Exponential Sums Related to Maass Forms

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Abstract

We estimate short exponential sums weighted by the Fourier coefficients of a Maass form. This requires working out a certain transformation formula for non-linear exponential sums, which is of independent interest. We also discuss how the results depend on the growth of the Fourier coefficients in question.

The short estimates allow us to reduce smoothing errors. In particular, we prove an analogue of the approximate functional equation previously proven for holomorphic cusp form coefficients.

As an application of these, we remove the logarithm from the classical upper bound for long linear sums weighted by Fourier coefficients of Maass forms, the resulting estimate being the best possible. This also involves improving the upper bounds for long linear sums with rational additive twists, the gains again allowed by the estimates for the short sums.

1 Introduction and the main results

1.1 Maass forms

Let \( \psi \) be a Maass form for the full modular group, corresponding to the eigenvalue \( 1/4 + \kappa^2 \) of the hyperbolic Laplacian, with the Fourier expansion

\[
\psi(x + yi) = y^{1/2} \sum_{n \neq 0} t(n) K_{1/2}(2\pi |n| y) e(nx),
\]

where \( x \in \mathbb{R} \) and \( y \in \mathbb{R}_+ \). We may assume without loss of generality that \( \psi \) is even or odd, i.e. that \( t(-n) = t(n) \) for all \( n \in \mathbb{Z}_+ \), or that \( t(-n) = -t(n) \) for all \( n \in \mathbb{Z}_+ \).

The Fourier coefficients \( t(n) \) satisfy a bound of the kind

\[
t(n) \ll n^{\vartheta + \varepsilon}
\]

for some \( \vartheta \in \mathbb{R}_+ \). The best known exponent \( \vartheta = \frac{7}{64} \) is due to Kim and Sarnak [25]. The Ramanujan–Petersson conjecture for Maass forms declares that \( \vartheta = 0 \) is admissible. On average, the Fourier coefficients are of constant size. In particular, we have a Rankin–Selberg type estimate for the Fourier coefficients. One such result, more than sufficient for our purposes is the following (see e.g. [15], Chapter 8):

\[
\int_0^1 \left| \sum_{n \leq M} t(n) e(n \alpha) \right|^2 \, d\alpha = \sum_{n \leq M} |t(n)|^2 = AM + O(M^{7/8}), \tag{1}
\]
where $A$ is a positive real constant depending on $\psi$.

1.2 Objects of study and motivation

In the following we will consider linear exponential sums of the form

$$\sum_{M \leq n \leq M + \Delta} t(n) e(n\alpha),$$

where $M \in [1, \infty[$, $\Delta \in [1, M]$ and $\alpha \in \mathbb{R}$. When $\Delta = o(M)$, we call such sums short.

The reasons for considering such sums are manifold. First of all, the Fourier coefficients $t(n)$ are interesting mathematical objects which are not as well understood as one might wish. The exponential sums above contain all the information about the Fourier coefficients and thus provide an interesting window into their behaviour.

When $\alpha$ is a rational number $h/k$, the problem of estimating long sums with $\Delta = M$ is very analogous to classical problems in analytic number theory, such as the problems of estimating the error terms in the circle and Dirichlet divisor problems. Furthermore, the problem of estimating such sums with $\Delta = o(M)$ provides an analogue for problems such as studying the behaviour of the aforementioned error terms in short intervals. For further information about these classical topics, see e.g. Chapter 13 of [12] or [33].

Finally, good estimates for the short exponential sums above can sometimes be used to reduce smoothing error. An example of such an application is given e.g. by Theorems 3 and 4 below.

For holomorphic cusp forms, short exponential sums have been studied by Jutila [19], and the best known bounds are due to Ernvall-Hytönen and Karpinnen [9, 4].

It is interesting to study how sensitive the arguments used for holomorphic cusp forms are to value of $\vartheta$. In a sense, the strictly positive value of $\vartheta$ is the main difference between the holomorphic and non-holomorphic cases: even though the Voronoi summation formulae have a different appearance, what remains after the Bessel functions have been cashed in in terms of their asymptotics is very similar.

1.3 The results: Bounds for short exponential sums with applications

**Theorem 1.** Let $M \in [1, \infty]$ and let $\Delta \in \mathbb{R}_+$ be such that $\Delta \ll M^{5/8}$. Then

$$\sum_{M \leq n \leq M + \Delta} t(n) e(n\alpha) \ll \Delta^{1/6} M^{1/3+\vartheta+\varepsilon},$$

uniformly for $\alpha \in \mathbb{R}$. This is better than estimating via absolute values when $M^{2/(5+6\vartheta)} \ll \Delta \ll M^{5/8}$.

When $\Delta = M^{5/8}$ this gives the upper bound $\ll M^{3d/8+21/48+\varepsilon}$, and so splitting a longer sum into sums of this length and estimating the subsums separately gives the following bound for longer sums.
Corollary 2. Let $M \in [1, \infty]$ and let $\Delta \in \mathbb{R}_+$ be such that $M^{5/8} \ll \Delta \ll M$. Then

$$\sum_{M \leq n \leq M+\Delta} t(n) e(n\alpha) \ll \Delta M^{3\theta/8-3/16+\varepsilon}.$$ 

This is better than the bound $\ll M^{1/2+\varepsilon}$ when $M^{5/8} \ll \Delta \ll M^{(11-6\theta)/16}$.

The proof of Theorem 4.1 in Ernvall-Hytonen and Karppinen’s [9]. Fortunately, the proof in [9] works verbatim in our case, except that when estimating individual Fourier coefficients the extra factor $M^\theta$ appears. On the other hand, the proof of the non-linear estimate requires a transformation formula of a certain shape for smoothed exponential sums, and this particular result does not seem to have been worked out before yet. Thus, in Section 4, we will give an analogue of the relevant Theorem 3.4 of Jutila’s monograph [18], which considers smooth sums with holomorphic cusp form coefficients, with full details for Maass forms. An analogue of Theorem 3.2 of [18] has been given by Meurman in [27].

The following provides a concrete example of how estimates for short sums allow one to reduce smoothing errors thereby leading to improved upper bounds.

Theorem 3. Let $M \leq 1$, $\Delta \in [0, M]$, $h \in \mathbb{Z}$, $k \in \mathbb{Z}_+$ and $(h, k) = 1$. Then

$$\sum_{n \leq M} t(n) e\left(\frac{nh}{k}\right) \ll k^{2/3} M^{1/3+\theta/3+\varepsilon}.$$ 

When $M^{3/(5+6\theta)-1/2+\theta} \ll k \ll M^{1/4+3\theta/8}$, we have the upper bound

$$\sum_{n \leq M} t(n) e\left(\frac{nh}{k}\right) \ll k^{(1-6\theta)/(4-6\theta)} M^{3/(8-12\theta)+\varepsilon}.$$ 

The case $k = 1$ was considered by Hafner and Ivić [11] who essentially obtained the bound $\ll M^{1/3+\theta/3}$. Similar reduction for certain ranges of $k$ in the case of holomorphic cusp forms have recently been proved by Vesalainen [34].

It is of interest to note here that for small enough $k$, the rationally twisted sum has on average (in the mean square sense) the order of magnitude $k^{1/2} M^{1/4}$. This kind of result was first proven by Cramér [1] for the error term in the Dirichlet divisor problem. Jutila [17] extended this to the divisor problem with rational additive twists, and in [18] Jutila proved the analogous result for holomorphic cusp forms.

1.4 The results: an approximate functional equation and applications

Wilton [36] proved an approximate functional equation for exponential sums involving the divisor function. Jutila [17] extended this to sums with additive twists, and in [19] he proved an analogue for holomorphic cusp forms. In [4] Ernvall-Hytonen improved the error term. The following is an analogue of Ernvall-Hytonen’s result. We write $\overline{h}$ for an integer such that $hh \equiv 1 \pmod{k}$. Also, to simplify the notation, we write

$$T(M, \Delta; \alpha) = \sum_{M \leq n \leq M+\Delta} t(n) e(n\alpha).$$
Theorem 4. Let $\alpha \in \mathbb{R}$ have the rational approximation $\alpha = \frac{h}{k} + \eta$, where $h$ and $k$ are coprime integers with $1 \leq k \leq M^{1/4}$ and $|\eta| \leq \frac{1}{kM^{1/4}}$. Furthermore, let $M \geq 1$ and $1 \leq \Delta \leq M$. If $k^2\eta^2 M \gg 1$, then

$$T(M, \Delta; \alpha) \approx \frac{T(k^2\eta^2 M, k^2\eta^2 \Delta; \beta)}{(k^2\eta^2 M)^{1/2}} + O((k^2\eta^2 M)^{9/2-1/12+\varepsilon}),$$

where $\beta = \frac{h}{k} - \frac{1}{k^2\eta}$.

Wilton [35] proved that for the normalized Fourier coefficients $a(n)$ of a fixed holomorphic cusp form,

$$\sum_{n \leq x} a(n) e(n\alpha) \ll x^{1/2} \log x,$$

uniformly in $\alpha \in \mathbb{R}$. The Rankin–Selberg bound on the mean square of Fourier coefficients implies that

$$\int_0^1 \left| \sum_{n \leq M} a(n) e(n\alpha) \right|^2 d\alpha = \sum_{n \leq M} |a(n)|^2 = AM + O(M^{3/5+\varepsilon}),$$

for a certain positive real constant $A$ depending on the underlying cusp form, and so at most the logarithm can be removed from Wilton’s estimate, and this indeed was done by Jutila [19]. For Maass forms, the estimate analogous to Wilton’s was proved by Epstein, Hafner and Sarnak [3, 10]. The following is an analogue of Jutila’s logarithm removal.

Theorem 5. We have

$$\sum_{n \leq x} t(n) e(n\alpha) \ll x^{1/2},$$

uniformly in $\alpha \in \mathbb{R}$.

This is sharp in view of [1].

1.5 $\Omega$-results

Finally, it is naturally interesting to consider what are the limits of estimating short sums. In [6] Ernvall-Hytönen proved that, if $d \in \mathbb{Z}^+$ is a fixed integer such that $t(d) \neq 0$, then

$$\sum_{M \leq n \leq M+\Delta} t(n) e(n\alpha) w(n) \asymp \Delta M^{-1/4},$$

where $w$ is a suitable weight function, $\alpha = \sqrt{d}/\sqrt{M}$, and $M^{1/2+\varepsilon} \ll \Delta \leq d^{-1/2} M^{3/4}$. This immediately implies that for this range of lengths $\Delta$,

$$\sum_{M \leq n \leq M+\Delta} t(n) e\left(\frac{n\sqrt{d}}{\sqrt{M}}\right) = \Omega(\Delta M^{-1/4}).$$

This result also has counterparts for the divisor function and Fourier coefficients of holomorphic cusp forms in the papers of Ernvall-Hytönen and Karppinen [9].
and Ernvall-Hytonen \cite{15}. We would also like to mention that recently, Ernvall-Hytonen \cite{8} has considered the mean square of longer short sums with rational additive twists.

For sums of length $\Delta \ll M^{1/2}$, it turns out that square root cancellation is the best that could be hoped for. Essentially, combining the truncated Voronoi identity of Meurman \cite{28} with the arguments of Jutila \cite{16}, one gets the following mean square asymptotics

$$\int_M^{2M} \left| \sum_{x \leq n \leq x+\Delta} t(n) \right|^2 \, dx \asymp \Delta M,$$

for $M^\varepsilon \ll \Delta \ll M^{1/2-\varepsilon}$. In fact, a sharper result could be obtained, but this is enough for the relevant $\Omega$-result. In \cite{16} Jutila actually considered the behaviour of the error terms in the Dirichlet divisor problem and the second moment for the Riemann $\zeta$-function in short intervals, but the proof for the divisor function carries through fairly easily for Fourier coefficients of holomorphic cusp forms or Maass forms.

1.6 Notation

All the implicit constants are allowed to depend on the underlying Maass form, and $\varepsilon$, which denotes an arbitrarily small fixed positive number, which is not the same on each occurrence. Implicit constants depend also on chosen positive integers $J$ and $K$, when they appear.

The symbols $\ll$, $\gg$, $\asymp$, and $O$ are used for the usual asymptotic notation: for complex valued functions $f$ and $g$ in some set $\Omega$, the notation $f \ll g$ means that $|f(x)| \leq C |g(x)|$ for all $x \in \Omega$ for some implied constant $C \in \mathbb{R}_+$. When the implied constant depends on some parameters $\alpha, \beta, \ldots$, we use $\ll_{\alpha, \beta, \ldots}$ instead of mere $\ll$. The notation $g \gg f$ means $f \ll g$, and $f \asymp g$ means $f \ll g \ll f$.

Let us point out one notation which is non-standard: the characteristic function of the set $B$ is denoted by $\chi_B$.

2 The Voronoi type summation formula for Maass forms

The main tool is a Voronoi type summation formula for Maass forms with rational additive twists, proved by Meurman \cite{28}. The following result is Theorem 2 in \cite{28}.

**Theorem 6.** For a function $f \in C^1([a, b])$, and a positive integer $k$ and an integer $h$ coprime to $k$, we have

$$\sum_{a \leq n \leq b} t(n) e\left( \frac{nh}{k} \right) f(n) = \frac{\pi i}{k \sin \pi \kappa} \sum_{n=1}^{\infty} t(n) e\left( -\frac{nh}{k} \right) \int_{a}^{b} \left( J_{2i\kappa} \left( \frac{4\pi \sqrt{nx}}{k} \right) - J_{-2i\kappa} \left( \frac{4\pi \sqrt{nx}}{k} \right) \right) f(x) \, dx$$

$$+ \frac{4 \cosh \pi \kappa}{k} \sum_{n=1}^{\infty} t(-n) e\left( \frac{nh}{k} \right) \int_{a}^{b} K_{2i\kappa} \left( \frac{4\pi \sqrt{nx}}{k} \right) f(x) \, dx.$$
The following upper bound for the $K$-Bessel function will be enough for estimating all the integrals involving it:

$$K_\nu(x) \ll x^{-1/2} e^{-x} \ll_A x^{-A},$$

where $A > 0$ is fixed and $x \gg 1$. This follows from (5.11.9) in [26]. Here $\nu$ is fixed. In particular, we may estimate

$$K_{2\kappa}\left(\frac{4\pi\sqrt{n\pi}}{k}\right) \ll_A k^A n^{-A/2} x^{-A/2}. \quad (2)$$

For the $J$-Bessel function, we will actually need two main terms:

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + C x^{-3/2} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-5/4}).$$

Here $\nu$ is again fixed, and, though it is not important for us, the value of $C$ is actually $(1 - 4\nu^2)/8$. This asymptotic formula follows form (5.11.6) of [26]. In practice, it will often be useful to replace the cosine and sine terms by exponential functions. In particular, the $J$-Bessel expression appearing in the Voronoi-type summation formula has the asymptotics, for $x \gg 1$,

$$J_{2\kappa}(x) - J_{-2\kappa}(x) = \sqrt{\frac{2}{\pi x}} \sinh(\kappa\pi) \left( e\left(-\frac{1}{8}\right) e^{ix} - e\left(\frac{1}{8}\right) e^{-ix} \right) + C_1 x^{-3/2} e^{ix} + C_2 x^{-3/2} e^{-ix} + O(x^{-5/2}).$$

or more usefully, for $nx \gg k^2$,

$$J_{2\kappa}\left(\frac{4\pi\sqrt{n\pi}}{k}\right) - J_{-2\kappa}\left(\frac{4\pi\sqrt{n\pi}}{k}\right) = k^{1/2} \frac{\sinh \pi \kappa}{\pi \sqrt{2}} n^{-1/4} x^{-1/4} \left( e\left(-\frac{1}{8}\right) e\left(\frac{2\sqrt{n\pi}}{k}\right) - e\left(\frac{1}{8}\right) e\left(-\frac{2\sqrt{n\pi}}{k}\right) \right) + k^{3/2} n^{-1/4} x^{-3/4} \left( C_1 e\left(\frac{2\sqrt{n\pi}}{k}\right) + C_2 e\left(-\frac{2\sqrt{n\pi}}{k}\right) \right) + O(k^{5/2} n^{-5/4} x^{-5/4}). \quad (3)$$

Sometimes we will use $J$-Bessel asymptotics in the following form: For every $K \in \mathbb{Z}_+$, we have the asymptotics, again for $nx \gg k^2$,

$$J_{2\kappa}\left(\frac{4\pi\sqrt{n\pi}}{k}\right) - J_{-2\kappa}\left(\frac{4\pi\sqrt{n\pi}}{k}\right) = k^{1/2} \frac{\sinh \pi \kappa}{\pi \sqrt{2}} n^{-1/4} x^{-1/4} + \sum_{\ell = 1}^K e\left(\pm \frac{1}{8} \pm \frac{2\sqrt{n\pi}}{k}\right) \left( 1 + \sum_{\ell = 1}^K e^{\pm k^\ell n^{-\ell/2} x^{-\ell/2}} \right) + O_K\left(k^{1/2+(K+1)/4} n^{-1/4-(K+1)/2} x^{-1/4-(K+1)/2}\right). \quad (4)$$
3 Theorems on exponential integrals

The use of the Voronoi summation formula leads to many exponential integrals. Some of them will have saddle points. The saddle point result in Theorem 7 below will be used in the proofs of the transformation formula and the approximate functional equation. It is Theorem 2.2 from [15].

Let us consider an interval \([M_1, M_2] \subseteq \mathbb{R}_+\), and let \(U \in \mathbb{R}_+\) and \(J \in \mathbb{Z}_+\) be such that \(2 J U < M_2 - M_1\). Following [15], we introduce weight function \(\eta_J\) by requiring that

\[
\int_{M_1}^{M_2} \eta_J(x) h(x) \, dx = U^{-J} \int_{0}^{U} \cdots \int_{0}^{U} h(x) \, dx \, du_1 \cdots du_J \tag{5}
\]

for any integrable function \(h\) on \(\mathbb{R}\). It is not too difficult to see that actually \(\eta_J\) is given by the convolution

\[
\eta_J = \frac{1}{U} \chi_{[0,U]} \ast \frac{1}{U} \chi_{[0,U]} \ast \cdots \ast \frac{1}{U} \chi_{[0,U]} \ast \chi_{[M_1,M_2-JU]},
\]

with \(U^{-1} \chi_{[0,U]}\) appearing \(J\) times. In particular, \(\eta_J\) is \(J - 1\) times continuously differentiable on \(\mathbb{R}\), and supported in \([M_1, M_2]\).

**Theorem 7.** Let us consider an interval \([M_1, M_2] \subseteq \mathbb{R}_+\), let \(\mu \in \mathbb{R}_+\), and let \(D\) stand for the domain

\[
D = \{ z \in \mathbb{C} \mid |z - x| < \mu \text{ for some } x \in [M_1, M_2] \}.
\]

Let \(f, g : D \rightarrow \mathbb{C}\) be holomorphic, let \(F, G \in \mathbb{R}_+\), and assume that

\[
f(x) \in \mathbb{R}, \quad f''(x) > 0 \quad \text{and} \quad f''(x) \gg F \mu^{-2},
\]

for \(x \in [M_1, M_2]\), and that

\[
f'(z) \ll F \mu^{-1} \quad \text{and} \quad g(z) \ll G
\]

for \(z \in D\).

Next, let \(U \in \mathbb{R}_0\) and \(J \in \mathbb{Z}_+\) be such that \(2 J U < M_2 - M_1\), and let \(\eta_J\) denote the weight function defined as above, namely the convolution

\[
\eta_J = \frac{1}{U} \chi_{[0,U]} \ast \frac{1}{U} \chi_{[0,U]} \ast \cdots \ast \frac{1}{U} \chi_{[0,U]} \ast \chi_{[M_1,M_2-JU]},
\]

with \(U^{-1} \chi_{[0,U]}\) appearing \(J\) times.

Finally, let \(\alpha \in \mathbb{R}_+\), and let \(x \in [M_1, M_2]\) be such that \(f'(x_0) + \alpha = 0\). Then

\[
\int_{M_1}^{M_2} g(x) e(f(x) + \alpha x) \eta_J(x) \, dx
\]

\[
= \xi J(x_0) g(x_0) f''(x_0)^{-1/2} e(f(x_0) + \alpha x_0 + 1/8) + \text{error},
\]

where the error is

\[
\ll (M_2 - M_1) \left( 1 + \mu^J U^{-J} \right) G e^{-A|\alpha|_U - A F}
\]
Here $A$ is some positive real constant independent of $f$, $g$, $\alpha$, and $[M_1, M_2]$, the symbol $E_J(x)$ stands for
\[
E_J(x) = \frac{G}{(|f'(x) + \alpha| + f''(x)^{1/2})^{J+1}},
\]
and the factor $\xi_J(x_0)$ is as follows:
1. If $M_1 + JU < x_0 < M_2 - JU$, then
   \[
   \xi_J(x_0) = 1.
   \]
2. If $M_1 < x_0 < M_1 + JU$, then
   \[
   \xi_J(x_0) = (J! U^j)^{-1} \sum_{j=0}^{j_1} \binom{j}{j} (-1)^j \sum_{0 \leq \nu \leq j/2} c_\nu f''(x_0)^{-\nu} (x_0 - a - jU)^{J-2\nu},
   \]
   where $j_1$ is the largest integer with $a + j_1U < x_0$.
3. If $M_2 - JU < x_0 < M_2$, then
   \[
   \xi_J(x_0) = (J! U^j)^{-1} \sum_{j=0}^{j_2} \binom{j}{j} (-1)^j \sum_{0 \leq \nu \leq j/2} c_\nu f''(x_0)^{-\nu} (b - jU - x_0)^{J-2\nu},
   \]
   where $j_2$ is the largest integer with $b - j_2U > x_0$.

The coefficients $c_\nu$ are fixed numerical constants only depending on $J$.

Some of the exponential integrals we will meet will not have saddle points. They can be handled with the following theorem, which is Theorem 2.3 in [18].

**Theorem 8.** Let us consider an interval $[M_1, M_2] \subseteq \mathbb{R}_+$, let $\mu \in \mathbb{R}_+$, and let $D$ stand for the domain
\[
D = \{ z \in \mathbb{C} \mid |z - x| < \mu \text{ for some } x \in [M_1, M_2] \}.
\]
Let $f, g: D \to \mathbb{C}$ be holomorphic, let $F, G \in \mathbb{R}_+$, and assume that
\[
f(x) \in \mathbb{R}, \quad \text{and} \quad f'(x) \asymp F \mu^{-1},
\]
for $x \in [M_1, M_2]$, and that
\[
f'(z) \ll F \mu^{-1} \quad \text{and} \quad g(z) \ll G
\]
for $z \in D$.

Next, let $U \in \mathbb{R}_+$ and $J \in \mathbb{Z}_+$ be such that $2JU < M_2 - M_1$, and let $\eta_J$ denote the weight function defined as above, namely the convolution
\[
\eta_J = \frac{1}{U} \chi[0,U] * \frac{1}{U} \chi[0,U] * \cdots * \frac{1}{U} \chi[0,U] * \chi[M_1, M_2 - JU],
\]
with $U^{-1} \chi_{[0,1]}$ appearing $J$ times.

Finally, let $\alpha \in \mathbb{R}$. Then

$$\int_{M_1}^{M_2} g(x) e(f(x) + \alpha x) \eta J(x) \, dx \leq U^{-J} G \mu^{J+1} F^{-J-1} + \left( \mu J U^{1-J} + M_2 - M_1 \right) G e^{-A F}.$$ 

Here $A$ is some positive real constant independent of $f$, $g$, $\alpha$, and $[M_1, M_2]$.

We will also use the following lemma for estimating exponential integrals. It is Lemma 6 in [22].

**Lemma 9.** Let $M_1, M_2 \in \mathbb{R}^+$ and $M_1 < M_2$, let $J \in \mathbb{Z}^+$, and let $g \in C_c^J(\mathbb{R}^+)$ with $\text{supp} \, g \subseteq [a, b]$, and let $G_0$ and $G_1$ be such that

$$g^{(\nu)}(x) \ll \nu G_0 G_1^{-\nu}$$

for all $x \in \mathbb{R}^+$ for each $\nu \in \{0, 1, 2, \ldots, J\}$. Also, let $f$ be holomorphic function defined in $D \subseteq \mathbb{C}$, which consists all points in the complex plane with distance smaller than $\mu \in \mathbb{R}^+$ from the interval $[M_1, M_2]$ of the real axis. Assume that $f$ is real-valued on $[M_1, M_2]$ and let $F_1 \in \mathbb{R}^+$ be such that

$$F_1 \ll |f'(z)|$$

for all $z \in D$. Then, for all all $P \in \{1, 2, \ldots, J\}$,

$$\int_{M_1}^{M_2} g(x) e(f(x)) \, dx \ll \mu P G_0 (G_1 F_1)^{-P} \left(1 + \frac{G_1}{\mu} \right)^P (M_2 - M_1).$$

4 A transformation formula for smoothed exponential sums

4.1 Statement of the transformation formula

In the following theorem $\delta_1, \delta_2, \ldots$ denote positive constants which may be supposed to be arbitrarily small. Further, we write $L$ for $\log M_1$.

**Theorem 10.** Let $2 \leq M_1 < M_2 \leq 2M_1$. We assume that $M_1$ is sufficiently large, the notion of sufficiently large depending on the implicit constants in the assumptions below and on $\delta_1$. Let $f$ and $g$ be holomorphic functions in the domain

$$D = \{ z \in \mathbb{C} \mid |z - x| < c M_1 \text{ for some } x \in [M_1, M_2] \},$$

where $c$ is a positive constant. Suppose that $f(x)$ is real for $x \in [M_1, M_2]$. Suppose also that, for some positive numbers $F$ and $G$,

$$g(z) \ll G, \quad f'(z) \ll \frac{F}{M_1}$$

for $z \in D$, and that

$$f''(x) > 0 \quad \text{and} \quad f'''(x) \gg \frac{F}{M_1^2} \quad \text{for} \quad x \in [M_1, M_2].$$
Let \( r = h/k \) be a rational number such that \((h, k) = 1\),
\[
1 \leq k \ll M_1^{1/2-\delta_1},
\]
\[
r \gg M_1^{-1}
\]
and
\[
f'(M(r)) = r
\]
for a certain number \( M(r) \in ]M_1, M_2[\). Write
\[
M_j = M(r) + (-1)^j m_j, \quad j = 1, 2.
\]
Suppose that \( m_1 \approx m_2 \), and that
\[
M_1^\delta_2 \max \left\{ M_1 F^{-1/2}, |hk| \right\} \ll m_1 \ll M_1^{1-\delta_3}.
\]
Define for \( j = 1, 2 \)
\[
p_{j,n} = f(x) - rx + (-1)^{j-1} \left( \frac{2\sqrt{nx}}{k} - \frac{1}{8} \right),
\]
\[
n_j = (r - f'(M_j))^2 k^2 M_j,
\]
and for \( n < n_j \) let \( x_{j,n} \) be the (unique) zero of \( p_{j,n}(x) \) in the interval \([M_1, M_2]\).
Also, let \( J \) be a fixed positive integer and sufficiently large depending on \( \delta_2 \) and \( \delta_4 \). Let
\[
U \gg F^{-1/2} M_1^{1+\delta_4} \approx F^{1/2} r^{-1} M_1^{\delta_4},
\]
where \( \delta_1 > \delta_2 \), and assume also that
\[
JU < M_2 - M_1/2.
\]
Write for \( j = 1, 2 \)
\[
M'_j = M_j + (-1)^{j-1} JU = M(r) + (-1)^j m'_j,
\]
and suppose that \( m'_j \approx m_j \). Define
\[
n'_j = (r - f'(M'_j))^2 k^2 M'_j.
\]
Then we have
\[
\sum_{M_1 \leq m \leq M_2} \eta_J(m) t(m) g(m) e(f(m)) = \sum_{j=1}^{2} \sum_{n < n_j} w_j(n) t(n) e \left( -\frac{nh}{k} \right) n^{-1/4} x_{j,n}^{-1/4}
\]
\[
\cdot g(x_{j,n}) p_{j,n}'(x_{j,n})^{-1/2} e \left( p_{j,n}(x_{j,n}) + \frac{1}{8} \right) + O \left( k^{-1/2} m_1^{1/2} F^{-1} |h|^{3/2} U F^{1/4} M_1^3 \right),
\]
\[
\]
where
\[ w_j(n) = 1 \quad \text{for} \quad n < n'_j, \]
\[ w_j(n) \ll 1 \quad \text{for} \quad n < n_j, \]

\( w_j(y) \) and \( w'_j(y) \) are piecewise continuous functions in the interval \([n'_j, n_j]\) with at most \( J - 1 \) discontinuities, and
\[ w'_j(y) \ll (n_j - n'_j)^{-1} \quad \text{for} \quad n'_j < y < n_j \]
whenever \( w'_j(y) \) exists.

4.2 The proof

A word on the notation: In the following \( j \in \{1, 2\} \). This parameter comes about as follows: After applying the Voronoi summation formula and replacing the \( J \)-Bessel function by a simpler asymptotic expression, the cosine is replaced by the sum of two exponentials with phase factors of opposite signs. The value \( j = 1 \) corresponds to the \( + \)-sign, and \( j = 2 \) corresponds to the \( - \)-sign. For simplicity, we consider the various errors with fixed \( j \); i.e. we omit the summation symbol \( \sum_{j=1}^{2} \).

4.2.1 Sizes of the parameters

Suppose, to be specific, that \( r > 0 \), and thus \( h > 0 \). The proof is similar for \( r < 0 \).

The assertion (6) should be understood as an asymptotic result, in which \( M_1 \) and \( M_2 \) are large.

We observe that since,
\[ h k M_1^{\delta_2} \ll M_1^{1-\delta_3}, \]
we have \( h \ll M_1^{1-\delta_2-\delta_3} \).

On the size of \( F \). The number \( F \) will be large. In fact,
\[ F \gg M_1 r \gg k^{-1} M_1 \gg M_1^{1/2+\delta_1}. \]

Before proving the latter, we observe that, in \([M_1, M_2],\)
\[ f''(x) = \int_{\partial B(x,cM_1/2)} f'(z) \frac{dz}{(z-x)^2} \ll M_1 \cdot \frac{F M_1^{-1}}{M_1^2} = F M_1^{-2}, \]
so that \( f''(x) \approx F M_1^{-2} \). Here \( c \) is the positive constant from the definition of \( D \). The same argument shows in fact more: we have \( f''(z) \ll F M_1^{-2} \) for \( z \) in, say, \( D(M_1, M_2, cM_1/4) \).

We should also point out that \( F \) is not very large either: Since
\[ h k \ll M_1^{1-\delta_3-\delta_2} \quad \text{and} \quad \frac{F}{M_1} \approx \frac{h}{k}, \]
we have
\[ F \ll \frac{M_1 h}{k} \ll \frac{M_{1}^{2 - \delta_3 - \delta_2}}{k^2} \ll M_{1}^{2 - \delta_3 - \delta_2}. \]

Very crudely, but more simply, \( F \) is bounded from above and below by powers of \( M_1 \):
\[ M_1^{1/2} \ll F \ll M_1^2. \]

On the sizes of \( n_j \) and \( n_j' \). The number \( n_j \) will also be large:
\[ n_j \gg h k M_1^{2\delta_2}. \]

Now
\[ \frac{h}{k} - f'(M_j) \sim \int_{M_j}^{M_{(r)}} f''(x) \, dx \asymp m_j F M_1^{-2}, \]
so that
\[ n_j = \left( \frac{h}{k} - f'(M_j) \right)^2 k^2 M_1 \asymp m_j^2 F^2 M_1^{-4} k^2 M_1 \asymp F^{-1} h^3 k^{-1} m_j^2 \]
\[ \gg F^{-1} h^3 k^{-1} M_{1}^{2 + 2\delta_2} F^{-1} \]
\[ \gg k^2 h^{-2} M_1^{-2} h^3 k^{-1} M_{1}^{2 + 2\delta_2} = h k M_1^{2\delta_2}. \]

By using the estimate derived in the previous calculation we get
\[ n_j \asymp F^{-1} h^3 k^{-1} m_j^2 \asymp kh^{-1} M_1 h^3 k^{-1} m_j^2 \]
\[ \ll M_1^{-1} h^2 m_j^2 \ll M_1^{3 - 2\delta_2}, \]
since \( h \ll M_1 \). In particular,
\[ \log n_j \ll \log M_1. \]

We also have a simple estimate
\[ n_j \asymp F^{-1} h^3 k^{-1} m_j^2 \asymp F^{-1} h^3 k^{-1} m_j' \asymp n_j' \]
due to the fact \( m_j \asymp m_j' \).

4.2.2 The behaviour of \( p_{j,n} \)

After the application of Voronoi’s summation formula and replacing the \( J \)-Bessel function by its asymptotics, the phase functions in the individual integrals will be given by
\[ p_{j,n}(x) = f(x) - \frac{hx}{k} + (-1)^{j-1} \left( \frac{2\sqrt{nx}}{k} + \frac{1}{8} \right), \]
where \( x \) ranges over \( [M_1, M_2] \).

The parameter \( n_j \) (which, despite the notation, is not necessarily an integer) is chosen so that
\[ p_{j,n_j}'(M_j) = 0. \]
As the derivative is
\[ p'_{j,n}(x) = f'(x) - \frac{h}{k} + (-1)^{j-1} \frac{\sqrt{n}}{k\sqrt{x}} \]
this simplifies to
\[ n_j = \left( \frac{h}{k} - f'(M_j) \right)^2 k^2 \sqrt{M_j}. \]

Now the first salient feature of the function \( p_{j,n} \) is that \( p'_{j,n} \) has a unique zero \( x_{j,n} \) in the interval \( [M_1, M_2] \) for \( n < n_j \), and no zero in the same interval when \( n \geq n_j \). The second feature is that \( p'_{j,n}(x) \) has no zero in \( [M_1, M_2] \) when \( n \geq n_j \).

The existence of a zero in \( [M_1, M_2] \) when \( n < n_j \) is easily seen from the inequalities
\[ (-1)^{j} p'_{j,n}(M(r)) < 0 \]
and
\[ (-1)^{j} p'_{j,n}(M_j) > (-1)^{j} p'_{j,n}(M_j) = 0, \]
where the latter follows from the fact that \( p'_{j,n} \) behaves monotonically with respect to \( n \). Furthermore, the zero \( x_{j,n} \), whose existence is guaranteed when \( n < n_j \), lies on \( [M_1, M(r)] \) when \( j = 1 \), and on \( [M(r), M_2] \) when \( j = 2 \).

When \( j = 2 \), the derivative \( p'_{2,n}(x) \) is monotonically increasing and therefore it is clear that \( x_{2,n} \) is unique for \( n < n_2 \), and that there is no zero when \( n \geq n_j \) as
\[ p'_{2,n}(M_2) < p'_{2,n_2}(M_2) = 0. \]

We mention in passing that, in fact,
\[ p''_{2,n}(x) = f''(x) + \frac{\sqrt{n}}{2k\sqrt{x}} \approx F M_1^{-2} \]
on the interval \([M_1, M_2]\).

The case \( j = 1 \) is slightly less obvious. The main point is that by inspecting \( p''_{1,n}(x) \), we will see that \( p'_{1,n} \) is strictly increasing in \([M_1, M_2]\) when \( n \leq n_j \), which will guarantee the uniqueness of \( x_{1,n} \) and the non-existence of zero \( n = n_1 \) (if \( n_1 \) happens to be an integer) as
\[ p'_{1,n_1}(M_1) = 0. \]

In particular, \( p'_{1,n_1} \) takes only non-negative values in \([M_1, M_2]\) and we get, for \( n > n_1 \), that
\[ p'_{1,n}(x) > p'_{1,n_1}(x) \geq 0, \]
thereby excluding the possibility of zeros.

Now it only remains to show that \( p''_{1,n}(x) \) is positive for \( n \leq n_1 \), when \( M_1 \) is supposed to be sufficiently large. Since
\[ p''_{1,n}(x) = f''(x) - \frac{1}{2} k^{-1} n^{1/2} x^{-3/2}, \]
and
\[ \frac{1}{2} k^{-1} n^{1/2} x^{-3/2} \ll k^{-1} n_1^{1/2} M_1^{-3/2} \]
\[ 13 \]
\[
\asymp k^{-1} m_1 F M_1^{-2} k M_1^{1/2} M_1^{-3/2} = m_1 F M_1^{-3},
\]
as well as \(m_1 \ll M_1^{1-\delta_1}\), we indeed have \(p''_n(x) \asymp f''(x) \asymp F M_1^{-2}\) if only \(M_1\) is sufficiently large, depending (at most) on the implicit constants in the assumptions of the theorem and \(\delta_1\).

The reason for introducing the numbers \(n'\) is the following: when \(n < n'\), the corresponding saddle-points \(x_{j,n}\) lie on the interval \([M_1, M_2]\). This is not hard to see: for \(j = 1\) the saddle-point \(x_{1,n}\) decreases strictly monotonically as \(n\) increases, and the value \(n'\) corresponds to the situation where \(x_{1,n}\) lies precisely at \(M_1\). For \(j = 2\) things work similarly, except that \(x_{2,n}\) increase monotonically as \(n\) increases. The monotonicity of \(x_{j,n}\) with respect to \(n\) follows from the fact that the expression for \(p'_{j,n}(x)\) depends strictly monotonically on \(n\).

### 4.2.3 Voronoi summation and Bessel asymptotics

We begin the transformation of the exponential sum by applying the Voronoi-type summation formula for Maass forms:

\[
\sum_{M_1 \leq m \leq M_2} \eta_J(m) t(m) g(m) e(f(m)) = \sum_{M_1 \leq m \leq M_2} \eta_J(m) t(m) g(m) e\left(\frac{mh}{k}\right) e\left(f(m) - \frac{mh}{k}\right)
\]

\[
= \frac{\pi i}{k \sinh \pi \kappa} \sum_{n=1}^{\infty} t(n) e\left(\frac{-nh}{k}\right)
\cdot \int_{M_1}^{M_2} \left(J_{2\kappa}\left(\frac{4\pi \sqrt{nx}}{k}\right) - J_{-2\kappa}\left(\frac{4\pi \sqrt{nx}}{k}\right)\right) \eta_J(x) g(x) e\left(f(x) - \frac{hx}{k}\right) \, dx
\]

\[
+ \frac{4 \cosh \pi \kappa}{k} \sum_{n=1}^{\infty} t(-n) e\left(\frac{nh}{k}\right)
\cdot \int_{M_1}^{M_2} K_{2\kappa}\left(\frac{4\pi \sqrt{nx}}{k}\right) \eta_J(x) g(x) e\left(f(x) - \frac{hx}{k}\right) \, dx.
\]

Using the asymptotics of \(J\)- and \(K\)-Bessel functions we combine above calculations to

\[
\sum_{M_1 \leq m \leq M_2} \eta_J(m) t(m) g(m) e(f(m)) = \sum_{j=1}^{2} \sum_{n=1}^{\infty} t(n) n^{-1/4} e\left(\frac{-nh}{k}\right)
\cdot \int_{M_1}^{M_2} x^{-1/4} e(p_{j,n}(x)) \left(1 + \frac{K}{\ell} \sum_{\ell=1}^{\infty} c_{\ell}^{(j)} k^{\ell/2} n^{-\ell/2} x^{-\ell/2}\right) \eta_J(x) g(x) \, dx
\]

\[
+ O\left(\frac{1}{k} \sum_{n=1}^{\infty} |t(n)| k^{1/2+K+1} n^{-1/4-(K+1)/2} M_2 \int_{M_1}^{M_2} x^{-1/4-(K+1)/2} \eta_J(x) |g(x)| \, dx\right)
\]

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for any $K \in \mathbb{Z}_+$ and fixed $A > 0$. Fixing large enough $K$ depending on the Maass form in question, the first error term on the right-hand side can be absorbed in the error term on the right-hand side of \( (9) \). Also, clearly the second error term is negligible in view of the error term by choosing large enough $A$.

Next, we will estimate the integral
\[
\int_{M_1}^{M_2} x^{-1/4} e(p_{j,n}(x)) \left( 1 + \sum_{\ell=1}^{K} c_{\ell}^{(j)} k^\ell n^{-\ell/2} x^{-\ell/2} \right) \eta_J(x) g(x) \, dx.
\]  

\[ (8) \]

4.2.4 Large frequencies

When $n > 2n_j$, the integrals are estimated using Theorem \[S\] with
\[
\mu \asymp m_1, \quad F^{-1} := k^{-1} M_1^{-1/2}, n^{1/2} \gg m_1 F M_1^{-2},
\]
and
\[
G := M_1^{-1/4} G.
\]
Of the conditions of the theorem, only the ones related to the size of $p_{j,n}(x)$ are not immediately checked. Also, the parameter $\mu \asymp m_1$ instead of, say, $\asymp M_1$, in order for $p_{j,n}(x)$ to be satisfy these conditions.

Since $f''(z) \ll F M_1^{-2}$ in $D(M_1, M_2, cM_1/2)$, we have
\[
f'(z) - \frac{h}{K} \asymp \int_{M(r)} f''(z) \, dz \ll m_1 F M_1^{-2}
\]
for $z \in D(M_1, M_2, \mu)$, where it is best to integrate along the straight line segment connecting $M(r)$ and $z$. Thus we have
\[
p_{j,n}'(z) = f'(z) - \frac{h}{K} + (-1)^{j-1} \sqrt{n} \frac{\sqrt{\eta_J}}{k \sqrt{x}} \ll m_1 F M_1^{-2} + k^{-1} M_1^{-1/2} n^{1/2} \ll M.
\]

The conclusion that $p_{j,n}(x) \asymp M$ on the interval $[M_1, M_2]$, when $n > 2n_j$, can be obtained by comparing $p_{j,n}(x)$ with $p_{j,n_1}(x)$. More precisely, when $j = 1$, the function $p_{j,n_0}(x)$ is non-negative, bounded from above by $\ll M$ by estimates similar to the ones above, and the difference $p_{j,n}(x) - p_{j,n_1}(x)$ is
\[
\frac{\sqrt{n} - \sqrt{n}_1}{k \sqrt{x}} \asymp \frac{\sqrt{n}}{k \sqrt{M_1}} \asymp M.
\]

When $j = 2$, the conclusion is obtained in the same way, except that now $p_{j,n_0}(x)$ is non-positive, and the difference $p_{j,n}(x) - p_{j,n_1}(x)$ has the opposite sign.

Now that the assumptions of the theorem certainly hold, the estimate will be
\[
\int_{M_1}^{M_2} x^{-1/4} e(p_{j,n}(x)) \left( 1 + \sum_{\ell=1}^{K} c_{\ell}^{(j)} k^\ell n^{-\ell/2} x^{-\ell/2} \right) \eta_J(x) g(x) \, dx
\]
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for any positive integer $B$ we may estimate

$$
\ll U^{-J} M_1^{-1/4} G \left( k^{-1} M_1^{-1/2} n^{1/2} \right)^{-J-1}
\quad + \left( m_1^J U^{-1-J} + m_1 \right) M_1^{-1/4} G \exp \left( -A \left( k^{-1} M_1^{-1/2} n^{1/2} \right) m_1 \right)
$$

$$
\ll k^{J+1} U^{-J} G M_1^{J/2+1/4} n^{-J/2-1/2}
\quad + \left( m_1^J U^{-1-J} + m_1 \right) M_1^{-1/4} G \exp \left( -A \left( k^{-1} M_1^{-1/2} n^{1/2} \right) m_1 \right)
$$

The terms with $n > 2n_j$ contribute

$$
\ll k^{-1/2} \sum_{n>2n_j} |t(n)| n^{-1/4} \left( k^{J+1} U^{-J} G M_1^{J/2+1/4} n^{-J/2-1/2}
\quad + \left( m_1^J U^{-1-J} + m_1 \right) M_1^{-1/4} G \exp \left( -A \left( k^{-1} M_1^{-1/2} n^{1/2} \right) m_1 \right) \right).
$$

The first error term. The error from the first error term is

$$
\ll k^{J+1/2} U^{-J} G M_1^{J/2+1/4} \sum_{n>2n_j} |t(n)| n^{-J/2-3/4}
\ll k^{J+1/2} U^{-J} G M_1^{J/2+1/4} n_j^{-J/2+1/4}
\ll k^{J+1/2} U^{-J} G M_1^{J/2+1/4} \left( m_1^2 F^2 M_1^{-4} k^2 M_1 \right)^{-J/2+1/4}
\ll k U^{-J} G F^{-J+1/2} M_1^{1/2} m_1^{-J+1/2}
\ll m_1^{1/2} k U^{-J} G F^{-J+1/2} M_1^{1/2} F^{J/2} M_1^{-J-5/2}
\ll m_1^{1/2} k U^{-J} G F^{-J+2+5/4} F^{-1} M_1^{-J-5/2} F^{1/2} M_1^{J+1/2}
\ll m_1^{1/2} F^{-1} k U^{-J} G F^{-J+2+5/4} M_1^{1/2-5/4} M_1^{-J-5/2} F^{1/2} M_1^{J+1/2}
\ll m_1^{1/2} F^{-1} k^{-J+2+9/4} U^{-J} G M_1^{-J/2-3/4-5/2},
$$

and this is, provided that $J \geq 6$,

$$
\ll F^{-1} G h^{3/2} k^{-1/2} m_1^{1/2} U L,
$$

which is small enough.

The second error term. Since

$$
k^{-1} M_1^{-1/2} n^{1/2} m_1 \gg k^{-1} M_1^{-1/2} \left( m_1^2 F^2 M_1^{-4} k^2 M_1 \right)^{1/2} m_1
\gg F M_1^{-2} m_1^2
\gg F M_1^{-2} \left( M_1^{1+5/2} F^{-1/2} \right)^2
\gg M_1^{2+5/2} \gg 1,
$$

we may estimate

$$
\exp \left( -A k^{-1} M_1^{-1/2} n^{1/2} m_1 \right) \ll_B \left( k^{-1} M_1^{1/2} n^{1/2} m_1 \right)^{-B}
$$

for any positive integer $B$. 

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The error from the “middle terms” (i.e. the terms involving $U^{1-J}$) is provided that $k^{-1}M_1^{-1/2}n^{1/2}m_1 \gg 1$ for $n > 2n_j$,

$$\ll_B k^{-1/2}GM_1^{-1/4}m_1 U^{1-J} \sum_{n > 2n_j} |t(n)|n^{-1/4}(k^{-1}M_1^{-1/2}n^{1/2}m_1)^{-2B}$$

$$\ll F^{-1}Gk^{-1/2}m_1^{1/2}U \cdot Fk^2B \cdot F^{-1}U$$

$$\ll F^{-1}GM_1^{-1/2}m_1^{1/2}U \cdot Fk^2B \left(M_1^{1+4}\cdot F^{-1/2}\right)^{-J}$$

$$\ll F^{-1}GM_1^{-3/4}m_1^{1/2-2B} \left(M_1^{1+4}k^2M_1\right)^{-B+3/4}$$

$$\ll F^{-1}GM_1^{-1/2}m_1^{1/2-2B} \cdot k^{3/2}F^{1+J/2-2B+3/2}M_1^{-J-\delta_4J+1/3-3B-3}m_1^{1+J-2B}$$

$$\ll F^{-1}GM_1^{-1/2}m_1^{1/2}U \cdot k^{3/2}F^{1+J/2-2B+3/2}M_1^{-J-\delta_4J+1/3-3B-3}m_1^{1+J-2B}$$

Choosing here $B = 2$ (which is sufficiently large to make everything finite) gives

$$\ll F^{-1}GM_1^{-3/4}m_1^{1/2}U \cdot M_1^{-3\delta_4J+1/2-\delta_2J-\delta_4J+5/4-3\delta_2},$$

and this is $\ll k^{-1/2}m_1^{1/2}F^{-1}GM_1^3U$, provided that $J$ is so large that the exponent of $M_1$ is not positive. Thus, the lower bound for $J$ depends on $\delta_1, \delta_2$ and $\delta_4$, and we must have $\delta_4 > \delta_2$.

The error from the “last term” (not involving $U$ at all) is

$$\ll_B k^{-1/2}GM_1^{-1/4}m_1 \sum_{n > 2n_j} |t(n)|n^{-1/4}(k^{-1}M_1^{-1/2}n^{1/2}m_1)^{-2B}$$

$$\ll F^{-1}GM_1^{-3/4}m_1^{1/2}U \cdot k^{2B}F^{1+J/2-2B+3/2}M_1^{1+J-2B-3}m_1^{1+J-3B-3}$$

$$\ll F^{-1}GM_1^{-3/4}m_1^{1/2}U \cdot k^{3/2}F^{1+J/2-2B+3/2}M_1^{1+J-2B-3}m_1^{1+J-3B-3}$$

$$\ll F^{-1}GM_1^{-3/4}m_1^{1/2}U \cdot k^{3/2}F^{1+J/2-2B+3/2}M_1^{1+J-2B-3}m_1^{1+J-3B-3}$$

Here any value $B \geq 1$ is allowed.

### 4.2.5 Applying the saddle-point theorem: the error terms

In this section we shall treat the error terms coming from the saddle point lemma. Here the saddle-point lemma in question is the second saddle-point theorem. It is applied with the parameters

$$G := GM_1^{-1/4}, \quad F := F, \quad \mu := \frac{1}{\mu}GM_1.$$
The first error term. For a single integral, the first error term arising from the saddle-point lemma is, in view of the estimates
\[ 1 \ll \frac{M_1}{m_1} \ll \frac{M_1}{U} \ll F^{1/2} M_1^{-\delta_4}, \]
at most
\[ \ll m_1 \left(1 + \frac{M_1}{U} \right) M_1^{-1/4} e^{-Ar M_1 - AF} \]
The total error is then
\[ \ll m_1 F^{1/2} M_1^{-1/4-\delta_4} G e^{-AF} \]
The first term is
\[ \ll k^{-1/2} \sum_{n \leq 2n_j} |t(n)| n^{-3/4} m_1 F^{1/2} M_1^{-1/4-\delta_4} G e^{-AF} \]
and since
\[ m_1^2 M_1^{-1-\delta_4} F^{1/2} e^{-AF} \ll_B M_1^{1+\delta_4} F^{1+J/2} e^{-B} \]
we are done once we choose \( B \) to be sufficiently large depending on \( \delta_4 \) and \( J \).

The second error term. We start by estimating \( n_j - n_j' \):
\[ n_j - n_j' = (r - f'(M_j))^2 k^2 M_j - (r - f'(M_j'))^2 k^2 M_j' \]
\[ = k^2 \left( (r - f'(M_j))^2 - (r - f'(M_j'))^2 \right) M_j \]
\[ + k^2 \left( r - f'(M_j') \right)^2 (M_j - M_j') \]
The first term is
\[ = k^2 \left( r - f'(M_j) + r - f'(M_j') \right) (f'(M_j) - f'(M_j')) \]
\[ \ll k^2 m_j F M_1^{-2} U F M_1^{-2} \ll k^2 m_j F^2 U M_1^{-3}. \]
The second one is
\[ \ll k^2 U m_j^2 F^2 M_1^{-4} \ll k^2 U m_j F^2 M_1^{-3}. \]
Combining these estimates gives
\[ n_j - n_j' \ll k^2 m_j F^2 U M_1^{-3}. \]
When \( n < n_j' \), the saddle point \( x_{j,n} \) lies inside the interval \( ]M_j', M_j[ \) and a single second error term coming from the saddle point theorem is
\[ \ll G M_1^{-1/4} M_1 F^{-3/2}. \]
In total these contribute
\[ \ll k^{-1/2} \sum_{n \leq n_j'} |t(n)| n^{-1/4} G F^{-3/2} M_1^{3/4} \]
\[ \ll k^{-1/2} G F^{-3/2} M_1^{3/4} n_j^{-6/4} \]
\[ \ll k^{-1/2} G F^{-3/2} M_1^{3/4} m_j^{3/2} F^{3/2} M_1^{-9/4} k^{3/2} \]
\[ \ll k^{-1/2} G F^{-1} m_1^{1/2} F m_1 M_1^{-3/2} k^{3/2} \]
\[ \ll k^{-1/2} G F^{-1} m_1^{1/2} F m_1 h^{3/2} F^{-3/2}, \]
and since \( m_1 F^{-1/2} \ll M_1 F^{-1/2} \ll U \ll U L, \) this is
\[ \ll k^{-1/2} G F^{-1} m_1^{1/2} h^{3/2} U L \]
as required.

When \( n_j' \leq n < n_j, \) the saddle point \( x_{j,n} \) is in the range \( |M_1, M_j'| \cup |M_2', M_2|, \) and a single second error term is
\[ \ll G M_1^{3/4} F^{-1}. \]
Since \( t(n) \ll n^{\delta + \varepsilon} \) for all \( n, \) the total contribution is
\[ \ll k^{-1/2} \sum_{n_j' \leq n < n_j} |t(n)| n^{-1/4} G M_1^{3/4} F^{-1} \]
\[ \ll k^{-1/2} G F^{-1} M_1^{3/4} n_j^{-1/4 + \delta} (n_j - n_j') L \]
\[ \ll k^{-1/2} G F^{-1} L M_1^{3/4} M_1^{\delta_2(\delta-1/4)} |h|^{\delta-1/4} k^{\delta-1/4} k^2 m_1 F^2 U M_1^{-3} \]
\[ \ll k^{-1/2} G F^{-1} L U m_1^{1/2} |h|^{3/2} |h|^7/8 k^{7/8} F^2 M_1^{3/4-3} m_1^{1/2} M_1^{\delta_2(\delta-1/4)} |h|^\delta k^\delta \]
\[ \ll k^{-1/2} G F^{-1} L U m_1^{1/2} |h|^{3/2} . F^{1/4} M_1^{-1/2} m_1^{-1/2} |h|^\delta k^\delta \]
\[ \ll k^{-1/2} G F^{-1} L U m_1^{1/2} |h|^{3/2} . F^{1/4} M_1^{\delta} \]
which is the desired error term.

The third error term. A single last error term is
\[ \ll U^{-J} \sum_{\ell=0}^J \left( E_J(a + \ell U) + E_J(b - \ell U) \right), \]
where
\[ E_J(x) = G M_1^{-1/4} \left( |r - f'(x)| + f''(x) x^{1/2} \right)^{-J-1}. \]
Since
\[ r - f'(x) \asymp |x - M(r)| F M_1^{-2} \ll m_1 F M_1^{-2} \]
for the values of \( x \) appearing in the error term, and since
\[ m_1 F^{-1/2} \gg F^{-1/2} M_1^{1+\delta_2} F M_1^{-2} \gg F M_1^{-2}, \]
the first term in the parentheses dominates, and we have
\[ E_J(x) \asymp G M_1^{-1/4} (m_1 F M_1^{-3})^{-J-1}. \]
The total error from these error terms is
\[ \ll k^{-1/2} \sum_{n < 2n_j} |t(n)| n^{-1/4} U^{-J} G M_1^{-1/4} (m_1 F M_1^{-2})^{-J-1} \]
\[ \ll k^{-1/2} G U n_j^{3/4} M_1^{-1/4} (m_1 F U M_1^{-2})^{-J-1} \]
\[ \ll k^{-1/2} G U m_1^{3/2} F^{3/2} k^{3/2} M_1^{-1/4} (m_1 F U M_1^{-2})^{-J-1} \]
\[ \ll k^{-1/2} G U m_1^{1/2} h^{3/2} F^{-1} \cdot m_1 F M_1^{1/2} (m_1 F U M_1^{-2})^{-J-1} \]
\[ \ll k^{-1/2} G U m_1^{1/2} h^{3/2} F^{-1} \cdot M_1^{3} (M_1^{δ_2+δ_4})^{-J-1}. \]

Now, if \( J \) is sufficiently large with respect to \( δ_2 \) or \( δ_4 \), then this is
\[ \ll k^{-1/2} G U m_1^{1/2} h^{3/2} F^{-1} \]
and we are done.

4.2.6 Applying the saddle-point theorem: the main terms

Obtaining the main terms. For each \( n < n_j \) in the integral \( \mathcal{S} \) we get a saddle-point term
\[ \xi_J(x_{j,n}) x_{j,n}^{-1/4} \left( 1 + \sum_{\ell=1}^{K} \epsilon^{(j)}_\ell \kappa^{\ell/2} x_{j,n}^{-\ell/2} \right) g(x_{j,n}) p''_{j,n}(x_{j,n})^{-1/2} e \left( p_{j,n}(x_{j,n}) + \frac{1}{8} \right). \]

Substituting this back to \( \mathcal{S} \) gives
\[ i 2^{-1/2} k^{-1/2} \sum_{j=1}^{2} (-1)^{j-1} \sum_{n < n_j} \xi_J(x_{j,n}) t(n) n^{-1/4} e \left( -\frac{n\hbar}{k} \right) x_{j,n}^{-1/4} g(x_{j,n}) \]
\[ \cdot \left( 1 + \sum_{\ell=1}^{K} \epsilon^{(j)}_\ell \kappa^{\ell/2} x_{j,n}^{-\ell/2} \right) p_{j,n}(x_{j,n})^{-1/2} e \left( p_{j,n}(x_{j,n}) + \frac{1}{8} \right) \]
This is exactly what it should be except for the term in brackets involving a sum over \( \ell \), the removal of which gives an error (for each \( j \) and \( \ell \))
\[ \ll k^{-1/2} \sum_{n < n_j} |t(n)| n^{-1/4} M_1^{-1/4} G n^{-\ell/2} M_1^{-\ell/2} k^{\ell/2} F^{-1/2} M_1 \]
\[ \ll k^{-1/2} n_j^{1/4} G F^{-1/2} k M_1^{1/4} \]
\[ \ll k^{-1/2} m_1^{1/2} F^{1/2} k^{1/2} M_1^{-3/4} G F^{-1/2} k M_1^{1/4} \]
\[ \ll k^{-1/2} m_1^{1/2} F^{-1} G \cdot k^{3/2} F^{3/2} F^{-1/2} M_1^{-1/2} \]
\[ \ll k^{-1/2} m_1^{1/2} F^{-1} G h^{3/2} M_1 F^{-1/2} \]
\[ \ll k^{-1/2} m_1^{1/2} F^{-1} G h^{3/2} U, \]
which is small enough.
The new weight functions \( w_j(n) \). In this subsection we show that the function \( w_j(n) = \xi(x_{j,n}) \), where \( n < n_j \), satisfies the properties of the statement of the theorem.

The first property follows at once from property 1 of the function \( \xi_J(x) \) on p. 8 for \( M' < x_{j,n} < M'' \). To prove the second one, we have three cases to consider. If \( M' < x_{j,n} < M'' \) the claim is trivial by the property 1.

On the other hand, if \( M_1 < x_{j,n} \leq M' \) the claim follows from property 2 of \( \xi_J(x) \) using the estimates \( p''_{j,n} \approx F M^{-2} \) and \( U \gg F^{-1} M^{-1} \).

Finally, the case \( M'' < x_{j,n} < M_2 \) is similar; we have by property 3 on p. 8 that

\[
\sum_{M_1 < m < M_2} t(m) g(m) e\left(\eta m + Bm^{1/2}\right) \ll \Delta^{5/6} (G + \Delta G') M^{9/3} F^{1/3+\varepsilon}.
\]

5 Estimates for non-linear sums

The savings in the estimates for short sums depend on an estimate for the kind of nonlinear sums that appear after the application of the Voronoi type summation formula. In the following theorem, it is essential that the estimate is better when shorter sums are considered.

Theorem 11. Let \( M \in [1, \infty[, \eta \in \mathbb{R}, B \in \mathbb{R}, \) and \( \Delta \in [1, M] \). Denote \( F = |B| M^{1/2} \), and assume that

\[
M^2 \ll \Delta F.
\]

Let \( g \) be a \( C^1 \)-function on the interval \([M, M + \Delta]\) satisfying bounds

\[
g(x) \ll G \quad \text{and} \quad g'(x) \ll G'
\]

on \([M, M + \Delta]\) for some positive real numbers \( G \) and \( G' \). Then

\[
\sum_{M \leq m \leq M + \Delta} t(m) g(m) e\left(\eta m + Bm^{1/2}\right) \ll \Delta^{5/6} (G + \Delta G') M^{9/3} F^{1/3+\varepsilon}.
\]
Proof. This is analogous to Theorem 4.1 in [9], and in fact, the proof given
in [9] works verbatim in our case, except that now we use Theorem 10 instead
of the corresponding result for holomorphic cusp form (i.e. Theorem 3.4 in [18],
and naturally, when smoothing error is to be estimated, an extra $M^{\vartheta}$ appears
in a few places. There is only one point which requires extra clarification, the
error term in Theorem 10 has the extra factor $F^{1/4} M^{\vartheta}$; this time the total error
from the error terms coming from using the transformation formula contributes
\[
\ll \frac{\Delta}{M} M^{1/2+\vartheta} F^{1/4+\varepsilon} \ll \Delta^{5/6} M^{\vartheta-1/3} F^{1/4+\varepsilon},
\]
which is smaller than the desired upper bound.

It turns out that for long sums, the $\vartheta$ in the upper bound may be erased. This
was proved by Karppinen in [23] by considering the mean value of the relevant
exponential sums. Earlier, Jutila [20, 21] had considered similar mean values
for holomorphic cusp forms and the divisor function. The following estimate is
Theorem 8.2 in [23].

Theorem 12. Let $M \in [1, \infty]$, $\eta \in \mathbb{R}$, $B \in \mathbb{R}$, and $\Delta \in [1, M]$. Denote
$F = |B| M^{1/2}$, and assume that $M \ll F$. Let $g$ be a $C^1$-function on the interval
$[M, M + \Delta]$ satisfying the bounds
\[
g(x) \ll G \quad \text{and} \quad g'(x) \ll G'
\]
on $[M, M + \Delta]$ for some positive real numbers $G$ and $G'$. Then
\[
\sum_{M \leq m \leq M + \Delta} t(m) g(m) e\left(\eta m + Bm^{1/2}\right) \ll (G + G') M^{1/2} F^{1/3+\varepsilon}.
\]

6 Proof of Theorem 1

We shall prove Theorem 1 by first proving estimates for smooth short exponen-
tial sums. For this purpose, we shall use a wide weight function $w \in C^\infty_c(\mathbb{R}_+)$
taking only values from $[0, 1]$, supported in $[M, M + \Delta]$, and for which
\[
w^{(\nu)}(x) \ll x^{-\nu},
\]
for every nonnegative integer $\nu$. The following estimates give an analogue of
Theorem 5.1 of [9].

Theorem 13. Let $M \in [1, \infty]$, and let $\Delta \in [1, M]$ with $\Delta \gg M^\alpha$ for
some arbitrarily small fixed $\alpha \in \mathbb{R}_+$. Furthermore, let $\alpha \in \mathbb{R}$, and let $h \in \mathbb{Z}, k \in \mathbb{Z}_+$
and $\eta \in \mathbb{Z}$ be such that
\[
\alpha = \frac{h}{k} + \eta, \quad (h,k) = 1, \quad k \leq K, \quad |\eta| \leq \frac{1}{kK},
\]
where $K = \Delta^{1/2-\delta}$ for an arbitrarily small fixed $\delta \in \mathbb{R}_+$.

1. If $|\eta| \ll \Delta^{-1+\delta}$, then
\[
\sum_{M \leq n \leq M + \Delta} t(n) e(n\alpha) w(n) \ll_{\alpha, \varepsilon} \Delta^{1/6} M^{1/3+\varepsilon}.
\]
2. If $\Delta^{-1+\delta} \ll |\eta|$ and $k^2 \eta^2 M < 1/2$, then

$$\sum_{M \leq n \leq M+\Delta} t(n) e(\nu n) w(n) \ll_{\alpha, \delta} 1.$$ 

3. If $\Delta^{-1+\delta} \ll \eta \ll M \Delta^{-2}$, $k^2 \eta^2 M \gg 1$ and $k^2 \eta M \Delta^{-1+\delta} \ll 1$, then

$$\sum_{M \leq n \leq M+\Delta} t(n) e(\nu n) w(n) \ll_{\alpha, \delta} 1 + k^{-1/2} \Delta^{-1/4} (k^2 \eta^2 M)^{-1/4+c}.$$

4. If $\Delta^{-1+\delta} \ll \eta \ll M \Delta^{-2}$, $k^2 \eta^2 M \gg 1$ and $k^2 \eta M \Delta^{-1+\delta} \gg 1$, then

$$\sum_{M \leq n \leq M+\Delta} t(n) e(\nu n) w(n) \ll_{\alpha, \delta} (k^2 \eta^2 M)^{\theta} \Delta^{1/6} M^{1/3+c}.$$

**Proof.** We begin by applying the Voronoi summation formula to the sum under study. The proof will soon split into two cases depending on whether $\eta$ is smaller than larger than $\Delta^{-1+\delta}$. Voronoi summation yields

$$\sum_{M \leq n \leq M+\Delta} t(n) e(\nu n) w(n)$$

$$= \frac{\pi}{k \sinh \pi k} \sum_{n=1}^{\infty} t(n) e\left(\frac{-n\hbar}{k}\right) \int_{M}^{M+\Delta} \left(J_{2\alpha \kappa} - J_{-2\alpha \kappa}\right) \left(\frac{4\pi \sqrt{n\lambda}}{k}\right) e(\eta x) w(x) \, dx$$

$$+ \frac{4 \cosh \pi k}{k} \sum_{n=1}^{\infty} t(-n) e\left(\frac{n\hbar}{k}\right) \int_{M}^{M+\Delta} K_{2\alpha \kappa} \left(\frac{4\pi \sqrt{n\lambda}}{k}\right) e(\eta x) w(x) \, dx.$$

Using the asymptotics for the $K$-Bessel function, and picking some large $A \in \mathbb{R}_+$, the $K$-series can be estimated by

$$\ll_A \frac{1}{k} \sum_{n=1}^{\infty} |t(n)| \int_{M}^{M+\Delta} k^{-A} n^{-A/2} \zeta^{-A/2} w(x) \, dx.$$ 

This is $\ll_A k^{A-1} M^{-A/2}$, provided that $A > 2$, and since $k \ll M^{1/2-\delta}$, it is furthermore $\ll_{\delta} 1$, provided that $A \gg \delta$. Similarly, by replacing the $J$-Bessel expression by the asymptotics given in [1] with some $K \in \mathbb{Z}_+$, the resulting $O$-terms contribute

$$\ll_K \frac{1}{k} \sum_{n=1}^{\infty} |t(n)| \int_{M}^{M+\Delta} k^{1/2+(K+1)} n^{-1/4-(K+1)/2} x^{-1/4-(K+1)/2} w(x) \, dx,$$

and this is again $\ll_{\delta} 1$ for a fixed $K \gg \delta$. Thus, we are led to

$$\sum_{M \leq n \leq M+\Delta} t(n) e(\nu n) w(n)$$

$$= O(1) + \frac{A}{k} \sum_{n=1}^{\infty} t(n) e\left(\frac{-n\hbar}{k}\right).$$
\[ \cdot \int_M^{M+\Delta} \frac{k^{1/2} A'}{n^{1/4} x^{1/4}} \sum_k e\left( \pm \frac{2 \sqrt{nx}}{k} \right) e\left( \mp \frac{1}{8} \right) g_{\pm}(x; n, k) e(\eta x) w(x) \, dx, \]

where

\[ g_{\pm}(x; n, k) = 1 + \sum_{\ell=1}^{K} c^\pm_k n^{-\ell/2} x^{-\ell/2}. \]

6.1 The case \( \eta \ll \Delta^{-1+\delta} \)

Write \( X = k^2 M^{1+\delta} \Delta^{-2} \). We shall handle separately the terms with \( n > X \) and the terms with \( n \leq X \).

The high-frequency terms with \( n > X \) contribute

\[ \ll \frac{1}{k} \sum_{n > X} t(n) k^{1/2} n^{-1/4} e\left( -\frac{n \eta}{k} \right) \cdot \int_M^{M+\Delta} x^{-1/4} g_{\pm}(x; n, k) e\left( \pm \frac{2 \sqrt{nx}}{k} + \eta x \right) w(x) \, dx. \]

Since we now have \( X^{1/2} k^{-1} M^{-1/2} \gg \eta \), Lemma \( \square \) says that the integral here is

\[ \int_M^{M+\Delta} \ldots \, dx \ll_P M^{-1/4} \left( \Delta n^{1/2} k^{-1} M^{-1/2} \right)^{-P} \Delta. \]

Provided that \( P \geq 2 \), the contribution from these high-frequency terms is

\[ \ll \frac{1}{k} \sum_{n > X} |t(n)| k^{1/2} n^{-1/4} M^{-1/4} \left( \Delta n^{1/2} k^{-1} M^{-1/2} \right)^{-P} \Delta \]

\[ \ll k^{-P/2} \Delta M^{P/2-1/4} \sum_{n > X} |t(n)| n^{-1/4-P/2} \]

\[ \ll k^{-P/2} \Delta^{1-P} M^{P/2-1/4} X^{3/4-P/2}, \]

and for a fixed \( P \gg \delta \) 1, this is \( \ll \delta \) 1.

Let us consider next the low-frequency terms with \( n \leq X \). These contribute

\[ \ll \frac{1}{k} \sum_{n \leq X} t(n) k^{1/2} n^{-1/4} e\left( -\frac{n \eta}{k} \right) \cdot \int_M^{M+\Delta} x^{-1/4} g_{\pm}(x; n, k) e\left( \pm \frac{2 \sqrt{nx}}{k} + \eta x \right) w(x) \, dx \]

\[ = k^{-1/2} \sum_{L \leq X/2} \int_M^{M+\Delta} x^{-1/4} w(x) \]

\[ \sum_{L < n \leq 2L} t(n) n^{-1/4} g_{\pm}(x; n, k) e\left( \pm \frac{2 \sqrt{nx}}{k} - \frac{n \eta}{k} x + \eta x \right) \, dx. \]
By Theorem 12, the conditions of which are met under the present circumstances, the sum $\sum_{L<n \leq 2L}$ can be estimated by

$$\ll L^{-1/4} L^{1/2} \left( M^{1/2} k^{-1} L^{1/2} \right)^{1/3+\varepsilon} \ll L^{5/12} M^{1/6+\varepsilon} k^{-1/3}.$$ 

Thus, the low-frequency terms contribute

$$\ll k^{-1/2} \sum_{L \leq X/2} \Delta M^{-1/4} L^{5/12} M^{1/6+\varepsilon} k^{-1/3}$$

$$\ll k^{-5/6} \Delta M^{-1/12+\varepsilon} X^{5/12} = \Delta^{1/6} M^{1/3+\varepsilon},$$

and we are finished with the case $\eta \ll \Delta^{-1+\delta}$.

### 6.2 The case $\eta \gg \Delta^{-1+\delta}$

This time we will choose $X = k^2 \eta^2 M$. The high-frequency terms with $n > 2X$ are again handled in the same way as in the case $\eta \ll \Delta^{-1+\delta}$. For an integer $P \geq 2$, we have

$$\ll \frac{1}{k} \sum_{n \geq 2X} t(n) k^{1/2} n^{-1/4} e \left( \frac{-n\eta}{k} \right)$$

$$\cdot \int_{M} x^{-1/4} y_{\pm}(x; n, k) e \left( \pm \frac{2\sqrt{x} \eta}{k} + \eta x \right) w(x) \, dx$$

$$\ll_p \frac{1}{k} \sum_{n \geq 2X} |t(n)| \left| k^{1/2} n^{-1/4} M^{-1/4} \right| \left( \Delta n^{1/2} k^{-1} M^{-1/2} \right)^{-P} \Delta$$

$$\ll_p k^{-1/2+P} \Delta^{1-P} M^{P/2-1/4} X^{3/4-P/2}.$$ 

For $P \gg \delta 1$, this contribution is again $\ll \delta 1$.

If $X < 1/2$, then the above already proves case 2 of the theorem, so let us assume that $X \gg 1$. The remaining terms, the ones with $n \leq 2X$, are then partitioned into two sets: those with $|n - X| \geq W$ and those with $|n - X| < W$, where $W = k^2 M \eta \Delta^{-1+\delta}$.

So, let us consider the terms with $n \leq 2X$ and $|n - X| \geq W$. The crucial observations here are that

$$\frac{\sqrt{X}}{k \sqrt{x}} - \frac{\sqrt{n}}{k \sqrt{x}} \geq \frac{1}{k \sqrt{M}} \int_{n}^{X} \frac{dt}{t^{1/2}} \geq \frac{|n - X|}{k \sqrt{M} \sqrt{x}} \geq \frac{W}{k \sqrt{M} \sqrt{x}} = \Delta^{-1+\delta},$$

and that, thanks to the assumption $\eta \ll M \Delta^{-2}$,

$$|\eta| - \frac{\sqrt{X}}{k \sqrt{x}} = \frac{\sqrt{X}}{k \sqrt{M}} - \frac{\sqrt{X}}{k \sqrt{x}} \geq \frac{\sqrt{X} \Delta}{k} \int_{M}^{x} \frac{dt}{t^{1/2}} \ll \frac{\sqrt{X} \Delta}{k M^{3/2}} = \frac{\eta \Delta}{M} \ll \Delta^{-1}.$$ 

Using these appropriately (depending on the sign of $\eta$), we conclude that

$$\frac{d}{dt} \left( \pm \frac{2\sqrt{n} x}{k} + \eta x \right) = \pm \frac{\sqrt{n}}{k \sqrt{x}} + \eta \gg \Delta^{-1+\delta},$$

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and so, by Lemma 9, the terms under consideration contribute
\[
\ll_P \frac{1}{k} \sum_{2X < n \leq X \iff n-X \geq W} |t(n)| k^{1/2} n^{-1/4} M^{-1/4} \Delta^{-\delta} P \Delta
\]
\[
\ll_P k^{-1/2} X^{-1/4} M^{-1/4} \Delta^{-\delta} P \Delta,
\]
and for a fixed \( P \gg \delta \) this is again \( \ll \delta \).

Next, if \( W \ll 1 \), then the remaining terms, the ones with \( |n - X| < W \), contribute
\[
\ll k^{-1/2} X^{\frac{d-1/4+\varepsilon}{d}} M^{-1/4} \ll k^{-1/2} (k^2 \eta^2 M)^{\frac{d-1/4+\varepsilon}{d}} M^{-1/4},
\]
and we have established case 3. Finally, only case 4 remains.

So, let us assume that \( W \gg 1 \). The idea now is to exchange integration and summation, apply Theorem 11 to the integrand with the parameters
\[
M = X, \quad \Delta = W, \quad \text{and} \quad B = \frac{\sqrt{X}}{k},
\]
observing that the condition \( \Delta F \gg M^2 \) of Theorem 11 holds, since it reduces to
\[
W \cdot \frac{\sqrt{X} \sqrt{M}}{k} \gg X,
\]
which follows from \( k^2 \eta \ll 1 \ll W \). The remaining terms are then seen to contribute
\[
\ll \frac{1}{k} \sum_{X-W < n < X+W} t(n) k^{1/2} n^{-1/4} e \left( -\frac{nk}{k} \right)
\]
\[
\cdot \int \frac{x^{-1/4}}{M} g(x; n, k) e \left( \pm \frac{2\sqrt{nx}}{k} + \eta x \right) w(x) \, dx
\]
\[
\ll k^{-1/2} \int \frac{x^{-1/4}}{M} e(\eta x) w(x)
\]
\[
\cdot \sum_{X-W < n < X+W} t(n) n^{-1/4} e \left( -\frac{nk}{k} \pm \frac{2\sqrt{nx}}{k} \right) \, dx
\]
\[
\ll k^{-1/2} \int \frac{x^{-1/4}}{M} w(x) \left( \frac{W}{X} \right)^{-5/6} X^{-1/4} X^{1/2+\varepsilon} \left( \frac{\sqrt{X} \sqrt{M}}{k} \right)^{1/3+\varepsilon} \, dx
\]
\[
\ll k^{-1/2} M^{-1/4} \left( \frac{k^2 \eta M \Delta^{-1+\delta}}{k^2 \eta^2 M} \right)^{5/6} \left( k^2 \eta^2 M \right)^{1/4+\varepsilon} \left( \frac{k \eta M}{k} \right)^{1/3+\varepsilon}
\]
\[
\ll k^{-1/2} M^{-1/4} \left( \frac{\Delta^{-1+\delta}}{\eta} \right)^{5/6} k^{1/2+2\theta} \eta^{1/2+2\theta} M^{1/4+\varepsilon} \eta^{1/3} M^{1/3+\varepsilon}
\]
\[
\ll (k^2 \eta^2 M)^{\theta} \Delta^{1/6} M^{1/3+\varepsilon},
\]
and we are done.
Proof of Theorem 1. We can now remove the weight function \( w \) from the estimates for short sums. For this purpose we shall introduce a partition of unity of \( M, M + \Delta \). Let us define a set of points \( M_\ell \) for \( \ell \in \mathbb{Z} \) by
\[
M_0 = M + \frac{\Delta}{2},
\]
and for each \( \ell \in \mathbb{Z}_+ \) we set
\[
M_{\pm \ell} = M + \frac{\Delta}{2} \pm \left( \frac{\Delta}{4} + \frac{\Delta}{8} + \ldots + \frac{\Delta}{2^{\ell+1}} \right).
\]
We pick functions \( w_\ell \in C^\infty_c(\mathbb{R}) \) such that each \( w_\ell \) only takes values from \([0,1] \), \( w_\ell \) is supported on \( [M_\ell, M_\ell + \Delta] \), and \( w_\ell \equiv 1 \) on \( [M_\ell, M_{\ell+1}] \), and \( w'_\ell \ll 2^\ell |\ell| \Delta^{-|\ell|} \).

Let \( L \in \mathbb{Z}_+ \) be such that
\[
\Delta 2^{-L} = M^{2/(5+6\delta)}.
\]
Then we shall have, by Theorem 13,
\[
\sum_{\ell=-L}^{L} \sum_{n \in \mathbb{Z}} t(n) e(n\alpha) w_\ell(n) \ll \sum_{\ell=-L}^{L} \left( \frac{\Delta}{2|\ell|} \right)^{1/6-\delta} M^{1/3+\delta+\epsilon} \ll \Delta^{1/6-\delta} M^{1/3+\delta+\epsilon},
\]
and estimating by absolute values,
\[
\sum_{M \leq n \leq M + \Delta} t(n) e(n\alpha) \left( 1 - \sum_{\ell=-L}^{L} w_\ell(n) \right) \ll M^{2/(5+6\delta)} M^{\delta+\epsilon} \ll \Delta^{1/6-\delta} M^{1/3+\delta+\epsilon}.
\]

Proposition 14. Let \( M \in [1, \infty] \) and let \( \Delta \in [1, M] \) satisfy \( \Delta \gg M^{5/8} \). Also, let \( \alpha \in \mathbb{R} \) have a rational approximation \( \alpha = h/k + \eta \), where \( h \) and \( k \) are coprime integers with \( 1 \leq k \ll M^{5/16-\epsilon} \), and where \( \eta \in \mathbb{R} \) satisfies \( \eta \ll k^{-1} M^{-5/16} \) and \( \eta \ll M \Delta^{-2} \). Then
\[
\sum_{M \leq n \leq M + \Delta} t(n) e(n\alpha) \ll \Delta^{1/6-\delta} M^{1/3+\delta+\epsilon} + k^{-1/2} \Delta M^{-1/4} \left( k^2 \eta^2 M \right)^{\delta-1/4+\epsilon}.
\]

Proof. This is very similar to the proof of Theorem 1 above. In particular, we may use the same weight functions \( w_\ell \), and we simply have an extra term on the right-hand side.

7 Proof of Theorem 3

Let \( U \in \mathbb{R}_+ \). We shall pick a weight function \( w \in C^\infty_c(\mathbb{R}_+) \) taking only nonnegative real values, supported in \([M, M + \Delta] \), identically equal to 1 in \([M + U, M + \Delta - U] \), for which
\[
w^{(\nu)}(x) \ll U^{-\nu},
\]
and whose derivatives are supported in \([M, M + U] \cup [M + \Delta - U, M + \Delta] \). Sums with this weight function can be estimated rather nicely:
Lemma 15. Let $X \in [1, \infty]$, let $M \in [1, \infty]$, and let $U$ and $w$ as above. Also, let $h$ and $k$ be coprime integers with $1 \leq k \ll M^{1/2 - \delta}$, where $\delta$ is a fixed positive real number. Then

$$\sum_{M \leq n \leq M + \Delta} t(n) e\left(\frac{nh}{k}\right) w(n) \ll \delta k^{1/2} X^{1/4} M^{1/4} + k^{3/2} X^{-1/4} M^{3/4} U^{-1}.$$ 

Furthermore, if we select $X = 1/2$, we can forget the first term.

Proof of Theorem 3. Introducing the above weight function $w$ gives

$$\sum_{M \leq n \leq M + \Delta} t(n) e\left(\frac{nh}{k}\right) \ll U M^{\theta + \epsilon} + \sum_{M \leq n \leq M + \Delta} t(n) e\left(\frac{nh}{k}\right) w(n).$$

If we select $U = k^{2/3} M^{1/3 - 2\theta/3}$ and $X = k^{2/3} M^{1/3 + 4\theta/3}$ in Lemma 15, we obtain

$$\sum_{M \leq n \leq M + \Delta} t(n) e\left(\frac{nh}{k}\right) \ll U M^{\theta + \epsilon} + k^{1/2} X^{1/4} M^{1/4} + k^{3/2} X^{-1/4} M^{3/4} U^{-1} \ll k^{2/3} M^{1/3 + \theta/3 + \epsilon},$$

as required.

When $M^{1/(5+6\theta)-1/2+\theta} \ll k < M^{1/4+3\theta/8}$ we argue similarly, except that now the smoothing error is estimated by Theorem 1 to be $\ll U^{1/6-\theta} M^{1/3+\theta+\epsilon}$, and we choose $X = k M U^{-1}$ and $U = k^{3/(2-3\theta)} M^{(1-6\theta)/(4-6\theta)}$. We observe that Theorem 1 is applicable here since a little simplification shows that

$$U \ll \left(M^{1/4+3\theta/8}\right)^{3/(2-3\theta)} M^{(1-6\theta)/(4-6\theta)} = M^{5/8}.$$

This choice of $X$ and $U$ leads to

$$\sum_{n \leq M} t(n) e\left(\frac{nh}{k}\right) \ll U^{1/6-\theta} M^{1/3+\theta+\epsilon} + k^{1/2} X^{1/4} M^{1/4} + k^{3/2} X^{-1/4} M^{3/4} U^{-1} \ll k^{1/(6\theta)/(4-6\theta)} M^{3/(8-12\theta)+\epsilon},$$

as required.

Proof of Lemma 15. We shall feed the sum in question to the Voronoi type summation formula cited in Theorem 6 with the choice $f = w$. The series involving the $K$-Bessel function will be negligible: Pick any $A \in ]2, \infty[$. Then the series involving the $K$-Bessel function can estimated as follows

$$\ll A \frac{1}{k} \int_{M}^{M + \Delta} \sum_{n = 1}^{\infty} \left| t(n) \right| w(x) k^{A} n^{-A/2} x^{-A/2} \, dx \ll A \frac{1}{k} \Delta (k M^{-1/2})^{A} \ll \frac{1}{k} \Delta M^{-A}. $$

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For $A \gg \delta$, this is $\ll \delta$.

In the series involving the $J$-Bessel function, we apply (4) with $K = 2$. The terms involving the error term contribute only

$$\ll \frac{1}{k} \sum_{n=1}^{\infty} |t(n)| \int_{M}^{M+\Delta} w(x) k^{5/2} n^{-5/4} x^{-5/4} \, dx \ll k^{3/2} M^{-5/4}.$$

We shall consider the series involving the $J$-function in two parts according to whether $n \leq X$ or $n > X$. The high-frequency terms $n > X$ are again treated by integrating by parts twice. However, here there will be a slight twist: the bound for the integral

$$\int_{M}^{M+\Delta} w(x) k^{1/2} n^{-1/4} x^{-1/4} (1 + C \pm k n^{-1/2} x^{1/2}) e \left( \pm \frac{2\sqrt{n}x}{k} \right) \, dx$$

will be

$$\ll k^{5/2} n^{-5/4} M^{3/4} (M^{-2} \Delta + U^{-1}) \ll k^{5/2} n^{-5/4} M^{3/4} U^{-1},$$

instead of $\ll k^{5/2} n^{-5/4} M^{3/4} U^{-2} \Delta$. The reason for this is that after having integrated by parts twice, the resulting integral is estimated by absolute values, and most of the terms in the integrands will be supported on $\text{supp } w'$ which is a set of length $\ll U$. The only terms in which the integrand is supported in a larger set are those, which still feature $w(x)$ after differentiation, but here the other factors all give an extra $M^{-1}$ instead of mere $U^{-1}$ upon differentiation.

Substituting the bound from integration by parts back into the series, we see that the contribution from the high-frequency terms is

$$\ll \frac{1}{k} \sum_{n > X} |t(n)| k^{5/2} n^{-5/4} M^{3/4} U^{-1} \ll k^{3/2} X^{-1/4} M^{3/4} U^{-1}.$$

With the low-frequency terms, we estimate the integral in question by the first derivative test to get

$$\int_{M}^{M+\Delta} \ldots \, dx \ll k^{1/2} n^{-1/4} M^{-1/4} \frac{k \sqrt{M}}{\sqrt{n}},$$

and so the contribution from the low-frequency terms is

$$\ll \frac{1}{k} \sum_{n \leq X} |t(n)| k^{1/2} n^{-1/4} M^{-1/4} k M^{1/2} n^{-1/2} \ll k^{1/2} X^{1/4} M^{1/4}.$$

8 Proof of Theorem 4

Let $J \in \mathbb{Z}_+$. In order to be able to apply the Voronoi summation formula, we shall consider the smoothed exponential sum

$$\sum_{M-1 \leq n \leq M} t(n) \, w(n) \, e(\alpha n),$$

(9)
Lemma 16. Let $M \in [1, \infty]$, and let $\alpha, h \in \mathbb{Z}$, $k \in \mathbb{Z}_+$ and $\eta \in \mathbb{R}$ be such that
\[ \alpha = \frac{h}{k} + \eta, \quad k \leq M^{1/4}, \quad \eta \in \mathbb{R}, \quad |\eta| \leq \frac{1}{k M^{1/4}}. \]
Furthermore, write $U = M^{1/2} \eta^{-1/2} (k^2 \eta^2 M)^d$, where $d \in \mathbb{R}_+$. Then, given $\epsilon \in \mathbb{R}_+$, we have
\[ \sum_{M \leq n \leq M + U} t(n) e(na) \ll M^{1/2} (k^2 \eta^2 M)^{\theta/2 - 1/12 + \epsilon}, \]
for fixed $d \ll \epsilon$.

Now, by partial summation and Lemma 15 we have
\[ \sum_{M - 1 \leq n \leq M} t(n) e(n\alpha) w(n) + \sum_{M_1 \leq n \leq M_2} t(n) e(n\alpha) w(n) \ll M^{1/2} (k^2 \eta^2 M)^{\theta/2 - 1/12 + \epsilon}. \] (10)

Proof of Lemma 16. Let us first dispose of the case $U \ll M^{5/8}$. In this case we have, by Theorem 1
\[ \sum_{M \leq n \leq M + U} t(n) e(na) \ll U^{1/6 - \theta} M^{1/3 + \theta + \epsilon} \ll \left(M^{1/2} \eta^{-1/2} (k^2 \eta^2 M)^d\right)^{1/6 - \theta} M^{1/3 + \theta + \epsilon} \ll M^{1/2} (k^2 \eta^2 M)^{\theta/2 - 1/12 + d/6 - d\theta} M^z k^{1/6 - \theta} \eta^{1/12 - \theta/2}. \]
If $k^2 \eta^2 M \gg M^{1/4}$, then certainly
\[ M^z k^{1/6 - \theta} \eta^{1/12 - \theta/2} \ll M^z \ll (k^2 \eta^2 M)^z. \]
If $k^2 \eta^2 M \ll M^{1/4}$, then
\[ k^{1/12 - \theta/2} \eta^{1/12 - \theta/2} \ll \left(M^{-3/8}\right)^{1/12 - \theta/2} = M^{-1/32 + 3\theta/16}, \]
so that
\[ M^z k^{1/12 - \theta/2} \eta^{1/12 - \theta/2} \ll M^z k^{1/12 - \theta/2} M^{-1/32 + 3\theta/16} \]
where $w$ is the weight function $\eta_J$ (see Section 3) which corresponds to the interval $[M_1, M_2]$ with parameter $U \in \mathbb{R}_+$, which is defined as follows: Let $d \in \mathbb{R}_+$ be a small constant and write $U = M^{1/2} \eta^{-1/2} (k^2 \eta^2 M)^d$. Let $M_1 = M - JU$, $M_2 = M + \Delta$, and $M_3 = M + \Delta + JU$. Also, we define $N_i = k^2 \eta^2 M_i$ for $i = -1, 1, 2$ and $N = k^2 \eta^2 M$.
Thus, in either case the sums of length \( U \ll M^{5/8} \) are sufficiently small. The same argument also takes care of all the later terms which have the shape \( U^{1/6-\delta} M^{1/3+\delta/2} \).

Let us next focus on the case \( U \gg M^{5/8} \). Let us first assume that \( h/k \) is a Farey fraction of order \( U^{1/2-\delta} \) for some small \( \delta \in \mathbb{R}_+ \), i.e., that \( |\eta| \ll k^{-1} U^{-1/2-\delta} \). Then the second error term from Proposition 14 contributes

\[
\ll k^{-1/2} U M^{-1/4} (k^2 \eta^2 M)^{\varepsilon-\delta/2-1/12},
\]

provided that \( \eta \ll M U^{-2} \). But this condition holds since it reduces to

\[
\eta \ll \frac{1}{M \eta^{-1} (k^2 \eta^2 M)^d},
\]

and we have

\[
\eta^2 \ll \frac{1}{k^2 M^{1/2}} \ll (k^2 \eta^2 M)^{-d}
\]

for sufficiently small \( d \).

Let us observe next that if \( U \gg M^{5/6} \), then \( M^{5/4} \ll U^{3/2} \). Further,

\[
k M (k^2 \eta^2 M)^{2d} \ll U^{3/2},
\]

so that, by the definition of \( U \),

\[
\eta = M U^{-2} (k^2 \eta^2 M)^{2d} \ll \frac{1}{k U^{1/2}}.
\]

Thus, if \( U \gg M^{5/6} \), then \( h/k \) is indeed a Farey fraction of order \( \Delta^{1/2-\varepsilon} \) and everything is fine.

The remaining length range is \( M^{5/8} \ll U \ll M^{5/6} \), and the only problematic case is the one in which \( \eta \gg k^{-1} U^{\delta-1/2} \). In this case Proposition 14 involves a Farey approximation different from \( h/k \). The right Farey approximation will be

\[
\alpha = \frac{h_1}{k_1} + \eta_1,
\]

where \( h_1 \in \mathbb{Z} \), \( k_1 \in \mathbb{Z}_+ \), \( \eta_1 \in \mathbb{R}_+ \), \( (h_1, k_1) = 1 \), \( k \ll U^{1/2-\delta} \) for some small \( \delta \in \mathbb{R}_+ \), and \( |\eta| \ll k^{-1} U^{\delta-1/2} \).

Let us observe that if we had \( k_1 \leq M^{1/4}/2 \), then we would have

\[
\frac{1}{k k_1} \leq \left| \frac{h_1}{k_1} - \frac{h}{k} \right| \leq |\eta| + |\eta_1| \leq \frac{1}{k M^{1/4}} + \frac{1}{k_1 U^{1/2-\delta}},
\]

so that

\[
1 \leq k_1 M^{-1/4} + k U^{\delta-1/2} \leq \frac{1}{2} + M^{1/4+\varepsilon-5/16} = \frac{1}{2} + M^{\varepsilon-1/16} = \frac{1}{2} + o(1),
\]

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which is impossible. Thus, we must have $k_1 \gg M^{1/4}$.

Again, the term $U^{1/6-\theta} M^{1/3+\theta+\varepsilon}$ does not pose any problems. Thus, it is enough to consider the other term $k_1^{-1/2} U M^{-1/4} \left( k_1^2 \eta_1^2 M \right)^{\theta}$ which only arises when

$$k_1^2 \eta_1 M U^{-1+\delta} \ll 1 \quad \text{and} \quad k_1^2 \eta_1^2 M \gg 1.$$  

Let us check that the condition $\eta_1 \ll M U^{-2}$ required in this case holds. Namely, since

$$k_2^1 \eta_1 M U^{-1+\delta} \ll 1,$$

we have

$$\eta_1 \ll \frac{U^{1-\delta}}{k_1^2 M}.$$  

It is therefore enough that

$$\frac{U^{1-\delta}}{k_1^2 M} \ll \frac{M}{U^2},$$  

i.e. that $U^{3-\delta} \ll k_1^2 M^2$. Since $k_1 \gg M^{1/4}$, it is enough that $U \ll M^{5/6+\delta/3}$, which is indeed true.

Finally, we need to check that the term $k_1^{-1/2} U M^{-1/4} \left( k_1^2 \eta_1^2 M \right)^{\theta}$ is small enough. We have

$$k_1^{-1/2} U M^{-1/4} \left( k_1^2 \eta_1^2 M \right)^{\theta} \ll M^{-1/8} U M^{-1/4} U^{-\theta+\delta \theta} M^{\theta} \ll U^{1-\theta} M^{-3/8+\varepsilon}.$$  

We have

$$U^{1-\theta} M^{3-3/8} \ll U^{1/6-\theta} M^{1/3+\theta}$$  

if and only if

$$U^{5/6} \ll M^{17/24},$$  

or equivalently, $U \ll M^{17/20}$. But this holds, since $5/6 \ll 17/20$, and we are done.

### 8.2 Voronoi summation formula and saddle-points: the main terms

Now we consider the smoothed sum \ref{eq:smoothsum}. Applying Voronoi-summation formula and replacing Bessel-functions by their asymptotic expressions we see that the following terms needs to be considered:

$$A_{\pm} = \frac{i}{\sqrt{2}} \sum_{n=1}^{\infty} t(n) e\left( -\frac{n \eta_1}{k} \right)$$

$$\cdot \int_{M_{-1}}^{M_{2}} k_1^{-1/2} n^{-1/4} x^{-1/4} e\left( \pm \frac{1}{8} \pm \frac{2 \sqrt{n x}}{k} \right) w(x) e(x \eta) \, dx,$$
\[ B_{\pm} = \frac{\pi i}{k \sinh \pi k} \sum_{n=1}^{\infty} t(n) e\left(-\frac{n\eta}{k}\right) \int_{M-1}^{M} k^{3/2} n^{3/4} x^{-3/4} e\left(\pm \frac{2\sqrt{n}x}{k}\right) w(x) e(x\eta) \, dx, \]

\[ C = k^{-1} \sum_{n=1}^{\infty} n^{\vartheta+\varepsilon} \int_{M-1}^{M} k^{5/2} (nx)^{\vartheta/4} \, dx, \]

and

\[ D = k^{-1} \sum_{n=1}^{\infty} t(-n) e\left(-\frac{n\eta}{k}\right) \int_{M-1}^{M} k^{E} n^{-E/2} x^{-E/2} \, dx, \]

where \( E > 0 \) is an arbitrary constant.

Each of the terms \( A_{+} \) and \( B_{\pm} \) contribute \( \ll 1 \) similarly as in [4], except we estimate

\[ k^{1/2} \sum_{1 \leq n \leq cN} t(n) n^{-3/4} \int_{M-1}^{M} w(x) x^{-3/4} e\left(\eta x \pm \frac{\sqrt{n}x}{k}\right) \, dx \ll k (k^{2} \eta^{2} M)^{\vartheta+\varepsilon} \ll M^{1/2} (k^{2} \eta^{2} M)^{e} \]

instead of using Lemma 2.5 of [4]. Also, \( C \ll 1 \) as in [4] since \( -5/4 + \vartheta \leq -5/4 + 7/64 < -1 \). The term can be estimated easily:

\[ D \ll k^{E-1} \sum_{n=1}^{\infty} n^{-E/2+\vartheta+\varepsilon} M^{1-E/2} \ll M^{E/4+3/4} \ll M^{1/8} \]

for sufficiently large \( E \). The term \( A_{-} \) is handled differently as in the case of a holomorphic cusp form. The terms with \( n \gg N \) contribute \( \ll 1 \) by Lemma 9.

Rest of the term \( A_{-} \) is treated using the first saddle point lemma, Theorem 7.

The arguments in [4] apply here nicely. For \( 1 \leq n \ll N \) we get

\[
\int_{M-1}^{M} e\left(x\eta - \frac{2\sqrt{n}x}{k}\right) w(x)x^{-1/4} \, dx = \xi(n) \cdot \frac{\sqrt{2}n^{1/4}}{\sqrt{k\eta}} e\left(-\frac{n}{k^{2}\eta} + \frac{1}{8}\right) + O\left(\frac{k^{3/2}}{n^{3/4}} + \delta(n) \frac{M^{1/2}k}{\sqrt{n}}\right) + O\left(M^{-1/4}U^{-J} \sum_{j=0}^{J} \left(\frac{\eta - \frac{\sqrt{n}}{\sqrt{M-jUk} + \frac{n^{1/4}}{\sqrt{kM^{3/4}}}}}{\sqrt{M-jUk} + \frac{n^{1/4}}{\sqrt{kM^{3/4}}}}\right)^{-j-1}\right) + O\left(M^{-1/4}U^{-J} \sum_{j=0}^{J} \left(\frac{\eta - \frac{\sqrt{n}}{\sqrt{M+\Delta+jUk} + \frac{n^{1/4}}{\sqrt{kM^{3/4}}}}}{\sqrt{M+\Delta+jUk} + \frac{n^{1/4}}{\sqrt{kM^{3/4}}}}\right)^{-j-1}\right),
\]

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where
\[
\begin{align*}
\xi(n) = 0 \text{ and } \delta(n) = 0 & \quad \text{if } n \leq N - 1 \text{ or } n \geq N_2 \\
\xi(n) = 1 \text{ and } \delta(n) = 0 & \quad \text{if } N \leq n \leq N_1 \\
\xi(n) \ll 1, \xi'(n) \ll (k^2 \eta^2 U)^{-1} \text{ and } \delta(n) = 1 & \quad \text{otherwise}.
\end{align*}
\]

The main term on the right-hand side produces total contribution
\[
\frac{1}{k \eta} \sum_{n=N}^{N_1} t(n)e \left( -\frac{n\eta}{k} - \frac{n}{k^2 \eta} \right) + \frac{1}{k \eta} \sum_{n=N_{-1}}^{N} t(n)\xi(n)e \left( -\frac{n\eta}{k} - \frac{n}{k^2 \eta} \right)
+ \frac{1}{k \eta} \sum_{n=N_1}^{N_2} t(n)\xi(n)e \left( -\frac{n\eta}{k} - \frac{n}{k^2 \eta} \right).
\]

The first term is exactly what appears in the statement of the theorem. Let us estimate the contribution of other main terms.

### 8.3 The error terms from the saddle point theorem

The first error term contributes
\[
k^{-1/2} \sum_{1 \leq n \leq N_2} \frac{t(n)}{n^{1/4}} M^{-1/4} U^{-J} \left( \left| \eta - \frac{n^{1/4}}{\sqrt{Tk}} \right| + \frac{n^{1/4}}{\sqrt{kM^{3/4}}} \right) \ll M(k^2 \eta^2 M)^{d+\epsilon}.
\]

The other two error terms are estimated by the following lemma.

**Lemma 17.** Let \( c \) be any given constant and \( \epsilon > 0 \) be arbitrary. Let \( T \) be any number of type \( M \pm jU \), where \( 0 \leq j \leq J \). Then
\[
k^{-1/2} \sum_{1 \leq n \leq cN} \frac{t(n)}{n^{1/4}} M^{-1/4} U^{-J} \left( \left| \eta - \frac{n^{1/4}}{\sqrt{Tk}} \right| + \frac{n^{1/4}}{\sqrt{kM^{3/4}}} \right) \ll M(k^2 \eta^2 M)^{d-\epsilon}.
\]

**Proof.** Clearly the left-hand side is \( \ll S_1 + S_2 + S_3 \), where
\[
S_1 = k^{-1/2} M^{-1/4} U^{-J} \sum_{|n-k^2 \eta^2 T| \leq \sqrt{N}} n^{\theta + \epsilon - 1/4} \left( \frac{n^{1/4}}{\sqrt{kM^{3/4}}} \right)^{-J-1},
\]
\[
S_2 = k^{-1/2} M^{-1/4} U^{-J} \sum_{1 \leq n \leq k^2 \eta^2 T - \sqrt{N}} n^{\theta + \epsilon - 1/4} \left| \eta - \frac{n}{\sqrt{Tk}} \right|^{-J-1},
\]
and
\[
S_3 = k^{-1/2} M^{-1/4} U^{-J} \sum_{k^2 \eta^2 T + \sqrt{N} \leq n \leq cN} n^{\theta + \epsilon - 1/4} \left| \eta - \frac{n}{\sqrt{Tk}} \right|^{-J-1}.
\]

We compute the claimed upper bound for each of them. Observe that
\[
S_1 = M^{1/2}(k^2 \eta^2 M)^{-J\epsilon} k^{J/2} M^{J/4} \eta^{1/2} \sum_{|n-k^2 \eta^2 T| \leq \sqrt{N}} n^{\theta + \epsilon - 1/2 - J/4}
\]

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If \( k \) unless \( k \in \) the summation range we have
\[ \text{as desired.} \]

Now, by a straightforward calculation, we establish

\[ 8.4 \text{ Removing the weight function } \xi \]

By partial summation it is enough to deal with the case \( \xi(n) \equiv 1 \). Observe that
\[ N - N_1 = N_2 - N_1 = k^2 \eta^2 U \]
and
\[ k^2 \eta^2 U = (k^2 \eta^2 M)^{d+1/2} k \eta^{1/2} \ll (k^2 \eta^2 M)^{5/8} = N_{5/8} \]
for sufficiently small \( d \).
In particular, with the exponent \( \vartheta \leq 1 \)

Choosing Lemma 18.

9.1 Logarithm removal near rational points

point or the sum in question has become shorter than some given constant.

We shall prove theorem 5 first near rational points, and then iterate approximate functional equation in the remaining cases until either we end up near a rational point or the sum in question has become shorter than some given constant.

Proof of Theorem 5

At this point we have proved that

\[
\sum_{M-1 \leq n \leq M_2} t(n) w(n) e(\alpha n) = \frac{1}{k \eta} \sum_{N \leq n \leq N_1} t(n) e(-\beta n) + O(M^{1/2}(k^2 \eta^2 M)^{\varepsilon - Jd}) + O(M^{1/2}(k^2 \eta^2 M)^{\varepsilon + d/6 - 1/12 + \theta/2}).
\]

Furthermore, using (10), this tells that

\[
\sum_{M \leq n \leq M + \Delta} t(n) e(\alpha n) = \frac{1}{k \eta} \sum_{N \leq n \leq N_1} t(n) e(-\beta n) + O(M^{1/2}(k^2 \eta^2 M)^{\varepsilon - Jd}) + O \left( M^{1/2}(k^2 \eta^2 M)^{\varepsilon + d/6 - 1/12 + \theta/2} \right) + O \left( M^{1/2}(k^2 \eta^2 M)^{\theta/2 - 1/12 + \varepsilon} \right).
\]

Choosing \( J = 1/(12d) \), and letting \( d \in \mathbb{R}_+ \) to be arbitrarily small finishes the proof.

9 Proof of Theorem 5

We shall prove theorem first near rational points, and then iterate approximate functional equation in the remaining cases until either we end up near a rational point or the sum in question has become shorter than some given constant.

9.1 Logarithm removal near rational points

Lemma 18. Let \( M \in [1, \infty] \), let \( \alpha \in \mathbb{R} \), and \( h \in \mathbb{Z} \) and \( k \in \mathbb{Z} \) be coprime with \( 1 \leq k \leq M^{1/4} \), and \( \alpha = h/k + \eta \) with \( |\eta| \leq k^{-1} M^{-1/4} \). If \( k^2 \eta^2 M < 1/2 \), then

\[
\sum_{M \leq n \leq 2M} t(n) e(n\alpha) \ll k^{(1-6\theta)/(4-6\theta)} M^{3/(8-12\theta)+\varepsilon}.
\]

In particular, with the exponent \( \theta = 7/64 \) the upper bound is \( \ll M^{203/428+\varepsilon} \ll M^{1/2} \)
The following Voronoi type identity for Maass forms can be found in Section 12 of Meurman’s paper [28].

**Theorem 19.** Let \( x \in [1, \infty], \) and let \( h \) and \( k \) be coprime integers with \( k \leq 1 \). Then

\[
\sum_{n \leq x} t(n) e\left(\frac{nh}{k}\right) = \frac{2\pi}{k \sinh \pi \kappa} \sum_{n=1}^{\infty} t(n) e\left(-\frac{nh}{k}\right) \int_{0}^{x} \Re \left( i J_{2\kappa} \left( \frac{4\pi \sqrt{n\nu}}{k} \right) \right) \, dv
\]

\[
+ \frac{4 \cosh \pi \kappa}{k} \sum_{n=1}^{\infty} t(-n) e\left(-\frac{nh}{k}\right) \int_{0}^{x} K_{2\kappa} \left( \frac{4\pi \sqrt{n\nu}}{k} \right) \, dv,
\]

and where the series are boundedly convergent for \( x \) restricted in any bounded subinterval of \([1, \infty]\).

The integrals involving the \( J \)-Bessel function will have an asymptotic expansion reminiscent of those for the \( J \)-Bessel function itself. The following asymptotics for the \( J \)-Bessel function integral are obtained from Section 6 of [28], and the asymptotics for the integral involving the \( K \)-Bessel function is easily obtained from the asymptotic properties of \( K_{2\kappa} \).

**Lemma 20.** Let \( n \in \mathbb{Z}_{+}, x \in [1, \infty], \) and \( k \in \mathbb{Z}_{+} \). If \( nx \gg k^{2} \), then we have an asymptotic expansion

\[
\int_{0}^{x} \Re \left( i J_{2\kappa} \left( \frac{4\pi \sqrt{n\nu}}{k} \right) \right) \, dv
\]

\[
= k^{3/2} n^{-3/4} x^{1/4} \sum_{\pm} A_{1, \pm} e\left( \pm \frac{2\sqrt{n\nu}}{k} \right) + A_{2} k^{2} n^{-1}
\]

\[
+ k^{5/2} n^{-5/4} x^{-1/4} \sum_{\pm} A_{3, \pm} e\left( \pm \frac{2\sqrt{n\nu}}{k} \right) + O_{\kappa}(k^{7/2} n^{-7/4} x^{-3/4}),
\]

where \( A_{1, +}, A_{1, -}, A_{2}, A_{3, +} \) and \( A_{3, -} \) are some constants only depending on \( \kappa \), and the implicit constant in the lower bound \( nx \gg k^{2} \). Similarly, we have the asymptotic expansion

\[
\int_{0}^{x} K_{2\kappa} \left( \frac{4\pi \sqrt{n\nu}}{k} \right) \, dv = B_{2} k^{2} n^{-1} + O_{\kappa, C}(k^{2+C} n^{-1-C/2} x^{-C/2}),
\]

where \( C \in \mathbb{R}_{+} \) is arbitrary and \( B_{2} \) is a constant only depending on \( \kappa \) and the implicit constant in \( nx \gg k^{2} \).

**Proof of Lemma 18.** We begin by integrating by parts:

\[
\sum_{M \leq n \leq 2M} t(n) e(\eta x) \sum_{n \leq x} t(n) e\left(\frac{nh}{k}\right) \bigg|_{x=2M}^{x=2M}
\]

\[
- 2\pi i \eta \int_{M}^{2M} e(\eta x) \sum_{n \leq x} t(n) e\left(\frac{nh}{k}\right) \, dx.
\]

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As $k \leq M^{1/4}$, Theorem [3] immediately tells us that the substitution terms are $\ll k^{(1-6\delta)/(4-\delta)} M^{3/(8-12\delta)+\varepsilon}$. We will prove that the term involving the integral is actually $\ll k^{1/2} M^{1/4}$. The full Voronoi identity for Maass forms tells us that

$$n \int_0^{2M} e(\eta x) \sum_{n \leq x} t(n) e\left(\frac{nh}{k}\right) \, dx$$

$$= \frac{2\pi \eta}{k \sin \pi \kappa} \sum_{n=1}^{\infty} t(n) e\left(-\frac{nh}{k}\right) \int_0^{2M} e(\eta x) \int_0^x \Re\left(i J_{2\kappa}\left(\frac{4\pi \sqrt{nu}}{k}\right)\right) \, dv \, dx$$

$$+ \frac{4\eta \cosh \pi \kappa}{k} \sum_{n=1}^{\infty} t(-n) e\left(\frac{nh}{k}\right) \int_0^{2M} e(\eta x) \int_0^x K_{2\kappa}\left(\frac{4\pi \sqrt{nu}}{k}\right) \, dv \, dx.$$

We emphasize that termwise integration of the series is allowed since the series converge boundedly. We note that the integral $\int_0^x \Re(i \ldots) \, dv$ by the asymptotics given by Lemma [20]. We start with the contribution from either of the first main terms. Since $k^2 \eta^2 M < 1/2$, we have

$$\frac{d}{dx} \left(\pm \frac{2\sqrt{nx}}{k} + \eta\right) = \pm \frac{\sqrt{n}}{k \sqrt{x}} + \eta \approx \frac{\sqrt{n}}{k \sqrt{x}} n^{1/2} k^{-1} M^{-1/2}.$$

Thus, using the first derivative test, the contribution from these terms is

$$\ll \frac{\eta}{k} \sum_{n=1}^{\infty} \left|t(n)\right| \int_0^{2M} \frac{\eta^{1/4} x^{1/4}}{k^{1/2}} e\left(\frac{2\sqrt{nx}}{k}\right) \, dx$$

$$\ll \frac{1}{k^2 M^{1/2}} \sum_{n=1}^{\infty} \left|t(n)\right| \frac{k^2}{n} \frac{n^{1/4} M^{1/2}}{k^{1/2}} \frac{k M^{1/2}}{n^{1/2}} \ll k^{1/2} M^{1/4}.$$

The contribution from the constant term of the asymptotics is

$$\ll \frac{\eta}{k} \sum_{n=1}^{\infty} t(n) e\left(-\frac{nh}{k}\right) \frac{k^2}{n} \int_0^{2M} e(\eta t) \, dt$$

$$\ll k \sum_{n=1}^{\infty} \frac{t(n)}{n} e\left(-\frac{nh}{k}\right) \ll k \ll M^{1/4}.$$

The contribution from the third main terms is clearly smaller than that from the first main terms since $k n^{-1/2} x^{-1/4} \ll 1$. Finally, the contribution from the $O$-term of the asymptotics contributes

$$\ll \frac{\eta}{k} \sum_{n=1}^{\infty} \left|t(n)\right| \frac{k^{7/2} n^{-7/4}}{x^{3/4}} \int_0^{2M} x^{-3/4} \, dx$$

$$\ll k^{-1} M^{-1/2} k^{5/2} M^{1/4} = k^{3/2} M^{-1/4} \ll k^{1/2},$$

and we are done.
9.2 Away from rational points; applying the approximate functional equation

When $k^2 \eta^2 M \gg 1$, the logarithm removal is implemented quite easily using the approximate functional equation. The result will be as follows:

Lemma 21. Let $M \in [1, \infty[$, let $\alpha \in \mathbb{R}$, let $h$ and $k$ be coprime integers with $1 \leq k \leq M^{1/4}$, and let $\alpha = h/k + \eta$ with $|\eta| \leq k^{-1} M^{-1/4}$. If $k^2 \eta^2 M \gg 1$, then

$$\sum_{M \leq n \leq 2M} t(n) e(n\alpha) \ll M^{1/2}.$$ 

We start by applying the approximate functional equation, obtaining:

$$\frac{1}{M^{1/2}} \sum_{M \leq n \leq 2M} t(n) e(n\alpha) = \frac{1}{(k^2 \eta^2 M)^{1/2}} \sum_{k^2 \eta^2 M \leq n \leq 2k^2 \eta^2 M} t(n) e(n\beta) + O((k^2 \eta^2 M)^{9/2 - 1/12 + \varepsilon}),$$

where $\beta = -h/k - (k^2 \eta)^{-1}$. Write for $\beta$ a rational approximation $\beta = h_1/k_1 + \eta_1$ with $h_1$ and $k_1$ coprime and $1 \leq k_1 \leq (k^2 \eta^2 M)^{1/4}$ and with remainder satisfying $|\eta_1| \leq k_1^{-1} (k^2 \eta^2 M)^{-1/4}$. If $k_1^2 \eta_1^2 (k^2 \eta^2 M) < 1/2$, then the first term on the right-hand side is $\ll 1$ by Lemma 18, and the error term is clearly $\ll 1$, and we are done.

If instead $k_1^2 \eta_1^2 (k^2 \eta^2 M) \gg 1$, then we apply the approximate functional equation again to the right-hand side, and iterate the above argument as many times as necessary. Since the length of the new sum from the approximate functional equation is at most the square root of the length of the previous sum, the exponential sum term will eventually be covered by Lemma 18 or become shorter than some constant length. In either case, the sum will ultimately be $\ll 1$, and the error terms will form a nice geometric progression which sums up to $\ll 1$.

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