A MINKOWSKI TYPE INEQUALITY IN SPACE FORMS

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ABSTRACT. In this note we apply the general Reilly formula established in [8] to the solution of a Neumann boundary value problem to prove an optimal Minkowski type inequality in space forms.

1. Introduction

Let $(\Omega^n, g)$ be an $n$-dimensional compact Riemannian manifold with smooth boundary $\partial \Omega = M$. Let $H$ be the (normalized) mean curvature and $h$ be the second fundamental form of $M \subset \Omega$ respectively. In the paper [8], we (joint with Qiu) have proved the following generalization of Reilly’s formula. We use the same notations as in [8].

Theorem A. (Qiu-Xia [8]) Let $V : \Omega \to \mathbb{R}$ be a given a.e. twice differentiable function. Given a smooth function $f$ on $\Omega$, we denote $z = f|_M$ and $u = \nabla \nu f$. Let $K \in \mathbb{R}$. Then we have the following identity:

$$
\int_{\Omega} V \left( (\Delta f + Knf)^2 - |\nabla^2 f + Kfg|^2 \right) d\Omega
$$

$$
= \int_{M} V \left( 2u\Delta z + (n-1)Hu^2 + h(\nabla z, \nabla z) + (2n-2)Kuz \right) dA
$$

$$
+ \int_{M} \nabla \nu V \left( |\nabla z|^2 - (n-1)Kz^2 \right) dA
$$

$$
+ \int_{\Omega} \left( \nabla^2 V - \Delta V g - (2n-2)KV g + V \text{Ric} \right)(\nabla f, \nabla f) d\Omega
$$

$$
+ (n-1) \int_{\Omega} (K\Delta V + nK^2 V)f^2 d\Omega.
$$

When $V \equiv 1$ and $K = 0$, (1) reduces to the classical Reilly’s formula [9, 10]. Reilly’s original formula has numerous applications, see for example [10, 3, 7, 12, 11]. In [8], we successfully apply the general Reilly formula (1) to prove a new Heintze-Karcher type inequality for compact manifolds with mean convex boundary and sectional curvature bounded below. In this paper, we continue to explore other applications of (1).

Reilly [11] used his formula to prove the following Minkowski inequality for compact Riemannian manifolds with nonnegative Ricci curvature and convex boundary.

Research of CX is supported in part by the Fundamental Research Funds for the Central Universities (Grant No. 20720150012), NSFC (Grant No. 11501480) and CRC Postdoc Fellowship.
Theorem B. (Reilly [11]) Let \((\Omega^n, g)\) be a compact \(n\)-dimensional Riemannian manifold with smooth convex boundary \(M\) and non-negative Ricci curvature. Then

\[
\text{Area}(M)^2 \geq n \text{Vol}(\Omega) \int_M H \, dA.
\]

The equality in (2) holds if and only if \(\Omega\) is isometric to an Euclidean ball.

When \(\Omega \subset \mathbb{R}^n\), inequality (2) is exactly a special case of Minkowski’s inequality for mixed volumes in the theory of convex bodies, see [13], Theorem 7.2.1. A different proof of Theorem B was given by Wang-Zhang [15], based on the Alexandrov-Bakelman-Pucci estimate.

Reilly’s proof is based on the solvability of the following Neumann problem

\[
\begin{aligned}
\Delta f &= 1 \quad \text{in } \Omega, \\
u &= c \quad \text{on } \partial\Omega,
\end{aligned}
\]

for \(c = \frac{\text{Vol}(\Omega)}{\text{Area}(\partial M)}\). He applied his formula (1) (for \(K = 0\) and \(V \equiv 1\)) to the solution of (3) to derive

\[
\frac{n-1}{n} \text{Vol}(\Omega) \geq c^2 \int_M H \, dA,
\]

which is (2).

The topic of geometric inequalities for curvature integrals in non-Euclidean space forms attracts many attentions in recent years, see for example [14] and references therein. Curvature integral with “weight” seems quite natural in the general relativity, especially in the hyperbolic space. Quite recently, Brendle-Hung-Wang [2] established a Minkowski type inequality between “weighted” mean curvature integral and “weighted” volume for hypersurfaces in anti-de Sitter-Schwarzschild manifolds by using a “weighted” Heintze-Karcher inequality by Brendle [1]. See also [4, 5, 6, 8] for related works.

In this short note, based on Theorem A, we prove an analog of Minkowski’s inequality for “weighted mixed volumes” in non-Euclidean space forms. We use \(\mathbb{H}^n\) to denote the hyperbolic space with curvature \(-1\) and \(\mathbb{S}^n_+\) to denote the open hemisphere with curvature 1.

Theorem 1.1. Let \(\Omega^n \subset \mathbb{H}^n\) (\(\mathbb{S}^n_+\) resp.) be a compact \(n\)-dimensional domain with smooth boundary \(M\). Let \(V(x) = \cosh r \, (\cos r \, \text{resp.})\), where \(r(x) = \text{dist}(x, p)\) for some fixed point \(p \in \mathbb{H}^n\) (\(p \in \mathbb{S}^n_+\) resp.). We further assume the second fundamental form of \(M\) satisfies

\[
h_{ij} \geq \nabla_v \log V g_{ij}.
\]

Then we have

\[
\left(\int_M V \, dA\right)^2 \geq n \int_\Omega V d\Omega \int_M HV dA.
\]

The equality in (5) holds if and only if \(\Omega\) is a geodesic ball \(B_R(q)\) for some point \(q \in \mathbb{H}^n\) (\(q \in \mathbb{S}^n_+\) resp.). In particular, (5) holds true when \(M\) is horo-spherical convex in the case \(\Omega \subset \mathbb{H}^n\) or \(M\) is convex and \(p \in \Omega\) in the case \(\Omega \subset \mathbb{S}^n_+\).

The horo-spherical convexity of \(M \subset \mathbb{H}^n\) means that all the principal curvatures are bigger than or equal to 1. Condition (4) seems like some kind of convexity for \(M\). Particularly, when \(\Omega \subset \mathbb{R}^n\) and \(V \equiv 1\), this is the usual convexity. Moreover,
horo-convexity in $\mathbb{H}^n$ and convexity in $S^m_+$ imply condition (4). This follows because $\nabla_{\nu} V < V$ in the case $\Omega \subset \mathbb{H}^n$ and $\nabla_{\nu} V \leq 0$ in the case $p \in \Omega \subset S^m_+$. We remark that, the equality in (5) holds for not only geodesic balls centered at $p$ but all geodesic balls.

In the Euclidean space, Theorem B is equivalent to say that

\[ d^2 dt^2 \left( \int_{\Omega_t} V d\Omega \right) \leq 0, \]

where $\Omega_t = \Omega + tB = \{ x \in \mathbb{R}^n | \text{dist}(x, \Omega) \leq t \}$. Similarly, Theorem 1 can be interpreted as the following equivalent statement.

**Theorem 1.2.** Let $\Omega^n \subset \mathbb{H}^n$ ($S^m_+$ resp.) and $V$ be as in Theorem 1.1. Let $K = -1$ ($K = 1$ resp.). Denote $\Omega_t := \{ x \in \mathbb{H}^n(S^m_+ \text{ resp.}) | \text{dist}(x, \Omega) \leq t \}$. For the case $S^m_+$ we assume $t \in [0, T)$ for which $\Omega_t \subset S^m_+$. Then

\[ \frac{d^2}{dt^2} \left( \int_{\Omega_t} V d\Omega \right) + K \left( \int_{\Omega_t} V d\Omega \right) \leq 0. \]

The idea to prove (5) is parallel to Reilly’s. We will utilize the solution to a Neumann boundary value problem (7) and the general Reilly formula. However, the computation is much more complicated due to the complication of the boundary terms in the general Reilly formula.

## 2. Proof of Theorem 1.1 and 1.2

Let $V = \cosh r, K = -1$ or $V = \cos r, K = 1$ in (1) for the case $\mathbb{H}^n$ or $S^m_+$ respectively, where $r(x) = \text{dist}(x, p)$. The function $f$ is the solution to the following Neumann boundary value problem:

\[ \begin{cases} \Delta f + K f = 1 & \text{in } \Omega, \\ V f_{\nu} - V f = cV & \text{on } \partial \Omega, \end{cases} \]

where $V_{\nu} := \nabla_{\nu} V$ and $c = \frac{\int_{\Omega} V}{\int_{\partial \Omega} V}$. We claim that there exists a unique solution $f \in C^\infty(\Omega)$ to (7), up to an additive $\alpha V$ for constants $\alpha \in \mathbb{R}$. In fact, it follows from the Fredholm alternative that there exists a unique solution $w \in C^\infty(\Omega)$ (up to an additive constant) to the following Neumann boundary value problem

\[ \begin{cases} \text{div}(V^2 w) = V & \text{in } \Omega, \\ V^2 w_{\nu} = cV & \text{on } \partial \Omega, \end{cases} \]

if and only if $c = \frac{\int_0 V}{\int_{M} V}$. Using (9) below, one checks readily that $f = wV$ solves (7).

For simplicity, we omit the volume form $d\Omega$ and the area form $dA$ in the integrations.

It is well-known that $V$ satisfies

\[ \nabla^2 V = -KV g, \]

which will be used frequently in the following.

We will use the solution $f$ of (7) in the general Reilly formula (11). For our choice of $K$ and $V$, we see from (9) that the integrand in last two lines of (11) vanishes. By
using Hölder’s inequality and the equation in (7), we have from (1) that

\begin{align*}
\frac{n-1}{n} \int_{\Omega} V & \geq \int_{\Omega} V \left( (\Delta f + Knf)^2 - |\nabla^2 f + Kfg|^2 \right) \\
& = \int_{M} V \left( 2u\Delta z + (n-1)Hu^2 + h(\nabla z, \nabla z) + (2n-2)Kuz \right) \\
& + \int_{M} V_\nu (|\nabla z|^2 - (n-1)Kz^2).
\end{align*}

(10)

Let us investigate the RHS of (10). By using the Gauss-Weigarten formula and (9), we see

\begin{align*}
\nabla_i V_\nu = \nabla_i \nabla_\nu V + h_{ij} V_j = h_{ij} V_j,
\end{align*}

(11)

\begin{align*}
\Delta V = \nabla^2 V - \nabla_\nu \nabla_\nu V - (n-1)HV_\nu = -(n-1)KV - (n-1)HV_\nu.
\end{align*}

(12)

Using the Neumann boundary condition in (7), integration by parts, (11) and (12), we have

\begin{align*}
\int_{M} 2Vu\Delta z = \int_{M} 2(V_\nu z + cV)\Delta z \\
= \int_{M} -2V_\nu |\nabla z|^2 - 2z\nabla V_\nu \nabla z + 2cz\Delta V \\
= \int_{M} -2V_\nu |\nabla z|^2 - 2zh_{ij}V_i z_j + 2(n-1)cz(-KV - HV_\nu),
\end{align*}

(13)

\begin{align*}
\int_{M} (n-1)Hu^2 V = \int_{M} (n-1)HV \left( c + \frac{V_\nu}{V} \frac{V}{z} \right)^2 \\
= \int_{M} (n-1)c^2 HV + 2c(n-1)HV_\nu z + (n-1)H \frac{V_\nu^2}{V} z^2,
\end{align*}

(14)

\begin{align*}
\int_{M} (2n-2)KuzV = \int_{M} 2(n-1)Kz (cV + V_\nu z).
\end{align*}

(15)

Inserting (13)-(15) into (10), we have

\begin{align*}
\frac{n-1}{n} \int_{\Omega} V & \geq \int_{M} V_\nu |\nabla z|^2 - 2zh_{ij}V_i z_j + (n-1)c^2 HV \\
& + (n-1)H \frac{V_\nu^2}{V} z^2 + h(\nabla z, \nabla z)V + (n-1)KV_\nu z^2.
\end{align*}

(16)
Multiplying \(-\frac{V}{\nu}z^2\) to both side of (12), integrating by parts and using (11), we have

\[
\int_M (n-1)H\frac{V^2}{V}z^2 + (n-1)KV\nu z^2 = \int_M -\frac{V^2}{V}z^2\Delta V = \int_M \frac{\nabla V\nabla V}{V}z^2 + \frac{2z\nabla z\nabla V\nu}{V} - \frac{V\nu z^2}{V^2}|\nabla V|^2
\]

(17)

Inserting (17) into (16), we obtain

\[
\frac{n-1}{n} \int_\Omega V \geq \int_M -V\nu|\nabla z|^2 - 2z\nabla z\nabla V\nu + (n-1)c^2HV + h(\nabla z, \nabla V) + \frac{h_{ij}V_iV_j}{V} + \frac{2z\nabla z\nabla V\nu}{V} - \frac{V\nu z^2}{V^2}|\nabla V|^2 = \int_M (n-1)c^2HV + Vh_{ij} \left( z_i - \frac{V_i z}{V} \right) \left( z_j - \frac{V_j z}{V} \right) - V\nu |\nabla z - \nabla V\nu|^2.
\]

(18)

By the assumption (1), the last line in (18) is nonnegative. Therefore, we derive from (18) that

\[
\frac{n-1}{n} \int_\Omega V \geq \int_M (n-1)c^2HV = \frac{\left(\int_\Omega V\right)^2}{\left(\int_M V\right)^2} \int_M (n-1)HV.
\]

(19)

It follows that

\[
\left(\int_M VdA\right)^2 \geq n \int_\Omega Vd\Omega \int_M HVdA.
\]

(20)

Let us explore the equality case in (20). We consider the case \(\Omega \subset \mathbb{H}^n\). First, for a geodesic ball \(B_R(p) \subset \mathbb{H}^n\), centered at \(p\), \(V = \cosh R\) and \(H = \coth R\) are constants on \(\partial B_R(p)\). Thus \(\int_{\partial B_R(p)} VdA = \omega_{n-1} \cosh R \sinh^{n-1} R\) and \(\int_M HVdA = \omega_{n-1} \cosh^2 R \sinh^{n-2} R\). On the other hand,

\[
\int_{B_R(p)} \cosh r(x)d\Omega(x) = \int_0^R \omega_{n-1} \cosh \rho \sinh^{n-1} \rho d\rho = \frac{\omega_{n-1}}{n} \sinh^n R.
\]

Thus equality in (20) holds when \(\Omega = B_R(p)\). Second, for a geodesic ball \(B_R(q) \subset \mathbb{H}^n\), centered at \(q \in \mathbb{H}^n\), not necessarily \(p\), \(H = \coth R\) is constant on \(\partial B_R(q)\) while \(V\) is not. Nevertheless, we still have the quality. Indeed, by Minkowski's formula and the constancy of \(H\),

\[
\int_{\partial B_R(q)} VdA = \int_{\partial B_R(q)} HV\nu dA = H \int_{\partial B_R(q)} V\nu dA = nH \int_{B_R(q)} Vd\Omega.
\]
Thus
\[
\left( \int_{\partial B_R(q)} V dA \right)^2 = nH \int_{B_R(q)} V d\Omega \int_{\partial B_R(q)} V dA = n \int_{B_R(q)} V d\Omega \int_{\partial B_R(q)} HV dA.
\]

Conversely, if the equality in (20) holds, then by checking the equality in (16) and (19) we see
\[
\begin{cases}
\nabla_{ij}f - fg_{ij} = \frac{1}{n}g_{ij} & \text{in } \Omega, \\
\nabla z - \frac{zV}{\Omega} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
The boundary identity means \( z = \alpha V \) for some constant \( \alpha \in \mathbb{R} \). Thus the function \( \tilde{f} = f - \alpha V + \frac{1}{n} \) satisfies\]
\[
\begin{cases}
\nabla_{ij}^2 \tilde{f} - \tilde{f} g_{ij} = 0 & \text{in } \Omega, \\
\tilde{f}|_{\partial \Omega} = \frac{1}{n} & \text{on } \partial \Omega.
\end{cases}
\]

It follows from an Obata type result (see Reilly [11]) that \( \Omega \) must be some geodesic ball.

The case \( \Omega \subset S^n_+ \) is similar. We finish the proof of Theorem 1.1. \( \square \)

**Proof of Theorem 1.2.** \( \Omega_t \) can be viewed as the normal flow
\[
\partial_t X(x, t) = \nu(x, t), \quad x \in \partial \Omega.
\]
The variational formulas give
\[
\frac{d}{dt} \int_{\Omega_t} V d\Omega = \int_{\partial \Omega} V dA_t,
\]
\[
\frac{d}{dt} \int_{\partial \Omega} V dA_t = \int_{\partial \Omega} V_{\nu} + (n-1)HV dA_t \]
\[
= \int_{\partial \Omega} (n-1)HV dA_t - nK \int_{\Omega_t} V d\Omega.
\]
Using (21), (22) and Theorem 5 we deduce
\[
\frac{d^2}{dt^2} \left( \int_{\Omega_t} V d\Omega \right)^{\frac{1}{n}} = \frac{1}{n} \left( \int_{\Omega_t} V d\Omega \right)^{\frac{1}{n} - 1} \left( \int_{\partial \Omega_t} (n-1)HV dA_t - nK \int_{\Omega_t} V d\Omega \right)
\]
\[
+ \frac{1}{n} \left( \frac{1}{n} - 1 \right) \left( \int_{\Omega_t} V d\Omega \right)^{\frac{1}{n} - 2} \left( \int_{\partial \Omega_t} V dA_t \right)^2 \leq -K \left( \int_{\Omega_t} V d\Omega \right)^{\frac{1}{n}}.
\]
We complete the proof. \( \square \)

**Remark 2.1.** It is well known that one may derive the isoperimetric inequality from (6) in the Euclidean space. Indeed, using the ODE comparison, we obtain
\[
\operatorname{Vol}(\Omega_t)^{\frac{1}{n}} \leq \operatorname{Vol}(\Omega)^{\frac{1}{n}} + \frac{1}{n} \operatorname{Vol}(\Omega)^{\frac{1}{n} - 1} \operatorname{Area}(\partial \Omega)t.
\]
Dividing both sides of (23) by \( t \) and letting \( t \to \infty \), we obtain
\[
\frac{1}{n} \text{Vol}(\Omega)^{\frac{1}{n}} \approx \frac{1}{n} \frac{\text{Area}(\partial \Omega)}{\text{Vol}(\Omega)} = V(\Omega)^{\frac{1}{n}} = V(B)^{\frac{1}{n}},
\]
which is the classical isoperimetric inequality. Similarly, in \( \mathbb{H}^n \), we have
\[
\left( \int_{\Omega_t} V d\Omega \right)^{\frac{1}{n}} \leq \left( \int_{\Omega} V d\Omega \right)^{\frac{1}{n}} \cosh t + \left( \frac{1}{n} \left( \int_{\Omega} V d\Omega \right)^{\frac{1}{n} - 1} \int_{\partial \Omega} V dA \right) \sinh t.
\] (24)

However, we are not able to derive an optimal inequality between \( \int_{\partial \Omega} V dA \) and \( \int_{\Omega} V d\Omega \) from (24) because \( \lim_{t \to \infty} \frac{1}{n} \frac{1}{\sinh t} \left( \int_{\Omega_t} V d\Omega \right)^{\frac{1}{n}} \) is not a dimensional constant in this case. In a forthcoming paper, we will use the flow approach to establish such kind of optimal inequality.

**Acknowledgements:** I would like to thank Professor Pengfei Guan for stimulating discussions and Prof. Guofang Wang for useful comments and for their constant supports.

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