RIESZ’S AND BESSEL’S OPERATORS IN BILATERAL GRAND
LEBESGUE SPACES

E. Ostrovsky
e-mail: galo@list.ru

E. Rogover
e-mail: rogovee@gmail.com

L. Sirota
e-mail: sirota@zahav.net.il

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel.

Abstract.
In this paper we obtain the non-asymptotic estimations for Riesz’s and Bessel’s potential integral operators in the so-called Bilateral Grand Lebesgue Spaces. We also give examples to show the sharpness of these inequalities.

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1. INTRODUCTION

The linear integral operator $I_\alpha f(x)$, or, more precisely, the family of operators of a view

$$u(x) = I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} \, dy$$  \hspace{0.5cm} (0)

is called Riesz’s integral operator, or simply Riesz’s potential, or fractional integral.

Here $\alpha = \text{const} \in (0, d)$ and $|y|$ denotes usually Euclidean norm of the $d-$ dimensional vector $y$; $y \in \mathbb{R}^d$, $d = 1, 2, \ldots$.

In the case $d \geq 2$ and $\alpha = 1$ $u(x)$ coincides with the classical Newton’s potential.

These operators are used in the theory of Fourier transform, theory of PDE, probability theory (study of potential functions for Markovian processes and spectral densities for stationary random fields), in the functional analysis, in particular, in the theory of interpolation of operators etc., see for instance [2], [17].

We denote as usually the classical $L_p$ Lebesgue norm

$$|f|_p = \left( \int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p}; \quad f \in L_p \iff |f|_p < \infty,$$  \hspace{0.5cm} (1)

and denote $L(a, b) = \cap_{p \in (a, b)} L_p$. 

We will consider in the first two sections only the values \( p \) from the open interval \( p \in (1, d/\alpha) \) and denote \( q = q(p) = pd/(d - \alpha p) \); or equally
\[
\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d},
\]
evidently, \( q \in (d/(d - \alpha), \infty) \).

The inverse function to the function \( q = q(p) \) has a view \( p = p(q) = dq/(d + \alpha q) \); note that if \( p \to 1 + 0 \Rightarrow q \to d/(d - \alpha) + 0 \) and if \( p \to d/\alpha - 0 \Rightarrow q \to \infty \).

More detail,
\[
p - 1 = \frac{(d - \alpha)(q - d/(d - \alpha))}{d + \alpha q},
\]
and
\[
\frac{d}{\alpha} - p = \frac{d^2}{\alpha(d + \alpha q)}.
\]

It is proved, see, e.g. in the book [17], Chapter 5, p. 117 - 121 that the following estimation holds for the Riesz’s integral operator (0):
\[
|I_\alpha f|_q \leq A(p) |f|_p, \quad \forall p \in (1, d/\alpha) A(p) \in (0, \infty).
\]

Our aim is a generalization of the estimation (3) on the so-called Bilateral Grand Lebesgue Spaces \( BGL = BGL(\psi) = G(\psi) \), i.e. when \( f(\cdot) \in G(\psi) \).

We recall briefly the definition and needed properties of these spaces. More details see in the works [4], [5], [6], [7], [14], [15], [10], [8], [9] etc. More about rearrangement invariant spaces see in the monographs [2], [11].

For \( a \) and \( b \) constants, \( 1 \leq a < b \leq \infty \), let \( \psi = \psi(p), \, p \in (a, b) \), be a continuous positive function such that there exists a limits (finite or not) \( \psi(a + 0) \) and \( \psi(b - 0) \), with conditions \( \inf_{p \in (a, b)} > 0 \) and \( \min\{\psi(a + 0), \psi(b - 0)\} > 0 \). We will denote the set of all these functions as \( \Psi(a, b) \).

The Bilateral Grand Lebesgue Space (in notation BGLS) \( G(\psi; a, b) = G(\psi) \) is the space of all measurable functions \( f : R^d \to R \) endowed with norm
\[
||f||_{G(\psi)} \overset{def}{=} \sup_{p \in (a, b)} \left[ \frac{|f|_p}{\psi(p)} \right],
\]
if it is finite.

In the article [15] there are many examples of these spaces. For instance, in the case when \( 1 \leq a < b < \infty, \beta, \gamma \geq 0 \) and
\[
\psi(p) = \psi(a, b; \beta, \gamma; p) = (p - a)^{-\beta}(b - p)^{-\gamma};
\]
we will denote the correspondent \( G(\psi) \) space by \( G(a, b; \beta, \gamma) \); it is not trivial, non-reflexive, non-separable etc. In the case \( b = \infty \) we need to take \( \gamma < 0 \) and define
\[
\psi(p) = \psi(a, b; \beta, \gamma; p) = (p - a)^{-\beta}, \quad p \in (a, h);
\]
\[
\psi(p) = \psi(a, b; \beta, \gamma; p) = p^{-\gamma} = p^{-|\gamma|}, \quad p \geq h,
\]
where the value \( h \) is the unique solution of a continuity equation.
\[(h - a)^{-\beta} = h^{-\gamma}\]

in the set \(h \in (a, \infty)\).

The \(G(\psi)\) spaces over some measurable space \((X, F, \mu)\) with condition \(\mu(X) = 1\) (probabilistic case) appeared in [10].

The BGLS spaces are rearrangement invariant spaces and moreover interpolation spaces between the spaces \(L_1(R^d)\) and \(L_\infty(R^d)\) under real interpolation method [3], [8].

It was proved also that in this case each \(G(\psi)\) space coincides with the so-called exponential Orlicz space, up to norm equivalence. In others quoted publications were investigated, for instance, their associate spaces, fundamental functions \(\phi(G(\psi; a, b); \delta)\), Fourier and singular operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.

**Remark 1.** If we introduce the *discontinuous* function

\[
\psi_r(p) = 1, \quad p = r; \quad \psi_r(p) = \infty, \quad p \neq r, \quad p, r \in (a, b)
\]

and define formally \(C/\infty = 0\), \(C = \text{const} \in R^1\), then the norm in the space \(G(\psi_r)\) coincides with the \(L_r\) norm:

\[
||f||_{G(\psi_r)} = |f|_r.
\]

Thus, the Bilateral Grand Lebesgue spaces are direct generalization of the classical exponential Orlicz’s spaces and Lebesgue spaces \(L_r\).

The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [4], [6], theory of probability in Banach spaces [12], [10], [14], in the modern non-parametrical statistics, for example, in the so-called regression problem [14].

The article is organized as follows. In the next section we obtain the main result: upper bounds for Riesz’s operators in the Bilateral Grand Lebesgue spaces. In the third section we construct some examples in order to illustrate the precision of upper estimations.

In the fourth section we investigate the so-called Bessel’s potential operator and establish its boundedness in the Grand Lebesgue Spaces.

In the fifth section we investigate the so-called truncated Riesz’s operator and establish its boundedness in the Grand Lebesgue Spaces, study the sharpness of the obtained results by the building of the suitable examples.

The last section contains some slight generalizations of obtained results.

We use symbols \(C(X, Y), C(p, q, \psi)\), etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like \(C_1(X, Y)\) and \(C_2(X, Y)\). The relation \(g(\cdot) \asymp h(\cdot), \quad p \in (A, B)\), where \(g = g(p), \quad h = h(p), \quad g, h : (A, B) \to R_+\), denotes as usually

\[
0 < \inf_{p \in (A, B)} h(p)/g(p) \leq \sup_{p \in (A, B)} h(p)/g(p) < \infty.
\]

The symbol \(~\) will denote usual equivalence in the limit sense.

We will denote as ordinary the indicator function

\[
I(x \in A) = 1, x \in A, \quad I(x \in A) = 0, x \notin A;
\]
here $A$ is a measurable set.
All the passing to the limit in this article may be grounded by means of Lebesgue dominated convergence theorem.

2. Main result: upper estimations for Riesz potential

A. Estimations for Riesz potential.

Let $\psi(\cdot) \in \Psi(a, b)$, where $1 \leq a < b \leq d/\alpha$. Define for the arbitrary function $\psi \in \Psi(a, b)$ the auxiliary function of the variable $q$, $q \in (d/(d - \alpha), \infty)$:

$$\psi_{\alpha,d}(p) = [(p - 1) (d/\alpha - p)]^{-1 + \alpha/d} \psi(p),$$

where $p$ is the following function of variable $q$:

$$p = p(q) = \frac{dq}{d + \alpha q},$$

or up to equivalence:

$$\psi_{\alpha,d}(p(q)) \asymp \zeta_{\alpha,d}(q) \overset{def}{=} \left[\frac{q^2}{q - d/(d - \alpha)}\right]^{1 - \alpha/d} \psi \left(\frac{dq}{d + \alpha q}\right).$$

Notice that the values $p = 1 + 0$ and $p = d/\alpha - 0$ are critical points; another points are not interest.

**Theorem 1.** Let $f \in G(\psi)$. Then

$$||I_{\alpha} f||G(\zeta_{\alpha,d}) \leq C(\alpha, d, \psi) ||f||G(\psi).$$

**Example 1.** When $a = 1, b = d/\alpha, \beta, \gamma > 0$, and $f \in G(a, b; \beta, \gamma)$, $f \not= 0$, then

$$I_{\alpha} f(\cdot) \in G(d/(d - \alpha), \infty; \beta + 1 - \alpha/d), -\gamma - 1 + \alpha/d).$$

**Lemma 1.** The constant $A(p)$ from the inequality (3) may be estimates as follows:

$$A(p) \leq C_1(d, \alpha) \frac{\alpha^{-1}(d - \alpha)^{-1}}{[(p - 1) (d/\alpha - p)]^{1 - \alpha/d}}, p \in (1, d/\alpha),$$

where $C_1(d, \alpha)$ is bounded continuous function of the variable $\alpha$ on the closed interval $\alpha \in [0, d]$.

**Proof** of lemma 1.

It follows from the mentioned monograph of E.M.Stein [17], Chapter 5, pp. 117 - 121 and from the [2], Chapter 4, section 4, pp. 216 - 230 a weak proposition, after some calculations.

In detail, let us denote

$$\phi(x) = |x|^{\alpha-d};$$

then the Riesz’s transform may be rewritten as a convolution

$$I_{\alpha} f = f \ast \phi.$$
The distribution function \( m_\phi(\lambda) \) for the positive function \( \phi \) has a view

\[
m_\phi(\lambda) = C_2(d, \alpha) \lambda^{d/(d-\alpha)}, \ \lambda \in (0, \infty).
\] (11)

Therefore, the function \( \phi(\cdot) \) belongs to the generalized Lebesgue space \( L_{d/(d-\alpha), \infty} \) and following the operator \( I_\alpha \) is of weak type \((1,d/(d - \alpha))\):

\[
|I_\alpha f|_{d/(d-\alpha), \infty} \leq |\phi|_{d/(d-\alpha)} |f|_1 \overset{\text{def}}{=} M_0 |f|_1.
\] (12)

The operator \( I_\alpha \) is also the the weak type \((d/(\alpha), \infty)\):

\[
|I_\alpha f| \leq M_1 \int_0^\infty f^*(t) t^{\alpha/d-1} dt,
\] (13)

where \( f^* \) denotes the inversion function to the distribution function for the function \( f \).

More exact calculation show us that

\[
\max(M_0, M_1) \leq C(d) / \min(\alpha, d - \alpha).
\]

We will use the classical Marcinkiewicz’s interpolation theorem, see e.g. [2], Chapter 4, corollary 4.14, p. 225. Finding the parameter \( \theta \) from the equation

\[
\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{d/\alpha},
\]

we have:

\[
\theta = \frac{d - p - 1}{d - \alpha}
\]

and correspondingly

\[
1 - \theta = \frac{\alpha - d/\alpha - p}{d - \alpha}
\]

From the Marcinkiewicz’s interpolation theorem follows that the operator \( I_\alpha \) is bounded as operator from the space \( L_p \) into the space \( L_q \) with the following norm estimation:

\[
||I_\alpha||_{(L_p \rightarrow L_q)} \leq C_2(d, \alpha) \frac{\max(M_0, M_1)}{\theta(1-\theta)} \leq C_3(d, \alpha)(p-1)^{-1} (d/\alpha - p)^{-1}.
\]

But in the book [1], pp. 49 - 54 there is described other approach, which used the so-called maximal operator

\[
Mf(x) = \sup_{r > 0} \left[ r^{-d} \int_{|y-x| \leq r} |f(y)| \, dy \right].
\]

It is known ( [17], p. 173 - 188) that

\[
|Mf|_p \leq C(d) \frac{p}{p-1} |f|_p, \ \ p \in (1, \infty),
\]

and if the variable \( p \) changed in the finite interval \( p \in (1, d/\alpha) \), then

\[
|Mf|_p \leq C(\alpha, d) \frac{1}{p-1} |f|_p.
\]

The exact computation following [1] show us that
\[ |I_\alpha f|_q \leq \frac{C |f|_p}{[(p-1) (d/\alpha - p)]^{1-\alpha/d}}. \] (14)

Further we will show that the last estimation (14) is sharp and will prove more general proposition.

**Proof** of the theorem 1. Denote for the simplicity \( u = I_\alpha f; \ u : R^d \to R. \)

We can assume without loss of generality that \( ||f||G(\psi) = 1; \) this means that

\[ \forall p \in (a, b) \Rightarrow |f|_p \leq \psi(p). \]

Using the inequality (3) with the estimation (14), we obtain:

\[ |u|_q \leq A(p) \psi(p) \leq C_4(d, \alpha) [(p-1) (d/\alpha - p)]^{-1+\alpha/d} \psi(p) \]

\[ \leq C_4(d, \alpha) \psi_{\alpha,d}(p) \ ||f||G(\psi). \] (15)

The assertion of theorem 1 follows after replacing \( p = dq/(d+\alpha q), \) dividing on the \( \zeta_{\alpha,d}(q) \) and on the basis of the definition of the \( G(\psi) \) spaces.

**B.** Derivatives of Riesz’s potential.

Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_d) \) be a non-negative integer multiindex: \( \xi_j \geq 0, \xi_j = 0, 1, 2, \ldots \) We define for such a indices \( |\xi| = \sum_{j=1}^d \xi_j. \)

We restrict the values \( \xi \) as follows: \( |\xi| < \alpha \) and denote

\[ \alpha(\xi) = \alpha - |\xi|; \ \alpha(0) = \alpha; \ \alpha(\xi) \in (0, \alpha]; \]

\[ u^{(\xi)}(x) \overset{def}{=} D^{(\xi)}u(x) = \frac{\partial^\xi u}{\partial x_1^{\xi_1} \partial x_2^{\xi_2} \ldots \partial x_d^{\xi_d}}, \]

here as before \( u(x) = I_\alpha f(x). \)

In this subsection we assume that the value \( p \) belongs to the interval \( p \in (1, d/(\alpha(\xi))) = (1, d/(\alpha - |\xi|)) \) and denote

\[ q^{(\xi)} = q^{(\xi)}(p) = \frac{dp}{d - (\alpha - |\xi|)p} \in \left( \frac{d}{d - \alpha(\xi)}, \infty \right); \]

\[ A^{(\xi)}(p) = \frac{\alpha^{-1}(d - \alpha)^{-1}}{[(p-1) (d/\alpha(\xi) - p)]^{1-\alpha(\xi)/d}}, \ p \in (1, d/\alpha(\xi)). \] (16)

**Theorem 2.**

\[ \left| u^{(\xi)} \right|_q \leq C A^{(\xi)}(p) |f|_p, \] (17)

and if \( f \in G(\psi(1, d/\alpha(\xi))), \) then \( \left| u^{(\xi)} \right|_q \leq \)

\[ C \ ||f||G\Psi \times \left[ \frac{q}{d/\alpha + |\xi|} \right]^{1-(\alpha-|\xi|)/d} \times \psi \left( \frac{dq}{d + q(\alpha - |\xi|)} \right). \] (18)

**Proof.** We follow the [II, p. 58 - 59. The expression for the function function \( u = I_\alpha f \) may be rewritten as the convolution \( u = I_\alpha \ast f, \) where \( I_\alpha(x) = |x|^{d-\alpha}. \) Therefore,
\[ D^{(\xi)}(I_\alpha \ast f) = (D^{(\xi)}I_\alpha \ast f). \]

The estimation

\[ |D^{(\xi)}I_\alpha(x)| \leq A(\xi, d, \alpha) I_{\alpha-\xi}(x) \quad (19) \]

see in \[\Pi\], p. 57; hence

\[ |u^{(\xi)}| \leq C I_{\alpha-\xi} \ast f. \quad (20) \]

The end of the proof of theorem 2 is at the same as the proof of theorem 1.

Example 2. Newton potential.

Recall that if \( d \geq 3 \) and \( f(\cdot) \in \cup_{p \in (1, d/2)} L_p(R^d) \), then

\[ f(x) = \frac{-1}{(d-2) \omega(d-1)} \int_{R^d} \frac{\Delta f(y) \, dy}{|x-y|^{d-2}}; \]

here \( \Delta \) denotes the Laplace operator and \( \omega(d-1) = 2\pi^{d/2}/\Gamma(d/2) \), or

\[ f(x) = \frac{1}{\omega(d-1)} \int_{R^d} \frac{\left( \text{grad} \ f(y) \cdot (x-y) \right) \, dy}{|x-y|^d}. \]

We obtain noticing that

\[ f(x) = C(d) I_2(\Delta f), \quad f(x) = C_1 I_1((\text{grad} \ f(x) \cdot x))) : \]

\[ |f|_q \leq C_1(d) \frac{\left| \Delta f \right|_p}{\left[ (p-1)(d/2-p) \right]^{1-2/d}}, \quad q = 1/(1/p - 2/d), \quad p \in (1, d/2). \]

or

\[ |f|_q \leq C_2(d) \frac{\left| \text{grad} f \right|_p}{\left[ (p-1)(d-p) \right]^{1-1/d}}, \quad q = 1/(1/p - 1/d), \quad p \in (1, d). \]

\[ \square \]

3. Low bounds for Riesz potential.

In this section we built some examples in order to illustrate the exactness of upper estimations. We consider only the case \( a = 1, \ b = d/\alpha \); another cases are trivial.

It is sufficient for constructing the low estimations to consider only the one-dimensional case: \( d = 1 \), i.e. \( x, y \in R^1 \).

Let us denote for the mentioned values \( p, q(p), f \in L_p, f \neq 0 \)

\[ V_{\alpha,d}(f;p) = V(f,p) = \frac{|I_\alpha f|_q \cdot [(p-1) (d/\alpha - p)]^{1-\alpha/d}}{|f|_p}, \quad (21) \]

where as before \( p \in (1, d/\alpha), \ q = q(p) = pd/(d - \alpha p) \in (d/(d - \alpha), \infty) \).

From the inequality (3) with the concrete values of parameters from (14) follows that for some non-trivial values \( C^{(1)} = C^{(1)}(\alpha, d), C^{(2)} = C^{(2)}(\alpha, d) \)

\[ \sup_{f \in L(1, d/\alpha), f \neq 0} \lim_{p \rightarrow 1+0} V(f, p) \leq C^{(1)}, \]
\[
\sup_{f \in L(1,d/\alpha), f \neq 0} \lim_{p \to d/\alpha - 0} V(f, p) \leq C^{(2)}.
\]

We intend to prove an inverse inequality at both the critical points \( p \to 1 + 0 \) and \( p \to d/\alpha - 0 \).

**Theorem 3.** For all the values \( \alpha \in (0, 1) \) there exist a constants \( C_1(\alpha), C_2(\alpha) \in (0, \infty) \) for which

\[
\sup_{f \in L(1,1/\alpha), f \neq 0} \lim_{p \to 1+0} V_{\alpha,1}(f, p) \geq C_1, \tag{22}
\]

\[
\sup_{f \in L(1,1/\alpha), f \neq 0} \lim_{p \to 1/\alpha - 0} V_{\alpha,1}(f, p) \geq C_2. \tag{23}
\]

**Proof.** First of all we consider the case \( p \to 1 + 0 \); for definiteness we restrict the value \( p \) in the semi-closed interval \( p \in (1, 0.5(1 + 1/\alpha)) \).

Let us consider a function (more exactly, a family of the functions) of a view

\[
g(x) = g_\Delta(x) = x^{-1}|\log x|^\Delta, \quad x \in (e, \infty)
\]

and \( g_\Delta(x) = 0 \) otherwise. Here the value \( \Delta \) is fixed constant number \( \Delta = \text{const} \geq 0 \).

We have:

\[
|g_\Delta|_p^p = \int_e^\infty x^{-p} (\log x)^\Delta y^\Delta \, dy \sim \int_1^{\infty} \exp(-(p-1)y) y^\Delta dy \sim \int_0^{\infty} \exp(-(p-1)y) y^\Delta dy = (p-1)^{-\Delta-1} \Gamma(\Delta p + 1), \tag{24}
\]

where \( \Gamma(\cdot) \) denotes usually Gamma-function.

It follows from the equality (24) that as \( p \to 1 + 0 \)

\[
|g_\Delta|_p \asymp (p-1)^{-\Delta-1} \asymp \left[1/(1-\alpha) - q\right]^{\Delta+1}.
\]

Further, let us investigate the behavior of the corresponding function \( u_\Delta = I_\alpha g_\Delta \) as \( x \to \infty \), \( x > 1 \):

\[
u_\Delta(x) = \int_e^\infty \frac{y^{-1} (\log y)^\Delta \, dy}{|x - y|^{1-\alpha}} = x^{\alpha-1} \int_{e/x}^\infty \frac{z^{-1} (\log x + \log z)^\Delta \, dz}{|z-1|^{1-\alpha}} \sim
\]

\[
x^{\alpha-1} (\log x)^\Delta \int_{1/x}^1 z^{-1} |z-1|^{\alpha-1} \, dz \sim x^{\alpha-1} (\log x)^\Delta \int_{1/x}^1 z^{-1} |z-1|^{\alpha-1} \, dz \sim
\]

\[
x^{\alpha-1} (\log x)^\Delta \int_{1/x}^1 z^{-1} \, dz \sim x^{\alpha-1} (\log x)^{\Delta+1},
\]

therefore as \( q \to 1/(1-\alpha) + 0 \)

\[
|u_\Delta|_q^q \asymp \int_e^\infty x^{q(\alpha-1)} (\log x)^{q(\Delta+1)} \, dx \sim \int_1^\infty x^{q(\alpha-1)} (\log x)^{q(\Delta+1)} \, dx =
\]

\[
\left[q - \frac{1}{1-\alpha}\right]^{-q(\Delta+1)-1} \Gamma(q(\Delta + 1) + 1),
\]
following
\[ |u_\Delta|^q \lesssim \left[ q - \frac{1}{1 - \alpha} \right]^{-\Delta - 1 - 1/q} \lesssim \left[ q - \frac{1}{1 - \alpha} \right]^{-\Delta - 2 + \alpha} \lesssim (p - 1)^{-\Delta - 2 + \alpha}. \]

Substituting into the expression for the functional \( V_{\alpha,1} \), we obtain for some constant \( C_1 = C_1(\alpha, 1, \Delta) \) and for the value \( p \) tending to \( 1 + 0 \):
\[ V_{\alpha,1}(g_\Delta, p) \lesssim \frac{(p - 1)^{-\Delta - 2 + \alpha}}{(p - 1)^{1 - \alpha}} \lesssim C_1. \]

We conclude for all the the values \( \Delta \geq 0 \)
\[ \lim_{p \to 1+0} V_{\alpha,1}(g_\Delta, p) \geq C_1. \]

We prove now the second assertion of theorem 3. Let us consider now the case \( p \to 1/\alpha - 0 \), then \( q \to \infty \). We introduce as an example the family of a functions

\[ f_\Delta(x) = x^{-\alpha} |\log x|^\Delta, \quad x \in (0, 1/e) \]
and \( f_\Delta(x) = 0 \) otherwise. As before, \( \Delta \) is arbitrary non-negative constant parameter.

We calculate:
\[ |f_\Delta|^p = \int_0^{1/e} x^{-\alpha p} |\log x|^\Delta p \, dx \approx C(\alpha) (1/\alpha - p)^{-\Delta^{-1}}; \]
\[ |f_\Delta|^p \approx (1/\alpha - p)^{-\Delta^{-\alpha}}. \]

Further, let us denote \( v_\Delta(x) = I_\alpha f_\Delta(x) \). We have as \( x \to 0+, x \in (0, 1/e) \):
\[ v_\Delta(x) = \int_0^{1/e} y^{-\alpha} \frac{|\log y|^\Delta}{|x - y|^{1-\alpha}} \, dy \sim |\log x|^\Delta \int_0^{1/e} \frac{z^{-\alpha}}{|z - 1|^{1-\alpha}} \, dz \approx |\log x|^\Delta \int_0^{1/e} \frac{dz}{z} = |\log x|^\Delta; \]
\[ |v_\Delta|^q \approx \int_0^1 |\log x|^q(\Delta + 1) \, dx = \Gamma(q(\Delta + 1) + 1), \]
\[ |v_\Delta|^q \approx q^{\Delta + 1}. \]

We used the Stirling’s formula as \( q \to \infty \) and correspondingly \( p \to 1/\alpha - 0 \).

Substituting into the expression for the functional \( V_{\alpha,1} \), we obtain for suitable constant \( C_2 = C_2(\alpha, 1, \Delta) \) and for the value \( p \) tending to \( 1/\alpha - 0 \):
\[ V_{\alpha,1}(f_\Delta, p) \lesssim \frac{(1/\alpha - p)^{-\Delta^{-1}} (1/\alpha - p)^{1-\alpha}}{(1/\alpha - p)^{-\Delta^{-\alpha}}} \lesssim C_2. \]

This completes the proof of the second assertion of theorem 3.

\[ \square \]

**Remark 2.** If we want to construct a single example for the proof of theorem 3, we can choose the function
\[ h_\Delta(x) = f_\Delta(x) + g_\Delta(x). \]
Corollary 1. If we choose as a function $\psi(p) = \psi_\Delta(p)$ as follows:

$$\psi_\Delta(p) = |h_\Delta|_p, \ p \in (1, d/\alpha),$$

we see that as long as the supports of a functions $f_\Delta$ and $g_\Delta$ are disjoint, for both the sets of a values $p: p \in (1, 0.5(1 + d/\alpha))$ and $p \in (0.5(1 + d/\alpha), d/\alpha)$

$$V_{a,1}(h_\Delta, p) \approx C.$$

Therefore, the proposition of theorem 1 is unprovable.

Remark 3. The ”extremal” case $\alpha = 0$, in which the integral operator $R_j f(x)$ understood as a principal value with ”signed kernel”:

$$R_j f(x) = p.v. \int_{R^d} \frac{x_j - y_j}{|x - y|^{d+1}} dy,$$

and is called a Riesz’s singular integral transform, is considered in [15]; there is obtained the exact $G(\psi)$ estimations for the norm of $R_j$ and is showed its exactness.

Namely, in the book [17], pp. 26 - 48 and 51 is proved the following estimation:

$$|R_j f|_p \leq K(p) |f|_p, \ p \in (1, \infty), \quad (25)$$

where

$$K(p) \leq C_5(d) \frac{p^2}{p - 1}. \quad (26)$$

Notice that the inequalities (25) and (26) are ”extremal” cases for the estimations for the Riesz’s transform: $\lim_{\alpha \to 0^+} q(p) = p$ and the interval for the values $p$ tends to whole semi-axis $(1, \infty)$.

The estimations (25) and (26) are true for more general singular integral operators and more generally for pseudodifferential operators.

From the inequalities (25) and (26) follows that if $f \in G(\psi; a, b)$, where $1 \leq a < b \leq \infty$, then $R_j f \in G(\psi^{(1)})$, where

$$\psi^{(1)}(p) = \frac{p^2}{p - 1} \cdot \psi(p):$$

$$||R_j f||G(\psi^{(1)}) \leq C_5(d) ||f||G(\psi), \quad (27)$$

and the last estimation (27) is exact still in the one-dimensional case, see [15], in which the Riesz’s transform coincides with the classical Hilbert transform.

4. The case of Bessel’s potential

A. Estimation of Bessel’s potential.

We consider here a so-called Bessel’s integral operator or equally Bessel potential:

$$v(x) = L_\alpha f = G_\alpha * f(x), \ G_\alpha(x) = |x|^{-(d-\alpha)/2} K_{\frac{d-\alpha}{2}}(|x|),$$

where $K_\xi(x)$ is a modified Bessel’s function of the third kind with index $\xi$, $\xi \geq 0$, or a Macdonald’s function.
It is true the following representation for the function $G_\alpha(\cdot)$:

$$G_\alpha(x) = C_{11}(\alpha, d) \int_{\mathbb{R}^d} \frac{\exp(ixy)}{(1 + |y|^2)^{\alpha/2}} dy,$$

which is used in the theory of Partial Differential Equations.

As long as

$$|L_\alpha f|_p \leq C_{12}(\alpha, d)|I_\alpha f|_p,$$

see [1], chapter 3, section 3.6, page 56, we conclude that for the operator $L_\alpha$ are true the upper estimations as for the operator $I_\alpha$.

Following,

$$|L_\alpha f|_q \leq \frac{C(\alpha, d) |f|_p}{[(p-1)(d/\alpha - p)]^{1-\alpha/d}};$$

recall that $1/q = 1/p - \alpha/d$, $p \in (1, d/\alpha)$, $q = q(p)$.

Aside from we have from the Young inequality:

$$|L_\alpha f|_p \leq C_1(\alpha, d) |f|_p,$$

see [1], p. 56.

It follows from the H"older inequality that for all the values $t$ from the closed interval $t \in [p, dp/(d - \alpha p)]$

$$|L_\alpha f|_t \leq \frac{C(\alpha, d) |f|_p}{[(p-1)(d/\alpha - p)]^{1-\alpha/d}}.$$

The last inequality may be rewritten as follows. We define the function

$$\psi_0(t) = 1, \quad t \in [p, dp/(d - \alpha p)], \quad \psi(\cdot) \in \Psi(p, dp/(d - \alpha p));$$

then

$$||L_\alpha f||_{G_0(\psi_0)} \leq C_2(\alpha, d) \inf_{p \in (1, d/\alpha)} \frac{C(\alpha, d) |f|_p}{[(p-1)(d/\alpha - p)]^{1-\alpha/d}}.$$

On the other hands, if for some $p_0 \in (1, d/\alpha)$ $|f|_{p_0} < \infty$, then $L_\alpha f \in G(\psi_0)$.

**B. Estimations of derivatives of Bessel's potential.**

We estimate in this subsection the norm in some Grand Lebesgue Space the derivatives of Bessel’s potential function

$$v^{(\xi)} = D^\xi v(x), \quad v = L_\alpha f(x) = G_\alpha * f(x).$$

We suppose as before $|\xi| < \alpha$ and define for the values $m \in (\max(2\alpha/|\xi|, 1), \infty)$ the function

$$\theta_{\alpha,\xi,d}(m) = \theta(m) = \frac{2m|\xi|}{2m|\xi| - \alpha} \cdot \psi_{|\xi|/2\alpha}(\frac{2m|\xi|}{\alpha}) \cdot \psi_{(\alpha-|\xi|)/(2\alpha)}(2m (1 - |\xi|/\alpha)),$$

where $\psi(\cdot) \in \Psi(2|\xi|/\alpha, \infty)$.

**Theorem 4.**
\[ \|v^{(\xi)}\|G(\theta) = \|D^\xi(G_\alpha * f)\|G(\theta) \leq C_3(\alpha, \xi, d) \|f\|G(\psi). \quad (28) \]

**Proof.** Let \( f \in G(\psi) \), otherwise is nothing to prove. We can and will suppose \(|\xi| \geq 1\) and \(\|f\|G(\psi) = 1\), then \(|f|_p \leq \psi(p)\).

We start from the inequality (see [1], p. 57):

\[ |v^{(\xi)}(x)| \leq C (Mf(x))^{\xi/\alpha} |f(x)|^{1-|\xi|/\alpha}. \quad (29) \]

We obtain after simple calculations using Cauchy - Schwartz inequality and denoting for simplicity: \( A = \alpha p/(2 |\xi|) \):

\[ [(|v|_A)^A]^2 = \left[ \int_{R^d} |v^{(\xi)}(x)|^A \, dx \right]^2 \leq \int_{R^d} |Mf(x)|^p \, dx \times \]
\[ \int_{R^d} |f(x)|^{2A(1-|\xi|/\alpha)} \, dx \leq C |f|_p^p \cdot \left[ \frac{p}{p-1} \right]^p \times \]
\[ \int_{R^d} |f(x)|^{p(\alpha-|\xi|)/|\xi|} \, dx = C |f|_p^p \left[ \frac{p}{p-1} \right]^p \cdot |f|_{p^{(\alpha-|\xi|)/|\xi|}}^p \]

therefore

\[ |v^{(\xi)}|_m = |v^{(\xi)}|_A = |v^{(\xi)}|_{o_p/2|\xi|} \leq C \cdot \left( \frac{p}{p-1} \right)^{\xi/(2\alpha)} \cdot |f|_p^{\xi/(2\alpha)} \cdot |f|_{p^{(\alpha-|\xi|)/|\xi|}} \leq \]
\[ C \cdot \frac{2m|\xi|}{2m|\xi| - \alpha} \cdot \psi^{\xi/(\alpha-\xi)/2\alpha} \left( \frac{2m|\xi|}{\alpha} \right) \cdot \psi^{(\alpha-|\xi|)/2\alpha} \left( \frac{2m \alpha - |\xi|}{\alpha} \right) = C \theta(m). \]

This competes the proof of the last theorem.

5. **The Case of a Bounded Domain**

We consider in this section the truncated Riesz’s operator

\[ u^{(B)} = u^{(B)}(x) = I^{(B)}_\alpha f(x) = \int_B \frac{f(x-y)}{|y|^{d-\alpha}} \, dy, \quad (30) \]

where \( B \) is open bounded domain in \( R^d \) contained the origin and such that

\[ 0 < \inf_{x \in \partial B} |x| \leq \sup_{x \in \partial B} |x| < \infty, \quad (31) \]

\( \partial B \) denotes boundary of the set \( B \).

It is known (see, e.g. [13], p.90), that if \( f \max(1, \log f) \in L_1(B), \) then \( f \in L_q(B) \).

We can and will assume further without loss of generality that the set \( B \) is unit ball in the space \( R^d \):

\[ B = \{ x, \, x \in R^d, \, |x| < 1 \}. \quad (32) \]

Let us denote \( p' = p/(p-1) \) and for the function \( \psi(\cdot) \in \Psi(1,d/\alpha) \) define

\[ \nu(r) = \nu_\psi(r) = \inf_{p \in [1,d/(d-\alpha)]} \left[ \left( \frac{d}{d-\alpha} - p \right)^{-1+\alpha/d} \cdot \psi \left( \frac{rp'}{r + rp'} \right) \right]. \quad (33) \]
Note that the function $\nu(r)$ may be simply estimated as follows: tacking the value $p = p_0 = \frac{d}{d - \alpha} - C_1/r$, we conclude:

$$\nu_\psi(r) \leq C_2 \ r^{1-\alpha/d} \ \psi(d/\alpha - C_3/r), \ r \geq C_4.$$

**Theorem 5.** Let $\psi \in \Psi(1, d/\alpha)$. Then

$$||I^{(B)}_\alpha f||G(\nu_\psi) \leq C_6(\alpha, d) ||f||G(\psi).$$ (34)

**Proof.** As before, we suppose $||f||G(\psi) = 1$, hence

$$|f|_p \leq \psi(p), \ p \in [1, d/\alpha).$$

We obtain by the direct computation for the values $p$ from the interval $p \in [1, d/(d - \alpha))$:

$$|\phi \cdot I(x \in B)|_p^p \simeq \left( \frac{d}{d - \alpha} - p \right)^{-1},$$

or equally

$$|\phi \cdot I(x \in B)|_p^p \simeq \left( \frac{d}{d - \alpha} - p \right)^{-1+\alpha/d}.$$ As long as the function $u^{(B)}$ may be written as a convolution

$$u^{(B)} = [\phi \cdot I(x \in B)] \ast f,$$

we can use the classical Young’s inequality

$$|u|_r \leq | [\phi \cdot I(x \in B)]|_p \cdot |f|_k,$$

where

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{p'},$$

or equally

$$k = \frac{rp'/r}{r + p'};$$

and following

$$|u|_r \leq C \left[ \left( \frac{d}{d - \alpha} - p \right)^{-1+\alpha/d} \right] |f|_{rp'/(r+p')}.$$ (35)

We obtain substituting into (34) and using the equality $||f||G(\psi) = 1$:

$$|u|_r \leq C \left[ \left( \frac{d}{d - \alpha} - p \right)^{-1+\alpha/d} \cdot \psi \left( \frac{rp'}{r + p'} \right) \right].$$ (36)

Since the value $p$, $p \in [1, d/(d - \alpha))$ is arbitrary, we can minimize the right - side of inequality (35) over $p$:
\[ |u|_r \leq C \inf_{p \in [1, d/(d - \alpha))] \left[ \left( \frac{d}{d - \alpha} - p \right)^{-1 + \alpha/d} \cdot \psi \left( \frac{rp}{r + p} \right) \right] = C \nu_\psi(r) = C \nu_\psi(r) \|f\|G(\psi). \]

This completes the proof of theorem 5. \[ \square \]

Example 3. Let \( \psi \in G\Psi(1, d/\alpha; 0, \gamma) \); recall that this means:

\[ \psi_\gamma(p) = (d/\alpha - p)^{-\gamma}, \ p \in [1, d/\alpha), \ \gamma = \text{const} > 0. \]

Let also \( f(\cdot) \) be a (measurable) function, \( f : \mathbb{R}^d \to \mathbb{R} \) such that

\[ |f|_p \asymp \psi_\gamma(p), \ p \in [1, d/\alpha). \]

This condition is satisfied, e.g., for the function

\[ f(x) = I(|x| > 1) \cdot |x|^{-\alpha} \cdot \log |x|^{\gamma - \alpha/d}, \ \gamma \geq \alpha/d. \]

The expression

\[ Z(p, r) = \psi_\gamma \left( \frac{rp}{r + p'} \right) \cdot \left( \frac{d}{d - \alpha} - p \right)^{-1 + \alpha/d} \]

achieves the asymptotical as \( r \to \infty \) minimal value at the point

\[ p_0 = \frac{d}{d - \alpha} - 0.5 \left( \frac{d}{d - \alpha} \right)^2 \frac{1}{r}, \]

and we obtain the following upper estimation for the \( L_r \) norm of the function \( u(\cdot) \):

\[ |u|_r \leq C \inf_{p \in [1, d/(d - \alpha))] Z(p, r) \leq C Z(p_0, r) \asymp r^{1 + \gamma - \alpha/d}, \ r \in [1, \infty). \] (37)

Note that the last inequality (37) denotes that the function \( u(\cdot) \) belongs to the Orlicz’s space over \( \mathbb{R}^d \) equipped with the \( N \) – function

\[ N(t) = \exp(|t|^m) - 1, \ 1/m = 1 + \gamma - \alpha/d, \ m \in (0, \infty). \]

We intend now to establish the exactness of assertion of theorem 5, also in the one - dimensional case \( d = 1 \).

Theorem 6. The proposition (37) is exact, i.e. there exists a function \( f_0 \in G(\psi_\gamma) \) for which

\[ |u|_r \geq C_2 \ r^{1 + \gamma - \alpha/d} = C_2 \ r^{1 + \gamma - \alpha}, \ r \geq 1. \]

Notice that the value \( r_0 = \infty \) in the unique ”critical” point for the function \( r \to |u|_r \), if \( f \in G(\psi_\gamma) \).

Proof. Let us denote \( \Delta = \gamma - \alpha/d \).

As before, it is enough to investigate only the one - dimensional case \( d = 1 \).

Let us choose

\[ f_0(x) = x^{-\alpha} |\log x|^\Delta, \ x \in (0, 1/e) \]
and $f_0(x) = 0$ otherwise; $\Delta = \text{const} > 0$.

We find by the direct calculation: $f_0 \in \cap_{p \in [1,1/\alpha]} L_p$ and

$$|f_0|_p \asymp (\alpha^{-1} - p)^{-\Delta - \alpha} = (\alpha^{-1} - p)^{-\gamma},$$
i.e. $f_0 \in G(\psi, \gamma)$.

Further, we have for the values $x \in (0, 1/e)$

$$v_0(x) \overset{\text{def}}{=} I^{(B)}_{\alpha, \beta} f_0(x) = \int_{x}^{1/e} |x - y|^{-\alpha} \log |x - y| \, |y|^{\alpha - 1} \, dy$$

$$\sim \int_{x}^{1} |x - y|^{-\alpha} \log |x - y| \, |y|^{\alpha - 1} \, dy$$

$$= \int_{1}^{1/x} |\log x| + |\log(z - 1)| \, z^{\alpha - 1} \, dz \sim \int_{e}^{1/x} |\log x| + |\log(z - 1)| \, z^{\alpha - 1} \, dz$$

$$\sim |\log x|^\Delta \int_{1}^{1/x} (z - 1)^{-\alpha} \, z^{\alpha - 1} \, dz \asymp |\log x|^\Delta \int_{e}^{1} z^{-\alpha} \, dz \sim |\log x|^{\Delta + 1};$$

$$|v_0|_r \asymp \int_{0}^{1/e} |\log x|^{r(\Delta + 1)} \, dx \sim \int_{0}^{1} |\log x|^{r(\Delta + 1)} \, dx = \Gamma(r(\Delta + 1));$$

$$|v_0|_r \sim C_6 \, r^{\Delta + 1} = C_6 \, r^{1 + \gamma - \alpha}.$$

This completes the proof of theorem 6.

\[ \square \]

6. Concluding remarks

A. We consider in this subsection some generalization of the Riesz’s potential operator of a view

$$I_{\alpha, \beta} f(x) = \int_{\mathbb{R}^d} \frac{f(y) \log |x - y| |y|^{\beta} \, dy}{|x - y|^{d-\alpha}},$$

$\alpha = \text{const} \in (0, d), \quad \beta = \text{const} > 0$, or equally

$$I_{\alpha, \beta} f(x) = \int_{\mathbb{R}^d} \frac{f(y) [1 + |\log |x - y||] |y|^{\beta} \, dy}{|x - y|^{d-\alpha}},$$
or more generally

$$I^{(S)}_{\alpha, \beta} f(x) = \int_{\mathbb{R}^d} \frac{f(y) \log |x - y| |y|^{\beta} S(|\log |x - y||) \, dy}{|x - y|^{d-\alpha}},$$

where $\alpha = \text{const} \in (0, d), \quad \beta = \text{const} > 0$, and $S(z)$ is a slowly varying as $z \to \infty$ continuous positive function:

$$\forall \lambda > 0 \Rightarrow \lim_{z \to \infty} S(\lambda z) / S(z) = 1.$$
Lemma 2.

\[ |I_{\alpha, \beta} f|_q \leq \frac{C |f|_p}{(p - 1) \left( \frac{d}{\alpha} - p \right)^{1 + \frac{\beta}{d}}} \]  

(38)

and the last inequality is sharp.

\textbf{Proof} of the first assertion. We use the method described in [1], pp. 49 - 54.

Let \( \chi = \chi(z) \) be a positive at \( z \in (0, \infty) \) continuous decreasing function such that \( \phi(\infty) \overset{\text{def}}{=} \lim_{z \to \infty} \phi(z) = 0 \). We define also a function

\[ \Phi(z) = \int_z^{\infty} \chi(t) \, dt, \]

if there exists.

We have for the values \( \delta \in (0, \infty) \) analogously to the assertion in [1], p. 49 - 51:

\[ \int_{y: |x - y| < \delta} \Phi(|x - y|) f(y) dy = \int_0^\delta \chi(r) \int_{y: |x - y| < r} f(y) \, dy + \Phi(\delta) \int_{y: |x - y| < \delta} f(y) \, dy. \]

Without loss of generality we can assume that the function \( f(\cdot) \) is non-negative.

As long as

\[ \int_{y: |x - y| \leq r} f(y) \, dy \leq C(d) \, r^d \, Mf(x), \]

\[ \int_{y: |x - y| \leq \delta} f(y) \, dy \leq C(d) \, \delta^d \, Mf(x), \]

we obtain the estimate

\[ \int_{y: |x - y| < \delta} \Phi(|x - y|) f(y) dy \leq C(d) \, Mf(x) \cdot \left[ \int_0^\delta r^d \chi(r) \, dr + \delta^d \Phi(\delta) \right] \overset{\text{def}}{=} Mf(x) \, A_{\chi, d}(\delta). \]

Further, we have denote \( s = p/(p - 1) \) and use the Hölder inequality:

\[ \int_{y: |x - y| \geq \delta} \Phi(|x - y|) f(y) \, dy \leq |f|_p \cdot \left[ \int_{y: |x - y| > \delta} \Phi^s(|x - y|) \, dy \right]^{1/s} = C_1(d) \, |f|_p \left[ \int_{\delta}^{\infty} r^{d-1} \Phi^s(r) \, dr \right]^{1/s} = D_{\chi}(p, \delta) \, |f|_p, \]

where

\[ D_{\chi}(p, \delta) \overset{\text{def}}{=} C_1(d) \left[ \int_{\delta}^{\infty} r^{d-1} \Phi^s(r) \, dr \right]^{1/s}, \]

if there exists for some values \( s \) from some non-trivial interval \( s \in (d/(d - \alpha), s_0) \); if \( s_0 < \infty \), then we define formally \( D_{\chi}(p, \delta) = +\infty \).

We conclude tacking into account the partition

\[ \int_{R^d} \Phi(|x - y|) f(y) \, dy = \int_{y: |x - y| < \delta} \Phi(|x - y|) f(y) \, dy \]
\begin{align*}
+ \int_{|x - y| \geq \delta} \Phi(|x - y|) f(y) \, dy : \\
\int_{R^d} \Phi(|x - y|) f(y) \, dy & \leq M f(x) \, A_{\chi,d}(\delta) + D_{\chi}(p,\delta) \, |f|_p. \\
\end{align*}

Therefore,
\begin{align*}
\int_{R^d} \Phi(|x - y|) f(y) \, dy & \leq \inf_{\delta > 0} \left[ M f(x) \, A_{\chi,d}(\delta) + D_{\chi}(p,\delta) \, |f|_p \right] \\
& \overset{\text{def}}{=} H(p, M f(x), |f|_p).
\end{align*}

Solving the last inequality, we obtain denoting
\begin{equation*}
w(x) = \int_{R^d} \Phi(|x - y|) f(y) \, dy :
\end{equation*}
the inequality of a view
\begin{equation*}
G(p, w(x), |f|_p) \leq (M f(x))^p,
\end{equation*}
and after the integration
\begin{align*}
\int_{R^d} G(p, w(x), |f|_p) \, dx & \leq |M f(x)|^p_p \leq C_p^p(\alpha, d) \, |f|_p^p (p - 1)^{-p}, \\
\left[ \int_{R^d} G(p, w(x), |f|_p) \, dx \right]^{1/p} & \leq |M f(x)|_p \leq C(\alpha, d, \chi) \, |f|_p (p - 1).
\end{align*}

Since the relation between the functions \( f(\cdot) \) and \( w(\cdot) \) is linear, the last inequality has a view
\begin{equation*}
|w|_q \leq C_2(\alpha, p, \chi(\cdot)) \, |f|_p.
\end{equation*}

Choosing the function \( \Phi(r) \) as follows:
\begin{equation*}
\Phi(r) = r^{\alpha - d} \, |\log r|^{\beta},
\end{equation*}
we obtain the first assertion of lemma 2.

**Proof** of the second assertion of the lemma 2.

Notice that it may be proved a more general assertion:
\begin{equation*}
|I_{\alpha,\beta}^{(S)} f|_q \leq \frac{C \, |f|_p}{(p - 1) \cdot (d/\alpha - p)]^{1+\beta-\alpha/d} \cdot S(q(d - \alpha) - d)^{-1} S(q),
\end{equation*}
and the last estimation is asymptotically exact: at the same examples as at the proof theorem 2 can be used by the proof of inverse inequalities.

More detail: let \( d = 1, \ p \in (1,0.5(1 + d/\alpha)) \), \( p \rightarrow 1 + 0 \), or equally \( q \rightarrow 1/(1 - \alpha) + 0 \) and let us choose
\begin{equation*}
f(x) = x^{-1} \, (\log x)^\Delta \, I(x > e), \ \Delta = \text{const} \geq 0.
\end{equation*}
The value \( |f|_p \) was calculated before, let us estimate the \( L_q \) norm of a function
\begin{equation*}
u(x) = I_{\alpha,\beta}^{(S)} f(x).
\end{equation*}
We have using the properties of slowly varying functions as \(x \to \infty, \, x > e\): 

\[
\int_e^\infty |x - y|^{\alpha-1} y^{-1} (\log y)^\Delta |\log |x - y||^{\beta} S(|\log |x - y||) \, dy = x^{\alpha-1} \times 
\]

\[
\int_{e/x}^{\infty} z^{-1} |z - 1|^{1-\alpha} x^{\alpha-1} |\log x + \log z|^\Delta [\log x + \log(|z - 1|)]^{\beta} S(\log x + \log(|z - 1|)) \, dz \asymp 
\]

\[
x^{\alpha-1} [\log x]^{\Delta+\beta} S(\log x) \int_{e/x}^{\infty} z^{-1} |z - 1|^{\alpha-1} \, dz \asymp 
\]

\[
x^{\alpha-1} [\log x]^{\Delta+\beta} S(\log x) \int_{e/x}^{1/e} z^{-1} |z - 1|^{\alpha-1} \, dz \asymp 
\]

\[
x^{\alpha-1} [\log x]^{\Delta+\beta} S(\log x) \int_{e/x}^{1/e} z^{-1} \, dz \asymp x^{\alpha-1} [\log x]^{\Delta+\beta+1} S(\log x); 
\]

following,

\[
|u|_q^p \asymp \int_e^\infty x^{-q(1-\alpha)} |\log x|^{q(\Delta+\beta+1)} S^q(\log x) \, dx \asymp 
\]

\[
\int_0^\infty e^{-y[q(1-\alpha)-1]} y^{q(\Delta+\beta+1)} S^q(y) \, dy = 
\]

\[
[q - 1/(1 - \alpha)]^{-q(\Delta+\beta+1)-1} \int_0^\infty e^{-z} z^{q(\Delta+\beta+1)} S^q \left( \frac{z}{q(1 - \alpha) - 1} \right) \, dz = 
\]

\[
[q - 1/(1 - \alpha)]^{-q(\Delta+\beta+1)-1} S^q \left( \frac{1}{q - 1/(1 - \alpha)} \right). 
\]

Therefore we have as \(q \to 1/(1 - \alpha) + 0\)

\[
|u|_q \asymp [q - 1/(1 - \alpha)]^{-\Delta-\beta-2+\alpha} S \left( \frac{1 - \alpha}{q(1 - \alpha) - 1} \right) \asymp 
\]

\[
[p - 1]^{-\Delta-\beta-2+\alpha} S \left( \frac{1 - \alpha}{q(1 - \alpha) - 1} \right). 
\]

Substituting into the expression for the left hand side of inequality (48), we can see that the relation between the left hand side to the right hand side calculated for the function \(f\) is bounded from below as \(p \to 1 + 0\).

The case \(p \to 1/\alpha - 0\) or equally \(q \to \infty\) may be considered analogously, by means of example of a function

\[
g(x) = x^{-\alpha} |\log x|^\Delta I(x \in (0, 1/e)). 
\]

We have in this case:

\[
|g|_p \asymp (1/\alpha - p)^{-\Delta-\alpha}; 
\]

\[
v(x) := I(x > 0) g(x) \asymp |\log x|^{\Delta+\beta+1} S(|\log x|), \, x \to 0+. 
\]
\[ |v|_q \asymp q^{\Delta + \beta + 1} S(q) \asymp (1/\alpha - p)^{-\Delta - \beta - 1} S\left(\frac{1}{1/\alpha - p}\right) \]

eqb

Let us denote for arbitrary function \( \psi(\cdot) \) from the class \( \Psi(1, d/\alpha) \)
\[ \zeta^{(S)}_{\alpha, \beta}(q) = \psi(p) \frac{S((p-1)^{-1}) S(q(d-\alpha) - d)^{-1}}{[(p-1)(d/\alpha - p)]^{1+\beta - \alpha/d}}. \]

Here as before
\[ p \in (1, d/\alpha), \quad q^{-1} = q^{-1}(p) = (p^{-1} - \alpha/d)^{-1} \in (d/(d-\alpha), \infty). \]

We find analogously the assertion of theorem 1:

**Theorem 7.**

\[ \|I^{(S)}_{\alpha, \beta}f\|_G \left( \zeta^{(S)}_{\alpha, \beta} \right) \leq C(\alpha, \beta, d, S(\cdot)) \|f\|_G(\psi). \]

**B.** Let us consider in this subsection the *generalized truncated* Riesz’s operator

\[ u^{(B)}_\beta = u^{(B)}_\beta(x) = I^{(B)}_{\alpha, \beta} f(x) = \int_B \frac{f(x-y) |\log |y||^\beta \, dy}{|y|^{d-\alpha}}, \]

where \( \beta = \text{const} > 0, \ B \) is open bounded domain in \( R^d \) contained the origin and such that

\[ 0 < \inf_{x \in \partial B} |x| \leq \sup_{x \in \partial B} |x| < \infty, \]

\( \partial B \) denotes boundary of the set \( B \).

We can and will assume further without loss of generality that the set \( B \) is the ball of a radius \( 1/e \) in the space \( R^d \) :

\[ B = \{x, \ x \in R^d, \ |x| < 1/e\}. \]

Let us denote for the function \( \psi(\cdot) \in \Psi(1, d/\alpha) \) define

\[ \nu^{(\beta)}(r) = \nu^{(\beta)}_{\psi}(r) = \inf_{p \in [1,d/(d-\alpha)]} \left( \frac{d}{d-\alpha} - p \right)^{-1-\beta + \alpha/d} \cdot \psi \left( \frac{r p^\prime}{r + p^\prime} \right). \] (39)

**Theorem 8.** Let \( \psi \in \Psi(1, d/\alpha) \). Then

\[ \|I^{(B)}_{\alpha, \beta}f\|_G(\nu^{(\beta)}_{\psi}) \leq C_9(\alpha, d) \|f\|_G(\psi). \] (40)

The **proof** is at the same as in theorem 5; it based on the following equality on the function

\[ \phi_\beta(x) = I(x \in B) |x|^{\alpha - d} |\log |x||^\beta : \]
\[ |\phi_{\beta}(\cdot)|_p \leq \left(\frac{d}{d - \alpha} - p\right)^{-\beta - 1 + \alpha/d}, \quad (41) \]

\( p \in [1, d/(d - \alpha)] \); hence when \( \psi \in \Psi(1, d/\alpha) \), then

\[ \nu^{(\beta)}(r) = \nu^{(\beta)}_{\psi}(r) = \inf_{p \in [1, d/(d - \alpha)]} \left[ \left(\frac{d}{d - \alpha} - p\right)^{-\beta - 1 + \alpha/d} \cdot \psi \left(\frac{rp'/r}{r + p'}\right) \right]. \]

As in the section 3, we conclude that the estimation (40) is sharp.

We can consider the more general case when the integral operator has a view:

\[ I^{(B,S)} f = \phi^{(S)}_{\alpha, \beta} * f, \quad \alpha \in (0, d), \beta \geq 0, \quad (42) \]

where

\[ \phi^{(S)}_{\alpha, \beta}(x) = I(x \in B) \cdot |x|^{-\alpha} \cdot \log |x|^\alpha S(|\log |x| |), \quad (43) \]

\( S(z) \) is a slowly varying as \( z \to \infty \) continuous positive function.

Let us denote

\[ \nu^{(S)}_{\alpha, \beta}(r) = \inf_{p \in [1, d/(d - \alpha)]} \left[ \left(\frac{d}{d - \alpha} - p\right)^{-1 - \beta + \alpha/d} \cdot S \left(\frac{d - \alpha}{d - p(d - \alpha)}\right) \cdot \psi \left(\frac{rp'/r}{r + p'}\right) \right]. \]

We assert analogously to the theorem 7:

**Theorem 9.**

\[ ||I^{(B,S)}_{\alpha, \beta} f||G(\nu^{(S)}_{\alpha, \beta}) \leq C_9(\alpha, d) ||f||G(\psi). \quad (44) \]

**Proof.** It is sufficient for the proof of the last assertion to calculate all the moments of a function

\[ R(x) = I(x \in B) \cdot |x|^{-\alpha} \cdot \log |x| \cdot S(|\log |x| |). \]

We have for the values \( p \) from the set \( p \in [1, d/\alpha) \) (for the values \( p \) greatest than \( d/\alpha \) : \( p > d/\alpha \) \( |R|_p = \infty \)) using the multidimensional spherical coordinates:

\[ |R|_p^p \leq \int_0^1 r^{d - 1 - \alpha p} |\log r|^{\Delta p} S^p(|\log r|) \, dr = \int_0^\infty e^{-y(d - \alpha p)} y^\Delta p S^p(y) \, dy = \]

\[ (d - \alpha p)^{-\Delta p - 1} \int_0^\infty e^{-z} z^{\Delta p} S^p \left(\frac{z}{d - \alpha p}\right) \, dz \sim (d - \alpha p)^{-\Delta p - 1} \times \]

\[ S^p \left(\frac{1}{d - \alpha p}\right) \int_0^\infty e^{-z} z^{\Delta p} \, dz = (d - \alpha p)^{-\Delta p - 1} S^p \left(\frac{\alpha}{d/\alpha - p}\right) \Gamma(\Delta p + 1), \]

as long as the function \( S(\cdot) \) is slowly varying.

Further,

\[ |R|_p \leq (d - \alpha p)^{-\Delta - \alpha/d} \cdot S \left(\frac{\alpha}{d/\alpha - p}\right). \]

B.
We consider now a so-called fractional sublinear maximal operator:

\[ M_\alpha f = \sup_{x \in \mathbb{R}^d} \sup_{\rho > 0} \left[ \rho^{\alpha - d} \int_{|y:|x-y| \leq \rho} |f(y)| \, dy \right]. \]

As long as

\[ C_{10}(\alpha, d) \| M_\alpha f \|_p \leq \| I_\alpha f \|_p \leq C_{11}(\alpha, d) \| M_\alpha f \|_p, \quad p \in (1, d/\alpha), \]

see [1], chapter 3, section 3.6, we conclude that for the operator \( M_\alpha \) are true at the same estimations as for the operator \( I_\alpha \).

E. The \( L_p \to L_q \) estimations for many examples of integral operators (regular and singular) in the \textit{weighted} Lebesgue spaces \( L_p(\mathbb{R}^d, w) \) with exact values of its norm see, for example, in [16].

This estimations allow to generalize described results on the weighted spaces.

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