ON THE DECOMPOSITION OF HECKE POLYNOMIALS OVER PARABOLIC
HECKE ALGEBRAS

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Abstract. We generalize a classical result of Andrianov on the decomposition of Hecke polynomials. Let \( \mathcal{F} \) be a non-archimedean local field. For every connected reductive group \( G \), we give a criterion for when a polynomial with coefficients in the spherical parahoric Hecke algebra of \( G(\mathcal{F}) \) decomposes over a parabolic Hecke algebra associated with a non-obtuse parabolic subgroup of \( G \). We classify the non-obtuse parabolics. This then shows that our decomposition theorem covers all the classical cases due to Andrianov and Gritsenko. We also obtain new cases when the relative root system of \( G \) contains factors of types \( E_6 \) or \( E_7 \).

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1. Introduction

1.1. Motivation. The problem to decompose Hecke polynomials emerged in the theory of Hecke operators acting on spaces of Siegel modular forms, see, e.g., [And77, And79, Gri84]. One of the principal tasks is to find and study relations between Fourier coefficients of eigenforms of

2020 Mathematics Subject Classification. 11C08, 20C08, 20G25.
Hecke operators and the corresponding eigenvalues. It is instructive to work through an example to see how decomposing Hecke polynomials helps to find such relations. Consider the modular group $\Gamma = \text{SL}_2(\mathbb{Z})$. Recall that a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ is called a modular form of weight $k$ if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and all $z \in \mathbb{H}$ it satisfies

$$f(z) = (f|\gamma)(z) := (cz + d)^{-k}f \left( \frac{az + b}{cz + d} \right),$$

and if it admits a Fourier expansion of the form $f(z) = \sum_{n=0}^{\infty} \alpha_f(j) \cdot e^{2\pi ijz}$. Denote $\mathcal{M}_k$ the $\mathbb{C}$-vector space of modular forms of weight $k$. Let $S$ be the set of $2 \times 2$-matrices with integral entries and positive determinant. Then the algebra of Hecke operators $\mathcal{H} := \mathcal{H}_{C}(\Gamma, S)$ naturally acts on $\mathcal{M}_k$: A double coset $(\Gamma g \Gamma) \in \mathcal{H}$ acts on $f$ via $f|_{\Gamma g \Gamma} := \sum_{\gamma \in \Gamma \cap \Gamma g \Gamma} f|_{\gamma}$. If $f$ is a Hecke eigenform, we write $\lambda_f : \mathcal{H} \to \mathbb{C}$ for the corresponding eigenvalue. Then $f$ is a Hecke eigenform if and only if $\alpha_{f|T}(j) = \lambda_f(T) \cdot \alpha_f(j)$, for all $T \in \mathcal{H}$, $j \in \mathbb{Z}_{\geq 0}$.

Fix a prime number $p$ and consider the Hecke polynomial

$$Q_p(t) = 1 - T_1 t + pT_2 t^2,$$

where $E_2$ is the $2 \times 2$ identity matrix. There is a natural embedding of $\mathcal{H}$ into the parabolic Hecke algebra $\mathcal{H}^0 := \mathcal{H}_{C}(\Gamma_0, S_0)$, where $\Gamma_0$ (resp. $S_0$) is the subgroup of upper triangular matrices in $\Gamma$ (resp. $S$). For example, one has $T_1 = T_1^+ + T_1^-$ in $\mathcal{H}_{C}(\Gamma_0, S_0)$, where $T_1^+ = (\Gamma_0 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix})$ and $T_1^- = (\Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix})$. Over the ring $\mathcal{H}_{C}(\Gamma_0, S_0)$ the polynomial $Q_p(t)$ decomposes as

$$Q_p(t) = (1 - T_1^+ t) \cdot (1 - T_1^- t).$$

Right multiplication with the inverse power series of $1 - T_1^- t$ yields

$$Q_p(t) \cdot \sum_{l=0}^{\infty} (T_1^-)^l t^l = 1 - T_1^+ t \quad \text{in } \mathcal{H}^0[t].$$

Note that $\mathcal{H}^0$ acts naturally on the space $\mathcal{M}^0_k$ of holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ satisfying (1) for all $\gamma \in \Gamma_0$ and admitting a Fourier expansion as above. Then $\mathcal{M}_k \subseteq \mathcal{M}^0_k$ and $\mathcal{H}^0[t]$ acts naturally on $\mathcal{M}^0_k[t]$. Given $f \in \mathcal{M}^0_k$, one computes $\alpha_{f|T_1^+}(j) = \alpha_f(j/p)$ and $\alpha_{f|T_1^-}(j) = p^{1-k} \alpha_f(pj)$, for all $j \geq 0$. (Here, we define $\alpha_f(j/p) = 0$ if $p \nmid j$.)

Let now $f \in \mathcal{M}_k$ be a Hecke eigenform and consider the complex polynomial

$$Q_{p,f}(t) = 1 - \lambda_f(T_1) \cdot t + p \cdot \lambda_f(T_2) \cdot t^2 \in \mathbb{C}[t].$$

Letting (3) act on $f$, we obtain on the level of Fourier coefficients the relations

$$Q_{p,f}(t) \cdot \sum_{l=0}^{\infty} p^{l(1-k)} \alpha_f(p^j l) t^l = \alpha_f(j) - \alpha_f(j/p) t \quad \text{in } \mathbb{C}[t], \text{ for each } j \geq 0.$$

This method of decomposing a Hecke polynomial over a parabolic Hecke algebra proved to be very fruitful in the more general context of Siegel modular forms. Andrianov proved a general decomposition theorem of type (2) in the context of Siegel modular forms, cf. [And77]. In this case the modular group $\text{SL}_2(\mathbb{Z})$ is replaced by $\text{Sp}_{2n}(\mathbb{Z})$, for some $n \in \mathbb{Z}_{>1}$, and one considers certain holomorphic functions on the Siegel upper half-space $\mathbb{H}_n$. The subgroup $\Gamma_0$ of upper triangular matrices is replaced by the “Siegel parabolic” in $\text{Sp}_{2n}(\mathbb{Z})$, that is, the subgroup of matrices whose lower left quadrant is zero.

It is then natural to ask whether a decomposition of type (2) also holds for more general groups. Since the problem is local in nature, one may replace $\mathbb{Z}$ with the ring of integers of a non-archimedean
local field \( \mathfrak{F} \). In this context, Gritsenko proved a decomposition theorem for \( \text{GL}_n(\mathfrak{F}) \) (and all parabolics) \cite{Gr88, Gr92} and for the classical groups \( \text{Sp}_{2n}(\mathfrak{F}) \), \( \text{SU}_n(\mathfrak{F}) \), and \( \text{SO}_n(\mathfrak{F}) \) (for the parabolics fixing a line in the standard representation) \cite{Gr90}.

The main result in \cite{Gr92} found an application in the theory of \( p \)-adic \( L \)-functions, where it was recently used by Januszewski \cite{Jan14} to define a projection map in order to obtain simultaneous eigenforms for certain Hecke operators. It is therefore reasonable to hope that a decomposition theorem for more general reductive groups will have applications in the theory of \( p \)-adic \( L \)-functions.

The aim of this paper is to generalize the theory developed by Andrianov \cite{And77} to the group \( G \) of \( \mathfrak{F} \)-rational points of a connected reductive \( \mathfrak{F} \)-group.

1.2. Main results. Let \( \mathfrak{F} \) be a non-archimedean local field of residue characteristic \( p > 0 \). Let \( G \) be a connected reductive group over \( \mathfrak{F} \), let \( B \) be a minimal parabolic \( \mathfrak{F} \)-subgroup of \( G \) with Levi decomposition \( B = ZU \). In this article a parabolic subgroup of \( G \) is a standard parabolic \( \mathfrak{F} \)-subgroup with respect to \( B \). Fix a special parahoric subgroup \( K := G(\mathfrak{F}) \) corresponding to a special point in the apartment determined by \( Z \). Then \( G = KB \), where \( B := B(\mathfrak{F}) \). For any subgroup \( X \subseteq G \) we put \( K_X = K \cap X \). Let \( P \) be a parabolic subgroup of \( G \) and put \( P := P(\mathfrak{F}) \).

Let \( R \) be a commutative \( \mathbb{Z}[1/p] \)-algebra, considered as a ring of coefficients.

The Hecke ring \( \mathcal{H}_R(K_\mathcal{F}, Z) \), where \( Z := \mathbb{Z}(\mathfrak{F}) \), identifies with the group algebra \( R[Z/K_\mathcal{F}] \), and there are natural embeddings \( \mathcal{H}_R(K, G) \subseteq \mathcal{H}_R(K, P) \subseteq \mathcal{H}_R(K_B, B) \). There is a natural algebra homomorphism

\[
\Theta^B_Z : \mathcal{H}_R(K_B, B) \rightarrow R[Z/K_\mathcal{F}]
\]

induced by the canonical projection map \( B \rightarrow Z \). The restriction of \( \Theta^B_Z \) to \( \mathcal{H}_R(K_G, G) \) is called the (unnormalized) Satake homomorphism. It is well-known that \( \mathcal{H}_R(K, G) \) is commutative. Besides \( \mathcal{H}_R(K, G) \), the parabolic Hecke algebra \( \mathcal{H}_R(K_P, P) \) contains another commutative algebra \( C_P \), which is constructed as the centralizer of a certain element of \( \mathcal{H}_R(K_P, P) \).

Consider the \( R \)-submodule \( \Theta^P_P := \mathcal{H}_R(K, G)C_P \) of \( \mathcal{H}_R(K_P, P) \). In order to develop a reasonable theory one needs to make the following assumption:

**Hypothesis 1.1.** The restriction of \( \Theta^B_Z \) to \( \Theta^P_P \) is injective.

It should be remarked that there is no example known where Hypothesis 1.1 fails. There is one maximal parabolic subgroup in \( \text{Sp}_6(\mathbb{Q}_p) \) for which we do not know whether Hypothesis 1.1 is satisfied. One can show (see Proposition 5.10) that, if Hypothesis 1.1 is satisfied for two parabolics \( P \) and \( Q \), then it is also satisfied for \( P \cap Q \). Hence, it would suffice to verify Hypothesis 1.1 for every maximal parabolic subgroup of \( G \). For practical reasons we work with an equivalent form of Hypothesis 1.1 see Hypothesis 5.10 on p. 33.

**Theorem** (Theorem 5.12). Assume that Hypothesis 1.1 is satisfied. Let \( d(t) \in \mathcal{H}_R(K, G)[t] \) be a polynomial such that \( d^\Theta^B_Z(t) \), the polynomial obtained by applying \( \Theta^B_Z \) to the coefficients of \( d(t) \), decomposes in \( R[Z/K_\mathcal{F}][t] \) as

\[
d^\Theta^B_Z(t) = \tilde{f}(t) \cdot \tilde{g}(t)
\]

such that \( \tilde{g}(t) \) has coefficients in \( \Theta^B_Z(C_P) \) with constant term 1. Then there exist polynomials \( f(t), g(t) \in \mathcal{H}_R(K_P, P)[t] \) with the following properties:

- \( \deg f(t) = \deg \tilde{f}(t) \) and \( f^\Theta^B_Z(t) = \tilde{f}(t) \);
- \( \deg g(t) = \deg \tilde{g}(t) \) and \( g^\Theta^B_Z(t) = \tilde{g}(t) \);
- \( d(t) = f(t) \cdot g(t) \in \mathcal{H}_R(K_P, P)[t] \).

The proof is merely a straightforward extension of the arguments in \cite{And77}. However, it is in general very hard to decide for which \( P \) Hypothesis 1.1 is satisfied. The main contribution of this paper is to single out a class of maximal parabolic subgroups in \( G \) for which this hypothesis holds. We make the following important definition:

**Definition** (See 33). Let \( P \) be a maximal parabolic subgroup of \( G \) with unipotent radical \( U_P \). Then \( P \) is called non-obtuse if any two relative roots that occur in \( U_P \) span a non-obtuse angle.
Our main result is then:

**Theorem 1.2** *(Theorem 5.11)*. Assume that $P$ is a non-obtuse parabolic subgroup of $G$. Then Hypothesis [1,5] is satisfied for $P$.

The obvious question then is whether there exist non-obtuse parabolics and if they can be classified. In Proposition 5.5 we achieve a complete classification of non-obtuse parabolic subgroups and formulate several equivalent conditions. The maximal parabolic subgroups correspond bijectively to the vertices in the Dynkin diagram of the relative root system of $G$, and hence it makes sense to say when a vertex of the Dynkin diagram is non-obtuse. The classification shows that all vertices in type $A_n$, the terminal vertices in types $B_n$, $C_n$, and $D_n$, two terminal vertices in type $E_6$, and one terminal vertex in type $E_7$ are non-obtuse. It also shows that there are no non-obtuse vertices in types $E_8$, $F_4$, and $G_2$, cf. Figure 1 on p. 13.

The proof of Theorem 1.2 requires us to investigate intersections of Cartan and Iwasawa double cosets. This problem is well-known and arises, for example, in the study of the Satake homomorphism, but here it is of a different flavor. More precisely, let $z, z' \in Z$ be such that $Uz K \cap K z K \neq \emptyset$. It is well-understood how $z'$ and $z$ relate. However, so far almost nothing is known about the $u \in U$ for which $uz' \in K z K$.

To state our main technical result in this direction, let $\varphi$ be the point in the (adjoint) Bruhat–Tits building of $G$ corresponding to $K$. By assumption, $\varphi$ lies in the apartment $\Delta$ corresponding to a torus $T \subseteq Z$ which is maximal $\mathfrak{g}$-split in $G$. By definition, $\varphi$ defines valuations, denoted $\varphi_\alpha$, on the root groups $U_\alpha$. There is a canonical homomorphism $\nu: Z \to V$ into the underlying $\mathbb{R}$-vector $V$ space of $\Delta$ containing the coroots with respect to $T$. Fix a strictly positive element $a \in Z$ so that $\langle \alpha, \nu(\alpha) \rangle < 0$ for all simple roots $\alpha$ (with respect to $B$). Choose the representative $z$ of $KzK$ such that $z^{-1} \cdot (K \cap U) \cdot z^{-1} \subseteq K \cap U$, where $U := U(\mathfrak{g})$. Denote $U_P := U_P(\mathfrak{g})$ the group of $\mathfrak{g}$-points of the unipotent radical of $P$. We prove the following technical result, which might be of independent interest:

**Theorem 1.3** *(Theorem 4.13)*. Assume that $P$ is non-obtuse and that $-\nu(az)$ is a sum of simple coroots. Let $z' \in Z$ and $u \in U_P$ with $uz' \in K z K$. Then one has $az' \cdot (K \cap U_P) \cdot (az')^{-1} \subseteq K \cap U_P$ and $aua^{-1} \in K$.

The condition $aua^{-1} \in K$ is the difficult part of the theorem and can be interpreted as follows: write $u = u_1 \cdots u_r$, for certain $u_i \in U_\alpha_i$. Then $\varphi_\alpha(u_i) \geq \langle \alpha_i, \nu(\alpha) \rangle$ for all $i$, that is, we obtain a lower bound for the valuations of the $u_i$. In order to obtain this bound, we describe an algorithm, see [31] which produces a sequence of left cosets $u_0 z_0 K, \ldots, u_r z_r K$ in $K z K$ (with $u_i \in U$, $z_i \in Z$) such that $u_0 z_0 = u z'$ and $u_r = 1$ (so that $\nu(z_r)$ lies in the orbit of $\nu(z)$ under the action of the finite Weyl group). As a byproduct, by a careful analysis, we can estimate the valuations of the root group elements $u_i$.

Finally, it should be mentioned that there is another possible approach to study intersections of Cartan and Iwasawa double cosets, which we do not follow here. For $p$-adic Chevalley groups Dąbrowski describes in [Dab94] the intersections of the form $U r I \cap I s I$ in terms of “good subexpressions”, where $I$ is an Iwahori subgroup and $r, s$ are elements of the affine Weyl group. By adapting the methods of [Lan01] it seems plausible that one could in this way explicitly describe the intersections $U z' K \cap K z K$.

1.3. **Structure of the paper.** In §2 we fix notations (§2.1) and recall some notions about reductive groups (§2.2–§2.4). In §2.5–§2.7 we discuss positive elements, the Cartan and Iwasawa decompositions, and abstract Hecke rings.

In §3 we study non-obtuse parabolics. In Proposition 3.2 we classify non-obtuse parabolic subgroups and provide equivalent characterizations.

In §4 we present the Algorithm 4.2. Although it will not be used in the sequel it is worth to mention that it always terminates (Proposition 4.3). Our main technical result is Theorem 4.4.

Finally, in §5 we develop the theory leading to the decomposition Theorem 5.17. The §5.1 introduces parabolic Hecke algebras and defines the unnormalized version of the Satake homomorphism.
In §5.2 we give another presentation of the twisted action due to Henniart–Vignéras [HV15]. Then in §5.3 we translate the main theorem of [HV15] into our context. In §5.4 we recall the commutative algebra $C_F$. In §5.5 we work out an explicit example of a parabolic Hecke algebra. We provide an explicit presentation of the parabolic Hecke algebra attached to $GL_2(\mathfrak{F})$ in terms of generators and relations. The straightforward proof is given in an appendix. We also explicitly compute the Satake homomorphism for illustrative purposes. Given a strictly positive element $a_F$, we construct in §5.6 a certain Hecke polynomial $\chi_{a_F}(t)$. With Hypothesis 5.10 we impose that $(K_{p,p})$ is a “left root” of $\chi_{a_F}(t)$. This hypothesis is crucial for proving the decomposition theorem. Theorem 5.11 shows that this hypothesis is satisfied provided that the parabolics $H$ and $V$ lists several conditions which are equivalent to Hypothesis 5.10. Proposition 5.16 shows that, in principle, it would suffice to verify Hypothesis 5.10 for maximal parabolics.

1.4. Acknowledgments. This article constitutes a part of my doctoral dissertation [Hey19] written at the Humboldt University in Berlin. I want to express my deep gratitude to my advisor Elmar Große-Klönne for his support. I also thank Peter Schneider for inviting me to present these results in the Mittagsseminar at the University of Münster. My thanks also goes to the organizers of the conference “Representation Theory and $D$-Modules”, held in June 2019 in Rennes, for giving me the chance to present a poster about this research. Finally, I thank the anonymous referee for a careful reading of the paper and for numerous remarks, questions, and suggestions. During the write-up of this article I was funded by the University of Münster and Germany’s Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics–Geometry–Structure.

2. Preliminaries

2.1. Notations. We fix a locally compact non-archimedean field $\mathfrak{F}$ with residue field $\mathbb{F}_q$ of characteristic $p$ and normalized valuation $\text{val}_\mathfrak{F}: \mathfrak{F} \to \mathbb{Z} \cup \{\infty\}$.

If $H$ is an algebraic group defined over $\mathfrak{F}$, we denote by the corresponding lightface letter $H := H(\mathfrak{F})$ its group of $\mathfrak{F}$-rational points. The topology on $\mathfrak{F}$ makes $H$ into a topological group.

Let $G$ be a connected reductive group defined over $\mathfrak{F}$. We choose a maximal $\mathfrak{F}$-split torus $T$ in $G$ and write $X^*(T)$ (resp. $X_*(T)$) for the group of algebraic $\mathfrak{F}$-characters (resp. algebraic $\mathfrak{F}$-cocharacters) of $T$.

We denote by $Z := Z_G(T)$ the centralizer and by $N := N_G(T)$ the normalizer of $T$ in $G$. We call $W_0 := N/Z$ the finite Weyl group of $G$.

The (relative) root system of $(G, T)$ is denoted by $\Phi \subseteq X^*(T)$; it need not be reduced if $G$ is non-split. The finite Weyl group $W_0$ identifies with the Weyl group of the root system $\Phi$. We denote

$$\Phi_{\text{red}} = \{\alpha \in \Phi \mid \alpha/2 \notin \Phi\}$$

the subroot system of reduced roots. The set of coroots is denoted $\Phi^\vee \subseteq X_*(T)$.

We consider the root group $U_{\alpha}$ attached to $\alpha \in \Phi$. Then $U_{2\alpha} \subseteq U_{\alpha}$ whenever $\alpha, 2\alpha \in \Phi$.

We fix a minimal parabolic $\mathfrak{F}$-subgroup $B$ of $G$ containing $T$. It then admits a Levi decomposition

$$B = UZ.$$ 

This choice fixes a system of positive roots $\Phi^+$ in $\Phi$ (resp. positive coroots $(\Phi^\vee)^+$ in $\Phi^\vee$), and the unipotent radical $U$ of $B$ decomposes as

$$U = \prod_{\alpha \in \Phi^+} U_{\alpha},$$

where $\Phi^+_{\text{red}} = \Phi_{\text{red}} \cap \Phi^+$ is the set of reduced positive roots.

All parabolic subgroups are taken to be standard with respect to $B$. 

2.2. The standard apartment. If $C$ denotes the connected center of $G$, we consider the finite-dimensional $\mathbb{R}$-vector space

$$V := \mathbb{R} \otimes_{\mathbb{R}} \left( X_*(T)/X_*(C) \right).$$

We view the set of coroots $\Phi^\vee$ as a subset of $V$ via the natural map. Note that $\Phi^\vee$ generates $V$ as an $\mathbb{R}$-vector space. On $V$ there is the following partial ordering: given $v, w \in V$, we write

$$v \leq w$$

if $w - v$ is a linear combination of simple coroots with non-negative coefficients.

The conjugation action of $W_0$ on $T$ induces an action on $V$ such that the natural pairing

$$(\cdot, \cdot) : V^* \times V \longrightarrow \mathbb{R}$$

is non-degenerate and $W_0$-equivariant. Here, $V^* = \text{Hom}_\mathbb{R}(V, \mathbb{R})$ denotes the $\mathbb{R}$-linear dual of $V$. We view $\Phi$ as a generating subset of $V^*$. We fix a $W_0$-invariant scalar product $(\cdot, \cdot)$ on $V$ so that $V$ becomes a Euclidean vector space. We denote $\| \cdot \|$ the norm induced by $(\cdot, \cdot)$. The scalar product $(\cdot, \cdot)$ induces a $W_0$-invariant scalar product on $V^*$ which we again denote $(\cdot, \cdot)$.

By [BT84] 1.1.13 the tuple $(Z, (U_\alpha)_{\alpha \in \Phi})$ is a generating root group datum of $G$ in the sense of [BT72] (6.1.1). In particular, this means that the root groups $U_\alpha$ satisfy the following condition:

(1) For all $\alpha, \beta \in \Phi$, the commutator group $[U_\alpha, U_\beta]$ is contained in the group generated by $U_{\alpha + m\beta}$, where $n, m \in \mathbb{Z}_{>0}$ are such that $n\alpha + m\beta \in \Phi$.

The standard apartment $A$ in the adjoint building $\mathcal{B}(G)$ of $G$ is an affine space under $V$ consisting of certain valuations $\mathcal{B}(G)$ (6.2.1) of $(Z, (U_\alpha)_{\alpha \in \Phi})$. Since valuations will be instrumental later on, we will recall their definition. For the moment let $L_\alpha$, for $\alpha \in \Phi$, be the subgroup generated by $U_\alpha, Z$, and $U_{-\alpha}$, and put

$$M_\alpha := \{ x \in L_\alpha \mid xU_\alpha x^{-1} = U_{-\alpha} \text{ and } xU_{-\alpha}x^{-1} = U_\alpha \} \subseteq N.$$

Then $M_\alpha$ is a left and right coset under $Z$ with image $\{ s_\alpha \}$ in $W_0$. We record the following useful lemma:

**Lemma 2.1** ([BT72] (6.1.2) (2)]. Let $\alpha \in \Phi$ and $u \in U_\alpha := U_\alpha \setminus \{ 1 \}$. Then there exists a unique triple $(u', m(u), u'') \in U_{-\alpha} \times G \times U_{-\alpha}$ such that $u = u'm(u)u''$, $m(u)U_{-\alpha}m(u)^{-1} = U_\alpha$, and $m(u)U_\alpha m(u)^{-1} = U_{-\alpha}$. Moreover, one has $m(u) \in M_\alpha$ and $u', u'' \neq 1$.

**Definition** ([BT72] (6.2.1)]. A valuation on $(Z, (U_\alpha)_{\alpha \in \Phi})$ is a tuple $\psi = (\psi_\alpha)_{\alpha \in \Phi}$ of functions $\psi_\alpha : U_\alpha \rightarrow \mathbb{R} \cup \{ \infty \}$ satisfying the following conditions:

1. **(V0)** The image of $\psi_\alpha$ contains at least three elements;
2. **(V1)** For each $\alpha \in \Phi$ and $r \in \mathbb{R} \cup \{ \infty \}$ the set $U_{\alpha, r} = \psi_\alpha^{-1}([r, \infty])$ is a subgroup of $U_\alpha$, and $U_{\alpha, \infty} = \{ 1 \}$;
3. **(V2)** For each $\alpha \in \Phi$ and each $m \in M_\alpha$ the function

$$U_\alpha^* = U_{-\alpha} \setminus \{ 1 \} \longrightarrow \mathbb{R}, \quad u \mapsto \psi_{-\alpha}(u) - \psi_\alpha(mu^{-1})$$

is constant.
4. **(V3)** Given $\alpha, \beta \in \Phi$ with $\beta \notin \mathbb{R}_{\geq 0}\alpha$, and $r, s \in \mathbb{R}$, the commutator group $[U_{\alpha, r}, U_{\beta, s}]$ is contained in the group generated by the groups $U_{n\alpha + m\beta, nr + ms}$, for $n, m \in \mathbb{Z}_{>0}$ such that $n\alpha + m\beta \in \Phi$;
5. **(V4)** If $\alpha, 2\alpha \in \Phi$, then $\psi_{2\alpha}$ is the restriction of $2\psi_\alpha$ to $U_{2\alpha} \subseteq U_\alpha$;
6. **(V5)** Given $\alpha \in \Phi$, $u \in U_\alpha$, and $u', u'' \in U_{-\alpha}$ such that $u'uu'' \in M_\alpha$, one has $\psi_{-\alpha}(u') = \psi_{-\alpha}(u'') = -\psi_\alpha(u)$.

The space of valuations on $(Z, (U_\alpha)_{\alpha \in \Phi})$ admits the following two actions $\mathcal{B}(G)$ (6.2.5):

- Given a valuation $\psi = (\psi_\alpha)_{\alpha \in \Phi}$ and $v \in V$, the tuple

$$\psi + v = (\psi_\alpha + \langle \alpha, v \rangle)_{\alpha \in \Phi}$$

is again a valuation.
Let \( \psi = (\psi_\alpha)_{\alpha \in \Phi} \) be a valuation and \( n \in N \). Denote \( w \) the image of \( n \) under the canonical projection \( N \rightarrow N/Z = W_0 \). We obtain a new valuation \( n.\psi = ((n.\psi)_\alpha)_{\alpha \in \Phi} \) defined by

\[
(n.\psi)_\alpha(u) = \psi_{\alpha^{-1}}(n^{-1}un), \quad \text{for all } u \in U_\alpha.
\]

In this way, the group \( N \) acts on the space of valuations.

By [BT84] 5.1.20 Theorème and 5.1.23 Proposition there exists a valuation

\[
\varphi = (\varphi_\alpha : U_\alpha \rightarrow \mathbb{R} \cup \{\infty\})_{\alpha \in \Phi} \text{ of } (Z,(U_\alpha)_{\alpha \in \Phi})
\]

which is discrete [BT84 (6.2.21)], special [BT84 (6.2.13)] and compatible with the valuation \( \varphi \) [BT84 4.2.8 Définition]. The standard apartment \( \mathcal{A} \) is the Euclidean affine space under \( V \) given by

\[
\mathcal{A} = \{ \varphi + v \mid v \in V \}.
\]

The action of \( N \) restricts to an action on \( \mathcal{A} \) by Euclidean affine automorphisms and the subgroup \( Z \) acts by translations [BT84 (6.2.10)]. More concretely, it follows from [V2] that there exists a unique group homomorphism \( \nu : Z \rightarrow V \) that satisfies for all \( u \in U_\alpha \), all \( \alpha \in \Phi \)

\[
\varphi_\alpha(z^{-1}uz) = \varphi_\alpha(u) + \langle \alpha, \nu(z) \rangle, \quad \text{for all } u \in U_\alpha.
\]

The fact that \( \varphi \) is compatible with \( \varphi \) then expresses the condition \( (\chi(\nu(z))) = -\varphi_\delta(\chi(z)) \), for all \( \chi \in X^*(Z) \).

The affine action of \( N \) on \( \mathcal{A} \) induces a linear action of \( W_0 = N/Z \) on \( V \), obtained by composing the action map \( N \rightarrow \text{Aff}(\mathcal{A}) \) (where \( \text{Aff}(\mathcal{A}) \) denotes the group of affine automorphisms of \( \mathcal{A} \)) with the canonical projection \( \text{Aff}(\mathcal{A}) \rightarrow \text{GL}_\mathbb{R}(V) \). This action coincides with the natural action of \( W_0 \) on \( V \).

Given \( \alpha \in V^* \) and \( r \in \mathbb{R} \), we consider the hyperplane

\[
H_{\alpha,r} := \{ \varphi + v \in \mathcal{A} \mid \langle \alpha, v \rangle + r = 0 \}
\]

and put

\[
\mathcal{H}_{\alpha,r} := \{ H_{\alpha,r} \mid \alpha \in \Phi_{\text{red}} \text{ and } r \in \varphi_\alpha(U^*_\alpha) \}.
\]

Then \( N \) acts on \( \mathcal{H} \) via

\[
n.H_{\alpha,r} = H_{w(\alpha),r - \langle w(\alpha), n.\varphi - \varphi \rangle}, \quad \text{for } n \in N \text{ with image } w \in W_0.
\]

The groups \( U_{\alpha,r} = \varphi_\alpha^{-1}([r, \infty]) \), for \( r \in \mathbb{R} \), are a neighborhood basis of \( 1 \in U_\alpha \) consisting of compact open subgroups, and we have

\[
n.U_{\alpha,r}^{-1} = U_{w(\alpha),r - \langle w(\alpha), n.\varphi - \varphi \rangle}, \quad \text{for } n \in N \text{ with image } w \in W_0.
\]

2.3. The associated reduced root system. We denote by \( S(\mathcal{H}) \) the set of orthogonal reflections \( s_H \) through \( H \in \mathcal{H} \). Conversely, we denote \( H_\delta \) the hyperplane in \( \mathcal{A} \) fixed by \( s \in S(\mathcal{H}) \). This exhibits a canonical bijection \( \mathcal{H} \cong S(\mathcal{H}) \).

The group \( W^{\text{aff}} \) generated by \( S(\mathcal{H}) \) (inside the group of affine automorphisms of \( \mathcal{A} \)) is called the affine Weyl group of \( G \). The stabilizer of \( \varphi \) in \( W^{\text{aff}} \) identifies with \( W_0 \), since \( \varphi \) is special. This yields a semidirect product decomposition

\[
W^{\text{aff}} = (W^{\text{aff}} \cap V) \rtimes W_0,
\]

and \( W^{\text{aff}} \cap V \) is generated by the translations \( r\alpha^\vee \), for \( \alpha \in \Phi_{\text{red}} \) and \( r \in \varphi_\alpha(U^*_\alpha) \) [BT84 (6.2.19)]. In particular, \( W^{\text{aff}} \cap V \) is a lattice of rank \( \dim V \) in \( V \). Now, [Bou81] Ch. VI, §2, no. 5, Prop. 8] shows that there exists a unique reduced root system

\[
\Sigma \subseteq V^*
\]

such that \( W^{\text{aff}} \) is the affine Weyl group of \( \Sigma \). This means that \( W^{\text{aff}} \) coincides with the group generated by the reflections \( s_{\alpha,k} \), for \( (\alpha,k) \in \Sigma^{\text{aff}} := \Sigma \times \mathbb{Z} \), defined by

\[
s_{\alpha,k}(x) = x - ((\alpha, x - \varphi) + k) \cdot \alpha^\vee, \quad \text{for } x \in \mathcal{A}.
\]

We write simply \( s_\alpha \) instead of \( s_{\alpha,0} \) and view it as an element of \( W_0 \).
We note the following properties:

- which induces a bijection $\Phi : Z \rightarrow \Lambda$ for $\alpha \in \Phi$,

the projection $\Lambda$ strictly positive, the group $Z$ and denote $\Lambda$.

2.5. The positive monoid. We define

$\Lambda^+ := \Lambda^{+,G} := \{ \lambda \in \Lambda \mid \langle \alpha, \nu(\lambda) \rangle \leq 0 \text{ for all } \alpha \in \Sigma^+ \}$

and denote $Z^+$ (or $Z^{+,G}$ if we want to emphasize the dependence on $G$) the preimage of $\Lambda^+$ under the projection $Z \rightarrow \Lambda$. We refer to $Z^+$ as the positive monoid. The negative monoid is defined as $Z^- := (Z^+)^{-1}$. We also write $\Lambda^- := -\Lambda^+$.

An element $\lambda \in \Lambda^+$ is called strictly positive if $\langle \alpha, \nu(\lambda) \rangle < 0$, for all $\alpha \in \Sigma^+$. Note that if $\lambda$ is strictly positive, the group $\Lambda$ is generated (as a monoid) by $\Lambda^+$ and $-\lambda$.

More generally, let $P = U_P M$ be a parabolic subgroup of $G$. We denote

$M^+ := \{ m \in M \mid mK_{U_P}m^{-1} \subseteq K_{U_P} \}$
the monoid of $M$-positive elements. Note that $K_M \subseteq M^+ \cap (M^+)^{-1}$. We define
\begin{equation}
\mu_{U^p}(m) := [K_{U^p} : K_{U^p} \cap m^{-1}K_{U^p}m] \in \mathbb{Z}_{\geq 0}.
\end{equation}
Clearly, $m \in M$ is $M$-positive if and only if $\mu_{U^p}(m) = 1$. The integers $\mu_{U^p}(m)$ have been studied in [Hey21b] §3.4.

An element $\lambda \in \Lambda$ is called strictly $M$-positive if $\langle \Sigma_M, \nu(\lambda) \rangle = 0$ and $\langle \alpha, \nu(\lambda) \rangle < 0$ for all $\alpha \in \Sigma^+ \setminus \Sigma_M$. Note that by (6), the monoid
\begin{equation}
\Lambda_{M^+} := \{ \lambda \in \Lambda \mid \langle \alpha, \nu(\lambda) \rangle \leq 0 \text{ for all } \alpha \in \Sigma^+ \setminus \Sigma_M \}
\end{equation}
coincides with the image of $Z \cap M^+$ in $\Lambda$.

We call $a \in Z$ strictly $M$-positive if $a$ lies in the center of $M$ and $aK_Z \subseteq \Lambda$ is strictly $M$-positive. We remark that by [BK98, (6.14)] there exist strictly $M$-positive elements.

2.6. Double coset decompositions. We recall here the Cartan and the Iwasawa decomposition of $G$. In [3] we will study intersections between Cartan and Iwasawa double cosets.

Cartan decomposition 2.2 ([HR10] Theorem 1.0.3]). The inclusion $Z \subseteq G$ induces a bijection
$$\Lambda/W_0 \cong K\backslash G/K.$$ 

Remark. (a) The monoids $\Lambda^+$ and $\Lambda^-$ are representatives for the $W_0$-orbits of $\Lambda$ [HV15] 6.3 Lemma. Therefore, the inclusion $Z \subseteq G$ induces bijections $\Lambda^+ \cong K\backslash G/K$ and $\Lambda^- \cong K\backslash G/K$. These are also referred to as the Cartan decomposition.

(b) The Cartan decomposition implies that if $KzK = Kz'K$, for some $z, z' \in Z$, then there exists $w \in W_0$ such that $w(zK_Z) = z'K_Z$ (and hence also $w.\nu(z) = \nu(z')$).

Iwasawa decomposition 2.3. The inclusion $Z \subseteq G$ induces a bijection
$$\Lambda \cong U\backslash G/K.$$ 

This decomposition is often written as $G = BK = UZK = ZUK$.

Remark 2.4. It is of general interest to study intersections of Cartan and Iwasawa double cosets. We recall some well-known results. Let $z \in Z^-$ and $z' \in Z$ such that $Uz'K \cap KzK \neq \emptyset$. Then:

(a) $\nu(z') \leq \nu(z)$, see [HR10] Lemma 10.2.1 or [HV15] 6.10 Proposition.

(b) If $\nu(z) = \nu(z')$, then $zK_Z = z'K_Z$, see [HV15] 6.10 Proposition.

(c) $w.\nu(z') \leq \nu(z)$, for all $w \in W_0$. This follows from properties of the Satake homomorphism as in [Rap00] Lemma 2.1 but using [HV15] 7.13 Theorem. This argument is also spelled out in Remark 5.5.

This inequality is equivalent to saying that $\nu(z')$ lies inside the convex polytope spanned by the $W_0$-orbit of $\nu(z)$, cf. [Mac03] (2.6.2).

In [4] we give an algorithm which yields also information about the $u \in U$ with $uz' \in KzK$.

2.7. Abstract Hecke rings. We briefly discuss abstract Hecke rings. The references below refer to, and details can be found in [AZ95] Ch. 3, §1.

Let $G$ be a topological group and $\Gamma \subseteq G$ a compact open subgroup. Let $\Gamma \subseteq S \subseteq G$ be a submonoid. The pair $(\Gamma, S)$ is called a Hecke pair. Let
$$Z[\Gamma \backslash S] = \bigoplus_{\gamma \in \Gamma \backslash S} Z.(\Gamma_s)$$
be the free $\mathbb{Z}$-module on the set of right cosets $\Gamma_s$. It admits a natural right $S$-action by $(\Gamma_s) \cdot s' = (\Gamma ss')$, for $s, s' \in S$. Clearly, $Z[\Gamma \backslash S]$ is a left module under the ring
$$\mathcal{H}(\Gamma, S) := \text{End}_S(Z[\Gamma \backslash S]).$$

We usually make the identification
$$\mathcal{H}(\Gamma, S) \xrightarrow{\cong} Z[\Gamma \backslash S]^\Gamma,$$
$$T \mapsto T((\Gamma)).$$
The submodule \( \mathbb{Z}[\Gamma \backslash S]^\Gamma \) of \( \Gamma \)-invariants is a free \( \mathbb{Z} \)-module on the set of double cosets \( \Gamma \backslash S / \Gamma \). Concretely, it admits \( \{(s)_\Gamma \mid \Gamma s \Gamma \in \Gamma \backslash S / \Gamma \} \) as a basis, where

\[
(s)_\Gamma := \sum_{\Gamma s' \leq \Gamma s \Gamma} (\Gamma s').
\]

The sum runs through all right cosets contained in \( \Gamma s \Gamma \). Note that the sum is finite, because \( \Gamma \) is compact open, so that the set \( (\Gamma \cap s^{-1} \Gamma s) \Gamma \) is finite, and the map

\[
(\Gamma \cap s^{-1} \Gamma s) \backslash \Gamma \ni s \mapsto (\Gamma s \Gamma) \gamma
\]

is bijective. The multiplication on \( \mathbb{Z}[\Gamma \backslash S]^\Gamma \) is concretely given by

\[
\left( \sum_{i} n_i \cdot (\Gamma s_i) \right) \cdot \left( \sum_{j} m_j \cdot (\Gamma t_j) \right) = \sum_{i,j} n_i m_j \cdot (\Gamma s_i t_j).
\]

For an explicit description of the multiplication in terms of double cosets, see [AZ95] Lemma 1.5.

The following two results are frequently useful:

**Proposition 2.5** ([AZ95] Prop. 1.9]). Let \( (\Gamma, S) \) and \( (\Gamma_0, S_0) \) be two Hecke pairs satisfying

\[
\Gamma_0 \subseteq \Gamma, \quad S \subseteq \Gamma S_0, \quad \text{and} \quad \Gamma \cap S_0 \cdot S_0^{-1} \subseteq \Gamma_0.
\]

Then the map

\[
\varepsilon: \mathcal{H}(\Gamma, S) \longrightarrow \mathcal{H}(\Gamma_0, S_0),
\]

\[
\sum n_i \cdot (\Gamma s_i) \longrightarrow \sum n_i \cdot (\Gamma_0 s_i),
\]

where the \( s_i \) are chosen in \( S_0 \), is an injective ring homomorphism.

**Proposition 2.6** ([AZ95] Prop. 1.11]). Let \( (\Gamma, S) \) be a Hecke pair. Then \( (\Gamma, S^{-1}) \) is also a Hecke pair, and the map

\[
\zeta_S: \mathcal{H}(\Gamma, S) \longrightarrow \mathcal{H}(\Gamma, S^{-1}),
\]

\[
(s)_\Gamma \longmapsto (s^{-1})_\Gamma
\]

is an anti-isomorphism of rings.

**Lemma 2.7** ([AZ95] Lem. 1.13]). Let \( (\Gamma, S) \) and \( (\Gamma_0, S_0) \) be two Hecke pairs satisfying (9) such that \( (\Gamma, S^{-1}) \) and \( (\Gamma_0, S_0^{-1}) \) also satisfy (9). Then the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{H}(\Gamma, S) & \longrightarrow & \mathcal{H}(\Gamma_0, S_0) \\
\varepsilon \downarrow & & \varepsilon \downarrow \\
\mathcal{H}(\Gamma, S^{-1}) & \longrightarrow & \mathcal{H}(\Gamma_0, S_0^{-1}).
\end{array}
\]

If \( R \) is a commutative ring with 1, we put

\[
\mathcal{H}_R(\Gamma, S) := R \otimes_{\mathbb{Z}} \mathcal{H}(\Gamma, S).
\]

It is clear that Propositions 2.5 and 2.6 and Lemma 2.7 remain valid for Hecke rings over \( R \).

3. Non-obtuse parabolics

We fix a maximal parabolic subgroup \( P = U_P M \) of \( G \). Recall from 2.2 the Euclidean vector space \( (V^*, \langle \cdot, \cdot \rangle) \), on which the finite Weyl group \( W_0 \) acts, and the special point \( \varphi \in \mathfrak{a}^* \) from 2.2. We view \( \varphi \) as a valuation on the root group datum \( (Z, (U_a)_{a \in \Phi}) \). Also recall from 2.3 the reduced root system \( \Sigma \subseteq V^* \). The system \( \Sigma^+ \) of positive roots determines a unique basis \( \Delta \) of \( \Sigma \). Since \( M \) is reductive, all these objects have an analogue for \( M \), and we denote them by adding the subscript ‘\( M \)’. For example, we write \( \Sigma_M, W_{0,M}, \Delta_M \) etc.
Definition. The parabolic $P$ is called non-obtuse if
\[ \langle \alpha, \beta^\vee \rangle \geq 0, \quad \text{for all } \alpha, \beta \in \Sigma^+ \setminus \Sigma_M. \]

Remark. For $P$ to be non-obtuse it is equivalent to say that
\[ (\alpha, \beta) \geq 0, \quad \text{for all } \alpha, \beta \in \Sigma^+ \setminus \Sigma_M. \]

This more geometric definition explains the term “non-obtuse”: it means that any two roots in $\Sigma^+ \setminus \Sigma_M$ span a non-obtuse angle.

During the whole section we assume that $P$ is non-obtuse. The main result of this section is a complete classification of the non-obtuse parabolic subgroups of $G$. First, we prove an important technical result which explains our interest in non-obtuse parabolics.

Lemma 3.1. Let $\lambda, \mu \in \Lambda$ such that $\lambda$ is strictly $M$-positive. Assume that $\nu(w(\mu)) \leq \nu(-\lambda)$ for all $w \in W_0$. Then one has
\[ \langle \alpha, \nu(\lambda + \mu) \rangle \leq 0, \quad \text{for all } \alpha \in \Sigma^+ \setminus \Sigma_M. \]

Example. Before giving the proof, let us look at the two examples in Figure 1. Example (A) explains why Lemma 3.1 should be expected to hold true for non-obtuse parabolics, while (B) explains why Lemma 3.1 fails otherwise.

![Figure 1](image)

(a) Type $A_2$: The translate of the dotted region by $\nu(\lambda)$ fits into the shaded area. Lemma 3.1 holds in this case.
(b) Type $G_2$: The translate of the dotted region by $\nu(\lambda)$ does not fit into the shaded area. Lemma 3.1 fails.

Proof of Lemma 3.1. Recall that $P$ is non-obtuse so that $\langle \alpha, \beta^\vee \rangle \geq 0$ for all $\alpha, \beta \in \Sigma^+ \setminus \Sigma_M$. We proceed in two steps.

Step 1: Assume $\mu = w(-\lambda)$, for some $w \in W_0$. We do an induction on the length $\ell(w)$ of $w$. If $\ell(w) = 1$, we write $w = s_\beta$ for some simple root $\beta \in \Delta$. For each $\alpha \in \Sigma^+ \setminus \Sigma_M$ we compute
\[
\langle \alpha, \nu(\lambda + s_\beta(-\lambda)) \rangle = \langle \alpha, \nu(\lambda) - s_\beta(\nu(\lambda)) \rangle = \langle \alpha, \nu(\lambda) \rangle - \langle \alpha, s_\beta(\nu(\lambda)) \rangle = \langle \beta, \nu(\lambda) \rangle \cdot \langle \alpha, \beta^\vee \rangle \leq 0,
\]
where in the last step we have used $\langle \beta, \nu(\lambda) \rangle < 0$ and that $P$ is non-obtuse. Now assume $\ell(w) > 1$, and let $\beta \in \Delta$ with $\ell(s_\beta w) < \ell(w)$. We write
\[
\nu(\lambda + w(-\lambda)) = \nu(\lambda + s_\beta(-\lambda)) + s_\beta(\nu(\lambda + s_\beta w(-\lambda))).
\]
We distinguish two cases:
If $\beta \in \Delta_M$, then we have $s_\beta(\Sigma^+ \setminus \Sigma_M) = \Sigma^+ \setminus \Sigma'_M$. For each $\alpha \in \Sigma^+ \setminus \Sigma_M$, the induction hypothesis (applied to $s_\beta w$) yields:

$$\langle \alpha, s_\beta(\nu(\lambda + s_\beta w(-\lambda))) \rangle = \langle s_\beta(\alpha), \nu(\lambda + s_\beta w(-\lambda)) \rangle \leq 0.$$

Together with (11) and the base case the statement follows.

- If $\beta \in \Delta \setminus \Delta_M$, then we compute for each $\alpha \in \Sigma^+ \setminus \Sigma_M$:

$$\langle \alpha, \nu(\lambda + w(-\lambda)) \rangle = \langle \alpha, \nu(\lambda + s_\beta(\lambda)) \rangle + \langle \alpha, s_\beta(\nu(\lambda + s_\beta w(-\lambda))) \rangle$$

$$= \langle \alpha, \langle \beta, \nu(\lambda) \rangle \cdot \beta' \rangle + \langle \beta, \nu(\lambda) \rangle$$

$$= \langle \beta, \nu(\lambda) \rangle \cdot \langle \alpha, \beta' \rangle + \langle \alpha - \beta', \beta, \nu(\lambda + s_\beta w(-\lambda)) \rangle$$

$$= \langle \beta, \nu(\lambda + s_\beta w(-\lambda)) \rangle - \langle \alpha, \beta' \rangle$$

$$= \langle \alpha, \nu(\lambda + s_\beta w(-\lambda)) \rangle + \langle \alpha, \beta' \rangle$$

$$\leq 0,$$

where the last step uses the induction hypothesis and that $(s_\beta w)^{-1}(\beta) \in \Sigma^+$, which in turn follows from $\ell((s_\beta w)^{-1}) > \ell((s_\beta w)^{-1})$ (see, e.g., [Hum90, 1.6 Lemma]).

This finishes the induction step.

**Step 2:** Let $\mu$ be general. Take an arbitrary $\alpha \in \Sigma^+ \setminus \Sigma_M$. Let $\alpha_0 = w(\alpha)$ be a root of maximal height in the $W_0$-orbit of $\alpha$. (If we write $\alpha_0 = \sum_{\beta \in \Delta} n_\beta \beta$, the *height* of $\alpha_0$ is $\sum_{\beta \in \Delta} n_\beta$.) Then we have $\langle \alpha_0, \beta' \rangle \geq 0$, for all $\beta \in \Sigma^+$, since otherwise $s_\beta(\alpha_0) = \alpha_0 - \langle \alpha_0, \beta' \rangle \cdot \beta$ would have greater height than $\alpha_0$. By the hypothesis, $\nu(-\lambda) - \nu(w(\mu))$ is a linear combination of simple coroots with non-negative coefficients. Therefore,

$$\langle \alpha, \nu(w^{-1}(\lambda) + \mu) \rangle = \langle w(\alpha), \nu(\lambda + w(\mu)) \rangle = -\langle \alpha_0, \nu(-\lambda) - \nu(w(\mu)) \rangle \leq 0.$$

By Step 1 we have

$$\langle \alpha, \nu(\lambda + w^{-1}(-\lambda)) \rangle \leq 0.$$

Finally, (12) and (13) imply

$$\langle \alpha, \nu(\lambda + w^{-1}(-\lambda)) \rangle = \langle \alpha, \nu(\lambda + w^{-1}(\lambda) + \mu) \rangle \leq 0.$$

We now turn to the classification of non-obtuse parabolics. Since $P$ is assumed to be maximal, the roots in $\Sigma^+ \setminus \Sigma_M$ are contained in an irreducible component of $\Sigma$. Without loss of generality we may therefore assume that $\Sigma$ is irreducible.

The maximal parabolics of $G$ are in one-to-one correspondence with the elements of the basis $\Delta$ of $\Sigma$. We write $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and denote $s_r := s_{\alpha_r}$ the simple reflection attached to $\alpha_r$. Let $P_r = U_{P_r} M_r$ be the maximal parabolic subgroup corresponding to $\alpha_r$ so that $\Delta_{M_r} = \Delta \setminus \{\alpha_r\}$. We put

$$\Sigma_{U_{P_r}} := \Sigma^+ \setminus \Sigma_{M_r}.$$

Then $\alpha_r$ is the unique element in $\Sigma_{U_{P_r}} \cap \Delta$. We say that $\alpha_r$ is *non-obtuse* if $P_r$ is.

**Proposition 3.2.** The classification of non-obtuse parabolic subgroups of $G$ is given in terms of the Dynkin diagram of $\Sigma$ in Figure [1]. Moreover, the following conditions are equivalent:

(i) $\alpha_r$ is non-obtuse.

(ii) $\langle \alpha_r, \beta' \rangle \geq 0$ for all $\beta \in \Sigma_{U_{P_r}}$.

(iii) The Weyl group $W_{0,M_r}$ acts transitively on the roots in $\Sigma_{U_{P_r}}$ of the same length.

(iv) In the notation of [14] below we have $c_r(\alpha'_n) = 1$, where $\alpha'_n$ is the highest root of the same length as $\alpha_r$. 


Proof. We consider each type separately. The equivalences of (i)-(iv) are obtained along the way. For the concrete description of the roots systems we follow [Bou81, Planches I–IX].

As usual we denote $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$, endowed with the canonical scalar product $(\cdot, \cdot)$. Given a root $\beta \in \Sigma$, we write

$$
\beta = \sum_{i=1}^{n} c_i(\beta) \cdot \alpha_i.
$$

Observe that $c_i(s_j(\beta)) = c_i(\beta)$, for $i \neq j$. In particular, the action of $W_{0,M_r}$ does not affect $c_r(\beta)$. Observe that $\Sigma_{U_{P_r}}$ consists of those $\beta \in \Sigma^+$ with $c_r(\beta) > 0$.

(A) Inside $V = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0\}$ the root system of type $A_n$ is

$$
\Sigma = \{ \pm(e_i - e_j) \mid 1 \leq i < j \leq n+1 \}.
$$

The simple roots are given by $\alpha_i = e_i - e_{i+1}$, for $i = 1, \ldots, n$, and the roots $e_i - e_j = \sum_{k=i}^{j-1} \alpha_k$, for $1 \leq i < j \leq n+1$, are positive. The Weyl group $W_0$ is the symmetric group $\mathfrak{S}_{n+1}$ acting on $e_1, \ldots, e_{n+1}$. Fix $1 \leq r \leq n$. Then we have

$$
\Sigma_{U_{P_r}} = \{ e_i - e_j \mid 1 \leq i \leq r < j \leq n+1 \}.
$$

If $e_i - e_j, e_a - e_b \in \Sigma_{U_{P_r}}$, then we have $a \neq j$ and $i \neq b$. Hence, $(e_i - e_j, e_a - e_b) = \delta_{ia} + \delta_{jb} \geq 0$ and $\alpha_r$ is non-obtuse. The Weyl group $W_{0,M_r}$ identifies with $\mathfrak{S}_r \times \mathfrak{S}_{n+1-r}$ with the first factor acting on $\{e_1, \ldots, e_r\}$ and the second on $\{e_{r+1}, \ldots, e_{n+1}\}$. It clearly acts transitively on $\Sigma_{U_{P_r}}$.
(B_n) Inside $V = \mathbb{R}^n$ the root system of type $B_n$ is

$$ \Sigma = \{ \pm e_i \mid 1 \leq i \leq n \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \}.$$

A basis is given by $\alpha_i = e_i - e_{i+1}$, for $1 \leq i < n$, and $\alpha_n = e_n$. The positive roots are

$$ \begin{cases} 
 e_i & \text{for } 1 \leq i \leq n; \\
 e_i - e_j & \text{for } 1 \leq i < j \leq n; \\
 e_i + e_j & = 2 \sum_{k=i}^{n-1} \alpha_k, \text{ for } 1 \leq i < j \leq n. 
\end{cases}$$

The Weyl group $W_0$ identifies with $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ with $S_n$ permuting the $e_i$, and $(\mathbb{Z}/2\mathbb{Z})^n$ acting by changing the signs of the $e_i$. Consider three cases:

- Assume $r = 1$. Then

$$ \Sigma_{U_{\mathcal{P}_1}} = \{ e_i \mid 2 \leq i \leq n \} \cup \{ e_i \pm e_j \mid 1 \leq i < j \leq n \}.$$ 

Given $e_1 + \varepsilon_1 e_i, e_1 + \varepsilon_2 e_j \in \Sigma_{U_{\mathcal{P}_1}}$, with $\varepsilon_1, \varepsilon_2 \in \{-1, 0, 1\}$, we compute

$$ (e_1 + \varepsilon_1 e_i, e_1 + \varepsilon_2 e_j) = 1 + \varepsilon_1 \varepsilon_2 \delta_{ij} \geq 0. $$

Hence, $\alpha_1$ is non-obtuse. The Weyl group $W_{0,M_1}$ identifies with $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_{n-1}$ with both groups acting on $\{ \pm e_2, \ldots, \pm e_n \}$, leaving $e_1$ fixed. It clearly acts transitively on $\{ e_1 \pm e_i \mid 2 \leq i \leq n \}$ (and on $\{ e_1 \}$).

- Assume $r = n$. Then

$$ \Sigma_{U_{\mathcal{P}_n}} = \{ e_i \mid 1 \leq i \leq n \} \cup \{ e_i \pm e_j \mid 1 \leq i < j \leq n \}. $$

It is obvious that $\alpha_n$ is non-obtuse. The Weyl group $W_{0,M_n}$ identifies with $S_n$ acting on $e_1, \ldots, e_n$. It clearly acts transitively on both $\{ e_1, \ldots, e_n \}$ and $\{ e_i \mid 1 \leq i < j \leq n \}$.

- Assume $1 < r < n$; in particular, $n \geq 3$. Note that both $\alpha_r = e_r - e_{r+1}$ and $e_{r-1} + e_{r+1}$ lie in $\Sigma_{U_{\mathcal{P}_n}}$ and satisfy $(e_r - e_{r+1}, e_{r-1} + e_{r+1}) = -1$. Hence, $\alpha_r$ is non-obtuse. The highest root is $\alpha_0 = e_1 + e_2 = \alpha_1 + 2 \sum_{k=2}^{n} \alpha_k$. Notice that $\alpha_0$ and $\alpha_r$ both lie in $\Sigma_{U_{\mathcal{P}_n}}$ and have the same length. But since $c_r(\alpha_0) = 2 \neq 1 = c_r(\alpha_r)$, it follows that $\alpha_0$ does not lie in the $W_{0,M_n}$-orbit of $\alpha_r$.

(C_n) Inside $V = \mathbb{R}^n$ the root system of type $C_n$ is

$$ \Sigma = \{ \pm 2e_i \mid 1 \leq i \leq n \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \}. $$

A basis is given by $\alpha_i = e_i - e_{i+1}$, for $1 \leq i < n$, and $\alpha_n = 2e_n$. The positive roots are

$$ \begin{cases} 
 e_i - e_j & = \sum_{k=i}^{j-1} \alpha_k, \text{ for } 1 \leq j \leq n; \\
 e_i + e_j & = \sum_{k=i}^{j-1} \alpha_k + 2 \sum_{k=j}^{n-1} \alpha_k + \alpha_n, \text{ for } 1 \leq i \leq j \leq n; \\
 2e_i & = 2 \sum_{k=i}^{n-1} \alpha_k + \alpha_n, \text{ for } 1 \leq i \leq n. 
\end{cases}$$

The Weyl group $W_0$ identifies with $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ as for type $B_n$. Consider three cases:

- Assume $r = 1$. Then

$$ \Sigma_{U_{\mathcal{P}_1}} = \{ 2e_i \mid 2 \leq i \leq n \} \cup \{ e_i \pm e_j \mid 2 \leq i \leq n \}. $$

Given $e_1 + \varepsilon_1 e_i, e_1 + \varepsilon_2 e_j \in \Sigma_{U_{\mathcal{P}_1}}$, with $i, j \neq 1$ and $\varepsilon_1, \varepsilon_2 \in \{ \pm 1 \}$, we compute

$$ (e_1 + \varepsilon_1 e_i, e_1 + \varepsilon_2 e_j) = 1 + \varepsilon_1 \varepsilon_2 \delta_{ij} \geq 0. $$

Since also $(2e_1, e_1 \pm e_i) = 2 \geq 0$ (for $i \neq 1$), the root $\alpha_1$ is non-obtuse. The Weyl group $W_{0,M_1}$ identifies with $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_{n-1}$ with both groups acting on $\{ \pm e_2, \ldots, \pm e_n \}$ leaving $e_1$ fixed. It clearly acts transitively on $\{ e_1 \pm e_i \mid 2 \leq i \leq n \}$ (and on $\{ 2e_1 \}$).

- Assume $r = n$. Then

$$ \Sigma_{U_{\mathcal{P}_n}} = \{ 2e_i \mid 1 \leq i \leq n \} \cup \{ e_i + e_j \mid 1 \leq i < j \leq n \}. $$

It is obvious that $\alpha_n$ is non-obtuse. The Weyl group $W_{0,M_n}$ identifies with $S_n$ acting on $e_1, \ldots, e_n$. It clearly acts transitively on $\{ 2e_1, \ldots, 2e_n \}$ and on $\{ e_i + e_j \mid 1 \leq i < j \leq n \}$. 

\[ \]
Assume $1 < r < n$; in particular, $n \geq 3$. Note that both $\alpha_r = e_r - e_{r+1}$ and $e_{r-1} + e_{r+1}$ lie in $\Sigma_{U_{P_0}}$ and satisfy $(e_r - e_{r+1}, e_{r-1} + e_{r+1}) = -1$. Hence, $\alpha_r$ is non-obtuse.

Consider the root $\alpha_0' = e_1 + e_2 = \alpha_1 + 2 \sum_{k=2}^{n-1} \alpha_k + \alpha_n$. Then $\alpha_0'$ and $\alpha_r$ both lie in $\Sigma_{U_{P_0}}$ and have the same length. But since $c_r(\alpha_0') = 2 \neq 1 = c_r(\alpha_r)$, it follows that $\alpha_0'$ does not lie in the $W_{0,M_1}$-orbit of $\alpha_r$.

$(D_n)$ Inside $V = \mathbb{R}^n$ the root system of type $D_n$ is

$$\Sigma = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \}.$$ 
A basis is given by $\alpha_i = e_i - e_{i+1}$, for $1 \leq i < n$, and $\alpha_n = e_{n-1} + e_n$. The positive roots are

$$\begin{align*}
\{ e_i - e_j = \sum_{k=1}^{j-1} \alpha_k, & \quad \text{for } 1 \leq i < j \leq n; \\
\{ e_i + e_n = \sum_{k=i}^{n-2} \alpha_k + \alpha_n, & \quad \text{for } 1 \leq i < n; \\
\{ e_i + e_j = \sum_{k=i}^{j-1} \alpha_k + 2 \sum_{k=j}^{n-2} \alpha_k + \alpha_{n-1} + \alpha_n, & \quad \text{for } 1 \leq i < j < n.
\end{align*}$$

The Weyl group $W_0$ identifies with $\Gamma \times \mathfrak{S}_n$, where $\Gamma$ is the kernel of the map $(\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$.

$(E_n)$ Inside $V = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_5 = x_7 = -x_8 \}$ the root system of type $E_6$ is

$$\Sigma = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq 5 \}
\cup \left\{ \pm \frac{1}{2} \left( e_8 - e_8 - e_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e_i \right) \mid \nu(i) \in \mathbb{Z}/2\mathbb{Z}, \sum_{i=1}^{5} \nu(i) = 0 \right\}.$$ 
A basis is given by $\alpha_1 = \frac{1}{2}(e_1 + e_3) - \frac{1}{2}(e_2 + e_3 + \cdots + e_7)$, $\alpha_2 = e_2 + e_1$, and $\alpha_i = e_{i-1} - e_{i-2}$, for $3 \leq i \leq 6$. We distinguish the following cases:

- Assume $r = 1$ or $r = 6$. By the symmetry of the Dynkin diagram for $E_6$ it suffices to consider the case $r = 1$. We have

$$\Sigma_{U_{P_1}} = \left\{ \frac{1}{2} (e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e_i) \mid \nu(i) \in \mathbb{Z}/2\mathbb{Z}, \sum_{i=1}^{5} \nu(i) = 0 \right\}.$$ 

To ease the notation we write $\alpha_\nu := \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e_i)$, for each $\nu = (\nu(i)) \in (\mathbb{Z}/2\mathbb{Z})^5$. Let $\nu, \mu \in (\mathbb{Z}/2\mathbb{Z})^5$ such that $\sum_{i=1}^{5} \nu(i) = \sum_{i=1}^{5} \mu(i) = 0$. Since $\sum_{i=1}^{5} (\nu(i) + \mu(i)) = 0$, we observe that the cardinality of the set $\{ 1 \leq i \leq 5 \mid \nu(i) \neq \mu(i) \}$
is even, hence equals 0, 2, or 4. But then $|\{1 \leq i \leq 5 \mid \nu(i) = \mu(i)\}|$ is either 1, 3, or 5. Thus, we compute

$$(\alpha_{\nu}, \alpha_{\mu}) = \frac{1}{4}(3 + |\{i \mid \nu(i) = \mu(i)\}| - |\{i \mid \nu(i) \neq \mu(i)\}|) \geq 0.$$ 

Therefore, $\alpha_1$ is non-obtuse. The Weyl group $W_{0, M_1}$ is the group $\Gamma \rtimes \mathfrak{S}_5$ of type $D_5$ described in $[D_n]$ it acts on $e_1, \ldots, e_5$, leaving $e_6, e_7,$ and $e_8$ fixed. Given $\nu, \mu \in (\mathbb{Z}/2\mathbb{Z})^6$ with $\alpha_{\nu}, \alpha_{\mu} \in \Sigma_{U_{P_i}}$, we may view $\mu - \nu$ as an element of $\Gamma$ which maps $\alpha_\nu$ to $\alpha_{\mu}$. Therefore, $W_{0, M_1}$ acts transitively on $\Sigma_{U_{P_i}}$.

- Assume $1 < r < 6$. The roots $\beta_1 = \sum_{k=1}^5 \alpha_k = \frac{1}{2}(e_8 - e_7 - e_6 + e_1 + e_2 - e_3 + e_4 - e_5)$ and $\beta_2 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 = e_5 + e_3$ both lie in $\Sigma_{U_{P_i}}$, and satisfy $(\beta_1, \beta_2) = -1$. Hence, $\alpha_r$ is not non-obtuse. Notice that $\beta_1 = w_r(\alpha_r)$, where $w_r \in W_{0, M_r}$ is given by

$$w_2 = s_5s_1s_3s_4,$$

$$w_3 = s_5s_1s_2s_4,$$

$$w_4 = s_5s_1s_3s_2,$$

$$w_5 = s_1s_2s_3s_4.$$ 

Clearly, $w_r^{-1}(\beta_2) \in \Sigma_{U_{P_i}}$, and $(\alpha_r, w_r^{-1}(\beta_2)) = (\beta_1, \beta_2) = -1$.

The highest root is $\alpha_0 = \frac{1}{2}(e_8 - e_7 - e_6 + e_1 + e_2 + e_3 + e_4 + e_5) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. Then $\alpha_0$ and $\alpha_r$ both lie in $\Sigma_{U_{P_i}}$ and have the same length. But since $c_r(\alpha_0) > 1 = c_r(\alpha_r)$, it follows that $\alpha_0$ does not lie in the $W_{0, M_r}$-orbit of $\alpha_r$.

(E7) Inside $V = \{ (x_1, \ldots, x_8) \in \mathbb{R}^8 \mid x_7 = -x_8 \}$ the root system of type $E_7$ is

$$\Sigma = \{ e_i \pm e_j \mid 1 \leq i < j \leq 6 \} \cup \{ e_8 - e_7 \}$$

$$\cup \left\{ \pm \frac{1}{2} \begin{pmatrix} e_8 - e_7 + \sum_{i=1}^6 (-1)^{\nu(i)} e_i \end{pmatrix} \nu(i) \in \mathbb{Z}/2\mathbb{Z}, \sum_{i=1}^6 \nu(i) \neq 0 \right\}.$$ 

A basis is given by $\alpha_1 = \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + \cdots + e_7), \alpha_2 = e_2 + e_1,$ and $\alpha_3 = e_i - e_i - e_i - e_i - 2$, for $3 \leq i \leq 7$. We distinguish the following cases:

- Assume $r = 7$. Then $\Sigma_{U_{P_r}} = \{ e_6 \pm e_i \mid 1 \leq i \leq 5 \} \cup \{ e_8 - e_7 \}$

$$\cup \left\{ \pm \frac{1}{2} \begin{pmatrix} e_8 - e_7 + e_6 + \sum_{i=1}^5 (-1)^{\nu(i)} e_i \end{pmatrix} \nu(i) \in \mathbb{Z}/2\mathbb{Z}, \sum_{i=1}^5 \nu(i) \neq 0 \right\}.$$ 

(These are all the positive roots not lying in the subroot system of type $E_6$.) To ease the notation we write $\alpha_\nu := \frac{1}{2}(e_8 - e_7 + \sum_{i=1}^6 (-1)^{\nu(i)} e_i)$, for each $\nu = (\nu(i))_i \in (\mathbb{Z}/2\mathbb{Z})^6$. For all $1 \leq i, j \leq 5$ and $\nu \in (\mathbb{Z}/2\mathbb{Z})^6$ with $\alpha_\nu \in \Sigma_{U_{P_r}}$ we compute

$$(e_6 \pm e_i, e_6 \pm e_j) = 1 \pm \delta_{ij} \geq 0,$$

$$(e_6 \pm e_i, e_8 - e_7) = 0,$$

$$(e_6 \pm e_i, \alpha_\nu) = \frac{1}{2}(1 \pm (-1)^{\nu(i)}) \geq 0,$$

$$(e_8 - e_7, \alpha_\nu) = 1.$$ 

Let now $\nu, \mu \in (\mathbb{Z}/2\mathbb{Z})^6$ with $\nu(6) = \mu(6) = 0$ and $\sum_{i=1}^5 \nu(i) = \sum_{i=1}^5 \mu(i) \neq 0$. As $\sum_{i=1}^5 (\nu(i) + \mu(i)) = 0$, we observe that the cardinality of the set $\{ 1 \leq i \leq 6 \mid \nu(i) \neq \mu(i) \}$ is even, but not 6, hence equals 0, 2, or 4. But then $|\{ 1 \leq i \leq 6 \mid \nu(i) = \mu(i) \}|$ is either 2, 4, or 6. Thus, we compute

$$(\alpha_\nu, \alpha_\mu) = \frac{1}{4}(2 + |\{ i \mid \nu(i) = \mu(i) \}| - |\{ i \mid \nu(i) \neq \mu(i) \}|) \geq 0.$$ 

Therefore, $\alpha_7$ is non-obtuse. The Weyl group $W_{0, M_7}$ is the group generated by $s_1$ and the group $\Gamma \rtimes \mathfrak{S}_5$ of type $D_5$ acting on $\{ \pm e_1, \ldots, \pm e_5 \}$ while leaving $e_6, e_7,$ and
$e_8$ fixed). Given $\nu, \mu \in \Sigma_{U_{r^*}}$, we may view $\mu - \nu$ as an element of $\Gamma$ (by forgetting the last entry) which maps $\alpha_{\nu}$ to $\alpha_{\mu}$. Moreover, $\Gamma \rtimes G_5$ clearly acts transitively on \{ $e_0 \pm e_i \mid 1 \leq i \leq 5$ \}. Together with

$$s_1(e_0 - e_1) = e_0 - e_1 + \alpha(0,1,1,1,1,1) = \alpha(1,1,1,1,1,0),$$

and

$$s_1(\alpha(1,0,0,0,0,0)) = \alpha(1,0,0,0,0,0) + \alpha(0,1,1,1,1,1) = e_8 - e_7,$$

and the fact that $\Gamma$ acts transitively on the set of $\alpha_{\nu}$ with $\sum_{i=1}^5 \nu(i) \neq 0$ and $\nu(6) = 0$, it follows that $\Sigma_{U_{r^*}}$ is the $W_{0,M_r}$-orbit of $\alpha_7$. Hence, $W_{0,M_r}$ acts transitively on $\Sigma_{U_{r^*}}$.

Assume $1 \leq r < 7$. The roots $\beta_1 = \sum_{k=1}^6 \alpha_k = \alpha(0,0,1,1,0,1)$ and $\beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 = \alpha(1,1,0,0,1,0)$ both lie in $\Sigma_{U_{r^*}}$ and satisfy $(\beta_1, \beta_2) = -1$. Hence, $\alpha_r$ is not non-obtuse. Note that $\beta_1 = w_r(\alpha_r)$, where $w_r \in W_{0,M_r}$ is given by

$$w_1 = s_6 s_5 s_2 s_8 4 s_3,$$
$$w_2 = s_6 s_5 s_1 s_3 s_4,$$
$$w_3 = s_6 s_5 s_2 s_4 s_1,$$
$$w_4 = s_6 s_5 s_1 s_3 s_2,$$
$$w_5 = s_6 s_1 s_3 s_2 s_4,$$
$$w_6 = s_1 s_2 s_3 s_4 s_5.$$

Clearly, $w_r^{-1}(\beta_2) \in \Sigma_{U_{r^*}}$ and $(\alpha_r, w_r^{-1}(\beta_2)) = (\beta_1, \beta_2) = -1$.

The highest root is $\alpha_0 = e_8 - e_7 = 2\alpha_1 + \alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. Then $\alpha_0$ and $\alpha_r$ both lie in $\Sigma_{U_{r^*}}$ and have the same length. But since $c_r(\alpha_0) > 1 = c_r(\alpha_r)$, it follows that $\alpha_0$ does not lie in the $W_{0,M_r}$-orbit of $\alpha_r$.

$(E_8)$ Inside $V = \mathbb{R}^8$ the root system of type $E_8$ is

$$\Sigma = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq 8 \} \cup \left\{ \pm \frac{1}{2} \left( e_8 + \sum_{i=1}^7 (-1)^{\nu(i)} e_i \right) \mid \nu(i) \in \mathbb{Z}/2\mathbb{Z}, \sum_{i=1}^7 \nu(i) = 0 \right\}.$$

A basis is given by $\alpha_1 = \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + \cdots + e_7)$, $\alpha_2 = e_2 + e_1$, and $\alpha_i = e_{i-1} - e_{i-2}$, for $3 \leq i \leq 8$.

Let $1 \leq r \leq 8$. The roots $\beta_1 = \sum_{k=1}^8 \alpha_k = \frac{1}{2}(e_8 + e_7 - e_6 + e_1 + e_2 - e_3 - e_4 - e_5)$ and $\beta_2 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + e_8 = \frac{1}{2}(e_8 + e_2 + e_6 - e_1 - e_2 + e_3 + e_4 + e_5)$ both lie in $\Sigma_{U_{r^*}}$ and satisfy $(\beta_1, \beta_2) = -1$. Hence, $\alpha_r$ is not non-obtuse. Note that $\beta_1 = w_r(\alpha_r)$, where $w_r \in W_{0,M_r}$ is given by

$$w_1 = s_8 s_7 s_6 s_5 s_2 s_8 4 s_3,$$
$$w_2 = s_8 s_7 s_6 s_5 s_1 s_3 s_4,$$
$$w_3 = s_8 s_7 s_6 s_5 s_2 s_4 s_1,$$
$$w_4 = s_8 s_7 s_6 s_5 s_1 s_3 s_2,$$
$$w_5 = s_8 s_7 s_6 s_2 s_1 s_3 s_4,$$
$$w_6 = s_8 s_7 s_2 s_1 s_3 s_4 s_5,$$
$$w_7 = s_8 s_2 s_1 s_3 s_4 s_5 s_6,$$
$$w_8 = s_2 s_1 s_3 s_4 s_5 s_6 s_7.$$

Clearly, $w_r^{-1}(\beta_2) \in \Sigma_{U_{r^*}}$ and $(\alpha_r, w_r^{-1}(\beta_2)) = (\beta_1, \beta_2) = -1$.

The highest root is $\alpha_0 = e_8 + e_7 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$. Then $\alpha_r$ and $\alpha_0$ both lie in $\Sigma_{U_{r^*}}$ and have the same length. But since $c_r(\alpha_0) > 1 = c_r(\alpha_r)$, it follows that $\alpha_0$ does not lie in the $W_{0,M_r}$-orbit of $\alpha_r$.

$(F_4)$ Inside $V = \mathbb{R}^4$ the root system of type $F_4$ is

$$\Sigma = \{ \pm e_i \mid 1 \leq i \leq 4 \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq 4 \} \cup \{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \}.$$
A basis is given by \( \alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \text{ and } \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4) \). We have

\[
(\alpha_1, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4) = (e_2 - e_3, e_1 + e_3) = -1, \\
(\alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4) = (e_3 - e_4, e_1 - e_3) = -1, \\
(\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4) = (e_4, e_1 - e_4) = -1, \\
(\alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4) = \frac{1}{4} \cdot (e_1 - e_2 - e_3 - e_4, e_1 + e_2 + e_3 + e_4) = -\frac{1}{2}.
\]

Hence, none of the \( \alpha_1, \ldots, \alpha_4 \) is non-obtuse. Consider the following cases:

- Assume \( r = 1 \) or \( r = 2 \). The highest root is \( \alpha_0 = e_1 + e_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \). Then both \( \alpha_0 \) and \( \alpha_r \) lie in \( \Sigma_{U_{r'}} \) and have the same length. But since \( c_r(\alpha_0) > 1 = c_r(\alpha_r) \), it follows that \( \alpha_0 \) does not lie in the \( W_{0,M_r} \)-orbit of \( \alpha_r \).

- Assume \( r = 3 \) or \( r = 4 \). The highest short root is \( \alpha_0' = e_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \). Then \( \alpha_0' \) and \( \alpha_r \) both lie in \( \Sigma_{U_{r'}} \) and have the same length. But since \( c_r(\alpha_0') > 1 = c_r(\alpha_r) \), it follows that \( \alpha_0' \) does not lie in the \( W_{0,M_r} \)-orbit of \( \alpha_r \).

\((G_2)\) Inside \( V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\} \) the root system of type \( G_2 \) is

\[
\Sigma = \{ -e_1 - e_2, -e_3, e_1 - e_3, 2e_1 - e_2 - e_3, 2e_2 - e_1 - e_3, 2e_3 - e_1 - e_2 \}.
\]

A basis is given by \( \alpha_1 = e_1 - e_2 \) and \( \alpha_2 = -2e_1 + e_2 + e_3 \). Since

\[
(\alpha_1, \alpha_1 + \alpha_2) = (e_1 - e_2, e_3 - e_1) = -1 \\
(\alpha_2, 3\alpha_1 + \alpha_2) = (-2e_1 + e_2 + e_3, e_1 - 2e_2 + e_3) = -3,
\]

neither \( \alpha_1 \) nor \( \alpha_2 \) are non-obtuse.

The highest root is \( \alpha_0 = 3\alpha_1 + 2\alpha_2 = -e_1 - e_2 + 2e_3 \). Then both \( \alpha_0 \) and \( \alpha_2 \) lie in \( \Sigma_{U_{r'}} \) and have the same length. But since \( c_2(\alpha_0) = 2 \neq 1 = c_2(\alpha_2) \), it follows that \( \alpha_0 \) does not lie in the \( W_{0,M_r} \)-orbit of \( \alpha_2 \).

Similarly, the highest short root is \( \alpha_0' = 2\alpha_1 + \alpha_2 = e_3 - e_2 \), and both \( \alpha_0' \) and \( \alpha_1 \) lie in \( \Sigma_{U_{r'}} \) and have the same length. But since \( c_1(\beta) = 2 \neq 1 = c_1(\alpha_1) \), it follows that \( \alpha_0' \) does not lie in the \( W_{0,M_r} \)-orbit of \( \alpha_1 \).

\( \square \)

We end this section by applying the previous analysis to prove a result on the ordering of positive roots that will become useful later. First, we need a preliminary lemma which also appeared in \textbf{Mac71} Lemma (2.1.1). For the convenience of the reader we supply the simple proof.

We make the following convention: if \( \Delta' \) is a basis of \( \Sigma \), we denote by \( \Sigma^+_{\Delta'} \) (resp. \( \Sigma^-_{\Delta'} \)) the system of positive (resp. negative) roots with respect to \( \Delta' \).

Recall, \textbf{Bou81} Ch. VI, §1.6, Cor. 3 of Prop. 17, that there exists a unique longest element \( w_0 \in W_0 \). It satisfies \( w_0^2 = 1 \) and \( \ell(ww_0) = \ell(w_0) - \ell(w) \), for all \( w \in W_0 \).

\textbf{Lemma 3.3.} \textit{Let} \( w_0 = s_{i_1} \cdots s_{i_r} \) \textit{be a reduced decomposition and put} \( \beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \), \textit{for} \( j = 1, \ldots, r \). \textit{Then} one has

\[
\Sigma^+_{s_{i_1} \cdots s_{i_j}(\Delta)} = \{ \beta_{j+1}, \beta_{j+2}, \ldots, \beta_r, -\beta_1, -\beta_2, \ldots, -\beta_j \}, \quad \text{for all} \ 0 \leq j \leq r.
\]

\textit{Proof.} Write \( w_j = s_{i_1} \cdots s_{i_j} \). Applying \textbf{Bou81} Ch. VI, §1.6, Cor. 2 of Prop. 17 \textit{to} \( w_j^{-1} = s_{i_r} \cdots s_{i_1} \) \textit{yields}

\[
\Sigma^+_{\Delta} \cap \Sigma^-_{w_j(\Delta)} = \Sigma^+_{\Delta} \cap w_j \Sigma^-_{\Delta} = \{ \beta_1, \beta_2, \ldots, \beta_r \}.
\]

\textbf{Note} that \( \Sigma^+_{\Delta} = \Sigma^+_{\Delta} \cap \Sigma^-_{w_0(\Delta)} = \{ \beta_1, \beta_2, \ldots, \beta_r \} \). Hence, the assertion follows from

\[
\Sigma^+_{w_j(\Delta)} = (\Sigma^-_{\Delta} \cap \Sigma^+_{w_j(\Delta)}) \cup (\Sigma^+_{\Delta} \cap \Sigma^+_{w_j(\Delta)}) = -(\Sigma^-_{\Delta} \cap \Sigma^-_{w_0(\Delta)}) \cup (\Sigma^+_{\Delta} \cap \Sigma^+_{w_0(\Delta)}).
\]

\( \square \)

\textbf{Example.} It is instructive to visualize an example. The orderings of the positive roots in \( \Sigma^+ \) obtained in \textbf{Lemma 3.3} generalize the “circular orderings” one has for root systems of rank 2. Assume \( \Sigma \) is
of type $G_2$ with basis $\{\alpha_1, \alpha_2\}$ such that $\alpha_2$ is the long root. The ordering of $\Sigma^+$ corresponding to the reduced decomposition $w_0 = s_1 s_2 s_1 s_2$ is shown in Figure 2.

![Figure 2. The circular ordering in type $G_2$](image)

**Corollary 3.4.** Let $\alpha_i \in \Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a non-obtuse simple root, and let $\alpha \in \Sigma_{U_P}$ such that $\alpha$ and $\alpha_i$ have the same length. There exists a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_r}$ such that, if we put $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$, there exists $0 \leq l < r$ with $\beta_1, \ldots, \beta_l \in \Sigma_M$ and $\beta_{l+1} = \alpha$. In particular,

$$\Sigma_{U_P} \setminus \{\alpha\} \subseteq \Sigma^+_{\beta_1, \ldots, \beta_{l+1}} = \{\beta_{l+2}, \ldots, \beta_r, -\beta_1, \ldots, -\beta_l, -\alpha\}.$$ 

**Remark.** Corollary 3.4 says geometrically that, for non-obtuse $\alpha_i$, the roots of length $\|\alpha_i\|$ are extremal in the cone generated by $\Sigma_{U_P}$.

The statement of Corollary 3.4 is generally false if $\alpha_i$ is not non-obtuse, see Figure 2.

**Proof of Corollary 3.4.** Denote $w_{0,M}$ the longest element in $W_{0,M}$. Since $\alpha_i$ is non-obtuse, we find by Proposition 3.2 an element $w \in W_{0,M}$ with $w(\alpha_i) = \alpha$. Choose reduced decompositions

$$w = s_{i_1} \cdots s_{i_l} \quad \text{and} \quad w_{0,M} w_0 = s_{i_{l+1}} \cdots s_{i_r}.$$ 

For each $v \in W_{0,M}$, we compute

$$\ell(v w_{0,M} w_0) = \ell(w_0) - \ell(v w_{0,M}) = \ell(w_0) - \ell(w_{0,M}) + \ell(v) = \ell(v) + \ell(w_{0,M} w_0).$$

In particular, $w w_{0,M} w_0 = s_{i_1} \cdots s_{i_r}$ is a reduced decomposition. We further observe $s_{i_{l+1}} = s_{i_r}$ for otherwise we would have $s_{i_{l+1}} \in W_{0,M}$ and $\ell(s_{i_{l+1}} w_{0,M} w_0) < \ell(w_{0,M} w_0)$, contradicting (15) for $v = s_{i_{l+1}}$. Now, if we pick a reduced decomposition $(w w_{0,M} w_0)^{-1} w_0 = s_{i_{l+1}} \cdots s_{i_r}$, then it is clear that we obtain a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_r}$. From the construction it is clear that $\beta_1, \ldots, \beta_l \in \Sigma_M$ and $\beta_{l+1} = \alpha$. The last statement is a consequence of Lemma 3.3. □

4. The algorithm

Recall the special parahoric subgroup $K$ of $G$ associated with $\varphi$. Given $z \in Z^-$ and $z' \in Z$, it is of general interest to understand the intersection of the Iwasawa double coset $U z' K$ and the Cartan double coset $K z K$. For example, it is well-known that $\nu(z') \leq \nu(z)$ provided $U z' K \cap K z K \neq \emptyset$, see 3.2.6.

There is, however, very little known about the $u \in U$ such that $u z' \in K z K$. One of the main goals of this article is to study the following question: if $u z' \in K z K$ and if we write $u = u_{\gamma_1} \cdots u_{\gamma_n}$, with $u_{\gamma_i} \in U_{\gamma_i}$, what can be said about the valuations $\varphi_{\gamma_i}(u_{\gamma_i})$? We will prove that for each strictly positive element $a \in Z$ with $\nu(a) \leq \nu(a^{-1})$, the valuation $\varphi_{\gamma_i}(u_{\gamma_i})$ is bounded below by $\langle \gamma_i, \nu(a) \rangle$, see Theorem 4.3.

In this section we present an algorithm that gives information about the $\varphi_{\gamma_i}(u_{\gamma_i})$. First, we need to set up some notation. We fix a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_r}$ of the longest element $w_0$ of $W_0$.

**Notation 4.1.** Recall the reduced root system $\Sigma$ associated with $\Phi$. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the fixed basis of $\Sigma$. 
(a) Put $\beta_j := s_{i_1} \cdots s_{i_{j-1}} (\alpha_{i_j})$ and $\Delta^{(j)} = s_{i_1} \cdots s_{i_j} (\Delta)$, for $1 \leq j \leq r$. Thanks to Lemma 4.3 we have
\[ \Sigma^{(j)} := \Sigma^{+}_{\Delta^{(j)}} = \{ \beta_{j+1}, \beta_{j+2}, \ldots, \beta_r, -\beta_1, -\beta_2, \ldots, -\beta_j \}, \quad \text{for } 0 \leq j \leq r. \]
Given $k \geq 0$, we let $j(k) \in \{1, \ldots, r\}$ be the unique integer with $k \equiv j(k) \pmod{r}$. Let
\[ \varepsilon_k := \begin{cases} 1, & \text{if } k \equiv j(k) \pmod{2r}, \\ -1, & \text{otherwise}. \end{cases} \]
Put $\beta_k := \varepsilon_k \beta_{j(k)}$ and $\Delta^{(k)} := \varepsilon_k \Delta^{(j(k))}$, and then
\[ \Sigma^{(k)} := \varepsilon_k \Sigma^{(j(k))} = \Sigma^{+}_{\Delta^{(j)}} = \{ \beta_{k+1}, \beta_{k+2}, \ldots, \beta_{k+r} \}. \]
Then $\Sigma^{(0)} = \Sigma$ and the sequence $(\Sigma^{(k)})_{k \in \mathbb{Z}_{\geq 0}}$ is 2r-periodic. See also Figure 3 below.

(b) Let $\alpha \in \Sigma$ and write $\alpha = \varepsilon_\beta \beta$ for the unique $\beta \in \Phi_{\text{red}}$. Define
\[ \varphi_\alpha := \varepsilon_\beta \varphi_\beta : U_\alpha^* \to \mathbb{Z}. \]
(c) For each $\alpha \in \Sigma$ we fix a lift $n_\alpha \in N \cap K$ of $s_\alpha \in W_0$.
(d) Given a basis $\Delta'$ of $\Sigma$, we denote by $U_{\Delta'}$ the group generated by $\bigcup_{\alpha \in \Sigma_{\Delta'}} U_\alpha$.

![Figure 3](image-url)

**Figure 3.** A visual aid for the sets $\Sigma^{(k)}$ relative to the reduced decomposition $w_0 = s_1 s_2 s_1$, where $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_3$.

Recall from Section 4.2 that $K \cap U_\alpha = U_{(\alpha, 0)}$, for all $\alpha \in \Sigma$.

**Algorithm 4.2.** Let $z, z' \in Z$ and $u \in U$ such that $uz' \in KzK$. We define sequences $(u^{(k)})_{k \geq 0}$ and $(z^{(k)})_{k \geq 0}$ with the following properties:
- $u^{(0)} = u$ and $z^{(0)} = z'$;
- $u^{(k)} \in U_{\Delta^{(k)}} \cap U_{\Delta^{(k-1)}}$ and $z^{(k)} \in Z$, for all $k \geq 1$;
- $u^{(k)} z^{(k)} \in KzK$, for all $k \geq 0$.

Suppose we have constructed $u^{(k)}$ and $z^{(k)}$ for some $k \geq 0$. Write
\[ u^{(k)} = u_{\beta_{k+r}}^{(k)} \cdots u_{\beta_{k+r-1}}^{(k)} u_{\beta_{k+1}}^{(k)} \in \begin{cases} U_{\Delta^{(k)}}, & \text{if } k = 0, \\ U_{\Delta^{(k)}} \cap U_{\Delta^{(k-1)}}, & \text{if } k \geq 1, \end{cases} \]
for uniquely determined $u_{\beta_{k+1}}^{(k)} \in U_{\beta_{k+1}}$. Depending on $\varphi_{\beta_{k+1}} (u_{\beta_{k+1}}^{(k)})$ we distinguish three cases:

\[ \text{Note that } u_{\beta_{k+1}}^{(k)} = 1 \text{ unless } k = 0. \]
Case \(\varphi_{\beta_{k+1}}(u_{\beta_{k+1}}^{(k)}) + \langle \beta_{k+1}, \nu(z^{(k)}) \rangle \geq 0\). By (i) this is equivalent to

\[ x := (z^{(k)})^{-1} \cdot u_{\beta_{k+1}}^{(k)} \cdot z^{(k)} \in U_{(\beta_{k+1}, 0)} = U_{\beta_{k+1}} \cap K. \]

We then define

\[ z^{(k+1)} := z^{(k)}, \]

\[ u^{(k+1)} := u^{(k)} \cdot (u_{\beta_{k+1}}^{(k)})^{-1} \in U_{\Delta(k+1)} \cap U_{\Delta(k)}. \]

Then \(u^{(k+1)} z^{(k+1)} = u^{(k)} z^{(k)} x^{-1} \in K z K\).

Case \(\varphi_{\beta_{k+1}}(u_{\beta_{k+1}}^{(k)}) \geq 0\) and not (Alg-1). Then \(u_{\beta_{k+1}}^{(k)} \in K\) and we define

\[ z^{(k+1)} := z^{(k)}, \]

\[ u^{(k+1)} := (u_{\beta_{k+1}}^{(k)})^{-1} \cdot u^{(k)} \in U_{\Delta(k+1)} \cap U_{\Delta(k)}. \]

The fact that \(u^{(k+1)} \in U_{\Delta(k+1)}\) follows from (DR2) and \(\Sigma^{(k)} \setminus \{ \beta_{k+1} \} = \Sigma^{(k+1)} \cap \Sigma^{(k)}\).

Case \(f_k := \varphi_{\beta_{k+1}}(u_{\beta_{k+1}}^{(k)}) < 0\) and not (Alg-1). Then \(u_{\beta_{k+1}}^{(k)} \notin K\). By Lemma 2.1 there exist unique \(u', u'' \in U_{-\beta_{k+1}}\) such that

\[ m^{(k)} := u' u_{\beta_{k+1}}^{(k)} u'' \in N. \]

By [BT72] Proposition (6.2.10) (ii) the element \(m^{(k)}\) acts on \(\mathcal{A}\) as the orthogonal reflection \(s_{\beta_{k+1}, f_k}\) in the hyperplane \(H_{\beta_{k+1}, f_k}\). Observe that the element

\[ z^{(k+1)} := m^{(k)} z^{(k)} \in \beta_{k+1} \]

lies in \(Z\), because its image in \(W_0 = N/Z\) is trivial. Considering how \(z^{(k+1)}\) acts on \(\varphi \in \mathcal{A}\), we deduce

\[ \nu(z^{(k+1)}) = s_{\beta_{k+1}, f_k}(\nu(z^{(k)})) = \nu(z^{(k)}) - (\langle \beta_{k+1}, \nu(z^{(k)}) \rangle + f_k) \cdot \beta_{k+1}. \]

Applying \(\langle \beta_{k+1}, - \rangle\) to this equation, and rearranging, we obtain

\begin{equation}
\varphi_{\beta_{k+1}}(u_{\beta_{k+1}}^{(k)}) = f_k = -\frac{1}{2} \cdot (\langle \beta_{k+1}, \nu(z^{(k)}) \rangle + \langle \beta_{k+1}, \nu(z^{(k+1)}) \rangle).
\end{equation}

By (V5) and (i) we have

\[ \varphi_{-\beta_{k+1}}(u') = -f_k > 0 \quad \text{and} \quad \varphi_{-\beta_{k+1}}((z^{(k)})^{-1} u'' z^{(k)}) = -f_k - \langle \beta_{k+1}, \nu(z^{(k)}) \rangle > 0. \]

This entails that \(u'\) and \((z^{(k)})^{-1} u'' z^{(k)}\) lie in \(K\). We now define

\[ u^{(k+1)} := u' \cdot u^{(k)} \cdot (u_{\beta_{k+1}}^{(k)})^{-1} \cdot (u')^{-1} \in U_{\Delta(k+1)} \cap U_{\Delta(k)}. \]

We remark that \(u_{\beta_{k+2}}^{(k+1)} = u_{\beta_{k+2}}^{(k)}\) as can be seen from [V3] and the fact that \(\beta_{k+2}\) is extremal in \(\Sigma^{(k+1)}\). Finally, we compute

\[ u^{(k+1)} z^{(k+1)} = u' \cdot u^{(k)} \cdot (u_{\beta_{k+1}}^{(k)})^{-1} \cdot (u')^{-1} \cdot m^{(k)} \cdot z^{(k)} \cdot n_{\beta_{k+1}} = u' \cdot u^{(k)} z^{(k)} \cdot (z^{(k)})^{-1} u'' z^{(k)} \cdot n_{\beta_{k+1}} \in K z K. \]

Note that, as a byproduct, at each step the algorithm provides a lower bound for \(\varphi_{\beta_{k+1}}(u_{\beta_{k+1}}^{(k)})\).

It is because of this property that Algorithm 2.2 will be useful for us later.

The next result will not be used in the sequel.

**Proposition 4.3.** Algorithm 2.2 terminates, that is, there exists \(l \geq 0\) such that \(u^{(l)} = 1\). Moreover, \(\nu(z^{(l)})\) lies in the \(W_0\)-orbit of \(\nu(z)\).
Proof. Let \( u \in U, z', z \in Z \) with \( uz' \in KzK \). Suppose we are in case \([\text{Alg-3}]\) at the \( k \)-th step. Then \( \nu(z^{(k+1)}) \) is obtained by reflecting \( \nu(z^{(k)}) \) along the hyperplane \( H_{\beta_{k+1}, f_k} \). But since \( \langle \beta_{k+1}, 0 \rangle + f_k = f_k < 0 \) and \( \langle \beta_{k+1}, \nu(z^{(k)}) \rangle + f_k < 0 \), it follows that \( 0 \) and \( \nu(z^{(k)}) \) are on the same side of \( H_{\beta_{k+1}, f_k} \), whereas \( \nu(z^{(k+1)}) \) lies on the other. An elementary argument in Euclidean geometry now shows \( \| \nu(z^{(k)}) \| < \| \nu(z^{(k+1)}) \| \).

As \( \nu(Z) \) is a lattice in \( V \), its intersection with the convex polytope \( C \) spanned by the \( W_0 \)-orbit of \( \nu(z) \) is finite. Since \( u^{(k)}z^{(k)} \in KzK \), we have \( \nu(z^{(k)}) \in C \), for all \( k \geq 0 \), see \([\text{Alg-2}]\). As the \( z^{(k)} \) remain unchanged in the cases \([\text{Alg-1}]\) and \([\text{Alg-2}]\) the above discussion shows that there are only finitely many instances of case \([\text{Alg-3}]\).

Let \( k \geq 0 \) such that \( \| \nu(z^{(k)}) \| \) is maximal. As only the cases \([\text{Alg-1}]\) and \([\text{Alg-2}]\) occur, it follows that for all \( j \geq 0 \) we have \( z^{(k+j)} = z^{(k)} \) and \( u^{(k+j)} \in U_{\Delta(\alpha+\beta) \cap U_{\Delta(\beta)}} \). In particular, we have \( u^{(k+r)} \in U_{\Delta(\alpha+\beta) \cap U_{\Delta(\beta)}} \) for all \( k \geq r \). Moreover, we have \( Kz^{(k+r)}K = KzK \) by the construction in Algorithm \([\text{Alg-2}]\) and the fact that \( u^{(k+r)} = 1 \), and hence the last assertion follows from the Cartan decomposition \([\text{Alg-2}]\). \( \qed \)

We are now ready to prove our main technical result, which may be of independent interest.

**Theorem 4.4.** Let \( P = U_{P \Phi} \) be a non-obtuse parabolic. Let \( \alpha \in Z \) be strictly \( M \)-positive. Let \( u \in U_P, z \in Z^-, \) and \( z' \in Z \) such that \( \nu(z) \leq \nu(a^{-1}) \) and \( uz' \in KzK \). Then the following assertions hold:

(i) \( az' \in M^+; \)
(ii) \( aua^{-1} \in K_P = K \cap P. \)

Proof. Note that \( uz' \in KzK \) implies \( w, \nu(z') \leq \nu(z) \) for all \( w \in W_0 \), see Remark 2.4(c). Let \( \lambda \) (resp. \( \mu \)) be the image of \( a \) (resp. \( z' \)) in \( \Lambda \). Then \([\text{Alg-2}]\) is equivalent to

\[ \langle \alpha, \nu(\lambda + \mu) \rangle \leq 0, \quad \text{for all } \alpha \in \Sigma^+ \setminus \Sigma_M \]

(cf. \([\text{Alg-2}]\)). But this follows from Lemma 3.1 since by assumption \( \lambda \) is strictly \( M \)-positive and \( \nu(w(\mu)) \leq \nu(z) \leq \nu(-\lambda) \) for all \( w \in W_0 \).

We now prove (ii). As \( P \) is maximal parabolic, the roots appearing in \( U_P \) are contained in a single irreducible component \( \Phi_1 \) of \( \Phi \). Since all computations will be done in the subgroup of \( G \) generated by \( Z \) and \( U_\alpha \), for \( \alpha \in \Phi_1 \), we may assume for notational convenience that \( \Phi \) (and hence \( \Sigma \)) is irreducible. As in \([\text{Alg-2}]\) we write \( \alpha_1, \ldots, \alpha_n \) for the simple roots in \( \Sigma \) and put \( \Sigma_{U_P} = \Sigma^+ \setminus \Sigma_M \).

Let \( w_0 = \prod_{\alpha \in \Sigma_{U_P}} u_\alpha \) be the longest element in \( W_0 \) (resp. \( W_{0,M} \)). Denote \( \alpha_0 = \Sigma_{U_P} \cap \Sigma_M \) the highest root of \( \Sigma \) and write \( \alpha_0 = \sum_{i=1}^n c_i(\alpha_0) \).

Write \( u = \prod_{\alpha \in \Sigma_{U_P}} u_\alpha \) for some ordering of the factors (to be specified later). Since we have \( K_P \cap U_P \in \Sigma_{U_P} U(a_0, 0) \), and because \( U_{\beta,0} \cap U_{2\beta} = U_{2\beta,0} \) whenever \( \beta, 2\beta \in \Phi \) (by \([\text{V4}]\)), it suffices to prove \( \varphi_\alpha(au_\alpha a^{-1}) = \varphi_\alpha(au) - \langle \alpha, \nu(a) \rangle \geq 0, \) that is,

\[ \varphi_\alpha(au_\alpha) \geq \langle \alpha, \nu(a) \rangle, \quad \text{for all } \alpha \in \Sigma_{U_P}. \]

The general procedure is as follows: We fix an ordering \( o \) of \( \Sigma_{U_P} \) with respect to which we write \( u = \prod_{\alpha \in \Sigma_{U_P}} u_\alpha \). For each \( \alpha \in \Sigma_{U_P} \) we apply Algorithm \([\text{Alg-2}]\) in order to estimate \( \varphi_\alpha(au_\alpha) \). This necessitates to temporarily consider a different ordering, and we need to ensure that in the notation of Algorithm \([\text{Alg-2}]\) we have \( \varphi_\alpha(au_\alpha) = \varphi_\alpha(u^{(k)}_{\beta_{k+1}}) \), for the minimal \( k \geq 0 \) for which \( \beta_{k+1} = \alpha \). (In many cases we will even have \( u_\alpha = u^{(k)}_{\beta_{k+1}} \).) The next step in the algorithm then provides the desired estimate for \( \varphi_\alpha(au_\alpha) \). Finally, we go back to the initial ordering \( o \) and repeat this procedure with another root of \( \Sigma_{U_P} \).

(a) Let \( w_0 = s_{i_1} \cdots s_{i_r} \) be a reduced decomposition and apply Algorithm \([\text{Alg-2}]\). At the \( k \)-th step we have

\[ u^{(k)} = u^{(k)}_{\beta_{k+1}} u^{(k)}_{\beta_{k+1} \cdots} u^{(k)}_{\beta_{k+1}}. \]
Note that, by (i), we have $a_\beta^{(k)} \in M^+$ for all $k \geq 0$. Assume $\beta_{k+1} \in \Sigma_{U_P}$, so that $\langle \beta_{k+1}, \nu(a_\beta^{(k)}) \rangle \leq 0$. In case (Alg-1) this implies

$$\varphi_{\beta_{k+1}}(u_{\beta_{k+1}}^{(k)}) \geq -\langle \beta_{k+1}, \nu(z^{(k)}) \rangle \geq \langle \beta_{k+1}, \nu(a) \rangle.$$  

In case (Alg-2) we estimate $\varphi_{\beta_{k+1}}(u_{\beta_{k+1}}^{(k)}) \geq 0 \geq \langle \beta_{k+1}, \nu(a) \rangle$. If, however, we are in case (Alg-3) then (16) implies

$$\varphi_{\beta_{k+1}}(u_{\beta_{k+1}}^{(k)}) = -\frac{1}{2} \left( \langle \beta_{k+1}, \nu(z^{(k)}) \rangle + \langle \beta_{k+1}, \nu(z^{(k+1)}) \rangle \right) \geq \langle \beta_{k+1}, \nu(a) \rangle.$$  

Thus, whenever $\beta_{k+1} \in \Sigma_{U_P}$, we have

$$\varphi_{\beta_{k+1}}(u_{\beta_{k+1}}^{(k)}) \geq \langle \beta_{k+1}, \nu(a) \rangle.$$  

(b) Let $\alpha_{i_0}$ be the unique simple root in $\Sigma_{U_P}$. Let $\alpha \in \Sigma_{U_P}$ with the same length as $\alpha_{i_0}$. By Corollary 3.3 we find a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_r}$ such that for some $0 \leq l < r$ we have $\beta_1, \ldots, \beta_l \in \Sigma_M$ and $\beta_{l+1} = \alpha$. We apply Algorithm 4.2 to this reduced decomposition. Note that $u_{\beta_1}^{(0)} = \cdots = u_{\beta_l}^{(0)} = 1$. As $\alpha = \beta_{l+1}$ is a simple root in $\Sigma_{s_{i_1} \cdots s_{i_l}(\Delta)}$ (which contains $\Sigma_{U_P}$) it follows that $\alpha$ cannot be expressed as the sum of two or more roots in $\Sigma_{U_P}$. Hence, (DR2) implies $u_\alpha = u_{\beta_{l+1}}^{(0)}$. Now, case (Alg-1) applies for the first $l$ steps. Consequently, we have $u_{\beta_{l+1}}^{(0)} = u_\alpha$. Hence, (13) shows $\varphi_\alpha(u_\alpha) \geq \langle \alpha, \nu(a) \rangle$.

This proves (17) in the case where $\alpha$ and $\alpha_{i_0}$ have the same length. When $\Sigma$ is simply-laced, that is, of type $ADE$, then all roots have the same length. This proves (ii) in this case.

It remains to study the cases where $\Sigma$ is of type $B_n$ or $C_n$, and where $\alpha \in \Sigma_{U_P}$ and $\alpha_{i_0}$ have different lengths.

(c) Suppose that $\Sigma$ of type $B_n$ and that $P$ corresponds to $\alpha_n = e_n$ in the notation of $[B_n]$ in the proof of Proposition 3.2. (Note that $\Phi$ is not necessarily reduced.)

We have

$$\Sigma_{U_P} = \{ e_i \mid 1 \leq i \leq n \} \cup \{ e_i + e_j \mid 1 \leq i < j \leq n \}.$$  

We now choose a specific ordering of the factors as follows: let $\sigma: \Sigma_{U_P} \to \{ 1, 2, \ldots, |\Sigma_{U_P}| \}$ be a bijection such that, writing $u = \prod_{i=1}^{\sigma_{i-1}} u_{\sigma_{i-1}}^{(i)}$ with $u_{\sigma_{i-1}}^{(i)} \in U_{\sigma_{i-1}}^{(i)}$, we have:

$$\varphi_\sigma(u_{\sigma_{i-1}}^{(i)}) < \varphi_\sigma(u_{\sigma_{i-1+1}}^{(i)})$$

implies $o(e_i) > o(e_j)$. With this choice of ordering we will prove (17).

For each $\alpha \in \Sigma_{U_P}$ we will apply Algorithm 4.2 to estimate $\varphi_\alpha(u_\alpha)$. As the algorithm changes the ordering, we have to ensure that $\varphi_\alpha(u_\alpha) = \varphi_\alpha(u_\alpha^{(0)})$.

Observe that $e_i$ cannot be written as the sum of two or more roots in $\Sigma_{U_P}$. An application of (DR2) shows that for any ordering the $e_i$-component of $u$ coincides with $u_{e_i}$. Therefore, the estimate for $\varphi_{e_i}(u_{e_i})$ is provided by (b).

Note that, given $\gamma_1, \gamma_2 \in \Sigma_{U_P}$, we have $e_i + e_j = \gamma_1 + \gamma_2$ only if $\{ e_i, e_j \} = \{ \gamma_1, \gamma_2 \}$. An application of (DR2) shows that the $(e_i + e_j)$-component of $u$ in a reordering depends only on the relative position of $u_{e_i}$ and $u_{e_j}$. In order to estimate $\varphi_{e_i + e_j}(u_{e_i + e_j})$, we thus have to ensure that the reordering needed for applying Algorithm 4.2 does not change the relative position of $u_{e_i}$ and $u_{e_j}$.

Observe that every reduced decomposition of $w_0 w_0$ necessarily starts with $s_{n_i} s_{n_1-1} \cdots$; indeed, this follows, since $s_1, \ldots, s_{n-1} \in W_0$ and $\ell(ww_0 w_0) = \ell(w) + \ell(w_0 w_0)$, for all $w \in W_0$, and $s_1 s_{n-1} = s_{n-1} s_1$ for all $1 \leq i \leq n - 2$. Fix $1 \leq i < n$ with $o(e_i) > o(e_j)$ and choose $w \in W_0$ such that $w(e_i) = e_i$ and $w(e_{i-1}) = e_j$. As in the proof of Corollary 3.3 we find a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_r}$ such that $s_{i_1} \cdots s_{i_r}$ is a reduced decomposition of $w$ (for some $0 \leq l \leq r - 2$) and $u_{\beta_{l+1}}^{(0)} = s_n$ and $s_{i_{l+2}}^{(l+2)} = s_{n-1}$. In particular, we have $\beta_1, \ldots, \beta_l \in \Sigma_M$ and $\beta_{l+1} = e_i$. Since $s_n(e_{n-1} - e_n) = e_{n-1} + e_n$, we also deduce $\beta_{l+2} = e_i + e_j$. Note that $e_j = \beta_{l'}$ for some $l' > l + 2$. 

We apply Algorithm 4.2 to this reduced decomposition and observe that, by construction, the relative position of \( u_{e_i} \), \( u_{e_j} \) and \( u^{(0)}_{i} \), \( u^{(0)}_{j} \) is the same; therefore, we have \( u_{e_i + e_j} = u^{(0)}_{e_i + e_j} \).

Note that \( u^{(l)} = u^{(0)} \) in \( U_P \), and hence \( u^{(l)}_{j+1} = u^{(0)}_{e_i} = u_{e_i} \) and \( u^{(l)}_{j+2} = u^{(0)}_{e_i + e_j} = u_{e_i + e_j} \). We now prove

\[
\varphi_{e_i + e_j}(u_{e_i + e_j}) \geq \langle e_i + e_j, \nu(a) \rangle .
\]

In cases \( \text{Alg-1} \) and \( \text{Alg-3} \) we have \( u^{(l+1)}_{j+2} = u^{(l)}_{j+2} = u_{e_i + e_j} \). Therefore, (19) follows from (18). Assume that we are in case \( \text{Alg-2} \), so that \( \varphi_{e_i}(u_{e_i}) \geq \varphi_{e_i}(u_{e_i}) \geq 0 \). Then

\[
u(e_i u_{e_j} = u_{e_i} u_{e_j}^{-1} [u_{e_i}, u_{e_j}^{-1}]
\]

with \([u_{e_i}, u_{e_j}^{-1}] \in U_{e_i + e_j, 0}, \) by (V3). But this means \( u^{(l+1)}_{j+2} = [u_{e_i}, u_{e_j}^{-1}] \cdot u_{e_i + e_j} \). Therefore, we have either \( \varphi_{e_i + e_j}(u_{e_i + e_j}) \geq 0 \geq \langle e_i + e_j, \nu(a) \rangle \), or we have \( \varphi_{e_i + e_j}(u^{(l+1)}_{j+2}) = \varphi_{e_i + e_j}(u_{e_i + e_j}) \).

In the latter case, (19) follows, again, from (18). This proves (19) in case \( \Sigma \) is of type \( B_n \) and \( P \) corresponds to \( \alpha_n \).

(d) If \( \Sigma \) is of type \( C_n \) and \( P \) corresponds to \( \alpha_n = 2e_n \), then a similar argument as in (c) applies. The argument becomes easier, though, since \( U_P \) is commutative (use (DR2) and the fact that \( c_n(a_0) = 1 \).

(e) Assume that \( \Phi \) is of type \( BC_n \) and \( P \) corresponds to \( \alpha_1 = e_1 - e_2 \) in the notation of (B_n) in the proof of Proposition 3.2. The other cases, where \( \Phi \) is of type \( B_n \) or \( C_n \) (and where \( P \) corresponds to \( \alpha_1 \)) are proved in essentially the same way.

Note that \( \Sigma \) is of type \( B_n \) and we have

\[
\Sigma U_P = \{ e_1 \} \cup \{ e_1 \pm e_i \; | \; 2 \leq i \leq n \}.
\]

We remark that, again by (DR2) the \( u_{e_i} \), do not depend on the ordering of the factors. By \( \text{[b]} \) we have \( \varphi_{e_1 \pm e_i}(u_{e_1 \pm e_i}) \geq \langle e_1 \pm e_i, \nu(a) \rangle \). It remains to prove

\[
\varphi_{e_1}(u_{e_1}) \geq \langle e_1, \nu(a) \rangle.
\]

Note that if \( 2 \varphi_{e_1}(u_{e_1}) \geq \varphi_{e_1 - e_i}(u_{e_1 - e_i}) + \varphi_{e_1 + e_i}(u_{e_1 + e_i}) \), for some \( 2 \leq i \leq n \) and some ordering of the factors, then we easily obtain (20). Therefore, we assume from now on

\[
2 \varphi_{e_1}(u_{e_1}) < \varphi_{e_1 - e_i}(u_{e_1 - e_i}) + \varphi_{e_1 + e_i}(u_{e_1 + e_i})
\]

for all \( 2 \leq i \leq n \) and all orderings of the factors.

Given \( v \in V \), we denote \( s_v \), the orthogonal reflection in the hyperplane orthogonal to \( v \).

Claim 1. The decomposition \( s_{e_1} = (s_1 s_2 \cdots s_{n-1}) s_n (s_{n-1} s_{n-2} \cdots s_1) \) is reduced.

Proof. We write this decomposition as \( s_i \cdots s_{i_2n-1} \) and put \( \beta_j := s_i \cdots s_{i_1}(a_i) \), for all \( 1 \leq j \leq 2n - 1 \). Then we have

\[
\beta_j = \begin{cases} s_1 \cdots s_{j-1}(e_j - e_{j+1}) = e_1 - e_{j+1}, & \text{for } 1 \leq j \leq n - 1; \\ s_1 \cdots s_{n-1}(e_1) = e_1, & \text{for } j = n. \end{cases}
\]

For \( 1 \leq j \leq n - 1 \) we compute

\[
\beta_{2n-j} = s_1 s_2 \cdots s_j s_{j+1} \cdots s_{n-1} s_{j+1} (e_j - e_{j+1})
\]

\[
= s_1 s_2 s_{j+1} (e_j - e_{j+1})
\]

\[
= s_1 s_j (e_j + e_{j+1})
\]

\[
= e_1 + e_{j+1}.
\]

Therefore, the elements \( \beta_1, \ldots, \beta_{2n-1} \) are pairwise distinct and \( \text{[Bou81]} \) Ch. IV, §1, no. 4, Lemma 2] shows that \( \ell(s_{e_1}) = 2n - 1 \).
We fix a reduced decomposition $s_{i_1} \cdots s_{i_r}$ of $w_0$ whose initial piece is $s_1 s_2 \cdots s_n s_{n-1} \cdots s_1$. Since the $u_{e_i}$ for $\gamma \in \Sigma_{\{e_1\}}$, are independent of the chosen ordering, we are free to choose a convenient ordering in order to estimate $\varphi_{e_i}(u_{e_i})$. We take the ordering given by the fixed reduced decomposition of $w_0$, so that $u_{e_i} = u_{e_i}^{(0)}$, and apply Algorithm 1.2. We need to study the support of $w^{(k)}$ that is, the set $\{ \gamma \in \Sigma \mid u^{(k)}_{\gamma} \neq 1 \}$. We define recursively $\Psi^{(0)} = \Sigma_{\{e_1\}}$ and then $\Psi^{(k)}$ as the closed subset of $\Sigma^{(k)}$ generated by $\Psi^{(k-1)} \setminus \{e_1 - e_{k+1}\}$ and $e_{k+1} - e_1$, for $1 \leq k \leq n - 1$. Concretely, we have for all $0 \leq k \leq n - 1$:

$\Psi^{(k)} = \{ e_1 \pm e_i \mid k + 2 \leq i \leq n \} \cup \{ e_i \pm e_j \mid 2 \leq i \leq k + 1 \}$

$\cup \{ e_i \pm e_j \mid 2 \leq i \leq k + 1, \ i \neq j \leq n \} \cup \{ e_i \mid 1 \leq i \leq k + 1 \}$ .

By construction, the support of $u^{(k)}_{e_i}$ is contained in $\Psi^{(k)}$.

Under the addition map $\Psi^{(k)} \times \Psi^{(k)} \to \mathbb{R}^n$ the preimage of $\{e_1, 2e_1\}$ is the set of pairs $(e_1 \pm e_i, e_1 \mp e_i)$, for $k + 2 \leq i \leq n$. The preimage of $e_1 \pm e_i$ is empty for $k + 2 \leq i \leq n$. Together with our assumption (21) we show that this implies the following claim:

**Claim 2.** For all $0 \leq k \leq n - 1$ one has:

1. $\varphi_{e_i}(u^{(k)}_{e_1}) = \varphi_{e_i}(u^{(k-1)}_{e_1})$;
2. $u^{(k)}_{e_1 \pm e_i} = u^{(k-1)}_{e_1 \pm e_i}$, for all $k + 2 \leq i \leq n$.

(We put $u^{(-1)}_{e_1} := u_{e_1}$, for $\gamma \in \Sigma$.)

**Proof.** We prove the claim by induction on $k$, the case $k = 0$ being trivial. Assume the claim holds for all $0 \leq j \leq k$, for some $0 \leq k \leq n - 2$. Recall that $\beta_{k+1} = e_1 - e_{k+2}$.

Assume we are in case [Alg-1] so that $\varphi_{e_1 - e_{k+2}}(u^{(k)}_{e_1 - e_{k+2}}) \geq - \langle \beta_{k+1}, \nu^{(k)} \rangle$. In this case, we clearly have $u^{(k+1)}_{e_1} = u^{(k)}_{e_1}$, for all $\gamma \in \Psi^{(k)} \setminus \{e_1 - e_{k+2}\}$, which proves the induction step in this case.

Suppose we are in case [Alg-3]. Then $u^{(k+1)}_{e_1} = u^{(k)}_{e_1}$, for all $k + 3 \leq i \leq n$, and $u^{(k+1)}_{e_1 \pm e_i} = u^{(k)}_{e_1 \pm e_i}$. This shows the induction step in this case.

Finally, assume we are in case [Alg-2] so that $\varphi_{e_1 - e_{k+2}}(u^{(k)}_{e_1 - e_{k+2}}) \geq 0$. We then have $u^{(k+1)}_{e_1 \pm e_i} = u^{(k)}_{e_1 \pm e_i}$, for all $k + 3 \leq i \leq n$. Moreover, we have

$$u^{(k+1)}_{e_1} = u^{(k)}_{e_1} \cdot [u^{(k)}_{e_1 - e_{k+2}}, (u^{(k)}_{e_1 + e_{k+2}})^{-1}] \in U_{e_1}.$$ 

The induction hypothesis implies $\varphi_{e_1}(u^{(k)}_{e_1}) = \varphi_{e_1}(u^{(k)}_{e_1})$ and $u^{(k)}_{e_1 \pm e_{k+2}} = u^{(k)}_{e_1 \pm e_{k+2}}$. Using [V4] and [V3] and (21), we compute

$$2\varphi_{e_1}([u^{(k)}_{e_1 - e_{k+2}}, (u^{(k)}_{e_1 + e_{k+2}})^{-1}]) = \varphi_{e_1}([u^{(k)}_{e_1 - e_{k+2}}, (u^{(k)}_{e_1 + e_{k+2}})^{-1}])$$

$$\geq \varphi_{e_1 - e_{k+2}}(u^{(k)}_{e_1 - e_{k+2}}) + \varphi_{e_1 + e_{k+2}}(u^{(k)}_{e_1 + e_{k+2}}) > 2\varphi_{e_1}(u^{(k)}_{e_1}).$$

Therefore, we conclude $\varphi_{e_1}(u^{(k+1)}_{e_1}) = \varphi_{e_1}(u^{(k)}_{e_1})$. This proves the induction step in this case and finishes the proof.

Now, Claim 2 and (18) show $\varphi_{e_1}(u^{(n-1)}_{e_1}) = \varphi_{e_1}(u^{(n-1)}_{e_1}) \geq \langle e_1, \nu(a) \rangle$. This shows (20) and finishes the proof.

5. Decomposition of Hecke polynomials

We fix a commutative ring $R$ with 1. In 5.2 and 5.6 we will assume that $p$ is invertible in $R$.  

2 A subset $X \subseteq \Sigma^{(k)}$ is called closed if $\gamma, \delta \in X$ with $\gamma + \delta \in \Sigma^{(k)}$ implies $\gamma + \delta \in X$. 

5.1. **Parabolic Hecke algebras.** Parabolic Hecke algebras for the general linear and the symplectic group were introduced and studied by Andrianov, see [And77], [And79], and the book [AZ95].

**Definition.** Let $P$ be a parabolic subgroup of $G$. Then

$$\mathcal{H}_R(K_P, P)$$

is called a **parabolic Hecke algebra**.

**Lemma 5.1.** Let $P$ and $Q$ be (not necessarily proper) parabolic subgroups of $G$ with $P \subseteq Q$. Then the map

$$\varepsilon_{P,Q} : \mathcal{H}_R(K_Q, Q) \hookrightarrow \mathcal{H}_R(K_P, P),$$

$$\sum_i r_i \cdot (K_Q g_i) \mapsto \sum_i r_i \cdot (K_P g_i),$$

where one may choose $g_i \in P$, is a well-defined injective $R$-algebra homomorphism. Moreover, the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{H}_R(K, G) & \xrightarrow{\varepsilon_{Q,G}} & \mathcal{H}_R(K_Q, Q)
\\
\varepsilon_{P,G} \downarrow & & \downarrow \varepsilon_{P,Q}
\\
\mathcal{H}_R(K_P, P) & & \mathcal{H}_R(K_P, P).
\end{array}
$$

**Proof.** Clearly, we have $K_P \subseteq K_Q$ and $K_Q \cap P = K_P$. The Iwasawa decomposition implies $Q = K_Q P$. Therefore, the conditions (9) for $(\Gamma, S) = (K_Q, Q)$ and $(\Gamma_0, S_0) = (K_P, P)$ are satisfied and the first statement follows from Proposition 2.5. The commutativity of the diagram is obvious. □

Let $P = U_PM$ be a parabolic subgroup of $G$. Let $pr_M : P \to M$ be the canonical projection. Note that $K_M = K \cap M$ is a special parahoric subgroup of $M$ (see §2.4). The map

$$\Theta^G_M : \mathcal{H}_R(K_P, P) \to \mathcal{H}_R(K_M, M),$$

$$\sum_i r_i \cdot (K_P g_i) \mapsto \sum_i r_i \cdot (K_M pr_M(g_i))$$

is a homomorphism of $R$-algebras.

**Definition.** The composition

$$S_M^G : \mathcal{H}_R(K, G) \xrightarrow{\varepsilon_{P,G}} \mathcal{H}_R(K_P, P) \xrightarrow{\Theta^G_M} \mathcal{H}_R(K_M, M)$$

is called the **(partial) Satake homomorphism**.

If $P = B$ and $M = Z$, then the subgroup $K_Z$ is normal in $Z$ and hence $\mathcal{H}_R(K_Z, Z)$ identifies with the commutative group algebra $R[K_Z \setminus Z] = R[\Lambda]$. In this case, the Satake homomorphism takes the form

$$S_Z^G : \mathcal{H}_R(K, G) \to R[\Lambda].$$

**Lemma 5.2.** Let $Q = U_QL$ and $P = U_PM$ be parabolic subgroups of $G$ and assume that $Q \subseteq P$. The diagram

$$
\begin{array}{ccc}
\mathcal{H}_R(K_P, P) & \xrightarrow{\Theta^G_M} & \mathcal{H}_R(K_M, M)
\\
\varepsilon_{Q,P} \downarrow & & \downarrow S_L^M
\\
\mathcal{H}_R(K_Q, Q) & \xrightarrow{\Theta^G_L} & \mathcal{H}_R(K_L, L)
\end{array}
$$

is commutative. In particular, one has $S_L^G = S_L^M \circ S_M^G$. 

Proof. Note that $Q \cap M$ is a parabolic subgroup of $M$ with Levi $L$. Given $b \in Q$, we have $pr_M(b) \in Q \cap M$ and $pr_L(pr_M(b)) = pr_L(b)$. Hence, for all $\sum_i r_i \cdot (K_p b_i) \in \mathcal{H}_R(K_p, P)$, where, by the Iwasawa decomposition, we may choose $b_i \in Q$, we compute

\[
S_L^M(\Theta_M^G(\sum_i r_i \cdot (K_p b_i))) = S_L^M(\sum_i r_i \cdot (K_M pr_M(b_i))) = \sum_i r_i \cdot (K_L pr_L(b_i) = \Theta_L^Q(\sum_i r_i \cdot (K_Q b_i))
\]

In particular, in view of Lemma 5.1, we have

\[
S_L^M \circ S_M^G = S_L^M \circ \Theta_M^G \circ \varepsilon_{P,G} = \Theta_L^Q \circ \varepsilon_{Q,P} \circ \varepsilon_{P,G} = \Theta_L^Q \circ \varepsilon_{Q,G} = S_L^G.
\]

5.2. The twisted action. Assume that $R$ is a $\mathbb{Z}/p\mathbb{Z}$-algebra. The twisted action of $W_0$ on $R[\Lambda]$ was defined by Henniart-Vignéras, [HV15 7.11, 7.12], in order to describe the image of the integral Satake homomorphism. We give a slightly different presentation.

Given $b \in B$, we consider the integers (see (6) in §2.5)

\[
\mu_U(b) := [K_U : K_U \cap b^{-1} K_U b].
\]

Observe that $\mu_U$ is constant on $K_Z$-cosets, since $K_Z$ normalizes $K_U$. Therefore, we obtain an induced map

\[
\mu_U : \Lambda \rightarrow q^{Z_{\geq 0}}.
\]

Note that $\mu_U(\lambda) = 1$ if and only if $\lambda \in \Lambda^+$. We employ the exponential notation $e^\lambda$ when we view $\lambda \in \Lambda$ as an element of $R[\Lambda]$.

**Definition.** The twisted action of $W_0$ on $R[\Lambda]$ is defined by

\[
 w \star e^\lambda := \frac{\mu_U(w(\lambda))}{\mu_U(\lambda)} \cdot e^{w(\lambda)}, \quad \text{for } \lambda \in \Lambda, w \in W_0.
\]

In order to describe the relation with the twisted action in [HV15 7.11], we recall the modulus character

\[
\delta : B \rightarrow q^Z, \quad \delta(b) := [bK_U b^{-1} : K_U] = \mu_U(b)/\mu_U(b^{-1}),
\]

where $[bK_U b^{-1} : K_U] := \frac{[bK_U b^{-1} bK_U b^{-1} \cap K_U]}{[K_U : bK_U b^{-1} \cap K_U]}$ denotes the generalized index. Similar to the above, $\delta$ induces a character

\[
\delta : \Lambda \rightarrow q^Z.
\]

**Lemma 5.3.** For all $w \in W_0$ and $\lambda \in \Lambda$, one has

\[
\frac{\delta(w(\lambda))}{\delta(\lambda)} = \left( \frac{\mu_U(w(\lambda))}{\mu_U(\lambda)} \right)^2 = \left( \frac{\mu_U(-\lambda)}{\mu_U(-w(\lambda))} \right)^2.
\]

**Proof.** Note that $\delta(w(\lambda))/\delta(\lambda) = \mu_U(w(\lambda))/\mu_U(\lambda)$, Therefore, for both equalities it suffices to show

\[
\mu_U(\lambda) \cdot \mu_U(-\lambda) = \mu_U(w(\lambda)) \cdot \mu_U(-w(\lambda)).
\]

But this follows from $\mu_U(\lambda)\mu_U(-\lambda) = q_\lambda$ (cf. [Hey21b Proposition 3.14.(a)]) and $q_\lambda = q_{\omega(\lambda)}$ (cf. [Vig16 Proposition 5.13]).
5.3. The Satake isomorphism. Given $\lambda \in \Lambda$, we denote $W_{0,\lambda}$ the stabilizer of $\lambda$ under the (usual) $W_0$-action on $\Lambda$. Then $W_{0,\lambda}$ is also the stabilizer of $e^\lambda$ under the twisted action of $W_0$ on $R[\Lambda]$.

Note that, if $R = \mathbb{Z}[1/p]$ and $\lambda \in \Lambda^+$, one has

$$S_\lambda := \sum_{w \in W_0/W_{0,\lambda}} w \cdot e^\lambda \in \mathbb{Z}[\Lambda].$$

With our notations, the main result of [HV15] is the following:

**Theorem 5.4.** Let $R$ be a commutative ring with 1 and consider the Satake homomorphism $S^G : \mathcal{H}_R(K, G) \to R[\Lambda]$.

(i) $S^G$ is injective.

(ii) The image of $S^G$ is a free $R$-module with basis $\{1 \otimes S_\lambda \mid \lambda \in \Lambda^+\}$. If $p \in R^\times$, then the image coincides with $R[\Lambda]^{W_0,*}$, the algebra of $W_0$-invariants under the twisted action.

(iii) Both $R[\Lambda]$ and $\mathcal{H}_R(K, G)$ are commutative algebras of finite type over $R$.

**Proof.** We briefly explain how our notations relate to the notations in [HV15]. Consider the space $C_c^\infty(K \backslash G/K, R)$ of compactly supported $K$-biinvariant functions $G \to R$ with product given by convolution:

$$(f_1 * f_2)(g) = \sum_{h \in G/K} f_1(h) \cdot f_2(h^{-1}g), \quad \text{for } f_1, f_2 \in C_c^\infty(K \backslash G/K, R) \text{ and } g \in G,$$

where “$h \in G/K$” means that $h$ runs through a set of representatives for the left cosets in $G/K$. The map

$$\rho_G : C_c^\infty(K \backslash G/K, R) \longrightarrow \mathcal{H}_R(K, G), \quad f \longmapsto \sum_{g \in K \backslash G} f(g^{-1}) \cdot (Kg)$$

is an anti-isomorphism of $R$-algebras. Following [Her11], the Satake homomorphism in [HV15] is defined as

$$S' : C_c^\infty(K \backslash G/K, R) \longrightarrow C_c^\infty(Z \backslash KZ, R) \cong R[\Lambda],$$

where $K_U = K \cap U$. Now, the diagram

\[
\begin{array}{ccc}
C_c^\infty(K \backslash G/K, R) & \xrightarrow{S'} & C_c^\infty(Z \backslash KZ, R) \\
\downarrow \rho_G & & \downarrow \rho_Z \\
\mathcal{H}_R(K, G) & \xrightarrow{S^G} & \mathcal{H}_R(K \backslash Z)
\end{array}
\]

commutes: fix a representing system $\Gamma \subseteq Z$ for the coset space $K \backslash Z$, so that $K \backslash U \times \Gamma \cong K \backslash G$ via $(K_Uu, z) \mapsto Kuz$. Then for each $f \in C_c^\infty(K \backslash G/K, R)$ we compute

$$\rho_Z(S'(f)) = \sum_{z \in \Gamma} S'(f)(z^{-1}) \cdot (K_Zz) = \sum_{z \in \Gamma} \sum_{u \in U/K_U} f(z^{-1}u) \cdot (K_Zz)$$

$$= \sum_{z \in \Gamma} \sum_{u \in K_U \setminus U} f((uz)^{-1}) \cdot (K_Zz) = \sum_{uz \in K \backslash G} f((uz)^{-1}) \cdot (K_Zz)$$

$$= S^G \left( \sum_{uz \in K \backslash G} f((uz)^{-1}) \cdot (Kuz) \right) = S^G(\rho_G(f)).$$

---

3As $C_c^\infty(K \backslash G/K, R)$ turns out to be commutative, $\rho_G$ is in fact a homomorphism.
If \( p \in R^x \), then the twisted action of \( W_0 \) on \( C_c^\infty(Z/K_Z, R) \) is defined by
\[
w \circ e^\lambda := \delta^{1/2}(\lambda - w(\lambda)) \cdot e^{w(\lambda)},
\]
where \( \delta^{1/2} \) is a square root of \( \delta \). This is indeed defined over \( R \), since Lemma 5.3 shows that \( \delta^{1/2}(\lambda - w(\lambda)) \) actually lies in \( q^R \). The same lemma also shows that \( \rho_Z(w \circ e^\lambda) = w \ast e^{-\lambda} \).

Therefore, we have \( \rho_Z(\sum_{w \in W_0} w \circ e^\lambda) = S_\lambda \) in \( R[\Lambda] \), for all \( \lambda \in \Lambda^- \).

Now, (i) and (ii) are [HV15, 7.15 Thm. and 7.13 Cor.], and (iii) is [HV15, 7.16]. \( \square \)

**Remark 5.5.** Let \( z \in Z^- \) and put \( \lambda = zK_Z \in \Lambda \). It follows from Remark 2.4(a) and (b) that
\[
S^G((z)_K) = e^\lambda + \sum_{\mu \in \Lambda^+, \nu(\mu) < \nu(\lambda)} a_\mu \cdot e^\mu \in Z[1/p][\Lambda].
\]

By Theorem 5.4, \( S^G((z)_K) \) is invariant under the twisted action of \( W_0 \). Therefore, \( \frac{a_\mu}{\nu(\mu)} = \frac{a_{\nu(\mu)}}{\nu(\mu)w(\mu)} \), for all \( \mu \in \Lambda \) and \( \nu(\mu) < \nu(\lambda) \). In particular, \( a_\mu \neq 0 \) if and only if \( w(\mu) \neq 0 \) for all \( \mu \in \Lambda \). Now, (22) shows \( w(\mu) \leq \lambda \), for all \( \mu \in \Lambda \) with \( \mu \leq \lambda \). This explains Remark 2.4(c).

### 5.4. Centralizers in parabolic Hecke algebras

Let \( P = U_PM \) be a parabolic subgroup of \( G \). We choose a strictly \( M \)-positive element \( a_P \in Z \), see (23). This means that \( a_P \) lies in the center of \( M \) and satisfies
\[
(\alpha, \nu(a_P)) < 0, \quad \text{for all} \quad \alpha \in \Sigma^+ \setminus \Sigma_M.
\]

Note that \( K_P a_P K_P = K_P a_P \) and hence \( (a_P)_{K_P} = (K_P a_P) \) in \( \mathcal{H}_R(K_P, P) \). We consider the centralizer algebra
\[
C^+_P := \{ X \in \mathcal{H}_R(K_P, P) \mid X \cdot (a_P)_{K_P} = (a_P)_{K_P} \cdot X \}.
\]

The algebra \( C^+_P \) was originally studied by Andrianov when \( P \) is the “Siegel parabolic” of a symplectic group, see [And77, And79].

**Lemma 5.6.** The following statements hold true:

(i) \( C^+_P = \{ X \in \mathcal{H}_R(K_P, P) \mid \lambda \cdot X = \sum r_i \cdot (K_P m_i) \text{ with } m_i \in M \text{ and } r_i \in R \} \).

(ii) For all \( X \in \mathcal{H}_R(K_P, P) \), there exists \( n > 0 \) such that \( (a_P)^n \cdot X \in C^+_P \).

(iii) The map \( \Theta^P_M \) induces by restriction an isomorphism \( C^+_P \cong \mathcal{H}_R(K_M, M^+) \). In particular, \( C^+_P \) is commutative.

**Proof.** See [Hey21a, Lemma 4 and Corollary 5]. The last assertion in (iii) follows from the fact that \( \mathcal{H}_R(K_M, M) \) is commutative by Theorem 5.4 applied to \( M \). \( \square \)

Note that Lemma 5.2(iii) shows that \( C^+_P \) is independent of the choice of \( a_P \).

Recall the anti-involution \( \zeta_P \) on \( \mathcal{H}_R(K_P, P) \), cf. [10], which is given by \( \zeta_P((g)_{K_P}) = (g^{-1})_{K_P} \).

**Remark.** Let \( a_P \in Z \) be a strictly \( M \)-positive element. Then \( a_P^{-1} \) is strictly \( M \)-negative and
\[
\zeta_P(C_P^{-1}) = C_P := \{ X \in \mathcal{H}_R(K_P, P) \mid X \cdot (a_P)^{-1} \cdot K_P = (a_P^{-1}) \cdot K_P \cdot X \}.
\]

The analog of Lemma 5.6 for \( C_P^{-1} \) also holds.

**Lemma 5.7.** One has
\[
\zeta_P(\ker \Theta^P_M) = \ker \Theta^P_M.
\]

**Proof.** Note that, for each \( g \in P \), the index \( \mu(g) := [K_P : K_P \cap g^{-1} K_P g] \) counts the number of right \( K_P \)-cosets in \( K_P g K_P \), by [5]. Similarly, \( \mu_M(m) := [K_M : K_M \cap m^{-1} K_M m] \) counts the number of right \( K_M \)-cosets in \( K_M m K_M \). Moreover,
\[
\delta : P \to q^R, \quad \delta(g) := [gK_P g^{-1} : K_P] = \frac{\mu(g)}{\mu(g^{-1})}.
\]

\[\text{However, note that it is not } \Theta^P_M \text{ which induces an isomorphism } C^+_P \cong \mathcal{H}_R(K_M, M^+), \text{ but rather } \zeta_M \circ \Theta^P_M \circ \zeta_P.\]
is the modulus character of $P$. As every element in $U_P$ is contained in a compact group, we have $\delta|_{U_P} = 1$, by [Vig96, Ch. I, 2.7]. Therefore,

\begin{equation}
\delta(g) = \delta(\text{pr}_M(g)), \quad \text{for all } g \in P.
\end{equation}

Note that $\Theta_P^M((g)_{K_P}) = \frac{\mu(g)}{\mu(\text{pr}_M(g))} \cdot (\text{pr}_M(g))_{K_M}$ by [Hey21b, Proposition 4.3], and hence $\text{Ker } \Theta_P^M$ is generated by elements of the form $(g)_{K_P} - \frac{\mu(g)}{\mu(\text{pr}_M(g))} \cdot (\text{pr}_M(g))_{K_P}$. We compute

\begin{equation*}
\Theta_P^M(\zeta_P((g)_{K_P}) - \frac{\mu(g)}{\mu(\text{pr}_M(g))} \cdot (\text{pr}_M(g))_{K_P}) = \frac{\mu(g)}{\mu(\text{pr}_M(g))} \cdot (\text{pr}_M(g^{-1}))_{K_M} - \frac{\mu(g)}{\mu(\text{pr}_M(g))} \cdot (\text{pr}_M(g^{-1}))_{K_M} = 0,
\end{equation*}

This shows $\zeta_P(\text{Ker } \Theta_P^M) \subseteq \text{Ker } \Theta_P^M$. The assertion follows from $\zeta_P^2 = \text{id}$. \hfill $\square$

Remark. One has $\Theta_P^M \circ \zeta_P \neq \zeta_P \circ \Theta_P^M$ (apply both maps to $(a_P)_{K_P}$ for a strictly $M$-positive $a_P$).

5.5. Example of a parabolic Hecke algebra. The purpose of this section is to work out an example of the setup so far.

Let $O_{\mathfrak{B}}$ be the valuation ring of $\mathfrak{B}$ and fix a uniformizer $\pi \in O_{\mathfrak{B}}$. Consider the group $G = \text{GL}_2(\mathfrak{B})$ and the maximal compact subgroup $K = \text{GL}_2(O_{\mathfrak{B}})$. Let $B \subseteq G$ be the subgroup of upper triangular matrices and $Z \subseteq B$ the subgroup of diagonal matrices. Fix a coefficient ring $R$. A variant of the parabolic Hecke algebra $\mathcal{H}_R(K_B, B)$ is briefly discussed in Vienney’s thesis [Vie12, p. 102]. We prove the following structure result of $\mathcal{H}_R(K_B, B)$ in Appendix A.

**Theorem 5.8.** The $R$-algebra $\mathcal{H}_R(K_B, B)$ is generated by the elements

\begin{align*}
X_+ := \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}_{K_B}, \quad X_- := \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix}_{K_B}, \quad Y := (\pi E_2)_{K_B}, \quad Y^{-1} := (\pi^{-1} E_2)_{K_B},
\end{align*}

where $E_2$ is the $2 \times 2$ identity matrix, subject only to the following relations:

\begin{align*}
YY^{-1} &= Y^{-1}Y = 1, \\
YX_+ &= X_+Y, \\
YX_- &= X_-Y, \\
X_+X_- &= q \cdot 1.
\end{align*}

In particular, $\mathcal{H}_R(K_B, B)$ is non-commutative.

Remark. It follows from Theorem 5.8 that $X_+$ is a left zero-divisor (resp. $X_-$ is a right zero-divisor), because

\[ X_+ \cdot (X_-X_+ - q \cdot 1) = (X_-X_+ - q \cdot 1) \cdot X_- = 0. \]

If $q$ is invertible in $R$, then $X_+$ is right invertible (resp. $X_-$ is left invertible), since

\[ X_+ \cdot q^{-1}X_- = q^{-1}X_+ \cdot X_- = 1. \]

Since $\Lambda \cong \mathbb{Z}^2$, the Hecke algebra $\mathcal{H}_R(K_Z, Z)$ identifies with $R[x^\pm 1, y^\pm 1]$ via $((\pi 0))_{K_Z} \mapsto x$ and $((0 \pi))_{K_Z} \mapsto y$. Then the map $\Theta_P^B$ is given by

\begin{align*}
\Theta_P^B : \mathcal{H}_R(K_B, B) &\rightarrow \mathcal{H}_R(K_Z, Z), \\
X_+ &\mapsto x, \\
X_- &\mapsto qx^{-1}, \\
Y &\mapsto xy.
\end{align*}

The kernel of $\Theta_P^B$ is the two-sided ideal generated by $X_-X_+ - q \cdot 1$.

Assume $q \in R^\times$. The twisted action of $W_0 = \{1, w_0 = (\tfrac{0}{\pi})_{K_Z} \}$ on $R[x^\pm 1, y^\pm 1]$ is given by

\[ w_0 \ast x = qy \quad \text{and} \quad w_0 \ast y = q^{-1}x. \]
For all \( a, b, c \in \mathbb{Z} \) with \( a > b \), the elements

\[
S_{a,b} := x^a y^b + q^{a-b} x^b y^a, \quad S_{c,c} := (xy)^c
\]

are \( W_0 \)-invariant with respect to the twisted action; in fact they constitute an \( R \)-basis of \( R[\Lambda]^{W_0} \).

Moreover, the relations

\[
S_{c,c} = S_{1,1}^c, \quad S_{a,b} = S_{b,b} \cdot S_{a-b,0} = S_{a-b,0} \cdot S_{b,b}, \quad S_{n,0} \cdot S_{1,0} = S_{n+1,0} + q \cdot S_{1,1} \cdot S_{n-1,0},
\]

are immediate. This easily implies \( R[\Lambda]^{W_0} = R[S_{1,0}, S_{1,1}^\pm] = R[x + qy, (xy)^\pm] \).

We identify \( \Lambda^+ \) with \( \{(a, b) \in \mathbb{Z}^2 \mid a \geq b \} \). By the Cartan decomposition \[2.2\] every double coset in \( G \) is uniquely of the form \( K \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix} K \) with \( (a, b) \in \Lambda^+ \). Let us compute the Satake homomorphism \( S^G : \mathcal{H}_R(K, G) \to R[\Lambda] \). Let \( A \subseteq \mathcal{O}_F \) be a complete system of representatives for the residue class field of \( F \) such that \( 0 \in A \). Note that \( |A| = q \). For each \( (a, b) \in \Lambda^+ \) we have

\[
K \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix} K = K \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix} \sqcup \bigcup_{c=1}^{a-b-1} \bigcup_{\beta_b \neq 0} K \begin{pmatrix} \pi^a & \pi^{b+c} \beta_i \pi^i \\ 0 & \pi^b \end{pmatrix} \bigcup \bigcup_{\beta_b \neq 0} K \begin{pmatrix} \pi^b & \sum_{i=1}^{a-c} \beta_i \pi^i \\ 0 & \pi^a \end{pmatrix}.
\]

Let \( \gamma := \lfloor \frac{a-b}{2} \rfloor \) be the largest integer \( \leq \frac{a-b}{2} \). We deduce

\[
S^G \left( \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix}_K \right) = x^a y^b + \sum_{c=1}^{a-b-1} (q-1) q^{c-1} x^{a-c} y^{b+c} + q^{a-b} x^b y^a,
\]

\[
\begin{align*}
&= S_{a,b} + (q-1) \sum_{c=1}^{\gamma} q^{c-1} S_{a-c,b+c} + \epsilon \cdot (q-1) q^{\gamma-1} S_{\gamma,\gamma}, \\
&= S_{a,b} + (q-1) \sum_{c=1}^{\gamma} q^{c-1} S_{a-c,b+c} + \epsilon \cdot (q-1) q^{\gamma-1} S_{\gamma,\gamma},
\end{align*}
\]

and where \( \epsilon = 1 \) if \( a-b \) is even and non-zero, and \( \epsilon = 0 \) otherwise. Consider on \( \Lambda \) the partial ordering defined by \( (c, d) \leq (a, b) \) if \( c \leq a \) and \( c + d = a + b \). As usual we write \( (c, d) < (a, b) \) if \( (c, d) \leq (a, b) \) and \( (c, d) \neq (a, b) \). Note that for each \( (a, b) \in \Lambda^+ \) there are only finitely many elements \( (c, d) \in \Lambda^+ \) satisfying \( (c, d) < (a, b) \). Then we have shown

\[
S^G \left( \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix}_K \right) \in S_{a,b} + \sum_{(c,d) \in \Lambda^+, (c,d) < (a,b)} \mathbb{Z} S_{c,d}.
\]

By a “triangular argument” it follows that \( S^G \) is injective with image \( R[\Lambda]^{W_0} \). In particular, \( \mathcal{H}_R(K, G) \) is commutative. Moreover,

\[
S^G \left( \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}_K \right) = S_{1,0} = x + qy, \\
S^G \left( \begin{pmatrix} \pi E_2 & 0 \\ 0 & 1 \end{pmatrix}_K \right) = S_{1,1} = xy,
\]

which shows that \( \mathcal{H}_R(K, G) \) identifies with the polynomial ring generated by \( \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \) and \( (\pi E_2)_K \), with \( (\pi E_2)_K \) invertible. We have verified Theorem \[5.4\] in this specific example.
We can also view $\mathcal{H}_R(K, G)$ as a subalgebra of $\mathcal{H}_R(K_B, B)$ via the embedding

$$
\varepsilon_{B,G}: \mathcal{H}_R(K, G) \rightarrow \mathcal{H}_R(K_B, B),
$$

$$(\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix})_K \mapsto X_+ + X_Y,
(\pi E_2)_K \mapsto Y.
$$

Note that $C_B^+$ is the centralizer of $X_+$. Explicitly, $C_B^+$ is the polynomial algebra

$$
C_B^+ = \mathcal{H}[X_+, Y^{\pm 1}] \subseteq \mathcal{H}_R(K_B, B).
$$

The anti-involution $\zeta_B$ on $\mathcal{H}_R(K_B, B)$ is determined by $\zeta_B(X_+) = X_-$ and $\zeta_B(Y) = Y^{-1}.$

Consider the polynomial

$$
Q(t) = 1 - (\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix})_K \cdot t + q \cdot (\pi \cdot E_2)_K \cdot t^2 \in \mathcal{H}_R(K, G)[t].
$$

Applying $S^G$ to the coefficients of $Q$, the resulting polynomial $Q^{S^G}(t)$ decomposes as follows:

$$
Q^{S^G}(t) = 1 - (x + qy) \cdot t + qxy \cdot t^2 = (1 - xt) \cdot (1 - qyt) \in \mathcal{H}[x^{\pm 1}, y^{\pm 1}][t].
$$

One may ask whether this decomposition can be lifted to a decomposition of $Q(t)$ in $\mathcal{H}_R(K, G)[t]$. Unfortunately, this is false. But it turns out that one can find a decomposition of $Q(t)$ over the parabolic Hecke algebra $\mathcal{H}_R(K_B, B)$: applying $\varepsilon_{B,G}$ to the coefficients of $Q(t)$, we obtain

$$
Q^{\varepsilon_{B,G}}(t) = 1 - (X_+ + X_Y) \cdot t + qYt^2 = (1 - X_+) \cdot (1 - X_Yt) \in \mathcal{H}_R(K_B, B)[t].
$$

Here, the free variable $t$ in $\mathcal{H}_R(K_B, B)[t]$ commutes with the elements in $\mathcal{H}_R(K_B, B)$. Note that the order of the factors is important, because $\mathcal{H}_R(K_B, B)$ is non-commutative.

In the next subsection we prove a general decomposition theorem following the ideas of Andrianov [And77] [AZ95].

5.6. The decomposition theorem. From now on we assume that $R$ is a $\mathbb{Z} [1/p]$-algebra. Let $P = U_P M$ be a parabolic subgroup of $G$. We view the embeddings $\varepsilon_{P,G}: \mathcal{H}_R(K, G) \rightarrow \mathcal{H}_R(K_P, P)$ and $\varepsilon_{B,P}: \mathcal{H}_R(K_P, P) \rightarrow \mathcal{H}_R(K_B, B)$ as inclusions.

Let $a_P \in Z$ be a strictly $M$-positive element and denote its image in $\Lambda$ by $\lambda_P$. Choose a representing system $W_0^M$ of $W_0/W_0M$ in $W_0$. Consider the polynomial

$$
\chi_{a_P}(t) := \prod_{w \in W_0^M} (1 - w \ast e^{-\lambda_P} \cdot t) \in 1 + t\mathcal{H}[\Lambda][t].
$$

This definition does not depend on the choice of $W_0^M$, because $W_0M$ fixes $e^{-\lambda_P} \in R[\Lambda]$ with respect to the twisted action. Note that, by construction, we have $\chi_{a_P}(e^{\lambda_P}) = 0$, and the coefficients of $\chi_{a_P}(t)$ are $W_0$-invariant for the twisted action. By Theorem 5.3(ii), the coefficients of $\chi_{a_P}(t)$ lie in the image of the Satake map $S^G$. Since, by Theorem $\text{red}(i)$, $S^G$ is injective, there exists a unique polynomial

$$
\chi_{a_P}(t) = \sum_{i=0}^{\lfloor W_0^M \rfloor} X_i \cdot t^i \in 1 + t\mathcal{H}(K, G)[t]
$$

with $\sum_i S^G(X_i) \cdot t^i = \chi_{a_P}(t).$ (Note that $X_0 = 1$.) Explicitly, we have, for all $0 \leq i \leq \lfloor W_0^M \rfloor$,

$$
S^G(X_i) = (-1)^i \mu_U (-\lambda_P)^{-1} \cdot \sum_{J \subseteq W_0^M} \mu_U (w(-\lambda_P)) \cdot e^{\sum_{w \in J} w(-\lambda_P)} \in R[\Lambda].
$$

(25)

**Lemma 5.9.** One has

$$
\chi_{a_P}((a_P)_K^P) := \sum_{i=0}^{\lfloor W_0^M \rfloor} (a_P)_K^P \cdot X_i \in \text{Ker} \Theta_M^P.
$$
Assume that $\zeta$ with Lemma 5.6.(i) implies reductive groups and non-obtuse $= \text{Sp}$ $H$ of quadrant is zero. Gritsenko then adopted the methods of Andrianov to prove it for $\mathcal{H}_R(K, G)$ coincides with $\mathcal{S}^G$. We compute 
\[
(S^M \circ \Theta^P_M)(\chi_{a_P}((a_P)_{K_P})) = \sum_i e^{i\chi_{a_P}} \cdot S^G(X_i) = \chi_{a_P}(e^{i\chi_{a_P}}) = 0.
\]
Since by Theorem 5.3(0) the map $S^M$ is injective, the assertion follows.

Proof. By Lemma 5.2 we have $S^M(\Theta^P_M((a_P)_{K_P})) = \Theta^P_M((a_P)_{K_P}) = e^{i\lambda_p}$, and the restriction of $\mathcal{S}^M \circ \Theta^P_M$ to $\mathcal{H}_R(K, G)$ coincides with $\mathcal{S}^G$. We compute 
\[
(S^M \circ \Theta^P_M)(\chi_{a_P}((a_P)_{K_P})) = \sum_i e^{i\chi_{a_P}} \cdot S^G(X_i) = \chi_{a_P}(e^{i\chi_{a_P}}) = 0.
\]
In order for the theory to work, one needs to assume the following strengthening of Lemma 5.9

Hypothesis 5.10. The element $(a_P)_{K_P}$ is a left root of $\chi_{a_P}(t)$, meaning that
\[
\chi_{a_P}((a_P)_{K_P}) = 0 \quad \text{in } \mathcal{H}_R(K_P, P).
\]
This hypothesis has been verified in many cases: Andrianov [And77] essentially proved it for $G = \text{Sp}_{2n}(\mathfrak{F})$ with $P$ being the “Siegel parabolic”, i.e., the subgroup of matrices whose lower left quadrant is zero. Gritsenko then adopted the methods of Andrianov to prove it for $G = \text{GL}_n(\mathfrak{F})$ and all parabolics [Gri88, Gri92]. Finally, Gritsenko verified Hypothesis 5.10 for the classical groups $\text{Sp}_{2n}(\mathfrak{F})$, $\text{SU}_n(\mathfrak{F})$, and $\text{SO}_n(\mathfrak{F})$, for the parabolics fixing a line in the standard representation, see [Gri90].

The main contribution of this article is the verification of Hypothesis 5.10 for general connected reductive groups and non-obtuse parabolics, cf. [3]. This covers, in particular, all the cases mentioned above.

Theorem 5.11. Assume that $P$ is non-obtuse. Then Hypothesis 5.10 holds true.

Proof. Lemma 5.6(iii) shows $C^+_P \cap \ker \Theta^P_M = \{0\}$. Hence, in view of Lemma 5.9 it suffices to prove $(a_P)_{K_P} X_i \in C^+_P$, for all $0 \leq i \leq |W^+_M|$

Let $i$ be arbitrary but fixed. Recall the explicit description [29, 30]. Note that $\sum_{w_j \in J} \nu(w(-\lambda_P)) \leq \nu(-i\lambda_P)$, where $J \subseteq W^+_M$ is such that $|J| = i$, see [Bou81] Ch. VI, §1, no. 6, Prop. 18. If we write
\[
X_i = \sum_{j=1}^n c_j \cdot (z_j)_K \quad \text{in } \mathcal{H}_R(K, G),
\]
then this and [22] imply $\nu(z_j) \leq \nu(a_{i}^{-1})$, for all $1 \leq j \leq n$. In view of the Cartan decomposition [2, 2] we may choose $z_j \in Z^-$.

By the Iwasawa decomposition [2, 3] for $G$ and the Cartan decomposition [2, 2] for $M$, we have
\[
G = KP = KU_P M = KU_P K_M Z^+, M \subseteq KU_P Z^+, M \subseteq K_M,
\]
because $K_M$ normalizes $U_P$. Therefore, we can write, for every $j$,
\[
(z_j)_K = \sum_{l=1}^{n_j} c_{jl} \cdot (K_{u_{jl}} z'_{jl} k_{jl}),
\]
with $u_{jl} \in U_P$, $z'_{jl} \in Z^+$, and $k_{jl} \in K_M$. Note that $u_{jl} z'_{jl} \in K z_j K$, for all $j, l$.

Since $P$ is non-obtuse, Theorem 3.3 implies $a_{i_P} u_{jl} a_{i_P}^{-1} \in K_P$, for all $j, l$. Now, since $a_{i_P} z'_{jl} k_{jl} \in M$, Lemma 5.6(i) implies
\[
(a_P)_{K_P} \cdot X_i = \sum_{j=1}^n \sum_{l=1}^{n_j} c_{jl} (K_P a_{i_P} z'_{jl} k_{jl}) \in C^+_P.
\]
Consider the submodules
\[
\Theta^+_P := C^+_P \cdot \mathcal{H}_R(K, G) := \left\{ \sum_i Y_i Z_i \in \mathcal{H}_R(K_P, P) \mid Y_i \in C^+_P, Z_i \in \mathcal{H}_R(K, G) \right\},
\]
and
\[
\Theta^-_P := \zeta_P(\Theta^+_P) = \mathcal{H}_R(K, G) C^+_P
\]
of $\mathcal{H}_R(K_P, P)$. (Note that $\zeta_P$ preserves $\mathcal{H}_R(K, G)$ by Lemma 2.7.)
**Definition.** Assume that Hypothesis 5.10 is satisfied. For every \( n \in \mathbb{Z}_{\geq 1} \) we define recursively the “negative powers” of \((a_P)_{K_P}\) as

\[
(a_P)^{-n}_{K_P} := - \sum_{i=1}^{\log a_P} (a_P)^{-n}_{K_P} \cdot X_i \in \mathcal{O}_P^+.
\]

It should be noted that \((a_P)_{K_P}\) is not invertible in \( \mathcal{H}_R(K_P, P) \). In fact, for \( n > 1 \), one even has \(((a_P)^{-1}_{K_P})^{-n} \neq (a_P)^{-n}_{K_P} \). However, by a simple induction on \( d \) we have

\[
(a_P)^d_{K_P} \cdot (a_P)^{n+d}_{K_P} = (a_P)^{n+d}_{K_P}, \quad \text{for all } n \in \mathbb{Z}_{\geq 0} \text{ and } d \in \mathbb{Z}.
\]

**Example 5.12.** Let us compute “negative powers” for \( G = \text{GL}_2(\mathfrak{f}) \). The notations are the same as in §5.5. Choose the strictly positive element \( a_B = (\frac{5}{3}, \frac{1}{3}) \), so that \((a_B)_{K_B} = X_+ \in \mathcal{H}_R(K_B, B) \). The polynomial

\[
\tilde{\chi}_{a_B}(t) = (1 - x^{-1}t) \cdot (1 - (qy)^{-1}t) = 1 - (x^{-1} + (qy)^{-1}) \cdot t + (qxy)^{-1} \cdot t^2 \in R[x^{\pm 1}, y^{\pm 1}][t]
\]

annihilates \( \Theta^B_Z((a_B)_{K_B}) = x \). We compute

\[
S^G(q^{-1}(\pi E_2)_{K_P}^{-1} \cdot \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = q^{-1} \cdot (xy)^{-1} \cdot (x + qy) = x^{-1} + (qy)^{-1}.
\]

and

\[
\varepsilon_{B,G}(q^{-1} \cdot (\pi E_2)_{K_P}^{-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = q^{-1} \cdot (X_+ + X_-Y) = q^{-1} \cdot (X_+Y^{-1} + X_-),
\]

\[
\varepsilon_{B,G}(q^{-1} \cdot (\pi E_2)_{K_P}^{-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = q^{-1}Y^{-1}.
\]

Therefore,

\[
\chi_{a_B}(t) = 1 - q^{-1} \cdot (X_+Y^{-1} + X_-) \cdot t + q^{-1}Y^{-1} \cdot t^2.
\]

It is clear that \( X_+ \) is a left root of \( \chi_{a_B}(t) \). We compute

\[
X_+^{-1} = q^{-1} \cdot (X_+Y^{-1} + X_-) - q^{-1}X_+Y^{-1} = q^{-1}X_-,
\]

\[
X_+^{-2} = q^{-1}X_+^{-1} \cdot (X_+Y^{-1} + X_-) - q^{-1}X_+Y^{-1} = q^{-1}X_-^2 + q^{-2}Y^{-1}(X_-X_+ - q \cdot 1)
\]

\[
X_+^{-3} = q^{-1}X_+^{-2} \cdot (X_+Y^{-1} + X_-) - q^{-1}X_+Y^{-1} = q^{-1}X_-^3 + q^{-3} \cdot (X_-X_+ - q \cdot 1)X_+Y^{-2} + q^{-3}X_-(X_-X_+ - q \cdot 1)Y^{-1}.
\]

Recall from Lemma 5.10(ii) that for every \( X \in \mathcal{O}_P^+ \) there exists \( n > 0 \) such that \((a_P)^n_{K_P} X \in C_P^+\). Using “negative powers” it is possible to reconstruct \( X \) from \((a_P)^n_{K_P} X\).

**Lemma 5.13.** Assume that Hypothesis 5.10 is satisfied (for \( P \)).

(i) For every \( X \in \mathcal{H}_R(K, \mathcal{O}) \) and every \( n \geq 0 \) with \((a_P)^n_{K_P} X \in C_P^+ \) we have

\[
(a_P)^n_{K_P} \cdot X \cdot (a_P)^d_{K_P} = (a_P)^{n+d}_{K_P} \cdot X, \quad \text{for all } d \in \mathbb{Z}.
\]

(ii) Let \( \mathcal{Q} \) be a parabolic contained in \( \mathcal{P} \). For every \( X \in \mathcal{O}_Q^+ \) we have, inside \( \mathcal{H}_R(K_Q, \mathcal{Q}) \),

\[
(a_P)^n_{K_P} \cdot X \cdot (a_P)^n_{K_P} = X, \quad \text{for all } n > 0.
\]

**Proof.** (i) Let \( X \in \mathcal{H}_R(K, \mathcal{O}) \) and let \( n \geq 0 \) such that \((a_P)^n_{K_P} X \in C_P^+\). We prove the assertion by descending induction on \( d \). If \( d \geq 0 \), this follows from the assumption that \((a_P)^n_{K_P} X\)
centralizes \((a_P)_{K_P}\). Now assume \(d < 0\). Since \(\mathcal{H}_R(K, G)\) is commutative by Theorem 5.4, we compute

\[
(a_P)_{K_P}^n \cdot X \cdot (a_P)_{K_P}^d = -\sum_{i=1}^{\mid W_O^M \mid} (a_P)_{K_P}^n \cdot X \cdot (a_P)_{K_P}^{i+d} \cdot X_i
\]

\[
= -\sum_{i=1}^{\mid W_O^M \mid} (a_P)_{K_P}^{n+i+d} \cdot XX_i
\]

(\text{induction hypothesis})

\[
= \left( -\sum_{i=1}^{\mid W_O^M \mid} (a_P)_{K_P}^{n+i+d}X_i \right) \cdot X
\]

(\(\mathcal{H}_R(K, G)\) is commutative)

\[
= (a_P)_{K_P}^{n+d} \cdot X.
\]

This finishes the induction step.

(ii) Write \(X = \sum_j Y_j Z_j\) with \(Y_j \in C_Q^+\) and \(Z_j \in \mathcal{H}_R(K, G)\). Choose \(n \in \mathbb{Z}_{\geq 0}\) such that \((a_P)_{K_P}^n Z_j \in C_Q^+\) for all \(j\). From Lemma 5.6.(i) applied to \(Q\), it follows easily that \((a_P)_{K_P}^n\) centralizes \(C_Q^+\). Using (i), we compute

\[
(a_P)_{K_P}^n \cdot X \cdot (a_P)_{K_P}^{-n} = \sum_j Y_j \cdot (a_P)_{K_P}^n Z_j (a_P)_{K_P}^{-n} = \sum_j Y_j Z_j = X.
\]

\(\square\)

We are now able to prove the following characterization of Hypothesis 5.10.

**Proposition 5.14.** The following assertions are equivalent:

(i) Hypothesis 5.10 is satisfied.

(ii) If \(X \in \mathcal{O}_P^+\) and \(n \geq 0\) are such that \((a_P)_{K_P}^n X = 0\), then \(X = 0\).

(iii) \(\mathcal{O}_P^+ \cap \text{Ker } \Theta_{M_P}^\prime = \{0\}\).

(iv) If \(X \in \mathcal{O}_P^+\) and \(n \geq 0\) are such that \(X \cdot (a_P^{-1})_{K_P}^n = 0\), then \(X = 0\).

(v) \(\mathcal{O}_P^- \cap \text{Ker } \Theta_{M_P}^\prime = \{0\}\).

**Proof.** The equivalences \([\text{ii}] \iff [\text{iv}]\) and \([\text{iii}] \iff [\text{v}]\) follow from Lemma 5.7 and \(\zeta_P(\mathcal{O}_P^+) = \mathcal{O}_P^-\).

Since \(\chi_{a_P}((a_P)_{K_P}) \in \mathcal{O}_P^+ \cap \text{Ker } \Theta_{M_P}^\prime\) by Lemma 5.9, it follows that \([\text{iii}]\) implies \([\text{i}]\).

Assume that \([\text{ii}]\) holds. Let \(X \in \mathcal{O}_P^+\) and \(n \geq 0\) such that \((a_P)_{K_P}^n X = 0\). After possibly enlarging \(n\), it follows from Lemma 5.8.(ii) that \(X = (a_P)_{K_P}^n X (a_P)_{K_P}^{-n}\), hence \((a_P)_{K_P}^n X = 0\). By Lemma 5.8.(iii) it implies \([\text{iii}]\).

Assume that \([\text{iii}]\) holds. Let \(X \in \mathcal{O}_P^+\) such that \(\Theta_{M_P}^\prime(X) = 0\). By Lemma 5.8.(ii) there exists \(n \geq 0\) such that \((a_P)_{K_P}^n X \in C_P^+\). But we also have \((a_P)_{K_P}^n X \in \text{Ker } \Theta_{M_P}^\prime\); hence Lemma 5.8.(iii) implies \([\text{iii}]\).

\(\square\)

**Corollary 5.15.** Assume that Hypothesis 5.10 is satisfied. Then

\(\mathcal{H}_R(K_P, P) = \mathcal{O}_P^+ \oplus \text{Ker } \Theta_{M_P}^\prime = \mathcal{O}_P^- \oplus \text{Ker } \Theta_{M_P}^\prime\).

**Proof.** In view of Lemma 5.7 it suffices to prove the first equality. By Proposition 5.14 it remains to prove \(\mathcal{H}_R(K_P, P) \subseteq \mathcal{O}_P^+ \oplus \text{Ker } \Theta_{M_P}^\prime\). So let \(X \in \mathcal{H}_R(K_P, P)\) and choose \(n \in \mathbb{Z}_{\geq 0}\) such that \((a_P)_{K_P}^n X \in C_P^+\). Then \((a_P)_{K_P}^n X (a_P)_{K_P}^{-n} \in \mathcal{O}_P^+\) and \(X - (a_P)_{K_P}^n X (a_P)_{K_P}^{-n} \in \text{Ker } \Theta_{M_P}^\prime\). \(\square\)

The next result implies that it suffices to verify Hypothesis 5.10 for maximal parabolics only.

**Proposition 5.16.** Let \(Q\) be another parabolic subgroup of \(G\) with Levi \(L\). Assume that Hypothesis 5.10 is satisfied for \(P\) and \(Q\). Then Hypothesis 5.10 is satisfied for \(P \cap Q\).

**Proof.** Let \(a_Q \in Z\) be a strictly \(L\)-positive element. Then \(a_P a_Q = (a_P)_{K_P} a_Q\) is strictly \(L \cap M\)-positive. The elements \((a_P)_{K_P}\) and \((a_Q)_{K_Q}\) commute inside \(\mathcal{H}_R(K_P, Q, P \cap Q)\) and we have \((a_P a_Q)_{K_P} = (a_P)_{K_P} (a_Q)_{K_Q}\) in \(\mathcal{H}_R(K_P, P \cap Q, P \cap Q)\). Let \(X \in \mathcal{O}_P^+\) and let \(n \geq 0\) such that \((a_P a_Q)^n X = 0\).
After enlarging \( n \) if necessary, Lemma \ref{5.13} (ii) shows

\[
0 = (a_{pq})^n_{K_{p,q}} \cdot (a_q)^{-n}_{K_{q}} (a_p)^{-n}_{K_{p}}
= (a_p)^n_{K_{p}} \cdot (a_q)^{-n}_{K_{q}} \cdot X \\
= (a_p)^n_{K_{p}} X \cdot (a_q)^{-n}_{K_{p}}
= X.
\]

The assertion now follows from \( \Box \text{(ii)} \Rightarrow \Box \text{(i)} \) in Proposition \ref{5.14} \( \Box \)

**Notation.** If \( \psi : A \to B \) is a homomorphism of \( R \)-algebras and \( f(t) = \sum_i a_i t^i \in A[t] \) is a polynomial, we denote \( f^\psi (t) := \sum_i \psi (a_i) t^i \in B[t] \) the polynomial obtained by applying \( \psi \) to the coefficients of \( f(t) \).

We now prove the decomposition theorem for Hecke polynomials. Recall the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}_R(K_B, B) & \xrightarrow{\Theta_B^p} & \mathcal{H}_R(K_M, M) \\
\downarrow{\varepsilon_{B,P}} & & \downarrow{S^M} \\
\mathcal{H}_R(K_P, P) & \xrightarrow{\Theta_P^p} & R[\Lambda]
\end{array}
\]

**Theorem 5.17 (Decomposition Theorem).** Assume that Hypothesis \ref{5.10} holds true. Let \( d(t) \in \mathcal{H}_R(K_G)[t] \) be a polynomial and assume that there is a decomposition

\[
d^S(t) = f(t) \cdot g(t) \quad \text{in } R[\Lambda][t],
\]

such that one of the following properties is satisfied:

(a) \( f(t) \) has coefficients in \( (S^M \circ \Theta_P^p)(C^+_p) \) with constant term 1, or

(b) \( g(t) \) has coefficients in \( (S^M \circ \Theta_P^p)(C^+_p) \) with constant term 1.

Then there exist polynomials \( f(t), g(t) \in \mathcal{H}_R(K_P, P)[t] \) such that

\[
\deg f(t) = \deg f(t), \quad f^S \circ \Theta_P^p(t) = f(t), \\
\deg g(t) = \deg g(t), \quad g^S \circ \Theta_P^p(t) = g(t),
\]

and

\[
d(t) = f(t) \cdot g(t) \quad \text{in } \mathcal{H}_R(K_P, P)[t].
\]

**Proof.** Note that \( \Theta_B^p|_{\mathcal{H}_R(K_P, P)} = S^M \circ \Theta_P^p \) and \( \Theta_Z^p|_{\mathcal{H}_R(K_G)} = S^G \).

Assume that (a) is satisfied. The case (b) is completely analogous. By Proposition \ref{5.14} the restriction of \( \Theta_M \) to \( \Theta_P^p \) is injective. As \( S^M \) is injective by Theorem \ref{5.14}, it follows that the restriction of \( \Theta_Z^p \) to \( \Theta_P^p \) is injective. By the assumption there exists a unique polynomial \( f(t) \in C^+_p[t] \) satisfying \( f^\Theta_Z^p(t) = f(t) \). Its constant term is necessarily 1, and hence there exists a power series \( h(t) = \sum_{i=0}^\infty h_i t^i \in C^+_p[t] \) with \( h(t) \cdot f(t) = f(t) \cdot h(t) = 1 \). Now, set

\[
g(t) := h(t) \cdot d(t) \in \Theta_P^p[t].
\]

Then \( g^\Theta_Z^p(t) = h^\Theta_Z^p(t) \cdot d^\Theta_Z^p(t) = h^\Theta_Z^p(t) \cdot f(t) \cdot g(t) = g(t) \). As the restriction of \( \Theta_B^p \) to \( \Theta_P^p \) is injective, it follows that \( g(t) \) is a polynomial of degree \( \deg g(t) \). Moreover, we have \( f(t) \cdot g(t) = f(t) \cdot h(t) \cdot d(t) = d(t) \) in \( \mathcal{H}_R(K_P, P)[t] \).

\( \Box \)

**Remark.** Suppose we are in the situation of Theorem 5.17

(i) The polynomial \( d^S(t) \) decomposes in \( \mathcal{H}_R(K_M, M)[t] \).
(ii) In case [a] the proof shows that one can choose \( f(t) \in C^+_P[t] \) and \( g(t) \in \mathcal{O}^+_P[t] \). Since, by Proposition 5.14, the restriction of \( \Theta^P_M \) to \( \mathcal{O}^+_P \) is injective, it follows that \( f(t) \) and \( g(t) \) are unique with these properties.

(iii) Similarly, in case [b] one can choose \( f(t) \in \mathcal{O}^-_P[t] \) and \( g(t) \in C^-_P[t] \), and both polynomials are unique with these properties.

We draw some consequences of Theorem 5.17, cf. also [And77, Thm. 6.2 and Cor.].

Corollary 5.18. Assume that Hypothesis 5.10 holds true.

(i) Let \( f(t) \in C^+_P[t] \) be a polynomial with constant term 1. Then there exists a polynomial \( g(t) \in \mathcal{H}_R(K_P, P)[t] \) with constant term 1 and \( \deg g(t) \leq \deg f(t) \cdot (|W^M_0| - 1) \) such that

\[
f(t) \cdot g(t) \in \mathcal{H}_R(K, G)[t].
\]

(ii) Let \( X \in C^+_P \). Then there exists a monic polynomial \( d(t) = \sum_i d_i t^i \in \mathcal{H}_R(K, G)[t] \) such that \( \deg d(t) \leq |W^M_0| \) and

\[
\sum_i X^i d_i = 0.
\]

(a') Let \( g(t) \in C^-_P[t] \) be a polynomial with constant term 1. Then there exists a polynomial \( f(t) \in \mathcal{H}_R(K_P, P)[t] \) with constant term 1 and \( \deg f(t) \leq \deg g(t) \cdot (|W^M_0| - 1) \) such that

\[
f(t) \cdot g(t) \in \mathcal{H}_R(K, G)[t].
\]

(ii') Let \( X \in C^-_P \). Then there exists a monic polynomial \( d(t) = \sum_i d_i t^i \in \mathcal{H}_R(K, G)[t] \) such that \( \deg d(t) \leq |W^M_0| \) and

\[
\sum_i d_i X^i = 0.
\]

Proof. Note that, using the anti-automorphism \( \zeta_P \) on \( \mathcal{H}_R(K_P, P) \), which on \( \mathcal{H}_R(K, G) \) restricts to \( \zeta_G \) by Lemma 2.7, one can easily deduce [i] and [ii] from [i] and [ii] respectively.

Let us prove [ii]. So let \( f(t) \) be a polynomial with coefficients in \( C^+_P \) and constant term 1. Write \( f^{\Theta^P_B} = \tilde{f}(t) = \sum_i f_i t^i \in R[N][t] \). Since \( S^M \circ \Theta^P_M = \Theta^B_P \circ \varepsilon \) by Lemma 5.2, the coefficients of \( \tilde{f}(t) \) lie in \( S^M(\mathcal{H}_R(K_M, M)) \). Hence, the coefficients of \( f(t) \) are invariant under the twisted action of \( W_0M \). Given \( w \in W_0 \), write \( \tilde{f}^w(t) = \sum_i (w \ast f_i) \cdot t^i \). The polynomial \( \tilde{d}(t) \) is then \( W_0 \)-invariant with respect to the twisted action and hence has coefficients in \( S^G(\mathcal{H}_R(K, G)) \).

Moreover, it factors as \( \tilde{d}(t) = \tilde{f}(t) \cdot \tilde{g}(t) \) for some \( \tilde{g}(t) \in R[N][t] \) with constant term 1 and \( \tilde{g}(t) \leq \deg f(t) \cdot (|W^M_0| - 1) \). The existence of the polynomial \( g(t) \) with the desired properties now follows from Theorem 5.17.

We now prove [ii]. Let \( X \in C^+_P \). Applying [i] to the polynomial \( f(t) := 1 - Xt \), we find a polynomial \( g(t) = \sum_{i=0}^{r-1} g_i t^i \in \mathcal{H}_R(K_P, P)[t] \) with \( g_0 = 1 \) and \( r \leq |W^M_0| \) and such that

\[
f(t) \cdot g(t) = \sum_{i=0}^{r-1} d_{r-i} t^i \in \mathcal{H}_R(K, G)[t].
\]

Since \( f(t) \cdot g(t) = 1 + \sum_{i=1}^{r-1} (g_i - Xg_{i-1})t^i - Xg_{r-1}t^r \), a comparison of coefficients shows that \( d_0 = -Xg_{r-1}, d_i = g_{r-i} - Xg_{r-(i+1)} \) for \( 1 \leq i \leq r-1 \), and \( d_r = g_0 = 1 \). Therefore, the polynomial \( \sum_{i=0}^{r} d_i t^i \in \mathcal{H}_R(K, G)[t] \) is monic, of degree \( r \leq |W_0^M| \), and satisfies

\[
\sum_{i=0}^{r} X^i d_i = -Xg_{r-1} + \sum_{i=1}^{r-1} (X^i g_{r-i} - X^{i+1} g_{r-(i+1)}) + X^r = 0.
\]

Example 5.19. We continue Example 5.12 and apply Theorem 5.17 to the polynomial

\[
\chi_{\alpha_B}(t) = 1 - q^{-1} \cdot (X_+ Y^{-1} + X_-) \cdot t + q^{-1} Y^{-1} t^2 \in \mathcal{H}_R(K, G)[t].
\]
Then \( \chi_{\alpha_{B}^{t}}(t) = 1 - ((qy)^{-1} + x^{-1}) \cdot t + (qxy)^{-1}t^2 \) decomposes in \( R[x^\pm 1, y^\pm 1][t] \) as \( \tilde{f}(t) \cdot \tilde{g}(t) \) with 
\[
\tilde{f}(t) = 1 - (qy)^{-1}t \quad \text{and} \quad \tilde{g}(t) = 1 - x^{-1}t.
\]
Then \( f(t) := 1 - q^{-1}X + Y - t \in C_B^{+}[t] \) is the unique polynomial with \( f_{\beta \gamma}^B(t) = \tilde{f}(t) \). Let \( h(t) := \sum_{i=0}^{\infty}(q^{-1}X + Y - 1)^i t^i \in C_B^{+}[t] \) be the inverse power series. Then

\[
g(t) := h(t) \cdot \chi_{\alpha_{B}}(t) = h(t) - h(t) \cdot q^{-1}X + Y - 1 \cdot t + h(t) \cdot q^{-1}Y - 1 \cdot t^2 = 1 - q^{-1}X - 1 \cdot t - \sum_{i=1}^{\infty}(q^{-1}X + Y - 1)^{i-1} \cdot q^{-1}Y - 1 \cdot t^{i+1} + \sum_{i=0}^{\infty}(q^{-1}X + Y - 1)^i \cdot q^{-1}Y - 1 \cdot t^{i+2} = 1 - q^{-1}X - 1 \cdot t.
\]

Hence, \( \chi_{\alpha_{B}}(t) \) decomposes in \( \mathcal{H}_R(K_B, B)[t] \) as

\[
\chi_{\alpha_{B}}(t) = (1 - q^{-1}X + Y - 1) \cdot (1 - q^{-1}X - 1) \cdot (1 - q^{-1}Y - 1).
\]

**APPENDIX A. PROOF OF THEOREM**

Let \( B \subseteq \text{GL}_2(\mathfrak{O}) \) be the subgroup of upper triangular matrices and let \( K_B \) be the subgroup of \( B \) with entries in \( \mathcal{O}_3 \). Fix a uniformizer \( \pi \in \mathcal{O}_3 \) and recall that \( q \) denotes the cardinality of the residue field of \( \mathfrak{O} \). Let \( R \) be a coefficient ring.

We describe the \( R \)-algebra \( \mathcal{H}_R(K_B, B) = R \otimes_{\mathbb{Z}} \mathcal{H}(K_B, B) \) in terms of generators and relations. As \( R \otimes_{\mathbb{Z}} \) is right exact, we may reduce to the case \( R = \mathbb{Z} \).

First, we need to understand the double cosets of \( B \) with respect to \( K_B \). Let \( A \) be a complete system of representatives for \( \mathcal{O}_3 / (\pi) \) with \( 0 \in A \). Then \( A_B := \{ \sum_{i=1}^{n} a_i \pi^{-1} \mid n \in \mathbb{Z}_{>0}, a_i \in A \} \) is a complete system of representatives for \( \mathfrak{O} / \mathcal{O}_3 \).

**Lemma A.1.** As a set \( B \) decomposes as

\[
B = \bigsqcup_{\substack{a,b,c \in \mathbb{Z}, \\ b \leq \min\{a,c\}}} K_B \left( \begin{array}{cc} \pi^a & \pi^b \\ 0 & \pi^c \end{array} \right) K_B,
\]

and for all \( a, b, c \in \mathbb{Z} \) with \( b \leq \min\{a, c\} \) one has the decomposition

\[
K_B \left( \begin{array}{cc} \pi^a & \pi^b \\ 0 & \pi^c \end{array} \right) K_B = \bigsqcup_{\substack{\beta \in A_B \pi, \\ \text{val}_3(\beta) = b \\ \text{or}\ \text{val}_3(\beta) \geq a \\ \text{and} \\ \text{val}_3(\beta) \leq a \\ \text{or} \\ \text{val}_3(\beta) = 0 \\ \text{and} \\ \text{val}_3(\beta) \geq a}} K_B \left( \begin{array}{cc} \pi^a & \beta \pi^c \\ 0 & \pi^c \end{array} \right),
\]

if \( b \leq \min\{a, c\} \);

\[
K_B \left( \begin{array}{cc} \pi^a & \beta \pi^c \\ 0 & \pi^c \end{array} \right) K_B = \bigsqcup_{\substack{\beta \in A_B \pi, \\ \text{val}_3(\beta) = b \\ \text{or}\ \text{val}_3(\beta) \geq a \\ \text{and} \\ \text{val}_3(\beta) \leq a \\ \text{or} \\ \text{val}_3(\beta) = 0 \\ \text{and} \\ \text{val}_3(\beta) \geq a}} K_B \left( \begin{array}{cc} \pi^a & \beta \pi^c \\ 0 & \pi^c \end{array} \right),
\]

if \( b = \min\{a, c\} \).

**Proof.** Let \( \left( \begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) \in B \). Write \( \alpha = \alpha_0 \pi^{a} \) and \( \gamma = \gamma_0 \pi^{c} \), where \( \alpha_0, \gamma_0 \in \mathcal{O}_3^{\pm} \), \( a, c \in \mathbb{Z} \), and write \( \frac{\beta}{\alpha_0 \pi^{a}} = \beta' + x \), with \( \beta' \in A_B \) and \( x \in \mathcal{O}_3 \). Then \( \beta' = 0 \) or \( \text{val}_3(\beta) = \text{val}_3(\beta' \pi^{c}) \). Moreover,

\[
\left( \begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) = \left( \begin{array}{cc} \alpha_0 & \alpha_0 x \\ 0 & \gamma_0 \end{array} \right) \cdot \left( \begin{array}{cc} \pi^a & \beta' \pi^c \\ 0 & \pi^c \end{array} \right) \in K_B \left( \begin{array}{cc} \pi^a & \beta' \pi^c \\ 0 & \pi^c \end{array} \right).
\]

Let now \( a, a', c, c' \in \mathbb{Z} \) and \( \beta, \beta' \in A_B \) with

\[
\left( \begin{array}{cc} \pi^a & \beta \pi^c \\ 0 & \pi^c \end{array} \right) \cdot \left( \begin{array}{cc} \pi^{a'} & \beta' \pi^{c'} \\ 0 & \pi^{c'} \end{array} \right)^{-1} = \left( \begin{array}{cc} \beta \pi^{c-a'} & 0 \\ 0 & \beta \pi^{c-c'} \end{array} \right) \in K_B.
\]

Then \( a = a' \) and \( c = c' \), and then \( \beta - \beta' \in \mathcal{O}_3 \), that is, \( \beta = \beta' \). This shows that \( B \) is the disjoint union of the right cosets \( K_B \left( \begin{array}{cc} \pi^a & \beta \pi^c \\ 0 & \pi^c \end{array} \right) \), where \( a, c \in \mathbb{Z} \) and \( \beta \in A_B \).
Let now \( a, c \in \mathbb{Z} \). Take any \( 0 \neq \beta = \beta_0 \pi^{\text{val}_\beta(\beta)} \in A_B \pi^c \) with \( \beta_0 \in \mathcal{O}_\beta^\times \). If \( \text{val}_\beta(\beta) < \min\{a, c\} \), then
\[
\begin{pmatrix}
\pi^a & \beta \\
0 & \pi^c
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \beta_0^{-1} \end{pmatrix} \cdot \begin{pmatrix}
\pi^a & \pi^{\text{val}_\beta(\beta)} \\
0 & \pi^c
\end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \beta_0 \end{pmatrix} \in K_B \begin{pmatrix}
\pi^a & \pi^{\text{val}_\beta(\beta)} \\
0 & \pi^c
\end{pmatrix} K_B.
\]
If \( \text{val}_\beta(\beta) \geq \min\{a, c\} \), then \( \text{val}_\beta(\beta) \geq a \), because \( \text{val}_\beta(\beta) < c \) always holds true. Hence,
\[
\begin{pmatrix}
\pi^a & \beta \\
0 & \pi^c
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-a} \end{pmatrix} \cdot \begin{pmatrix} \beta \pi^{-a} & 0 \\ 0 & 1 \end{pmatrix} \in K_B \begin{pmatrix}
\pi^a & 0 \\
0 & \pi^c
\end{pmatrix} K_B.
\]
The lemma follows. \( \square \)

We now prove Theorem 5.8. Let \( \mathcal{H} \) be the \( \mathbb{Z} \)-algebra generated by the variables \( X_+, X_-, Y, \) and \( Y^{-1} \), subject to the relations \( (24) \). This means that \( Y \) is central and invertible and we have \( X_+ X_- = q \cdot 1 \) in \( \mathcal{H} \). We show that
\[
\rho : \mathcal{H} \rightarrow \mathcal{H}(K_B, B),
\]
\[
X_+ \mapsto \left( \begin{array}{cc} \pi & 0 \\ 0 & 1 \end{array} \right)_{K_B},
\]
\[
X_- \mapsto \left( \begin{array}{cc} \pi^{-1} & 0 \\ 0 & 1 \end{array} \right)_{K_B},
\]
\[
Y \mapsto (\pi E_2)_{K_B}
\]
gives a well-defined isomorphism of algebras.

It is clear that \( (K_B) \) is the unit in \( \mathcal{H}(K_B, B) \), and that \( (\pi E_2)_{K_B} \) is central and invertible with inverse \( (\pi^{-1} E_2)_{K_B} \). In addition, using Lemma \( 4.4 \) we compute
\[
\left( \begin{array}{cc} \pi & 0 \\ 0 & 1 \end{array} \right)_{K_B} \cdot \left( \begin{array}{cc} \pi^{-1} & 0 \\ 0 & 1 \end{array} \right)_{K_B} = \left( K_B \begin{array}{cc} \pi & 0 \\ 0 & 1 \end{array} \right) \cdot \sum_{\beta \in A} \left( K_B \begin{array}{cc} \pi^{-1} & \beta \pi^{-1} \\ 0 & 1 \end{array} \right)
\]
\[
= \sum_{\beta \in A} \left( K_B \begin{array}{cc} 1 & \beta \\ 0 & 1 \end{array} \right) = q \cdot (K_B).
\]
Therefore, \( \rho \) is a well-defined algebra homomorphism.

Observe that \( \rho(X_+^m) = \left( \begin{array}{cc} \pi^m & 0 \\ 0 & 1 \end{array} \right)_{K_B} \) and \( \rho(Y^k) = (\pi^k E_2)_{K_B} \), for all \( m \in \mathbb{Z}_{\geq 0} \) and \( k \in \mathbb{Z} \). For each \( n \in \mathbb{Z}_{\geq 0} \) we compute
\[
\rho(X^n) = \rho(X_-)^n = \left( \sum_{\beta \in A} \left( K_B \begin{array}{cc} \pi^{-1} & \beta \pi^{-1} \\ 0 & 1 \end{array} \right) \right)^n
\]
\[
= \sum_{\beta_1, \ldots, \beta_n \in A} \left( K_B \begin{array}{cc} \pi^{-n} & \sum_{i=1}^{n} \beta_i \pi^{-i} \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} \pi^{-n} & 0 \\ 0 & 1 \end{array} \right)_{K_B}.
\]
Given \( m, n \in \mathbb{Z}_{\geq 0} \) and \( k \in \mathbb{Z} \), this shows
\[
\rho(X_+^m X_+^n Y^k) = \left( \begin{array}{cc} \pi^{-n} & 0 \\ 0 & 1 \end{array} \right)_{K_B} \cdot \left( \begin{array}{cc} \pi^m & 0 \\ 0 & 1 \end{array} \right)_{K_B} \cdot (\pi^k E_2)_{K_B}
\]
\[
= \sum_{\beta_1, \ldots, \beta_n \in A} \left( K_B \begin{array}{cc} \pi^{m+k-n} & \pi^k \sum_{i=1}^{n} \beta_i \pi^{-i} \\ 0 & \pi^k \end{array} \right)
\]
\[
= \sum_{b=k-n}^{\min\{k, m+k-n\}} \left( \begin{array}{cc} \pi^{m+k-n} & \pi^k \\ 0 & \pi^k \end{array} \right)_{K_B}.
\]
For \( m, n \in \mathbb{Z}_{\geq 1} \) and \( k \in \mathbb{Z} \) this implies
\[
\rho(X_+^n X_+^m Y^k - X_+^{n-1} X_+^{m-1} Y^k) = \left( \begin{array}{cc} \pi^{m+k-n} & \pi^{k-n} \\ 0 & \pi^k \end{array} \right)_{K_B}.
\]
Now notice that
\[
\left\{ X^m Y^k \mid m \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z} \right\} \cup \left\{ X^n Y^k \mid n \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z} \right\} \\
\cup \left\{ X^n X^m Y^k - X^{n-1} X^{m-1} Y^k \mid m, n \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z} \right\}
\]
generates \( \mathcal{H} \) as a \( \mathbb{Z} \)-module and identifies via \( \rho \) with the double coset basis of \( \mathcal{H}(K_B, B) \). It follows that (26) is in fact \( \mathbb{Z} \)-linearly independent, hence a \( \mathbb{Z} \)-basis of \( \mathcal{H} \). Consequently, \( \rho \) is an isomorphism.

References

[AAnd77] Anatoli N. Andrianov. On factorization of Hecke polynomials for the symplectic groups of genus \( n \). Mathematics of the USSR-Sbornik, 33(3):343–373, 1977. doi:10.1070/SM1977v033n03ABEH002428

[AAnd79] Anatoli N. Andrianov. The multiplicative arithmetic of Siegel modular forms. Russian Mathematical Surveys, 34(1):75–148, 1979. doi:10.1070/RM1979v034n01ABEH002872

[AZ95] Anatoli N. Andrianov and Vladimir G. Zhuravlev. Modular Forms and Hecke Operators. American Mathematical Society, 1995.

[BK98] Colin J. Bushnell and Philip C. Kutzko. Smooth representations of reductive \( p \)-adic groups: structure theory via types. Proceedings of the London Mathematical Society, 77(3):582–634, 1998. doi:10.1112/S0024611598005074

[Bou81] Nicolas Bourbaki. Groupes et algèbres de Lie: Chapitres 4, 5 et 6. Éléments de Mathématiques. Masson, 1981.

[BT72] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local: I. données radicielles valuées. Publications Mathématiques de l'IHÉS, 41:5–251, 1972. URL: http://eudml.org/doc/104001

[BT84] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local: II. Schémas en groupes. Existence d'une donnée radicielle valuée. Publications Mathématiques de l'IHÉS, 60:5–184, 1984. URL: http://eudml.org/doc/103918

[Dab94] Romuald Dąbrowski. Comparison of the Bruhat and the Iwahori Decompositions of a \( p \)-Adic Chevalley Group. Journal of Algebra, 167(3):704–723, 1994. doi:10.1006/jabr.1994.1208

[Gri84] Gritsenko, Valeri A. The action of modular operators on the Fourier–Jacobi coefficients of modular forms. Mathematics of the USSR-Sbornik, 47(1):237–268, Feb 1984. doi:10.1070/SM1979v047n01ABEH002640

[Gri88] Valeri A. Gritsenko. Parabolic extensions of a Hecke ring of the general linear group. Journal of Soviet Mathematics, 43(4):2533–2540, Nov 1988. doi:10.1007/BF01374963

[Gri90] Valeri A. Gritsenko. Expansion of Hecke polynomials of classical groups. Mathematics of the USSR-Sbornik, 65(2):333–356, 1990. doi:10.1070/SM1990v065n02ABEH001145

[Gri92] Valeri A. Gritsenko. Parabolic extensions of the Hecke ring of the general linear group. II. Journal of Soviet Mathematics, 62(4):2869–2882, Dec 1992. doi:10.1007/BF01098922

[Her11] Florian Herzig. A Satake isomorphism in characteristic \( p \). Compositio Mathematica, 147(1):263–283, 2011. doi:10.1112/S0010437X10004951

[Hey19] Claudius Heyer. Applications of parabolic Hecke algebras: parabolic induction and Hecke polynomials. PhD thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät, 2019. doi:http://dx.doi.org/10.18452/20137

[Hey21a] Claudius Heyer. Localization of the Parabolic Hecke Algebra at a Strictly Positive Element, 2021. arXiv:2103.16949

[Hey21b] Claudius Heyer. Parabolic induction via the parabolic pro-\( p \) Iwahori–Hecke algebra. Represent. Theory, 25:807–843, 2021. doi:10.1090/ert/585

[HR10] Thomas Haines and Sean Rostami. The Satake isomorphism for special maximal parahoric Hecke algebras. Representation Theory of the American Mathematical Society, 14(7):264–284, 2010. doi:10.1090/S1088-4165-10-00370-5

[Hum90] James E. Humphreys. Reflection Groups and Coxeter Groups. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1990. doi:10.1017/CBO9780511625646

[HV15] Guy Henniart and Marie-France Vignéras. A Satake isomorphism for representations modulo \( p \) of reductive groups over local fields. Journal für die reine und angewandte Mathematik, 701:33–75, 2015. doi:10.1515/crelle-2013-0021

[Jan14] Fabian Januszewski. On \( p \)-adic L-functions for \( GL(n) \times GL(n − 1) \) Over Totally Real Fields. International Mathematics Research Notices, 2015(17):7884–7949, 2014. doi:10.1093/imrn/rnu181

[Lan01] Joshua M. Lansky. Decomposition of double cosets in \( p \)-adic groups. Pacific Journal of Mathematics, 197(1):97–117, 2001. doi:10.2140/pjm.2001.197.97

[Mac71] Ian G. Macdonald. Spherical functions on a group of \( p \)-adic type. Ramanujan Institute, University of Madras, Madras 5, India, nov 1971.

[Mac03] I. G. Macdonald. Affine Hecke Algebras and Orthogonal Polynomials. Cambridge Tracts in Mathematics. Cambridge University Press, 2003. doi:10.1017/CBO9780511542824

[Rap00] Michael Rapoport. A positivity property of the Satake isomorphism. manuscripta mathematica, 101(2):153–166, 2000. doi:10.1007/s002290050019
[SS97] Peter Schneider and Ulrich Stuhler. Representation theory and sheaves on the Bruhat-Tits building. Publications Mathématiques de l’Institut des Hautes Études Scientifiques, 85(1):97–191, 1997.

[Vie12] Mathieu Vienney. Construction de $(\varphi, \Gamma)$-modules en caractéristique $p$. PhD thesis, Ecole normale supérieure de Lyon – ENS LYON, 2012.

[Vig96] Marie-France Vignéras. Représentations $l$-modulaires d’un groupe réductif $p$-adique avec $l \neq p$. Progress in Mathematics, 137, Birkhäuser: Boston, 1996.

[Vig16] Marie-France Vignéras. The pro-$p$ Iwahori Hecke algebra of a reductive $p$-adic group I. Compositio Mathematica, 152(4):693–753, 2016. [doi:10.1112/S0010437X15007666]

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