Partial-Approximate Controllability of Nonlocal Fractional Evolution Equations via Approximating Method

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In this paper we study partial-approximate controllability of semilinear nonlocal fractional evolution equations in Hilbert spaces. By using fractional calculus, variational approach and approximating technique, we give the approximate problem of the control system and get the compactness of approximate solution set. Then new sufficient conditions for the partial-approximate controllability of the control system are obtained when the compactness conditions or Lipschitz conditions for the nonlocal function are not required. Finally, we apply our abstract results to the partial-approximate controllability of the semilinear heat equation and delay equation.

1. Introduction

Controllability concepts are important properties of a control system that plays a substantial role in many engineering problems, such as stabilizing unstable systems using feedback control. Therefore, in recent years, controllability problems for various types of linear and semilinear (fractional) dynamical systems have been studied in many articles (see [1]-[31] and the references therein). From the mathematical point of view it is necessary to distinguish between problems of exact and approximate controllability. Exact controllability allows to move the system to an arbitrary final state, while approximate controllability means that the system can be moved to an arbitrary small neighborhood of the final state. In particular, approximately controlled systems are more predominant, and very often the approximate controllability is quite adequate in applications. There are some interesting and important approximate controllability results for semilinear evolution systems in abstract spaces with a Caputo fractional derivative. Sakthivel et. al. [23] initiated to study approximate controllability of fractional differential systems in Hilbert spaces. Meanwhile, Sakthivel and Ren [24], Debbouche and Torres [8], Mahmudov [15], [16] pay attention to the study of approximate controllability for various types of (fractional) evolution systems in abstract spaces. On the other hand, existence and approximate controllability of nonlocal fractional equations were studied in [11], [12], [14], [17].

This paper is devoted to the study of partial-approximate controllability of the semilinear fractional evolution equation with nonlocal conditions,

\[ C_{D}^{q}y(t) = Ay(t) + Bu(t) + f(t, y(t)), \quad 0 < t \leq T, \]
\[ y(0) = y_0 - g(y), \]  

where the state variable \( y(\cdot) \) takes values in the Hilbert space \( X \), \( C_{D}^{q} \) is the Caputo fractional derivative of order \( q \) with \( \frac{1}{2} < q \leq 1 \), \( A : D(A) \subset X \to X \) is a family of closed and bounded linear operators generating a strongly continuous semigroup \( S : [0, b] \to \mathcal{L}(X) \), where the domain \( D(A) \subset X \) is dense in \( X \), the control function \( u(\cdot) \) is given in \( L^{2}([0, b], U) \), \( U \) is a Hilbert space, \( B \) is a bounded linear operator from \( U \) into \( X \), \( f : [0, b] \times X \to X \), \( g : C([0, b], X) \to X \) are given functions satisfying some assumptions to be specified later and \( y_0 \) is an element of the Hilbert space \( X \).

One of the purposes of this article is to investigate the partial-approximate controllability of the system \( (1) \) without Lipschitz continuous or compact assumptions on the non-local term \( g \). In fact, \( g \) is supposed to be continuous and is completely defined by \( [\delta, b] \) for some small \( \delta > 0 \). Meanwhile, in order to obtain the existence of solutions of the control system \( (1) \), we construct an approximate problem of the system \( (1) \) and obtain the compactness of the set of approximate solutions. It differs from the usual approach
that the fixed-point theorem is applied directly to the corresponding solution operator. Our results, therefore, can be seen as the extension and development of existing results.

It has not yet been reported work on the partial-approximate controllability of fractional semilinear evolution equations with nonlocal conditions. Inspired by the aforementioned recent contributions, we propose to discuss the partial-approximate controllability of fractional evolution systems in Hilbert spaces with classical nonlocal conditions. We first impose a partial-approximate controllability of the associated linear system. Then we develop a variational approach in [17], [16] and approximating method in [14], and rewrite our control problem as a sequence of fixed point problems. Next using the Schauder fixed point theorem we get the existence of fixed points and show that these solutions (fixed points) steers the system to an arbitrary small neighborhood of the final state in a closed subspace.

We organize the article as follows. In Section 2, we provide preliminaries, assumptions and formulate the main result on partial-approximate controllability. In Section 3, we extend the variational method to construct approximating control for the approximating controllability problem and use the Schauder fixed point theorem to show existence of a solution for the approximating controllability problem. The partial-approximate controllability result is proved in Section 4. In this section we solve the difficulty concerning the compactness of solution operator by means of approximate solution set constructed in Section 3. The difficulty is due to the fact that a compact semigroup $S(t)$ at $t = 0$ by means of approximate solution set is not compact at $t = 0$ in an infinite dimensional space. Finally, we provide two examples to illustrate the application of the abstract results.

2. Statement of results

Throughout this paper, let $N, \mathbb{R}, \mathbb{R}_+$ be the set of positive integers, real numbers and positive real numbers, respectively. We denote by $X$ a Hilbert space with norm $\| \cdot \|$, $C([0,b], X)$ the space of all $X$-valued continuous functions on $[0, b]$ with the norm $\| \cdot \|_{C}$, $L^2([0,b], U)$ the space of all $U$-valued square integrable functions on $[0,b]$, $\mathcal{L}(X)$ the space of all bounded linear operators from $X$ to $X$ with the usual norm $\| \cdot \|_{\mathcal{L}(X)}$, let $A$ be the infinitesimal generator of $C_0$-semigroup $\{S(t) : t \geq 0\}$ of uniformly bounded linear operators on $X$. Clearly, $M_S := \sup \{\|S(t)\|_{\mathcal{L}(X)} : t \geq 0\} < \infty$. Let $E$ be a closed subspace of $X$ and denote by $\Pi$ the projection from $X$ onto $E$.

We recall some notations, definitions and results on fractional derivative and fractional differential equations.

**Definition 1** [22] The Riemann-Liouville fractional order derivative of $f : [0, \infty) \to X$ of order $q \in \mathbb{R}_+$ is defined by

$$RLD^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-q-1} f(s) \, ds,$$

where $q \in (n-1, n)$, $n \in \mathbb{N}$.

**Definition 2** [22] The Caputo fractional order derivative of $f : [0, \infty) \to X$ of order $q \in \mathbb{R}_+$ is defined by

$$RLD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) \, ds,$$

where $q \in (n-1, n)$, $n \in \mathbb{N}$.

Now, we use the probability density function to give the following definition of mild solutions to (1).

**Definition 3** [23] A solution $y(\cdot; u) \in C([0,b], X)$ is said to be a mild solution of (1) if for any $u \in L^2([0,b], U)$ the integral equation

$$y(t) = S_q(t) (y_0 - g(y)) + \int_0^t (t-s)^{q-1} T_q(t-s)[Bu(s) + f(s, y(s))]) \, ds, \quad 0 \leq t \leq b,$$

(2)
Theorem 7 Assume the following conditions:

$$S_q(t) = \int_0^\infty \omega_\alpha(\theta) S(t^\alpha \theta) d\theta, \quad T_q(t) = \alpha \int_0^\infty \theta \omega_\alpha(\theta) S(t^\alpha \theta) d\theta, \quad t \geq 0,$$

$$\omega_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\alpha/\alpha} \omega_\alpha(\theta^{-1/\alpha}) \geq 0,$$

$$\omega_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n-\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty),$$

where is a probability density function defined on $(0, \infty)$, that is

$$\omega_\alpha(\theta) \geq 0, \quad \theta \in (0, \infty), \quad \int_0^\infty \omega_\alpha(\theta) d\theta = 1.$$

We present some basic properties of $S_q$ and $T_q$ which will be used in the sequel [28].

- For any fixed $t \geq 0$ and any $y \in X$, $\|S_q(t) y\| \leq M_S \|y\|$ and $\|T_q(t) y\| \leq \frac{M_S}{\Gamma(q)} \|y\|$.

- $\{S_q(t) : t \geq 0\}$ and $\{T_q(t) : t \geq 0\}$ are strongly continuous.

- $\{S_q(t) : t > 0\}$ and $\{T_q(t) : t > 0\}$ are compact operators provided that $\{S(t) : t > 0\}$ is compact.

Let $y(b; u)$ be the state value of (1) at terminal time $b$ corresponding to the control $u$.

Definition 4 [2] Given $b > 0$, $y_0 \in X$, $y_6 \in E$ and $\varepsilon > 0$. The system (1) is said to be partial-approximately controllable on $[0, b]$ if there exists a control $u_\varepsilon \in L^2([0, b], U)$ such that the corresponding solution $y(t; u_\varepsilon)$ of (1), satisfies the conditions

$$\|y(b; u_\varepsilon) - y_6\| < \varepsilon. \quad (3)$$

Remark 5 In particular, if $E = X$, the concept of partial-approximate controllability coincide with the well known concept of approximate controllability.

Remark 6 It is known that the system (1) approximately controllable on $[0, b]$ if and only if the condition $B^*T_q^*(b-s)\Pi^*\varphi = 0$, $0 < s < b$ implies that $\varphi = 0$, see [2].

Our main result is as follows:

Theorem 7 Assume the following conditions:

(S) $S(t)$, $t > 0$ is compact operator;

(F) The function $f : [0, b] \times X \rightarrow X$ satisfies the following

(a) $f(\cdot, \cdot) : X \rightarrow X$ is jointly continuous;

(b) there is a positive continuous function $n \in C([0, b], \mathbb{R}_+)$ such that for every $(t, y) \in [0, b] \times X$, we have

$$\|f(t, y)\| \leq n(t).$$

(G) The function $g : C([0, b], X) \rightarrow X$ satisfies the following

(a) $g$ is continuous, there exists a positive constant $\Lambda_g$ such that for all $y \in X$,

$$\|g(y)\| \leq \Lambda_g.$$

(b) There is a $\delta \in (0, b)$ such that for any $y, z \in C([0, b], X)$ satisfying $y(t) = z(t), \ t \in [\delta, b]$,

$$g(y) = g(z).$$
(B) $B : U \to X$ is a linear continuous operator with $M_B := \|B\|$.

(AC) The linear system

$$y(t) = S_q(t) y_0 + \int_0^t (t-s)^{q-1} T_q(t-s) B u(s) \, ds$$

(4)

is partial-approximately controllable in $[0,b]$.

Then the fractional control system (2) is partial-approximately controllable on $[0,b]$.

**Remark 8** Our results are new for approximate controllability, that is for the case $E = X$.

**Remark 9** Our results are new even for the classical evolution control system with nonlocal conditions (the case $q = 1$).

**Remark 10** One may expect the results of this paper to hold for a class of problems governed by different type of evolution systems such as Riemann-Liouville FDEs, stochastic (fractional) DEs, FDEs with infinite delay and so on.

3. Auxiliary lemmas

For $\varepsilon > 0$ and $n \geq 1$ we introduce the following functional

$$J_{\varepsilon,n} (\varphi ; z) = \frac{1}{2} \int_0^b \int_0^{b-\eta} (b-s)^{q-1} \| B^* T_q^* (b-s) \Pi^* \varphi \|^2 \, ds + \varepsilon \| \varphi \| - \langle \varphi, h_n (z) \rangle ,$$

(5)

where

$$h_n (z) = \Pi S_q (b) \left( y_0 - S \left( \frac{1}{n} \right) g(z) \right) + \int_0^b \Pi T_q (b-s) f(s,z(s)) \, ds - y_b.$$

and the following approximating operators $Q : C ([0,b],X) \to E$

$$Q (z) := \Pi S_q (b) \left( y_0 - S \left( \frac{1}{n} \right) g(z) \right) + \int_0^b (b-s)^{q-1} \Pi T_q (b-s) f(s,z(s)) \, ds$$

$$= : Q_1 (z) + Q_2 (z) .$$

Set

$$B (0;r) = \{ x \in C ([0,b],X) : \| x \|_C \leq r \} .$$

**Lemma 11** The mapping $Q : B (0;r) \to E$ is compact.

**Proof.** For any $\eta \in (0,b)$ and $\delta > 0$, we define an operator $Q_2^{\eta,\delta}$ on $B (0;r)$ by the formula

$$Q_2^{\eta,\delta} (z) = q \int_0^{b-\eta} \int_\delta^\infty (b-s)^{q-1} \theta \omega_q (\theta) \Pi S ((b-s)^q \theta) f(s,z(s)) \, d\theta ds$$

$$= \Pi S (\eta^q \delta) q \int_0^{b-\eta} \int_\delta^\infty (b-s)^{q-1} \theta \omega_q (\theta) S ((b-s)^q \theta - S (\eta^q \delta)) f(s,z(s)) \, d\theta ds$$

$$=: \Pi S (\eta^q \delta) Z (\eta) .$$

By the assumption (F)

$$\| Z (\eta) \| \leq b M_S \| n \|_C .$$
Then from the compactness of \( S (\eta, \delta) \) and boundedness of \( Z (\eta) \), we obtain that the set \( Q_2^{\eta, \delta} : B (0; r) \to E \) is compact. Moreover, for every \( z \in B (0; r) \), we have

\[
\left\| Q_2 (z) - Q_2^{\eta, \delta} (z) \right\|
\leq q \int_0^b \int_0^\delta (b - s)^{q - 1} \theta\omega_q (\theta) \left\| \Pi S ((b - s)^q \theta) \right\| \| f (s, z (s)) \| \, d\theta ds
\]

\[
+ q \int_{b - \eta}^b \int_0^\infty (b - s)^{q - 1} \theta\omega_q (\theta) \left\| \Pi S ((b - s)^q \theta) \right\| \| f (s, z (s)) \| \, d\theta ds
\]

\[
\leq \| \Pi \| M_S \left( b^q \int_0^\delta \theta\omega_q (\theta) \, d\theta + \frac{\eta^q}{\Gamma (q + 1)} \right) \| n \|_C \to 0,
\]

as \( \eta, \delta \to 0 \). Hence, there exist relatively compact operator that can be arbitrarily close to \( Q_2 \). Then, \( Q_2 \) is compact.

Since \( S \left( \frac{1}{n} \right) \), \( n \geq 1 \), is compact, \( \Pi \) is linear bounded and the assumption (G) holds, it is easily seen that, the set

\[
\left\{ \Pi S_q (b) \left( y_0 - S \left( \frac{1}{n} \right) g (z) \right) : z \in B (0; r) \right\}
\]

is relatively compact in \( X \). Thus the operator \( Q = Q_1 + Q_2 \) is compact.

**Lemma 12** For any \( n \geq 1 \), \( h_n : B (0; r) \to E \) is a continuous function.

**Proof.** For any \( z_l, z \in B (0; r) \) with \( \lim_{l \to \infty} \| z_l - z \|_C = 0 \). By the condition (F1), we have we can conclude that \( \lim_{l \to \infty} \| f (\cdot, z_l (\cdot)) - f (\cdot, z (\cdot)) \|_C = 0 \). On the other hand

\[
\left\| h_n (z_l) - h_n (z) \right\|_E \leq \left\| \Pi S_q (b) \left( S \left( \frac{1}{n} \right) g (z_l) - g (z) \right) \right\|
\]

\[
+ \int_0^b \left\| (b - s)^{q - 1} \Pi T_q (b - s) \left[ f (s, z_l (s)) - f (s, z (s)) \right] \right\| \, d\theta ds
\]

\[
\leq M_S^2 \| \Pi \| \| g (z_l) - g (z) \|
\]

\[
+ \frac{M_S b^q}{\Gamma (q)} \| \Pi \| \| f (\cdot, z_l (\cdot)) - f (\cdot, z (\cdot)) \|_C,
\]

which implies that

\[
\left\| h_n (z_l) - h_n (z) \right\|_E \to 0 \quad \text{as} \quad l \to \infty.
\]

**Lemma 13** For any \( B (0; r) \),

\[
\lim_{\| \varphi \| \to \infty} \inf_{z \in B (0; r)} \frac{J_{\varepsilon, n} (\varphi; z)}{\| \varphi \|} \geq \varepsilon.
\]

**Proof.** In order to prove (6), suppose that it is not the case. Then there exists sequences \( \{ \varphi_l \} \subset X \), \( \{ z_l \} \subset B (0; r) \), with \( \| \varphi_l \| \to \infty \), such that

\[
\lim_{l \to \infty} \frac{J_{\varepsilon, n} (\varphi_l; z_l)}{\| \varphi_l \|} < \varepsilon.
\]

Without loss of generality, we may assume that

\[
h_n (z_l) \to h_n, \quad \text{strongly in} \quad X,
\]
for some $h_n \in X$. In fact, $\{h_n (z_i) : l \geq 1\} \subset \text{Im} Q$ is relatively compact in $X$; thus, we may assume this by picking a subsequence.

Next, we normalize $\varphi_l : \tilde{\varphi}_l = \frac{\varphi_l}{\|\varphi_l\|}$. Since $\|\tilde{\varphi}_l\| = 1$, we can extract a subsequence (still denoted by $\tilde{\varphi}_l$), which weakly converges in $X$ to an element $\tilde{\varphi}$ in $X$. Consequently, because $S(t), t > 0$, is a compact semigroup, we see that

$$B^* S^* (b - \cdot) \Pi^* \tilde{\varphi}_l \rightarrow B^* S^* (b - \cdot) \Pi^* \tilde{\varphi}, \quad \text{strongly in } C([0, b), X), \quad \text{as } l \rightarrow \infty. \quad (9)$$

From (5), it follows that

$$\frac{J_{\varepsilon, n} (\varphi_l; z_l)}{\|\varphi_l\|} = \frac{\|\varphi_l\|}{2} \int_0^b (b - s)^{q-1} \left\| B^* T_q^* (b - s) \Pi^* \varphi \right\|^2 ds + \varepsilon \|\tilde{\varphi}_l\| - \langle \tilde{\varphi}_l, h_n (z_l) \rangle.$$

Thus, noting that $\|\varphi_l\| \rightarrow \infty$, by (7)-(9) and the Fatou lemma

$$\int_0^b (b - s)^{q-1} \left\| B^* T_q^* (b - s) \Pi^* \tilde{\varphi}_l \right\|^2 ds \leq \lim_{l \rightarrow \infty} \int_0^b (b - s)^{q-1} \left\| B^* T_q^* (b - s) \Pi^* \tilde{\varphi}_l \right\|^2 ds = 0.$$

By assumption (AC) we have $\tilde{\varphi} = 0$, and we deduce that

$$\tilde{\varphi}_l \rightarrow 0 \quad \text{weakly in } X \quad \text{as } l \rightarrow \infty.$$

Hence

$$\lim_{l \rightarrow \infty} \frac{J_{\varepsilon, n} (\varphi_l; z_l)}{\|\varphi_l\|} \geq \lim_{l \rightarrow \infty} (\varepsilon \|\tilde{\varphi}_l\| - \langle \tilde{\varphi}_l, h_n (z_l) \rangle) = \varepsilon,$$

which contradicts (7) and proves the claim (8).

For any $z \in C([0, b], X)$, the functional $J_{\varepsilon, n} (\cdot, z)$ admits a unique minimum $\tilde{\varphi}_{\varepsilon, n}$ that defines a map $\Phi_{\varepsilon, n} : C([0, b], X) \rightarrow X$. $\Phi_{\varepsilon, n}$ has the following properties.

**Lemma 14** There exists $R_\varepsilon > 0$ such that $\|\Phi_{\varepsilon, n} (z)\| < R_\varepsilon$ for any $z \in B(0; r), n \geq 1$.

**Proof.** Let $z \in B(0; r)$. From Lemma 14 (b), we see that there exists a constant $R_\varepsilon > 0$, such that

$$\inf_{z \in B(0; r)} \frac{J_{\varepsilon, n} (\varphi; z)}{\|\varphi\|} \geq \frac{\varepsilon}{2}, \quad \|\varphi\| \geq R_\varepsilon. \quad (10)$$

On the other hand, by the definition of $\Phi_\varepsilon$,

$$J_{\varepsilon, n} (\Phi_{\varepsilon, n} (z); z) \leq J_{\varepsilon, n} (0; z) = 0. \quad (11)$$

Hence, combining (10) and (11), we have

$$\|\Phi_{\varepsilon, n} (z)\| < R_\varepsilon, \quad \text{for all } z \in B(0; r).$$

**Lemma 15** For any $z_l, z \in B(0; r)$ such that

$$z_l \rightarrow z, \quad \text{in } C([0, b], X),$$

it holds that

$$\lim_{l \rightarrow \infty} \|\Phi_{\varepsilon, n} (z_l) - \Phi_{\varepsilon, n} (z)\| = 0.$$
Proof. By Lemma 13 we have boundedness of \( \hat{\varphi}_{\varepsilon,n,l} = \Phi_{\varepsilon,n}(z_l) \). Consequently, we may assume that \( \hat{\varphi}_{\varepsilon,n,l} \xrightarrow{w} \tilde{\varphi}_{\varepsilon,n} \). Thus, by the definition of \( J_{\varepsilon,n} \) and the optimality of both \( \hat{\varphi}_{\varepsilon,n,l} = \Phi_{\varepsilon,n}(z_l) \) and \( \hat{\varphi}_{\varepsilon,n} = \Phi_{\varepsilon,n}(z) \), one has

\[
J_{\varepsilon,n}(\hat{\varphi}_{\varepsilon,n}; z) \leq J_{\varepsilon,n}(\tilde{\varphi}_{\varepsilon,n}; z) \leq \lim_{l \to \infty} J_{\varepsilon,n}(\hat{\varphi}_{\varepsilon,n,l}; z_l) \leq \lim_{l \to \infty} J_{\varepsilon,n}(\tilde{\varphi}_{\varepsilon,n,l}; z_l) = J_{\varepsilon,n}(\tilde{\varphi}_{\varepsilon,n}; z).
\]

Hence, the equalities hold in the above. That means that \( \tilde{\varphi}_{\varepsilon,n} \) is also a minimum of \( J_{\varepsilon,n}(\cdot; z) \). By the uniqueness of the minimum, it is necessary that \( \hat{\varphi}_{\varepsilon,n} = \tilde{\varphi}_{\varepsilon,n} \). Therefore

\[
\lim_{l \to \infty} J_{\varepsilon,n}(\hat{\varphi}_{\varepsilon,n,l}; z_l) = J_{\varepsilon,n}(\tilde{\varphi}_{\varepsilon,n}; z),
\]

\[
\lim_{l \to \infty} \int_0^b (b - s)^{q-1} \|B^*T_s(b - s)\Pi^*\hat{\varphi}_{\varepsilon,n,l}\|^2 ds = \int_0^b (b - s)^{q-1} \|B^*T_s(b - s)\Pi^*\tilde{\varphi}_{\varepsilon,n}\|^2 ds,
\]

\[
\lim_{l \to \infty} \langle \hat{\varphi}_{\varepsilon,n,l}, h_n(z_l) \rangle = \langle \tilde{\varphi}_{\varepsilon,n}, h_n(z) \rangle, \quad \|\tilde{\varphi}_{\varepsilon,n}\| \leq \lim_{l \to \infty} \|\hat{\varphi}_{\varepsilon,n,l}\|.
\]

These relations imply that

\[
\lim_{l \to \infty} \|\hat{\varphi}_{\varepsilon,n,l}\| = \|\tilde{\varphi}_{\varepsilon,n}\|.
\]

Because \( X \) is Hilbert space, from \( \hat{\varphi}_{\varepsilon,n,l} \xrightarrow{w} \tilde{\varphi}_{\varepsilon,n} \) and (12), we obtain the strong convergence of \( \hat{\varphi}_{\varepsilon,n,l} \) to \( \tilde{\varphi}_{\varepsilon,n} \)\n.

For fixed \( n \geq 1 \), set \( \Theta_{\varepsilon,n} : C([0,b],X) \to C([0,b],X) \) defined by

\[
(\Theta_{\varepsilon,n}z)(t) = S_q(t)\left(y_0 - S\left(\frac{1}{n}\right)g(z)\right) + \int_0^t (t - s)^{q-1} T_s(t - s) \left[BU_{\varepsilon,n}(s,z) + f(s,z(s))\right] ds,
\]

(13)
with

\[
u_{\varepsilon,n}(s,z) = B^*T_s^*(b - s)\Pi^*\hat{\varphi}_{\varepsilon,n} = B^*T_s^*(b - s)\Pi^*\Phi_{\varepsilon,n}(z).
\]

(14)

We will prove \( \Theta_{\varepsilon,n} \) has a fixed point by using Schauder’s fixed point theorem.

Theorem 16 Assume that the hypotheses of Theorem 7 are satisfied. Then for \( n \geq 1 \), the approximate control operator \( \Theta_{\varepsilon,n} \) has at least one fixed point in \( C([0,b],X) \).

Proof. Step 1: For any \( n \geq 1 \), \( \Theta_{\varepsilon,n} \) is continuous on \( C([0,b],X) \).

Let \( \{z_m : m \geq 1\} \) be a sequence in \( C([0,b],X) \) with \( \lim_{m \to \infty} z_m = z \) in \( C([0,b],X) \). By the continuity of \( f \) and \( u_{\varepsilon,n} \), we deduce that \( (f(s,z_m(s)), u_{\varepsilon,n}(s,z_m)) \) converges to \( (f(s,z(s)), u_{\varepsilon,n}(s,z)) \) uniformly for \( s \in [0,b] \), and we have

\[
\|(\Theta_{\varepsilon,n}z_m)(t) - (\Theta_{\varepsilon,n}z)(t)\|
\leq M_3 \|g(z_m) - g(z)\| + \frac{M_3 b^q}{\Gamma(q + 1)} \|f(\cdot,z_m(\cdot)) - f(\cdot,z(\cdot))\|_C
\]

\[
+ \frac{M_3 b^q}{\Gamma(q + 1)} \|u_{\varepsilon,n}(\cdot,z_m) - u_{\varepsilon,n}(\cdot,z)\|_C \to 0, \quad \text{as} \quad n \to \infty,
\]

which implies that the mapping \( \Theta_{\varepsilon,n} \) is continuous on \( C([0,b],X) \).

Step 2: There is a positive number \( r(\varepsilon) > 0 \) such that \( \Theta_{\varepsilon,n} \) maps \( B(0;r(\varepsilon)) \) into itself.

We see that

\[
\|(\Theta_{\varepsilon,n}z_k)(t_k)\|
\leq M_3^2 A_g + \frac{M_3 b^q}{\Gamma(q + 1)} (\|n\|_C + M_3 M_2 M_B R_\varepsilon) := r(\varepsilon).
\]
Step 3: For any $n \geq 1$, $\Theta_{\varepsilon,n}$ is compact.
At the end, applying the Schauder fixed point theorem we obtain that for each $n \geq 1$, $\Theta_{\varepsilon,n}$ has at least one fixed point in $B(0; r(\varepsilon))$. 

Assume that $z_{\varepsilon,n} \in B(0; r(\varepsilon)) \subset C([0, b], X)$ is a fixed point of $\Theta_{\varepsilon,n}$

$$\Theta_{\varepsilon,n} z_{\varepsilon,n} = z_{\varepsilon,n}$$

and $\Phi_{\varepsilon,n}(z_{\varepsilon,n})$ is minimizer of $J_{\varepsilon,n}(\varphi; z_{\varepsilon,n})$ and

$$u_{\varepsilon,n}(s, z_{\varepsilon,n}) = B^*T^*_q(b - s) \Pi^*\Phi_{\varepsilon,n}(z_{\varepsilon,n}),$$

is the corresponding control. Moreover, assume that

$$z_{\varepsilon,n} \xrightarrow{n \to \infty} z_{\varepsilon}$$

strongly in $C([0, b], X)$, as $n \to \infty$.

$\Phi_{\varepsilon}(z_{\varepsilon})$ is minimizer of $J_{\varepsilon}(\varphi; z_{\varepsilon})$ and

$$u_{\varepsilon}(s, z_{\varepsilon}) = B^*T^*_q(b - s) \Pi^*\Phi_{\varepsilon}(z_{\varepsilon}),$$

is the corresponding control.

Lemma 17 Assume that

$$\lim_{n \to \infty} \|z_{\varepsilon,n} - z_{\varepsilon}\|_C = 0.$$ 

Then

$$\Phi_{\varepsilon,n}(z_{\varepsilon,n}) \rightharpoonup \Phi_{\varepsilon}(z_{\varepsilon}) \text{ weakly in } X,$$

$$\lim_{n \to \infty} \|u_{\varepsilon,n}(s, z_{\varepsilon,n}) - u_{\varepsilon}(s, z_{\varepsilon})\|_C = 0.$$

Proof. By definition of the minimizing functional $\Phi_{\varepsilon,n}(z_{\varepsilon,n})$ and $\Phi_{\varepsilon}(z_{\varepsilon})$ are minimizers of

$$J_{\varepsilon,n}(\varphi; z_{\varepsilon,n}) = \frac{1}{2} \int_0^b (b - s)^{n-1} \|B^*T^*_q(b - s) \Pi^*\varphi\|^2 ds + \varepsilon \|\varphi\| - \langle \varphi, h_n(z_{\varepsilon,n}) \rangle,$$

$$J_{\varepsilon}(\varphi; z_{\varepsilon}) = \frac{1}{2} \int_0^b (b - s)^{n-1} \|B^*T^*_q(b - s) \Pi^*\varphi\|^2 ds + \varepsilon \|\varphi\| - \langle \varphi, h(z_{\varepsilon}) \rangle,$$

correspondingly. By Lemma 14 we have boundedness of $\Phi_{\varepsilon,n}(z_{\varepsilon,n})$. Consequently, we may assume that $\Phi_{\varepsilon,n}(z_{\varepsilon,n}) \rightharpoonup \Phi_{\varepsilon}$ weakly in $X$. Thus, by the definition of $J_{\varepsilon,n}$ and the optimality of both $\Phi_{\varepsilon,n}(z_{\varepsilon,n})$ and $\Phi_{\varepsilon}(z_{\varepsilon})$, one has

$$J_{\varepsilon}(\Phi_{\varepsilon}(z_{\varepsilon}); z_{\varepsilon}) \leq J_{\varepsilon}(\Phi_{\varepsilon}; z_{\varepsilon}) \leq \lim_{n \to \infty} J_{\varepsilon}(\Phi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon}),$$

(15)

and

$$\lim_{n \to \infty} J_{\varepsilon,n}(\Phi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n}) \leq \lim_{n \to \infty} J_{\varepsilon,n}(\Phi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n}) \leq \lim_{n \to \infty} J_{\varepsilon,n}(\Phi_{\varepsilon}(z_{\varepsilon}); z_{\varepsilon,n}) = J_{\varepsilon}(\Phi_{\varepsilon}(z_{\varepsilon}); z_{\varepsilon}).$$

(16)

Noting that $\lim_{n \to \infty} J_{\varepsilon}(\Phi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n}) = \lim_{n \to \infty} J_{\varepsilon,n}(\Phi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n})$ and combining (15) and (16) we have

$$J_{\varepsilon}(\Phi_{\varepsilon}(z_{\varepsilon}); z_{\varepsilon}) = J_{\varepsilon}(\Phi_{\varepsilon}; z_{\varepsilon}),$$

$$\lim_{n \to \infty} J_{\varepsilon,n}(\Phi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n}) = \lim_{n \to \infty} J_{\varepsilon,n}(\Phi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n}) = J_{\varepsilon}(\Phi_{\varepsilon}(z_{\varepsilon}); z_{\varepsilon}).$$
That means that $\Phi_\varepsilon$ is also a minimum of $J_\varepsilon (\cdot | z_\varepsilon)$. By the uniqueness of the minimum, it is necessary that $\Phi_\varepsilon = \Phi_\varepsilon (z_\varepsilon)$. Therefore

$$\lim_{n \to \infty} J_{\varepsilon,n} (\Phi_{\varepsilon,n} (z_{\varepsilon,n}) ; z_{\varepsilon,n}) = J_\varepsilon (\Phi_\varepsilon (z_\varepsilon) ; z_\varepsilon),$$

$$\lim_{n \to \infty} \int_0^b (b-s)^{q-1} \| B^* T_q^* (b-s) \| \Pi^* \Phi_{\varepsilon,n} (z_{\varepsilon,n}) \| ds = \int_0^b (b-s)^{q-1} \| B^* T_q^* (b-s) \Pi^* \Phi_\varepsilon (z_\varepsilon) \| ds,$$

$$\lim_{n \to \infty} \langle \Phi_{\varepsilon,n} (z_{\varepsilon,n}) , h_n (z_{\varepsilon,n}) \rangle = \langle \Phi_\varepsilon (z_\varepsilon) , h (z_\varepsilon) \rangle , \quad \| \Phi_\varepsilon (z_\varepsilon) \| \leq \lim_{n \to \infty} \| \Phi_{\varepsilon,n} (z_{\varepsilon,n}) \| .$$

These relations imply that

$$\lim_{n \to \infty} \| \Phi_{\varepsilon,n} (z_{\varepsilon,n}) \| = \| \Phi_\varepsilon (z_\varepsilon) \| . \quad (17)$$

As $X$ is Hilbert space, from $\Phi_{\varepsilon,n} (z_{\varepsilon,n}) \to \Phi_\varepsilon (z_\varepsilon)$ weakly in $X$ and (17), we obtain the strong convergence of $\Phi_{\varepsilon,n} (z_{\varepsilon,n})$ to $\Phi_\varepsilon (z_\varepsilon)$. □

4. Proof of the main result

Let $y_0 \in X$, $y_b \in E$ be any given two points and $\varepsilon > 0$ be any given accuracy. Then it is seen that for all $t \in [0, b]$

$$(\Theta_\varepsilon z) (t) = S_q (t) (y_0 - g (z)) + \int_0^t (t-s)^{q-1} T_q (t-s) [B u_\varepsilon (s,z) + f (s,z(s))] ds,$$

$$u (s,z) = B^* T_q^* (b-s) \Pi^* \tilde{\omega}_\varepsilon = B^* T_q^* (b-s) \Pi^* \Phi_\varepsilon (z_\varepsilon).$$

**Theorem 18** Assume that the hypotheses of Theorem 7 are satisfied. Then the control operator $\Theta_\varepsilon$ has at least one fixed point in $C ([0,b], X)$.

**Proof.** Let $n \geq 1$ be fixed. By Theorem 13 the approximate operator $\Theta_{\varepsilon,n}$ defined by (13) has a fixed point, say $y_{\varepsilon,n}$. We define the approximate solution set $D$ by

$$D = \{ y_{\varepsilon,n} \in C ([0,b], X) : \Theta_{\varepsilon,n} y_{\varepsilon,n} = y_{\varepsilon,n}, \ n \geq 1 \} .$$

Step 1: $D (0)$ is relatively compact in $X$.

For $y_{\varepsilon,n} \in D$, $n \geq 1$, define

$$y_{\varepsilon,n} (t) = \begin{cases} y_{\varepsilon,n} (t) , & \text{if } \delta \leq t \leq b, \\ y_{\varepsilon,n} (\delta) , & 0 \leq t \leq \delta, \end{cases}$$

where $\delta$ comes from the condition (G). It is easily seen that $\{ y_{\varepsilon,n} : n \geq 1 \}$ is relatively compact in $C ([0,b], X)$. Without loss of generality, we may suppose that $\tilde{y}_{\varepsilon,n} \to \tilde{y}_\varepsilon \in C ([0,b], X)$ as $n \to \infty$. By assumption (G), we get that $g (y_{\varepsilon,n}) = g (\tilde{y}_{\varepsilon,n}) \to g (\tilde{y}_\varepsilon)$. Thus by the continuity of $S_q (t)$ and $g$, we have

$$\| y_{\varepsilon,n} (0) - (y_0 - g (\tilde{y}_\varepsilon)) \| = \left\| S \left( \frac{1}{n} \right) g (y_{\varepsilon,n}) - g (\tilde{y}_\varepsilon) \right\|$$

$$\leq \left\| S \left( \frac{1}{n} \right) g (y_{\varepsilon,n}) - S \left( \frac{1}{n} \right) g (\tilde{y}_\varepsilon) \right\| + \left\| S \left( \frac{1}{n} \right) g (\tilde{y}_\varepsilon) - g (\tilde{y}_\varepsilon) \right\|$$

$$\leq M_\varepsilon \| g (y_{\varepsilon,n}) - g (\tilde{y}_\varepsilon) \| + \left\| S \left( \frac{1}{n} \right) g (\tilde{y}_\varepsilon) - g (\tilde{y}_\varepsilon) \right\| \to 0,$$

as $n \to \infty$. So, $D (0) = \{ y_{\varepsilon,n} (0) = y_0 - S \left( \frac{1}{n} \right) g (y_{\varepsilon,n}) \}$ is relatively compact in $X$.

Step 2: For each $t \in (0,b]$ the set $D (t) := \{ y_{\varepsilon,n} (t) : n \geq 1 \}$ is relatively compact in $X$.

Step 3: $D$ is equicontinuous at $t = 0$.

First of all note that

$$\sup \{ \| B u_{\varepsilon,n} (s,y_{\varepsilon,n}) + f (s,y_{\varepsilon,n} (s)) \| : s \in [0,b], y_{\varepsilon,n} \in D \}$$

$$\leq M_\varepsilon M_S R_\varepsilon + \| n \|_C .$$
For $0 < t < b$, we have

\[
\|y_{\varepsilon,n}(t) - y_{\varepsilon,n}(0)\| = \left\| S_q(t) S \left( \frac{1}{n} \right) g(y_{\varepsilon,n}) - S \left( \frac{1}{n} \right) g(y_{\varepsilon,n}) \right\|
\]

\[
+ \frac{M_S}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \sup \{ \| Bu_{\varepsilon,n}(s,y_{\varepsilon,n}) + f(s,y_{\varepsilon,n}(s)) \| : s \in [0,b], y_{\varepsilon,n} \in D \}
\]

\[
\leq \left\| (S_q(t) - I) S \left( \frac{1}{n} \right) g(y_{\varepsilon,n}) \right\|
\]

\[
+ \frac{M_S t^q}{\Gamma(q+1)} (M_B^2 M_S R_\varepsilon + \| n \|_C)
\]

approaches zero uniformly as $t \to 0$, since $\left\{ S \left( \frac{1}{n} \right) g(y_{\varepsilon,n}) : n \geq 1 \right\}$ is relatively compact in $X$. Then we obtain the set $D$ is equicontinuous at $t = 0$.

Step 4: $D$ is equicontinuous on $(0,b]$.

Let $0 < t_1 < t_2 \leq b$ and choose $\eta > 0$ such that $t_1 - \eta > 0$. Then, for any $y_{\varepsilon,n} \in D$, we have

\[
\|y_{\varepsilon,n}(t_2) - y_{\varepsilon,n}(t_1)\|
\]

\[
\leq \left\| (S_q(t_2) - S_q(t_1)) \left( y_0 - S \left( \frac{1}{n} \right) g(y_{\varepsilon,n}) \right) \right\|
\]

\[
+ \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} T_q(t_2 - s) (Bu_{\varepsilon,n}(s,y_{\varepsilon,n}) + f(s,y_{\varepsilon,n}(s))) ds \right\|
\]

\[
+ \left\| \int_{0}^{t_1} (t_2 - s)^{q-1} - (t_1 - s)^{q-1} T_q(t_2 - s) (Bu_{\varepsilon,n}(s,y_{\varepsilon,n}) + f(s,y_{\varepsilon,n}(s))) ds \right\|
\]

\[
+ \left\| \int_{0}^{t_1} (t_1 - s)^{q-1} (T_q(t_2 - s) - T_q(t_1 - s)) (Bu_{\varepsilon,n}(s,y_{\varepsilon,n}) + f(s,y_{\varepsilon,n}(s))) ds \right\|
\]

\[
=: I_1 + I_2 + I_3 + I_4.
\]

From assumption (G) and norm continuity of $S_q(t)$, $t > 0$, it follows that

\[
I_1 = \left\| (S_q(t_2) - S_q(t_1)) \left( y_0 - S \left( \frac{1}{n} \right) g(y_{\varepsilon,n}) \right) \right\| \to 0,
\]

as $t_2 - t_1 \to 0$, uniformly for all $y_{\varepsilon,n} \in D$. By direct calculation to $I_2$, we obtain that

\[
I_2 \leq \frac{M_S}{\Gamma(q)} \left( M_B^2 M_S R_\varepsilon + \| n \|_C \right)
\]

As $t_1 > 0$, we obtain

\[
I_3 \leq \frac{M_S}{\Gamma(q)} \int_{0}^{t_1} (t_2 - s)^{q-1} - (t_1 - s)^{q-1} ds \left( M_B^2 M_S R_\varepsilon + \| n \|_C \right)
\]

Next, for $t_1 - \eta > 0$, we have

\[
I_4 \leq \left\| \left( \int_{0}^{t_1} T_q(t_2 - s) - T_q(t_1 - s) \right) (Bu_{\varepsilon,n}(s,y_{\varepsilon,n}) + f(s,y_{\varepsilon,n}(s))) ds \right\|
\]

\[
\leq \frac{t_2 - t_1}{t_2} \sup \{ \| Bu_{\varepsilon,n}(s,y_{\varepsilon,n}) + f(s,y_{\varepsilon,n}(s)) \| : s \in [0,b], y_{\varepsilon,n} \in D \} \sup_{0 \leq s \leq t_1 - \eta} \| T_q(t_2 - s) - T_q(t_1 - s)\|
\]

\[
+ \frac{2 M_S \eta^q}{\Gamma(q+1)} (M_B^2 M_S R_\varepsilon + \| n \|_C)
\]

Thus, combining the above inequalities $(18)$–(21) with the continuity of $T_q(t), t > 0$, in the uniform operator topology, we obtain the equicontinuity of the set $D$ on $(0,b]$. 
Therefore, the set $D$ is relatively compact in $C([0, b], X)$ and we may assume $y_{ε,n} → y_ε$ for some $y_ε ∈ C([0, b], X)$ as $n → ∞$. On the other hand, by Lemma 17, $u_{ε,n} (s, y_{ε,n}) → u_ε (s, y_ε)$ in $C([0, b], X)$ as $n → ∞$. By taking the limit as $n → ∞$ in $Θ_ε,n y_{ε,n} = y_{ε,n}$ and using the Lebesgue dominated convergence theorem, we obtain that

$$y_ε (t) = S_q (t) (y_0 - g (y_ε)) + \int_0^t (t - s)^{q-1} T_q (t - s) [B u_ε (s, y_ε) + f (s, y_ε (s))] ds,$$

for $t ∈ [0, b]$, which implies that $y_ε$ is a mild solution of semilinear fractional control system (13). This completes the proof.

In view of Theorem 11, for any $ε > 0$ there exists $y_ε ∈ C([0, b], X)$ such that

$$y_ε (t) = S_q (t) (y_0 - g (y_ε)) + \int_0^t (t - s)^{q-1} T_q (t - s) [B u_ε (s, y_ε) + f (s, y_ε (s))] ds,$$

where $u (s, y_ε) = B^* S^* (b - s) Φ_ε (y_ε)$.

Now we prove our main result.

**Proof of Theorem 17.** By Lemma 13(a) we know that $J_ε$ is strictly convex. Then $J_ε (φ; y_ε)$ has a unique critical point which is its minimizer:

$$\hat{φ}_ε ∈ X : J_ε (\hat{φ}_ε; y_ε) = \min_{φ \in X} J_ε (φ; y_ε).$$

Given any $ψ \in X$ and $λ ∈ R$ we have

$$J_ε (\hat{φ}_ε; y_ε) ≤ J_ε (\hat{φ}_ε + λψ; y_ε)$$

or, in other words,

$$ε \| \hat{φ}_ε \|
\leq \frac{λ^2}{2} \int_0^b (b - s)^{q-1} \| B^* T_q^* (b - s) Π^* ψ \|^2 ds + λ \int_0^b (b - s)^{q-1} \langle B^* T_q^* (b - s) Π^* \hat{φ}_ε, B^* T_q^* (b - s) Π^* ψ \rangle ds
+ ε \| \hat{φ}_ε + λψ \| - λ \langle ψ, h (y_ε) \rangle.$$

Dividing this inequality by $λ > 0$ and letting $λ → 0^+$ we obtain that

$$\langle ψ, h (y_ε) \rangle ≤ \int_0^b (b - s)^{q-1} \langle B^* T_q^* (b - s) Π^* \hat{φ}_ε, B^* T_q^* (b - s) Π^* ψ \rangle ds
+ ε \lim_{λ → 0^+} \inf_{\lambda → 0^+} \frac{\| \hat{φ}_ε + λψ \| - \| \hat{φ}_ε \|}{λ}
\leq \int_0^b (b - s)^{q-1} \langle B^* T_q^* (b - s) Π^* \hat{φ}_ε, B^* T_q^* (b - s) Π^* ψ \rangle ds + ε \| ψ \|.$$

Repeating this argument with $λ < 0$ we obtain finally that

$$\int_0^b (b - s)^{q-1} \langle B^* T_q^* (b - s) Π^* \hat{φ}_ε, B^* T_q^* (b - s) Π^* ψ \rangle ds - \langle ψ, h (y_ε) \rangle ≤ ε \| ψ \|.$$

On the other hand, with $u_ε = B^* T_q^* (b - s) Π^* \hat{φ}_ε$ we have

$$\int_0^b (b - s)^{q-1} \langle Π T_q (b - s) BB^* T_q^* (b - s) Π^* \hat{φ}_ε ds - h (y_ε), ψ \rangle = \langle Π y_ε (b - y_ε), ψ \rangle,$$

$$h (y_ε) = y_ε - Π S_q (b) (y_0 - g (y_ε)) - \int_0^b (b - s)^{q-1} Π T_q (b - s) f (s, y_ε (s)) ds.$$
Then, combining \((22)\) and \((23)\) we obtain that

\[
|\langle \Pi_{\varepsilon} y \varepsilon (b) - y_b, \psi \rangle| \leq \varepsilon \| \psi \|
\]

holds for any \(\psi \in X\). Thus

\[
\| \Pi_{\varepsilon} y \varepsilon (b) - y_b \| \leq \varepsilon.
\]

5. Applications

**Example 1:** Consider the following initial-boundary value problem of fractional parabolic control system with Caputo fractional derivatives, based on example 2 in [7]:

\[
\begin{aligned}
\frac{C}{\partial_{0,t}^2/3} x(t, \theta) &= \frac{\partial^2}{\partial \theta^2} x(t, \theta) + Bu(t, \theta) + f(t, x(t, \theta)), & t \in [0, b], \quad \theta \in [0, \pi], \\
x(t, 0) &= x(t, \pi) = 0, & t \in [0, b], \\
x(0, \theta) &= x_0(\theta) - \sum_{i=0}^{m} k(\theta, s) x(t_i, s) ds, & t \in [0, b], \quad \theta \in [0, \pi].
\end{aligned}
\]

(24)

Here, \(f\) is a given function, \(m\) is a positive integer, \(0 < t_0 < t_1 < \ldots < t_m < b\), \(k(\cdot, \cdot) \in L^2([0, \pi] \times [0, \pi], \mathbb{R}^+)\).

Take \(X = L^2[0, \pi]\) and the operator \(A : D(A) \subset X \rightarrow X\) is defined by

\[
Ay = y'',
\]

where the domain \(D(A)\) is defined by

\[
\{ x \in X : x, x' \text{ are absolutely continuous, } x'' \in X, \ x(0) = x(\pi) = 0 \}.
\]

Then, \(A\) can be written as

\[
Ay = -\sum_{n=1}^{\infty} n^2 \langle y, e_n \rangle e_n, \quad y \in D(A),
\]

where \(e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx\) is an orthonormal basis of \(X\). It is well known that \(A\) is the infinitesimal generator of a compact, analytic and self-adjoint semigroup \(S(t), t > 0,\) in \(X\) given by

\[
S(t) y = \sum_{n=1}^{\infty} \exp(-n^2 t) \langle y, e_n \rangle e_n, \quad y \in X.
\]

Now, define an infinite-dimensional space \(U\) by

\[
U = \left\{ u = \sum_{n=2}^{\infty} u_ne_n(\theta) : \sum_{n=2}^{\infty} u_n^2 < \infty \right\},
\]

with the norm \(\|u\|^2 = \sum_{n=2}^{\infty} u_n^2\). Define the operator \(B : U \rightarrow X\) as follows

\[
Bu = 2u_2 e_1(\theta) + \sum_{n=2}^{\infty} u_ne_n(\theta).
\]

The system \((24)\) can be reformulated as the following nonlocal controllability problem in \(X\).

\[
\frac{C}{\partial_{0,t}^{2/3}} y(t) = Ay(t) + Bu(t) + f(t, y(t)), \\
y(0) = y_0 - g(y),
\]

(25)
where \( y(t) = x(t, \cdot) \), \( f : [0, b] \times X \to X \) is given by \( f(t, y(t)) = f(t, x(t, \cdot)) \) and the function \( g : C([0, b], X) \to X \) is given by

\[
g(y) = \sum_{i=0}^{m} \int_{0}^{\sigma} k(\cdot, s) x(t_i, s) \, ds
\]

This implies that the condition \((AC)\) is satisfied. If we assume that \( f \) and \( g \) satisfy the assumptions \((F)\) and \((G)\), then by Theorem \([14]\) we obtain that the system \((24)\) is approximately controllable on \([0, T]\).

**Example 2:** Consider the following affine hereditary differential system in the Hilbert space \( E \)

\[
y'(t) = Ay(t) + Ny(t - h) + \int_{-h}^{0} M(\theta) y(t + \theta) \, d\theta + Bu(t) + f(t, y(t)),
\]

\[
y(0) = \xi + \int_{0}^{b} h(s, y(s)) \, ds, \quad y(\theta) = \eta(\theta), \quad -h \leq \theta < 0, \; h > 0, \; \delta > 0,
\]

where \( A \) is the generator of the strongly continuous compact semigroup \( S(t) : E \to E, \; t > 0, \; N \in \mathcal{L}(E), \; M \) is a Lebesgue measurable and essentially bounded \( \mathcal{L}(E) \)-valued function on \([-h, 0], B \in \mathcal{L}(U, E), \; \xi \in E, \; \eta \in L^2([-h, 0], E) \). Introduce \( M^2([-h, 0], E) := E \times L^2([-h, 0], E) \) and \( A : D(\overline{A}) \subset M^2([-h, 0], E) \to M^2([-h, 0], E) \) defined by

\[
\overline{A} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} A\xi + N\eta(-h) + \int_{-h}^{0} M(\theta) \eta(\theta) \, d\theta \\ \frac{d}{d\theta} (\eta - \xi) \end{bmatrix},
\]

where

\[
D(\overline{A}) = \left\{ \begin{bmatrix} \xi \\ \eta \end{bmatrix} : \xi \in E, \; \eta \in L^2([-h, 0], E), \; \frac{d}{d\theta} \eta \in L^2([-h, 0], E), \; \eta(0) = \xi \right\}.
\]

\[
\overline{\eta} = \begin{bmatrix} y \\ \eta \end{bmatrix}, \quad \overline{\eta}_0 = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \overline{g}(\overline{\eta}) = \int_{0}^{b} h(s, y(s)) \, ds, \quad \overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \overline{f} = \begin{bmatrix} f \\ 0 \end{bmatrix}.
\]

The system \((27)\) can be written in the standard form

\[
\overline{y}'(t) = \overline{A}\overline{y}(t) + \overline{B}u(t) + \overline{f}(t, \overline{y}(t)),
\]

\[
\overline{y}(0) = \overline{y}_0 + \overline{g}(\overline{y}).
\]

Following \([10]\), we say that the system \((27)\) is

- (partial-)approximately controllable on \([0, T]\) if \( \{ y(T; u) : u \in L^2([0, T], U) \} = E \),
- approximately \( M^2 \)-controllable on \([0, T]\) if \( \{ y(T; u) : u \in L^2([0, T], U) \} = M^2([-h, 0], E) \).

Assume that \( \Phi(t) \) is a solution of the equation

\[
\Phi'(t) = A\Phi(t) + N\Phi(t - h) + \int_{-h}^{0} M(\theta) \Phi(t + \theta) \, d\theta, \; t \geq h,
\]

\[
\Phi'(t) = A\Phi(t), \quad t < h, \quad \Phi(0) = I.
\]

It is shown in \([10], [5]\) that the linear system corresponding to \((27)\) is partial-approximately controllable if

\[
B^* \Phi^*(t - t) y = 0, \; 0 < t < T \implies y = 0.
\]

Hence, the system \((27)\) is partial-approximately controllable on \([0, T]\) provided that all conditions of Theorem \([7]\) are satisfied.
6. Conclusion

In this paper, partial-approximate controllability of semilinear evolution systems in Hilbert spaces with nonlocal conditions have been investigated. Sufficient condition for the partial-approximate controllability of such systems have been established. Compared with some existing results, it can be found that the variational approach, together with the approximating technique, has been extended to consider the partial-approximate controllability of more general systems.

The key hypothesis on the nonlocal function $g$ is the assumption (G), which means that $g$ depends only on the value of $y$ in the interval $[\delta, b]$, $\delta > 0$. This hypothesis covers the situation that the nonlocal function $g$ is given by $g(y) = \sum_{k=1}^{p} c_k y(t_k)$, where $0 < t_1 < ... < t_p < b$, $c_1, ... c_p$ are given constants or by $g(y) = \int_{\delta}^{b} h(s, y(s)) \, ds$, $0 < \delta < b$. However, if the nonlocal function $g$ depends on the value of $y$ on the whole interval $[\delta, b]$, such as the nonlocal function $g$ is given by $g(y) = \int_{0}^{b} h(s, y(s)) \, ds$, then even the question of existence of a mild solution and corresponding controllability issues are still open. For this reason, we are committed to studying such problems in the future.

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