Mathematical aspects of the nuclear glory phenomenon; backward focusing and Chebyshev polynomials

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Abstract

The angular dependence of the cumulative particles production off nuclei near the kinematical boundary for multistep process is defined by characteristic polynomials in angular variables, describing spatial momenta of the particles in intermediate and final states. Physical argumentation, exploring the small phase space method, leads to the appearance of equations for these polynomials in $\cos(\theta/N)$, where $\theta$ is the polar angle of the momentum of final (cumulative) particle, the integer $N$ being the multiplicity of the process (the number of interactions). It is shown explicitly how these equations appear, and the recurrent relations between polynomials with different $N$ are obtained. Factorization properties of characteristic polynomials found previously, are extended, and their connection with known in mathematics Chebyshev polynomials of 2-d kind is established. As a result, differential cross section of the cumulative particle production has characteristic behaviour $d\sigma \sim 1/\sqrt{\pi - \theta}$ near the strictly backward direction ($\theta = \pi$, the backward focusing effect). Such behaviour takes place for any multiplicity of the interaction, beginning with $n = 3$, elastic or inelastic (with resonance excitations in intermediate states), and can be called the nuclear glory phenomenon, or 'Buddha’s light' of cumulative particles.

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1 Introduction

One of directions of studies in high energy nuclear physics are studies of the high energy particles (nuclei) interactions with nuclei. Intensive studies of the particles production processes in high energy interactions of different projectiles with nuclei, in regions forbidden by kinematics for the interaction with a single free nucleon, or cumulative particles production, started at 70-th mostly in Dubna (JINR), beginning with the paper [1], and in Moscow (ITEP) [2, 3, 4] (some restricted review of data can be found in [5, 6], and also in [7]-[16]).

The main goal of these studies was to find features or peculiarities of the nuclear structure, which make the nucleus \( A \) different from a collection of corresponding number (\( A \)) of weakly interacting nucleons, as it was established previously. The interpretation of these phenomena as being manifestation of internal structure of nuclei assumes that the secondary interactions, or, more generally, multiple interactions processes (MIP) do not play a crucial role in such production. Generally, the role of secondary interactions in the particles production off nuclei is at least two-fold: they decrease the amount of produced particles in the regions, where it is large (it is, in particular, the screening phenomenon), and increase the production probability in regions where it was small; so, they smash out the whole production picture.

In the case of the large angle particle production the background processes which mask the possible manifestations of nontrivial features of nuclear structure, are subsequent multiple interactions with nucleons inside the nucleus leading to the particles emission in the ”kinematically forbidden” region (KFR). The problem of the background is a real problem in most of physical experiments aimed to find new phenomena. In the case of cumulative particles production studies of multistep, or cascade processes have been unpopular among physisists \(^1\), because the main goal of experiments in this field was to reveal manifestations of new nontrivial effects in nuclear structure. However, it has been proved [10, 11, 12, 15, 16], that multistep processes provide not negligible contribution to the cumulative production cross sections, although other important contributions are not excluded and remain to be the main purpose of further studies.

The small phase space method developed previously in [11, 12, 15] allows to get analytical expressions for the probability of the multiple interaction processes near the corresponding kinematical boundaries. The quadratic form in angular variables (deviations from the optimal kinematics) plays the key role in this approach. The recurrent relations for the characteristic polynomials in polar angles deviations have been obtained in [6] and are reproduced in present paper. The connection of these polynomials with Chebyshev polynomials of 2-d kind, known in mathematics since middle of 19-th century [17, 18, 19] and used in approximation theory, has been established. It is an example of interest when physics arguments have led to some results in mathematics.

The aim of present paper is to provide mathematically complete explanation of essential details of the method used in [11, 12, 15], which were not complete just from mathematical point of view. In the next section the peculiarities of kinematics of the processes in KFR are recalled, in section 3 the small phase space method of the MIP contributions calculation to the particles production cross section in KFR is described according to [11, 12, 15, 16]. In section 4 the characteristic polynomials in polar angle deviations from the optimal kinematics which define the angular dependence of the cross section on the emission angle of the final

\(^1\)Moreover, such studies have been absolutely out of fashion during long period.
(cumulative) particle and lead to the backward focusing effect, similar to the known in optics glory phenomenon. The recurrent relations between polynomials with different $N$ are obtained, and their connection with Chebyshev polynomials of 2-d kind (Chebyshev-Korkin-Zolotarev, or CKZ-polynomials) is established. Some generalizations for the case of inelastic rescatterings are presented as well. Final section contains discussion of problems and conclusions. Appendix contains some technical details which can be useful for careful reading of the paper.

2 Features of kinematics of the processes in KFR

When the particle with 4-momentum $k_0$ interacts with the nucleus with the mass $m_t \simeq A m_N$, and the final particle of interest has the 4-momentum $k_f$ the basic kinematical relation is

$$(k_0 + p_t - k_f)^2 \geq M_f^2,$$ (2.1)

where $M_f$ is the sum of the final particles masses, except the detected particle of interest. At large enough incident energy, $\omega_0 = k_0^0 \gg M_f$, we obtain easily

$$\omega_f - z k_f \leq m_t,$$ (2.2)

which is the basic restriction for such processes. Here $k_f = |\vec{k}_f|$, $z = \cos \theta < 0$ for particle produced in backward hemisphere. The quantity $(\omega_f - z k_f)/m_N$ has been called the cumulative number (more precise, the integer part of this ratio plus 1).

Let us recall some peculiarities of the multistep processes kinematics established first in [11, 12] and described in details in [15].

2.1 Rescatterings

For light particles (photon, also $\pi$-meson) iteration of the Compton formula

$$\frac{1}{\omega_n} - \frac{1}{\omega_{n-1}} \simeq \frac{1}{m} [1 - \cos(\theta_n)]$$ (2.3)

allows to get the final energy in the form

$$\frac{1}{\omega_N} - \frac{1}{\omega_0} = \frac{1}{m} \sum_{n=1}^{N} [1 - \cos(\theta_n)]$$ (2.4)

The maximal energy of final particle is reached for the coplanar process when all scattering processes take place in the same plane and each angle equals to $\theta_k = \theta/N$. As a result we obtain

$$\frac{1}{\omega_N^{\text{max}}} - \frac{1}{\omega_0} = \frac{1}{m} N [1 - \cos(\theta/N)]$$ (2.5)

Already at $N > 2$ and for $\theta \leq \pi$ the expansion can be made

$$1 - \cos(\theta/N) \simeq \theta^2/2N^2$$ (2.6)
and for large enough $\omega_0$ we obtain

$$\omega^\text{max}_N \simeq N \frac{2m}{\theta^2}. \tag{2.7}$$

This means that the kinematically forbidden for interaction with one nucleon region is partly filled up due to elastic rescatterings. Remarkably, that this rather simple property of rescattering processes has not been even mentioned in pioneer papers [1] - [3].

In the case of the nucleon-nucleon scattering (scattering of particles with equal nonzero masses in general case) it is convenient to introduce the factor

$$\zeta = \frac{p}{E + m}, \tag{2.8}$$

where $p$ and $E$ are spatial momentum and total energy of the particle with the mass $m$. When scattering takes place on the particle which is at rest in the laboratory frame, the $\zeta$ factor of scattered particle is multiplied by $\cos \theta$, where $\theta$ is the scattering angle in the laboratory frame. So, after $n$ rescatterings we obtain the $\zeta$ factor

$$\zeta_N = \zeta_0 \cos \theta_1 \cos \theta_2 \ldots \cos \theta_N. \tag{2.9}$$

As in the massless case, the maximal value of final $\zeta_N$ is obtained when all scattering angles are equal

$$\theta_1 = \theta_2 = \ldots = \theta_N = \theta/N, \tag{2.10}$$

and the process proceeds in one plane. So, we have

$$\zeta_N^\text{max} = \zeta_0 \left[\cos (\theta/N)\right]^N. \tag{2.11}$$

The final momentum is from here

$$p^\text{max}_N = 2m \frac{\zeta_N^\text{max}}{1 - (\zeta_N^\text{max})^2}. \tag{2.12}$$

Again, at large enough $N$ and large incident energy ($\zeta_0 \rightarrow 1$) the expansion can be made at $k \gg m$, and we obtain

$$p^\text{max}_N \simeq N \frac{2m}{\theta^2}. \tag{2.13}$$

which coincides with previous result for the rescattering of light particles. However, the preasymptotic corrections to this result are greater than in former case of the light particle.

The normal Fermi motion of nucleons inside the nucleus makes these boundaries wider [15]:

$$p^\text{max}_N \simeq N \frac{2m}{\theta^2} \left[1 + \frac{p_F^\text{max}}{2m} \left(\theta + \frac{1}{\theta}\right)\right], \tag{2.14}$$

at $\theta \sim \pi$, where it is supposed that the final angle $\theta$ is large. For numerical estimates we took the step function for the distribution in the fermi momenta of nucleons inside of nuclei, with $p_F^\text{max} / m \simeq 0.27$ [15] and references there.

There is characteristic decrease (down-fall) of the cumulative particle production cross section due to simple rescatterings, near the strictly backward direction. However, inelastic processes with excitations of intermediate particles, i.e. with intermediate resonances, are able to fill up the region at $\theta \sim \pi$.

\[2\] This property was well known, however, to V.M.Lobashev, who observed experimentally that the energy of the photon after 2-fold interaction can be substantially greater than the energy of the photon emitted at the same angle in 1-fold interaction.
2.2 Resonance excitations in intermediate states

The elastic rescatterings themselves are only the “top of the iceberg”. Excitations of the rescattered particles, i.e. production of resonances in intermediate states which go over again into detected particles in subsequent interactions, provide the dominant contribution to the production cross section. Simplest examples of such processes may be $NN \rightarrow NN^* \rightarrow NN$, $\pi N \rightarrow \rho N \rightarrow \pi N$, etc.

To produce the final particle near the absolute boundary one needs to have the masses of intermediate resonances (or some other object) of the order of incident energy, $s \sim k_0^2 m_A$. As a first example let us consider the interaction with the deuteron.

The mass of the object in intermediate state equals

$$M^2_1 = (k_0 - k_2)d/2 + (k_0^2 + k_2^2)/2 + p_2^2 - d^2/4,$$  \hspace{1cm} (2.14)$$

where $k_0$ and $k_2$ are the 4-momenta of the incident and final particles, $p_2$ is the 4-momentum each of final nucleons, $d$ - the deuteron 4-momentum. The motion of nucleons inside the deuteron leads to considerable decrease of the mass $M_1$ necessary to produce the particle at the absolute kinematical boundary.

It is not difficult to find the necessary masses of intermediate states in the extreme case when the emitted particle is just on the absolute boundary for the interaction with arbitrary nucleus of atomic number $A$ \cite{12, 15}. Calculation of contributions of such processes to the production cross section of cumulative particle of interest is not possible due to lack of information about elementary processes amplitudes of resonances excitation - deexcitation at necessary energies and momentum transfers.

What is very important: at arbitrary high incident energy the kinematics of all subsequent processes is defined by the momentum and the angle of the outgoing particle. The number of different processes (with different resonances excited in intermediate states) is very large and grows exponentially with increasing $N$ - the number of interactions (like $(N_R + 1)^{(N-1)}$, where $N_R$ is the number of resonances. Therefore, for any final (cumulative particle) momentum $k$ at any emission angle $\theta$ there are processes, which kinematical boundary is just near this value of $k$, and our consideration can be applied.

3 The small phase space method for the MIP probability calculations

This method, most adequate for analytical and semi-analytical calculations of the MIP probabilities, has been proposed in \cite{12} and developed later in \cite{15}. It is based on the fact that, according to established in \cite{11, 12} and presented in previous section kinematical relations, there is a preferable plane of the whole MIP leading to the production of energetic particle at large angle $\theta$, but not strictly backwards. Also, the angles of subsequent rescatterings are close to $\theta/N$. Such kinematics has been called optimal, or basic kinematics. The deviations of real angles from the optimal values are small, they are defined mostly by the difference $k_N^{max} - k$, where $k_N^{max}(\theta)$ is the maximal possible momentum reachable for definite MIP, and $k$ is the final momentum of the detected particle. $k_N^{max}(\theta)$ should be calculated taking into account normal Fermi motion of nucleons inside the nucleus, and also resonances excitation — deexcitation in
the intermediate state. Some high power of the difference \((k_{N}^{\text{max}} - k)/k_{N}^{\text{max}}\) enters the resulting probability.

Within the quasiclassical treatment adequate for our case, the probability product approximation is valid, and the following starting expression for the inclusive cross section of the particle production at large angles takes place (see, e.g., Eq. (4.11) of [15]):

\[
\frac{d^{3}k}{\omega} f_{N} = \pi R_{A}^{2} G_{N}(R_{A}, \theta) \int \frac{f_{1}(\vec{k}_{1}) d^{3}k_{1}}{\sigma_{1}^{\text{leav}}(m)} \prod_{l=2}^{N} M_{k}^{2}(s_{k}, t_{k}) \delta(m + \omega_{l-1} - \omega_{l} - \omega_{l-1}) \frac{d^{3}k_{l}}{(8\pi)^{2}\sigma_{l}^{\text{leav}}m k_{l-1} \omega_{l-1}}
\]

(3.1)

Here \(\sigma_{l}^{\text{leav}}\) is the cross section defining the removal (or leaving) of the rescattered object at the corresponding section of the trajectory. It includes all inelastic cross section, the part of elastic cross section and the part of the resonance production cross sections, and can be considerably smaller than the total interaction cross section of the \(l\)-th intermediate particle with nucleon. \(G_{N}(R_{A}, \theta)\) is the geometrical factor defining the probability of the \(N\)-fold interaction with definite trajectory of the interacting particles (resonances) inside the nucleus. This trajectory is defined mostly by the final values of \(\vec{k}, \theta\), according to the kinematical relations of previous section. \(f_{1} = \omega_{1} d^{3}\sigma_{1}/d^{3}k_{1}, \omega_{N}' = \omega\) — the energy of the observed particle.

After some evaluation and introducing the differential cross sections of binary reactions \(d\sigma_{l}/dt_{l}(s_{l}, t_{l})\) instead of the matrix elements of binary reactions \(M_{k}^{2}(s_{l}, t_{l})\), we came to the formula for the production cross section due to the \(N\)-fold MIP [12, 15]

\[
f_{N}(\vec{p}_{0}, \vec{k}) = \pi R_{A}^{2} G_{N}(R_{A}, \theta) \int \frac{f_{1}(\vec{p}_{0}, \vec{k}_{1}) (k_{1}^{0})^{3} x_{1}^{2} d\Omega_{1} \prod_{l=2}^{N} \left( \frac{d\sigma_{l}(s_{l}, t_{l})}{dt_{l}} \right)}{\omega_{l}^{N-1} \prod_{l=2}^{N-1} k_{l}(m + \omega_{l-1} - z_{l} \omega_{l} k_{l-1})} \frac{1}{\omega_{N}' \delta(m + \omega_{N-1} - \omega_{N} - \omega_{N}')}.
\]

(3.2)

### 3.1 Rescattering of the light particle (\(\pi\)-meson)

Further details depend on the particular process. For the case of the light particle rescattering, \(\pi\)-meson for example, \(\mu_{l}^{2}/m^{2} \ll 1\), we have

\[
\frac{1}{\omega_{N}'} \delta(m + \omega_{N-1} - \omega_{N} - \omega_{N}') = \frac{1}{kk_{N-1}} \delta \left[ \frac{m}{k} - \sum_{l=2}^{N} (1 - z_{l}) - \frac{1}{x_{1}} \left( \frac{m}{p_{0}} + 1 - z_{l} \right) \right]
\]

(3.3)

To obtain this relation, one should use the equality (energy-momentum conservation in the last interaction act)

\[
\omega_{N}' = \sqrt{m^{2} + (\vec{k}_{N-1}^{2} - \vec{k})^{2}}
\]

and the known rules for manipulations with the \(\delta\)-function, see Appendix. When the final angle \(\theta\) is considerably different from \(\pi\), there is a preferable plane near which the whole multiple interaction process takes place, and only processes near this plane contribute to the final output. At the angle \(\theta = \pi\), strictly backwards, there is azimuthal symmetry, and the processes from the whole interval of azimuthal angle \(\phi\), defining the plane of the process, \(0 < \phi < 2\pi\), provide contribution to the final output (azimuthal focusing, see next section). A necessary step is to
introduce azimuthal deviations from this optimal kinematics, \( \varphi_k, \, k = 1, \ldots, N - 1 \). \( \varphi_N = 0 \) by definition of the plane of the process, \((\vec{p}_0, \vec{k})\). Polar deviations from the basic values, \( \theta/N \), are denoted as \( \vartheta_k \), obviously, \( \sum_{k=1}^{N} \vartheta_k = 0 \). The direction of the momentum \( \vec{k}_l \) after \( l \)-th interaction, \( \vec{m}_l \), is defined by the azimuthal angle \( \varphi_l \) and the polar angle \( \theta_l = (\theta/N) + \vartheta_1 + \ldots + \vartheta_l \).

Then we obtained [12, 15] making the expansion in \( \varphi_l, \, \vartheta_l \) and including quadratic terms in these variables:

\[
z_k = (\vec{m}_k \vec{m}_{k-1}) \simeq \cos(\theta/N)(1 - \vartheta_k^2/2) - \sin(\theta/N)\vartheta_k + \sin(k\theta/N) \sin((k - 1)\theta/N)(\varphi_k - \varphi_{k-1})^2/2. \tag{3.4}
\]

In the case of the rescattering of light particles the sum enters the phase space of the process

\[
\sum_{k=1}^{N} (1 - \cos\vartheta_k) = N[1 - \cos(\theta/N)] + \cos(\theta/N) \sum_{k=1}^{N} \left[ - \varphi_k^2 \sin^2(k\theta/N) + \frac{1}{\cos(\theta/N)} \sin(k\theta/N) \sin((k - 1)\theta/N) \right] - \frac{\cos(\theta/N)}{2} \sum_{k=1}^{N} \vartheta_k^2 \tag{3.5}
\]

To derive this equality we used that \( \varphi_N = \varphi_0 = 0 \) — by definition of the plane of the MIP, and the mentioned relation \( \sum_{k=1}^{N} \vartheta_k = 0 \). We used also the identity, valid for \( \varphi_N = \varphi_0 = 0 \):

\[
\frac{1}{2} \sum_{k=1}^{N} \left( \varphi_k^2 + \varphi_{k-1}^2 \right) \sin(k\theta/N) \sin((k - 1)\theta/N) = \cos(\theta/N) \sum_{k=1}^{N} \varphi_k^2 \sin^2(k\theta/N). \tag{3.5a}
\]

It is possible to present it in the canonical form and to perform integration easily, see Appendix and Eq. (4.23) of [15]. As a result, we have the integral over angular variables of the following form:

\[
I_N(\varphi, \vartheta) = \int \delta(\Delta^\text{ext} - z_N \left( \sum_{k=1}^{k=N} \varphi_k^2 - \varphi_k \varphi_{k-1}/z_N + \vartheta_k^2/2 \right)) \prod_{l=1}^{N-1} d\varphi_l d\vartheta_l = \frac{(\Delta^\text{ext})^{N-2} (\sqrt{2})^{N-1}}{J_N(z_N) \sqrt{N(N - 2)!z_N^{N-1}}}, \tag{3.6}
\]

Since the element of a solid angle \( d\Omega_l = \sin(\theta l/N) d\vartheta_l d\varphi_l \), we made here substitution \( \sin(\theta l/N) d\varphi_l \rightarrow d\varphi_l, \, z_N = \cos(\theta/N), \, \Delta^\text{ext} \simeq m/k - m/p_0 - N(1 - z_N) + (1 - x_1)m/p_0 \).

\( \Delta^\text{ext}_N \) defines the distance of the momentum (energy) of the emitted particle \( \vec{k}, \, \omega \) from the kinematical boundary for the whole \( N \)-fold MIP.

\[
J_N^2(z) = Det \|a_N\|, \tag{3.7}
\]

where the matrix \( \|a\| \) defines the quadratic form \( Q_N(z) \) which enters the argument of the \( \delta \)-function in Eq. (3.6):

\[
Q_N(z, \varphi_k) = a_{kl}\varphi_k \varphi_l = \sum_{k=1}^{k=N} \varphi_k^2 - \frac{\varphi_k \varphi_{k-1}}{z}. \tag{3.8}
\]

For example:

\[
Q_2 = \varphi_1^2, \quad Q_3 = \varphi_1^2 + \varphi_2^2 - \frac{\varphi_1 \varphi_2}{\cos(\theta/3)}; \quad Q_4 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 - \frac{\varphi_1 \varphi_2}{\cos(\theta/4)} - \frac{\varphi_2 \varphi_3}{\cos(\theta/4)}; \ldots
\]
Properties of these quadratic forms are considered in the next section.

The phase space of the process (3.3) which depends strongly on $\Delta_N^{ext}$ after integration over angular variables takes the form

$$\Phi_{nucleons}^{pions} = \frac{1}{\omega_N} \delta(m + \omega_{N-1} - \omega_N - \omega'_N) \prod_{l=1}^{N} d\Omega_l = \frac{(\sqrt{2\pi})^{N-1}(\Delta_N^{ext})^{N-2}}{kk_{N-1}(N-2)!\sqrt{N}J_N(z_N)z_N^{N-1}}$$

\[ (3.15) \]

### 3.2 Rescattering of the nucleon

For the case of nucleons rescattering there are some important differences from the light particle case, but the quadratic form which enters the phase space of the process is essentially the same:

$$\Phi_{nucleons}^{\text{nucl}} = \frac{1}{\omega_N} \delta(m + \omega_{N-1} - \omega_N - \omega'_N) \prod_{l=1}^{N} d\Omega_l = \int \delta \left[ \Delta_N^{\text{nucl}} - z_N^NQ_N(\varphi_k) - \frac{z_N^{N-2}}{2} \sum_{l=1}^{N} \vartheta_l^2 \right] =$$

$$= \frac{(\sqrt{2\pi})^{N}(\Delta_N^{ext})^{N-2}}{kk_{N-1}(N-2)!\sqrt{N}J_N(z_N)z_N^{N-1}}$$

\[ (3.16) \]

The normal Fermi motion of target nucleons inside of the nucleus increases the phase space considerably [12, 15]:

$$\Delta_N^{ext} = \Delta_N^{ext}|_{p_f=0} + \vec{p}_f^2/2m,$$

where $\vec{r}_l = 2m(\vec{k}_l - \vec{k}_{l-1})/k_lk_{l-1}$. A reasonable approximation is to take vectors $\vec{r}_l$ according to the optimal kinematics for the whole process, and the Fermi momenta distribution of nucleons inside of the nucleus in the form of the step function.

For the case of the nucleons rescattering some difference from the case of the light particle rescattering takes place, but the axial focusing effect persists.

$$\Phi_N = \frac{1 + \zeta_1^2}{k(m + \omega_{N-1})\zeta_N(1 - \zeta_1^2)} \frac{\sqrt{2\pi}}{\zeta_N} \left[ \frac{\Delta_N - k/(\omega + m)]}{\zeta_N}\zeta_N^N(3N-1)! \right]$$

\[ (3.17) \]

We obtained for this case, taking into account the normal Fermi motion of nucleons in the nucleus ( [12] and Eq. (4.25) of [15]):

$$\Phi_{nucleons} = \int_0^{1/\omega_N} \delta(m + \omega_{N-1} - \omega_N - \omega'_N) \prod_{l=1}^{N} d\Omega_l \rho(\vec{p}_F) d^3\vec{p}_F =$$

$$= \frac{z_N^{N^2+N-1}(1 + \zeta_0^2)(\sqrt{2\pi})^{N-1}(3/2\zeta_0^2)^N(3N-1)! \prod_{i=1}^{N} 1 - \Delta_i^2}{k(m + \omega_{N-1})J_N(z_N)\zeta_0^{N-1}(1 - \zeta_0^2)\sqrt{N}(3N-1)!}$$

\[ (3.18) \]

with $b = p_F^{max}/2m$, $r_i^{\text{nucl}} = 2m|\vec{k}_i - \vec{K}_{i-1}|/k_i(m + \omega_{i-1})$. At high enough incident energy substitution $\zeta_0 \rightarrow 1$ can be done.
4 Quadratic form in angular deviations, characteristic polynomials and their properties

4.1 Relations between characteristic polynomials

The obvious recurrent relation takes place for the quadratic form in azimuthal deviations:

$$Q_{N+1}(z, \varphi_k, \varphi_l) = Q_N(z, \varphi_k, \varphi_l) + \varphi_N^2 - \varphi_N \varphi_{N-1}/z,$$  \hspace{1cm} (4.1)

where $z = \cos[\theta/(N + 1)]$, has the same value in both sides of this equation, $\varphi_{N+1} = 0$ by definition of the plane of the process. Relation (4.1) plays a key role to establish the connection of characteristic polynomials (4.3), (4.5) with Chebyshev polynomials of 2-d kind, noted at first in [16].

Let $t$ be the transformation (matrix) which brings our quadratic form to the canonical form:

$$\tilde{t}a t = I,$$  \hspace{1cm} (4.2)

where $I$ is the unit matrix $n \times n$, and $\tilde{t}_{kl} = t_{lk}$. Then the equality takes place for the Jacobian of this transformation

$$(\det ||t||)^{-2} = J_a^2(z) = \det ||a||, \hspace{1cm} (\det ||t||)^{-1} = J_a(z) = \sqrt{\det ||a||}. \hspace{1cm} (4.3)$$

The matrices $||a_{ij}||^N$ can be presented in the following symmetric form:

$$||a_{ij}||^{N=3} = \begin{vmatrix} 1 & -\frac{1}{2\cos(\theta/3)} & 0 \\ -\frac{1}{2\cos(\theta/3)} & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \hspace{1cm} \text{Det} ||a_{ij}||^{N=3} = 1 - \frac{1}{4\cos^2\theta/3}, \hspace{1cm} (4.4)$$

$$||a_{ij}||^{N=4} = \begin{vmatrix} 1 & -\frac{1}{2\cos(\theta/4)} & 0 & 0 \\ -\frac{1}{2\cos(\theta/4)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \hspace{1cm} \text{Det} ||a_{ij}||^{N=4} = 1 - \frac{1}{4\cos^2\theta/4} + \frac{1}{2\cos\theta/4} \left( -\frac{1}{2\cos\theta/4} \right) = 1 - \frac{1}{2\cos^2\theta/4}. \hspace{1cm} (4.5)$$

For arbitrary $N$ the following general expression for the matrix $||a_{ij}||^N$ can be written:

$$||a_{ij}||^N = \begin{vmatrix} 1 & -\frac{1}{2\cos(\theta/N)} & 0 & \ldots & 0 & 0 \\ -\frac{1}{2\cos(\theta/N)} & 1 & -\frac{1}{2\cos(\theta/N)} & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & -\frac{1}{2\cos(\theta/N)} \\ 0 & 0 & 0 & \ldots & -\frac{1}{2\cos(\theta/N)} & 1 \end{vmatrix}. \hspace{1cm} (4.7)$$

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According to well known rules for the calculations of the determinants of matrices, the relation can be easily established

\[ \text{Det} |a_{ij}|^N = J_N^2 = J_{N-1}^2 - \frac{1}{4 \cos^2(\theta/N)} J_{N-2}^2 \]  

(4.8)

Using this recurrent relation, it is possible to present the quadratic form in \( \phi_k, \phi_l \), which enters the \( \delta \)-function as the sum of squares of certain combinations of deviations \( \varphi_k \), which will be convenient in future applications:

\[ Q_N(\varphi_k, \varphi_l) = J_2^2 \left( \varphi_1 - \frac{\varphi_2}{2z J_2^2} \right)^2 + J_3^2 \left( \varphi_2 - \frac{J_2^2 \varphi_3}{2z J_3^2} \right)^2 + \ldots \]

\[ \ldots + J_{N-1}^2 \left( \varphi_{N-2} - \frac{J_{N-2}^2 \varphi_{N-1}}{2z J_{N-1}^2} \right)^2 + \frac{J_N^2}{J_{N-1}^2} \varphi_{N-1}^2 \]  

(4.9)

with \( J_2^2 = 1 \).

The following general formula for \( J_N^2(z_N) \) has been obtained in [15], Eq. (4.23)\(^3\):

\[ \text{Det} |a_{kl}| = J_N^2(z_N) = J_N^2(\theta/N) = 1 + \sum_{m=1}^{m<N/2} \left( -\frac{1}{4z_N^2} \right)^m \frac{\prod_{k=1}^m (N - m - k)}{m!} = \]

\[ = 1 + \sum_{m=1}^{m<N/2} \left( -\frac{1}{4z_N^2} \right)^m C_{N-m-1}^m, \]  

(4.10)

\( z_n = \cos(\theta/n), \text{Det} |a_{kl}| \) is the determinant of the matrix \(|a|\), \( C_n^m \) is the number of combinations.

As it became clear to us recently, the polynomials (4.7) coincide, up to some factor depending on \( z = \cos(\theta/N) \) with Chebyshev polynomials of 2-d kind, discovered in the middle of 19-th century [17, 18, 19]. The connection of characteristic polynomials \( J_N^2(\cos(\theta/N)) \) with Chebyshev polynomials of 2-d kind (CKZ-polynomials) will be described in next section.

The angle \( \theta_1 \) - the polar angle of the momentum of the particle in intermediate state - is different from \( \theta/N \) if the resonance excitation takes place. It can take place in the first interaction act, and in any of subsequent interactions. But the optimal kinematics is changed in any of these processes. We can write then \( \theta_1 = \theta/N + \delta_1 \), assuming that \( \delta_1 \ll \theta/N \).

### 4.2 Algebra of characteristic polynomials

The condition \( J_N(\pi/N) = 0 \) leads to the equation for \( z_N \) which solution (one of all possible roots) provides the value of \( \cos(\pi/N) \) in terms of radicals. The following expressions for these jacobians take place [12]:

\[ J_2^2(z) = 1; \quad J_3^2(z) = 1 - \frac{1}{4z^2}; \quad J_4^2(z) = 1 - \frac{1}{2z^2}; \quad J_5^2 = 1 - \frac{3}{4z^2} + \frac{1}{16z^4} \]  

(4.11)

\(^3\)In the paper [12], Eq (15) this formula has been presented for \( N \) up to \( N = 5 \).
The given text is a mathematical exposition, discussing the properties of Bessel functions. The text is presented in a narrative form, explaining the behavior of these functions under certain conditions.

For instance, it states:

\[ J_3(\pi/3) = J_3(z = 1/2) = 0, \quad J_4(\pi/4) = I_4(z = 1/\sqrt{2}) = 0, \quad \cos^2(\pi/5) = (3 + \sqrt{5})/8. \]

The case \( N = 2 \) is a special one, because \( J_2(z) = 1 \) is a constant. But in this case the 2-fold process at \( \theta = \pi \) (strictly backwards) has no advantage in comparison with the direct one, if we consider the elastic rescatterings.

For \( N = 5 \)

\[ J^2 = 1 - \frac{3}{4z^2} + \frac{1}{16z^4}, \quad (4.12) \]

Several other examples were given later in [15], and recently in [5, 6].

At \( N = 6 \)

\[ J^2_6 = 1 - \frac{1}{z^2} + \frac{3}{16z^4} = J^2_3 \left(1 - \frac{3}{4z^2}\right). \quad (4.13) \]

For \( N = 7 \)

\[ J^2_7 = 1 - \frac{5}{4z^2} + \frac{3}{8z^4} - \frac{1}{64z^6}. \quad (4.14) \]

\( J_7(\pi/7) = 0 \). One of the solutions of equation \( J^2(z) = 0 \) is \( z = \cos(\pi/7) \), which can be verified using the method by Ferro and Tartaglia (published by Cardano, see e.g. [17]). Other two solutions can be found easily, when we divide the polynomial \( J^2(z) \) by \( z - \cos(\pi/7) \).

\[ J^2_8 = 1 - \frac{3}{2z^2} + \frac{5}{8z^4} - \frac{1}{16z^6} = J^2_4 \left(1 - \frac{1}{z^2} + \frac{1}{8z^4}\right), \quad (4.15) \]

\( J_8(\pi/8) = 0 \). For arbitrary \( N \), \( J^2_N \) is a polynomial in \( 1/4z^2 \) of the power \(|(N - 1)/2| \) (integer part of \((N - 1)/2\)). Since the solutions of the equations \( J^2_N(z) = 0 \) are not unknown in general form when the power of the polynomial is greater than 5, the knowledge of at least one solution, \( z = \cos(\pi/N) \), can be helpful.

The relation can be obtained from here

\[ J^2_N(z) = J^2_{N-k}(z)J^2_{k+1}(z) - \frac{1}{4z^2}J^2_{N-k-1}(z)J^2_k(z) \quad (4.16) \]

which, at \( N = 2m, \ k = m \) (\( m \) is the integer), leads to remarkable relation

\[ J^2_{2m}(z) = J^2_m(z) \left(J^2_{m+1}(z) - \frac{1}{4z^2}J^2_{m-1}(z)\right) = J^2_m(z) \left(J^2_m(z) - \frac{1}{2z^2}J^2_{m-1}(z)\right). \quad (4.17) \]

This relation can be rewritten in another form, convenient for further investigations:

\[ J^2_{m+n}(z) = J^2_m(z)J^2_n(z) - \frac{1}{4z^2}J^2_{m-1}(z)J^2_n(z) = J^2_{m+1}(z)J^2_n(z) - \frac{1}{4z^2}J^2_m(z)J^2_{n-1}(z) \quad (4.18) \]

This relation can be verified easily for \( J^2_1, J^2_6 \) and \( J^2_8 \), see section 4. It follows from here that at \( N = 2m \) not only \( J_N(\pi/N) = 0 \), but also \( J_N(2\pi/N) = 0 \) which has quite simple explanation (\( m \geq 3 \); the case of \( m = 2 \) is an exception).

For \( N = 3m \) we obtain

\[ J^2_{3m}(z) = J^2_m \left\{ (J^2_m)^2 \left(1 - \frac{1}{4z^2}\right) - \frac{3}{4z^2}J^2_mJ^2_{m-1} + \frac{3}{16z^4} (J^2_{m-1})^2 \right\} \quad (4.19) \]

or
\[ J_{3m}^2(z) = J_m^2 \left\{ \left( J_{m+1}^2 \right)^2 - \frac{1}{4z^2} \left[ J_{m-1}^2 J_{m+1}^2 + \left( J_m^2 \right)^2 \right] + \frac{1}{16z^4} \left( J_{m-1}^2 \right)^2 \right\}. \] (4.20)

For arbitrary odd values of \( N \) another useful factorization property takes place:

\[ J_{2m+1}^2(z) = \left( J_{m+1}^2(z) \right)^2 - \frac{1}{4z^2} \left( J_m^2(z) \right)^2 = \left( J_{m+1}^2(z) - \frac{1}{2z} J_m^2(z) \right) \left( J_{m+1}^2(z) + \frac{1}{2z} J_m^2(z) \right), \] (4.21)

which can be easily verified for \( J_7^2 \) and \( J_5^2 \) given in section 4.

Evidently, it can be shown by induction, that if \( N = pm \) - the product of two integers \( p \) and \( m \), then

\[ J_{pm}^2 \sim J_m^2. \]

Indeed, we have from relation (4.19) at \( N = pm, k = m - 1: \)

\[ J_{pm}^2 = J_m^2 J_{pm-m}^2 - \frac{1}{4z^2} J_{m-1}^2 J_{(p-1)m}. \] (4.22)

It follows from this equality, that if \( J_{(p-1)m}^2 \sim J_m^2, p - 1 > 1 \), then \( J_{pm}^2 \sim J_m^2 \) as well. Obviously, at the same time

\[ J_{pm}^2 \sim J_p^2. \]

This means, that \( J_{pm}^2 = 0 \) not only for \( z = \cos(\pi/pm) \), but also for \( z = \cos(\pi/p) \) and for \( z = \cos(\pi/m) \). This factorization property allows to simplify the calculation of the roots of these polynomials.

The connection of polynomials \( J_N^2(z) \) with Chebyshev polynomials of 2-d kind, or CKZ-polynomials (Chebyshev-Korkin-Zolotarev, see further), explains naturally their factorization properties.

Some further examples are of interest.

\[ J_{12}^2 = J_3^2 J_4^2 \left( 1 - \frac{3}{4z^2} \right) \left( 1 - \frac{1}{z^2} + \frac{1}{16z^4} \right). \] (4.23)

\( \cos(\pi/12) \) can be found as a root of the equation \( (1 - 1/z^2 + 1/16z^4) = 0 \) which can be solved easily, \( \cos(\pi/12) = \sqrt{2 + \sqrt{3}}/2. \)

Another simple example:

\[ J_{15}^2 = J_5^2 \left\{ (J_6^2)^2 - \frac{1}{4z^2} \left[ J_4^2 J_6^2 + J_9^2 \right] \right\}. \] (4.24)

Now we can use the above relations \( J_6^2 = J_3^2(1 - 3/(4z^2)) \) and \( J_9^2 = J_3^2(1 - 3/(2z^2) + 9/(16z^4)) - 1/(64z^6) \) to obtain

\[ J_{15}^2 = J_3^2 J_5^2 \left( 1 - \frac{9}{4z^2} + \frac{13}{8z^4} - \frac{3}{8z^6} + \frac{1}{256z^8} \right). \] (4.25)

Simple explanation of the properties of polynomials \( J_N^2 \) is presented in the next section.
Connection with Chebyshev polynomials of 2-d kind

The following useful relations have been found, which can be easily verified:

\[(2z^\theta_N)^{N-1} J_N^2 \left( z^\theta_N \right) \sin \frac{\theta}{N} = \sin \theta. \tag{5.1} \]

or

\[J_N^2(z) = \frac{1}{(2z^\theta_N)^{N-1}} \frac{\sin \theta}{\sin(\theta/N)} \tag{5.1a} \]

It follows immediately, that zeros of \(J_N(z)\) occur at \(\theta = m\pi\), \(m\) being any integer. Roots of \(J_N^2(z)\) are \(\cos(\theta/N)\).

Obviously, the right side of these equalities equals zero at \(\theta = \pi\), but \(\sin(\pi/N)\) is different from zero for any integer \(N \geq 2\). Therefore, the polynomial in \(\cos(\pi/N)\) in the left side of these equalities should be equal to zero. These relations provide the link between the general case considered at the beginning of section 4 and the particular case of the optimal kinematics with all scattering angles equal to \(\theta/N\).

The known in mathematics Chebyshev polynomials of 2-d kind [17, 18, 19] are defined as

\[U_n[\cos \theta] = \frac{\sin(n+1)\theta}{\sin \theta}. \tag{5.2} \]

For any number \(n\) these polynomials are defined as function of common variable \(x = \cos \theta\) which is confined in the interval \(-1 \leq x \leq 1\). The recurrent relation

\[U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \tag{5.3} \]

with \(x = \cos \theta\), can be easily checked using this definition. Indeed, (5.2) can be written as

\[U_{n+1}(x) + U_{n-1}(x) = 2xU_n(x), \tag{5.3a} \]

or

\[\sin(\alpha + \theta) + \sin(\alpha - \theta) = 2\sin \alpha \cos \theta \tag{5.3b} \]

with \(\alpha = (n+1)\theta\), which is well known trigonometrical relation. Several examples are presented in the table. Different equivalent general expressions for the CKZ-polynomials are presented in [18]:

\[U_n(x) = \sum_{k=0}^{[n/2]} C^k_{n-k} (2x)^{n-2k} = \]

\[= \sum_{k=0}^{[n/2]} C^2k+1_{n+1}(x^2-1)^k x^{n-2k} = \sum_{k=0}^{[n/2]} C^k_{2k-n-1}(2x)^{n-2k}, \]

where \(n > 0\), \([n/2]\) is the integer part of \(n/2\), \(C^m_n = n!/m!(n-m)!\) is the number of combinations. The first of these formulas coincides with the expression, presented in [15] up to some coefficient.

\[\text{According to [19], the Chebyshev polynomials of 2-d kind have been considered first by his pupils A.Korkin and E.Zolotarev and were named in honor of their teacher. Therefore, it is correct to name these polynomials Chebyshev-Korkin-Zolotarev, or CKZ-polynomials.} \]
It follows from above definition that zeros (roots) of polynomials take place when \(\sin(n+1)\theta = 0\), but \(\sin\theta\) is different from zero. So, there are \(n\) roots at

\[
\theta = \frac{\pi}{n+1}, \quad \theta = \frac{2\pi}{n+1}, \ldots \quad \theta = \frac{n\pi}{n+1},
\]

therefore we have the equations which define the values of \(\cos(k\pi/n)\) at arbitrary integer \(n\) and \(k\).

The orthonormality conditions for the CKZ polynomials have the form

\[
\int_{-1}^{1} U_m(x) U_n(x) = \pi \delta_{nm}
\]

which can be easily verified using the trigonometrical definition of these polynomials.

The relation between characteristic polynomials \(J^2(\cos(\theta/N))\) and CKZ-polynomials \(U_N(\cos(\theta/N))\) takes place

\[
\left(2z_N^\theta\right)^{N-1} J_N^2 = U_{N-1}(z_N^\theta) = \frac{\sin \theta}{\sin(\theta/N)}
\]

with \(z_N^\theta = \cos(\theta/N)\). It is sufficient to check this relation for some small values of \(N\), e.g. \(N = 2\) and \(3\) we have on the left side \(2\cos(\theta/2)\) and \(4\cos^2(\theta/3)\), and on the right side we have \(\sin\theta/\sin(\theta/2) = 2\cos(\theta/2)\) and \(\sin\theta/\sin(\theta/3) = 4\cos^2\theta - 1\). For greater \(N\) the equality will be valid because recurrency relations for the right and left sides are essentially the same. This connection was written explicitly in [16].

| \(N\) | \(J_N^2(z_N)\) | \(U_{N-1}(x)\) |
|------|----------------|----------------|
| 3    | 1 - 1/4x^2     | 4x^2 - 1       |
| 4    | 1 - 1/2x^2     | 8x^4 - 4x     |
| 5    | 1 - 3/4x^2 + 1/16x^4 | 16x^4 - 12x^2 + 1 |
| 6    | 1 - 1/x^2 + 3/16x^4 | 32x^6 - 32x^4 + 6x  |
| 7    | 1 - 5/4x^2 + 3/8x^4 - 1/64x^6 | 64x^6 - 80x^4 + 24x^2 - 1 |
| 8    | 1 - 3/2x^2 + 5/8x^4 - 1/16x^6 | 128x^8 - 192x^6 + 80x^4 - 8x |
| 8    | 1 - 7/4x^2 + 15/16x^4 - 5/32x^6 + 1/256x^8 | 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1 |

Table. Characteristic polynomials \(J_N^2(z)\) presented in [5, 6] and Chebyshev polynomials of 2-d kind \(U_{N-1}\) given in literature [17, 18, 19]. \(x = \cos(\theta/N)\). The connection \(U_{N-1}(x) = (2x)^{N-1} J_N^2(x)\) can be easily verified.

Evidently, zeros of \(J_N(z)\) occur at \(\theta = m\pi\), \(m\) being any integer, and roots of \(J_N^2(z)\) are \(\cos(m\pi/N)\).

It follows from here that \(\lim J_N^2|_{\theta \to 0} = N/2^{N-1}\).

Now, with the help of Eq. (5.6), we can cross-check the relations obtained in previous section, using known trigonometrical identities. Let us denote \(z = \cos\phi\). Then, Eq. (4.20) can be rewritten in trigonometrical form

\[
\frac{\sin N\phi}{\sin(2\cos\phi)^{N-1}} = \frac{\sin(N-k)\phi}{\sin(2\cos\phi)^{N-k-1}} \cdot \frac{\sin(k+1)\phi}{\sin(2\cos\phi)^k} - \frac{\sin(N-k-1)\phi}{\sin(2\cos\phi)^{N-k}} \cdot \frac{\sin k\phi}{\sin(2\cos\phi)^{k-1}}.
\]
After removal of some common factors in left and right sides, we come to the equality to be checked

\[ \sin N\phi \sin \phi = \sin (N - k)\phi \sin (k + 1)\phi - \sin (N - k - 1)\phi \sin k\phi, \]

which can be checked using known relations \( \sin (N - k)\phi = \sin N\phi \cos k\phi - \cos N\phi \sin k\phi, \) etc.

We can check other relations in similar way. From

\[ J_{2m}^2 = \frac{\sin(2m\phi)}{\sin \phi (2\cos \phi)^{2m-1}}; \quad J_{m+1}^2 = \frac{\sin(m + 1)\phi}{\sin \phi (2\cos \phi)^m}; \quad J_{m-1}^2 = \frac{\sin((m - 1)\phi)}{\sin \phi (2\cos \phi)^{m-2}} \]

and takes the form

\[ \frac{\sin(2m\phi)}{\sin \phi (2\cos \phi)^{2m-1}} = \frac{\sin(m\phi)}{\sin \phi (2\cos \phi)^{m-1}} \left[ \frac{\sin(m + 1)\phi}{\sin \phi (2\cos \phi)^m} - \frac{\sin(m - 1)\phi}{\sin \phi (2\cos \phi)^m} \right], \]

and after cancellation of common factors we come to

\[ \sin(2m\phi) = \frac{\sin(m\phi)}{\sin \phi} [\sin(m + 1)\phi - \sin(m - 1)\phi], \]

which can be easily verified using well known trigonometrical relations.

This relation can be cross-checked in similar way.

6 The backward focusing effect (Buddha’s light of cumulative particles)

This is the sharp enhancement of the production cross section at the strictly backward direction, \( \theta = \pi. \) This effect has been noted first experimentally in Dubna (incident protons, final particles protons and deuterons) [21] and somewhat later by Leksin’s group (incident protons of 7.5 Gev/c, emitted protons of 0.5 Gev/c) [22]. This striking effect was not well studied previously, both experimentally and theoretically. In the papers [12, 15] where the small phase space method has been developed, it was noted that this effect can appear due to multiple interaction processes. However, the consideration of this effect was not detailed enough, and estimates have not been made \(^5\).

Mathematically the focusing effect comes from the consideration of the phase space of the whole process in the method of the small phase space adequate in this case. It takes place

\(^5\)At 80-th I have discussed mechanisms of cumulative production with professor Ya.A.Smorodinsky who noted its analogy with known optical phenomenon - glory, or ”Buddha’s light”. The glory effect has been mentioned by Leksin and collaborators [24], however, it was not clear to authors of [24], can it be related to cumulative production, or not. In the case of the optical (atmospheric) glory phenomenon the light scatterings take place within droplets of water, or another liquid. A variant of the atmospheric glory theory can be found in [26]. However, the optical glory is still not fully understood, existing explanation is still incomplete, see, e.g. http://www.atoptics.co.uk/droplets/glofeat.htm. In nuclear physics the glory-like phenomenon due to Coulomb interaction has been studied in [27] for the case of low energy antiprotons (energy up to few KeV) interacting with heavy nuclei.
for any multiple interaction process, regardless the particular kind of particles or resonances in
the intermediate states. When the angle of cumulative particle emission is large, but different
from $\theta = \pi$, there is a preferred plane for the whole process, as it was explained in previous
section. The deviations of real angles of particles in intermediate states, including all azimuthal
angles, from the optimal, or basic kinematics with $\phi_k = 0$, $\theta_k = \theta/N$ are small. When the final
angle $\theta = \pi$, then integration over one of azimuthal angles takes place for the whole interval
$[0, 2\pi]$, which leads to the rapid increase of the resulting cross section.

General case is of special interest. The particular values of the polar angles of the
momenta in the rescattering process - which correspond to the maximal momentum of the final
emitted particle - depend on the masses of the rescattered objects. It can be proved, however,
that the backward focusing effect takes place for any values of the polar angles. For arbitrary
polar angles $\theta_k$

$$z_k = (\vec{n}_k \vec{n}_{k-1}) \simeq \cos(\theta_k - \theta_{k-1})(1 - \vartheta_k^2/2) - \sin((\theta_k - \theta_{k-1}))\varphi_k +$$

$$+ \sin(\theta_k) \sin\theta_{k-1}(\varphi_k - \varphi_{k-1})^2/2. \quad (6.1)$$

After substitution $\sin\theta_k \varphi_k \to \varphi_k$ we obtain

$$z_k = (\vec{n}_k \vec{n}_{k-1}) \simeq \cos(\theta_k - \theta_{k-1})(1 - \vartheta_k^2/2) - \sin((\theta_k - \theta_{k-1}))\varphi_k + \frac{s_{k-1}}{2s_k} \varphi_k^2 +$$

$$+ \frac{s_k}{2s_{k-1}}(\varphi_{k-1} - \varphi_{k-1})\varphi_k, \quad (6.2)$$

where we introduced shorter notations $s_k = \sin\theta_k$.

The quadratic form depending on the azimuthal angles $\varphi_k$ for the $N$-fold process is

$$Q_N(\varphi, \varphi') = \frac{s_2}{s_1} \varphi_1^2 + \frac{s_1 + s_3}{s_2} \varphi_2^2 + \frac{s_2 + s_4}{s_3} \varphi_3^2 + \ldots + \frac{s_{N-2} + s_N}{s_{N-1}} \varphi_{N-1}^2 -$$

$$- 2\varphi_1 \varphi_2 - 2\varphi_2 \varphi_3 - \ldots - 2\varphi_{N-2} \varphi_{N-1},$$

with $s_N = \sin\theta$.

E.g., for $N = 5$ we have the matrix

$$||a||_{N=5}(\theta_1, \theta_2, \theta_3, \theta_4) = \begin{bmatrix}
\frac{s_{\theta_2}}{s_{\theta_4}} & -1 & 0 & 0 \\
-1 & \frac{(s_{\theta_1} + s_{\theta_3})}{s_{\theta_2}} & -1 & 0 \\
0 & -1 & \frac{(s_{\theta_2} + s_{\theta_4})}{s_{\theta_3}} & -1 \\
0 & 0 & -1 & \frac{(s_{\theta_3} + s_{\theta_4})}{s_{\theta_4}}
\end{bmatrix}, \quad (6.3)$$

Determinant of this matrix can be easily calculated:

$$\text{Det} (||a||_{N=5}) = J_5^2 = \frac{s_{\theta}}{s_{\theta_1}} \quad (6.4)$$

It can be shown further by induction that at arbitrary $N$

$$\text{Det} (||a||_{N}) = J_N^2 = \frac{s_{\theta}}{s_{\theta_1}} \quad (6.5)$$
It follows from the expression (3.2) for the matrix $||a||$

$$Det||a||_{N+1}(\theta) = \frac{s_{N-1} + s_\theta}{s_N} Det(||a||_N(s_N) - Det(||a||_{N-1}) (s_{N-1})$$

(6.6)

Since $Det(||a||_N(s_N) = s_N/s_1$ and $Det(||a||_{N-1}) = s_{N-1}/s_1$, we obtain easily

$$Det||a||_{N+1}(\theta) = \frac{s_{N-1} + s_\theta s_N}{s_N} - \frac{s_{N-1}}{s_1} = \frac{s_\theta}{s_1}$$

(6.7)

7 Cross section at singular point $\theta = \pi$

For particles emitted strictly backwards the phase space has different form, instead of $J_N(\theta/N)$ enters $J_{N-1}(\theta/N)$ which is different from zero at $\theta = \pi$, and we have instead of Eq. (3.6)

$$I_N(\varphi, \vartheta) = \int \delta \left[ \Delta^{ext} - z_N \left( \sum_{k=1}^{k=N} \varphi_k^2 - \varphi_k \varphi_{k-1}/z_N + \vartheta^2/2 \right) \right] \left[ \prod_{l=1}^{N-2} d\varphi_l d\vartheta_l \right] 2\pi d\vartheta_{N-1} =$$

$$= \frac{(\Delta^{ext})^{N-5/2} (2\sqrt{2\pi})^{N-1}}{J_{N-1}(z_N) \sqrt{N(2N-5)!} z_N^{N-3/2}}$$

(7.1)

This follows from the expression for the quadratic form in azimuthal angles, where $J_N(\pi/N) = 0$, and integration over $\phi_{N-1}$ takes place over the whole interval $[0, 2\pi]$.

To illustrate the axial focusing which takes place near $\theta = \pi$ the ratio is useful of the phase spaces at $\theta = \pi$ and near the backward direction. The ratio of the observed cross sections in the interval of several degrees slightly depends on the elementary cross sections and is defined mainly by this ratio. It is

$$R_N = \frac{\Phi(z)}{\Phi(\theta = \pi)} = \sqrt{\frac{\Delta^{ext}}{z_N}} \frac{(2n-5)! \sqrt{N} J_{N-1}(z_N)}{2^{N-1} \sqrt{N} J_N(z)}$$

(7.2)

Near $\theta = \pi$ we use that

$$J_N(z) \simeq \sqrt{[J_N^2]'(z_N) \sin \frac{\pi}{N} \frac{\pi - \theta}{N}}$$

(7.3)

and thus we get

$$R_N = C_N \sqrt{\frac{\Delta^{ext}}{\pi - \theta}}$$

(7.4)

with

$$C_N = \frac{J_{N-1}(z_N)}{[(J_N^2)'(z_N) \sin (\pi/N)]^{1/2}} \frac{\sqrt{N(2N-5)!}}{\sqrt{z_N(N-2)!} 2^{N-1}}$$

(7.5)

The phase space for the case of nucleons

$$\Phi_N(\theta = \pi) = \frac{1 + \zeta_t^2}{k(m + \omega_{N-1})} \frac{(\sqrt{2\pi})^{N-1}}{I_{N-1}(z_N) \sqrt{N(6N-3)!}} \left[ \zeta_N - k/(\omega + m) \right]^{N-1}$$

(7.6)

There are other data where the glory like effect is clearly seen. There are also data where the cross section enhancement near the final angle $\theta = \pi$ is not observed, but in such experiments the deviation of the final angle from 180 deg. usually is large, as e.g. in [22].
8 Conclusions

The origin of cumulative particles is not quite clear yet. The well known statement could characterize situation with these studies: "never had so many people in such a long period of time done so little success..."

The glory-like backward focusing effect, noted first experimentally at JINR [21], somewhat later observed at ITEP [22, 23], and studied in more details in [24, 25], was considered as a puzzle. We have shown that just consideration of the MIP contributions, very unpopular (absolutely out of fashion) during many years, allows, in principle, to explain this effect. Relatively simple physics argumentation, based on the small phase space method for description of the MIP probability in so called "kinematically forbidden regions" (cumulative particles production), leads to appearance of the characteristic polynomials in polar angles variables.

Some correct expressions for the probability of the MIP were obtained about 40 years ago [12], characteristic polynomials $J_N^2$ have been obtained here for the multiplicity $N \leq 5$. General expression for $J_N^2[\cos(\theta/N)]$ was presented in [15] more than 30 years ago. This is somewhat painstaking work, which did not attract much attention of theorists, working in the field of high energy nuclear physics. And only recently connection between $J_N^2$ and known in mathematics Chebyshev polynomials of 2-d kind became clear [16]. These polynomials, which should be called also the Chebyshev-Korkin-Zolotarev (CKZ) polynomials, are widely used in approximation theory and in other fields.

The cross section enhancement near the backward direction can be explained as a result of multistep processes contributions just because of remarkable properties of the characteristic polynomials in angular variables, which enter the denominator of the expression for the probability of the process ($d\sigma \sim 1/J_N[\sin(\theta/N)]$). There is huge amount of different kinds of MIP, elastic and inelastic, kinematical boundary some of them is near to the momentum of observed cumulative particle, and just such MIP will provide a glory-like enhancement of the cross section. An actual problem is to observe this effect for different kinds of particles — cumulative kaons, hyperons, etc, not only pions and nucleons, and for different projectiles.

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6Chebyshev polynomials are heavily used in numerical solutions. One of well known applications is in electrical filters. The Chebyshev filter is arguably one of the most common filters used in any electrical circuit that demands a filter. There is a 90% probability that the mobile device in our pocket uses a Chebyshev filter in either the audio or RF filter section. The benefit of the Chebyshev polynomial in a filter design is that it allows a much faster rolloff of the filter "skirts" compared to other filter polynomials (e.g. a Butterworth) by allowing some passband response ripple. Chebyshev polynomials can be applied to the optimal control of time-varying linear systems.
9 Appendix. Useful mathematical relations and formulas

We prove here the relation (see Eq. (3.3) )

\[ \frac{1}{\omega_N} \delta(m + \omega_{N-1} - \omega - \omega'_N) = \frac{1}{kk_{N-1}} \delta \left( \frac{m}{k} - \sum_{n=1}^{N} (1 - \cos \theta_n) \right), \]  

(A.1)

\( \omega = \omega_N \) is the energy of the final (cumulative) particle.

We recall that for the case of the light particle \( \omega = k = |\vec{k}| \), and \( k_N = k \). This relation converts the energy conservation in the last (\( N \)-th) interaction act into the restriction on the angular deviations from the optimal kinematics of the MIP (the constraint for the quadratic form in angular deviations), which is crucially important in further treatment. The energy-momentum conservation takes place in each of interaction acts.

The recoil energy in the last interaction act equals

\[ \omega'_N = \sqrt{m^2 + (k_{N-1} - \vec{k})^2} = \sqrt{m^2 + k_{N-1}^2 + k^2 - 2kk_{N-1}z_N} \]  

(A.2)

and we can find the value \( z_N^0 \) which satisfies the above \( \delta \)-function, i.e. the equality

\[ m + \omega_{N-1} = \omega + \omega'_N \]  

(A.3)

or

\[ \omega_N^2 = m^2 + k_{N-1}^2 + k^2 - 2kk_{N-1}z_N = (m + \omega_{N-1} - \omega)^2. \]  

(A.4)

Taking into account that \( \omega_{N-1} = k_{N-1} \), \( \omega = k \), we obtain easily

\[ 1 - z_N^0 = \frac{m(k_{N-1} - k)}{kk_{N-1}}. \]  

(A.5)

When \( z_N \) is near \( z_N^0 \), we can write

\[ \delta[f(z_N)] = \delta[f'_{z_N}(z_N = z_N^0)] = \frac{\omega'_N}{kk_{N-1}} \delta(z_N - z_N^0), \]  

(A.6)

where function \( f = m + \omega_{N-1} - \omega - \omega'_N \), and

\[ f'_{z_N}(z_N = z_N^0) = \frac{kk_{N-1}}{\omega'_N}, \]  

(A.7)

and we come to

\[ z_N - z_N^0 = \frac{m}{k} - \frac{m}{k_{N-1}} + z_N - 1. \]

Taking into account that

\[ \frac{m}{k_{N-1}} = \sum_{l=1}^{N-1} (1 - z_l) \]

we come to eq. (A.1). The elastic rescattering in the first interaction act was assumed here. Not essential change should be done for the inelastic interaction.
Here we present for the readers convenience some formulas and relations which have been used in present paper (sections 3 and 4). The nuclear glory phenomenon is an example when solving the physics problem leads to mathematical consequences of interest.

In sections 3 and 4 the following integrals of the $\delta$-functions have been used:

\[ I_n(\Delta) = \int \delta(\Delta - x_1^2 - \ldots - x_n^2) dx_1 \ldots dx_n = \pi \frac{(2\pi)^{(n-2)/2}}{(n-2)!!} \Delta^{(n-2)/2} \]  

(A.8)

for integer even $n$.

\[ I_n(\Delta)_n = \int \delta(\Delta - x_1^2 - \ldots - x_n^2) dx_1 \ldots dx_n = \frac{(2\pi)^{(n-1)/2}}{(n-2)!!} \Delta^{(n-2)/2} \]  

(A.9)

for integer odd $n$. Relations

\[ \int_0^\pi \sin^{2m} \theta d\theta = \pi \frac{(2m-1)!!}{(2m)!!}; \quad \int_0^\pi \sin^{2m-1} \theta d\theta = 2 \frac{(2m-2)!!}{(2m-1)!!}, \]  

(A.10)

$m$ — integer, allow to check (6.1) and (6.2) easily.

The equality takes place

\[ \int \delta(\Delta - x_1^2 - \ldots - x_n^2) \delta(x_1 + x_2 + \ldots + x_n) dx_1 \ldots dx_{n-1} dx_n = \frac{1}{\sqrt{n}} I_{n-1}(\Delta) \]  

(A.11)

More generally, for any quadratic form in variables $x_k$, $k=1, \ldots n$ after diagonalization we obtain

\[ \int \delta(\Delta - a_{kl} x_k x_l) dx_1 \ldots dx_n = \int \delta(\Delta - x_1^2 - \ldots - x_n^2) \frac{dx_1' \ldots dx_n'}{\sqrt{det||a||}} = \frac{1}{\sqrt{det||a||}} I_n(\Delta). \]  

(A.12)

The polynomials $J_N^2$ and equations for $z_N^\pi = \cos(\pi/N)$ can be obtained in more conventional way. There is an obvious equality

\[ \exp(i\pi/N)^N = \exp(i\pi) = -1 \]  

(A.13)

It can be written in the form

\[ \cos(\pi/N) + isin(\pi/N)^N = -1, \]  

(A.14)

or separately for the real and imaginary parts

\[ \text{Re} \left\{ \cos(\pi/N) + isin(\pi/N)^N \right\} = -1, \quad \text{Im} \left\{ \cos(\pi/N) + isin(\pi/N)^N \right\} = 0. \]  

(A.15)

The polynomials in $z_N^\pi = \cos(\pi/N)$ which are obtained in the left side of (6.13) coincide with polynomials obtained in section 4. However, some further efforts are necessary to get recurrent relations (6.9), (6.10).

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