QCD sum rules for $\rho$ mesons in nuclear matter

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We investigate QCD sum rules for vector currents in the rho meson channel in the nuclear medium. For increased sensitivity, we subtract out the vacuum contributions. With a saturation scheme often considered in the literature, we find these “subtracted” sum rules to be unstable. It indicates the importance of other contributions neglected in the saturation scheme. These include more Landau singularities at low energy, additional operators in the medium, and possibly the in-medium width of the rho meson.

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I. INTRODUCTION

It was quite some time ago that the original vacuum QCD sum rules [1] were first extended to a medium at finite temperature and density [2,3]. Since then much work has been devoted to extracting results from them. Typically one tries to predict the medium dependence of the parameters of particles and resonances in the low energy region. However, these in-medium sum rules have several new features. These must be included quantitatively in the saturation scheme for reliable predictions.

One such feature, recognized soon after their initial formulation, is that the in-medium sum rules have contributions from new operators over those already present in the vacuum sum rules [4,5]. The velocity four-vector of the medium allows one to construct additional Lorentz scalar operators. (In the rest frame of the medium they are simply the time components of Lorentz tensor operators.) At higher dimensions there arises a multitude of such operators.

The other features relate to the hadron spectral function. A communicating single particle state may acquire a large width (in addition to a possible shift in mass) due to its scattering from the particles in the medium, so that it may not be possible to treat it in the narrow width approximation as in the vacuum. Also multiparticle states in the medium may give rise to several nearby singularities of the Landau type.

In this work we reexamine the sum rules in the $\rho$-meson channel in the nuclear medium at zero temperature. Our saturation scheme is similar to one mostly employed in the literature, which includes only some of these features [6–8]. In the operator product expansion we include properly all the operators up to dimension 4 and only the four-quark operators at dimension 6. In the spectral function we take the $\rho$ meson in the narrow width approximation and the $N\bar{N}$ state.

A complete or “full” in-medium sum rule is actually the corresponding vacuum sum rule, perturbed by small, density dependent terms. Such a “full” sum rule, when evaluated, is guaranteed to be stable, more or less, irrespective of these small contributions, simply because the vacuum sum rule is so. To get rid of this insensitivity, we subtract out the vacuum sum rule from the “full” one, retaining on both sides only the terms to first order in the density.

Though well-known, these sum rules are derived here to bring out some technical points. We start with a broader framework, namely the ensemble average of the correlation function of currents at finite temperature and chemical potential. The special cases of sum rules at finite temperature and at finite density may then be recovered from the general formulation by setting respectively the chemical potential and the temperature to zero. We derive the mixing of two operators of dimension 4, namely the quark and the gluon parts of the energy momentum tensor, a result known in the context of deep inelastic scattering. We also present a field theoretic derivation for the $N\bar{N}$ contribution to the spectral function.

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1 Actually there could be further density independent contributions in the continuum, but they are eliminated in the subtracted sum rules we shall be considering.
On evaluation we find that the sum rules as saturated above do not yield stable results, as we have already reported recently [1]. This instability points to the importance of other contributions we did not include, namely, the Lorentz nonscalar operators at dimension 6 and the Landau singularities besides the one from the $NN$ state. The neglect of the (large) in-medium width of the $\rho$-meson also possibly adds to this instability [10,11].

In Sec. [II] we introduce the kinematic decomposition of the two point function of vector currents. In Sec. [II] a simple derivation of the operator product expansion is presented in coordinate space. Then in Sec. [IV] we construct the spectral representation from $\rho$ and $NN$ intermediate states at finite density. The sum rules are written and evaluated in Sec. [V]. We conclude in Sec. [VI].

II. KINEMATICS

Consider the ensemble average of the time ordered $(T)$ product of two currents

$$T_{\mu\nu}^{ab}(q) = i \int d^4x e^{i\tau q \cdot x} \langle T \{ V_\mu^a(x) V_\nu^b(0) \} \rangle .$$

Here $V_\mu^a(x)$ is the vector current (in the $\rho$-meson channel) in QCD,

$$V_\mu^a(x) = \bar{q}(x)\gamma_\mu \tau^a q(x),$$

$q(x)$ being the field of the $u$ and $d$ quark doublet and $\tau^a$ the Pauli matrices. The ensemble average of an operator $O$ is denoted by $\langle O \rangle$,

$$\langle O \rangle = \text{Tr} \left[ e^{-\beta(H-\mu N)} O \right] / \text{Tr} \left[ e^{-\beta(H-\mu N)} \right] ,$$

where $H$ is the QCD Hamiltonian, $\beta$ is the inverse of the temperature $T$, and Tr denotes the trace over any complete set of states. Keeping our application in mind, we have introduced the nucleon chemical potential $\mu$ with the corresponding number operator $N$.

For explicit calculations one generally chooses the rest frame of the medium, where the temperature is defined. This breakdown of Lorentz invariance may be restored if we let the medium have an arbitrary four-velocity $u_\mu$ [4]. [In the matter rest frame $u_\mu = (1,0,0,0)$.] The time and space components of $q_\mu$ are then raised to the Lorentz scalars, $w = u \cdot q$ and $\bar{q} = \sqrt{w^2 - q^2}$.

There are two independent, conserved kinematic covariants [13], in which the two point function from Eq. (2.1) may be decomposed. In choosing their forms we must note that the dynamical singularities extend up to $q^2 = 0$. We thus have to ensure that the kinematic covariants do not have any singularity in this region, particularly at $q^2 = 0$. The covariants $P_{\mu\nu}$ and $Q_{\mu\nu}$, defined by

$$P_{\mu\nu} = -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} - \frac{q^2}{q^2} \bar{u}_\mu \bar{u}_\nu, \quad Q_{\mu\nu} = \frac{q^4}{q^2} \bar{u}_\mu \bar{u}_\nu,$$

where $\bar{u}_\mu = u_\mu - w q_\mu / q^2$, are indeed so, as can be seen by inspecting their components for space and time indices. The invariant decomposition of $T_{\mu\nu}^{ab}$ is then given by

$$T_{\mu\nu}^{ab}(q) = \delta^{ab}(Q_{\mu\nu} T + P_{\mu\nu} T_t),$$

where the invariant amplitudes $T_{1,4}$ are functions of $q^2$ and $w$.

The kinematic covariants are still not regular: they have a dependence on the direction of the three-vector $\bar{q}$ even as $|\bar{q}| \to 0$. Thus the spatial components of $T_{\mu\nu}^{ab}$ in this limit become

$$T_{ij}^{ab}(q) = \delta^{ab} \left[ \delta_{ij} T_t - n_i n_j (T_t - q_0^2 T_t) \right],$$

where $n_i$ is the $i$th component of the unit vector along $\bar{q}$. To eliminate this direction dependence, we must require that [2]

$$T_t(q_0, |\bar{q}| = 0) = q_0^2 T_t(q_0, |\bar{q}| = 0).$$
III. OPERATOR PRODUCT EXPANSION

Here we outline the derivation of the short distance expansion of the product of two current operators in Eq. (2.1). As already stated, the availability of the velocity four-vector \( u^\mu \) of the medium allows one to construct new, independent Lorentz scalar operators. Thus up to dimension 4 we have, in addition to the old ones \( 1, \bar{q}q \equiv \bar{u}u + \bar{d}d \), \( G^2 \equiv (\alpha_s/\pi)G_{\mu\nu}G^{\mu\nu} \), appearing in the vacuum expectation value, two new ones \( \Theta^\mu \equiv u^\mu \Theta^\mu_{\nu\nu} u^\nu \) and \( \Theta^q \equiv u^\mu \Theta^q_{\mu\nu} u^\nu \) for the ensemble average. Here \( G_{\mu\nu} \), \( c = 1, \ldots, 8 \) are the gluon field strengths and \( \alpha_s = g_s^2/4\pi \), \( g_s \) being the QCD coupling constant. \( \Theta^\mu_{\nu\nu} \) are the quark and the gluon parts of the traceless energy momentum tensor

\[
\Theta^\mu_{\nu\nu} = \bar{q}i\gamma_\mu D_\nu q - \frac{\hat{m}}{4} g_{\mu\nu}\bar{q}q, \\
\Theta^q_{\mu\nu} = -G_{\mu\alpha}^c G_{\nu}^{c\lambda} + \frac{1}{4} g_{\mu\nu} G_{\alpha\beta}^c G^{\alpha\beta c}, \tag{3.1}
\]

where \( \hat{m} \) is the quark mass in the limit of SU(2) symmetry.

We now obtain the (singular) coefficients of the operators in coordinate space \([4, 13]\). In the free field theory these are obtained from the Wick decomposition of the operator product. If we treat the gauge fields as external, the same form of decomposition also holds for the interacting theory

\[
T \{ V^a_\mu(x)V^a_\nu(0) \} = \text{tr}[\gamma_\mu S(x,0)\gamma_\nu S(0,x)\tau^a \tau^b /4] \\
- \bar{i}\bar{q}(x)\gamma_\mu S(x,0)\gamma_\nu (\tau^a \tau^b /4)q(x) - \bar{i}\bar{q}(x)\gamma_\nu S(x,0)\gamma_\mu (\tau^a \tau^b /4)q(x) \\
+ \text{a regular term.} \tag{3.2}
\]

The trace in the first term is over all the indices, namely spin, flavor and color of the quark field. The quark propagator \( S(x,0) \) is now in the presence of the (matrix valued) gauge potential \( A_\mu(x) \),

\[
\{ -i\gamma^\mu[\partial_\mu - igA_\mu(x)] + \hat{m} \} S(x,0) = \delta^4(x). \tag{3.3}
\]

An important technical point here is that quantities can be put in a gauge covariant form from the beginning, if we use the Fock-Schwinger gauge, in which, \( x^\mu A_\mu(x) = 0 \). In this gauge

\[
A_\mu(x) = (1/2)x^\alpha G_{\alpha\mu}(0) + \cdots , \\
q(x) = q(0) + x^\mu D_\mu q(0) + \cdots . \tag{3.4}
\]

Equation (3.3) may be solved for the quark propagator in a series of increasing number of gluon field strengths.

The unit and the quark operators \( 1, \bar{q}q \), and \( \Theta^q \) reside in the first and the next two terms in Eq. (1.2), respectively. To calculate their coefficients, it suffices to use the free quark propagator

\[
S_0(x,0) = \frac{1}{4\pi^2} \left( -\frac{2}\eta \frac{2}{x^2 - i\epsilon} \right) - \frac{i\hat{m}}{x^2 - i\epsilon}, \tag{3.5}
\]

where we retain only the leading singular term in \( \hat{m} \). The quark operators in Eq. (3.2) can then be projected on to the scalar operators by using the formulæ

\[
\langle \bar{q}^j_A q^k_B \rangle_A = \frac{1}{8} \delta^{jk}(\delta_{BA}\langle \bar{q}q \rangle + \langle q \rangle_{BA}\langle \bar{q}q \rangle), \tag{3.6}
\]

\[
\langle \bar{q}^j_A (D_\mu q)_B^k \rangle = \frac{1}{2} \gamma^\mu \left( \frac{1}{16} \delta_{BA} \hat{m} \langle \bar{q}q \rangle + \langle \gamma_{BA} \rangle \right) + \frac{1}{12} \langle \Theta^q \rangle + u_\mu \langle \bar{q} \rangle_{BA} \frac{1}{3} \langle \Theta^q \rangle. \tag{3.7}
\]

The flavor and spinor indices of the quark field have been denoted by \( (j,k) \) and \( (A,B) \), respectively. A sum over color indices on both sides is understood. These projection formulæ are easily obtained by expanding the quark bilinear matrix in terms of the complete set of Dirac matrices, noting parity conservation, and using the free equation of motion for the quark field. (Chirality forbids \( \langle \bar{q}q \rangle \) to appear in the vector-vector correlation function.)

The two-gluon operators \( G^2 \) and \( \Theta^q \) arise only from the first term in Eq. (3.2). To get their coefficients we have to work out the quark propagator up to two gluons. The general fourth rank tensor with two gluon fields may be projected on to the scalar operators by
\[
\langle \text{tr}^c G_{\alpha\beta} G_{\lambda\sigma} \rangle = \frac{1}{6} (g_{\alpha\lambda} g_{\beta\sigma} - g_{\alpha\sigma} g_{\beta\lambda}) \left( \frac{1}{4} (G_{\mu\nu} G^{\mu\nu}) + \langle \Theta^g \rangle \right) \\
- \frac{1}{3} (u_\alpha u_\lambda g_{\beta\sigma} - u_\alpha u_\sigma g_{\beta\lambda} - u_\beta u_\lambda g_{\alpha\sigma} + u_\beta u_\sigma g_{\alpha\lambda}) \langle \Theta^g \rangle ,
\] (3.8)

\text{tr}^c \text{ indicating trace over the color matrices. It turns out that if we insert this expression into the two gluon terms present in } S, \text{ they do not contribute to } G^2. \text{ Thus the calculation of its coefficient reduces to inserting } S \text{ up to the one-gluon term}

\[ S(x,0) = S_0(x,0) - \frac{ig_{\epsilon\mu} \epsilon\nu G_{\mu\nu}}{16\pi^2(x^2 - i\epsilon)}, \] (3.9)
in the first term of Eq. (3.2) and then using Eq. (3.8). However the two-gluon terms in } S \text{ do contribute to } \Theta^g. \text{ But it is not necessary to calculate this coefficient. As we show below, the renormalization group improvement automatically brings it in.}

Having indicated the procedure to calculate the coefficients in coordinate space, we take the Fourier transform and write the result in momentum space \((Q^2 \equiv -q^2),\)

\[
i \int d^4 x e^{ix \cdot q} T \{ V^a_\mu(x)V_\nu^b(0) \} = \delta^{ab} \left[ Q_{\mu\nu} \left\{ -\frac{1}{8\pi^2} \ln(Q^2/\mu^2)1 + \frac{1}{Q^4} \left( \frac{1}{2} \tilde{m}q + \frac{1}{24} G^2 + \frac{2}{3} \Theta^g \right) \right\} \\
+ P_{\mu\nu} \left[ \frac{Q^2}{8\pi^2} \ln(Q^2/\mu^2)1 - \frac{1}{Q^4} \left( \frac{1}{2} \tilde{m}q + \frac{1}{24} G^2 + \frac{2}{3} \left( 1 - \frac{2g^2}{Q^2} \right) \Theta^g \right) \right] \right].
\] (3.10)

Here \(\mu\) is the renormalization scale, which we take to be 1 GeV as the natural scale to normalize the operators. But this scale is not convenient for the coefficients. These, calculated above only to the lowest order in \(\alpha_s\), may contain powers of \(\ln(Q^2/\mu^2)\) in higher orders. To get rid of these large logarithms, \(^2\) we must shift the renormalization point for the coefficients from \(\mu^2\) to \(Q^2\).

Now the left hand side of Eq. (3.10) is independent of the renormalization scale. So are all the operators \(1, \tilde{m}q, G^2\) and hence their coefficients. But it is not so for \(\Theta^g\) and \(\Theta^f\). Indeed, it is well-known in the context of deep inelastic scattering that \(\Theta^g\) and \(\Theta^f\) mix under a change of scale \(^3\). Let us denote the contribution of these operators generally as

\[ C_q \mid \mu \Theta^q \mid \mu + C_g \mid \mu \Theta^g \mid \mu = \sum_i C_i \mid \mu \Theta^i \mid \mu , \] (3.11)

where only the dependence on the scale \(\mu\) is shown explicitly and the index \(i\) runs over \(q\) and \(g\). The coefficients \(C_i\) satisfy a coupled set of renormalization group equations having the solution

\[ C_i \mid \mu = C_j \mid Q \left( e^{-\int_\mu^Q \gamma(t') dt'} \right)_{ji} , \] (3.12)

where \(t = \frac{1}{2} \ln(Q^2/\mu^2)\) and \(\gamma\) is the \(2 \otimes 2\) anomalous dimension matrix. In the basis we are using it is

\[ \gamma = -\frac{g^2}{(4\pi)^2} \left( \begin{array}{cc} \frac{64}{3} & \frac{4}{3} \nu_f \\ -\frac{4}{3} \nu_f & -\frac{4}{3} \nu_f \end{array} \right) , \] (3.13)

where \(\nu_f(=2)\) is the number of flavors. The exponentiation in Eq. (3.12) can be easily worked out by a spectral decomposition of the matrix \(\gamma = \sum \lambda_i P_i\), where \(\lambda_i\) and \(P_i\) are the eigenvalues and the corresponding projection operators \(^{19}\). The eigenvalues are

\[ \lambda_1 = 0, \quad \lambda_2 = \frac{g^2}{(4\pi)^2} \frac{4}{3} \left( \frac{16}{3} + \frac{4}{3} \nu_f \right) , \] (3.14)

\(^2\)The logarithm in the coefficient of the unit operator in Eq. (3.10) is of different origin. See the first footnote in Sec. 4.6 of Ref. \(^{18}\).

\(^3\)If we had calculated the coefficient of \(\Theta^g\), we would have found it to be proportional to \(\alpha_s \ln(Q^2/\mu^2)\). It signals mixing with other operators.
For the vacuum amplitudes, these are the Källen-Lehmann representations in \( \vec{q} \) decomposition in \( A \). At finite temperature and chemical potential, it is convenient to use the Landau representation, which is a spectral approximation of ground state saturation, its contribution reduces to the familiar expression involving \( \Theta \)'s.

\[
\sum_i C_i \left| \mu \Theta_i \right|_\mu = \frac{1}{(\frac{16}{3} + n_f)} \left\{ \left( n_f C_q + \frac{16}{3} C_g \right) \Theta + \lambda(Q^2)(C_q - C_g) \mid_\mu \left( \frac{16}{3} \Theta^g - n_f \Theta^g \right) \right\},
\]

where \( \Theta = \Theta^g + \Theta^q \) and the anomalous dimension of the second operator resides in \( \lambda(Q^2) \),

\[
\lambda(Q^2) = \left( \frac{\alpha_s(\mu^2)}{\alpha_s(Q^2)} \right)^{-d/2b}, \quad d = \frac{4}{3} \left( \frac{16}{3} + n_f \right), \quad b = 11 - \frac{2}{3} n_f.
\]

We will use \( \alpha_s(\mu^2 = 1 \text{ GeV}^2) = 0.4 \) in our numerical evaluations below.

Once the renormalization group improvement is made, we can replace the coefficient functions by their lowest order values. These may be read off from Eq. (3.10) for the amplitude \( T_1 \), for example, as \( C_f \mid_{Q^2} = 2/(3Q^2) \), \( C_g \mid_{Q^2} = 0 \). Thus the renormalization group improved result for \( T_1 \) reads (with \( n_f = 2 \))

\[
T_1 = -\frac{1}{8\pi^2} \ln(Q^2/\mu^2) + \frac{1}{Q^4} \left\{ \frac{1}{2} \hat{m}(\bar{q}q) + \frac{1}{24} \left( \langle \Theta \rangle + \lambda(Q^2) \langle \frac{8}{3} \Theta^g - \Theta^g \rangle \right) \right\}.
\]

Notice that if \( \lambda(Q^2) \) is set equal to unity, it collapses to the previous result in Eq. (3.10). The amplitude \( T_1 \) is also given by the same expression, except for an overall factor of \( -Q^2 \) and a factor of \( (1 - 2q^2/Q^2) \) multiplying the term with \( \Theta \)'s.

Let us summarize the effect of mixing of \( \Theta^q \) and \( \Theta^g \) under the renormalization group. It introduces the multiplicative factor, \( \lambda(Q^2) \), which is nonperturbative in the sense that it is a sum over all the leading logarithms in the perturbation expansion. It is true that the coefficient of \( \Theta^q \) is smaller than \( \Theta^g \) by a factor of \( g_s^2 \). We have left out this smaller coefficient, as did the earlier authors, but retained the nonperturbative factor. Numerically the factor is significant for \( Q^2 \) away from \( \mu^2 = 1 \text{ GeV}^2 \).

At dimension 6 we include only the Lorentz scalar operators in vacuum, namely the four-quark operators \([1]\). In the approximation of ground state saturation, its contribution reduces to the familiar expression involving \( \langle \bar{q}q \rangle^{-2} \).

### IV. SPECTRAL REPRESENTATION

QCD sum rules make contact with the physical states through the spectral representation of the two point function. For the vacuum amplitudes, these are the Källen-Lehmann representations in \( q^2 \), the only variable available there. At finite temperature and chemical potential, it is convenient to use the Landau representation, which is a spectral decomposition in \( q_0 \) at fixed \( |q| \) \([2]\). For the invariant amplitudes they are

\[
T_{l,s}(q_0, |q|) = \frac{1}{\pi} \int^{-\infty}_{\infty} dq_0' \frac{\text{Im} T_{l,s}(q_0', |q|)}{q_0' - q_0 - i\epsilon} \tanh(\beta q_0'/2),
\]

up to subtraction terms. The imaginary parts from different intermediate states can be written as phase space integrals over the matrix elements of currents. But we shall not write these expressions directly. Instead, we first calculate the full amplitudes with these intermediate states in the real time formulation of field theory at finite temperature and density \([21]\) and then read off their imaginary parts. Since we have to calculate the amplitudes at the tree level, only the 11-component of the propagators will be needed.

We first consider the \( \rho \)-meson pole, see Fig. \( \text{a(a)} \), which would dominate the absorptive part of the vacuum correlation function.

![Fig. 1](image)

**FIG. 1.** Contributions to the spectral function from (a) the \( \rho \) meson and (b) the \( NN \) state.
In the vacuum, its coupling to the current $V^a_{\mu}(x)$ is given by
\begin{equation}
\langle 0|V^a_{\mu}(0)|\rho^b \rangle = \delta^{ab}F_\rho m_\rho \epsilon_\mu ,
\end{equation}
where $\epsilon_\mu$ and $m_\rho$ are the polarization vector and the mass of the $\rho$. Experimentally $F_\rho = 153.5$ MeV. This matrix element suggests that, as far as the $\rho$-meson contribution is concerned, we may write the operator relation
\begin{equation}
V^a_{\mu}(x) = m_\rho F_\rho \rho^a_\mu (x). \tag{4.3}
\end{equation}
In the medium such a relation is generally modified in two ways. First, the parameters $m_\rho$ and $F_\rho$ must be replaced by their effective values in the medium, to be denoted below by a star. Secondly, the space and the time components have different constants of proportionality. However, the latter modification takes place in higher orders in density and temperature \cite{22}. As we work to the lowest order, the operator relation (4.3) changes in the medium simply to
\begin{equation}
V^a_{\mu}(x) = m_\rho^* F_\rho^* \rho^a_\mu (x). \tag{4.4}
\end{equation}
Then the $\rho$-meson contribution to the correlation function is given essentially by its propagator in the medium. As before, we get
\begin{equation}
\left( \frac{\text{Im} T_i(q_0, |q|^2)}{\text{Im} T_i(q_0, |q|^2)} \right) = \left( \frac{1}{m_\rho^*} \right) \pi F_\rho^* \delta \left( q_0^2 - (|q|^2 + m_\rho^* \gamma^2) \right) \coth(\beta q_0/2). \tag{4.5}
\end{equation}
In the medium there are other equally important contributions. These are from the $2\pi$ and, if the medium is nuclear, also the $N\bar{N}$ intermediate state, see Fig. 4(b). As we restrict this work to sum rules at finite density but at zero temperature, the $2\pi$ contribution would be small and close to its vacuum value. The $N\bar{N}$ contribution in vacuum would also be small, beginning with a cut at the threshold $q^2_0 = 4m_N^2 + |q|^2$. However, in the nuclear medium the current interacts with the real nucleons to give rise to a short cut around the origin, $-|q| < q_0 < +|q|$. Below we evaluate this contribution.

The contribution of the nucleon isodoublet, spinor field $N_A^i(x)$, $i = 1, 2$, $A = 1, \ldots, 4$, to the vector current is
\begin{equation}
V^a_{\mu}(x) = \bar{N}(x) \gamma_\mu \frac{\tau^a}{2} N(x). \tag{4.6}
\end{equation}
The $N\bar{N}$ contribution to the two point function is then
\begin{equation}
T_{\mu\nu}^{ab}(q) = \frac{i}{2} \delta^{ab} \int d^4xe^{iqx} \text{tr}[\gamma_\mu S(x)\gamma_\nu S(-x)], \tag{4.7}
\end{equation}
where the trace is over the gamma matrices and $iS(x)$ is the 11-component of the nucleon propagator in the medium
\begin{equation}
iS(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-i(k-x)y} (\not{k} + m_N) \left\{ \frac{i}{k^2 - m_N^2 + i\epsilon} - 2\pi \delta(k^2 - m_N^2) \{ n^{-}\theta(k_0) + n^{+}\theta(-k_0) \} \right\}, \tag{4.8}
\end{equation}
the distribution functions $n^{\mp}$ for the nucleon and the anti-nucleon being given by $n^{\mp}(\omega) = (e^{\beta(\omega^{-}\mu}) + 1)^{-1}$ with $\omega = \sqrt{k^2 + m_N^2}$.

Our aim is to integrate out the time component of the loop momentum, when the imaginary part of $T_{\mu\nu}^{ab}(q)$ can be read off from the resulting expression. However, the above expression for the propagator is not convenient for this purpose because of the presence of $\theta(\pm k_0)$. So we carry out the $k_0$ integration in the propagator itself, getting
\begin{align}
iS(x) &= \int \frac{d^3k}{(2\pi)^3 2k_0} \left[ \theta(x^0) \left\{ (\not{k} + m_N)(1-n^-)e^{-ikx} - (-\not{k} + m_N)n^+e^{ikx} \right\} \\
&\hspace{1cm} + \theta(-x^0) \left\{ -(\not{k} + m_N)n^-e^{-ikx} + (-\not{k} + m_N)(1-n^+)e^{ikx} \right\} \right]. \tag{4.9}
\end{align}
Here the time component of $k_\mu$ is understood to be given by $\omega$. We can now integrate over $x^\mu$ in Eq. (4.7) to get $\delta$-functions in three-momentum and energy denominators in the time components. After some algebra we get for the imaginary part.
\[ \text{Im} T_{\mu\nu}^{ab}(q) = -\frac{\pi}{2} \delta^{ab} \coth(\beta q_0 / 2) \{ I_{\mu\nu}(q) + I_{\mu\nu}(-q) \}, \tag{4.10} \]

where

\[ I_{\mu\nu}(q) = \int \frac{d^3k}{(2\pi)^3 4\omega_1 \omega_2} \left[ (1 - n_1^- - n_2^+) \delta(q_0 - \omega_1 - \omega_2) - (n_2^- - n_1^+) \delta(q_0 - \omega_1 + \omega_2) \right], \tag{4.11} \]

with

\[ \Lambda_{\mu\nu} \equiv \text{tr} \{ \gamma_{\mu} (\bar{k} + m_N) \gamma_{\nu} (\bar{k} - \bar{q} + m_N) \}, \]
\[ = 4 [2k_{\mu} k_{\nu} - (k_{\mu} q_{\nu} + k_{\nu} q_{\mu}) - g_{\mu\nu} (k \cdot (k - q) - m_N^2)]. \tag{4.12} \]

Here the subscripts on \( n^\pm \) serve to indicate that their arguments are \( \omega_1 = \sqrt{k^2 + m_N^2} \) and \( \omega_2 = \sqrt{(\bar{k} - \bar{q})^2 + m_N^2} \), respectively. The first and the second terms in Eq. (4.11) are nonvanishing in the timelike \((q^2 > 4m_N^2)\) and spacelike \((q^2 < 0)\) regions, respectively. Working out the angular integration, it becomes

\[ I_{\mu\nu}(q) = \frac{1}{4|q|} \int_{\omega_-}^{\omega_+} \frac{d\omega_1}{(2\pi)^2} \Lambda_{\mu\nu} (1 - n_1^- - n_2^+) \theta(q^2 - 4m_N^2) \]
\[ - \frac{1}{4|q|} \int_{\omega_+}^{\infty} \frac{d\omega_1}{(2\pi)^2} \Lambda_{\mu\nu} (n_2^- - n_1^-) \theta(-q^2), \tag{4.13} \]

where the integration limits are given by \( \omega_\pm = |q_0 \pm |q| v(q^2)|/2, \) with \( v(q^2) = \sqrt{1 - 4m_N^2/q^2}. \)

To project the tensor \( I_{\mu\nu} \) on the imaginary parts of the invariant amplitudes it is convenient to form the scalars \( I_\mu^\alpha \) and \( u^\mu I_{\mu\nu} u^\nu \) and relate them to the invariant amplitudes. Changing the variable \( \omega_1 \) to \( x \) given by \( \omega_1 = \frac{1}{2}(q_0 + |q| x), \) we finally get for the contribution from the \( N\bar{N} \) state

\[ \begin{pmatrix} \text{Im} T_\mu(q_0, |q|^2) \\ \text{Im} T_\mu(q_0, |q|^2) \end{pmatrix} = -\frac{\pi}{2} \coth(\beta q_0 / 2) \left[ -\frac{v(3 - v^2)}{24\pi^2} \left( \frac{1}{q^2} \right) + \left( \frac{I_+^\mu}{I_+^\mu} \right) \right], \quad \text{for } q^2 > 4m_N^2, \tag{4.14} \]

with

\[ \begin{pmatrix} I_+^\mu \\ I_+^\mu \end{pmatrix} = -\frac{1}{32\pi^2} \int_{-\infty}^{\infty} dx \left( -q^2(2 - v^2 + x^2) \right) \{ n(3 + |q| x / 2 - \mu) - n((q_0 - |q| x / 2 + \mu) \}, \tag{4.15} \]

and

\[ \begin{pmatrix} \text{Im} T_\mu(q_0, |q|^2) \\ \text{Im} T_\mu(q_0, |q|^2) \end{pmatrix} = -\frac{\pi}{2} \coth(\beta q_0 / 2) \left( \frac{I_-^\mu}{I_-^\mu} \right), \quad \text{for } q^2 < 0, \tag{4.16} \]

with

\[ \begin{pmatrix} I_-^\mu \\ I_-^\mu \end{pmatrix} = -\frac{1}{32\pi^2} \int_{-\infty}^{\infty} dx \left( -q^2(2 - v^2 + x^2) \right) \{ n(3 + |q| x / 2 + \mu) - n((q_0 + |q| x / 2 - \mu) \}. \tag{4.17} \]

The superscripts (±) on \( I_\pm \) denote timelike and spacelike \( q_\mu \), respectively. The distribution function \( n(\varepsilon) \) appearing in Eqs. (4.15) and (4.17) is defined by \( n(\varepsilon) = (e^{\beta \varepsilon} + 1)^{-1} \).

Still higher continuum contributions, at least their leading parts, are density independent, as is shown by the coefficient of the unit operator. They will drop out in our subtraction process.

**V. SUM RULES**

We now go to the spacelike region in \( q_\mu, Q_0^2 = -q_0^2 > 0 \) at fixed \(|\bar{q}|\) and take the Borel transform of both the spectral representation and the operator product expansion with respect to \( Q_0^2 \). The sum rules are obtained by equating these transforms at sufficiently high \( M^2 \). For \( T_\mu \) it is
respectively. We thus finally arrive at the following sum rules at finite nuclear density:

\[
F_p^2 e^{-m_p^2/M^2} + \frac{1}{48\pi^2} \int_{4m_N^2}^{\infty} dq_0^2 e^{-q_0^2/M^2} \nu_0(q_0^2 + |\vec{q}|^2)
\]

\[+ e^{(\vec{q})^2/M^2} \left(\int_{4m_N^2+|\vec{q}|^2}^{\infty} dq_0^2 e^{-q_0^2/M^2} I_1^+(q_0, |\vec{q}|) + \int_0^{|\vec{q}|^2} dq_0^2 e^{-q_0^2/M^2} I_1^-(q_0, |\vec{q}|)\right)\]

\[= M^2 \frac{8\pi^2}{2M^4} + \frac{(O)}{M^2} - \frac{(O')}{2M^4},\]

(5.1)

where \((O)\) and \((O')\) represent the contributions of all dimension 4 operators and of dimension 6, scalar operators in vacuum, respectively.

\[\langle O \rangle = \frac{1}{2} \bar{m}(\bar{q}q) + \frac{G^2}{24} + \frac{2}{11} \left\{(\Theta) + \lambda(M^2) \left(\frac{8}{3} \Theta^q - \Theta^q\right)\right\}, \quad \langle O' \rangle = \frac{7g^2}{81} (\bar{q}q)^2.\]

(5.2)

The amplitude \(T_t\) satisfies a similar sum rule.

Considerable simplification results if we take \(|\vec{q}| \to 0\). This limit may be taken inside the integrals in Eq. (5.1), except for the last one, where the integrand diverges, while the range of integration shrinks to zero. A careful evaluation gives a result resembling a pole at \(q^2 = 0\). Then the above sum rule for \(T_t\) becomes

\[F_p^2 e^{-m_p^2/M^2} + \frac{1}{48\pi^2} \int_{4m_N^2}^{\infty} dq_0^2 \nu_0(3 - v_0^2) \left[n(q_0/2 - \mu) + e^{-q_0^2/M^2} \left\{1 - n(q_0/2 - \mu) - n(q_0/2 + \mu)\right\}\right]\]

\[= M^2 \frac{8\pi^2}{2M^4} + \frac{(O)}{M^2} - \frac{(O')}{2M^4},\]

(5.3)

while the sum rule for \(T_t\) in this limit may similarly be found to be

\[m_p^2 F_p^2 e^{-m_p^2/M^2} + \frac{1}{24\pi^2} \int_{4m_N^2}^{\infty} dq_0^2 \nu_0(3 - v_0^2) e^{-q_0^2/M^2} \left\{1 - n(q_0/2 - \mu) - n(q_0/2 + \mu)\right\}\]

\[= M^4 \frac{8\pi^2}{2M^4} - \langle O \rangle + \frac{(O')}{M^2}.\]

(5.4)

where \(v_0 = (1 - 4m_N^2/\bar{q}^2)^{1/2}\). Note that the contribution of the short cut is nonvanishing in this limit only in \(T_t\).

As they stand, the above sum rules cannot be well saturated, since the \(2\pi\) and the continuum contributions on the spectral side are missing. However, if we subtract the corresponding vacuum sum rules, the continuum contribution will practically drop out. We now go to the limit of zero temperature, when the \(2\pi\) contribution reduces to zero. With \(\mu > 0\), the nucleon and the antinucleon distribution functions reduce in this limit to a \(\theta\)-function and zero, respectively. We thus finally arrive at the following sum rules at finite nuclear density:

\[
\bar{m}^2 F_p^2 e^{-m_p^2/M^2} + \frac{1}{24\pi^2} \int_{4m_N^2}^{\infty} ds(1 - e^{-s/M^2}) \sqrt{1 - 4m_N^2/s}(1 + 2m_N^2/s) = \langle O \rangle - \frac{(O')}{2M^4},\]

(5.5)

\[
m_p^2 F_p^2 e^{-m_p^2/M^2} - \frac{1}{24\pi^2} \int_{4m_N^2}^{\infty} ds e^{-s/M^2} \sqrt{1 - 4m_N^2/s}(1 + 2m_N^2/s) = -\langle O \rangle + \langle O' \rangle,\]

(5.6)

where the bar over a quantity denotes subtraction of its vacuum value, e.g., \(\langle O \rangle = \langle O \rangle - \langle 0|O|0\rangle\).

The spectral integrals in the sum rules can be expanded in powers of the Fermi momentum \(p_F = \sqrt{\mu^2 - m_N^2}\), or the nucleon number density 4

\[
n = 4 \int \frac{d^3p}{(2\pi)^3} \theta(p_F - |\vec{p}|) = \frac{2p_F^3}{3\pi^2}.
\]

(5.7)

The operator matrix elements can also be expanded in powers of the nucleon number density. With the normalization of the one nucleon state \(\langle p, \alpha | p', \alpha' \rangle = (2\pi)^3 2p_0 \delta_{\alpha,\alpha'} \delta^3(\vec{p} - \vec{p}')\), \(\alpha, \alpha'\) denoting the spin and isospin degrees of freedom, we have the expansion for any operator \(\tilde{R}\) up to first order as

4The situation is similar to that at finite temperature. See also Ref. 24.
\[ \langle R \rangle = \langle 0 | R | 0 \rangle + \int \frac{d^3p}{(2\pi)^3 2p_0} \sum_\alpha \langle p, \alpha | R | p, \alpha \rangle \theta (p_F - |\vec{p}|). \]  

(5.8)

Denoting the constant, averaged matrix element by \( \langle p | R | p \rangle \), we get

\[ \langle R \rangle = \langle 0 | R | 0 \rangle + \frac{\langle p | R | p \rangle}{2m_N} \tilde{n}. \]  

(5.9)

Thus

\[ \langle \hat{m} \bar{q} q \rangle = \langle 0 | \hat{m} \bar{q} q | 0 \rangle + \sigma \tilde{n}, \]  

(5.10)

where the \( \sigma \) matrix element is defined by \( \sigma = \langle p | \hat{m} (\bar{u}u + \bar{d}d) | p \rangle / (2m_N) \simeq 45 \text{ MeV} \). The two-gluon operator is determined by the trace anomaly

\[ \Theta^\mu_\mu = - \frac{9}{8} \alpha_s \pi G_{\mu\nu} G^{\mu\nu} + \hat{m} \bar{q} q. \]  

(5.11)

We then get

\[ \langle G^2 \rangle = \langle 0 | G^2 | 0 \rangle - \frac{8}{9} (m_N - \sigma) \tilde{n}. \]  

(5.12)

To determine the quark and the gluon energy densities in the medium, we need their nucleon matrix elements. More generally one may write the traceless \( \Theta^{q,g}_{\mu\nu} \) as

\[ \langle p | \Theta^{q,g}_{\mu\nu} | p \rangle = 2 A^{q,g} \left( p_\mu p_\nu - \frac{1}{4} g_{\mu\nu} p^2 \right). \]  

(5.13)

The factor 2 ensures that the constants satisfy \( \sum_j A_j^q + A_j^g = 1 \). These constants individually may be obtained from fits with the data on deep inelastic scattering. Based on the parametrizations for the quark and the gluon distribution functions in Ref. [25], these are found at \( Q^2 = 1 \text{ GeV}^2 \) to be

\[ A^q = 0.62, \quad A^g = 0.35. \]  

(5.14)

Collecting the above results we get

\[ \overline{O} = C \tilde{n}, \quad C = \frac{\sigma}{2} - \frac{1}{27} (m_N - \sigma) + \frac{3}{22} m_N \left\{ A^q + A^g + \lambda (M^2) \left( \frac{8}{3} A^q - A^g \right) \right\}, \]  

(5.15)

\[ \overline{O'} = D \tilde{n}, \quad D = \frac{56}{81} \pi \alpha_s (\langle 0 | \bar{q} q | 0 \rangle \frac{\sigma}{m}), \]  

(5.16)

where the quark condensate in vacuum is given by \( \langle 0 | \bar{q} q | 0 \rangle / 2 = \langle 0 | \bar{u} u | 0 \rangle = -(225 \text{ MeV})^3 \). Furthermore, we will use \( \hat{m} = 7 \text{ MeV} \) below.

The sum rules now determine the nuclear density dependence of the parameters of the \( \rho \) meson. To first order in the number density, we may expand these as

\[ m_\rho^* = m_\rho \left( 1 + a \frac{\tilde{n}}{n_s} \right), \quad F_\rho^* = F_\rho \left( 1 + b \frac{\tilde{n}}{n_s} \right), \]  

(5.17)

where \( n_s \) is the saturation nucleon number density, \( n_s = (110 \text{ MeV})^3 \). From the sum rules [5,9] and [5,8] we then obtain the following expressions for the coefficients \( a \) and \( b \):

\[ a = - \frac{n_s}{2 F_\rho^2} e^{m_\rho^*/M^2} \left[ C \left( \frac{1}{m_\rho^2} + \frac{1}{M^2} \right) - \frac{D}{M^2} \left( \frac{1}{m_\rho^2} + \frac{1}{2M^2} \right) - \frac{1}{4m_N} \left( \frac{m_N}{m_\rho^2} - \frac{1}{4m_N} \right) e^{-4m_N^2/M^2} \right], \]  

(5.18)

\[ b = - \frac{n_s}{2 F_\rho^2} e^{m_\rho^*/M^2} \left[ C \frac{m_\rho^2}{M^4} - \frac{D}{2M^2} \left( 1 + \frac{m_\rho^2}{M^2} \right) - \frac{1}{4m_N} \left( \frac{m_N}{M} - \frac{1}{4m_N} \right) e^{-4m_N^2/M^2} \right]. \]  

(5.19)
The resolution of this contradiction lies in the fact that the “full” sum rules considered by these authors are dominated by vacuum contributions, the density dependent terms serving as small perturbations. As stated already in the Introduction, their stability follows essentially from that of the vacuum sum rules. The variation of some free parameters such as $s_0$, the beginning of the continuum, in these works would further contribute to this apparent stability.

It would appear from earlier works that QCD sum rules can accommodate different saturation schemes, each one giving rise to a new set of results [11]. By working with a more sensitive version of the sum rules, we show that the situation may not actually be so: a simple saturation scheme, such as here, is rejected and a more realistic one is called for.

Let us finally discuss the neglected contributions, whose inclusion should restore the stability of these sum rules. At dimension 6, five more operators would have contributed. These contributions can be evaluated using the data on deep inelastic scattering [26]. To get an idea of the convergence of the series of operators, we note that the contributions of a family of similar operators, differing only in the number of Lorentz indices, form a series in powers of $(m_N/M)$ with decreasing coefficients, given by the corresponding moments of the structure functions.

In the spectral function, it is not just the $N\bar{N}$ state, which can give rise to the Landau cut in the low $q^2$ region. Any other state $NN^*$, where $N^*$ is a resonance communicating with the $\rho N$ channel, can contribute a cut for $q^2 \leq (m_N - m_{N^*})^2$. The two resonances $\Delta(1232)$ and $N(1440)$ would presumably contribute significantly. To include these contributions we need, however, know the current-$NN^*$ couplings. Also the increased width of the $\rho$ meson in the medium is another source of altered contributions to the sum rules. We note that a study of such “subtracted” sum rules for $\rho$ mesons at finite temperature showed no such instabilities as mentioned above [27].

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