DISTRIBUTION OF A RANDOM FUNCTIONAL OF A FERGUSON-
DIRICHLET PROCESS OVER THE UNIT SPHERE

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Abstract
Jiang, Dickey, and Kuo [12] gave the multivariate $c$-characteristic function and showed that it has properties similar to those of the multivariate Fourier transformation. We first give the multivariate $c$-characteristic function of a random functional of a Ferguson-Dirichlet process over the unit sphere. We then find out its probability density function using properties of the multivariate $c$-characteristic function. This new result would generalize that given by [11].

1 Introduction

Ferguson [5] introduced the Ferguson-Dirichlet process and studied its applications to nonparametric Bayesian inference. He also showed that when the prior distribution is a Ferguson-Dirichlet process with parameter $\mu$, then the posterior distribution, given the sample $s_1, s_2, \ldots, s_n$, is also a Ferguson-Dirichlet process having parameter $\mu + \sum_{j=1}^n \delta_{s_j}$, where $\delta_{s_j}$ denotes point mass at $s_j$. The most natural use of random functionals of a Ferguson-Dirichlet process is to make Bayesian inferences concerning the parameters of a statistical population. Hence, the expression for the probability density function of any random functional of a Ferguson-Dirichlet process can be employed both for prior and posterior Bayesian analyses. Further applications related to the random functional can be seen in [3] and other references. For example, random means and random variances of a Ferguson-Dirichlet process can be used for smooth Bayesian nonparametric density estimation (see [15]) and for quality control problems (see [4] for further discussions), respectively.

Research on the distribution of a random functional of a Ferguson-Dirichlet process has been ongoing for decades. A partial list of papers in this area are [2, 3, 8, 9, 11, 12, 14, 16, 17]. In particular, [11] gave the distribution of a random functional of a Ferguson-Dirichlet process over the unit circle. In this paper, we shall use the multivariate $c$-characteristic function, a tool given
by [12], to generalize the result to the case over the unit sphere in three-dimension.
In Section 2, we first review the definition of the multivariate c-characteristic function and some of its properties. We then compute a multivariate c-characteristic function of an interesting distribution. The multivariate c-characteristic function of the random mean of a Ferguson-Dirichlet process over the unit sphere is given in Section 3. Using the uniqueness property of the multivariate c-characteristic function, we then determine the distribution of the random mean of a Ferguson-Dirichlet process over the unit sphere. Conclusions are given in Section 4.

2 Multivariate c-characteristic function

Jiang [10] first gave a univariate c-characteristic function. Jiang, Dickey, and Kuo [12] generalized it to a multivariate c-characteristic function, which can be very useful when a distribution is difficult to deal with by traditional characteristic function. See [12] for detailed results. First, we state the definition of the multivariate c-characteristic function.

**Definition 1.** If \( u = (u_1, \ldots, u_L) \) is a random vector on a subset \( S \) of \( A = [-a_1, a_1] \times \cdots \times [-a_L, a_L] \), its multivariate c-characteristic function is defined as

\[
g(t; u, c) = E[(1 - i t \cdot u)^c], \quad |t| < a^{-1},
\]

where \( c > 0, a = \sqrt{\sum_{i=1}^L a_i^2}, \ t' = (t_1, \ldots, t_L), \ |t| = \sqrt{\sum_{i=1}^L t_i^2}, \) and \( t \cdot u \) is the inner product of \( t \) and \( u \).

The above assumptions that \( c \) is positive and \( u \) has a bounded support are needed in [12] Lemma 2.2, which shows that, for any positive \( c \), there is a one-to-one correspondence between \( g(t; u, c) \) and the distribution of \( u \).

Next, we give the multivariate c-characteristic function of an interesting distribution in the next lemma.

**Lemma 2.** Let \( u = (u_1, u_2, u_3) \) be a distribution on the inside of a unit ball, i.e., \( \{(u_1, u_2, u_3) \mid u_1^2 + u_2^2 + u_3^2 < 1\} \), with the probability density function

\[
f(u) = \frac{-e}{4\pi^2 r} (1 + r)^{-(1+r)/2}(1 - r)^{-(1-r)/2} \left( -\pi \sin \frac{\pi r}{2} + \ln \frac{1 - r}{1 + r} \cos \frac{\pi r}{2} \right),
\]

where \( r = |u| \). Then the multivariate 1-characteristic function of \( u \) is

\[
g(t; u, 1) = \exp \left( \sum_{n=1}^\infty \frac{(-t_1^2 - t_2^2 - t_3^2)^n}{2n(2n+1)} \right). \tag{1}
\]

**Proof.** Let \( C = \{(u_1, u_2, u_3) \mid u_1^2 + u_2^2 + u_3^2 < 1\} \). Eq. (1) is equivalent to the following identity

\[
\int_C (1 - i t \cdot u)^{-1} f(u) \, du = \exp \left( \sum_{n=1}^\infty \frac{(-t_1^2 - t_2^2 - t_3^2)^n}{2n(2n+1)} \right).
\]

To prove the above identity, we establish the following four equations first. From [7] p. 105, we have

\[
\int_0^{2\pi} (a \cos \alpha + b \sin \alpha)^n \, d\alpha = \begin{cases} 
\frac{(1/2n)(2\sqrt{a^2+b^2})^{n/2} \pi}{(n/2)!}, & n \text{ is even}, \\
0, & n \text{ is odd},
\end{cases} \tag{2}
\]
where \( a \) and \( b \) are real numbers and \((a,k) = a(a+1) \cdots (a+k-1)\). We also can obtain the following equation from [6 Eq. 3.621.5],

\[
\int_0^\pi \sin^{a-1} x \cos^{b-1} x \, dx = \begin{cases} \frac{B(a/2,b/2)}{2}, & \text{Re } a > 0, \ b > 0 \text{ is odd}, \\ 0, & \text{Re } a > 0, \ b > 0 \text{ is even}. \end{cases}
\] (3)

Using integration by parts, we have the following identity,

\[
\int_0^1 r^{2n+1}(1+r)^{-(1+r)/2}(1-r)^{-(1-r)/2} \left( -\pi \sin \frac{\pi r}{2} \right) \, dr = \int_0^1 r^{2n+1}(1+r)^{-(1+r)/2}(1-r)^{-(1-r)/2} \ln \frac{1-r}{1+r} \cos \frac{\pi r}{2} \, dr
\] (4)

Using [13 Lemma 8 and Example 2], we can obtain the following equality:

\[
\exp \left( -\int_{-1}^1 \ln(1-itx) \frac{1}{2} \, dx \right) = \int_{-1}^1 (1-itx)^{-1} \frac{e}{\pi} (x+1)^{-x+1/2}(1-x)^{-x+1/2} \cos \frac{\pi x}{2} \, dx.
\]

Since

\[
\exp \left( -\int_{-1}^1 \ln(1-itx) \frac{1}{2} \, dx \right) = \exp \left( \sum_{n=1}^\infty \frac{(-t^2)^n}{2n(2n+1)} \right)
\]

and

\[
\int_{-1}^1 (1-itx)^{-1} \frac{e}{\pi} (x+1)^{-x+1/2}(1-x)^{-x+1/2} \cos \frac{\pi x}{2} \, dx = \sum_{n=0}^\infty \int_{-1}^1 \frac{e^{-itx^n}}{\pi} x^n(x+1)^{-x+1/2}(1-x)^{-x+1/2} \cos \frac{\pi x}{2} \, dx,
\]

and by the fact that the function \((x+1)^{-x+1/2}(1-x)^{-x+1/2} \cos \frac{\pi x}{2}\) is symmetric at \(x = 0\), we have

\[
\exp \left( \sum_{n=1}^\infty \frac{(-t^2)^n}{2n(2n+1)} \right) = \frac{2e}{\pi} \sum_{n=0}^\infty (-t^2)^n \int_0^1 x^{2n}(x+1)^{-x+1/2}(1-x)^{-x+1/2} \cos \frac{\pi x}{2} \, dx. \] (5)

Setting

\[
g(r) = \frac{-er}{4\pi^2} (1+r)^{-(1+r)/2}(1-r)^{-(1-r)/2} \left( -\pi \sin \frac{\pi r}{2} + \ln \frac{1-r}{1+r} \cos \frac{\pi r}{2} \right),
\]
A random functional of a Ferguson-Dirichlet process over the unit sphere

and using the spherical coordinate transformation, we have

\[
\begin{align*}
\int_C (1 - it \cdot u)^{-1} f(u) \, du \\
&= \int_0^1 \int_0^{2\pi} \int_0^\pi (1 - it_1 r \cos \theta \sin \phi - it_2 r \sin \theta \sin \phi - it_3 r \cos \phi)^{-1} \sin \phi g(r) \, d\phi \, d\theta \, dr \\
&= \int_0^1 \sum_{n=0}^{\infty} (ir)^n g(r) \int_0^{2\pi} \int_0^\pi (t_1 \cos \theta \sin \phi + t_2 \sin \theta \sin \phi + t_3 \cos \phi)^n \sin \phi \, d\phi \, d\theta \, dr \\
&= \int_0^1 \sum_{n=0}^{\infty} (ir)^n g(r) \int_0^{2\pi} \int_0^\pi \sum_{k=0}^n \binom{n}{k} (t_1 \cos \theta + t_2 \sin \theta)^k t_3^{n-k} \sin^{n-k+1} \phi \cos \phi \sin^k \phi \, d\phi \, d\theta \, dr \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{4\pi(-t_1^2 - t_2^2 - t_3^2)^n r^{2n}}{2n + 1} g(r) \, dr \\
&= \frac{2e}{\pi} \sum_{n=0}^{\infty} \frac{(-t_1^2 - t_2^2 - t_3^2)^n}{2n(2n+1)} \int_0^1 r^{2n}(1 + r)^{-(1+r)/2}(1 - r)^{-(1-r)/2} \cos \frac{\pi r}{2} \, dr \\
&= \exp \left( \sum_{n=1}^{\infty} \frac{(-t_1^2 - t_2^2 - t_3^2)^n}{2n(2n+1)} \right). \tag{8}
\end{align*}
\]

Identity (5) can be obtained by Eqs. (2) and (3). Identities (7) and (8) follow from Eq. (4) and Eq. (5), respectively. \hfill \square

3 Distribution of a random functional of a Ferguson-Dirichlet process over the unit sphere

Ferguson [5] first defined the Ferguson-Dirichlet process. Let \( \mu \) be a finite non-null measure on \((Y,A)\), where \( Y \) is a Borel set in Euclidean space \( \mathbb{R}^n \) and \( A \) is the \( \sigma \)-field of Borel subsets of \( Y \), and let \( U \) be a stochastic process indexed by elements of \( A \). We say that \( U \) is a Ferguson-Dirichlet process with parameter \( \mu \), if for every finite measurable partition \( \{B_1, \ldots, B_m\} \) of \( Y \), the random vector \((U(B_1), \ldots, U(B_m))\) has a Dirichlet distribution with parameter \((\mu(B_1), \ldots, \mu(B_m))\), where \( \mu(B_j) > 0 \) for all \( j = 1, \ldots, m \). A random vector \( \mathbf{v} = (v_1, \ldots, v_m)' \) is said to have a Dirichlet distribution with parameter \( \mathbf{b} = (b_1, \ldots, b_m)' \) where each \( b_j > 0 \), if \( \mathbf{v} \) has the probability density function

\[
f(\mathbf{v}; \mathbf{b}) = \frac{\Gamma(b_1 + \cdots + b_m)}{\prod_{j=1}^m \Gamma(b_j)} \prod_{j=1}^m v_j^{b_j - 1},
\]

for all \( \mathbf{v} \) in the probability simplex \( \{ \mathbf{v} \mid \text{each } v_j \geq 0, v_1 + \cdots + v_m = 1 \} \).

First, we give a trivariate \( c \)-characteristic function expression of any trivariate random functional of a Ferguson-Dirichlet process over a Borel set \( Y \) in Euclidean space in the next lemma.

Lemma 3. Let \( \mathbf{w} = \int \mathbf{h(x)} \, dU(x) \) be a random functional where \( \mathbf{h(x)} = (h_1(x), h_2(x), h_3(x))' \) is a bounded measurable function defined on a Borel set \( Y \) in Euclidean space \( \mathbb{R}^n \), and \( U \) is a Ferguson-Dirichlet process with parameter \( \mu \) on \((Y,A)\). Then the trivariate \( c \)-characteristic function of \( \mathbf{w} \) can
be expressed as

\[ g(t; w, c) = \exp \left( - \int_Y (1 - it \cdot h(x)) d\mu(x) \right), \text{ where } c = \mu(Y). \]

**Proof.** For any \( k \geq 2 \), let \( \{B_{kj}\} \) be a partition of \( Y \), \( b_{kj} \in B_{kj}, v_k = \max\{\text{volume}(B_{kj})\} \)

\( 1 \leq j \leq k \), and \( \lim_{k \to \infty} v_k = 0 \). Then \( (U(B_{k1}), \ldots, U(B_{kk})) \) follows a Dirichlet distribution with parameter \((\mu(B_{k1}), \ldots, \mu(B_{kk}))\). In addition, \( \sum_{j=1}^k U(B_{kj}) = 1 \) for all \( k \geq 2 \). Define \( g_k(x) = \sum_{j=1}^k h(b_{kj})\delta_{b_{kj}}(x) \) and \( w_k = \int_Y g_k(x) dU(x) \), where \( \delta_{b_{kj}}(x) \) is 1, for \( x \in B_{kj}; \) and is 0, otherwise. Then \( \lim_{k \to \infty} g_k(x) = h(x) \) for all \( x \in Y \), and \( w_k = \sum_{j=1}^k g_k(b_{kj})U(B_{kj}) \). The trivariate \( c \)-characteristic function of \( w_k \) can be expressed as

\[
g(t; w_k, c) = E(1 - it \cdot w_k)^{-c} = E \left( 1 - i \sum_{j=1}^k [t \cdot g_k(b_{kj})]U(B_{kj}) \right)^{-c} = E \left( \sum_{j=1}^k U(B_{kj})[1 - it \cdot g_k(b_{kj})] \right)^{-c} = R^{-c}(\mu(B_{k1}), \ldots, \mu(B_{kk})); 1 - it \cdot g_k(b_{k1}), \ldots, 1 - it \cdot g_k(b_{kk})) = \prod_{j=1}^k (1 - it \cdot g_k(b_{kj}))^{-\mu(b_{kj})},
\]

where \( R \) is a Carlson’s multiple hypergeometric function \((1\text{I})\), and the last equality can be obtained by \([1\text{I}] \text{ formula 6.6.5}\). Therefore, the limit of the trivariate \( c \)-characteristic function of \( w_k \)’s, as \( k \) approaches \( \infty \), is

\[
\lim_{k \to \infty} g(t; w_k, c) = \exp \left( \lim_{k \to \infty} \sum_{j=1}^k -\mu(B_{kj})\ln(1 - it \cdot g_k(b_{kj})) \right) = \exp \left( - \int_Y \ln(1 - it \cdot h(x)) d\mu(x) \right).
\]

In addition, by the Dominated Convergence Theorem, we have \( \lim_{k \to \infty} w_k = w \). By \([1\text{I2}] \text{ Theorem 2.4}\), we conclude that

\[ g(t; w, c) = \exp \left( - \int_Y \ln(1 - it \cdot h(x)) d\mu(x) \right). \]

\[ \square \]

In the rest of this section, we study the random functional \( u = \int_X x dU(x) \), where \( X \) is the unit sphere in \( \mathbb{R}^3 \). We use Lemma 3 in the following theorem to first establish the trivariate \( c \)-characteristic function of \( u \).

**Theorem 4.** Let \( X = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\} \), and \( U \) be a Ferguson-Dirichlet process over \( X \) with uniform measure \( \mu \) as its parameter, where \( \mu(X) = c \). Then the trivariate \( c \)-characteristic function of \( u \).
function of the random functional \( u = \int_{X} x \, dU(x) \) can be expressed as

\[
g(t; u, c) = \exp \left( \sum_{n=1}^{\infty} \frac{c}{2n(2n+1)} (-t_1^2 - t_2^2 - t_3^2)^n \right), \text{ where } t = (t_1, t_2, t_3)^t.
\]

**Proof.** First, we give the following two equations, which are about Appell's notations and can be shown easily.

\[
\Gamma(a + n) = \Gamma(a)(a, n), \quad (a, 2n) = 2^n \left( \frac{a}{2}, n \right) \left( \frac{a+1}{2}, n \right).
\]

By Lemma \[3\] we have

\[
g(t; u, c) = \exp \left( \frac{-c}{4\pi} \int_{X} \ln(1 - it \cdot x) \, dx \right)
= \exp \left( \frac{-c}{4\pi} \int_{0}^{\pi} \int_{0}^{2\pi} \ln(1 - it_1 \cos \theta_1 - it_2 \sin \theta_1 \cos \theta_2 - it_3 \sin \theta_1 \sin \theta_2) \, d\theta_1 \, d\theta_2 \right)
= \exp \left( \frac{c}{4\pi} \sum_{n=1}^{\infty} \frac{i^n}{n} \int_{0}^{2\pi} (t_1 \cos \theta_1 + t_2 \sin \theta_1 \cos \theta_2 + t_3 \sin \theta_1 \sin \theta_2)^n \, d\theta_1 \, d\theta_2 \right)
= \exp \left( \frac{c}{4\pi} \sum_{n=1}^{\infty} \frac{i^n}{n} \sum_{k=0}^{n} \binom{n}{k} \int_{0}^{2\pi} (t_1 \cos \theta_1)^k \sin^{n-k+1} \theta_1 (t_2 \cos \theta_2 + t_3 \sin \theta_2)^{n-k} \, d\theta_1 \, d\theta_2 \right)
= \exp \left( \frac{c}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \sum_{k=0}^{n} \binom{2n}{2k} (1/2, n-k)(t_2^2 + t_3^2)^{n-k} \frac{t_1^{2k} B(n-k+1, k+1/2)}{(n-k)!} \right)
= \exp \left( \sum_{n=1}^{\infty} \frac{c}{2n(2n+1)} (-t_1^2 - t_2^2 - t_3^2)^n \right).
\]

The fifth identity can be obtained by Eqs. (2) and (3). The last identity follows from Eqs. (9) and (10).

By [12] Lemma 2.2, Lemma 2 and Theorem 4, we can obtain the following corollary.

**Corollary 5.** The probability density function of \( u = \int_{X} x \, dU(x) \), where \( U \) is a Ferguson-Dirichlet process over the unit sphere \( X \) with uniform probability measure as its parameter, is

\[
f(u) = \frac{-e}{4\pi^2 r} (1 + r)^{-(1+r)/2} (1 - r)^{-(1-r)/2} \left( -\pi \sin \frac{\pi r}{2} + \ln \frac{1 - r}{1 + r} \cos \frac{\pi r}{2} \right),
\]

where \( r = \sqrt{u_1^2 + u_2^2 + u_3^2} \) and \( u_1^2 + u_2^2 + u_3^2 < 1 \).

**4 Conclusions**

In this paper, we obtain the trivariate \( c \)-characteristic function expression for a random functional of a Ferguson-Dirichlet process over any finite three-dimensional space. We also obtain the probability density function of the random functional of a Ferguson-Dirichlet process with uniform probability measure parameter over the unit sphere. This generalizes [11] Theorem 2.
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