Closure of Constraints for Plane Gravity Waves

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Abstract

The metric for gravitational plane waves has very high symmetry (two spacelike commuting Killing vectors). For this high symmetry, a simple renormalization of the lapse function is found which allows the constraint algebra for canonical quantum gravity to close; also, the vector constraint has the correct form to generate spatial diffeomorphisms. A measure is constructed which respects the reality conditions, but does not yet respect the invariances of the theory.
1 Introduction

The connection-triad variables introduced by Ashtekar have simplified the constraint equations of quantum gravity; further, these variables suggest that in the future we may be able to reformulate gravity in terms of non-local holonomies rather than local field operators. However, the new variables are unfamiliar, and it is not always clear what they mean physically and geometrically. For example, it is not clear what operators or structures correspond to gravity waves. On the other hand, there has been great progress in constructing diffeomorphism-invariant volume and area operators; these operators turn out to be essentially counting operators for numbers of loops and loop intersections. The quantum constraint equations are much simpler in the new variables, and solutions to these equations have been found. However, these solutions correspond to some metric, and it is not always clear what that metric is (the problem of physical interpretation again). In order to investigate the metric, one needs a measure, so as to be able to form dot products and take expectation values. Some progress has been made in constructing a measure.

The new approach is not yet truly non-local, and it shares the same renormalization and regularization difficulties which plague other local field theories such as QCD. For gravity, there are some new twists to the old story. The theory is hard to regulate, because the regulators do not always respect diffeomorphism invariance. However, the theory is astoundingly easy to renormalize. (Compare to QCD, which is easy to regulate, whereas renormalization is difficult.)

Also, it is not always possible to operator-order the gravitational constraints so that both the constraint algebra closes (commutator of two constraints = sum of constraints) and the vector constraints generate spatial diffeomorphisms. (This is a difficulty which the Ashtekar approach shares with more traditional approaches.) The constraint closure is not only essential to the consistency of the Dirac quantization procedure; closure is important even classically. When the $3 + 1$ splitup is integrated forward in time, to construct the entire spacetime, the theory will not be invariant under the full four-dimensional diffeomorphism group unless the constraint alge-
bra closes. Thus an invariance of the classical theory is lost if commutator closure is neglected.

In this paper we consider the application of the Ashtekar formalism to the problem of plane gravitational waves. "Plane" means the metric possesses two commuting spacelike Killing vectors, and we shall choose coordinates so that these vectors are unit vectors pointing in the x and y directions.

\[ k^{(x)} = \partial_x; \quad k^{(y)} = \partial_y. \]  

(1)

We begin the quantization procedure in section 2 by choosing a factor ordering and verifying closure of the constraint algebra. We find a rather surprising result: in the plane wave case, one and the same factor ordering makes the vector constraint into a diffeomorphism generator and allows the algebra to close. This result requires the high symmetry (the two spatial Killing vectors).

In section 3 we tackle the problem of constructing a measure. Here we attain some partial success. Our measure respects the reality constraints obeyed by the Ashtekar connections. However, our measure is not invariant under the constraints.

The literature on plane waves is vast, but we single out two papers which are especially close to the present paper. Husain and Smolin [12] were the first to apply the Ashtekar formalism to the two-Killing vector case. Neville [13], working with the traditional geometrodynamical variables, found the transformations which reduce the Hamiltonian to parameterized free field form and constructed the classical constants of the motion. This paper will be referred to as I.

Reference I studied waves which were unidirectional as well as plane, where "unidirectional" in the present coordinates means the waves are moving only in the +z direction. Since unidirectional waves are known to obey a superposition principle, they do not scatter, except off waves moving in the -z direction. The present paper does not use the unidirectional assumption. All results on closure and the measure hold in the presence of scattering.

Our notation is typical of papers based upon the Hamiltonian approach with concomitant 3 + 1 splitup. Upper case indices A, B, ...I, J, K, ... denote local Lorentz indices ("internal" SU(2)
indices) ranging over \(X, Y, Z\) only. Lower case indices \(a, b, \ldots, i, j, \ldots\) are also three-dimensional and denote global coordinates on the three-manifold. Occasionally the formula will contain a field with a superscript (4), in which case the local Lorentz indices range over \(X, Y, Z, T\) and the global indices are similarly four-dimensional; or a (2), in which case the local indices range over \(X, Y\) (and global indices over \(x, y\)) only. The (2) and (4) are also used in conjunction with determinants; e.g., \(g\) is the usual 3x3 spatial determinant, while \(e^2\) denotes the determinant of the 2x2 \(X, Y\) subblock of the triad matrix \(e^a_\alpha\). We use Levi-Civita symbols of various dimensions: 

\[
\epsilon_{XXYZ} = \epsilon_{XYZ} = \epsilon_{XY} = +1.
\]

The basic variables of the Ashtekar approach are an inverse densitized triad \(\tilde{E}^a_\alpha\) and a complex SU(2) connection \(A^A_a\).

\[
\tilde{E}^a_\alpha = \sqrt{g} e^a_\alpha; \tag{2}
\]

\[
[\tilde{E}^a_\alpha, A^B_b] = -h \delta(x - x') \delta^B_A \delta^a_b. \tag{3}
\]

The local Lorentz indices are vector rather than spinor; strictly speaking the internal symmetry is \(O(3)\) rather than \(SU(2)\), gauge-fixed to \(O(2)\) rather than \(U(1)\).

## 2 Closure of Constraints

Since the result that the constraints do close is surprising, and since any proof of closure is bound to be detailed, it would be helpful if we could state some simple reason why the constraints close, before becoming enmeshed in the details. This is not too hard to do. We consider both the geometrodynamical and Ashtekar approaches, since closure should be independent of choice of basis variables.

We begin by establishing some notation. In the coordinate system (1.1), metric components are independent of \(x\) and \(y\), and it is possible to bring the metric to a block diagonal form \([4]\). One 2x2 subblock connects only \(t\) and \(z\) components of the metric; another 2x2 subblock connects only \(x\) and \(y\) components. The \(x,y\) subblock may be parameterized using variables suggested by Szekeres \([5]\):

\[
ds^2 = e^A [dx^2 e^B \cosh W + dy^2 e^{-B} \cosh W - 2dxdy \sinh W]
\]
\[ e^{D^{\Lambda}}/2 \{ - (N')^2 + (N^z)^2 \} dt^2 + 2N^z dz dt + dz^2 \}. \] 

(4)

\( N' \) is not quite the usual lapse \( N \).

\[
N' = N / \sqrt{g_{zz}}.
\] 

(5)

Now recall the usual reason why the constraints do not close. Let the Hamiltonian density be \( NH_0 + N^iH_i \), with \( H_0 \) the scalar constraint and \( H_i \) the vector constraints. Then the commutator of two scalar constraints, 

\[
[NH_0, MH_0] = \delta(x - x') \{ M\partial_i N - N\partial_i M \} g^{ij} \]

(6) 

does not close because the final inverse metric factor prevents \( H_j \) from annihilating the wave functional, implicitly assumed to stand to the right on both sides of equation (5). In the present case, from equation (5), the inverse spatial metric is block diagonal, with a 1x1 subblock containing only \( g_{zz} \). Also, \( x \) and \( y \) vector constraints have been gauged away in the process of bringing the metric to block diagonal form, so that the problem contains only a \( z \) vector constraint, and the \( g^{ij} \) in equation (5) collapses to \( g^{zz} = 1/g_{zz} \). Thus we can completely eliminate the \( g^{ij} \) problem by absorbing a factor of \( \sqrt{1/g_{zz}} \) into each of the lapse functions \( N \) and \( M \), as at equation (5). This is the simple reason why the constraints close. The innocuous-looking renormalization (5) is the key step.

What would the renormalization (5) look like in the Ashtekar notation? In that formalism the 3-metric and its conjugate momenta are replaced by complex connection fields \( A^A \) and densitized inverse triads \( \tilde{E}_A^a \), where \( A = X,Y,Z \) is a local Lorentz index. Besides the vector and scalar constraints, there are three new constraints, which generate local SU(2) rotations (or in our case, local O(3) rotations, since \( A \) is a vector rather than a spinor index.), The 3x3 \( \tilde{E}_A^a \) matrix may be brought to block diagonal form, with one 1x1 subblock plus one 2x2 subblock, exactly as for the 3x3 spatial subblock of the metric \( g^{ij} \). The \( \tilde{E}_A^a \) matrix is not symmetric, however, so contains more independent components than the metric. To bring the inverse triad matrix to block diagonal form, one must gauge away \( 9 - (1 + 4) = 4 \) inverse triad components, whereas in the metric case one
removes only $6 - (1 + 3) = 2$ components. To remove the extra two components one must fix (the x and y vector constraints, as before, plus) two more constraints, the X and Y SU(2) or O(3) generators. The 1x1 subblock of the $\tilde{E}_A^a$ matrix is occupied by $\tilde{E}_Z^a$, while the 2x2 subblock contains all $\tilde{E}_A^a$ with $a = x,y$ and $A = X,Y$. The Ashtekar scalar constraint $H$ is density weight 2, and the new Lagrange multiplier for the scalar constraint is a weight -1 lapse $\bar{N}$ which is related to $N$ as follows:

$$\bar{N}\tilde{E}_Z^a = \left(\frac{N}{\sqrt{g}}\right)\left(\sqrt{\bar{g}}\delta_Z^a\right)$$

$$= N \sqrt{g^{ZZ}}$$

$$= N'. \quad (7)$$

Hence in the Ashtekar approach we will be absorbing factors of $\tilde{E}_Z^a$ into the densitized lapse.

Since $\bar{N}H = N'(H/\tilde{E}_Z^a)$, the new scalar constraint $H/\tilde{E}_Z^a$ becomes a rational function of the basic fields, rather than a polynomial function. This complication is a price which must be paid in order to secure closure. We consider it a small price. Firstly, the constraints should close as a matter of principle, in order to have a consistent quantization and full four-dimensional diffeomorphism invariance.

Secondly, the presence of the $1/\tilde{E}_Z^a$ factor actually makes the Hamiltonian less singular. $H$ contains products of several operators, all evaluated at the same point $z$, which can lead to undefined $\delta(z - z)$ factors when $H$ acts upon a wave functional. To avoid such factors, Husain and Smolin [12] must regulate $H$ by point-splitting. However, most of the terms in $H$ contain a factor of $\tilde{E}_Z^a$, which is removed by the $1/\tilde{E}_Z^a$, leaving behind a simpler operator which cannot produce a $\delta(z - z)$ and does not have to be regulated. To study the term where the $1/\tilde{E}_Z^a$ does not cancel, we recall that $\tilde{E}_Z^a$ is conjugate to the complex Ashtekar connection $A_a^A$,

$$[\tilde{E}_A^a, A_b^B] = -\hbar\delta(z - z')\delta_b^a\delta_A^B, \quad (8)$$

Therefore the action of $\tilde{E}_Z^a$ on a wave functional $\psi = \psi[A]$ is $\tilde{E}_Z^a\psi = -\hbar\delta\psi/\delta A_a^Z$. We expect the $A_a^Z$ dependence of $\psi$ to be holonomic.
\[
\psi = \cdots X(z_3) \exp[i \int_{z_2}^{z_3} A^Z_z S \, dz] X(z_2) \exp[i \int_{z_1}^{z_2} A^Z_z S \, dz] \cdots \\
:= \cdots X(z_3) U(z_3, z_2) X(z_2) U(z_2, z_1) \cdots .
\]

The \( X \) are assumed to be operators in the Lie algebra of SU(2), and independent of \( A^Z_z \). Explicit factors of \( i \) are present, because we use the usual, Hermitian SU(2) generators \( S \). Then the direct action of \( \tilde{E}^z \) on the wave functional is

\[
\tilde{E}^z(z) \psi = \cdots + \cdots X(z_3) [\hbar \theta(z_3 - z) \theta(z - z_2) (\theta(z_3 - z) \theta(z - z_2) - i S^Z) X(z_2) U \cdots + \cdots ,
\]

(10)
one such term for each \( U \). To find the action of \( [1/\tilde{E}^z(z)] \), multiply both sides of equation (10) by \( \theta(z_3 - z) \theta(z - z_2) \), in order to project out the term exhibited explicitly on the right; then multiply both sides of equation (10) by \( \hbar/\tilde{E}^z(z) \).

\[
[\hbar/\tilde{E}^z(z)] \psi = \cdots + \cdots X(z_3) [\hbar \theta(z_3 - z) \theta(z - z_2) (\theta(z_3 - z) \theta(z - z_2) - i S^Z) X(z_2) U \cdots + \cdots + \text{const.}
\]

(11)

In order to make this look like \( \hbar/\tilde{E}^z(z) \) acting upon \( \psi \), multiply \( \psi \) by the following partition of unity:

\[
1 = [\theta(z - z_n) + \theta(z_n - z) \theta(z - z_{n-1}) + \cdots + \theta(z_3 - z) \theta(z - z_2) + \cdots].
\]

(12)

Therefore

\[
[\hbar/\tilde{E}^z(z)] \psi = \cdots + \cdots X(3) [\hbar \theta(z_3 - z) \theta(z - z_2) (\theta(z_3 - z) \theta(z - z_2) - i S^Z)^{-1} X(z_2) U \cdots + \cdots + \text{const.}
\]

(13)

Evidently the action of \( \hbar/\tilde{E}^z(z) \) on \( \psi \) is quite mild. There is not even a \( \delta(z - z') \) factor, let alone a \( \delta(z - z) \).

If the \( X \) are helicity-changing operators, the eigenvalue of \( S^Z \) in equation (13) will vary from one \( U \) to the next. If we use the 2x2 Pauli representation, the eigenvalue never vanishes and \( (S^Z)^{-1} \) is always finite. However, in future work we shall use the \((2j+1)\times(2j+1)\)
representation, where $S_Z$ can have a zero eigenvalue if $j$ is integer. When $S_Z$ vanishes, so that $U(z_3, z_2)$ is unity, one may replace the square bracket in equation (13) by

$$[-\theta(z_3 - z)\theta(z - z_2)U(z_3, z_2) \int_2^3 A_Z^2 dz]. \quad (14)$$

As a check, when equation (13) is inverted by multiplying both sides by $\tilde{E}_z Z/h$, the square bracket (14) gives the same answer for vanishing $S_Z$ as the square bracket (13) gives for finite $S_Z$.

We should also check that $\tilde{E}_z^x$ is not a gauge artifact (which can be gauged to zero!). Despite its contravariant $z$ index, $\tilde{E}_Z^x$ is a scalar function under diffeomorphisms,

$$\delta \tilde{E}_Z^x = N^z \partial_z \tilde{E}_Z^x \quad (15)$$

The $\tilde{E}_Z^x$ field is both contravariant and weight 1, and in effect the two transformation properties cancel each other in one space dimension, leaving an ordinary scalar. Another way to see the scalar behavior is to relate $\tilde{E}_Z^x$ to the metric variables $A$, $B$, $W$, and $D$ introduced at equation (4). Using the same relationships as at equation (7), we find

$$\tilde{E}_Z^x = \exp(A). \quad (16)$$

Since $A$ occurs in the $x,y$ sector of the metric (4), and this sector transforms as a scalar under diffeomorphisms, the function $A$ is a scalar. Hence $\tilde{E}_Z^x$ cannot be gauged away.

We now pass to the details. In a coordinate system where both $\tilde{E}_A$ and $A_A^A$ fields are block diagonal, the total Hamiltonian reduces to

$$H_T = N'[\epsilon_{MN} \tilde{E}_M^x \tilde{E}_N^y (\tilde{E}_Z^x)^{-1}\epsilon_{AB} A_A^x A_B^y + \epsilon_{MN} \tilde{E}_M^b F_{2b}^N]$$

$$+i N^x \tilde{E}_M^b F_{2b}^M$$

$$-i N^x [\partial_z \tilde{E}_Z^x - \epsilon_{1I} \tilde{E}_1^a A_a^I]$$

$$\equiv N^x H_S + N^x H_x + N_G H_G, \quad (17)$$

where

$$F_{2b}^N = \partial_z A_b^N - \epsilon_{NQ} A_Z^A A_b^Q, \quad (18)$$
and the $N_G$ term (the Gauss constraint) is the generator of local SU(2) rotations around the $Z$ axis. The lapse has been renormalized as at equation (7), by removing a factor $\tilde{E}_Z$ from the scalar constraint. From now on the ”scalar constraint” will mean the expression $H_S$ multiplying $N'$ in equation (17), namely, the usual Ashtekar scalar constraint $H$ divided by $\tilde{E}_Z$. We have operator-ordered equation (17) in a way which anticipates the following section, where we shall consider solutions $\psi$ which depend on $A^Z$ and the four $\tilde{E}$ in the $2x2$ sector. If we call these five commuting variables the $Q$ variables ($\psi = \psi(Q)$) and the five conjugate variables the $P$ variables, then we have ordered $P$’s to the right, $Q$’s to the left in $H$. We shall carry out the proof of closure for this specific choice of $Q$’s and this specific ordering, but the proof would also go through for the other popular choice of $Q$’s, in which the five $A$’s are chosen as $Q$’s (and the $A$’s are ordered to the left in $H$).

Now let us ask which commutators, or which parts of which commutators, are likely to give trouble. First of all, it is easy to check that the classical commutators (or rather, the classical Poisson brackets) close on pure constraints, with no undesirable factors of $g^{zz}$ or the Ashtekar analog of $g^{zz}$. Since the quantum commutators are designed to reproduce exactly the same fields as the classical Poisson brackets, there will be no factor of $g^{zz}$ in the quantum case either. We will get the same fields; and the only thing which can go wrong is that the commutator yields a $P$ field to the left of a $Q$ field. Remembering that each constraint is a sum of terms of the form $f(Q)g(P)$, we want, schematically,

$$[f_1(Q)g_1(P), f_2(Q)g_2(P)] = f_3(Q)g_3(P). \quad (19)$$

There would be trouble if $P$’s occured to the left of $Q$’s on the right-hand side, for example $Pf_3(Q)g_3(P)$.

We now show that almost all the terms in a typical constraint commutator $[H_i, H_j]$ will give no trouble. Each term in this commutator will look like the left-hand side of equation (19). Write this term out using the identity

$$[AB, CD] = AC[B, D] + A[B, C]D +$$
On the right-hand side, the $f$ and $g$ factors outside the commutators are in the correct order ($Q$’s to the left). Therefore if the commutators on the right yield fields which commute among themselves, the entire expression on the right can be ordered correctly, and the term gives no trouble. In particular, if either $f$ or $g$ is a monomial, say $g \sim P$, then the commutator of any $f$ with $g$ yields only $Q$ fields, and the term can be ordered correctly. Examination of equation (17) shows that all the terms in $H_T$, except the $(\tilde{E}_z)^{-1}$ term, are monomials in the $P$’s, of the form $Q^2 P$ or $QP$; and one term is independent of the $P$’s (pure $Q$) which is even better. Therefore the only commutators we have to check are the ones involving the $(\tilde{E}_z)^{-1}$ term,

\[
H_E := N' \epsilon_{MN} \tilde{E}_M^x \tilde{E}_N^x (\tilde{E}_z)^{-1} \epsilon_{AB} A_x^A A_y^B
\sim Q^2 (1/P) P^2.
\] (22)

We now investigate commutators of $H_E$ with $Q$ terms, $QP$ terms, $Q^2 P$ terms, and finally commutators of $H_E$ with itself. (a). Commutators of $H_E$ with $Q$ and $QP$ terms. For this case, in commutator (19), both $f_2$ and $g_2$ are monomials or constants. It follows immediately that both commutators on the right-hand side of equation (21) involve at least one monomial and can be correctly ordered. Even if $f_2$ is $A_z^Z$, there is no problem, since the commutator with $(\tilde{E}_z)^{-1}$,

\[
[(\tilde{E}_z)^{-1}, A_z^Z] = \hbar \delta (z - z') (\tilde{E}_z)^{-2},
\] (23)
yields factors which commute among themselves. Equation (23) may be proven by multiplying it on left and right by $\tilde{E}_z^Z$.

(b). Commutators of $H_E$ with $Q^2 P$ terms. These are terms of the form $\tilde{E}_M^x A_x^Z A_y^Q$ coming from the $F$’s (field strengths) in the scalar and vector constraints, equations (17)-(18). (b1). When commuting the scalar constraint with itself, for example, we get commutators of the form

\[
+C[A, D]B + [A, C]DB. \quad (20)
\]

\[
[f_1 g_1, f_2 g_2] = 0 + f_1 [g_1, f_2] g_2 + f_2 [f_1, g_2] g_1 + 0. \quad (21)
\]
where subscripts E denote fields coming from $H_E$, and $\cdots$ indicates harmless commutators involving the monomial $P$. The two commutators on the last line differ by a minus sign and an interchange of $z$ and $z'$. The interchange affects nothing, since the commutator is proportional to $\delta(z - z')$. Therefore the two commutators cancel each other, and the term is harmless. (b2). When commuting the scalar constraint with the vector constraint, we get

\[
[Q^2 P(z), Q^2 (1/P) P^2 (z')] = \cdots + Q^2 [(1/P) P^2 (z), Q^2 (z')] P +
+ Q^2 [Q^2 (z), (1/P) P^2 (z')] P,
\]

On the third line we have used the ABCD identity, equation (20). On the last line, we can commute the $iA_b^N$ factor to the left, until it forms the expression $i \tilde{E} b A^{N} \epsilon_{MN}$. We can subtract from this expression the Gauss constraint $H_G$, equation (17). This causes no problems, since the Gauss constraint commutes with everything to the right of $i \tilde{E} b A^{N} \epsilon_{MN}$, hence can be commuted to the far right where it will eventually annihilate the wave functional. Since $i \tilde{E} b A^{N} \epsilon_{MN} \div H_G = i \partial_x \tilde{E} z$, the last line becomes

\[
\cdots + i (2) \tilde{E} [h \delta(z - z') \partial_x (\tilde{E} z)^{-1}] \epsilon_{AB} A^A A^B.
\]

This now has the same form as the corresponding term in the classical calculation, and moreover the operators are correctly ordered (Q's to the left).

The calculation just completed, however, suggests a new way in which the constraints might fail to close. Suppose that at some point in the calculation it is necessary to insert the Gauss constraint in

\[11\]
the middle of a term (as was done just above); if the $H_G$ factor cannot be commuted to the far right, then closure will be spoiled. Fortunately, the classical calculation once again comes to the rescue. Since the classical calculation has the same pattern of fields as the quantum calculation, a Gauss insertion is necessary in the quantum calculation if and only if a Gauss insertion is necessary at the same point in the classical calculation. It turns out that the only point where a Gauss insertion occurs, classically, is in the $[\text{vector,scalar}]$ commutator term just considered, and this insertion is harmless.

(c). Finally, we consider the commutator of $H_E$ with itself. Using the ABCD identity (20), we get

$$[H_E(z), H_E(z')] = (2) \tilde{E}(z)(\tilde{E}_z)^{-1}[\epsilon_{AB} A_x^A A_y^B(z), (2) \tilde{E}(z')(\tilde{E}_z')^{-1}] \epsilon_{CD} A_x^C A_y^D(z') +$$

$$(2) \tilde{E}(z')(\tilde{E}_z')^{-1}[2 \tilde{E}(z)(\tilde{E}_z)^{-1}, \epsilon_{CD} A_x^C A_y^D(z')] \epsilon_{AB} A_x^A A_y^B(z).$$

The two commutators cancel, after a relabeling $AB \leftrightarrow CD$ in the second commutator. This completes the proof that the constraints close.

It is also easily verified that the constraint $H_z$ generates diffeomorphisms in the $z$ direction. (In fact $H_z$ fails to generate diffeomorphisms only when the $P'$s are ordered to the left [5].) $H_z$ is not quite the diffeomorphism generator, but differs from that generator by a term of the form $N^A A^B z H_G$, $H_G$ the Gauss constraint. A term of this form can be added to $H_z$ without affecting closure, since the added term is linear in $P$ and therefore harmless.

Since there is no factor of $g^{zz}$ or other fields in the constraint algebra, it is a true Lie algebra (structure “constants” are at most $\delta$ functions, not functions of fields). The structure of this Lie algebra is very simple. It breaks up into two commuting subalgebras generated by $(H_z \pm H_S)/2$,

$$\int dz \int dz'[M(z)(H_z \pm H_S)/2, N(z')(H_z \pm H_S)/2] = i \hbar \int dz (M \partial_z N - N \partial_z M)(H_z \pm H_S)/2.$$

(24)

Presumably these generators may be interpreted physically as displacements along the light cone, in directions $(z \pm ct)$. 

12
3 The Reality Constraints

Since the Ashtekar connections are complex, they obey reality constraints of the form \( A + A^* = 2 \text{ Re } A = \text{ known function of the } \tilde{E}. \) For completeness, and to establish certain detailed formulas which we will need later, we sketch a derivation of these constraints. The derivation ends at equation (38). At equation (39) we propose a measure to enforce these constraints.

We start from the four-dimensional connection

\[
2G^{(4)}A_a^{IJ} = \omega_a^{IJ} + i\epsilon_{MNa}^M \omega_a^{MN}, \tag{25}
\]

where \( G \) is the Newtonian constant and \( \omega \) is the SL(2,C) Lorentz connection. After the 3 + 1 splitup \([17, 18]\), one obtains the SU(2) connection which is canonically conjugate to \( \tilde{E}_a^I \) (equation (3)).

\[
2GA_a^S = \epsilon_{MSN}^{(4)}A_a^{MN} = \epsilon_{MSNa}^M \omega_a^{MN} - 2i\omega_a^{TS}. \tag{26}
\]

From this, the real part of \( A \) is

\[
G[A_a^S + A_a^{S*}] = \epsilon_{MSN}^{(4)}a^{MN}, \tag{27}
\]

or when these equations are written out for the 1x1 and 2x2 subblocks,

\[
G[A_z^Z + A_z^{Z*}] = -2\omega_z^{XY}; \tag{28}
\]

\[
G[A_i^I + A_i^{I*}] = 2\epsilon_{IJ} \omega_i^Z. \tag{29}
\]

These “reality constraints” relate the \( A \)'s to the \( \tilde{E} \)'s, since \( \omega = \omega(\tilde{E}) \). The next step is to exhibit this \( \tilde{E} \) dependence. This is done by first relating the \( \omega \)'s to the triads \( e^i_I \) and inverse triads \( e^i_I \), then relating the \( e^i_I \) and \( e^i_I \) to the \( \tilde{E} \). The requirement that the triads have zero covariant derivative leads to

\[
\omega_a^{IJ} = [\Omega_I[ja] + \Omega_a[ai] - \Omega_a[ij]a^Ia^J], \tag{30}
\]

where
\[ \Omega_{[ja]} = e_{IM}[\partial_j e^M_a - \partial_a e^M_j]/2. \]  

(31)

In the present case the triad matrix is block diagonal, with 2x2 and 1x1 subblocks, and these equations simplify considerably. Also, from equations (29) and (24) we shall need only

\begin{align*}
\omega^X_Y &= [e^X_i \partial_z e^Y_i - e^Y_i \partial_z e^X_i]; \\
\omega^Z_I &= -\partial_z g_{ij} e^Z_j e^{ij}/2,
\end{align*}

(32)

(33)

where the indices i,j range over x,y only.

Now we replace metric, triads, and inverse triads by \(\tilde{E}\) fields. The inverse triad fields are easiest to replace. From equation (3)

\[ e^a_A = \tilde{E}^a_A/\sqrt{g} = \tilde{E}^a_A/\sqrt{\tilde{E}^z_Z/\tilde{E}}. \]

(34)

For the metric and triad fields, the strategy is to express them in terms of the inverse triads, then replace the latter. For example in the 2x2 subblock,

\[ (2)g_{ab} = \epsilon_{am}\epsilon_{bn} (2)g^{mn} (2)g \\
= \epsilon_{am}\epsilon_{bn} \tilde{E}^m_M \tilde{E}^n_M (2)g/g \\
= \epsilon_{am}\epsilon_{bn} \tilde{E}^m_M \tilde{E}^n_M \tilde{E}^z_Z/(2)\tilde{E}; \]

(35)

\[ e^M_m = \epsilon_{MN}\epsilon_{mn} (2)e \\
= \epsilon_{MN}\epsilon_{mn} \tilde{E}^n_N \sqrt{\tilde{E}^z_Z/(2)\tilde{E}}. \]

(36)

We make these replacements in equations (32) and (33) and obtain

\begin{align*}
\omega^X_Y &= -[\epsilon_{mn} \tilde{E}^m_M \partial_z \tilde{E}^n_M]/2(2)\tilde{E}; \\
\omega^Z_I &= -[\tilde{E}^j_J/(2)\tilde{E}] \partial_z [\epsilon_{im} \epsilon_{jn} \tilde{E}^m_M \tilde{E}^n_N \tilde{E}^z_Z/(2)\tilde{E}] .
\end{align*}

(37)

(38)
After these algebraic preliminaries, we are ready to consider the measure. Since our complete set of commuting observables are the four $\hat{E}$ in the 2x2 X,Y sector, plus the complex connection $A^Z_z$, we try a dot product of the form

$$<\phi|\psi> = \int \phi^*\psi \mu d^4\tilde{E} d^2A,$$  

(39)

where $d^2A \equiv d\text{Re}A^Z_z d\text{Im}A^Z_z$. The measure $\mu$ must satisfy several requirements. (i) It must guarantee the quantum form of the reality constraints.

$$<\phi|A\psi> + <A\phi|\psi> = 2<\phi|\text{Re}A\psi>.$$  

(40)

(ii) It must guarantee the invariance of $\mu d^4\tilde{E} d^2A$ under transformations generated by the scalar, vector, and Gauss constraints. (iii) It must contain enough gauge-fixing delta functions to remove the usual unbounded integrations over infinite numbers of gauge copies. Note that (ii) requires only invariance under the constraints, not invariance under four-dimensional diffeomorphisms. In a $3+1$ formalism, one does not have the proper set of fields to implement the latter invariance, essentially because all fields are evaluated on a constant time hyperslice, whereas four-dimensional diffeomorphisms move fields off the hyperslice [20].

As yet we do not know how to satisfy requirements (ii)-(iii). We shall, however, propose a $\mu$ which will satisfy requirement (i). We set

$$\mu = \delta[G(A^Z_z + A^{Z*}_z) + 2\omega^{XY}_z].$$  

(41)

From equation (29), this delta function enforces the $A^Z_z$ reality constraint. The surprising fact is that it also enforces the remaining reality constraints (29) as well, as we shall prove now.

First we should clarify our notation. Since our integration variables are $\text{Re}A^Z_z$ and $\text{Im}A^Z_z$, not $A$ and $A^*$, the $A^Z_z$ functional derivative really means

$$\delta/\delta A^Z_z := [\delta/\delta \text{Re}A^Z_z + (1/i)\delta/\delta \text{Im}A^Z_z]/2,$$  

(42)

which follows from $\text{Re}A = (A + A^*)/2$, etc. We can then check that, since the $\phi$ in the bra, equation (40), is complex conjugated,
\[
\frac{\delta \phi}{\delta A_2^x} = \frac{\delta \phi [A_*]}{\delta A}
\]
\[
= \int dz' |\delta \phi / \delta A_2^x(z')| \delta A_2^x(z') / \delta \text{Re} A_2^x + (1/i) \delta A_2^x / \delta \text{Im} A_2^x / 2 = 0,
\]
as expected. Now write
\[
< \phi | A_1^2(z) \psi > = -\hbar \int \phi^\ast \mu \delta \psi / \delta \tilde{E}_1^i(z) = +\hbar \int [\delta \phi / \delta \tilde{E}_1^i \mu \psi + \phi^\ast \delta \mu / \delta \tilde{E}_1^i \psi]
\]
\[
= - < A_1^1 \phi | \psi > + \hbar \int \phi^\ast \delta [G(A + A*) + 2 \omega_z^{XY}] / \delta A_2^z(z') \times
\]
\[
\times [2 \delta \omega_z^{XY} (z') / \delta \tilde{E}_1^i(z)] dz' \psi
\]
\[
= - < A_1^1 \phi | \psi > -(\hbar / G) \int \phi^\ast [2 \delta \omega_z^{XY} (z') / \delta \tilde{E}_1^i(z)] dz' \delta \psi / \delta A_2^z(z')
\]
\[
= - < A_1^1 \phi | \psi > -(1/G) \int \phi^\ast [2 \delta \omega_z^{XY} (z') / \delta \tilde{E}_1^i(z)] dz' \tilde{E}_2^z(z') \psi.
\]
Again, the \( \delta / \delta A_2^x \) is really a sum of ReA and ImA functional derivatives, as at equation (12), with \( \delta \mu / \delta \text{Im} A_2^x \) vanishing. The last square bracket can be rewritten using equation (38) and (2) \( \tilde{E} = \epsilon_{ij} \tilde{E}_1^i \tilde{E}_1^j / 2 \).

\[
\tilde{E}_2^z(z')2 \delta \omega_z^{XY} (z') / \delta \tilde{E}_1^i(z) = \tilde{E}_2^z[(\delta \epsilon_{in} \partial_n \tilde{E}_1^n + \epsilon_{mi} \tilde{E}_1^m \partial_n \delta) / (2) \tilde{E} - \epsilon_{mn} \tilde{E}_1^m \partial_n \tilde{E}_1^n \tilde{E}_1^i / (2 \tilde{E})^2]
\]
where \( \delta = \delta (z-z') \). Now use \( \epsilon_{mn} \epsilon_{ij} = \epsilon_{mi} \epsilon_{nj} + \epsilon_{mj} \epsilon_{in} \) in the last term of equation (15). The two terms involving \( \epsilon_{in} \) cancel. After integrating by parts to remove the derivative from the delta function in the second term, one may replace the \( \tilde{E}_1^n \) by \( \tilde{E}_1^m \delta \tilde{E}_1^i \). We then have

\[
\tilde{E}_2^z2 \delta \omega_z^{XY} (z') / \delta \tilde{E}_1^i(z) = -\partial_n [\tilde{E}_2^z \epsilon_{mi} \epsilon_{nj} \tilde{E}_1^m \tilde{E}_1^n / (2 \tilde{E}) \delta (z-z') \epsilon_{ij} \tilde{E}_1^j / (2 \tilde{E})
\]
\[
= 2 \omega_z^{XY} \epsilon_{ij} \tilde{E}_1^i \delta (z-z')
\]
\[
= 2G \text{Re} A_1^i \delta (z-z'),
\]
(46)
where the third line uses equation (38) and the last line uses equation (29). Inserting the result (46) into equation (44), we obtain the reality condition for the $A_1^i$ field, QED.

4 Directions for Further Research.

For the plane wave problem, we have constructed a constraint algebra which closes after a simple renormalization of the lapse function. We have argued that the cost of this renormalization (rational, rather than polynomial constraints) is small compared to the benefits (consistent constraints in the quantum-mechanical theory; full diffeomorphism invariance in the classical theory). We have also made modest progress toward constructing a measure.

It is a standard result that the Hamiltonian has surface terms whenever the spatial manifold is non-compact [16]. In future work, we intend to describe these surface terms. They are surprising: in the plane wave case, it is not automatically true that the Gauss constraint term in the Hamiltonian falls to zero at infinity.

Because of these additional terms at infinity, one must exercise care when interpreting any wavefunctional involving holonomies defined over open contours. In the planar case a holonomy $\exp(i \int A^Z_z S_z)$ integrated over a closed contour is necessarily zero, since the $z$ contour must retrace itself. Therefore it is natural to consider open contours and extend the endpoints to $z = \pm \infty$, to respect spatial diffeomorphism invariance. It is possible to enrich this elemental holonomic structure in two ways. First, insert $\tilde{E}$ operators at various points along the holonomy; the wavefunctional then looks like a Rovelli-Smolin $T^n$ operator [4] (defined over an open rather than closed contour). Second, replace the usual 2x2 $S_z$ matrices in the holonomy and in $\tilde{E}$ by the $(2j + 1) \times (2j + 1)$ spin-$j$ representation. The resultant structure is reminiscent of a symmetric state, or spin network state [11], with the holonomies corresponding to flux lines of spin $j$ and the $E$ operators corresponding to vertices. For appropriate choice of the $\tilde{E}$, a wavefunctional constructed in this manner is annihilated by the Hamiltonian at finite values of $z$.

In three spatial dimensions, this would be essentially the end of the story: the flux exiting at infinity is irrelevant, since Gauss
rotations at infinity are not allowed. The surviving surface terms in the Hamiltonian simply give the ADM energy. In one spatial dimension, however, the Gauss term contributes at infinity. One could add Fermionic matter to the theory [22, 21] and terminate the flux lines on Fermions at $\pm \infty$. However, adding Fermions probably complicates the theory unnecessarily. In one-dimensional QED, for example, one can learn quite a bit by studying electromagnetic plane waves at finite $z$, while ignoring what happens to the wave at infinity. In a future publication we will adopt this philosophy and study the finite $z$ properties of the open flux line solutions. Work is also in progress on solutions involving closed flux lines.

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