Berry’s Phase for Ultracold Atoms in an Accelerated Optical Lattice

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Berry’s phase is investigated for ultracold atoms in a frequency modulated optical lattice. It is shown that Berry’s phase appears due to Bloch oscillation and the periodic motion of the optical lattice. Particularly, Berry’s phase for ultracold atoms under the gravitational force in an oscillating tight-binding optical lattice is calculated analytically. It is found that the Berry’s phase depends linearly on the amplitude of the oscillation of the optical lattice.

INTRODUCTION

The dynamics of a charged particle in a one-dimensional spatially periodic potential in the presence of a uniform electric field has attracted considerable attention over the years. The dynamical response of the particle to the electric field is the well-known phenomena of Bloch oscillations. The researchers were unable to observe Bloch oscillation in natural crystals since the scattering time of the electrons by the lattice defects is much shorter than the Bloch period. The fabrication of semiconductor superlattices played an important role for the experimental verification of this pure theoretical problem. The first experimental observation of Bloch oscillations was reported for superlattices in 1992 [1]. With the advent of laser cooling and trapping techniques, it became possible to observe Bloch oscillation and to check some other theoretical predictions experimentally in an unprecedented way [2, 3]. For example, Bloch oscillations were observed both with ultracold cesium atoms [2] and with a Bose-Einstein condensate [3] in an optical lattice. The experiment [4] was the first one with a Bose-Einstein condensate (BEC) of Rubidium atoms in an optical lattice with gravity acting as the static field. The persistent Bloch oscillations were observed for \( t = 10 \) s using laser-cooled strontium atoms \(^{88}\)Sr in optical lattices [5]. Wannier-Stark ladders [6] and Landau-Zener tunneling were also observed experimentally [7].

In contrast to natural crystals, lattice constant, potential well depth and lattice motion are controllable for optical lattices, where standing laser waves and cold neutral atoms play the role of the crystal lattices and the electrons, respectively. The motion of the lattice is possible by generating frequency shift between the two laser beams [6, 23]. If the phase difference is modulated periodically in time, the wells in the optical lattice can be periodically shaken back and forth. In [6], the motion of ultracold atoms in an accelerating potential of the form \( V_0 \cos(2k_L (z - g(t))) \), where \( g(t) = c_1 t^2 + c_2 \cos(\omega t) \), \( c_1, c_2, \omega \) are constants was considered. In the reference frame of the standing wave, this potential becomes \( V_0 \cos(2k_L z) + z + z \cos(\omega t) \). In the rest frame of the optical lattice, there will be an additional force acting on the condensate atoms due to the oscillation of the lattice. The rocking optical lattice was also used to demonstrate the dynamical localization of matter-wave packets [9], which was originally proposed by Dunlap and Kenkre for a charged particle in a tight-binding lattice driven by a sinusoidal electric field [24].

In the experiment [21], Wannier-Stark intraband transitions was observed for ultracold atoms influenced by the gravitational force in a vertically oriented oscillating optical lattice. In this paper, we will investigate one another physical phenomena for an accelerated optical lattice. This is the Berry’s phase [25, 27]. The concept of the Berry’s phase is of great interest in a variety of different branches of physics. Of particular example in solid state physics is the Berry’s phase for the problem of motion of an electron in a periodic potential and a time-dependent electric field [28, 31]. Here we will derive a formula for the Berry’s phase for neutral atoms placed in a periodically shaken optical lattice.

The paper is structured as follows. The following section briefly reviews the concept of the Berry’s phase. The final section studies the Berry’s phase for ultracold atoms in an accelerated optical lattice.

BERRY’S PHASES

In 1984, Berry reinvestigated an old problem of adiabatic evolution of a quantum state when the time dependent external parameters change periodically [25]. He considered a time periodic Hamiltonian and supposed that the external parameters are slow enough so that there is no transition to the higher energy levels. He found that in the absence of degeneracy, the eigenstate for a cyclic Hamiltonian comes back to itself after a period but takes an extra phase difference. This phase difference, later commonly called Berry’s phase, is gauge invariant. The gauge invariance property makes the Berry’s phase physical. The Berry’s phase is also geometrical, that is, it doesn’t depend on the exact rate of change of the external parameters in the Hamiltonian. So, the Berry’s phase is expressed in terms of local geometrical quantities.

Let us introduce the basic concepts of the Berry phase arising from the adiabatic evolution of a quantum state. Consider a physical system described by a Hamiltonian...
that depends on time. Suppose that the Hamiltonian is cyclic with a period $T$.

$$H(t + T) = H(t) .$$  \hspace{1cm} (1)

We are interested in the adiabatic evolution of the system. Hence, the system initially in one of the eigenstates of the Hamiltonian $H(0)$ will stay as an instantaneous eigenstate of the Hamiltonian throughout the process. The only difference is the phase difference between the initial and final quantum states. The wave function for a time periodic Hamiltonian at time $t$ reads

$$\Psi(z, t) = \exp \left( i\gamma(t) - \frac{i}{\hbar} \int E dt \right) \phi(z, t) ,$$  \hspace{1cm} (2)

where $E$ is the corresponding time dependent energy, the function $\phi(z, t)$ satisfies

$$H \phi(z, t) = E \phi(z, t) ,$$  \hspace{1cm} (3)

and $\gamma(t)$ is the Berry’s phase, which was usually neglected in the theoretical treatment of time-dependent problem until Berry reconsidered the cyclic evolution of the system. Substitution of (2) into the corresponding Schrodinger equation yields

$$\dot{\gamma} = i\hbar < \phi(z, t) \frac{\partial \phi(z, t)}{\partial t} > ,$$  \hspace{1cm} (4)

where dot denotes time derivation. It was generally assumed that $\gamma(t)$ could be eliminated by redefining the phase of the eigensate. Berry, however, realized that such a phase is observable when the system comes to its initial state.

$$\gamma = i\hbar \int dt < \phi(z, t) \frac{\partial \phi(z, t)}{\partial t} > .$$  \hspace{1cm} (5)

Having briefly reviewed the Berry’s phase, let us study the Berry’s phase for ultracold atoms in an accelerated optical lattice.

**FORMALISM**

Consider an atom influenced by a time dependent linear potential in an optical lattice. Suppose that the optical lattice is shaken by a frequency shift between the two lattice beams. We will investigate the Berry’s phase for such a system. The optical lattice is no longer stationary in space. We assume that the transverse motion of the atoms is frozen (i.e., we are dealing with a 1-D problem). Then the corresponding Hamiltonian reads

$$H = \frac{p^2}{2m} + V_0 \cos \{2k_L (z - z_0(t))\} + f(t) z ,$$  \hspace{1cm} (6)

where $m$ is the atomic mass, $V_0$ is the lattice depth, $k_L$ is the optical lattice wave number, $f(t)$ is, in general, a time-dependent force and $z_0(t)$ is a time-dependent periodic phase.

We are interested in the adiabatic evolution of the system. In other words, the rate of change of $z_0(t)$ and $f(t)$ are much slower than frequency between the bands so that it doesn’t induce interband transition. From the experimental point of view, the requirement for $z_0(t)$ is practically met by having a much smaller frequency shift between the lattice beams than the relevant band gaps. The requirement for $f(t)$ can also be experimentally reached. As a particular example, the gravitational force acting on neutral atoms is a good choice to observe Berry’s phase experimentally.

We further assume that $f(t + T) = f(t)$ and $z_0(t + T) = z_0(t)$, where $T$ is the period so that the Hamiltonini is time periodic. We emphasize that there exists one another parameter that must be periodic with the same period $T$. It is the quasimomentum, $k$. The periodicity of $k$ with the period $T$ is required since the path must be closed to make Berry’s phase a gauge-invariant quantity with physical significance. The linear potential in (6) makes the quasimomentum $k$ vary over the entire Brillouin zone and generates a closed path in the momentum space. Here we are mainly interested in the dynamics of Bloch oscillations and therefore use a weak force and initial states populating the lowest Bloch band. Unless the acceleration and the force is large enough for the atoms to undergo a Landau-Zener tunneling, atoms will remain in the first band.

The time-dependent Schrodinger equation corresponding to the Hamiltonian (6) is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V_0 \cos \{2k_L (z - z_0(t))\} + f(t) z \right) \Psi .$$  \hspace{1cm} (7)

In the adiabatic limit, the wave function $\Psi(z, t)$ is given by the equation (2). Now, we need $\phi(z, t)$, which satisfies the equation (3) with the Hamiltonian (6). The energy spectrum of an atom in a periodic potential consists of the Bloch bands. According to Bloch’s theorem, the eigenstates of a periodic Hamiltonian is given by

$$\phi(z, t) = e^{ikz} u_{n,k}(z, t) .$$  \hspace{1cm} (8)

where $k$ is the quasimomentum and $u_{n,k}(z, t)$ satisfies the following equation

$$\left( \frac{(p + \hbar k)^2}{2m} + V_0 \cos \{2k_L (z - z_0(t))\} + f(t) z \right) u = E u .$$  \hspace{1cm} (9)

Note that the energy $E$ is, in general, time dependent. From now on, we will drop the index $n$.

The Berry’s phase for our system becomes

$$\gamma = i\hbar \int dt < \phi | \frac{\partial \phi}{\partial t} > = i\hbar \int_0^T dt < \phi | \frac{\partial \phi}{\partial t} > .$$  \hspace{1cm} (10)

Note that the Berry’s phase is independent of how the time dependent periodic functions, $z_0(t)$ and $f(t)$, vary
in time.
Since the system is oscillating in time, one should be
careful when calculating the scalar product \( \langle \phi | \partial \phi / \partial t \rangle \) in
the above integral. It is preferable to transform the mov-
ing frame to the stationary lattice frame. The coordinate
transformation is
\[
z' = z - z_0(t) . \tag{11}
\]
Let us rewrite the equation \( \text{(9)} \) in the lattice coordinate
frame. It becomes
\[
\left( \frac{(p + \hbar k)^2}{2m} + V_0 \cos (2kLz') + f(t)z' \right) u' = E' u' . \tag{12}
\]
where \( E' = E - z_0(t)f(t) \) and \( u' = u(z - z_0(t)) \).
In the absence of the force, \( f(t) \to 0, k \) is a conserved
quantity and the Bloch function is specified by a band
index \( n \) and \( k \). In the presence of the additional force
\( f(t) \), the wave packet can be characterized by a single
mean quasimomentum \( k(t) \) at time \( t \) if the width of the
wave packet in quasimomentum space is small. The wave
packet prepared with a well-defined quasimomentum will
oscillate in position. Applying a perturbation to the atom
make \( k \) vary on a closed path in the Brillouin zone ac-
cording to the equation
\[
k = \frac{f(t)}{\hbar} . \tag{13}
\]
In a one dimensional system, \( k \) sweeps the interval
\((-\pi/a, \pi/a)\), where \( a \) is the lattice constant.
Under the coordinate transformation \( \text{(11)} \), Berry’s phase
is also transformed. Note that the time derivative op-
erator transforms as \( \partial / \partial t \to \partial / \partial z' \). Hence the Berry’s phase
\( \text{(10)} \) is transformed according to
\[
\gamma = i\hbar \int dt < \phi' | \partial \phi'/\partial z' > - i\hbar \int dt z_0 < \phi' | \partial \phi'/\partial z' > , \tag{14}
\]
where \( \phi' = e^{ikz} u' \). Using \( \text{(13)} \) and \( \int f(t) \ dt = 0 \), we get
\[
\gamma = i\hbar \int dt < u' | \partial u'/\partial t > - i\hbar \int dt z_0 < \phi' | \partial \phi'/\partial z' > , \tag{15}
\]
The scalar products in the above integral will be calcu-
lated in the stationary lattice frame. Let us study the
Berry’s phase \( \text{(15)} \) in detail. There are two terms in the
Berry’s phase. The first term appears because the quasi-
momentum \( k \) is made to vary across the Brillouin zone.
The first term exists even if the optical lattice is station-
ary. It was studied before and known as Zak’s phase \( \text{[28]} \).
The value of the Zak’s phase depends upon the symme-
try of the lattice. It is usually found to be either zero or
\( \pi \) in the presence of inversion symmetry. Defining
\[
q_n = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} \chi_{nn}(k) \ dk . \tag{16}
\]
where
\[
\chi_{nn}(k) = i\hbar \frac{2\pi}{a} \int_0^a u_{nk}^* \partial u_{nk} / \partial k \ dx . \tag{17}
\]
the Zak’s phase is given by \( \text{[28]} \)
\[
\gamma_{Zak} = i\hbar \int dt < \phi' | \partial \phi'/\partial k > = \frac{2\pi}{a} q_n . \tag{18}
\]
The second term in \( \text{(15)} \) is due to the fact that the optical
lattice is oscillating in time. It can be further simplified
if we use the relation; \(-i\hbar < \phi' | \partial \phi'/\partial t > = p'_{A} \), where \( p'_{A} \)
is the wavepacket momentum for atoms in the stationary
frame; \( p'_{A} = \partial E'/\partial k \). The rate of change of the quasi
momentum is given by the external force \( \text{(13)} \), while the rate
of change of the momentum is given by the total force
including the influence of the periodic potential. Using
\( p'_{A} = mv'_{L} \) and \( v_{L} = z_0 \), where \( v_{L} \) is the velocity of
the optical lattice, the equation \( \text{(15)} \) is reduced to
\[
\gamma = \gamma_{Zak} + m \int dt v_{L} v'_{A} . \tag{19}
\]
It is the Berry’s phase for the Hamiltonian \( \text{(6)} \). The addi-
tional Berry’s phase due to the oscillation of the optical
lattice is equal to the time integral of the momentum of
atoms times the velocity of the optical lattice. This
second term is, in general, nonzero as can be seen for a
specific example given below.
As an example, let us suppose that the force is equal
to a constant, \( f(t) = f_0 = mg \), where \( m \) and \( g \) are mass
and the gravitational acceleration, respectively. Such a
constant force always exists in the optical lattice because
the atoms feels the gravitational force. The solution of
\( \text{(13)} \) gives the time evolution of the quasimomentum. It
is given by
\[
k(t) = k_0 + \frac{f_0}{\hbar} t , \tag{20}
\]
where \( k_0 \) is the initial value of the quasimomentum. The
quasi momentum varies linearly with time for the con-
stant force. The period of the motion in \( k \)-space is given
by \( T = 2\pi\hbar / a f_0 \), where \( a \) is the lattice constant.
Suppose now that the optical lattice is periodically
kicked, that is
\[
z_0(t) = L \sin \omega t . \tag{21}
\]
where the amplitude \( L \) is a constant and the angular fre-
cquency \( \omega = a f_0 / \hbar \) ( the period of the oscillation is equal
to the period of the motion in \( k \)-space). The system
comes to its initial position after a period \( T \). The veloci-
ty of the optical lattice is \( v_{L} = \omega L \cos \omega t \).
We need the momentum, \( p'_{A} \), to calculate the Berry’s phase analytically \( \text{(19)} \). Let us use the tight binding
approximation. In this case, the energy is given by $E' = E_0 - 2\delta \text{cos } ka$, where $E_0$ and $\delta$ are constants. Under the action of the force, the particle state changes continually, so does the momentum $p'_A = -2a\delta \text{sin } ka$. If we use the relation (20), we get $p'_A = -2a\delta \text{sin } (k_0 a + \omega t)$. Substitution of $v_L$ and $p'_A$ expressions into the equation (19) yields 

$$\gamma = \gamma_{\text{Zak}} + \gamma_{\text{osc}}. \quad (22)$$

where $\gamma_{\text{osc}}$ is the Berry’s phase due to the oscillation of the optical lattice

$$\gamma_{\text{osc}} = -2\pi m\delta a \sin(ka) \quad (23)$$

This is the Berry’s phase for the constant force in a periodically kicked tight-binding optical lattice. When there is no oscillation, $L \to 0$, the Berry’s phase is just equal to the Zak’s phase. In the rocking optical lattice, the Bloch state picks up an additional phase (23). The Berry’s phase, which increases linearly with the amplitude $L$, can be controlled by changing the amplitude of the oscillation. However, it does not depend on how $z_0(t)$ changes in time. It depends on the geometry: the lattice length $a$, the amplitude $L$ and the initial value of the quasi-momentum $k_0$.

We would like to point out that different energy bands have different Berry’s phases. In principle, any interference experiment with neutral atoms for two such energy bands enables one to measure the phase difference. One can eliminate the Zak’s phase by using two independent Bose-Einstein condensates of the same atomic species and the same band indexes [31]. In this case, Zak’s phases for the two independent condensates are the same, while the Berry’s phases due to the oscillation, $\gamma_{\text{osc}}$, are, in general, different. Particularly, if the two independent condensates are oscillated with different amplitudes, the phase difference due to the oscillation of the lattice can be measured. So, the theoretical results for $\gamma_{\text{osc}}$ can be checked by such an interference experiment. Particularly, $^{88}\text{Sr}$ atoms is a good choice for such an experiment because it has remarkably small atom-atom interactions and in the ground state it has zero orbital, spin and nuclear angular momentum.

We would like to prove that the Berry’s phase (22) is gauge invariant. By a gauge transformation, $\Psi = e^{-iA z}/\hbar \psi$, where $A = f_{0}\delta t$ is the gauge potential, the equation (7) is transformed to

$$i\hbar \frac{\partial \psi}{\partial t} = \left( \frac{(p - A)^2}{2m} + V_0 \cos \{2kL \ (z - z_0(t))\} \right) \psi. \quad (24)$$

Therefore the Berry’s phase (10) is transformed according to

$$\gamma' = \gamma + f_0 \oint dt < z > \quad (25)$$

where $\gamma'$ is the transformed Berry’s phase under the gauge transformation and $< z > = < \phi | z | \phi >$. Using the relation $< z > = < z' > + z_0$, where $\frac{d < z' >}{dt} = mv'_A$ and the expression for $z_0(t)$ (21), one can readily prove that the second term in the right hand side of (24) vanishes. Hence,

$$\gamma' = \gamma. \quad (26)$$

The gauge invariance of the Berry’s phase guarantees that it can be observable.

In conclusion, we have shown that there exists non-vanishing Berry’s phase for the rocking optical lattice with the linear potential. The Berry’s phase consists of the two terms, $\gamma_{\text{Zak}}$ and $\gamma_{\text{osc}}$. The first one was already studied by Zak. In this case, the parameter space (the Brillouin zone) exists naturally [28]. The second one, $\gamma_{\text{osc}}$, arises due to the oscillation of the optical lattice. As an example, the Berry’s phase for ultracold atoms in a periodically kicked tight-binding optical lattice is calculated analytically. It is found that the Berry’s phase $\gamma_{\text{osc}}$ can be controlled by the amplitude of the oscillation. However, the frequency of the oscillation has no effect on it. We have also discussed that the Berry’s phase can be measured with the current technology by an interference experiment with two independent optical lattices. Here, our investigation is restricted to the one dimension. This is not only for simplification. The first example that Berry’s phase appears in one dimension was given by Zak. Here, another example in one dimension has been given. The generalization of the results to three dimensional systems is straightforward. Finally, we have proven that the Berry’s phase remains invariant under a gauge transformation.

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