Weak law of large numbers for iterates of random-valued functions

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To the memory of Professor Marek Kuczma and Professor György Targoński.

Abstract. Given a probability space \((\Omega, \mathcal{A}, P)\), a complete and separable metric space \(X\) with the \(\sigma\)-algebra \(\mathcal{B}\) of all its Borel subsets and a \(\mathcal{B} \otimes \mathcal{A}\)-measurable \(f : X \times \Omega \to X\) we consider its iterates \(f^n\) defined on \(X \times \Omega^N\) by \(f^0(x, \omega) = x\) and \(f^n(x, \omega) = f(f^{n-1}(x, \omega), \omega_n)\) for \(n \in \mathbb{N}\) and provide a simple criterion for the existence of a probability Borel measure \(\pi\) on \(X\) such that for every \(x \in X\) and for every Lipschitz and bounded \(\psi : X \to \mathbb{R}\) the sequence \(\left(\frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k(x, \cdot))\right)_{n \in \mathbb{N}}\) converges in probability to \(\int_X \psi(y) \pi(dy)\).

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1. Introduction

Fix a probability space \((\Omega, \mathcal{A}, P)\) and a complete and separable metric space \((X, \rho)\).

Let \(\mathcal{B}\) denote the \(\sigma\)-algebra of all Borel subsets of \(X\). We say that \(f : X \times \Omega \to X\) is a random-valued function (shortly: an rv-function) if it is measurable with respect to the product \(\sigma\)-algebra \(\mathcal{B} \otimes \mathcal{A}\). The iterates of such an rv-function are given by

\[
f^0(x, \omega_1, \omega_2, \ldots) = x, \quad f^n(x, \omega_1, \omega_2, \ldots) = f(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n)
\]

for \(n \in \mathbb{N}\), \(x \in X\) and \((\omega_1, \omega_2, \ldots)\) from \(\Omega^\infty\) defined as \(\Omega^N\). Note that \(f^n : X \times \Omega^\infty \to X\) is an rv-function on the product probability space \((\Omega^\infty, \mathcal{A}^\infty, P^\infty)\). More exactly, for \(n \in \mathbb{N}\) the \(n\)th iterate \(f^n\) is \(\mathcal{B} \otimes \mathcal{A}_n\)-measurable, where \(\mathcal{A}_n\)
denotes the \( \sigma \)-algebra of all sets of the form
\[
\{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \ldots, \omega_n) \in A\}
\]
with \( A \) from the product \( \sigma \)-algebra \( \mathcal{A}^n \). (See [4], [5, Sec. 1.4].)

A result on the a.s. convergence of \( (f^n(x, \cdot))_{n \in \mathbb{N}} \) for \( X \) being the unit interval may be found in [5, Sec. 1.4B]. The paper [4] by Rafal Kapica brings theorems on the convergence a.s. and in \( L^1 \) of those sequences of iterates in the case where \( X \) is a closed subset of a Banach lattice. A simple criterion for the convergence in law of \( (f^n(x, \cdot))_{n \in \mathbb{N}} \) to a random variable independent of \( x \in X \) was proved in [1] and applied to the equation
\[
\varphi(x) = \int_\Omega \varphi(f(x, \omega)) P(d\omega) + F(x)
\]
with \( \varphi \) as the unknown function. In [2] this criterion was applied to the equation
\[
\varphi(x) = F(x) - \int_\Omega \varphi(f(x, \omega)) P(d\omega).
\]

In the present paper it is strengthened and applied to get a weak law of large numbers for iterates of random-valued functions.

2. Wasserstein metric

By a distribution (on \( X \)) we mean any probability measure defined on \( \mathcal{B} \). Recall that a sequence \( (\pi_n)_{n \in \mathbb{N}} \) of distributions converges weakly to a distribution \( \pi \) if
\[
\lim_{n \to \infty} \int_X u(x)\pi_n(dx) = \int_X u(x)\pi(dx)
\]
for any continuous and bounded \( u : X \to \mathbb{R} \). It is well known (see [3, Th. 11.3.3]) that this convergence is metrizable by the (Fortet–Mourier, Lévy–Prohorov, Wasserstein) metric
\[
\|\pi_1 - \pi_2\|_W = \sup \left\{ \left| \int_X ud\pi_1 - \int_X ud\pi_2 \right| : u \in \text{Lip}_1(X), \|u\|_\infty \leq 1 \right\},
\]
where
\[
\text{Lip}_1(X) = \{ u : X \to \mathbb{R} | |u(x) - u(z)| \leq \rho(x, z) \text{ for } x, z \in X \}
\]
and \( \|u\|_\infty = \sup\{|u(x)| : x \in X\} \) for a bounded \( u : X \to \mathbb{R} \).

3. Convergence in law

Fix an rv-function \( f : X \times \Omega \to X \) and let \( \pi_n(x, \cdot) \) denote the distribution of \( f^n(x, \cdot), \) i.e.,
\[
\pi_n(x, B) = P^\infty(f^n(x, \cdot) \in B) \quad \text{for } n \in \mathbb{N} \cup \{0\}, \ x \in X \text{ and } B \in \mathcal{B}.
\]
The above mentioned strengthening of [1, Th. 3.1] reads as follows.
Theorem 3.1. If
\[ \int_{\Omega} \varrho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \lambda \varrho(x, z) \quad \text{for } x, z \in X \] (1)
with a \( \lambda \in (0, 1) \), and
\[ \int_{\Omega} \varrho(f(x, \omega), x) P(d\omega) < \infty \quad \text{for } x \in X, \] (2)
then there exists a distribution \( \pi \) on \( X \) such that for every \( x \in X \) the sequence \( (\pi_n(x, \cdot))_{n \in \mathbb{N}} \) converges weakly to \( \pi \); moreover,
\[ \| \pi_n(x, \cdot) - \pi \|_W \leq \frac{\lambda^n}{1 - \lambda} \int_X \varrho(f(x, \omega), x) P(d\omega) \quad \text{for } x \in X \text{ and } n \in \mathbb{N} \] (3)
and
\[ \int_X \varrho(x, y) \pi(dy) < \infty \quad \text{for } x \in X. \] (4)

Proof. It follows from [1, Th. 3.1] that there exists a distribution \( \pi \) on \( X \) such that (3) holds. We shall show that (4) is also satisfied. To this end note first that by (1) we have
\[ \int_{\Omega^\infty} \varrho(f^n(x, \omega), f^n(z, \omega)) P^\infty(d\omega) \leq \lambda^n \varrho(x, z) \quad \text{for } x, z \in X \text{ and } n \in \mathbb{N}. \] (5)

Fix \( x \in X \) and for every \( n \in \mathbb{N} \) define \( \tau_n : [0, \infty) \rightarrow [0, \infty) \) by
\[ \tau_n(t) = \min\{t, n\}. \]
Since, by (3),
\[ \left| \int_X \tau_n(\rho(x, y)) \pi_n(x, dy) - \int_X \tau_n(\rho(x, y)) \pi(dy) \right| \leq n \| \pi_n(x, \cdot) - \pi \|_W \]
\[ \leq \frac{n \lambda^n}{1 - \lambda} \int_X \varrho(f(x, \omega), x) P(d\omega) \]
for \( n \in \mathbb{N} \) and by the monotone convergence theorem
\[ \int_X \rho(x, y) \pi(dy) = \lim_{n \to \infty} \int_X \tau_n(\rho(x, y)) \pi(dy), \]
it is enough to prove that the sequence \( \left( \int_X \tau_n(\rho(x, y)) \pi_n(x, dy) \right)_{n \in \mathbb{N}}, \) i.e., the sequence \( \left( \int_{\Omega^\infty} \tau_n(\rho(x, f^n(x, \omega))) P^\infty(d\omega) \right)_{n \in \mathbb{N}}, \) is bounded.
To show it observe that for every \( n \in \mathbb{N} \) and \( (\omega_1, \omega_2, \ldots) \in \Omega^\infty \) we have
\( \tau_n(\rho(f^n(x, \omega_1, \omega_2, \ldots), x)) \leq \rho(f^n(x, \omega_1, \omega_2, \ldots), x) \)
\[ = \rho(f^{n-1}(f(x, \omega_1), \omega_2, \omega_3, \ldots), x) \]
\[ \leq \sum_{k=1}^{n} \rho(f^{n-k}(f(x, \omega_k), \omega_{k+1}, \omega_{k+2}, \ldots), f^{n-k}(x, \omega_{k+1}, \omega_{k+2}, \ldots)) \]
and for every $y \in X$ the value $f^n(y, \omega_1, \omega_2, \ldots)$ depends only on $y$ and on $(\omega_1, \ldots, \omega_n)$. Hence, applying the Fubini theorem and (5), for every $n \in \mathbb{N}$ we get

$$
\int_{\Omega} \tau_n \left( \rho \left( f^n(x, \omega), x \right) \right) P^\infty(d\omega) \\
\leq \sum_{k=1}^{n} \int_{\Omega} \rho \left( f^{n-k} \left( f(x, \omega_1), \omega_2, \omega_3, \ldots \right), f^{n-k} \left( x, \omega_2, \omega_3, \ldots \right) \right) P^\infty(d(\omega_1, \omega_2, \ldots)) \\
\leq \sum_{k=1}^{n} \lambda^{n-k} \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) \leq \frac{1}{1 - \lambda} \int_{\Omega} \rho(f(x, \omega), x) P(d\omega).
$$

□

Remark 3.2. If (1) holds with a $\lambda \in (0, \infty)$ and (2) is satisfied, then the function $\nu : X \to [0, \infty)$ defined by

$$
\nu(x) = \int_{\Omega} \rho(f(x, \omega), x) P(d\omega)
$$

is Lipschitz.

4. Weak law of large numbers

Theorem 4.1. If (1) holds with a $\lambda \in (0, 1)$ and (2) is satisfied, then there exists a distribution $\pi$ on $X$ such that for every $x \in X$ and for every Lipschitz and bounded $\psi : X \to \mathbb{R}$ the sequence $\left( \frac{1}{n} \sum_{k=0}^{n-1} \psi \circ f^k(x, \cdot) \right)_{n \in \mathbb{N}}$ converges in probability to $\int_X \psi(y) \pi(dy)$.

Proof. Making use of Theorem 3.1 let $\pi$ be a distribution on $X$ such that (3) and (4) hold. It follows from Remark 3.2 and (4) that

$$
\int_X \nu(y) \pi(dy) < \infty.
$$

(7)

Fix $x_0 \in X$, a Lipschitz and bounded $\psi : X \to \mathbb{R}$ and an $\epsilon \in (0, \infty)$. Put

$$
\xi_n = \psi \circ f^n(x_0, \cdot) \quad \text{for } n \in \mathbb{N}, \quad c = \int_X \psi(y) \pi(dy).
$$

(8)

We shall show that

$$
\lim_{n \to \infty} P^\infty \left( \left| \frac{1}{n} \sum_{k=0}^{n-1} \xi_k - c \right| \geq \epsilon \right) = 0.
$$
Since by Chebyshev’s inequality
\[ P^\infty \left( \left| \frac{1}{n} \sum_{k=0}^{n-1} \xi_k - c \right| \geq \epsilon \right) \leq \frac{1}{n^2 \epsilon^2} \int_{\Omega^\infty} \left( \sum_{k=0}^{n-1} (\xi_k - c) \right)^2 \, dP^\infty \quad \text{for } n \in \mathbb{N}, \]
it is enough to prove that
\[ \lim_{n \to \infty} \frac{1}{n^2} \int_{\Omega^\infty} \left( \sum_{k=0}^{n-1} (\xi_k - c) \right)^2 \, dP^\infty = 0. \]

We may assume that
\[ \psi \in \text{Lip}_1(X) \quad \text{and} \quad \|\psi\|_\infty \leq 1. \quad (9) \]

We shall prove that
\[ \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \int_{\Omega^\infty} \xi_k \xi_l \, dP^\infty = \frac{c^2}{2}, \quad (10) \]
\[ \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \int_{\Omega^\infty} \xi_k^2 \, dP^\infty = 0, \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{\Omega^\infty} \xi_k \, dP^\infty = c. \quad (11) \]

Since
\[
\int_{\Omega^\infty} \left( \sum_{k=0}^{n-1} (\xi_k - c) \right)^2 \, dP^\infty = 2 \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \int_{\Omega^\infty} \xi_k \xi_l \, dP^\infty \\
+ \sum_{k=0}^{n-1} \int_{\Omega^\infty} \xi_k^2 \, dP^\infty - 2nc \sum_{k=0}^{n-1} \int_{\Omega^\infty} \xi_k \, dP^\infty + n^2 \epsilon^2
\]
for every integer \( n \geq 2 \), it will complete the proof.

Fix integers \( n \geq 2 \), \( k \in [1, n-1] \) and \( l \in [0, k-1] \). Then
\[ f^k(x_0, \omega_1, \omega_2, \ldots) = f^{k-l}(f^l(x_0, \omega_1, \omega_2, \ldots), \omega_{l+1}, \omega_{l+2}, \ldots) \]
for \((\omega_1, \omega_2, \ldots) \in \Omega^\infty\). Hence, by (8) and the Fubini theorem,
\[ \int_{\Omega^\infty} \xi_k \xi_l \, dP^\infty = \int_{\Omega^\infty} \left( \int_X \psi \left( f^{k-l}(y, \omega) \right) \psi(y) \pi_1(x_0, dy) \right) P^\infty(d\omega). \]

It follows from (9) and (5) that the function
\[ x \mapsto \int_{\Omega^\infty} \psi \left( f^{k-l}(x, \omega) \right) P^\infty(d\omega), \quad x \in X, \]
has values in \([-1, 1]\) and is Lipschitz with a Lipschitz constant \( \lambda^{k-l} \), whence the function
\[ x \mapsto \psi(x) \int_{\Omega^\infty} \psi \left( f^{k-l}(x, \omega) \right) P^\infty(d\omega), \quad x \in X, \]
has value in $[-1, 1]$ and is Lipschitz with a Lipschitz constant $1 + \lambda^{k-l}$. Hence and from (3) and (6) we infer that

$$\left| \int_X \psi(y) \left( \int_{\Omega^\infty} \psi \left( f^{k-l}(y, \omega) \right) P^\infty(d\omega) \right) \pi_l(x_0, dy) - \int_X \psi(y) \left( \int_{\Omega^\infty} \psi \left( f^{k-l}(y, \omega) \right) P^\infty(d\omega) \right) \pi(dy) \right| \leq 2\|\pi_l(x_0, \cdot) - \pi\|_W \leq \frac{2\lambda^l}{1 - \lambda} v(x_0).$$

Consequently, for every integer $n \geq 2$,

$$\left| \sum_{k=1}^{n-1} k \sum_{l=0}^{k-1} \left( \int_{\Omega^\infty} \xi_k \xi_l dP^\infty - \int_X \psi(y) \left( \int_X \psi(z) \pi_{k-l}(y, dz) \right) \pi(dy) \right) \right| \leq \sum_{k=1}^{n-1} k \sum_{l=0}^{k-1} \frac{2\lambda^l}{1 - \lambda} v(x_0) = \frac{2\lambda^l}{1 - \lambda} \sum_{k=1}^{n-1} \frac{1 - \lambda^k}{1 - \lambda} \leq \frac{2(n - 1) v(x_0)}{(1 - \lambda)^2}.$$ 

It shows that

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n-1} k \sum_{l=0}^{k-1} \left( \int_{\Omega^\infty} \xi_k \xi_l dP^\infty - \int_X \psi(y) \left( \int_X \psi(z) \pi_{k-l}(y, dz) \right) \pi(dy) \right) = 0.$$ 

(12)

Further, for every integer $n \geq 2$,

$$\sum_{k=1}^{n-1} k \sum_{l=0}^{k-1} \int_X \psi(y) \left( \int_X \psi(z) \pi_{k-l}(y, dz) \right) \pi(dy) = \sum_{k=1}^{n-1} (n - k) \int_X \psi(y) \left( \int_X \psi(z) \pi_k(y, dz) \right) \pi(dy)$$

and, by (9), (3) and (6),

$$\left| \int_X \psi(z) \pi_k(y, dz) - \int_X \psi(z) \pi(dz) \right| \leq \|\pi_k(y, \cdot) - \pi\|_W \leq \frac{\lambda^k}{1 - \lambda} v(y)$$
for $y \in X$ and $k \in \mathbb{N}$, whence
\[
\left| \sum_{k=0}^{n-1} (n-k) \int_X \psi(y) \left( \int_X \psi(z) \pi_k(y, dz) \right) \pi(dy) \right| \\
- \sum_{k=1}^{n-1} (n-k) \int_X \psi(y) \left( \int_X \psi(z) \pi(dz) \right) \pi(dy) \\
\leq \sum_{k=1}^{n-1} (n-k) \int_X |\psi(y)| \left| \int_X \psi(z) \pi_k(y, dz) - \int_X \psi(z) \pi(dz) \right| \pi(dy) \\
\leq \sum_{k=1}^{n-1} (n-k) \int_X \lambda^k \frac{1}{1-\lambda} \psi(y) \pi(dy) \leq \frac{n-1}{1-\lambda} \int_X \psi(y) \pi(dy) \sum_{k=1}^{n-1} \lambda^k \\
= \frac{(n-1)\lambda(1-\lambda^{n-1})}{(1-\lambda)^2} \int_X \psi(y) \pi(dy).
\]
Since, by (8),
\[
\sum_{k=1}^{n-1} (n-k) \int_X \psi(y) \left( \int_X \psi(z) \pi(dz) \right) \pi(dy) = \frac{n(n-1)}{2} c^2,
\]
jointly with (7), it gives
\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \int_X \psi(y) \left( \int_X \psi(z) \pi_{k-l}(y, dz) \right) \pi(dy) = \frac{c^2}{2}.
\]
Hence and from (12) we have (10).

From the weak convergence of $(\pi_n(x_0, \cdot))_{n \in \mathbb{N}}$ to $\pi$ it follows that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{\Omega} \xi_k dP^\infty = \int_X \psi(y) \pi(dy) = c
\]
and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{\Omega} \xi_k^2 dP^\infty = \int_X \psi(y)^2 \pi(dy),
\]
which shows that (11) also holds and ends the proof. \(\Box\)

Since continuous real functions defined on a compact metric space can be uniformly approximated by Lipschitz functions (see [3, Theorem 11.2.4]), Theorem 4.1 implies the following corollary.

**Corollary 4.2.** Assume $(X, \rho)$ is a compact metric space. If (1) holds with a $\lambda \in (0, 1)$ and (2) is satisfied, then there exists a distribution $\pi$ on $X$ such that for every $x \in X$ and for every continuous $\psi : X \to \mathbb{R}$ the sequence $(\frac{1}{n} \sum_{k=0}^{n-1} \psi \circ f^k(x, \cdot))_{n \in \mathbb{N}}$ converges in probability to $\int_X \psi(y) \pi(dy)$. 
Remark 4.3. In the results presented we cannot replace the sequence of means 
\[ \left( \frac{1}{n} \sum_{k=0}^{n-1} \psi \circ f^k(x, \cdot) \right)_{n \in \mathbb{N}} \] 
by \( \left( \psi \circ f^n(x, \cdot) \right)_{n \in \mathbb{N}} \).

To see it fix a \( \lambda \in (0, 1) \) and an \( \mathcal{A} \)-measurable \( \xi : \Omega \to [0, 1 - \lambda] \), and consider the rv-function \( f : [0, 1] \times \Omega \to [0, 1] \) given by
\[ f(x, \omega) = \lambda x + \xi(\omega). \]

We shall show that if \( \left( \psi \circ f^n(x, \cdot) \right)_{n \in \mathbb{N}} \) converges in probability for an \( x \in [0, 1] \) and for a Borel \( \psi : [0, 1] \to \mathbb{R} \) such that
\[ c|\pi - z| \leq |\psi(x) - \psi(z)| \quad \text{for} \quad x, z \in [0, 1] \]
with a \( c \in (0, \infty) \), then \( \xi \) is a.s. constant.

Proof. For every \( n \in \mathbb{N} \) we have
\[ f^n(x, \cdot) = \lambda f^{n-1}(x, \cdot) + \xi_n, \]
where
\[ \xi_n(\omega_1, \omega_2, \ldots) = \xi(\omega_n) \quad \text{for} \quad (\omega_1, \omega_2, \ldots) \in \Omega^\infty, \]
and
\[ c|f^n(x, \omega) - f^{n-1}(x, \omega)| \leq |\psi(f^n(x, \omega)) - \psi(f^{n-1}(x, \omega))| \quad \text{for} \quad \omega \in \Omega^\infty, \]
which implies that the sequence \( \left( f^{n-1}(x, \cdot) + \frac{1}{\lambda - 1} \xi_n \right)_{n \in \mathbb{N}} \) converges in probability to zero. Since
\[ f^n(x, \cdot) + \frac{1}{\lambda - 1} \xi_{n+1} = \lambda \left( f^{n-1}(x, \cdot) + \frac{1}{\lambda - 1} \xi_n \right) + \frac{1}{\lambda - 1} (\xi_{n+1} - \xi_n) \]
for \( n \in \mathbb{N} \), it proves that the sequence \( (\xi_{n+1} - \xi_n)_{n \in \mathbb{N}} \) converges in probability to zero. But \( (\xi_n)_{n \in \mathbb{N}} \) is a sequence of independent and identically distributed random variables, the distribution of \( \xi_n \) is just the distribution of \( \xi \) for every \( n \in \mathbb{N} \), whence (cf. [3, Theorem 9.1.3])
\[ (\mu_\xi * \mu_{-\xi})((-\infty, -\epsilon] \cup [\epsilon, \infty)) = 0 \quad \text{for} \quad \epsilon \in (0, \infty), \]
where \( \mu_\xi \) and \( \mu_{-\xi} \) denote the distributions of \( \xi \) and \( -\xi \), respectively. Consequently,
\[ (\mu_\xi * \mu_{-\xi})(\mathbb{R} \setminus \{0\}) = 0, \]
from which
\[ 1 = (\mu_\xi * \mu_{-\xi})(\{0\}) = \int_\mathbb{R} \mu_{-\xi}(-z) \mu_\xi(dz) = \int_\mathbb{R} \mu_\xi(z) \mu_\xi(dz), \]
and so \( \xi \) is a.s. constant. \( \square \)
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