Nonlinear Gravitational Clustering in Expanding Universe

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Abstract. The gravitational clustering of collisionless particles in an expanding universe is modelled using some simple physical ideas. I show that it is possible to understand the nonlinear clustering in terms of three well defined regimes: (1) linear regime; (2) quasilinear regime which is dominated by scale-invariant radial infall and (3) nonlinear regime dominated by nonradial motions and mergers. Modelling each of these regimes separately I show how the nonlinear two point correlation function can be related to the linear correlation function in hierarchical models. This analysis leads to results which are in good agreement with numerical simulations thereby providing an explanation for numerical results. Using this model and some simple extensions, it is possible to understand the transfer of power from large to small scales and the behaviour of higher order correlation functions. The ideas presented here will also serve as a powerful analytical tool to investigate nonlinear clustering in different models.

1. Introduction

Consider a collection $N$ point particles, interacting with each other by the Newtonian gravity, in an expanding background characterized by a scale factor $a(t)$. What can we say about the time evolution of such a system?

This problem is of considerable interest for several reasons. To begin with, the behaviour of large number of particles interacting via Newtonian gravity poses a formidable challenge to the usual methods of statistical mechanics (T. Padmanabhan, 1990). So, purely from an academic point of view, this seems to be a challenging but solvable problem.

Secondly, this problem might even have some practical interest. There is considerable evidence that the universe is dominated by collisionless non-baryonic dark matter particles. In that case, they will play a key role in the formation of large scale structures. If the length scales of interest are (i) small compared to Hubble radius but (ii) large compared to the scales at which non-gravitational processes are significant, then the system of dark matter particles constitutes an example in which the question raised in the first paragraph becomes relevant. In fact, most of the work in this subject has been inspired by considerations of structure formation.
A brute force method for solving this problem relies on numerical simulations. In such an approach one starts with large number of particles distributed nearly uniformly and calculates the future trajectories by a suitable numerical algorithm. This approach, however, does not lead to genuine understanding unless we supplement it with some analytic modelling. In this talk, I shall outline how one can make analytic progress in the problem of nonlinear gravitational clustering and thereby reproduce the key features of numerical simulations (Padmanabhan, 1996 a,b).

2. Modelling the gravitational clustering

Consider a system of particles distributed homogeneously, on the average, with a mean density $\bar{\rho}(t)$. This uniform density will cause an expansion of the universe and the proper distance $r = a(t)\mathbf{x}$ between particles will increase with time. If the distribution was not strictly uniform, then the perturbations in the density will act as local centres of clustering. A region with overdensity will accrete matter around it while an underdense region will repel matter in its surroundings. As a result, perturbations in density will tend to grow and, when the density contrast is of order unity, these cluster centres will exert significant influence on the evolution. Particles in a highly overdense regions will evolve essentially under their own self-gravity and will tend to form gravitationally bound systems.

When these density perturbations are small, it is possible to study their evolution using linear theory. But once the density contrast becomes comparable to unity, linear perturbation theory breaks down and one must use N-body simulations to study the growth of perturbations. While these simulations are of some value in making concrete predictions for specific models, they do not provide clear physical insight into the process of non-linear gravitational dynamics. To obtain such an insight into this complex problem, it is necessary to model the gravitational clustering of collisionless particles using simple physical concepts. I shall develop one such model in this section, which - in spite of extreme simplicity - reproduces the simulation results for hierarchical models fairly accurately. Further, this model also provides insight into the clustering process and can be modified to take into account more complicated situations.

The paradigm for understanding the clustering is based on the well known behaviour of a spherically symmetric overdense region in the universe. In the behaviour of such a region, one can identify three different regimes of interest: (1) In the early stages of the evolution, when
the density contrast is small, the evolution is described by linear theory. (2) Each of the spherical shells with an initial radius \( x_i \) can be parametrised by a mass contained inside the shell, \( M(x_i) \), and the energy, \( E(x_i) \) for the particular shell. Each shell will expand to a maximum radius \( x_{\text{max}} \propto M/|E| \) and then turn around and collapse. Such a spherical collapse and resulting evolution allows a self similar description (Filmore & Goldreich, 1984; Bertshinger, 1985) in which each shell acts as though it has an effective radius proportional to \( x_{\text{max}} \). This will be the quasilinear phase. (3) The spherical evolution will break down during the later stages due to several reasons. First of all, non radial motions will arise due to amplification of deviations from spherical symmetry. Secondly, the existence of substructure will influence the evolution in a non-spherically symmetric way. Finally, in the real universe, there will be merging of such clusters [each of which could have been centres of spherical overdense regions in the beginning] which will again destroy the spherical symmetry. This will be the nonlinear phase.

The description given above is sufficiently well known that one may suspect it can not lead to any insight into the problem. In particular, structures observed in the real universe are hardly spherical. I will show that it is, however, possible to model the above process in a manner which allows direct generalisation to the real universe.

To do this we will begin by studying the evolution of system starting from a gaussian initial fluctuations with an initial power spectrum, \( P_{in}(k) \). The fourier transform of the power spectrum defines the correlation function \( \xi(a, x) \) where \( a \propto t^{2/3} \) is the expansion factor in a universe with \( \Omega = 1 \). It is more convenient for our purpose to work with the average correlation function inside a sphere of radius \( x \), defined by

\[
\bar{\xi}(a, x) \equiv \frac{3}{x^3} \int_0^x \xi(a, y)y^2 dy
\]  

In the linear regime we have \( \bar{\xi}_L(a, x) \propto a^2 \xi_{in}(a_i, x) \). In the quasilinear and nonlinear regimes, we would like to have prescription which relates the exact \( \bar{\xi} \) to the mean correlation function calculated from the linear theory. One might have naively imagined that \( \bar{\xi}(a, x) \) should be related to \( \bar{\xi}_L(a, x) \). But one can convince oneself that the relationship is likely to be nonlocal by the following analysis:

Recall that, the conservation of pairs of particles, gives an exact equation satisfied by the correlation function (Peebles, 1980):

\[
\frac{\partial \xi}{\partial t} + \frac{1}{a x^2} \frac{\partial}{\partial x} [x^2 (1 + \xi) v] = 0
\]

where \( v(t, x) \) denotes the mean relative velocity of pairs at separation \( x \) and epoch \( t \). Using the mean correlation function \( \bar{\xi} \) and a dimensionless
pair velocity \( h(a, x) \equiv -(v/\dot{a}x) \), equation (2) can be written as

\[
\left( \frac{\partial}{\partial \ln a} - h \frac{\partial}{\partial \ln x} \right) (1 + \bar{\xi}) = 3h(1 + \bar{\xi})
\] (3)

This equation can be simplified by first introducing the variables

\[ A = \ln a, \quad X = \ln x, \quad D(X, A) = \ln(1 + \bar{\xi}) \] (4)

in terms of which we have (Nityananda and Padmanabhan, 1994)

\[
\frac{\partial D}{\partial A} - h(A, X) \frac{\partial D}{\partial X} = 3h(A, X)
\] (5)

Introducing further a variable \( F = D + 3X \), (5) can be written in a remarkably simple form as

\[
\frac{\partial F}{\partial A} - h(A, X) \frac{\partial F}{\partial X} = 0
\] (6)

The characteristic curves to this equation - on which \( F \) is a constant - are determined by \((dX/dA) = -h(X, A)\) which can be integrated if \( h \) is known. But note that the characteristics satisfy the condition

\[
F = 3X + D = \ln[x^3(1 + \bar{\xi})] = \text{constant}
\] (7)

or, equivalently,

\[
x^3(1 + \bar{\xi}) = l^3
\] (8)

where \( l \) is another length scale. When the evolution is linear at all the relevant scales, \( \bar{\xi} \ll 1 \) and \( l \approx x \). As clustering develops, \( \bar{\xi} \) increases and \( x \) becomes considerably smaller than \( l \). It is clear that the behaviour of clustering at some scale \( x \) is determined by the original linear power spectrum at the scale \( l \) through the “flow of information” along the characteristics. This suggests that we should actually try to express the true correlation function \( \bar{\xi}(a, x) \) in terms of the linear correlation function \( \bar{\xi}_L(a, l) \) evaluated at a different point.

Let us see how we can do this starting from the quasilinear regime. Consider a region surrounding a density peak in the linear stage, around which we expect the clustering to take place. It is well known that density profile around this peak can be described by

\[
\rho(x) \approx \rho_{bg}[1 + \xi(x)]
\] (9)

Hence the initial mean density contrast scales with the initial shell radius \( l \) as \( \delta_i(l) \propto \xi_L(l) \) in the initial epoch, when linear theory is valid. This shell will expand to a maximum radius of \( x_{\text{max}} \propto l/\delta_i \propto l/\xi_L(l) \). In
scale-invariant, radial collapse, models each shell may be approximated as contributing with an effective radius which is proportional to $x_{\text{max}}$. Taking the final effective radius $x$ as proportional to $x_{\text{max}}$, the final mean correlation function will be

$$\bar{\xi}_{QL}(x) \propto \rho \propto \frac{M}{x^3} \propto \frac{l^3}{(l^3/\bar{\xi}_L(l))^3} \propto \bar{\xi}_L(l)^3$$

(10)

That is, the final correlation function $\bar{\xi}_{QL}$ at $x$ is the cube of initial correlation function at $l$ where $l^3 \propto x^3 \bar{\xi}_L \propto x^3 \bar{\xi}_{QL}(x)$. This is in the form demanded by (8) if $\bar{\xi} \gg 1$. Note that we did not assume that the initial power spectrum is a power law to get this result.

In case the initial power spectrum is a power law, with $\bar{\xi}_L \propto x^{-(n+3)}$, then we immediately find that

$$\bar{\xi}_{QL} \propto x^{-3(n+3)/(n+4)}$$

(11)

[If the correlation function in linear theory has the powerlaw form $\bar{\xi}_L \propto x^{-\alpha}$ then the process described above changes the index from $\alpha$ to $3\alpha/(1 + \alpha)$. We shall comment more about this aspect later].

For the power law case, the same result can be obtained by more explicit means. For example, in power law models the energy of spherical shell with mean density $\bar{\delta}(x_i) \propto x_i^{-b}$ will scale with its radius as $E \propto G\bar{\delta}M(x_i)/x_i \propto G\delta x_i^2 \propto x_i^{2-b}$. Since $M \propto x_i^3$, it follows that the maximum radius reached by the shell scales as $x_{\text{max}} \propto (M/E) \propto x_i^{1+b}$. Taking the effective radius as $x = x_{\text{eff}} \propto x_i^{1+b}$, the final density scales as

$$\rho \propto \frac{M}{x^3} \propto \frac{x_i^3}{x_i^{3(1+b)}} \propto x_i^{-3b} \propto x^{-3b/(1+b)}$$

(12)

In this quasilinear regime, $\bar{\xi}$ will scale like the density and we get $\bar{\xi}_{QL} \propto x^{-3b/(1+b)}$. The index $b$ can be related to $n$ by assuming the evolution starts at a moment when linear theory is valid. Since the gravitational potential energy [or the kinetic energy] scales as $E \propto x_i^{-(n+1)}$ in the linear theory, it follows that $b = n + 3$. This leads to the correlation function in the quasilinear regime, given by (11).

If $\Omega = 1$ and the initial spectrum is a power law, then there is no intrinsic scale in the problem. It follows that the evolution has to be self similar and $\bar{\xi}$ can only depend on the combination $q = xa^{-2/(n+3)}$. This allows to determine the $a$ dependence of $\bar{\xi}_{QL}$ by substituting $q$ for $x$ in (11). We find

$$\bar{\xi}_{QL}(a, x) \propto a^{6/(n+4) - 3(n+3)/(n+4)}$$

(13)
We know that, in the linear regime, $\bar{\xi} = \bar{\xi}_L \propto a^2$. Equation (13) shows that, in the quasilinear regime, $\bar{\xi} = \bar{\xi}_{QL} \propto a^{6/(n+4)}$. Spectra with $n < -1$ grow faster than $a^2$, spectra with $n > -1$ grow slower than $a^2$ and $n = -1$ spectrum grows as $a^2$.

Direct algebra shows that
\[
\bar{\xi}_{QL}(a, x) \propto [\bar{\xi}_L(a, l)]^3
\] (14)

reconfirming the local dependence in $a$ and nonlocal dependence in spatial coordinate. This result has no trace of original assumptions [spherical evolution, scale-invariant spectrum ...] left in it and hence once would strongly suspect that it will have far general validity.

Let us now proceed to the third and nonlinear regime. If we ignore the effect of mergers, then it seems reasonable that virialised systems should maintain their densities and sizes in proper coordinates, i.e. the clustering should be “stable”. This would require the correlation function to have the form $\bar{\xi}_{NL}(a, x) = a^3 F(ax)$. [The factor $a^3$ arising from the decrease in background density]. From our previous analysis we expect this to be a function of $\bar{\xi}_L(a, l)$ where $l^3 \approx x^3 \bar{\xi}_{NL}(a, x)$. Let us write this relation as
\[
\bar{\xi}_{NL}(a, x) = a^3 F(ax) = U[\bar{\xi}_L(a, l)]
\] (15)

where $U[z]$ is an unknown function of its argument which needs to be determined. Since linear correlation function evolves as $a^2$ we know that we can write $\bar{\xi}_L(a, l) = a^2 Q[l^3]$ where $Q$ is some known function of its argument. [We are using $l^3$ rather than $l$ in defining this function just for future convenience of notation]. In our case $l^3 = x^3 \bar{\xi}_{NL}(a, x) = (ax)^3 F(ax) = r^3 F(r)$ where we have changed variables from $(a, x)$ to $(a, r)$ with $r = ax$. Equation (13) now reads
\[
a^3 F(r) = U[\bar{\xi}_L(a, l)] = U[a^2 Q[l^3]] = U[a^2 Q[r^3 F(r)]]
\] (16)

Consider this relation as a function of $a$ at constant $r$. Clearly we need to satisfy $U[c_1 a^2] = c_2 a^3$ where $c_1$ and $c_2$ are constants. Hence we must have
\[
U[z] \propto z^{3/2}.
\] (17)

Thus in the extreme nonlinear end we should have
\[
\bar{\xi}_{NL}(a, x) \propto [\bar{\xi}_L(a, l)]^{3/2}
\] (18)

[Another way deriving this result is to note that if $\bar{\xi} = a^3 F(ax)$, then $h = 1$. Integrating (3) with appropriate boundary condition leads to]
Once again we did not need to invoke the assumption that the spectrum is a power law. If it is a power law, then we get,

$$\bar{\xi}_{NL}(a, x) \propto a^{(3-\gamma)x^{-\gamma}}; \quad \gamma = \frac{3(n+3)}{(n+5)} \quad (19)$$

This result is based on the assumption of “stable clustering” and was originally derived by Peebles (Peebles, 1965). It can be directly verified that the right hand side of this equation can be expressed in terms of $q$ alone, as we would have expected.

Putting all our results together, we find that the nonlinear mean correlation function can be expressed in terms of the linear mean correlation function by the relation:

$$\bar{\xi}(a, x) = \begin{cases} 
\bar{\xi}_L(a, l) & \text{(for } \bar{\xi}_L < 1, \bar{\xi} < 1) \\
\bar{\xi}_L(a, l)^3 & \text{(for } 1 < \bar{\xi}_L < 5.85, 1 < \bar{\xi} < 200) \\
14.14\bar{\xi}_L(a, l)^{3/2} & \text{(for } 5.85 < \bar{\xi}_L, 200 < \bar{\xi})
\end{cases} \quad (20)$$

The numerical coefficients have been determined by continuity arguments. We have assumed the linear result to be valid up to $\bar{\xi} = 1$ and the virialisation to occur at $\bar{\xi} \approx 200$ which is result arising from the spherical model. The exact values of the numerical coefficients can be obtained only from simulations.

The true test of such a model, of course, is N-body simulations and remarkably enough, simulations are very well represented by relations of the above form. The simulation data for CDM, for example, is well fitted by (Padmanabhan et al., 1996):

$$\bar{\xi}(a, x) = \begin{cases} 
\bar{\xi}_L(a, l) & \text{(for } \bar{\xi}_L < 1.2, \bar{\xi} < 1.2) \\
\bar{\xi}_L(a, l)^3 & \text{(for } 1 < \bar{\xi}_L < 5, 1 < \bar{\xi} < 125) \\
11.7\bar{\xi}_L(a, l)^{3/2} & \text{(for } 5 < \bar{\xi}_L, 125 < \bar{\xi})
\end{cases} \quad (21)$$

which is fairly close to the theoretical prediction. [The fact that numerical simulations show a correlation between $\bar{\xi}(a, x)$ and $\bar{\xi}_L(a, l)$ was originally pointed out by Hamilton et al. (1991) who, however, tried to give a multiparameter fit to the data. This fit has somewhat obscured the simple physical interpretation of the result though has the virtue of being very accurate for numerical work.]

A comparison of (20) and (21) shows that the physical processes which operate at different scales are well represented by our model. In other words, the processes described in the quasilinear and nonlinear regimes for an individual lump still models the average behaviour of the universe in a statistical sense. It must be emphasised that the key point is the “flow of information” from $l$ to $x$ which is an exact result. Only
when the results of the specific model are recast in terms of suitably chosen variables, we get a relation which is of general validity. It would have been, for example, incorrect to use spherical model to obtain relation between linear and nonlinear densities at the same location or to model the function $h$.

It may be noted that to obtain the result in the nonlinear regime, we needed to invoke the assumption of stable clustering which has not been deduced from any fundamental considerations. In case mergers of structures are important, one would consider this assumption to be suspect (see Padmanabhan et al., 1996). We can, however, generalise the above argument in the following manner: If the virialised systems have reached stationarity in the statistical sense, the function $h$ - which is the ratio between two velocities - should reach some constant value. In that case, one can integrate (11) and obtain the result $\bar{\xi}_{NL} = a^{3h} F(a^{h} x)$. A similar argument will now show that

$$
\bar{\xi}_{NL}(a, x) \propto [\bar{\xi}_L(a, l)]^{3h/2}
$$

in the general case. For the power law spectra, one would get

$$
\bar{\xi}(a, x) \propto a^{(3-\gamma)h} x^{-\gamma}; \quad \gamma = \frac{3h(n+3)}{2 + h(n+3)}
$$

Simulations are not accurate enough to fix the value of $h$; in particular, the asymptotic value of $h$ could depend on $n$ within the accuracy of the simulations. It may be possible to determine this dependence by modelling mergers in some simplified form.

If $h = 1$ asymptotically, the correlation function in the extreme nonlinear end depends on the linear index $n$. One may feel that physics at highly nonlinear end should be independent of the linear spectral index $n$. This will be the case if the asymptotic value of $h$ satisfies the scaling

$$
h = \frac{3c}{n+3}
$$

in the nonlinear end with some constant $c$. Only high resolution numerical simulations can test this conjecture that $h(n+3) = \text{constant}$.

It is possible to obtain similar relations between $\xi(a, x)$ and $\xi_L(a, l)$ in two dimensions as well. In 2-D the scaling relations turn out to be

$$
\bar{\xi}(a, x) \propto \begin{cases} 
\bar{\xi}_L(a, l) & \text{(Linear)} \\
[\bar{\xi}_L(a, l)]^2 & \text{(Quasi-linear)} \\
\xi_L(a, l) & \text{(Nonlinear)} 
\end{cases}
$$

For power law spectrum the nonlinear correction function will $\bar{\xi}_{NL}(a, x) = a^{2-\gamma} x^{-\gamma}$ with $\gamma = 2(n+2)/(n+4)$. 
If we generalize the concept of stable clustering to mean constancy of $h$ in the nonlinear epoch, then the correlation function will behave as $\xi_{NL}(a, x) = a^{2h}F(a^h x)$. In this case, if the spectrum is a power law then the nonlinear and linear indices are related to

$$\gamma = \frac{2h(n + 2)}{2 + h(n + 2)}$$

(26)

All the features discussed in the case of 3 dimensions are present here as well. For example, if the asymptotic value of $h$ scales with $n$ such that $h(n + 2) = \text{constant}$ then the nonlinear index will be independent of the linear index. (Numerically it would be lot easier to test this result in 2-D rather than in 3-D; work is in progress to test these results).

We shall now consider some applications and further generalisations of our model.

3. Critical Index and power transfer

Given a model for the evolution of the power spectra in the quasilinear and nonlinear regimes, one could explore whether evolution of gravitational clustering possesses any universal characterisitics. For example one could ask whether a complicated initial power spectrum will be driven to any particular form of power spectrum in the late stages of the evolution.

One suspects that such a possibility might arise because of the following reason: We saw in the last section that [in the quasilinear regime] spectra with $n < -1$ grow faster than $a^2$ while spectra with $n > -1$ grow slower than $a^2$. This feature could drive the spectral index to $n = n_c \approx -1$ in the quasilinear regime irrespective of the initial index. Similarly, the index in the nonlinear regime could be driven to $n \approx -2$ during the late time evolution. So the spectral indices $-1$ and $-2$ are some kind of “fixed points” in the quasilinear and nonlinear regimes. Speculating along these lines, we would expect the gravitational clustering to lead to a “universal” profile which scales as $x^{-1}$ at the nonlinear end changing over to $x^{-2}$ in the quasilinear regime.

This effect can be understood better by studying the “effective” index for the power spectra at different stages of the evolution. These are plotted in figure 1. The three panels of figure 1 illustrate features related to the existence of fixed points in a clear manner. In the top panel we have plotted index of growth $n_a \equiv (\partial \ln \xi(a, x) / \partial \ln a)_x$ as a function of $\xi$ in the quasilinear regime obtained from our scaling relations. Curves correspond to an input spectrum with index $n = -2, -1, 1$. The dashed horizontal line at $n_a = 2$ represents the linear
growth rate. An index above this dashed horizontal line will represent a rate of growth faster than linear growth rate and the one below will represent a rate which is slower than the linear rate. It is clear that – in the quasilinear regime – the curve for \( n = -1 \) closely follows the linear growth while \( n = -2 \) grows faster and \( n = 1 \) grows slower; so the critical index is \( n_c \approx -1 \). The curves are based on the fitting formula due to Hamilton et al, 1991.

The second panel of figure 1 shows the effective index \( n_a \) as a function of the index \( n \) of the original linear spectrum at different levels of nonlinearity labelled by \( \bar{\xi} = 1, 5, 10, 50, 100 \). We see that in the quasilinear regime, \( n_a > 2 \) for \( n < -1 \) and \( n_a < 2 \) for \( n > -1 \).

The lower panel of figure 1 shows the slope \( n_x = -3 - (\partial \ln \bar{\xi}/\partial \ln x)_a \) of \( \bar{\xi} \) for different power law spectra. It is clear that \( n_x \) crowds around \( n_x \approx -1 \) in the quasilinear regime. If perturbations grow by gravitational instability, starting from an epoch at which \( \bar{\xi}_{initial} \ll 1 \) at all scales, then it can be shown that \( n_x \) at any epoch must satisfy the inequality

\[
n_x \leq \left( \frac{3}{\bar{\xi}} \right).
\]

This bounding curve is shown by a dotted line in the figure. This powerful inequality shows that regions of strong nonlinearity [with \( \bar{\xi} \gg 1 \)] should have effective index which is close to or less than zero.

The index \( n_c = -1 \) corresponds to the isothermal profile with \( \bar{\xi}(a, x) = a^2x^{-2} \) and has two interesting features to recommend it as a candidate for fixed point:

(i) For \( n = -1 \) spectra each logarithmic scale contributes the same amount of correlation potential energy. If the regime is modelled by scale invariant radial flows, then the kinetic energy will scale in the same way. It is conceivable that flow of power leads to such an equipartition state as a fixed point though it is difficult prove such a result in any generality.

(ii) It can be shown that scale invariant spherical collapse will change the density profile \( x^{-b} \) with an index \( b \) to another profile with index \( 3b/(1 + b) \). Such a mapping has a nontrivial fixed point for \( b = 2 \) corresponding to the isothermal profile and an index of power spectrum \( n = -1 \) (see Padmanabhan, 1996a).

These considerations also allow us to predict the nature of power transfer in gravitational clustering. Suppose that, initially, the power spectrum was sharply peaked at some scale \( k_0 = 2\pi/L_0 \) and has a small width \( \Delta k \). When the peak amplitude of the spectrum is far less than unity, the evolution will be described by linear theory and there will be no flow of power to other scales. But once the peak approaches a value close to unity, power will be generated at other scales due to nonlinear couplings even though the amplitude of perturbations in these scales
Figure 1. The top panel shows exponent of rate of growth of density fluctuations as a function of amplitude. We have plotted the rate of growth for three scale invariant spectra $n = -2, -1, 1$. The dashed horizontal line indicates the exponent for linear growth. For the range $1 < \delta < 100$, the $n = -1$ spectrum grows as in linear theory; $n < -1$ grows faster and $n > -1$ grows slower. The second panel shows exponent of rate of growth as a function of linear index of the power spectrum for different values of $\bar{\xi}$ (1, 5, 10, 50, 100). These are represented by thick, dashed, dot-dashed, dotted and the dot-dot-dashed lines respectively. It is clear that spectra with $n_{\text{lin}} < -1$ grow faster than the rate of growth in linear regime and $n_{\text{lin}} > -1$ grow slower. The lower panel shows the evolution of index $n_x = -3 - (\partial \ln \bar{\xi} / \partial \ln x)_{\|}$ with $\bar{\xi}$. Indices vary from $n = -2.5$ to $n = 4.0$ in steps of 0.5. The tendency for $n_x$ to crowd around $n_c = -1$ is apparent in the quasilinear regime. The dashed curve is a bounding curve for the index ($n_x < 3/\bar{\xi}$) if perturbations grow via gravitational instability.
are less than unity. Mathematically, this can be understood from the evolution equations for the density contrast - written in Fourier space - as:

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = 4\pi G \bar{\rho}\delta_k + Q$$

(28)

where $\delta_k(t)$ is the Fourier transform of the density contrast, $\bar{\rho}$ is the background density and $Q$ is a nonlocal, nonlinear function which couples the mode $k$ to all other modes $k'$ (Peebles, 1980). Coupling between different modes is significant in two cases. The obvious case is one with $\delta_k \geq 1$. A more interesting possibility arises for modes with no initial power [or exponentially small power]. In this case nonlinear coupling provides the only driving terms, represented by $Q$ in equation (28). These generate power at the scale $k$ through mode-coupling, provided power exists at some other scale. Note that the growth of power at the scale $k$ will now be governed purely by nonlinear effects even though $\delta_k \ll 1$.

Physically, this arises along the following lines: If the initial spectrum is sharply peaked at some scale $L_0$, first structures to form are voids with a typical diameter $L_0$. Formation and fragmentation of sheets bounding the voids lead to generation of power at scales $L < L_0$. First bound structures will then form at the mass scale corresponding to $L_0$. In such a model, $\xi_{\text{lin}}$ at $L < L_0$ is nearly constant with an effective index of $n \approx -3$. Assuming we can use equation (24) with the local index in this case, we expect the power to grow very rapidly as compared to the linear rate of $a^2$. [The rate of growth is $a^6$ for $n = -3$ and $a^4$ for $n = -2.5$.] Different rate of growth for regions with different local index will lead to steepening of the power spectrum and an eventual slowing down of the rate of growth. In this process, which is the dominant one, the power transfer is mostly from large scales to small scales. [There is also a generation of the $k^4$ tail at large scales which we shall not discuss here; see Bagla and Padmanabhan, 1996].

From our previous discussion, we would have expected such an evolution to lead to a “universal” power spectrum with some critical index $n_c \approx -1$ for which the rate of growth is that of linear theory - viz., $a^2$. In fact, the same results should hold even when there exists small scale power; recent numerical simulations dramatically confirm this prediction and show that - in the quasilinear regime, with $1 < \delta < 100$ - power spectrum indeed has a universal slope [see Bagla and Padmanabhan, 1996].
4. Further generalizations

The ideas presented here can be generalised in two obvious directions (see Munshi and Padmanabhan, 1996): (i) By considering peaks of different heights, drawn from an initial gaussian random field, and averaging over the probability distribution one can obtain a more precise scaling relation. (ii) By using a generalised ansatz for higher order correlation functions, one can attempt to compute the \( S_N \) parameters in the quasilinear and nonlinear regimes. I shall briefly comment on the results of these two generalisations.

(i) The basic idea behind the model used in section 2 can be described as follows: Consider the evolution of density perturbations starting from an initial configuration, which is taken to be a realisation of a Gaussian random field with variance \( \sigma \). A region with initial density contrast \( \delta_i \) will expand to a maximum radius \( x_f = x_i/\delta_i \) and will contribute to the two-point correlation function an amount proportional to \( (x_i/x_f)^3 = \delta_i^3 \). The initial density contrast within a randomly placed sphere of radius \( x_i \) will be \( \nu \sigma(x_i) \) with a probability proportional to \( \exp(-\nu^2/2) \). On the other hand, the initial density contrast within a sphere of radius \( x_i \), centered around a peak in the density field will be proportional to the two-point correlation function and will be \( \nu^2 \xi(x_i) \) with a probability proportional to \( \exp(-\nu^2/2) \). It follows that the contribution from a typical region will scale as \( \bar{\xi} \propto \bar{\xi}_i^{3/2} \) while that from higher peaks will scale as \( \xi \propto \bar{\xi}_i^3 \). In the quasilinear phase, most dominant contribution arises from high peaks and we find the scaling to be \( \bar{\xi}_{QL} \propto \bar{\xi}_i^3 \). The non-linear, virialized, regime is dominated by contribution from several typical initial regions and has the scaling \( \bar{\xi}_{NL} \propto \bar{\xi}_i^{3/2} \). This was essentially the result obtained in section 2 except that we took \( \nu = 1 \). To take into account the statistical fluctuations of the initial Gaussian field we can average over different \( \nu \) with a Gaussian probability distribution.

Such an analysis leads to the following result. The relationship between \( \bar{\xi}(a, x) \) and \( \bar{\xi}_L(a, l) \) becomes

\[
\bar{\xi}(a, x) = A [\bar{\xi}_L(a, l)]^{3h/2}; A = \left( \frac{2}{\lambda} \right)^{3h} \left[ \frac{\Gamma \left( \frac{a+1}{2} \right)}{2\sqrt{\pi}} \right]^{3h/\alpha}
\]  

(29)

where

\[
\alpha = \frac{6h}{2 + h(n + 3)}
\]

(30)

and \( \lambda \approx 0.5 \) is the ratio between the final virialized radius and the radius at turn-around. In our model, \( h = 2 \) in the quasi-linear regime.
and $h = 1$ in the non-linear regime. However, the above result holds for any other value of $h$. Equation (29) shows that the scaling relations (20) acquire coefficients which depend on the spectral index $n$ when we average over peaks of different heights. This effect is seen in simulations and equation (29) correctly accounts for the numerical results (Munshi and Padmanabhan, 1996).

(ii) In attempting to generalize our results to higher order correlation functions, it is important to keep the following aspect in mind. The $N$th order correlation function will involve $N - 1$ different length scales. To make progress, one needs to assume that, although there are different length scales present in reduced $N$-point correlation function, all of them have to be roughly of the same order to give a significant contribution. If the correlation functions are described by a single scale, then a natural generalisation of equation (30), will be

$$
\bar{\xi}_N \approx \langle x_i^{3(N-1)} \rangle / x_3^{3(N-1)}
$$

(31)

Given such an ansatz for the $N$ point correlation function, one can compute the $S_N$ coefficients defined by the relation $S_N \equiv \xi_N / \xi_2^{N-1}$ in a straightforward manner. We find that

$$
S_N = (4\pi)^{(N-2)/2} \frac{\Gamma \left( \frac{\alpha(N-1)+1}{2} \right)}{\left[ \Gamma \left( \frac{\alpha+1}{2} \right) \right]^{N-1}}
$$

(32)

where $\alpha$ is defined in equation (30). Given the function $h(\bar{\xi})$, this equation allows one to compute (approximately) the value of $S_N$ parameters in the quasi-linear and non-linear regimes. In our model $h = 2$ in the quasi-linear regime and $h = 1$ in the non-linear regime. The numerical values of $S_N$ computed for different power spectra agrees reasonably well with simulation results. (For more details, see Munshi and Padmanabhan, 1996.)

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