Hyperbolic spin Ruijsenaars-Schneider model from Poisson reduction

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ABSTRACT: We derive a Hamiltonian structure for the \( N \)-particle hyperbolic spin Ruijsenaars-Schneider model by means of Poisson reduction of a suitable initial phase space. This phase space is realised as the direct product of the Heisenberg double of a factorisable Lie group with another symplectic manifold that is a certain deformation of the standard canonical relations for \( N \ell \) conjugate pairs of dynamical variables. We show that the model enjoys the Poisson-Lie symmetry of the spin group \( \text{GL}_\ell(\mathbb{C}) \) which explains its superintegrability. Our results are obtained in the formalism of the classical \( r \)-matrix and they are compatible with the recent findings on the different Hamiltonian structure of the model established in the framework of the quasi-Hamiltonian reduction applied to a quasi-Poisson manifold.

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1 Introduction

The Ruijsenaars-Schneider (RS) integrable models [1, 2] continue to deliver rich mathematical structures that are worth further exploring. One particular aspect concerns the introduction of spin degrees of freedom. Recall that a spin generalisation of the RS model with the most general elliptic potential was proposed in [3] as a dynamical system describing the evolution of poles of elliptic solutions of the non-abelian 2d Toda chain. This is a system of \( N \) particles on a line with internal degrees of freedom represented by two \( \ell \)-dimensional vectors attached to each of the particles. The proposed spin RS model is given in terms of equations of motion for the particle coordinates \( q_i, i = 1, \ldots, N \) and the spin variables \( a_{i\alpha} \) and \( c_{\alpha i} \), where \( \alpha = 1, \ldots, \ell \). The knowledge of the equations of motion contains but unfortunately does not immediately yield the Hamiltonian structure behind this dynamical system.

In [4] we established the underlying Hamiltonian structure for the case of rational degeneration of the elliptic spin RS model. This was done by relaying on the observation that goes back to [5] and further developed in [6]-[15] that the Calogero-Moser-Sutherland and Ruijsenaars-Schneider models can be obtained by means of the Hamiltonian or Poisson reduction procedure applied to a suitably chosen initial phase space. In the case of the rational spin RS model the suitable initial phase space \( \mathcal{M} \) appears to be the direct product \( \mathcal{M} = T^*G \times \Sigma \), where \( T^*G \) is the cotangent bundle to a Lie group \( G \) with the Lie algebra \( g \) and \( \Sigma \) is the symplectic manifold of \( N\ell \) pairs of canonical variables (oscillators). This phase space is a Poisson manifold which carries the Hamiltonian action of \( G \). Choosing \( G = GL_N(\mathbb{C}) \) the Hamiltonian reduction of \( \mathcal{M} \) by the action of \( G \) yields the desired Poisson structure of the spin RS model [4]. The Poisson brackets of the invariant spin variables appear rather involved. Although it was possible to guess a natural generalisation of the Poisson structure for “collective” spin variables \( f_{ij} = \sum_\alpha a_{i\alpha} c_{\alpha j} \) to the hyperbolic spin RS model, the progress of finding the Poisson structure of individual spins in the hyperbolic case was delayed for years. Quite recently this structure has been

\(^1\)We follow the notation of [4].
found [16] confirming the conjecture in [4] on the brackets of collective spin variables. The approach of [16], see also [17, 18], is based on the quasi-Hamiltonian reduction procedure, where one starts from an initial manifold \( \mathcal{M} \) supplied with a quasi-Poisson structure and which carries a free action of a Lie group \( G \). Although \( \mathcal{M} \) is not Poisson, the quotient \( \mathcal{M}/G \) inherits the well-defined Poisson structure from the quasi-Poisson structure on \( \mathcal{M} \). Picking as \( \mathcal{M} \) a representation space of a framed Jordan quiver, it was shown in [16] that the reduction of this by \( G \) yields the Poisson structure of invariant spins that perfectly fits the hyperbolic (trigonometric complex) spin RS model. The Liouville integrability and superintegrability (degenerate integrability) of the spin RS model also follow from this approach.

Having established these nice results, one still may wonder if there would exist a conventional way of getting the spin hyperbolic RS model by the usual Poisson reduction but applied to a more complicated initial phase space being the next in the deformation hierarchy after \( T^*G \times \Sigma \) responsible for the rational model. Indeed, the spinless hyperbolic RS model follows from the Poisson reduction applied to the Heisenberg double \( D_+(G) \) of \( G \), as has been recently discussed in [19]. The Poisson structure of the Heisenberg double [20] is a deformation of the one of \( T^*G \). From the point of view of the deformation theory, it is then natural to replace the moment map on \( \Sigma \), taking values into the dual Lie algebra \( g^* \), with a non-abelian moment map defined on a suitable deformation of \( \Sigma \) and which takes values in the dual Poisson-Lie group \( G^* \). The main question is how to realise the quadratic Poisson structure of \( G^* \) in terms of \( N\ell \)-pairs of oscillators that should replace those used to represent the linear Kirillov-Kostant bracket in the rational case. In this paper we solve this problem and reconstruct the spin hyperbolic RS model in the standard framework of the Poisson reduction.

The main tool in our approach is a Poisson pencil of a constant and quadratic Poisson structures on an oscillator manifold \( \Sigma_{N,\ell} \) spanned by \( 2N\ell \) dynamical variables \( a_{i\alpha}, b_{\alpha i} \). When the coefficient \( \kappa \) in front of the quadratic structure vanishes, one obtains the standard canonical relations of the \( N\ell \) conjugate pairs. In fact there are two different quadratic structures, to distinguish between them we label the corresponding Poisson manifolds as \( \Sigma_{N,\ell}^{\pm} \). These Poisson manifolds carry Poisson actions of two different Poisson-Lie groups – the particle group \( \text{GL}_N(\mathbb{C}) \) and the spin group \( \text{GL}_\ell(\mathbb{C}) \), acting by linear transformations on the oscillator indices \( i \) and \( \alpha \), respectively. Starting from the initial phase space \( \mathcal{M} = D_+(G) \times \Sigma_{N,\ell}^{\pm} \) and reducing this manifold by the action of the particle group, we obtain the spin RS model with the Poisson structure inherited from that on \( \mathcal{M} \). The equations of motion for the spins are the same regardless of which manifold \( \Sigma_{N,\ell}^{\pm} \) we use, and they coincide with those that follow from the Poisson structure of spins found in [16] through the quasi-Hamiltonian reduction. The construction of conserved quantities, both Poisson commutative and non-commutative, is straightforward and follows the same pattern as in the rational case. The spin group continues to act on the reduced phase space as a Poisson-Lie symmetry and its presence explains the superintegrability of the model. In fact, there are higher symmetries whose generators are polynomial in the spin variables and which arise from conjunction of the spin symmetries with abelian symmetries generated by higher commuting charges. We show that the Poisson structure of the currents encoding these symmetries is a quadratic deformation of the linear bracket of the rational model. This quadratic part appears as an affine version of the Poisson-Lie structure on \( G^* \).

Concluding the brief discussion of our approach, we point out that it would be interesting to extend it to account for the most general elliptic spin model. Also, since we are building on the classical \( r \)-matrix formalism, the recognition of various \( r \)-matrix structures might help to pave the way for quantising the spin model which currently remains another open problem.

The paper is organised as follows. In the next section we recall the necessary facts about the Heisenberg double. In section 3 we introduce the oscillator manifold. In section 4 we discuss the
Poisson action of a Poisson-Lie group on the product of two manifolds. In section 5 we solve the moment map equation obtaining the Lax matrix of the spin RS model on the reduced phase space. The Poisson brackets of $G$-invariant variables are studied in section 6 and section 7 is devoted to the discussion of symmetries of the model responsible for its superintegrable status. We conclude this section by showing what superintegrability implies for solvability of the equations of motion. Some technical details are collected in appendix. All the considerations in the paper are done in the context of holomorphic integrability.

2 Heisenberg Double

We start with recalling the construction of the double of a factorisable Lie bialgebra. Let $G$ be a Lie group with the Lie algebra $g$. Denote by $g^*$ the dual of $g$. We assume that $(g, g^*)$ is a factorisable Lie bialgebra and we use the corresponding invariant form on $g$ to identify $g^* \simeq g$. The double $D$ of $(g, g^*)$ can be identified with $D = g \oplus g$ supplied with the Lie algebra structure of the direct sum of two copies of the Lie algebra. The Lie algebra $g \subset D$ is embedded in $D$ as the diagonal subalgebra, while the Lie subalgebra $g^*$ is identified inside $D$ as a subset

$$(X_+, X_-) = (\hat{i}_+ X, \hat{i}_- X) \subset D, \quad \forall X \in g^* \simeq g.$$ 

Here $\hat{i}_\pm = \hat{i} \pm \frac{1}{2} \mathbb{I}$ are two linear operators, $\hat{i}_\pm : g^* \to g_\pm \subset g$, constructed from a skew-symmetric split solution $\hat{\iota} \in g \wedge g$ of the modified Yang-Baxter equation. Any $X \in g$ has a unique decomposition $X = X_+ - X_-$. Let $D = G \times G$ be the double Lie group corresponding to $D$. The connected Lie group $G^*$ corresponding to the Lie algebra $g^*$ is embedded in $D$ as $G^* \simeq (u_+, u_-) \subset D$ by extending the Lie algebra homomorphisms given by $\hat{i}_\pm$. Here $u_\pm \in G_\pm$, where $G_\pm$ are the corresponding subgroups of $G$. In the following we assume the existence of a global diffeomorphism $\sigma : G^* \simeq G$,

$$\sigma(u_+, u_-) = u_+ u_-^{-1} = u,$$  \hspace{1cm} (2.1)

such that the factorisation problem (2.1) has a unique solution for any $u \in G$.

Now we introduce the Heisenberg double $D_+(G)$ of $G$. Consider a pair of matrices $(A, B) \in D$, $A, B \in G$. The entries of $A, B$ can be regarded as generators of the coordinate ring of the algebra or regular functions on $D$. The Heisenberg double $D_+(G)$ is $D$ viewed as a Poisson manifold with the following Poisson relations between the generators

$$\begin{align*}
\frac{1}{\kappa} \{A_1, A_2\} &= -\tau_- A_1 A_2 - A_1 A_2 \tau_+ + A_1 \tau_- A_2 + A_2 \tau_+ A_1, \\
\frac{1}{\kappa} \{B_1, B_2\} &= -\tau_- B_1 B_2 - B_1 B_2 \tau_+ + B_1 \tau_- B_2 + B_2 \tau_+ B_1, \\
\frac{1}{\kappa} \{A_1, B_2\} &= -\tau_- A_1 B_2 - A_1 B_2 \tau_+ + A_1 \tau_- B_2 + B_2 \tau_+ A_1, \\
\frac{1}{\kappa} \{B_1, A_2\} &= -\tau_- B_1 A_2 - B_1 A_2 \tau_+ + B_1 \tau_- A_2 + A_2 \tau_+ B_1. 
\end{align*}$$  \hspace{1cm} (2.2)

where $\kappa$ is a complex parameter. Here $\tau_\pm$ are two canonical solutions of the classical Yang-Baxter equation associated with the factorisable Lie algebra $g$; they correspond to the operators $\hat{i}_\pm$.

In this work we are primarily interested in the case $G = GL_N(\mathbb{C})$ for which the matrices $\tau_\pm$ are

$$\tau_\pm = \pm \frac{1}{2} \sum_{i=1}^N E_{ii} \otimes E_{ii} \pm \sum_{i < j}^N E_{ij} \otimes E_{ji}. \hspace{1cm} (2.3)$$
Here $E_{ij}$ are the standard matrix unities, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. We also recall that
\[ \tau_{\pm 21} = -\tau_{\mp 12}, \quad \tau_+ - \tau_- = C_{12} = \sum_{i,j=1}^N E_{ij} \otimes E_{ji}, \] (2.4)

and introduce $\tau = \frac{1}{2}(\tau_+ + \tau_-)$, which is a skew-symmetric split solution to the modified classical Yang-Baxter equation mentioned above.

The Heisenberg double (2.2) carries a Poisson action of a Poisson-Lie group $G$
\[ A \to hAh^{-1}, \quad B \to hBh^{-1}, \quad h \in G. \] (2.5)
The Poisson-Lie structure of $G$ is given in terms of the Sklyanin bracket
\[ \{ h_1, h_2 \} = -\kappa [\tau_\pm, h_1 h_2], \quad h \in G. \] (2.6)
The non-abelian moment map for this action $(m_+, m_-)$ takes values in the group $G^*$. Under $\sigma$ it maps onto an element $m = m_+ m_-^{-1} \in G$, where
\[ m = BA^{-1}B^{-1}A. \] (2.7)
The Poisson algebra between the entries of $m$ is
\[ \frac{1}{\kappa} \{ m_1, m_2 \} = -\tau_+ m_1 m_2 - m_1 m_2 \tau_- + m_1 \tau_- m_2 + m_2 \tau_+ m_1. \] (2.8)

The Poisson algebra (2.2) has two obvious involutive subalgebras - one is generated by $\text{Tr}A^k$ and the other by $\text{Tr}B^k$, where $k \in \mathbb{Z}$. There is yet another involutive family which plays an essential role in this work, namely,
\[ H_k = \text{Tr}(BA^{-1})^k = \text{Tr}(A^{-1}B)^k, \quad k \in \mathbb{Z}. \] (2.9)
The fact that $\{ H_k, H_m \} = 0$ for any $k, m \in \mathbb{Z}$ can be verified by direct computation. A deeper observation is that the map
\[ A \to A, \quad B \to BA^{-1}, \] (2.10)
is a canonical transformation, i.e. under this map the Poisson structure (2.2) remains invariant. Note that all the involutive families mentioned above are generated by invariants of the adjoint action (2.5).

In the following we need two facts about the group $G^*$. First, $G^*$ is a Poisson-Lie group. In terms of the generators $u_{\pm} \in G_{\pm} \subset G$ the corresponding Poisson-Lie structure is given by the following Poisson brackets
\[ \frac{1}{\kappa} \{ u_{\pm 1}, u_{\pm 2} \} = -[\tau, u_{\pm 1} u_{\pm 2}], \quad \frac{1}{\kappa} \{ u_{\pm 1}, u_{\mp 2} \} = -[\tau, u_{\pm 1} u_{\mp 2}]. \] (2.11)
Under the map (2.1), these brackets endow $G$ with the structure of a Poisson manifold given by the Semenov-Tian-Shansky bracket [20]
\[ \frac{1}{\kappa} \{ u_1, u_2 \} = -\tau_+ u_1 u_2 - u_1 u_2 \tau_- + u_1 \tau_- u_2 + u_2 \tau_+ u_1. \] (2.12)
Comparing (2.8) with (2.12) shows that the Poisson algebra of $m$ is given by the Semenov-Tian-Shansky bracket.
The product in $G^*$ induces under (2.1) a new product in $G$ which we denote by $\star$. For any $u, v \in G^*$ it is defined as

$$v \star u = v_+ u_+ u_+^{-1} v_-^{-1} = v_- u_+^{-1}.$$  \hspace{1cm} (2.13)

where $u_\pm$ and $v_\pm$ are solutions of the factorisation problems $u = u_+ u_-^{-1}$ and $v = v_+ v_-^{-1}$. The Poisson-Lie structure of $G^*$ is then encoded in the following relation

$$\{v_1 u_1, v_2 u_2\} = \{v_1 u_1 v_-^{-1}, v_2 u_2 v_-^{-1}\} = \{u_1, u_2\} (v \star u),$$

where the bracket of $u$'s is (2.12), while the brackets of $v_\pm$ are evaluated according to (2.11).

Second, the Poisson-Lie group $G$ acts on $G^*$ by dressing transformations [20]. Modelling $G^*$ over $G$, these transformations take the form of the adjoint action

$$u \rightarrow h u h^{-1}, \quad h \in G, \hspace{1cm} (2.14)$$

and they are Poisson maps of the Semenov-Tian-Shansky bracket provided the Poisson-Lie structure on $G$ is given by (2.6). The non-abelian moment map of this action is $u$. It is well known that the symplectic leaves of (2.12) coincide with the orbits of (2.14).

3 Oscillator manifold

As the next step, we introduce a manifold $\Sigma_{N,\ell}$ as the product of two linear spaces of all rectangular $N \times \ell$-matrices

$$\Sigma_{N,\ell} = \text{Mat}_{N,\ell}(C) \times \text{Mat}_{\ell,N}(C),$$

where $N$ is the number of particles of the model and $\ell$ is the length of spin vectors. Let $(a, b)$ be two arbitrary $N \times \ell$- and $\ell \times N$-matrices. Their entries

$$a_{i\alpha} \equiv (a)_{i\alpha}, \quad b_{\alpha j} \equiv (b)_{\alpha j} \quad i = 1, \ldots, N, \quad \alpha = 1, \ldots, \ell.$$  \hspace{1cm} (3.2)

provide a global coordinate system on $\Sigma_{N,\ell}$. We call $a_{i\alpha}$ and $b_{\alpha j}$ oscillators and refer to $\Sigma_{N,\ell}$ as to an oscillator manifold.

Now we endow $\Sigma_{N,\ell}$ with two different $\pm$-structures of a Poisson manifold $\Sigma_{N,\ell}^\pm$ by defining the following Poisson brackets $\{ , \}$ between oscillators

$$\{a_{i\alpha}, a_{j\beta}\}_\pm = \kappa ( \pm a_{i\alpha} a_{j\beta} - a_{i\alpha} a_{j\beta} \rho) ,$$

$$\{b_{\alpha i}, b_{\beta j}\}_\pm = \kappa ( b_{\alpha i} b_{\beta j} \mp \rho b_{\alpha i} b_{\beta j} ) ,$$

$$\{a_{i\alpha}, b_{\beta j}\}_\pm = \kappa ( -b_{\beta j} b_{\alpha i} \pm a_{i\alpha} \rho b_{\beta j} ) - C^{\text{rec}}_{12} ,$$

$$\{b_{\alpha i}, a_{j\beta}\}_\pm = \kappa ( -b_{\alpha i} a_{j\beta} \pm a_{j\beta} \rho b_{\alpha i} ) + C^{\text{rec}}_{21} .$$

Here we have introduced a “rectangular split Casimir”

$$C^{\text{rec}}_{12} = \sum_{i=1}^N \sum_{\alpha=1}^\ell E_{i\alpha} \otimes E_{\alpha i} ,$$

\hspace{1cm} (3.3)

\hspace{1cm} (3.4)

This is in fact the coadjoint action of $G$ on $G^*$.
where \((E_{i\alpha})_{j\beta} = \delta_{ij}\delta_{\alpha\beta}\). The matrices \(\rho_\pm\) are the following analogues of \(r_\pm\) in the spin space
\[
\rho_\pm = \pm \frac{1}{2} \sum_{\alpha=1}^\ell E_{\alpha\alpha} \otimes E_{\alpha\alpha} \pm \sum_{\alpha\leq \beta} E_{\alpha\beta} \otimes E_{\beta\alpha}
\] (3.5)
and \(\rho = \frac{1}{2}(\rho_+ + \rho_-)\). One also has
\[
\rho_+ - \rho_- = C_{12}^a = \sum_{\alpha,\beta=1}^\ell E_{\alpha\beta} \otimes E_{\beta\alpha} .
\] (3.6)

For \(\kappa = 0\) the brackets (3.3) turn into the standard oscillator algebra formed by \(N\ell\) pairs of canonically conjugate variables
\[
\{a_{i\alpha}, b_{j\beta}\} = -\delta_{ij}\delta_{\alpha\beta} .
\] (3.7)
The brackets (3.3) satisfy the Jacobi identity for any \(\kappa\), i.e. the constant and quadratic structures in (3.3) form a Poisson pencil being a one-parametric deformation of the canonical relations (3.7). It remains to note that if we define
\[
\omega = 1 + \kappa ab ,
\] (3.8)
where \(ab\) is an \(N \times N\)-matrix being a natural product of two rectangular matrices, then due to (3.3), \(\omega\) will satisfy the Poisson algebra
\[
\frac{1}{\kappa}\{\omega_1, \omega_2\} = (r_+\omega_1\omega_2 + \omega_1\omega_2 r_- - \omega_1 r_-\omega_2 - \omega_2 r_+\omega_1 ,
\] (3.9)
which is different from (2.12) by an overall sign only. In particular, the contribution of the spin matrices \(\rho, \rho_\pm\) completely decouples. Thus, formulæ (3.8) give a realisation of the Semenov-Tian-Shansky bracket in terms of the oscillator algebra (3.3). We also point out the Poisson relations between \(\omega\) and oscillators
\[
\frac{1}{\kappa}\{\omega_1, a_2\} = (r_+\omega_1 - \omega_1 r_-)a_2 , \quad \frac{1}{\kappa}\{\omega_1, b_2\} = -b_2 (r_+\omega_1 - \omega_1 r_-) .
\] (3.10)
In deriving (3.9) and (3.10) one has to use the relations
\[
a_1 C_{21}^{rec} = C_{12}a_2 , \quad C_{12}^{rec}b_1 = b_2 C_{12} , \quad C_{12}^{rec}b_2 = b_1 b_2 C_{12} .
\]

Importantly, one can now verify that if we allow \(G\) to act infinitesimally on oscillators as
\[
\delta_X a_{i\alpha} = (Ad^*_X a)_{i\alpha} , \quad \delta_X b_{i\alpha} = -(b Ad^*_X)_{i\alpha} , \quad X \in \mathfrak{g} ,
\] (3.11)
then this action \(G \times \Sigma_{N,\ell}^+ \to \Sigma_{N,\ell}^+\) is a mapping of Poisson manifolds provided \(G\) is equipped with the Sklyanin bracket (2.6). Here \(Ad^*_X\) for \(g = (g_+, g_-) \in G^+\) is the coadjoint (dressing) action of \(G^+\) on the Lie algebra \(\mathfrak{g}\). If we factorise \(\omega = \omega_+\omega_-^{-1}\) according to (2.1), then \((\omega_+^{-1}, \omega_-^{-1}) \in G^+\) is the moment map for the Poisson action (3.11). Under (2.1) it defines the following element of \(G\)
\[
N = \omega_+^{-1}\omega_- \in G .
\] (3.12)
The fact that \(N\) generates the action (3.11) can be deduced from the Poisson brackets (3.10) together with the fact that \(\omega \ast \{N, \_\} = \{\omega, \_\} \ast N\). The Poisson algebra of \(N\) coincides with (2.12).

Further, the oscillator manifolds carries an action of the spin Poisson-Lie group \(S = GL_\ell(\mathbb{C})\)
\[
\begin{align*}
a_{i\alpha} &\mapsto (ag)_{i\alpha} , & \quad b_{i\alpha} &\mapsto (g^{-1}b)_{i\alpha} , & \quad g &\in S .
\end{align*}
\] (3.13)
This action is Poisson provided the Poisson-Lie structure on \(S\) is taken for \(\Sigma_{N,\ell}^+\) to be
\[
\{g_1, g_2\} = \pm \kappa [\rho, g_1g_2] .
\] (3.14)
4 Poisson-Lie group action on a product manifold

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Poisson manifolds with brackets $\{\cdot, \cdot\}_{\mathcal{M}_1}$ and $\{\cdot, \cdot\}_{\mathcal{M}_2}$ that carry the Poisson action of a Poisson-Lie group $G$. Let $\mathcal{M}_i : \mathcal{M}_i \to G^*$ be the corresponding non-abelian moment maps which are assumed to be Poisson. Then, one can define the Poisson action of $G$ on the product manifold $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ by taking the product of the moment maps $\mathcal{M} = \mathcal{M}_1 \mathcal{M}_2$, and allowing it to act on functions $f$ by means of the formula

$$\xi_X f = \{X, \{M, f\}_{\mathcal{M}}\}_{\mathcal{M}^{-1}} m_1 m_2^{-1}, \quad f \in \text{Fun}(\mathcal{M}),$$

(4.1)

where $\xi_X$ is a vector field corresponding to $X \in \mathfrak{g}$ and $\{\cdot, \cdot\}$ is the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. We have

$$\xi_X f = \{X, \{m_1, f\}_{\mathcal{M}_1}\}_{\mathcal{M}_1} m_1^{-1} m_2 \{m_2, f\}_{\mathcal{M}_2} m_2^{-1} m_1^{-1}. \quad (4.2)$$

Let $\xi_X^{(1)}$ and $\xi_X^{(2)}$ be the fundamental vector fields induced by the group action on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Formula (4.2) is equivalent to the statement that at a point $x = (x_1, x_2) \in \mathcal{M}$, where $x_1 \in \mathcal{M}_1$ and $x_2 \in \mathcal{M}_2$, the vector field $\xi_X$ is defined as

$$\xi_X(x) = \xi_X^{(1)}(x_1) + \xi_X^{(2)}(x_2), \quad (4.3)$$

where $\text{Ad}^*_{h'} h \in G^*$ is the coadjoint action of $G^*$ on $G$ which is also an example of dressing transformations [20]. One can show that the map $X \to \xi_X$, where $\xi_X$ is defined by (4.3), is the Lie algebra homomorphism, so that $\xi_X$ is the fundamental vector field of the group action on $G$ [21, 22]. Since $G^*$ is a Poisson-Lie group, $\mathcal{M}$ will have the same Poisson brackets between its entries as $\mathcal{M}_1$ or $\mathcal{M}_2$.

To construct the Hamiltonian structure of the spin RS model, we take the product of symplectic manifolds $\mathcal{M}_1 = D_+(G)$ and $\mathcal{M}_2 = \Sigma^\pm N, \ell$, $\mathcal{M} = D_+(G) \times \Sigma^\pm N, \ell. \quad (4.4)$

Here the Poisson structure on the Heisenberg double $D_+(G)$ is given by (2.2) and that on the oscillator manifold is (3.3). We define the Poisson action of $G$ on $\mathcal{M}$ through its moment map

$$m \star n = m_n m_{n^1}, \quad (4.5)$$

where $n$ is the moment map (3.12) of the action (3.11) and $m$ is (2.7). Since $m$ and $n$ are elements of $G^*$ modelled by $G$, we multiply them with the star product. To obtain the RS model on the reduced phase space, we fix the moment map to the following value

$$m \star n = q \mathbb{1}, \quad (4.6)$$

where $\mathbb{1}$ is the group identity in $G$ and $q$ is the coupling constant. Since the right hand side of (4.6) is proportional to the identity, the stability group of the moment map coincides with the whole group $G$.

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3The product is naturally taken in $G^*$.
4We are grateful to László Fehér for drawing our attention to this work.
and, therefore, all the entries of $M \star N$ are constraints of the first class. Equation (4.6) can be written as the following equation in $G$

$$M = q \omega_+ \omega_-^{-1} = q \omega.$$  (4.7)

Some comments are in order. The choice of the initial manifold (4.4), as well as the use of relevant reduction techniques to obtain the spin RS models on the reduced phase space was already suggested earlier, see e.g. [4, 23]. Also, a similar construction was developed in [15], where $G$ was taken to be the compact Lie group $U(N)$. In this case the underlying Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is not factorisable and the corresponding double $\mathcal{D}$ can be identified with the complexification of $\mathfrak{g} = \mathfrak{su}(N)$. The dynamical system one finds on the reduced phase space coincides with the trigonometric spin RS model. The point, however, is that working with the collective spin variable $\omega$ alone leaves invisible the evolution of individual spin components of a spin vector associated to each particle. The aim of our present construction is to further resolve $\omega \in G^*$ in terms of internal spin degrees of freedom and obtain the dynamical equations for individual spins, as in [3].

5 Reduction

We can now develop the reduction procedure starting from the initial phase space (4.4)

$$\mathcal{M} = D_+(G) \times \Sigma_{N, \ell}^+. \quad (5.1)$$

The moment map equation (4.7) takes the form

$$BA^{-1}B^{-1}A = q(1 + x_{ab}). \quad (5.2)$$

The reduced phase space $\mathcal{P}$ is obtained by factoring solutions of (5.2) by the action of the group $G$

$$\mathcal{P} = \{\text{Solutions of (5.2)}\}/G. \quad (5.3)$$

Note that for our reduction procedure the parameter $x$ controlling the Poisson brackets (2.2) of the Heisenberg double and the brackets (3.3) of the oscillator manifold is chosen to be the one and the same.

We point out that under the Poisson action on the product manifold (5.1) the transformation of oscillators get simplified over the hypersurface defined by (5.2). Indeed recalling (4.3) and (3.11), we get

$$\delta_X a_{\alpha} = (\text{Ad}_{\omega \star M^{-1}} X) a_{\alpha}, \quad \delta_X b_{\alpha} = - (b \text{Ad}_{\omega \star M^{-1}} X)_{\alpha}, \quad X \in \mathfrak{g}, \quad (5.4)$$

and since $\omega \star M^{-1} = \omega_+ m^{-1}_+ \omega_-^{-1} \equiv q^{-1} 1$ the action of $\text{Ad}_{\omega \star M^{-1}}$ is ineffective and the oscillators transform as

$$a_{\alpha} \rightarrow (h a)_{\alpha}, \quad b_{\alpha} \rightarrow (b h^{-1})_{\alpha}, \quad h = e^X \in G. \quad (5.5)$$

The most efficient way to factor out solutions by the action of $G$ is to reformulate and solve the moment map equation (5.2) in terms of gauge-invariant variables. To this end, following [19] we introduce a new coordinate system on the diagonalisable locus of the Heisenberg double

$$A = T Q T^{-1}, \quad B = U P^{-1} T^{-1}, \quad (5.6)$$

$$-8-$$
where \( Q \) and \( P \) are diagonal matrices with entries
\[
Q_{ij} = \delta_{ij} Q_j, \quad P_{ij} = \delta_{ij} P_j.
\] (5.6)

The matrices \( T, U \) are Frobenius, i.e. they are subjected to the following constraints
\[
\sum_{j=1}^{N} T_{ij} = \sum_{j=1}^{N} U_{ij} = 1, \quad \forall i = 1, \ldots, N.
\] (5.7)

Imposition of these constraints renders decomposition (5.5) unique.

Under the transformations (3.11) the new variables transform as follows
\[
Q \rightarrow Q, \quad P \rightarrow P d_T^{-1} d_U, \quad T \rightarrow h T d_T, \quad U \rightarrow h U d_U,
\] (5.8)

where \((d_X)_{ij} = \delta_{ij} \sum_{k=1}^{N} (hX)_{ik}\) for any \(X \in \text{GL}_N(\mathbb{C})\). In particular, \(Q\) is invariant under the \(G\)-action.

Substituting (5.5) into (5.2), we will get
\[
U Q^{-1} U^{-1} T Q T^{-1} = q(1 + \kappa ab),
\]
where, in particular, the momentum variable \(P\) has completely decoupled. There are different ways to solve the above equation, we follow the one which relies on the simplest invariant spin variables.

We have
\[
T^{-1} U Q^{-1} = q(Q^{-1} T^{-1} U + \kappa T^{-1} ab T Q^{-1} T^{-1} U),
\]

Following the spinless pattern in [10, 19], we introduce the Frobenius matrix \(W = T^{-1} U\) and reintroduce the momentum \(P\) by multiplying from the right both sides of the equations above by \(P^{-1}\),
\[
WP^{-1} Q^{-1} - qQ^{-1} WP^{-1} = q\kappa T^{-1} ab A^{-1} BT,
\] (5.9)

Note that under (3.11) the variable \(WP^{-1}\) is not invariant, rather it transforms as
\[
WP^{-1} \rightarrow d_T^{-1}(WP^{-1}) d_T.
\]

On the other hand, a matrix \(T^{-1} a\) transforms as
\[
T^{-1} a \rightarrow d_T^{-1} T^{-1} h^{-1} h a = d_T^{-1} T^{-1} a,
\]
where we have taken into account the transformation law (3.11) for the spin variables. This suggests to introduce a diagonal matrix \(t\) with entries
\[
t_{ij} = \delta_{ij} \sum_{\alpha=1}^{t} (T^{-1} a)_{i\alpha}.
\] (5.10)

Multiplying (5.9) from the left and from the right by \(t^{-1}\) and \(t\), respectively, projects the moment map equation of the space of \(G\)-invariants
\[
t^{-1} WP^{-1} t Q^{-1} - q Q^{-1} t^{-1} WP^{-1} t = q\kappa t^{-1} T^{-1} ab A^{-1} BT t.
\]

Introducing the \(G\)-invariant combinations
\[
L = t^{-1} WP^{-1} t Q^{-1}, \quad a = t^{-1} T^{-1} a, \quad c = b A^{-1} BT t,
\] (5.11)
we rewrite the moment map equation in its final invariant form
\[ L - qQ^{-1}LQ = q\mathcal{X} \mathbf{a}\mathbf{c}. \] (5.12)
The last equation is elementary solved for \( L \)
\[ L = q\mathcal{X} \sum_{i,j=1}^{N} \frac{Q_i}{Q_i - qQ_j} (\mathbf{a}\mathbf{c})_{ij} E_{ij}. \] (5.13)
The quantity (5.13) is the Lax matrix of the hyperbolic spin RS model, as can be seen by introducing the following parametrisation
\[ q = e^{-2\gamma}, \quad Q_i = e^{2q_i}, \quad q_{ij} = q_i - q_j, \] (5.14)
so that \( L \) takes the familiar form
\[ L = \mathcal{X} e^{-2\gamma} \sum_{i,j=1}^{N} \frac{e^{q_{ij} + \gamma}}{2\sinh(q_{ij} + \gamma)} f_{ij} E_{ij}, \quad f_{ij} \equiv (\mathbf{a}\mathbf{c})_{ij}. \]
Computing the trace of \( L^k \),
\[ \text{Tr} L^k = \text{Tr}(WP^{-1}Q^{-1})^k = \text{Tr}(UP^{-1}T^{-1}TQ^{-1}T^{-1})^k = \text{Tr}(BA^{-1})^k, \] (5.15)
we recognise that \( \text{Tr} L^k \) originate from the \( G \)-invariant involutive family (2.9). Thus, \( \text{Tr} L^k \) are in involution. We take \( H = H_1 \) as the Hamiltonian.

6 Poisson brackets of \( G \)-invariants

As we have found, the reduced phase space \( \mathcal{P} \) has a natural parametrisation in terms of the following \( G \)-invariant variables
\[ a_{i\alpha}, c_{\alpha i}, Q_i, \quad i = 1, \ldots, N, \quad \alpha = 1, \ldots, \ell. \] (6.1)
Note that by construction the spin variables \( a_{i\alpha} \) are constrained to satisfy
\[ \sum_{\alpha=1}^{\ell} a_{i\alpha} = 1, \] (6.2)
which can be regarded as the Frobenius condition in the spin space. The Lax matrix (5.13) depends on the collective spin variables \( f_{ij} \) only, which allows to perform the \( \text{GL}_\ell(\mathbb{C}) \)-rotations
\[ a_{i\alpha} \rightarrow \frac{1}{u_i} a_{i\beta} (g^{-1})_{\alpha}^{\beta}, \quad c_{\alpha i} \rightarrow u_i g_{\alpha\beta} c_{\beta i}, \quad u_i = \sum_{\alpha,\beta=1}^{\ell} a_{i\beta} (g^{-1})_{\alpha}^{\beta}, \quad g \in \text{GL}_\ell(\mathbb{C}), \]
without changing \( f_{ij} \) and preserving the Frobenius condition (6.2).

Now we are in a position to determine the Poisson brackets between the variables (6.1) constituting the phase space. For that we need the Poisson brackets between \( T, U, Q \) and \( P \) variables of the double. They have been already found in our previous work [19] and for the reader convenience we collect them in appendix A. The brackets between invariant spins and \( Q \) are then
\[ \{ Q_i, a_{j\alpha} \} = 0, \quad \{ Q_i, c_{\alpha j} \} = \delta_{ij} c_{\alpha j} Q_j. \] (6.3)
For the brackets of spins between themselves we find

\[
\{a_1, a_2\}_\pm = \kappa (r^\pm + Y) a_1 a_2 \mp a_1 a_2 \rho \mp a_1 X_{21} a_2 \pm a_2 X_{12} a_1 ,
\]

\[
\{a_1, c_2\}_\pm = \kappa [c_2 (r^\pm + Y) a_1 \pm a_1 \rho \pm c_2 \pm a_1 c_2 X_{21} \mp X_{12} a_1 c_2 , + K_{21} a_1 \{c_2, X_{21} a_1\} + C_{12} Z_2 ,
\]

\[
\{c_1, a_2\}_\pm = \kappa [c_1 (-r^\pm + Y) a_2 \pm a_2 \rho \pm c_1 \mp a_2 c_1 X_{12} \mp X_{21} a_2 c_1 , - K_{12} a_2 Z_1 + C_{21} Z_1 ,
\]

\[
\{c_1, c_2\}_\pm = \kappa [c_1 c_2 (r^\pm + Y) \mp c_1 c_2 \pm c_1 X_{12} c_2 \pm c_2 X_{21} c_1 , + c_2 K_{12} Z_1 - c_1 K_{21} Z_2 ,
\]

where we introduced the matrices \(Z = Q^{-1} LQ\) and

\[
X_{12} = \sum_{i,j} (a_1 \rho)_{i\beta \delta} E_{ii} \otimes E_{\sigma \delta} , \quad X_{12}^\pm = \sum_{i,j} (a_1 \rho^\pm)_{i\beta \delta} E_{ii} \otimes E_{\sigma \delta} , \quad K_{12} = \sum_{i} E_{\sigma i} \otimes E_{ii} , \quad Y_{12} = \sum_{i} (a_1 a_2 \rho)_{i\beta \delta k} E_{ii} \otimes E_{kk} .
\]

While the matrices \(r^*, r^*, r^\circ\) depend on coordinates \(Q_i\) and they are defined as follows:

\[
r^* = \frac{1}{2} \sum_{i,j=1}^N \frac{Q_i + Q_j}{Q_i - Q_j} (E_{ii} - E_{ij}) \otimes (E_{jj} - E_{ji}) ,
\]

\[
r^* = \frac{1}{2} \sum_{i,j=1}^N \frac{Q_i + Q_j}{Q_i - Q_j} (E_{ii} - E_{ii}) \otimes E_{jj} , \quad r^\circ = \frac{1}{2} \sum_{i,j=1}^N \frac{Q_i + Q_j}{Q_i - Q_j} (E_{ii} \otimes E_{jj} - E_{ij} \otimes E_{ji}) .
\]

Writing the brackets (6.4) for the choice “−” in components one finds that for \(N = 1, 2\) and any spin \(\ell\), either \(\ell = 1, 2\) and any number of particles \(N\), it coincides with the result obtained in [16] by means of a quasi-Hamiltonian reduction.5 There are further immediate consequences of our findings. First, the rational limit of (6.4), which consists in rescaling \(q_i \rightarrow \kappa q_i, \gamma \rightarrow \kappa \gamma\) with further sending \(\kappa\) to zero, reproduces the Poisson structure of invariant spins established in [4]. Second, the Poisson algebra of collective spin variables \(f_{ij}\) that follows from (6.4) is in general different from the result conjectured in [4], and their difference written in the matrix form is

\[
\mp f_1 f_2 Y \mp Y f_1 f_2 \pm f_1 Y f_2 \pm f_2 Y f_1 .
\]

As a result, the Lax matrix (5.13) does not satisfies the same Poisson algebra as in the spinless case, due to the contributions of \(Y_{12}\). The Poisson bracket between Lax matrices reads

\[
\frac{1}{\kappa} \{L_1, L_2\}_\pm = (r_{12} + Y) L_1 L_2 - L_1 L_2 (r_{12} + Y) + L_1 (r_{21} + Y) L_2 - L_2 (r_{12} - Y) L_1 ,
\]

where the dynamical \(r\)-matrices are [19]

\[
r = \sum_{i \neq j} \left( \frac{Q_i}{Q_{ij}} (E_{ii} - E_{ij}) \otimes (E_{jj} - E_{ji}) , \right.
\]

\[
\tilde{r} = \sum_{i \neq j} \frac{Q_i}{Q_{ij}} (E_{ii} - E_{ij}) \otimes E_{jj} , \quad r = \sum_{i \neq j} \frac{Q_i}{Q_{ij}} (E_{ij} \otimes E_{ji} - E_{ii} \otimes E_{jj}) ,
\]

where similarly to the rational case we introduced the notation \(Q_{ij} = Q_i - Q_j\).

The bracket (6.7) has the general form of the \(r\)-matrix structure compatible with involutivity of the spectral invariants of \(L\), but the \(Q\)-dependent \(r\)-matrices of the spinless case receive now an extra contribution from the spin variables. As to the Poisson structure of [16], the corresponding \(LL\)-algebra is given by (6.7) where \(Y\) should be taken to zero.

5We thank to Maxime Fairon for pointing out the difference between the Poisson brackets (6.4) and those of [16] for a generic choice of \(N\) and \(\ell\).
7 Superintegrability

Here we explain how superintegrability of the spin RS model follows from our approach. Consider the following two families of functions on the Heisenberg double

\[J_n^+ = \text{Tr}[S(BA^{-1})^n], \quad J_n^- = \text{Tr}[S(A^{-1}B)^n], \quad n \in \mathbb{Z},\]

where \(S\) is an arbitrary \(N \times N\)-matrix which has a vanishing Poisson bracket with both \(A\) and \(B\). Using (2.2), it is elementary to find \(\{H_m, J_n^\pm \} = 0\), where \(H_m = \text{Tr}(BA^{-1})^m\) constitute a commutative family containing the Hamiltonian \(H_1\). Thus, \(J_n^\pm\) are integrals of motion. We take as \(S\) a matrix \(S^{\alpha\beta}\) with entries \((S^{\alpha\beta})_{ij} = a_{i\alpha}b_{j\beta}\). Thus, on the initial phase space \(\mathcal{M}\) we have two families of integrals

\[J_n^{+\alpha\beta} = \text{Tr}[S^{\alpha\beta}(BA^{-1})^n], \quad J_n^{-\alpha\beta} = \text{Tr}[S^{\alpha\beta}(A^{-1}B)^n], \quad \forall \alpha, \beta = 1, \ldots, \ell. \quad (7.1)\]

These integrals are actually functions on the reduced phase space \(\mathcal{P}\) as they can be expressed in terms of gauge-invariant variables. Indeed, we have \(BA^{-1} = TtL^{-1}t^{-1}\) and \(A^{-1}B = Tt(Q^{-1}LQ)t^{-1}t^{-1}\), so that

\[BA^{-1} = A^{-1}B(B^{-1}ABA^{-1}) = A^{-1}BTt(Q^{-1}L^{-1}Q)t^{-1}t^{-1}\]

and, therefore,

\[J_n^{+\alpha\beta} = \text{Tr}[S^{\alpha\beta}(A^{-1}B)^n], \quad J_n^{-\alpha\beta} = \text{Tr}[S^{\alpha\beta}Q^{-1}L^{-1}Q^n],\]

where the matrix \(S^{\alpha\beta}\) comprises invariant spins \((S^{\alpha\beta})_{ij} = a_{i\alpha}c_{j\beta}\). Clearly, \(J_0^{+\alpha\beta} = J_0^{-\alpha\beta} = \text{Tr} S^{\alpha\beta}\).

Because \(J_n^{+\alpha\beta}\) are gauge invariants, their Poisson algebra computed on \(\mathcal{M}\) straightforwardly descends to the reduced phase space. To compute the Poisson brackets of the integrals, we start with

\[
\frac{1}{\kappa} \{S_1^{\alpha\beta}, S_2^{\gamma\delta}\}_{\pm} = \frac{1}{\kappa} C_{12}(\delta^{\beta\gamma}S_2^{\alpha\delta} - \delta^{\alpha\delta}S_2^{\beta\gamma}) + \tau S_1^{\alpha\beta}S_2^{\gamma\delta} - S_2^{\alpha\delta}S_1^{\gamma\beta} + S_1^{\alpha\delta}S_2^{\beta\gamma} - S_2^{\alpha\beta}S_1^{\gamma\delta} - S_1^{\gamma\delta}S_2^{\alpha\beta} - S_2^{\gamma\beta}S_1^{\alpha\delta} \\
\pm \left[ \rho_{\alpha\mu, \gamma\nu} S_1^{\mu\beta}S_2^{\nu\delta} + S_1^{\alpha\mu}S_2^{\nu\gamma} - S_2^{\alpha\nu}S_1^{\mu\gamma} - S_1^{\alpha\mu}S_2^{\mu\nu} - S_1^{\alpha\nu}S_2^{\mu\mu} \right], \quad (7.2)\]

where the indices 1, 2 are associated to the \(N \times N\) matrix spaces. In deriving the last formula we used the properties of the spin \(\rho\)-matrices \(\rho^T = -\rho\) and \(\rho_{\mu\nu}^T = -\rho_{\nu\mu}\), where \(T\) means transposition.

To present further results in a concise manner, we introduce a unifying notation

\[J_n^{\alpha\beta} = \text{Tr}(S^{\alpha\beta} \mathcal{W}^n), \quad (7.3)\]

where \(\mathcal{W}\) should be identified with \(\mathcal{W}^+ = BA^{-1}\) or with \(\mathcal{W}^- = A^{-1}B\). The Poisson brackets between the entries of \(\mathcal{W}^\pm\) are then

\[
\frac{1}{\kappa} \{\mathcal{W}^+_{1}, \mathcal{W}^\pm_{2}\} = -\tau_{\pm} \mathcal{W}^+_{1} \mathcal{W}^\pm_{2} - \mathcal{W}^+_{1} \mathcal{W}^\pm_{2} \pm_{\pm} + \mathcal{W}^{-\pm}_{1} \pm_{\pm} + \mathcal{W}^{-\pm}_{2} \pm_{\pm} \mathcal{W}^+_{1}. \quad (7.4)\]

By straightforward computation we then find the following result

\[
\frac{1}{\kappa} \{J_n^{\alpha\beta}, J_m^{\gamma\delta}\} = \frac{1}{\kappa} (\delta^{\beta\gamma}J_n^{\alpha\delta} - \delta^{\alpha\delta}J_n^{\beta\gamma}) \\
\pm \left[ \rho_{\alpha\mu, \gamma\nu} J_n^{\mu\beta}J_m^{\nu\delta} + J_n^{\alpha\mu}J_m^{\nu\gamma} - J_m^{\nu\rho}J_n^{\mu\beta} \rho_{\alpha\mu, \nu\rho}J_m^{\gamma\delta} - J_n^{\alpha\rho} \rho_{\alpha\rho, \gamma\delta}J_m^{\nu\delta} \right] \\
\pm \left[ -\frac{1}{2} (J_n^{\alpha\beta} J_m^{\gamma\delta} - J_n^{\alpha\delta} J_m^{\beta\gamma}) + \sum_{p=0}^{m} (J_n^{\alpha\delta} J_{m-p}^{\beta\gamma} - J_n^{\alpha\gamma} J_{m-p}^{\beta\delta}) \right].
\]
Define for both choices of the sign in the last formula the quantity bracket in the spin space.

We then see that the Poisson bracket for the entries of $H$ with abelian symmetries generated by $J$ variables, it is enough to consider one of these families. As is clear from (7.5), the Poisson algebra of conserved quantities $P$ is simpler because a distinguished contribution of zero modes in the last line of (7.5) decouples.

Further, we note that the zero modes $J_{0}^{\alpha\beta}$ form a Poisson subalgebra

$${\{J_{0}^{\alpha\beta}, J_{0}^{\gamma\delta}\} = \delta^{\beta\gamma} J_{0}^{\alpha\delta} - \delta^{\alpha\delta} J_{0}^{\gamma\beta},}$$

$${\pm \varkappa \left[ \rho_{\mu\rho,\nu\delta} J_{0}^{\alpha\beta} J_{0}^{\mu\delta} + J_{0}^{\alpha\mu} J_{0}^{\nu\rho} \rho_{\mu\beta,\nu\delta} - J_{0}^{\gamma\nu} \rho_{\pm\alpha\mu,\nu\delta} J_{0}^{\mu\delta} - J_{0}^{\alpha\mu} J_{0}^{\rho_{\mp\beta,\gamma\nu}} J_{0}^{\mu\delta} \right].}$$

Define for both choices of the sign in the last formula the quantity

$${\varpi}^{\alpha\beta} = \delta^{\alpha\beta} \pm \varkappa J_{0}^{\alpha\beta}. \quad (7.6)$$

We then see that the Poisson bracket for the entries of $\varpi$ is nothing else but the Semenov-Tian-Shansky bracket in the spin space

$${\{\varpi_{1}, \varpi_{2}\} = \pm (\rho \varpi_{1} \varpi_{2} + \varpi_{1} \varpi_{2} \rho - \varpi_{2} \rho_{\pm} \varpi_{1} - \varpi_{1} \rho_{\mp} \varpi_{2}).} \quad (7.7)$$

We therefore recognise that $\varpi$ is the non-abelian moment map for the Poisson actions (3.13) of the spin Poisson-Lie group (3.14) on $\Sigma_{N,\ell}^{\pm}$. Thus, $J_{\alpha\beta}$ generates infinitesimal spin transformations, while the conserved quantities $J_{\alpha\beta}$ generate higher symmetries arising from conjunction of spin transformations with abelian symmetries generated by $H_{k}$.

Since on $\mathcal{P}$ the passage from $J_{-}^{n}$ to $J_{+}^{n}$ can be understood as a redefinition of invariant spin variables, it is enough to consider one of these families. As is clear from (7.5), the Poisson algebra of $J_{n}^{\alpha\beta}$ is simpler because a distinguished contribution of zero modes in the last line of (7.5) decouples.

Introducing a generating function of the corresponding modes

$${J(\lambda) = \sum_{n=0}^{\infty} J_{n}^{\lambda} \lambda^{-n-1},} \quad (7.8)$$

we then convert (7.5) into the Poisson bracket between the currents. In the matrix notation this bracket reads as

$${\{J_{1}(\lambda), J_{2}(\mu)\} = \frac{1}{\lambda - \mu} [C_{12}^{1}, J_{1}(\lambda) + J_{2}(\mu)]}$$

$$\pm \varkappa \left[ \rho_{\pm}(\lambda, \mu) J_{1}(\lambda) J_{2}(\mu) + J_{1}(\lambda) J_{2}(\mu) \rho_{\mp}(\lambda, \mu) - J_{2}(\mu) \rho_{\pm}(\lambda, \mu) - J_{1}(\lambda) \rho_{\mp}(\lambda, \mu) \right]. \quad (7.9)$$

Here we have introduce two spectral dependent $r$-matrices in the spin space

$${\rho}_{\pm}(\lambda, \mu) = \rho \pm \frac{1}{2} \frac{\lambda + \mu}{\lambda - \mu} C_{12}^{1} = \frac{\lambda \rho_{\pm} + \mu \rho_{\mp}}{\lambda - \mu}. \quad (7.10)$$
which are the standard solutions of the trigonometric Yang-Baxter equation with properties
\[ \rho_\pm(\mu, \lambda) = \rho_\pm(\lambda, \mu), \quad P\rho_\pm(\lambda, \mu)P = -\rho_\pm(\mu, \lambda), \]
where \( P = C^\alpha \) is the permutation in the spin space. Note also that \( \rho_\pm(\lambda, 0) = \rho_\pm \).

Formula (7.9) is the symmetry algebra of non-abelian integrals of the hyperbolic spin RS model. In the rational limit \( \kappa \to 0 \) the bracket linearises and coincides with the defining relations of the positive-frequency part of the \( \text{GL}(\ell) \)-current algebra [4]. The quadratic piece of (7.9) is the affine version of the Semenov-Tian-Shansky bracket that extends the Poisson algebra of zero modes, while the whole bracket is the Poisson pencil of the linear and quadratic structures. The algebra (7.9) has an abelian subalgebra spanned by \( \text{Tr} J(\lambda) \), \( n \in \mathbb{Z}_+ \), where the trace is taken over the spin space.

Finally, we note that the superintegrable structure of the model is ultimately responsible for the possibility to solve the equations of motion for invariant spins. Indeed, the equations of motion on \( \mathcal{M} \) triggered by \( H_1 \) are
\[ \dot{A} = -B, \quad \dot{B} = -BA^{-1}B, \quad \dot{a} = 0 = \dot{b}. \]

These equations imply that \( BA^{-1} = I \) is an integral of motion and also \( a = \text{const}, \ b = \text{const} \). Thus, equations for \( A \) and \( B \) are elementary integrated
\[ A(\tau) = e^{-\tau L}A(0), \quad B(\tau) = Ie^{-\tau L}A(0). \]
(7.11)

We assume that at the initial moment of time \( \tau = 0 \) the system is represented by a point on the reduced phase space \( \mathcal{M} \). In particular, at this moment of time coordinates of particles constitute a diagonal matrix \( A(0) \equiv Q \) and the variables \( a_{i\alpha}(0) \equiv a_{i\alpha} \) obey the Frobenius condition \( \sum a_{i\alpha} = 1 \) for any \( i \). With this assumption, it is easy to see that \( I = L(0) \), where \( L \) is the Lax matrix containing the dependence on the initial data. Then, the positions of particles at time \( \tau \) are given by the solution \( Q(\tau) \) of the factorisation problem \( e^{-L(0)\tau}Q = T(\tau)Q T(\tau)^{-1} \), where \( T(\tau) \) is the Frobenius matrix satisfying the initial condition \( T(0) = 1 \). Equations of motion for invariant spins \( a_{i\alpha}(\tau) \) are then solved with the help of \( T(\tau) \)
\[ a_{i\alpha}(\tau) = \frac{T(\tau)^{-1} a_{j\alpha}}{\sum \beta T(\tau)^{-1} a_{j\beta}} = T(\tau)^{-1} a_{j\alpha}. \]

A similar solution can be given for invariant spins \( c_{\alpha i} \). While oscillators \( a_{i\alpha} \) mix under the time evolution with respect to their “particle” index \( i \), the “spin” index \( \alpha \) remains essentially untouched and the solution above is written for the whole \( \ell \)-dimensional vector. This situation is, of course, a consequence of the spin symmetry commuting with the evolution flow.

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A Poisson structure of the Heisenberg double

In order to compute the Poisson structure of invariant spins (6.4), one needs to compute the brackets on the Heisenberg double in terms of the parametrisation \((T, U, P, Q)\), used for the reduction. Indeed, recalling the expression of spins (5.11)

\[
a = t^{-1} T^{-1} a, \quad c = b A^{-1} B T t = b T Q^{-1} T^{-1} U P^{-1} t,
\]

the needed brackets are \(\{T_1, T_2\}\) for \(\{a_1, a_2\}\) and also \(\{T_1, U_2\}\), \(\{T_1, P_2\}\), \(\{T_1, Q_2\}\) for \(\{a_1, c_2\}\). Moreover, the computation of \(\{c_1, c_2\}\) requires the knowledge of \(\{Q_1, Q_2\}\) and \(\{(A^{-1}B)_1, (A^{-1}B)_2\}\), which can be straightforwardly obtained from (2.2).

We introduce the following notation for \(r\)-matrices (2.3) dressed by generic \(M, K \in \text{GL}_N(\mathbb{C})\)

\[
r_{MK}^+ = M_1^{-1} K_2^{-1} r_\pm M_1 K_2,
\]

and two projectors on a generic \(M_{12} \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N)\)

\[
\pi(M)_{ijkl} = M_{ijkl} - \delta_{ij} \sum_{j=1}^{N} M_{ijkl} - \delta_{kl} \sum_{i=1}^{N} M_{ijkl} + \delta_{ij} \delta_{kl} \sum_{j=1}^{N} M_{ijkl},
\]

\[
\bar{\pi}(M)_{ijkl} = \delta_{kl} \sum_{s=1}^{N} \left(M_{ijks} - \delta_{ij} \sum_{j=1}^{N} M_{ijks}\right).
\]

Starting from (2.2) and (5.5), one can compute the required brackets \(\{T_1, Q_2\} = 0 = \{Q_1, Q_2\}\) and

\[
\begin{align*}
\{T_1, T_2\} &= T_1 T_2 r_{Q} - T_1 T_2 \pi(r_{TT}), \\
\{T_1, U_2\} &= -T_1 U_2 \bar{\pi}(r_{TU}), \\
\{T_1, P_2\} &= -T_1 P_2 \bar{\pi}(r_{TT}) + T_1 P_2 \pi(r_{TT}),
\end{align*}
\]

where we introduced

\[
\begin{align*}
\tau_Q &= \sum_{i \neq j}^{N} \frac{Q_{ij}}{Q_{ij}} (E_{ii} - E_{ij}) \otimes (E_{jj} - E_{ji}), \\
\bar{\tau}_Q &= \sum_{i \neq j}^{N} \frac{Q_{ij}}{Q_{ij}} (E_{ii} - E_{ij}) \otimes E_{jj}.
\end{align*}
\]

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