Smooth approximation for classifying spaces of diffeomorphism groups

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Abstract

We prove a smooth approximation theorem for classifying spaces of certain infinite-dimensional smooth groups. More precisely, using the framework of diffeological spaces, we show that the smooth singular complex of a classifying space $BG$ is weakly homotopy equivalent to the (continuous) singular complex of $BG$ when $G$ is a diffeomorphism group of a compact smooth manifold. In particular, the smooth homotopy groups of $BG$ are naturally isomorphic to the usual (continuous) homotopy groups of $BG$. On top of a computation of homotopy groups, our methods yield a way to construct homotopically coherent actions of $G$ using $\infty$-categorical techniques. We discuss some generalizations and consequences of this result with an eye toward [OT19], where we show that higher homotopy groups of symplectic automorphism groups map to Fukaya-categorical invariants, and where we prove a conjecture of Teleman from the 2014 ICM in the Liouville and monotone settings.

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1 Introduction

Let $G$ be a topological group. Then the classifying space $BG$ is well-understood—we can compute its homotopy groups in terms of those of $G$, we have useful models for its homotopy type, and we also know that (homotopy classes of) maps into $BG$ classify (isomorphism classes of) principal $G$-bundles.

When $G$ is an infinite-dimensional “smooth” group, we know that the framework of diffeological spaces allows us to conclude that smooth $G$-bundles are classified by smooth maps to $BG$ [CW17]. Given this power, we would like at least a homotopical version of smooth approximation. For example, the smooth homotopy groups $\pi_n^{C\infty}$ of $BG$ (defined by smooth homotopy classes of smooth maps from spheres) should be isomorphic to the usual homotopy groups $\pi_n$ of $BG$. This would allow us to deduce facts about the homotopy types of infinite-dimensional entities using smooth constructions. Of course, exhibiting a natural isomorphism for homotopy groups of $BG$ is equivalent to exhibiting a natural isomorphism for the homotopy groups of $G$ itself. The latter is standard when $G$ is finite-dimensional, but in this paper we deal with infinite-dimensional groups such as diffeomorphism groups.

Using the framework of diffeological spaces [IZ13], we prove precisely such a theorem for diffeomorphism groups:

**Theorem 1.1.** Let $Q$ be a compact, smooth manifold. We let $G = \text{Diff}(Q)$ be the topological space of diffeomorphisms of $Q$, while we let $\widehat{\text{Diff}}(Q)$ be the diffeological group of diffeomorphisms. Then the inclusion of smooth maps into continuous maps induce isomorphisms

$$\pi_n^{C\infty}(\widehat{\text{Diff}}(Q)) \cong \pi_n(\text{Diff}(Q))$$

between the smooth and continuous homotopy groups.

By invoking the standard long exact sequence for smooth and for continuous homotopy groups, we conclude:

**Theorem 1.2.** Let $B\widehat{\text{Diff}}(Q)$ be the Milnor classifying space, equipped with a diffeological space structure as in [CW17]. Then

1. The inclusion of smooth maps into continuous maps induce isomorphisms

$$\pi_n^{C\infty}(B\widehat{\text{Diff}}(Q)) \cong \pi_n(B\text{Diff}(Q)).$$

Moreover, let $\text{Sing}^{C\infty}(B\widehat{\text{Diff}}(Q))$ be the simplicial set of smooth extended simplices mapping to $B\widehat{\text{Diff}}(Q)$, and let $\text{Sing}(B\text{Diff}(Q))$ be the usual simplicial set of continuous simplices mapping to $B\text{Diff}(Q)$. Then

2. The inclusion of smooth simplices into continuous simplices

$$\text{Sing}^{C\infty}(B\widehat{\text{Diff}}(Q)) \rightarrow \text{Sing}(B\text{Diff}(Q))$$

is a weak homotopy equivalence.

**Remark 1.3.** It is known that an isomorphism between $\pi_n^{C\infty}$ and $\pi_n$ cannot exist in full generality, even in finite dimensions. For instance, the irrational torus as a topological space has trivial homotopy groups, while the irrational torus as a diffeological space has smooth fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (see Remark 2.26). One can thus view the theorems above as witnessing a “regularity” of diffeomorphism groups, in contrast to other diffeological groups.
Remark 1.4. We do not formally identify the class of diffeological groups $\hat{G}$ for which Theorem 1.1 holds, but let us remark that if $\hat{G}$ acts faithfully and smoothly on a manifold, our proofs will carry through so long as the action of each $g \in G$ is controlled outside some compact subset of said manifold. (The compact subset need not be uniformly chosen across all of $G$.) In fact, our main diffeological group of interest is the automorphism group of a Liouville sector, and the above results hold for this choice of $G$.

Though the above statements are stated as theorems, the proof methods are rather elementary. We might file these theorems under “things that should have been proven already.” Our interest in establishing these facts is to open the door to utilizing modern homotopy theory in the study of infinite-dimensional smooth groups.

For example, by utilizing a categorical version of the manifest homotopy equivalence between a simplicial complex and its barycentric subdivision, the following emerges from the above results:

**Theorem 1.5.** Let

$$\text{Simp}(\hat{B}\text{Diff}(Q))$$

be the category whose objects are smooth simplices $|\Delta_k^n| \to \text{Simp}(\hat{B}\text{Diff}(Q))$. Its ($\infty$-categorical) localization along all morphisms—i.e., its $\infty$-groupoid completion—is equivalent to $\text{Sing}(B\text{Diff}(Q))$.

Theorem 1.5 is the main result of this paper. We refer to Section 1.1 for one application we have in mind.

We mentioned in Remark 1.4 that our main interest is not in diffeomorphism groups, but in Liouville automorphisms. (This example also illustrates what we mean by “control” near infinity in that same remark.) We collect the Liouville analogues of our preceding theorems as follows:

**Theorem 1.6.** Let $M$ be a Liouville sector and $G = \text{Aut}^o(M)$ the group of Liouville automorphisms of $M$. We endow $G$ with the natural diffeological space structure, and with the strong Whitney topology. Then

1. The natural map $\pi_C^\infty_n(G) \to \pi_n(G)$ is a bijection for $n = 0$, and an isomorphism for $n \geq 1$ for any choice of basepoint.

2. Likewise for the natural map $\pi_C^\infty_n(BG) \to \pi_n(BG)$. Moreover,

3. The inclusion of smooth simplices into continuous simplices

$$\text{Sing}^C^\infty(\hat{B}G) \to \text{Sing}(BG)$$

is a weak homotopy equivalence.

4. Let $\text{Simp}(B\text{Aut}^o(M))$ denote the category of smooth simplices in $B\text{Aut}^o(M)$. Then the simplicial set $\text{Sing}(B\text{Aut}^o(M))$ of continuous simplices in $B\text{Aut}^o(M)$ is the $\infty$-categorical localization of $\text{Simp}(B\text{Aut}^o(M))$ (along all morphisms).

We refer the reader to Section 5 on the precise definition of Liouville automorphisms. In fact, the techniques here can be used to study homotopical enrichments of smooth groups as well—for example, to study not just diffeomorphisms of a manifold $Q$, but diffeomorphisms equipped with homotopical data respecting some tangential structures. This will be applied in [OT19].
1.1 Application

Let us explain the strength of Theorem 1.5. If one can construct a functor from $\text{Simp}$ to some other $\infty$-category $\mathcal{D}$, and if every morphism of $\text{Simp}$ is sent to an equivalence under this functor, one may formally conclude the existence of a map from $\text{Sing}(B\text{Aut}^\circ(M))$ to $\mathcal{D}$. In other words, we conclude that there is some object of $\mathcal{D}$ receiving a homotopically coherent action of $G$. For us, value of this result emerges from the ease with which one can construct functors out of a 1-category sending morphisms to equivalences.

Here is a basic application of Theorem 1.5. As an input, assume that one can construct a functorial invariant of $G$-fiber bundles. This means every $G$ fiber bundle is assigned some object (the invariant), and one can construct functorial maps between these invariants whenever bundles can be pulled back. If these maps induce equivalences of invariants for all $G$-bundles over simplices, then one has an induced action of $G$ on the invariant associated to a $G$-bundle over a point. In fact, one can produce such invariants for bundles associated to $G$-bundles, and that is precisely our application for Result (4) in Theorem 1.6:

A construction of an invariant as above is carried out for wrapped Fukaya categories. Namely, because $\hat{B}\text{Aut}^\circ(M)$ classifies smooth Liouville bundles with fiber $M$, any object of $\text{Simp}(\hat{B}\text{Aut}^\circ(M))$ is the data of a Liouville bundle over an $n$-simplex with fiber $M$. One can construct a version of the wrapped Fukaya category for this fiber bundle, and show that morphisms in $\text{Simp}(\hat{B}\text{Aut}^\circ(M))$ induce equivalences of these wrapped Fukaya categories. In this way, Theorem 1.6(4) produces a representation of $\text{Aut}^\circ(M)$ on the wrapped Fukaya category of $M$; this proves a conjecture of Teleman from the 2014 ICM in the Liouville setting [Tel14]. We provide details and applications of this construction in [OT20b] and [OT19].

Remark 1.7. The ability to consider smooth bundles—i.e., induced by smooth maps $|\Delta^n| \to \hat{B}\text{Aut}^\circ(M)$, as opposed to continuous maps $|\Delta^n| \to B\text{Aut}^\circ(M)$—is precisely what makes the construction of Fukaya-type invariants possible, as the construction must utilize tools of the smooth world (transversality, connections, et cetera).

1.2 Notation

Let us set some notation and conventions. Throughout this work, we will assume that the reader is familiar with simplicial and $\infty$-categorical constructions, including the notions of coCartesian fibrations and Cartesian fibrations. For background on (co)Cartesian fibrations, we refer the reader to Section 2.9 of [OT20a], Section 4 of [Tan19], and Section 3.2 of [Lur09].

Because it will be important to distinguish between a topological space and a choice of smooth structure on it, we hereby enact the following:

Notation 1.8 (Smooth players wear hats.). We will refer to an object with smooth structure by $\hat{B}$ (i.e., by making the symbol wear a hat). $B$ will often denote an underlying set, or space, associated to $\hat{B}$. For example, $\hat{B}G$ is a diffeological space, while $B\text{G}$ is the Milnor classifying space associated to the topological group $G$. (These have the same underlying set.)

Notation 1.9 (The nerve $\text{N}(\mathbb{C})$). As usual, if $\mathbb{C}$ is a category, the nerve of $\mathbb{C}$ is a simplicial set whose $k$-simplices consist of commutative diagrams in $\mathbb{C}$ in the shape of a $k$-simplex. We let $\text{N}(\mathbb{C})$ denote the nerve.

Notation 1.10 (The combinatorial $n$-simplex). As usual we let $\Delta^n$ denote the simplicial set represented by the poset $[n]$. 

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2 Reminders on diffeological spaces

We collect various results, many of which are due to the papers of Christensen-Sinnamon-Wu, Christensen-Wu, and Magnot-Watts [CSW14, CW14, CW17, MW17].

Notation 2.1 (Mfld and Euc.). Let Mfld denote the category of smooth manifolds—its objects are smooth manifolds, and morphisms are smooth maps. We let Euc ⊂ Mfld denote the full subcategory of those manifolds that are diffeomorphic to an open subset of $\mathbb{R}^n$ for some $n$.

When defining a geometric object, one can take a Lawvere-type approach to define functions on that object, or one can take a functor-of-points approach to define functions into that object. A diffeological space is defined by the latter approach: We will often define a diffeological space by beginning with the data of a set $X$, and then for all $U \in \text{Ob Euc}$, specifying which functions $U \to X$ are “smooth.” This defines a functor $\hat{X} : \text{Euc}^{\text{op}} \to \text{Sets}$ as in the following definition:

Definition 2.2 (Diffeological space). Fix a functor $\hat{X} : \text{Euc}^{\text{op}} \to \text{Sets}$ (i.e., a presheaf on Euc). We say that $\hat{X}$ is a diffeological space if the following two conditions hold:

1. For any $U \in \text{Euc}$, the function
   $$\hat{X}(U) \to \text{hom}_{\text{Sets}}(\text{hom}_{\text{Euc}}(\mathbb{R}^0, U), \hat{X}(\mathbb{R}^0))$$
   is an injection. (That is, functions are determined by their values on points of $U$.)

2. $\hat{X}$ is a sheaf (with the usual notion of open cover on smooth manifolds).

A map of diffeological spaces—also known as a smooth map of diffeological spaces—is a map of presheaves.

Remark 2.3. The map in 1. is induced by the structure map for presheaves

$$\hat{X}(U) \times \text{hom}_{\text{Euc}}(\mathbb{R}^0, U) \to \hat{X}(\mathbb{R}^0).$$

Notation 2.4 (Underlying set $X$). Let $\hat{X}$ be a diffeological space. Then we say that $\hat{X}(\mathbb{R}^0)$ is the underlying set of $\hat{X}$, and we denote it $X$. Note that by 1., every element of $\hat{X}(U)$ determines a function $f : U \to X$. If $f$ is in the image of the map in 1., we say that $f$ is a smooth map from $U$ to $X$. (In the literature, this is also called a plot.)

Unwinding the definitions, we thus see that a diffeological space is equivalent to the data of a set $X$, and for every $U \in \text{Ob Euc}$, a subset $\hat{X}(U) \subset \text{hom}_{\text{Sets}}(U, X)$, subject to the following properties:

- $\hat{X}(U)$ contains all the constant maps.
Concreteness is precisely equivalent to condition 1 of Definition 2.2.

The category of diffeological spaces is equivalent to the category of so-called "concrete" sheaves on a site.

If there is an open cover \( \{ U_i \} \) of \( U \) such that the function \( U \to X \) factors as \( U_i \to X \), and each \( U_i \to X \) is in \( \widehat{X}(U_i) \), then the function \( U \to X \) is in \( \widehat{X}(U) \).

**Definition 2.5 (Smoothness of maps).** Let \( \widehat{X} \) and \( \widehat{Y} \) be diffeological spaces. A function \( X \to Y \) of underlying sets is called smooth if it is induced by a map of diffeological spaces.

**Remark 2.6 (D-topology).** Let \( \widehat{X} \) be a diffeological space and \( X \) its underlying set. One can endow \( X \) with the finest topology for which every smooth function \( f : U \to X \) determined by \( \widehat{X} \) is continuous. This is called the D-topology in the literature.

**Warning 2.7.** However, we will almost never make use of the D-topology, and in fact our main example \( X = \text{Aut}^0(M) \) will not be endowed with the D-topology. So the reader should not assume that the underlying set \( X \) of a diffeological space \( \widehat{X} \) is endowed with the D-topology.

**Example 2.8 (Smooth manifolds).** Let \( X \) be a smooth manifold. Then one can define a diffeological space by declaring \( \widehat{X}(U) = \text{hom}_{\text{Mfd}}(U, X) \); note that \( X \) is indeed the underlying set of \( \widehat{X} \) as implied by our notation, and the D-topology coincides with the usual one. This construction gives a fully faithful embedding of the category of smooth manifolds into the category of diffeological spaces.

**Example 2.9 (Subspaces).** Let \( \widehat{X} \) be a diffeological space, and let \( A \subset X \) be a subset. Then \( A \) determines a subsheaf \( \widehat{A} \subset \widehat{X} \) where \( \widehat{A}(U) \) consists of all those elements \( f : U \to X \) whose image lies in \( A \). We call this the subspace diffeology on \( A \). Note this is an example where there is ambiguity in the topology of \( A \)—one could give it the subspace topology with respect to the D-topology on \( X \), or give it the D-topology induced by the diffeological structure \( \widehat{A} \). These topologies do not always coincide.

**Example 2.10 (Function spaces).** Let \( X \) and \( Y \) be smooth manifolds, and let \( C^\infty(X, Y) \) be the set of smooth functions. We can endow this set with a diffeological space structure by declaring a function \( U \to C^\infty(X, Y) \) to be smooth if and only if the adjoint map \( U \times X \to Y \) is smooth. In general, the D-topology of this set is finer than the compact-open topology, finer than the weak Whitney topology, but coarser than the strong Whitney topology.

**Remark 2.11.** The category of diffeological spaces has all limits and colimits; in fact, the functor sending a diffeological space to its underlying set has both left and right adjoints, so the underlying sets of limits and colimits can be understood in the usual way. Moreover, the functor sending a diffeological space to its underlying space (with the D-topology) admits a right adjoint, so the D-topologized topological space of colimits can be understood in terms of colimits of spaces in the usual way. We also have an explicit description of the colimit diffeology: A function \( U \to \text{colim} f \) is smooth if and only if there is an open cover \( \{ U_i \} \) of \( U \), and an object \( f(j) \) in the diagram given by \( f \), such that the function factors \( U_i \to f(j) \to \text{colim} f \) with \( U_i \to f(j) \) being smooth.

One can also show that the category of diffeological spaces is Cartesian closed. The hom-objects are precisely the function spaces with the diffeological space structure of Example 2.10.

We also remark that these observations follow straightforwardly from the fact that the category of diffeological spaces is equivalent to the category of so-called “concrete” sheaves on a site. (Concreteness is precisely equivalent to condition 1 of Definition 2.2.)
2.1 Diffeological groups

Definition 2.12. A diffeological group is a group object in the category of diffeological spaces. Concretely, this is the data of a diffeological space $\hat{G}$, together with a group structure whose inverse and multiplication operations are smooth.

Example 2.13. For any smooth manifold $X$, the diffeomorphism group $\text{Diff}(X)$ is a diffeological space. It is in fact a diffeological group (Definition 2.12). But this diffeology is not induced by the subspace diffeology of Examples 2.9 and 2.10; one must reduce the number of smooth maps to guarantee that the inverse function is smooth. (For example, a map $\text{the subspace diffeology of Examples 2.9 and 2.10};$ one must reduce the number of smooth maps to $X$. It is in fact a diffeological group (Definition 2.12). But this diffeology is not induced by $\pi_\text{homotopy groups}$. The usual notion $\text{Concretely, this is the data of a diffeological space } \hat{G}, \text{ together with a group structure whose inverse and multiplication operations are smooth.}$

Example 2.14. In particular, for any Liouville sector $M$, the group $\text{Aut}^0(M) \subset \text{Diff}(M)$ is a diffeological group.

2.2 Some homotopy theory of diffeological spaces

One of the most useful tools in homotopy theory is the ability to convert any topological space into a simplicial set. We recall the analogue of this for diffeological spaces.

Definition 2.15 ($|\Delta^k|$. Let $|\Delta^k| \subset \mathbb{R}^{k+1} \cong \text{hom}(k, \mathbb{R})$ denote the affine hyperplane defined by the equation $\sum_{i=0}^k t_i = 1$. We refer to $|\Delta^k|$ as the extended $k$-simplex, and consider it a smooth manifold in the obvious way. (It is diffeomorphic to the standard Euclidean space $\mathbb{R}^k$.)

We will refer to a map $|\Delta^k| \to |\Delta^{k'}|$ as simplicial if it is the restriction of the linear map $\mathbb{R}^k \to \mathbb{R}^{k'}$ induced by some map of sets $[k] \to [k']$. (This map need not respect order.)

In the diffeological space setting, we make the following definition:

Definition 2.16 (Smooth homotopy groups). The $n$th smooth homotopy group

$$\pi_n^{C^\infty}(\hat{X}, x_0)$$

for $x_0 \in X$ is the group of smooth homotopy classes of maps $f : |\Delta^n_0| \to \hat{X}$ satisfying the condition that $f(y) = x_0$ for any $y \in |\Delta^n_0|$ for which $y$ has some coordinate equal to zero. The homotopy classes of maps are taken relative to the subset of those $y \in |\Delta^n_0|$ with at least one coordinate equal to zero.

Remark 2.17. The above model for $\pi_n^{C^\infty}$ is equivalent to many others. (For example, one could take $\pi_n^{C^\infty}$ to be defined as smooth homotopy classes of smooth, pointed maps from the standard smooth $n$-sphere.) See Theorem 3.2 of [CW14], where $\pi_n^{C^\infty}$ is written as $\pi_n^{D}$.

In the usual homotopy theory of topological spaces, we can compare two different notions of homotopy groups. The usual notion $\pi_n$ is defined by based homotopy classes of continuous maps $S^n \to X$, and the combinatorial definition is defined by classes of maps from $\Delta^n$ to the simplicial set $\text{Sing}(X)$. Let us explain the analogue of $\text{Sing}$ in the diffeological setting.

Remark 2.18. Since any smooth manifold is a diffeological space (Example 2.8), the assignment $[k] \mapsto |\Delta^k|$ defines a cosimplicial object in the category of diffeological spaces.

Notation 2.19. Let $\hat{X}$ be a diffeological space. We let $\text{Sing}^{C^\infty}(\hat{X})$ denote the simplicial set

$$\text{Sing}^{C^\infty}(\hat{X}) : \Delta^{op} \to \text{Sets}, \quad [k] \mapsto \text{hom}^{C^\infty}(|\Delta^k|, \hat{X})$$

whose $k$-simplices consist of maps (of diffeological spaces) from extended $k$-simplices to $\hat{X}$. 8
Remark 2.20. In [CW14], the notation $S^D(X)$ is used to denote what we write as $\text{Sing}^{C\infty}(\tilde{X})$. Also in loc. cit., the $\tilde{X}$ notation is not used to distinguish a diffeological space from its underlying set.

Example 2.21. Let $X$ be a smooth manifold. We let $\text{Sing}(X)$ denote the usual simplicial set of continuous simplices $|\Delta^k| \to X$, and let $\tilde{X}$ denote the associated diffeological space (Example 2.8). The natural map $\text{Sing}^{C\infty}(\tilde{X}) \to \text{Sing}(X)$ is a weak homotopy equivalence by the smooth approximation theorem (sometimes called the Whitney approximation theorem in this generality).

Warning 2.22. The simplicial set $\text{Sing}^{C\infty}(\tilde{X})$ need not be a Kan complex. (In contrast: For a space $X$, $\text{Sing}(X)$ is always a Kan complex.)

In general, it seems that a “homogeneity” property is needed to conclude that $\text{Sing}^{C\infty}(\tilde{X})$ is a Kan complex. For example, when $\tilde{X}$ is the diffeological space associated to a smooth manifold, $\text{Sing}^{C\infty}(\tilde{X})$ is a Kan complex (Corollary 4.36 of [CW14]). But when $\tilde{X}$ is associated to a smooth manifold with non-empty boundary, this is no longer true (Corollary 4.47 of ibid.).

Following the theme of “homogeneity implies Kan,” we have the following.

Proposition 2.23 (Proposition 4.30 of [CW14]). For any diffeological group $\tilde{G}$ (see Definition 2.12), $\text{Sing}^{C\infty}(\tilde{G})$ is a Kan complex.

Remark 2.24. Let us assume $\tilde{X}$ is a diffeological space for which $\text{Sing}^{C\infty}(\tilde{X})$ is a Kan complex. Then the natural maps

$$\pi_n^{C\infty}(\tilde{X}, x_0) \to \pi_n(\text{Sing}^{C\infty}(\tilde{X}), x_0).$$

(2.1)

are bijections for $\pi_0$, and are isomorphisms for any choice of $x_0 \in X$. (Theorem 4.11 of [CW14]).

In particular, this holds for diffeological groups by Proposition 2.23. Let us record this:

Proposition 2.25. Let $\tilde{G}$ be a diffeological group. Then the map $\pi_0^{C\infty}(\tilde{G}) \to \pi_0(\text{Sing}^{C\infty}(\tilde{G}))$ is a group isomorphism. Moreover, the map (2.1) is an isomorphism for any choice of $x_0 \in G$.

Remark 2.26. It is not always true that the (continuous) homotopy groups with respect to the D-topology are isomorphic to the smooth homotopy groups. A counterexample is the irrational torus, see Example 3.20 of [CW14].

3 Smooth approximation

We have seen that smooth approximation of homotopy groups holds for some diffeological spaces (Example 2.21), but not others (Remark 2.26). We turn our attention to the example of diffeomorphism groups.

We first relate smoothness in the diffeological sense to continuity in the usual sense. The following is a standard exercise, so we omit its proof:

Lemma 3.1. Let $S$ be compact, and fix a function $f : S \to \text{Diff}(Q)$ such that the adjoint map $S \times Q \to Q$ is smooth. Then $f$ is continuous in the strong Whitney topology.

3.1 For $\text{Diff}(Q)$

Let $Q$ be a compact smooth manifold and $\text{Diff}(Q)$ the topological space of diffeomorphisms with the strong Whitney topology. We let $\widehat{\text{Diff}(Q)}$ denote the diffeological space as defined in Example 2.13.
Lemma 3.2. Let $\alpha : |\Delta^n| \to \text{Diff}(Q)$ be a continuous function from the standard $n$-simplex with image contained in a very small open set. Then

1. $\alpha$ is continuously homotopic to a map $\beta : |\Delta^n| \to \text{Diff}(Q)$ for which $\beta$ is smooth with respect to the diffeology on $\text{Diff}(Q)$ (Example 2.13). Moreover, if $\alpha$ is already smooth on some open neighborhood of the boundary $\partial |\Delta^n|$, then the homotopy may be chosen to be constant on this neighborhood.

2. Moreover, $\beta$ may be assumed to be collared near the boundary of $|\Delta^n|$. 

Proof. 1. Without loss of generality (for example, we may multiply $\alpha$ by an appropriate element of $\text{Diff}(Q)$) we may assume that the image of $\alpha$ is contained in a small neighborhood of the identity $\text{id}_Q \in \text{Diff}(Q)$. Because $\alpha$ lands in the identity component of $\text{Diff}(Q)$, we may assume that for every $s \in |\Delta^n|$, $\alpha_s$ is the time-one flow of a time-dependent vector field $X_{s,t}$ on $Q$. Moreover, because $\alpha$ has image contained in a tiny neighborhood of $\text{id}_Q$, we may assume that $\alpha$ is induced by a continuous map

$$\tilde{\alpha} : |\Delta^n| \to \Gamma(Q \times \mathbb{R}, TQ) \tag{3.1}$$

from the $n$-simplex to the space of time-dependent vector fields on $Q$. (That is, for every $s \in |\Delta^n|$, $\alpha_s$ is the time-one flow of $\tilde{\alpha}_s$.) By the usual Whitney approximation theorem, we may choose a continuous homotopy of $\tilde{\alpha}$ to a smooth map

$$\tilde{\beta} : |\Delta^n| \to \gamma(Q \times \mathbb{R}, TQ).$$

It follows from the usual Whitney approximation theorem’s proof that this homotopy may be chosen to be constant on a neighborhood of $\partial |\Delta^n|$ if $\tilde{\alpha}$ is already smooth there.

By smooth dependence of solutions to ODEs, it follows that the induced time-one-flow map

$$\beta : |\Delta^n| \to \text{Diff}(Q), \quad s \mapsto \text{Flow}_{\tilde{\beta}_s}^{t=1}$$

has a smooth adjoint $|\Delta^n| \times Q \to Q$. Moreover, because any time-one flow of a time-dependent vector field $\tilde{\beta}_s$ has an obvious inverse —given by the time-one flow of the time-dependent vector field $t \mapsto -\tilde{\beta}_s(1-t)$—the map

$$|\Delta^n| \times Q \to Q, \quad (s,q) \mapsto \tilde{\beta}_s^{-1}(q)$$

is also smooth. This shows that $\beta$ is a smooth map in the diffeology of $\text{Diff}(Q)$, and completes the proof of the first claim.

2. Let $U \subset |\Delta^n|$ be an open neighborhood of the boundary $\partial |\Delta^n|$, and choose any smooth map $j : |\Delta^n| \to |\Delta^n|$ which restricts to a strong deformation retraction of $U$ onto $\partial |\Delta^n|$. We pull back $\beta$ along $j$, noting that this pullback admits a continuous homotopy to $\beta$ itself. 

Proof of Theorem 1.1. Suppose we have a continuous map $f : |\Delta^k| \to \text{Diff}(Q)$ such that $\partial |\Delta^k|$ is sent to the identity. We will first prove that $f$ is continuously homotopic to the restriction of a smooth map $(\Delta^k) \to \text{Diff}(Q)$ through maps constant along $\partial |\Delta^k|$. 

First, we may as well assume that $f$ is not only constant on $\partial |\Delta^k|$, but also constant on a neighborhood of $\partial |\Delta^k|$. (By retracting a small neighborhood of $\partial |\Delta^k|$, for example.) In particular, we may assume that $f$ is smooth on a neighborhood of $\partial |\Delta^k|$. 

Next, we take an iterated barycentric subdivision $S$ of $|\Delta^k|$ fine enough so that, for every simplex $E$ in the subdivision, the image of the open star about $E$, $f(\text{Star}(E))$, is contained in a
tiny neighborhood. Now we may apply Lemma 3.2 repeatedly to all the simplices in \( S \) to arrive at the desired result.

Here are some details: For \( i = 0 \), homotope \( f \) to a map \( f_0 \) satisfying the following property: For every vertex \( v \in S \), \( f_0 \) is constant in a neighborhood of \( v \), and \( f_0 \) is homotopic to \( f \).

Inductively, we homotope \( f_{i-1} \) to \( f_i \), with \( f_i \) satisfying the following: (i) \( f_i \) is smooth in a neighborhood of every \( i \)-simplex of \( S \), (ii) \( f_i \) equals \( f_{i-1} \) in a small neighborhood of each \((i-1)\)-simplex.

Because \( \text{Diff}(Q) \) is a diffeological group, its smooth singular complex is a Kan complex (Proposition 2.23). By Remark 2.24, we conclude:

**Corollary 3.3.** The map of simplicial sets

\[
\text{Sing}^{\infty}(\widehat{\text{Diff}}(Q)) \to \text{Sing}(\text{Diff}(Q))
\]

is a homotopy equivalence.

### 3.2 The classifying space \( \widehat{BG} \)

Now let us recall some constructions of classifying spaces for diffeological groups. We follow [MW17] and [CW17].

Let \( G \) be a diffeological group. Then in [MW17] and [CW17], the authors construct two diffeological spaces \( \widehat{EG} \) and \( \widehat{BG} \), together with a smooth map \( \widehat{EG} \to \widehat{BG} \). We employ the diffeology of [CW17], as the resulting statements are a bit more general.

**Remark 3.4.** The constructions of \( \widehat{EG} \) and \( \widehat{BG} \) below are modeled on Milnor’s join construction [Mil56]. One obvious reason to prefer Milnor’s construction as opposed to the usual simplicial space construction is that the map \( \widehat{EG} \to \widehat{BG} \) need not be locally trivial for the latter; this was pointed out as early as [Seg68].

**Construction 3.5 (\( \widehat{EG}. \)).** Let \( |\Delta^\omega| \) denote the infinite-dimensional simplex. That is, it is the set of those \((t_i), i \in \mathbb{Z}_{\geq 0} \in \oplus_\omega \mathbb{R} \) for which only finitely many \( t_i \) are non-zero, and \( \sum t_i = 1 \). As a diffeological space, we have that

\[
|\Delta^\omega| \cong \text{colim}_{i \geq 0} |\Delta^i|
\]

where \( i \) is the standard \( i \)-dimensional simplex, given the subspace diffeology from \( \mathbb{R}^{i+1} \).

Then \( |\Delta^\omega| \times \prod_\omega G \) can be given the product diffeology, and we define \( \widehat{EG} \) to be the quotient by identifying \((t_i, g_i) \sim (t'_i, g'_i)\) when the following two conditions hold:

1. \( t_i = t'_i \) for all \( i \), and
2. If \( t_i = t'_i \neq 0 \), then \( g_i = g'_i \).

Obviously \( \widehat{EG} \) retains the projection map to \( |\Delta^\omega| \); its fibers above an element \((t_i)\) can be identified with the product \( \prod_{i \text{ s.t. } t_i \neq 0} G \).

Of course, \( \widehat{EG} \) is a diffeological space by virtue of the category of diffeological spaces having all limits and colimits (Remark 2.11). Likewise, the following is a diffeological space:

**Construction 3.6 (\( \widehat{BG}. \)).** We let \( \widehat{BG} \) be the quotient by the natural action

\[
\widehat{EG} \times G \to \widehat{EG}, ([t_i, g_i]) \cdot g = ([t_i, g_i g]).
\]
These satisfy a series of properties that we state as a single theorem.

**Theorem 3.7.** The following are true:

1. The underlying set of $\widehat{BG}$ is the Milnor construction of the classifying space $BG$ [Mil56].
2. $\widehat{EG}$ is smoothly contractible. (Corollary 5.5 of [CW17].)
3. The map $\widehat{EG} \to \widehat{BG}$ is a diffeological principal $\widehat{G}$-bundle. (Theorem 5.3 of [CW17].)
4. For any diffeological space $\widehat{X}$, pull-back induces a bijection between smooth homotopy classes of maps $\widehat{X} \to \widehat{BG}$, and principal $\widehat{G}$-bundles over $\widehat{X}$. (Theorem 5.10 of [CW17].)

**Remark 3.8.** That $\widehat{EG}$ is smoothly contractible means that we can find a smooth map $\Delta^k \times \widehat{EG} \to \widehat{EG}$ which interpolates between a constant map and the identity map. By Lemma 4.10 of [CW14], this induces a simplicial homotopy between the identity map of $\text{Sing}^{C^\infty}(\widehat{EG})$ and a constant map—that is, $\text{Sing}^{C^\infty}(\widehat{EG})$ is weakly contractible as a simplicial set, hence weakly homotopy equivalent to a point.

**Remark 3.9.** Let $\widehat{E} \to \widehat{B}$ be a diffeological bundle, and assume that the fibers are diffeological spaces for whom $\text{Sing}^{C^\infty}$ is a Kan complex. Then the induced map $\text{Sing}^{C^\infty}(\widehat{E}) \to \text{Sing}^{C^\infty}(\widehat{B})$ is a Kan fibration. (See Proposition 4.28 of [CW14].)

As a result, Theorem 3.7(3) implies that we have a Kan fibration sequence of simplicial sets

$$\text{Sing}^{C^\infty}(\widehat{G}) \to \text{Sing}^{C^\infty}(\widehat{EG}) \to \text{Sing}^{C^\infty}(\widehat{BG}).$$

### 3.3 For $B\text{Diff}(Q)$

Now we prove smooth approximation (of homotopy groups) for $B\text{Diff}(Q)$, both for combinatorially defined smooth homotopy groups, and for smooth homotopy groups. (This a posteriori allows us to show that these two versions of smooth homotopy groups are isomorphic. An a priori deduction would result from proving that $\text{Sing}^{C^\infty}(B\text{Diff}(Q))$ is a Kan complex, but we do not investigate whether this simplicial set is Kan in this work.)

For a diffeological space $\widehat{X}$, consider the restriction map

$$\left(f: |\Delta_k| \to \widehat{X}\right) \mapsto \left(f|_{\Delta^k_i}: |\Delta^k| \to X\right).$$

**Lemma 3.10.** For $\widehat{X} = \widehat{\text{Diff}(Q)}, \widehat{E\text{Diff}(Q)}, \widehat{B\text{Diff}(Q)}$, the restriction map induces a commutative diagram of simplicial sets

$$\begin{array}{ccc}
\text{Sing}^{C^\infty}(\widehat{\text{Diff}(Q)}) & \longrightarrow & \text{Sing}^{C^\infty}(\widehat{E\text{Diff}(Q)}) \\
\downarrow & & \downarrow \\
\text{Sing}(\text{Diff}(Q)) & \longrightarrow & \text{Sing}(E\text{Diff}(Q)) \\
\downarrow & & \downarrow \\
\text{Sing}(B\text{Diff}(Q)). & \longrightarrow & \text{Sing}(B\text{Diff}(Q)).
\end{array}$$

**Remark 3.11.** On the bottom row, we have topologized $\text{Diff}(Q)$ with the strong Whitney topology, and we have used the induced topology on $E\text{Diff}(Q), B\text{Diff}(Q)$ (these are induced by Milnor’s join constructions for these spaces—see Constructions 3.5 and 3.6).
**Proof of Lemma 3.10.** For brevity, let us set \( G = \text{Diff}(Q) \).

It is obvious that the diagram commutes, so one need only check that the restriction of \( f \) to \(|\Delta^k|\) is continuous for every choice of \( \hat{X} \).

For \( \hat{X} = \hat{G} \), this is the content of Lemma 3.1.

For \( \hat{X} = \hat{EG} \), the definition of the colimit diffeology guarantees that there is some cover \(|\Delta^k|\) and smooth maps \( f_i : U_i \to |\Delta^{|n_i}| \times G^{n_i} \) such that \( f \) factors as

\[
\begin{array}{ccc}
U & \xrightarrow{f} & \hat{EG} \\
\uparrow & & \uparrow \\
\bigcup_i U_i & \xrightarrow{\bigcup_i f_i} & \bigcup_i |\Delta^{n_i}| \times G^{n_i} \xrightarrow{\sim} |\Delta^\omega| \times \prod_\omega G
\end{array}
\]

where the bottom-right horizontal arrow is the (union of the) obvious inclusion map. (The inclusion map identifies \( G^{n_i} \) with the subspace of \( \prod_\omega G \) consisting of those \( (g_j) \) whose components for \( j > n_i \) are all equal to the identity element of \( G \).)

Now replacing \( U_i \) by (a possibly greater quantity of) smaller disks and taking their closures, we may assume each \( U_i \) is compact. Then the same argument as in Lemma 3.1 shows that each \( f_i \) may be assumed continuous. In particular, since \(|\Delta^{n_i}| \times G^{n_i} \to |\Delta^\omega| \times \prod_\omega G \) is obviously continuous, as is the projection map to \( EG \), we conclude that \( f \) is continuous. So the middle vertical arrow in the statement of the lemma indeed lands in \( \text{Sing}(EG) \).

A similar argument shows that the right vertical arrow also lands in \( \text{Sing}(BG) \). \( \Box \)

**Proof of Theorem 1.2.** 1. First, let us recall that because \( \hat{EG} \to \hat{BG} \) is a diffeological \( G \)-bundle, we obtain a long exact sequence in \( \pi_n^{C\infty} \) (See 8.21 of [IZ13]). We note that the same techniques as in the proof of Lemma 3.10 show that the restriction map induces homomorphisms \( \pi_n^{C\infty}(\hat{BG}) \to \pi_n(BG) \), \( \pi_n^{C\infty}(\hat{EG}) \to \pi_n(EG) \), and that moreover, these are compatible with the long exact sequences of homotopy groups. Thus we have a commutative diagram

\[
\begin{array}{cccccc}
\cdots & \to & \pi_n^{C\infty}(G) & \to & \pi_n^{C\infty}(\hat{EG}) & \to & \pi_n^{C\infty}(\hat{BG}) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \cdots \\
\cdots & \to & \pi_n(G) & \to & \pi_n(EG) & \to & \pi_n(BG) & \to & \cdots
\end{array}
\]

We know that \( \hat{EG} \) is smoothly contractible, and that the homotopy groups of \( \text{Diff}(Q) \) are isomorphic by Theorem 1.1. So we conclude using the five lemma.

2. The proof for the combinatorially defined smooth homotopy groups is immediate from Lemma 3.10; one simply examines the induced map of long exact sequences

\[
\begin{array}{cccccc}
\cdots & \to & \pi_n\text{Sing}^{C\infty}(G) & \to & \pi_n\text{Sing}^{C\infty}(\hat{EG}) & \to & \pi_n\text{Sing}^{C\infty}(\hat{BG}) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \cdots \\
\cdots & \to & \pi_n\text{Sing}(G) & \to & \pi_n\text{Sing}(EG) & \to & \pi_n\text{Sing}(BG) & \to & \cdots
\end{array}
\]

and applies the Five Lemma again. \( \Box \)
4 Localization

**Notation 4.1** \((\text{Simp}(\hat{B}))\). Let \(\hat{B}\) be any diffeological space. We let \(\text{Simp}(\hat{B})\) denote the category of smooth, extended simplices of \(\hat{B}\). That is, an object of \(\text{Simp}(\hat{B})\) is the data of a smooth map \(j : |\Delta^n_e| \to \hat{B}\). (See Definition 2.15 for \(|\Delta^n_e|\).) A morphism is a commutative diagram

\[
\begin{array}{ccc}
|\Delta^n_e| & \xrightarrow{j} & |\Delta^n'_{e'}| \\
\downarrow{\Delta^n_e} & & \downarrow{\Delta^n'_{e'}} \\
\hat{B} & \xrightarrow{j'} & \hat{B}
\end{array}
\]

where the map \(|\Delta^n_e| \to |\Delta^n'_{e'}|\) is (induced by) an injective, order-preserving simplicial map.

Equivalently, let \((\Delta_{\text{inj}})^{/\text{Sing}^{C\infty}(\hat{B})}\) be the slice category over \(\text{Sing}^{C\infty}(\hat{B})\). Then

\[
\text{Simp}(\hat{B}) \cong (\Delta_{\text{inj}})^{/\text{Sing}^{C\infty}(\hat{B})}.
\]

**Notation 4.2** \((\text{subdiv}(X))\). More generally, if \(X\) is any simplicial set, we let \(\text{subdiv}(X)\) denote the barycentric subdivision of \(X\). This can be realized as follows: There exists a category of simplices of \(X\), where an object is a map \(f : \Delta^n \to X\) of simplicial sets, and a morphism from \(f\) to \(f'\) is a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{f} & \Delta^n' \\
\downarrow{\Delta^n} & & \downarrow{\Delta^n'} \\
X & \xrightarrow{f'} & X
\end{array}
\]

where \(\Delta^n \to \Delta^n'\) is induced by an injection \([n] \to [n']\).

Equivalently, let \((\Delta_{\text{inj}})^{/X}\) be the slice category over \(X\). Then \(\text{subdiv}(X)\) is the nerve of this category:

\[
\text{subdiv}(X) = N((\Delta_{\text{inj}})^{/X}) \quad (4.1)
\]

**Remark 4.3.** We have a natural isomorphism of simplicial sets

\[
N(\text{Simp}(\hat{B})) \cong \text{subdiv}(\text{Sing}^{C\infty}(\hat{B})).
\]

4.1 Realizations of subdivisions

The following lemma illustrates one power of localization: It turns (the nerve of) a strict category into a homotopically rich object.

**Lemma 4.4.** Let \(X\) be a Kan complex. Then the Kan completion of \(\text{subdiv}(X)\) is homotopy equivalent to \(X\).

We give two proofs for the reader’s edification. The second proof has the advantage that one sees an explicit map leading to the homotopy equivalence.

*Proof of Lemma 4.4 using coCartesian fibrations.* By construction (4.1), \(\text{subdiv}(X)\) is the total space of a Cartesian fibration over \(N(\Delta_{\text{inj}})\) with discrete fibers; in particular, the opposite category is a coCartesian fibration over \(\Delta_{\text{inj}}^{op}\). This coCartesian fibration classifies the functor

\[
\Delta^{op} \to \text{Sets} \subset \text{Spd}_{\infty} \subset \text{Cat}_{\infty}, \quad [n] \mapsto \{[n] \to X\},
\]

14
otherwise known as $X$. Recall that the colimit of a diagram of ∞-categories is computed by localizing the total space of the corresponding coCartesian fibration along coCartesian edges. Thus we have an equivalence of ∞-categories
\[
\colim_{\Delta^{op}} X \to \text{subdiv}(X)[C^{-1}]
\]
where $C$ is the collection of coCartesian edges. Because the inclusion $\text{spd}_\infty \subset \text{Cat}_\infty$ admits a right adjoint, the colimit of this functor into $\text{Cat}_\infty$ may be computed via the colimit in $\text{spd}_\infty$, but this is of course the usual geometric realization because all of the relevant ∞-groupoids in the simplicial diagram are discrete. So the domain of the equivalence is homotopy equivalent to the singular complex of the geometric realization $X$; this is equivalent to $X$ because $X$ is a Kan complex.

On the other hand, the localization on the righthand side is precisely the Kan completion of $\text{subdiv}(X)^{op}$ (because every edge is coCartesian), and hence the Kan completion of the non-opposite category.

Here is another proof.

**Notation 4.5** (max). Fix a simplicial set $X$. ($X$ need not be a Kan complex.) We denote by
\[
\max : \text{subdiv}(X) \to X
\]
the map of simplicial sets given by evaluating $j$ at the maximal vertex of $\Delta^n$.

**Proposition 4.6.** If $X$ is a Kan complex, max exhibits $X$ as the Kan completion of $\text{subdiv}(X)$. More generally, even if $X$ is not a Kan complex, max is a weak homotopy equivalence, so the Kan completion of $X$ is homotopy equivalent to the Kan completion of $\text{subdiv}(X)$.

**Proof.** It suffices to show that the induced map of geometric realizations $|\max| : |\text{subdiv}(X)| \to |X|$ is a homotopy equivalence; in fact, it is a homeomorphism. This is a classical result, as $\text{subdiv}(X)$ is nothing more than the barycentric subdivision of $X$. See for example III.4 of [GJ09].

**Remark 4.7.** If $B$ is any topological space, one can analogously define the strict category $\text{Simp}(B)$ of continuous simplices in $B$. The proof of Lemma 4.4 adapts straightforwardly to show that the Kan completion of $N(\text{Simp}(B))$ is homotopy equivalent to $\text{Sing}(B)$.

**Proof of Lemma 4.4 using max map.** Immediate from Proposition 4.6.

**Proof of Theorem 1.5.** For brevity, let $G = \text{Diff}(Q)$. Also for brevity, given a simplicial set $X$, we let $|X|$ denote its Kan completion. We have the string of maps
\[
N(\text{Simp}(\widehat{BG})) \to \text{subdiv}(\text{Sing}^C(\widehat{BG}))
\]
\[
\to |\text{Sing}^C(\widehat{BG})|
\]
\[
\to |\text{Sing}(BG)|.
\]
The first map is an isomorphism by Remark 4.3. The next map exhibits $|\text{Sing}^C(\widehat{BG})|$ as a Kan completion of $\text{subdiv}(\text{Sing}^C(\widehat{BG}))$ by Proposition 4.6.

The last map is a weak homotopy equivalence by Theorem 1.2(2).
5 The case of Liouville automorphisms

5.1 Definitions

By a Liouville sector, we mean a smooth manifold $M$ with boundary, equipped with a smooth 1-form $\theta$ satisfying the following:

1. $d\theta$ is a symplectic form, and
2. The boundary of $M$ is convex, and
3. The Liouville flow (i.e., the flow associated to the vector field symplectically dual to $\theta$) exhibits $M$ as a completion of a Liouville domain with convex boundary.

We refer to 2.4 of [OT19] for more details. In practice, we are only interested in an equivalence class of $\theta$—we declare $\theta$ and $\theta'$ to be equivalent if their difference is the de Rham derivative of a compactly supported smooth function on $M$.

By a Liouville automorphism, we mean a smooth map $\phi : M \to M$ such that $\phi^*(\theta)$ is in the same equivalence class as $\theta$.

Notation 5.1 ($\text{Aut}^o(M)$). We let

$$\text{Aut}^o(M) \subset \text{Diff}(M)$$

denote the subgroup of Liouville automorphisms, with the induced diffeological group structure.

5.2 Without decorations

Our goal is to prove that the smooth homotopy groups of $\widehat{\text{Aut}}^o(M)$ are isomorphic to the usual (continuous) homotopy groups of $\text{Aut}^o(M)$. We begin with an analogue of Lemma 3.1:

**Lemma 5.2.** Let $S$ be compact, and fix a function $f : S \to \text{Aut}^o(M)$ such that the induced map $S \times M \to M$ is smooth. Then $f$ is continuous in the strong Whitney topology.

**Proof.** For this proof, we fix a 1-form $\theta$ on $M$ realizing $M$ as a Liouville completion.

Let $g : S \times M \to M$ be the adjoint map, and for every $s \in S$, let $g_s : M \to M$ be the obvious function. Because $S$ is compact and because $f$ has image in $\text{Aut}^o(M)$, we know that there is some compact subspace $M_0 \subset M$ such that for all $s \in S$, we have $g_s(M_0) \subset M_0$, and that $g_s|_{M \setminus M_0}$ respects $\theta$. Note that there is a uniform bound on the variation in the derivatives of $g$ on $M_0$ by the compactness of $M_0$ and of $S$. This shows that the smoothness of $g$ guarantees that $f$ is a continuous map in the strong Whitney topology. $\square$

**Proof of Theorem 1.6.** 1. The steps for proving Theorem 1.1 may be followed nearly verbatim by invoking a version of Lemma 3.2. The only change is that for Liouville automorphisms, one may assume that the vector fields $\tilde{\alpha}$ arise from time-dependent, linear-near-infinity Hamiltonian functions.

2. Invoke the long exact sequence of smooth homotopy groups, the fact that $\widehat{EG}$ is contractible, and the Five Lemma.

3 This proof is identical to the proof of Theorem 1.2(2): We observe that $\widehat{\text{Aut}}^o(M)$ is a diffeological group and apply Remark 2.24 to the induced map of long exact sequences.

4. This proof is identical to the proof of Theorem 1.5 by taking $G = \text{Aut}^o(M)$.

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1See the proof of Lemma 3.2 for the $\tilde{\alpha}$ notation.
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