On infinite Jacobi matrices with a trace class resolvent

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Abstract
Let \( \{\hat{P}_n(x)\} \) be an orthonormal polynomial sequence and denote by \( \{w_n(x)\} \) the respective sequence of functions of the second kind. Suppose the Hamburger moment problem for \( \{\hat{P}_n(x)\} \) is determinate and denote by \( J \) the corresponding Jacobi matrix operator on \( \ell^2 \). We show that if \( J \) is positive definite and \( J^{-1} \) belongs to the trace class then the series on the right-hand side of the defining equation
\[
\mathfrak{F}(z) := 1 - z \sum_{n=0}^{\infty} w_n(0) \hat{P}_n(z)
\]
converges locally uniformly on \( \mathbb{C} \) and it holds true that \( \mathfrak{F}(z) = \prod_{n=1}^{\infty} (1 - z/\lambda_n) \) where \( \{\lambda_n; n = 1, 2, 3, \ldots\} = \text{spec} J \). Furthermore, the Al-Salam–Carlitz II polynomials are treated as an example of orthogonal polynomials to which this theorem can be applied.

Keywords: infinite Jacobi matrix; orthogonal polynomials; functions of the second kind; spectral problem; Al-Salam–Carlitz II polynomials

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1 Introduction
Denote for a while by \( \mathfrak{A}_\infty \subset \mathbb{C}^{\infty,\infty} \) the unital algebra whose elements are semi-infinite matrices \( \mathcal{A} = (A_{m,n})_{m,n \geq 0} \) obeying the condition
\[
\exists k = k_{\mathcal{A}} \in \mathbb{Z}_+ \text{ s.t. } \forall m, n \in \mathbb{Z}_+, A_{m,n} = 0 \text{ if } n > m + k.
\]
Here \( \mathbb{Z}_+ \) stands for non-negative integers. Clearly, if \( \mathcal{A}, \mathcal{B} \in \mathfrak{A}_\infty \) and \( A_{m,n} = 0 \) for \( n > m + k \), \( B_{m,n} = 0 \) for \( n > m + \ell \) then \( (\mathcal{A}\mathcal{B})_{m,n} = 0 \) for \( n > m + k + \ell \). The product \( \mathcal{A}\mathcal{B} \) makes good sense even for any \( \mathcal{A} \in \mathfrak{A}_\infty \) and \( \mathcal{B} \in \mathbb{C}^{\infty,\infty} \).
Regarding $\mathbb{C}^\infty$ as a linear space of semi-infinite column vectors every $A \in \mathfrak{A}_\infty$ becomes naturally a linear operator on $\mathbb{C}^\infty$. The Hilbert space $\ell^2 \equiv \ell^2(\mathbb{Z}_+)$ may be viewed as a linear subspace of $\mathbb{C}^\infty$. Denote by $\{e_n\}$ the canonical basis in $\ell^2$. If every column of $A \in \mathfrak{A}_\infty$ is square summable then $A$ can be regarded as well as an operator in $\ell^2$ with $\text{Dom } A := \text{span}\{e_n\}$. In this case we say that $f \in \mathbb{C}^\infty$ is a formal eigenvector of $A$ if it is an eigenvector of $A$ in $\mathbb{C}^\infty$.

Furthermore, we shall extend the usual scalar product $\langle \cdot, \cdot \rangle$ on $\ell^2$ to $\mathbb{C}^\infty$ by letting $\langle f, g \rangle := \sum_{n=0}^{\infty} f_n g_n$ for any two vectors $f, g \in \mathbb{C}^\infty$ such that the series converges. Particularly this is the case if one of the vectors has only finitely many non-vanishing components.

This point of view is applicable to any Jacobi (tridiagonal) matrix which is in the sequel always supposed to be semi-infinite, symmetric, real and non-decomposable. A Jacobi matrix $J$ will be written in the form

$$J = \begin{pmatrix}
\beta_0 & \alpha_0 & 0 & 0 & 0 & \cdots \\
\alpha_0 & \beta_1 & \alpha_1 & 0 & 0 & \cdots \\
0 & \alpha_1 & \beta_2 & \alpha_2 & 0 & \cdots \\
0 & 0 & \alpha_2 & \beta_3 & \alpha_3 & \cdots \\
0 & 0 & 0 & \alpha_3 & \beta_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}.$$  

Without loss of generality we can even suppose that $\alpha_n > 0$ for all $n$.

If $A \in \mathfrak{A}_\infty$ regarded as an operator in $\ell^2$ is bounded it extends as a bounded operator unambiguously to the whole Hilbert space and therefore, with some abuse of notation, we can use the same symbol for a bounded operator on $\ell^2$ and its matrix (expressed in the canonical basis). In particular, the symbol $I$ will denote, at the same time, the unit operator on $\ell^2$ and the unit matrix in $\mathbb{C}^\infty$. With a general Jacobi matrix one has to be more careful, however.

Recall that with every Jacobi matrix $J$ one associates a sequence of moments

$$m_k := \langle e_0, J^k e_0 \rangle, \quad k \geq 0. \tag{1}$$

The following definition may simplify some formulations.

**Definition 1.** We shall say that a Jacobi matrix $J$ is Hamburger determinate if the Hamburger moment problem for the sequence of moments $\{m_k\}$ is determinate.

Given a Jacobi matrix $J$ let us re-denote $J$ as $\hat{J}$ if treated as an operator on $\ell^2$. Hence $\text{Dom } \hat{J} := \text{span}\{e_n\}$. Clearly, $\hat{J}$ is symmetric. As is well known, $J$ is Hamburger determinate if and only if $\hat{J}$ is essentially self-adjoint $[2]$. If this happens we shall denote by $J := \overline{\hat{J}}$ the unique self-adjoint extension of $\hat{J}$. Then

$$\text{Dom } J = \{f \in \ell^2; Jf \in \ell^2\}. \tag{2}$$

The symbol $\varrho(J)$ will stand for the resolvent set of $J$ and the spectral decomposition of $J$ will be written in the form $J = \int \lambda \, dE_\lambda$, with $E$ being a projection-valued measure. Then the probability measure on $\mathbb{R}$,

$$\mu(\cdot) := \langle e_0, E(\cdot)e_0 \rangle, \tag{3}$$
is the only solution to the Hamburger moment problem. In particular, \( \text{supp } \mu = \text{spec } J \) and we have

\[ m_k = (e_0, J^k e_0) = \int \lambda^k d\mu(\lambda), \quad k \geq 0. \]

We shall focus on the case when \( \dot{J} \) is is essentially self-adjoint and positive definite, i.e.

\[ \forall f \in \text{span}\{e_n\}, \quad \langle f, J f \rangle \geq \gamma \| f \|^2 \]

for some \( \gamma > 0 \). Moreover, we shall assume that \( J^{-1} \) is a trace class operator. If so, denote by \( \{\lambda_n; n \geq 1\} \) the sequence of eigenvalues of \( J \) ordered increasingly,

\[ 0 < \gamma < \lambda_1 < \lambda_2 < \lambda_3 < \cdots. \]

Of course, all eigenvalues \( \lambda_n \) are simple. Then \( \mu \) is supported on the discrete set \( \{\lambda_n; n \geq 1\} = \text{spec } J \). Furthermore, the function \( z \mapsto \prod_{n=1}^{\infty} (1 - z/\lambda_n) \) is entire and its zero set coincides with \( \text{spec } J \).

An indispensable companion of \( J \) is an orthonormal polynomial sequence \( \{\hat{P}_n(x)\} \) obeying the three-term recurrence equation

\[ \begin{align*}
\alpha_0 \hat{P}_1(x) + (\beta_0 - x) \hat{P}_0(x) &= 0, \\
\alpha_n \hat{P}_{n+1}(x) + (\beta_n - x) \hat{P}_n(x) + \alpha_{n-1} \hat{P}_{n-1}(x) &= 0, \quad n \geq 1,
\end{align*} \]

with the initial data \( \hat{P}_0(x) = 1 \). Whenever convenient we shall treat the sequence \( \{\hat{P}_n(x)\} \) as a column vector \( \hat{P}_*(x) \in \mathbb{C}^\infty \),

\[ \hat{P}_*(x)^T := (\hat{P}_0(x), \hat{P}_1(x), \hat{P}_2(x), \ldots). \]

An analogous notation will be used in case of other polynomial sequences as well. The recurrence (4) in fact means exactly that \( \hat{P}_*(x) \) is a formal eigenvector of \( J \) corresponding to the eigenvalue \( x \),

\[ \mathcal{J} \hat{P}_*(x) = x \hat{P}_*(x), \]

with the initial component \( \hat{P}_0(x) = 1 \). Recall also the following relation which is not only easy to verify but also quite useful:

\[ \forall n \geq 0, \quad \hat{P}_n(J) e_0 = e_n. \]

If \( \mathcal{J} \) is Hamburger determinate then \( \{\hat{P}_n(x)\} \) is an orthonormal basis in \( L^2(\mathbb{R}, d\mu(x)) \) and from (2) it is seen that \( \sum_{n=0}^{\infty} |\hat{P}_n(z)|^2 = \infty \) for all \( z \in \mathbb{C} \) unless \( z \in \mathbb{R} \) is an eigenvalue of \( J \).

Another useful tool which can be applied in case of a Hamburger determinate Jacobi matrix \( \mathcal{J} \) is that of the sequence of functions of the second kind \( \hat{P}_*(z) \). Let us define an analytic vector-valued function \( w(z) \) on \( \varrho(J) \) with values in \( \ell^2 \) by the equation

\[ \forall z \in \varrho(J), \quad w(z) := (J - z)^{-1} e_0 = \int \frac{\hat{P}_*(\lambda)}{\lambda - z} d\mu(\lambda). \]
Its components $w_k(z)$ are called functions of the second kind. Hence
\[ \forall z \in \varrho(J), \forall k \geq 0, \ w_k(z) = \int \frac{\hat{P}_k(\lambda)}{\lambda - z} \, d\mu(\lambda). \]

The first component is also called the Weyl function of $J$,
\[ \forall z \in \varrho(J), \ w(z) \equiv w_0(z) = \langle e_0, (J - z)^{-1} e_0 \rangle = \int \frac{d\mu(\lambda)}{\lambda - z}, \]
and it is, up to a sign, the Stieltjes transform of the measure $\mu$. Note that, in $\mathbb{C}^\infty$, $(J - z)^n w(z) = e_0$ with the initial data $\langle e_0, w(z) \rangle = w(z)$. This is in fact a recurrence relation which along with the initial data determines the vector $w(z)$ unambiguously.

We claim that if $J$ is Hamburger determinate then
\[ \forall z \in \varrho(J), \forall n \geq 0, \ w_n(z) \hat{P}_n(z) = \langle e_n, (J - z)^{-1} e_n \rangle. \]

Indeed,
\[
\begin{align*}
w_n(z) \hat{P}_n(z) &= \hat{P}_n(z) \int \frac{\hat{P}_n(\lambda)}{\lambda - z} \, d\mu(\lambda) + \int \hat{P}_n(\lambda) \frac{\hat{P}_n(\lambda) - \hat{P}_n(z)}{\lambda - z} \, d\mu(\lambda) \\
&= \int \frac{\hat{P}_n(\lambda)^2}{\lambda - z} \, d\mu(\lambda) \\
&= \langle e_n, (J - z)^{-1} e_n \rangle.
\end{align*}
\]
In the first equation we have used the orthogonality of the polynomial sequence and that $(\hat{P}_n(\lambda) - \hat{P}_n(z))/(\lambda - z)$ is a polynomial in $\lambda$ of degree less than $n$. The last equality follows from (3) and (6).

Moreover, note that $\int \hat{P}_n(\lambda)^2 \, d\mu(\lambda) = 1$ and so the function $w_n(z) \hat{P}_n(z)$ is, up to a sign, the Stieltjes transform of a probability measure and as such it is known to have no zeros outside the convex hull of the support of that measure [10]. Hence $w_n(z) \hat{P}_n(z)$ has no zeros outside the convex hull of $\text{supp } \mu$. Consequently, if $J$ is Hamburger determinate and $J$ is positive definite then $w_n(0) \hat{P}_n(0) > 0$ for all $n \geq 0$ and $J^{-1}$ belongs to the trace class if and only if
\[ \text{tr } J^{-1} = \sum_{n=0}^{\infty} w_n(0) \hat{P}_n(0) < \infty. \]  

**Theorem 2.** Let $J$ be a Hamburger determinate Jacobi matrix which is positive definite on $\text{span}\{e_n; \ n \geq 0\}$ and $J$ be the respective Jacobi matrix operator on $\ell^2$. If $J^{-1}$ is a trace class operator then the series $\sum_{n=0}^{\infty} w_n(0) \hat{P}_n(z)$ converges locally uniformly in $z \in \mathbb{C}$ and
\[ \mathcal{F}(z) := 1 - z \sum_{n=0}^{\infty} w_n(0) \hat{P}_n(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{\lambda_n} \right) \]
where $\{\lambda_n; \ n = 1, 2, 3, \ldots\} = \text{spec } J$.

Particularly this means that the zeros of the entire function $\mathcal{F}(z)$ are simple and the zero set coincides with the spectrum of $J$. 

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Remark 3. An earlier result derived in [11] provides another formula for a characteristic function of $J$ written in a factorized form. This has been done under the assumption

$$\sum_{n=0}^{\infty} \frac{\alpha_n^2}{\beta_n \beta_{n+1}} < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty. \tag{9}$$

The both results are in principle independent. The Jacobi matrix treated as an example below in Section 4 does not comply with (9) and thus the method of [11] fails in this case. On the other hand, conditions (9) guarantee only that $J^{-1}$ belongs to the Hilbert-Schmidt class but not necessarily to the trace class as demonstrated by Example 4.1 in [11]. Therefore this example is not covered by Theorem 2.

The paper is organized as follows. In Section 2 we provide a brief summary of some known results which are substantial for the proof of Theorem 2. They are mostly concerned with the associated orthogonal polynomials and the functions of the second kind. Here we also point out an application of the Green function of a Jacobi matrix which is of importance for our purposes. Section 3 is entirely devoted to the proof of Theorem 2. In Section 4 we treat the Al–Salam–Carlitz II polynomials as an example of an orthogonal polynomial sequence to which our approach can be successfully applied.

2 Preliminaries

2.1 Associated polynomials, functions of the second kind and the Weyl function

Here we summarize some known results concerning the associated orthogonal polynomials and the associated functions which are substantial for the proof of Theorem 2. We may provide also a few additional observations. Further details can be found, for instance, in [3, 12].

Given an arbitrary Jacobi matrix $J$, the so called sequence of polynomials of the second kind, $\{Q_n(x)\}$, is again determined by the three-term recurrence

$$\alpha_n Q_{n+1}(x) + (\beta_n - x)Q_n(z) + \alpha_{n-1}Q_{n-1}(x) = 0, \quad n \geq 1, \tag{10}$$

with the initial data $Q_0(x) = 0, Q_1(x) = 1/\alpha_0$. Since $\{\hat{P}_n(z)\}$ and $\{Q_n(z)\}$ obey the same recurrence equation it is a matter of routine manipulations to derive that

$$Q_n(z) = \left(\sum_{j=0}^{n-1} \frac{1}{\alpha_j P_j(z) \hat{P}_{j+1}(z)}\right) \hat{P}_n(z), \quad n = 0, 1, 2, \ldots, \tag{11}$$

provided that $\forall n \geq 1$, $\hat{P}_n(z) \neq 0$. Actually, assuming [4] one readily verifies that [11] is a solution of (10) and fulfills the required initial condition. If $J$ is Hamburger determinate and $z \in \mathbb{C}$ does not belong to the convex hull of $\text{supp} \mu$ then, indeed, $\hat{P}_n(z) \neq 0$ for all $n \geq 1$. 

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Note that (4) and (10) can be respectively rewritten as vector equations in $\mathbb{C}^\infty$, namely

$$(J - z)\hat{P}_s(z) = 0, \quad (J - z)Q_s(z) = e_0,$$

with the initial data $\hat{P}_0(z) = 1$, $Q_0(z) = 0$. Consequently,

$$(J - z)(w(z)\hat{P}_s(z) + Q_s(z)) = e_0, \quad (e_0, w(z)\hat{P}_s(z) + Q_s(z)) = w(z).$$

In the Hamburger determinate case the uniqueness of the solution implies that

$$\forall z \in \mathcal{g}(J), \quad w(z)\hat{P}_s(z) + Q_s(z) = (J - z)^{-1}e_0 = w(z) \in \ell^2. \quad (12)$$

As a matter of fact, note that for any $z \in \mathcal{g}(J)$, $w(z)$ is the unique complex number such that $w(z)\hat{P}_s(z) + Q_s(z)$ is square summable.

In [4], a generalization of Markov’s Theorem has been derived which is applicable to any Hamburger determinate Jacobi matrix $J$. Denote by $Z_n$ the set of roots of $\hat{P}_n(x)$ and put

$$\Lambda := \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty Z_n. \quad (13)$$

It is well known that $\text{supp } \mu \subset \Lambda$. Then for any $z \in \mathbb{C} \setminus \Lambda \subset \mathcal{g}(J)$ the limit $\lim_{n \to \infty} Q_n(z)/\hat{P}_n(z)$ exists. In that case it follows from (12) that necessarily

$$\lim_{n \to \infty} \frac{Q_n(z)}{\hat{P}_n(z)} = -w(z) = -\langle e_0, (J - z)^{-1}e_0 \rangle = \int \frac{d\mu(\lambda)}{z - \lambda}.$$

With $J$ one can associate a sequence of Jacobi matrices $J^{(k)}$, $k \geq 0$. $J^{(k)}$ is obtained from $J$ by deleting the first $k$ rows and columns. In particular, $J^{(0)} = J$. Every Jacobi matrix $J^{(k)}$ again determines an orthonormal polynomial sequence $\{\hat{P}_n^{(k)}(x); n \geq 0\}$ unambiguously defined by $J^{(k)}\hat{P}_n^{(k)}(x) = x\hat{P}_n^{(k)}(x)$ and $\hat{P}_0^{(k)}(x) = 1$. Polynomials $\hat{P}_n^{(k)}(x)$ are called associated orthogonal polynomials. Note that

$$\hat{P}_n^{(1)}(x) = Q_{n+1}(x)/Q_1(0), \quad n \geq 0. \quad (14)$$

If convenient, we shall identify $\hat{P}_n(x) \equiv \hat{P}_n^{(0)}(x)$.

**Proposition 4.** If a Jacobi matrix $J$ is determinate then the associated Jacobi matrices $J^{(k)}$ are determinate for all $k \geq 1$.

**Proof.** Obviously it is sufficient to verify only the case $k = 1$. The Weyl function is, up to a sign, the Stieltjes transform of the probability measure $\mu$ and as such it is known to have no zeros outside the convex hull of $\text{supp } \mu$ [10]. Taking into account that $w(z)\hat{P}_s(z) + Q_s(z) \in \ell^2$ but $\hat{P}_s(z) \notin \ell^2$ we deduce that $Q_s(z) \notin \ell^2$, equivalently $\hat{P}_s^{(1)}(z) \notin \ell^2$, for all $z \in \mathbb{C} \setminus \mathbb{R}$. Hence the deficiency indices of $J^{(1)}$ are $(0, 0)$ meaning that $J^{(1)}$ is essentially self-adjoint. \qed

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Particularly, \( J \) terminate. Again, \( J^{(k)} \) is the unique self-adjoint operator in \( \ell^2 \) corresponding to \( J^{(k)} \). Assume additionally that 

Then \( Q_n^{(0)}(x) \equiv Q_n(x) \) and 

\[
\hat{P}_n^{(k)} = Q_{n+k}^{(k-1)}(x)/Q_k^{(k-1)}(0), \quad n \geq 0, \quad k \geq 1.
\]  

Up to the end of the current subsection \( J \) is supposed to be Hamburger determinate. Again, \( J^{(k)} \) is the unique self-adjoint operator in \( \ell^2 \) corresponding to \( J^{(k)} \). Assume additionally that 

\[
J \geq \gamma > 0.
\]

Then it is clear that \( J^{(k)} \geq \gamma > 0 \) for all \( k \geq 1 \).

For any \( k \geq 0 \) and \( z \in \mathcal{D}(J) \) define 

\[
w^{(k)}(z) := (J - z)^{-1}e_k \in \ell^2.
\]

Particularly, \( w^{(0)}(z) \equiv w(z) \). The column vector \( w^{(k)}(z) \) solves the equations 

\[
(J - z)w^{(k)}(z) = e_k \text{ in } \mathbb{C}^\infty, \quad \langle e_0, w^{(k)}(z) \rangle = w_k(z).
\]

By a similar reasoning as above, 

\[
\forall z \in \mathcal{D}(J), \quad w^{(k)}(z) = w_k(z)\hat{P}_n(z) + Q_n^{(k)}(z) \in \ell^2.
\]

Since \( \hat{P}_n(z) \notin \ell^2 \) the complex number \( w_k(z) \) is unique such that \( w_k(z)\hat{P}_n(z) + Q_n^{(k)}(z) \) is square summable.

Referring again to [11] we know that the limit \( \lim_{n \to \infty} \hat{P}_n^{(k+1)}(z)/\hat{P}_n(z) \) exists for all \( k \geq 0 \) and \( z \in \mathbb{C} \setminus [\gamma, \infty) \). In view of [16] one finds that the limit \( \lim_{n \to \infty} Q_n^{(k)}(z)/\hat{P}_n(z) \) exists as well. Then, in regard of [19], necessarily 

\[
\lim_{n \to \infty} \frac{Q_n^{(k)}(z)}{\hat{P}_n(z)} = -w_k(z), \quad k \geq 0, \quad z \in \mathbb{C} \setminus [\gamma, \infty).
\]

This is a well known generalization of Markov’s theorem, see [12 Theorem 2].

In view of (18), (19) and (15) it holds true that 

\[
\forall z \in \mathcal{D}(J), \forall m \leq n, \quad \langle e_m, (J - z)^{-1}e_n \rangle = \langle e_m, w_n(z)\hat{P}_n(z) + Q_n^{(k)}(z) \rangle = \hat{P}_m(z)w_n(z).
\]

On the other hand, for \( m > n \) we have 

\[
\langle e_m, (J - z)^{-1}e_n \rangle = \hat{P}_m(z)w_n(z) + Q_m^{(n)}(z)
\]

and, at the same time, 

\[
\langle e_m, (J - z)^{-1}e_n \rangle = \langle e_m, (J - z)^{-1}e_m \rangle = w_m(z)\hat{P}_n(z).
\]
Whence
\[ \forall m > n, \ Q_m(z) = w_m(z)\hat{P}_n(z) - \hat{P}_m(z)w_n(z). \tag{21} \]
Letting \( n = 0 \) we again get the known equation
\[ \forall m \geq 1, \ Q_m(z) = w_m(z) - w(z)\hat{P}_m(z). \tag{22} \]
Combining (21) and (22) we obtain
\[ \forall m > n, \forall z \in \mathbb{C}\setminus[\gamma, \infty), \ Q_m(z) = Q_m(z)\hat{P}_n(z) - \hat{P}_m(z)Q_n(z). \tag{23} \]

If \( J \) is positive definite then this is particularly true for \( z = 0 \). All these relations are well known, see [12].

(17) is still supposed to be true. It is well known that all roots of the polynomials \( \hat{P}_n(x) \) and \( Q_n(x) \) lie in \([\gamma, \infty)\) and therefore \( \Lambda \), as defined in (13), is a subset of \([\gamma, \infty)\). From (11) it follows that for \( z \in \mathbb{C}\setminus[\gamma, \infty) \),
\[ w(z) = -\lim_{n \to \infty} \frac{Q_n(z)}{P_n(z)} = -\sum_{j=0}^{\infty} \frac{1}{\alpha_j \hat{P}_j(z)\hat{P}_{j+1}(z)} \]
and
\[ w_n(z) = w(z)\hat{P}_n(z) + Q_n(z) = -\left( \sum_{j=n}^{\infty} \frac{1}{\alpha_j \hat{P}_j(z)\hat{P}_{j+1}(z)} \right)\hat{P}_n(z), \quad n \geq 0. \tag{23} \]

Furthermore the polynomials \( \hat{P}_n(x) \) and \( Q_n(x) \) have all their leading coefficients positive and since they do not change the sign on \((-\infty, \gamma)\) we have
\[ \text{sgn} \hat{P}_n(0) = (-1)^n, \quad n \geq 0; \quad \text{sgn} Q_n(0) = (-1)^{n+1}, \quad n \geq 1. \]
Moreover, from (23) it is seen that \( w_n(0)/\hat{P}_n(0) > 0 \) for all \( n \geq 0 \).

2.2 A relation between an orthogonal polynomial sequence and the respective Green function

Given a Jacobi matrix \( J \) (not necessarily Hamburger determinate) let \( G \) be a strictly lower triangular matrix with the entries
\[ G_{m,n} = Q_m^{(n)}(0) \quad \text{for} \ m,n \geq 0, \quad G_{m,n} = 0 \quad \text{otherwise}, \tag{24} \]
with \( \{Q_m^{(n)}(x)\} \) being defined in (15). Then \( G \) is a right inverse of \( J \) in \( \mathbb{C}^\infty \), \( JG = I \), and \( G \) is obviously the unique strictly lower triangular matrix with this property. \( G \) can be interpreted as the Green function of the Jacobi matrix \( J \) [8].

**Theorem 5.** Let \( J \) be a Jacobi matrix, \( \{\hat{P}_n(x)\} \) be the respective orthonormal polynomial sequence and \( G \) be the Green function of \( J \). Then
\[ \hat{P}_n(x) = (I - xG)^{-1}\hat{P}_n(0). \tag{25} \]
Conversely, equation (25) determines the strictly lower triangular matrix \( G \) unambiguously.
Lemma 6. If $M \in \mathbb{C}^{\infty,\infty}$ is a lower triangular matrix and $M \hat{P}_s(x) = 0$ holds in $\mathbb{C}[x]^{\infty}$ then $M = 0$.

**Proof.** Since for every $n$, $\hat{P}_n(x)$ is a polynomial of degree $n$ with with a nonzero leading coefficient there exists a unique lower triangular matrix $\Pi$ with no zeros on the diagonal such that

$$\hat{P}_s(x) = \Pi v(x) \quad \text{where} \quad v(x)^T = (1, x, x^2, \ldots).$$

Then $M \hat{P}_s(x) = 0$ is equivalent to $M \Pi = 0$. But $\Pi$ is invertible (in $\mathbb{C}^{\infty}$) and therefore $M = 0$. 

**Proof of Theorem 5.** Owing to (5) and since $J G = I$ we have $J (I - xG) \hat{P}_s(x) = 0$. Obviously, if a column vector $v$, $v^T = (v_0, v_1, v_2, \ldots)$, obeys $J v = 0$ and $v_0 = 0$ then $v = 0$. In particular, this observation is applicable to

$$v = (I - xG) \hat{P}_s(x) - \hat{P}_s(0),$$

and (25) follows.

Conversely, suppose $\hat{P}_s(x) = (I - xL)^{-1} \hat{P}_s(0)$ holds for a strictly lower triangular matrix $L$. Then $\hat{P}_s(x) = \hat{P}_s(0) + xL \hat{P}_s(x)$. In regard of (5) we obtain

$$x \hat{P}_s(x) = J \hat{P}_s(x) = J (\hat{P}_s(0) + xL \hat{P}_s(x)) = xJ L \hat{P}_s(x)$$

whence $(I - J L) \hat{P}_s(x) = 0$. Observing that $I - J L$ is lower triangular we have, by Lemma 6, $J L = I$. But this equation determines $L$ unambiguously.

Finally, note that in the case $J$ is Hamburger determinate and $J$ is positive definite (20) tells us that

$$\lim_{m \to \infty} \frac{G_{m,n}}{\hat{P}_m(0)} = -w_n(0) = -\langle e_n, J^{-1} e_0 \rangle. \quad (26)$$

**3 Proof of the main theorem**

**Proof of Theorem 2.** First, we shall show that the series in (8) converges locally uniformly on $\mathbb{C}$ and, moreover, $\mathfrak{F}(z)$ is a locally uniform limit of the sequence $\{\hat{P}_n(z)/\hat{P}_n(0)\}$.

Let

$$\kappa_n := w_n(0) \hat{P}_n(0).$$

By the assumptions, see (7), $\kappa_n$ are all positive and $\{\kappa_n\} \in \ell^1$. Equation (25) means that

$$\forall n \geq 0, \quad \hat{P}_n(z) = \hat{P}_n(0) + \sum_{\ell=1}^{n} z^\ell \sum_{0 \leq k_1 < k_2 < \ldots < k_\ell < n} G_{nk_1} G_{k_1 k_{k_1-1}} \cdots G_{k_\ell k_1} \hat{P}_{k_1}(0). \quad (27)$$

Let us introduce, for the purpose of the proof, a strictly lower triangular matrix $K$ and a vector $k \in \ell^2$:

$$\forall m, n \geq 0, \quad K_{m,n} := \frac{\hat{P}_n(0)}{\hat{P}_m(0)} \sqrt{\frac{\kappa_m}{\kappa_n}} G_{m,n}; \quad k^T := (\sqrt{\kappa_0}, \sqrt{\kappa_1}, \sqrt{\kappa_2}, \ldots). \quad (28)$$
Let us show that
\[ \forall m > n, \quad \left| \frac{\hat{P}_n(0)}{\hat{P}_m(0)} G_{m,n} \right| \leq \kappa_n \quad \text{and} \quad |K_{m,n}| \leq \sqrt{\kappa_m \kappa_n}. \] (29)

In fact, the latter estimate is obviously a consequence of the former one. Concerning the former estimate we have, in view of (24) and (21),
\[ \left| \frac{\hat{P}_n(0)}{\hat{P}_m(0)} G_{m,n} \right| = \hat{P}_n(0)^2 \left| \frac{w_m(0)}{\hat{P}_m(0)} - \frac{w_n(0)}{\hat{P}_n(0)} \right|. \]

Remember that
\[ w_n(0) = -\infty \sum_{j=n}^\infty \alpha_j \hat{P}_j(0) \hat{P}_{j+1}(0) > 0 \]
is a decreasing sequence in \( n \), see (23). Hence, still assuming that \( m > n \),
\[ \left| \frac{\hat{P}_n(0)}{\hat{P}_m(0)} G_{m,n} \right| \leq \hat{P}_n(0)^2 \max \left\{ \frac{w_m(0)}{\hat{P}_m(0)}, \frac{w_n(0)}{\hat{P}_n(0)} \right\} \leq \hat{P}_n(0)^2 \frac{w_n(0)}{\hat{P}_n(0)} = \kappa_n. \]

The estimate in particular means that the matrix \( K \) can be regarded as a Hilbert-Schmidt operator on \( \ell^2 \).

We can rewrite (27) as
\[ \frac{\hat{P}_n(z)}{\hat{P}_n(0)} = 1 + \sum_{j=1}^n \frac{z^j}{\sqrt{\kappa_n}} \langle e_n, K^j k \rangle. \] (30)

For all \( j, n \), \( 1 \leq j \leq n \), we can estimate
\[ \left| \frac{1}{\sqrt{\kappa_n}} \langle e_n, K^j k \rangle \right| = \left| \sum_{0 \leq k_1 < k_2 < \ldots < k_j < n} \frac{1}{\sqrt{\kappa_n}} K_{nk_j} K_{k_j k_{j-1}} \cdots K_{k_2 k_1} \sqrt{\kappa_{k_1}} \right| \leq \sum_{0 \leq k_1 < k_2 < \ldots < k_j < n} \kappa_{k_1} \kappa_{k_2} \cdots \kappa_{k_j} \kappa_{k_1} \leq \frac{1}{j!} \left( \sum_{s=1}^\infty \kappa_s \right)^j. \] (31)

Consequently,
\[ \forall j \geq 0, \quad \| K^j k \| \leq \frac{1}{j!} \left( \sum_{s=1}^\infty \kappa_s \right)^{j+1/2}. \] (32)

Very similarly, for \( m > n \) and \( j \geq 1 \),
\[ \langle e_m, K^j e_n \rangle \leq \sqrt{\kappa_m \kappa_n} \sum_{n < k_1 < k_2 < \ldots < k_j < m} \kappa_{k_j} \cdots \kappa_{k_2} \kappa_{k_1} \leq \frac{\sqrt{\kappa_m \kappa_n}}{(j - 1)!} \left( \sum_{s=1}^\infty \kappa_s \right)^{j-1}. \]
It follows that
\[ \forall j \geq 1, \|K^j\| \leq \frac{1}{2(j-1)!} \left( \sum_{s=1}^{\infty} \kappa_s \right)^j. \]
Hence the spectral radius of $K$ equals 0. Estimate (31) implies that, in addition,
\[ \forall z \in \mathbb{C}, \forall n \geq 0, |\langle e_n, (I - zK)^{-1}k \rangle| \leq \exp \left( |z| \sum_{s=1}^{\infty} \kappa_s \right) \sqrt{\kappa_n}. \]  
(33)

Furthermore, (26) means that
\[ \lim_{n \to \infty} \hat{K}_{n,k} \sqrt{\kappa_n} = -w_k(0) \hat{P}_n(z), \]
and we have
\[ \frac{1}{\sqrt{\kappa_n}} \langle e_n, K^j k \rangle = \sum_{k=0}^{\infty} K_{n,k} \frac{\sqrt{\kappa_n}}{\sqrt{\kappa_k}} \langle e_k, K^{j-1} k \rangle \]
along with the estimate, as it follows from (31),
\[ \left| \frac{K_{n,k}}{\sqrt{\kappa_n}} \langle e_k, K^{j-1} k \rangle \right| \leq \frac{\kappa_k}{(j-1)!} \left( \sum_{s=1}^{\infty} \kappa_s \right)^{j-1}. \]
Since this upper bound is independent of $n$ and summable in $k$,
\[ \lim_{n \to \infty} \frac{1}{\sqrt{\kappa_n}} \langle e_n, K^j k \rangle = -\sum_{k=0}^{\infty} \frac{w_k(0) \hat{P}_k(0)}{\sqrt{\kappa_k}} \langle e_k, K^{j-1} k \rangle = -\langle k, K^{j-1} k \rangle. \]
Similarly, the upper bound in (31) is independent of $n$ and we obtain from (30)
\[ \lim_{n \to \infty} \frac{\hat{P}_n(z)}{\hat{P}_n(0)} = 1 - z \sum_{j=0}^{\infty} z^j \langle k, K^j k \rangle = 1 - z \langle k, (I - zK)^{-1} k \rangle. \]
The estimate guarantees that the limit is locally uniform on $\mathbb{C}$ and, in view of (32), the resulting series, too, converges uniformly on every compact subset of $\mathbb{C}$.

From defining equations (28) along with (25) one finds that
\[ \sqrt{\kappa_n} \langle e_n, (I - zK)^{-1} k \rangle = w_n(z) \hat{P}_n(z). \]
Hence
\[ 1 - z \langle k, (I - zK)^{-1} k \rangle = 1 - z \sum_{n=0}^{\infty} w_n(z) \hat{P}_n(z). \]
Moreover, referring to (33) we have the bound
\[ |w_n(0) \hat{P}_n(z)| = \sqrt{\kappa_n} |\langle e_n, (I - zK)^{-1} k \rangle| \leq \exp \left( |z| \sum_{s=1}^{\infty} \kappa_s \right) \kappa_n \]  
(34)
showing that the series converges locally uniformly on \( \mathbb{C} \).

Second, we shall show that \( \mathfrak{F}^{-1}(0) = \text{spec } J \) with all zeros of \( \mathfrak{F}(z) \) being simple. We assume that \( J \) is positive definite, and we have already proved that \( J^{-1} \) is a compact operator. Hence \( \text{spec } J \) constitutes of simple eigenvalues which can be ordered increasingly. We shall use the notation

\[
\text{spec } J = \{ \lambda_n; n \geq 1 \}, \quad 0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots ,
\]

and we have \( \lim \lambda_n = +\infty \).

As explained for instance in [6, §II.4], each eigenvalue \( \lambda_j \) can be obtained as the limit of a sequence of roots of the polynomials \( \hat{P}_n(x) \). Denote by \( x_{n,j} \) the \( j \)th root of the polynomial \( \hat{P}_n(x) \), \( n \geq 1, j = 1, 2, \ldots , n \), while supposing that the roots are numbered in increasing order. Then, for a fixed \( j \in \mathbb{N} \), the sequence \( \{ x_{n,j}; n = j, j+1, j+2, \ldots \} \) is strictly decreasing and

\[
\lim_{n \to \infty} x_{n,j} = \lambda_j.
\]

Using the already proven convergence, \( \hat{P}_n(z)/\hat{P}_n(0) \Rightarrow \mathfrak{F}(z) \) locally on \( \mathbb{C} \), one can determine the zero set \( \mathcal{F}^{-1}(0) \). It is a basic fact that \( \hat{P}_n'(z)/\hat{P}_n(0) \Rightarrow \mathfrak{F}'(z) \) locally on \( \mathbb{C} \), and since the entire function \( \mathfrak{F}(z) \) is not equal to zero identically its zero set has no limit points.

Consider a positively oriented circle \( C(\lambda_j, \epsilon) \) centered at some eigenvalue \( \lambda_j \) and with a radius \( \epsilon > 0 \) sufficiently small so that the distance of \( \lambda_j \) from the rest of the spectrum of \( J \) is greater than \( \epsilon \) and such that \( \mathfrak{F}(z) \) has no zeros on \( C(\lambda_j, \epsilon) \). In view of (33), \( \hat{P}_n(z) \) has exactly one root in the interior of \( C(\lambda_j, \epsilon) \) for all sufficiently large \( n \). Then the number of zeros of \( \mathfrak{F}(z) \) located in the interior of \( C(\lambda_j, \epsilon) \) equals

\[
\frac{1}{2\pi i} \int_{C(\lambda_j, \epsilon)} \frac{\mathfrak{F}'(z)}{\mathfrak{F}(z)} \, dz = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{C(\lambda_j, \epsilon)} \frac{\hat{P}_n'(z)}{\hat{P}_n(z)} \, dz = 1.
\]

Since \( \epsilon \) can be made arbitrarily small it is clear that \( \lambda_j \) must be a simple root of \( \mathfrak{F}(z) \). Moreover, by a very analogous reasoning, if \( z_0 \in \mathbb{C} \) is not an eigenvalue of \( J \) then it cannot be a root of \( \mathfrak{F}(z) \).

Third, let us verify (8). With the above estimates it is immediate to show that the order of \( \mathfrak{F} \) does not exceed one. Actually, from (8) and (34) it is seen that

\[
|\mathfrak{F}(z)| \leq 1 + |z| \exp \left( |z| \sum_{j=0}^{\infty} \kappa_j \right) \sum_{j=0}^{\infty} \kappa_j \leq \exp(C|z|)
\]

for some constant \( C > 0 \). The assertion readily follows. Note that by Hadamard’s Factorization Theorem and since \( \mathfrak{F}(0) = 1 \) there exists \( c \in \mathbb{R} \) such that

\[
\mathfrak{F}(z) = e^{cz} \prod_{n=1}^{\infty} \left( 1 - \frac{z}{\lambda_n} \right)^{e^{z/\lambda_n}}.
\]
But by our assumptions $\text{tr} \, J^{-1} = \sum_{n=1}^{\infty} \lambda_n^{-1} < \infty$ and therefore we even have

$$\mathfrak{F}(z) = e^{dz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right)$$  \hspace{1cm} (37)

where $d = c + \sum_{n=1}^{\infty} \lambda_n^{-1}$. To determine $d$ one can calculate $\mathfrak{F}'(0)$ using (8) and (37),

$$\mathfrak{F}'(0) = -\sum_{n=0}^{\infty} w_n(0) \hat{P}_n(0) = d - \sum_{n=1}^{\infty} \frac{1}{\lambda_n}.$$  \hspace{1cm} (38)

According to (7),

$$\sum_{n=0}^{\infty} w_n(0) \hat{P}_n(0) = \text{tr} \, J^{-1} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}.$$  \hspace{1cm} (39)

Hence $d = 0$. \hfill \Box

Remark 7. Comparing (13), where we let $\mathcal{Z}_n = \{x_{n,j}; j = 1, 2, \ldots, n\}$, to the description of the spectrum of $J$ given in (35), (36) we find that in this case $\Lambda = \text{spec} \, J$.

4 An example: the Al-Salam–Carlitz II polynomials

Everywhere in what follows, $0 < q < 1$. The Jacobi matrix $J$ corresponding to the Al-Salam–Carlitz II polynomials [1] has the entries ($\alpha_n > 0$)

$$\alpha_n^2 = aq^{-2n-1}(1 - q^{n+1}), \; \beta_n = (a + 1)q^{-n},$$

where $a > 0$ is a parameter.

It is straightforward to verify that

$$J = WW^T + I$$

where $W$ is a lower triangular matrix with the entries

$$W_{n,n} = \sqrt{a} q^{-n/2}, \; W_{n+1,n} = q^{-(n+1)/2} \sqrt{1 - q^{n+1}}, \; n = 0, 1, 2, \ldots,$$

and $W_{m,n} = 0$ otherwise. Hence $J \geq 1$ on span$\{e_n; n = 0, 1, 2, \ldots\}$.

The Al-Salam–Carlitz II polynomials can be expressed as [9]

$$V_n^{(a)}(x; q) = (-a)^n q^{-n(n-1)/2} 2\phi_0 \left( q^{-n}, x; -; q, \frac{q^n}{a} \right)$$

$$= (-a)^n q^{-n(n-1)/2} \left( q; q \right)_n \sum_{k=0}^{n} \frac{(x; q)_k a^{-k}}{(q; q)_{n-k}(q; q)_k}, \; n \in \mathbb{Z}_+.$$  \hspace{1cm} (38)
The respective orthonormal polynomial sequence takes the form

$$\hat{P}_n(x) = \frac{q^{n^2/2}a^{-n/2}}{\sqrt{(q;q)_n}} V_n^{(a)}(x; q), \quad n \in \mathbb{Z}_+. \quad (39)$$

Values at the point $x = a$ have particularly simple form. Using the identity [9, Eq. (1.11.7)]

$$2\varphi_0\left(q^{-n}; a; -q, \frac{q^n}{a}\right) = a^{-n}, \quad n \in \mathbb{Z}_+,$$

we obtain

$$\hat{P}_n(a) = \frac{(-1)^n}{\sqrt{(q;q)_n}} \left(\frac{q}{a}\right)^{n/2}. \quad (40)$$

For the orthogonal polynomials of the second kind (11) we get the value

$$Q_n(a) = \left(\sum_{j=0}^{n-1} \frac{1}{\alpha_j \hat{P}_j(a) \hat{P}_{j+1}(a)}\right) \hat{P}_n(a) = \frac{(-1)^{n+1}}{\sqrt{(q;q)_n}} \left(\frac{q}{a}\right)^{n/2} \sum_{j=0}^{n-1} (q; q)_j a^j. \quad (41)$$

Note that

$$C_1(qa)^n \leq |Q_n(a)| \leq C_2 \left(\frac{q}{a}\right)^{n/2} \frac{a^n - 1}{a - 1} \quad (42)$$

where the constants $C_1, C_2$ do not depend on $n$.

With the additional assumption $a < 1$ we can also evaluate functions of the second kind at $x = a$. In particular the value of the Weyl function equals

$$w(a) = w_0(a) = -\lim_{n \to \infty} \frac{Q_n(a)}{\hat{P}_n(a)} = \sum_{j=0}^{\infty} (q; q)_j a^j.$$ For a general index $n$ we have (22)

$$w_n(a) = w(a) \hat{P}_n(a) + Q_n(a) = \frac{(-1)^n}{\sqrt{(q;q)_n}} \left(\frac{q}{a}\right)^{n/2} \sum_{j=0}^{\infty} (q; q)_j a^j. \quad (43)$$

From (40), (41) it is immediately seen, as is well known [9, 5], that the Hamburger moment problem is indeterminate if and only if $q < a < q^{-1}$. To this end, it suffices to recall the classical result [2] asserting that a Hamburger moment problem is indeterminate if and only if the sequences $\{\hat{P}_n(w)\}$ and $\{Q_n(w)\}$ are both square summable where $w$ is an arbitrary fixed point in $\mathbb{C}$.

From now on we assume that $0 < a \leq q$. Hence the Hamburger moment problem is determinate. We wish to apply our method to evaluate the characteristic function $\Phi(z)$ for the Al-Salam–Carlitz II polynomials. This goal can be achieved if we shift the Jacobi matrix by a constant considering $J - aI$ instead of $J$. The orthonormal polynomial sequence corresponding to the shifted matrix is $\{\hat{P}_n(z + a)\}$ and, similarly, the sequence of functions of the second kind after the shift has the form $\{w_n(z + a)\}$. 

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It is straightforward to verify the assumptions of Theorem 2 for the shifted Jacobi matrix. First note that by our assumptions $0 < a \leq q < 1$ and hence $J - a$ is still positive definite on span$\{e_n; n = 0, 1, 2, \ldots\}$. Referring to (40), (42) we have
\[
\sum_{n=0}^{\infty} w_n(a) \hat{P}_n(a) = \sum_{n=0}^{\infty} \left(\frac{q}{a}\right)^n \sum_{j=n}^{\infty} (q; q)_j a^j \leq \frac{1}{(q; q)^{\infty}} \sum_{n=0}^{\infty} \left(\frac{q}{a}\right)^n \frac{a^n}{1-a} < \infty.
\]
Consequently, $(J - a)^{-1}$ belongs even to the trace class.

Put
\[
\tilde{\Phi}^{(a)}(z) := 1 - z \sum_{n=0}^{\infty} w_n(a) \hat{P}_n(z + a).
\]
In view of (8) we have
\[
\tilde{\Phi}^{(a)}(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n - a}\right)
\]
whence
\[
\tilde{\Phi}^{(a)}(z - a) = 1 - (z - a) \sum_{n=0}^{\infty} w_n(a) \hat{P}_n(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) / \prod_{n=1}^{\infty} \left(1 - \frac{a}{\lambda_n}\right).
\]

**Proposition 8.** For $0 < a \leq q$ we have
\[
1 - (z - a) \sum_{n=0}^{\infty} w_n(a) \hat{P}_n(z) = \frac{z; q)_\infty}{(a; q)_\infty}.
\]
Consequently, $\prod_{n=1}^{\infty} (1 - z/\lambda_n) = (z; q)_\infty$.

**Lemma 9.** For all $a, c \in \mathbb{C}$, $c \neq 1, q^{-1}, q^{-2}, \ldots$, it holds true that
\[
2\phi_1(a, q; q; c, q) = \frac{1 - c}{a - c} \left(1 - \frac{(a; q)_\infty}{(c; q)_\infty}\right).
\]

**Proof.** Let
\[
F(a, c) = 1 - \frac{a - c}{1 - c} 2\phi_1(a, q; q; c, q).
\]
We have to show that $F(a, c) = (a; q)_\infty/(c; q)_\infty$. Since
\[
2\phi_1(a, q; b; q, z) = 1 + \frac{(1 - a)z}{1 - b} 2\phi_1(qa, q; qb; q, q)
\]
one readily verifies that $F(a, c) = ((1 - a)/(1 - c)) F(qa, qc)$ and, by mathematical induction,
\[
\forall n \in \mathbb{Z}_+, F(a, c) = \frac{(a; q)_n}{(c; q)_n} F(q^n a, q^n c).
\]
Taking the limit $n \to \infty$ gives the result. \hfill \square
Proof of Proposition 8. In view of (39) we can write
\[
\sum_{n=0}^{\infty} w_n(a) \hat{P}_n(z) = \sum_{j=0}^{\infty} X_j a^j \quad \text{where} \quad X_j := \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{(q; q)_{j+n}}{(q; q)_n} V_n^{(a)}(z; q).
\]
Using (38) we can express
\[
X_j = \sum_{n=0}^{\infty} (qa)^n (q; q)_j n q^n \frac{(z; q)_n}{(q; q)_n} V_n^{(a)}(z; q) = \sum_{k=0}^{\infty} \frac{(z; q)_k}{(q; q)_k} \sum_{j=0}^{\infty} \frac{(q; q)^j}{(q; q)_j} (qa)^j.
\]
In the last step we have used the \(q\)-Binomial Theorem [7, Eq. (II.3)]
\[
\sum_{\ell=0}^{\infty} \frac{(u; q)_\ell}{(q; q)_\ell} z^\ell = \frac{(zu; q)_\infty}{(z; q)_\infty}, \quad |z| < 1.
\]
Now an application of the \(q\)-Gauss summation [7, Eq. (II.7)]
\[
\binom{2}{1} \left( a, b; c; q, \frac{c}{ab} \right) = \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/(ab); q)_\infty}
\]
gives
\[
\sum_{j=0}^{\infty} \frac{(q; q)_{j+k}}{(q; q)_j} a^j = \frac{1 - q^{k+1} a}{1 - a}
\]
and therefore
\[
\sum_{n=0}^{\infty} w_n(a) \hat{P}_n(z) = \sum_{j=0}^{\infty} X_j a^j = \frac{1}{1 - a} \sum_{k=0}^{\infty} \frac{(z; q)_k}{(qa; q)_k} q^k = \frac{2 \binom{2}{1}(z; q; qa; q, q)}{1 - a}. \quad (44)
\]
Combining (44) and (43) we obtain the desired equation.

\[\square\]

Referring to Proposition 8 we conclude that an application of Theorem 2 in this example shows that the zero set of the function \(z \mapsto (z; q)_\infty\) coincides with the spectrum of \(J\) and, at the same time, with the support of the orthogonality measure for the Al-Salam–Carlitz II polynomials. This is actually the correct answer as the orthogonality measure in this case is known to be supported on the set \(\{q^{-n}; n \in \mathbb{Z}_+\}\) [1, 9].

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