Strong Insertion of a Contra-Continuous Function between Two Comparable Contra-Precontinuous (Contra-Semi–Continuous) Functions

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Abstract: Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.

1. INTRODUCTION

The concept of a preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [4]. A subset $A$ of a topological space $(X, \tau)$ is called preopen or locally dense or nearly open if $A \subseteq \text{Int}(\text{Cl}(A))$. A set $A$ is called preclosed if its complement is preopen or equivalently if $\text{Cl}(\text{Int}(A)) \subseteq A$.

The term preopen, was used for the first time by A.S. Mashhour, M.E. Abd El Monsef and S.N. El-Deeb [20], while the concept of a locally dense, set was introduced by H.H. Corson and E. Michael [4].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [17]. A subset $A$ of a topological space $(X, \tau)$ is called semiopen [10] if $A \subseteq \text{Cl}(\text{Int}(A))$. A set $A$ is called semiclosed if its complement is semi-open or equivalently if $\text{Int}(\text{Cl}(A)) \subseteq A$.

A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V$–sets. Complements of $V$–sets, i.e., sets that are intersection of open sets are called $A$–sets [19].

Recall that a real-valued function $f$ defined on a topological space $X$ is called $A$–continuous [25] if the preimage of every open subset of $R$ belongs to $A$, where $A$ is a collection of subsets of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A$–continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contracontinuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 12, 13, 24].

Hence, a real-valued function $f$ defined on a topological space $X$ is called contra-continuous (resp. contra-semi–continuous, contra-precontinuous) if the preimage of every open subset of $R$ is closed (resp. semi–closed, preclosed) in $X$[6].

Results of Kat’etov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable realvalued functions on such topological spaces that $A$–sets or kernel of sets are open [19].

If $g$ and $f$ are real-valued functions defined on a space $X$, we write $g \leq f$ in case $g(x) \leq f(x)$ for all $x$ in $X$.

The following definitions are modifications of conditions considered in [16].

A property $P$ defined relative to a real-valued function on a topological space is a $cc$–property provided that any constant function has property $P$ and provided that the sum of a function with property $P$ and...
any contraccontinuous function also has property \( P \). If \( P_1 \) and \( P_2 \) are \( cc \)-properties, the following terminology is used:(i) A space \( X \) has the weak \( cc \)-insertion property for \((P_1, P_2)\) if and only if for any functions \( g \) and \( f \) on \( X \) such that \( g \leq f, g \) has property \( P_1 \) and \( f \) has property \( P_2 \), then there exists a contraccontinuous function \( h \) such that \( g \leq h \leq f.\)(ii) A space \( X \) has the strong \( cc \)-insertion property for \((P_1, P_2)\) if and only if for any functions \( g \) and \( f \) on \( X \) such that \( g \leq f, g \) has property \( P_1 \) and \( f \) has property \( P_2 \), then there exists a contrac-continuous function \( h \) such that \( g \leq h \leq f \) and if \( g(x) < f(x) \) for any \( x \) in \( X \), then \( g(x) < h(x) < f(x) \).

In this paper, for a topological space whose \( \Lambda \)-sets or kernel of sets are open, is given a sufficient condition for the weak \( cc \)-insertion property.

Also for a space with the weak \( cc \)-insertion property, we give necessary and sufficient conditions for the space to have the strong \( cc \)-insertion property. Several insertion theorems are obtained as corollaries of these results.

2. THE MAIN RESULT

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.

The abbreviations \( cc \), \( cpc \) and \( csc \) are used for contra-continuous, contraprecontinuous and contra-semi-continuous, respectively.

**Definition 2.1.** Let \( A \) be a subset of a topological space \((X, \tau)\). We define the subsets \( A^\wedge \) and \( A^V \) as follows:

\[ A^\wedge = \cap \{ O : O \supseteq A, O \in (X, \tau) \} \quad \text{and} \quad A^V = \cup \{ F : F \subseteq A, F \in (X, \tau) \}. \]

In [7, 18, 23], \( A^\wedge \) is called the *kernel* of \( A \).

The family of all preopen, preclosed, *semi*–open and *semi*–closed will be denoted by \( pO(X, \tau), pC(X, \tau), sO(X, \tau) \) and \( sC(X, \tau) \), respectively.

We define the subsets \( p(A^\wedge), p(A^V), s(A^\wedge) \) and \( s(A^V) \) as follows: \( p(A^\wedge) = \cap \{ O : O \supseteq A, O \in pO(X, \tau) \}, \)

\[ p(A^V) = \cup \{ F : F \subseteq A, F \in pC(X, \tau) \}, \quad s(A^\wedge) = \cap \{ O : O \supseteq A, O \in sO(X, \tau) \} \quad \text{and} \quad s(A^V) = \cup \{ F : F \subseteq A, F \in \}

\[ sC(X, \tau) \}. \]

\( p(A^\wedge) \) (resp. \( s(A^\wedge) \)) is called the *prekernel* (resp. *semi*–kernel) of \( A \).

The following first two definitions are modifications of conditions considered in [14, 15].

**Definition 2.2.** If \( \rho \) is a binary relation in a set \( S \) then \( \rho^- \) is defined as follows: \( x \rho^- y \) if and only if \( y \rho \) \( v \) implies \( x \rho v \) and \( u \rho x \) implies \( u \rho y \) for any \( u \) and \( v \) in \( S \).

**Definition 2.3.** A binary relation \( \rho \) in the power set \( P(X) \) of a topological space \( X \) is called a *strong binary relation* in \( P(X) \) in case \( \rho \) satisfies each of the following conditions:

- If \( A, B \in P(X) \), for any \( i \in \{1,...,r\} \) and for any \( j \in \{1,...,n\} \), then there exists a set \( C \in P(X) \) such that \( A \rho C \) and \( C \rho B \) for any \( i \in \{1,...,r\} \) and any \( j \in \{1,...,n\} \).
- If \( A \subseteq B \), then \( A \rho^- B \).
- If \( A \rho B \), then \( A^\wedge \subseteq B \) and \( A \subseteq B^V \).

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 2.4.** If \( f \) is a real-valued function defined on a space \( X \) and if \( \{ x \in X : f(x) < \ell \} \subseteq A(f, \ell) \subseteq \{ x \in X : f(x) \leq \ell \} \) for a real number \( \ell \), then \( A(f, \ell) \) is called a *lower indefinite cut set* in the domain of \( f \) at the level \( \ell \).

We now give the following main result:

**Theorem 2.1.** Let \( g \) and \( f \) be real-valued functions on the topological space \( X \), in which kernel sets are open, with \( g \leq f \). If there exists a strong binary relation \( \rho \) on the power set of \( X \) and if there exist lower indefinite cut sets \( A(f, t_i) \) and \( A(g, t_j) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_i < t_j \) then \( A(f, t_i) \rho A(g, t_j) \), then there exists a contra-continuous function \( h \) defined on \( X \) such that \( g \leq h \leq f \). **Proof.** Theorem 2.1. of [22].

**Theorem 2.2.** Let \( P_1 \) and \( P_2 \) be \( cc \)-property and \( X \) be a space that satisfies the weak \( cc \)-insertion property for \((P_1, P_2)\). Also assume that \( g \) and \( f \) are functions on \( X \) such that \( g \leq f, g \) has property \( P_1 \) and \( f \) has property \( P_2 \). The space \( X \) has the strong \( cc \)-insertion property for \((P_1, P_2)\) if and only if there exist
lower cut sets \( A(f - g, 2^{-n}) \) and there exists a sequence \( \{ H_n \} \) of subsets of \( X \) such that (i) for each \( n \), \( H_n \) and \( A(f - g, 2^{-n}) \) are completely separated by contra-continuous functions, and (ii) \( \{ x \in X : (f - g)(x) > 0 \} = \bigcup_{n=1}^{\infty} H_n \).

**Proof.** Theorem 3.1, of [21].

**Theorem 2.3.** Let \( P_1 \) and \( P_2 \) be cc—properties and assume that the space \( X \) satisfied the weak cc—insertion property for \((P_1, P_2)\). The space \( X \) satisfies the strong cc—insertion property for \((P_1, P_2)\) if and only if \( X \) satisfies the strong cc—insertion property for \((P_1, cc)\) and for \((cc, P_2)\). **Proof.** Theorem 3.2, of [21].

### 3. Applications

Before stating the consequences of Theorems 2.1, 2.2, and 2.3 we suppose that \( X \) is a topological space whose kernel sets are open.

**Corollary 3.1.** If for each pair of disjoint preopen (resp. semi—open) sets \( G_1, G_2 \) of \( X \), there exist closed sets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1 \), \( G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \) then \( X \) has the weak cc—insertion property for \((cpc, cpc)\) (resp. \((csc, csc)\)).

**Proof.** Corollary 3.1, of [22].

**Corollary 3.2.** If for each pair of disjoint preopen (resp. semi—open) sets \( G_1, G_2 \), there exist closed sets \( F_1 \) and \( F_2 \) such that \( G_1 \subseteq F_1 \), \( G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \) then every contra-precontinuous (resp. contra-semi—continuous) function is contra-continuous.

**Proof.** Corollary 3.2, of [22].

**Corollary 3.3.** If for each pair of disjoint preopen (resp. semi—open) sets \( G_1, G_2 \) of \( X \), there exist closed sets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1 \), \( G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \) then \( X \) has the cc—insertion property for \((cpc, cpc)\) (resp. \((csc, csc)\)).

**Proof.** Corollary 3.3, of [22].

**Corollary 3.4.** If for each pair of disjoint subsets \( G_1, G_2 \) of \( X \), such that \( G_1 \) is preopen and \( G_2 \) is semi—open, there exist closed subsets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1 \), \( G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \) then \( X \) have the weak cc—insertion property for \((cpc, csc)\) and \((csc, cpc)\).

**Proof.** Corollary 3.4, of [22].

Before stating consequences of Theorem 2.2, 2.3 we state and prove the necessary lemmas.

**Lemma 3.1.** The following conditions on the space \( X \) are equivalent:

- For each pair of disjoint subsets \( G_1, G_2 \) of \( X \), such that \( G_1 \) is preopen and \( G_2 \) is semi—open, there exist closed subsets \( F_1, F_2 \) of \( X \) such that \( G_1 \subseteq F_1 \), \( G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \).

- If \( G \) is a semi—open (resp. preopen) subset of \( X \) which is contained in a preclosed (resp. semi—closed) subset \( F \) of \( X \), then there exists a closed subset \( H \) of \( X \) such that \( G \subseteq H \subseteq H^c \subseteq F \).

**Proof.** Lemma 3.1, of [22].

**Lemma 3.2.** Suppose that \( X \) is a topological space. If each pair of disjoint subsets \( G_1, G_2 \) of \( X \), where \( G_1 \) is preopen and \( G_2 \) is semi—open, can be separated by closed subsets of \( X \) then there exists a contra-continuous function \( h : X \rightarrow [0, 1] \) such that \( h(G_2) = \{0\} \) and \( h(G_1) = \{1\} \). **Proof.** Lemma 3.2, of [22].

**Lemma 3.3.** Suppose that \( X \) is a topological space. If each pair of disjoint subsets \( G_1, G_2 \) of \( X \), where \( G_1 \) is preopen and \( G_2 \) is semi—open, can separate by closed subsets of \( X \), and \( G_1 \) (resp. \( G_2 \)) is a closed subsets of \( X \) then there exists a contra-continuous function \( h : X \rightarrow [0, 1] \) such that \( h^{-1}(0) = G_1 \) (resp. \( h^{-1}(0) = G_2 \)) and \( h(G_2) = \{1\} \) (resp. \( h(G_1) = \{1\} \)).
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Proof. Suppose that $G_1$ (resp. $G_2$) is a closed subset of $X$. By Lemma 3.2, there exists a contra-continuous function $h : X \to [0,1]$ such that, $h(G_1) = \{0\}$ (resp. $h(G_2) = \{1\}$) and $h(X \setminus G_1) = \{1\}$ (resp. $h(X \setminus G_2) = \{0\}$). Hence, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and since $G_2 \subseteq X \setminus G_1$ (resp. $G_1 \subseteq X \setminus G_2$), therefore $h(G_2) = \{1\}$ (resp. $h(G_1) = \{0\}$).

Lemma 3.4. Suppose that $X$ is a topological space such that every two disjoint semi–open and preopen subsets of $X$ can be separated by closed subsets of $X$. The following conditions are equivalent:

- For every two disjoint subsets $G_1$ and $G_2$ of $X$, where $G_1$ is preopen and $G_2$ is semi–open, there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and $h^{-1}(1) = G_2$ (resp. $h^{-1}(1) = G_1$).
- Every preopen (resp. semi–open) subset of $X$ is a closed subsets of $X$.
- Every preclosed (resp. semi–closed) subset of $X$ is an open subsets of $X$.

Proof. (i) ⇒ (ii) Suppose that $G$ is a preopen (resp. semi–open) subset of $X$. Since $\emptyset$ is a semi–open (resp. preopen) subset of $X$, by (i) there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G$. Set $F_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, $F_n$ is a closed subset of $X$ and \( \bigcap_{n=1}^{\infty} F_n = \{x \in X : h(x) = 0\} = G \).

(ii) ⇒ (i) Suppose that $G_1$ and $G_2$ are two disjoint subsets of $X$, where $G_1$ is preopen and $G_2$ is semi–open. By Lemma 3.3, there exists a contra-continuous function $f : X \to [0,1]$ such that, $f^{-1}(1) = G_1$ and $f(G_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two open subsets of $X$ and $(G \cup F) \cap (H \cup F) = \emptyset$. By Lemma 3.3, there exists a contra-continuous function $g : X \to [\frac{1}{2},1]$ such that, $g^{-1}(1) = G_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define $h$ by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then $h$ is welldefined and a contra-continuous function, since $(G \cup F) \cap (H \cup F) = \emptyset$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence $h$ defined on $X$ and maps to $[0,1]$. Also, we have $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$.

(ii) ⇔ (iii) By De Morgan law and noting that the complement of every open subset of $X$ is a closed subset of $X$ and complement of every closed subset of $X$ is an open subset of $X$, the equivalence is hold.

Corollary 3.5. If for every two disjoint subsets $G_1$ and $G_2$ of $X$, where $G_1$ is preopen (resp. semi–open) and $G_2$ is semi–open (resp. preopen), there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$ then $X$ has the strong cc–insertion property for $(cpc,cc)$ (resp. $(csc,csc)$).

Proof. Since for every two disjoint subsets $G_1$ and $G_2$ of $X$, where $G_1$ is preopen (resp. semi–open) and $G_2$ is semi–open (resp. preopen), there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, define $F_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $F_2 = \{x \in X : h(x) > \frac{1}{2}\}$.

Then $F_1$ and $F_2$ are two disjoint closed subsets of $X$ that contain $G_1$ and $G_2$, respectively. Hence by Corollary 3.4, $X$ has the weak cc–insertion property for $(cpc,cc)$ and $(csc,csc)$. Now, assume that $g$ and $f$ are functions on $X$ such that $g \leq f$ is $cpc$ (resp. $csc$) and $f$ is $cc$. Since $f - g$ is $cpc$ (resp. $csc$), therefore the lower cut set $A(f-g,2^n) = \{x \in X : (f-g)(x) \leq 2^n\}$ is a preopen (resp. semi–open) subset of $X$. Now setting $H_n = \{x \in X : (f-g)(x) > 2^n\}$ for every $n \in \mathbb{N}$, then by Lemma 3.4, $H_n$ is an open subset of $X$ and we have $\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} H_n$ and for every $n \in \mathbb{N}$, $H_n$ and $A(f-g,2^n)$ are disjoint subsets of $X$. By Lemma 3.2, $H_n$ and $A(f-g,2^n)$ can be completely separated by contra-continuous functions. Hence by Theorem 2.2, $X$ has the strong cc–insertion property for $(cpc,cc)$ (resp. $(csc,cc)$).

By an analogous argument, we can prove that $X$ has the strong cc–insertion property for $(cc,csc)$ (resp. $(cc,cpc)$). Hence, by Theorem 2.3, $X$ has the strong cc–insertion property for $(cpc,csc)$ (resp. $(csc,cpc)$).

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