BIRATIONAL MAPS AND NORI MOTIVES

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ABSTRACT. The monograph [HuM-St17] contains a systematic exposition of Nori motives that were developed and studied as the “universal (co)homology theory” of algebraic varieties (or schemes), according to the prophetic vision of A. Grothendieck. Since then, some research was dedicated to application of Nori motives in various domains of algebraic geometry: geometries in characteristic 1 ([LieMaMar19], [MaMar18]), absolute Galois group ([MaMar19-2]), persistence formalism ([MaMar19-1]).

In this note, we sketch an approach to the problems of equivariant birational geometry developed by M. Kontsevich and Yu. Tschinkel in [KTsch20], where Burnside invariants were introduced. We are making explicit the role of Nori constructions in this environment.

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0. INTRODUCTION AND SUMMARY

0.1. Birational maps and their symmetries. Our main objects of study here are stable birational maps, mostly between algebraic varieties defined over a subfield of \( \mathbb{C} \). The general restriction of stability is discussed in [AVi02], and main results about natural categories/“towers” of birational maps we use here are given in [AT16] and [BeRy19].

Symmetries of stable maps and their moduli spaces appear in various contexts. The celebrated Grothendieck’s approach to the study of absolute Galois group \( G_\mathbb{Q} \) of the field of all algebraic numbers \( \overline{\mathbb{Q}} \) bridged geometry and arithmetic via the tower of stable étale maps to \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \). More generally, for any integral scheme \( X \), the exact sequence

\[
1 \to \pi_1(X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \to \pi_1(X) \to G_\mathbb{Q} \to 1
\]

merges actions of what we call arithmetic and geometric symmetries.

In [CMMar20], in place of étale maps \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) we considered moduli spaces of stable curves of genus zero with marked points, and later demonstrated that their geometric symmetries can be approached through the Nori motivic structures.
of these moduli spaces. In this context, the exact sequence of fundamental groups above is replaced by a subextension of the “motivic fundamental groups”

$$1 \to G_{\text{mot}}(\overline{\mathbb{Q}}, \mathbb{Q}) \to G_{\text{mot}}(\mathbb{Q}, \mathbb{Q}) \to G_{\mathbb{Q}} \to 1$$

Here we focus on similar constructions, but starting with towers of birational maps replacing towers of stable moduli spaces above.

0.2. Burnside groups and Nori diagrams. In [AT16], [BeRy19], and other papers it was shown that the problem of classification of birational maps can use information encoded in the tower of natural maps structurally similar to the tower of stable genus zero modular spaces $M_{0,n}$. Using these natural maps, in the recent article [KrTsch20], A. Kresch and Yu. Tschinkel are imposing an additional geometric symmetry group from the start and then showing that this geometric symmetry can be encoded by certain analogues of “modular symbols”.

In this note we demonstrate that the Kresch–Tschinkel modular symbols defined through Burnside groups, also have a natural description in terms of Nori motivic structures. Briefly, we prove the following theorem:

0.2.1. Theorem. Nori stratifications of appropriate towers of birational maps can be enriched to the homological Nori geometric diagram by Burnside groups of strata.

For a more precise statement, see Theorem 3.3.

0.3. Quantum statistical mechanics via Burnside groups. Finally, in the last section we consider the Bost–Connes formalism connecting arithmetic zeta–functions of fields of algebraic numbers with physicists’ studies of classical and quantum behaviour of physical systems with infinite number of degrees of freedom: see [BoCon95]; Sec. 5 of [MaMar18], and references therein.

It was in this Sec. 5, [MaMar18], that we have sketched the enrichment of Bost–Connes via Burnside groups which in more detail will be presented here.

0.4. Summary. Many preliminaries and necessary definitions in this paper are distributed as follows. Sec. 1 contains a short dictionary of graphic presentations of categories and functors. Nori diagrams and representations are discussed in Sec. 2; Burnside groups in Sec. 3, containing also the last steps of the proof of Theorem 0.2.1. Section 4 recalls a construction of [LieMaMar19] of a lifting of the
Bost–Connes algebra to the relative \( \mathbb{Z} \)-equivariant Kontsevich-Tschinkel Burnside group and to its underlying assembler category and spectrum, and to equivariant Nori motives. Motivated by this construction, we then show that a similar Bost-Connes-type structure is also present in the Kontsevich-Pestun-Tschinkel modular symbols.

1. BACKGROUND

1.1. Categories and their diagrams. A diagram \( D \) is family \((V(D), E(D), \partial)\) where \( \partial \) (boundary map) is an embedding of \( E(D) \) (edges) into \( V(D) \times V(D) \) (ordered pairs of vertices). Each category defines its diagram, whose edges are its morphisms, and vertices are its objects.

Conversely, diagrams themselves form objects of category, whose morphisms imitate functors.

For a more detailed discussion of combinatorics of categories based upon diagrams, we refer the reader to Sec. 0.2 of [MaMar19-1] and Sec. 5 of [CM19]. Here we remind only the definition of posets in groupoids ([CM19], Def. 5.2).

1.2. Definition. A category \( \mathcal{PG} \) is called a poset in groupoids, if

(a) For any object \( X \) of \( \mathcal{PG} \), the full subcategory consisting of all objects isomorphic \( X \), is a groupoid, that is, all morphisms in it are isomorphisms.

(b) Whenever \( X_1 \) and \( X_2 \) are not isomorphic and \( \text{Hom}(X_1, X_2) \neq \emptyset \), then \( \text{Hom}(X_1, X_2) \) has a single orbit with respect to the left action of the group

\[ \text{Hom}(X_1, X_1) \times \text{Hom}(X_2, X_2)^{op} \]

combining precomposition and postcomposition.

A part of posets in groupoids consists of thin categories \( \mathcal{C} \): such that if any set \( \text{Hom}_\mathcal{C}(X_1, X_2) \) has cardinality \( \leq 1 \), and if both \( \text{Hom}_\mathcal{C}(X_1, X_2) \) and \( \text{Hom}_\mathcal{C}(X_2, X_1) \) are non-empty, then \( X_1 = X_2 \).

1.3. Diagrams of effective pairs. ([HuM-St17], Ch. 9, Def. 9.1.1.) Fix a subfield \( k \) of \( \mathbb{C} \) and define the diagram \( \text{Pairs}^{\text{eff}} \) of effective pairs over \( k \) in the following way.

(a) Vertices of \( \text{Pairs}^{\text{eff}} \) are triples \((X, Y, i)\) where \( X \) is a variety over \( k \), \( Y \subset X \) is a closed subvariety, and \( i \in \mathbb{Z} \).
(b) There are two types of edges of $\text{Pairs}^{\text{eff}}$: functoriality edges and coboundary edges:

(b1) Each morphism $f: X \to X'$ with $f(Y) \subset Y'$ determines edges denoted $(f^*, i)$ starting at $(X, Y, i)$ and landing at $(X', Y', i + 1)$ for every $i \in \mathbb{Z}$.

(b2) Each ladder $Z \subset Y \subset X$ of closed subvarieties determines edges $(\partial, i)$ starting at $(Y, Z, i)$ and ending at $(X, Y, i + 1)$ for every $i \in \mathbb{Z}$.

Later, whenever we will have to consider tensor structures on the categories of diagrams, we will have to consider (super)gradings of such diagrams which are discussed in [HuM-St17], Ch. 8.

1.4. Categories of blowings up. We will now describe some categories of good blowings up, following Sec. 1.3 of [AT16] (with somewhat changed terminology and notation).

We will call a “good” scheme what in the Introduction to [AT16] is called a “noetherian quasi excellent (qe) regular scheme”.

Consider a morphism of good schemes $\varphi: X_1 \to X_2$, which is the blowing up of a coherent sheaf of ideals $I \subset \mathcal{O}_{X_2}$ (for the relevant definitions in this context, see [AT16], Sec. 2.1.8). Alternatively, we will call such $\varphi$ the blowing up of the closed subscheme defined by equations the $f = 0$ for all $f \in I$.

Assume also given normal crossings divisors $D_i \subset X_i$ such that $D_1 = \varphi^{-1}(D_2)$.

Let $U$ be the maximal open subscheme of $X_2$ upon which the restriction of $I$ is its structure sheaf. It follows that $\varphi$ induces an isomorphism $\varphi^{-1}(U) \to U$.

We will call a good morphism the structure represented by a set of data $(X_i, D_i, I, \varphi)$ as above. In particular, identical morphisms, and generally, automorphisms, are good: for them $D_1 = D_2 = \emptyset$.

Consider a finite connected poset in groupoids $\mathcal{M}$ whose objects are data $(X, D, I)$ and morphisms are good morphisms $\varphi$ in the sense of Sec. 0.1 above.

Consider the diagram (with identity) $D(\mathcal{M})$ whose vertices are objects of $\mathcal{M}$, oriented edges are morphisms of $\mathcal{M}$, and orientation of $X \to Y$ is from $X$ to $Y$. Call edges corresponding to isomorphisms (in particular, identities) horizontal ones, and other edges vertical ones.

Form also the following quotient of $D(\mathcal{M})$ which we denote $T(\mathcal{M})$: vertices of $T(\mathcal{M})$ are isomorphism classes of $\mathcal{M}$, and oriented edges of $T(\mathcal{M})$ are orbits of non-empty sets $\text{Hom}_{\mathcal{M}}(X_1, X_2)$ with respect to the left action of the group

$$\text{Hom}(X_1, X_1) \times \text{Hom}(X_2, X_2)^{\text{op}}$$
combining precomposition and postcomposition, as in Def. 1.2 above. We omit edges corresponding to identities.

Call an edge $X_1 \to X_2$ in $T(M)$ \textit{indecomposable}, if the respective morphism cannot be expressed as composition of other morphisms.

In an important particular case, the diagram $T(M)$ is in fact a tree oriented downwards. More precisely, starting with any its vertex $X_0$, we may consider the longest sequence of vertices ("a path down")

$$X_0 \to X_1 \to \cdots \to X_h.$$  

Assume that there is only one vertex from which any longest path down can start.

1.4.1. Definition. One object of the category Bl$_{rs}$ \textit{(the regular surjective category of blowings up}, cf. Sec. 1.3 of [TM16]) is a triple $(X_2, I, D_2)$ constituting a part of good morphism $\varphi$ as above.

One morphism between such objects of Bl$_{rs}$

$$(X'_i, D'_i, I', \varphi') \to (X_i, D_i, I, \varphi)$$

is represented by a regular and surjective morphism

$$g : X'_2 \to X_2$$

satisfying the following conditions:

$$g^{-1}(D_2) = D'_2, \quad g^*(I) = I'.$$

Remark. From this definition one can deduce, that $g$ induces a canonical isomorphism

$$X'_1 \to X_1 \times_{X_2} X'_2,$$

and moreover, $D'_1$ is the inverse image of $D_1$ with respect to the composition of this isomorphism and projection $X_1 \times_{X_2} X'_2 \to X_1$ (cf. Definition 1.3.1 of [AT16]). This presentation might me helpful for defining and studying compositions of morphisms between objects of Bl$_{rs}$.

In the next Sections, we will be studying posets of groupoids as above from the viewpoint of Nori theory, as presented in Sec.1 of [MaMar19-1]. But before starting it, we need one more definition.
1.5. Simple normal crossings divisors. Let $S$ be a finite set. Consider a family $\{D_s \subset X | s \in S\}$ of closed immersions. Following Definition (3.1) of [BeRy19], we will call it an \textit{$S$–labelled simple normal crossings divisor on $X$}, if for any finite subset $S' \subseteq S$ the intersection $\bigcap_{s \in S'} D_s$ is smooth of codimension $\text{card} \, S'$.

2. NORI GEOMETRIC DIAGRAMS OF BLOWINGS UP

2.1. Categories $\mathcal{M}$. We will now introduce a class of geometric categories that will be the starting point for our enrichment of birational maps by Nori motives. Notation $\mathcal{M}$ for a generic member of this class should remind the reader that we generalise here the basic example of stable modular spaces of genus zero and their canonical stratifications studied in [CMMar20]. Here are the basic restrictions imposed upon $\mathcal{M}$.

(a) Objects of $\mathcal{M}$ are some objects of $\text{Bl}_{\text{rs}}$.

(b) For any object $(X_2, I, D_2)$ (cf. Def. 1.4.1 above) of $\mathcal{M}$, the divisor $D_2$ is a simple normal crossings divisor. Sets of labels $S$, together with their functorial behaviour, may be included as separate elements of the structure of $\mathcal{M}$.

2.2. Two classes of morphisms in $\mathcal{M}$. Fix a category $\mathcal{M}$ as above. Let $\mathcal{X} := (X_2, I_X, D_1)$ and $\mathcal{Y} := (Y_2, I_Y, D_2)$ be two objects of $\mathcal{M}$.

Assume that we have a locally closed embedding $Y_2 \hookrightarrow X_2$ which extends in a natural way to a morphism between some blowings up of $X$, resp $Y$. The resulting commutative diagrams will be declared some new morphisms in $\mathcal{M}$, “morphisms of closed embeddings”: cf. Sec. 1.5 of [CMMar20].

Similarly, “morphisms of complements to locally closed embeddings” are extensions of this definition to $X_2 \setminus \overline{Y}_2 \hookrightarrow X_2$ where $\overline{Y}_2$ denotes the closure of $Y_2$ in $X_2$.

They are presented below as left and right sides of the commutative diagram:

$$
\begin{array}{ccc}
Y_1 & \hookrightarrow & X_1 \\
\downarrow \varphi_Y & & \downarrow \varphi_X \\
Y_2 & \hookrightarrow & X_2
\end{array}
$$

$$
\begin{array}{ccc}
\downarrow \varphi_X \setminus Y & & \downarrow \varphi_X \setminus Y \\
\overline{Y}_1 & \hookrightarrow & \overline{Y}_2
\end{array}
$$

Subschemes $Y_i \hookrightarrow X_i$ (resp. $X_i \setminus \overline{Y}_i \hookrightarrow X_i$), $i = 1, 2$, will be called \textit{locally closed} (resp. \textit{open}) strata of $X_i$. 
2.3. **Example: Kapranov’s presentation of** $\overline{M}_{0,n}$, $n \geq 3$. This presentation of $\overline{M}_{0,n}$, $n \geq 3$, as a result of successive blowings of projective subspaces in $\mathbb{P}^{n-3}$ was given in [Ka93], and then used in [BrMe13] for calculating of regular automorphisms of these stable modular spaces.

2.4. **Example: Connes–Kreimer Hopf algebras from rooted trees.** The Connes–Kreimer Hopf algebras were introduced in [CoKr00]. Later, using the operadic formalism, F. Chapoton and M. Livernet have shown that they appear as well in a geometric environment as above: see iSec. 6 of [ChaLiv07].

3. **NORI MOTIVES AND BURNSIDE GROUPS**

Below we work over a fixed field $k$ of characteristic 0.

3.1. **Definition.** ([KoTsch19], Sec. 4, Def. 10). Let $\mathcal{B}$ (“base scheme”) be a separated scheme of finite type over $k$.

Consider a smooth $\mathcal{B}$–scheme $f : X \to \mathcal{B}$. If $U \hookrightarrow X$ is an open embedding with $\overline{U} = X$, then $f|_U : U \to \mathcal{B}$ is also a smooth $\mathcal{B}$–scheme.

(a) Define the set $\text{Burn}_+(\mathcal{B})$ as the set of equivalence classes of smooth $\mathcal{B}$–schemes modulo equivalence relation generated by $f \sim f|_U$ as above. We may denote the respective equivalence class by $[f : X \to \mathcal{B}]$, or simply $[f]$.

(b) Define the monoid structure $+$ upon $\text{Burn}_+(\mathcal{B})$ as generated by disjoint union of smooth $\mathcal{B}$–schemes.

It generates the respective Grothendieck group $\text{Burn}(\mathcal{B})$.

(c) Both Burnside group and Burnside monoid a naturally graded: class of $X$ of pure dimension $n$ belongs to $\text{Burn}_{+,n}(\mathcal{B})$ and $\text{Burn}_n(\mathcal{B})$.

These constructions are covariant functors of $\mathcal{B}$: a morphism $g : \mathcal{B}' \to \mathcal{B}$ induces maps $g_*[f] := [g \circ f]$.

3.2. **Boundary homomorphisms.** Start with a pair $Z \subset X$, in which $X$ is an equidimensional algebraic variety, and $Z$ its closed subvariety of strictly lesser dimension. Moreover, we will assume that $X$ is reduced and separated, but nothing more.

3.2.1. **Theorem.** ([KoTsch19], Sec. 4, Theorem 11). On the set of Burnside groups of members of such pairs $Z \subset X$ one can define graded boundary elements
\[ \partial_Z(X) \in \text{Burn}_{\dim(X)-1}(Z) \]
satisfying two requirements:

(a) For any proper surjective morphism \( g : X' \to X \) inducing birational equivalence between \( X \) and \( X' \) and such that \( Z' = g^{-1}(Z) \), we have

\[ \partial_Z(X) = (g|_Z)_\ast(\partial_{Z'}(X')). \]

(b) If \( X \) is smooth, and \( Z \) is an \( S \)-labelled simple normal crossings divisor in the sense explained in Sec. 1.4 above, then we have an explicit presentation

\[ \partial_Z(X) = -\sum_{\emptyset \neq T \subseteq S} (-1)^{\text{card } T}[f_T] \]

where \( f_T : D_T \times A^{\text{card } T-1} \to Z \) is the composition of projection to \( D_T \) and its inclusion into \( Z \).

Moreover, these two requirements uniquely determine boundary elements.

We can now state precisely and prove Theorem 0.2.1.

Consider a category \( \mathcal{M} \) as in Sec. 2.1 above, with its objects graded by dimension. For each object \( \mathcal{B} \) of \( \mathcal{M} \), construct its Grothendieck–Burnside group \( \text{Burn}(\mathcal{B}) \). Denote by \( \text{GrAb} \) the category of graded abelian groups.

3.3. Theorem. The natural degree zero map \( \text{Ob}(\mathcal{M}) \to \text{Ob}(\text{GrAb}) : \mathcal{B} \mapsto \text{Burn}(\mathcal{B}) \) extends to the homological Nori geometric diagram, in which boundary edges correspond to three step towers of closed embeddings \( Z \subset Y \subset X \). The morphism

\[ (X,Y,i+1) \to (Y,Z,i) \]
sends boundary element \( \partial_Y(X) \) to \( \partial_Z(Y) \).

See [MaMar18], subsections 4.7–4.8.

4. BOST–CONNES SYSTEMS AND BURNSIDE GROUPS

4.1 Assemblers. I. Zakharevich introduced and developed in [Za14], [Za15] a categorical formalism useful for our study.

Briefly, an assembler \( \mathcal{C} \) is a small category endowed with Grothendieck topology and an initial object \( \emptyset \) satisfying the following restrictions:
(a) All morphisms in it are monomorphisms.

(b) For each object $X$ in $C$ and any two finite disjoint covering families of $X$ have a common refinement which is also a disjoint covering family.

Here two morphisms $f : Y \to X$ and $g : Z \to X$ are called disjoint if $Y \times_X Z = \emptyset$.

An epimorphic assembler $C$ with a sink object $S$ is an assembler, such that

(c) For each object $X$ of $C$, $\text{Hom}_C(X, S)$ is non-empty.

(d) Each morphism $f : X \to Y$ in $C$ with non-initial $X$ is an epimorphism, and this epimorphism is a covering family.

(e) If $Y, Z \neq \emptyset$, then no two morphisms $Y \to X$ and $Z \to X$ are disjoint.

In [MaMar18] it was shown that there is an assembler category, in the sense of [Za14], underlying the Kontsevich–Tschinkel Burnside group, and its equivariant version. Moreover, there is a Bost–Connes system of endomorphisms acting on the $\hat{\mathbb{Z}}$–equivariant version of the Kontsevich–Tschinkel Burnside ring. This is a lift of the integral Bost–Connes algebra of [CoConMar08], through the map to the graded ring associated to the filtration of the Grothendieck ring by dimension, which in turn maps to the integral Bost–Connes algebra through the equivariant Euler characteristic. Moreover, this Bost–Connes structure can be lifted higher, to endofunctors of the assembler category and endomorphisms of the associated homotopy–theoretic spectrum.

On the other hand, in [LiMaMar19] it was also shown that the lift of the Bost–Connes algebra to the $\hat{\mathbb{Z}}$–equivariant Grothendieck ring of varieties can be lifted higher, to a $\hat{\mathbb{Z}}$–equivariant category of Nori motives, where it maps through the fibre functor to the categorification of the Bost–Connes system constructed in [MarTa14].

We review here briefly the setting of [LiMaMar19] and [MaMar18], and then we show that a similar categorification of the Bost–Connes system via Nori motives can be constructed in the case of the Kontsevich–Tschinkel Burnside group. As was done in [LieMaMar19] for the Grothendieck ring, we work here with the relative version of the Kontsevich–Tschinkel Burnside ring considered in the previous sections, and with its $\hat{\mathbb{Z}}$–equivariant version.

For much more detailed expositions, see [Za14], [Za15], and Sec. 4 of [MaMar18].

4.2. Relative equivariant Kontsevich–Tschinkel Burnside group. In the previous sections we considered a base scheme $\mathcal{B}$ and the Kontsevich–Tschinkel Burnside group $\text{Burn}(\mathcal{B})$. 

Here we introduce its equivariant version $\text{Burn}^\mathbb{Z}(\mathcal{B})$ where now $\mathcal{B}$ is endowed with a residually finite action of $\mathbb{Z}$. The last condition means that this action factors through some finite $\mathbb{Z}/N\mathbb{Z}$–quotient of $\mathbb{Z}$.

The Burnside group $\text{Burn}^\mathbb{Z}(\mathcal{B})$ then is generated by equivalence classes of smooth $\mathcal{B}$–schemes $f : X \to \mathcal{B}$ where $X$ is also endowed with a residually finite action of $\mathbb{Z}$, and $f$ is $\mathbb{Z}$–equivariant. Equivalence classes correspond to the equivalence relation $f \sim f|_U$ where $U \hookrightarrow X$ is a $\mathbb{Z}$–equivariant dense open embedding.

As in [LiMaMar19], we will adopt the notation $(\mathcal{B}, \alpha_\mathcal{B})$ for the base scheme endowed with action $\alpha_\mathcal{B} : \mathbb{Z} \times \mathcal{B} \to \mathcal{B}$, and we similarly write $(X, \alpha_X)$ and $f : (X, \alpha_X) \to (\mathcal{B}, \alpha_\mathcal{B})$ to keep track explicitly of the $\mathbb{Z}$–actions.

In [MaMar18] we constructed an assembler category and the associated homotopy theoretic spectrum underlying the Kontsevich–Tschinkel Burnside group $\text{Burn}^\mathbb{Z}(K)$, where $K$ a field. The relative case we consider here is similar. We give a brief description of the resulting assembler.

4.3. Proposition. Consider the family $\mathcal{C}^\mathbb{Z}_{(\mathcal{B}, \alpha_\mathcal{B})}$ of epimorphic assemblers with sink whose objects are $\mathbb{Z}$–equivariant dense open embeddings $\iota : (U, \alpha_U) \hookrightarrow (X, \alpha_X)$. Consider the coproduct over equivalence classes of $\mathcal{B}$–schemes $f : (X, \alpha_X) \to (\mathcal{B}, \alpha_\mathcal{B})$ of members of this family.

The set $\pi_0$ of the associated spectrum is the relative equivariant Kontsevich–Tschinkel set $\text{Burn}^\mathbb{Z}(\mathcal{B})$.

More generally, the spectra of these assemblers satisfy

$$K(\mathcal{C}^\mathbb{Z}_{(\mathcal{B}, \alpha_\mathcal{B})}(X, \alpha_X)) \simeq \Sigma^\infty_+ B\mathcal{G}^\mathbb{Z}_{(\mathcal{B}, \alpha_\mathcal{B})}(X, \alpha_X),$$

where $\mathcal{G}^\mathbb{Z}_{(\mathcal{B}, \alpha_\mathcal{B})}(X, \alpha_X)$ is the group of $\mathbb{Z}$–equivariant birational automorphisms.

Proof. According to Theorem 5.3 of [MarTa14] and Lemma 4.5 of [MaMar18], $\mathcal{C}^\mathbb{Z}_{(\mathcal{B}, \alpha_\mathcal{B})}(X, \alpha_X)$ is indeed an epimorphic assembler with sink.

Therefore, in view of Theorem 4.8 of [Za14], we have

$$\mathcal{C}^\mathbb{Z}_{(\mathcal{B}, \alpha_\mathcal{B})}(X, \alpha_X) \simeq \Sigma^\infty_+ B\mathcal{G}^\mathbb{Z}_{(\mathcal{B}, \alpha_\mathcal{B})}(X, \alpha_X),$$

where $\mathcal{G}^\mathbb{Z}_{(\mathcal{B}, \alpha_\mathcal{B})}(X, \alpha_X)$ is the group of $\mathbb{Z}$–equivariant birational automorphisms of the $\mathcal{B}$–scheme $f : (X, \alpha_X) \to (\mathcal{B}, \alpha_\mathcal{B})$. 
The remaining statements also follow from Lemma 4.5 of [MaMar18].

4.4. Lifting the Bost–Connes system. We will use notation and conventions of [LieMaMar19], which are briefly repeated below.

Put $Z_n := \text{Spec } K^n$ where $K$ is our ground field.

Given $(\mathcal{B}, \alpha_B)$ as above, denote by $\Phi(\alpha_B)$ the $\widehat{\mathbb{Z}}$-action upon $\mathcal{B} \times Z_n$ through composition with geometric Verschiebung map: see [LieMaMar19], (2.12) and (2.13).

This action is residually finite as well: if the action $\alpha_B$ of $\widehat{\mathbb{Z}}$ factors through $\mathbb{Z}/N\mathbb{Z}$, then the action $\Phi(\alpha_B)$ factors through $\mathbb{Z}/Nn\mathbb{Z}$. Denote by $\sigma_n$ the endomorphism of multiplication by $n$.

Now, by lifting $\sigma_n$ (and avoiding extra notation), define the maps $\sigma_n$ and $\tilde{\rho}_n$ as follows:

$$\sigma_n : (f : (X, \alpha_X) \to (\mathcal{B}, \alpha_B)) \mapsto (f : (X, \alpha_B \circ \sigma_n) \to (\mathcal{B}, \alpha_B \circ \sigma_n)),$$

$$\tilde{\rho}_n : (f : (X, \alpha_X) \to (\mathcal{B}, \alpha_B)) \mapsto (f \times id : (X \times Z_n, \Phi_n(\alpha_X) \to (\mathcal{B} \times Z_n, \Phi_n(\alpha_B))).$$

4.4.1. Proposition. The maps above can be lifted to functors of respective assembler categories with sinks that we will denote by the same letters $\sigma_n$ and $\tilde{\rho}_n$.

These functors induce ring homomorphisms

$$\sigma_n : \text{Burn} \hat{\mathbb{Z}}(\mathcal{B}, \alpha_B) \to \text{Burn} \hat{\mathbb{Z}}(\mathcal{B}, \alpha_B \circ \sigma_n)$$

and group homomorphisms

$$\tilde{\rho}_n : \text{Burn} \hat{\mathbb{Z}}(\mathcal{B}, \alpha_B) \to \text{Burn} \hat{\mathbb{Z}}(\mathcal{B} \times Z_n, \Phi_n(\alpha_B))$$

by the induced maps on the $\pi_0$ of the associated spectra.

For proofs, see [MaMar18], Proposition 4.7, and LieMaMar19, Theorem 3.15.

The functors $\sigma_n$ and $\tilde{\rho}_n$ of Proposition 4.4.1 and the induced morphisms of the Kontsevich–Tschinkel Burnside set lift the maps $\sigma_n$ and $\tilde{\rho}_n$ of the integral Bost–Connes algebra of [ConCoMar08] in the following way.

**Proposition 4.5.** The action of the $\sigma_n$ and $\tilde{\rho}_n$ on the $\text{Burn} \hat{\mathbb{Z}}(\mathcal{B}, \alpha_B)$ is compatible with the $\sigma_n$ and $\tilde{\rho}_n$ constructed in [LieMaMar19] on the Grothendieck rings.
$K_0^\hat{Z}(B,\alpha_B)$, through the morphism $\text{Burn}^\hat{Z}(B,\alpha_B) \to \text{gr}K_0^\hat{Z}(B,\alpha_B)$ to the associated graded object with respect to filtration by dimension.

Proof. The maps $\sigma_n$ and $\tilde{\rho}_n$ on $K_0^\hat{Z}(B,\alpha_B)$ are constructed in [LieMaMar19] using the same formulas as above. These preserve the filtration by dimension, since the schemes $Z_n$ are zero dimensional. Thus, the map $\text{Burn}^\hat{Z}(B,\alpha_B) \to \text{gr}K_0^\hat{Z}(B,\alpha_B)$ intertwines the action of the $\sigma_n$ and $\tilde{\rho}_n$ on the Burnside ring and on the Grothendieck ring.

It is also shown in [LieMaMar19], Theorem 2.11, that the actions of $\sigma_n$ and $\tilde{\rho}_n$ on $K_0^\hat{Z}(B,\alpha_B)$ are compatible with the $\sigma_n$ and $\tilde{\rho}_n$ acting on $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, through an equivariant Euler characteristic map.

4.6. Equivariant Nori motives. Given a base scheme $B$ with a residually finite $\hat{Z}$-action $\alpha_B$, we can consider, as in [LieMaMar19], Sec. 7, a category $\mathcal{M}_{(B,\alpha_B)}$ of $\hat{Z}$-equivariant Nori motivic sheaves, in the sense of [Ar08]. This is defined as the abelian category constructed from a Nori diagram $\mathcal{D}(B,\alpha_B)$ with the following vertices and edges:

- Vertices are given by elements of the form 
  
  $$(f : (X,\alpha_X) \to (B,\alpha_B), (Y,\alpha_X|_Y), i, w)$$

  where $f : (X,\alpha_X) \to (B,\alpha_B)$ is a smooth $\hat{Z}$-equivariant $B$-scheme, with a $\hat{Z}$-equivariant embedding $(Y,\alpha_X|_Y) \hookrightarrow (X,\alpha_X)$, and $i \in \mathbb{N}, w \in \mathbb{Z}$.

- Edges are of three types:
  1. edges $h^* : (X' \to B, Y', i, w) \to (X \to B, Y, i, w)$ associated to morphisms $h : (X \to B, Y) \to (X' \to B, Y')$;
  2. connecting morphisms $\partial : (Y \to B, Z, i, w) \to (X \to B, Y, i + 1, w)$ associated to a chain of embeddings $Z \hookrightarrow Y \hookrightarrow X$;
  3. twisted projections $(X, Y, i, w) \to (X \times \mathbb{P}^1, Y \times \mathbb{P}^1 \cup X \times \{0\}, i + 2, w + 1)$.

As was shown in [LieMaMar19], the maps $\sigma_n$ and $\tilde{\rho}_n$ from above induce functors of the categories of Nori motivic sheaves considered above, with

$$\sigma_n : \mathcal{M}_{(B,\alpha_B)} \to \mathcal{M}_{(B,\alpha_B \circ \sigma_n)}$$

$$\tilde{\rho}_n : \mathcal{M}_{(B,\alpha_B)} \to \mathcal{M}_{(B \times \mathbb{Z}_n, \Phi_n(\alpha_B))}.$$
The functors $\sigma_n$ are compatible with the monoidal structure but $\tilde{\rho}_n$, are not, as discussed in [LieMaMar19].

By construction, the Bost–Connes structure on the categories $\mathcal{M}_{(B, \alpha_B)}$ of Nori motives is compatible with those discussed above on $K^\mathbb{Z}_0(B, \alpha_B)$ and on $\text{Burn}^\mathbb{Z}(B, \alpha_B)$. In particular, the same construction can also be applied to the $\mathbb{Z}$-equivariant version of the Nori motives considered in relation to the categories of blowups.

4.7. The Bost–Connes structure of the Kontsevich-Pestun-Tschinkel modular symbols

A summarized above, in [LieMaMar19] and [MaMar18] we considered various lifts of the Bost–Connes algebra to Grothendieck rings, assembler, spectra, and Nori motives, based on varieties with good actions of $\mathbb{Z}$. This setting includes the case of equivariant Kontsevich–Tschinkel Burnside ring and associated assembler and Nori motives described here in the previous subsections. There is another setting in birational geometry where the Bost–Connes structure naturally appears, which we discuss in this subsection, namely the Kontsevich-Pestun-Tschinkel modular symbols of [KPT19].

As in [KPT19] we consider, for $n \in \mathbb{N}$ and $G$ a finite abelian group, the $\mathbb{Z}$-modules $\mathcal{M}_n(G)$, generated by symbols $\langle a_1, \ldots , a_n \rangle$ with $a_i \in A = G' = \text{Hom}(G, \mathbb{C}^*)$, such that $a_1, \ldots , a_n$ generate $A$, with relations:

1. $\langle a_{\sigma(1)}, \ldots , a_{\sigma(n)} \rangle = \langle a_1, \ldots , a_n \rangle$, for all permutations $\sigma \in S_n$;

2. for all $2 \leq k \leq n$ and all $a_1, \ldots , a_k$ and $b_1, \ldots , b_{n-k}$ in $A$ satisfying

$$\sum_i \mathbb{Z}a_i + \sum_j \mathbb{Z}b_j = A$$

one has the relation

$$\langle a_1, \ldots , a_k, b_1, \ldots , b_{n-k} \rangle = \sum_{1 \leq i \leq k} \langle a_1 - a_i, \ldots , a_i, \ldots , a_k - a_i, b_1, \ldots , b_{n-k} \rangle,$$

with $a_i$ in the $i$th place.

As shown in [KPT19] these relations reflect certain scissor-congruence relations on convex cones in lattices. We also consider as in [KPT19] the quotient $\mathcal{M}_n^-(G)$ of $\mathcal{M}_n(G)$ by the further relation $\langle -a_1, \ldots , a_n \rangle = -\langle a_1, \ldots , a_n \rangle$.

There is a $\mathbb{Z}$-bilinear multiplication, for $n = n' + n''$,

$$\nabla : \mathcal{M}_{n'}(G') \otimes \mathcal{M}_{n''}(G'') \to \mathcal{M}_n(G)$$
associated to an exact sequence of finite abelian groups

$$0 \to G' \to G \to G'' \to 0,$$

with both $G'$ and $G''$ nontrivial, given by

$$\nabla : \langle a_1, \ldots, a_{n'} \rangle \otimes \langle b_1, \ldots, b_{n''} \rangle \mapsto \sum \langle \tilde{a}_1, \ldots, \tilde{a}_{n'}, \tilde{b}_1, \ldots, \tilde{b}_{n''} \rangle,$$

where the sum is over all the lifts $\tilde{a}_i$ in $A$, in the dual exact sequence

$$0 \to A'' \to A \to A' \to 0,$$

while the $\tilde{b}_j$ are the images in $A$ of the $b_j$ under the embedding $A'' \to A$. The multiplication map descends to the $M^{-n}(G)$ in the same form.

There is similarly a $\mathbb{Z}$-bilinear comultiplication, for $n = n' + n''$, determined by

$$\Delta : M_n(G) \to M_{n'}(G') \otimes M_{n''}(G'')$$

for a sequence as above with $G''$ nontrivial, given by

$$\Delta : \langle a_1, \ldots, a_n \rangle \mapsto \sum \langle a_{I'} \mod A'' \rangle \otimes \langle a_{I''} \rangle^-,$$

for $\{1, \ldots, n\} = I' \sqcup I''$ with $\#I' = n'$ and $\#I'' = n''$, and $a_{I'} = a_{i_1}, \ldots, a_{i_{n'}}$ and $a_{I''} = a_{j_1} \ldots a_{j_{n''}}$ for $I' = \{i_1, \ldots, i_{n'}\}$ and $I'' = \{j_1, \ldots, j_{n''}\}$, such that all the $a_{j_k}$ are in $A'' \hookrightarrow A$. Here taking the quotient $\mathcal{M}_{n''}(G'')$ instead of $\mathcal{M}_{n''}(G'')$ is necessary because of the second type of relations in $\mathcal{M}_n(G)$, see Proposition 9 of [KPT19]. The comultiplication also descends to $\mathcal{M}_n^{-}(G)$.

In particular, we will focus here on the case of $\mathcal{M}_{n,N} := \mathcal{M}_n(\mathbb{Z}/N\mathbb{Z})$. We write $G_N = \mathbb{Z}/N\mathbb{Z}$ and $A_N$ for the characters and we consider the projective system of the $A_N$ ordered by divisibility, with the maps $\sigma_k : A_N \to A_M$ for $M | N$ with $N = Mk$, given by $\sigma_k : \zeta \mapsto \zeta^k$ when we identify $A_N$ with the group of $N$th roots of unity (multiplication by $k$ if written additively). Dually we have the injective system of the $G_N$ with the corresponding inclusions $j_k : G_M \to G_N$, so that

$$\lim G_N = \mathbb{Q}/\mathbb{Z}, \quad \lim A_N = \hat{\mathbb{Z}}.$$
We define $\mathcal{M}_n(\mathbb{Q}/\mathbb{Z})$ as the $\mathbb{Z}$-module spanned by the $\mathbb{Z}$-module $\mathcal{M}_n(G_N)$ for all $N$. We write elements of $\mathcal{M}_n(\mathbb{Q}/\mathbb{Z})$ in the form $\sum_i c_i \langle a_i \rangle$ where we use the shorthand notation $\langle a_i \rangle := \langle a_{i,1}, \ldots, a_{i,n} \rangle$ with $a_{i,k} \in A_N$. We also write $\mathcal{M}_n(\mathbb{Q}/\mathbb{Z}) := \mathcal{M}(\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z} \mathbb{Q}$. We also write $\mathcal{M}_n(\mathbb{Q}/\mathbb{Z})^-$ for the span of the $\mathcal{M}_n(G_N)^-$. Let $\sigma_k : \mathcal{M}_n(\mathbb{Q}/\mathbb{Z}) \to \mathcal{M}_n(\mathbb{Q}/\mathbb{Z})$ be the map of $\mathbb{Z}$-modules determined by $\langle a \rangle \mapsto \langle \sigma_k(a) \rangle$, for $\langle a \rangle = \langle a_1, \ldots, a_n \rangle$ in $\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z})$ and $\langle \sigma_k(a) \rangle = \langle \sigma_k(a_1), \ldots, \sigma_k(a_n) \rangle$. We set $\sigma_k \langle a \rangle = 0$ whenever $\langle \sigma_k(a) \rangle$ would not be an acceptable symbol: for instance this happens for $\langle a \rangle \in \mathcal{M}_n(\mathbb{Z}/k\mathbb{Z})$ when all $\sigma_k(a_i) = 0 \in \mathbb{Z}/k\mathbb{Z}$. We also consider the maps of $\mathbb{Z}$-modules $\rho_k : \mathcal{M}_n(G_M) \to \mathcal{M}_n(G_N)$ that maps a symbol to the sum over preimages $\rho_k : \langle a_1, \ldots, a_n \rangle \mapsto \sum_{\sigma_k(b_i) = a_i} \langle b_1, \ldots, b_n \rangle$. With a slight abuse of notation we will write for convenience $\langle \rho_k(a) \rangle := \sum_{\sigma_k(b_i) = a_i} \langle b_1, \ldots, b_n \rangle$, so that we can write $\tilde{\rho}_k \langle a \rangle = \langle \tilde{\rho}_k(a) \rangle$. On $\mathcal{M}_Q(\mathbb{Q}/\mathbb{Z})$ we also consider the $\mathbb{Q}$-linear maps $\hat{\rho}_k = \frac{1}{k^n} \tilde{\rho}_k : \mathcal{M}_Q, n(\mathbb{Q}/\mathbb{Z}) \to \mathcal{M}_Q, n(\mathbb{Q}/\mathbb{Z})$, $\hat{\rho}_k : \langle a_1, \ldots, a_n \rangle \mapsto \frac{1}{k^n} \sum_{\sigma_k(b_i) = a_i} \langle b_1, \ldots, b_n \rangle$. Notice that the notation we are using here differs from [ConCoMar08], [LieMa-Mar19], [MaMar18] where we used $\tilde{\rho}_k$ for what is here called $\rho_k$ and $\rho_k$ for what is here called $\hat{\rho}_k$. The reason for this change of notation is that, as maps on the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, the $\tilde{\rho}_k$ of [CCM] are not ring homomorphisms, while the $\rho_k$ of [CCM] are algebra homomorphisms of $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ and are the original Bost–Connes maps. Here however, since we will be considering the multiplication structure on
the \( \mathcal{M}_n(G) \) as in [KPT19] rather than the group ring \( \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \), it will turn out that the \( \tilde{\rho}_k \) of [ConCoMar08] are the fundamental maps, which we call simply \( \rho_k \).

**Lemma 4.8.** The maps \( \sigma_k \) and \( \rho_k \) on \( \mathcal{M}_n(\mathbb{Q}/\mathbb{Z}) \) defined as above satisfy the relations \( \sigma_k \sigma_\ell = \sigma_{k\ell} \), \( \rho_k \rho_\ell = \rho_{k\ell} \). Moreover, if \( (k, \ell) = \gcd\{k, \ell\} = 1 \), then

\[
\sigma_k \circ \rho_\ell = \rho_\ell \circ \sigma_k,
\]

while

\[
\sigma_k \circ \rho_k = k \cdot \text{id}, \quad \rho_k \circ \sigma_k = e_k,
\]

with the map \( e_k : \mathcal{M}_n(\mathbb{Q}/\mathbb{Z}) \to \mathcal{M}_n(\mathbb{Q}/\mathbb{Z}) \) given by

\[
e_k \langle a_1, \ldots, a_n \rangle = \sum_{k s_i = 0} (a_1 + s_1, \ldots, a_n + s_n).
\]

**Proof.** The compositions \( \sigma_k \sigma_\ell = \sigma_{k\ell} \) and \( \rho_k \rho_\ell = \rho_{k\ell} \) follow directly from the definition. For \( (k, \ell) = 1 \) we have

\[
\sigma_k \sum \langle b_1, \ldots, b_n \rangle = \sum \langle kb_1, \ldots, kb_n \rangle
\]

\[
= \sum \langle kb_1, \ldots, kb_n \rangle = \rho_\ell \langle ka_1, \ldots, ka_n \rangle.
\]

On the other hand \( \sigma_k \sum_{kb_i = a_i} \langle b_1, \ldots, b_n \rangle = \sum_{kb_i = a_i} \langle a_1, \ldots, a_n \rangle = k \cdot \langle a_1, \ldots, a_n \rangle \) and \( \rho_k \langle ka_1, \ldots, ka_n \rangle = \sum_{kb_i = ka_i} \langle b_1, \ldots, b_n \rangle = \sum_{k s_i = 0} (a_1 + s_1, \ldots, a_n + s_n) \). The more general case with \( (k, \ell) \neq 1 \) is also obtained by combining the relations above.

Note that, unlike in the case of the maps \( \sigma_n \) and \( \rho_n \) acting on \( \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \) in the Bost–Connes system, the \( e_k \) here are not idempotents with respect to \( \nabla \).

The following lemma is a direct consequence of the explicit expression for the multiplication and comultiplication and of the maps \( \sigma_k \) and \( \rho_k \) defined above.

**Lemma 4.9.** The multiplication \( \nabla \) induces a multiplication

\[
\nabla : \mathcal{M}_n(\mathbb{Q}/\mathbb{Z}) \otimes \mathcal{M}_n(\mathbb{Q}/\mathbb{Z}) \to \mathcal{M}_{n+n}(\mathbb{Q}/\mathbb{Z}),
\]

which is given by

\[
\nabla(\langle a \rangle, \langle b \rangle) = \langle \rho_\ell(a), b \rangle,
\]
for $\langle b \rangle \in \mathcal{M}_n(\mathbb{Z}/\ell\mathbb{Z})$. The comultiplication $\Delta$ induces a comultiplication

$$\Delta : \mathcal{M}_n(\mathbb{Q}/\mathbb{Z}) \to \oplus_{n'+n''=n} \mathcal{M}_{n'}(\mathbb{Q}/\mathbb{Z}) \otimes \mathcal{M}_{n''}(\mathbb{Q}/\mathbb{Z})$$

given by

$$\Delta(\langle a \rangle) = \oplus \langle \sigma_\ell(a_{I'}) \rangle \otimes \langle a_{I''} \rangle^-,$$

where for $\langle a \rangle \in \mathcal{M}_n(G_N)$ the sum is over $\ell|N$ and over all partitions

$$I' \sqcup I'' = \{1, \ldots, n\},$$

with $\langle a_{I''} \rangle$ in $\mathcal{M}_n(Ker(\sigma_\ell))$.

Thus, we can consider the graded ring $\mathcal{M}(\mathbb{Q}/\mathbb{Z}) = \bigoplus_n \mathcal{M}_n(\mathbb{Q}/\mathbb{Z})$ endowed with this multiplication, and the $\mathbb{Q}$-algebra $\mathcal{M}(\mathbb{Q}/\mathbb{Z})$.

**Proposition 4.10.** The maps $\rho_k : \mathcal{M}(\mathbb{Q}/\mathbb{Z}) \to \mathcal{M}(\mathbb{Q}/\mathbb{Z})$ are ring homomorphisms, while the maps $\sigma_k : \mathcal{M}(\mathbb{Q}/\mathbb{Z}) \to \mathcal{M}(\mathbb{Q}/\mathbb{Z})$ are coalgebra homomorphisms.

**Proof.** As mentioned above, the multiplication in $\mathcal{M}(\mathbb{Q}/\mathbb{Z})$ is given by $\nabla(\langle a \rangle, \langle b \rangle) = \langle \rho_\ell(a), b \rangle$, hence we have

$$\rho_k \circ \nabla(\langle a \rangle, \langle b \rangle) = \langle \rho_k \rho_\ell(a), \rho_k(b) \rangle = \langle \rho_\ell \rho_k(a), \rho_k(b) \rangle = \nabla \circ \rho_k(\langle a \rangle, \langle b \rangle),$$

while the comultiplication is given by $\Delta(\langle a \rangle) = \oplus \langle \sigma_\ell(a_{I'}) \rangle \otimes \langle a_{I''} \rangle^-$ so that

$$\sigma_k \circ \Delta(\langle a \rangle) = \oplus \langle \sigma_k \sigma_\ell(a_{I'}) \rangle \otimes \langle a_{I''} \rangle^-.$$

The sum is over all $\ell|N$, for $\langle a \rangle \in \mathcal{M}_n(\mathbb{Z}/N\mathbb{Z})$ and over all decompositions $I', I''$ where $\langle a_{I''} \rangle$ is in $\mathcal{M}_n(\mathbb{Z}/\ell\mathbb{Z})$. If $(k, N) = 1$, then also $(k, \ell) = 1$ and we have $\sigma_\ell : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ and $\langle a_{I''} \rangle$ is in $\mathcal{M}_n(\mathbb{Z}/\ell\mathbb{Z})$ iff the same holds for $\langle \sigma_k(a_{I''}) \rangle$, hence the sum on the right hand side is the same as $\Delta \circ \sigma_k(\langle a \rangle)$. It suffices then to check the case where $k|N$, with $N = kM$. Then $\langle \sigma_k(a) \rangle \in \mathcal{M}_n(\mathbb{Z}/M\mathbb{Z})$ and $\Delta(\sigma_k(a))$ is the sum over all the $\ell'|M$ and the partitions $I', I''$ with $\langle b_{I''} \rangle \in \mathcal{M}_n(\mathbb{Z}/\ell''\mathbb{Z})$, up to terms that are mapped to zero by $\sigma_k$ when $a_{I''} \in \mathbb{Z}/k\mathbb{Z}$. Thus, the maps $\rho_k$ are compatible with the multiplication $\nabla$ and the maps $\sigma_k$ are compatible with the comultiplication $\Delta$.

Note that the maps $\sigma_k$ are not compatible with multiplication since

$$\sigma_k \nabla(\langle a \rangle, \langle b \rangle) = \langle \sigma_k \rho_\ell(a), \sigma_k(b) \rangle.$$
For \((k, \ell) = 1\) one has \(\sigma_k \rho_\ell = \rho_\ell \sigma_k\), which gives the right compatibility, but in the case where \((k, \ell) \neq 1\) this is not the case. For simplicity, consider the case where \(\ell = k\). where \(\langle \sigma_k \rho_k(a), \sigma_k(b) \rangle = \langle a, \sigma_k(b) \rangle\) while \(\langle \rho_k \sigma_k(a), \sigma_k(b) \rangle = \sum_{k_i = 0}^{k_i = 1} \langle a + s, \sigma_k(b) \rangle\), with \(a + s = (a_1 + s_1, \ldots, a_n + s_n)\). Similarly, the maps \(\rho_k\) are not compatible with comultiplication for exactly the same reason.

The data \((\mathcal{M}(\mathbb{Q}/\mathbb{Z}), \nabla, \Delta, \rho_k, \sigma_k)\) provide the Bost–Connes structure associated to the Kontsevich-Pestun-Tschinkel modular symbols.

In the original Bost–Connes quantum statistical mechanical system, the maps \(\rho_k\) have a geometric interpretation in terms of lattices and commensurability, see Chapter 3 of [CoMa08]. One defines \(\mathbb{Q}\)-lattices as pairs \((L, \phi)\) of a rank \(n\)-lattice \(L\) (with \(n = 1\) in the original Bost–Connes case) and a group homomorphism \(\phi : \mathbb{Q}^n / \mathbb{Z}^n \to \mathbb{QL}/L\). Two \(\mathbb{Q}\)-lattices are commensurable if \(\mathbb{QL}_1 = \mathbb{QL}_2\) and \(\phi_1 = \phi_2\) modulo \(L_1 + L_2\). In the rank one case, the commensurability relation is precisely captured by the maps \(\rho_k\) of the Bost–Connes system.

In the case of the Kontsevich-Pestun-Tschinkel modular symbols, it is shown in [KPT19] that the \(\mathcal{M}_n(G)\) can also be described in terms of lattices and cones. For \(\mathcal{M}_n(G)\), one considers isomorphism classes of data \((L, \chi, \Lambda)\) with \(L \simeq \mathbb{Z}^n\) a lattice of rank \(n\), with an element \(\chi \in L \otimes A\) for which the induced homomorphism \(L^\vee \to A\) is surjective, and a strictly convex cone \(\Lambda \subset L_R\) spanned by a basis of \(L\). To a datum \((L, \chi, \Lambda)\) one associates a symbol \(\psi(L, \chi, \Lambda) = \langle a_1, \ldots, a_n \rangle\) obtained by a choice of a basis \(\{e_i\}\) of \(L\) with respect to which \(\chi = \sum_i e_i \otimes a_i\). The relations of \(\mathcal{M}_n(G)\) are then described geometrically in terms of scissor congruence relations \(\psi(L, \chi, \Lambda) = \sum_i \psi(L, \chi, \Lambda_i)\) determined by possible decompositions of the cone.

To see the compatibility between these two settings, note that a lattice \(L\) with the choice of a basis \(\{e_i\}\) determines an invertible \(\mathbb{Q}\)-lattice of rank \(n\), by taking \(\phi : \mathbb{Q}^n / \mathbb{Z}^n \to \mathbb{QL}/L\) that maps \(\phi : (a_1, \ldots, a_n) \mapsto \sum_i a_i e_i\). To a (not necessarily invertible) \(\mathbb{Q}\)-lattice with \(\phi : \mathbb{Q}^n / \mathbb{Z}^n \to \mathbb{QL}/L\), we can assign an element \(\chi \in L \otimes \mathbb{Q}/\mathbb{Z}\) by \(\chi = \sum_i \phi(e_i) \otimes a_i\) with \(e_i\) the tuple in \((\mathbb{Q}/\mathbb{Z})^n\) with 1 in the \(i\)th position and zeroes elsewhere. For a fixed \(\mathbb{Z}/N\mathbb{Z}\), the restriction of \(\phi\) to \((\mathbb{Z}/N\mathbb{Z})^n\) induces on those \(n\)-tuples that determine corresponding symbols \(\langle a_1, \ldots, a_n \rangle\), a map \(\langle a_1, \ldots, a_n \rangle \mapsto \sum_i e_i \otimes a_i\) that gives rise to an associated \(\chi\) as above. The maps \(\rho_k\) are then implementing the commensurability relation as in the Bost–Connes case on the rank one \(\mathbb{Q}\)-lattices obtained by projecting along the \(e_i\) directions.
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