Optimal Battery Control Under Cycle Aging Mechanisms
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Abstract—We study the optimal control of battery energy storage under a general “pay-for-performance” setup such as providing frequency regulation and renewable integration. Batteries need to carefully balance the trade-off between following to the instruction signals and their degradation costs in real-time. Existing battery control strategies either do not consider the uncertainty of future signals, or cannot accurately account for battery cycle aging mechanism during operation. In this work, we take a different approach to the optimal battery control problem. Instead of attacking the complexity of battery degradation function or the lack of future information one at a time, we address these two challenges together in a joint fashion. In particular, we present an electrochemically accurate and trackable battery degradation model called the rainfall cycle-based model. We prove the degradation cost is convex. Then we propose an online control policy with a simple threshold structure and show it achieve near-optimal performance with respect to an offline controller that has complete future information. We explicitly characterize the optimality gap and show it is independent to length of the time of operations. Simulation results with both synthetic and real regulation traces are conducted to illustrate the theoretical results.

I. INTRODUCTION

A confluence of industry drivers— including increased deployment of renewable generation, the high capital cost of managing grid peak demands, and large investments in grid infrastructure for reliability — has created keen interest in building and employing more energy storage systems [1]. Plenty of energy storage technologies have been developed to serve different applications, such as pumped hydro-power, compressed air energy storage, batteries, flywheels and many more [2]. Among these different technologies, battery energy storage (BES) (e.g., lithium-ion batteries) features quick response time, high round-trip efficiency, pollution-free operation, and flexible power/energy ratings. These characteristics make it an ideal choice for a wide range of power system applications, including integration of renewable resources [3], grid frequency regulation [4] and behind-the-meter load management of commercial and residential users [5]. For example, in 2015, there are 153.5MW newly installed battery energy storage devices in the US [6], which is roughly four times the amount of BES installment in 2014. It’s worth to mention that over 80% of the installed capacity in 2015 occurred within the territory of PJM Independent System Operator (ISO), and the predominant use was frequency regulation. The focus of the paper is on the optimal control of battery energy storage under a general “pay for performance” setup: a battery is incentivized to follow certain instruction signals and is penalized when it cannot. For example, a battery participating in frequency regulation would receive a signal and is paid based on how well it follows the signal. Another important application that falls under this setup is a battery used by customers with onsite renewable generations, where the customers may need to purchase more expensive power from the grid if the battery cannot smooth out the local net demand. The common theme of the problems under the pay for performance setup is that the signal the battery should follow is inherently random and the control decisions must be made in real-time. Furthermore, battery storage naturally couples the decisions across time because of its finite energy and power capacities. Therefore, finding the optimal control policy for a battery is essentially a constrained online stochastic control problem [7].

This online problem is challenging for two main reasons: 1) battery degradation and 2) lack of future information. The first arises because a vital aspect of energy storage operation is to accurately model the operational cost of battery, which mainly comes from battery cell degradation [8]. Analogous to cell phone batteries losing their capacity after several years of use, larger batteries used in the grid also loses their capacity with every charge and discharge actions (sometimes called capacity fading) [9]. In fact, overly aggressive use of batteries can often deplete their useful capacities in a matter of months [10]. However, battery degradation is a complex process governed by electrochemical reactions and depends multiple environmental and utilization factors. The second challenge of the lack of information is common to all stochastic control problems. At any given time, a decision must be made without knowing the future signals. This is further complicated by the coupling constrains introduced by battery.

These two challenges are illustrated well in the the fast frequency regulation problem. Frequency regulation is a mechanism used by power system operators to correct the short timescale imbalance between generation and demand in the overall grid. In fast regulation (e.g., regD in PJM), a signal representing the imbalance is broadcasted every 2 or 4 seconds. Having enough energy to follow the regulation signals is critical to the function of the power system, especially as renewables increase the uncertainties in both generation and supply. Frequency regulation is also a natural application for batteries because of the fast variations and roughly zero-mean nature of the regulation signal. By participating in regulation, a battery receives a fixed payment ahead of the time. However, if
it cannot follow the regulation signal, then a penalty is charged based on the mismatch. Therefore, at every time step, a battery must balance its degradation from following the regulation signal with the penalty of doing so, while not knowing the future value of the signal.

In the past, many studies have attacked the battery control problem by focusing on one of the challenges. On one hand, by assuming the degradation of batteries is a quadratic function of the charge/discharge powers, we recover a type of constrained stochastic quadratic regret problem where the key challenge is the lack of future information [11], [12]. On the other hand, one can focus on the degradation of the batteries, by employing accurate electrochemical models while assuming full knowledge of the future [13], [14]. Both directions have led to significant advances by still remaining unsatisfactory. Even by assuming the signal that a battery faces is Gaussian, a constrained linear quadratic Gaussian problem is still extremely challenging to solve and provide any theoretical performance guarantees. Similarly, solving the optimization problem with accurate electrochemical models is by no means trivial even under full knowledge, and it is usually difficult to adapt the solutions to an online form. Given these difficulties, batteries still only serve as emergency backup, or used actively in grid services when they are owned by the utilities and are subsidized under renewable portfolio incentives.

In this paper, we take a different approach to the battery control problem. Instead of attacking the complexity of the degradation function or the lack of future information one at a time, we address these two challenges together in a joint fashion. Surprisingly, we provide a provably near optimal online algorithm for battery control. In particular, we show that under a form of so-called cycle based degradations, there is an online strategy that is within a constant additive gap of the optimal offline strategy under all possible future signals. We explicitly characterize this gap and relate it to the set of possible future signals. The key insight of this result comes from a better understanding of the degradation of electrochemical batteries and how it relates to the control problem. At a high level, capacity fading of these batteries due to charging and discharging is similar to the fatigue process of materials subjected to cyclic loading [15]–[17]. For each cycle, the capacity fades as a function of the depth of that cycle. In past approaches, these cycles were studied in the time domain, leading to complex optimization problems. In contrast, we look at the problem in the cycle-domain, where the problem naturally decouples according to each cycle of the charge/discharge profile. This approach has a loose analogy with time/frequency duality, where some problems are much simpler in the frequency domain than in the time domain. Altogether, our work makes three contributions to the current state-of-art in battery control:

1) We present an electrochemically accurate and trackable battery degradation model, called the rainflow cycle-based degradation. We prove the cost model is convex, which enables it to be easily used in various battery optimization problems and guarantees the solution quality.

2) We provide a subgradient algorithm to solve the optimization problem efficiently and optimally for offline battery planning and dispatch.

3) We offer an online battery control policy with a simple threshold structure, and achieve near-optimal performance with respect to an offline controller that has complete future information.

The online control policy proposed in this paper takes a simple threshold structure which limits a battery’s state of charge (SoC). It reacts to new battery instructions without having to solve new optimization problems, leading to better computational performances than algorithms based on model predictive control and dynamic programming. Compared with traditional threshold control strategies, such as pre-fixed SoC bounds [18], or proportional integral (PI) controller [19], our policy incorporates the application market prices and battery aging model into the SoC threshold calculations, which improves the model accuracy and making it applicable to most electrochemical battery cells. These considerations allows us to derive performance guarantees in form of a bounded constant gap to the full information optimal solution. This battery control policy can be applied to any power system applications that face stochastic signals and have constant prices over a specific period, such as frequency regulation and behind-the-meter peak shaving.

The rest of the paper is organized as follows. Section II covers the background and prior works on the battery control problem. Section III describes the proposed rainflow cycle-based degradation model. Section V sketches the convexity proof and the subgradient algorithm for solving the offline problem. Section VI describes the proposed online control strategy and the optimality proof. We provide a case study in Section VII using real data from PJM frequency regulation market, and demonstrate the effectiveness of the proposed control algorithm in maximizing profits as well as extending battery lifetime. Finally, Section VIII concludes the paper and outlines directions for future work.

II. BACKGROUND AND PRIOR WORKS

The operation of battery energy storage has received much recent research attention because of the importance of batteries to a power system with high penetration of renewables and maturing technologies [3]–[5], [8], [20]–[23]. In these works, the degradation cost of the batteries are modeled in different ways. The authors of [3], [8], [20], [21] assume battery has a fixed lifetime and ignore the degradation cost. This assumption works well when batteries are used sparingly, but tend to lead to overly aggressive actions for finer time resolution applications such as frequency regulation. Other energy storage control studies include degradation models either based on battery charging/discharging power [4], [5] or energy throughput [22], [23]. For example, [5] assumes a convex degradation cost model based on battery charging/discharging power for households demand response, and [22], [23] assign a constant price $2/MWh based on battery energy throughput. These degradation models are convenient to be incorporated in existing optimization problems, at a cost of losing accuracy in quantifying the actual degradation cost. For example, a Lithium Nickel Manganese Cobalt Oxide (NMC) battery has
ten times more degradation when operated at near 100% cycle depth of discharge (DoD) compared to operated at 10% DoD for the same amount of charged power or energy throughout [24]. However, the impact of cycle depth is difficult to capture using power or energy based degradation functions.

The battery aging process is fundamentally described by a set of partial differential and algebraic equations [13], [25], however, they are in some sense too detailed to be used in power system applications. Even with dedicated state-of-the-art algorithms, these equations take several seconds to solve, making them too slow to be used in applications like frequency regulation where one receives a signal every 2 or 4 seconds. To mitigate these difficulties, we use a semi-empirical degradation model that combines theoretical battery aging mechanism with experimental observations. This model is motivated by viewing battery capacity fading as a material fatigue process, where a deeper charge/discharge cycle stresses battery much more than an equivalent number of shallower cycles. Then the relationship between cycle depth and battery degradation is defined by the cycle depth-number curve (Fig. 1), which are normally provided by battery manufacturers or can be estimated from field measurements.

Under this cycle aging model, each cycle causes independent stress, and the loss of battery life is the accumulation of degradation from all cycles. A natural question is how one should count the number of cycles in a general profile, since all of them would be of heterogenous depth. Here we use the “rainflow” algorithm [27], which is the most widely adopted algorithm for cycle identification in material fatigue analysis [27]–[29] as well as for battery degradation [30]–[32]. We show that this electrochemical accurate degradation model is actually convex, which is a key step in deriving the online control algorithm and is of independent interest to many other battery applications.

The online nature of the battery control problem has perhaps received more attention from the control community. Multiple types of approaches have been developed, including model predictive control [20], [33], [34], stochastic and dynamic programming [3], [35]–[38]. The authors of [33] derive a model predictive control (MPC)-based for battery energy storage and wind integration, although without any performance guarantees. Recent works [20], [34] do include results that bound the performance gap of online algorithms, but it is difficult to evaluate the quality of these bounds since they are either quite loose or depend on complicated optimization problems themselves that grows with the time of operation. In addition, none of these bounds considers a cycle-based degradation problem. Our results in this paper provide an online algorithm with a constant gap to the offline optimal that is independent to the length of the operation time.

In addition to MPC type of algorithms, another widely used strategy is dynamic programming (DP). For example, [3] and [35] consider using DP for storage operation with a co-located wind farm, [36] and [37] for operating storage with end-user demands, and [38] for storage with demand response. However, for real-time control problems, the battery state space, action space and the instruction signal are all continuous. Standard DP discretization approaches tend to cause the dimension of the problem to grow exponentially. Also, implementing these algorithms requires the distributional information of the random instruction signal, which may not be readily available. In contrast, our algorithm does not require any distributional information.

Remark 1. In this paper we focus on the impact of cycle-depth on the capacity lifetime of batteries. In addition to cycle-depth, there are a host of factors that contributes for capacity fading. For example, the temperature of the battery has a dramatic influence in its lifetime. However, in grid applications, the temperature of the cells are normally controlled to be within a narrow band. Similarly, other factors such as extremely high C-rate and unbalanced battery cells either do not come into play for grid applications or are controlled by lower level power electronics [10]. Therefore, in this paper we focus on the most relevant factor to degradation: the depth of charge and discharge cycles.

III. Model

In this section we describe the battery operation model, the rainflow cycle-based battery aging cost, and the pay for performance market setup. Then we state the main optimization problem on how to balance profit from frequency regulation and the degradation cost of battery operation in an online fashion.

A. Battery Operations

We consider an operation defined over finite discrete control time steps $t \in \{1, \ldots, T\}$, and each control time interval has a duration of $\tau$. Let $x_t$ be the energy stored in the battery—the state of charge (SoC)—at time $t$. By convention, $x_t$ is a normalized quantity between 0 (empty battery) and 1 (full battery). At any time interval, the battery can either charge with power $c_t$ (in units of kW) or discharge with power $d_t$ (in units of kW) or remain idle. The state of charge at time $t$ can be expressed as $x_{t+1} = x_t + \min(\max(x_t + \tau \cdot c_t - \tau \cdot d_t, 0), 1)$.

In practice, $\tau$ is set by the power electronic based battery management system, and is normally in the scale of milliseconds [29].
units of kW. Then its state of charge evolves according to the following linear difference equation:

\[ x_{t+1} - x_t = \frac{\tau \eta_c}{E} c_t - \frac{\tau}{\eta_d} d_t, \]

(1)

where \( \eta_c \) and \( \eta_d \) are the charging and discharging efficiency\(^2\), and \( E \) (in units of kWh) is the rated energy capacity of battery. We use bold symbols \( \mathbf{x}, \mathbf{c}, \) and \( \mathbf{d} \) to denote the vector version of SoC, charging powers and discharging powers, respectively.

For a given battery, it has three types of operational constraints. The first is the limits its SoC, where the stored energy in the battery is constrained to be within a particular region. This constraint can arise from either health concerns since batteries like lithium-ion should not be charged completely full or discharged to be completely empty. It can also arise if batteries are used for other applications such as backup. In this paper, we assume that the SoC limits are given. The other two constraints on battery operation are the rate constraints on the charging and discharging powers. These constraints are written as:

\[ \underline{x} \leq x_t \leq \bar{x}, \quad 0 \leq c_t \leq P, \quad \text{and} \quad 0 \leq d_t \leq P, \]

where \( \underline{x} \) and \( \bar{x} \) is the minimum and maximum SoC of the battery, respectively; \( P \) is the battery power rating.

We consider an optimization problem where a battery is incentivized to follow an instruction signal \( \mathbf{r} \). Suppose the operation revenue is \( R(c, d, \mathbf{r}) \), a function of battery power output \( c, d \) and the instruction signal. The operational cost comes from the battery degradation, denoted here by \( f(c, d) \), a function of battery charging/discharging responses. The exact form of \( f(\cdot) \), namely the rainflow cycle-based degradation function, is introduced in the next section. The optimization objective is to maximize the net utility of the battery, which equals to operational revenue subtracting the cost. The overall problem is:

\[
\begin{align*}
\max_{\mathbf{c}, \mathbf{d}} \quad & R(\mathbf{c}, \mathbf{d}, \mathbf{r}) - f(\mathbf{c}, \mathbf{d}) \quad \text{(2a)} \\
\text{s.t.} \quad & x_{t+1} = x_t + \frac{\tau \eta_c}{E} c_t - \frac{\tau}{\eta_d} d_t, \quad \text{(2b)} \\
& \underline{x} \leq x_t \leq \bar{x}, \quad \text{(2c)} \\
& 0 \leq c_t \leq P, \quad \text{(2d)} \\
& 0 \leq d_t \leq P, \quad \text{(2e)}
\end{align*}
\]

where (2b) is the state evolution equation, (2c) is the SoC constraint, (2d) and (2e) are the power constraints. Component-wise inequality between two vectors is denoted by \( \preceq \). Note here we may include a constraint that storage cannot charge and discharge at the same time\([10]\), but it turns out that this condition will always be satisfied in our setting.

Solving (2) has proven to be difficult for two reasons. The first is that most realistic cycle-based degradation functions are not well understood (e.g., they are not known to be convex), making the deterministic version of (2) nontrivial\([10]\). The second is that in real-time applications such as frequency response, the signal \( \mathbf{r} \) is inherently random and difficult to forecast\([41],[42]\), while the state of the problem \( x_t \) is constrained and coupled over time. Therefore, even for relatively simple forms of \( f \) (e.g. \( f = \sum c_i^2 + d_i^2 \)), there are no optimal or provably suboptimal online algorithms. The next section describes the rainflow cycle-based degradation model in detail, and the rest of the paper shows that rather surprisingly, this realistic will lead to a simple provable optimal online algorithm.

B. Cycle Counting via Rainflow

To model the battery degradation cost \( f(c, d) \), we take the rainflow cycle-based method, the most widely used semi-empirical method used in practice\([10],[43],[46]\). The cycle aging of electrochemical battery cells is evaluated based on

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\(^2\)By convention, \( \eta_c \) is between 0 and 1 and \( \eta_d \) is larger than 1.
stress cycles, and the rainflow method identify cycles from local extrema in a SoC profile. A local extrema point indicates that the battery switched from charge mode to discharge mode, or vice versa. We use \( s_1, s_2, \ldots \) to denote the extrema, including both minima and maxima. Figure 2a gives an example of a profile with seven extrema.

Given a SoC profile with its local extrema, the goal of the rainflow algorithm is to extract the cycle depths between the local maxima and minima of the profile. The key feature of the algorithm is that it does not necessarily calculate the depth between two successive extrema. Rather, it first finds all cycle depths \( s_1 \) and \( s_2 \) (between \( s_3 \) and \( s_4 \) (40% SoC)), the other two cycles are both of 10% SoC (between \( s_3 \) and \( s_4 \) in Fig. 2a) and between \( s_5 \) and \( s_6 \).

For a general SoC profile, the rainflow counting method is given below in Algorithm 1. Note there are multiple equivalent algorithms, and the one we adopt here is based on [25].

**Algorithm 1: Rainflow Counting Algorithm.**

**Data:** A SoC profile with a finite number of local extrema \( S = \{s_1, s_2, \ldots \} \).

**Result:** A set of charging depths \( V = \{v_1, v_2, \ldots \} \) and a set of discharging depths \( W = \{w_1, w_2, \ldots \} \).

**Start from the beginning of the profile;**

while \( S \) is not empty do

if there are more than three points in \( S \) then

Calculate \( \Delta s_1 = |s_1 - s_2|, \Delta s_2 = |s_2 - s_3|, \Delta s_3 = |s_3 - s_4| \);

if \( \Delta s_2 \leq \Delta s_1 \) and \( \Delta s_2 \leq \Delta s_3 \) then

A full cycle of depth \( \Delta s_2 \) associated with \( s_2 \) and \( s_3 \) has been identified. Add \( \Delta s_2 \) to both \( V \) and \( W \);

Remove \( s_2 \) and \( s_3 \) from the profile, set \( S = \{s_1, s_2, s_5, s_6, \ldots \} \);

else

Shift the identification forward and repeat with \( S = \{s_1, s_3, s_4, s_5, s_6, \ldots \} \);
endif
endif

if \( s_1 \leq s_2 \leq s_3 \) then

Add \( s_3 - s_1 \) to \( V \);
else if \( s_1 \geq s_2 \geq s_3 \) then

Add \( s_1 - s_3 \) to \( W \);
else if \( s_1 \geq s_2, s_2 \leq s_3 \) then

Add \( s_3 - s_2 \) to \( V, s_1 - s_2 \) to \( W \);
else

Add \( s_2 - s_1 \) to \( V, s_2 - s_3 \) to \( W \);
endif

Set \( S \) to the empty set.
end

This algorithm essentially moves through the extrema to look for charging and discharging half-cycles [47]. After a cycle is identified, its associated end points are removed, and the process is repeated until no points are left. Figure 2 shows the progression of Algorithm 1 through an example profile.

Let Rainflow be the functional form of the rainflow counting algorithm in Algorithm 1, then it takes a SoC profile as an input and outputs the cycle depths:

\[
(v, w) = \text{Rainflow}(x)
\]

where \( v \) is the vector of charging half cycles and \( w \) is the vector of discharging half cycles. Since cycle depths only depend on the relative differences of the turning point of the SoC profile and not on the initial SoC value, they can be calculated from \( (c, d) \)

\[
(v, w) = \text{Rainflow}\left(\frac{\tau}{E^c - \frac{\tau}{\eta_d E^d}}\right).
\]

**C. Battery Degradation Cost**

After counting the cycles, a cycle depth stress function \( \Phi(u) : [0, 1] \to \mathbb{R}^+ \) is used to model the life loss from a single cycle of depth \( u \) measured in terms of (normalized) changes in the SoC. This function indicates that if a battery cell is repetitively cycled with depth \( u \) then it can operate \( 1/\Phi(u) \) number of cycles before reaching its end of life. In practice, this function can be estimated through empirical measurements in Fig. 1 is normally included by battery manufacturers. For most electrochemical batteries, \( \Phi(u) \) is a convex function [24], [43]–[46], popularly parameterized as a power function \( \alpha u^\beta \) or exponential functions \( \alpha e^{\beta u} \) [17]. Because cycle aging is a cumulative fatigue process [24], [43], the total life loss \( \Delta L \) is the sum of the life loss from all half cycles:

\[
\Delta L(v, w) = \frac{\sum_{i=1}^{\|v\|} \Phi(v_i)}{2} + \frac{\sum_{i=1}^{\|w\|} \Phi(w_i)}{2},
\]

where \( \| \cdot \| \) is the cardinality of a vector. By convention, the factor \( 1/2 \) is included. Here we assumed that a charging half cycle and a discharging half cycle has the same stress function \( \Phi \), but our results do not change if different functions are used. For example, to calculate cycle aging for the profile in Fig. 2a we set \( v_1 = 0.1, v_2 = 0.1, v_3 = 0.4 \) and \( w_1 = 0.1, w_2 = 0.1, w_3 = 0.2 \). If we substitute the rainflow algorithm as in (4) into (5), the incremental cycle aging can therefore be written as a function of the control actions \( c \) and \( d \). To convert the loss of life to a cost, let \( B \) be the battery cell replacement unit cost in $/kWh and \( E \) be the capacity of the battery in kWh. Then the cycle aging cost function \( f(c, d) \) is

\[
f(c, d) = \Delta L(c, d) \cdot E \cdot B.
\]

**D. Revenue Model**

Here we describe a version of a pay for performance mechanism widely used by system operators. This mechanism has a two-stage structure. In the first stage, ahead of real-time, a payment \( C \) (in units of $) is provided to the participant. Here, we assume that this payment is known and given and focus on the second stage. The second stage occurs in real-time, where a participant is given a signal \( r \) and faces a penalty if it cannot follow the signal. That is, it pays a over-response
price \( \theta \in R^+ \) ($/MWh) for surplus injections or deficient demands during each dispatch interval, and a under-response price \( \pi \in R^+ \) ($/MWh) for deficient injections or surplus demands. Then the total revenue is:

\[
R(c, d, r) = C - \tau \theta \sum_{t=1}^{T} |\eta_t c_t - \frac{d_t}{\eta_d} - r_t|^+ - \tau \pi \sum_{t=1}^{T} |r_t - \eta_t c_t + \frac{d_t}{\eta_d}|^+ ,
\]

where \( \eta_t c_t - \frac{d_t}{\eta_d} \) is the net charging power, \( r_t \in [-P, P] \) is the instructed regulation dispatch set-point for the dispatch time step \( t \), with the convention positive values in \( r_t \) represents charging instructions.

This model captures the essence of two important applications of storage in the grid: frequency regulation and the demand shaping. In frequency regulation, \( C \) is the capacity payment and \( r_t \) is the regulation signal sent every 2 to 4 second by the system operator. The penalty prices \( \theta \) and \( \pi \) are published values. In demand shaping, a battery would enter into an agreement with an utility to keep demand of a customer at prescribed levels at payment \( C \) and \( r_t \) can be thought as the net time-varying demand of the user. Here the penalty prices are also determined ahead of time. An important future direction is to extend our results to settings where the penalty prices are random in themselves, such as real-time arbitrage [48], [49].

E. Optimization Problem

Summarizing the previous sections, we are left with the following optimization problem:

\[
\begin{align*}
\min_{c,d} & \quad \tau \sum_{t=1}^{T} \left( \theta |\eta_t c_t - \frac{d_t}{\eta_d} - r_t|^+ - \pi |r_t - \eta_t c_t + \frac{d_t}{\eta_d}|^+ \right) \\
& \quad + \left( \sum_{i=1}^{W} \frac{\Phi(v_i)}{2} + \sum_{i=1}^{W} \Phi(w_i) \right) \cdot B \cdot E \\
\text{s.t.} & \quad x_{t+1} = x_t + \frac{\tau \eta_t c_t}{E} - \frac{\tau}{\eta_d E} d_t , \\
& \quad 0 \leq x \leq x \leq \bar{x}, \\
& \quad 0 \leq c \leq P, \\
& \quad 0 \leq d \leq P. \\
& \quad (v, w) = \text{Rainflow(x)}. 
\end{align*}
\]

We are interested in solving (8) in two settings:

**Offline:** In the off-line setting, the entire sequence of the signal \( r \) is given. This is important in many planning and validation problems.

**Online:** Here, we only allow \( c_t \) and \( d_t \) to depend on the current and past information: \( \{r_t, r_{t-1}, \ldots, r_1\} \). This models the real-time decisions that batteries need to make for charging and discharging.

3In practice, different systems operators have slightly different rules for frequency response. Instead of cumbersome accounting for these rules, we focus on the general structure which is given in [4].

IV. MAIN RESULTS

The main contributions of this paper provide positive results to both the offline and online solutions of (8). For the offline setting, we have the following theorem:

**Theorem 1. Convexity.** Suppose the battery cycle aging stress function \( \Phi \) is convex. Then the offline version of the optimization problem in (8) is convex in the charge and discharge variables.

This theorem settles an open question about cycle-based degradation cost functions [47], [50] and is used in the proof of the optimality of the online policy. The penalty term in the objective function (8a) is clearly convex in \( c \) and \( d \), but the convexity of the term associated with the cycle stress functions is not obvious because of the nonlinear Rainflow(\cdot) function in (8d). Previously, problems like the one in (8) are solved via generic optimization programs (e.g., fmincon in Matlab), which can be extremely computationally expensive even for small problems. As we illustrate in Section VII, frequency regulation problems with time horizon longer than 4 hours take longer than 8 hours to solve using existing approaches, but can be solved in less than 3 minutes using a subgradient algorithm specifically developed for (8). Of course, by Theorem 1 the algorithm is optimal.

Next we state the optimality result with respect to the online optimization problem. Let \( J_g \) denote the of the value of (8) under an online algorithm \( g \), and \( J^* \) denote the offline optimal where all information are known at the beginning. Then we have:

**Theorem 2. Online optimality.** Suppose the battery cycle aging stress function \( \Phi \) is strictly convex. There exist a threshold online algorithm that has a constant worst-case optimality gap that is independent of the operation time duration \( T \).

The control policy \( g \) in Theorem 2 is a fairly straightforward threshold policy and is given as Algorithm 2 in Section VII. The bound in the theorem is much tighter compared to standard bounds for online optimization problems. Normally, one would compare the averaged regret, namely \( \lim_{T \to \infty} \frac{1}{T} (J_g - J^*) \) and a sublinear regret is considered to be “good” [51], [52]. Here, our result essentially shows that one can solve the online version of (8) with zero regret, since the constant \( \epsilon \) do not depend on \( T \). In contrast, most existing algorithms cannot even achieve sublinear regret. Again, the key to our result is to explore the particular cyclic structure of the rainflow based cost functions. By a case study on PJM frequency regulation market in Section VII we show that the proposed control algorithm could significantly improve the operational revenue up to 30% and the battery can last as much as 4 times longer. A useful corollary of Theorem 2 showing when the gap is 0:

**Corollary 1. Zero-optimality Gap** If \( \pi \eta_d = \theta / \eta_c \), then there is no gap between the online algorithm and the optimal offline algorithm with full information.

For example, this corollary holds if the battery has the same
charging and discharging efficiency $\eta$ and the penalty prices for over and under injections are the same, then there exists an optimal online algorithm.

V. CONVEXITY AND SUBGRADIENT ALGORITHM

In this section, we sketch the proof of Theorem 1 to provide some intuitions and then provide the subgradient algorithm. A reader more interested in the online algorithm can directly proceed to the next section.

A. Proof of Theorem 1

Here we sketch the proof. Without loss of generality, we only consider the cost of charging cycles given the interchangeable and symmetric nature of charging/discharging variables. A detailed proof is given in Appendix.

To prove Theorem 1, it suffices to show that the mapping from the SoC profile $x$ to degradation cost:

$$f(x) = \left[\sum_{i=1}^{|x|} \Phi(u_i) + \sum_{i=1}^{|w|} \Phi(w_i)\right] / 2$$

is convex in terms of $x$ given the cycle stress function $\Phi(\cdot)$ convex. That is, for any two SoC time series $x, y \in \mathbb{R}^T$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \forall \lambda \in [0, 1]. \quad (9)$$

Intuitively, given two SoC series $x$ and $y$, if they change in different directions, the two cancel each other out so that the left hand side of (9) is less than the right hand side by the convexity of $\Phi$. When $x$ changes in exactly the same direction as $y$ for all time steps, the equality holds. The difficulty of proving this result lies in the fact that the rainfall function in Algorithm 1 is a many-to-many function that maps a sequence in $\mathbb{R}^T$ to a set of cycle depth of indeterminate length. The proof uses induction as described in the rest of this section.

1) Unit step decomposition: First, we introduce the step function decomposition of SoC signal. Any SoC series $x$ could be written out as a finite sum of step functions, where

$$x = \sum_{t=1}^{T} P_t U_t, \quad (10)$$

where $U_t$ is a unit step function with a jump at time $t$ defined as:

$$U_t(\tau) = \begin{cases} 
1 & \tau \geq t \\
0 & \text{otherwise}.
\end{cases}$$

Fig. 3 gives an example of step function decomposition of $x$. We use this decomposition to write out $x$, $y$ and $\lambda x + (1 - \lambda)y$ as finite sum of step functions, where

$$x = \sum_{t=1}^{T} P_t U_t, \quad y = \sum_{t=1}^{T} Q_t U_t, \quad (11)$$

$$\lambda x + (1 - \lambda)y = \sum_{t=1}^{T} Z_t U_t. \quad (12)$$

$K$ steps.

B. Initial case

We first show that $f(x)$ is convex when a profile has only one non-zero step change as shown in Fig. 4.

**Lemma 1.** Under the conditions in Theorem 1, the rainfall cycle-based cost function $f$ satisfies

$$f(\lambda x + (1 - \lambda)Q_t U_t) \leq \lambda f(x) + (1 - \lambda)f(Q_t U_t), \forall \lambda \in [0, 1],$$

where $x \in \mathbb{R}^T$, and $Q_t U_t$ is a step function with a jump happens at time $t$ with amplitude $Q_t$.

Fig. 4: Base case of the induction, where one of the profiles consists of a single step.

The proof of this initial case requires analyzing the impact on all cycle depths from the single step and is given in Appendix A.

C. Induction Steps

Assuming Theorem 1 is true if one of the two profiles $x$ or $y$ has a single non-zero step. Now, assume $f$ is convex up to the sum of $K$ step changes (arranged by time index):

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \lambda \in [0, 1]$$

if $y$ has $K$ non-zero step changes ($K < T$). We need to show $f$ is convex up to the sum of $K + 1$ step changes (i.e., $y$ is of length $K + 1$).

The induction step proof relies on a case-by-case analysis. It contains three major conditions, where 1) $Z_{K+1}$ (the amplitude of $K$+1 step of the combined profile) and $Z_K$...
SoC \quad v

when using a log-barrier function \[53\]: converge to the optimal solution with a user-defined precision problems. Here we provide an efficient subgradient algorithm. With difficulty in solving (8) even for planning and evaluation problems. Appendix A. the overall convexity proof and the detailed reasoning is given in Appendix A.

are in the same direction 2) \(Z_K\) and \(Z_{K+1}\) are in different directions, with \(|Z_K| \geq |Z_{K+1}|\) or 3) \(Z_K\) and \(Z_{K+1}\) are different directions, with \(|Z_K| < |Z_{K+1}|\). Each major category may contain some further sub-cases and requires careful accounting. Showing convexity for each sub-case finishes the overall convexity proof and the detailed reasoning is given in Appendix A.

D. Subgradient Algorithm

The convexity of the offline problem in (8) suggests that it can solved efficiently. However, the degradation cost term \(f(c, d)\) is not continuously differentiable (not differentiable at cycle junction points). This has contributed to the current difficulty in solving (8) even for planning and evaluation problems. Here we provide an efficient subgradient algorithm. With proper step size, the subgradient algorithm is guaranteed to converge to the optimal solution with a user-defined precision level [53].

To begin with, we re-write the constrained optimal battery control problem in (8) as an unconstrained optimization problem using a log-barrier function [53]:

\[
\min_{c, d} J(\cdot) := \tau \sum_{t=1}^{T} \left[ \theta |\eta_c c_t - \frac{d_t}{\eta_d} - r_t|^+ + \pi |r_t - \eta_c c_t + \frac{d_t}{\eta_d}|^+ \right] \\
+ \left[ \sum_{i=1}^{|v|} \frac{\Phi(v_i)}{2} + \sum_{j=1}^{|w|} \frac{\Phi(w_j)}{2} \right] EB \\
- \frac{1}{\lambda} \left\{ \sum_{t=1}^{T} \log[\pi - x(t)] + \sum_{t=1}^{T} \log[x(t) - \pi] \\
+ \sum_{t=1}^{T} \log[P - c_t] + \sum_{t=1}^{T} \log[c_t] \\
+ \sum_{t=1}^{T} \log[P - d_t] + \sum_{t=1}^{T} \log[d_t] \right\}
\]

(13)

when \(\lambda \rightarrow +\infty\), the unconstrained problems (13) becomes equivalent to the original constrained problem.

The major challenge of solving Eq. (13) lies in the second term. We need to find the mathematical relationship between charging cycle depth \(v_i\) and charging power \(c_t\), as well as the relationship between discharging cycle depth \(w_j\) and discharging power \(d_t\). Recall that the rainflow cycle counting algorithm introduced in Section III each time index is mapped to at least one charging half cycle or at least one discharging half cycle. Some time steps sit on the junction of two cycles. For example, in Fig. 2 \(s_2\) lies on the junction of charge half cycle \(s_1 - s_4\) and charge half cycle \(s_3 - s_4\). No time step belongs to more than two cycles.

Let \(T_v\) be all the time indexes that belong to the charging half cycle \(i\) and let the time indexes belonging to the discharging half cycle \(j\) be set \(T_w\). Then

\[
T_v \cup \ldots \cup T_{v_i} \cup T_{w_1} \cup \ldots \cup T_{w_{|w|}} = \{1, \ldots, T\},
\]

(14)

\[
T_{v_i} \cap T_{w_j} = \emptyset, \forall i, j.
\]

(15)

Eq. (15) shows there is no overlapping between a charging and a discharging cycle. That is, each half-cycle is either charging or discharging. The cycle depth therefore equals to the sum of battery charging or discharging within the cycle time frame,

\[
v_i = \sum_{t \in T_{v_i}} \frac{\tau \eta_c}{E} c_t,
\]

(16)

\[
w_j = \sum_{t \in T_{w_j}} \frac{\tau}{\eta_d E} d_t.
\]

(17)

The rainflow cycle cost \(f(x)\) is not continuously differentiable. At each cycle junction point, it has more than one subgradient. We use \(\partial f(x)|_{c_t}\) to denote a subgradient at \(c_t\). Since the SoC profile \(x\) is a function of \(c\), by the chain rule, we have

\[
\partial f(x)|_{c_t} = \Phi(v_i) \frac{B_T \eta_c}{2} , t \in T_{v_i},
\]

(18)

where \(v_i\) is the depth of cycle that \(c_t\) belongs to. Note, at junction points, \(c_t\) belongs to two cycles so that the subgradient is not unique. We can set \(v_i\) to any value between \(v_{i_1}\) and \(v_{i_2}\), where \(v_{i_1}\) and \(v_{i_2}\) are the depths of two junction cycles \(c_t\) belongs to.

Similarly for discharging cycle, a subgradient at \(d_t\) is

\[
\partial f(x)|_{d_t} = \Phi(w_j) \frac{B_T}{2 \eta_d} , t \in T_{w_j}
\]

(19)

where \(w_j\) is the depth of the cycle that \(d_t\) belongs to. At the junction point, \(w_j\) could be set to any value between \(w_{j_1}\) and \(w_{j_2}\), which are the two junction cycles \(d_t\) belongs to.

Therefore, we write the subgradient of \(J(\cdot)\) with respect to \(c_t\) and \(d_t\) as \(\partial J|_{c_t}\) and \(\partial J|_{d_t}\), where

\[
\partial J|_{c_t} = -\frac{\partial R(c, d, r)}{\partial c_t} + \Phi(v_i) \frac{B_T \eta_c}{2} - \frac{1}{\lambda} \left\{ \sum_{k=1}^{T} \frac{1}{x(k) - \pi} \frac{\tau \eta_c}{E} \right\} \\
+ \sum_{k=t}^{T} \frac{1}{x(k) - \pi} \frac{\tau \eta_c}{E} + \frac{1}{c_t - P} + \frac{1}{\lambda} \right\}, t \in T_{v_i}
\]

(20)

\[
\partial J|_{d_t} = -\frac{\partial R(c, d, r)}{\partial d_t} + \Phi(w_j) \frac{B_T}{2 \eta_d} - \frac{1}{\lambda} \left\{ \sum_{k=1}^{T} \frac{1}{x(k) - \pi} \frac{\tau \eta_d}{E} \right\} \\
- \sum_{k=t}^{T} \frac{1}{x(k) - \pi} \frac{\tau \eta_d}{E} + \frac{1}{d_t - P} + \frac{1}{\lambda} \right\}, t \in T_{w_j}
\]

(21)

The update rules for \(c_t\) and \(d_t\) at the \(k\)th iteration are,

\[
c_t(k) = c_t(k-1) - \alpha_k \cdot \partial J|_{c_t(k-1)}(t),
\]

(22)

\[
d_t(k) = d_t(k-1) - \alpha_k \cdot \partial J|_{d_t(k-1)}(t),
\]

(23)
\[ d_{(k)}(t) = d_{(k-1)}(t) - \alpha_k \cdot \partial J |_{d_{(k-1)}(t)} \]

where \( \alpha_k \) is the step length at \( k \)th iteration. Since the subgradient method is not a decent method \([\text{33}]\), it is common to keep track of the best point found so far, i.e., the one with smallest function value. At each step, we set

\[ J^\text{best}_{(k)} = \min \{ J^\text{best}_{(k-1)}, J(c_{(k)}, d_{(k)}) \} \]

Since the \( J(\cdot) \) is convex, choosing an appropriate step size guarantees convergence.

VI. ONLINE POLICY

In this section, we introduce the proposed online battery control policy which balances the cost of deviating from the instruction signal and the cycle aging cost of batteries while satisfying operation constraints. This policy takes a threshold form and achieves an optimality gap that is independent of the total number of time steps. Therefore in term of regret, this policy achieves the strongest possible result: the regret do not grow with time. Note we assume the regulation capacity has already been fixed in the previous capacity settlement stage.

A. Control Policy Formulation

The key part of the control policy is to calculate thresholds that bounds the SoC of the battery as functions of the deviation penalty and degradation cost. Let \( \hat{u} \) denote this bound on the SoC and it is given by:

\[ \hat{u} = \Phi^{-1}(\frac{\pi \eta_4 + \theta/\eta_c}{B}) \]  

where \( \Phi^{-1}(\cdot) \) is the inverse function of the derivative of the cycle stress function \( \Phi(\cdot) \).

Algorithm 2: Proposed Control Policy

\begin{enumerate}
\item[Result:] Determine battery dispatch point \( c_t, d_t \)
\item[\textit{// initialization}]
\begin{itemize}
\item set \( \Phi \left( \frac{\pi \eta_4 + \theta/\eta_c}{B} \right) \rightarrow \hat{u}, x_0 \rightarrow x_0^{\max}, x_0 \rightarrow x_0^{\min} \)
\end{itemize}
\item[\textit{// read} \( t \leq T \) do
\begin{itemize}
\item \textit{// read} \( x_t \) and update controller
\item set \( \max\{x_{t-1}^{\max}, x_t\} \rightarrow x_t^{\max}, \min\{x_{t-1}^{\min}, x_t\} \rightarrow x_t^{\min} \)
\item set \( \min\{\hat{x}, x_t^{\min} + \hat{u}\} \rightarrow \hat{x} \)
\item set \( \max\{\hat{x}, x_t^{\max} - \hat{u}\} \rightarrow \hat{x} \)
\item \textit{// read} \( r_t \) and enforce soc bound
\item if \( r_t \geq 0 \) then
\begin{itemize}
\item set \( \min\left\{ \frac{E}{\eta_c}(\pi_t - x_t), r_t \right\} \rightarrow c_t, 0 \rightarrow d_t \)
\end{itemize}
\item else
\begin{itemize}
\item set \( 0 \rightarrow c_t, \min\left\{ \frac{E}{\eta_c}(x_t - x_{\hat{L}}), r_t \right\} \rightarrow d_t \)
\end{itemize}
\end{itemize}
\item[\textit{// wait until next control interval}]
\begin{itemize}
\item set \( t + 1 \rightarrow t \)
\end{itemize}
\end{enumerate}

The proposed control policy is summarized in Algorithm 2 and Fig. 5 shows a control example of the proposed policy, in which the battery follows the regulation instruction until the distance between its maximum and minimum SoC reaches \( \hat{u} \).

The detailed formulation is as follows. We assume at a particular control step \( t \), \( x_t \) (battery state of charge) and \( r_t \) (frequency regulation signal) are observed, and the proposed regulation policy has the following form:

\[ g_t(x_t, r_t) = [c_t, d_t] \]

The control policy employs the following strategy

\[
\begin{align*}
\text{If } r_t &\geq 0, c_t = \min \left\{ \frac{E}{\tau \eta_c} (\pi_t - x_t), r_t \right\} \\
\text{If } r_t &< 0, d_t = \min \left\{ \frac{E \eta_4}{\tau} (x_t - \zeta), r_t \right\}
\end{align*}
\]

where \( \pi_t \) and \( \zeta \) are the upper and lower storage energy level bound determined by the controller at the control interval \( t \) for enforcing the SoC band \( \hat{u} \).

\[
\begin{align*}
\pi_t &= \min\{\pi, x_t^{\min} + \hat{u}\} \\
\zeta &= \max\{\pi, x_t^{\max} - \hat{u}\}
\end{align*}
\]

and \( x_t^{\max}, x_t^{\min} \) is the current maximum and minimum battery storage level since the beginning of the operation, which are updated at each control step as

\[
\begin{align*}
x_t^{\max} &= \max\{x_{t-1}^{\max}, x_t\} \\
x_t^{\min} &= \min\{x_{t-1}^{\min}, x_t\}
\end{align*}
\]

B. Optimality Gap to Offline Problem

Theorem 2 states that the gap between the online policy in Algorithm 2 and an offline optimal solution is bounded by a constant. This constant can be explicitly characterized. To do this, we define three new functions:

\[
\begin{align*}
J_u(u) &= EB\Phi(u) + E(\theta/\eta_c + \pi \eta_4) u \\
J_v(v) &= (1/2)EB\Phi(u) + (E/\eta_c) \theta v \\
J_w(w) &= (1/2)EB\Phi(u) + E\eta_4 \tau w
\end{align*}
\]
where \( J_u \) is the cost associated with a full cycle (made up of a charging half cycle and a discharging half cycle with equal magnitude), \( J_v \) for a charge half cycle, and \( J_w \) for a discharge half cycle. The detailed transforming procedure is discussed in the Appendix II-A.

If the cycle depth stress function \( \Phi(\cdot) \) is strictly convex, then it is easy to see that \((22)\) is the unconstrained minimizer to \((27a)\). Similarly, the unconstrained minimizers of \((27b)\) and \((27c)\) are:

\[
\hat{\nu} = \Phi^{-1}\left(\frac{\theta}{\eta_c} \right), \quad \hat{\nu} = \Phi^{-1}\left(\frac{\pi \eta_d}{B} \right). \tag{28}
\]

The following theorem offers the analytical expression for \( \epsilon \).

**Theorem 3.** If function \( \Phi(\cdot) \) is strictly convex, then the worst-case optimality gap for the proposed policy \( g(\cdot) \) in Theorem 2 is

\[
\epsilon = \begin{cases} 
\epsilon_w & \text{if } \pi \eta_d > \theta/\eta_c \\
0 & \text{if } \pi \eta_d = \theta/\eta_c \\
\epsilon_v & \text{if } \pi \eta_d < \theta/\eta_c.
\end{cases} \tag{29}
\]

where

\[
\epsilon_w = J_w(\hat{u}) + 2J_v(\hat{u}) - J_w(\hat{w}) - 2J_v(\hat{v}) \tag{30}
\]

\[
\epsilon_v = 2J_w(\hat{u}) + J_v(\hat{u}) - 2J_w(\hat{w}) - J_v(\hat{v}). \tag{31}
\]

Note that Corollary 1 follows from Theorem 3 directly. We defer the proof of the latter to Appendix A. The intuition is that battery operations consist mostly full cycles due to limited storage capacity because the battery has to be charged up before discharged, and vice versa. Enforcing \( \hat{w} \) the optimal full cycle depth calculated from penalty prices and battery coefficients—ensures optimal responses in all full cycles. In cases that \( \pi \eta_d = \theta/\eta_c \), \( \hat{u} \) is also the optimal depth for half cycles, and the proposed policy achieves optimal control. In other cases, the optimality gap is caused by half cycles because they have different optimal depths. However, half cycles have limited occurrences in a battery operation because they are incomplete cycles \((29)\), so that the optimality gap is bounded as stated in Theorem 3. Fig. 7 shows examples of the policy optimality when responding to the regulation instruction (Fig 7a) under different price settings. The proposed policy has the same control action in all three price settings because of the same \( \hat{u} \). The policy achieves optimal control in Fig 7b because \( \hat{u} \) is the optimal depth for all cycles. In Fig 7c and Fig 7d, half cycles have different optimal depths and the policy is only near-optimal. However, the offline result also selectively responses to instructions with a zero penalty price (charge instructions in Fig 7c, discharge instructions in Fig 7d), because it returns the battery to a shallower cycle depth with smaller marginal cost.

**VII. Simulation Results**

**A. Simulation Setting**

We compare the proposed control policy with the offline optimal result and a simple control policy proposed in \([3]\). The maximum state of charge (SoC) level \( \pi \) is set to 95% and the minimum SoC level \( \underline{\pi} \) is set to 10%. This assumed battery storage consists of lithium-ion battery cells that can perform 3000 cycles at 80% cycle depth before reaching end of life, and these cells have a polynomial cycle depth stress function concluded from lab tests \([17]\): \( \Phi(u) = (5.24 \times 10^{-4})u^{2.03} \), and the cell replacement price is set to 300 $/kWh. Suppose the battery power rating is 1MW and energy rating is 1MWh.

**B. Optimality Gap**

We simulate regulation using random generated regulation traces to exam the optimality of the proposed policy and to validate Theorem 2, 1, and 3. We generate 100 regulation signal traces assuming a uniform distribution between \([-1, 1]\), and design nine test cases. Each test case has different market prices and battery round-trip efficiency \( \eta = \eta_d \eta_c \). In order to demonstrate the time-invariant property of the optimality gap,
### Simulation with Random Generated Regulation Signals.

| Case | $\theta$ [$/\text{MWh}$] | $\pi$ [$/\text{MWh}$] | $\eta$ [%] | $\hat{u}$ [%] | $\epsilon$ [$/\text{MWh}$] | Maximum optimality gap [$/\text{MWh}$] | Average objective value [$/\text{MWh}$] |
|------|----------------|----------------|--------|------------|----------------|----------------|----------------|
| 1 | 50 | 50 | 100 | 100 | 11.1 | 0.00 | 0.00 | 183.9 | 117.4 | 117.4 | 200.2 |
| 2 | 100 | 100 | 100 | 100 | 21.9 | 0.00 | 0.00 | 127.5 | 168.7 | 168.7 | 209.0 |
| 3 | 200 | 200 | 100 | 100 | 42.8 | 0.00 | 0.00 | 47.9 | 219.4 | 219.4 | 226.7 |
| 4 | 50 | 50 | 85 | 100 | 11.2 | 0.06 | 0.06 | 184.9 | 117.2 | 117.3 | 202.9 |
| 5 | 80 | 20 | 85 | 100 | 11.7 | 3.83 | 3.83 | 181.4 | 108.0 | 110.7 | 198.9 |
| 6 | 20 | 80 | 85 | 100 | 10.6 | 2.19 | 2.19 | 192.6 | 122.4 | 123.8 | 206.8 |
| 7 | 50 | 50 | 85 | 100 | 11.2 | 0.06 | 0.06 | 408.8 | 235.6 | 235.7 | 388.3 |
| 8 | 80 | 20 | 85 | 200 | 11.7 | 3.83 | 3.83 | 400.6 | 219.5 | 222.2 | 375.4 |
| 9 | 20 | 80 | 85 | 200 | 10.6 | 2.19 | 2.19 | 421.4 | 247.6 | 248.9 | 401.1 |

**Fig. 9:** Regulation operating cost break-down comparison between the proposed policy and the simple policy. Although the proposed policy has higher penalties, the cost of cycle aging is significantly smaller, so it achieves better trade-offs between degradation and mismatch penalty.

Table II shows the difference of computation time between the subgradient algorithm in Section V-D and a standard numerical solver implemented using fmincon in Matlab. It turns out that the latter does not converge for problem horizon of longer than 4 hours. All experiments conducted on a Macbook Pro with 2.5 GHz Intel Core i7, 16 GB 1600 MHz DDR3.

### Computation Time

| Time horizon (min) | 60 | 120 | 240 | 720 | 1440 |
|--------------------|----|-----|-----|-----|-----|
| Subgradient solving time(s) | 23.9 | 62.5 | 156.3 | 673.5 | 2522 |
| fmincon solving time(s) | 264 | 2006 | 29800 | ~ | ~ |

We repeat the simulation using different penalty prices. We set $\theta = \pi$ in each test case and set the round-trip efficiency to 85%. Table II summarizes the results.

### Conclusion and Future Work

We consider the optimal control of battery energy storage under a general “pay-for-performance” setup, where batteries need to trade-off between following instruction signals and the impact of degradation from charging and discharging actions. We show that under electrochemically accurate cycle-based degradation models, the battery control problem can be much as 4 times longer compared to the simple policy case.

VIII. CONCLUSION AND FUTURE WORK

We consider the optimal control of battery energy storage under a general “pay-for-performance” setup, where batteries need to trade-off between following instruction signals and the impact of degradation from charging and discharging actions. We show that under electrochemically accurate cycle-based degradation models, the battery control problem can be much as 4 times longer compared to the simple policy case.
formulated as a convex online optimization problem. Based on this result, we developed an online control policy that has a bounded time-invariant worst-case optimality gap, and is strictly optimal under certain market scenarios. From the case study in PJM regulation market, we verified the proposed degradation model and online control policy can significantly reduce operation cost and extend battery lifetime.

REFERENCES

[1] D. Rastler, *Electricity energy storage technology options: a white paper primer on applications, costs and benefits*. Electric Power Research Institute, 2010.

[2] B. Dunn, H. Kamath, and J.-M. Tarascon, “Electrical energy storage for the grid: a battery of choices,” *Science*, vol. 334, no. 6058, pp. 928–935, 2011.

[3] E. Bitar, R. Rajagopal, P. Khargonekar, and D. Kirschen, “The role of co-located storage for wind power producers in conventional electricity markets,” in *American Control Conference (ACC)*, 2011. IEEE, 2011, pp. 3886–3891.

[4] S. Shi, E. Xu, B. Zhang, and D. Wang, “Leveraging energy storage to optimize data center electricity cost in emerging power markets,” in *Proceedings of the Seventh International Conference on Future Energy Systems (e-Energy)*. ACM, 2016, p. 18.

[5] N. Li, L. Chen, and S. H. Low, “Optimal demand response based on utility maximization in power networks,” in *Power and Energy Society General Meeting*, 2011 IEEE. IEEE, 2011, pp. 1–8.

[6] H. David and S. Alfred, “Deployment of grid-scale batteries in the united states,” in *Prepared for Office of Energy Policy and Systems Analysis, U.S. Department of Energy*, 2016.

[7] D. P. Bertsekas, D. P. Bertsekas, D. P. Bertsekas, and D. P. Bertsekas, *Dynamic programming and optimal control*. Athena scientific Belmont, MA, 1995, vol. 1.

[8] B. Xu, Y. Shi, D. S. Kirschen, and B. Zhang, “Optimal regulation response of batteries under cycle aging mechanisms,” Accepted by IEEE CDC 2017, 2017.

[9] P. Arora, R. E. White, and M. Doyle, “Capacity fade mechanisms and side reactions in lithium-ion batteries,” *Journal of the Electrochemical Society*, vol. 145, no. 10, pp. 3647–3667, 1998.

[10] B. Xu, A. Oudalov, A. Ulbig, G. Andersson, and D. Kirschen, “Modeling of lithium-ion battery degradation for cell life assessment,” *IEEE Transactions on Smart Grid*, vol. PP, no. 99, pp. 1–1, 2016.

[11] E. Chemali, L. McCurlie, B. Howey, T. Stiene, M. M. Rahman, V. Muenzel, J. de Hoog, M. Brazil, A. Vishwanath, and S. Kalyanaraman, “A multi-factor battery cycle life prediction methodology for optimal battery management,” in *Proceedings of the 2015 ACM Sixth International Conference on Future Energy Systems*. ACM, 2015, pp. 57–66.

[12] T. Dragičević, H. Pandžić, D. Škrlec, I. Kuzle, J. M. Guerrero, and D. S. Kirschen, “Capacity optimization of renewable energy sources and battery storage in an autonomous telecommunication facility,” *IEEE Transactions on Sustainable Energy*, vol. 5, no. 4, pp. 1367–1378, 2014.

[13] M. Musallam and C. M. Johnson, “An efficient implementation of the rainbow counting algorithm for life consumption estimation,” *IEEE Transactions on Reliability*, vol. 61, no. 4, pp. 978–986, 2012.

[14] L. Xie, Y. Gu, A. Eskandari, and M. Ehsani, “Fast mpc-based coordination of wind power and battery energy storage systems,” *Journal of Energy Engineering*, vol. 138, no. 2, pp. 43–53, 2012.

[15] J. Qin, Y. Chow, J. Yang, and R. Rajagopal, “Distributed online modified greedy algorithm for storage control under uncertainty,” *IEEE Transactions on Smart Grid*, vol. 7, no. 2, pp. 1106–1118, 2016.

[16] J. H. Kim and W. B. Powell, “Optimal energy commitments with storage and intermittent supply: Operations research,” *IEEE Transactions on Power Systems*, vol. 30, no. 6, pp. 3122–3130, 2015.

[17] T. Borsche, A. Ulbig, M. Koller, and G. Andersson, “Power and energy capacity requirements of storages providing frequency control reserves,” in *IEEE PES General Meeting*, 2013.

[18] J. Qin, Y. Chow, J. Yang, and R. Rajagopal, “Online modified greedy algorithm for storage control under uncertainty,” *IEEE Transactions on Power Systems*, vol. 31, no. 3, pp. 1729–1734, 2016.

[19] H. Pandžić, Y. Wang, T. Qiu, Y. Dvorkin, and D. S. Kirschen, “Near-optimal method for siting and sizing of distributed storage in a transmission network,” *Power Systems*, IEEE Transactions on, vol. 30, no. 5, pp. 2288–2300, 2015.

[20] A. A. Akhil, G. Huff, A. B. Currier, B. C. Kaun, D. M. Rastler, S. B. Chen, A. L. Cotter, D. T. Bradshaw, and W. D. Gauntlett, *DOE/EPR 13 electricity storage handbook in collaboration with NRECA*. Sandia National Laboratories Albuquerque, NM, 2013.

[21] B. Zakeri and S. Syri, “Electrical energy storage systems: A comparative life cycle cost analysis,” *Renewable and Sustainable Energy Reviews*, vol. 42, pp. 569–596, 2015.

[22] M. Ecker, N. Nieto, S. Kubitza, J. Schmalstieg, H. Blanke, A. Warnecke, and D. U. Sauer, “Calendar and cycle life study of li (nmc) o 2-based 18650 lithium-ion batteries,” *Journal of Power Sources*, vol. 248, pp. 839–851, 2014.

[23] V. Ramadesigan, P. W. Northrop, S. De, S. Santhanagopalan, R. D. Braatz, and V. R. Subramanian, “Modeling and simulation of lithium-ion batteries from a systems engineering perspective,” *Journal of The Electrochemical Society*, vol. 159, no. 3, pp. R23–R45, 2012.

[24] “Northern arizona wind & sun inc. battery cycles v.s. lifespan plot.” [Online]. Available: https://www.solar-electric.com/learning-center/batteries-and-charging/deep-cycle-battery-faq.html#Cycles%20v%20Life

[25] S. D. Downing and D. Socie, “Simple rainfall counting algorithms,” *International Journal of Fatigue*, vol. 4, no. 1, pp. 31–40, 1982.

[26] I. Rychlik, “A new definition of the random cycle counting method,” *International Journal of Fatigue*, vol. 9, no. 2, pp. 119–121, 1987.

[27] C. Azzallag, J. Gery, J. Robert, and J. Bahauaud, “Standardization of the rainfall counting method for fatigue analysis,” *International Journal of Fatigue*, vol. 16, no. 4, pp. 287–293, 1994.

[28] V. Muenzel, H. de Hoog, M. Brazil, A. Vishwanath, and S. Kalyanaraman, “A multi-factor battery cycle life prediction methodology for optimal battery management,” in *Proceedings of the 2015 ACM Sixth International Conference on Future Energy Systems*. ACM, 2015, pp. 57–66.

[29] T. Dragičević, H. Pandžić, D. Škrlec, I. Kuzle, J. M. Guerrero, and D. S. Kirschen, “Capacity optimization of renewable energy sources and battery storage in an autonomous telecommunication facility,” *IEEE Transactions on Sustainable Energy*, vol. 5, no. 4, pp. 1367–1378, 2014.

[30] M. Musallam and C. M. Johnson, “An efficient implementation of the rainbow counting algorithm for life consumption estimation,” *IEEE Transactions on Reliability*, vol. 61, no. 4, pp. 978–986, 2012.

[31] L. Xie, Y. Gu, A. Eskandari, and M. Ehsani, “Fast mpc-based coordination of wind power and battery energy storage systems,” *Journal of Energy Engineering*, vol. 138, no. 2, pp. 43–53, 2012.

[32] J. Qin, Y. Chow, J. Yang, and R. Rajagopal, “Distributed online modified greedy algorithm for networked storage operation under uncertainty,” *IEEE Transactions on Smart Grid*, vol. 7, no. 2, pp. 1106–1118, 2016.

[33] J. H. Kim and W. B. Powell, “Optimal energy commitments with storage and intermittent supply: Operations research,” *IEEE Transactions on Power Systems*, vol. 30, no. 6, pp. 1347–1360, 2011.

[34] Y. Xu and L. Tong, “On the value of storage at consumer locations,” in *PES General Meeting—Conference & Exposition, 2014 IEEE*. IEEE, 2014, pp. 1–5.

[35] P. M. van de Ven, N. Hegde, L. Massoulie, and T. Salonidis, “Optimal control of end-user energy storage,” *IEEE Transactions on Smart Grid*, vol. 4, no. 2, pp. 789–797, 2013.

[36] J. L. Melo, T. J. Lim, and S. Sun, “Online demand response strategies for non-deferrable loads with renewable energy,” *IEEE Transactions on Smart Grid*, 2017.

[37] V. Pop, H. J. Bergveld, D. Danilov, P. P. Regtien, and P. H. Notten, *Battery management systems: Accurate state-of-charge indication for battery-powered applications*. Springer Science & Business Media, 2008, vol. 9.

[38] M. R. Almassalkhi and I. A. Hiskens, “Model-predictive cascade mitigation in electric power systems with storage and renewablespart i: Theory and implementation,” *IEEE Transactions on Power Systems*, vol. 30, no. 1, pp. 67–77, 2015.

[39] J. Zhang and A. Domínguez-García, “On the impact of measurement errors on power system automatic generation control,” *IEEE Transactions on Smart Grid*, 2016.
Proposition 2. Let $g(\cdot)$ be a convex function where $g(0) = 0$. Let $r_1, r_2$ be positive real numbers, and $r_1 \geq r_2$. Then
$$g(r_1 - r_2) \leq g(r_1) - g(r_2),$$
(33)

Proof. By Proposition 1
$$g(\alpha + \beta) \geq g(\alpha) + g(\beta), \forall \alpha, \beta > 0,$$

Let $\alpha = r_1 - r_2 > 0$, $\beta = r_2 > 0$, so that
$$g(r_1 - r_2 + r_2) \geq g(r_1 - r_2) + g(r_2).$$

Proposition 3. Let $g(\cdot)$ be a convex function where $g(0) = 0$. Let $\lambda r_1 \geq r_2 > 0$ be positive real numbers. Then
$$g\left(\frac{1}{2} r_1 - \frac{1}{2} r_2\right) \leq \frac{1}{2} g(r_1) - \frac{1}{2} g(r_2),$$
(34)

Proof. From Proposition 2
$$g\left(\frac{1}{2} r_1 - \frac{1}{2} r_2\right) \leq \frac{1}{2} g(r_1) - \frac{1}{2} g(r_2), \forall r_1 \geq r_2 > 0$$

Therefore, it suffices to show
$$\frac{1}{2} g(r_1) - \frac{1}{2} g(r_2) \geq \frac{1}{2} g(r_1) - \frac{1}{2} g(r_2),$$

Define $h(z) = g\left(\frac{1}{2} z\right) - \frac{1}{2} g(z)$,
$$h'(z) = \frac{1}{2} g'\left(\frac{1}{2} z\right) - \frac{1}{2} g(z) = \frac{1}{2} g'\left(\frac{1}{2} z\right) - g'(z) < 0,$$
$h(\cdot)$ is a monotone decreasing function. For $r_1 \geq r_2 > 0$,
$$h(r_1) \leq h(r_2),$$

$$\frac{1}{2} g(r_1) - \frac{1}{2} g(r_2) \leq \frac{1}{2} g(r_1) - \frac{1}{2} g(r_2).$$

If $g(\cdot)$ is continuous, we can generalize the midpoint property to
$$g(\lambda r_1 - (1-\lambda) r_2) \leq \lambda g(r_1) - (1-\lambda) g(r_2), \forall \lambda r_1 \geq (1-\lambda) r_2 > 0.$$

Proposition 4. Let $g(\cdot)$ be a convex function where $g(0) = 0$. Let $r_1, r_2, r_3$ be positive real numbers, which satisfy that $r_1 + r_2 - r_3 \geq 0$, and $r_1 \leq r_1 + r_2 - r_3, \forall i \in [1, 2, 3]$. Then
$$g(r_1 + r_2 - r_3) \geq g(r_1) + g(r_2) - g(r_3),$$
(35)

Proof. From $r_1 \leq r_1 + r_2 - r_3$ we have $r_2 \geq r_3$. From $r_2 \leq r_1 + r_2 - r_3$, we have $r_1 \geq r_3$

Let’s further assume $r_1 \geq r_2$,
$$g(r_1 + r_2 - r_3) - g(r_1) = (r_2 - r_3) \cdot g'(\theta_1), \theta_1 \in [r_1, r_1 + r_2 - r_3],$$
$$g(r_2) - g(r_3) = (r_2 - r_3) \cdot g'(\theta_2), \theta_2 \in [r_3, r_2].$$

Since $g(\cdot)$ is a convex function, for $\theta_2 \leq r_2 \leq r_1 \leq \theta_1$, we have $g'(\theta_2) \leq g'(\theta_1)$. Therefore,
$$g(r_2) - g(r_3) \leq g(r_1 + r_2 - r_3) - g(r_1),$$
$$g(r_1 + r_2 - r_3) \geq g(r_1) + g(r_2) - g(r_3).$$
If \( r_1 < r_2 \), similarly we have
\[
g(r_1) - g(r_3) \leq g(r_1 + r_2 - r_3) - g(r_2),
\]
\[
g(r_1 + r_2 - r_3) \geq g(r_1) + g(r_2) - g(r_3).
\]

**Proposition 5.** Let \( g(\cdot) \) be a convex function where \( g(0) = 0 \). Let \( r_1, r_2, r_3, ..., r_n \) be real numbers, suppose

- \( \sum_{i=1}^{n} r_i = D > 0 \)
- \( |r_i| \leq D, \forall i \in \{1, 2, 3, ..., n\} \)

Then,
\[
g\left(\sum_{i=1}^{n} r_i\right) \geq \sum_{i:r_i \geq 0} g(r_i) - \sum_{i:r_i < 0} g(|r_i|). \tag{36}
\]

**Proof.** (1) If all \( r_n \)'s are positive, it is trivial to show \( g\left(\sum_{i=1}^{n} r_i\right) \geq \sum_{i} g(r_i) \) by Proposition 4.8

(2) If \( r_n \) contains both positive and negative numbers, we order them in an ascending order and renumber them as,
\[
r_1 \leq r_2 \leq ... \leq 0 \leq ... \leq r_n,
\]
Pick \( r_1 \) (the most negative number), and find some positive \( r_i, r_{i+1} \) such that
\[
r_{i+1} \geq r_i \geq |r_1| > 0,
\]
Applying Proposition 4.8 we have
\[
g(r_{i+1}+r_i+r_1) = g(r_{i+1}+r_i-|r_1|) \geq g(r_{i+1})+g(r_i)-g(|r_1|),
\]
Note, if we can not find such \( r_i, r_{i+1} \), eg. \( r_n \leq |r_1| \). We could group a bunch of positive \( r_i \)'s to form two new variables \( s_1 = \sum_{i \in N_1} r_i, s_2 = \sum_{i \in N_2} r_i \) where \( N_1 \cap N_2 = \emptyset \). For sure there exists such \( s_1 \geq s_2 \geq |r_1| \), since
\[
\left|\sum_{i=1}^{n} r_i\right| = \left| \sum_{i:r_i \geq 0} r_i + \sum_{j:r_j < 0, j \neq 1} r_j + r_1 \right|
= \sum_{i:r_i \geq 0} r_i - \left| \sum_{j:r_j < 0, j \neq 1} r_j - |r_1| \right|
= D \sum_{i:r_i \geq 0} r_i = \left| \sum_{j:r_j < 0, j \neq 1} r_j + |r_1| + D \geq 2|r_1| \right|
\]
Applying Proposition 4.8
\[
g(s_1 + s_2 + r_1)
= g(s_1 + s_2 - |r_1|)
\geq g(s_1) + g(s_2) - g(|r_1|)
\geq \sum_{i \in N_1} g(r_i) + \sum_{j \in N_2} g(r_j) - g(|r_1|), N_1 \cap N_2 = \emptyset
\]
Define \( r'_i = r_{i+1} + r_i + r_1 > 0 \) and re-order \( r'_1, r'_2, r'_3, ..., r'_{i-1}, r'_{i+1}, ..., r_n \). Or define \( r_1 = y_1 + y_2 + r_1 > 0 \), re-order \( \{ r_i : i \neq 1, i \notin N_1 \cup N_2 \}, r_1 \). Repeat the above steps till all \( r'_i \) are positive finishes the proof.

**Proposition 6.** Consider a step change added to \( x \), where \( x(t) = x(t) + Q_t U_t, t \in [0, T] \). Suppose \( Q_t \) is positive
\[
\text{then the rainflow cycle decomposition results (only considering charging cycles) for } x \text{ and } x' \text{ are,}
\]
\[
x : v_1, v_2, ..., v_M, 0, 0, ..., \]
\[
x' : v'_1, v'_2, ..., v'_N, 0, 0, ..., \]
Define \( L = \max(M, N) \), we could re-write the cycles in \( x \) and \( x' \) as,
\[
x : v_1, v_2, ..., v_M, 0, 0, ..., \]
\[
x' : v'_1, v'_2, ..., v'_N, 0, 0, ..., \]
Define \( \Delta v_i \) such that, \( v'_i = v_i + \Delta v_i, \forall i = 1, 2, ..., L \)
The following relations always holds,
\[
\left| \sum_{i=1}^{L} \Delta v_i \right| \leq Q_t, \tag{37}
\]
\[
|\Delta v_i| \leq Q_t, \tag{38}
\]

**Proof.** There exists a small enough \( \Delta Q \) such that only one cycle depth \( v_i \) will change: \( |\Delta v_i| \leq \Delta Q \) and \( -\Delta Q \leq \Delta v_i \leq \Delta Q \).
Consider \( Q_t \) as a cumulation of small \( \Delta Q \), by the principle of integration, we have
\[
-\int \Delta Q dq \leq \sum_{i=1}^{L} \Delta v_i \leq \int \Delta Q dq ,
\]
Such that,
\[
\left| \sum_{i=1}^{L} \Delta v_i \right| \leq Q_t
\]
\[
|\Delta v_i| \leq Q_t \text{ holds for the worst case where all cycle depth changes happen at one certain cycle. Therefore, it is trivial to show that } |\Delta v_i| \leq Q_t \text{ hold in all conditions.}
\]

By propositions 1-6, we get the proof of Lemma 1 below.

**Proof.** Let’s consider \( x' = \lambda x + (1 - \lambda)Q_t U_t \). Then the rainflow cycle decomposition results for \( \lambda x \) and \( x' \) are
\[
\lambda x : \lambda v_1, \lambda v_2, ..., \lambda v_M, 0, 0, ..., \]
\[
x' : v'_1, v'_2, ..., v'_N, 0, 0, ..., \]
Define \( \Delta v_i \) such that,
\[
v'_i = v_i + (1 - \lambda)\Delta v_i, \forall i = 1, 2, ..., L
\]
\[\text{5The proof for negative } Q_t \text{ is the same, just change } Q_t \text{ to } |Q_t|\]
\[ f(\lambda x + (1 - \lambda)Q_t U_t) \]
\[ = \sum_{i=1}^{L} \Phi(\lambda v_i + (1 - \lambda)\Delta v_i) \]
\[ = \sum_{i=1}^{L^+} \Phi(\lambda v_i + (1 - \lambda)\Delta v_i) + \sum_{i=1}^{L^-} \Phi(\lambda v_i - (1 - \lambda)|\Delta v_i|) \]
\[ \leq \sum_{i=1}^{L^+} [\lambda \Phi(v_i) + (1 - \lambda)\Phi(\Delta v_i)] + \sum_{i=1}^{L^-} \lambda \Phi(v_i) - (1 - \lambda)\Phi(|\Delta v_i|) \]
\[ \leq \lambda \sum_{i=1}^{L} \Phi(v_i) + (1 - \lambda)\Phi(Q_t) \] (39)

To continue the proof in (39), and derive the final relation, we separate the whole variable space to two cases based on equations (37) and (38).

1. Assume \( \sum_{i=1}^{L} \Delta v_i = Q_t, |\Delta v_i| \leq Q_t. \) By Proposition 5, it follows that

\[ f(\lambda x + (1 - \lambda)Q_t U_t) \]
\[ \leq \lambda \sum_{i=1}^{L} \Phi(v_i) + (1 - \lambda)|\sum_{i=1}^{L} \Phi(\Delta v_i) - \sum_{i=1}^{L^-} \Phi(\Delta v_i)| \]
\[ \leq \lambda \sum_{i=1}^{L} \Phi(v_i) + (1 - \lambda)\Phi(Q_t) \]

2. Assume \(-Q_t \leq \sum_{i=1}^{L} \Delta v_i < Q_t, |\Delta v_i| \leq Q_t.\)

Add some “virtual cycles” \( v'_{L+1}, v'_{L+2}, ..., v'_{L+K} \) at the end of \( x' \), each \( v'_{L+i} \) is positive and satisfies that \( |v'_{L+i}| \leq Q_t \). So that \( \sum_{i=1}^{L+K} \Delta v_i = Q_t, |\Delta v_i| \leq Q_t, \forall i \in [1, 2, ..., L + K]. \) Write 0 at the end of \( \lambda x \) to achieve the same cycle number.

\[ \lambda x : \lambda v_1, \lambda v_2, ..., \lambda v_M, 0, 0, 0, ..., 0 \]
\[ x' : v_1', v_2', ..., v_N', 0, 0, 0, 0, v'_{L+1}, v'_{L+2}, ..., v'_{L+K} \]

\[ f(\lambda x + (1 - \lambda)Q_t U_t) \]
\[ \leq \lambda \sum_{i=1}^{L} \Phi(v_i) + (1 - \lambda)|\sum_{i=1}^{L} \Phi(\Delta v_i) - \sum_{i=1}^{L+K} \Phi(\Delta v_i)| \]
\[ \leq \lambda \sum_{i=1}^{L} \Phi(v_i) + (1 - \lambda)\Phi(Q_t) \]

To sum up,

\[ f(\lambda x + (1 - \lambda)Q_t U_t) \leq \lambda \sum_{i=1}^{L} \Phi(v_i) + (1 - \lambda)\Phi(Q_t) = \lambda f(x) + (1 - \lambda)f(Q_t U_t), \] (40)

where \( \lambda \in [0, 1]. \)

Lemma 1 shows that \( f(x) \) is convex up to every step change in \( x \). Next, we will prove the general rainflow convexity by induction.

B. General rainflow cycle life loss convexity

We will prove the general rainflow convexity by induction. By lemma 1, we already proved the base case convexity. When \( K = 1, \)

\[ f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \lambda \in [0, 1] \]

Next we need to show the induction relation. Suppose that, \( f(x) \) is convex up to the sum of \( K \) step changes (arranged by time index)

\[ f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \lambda \in [0, 1], x, y \in \mathbb{R}^K \]

Then we prove \( f(x) \) is convex up to the sum of \( K + 1 \) step changes (see Fig. 10).

\[ f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \lambda \in [0, 1], x, y \in \mathbb{R}^{K+1} \]

The following proposition is needed for the proof.

\textbf{Proposition 7.}

\[ f(\sum_{t=1}^{K} P_t U_t) \geq f(\sum_{t=1}^{i-1} P_t U_t + (P_i + P_{i+1})U_t + \sum_{t=i+2}^{K} P_t U_t), \]

\[ (41) \]

In other words, the cycle stress cost will reduce if combining adjacent unit changes.

\textbf{Proof.} The rainflow cycle counting algorithm only considers local extreme points.

I) If \( P_i \) and \( P_{i+1} \) are the same direction, combining them doesn’t affect the value of local extreme points. Therefore the left side cost equals right side cost.

II) If \( P_i \) and \( P_{i+1} \) are in different directions, suppose \( P_i \) is negative and \( P_{i+1} \) positive (otherwise the same). Time \( t = i \) makes a local minimum point.

- Case a: If \( |P_{i+1}| \leq |P_i| \), combining them will raise the value of local minimum point \( i \), thus reducing the
depth of cycles which contains \( i \). Therefore, the cost after combining is less than the original cost.

- Case b: If \(| P_{i+1} | > | P_i |\), combining them will lead to the removal of local minimum point \( i \).
  In one case, if \( P_{i-1} \) and \( P_i \) are the same direction, time \( t = i - 1 \) will make a local minimum point taking the place of time \( t = i \). Therefore, the magnitude of the local minimum point decreases, similar to case (a), the total cost after combining is less than the original cost.
  In the other case, if \( P_{i-1} \) and \( P_i \) are different directions, we lose a full cycle with depth \(| P_i |\) after combining. So the cost after combining \( P_i \), \( P_{i+1} \) is also less than the original.

Recall the step function decomposition results for \( x \), \( y \) and \( \lambda x + (1 - \lambda) y \), where

\[
x = \sum_{t=1}^{T} P_t U_t, \quad y = \sum_{t=1}^{T} Q_t U_t,
\]

\[
\lambda x + (1 - \lambda) y = \sum_{t=1}^{T} Z_t U_t,
\]

There are three cases when \( T \) goes from \( K \) to \( K+1 \), classified by the value and signs of \( Z_K \), \( Z_{K+1} \).

**Case 1:** \( Z_K \) and \( Z_{K+1} \) are same direction.

If \( Z_{K+1} \) and \( Z_K \) are same direction, we could move \( Z_{K+1} \) to the previous step without affecting the total cost \( f(x + (1 - \lambda) y) \). Then we prove the \( K+1 \) convexity by applying Proposition [7]

\[
f(\lambda x + (1 - \lambda) y) = f(\lambda x^K + (1 - \lambda) y^K + Z_{K+1} U_K) \leq \lambda f(x^K + P_{K+1} U_{K+1} + (1 - \lambda) f(y^K + Q_{K+1} U_{K+1}) \leq \lambda f(x) + (1 - \lambda) f(y) \quad \text{(42)}
\]

where \( x^K \) and \( y^K \) denote the first \( K \) elements of \( x \) and \( y \).

**Case 2:** \( Z_K \) and \( Z_{K+1} \) are different directions, with \( |Z_K| \geq |Z_{K+1}| \). In this case, the last step \( Z_{K+1} \) could be separated out from the previous SoC profile. Therefore,

\[
f(\lambda x + (1 - \lambda) y) = f(\lambda x^K + (1 - \lambda) y^K + \Phi(Z_{K+1} U_{K+1})) \leq \lambda f(x^K + P_{K+1} U_{K+1} + (1 - \lambda) f(y^K + Q_{K+1} U_{K+1}) \leq \lambda f(x) + (1 - \lambda) f(y) \quad \text{(43)}
\]

**Case 3:** \( Z_K \) and \( Z_{K+1} \) are different directions, with \( |Z_K| < |Z_{K+1}| \). In such condition, \( Z_{K+1} \) is not easily separated out from previous SoC. To derive the induction relation, we analyze in three further sub-cases.

- **Case 3a:** \( Z_{K+1} \) and \( Z_K \) are the same direction. In this sub-case, we could use the same "trick" in Case 1 to combine step \( K - 1 \) and \( K \). Proof is trivial for this case.
- **Case 3b:** \( Z_{K+1} \) and \( Z_K \) are different directions, while \( Z_K \) and \( Z_{K+1} \) together form a cycle that is separable from the rest of signal (eg. it is the deepest cycle). We can separate \( Z_{K+1} \) out, and proof will be similar to Case 2.
- **Case 3c:** \( Z_{K-1} \) and \( Z_K \) are different directions, and \( Z_K \), \( Z_{K+1} \) do not form a separate cycle. This case is the most complicated case, since it's hard to move \( Z_{K+1} \) to the previous step, or separate it out. Therefore, we need to look into \( P_K \), \( P_{K+1} \), \( Q_K \), \( Q_{K+1} \) in order to show the \( K+1 \) step convexity. It contains four more situations (Fig. 11), for simplicity we only consider the cost of charging cycles. Showing convexity for each situation finishes the overall convexity proof.

Case a) \( P_K \) and \( P_{K+1} \) are in different directions, with \(|P_K| < |P_{K+1}|\). \( Q_K \) and \( Q_{K+1} \) are also in different directions, with \(|Q_K| < |Q_{K+1}|\). In such condition, the extra charging half cycle \( \Delta_{K+1} \) could be decomposed into two charging half cycles in \( x \) and \( y \) respectively.

\[
f(\lambda x + (1 - \lambda) y) = f(\lambda x^K + (1 - \lambda) y^K + Z_{K+1} U_{K+1}) \leq \lambda f(x^K + P_{K+1} U_{K+1} + (1 - \lambda) f(y^K + Q_{K+1} U_{K+1}) + \Phi(\Delta_{K+1}) \leq \lambda f(x) + (1 - \lambda) f(y) + \Phi(\Delta_{P,K+1} + (1 - \lambda) \Delta_{Q,K+1}) \leq \lambda f(x) + (1 - \lambda) f(y) + \Phi(\Delta_{P,K+1}) + \Phi(\Delta_{Q,K+1}) \leq \lambda f(x) + (1 - \lambda) f(y) \quad \text{(44)}
\]

Case c) \( P_K \) and \( P_{K+1} \) are in the same direction. \( Q_K \) and
Given real numbers, because \( P_{K+1} \) is a small separation cycle, so that \( v_{N-1} \) is the charging cycle that the \( K+1 \) step of \( v_N \), denoted as \( y(K+1) \) belongs to. By assumption that \( Q_K \) and \( Q_{K+1} \) do not form a separate cycle, so that \( v_{N-1} \geq Q_{K+1} \). By assumption, \( \frac{\lambda}{1-\lambda}(\bar{P}_K - P_{K+1}) > 0 \). Let \( \delta = \frac{1}{1-\lambda}(\bar{P}_K - P_{K+1}) \), and since \( P_{K+1} + Q_{K+1} \geq \bar{P}_K + Q_K \) in case b), \( Q_K + \delta \leq Q_{K+1} \leq v_{N-1} \). Therefore applying Proposition 8, \( a = v_{N-1} \) and \( b = Q_K, \delta = \frac{1}{1-\lambda}(P_K - P_{K+1}) \), we have the desired result.

\[ f(y^K + (Q_{K+1} - \frac{\lambda}{1-\lambda}(\bar{P}_K - P_{K+1}))U_{K+1}) \leq f(y) \]

We can write out the cost of charging cycles in \( y \) as \( f(y) \),

\[ f(y) = \sum_{i=1}^{N-1} \Phi(v_i) + \Phi(\bar{Q}_K), \]

where \( v_{N-1} \) is the charging cycle that the \( K+1 \) step of \( v_N \), denoted as \( y(K+1) \) belongs to.

By assumption that \( Q_K \) and \( Q_{K+1} \) do not form a separate cycle, so that \( v_{N-1} \geq Q_{K+1} \). By assumption, \( \frac{\lambda}{1-\lambda}(\bar{P}_K - P_{K+1}) > 0 \). Let \( \delta = \frac{1}{1-\lambda}(\bar{P}_K - P_{K+1}) \), and since \( P_{K+1} + Q_{K+1} \geq \bar{P}_K + Q_K \) in case b), \( Q_K + \delta \leq Q_{K+1} \leq v_{N-1} \). Therefore applying Proposition 8, \( a = v_{N-1} \) and \( b = Q_K, \delta = \frac{1}{1-\lambda}(P_K - P_{K+1}) \), we have the desired result.

\[ f(y^K + (Q_{K+1} - \frac{\lambda}{1-\lambda}(\bar{P}_K - P_{K+1}))U_{K+1}) \leq f(y) \]

Since \( Q_{K+1} = \bar{Q}_K + \delta_{Q,K+1} \), we re-write the above inequality as,

\[ f(y^K + (Q_{K+1} - \delta_{Q,K+1} - \frac{\lambda}{1-\lambda}(\bar{P}_K - P_{K+1}))U_{K+1}) \leq f(y) \]
Denote $\delta = \delta_{Q,K+1} - \frac{\lambda}{1-\lambda}(\hat{P}_K - P_{K+1}) > 0$

$$f(y) = \sum_{i=1}^{N-2} \Phi(v_i) + \Phi(v_{N-1}) + \Phi(Q_{K+1}),$$

$v_{N-1}$ is the deepest charging cycle where its ending SoC equals to $Q_K$’s starting SoC, and $v_{N-1} \leq Q_{K+1}$ since $Q_{K+1}$ forms a separate cycle. Applying Proposition 8 by setting $a = Q_{K+1}$, $b = v_{N-1}$, we have the desired result.

**APPENDIX**

**Proof of Theorem [3]**

### A. Model Reformulation

Both Theorem 2 and Corollary 1 follows directly from Theorem 3. To prove Theorem 2 we rewrite the optimization problem (8) in Section III.E as,

$$(c^*, d^*) \in \arg\min_{c,d} f(c, d) - \tau \sum_{t=1}^{T} [\theta c_t + \pi d_t] \quad (47a)$$

subject to (8b), (8c), and

$$0 \leq c_t \leq [r_c]^+ \quad (47b)$$

$$0 \leq d_t \leq [-r_d]^+ \quad (47c)$$

by observing that a battery’s actions would never exceed the regulation signals. $f(c, d)$ defines the rainflow cycle-based degradation cost.

We utilize the rainflow algorithm to transform the problem into a cycle-based form. The rainflow method maps the entire operation uniquely to cycles, the sum of all charge and discharge power can be represented as the sum of cycle depths as (recall that a full cycle has symmetric depth for charge and discharge)

$$\sum_{i=1}^{u}\sum_{j=1}^{v} u_i + \sum_{i=1}^{w}\sum_{j=1}^{v} v_i = \frac{\tau \eta_c}{E} \sum_{t=1}^{T} c_t \quad (48)$$

$$\sum_{i=1}^{u}\sum_{j=1}^{w} u_j + \sum_{i=1}^{w}\sum_{j=1}^{w} v_i = \frac{\tau \eta_d}{E} \sum_{t=1}^{T} d_t . \quad (49)$$

We substitute (48) and (49) into the reformulated objective function (47a) to replace $c_t$ and $d_t$ with cycle depths

$$J_{cycle}(c, d) + J_{reg}(c, d, r) = \sum_{i=1}^{u} J_u(u_i) + \sum_{i=1}^{v} J_v(v_i) + \sum_{i=1}^{w} J_w(w_i). \quad (50)$$

### B. Proof for Theorem [3]

The following lemmas support the proof for Theorem 3.

**Lemma 2.** Suppose an minimizer $(c^*, d^*)$ of (8) in the offline setting has the corresponding cycle depths $(u^*, v^*, w^*)$. Then the depth of each cycle in this result either reaches the optimal cycle depth or bounded by the operation constraints as

$$u_i^* = \min(\hat{u}, \pi_u) \quad (51a)$$

$$v_i^* = \min(\hat{v}, \pi_v) \quad (51b)$$

$$w_i^* = \min(\hat{w}, \pi_w) \quad (51c)$$

where $\pi_u$, $\pi_v$, $\pi_w$ denote constraint bounds including the regulation instruction signal and battery energy limit.

**Lemma 3.** A cycle depth in the control action of $g(\cdot)$ either reaches the depth of $\hat{u}$ or is bounded by the operation constraints.

**Lemma 4.** There exists one and only one half cycle with the largest depth in a rainflow residue profile. Other half cycles are in strictly decreasing order either to the left- or to the right-hand side direction of this largest half cycle.

It is easy to see now from Lemma 2 and Lemma 3 that the proposed control policy achieves optimal control result for all full cycles, and the optimality gap is caused by half cycle results. Consider the following relationship in a rainflow residue profile as in Lemma 4 assuming the largest half cycle is in the discharging direction

$$\ldots < u_{j-1}^* < v_{j-1}^* < u_j^* < v_j^* > w_{j+1}^* > \ldots \quad (52)$$

and substitute Lemma 3 into (52)

$$\ldots \min\{\hat{v}, \pi_j\} < \min\{\hat{w}, \pi_{j+1}\} \ldots \quad (53)$$

It is easy to see now that if $\hat{w} > \hat{v}$, then the largest possible value for $u_j^*$ is $\hat{w}$, and the largest possible value for $v_j^*$ and $v_{j-1}^*$ is $\hat{v}$, the rest half cycles in (52) must have depths smaller than $\hat{v}$, which indicates that their depths are bounded by operation. If $\hat{v} > \hat{w}$, then the largest possible value for $w_j^*$ is $\hat{w}$, and the rest half cycles must have depths smaller than $\hat{w}$. We repeat this analysis for cases that $v_j^*$ is the largest cycle, and summarize the half cycle conditions in Table III. Hence, the worst-case optimality gap is caused by that some half cycles have depth $\hat{u}$ or $\hat{w}$, while the control policy enforces $\hat{u}$ as the depth of all cycles unbounded by operation. The gap in Theorem 3 is therefore calculated using half cycle depth conditions in Table III.

**Proof for Lemma 2** Since cycles are linear combinations of charge and discharge power, and constraints (47b), (47c),

![Fig. 12: Illustration for Lemma 4](image-url)

The largest half cycle is between $s_4$ and $s_5$, other half cycles are in strictly decreasing order either to the left- or to the right-hand side direction of this largest half cycle.
can be transformed into linear constraints with respect to cycle depths. From Theorem 1, the transformed cycle-based problem is also has a convex objective function with linear constraints. Although exact formulations of the transformed constraints are complicated to express, we use $\pi_i$, $\nu_i$, and $\omega_i$ to denote these binds, which are sufficient for the proof for Theorem 3.

**Proof for Lemma 3** The rainflow method always identify the largest cycle as between the minimum and the maximum SoC point. In the proposed policy, any operation that goes outside the defined operation zone will cause the largest cycle depth to change instead of the depth of the cycle it was previous in. For example, in Fig. 13 the maximum cycle is between SoC $s$ and $s + u$, and the battery is at time $t_4$. If the battery continue to charge and the SoC goes about $s + u$, then this operation will increase the largest cycle depth instead of the shallower cycles assoicated with extrema $s_2$, $s_3$ and $s_4$.

**Proof for Lemma 4** Because the rainflow method identifies a cycle from extrema distances if $\Delta s_{i-1} \geq \Delta s_i \leq \Delta s_{i+1}$, then all extrema in the rainflow residue must satisfy either $\Delta s_{i-1} < \Delta s_i < s_{i+1}$ or $\Delta s_{i-1} < s_i < s_{i+1}$ or $\Delta s_{i-1} > \Delta s_i > s_{i+1}$, which proofs this lemma.