Classical long-range interacting N-particle configurations and its applications.

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Abstract

We consider classical N-particle system with arbitrary central pair potential. Mechanical equilibrium condition in spherically-symmetric case leads to a nonlinear integro-differential equation for concentration \( n(r) \). For special state equation \( p = an^\gamma(1 + bn) \), original integro-differential equation transforms into purely integral one. Its solution (under \( b = 0 \)) is written through a row over interaction parameter. Physical conditions for its convergency is discussed. For power-like potential kernal of integral operator is calculated in apparent kind. The cases of Coulomb and harmonic potentials are considered separately up to a third order. General scheme of application of the theory for some astrophysical and cosmological problems is presented. Model system with spherical potential pits is considered. By perturbation theory first virial correction \( \delta n \) is calculated.

1 Introduction

Imperfect systems, whose interparticle (quasiparticle) interactions principally determine its physical properties, are of great interest today. Their distinctive peculiarities — nonlinearity, nonadditivity of equilibrium thermodynamic potentials and parameters, critical phenomenas, selfordering effects etc — endow this sort of system by nontrivial dynamics and interesting observable effects. The last from the listed properties relates mainly to the imperfect systems with long-range interactions, which can provide orderization on scales about size of the system. Here and below the term "long-range" will mean order of magnitude of interactions radius equal or more then size of a system. The principal difficulty of statistical investigation of long-range interaction system (LRIS) concerns with a very complicated kind of statistical sum (integral) for long-range potentials. The well known virial decomposition \[ p = - \left( \frac{\partial E}{\partial V} \right)_S \] becomes invalid or, more exactly, defines "generalized pressure". It is expressed by the sum: \( p = p^{(l)} + p^{(g)} \), where first term is ordinary local (contact) pressure, borned by classical or quantum motion of particles, while second one is determined by a long range interaction of some picked volume with environment. This decomposition corresponds to the subdividing of internal energy \( E = E^{(l)} + E^{(g)} \), where \( E^{(l)} \) — local internal energy, \( E^{(g)} \) — interaction energy of a small picked subsystem with environment. Now one can’t understand \( p \) simply as a component of stress tensor of continuum media, if stress tensor of long-range field is not also included into consideration. The force components then will be

\[
F_i = \iint_{\Sigma = \partial V} (\sigma_{ik}^{(l)} + t^{(l)}_{ik}) \, d\Sigma^k = \int_V \text{div} (\sigma^{(l)} + t^{(l)}) \, dV,
\]
where $\sigma^{(l)}$ — ordinary continuum media stress tensor, $t^{(f)}$ — field stress tensor, $V$ — arbitrary volume inside the system. Mechanical equilibrium condition reads:

$$\text{div}(\sigma^{(l)} + t^{(f)}) = 0,$$

that means intercompensation of local and global forces. But this resulting force depends on some internal properties of the system, for example — density of "generalized charge" distribution and we unavoidably go to the self-consistent problem.

Attempts of investigations of LRIS have been made in a different areas: for dielectrics in electrostatic by thermodynamical methods in [4], for Newtonian gravity in balls stellar clusters by virial theorem [5], by variational methods with fields equation as constraints — [4, 6], for quantum system — [6]-[10]. General problems are both choice of interparticle interaction or field equation, and solving of self-consistent equations, which are in general case nonlinear and integral.

In present paper we don’t restrict ourself to any concrete physical system and consider abstract centrally-interacting structureless classical particles. It is assumed, that pair potential satisfies some linear field equation, so that superposition principle is valid. We consider the case of identical particles, with pair potential:

$$\phi = g(r),$$

where $\alpha$ — generalized charge of a particle (the source of a long-range field ), which is the same for all particles, $r$ — distance between the particle-source and point of observation, $g$ — universal for all particles function, depending on the distance and some universal constant parameters. It is postulated, that the system after a long time goes in equilibrium state, where all macroscopic parameters become constant. We are interested by practically important case of macroscopic spherically symmetry, when all macroscopic parameters (for example concentration) depends only on a distance from a center of the symmetry.

Let in equilibrium state the system takes ball with radius $R$ and let $n(r)$ — concentration of particle, normalizing by the condition:

$$\int_{V_R} n dV = 4\pi \int_0^R r^2 n(r) dr = N,$$

where $N \gg 1$ — full number of particles. In (4) central symmetry is used.

Let’s consider arbitrary point inside the system with radius-vector $\vec{r}$. Potential $d\phi$ at the point, producing by the particles inside some remoted volume $dV'$ in neighborhood of some another point $\vec{r}'$ of the system accordingly to (3) is equal:

$$d\phi(r) = g(|\vec{r} - \vec{r}'|) dL',$$

where $dL' \equiv \alpha \cdot n(r') \cdot dV'$ — total charge of $dV'$. Superposition principle for the full potential at $\vec{r}'$ gives

$$\varphi(r) = \int_{V_R} g(|\vec{r} - \vec{r}'|) dL'.$$

Taking into account apparent kind $|\vec{r} - \vec{r}'| = \sqrt{r^2 - 2rr'\cos\theta + r'^2}$ and carrying out angle integration, we get

$$\varphi(r) = 2\pi\alpha \int_0^R G(r, r') n(r') r'^2 dr' \equiv 2\pi\alpha \hat{K}[n]$$

where

$$G(r, r') = \int_{-1}^1 g(\sqrt{r^2 - 2rr'\xi + r'^2}) d\xi = G(r', r)$$

determines kernal $\hat{K}$ of integral operator $\hat{K}$.

\footnote{Rigorously speaking kernal in (3) is $G(r', r)r'^2$, but for the brevity we’ll understand as kernal the function $G$. Note, that $G(r', r)$ is, in fact, Green function for the linear equation, which pair potential $\phi$ obeys to. So by (3) we postulate special Green function, that corresponds to a wide class of linear differential field operators.}
Attractive force, acting on a volume element \( dV \) from the effective field of the system is

\[
dF = -\frac{\partial \varphi}{\partial r} dL = -\frac{\partial \varphi}{\partial r} \cdot \alpha \cdot n(r) \cdot dV,
\]

and its volume density

\[
f = \frac{dF}{dV} = -\frac{\partial \varphi}{\partial r} \alpha \cdot n(r) = -2\pi \alpha^2 n(r) \int_0^r \frac{\partial G(r, r')}{\partial r} r'^2 n(r') \, dr'
\]

Equation (3) of mechanical equilibrium, which in our case is reduced to \(-dp/dr + f = 0\), with local pressure \( p \), takes the form:

\[
\frac{dp}{dr} + 2\pi \alpha^2 n(r) \int_0^r \frac{\partial G(r, r')}{\partial r} r'^2 n(r') \, dr' = 0
\]

The obtained equation belongs to a class of integro-differential ones. Really, pressure in general case is connected with concentration by thermal state equation \( p = p(n) \), which should be accounted by statistical methods. In our case it is determined also by long-range potential (its derivatives), whose influence on state equation is very complicated.

2 Simplifying of equilibrium equation

To make analytical investigation of eq. (10) some simplifying assumptions are necessary:

1) Assume the local field has no influence on local state equation. This assumption will be valid in case of not strong fields and its gradients. In fact our consideration concerns with zero-th term of Taylor row of \( p \) over \( \varphi \) (or, more strictly its derivatives).

2) Lets restrict our attention to a wide class of state equations of the form:

\[
p = an^\gamma (1 + bn)
\]

where \( a, b \) and \( \gamma \) — phenomenological constants. Note, that \( a \) has a sense of temperature multiplier, which constantness within the system automatically implies its heat equilibrium. Expression (11) should be understood as the first two terms of effective virial expansion. For a not large concentrations second term much less then first one and so it influence on equilibrium distribution can be accounted by perturbations methods, with background, corresponding to the state equation \( p = an^\gamma \).

Stress, that thermal state equation (11) is postulated by us, but not calculated as it should be done in fundamental theory. So our approach should be called semi-self-consistent: we take into account influence of long-range potential on concentration, but not on state equation.

3) Lets take function \( g(r) \) from a ring of a generalized polynoms:

\[
g(r) = \sum_i \varepsilon_i r^{\lambda_i},
\]

where \( \varepsilon_i, \lambda_i \) — arbitrary, but similar for all particles real parameters of potential.

The first and second assumptions (under \( b = 0 \)) in (11) lead to equation:

\[
a \gamma n^{\gamma-1} n' + 2\pi \alpha^2 n(r) \int_0^r \frac{\partial G(r, r')}{\partial r} r'^2 n(r') \, dr' = 0
\]

or, after integration over \( r \):

\[
n^{\gamma-1} = -s \bar{K}[n] + A
\]

in case \( \gamma \neq 1 \), and

\[
n = \bar{A}e^{-i\bar{K}[n]}, \quad \bar{A} = \bar{K}[n]/(1 - s)
\]

\[\varepsilon_i \text{ will mean ordinary local pressure.}\]
in case $\gamma = 1$. Here $A$ and $\bar{A}$ — yet undefined integration constants, $s = 2\pi\alpha^2(\gamma - 1)/a\gamma$,

$$\bar{s} = 2\pi\alpha^2/a. \tag{15}$$

It easily to see, that equation (14) is Boltzmann distribution written in self-consistent form. Really identifying (14) with $n = n_0\exp(-U/kT)$ under $\bar{A} = n_0$,

$$a = kT, \tag{16}$$

$U = 2\pi\alpha^2\tilde{K}[n]$ (see [7], [13]), we obtain that this is true.

The third simplifying assumption lets us by elementary integration calculate $G$. By linearity $G$ over $g$ it is sufficiently to carry out all calculations for one arbitrary power $\lambda$ in (12). The result is:

$$G(r, r') = \int_{-1}^{1} \varepsilon(r^2 - 2rr'\xi + r'^2)^{\lambda/2} d\xi = \begin{cases} \frac{\varepsilon}{(\lambda + 2)rr'}[(r + r')^{\lambda + 2} - |r - r'|^{\lambda + 2}], & \lambda \neq -2; \\ \frac{\varepsilon}{rr'} \ln |r + r'|, & \lambda = -2 \end{cases} \tag{17}$$

### 3 General properties of equilibrium equations

In this section we present some general consequences of (13) and (14), which are independent of any assumptions concerning with perturbation theory (see following section). More exactly we are interested by possibility of cusps and finite radius $R$ of the system, that is actual in galaxy and stellar dynamics [1, 2, 13, 4]. The necessary conditions can be formulated by the two following simple theorems:

**Theorem 1.** If $g(r)$ is bounded from the up, then for $\gamma \leq 1$ configuration with finite $R$ is impossible.

**Theorem 2.** If $g(r)$ is bounded from the down, then for $\gamma \geq 1$ configuration with cusp is impossible.

**Proof.** Condition for finite $R$ is $n(R) = 0 \ (R < \infty)$. From (13) and (14) under $\gamma \leq 1$ it is followed, that this possible if (and only if) $\tilde{K}[n]_{r=R} \to +\infty$. Bounded from the up potential $g$ can be made negative on its whole domain of definition by finite shift: it only will redefine integration constants $A$ and $\bar{A}$. Such negative $g$ will give negative $G$ and negative $\tilde{K}[n]$, since $n(r) \geq 0$. So, $\tilde{K}[n]$, even it will have singular points will negative there: $\tilde{K}[n]_{r=R} \to -\infty$, that means impossibility of finite radius.

By the similar manner one can proof that under $\gamma \geq 1$ and $g$ bounded from the down $\tilde{K}[n]$ at best gives: $\tilde{K}[n]_{r=0} \to +\infty$, that is opposite to that is required by cusps condition $\tilde{K}[n]_{r=0} \to -\infty$.

The important consequence from the theorems is: finite configuration with cusp in case $\gamma = 1$ is possible, when $g$ is unbounded both from the up, and from the down. This fact suggests, that the so called force softening procedure [5], introducing in N-body simulation scheme to avoid local force infinities under collisions and to short computer time, can (under certain circumstances) drastically change global properties of investigated system.

### 4 General form of solutions

Let build at first a general solution for the case (14). For this purpose we write it through the formal row over powers of interaction parameter $\bar{s}$:

$$n = X_0 + X_1\bar{s} + X_2\bar{s}^2 + \ldots, \tag{18}$$

where $X_i$ — yet unknown functions $r$. Expand then the exponent on $\bar{K}$ in (14) in the Taylors row:

$$e^{-\bar{s}\tilde{K}[n]} = 1 - \frac{\bar{s}\tilde{K}[n]}{1!} + \frac{\bar{s}^2\tilde{K}^2[n]}{2!} + \ldots = \tag{19}$$

\footnote{Inspite of dimension character of $\bar{s}$ we prefer it to dimensionless ordering parameter $\xi = \alpha^2R^\lambda/a$ (see below), since $R$ can be infinite.}
Here to reexpand every term of (19) over \( \bar{s} \). To equate coefficients from the left and right in (14) after substituting (18) and (19), it is necessary to equate coefficients in (18):

\[
B^p_{\vec k, \vec n, \vec a, \vec b} = \left( \frac{1}{p!} \right)^{k_1} \frac{1}{(p-k_1)!} \frac{1}{(p-k_2)!} \cdots \frac{1}{(p-k_n)!} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}.
\]

In this expression \( \vec k \equiv (k_1, \ldots, k_n), \vec n \equiv (1, \ldots, n) \) — integer-values \( n \)-dimensional vector, \( (\vec a, \vec b) \) — euclidian scalar product, coefficients \( B^p_{\vec k, \vec n, \vec a, \vec b} \) can be calculated with the help of recurrent formulae, given in Appendix 1. For an integer positive \( p \) summation is carried out only for these \( \vec k \), which satisfies the nonequality \( p - k_1 - \cdots - k_n \geq 0 \). It is easily to check that for particular cases \( p \in \mathbb{N} \) and \( p = -1 \) under \( x_i = 0 \) for all \( i = 2, \ldots \), (20) reproduces well known Newton binom and formula for infinite geometrical progression correspondingly.

Substituting (18) and (19) into (14), taking into account (20) and equating coefficients for equal powers \( \bar{s} \) in (14) from the left and right, we get the system of recurrent relations on unknown coefficients in (18):

\[
X_0 = \bar{A}; \quad X_1 = -\bar{A} \dot{K}[X_0];
\]

\[
X_j = \bar{A} \left[ \sum_{p=1}^{j-1} \frac{(-1)^p}{(p-j)!} \sum_{(\vec k, \vec n, \vec a, \vec b)} \frac{(p-k_1)!! \cdots (p-k_j)!!}{(p-k_1)! \cdots (p-k_j)!} B^p_{\vec k, \vec n, \vec a, \vec b} X_0^{p-k_1-\cdots-k_n} X_1^{k_1} \cdots X_n^{k_n} \right] \left( \dot{K}[X_0] \right)^{(j-1)} + \frac{(-1)^j}{j!} (\dot{K}[X_0])^j.
\]

For the first four equations of the infinite chain (23) gives:

\[
X_0 = \bar{A};
\]

\[
X_1 = -\bar{A} \dot{K}[X_0] = -\bar{A}^2 K;
\]

\[
X_2 = \bar{A} (-\dot{K}[X_0] + \frac{1}{2} \ddot{K}[X_0]) = \bar{A}^3 \left[ K_2 + \frac{1}{2} \dot{K}^2 \right];
\]

\[
X_3 = \bar{A} (-\dot{K}[X_2] + \dot{K}[X_0] \cdot \dot{K}[X_1] - \frac{1}{3} \ddot{K}^2 \dot{K}[X_0]) = -\bar{A}^4 \left[ K_3 + \frac{1}{4} \ddot{K}[K^2] + K \cdot K_2 + \frac{1}{3} \dot{K}^3 \right]
\]

\[
\vdots
\]

Here \( \bar{K} \equiv \dot{K}[1]; \ K_n \equiv \dot{K}^n[1]. \)

The same way is appropriate for the more general case (13) of equilibrium equation. Substitution of decomposition (18) (we omit a bar over letters) and equation of coefficients for all equal powers of \( s \) with (23) gives:

\[
X_0^{-1} = A;
\]

\[
\frac{1}{n!} \sum_{(\vec k, \vec n, \vec a, \vec b)} \frac{(\gamma - 1)!!(1)!! \cdots (n-1)!!}{(\gamma - 1 - k_1 - \cdots - k_n)!} B^p_{\vec k, \vec n, \vec a, \vec b} X_0^{\gamma-1-k_1-\cdots-k_n} X_1^{k_1} \cdots X_n^{k_n} = -K X_{n-1},
\]

or in resolved form:

\[
X_n = -X_n^{-\gamma} \left( \frac{\dot{K}[X_{n-1}]}{(\gamma - 1)} + \frac{1}{n!} \sum_{(\vec k, \vec n, \vec a, \vec b)} \frac{(\gamma - 1)!!(1)!! \cdots (n-1)!!}{(\gamma - 1 - k_1 - \cdots - k_{n-1})!} B^p_{\vec k, \vec n, \vec a, \vec b} X_0^{\gamma-1-k_1-\cdots-k_{n-1}} X_1^{k_1} \cdots X_{n-1}^{k_{n-1}} \right).
\]
All noninteger factorials in (23) should be understood as infinite products $a! \equiv a(a-1)(a-2) \cdots$. The first four equations of the infinite chain have the form:

$$
\begin{align*}
X_0 &= A^\gamma \equiv \tilde{X}_0; \\
X_1 &= -\frac{\tilde{X}_0^{3-\gamma}}{\gamma - 1} K; \\
X_2 &= \frac{\tilde{X}_0^{-2\gamma}}{(\gamma - 1)^2} \left[ K_2 - \frac{1}{2} (\gamma - 2) K^2 \right]; \\
X_3 &= -\frac{\tilde{X}_0^{-3\gamma}}{(\gamma - 1)^3} \left[ K_3 - \frac{1}{2} (\gamma - 2) \tilde{K} K^2 + \frac{(\gamma - 2)(2\gamma - 3)}{6} K^3 - (\gamma - 2) K \cdot K_2 \right].
\end{align*}
$$

(24)

At the end of the paragraph, let's discuss some general physical considerations, relating to the convergency question of the row (18). If generalized charges of a system will be too large or potential will be too rapidly fall in neighborhood of a particle (see following paragraph), then the configuration can not be equilibrium: there is singular condensate of the particles (with zero classical entropy) and row (18) is diverging. As a characteristic parameter, which is responsible for the such condensate forming, the value

$$
\xi = \frac{\alpha^2 R^\lambda}{a} \sim \frac{u_R}{kT},
$$

should be taken, where $u_R$ is potential energy of a pair particles at a distance of size of the system. The parameter $\xi$ is also characterize the influence of a long-range ordering forces, in comparison with influence of disordering heat chaotic particles motion inside the system. So, $\xi$ is natural to be called ordering parameter.

On the other hand, an energy of long-range interaction of a particle with the whole system must be sufficiently large to provide bound state of the particles. In opposite case the system will be diffused. Corresponding parameters is

$$
\Xi = \frac{U_R}{a} \sim \frac{U_R}{kT},
$$

where $U_R$ is potential energy of interaction of a particle with a whole system. So reasonable physical conditions of equilibrium (and convergency of the rows) are:

$$
\xi \ll 1 \ll \Xi.
$$

5 The cases $\lambda = -1$ and $\lambda = 2$

Due to simplicity and practical importance particular interest have particles with Coulomb and harmonic potentials. Lets calculate the first nontrivial approximation $X_1$ for $\gamma = 1$. As it is seen from (22) the matter is reduced to a calculation of $K$. Simple analysis shows, that $K \to \infty$ under $\lambda \leq -3$. In this case all terms of the formal row (18) become infinite, that expresses nonstability of configuration with these potentials: all particles falls onto the center. In case $\lambda > -3$ elementary integration gives:

$$
K = \begin{cases}
\frac{\varepsilon}{(\lambda + 2)r} \left\{ \frac{1}{\lambda + 4} ((R + r)^{\lambda+4} - (R - r)^{\lambda+4}) - \frac{r}{\lambda + 3} ((R + r)^{\lambda+3} + (R - r)^{\lambda+3}) \right\}, \\
\varepsilon \left( \frac{1}{2} \right)^{\lambda} \left[ (R^2 - r^2) \ln \frac{R}{R - r} + 2Rr \right], \\
\lambda = -2.
\end{cases}
$$

(25)

*Note, that direct mathematical investigation of the rows (21)-(23) convergency is problematic due to a very complicated structure of its coefficients. Even the evaluation of a number of nonzero $B^n$ for arbitrary fixed $n$, presented in Appendix 1, requires rather long calculations on n-dimensional integer-valued lattice.*
For $\lambda = -1$ we have $K = \varepsilon R^2 (1 - z^2 / 3)$. For $\lambda = 2$, $K = \varepsilon (2R^5 / 5)(1 + 5z^2 / 3)$, $z = r/R$. In the first case one should put $\varepsilon = -1$, in second $\varepsilon = 1$ to provide attractive character of interaction. For the former accordingly to (18) and (22) expression for $n(r)$ takes the form:

$$n \approx n_0 - n_0^2 s K = n_0 + n_0^2 s R^2 \left( 1 - \frac{z^2}{3} \right),$$  \hspace{1cm} (26)

which must vanish at $z = 1$. It gives $n_0 = -3/2R^2 s$. Substituting in (26), we get

$$n_c = \frac{3}{4R^2 s} \left( 1 - z^2 \right).$$  \hspace{1cm} (27)

Writing normalizing condition (1) for (27) we get the following first approximation for relation between equilibrium size of a system and number of particles:

$$R_c = \frac{5s}{2\pi} N.$$  \hspace{1cm} (28)

Similarly for the case of harmonic pair potential we obtain:

$$n_h = \frac{75}{128sR^5} \left( 1 - z^2 \right)$$  \hspace{1cm} (29)

and

$$R_h = \sqrt{\frac{5\pi}{16sN}}.$$  \hspace{1cm} (30)

Schematic curve for the both cases is shown in fig.1(a).

Figure 1: (a) — schematic dependency of $n(r)$ for interparticle Coulomb and harmonic potentials in the first approximation; (b) — theoretical radial profile of luminosity for galaxies for the same potentials.

The obtained expressions for equilibrium sizes of a system shows its strong nonadditivity, that has been mentioned in introduction. It is interesting, that equilibrium size of the system of bounded harmonic oscillators is decrease, when $N$ increase, as it follows from (30).

Note, that within central two-body problem there is interesting symmetry between this types of potentials: trajectories can be transformed to each other by the so called Bohlin’s transformation [16]. As it can be seen from (27) and (29) the similar symmetry is revealed also within N-body problem at least in first approximation. As it follows from the theorems of the previous section, Coulomb system has $R = \infty$ and finite $R$ in (28) is induced by approximation, while harmonic system has’nt cusp at the center. In fact, it means that Coulomb system of $N < \infty$ attractive particles totally diffuse in equilibrium. It theoretically support the fact, that galaxy and stars formations are looked as essentially nonequilibrium systems. Futher we’ll use expressions (27)-(29) only to demonstrate the method, without any aims to reproduce observations.

Using integrals $K_n$ from Appendix2, we can calculate the following approximation for $n$. The third one has the form

$$n \approx \frac{11}{R^2 s} \left( 1 - 1.26z^2 + 0.29z^4 - 0.03z^6 \right); \hspace{1cm} R \approx \frac{sN}{5.2\pi}$$

and for harmonic potential:

$$n \approx \frac{0.52}{sR^5} \left( 1 - 0.95z^2 - 0.14z^4 - 0.05z^6 \right); \hspace{1cm} R \approx \sqrt{\frac{0.24\pi}{sN}}.$$  

\hspace{1cm} (27)

\hspace{1cm} (29)
6 Application in cosmology

Our model may be relevant to the systems of a large \((N \gg 1)\) classical interacting objects, without any reference to its real scale. As an example, let consider cosmological application of the theory — E0 galaxies or ball clusters. Statistical approach to such systems has been applied in [11]. Rough consideration, based on virial theorem was made by Einstein in [3].

For a comparing the theory with observable datas it is necessary to translate it in terms of visual optical values, for example absolute surface luminosity. For a model, where all stars have the same overage characteristics it is easily to obtain the following formula, relating surface luminosity with concentration:

\[
L = l_0 \sqrt{R^2 - \rho^2} \int_{-\sqrt{R^2 - \rho^2}} n(\sqrt{\rho^2 + \xi^2}) d\xi,
\]

(31)

where \(l_0\) — absolute luminosity of a single star, \(\rho\) — polar radius in the coordinate frame, related to the picture plane.

Substituting of (27) or (29) in (31) leads to the following expression:

\[
L = A_0 (1 - \zeta^2)^{3/2}, \quad \zeta = \frac{\rho}{R},
\]

(32)

where

\[
A_{\text{coul}} = \frac{l_0}{R^3}; \quad A_{\text{harm}} = \frac{25l_0}{32R^4},
\]

which should be compared with experimental functions \(L(\rho)\). In fig.1(b) the curve (32) is schematically shown. This curve is well applied for a central part of cluster, where number of stars is sufficient to form self-consistent field, but individual stars are well resolved. Boundary region of a cluster is turn off from our consideration etc, since there are large fluctuations. The problem of separation and description of clusters core and boundary is considered in details in [11].

The method is also applicable for an investigation of a dark matter problem [17], which had been mentioned yet in Einstein cited work [3]. The problem is inverse with respect to preceding one: using photometric datas one should reconstruct pair potential of interaction of astrophysical objects. Below we formulate general rough scheme of the approach, deferring details and corrections for a future investigations.

As it is well known from rotational curves analysis and from another astrophysical datas luminous mass of a galaxy presents it smaller part, while remained one (about 90%) is confined in a dark halo, which extends to a distances about some radiuses of the visual central part. We assume, that equilibrium properties of a dark matter — its distribution and equilibrium size are practically independent from the luminous ones, so that the last can be viewed as probe subsystem at the dark background. Lets dark matter particles interact with each other by means of potential \(\tilde{\phi}\), where “tilde” will denote values, related to dark matter. Then using general formulas of our approach, one can calculate dark matter distribution \(\tilde{n}\). This last create at every point additional potential for luminous matter, which is self interacting with Coulomb potential. The self-consistent ”dark potential” \(\tilde{\varphi}\), should be taken into account in expression (1) for a total potential:

\[
\varphi(r) = 2\pi\alpha \int_0^R G(r, r') n(r') r'^2 dr' + \tilde{\varphi} \equiv 2\pi\alpha \tilde{K}[n] + \tilde{\varphi}.
\]

(33)

The potential \(\tilde{\varphi}\) when \(\tilde{n}\) is known can be calculated with the help of (1), with interaction and temperature parameters \(\tilde{a}, \tilde{\alpha}\) and kern \(\tilde{G}\), taken in general form (17) with parameters \(\tilde{\lambda}\) and \(\tilde{\varepsilon}\). These last are, in fact, to be found. For the above considered Coulomb and harmonic potentials simple calculations give:

\[
\tilde{\varphi}_{\text{coul}} = \frac{3\tilde{a}}{4\tilde{\alpha}} \left( \frac{\tilde{z}^4}{10} - \frac{\tilde{z}^2}{3} + \frac{1}{2} \right) + \text{const}; \quad \tilde{\varphi}_{\text{harm}} = \frac{15\tilde{a}}{224\tilde{\alpha}} \left( 1 + \frac{7}{3}\tilde{z}^2 \right) + \text{const},
\]

(34)

where \(\tilde{z} = r/R\), \(\tilde{R}\) — dark halo radius. Integration of equilibrium equation for luminous matter with potential (33) in case \(\gamma = 1\) leads to the equation, generalizing (14) (we omit bar over letters):

\[
n = A e^{-s\tilde{K}[n]-\kappa\tilde{\varphi}},
\]

(35)
where $\kappa = \alpha'/a$, $\alpha'$ — new interaction constant of luminous and dark matters. Introducing new function $n^* = ne^{\kappa \hat{\varphi}}$, and new kernal $G^*(r, r') = G(r, r')e^{-\kappa \hat{\varphi}(r)}$, one can put (33) to the form (14):

$$n^* = Ae^{-sK^*[n^*]}.$$  \hspace{1cm} (36)

If luminous particles are interact with each other through Coulomb potential then with equality $K^* = e^{-\kappa \hat{\varphi}}K$, first approximation over $s$ gives:

$$n = F_2(\hat{\varphi}) \frac{3}{4R^2s} \left(1 - \frac{r^2}{R^2}F_1(\hat{\varphi})\right)$$  \hspace{1cm} (37)

— expression for luminous matter distribution with correction formfactors. These last are $F_1(\hat{\varphi}) = (3 - 2e^{2\kappa \hat{\varphi}})^{-1}$, $F_2 = e^{-2\kappa \hat{\varphi}}/F_1 = 3e^{-2\kappa \hat{\varphi}} - 2e^{-\kappa \hat{\varphi}}$, where norming of potential is $\hat{\varphi}(R) = 0$. Under $\kappa = 0$ (interaction between luminous and dark matter is absent) (37) transforms into (27). Then one should substitute expression (37) into (31) for surface luminosity calculation. The result can be represented in the following convenient for comparison form:

$$L(\zeta) = L_0(\zeta) + \Delta L(\zeta),$$

where correction

$$\Delta L = -\frac{3l_0}{4Rs} \int \sqrt{1-\zeta^2} \left(1 + z^2(F_1F_2 - 1) - F_2\right) d\xi |_{z=\sqrt{\zeta^2+\xi^2}}$$  \hspace{1cm} (38)

and $L_0$ — surface luminosity (22) in absence of dark matter. For example, we present expression for $\Delta L$, calculated on a dark matter background with Coulomb and harmonic potentials. Under the approximations $R/R = \epsilon \ll 1$, which very simplifies final expression we have:

$$\Delta L = \tilde{A} \cdot \frac{l_0 \nu \epsilon^2}{Rs} (1 - \epsilon^2)^{3/2} \left(1 - \frac{4}{9} \epsilon^2\right); \hspace{0.5cm} \tilde{A}_{\text{coul}} = \frac{9}{10}; \hspace{0.5cm} \tilde{A}_{\text{harm}} = \frac{9}{16}. \hspace{1cm} (39)$$

where $\nu = (a\ddot{a}/\dot{a}a)$. Comparison (39) with observations gives us possibility for estimations of some combination of interaction parameters and kind of potential. Note, that proposed approach can be complement to the so called MOND (Modified Newton Dynamics), where one deals with modified Newtonian pair potential too [18]. Obviously, the presented scheme is too rough and demands its following improving: account of a back reaction of a luminous matter on a dark one, more wide class of model potential, introducing of multicomponent systems etc..

### 7 The model with potential pits.

Lets consider a system with potential of the following form:

$$g = \begin{cases} 
-1, & r < \rho; \\
0, & r > \rho,
\end{cases}$$  \hspace{1cm} (40)

where $\rho$ — length parameter, having the sense of effective interaction radius. The formulae (13) in this case is invalid, and direct calculation by (8) gives:

$$G(r, r') = -\left(\frac{\rho^2 - (r - r')^2}{2rr'}\right).$$  \hspace{1cm} (41)

The integral $K$ for the first approximation (23) has the form:

$$K = -\frac{1}{2r} \left(\frac{R^2}{2}(\rho^2 - \frac{R^2}{2}) - \frac{r^2R^2}{2} + \frac{2rR^3}{3}\right).$$  \hspace{1cm} (42)

Requirement of vanishing of expression $n = n_0 - n_0^2 \tilde{s}K$ under $r = R$ gives $n_0 = -4/\tilde{s}R(\rho^2 - R^2/6)$. Under $\rho^2 = R^2/6$ following approximations should be accounted. Expression for $n$ after some algebra takes the form:

$$n = -\frac{4}{\tilde{s}Rr(\rho^2 - R^2/6)^2} \left(r^2R + r(\rho^2 - \frac{3}{2}R^2) - R(\rho^2 - R^2/2)\right).$$  \hspace{1cm} (43)
Its schematic kind is shown in fig.2(a). Note, that in accordance with theorem 2, cusp — is induced by approximation.

Nonnegativity condition for (43) will be nonequality:

$$ R \leq \sqrt{2} \rho, \tag{44} $$

that physically (in this approximation) means coincidence of orders magnitudes of size of the system and radius of pair interaction. Normalizing condition (4) then gives:

$$ \frac{8\pi}{3\bar{s}} \frac{R^2 \rho^2}{(\rho^2 - R^2/6)^2} = N. \tag{45} $$

With (44) the (45) leads to the following restriction on a number $N$:

$$ \bar{s} N \leq 12\pi \tag{46} $$

Physical sense of (46) (which should remain valid, in principle, for any approximation) is concluded in the fact, that when interaction constant $\bar{s}$ is fixed, there is no equilibrium state for infinitely large $N$, since adding new particles we don’t change attractive interaction intensity, while repulsive pressure is increased. This property simulate well known ”saturation” phenomenas of water steam or nuclear systems.

Dependency of equilibrium size $d = R/\rho$ on the interaction parameter $\sigma = 3\bar{s}N/8\pi$, $(0 \leq \sigma < 9/2)$ has the kind:

$$ d = 3\sqrt{2} \sqrt{1 + \frac{\sigma}{3} - \sqrt{1 + \frac{2}{3}\sigma}} $$

and is shown in fig.2(b).

8 Virial correction calculation.

Lets go backward to the expression (10) and lets substitute in it the more general state equation (11). Using equality $p'_n = a\gamma n^{\gamma - 1} + ab(\gamma + 1)n^{\gamma}$, we get:

$$ a\gamma n^{\gamma - 1}n' + ab(\gamma + 1)n^{\gamma}n' + 2\pi\alpha^2n \frac{\partial \dot{K}}{\partial r}[a] = 0. \tag{47} $$

Dividing the both sides over $n$ and integrating over $r$ we go to equations generalizing (13) and (14):

$$ \frac{a\gamma}{\gamma - 1} n^{\gamma - 1} + ab \frac{\gamma + 1}{\gamma} n^{\gamma} + 2\pi\alpha^2 \dot{K}[a] + C = 0; \quad (\gamma \neq 1); \tag{48} $$

$$ a n + 2abn + 2\pi\alpha^2 K + C = 0, \quad (\gamma = 1). \tag{49} $$

We’ll find solution to (48)-(49) by perturbation theory with small nonlinearity parameter $b$:

$$ n(r) = \bar{n}(r) + b\delta_1 n + b^2\delta_2 n \ldots, \tag{50} $$
where \( \hat{n} \) — solution to \([18]-[19]\) under \( b = 0 \), which has been obtained earlier, \( \delta_1 n, \delta_2 n, \ldots \) — unknown functions, defining corrections of different orders. Substituting \([50]\) in \([18]-[19]\) and holding terms of a first order over \( b \) from the both side we get:

\[
\delta_1 n + \frac{\gamma + 1}{\gamma^2} \hat{n}^2 + \frac{2\pi a^2}{\gamma a n \gamma^2} \tilde{K}\hat{n} = 0, \tag{51}
\]

— linear nonhomogeneous integral Fredholm equation on \( \delta_1 n \). This equation is valid for the case \( \gamma = 1 \) too.

Let's find virial corrections for the considered earlier potentials in case \( \gamma = 1 \).

a) **Coulomb potential.** Expression for the kernel \( G \) of \( \tilde{K} \), as it follows from general formulae \([17]\) has the form:

\[
G(r, r') = \begin{cases} 
-2/r, & r' < r, \\
-2/r', & r' > r.
\end{cases} \tag{52}
\]

Equation \((51)\) with the kernel \((52)\) and function \( \hat{n} \) from \([17]\) takes the form:

\[
\delta_1 n + \frac{9}{8s^2R^2} \left( 1 - \frac{r^2}{R^2} \right)^2 + \frac{3}{4R^2} \left( 1 - \frac{r^2}{R^2} \right) \left\{ -2 \int_0^r \frac{r'^2 \delta_1 n \cdot dr'}{r} - 2 \int_0^R \frac{r' \delta_1 n \cdot dr'}{r} \right\} = 0. \tag{53}
\]

It is convenient to go to the dimensionless variables \( z = r/R, y = \delta_1 n \cdot R^6 \) and to introduce dimensionless parameter \( \xi^2 = R^2/s^2 \). Then equation \((53)\) will take the following more simple form:

\[
y + \frac{9}{8s^2(1 - z^2)^2} + \frac{3}{4(1 - z^2)} \left\{ -2 \int_0^z z'^2 y \cdot dz' - 2 \int_0^1 z'y \cdot dz' \right\} = 0. \tag{54}
\]

Dividing over \(-3(1 - z^2)/2z\) and differentiating twice over \( z \), we get differential consequence of \((54)\), which is differential equation of second order:

\[
u'' + \frac{3}{2}(1 - z^2)u + \frac{9}{2}\xi^2 z = 0, \tag{55}
\]

where \( u = -2zy/3(1 - z^2) \). Its solution is given by the power row

\[
u(\xi, z) = \sum_{k=0}^{\infty} a_k z^k, \tag{56}
\]

where coefficients are defined by the relations:

\[
a_2 = -\frac{3}{4}a_0, \ a_3 = -\frac{1}{4}(a_1 + 3\xi^2), \ a_{k+2} = \frac{3(a_{k-2} - a_k)}{2(k + 1)(k + 2)}, \ k = 2, \ldots \tag{57}
\]

Here \( a_0 \) and \( a_1 \) — yet unknown integration constants. To find ones it is necessary to restrict obtained general solution to a subclass, satisfying \((51)\). Substituting \((56)\) in \((54)\) and integrating term by term, we equate coefficients for the same powers of \( z \). This, apart from \((57)\), will give additional equalities:

\[
a_0 = 0, \ a_1 = \frac{3}{4}\xi^2 + 3 \sum_{n=0}^{\infty} \frac{a_n}{(n + 1)(n + 3)}, \tag{58}
\]

that fully defined our solution. Since \( a_0 = 0 \) implies by \((57)\) vanishing of all even coefficients in \((56)\), we finally get:

\[
\delta_1 n = -\frac{3(1 - z^2)}{2R^6} \sum_{n=0}^{\infty} \beta_n z^{2n}, \tag{58}
\]

\[
\beta_0 = \frac{3}{4} \left( \xi^2 + \sum_{n=0}^{\infty} \frac{\beta_n}{(n + 1)(n + 2)} \right), \quad \beta_1 = -\frac{1}{4}(\beta_0 + 3\xi^2), \quad \beta_n = \frac{3(\beta_{n-2} - \beta_{n-1})}{4n(2n + 1)}.
\]

From the \((58)\) after some analysis we conclude that under \( b > 0 \), (that takes place for a majority of real systems and means additional repulsion) the correction \( b\delta_1 n \) is always negative: system become more rared and its equilibrium size increased, as it is intuitively obviously.
b) **Harmonic potential.** Taking expression for \( \bar{n} \) from (29) and substituting it in (51) we get
\[
y + 2k^2\xi^2(1-z^2)^2 + k(1-z^2)K[y] = 0,
\]
where \( y \) and \( z \) — are the same, that in previous case, \( k = 75/128, \xi = 1/(sR^2) \). Formulae (17) for \( G \) in case \( \lambda = 2 \) gives:
\[
G(r, r') = 2(r^2 + r'^2),
\]
so for \( K[y] \) we get
\[
K[y] = 2\int_0^1 z^2(z^2 + z'^2)y(z')dz' = Az^2 + B,
\]
where \( A = 2\int_0^1 yz^2dz', \ B = 2\int_0^1 yz^4dz' \). So the equation (59) on \( y \) is purely algebraic and its solution is reduced to finding of unknown \( A \) and \( B \). To do it we substitute \( y \) from (53) in expressions for \( A \) and \( B \) and after some transformations we go to the following linear system:
\[
\begin{align*}
(2kI_{41} + 1)A + 2kI_{21}B &= -4k^2\xi^2I_{22}; \\
2kI_{01}A + (2kI_{41} + 1)B &= -4k^2\xi^2I_{42};
\end{align*}
\]
where
\[
I_{pq} \equiv \int_0^1 x^p(1-x^2)^qdx = \sum_{k=0}^q (-1)^q-k C_k^q \frac{1}{2(q-k)+p+1}.
\]
For the integrals in (61) the (62) gives the following expressions:
\[
I_{21} = \frac{2}{15}; \quad I_{22} = \frac{8}{105}; \quad I_{41} = \frac{2}{35}; \quad I_{42} = \frac{8}{315}; \quad I_{61} = \frac{2}{63}.
\]
Substituting it into (61) and expressing \( A \) and \( B \) by Kramers rule, we get
\[
A = -\frac{375}{15808}\xi^2 \approx -0.024\xi^2; \quad B = -\frac{8025}{15808}\xi^2 \approx -0.51\xi^2;
\]
Finally,
\[
\delta_1 n = -\frac{0.39}{s^4R^4}(1 - 2.8z^2 + 1.8z^4).
\]
As it has been in the previous case, the case \( b > 0 \) rares the system.

### 9 Conclusion.

So, the considered approach is relevant to the following problems:
1) Calculating of equilibrium distribution of any LRIS;
2) Finding of potential parameters by comparing calculated distribution with observation (inverse to 1-st), that is more important from theoretical point of view, than 1-st.
3) Calculation of virial corrections.
The following obvious generalizations are possible:
1) generalization on the case of coexisting chemical phases;
2) generalization on rotating systems;
3) generalization on identical quantum particles. This last case will involve nonlinear self-consistent multiparticle Shreidingger equation.

In conclusion I will be grateful to express many thanks to S.B.Moskovskiy for the interest and useful critical comments.

### A Appendix 1: coefficients \( B_{k_1...k_n}^n \).

Coefficients \( B_{k_1...k_n}^n \), appearing in the rows (20) can be calculated for any \( n, k_1, \ldots k_n \) with the help of the following recurrent formulae:
\[
B_{k_1...k_{n-1}}^{n} = B_{k_1-1k_2...k_{n-1}}^{n-1} + \sum_{j=1}^{n-2}(k_j + 1)B_{k_1...k_j+1k_{j+1}-1...k_{n-1}}^{n-1},
\]
\[
B_{00...01}^n = 1.
\]
All other coefficients are zero, and beginning value $B_1^1 = 1$. For a number $f(n)$ of nonzero $B^n$ for a given $n$ the following estimate can be obtained:

$$f(n) < \frac{e^n}{\sqrt{2\pi n}}.$$

Let's give the nonzero $B^n_{k_1...k_n}$ for $n = 1, 5$ includingly.

\begin{align*}
n = 1 & \quad B_1^1 = 1 \\
n = 2 & \quad B_{20}^2 = 1; \quad B_{01}^2 = 1; \\
n = 3 & \quad B_{300}^3 = 1; \quad B_{110}^3 = 3; \quad B_{001}^3 = 1; \\
n = 4 & \quad B_{4000}^4 = 1; \quad B_{2100}^4 = 6; \quad B_{1010}^4 = 4; \quad B_{0200}^4 = 3; \quad B_{0001}^4 = 1; \\
n = 5 & \quad B_{50000}^5 = 1; \quad B_{51000}^5 = 10; \quad B_{52000}^5 = 10; \quad B_{52000}^5 = 15; \quad B_{51000}^5 = 5; \\
& \quad B_{01100}^5 = 10; \quad B_{00001}^5 = 1
\end{align*}

**B Appendix 2: calculation of $K_n$**

Let's show the results of calculations of integrals, which are necessary for third correction calculation.

For a Coulomb potential:

\begin{align*}
K_1 &= \frac{r^2}{3} - R^2; \\
K_2 &= \frac{r^4}{30} - \frac{r^2 R^2}{3} + \frac{5 R^4}{6}; \\
K_3 &= \frac{r^6}{630} - \frac{R^2 r^4}{30} + \frac{5 r^2 R^4}{18} - \frac{61 R^6}{90}; \\
K[K^2] &= \frac{r^6}{189} - \frac{R^2 r^4}{15} + \frac{r^2 R^4}{3} + \frac{19 R^6}{27}.
\end{align*}

For harmonic potential:

\begin{align*}
K_1 &= \frac{2 r^2 R^3}{3} + \frac{2 R^5}{5}; \\
K_2 &= \frac{8 R^8 r^2}{15} + \frac{184 R^{10}}{525}; \\
K_3 &= \frac{704 R^{13} r^2}{1575} + \frac{768 R^{15}}{2625}; \\
K[K^2] &= \frac{704 R^{13} r^2}{1575} + \frac{22336 R^{15}}{70875}.
\end{align*}
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