FROM MASSIVE GRAVITY TO DARK MATTER DENSITY II

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Abstract

As previously observed the massless limit of massive gravity leads to a modification of general relativity. Here we study spherically symmetric solutions of the modified field equations which contain normal matter together with a dark energy density. If the dark density profile is assumed to be known, the whole problem is reduced to a linear first order differential equation which can be solved by quadratures.

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1 Introduction

In a previous paper with the same title [1] we have made the thousand and first proposal to explain dark matter density. In contrast to the other thousand proposals we came to this subject by accident, that means not by looking for an explanation of dark matter. As the title indicates we have studied massive quantum gravity which is gravity with a massive graviton. To have a massive spin-2 gauge theory the so-called vector graviton field \( v_\lambda \) is indispensable. The crucial observation was that in the limit \( m \to 0 \) of vanishing graviton mass this field \( v_\lambda \) does not decouple from the symmetric tensor field \( h^{\mu \nu} \) which is equivalent to Einstein’s \( g^{\mu \nu} \). In the classical limit \( v_\lambda \) acts as four scalar fields \( v_n \). As a consequence the massless limit of massive gravity is different from general relativity. There remains the additional coupling to the four (now massless) scalar fields \( v_n \). In the resulting modified Einstein’s equations this gives additional terms in the energy-momentum tensor. In the 00-component the new contribution looks as if it comes from a dark matter density, but there are also peculiar modifications of the pressure components in the \( jj \)-equations.

The paper is organized as follows. In the next section we review the modified general relativistic equations, for their derivation we refer to [1]. We then consider static spherically symmetric solutions including normal matter which is described by a mass density \( q(r) \) and an isotropic pressure \( p(r) \). By expanding the solution for large distance \( r \) we get already an important result: the dark density profile must decrease as \( 1/r^4 \) as already found in [1] for the solution without matter. This tail contradicts the widely discussed Navarro - Frenk - White (NFW) profile [2] [3]

\[
\rho(r) = \frac{\rho_s}{(r/r_s)(1 + (r/r_s))^2}
\]

which has a \( 1/r^3 \) tail. However, this profile must be modified for large \( r \) anyway to get a finite total mass for the dark halo. A second conclusion can be drawn from the expansion around \( r = 0 \). In general relativity there exists the inner Schwarzschild solution which is finite in all quantities including mass density and pressure. This is of course also a solution of our modified theory with vanishing dark density. We can now test whether there is a corresponding finite solution with a dark density different from zero. The answer is no. This may explain the fact that a single star like the sun which is described by this finite solution does not have a dark halo. In order to describe dark halos we have to study solutions with some singularity for
small $r$. In [1] we have investigated the vacuum solutions, here we start the analysis of solutions with normal matter.

The modified general relativity has the following nice property. To specify a solution one quantity must be given which usually is the mass density $q(r)$ of normal matter. In this case one has to solve non-linear differential equations. Alternatively, if the dark profile $\rho(r)$ is considered to be known, the remaining differential equation is linear and first order. It can be solved by quadratures. This is a machine which produces solutions at low cost. In this way we construct a simple singular solution in section 3. This solution with a dark halo has a singularity at some finite radius $r_s$. There is no horizon at $r > r_s$ so that the singularity is a naked one.

2 Massless limit of massive gravity

The basic classical field equations which follow from massive gravity in the limit of vanishing graviton mass [1] are the modified Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{8\pi G}{c^4} t_{\mu\nu} = \frac{16\pi G}{c^3} \left( -\partial_\mu v_n \partial_\nu v^n + \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha v_n \partial_\beta v^n \right).$$

(2.1)

together with the Laplace-Beltrami wave equation for the four scalar fields $v^n, n = 0, 1, 2, 3$

$$\partial_\alpha (\sqrt{-g} g^{\alpha\beta} \partial_\beta v^n) = 0.$$  

(2.2)

As discussed in [1] the Latin index $n$ of $v^n$ is raised and lowered with the Minkowski tensor diag$(1, -1, -1, -1)$, in contrast to the Greek indices which are changed with the metric tensor $g^{\mu\nu}$. Since upon $v^n$ we act by partial derivatives not by covariant ones, this means that the $v_n$ are four scalar fields, indeed. The new contribution on the right-hand side of (2.1) is the possible origin of the dark density.

We want to study static spherically symmetric solutions of these field equations. As in [1] we write the metric as

$$ds^2 = e^\nu c^2 dt^2 - e^\lambda dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$$

(2.3)

where $\nu$ and $\lambda$ are functions of $r$ only. We take the coordinates $x^0 = ct$, $x^1 = r$, $x^2 = \vartheta$, $x^3 = \varphi$ such that

$$g_{00} = e^\nu, \quad g_{11} = -e^\lambda,$$

$$g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \vartheta.$$  

(2.4)
and zero otherwise. The components with upper indices are the inverse of this. The determinant comes out to be

\[ g = \det g_{\mu\nu} = -e^{\nu + \lambda} r^4 \sin^2 \vartheta. \] (2.5)

The energy momentum tensor in (2.1) is assumed in the simple form

\[ t^\beta_\alpha = \text{diag}(qc^2, -p, -p, -p) \] (2.6)

where \( q(r) \) is the ordinary mass density and \( p(r) \) is an isotropic pressure. Now the following modified radial Einstein equations must be solved

\[ e^{-\lambda} \left( \frac{\nu'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{8\pi G}{c^3} (qc + w_0(r)) \] (2.7)

\[ e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{8\pi G}{c^3} \left( \frac{p(r)}{c} - w_0(r) \right) \] (2.8)

\[ e^{-\lambda} \left( \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu' \lambda'}{2r} + \frac{\nu'}{2r^2} \right) = \frac{8\pi G}{c^3} \left( \frac{p(r)}{c} + w_0(r) \right), \] (2.9)

where \( w_0 \) is the contribution of the four scalar fields \( v^n \). As discussed in [1] in the spherically symmetric case this is of the form

\[ w_0 = \frac{\rho_0}{r^4} e^{-\nu(r)}. \] (2.10)

First by suitable combination we simplify the equations. Adding (2.7) to (2.8) we get

\[ \frac{e^{-\lambda}}{r} (\lambda' + \nu') = g(qc + \frac{p}{c}) \] (2.11)

where

\[ g = \frac{8\pi G}{c^3}. \] (2.12)

Eliminating \( p \) from (2.8) and (2.9) we obtain

\[ \nu'' = 2e^{\lambda} \left( -\frac{1}{r^2} + 2g \frac{w_0}{c} \right) + \frac{\nu' \lambda'}{2} - \frac{\nu'^2}{2} + \frac{\lambda' + \nu'}{r} + \frac{2}{r^2}. \] (2.13)

Next we differentiate (2.8) with respect to \( r \) and use (2.13):

\[ g \left( \frac{p'}{c^2} - \frac{w_0'}{c} \right) = -e^{-\lambda} \frac{\nu'}{2r} (\lambda' + \nu') + 4 \frac{g}{cr} w_0. \] (2.14)
Substituting (2.11) in here we finally arrive at

\[
\frac{p'}{c^2} = -\frac{\nu'}{2} \left(q + \frac{p}{c^2} + \frac{2}{c}w_0 \right)
\]  

(2.15)

where (2.10) has been used. This differential equation for the pressure will be used instead of the second order equation (2.9) in the following.

To study the solution for large \(r\) where it should approach flat space, we set up an expansion in powers of \(1/r\):

\[
e^{-\lambda} = 1 - \frac{a_1}{r} + \frac{a_2}{r^2} + \ldots
\]

(2.16)

\[
\nu = \frac{b_1}{r} + \frac{b_2}{r^2} + \ldots
\]

(2.17)

\[
q = \frac{q_4}{r^4} + \frac{q_5}{r^5} + \ldots
\]

(2.18)

\[
p = \frac{p_5}{r^5} + \frac{p_6}{r^6} + \ldots
\]

(2.19)

We find

\[
a_1 = -b_1
\]

(2.20)

\[
a_2 = g(q_4 + \rho_1), \quad \rho_1 = \frac{\rho_0}{c}
\]

(2.21)

\[
q_4 = \frac{1}{g}(4b_2 + b_1^2) - 2\rho_1
\]

(2.22)

\[
\frac{p_5}{c^2} = -\frac{b_1}{10}(q_4 + 2\rho_1) = -\frac{b_1}{10g}(4b_2 + b_1^2).
\]

(2.23)

If the dark density \(w_0(r)\) is given (2.10), i.e. \(\rho_0\) and \(\nu(r)\) are known, we have a unique solution. The equation of state of the normal matter is then determined.

In ordinary general relativity there exists a finite inner solution which can be expanded around \(r = 0\) in the form

\[
e^{-\lambda} = 1 + a_1r + a_2r^2 + a_3r^3 + \ldots
\]

\[
\nu = b_0 + b_1 + b_2r^2 + b_3r^3 + \ldots
\]

(2.24)

\[
qc = q_0 + q_1r + q_2r^2 + \ldots
\]

\[
\frac{p}{c} = p_0 + p_1r + p_2r^2 + \ldots
\]

.
Substituting this into (2.7-9) with $w_0 = 0$ we find

$$a_1 = 0, \quad a_2 = -\frac{g}{3}q_0, \quad a_3 = -\frac{g}{4}q_1 \quad (2.25)$$

$$b_1 = 0, \quad b_2 = \frac{g}{2}(p_0 + \frac{q_0}{3}), \quad b_3 = \frac{g}{12}q_1 \quad (2.26)$$

$$p_1 = 0, \quad p_2 = -\frac{g}{4}(p_0 + \frac{q_0}{3})(p_0 + q_0) \quad (2.27)$$

$$p_3 = -\frac{g}{6}p_0 + \frac{7}{12}q_0q_1.$$

Now we can check whether this finite solution has a counterpart with non-vanishing $w_0$. Expanding (2.10)

$$w_0(r) = \frac{\rho_0}{r^4}e^{-\nu} = \rho_0 e^{-b_0}\left(\frac{1}{r^4} - \frac{b_2}{r^2} + \ldots\right),$$

we easily see from (2.7) that $\rho_0$ must vanish. A solution finite at $r = 0$ cannot have a dark halo. Therefore we have to look for singular solutions.

3 A singular solution

To construct a singular solution with non-vanishing dark density $w_0(r)$ we proceed as follows. We eliminate $p(r)$ in the two equations (2.8) (2.9):

$$-\lambda' e^{-\lambda}\left(\frac{\nu'}{4} + \frac{1}{2r}\right) = e^{-\lambda}\left(-\frac{\nu''}{2} - \frac{r'}{4} + \frac{r'}{2r} + \frac{1}{r^2} - \frac{1}{r^2} + 2gw_0\right).$$

Introducing

$$y = e^{-\lambda}$$

we have obtained a linear first order differential equation for $y(r)$:

$$f_1(r)y'(r) = f_2(r)y + f_3(r) \quad (3.3)$$

where

$$f_1 = \frac{\nu'}{4} + \frac{1}{2r} \quad (3.4)$$

$$f_2 = -\frac{\nu''}{2} - \frac{\nu^2}{4} + \frac{\nu'}{2r} + \frac{1}{r^2} \quad (3.5)$$
\[ f_3 = -\frac{1}{r^2} + 2gw_0. \]  
\( (3.6) \)

Considering the dark density \( w_0(r) \) (2.10) as given we can compute
\[ \nu(r) = \log \rho_0 - \log w_0 - 4\log r. \]  
\( (3.7) \)

This enables us to express the coefficients \( f_1, f_2, f_3 \) in terms of \( w_0 \):
\[ f_1 = -\frac{w'_0}{4w_0} - \frac{1}{2r} \]  
\( (3.8) \)
\[ f_2 = \frac{w''_0}{2w_0} - \frac{3w'^2_0}{4w^2_0} - \frac{5}{2} \frac{w'_0}{rw_0} - \frac{7}{r^2} \]  
\( (3.9) \)
\[ f_3 = -\frac{1}{r^2} + 2gw_0. \]  
\( (3.10) \)

The remaining equation (2.7) then determines the mass density \( q(r) \) of the normal matter.

The linear equation (3.3) can be simply solved by quadratures, so we have full control of the solution. The general solution \( y(r) \) is the sum of a particular solution \( y_1(r) \) of the inhomogeneous equation (3.3) plus a solution \( y_0(r) \) of the homogeneous equation. We will soon realize that the solution is uniquely fixed by requiring that it approaches flat space for \( r \to \infty \). To see this we set up an expansion of the form
\[ y = 1 - \frac{a_1}{r} + \frac{a_2}{r^2} + \ldots \]  
\( (3.11) \)

In agreement with the expansion (2.16-19) above the dark density (2.10) must start as follows
\[ w_0 = \rho_0 r^4 + \frac{d_1}{r^5} + \frac{d_2}{r^6} + \ldots \]  
\( (3.12) \)

A \( 1/r^3 \) tail is in contradiction with the flat space asymptotic \( y \to 1 \) for \( r \to \infty \). From (3.12) and (3.8) we find
\[ f_1 = \frac{1}{2r} + \frac{d_1}{4\rho_0} \frac{1}{r^2} + \left( \frac{d_2}{2\rho_0} - \frac{d_1^2}{4\rho^2_0} \right) \frac{1}{r^3} + \ldots \]  
\( (3.13) \)
and similarly
\[ f_2 = \frac{1}{r^2} + \frac{3d_1}{2\rho_0} \frac{1}{r^3} + \left( \frac{4d_2}{\rho_0} - \frac{9d_1^2}{4\rho^2_0} \right) \frac{1}{r^4} + \ldots \]  
\( (3.14) \)
\begin{equation}
  f_3 = -\frac{1}{r^2} + 2g\left(\frac{\rho_0}{r^4} + \frac{d_1}{r^5} + \frac{d_2}{r^6} + \ldots\right).
\end{equation}

Substituting all this into (3.3) we can determine the coefficients in (3.11). The result is

\begin{equation}
  y = 1 - \frac{d_1}{\rho_0 r} + \left(\frac{2d_1^2}{\rho_0^2} - \frac{2d_2}{\rho_0} - g\rho_0\right)\frac{1}{r^2} + \ldots
\end{equation}

To carry out all integrations in terms of elementary functions we choose the following simple dark density profile

\begin{equation}
  w_0 = \frac{w_1}{1 + (r/r_s)^4}. \quad (3.17)
\end{equation}

It contains two parameters \(w_1\) and \(r_s\) as most phenomenological dark matter profiles in the literature and has the correct asymptotic behavior. First we want to calculate the solution \(y_0(r)\) of the homogeneous equation

\begin{equation}
  f_1 y_0' = f_2 y_0, \quad (3.18)
\end{equation}

which is given by

\begin{equation}
  y_0(r) = \exp \int \frac{f_2}{f_1} dx. \quad (3.19)
\end{equation}

Now from (3.8) and (3.17) we get

\begin{equation}
  f_1 = \frac{r^4 - r_s^4}{2r(r^4 + r_s^4)}. \quad (3.20)
\end{equation}

In view of the integral in (3.19) it is convenient to use the original definitions (3.4) (3.5) which yield

\begin{equation}
  \frac{f_2}{f_1} = -2\frac{f_1'}{f_1} - \nu' + \frac{4}{r} - \frac{2}{r^2 f_1}. \quad (3.21)
\end{equation}

In our special case (3.17) this is equal to

\begin{equation}
  \frac{f_2}{f_1} = -2\frac{f_1'}{f_1} - \nu' - \frac{8r_s^4}{r(r^4 - r_s^4)}. \quad (3.22)
\end{equation}

This can be easily integrated and exponentiated using (2.10)

\begin{equation}
  e^{-\nu} = r^4 \frac{w_0}{\rho_0}. \quad (2.10)
\end{equation}
In this way we find the following solution (3.19)

\[ y_0(r) = \text{const.} \frac{r^{14} (r^4 + r_s^4)}{(r^4 - r_s^4)^4}. \]  

(3.23)

The solution of the linear homogeneous equation contains a free prefactor, of course. For \( r \to \infty \) \( y_0(r) \) increases as \( r^2 \) which is in conflict with the asymptotic flatness \( y \sim \text{const} \). Consequently, the prefactor of \( y_0 \) has to be zero so that we get a unique solution of our problem as in the expansion (2.16) in the last section.

From \( y_0(r) \) one finds the solution \( y_1(r) \) of the inhomogeneous equation by the well-known method of variation of the constants. One makes the ansatz

\[ y_1(r) = a(r)y_0(r), \]  

(3.24)

where \( y_0 \) is given by (3.23) with prefactor 1. Substituting this into (3.3) one obtains \( a(r) \) by another integration:

\[ a(r) = \int \frac{f_3(x)}{f_1(x)y_0(x)} \, dx, \]  

(3.25)

where a constant of integration, i.e. the lower limit of the integral is free. The integration of the rational function (3.25) is elementary. The final result is

\[ a(r) = \frac{1}{r^2} - \frac{r_s^4}{r^6} + \frac{3}{5} \frac{r_s^8}{r^{10}} - \frac{r_s^{12}}{14 r^{14}} + \]

\[ + 4gw_1 \left[ -2 \log \left( 1 + \frac{1}{r^4} \right) + \frac{7}{4} \frac{r_s^4}{r^4} - \frac{1}{2} \frac{r_s^8}{r^8} + \frac{1}{12} \frac{r_s^{12}}{r^{12}} \right]. \]  

(3.26)

Substituting this back into (3.24) we get the desired unique solution which we denote by \( y(r) \) again:

\[ y(r) = \frac{r^4 + r_s^4}{(r^4 - r_s^4)^4} \left\{ r^{12} - r_s^{12}r^8 + \frac{3}{5} r_s^8 r^4 - \frac{1}{r_s^2} r^4 + \right. \]

\[ + gw_1 \left[ -8r_s^{14} \log (1 + \frac{r_s^4}{r^4}) + 7r_s^4 r^{10} - 2r_s^8 r^6 + \frac{1}{3} r_s^{12} r^2 \right] \}. \]  

(3.27)

The solution \( y(r) \) has the correct asymptotic behavior \( y \to 1 \) for \( r \to \infty \). However, there is a singularity at \( r = r_s \). We ask whether this singularity is hidden by a horizon. A horizon corresponds to \( y = \exp(-\lambda) = 0 \). So we look for a zero of the curly bracket in (3.27). If we insert realistic numbers
for the dark matter density $w_1$ as discussed in the next section, we find that the second line in (3.27) is completely neglegible compared to the first one due to the smallness of $g$. The polynomial
\[ r^{12} - r_s^{4}r^8 + \frac{3}{5}r_s^8r^4 - \frac{1}{7}r_s^{12} \]
has only one real zero $r_0 < r_s$. Consequently the singularity can be seen from the outside, that means it is naked.

Finally the mass density $q(r)$ of normal matter is obtained from (2.7) according to
\[ cq(r) = \frac{1}{g}\left(-\frac{y'}{r} - \frac{y}{r^2} + \frac{1}{r^2}\right) - w_0. \tag{3.28} \]
This also becomes singular at $r = r_s$.

## 4 Comparison with observations

At first side it is questionable whether our solution with a spherically symmetric density distribution $q(r)$ of the normal matter can be compared to observations of spiral galaxies. However, since the dark halo is approximately spherically symmetric, at least the gas component in the outer part of the halo can be assumed to be also spherically symmetric [4]. If we chose a galaxy which is dark matter dominated where the normal matter is only 10 percent of the total mass, say, then a spherically symmetric description is not a bad approximation. The spiral galaxy M33 is a good example for our purpose. Edvige Corbelli [5] has obtained good fits to the data which give (via rotation curves) the density profiles of dark and visible matter. The data are compatible with very different forms of the dark density profile. Beside the NFW profile (1.1) Corbelli considers the isothermal profile
\[ \rho(r) = \frac{\rho_i}{(1 + (r/r_i))^2} \tag{4.1} \]
and the Burkert profile [6]
\[ \rho(r) = \frac{\rho_B}{(1 + r/r_B)(1 + (r/r_B)^2)^2}, \tag{4.2} \]
which both have a central constant - density core like our profile (3.17). The data are not much constraining the profile in the central region, and in the tail region $r > 17$ kpc there are no data. Consequently it is well possible to
fit our profile (3.17) to the best fit of Corbelli between, say, 4 and 14 kpc. However, the resulting singular radius $r_s$ comes out to be around 9 kpc. This is too big. We could except a value $r_s < 0.5$ kpc which is the radius of the bright “nucleus”[5], where something unknown is going on.

A similar conclusion can be drawn from the analysis of the following 3-parameter profile

$$w_0 = \frac{w_1}{r^2(1 - a_1 r + a_2 r^2)}. \quad (4.3)$$

Then the solution of the homogeneous equation (3.18) is given by

$$y_0(r) = \frac{r^{-2-16a_2/a_1^2}(1 - a_1 r + a_2 r^2)}{|2a_2 r - a_1|^{16a_2/a_1^2}} \exp\left(-\frac{8}{a_1 r}\right). \quad (4.4)$$

This produces a singularity at $r_s = a_1/2a_2$. Again fitting (4.3) to Corbelli’s result for the dark matter profile we get $r_s$ around 10 kpc which is unacceptable.

The occurrence of a singularity seems to be a general feature of the simple equation (3.3). It is a consequence of the integrability of the problem which is due to the assumption of an isotropic pressure $p(r)$ in (2.6). On the other hand the dark contributions $\mp w_0(r)$ in (2.8) and (2.9) appear with different signs. Since the normal matter is probably following the dark in a certain way, the pressure components $p_j(r)$ in (2.8) and (2.9) should not be the same. We shall investigate this more general situation in the next part of this series.

References

[1] G. Scharf, From massive gravity to dark matter density hep-th/0711.3749, Romanian Journ. Phys 53 (2008) 1199

[2] J.F. Navarro, C.S. Frenk, S.D.M. White, Astrophys. J. 462 (1996) 563

[3] J.F. Navarro, E.Hayashi, C.Power, A.R.Jenkins, C.S. Frenk, S.D.M. White, V.Springel, J.Stadel, T.R. Quinn, MNRAS 349 (2004) 1039, arXiv:astro-ph/0311231

[4] E. Komatsu, U. Seljak, MNRAS 39 (9) (2001) arXiv:astro-ph/0106151

[5] E.Corbelli, Mon.Not.Roy.Astron.Soc. 342 199 (2003), astro-ph/0302318

[6] A. Burkert, ApJ 447 L25 (1995)