Taming the end-point singularities in heavy-to-light decays

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Abstract

It is argued that there is a fundamental momentum cutoff in heavy-to-light transitions, which is caused by possible Cherenkov gluon radiation when an energetic light parton travels through the "brown muck". The soft-overlap contributions where the partonic momenta configuration is highly asymmetric are disfavored, and the problematic end-point singularities and the double countings are absent in this framework. A simple calculation with a natural scale for the cutoff gives a plausible result for the $B \to \pi$ form factor.

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Heavy-to-light transition is one of the most important processes in flavor physics. With the successful running of B factories BABAR and BELLE, semileptonic B decays like $B \to \pi(\rho)\ell\nu$ provide the information about the least known Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{ub}|$, for which $B \to \pi$ transition form factors are crucial. The same form factors enter the nonleptonic decays such as $B \to \pi\pi$, which is important not only to allow us to access the angles of CKM unitarity triangle, but also to check the factorization.

But the theoretical understanding of heavy-to-light transition is rather poor and still controversial. In this paper we only deal with $B \to \pi$ transition for concentration; its generalization to other processes is straightforward. In the standard convention, $B \to \pi$ transition is completely described by three form factors, $f_+, f_0$, and $f_T$. At large recoil limit where the pion energy $E$ is sufficiently large, these form factors are not independent, and only one form factor remains $[1]$. This is known as the spin-symmetry relations. It is a common lore that there are two kinematically distinctive, "soft overlap (or Feynman mechanism)" and "hard scattering" contributions to the form factors. In the former picture, one of the partons in the daughter meson ($\pi$) carries almost all the momentum. In the latter case, by exchanging hard gluons, none of the partons in the daughter meson is in the end-point region of momentum configuration.

One of the main issue here is the end-point singularity. Contributions of gluon exchange with the spectator quarks are described by the convolution integrals involving meson distribution amplitudes (DA) and some kernel. At the heavy quark limit, the kernel behaves like $\sim 1/x^2$ where $x$ is some momentum fraction while the meson DAs do as $\sim x$ in its asymptotic form. Thus the resulting convolution integral diverges, and this is called the end-point singularity.

But the end-point singularity has been dealt in many different ways in different theoretical frameworks. In perturbative QCD (pQCD) approach $[2, 3]$, the end-point singularity is absent due to the Sudakov suppression near the end-point region. But subsequently it is argued that the Sudakov suppression is not severe at the heavy quark scale $m_B \sim 5.3$ GeV $[4, 5, 6]$. In Ref. $[7]$, the problem of end-point singularity is avoided by absorbing the terms with singularities into the soft form factor. Here the heavy-light form factors are compactly written as

$$f_i(q^2) = C_i \xi_\pi(E) + \phi_B \otimes T_i \otimes \phi_\pi,$$

where $\xi_\pi$ is the soft form factor with $E$ being the pion energy, and $C_i$ are the hard vertex renormalization; $T_i$ are hard kernels which are convoluted ($\otimes$) with the meson DAs $\phi_B$ and $\phi_\pi$. It was also shown that the soft form factor $\xi_\pi$ satisfies the spin-symmetry relations mentioned before. The second contribution arises from the hard spectator interactions. Terms involving end-point singularities are already absorbed into $\xi_\pi$, so the remaining convolutions are end-point finite. And they are shown to break the spin-symmetry relations. From the numerical analysis, the authors of Ref. $[7]$ found that the symmetry breaking corrections contribute about 10%; heavy-light form factors are largely from the soft form factor.

The advent of the soft-collinear effective theory (SCET) $[8]$ shed new lights on the heavy-to-light decays. In this framework, the heavy-light form factor is described as $[6, 9, 10]$

$$f_+ = T^{(+)}(E) \xi^{B\pi}(E) + N_0 \phi_B \otimes C^{(+)}_J \otimes J \otimes \phi_\pi,$$

where $T^{(+)}$ and $C^{(+)}_J$ are the hard functions and $J$ is the jet function, and $N_0 = f_B f_\pi m_B (4E^2)$ with $f_B, f_\pi$ being the meson decay constants. Here $\xi^{B\pi}$ is the SCET version of the "soft" form factor. Just as in $[7]$, the end-point singular terms are absorbed into
\[ \zeta^{B\pi}, \text{ and as a whole satisfy the spin-symmetry relations. From the fact that the same form factors enter the nonleptonic two-body } B \text{ decays, Ref. [11] showed that the two contributions of Eq. (2) are comparable in size. This point differs from the QCD factorization (QCDF) analysis [12], where the hard scattering contribution is very small.} \]

Although the brief summary above shows impressive achievements in heavy-to-light transitions, there are still some ambiguities and confusions. First of all, the quantity \( \zeta^{B\pi} \) in Eq. (2) is not "soft" in the sense that the involved quarks are not in the asymmetric momentum configuration; it is defined by the collinear quarks. In this context, the SCET description of Eq. (2) is much closer to the pQCD prediction where the contributions from the asymmetric momentum configuration are highly suppressed. And the Sudakov suppression in the end-point region is still disputable [13].

In this paper, we present a new viewpoint on the heavy-to-light transition. It will be argued that there is a fundamental cutoff for the momentum of outgoing quark from the weak vertex. This is due to the Cherenkov gluon radiation inside the hadron when the heavy quark is changed into an energetic light quark to propagate through the "brown muck". The existence of the fundamental cutoff naturally cures the problem of the end-point singularity and does not allow the contributions from a highly asymmetric momentum configuration.

Let us first consider the \( B \to \pi \) form factors. The standard definition is \[ \langle \pi | \bar{q} \gamma^\mu b | \bar{B}(p_B) \rangle = f_+(q^2) \left[ p_b^\mu + p^\mu - \frac{m_B^2}{q^2} q^\mu \right] + f_0(q^2) \frac{m_B^2}{q^2} q^\mu, \]

where the pion mass squared \( m_\pi^2 \) terms are neglected. The end-point singularity appears when one considers the gluon exchange diagrams with the spectator quarks (Fig. 1). Their amplitude is proportional to the scattering kernel \[ T_{ij\ell m} = -g^2 C_F \frac{\gamma^\mu_{ij\ell m}}{(p_2 - k)^2} \left[ \frac{p_b^\mu + p^\mu - m_b^2}{(p_b + k - p_2)^2} \right] \gamma_\mu + \gamma_\mu \frac{p_1^\mu - k^\mu + p_2^\mu}{(p_1 - k + p_2)^2} \right] \Gamma_{ij}, \]

which will be convoluted with the meson DAs to give the full amplitude. At the heavy quark and large recoil limit, the kernel reduces to

\[ T_{ij\ell m} \simeq -g^2 C_F \gamma^\mu_{ij\ell m} \left[ \frac{m_B(1 + \gamma)}{4\bar{u}m_bE^2k^+} \gamma_\mu + \gamma_\mu \frac{E\gamma - k^\mu}{4\bar{u}E^2(k^+)^2} \right] \Gamma_{ij}, \]
where \( p_b = m_b v \), \( p = E_n, p_1 = u p \), \( p_2 = \bar{u} p \equiv (1 - u)p \), and \( k^+ = n \cdot k \) with \( n \) being a light-like vector. The end-point singularity occurs when \( \bar{u} \to 0 \) (i.e., \( u \to 1 \)) or \( k^+ \to 0 \) unless the meson DAs fall fast enough. Actually, the asymptotic form of the pion DA is \( \phi_\pi(u) \sim u \bar{u} \). As for \( B \) DA, the specific form depends on models, but it can also suffer from the end-point singularity with \( 1/(k^+)^2 \). In this work, we just concentrate on the problem of \( \bar{u} \to 0 \) for simplicity, which is conceptually more important.

From this observation, it is commonly argued that \( B \to \pi \) transition is dominated by the ”soft” physics where \( \bar{u} \to 0 \). And the troublesome divergent terms are absorbed into the nonperturbative ”soft” form factor. Dominance of the soft-overlap contribution is then supported by the phenomenological fitting within this parametrization. Note that near the end point where \( \bar{u} \to 0, p_1 \to p \) and \( p_2 \to 0 \); the parton momenta of the outgoing pion are highly asymmetric. One crucial drawback of this kind of argument lies in the fact that we cannot extrapolate the nonperturbative physics from the perturbative analysis. When \( \bar{u} \) is very close to 0, the offshellness of the exchanged gluon momentum \( (p_2 - k)^2 \sim -2\bar{u} E k^+ \) is vanishing, so we are entering the nonperturbative regime. In this case the neglected terms of order \( \sim O(\Lambda_{\text{QCD}}) \) become significant, and thus Eq. (5) is not reliable any longer.

It is quite interesting to see how the ”soft” form factor is treated in SCET. In SCET the soft form factor is defined by a series of operators containing collinear quarks \[9\]. In other words, energetic pion is described solely by the collinear quarks; soft+collinear combination is not allowed. That’s the reason why the defining operators have the interaction Lagrangians \( \mathcal{L}_{\xi q} \) or messenger modes which convert the soft spectator into collinear one through the exchange of collinear gluons or into the soft-collinear quark. In this sense, (due to the large rapidity gap \[10\]) there is no soft-overlap contributions in SCET \emph{a priori}.

This makes a sharp contrast with the works of \[5, 14, 15\] where the soft-overlap contribution plays an important role. Also in many literatures the end-point singular terms are absorbed into the ”soft” form factor as in Eq. (11) \[7, 16\]. Thus the soft form factor \( \xi_{\pi} \) contains both soft overlap and hard spectator interactions. This parametrization is not bad in a viewpoint of the spin symmetry because all the terms in the soft form factor satisfy the spin-symmetry relations. But in some cases it causes many confusions; for example, it is not adequate to directly compare \( \xi_{\pi} \) with \( \zeta^{B\pi} \) since the soft overlap is included in the former while not in the latter from the construction. As pointed out in \[16\], it might be unfruitful to extract the hard scattering effects from \( \xi_{\pi} \) to leave it purely nonperturbative. However, it is at least conceptually important to separate the soft overlap from the hard scattering when which of the two is dominant matters.

One more profound matter in SCET regarding the end-point singularity is double counting. The phase space region where \( \bar{u} \to 0 \) corresponds to the zero-bin of the collinear momentum, which must be subtracted to avoid double counting \[10\]. When \( \bar{u} \) is very close to 0, then the ”collinear” momentum \( p_2 \) is no longer collinear; it becomes a soft mode which cannot participate in forming the collinear pion, as mentioned above. This is very similar to the case pointed out earlier, where the nonperturbative region is extrapolated from the perturbative analysis.

It is very helpful to see how the soft overlap is identified in the light-cone sum rules (LCSR). At tree level after the Borel transformation, the weak form factor is proportional to \[14\]

\[
f^\text{tree}_+ \sim \int_{u_0}^{1} du \phi_\pi(u) T_H(u) ,
\]

where \( T_H \) is the process-dependent amplitude. Here \( u_0 \equiv (m_b^2 - q^2)/(s_0 - q^2) \) where \( s_0 \)
is the continuum threshold is a lower limit of the convolution integral. It scales as \( u_0 \sim 1 - \Lambda_{\text{QCD}}/m_b \); thus only the highly asymmetric momentum configuration is relevant. This is nothing but the exact meaning of the soft overlap. Hard spectator interactions as well as the vertex corrections appear at \( \mathcal{O}(\alpha_s) \). Numerically Eq. (6) is dominant compared to the \( \mathcal{O}(\alpha_s) \) contributions.

We now propose a new possibility that there is an upper limit on the momentum of the outgoing quark from the weak vertex, less than the maximum recoil energy of \( m_b/2 \). When the heavy quark is changed into the light quark with very high energy via weak interaction, it suddenly moves through the "brown muck" consisting of the light degrees of freedom. The situation is very similar to the case when a fast electron goes through a dense medium, where the Cherenkov radiation should occur. Much more similar processes have been studied in the heavy-ion collisions recently. Here an energetic parton enters through a dense hadronic medium, and possible Cherenkov gluon radiation has been studied extensively \[17, 18\].

The necessary condition for the Cherenkov radiation is \( \text{Re}[n(\epsilon)] > 1 \), where \( n(\epsilon) \) is the index of refraction. Analogous to the photon case, \( n(\epsilon) - 1 \) is proportional to the forward scattering amplitudes \( F(\epsilon) \). At low energies, \( \text{Re}[F(\epsilon)] > 0 \) if \( \epsilon > \epsilon_R \) for the Breit-Wigner resonance \( F(\epsilon) \sim (\epsilon - \epsilon_R + i\Gamma/2)^{-1} \) where \( \epsilon_R \) is the resonant energy and \( \Gamma \) is the decay width \[18\]. Since the light mesons are possible intermediate resonances of the brown muck, the necessary condition can be easily satisfied also in \( B \to \pi \) transition. In inclusive decays, the Cherenkov gluons will appear as cone-like events while in exclusive decays they will eventually couple and transfer the energy to the light degrees of freedom to make the final state meson.

The energy loss due to the Cherenkov radiation is given by

\[
\frac{dE_c}{dx} = 4\pi\alpha_s \int_{n(\epsilon)>1} d\epsilon \, \epsilon \left[ 1 - \frac{1}{n^2(\epsilon)} \right]. \tag{7}
\]

The nonperturbative nature is encoded in \( n(\epsilon) \). The amount of energy loss for heavy-ion collisions varies around \( 0.1 \sim 1 \) GeV/fm up to the model. Roughly speaking, the Cherenkov energy loss is about \[17\] \( dE_c/dx \sim 4\pi\alpha_s \ell_0^2/2 \), where \( \ell_0 \sim \mathcal{O}(\Lambda_{\text{QCD}}) \) is the gluon energy. The total energy loss might be

\[
\frac{E_c}{E} \sim 4\pi\alpha_s \frac{\ell_0^2 L}{2E} \sim \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}}{m_B} \right), \tag{8}
\]

where \( L \sim 1/\Lambda_{\text{QCD}} \) is the flight length of the energetic parton during the formation of \( \pi \). More precise estimation requires the detailed structure of the index of refraction \( n(\epsilon) \). But this naive power counting is enough to give an important message for the soft overlap. If we take into account the Cherenkov energy loss, the convolution integral of Eq. (6) will be changed into

\[
f_+^{\text{tree}} \sim \int_{u_0}^{1 - E_c/E} du \, \phi_\pi(u) T_H(u). \tag{9}
\]

Since \( 1 - u_0 \sim \mathcal{O}(\Lambda_{\text{QCD}}/m_B) \sim E_c/E \), the integration domain shrinks severely. Consequently the soft overlap is highly suppressed. It will be a good phenomenological trade to introduce the cutoff \( \bar{u}_c \equiv 1 - u_c \equiv 1 - E_c/E \) for the nonperturbative \( n(\epsilon) \). We stress that the existence of \( \bar{u}_c \) is fundamental for heavy-to-light decays. In numerical calculations, however, the value of \( u_0 \) is not so close to 1. Typically, \( u_0 \approx 0.65 \sim 0.70 \). Its deviation from unity is much larger than the usual \( \Lambda_{\text{QCD}}/m_B \approx 0.04 \) for \( \Lambda_{\text{QCD}} \sim 200 \) MeV. It is very difficult and ambiguous
to determine what portion of momentum should be transferred to insure the soft overlap configuration, or to make soft quark collinear. But it is quite true that Eq. (20) contains more than ”soft overlap” with \( u_0 \approx 0.65 \sim 0.70 \) in numerics. Furthermore, the approximation in \[15\]

\[ f_+^{\text{free}} \sim \int_{u_0}^{1} du \phi_+(u) \approx -\frac{1}{2} \phi_+(1) u_0^2 \approx 0.35 \]  

(10)
tends to increase the numerical value compared to the original integral (\( \approx 0.27 \)). This is because \( \bar{u}_0 \equiv 1 - u_0 \approx 0.34 \) is not sufficiently small. In short, the soft overlap contribution is overestimated in LCSR. The pure soft overlap contribution comes from the much narrower range of momentum fraction, which would be shrunken again by the Cherenkov radiation.

In what follows, we assume that the soft overlap is negligible. This approach is on the same line as pQCD or SCET where the soft overlap is ignored. The \( B \to \pi \) form factors are given by the hard gluon exchange processes. There is now one nonperturbative parameter \( \bar{u}_c \) which regulates the divergent convolution as a cutoff. Explicitly \[7\],

\[ f_+ = \left( \frac{\alpha_s c_F}{4\pi} \right) \left( \frac{\pi^2 f_B \pi m_B}{N_c E^2} \right) \int_{u_c}^{\bar{u}_c} du \int_0^\infty dk^+ \left\{ \frac{4E - m_b}{m_b} \frac{\phi_+(u) \phi_+^B(k^+)}{\bar{u} k^+} + \frac{1}{\bar{u}^2 k^+} \phi_+^B(k^+) \right\} + \frac{\mu_\pi}{2E} \left\{ \frac{1}{\bar{u}^2 k^+} \left( \phi_\rho(u) - \frac{\phi_\rho'(u)}{6} \right) + \frac{4E}{\bar{u}(k^+)} \phi_\rho(u) \right\} \phi_+^B(k^+) \],

(11)

where \( N_c = 3, \mu_\pi = m_\pi^2/(m_u + m_d) \), and \( \phi_{\rho,\sigma}(u) \) are the higher twist DAs. Note that the integration domain is changed into \( \int_0^1 du \to \int_{u_c}^{\bar{u}_c} du \) to avoid the soft overlap region. The integral over \( k^+ \) will cause another divergence at \( k^+ = 0 \). We simply introduce an IR cutoff \( \bar{\Lambda} = m_B - m_b \) for \( B \) meson sector. To get the numerical estimate, we use the asymptotic form of \( \phi_\sigma \), and

\[ \phi_\sigma^B(\omega) = \frac{\omega}{\omega_0} e^{-\omega/\omega_0}, \quad \phi_\sigma^B(\omega) = \frac{1}{\omega_0} e^{-\omega/\omega_0} , \]  

(12)

where \( \omega_0 \) is a model parameter \[19\]. In Table I some values of \( f_{+,T} \) are given for different \( u_c \). The value of \( f_+ = 0.212 \) at \( u_c = \pi/m_B \) is very close to the other approaches, e.g. \( f_+ = 0.258 \pm 0.031 \) from LCSR \[14\], or \( f_+ = 0.23 \pm 0.02 \) \[20\], \( f_+ = 0.251 \pm 0.015 \) \[21\] from the lattice calculations.

In conclusion, we propose a fundamental cutoff for the heavy-to-light transitions due to possible Cherenkov gluon radiation. It naturally avoids the end-point singularity and double counting problems. In this picture the soft overlap contribution is highly suppressed; heavy-to-light decay is dominated by the hard scattering processes. Though the numerical values of the weak form factors are very sensitive to the choice of the cutoff, for a natural scale of

| \( u_c \) | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 | 0.1 |
| \( f_B \) | 0.515 | 0.273 | 0.190 | 0.148 | 0.123 | 0.105 | 0.093 | 0.083 | 0.075 | 0.069 |
| \( f_T \) | 0.495 | 0.247 | 0.163 | 0.21 | 0.095 | 0.078 | 0.065 | 0.056 | 0.049 | 0.043 |

TABLE I: Some values of \( f_{+,T} \) for different \( u_c \) at \( q^2 = 0 \). \( \omega_0 \) is chosen to be \( 2\bar{\Lambda}/3, \) and \( \mu = \sqrt{m_B\Lambda_{QCD}} \approx 1.47 \) GeV is taken for \( \alpha_s \).
$u_c \simeq m_\pi/m_B$ one gets a compatible result with other approaches. Current framework can be applied straightforwardly to the $B$ to light vector meson decays.

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