Bose–Einstein condensation of quasiparticles in graphene

Oleg L Berman¹, Roman Ya Kezerashvili¹,² and Yurii E Lozovik³

¹ Physics Department, New York City College of Technology, The City University of New York, Brooklyn, NY 11201, USA
² The Graduate School and University Center, The City University of New York, New York, NY 10016, USA
³ Institute of Spectroscopy, Russian Academy of Sciences, 142190 Troitsk, Moscow Region, Russia

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Abstract
The collective properties of different quasiparticles in various graphene-based structures in a high magnetic field have been studied. We predict Bose–Einstein condensation (BEC) and the superfluidity of 2D spatially indirect magnetoexcitons in a two-layer graphene. The superfluid density and the temperature of the Kosterlitz–Thouless phase transition are shown to be increasing functions of the excitonic density but decreasing functions of a magnetic field and the interlayer separation. The instability of the ground state of the interacting 2D indirect magnetoexcitons in a slab of superlattice with alternating electron and hole graphene layers (GLs) is established. The stable system of indirect 2D magnetobiexcitons, consisting of a pair of indirect excitons with antiparallel dipole moments, is considered in a graphene superlattice. The superfluid density and the temperature of the Kosterlitz–Thouless phase transition for magnetobiexcitons in a graphene superlattice are obtained. Moreover, the BEC of excitonic polaritons in a GL embedded in a semiconductor microcavity in a high magnetic field is predicted. While the superfluid phase in this magnetoexciton polariton system is absent due to a vanishing magnetoexciton–magnetoexciton interaction in a single layer in the limit of a high magnetic field, the critical temperature of the BEC formation is calculated. The observation of the BEC and superfluidity of 2D quasiparticles in graphene in a high magnetic field would be interesting confirmation of the phenomena we have described.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
The production of graphene, a two-dimensional (2D) honeycomb lattice of carbon atoms that form the basic planar structure in graphite, has been achieved recently [1, 2]. Electronic properties of graphene, caused by unusual properties of the band structure, have became the object of many recent experimental and theoretical studies [1–7]. Graphene is a gapless semiconductor with massless electrons and holes, described as Dirac fermions [8]. Various studies of the unique electronic properties of graphene in a magnetic field have been performed recently [9–12]. The energy spectrum and the wavefunctions of magnetoexcitons, or electron–hole pairs in a magnetic field, in graphene have been calculated recently [13, 14].

The 2D electron system was studied in quantum wells (QWs) [15]. Spatially indirect excitons (or pairs of electrons and holes spatially separated in different QWs) in the system of coupled quantum wells (CQWs), with and without a magnetic field were studied in [16–23]. The experimental and theoretical motivation to study these systems is particularly due to the possibility of the BEC and superfluidity of indirect excitons, which can manifest in the CQWs as persistent electrical currents in each well and also through coherent optical properties and Josephson phenomena [16, 18–20, 22, 23]. Outstanding experimental success has now been achieved in this field [24–27]. Electron–hole pair condensation in graphene-based bilayers has been studied in [28–31].

The collective properties of Bose quasiparticles, such as excitons, biexcitons, and polaritons, in various graphene-based structures in a high magnetic field are very interesting with relevance to the BEC and superfluidity, since the random field in graphene is weaker than in a QW, particularly, because in
a QW the random field is generated due to fluctuations of the width of the QW. Let us mention that if the interaction of bosons with the random field is stronger, the BEC critical temperature is lower [32]. In this paper we present our studies of the superfluidity of magnetoexcitons in bilayer graphene, the instability of the system of magnetoexcitons in a superlattice formed by many GLs, the superfluidity of magnetobexcitons in a graphene superlattice, and the BEC of polaritons in a GL embedded in a trapped optical microcavity. All these systems of quasiparticles are considered in a high magnetic field. The BEC of magnetoexcitons in graphene layers can exist at much lower magnetic field than in QWs, because the distance between electron Landau levels in graphene is much higher than in a QW in the same magnetic field, and, therefore, a lower magnetic field is required in graphene than in a QW in order to effect the electron transitions between the Landau levels.

2. Effective Hamiltonian of magnetoexcitons and photons in a microcavity in a high magnetic field

Recently, Bose coherent effects of 2D exciton polaritons in a quantum well embedded in an optical microcavity have been the subject of theoretical [33] and experimental [34, 36, 37] studies. To obtain polaritons, two mirrors placed opposite each other form a microcavity, and quantum wells are embedded within the cavity at the antinodes of the confined optical mode. The resonant exciton–photon interaction results in the Rabi splitting of the excitation spectrum. Two polariton branches appear in the spectrum due to the resonant exciton–photon coupling. The lower polariton (LP) branch of the spectrum has a minimum at zero momentum. The effective mass of the lower polariton is extremely small, and lies in the range $10^{-5}$–$10^{-4}$ of the free electron mass. These lower polaritons form a 2D weakly interacting Bose gas. The extremely light mass of these bosonic quasiparticles, which corresponds to experimentally achievable excitonic densities, results in a relatively high critical temperature for superfluidity, of 100 K or even higher. The reason for such a high critical temperature is that the 2D thermal de Broglie wavelength is proportional to the inverse square root of the mass of the quasiparticle.

While at finite temperature there is no true BEC in any infinite untrapped 2D system, a true 2D BEC can exist in the presence of a confining potential [38, 39]. Recently, the polaritons in a harmonic potential trap have been studied experimentally in a GaAs/AlGaAs quantum well embedded in a GaAs/AlGaAs microcavity [40], where the energy of the trapped exciton is shifted using stress. In this system, evidence for the BEC of polaritons in a QW has been observed [41]. The theory of the BEC and superfluidity of excitonic polaritons in a QW without magnetic field in a parabolic trap has been developed in [42]. The Bose condensation of polaritons is caused by their bosonic character [41–43]. However, while exciton polaritons have been studied in a QW, the formation of polaritons in graphene in a high magnetic field has not yet been considered. Following [35], we consider a 2D system of polaritons in graphene layers embedded in a microcavity in a high magnetic field from the point of view of the existence of the BEC within it.

The most general case is when the superlattice with alternating electronic and hole parallel GLs in the external field is embedded in an optical microcavity in a high magnetic field. We consider magnetoexcitons in the superlattices with alternating electronic and hole GLs. We suppose that recombination times can be much greater than relaxation times $\tau_n$ due to a small overlapping of the spatially separation of electron and hole wavefunctions. In GLs, this can electrons and holes be characterized by different quasi-equilibrium chemical potentials. Then, in the system of indirect excitons in superlattices, as in QW [16, 19], quasi-equilibrium phases appear. No external field applied to the slab of the superlattice is assumed. If ‘electron’ and ‘hole’ quantum wells alternate, there are excitons with parallel dipole moments in one pair of wells, but the dipole moments of excitons in the other neighboring pairs of wells have the antiparallel direction. This fact leads to the essential distinction of the properties of the e–h system in superlattices from the one for coupled quantum wells with spatially separated electrons and holes, in which the indirect exciton system is stable due to the dipole–dipole repulsion of all excitons. This difference manifests itself already beginning with the three-layer e–h–e or h–e–h systems. We assume that alternating e–h–e layers can be formed by independent gating, with the corresponding potentials which shift chemical potentials in neighboring layers up and down, or by alternating doping (by donors and acceptors, respectively). At small densities $n$, the system of indirect excitons at low temperature is a two-dimensional weakly nonideal Bose gas with dipole moments $d$ normal to the layers in the ground state ($d = eD$, $e$ is the charge of an electron, $D$ is the interlayer separation). In contrast to ordinary excitons, for the low-density spatially indirect magnetoexciton system the main contribution to the energy originates from the dipole–dipole interactions of magnetoexcitons with parallel (figure 1) and antiparallel (figure 2) dipoles. The potential energies of interaction between two indirect magnetoexcitons $U_-(R)$ for antiparallel and $U_+(R)$ for parallel dipoles are functions of the distance $R$ between indirect magnetoexcitons along the GLs and are given as

$$U_+(R) = \frac{2e^2}{\epsilon R} - \frac{2e^2}{\epsilon \sqrt{R^2 + D^2}},$$

$$U_-(R) = \frac{\epsilon^2}{\epsilon R} - \frac{2e^2}{\epsilon \sqrt{R^2 + D^2}} + \frac{\epsilon^2}{\epsilon \sqrt{R^2 + 4D^2}}.$$  \hspace{1cm} (1)
where $e$ is the charge of an electron and $\epsilon$ is the dielectric constant. The behavior of the potential energies $U_+(R)$ and $U_-(R)$ as the functions of the distance between two excitons $R$ is shown in Figure 3.

The Hamiltonian of magnetoexcitons and photons in the strong magnetic field is given by

$$\hat{H}_{\text{tot}} = \hat{H}_{\text{mex}} + \hat{H}_{\text{ph}} + \hat{H}_{\text{mex-\text{ph}}}$$

(2)

where $\hat{H}_{\text{mex}}$ is the magnetoexcitonic Hamiltonian, $\hat{H}_{\text{ph}}$ is the photonic Hamiltonian, and $\hat{H}_{\text{mex-\text{ph}}}$ is the Hamiltonian of the magnetoexciton–photon interaction.

Let us analyze each term of the Hamiltonian (2). The effective Hamiltonian and the energy dispersion for magnetoexcitons in graphene layers in a high magnetic field $B$ in the infinite system was derived in [44]. The effective Hamiltonian $\hat{H}_{\text{mex}}$ of the low-density system of the indirect magnetoexcitons in a high magnetic field in the superlattice in the subspace of the lowest Landau level is given by [44] (we neglect the electron transitions between different Landau levels due to electron–hole Coulomb attraction)

$$\hat{H}_{\text{mex}} = \hat{H}_0 + \hat{H}_{\text{int}}.$$  

(3)

Here $\hat{H}_0$ is the effective Hamiltonian of the system of non-interacting trapped magnetoexcitons in a high magnetic field:

$$\hat{H}_0 = \sum_p \varepsilon_{\text{mex}}(p)(a_p^+ a_p + b_p^+ b_p + a_{p\downarrow}^+ a_{p\downarrow} - p + b_{p\uparrow}^+ b_{p\uparrow} - p),$$

(4)

$$\varepsilon_{\text{mex}}(P) = E_{\text{band}}(r) - E_{\text{B}}^{(b)} + \varepsilon_0(P),$$

where $a_p^+$, $a_p$, $b_p^+$, $b_p$ are creation and annihilation operators of the magnetoexcitons with up and down polarizations. In equation (4), $E_{\text{band}}(r)$ is the band gap energy, which can depend on the position of the magnetoexciton in space in the presence of the trap, $E_{\text{B}}^{(b)}$ is the binding energy of a magnetoexciton, and $\varepsilon_0(P) = P^2/(2m_B)$, where $m_B$ is the effective magnetic mass of a magnetoexciton. Similarly to the Bose atoms in a trap in the case of a slowly varying external potential [45], we can make the quasiclassical approximation, assuming that the effective magnetoexciton mass does not depend on the characteristic size $l$ of the trap and that $l$ is a constant within the trap. This quasiclassical approximation is valid if $P \gg \hbar/l$. The harmonic trap is formed by the two-dimensional planar potential in the plane of graphene. The potential trap can be produced in two different ways. One way is when the potential trap can be produced by applying an external inhomogeneous electric field. The spatial dependence of the external field potential $V(r)$ is caused by shifting the magnetoexciton energy by applying an external inhomogeneous electric field. The photonic states in the cavity are assumed to be unaffected by this electric field. In this case the band energy is given by $E_{\text{band}}(r) = E_{\text{band}}(0) + V(r)$ ($E_{\text{band}}(0) = \sqrt{2\hbar v_F}/r_B$ is the band gap energy, which is the difference between Landau levels 1 and 0, $v_F$ is the Fermi velocity of electrons in graphene [46], $r_B = \sqrt{\hbar c/(e B)}$ is the magnetic length). Near the minimum of the magnetoexciton energy, $V(r)$ can be approximated by the planar harmonic potential $\gamma r^2/2$, where $\gamma$ is the spring constant, $r$ is the distance between the center of mass of the magnetoexciton and the center of the trap. Note that a high magnetic field does not change the trapping potential in the effective Hamiltonian [47]. Let us mention that the quasiparticles in GLs and QWs in a high magnetic field are described by the same effective Hamiltonian, with the only difference being in the effective magnetic mass of magnetoexciton. This difference is caused by the four-component spinor structure of the magnetoexciton wavefunction in GL, while the magnetoexciton in QWs and CQWs is characterized by a one-component scalar wavefunction.
The Hamiltonian which describes the interaction between magnetoexcitons is
\[
\hat{H}_{\text{int}} = (2S)^{-1} \sum_{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3} (U_+ (\hat{a}_{\mathbf{P}_1}^{\dagger} \hat{a}_{\mathbf{P}_2} + \hat{a}_{\mathbf{P}_2}^{\dagger} \hat{a}_{\mathbf{P}_1}) + U_- \hat{a}_{\mathbf{P}_1}^{\dagger} \hat{a}_{\mathbf{P}_2}^{\dagger} \hat{a}_{\mathbf{P}_3} \hat{a}_{\mathbf{P}_3}^{\dagger}),
\]
where \( U_+ \) and \( U_- \) are the 2D Fourier images of \( U_+ (R) \) and \( U_- (R) \), respectively, and \( S \) is the surface of the system.

The projection of the electron–hole Hamiltonian for CQWs and GLs in a magnetic field onto the lowest Landau level results in the effective Hamiltonian (3) with a renormalized mass and where the term related to the vector potential is missing [44]. The magnetic field in the effective Hamiltonian (3) enters in the renormalized mass of the magnetoexciton \( m_B \). Therefore, the Hamiltonian for spatially separated electrons and holes in a two-layer system for CQWs and for the bilayer graphene can be reduced in a high magnetic field to the effective Hamiltonian (3). The magnetic field \( B \) is reflected by the effective Hamiltonian (3) only through the effective magnetic mass of a magnetoexciton \( m_B \). The only difference in the effective Hamiltonian (3) for the CQWs and for the bilayer graphene can be reduced in a high magnetic field without an external electric field is in the form of the Coulomb electron–hole attraction for the large electron–hole separation \( D \gg r_0 \) can be neglected, if the following condition is valid:

\[
E_b = \frac{e^2}{\epsilon_B(D)} \ll \hbar \omega \approx \hbar \omega_B + \epsilon_{\text{ex}}(P) / (2m_B r_0^2 c^2) \text{ for the CQWs and } E_b = 4e^2 / (\epsilon_D D) \ll \hbar \nu_F / r_B \text{ for the GLs, where } \nu_F \text{ is the Fermi velocity of electrons [46], and } E_b \text{ and } \omega_B \text{ are the magnetoexcitonic binding energy and the cyclotron frequency, respectively. This corresponds to a high magnetic field } B, \text{ a large interlayer separation } D, \text{ and a large dielectric constant of the insulator layer between the GLs.}
\]

3. Bose–Einstein condensation of polaritons in graphene in a high magnetic field

In this section we consider trapped polaritons in a single graphene layer embedded into an optical microcavity in a high magnetic field. When an undoped electron system in graphene in a magnetic field without an external electric field is in the ground state, half of the zeroth Landau level is filled with electrons, all Landau levels above the zeroth one are empty, and all levels below the zeroth one are filled with electrons. We suggest using the gate voltage to control the chemical potential in graphene in two ways: to shift it above the zeroth level so that it is between the zeroth and first Landau levels (case 1) or to shift the chemical potential below the zeroth level so that it is between the first negative and zeroth Landau levels (case 2). Therefore, in the first case magnetoexcitons are formed in graphene by the electron on the first Landau level and the hole on the zeroth Landau level, or in the second case by the electron on the zeroth Landau level and the hole on Landau level \(-1\). Note with an appropriate gate potential we can also use any other neighboring Landau levels \( n \) and \( n + 1 \). In both cases, all Landau levels below the chemical potential are completely filled and all Landau levels above the chemical potential are completely empty. Based on the selection rules for optical transitions between the Landau levels in single-layer graphene [54], in the first case, there are allowed transitions between the zeroth and the first Landau levels, while in the
second case, there are allowed transitions between the first negative and zeroth Landau levels.

For a relatively high dielectric constant of the microcavity, \( \epsilon \gg e^2/(\hbar v_F) \approx 2 \), the magnetoexciton energy in graphene can be calculated by applying perturbation theory with respect to the strength of the Coulomb electron–hole attraction, analogously to what was done in [17] for 2D quantum wells in a high magnetic field with non-zero electron \( m_e \) and hole \( m_h \) masses. This approach allows us to obtain the spectrum of an isolated magnetoexciton with the electron on Landau level 1 and the hole on Landau level 0 in a single graphene layer, and it will be exactly the same as for the magnetoexciton with the electron on Landau level 0 and the hole on Landau level \(-1\). The characteristic Coulomb electron–hole attraction for the single graphene layer is \( e^2/(\epsilon r_B) \). The energy difference between the first and zeroth Landau levels in graphene is \( \hbar v_F/r_B \). For graphene, the perturbative approach with respect to the strength of the Coulomb electron–hole attraction is valid when \( e^2/(\epsilon r_B) \ll \hbar v_F/r_B \) [17]. This condition can be fulfilled with all magnetic fields \( B \) if the dielectric constant of the surrounding media satisfies the condition \( e^2/(\epsilon r_B) \ll 1 \). Therefore, we claim that the energy difference between the first and zeroth Landau levels is always greater than the characteristic Coulomb attraction between the electron and the hole in the single graphene layer at any \( B \) if \( \epsilon \gg e^2/(\hbar v_F) \approx 2 \). Thus, applying perturbation theory with respect to the weak Coulomb electron–hole attraction in graphene embedded in the GaAs microcavity (\( \epsilon = 12.9 \)) is more accurate than for graphene embedded in the SiO\(_2\) microcavity (\( \epsilon = 4.5 \)). However, the magnetoexcitons in graphene exist in a high magnetic field. Therefore, we restrict ourselves by consideration of high magnetic fields.

Polaritons are linear superpositions of excitons and photons. In high magnetic fields, when magnetoexcitons may exist, the polaritons become linear superpositions of magnetoexcitons and photons. Let us define the superpositions of magnetoexcitons and photons as magneto-polaritons. It is obvious that magneto-polaritons in graphene are two-dimensional, since graphene is a 2D structure. The Hamiltonian of magneto-polaritons in the strong magnetic field is given by equation (2). It can be shown that the interaction between two direct 2D magnetoexcitons in graphene with the electron on Landau level 1 and the hole on Landau level 0 can be neglected in a strong magnetic field, in analogy to what is described in [17] for 2D magnetoexcitons in a quantum well. Thus, the Hamiltonian \( H_{\text{tot}} \) (2) does not include the term corresponding to the interaction between two direct magnetoexcitons in a single graphene layer. So, in a high magnetic field there is the BEC of the ideal magnetoexcitonic gas in graphene. Therefore, in a single graphene layer in a high magnetic field we assume \( H_{\text{tot}} = 0 \) in equation (3).

The binding energy \( \epsilon_B^{(b)} \) and effective magnetic mass \( m_B^{(b)} \) of a magnetoexciton in graphene, obtained using the first order perturbation with respect to the electron–hole Coulomb attraction, similarly to the case of a single quantum well [17], are given by

\[
\epsilon_B^{(b)} = \sqrt{\frac{\pi}{2} \frac{e^2}{\epsilon r_B}}, \quad m_B^{(b)} = \frac{2^{7/2} e \hbar^2}{\sqrt{\pi} e^2 r_B}.
\]

We obtain the effective Hamiltonian of polaritons by applying the standard procedure [49–52], where we diagonalize the Hamiltonian \( H_{\text{tot}} \) (2) by using Bogoliubov transformations. If we measure the energy relative to the \( P = 0 \) lower magneto-polariton energy \( (\epsilon_B^{(b)}/n)\hbar \pi L_c^{-1} - |\hbar \Omega_R| \), we obtain the resulting effective Hamiltonian for trapped magneto-polaritons in graphene in a magnetic field. At small momenta \( \alpha \ll 1 \) (\( L_c = \hbar c/\pi (E_{\text{band}} - E_B^{(b)})^{-1} \)) and weak confinement \( \beta \ll 1 \), this effective Hamiltonian is

\[
\hat{H}_{\text{eff}} = \sum_F \left( \frac{p_F^2}{2 M_{\text{eff}}(B)} + \frac{1}{2} V(r) \right) \hat{p}_F^2 \hat{p}_F.
\]  

and the effective magnetic mass of a magneto-polariton is given by

\[
M_{\text{eff}}(B) = 2 \left( m_B^{-1} + \frac{e L_c(B)}{\hbar \pi} \right)^{-1}.
\]

According to equations (11) and (9), the effective magneto-polariton mass \( M_{\text{eff}} \) increases with an increase of the magnetic field as \( B^{1/2} \). Let us emphasize that the resulting effective Hamiltonian for magneto-polaritons in graphene in a magnetic field for the parabolic trap is given by equation (10) for both physical realizations of confinement represented by case 1 and case 2.

Neglecting anharmonic terms for the magnetoexciton–photon coupling, the Rabi splitting constant \( \Omega_R \) can be estimated quasiclassically as

\[
|\hbar \Omega_R|^2 = \langle \langle |\hat{H}_{\text{int}}| \rangle \rangle^2, \quad E_{\text{ph0}} = \left( \frac{8 \pi \hbar \omega}{\epsilon W} \right)^{1/2}, \quad \hat{H}_{\text{int}} = \frac{v_F e^2}{c} \hat{A} = \frac{v_F e^2}{\omega_0} \hat{\sigma} \cdot \hat{E}_{\text{ph0}},
\]

where \( \hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \), \( \sigma_x \) and \( \sigma_z \) are Pauli matrices, \( \hat{H}_{\text{int}} \) is the Hamiltonian of the electron–photon interaction corresponding to the electron in graphene described by Dirac dispersion, \( E_{\text{ph0}} \) is the electric field corresponding to a single cavity photon, \( W \) is the volume of microcavity, and \( \omega \) is the photon frequency. The initial \( |i \rangle \) electron state corresponds to the completely filled Landau level 0 and completely empty Landau level 1. The final \( |f \rangle \) electron state corresponds to creation of one magnetoexciton with the electron on Landau level 1 and the hole on Landau level 0. The transition dipole moment corresponding to the process of creation of this magnetoexciton is given by \( d_{12} = e r_B/4 \). Let us note that in equation (12) the energy of photon absorbed at the creation of the magnetoexciton is given by \( \hbar \omega = \epsilon_1 - \epsilon_0 = \sqrt{2} \hbar v_F/r_B \) (we assume that \( \epsilon_B^{(b)} \ll \epsilon_1 - \epsilon_0 \)). Substituting the photon energy and the transition dipole moment into equation (12), we obtain the Rabi splitting corresponding to the creation of a magnetoexciton with the electron on Landau level 1 and the hole on Landau level 0 in graphene: \( \hbar \Omega_R = 2e(\pi \hbar v_F r_B / \sqrt{\epsilon W})^{1/2} \).

Therefore, the Rabi splitting in graphene can be controlled by the external magnetic field. It is easy to show that the
Rabi splitting related to the creation of the magnetoexciton, the electron on Landau level 0 and the hole on Landau level \(-1\) will be exactly the same as for the magnetoexciton with the electron on Landau level 1 and the hole on Landau level 0. Let us mention that dipole optical transitions from Landau level \(-1\) to Landau level 0, as well as from Landau level 0 to Landau level 1, are allowed by the selection rules for optical transitions in single-layer graphene [54].

Although Bose–Einstein condensation cannot take place in a 2D homogeneous ideal gas at non-zero temperature, as discussed in [38], in a harmonic trap the BEC can occur in two dimensions below a critical temperature \(T_c^0\). In a harmonic trap at a temperature \(T\) below a critical temperature \(T_c^0 (T < T_c^0)\), the number \(N(T, B)\) of non-interacting magnetopolaritons in the condensate is given in [38]. Applying the condition \(N_0 = 0\), and assuming that the magnetopolariton effective mass is given by equation (11), we obtain the BEC critical temperature \(T_c^{(0)}\) for the ideal gas of magnetopolaritons in a single graphene layer in a magnetic field:

\[
T_c^{(0)}(B) = \frac{1}{k_B} \left( \frac{3 \hbar^2 \gamma N}{\pi (g_s^{(e)} g_v^{(e)} + g_s^{(h)} g_v^{(h)}) M_{\text{eff}}(B)} \right)^{1/2}, \tag{13}
\]

where \(N\) is the total number of magnetopolaritons, \(g_s^{(e), (h)}\) and \(g_v^{(e), (h)}\) are the spin and graphene valley degeneracies for an electron and a hole, respectively, and \(k_B\) is the Boltzmann constant. At temperatures above \(T_c^{(0)}\), the BEC of magnetopolaritons in a single graphene layer does not exist. \(T_c^{(0)}/\sqrt{N}\) as a function of magnetic field \(B\) and spring constant \(\gamma\) is presented in figure 4. In our calculations, we used \(g_s^{(e)} = g_s^{(h)} = g_v^{(e)} = g_v^{(h)} = 2\). According to equation (13), the BEC critical temperature \(T_c^{(0)}\) decreases with the magnetic field as \(B^{-1/4}\) and increases with the spring constant as \(\gamma^{1/2}\). Note that we assume that the quality of the cavity is sufficiently high, so that the time of the relaxation to the Bose condensate quasiequilibrium state is smaller than the lifetime of the photons in the cavity.

Above we discussed the BEC of the magnetopolaritons in a single graphene layer placed within a strong magnetic field. What would happen in a multilayer graphene system in a high magnetic field? Let us mention that the magnetopolaritons formed by the microcavity photons and the indirect excitons with spatially separated electrons and holes in different parallel graphene layers embedded in a semiconductor microcavity can exist only at very low temperature \(k_B T \ll \hbar \Omega_R\). For the case of the spatially separated electrons and holes, the Rabi splitting \(\Omega_R\) is very small in comparison to the case of electrons and holes placed in a single graphene layer. This is because \(\Omega_R \sim d_{12}\) and the matrix element of magnetoexciton generation transition \(d_{12}\) is proportional to the overlapping integral of the electron and hole wavefunctions, which is very small if the electrons and holes are placed in different graphene layers. Therefore, we cannot predict the effect of relatively high BEC critical temperature for electrons and holes placed in different graphene layers.

![Figure 4](image)

**Figure 4.** The ratio of the BEC critical temperature to the square root of the total number of magnetopolaritons \(T_c^{(0)}/\sqrt{N}\) as a function of magnetic field \(B\) at different spring constants \(\gamma\). We assume the environment around graphene is GaAs with \(\epsilon = 12.9\). Figure from [35].

### 4. Superfluidity of magnetoexcitons in bilayer graphene

We consider two parallel graphene layers separated by an insulating slab of dielectric (for example, SiO\(_2\)) [53]. The spatial separation of electrons and holes in different GLs can be achieved by applying an external electric field. Therefore, spatially separated electrons and holes can be created by applying a bias voltage between two GLs or between two gates located near the corresponding graphene sheets and, thus, varying the chemical potential. The equilibrium system of local pairs of spatially separated electrons and holes can be created by varying the chemical potential (case A) (for simplicity, we also call these equilibrium local e–h pairs in a magnetic field \(B\) as indirect magnetoexcitons). Magnetoexcitons with spatially separated electrons and holes can also be created by laser pumping (far infrared in graphene) and by applying a perpendicular electric field, as for CQWs [24, 25, 27] (case B). In case A, a magnetoexciton is formed by an electron on Landau level 1 and a hole on Landau level \(-1\). In case B, a magnetoexciton is formed by an electron on Landau level 1 and a hole on Landau level 0. In case B we assume the system is in a quasiequilibrium state. Below, we consider the low-density regime for magnetoexcitons, i.e. magnetoexciton radius \(a < n^{-1/2}\).

In a strong magnetic field at low densities, \(n \ll r_B^{-2}\) (\(r_B = (\hbar c/eB)^{1/2}\) is the magnetic length, \(e\) is the electron charge, \(c\) is the speed of light), indirect magnetoexcitons repel as parallel dipoles, and we have for the pair interaction potential:

\[
U(|\mathbf{R}_1 - \mathbf{R}_2|) \simeq \frac{\epsilon^2 D^2}{e|\mathbf{R}_1 - \mathbf{R}_2|^3}, \tag{14}
\]

where \(D\) is the interlayer separation, \(\epsilon\) is the dielectric constant of the insulator between two layers, and \(\mathbf{R}_{1(2)}\) are the radius vectors of the center of mass of the two magnetoexcitons.
Since typically the value of $r$ is $r_0$ and $P \ll \hbar/r_0$ in this approximation, the effective Hamiltonian $\hat{H}_{\text{exc}}$ in the magnetic momentum representation $P$ in the subspace of the lowest Landau level has the same form (compare with [19]) as for the two-dimensional boson system without a magnetic field, but with the magnetoexciton magnetic mass $m_B$ (which depends on $B$ and $D$; see below) instead of the exciton mass ($M = m_s + m_p$) and magnetic moments instead of ordinary momenta. We can obtain the effective Hamiltonian for bilayer graphene without confinement if we consider only two graphene layers in the magnetoexciton effective Hamiltonian without a trap (3):

$$\hat{H}_{\text{exc}} = \sum_P \varepsilon_0(P) \hat{a}_P^\dagger \hat{a}_P + \frac{1}{2} \sum_{P_1, P_2} \left\langle P_1, P_2 \right| \hat{U}(P_3, P_4) \hat{a}_{P_1}^\dagger \hat{a}_{P_2}^\dagger \hat{a}_{P_2} \hat{a}_{P_1},$$

where the matrix element $\left\langle P_1, P_2 \right| \hat{U}(P_3, P_4)$ is the Fourier transform of the pair interaction potential $U(R) = e^2 D^2/e R^3$ and for the lowest Landau level we denote the spectrum of the single exciton $\varepsilon_0(P) \equiv \varepsilon_{01}(P)$. For an isolated magnetoexciton on the lowest Landau level at the small magnetic momenta under consideration, $\varepsilon_0(P) \approx P^2/(2m_B)$, where $m_B$ is the effective magnetic mass of a magnetoexciton in the lowest Landau level and is a function of the distance $D$ between $e$- and $h$-layers and the magnetic field $B$ (see [21]). In strong magnetic fields at $D \gg r_B$, the exciton magnetic mass is $m_B(D) = \varepsilon D^3/(e^2 r_B^2)$ for the QWs [21] and $m_B(D) = \varepsilon D^3/(4e^2 r_B^2)$ for the GLs [53]. We next study the magnetoexciton–magnetoexciton scattering, applying the theory of weakly interacting 2D Bose gas [16, 19]. The chemical potential $\mu$ of two-dimensional dipole magnetoexcitons in the graphene bilayer system, in the ladder approximation, has the form (compare to [16, 19]):

$$\mu = \frac{k^2}{2m_B} = \frac{\pi \hbar^2 n}{sm_B \log \left[ \frac{sh \varepsilon^2}{2\pi nm_B^2 \varepsilon^2 D^3} \right]},$$

where $s = \pm 1$ is the spin and valley degeneracy factor for a magnetoexciton in the graphene bilayer, and $n$ is the 2D density of magnetoexcitons.

At small momenta, the collective spectrum of the magnetoexciton system is the sound-like $\varepsilon(p) = c_s p$ ($c_s = \sqrt{\mu/(2m_B)}$ is the sound velocity) and satisfies the Landau criterion for superfluidity. The density of the superfluid component $n_s(T)$ for a two-dimensional system with the sound spectrum can be estimated as [32]:

$$n_s = n/(4s) - \frac{3\xi(3) \kappa B^2 p^3}{2\pi \hbar^2 c^2 m_B},$$

where $\xi(z)$ is the Riemann zeta function, and $\xi(3) \simeq 1.202$. The second term in equation (17) is the temperature dependent normal density, which takes into account the gas of phonons (‘bogolons’) with the dispersion law $\varepsilon(p) = \sqrt{\mu/(2m_B)} p$, where $\mu$ is given by equation (16).

In a 2D system, superfluidity of magnetoexcitons appears below the Kosterlitz–Thouless transition temperature $T_c$, which is obtained as the solution of the equation [55]:

$$T_c = \frac{\pi n_s(T_c)^2/(2m_B)}{\varepsilon(p)} \text{ at } T = T_c,$$

where $n_s(T_c)$ is obtained as the solution of the equation [55]:

$$T_c = \left( \frac{\pi n_s(T_c)^2}{2m_B} \right)^{1/2} = \frac{\pi n_s(T_c)^2/(2m_B)}{\varepsilon(p)} \text{ at } T = T_c.$$

5. Instability of dipole magnetoexcitons and superfluidity of magnetobiegictions in graphene superlattices

Let us show that the low-density system of weakly interacting two-dimensional indirect magnetoexcitons in superlattices is unstable, contrary to the two-layer system in the CQW. At small density $n$, the system of indirect excitons at low temperature has two-dimensional weakly nonideal Bose gas dipole moments perpendicular to the layers $d$ in the ground state.

We can obtain the effective Hamiltonian for a graphene superlattice without confinement if we consider only the magnetoexciton effective Hamiltonian (3) without a trap. Let us apply the Bogoliubov approximation to analyze the stability of the ground state of the weakly nonideal Bose gas of indirect
excitons in superlattices. We assume $U_+$ and $U_-$ are the 2D Fourier images of $U_+(R)$ and $U_-(R)$ at $P = 0$, respectively, and $S$ is the surface of the system. Let us mention that the appropriate cut-off parameter for this Fourier transform is the classical turning point of the dipole–dipole interaction. Note that the cut-off parameter $R_0$ for the potential $U_+(R)$ is much greater than for $U_-(R)$ (the cut-off parameters $R_0$ for the both potentials can be represented in figure 3 by the points where the curves corresponding to $U_+(R)$ and $U_-(R)$ are crossed by the chemical potential $\mu_{\epsilon}$, represented by the horizontal straight line placed right above but close to $U_{1\pm}(R) = 0$). Therefore, we claim that $U_+ > 0$, $U_- < 0$, and $|U_-| > |U_+|$.

Applying the unitary Bogoliubov transformations to the magnetoexciton operators $a_p$, $a_p^\dagger$, $b_p$, and $b_p^\dagger$, we diagonalize the Hamiltonian $\hat{H}_{\text{tot}}$ in the Bogoliubov approximation [56]. Finally, we obtain

$$\hat{H}_{\text{tot}} = \sum_{\rho \neq 0} \varepsilon(p)(a_\rho a_{\rho}^\dagger + b_{\rho} b_{\rho}^\dagger)$$

with the spectrum of quasiparticles $\varepsilon(p)$:

$$\varepsilon_1^2(p) = \varepsilon_0^2(p) + 2nU_+ \varepsilon_0(p),$$

$$\varepsilon_2^2(p) = \varepsilon_0^2(p) + 2n(U_+ + U_-) \varepsilon_0(p).$$

Since $U_+ > 0$ and $U_- < 0$, we have $\varepsilon_1^2(p) > \varepsilon_2^2(p)$ at $P > 0$. Therefore, at low temperature only the quasiparticles with the spectrum $\varepsilon_1^2(p)$ will be excited, since the excitations of these quasiparticles requires less energy than for the quasiparticles with the spectrum $\varepsilon_2^2(p)$. Since $U_+ + U_- < 0$, it is easy to see from equation (19) that for the small momenta $P \ll (\sqrt{4m_B n} |U_+ + U_-|)$ the spectrum of excitations becomes imaginary. Hence, the system of weakly interacting indirect magnetoexcitons in the slab of the superlattice is unstable. It can be seen that the condition of the instability of magnetoexcitons becomes stronger for larger magnetic field, because $m_B$ increases with the magnetic field and, therefore, the region of $P$, resulting in the imaginary collective spectrum, increases as $B$ increases.

For the ground state of the system, we consider a low-density weakly nonideal gas of two-dimensional indirect magnetobiexcitons, created by indirect magnetoexcitons with antiparallel dipoles in neighboring pairs of wells (figure 2). The mean dipole moment of the indirect magnetoexciton is equal to zero. However, the quadrupole moment is non-zero and equal to $Q = 3eD^2$ (the large axis of the quadrupole is normal to the quantum wells/graphene layers). So, indirect magnetobiexcitons interact at long distances $R \gg D$ as parallel quadrupoles: $U(R) = 9e^2 D^2(\varepsilon R^3)$.

We apply the theory of a weakly interacting 2D Bose gas [16] to study the magnetobiexciton–magnetobiexciton repulsion. The chemical potential $\mu$ of two-dimensional biexcitons, repulsed by the quadrupole law, in the ladder approximation, has the form (compare to [16, 19]):

$$\mu = \frac{4\pi h^2 n_{\text{hex}}}{m_B^b} \log \left[ h^{\gamma_3} e^{2/3} / (8\pi (18n_B e^2 D^2)^{2/3} n_{\text{hex}}) \right].$$

where $n_{\text{hex}} = n/8$ is the density of magnetobiexcitons in graphene layers and we consider that the magnetic mass of a magnetobiexciton is twice the magnetic mass of a magnetobiexciton, i.e. $2m_B$.

At small momenta the collective spectrum of the magnetobiexciton system is the sound-like $\varepsilon(p) = c_p (c_p = \sqrt{\mu/(2m_B)}$ is the sound velocity) and satisfies the Landau criterion for superfluidity. The density of the superfluid component $n_S(T)$ for a two-dimensional system with the sound spectrum is [32]

$$n_S(T) = n_{\text{hex}} - \frac{3\zeta(3) k_B^3 T^3}{4\pi h^2 m_B c_s^2}.$$  

In a 2D system, superfluidity of magnetobiexcitons appears below the Kosterlitz–Thouless transition temperature $T_c = T_c(B)$ for a superlattice consisting of QWs for GaAs/AlGaAs, $\epsilon = 13$, and for GLs separated by a layer of SiO$_2$ with $\epsilon = 4.5$, on the magnetobiexciton density $n = 10$ nm at different magnetic fields. The solid, dashed and thin solid curves are for the QWs, and the dotted, dashed–dotted and thin dotted curves are for the GLs at $B = 20$ T, $B = 15$ T and $B = 10$ T, respectively. Figure from [44].

![Figure 6. Dependence of the Kosterlitz–Thouless transition temperature $T_c$ for a superlattice consisting of QWs for GaAs/AlGaAs, $\epsilon = 13$, and for GLs separated by a layer of SiO$_2$ with $\epsilon = 4.5$, on the magnetobiexciton density $n = 10$ nm at different magnetic fields. The solid, dashed and thin solid curves are for the QWs, and the dotted, dashed–dotted and thin dotted curves are for the GLs at $B = 20$ T, $B = 15$ T and $B = 10$ T, respectively.](image-url)
superlattice consisting of quantum wells, and this difference is stronger for weaker magnetic fields.

6. Conclusions

We have obtained the effective Hamiltonian of quasiparticles in graphene structures in a high magnetic field: indirect magnetoe excitons in a graphene bilayer, magnetobiloe excitons in graphene superlattices and magnetopolaritons in a graphene layer embedded in an optical microcavity. It was shown that the gas of magnetoe excitons in a graphene superlattice is unstable due to the attraction between magnetoe excitons with parallel dipoles, while the system of magnetobiloe excitons in the graphene superlattice is stable. We have shown that the quasiparticles in GLs and QWs in a high magnetic field are described by the same effective Hamiltonian with the only difference being the effective magnetic mass of the magnetoe exciton. This difference is caused by the four-component spinor structure of the magnetoe exciton wavefunction in GL, while the magnetoe exciton in QWs and CWQs is characterized by the one-component scalar wavefunction. Moreover, we show that the magnetoe exciton system in a graphene bilayer and the magnetobiloe exciton system in a graphene superlattice can be described as a 2D weakly interacting Bose gas, which is a superfluid below the Kosterlitz–Thouless phase transition temperature. We have calculated the density of the superfluid component and the Kosterlitz–Thouless temperature, for systems of magnetoe excitons and magnetobiloe excitons, as functions of the magnetic field $B$, interlayer separation $D$, and magnetoe exciton density $n$. In contrast to magnetoe excitons in a graphene bilayer and magnetobiloe excitons in a graphene superlattice, the magnetopolaritons in a GL embedded in an optical cavity in the limit of high magnetic field is an ideal Bose gas without interparticle interactions and, therefore, the magnetobiloe exciton gas is not a superfluid. However, there is BEC at temperatures below the critical temperature in this system in a trap. We have calculated the critical temperature of magnetopolariton BEC in a GL embedded in an optical microcavity in a trap as a function of the magnetic field and the curvature $\gamma$ of the trap. Note that taking into account the virtual transitions of electrons and holes between Landau levels results in weak (at large $e$) interactions between magnetoe excitons [17]. In turn, this leads to the possibility of superfluidity of the magnetopolariton system.

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