
\]

\[
\vartheta_1(z) - \eta(z)\vartheta_1(\xi) + \vartheta_2(\phi z) = 0
\]

with solution \( \vartheta_1 = \vartheta_2 \circ \phi = 0 \). Then the trace of (57) with respect to \( x = \phi e_i, y = e_i \) yields the vanishing of \( \vartheta_2 \). Therefore, (58) imply

\[
\chi_1(z) = [1 - \exp(w)] \cosh(2v)\eta(z), \quad \chi_2(z) = [1 - \exp(w)] \sinh(2v)\eta(z). \tag{59}
\]

By virtue of (21), (51) and (59), we derive

**Proposition 5.2** Let \((M, \phi, \xi, \eta, g)\) be a para-Sasaki-like Riemannian manifold. Then the structure \((\phi, \bar{\xi}, \bar{\eta}, \bar{g})\) defined by (50) is para-Sasaki-like if and only if the smooth functions \(u, v, w\) satisfy the following conditions

\[
dw \circ \phi = 0, \quad du - dv \circ \phi = 0, \quad du \circ \phi - dv = [1 - \exp(w)]\eta. \tag{60}
\]
Consequently we have
\[ du(\xi) = 0, \quad dv(\xi) = \exp(w) - 1. \]

In the case \( w = 0 \), the 1-forms \( du \) and \( dv \) do not depend on \( \xi \) and the global smooth functions \( u \) and \( v \) are locally defined on the paracomplex submanifold \( N^{2n} \). Then the paracomplex-valued function \( u + e^v \), where \( e^2 = 1 \), is a paraholomorphic function on \( N^{2n} \).

**Proof** Equality (56) is the first part of (60). Solving the linear system (59), we obtain the latter two equalities in (60). If \( w = 0 \), we obtain from (60) that
\[ du - dv \circ \phi = 0, \quad du \circ \phi - dv = 0, \]
which shows that the paracomplex function \( u + e^v \) on \( N^{2n} \) is paraholomorphic.

### 5.1 Paracontact homothetic transformations

We consider paracontact homothetic transformations of a para-Sasaki-like Riemannian manifold \((M, \phi, \xi, \eta, g)\). Since the functions \( u, v, w \) are constant, the Koszul formula, (50) and (52) imply that the Levi-Civita connections \( \nabla \) and \( \bar{\nabla} \) of the metrics \( \bar{g} \) and \( g \), respectively, are related by the formula
\[
\bar{\nabla}_x y = \nabla_x y - \exp(2u - 2w) \sinh(2v) g(\phi y, \phi z) \eta(x) \xi \\
+ [1 - \exp(2u - 2w) \cosh(2v)] g(x, \eta(y) \xi).
\]
Consequently, the corresponding curvature tensors \( \bar{R} \) and \( R \) are related by
\[
\bar{R}(x, y) z = R(x, y) z \\
+ \{1 - \exp(2u - 2w) \cosh(2v)\} \{g(\phi y, \phi z) \eta(x) \xi - g(\phi x, \phi z) \eta(y) \xi\} \\
+ g(y, \phi z) \phi x - g(x, \phi z) \phi y \\
- \exp(2u - 2w) \sinh(2v) \{g(\phi y, \phi z) \eta(x) \xi - g(x, \phi z) \eta(y) \xi\} \\
+ g(\phi y, \phi z) \phi x - g(\phi x, \phi z) \phi y\}.
\]

**Proposition 5.3** The Ricci tensor of a para-Sasaki-like Riemannian manifold is invariant under a paracontact homothetic transformation,
\[
\bar{Ric} = Ric.
\]
Moreover, we get
\[
\bar{\text{Scal}} = \exp(-2u) \cosh(2v) \text{Scal} - \exp(-2u) \sinh(2v) \text{Scal}^* \\
- 2n \{\exp(-2w) - \exp(-2u) \cosh(2v)\},
\]
\[
\bar{\text{Scal}}^* = \exp(-2u) \cosh(2v) \text{Scal}^* - \exp(-2u) \sinh(2v) \text{Scal}.
\]

**Proof** We obtain (62) by taking the trace of (61). Consequently, the traces in (62) imply (63).

**Remark 5.4** Note that under a paracontact homothetic transformation of a para-Sasaki-like Riemannian manifold, the resulting space is not generally a para-Sasaki-like Riemannian manifold. In fact, condition (60) is not true for constants \( u, v, w \neq 0 \), but it is satisfied for constants \( u, v, w = 0 \).

Using Proposition 5.3, we can make Proposition 4.2 a little stronger as follows
Proposition 5.5 A para-Sasaki-like Riemannian manifold \((M, \phi, \xi, \eta, g)\) is paracontact homothetic to an Einstein para-Sasaki-like Riemannian manifold if and only if the underlying $\text{phpcR}$ manifold \((N^{2n}, T N^{2n} = H, P, \bar{h})\) is an Einstein manifold of negative scalar curvature.

Proof We consider a paracontact homothetic transformation with \(v = w = 0\). Then \((M, \phi, \xi, \eta, \bar{g})\), where \(\bar{g} = \exp(2u) g + \{1 - \exp(2u)\} \eta \otimes \eta\), is also a para-Sasaki-like Riemannian manifold according to Proposition 5.2. We get the following sequence of equalities, applying Proposition 5.3 and (38)

\[
\overline{\text{Ric}}^{\bar{h}} = \overline{\text{Ric}}|_{H} = \overline{\text{Ric}}|_{H} = \overline{\text{Ric}}^{h} = \frac{1}{2n} \text{Scal}^{h} \bar{g}|_{H} = \frac{1}{2n} \exp(-2u) \text{Scal}^{h} \bar{g}|_{H}.
\]

The above equalities show that the underlying $\text{phpcR}$ manifold \((N^{2n}, T N^{2n} = H, P, \bar{h})\) is an Einstein manifold of scalar curvature \(\text{Scal}^{h} = \exp(-2u) \text{Scal}^{h}\). Since \(\text{Scal}^{h}\) is negative, we can take \(u = -\frac{1}{4} \ln \left(\frac{4n^{2}}{-\text{Scal}^{h}}\right)\) to get \(\text{Scal}^{h} = -4n^{2}\) and Proposition 4.2 shows that \((M, \phi, \xi, \eta, \bar{g})\) is an Einstein para-Sasaki-like Riemannian manifold.

Let \((M, \phi, \xi, \eta, g)\) be an Einstein para-Sasaki-like Riemannian manifold, \(\text{Ric} = -2ng\). Consider the paracontact homothetic transformation

\[
\overline{\phi} = \phi, \quad \overline{\xi} = \xi, \quad \overline{\eta} = \eta, \quad \overline{g}(x, y) = p g(x, y) + q g(x, \phi y) + (1 - p) \eta(x) \eta(y),
\]

where \(p > 0\) and \(q\) are constants satisfying \(p^{2} - q^{2} > 0\).

Using Proposition 5.3, we obtain

\[
\overline{\text{Ric}}(x, y) = \text{Ric}(x, y) = -2n g(x, y)
\]

\[
= -\frac{2n}{p^{2} - q^{2}} \left\{ p \overline{g}(x, y) - q \overline{g}(x, \phi y) + (p^{2} - q^{2} - p) \eta(x) \eta(y) \right\}.
\]

We call a para-Sasaki-like Riemannian manifold whose Ricci tensor satisfies (64) an \(\eta\)-paracomplex-Einstein para-Sasaki-like Riemannian manifold. In particular, it is an almost Einstein space [9]. Thereby, we have shown the following

Proposition 5.6 Each \(\eta\)-paracomplex-Einstein para-Sasaki-like Riemannian manifold is paracontact homothetic to an Einstein para-Sasaki-like Riemannian space.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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