Remarks on the density of the law of the occupation time for Bessel bridges and stable excursions

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February 1, 2008

Abstract

Smoothness and asymptotic behaviors are studied for the densities of the law of the occupation time on the positive line for Bessel bridges and the normalized excursion of strictly stable processes. The key role is played by these properties for functions defined by Riemann–Liouville fractional integrals.

1 Introduction

For a standard one-dimensional Brownian motion $X = (X(t))$, the occupation time $A_+ = \int_0^1 1_{\{X(s) > 0\}} \, ds$ has a density $f_+(x) = 1/\{\pi \sqrt{x(1-x)}\}$, which is well-known as Lévy’s arc-sine law. In his formula, we find out that the density $f_+(x)$ is smooth in $(0,1)$ and diverges at $x = 0+$ as $f_+(x) \sim (1/\pi)x^{-1/2}$. This implies the paradoxical fact that $A_+$ is more likely to take values near the extreme values $0$ and $1$ than near the central value $1/2$.

There have been a lot of attempts to study the law of the occupation time $A_+$ for other processes $X$. For Bessel processes of dimension $0 < d < 2$, Barlow–Pitman–Yor \cite{BPY} have found out that the occupation time $A_+$ has the same law as that introduced by Lamperti \cite{Lan}, and hence we see that the law has a density which is smooth in the interior of its domain and diverges at the origin of order $x^{-d/2}$.

For a Brownian bridge, it is also well-known as Lévy’s theorem that the law of the occupation time is uniform. For Bessel bridges, the second author \cite{Yan} has obtained an expression of the distribution function in terms of the Riemann–Liouville fractional integral. We encounter a similar situation in the case of the normalized excursion of a strictly stable process (we call it a stable excursion in short) whose law has been characterized by Fitzsimmons–Getoor \cite{FG}. These expressions are so implicit that it is worth discussing regularity and asymptotic behaviors of their densities.

In the present paper, we study the law of the occupation time for Bessel bridges and for stable excursions. We show that the distribution functions are smooth in the interior of the domain and determine the asymptotic behaviors of their densities at the extreme values. For these purposes, we study these properties for functions which are expressed by the Riemann–Liouville fractional integrals. Although the arguments are rather elementary, we could not find the results in the literature, and so it is still useful to the readers to deal with them to the extent which we need for our purposes.

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According to Theorem 2.1, which is one of our main theorems, the law of the occupation time $A_+$ for a Bessel bridge $X$ of dimension $0 < d < 2$ has a smooth density $f_+(x)$ such that $f_+(x) \sim Cx^{1-d}$ as $x \to 0+$ for some constant $C$, which generalizes Lévy’s theorem for a Brownian bridge to Bessel bridges. We remark that the density $f_+(x)$ diverges at $x = 0+$ when $1 < d < 2$ while it vanishes there when $0 < d < 1$.

The present paper is organized as follows. In Section 2, we give a brief summary of the preceding results about the law of the occupation time for several processes. We also state our main theorems there. In Section 3, we give the precise definition of the Riemann–Liouville fractional integrals for a certain class of functions. We prove smoothness of our distribution functions in the end of that section. Section 4 is devoted to the proof of asymptotic behaviors of our distribution functions.

2 Backgrounds and Main theorems

Throughout the present paper, we suppose that the process considered which we will denote by $X = (X(t) : 0 \leq t \leq 1)$ takes values in $\mathbb{R}$ and starts from 0 and we will denote its occupation time up to time 1 on the positive line by $A_+ = A_+(1) = \int_0^1 1_{\{X(s) > 0\}}ds$. By a density of the law of $A_+$ we always mean one with respect to the Lebesgue measure and will denote it by $f_+$. For two functions $f, g$ defined on $(0, \varepsilon)$ for some $\varepsilon > 0$ such that $f$ and $g$ does not vanish on $(0, \varepsilon)$, we say that $f(x) \sim g(x)$ if $\lim_{x \to 0^{+}} f(x)/g(x) = 1$.

1°). When $X$ is a standard Brownian motion, Lévy’s arc-sine law (see, e.g., [10, pp. 57]) asserts that $P(A_+ \leq x) = \frac{2}{\pi} \sin^{-1} \sqrt{x}$, $0 \leq x \leq 1$, i.e.,

$$f_+(x) = \frac{1}{\pi \sqrt{x(1-x)}}, \quad 0 < x < 1.$$  

Here we remark that (i) the density $f_+$ is smooth in $(0, 1)$, and (ii) it has an asymptotic $f_+(x) \sim \frac{1}{\pi} x^{-1/2}$ as $x \to 0+$.

When $X$ is a skew Bessel processes of dimension $0 < d < 2$ with skewness parameter $0 < p < 1$, Barlow–Pitman–Yor [11] (see also Watanabe [14]) have found out that $A_+$ has the same law as a random variable $Y_{\alpha,p}$ for $\alpha = 1 - d/2 \in (0, 1)$ whose law is characterized by the Stieltjes transform:

$$E \left[ \frac{1}{\lambda + Y_{\alpha,p}} \right] = \frac{p(\lambda + 1)^{\alpha-1} + (1 - p)\lambda^{\alpha-1}}{p(\lambda + 1)^{\alpha} + (1 - p)\lambda^{\alpha}}, \quad \lambda > 0.$$  

The class $Y_{\alpha,p}$ of random variables was introduced by Lamperti [13] as the possible limit distributions of the occupation time of random walks (see also Fujihara–Kawamura–the second author [17]). By inverting the Stieltjes transform, we see that the law of $Y_{\alpha,p}$ has a density $f_+ = f_{\alpha,p}$ which is given by

$$f_{\alpha,p}(x) = \frac{\sin \alpha \pi}{\pi} \cdot \frac{p(1-p)x^{\alpha-1}(1-x)^{\alpha-1}}{p^2(1-x)^{2\alpha} + (1-p)^2x^{2\alpha} + 2p(1-p)x^{\alpha}(1-x)^{\alpha} \cos \alpha \pi}$$  

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for $0 < x < 1$. Here we also remark that (i) the density $f_{\alpha,p}$ is smooth in $(0,1)$, and (ii) it has an asymptotic

\begin{equation}
(2.4) \quad f_{\alpha,p}(x) \sim \frac{\sin \alpha \pi}{\pi} \cdot \frac{1 - p}{p} \cdot x^{\alpha - 1} \quad \text{as } x \to 0^+.
\end{equation}

For general diffusion processes $X$, Kasahara–the second author [11] have studied the asymptotic behavior of the distribution function of $A_+$ under a certain regular variation assumption on the speed measure at the origin. Motivated by the result, Watanabe–the authors [15] have proved that the law of $A_+$ has a density $f_{\alpha,p}$ in quite a general case and also proved that the density is continuous in $(0,1)$ and has an asymptotic $f_{\alpha,p}(x) \sim f_{\alpha,p}(x)$ as $x \to 0^+$ for some $\alpha, p$ under the regular variation assumption.

We also remark on a mysterious resemblance between the law of $Y_{\alpha,1/2}$ and that of $J_\alpha$ for $0 < \alpha < 1$ which is characterized by its Stieltjes transform as

\begin{equation}
(2.5) \quad E \left[ \frac{1}{\lambda + J_\alpha} \right] = \frac{\alpha}{1 - \alpha} \cdot \frac{\lambda^{\alpha - 1} - (1 + \lambda)^{\alpha - 1}}{(1 + \lambda)^\alpha - \lambda^\alpha}, \quad \lambda > 0
\end{equation}

and which has a density $f_{J_\alpha}$ on $(0,1)$ with respect to the Lebesgue measure given by

\begin{equation}
(2.6) \quad f_{J_\alpha}(x) = \frac{\sin \alpha \pi}{\pi} \cdot \frac{\alpha}{1 - \alpha} \cdot \frac{x^{\alpha - 1} (1 - x)^{\alpha - 1}}{x^{2\alpha} + (1 - x)^{2\alpha} - 2x^{\alpha} (1 - x)^\alpha \cos \alpha \pi}
\end{equation}

for $0 < x < 1$. The class of random variables $J_\alpha$ has been introduced by Bertoin–Fujita–Roynette–Yor [2, Theorem 1.1] to characterize the Lévy measure of the duration of the excursion straddling an independent standard exponential time for a Bessel process of dimension $d = 2 - 2\alpha$. They also introduced a two-parameter family of random variables $J_{\alpha,\beta}$ such that $J_{\alpha,\alpha} \overset{\text{law}}{=} J_\alpha$ and $J_{\alpha,1-\alpha} \overset{\text{law}}{=} Z_\alpha$ where $Z_\alpha$ will be defined below.

In what follows, by a bridge process we mean the process obtained from a process $Y$ for which all points are regular by conditioning on $Y(1) = 0$. We refer to [6] for the precise definition.

When $X$ is the bridge process of a Brownian motion (or a Brownian bridge), Lévy’s theorem (see, e.g., [10, pp. 58]) asserts that the law of $A_+$ is uniform on $(0,1)$. We point out that the law of $A_+$ has a constant density, which is a completely different situation from that of a Brownian motion. This result has been generalized to the bridge process of a Lévy process (or a Lévy bridge) for which all points are regular, by Fitzsimmons–Getoor [5] and Knight [12] independently who showed that the law of $A_+$ is uniform on $(0,1)$.

Contrary to the case of Lévy bridges, we encounter a drastically different situation in the case of diffusion bridges. When $X$ is the bridge process of a skew Bessel process (or a skew Bessel bridge) of dimension $0 < d < 2$ with skewness parameter $0 < p < 1$, the second author [16, Theorem 3.1] has proved that the distribution function of $A_+$ coincides with $G_{1-2/d,p}(x)$ where $G_{\alpha,p}(x)$ for $0 < \alpha < 1$ is a function on $(0,1)$ characterized by its generalized Stieltjes transform of index $\alpha$ as

\begin{equation}
(2.7) \quad \int_0^1 \frac{dG_{\alpha,p}(x)}{(\lambda + x)^\alpha} = \frac{1}{p(1 + \lambda)^\alpha + (1 - p)\lambda^\alpha}, \quad \lambda > 0.
\end{equation}
Inverting the transform in the formula (2.7), she obtained the following expression of $G_{\alpha,p}$ in terms of the Riemann–Liouville fractional integral (see [16, Theorem 4.1]):

\begin{equation}
G_{\alpha,p}(x) = \int_0^x (x-t)^{\alpha-1} g_{\alpha,p}(t) dt, \quad 0 \leq x \leq 1
\end{equation}

where

\begin{equation}
g_{\alpha,p}(t) = \frac{\sin \alpha \pi}{\pi} \cdot \frac{(1-p)t^{\alpha}}{p^2(1-t)^{2\alpha} + (1-p)^2 t^{2\alpha} + 2p(1-p)t^{\alpha}(1-t)^{\alpha} \cos \alpha \pi}
\end{equation}

for $0 < t < 1$. From (2.8) and Lemma 3.2, it follows that the density $f_+$ is given by

\begin{equation}
G_{\alpha,p}'(x) = \int_0^x (x-t)^{\alpha-1} g_{\alpha,p}'(t) dt, \quad 0 < x < 1.
\end{equation}

We point out here that the integral $\int_0^x (x-t)^{\alpha-1} g_{\alpha,p}''(t) dt$ is meaningless because of the asymptotic $g_{\alpha,p}''(t) \sim C_1 t^{\alpha-2}$ as $t \to 0^+$ for some constant $C_1$. Nevertheless, the first one of our main theorems is

**Theorem 2.1.** The distribution function $G_{\alpha,p}(x)$ is infinitely differentiable in $(0,1)$ and its derivative has an asymptotic

\begin{equation}
G_{\alpha,p}'(x) \sim \frac{\sin \alpha \pi}{\pi} \cdot \frac{1-p}{p^2} \cdot \frac{\alpha \Gamma(\alpha)^2}{\Gamma(2\alpha)} x^{2\alpha-1} \quad \text{as } x \to 0^+.
\end{equation}

The proof of Theorem 2.1 will be given in Sections 3 and 4. We remark that the asymptotic behavior (2.11) of the density function $G_{\alpha,p}'(x)$ generalizes the asymptotic result [16, pp.795] of the distribution function $G_{\alpha,p}(x)$.

3°). When $X$ is a Lévy process such that $P(X(t) > 0)$ for $t > 0$ is a constant $0 < c < 1$, Getoor–Sharpe [9] has proved that $A_+$ has the same law as a random variable $Z_c$ whose law has a density $f_{Z_c}$ given by

\begin{equation}
f_{Z_c}(x) = \frac{\sin c \pi}{\pi} \cdot x^{c-1}(1-x)^{-c}, \quad 0 < x < 1.
\end{equation}

It is proved by Getoor–Sharpe [8] and Bertoin–Yor [3] that the occupation time of such a process $X$ time-changed by the inverse local time of an independent Markov process has the same law as $Z_c$.

We encounter a drastically different situation again in the case of Lévy excursions. Suppose that $X$ obeys the conditional law of the excursion measure of an $\alpha$-stable process of index $1 < \alpha < 2$ given that the lifetime equals one. The law is that of the normalized excursion and is $P^*_1$ in the notation of Fitzsimmons–Getoor [5] to which we refer for the details. Then Fitzsimmons–Getoor [5, eq. (4.26)] has proved the distribution function of $A_+$ which we denote by $H_{1/\alpha}(x)$ where $H_{1/\gamma}$ for $1/2 < \gamma < 1$ is a function characterized by its generalized Stieltjes transform of index $\gamma - 1$ as

\begin{equation}
\int_0^1 (\lambda + x)^{1-\gamma} dH_{\gamma}(x) = \frac{\gamma}{(\lambda + 1)^{\gamma} - \lambda^{\gamma}}, \quad \lambda > 0.
\end{equation}
Integrating by parts in the LHS of (2.13) and then using the inversion formula [16, Theorem 4.1], we obtain

\begin{equation}
\int_0^x H_\gamma(y)dy = \int_0^x (x-t)^{\gamma-1} h_\gamma(t)dt
\end{equation}

where

\begin{equation}
h_\gamma(t) = \frac{\sin \gamma \pi}{\pi} \cdot \frac{\gamma}{1-\gamma} \cdot \frac{t^\gamma}{(1-t)^{2\gamma} + t^{2\gamma} - 2t^\gamma (1-t)^\gamma \cos \gamma \pi}.
\end{equation}

We remark again on a mysterious resemblance between \(G_{\alpha,1/2}\) and \(H_\alpha\). From (2.14) and Lemma 3.2 it follows that the distribution function is given by

\begin{equation}
H_\gamma(x) = \int_0^x (x-t)^{\gamma-1} h'_\gamma(t)dt, \quad 0 < x < 1.
\end{equation}

We point out here again that the integral \(\int_0^x (x-t)^{\gamma-1} h''_\gamma(t)dt\) is meaningless because of the asymptotic \(h''_\gamma(t) \sim C_2 t^{-2}\) as \(t \to 0^+\) for some constant \(C_2\). Nevertheless, the second one of our main theorems is

**Theorem 2.2.** Let \(1/2 < \gamma < 1\). Then the function \(H_\gamma(x)\) is infinitely differentiable in \((0,1)\). In particular, its derivative is given by

\begin{equation}
H'_\gamma(x) = \frac{1}{x} \int_0^x (x-t)^{\gamma-1} \left\{ \gamma h'_\gamma(t) + t h''_\gamma(t) \right\} dt, \quad 0 < x < 1.
\end{equation}

and it has an asymptotic

\begin{equation}
H'_\gamma(x) \sim \frac{\sin \gamma \pi}{\pi} \cdot \frac{\Gamma(\gamma+1)^2}{\Gamma(2\gamma)} \cdot \frac{2\gamma-1}{1-\gamma} \cdot x^{2\gamma-2} \quad \text{as } x \to 0^+.
\end{equation}

The proof of Theorem 2.2 will be given in Sections 3 and 4.

### 3 Differentiability of Riemann–Liouville fractional integrals

We denote by \(C(0,1)\) (resp. \(C_0(0,1)\)) the class of continuous (resp. and bounded) functions on \((0,1)\), and by \(C^n(0,1)\) the class of \(n\)-times differentiable functions. We denote by \(L^1(0,r)\) for \(0 < r < 1\) the class of Borel measurable functions on \((0,1)\) which are integrable on \((0,r)\) with respect to the Lebesgue measure, and define \(L^1(0,1) = \cap_{0<r<1} L^1(0,r)\).

For \(\alpha > 0\), we define a linear operator \(I^\alpha\) on \(C_0(0,1)\) by

\begin{equation}
I^\alpha[f](x) = \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt, \quad 0 < x < 1, \quad f \in C_0(0,1)
\end{equation}

where the integral \(\int_0^x dt\) stands for \(\int_0^1 dt_1 f(t_1)(t)\) in the sense of the Lebesgue integral. We remark that, if \(\alpha\) is a positive integer \(\alpha = n\), then \(I^n[f]\) is the \(n\)-th multiple integral of \(f\).

It is obvious that

\begin{equation}
\int_0^r |I^\alpha[f](x)| dx \leq \frac{r^\alpha}{\Gamma(\alpha+1)} \int_0^r |f(x)| dx, \quad 0 < r < 1, \quad f \in C_0(0,1).
\end{equation}
Thus the operator $I^\alpha$ on $C_b(0,1)$ extends to a linear operator on $L^1(0,1)$ which is continuous when restricted on $L^1(0,r)$ for all $0 < r < 1$. We will denote the extension by the same symbol $I^\alpha$ and we call it the Riemann–Liouville fractional integral of order $\alpha$. The following lemma asserts that $I^\alpha$ preserves the space of continuous functions:

**Lemma 3.1.** Let $\alpha > 0$. For $f \in C_b(0,1)$ and for $0 < a < b < 1$, the inequality

\begin{equation}
|I^\alpha[f](x)| \leq C_{a,a,b}^1 \int_0^a |f(x)|dx + C_{a,a,b}^2 \sup_{x \in [a,b]} |f(x)|, \quad x \in (a,b)
\end{equation}

holds where $C_{a,a,b}^1 = \max\{(b-a)^{\alpha-1}, a^{\alpha-1}\}/\Gamma(\alpha)$ and $C_{a,a,b}^2 = (b-a)^\alpha/\Gamma(\alpha + 1)$. In particular, if $f \in L^1(0,1-) \cap C(0,1)$, then $I^\alpha[f]$ possesses a continuous version on $(0,1)$.

**Proof.** The inequality (3.3) is immediate from the following obvious inequality:

\begin{equation}
|I^\alpha[f](x)| \leq \int_0^\infty \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} |f(t)|dt + \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} |f(t)|dt
\end{equation}

for all $x \in (a,b)$.

Let $f \in L^1(0,1-) \cap C(0,1)$ and let $0 < a < b < 1$. Take a sequence $f_n \in C_b(0,1)$ which approximates $f$ in $L^1$ on the interval $(0,a)$ and uniformly on the interval $[a,b]$. Then, by the inequality (3.3), we see that $I^\alpha[f_n]$ converges to $I^\alpha[f]$ uniformly on each compact subset of $(a,b)$, which shows that $I^\alpha[f]$ possesses a continuous version on $(a,b)$. Since $a$ and $b$ are arbitrary, we obtain the second assertion. \(\square\)

If $f \in L^1(0,1-) \cap C(0,1)$, we always assume that $I^\alpha[f]$ stands for its continuous version. We know the well-known identity

\begin{equation}
I^\beta[I^\alpha[f]] = I^{\beta+\alpha}[f], \quad \alpha, \beta > 0, \ f \in L^1(0,1-).
\end{equation}

This fact implies the following immediately: If $f \in L^1(0,1-)$ and if $\alpha > 1$, then $I^\alpha[f]$ is differentiable and

\begin{equation}
\frac{d}{dx} I^\alpha[f](x) = I^{\alpha-1}[f](x)
\end{equation}

holds. As a sufficient condition for differentiability of $I^\alpha[f]$ which is valid for all $\alpha > 0$, the following lemma is also immediate:

**Lemma 3.2.** Let $\alpha > 0$. If $f \in C^1(0,1)$ and if $f' \in L^1(0,1-)$, then $I^\alpha[f] \in C^1(0,1)$ and its derivative is given by

\begin{equation}
\frac{d}{dx} I^\alpha[f](x) = \frac{f(0^+)}{\Gamma(\alpha)} x^{\alpha-1} + I^\alpha[f'](x), \quad 0 < x < 1.
\end{equation}

**Proof.** Since $f' \in L^1(0,1-)$, the right-hand limit $f(0^+)$ exists. Since $f(x) = f(0^+) + I^1[f'](x)$, we have

\begin{equation}
I^\alpha[f](x) = f(0^+)I^\alpha[1](x) + I^{\alpha+1}[f'](x)
\end{equation}

\begin{equation}
= f(0^+) \frac{x^{\alpha}}{\Gamma(\alpha + 1)} + \int_0^x I^\alpha[f'](t)dt.
\end{equation}

This proves (3.7). \(\square\)
We can apply Lemma 3.2 to \( f = g_{\alpha,p} \) (resp. \( f = h_{\gamma} \)) which is introduced in (2.9) (resp. (2.15)) and obtain (2.10) (resp. (2.16)). However, the integrability assumption of \( f' \) at the origin is not satisfied by \( f = g_{\alpha,p} \) nor by \( f = h_{\gamma} \) because \( g_{\alpha,p}''(t) \sim C_1 t^{\alpha-2} \) and \( h_{\gamma}''(t) \sim C_2 t^{\gamma-2} \) as \( t \to 0^+ \). We would like to relax the integrability assumption of \( f' \) at the origin for differentiability of \( I_\alpha[f] \).

For \( f \in C^1(0,1) \), we define

\[
(\delta f)(x) = xf'(x), \quad x \in (0,1).
\]

Now we obtain the key proposition as follows:

**Proposition 3.3.** Let \( \alpha > 0 \). Suppose that \( f \in C^1(0,1) \) and that \( \delta f \in L^1(0,1) \). Then \( f \in L^1(0,1) \) and \( I_\alpha[f] \in C^1(0,1) \). Moreover, the following relation holds:

\[
\delta(I_\alpha[f]) = I_\alpha[\alpha f + \delta f].
\]

**Proof.** For \( 0 < r < 1 \), we have

\[
\int_0^r |f(x)| \, dx = \int_0^r dx \left| f(r) - \int_x^r f'(t) \, dt \right|
\leq r |f(r)| + \int_0^r |\delta f(t)| \, dt < \infty.
\]

This proves that \( f \in L^1(0,1) \).

Set \( g(t) = tf(t) \). Then we have \( g \in L^1(0,1) \cap C^1(0,1) \) by the assumptions and we have \( g' = f + \delta f \). Since \( g \) satisfies the assumptions of Lemma 3.2 we see that \( I_\alpha[g] \in C^1(0,1) \) and that

\[
\frac{d}{dx}I_\alpha[g](x) = I_\alpha[f + \delta f](x).
\]

We note that

\[
\alpha I_\alpha^{\alpha+1}[f] = \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} (x-t)f(t) \, dt = xI_\alpha[f](x) - I_\alpha[g](x).
\]

Hence we have

\[
I_\alpha[f](x) = \frac{1}{x} \left\{ \alpha I_\alpha^{\alpha+1}[f](x) + I_\alpha[g](x) \right\}.
\]

Now we conclude that \( I_\alpha[f] \in C^1(0,1) \), and using (3.6), (3.14) and (3.15), we obtain

\[
\frac{d}{dx}I_\alpha[f](x) = \frac{1}{x} I_\alpha[\alpha f + \delta f](x).
\]

This completes the proof. \( \square \)

Immediately from Proposition 3.3 we obtain the following
Corollary 3.4. Let $\alpha > 0$. Suppose that $f \in C^n(0,1)$ and $\delta^n f \in L^1(0,1-)$ for all $n \geq 1$. Then $I^\alpha[f] \in C^n(0,1)$ for all $n \geq 1$ and its $n$-th derivative for each $n \geq 1$ is given by

\begin{equation}
(3.18) \quad \frac{d^n}{dx^n} I^\alpha[f](x) = \frac{1}{x^n} I^\alpha[p_n(\alpha)(\delta^n f)(x)]
\end{equation}

where $p_n(t) = t(t-1) \cdots (t-n+1)$.

Proof. We prove the assertion by induction. The assertion for $n = 1$ is nothing but Proposition 3.3. Suppose that we have $I^\alpha[f] \in C^n(0,1)$ and (3.18) for a fixed $n \geq 1$. We note that $p_n(\alpha + \delta)f \in C^1(0,1)$ and $\delta p_n(\alpha + \delta)f \in L^1(0,1-)$ by the assumptions that $f \in C^n(0,1)$ and $\delta^n f \in L^1(0,1-)$ for all $n \geq 1$. Hence it follows from Proposition 3.3 that $I^\alpha[p_n(\alpha + \delta)f] \in C^1(0,1)$ and we have

\begin{equation}
(3.19) \quad \delta I^\alpha[p_n(\alpha + \delta)f] = I^\alpha[(\alpha + \delta)p_n(\alpha + \delta)f].
\end{equation}

Differentiating both sides of (3.19) in $x$, we obtain

\begin{align}
(3.20) \quad \frac{d^{n+1}}{dx^{n+1}} I^\alpha[f](x) &= \frac{1}{x^{n+1}} \left\{ \delta I^\alpha[p_n(\alpha + \delta)f](x) - nI^\alpha[p_n(\alpha + \delta)f](x) \right\} \\
(3.21) &= \frac{1}{x^{n+1}} I^\alpha[(\alpha + \delta - n)p_n(\alpha + \delta)f](x) \\
(3.22) &= \frac{1}{x^{n+1}} I^\alpha[p_{n+1}(\alpha + \delta)f](x).
\end{align}

This shows that the assertion holds for $n + 1$, which completes the proof.

Now let us prove smoothness of our distribution functions $G_{\alpha,p}$ and $H_\gamma$ which has been introduced in (2.8) and (2.16). For this purpose, we introduce a class of functions $F$. Define $P$ by the class of functions on $(0,1)$ which are linear combinations of $t^\beta v(t)$ for some $\beta > 0$ and some infinitely differentiable function $v$ on $(0,1)$ whose derivatives of all orders are bounded near $t = 0$. Now define $F$ by the class of functions on $(0,1)$ spanned by functions $g$ of the form $g(t) = t^{\beta-1}u(t)/v(t)$ for some $\beta > 0$ and $u, v \in P$. Note that $F$ is a linear subspace of $L^1(0,1-) \cap C(0,1)$.

Lemma 3.5. If $g \in F$, then $\delta g \in F$. In particular, if $g \in F$, then $I^\alpha[g]$ for $\alpha > 0$ is infinitely differentiable on $(0,1)$.

Proof. Let $g(t) = t^{\beta-1}u(t)/v(t)$ for some $\beta > 0$ and $u, v \in P$. Then

\begin{equation}
(3.23) \quad \delta g(t) = \frac{t^{\beta-1} \{ (\beta-1)u(t)v(t) + \delta u(t)v(t) - u(t)\delta v(t) \}}{v(t)^2}.
\end{equation}

We remark the following facts: (i) $u \in P$ implies $\delta u \in P$; (ii) $u, v \in P$ implies $uv \in P$. Hence, by (3.23), we see that $\delta g \in F$.

Now it follows by induction that $\delta^n g \in F$ for $n \geq 1$. Therefore we obtain $\delta^n g \in L^1(0,1-)$ for $n \geq 1$. This proves the second assertion by Corollary 3.4.

Now we obtain the following result, which is the former halves of the statements of Theorems 2.1 and 2.2. 

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Theorem 3.6. The distribution functions $G_{\alpha,p}$ and $H_{\gamma}$ are infinitely differentiable in $(0,1)$. In particular, the densities $G_{\alpha,p}'$ and $H_{\gamma}'$ are given by (2.10) and (2.17), respectively.

Proof. We may rewrite (2.8) and (2.14) as $G_{\alpha,p} = \Gamma(\alpha)I^\alpha[g_{\alpha,p}]$ and $I^1[H_{\gamma}] = \Gamma(\gamma)I^\gamma[h_{\gamma}]$, respectively. We can easily see that $g_{\alpha,p}, h_{\gamma} \in \mathcal{F}$, and therefore we obtain $G_{\alpha,p}, H_{\gamma} \in C^n(0,1)$ for all $n \geq 1$ by Lemma 3.5.

Using Lemma 3.2 we obtain the formulae

\begin{equation}
G_{\alpha,p}' = \Gamma(\alpha)I^\alpha[g_{\alpha,p}']
\end{equation}

and

\begin{equation}
H_{\gamma}' = \Gamma(\gamma)I^\gamma[h_{\gamma}'].
\end{equation}

Using Proposition 3.3 we obtain the formula

\begin{equation}
H_{\gamma}'(x) = \frac{1}{x}\Gamma(\gamma)I^\gamma[\gamma h_{\gamma}' + \delta h_{\gamma}'](x).
\end{equation}

Now the proof is completed. \qed

4 Asymptotic behaviors of Riemann–Liouville fractional integrals

In this section, we study an asymptotic property for a function expressed by the Riemann–Liouville fractional integral. For this purpose, we use Karamata’s theory of regular variations; see, e.g., [4] for the details.

Proposition 4.1. Let $\alpha > 0$ and $f \in L^1(0,1) \cap C(0,1)$. Suppose that

\begin{equation}
f(x) \sim x^{\beta-1}K(x) \quad \text{as} \quad x \to 0+
\end{equation}

for some $\beta > 0$ and some slowly varying function $K(x)$ at $x = 0$. Then

\begin{equation}
I^\alpha[f](x) \sim \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)}x^{\alpha+\beta-1}K(x) \quad \text{as} \quad x \to 0+.
\end{equation}

Proof. Since $f \in C(0,1)$ and $f$ satisfies the assumption (4.1), the integral

\begin{equation}
I^\alpha[f](x) = \int_0^x \frac{(x - t)^{\alpha-1}}{\Gamma(\alpha)}f(t)dt
\end{equation}

makes sense in the sense of the Riemann integral. Changing variables to $s = t/x$, we have

\begin{equation}
I^\alpha[f](x) = x^\alpha \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)}f(xs)ds.
\end{equation}

By the assumption (4.1), it is obvious that

\begin{equation}
\frac{I^\alpha[f](x)}{x^{\alpha+\beta-1}K(x)} \to \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1}s^{\beta-1}ds = \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)}
\end{equation}

as $x \to 0+$. This completes the proof. \qed
Proof of Theorems 2.1 and 2.2. It is easy to see by (2.9) that

\[ g'_{\alpha,p}(t) \sim \frac{\sin \alpha \pi}{\pi} \cdot \frac{1-p}{p^2} \cdot \alpha t^{\alpha} \quad \text{as } t \to 0+ \]

and by (2.15) that

\[ \gamma h'_{\gamma}(t) + \delta h'_{\gamma}(t) \sim \frac{\sin \gamma \pi}{\pi} \cdot \frac{\gamma^2 (2\gamma - 1)}{1 - \gamma} \cdot t^{\gamma-1} \quad \text{as } t \to 0+ . \]

Now we apply Proposition 4.1 to (3.24) and (3.26), we obtain the desired results. \( \square \)

Acknowledgements: The present study started when the first author had a short stay at University of California, San Diego in the winter of 2006. He expresses his sincere thanks to Professor Patrick J. Fitzsimmons for the fruitful discussions and his hospitality during that stay. Both of the authors would like to express their hearty gratitude to Professor Marc Yor for the stimulating discussions and his hospitality during their short stay in Université Paris VI in May 2007.

References

[1] M. Barlow, J. Pitman, and M. Yor. Une extension multidimensionnelle de la loi de l’arc sinus. In Séminaire de Probabilités, XXIII, volume 1372 of Lecture Notes in Math., pages 294–314. Springer, Berlin, 1989.

[2] J. Bertoin, T. Fujita, B. Roynette, and M. Yor. On a particular class of self-decomposable random variables: the durations of bessel excursions straddling independent exponential times. Prob. Math. Stat., 26(2):315–366, 2006.

[3] J. Bertoin and M. Yor. Some independence results related to the arc-sine law. J. Theoret. Probab., 9(2):447–458, 1996.

[4] N. H. Bingham, C. M. Goldie, and J. L. Teugels. Regular variation, volume 27 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1989.

[5] P. J. Fitzsimmons and R. K. Getoor. Occupation time distributions for Lévy bridges and excursions. Stochastic Process. Appl., 58(1):73–89, 1995.

[6] P. J. Fitzsimmons, J. Pitman, and M. Yor. Markovian bridges: construction, Palm interpretation, and splicing. In Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992), volume 33 of Progr. Probab., pages 101–134. Birkhäuser Boston, Boston, MA, 1993.

[7] E. Fujihara, Y. Kawamura, and Y. Yano. Functional limit theorems for occupation times of Lamperti’s stochastic processes in discrete time. J. Math. Kyoto Univ, to appear.
[8] R. K. Getoor and M. J. Sharpe. Local times on rays for a class of planar Lévy processes. *J. Theoret. Probab.*, 7(4):799–811, 1994.

[9] R. K. Getoor and M. J. Sharpe. On the arc-sine laws for Lévy processes. *J. Appl. Probab.*, 31(1):76–89, 1994.

[10] K. Itô and H. P. McKean, Jr. *Diffusion processes and their sample paths*. Springer-Verlag, Berlin, 1974. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125.

[11] Y. Kasahara and Y. Yano. On a generalized arc-sine law for one-dimensional diffusion processes. *Osaka J. Math.*, 42(1):1–10, 2005.

[12] F. B. Knight. The uniform law for exchangeable and Lévy process bridges. *Astérisque*, (236):171–188, 1996. Hommage à P. A. Meyer et J. Neveu.

[13] J. Lamperti. An occupation time theorem for a class of stochastic processes. *Trans. Amer. Math. Soc.*, 88:380–387, 1958.

[14] S. Watanabe. Generalized arc-sine laws for one-dimensional diffusion processes and random walks. In *Stochastic analysis (Ithaca, NY, 1993)*, volume 57 of *Proc. Sympos. Pure Math.*, pages 157–172. Amer. Math. Soc., Providence, RI, 1995.

[15] S. Watanabe, K. Yano, and Y. Yano. A density formula for the law of time spent on the positive side of one-dimensional diffusion processes. *J. Math. Kyoto Univ.*, 45(4):781–806, 2005.

[16] Y. Yano. On the occupation time on the half line of pinned diffusion processes. *Publ. Res. Inst. Math. Sci.*, 42(3):787–802, 2006.