A FUNDAMENTAL THEOREM FOR THE $K$-THEORY OF CONNECTIVE $S$-ALGEBRAS

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1. Introduction

The Fundamental Theorem of $K$-Theory (first formulated by Bass in low dimensions and later extended by Quillen to all dimensions [5]) yields an isomorphism

$$K_*(R[t, t^{-1}]) \cong K_*(R) \oplus K_{*-1}(R) \oplus NK_*^+(R) \oplus NK_*^-(R)$$

where $R$ is a discrete ring, $K_*(-)$ denotes its Quillen $K$-groups, and

$$NK_*^\pm(R) \cong NK_*(R) := \ker(K_*(R[t]) \xrightarrow{t \to 0} K_*(R))$$

The groups here are possibly non-zero in negative degrees, given that they are computed as the homotopy groups of a (potentially) non-connective delooping of the Quillen $K$-theory space, arising from a spectral formulation of this result [11]. The nil-groups $NK_*(R)$ capture subtle “tangential” information about $R$, and are remarkably difficult to compute. In this short paper we extend this fundamental theorem to the Waldhausen $K$-theory of connective $S$-algebras using the recent result of [9], and note the corresponding nil-groups are nontrivial even for the sphere spectrum. Their structure will be investigated more thoroughly in future work. We remark that our fundamental theorem recovers the main result of [6] as a special case.

To state the result, let $\mathcal{CSA}$ denote the category of connective (i.e., $(-1)$-connected) $S$-algebras and $S$-algebra homomorphisms, in the sense of [4]. Write $K(-)$ for the functor which associates to an $S$-algebra its (connective) Waldhausen $K$-theory spectrum.

**Theorem** (Fundamental Theorem of $K$-Theory for connective $S$-algebras). For a connective $S$-algebra $A$, there is a map of spectra

$$K(A) \longrightarrow \Sigma^{-1} \text{hocolim} (K(A[t]) \vee_{K(A)} K(A[t^{-1}]) \to K(A[t, t^{-1}]))$$

which is functorial in $A$ and induces an equivalences between $K(A)$ and the $(-1)$-connected cover of the target.

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2. Proof of the main result

If $A$ is an $S$-algebra, then $A[t], A[t^{-1}], A[t, t^{-1}]$ admit $S$-algebra structures induced by that on $A$ in a natural way. Following [11] IV.10, define functors from ($S$-algebras) to (spectra)* by

$$F_{0,0}(A) := K(A)$$

$$F_{0,1}(A) := K(A[t]) \vee_{K(A)} K(A[t^{-1}]) = F_{0,0}(A[t]) \vee_{F_{0,0}(A)} F_{0,0}(A[t^{-1}])$$

$$F_{0,2}(A) := K(A[t, t^{-1}]) = F_{0,0}(A[t, t^{-1}])$$

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There is an obvious transformation \( F_{0,1}(-) \to F_{0,2}(-) \) induced by the inclusions of \( A[t] \) and \( A[t^{-1}] \) as subalgebras of \( A[t, t^{-1}] \), and we set \( F_{0,3}(-) := \text{hocofib}(F_{0,1}(-) \to F_{0,2}(-)) \). For a spectrum \( T \), write \( \Sigma^{-1}T \) for the desuspension of \( T \). In terms of these functors, the Fundamental Theorem is equivalent to

**Theorem 1.** For a connective \( S \)-algebra \( A \), there is a map of spectra \( F_{0,0}(A) \to \Sigma^{-1}F_{0,3}(A) \), functorial in \( A \), which induces an equivalence between \( F_{0,0}(A) \) and the \((-1)\)-connected cover \( \Sigma^{-1}F_{0,3}(A)(-1) \) of \( \Sigma^{-1}F_{0,3}(A) \).

We prove the theorem using two lemmas and a density argument inspired by \([3]\).

**Lemma 2.** The theorem is true for simplicial rings \( A \).

**Proof.** Modules over a simplicial ring form an additive category. The results of \([9]\) prove the result for both \( A \) as a simplicial ring and, equivalently, \( HA \) as a spectrum that lies in the image of simplicial rings under the Eilenberg–MacLane construction. \( \square \)

**Lemma 3.** If \( S_1 \to S_2 \to S_3 \) and \( T_1 \to T_2 \to T_3 \) are cofiber sequences of \((-1)\)- and 
\((-2)\)-connected spectra (respectively), and \( \phi_i : S_i \to T_i \) are maps respecting these cofiber sequences

\[
\begin{array}{ccc}
S_1 & \longrightarrow & S_2 \\
\downarrow \phi_1 & & \downarrow \phi_2 \\
T_1 & \longrightarrow & T_2 \\
\end{array}
\]

then if \( \phi_1 \) and \( \phi_3 \) are equivalences of \((-1)\)-connected covers, then so is \( \phi_2 \).

**Proof.** Take homotopy cofibers of the maps \( \phi_i \) to produce a new cofiber sequence \( \text{hocofib}(\phi_1) \to \text{hocofib}(\phi_2) \to \text{hocofib}(\phi_3) \). By connectivity of the maps \( \phi_1 \) and \( \phi_2 \), \( \text{hocofib}(\phi_1) \) and \( \text{hocofib}(\phi_2) \) have homotopy groups concentrated in degrees strictly below \((-1)\). Hence \( \text{hocofib}(\phi_2) \) does as well and the result follows. \( \square \)

**Proof of theorem**\([4]\). The canonical element \([t] \in K_1(S[t, t^{-1}])\) represented by the unital element \( t \) induces a map

\[
S^1 \wedge K(A) = S^1 \wedge F_{0,0}(A) \to K(S[t, t^{-1}]) \wedge F_{0,0}(A) \to K(A[t, t^{-1}]) = F_{0,2}(A) \to F_{0,3}(A)
\]

whose adjoint provides the transformation \( F_{0,0}(-) = K(-) \to \Sigma^{-1}F_{0,3}(-) \).

Following \([3, 3.1.10]\), we can resolve our \( S \)-algebra \( A \) by an \( n \)-cube \( (A)_S \) (where \( S \) lies in \( \mathcal{P}_n \), the poset of subsets of \( \{1, 2, \ldots, n\} \), \( \mathcal{P} \to \text{SAlg} \) with three crucial properties:

- the \( n \)-cube is id-cartesian,
- each vertex of the \( n \)-cube is the Eilenberg-MacLane spectrum of a simplicial ring except for \( A_\varnothing = A \), and
- after puncturing \( (A)_S \) by restricting to \( S \neq \varnothing \), the remaining maps all arise from maps of simplicial rings save in one direction.

For notational convenience, we will assume that not-so-nice direction in the cube are maps \( S' \to S' \cup \{n\} \) with \( S' \in P_n \). We will also assume \( n \geq 2 \) for the following argument.

We form two new cartesian \( n \)-cubes and by applying \( F_{0,0}(-) \) and \( F_{0,3}(-) \) to the punctured cube \( (A)_S|_{S \neq \varnothing} \) and then completing the diagrams by forming homotopy limits. Specifically, define \( X_\varnothing = F_{0,0}(A)_S \) and \( X_\varnothing = \text{holim}_{S \neq \varnothing} F_{0,0}(A)_S \) and likewise \( Y_\varnothing = F_{0,3}(A)_S \) and \( Y_\varnothing = \text{holim}_{S \neq \varnothing} F_{0,3}(-) \).

When \( n = 2 \), we have the following two homotopy pullback cubes.

\[
\begin{array}{ccc}
X_\varnothing & \longrightarrow & X_{\{1\}} = F_{0,0}(A)_{\{1\}} \\
\downarrow & & \downarrow \\
X_{\{2\}} = F_{0,0}(A)_{\{2\}} & \longrightarrow & X_{\{1,2\}} = F_{0,0}(A)_{\{1,2\}} \\
\end{array}
\quad
\begin{array}{ccc}
Y_\varnothing & \longrightarrow & Y_{\{1\}} = F_{0,3}(A)_{\{1\}} \\
\downarrow & & \downarrow \\
Y_{\{2\}} = F_{0,3}(A)_{\{2\}} & \longrightarrow & Y_{\{1,2\}} = F_{0,3}(A)_{\{1,2\}}
\end{array}
\]

\[2\]
The aforementioned natural transformation $\Sigma F_{0,0} \to F_{0,3}$ induces a map of cubes $\Sigma X_S \to Y_S$. Whenever $S \neq \emptyset$, the vertices are simplicial rings and $\Sigma P_S \to Q_S$ is an equivalence of $(-1)$-connected covers by lemma 2.

Write $P_{\text{top}}$ for the subcategory $P_{n-1}$ of $P_n$ where $n \notin S$. Write $P_{\text{bot}}$ for the subcategory of $P_n$ with $n \in S$. Note that $\{n\}$ is the initial object in $P_{\text{bot}}$. Since $X$ and $Y$ are both cartesian, the maps

$$\text{tohofib}_{S \in P_{\text{top}} - \emptyset} \Sigma X_S \to \text{tohofib}_{S \in P_{\text{bot}} - \{n\}} \Sigma X_S$$

and

$$\text{tohofib}_{S \in P_{\text{top}} - \emptyset} Y_S \to \text{tohofib}_{S \in P_{\text{bot}} - \{n\}} Y_S$$

between total homotopy fibers are weak equivalences. We note that $\Sigma$ maps $\{n\}$ with $n$ with $n$ with $n$.

We are left with the following diagram of fiber sequences.

$$\text{tohofib}_{S \in P_{\text{top}} - \emptyset} \Sigma X_S \longrightarrow \Sigma X_S \longrightarrow \text{holim}_{S \in P_{\text{top}} - \emptyset} \Sigma X_S$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\text{tohofib}_{S \in P_{\text{top}} - \emptyset} Y_S \longrightarrow Y_S \longrightarrow \text{holim}_{S \in P_{\text{top}} - \emptyset} Y_S$$

The left and right vertical maps are equivalences on $(-1)$-connected covers by lemma 2 so the middle map is as well by lemma 3.

All that remains is to compare the result on the $n$-cubes to the desired result on $A$. [3] Thm. 3.2.1 shows that $K$-theory takes id-cartesian $n$-cubes to $(n+1)$-cartesian $n$-cubes. Hence, $F_{0,0}(A) \to X_S$ and $F_{0,3}(A) \to Y_S$ are $(n+1)$-connected. As we take $n$ to infinity by including $P_n \subset P_{n+1}$, we observe that these become weak equivalences. This extends the desired result from the cubes constructed from simplicial rings to the $S$-module $A$.

This construction may be iterated. Assume $F_{n-1,i}(A)$ has been defined $(0 \leq i \leq 3)$, $n \geq 1$. Let

$$F_{n,0}(A) := F_{n-1,3}(A)$$

$$F_{n,1}(A) := F_{n,0}(A[t]) \bigvee_{F_{n,0}(A)} F_{n,0}(A[t^{-1}])$$

$$F_{n,2}(A) := F_{n,0}(A[t,t^{-1}])$$

$$F_{n,3}(A) := \text{hocofib}(F_{n,1}(A) \to F_{n,2}(A))$$

**Theorem 4.** For all $n \geq 0$, there is an equivalence

$$F_{0,0}(A) \simeq \Sigma^{-n-1} F_{n,3}(A)(-1)$$

natural in $A$.

**Proof.** By induction on $n$, we may assume that, for $n \geq 1$, there is a natural equivalence

$$F_{0,0}(A) \simeq \Sigma^{-n} F_{n-1,3}(A)(-1) = \Sigma^{-n} F_{n,0}(A)(-1)$$

Noting that the statement is true for simplicial rings, we may now repeat the argument used in the proof of the previous theorem to conclude that there is an equally natural equivalence

$$F_{n,0}(A) \simeq \Sigma^{-1} F_{n,3}(A)(-1)$$

The result follows.

**Remark 5.** In [2, §9], Blumberg and Mandell coin the term Bass functor for homotopy functors exhibiting the above type of behavior. In particular, they show that the topological Dennis trace $K(-) \to THH(-)$ is a transformation of Bass functors, at least for discrete rings. The above suggests that this particular result of theirs extends to the category of $S$-algebras.
A consequence of this last theorem is that the usual machinery associated with a spectral interpretation of the Fundamental Theorem produces a natural non-connective delooping of the $K$-theory functor $A \mapsto K(A)$ on the category $\mathcal{CS}\mathcal{A}$, via iterated application of the natural transformation $K(-) \to \Sigma^{-1}F_3(-)$. The result is a (potentially) non-connective functor

$$A \mapsto K^B(A)$$

differing from the deloopings arising from the “plus” construction $[4]$, or iterations of Waldhausen’s $wS_\bullet$-construction $[10]$, which are always connective. By construction, this non-connective $K$-theory functor agrees with Lueck’s for simplicial rings.

We can use a similar argument to show that, at least for connective $\mathbf{S}$-algebras, the negative $K$-groups arising from iteration of the above construction $[11]$ Cor. IV.10.3] depend only on $\pi_0(A)$. This result appears as $[1]$ Thm. 9.53] but our proof is independent of that result and substantially more direct.

**Theorem 6.** For any connective $\mathbf{S}$-algebra, the augmentation $A \mapsto \pi_0(A)$ induces an isomorphism

$$K_n(A) = K_n(\pi_0(A)), \quad n \leq 1.$$

**Proof.** For simplicial rings $R$, the map $R \to \pi_0(R)$ is 1-connected, so $K^B(R) \to K^B(\pi_0(R))$ is 2-connected. We can extend this result to connective $\mathbf{S}$-algebras $A$ by resolving $K^B(A)$ and $K^B(\pi_0(A))$ by simplicial rings as in the proof of theorem $[4]$. Let $X_S$ be the resolution $n$-cube for $K^B(A)$ completed to a cartesian $n$-cube, and $Y_S$ likewise for $K^B(\pi_0(A))$. When $n = 2$, we arrive at the following diagram for $X_S$.

$$K^B(A) = K^B(A_\emptyset) \longrightarrow X_\emptyset \longrightarrow X_{\{1\}} = K^B(A)_{\{1\}} \downarrow \quad \downarrow$$

$$X_{\{2\}} = K^B(A)_{\{2\}} \longrightarrow X_{\{1,2\}} = K^B(A)_{\{1,2\}}$$

We know that $X_S$ and $Y_S$ are simplicial rings when $S \neq \emptyset$ so the maps $X_S \to Y_S$ are 2-connected. Following the proof of theorem $[4]$ we extend the desired result to $X_\emptyset \to Y_\emptyset$ by analyzing the induced maps between the fiber sequences.

$$\text{tohofib}_{S \in P_{\text{op}} - \emptyset} X_S \longrightarrow X_\emptyset \longrightarrow \lim_{S \in P_{\text{op}} - \emptyset} X_S$$

$$\text{tohofib}_{S \in P_{\text{op}} - \emptyset} Y_S \longrightarrow Y_\emptyset \longrightarrow \lim_{S \in P_{\text{op}} - \emptyset} Y_S$$

Here, the left and right maps are $\pi_n$-isomorphisms for $n \leq 1$ and surjections on $n = 2$ from the simplicial ring case. The long exact sequence in homotopy groups shows that the middle is a $\pi_n$-isomorphism for $n \leq 1$.

Finally, $K$-theory carries id-cartesian $n$-cubes of $\mathbf{S}$-algebras to $(n+1)$-cartesian cubes $[3]$ Thm. 3.2.1], so the comparison maps $K^B(A) \to X_\emptyset$ and $K^B(\pi_0(A)) \to Y_\emptyset$ will be $(n + 1)$-connected. Even just at $n = 2$, this extends the result to $K^B(A) \to K^B(\pi_0(A))$ as desired. \hfill $\Box$

**Definition 7.** The NK-spectrum of an $\mathbf{S}$-algebra $A$ is $NK(A) := \text{hofib}(K^B(A[t]) \to K^B(A))$.

To make the notation correspond with convention, we should set $NK^+(A) := NK(A)$ as just defined, and $NK^-(A) := \text{hofib}(K^B(A[t^{-1}]) \to K^B(A))$. In this way, we arrive at a more conventional formulation of Theorem $[4]$

**Theorem 8.** For a connective $\mathbf{S}$-algebra $A$, there is a functorial splitting of spectra

$$K^B(A[t,t^{-1}]) \simeq K^B(A) \vee \Omega^{-1}(K^B(A)) \vee NK^+(A) \vee NK^-(A)$$
where $\Omega^{-1}(K^B(A))$ denotes the non-connective delooping of $K^B(A)$ indicated above. Moreover, the involution $t \mapsto t^{-1}$ induces an involution on $K^B(A[t,t^{-1}])$ which acts as the identity on the first two factors and switches the second two factors.

In the particular case $A = \Sigma^\infty(\Omega(X)_+)$ for a connected pointed space $X$, we recover the main results of [6],[7].

Given the difficulty of computing $NK_*(R)$ for discrete rings, it is not surprising that not much is known about $NK(A)$ for general $S$-algebras $A$. In the discrete setting, it is a classical result of Quillen that $R$ Noetherian regular implies $NK(R) \simeq \ast$. This fact led to the notion of $NK$-regularity; rings whose $NK$-spectrum was contractible. Via the above discussion, the same notion of $NK$-regularity may be extended to arbitrary $S$-algebras.

It has been shown by Klein and Williams [8] that the map of Waldhausen spaces arising from the Fundamental Theorem of [6] (and temporarily writing $A(X)$ for the Waldhausen $K$-theory of the space $X$)

$$A(*) \vee \Omega^{-1}A(*) \to A(S^1)$$

is the inclusion of a summand but not an equivalence. In the notation used here, $A(*) = K(S)$ and $A(S^1) = K(S[t,t^{-1}])$, where $S$ denotes the sphere spectrum. Thus (unlike the case of the discrete ring $\mathbb{Z}$), one has

**Corollary 9.** The sphere spectrum $S$ is not $NK$-regular.

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