ON DUAL CANONICAL BASES

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ABSTRACT. The dual basis of the canonical basis of the modified quantized enveloping algebra is studied, in particular for type A. The construction of a basis for the coordinate algebra of the $n \times n$ quantum matrices is appropriate for the study the multiplicative property. It is shown that this basis is invariant under multiplication by certain quantum minors including the quantum determinant. Then a basis of quantum $SL(n)$ is obtained by setting the quantum determinant to one. This basis turns out to be equivalent to the dual canonical basis.

1. INTRODUCTION

Throughout the paper, the base field is $K = \mathbb{Q}(q)$, i.e., the field of quotients of polynomials in the indeterminate $q$ with rational coefficients. Let $A$ be an algebra over $K$. Two elements $b, b' \in A$ are called equivalent (denoted by $b \sim b'$) if there exists $m \in \mathbb{Z}$ such that $b' = q^m b$. Two elements $b, b'$ are called $q$-commuting if $bb' \sim b' b$.

Let $g$ be the Kac-Moody algebra associated to a $n \times n$ symmetrizable Cartan matrix $A$. Let $U_q(g)$ be the quantized enveloping algebra associated to $g$, with its two usual subalgebras $U_q(n^+)$ and $U_q(n^-)$ (see section 2 for details). The dual basis of the canonical basis of $U_q(n^-)$ has been widely studied in literature. In [6], a conjecture posed by Berenstein and Zelevinsky is stated as follows: Two elements $b_1, b_2$ of the dual canonical basis are $q$-commuting with each other, if and only if $b_1 b_2 \sim b$ for some $b$ in the dual canonical basis. This property of basis is called the multiplicative property. By use of the Hall algebra technique, the multiplicative property of the dual canonical basis of $U_q(n^-)$ is studied in [14]. In [8], counter-examples are given for the Berenstein-Zelevinsky conjecture by finding some so-called imaginary vectors. There are many connections between the irreducible representations of Hecke algebras of $A$ type and the multiplicative property of the dual canonical basis, see [8] and [9].

Let $L(\lambda)$ be an irreducible highest weight module for $U_q(g)$ and let $L^*(\lambda)$ be its graded dual. In [10], Lusztig constructed a canonical basis of the tensor product $U(\lambda, \mu) := L(\lambda) \otimes L^*(\mu)$ which can be lifted to a canonical basis $\tilde{B}$ of the so-called
modified quantized enveloping algebra \( \tilde{U}_q(g) \). In the present paper we will show that the module \( L(\lambda) \otimes L^*(\mu) \) is absolutely indecomposable if the Kac-Moody algebra \( g \) is of affine or indefinite type. Next, we focus on the case of type \( A \). By constructing a basis of the coordinate algebra \( O_q(M(n)) \) of the \( n \times n \) quantum matrices, we get a basis of \( O_q(SL(n)) \) which turns out to be equivalent to the dual canonical basis. A pleasant aspect of this construction is that it is appropriate to study the multiplicative property of the basis.

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2. Kashiwara’s construction

Let \( g \) be the Kac-Moody algebra associated to a \( n \times n \) symmetrizable Cartan matrix \( A \). One can choose a bilinear form such that the integral weight lattice is an even integral lattice. Let \( \Pi = \{ \alpha_1, \alpha_2, \cdots, \alpha_n \} \) and \( \Pi^v = \{ \alpha_1^v, \alpha_2^v, \cdots, \alpha_n^v \} \) be the set of simple roots and the set of simple coroots respectively. Let \( U_q(g) \) be the quantized enveloping algebra associated to \( g \) with generators \( E_1, \cdots, E_n, F_1, \cdots, F_n, K_1, K_1^{-1}, \cdots, K_n, K_n^{-1} \) with the usual defining relations (see e.g. [7]) by replacing \( q \) by \( q^2 \) because we do not want to use the square root of \( q \) later.

Let \( U_q(n^+) \) (resp. \( U_q(n^-) \)) be the subalgebra generated by \( E_1, \cdots, E_n \) (resp. \( F_1, \cdots, F_n \)). For any dominant weight \( \lambda \), denote by \( L(\lambda) \) the irreducible highest weight module over \( U_q(g) \) with the highest weight \( \lambda \). Denote by \( L^*(\lambda) \) the graded dual of \( L(\lambda) \) which is an irreducible lowest weight module with the lowest weight \(-\lambda \). Let \( - \) be the automorphism of the algebra \( U_q(g) \) given by

\[
\bar{q} = q^{-1}, \quad \bar{E}_i = E_i, \quad \bar{F}_i = F_i, \quad \bar{K}_i = K_i^{-1}
\]

for all \( i \). Let \( v_\lambda \) (resp. \( v^*_\mu \)) be a highest weight vector of \( L(\lambda) \) (resp. the lowest weight vector of \( L^*(\mu) \)). Denote also by \( - \) the linear automorphism of the module \( L(\lambda) \) and of the module \( L^*(\mu) \) given by

\[
\bar{p}v_\lambda = p\bar{v}_\lambda, \quad \bar{p}v^*_\mu = \bar{p}v^*_\mu
\]

for \( p \in U_q(g) \).

Remark 2.1. Although we use \( - \) to denote several different automorphism of various spaces. One can identify the meaning of the \( - \) from the context.
In [10], Lusztig constructed a canonical basis of the tensor product \( U(\lambda, \mu) := L(\lambda) \otimes L^*(\mu) \) which can be lifted to a canonical basis \( \hat{B} \) of the modified quantized enveloping algebra \( \hat{U}_q(g) \).

We will not go in detail about the canonical basis of the module \( U(\lambda, \mu) \). However, we would like to show one remarkable fact about the module \( U(\lambda, \mu) \). It is known that if \( g \) is of finite type, \( U(\lambda, \mu) \) is of finite dimensional and is indecomposable if and only if one of \( \lambda \) and \( \mu \) is zero. However, if \( g \) is of affine or indefinite type, the situation changes dramatically.

**Theorem 2.2.** If \( g \) is of affine or indefinite type, then

\[
\text{End}_{U_q(g)} U(\lambda, \mu) \cong \mathbb{Q}(q).
\]

Hence, \( U(\lambda, \mu) \) is absolutely indecomposable.

**Proof:** Clearly, if \( \lambda \) or \( \mu \) is trivial, then \( U(\lambda, \mu) \) is a lowest weight module or a highest weight module and the theorem holds. Hence, we may assume that both \( \lambda \) and \( \mu \) are nontrivial.

It is known that \( U(\lambda, \mu) \) is a cyclic module and is generated by \( v_\lambda \otimes v^*_\mu \). For any \( \psi \in \text{End}_{U_q(g)} U(\lambda, \mu) \), then \( \psi(v_\lambda \otimes v^*_\mu) = u(v_\lambda \otimes v^*_\mu) \in U(\lambda, \mu)_{\lambda-\mu} \) for some \( u \in U_q(g) \) which is of weight zero. If \( u(v_\lambda \otimes v^*_\mu) \) is not a multiple of \( v_\lambda \otimes v^*_\mu \), then

\[
u(v_\lambda \otimes v^*_\mu) = s(v_\lambda \otimes v^*_\mu) + \sum_i u_i v_\lambda \otimes w_i v^*_\mu
\]

where \( s \in \mathbb{Q}(q) \), \( u_i \in U_q(n^-) \), \( w_i \in U_q(n^+) \), for all \( i \), and the set \( \{w_i v^*_\mu\}_i \) is linearly independent. Choose \( w_k v^*_\mu \), such that its weight is maximal among all of the weights of \( w_i v^*_\mu \) for all \( i \). Assume that \( u_k v_\lambda \in L(\lambda)_\Lambda \), where \( \Lambda \) must be smaller than \( \lambda \).

1. If the Cartan matrix \( A \) is of indefinite type, then there exists \( \alpha_i^v \) such that

\[
< \lambda - \Lambda, \alpha_i^v > < 0,
\]

i.e. \( < \lambda, \alpha_i^v > < < \Lambda, \alpha_i^v > \) and so \( F_i^{<,\lambda,\alpha_i^v>\geq+1} u_k v_\lambda \neq 0. \)

However, \( F_i^{<,\lambda,\alpha_i^v>+1} u(v_\lambda \otimes v^*_\mu) = \psi(F_i^{<,\lambda,\alpha_i^v>+1}(v_\lambda \otimes v^*_\mu)) = 0. \) On the other hand, \( F_i^{<,\lambda,\alpha_i^v>+1} u(v_\lambda \otimes v^*_\mu) = \sum_{m,j} c_i^{(m)} c_{ijj}^{(m)} F_i^{<,\lambda,\alpha_i^v>+1-m} u_j v_\lambda \otimes F_i^{m} w_j v^*_\mu \), where \( c_i^{(m)} \in \mathbb{Q}(q) \).

One can see easily that \( c_i^{(0)} = 1. \) Hence, \( F_i^{<,\lambda,\alpha_i^v>+1} u_k v_\lambda = 0. \) Contradiction!

2. Now, we may assume that the Cartan matrix \( A \) is of affine type. If there exists \( \alpha_i^v \) such that

\[
< \lambda - \Lambda, \alpha_i^v > < 0,
\]

then we can prove in the same way as above. If \( < \lambda - \Lambda, \alpha_i^v > \geq 0 \) for all \( i \), then we must have \( < \lambda - \Lambda, \alpha_i^v > = 0 \) for all \( i \). As there exists \( E_i \) such that \( E_i u_k v_\lambda \neq 0 \), we have again that \( F_i^{<,\lambda,\alpha_i^v>+1} u_k v_\lambda \neq 0. \)

Therefore, \( u(v_\lambda \otimes v^*_\mu) \) is a multiple of \( v_\lambda \otimes v^*_\mu \) and so \( \psi \) is a scalar endomorphism. Moreover, \( U(\lambda, \mu) \) is absolutely indecomposable. \qed
Let $U^Z(g)$ be the integral form of the quantized enveloping algebra which is a $\mathbb{Z}[q, q^{-1}]$-subalgebra of the quantized enveloping algebra $U_q(g)$ generated by the divided powers $E_i^{(s)} := E_i^s/[s]!, F_i^{(s)} := F_i^s/[s]!, K_i, K_i^{-1}$ for all $i$. The quantum integer is defined by $[s] = \frac{q^s - q^{-s}}{q - q^{-1}}$, (one may refer to [4] for terminologies and notations).

Denote by $U_q(g)^*$ the linear dual of the algebra $U_q(g)$. Since $U_q(g)$ is a $U_q(g)$ bi-module, $U_q(g)^*$ has an induced $U_q(g)$ bi-module structure. Let

$$A_q(g) = \{ f \in U_q(g)^* | \text{there exists } l \geq 0 \text{ such that } E_{i_1} \cdots E_{i_l} f = 0 \text{ for any } i_1, \cdots, i_l \}.$$ (2.1)

The quantum Peter-Weyl theorem was proved in [6]

**Theorem 2.3.** As $U_q(g)$ bi-modules

$$A_q(g) \cong \bigoplus_{\lambda \in P} L(\lambda) \otimes L^*(\lambda)$$

where $u \otimes v \in L(\lambda) \otimes L^*(\lambda)$ viewed as a linear function on $U_q(g)$ as follows:

$$(u \otimes v)(p) = \langle up, v \rangle, \text{ for } p \in U_q(g),$$

where $L(\lambda)$ and $L^*(\lambda)$ are viewed as right $U_q(g)$ module and left $U_q(g)$ module respectively.

Let $A$ be the subring of $\mathbb{Q}(q)$ consists of the rational functions of $q$ which are regular at $q = 0$. Let $-$ be the ring endmorphism of $\mathbb{Q}(q)$ sending $q$ to $q^{-1}$.

Let $M$ be an integral $U_q(g)$ module. Then

$$M = \bigoplus_{\lambda} F_i^{(n)}(\text{Ker}E_i \cap M_{\lambda}).$$

We define the lower Kashiwara operators $e_i, f_i$ of $M$ by

$$f_i(F_i^{(n)}u) = F_i^{(n+1)}u \text{ and } e_i(F_i^{(n)}u) = F_i^{(n-1)}u$$

for $u \in \text{Ker}E_i \cap M_{\lambda}$.

**Definition 2.4.** A pair $(L, B)$ is called a lower crystal base of $M$ if it satisfies the following conditions:

1. $L$ is a free sub-$A$-module of $M$ such that $M \cong \mathbb{Q}(q) \otimes A L$.
2. $B$ is a base of the $\mathbb{Q}$-vector space $L/qL$.
3. $e_i L \subset L$ and $f_i L \subset L$ for any $i$.
4. $e_i B \subset B \cup \{0\}$ and $f_i B \subset B \cup \{0\}$.
5. $L = \bigoplus_{\lambda \in P} L_{\lambda}$ and $B = \bigcup_{\lambda \in P} B_{\lambda}$, where $L_{\lambda} = L \cap M_{\lambda}, B_{\lambda} = B \cap L_{\lambda}/qL_{\lambda}$.
6. For any $b, b' \in B$, $b' = f_i b$ if and only if $b = e_i b'$. 
The upper Kashiwara operators $e'_i$ and $f'_i$ are defined as follows: for $u \in \text{Ker} E_i \cap M_\lambda$ and $0 \leq n \leq <\lambda, \alpha^v>$,

$$e'_i(F_i^{(n)}u) = \frac{[<\lambda, \alpha^v> - n + 1]}{[n]} F_i^{(n-1)}u,$$

and

$$f'_i(F_i^{(n)}u) = \frac{[n + 1]}{[<\lambda, \alpha^v> - n]} F_i^{(n+1)}u.$$

We say that $(L, B)$ is an upper crystal base if $(L, B)$ satisfies the conditions in the definition of lower crystal base with $e'_i, f'_i$ instead of $e_i, f_i$.

For $\lambda \in P$, we define $\psi_M \in \text{Aut} M$ by

$$\psi_M(u) = q^{-2(\lambda, \lambda)}u$$

for $u \in M_\lambda$. It is known that $\psi_M^{-1}e'_i\psi_M$ (resp. $\psi_M^{-1}f'_i\psi_M$) coincides with $e_i$ (resp. $f_i$) on $L/qL$.

In [5], Kashiwara proved that

**Lemma 2.5.** $(L, B)$ is a lower crystal base if and only if $\psi_M(L, B)$ is an upper crystal base.

Let $L(\lambda)$ be the upper crystal lattice which is the smallest $A$ submodule of $L(\lambda)$ containing $v_\lambda$ and is stable under the action of upper Kashiwara operators. Similarly, let $L^*(\lambda)$ be the upper crystal lattice which is the smallest $A$ submodule of $L^*(\lambda)$ containing $v_\lambda^*$ and is stable under the action of upper Kashiwara operators. Set

$$\mathcal{L}(A_q(g)) := \bigoplus_{\lambda \in P_+} \mathcal{L}(\lambda) \otimes \mathcal{L}^*(\lambda).$$

Define that

$$<\bar{u}, p> = <u, p>,$$

then one can check that $\bar{u} \otimes v = \bar{u} \otimes \bar{v}$ for $u \in L(\lambda)$ and $v \in L^*(\lambda)$. Hence

$$\overline{\mathcal{L}(A_q(g))} = \bigoplus_{\lambda \in P_+} \overline{\mathcal{L}(\lambda)} \otimes \overline{\mathcal{L}^*(\lambda)}.$$

Let

$$A^Z_q(g) = \{ f \in A_q(g) | <f, U^Z(g)> \subset \mathbb{Z}[q, q^{-1}] \}$$

Let $u \otimes v \in L(\lambda) \otimes L^*(\lambda)$, where $u$ (resp. $v$) is a weight vector of weight $\lambda_l$ (resp. $\lambda_r$). Then $u \otimes v$ is called a weight vector with left weight $\lambda_l$ and right weight $\lambda_r$. An element in $A_q(g)$ is called a refined weight vector if it is a linear combination of the elements $u \otimes v$ with the same left and right weights.

Let us recall the definition of balanced triple. Let $V$ be a vector space over $\mathbb{Q}(q)$, a $B$-lattice of $V$ is a $B$-submodule $M$ of $V$ such that $V \cong \mathbb{Q}(q) \otimes_B M$. Let $V_\mathbb{Z}$
be a $\mathbb{Z}[q, q^{-1}]$-lattice of $V$, $L$ an $A$-lattice of $V$, and $\overline{L}$ an $\overline{A}$-lattice of $V$. In [5], it was proved that

**Lemma 2.6.** Set $E = V_{\mathbb{Z}} \cap L \cap \overline{L}$. Then the following conditions are equivalent.

1. $E \rightarrow V_{\mathbb{Z}} \cap L \cap qL$ is an isomorphism.
2. $E \rightarrow V_{\mathbb{Z}} \cap \overline{L} \cap V_{\mathbb{Z}} \cap q^{-1}L$ is an isomorphism.
3. $V_{\mathbb{Z}} \cap qL \oplus V_{\mathbb{Z}} \cap L \rightarrow V_{\mathbb{Z}}$ is an isomorphism.
4. $A \otimes E \rightarrow L$, $\overline{A} \otimes E \rightarrow \overline{L}$, $\mathbb{Z}[q, q^{-1}] \otimes E \rightarrow V_{\mathbb{Z}}$, $\mathbb{Q}(q) \otimes E \rightarrow V$ are isomorphisms.

We call $(L, \overline{L}, V_{\mathbb{Z}})$ balanced if these equivalent conditions are satisfied. Let us denote by $G$ the inverse of the isomorphism $E \rightarrow V_{\mathbb{Z}} \cap \overline{L} \cap \overline{L}$. If $B$ is a base of $V_{\mathbb{Z}} \cap \overline{L} \cap \overline{L}$, then $\{G(b) | b \in B\}$ is a base of $V$.

In [6], it was proved that $(A_q^Z(g), \mathcal{L}(A_q(g)), \overline{\mathcal{L}}(A_q(g)))$ is a balanced triple. Hence there is a $\mathbb{Z}$ basis $B'$ of

$$(A_q^Z(g) \cap \mathcal{L}(A_q(g)) \cap \overline{\mathcal{L}}(A_q(g))).$$

In [7], it was shown that $B'$ is the dual basis of the canonical basis of the modified enveloping algebra $\tilde{U}_q(g)$ if $g$ if of finite type.

In the following, we always assume that $g$ is of finite type. We fix a reduced expression in the longest element of the Weyl group. Let $F_{\beta_1}, F_{\beta_2}, \cdots, F_{\beta_N}$ be the ordered root vectors given defined according to the chosen reduced expression of the longest element in the Weyl group, where $N$ is the length of the longest element in the Weyl group. For any $I = (i_1, i_2, \cdots, i_N) \in \mathbb{Z}_+^N$, denote by $F^I$ the monomial $F_{\beta_{i_1}}^{(i_1)} F_{\beta_{i_2}}^{(i_2)} \cdots F_{\beta_{i_N}}^{(i_N)}$ which form a PBW type basis of the subalgebra $U_q(n^-)$. The monomial $E^I$ is defined similarly which form a PBW type basis of the subalgebra $U_q(n^+)$. Let $B^-$ and $B^+$ be the canonical basis of $U_q(n^-)$ and $U_q(n^+)$ respectively. For any dominant weight $\lambda$, denote by

$$B^-_\lambda = \{ b \in B^- | bv_\lambda \neq 0 \}$$

and

$$B^+_\lambda = \{ b' \in B^+ | b'v^*_\lambda \neq 0 \}.$$ 

Note that each dual canonical basis element is a refined weight vector. Hence, we only need to consider the homogeneous part $\mathcal{L}(\lambda) \mu \otimes \mathcal{L}^*(\lambda)_{\gamma} \cap \overline{\mathcal{L}}(\lambda) \mu \otimes \mathcal{L}^*(\lambda)_{\gamma} \cap A_q^Z(g)$. 

It is well-known that any canonical basis element $b$ in $B^-$ is of the form

$$b = F^I + \sum_{I'} a_{I,I'} F^{I'}$$

where the coefficients $a_{I,I'} \in q\mathbb{Z}[q]$ and the element $b$ is $-\text{invariant}$. $F^I$ is called the leading term of $b$. The canonical basis elements in $B^+$ have the similar form.

Let

$$C^-_\lambda = \{ F^I \mid F^I \text{ is the leading term of an element } b \in B^- \lambda \}$$

and let

$$C^+_\lambda = \{ E^I \mid E^I \text{ is the leading term of an element } b' \in B^+ \lambda \}.$$

Then $C^-_\lambda v_\lambda$ (resp. $C^+_\lambda v^*_\lambda$) is the PBW-basis of $L(\lambda)$ (resp. $L(\lambda)^*$) with an order given by the chosen reduced expression of the longest element in the Weyl group.

We order the PBW type basis $\bigcup \lambda C^-_\lambda \otimes C^+_\lambda$ by lexicographic ordering.

**Theorem 2.7.** The basis $B'$ is characterized by the following two conditions.

1. $b' = F^I v_\lambda \otimes E^{I'} v^*_\lambda + \sum_{I_k, I'_k} a_{I_k, I'_k}^{I_k, I'_k} F^{I_k} v_\lambda \otimes E^{I'_k} v^*_\lambda$ where $a_{I_k, I'_k}^{I_k, I'_k} \in q\mathbb{Z}[q]$ and $a_{I_k, I'_k}^{I_k, I'_k} \neq 0$ only if $(I_k, I'_k) \leq (I, I')$, for any $b' \in B'$.
2. $\overline{b'} = b'$.

**Proof:** Clearly, each element $b v_\lambda \otimes b' v^*_\lambda$ satisfies the two conditions. The uniqueness can be proved in the same way as in [1].

The following result was proved in [6].

**Proposition 2.8.** Let $x$ and $y$ be refined weight vectors of weights $(\lambda_l, \lambda_r)$ and $(\mu_l, \mu_r)$ respectively. Then

$$\overline{x y} = q^{2(\lambda_r, \mu_r) - 2(\lambda_l, \mu_l)} \overline{y} \overline{x}.$$

By using the above Proposition, one can easily verify that

**Lemma 2.9.** The mapping

$$\phi : A_q(g) \longrightarrow A_q(g),$$

$$q \mapsto q^{-1},$$

$$u \otimes v \mapsto q^{(\lambda_l, \lambda_r) - (\lambda_l, \lambda_r)} \overline{u} \otimes \overline{v}.$$
Let \( b' \in B' \) with weights \( (\lambda_l, \lambda_r) \). Then the element \( b = q^{\frac{1}{2}(\lambda_l, \lambda_l) - (\lambda_r, \lambda_r)}b' \) is invariant under the anti automorphism \( \phi \). Let
\[
L^* = \{b | b' \in B'\}.
\]
Then \( L^* \) is also a \( \mathbb{Z}[q, q^{-1}] \) basis of \( A^Z(g) \).

It is clearly that the multiplicative properties of \( B' \) and \( L^* \) are the same.

**Proposition 2.10.** Let \( b_1, b_2 \in L^* \). Assume that \( b_1b_2 \sim b \) for some \( b \in L^* \). Then \( b_1b_2 \sim b_2b_1 \).

**Proof:** Assume that \( b_1b_2 = qa \) for some \( a \in \mathbb{Z} \). Applying the anti automorphism \( \phi \), we deduce that \( b_2b_1 = q^{-a}b \). Hence, \( b_1b_2 = q^2b_2b_1 \).

\[3.\text{ THE CONSTRUCTION OF THE BASIS OF } O_q(M(n))\]

The coordinate algebra \( O_q(M(n)) \) of the quantum matrix is an associative algebra, generated by elements \( Z_{ij}, i, j = 1, 2, \cdots, n \), subject to the following defining relations:

\[
Z_{ij}Z_{ik} = q^2Z_{ik}Z_{ij} \text{ if } j < k, \quad (3.1)
\]
\[
Z_{ij}Z_{kj} = q^2Z_{kj}Z_{ij} \text{ if } i < k, \quad (3.2)
\]
\[
Z_{ij}Z_{st} = Z_{st}Z_{ij} \text{ if } i > s, j < t, \quad (3.3)
\]
\[
Z_{ij}Z_{st} = Z_{st}Z_{ij} + (q^2 - q^{-2})Z_{it}Z_{sj} \text{ if } i < s, j < t. \quad (3.4)
\]

For any matrix \( A = (a_{ij})^{n}_{i,j=1} \in M_n(\mathbb{Z}_+)(\mathbb{Z}_+ = \{0, 1, \cdots\}) \) we define a monomial \( Z^A \) by
\[
Z^A = \prod_{i,j=1}^{n} Z_{ij}^{a_{ij}}, \quad (3.5)
\]
where the factors are arranged in the lexicographic order on \( I(n) = \{(i, j) \mid i, j = 1, \ldots, n\} \). It is well known that the set \( \{Z^A | A \in M_n(\mathbb{Z}_+)\} \) is a basis of the algebra \( O_q(M(n)) \).

From the defining relations (3.1) of the algebra \( O_q(M(n)) \), it is easy to show the following lemma.

**Lemma 3.1.** (1) The mapping
\[
- : Z_{ij} \mapsto Z_{ij} \quad (3.6)
\]
\[
q \mapsto q^{-1}
\]
extends to an algebra anti-automorphism of the algebra \( O_q(M(n)) \) as an algebra over \( \mathbb{Q} \).
(2) The mapping 

\[ \sigma : Z_{ij} \mapsto Z_{ji} \]  

(3.7) 

extends to an algebra automorphism of the algebra \( O_q(M(n)) \) as an algebra over \( K = \mathbb{Q}(q) \).

For any \( A = (a_{ij})_{n \times n} \in M_n(\mathbb{Z}_+) \). Let 

\[ \text{ro}(A) = (\sum_j a_{1j}, \ldots, \sum_j a_{nj}) = (r_1, r_2, \ldots, r_n) \] 

which is called the row sum of \( A \) and

\[ \text{co}(A) = (\sum_j a_{1j}, \ldots, \sum_j a_{jn}) = (c_1, c_2, \ldots, c_n) \] 

which is called the column sum of \( A \).

For any matrix \( A = (a_{ij})_{i,j=1}^n \in M_n(\mathbb{Z}_+) \), a monomial having the factors of \( Z^A \)in arbitrary order. Then its expansion in terms of monomials \( Z^B \) only involves terms where \( \text{ro}(B) = \text{ro}(A) \) and \( \text{co}(B) = \text{co}(A) \). Let \( Pr(A, s, t) = \sum_{i \leq s, j \leq t} a_{ij} \). Then \( Pr(A, s, t) \geq Pr(B, s, t) \) for any \( s, t \leq n \) and matrix \( B \) appeared in the expansion considered above.

From the defining relations (3.1) of the algebra \( O_q(M(n)) \), we have

\[ \overline{Z^A} = E(A)Z^A + \sum_B c_B(A)Z^B, \] 

(3.8)

where

\[ E(A) = q^{-2(\sum_i \sum_{j>k} a_{ij}a_{ik} + \sum_i \sum_{j>k} a_{ji}a_{ki})} \]

and \( B < A, \text{ro}(B) = \text{ro}(A), \text{co}(B) = \text{co}(A), c_B(A) \in \mathbb{Z}[q, q^{-1}], \leq \) is the lexicographic ordering.

For a pair of vectors \( R, C \in \mathbb{Z}_+^n \), denote by \( M(R, C) \) the subspace of \( O_q(M(n)) \) spanned by \( Z^A \) with \( \text{ro}(A) = R \) and \( \text{co}(A) = C \). Note that \( M(R, C) \) is invariant and \( O_q(M(n)) = \oplus_{R,C} M(R, C) \).

Let \( D(A) = q^{-\sum_i \sum_{j>k} a_{ij}a_{jk} - \sum_i \sum_{j>k} a_{ji}a_{ki}} \) and let \( Z(A) = D(A)Z^A \). Set

\[ L^* = \oplus_{A \in M_n(\mathbb{Z}_+)} \mathbb{Z}[q]Z(A). \]

**Theorem 3.2.** There is a unique basis \( B^* = \{ b(A) | A \in M_n(\mathbb{Z}_+) \} \) of \( L^* \) determined by the following conditions:

(1) \( \overline{b(A)} = b(A) \) for all \( A \).

(2) \( b(A) = Z(A) + \sum_{B<A} h_B(A)Z(B) \) where \( h_B(A) \in q\mathbb{Z}[q] \) and \( \text{ro}(B) = \text{ro}(A), \text{co}(B) = \text{co}(A) \).
Proof: We rewrite the equation (3.8) in terms of \(Z(A)\), then
\[
Z(A) = \sum_B a_{AB} Z(B),
\] (3.9)
where \(a_{AA} = 1\), \(a_{AB} \in \mathbb{Z}[q, q^{-1}]\) and \(a_{AB} = 0\) unless \(B \leq A\), where \(\leq\) is the lexicographic ordering. By Theorem 1.2 of [1], there is an IC-basis with respect to the triple (\(\{Z^A \mid A \in M_n(\mathbb{Z}_+)\}, -, \leq\)) determined by the relation stated in the context of the theorem. □

The quantum determinant \(\det_q\) is defined as follows:
\[
\det_q = \sum_{\sigma \in S_n} (-q^2)^{l(\sigma)} Z_{1\sigma(1)} Z_{2\sigma(2)} \cdots Z_{n\sigma(n)}. \tag{3.10}
\]
It is known that \(\det_q\) is a central element of the algebra \(O_q(M(n))\).

For later reference we now introduce some terminology. Let \(m \leq n\) be a positive integer. Given any two subsets \(I = \{i_1, i_2, \cdots, i_m\}\) and \(J = \{j_1, j_2, \cdots, j_m\}\) of \(\{1, 2, \cdots, n\}\), each having cardinality \(m\), it is clear that the subalgebra of \(O_q(M(n))\) generated by the elements \(Z_{i_r j_s}\) with \(r, s = 1, 2, \cdots, m\), is isomorphic to \(O_q(M(m))\), so we can talk about its determinant. Such a determinant is called a quantum minor, and will be denoted by \(\det_q(I, J)\).

Let \(I, J\) be two subsets of \(\{1, 2, \cdots, n\}\) with the same cardinality. Obviously, the dual canonical basis of the subalgebra generated by \(Z_{ij}\) for \(i \in I, j \in J\) is a subset of the basis \(B^*\) of the algebra \(O_q(M(n))\). More generally, if \((u, v) \leq (s, t)\), then the subalgebra \(O_q(M(n))_{(u,v)}^{(s,t)}\) generated by \(Z_{ij}\), for \((u, v) \leq (i, j) \leq (s, t)\), is invariant and one can construct a basis analogous to the construction of the basis considered in Theorem 3.2, and obviously the resulting basis of \(O_q(M(n))_{(u,v)}^{(s,t)}\) is a subset of the basis \(B^*\).

Lemma 3.3. The quantum determinant \(\det_q\) is an element of the basis \(B^*\). Furthermore, any quantum minor is also an element of the dual canonical basis.

Proof: We only need to show that \(\det_q\) is invariant. It is well known that the center of the algebra \(O_q(M(n))\) is generated by the quantum determinant [12]. Note that
\[
\overline{\det_q} Z_{ij} = \overline{Z_{ij} \det_q} = \det_q Z_{ij} = Z_{ij} \overline{\det_q}, \tag{3.11}
\]
for any \(i, j\). Hence, \(\overline{\det_q}\) is a polynomial of \(\det_q\). Therefore,
\[
\overline{\det_q} = \det_q
by comparing the leading terms.

**Corollary 3.4.** The basis $B^*$ is $\sigma$ invariant. More precisely,

$$\sigma(b(A)) = b(A^T),$$

for all $A \in M_n(\mathbb{Z}_+)$, where $A^T$ is the transposition of $A$.

*Proof:* Let $b(A)$ be an element of the dual canonical basis $B^*$ of the form given in Theorem 3.2 (2). Then it follows that all of the matrices $B$ appearing in the expansion of $b(A)$ are obtained from $A$ by a sequence of $2 \times 2$ submatrix transformations of the following form:

$$\begin{pmatrix} a_{ij} & a_{it} \\ a_{sj} & a_{st} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{ij} - 1 & a_{it} + 1 \\ a_{sj} + 1 & a_{st} - 1 \end{pmatrix},$$

(3.12)

if both $a_{ij}$ and $a_{st}$ are positive. Hence $B^T$ can be obtained from $A^T$ by a sequence of the submatrix transformations of the form:

$$\begin{pmatrix} a_{ji} & a_{ti} \\ a_{js} & a_{ts} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{ji} - 1 & a_{ti} + 1 \\ a_{js} + 1 & a_{ts} - 1 \end{pmatrix}.$$

(3.13)

Especially, $B^T \leq A^T$. Note that the monomials $Z^B$ and $\sigma(Z^B)$ have the same factors but could be in different order. However, two generators $Z_{ij}$ and $Z_{st}$ appear in the monomials but in different order must satisfy the third relation in (3.1). Hence, $Z^{B^T} = \sigma(Z^B)$

$$\sigma(b(A)) = Z(A^T) + \sum_B h_B(A)Z(B^T)$$

(3.14)

with $h_B(A) \in q\mathbb{Z}[q]$. Clearly,

$$\overline{\sigma(b(A))} = \sigma(b(A))$$

since $\sigma$ and $-$ commute with each other. □

Denote by $I_n$ the $n \times n$ identity matrix.

**Lemma 3.5.** For any $A \in M_n(\mathbb{Z}_+)$,

$$Z(A)\det_q = Z(A + I_n) \mod qL^*.$$

*Proof:* For $i < s, j < t$, we have

$$Z_{si}^m Z_{ij} = Z_{ij} Z_{st}^m + (q^{2m} - q^2) Z_{it} Z_{sj} Z_{st}^{m-1}.$$
Recall that
\[ \det_q = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} q^{2l(\sigma)} Z_{1\sigma(1)} Z_{2\sigma(2)} \cdots Z_{n\sigma(n)}. \]
When we compute \( Z(A) \det_q \), we only have to deal with those coefficients of the form \( q^{-2a} \) with \( a \) a positive integer. Assume that
\[ Z(A) \det_q = \sum a_B Z(B). \]
Clearly, \( a_B \in \mathbb{Z}[q, q^{-1}] \) and the leading term is \( Z(A + I_n) \) and those matrix \( B \) appeared in the expression has at least one nonzero entry in each row and each column. We need to compute \( Z(A)(-1)^{l(\sigma)} q^{2l(\sigma)} Z_{1\sigma(1)} Z_{2\sigma(2)} \cdots Z_{n\sigma(n)}, \) for all \( \sigma \in S_n \). From the expression of the quantum determinant we see that there are four possibilities to produce coefficients of the form \( q^{-2a} \) with \( a \) a positive integer.

**Case 1.** \( Z_{st}^m Z_{sj} = q^{-2m} Z_{sj} Z_{st} \) where \( t > j \) but no \( Z_{it} \) behind. Then \( q^{2m} \) will be absorbed by \( D(B) \) where \( Z(B) \) is the resulted term.

**Case 2.** \( Z_{st}^m Z_{it} = q^{-2m} Z_{it} Z_{st} \) where \( s > i \) but no \( Z_{sj} \) appeared before. Then \( q^{2m} \) will be absorbed by \( D(B) \) where \( Z(B) \) is the resulted term.

**Case 3.** Both \( Z_{st}^m Z_{sj} = q^{-2m} Z_{sj} Z_{st} \) where \( t > j \) and \( Z_{st}^m Z_{it} = q^{-2m} Z_{it} Z_{st} \) where \( s > i \) happened. Then we get \( q^{-4m} \). However, we will see that it will be cancelled by a term in the next case. To this end, we need to remember that the terms we are dealing with are from \( Z(A) Z_{1\sigma(1)} Z_{2\sigma(2)} \cdots Z_{n\sigma(n)} \). Note that \( l(\sigma(jt)) = l(\sigma) - 1 \).

**Case 4.** \( Z_{st}^m Z_{ij} = Z_{ij} Z_{st}^m + (q^{2-4m} - q^2) Z_{it} Z_{sj} Z_{st}^{m-1} \) where \( s > i, t > j \). Then the coefficient \( q^{2-4m} \) will be cancelled by a term in case 3.

Hence, the coefficients \( a_B \) are all in \( q\mathbb{Z}[q] \) except \( a_A \) which is 1. \( \square \)

The following proposition follows directly from the above lemma.

**Proposition 3.6.** The basis \( B^* \) is invariant under the multiplication of \( \det_q \). More precisely,
\[ b(A) \det_q = b(A + I_n) \]
for all \( A \in M_n(\mathbb{Z}_+) \).

By using this proposition, we can determine \( b(A) \), if \( A \) is a diagonal matrix. Let \( A = \text{diag}(a_1, a_2, \ldots, a_n) \). We may assume that \( a_1 \leq a_2 \leq \cdots \leq a_n \) without loss of generality. Then
\[ b(A) = \Pi_{i=1}^n \det_{q,i}^{a_i-a_{i-1}} \]
where \( \det_{q,i} \) is the quantum determinant of the subalgebra generated by \( Z_{st} \) for \( s, t = i, \cdots, n \), and where we put \( a_0 = 0 \).
4. SOME SUBALGEBRAS

In this section, we study the multiplicative property of the basis $B^*$. Similar to the proof of Proposition 2.9, we get

**Lemma 4.1.** Let $b_1, b_2 \in B^*$. If $b_1 b_2 \sim b$ for some $b \in B^*$, then $b_1 b_2 \sim b_2 b_1$.

Divide the matrix by a broken-line $\xi$ which consists lines determined by the equations $ax + by = m$ for $a, b \in \mathbb{Z}_+$ and $m \in \mathbb{N}$ (each line has non-positive slope). Recall that $I(n) = \{(i, j) \mid i, j = 1, \ldots, n\}$. Let

$$ I_1 = \{(x, y) \in I(n) \mid (x, y) \text{ is in the left upper side of the broken line } \xi\}, $$

and let $I_2$ be the complement of $I_1$ in $I(n)$.

Let $O_i$ be the subalgebra of $O_q(M(n))$ generated by $Z_{xy}$ for $(x, y) \in I_i, i = 1, 2$. One can easily see that $O_i$ is determined by the generators $Z_{xy}$ and relations (3.1).

Hence, the algebra $O_i$ is closed under the bar action and therefore there is a basis $B_i^*$ of the sub-lattice $L_i^*$ of the $\mathbb{Z}[q]$-lattice $L^*$ spanned by $\{Z(A) \mid A = (a_{xy}) \in M_n(\mathbb{Z}_+), a_{xy} = 0 \text{ if } (x, y) \in I_{3-i}\}$. Clearly, $B_i^*$ is a subset of $B^*$ consists of those $b(A)$ for $A = (a_{xy}) \in M_n(\mathbb{Z}_+), a_{xy} = 0 \text{ if } (x, y) \in I_{3-i}$.

Write

$$ A = A^+ + A^-, $$

Where the entries of $A^+$ in the left upper side of the broken line $\xi$ are zero and the entries of $A^-$ in the right lower side (including the broken line $\xi$) are zero. Then

**Theorem 4.2.** $b(A) \sim b(A^+)b(A^-)$ if and only if $b(A^+)b(A^-) \sim b(A^-)b(A^+)$

**Proof:** If $b(A) = q^a b(A^+)b(A^-)$ for some integer $a$, then $b(A^-)b(A^+) \sim b(A^+)b(A^-)$ by the above lemma.

For

$$ b(A^+) = Z(A^+) + \sum_{B^+} a_{B^+} Z(B^+), $$

and

$$ b(A^-) = Z(A^-) + \sum_{B^-} a_{B^-} Z(B^-), $$

where $a_{B^+}, a_{B^-} \in q\mathbb{Z}[q]$. Assume that $b(A^+) b(A^-) = q^a b(A^-) b(A^+)$, for some integer $a$ which can be computed by only considering the leading terms. From the defining relations (3.1), the integer $a$ must be even, say, $a = 2m$. Then $q^{-m} b(A^+) b(A^-)$ is bar-invariant with leading term $Z(A)$. Note that the coefficients we encounter only depend on the row sums and column sums. Actually, $m = \sum_j (r_j^+ r_j^- + c_j^+ c_j^-)$ where $(r_1^+, \ldots, r_n^+)$ and $(c_1^+, \ldots, c_n^+)$ (resp. $(r_1^-, \ldots, r_n^-)$...
and \((c_1, \cdots, c_n)\) are the row sum and column sum respectively of \(A^+\) (resp of \(A^-\)). Then all term produce the same \(m\). Therefore,

\[
b(A) = q^{-m}b(A^+)b(A^-)
\]

by Theorem 3.2.

\[\square\]

5. SOME QUANTUM MINORS

Let \(\det_q(t) = \det_q(\{1, \cdots, t\}, \{n - t + 1, \cdots, n\})\), for \(t = 1, 2, \cdots, n\).

Let \(M_t^- = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq t\) and \(1 \leq j \leq n - t\}\), \(M_t^+ = \{(i, j) \in \mathbb{N}^2 \mid t + 1 \leq i \leq n\) and \(n - t + 1 \leq j \leq n\}\), \(M_t^l = \{(i, j) \in \mathbb{N}^2 \mid t + 1 \leq i \leq n\) and \(1 \leq j \leq n - t\}\), and \(M_t^r = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq t\) and \(n - t + 1 \leq j \leq n\}\). The following result was proved in [3].

**Lemma 5.1.** For any \(i, j, t\),

\[
Z_{ij}\det_q(t) = \det_q(t)Z_{ij} \text{ if } (i, j) \in M_t^l \cup M_t^r, \tag{5.1}
\]

\[
Z_{ij}\det_q(t) = q^2\det_q(t)Z_{ij} \text{ if } (i, j) \in M_t^-, \text{ and}
\]

\[
Z_{ij}\det_q(t) = q^{-2}\det_q(t)Z_{ij} \text{ if } (i, j) \in M_t^+.
\]

Let \(E_t = \begin{pmatrix} 0 & I_t \\ 0 & 0 \end{pmatrix}\) and let \(q^{Z}B^* = \{q^ab(A)\mid\text{ for all } A \text{ and } a \in \mathbb{Z}\}\).

**Theorem 5.2.** The set \(q^{Z}B^*\) is invariant under the multiplication of the quantum minors \(\det_q(t)\) and \(\sigma(\det_q(t))\). More precisely,

\[
b(A)\det_q(t) = q^{r_1+\cdots+r_t-c_{n-t+1-\cdots-c_n}}b(A + E_t). \tag{5.2}
\]

\[
b(A)\sigma(\det_q(t)) = q^{c_1+c_2+\cdots+c_t-r_{n-t+1-\cdots-r_n}}b(A + E_t^T), \tag{5.3}
\]

where \(E_t^T\) is the transposition of the matrix \(E_t\).

**Proof:**

For any \(A \in M_n(\mathbb{Z}_+),\) write

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

where \(A_{11}\) is a \(t \times (n-t)\) submatrix, \(A_{12}\) is a \(t \times t\) submatrix, \(A_{21}\) is a \((n-t) \times (n-t)\) submatrix and \(A_{22}\) is a \((n-t) \times t\) submatrix. Then two monomials \(Z^A\) and
$Z^{A_{11}}Z^{A_{12}}Z^{A_{21}}Z^{A_{22}}$ have the same factors but could be in different order. However, the first monomial can be obtained from the second one by applying the third defining relation of the algebra $O_q(M(n))$. Hence,

$$Z^A = Z^{A_{11}}Z^{A_{12}}Z^{A_{21}}Z^{A_{22}},$$

and

$$Z(A)\det_q(t) = q^{-2} \sum_{i\geq t+1, j\geq n-t+1} a_{ij} D(A) Z^{A_{11}}Z^{A_{12}}\det_q(t)Z^{A_{21}}Z^{A_{22}}.$$  

Apply the above lemma to $Z^{A_{12}}\det_q(t)$, then we get

$$Z(A_{12})\det_q(t) = Z(A_{12} + I_t) + \sum_{B_{12}} h_{B_{12}}(A_{12})Z(B_{12}),$$

where $B_{12} \leq A_{12} + I_t$, $B_{12}$ and $A_{12} + I_t$ have the same row sums and column sums. Hence

$$Z(A)\det_q(t) = q^{-2} \sum_{i\geq t+1, j\geq n-t+1} a_{ij} D(A) D(A_{12})^{-1} \quad D(A_{12} + I_t) D(A + E_t)^{-1} Z(A + E_t) \quad + \sum_{B_{12}} h_{B_{12}}(A_{12}) q^{-2} \sum_{i\geq t+1, j\geq n-t+1} a_{ij} D(A) D(A_{12})^{-1} \quad D(B_{12}) D\left(\begin{pmatrix} A_{11} & B_{12} \\ A_{21} & A_{22} \end{pmatrix}\right)^{-1} Z\left(\begin{pmatrix} A_{11} & B_{12} \\ A_{21} & A_{22} \end{pmatrix}\right).$$

By direct computation, one can show that the dependence of

$$D(B_{12}) D\left(\begin{pmatrix} A_{11} & B_{12} \\ A_{21} & A_{22} \end{pmatrix}\right)^{-1}$$

on the matrix entries of $B_{12}$ is only a dependence on the row and column sums. Then one deduces that

$$Z(A)\det_q(t) = q^{r_1 + \cdots + r_t - c_{n-t+1} - \cdots - c_n} (Z(A + E_t) + \sum_{D, D < A + E_t} c(D, A) Z(D))$$

with $c(D, A) \in q\mathbb{Z}[q]$.

For a basis element $b(A)$ of the form given in Theorem 3.2, we then deduce that

$$b(A)\det_q(t) = q^{r_1 + \cdots + r_t - c_{n-t+1} - \cdots - c_n} \quad [(Z(A + E_t) + \sum_{D, D < A + E_t} c_D(A) Z(D)) \quad + \sum_{B, B < A} h_B(A)(Z(B + E_t) + \sum_{D, D < B + E_t} c_D(B) Z(D))]$$  

(5.4)
with \( c_D(A), c_D(B) \in q\mathbb{Z}[q] \). By
\[
b(A) \det_q(t) = q^{2(r_1 + \cdots + r_t - c_{n-t+1} - \cdots - c_n)} \det_q(t) b(A),
\]
we see that \((Z(A + E_t) + \sum_{D,D < A + E_t} c_D(A)Z(D)) + \sum_{B,B < A} h_B(A)(Z(B + E_t) + \sum_{D,D < B + E_t} c_D(B)Z(D))\) is \(-\) invariant, and it must be the basis element \( b(A + E_t) \).
Finally, apply the algebra automorphism \( \sigma \). Then the second statement follows from Corollary 3.4

\[
\text{Corollary 5.3. Let } A = \begin{pmatrix} a_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ c_2 & a_2 & b_2 & \cdots & b_{n-2} & b_{n-1} \\ c_3 & c_2 & a_3 & \cdots & b_{n-3} & b_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_2 & a_n \end{pmatrix}. \text{ Then the basis element}
\]
\[
b(A) \sim \Pi_{t=1}^{n-1} \det_q(t)^{b_{n-t+1}} \sigma(\det_q(t))^{c_{n-t+1}} b(\text{diag}(a_1, a_2, \cdots, a_n)).
\]

\textbf{Proof:} Successively peel off the off-diagonals of } \( A \) \text{ by (5.2) and (5.3).}

\textbf{Definition 5.4. A matrix } A = (a_{ij}) \in M_n(\mathbb{Z}_+) \text{ is called a ladder if } a_{ij} \geq a_{i+1,j+1} \text{ for all } i, j.

Let \( A \) be a ladder. Successively peel off the off-diagonals of \( A \) by (5.2) and (5.3), the basis element \( b(A) \) is equivalent to a product of the quantum minors \( \det_q(t) \) and \( \sigma(\det_q(t)) \) and a basis element \( b\left( \begin{pmatrix} A_{n-1} & 0 \\ 0 & 0 \end{pmatrix} \right) \), where \( A_{n-1} \) is a ladder of size \( n - 1 \). Repeatedly, the basis element \( b(A) \) can be written as a product of some quantum minors which are \( q \)-commuting with each other.

\section{6. The coincidence of two bases}

Let \( g \) be the finite dimensional simple Lie algebra of type \( A_{n-1} \) and let \( \Lambda_1, \cdots, \Lambda_{n-1} \) be the fundamental dominant weights. For any dominant weight \( \lambda \), the irreducible highest weight module \( L(\lambda) \) occurs as a sub-quotient of a suitable power of the natural representation \( L(\Lambda_1) \). The simple modules \( L(\Lambda_1) \) and \( L(\Lambda_{n-1}) \) are dual to each other and are of dimension \( n \). Let \( e_1, e_2, \cdots, e_n \) be the standard basis of \( L(\Lambda_1) \) and let \( e_1^*, e_2^*, \cdots, e_n^* \) be the dual basis of \( L(\Lambda_{n-1}) \). Then it is well-known that the matrix coefficients \( X_{ij} = e_i^* \otimes e_j \) satisfy the following relations:
\[ X_{ij}X_{ik} = q^2X_{ik}X_{ij} \text{ if } j < k, \quad (6.1) \]
\[ X_{ij}X_{kj} = q^2X_{kj}X_{ij} \text{ if } i < k, \quad (6.2) \]
\[ X_{ij}X_{st} = X_{st}X_{ij} \text{ if } i > s, j < t, \quad (6.3) \]
\[ X_{ij}X_{st} = X_{st}X_{ij} + (q^2 - q^{-2})X_{it}X_{sj} \text{ if } i < s, j < t, \quad (6.4) \]
\[ \Sigma_{\sigma \in S_n} (-q^2)^{l(\sigma)} X_{1\sigma(1)}X_{2\sigma(2)} \cdots X_{n\sigma(n)} = 1. \]

Since the basis \( B^* \) is invariant under the multiplication of the quantum determinant, we get a basis \( K^* \) of \( O_q(SL_n)(= A_q(g)) \), by setting the quantum determinant to one. Clearly, the anti-automorphism \( \phi \) induces the anti-automorphism \( \phi \) of \( O_q(SL(n)) \) (see Lemma 2.9). Let \( X(A) \) be the image of \( Z(A) \) in \( O_q(SL(n)) \). Then
\[ \{ X(A)|trA = 0 \} \]
is a basis of \( O_q(SL(n)) \).

**Lemma 6.1.** The matrix coefficients \( X_{ij} \) are both invariant under \(-\) (the bar action of \( A_q(g) \)) and \( \phi \).

**Proof:** It is known that \( \{e_1, e_2, \ldots, e_n\} \) (resp. \( \{e^*_1, e^*_2, \ldots, e^*_n\} \)) is the canonical basis of \( L(\Lambda_1) \) (resp. of \( L(\Lambda_{n-1}) \)). Therefore, \( e_i \) and \( e^*_j \) are invariant under the bar action of \( L(\Lambda_1) \) and \( L(\Lambda_{n-1}) \) respectively. Hence, the matrix coefficients \( X_{ij} \) are \(-\) invariant. Note that \( \Lambda_1 \) and \( \Lambda_{n-1} \) are minuscule dominant weights so the left weight \( \lambda_l \) (resp. the right weight \( \lambda_r \)) of \( X_{ij} \) is conjugate to \( \Lambda_1 \) (resp. \( \Lambda_{n-1} \)) under the action of the Weyl group which implies that \( (\lambda_l, \lambda_l) - (\lambda_r, \lambda_r) = (\Lambda_1, \Lambda_1) - (\Lambda_{n-1}, \Lambda_{n-1}) = 0. \)

The basis \( K^* \) can be described similarly to the theorem 3.2 by replacing \( Z_{ij} \) by \( X_{ij} \) and \(-\) by \( \phi \).

**Theorem 6.2.** There is a unique basis \( \tilde{B}^* = \{b(\tilde{A})|A \in M_n(\mathbb{Z}_+), trA = 0\} \) of \( \tilde{L}^* = \oplus A\mathbb{Z}[q]X(A) \) determined by the following conditions:

1. \( \phi b(\tilde{A}) = b(\tilde{A}) \) for all \( A \).
2. \( b(\tilde{A}) = X(A) + \Sigma_{B < A} h_B(A)X(B) \) where \( h_B(A) \in q\mathbb{Z}[q] \) and \( ro(B) = ro(A), co(B) = co(A) \).

Let \( \mathbb{R}^n \) be the \( n \) dimensional Euclidean space with standard orthogonal basis \( e_1, e_2, \ldots, e_n \). It is well-known that the root system of type \( A_{n-1} \) is a subset of \( \mathbb{R}^n \) with simple roots \( \alpha_i = \epsilon_i - \epsilon_{i+1} \), for \( i = 1, 2, \ldots, n-1. \)

The \( U_q(g) \) bi-module structure can be written explicitly (see also \([?]\)).
For homogeneous elements $x$, and $y$ with weights $(\lambda_l, \lambda_r)$ and $(\mu_l, \mu_r)$ respectively.

The left action is defined by

$$
E_i X_{st} = \delta_{is} X_{s-1,t}, \quad F_i X_{st} = \delta_{i,s+1} X_{s+1,t}, \quad K_i X_{st} = q^{2(\varepsilon_s, \alpha_i)} X_{st}
$$

with Leibniz rule

$$
E_i(xy) = E_i(x)y + q^{2(\lambda_l, \alpha_i)} x E_i(y),
$$

$$
F_i(xy) = x F_i(y) + q^{-2(\mu_l, \alpha_i)} F_i(x)y,
$$

$$
K_i(xy) = q^{2(\lambda_l + \mu_l, \alpha_i)} xy,
$$

The right action is defined by

$$
X_{st} E_i = \delta_{i,s+1} X_{s+1,t}, \quad X_{st} F_i = \delta_{i,s} X_{s-1,t}, \quad X_{st} K_i = q^{2(\varepsilon_s, \alpha_i)} X_{st}
$$

with Leibniz rule

$$
(xy) E_i = (x) E_i y + q^{2(\lambda_r, \alpha_i)} x (y) E_i,
$$

$$
(xy) F_i = x (y) F_i + q^{-2(\mu_r, \alpha_i)} (x) F_i y,
$$

$$
(xy) K_i = q^{2(\lambda_r + \mu_r, \alpha_i)} xy,
$$

Denote by the same notation the image of $\text{det}_q(i)$ in $O_q(SL(n))$.

For $\lambda = m_1 \Lambda_1 + m_2 \Lambda_2 + \cdots + m_{n-1} \Lambda_{n-1}$, where $\Lambda_1, \Lambda_2, \cdots, \Lambda_{n-1}$ are fundamental weights. The module $L(\lambda) \otimes L^*(\lambda)$ is cyclic on $v_\lambda \otimes v_\lambda^*$. The lattice $L(\lambda) \otimes L^*(\lambda)$ is generated by $v_\lambda \otimes v_\lambda^*$ (under the action of upper Kashiwara operators) which corresponds to

$$
\Pi_i \text{det}_q(i)^{m_i}
$$

which is an element in the basis $K^*$. The quantum minor $\text{det}_q(t)$ is annihilated by the left action of $E_i$ for $i = 1, 2, \cdots, t-1$ and the right action of $F_j$ for $j = n-1, n-2, \cdots, n-t+1$.

By the action of the upper kashiwara operators, the normalized monomial $X(A)$ and the element $q^{\frac{1}{2}(\langle \lambda_l, \alpha_i \rangle -\langle \lambda_r, \alpha_i \rangle)} B'$ are in the same $\mathbb{Z}[q]$-lattice. By the uniqueness of Lusztig’s construction. The bases $K^*$ and $L^*$ are coincide.
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