Josephson Frequency Singularity in the Noise of Normal Metal – Superconductor Junctions

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A singularity at the Josephson frequency in the noise spectral density of a disordered normal metal – superconductor junction is predicted for bias voltages below the superconducting gap. The non-stationary Aharonov-Bohm effect, recently introduced for normal metals [2], is proposed as a tool for detecting this singularity. In the presence of a harmonic external field, the derivative of the noise with respect to the voltage bias reveals jumps when the applied frequency is commensurate with the Josephson frequency associated with this bias. The height of these jumps is non-monotonic in the amplitude of the periodic field. The superconducting flux quantum enters this dependence. Additional singularities in the frequency dependent noise are predicted above gap.

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The observation of the Josephson effect typically requires two superconductors in contact. Here, we present two situations where a Josephson frequency can be observed in a normal metal – superconductor (NS) junction. Although the Josephson frequency does not manifest itself in the average current through the NS junction, the noise characteristics bears clearly its signature. Superconducting features have been predicted in NS junctions: the doubling of the shot noise and the crossover from thermal noise to excess noise at \(2k_B T = (2e)V\) (V applied bias) are examples [3]. Recent experiments in SNS junctions seem to be in qualitative agreement with this crossover. First we present a general framework to investigate the finite frequency noise of NS junctions, second, we propose an equivalent zero frequency, constant bias measurement in the presence of an harmonic perturbation to detect this analog of the Josephson effect. In the first case, the singularity at the Josephson frequency originates from the time oscillations of the current–current correlation function. In the second scheme, the superposition of an alternating field to the bias voltage leads to steps/singularities in the noise derivatives as a function of DC voltage. The strength of these singularities is non-monotonic with the amplitude of the harmonic perturbation. This so called “non-stationary Aharonov–Bohm effect” has been predicted [4] and observed experimentally [2] in normal mesoscopic samples. Similarly, the singular behavior of finite frequency noise is understood both theoretically and experimentally [2]. For an NS junction with a bias smaller than the gap, the physical quantities can in principle be obtained from the normal metal results by replacing everywhere the electron charge by the charge of a Cooper pair. In both the conductance and the excess noise of NS junctions, this doubling of the electron charge has to be divided by two in the prefactors, in order to account for the lack of spin degeneracy. However, no detailed calculation of these effects for finite frequencies is available, and the generalization to above gap voltages cannot be obtained from an effective carrier charge.

A general framework is provided for both the stationary and non-stationary problems. Consider the coherent normal metal – superconductor junction in Fig. 1. A steplike dependence of the gap is assumed at the boundary, and the one channel states for electrons and holes incident from the normal side are specified by a scattering matrix with elements \(s_{\alpha\beta}(\alpha, \beta = e, h)\). A time dependent phase \(\Phi(t)\) is accumulated on the normal side by electrons impinging on the superconductor (incident holes will bear the opposite phase). Normally reflected particles will not accumulate this external phase. No inelastic effects other than this perturbation are assumed close to the NS boundary. The probability for Andreev reflection [3] is \(R_A = |s_{e, h}|^2\). In general, a time dependent formulation [11] of the Bogolubov–de Gennes (BdG) equations is needed. Here, we assume that the time dependence is adiabatic, so that appropriate elements of the scattering matrix are simply multiplied by \(\exp(\pm 2i\Phi(t))\) in the BdG equations [11]. The electron and hole wave functions are given by:

\[
u_\alpha(x, t) \simeq \begin{cases}
\delta_{\alpha e} \left( e^{ik^+x} + s_{ee} e^{-ik^+x} \right) \\
\delta_{\alpha h} s_{eh} e^{-2i\Phi(t)} e^{-ik^+x} / \sqrt{\hbar v^+_s},
\end{cases}
\]

\[
u_\alpha(x, t) \simeq \begin{cases}
\delta_{\alpha h} \left( e^{-ik^-x} + s_{hh} e^{ik^-x} \right) \\
\delta_{\alpha e} s_{he} e^{2i\Phi(t)} e^{ik^-x} / \sqrt{\hbar v^-_s},
\end{cases}
\]

where \(v^\pm = \hbar k^\pm / m\) is the velocity of waves with \(\hbar k^\pm = \sqrt{2m(\mu_S \pm \varepsilon)}\). In what follows, we neglect the difference between the hole and electron wave numbers in Eq. 1 and 2, assuming the chemical potential \(\mu_S\) to be large.

Performing the Bogolubov transformation on the current operator, the statistical average of the current–current correlator is computed:

\[
\langle \langle I(t_1)I(t_2) \rangle \rangle = \frac{e^2 \hbar^2}{2m^2} \sum_{\alpha, \beta} \int_0^{+\infty} \int_0^{+\infty} d\varepsilon d\varepsilon' \{...
\]
\[ f_\alpha (1 - f_\beta) e^{i(\epsilon - \epsilon')(t_2 - t_1)/\hbar} \]
\[ \left[ (u^\beta \partial v_\alpha^*) t_1, (u^\beta \partial v_\alpha) t_2 + (v^\beta \partial v_\alpha^*) t_1, (v^\beta \partial v_\alpha) t_2 + (v^\beta \partial v_\alpha^*) t_1, (u^\beta \partial v_\alpha) t_2 \right] \]
\[ + f_\alpha f_\beta e^{-i(\epsilon + \epsilon')(t_2 - t_1)/\hbar} \]
\[ \left[ (u^\alpha \partial v_\beta) t_1, (u^\alpha \partial v_\beta) t_2 + (u^\beta \partial v^*_\alpha) t_1 \right] \]
\[ + (1 - f_\alpha)(1 - f_\beta) e^{i(\epsilon - \epsilon')(t_2 - t_1)/\hbar} \]
\[ \left[ (u_\alpha \partial v_\beta) t_1 + (u^\beta \partial v^*_\alpha) t_1 \right] \]
\[ \left[ (u^\alpha \partial v^*_\beta) t_2 \right] \] (3)

where \( u^\beta \partial v = u \partial_x v - v \partial_x u \) and \( \langle \rangle \) means that the square of the average current has been subtracted. The time dependence indicated with the indices \( t_1 \) and \( t_2 \) is explicit in the electron and hole wave functions of Eq. (6) and (9). \( f_\alpha (\epsilon) = f(\epsilon - eV) \) \((f_\beta (\epsilon) = 1 - f(-\epsilon - eV))\) denote the electron (hole) distribution function and \( f \) is the Fermi–Dirac function. While only the terms containing the product \( f_\alpha (1 - f_\beta) \) contribute to the low frequency noise [2][3], terms proportional to \( f_\alpha f_\beta \) and \( (1 - f_\alpha)(1 - f_\beta) \), which involve matrix elements between states which differ by two quasiparticles, contribute to the low frequency noise and to the non-stationary calculation below. The current–current correlator of Eq. (6) would have a spatial dependence if the exact wave vector of electrons and holes was included in this finite frequency calculation [1]. However, this spatial dependence is only relevant at frequencies large compared to the inverse time of flight through the sample, which are not considered here.

[FIG. 1. a) Zero order spectral density of noise for the non-stationary Josephson effect (top) with a finite linewidth, and for the NS junction (bottom). b) The NS boundary. The light shaded region determines where the electron/hole wave function may accumulate phase: schematic description of the two scattering processes, Andreev and normal reflection.]

In order to describe non-stationary situations, we introduce the double Fourier transform of the current–current correlator:

\[ \widehat{S}(\Omega_1, \Omega_2) = \int \int dt_1 dt_2 e^{i(\Omega_1 t_1 + \Omega_2 t_2)} \langle (I(t_1)I(t_2)) \rangle . \] (4)

If the translational invariance in time is broken either by an external alternating field (with frequency \( \Omega \)) or spontaneously, like in the non-stationary Josephson effect (NJES), this double Fourier transform can be written as:

\[ \widehat{S}(\Omega_1, \Omega_2) = \sum_{m=-\infty}^{+\infty} 2\pi \delta(\Omega_1 + \Omega_2 - m\Omega) S^{(m)}(\Omega_2) . \] (5)

In both cases the zero harmonic, which is proportional to \( \delta(\Omega_1 + \Omega_2) \) is the most standard quantity to study. In the presence of this time invariance this is the only term which is present. For the above mentioned NJES, the zeroth order spectral density \( S^{(0)}(\omega) \), defined in Eq. (6) is proportional to the spectral function line-shape \( D(\omega) \), shifted by the Josephson frequency: \( S^{(0)}(\omega) = D(\omega - 2eV/h) \) which is depicted in Fig. 1 a).

We first consider the finite frequency noise in the presence of a constant bias. Because of translational invariance in time, the Fourier transform reduces to:

\[ \widehat{S}(\Omega_1, \Omega_2) = 2\pi \delta(\Omega_1 + \Omega_2) S(\Omega_2) . \] (6)

where \( S(\Omega_2) \) is the “usual” frequency dependent noise. In what follows, we assume that the energy dependence of the scattering matrix coefficients can be neglected for biases small compared to the gap. At arbitrary temperature and frequency below the superconducting gap, the noise is:

\[ S(\omega) = \frac{8e^2}{h} R_A^2 \int_{-\infty}^{+\infty} d\epsilon f(\epsilon - eV)(1 - f(\epsilon - eV - h\omega)) + \frac{4e^2}{h} R_A (1 - R_A) F_V(\omega) , \] (7)

with

\[ F_V(h\omega) = \int_{-\infty}^{+\infty} d\epsilon \left[ f(\epsilon - eV)(1 - f(\epsilon + eV - h\omega)) + f(\epsilon + eV + h\omega)(1 - f(\epsilon - eV)) \right] . \] (8)

Note that the (constant) phase factor \( \Phi \) does not appear in the above because the geometry is open. The first term in Eq. (7) is important when thermal fluctuations are present. At zero temperature and for positive frequencies, only the second term contributes, this gives:

\[ S(\omega) = \frac{4e^2}{h} R_A (1 - R_A) (2eV - h\omega) \theta(2eV - h\omega) . \] (9)

with the convention \( eV > 0 \). The striking feature in Eq. (9) is the singularity at the Josephson frequency \( 2eV/h \). This frequency scale appears in this normal superconducting geometry because below the gap, the only available charge transfer process involves the conversion
of electrons into holes. To gain a better understanding of this singularity, remember that for a junction between two superconductors, the order parameter on each side oscillates as \( e^{-i2\mu_{S1}x/t}\) with \( \mu_{S1}\) and \( \mu_{S2}\) the chemical potentials on each side, leading to a current oscillation with frequency \( 2(\mu_{S2} - \mu_{S1})/\hbar\). For a junction between two normal metals, where the electrons wave functions on each side oscillate like \( e^{-i\mu_{1}x/t}\), a singularity in noise at \( (\mu_2 - \mu_1)/\hbar\) has been pointed out [7]. In the case of an NS junction, the electron and hole wave functions on the normal side have a time dependence \( e^{i\mu_{2}x/t}\), where \( \mu_{S}\) is the chemical potential of the superconductor, leading to the singular behavior in Eq. (1). A similar but weaker singularity has been pointed out in the fractional quantum Hall regime (FQHE) [15], at the "Josephson" frequency \( e^*/\hbar\) with \( e^*/e\) the electron filling factor.

At \( \omega = 0\), we recover the doubled shot noise of NS junctions although the current–current correlation function in Eq. (6) has not been symmetrized here. At zero frequency, the phase of considering \( \langle I(t_1)I(t_2)\rangle\) or its symmetrized analog does not matter. At finite frequency, depending on the measurement procedure, both correlators may occur [16]. Here, the time dependence of the symmetrized correlator at \( t > \hbar/eV\) is:

\[
\langle I(t)I(0) + I(0)I(t)\rangle = \frac{8e^2}{\pi^2} R_A (1 - R_A) \sin^2(eVt/\hbar) \frac{\sin^2(eVt/\hbar)}{t^2} .
\]

(10)

Up to our knowledge experimental techniques do not allow for the direct observations of these oscillations in time. Nevertheless, finite frequency measurements are possible [16].

We now turn to a situation where a time dependent vector potential is applied near the boundary (Fig. 1). Because low frequency measurements are more accessible than finite frequency ones, we suggest that a zero frequency noise analysis of a sinusoidally perturbed system is a more straightforward tool to investigate the presence of the Josephson frequency in this NS system. In addition, this setup gives us an extra opportunity to observe a superconducting behavior, i.e. a doubling of the electron charge, as shown below. The phase is chosen to be a periodic function of time \( \Phi(t) = \Phi_n \sin(\Omega t)\) with \( \Phi_n = 2\pi \int_{x_1}^{x_2} dx A_x/\phi_0\) where \( \phi_0 = hc/e\) is the normal flux quantum, \( [x_1, x_2] \) defines the interval where the vector potential is confined (Fig. 2). The effect of this perturbation on the average current is straightforward in the limit where \( R_A\) depends weakly on the energy: it brings a periodic modulation of the current \( \Delta I = (4e^2/\hbar)(2\Omega)R_A(h\Omega/e)\Phi_n \cos(\Omega t)\). In the current-current correlations, this modulation leads to a non-monotonic effect as a function of phase, in contrast with the electromotive force action on the current. Note that no closed topology is imposed, in contrast to the usual AB effect. Nevertheless, the phase is periodically modulated, and yields a non-zero contribution in the noise harmonics despite the open geometry.

The low frequency noise is computed by first performing the time integrals in Eq. (1) from the explicit time dependence of the current matrix elements. Using the generating function of the Bessel functions \( J_n\), one obtains:

\[
S^{(0)}(0) = \frac{4e^2}{\hbar} R_A (1 - R_A) \sum_{m=-\infty}^{+\infty} J_m^2(2\Phi_n) F_V(mh\Omega) + \frac{8e^2}{\hbar} R_A^2 k_B T .
\]

(11)

The phase \( 2\Phi\) (Fig. 1) accumulated in the Andreev process leads to a factor 2 in the argument of \( J_n\), which is reminiscent of the Cooper pair charge. The temperature dependence in Eq. (11) is specified by \( F_V(mh\Omega) = (2eV - mh\Omega)\coth[(2eV - mh\Omega)/2k_B T]\) determines how the steps in the noise derivative

\[
\frac{\partial S^{(0)}(0)}{\partial V} \approx \frac{8e^2}{\hbar} R_A (1 - R_A) \sum_{m=-M}^{+M} J_m^2(2\Phi_n)
\]

(12)

are smeared with temperature. In Eq. (12) the sum over harmonics has a cutoff at \( M = [2eV/h\Omega]\). In experiments [17] it is more convenient to characterize the non-monotonic dependence on voltage by taking the second derivative of the AB contribution to the noise. This is illustrated for two distinct temperatures in Fig. 2.

![FIG. 2. Non-stationary AB effect: plot of \( \partial^2 S/\partial V^2\), expressed in units of \( (8e^2/\pi^2 h^2 \Omega) R_A (1 - R_A)\), as a function of \( 2eV/h\Omega\), with the choice \( \Phi_n = 3\). For \( 2k_B T = 0.2h\Omega\) (full line) and for \( 2k_B T = h\Omega\) (dashed line) for 2eV/h\Omega. One observes oscillations as a function of 2eV/h\Omega, with a clustering of large amplitude peaks. For temperatures \( h\Omega < 2k_B T\), although individual peaks can no longer be identified, clusters of "large" steps in the noise derivative continue to appear.](image-url)
give an average contribution to the non-stationary AB effect. This robustness enhances the likelihood of experimental observation, which is addressed below.

In Ref. 5, a small alternative microwave voltage $V_1(t) = V_1 \sin(\Omega t)$ was superposed to the DC bias instead of applying locally a magnetic flux. If the drop in voltage occurs at the NS boundary then gauge invariance states that this is equivalent to the present proposal, with the substitution $\Phi_n = eV_1/4\Omega$. If this drop is more extended, the analogy is restricted as the time of flight of electrons and holes through the relevant region is increased. Moreover, the experiments of Ref. 6 were performed in the diffusive regime, with a diffusion time $\tau_D$ of the order of the $\Omega^{-1}$. Although this seems to be at the border of applicability of the present approach, the assumptions made in the superconducting case are similar, so one expects these structures to be rather robust.

Note that for these predictions to be valid, the harmonic perturbation should not affect significantly the distribution function of electrons in the reservoirs. In addition, for $\Delta \gg eV$, a multichannel generalization of these results is obtained straightforwardly with the substitution $R_A(1 - R_A) = \sum_n R_{An}(1 - R_{An})$, where $R_{An}$ are the eigenvalues of the Andreev reflection matrix.

The present calculations are limited to the subgap regime, but can be readily extended to $eV > \Delta$ provided that one takes into account quasiparticle scattering states in the superconductor. In turn, a specific model for the NS boundary has to be chosen in order to specify the energy dependence of the scattering matrix coefficients. Calculations using the BTK model of a step pair potential with a delta function impurity potential will be discussed elsewhere in detail. Results indicate additional singularities to the one at $\omega = 2eV/h$ due to the presence of new scattering channels: electron-like quasiparticle transmission and hole-like (Andreev) quasiparticle transmission in the superconductor, which give rise to cusps at $\omega = (eV \mp \Delta)/h$. These occur because of the presence of a sharp edge in the density of states of the superconductor. Note that contrarily to the subgap regime, these striking features cannot be interpreted by the substitution of an effective charge. The above gap regime provides an additional justification for the systematic study presented here: these frequency scales are tied to the energy intervals which are relevant for a given bias voltage, rather than an effective charge. Further increasing the bias $\Delta \ll eV$, the superconducting singularity $\omega = 2eV/h$ weakens, and the dominant singularities $(eV / \pm \Delta)/h \approx eV/h$ give the analog of the singularity of normal metals.

The non-stationary Aharonov–Bohm effect provides an extra tool for the study of dynamical effects in low frequency noise. Because the effective charge $2e$ appears both in the position of the steps in the noise deriva-

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