Bifurcation currents in holomorphic dynamics on $\mathbb{P}^k$

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Introduction

Potential theory has been introduced in one-dimensional rational dynamics by Brolin and Tortrat ([4], [30]) but it does not play a central role there. In higher dimension however, as the classical tools are not any longer efficient, pluri-potential theory has revealed itself to be essential. The fundamental works of Hubbard-Padapopol, Fornaess-Sibony, Briend-Duval, Bedford-Smillie (see [27] for precise references) enlighten the remarkable effectiveness of pluri-potential theory in holomorphic dynamics on $\mathbb{P}^k$ or $\mathbb{C}^k$. It is therefore tempting to study the parameter spaces in a similar way. More precisely, one would like to relate the bifurcations of a holomorphic family $\{ f_\lambda \}_{\lambda \in X}$ of endomorphisms of $\mathbb{P}^k$ to the powers of a certain current on the parameter space $X$.

Let us recall that in dimension $k = 1$, a bifurcation is said to occur at some point $\lambda_0 \in X$ if the Julia set of $f_\lambda$ does not move continuously around $\lambda_0$. The famous work of Mañé-Sad-Sullivan [16], which is based on the $\lambda$-lemma and the Fatou-Cremer-Sullivan classification, relates the bifurcations with the instability of the critical orbits. It also asserts that the bifurcations concentrate on the complement of an open dense subset of $X$ (for the quadratic family $\{ z^2 + \lambda \}_{\lambda \in \mathbb{C}}$, the bifurcation locus is precisely the boundary of the Mandelbrot set).

A seminal idea towards the application of potential theory to the study of bifurcations is due to Przytycki, who raised the following problem in the final remarks of his paper. Problem [23]: understand the connections between Lyapunov characteristic exponents and potential theory for rational mappings.

To support his point of view, Przytycki also analysed the following formula for a polynomial $p$ of degree $d$ on $\mathbb{C}$ (see also [18]):

$$ L(p) = \sum_j G_p(c_j) + \log d $$

(0.1)

where $L(p)$ is the Lyapunov exponent of $p$ with respect to its equilibrium measure, $G_p$ its Green function and $c_j$ are the critical points of $p$. More recently, DeMarco ([9], [8]) has obtained a generalization of this formula to rational maps of $\mathbb{P}^1$. She also used her formula to show that, for a holomorphic family $\{ f_\lambda \}_{\lambda \in X}$, the current $dd^c L(f_\lambda)$ is supported by the

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bifurcation locus.

The results of the present paper deal with Przytycki problem. Our first goal is to show that $d\bar{d}c\, L(f_\lambda)$ is a reasonable bifurcation current in any dimension. To this purpose, we prove the following theorem in Section 2.

**Theorem 2.2** Given a holomorphic family $\{f_\lambda\}_{\lambda \in X}$ of endomorphisms of $\mathbb{P}^k$, the function $L(f_\lambda)$, defined as the sum of Lyapunov exponents of $f_\lambda$ for its Green measure, is pluriharmonic on $X$ if the repulsive cycles of $f_\lambda$ move holomorphically on $X$.

Let us mention here that all what we need to know about $L$ in the paper is that $L(F) = \int_{C_{k+1}} \log|\det F'| \mu_F$ where $F$ is a lift of $f$ and $\mu_F$ is its Green measure.

With the goal of analysing the support of $d\bar{d}c\, L(f_\lambda)$ and its powers, we then generalize formula (0.1) to endomorphisms of $\mathbb{P}^k$. This is done in a very natural way by using an integration by part formula on a suitable line bundle. We obtain the following

**Formula** (see Theorem 4.1) $L(f) = \sum_{j=0}^{k-1} \int_{\mathbb{P}^k} g_{F_\lambda} (d\bar{d}c g_{F_\lambda} + \omega)^j \wedge \omega^{k-j-1} \wedge [C_X] -$

$- \log d + \int_{\mathbb{P}^k} \log||J_F||_0 \omega^k - (k+1)(d-1) \sum_{j=0}^{k} \int_{\mathbb{P}^k} g_{F_\lambda} (d\bar{d}c g_{F_\lambda} + \omega)^j \wedge \omega^{k-j}.$

For a holomorphic family $\{f_\lambda\}_{\lambda \in X}$, the above formula allows us to compute the bifurcation current $d\bar{d}c\, L(f_\lambda)$. We get the following synthetic statement:

**Theorem** (see Corollary 4.6) $d\bar{d}c\, L(f_\lambda) = p_\ast((d\bar{d}c g_{F_\lambda} + \omega)^k \wedge [C_X]), \quad (0.2)$

and on $X \times \mathbb{P}^k$

$$(d\bar{d}c g_{F_\lambda} + \omega)^{k+1} = 0. \quad (0.3)$$

In these formulas the operator $d\bar{d}c$ is acting on $X \times \mathbb{P}^k$, $p$ is the canonical projection from $X \times \mathbb{P}^k$ to $X$, $g_{F_\lambda}$ is the Green function of $f_\lambda$ on $\mathbb{P}^k$ associated to the lift $F_\lambda$ and $C_X$ is the hypersurface of $X \times \mathbb{P}^k$ defined by the equation $\det F'_\lambda(z) = 0$.

It is worth emphasize that there is a certain interaction between formulas (0.2) and (0.3), this may be seen in the example in Subsection 7.2 and in the Appendix (see formula (7.4)). Moreover, formula (0.3) is formally equivalent to the equation of geodesics on the space of Kähler metrics on $\mathbb{P}^k$; this leads to some examples of such geodesics, as discussed in Subsection 7.1. These results are established in Section 4 but we treat the case of dimension $k = 1$ separately in Section 3 since it is technically less involved and may help the reader to a better understanding. Let us also stress that in the one-dimensional case we get several explicit formulas for $L(f)$ (see Theorem 3.1). Moreover, our approach offers a much simpler proof of DeMarco’s formula. The equivalence between DeMarco’s formula and ours is a consequence of the following identity which may be of independent interest (see Proposition 4.9):

$$\int_{\mathbb{P}^1} g_F(\mu_F + \omega) = \frac{1}{2} (\log |\text{Res}(F)| - 1).$$

Sections 5 and 6 are devoted to the study of bifurcations for holomorphic families $\{f_\lambda\}_{\lambda \in X}$ of rational maps of $\mathbb{P}^1$. For a holomorphic family $\{f_\lambda\}_{\lambda \in X}$ of endomorphisms
of $\mathbb{P}^1$ we obtain a geometrical description of the support of the bifurcation currents $(dd^cL(f_\lambda))^p$ by means of certain complex hypersurfaces.

For $\theta \in \mathbb{R} \setminus \mathbb{Z}$, the set of all $\lambda \in X$ such that $f_\lambda$ has a periodic point of period $n$ and multiplier $e^{2i\pi\theta}$ is generically a complex hypersurface of $X$, denoted by $\text{Per}(X,n,e^{2i\pi\theta})$. Therefore, for $n$ fixed, $\bigcup_\theta \text{Per}(X,n,e^{2i\pi\theta})$ can be thought as a real hypersurface foliated by complex hypersurfaces. The union $Z_1(X)$ of all these hypersurfaces is dense in the support of the the bifurcation current $dd^cL(f_\lambda)$:

**Theorem** $Z_1(X) = \text{Supp}(dd^cL(f_\lambda))$.

We included this geometrical characterization of the bifurcation locus in the statement of Theorem 5.2 which may be interpreted as treating a substantial part of Mañé-Sad-Sullivan theory by potential-theoretic methods. The proof exploits the links between the vanishing of $dd^cL$, the motion of repulsive cycles and the stability of critical orbits. We point out that these links are revealed by formula (0.2) and the above quoted Theorem 2.2.

For the powers of $dd^cL(f_\lambda)$ the geometry is more involved. Taking all possible intersections between $p$ of the above complex hypersurfaces one gets a large family of codimension $p$ subvarieties of $X$; the union of this family, denoted by $Z_p(X)$, satisfies:

**Theorem** 5.5 For any $1 \leq p \leq \dim_{\mathbb{C}} X$, $\text{Supp}((dd^cL(f_\lambda))^p) \subset Z_p(X)$.

These results have some significant consequences considering the family $\mathcal{H}_d(\mathbb{P}^1)$ of all the rational maps of degree $d$. First of all one may show that:

**Proposition** 6.5 For $1 \leq p \leq 2d+1$ the bifurcation current $(dd^cL)^p$ has finite mass on $\mathcal{H}_d(\mathbb{P}^1)$.

This implies that $(dd^cL)^{2d-2}$ induces a measure $\mu$ of finite mass on the moduli space $\mathcal{M}_d$ of rational maps of degree $d$. We call it the bifurcation measure and show that its support contains all isolated Lattès maps. Using our description of $\text{Supp}((dd^cL)^p)$ we also obtain the following fact:

**Proposition** 6.8 Any open set of $\mathcal{M}_d$ intersecting the support of the bifurcation measure contains an uncountable set of chaotic mappings.

As a by-product of our investigation one sees that any non flexible Lattès map is generating quite complicated bifurcations.

1 Preliminaries

In all the paper $\omega$ denotes the Fubini-Study form in $\mathbb{P}^k$ and let $||| \ | ||$ be the Hermitian norm in $\mathbb{C}^{k+1}$.

1.1 The spaces $\mathcal{H}_d(\mathbb{C}^{k+1})$ and $\mathcal{H}_d(\mathbb{P}^k)$

Every holomorphic endomorphism $f$ of $\mathbb{P}^k$ has a lift $F : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$, that is: a homogeneous, non-degenerate, polynomial map such that $\pi \circ F = f \circ \pi$, where $\mathbb{C}^{k+1} \setminus \{0\} \overset{\pi}{\rightarrow} \mathbb{P}^k$ is the canonical projection. The degree $d$ of $F$ is, by definition, the algebraic degree of $f$, while $d^k$ is the topological degree of $f$. 


In the following it will always be assumed that $d \geq 2$.

Since a homogeneous polynomial of degree $d$ in $k + 1$ variables depends on \( \frac{(d+k)!}{d!k!} \) coefficients, such a lift $F = (F_0, \ldots, F_k)$ can be identified with an element of $\mathbb{C}^{N+1}$, where $N = (k + 1)\frac{(d+k)!}{d!k!} - 1$. With this identification, the space of all homogeneous, non-degenerate, polynomial maps of degree $d$ on $\mathbb{C}^{k+1}$ is an open subset of $\mathbb{C}^{N+1}$ which we denote by $\mathcal{H}_d(\mathbb{C}^{k+1})$. Denoting again by $\mathbb{C}^{N+1} \setminus \{0\}$ the canonical projection, we get $\pi(F) = f$ and that $\pi(\mathcal{H}_d(\mathbb{C}^{k+1}))$ is the space of all holomorphic endomorphisms of $\mathbb{P}^k$ of degree $d$, which we denote by $\mathcal{H}_d(\mathbb{P}^k)$.

The following Proposition shows that the complement of $\mathcal{H}_d(\mathbb{C}^{k+1})$ in $\mathbb{C}^{N+1}$ is an irreducible complex hypersurface $\Sigma_d = \{ \text{Res} = 0 \}$. Thus the projective hypersurface $\Sigma_d := \pi(\Sigma_d)$ is the complement of $\mathcal{H}_d(\mathbb{P}^k)$ in $\mathbb{P}^N$.

**Proposition 1.1** Let $F_0, \ldots, F_k$, be homogeneous polynomials of degree $d$, in $k + 1$ complex variables. There exists a unique polynomial $\text{Res}(F_0, \ldots, F_k)$ in the coefficients of $F_0, \ldots, F_k$, which is homogeneous of degree $(k + 1)d^k$, irreducible, and such that

(i) $\text{Res}(F_0, \ldots, F_k) = 0$ if and only if $F = (F_0, \ldots, F_k) : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ is degenerate,

(ii) $\text{Res}(z_0^d, \ldots, z_k^d) = 1$.

**Proof.** See [13] p. 427 and p. 105.

### 1.2 Green functions

To any $F \in \mathcal{H}_d(\mathbb{C}^{k+1})$ it is associated a Green function $G_F$ defined by

$$G_F := \lim_n d^{-n} \log \| F^n(z) \|.$$ 

Let us stress that $G_F$ is the limit of a sequence $\{G_{F,n}\}$ of p.s.h. and continuous functions on $\mathcal{H}_d(\mathbb{C}^{k+1}) \times (\mathbb{C}^{k+1} \setminus \{0\})$. The following Proposition summarizes the regularity properties of $G_F(z)$. The only novelty here is the Hölder-continuity in $F$.

**Proposition 1.2** i) For any compact subset $\mathcal{K}$ of $\mathcal{H}_d(\mathbb{C}^{k+1})$, the sequence $G_{F,n}(z)$ converges uniformly to $G_F(z)$ on $\mathcal{K} \times (\mathbb{C}^{k+1} \setminus \{0\})$. In particular $G_F(z)$ is p.s.h and continuous on $\mathcal{H}_d(\mathbb{C}^{k+1}) \times (\mathbb{C}^{k+1} \setminus \{0\})$. It satisfies the following homogeneity property:

$$G_F(tz) = \log |t| + G_F(z); \quad \forall t \in \mathbb{C}^*, \forall z \in \mathbb{C}^{k+1}$$

and the functional equation

$$G_F \circ F = dG_F.$$ (1.1)

In the definition of $G_F$, the norm $\| \|$ may be replaced by any continuous gauge function.

ii) The function $G_F(z)$ is Hölder-continuous on every compact subset of $\mathcal{H}_d(\mathbb{C}^{k+1}) \times (\mathbb{C}^{k+1} \setminus \{0\})$.

**Proof.** i) Let $N$ be any continuous gauge function on $\mathbb{C}^{k+1}$. Let $K$ be a compact subset of $\mathcal{H}_d(\mathbb{C}^{k+1})$ and $C > 1$ be a constant such that

$$\frac{1}{C} \leq N(F(z)) \leq C; \quad \forall F \in K, \forall z \in \{N = 1\}.$$
Then, by homogeneity we have:

\[
\frac{1}{C^{1+\ldots+d^n-1}} N(z)^{d^n} \leq N(F^n(z)) \leq C^{1+\ldots+d^n-1} N(z)^{d^n}; \quad \forall F \in \mathcal{K}, \forall z \in \mathbb{C}^{k+1} \setminus \{0\}, \forall n \in \mathbb{N}.
\]

After replacing \(z\) by \(F^m(z)\), taking log and dividing by \(d^{m+m}\), (1.2) gives:

\[
|G_{F,n+m}(z) - G_{F,n}(z)| \leq \frac{\log C}{d^{m(d - 1)}}; \quad \forall F \in \mathcal{K}, \forall z \in \mathbb{C}^{k+1} \setminus \{0\}.
\]

ii) The H"older continuity in \(z\) has been established by Briend-Duval ([3]). Inspecting the proof and taking into account the continuity of \(G_F(z)\) in \((F, z)\), it is not hard to see that the constants might be chosen uniformly in \(F\) (see [27] Théorème 1.7.1 and Remarque 1.7.2). More precisely, for any compact \(K \times K \subset \mathcal{H}_d(\mathbb{C}^{k+1}) \times (\mathbb{C}^{k+1} \setminus \{0\})\) there are constants \(C > 0\) and \(0 < \alpha < 1\) such that:

\[
|G_F(z) - G_F(z')| \leq C\|z - z'\|^{\alpha}; \quad \forall F \in \mathcal{K}, \forall z, z' \in K.
\]

We will show how this property may be transferred to \(F\). We may assume that \(\{G_F = 0\} \subset K\) for every \(F \in \mathcal{K}\). Let us pick \(F, F_0 \in \mathcal{K}\) and consider the gauge function \(N_0 := e^{G_{F_0}}\). By the Hölder-continuity of \(G_{F_0}\) we have

\[
|G_{F_0}(F(z)) - G_{F_0}(F_0(z))| \leq C_1\|F(z) - F_0(z)\|^{\alpha} \leq C_2\|F - F_0\|^{\alpha}; \quad \forall z \in K.
\]

When \(z \in \{N_0 = 1\}\) this inequality becomes \(\frac{1}{C_0} \leq N_0(F(z)) \leq C_0\) where \(C_0 := e^{C_2\|F - F_0\|^{\alpha}}\). Just like for (1.2), this implies

\[
\frac{1}{C^{1+\ldots+d^n-1}} N_0(z)^{d^n} \leq N_0(F^n(z)) \leq C^{1+\ldots+d^n-1} N_0(z)^{d^n}; \quad \forall F \in \mathcal{K}, \forall z \in \mathbb{C}^{k+1} \setminus \{0\}, \forall n \in \mathbb{N}.
\]

Taking log, dividing by \(d^n\) and making \(n \to \infty\), this yields (as \(G_F = \lim_n d^{-n} N_0(F^n(z))\)):

\[
|G_F(z) - G_{F_0}(z)| \leq \frac{C_2}{d-1}\|F - F_0\|^{\alpha}; \quad \forall z \in (\mathbb{C}^{k+1} \setminus \{0\}).
\]

\[
\square
\]

The Green function \(G_F\) induces a continuous, \(\omega\)-\(p.s.h\) function \(g_F\) on \(\mathcal{H}_d(\mathbb{P}^k) \times \mathbb{P}^k\) which will also be called a Green function of \(F\):

\[
g_F \circ \pi := G_F - \log \|\cdot\|.
\]

**Remark 1.3** It is straightforward to check that \(g_F(\pi(z)) \leq \frac{M}{d-1}\), where \(M := \sup_{\|z\|=1} \|F(z)\|\). In particular, for every compact subset \(K \subset \mathbb{C}^{N+1} = \mathcal{H}_d(\mathbb{C}^{k+1}) \cup \Sigma_d\), \(g_F\) is bounded from above on \((K \cap \mathcal{H}_d(\mathbb{C}^{k+1})) \times \mathbb{P}^k\).
1.3 Green currents and measures

Let \( f \in \mathcal{H}_d(P^k) \) and \( F \in \mathcal{H}_d(C^{k+1}) \) be a lift of \( f \). One defines a closed, positive \((1,1)\)-current \( T_f \) on \( P^k \) by setting:

\[
T_f := dd^c g_F + \omega. \tag{1.4}
\]

As (1.3) shows, this current may equivalently be defined by \( \pi^* T_f = dd^c G_F \). Since \( g_a F = \frac{1}{a^2} \log |a| + g_F \), this current does not depend on the choice of the lift \( F \) and will be called the Green current of \( f \). The functional equation (1.1) implies that:

\[
f^* T_f = dT_f. \tag{1.5}
\]

The Green measure \( \mu_f \) of \( f \) is defined by

\[
\mu_f := (T_f)^k.
\]

It is a probability measure with respect to which \( f \) has constant Jacobian: \( f^* \mu_f = d^k \mu_f \). It follows that \( \mu_f \) is \( f \)-invariant \((f^* \mu_f = \mu_f)\) and \( f \)-ergodic.

It will also be useful to consider the probability measures \( m \) and \( \mu_F \) defined on \( C^{k+1} \) by:

\[
m := \left( dd^c \log^+ || || \right)^{k+1} \quad \mu_F := \left( dd^c G_F^+ \right)^{k+1};
\]

these measures are respectively supported by \( \{ || = 1 \} \) and \( \{ G_F = 0 \} \); they are related to \( \omega^k \) and \( \mu_f \) by:

\[
\pi_* m = \omega^k \quad \pi_* \mu_F = \mu_f.
\]

1.4 Lyapunov exponents

Let \( f \in \mathcal{H}_d(P^k) \) and \( F \in \mathcal{H}_d(C^{k+1}) \) a lift of \( f \). We shall denote by \( L(F) \) the sum of Lyapunov exponents of \( F \) with respect to \( \mu_F \) and by \( L(f) \) the sum of Lyapunov exponents of \( f \) with respect to \( \mu_f \). The following facts hold:

(i) \( L(F) = \int_{C^{k+1}} \log |det F| \mu_F \);

(ii) \( L(F) = L(f) + \log d \) (see [14]);

(iii) \( L(F^n) = n L(F) \), for all \( n \in \mathbb{N}^* \) (use (i) and \( f^* \mu_f = \mu_f \));

(iv) \( L(f) \) is p.s.h. on \( \mathcal{H}_d(C^{k+1}) \), as it has been proved in the larger setting of polynomial like mappings (see [10]).

1.5 Green metric on \( O_{P^k}(D) \)

Let \( D \in \mathbb{N}^* \). The line bundle \( O_{P^k}(D) \) over \( P^k \) is conveniently seen as the quotient of \( (C^{k+1} \setminus \{0\}) \times C \) by the relation \((z,x) \equiv (uz, u^D x)\) for all \( u \in C^* \), denoting its elements by \([z,x]\). The canonical metric on \( O_{P^k}(D) \) may be expressed by:

\[
||[z,x]||_0 := e^{-D \log ||z|| \cdot |x|}.
\]
The homogeneity of \( G_F \) allows us to associate to any \( F \in \mathcal{H}_d(\mathbb{C}^{k+1}) \) a Green metric defined on \( \mathcal{O}_{\mathbb{P}^k}(D) \) by:
\[
\| [z, x] \|_{G_F} := e^{-DG_F(z)}|x|.
\]

Then we have the following lemma:

**Lemma 1.4** Let \( F \in \mathcal{H}_d(\mathbb{P}^k) \) and \( D = (k + 1)(d - 1) \). Let us endow \( \mathcal{O}_{\mathbb{P}^k}(D) \) with the canonical and the Green metrics. Then the following identities occur:

1) \( L(F) = \int_{\mathbb{C}^{k+1}} \log |detF'| \mu_F = \int_{\mathbb{P}^k} \log \| J_F \| F \mu_f \)
2) \( \int_{\mathbb{C}^{k+1}} \log |detF'| m = \int_{\mathbb{P}^k} \log \| J_F \| \omega^k. \)

**Proof.** 1) As \( G_F \) identically vanishes on the support of \( \mu_F \), we have \( \int_{\mathbb{C}^{k+1}} \log |detF'| \mu_F = \int_{\mathbb{C}^{k+1}} \log (e^{-DG_F|detF'|}) \mu_F = \int_{\mathbb{C}^{k+1}} \log \| J_F \circ \pi \| \mu_F \) and the conclusion follows from \( \pi_* \mu_F = \mu_f \).
2) We proceed in the same way, using the fact that \( \log^+ \| \cdot \| \) identically vanishes on the support of \( m \) and \( \pi_* m = \omega^k. \)

\[\square\]

### 2 A current detecting the holomorphic motion of repulsive cycles

Let \( \{f_\lambda\}_{\lambda \in \lambda} \) be a holomorphic family of endomorphisms of \( \mathbb{P}^k \) parametrized by a complex manifold \( X \). The p.s.h. function \( L(\lambda) = \lambda f_\lambda \) given by the sum of Lyapunov exponents of \( f_\lambda \), provides a closed, positive \((1,1)\)-current \( dd^cL \) on \( X \). In this section, we will show that \( dd^cL \) vanishes if the repulsive cycles of \( f_\lambda \) move holomorphically. Let us precisely state what we mean by this holomorphic motion.

**Definition 2.1** The repulsive cycles of period \( n \) of \( \{f_\lambda\}_{\lambda \in \lambda} \) move holomorphically over an open subset \( U \) of \( X \) if and only if there exists a collection of holomorphic mappings \( \alpha_{n,j} : U \to \mathbb{P}^k \) such that, for any \( \lambda \in U \), the set of \( n \)-repulsive cycles is given by \( \{\alpha_{n,j}(\lambda)\} \).

In dimension \( k = 1 \), it is well known that the Julia set of \( f_\lambda \) depends continuously on \( \lambda \in U \) if and only if the repulsive cycles of sufficiently high order of \( f_\lambda \) move holomorphically on \( U \) (see [19], Theorem 4.2).

Although such a phenomenon is far from being clear in higher dimension, we would like to motivate the study of \( dd^cL \) as a bifurcation current by the following result:

**Theorem 2.2** Let \( \{f_\lambda\}_{\lambda \in \lambda} \) be a holomorphic family of endomorphisms of \( \mathbb{P}^k \) with algebraic degree \( d \). If all repulsive cycles of \( f_\lambda \) of period \( n \geq n_0 \) move holomorphically on some open subset \( U \) of \( X \) then the sum \( L(f_\lambda) \) of Lyapunov exponents of \( f_\lambda \) is pluriharmonic on \( U \).
Proof. We may assume that $U$ is a small ball on which $\{f_\lambda\}_{\lambda \in U}$ lifts to some holomorphic family $\{F_\lambda\}_{\lambda \in U}$. Then it is not hard to see that the set of $n$-repulsive cycles of $F_\lambda$ is given by $\{a_{n,j}(\lambda) ; \lambda \in U\}$ where the maps $a_{n,j} : U \to \mathbb{C}^{k+1}$ are holomorphic. The number of elements of $\{a_{n,j}(\lambda) ; \lambda \in U\}$ does not depend on $\lambda$ and will be denoted by $N_n$.

By a theorem of Briand-Duval (see [3], Theorem 2), the Green measure $\mu_{F_\lambda}$ of $F_\lambda$ is the weak limit of a sequence of discrete measures:

$$\frac{1}{N_n} \sum_j \delta_{a_{n,j}(\lambda)} =: \mu_{F_\lambda} \to \mu_{F_\lambda}.$$ 

It is therefore natural to consider the sequence of pluriharmonic functions

$$L_n(\lambda) := \int_{\mathbb{C}^{k+1}} \log |\det F_\lambda'| \cdot \mu_{F_\lambda} = \frac{1}{N_n} \sum_j \log |\det F_\lambda' (a_{n,j}(\lambda))|$$

and to compare it with $L(\lambda) := \int_{\mathbb{C}^{k+1}} \log |\det F_\lambda'| \cdot \mu_{F_\lambda} = L(f_\lambda) + \log d$.

However, as the function $\log |\det F_\lambda'|$ is unbounded, this comparison is not immediate. As we shall see, the fact that the measures $\pi_\alpha \mu_{F_\lambda}$ have local $\alpha$-Hölder potentials is essential to overcome this difficulty.

Let us now enter into details and, to this purpose, fix a few notations. The Green function of $F_\lambda$ will be denoted by $G_\lambda$ and for any $\varepsilon > 0$ we set

$$W_{\lambda,\varepsilon} := \{G_\lambda = 0\} \cap \{|\det F_\lambda'| \leq \varepsilon\}.$$ 

We shall call $d_{n,j}(\lambda)$ the holomorphic function $\det F_\lambda' (a_{n,j}(\lambda))$ and introduce the following sequence of discrete measures:

$$L_{n,\lambda} := \frac{1}{N_n} \sum_j \log |d_{n,j}(\lambda)||\delta_{a_{n,j}(\lambda)}|.$$ 

Since $G_\lambda \circ F_\lambda^n = d^n G_\lambda$, the $F_\lambda^n$-fixed points $a_{n,j}(\lambda)$ belong to $\{G_\lambda = 0\}$ and thus, according to our notations, we have

$$L_n(\lambda) = L_{n,\lambda}(\mathbb{C}^{k+1}) = L_{n,\lambda}(W_{\lambda,\varepsilon}) + L_{n,\lambda}(W_{\lambda,\varepsilon}^c).$$

(2.1)

We now fix $\lambda_0 \in U$, a unit vector $z_0 \in \mathbb{C}^N \setminus \{0\}$ and $\rho > 0$ such that $u_\theta := \lambda_0 + \rho e^{i\theta} z_0$ belongs to $U$ for every $\theta \in \mathbb{R}$. Since the functions $L_n(\lambda)$ are pluriharmonic on $U$, the identity (2.1) may be rewritten as

$$L_n(\lambda_0) = \frac{1}{2\pi} \int_0^{2\pi} L_{n,u_\theta} \left(W_{u_\theta,\varepsilon}^c\right) d\theta + \frac{1}{2\pi} \int_0^{2\pi} L_{n,u_\theta} \left(W_{u_\theta,\varepsilon}\right) d\theta.$$ 

(2.2)

Since the function $L$ is p.s.h. (see Subsection 1.4), we simply have to deduce from (2.2) that $\frac{1}{2\pi} \int_0^{2\pi} L(u_\theta) d\theta \leq L(\lambda_0)$. This will require the following lemmas.

**Lemma 2.3** There exists an universal function $M : [0, \varepsilon_0] \to [0, 1]$ which tends to 0 at 0 and such that $\frac{1}{2\pi} \int_0^{2\pi} L_{n,u_\theta} \left(W_{u_\theta,\varepsilon}\right) d\theta \geq L_{n,\lambda_0} \left(W_{\lambda_0, M(\varepsilon)}\right)$ for every $n \in \mathbb{N}$ and every $0 < \varepsilon \leq \varepsilon_0$. 

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**Lemma 2.4** \( \lim_{\varepsilon \to 0} \left( \lim_{n} \mathcal{L}_{n,\lambda} \left( W_{\lambda,\varepsilon}^{c} \right) \right) = L(\lambda) \) for every \( \lambda \in X \).

Using Lemma 2.3 and the identities 2.1, 2.2 we get

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \mathcal{L}_{n,u_{\theta}} \left( W_{u_{\theta},\varepsilon}^{c} \right) d\theta = L_{n}(\lambda_{0}) - \frac{1}{2\pi} \int_{0}^{2\pi} \mathcal{L}_{n,u_{\theta}} \left( W_{u_{\theta},\varepsilon} \right) d\theta \leq L_{n}(\lambda_{0}) - \mathcal{L}_{n,\lambda_{0}} \left( W_{\lambda_{0},M(\varepsilon)} \right) = \mathcal{L}_{n,\lambda_{0}} \left( W_{\lambda_{0},M(\varepsilon)}^{c} \right)
\]

then, by Fatou’s theorem we have

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \liminf_{n} \mathcal{L}_{n,u_{\theta}} \left( W_{u_{\theta},\varepsilon}^{c} \right) d\theta \leq \liminf_{n} \mathcal{L}_{n,\lambda_{0}} \left( W_{\lambda_{0},M(\varepsilon)}^{c} \right).
\]

Thus, as \( \lim_{\varepsilon \to 0} M(\varepsilon) = 0 \), the inequality \( \frac{1}{2\pi} \int_{0}^{2\pi} L(u_{\theta}) d\theta \leq L(\lambda_{0}) \) immediately follows from Lemma 2.3 when \( \varepsilon \to 0 \). This ends the proof of Theorem 2.2.

**Proof of lemma 2.3** we shall use the following fact which is a direct consequence of Montel’s theorem and Hurwitz lemma.

**Fact:** Let \( 0 < \rho < r < R \) and \( S_{\varepsilon} := \{ \varphi \in \mathcal{O}(\Delta_{r},\Delta_{R}^{+}); \inf_{|z|=\rho} |\varphi(z)| = \varepsilon \} \). Let \( M(\varepsilon) := \sup_{\varphi \in S_{\varepsilon}} \sup_{|z|\leq \rho} |\varphi(z)| \). Then \( \lim_{\varepsilon \to 0} M(\varepsilon) = 0 \) and in particular \( M(\varepsilon) \leq 1 \) for \( 0 < \varepsilon \leq \varepsilon_{0} \).

Let us observe that the functions \( d_{n,j}(\lambda) \) are uniformly locally bounded. This follows from the continuity of \( G_{\lambda}(z) \) and the previous observation that \( \{ d_{n,j}(\lambda) \} \subset \{ G_{\lambda} = 0 \} \). According to our notations we have

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \mathcal{L}_{n,u_{\theta}} \left( W_{u_{\theta},\varepsilon} \right) d\theta = \frac{1}{N_{n}} \sum_{j} \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| d_{n,j}(u_{\theta}) \right| 1_{\{|d_{n,j}| \leq \varepsilon\}}(u_{\theta}) d\theta \tag{2.3}
\]

where \( \Sigma'_{j} \) indicates that we only consider the terms for which \( \inf_{\theta} |d_{n,j}(u_{\theta})| \leq \varepsilon \). By the Fact, all these terms satisfy \( |d_{n,j}(u_{\theta})| \leq M(\varepsilon) \leq 1 \) for \( \varepsilon \leq \varepsilon_{0} \) and \( |u_{\theta} - \lambda_{0}| \leq \rho \). In particular, \( \frac{1}{2\pi} \int_{0}^{2\pi} \log |d_{n,j}(u_{\theta})| 1_{\{|d_{n,j}| \leq \varepsilon\}}(u_{\theta}) d\theta \geq \frac{1}{2\pi} \int_{0}^{2\pi} \log |d_{n,j}(u_{\theta})| d\theta = \log |d_{n,j}(\lambda_{0})| \).

Thus (2.3) yields

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \mathcal{L}_{n,u_{\theta}} \left( W_{u_{\theta},\varepsilon} \right) d\theta \geq \frac{1}{N_{n}} \sum_{j} \log |d_{n,j}(\lambda_{0})|.
\]

Finally, as \( |d_{n,j}(\lambda_{0})| \leq M(\varepsilon) \leq 1 \) for all terms in \( \Sigma'_{j} \), we have

\[
\frac{1}{N_{n}} \sum_{j} \log |d_{n,j}(\lambda_{0})| \geq \frac{1}{N_{n}} \sum_{j} \log |d_{n,j}(\lambda_{0})| 1_{\{|d_{n,j}| \leq M(\varepsilon)\}}(\lambda_{0}) = \mathcal{L}_{n,\lambda_{0}} \left( W_{\lambda_{0},M(\varepsilon)} \right)
\]

and the conclusion follows.
Proof of lemma 2.4 Let us denote by \( \log_{e} x \) an increasing smooth function on \([0, +\infty[\) such that \( \log_{e} x \geq 2 \log_{e} \varepsilon \) for \( 0 \leq x < \varepsilon < 1 \) and \( \log_{e} x = \log x \) for \( x \geq \varepsilon \). Then

\[
0 \leq \mathcal{L}_{n,\lambda} \left( W_{x,\varepsilon}^{c} \right) - \int_{C_{k+1}} \log_{e} |\det F'_{\lambda}| \mu_{F_{\lambda},n} \leq -2(\log_{e} \varepsilon) \mu_{F_{\lambda},n} \left( \{|\det F'_{\lambda}| \leq \varepsilon\}\right). \tag{2.4}
\]

There are constants \( a, A > 0 \) such that \( \{|\det F'_{\lambda}| \leq \varepsilon\} \subset \pi^{-1}(V_{A_{x}}(C_{f_{\lambda}})) \) where \( V_{i}(C_{f_{\lambda}}) \) denotes a \( t \)-neighbourhood of \( C_{f_{\lambda}} \). Since \( \mu_{\lambda} := \pi_{*} \mu_{F_{\lambda}} \) has (local) \( \alpha \)-Hölder continuous potential we have \( \mu_{\lambda} (V_{A_{x}}(C_{f_{\lambda}})) \leq \text{cst} \varepsilon^{\alpha} \) (see the proof of Theorem 1.7.3 in [27]). Thus, for \( n \) big enough,

\[
\mu_{F_{\lambda},n} \left( \{|\det F'_{\lambda}| \leq \varepsilon\}\right) \leq 2 \mu_{F_{\lambda}} \left( \{|\det F'_{\lambda}| \leq 2\varepsilon\}\right) \leq \text{cst} \varepsilon^{\alpha}. \tag{2.5}
\]

From (2.4) and (2.5) we get

\[
0 \leq \liminf_{n} \mathcal{L}_{n,\lambda} \left( W_{x,\varepsilon}^{c} \right) - \int_{C_{k+1}} \log_{e} |\det F'_{\lambda}| \mu_{F_{\lambda}} \leq -\text{cst} \varepsilon^{\alpha} \log \varepsilon
\]

and the conclusion follows by making \( \varepsilon \to 0 \). \qed

3 Formulas for the Lyapunov exponent of a rational function

In this section we establish some formulas which relate the Lyapunov exponent to the critical points of a rational function.

Theorem 3.1 Let \( f \) be a rational function of degree \( d \) and \( F \) be one of its lifts to \( \mathbf{C}^{2} \). The Lyapunov exponent \( L(f) \) of \( f \) is given by one of the following formulas:

(i) \( L(f) + \log d = \int_{\mathbf{P}^{1}} g_{F} [C_{f}] - 2(d - 1) \int_{\mathbf{P}^{1}} g_{F} (\mu_{f} + \omega) + \int_{\mathbf{P}^{1}} \log ||J_{F}||_{0} \omega. \)

(ii) \( L(f) + \log d = \int_{\mathbf{P}^{1}} g_{F} [C_{f}] - 2(d - 1) \int_{\mathbf{P}^{1}} g_{F} (\mu_{f} + \omega) + \int_{\mathbf{C}^{2}} \log |\det F'| m. \)

(iii) If \( \tilde{c}_{1}, \ldots, \tilde{c}_{2d - 2} \) are chosen such that \( \det F'(z) = \prod_{j=1}^{2d - 2} \tilde{c}_{j} \wedge z \) one has:

\[
L(f) + \log d = \Sigma_{j} G_{F} (\tilde{c}_{j}) - (d - 1) \left( 1 + 2 \int_{\mathbf{P}^{1}} g_{F} (\mu_{f} + \omega) \right).
\]

Proof. Let us start with the first formula. We know that \( L(f) + \log d = L(F) \). We shall use the formalism introduced in the Subsection 1.5. By the first assertion of Lemma 1.3 and the definition of \( \mu_{f} \) we have

\[
L(F) = \int_{\mathbf{P}^{1}} \log ||J_{F}||_{G_{F} \mu_{f}} = \int_{\mathbf{P}^{1}} \log ||J_{F}||_{G_{F} dd^{e} g_{F}} + \int_{\mathbf{P}^{1}} \log ||J_{F}||_{G_{F} \omega}. \tag{3.1}
\]

After an integration by parts (see next section for a careful justification) the identity (3.1) yields

\[
L(F) = \int_{\mathbf{P}^{1}} g_{F} dd^{e} \log ||J_{F}||_{G_{F}} + \int_{\mathbf{P}^{1}} \log ||J_{F}||_{G_{F} \omega}. \tag{3.2}
\]
Using the Poincaré-Lelong equation $dd^c \log \|J_F\|_{G_F} = -(2d-2)\mu_f + |C_f|$, (3.2) becomes:

$$L(F) = \int_{\mathbb{P}^1} g_F[C_f] - (2d-2) \int_{\mathbb{P}^1} g_F\mu_f + \int_{\mathbb{P}^1} \log \|J_F\|_{G_F} \omega. \quad (3.3)$$

After observing that $\| \cdot \|_{G_F} = e^{-(2d-2)g_F} \| \cdot \|_0$ we may rewrite the last integral in (3.3) as:

$$\int_{\mathbb{P}^1} \log \|J_F\|_0 \omega - (2d-2) \int_{\mathbb{P}^1} g_F \omega$$

and this gives our first formula.

In order to establish the second formula, we simply transform the first one by using the second assertion of Lemma 1.2.

Let us finally prove the third formula. Picking $U_j \in \mathbb{U}(2, \mathbb{C})$ such that $U_j^{-1}(\tilde{c}_j) = (\|\tilde{c}_j\|, 0)$ we have $U_j z \wedge \tilde{c}_j = -z_2 \|\tilde{c}_j\|$. Since $\int_{C^2} \log |z_2|m = -\frac{1}{2}$, we get $\int_{C^2} \log |detF'|m = \Sigma_j \int_{C^2} \log |U_j z \wedge \tilde{c}_j|m = \Sigma_j \log \|\tilde{c}_j\| - (d-1)$.

On the other hand, $\int_{\mathbb{P}^1} g_F[C_f] = \Sigma_j g_F \circ \pi(\tilde{c}_j)$ is pluriharmonic on $\mathcal{M}$. We may consider a local holomorphic parametrization $\lambda \in \mathcal{M}$ such that $\lambda \in \mathcal{M}$.

The usefulness of our formulas consists in the fact that the function $B(F) := \int_{\mathbb{P}^1} g_F(\mu_f + \omega)$ is pluriharmonic on $\mathcal{H}_d \left(\mathbb{C}^2\right)$.

Theorem 3.2 The function $B(F) := \int_{\mathbb{P}^1} g_F(\mu_f + \omega)$ is pluriharmonic on $\mathcal{H}_d \left(\mathbb{C}^2\right)$.

Using the third formula of Theorem 3.1 and Theorem 3.2 we get the following corollary, previously obtained by DeMarco (see [9]). It allows to relate the pluriharmonicity of $L(F)$ with the stability of the dynamic of critical points. As we shall see in Section 5 this is a key point when approaching the Mañé-Sad-Sullivan theory via potential-theoretic methods.

Corollary 3.3 Let $\{f_\lambda\}_{\lambda \in X}$ be a holomorphic family of rational maps of degree $d$. Then $dd^c L(f_\lambda) = dd^c \sum_{j=1}^{2d-2} G_F(\tilde{c}_j)$.

Using Proposition 1.2 (ii), one also reads on the third formula of Theorem 3.1 that the Lyapunov exponent $L(F)$ is an Hölder-continuous function in $F$. The continuity was first proved by Mañé [17].

Corollary 3.4 The function $L(F)$ is p.s.h. and Hölder-continuous on $\mathcal{H}_d \left(\mathbb{C}^2\right)$.

Proof of Theorem 3.2 We may consider a local holomorphic parametrization $\lambda \mapsto F_\lambda$ of $\mathcal{H}_d \left(\mathbb{C}^2\right)$ defined on some open subset $U$ of $\mathbb{C}^{2d+2}$. We shall denote by $f_\lambda$ the induced map on $\mathbb{P}^1$ and set $B(\lambda) := B(F_\lambda)$. There exists an analytic subset $A$ of $U$ such that, for any $\lambda \in U \setminus A$, the critical points of $f_\lambda$ consist in $2d-2$ distinct, regular values of $f_\lambda$. As the function $B(F)$ is locally bounded, it suffices to show that it is pluriharmonic on any sufficiently small ball contained in $U \setminus A$.

On such a ball $B$, there are $2d-2$ holomorphic maps $\tilde{c}_j$ such that

$$detF_\lambda' = \prod_{j=1}^{2d-2} \tilde{c}_j(\lambda) \wedge z.$$
Moreover, for each $1 \leq j \leq 2d-2$, there are $d$ holomorphic maps $\tilde{c}_{j,i}$ such that $F_\lambda \circ \tilde{c}_{j,i}(\lambda) = \tilde{c}_{j}(\lambda)$ and therefore:

$$
det F_\lambda^{2d} = h(\lambda) \left( \prod_{j=1}^{2d-2} \pi_{i=1}^d \tilde{c}_{j,i}(\lambda) \wedge z \right) \left( \prod_{j=1}^{2d-2} \tilde{c}_{j}(\lambda) \wedge z \right)
$$

where $h$ is a non-vanishing holomorphic function on $B$. Let $N$ denote the degree of $\det F_\lambda^{2d}$, after setting $\tilde{c}_{j,i} = h^\frac{1}{N} \tilde{c}_{j,i}$ and $\tilde{c}_{j} = h^\frac{1}{N} \tilde{c}_{j}$ we get

$$
det F_\lambda^{2d} = \left( \prod_{j=1}^{2d-2} \pi_{i=1}^d \tilde{c}_{j,i}(\lambda) \wedge z \right) \left( \prod_{j=1}^{2d-2} \tilde{c}_{j}(\lambda) \wedge z \right).
$$

We are now in order to use the third formula of Theorem 3.1 for $f_\lambda^2$. Since $G_{F^2} = G_F$, $\mu_{F^2} = \mu_F$ and $B (F^2) = B (F)$, it yields:

$$
L \left( f_\lambda^2 \right) + \log d^2 + (d^2 - 1) (2B (F_\lambda) + 1) = \Sigma_{j=1}^{2d-2} \Sigma_{i=1}^d G_F \left( \tilde{c}_{j,i}(\lambda) \right) + \Sigma_{j=1}^{2d-2} G_F \left( \tilde{c}_{j}(\lambda) \right)
$$

$$
= \log |h(\lambda)| + \Sigma_{j=1}^{2d-2} \Sigma_{i=1}^d G_F \left( \tilde{c}_{j,i}(\lambda) \right) + \Sigma_{j=1}^{2d-2} G_F \left( \tilde{c}_{j}(\lambda) \right)
$$

$$
= \log |h(\lambda)| + 2 \Sigma_{j=1}^{2d-2} G_F \left( \tilde{c}_{j}(\lambda) \right).
$$

On the other hand, for $f_\lambda$, the same formula gives:

$$
L \left( f_\lambda^2 \right) + \log d^2 = 2 \left( L \left( f_\lambda \right) + \log d \right) = 2 \Sigma_{j=1}^{2d-2} G_F \left( \tilde{c}_{j}(\lambda) \right) - 2(d - 1) (2B (F_\lambda) + 1).
$$

By comparison we thus obtain

$$
2B(\lambda) + 1 = \frac{1}{(d-1)^2} \log |h(\lambda)|.
$$

\[ \square \]

Remark 3.5 By using its pluriharmonicity, one may show that the function $B(F)$ is given by $B(F) = \frac{1}{d(d-1)} \log |\text{Res}(F)| - \frac{1}{2}$. This gives again DeMarco’s formula ([9] Corollary 1.6) and will be proved in Proposition 4.9 in arbitrary dimension.

4 A formula for the sum of Lyapunov exponents of holomorphic endomorphisms of $\mathbb{P}^k$

Our aim here is to generalize the results of the previous section to endomorphisms of $\mathbb{P}^k$. We first establish a formula which relates the sum of the Lyapunov exponents $L(f)$ with the Green current and the current of integration on the critical set. This extends Theorem 3.1 (i). We then generalize Theorem 3.2 and, in particular, obtain an intrinsic expression for $dd^c L$.

Theorem 4.1 Let $f$ be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Let $F$ be one of the lifts of $f$ to $\mathbb{C}^{k+1}$ and $T_f = dd^c g_F + \omega$ be the Green current of $f$. Then the sum of the Lyapunov exponents $L(f)$ of $f$ is given by:

$$
L(f) + \log d = L(F) = H(F) - (k+1)(d-1)B(F)
$$

(4.1)
where

\[ H(F) := \sum_{j=0}^{k-1} \int_{\mathcal{P}_k} g_F T^j_f \wedge \omega^{k-j-1} \wedge [C_f] + \int_{\mathcal{P}_k} \log ||J_F||_0 \omega^k \]  

(4.2)

and

\[ B(F) := \sum_{j=0}^{k} \int_{\mathcal{P}_k} g_F T^j_f \wedge \omega^{k-j}. \]  

(4.3)

**Proof.** According to Lemma 1.4 we have

\[ L(f) + \log d = \int_{\mathcal{P}_k} \log ||J_F||_G T^k_f. \]

Let us start by showing that

\[ L(f) + \log d = \sum_{j=0}^{k-1} \int_{\mathcal{P}_k} \log ||J_F||_G \ dd^c g_F \wedge T^j_f \wedge \omega^{k-j-1} + \]

\[ + \int_{\mathcal{P}_k} \log ||J_F||_G \omega^k. \]  

(4.4)

To this purpose we first note that each term in the above sum is finite (this follows immediately from the Chern-Levine-Nirenberg inequalities) and then we observe that:

\[ dd^c g_F \wedge \left( \sum_{j=0}^{k-1} T^j_f \wedge \omega^{k-j-1} \right) = (T_f - \omega) \wedge \left( \sum_{j=0}^{k-1} T^j_f \wedge \omega^{k-j-1} \right) = \]

\[ = \sum_{j=0}^{k-1} T^{j+1}_f \wedge \omega^{k-j-1} - \sum_{j=0}^{k-1} T^j_f \wedge \omega^{k-j} = T^k_f - \omega^k. \]  

(4.5)

We shall now use the following integration by part property which will be proved separately. **Fact:** for \(0 \leq j < k,\)

\[ \int_{\mathcal{P}_k} \log ||J_F||_G \ dd^c g_F \wedge T^j_f \wedge \omega^{k-j-1} = \int_{\mathcal{P}_k} g_F \ dd^c \log ||J_F||_G \wedge T^j_f \wedge \omega^{k-j-1}. \]

This allows us to transform the identity (4.4) and get:

\[ L(f) + \log d = \sum_{j=0}^{k-1} \int_{\mathcal{P}_k} g_F \ dd^c \log ||J_F||_G \wedge T^j_f \wedge \omega^{k-j-1} + \]

\[ + \int_{\mathcal{P}_k} \log ||J_F||_G \omega^k. \]

Next, by the Poincaré-Lelong equation \( dd^c \log ||J_F||_G = [C_f] - (k+1)(d-1)T_f, \) we obtain:

\[ L(f) + \log d = \sum_{j=0}^{k-1} \int_{\mathcal{P}_k} g_F T^j_f \wedge \omega^{k-j-1} \wedge [C_f] - \]  

(4.6)
Proposition 4.3

Lemma 4.2 allows us to use monotone convergence theorem ([27] Theorem A.6.2), thus

\[-(k+1)(d-1)\sum_{j=0}^{k-1} \int_{\mathbb{P}^k} g_F T^j_f \wedge \omega^{k-j-1} + \int_{\mathbb{P}^k} \log ||J_F||_G \omega^k.\]

Finally, as \(||G = e^{-(k+1)(d-1)g_F}||_0\), we may replace the last integral in (4.6) by \(\int_{\mathbb{P}^k} \log ||J_F||_0 \omega^k - (k+1)(d-1) \int_{\mathbb{P}^k} g_F \omega^k\) and this immediately yields to the expected formula. \(\square\)

It remains to establish the Fact. We shall proceed by regularization and use the following lemma.

Lemma 4.2 Let \(\{\phi_n\}_{n\in\mathbb{N}^*}\) be a decreasing sequence of increasing smooth convex functions on \(\mathbb{R}\) such that:

(i) \(\phi_n(x) = -n\), on \([-\infty, -n + \frac{1}{n}]

(ii) \(\phi_n(x) = x\), on \([-n + \frac{1}{n}, +\infty[\]

Let \(\log_n x\) be defined by \(\log_n x := \phi_n(\log x)\). Then \(\{\log_n ||J_F||_0\}_{n\in\mathbb{N}^*}\) is a decreasing sequence of smooth functions, which converges to \(\log ||J_F||_0\). Moreover \(dd^c \log_n ||J_F||_0 + (k+1)(d-1)\omega \geq 0\) for all \(n \in \mathbb{N}^*\).

The proof is a straightforward computation and we omit it.

Proof of the Fact. Using the relation \(||G_F = e^{-(k+1)(d-1)g_F}||_0\), we get:

\[\int_{\mathbb{P}^k} \log ||J_F||_G dd^c g_F \wedge T^j_f \wedge \omega^{k-j-1} = \]

\[= \int_{\mathbb{P}^k} \log ||J_F||_0 dd^c g_F \wedge T^j_f \wedge \omega^{k-j-1} - (k+1)(d-1) \int_{\mathbb{P}^k} g_F dd^c g_F \wedge T^k_f \wedge \omega^{k-j-1}.\]

Lemma 4.2 allows us to use monotone convergence theorem ([27] Theorem A.6.2), thus

\[\int_{\mathbb{P}^k} \log ||J_F||_0 dd^c g_F \wedge T^j_f \wedge \omega^{k-j-1} = \lim_{n\to\infty} \int_{\mathbb{P}^k} \log_n ||J_F||_0 dd^c g_F \wedge T^j_f \wedge \omega^{k-j-1} = \]

\[= \lim_{n\to\infty} \int_{\mathbb{P}^k} g_F dd^c \log_n ||J_F||_0 \wedge T^j_f \wedge \omega^{k-j-1} = \int_{\mathbb{P}^k} g_F dd^c \log ||J_F||_0 \wedge T^j_f \wedge \omega^{k-j-1}.\]

\(\square\)

Our aim now is to compute \(dd^c L(f_\lambda)\) when \(\{f_\lambda\}_{\lambda\in X}\) is a holomorphic family of endomorphisms of \(\mathbb{P}^k\). We need the following technical Proposition which will be proved in the Appendix.

Proposition 4.3 Let \(X^m \xrightarrow{\pi} Y^n\) be a holomorphic submersion between complex manifolds. If \(R\) is a current on \(X\), for \(y \in Y\) the slice (if it exists) of \(R\) along the fiber \(\pi^{-1}(y)\) is denoted by \(R_y\). Let \(u_1, \ldots, u_h\) be almost plurisubharmonic, locally bounded functions on \(X\) and \(T\) be a positive, closed \((k,k)\)-current on \(X\), with \(h + k \leq m - n\). Thus, for a.e. \(y \in Y\),

\[(u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_h \wedge T)_y = u_1|_{\pi^{-1}(y)} dd^c (u_2|_{\pi^{-1}(y)}) \wedge \cdots \wedge dd^c (u_h|_{\pi^{-1}(y)}) \wedge T_y.\]
Let us recall that, for a \((k,k)\)-current \(R\) on \(X\), slicing is characterized (for a.e. \(y \in Y\)) by the following identity:

\[
\int_X R \wedge \psi \wedge \pi^* \phi = \int_Y \left( \int_{\pi^{-1}(y)} R_y \wedge \iota_y^* \psi \right) \phi
\]

for every smooth \((n,n)\)-form \(\phi\) on \(Y\) and for every smooth and compactly supported \((m-n-k, m-n-k)\)-form \(\psi\) on \(X\) (here \(\iota_y : \pi^{-1}(y) \to X\) is the inclusion.)

By Theorem [L.1] \(L(f_\lambda) + \log d = H(F_\lambda) - (k+1)(d-1)B(F_\lambda)\). We first compute \(dd^c H\):

**Proposition 4.4** Let \(\{f_\lambda\}_{\lambda \in X}\) be a holomorphic family of endomorphisms of \(P^k\) such that there is a holomorphic lift \(\{F_\lambda\}_{\lambda \in X}\) to \(C^{k+1}\). Then

\[dd^c H(F_\lambda) = p_\lambda((dd^c g_{F_\lambda} + \omega)^k \wedge [C_X])\]

where \(C_X\) is the hypersurface of \(X \times P^k\) defined by the equation \(\det F_\lambda(z) = 0\) and \(p : X \times P^k \to X\) is the canonical projection.

**Remark.** \(dd^c g_{F_\lambda}\) involves derivatives in both \(\lambda \in X\) and \(z \in P^k\).

**Proof.** Let \(q = \dim C X\), for a \((q - 1, q - 1)\)-form \(\phi\) with compact support on \(X\) we have

\[
< dd^c H, \phi > = \int_X \left( \int_{P^k} g_{F_\lambda} \sum_{j=0}^{k-1} (dd^c g_{F_\lambda} + \omega)^j \wedge \omega^{k-j-1} \wedge [C_{F_\lambda}] \right) dd^c \phi + \int_X \left( \int_{P^k} \log ||J_{F_\lambda}||_0 \omega^k \right) dd^c \phi.
\]

Since \([C_{F_\lambda}]\) is the slice of \([C_X]\) (see [29] (10.4)), by means of the Proposition [L.3] the first integral is

\[
\int_{X \times P^k} p^* \phi \wedge dd^c g_{F_\lambda} \wedge \sum_{j=0}^{k-1} (dd^c g_{F_\lambda} + \omega)^j \wedge \omega^{k-j-1} \wedge [C_X]
\]

By Poincaré - Lelong formula \([C_X] = dd^c \log ||J_{F_\lambda}||_0 + (k+1)(d-1) \omega\) one sees that \(\log ||J_{F_\lambda}||_0\) is almost plurisubharmonic and therefore locally summable. Thus the second integral is

\[
\int_{X \times P^k} p^* \phi \wedge dd^c \log ||J_{F_\lambda}||_0 \wedge \omega^k = \int_{X \times P^k} p^* \phi \wedge [C_X] - (k+1)(d-1) \int_{X \times P^k} p^* \phi \wedge \omega^{k+1}.
\]

But \(\omega^{k+1} = 0\) on \(X \times P^k\) and therefore, after summing up, we obtain:

\[
< dd^c H, \phi > = \int_{C_X} p^* \phi \wedge (dd^c g_{F_\lambda} + \omega)^k = < (dd^c g_{F_\lambda} + \omega)^k \wedge [C_X], p^* \phi > .
\]

\[\square\]

Now we can also extend Theorem [L.2] to the \(k\)-dimensional case. We shall use the same device, that is to compare formulas for \(F_\lambda\) and \(F^2_\lambda\).
**Theorem 4.5** The function $B(F)$ is pluriharmonic on $\mathcal{H}_d(\mathbb{C}^{k+1})$.

Proof. Let us start with a claim:

**Claim:** $H(F)$ is p.s.h. on $\mathcal{H}_d(\mathbb{C}^{k+1})$ and $dd^c H(F^2) = 2dd^c H(F)$.

Proof of the Claim. Let $X := \mathcal{H}_d(\mathbb{C}^{k+1})$, the projection $X \ni F \mapsto f \in \mathcal{H}_d(\mathbb{P}^k)$ defines a holomorphic family $\{f\}_{F \in X}$ and the plurisubharmonicity follows from Proposition 4.4.

Let $C_X$ be as above and denote by $C'_X$ the analogous critical set of the family $\{f^2\}_{F \in X}$. Considering the map $\Phi : X \times \mathbb{P}^k \to X \times \mathbb{P}^k$ defined by $\Phi(F, z) := (F, f(z))$, we get $[C'_X] = [C_X] + \Phi^*[C_X]$.

As $F^2$ and $F$ have the same Green function $g_F$, Proposition 4.4 gives

$$dd^c H(F^2) = p_* \left( (dd^c g_F + \omega)^k \land [C'_X] \right).$$

From $G_F \circ F = d.G_F$, it follows $\Phi^*(dd^c g_F + \omega) = d.(dd^c g_F + \omega)$, thus

$$dd^c H(F^2) = p_* \left( (dd^c g_F + \omega)^k \land [C_X] \right) + \frac{1}{d^k} p_* \Phi^* \left( (dd^c g_F + \omega)^k \land [C_X] \right).$$

But $p \circ \Phi = p$, thus $p_* = p_* \Phi_*$; moreover $\Phi_* \Phi^* = d^k \text{id}$, thus

$$\frac{1}{d^k} p_* \Phi^* \left( (dd^c g_F + \omega)^k \land [C_X] \right) = p_* \left( (dd^c g_F + \omega)^k \land [C_X] \right),$$

therefore $dd^c H(F^2) = 2dd^c H(F)$.

End of the proof of Theorem 4.5 Since $L(F^2) = 2L(F)$ we have

$$dd^c L(F^2) = 2dd^c L(F) = 2(2dd^c H(F) - (k + 1)(d - 1)dd^c B(F))$$

on the other hand, since $B(F^2) = B(F)$, we may use the Claim and get:

$$dd^c L(F^2) = dd^c H(F^2) - (k + 1)(d^2 - 1)dd^c B(F^2) =$$

$$= 2dd^c H(F) - (k + 1)(d^2 - 1)dd^c B(F).$$

By comparison we get $(d - 1)^2dd^c B(f) = 0$, thus $B$ is pluriharmonic on $\mathcal{H}_d(\mathbb{C}^{k+1})$.

**Corollary 4.6** Let $\{f_\lambda\}_{\lambda \in X}$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ with algebraic degree $d \geq 2$. Then

$$dd^c L(f_\lambda) = p_*((dd^c g_{F_\lambda} + \omega)^k \land [C_X])$$

and on $X \times \mathbb{P}^k$

$$(dd^c g_{F_\lambda} + \omega)^{k+1} = 0.$$
Remark 4.7 As we have already noted, the operator $dd^c$ in the above formula involves derivatives in both $\lambda \in X$ and $z \in \mathbb{P}^k$; thus the current $\bar{T} := dd^c g_{F_{\lambda}} + \omega$ is different from the Green current. The current $\bar{T}$ depends only on the family $\{F_{\lambda}\}$ and not on the local lift $\{\{F_{\lambda}\}\}$; moreover it is positive on $X \times \mathbb{P}^k$ since $G_{F_{\lambda}}(z)$ is p.s.h. on $X \times (\mathbb{C}^{k+1} \setminus \{0\})$ (see Proposition 4.4). Using this current we may express the formulas of Corollary 4.6 in a synthetic way and avoid any reference to the lift $\{F_{\lambda}\}$, which in general is only defined locally: 

$$dd^c L(f_{\lambda}) = p_*(\bar{T}^k \wedge |C_X|)$$
and
$$\bar{T}^{k+1} = 0.$$ 

Proof of Corollary 4.6 From Proposition 4.4 and Theorem 4.5 we get the first statement. 

We argue as in the proof of Proposition 4.4 (using again Proposition 4.3); choosing an open subset $V \subset X$ such that there is a holomorphic family of lifts $\{F_{\lambda}\}_{\lambda \in V}$, we have

$$<dd^c B, \phi> = \int_V \left( \int_{\mathbb{P}^k} g_{F_{\lambda}} \left( \sum_{j=0}^k (dd^c g_{F_{\lambda}} + \omega)^j \wedge \omega^{k-j} \right) \right) dd^c \phi =$$

$$= \int_{V \times \mathbb{P}^k} p^*(\phi) \wedge dd^c g_{F_{\lambda}} \wedge \left( \sum_{j=0}^k (dd^c g_{F_{\lambda}} + \omega)^j \wedge \omega^{k-j} \right).$$

Then, as in 4.3), we get

$$<dd^c B, \phi> = \int_{V \times \mathbb{P}^k} p^*(\phi) \wedge (\bar{T}^{k+1} - \omega^{k+1}) = \langle p_*(\bar{T}^{k+1}), \phi \rangle,$$

because $\omega^{k+1}$ vanishes. Finally, since $dd^c B = 0$ and $\bar{T}$ is positive, we get $\bar{T}^{k+1} = 0$. \qed 

By Theorem 4.3, the function $B$ is pluriharmonic: this suggests the existence of a simpler analytic expression for $B$, as indeed Proposition 4.9 states. Since $B$ is defined by means of dynamical quantities, this result seems of some interest. We shall need the following lemma.

Lemma 4.8 $H^1(\mathcal{H}_d(\mathbb{P}^k); R) = 0$. 

Proof. The Fubini-Study form $\omega$ generates $H^{2N-2}(\mathbb{P}^N; \mathbb{R})$ and $(\iota^*(\omega^{N-1}), \Sigma_d) = (\int_{\Sigma_d} \omega^{N-1} = vol(\Sigma_d) \neq 0$, where $\iota: \Sigma_d \rightarrow \mathbb{P}^N$ is the inclusion; therefore the map 

$R = H^{2N-2}(\mathbb{P}^N; \mathbb{R}) \rightarrow H^{2N-2}(\Sigma_d; R) = R$ 

is an isomorphism. Hence, from the exact sequence 

$$H^{2N-2}(\mathbb{P}^N; \mathbb{R}) \rightarrow H^{2N-2}(\Sigma_d; R) \rightarrow H^{2N-1}(\mathbb{P}^N; \Sigma_d; R) \rightarrow H^{2N-1}(\mathbb{P}^N; R) = 0,$$

it follows that $H^{2N-1}(\mathbb{P}^N, \Sigma_d; R) = 0$.

Observe that $\Sigma_d$ is an euclidean neighbourhood retract (see Prop. IV.8.2, VIII.6.12, VIII.7.2) thus $H^1(\mathbb{P}^N, \Sigma_d; R) = H_{2N-j}(\mathbb{P}^N \setminus \Sigma_d; R)$. In particular $0 = H^{2N-1}(\mathbb{P}^N, \Sigma_d; R) = H_1(\mathbb{P}^N \setminus \Sigma_d; R)$ and then also its dual space $H^1(\mathbb{P}^N \setminus \Sigma_d; R)$ vanishes. \qed 

Now we can establish:

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Proposition 4.9 There exists a constant $C_{d,k}$ such that, $\forall F \in \mathcal{H}_d(C^{k+1})$,

$$B(F) = \frac{1}{d^k(d-1)} \log |\text{Res}(F)| + C_{d,k}.$$ 

Proof. If $a \in \mathbb{C} \setminus \{0\}$, then $g_{aF} = \frac{1}{d-1} \log |a| + g_F$; moreover since $\int_{\mathbb{P}^1} T^j_f \wedge \omega^{k-j} = 1$ we have

$$B(aF) = \frac{(k+1)}{d-1} \log |a| + B(F).$$

The polynomial $\text{Res}(F)$ is homogeneous of degree $(k+1)d^k$, thus the function $d^k(d-1)B(F) - \log |\text{Res}(F)|$ is homogeneous of degree 0 and defines a pluriharmonic function $\Phi : \mathcal{H}_d(\mathbb{P}^k) \to \mathbb{R}$ such that

$$\forall F \in \mathcal{H}_d(C^{k+1}), \quad \Phi \circ \pi(F) = d^k(d-1)B(F) - \log |\text{Res}(F)|.$$ 

Let $\mathcal{P}\mathcal{H}$ be the sheaf of pluriharmonic functions, by means of Lemma 4.8, from the exact sequence $0 \to \mathbb{R} \xrightarrow{i} \mathcal{O} \xrightarrow{R\mathcal{E}} \mathcal{P}\mathcal{H} \to 0$ we get that $H^0(\mathcal{H}_d(\mathbb{P}^k), \mathcal{O}) \xrightarrow{R\mathcal{E}} H^0(\mathcal{H}_d(\mathbb{P}^k), \mathcal{P}\mathcal{H})$ is surjective; therefore there exists a holomorphic function $\varphi$ on $\mathcal{H}_d(\mathbb{P}^k)$ such that $R\mathcal{E}(\varphi) = \Phi$. Setting $\psi := e^\varphi$, we obtain

$$\log |\psi| = \Phi \quad \text{on} \quad \mathcal{H}_d(\mathbb{P}^k).$$

Using Remark 4.8 one sees that $B$ is bounded from above on $\mathcal{K} \cap \mathcal{H}_d(C^{k+1})$, for every compact $\mathcal{K} \subset \mathbb{C}^{N+1}$. It follows that $\text{Res}(F,\psi(\pi(F)))$ is locally bounded and thus can be extended to a holomorphic function $\chi$ on $\mathbb{C}^{N+1}$. But $\chi$ is clearly homogeneous with the same degree as $\text{Res}(F)$, thus $\chi$ is a polynomial on $\mathbb{C}^{N+1}$ and $\psi$ is a constant. \hfill \Box

Proposition 4.10 The constant $C_{d,k}$ does not depend on $d$, indeed

$$C_{d,k} = -\frac{1}{2} \left( k + \frac{k-1}{2} + \cdots + \frac{2}{k-1} + \frac{1}{k} \right).$$

Proof. See Appendix.

Remark 4.11 In the one-dimensional case Propositions 4.9 and 4.10 give a new proof of DeMarco’s formula (see \cite{DeMarco} Corollary 1.6) since

$$\int_{\mathbb{P}^1} g_F(\mu_f + \omega) = \frac{1}{2} (\log |\text{Res}(F)| - 1).$$
5 The bifurcation currents

In this section, we associate to any holomorphic family \( \{ f_\lambda \} \) \( \lambda \in X \) in \( \mathcal{H}_d(\mathbb{P}^1) \) a collection of bifurcation currents \( (dd^c L(f_\lambda))^p \) where \( 1 \leq p \leq \text{dim}_\mathbb{C}X \). Our main goal is to give a rather precise description of their supports and, more precisely, to compare them with the hypersurfaces consisting of mappings having neutral cycles. The extremal cases \( p = 1 \) and \( p = 2d - 2 \) are of special interest. For \( p = 1 \), we partially recover Mañé-Sad-Sullivan work. For \( p = 2d - 2 \), our description will become significant in the last section when introducing a bifurcation measure on the moduli space \( \mathcal{M}_d \). Let us notice that we shall proceed by induction on \( p \).

In order to state the results we must precise a few notations.

**Definition 5.1** We will consider a holomorphic family \( \{ f_\lambda \} \) \( \lambda \in X \) of elements of \( \mathcal{H}_d(\mathbb{P}^1) \) parametrized by an arbitrary complex manifold \( X \). We set \( D := 2d - 2 \) and denote by \( L(\lambda) \) the p.s.h. function on \( X \) defined by \( L(\lambda) := L(f_\lambda) \). Next we introduce the following subsets of \( X \):

\[
\mathcal{R} := \{ \lambda_0 \in X; \text{the repulsive cycles of sufficiently high period of } f_\lambda \text{ move holomorphically on a fixed neighbourhood } U_0 \text{ of } \lambda_0 \},
\]

\[
\mathcal{S} := \{ \lambda_0 \in X; \lambda \to f_\lambda^n(C_{f_\lambda}) \text{ is equicontinuous at } \lambda_0 \},
\]

\[
\text{Per}(X,n,e^{2i\pi \theta}) := \{ \lambda_0 \in X; f_{\lambda_0} \text{ has a cycle of period } n \text{ and multiplier } e^{2i\pi \theta} \}, \text{ where } \theta \in ]0,1[.
\]

It may happen that \( \text{Per}(X,n,e^{2i\pi \theta}) \) is empty or coincides with \( X \); otherwise it is a hypersurface of \( X \). The union of the irreducible components of codimension 1 of \( \text{Per}(X,n,e^{2i\pi \theta}) \) will be denoted by \( \text{Per}_1(X,n,e^{2i\pi \theta}) \). For any dense subset \( E \) of \( ]0,1[ \), we set

\[
\mathcal{Z}_1(X,E) = \bigcup_{n \in \mathbb{N}^*, \theta \in E} \text{Per}_1(X,n,e^{2i\pi \theta})
\]

Let us recall that the set \( \mathcal{R} \) has been implicitly considered in Theorem 2.2, which may be stated as \( \mathcal{R} \cap \text{Supp}(dd^c L) = \emptyset \). Note also that, in the definition of \( \mathcal{S} \), the maps \( \lambda \to f_\lambda^n(C_{f_\lambda}) \) are considered as finitely valued holomorphic maps from \( X \) to \( \mathbb{P}^1 \).

Our description of \( \text{Supp}(dd^c L) \) contains a substantial part of Mañé-Sad-Sullivan theory (see [16]). The originality here relies on the potential-theoretic nature of our proof.

**Theorem 5.2** Let \( E \) be a dense subset of \( ]0,1[ \). Let \( \{ f_\lambda \} \) \( \lambda \in X \) be a holomorphic family of rational maps of degree \( d \) on \( \mathbb{P}^1 \). Then

\[
\overline{\mathcal{Z}_1(X,E)} = \mathcal{R}^c = \text{Supp}(dd^c L) = \mathcal{S}^c.
\]

Let us briefly sketch the proof before entering into details. The inclusion \( \mathcal{S}^c \subset \text{Supp}(dd^c L) \) is a consequence of Corollary 3.3 and was already observed by DeMarco ([9], Theorem 1.1). The inclusion \( \text{Supp}(dd^c L) \subset \mathcal{R}^c \) was proved in Theorem 2.2 (we recall that the main ingredient was the equidistribution of repulsive cycles). The inclusions \( \mathcal{R}^c \subset \overline{\mathcal{Z}_1(X,E)} \subset \mathcal{S}^c \) are classical since Mañé-Mad-Sullivan work. Their proofs, which we
reproduce here for sake of completeness, only use the elementary fact that any attractive basin contains at least a critical point.

$S^c \subset \text{Supp}(dd^cL)$: Let $\Omega$ be an open ball in $X$ on which $L$ is pluriharmonic; we have to show that $\Omega \subset S$. Shrinking $\Omega$ if necessary, we find a $D$-valued holomorphic map $\lambda \mapsto \tilde{C}_f^\lambda$ from $\Omega$ to $\mathbb{C}^2 \setminus \{0\}$ such that $\pi \circ \tilde{C}_{f}\lambda = C_f\lambda$, and an analytic subset $A$ of $\Omega$ such that $\tilde{C}_f^\lambda = \{\tilde{c}_1(\lambda), \ldots, \tilde{c}_D(\lambda)\}$ where the $\tilde{c}_j(\lambda)$ are holomorphic maps on $\Omega \setminus A$. The product $\Pi(z \wedge \tilde{c}_j(\lambda))$ is a well defined $D$-homogeneous polynomial on $\mathbb{C}^2$ whose coefficients are bounded holomorphic functions on $\Omega \setminus A$. It therefore coincides with the restriction of a polynomial $H$ with holomorphic coefficients on $\Omega$. Moreover, as $H$ is obviously proportional to $\det F^\prime$, there exists a non-vanishing holomorphic function $\varphi(\lambda)$ on $\Omega$ such that $H = \varphi(\lambda) \det F^\prime$. Thus, after replacing $\tilde{c}_j(\lambda)$ by $(\varphi(\lambda))^{1/D} \tilde{c}_j(\lambda)$, we may assume that

$$H = \Pi(z \wedge \tilde{c}_j(\lambda)) = \det F^\prime, \ \forall z \in \mathbb{C}^2, \forall \lambda \in \Omega. \quad (5.1)$$

In the same way, we may construct a sequence of $D$-homogeneous polynomials $H_n$ of the form

$$H_n := h(\lambda)^{-d\theta} \Pi(z \wedge F^{n}\lambda(\tilde{c}_j(\lambda))) \quad (5.2)$$

where $h$ is a non-vanishing holomorphic function on $\Omega$. We will see that for a good choice of $h$ the coefficients of $H_n$ are uniformly bounded holomorphic functions on $\Omega$. As $\pi(\{H_n(\lambda, \cdot) = 0\}) = f^\prime_\lambda(C_{f\lambda})$, this implies that $\Omega \subset S$.

Let us construct $h$. Since $dd^cL = 0$ on $\Omega$, it follows from [5.1] and Corollary [3.3] that the function $\sum_{j=1}^{D} G_\lambda(\tilde{c}_j(\lambda))$ is pluriharmonic on $\Omega \setminus A$. As it is continuous on $\Omega$, it is actually pluriharmonic on $\Omega$ and therefore coincides with $\log |h(\lambda)|$ for some non-vanishing holomorphic function $h$.

It remains to show that the coefficients of $H_n$ are uniformly bounded, for this choice of $h$. Let us consider an arbitrarily small ball $B$ contained in $\Omega \setminus A$. We will show that, for all $\lambda \in B$, one has $H_n(\lambda, z) = e^{-id\theta B} \Pi(z \wedge A_j(\lambda))$ where $\theta B \in \mathbb{R}$ and $A_j(\lambda) \in \{G_\lambda = 0\}$. The conclusion will follow since $\cup_{\lambda \in \Omega} \{G_\lambda = 0\} \subset \subset \mathbb{C}^2$. As each term in the sum $\sum_{j=1}^{D} G_\lambda(\tilde{c}_j(\lambda))$ is p.s.h. on $B$, there are $D$ non-vanishing holomorphic functions $h_j$ such that $G_\lambda(\tilde{c}_j(\lambda)) = \log |h_j|$. Thus $\log |h| = \log \Pi|h_j|$ and $h = e^{id\theta B} \Pi h_j$ for some $\theta B \in \mathbb{R}$. Then, for any $\lambda \in B$, we get from (5.2):

$$H_n(\lambda, z) = e^{-id\theta B} \Pi \left( h_j(\lambda)^{-d\theta} z \wedge F^{n}_\lambda(\tilde{c}_j(\lambda)) \right) = e^{-id\theta B} \Pi \left( z \wedge F^{n}_\lambda \left( \frac{\tilde{c}_j(\lambda)}{h_j(\lambda)} \right) \right).$$

It finally suffices to set $A_j(\lambda) := \frac{\tilde{c}_j(\lambda)}{h_j(\lambda)}$ since, as desired, we have $G_\lambda(A_j(\lambda)) = G_\lambda(\tilde{c}_j(\lambda)) - \log |h_j(\lambda)| = 0$.

$\text{Supp}(dd^cL) \subset \mathbb{R}^c$: this is given by Theorem [2.2].

$\mathbb{R}^c \subset \mathbb{Z}_1(X, E)$: we shall use the following Lemma [2.2], Lemma VII.5.

**Lemma 5.3** Let $z_0 \in \mathbb{P}^1$ be a repulsive fixed point of $f^{n_0}_\lambda$ ($n_0$ being the period of the associated cycle) and $B$ be a ball centered at $\lambda_0$ in $X$. Let $z(\lambda)$ be a holomorphic map defined on some neighbourhood of $\lambda_0$ in $X$ such that $z(\lambda_0) = z_0$ and, for every $\lambda$, $z(\lambda)$ is a repulsive fixed point of $f^{n_0}_\lambda$ (the points $z(\lambda)$ are given by the implicit function theorem). Then: either
i) $z(\lambda)$ holomorphically extends to $B$ and $z(\lambda)$ is a repulsive fixed point of $f^{n_0}_{\lambda}$ which belongs to a cycle of period $n_0$, for every $\lambda \in B$,

or

ii) $z(\lambda)$ holomorphically extends to a neighbourhood of some path $\gamma$ joining $\lambda_0$ to $\lambda_1$ in $B$ and $z(\lambda_1)$ is an attracting fixed point of $f^{n_0}_{\lambda_1}$. In particular, there are infinitely many values of $\lambda'$ such that $z(\lambda')$ is a neutral fixed point of $f^{n_0}_{\lambda'}$ and the set of corresponding multiplier $s$ contains an open subset of $S^1$. Again every $z(\lambda)$ belongs to a cycle of period $n_0$.

If $\lambda_0 \in \mathcal{R}^c$ then, using the above lemma, we may find a non stationary sequence $\lambda_k \to \lambda_0$ such that $f_{\lambda_k}$ has a neutral cycle of period $n_k$ and a multiplier $e^{2i\pi \theta_k}$ with $\theta_k \in E$. Since by Fatou theorem $f_{\lambda_0}$ has at most 6d – 6 non-repulsive cycles, all but a finite number of $\Per(X, n_k, e^{2i\pi \theta_k})$ differ from $X$, this shows that $\lambda_0 \in \mathcal{Z}_1(X, E)$.

\[ \mathcal{Z}_1(X, E) \subset \mathcal{R}_c: \text{ we proceed by contradiction. Let } \lambda_0 \in \mathcal{Z}_1(X, E) \text{ and } B \text{ an open ball in } X \text{ such that } \lambda_0 \in B \subset S. \text{ Let } n_0 \in \mathbb{N}^* \text{ and } \theta_0 \in E \text{ such that } B \cap \Per_1(X, n_0, e^{2i\pi \theta_0}) \neq \emptyset. \] 

On a small ball $B' \subset B$ centered at some point $\lambda_1 \in \Per_1(X, n_0, e^{2i\pi \theta_0})$ there exists a holomorphic map $z(\lambda)$ such that $f_{\lambda_0}^{n_0}(z(\lambda)) = z(\lambda)$. Moreover, as the multiplier of $f_{\lambda_0}^{n_0}$ at $z(\lambda)$ is not constant near $\lambda_1$ (otherwise $\Per(X, n_0, e^{2i\pi \theta_0})$ would coincide with $X$), we may find $\lambda_2, \lambda_3 \in B'$ such that $z(\lambda_2)$ (resp. $z(\lambda_3)$) is attractive (resp.repulsive) for $f_{\lambda_2}$ (resp. $f_{\lambda_3}^{n_0}$). As the basin of $f_{\lambda_0}^{n_0}$ at $z(\lambda_2)$ contains a critical point, there exists $0 \leq i_0 \leq n_0$ such that the sequence $d[z(\lambda), f_{\lambda_2}^{k_0 - i_0}(C_{f_{\lambda_2}})]$ is converging to 0 around $\lambda_2$. Then, since $B' \subset B \subset S$, $d[z(\lambda), f_{\lambda_2}^{k_0 - i_0}(C_{f_{\lambda_2}})]$ actually converges to 0 in $B'$ which is impossible because $z(\lambda_3)$ is repulsive. \hfill \Box

**Remark 5.4** Two fundamental facts in Mañé-Sad-Sullivan theory are the density of $\mathcal{R}$ in $X$ and the emerging concept of hyperbolic component: two elements lying in the same connected component of $\mathcal{R}$ are either both hyperbolic or both non-hyperbolic. This plays an important role in the approach of Fatou’s conjecture on the density of hyperbolic rational maps. It turns out that these facts may be established by mean of elementary arguments similar to those used in the last steps of the proof of Theorem 5.2.

We now aim to generalize Theorem 5.2 to the case of powers $(dd^c L)^p$. To this purpose we have to discuss the intersection of $p$ hypersurfaces $\Per_1(X, n, e^{2i\pi \theta})$. For any $N_p := (n_1, \ldots, n_p) \in (\mathbb{N}^*)^p$ and $\Theta_p := (\theta_1, \ldots, \theta_p) \in E^p$ we define

\[ \Per(X, N_p, e^{2i\pi \Theta_p}) := \Per(X, n_1, e^{2i\pi \theta_1}) \cap \cdots \cap \Per(X, n_p, e^{2i\pi \theta_p}). \]

As previously, $\Per_p(X, N_p, e^{2i\pi \Theta_p})$ denotes the union of all the codimension $p$, irreducible components of $\Per(X, N_p, e^{2i\pi \Theta_p})$. We then set:

\[ \mathcal{Z}_p(X, E) := \bigcup_{N_p \in (\mathbb{N}^*)^p, \Theta_p \in E^p} \Per_p(X, N_p, e^{2i\pi \Theta_p}). \]

Our generalization may be stated as follows.
Theorem 5.5  Let $E$ be a dense subset of $]0,1[$. Let $\{f_\lambda\}_{\lambda \in X}$ be a holomorphic family of rational maps of degree $d$ on $\mathbb{P}^1$. Then for any $1 \leq p \leq \dim_\mathbb{C} X$:

$$\text{Supp}(dd^c L)^p \subset \overline{Z}_p(X,E).$$

Proof. We proceed by induction on $p$. Let us call $(\mathcal{H})_p$ the following assertion:

$$(\mathcal{H})_p: \text{ For any complex manifold } X \text{ of dimension } n \geq p \text{ and any holomorphic family } \{f_\lambda\}_{\lambda \in X} \text{ parametrized by } X \text{ we have Supp}(dd^c L)^p \subset \overline{Z}_p(X,E).$$

According to Theorem 5.5 $(\mathcal{H})_1$ is true. Let us show that $(\mathcal{H})_p$ implies $(\mathcal{H})_{p+1}$. To this end we shall combine the following fact with $(\mathcal{H})_1$.

Fact: Assume that $(\mathcal{H})_p$ is true. Let $U$ be an open set in $\mathbb{C}^n$ $(n > p)$ and $\{f_\lambda\}_{\lambda \in U}$ be a holomorphic family. If $L(f_\lambda)$ is pluriharmonic on every $\text{Per}_p(U,N_p,e^{2i\pi \Theta_p})$ then $(dd^c L)^{p+1} \equiv 0$ on $U$.

This fact, which is actually the heart of our proof, will be established later. It is useful to remark that a continuous function on an analytic set $Y$ is p.s.h. if and only if it is p.s.h. on the set of regular points of $Y$ (see [6], Theorem 1.7). We also recall that $L$ is continuous (see Corollary 3.5).

Let us consider a holomorphic family $\{f_\lambda\}_{\lambda \in X}$ and $\lambda_0 \in \text{Supp}(dd^c L)^{p+1}$. Pick an arbitrarily small open set $U$ such that $\lambda_0 \in U$. We have to show that $\overline{Z}_{p+1}(X,E) \cap U \neq \emptyset$. We may identify $U$ with an open set of $\mathbb{C}^n$. According to the above fact, there exist $N_p \in (\mathbb{N}^*)^p$ and $\Theta_p \in E^p$ such that $L$ is not pluriharmonic on $\text{Per}_p(U,N_p,e^{2i\pi \Theta_p})$. This implies the existence of some regular curve $\Gamma$ contained in $\text{Per}_p(U,N_p,e^{2i\pi \Theta_p})$ such that $dd^c(L|_\Gamma)$ does not vanish. Thus, Theorem 5.5 applied to the family $\{f_\lambda\}_{\lambda \in \Gamma}$ guarantees the existence of some $\text{Per}_1(\Gamma,N_{p+1},e^{2i\pi \Theta_{p+1}})$, $\Theta_{p+1} \in E$. Let $N_{p+1} := (N_p,N_{p+1})$ and $\Theta_{p+1} := (\Theta_p,\Theta_{p+1})$; since $\text{Per}_{p+1}^1(U,N_{p+1},e^{2i\pi \Theta_{p+1}}) \subset \overline{Z}_{p+1}(X,E) \cap U$, it is enough to observe that $\text{Per}_{p+1}^1(U,N_{p+1},e^{2i\pi \Theta_{p+1}}) \neq \emptyset$. 

Proof of the fact: By an elementary slicing argument, the positive current $(dd^c L)^{p+1}$ vanishes identically on $U$ as soon as the positive measures obtained by restriction on the $(p + 1)$-dimensional affine subspaces vanish. Let $S$ be the intersection of $U$ with such an affine subspace of $\mathbb{C}^n$ and set $L_0 := L|_S$, $\mu_0 := (dd^c L_0)^{p+1}$. We have to show that for every euclidean $(p + 1)$-dimensional, open ball $B \subset S$, the measure $\mu_0$ vanishes on $\frac{1}{2}B$.

To this end, we introduce the solution $\tilde{L}_0$ of the Dirichlet-Monge-Ampère problem with datum $L_0$ on $bB$. The function $\tilde{L}_0$ is continuous on $\overline{B}$, coincides with $L_0$ on $bB$ and is p.s.h. maximal on $B$ (see [11]). By maximality, $\tilde{L}_0 \geq L_0$ on $\overline{B}$. We also consider the set $\Sigma_\varepsilon \subset \frac{1}{2}B$ where $L_0$ and $\tilde{L}_0$ are $\varepsilon$-close:

$$\Sigma_\varepsilon := \{0 \leq \tilde{L}_0 - L_0 \leq \varepsilon\} \cap \frac{1}{2}B.$$  

A theorem of Briand-Duval (see [3] or [27] Theorem A.10.2) states that

$$\mu_0(\Sigma_\varepsilon) \leq C\varepsilon$$

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where the constant $C$ depends only on $L_0$ and $B$. It thus suffices to show that $\text{Supp}(\mu_0) \cap \frac{1}{2}B \subset \Sigma_\varepsilon$ for any $\varepsilon > 0$.

The set $Z_p(S, E)$ is an union of complex curves in $S$. Let $A$ be one of these curves, then $A$ is a component of $\text{Per}_p(S, N_p, e^{2i\pi \Theta_p})$ and is therefore contained in $S \cap \text{Per}_p(U, N_p, e^{2i\pi \Theta_p})$; since $S$ is an affine subspace this is easy to check on the regular part of $A$. Thus the function $L = L_0$ is, by assumption, harmonic on $A \cap B$. The function $\tilde{L}_0 - L_0$ is thus subharmonic on $A \cap B$ and, by the maximum principle, vanishes identically. Therefore $\tilde{L}_0 - L_0$ vanishes on $Z_p(S, E) \cap B$.

Of course $\text{Supp}(\mu_0) \subset \text{Supp}(dd^c L_0)^p$ and, as $(\mathcal{H}_p)$ is supposed be true, $\text{Supp}(dd^c L_0)^p \subset Z_p(S, E)$. Thus $\tilde{L}_0 - L_0$ vanishes on $\text{Supp}(\mu_0) \cap \frac{1}{2}B$, which is, therefore, contained in $\Sigma_\varepsilon$. \qed

6 The bifurcation measure

In this section we define the bifurcation measure $\mu$ on the moduli space $\mathcal{M}_d$ of rational maps $\mathbf{P}^1 \to \mathbf{P}^1$ and we establish some basic results about it. Although the section is mainly devoted to the one-dimensional case, the fact that the bifurcation currents $(dd^c L)^p$ have finite mass on $\mathcal{H}_d$ will be established in any dimension, that is in $\mathcal{H}_d(\mathbf{P}^k)$.

The group $\text{PSL}(2, \mathbb{C})$ of Möbius transformations acts on the space $\mathcal{H}_d(\mathbf{P}^1)$ by conjugation. Two conjugated rational functions $f_1, f_2 \in \mathcal{H}_d(\mathbf{P}^1)$ have the same dynamics, therefore in order to study the stability of holomorphic families of rational functions, one considers, instead of $\mathcal{H}_d(\mathbf{P}^1)$, the moduli space $\mathcal{M}_d := \mathcal{H}_d(\mathbf{P}^1)/\text{PSL}(2, \mathbb{C})$.

**Remark 6.1** The moduli space $\mathcal{M}_d$ is a normal, quasi-projective variety (see [28], Remark p.43); the proof requires some effort because $\text{PSL}(2, \mathbb{C})$ is not compact and its action on $\mathcal{H}_d(\mathbf{P}^1)$ is not free (indeed there is some $f \in \mathcal{H}_d(\mathbf{P}^1)$ whose isotropy group $\text{Aut}(f) := \{\varphi \in \text{PSL}(2, \mathbb{C}); \varphi^{-1} \circ f \circ \varphi = f\}$ is not trivial). Here we recall some useful facts about $\mathcal{M}_d$:

(i) the canonical projection $\Pi : \mathcal{H}_d(\mathbf{P}^1) \to \mathcal{M}_d$ is open;

(ii) for all $f \in \mathcal{H}_d(\mathbf{P}^1)$, the isotropy group $\text{Aut}(f)$ is finite and locally there is a complex submanifold $V$, invariant by the action of $\text{Aut}(f)$, transverse at $f$ to the orbit of $f$, such that $\Pi(V)$ is open in $\mathcal{M}_d$ and the canonical projection $\Pi$ induces a biholomorphism $V/\text{Aut}(f) \to \Pi(V)$;

(iii) the set of all $f \in \mathcal{H}_d(\mathbf{P}^1)$ such that $\text{Aut}(f) \neq \{\text{id}_{\mathbf{P}^1}\}$ is an analytic subset $Z$ of $\mathcal{H}_d(\mathbf{P}^1)$ and $\text{Sing}(\mathcal{M}_d) \subset \Pi(Z)$.

(In general $\text{Sing}(\mathcal{M}_d) \neq \Pi(Z)$, e.g. $\mathcal{M}_2 = \mathbb{C}^2$ is smooth, and $\Pi(Z)$ is a cubic curve of $\mathbb{C}^2 = \mathcal{M}_2$ (see [20], Corollary 5.3). But, for $d > 2$, $\mathcal{M}_d$ has singular points). It follows that if $f \notin Z$, then $\text{PSL}(2; \mathbb{C}) \times V \simeq \Pi^{-1}(\Pi(V))$, therefore

(iv) $\mathcal{H}_d(\mathbf{P}^1) \setminus Z \to \mathcal{M}_d \setminus \Pi(Z)$ is a principal bundle.

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Since $\dim_{\mathbb{C}} \text{PSL}(2, \mathbb{C}) = 3$ and all isotropy groups $\text{Aut}(f)$ are finite, hence each orbit is a complex 3-fold and $\dim_{\mathbb{C}} \mathcal{M}_d = \dim_{\mathbb{C}} \mathcal{H}_d(\mathbb{P}^1) - 3 = 2(d - 1)$. The Lyapunov exponent $L(f)$, $f \in \mathcal{H}_d(\mathbb{P}^1)$, which is invariant under the action of $\text{PSL}(2, \mathbb{C})$, is constant on the orbits and, if $p > 2(d - 1)$, the current $(dd^c L)^p$ vanishes identically on $\mathcal{H}_d(\mathbb{P}^1)$. Therefore in order to define a measure by means of Monge-Ampère operator on $L$, it is necessary to consider the function $\hat{L} : \mathcal{M}_d \to \mathbb{R}$ induced from $\mathcal{H}_d(\mathbb{P}^1) \rightarrow \mathbb{R}$.

**Proposition 6.2** The function $\hat{L}$ is continuous, bounded from below and p.s.h. on $\mathcal{M}_d$.

**Proof.** By [21], $L$ is bounded from below. Using Corollary 3.3 it is enough to notice that, by means of Remark 6.1 (iv), $\hat{L}$ is p.s.h. on $\mathcal{M}_d \setminus \Pi(Z)$ and thus (see [6], Theorem 1.7) on the whole $\mathcal{M}_d$. \hfill \square

Now the currents $(dd^c \hat{L})^p$, $1 \leq p \leq 2(d - 1)$, are well defined on $\mathcal{M}_d$. In particular, the measure $\mu := (dd^c \hat{L})^{2(d-1)}$ will be called **bifurcation measure**.

**Proposition 6.3** The bifurcation measure $\mu$ does not vanish identically, in particular any non-flexible Lattès map lies in the support of $\mu$.

**Proof.** For $d \geq 2$ fixed, let $f_0 \in \mathcal{H}_d(\mathbb{P}^1)$ be a non-flexible Lattès map (e.g. $f_0$ is the map associated to an imaginary quadratic number field, see [22] Lemma 5.4), then all Lattès maps which belong to a small neighbourhood of $f_0$ in $\mathcal{H}_d(\mathbb{P}^1)$ are conjugated to $f_0$. Let $V$ be a complex submanifold in a neighbourhood of $f_0$ as in Remark 6.1 (ii); since the function $L(f)$ takes its minimum value $\log \sqrt{d}$ exactly when $f$ is a Lattès map (see [15], [31]), hence $f_0$ is a point of strict minimum for $L|_V$. As $\dim_{\mathbb{C}} V = 2(d - 1)$ we shall see that $f_0 \in \text{Supp}(dd^c L|_V)^{2(d-1)}$, i.e. $\Pi(f_0) \in \text{Supp}(\mu)$. For every small, euclidean, open ball $B \subset V$ centered at $f_0$ there is a suitable constant $c$ such that $L(f_0) < c < L(f)$, for every $f \in B$; so the function $L|_V - c$ does not take its minimum on $\overline{B}$ at the boundary, therefore (see Theorem A in [1]) it is not maximal, that is $(dd^c L|_V)^{2(d-1)}$ does not vanish identically on $B$. \hfill \square

In order to see that $\mu$ has finite mass (see Proposition 6.3) we shall show that $L$ extends from $\mathcal{H}_d$ to the whole projective space across the hypersurface $\Sigma_d$ and that the powers of $dd^c L$ have finite mass on $\mathcal{H}_d$. Since these results hold for holomorphic maps $\mathbb{P}^k \to \mathbb{P}^k$, $k \geq 1$, we believe that it is useful to present them in this more general case.

First of all let us recall that $\mathcal{H}_d(\mathbb{C}^{k+1}) = \mathbb{C}^N \setminus \Sigma_d$, see Subsection 1.1 and that from (4.1) and Proposition 4.9 it follows:

$$L(F) = H(F) \frac{k + 1}{d^k} \log |\text{Res}(F)| + \text{cst.} \quad (6.1)$$

for every polynomial map $F \in \mathcal{H}_d(\mathbb{C}^{k+1}) \subset \mathbb{C}^{N+1}$.

**Proposition 6.4** The function $H$ extends from $\mathcal{H}_d(\mathbb{C}^{k+1})$ to a p.s.h. function on the whole $\mathbb{C}^{N+1}$ and the function $L(f)$ extends from $\mathcal{H}_d(\mathbb{P}^k)$ to a function $L_{\text{loc}}(\mathbb{P}^N)$. Moreover there is a $(1,1)$-current $R$, positive and closed on $\mathbb{P}^N$ such that $\pi^* R = dd^c H$ and

$$dd^c L = R - \frac{k + 1}{d^k} [\Sigma_d]. \quad (6.2)$$

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Proof. Let $K \subset \mathbb{C}^{N+1}$ be compact, let us check that $H$ is bounded from above on $K \cap \mathcal{H}_d(C^k)$. By definition (see (4.2)):

$$H(F) = \int_{\mathbb{P}^k} g_c \sum_{j=0}^{k-1} T_j^c \wedge \omega^{k-j-1} \wedge [C_f] + \int_{\mathbb{P}^k} \log \|J_F\|_0 \omega^c.$$ 

Then, as $g_c$ is locally bounded from above (see Remark 1.3), one concludes taking into account the following formulas: $\int_K \sum_{j=0}^{k-1} T_j^c \wedge \omega^{k-j-1} \wedge [C_f] = k \deg(C_f)$ and $\int_K \log \|J_F\|_0 \omega^c = \int_{C^{k+1}} \log |\det(F^c(z))| \nu$ (Lemma 1.4). From this, since $H$ is p.s.h. on $\mathcal{H}_d(C^{k+1})$ (see the Claim in the proof of Theorem 4.5), it follows that $H$ extends to a p.s.h. function on the whole $\mathbb{C}^{N+1}$. Thus the right hand side of (6.1) belongs to $L^1_{\text{loc}}(\mathbb{C}^{N+1})$ and extends $L(F)$ as a 0-homogeneous $L^1_{\text{loc}}$ function on $\mathbb{C}^{N+1}$. Thus $L(f)$ is well defined on the whole $\mathbb{P}^N$. Choosing a holomorphic section $U \xrightarrow{\sigma} \mathbb{C}^{N+1} \setminus \{0\}$ on an open subset $U$ of $\mathbb{P}^N$, we get $\forall f \in U$,

$$L(f) = H(\sigma(f)) - \frac{k + 1}{d^k} \log |\text{Res}(\sigma(f))| + \text{cst.} \quad (6.3)$$

As $dd^c(H \circ \sigma)$ does not depend on $\sigma$, it defines a positive, closed current $R$ on $\mathbb{P}^N$ such that $\pi^*R = dd^cH$. Then (6.2) follows from (6.3). \hfill \Box

The sum $L(f)$ of the Lyapunov exponents is bounded from below (see Theorem 1 in [3]), thus, as in the one-dimensional case, the powers of $dd^c(L|_{\mathcal{H}_d(\mathbb{P}^N)})$ are well defined; but to show that these currents have finite mass requires some work.

**Proposition 6.5** For $1 \leq p \leq N$,

$$\int_{\mathcal{H}_d(\mathbb{P}^k)} (dd^cL)^p \wedge \omega^{N-p} < \infty$$

and the trivial extension $S_{(p)}$ of $(dd^c(L|_{\mathcal{H}_d}))^p$ to the whole $\mathbb{P}^N$ is well defined.

**Remark.** We recall that, by definition of trivial extension, $S_{(p)}$ is the positive, closed, $(p,p)$-current on $\mathbb{P}^N$ characterized by

(i) $S_{(p)} = (dd^cL)^p$ on $\mathcal{H}_d(\mathbb{P}^k)$

(ii) $\chi_{\Sigma_d} \cdot S_{(p)} = 0$, where $\chi_{\Sigma_d}$ is the characteristic function.

**Proof.** From (6.2) it follows $dd^cL \leq R$, thus $dd^cL$ has finite mass on $\mathcal{H}_d(\mathbb{P}^k)$ and its trivial extension to $\mathbb{P}^N$ is the positive, closed current $S_{(1)} := (1 - \chi_{\Sigma_d})R$.

Now we shall argue by induction, assuming that the trivial extension $S_{(p)}$ of $(dd^c(L|_{\mathcal{H}_d}))^p$ to the whole $\mathbb{P}^N$ is well defined (and, of course, positive and closed). There is a smooth, closed $(1,1)$-form $\alpha$ on $\mathbb{P}^N$ such that $S_{(1)} - \alpha = dd^c\nu$ and, by means of the regularization theorem of Demailly ([2]), there are a sequence $\{\nu_n\}$ of smooth functions decreasing to $\nu$ and a sequence $\{\lambda_n\}$ of continuous functions decreasing to $\nu(S_{(1)},\cdot)$, such that $S_n := \alpha + dd^c\nu_n \to S_{(1)}$ and $S_n + \lambda_n \omega \geq 0$. 

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We can estimate the mass of \( S_{(p)} \wedge (S_n + \lambda_n \omega) \) (which is a positive current) as follows:

\[
||S_{(p)} \wedge (S_n + \lambda_n \omega)|| = \int_{\mathbb{P}^N} S_{(p)} \wedge \alpha \wedge \omega^{N-p-1} + \int_{\mathbb{P}^N} S_{(p)} \wedge \omega^{N-p} + \int_{\mathbb{P}^N} \lambda_n S_{(p)} \wedge \omega^{N-p}
\]

Let us look at the right hand side: the first term is constant, the second vanishes. Since \( L \) (and therefore \( H \)) is bounded from below on \( \mathbb{P}^N \setminus \Sigma_d = H_d(\mathbb{P}^k) \), the Lelong numbers of \( S_{(1)} \) vanish outside \( \Sigma_d \). Thus \( \lambda_n \) decreases to \( \nu(S_{(1)},.) \chi_{\Sigma_d} \), but \( \chi_{\Sigma_d} S_{(p)} = 0 \), therefore \( \lambda_n S_{(p)} \to 0 \).

This means that \( S_{(p)} \wedge (S_n + \lambda_n \omega) \) has bounded mass, thus we can assume that it converges to a positive current \( Q \). But \( S_{(p)} \wedge \lambda_n \omega \to 0 \), thus \( Q \) is closed. Therefore we can set \( S_{(p+1)} := (1 - \chi_{\Sigma_d})Q \).

\[\square\]

Coming back to one-dimensional case, we can establish:

**Proposition 6.6** The bifurcation measure \( \mu \) has finite mass on \( \mathcal{M}_d \).

**Proof.** Since \( \hat{L} \) is bounded from below, the measure \( \mu \) does not charge analytic subsets thus from Remark 6.1 (iv) and the previous Proposition we get:

\[
\int_{\mathcal{M}_d} \mu = \int_{\mathcal{M}_d \setminus \Pi(Z)} (dd^c \hat{L})^{2(d-1)} = \int_{\mathcal{H}_d(\mathbb{P}^1) \setminus Z} (dd^c L)^{2(d-1)} \wedge \omega^3 = \int_{\mathcal{H}_d(\mathbb{P}^1)} (dd^c L)^{2(d-1)} \wedge \omega^3 < \infty.
\]

\[\square\]

**Remark 6.7** Since the hypersurfaces \( Per(\mathcal{H}_d(\mathbb{P}^1), n, e^{2i\pi \theta}) \) are invariant under the action of \( PSL(2, \mathbb{C}) \), there are no difficulties in order to get, from Theorems 5.2 and 6.5, the corresponding statement for \( \hat{L} \). Actually the following claim holds: \( Supp(dd^c L) = \mathcal{Z}_1(\mathcal{M}_d) \) and for \( 1 < p \leq 2(d-1), \) \( Supp(dd^c \hat{L})^p \subset \mathcal{Z}_p(\mathcal{M}_d) \). In particular \( Supp(\mu) \subset \mathcal{Z}_{2(d-1)}(\mathcal{M}_d) \).

We say that a point \( x \in \mathcal{M}_d \) is chaotic if the Julia set of any \( f \in \Pi^{-1}(x) \) is \( \mathbb{P}^1 \).

**Proposition 6.8** In any neighbourhood of a point of \( Supp(\mu) \) there are uncountably many chaotic points.

**Proof.** Consider \( \{ \theta \in \mathbb{R}; \limsup_{n \to \infty} \log \log(1/|\theta^n - 1|) < \log d \} \) and use this open dense and uncountable subset of \( \mathbb{R} \) to define \( E \) and \( \mathcal{Z}_p(\mathcal{H}_d(\mathbb{P}^1), E) \) (see [21]). If \( z_0 \) is a periodic points of \( f \in \mathcal{H}_d(\mathbb{P}^1) \) with multiplier \( e^{2i\pi \theta}, \theta \in E \), then \( z_0 \) is a Cremer point; therefore any \( f \in \mathcal{Z}_{2(d-1)}(\mathcal{H}_d(\mathbb{P}^1), E) \) has \( 2(d-1) \) Cremer points and (see [26], Corollary 2) is chaotic.

\[\square\]

\[\text{1We would like to thank T.C. Dinh who told us the possibility to use Shishikura’s theorem here.}\]
7 Examples and applications

7.1 Geodesics on the space of Kähler metrics

Let \( M \) be a compact Kähler manifold of dimension \( k \) with a fixed Kähler metric \( \omega \). In order to discuss extremal (e.g. Einstein and of constant scalar curvature) Kähler metrics it is useful to consider the space \( H_\omega \) of Kähler metrics with the same Kähler class of \( \omega \) (see e.g. [5]). It can be thought also as the space of Kähler potentials, that is

\[
H_\omega := \{ \phi \in C^\infty(M); \omega + i\partial \bar{\partial} \phi > 0 \}/\sim
\]

where \( \phi_1 \sim \phi_2 \) if and only if \( \phi_1 - \phi_2 \) is a constant. Endowing \( H_\omega \) with a suitable metric it turns out that \( H_\omega \), as Riemannian manifold, is an infinite dimensional symmetric space and there is a (unique) Levi-Civita connection whose curvature is covariant constant (see [25] and [5]).

For such a connection the equation of geodesic is

\[
(i\partial \bar{\partial} \phi + \omega)^{k+1} = 0 \tag{7.1}
\]

This means that \( \phi \) is a smooth real function defined on \([0, 1] \times M\) (in this case one understands the \( \overline{\partial} \) operator as the one on the cylinder \([0, 1] \times S^1\) with its natural complex structure) or, for complex geodesics, it is more generally defined on \( X \times M \) where \( X \) is a Riemann surface. We point out that very few explicit examples of these geodesics are known, thus the following remark may have some interest.

Let \( M = P^k \) and \( \omega \) be the Fubini-Study metric, Corollary 4.6 says that any holomorphic family \( \{f_\lambda\}_{\lambda \in X} \) of endomorphisms of \( P^k \) defines a “geodesic” \( \phi := g_{F_\lambda} \). Of course the behaviour of \( dd^c(g_{F_\lambda} + \omega) \) is very far from the desired regularity, but there is at least one case in which holomorphic dynamics may give interesting examples: let \( M = P^1 \) and \( \{f_\lambda\}_{\lambda \in X} \) be a family of flexible Lattès maps (see [22], Ch. 8.3), then the functions \( g_{F_\lambda} : P^1 \to \mathbb{R} \) are smooth outside a finite set.

7.2 Attractors in \( P^2 \)

**Definition 7.1** Let \( \{f_\lambda\}_{\lambda \in X} \) be an one parameter holomorphic family (i.e. \( X \) is an open subset of \( \mathbb{C} \)) of endomorphisms of \( P^k \) and let \( Y \) be a complex subspace of \( X \times P^k \) of pure dimension \( q \). We shall say that the Green function \( G_\lambda \) is maximal on \( Y \) if and only if, for every holomorphic section \( P^k \supset U \xrightarrow{\sigma} \mathbb{C}^{k+1} \setminus \{0\}, \)

\[
(dd^c(G_\lambda \circ \sigma))^q = 0 \text{ on } Y \cap (X \times U).
\]

Although the Green function depends on the choice of the lift of \( f_\lambda \), the definition is well posed since \( dd^c(G_\lambda \circ \sigma) \) does not depend on the particular family of lifts \( \{F_\lambda\}_{\lambda \in W} \) chosen in an open subset \( W \) of \( X \).

With this definition we can give the following formulation of Corollary 4.6

**Proposition 7.2** Let \( \{f_\lambda\}_{\lambda \in X} \) be an one parameter holomorphic family of endomorphisms of \( P^k \). Then \( G_\lambda \) is maximal on \( X \times P^k \). Moreover \( G_\lambda \) is maximal on \( C_X \) if and only if \( L(f_\lambda) \) is harmonic.
Now we shall apply this Proposition to a particular case. For $\varepsilon \in \mathbb{C}$, consider the rational map $\mathbb{P}^2 \to \mathbb{P}^2$ defined by

$$f_\varepsilon = [P(z, w) : Q(z, w) : t^d + \varepsilon R(z, w)]$$

where $P, Q, R$ are homogeneous polynomials of degree $d \geq 2$ such that $(P, Q)$ is non degenerate and the induced rational function

$$f = [P(z, w) : Q(z, w)]$$

is strictly critically finite. It is useful to consider the line $\mathcal{R}_\infty := \{ t = 0 \}$ as the line at infinity of the complex plane $\mathbb{C}^2 \simeq \{ [z : w] \in \mathbb{P}^2 \}$. If $a \in \mathcal{R}_\infty$, we shall denote by $\mathcal{R}_a$ the line passing through the origin $[0 : 0 : 1]$ and $a$. The map $f_\varepsilon$ preserves lines through the origin and moves them in a chaotic way, since $f$ is chaotic, indeed identifying $\mathcal{R}_\infty$ with $\mathbb{P}^1$ we get $f_\varepsilon(\mathcal{R}_a) \cap \mathcal{R}_\infty = \{ f(a) \}$.

For $|\varepsilon| << 1$ the only Fatou component is the superattractive basin of the origin (see [12] Lemma 2.1). Moreover for $|\varepsilon| << 1$ the map $f_\varepsilon$ has an attractor $A$ contained in a neighbourhood of the line at infinity, which intersect any line passing through the origin (ibidem, Lemma 2.2).

Our aim is to show that, for $|\varepsilon| << 1$, the family $\{f_\varepsilon\}$ is stable in the following sense:

**Proposition 7.3** The function $\varepsilon \mapsto L(f_\varepsilon)$ is harmonic in a neighbourhood of $0 \in \mathbb{C}$.

**Proof.** Let $X := \mathbb{C}$ and $F_\varepsilon = (P, Q, t^d + \varepsilon R)$. A simple inspection on $F_\varepsilon'$ shows that $C_{F_\varepsilon} = \mathcal{R}_\infty \cup \left( \cup_{c \in C_f} C_c \right)$, thus the critical set does not depend on $\varepsilon$ and

$$C_X = (X \times \mathcal{R}_\infty) \cup \left( \cup_{c \in C_f} X \times \mathcal{R}_c \right).$$

Let $c \in C_f$; then by hypothesis, there exist $j, k \in \mathbb{N}^*$ such that, $a := f^j(c)$ and $f^k(a) = a$. This means that $\mathcal{R}_a$ is fixed by $f^k$. Thus we can consider the family $f^k_{\varepsilon|\mathcal{R}_a}$ as a family of endomorphisms of $\mathcal{R}_a$. From Proposition 7.2 it follows that, for this family, the Green function is maximal on $X \times \mathcal{R}_a$. But all the powers of $f_{\varepsilon}$ have the same Green function $G_{F_{\varepsilon}}$. That is, for every section $U \xrightarrow{\sigma} \mathbb{C}^2 \setminus \{0\}$, the function $G_{F_{\varepsilon}} \circ \sigma$ is maximal on $X \times (U \cap \mathcal{R}_a)$. Since $f^j(\mathcal{R}_c) = \mathcal{R}_a$ and $G_{F_{\varepsilon}} \circ \sigma \circ f^j_\varepsilon = G_{F_{\varepsilon}}(h, (F^j_\varepsilon \circ \sigma')) = d^j G_{F_{\varepsilon}} \circ \sigma' + \log |h|$, for a suitable section $\sigma'$ and a holomorphic, never vanishing, function $h$, we get that $G_{F_{\varepsilon}}$ is maximal on $X \times \mathcal{R}_c$.

Now we shall show that choosing $V = \{ |\varepsilon| << 1 \}$, the function $G_{F_{\varepsilon}}$ is maximal on $V \times \mathcal{R}_\infty$. Let $u \in \mathcal{R}_\infty \simeq \mathbb{P}^1$ be a periodic point for $f$ (that is $f^j(\mathcal{R}_u) = \mathcal{R}_u$ for some $j$). From the proof of Lemma 2.2 in [12] it follows that there is an open neighbourhood $W$ of $\mathcal{R}_\infty$ in $\mathbb{P}^2$ such that, if $|\varepsilon| << 1$, then $f_\varepsilon(W) \subset W$; thus the family $\{f_\varepsilon^j(u)\}_{j \in \mathbb{N}}$ is a normal in $V$ (as functions of $\varepsilon$). Therefore $\varepsilon \mapsto G_{F_{\varepsilon}}(\sigma(u))$ is harmonic on $V$. That’s enough since these points $u$ are dense in $\mathcal{R}_\infty$. \qed

### 7.3 The bifurcation measure on $\mathcal{M}_2$

As we have already recalled the moduli space $\mathcal{M}_2$ of the rational functions of degree 2 can be identified biholomorphically and in a canonical way with $\mathbb{C}^2$ (see [20] Remark 3.3). Such an identification involves the affine structure since, for every $\eta \in \mathbb{C}$, $\text{Per}(\mathcal{M}_2, 1, \eta)$ is a straight line of $\mathbb{C}^2$. In particular, the Mandelbrot family $\{z^2 + c; c \in \mathbb{C}\}$ coincides with the straightline $\text{Per}(\mathcal{M}_2, 1, 0)$ of rational functions with a superattractive fixed point.
**Proposition 7.4** The Mandelbrot family is disjoint from the support of the bifurcation measure \( \mu \) on \( \mathcal{M}_2 \).

**Proof.** Since every \( f_0 \in \text{Per}(\mathcal{M}_2, 1, 0) \) has a superattractive fixed point, there is an open subset \( V, \text{Per}(\mathcal{M}_2, 1, 0) \subset V \subset \mathcal{M}_2 \), such that every \( f \in V \) has an attracting fixed point.

For every \( f \in \mathcal{M}_2 \) the number of attracting or indifferent cycles is \( \leq 2 \) (see [26], Corollary 1), therefore \( V \cap \mathcal{Z}_2(\mathcal{M}_2, E) = \emptyset \). Thus from Theorem 5.6 it follows that \( V \) is disjoint from \( \text{Supp}(dd^c\hat{L})^2 = \text{Supp}(\mu) \).

**Remark 7.5** Let us point out that it is not necessary to use Shishikura’s result. First we can assume that if \( f \in V \), then the attractive fixed point depends holomorphically on \( f \). Consider a holomorphic disc \( \{f_\lambda\}_{\lambda \in \Delta} \) contained in \( V \cap \text{Per}_1(\mathcal{M}_2, n, e^{2i\pi \theta}) \). If \( \hat{L} \) is not harmonic on this disc, then from Theorem 5.2 there is a holomorphic function \( z(\lambda) \) on an open disc \( \Delta' \subset \Delta \) such that \( f_\lambda^*(z(\lambda)) = z(\lambda) \) and two values \( \lambda_1, \lambda_2 \in \Delta' \) such that \( z(\lambda_1) \) is repulsive and \( z(\lambda_2) \) is attractive. Thus \( f_{\lambda_2} \) has two attracting points, and it is stable in \( \mathcal{M}_2 \). This contradicts the fact that \( f_{\lambda_2} \in \text{Per}_1(\mathcal{M}_2, n, e^{2i\pi \theta}) \subset \text{Supp}(dd^c\hat{L}) \). Therefore \( \hat{L} \) is pluriharmonic on every \( \text{Per}_1(\mathcal{V}, n, e^{2i\pi \theta}) \); from the Fact in the proof of Theorem 5.6 it follows that \( (dd^c\hat{L})^2 = 0 \) on \( V \).

**Appendix**

**Proof of proposition 4.3**

Let \( u \) be a locally bounded function on \( X \); using (4.7) with \( R = u \), Fubini theorem and a suitable partition of the unity, it follows that \( u_y = u|_{\pi^{-1}(y)} \), for a.e. \( y \in Y \). Since \( T \) is positive and closed, then the slices of \( T \) exist (see [29] (10.3)); from (4.7), since slicing commutes with the operators \( d, \partial \) and \( \mathcal{F} \) (see [29] (10.4)), it follows that, for a.e. \( y \in Y \), \( T_y \) is a current on \( \pi^{-1}(y) \), positive and closed. Thus \( u_y T_y \) is well defined. By definition \( dd^c u \wedge T = dd^c (u \wedge T) \), thus we shall argue by recurrence and, in order to finish the proof, it is enough to show \( u_y T_y = (uT)_y \).

This obviously holds if \( u \in \mathcal{C}^\infty(X) \). Fix \( \phi \) and \( \psi \) as in (4.7) and let \( K \) be a compact set such that \( \text{Supp}(\psi) \subset K \). For a.e. \( y \in Y \)

\[
|\int_{\pi^{-1}(y)} u_y T_y \wedge \iota_y^* \psi| \leq \sup_K |u| \ C_{\psi} \ ||T_y||_{K \cap \pi^{-1}(y)},
\]

therefore

\[
|\int_Y (\int_{\pi^{-1}(y)} u_y T_y \wedge \iota_y^* \psi) \phi| \leq \sup_K |u| \ C_{\phi, \psi} \ ||T||_K
\]

and the left hand side is well defined.

That’s all, because this inequality shows that the operator \( \Phi(u) = \int_Y (\int_{\pi^{-1}(y)} T_y \wedge \iota_y^* (u \psi)) \phi \) can be continuously extended from \( \mathcal{C}^\infty(X) \) to \( L^\infty_{\text{loc}}(X) \).

**Proof of Proposition 4.10**

In order to compute the constant \( C_{d,k} \) it is enough to consider a particular \( F \); infact if we take \( F(z_0, \ldots, z_k) := \langle z_0^d, \ldots, z_k^d \rangle \), then \( \text{Res}(F) = 1 \) by Proposition 11 and, from
Proposition 4.9 it follows
\[ C_{d,k} = B(F). \] (7.2)

It turns out that
\[ G_F(z) = \log \max_{0 \leq j \leq k} |z_j|. \] (7.3)

Therefore \( G_F \) and (see (4.3)) \( B(F) = C_{d,k} \) do not depend on \( d \). We shall write \( C_k := C_{d,k} \).

Now a direct computation of \( B(F) \) is possible using (4.3) and (7.3), but the following proof is more elementary.

**Claim 1:** \( L(F) = (k + 1) \log d \).

**Proof of Claim 1.** Using 1.4(i) it is enough to compute \( \det F'(z) = d^{k+1}z_0^{d-1} \ldots z_k^{d-1} \) and remark that \( \text{Supp}(\mu_F) = \{ z \in \mathbb{C}^{k+1} : |z_0| = \ldots = |z_k| = 1 \} \). \( \square \)

**Claim 2:** \( H(F) = (k + 1)(d - 1) \left( C_{k-1} - \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{k}) \right) + (k + 1) \log d \)

**Proof of Claim 2.** The critical set \( C_F \) is the union of the projective hyperplanes \( H_s := \{ z_s = 0 \}, 0 \leq s \leq k, \) more precisely \( [C_F] = \{ d - 1 \} \sum_{s=0}^{k} [H_s] \). The restriction \( \tilde{F} \) of \( F \) to the hyperplane \( \{ z_k = 0 \} \) is of the same form: \( \tilde{F}(z_0, \ldots, z_{k-1}) = (z_0^d, \ldots, z_{k-1}^d) \). Hence \( B(\tilde{F}) = C_{k-1} \). Now from (4.3) it follows
\[
\int_{\mathbb{P}^k} g_F \sum_{j=0}^{k-1} T_j \wedge \omega^{k-1-j} \wedge [C_F] = (d - 1) \sum_{s=0}^{k} \int_{H_s} g_F \sum_{j=0}^{k-1} T_j \wedge \omega^{k-1-j} = (d - 1)(k + 1)B(\tilde{F}) = (d - 1)(k + 1)C_{k-1}. \] (7.4)

From Lemma 1.4(2)
\[
\int_{\mathbb{P}^k} \log ||J_F||_0 \omega^k = \int_{\mathbb{C}^{k+1}} \log |\det F'| m = (k + 1) \log d + (d - 1) \sum_{j=0}^{k} \int_{\mathbb{C}^{k+1}} \log |z_j| dm,
\]
and by means of an elementary computation
\[
\int_{\mathbb{P}^k} \log ||J_F(z)||_0 \omega^k = (k + 1) \log d - (k + 1)(d - 1) \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{k}). \] (7.5)

Putting (7.4)-(7.5) in (7.2) we get the Claim. \( \square \)

Now putting (7.2) and the two claims in (4.1) we get
\[ C_k = C_{k-1} - \frac{1}{2} \sum_{j=1}^{k} \frac{1}{j}. \]

Now to finish it is enough to find \( C_1 \); for \( k = 1 \) the map \( F \) is given by \( F(z_0, z_1) = (z_0^d, z_1^d) \) and in this case (4.1) gives
\[ L(F) = g_F(0 : 1) + \int_{\mathbb{P}^1} \log ||J_F(z)||_0 \omega - 2(d - 1)B(F); \]
now by Claim 1, \( L(F) = 2 \log d, \) by (7.5), \( \int_{\mathbb{P}^1} \log ||J_F(z)||_0 \omega = 2 \log d - (d - 1), \) and, by (7.3), \( g_F(0 : 1) = g_F(1 : 0) = 0, \) thus:
\[ -1/2 = B(F) = C_1. \] \( \square \)
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