Coherence length of neutron superfluids

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The coherence length of superfluid neutron matter is calculated from the microscopic BCS wavefunction of a Cooper pair in momentum space making use of the Bonn meson-exchange potential. We find that the coherence length is proportional to the Fermi momentum-to-pairing gap ratio, in good agreement with simple estimates used in the literature, and we establish the appropriate fitting constants using our numerical data. Our calculations can be applied to the problem of inhomogeneous superfluidity of hadronic matter in the crust of a neutron star.

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Calculations based on BCS theory with phenomenological nucleon-nucleon (NN) forces indicate that neutron matter is superfluid in a wide region of densities and temperatures. In particular, at the densities corresponding to the interior of a neutron star crust ($4 \times 10^{11}$ g cm\(^{-3}\) $< \rho < 10^{14}$ g cm\(^{-3}\)) neutrons couple in the singlet isotropic channel \(^1\text{S}_0\) while at larger densities they pair in the \(^3\text{P}_2\) state. Although many investigations have been devoted to the superfluidity and superconductivity of neutrons, \(\beta\)-stable and nuclear matter, little attention has been paid to possible effects of inhomogeneities in hadronic superfluids or superconductors. In fact the interior of neutron stars, the subject of most of such calculations, is often referred to as the only existing example in nature of infinite superfluid neutron or \(\beta\)-stable matter, since in an atomic nucleus the wavefunctions of the Cooper pairs are limited in extension by the potential well. On the other hand, superfluidity in a neutron star crust represents a case intermediate between the nucleus and the idealized infinite system, since superfluid neutrons in the inner crust occupy a region where a lattice of nuclei creates strong inhomogeneities in the medium. From the point of view of astrophysical observations, probably most of the visible effects of the presence of superfluids in a neutron star are due to phenomena in the crust.

An important length scale of the neutron superfluid is the coherence length. From a microscopic point of view the coherence length represents the squared mean distance of two paired particles (a Cooper pair of neutrons) on top of the Fermi surface. The magnitude of this quantity affects several of the physical properties of a neutron star crust. First of all, neutrons paired in a singlet state form quantized vortices induced by the rotational state of the star. These can pin to the nuclei present in the crust, possibly leading to the observed sudden release of angular momentum known as pulsar glitches. The magnitude of the pinning force depends on the size of the vortex cores, which is equal to the coherence length of the neutron superfluid. A second question is how properties of the neutron superfluid change due to the inhomogeneous environment of a neutron star crust, a problem related to the average thermodynamical property of neutron matter. In the inner crust, depending on density, nuclei of different shapes and sizes are present. At a density of $\sim 10^{14}$ g cm\(^{-3}\), spherical nuclei cease to be energetically favored and are replaced first by cylindrical nuclei, then slabs to end up with holes, where the roles of protons and neutrons are exchanged, see Refs. for further details. Only at higher densities, corresponding to what is called the core of the star, do nuclei merge into the uniform medium. The fact that neutron superfluidity in neutron star matter is actually a problem of inhomogeneous superfluidity in hadronic matter has been noticed quite recently. According to Anderson’s theorem the electron density of states in a superconductor is changed very little from a pure metal to an alloy of similar chemical properties. The physical situation we are examining here is quite different, since both the average density of states and the effective neutron-neutron matrix elements are changed compared to the uniform case when one considers the presence of nuclei.

The typical dimension of nuclei in the inner crust of a neutron star is $R_N \approx 4 - 6$ fm. This number is, in an appropriate range of densities, comparable to the coherence length $\xi$ as estimated from existing BCS calculations. If Anderson’s theorem holds also for neutron star matter, (limit where $R_N \ll \xi$) there will be no appreciable variations in the superfluid properties induced by the nuclei. On the other hand, if $R_N \gg \xi$ the superfluid will change its properties locally. This limit has been investigated in a recent series of papers where it was found that some thermodynamical properties like e.g. the neutron specific heat may change by a very large amount. Unfortunately, the situation is complicated by the fact that $R_N$ and $\xi$ are of the same order of magnitude.

Clearly, the coherence length represents a critical parameter by which one can establish the behavior of an inhomogeneous superfluid. It sets the scale for the possible spatial variation of the pairing properties of the system.
and thus plays a role if some inhomogeneities are present in the system at a length scale comparable to it.

A simple estimate of the coherence length is obtained by assuming constant matrix elements between particle states within a shell centered at the Fermi momentum and zero outside (see for example Ref. [3]). This gives

\[ \xi \approx K \frac{k_F}{\Delta_F} \]  

(1)

where \( k_F \) is the Fermi momentum and \( \Delta_F \) is the pairing gap at the Fermi momentum. The value of the constant \( K \) depends on the approximations used, and different values have been reported in the literature. However, the estimate [3] does not consider thoroughly the matrix elements of the particle-particle interactions, and in principle can be too rough if one wants to calculate the coherence length at all values of the density. In view of the important astrophysical considerations discussed above, it is desirable to calculate the coherence length from a microscopic study of the wavefunction of a Cooper pair.

The aim of this Brief Report is thus twofold: firstly we check the validity of the simple estimate [3] and secondly we determine the best value of \( K \) and check the dependence of that formula on the neutron interactions with the same potential, see Ref. [3] for details. The chemical potential \( \mu \) was set equal to the Fermi energy \( \epsilon(k_F) \).

We calculate the pairing gaps and wavefunctions in momentum space as functions of the momentum from the BCS gap equation [1]

\[ \Delta(p) = -\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \tilde{\nu}(p,k) \frac{\Delta(k)}{E(k)} \]  

(4)

for different values of the Fermi momentum. Here \( \tilde{\nu}(p,k) \) are the matrix elements of the \( ^1 S_0 \) NN interaction, obtained from the Bonn A potential, following the procedure outlined in Ref. [1]. The single-particle energies \( \epsilon(k) \) were obtained from Brueckner-Hartree-Fock calculations with the same potential, see Ref. [1] for details. The chemical potential \( \mu \) was set equal to the Fermi energy \( \epsilon(k_F) \).

Notice that for our purposes the pairing gap has to be calculated at all wavenumbers and not only at the Fermi surface, as in the simple formula [3]. The wavefunctions \( \chi(k) \) in momentum space for four different values of the Fermi momentum \( k_F = 0.1, 0.4, 0.8 \) and \( 1.2 \text{ fm}^{-1} \) \( k_F \) are shown in Fig. 1. The square of the derivatives of the respective wavefunctions are displayed in Fig. 2. It is seen that the spread of the wavefunction in momentum space \( < k^2 > \) varies considerably according to the value of the Fermi wavenumber \( k_F \). The distribution in momentum space is particularly wide for intermediate values of \( k_F \) and, since we have through the uncertainty relation that \( < k^2 > \approx \xi^2 \approx 1/ < r^2 > \), a smaller value of the coherence length for these values is expected. Due to the limited, but significant variation of the the pairing gap \( \Delta(k) \) within a single peak (each peak occurs at the Fermi momentum), the wavefunction is distorted with respect to a simple gaussian and therefore deviations from the simple relation [3] for a gaussian distribution should be expected. This becomes evident by looking at the squared derivative of the wavefunction \( (\partial^2 \chi(k)/\partial k^2) \) which shows two unequal peaks.

In Fig. 3 we present the squared wavefunctions \( r^2|\phi(r)|^2 \) for the same values of the Fermi momentum as in Figs. 1 and 2. The wavefunctions oscillate with a wavelength of the order \( k_F^{-1} \) and extend for quite long distances compared to the range of nuclear forces (a few fermis at most). Not surprisingly, this behavior is due to the relatively small value of the energy of the pair compared to the Fermi energy (a comparison can be made with another weakly bound state with a very spread wavefunction, the deuteron [3]).

In Fig. 4 we show our microscopic calculations of the coherence length for pure neutron matter performed using Eqs. [1],[3]. The microscopically calculated points, given by the solid line, can be fitted by a function linear in the parameter \( k_F/\Delta_F \), where \( \Delta_F \) is the pairing gap at the Fermi surface. We find as a best fit, see Ref. [1] for further details on the form for \( \xi \),

\[ \xi = \theta \frac{\hbar^2}{2m} \frac{k_F}{\Delta_F} \]  

(5)
where $x = m/m^*$ is the ratio between the bare and the effective mass, and the dimensionless parameter $\theta$ is found by a direct fit to our data to be $\theta = 0.814$ (for $x = 1$) and $\theta = 0.836$ (with the correct density dependent value of $x$ from the microscopic calculation). The choice $x = 1$ is commonly used in the literature.

The simple linear behavior of the momentum-to gap ratio is in very good agreement with Eq. (4) and shows that a Cooper pair represents a packet having a nearly well-defined relative momentum. As seen from Fig. 4, the use of $x = 1$ (effective mass equal to bare mass) does not change the picture very much, as is evident from the fact that at e.g., the relatively high Fermi momentum of $k_F = 1.0$ fm$^{-1}$ the effective mass differs from the bare one by less than 4%. However, deviations from Eq. (4) are found, as can be seen in Fig. 4, especially at higher densities. In Fig. 4 we have also plotted the values of the coherence length from an approximation often used in neutron star studies, namely the choice of a coefficient $\theta = 2/\pi = 0.637$ in Eq. (5). Our values were on the average more than 20% larger than the results obtained with this choice. It is therefore interesting to see that our microscopic calculation of the gap and single-particle properties through a complicated many-body scheme, yields a qualitative similar result as that of Eq. (1), derived originally in a solid state context.

A relevant point is whether the value of the coefficient $\theta$ depends on the interaction chosen. Besides, it might be that the good agreement with Eq. (4) gets worse with other interactions. To partially answer these questions, we repeated the same calculations making use of an effective force, the Gogny force, which pairing properties have been investigated in several works. The Gogny force is an effective interaction fitted to reproduce various nuclear data, and contains therefore effects of nucleon-nucleon correlations not included in a bare nucleon-nucleon interaction. It is therefore both qualitatively and quantitatively different from a bare NN interaction like the Bonn potential. We find that with the D1 parameterization of the Gogny force a very good fit can be made for Eq. (3) with a coefficient $\theta = 0.815$, very close to the value found with Bonn A. This near-equality of the coefficient $\theta$ for the two different interactions is not predictable on the basis of simple arguments. However, it ought to be stressed that we have not included so-called polarization effects in the calculations of the pairing gaps of Ref. [1]. Such effects are expected to reduce by at least a factor of two the value of the pairing gap, see Ref. [11]. How these many-body effects will change the coherence length remains to be investigated. However, using the weak-coupling approach of Eq. (3) for the coherence length, one sees that $\xi$ may be twice as large.

To conclude, we calculated microscopically the coherence length of superfluid neutron matter. Eq. (4) with the nearly interaction-independent coefficient $\theta = 0.83$ represents the main result of our calculations. The data reported in this brief report may be useful in quantitative calculations of the superfluid properties of neutron star crusts. Moreover, as can be seen from Fig. 4, the coherence length is for several Fermi momenta larger than the typical size of known nuclei. This poses several constraints on the use of local density approximations in the study of neutron star crust properties.

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[1] O. Elgarøy, L. Engvik, M. Hjorth-Jensen, and E. Osnes, Nucl. Phys. A604, 466 (1996); Nucl. Phys. A607, 425 (1996).
[2] C. J. Pethick and D. G. Ravenhall, Annu. Rev. Nucl. Part. Sci. 45, 429 (1995).
[3] R. A. Broglia, F. V. De Blasio, G. Lazzari, M. C. Lazzari and P. M. Pizzochero, Phys. Rev. D 50, 4781 (1994).
[4] C. P. Lorenz, D. G. Ravenhall and C. J. Pethick, Phys. Rev. Lett. 70, 379 (1993); K. Oyamatsu, Nucl. Phys. A561, 431 (1993).
[5] P. E. de Gennes, “Superconductivity in metals and alloys”, Benjamin, New York, (1966).
[6] R. Machleidt, Adv. Nucl. Phys. 19, 185 (1989).
[7] M. Hjorth-Jensen, T. T. S. Kuo, and E. Osnes, Phys. Rep. 261, 125 (1995).
[8] V. F. Weisskopf, Contemp. Phys. 22, 375 (1981).
[9] M. A. Alpar, P. W. Anderson, D. Pines, and J. Shaham, Astrophys. J. 278, 791 (1984).
[10] J. Dechargé and D. Gogny, Phys. Rev. C 21, 1568 (1980); H. Kucharek, P. Ring, P. Schuck, R. Bengtsson, and M. Girod, Phys. Lett. B216, 249 (1989).
[11] H.-J. Schulze, J. Cugnon, A. Lejeune, M. Baldo and U. Lombardo, Phys. Lett. B375, 1 (1996).
\[ k_F = 0.1 \text{ fm}^{-1} \quad --- \\
\quad k_F = 0.4 \text{ fm}^{-1} \quad --- \\
\quad k_F = 0.8 \text{ fm}^{-1} \quad ----- \\
\quad k_F = 1.2 \text{ fm}^{-1} \quad ----- \\
\]

FIG. 1. Wavefunctions in momentum space, \( \chi(k) \) for four different values of the Fermi momentum, \( k_F = 0.1 \text{ fm}^{-1}, k_F = 0.4 \text{ fm}^{-1}, k_F = 0.8 \text{ fm}^{-1} \) and \( k_F = 1.2 \text{ fm}^{-1} \). The wavefunctions peak at the corresponding value of the Fermi momentum.

\[ |\partial \chi(k)/\partial k|^2 \]

FIG. 2. Plot of \( |\partial \chi(k)/\partial k|^2 \) proportional to the spread in space of the Cooper pair. Legend as in Fig. 1.
FIG. 3. $r^2|\phi(r)|^2$ as function of $r$.

FIG. 4. Coherence length for neutron matter calculated numerically with the Bonn potential (solid line) using Eqs. (2)-(4). The line with long dashes are for a $\xi$ which fits the numerical points with $\theta = 0.814$ while for the short-dashed line $\theta = 0.836$, see text for further details. The dotted line represents $\xi$ given by an approximation frequently used in the vortex pinning literature [9].