GEOMETRIC SINGULAR PERTURBATION ANALYSIS OF DEGASPERIS-PROCESI EQUATION WITH DISTRIBUTED DELAY

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Abstract. In this paper we consider the Degasperis-Procesi equation, which is an approximation to the incompressible Euler equation in shallow water regime. First we provide the existence of solitary wave solutions for the original DP equation and the general theory of geometric singular perturbation. Then we prove the existence of solitary wave solutions for the equation with a special local delay convolution kernel and a special nonlocal delay convolution kernel by using the geometric singular perturbation theory and invariant manifold theory. According to the relationship between solitary wave and homoclinic orbit, the Degasperis-Procesi equation is transformed into the slow-fast system by using the traveling wave transformation. It is proved that the perturbed equation also has a homoclinic orbit, which corresponds to a solitary wave solution of the delayed Degasperis-Procesi equation.

1. Introduction. The Degasperis-Procesi (DP) equation
\[ u_t - u_{xxt} + 2ku_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R}, \quad t > 0, \eqno{(1)} \]
where \( k > 0 \) as a parameter related to the critical shallow water speed, was originally derived by Degasperis and Procesi \[6\] using the method of asymptotic integrability up to the third order as one of three integrable equation in the family of third-order dispersive PDE conservation laws of the form
\[ u_t - \alpha^2 u_{xxt} + c_0 u_x + \gamma u_{xxx} = c_1 uu_x + c_2 u_x u_{xx} + c_3 uu_{xxx}. \]
After rescaling and applying a Galilean transformation, the other two integrable equations in the family are the Korteweg-de Vries (KdV) equation \[13\],
\[ u_t + u_{xxx} + uu_x = 0, \]
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and the Camassa-Holm (CH) equation [5]

\[ u_t - u_{xxt} + 2k u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \]

The DP equation is also an approximation to the irrotational two dimensional Euler equation and its asymptotic accuracy is the same as that of the CH shallow-water equation [4], where the solution \( u(t, x) \) of (1) represents the horizontal component of the fluid velocity at height \( z_0 = \sqrt{23/36} \) after the re-scaling within \( 0 \leq z_0 \leq 1 \) at time \( t \) in the spatial \( x \)-direction.

The DP equation has an apparent similarity to the CH equation. Both of them are important model equations of shallow water waves with breaking phenomenon (i.e. the solution remains bounded while its slope becomes unbounded in finite time) [1, 2, 10, 14, 15]. However, there is much less known about qualitative properties and long-time dynamics of the DP equation than to the CH equation, due to major structural differences between the DP equation and the CH equation. For instance, the isospectral problems in the Lax pair for DP equation (1) is the third-order equation [7]

\[ \psi_x - \psi_{xxx} - \lambda m \psi = 0, \]

while the isospectral problems for CH equation has the second-order equation [5]

\[ \psi_{xx} - \frac{1}{4} \psi - \lambda m \psi = 0, \]

where \( m = u - u_{xx} \) in both cases. Furthermore, the CH equation is a re-expression of geodesic flow on the diffeomorphism group [3] or on the Bott-Virasoro group [16], while no such geometric derivation of the DP equation is available. Moreover and more interestingly, the DP equation admits shock wave while the CH does not, neither do their cubic counterparts, i.e. the Novikov equation or the modified CH equation (the FORQ equation).

Recently, Du, Li and Li [8] discussed the existence of solitary wave solution for the delayed CH equation

\[ u_t - u_{xxt} + 2k u_x + 3(f * u) u_x + \tau u_{xx} = 2u_x u_{xx} + uu_{xxx}, \]

by using twice the geometric singular perturbation theory. Here \( f * u \) is the spatial-temporal convolution representing distributed delay and \( \tau \) is a small constant. In this paper, we are interested in the existence of solitary wave solution for the delayed DP equation

\[ u_t - u_{xxt} + 2k u_x + 4(f * u) u_x + \tau u_{xx} = 3u_x u_{xx} + uu_{xxx} \]

where the convolution \( f * u \) is defined by

\[ f * u = \int_{-\infty}^{t} f(t - s)u(x, s)ds, \]

and

\[ f * u = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} f(x - y, t - s)u(y, s)dyds. \]

The first one is called local distributed delay and the second one is called nonlocal distributed delay.

Our main tool in proving existence of perturbed solitary wave is the geometric singular perturbation theory and Melnikov method. As for the existence of solitary wave, one big obstacle results from the introduction of delay which breaks the integrability. This is narrowly solved by a double singular perturbation reduction.
according to the special structure of the DP equation. The other obstacle lies on the totally different conservation laws of the DP equation from the CH equation, which makes the calculation and analysis of separation function for the DP equation much more difficult.

In Section 2, we provide the existence of solitary wave solutions for the original DP equation and the general theory of geometric singular perturbation. In Section 3, We study the systems with local distributed delay and nonlocal delay, and prove the existence of unique persistent solitary wave.

2. Preliminaries. In this section, we provide a short discussion on the original Degasperis-Procesi equation and preliminaries including geometric singular perturbation theory.

2.1. The original DP equation. Without delay and perturbation, (3) reduces to the original DP equation:

$$u_t - u_{xxx} + 2ku_x + 4uu_x = 3u_xu_{xx} + uu_{xxx}. \quad (4)$$

Definition 2.1. A traveling wave solution $u_{x,t} = \phi(x - ct) =: \phi(\xi)$ of the equation (4) is called a solitary wave solution if $\lim_{\xi \to \pm \infty} \phi(\xi) = 0$. Here $c > 0$ is the wave speed.

The corresponding solitary wave equation is

$$c\phi''' + (2k - c)\phi' + 4\phi\phi' - 3\phi'\phi'' - \phi\phi''' = 0, \quad (5)$$

where $' = \frac{d}{d\xi}$, and $\phi$ is as in Definition 2.1. Integrating the above equation with respect to $\xi$ yields the travelling wave equation

$$(c - \phi)\phi'' + (2k - c)\phi + 2\phi^2 - (\phi')^2 = 0, \quad (6)$$

taking into consideration that $\phi(\xi), \phi'(\xi), \phi''(\xi) \to 0$, as $\xi \to \pm \infty$. The above equation is equivalent to the following system of first-order equations

$$\begin{cases} 
\phi' = \psi, \\
(c - \phi)\psi' = (c - 2k)\phi - 2\phi^2 + \psi^2. 
\end{cases} \quad (7)$$

It has the first integral

$$H(\phi, \psi) = \frac{1}{2}(\phi - c)^2\psi^2 + (c - \frac{2k}{3})\phi^3 - \frac{1}{2}\phi^4 - \frac{c(c - 2k)}{2}\phi^2,$$

constructed by multiplying the integral factor $c - \phi$, from which and analysis of the phase space, we get the following

**Theorem 2.2.** If $c > 2k > 0$, then in the $(\phi, \psi)$ phase plane, system (7) has a homoclinic orbit to the critical point $(0, 0)$. This connection is confined to $0 < \phi \leq \phi^*$, where $\phi^* = (c - \frac{2}{3}k) - \sqrt{\frac{2}{3}ck + \frac{4}{3}k^2}$ (see Fig.1).
2.2. Geometric theory of singular perturbation. Consider the system

\[
\begin{align*}
    x' &= f(x, y, \varepsilon), \\
    y' &= \varepsilon g(x, y, \varepsilon),
\end{align*}
\]  

where \( \varepsilon = \frac{d}{dt}, \) \( x \in \mathbb{R}^n, \) \( y \in \mathbb{R}^l \) and \( \varepsilon \) is a real parameter and positive, the function \( f \) and \( g \) are \( C^\infty \) on a set \( U \times I \) where \( U \subseteq \mathbb{R}^{n+l} \) is open, and \( I \) is an open interval, containing 0. Furthermore, the \( x \) variables are called fast variables, and the \( y \) variables are called slow variables. Setting \( \tau = \varepsilon t \) gives the equivalent form

\[
\begin{align*}
    \varepsilon \dot{x} &= f(x, y, \varepsilon), \\
    \dot{y} &= g(x, y, \varepsilon),
\end{align*}
\]  

where \( \cdot = \frac{d}{d\tau} \). We refer to \( t \) as the fast time scale or fast time and to \( \tau \) as the slow time scale or slow time. Each of the scalings is naturally associated with a limit as \( \varepsilon \) tend to zero. These limits are respectively given by

\[
\begin{align*}
    x' &= f(x, y, 0), \\
    y' &= 0,
\end{align*}
\]  

and

\[
\begin{align*}
    0 &= f(x, y, 0), \\
    \dot{y} &= g(x, y, 0),
\end{align*}
\]  

The former is called the layer problem and the latter the reduced system.

**Definition 2.3.** (\cite{12}) A manifold \( M_0 \) on which \( f(x, y, 0) = 0 \) is called a critical manifold or slow manifold. A critical manifold \( M_0 \) is said to be normally hyperbolic if the linearization of system (8) at each point in \( M_0 \) has exactly \( l \) eigenvalues on the imaginary axis \( Re(\lambda) = 0 \).
Definition 2.4. ([12]) A set $M$ is locally invariant under the flow of (8) if it has neighborhood $V$ so that no trajectory can leave $M$ without also leaving $V$. In other words, it is locally invariant if for all $x \in M$, $x \cdot [0,t] \subseteq M$, similarly with $[t,0]$ replaced by $[t,0]$ when $t < 0$, where $x \cdot [0,t]$ denotes the application of a flow after time $t$ to the initial condition $x$.

Fenichel [9] established the following geometric theory of singular perturbation.

Theorem 2.5. Let $M_0$ be a compact, normally hyperbolic critical manifold given as a graph $\{(x,y) : y = h^0(y)\}$. Then for sufficiently small positive $\varepsilon$ and any $0 < r < +\infty$,

- there exists a manifold $M_\varepsilon$, which is locally invariant under the flow of (8) and $C^r$ in $x, y, \varepsilon$. Moreover, $M_\varepsilon$ is given as graph:

  $$M_\varepsilon = \{(x,y) : x = h^\varepsilon(y)\}$$

  for some $C^r$ function $h^\varepsilon(y)$;

- $M_\varepsilon$ possesses locally invariant stable and unstable manifold $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$ lying within $O(\varepsilon)$ and being $C^r$ diffeomorphic to the stable and unstable manifold $W^s(M_0)$ and $W^u(M_0)$ of the critical manifold $M_0$;

- $W^s(M_\varepsilon)$ is partitioned by moving invariant submanifolds $F^s(p_\varepsilon)$, which are $O(\varepsilon)$ close and diffeomorphic to $F^s(p_0)$, with base point $p_\varepsilon$ belonging to $M_\varepsilon$. Moreover, they are $C^r$ with respect to $p$ and $\varepsilon$. Moving invariance means the submanifold $F^s(p_\varepsilon)$ is mapped under the time $t$ flow to another submanifold $F^s(p_\varepsilon \cdot t)$ whose base point is the time $t$ evolution image of the taken base point $p_\varepsilon$;

- the dynamics on $M_\varepsilon$ is a regular perturbation of that generated by system (11).

Remark 1. A manifold $M_\varepsilon$, as obtained in the conclusion of Theorem 2.5 is called the slow manifold. As stated in the Theorem, the slow manifold $M_\varepsilon$ is smooth in $\varepsilon$. In general results of Fenichel [9] holds without the condition $M_0$ given as a graph. It’s also not necessary to require $M_0$ to be normally hyperbolic. Then the persistence results are in terms of existence of local center manifold of $M_0$ as well as local center stable, local center unstable manifold of $M_0$.

Definition 2.6. ([18]) Let $N$, $N_1$ and $N_2$ be invariant manifolds of a dynamical system. The orbit of a point $p$ is heteroclinic to $N_1$ and $N_2$ if $p$ lies in the unstable manifold of $N_1$ and in the stable manifold of $N_2$. The orbit of a point $p$ is homoclinic to $N$ if $p$ lies in both the unstable and the stable manifold of $N$.

3. Solitary wave solution for delayed DP equation. In this section, we establish the existence of solitary wave solutions for equation (3) in two cases: local distributed delay and nonlocal delay.

3.1. The model with local delay. We consider solitary wave solutions for equation (3) in the case that the convolution $f * u$ is defined by

$$f * u(x,t) = \int_0^\infty f(t - s)u(x,s)ds,$$

where the kernel $f : [0, \infty) \rightarrow [0, \infty)$ satisfies

$$\int_0^\infty f(t)dt = 1, \quad t f(t) \in L^1((0, \infty), R),$$

with the averaging delay for the kernel $f(t)$ defined as
\[ \tau = \int_0^\infty tf(t)dt. \]

The kernel
\[ f(t) = \frac{4t}{\tau^2}e^{-\frac{2t}{\tau}} \]
with averaging delay \( \tau \) is called the strong generic delay kernel and is frequently seen in the literature on the delay differential equation. In this section, we discuss the strong general delay kernel, i.e.
\[ f\ast u(x,t) = \int_0^\infty f(s)u(x,t-s)ds = \int_0^\infty \frac{4s}{\tau^2}e^{-\frac{2s}{\tau}}u(x,t-s)ds. \]

Consider the case of averaging delay \( 0 < \tau \ll 1 \). We seek traveling wave solutions of (3). These will be solutions of (3) that are functions of the single variable \( \xi = x - ct \). We are specifically interested in those that are asymptotic to the rest state \( u = 0 \) as \( \xi \to \pm\infty \), these will then be solitary waves. The wave \( u(x,t) = u(x-ct) =: \phi(\xi) \) must satisfy the ODE
\[ c\phi''' + (2k - c)\phi' + 4\phi'\eta - 3\phi'\phi'' - \phi\phi''' + \tau\phi'' = 0, \tag{12} \]
where \( \eta(\xi) = \int_0^\infty \frac{4t}{\tau^2}e^{-\frac{2t}{\tau}}\phi(\xi + ct)dt, ' = \frac{d}{d\xi} \). We obtain
\[ \frac{d\eta}{d\xi} = \frac{1}{c\tau}(2\eta - \zeta), \]
where \( \zeta = \int_0^\infty \frac{4}{\tau}e^{-\frac{2t}{\tau}}\phi(\xi + ct)dt. \]
Differentiating both sides with respect to \( \xi \), we obtain
\[ \frac{d\zeta}{d\xi} = \frac{2}{c\tau}(\zeta - 2\phi). \]

Therefore the solitary wave equation (12) is equivalent to the following slow-fast system
\[
\begin{cases}
\phi' = \psi, \\
\psi' = \nu, \\
(c - \phi)\nu' = (c - 2k)\psi - 4\psi\eta + 3\psi\nu - \tau\nu, \\
c\eta' = 2\eta - \zeta, \\
c\zeta' = 2(\zeta - 2\phi). 
\end{cases}
\tag{13}
\]

Setting \( \tau = 0 \), we get the critical manifold to be
\[ M_0 = \{(\phi, \psi, \nu, \eta, \zeta) \in \mathbb{R}^5 : \eta = \frac{1}{2}\zeta = \phi, -\frac{1}{2}k \leq \phi \leq c - \frac{1}{6}k \}. \]

Note that if \( \tau = 0 \), then \( \eta = \phi \) and (13) is equivalent to non-delay equation (5), where the homoclinic orbit of interests is restricted by the condition \( 0 \leq \phi \leq \phi^* = (c - \frac{2}{3}k) - \sqrt{\frac{2}{3}ck + \frac{4}{3}k^2}. \) To study the persistence of the homoclinic orbit for sufficiently small \( \tau > 0 \), it suffices to restrict to a neighborhood of the unperturbed
homoclinic orbit. Reparameterizing the ‘time’ with \( d\xi = (c - \phi)dz \) gives the corresponding slow system
\[
\begin{aligned}
\dot{\phi} &= (c - \phi)\psi, \\
\dot{\psi} &= (c - \phi)\nu, \\
\dot{\nu} &= (c - 2k)\psi - 4\psi\eta + 3\psi\nu - \tau\nu, \\
\tau\dot{\eta} &= \frac{1}{c}(2\eta - \zeta)(c - \phi), \\
\tau\dot{\zeta} &= \frac{2}{c}(\zeta - 2\phi)(c - \phi),
\end{aligned}
\]
(14)

where \( \dot{\cdot} = \frac{d}{dz} \). With \( z = \tau s \), the corresponding fast system is
\[
\begin{aligned}
\phi' &= \tau(c - \phi)\psi, \\
\psi' &= \tau(c - \phi)\nu, \\
\nu' &= \tau[(c - 2k)\psi - 4\psi\eta + 3\psi\nu - \tau\nu], \\
\eta' &= \frac{1}{c}(2\eta - \zeta)(c - \phi), \\
\zeta' &= \frac{2}{c}(\zeta - 2\phi)(c - \phi),
\end{aligned}
\]
(15)

where \( \dot{\cdot} = \frac{d}{ds} \). The above two systems are equivalent when \( \tau > 0 \). It is noted that we have restricted the discussion in a neighborhood of the unperturbed homoclinic orbit which satisfies \( 0 \leq \phi \leq \phi^* \). The linearization of (15) with \( \tau = 0 \) is
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{c}(c - \phi) & -\frac{1}{c}(c - \phi) \\
* & 0 & 0 & 0 & \frac{2}{c}(c - \phi)
\end{pmatrix}.
\]
Therefore the manifold \( M_0 \), on which it holds: \( c - \phi \geq \frac{1}{3}k \), is normally hyperbolic with two unstable normal directions. According to Theorem 2.5, \( M_0 \) persists for \( 0 < \tau \ll 1 \), i.e. there exists a slow manifold \( M_\tau \), which is locally invariant under the flow of (15) and \( C^1 O(\tau) \) close to \( M_0 \):
\[
M_\tau = \{(\phi, \psi, \nu, \eta, \zeta) \in \mathbb{R}^5 : \eta = \phi + g(\phi, \psi, \nu, \tau), \\
\zeta = 2\phi + h(\phi, \psi, \nu, \tau), -\frac{1}{3}k \leq \phi \leq c - \frac{1}{3}k \}
\]
(16)

where the functions \( g, h \) are smooth functions defined on a compact domain, and satisfy
\[
g(\phi, \psi, \nu, 0) = 0, \quad h(\phi, \psi, \nu, 0) = 0.
\]
We also know that when \( \tau \to 0 \), equation (14) reduces to the non-delay equation (5). Because \( M_\tau \) is locally invariant under the flow of (15), so
\[
\begin{aligned}
\eta' &= \phi' + D_1 g(\phi, \psi, \nu, \tau)\phi' + D_2 g(\phi, \psi, \nu, \tau)\psi' + D_3 g(\phi, \psi, \nu, \tau)\nu', \\
\zeta' &= \phi' + D_1 h(\phi, \psi, \nu, \tau)\phi' + D_2 h(\phi, \psi, \nu, \tau)\psi' + D_3 h(\phi, \psi, \nu, \tau)\nu',
\end{aligned}
\]
(17)
and
\[
\begin{aligned}
\eta' &= \frac{1}{c}(2\eta - \zeta)(c - \phi) = \frac{1}{c}(2g - h)(c - \phi), \\
\zeta' &= \frac{2}{c}(\zeta - 2\phi)(c - \phi) = \frac{2}{c}h(c - \phi).
\end{aligned}
\]
(18)
We substitute (15) into (17), to get
\[
\begin{align*}
\eta' &= \tau(c-\phi)\psi + O(\tau^2), \\
\zeta' &= \tau(c-\phi)\psi + O(\tau^2).
\end{align*}
\] (19)

To compute series expansions for $M_\tau$, let us do the Taylor expansion on the right-hand side of equation (18) with respect to $\tau$,
\[
\begin{align*}
\eta' &= \frac{1}{c}[(2D_4g - D_4h)(\phi, \psi, \nu, 0)\tau + O(\tau^2)](c-\phi), \\
\zeta' &= \frac{2}{c}[D_4h(\phi, \psi, \nu, 0)\tau + O(\tau^2)](c-\phi).
\end{align*}
\] (20)

Comparing the coefficients of (19) and (20), we have
\[
\begin{align*}
h &= \frac{1}{2}c\tau\psi + O(\tau^2), \\
g &= \frac{3}{4}c\tau\psi + O(\tau^2).
\end{align*}
\] (21)

Therefore restricted to $M_\tau$, (14) becomes the following system
\[
\begin{align*}
\dot{\phi} &= (c-\phi)\psi, \\
\dot{\psi} &= (c-\phi)\nu, \\
\dot{\nu} &= (c-2k)\psi - 4\psi\phi + 3\psi\nu - 3c\tau\psi^2 - \tau\nu + O(\tau^2).
\end{align*}
\] (22)

Because both normal directions are unstable, the persistent homoclinic orbit (if ever exists) must be completely contained in $M_\tau$. So (22) is exactly the system to study for the existence of a homoclinic orbit.

Note that, for the original system (13), it always possesses the line of equilibrium
\[
\{\psi = 0, \nu = 0, \eta = \phi, \zeta = 2\phi\},
\]
even for nonzero values of $\tau$. This property is inherited by the reduced system (22), i.e. it has a line of equilibrium given by
\[
\{\psi = 0, \nu = 0\}.
\]

**Remark 2.** This key information can not be drawn from solely investigation of system (22) because of the $O(\tau^2)$ term, but from the special structure of the DP equation. It is this property that makes possible another application of the method of singular perturbation.

Note that if $\phi(\xi), \phi'(\xi),$ and $\phi''(\xi) \to 0$ as $\xi \to \pm \infty$, integrating once for non-delayed equation (5) we can get (6). If only the conditions $\lim_{\xi \to \pm \infty} \phi(\xi) = 0$ and $\lim_{\xi \to \pm \infty} \phi'(\xi) = 0$ of the solitary wave is considered, integrating both sides of equation (5) leads the equation
\[
(c-\phi)\phi'' + (2k-c)\phi + 2\phi^2 - (\phi')^2 = C,
\]
where $C$ is a constant. That is to say
\[
(c-\phi)\nu + (2k-c)\phi + 2\phi^2 - \psi^2 = C.
\]

So we guess that when $\tau$ is sufficiently small, the righthand of the above equation is approximately a constant. Naturally, considering the following variable substitutions
\[
\begin{align*}
\hat{\phi} &= \phi, \\
\hat{\psi} &= \psi, \\
\hat{\nu} &= (\phi-c)\nu - [(2k-c)\phi + 2\phi^2 - \psi^2].
\end{align*}
\]
It turns out \( \tilde{\phi}, \tilde{\psi}, \tilde{\nu} \) satisfy the following slow-fast system:

\[
\begin{aligned}
\dot{\phi} &= (c - \phi) \tilde{\psi}, \\
\dot{\psi} &= (c - 2k)\tilde{\phi} - 2\phi^2 + \psi^2 - \nu, \\
\dot{\nu} &= 3c\tau(c - \phi)^2\psi^2 + \tau[(c - 2k)\tilde{\phi} - 2\phi^2 + \psi^2 - \nu] + O(\tau^2).
\end{aligned}
\]

For readability, we remove the superscript tilde:

\[
\begin{aligned}
\dot{\phi} &= (c - \phi)\psi, \\
\dot{\psi} &= (c - 2k)\phi - 2\phi^2 + \psi^2 - \nu, \\
\dot{\nu} &= 3c\tau(c - \phi)^2\psi^2 + \tau[(c - 2k)\phi - 2\phi^2 + \psi^2 - \nu] + O(\tau^2).
\end{aligned}
\]  

Since (22) has a line of equilibrium for nonzero values of \( \tau \), so does (23), with a curve of equilibrium being \( \{ \phi = \frac{(c - 2k) - \sqrt{(c - 2k)^2 - 8\nu}}{4}, \psi = 0, |\nu| \leq \delta \} \) for some small constant \( \delta > 0 \) independent of \( \tau \), which is easy to check the normal hyperbolicity with one stable and one unstable normal direction. From Theorem 2.5, \( \tilde{M}_0 \) persists for \( 0 < \tau \ll 1 \), denoted by \( \tilde{M}_\tau \), which is \( C^1 \) \( O(\tau) \) close to \( \tilde{M}_0 \). Because \( \tilde{M}_0 \) is composed of the equilibrium point of systems (23), \( \tilde{M}_\tau = \tilde{M}_0 \) for nonzero values of \( \tau \). Fenichels theory indicates the existence of two dimensional stable and unstable manifold \( W^s_\tau \) and \( W^u_\tau \) of \( \tilde{M}_\tau \), being \( C^1 \) \( O(\tau) \) close to corresponding stable and unstable manifolds \( W^s_0 \) and \( W^u_0 \) of \( \tilde{M}_0 \), respectively.

In order to study the separation of \( W^s_\tau \) and \( W^u_\tau \), it is convenient to change variable as \( dz = (c - \phi)dx \) which transforms (23) to:

\[
\begin{aligned}
\phi' &= (c - \phi)^2\psi, \\
\psi' &= (c - \phi)[(c - 2k)\phi - 2\phi^2 + \psi^2 - \nu], \\
\nu' &= 3c\tau(c - \phi)^2\psi^2 + \tau[(c - 2k)\phi - 2\phi^2 + \psi^2 - \nu] + O(\tau^2),
\end{aligned}
\]  

where \( ' = \frac{d}{dz} \). The \((\phi, \psi)\) sub-vector field is now divergence free. The effective separation function \( \text{Sep} \) is given by

\[
\text{Sep}(c, \tau) = \tau\Delta + O(\tau^2), \quad \Delta = \int^{+\infty}_{-\infty} (c - \phi)^3\psi \frac{\partial \nu}{\partial \tau} dx, \tag{25}
\]

where \( \phi, \psi \) are evaluated along \( \Gamma \), and \( \frac{\partial \nu}{\partial \tau} \) satisfies

\[
\begin{aligned}
\frac{d}{dz}(\frac{\partial \nu}{\partial \tau}) &= 3c(c - \phi)\psi^2 + (c - 2k)\phi - 2\phi^2 + \psi^2, \\
\frac{\partial \nu}{\partial \tau} &= 0 \quad \text{at} \quad z = 0, \tag{26}
\end{aligned}
\]

with \( \phi, \psi \) evaluated along \( \Gamma \). A detailed derivation of the above form of Melnikov integral is provided by [17]. See also [11] for applications.
By $dz = (c - \phi)dx$, the Melnikov function is here given by
\[ \Delta = \int_{-\infty}^{\infty} (c - \phi)^3 \frac{\partial \nu}{\partial \tau} dx = \int_{-\infty}^{\infty} (c - \phi)^2 \frac{\partial \nu}{\partial \tau} dz = \int_{\Gamma} (c - \phi) \frac{\partial \nu}{\partial \tau} d\phi, \] (27)
where $\Gamma$ satisfies
\[ \begin{cases} \dot{\phi} = (c - \phi) \psi, \\ \dot{\psi} = (c - 2k)\phi - 2\phi^2 + \psi^2, \end{cases} \] (28)
with $(c - \phi)^2 \psi^2 + 2(c - \frac{2}{3}k)\phi^3 - \phi^4 - c(c - 2k)\phi^2 = 0$. We integrate by parts
\[ \Delta = \phi(c - \frac{1}{2} \phi) \frac{\partial \nu}{\partial \tau} \bigg|_{-\infty}^{+\infty} - \int_{\Gamma} \phi(c - \frac{1}{2} \phi) \frac{d}{dz} \left( \frac{\partial \nu}{\partial \tau} \right) dz. \]
Since $\frac{d}{dz} \left( \frac{\partial \nu}{\partial \tau} \right) = 3c(c - \phi)\psi^2 + (c - 2k)\phi - 2\phi^2 + \psi^2$ is bounded and exponentially small at $\pm \infty$, $\frac{\partial \nu}{\partial \tau}$ increases at most sub-exponentially at $\pm \infty$. On the other hand, $\phi$ is exponentially small at $\pm \infty$, the first term in $\Delta$ vanishes. So
\[ \Delta = -\frac{1}{2} \int_{-\infty}^{+\infty} \phi(2c - \phi) \frac{d}{dz} \left( \frac{\partial \nu}{\partial \tau} \right) dz = -\frac{1}{2} \int_{-\infty}^{+\infty} \phi(2c - \phi)[3c(c - \phi)\psi^2 + (c - 2k)\phi - 2\phi^2 + \psi^2] dz. \]
Further, by (28) and another integration by parts, we have
\[ \Delta = -\int_{0}^{\phi^*} 3c\phi\psi(2c - \phi) d\phi + \int_{0}^{\phi^*} 2\psi(c - \phi) d\phi = -\int_{0}^{\phi^*} 3c^2 \phi \psi d\phi - \int_{0}^{\phi^*} (3c\phi - 2)(c - \phi) \psi d\phi
= -\int_{0}^{\phi^*} 3c^2 \frac{\phi^2}{c - \phi} \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)\phi} d\phi
- \int_{0}^{\phi^*} (3c\phi - 2)\phi \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)\phi} d\phi. \] (29)
It is noted that one can not take derivative with respect to $c$ directly because of the singularity in the denominator. Instead, introducing the change of variable
\[ z =: \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)} = \sqrt{(\phi - \phi_-)(\phi - \phi_+)}, \]
\[ \phi_+ = (c - \frac{2}{3}k) + \sqrt{\frac{2}{3}ck + \frac{4}{9}k^2}, \quad \phi_- = (c - \frac{2}{3}k) - \sqrt{\frac{2}{3}ck + \frac{4}{9}k^2} = \phi^*, \]
and denote:
\[ \beta =: \phi_- \phi_+, \quad \alpha_+ =: c - \frac{2}{3}k, \quad \alpha_- =: \sqrt{\frac{2}{3}ck + \frac{4}{9}k^2}. \]
Then
\[ d\phi = -\frac{2dz}{\alpha_+ - \phi}, \quad \alpha_+ - \phi = \sqrt{z^2 + \alpha_-^2}, \quad \beta = c^2 - 2ck = \alpha_+^2 - \alpha_-^2, \]
\[ (\alpha_+ - \phi_+)(\alpha_+ - \phi_-) = -\alpha_-^2, \quad 2\alpha_+ - \phi_+ - \phi_- = 0. \]

For notation convenience, denote
\[ A_1 = 2\alpha_+ - \frac{2}{3c}, \quad A_2 = -\frac{2}{3c}\alpha_+ + \beta, \]
\[ A_3 = (\frac{2}{3c} - 2\alpha_+ - \alpha_-^2, \quad A_4 = \alpha_+ (\frac{2}{3c} - \alpha_+ - \alpha_-^2, \]
\[ \frac{3}{4}A_2^2 + A_2 = c^2 - \frac{3}{2}ck + \frac{1}{3}k^2 - \frac{2}{3} + \frac{4k}{9c}, \quad A_3 + \alpha_+ A_1 = 0, \]
\[ \frac{1}{2} \left( \frac{3}{4} A_2^2 + A_2 \right) + A_4 = -\frac{1}{3} k (c + \frac{2}{3} k) (c^2 - \frac{7}{6} ck + \frac{5}{9} k^2 - \frac{2}{3} + \frac{4k}{9c}); \]

then
\[
-\int_0^{\phi^*} (3c\phi - 2) \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)} d\phi = -3c \int_0^{\sqrt{\sigma}} \left( \frac{1}{2} (\alpha_+ - \phi) - \frac{2}{3c} \alpha_+ - (\alpha_+ - \phi_+ - \phi_-) \right) \alpha_+ - \phi d\phi \\
\times \left[ (\alpha_+ - \phi) - (\alpha_+ - \phi_-) \right] dz \\
= -3c \int_0^{\sqrt{\sigma}} (\alpha_+ - \phi)^3 - A_1 (\alpha_+ - \phi)^2 + A_2 (\alpha_+ - \phi) - A_3 + \frac{A_4}{\alpha_+ - \phi} dz \\
= -3c \int_0^{\sqrt{\sigma}} (z^2 + \alpha_-^2)^2 - A_1 (z^2 + \alpha_-^2) + A_2 \sqrt{z^2 + \alpha_-^2} - A_3 + \frac{A_4}{\sqrt{z^2 + \alpha_-^2}} dz \\
= -3c \left\{ \frac{1}{4} z (z^2 + \alpha_-^2)^2 + \frac{1}{2} \left( \frac{3}{4} A_2^2 + A_2 \right) z \sqrt{z^2 + \alpha_-^2} - (A_3 + \alpha_+ A_1) z \\
\frac{1}{3} z^3 + \frac{1}{2} A_2 \left( \frac{3}{4} A_2^2 + A_2 \right) A_4 \log(z + \sqrt{z^2 + \alpha_-^2}) \right\} \bigg|_{z=0}^{z=\sqrt{\sigma}} \\
= -3c \left\{ \frac{1}{4} (c - \frac{2}{3}k)^3 \sqrt{c(c - 2k)} - \frac{1}{3} (2c - \frac{4}{3}k - \frac{2}{3} \frac{2}{3c}) [c(c - 2k)]^2 \\
+ \frac{1}{2} (c^2 - \frac{3}{2} \frac{2}{3}k + \frac{1}{3} k^2 - \frac{2}{3} + \frac{4k}{9c}) (c - \frac{2}{3}k) \sqrt{c(c - 2k)} \\
- \frac{1}{3} k (c + \frac{2}{3}k) (c^2 - \frac{7}{6} ck + \frac{5}{9} k^2 - \frac{2}{3} + \frac{4k}{9c}) \log \frac{c - \frac{2}{3}k + \sqrt{c(c - 2k)}}{\sqrt{c(c - 2k)}} \right\} \\
= \left( -\frac{1}{4} c^4 - \frac{7}{12} c^3 k - \frac{1}{3} c^2 k^2 + \frac{5}{9} c^3 k^2 + \frac{1}{3} c^2 + \frac{4k}{9c} \right) \sqrt{c(c - 2k)} \\
+ \left( c^4 k - \frac{1}{2} c^3 k^2 - \frac{2}{9} c^2 k^3 + \frac{10}{27} c k^4 - \frac{2}{3} c^2 k + \frac{8}{27} k^3 \right) \log \frac{c - \frac{2}{3}k + \sqrt{c(c - 2k)}}{\sqrt{c(c - 2k)}}.
and

\[ - \int_0^{\phi^*} \frac{\phi^2}{c - \phi} \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)} d\phi \]

\[ = \int_0^{\phi^*} (c + \phi) \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)} d\phi \]

\[ - c^2 \int_0^{\sqrt{\beta}} \frac{1}{c - \phi} \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)} d\phi \]

\[ = \left\{ \frac{1}{3} \left[ \phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k) \right]^{\frac{3}{2}} - (c - \frac{1}{3}) \times \left[ (\alpha_+ - \phi) \sqrt{(\alpha_+ - \phi)^2 - \alpha_+^2} - \alpha_+^2 \log(\alpha_+ - \phi + \sqrt{(\alpha_+ - \phi)^2 - \alpha_+^2}) \right] \right\}^{\phi^*}_0 \]

\[ - c^2 \sqrt{\beta} + \frac{2}{3} c^2 k \int_0^{\sqrt{\beta}} \frac{1}{\sqrt{z^2 + \alpha_+^2}} dz \]

\[ + \frac{2}{3} c^3 k \int_0^{\sqrt{\beta}} \frac{1}{\sqrt{z^2 + \alpha_+^2}} \times \left[ \sqrt{z^2 + \alpha_+^2} + (c - \alpha_+) \right] dz \]

\[ = (-\frac{1}{3} c^2 - \frac{1}{3} c k + \frac{2}{9} k^2) \sqrt{c(c - 2k)} + (-\frac{2}{9} c k^2 + \frac{4}{27} k^3) \log \frac{c - \frac{2}{3} k + \sqrt{c(c - 2k)}}{\sqrt{3} ck + \frac{5}{9} k^2} \]

\[ - \sqrt{\frac{8}{3}} c^2 k^2 \arctan \left( \frac{\sqrt{z^2 + \alpha_+^2} - z + \frac{2}{3} k}{\sqrt{\frac{2}{3} ck}} \right) \right|_0^{\sqrt{\beta}} \]

\[ = (-\frac{1}{3} c^2 - \frac{1}{3} c k + \frac{2}{9} k^2) \sqrt{c(c - 2k)} + (-\frac{2}{9} c k^2 + \frac{4}{27} k^3) \log \frac{c - \frac{2}{3} k + \sqrt{c(c - 2k)}}{\sqrt{3} ck + \frac{5}{9} k^2} \]

\[ + \sqrt{\frac{8}{3}} c^2 k^2 \left[ \arctan \left( \frac{\frac{2}{3} k + \sqrt{\frac{2}{3} ck + \frac{4}{9} k^2}}{\sqrt{\frac{2}{3} ck}} \right) - \arctan \left( \frac{c - \sqrt{c(c - 2k)}}{\sqrt{\frac{2}{3} ck}} \right) \right]. \]

Therefore

\[ \Delta = (c^4 k - \frac{7}{6} c^3 k + \frac{2}{9} c^2 k^3 - \frac{2}{3} c^2 k + \frac{10}{27} c k^4 + \frac{8}{27} k^5) \log \frac{c - \frac{2}{3} k + \sqrt{c(c - 2k)}}{\sqrt{3} ck + \frac{5}{9} k^2} \]

\[ + \sqrt{24} c^2 k^2 \left[ \arctan \left( \frac{\frac{2}{3} k + \sqrt{\frac{2}{3} ck + \frac{4}{9} k^2}}{\sqrt{\frac{2}{3} ck}} \right) - \arctan \left( \frac{c - \sqrt{c(c - 2k)}}{\sqrt{\frac{2}{3} ck}} \right) \right] \]

\[ + (-\frac{5}{4} c^4 - \frac{19}{12} c^3 k + \frac{1}{3} c^2 k^2 + \frac{1}{3} c k^3 + \frac{5}{9} k^4 + \frac{4}{9} k^5) \sqrt{c(c - 2k)}. \] (30)

Notice that $\phi^*$ tends to zero as $c$ tends to $2k$, according to equation (29), it indicates $\Delta$ is positive for $c$ close to $2k$. Further, by equation (30), we conclude that $\Delta$ and $\partial \Delta/\partial c$ tend to zero as $c$ tends to $2k$ and $\Delta$ tends to negative infinity as $c$ tends to positive infinity. However, because of the complexity of the $\Delta$ expression, it is difficult to get the monotonicity of $\Delta$. It is clear that $\Delta$ is continuous about
For $c > 2k$, whose figure can be well depicted by numerical simulation, see Fig. 2 as a typical example. Owing to this, we can obviously find the fact that when $c$ is greater than $2k$, $\Delta$ has a tendency to increase first and then decrease, which finally tends to negative infinity. As a result, there exists a unique $c^* > 2k$:

\[
\Delta = \Delta(c^*) = 0, \quad \text{and} \quad \frac{\partial \Delta}{\partial c}(c^*) < 0.
\]

Then by the implicit function theorem, for each small nonzero value of $\tau$, there is a unique value $c = c(\tau)$ such that $\text{Sep}(c(\tau), \tau) = 0$ in (25). We have the following

**Theorem 3.1.** For sufficiently small $\tau > 0$, there exists a unique speed $c$ such that equation (3) with the strong generic local delay kernel has a solitary wave solution in the sense that the corresponding solitary wave equation (12) has a solution $\Gamma_\tau$ which is heteroclinic to two equilibrium $O_1$ and $O_2$ lying $O(\tau)$ close to the origin, and that $\Gamma_\tau$ is $C^1 O(\tau)$ close to the unperturbed homoclinic $\Gamma$ and approaches $\Gamma$ in $C^1$ norm as $\tau \to 0$.

**Proof.** It is clear that, from the Melnikov method above, outside of a small $O(1)$ neighborhood $V$ of the slow manifold $\tilde{M}_\tau$, $\Gamma_\tau$ lies $C^1 O(\tau)$ close to the original homoclinic orbit $\Gamma$.

After entering the neighborhood $V$, $\Gamma_\tau$ intersects some submanifold $F^s(\nu_\tau)$ and follows the evolution of the submanifold, whose existence and properties are explained in Theorem 2.5. Also as a result of the third part of Theorem 2.5, $F^s(\nu_\tau)$ is $C^1 O(\tau)$ close to the $\Gamma$, which belongs to the stable manifold with base point $\nu_0 = 0$. Notice that the slow manifold $\tilde{M}_\tau$ consists of true equilibrium. It means that base points on $\tilde{M}_\tau$ is not moving at all, and the moving invariant submanifold $F^s(\nu_\tau)$ is actually invariant, so $\Gamma_\tau = F^s(\nu_\tau)$.

As a result, for all forward time, $\Gamma_\tau$ is $C^1 O(\tau)$ close to the unperturbed homoclinic $\Gamma$. The backward time is the same. \[\square\]

**Remark 3.** The perturbation term $\tau u_{xx}$ is necessary for the existence of the above solitary wave. Without the perturbation, the Melnikov integral (25) is calculated...
to be
\[ \Delta = -\int_0^{\phi^*} 3c\phi(2c - \phi) \times \frac{\phi}{c - \phi} \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)}d\phi, \]
which is clearly negative for admissible values of $c, k$. Easy calculation shows that if the original DP equation (4) is perturbed only by $\tau u_{xx}$ without delay, the corresponding Melnikov integral is
\[ \Delta = \int_0^{\phi^*} 2\phi \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)}d\phi > 0, \]
then there is no solitary wave solution, either. We can understand that the delay effect drives $\Delta$ negative while the perturbation effect drives $\Delta$ positive. The combined effect produces unique solitary waves.

3.2. The model with nonlocal delay. In this section, we consider solitary wave solutions for equation (3) with a nonlocal weak generic kernel. The convolution $f \ast u$ is defined by
\[ (f \ast u)(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} f(x-y,t-s)u(y,s)dyds, \]
where the kernel $f : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ satisfied the following normalization assumption
\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,t)dxdt = 1. \]
Equations of various types can be derived from equation (3) by using different delay kernel functions. For example, when taking the kernel to be $f(x,t) = \delta(x)\delta(t-\tau)$, where $\delta$ denotes Dirac’s function, equation (3) becomes the corresponding delayed DP equation,
\[ u_t - u_{xxt} + 2ku_x + 4u(x,t-\tau)u_x = 3u_xu_{xx} + uu_{xxx}. \]
At this section, we consider the special weak generic delay kernel
\[ f(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} e^{-\frac{t-\tau}{\tau}} \]
where the parameter $\tau > 0$ measures the average delay time.
Define for $0 < \tau \ll 1$:
\[ v(x,t) = (f \ast u)(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{1}{\tau} e^{-\frac{t-s}{\tau}} u(y,s)dyds, \]
then
\[ v_t - v_{xx} = \frac{1}{\tau} (u - v). \]
So (3) is equivalent to
\[ \begin{cases} u_t - u_{xxt} + 2ku_x + 4u_xu_x - 3u_xu_{xx} - uu_{xxx} + \tau u_{xx} = 0, \\ v_t - v_{xx} = \frac{1}{\tau} (u - v). \end{cases} \]
Let $u(x,t) = \phi(\xi), v(x,t) = \eta(\xi), \xi = x - ct$, the $\phi, \eta$ satisfy the following solitary wave system
\[ \begin{cases} c\phi'''' + (2k - c)\phi + 4\eta\phi' - 3\phi'\phi'''' - \phi\phi'''' + \tau \phi'' = 0, \\ -c\eta' - \eta'' = \frac{1}{\tau} (\phi - \eta), \end{cases} \]
where \( \frac{d}{ds} \). In terms of a system of first order ODEs, we get

\[
\begin{align*}
\phi' &= \psi, \\
\psi' &= \nu, \\
(\phi - c)\nu' &= (2k - c)\psi + 4\psi\eta - 3\psi\nu + \tau\nu, \\
\eta' &= \zeta, \\
\zeta' &= -c\zeta + \frac{1}{\tau}(\eta - \phi).
\end{align*}
\]

Letting \( \varepsilon = \sqrt{\tau} \), and \( \zeta = \varepsilon\tilde{\zeta} \), the above system is transformed to the following slow-fast systems

\[
\begin{align*}
\phi' &= \psi, \\
\psi' &= \nu, \\
(\phi - c)\nu' &= (2k - c)\psi + 4\psi\eta - 3\psi\nu + \varepsilon^2\nu, \\
\varepsilon\eta' &= \zeta, \\
\varepsilon\zeta' &= -c\varepsilon\zeta + \eta - \phi.
\end{align*}
\]

Note that if \( \tau = 0 \), then \( \eta = \phi \) and (13) becomes non-delay equation (5), where the homoclinic orbit of interests is restricted by the condition \( 0 \leq \phi \leq \phi^* = (c - \frac{2}{3}k) - \sqrt{\frac{2}{3}ck + \frac{4}{9}k^2} \). To study the persistence of the homoclinic orbit for sufficiently small \( \tau > 0 \), it suffices to restrict to a neighborhood of the unperturbed homoclinic orbit. Reparameterizing the ‘time’ with \( d\xi = (c - \phi)dz \) gives the corresponding slow system

\[
\begin{align*}
\dot{\phi} &= (c - \phi)\psi, \\
\dot{\psi} &= (c - \phi)\nu, \\
\dot{\nu} &= (c - 2k)\psi - 4\psi\eta + 3\psi\nu - \varepsilon^2\nu, \\
\varepsilon\dot{\eta} &= \zeta(c - \phi), \\
\varepsilon\dot{\zeta} &= (\eta - \phi - c\varepsilon\zeta)(c - \phi),
\end{align*}
\]

where \( \cdot = \frac{d}{ds} \). With \( z = \tau s \), the corresponding fast system is

\[
\begin{align*}
\phi' &= \varepsilon(c - \phi)\psi, \\
\psi' &= \varepsilon(c - \phi)\nu, \\
\nu' &= \varepsilon[(c - 2k)\psi - 4\psi\eta + 3\psi\nu - \varepsilon^2\nu], \\
\eta' &= \zeta(c - \phi), \\
\zeta' &= (\eta - \phi - c\varepsilon\zeta)(c - \phi),
\end{align*}
\]

where \( \cdot = \frac{d}{ds} \). The above two systems are equivalent when \( \varepsilon > 0 \). Setting \( \varepsilon = 0 \), we take the corresponding critical manifold is

\[
M_0 = \{(\phi, \psi, \nu, \eta, \zeta) \in \mathbb{R}^5 : \zeta = 0, \eta = \phi, -\frac{1}{2}k \leq \phi \leq c - \frac{1}{6}k \}.
\]
which is a three dimensional manifold of equilibrium for (33). The linearization is

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c - \phi \\
* & 0 & 0 & c - \phi & 0 \\
\end{pmatrix}.
$$

Therefore the manifold $M_0$, on which it hold: $c - \phi \geq \frac{1}{3}k$, is normally hyperbolic with one stable and one unstable normal directions. According to Theorem 2.5, $M_0$ persists for $0 < \varepsilon \ll 1$, there exists a slow manifold $M_\varepsilon$ is locally invariant under the flow of (32) and $C^1 O(\varepsilon)$ close to $M_0$:

$$
M_\varepsilon = \{(\phi, \psi, \nu, \eta, \zeta) \in \mathbb{R}^5 : \eta = \phi + g(\phi, \psi, \nu), \zeta = 0 + h(\phi, \psi, \nu), -\frac{1}{3}k \leq \phi \leq c - \frac{1}{3}k\},
$$

where the functions $g$, $h$ are smooth functions defined on a compact domain, and satisfy

$$
g(\phi, \psi, \nu, 0) = 0, \quad h(\phi, \psi, \nu, 0) = 0.
$$

Using the same calculation method in Section 3.1, $g$ and $h$ have the following asymptotic expansion:

$$
g = \varepsilon^2 (\nu + c\psi) + O(\varepsilon^3), \quad h = \varepsilon\psi + O(\varepsilon^3).
$$

Therefore restricted to $M_\varepsilon$, (32) is the following system of regular perturbation

$$
\begin{align*}
\dot{\phi} &= (c - \phi)\psi, \\
\dot{\psi} &= (c - \phi)\nu, \\
\dot{\nu} &= (c - 2k)\psi - 4\psi\phi + 3\psi\nu - 4(c\psi + \nu)\psi\varepsilon^2 - \nu\varepsilon^2 + O(\varepsilon^3).
\end{align*}
$$

Due to the normally hyperbolicity of $M_0$, the persistent homoclinic orbit (if ever exists) must be completely contained in $M_\varepsilon$. So we study (34) for the existence of a homoclinic orbit.

Let

$$
\begin{align*}
\tilde{\phi} &= \phi, \\
\tilde{\psi} &= \psi, \\
\tilde{\nu} &= (\phi - c)\nu - [(2k - c)\phi + 2\phi^2 - \psi^2].
\end{align*}
$$

We get the following system for variables $\tilde{\phi}$, $\tilde{\psi}$, $\tilde{\nu}$:

$$
\begin{align*}
\dot{\tilde{\phi}} &= (c - \tilde{\phi})\tilde{\psi}, \\
\dot{\tilde{\psi}} &= (c - 2k)\tilde{\phi} - 2\tilde{\phi}^2 + \tilde{\psi}^2 - \tilde{\nu}, \\
\dot{\tilde{\nu}} &= 4\varepsilon^2 c\tilde{\psi}^2 (c - \tilde{\phi}) + \varepsilon^2 (4\tilde{\psi} + 1)[(c - 2k)\tilde{\phi} - 2\tilde{\phi}^2 + \tilde{\psi}^2 - \tilde{\nu}] + O(\varepsilon^3).
\end{align*}
$$

Suppress the tilde again for readability:

$$
\begin{align*}
\dot{\phi} &= (c - \phi)\psi, \\
\dot{\psi} &= (c - 2k)\phi - 2\phi^2 + \psi^2 - \nu, \\
\dot{\nu} &= 4\varepsilon^2 c\psi^2 (c - \phi) + \varepsilon^2 (4\psi + 1)[(c - 2k)\phi - 2\phi^2 + \psi^2 - \nu] + O(\varepsilon^3),
\end{align*}
$$
and in the form involving $\tau$:
\[
\begin{align*}
\dot{\phi} &= (c - \phi)\psi, \\
\dot{\psi} &= (c - 2k)\phi - 2\phi^2 + \psi^2 - \nu, \\
\dot{\nu} &= 4c\tau\psi^2(c - \phi) + \tau(4\psi + 1)[(c - 2k)\phi - 2\phi^2 + \psi^2 - \nu] + O(\tau^2).
\end{align*}
\]
Following the line of last subsection, we estimate the corresponding Melnikov integral and we have:
\[
\Delta = -\frac{1}{2} \int_{-\infty}^{+\infty} \phi(2c - \phi) \frac{d}{dz} \left( \frac{\partial \nu}{\partial \tau} \right) dz
\]
\[
= -\frac{1}{2} \int_{-\infty}^{+\infty} \phi(2c - \phi) \{4c\psi^2(c - \phi) + (4\psi + 1)[(c - 2k)\phi - 2\phi^2 + \psi^2]\} dz
\]
\[
= -\int_{0}^{\infty} \phi(2c - \phi) \times 4c\psi d\phi + 4 \int_{0}^{\infty} \phi^2(c - \phi)d\phi + 2 \int_{0}^{\infty} \phi(c - \phi)d\phi
\]
\[
= -4c^2 \int_{0}^{\infty} \phi\psi d\phi + 4 \int_{0}^{\infty} (c - \phi)\psi d\phi - 2 \int_{0}^{\infty} (2c\phi - 1)(c - \phi)\psi d\phi
\]
\[
= -4c^2 \int_{0}^{\infty} \frac{\phi^2}{c - \phi} \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)} d\phi
\]
\[
+ 4 \int_{0}^{\infty} \frac{\phi^2}{c - \phi} d\phi - 2 \int_{0}^{\infty} (2c\phi - 1)\phi \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)} d\phi.
\]
(35)

After calculation, we can see that
\[
\Delta = \left( \frac{4}{3} c^4 k - \frac{14}{9} c^4 k^3 + \frac{8}{27} c^4 k^3 - \frac{2}{3} c^2 k + \frac{40}{81} c k^4 + \frac{8}{27} k^3 \right) \log \frac{c - \frac{2}{3} k + \sqrt{c(c - 2k)}}{\sqrt{\frac{2}{3} c k + \frac{4}{9} k^2}}.
\]

Notice that $\phi^*$ tends to 0 as $c$ tends to $2k$, according to equation (35), it indicates $\Delta$ is positive for $c$ close to $2k$. Further, by equation (36), we conclude that $\Delta$ and $\partial\Delta/\partial c$ tend to zero as $c$ tends to $2k$ and $\Delta$ tends to negative infinity as $c$ tends to positive infinity. However, because of the complexity of the $\Delta$ expression, it is difficult to get the monotonocity of $\Delta$. It is clear that $\Delta$ is continuous about $c$ for $c > 2k$, whose figure can be well depicted by numerical simulation, see Fig.3 as a typical example. Owing to this, we can obviously find the fact that when $c$ is greater
than $2k$, $\Delta$ has a tendency to increase first and then decrease, which finally tends to negative infinity. As a result, there exists a unique $c^* > 2k$

$$\Delta = \Delta(c^*) = 0, \quad \text{and} \quad \frac{\partial \Delta}{\partial c}(c^*) < 0.$$  

This implies the existence and uniqueness of zeros $\Delta$ as well as zeros of $\text{Sep}(\cdot, \tau)$. We have the following

**Theorem 3.2.** For sufficiently small $\tau > 0$, there exists a unique speed $c$ such that equation (3) with the weak generic nonlocal delay kernel has a generalized solitary wave solution in the sense that the corresponding solitary wave equation (31) has a solution $\Gamma_\tau$ which is heteroclinic to two equilibrium $O_1$ and $O_2$ lying $O(\tau)$ close to the origin, and that $\Gamma_\tau$ lies uniformly $C^1$ $O(\tau)$ close to the unperturbed homoclinic $\Gamma$ and approaches $\Gamma$ in $C^1$ norm as $\tau \to 0$.

The rest of the proof is exactly the same as that following Theorem 3.1.

**Remark 4.** Without the term $\tau u_{xx}$, the Melnikov integral is calculated to be:

$$\Delta = -4 \int_0^{c^*} \left[ c(2c - \phi) - \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)} \right]$$

$$\times \frac{\phi^2}{c - \phi} \sqrt{\phi^2 - 2(c - \frac{2}{3}k)\phi + c(c - 2k)} d\phi.$$  

If $k > \frac{\sqrt{3}}{2}$, then $c > \frac{\sqrt{3}}{2}$ and $\Delta$ is always negative. If $k$ is smaller, it is then not so clear whether $\Delta$ has zeros.

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