$k_T$ and threshold resummations

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Abstract

We demonstrate that both the $k_T$ and threshold resummations can be performed in the Collins-Soper resummation formalism by evaluating soft gluon emissions with infrared cutoffs for the longitudinal and transverse loop momenta, respectively. The reason the $k_T$ resummation for a parton distribution function leads to suppression in the large $b$ region, $b$ being the conjugate variable of parton transverse momentum $k_T$, and the threshold resummation leads to enhancement in the large $N$ limit, $N$ being the moment of a distribution function, is a consequence of opposite directions of double-logarithm evolutions. The $k_T$ and threshold resummations for an energetic final-state jet give suppression. The switch of the threshold resummation from enhancement to suppression is attributed to a nonvanishing jet invariant mass. In the same framework we derive a unification of the $k_T$ and threshold resummations for a parton distribution function by requiring infrared cutoffs for both longitudinal and transverse loop momenta. This unified resummation exhibits suppression at large $b$, similar to the $k_T$ resummation, and exhibits enhancement at small $b$, similar to the threshold resummation.
I. INTRODUCTION

Recently, we have applied the Collins-Soper (CS) resummation technique [1] to the $k_T$ resummation for a parton distribution function $\phi(\xi, k_T, p^+)$ associated with a hadron of momentum $p^\mu = p^+ \delta^{\mu+}$ [2], which describes the probability that a parton carries longitudinal momentum $\xi p^+$ and transverse momentum $k_T$. In this formalism the double logarithms $\ln^2(p^+ b)$ contained in $\phi$, $b$ being the conjugate variable of $k_T$, are organized. The result is a Sudakov suppression factor, quoted as [2]

$$\phi(\xi, b, p^+) = \exp \left[ - \int_{1/b}^{\xi p^+} \frac{dp}{p} \int_{1/b^2}^{p^2} \frac{d\mu^2}{\mu^2} \gamma_K(\alpha_s(\mu)) \right] \phi(\xi), \quad (1)$$

where the anomalous dimension $\gamma_K$ will be defined in Sec. II, and the initial condition $\phi(\xi)$ of the double-logarithm evolution coincides with the standard parton model. We have neglected the dependence of $\phi$ on the renormalization (or factorization) scale $\mu$, which is not relevant to our discussion here. The $\mu$ dependence denotes a single-logarithm evolution, which can be easily derived using renormalization-group (RG) equations.

Equation (1) has been employed to evaluate the $p_T$ spectrum of direct photon production, $p_T$ being the transverse momentum of the direct photon [3]. It was observed that the double-logarithm exponential, after Fourier transformed to $k_T$ space, provides the necessary $k_T$ smearing effect, which resolves the discrepancy between most data sets and next-to-leading-order QCD ($\alpha_s^2$) predictions [4]. This success implies the importance of the $k_T$ resummation for studies of hard semi-inclusive QCD processes, in which transverse degrees of freedom of final states are measured.

In a series of works [5] the threshold resummation [6–8] for dijet, direct photon and heavy quark productions in kinematic end-point regions has been performed. In this formalism a different type of double logarithms $\ln^2(1/N)$, which come from the Mellin transformation of the logarithmic corrections $\ln(1 - \xi)/(1 - \xi)_+$ to moment $N$ space, is organized. They are important logarithms, since $\xi \to 1$ corresponds to the large $N$ limit. On the contrary, the threshold resummation for a parton distribution function $\phi(\xi, p^+)$, where the transverse
degrees of freedom have been integrated out, results in enhancement:

\[ \phi(N, p^+) = \exp \left[ - \int_0^1 \frac{dz}{1-z} \frac{z^{N-1} - 1}{1-z} \int_0^{1-z} \frac{d\lambda}{\lambda} \gamma_K(\alpha_s(\sqrt{\lambda} p^+)) \right] \phi^{(0)}, \]

with the initial condition \( \phi^{(0)} \) and the same anomalous dimension \( \gamma_K \).

At the kinematic end points of QCD processes, energetic final-state jets may be produced, which also involve large double logarithms. Take deep inelastic scattering (DIS) as an example. The final-state quark scattered by the photon with momentum \( q \) carries the momentum \( p' = \xi p + q = ((\xi - x)p^+, q^-, 0) \) for the Bjorken variable \( x = -q^2/(2p \cdot q) = -q^+/p^+ \) in a frame where \( q \) possesses only the plus and minus components. As \( x \to 1 \), \( p' \) approaches the light cone with vanishing invariant \( p'^2 \), and the final-state quark forms a jet.

The threshold resummation for a jet function \( J(N, p'^-) \) gives Sudakov suppression [5],

\[ J(N, p'^-) = \exp \left[ \int_0^1 \frac{dz}{1-z} \frac{z^{N-1} - 1}{1-z} \int_0^{1-z} \frac{d\lambda}{\lambda} \gamma_K(\alpha_s(\sqrt{\lambda} p'^-)) \right] J^{(0)}, \]

opposite to the enhancement in Eq. (2). While the \( k_T \) resummation for a jet function \( J(b, p'^-) \) still gives Sudakov suppression similar to Eq. (3),

\[ J(b, p'^-) = \exp \left[ - \int_0^{p'^-} \frac{dp}{p} \int_0^{p^2} \frac{d\mu^2}{\mu^2} \gamma_K(\alpha_s(\mu)) \right] J^{(0)}. \]

The above expression has been intensively applied to the analyses of end-point spectra of heavy meson decays [6].

With the above comparison, it is worthwhile to explore the relation between the \( k_T \) and threshold resummations, and the difference between their results for a parton distribution function and for a jet function. In this paper we shall show that the threshold resummation can also be performed in the CS framework. The double logarithms \( \ln^2(p^+ b) \) and \( \ln^2(1/N) \) are summed simply by choosing appropriate infrared cutoffs in the evaluation of soft gluon corrections. If transverse degrees of freedom of a parton are included, \( 1/b \) will serve as a transverse infrared cutoff. Combined with the collinear logarithms \( \ln p^+ \), \( \ln^2(p^+ b) \) are generated. In the end-point region with \( x \to 1, \ p^+/N \) is a longitudinal infrared cutoff, implying the presence of \( \ln^2(1/N) \). Therefore, in the region away from end points, we
neglect the longitudinal cutoff, and sum $\ln^2(p^+b)$. As close to end points, we keep only the longitudinal cutoff, and integrate over transverse degrees of freedom of a parton in the same formalism. The logarithms $\ln^2(1/N)$ are then summed.

It will be shown, in the CS framework, that suppression from the $k_T$ resummation and enhancement from the threshold resummation for a parton distribution function is a consequence of opposite directions of double-logarithm evolutions: from the small $1/b$ to the large $p^+$ in the former case and from $p^+$ to $p^+/N \ll p^+$ in the latter case. The comparison of the two resummations for a jet function is subtler. In the $k_T$ resummation the jet momentum is parametrized as $p' = (0, p'^- , k_T)$, similar to the parton momentum $(\xi p^+, 0, k_T)$, and thus the same suppression effect is expected. In the threshold resummation the jet momentum is $p' = ((\xi - x)p^+, q^- , 0)$, different from the parton momentum $(\xi p^+, 0, 0)$. The former has a nonvanishing invariant mass $p'^2 = 2(\xi - x)p \cdot p'$, while the latter lies on the light cone. We shall demonstrate that this difference is responsible for the switch from enhancement for a parton distribution function to suppression for a jet function.

Once the two different resummations can be reproduced in the same formalism, it is possible to develop a unified resummation, in which both the logarithms $\ln^2(p^+b)$ and $\ln^2(1/N)$ are organized to all orders. The unification of the $k_T$ and threshold resummations for a parton distribution function are achieved simply by retaining the longitudinal and transverse cutoffs simultaneously. It will be observed that the result exhibits suppression at large $b$, similar to the $k_T$ resummation, and enhancement at small $b$, similar to the threshold resummation. The unified resummation is appropriate for studies of QCD processes, in which final states possess large rapidities (corresponding to large momentum fractions), and their transverse degrees of freedom are measured.

In Sec. II we review the application of the CS technique to the $k_T$ resummation for a quark distribution function. In Sec. III we perform the threshold resummation for a quark distribution function in the same formalism, comparing each step with that in the $k_T$ resummation. The threshold resummation for a jet function is derived in Sec. IV. The
unification of the two resummations is proposed in Sec. V. Section VI is the conclusion.

II. $k_T$ RESUMMATION

Consider a quark distribution function for a hadron in the minimal subtraction scheme,

$$
\phi(x, k_T, p^+) = \int \frac{dy^-}{2\pi} \int \frac{d^2y_T}{(2\pi)^2} e^{-ixp^+y^- + i k_T \cdot y_T} \langle p| \bar{q}(y^- , y_T) \frac{1}{2} \gamma^+ q(0) | p \rangle , \tag{5}
$$

where $\gamma^+$ is a Dirac matrix, and $|p\rangle$ denotes a hadron with the momentum $p^\mu = p^+ \delta^{\mu+}$. Averages over spin and color are understood. The above definition is given in the axial gauge $n \cdot A = 0$, where the gauge vector $n$ is assumed to be arbitrary with $n^2 \neq 0$. Though the definition is gauge dependent, physical observables, such as hadron structure functions and cross sections, are gauge invariant. It has been shown that the $n$ dependences cancel among convolution factors of a factorization formula, i.e., among parton distribution functions, final-state jets, and nonfactorizable soft gluon exchanges, in the factorization formula for a DIS structure function \[2\]. A resummation can also be performed in the covariant gauge $\partial \cdot A = 0$, and the result is the same as that from the axial gauge \[2\].

The essential step in the CS technique is to derive a differential equation $p^+ d\phi / dp^+ = C \phi$ \[1\], where the coefficient function $C$ contains only single logarithms as shown below, and can be treated by RG methods. In the axial gauge $n$ appears in the gluon propagator, $(-i/l^2)N^{\mu\nu}(l)$, with

$$
N^{\mu\nu}(l) = g^{\mu\nu} - \frac{n^{\mu}l^{\nu} + n^{\nu}l^{\mu}}{n \cdot l} + n^2 \frac{l^{\mu}l^{\nu}}{(n \cdot l)^2} . \tag{6}
$$

Because of the scale invariance of $N^{\mu\nu}$ in $n$, $\phi$ depends on $p^+$ through the ratio $(p \cdot n)^2 / n^2$, implying that the differential operator $d / dp^+$ can be replaced by $d / dn_\alpha$ using a chain rule,

$$
p^+ \frac{d}{dp^+} \phi = - \frac{n^2}{v \cdot n} v_\alpha \frac{d}{dn_\alpha} \phi , \tag{7}
$$

where $v = (1, 0, 0)$ is a dimensionless vector along $p$. The operator $d / dn_\alpha$ applies to $N^{\mu\nu}$, leading to
\[ -\frac{n^2}{v \cdot n} v_\alpha d\frac{d}{dn_\alpha} N^{\mu\nu} = \hat{v}_\alpha (N^{\mu\nu} l^\nu + N^{\alpha\nu} l^\mu), \]  
(8)

with the special vertex
\[ \hat{v}_\alpha = \frac{n^2 v_\alpha}{v \cdot n n \cdot l}. \]  
(9)

The momentum \( l^\mu \) \((l^\nu)\) is contracted with a vertex the differentiated gluon attaches, which is then replaced by a special vertex. For each type of vertices, there exists a Ward identity, which relates a diagram with the contraction of \( l^\mu \) \((l^\nu)\) to the difference of two diagrams \([10]\). A pair cancellation then occurs between the contractions with two adjacent vertices. Summing diagrams with different differentiated gluons, the special vertex moves to the outer end of a parton line. We arrive at the derivative,
\[ p^+ \frac{d}{dp^+} \phi(x, k_T, p^+) = 2\bar{\phi}(x, k_T, p^+), \]  
(10)

described by Fig. 1(a), where the square in the new function \( \bar{\phi} \) represents the special vertex \( \hat{v}_\alpha \). The coefficient 2 comes from the equality of the two new functions with the special vertex on either side of the final-state cut.

To obtain a differential equation of \( \phi \), we need to factorize subdiagrams containing the special vertex out of \( \bar{\phi} \). The factorization holds in the leading regions of the loop momentum \( l \) that flows through the special vertex. The collinear region of \( l \) is not leading because of the factor \( 1/(n \cdot l) \) in \( \hat{v}_\alpha \) with nonvanishing \( n^2 \). Therefore, the leading regions of \( l \) are soft and hard, in which the subdiagrams are factorized from \( \bar{\phi} \) into a soft function \( K \) and a hard function \( G \), respectively. The remaining part is the original distribution function \( \phi \). That is, \( \bar{\phi} \) is expressed as the convolution of the functions \( K \) and \( G \) with \( \phi \).

The lowest-order contribution to \( K \) from Fig. 1(b) is written as
\[ \bar{\phi}_s(x, k_T, p^+) = \bar{\phi}_{sv}(x, k_T, p^+) + \bar{\phi}_{sr}(x, k_T, p^+), \]  
(11)

with
\[ \bar{\phi}_{sv} = \left[ ig^2 C_F \mu^\epsilon \int \frac{d^4 l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \frac{\hat{v}^\beta v^\nu}{v \cdot l} \frac{1}{l^2} - \delta K \right] \phi(x, k_T, p^+), \]  
(12)
\[ \bar{\phi}_{sr} = ig^2 C_F \mu^\epsilon \int \frac{d^4 l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \frac{\hat{v}^\beta v^\nu}{v \cdot l} 2\pi i \delta(l^2) \phi(x, |k_T + l_T|, p^+), \]  
(13)
corresponding to virtual and real gluon emissions, respectively. $C_F = 4/3$ is a color factor, and $\delta K$ an additive counterterm. The ultraviolet pole in Eq. (12) is isolated using the dimensional regularization. To work out the loop integral in Eq. (13) explicitly, we employ the Fourier transformation from $k_T$ space to $b$ space. The convolution of the subdiagram with $\phi$ in the loop momentum $l_T$ is then simplified into a product. This is the reason the $k_T$ resummation should be performed in the space conjugate to $k_T$. The combination of Eqs. (12) and (13) then gives

$$\bar{\phi}_s(x, b, p^+) = K(1/(b\mu), \alpha_s(\mu))\phi(x, b, p^+),$$

with

$$K = ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon} N_{\nu\beta}(l)} \hat{v}_{\nu}^\alpha \hat{v}_{\gamma}^\beta \left[ \frac{1}{l^2} + 2\pi i \delta(l^2) e^{il_T \cdot b} \right] - \delta K,$$

where the factor $\exp(il_T \cdot b)$ is introduced by the Fourier transformation.

Note that we have applied the soft approximation to the real gluon emission:

$$\phi(x + l^+/p^+, |k_T + l_T|, p^+) \approx \phi(x, |k_T + l_T|, p^+)$$

when writing Eq. (13). This approximation implies that $1/b$ serves as an infrared cutoff for the evaluation of the subdiagrams shown in Eq. (15). Combined with the hard function $G$, which is characterized by the scale $p^+$, the logarithms $\ln(p^+ b)$ are formed. Hence, Eq. (16) is associated with the $k_T$ resummation. We shall show in the next section that the alternative soft approximation

$$\phi(x + l^+/p^+, |k_T + l_T|, p^+) \approx \phi(x + l^+/p^+, k_T, p^+)$$

implies the infrared cutoff $p^+/N$ in moment space. Combined with $G$, the logarithms $\ln(1/N)$ are produced, so that Eq. (17) is associated with the threshold resummation.

The lowest-order contribution to $G$ from Fig. 1(c) is given by

$$\bar{\phi}_h(x, b, p^+) = G(xp^+/\mu, \alpha_s(\mu)) \phi(x, b, p^+),$$

in $b$ space, with
\[ G = -ig^2 C_F \mu^\epsilon \int \frac{d^4-l}{(2\pi)^4} N_{\nu\beta}(l) \hat{\beta}^\nu \left[ \frac{x \cdot \mathbf{p} - l}{(xp-l)^2} \gamma^\nu + \frac{v^\nu}{v \cdot l} \right] - \delta G, \]  

(19)

where \( \delta G \) is an additive counterterm. The second term in the above expression, whose sign is opposite to that of \( \bar{\phi}_{sv} \), is a soft subtraction. This term avoids double counting, and ensures a hard momentum flow in \( G \). At intermediate \( x \), \( G \) is characterized by a large scale \( p^+ \) as stated above.

Using the definition \( \bar{\phi} = \bar{\phi}_s + \bar{\phi}_h \), Eq. (11) becomes

\[ p^+ \frac{d}{dp^+} \phi(x, b, p^+) = 2 \left[ K(1/(b\mu), \alpha_s(\mu)) + G(xp^+/\mu, \alpha_s(\mu)) \right] \phi(x, b, p^+), \]

(20)

where the sum \( K + G \) is the coefficient function \( C \) mentioned before. A straightforward calculation leads Eqs. (15) and (19) to

\[ K(1/(b\mu), \alpha_s(\mu)) = \frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{1}{b\mu}, \]

(21)

\[ G(xp^+/\mu, \alpha_s(\mu)) = -\frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{xp^+\nu}{\mu}, \]

(22)

with \( \nu^2 = (v \cdot n)^2/|n^2| \) being a gauge factor, where we have assumed \( n^2 < 0 \). Constants of order unity, which are irrelevant to our discussion, have been neglected. Equation (22) confirms our argument that \( \phi \) depends on \( p^+ \) via the ratio \( (p \cdot n)^2/n^2 \). For a detailed derivation of Eqs. (21) and (22), refer to [2].

The functions \( K \) and \( G \) possess ultraviolet divergences individually as indicated by their counterterms. These divergences, both from the virtual gluon contribution \( \bar{\phi}_{sv} \), cancel each other, such that the sum \( K + G \) is RG invariant. The single logarithms \( \ln(b\mu) \) and \( \ln(p^+/\mu) \), contained in \( K \) and \( G \), respectively, are organized by the RG equations

\[ \mu \frac{d}{d\mu} K = -\gamma_K = -\mu \frac{d}{d\mu} G. \]

(23)

The anomalous dimension of \( K \), \( \lambda_K = \mu d\delta K/d\mu \), is given, up to two loops, by [1]

\[ \gamma_K = \frac{\alpha_s}{\pi} C_F + \left( \frac{\alpha_s}{\pi} \right)^2 C_F \left[ C_A \left( \frac{67}{36} - \frac{\pi^2}{12} \right) - \frac{5}{18} n_f \right], \]

(24)

with \( n_f \) being the number of quark flavors, and \( C_A = 3 \) a color factor.

8
Solving Eq. (23), we have

\[ K\left(\frac{1}{b\mu}, \alpha_s(\mu)\right) + G(xp^+/\mu, \alpha_s(\mu)) = K(1, \alpha_s(1/b)) + G(1, \alpha_s(xp^+)) - \int_{1/b}^{xp^+} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)), \]

\[ = -\int_{1/b}^{xp^+} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)), \tag{25} \]

where the initial condition \( K(1, \alpha_s(1/b)) \) vanishes, and \( \ln \nu \) in \( G(1, \alpha_s(xp^+)) \) has been dropped, since the gauge factor will be cancelled as computing physical quantities. Inserting the above expression into Eq. (20), we obtain the solution

\[ \phi(x, b, p^+) = \Delta_k(b, xp^+)\phi(x) \tag{26} \]

with the Sudakov exponential from the \( k_T \) resummation,

\[ \Delta_k(b, xp^+) = \exp \left[ -\int_{1/b}^{xp^+} \frac{dp}{p} \int_{1/b^2}^{p^2} \frac{d\mu^2}{\mu^2} \gamma_K(\alpha_s(\mu)) \right]. \tag{27} \]

We have set the upper bound of the variable \( p \) to \( xp^+ \), which corresponds to the final condition \( \phi(x, b, p^+) \), and the lower bound to \( 1/b \), such that the initial condition \( \phi(x) \) does not contain the logarithms \( \ln(p^+b) \). This statement will become essential, when Eq. (27) is compared with the exponential from the threshold resummation for \( \phi \).

### III. THRESHOLD RESUMMATION

In this section we derive, using the CS formalism, the threshold resummation for the quark distribution function,

\[ \phi(x, p^+) = \int \frac{dy^-}{2\pi} e^{-ixp^+y^-} \langle p|\bar{q}(y^-)\frac{1}{2}\gamma^+q(0)|p \rangle, \tag{28} \]

which is obtained by integrating Eq. (2) over \( k_T \). Comparing each step of the derivation, the relation of the threshold resummation to the \( k_T \) resummation will be clear. The argument \( p^+ \) represents the logarithms \( \ln(1-x)p^+ \). According to the same reasoning, \( \phi \) depends on \( p^+ \) through the ratio \((p \cdot n)^2/n^2\), and the chain rule in Eq. (7) holds. Following Eqs. (8)
and (10), we have the factorization of $\bar{\phi}$ into the convolution of subdiagrams containing the special vertex with $\phi$. Similarly, subdiagrams with soft and hard loop momentum flows are absorbed into the functions $K$ and $G$, respectively.

The lowest-order contribution to $K$ from Fig. 1(b) is written as

$$\bar{\phi}_s(x, p^+) = \bar{\phi}_{sv}(x, p^+) + \bar{\phi}_{sr}(x, p^+) ,$$  \hspace{1cm} (29)

with

$$\bar{\phi}_{sv} = \left[ ig^2 C_F \mu^\epsilon \int \frac{d^{1-\epsilon} l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \frac{\hat{\phi}^{\beta\nu\nu}}{v \cdot l} \frac{1}{l^2} - \delta K \right] \phi(x, p^+) ,$$  \hspace{1cm} (30)

$$\bar{\phi}_{sr} = ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \frac{\hat{\phi}^{\beta\nu\nu}}{v \cdot l} 2\pi i \delta(l^2) \phi(x + l^+/p^+, p^+) ,$$  \hspace{1cm} (31)

corresponding to virtual and real gluon emissions, respectively. The virtual gluon contribution is the same as that in Eq. (12), and Eq. (31) is the consequence of the soft approximation in Eq. (17). If retaining transverse degrees of freedom of a parton at the beginning, Eq. (17) allows us to integrate out the $k_T$ dependences of $\bar{\phi}_{sr}$ and of $\phi$, and we still arrive at Eq. (31).

Inserting the identities

$$\int_0^1 d\xi \delta(\xi - x) = 1 , \quad \int_0^1 d\xi \delta(\xi - x - l^+/p^+) = 1 ,$$  \hspace{1cm} (32)

into Eqs. (30) and (31), respectively, Eq. (29) is reexpressed as

$$\bar{\phi}_s(x, p^+) = \int_x^1 \frac{d\xi}{\xi} K \left( \frac{1 - x}{\xi} \right) \left( \frac{p^+}{\mu} \right) \alpha_s(\mu) \phi(\xi, p^+) ,$$  \hspace{1cm} (33)

with

$$K = ig^2 C_F \mu^\epsilon \int \frac{d^{1-\epsilon} l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \frac{\hat{\phi}^{\beta\nu\nu}}{v \cdot l} \left[ \frac{\delta(1 - x/\xi)}{l^2} \right.$$

$$\left. + 2\pi i \delta(l^2) \delta \left( 1 - \frac{x}{\xi} - \frac{l^+}{p^+} \right) \right] - \delta K \delta \left( 1 - \frac{x}{\xi} \right) .$$  \hspace{1cm} (34)

To obtain the above expression, we have adopted the approximation

$$\delta \left( 1 - \frac{x}{\xi} - \frac{l^+}{\xi p^+} \right) \approx \delta \left( 1 - \frac{x}{\xi} - \frac{l^+}{p^+} \right) ,$$  \hspace{1cm} (35)

which is appropriate in the considered region with $x \to 1$. 

10
To work out the $\xi$ integration explicitly, we employ a Mellin transformation from momentum fraction ($x$) space to moment ($N$) space,

$$\bar{\phi}_s(N,p^+) \equiv \int_0^1 dx x^{N-1} \bar{\phi}_s(x,p^+) ,$$

$$= K(p^+/(N\mu),\alpha_s(\mu))\phi(N,p^+) .$$

(36)

with

$$K(p^+/(N\mu),\alpha_s(\mu)) = \int_0^1 dz z^{N-1} K((1-z)p^+/\mu,\alpha_s(\mu)) .$$

(37)

The convolution between $K$ and $\phi$ is then simplified into a product. This is the reason the threshold resummation should be performed in moment space. The Mellin transformation for the threshold resummation and the Fourier transformation for the $k_T$ resummation then play the same role. As shown in the Appendix, $K$ is given by

$$K(p^+/(N\mu),\alpha_s(\mu)) = \frac{\alpha_s(\mu)}{\pi} C_F \left( \int_0^1 dz z^{N-1} - 1 + \ln \frac{p^+\nu}{\mu} \right) ,$$

$$= \frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{p^+\nu}{N\mu} ,$$

(38)

with the same counterterm $\delta K$. To derive the second expression, we have identified the integral over $z$ as $\ln(1/N)$, which is valid up to corrections suppressed by $1/N$. Hence, soft real gluon emissions produce the logarithms $\ln(p^+/N)$ as stated before.

The lowest-order contribution to the hard function $G$ from Fig. 1(c) is written as

$$\bar{\phi}_h(N,p^+) = G(p^+/(\mu,\alpha_s(\mu))\phi(N,p^+) ,$$

(39)

in $N$ space, where the expressions of $G$ have been given in Eqs. (19) and (22) with $x = 1$. In conclusion, the functional forms of $K$ and $G$ in the threshold resummation are the same as Eqs. (21) and (22), but with the scales $1/b$ replaced by $p^+\nu/N$ and $xp^+$ by $p^+$, respectively.

Using $\bar{\phi} = \bar{\phi}_s + \bar{\phi}_h$, we have the differential equation

$$p^+ \frac{d}{dp^+} \phi(N,p^+) = 2 \left[ K(p^+/(N\mu),\alpha_s(\mu)) + G(p^+/(\mu,\alpha_s(\mu)) \right] \phi(N,p^+) .$$

(40)

Again, $K$ and $G$ contain ultraviolet divergences individually, but their sum $K + G$ is RG invariant. The RG solution of $K + G$ is given by
\[
K(p^+/N\mu, \alpha_s(\mu)) + G(p^+/\mu, \alpha_s(\mu)) = -\int_{p^+/N}^{p^+} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)).
\]  
(41)

To sum \(\ln(1/N)\) by means of Eq. (40), we make the replacement

\[
p^+ \frac{d\phi}{dp^+} = \frac{p^+}{N} \frac{d\phi}{d(p^+/N)}.
\]  
(42)

That is, the characteristic scale \(p^+\) of \(G\) is frozen, when solving Eq. (40). We then obtain

\[
\phi(N, p^+) = \Delta_t(N, p^+)^\phi(0),
\]  
(43)

with the exponential from the threshold resummation,

\[
\Delta_t(N, p^+) = \exp \left[-\int_{p^+/N}^{p^+} \frac{d\mu}{\mu} \int_{p^2}^{p^+} \frac{d\lambda^2}{\lambda^2} \gamma_K(\alpha_s(\lambda)) \right],
\]  
(44)

which exhibits enhancement.

We have set the upper bound of the variable \(p\) to \(p^+/N\), which corresponds to the final condition \(\phi(N, p^+)\) with \(p^+/p = N\), and the lower bound to \(p^+\), such that the initial condition \(\phi(0)\) does not contain \(\ln(1/N)\) because of \(p^+/p = 1\). In this way the \(N\) dependence of \(\phi\) is grouped into the exponential \(\Delta_t\). Contrary to Eq. (27), the lower bound of \(p\) is larger than the upper bound in \(\Delta_t\). Hence, the \(k_T\) resummation gives the evolution of a parton distribution function from the small scale \(1/b\) to the large scale \(p^+\), while the threshold resummation gives the evolution from the large \(p^+\) to the small \(p^+/N\). Since the directions of the double-logarithm evolutions are opposite, their resummation effects are also opposite.

Employing the variable change \(p = (1-z)p^+\) and \(\mu = \sqrt{\lambda}p^+\), Eq. (44) becomes

\[
\Delta_t(N, p^+) = \exp \left[-\int_0^{1-1/N} dz \frac{1}{1-z} \int_1^{1-z^2} \frac{d\lambda}{\lambda} \gamma_K(\alpha_s(\sqrt{\lambda}p^+)) \right].
\]  
(45)

It can be easily justified that the above expression is equivalent to

\[
\Delta_t(N, p^+) = \exp \left[-\int_0^1 dz \frac{z^{N-1} - 1}{1-z} \int_{(1-z)^2}^{1} \frac{d\lambda}{\lambda} \gamma_K(\alpha_s(\sqrt{\lambda}p^+)) \right],
\]  
(46)

up to \(O(1/N)\) corrections, which has appeared in [5].

IV. THRESHOLD RESUMMATION FOR A JET

12
As stated in the Introduction, the scattered quark in DIS carries the momentum $p' = \xi p + q = ((\xi - x)p^+, q^-, 0)$, where $\xi p$ is the initial quark momentum and $q$ the photon momentum. In the threshold region with $x \to \xi \to 1$, $p'$ possesses a large minus component $p'^- = q^-$ but a small invariant $p'^2 = 2(\xi - x)p \cdot q \equiv (\xi - x)s$. This scattered quark produces a jet of particles in this region, to which involved radiative corrections contain both collinear divergences from $l$ parallel to $p'$ and soft divergences from $l \to 0$. Because of the cancellation of soft divergences, these corrections are mainly collinear, and absorbed into a jet function $J$. This jet function $J$ depends on the ratio $p'^2/s = \xi - x$, and more precisely, on the momentum fraction $w = 1 - x/\xi$ in a factorization formula. For convenience, we parametrize the scattered quark momentum as $p' = (wp'^-, p'^-, 0)$, which can be achieved by choosing a frame with $p^+ + q^- = p'^-$. We perform the threshold resummation for $J(w, p'^-)$ in the CS framework. It will be confirmed below that the large scale $p'^-$ appears in the ratio

$$\frac{(n \cdot v')^2}{n^2 p'^-2}$$ (47)

in the case with nonvanishing jet invariant mass, where $v' = (w, 1, 0)$ is a dimensionless vector along $p'$. The above ratio differs from $(p \cdot n)^2/n^2$ in the massless case for the quark distribution function, which has been discussed in the previous section. We shall demonstrate that this difference turns the threshold resummation from enhancement for a parton distribution function to suppression for a jet function. Note that Eq. (47) does not approach $(p' \cdot n)^2/n^2$ as $w \to 0$. This explains why the threshold resummations for a jet function and for a parton distribution function do not coincide with each other in the large $N$ (small $w$ and large $x$) limit.

The chain rule for $J$, corresponding to Eq. (7) for $\phi$, becomes

$$-p'^- \frac{d}{dp'^-}J = -\frac{n^2}{v' \cdot n} v'^\alpha d_{n\alpha} J.$$ (48)

Obviously, the extra minus sign on the left-hand side of the above equation will lead to suppression. Following the same procedures as in Sec. III, we obtain the derivative
\[ -p' - \frac{d}{dp'} J(w, p') = 2 \bar{J}(w, p') , \]  

(49)

where the new function \( \bar{J} \) contains the special vertex

\[ \hat{v}'_\alpha = \frac{n^2 v'_\alpha}{v' \cdot n n \cdot l} . \]  

(50)

The coefficient 2, again, comes from the equality of the two new functions with the special vertex on either side of the final-state cut.

Similarly, the leading regions of the loop momentum \( l \) flowing through the special vertex are soft and hard, in which subdiagrams containing the special vertex are factorized out of \( \bar{J} \) into a soft function \( K \) and a hard function \( G \), respectively. The lowest-order subdiagrams for \( K \) and for \( G \) are basically the same as those in Figs. 1(b) and 1(c). The contribution from Fig. 1(b) is written as

\[ \bar{J}_s(w, p'^- ) = \bar{J}_{sv}(w, p'^- ) + \bar{J}_{sr}(w, p'^- ) , \]  

(51)

\[ \bar{J}_{sv} = \left[ i g^2 C_F \mu^\epsilon \int \frac{d^4-l}{(2\pi)^4-\epsilon} N_{\nu\beta}(l) \frac{\hat{v}'^\nu v'^\nu}{v' \cdot l} \frac{1}{l^2} - \delta K \right] J(w, p'^- ) , \]  

(52)

\[ \bar{J}_{sr} = i g^2 C_F \int \frac{d^4-l}{(2\pi)^4-\epsilon} N_{\nu\beta}(l) \frac{\hat{v}'^\nu v'^\nu}{v' \cdot l} 2\pi i \delta(l^2) J((p' - l)^2 / s, p'^- ) , \]  

(53)

corresponding to virtual and real gluon emissions, respectively. The virtual gluon contribution is the same as in Eq. (30) except for the substitution of the vector \( v' \) for \( v \), and thus the counterterm is also \( \delta K \). The first argument of \( J \) associated with the real gluon emission,

\[ \frac{(p' - l)^2}{s} = w - \frac{l^+}{p'^-} - \frac{w l^-}{p'^-} , \]  

(54)

differs from the corresponding one in Eq. (31).

Inserting the identities

\[ \int_0^w dy \delta(w - y) = 1 , \quad \int_0^w dy \delta \left( w - y - \frac{l^+}{p'^-} - \frac{w l^-}{p'^-} \right) = 1 , \]  

(55)

into Eqs. (52) and (53), respectively, Eq.(51) is reexpressed as
\[ \bar{J}_s(w, p^-) = \int_0^w \frac{dy}{1 - y} K \left( \frac{w - y p^-}{1 - y \mu}, \alpha_s(\mu) \right) J(y, p^-), \tag{56} \]

with

\[ K = i g^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l) \frac{\delta^{\nu\nu'}}{v \cdot l} \left[ \frac{1}{l^2} \delta \left( \frac{w - y}{1 - y} \right) + 2 \pi i \delta(l^2) \delta \left( \frac{w - y}{1 - y} \right) \right] - \delta K \delta \left( \frac{w - y}{1 - y} \right). \tag{57} \]

To obtain the above expression, we have adopted the approximation

\[ \delta \left( \frac{w - y}{1 - y} \right) \approx \delta \left( \frac{w - y}{1 - y} \right), \tag{58} \]

which is appropriate in the considered region with \( w \to 0 \).

To work out the \( y \) integration explicitly, we apply a Mellin transformation from momentum fraction \( (w) \) space to moment \( (N) \) space,

\[ \bar{J}_s(N, p^-) = \int_0^1 dw (1 - w)^{N-1} \bar{J}_s(w, p^-), \]

\[ = \int_0^1 \frac{dy}{1 - y} \int_y^1 dw (1 - w)^{N-1} K \left( \frac{w - y p^-}{1 - y \mu}, \alpha_s(\mu) \right) J(y, p^-), \tag{59} \]

Using the variable change \( w = z(1 - y) + y \), the above expression reduces to

\[ \bar{J}_s(N, p^-) = K(p^-/(N\mu), \alpha_s(\mu)) J(N, p^-), \tag{60} \]

with

\[ K(p^-/(N\mu), \alpha_s(\mu)) = \int_0^1 dz (1 - z)^{N-1} K(zp^-/\mu, \alpha_s(\mu)). \tag{61} \]

Replacing the variable \( z \) by \( 1 - z \), and following the procedures in the Appendix, the result of \( K \) is

\[ K(p^-/(N\mu), \alpha_s(\mu)) = \frac{\alpha_s(\mu)}{\pi} C_F \left( \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} + \ln \frac{p^-}{\nu' \mu} \right), \]

\[ = \frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{p^-}{N\nu' \mu}, \tag{62} \]
with the gauge factor \( \nu'^2 = (n \cdot v')^2/|n^2| \), which confirms that \( J \) depends on \( p'^{-} \) via the ratio \( (n \cdot v')^2/(n^2 p'^{-2}) \) in the massive case.

An alternative Mellin transformation adopted in \cite{5} is

\[
\bar{J}_s(N, p'^{-}) = \int_0^\infty dwe^{-Nw}\bar{J}_s(w, p'^{-})
= \int_0^\infty dy \int_y^\infty dwe^{-Nw}K((w - y)p'^{-}/\mu, \alpha_s(\mu))J(y, p'^{-}).
\]

Employing the variable change \( w = z + y \), the above expression reduces to Eq. (60), but with

\[
K(p'^{-}/(N\mu), \alpha_s(\mu)) = \int_0^\infty dze^{-Nz}K(zp'^{-}/\mu, \alpha_s(\mu)).
\]

It is easy to justify the equivalence between the transformation in Eq. (63) and that in Eq. (59).

The lowest-order contribution to the hard function \( G \) from Fig. 1(c) is given by

\[
\bar{J}_h(w, p'^{-}) = G(\sqrt{wp'^{-}/\mu, \alpha_s(\mu)})J(w, p'^{-}),
\]

with the hard function

\[
G = -ig^2C_F\mu^\epsilon \int \frac{d^4-l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l)\hat{\nu}^\beta l^2 \left[ \gamma'_{\nu} - \frac{2p'_{\nu}}{p'^2 - 2p' \cdot l} \right] - \delta G.
\]

Note the modified eikonal approximation for the quark propagator \((p' - l)^2 \approx p'^2 - 2p' \cdot l\), since we are considering the massive case with \( p'^2 \neq 0 \). As shown in the Appendix, the expression for \( G \) is

\[
G = -\frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{\sqrt{wp'^{-}}}{\nu'\mu}.
\]

As applying the Mellin transformation in Eq. (63) to both sides of Eq. (65), we simply replace the momentum fraction \( w \) in \( G \) by \( 1/N \) because of

\[
\int_0^\infty dwe^{-Nw} \ln w\bar{J}(w, p'^{-}) = \ln \frac{1}{N} \int_0^\infty dwe^{-Nw}\bar{J}(w, p'^{-}),
\]

which is valid up to \( O(1/N) \) corrections. Equation (65) then leads to
\[ \bar{J}_h(N, p') = G(p'/(\sqrt{N}\mu), \alpha_s(\mu))J(N, p') \quad . \] (69)

Using \( \bar{J} = \bar{J}_s + \bar{J}_h \), Eq. (49) becomes

\[ -\frac{d}{dp'}J(N, p') = 2[K(p'/(N\mu), \alpha_s(\mu)) + G(p'/(\sqrt{N}\mu), \alpha_s(\mu))]J(N, p') \quad . \] (70)

The RG analysis gives

\[ K(p'/(N\mu), \alpha_s(\mu)) + G(p'/(\sqrt{N}\mu), \alpha_s(\mu)) = -\frac{1}{2} \int_{p'/p^2} d\mu \frac{\mu^2}{p^2} \gamma_K(\alpha_s(\mu)) \quad . \] (71)

Solving the differential equation (70) under a modification of the derivative similar to Eq. (42), we derive

\[ J(N, p') = \Delta'_t(N, p')J^{(0)} \quad , \] (72)

with the exponential from the threshold resummation,

\[ \Delta'_t(N, p') = \exp \left[ \int_{p'/p^2} d\mu \frac{\mu^2}{p^2} \gamma_K(\alpha_s(\mu)) \right] \quad . \] (73)

We have set the upper bound of the variable \( p \) to \( p'/p^2 \) and the lower bound to \( p' \), which correspond to the final condition \( J(N, p') \) and to the initial condition \( J^{(0)} \), respectively.

At last, adopting the variable changes \( p = (1 - z)p' \) and \( \mu = \sqrt{\lambda p'} \), Eq. (73) is reexpressed as

\[ \Delta'_t(N, p') = \exp \left[ -\int_0^{1-1/N} dz \int_{1-z}^{1-z} \frac{d\lambda}{\lambda^2} \gamma_K(\alpha_s(\sqrt{\lambda p'})) \right] \quad , \] (74)

which is equivalent to (5)

\[ \Delta'_t(N, p') = \exp \left[ \int_0^1 dz \int_{1-z}^{1-z} \frac{d\lambda}{\lambda^2} \gamma_K(\alpha_s(\sqrt{\lambda p'})) \right] \quad , \] (75)

up to corrections suppressed by \( 1/N \).

V. UNIFIED RESUMMATION

In the previous sections we have shown that both the \( k_T \) and threshold resummations can be reproduced in the CS framework by introducing the transverse and longitudinal cutoffs
for soft real gluon emissions, respectively. It is then possible to derive a unification of the two resummations for a parton distribution function in the same formalism by keeping both cutoffs. Consider the quark distribution function defined by Eq. (5). Adopting the similar reasoning, we have Eq. (10), and the factorization of \( \bar{\phi} \) into the convolution of subdiagrams containing the special vertex with \( \phi \): the subdiagrams with soft and hard loop momentum flows are absorbed into the functions \( K \) and \( G \), respectively.

The lowest-order contribution to \( \bar{\phi}_s \) from Fig. 1(b) is written as Eq. (11), with \( \bar{\phi}_{sv}(x, k_T, p^+) \) the same as Eq. (12) and

\[
\bar{\phi}_{sr} = i g^2 C_F \mu^\epsilon \int \frac{d^4-l}{(2\pi)^4} N_{\nu\beta}(l) \frac{\partial^\nu}{v \cdot l} 2\pi i \delta(l^2) \phi(x + l^+ / p^+, |k_T + l_T|, p^+). \tag{76}
\]

Note that both the \( l^+ \) and \( l_T \) dependences of \( \phi \) in the integrand of \( \bar{\phi}_{sr} \) have been retained. To work out the \( l_T \) integration explicitly, we employ the Fourier transformation from \( k_T \) space to \( b \) space as shown in Eq. (15). Following the derivation of Eq. (34), \( \bar{\phi}_s \) becomes

\[
\bar{\phi}_s(x, b, p^+) = \int_x^1 \frac{d\xi}{\xi} K \left( \left( 1 - \frac{x}{\xi} \right) \frac{p^+}{\mu}, 1/(b \mu), \alpha_s(\mu) \right) \phi(\xi, b, p^+), \tag{77}
\]

with

\[
K = i g^2 C_F \mu^\epsilon \int \frac{d^4-l}{(2\pi)^4} N_{\nu\beta}(l) \frac{\partial^\nu}{v \cdot l} \left[ \frac{\delta(1-x/\xi)}{l^2} + 2\pi i \delta(l^2) \delta \left( 1 - \frac{x}{\xi} - \frac{l^+}{p^+} \right) e^{i b \cdot l} \right] \delta K \delta \left( 1 - \frac{x}{\xi} \right). \tag{78}
\]

Further applying the Mellin transformation to Eq. (77), we obtain

\[
\bar{\phi}_s(N, b, p^+) = K(p^+/(N \mu), 1/(b \mu), \alpha_s(\mu)) \phi(N, b, p^+), \tag{79}
\]

with

\[
K(p^+/(N \mu), 1/(b \mu), \alpha_s(\mu)) = \int_0^1 dz z^{N-1} K((1-z)p^+ / \mu, 1/(b \mu), \alpha_s(\mu)). \tag{80}
\]

The convolutions between \( K \) and \( \phi \) in \( l^+ \) and \( l_T \) are then completely simplified into a multiplication under the Mellin and Fourier transformations, respectively.

It is easy to work out the integration in Eq. (80), and the result is
\[ K(p^+/N\mu, 1/(b\mu), \alpha_s(\mu)) = \frac{\alpha_s(\mu)}{\pi} C_F \left[ \ln \frac{1}{b\mu} - K_0 \left( \frac{2\nu p^+ b}{N} \right) \right], \quad (81) \]

\( K_0 \) being the modified Bessel function. We examine the large \( p^+ b \) and \( N \) limits of the above expression. For \( p^+ b \gg N \), we have \( K_0 \to 0 \) and

\[ K \to \frac{\alpha_s}{\pi} C_F \ln \frac{1}{b\mu}, \quad (82) \]

which is the soft function for the \( k_T \) resummation in Eq. (21). For \( N \gg p^+ b \), we have \( K_0 \approx -\ln(\nu p^+ b/N) \) and

\[ K \to \frac{\alpha_s}{\pi} C_F \ln \frac{\nu p^+}{N\mu}, \quad (83) \]

which is the soft function for the threshold resummation in Eq. (38). Hence, Eq. (81) is indeed appropriate for the unification of the \( k_T \) and threshold resummations.

The lowest-order contribution to \( G \) from Fig. 1(c) is the same as in Eq. (39). Using \( \bar{\phi} = \bar{\phi}_s + \bar{\phi}_h \), Eq. (44) becomes

\[ p^+ \frac{d}{dp^+} \phi(N, b, p^+) = 2 \left[ K(p^+/N\mu, 1/(b\mu), \alpha_s(\mu)) + G(p^+/\mu, \alpha_s(\mu)) \right] \phi(N, b, p^+). \quad (84) \]

As solving the RG equations (23), we allow the variable \( \mu \) evolves from the characteristic scale of \( K \) to the scale of \( G \). Equation (81) implies the characteristic scale of order

\[ \frac{1}{b} \exp \left[ -K_0 \left( \frac{p^+ b}{N} \right) \right], \quad (85) \]

for the unified resummation. We discuss the cases for \( p^+ b \gg N \) and for \( N \gg p^+ b \) first, which will help the derivation of the unified resummation. The solution of \( K + G \) is written as

\[ K(p^+/N\mu, 1/(b\mu), \alpha_s(\mu)) + G(p^+/\mu, \alpha_s(\mu)) \]

\[ = -\int_{p^+}^{p^+/N} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)), \quad \text{for } p^+ b \gg N, \]

\[ -\int_{p^+}^{p^+/N} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)), \quad \text{for } N \gg p^+ b, \quad (86) \]
indicating that the distribution function $\phi(N, b, p^+)$ involves $\ln(p^+ b)$ and $\ln(1/N)$ in the $p^+ b \to \infty$ and $N \to \infty$ limits, respectively. This can be easily understood by ignoring the variation of $\gamma_K$, and performing the $\mu$ integration directly.

Inserting Eq. (86) into (84), we obtain the solutions

$$\phi(N, b, p^+) = \exp\left[-2 \int_{\exp[-K_0(p^+ b/N)/b]}^{p^+} \frac{dp}{p} \int_{1/b}^{p} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)) \right] \phi^{(0)} ,$$

(87)

$$\phi(N, b, p^+) = \exp\left[-2 \int_{\exp[-K_0(p^+ b/N)/b]}^{p^+} \frac{dp}{p} \int_{p^+}^{p} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)) \right] \phi^{(0)} ,$$

(88)

for $p^+ b \gg N$ and $N \gg p^+ b$, respectively. We have chosen the characteristic scale in Eq. (85) as the lower bound of the variable $p$. To unify the above expressions, we replace the lower bounds of $\mu$ by

$$\frac{1}{b} \exp[-K_0(p^+ b)] ,$$

(89)

that is motivated by Eq. (85). At last, the unified resummation is given by

$$\phi(N, b, p^+) = \Delta_u(N, b, p^+) \phi^{(0)} ,$$

(90)

with the exponential

$$\Delta_u(N, b, p^+) = \exp\left[-2 \int_{\exp[-K_0(p^+ b/N)/b]}^{p^+} \frac{dp}{p} \int_{\exp[-K_0(p^+ b/N)/b]}^{p} \frac{d\mu}{\mu} \gamma_K(\alpha_s(\mu)) \right] ,$$

(91)

which is appropriate for arbitrary $p^+ b$ and $N$. It is easy to justify that Eq. (91) approaches the $k_T$ resummation in Eq. (27) as $b \to \infty$, and approaches the threshold resummation in Eq. (44) as $b \to 0$.

VI. CONCLUSION

In this paper we have explored the relation between the $k_T$ and threshold resummations, and discussed the difference between their applications to a parton distribution function and to a jet function in the CS framework. It has been understood that the summations of the logarithms $\ln^2(p^+ b)$ and $\ln^2(1/N)$ are determined by the soft approximations for real
gluon emission in Eqs. (16) and (17), respectively. Suppression from the $k_T$ resummation, as indicated by $\Delta_k$ in Eq. (27), and enhancement from the threshold resummation, as indicated by $\Delta_t$ in Eq. (46), for a parton distribution function is the consequence of the opposite directions of their double-logarithm evolutions. While the switch of the threshold resummation from enhancement for a parton distribution function to suppression for a jet function is attributed to the change from a massless case to a massive case. The $k_T$ resummation for a jet function remains suppressive as shown in Eq. (4), since the jet momentum and the parton momentum are in the same form. Hence, we do not repeat the derivation here.

We have also unified the $k_T$ and threshold resummations by keeping simultaneously the longitudinal and transverse loop momenta in the parton distribution function $\phi(x + l^+/p^+, |k_T + l_T|, p^+)$ as shown in Eq. (76). Comparing the result in Eq. (91) with the $k_T$ resummation in Eq. (27), which gives suppression, and with the threshold resummation in Eq. (44), which gives enhancement, the unified resummation exhibits both behaviors: it is suppression in the large $b$ region ($p^+ b \gg N$), and turns into enhancement in the small $b$ region ($N \gg p^+ b$). That is, Eq. (91) displays the opposite effects of the $k_T$ and threshold resummations at different $b$. The behavior of the unified resummation can be understood as follows. For an intermediate $x$, virtual and real soft gluon corrections cancel exactly in the small $b$ region, since they have almost equal phase space. Hence, there are only single collinear logarithms, namely, no double logarithms. In this case the Sudakov exponential approaches unity as $b < 1/p^+$, indicating the soft cancellation stated above. However, at threshold ($x \to 1$), real gluon emissions still do not have sufficient phase space even as $b \to 0$, and soft virtual corrections are not cancelled exactly. In this case the double logarithms $\ln^2(1/N)$ persist and become dominant. Sudakov suppression then transits into enhancement, instead of unity, as $b$ decreases.

The unified resummation obtained in this work has important applications to studies of some QCD processes, such as dijet production [13]. In this experiment one jet (the trigger jet) is required to be in the central rapidity region, while the other jet (the probe jet) has any rapidity up to 3.0. With the large rapidity, dynamics of a hadron at higher $x$ values is
probed, so that the threshold resummation is necessary. On the other hand, the differential dijet cross section versus the transverse energy of the trigger jet is measured, which further demands the inclusion of the $k_T$ resummation. This subject will be discussed elsewhere.

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APPENDIX

In this Appendix we present the details of the loop calculations involved in this paper.

(1) Equation (38)

The virtual gluon contribution to the soft function $K$ associated with the threshold resummation for the quark distribution function is given by

$$K_s = ig^2 C_F \mu^\epsilon \int \frac{d^4l}{(2\pi)^4} N_{\nu \beta}(l) \frac{\hat{\theta}^\beta v^\nu}{v \cdot l} \frac{1}{l^2 - a^2} - \delta K \, ,$$

(92)

where the infrared regulator $a$ is introduced for convenience, and will approach zero at last. The first term $g_{\nu \beta}$ in $N_{\nu \beta}$ gives a vanishing contribution because of $v^2 = 0$. The contribution from the second term $-n_{\nu}l_{\beta}/n \cdot l$ cancels that from the fourth term $n^2l_{\nu}l_{\beta}/(n \cdot l)^2$. Hence, we concentrate only on the third term $-n_{\beta}l_{\nu}/n \cdot l$, which leads to

$$K_s = -ig^2 C_F \mu^\epsilon \int \frac{d^4l}{(2\pi)^4} N_{\nu \beta}(l) \frac{1}{n \cdot l^2 - a^2} - \delta K \, .$$

(93)

The above integral has been evaluated in [3], and thus we quote the result directly:

$$K_s = \frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{a}{\mu} \, ,$$

(94)

with the counterterm $\delta K = -\alpha_s C_F/(\pi \epsilon)$.

The soft real gluon emission contributes

$$K_r = ig^2 C_F \int \frac{d^4l}{(2\pi)^4} N_{\nu \beta}(l) \frac{\hat{\theta}^\beta v^\nu}{v \cdot l} 2\pi i \delta(l^2 - a^2) \delta(1 - z - l^+/p^+) \, ,$$

$$= g^2 C_F \int \frac{d^4l}{(2\pi)^4} \frac{n^2}{(n \cdot l)^2} 2\pi i \delta(l^2 - a^2) \delta(1 - z - l^+/p^+) \, ,$$

(95)
with \( z = x/\xi \), where \( \epsilon \) has been set to zero, since the integral is ultraviolet finite. Performing the integrations over \( l^- \) and \( l_T \), Eq. (95) reduces to

\[
K_r = \frac{\alpha_s(\mu)}{\pi} C_F \frac{1 - z}{(1 - z)^2 - a^2/(2p^\nu \nu)^2},
\]

(96)

with \( \nu^2 = (v \cdot n)^2/|n|^2 \). Combined with Eq. (94), the function \( K \) in moment space is, according to Eq. (37), written as

\[
K_{p^+/(N\mu), p^+} = \frac{\alpha_s(\mu)}{\pi} C_F \left[ \int_0^1 dz \frac{z^{N-1}(1 - z)}{(1 - z)^2 - a^2/(p^+ \nu^2)^2} + \ln \frac{a}{\mu} \right].
\]

(97)

A simple identity \( z^{N-1} = (z^{N-1} - 1) + 1 \) isolates the logarithmic term \( \ln a \) in the \( z \) integral, and the above expression becomes Eq. (38). As expected, the infrared regulator \( a \) has cancelled between the virtual and real gluon contributions.

(2) Equation (62)

We need to evaluate only the real gluon part,

\[
K_r = ig^2 C_F \int \frac{d^4l}{(2\pi)^4} N_{\nu\beta}(l) \tilde{g}^\beta \tilde{v}^\nu l \cdot l 2\pi i \delta(l^2 - a^2) \delta(z - l^+ / p^+ - w l^- / p^-),
\]

(98)

with \( z = (w - y)/(1 - y) \). The first term \( g_{\nu\beta} \) in \( N_{\nu\beta} \) gives a negligible contribution because of \( \nu^2 = 2w \to 0 \). The contribution from the second term \( -n_{\nu} l_{\beta} / n \cdot l \) cancels that from the fourth term \( n^2 l_{\nu} l_{\beta} / (n \cdot l)^2 \). Hence, we consider the third term \( -n_{\beta} l_{\nu} / n \cdot l \), which leads to

\[
K_r = g^2 C_F \int \frac{d^4l}{(2\pi)^3} \frac{n^2}{(n \cdot l)^2} \delta(l^2 - a^2) \delta(z - l^+ / p^+ - w l^- / p^-). \]

(99)

Performing the integration over \( l_T \) first, we obtain

\[
K_r = g^2 C_F \int \frac{dl^+ dl^-}{8\pi^2} \frac{n^2}{(n \cdot l)^2} \delta(z - l^+ / p^+ - w l^- / p^-),
\]

(100)

with the constraint \( 2l^+ l^- = l_T^2 + a^2 \geq a^2 \). Here we have assumed that the gauge vector \( n \) possesses only the plus and minus components, \( i.e., n = (n^+, n^-) \) for convenience. Using the relation \( l^+ = z p^- - w l^- \) from the \( \delta \)-function, the constraint determines the bounds of \( l^- \) in the integral

\[
K_r = \frac{\alpha_s(\mu)}{2\pi} C_F \int_{l^-_{\text{min}}}^{l^-_{\text{max}}} dl^- \frac{n^2 p^-}{[n^+ l^- + n^-(z p^- - w l^-)]^2},
\]

(101)
with

\[ l_{\text{min}} = \frac{zp' - \sqrt{z^2p'^2 - 2wa^2}}{2w}, \]
\[ l_{\text{max}} = \frac{zp' + \sqrt{z^2p'^2 - 2wa^2}}{2w}. \] (102)

The \( l^- \) integration gives

\[ K_r = \frac{\alpha_s(\mu)}{\pi} C_F \frac{wn^2}{n^+ - wn^-} \times \left\{ \frac{1}{(n^+ + wn^-)z - (n^+ - wn^-)\sqrt{z^2 - 2wa^2/p'^2}} - \frac{1}{(n^+ + wn^-)z + (n^+ - wn^-)\sqrt{z^2 - 2wa^2/p'^2}} \right\}. \] (103)

After a simple algebra, the above expression is simplified into

\[ K_r = \frac{\alpha_s(\mu)}{\pi} C_F \frac{\sqrt{z^2 - 2wa^2/p'^2}}{z^2 + (n^+ - wn^-)^2a^2/(n^2p'^2)}. \] (104)

Since the infrared regulator \( a \) will approach zero at last, we drop \( a \) in the numerator, which does not affect the infrared structure of \( K_r \). According to the definition

\[ \frac{(n^+ - wn^-)^2}{n^2} = \frac{(n \cdot v')^2}{n^2} - \nu'^2 \approx \frac{(n \cdot v')^2}{n^2} \equiv -\nu'^2 \] (105)

because of \( n^2 < 0 \), Eq. (104) becomes

\[ K_r = \frac{\alpha_s(\mu)}{\pi} C_F \frac{z}{z^2 - \nu'^2a^2/p'^2}. \] (106)

Note that we should neglect the variable \( w \) in \( \nu' \) in order to decouple the integrations over \( y \) and over \( w \) completely in Eq. (59).

Combined with the virtual gluon contribution in Eq. (94), we derive \( K \) in moment space,

\[ K_r(p'^-(N\mu), p'^-') = \frac{\alpha_s(\mu)}{\pi} C_F \left[ \int_0^1 dz \frac{z^{N-1}(1-z)}{(1-z)^2 - \nu'^2a^2/p'^2} + \ln \frac{a}{\mu} \right]. \] (107)

where the variable change \( z \to 1 - z \) has been inserted. A similar manipulation isolates \( \ln a \) in the \( z \) integral, and the above expression leads to Eq. (62).

(3) Equation (67)
We evaluate first the soft subtraction term in Eq. (66),
\[ G_s = ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}} N_{\nu \beta}(l) \frac{i^{\beta \alpha}}{l^2} \frac{2p^\nu}{p^2 - 2p' \cdot l} - \delta G , \]
\[ = -ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon} (n \cdot l)^2 l^2 (p^2 - 2p' \cdot l)} - \delta G . \]  
(108)

We assume that the coefficient \( n^+ \) of \( l^- \) in the denominator \((n \cdot l)^2 = (n^+ l^- + n^- l^+)^2\) is negative. Since the coefficient \(-2wp^-\) of \( l^- \) in the denominator \( p^2 - 2p' \cdot l = p^2 - 2wp^- l^- - 2p' l^+\) is also negative, a pole from the denominator \( l^2 = 2l^+ l^- - l_T^2\) exists in the \( l^- \) plane only for \( l^+ > 0 \). Performing the contour integration over \( l^- \) around the pole \( l^- = l_T^2/(2l^+)\), Eq. (108) gives
\[ G_s = 2g^2 C_F n^2 \mu^\epsilon \int \frac{d^{2-\epsilon}l_T}{(2\pi)^{3-\epsilon}} \int_0^\infty d l^+ \frac{l^+(2l^+ + w l_T^2)}{(2n - l^+ + w l_T^2)^2 (2l^+ - 2wp^- l^- + w l_T^2)} - \delta G . \]  
(109)

The integration over \( l^+ \) leads to
\[ G_s = g^2 C_F n^2 \mu^\epsilon \int \frac{d^{2-\epsilon}l_T}{(2\pi)^{3-\epsilon}} \frac{(n^+ - wn^-)^2}{l_T^2 (n^+ - wn^-)^2 + n^2 w^2 p'^{-2}} - \delta G . \]  
(110)

At last, performing the integration over \( l_T\), we derive
\[ G_s = -\frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{wp'^-}{\nu' \mu} , \]  
(111)

with the counterterm \( \delta G = \alpha_s C_F / (\pi \epsilon) \).

Next we evaluate the first term in Eq. (66),
\[ G_h = ig^2 C_F \int \frac{n^2}{\nu' \cdot n} \int \frac{d^4l}{(2\pi)^4 (n \cdot l)^2 l^2 (p' - l)^2} \, h . \]  
(112)

Note that we have set \( \epsilon \) to zero, because the integral is ultraviolet finite. It is easy to observe that poles of \( l^+ \) exist in the denominator \( l^2 \) for \( l^- < 0 \) and in \((p' - l)^2\) for \( l^- < p'^- \). Following the similar procedures, we can work out Eq. (112), though the calculation is much more complicated. The final expression of \( G_h \) is
\[ G_h = \frac{\alpha_s(\mu)}{2\pi} C_F \ln w , \]  
(113)

where constants of order unity have been dropped. Combining Eqs. (111) and (113), we arrive at Eq. (67).
A point that needs to be mentioned is the treatment of the gamma matrices appearing in the numerator of Eq. (112). We express $\gamma^+$ and $\gamma^-$ in terms of $\not p'$ and $\not n$:

$$
\gamma^+ = \frac{n^+ \not p' - wp^- \not n}{p'^-(n^+ - wn^-)}, \quad \gamma^- = \frac{p'^- \not n - n^- \not p'}{p'^-(n^+ - wn^-)}.
$$

(114)

Since the jet function contains the matrix structure proportional to $\not p'$, the products $\gamma^+ \not n$ and $\gamma^- \not n$ are written as

$$
\gamma^+ \not n = \frac{2n^+ p' \cdot n - wp' n^2}{p'^-(n^+ - wn^-)}, \quad \gamma^- \not n = \frac{p'^- n^2 - 2n^- p' \cdot n}{p'^-(n^+ - wn^-)},
$$

(115)

where we have used the relation $\not p' \not n = 2p' \cdot n - \not n \not p'$, and neglected the second term, which leads to a negligible invariant $p'^2 = wp'^-2 \approx 0$. 


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Figure Captions

Fig. 1. (a) The derivative $p^+ d\phi / dp^+$ in the axial gauge. (b) The $O(\alpha_s)$ function $K$. (c) The $O(\alpha_s)$ function $G$. 
\[ p^+ \frac{d}{dp^+} = 2 \]

(a)

(b)

(c)

FIG. 1