A Randomized Incremental Approach for the Hausdorff Voronoi Diagram of Non-crossing Clusters

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Abstract. In the Hausdorff Voronoi diagram of a set of point-clusters in the plane, the distance between a point \(t\) and a cluster \(P\) is measured as the maximum distance between \(t\) and any point in \(P\) while the diagram is defined in a nearest sense. This diagram finds direct applications in VLSI computer-aided design. In this paper, we consider “non-crossing” clusters, for which the combinatorial complexity of the diagram is linear in the total number \(n\) of points on the convex hulls of all clusters. We present a randomized incremental construction, based on point-location, to compute the diagram in expected \(O(n \log^2 n)\) time and expected \(O(n)\) space, which considerably improves previous results. Our technique efficiently handles non-standard characteristics of generalized Voronoi diagrams, such as sites of non-constant complexity, sites that are not enclosed in their Voronoi regions, and empty Voronoi regions.

1 Introduction

Given a set \(S\) of sites contained in some space, the Voronoi region of a site \(s \in S\) is the geometric locus of points in the given space that are closer to \(s\) than to any other site. In the classic Voronoi diagram, each site is a point and closeness is measured according to the Euclidean distance. In this work, we consider the Hausdorff Voronoi diagram. The containing space is \(\mathbb{R}^2\), each site is a cluster of points (i.e., a set of points), and closeness of a point \(t \in \mathbb{R}^2\) to a cluster \(P\) is measured by the farthest distance \(d_f(t, P) = \max_{p \in P} d(t, p)\), where \(d(\cdot, \cdot)\) is the Euclidean distance between two points. The farthest distance \(d_f(t, P)\) equals the Hausdorff distance between \(t\) and cluster \(P\), hence the name of the diagram.

Our motivation for investigating the Hausdorff Voronoi diagram comes from VLSI circuit design, where this diagram can be used to efficiently estimate the critical area of a VLSI layout for various types of open faults [19,20].

1.1 Previous Work

Let \(k\) be the number of clusters in the input family, and \(n\) be the total number of points on the convex hulls of all clusters. We denote by \(\text{conv}\ P\) the convex

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hull of cluster $P$ and by $\text{CH}(P)$ the sequence of points of $P$ on the boundary of the convex hull in counterclockwise order.

**Definition 1.** Two clusters $P$ and $Q$ are called non-crossing if the convex hull of $P \cup Q$ admits at most two supporting segments with one endpoint in $P$ and one endpoint in $Q$, or equivalently convex hulls of $P$ and $Q$ are pseudodisks. See Fig. 1.

![Fig. 1. Non-crossing and crossing clusters with supporting segments (dashed lines)](image)

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![Fig. 2. HVD of five 2-point clusters; region of $C = \{c_1, c_2\}$ (gray)](image)

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The combinatorial complexity (size) of the Hausdorff Voronoi diagram is $O(n + m)$, where $m$ is the number of supporting segments reflecting crossings between all pairs of crossing clusters, and this is tight [21]. In the worst case, $m$ is $\Theta(n^2)$. If all clusters are non-crossing ($m = 0$) the diagram has linear size. There are plane sweep [19] and divide and conquer [21] algorithms for constructing the Hausdorff Voronoi diagram of arbitrary clusters. Both algorithms have a $K \log n$ term in their time complexity, where $K$ is a parameter reflecting the number of pairs of clusters such that one is contained in a specially defined enclosing circle of the other, for example, the minimum enclosing circle [21]. However, $K$ can be $\omega(n)$ (superlinear) even in the case of non-crossing clusters. The Hausdorff Voronoi diagram is equivalent to an upper envelope of a family of lower envelopes of an arrangement of hyperplanes in $\mathbb{R}^3$ (each envelope corresponds to a cluster) [12]. Edelsbrunner et al. give a construction algorithm of $O(n^2)$ time complexity.\(^3\) Although the time complexity is optimal in the worst case, it remains quadratic even for non-crossing clusters, for which the size of the diagram is linear. A more recent parallel algorithm [9] constructs the Hausdorff Voronoi diagram of non-crossing clusters in $O(p^{-1}n \log^4 n)$ time with $p$ processors, which implies a divide and conquer sequential algorithm of time complexity $O(n \log^4 n)$ and space complexity $O(n \log^2 n)$.

The Hausdorff Voronoi diagram of a family of non-crossing clusters is an instance of abstract Voronoi diagrams [15]. Using the randomized incremental framework of Klein et al. [16], it can be computed in expected $O(bn \log n)$ time, where $b$ is the time it takes to construct the bisector between two clusters [1]. If there are clusters of linear size, then $b$ can be $\Theta(n)$. The framework was

\(^3\)The reported $O(n^2 \alpha(n))$ time complexity (where $\alpha(n)$ is the inverse Ackermann function) improves to $O(n^2)$ due to the $O(n^2)$ bound on the size of the diagram.
successfully applied to compute the Voronoi diagram of disjoint polygons [17] in $O(k \log n)$ time, where $k$ is the number of the sites, and $n$ is their total size. It is not easy, however, to apply a similar approach to the Hausdorff Voronoi diagram because of a fundamental difference between the farthest and the nearest distance from a point to a convex polygon [11].

The Hausdorff Voronoi diagram is a min-max diagram type of diagram, where every point $t$ in the plane lies in the region of the closest cluster with respect to the farthest distance. The “dual” max-min diagram is the farthest color Voronoi diagram [2,13]. For disjoint simple polygons, the farthest color Voronoi diagram can be constructed in $O(n \log^3 n)$ time where $n$ is the total size of the sites [7].

1.2 Our Contribution

In this paper we give a randomized incremental algorithm to compute the Hausdorff Voronoi diagram of a family of $k$ non-crossing clusters, based on point location. Clusters are inserted in random order one by one, while the diagram computed so far is maintained in a dynamic data structure, where generalized point location queries can be answered efficiently. To insert a cluster, a representative point in the new Voronoi region of this cluster is first identified and located, and then the new region is traced while the data structure is updated [6,10,14].

In case of the Hausdorff Voronoi diagram, a major technical challenge is to quickly identify a representative point that lies in the new Voronoi region. This is difficult because: (a) the region of the new cluster might not contain any of its points, (b) the region of the new cluster might be empty, and (c) sites have non-constant size and thus the computation of a bisector or answering an in-circle test require non-constant time. Furthermore, the addition of a new cluster may make an existing region empty.

The dynamic data structure that we use is a variant of the Voronoi hierarchy [14], which in turn is inspired by the Delaunay hierarchy [10], and which we augment with the ability to efficiently handle the difficulties listed above. We also exploit a technique by Aronov et al. [4] to efficiently query the static farthest Voronoi diagram of a cluster. The expected running time of our algorithm is $O(n \log n \log k)$ and the expected space complexity is $O(n)$. The augmentation of the Voronoi hierarchy introduced in this paper may be of interest for incremental constructions of other non-standard types of generalized Voronoi diagrams. Our algorithm can also be implemented in deterministic $O(n)$ space and $O(n \log^2 n(\log \log n)^2)$ expected running time, using the dynamic point location data structure by Baumgarten et al. [5], while applying a simplified type of parametric search similarly to Cheong et al. [7].

2 Preliminaries

Throughout this paper, we consider a family $F = \{C_1, \ldots, C_k\}$ of non-crossing clusters of points. We assume that no two clusters have a common point, and no four points lie on the same circle.
For a point \( c \in C \), the \textbf{farthest Voronoi region} of \( c \) is \( \text{freg}_C(c) = \{ p | \forall c' \in C \setminus \{c\} : d(p, c) > d(p, c') \} \). The farthest Voronoi diagram of \( C \) is denoted as \( \text{FVD}(C) \) and its graph structure as \( \mathcal{T}(C) \). If \( |C| > 1 \), \( \mathcal{T}(C) \) is a tree defined as \( \mathbb{R}^2 \setminus \bigcup_{c \in C} \text{freg}_C(c) \), and \( \mathcal{T}(C) = c \), if \( C = \{c\} \). A point at infinity along an arbitrary unbounded edge of \( \mathcal{T}(C) \) is treated as the root of \( \mathcal{T}(C) \), denoted as \( \text{root}(C) \).

For a cluster \( C \in F \), the \textbf{Hausdorff Voronoi region} of \( C \) is \( \text{hreg}_F(C) = \{ p | \forall C' \in F \setminus \{C\} : d_t(p, C) < d_t(p, C') \} \).

For a point \( c \in C \), \( \text{hreg}_F(C) = \text{hreg}_F(C) \cap \text{freg}_C(c) \). The closure of \( \text{freg}_C(c) \), \( \text{hreg}_F(C) \), and \( \text{hreg}_F(c) \) is denoted by \( \text{freg}_C(c) \), \( \text{hreg}_F(C) \), and \( \text{hreg}_F(c) \), respectively. When there is no ambiguity on the set under consideration, we omit the subscript from the above notation. The partitioning of the plane into non-empty Hausdorff Voronoi regions, together with their bounding edges and vertices, is called the \textbf{Hausdorff Voronoi diagram} of \( F \), and it is denoted as \( \text{HVD}(F) \). Below we review some useful definitions and properties of the Hausdorff Voronoi diagram, which appeared in previous work [21].

The Hausdorff Voronoi diagram is \textit{monotone}, that is, a region of the diagram can only shrink with the insertion of a new cluster. The structure of the Hausdorff Voronoi region of a point \( c \in C \) is shown in Fig. 3. Its boundary consists of two chains: (1) the \textbf{farthest boundary} that belongs to \( \mathcal{T}(C) \) and is internal to \( \text{hreg}(C) \), \( \text{bd} \text{hreg}(c) \cap \text{bd} \text{freg}(c) \); (2) the \textbf{Hausdorff boundary} \( \text{bd} \text{hreg}(c) \cap \text{bd} \text{freg}(C) \).

Neither chain can be empty, if \( \text{hreg}(C) \neq \emptyset \) and \( |C| > 1 \). There are three types of vertices on the boundary of \( \text{hreg}(c) \): (1) Standard Voronoi vertices that are equidistant from \( C \) and two other clusters, referred in this paper as \textit{pure} vertices. Pure vertices appear on the Hausdorff boundary of \( \text{hreg}(c) \). (2) \textit{Mixed} vertices that are equidistant to three points of two clusters (\( C \) and another cluster). The mixed vertices which are equidistant to two points of \( C \) and one point of another cluster are called \textit{C-mixed} vertices; there are exactly two of them on the boundary of \( \text{hreg}(c) \) and they delimit both the farthest boundary of \( c \) and the Hausdorff boundary of \( c \). (3) Vertices of \( \mathcal{T}(C) \) on the farthest boundary of \( c \).

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig3}
\caption{Features of the Hausdorff Voronoi region of a point \( c \in C \)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig4}
\caption{The 2-point cluster \( Q \) (red), forward limiting w.r.t. the 3-point cluster \( P \) (black) with \( P \)-circle \( \mathcal{K}_y \); portion \( \mathcal{K}_y' \) (shaded)}
\end{figure}
A line-segment $\overline{y_1y_2}$ is a chord of cluster $C$ if $c_1, c_2 \in CH(C)$ and $c_1 \neq c_2$. In Fig. 4, $\overrightarrow{PQ}$ is a chord of cluster $P$.

**Definition 2 (C-circle $K_y; K_y^l, K_y^r$ [21]).** Let $uv$ be an edge of $T(C)$ bisecting a chord $\overline{y_1y_2}$ of $C \in F$. A circle centered at $y \in uv$ of radius $d(y, c_1) = d_t(y, C)$ is called the C-circle of $y$ and is denoted as $K_y$. The chord $\overline{y_1y_2}$ partitions $K_y$ in two parts: $K_y^l$ and $K_y^r$, where $K_y^l$ is the part that encloses the two points of $C$ that define root($C$). In case $y$ and root($C$) are on the same edge of $T(C)$, $K_y^l$ is the portion of $K_y$ that is enclosed in the halfplane bounded by $c_1c_2$ which does not contain root($C$).

**Definition 3 (Rear/forward limiting cluster [21]).** A cluster $P \in F \setminus \{C\}$ is rear limiting with respect to $C$, if there is a C-circle $K_y$ such that $P$ is enclosed in $K_y^l \cup \text{conv}$. Similarly, $P$ is forward limiting with respect to $C$, if there is a C-circle $K_y$ such that $P$ is enclosed in $K_y^r \cup \text{conv}$. See Fig. 4.

**Properties.** (They can be directly derived from Lemma 2, Properties 2, 3 [21].)

1. If $\text{hreg}(C) \neq \emptyset$, then $\text{hreg}(C) \cap T(C)$ consists of exactly one non-empty connected component.
2. Consider a point $v$ of $T(P)$, such that $v \notin \text{hreg}(P)$. Let $Q$ be a cluster, which is closer to $v$ than $P$. Then, only one of the subtrees of $T(P)$ rooted at $v$, might contain points which are closer to $P$ than to $Q$.
3. Let $uv$ be an edge of $T(P)$. If both $u$ and $v$ are closer to $Q$ than to $P$ then $\text{hreg}_F(P)$ cannot intersect $uv$.
4. Region $\text{hreg}_F(P) = \emptyset$, if and only if there is a cluster $Q \subset \text{conv} P$, or there is a pair of clusters $\{Q, R\}$ such that $Q$ is rear limiting and $R$ is forward limiting with the same $P$-circle. Pair $\{Q, R\}$ is called a killing pair for $P$.

## 3 A Randomized Incremental Algorithm

Let $C_1, \ldots, C_k$ be a random permutation of the clusters in family $F$, and let $F_i = \{C_1, \ldots, C_i\}$ for $1 \leq i \leq k$. The algorithm iteratively constructs $\text{HVD}(F_1), \ldots, \text{HVD}(F_k) = \text{HVD}(F)$. The cluster $C_i$ is inserted in $\text{HVD}(F_{i-1})$ as follows:

1. Identify a point $t$ that is closer to $C_i$ than to any cluster in $F_{i-1}$ (i.e., $t \in \text{hreg}_{F_i}(C_i)$) or determine that no such point exists (i.e., $\text{hreg}_{F_i}(C_i) = \emptyset$).
2. If $t$ exists, grow $\text{hreg}_{F_i}(C_i)$ starting from $t$ and update $\text{HVD}(F_{i-1})$ to derive $\text{HVD}(F_i)$; otherwise, $\text{HVD}(F_i) = \text{HVD}(F_{i-1})$.

The main challenge is to perform Step 1 efficiently. Step 2 can be performed in linear time [21]. Details are given in Appendix A. Throughout this section, we skip the subscript $F_i$ and let $\text{hreg}(C_i)$ stand for $\text{hreg}_{F_i}(C_i)$.

To identify a representative point $t$ in $\text{hreg}(C_i)$ (Step 1) it is enough to search along $T(C_i)$, by Property 1. However, $\text{hreg}(C_i) \cap T(C_i)$ might not contain a vertex of $T(C_i)$, see e.g., the gray region in Fig. 2. In this case, $\text{hreg}(C_i)$ is either empty, or intersects exactly one edge of $T(C_i)$, which is called a candidate edge.
Definition 4. Let $uv$ be an edge of $\mathcal{T}(C_i)$. Let the clusters $Q^u, Q^v \in F_{i-1}$ be the clusters closest to $u$ and $v$ respectively. We call $uv$ a candidate edge if $Q^u \neq Q^v$ and $uv$ satisfies the following predicate:
\[
cand(uv) = d_t(u, Q^v) < d_t(u, C_i) < d_t(u, Q^v) \land d_t(v, Q^v) < d_t(v, C_i) < d_t(v, Q^v).
\]

By Properties 2 and 3 we derive the following.

Lemma 1. Suppose $hreg(C_i) \cap \mathcal{T}(C_i)$ does not contain any vertex of $\mathcal{T}(C_i)$. Then at most one edge $uv$ of $\mathcal{T}(C_i)$ can be a candidate edge, in which case $hreg(C_i) \cap \mathcal{T}(C_i) \subset uv$. Otherwise $hreg(C_i) = \emptyset$.

A high-level description of Step 1 is as follows: We traverse $\mathcal{T}(C_i)$ starting at root($C_i$), checking its vertices and pruning if possible appropriate subtrees according to Property 3. In this process we either determine $t$ as a vertex of $\mathcal{T}(C_i)$, or we determine a candidate edge $uv$, or $hreg(C_i) = \emptyset$. Pseudocode is given as Procedure 1 below, which should be run with $u = \text{root}(C_i)$.

In more detail, to check if a vertex $w$ suits as $t$, determine the cluster $Q^w \in F_{i-1}$, which is nearest to $w$ by point location in $\text{HVD}(F_{i-1})$. If $d_t(w, C_i) < d_t(w, Q^w)$, then $t = w$. To compute $d_t(w, P)$ for a cluster $P$, do point location in $\text{FVD}(P)$. If Procedure 1 identifies a candidate edge, the representative point $t$ is determined by performing parametric point location along the candidate edge in $\text{HVD}(F_{i-1})$.

Procedure 1 Tracing the subtree of $\mathcal{T}(C_i)$ rooted at $u$ (within Step 1)

Require: $d_t(u, C_i) > d_t(u, Q^v)$.

Locate $v$ and $w$ be children of $u$. If $d_t(v, C_i) < d_t(v, Q^v)$ or $d_t(w, C_i) < d_t(w, Q^v)$ then return $v$ or $w$ respectively. If either $uv$ or $uw$ is a candidate edge then return the $uv$ or $uw$ respectively.

\[
\begin{align*}
&\text{if } d_t(v, C_i) < d_t(v, Q^v) \text{ then} &&\text{Otherwise, prune the subtree of } w \\
&\quad \text{Set } u = w \text{ and recurse.} \\
&\text{if } d_t(w, C_i) < d_t(w, Q^v) \text{ then} &&\text{Otherwise, prune the subtree of } v \\
&\quad \text{Set } u = v \text{ and recurse.}
&\end{align*}
\]

Definition 5 (Parametric point location). Given $\text{HVD}(F_{i-1})$ and a candidate edge $uv \subset \mathcal{T}(C_i)$ determine the cluster $P_t \in F_{i-1}$ and the point $t \in uv$ such that $d_t(t, C_i) = d_t(t, P_t) = \min_{P \in F_{i-1}} d_t(t, P)$. If such point $p$ does not exist, return nil.

Parametric point location in the Hausdorff Voronoi diagram is performed using the data structure that stores the diagram. Its performance determines the time complexity of our algorithm. In Sections 4 and 5, we describe the data structures and the algorithms used to answer the necessary queries.
4 Separator Decomposition

In this section we describe a data structure to efficiently perform point location and answer so-called segment queries in a tree-type of planar subdivision such as a farthest Voronoi diagram.

It is well-known [18] that any tree with \( h \) vertices has a vertex called centroid, removal of which decomposes the tree into subtrees of at most \( h/2 \) vertices each. The centroid can be found in \( O(h) \) time [18]. Thus, the farthest Voronoi diagram of a cluster \( P \) can be organized as a balanced tree, whose nodes correspond to vertices of the diagram. This representation is called the separator decomposition, it is denoted as \( SD(P) \), and can be built as follows:

- Find a centroid \( c \) of \( T(P) \). Create a node for \( c \) and assign it as the root node.
- Remove \( c \) from \( T(P) \). Recursively build the trees for the remaining three connected components, and link them as subtrees of the root.

Point location in \( SD(P) \) for a query point \( q \) is performed as follows. Starting from the root of \( SD(P) \), perform a constant-time test of the query point \( q \) against a node of \( SD(P) \), to decide in which of the node’s subtrees to continue. When a leaf of \( SD(P) \) is reached, choose \( p \) among the owners of the three regions that are adjacent to the corresponding vertex of \( FVD(P) \). The test of \( q \) against a node \( \alpha \) of \( SD(P) \) is due to Aronov et al. [4]. In more detail, let the node \( \alpha \) correspond to a vertex \( w \) of \( FVD(P) \). Let the points \( p_1, p_2, p_3 \in P \) be the owners of the three regions of \( FVD(P) \), incident to \( w \). Consider the rays \( r_i, i = 1, 2, 3 \) with origin at \( w \) and direction \( \overrightarrow{p_iw} \) respectively. Each ray \( r_i \) lies entirely inside \( freg(p_i) \), and thus \( r_1, r_2 \) and \( r_3 \) subdivide the plane into three sectors with exactly one connected component of \( T(P) \setminus \{ w \} \) in each sector. Choose the sector that contains \( q \), and pick the corresponding subtree of \( \alpha \).

A segment query in a farthest Voronoi diagram is as follows. Let \( C, P \in F \). Given \( FVD(P) \) and a segment \( uv \subset T(C) \) such that \( d_t(u, C) < d_t(u, P) \) and \( d_t(v, C) > d_t(v, P) \), find the point \( x \in uv \), that is equidistant from both \( C \) and \( P \) (\( d_t(x, C) = d_t(x, P) \)).

If \( FVD(P) \) is represented as a separator decomposition, the segment query can be performed efficiently similarly to a point location query with the difference that we test a segment against a node of \( SD(P) \). In particular, consider a node of \( SD(P) \) corresponding to a vertex \( w \) of \( FVD(P) \). Let rays \( r_i, i = 1, 2, 3 \), be defined as above. Consider the (at most two) intersection points of \( uv \) with the rays \( r_i \). If any of these points is equidistant to \( C \) and \( P \), return it. Otherwise, since \( P \) and \( C \) are non-crossing, there is exactly one subsegment \( u'v' \subset uv \) such that \( d_t(u', C) < d_t(u', P) \) and \( d_t(v', C) > d_t(v', P) \), where \( u', v' \) can be any of \( u, v \), or the intersection points. The subsegment \( u'v' \) can be computed in constant time, together with one of the three sectors where \( u'v' \) is contained.

If we reached a leaf of \( SD(P) \), we are left with a single edge \( e \) of \( T(P) \). Suppose \( e \) bisects the chord \( \overline{p_1p_2} \) of \( P \), and the current \( u'v' \) bisects the chord \( \overline{c_1c_2} \) of \( C \). Then, return as point \( x \) the center of the circle passing through \( p, c_1, c_2 \), where \( p \) is the point among \( p_1, p_2 \) farthest from \( x \).
Lemma 2. The separator decomposition SD(P) of a cluster P ∈ F can be built in \( O(n_p \log n_p) \) time, where \( n_p \) is the number of vertices of FVD(P). Both the point location and the segment query in SD(P) require \( O(\log n_p) \) time.

5 Voronoi Hierarchy for the Hausdorff Voronoi Diagram

Consider a set \( S \) of \( k \) sites. The Voronoi hierarchy of \( S \) is a sequence of levels \( S = S^{(0)} \supseteq \ldots \supseteq S^{(h)} \). For \( \ell \in \{1, \ldots, h\} \), level \( S^{(\ell)} \) is a random sample of \( S^{(\ell-1)} \) according to a Bernoulli distribution with parameter \( \beta \in (0, 1) \). For each level \( S^{(\ell)} \) the data structure stores the Voronoi diagram of \( S^{(\ell)} \). The Voronoi hierarchy is inspired by the Delaunay hierarchy given by Devillers [10].

In the Hausdorff Voronoi diagram sites are clusters of non-constant size each. We first adapt some known properties of the hierarchy to be valid in such an environment. Then, we consider several enhancements of the hierarchy to handle efficiently the Hausdorff Voronoi diagram and its queries, such as point location through walks, dynamic updates, including the handling of empty Voronoi regions, and parametric point location along a segment.

Lemma 3. Let the underlying Voronoi diagram have size \( O(n) \), where \( n \) is the total size of the sites. Then for any set \( S \) of \( k \) sites of total size \( n \), the Voronoi hierarchy of \( S \) has \( O(n) \) expected size and \( O(\log k) \) expected number of levels.

Proof. Let \( \|S^{(\ell)}\| \) be the total complexity of the sites in \( S^{(\ell)} \). The probability that a site \( s \in S \) appears in family \( S^{(\ell)} \) is \( \beta^\ell \). Then \( \mathbb{E}[\|S^{(\ell)}\|] = \beta^\ell \|S\| = \beta^\ell n \) and the expected size of the Voronoi diagram at level \( \ell \) is within \( O(\beta^\ell n) \). The expected size of the hierarchy is

\[
\sum_{\ell=0}^{\infty} O(\mathbb{E}[\|S^{(\ell)}\|]) = \sum_{\ell=0}^{\infty} O(\beta^\ell n) = \frac{1}{1-\beta} O(n) = O(n).
\]

To perform point location in the Voronoi hierarchy for a query point \( q \), we start at level 1, and for each level \( \ell \), we determine the site \( s^{(\ell)} \in S^{(\ell)} \) that is closest to \( q \), by performing a walk. Each step of the walk moves from a site \( s \in S^{(\ell)} \) to a neighbor of \( s \), such that the distance to \( q \) is reduced. A walk at level \( \ell - 1 \) starts from \( s^{(\ell)} \). The answer to the query is \( s^{(0)} \).

Lemma 4. Let \( s^0, \ldots, s^r = s^\ell \) be the sequence of sites visited at level \( \ell \) during the point location of a query point \( q \). Assuming that \( d_i(q, s^i) < d(q, s^i_{\ell-1}) \), for \( i \in \{1, \ldots, r\} \), and either \( s^{\ell+1} = s^0 \), or \( d_1(q, s^0) < d_1(q, s^{\ell+1}) \), the expectation of the length \( r \) of the walk at level \( \ell \) is constant.

Proof. The following lemma is an adaptation of Lemma 9 [14].

Lemma 5. Let \( q \) be a point in \( \mathbb{R}^2 \) and let \( s^{\ell+1} \in S^{(\ell+1)} \) be the site nearest to \( q \) at level \( \ell + 1 \). The expected number of sites in \( S^{(\ell)} \), which are closer to \( q \) than \( s^{\ell+1} \), is constant.
Proof (Lemma 5). The probability that $s^{t+1}$ is the $t$-th nearest site to $q$, among all sites in $S^{(t)}$, is $\beta(1-\beta)^{t-1}$. This is because the probability that any site among the $t-1$ sites, nearest to $q$ in $S^{(t)}$, does not belong in $S^{(t+1)}$ is $(1-\beta)$, the probability that the $t$-th nearest site to $q$ in $S^{(t)}$ belongs in $S^{(t+1)}$ is $\beta$, and all these events are independent. Therefore, the expected number of sites in $S^{(t)}$, which are closer than $s^{t+1}$ to $q$, is

$$N_t < \sum_{t=1}^{|S^{(t)}|} t(1-\beta)^{t-1} \beta \leq \sum_{t=1}^{\infty} t(1-\beta)^{t-1} = \frac{1}{\beta},$$

which is constant.

We continue the proof of Lemma 4. The distance to $q$ during the walk is monotonically decreasing, and the walk starts either at $s^{t+1}$ or at a site closer to $q$ than $s^{t+1}$ is. Therefore, each of the sites $s_1, \ldots, s_r$ is closer to $q$ than $s^{t+1}$. By Lemma 5 the expected number of such sites is constant.

In the original Voronoi hierarchy for a set of disjoint convex objects [14], one step of the walk to determine the correct neighboring site consists of a binary search among the neighbors of the site. For a Hausdorff Voronoi diagram, however, there is no natural ordering for the set of neighbors of a site. In addition, the subset of points in a cluster that contribute to the diagram reduces over time.

A single step of the walk for the Hausdorff Voronoi diagram. Consider point location in the Voronoi hierarchy for a family $\mathcal{F}$ of non-crossing clusters and a query point $q$. Let $C \in \mathcal{F}^{(t)}$ be the current cluster being considered at level $t$. We need to determine a cluster $Q$ at level $t$ whose region neighbors the region of $C$ and whose distance from $q$ gets reduced. Let $C' \subset C$ denote the set of all active points $c \in C$ that contribute a face to $h_{\text{reg}}(\mathcal{F}^{(t)})(C)$ at the current level $t$ ($h_{\text{reg}}(\mathcal{F}^{(t)})(c) \neq \emptyset$). Let $h_{\text{reg}}(\mathcal{F}^{(t)})(\cdot)$ denote $h_{\text{reg}}(\mathcal{F}^{(t)}(\cdot))$.

The cluster $Q$ is determined as follows. Let $c \in C'$ be the active point that is farthest from query point $q$ ($q \in h_{\text{reg}}(C')(c)$). To determine point $c$ it is enough to draw the tangents from $q$ to CH($\hat{C}$). Let $v_1, \ldots, v_j$ be the pure vertices in $h_{\text{reg}}(\mathcal{F}^{(t)}(c))$ (see Fig. 5) in counterclockwise order, and let $Q^0, \ldots, Q^j, Q^{j+1}$ be their respective adjacent clusters. The rays $\overrightarrow{cv_1}, \ldots, \overrightarrow{cv_j}$ partition $h_{\text{reg}}(C)(c)$ into $j+1$ unbounded regions. The walk should move from $C$ to $Q^j$ such that the ray $\overrightarrow{cv_j}$ immediately follows $\overrightarrow{cv_i}$ or immediately precedes $\overrightarrow{cv_{i+1}}$. For example, in Fig. 5, $c \in \hat{C}^{(t)}$ is the farthest active point from $q$ ($d_t(q, \hat{C}^{(t)}) = d(q,c)$). Region $h_{\text{reg}}(C)(c)$ is shown gray and its boundary is drawn bold. The step in the walk should move from $C$ to $Q = Q^2$. We organize $\hat{C}$ as a sorted list of its points and for each point $c \in C$ we maintain a sorted list of all Voronoi vertices adjacent to $h_{\text{reg}}(\mathcal{F}^{(t)}(c))$. It can be shown that $d_t(q, \hat{Q}) \leq d_t(q, \hat{C})$, thus, the above procedure is correct. Note that $d_t(q, \hat{Q})$ may be greater than $d_t(q, \hat{C})$ because $d_t(q, \hat{C})$ may be different from $d_t(q, C)$ if $q \notin h_{\text{reg}}(C)$. We defer the proof of the correctness of the single step to Section 7.
Parametric point location in the Voronoi hierarchy. We are given $HVD(F_{i-1})$, stored as a Voronoi hierarchy, and the candidate edge $uv \in T(C_i)$. For each level $\ell$ of the Voronoi hierarchy, starting from the last level $h$, we search for the cluster $Q^\ell \in F_{i-1}^{(\ell)}$ and a point $u^\ell \in uv$ such that $u^\ell \in \text{hreg}_{F_{i-1}^{(\ell)}}(Q^\ell)$ and $d_i(u^\ell, C_i) = d_i(u^\ell, Q^\ell)$. If at some level there is no such point, return nil. Else return the cluster $Q^0$ and the point $u^0$ determined at level 0.

In more detail, suppose that $u^{\ell+1}$ and $Q^{\ell+1}$ have been computed, for some $\ell \in \{0, \ldots, h-1\}$. To compute $u^\ell$ and $Q^\ell$, we determine a sequence $u^{\ell+1} = a_0, a_1, \ldots, a_r = u^\ell$ of points on $uv$. Let $Q^{a_j}$ be the cluster in $F_{i-1}^{(\ell)}$ nearest to $a_j$. It is determined by a walk at level $\ell$ starting with $Q^{a_{\ell+1}}$. Then point $a_{\ell+1}$ is the point on $uv$, equidistant from $C_i$ and $Q^{a_{\ell+1}}$ ($d_i(a_{\ell+1}, C_i) = d_i(a_{\ell+1}, Q^{a_{\ell+1}})$). If $a_j$ is equidistant from $C_i$ and $Q^{a_j}$, we are done at level $\ell$; continue to level $\ell + 1$ with $u^\ell = a_j$ and $Q^\ell = Q^{a_j}$. Else, if $d_i(v, Q^{a_j}) \geq d_i(v, C_i)$, perform a segment query to determine $a_{\ell+1}$. Otherwise, report that a point $t$ does not exist.

Lemma 6. The expected number of visits of clusters at level $\ell$ during the parametric point location is $O(1)$.

Proof. The procedure of parametric point location for a candidate edge $uv$ at level $\ell$ consists of a successive search for points $u^{\ell+1} = a_0, a_1, \ldots, a_r = u^\ell$ as described before. For each such point $a_j$, a walk from $Q^{a_{j-1}}$ to $Q^{a_j}$ is performed.

We first claim that the expected length $r$ of the sequence $a_0, a_1, \ldots, a_r$ is constant.

Each point $a_j$, for $j \in \{1, \ldots, r\}$, is by construction equidistant from $C_i$ and $Q^{a_{j-1}}$. Cluster $Q^{a_{j-1}}$ is enclosed in disk $D_{a_j}$, centered at $a_j$ and with radius $d_i(a_j, C_i)$. Furthermore, since $d_i(v, Q^{a_{j-1}}) > d_i(v, C_i)$, $Q^{a_{j-1}} \subset D_{a_j} \cup \text{conv} C_i$, where $D_{a_j} = \text{conv}(K_{a_j})$ for the $C_i$-circle $K_{a_j}$ (assuming that $u$ is an ancestor of $v$ in $T(C_i)$). In other words, $Q^{a_{j-1}}$ is a rear limiting cluster with respect to $C_i$. Therefore, $Q^{a_{j-1}} \subset D_{a_j}$ (see [19, Lemma 1]). Note, that $Q^{a_r} = Q^{a_{r-1}}$. Thus, $Q^{a_j} \subset D_{a_j}$ for all $j \in \{0, \ldots, r\}$.

Let $Q^{\ell+1} \in F^{(\ell+1)}$ be the cluster nearest to $a_0 = u^{\ell+1}$ at level $\ell + 1$. By Lemma 5, the expected number of clusters in $F_{i-1}^{(\ell)}$ that are enclosed in $D_{a_j}$ is constant. Since all $Q^{a_j}$, for $j \in \{0, \ldots, r\}$, are enclosed in disk $D_{a_j}$, $r$ is expected to be constant.
Now, note that for each $j \in \{1, \ldots, r\}$, $d_{\ell}(a_j, Q^{(\ell-1)}) = d_{\ell}(a_j, C_i)$, and $a_j$ is closer to $C_i$ than to any cluster in $F^{(\ell-1)}$. Therefore, by Lemma 4, the expected length of the walk from $Q^{(\ell-1)}$ to $Q^{(\ell)}$ is constant.

During the parametric point location at level $\ell$ an expected constant number of walks is performed, of expected constant length each. The claim follows.

Handling the empty regions. After inserting $C_i$ at level 0 of the hierarchy for HVD($F_{i-1}$), we insert $C_i$ into the series of higher levels. When $C_i$ is inserted at a given level, however, a region of another cluster may become empty.

We call a cluster $P$ critical at level $\ell$ if $\operatorname{hreg}_{F_{\ell-1}}^{(\ell-1)}(P) \neq \emptyset$, $\operatorname{hreg}_{F_{\ell-1}}^{(\ell-1)}(P) = \emptyset$, and $\operatorname{hreg}_{F_{\ell-1}}^{(\ell)}(P) \neq \emptyset$. Such a cluster $P$ becomes an obstacle to correct point location in the Voronoi hierarchy for HVD($F_{\ell}$). Indeed, if a query point lies in $\operatorname{hreg}_{F_{\ell-1}}^{(\ell)}(P)$, we do not know where to continue the point location at level $\ell-1$.

To fix the problem, $P$ can be deleted from all levels, but this is computationally expensive. Instead, we link $P$ to at most two other clusters $Q, R \in F^{(\ell-1)}$, such that every point $q \in \mathbb{R}^2$ is closer to either $Q$ or to $R$ than to $P$. Property 4 guarantees that such a cluster or two clusters exist (a cluster contained in conv $P$ or a killing pair for $P$, respectively).

We now describe how to find a killing pair for $P$. While inserting $C_i$ at level $\ell-1$, we store all (deleted) $P$-mixed vertices of $\operatorname{hreg}_{F_{\ell-1}}^{(\ell-1)}(P)$ in a list $V$. At level $\ell$, for each $P$-mixed vertex $v$ of $\operatorname{hreg}_{F_{\ell-1}}^{(\ell)}(P)$, we check if $v$ is closer to $C_i$ or to $P$. If $d_{\ell}(v, C_i) \geq d_{\ell}(v, P)$, let $c$ be the point in $C_i$ for which $d_{\ell}(v, C_i) = d(v, c)$. Note that $c \notin \text{conv } P$, which will be useful. The linking is performed as follows:

- If all $P$-mixed vertices of $\operatorname{hreg}_{F_{\ell-1}}^{(\ell)}(P)$ are closer to $C_i$ than to $P$, link only to $C_i$. (This happens only if $C_i \notin F^{(\ell)}$.)
- Else find the cluster $K \in F^{(\ell-1)}$ such that $\{K, C_i\}$ is a killing pair for $P$. If $C_i \in F^{(\ell)}$, link only to cluster $K$. Otherwise, link to both $K$ and $C_i$.

What remains is to determine cluster $K$. To this aim, we use list $V$ and point $c$. Each vertex $u \in V$ is equidistant from two points $p_1, p_2 \in P$, and one point $q \in Q$, for some $Q \in F^{(\ell-1)}$. We simply check whether $c$ and $q$ are on different sides of the chord $\overline{pq}$. If they are, then we set $K = Q$ and we stop. Below we show that $\{K, C_i\}$ is a killing pair for $P$, and thus the linking is correct.

Lemma 7 (Correctness of linking). Let the cluster $P$ be critical at level $\ell$. If $P$ is linked to cluster $K \in F^{(\ell-1)}$ (case 2), then $K$ is a unique cluster at level $\ell-1$ that constitutes together with $C_i$ a killing pair for $P$.

Proof. Let $u$ be a $P$-mixed vertex of $\operatorname{hreg}_{F_{\ell-1}}^{(\ell)}(P)$ closer to $P$ than $C_i$, which reveals a point $c \in C_i$ for which $c \notin \text{conv } (P)$. Since we are in case 2, at least one such mixed vertex exists. By Property 2, the portion of $T(P)$ in $\operatorname{hreg}_{F_{\ell-1}}^{(\ell-1)}(P)$ is a connected subtree that can be regarded without loss of generality as a descendant of $u$ in $T(P)$. Then, $C_i$ is a forward limiting cluster w.r.t. $P$ with $P$-circle $\mathcal{K}_u$ for any $P$-mixed vertex $v$ of $\operatorname{hreg}_{F_{\ell-1}}^{(\ell)}(P)$, that is, $C_i \subset D_v$ and in
particular, \( C_i \subset D_f^y \cup \text{conv} \, P \); thus, \( c \in D_f^y \). Let \( w \) be the first \( P \)-mixed vertex of \( \text{hreg}^{(\ell-1)}_F(P) \) encountered as we traverse \( T(P) \) from \( u \) to its portion enclosed in \( \text{hreg}^{(\ell)}_F(P) \). Let \( Q \) be the cluster inducing \( w \) and let \( q \) be the point in \( Q \) for which \( d(w,q) = d_f(w,Q) \). By definition of \( w \), cluster \( Q \) must be rear limiting with respect to \( C_i \) with the \( C_i \)-circle \( K_w \), and thus \( q \in D_w^r \). Thus, \( \{Q, C_i\} \) is a killing pair for \( P \) and \( q, c \) lie at opposite sides of the chord of \( P \) inducing \( w \). All other mixed vertices \( v_i \) of \( \text{hreg}^{(\ell-1)}_F(P) \), where \( v_i \neq r \), must be induced by clusters \( Q_i \) that are forward limiting with respect with \( C_i \)-circle \( K_{v_i} \) (see [19, Lemma 2]). Thus, any point \( q_i \), inducing a \( P \)-mixed vertex \( v_i \), considered during our algorithm, other than \( q \), must lie on the same side of \( D_{v_i} \) as \( c \). Thus, our algorithm correctly sets \( K = Q \). Furthermore, there is no other cluster on level \( \ell - 1 \) that can form a killing pair with \( C_i \) for \( P \).

We summarize the result on the Voronoi hierarchy in the following lemma, which is easily derived from Lemmas 3 to 6 and the discussion in Section 5.

**Theorem 1.** The Voronoi hierarchy for the Hausdorff Voronoi diagram of a family of \( k \) clusters of total complexity \( n \) has expected size \( O(n) \). Both the point location query and the parametric point location take expected \( O(\log n \log k) \) time. Insertion of a cluster takes amortized \( O((N/k) \log n) \) time, where \( N \) is the total number of update operations in all levels during the insertion of all \( k \) clusters.

**Proof.** The expected space of the Voronoi hierarchy is analyzed by Lemma 3. The time complexity of the point location follows from the fact that the step of the walk reduces the distance, see Lemma 11. Since the distance is reduced, by Lemma 4 the walk at each level has constant length. One step of the walk is two binary searches, which take \( O(\log n) \) time, and the expected number of levels is within \( O(\log k) \). Therefore, the point location query takes expected \( O(\log n \log k) \) time.

The same argument (due to Lemma 6) holds in case of the parametric point location.

During the insertion of a cluster into Voronoi hierarchy, two procedures are performed: updates at all levels and the linking of disappeared regions. The former operations are all counted by \( N \). Consider the linking of the cluster \( P \), which is critical at level \( \ell \). We visit all \( P \)-mixed vertices of \( \text{HVD}(F^{(\ell-1)}_i) \), which are all deleted during the same step. We also visit all \( P \)-mixed vertices of \( \text{HVD}(F^{(\ell)}_i) \), which are not deleted. But these vertices can be visited at most twice: when \( P \) is critical at level \( \ell - 1 \) and when \( P \) is critical at level \( \ell \). Obviously, \( P \) can be critical at a certain level at most once. The cost of each of considered visits is within \( O(\log n) \). Thus, the claimed complexity follows.

### 6 Complexity Analysis

The running time of our algorithm depends on the number of update operations (insertions and deletions) during the construction of the diagram. Using the
Clarkson-Shor technique [8], we prove that the expectation of this number is linear, when clusters are inserted in random order. Note that in contrast to the standard probabilistic argument, our proof does not assume sites (clusters) to have constant size.

Theorem 2. The expected number of update operations is $O(n)$.

The proof of this theorem is given below in Section 6.1. Theorem 2 can be easily extended to all levels of the Voronoi hierarchy (see Section 6.2). The total time for the construction of the separator decomposition for all clusters is $O(n \log n)$ (see Lemma 2). For each cluster $C \in F$, we perform $O(|C|)$ point location queries and at most one parametric point location in Voronoi hierarchy. By Lemma 2 and Theorems 1 and 2, we conclude.

Theorem 3. The Hausdorff Voronoi diagram of non-crossing clusters can be constructed in $O(n \log n \log k)$ expected time and $O(n)$ expected space.

Deterministic $O(n)$ space could be achieved by using a dynamic point location data structure for a planar subdivision [3, 5]. On this data structure, the parametric point location can be performed as a simplified form of the parametric search, as described by Cheong et al. [7]. The time complexity of such a query is $t_q^2 q$, where $t_q$ is the time complexity of point location in the chosen data structure. In particular, the data structure by Baumgarten et al. [5] has $t_q \in O((\log n \log \log n)^2)$, which leads to the construction of the Hausdorff Voronoi diagram with expected running time $O(n \log^2 n (\log \log n)^2)$ and deterministic space $O(n)$.

6.1 Proof of Theorem 2 – Expected number of operations

We will associate the operations with features of the diagrams. Each feature (vertex, edge, face) of the diagram that appears during the incremental algorithm has been inserted by an operation. If a feature is deleted, then it cannot be inserted again in the future, because of the monotonicity of the Hausdorff Voronoi diagram. As a result, the number of deletion operations is bounded by the number of insertion operations. So, we intend to prove that the expected number of features that appear during the construction of the diagram is within $O(n)$. To that end, we can ignore features associated only with the farthest Voronoi diagrams of each cluster, because even their total worst case combinatorial complexity is within $O(n)$.

Configurations. We define some notions that are related with some features of the diagram.

Definition 6. A configuration is a triple of points $(p, q, r)$ such that $p$, $q$, $r$ lie on the boundary of a disk $D$ and $q$ is contained in the interior of the counterclockwise arc from $p$ to $r$. We call $D$ the disk of the configuration, its center the center of the configuration, and the counterclockwise arc $pr$ the arc of the configuration.
A configuration is pure if its three points belong to three different clusters of \( F \) and all other points of these three clusters are contained in the interior of the disk of the configuration.

A configuration is mixed if its three points belong to two different clusters of \( F \) and all other points of these two clusters are contained in the interior of the disk of the configuration.

From now on, configurations of our interest will be either pure or mixed. Therefore, each configuration is either associated with three (a pure one) or two (a mixed one) clusters.

**Definition 7.** A cluster \( C \) is in conflict with a configuration if (a) \( C \) does not contain any of the points in the configuration and (b) \( C \) is contained in the union of the interior of the disk of the configuration and the arc of the configuration.

The weight of a configuration is the number of clusters in conflict with it.

**Lemma 8.** The number of zero weight configurations of \( F \) is of the same order as the combinatorial complexity of the Hausdorff Voronoi diagram of \( F \).

**Proof.** Each zero weight configuration is associated with a vertex of the Hausdorff Voronoi diagram. Indeed the center of this configuration is at the vertex and the disk of the configuration contains the clusters associated with the configuration. Consider a vertex \( v \) of the Hausdorff Voronoi diagram. The degree of \( v \) in the arrangement equals the number of configurations with center \( v \) plus the number of some features that are associated just with farthest Voronoi diagrams (that we have claimed before that we can ignore). As a result, zero weight configurations estimate well the combinatorial complexity of the Hausdorff Voronoi diagram.

**Configurations of weight at most \( k \).** Let \( K_{0}^{\text{pure}}(F) \), \( K_{k}^{\text{pure}}(F) \), \( K_{\leq k}^{\text{pure}}(F) \) denote the sets of pure configurations of zero weight, weight equal to \( k \), and weight at most \( k \), of a family \( F \) of non-crossing clusters, respectively. Let \( N_{0}^{\text{pure}}(F) \), \( N_{k}^{\text{pure}}(F) \), \( N_{\leq k}^{\text{pure}}(F) \) denote the cardinality of the aforementioned sets, respectively. Define analogously the sets of mixed configurations \( K_{0}^{\text{mix}}(F) \), \( K_{k}^{\text{mix}}(F) \), \( K_{\leq k}^{\text{mix}}(F) \) and their cardinalities \( N_{0}^{\text{mix}}(F) \), \( N_{k}^{\text{mix}}(F) \), \( N_{\leq k}^{\text{mix}}(F) \), respectively. Both \( N_{0}^{\text{pure}}(F) \) and \( N_{0}^{\text{mix}}(F) \) are within \( O(\sum_{C \in F} |C|) = O(n) \) [19]. Then, using the Clarkson-Shor technique [8] with a random sample of the clusters in the family \( F \), we can obtain:

\[
N_{\leq k}^{\text{pure}}(F) \leq c_{\text{pure}}^{\text{pure}} \cdot nk^2 \quad \text{and} \quad N_{\leq k}^{\text{mix}}(F) \leq c_{\text{mix}}^{\text{mix}} \cdot nk,
\]

for \( k > 0 \) and some constants \( c_{\text{pure}}^{\text{pure}} \) and \( c_{\text{mix}}^{\text{mix}} \).

**Appearance of a feature.** Consider a configuration \( c \) of weight \( k \) in family \( F \) with \( m \) clusters. Assume the Hausdorff Voronoi diagram of \( F \) is constructed with the incremental algorithm and the clusters are inserted according to permutation \( \pi \). The feature corresponding to \( c \) appears at some stage of the incremental
A randomized incremental approach for the HVD of non-crossing clusters

Algorithm if and only if the clusters associated with \( c \) occur in \( \pi \) before the \( k \) clusters that conflict with configuration \( c \). This event happens with probability

\[
\Pr[\text{pure } c \text{ feature appears}] = \frac{3k!}{(k+3)!} = \frac{6}{(k+1)(k+2)(k+3)}
\]

for pure configurations and with probability

\[
\Pr[\text{mixed } c \text{ feature appears}] = \frac{2k!}{(k+2)!} = \frac{2}{(k+1)(k+2)}
\]

for mixed configurations.

The expected number of appearances of features corresponding to a pure configuration is therefore:

\[
\sum_{k=0}^{m-3} \sum_{c \in K_k^{\text{pure}}(F)} \Pr[\text{pure } c \text{ feature appears}] = \sum_{k=0}^{m-3} \sum_{c \in K_k^{\text{pure}}(F)} \frac{6}{(k+1)(k+2)(k+3)}
\]

\[
= 6 \sum_{k=0}^{m-3} \frac{N_k^{\text{pure}}(F)}{(k+1)(k+2)(k+3)} = N_0^{\text{pure}}(F) + 6 \sum_{k=1}^{m-3} \frac{N_{k-1}^{\text{pure}}(F) - N_k^{\text{pure}}(F)}{(k+1)(k+2)(k+3)}
\]

\[
= \frac{3}{4} N_0^{\text{pure}}(F) + 18 \sum_{k=1}^{m-4} \frac{N_{k-1}^{\text{pure}}(F)}{(k+1)(k+2)(k+3)(k+4)} + \frac{N_{m-2}^{\text{pure}}(F)}{(m-2)(m-1)m}
\]

\[
\leq \frac{3}{4} N_0^{\text{pure}}(F) + 18 \sum_{k=1}^{m-4} \frac{c_{\text{pure}} \cdot n k^2}{(k+1)(k+2)(k+3)(k+4)} + \frac{c_{\text{pure}} \cdot n (m-3)^2}{(m-2)(m-1)m}
\]

\[
\leq \frac{3}{4} N_0^{\text{pure}}(F) + 18 \cdot c_{\text{pure}} \cdot n \sum_{k=1}^{m-4} \frac{1}{k^2} + \frac{c_{\text{pure}} \cdot n}{m} = O(n)
\]

Similarly, the expected number of appearances of features corresponding to a mixed configuration is:

\[
\sum_{k=0}^{m-2} \sum_{c \in K_k^{\text{mix}}(F)} \Pr[\text{mixed } c \text{ feature appears}] = \sum_{k=0}^{m-2} \sum_{c \in K_k^{\text{mix}}(F)} \frac{2}{(k+1)(k+2)}
\]

\[
= 2 \sum_{k=0}^{m-2} \frac{N_k^{\text{mix}}(F)}{(k+1)(k+2)} = N_0^{\text{mix}}(F) + 2 \sum_{k=1}^{m-2} \frac{N_{k-1}^{\text{mix}}(F) - N_k^{\text{mix}}(F)}{(k+1)(k+2)}
\]

\[
= \frac{1}{2} N_0^{\text{mix}}(F) + 4 \sum_{k=1}^{m-3} \frac{N_{k-1}^{\text{mix}}(F)}{(k+1)(k+2)(k+3)} + \frac{c_{\text{mix}} \cdot n (m-2)}{(m-1)m}
\]

\[
\leq \frac{1}{2} N_0^{\text{mix}}(F) + 4 \cdot c_{\text{mix}} \cdot n \sum_{k=1}^{m-3} \frac{1}{k^2} + \frac{c_{\text{mix}} \cdot n}{m} = O(n)
\]

Therefore, we have proved the following, which implies Theorem 2.

**Lemma 9.** The expected number of features that appear during the incremental construction is within \( O(n) \).
6.2 Number of Update Operations on Voronoi Hierarchy

We argue about structural changes during the incremental construction in all Hausdorff Voronoi diagrams which are maintained (one per level) within the hierarchy. Recall from Section 6, that the expected number of structural changes is proportional to the expected number of appearing features, where features correspond to pure and mixed configurations.

Lemma 10. The expected number of features that appear at any level of the Hausdorff Voronoi hierarchy during the incremental construction is within $O(n)$.

Proof. The expected total number of points in $F^{(t)}$ is $\beta^t n$. Using Lemma 9, the expected number of features that appear during the incremental construction of $HVD(F^{(t)})$ at level $\ell$ is within $O(\beta^\ell n)$. Therefore, the expected number of features that appear at any level is $\sum_{\ell=0}^{\infty} O(\beta^\ell n) = O(n)$.

7 Details for the Single Step of a Walk in Voronoi Hierarchy

The Hausdorff Voronoi diagram of $F^{(t)}$ at any level $\ell \in \{0, \ldots, h\}$ of the Voronoi hierarchy, $HVD(F^{(t)})$ is equivalent to $HVD(F^{(t)}_{act})$, where $F^{(t)}_{act} = \{\hat{C}^{(t)}|\forall C \in F^{(t)}\}$. The distance from a point $t$ to a cluster $P \in F^{(t)}$ in this diagram is $d_t(t, \hat{C}^{(t)})$, denoted as $d_t^{(t)}(t, P)$.

Lemma 11. Let the cluster $Q \in F^{(t)}$ be obtained from $C$ and $q$ as described above. Assume that the cluster $C$ is not closest to $q$ among clusters in $F^{(t)}$. Then $d_{q}^{(t)}(q, Q) < d_{t}^{(t)}(q, C)$.

Proof. Let $D(x_0, R)$ denote the closed disk with center $x_0$ and radius $R$. Let $c \in \hat{C}^{(t)}$ be the farthest point to $q$ among points in $\hat{C}^{(t)}$. Define the closed disk $D_q = D(q, |cq|)$. Then, $D_q \supseteq \hat{C}^{(t)}$. We treat separately the following two cases:

1. Ray $\overrightarrow{cq}$ intersects the Hausdorff boundary of $c$. In that case, segment $cq$ intersects the Hausdorff boundary of $c$ at a single point $x$ in the interior of segment $cq$, i.e., $|cx| < |cq|$ (see Fig. 6). Point $x$ is equidistant from $C$ and $Q$. For the closed disk $D_x = D(x, |cx|)$, we have $D_x \supseteq Q$ and $D_q \supseteq D_x$. Moreover, the only point on the boundary of $D_q$ which lies also in $D_x$ is $c$ and $c \notin Q$ (because $c \in C$). Therefore, $Q$ is contained in the interior of $D_q$, which implies $d_t(q, Q) < d_t^{(t)}(q, C) \leq d_t(q, C)$. Since $d_t^{(t)}(q, Q) \leq d_t(q, Q)$, the claim follows.

2. Ray $\overrightarrow{cq}$ does not intersect the Hausdorff boundary of $c$. In that case, let $K = \text{bd} \text{freg}_{\hat{C}^{(t)}}(c)$. Then, the segment $cq$ intersects $K$ at a single point $y$ and thus $|cy| \leq |cq|$ (see Fig. 7). Consider the closed disk $D_y = D(y, |cy|)$. We have $\hat{C}^{(t)} \subseteq \text{conv} \hat{C}^{(t)} \subseteq D_q \subseteq D_q$. Moreover, since $y$ lies on the interior of an edge of $FVD(\hat{C}^{(t)})$, the boundary of $D_y$ contains exactly two points $c$ and $c^*$ of $\hat{C}^{(t)}$ and all other points of conv $\hat{C}^{(t)}$ are contained in the interior of
A randomized incremental approach for the HVD of non-crossing clusters

A randomized incremental approach for the HVD of non-crossing clusters

\[ c \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \]

\[ Q \rightarrow x \rightarrow D_q \]

Fig. 6. Ray \( \overrightarrow{cq} \) intersects the Hausdorff boundary of \( c \) at point \( x \). Closed disk \( D_x \) contains both \( C \) and \( Q \). Moreover, \( D_x \subset D_q \).

\[ c \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow q \rightarrow Q \rightarrow x \rightarrow D_q \rightarrow D_x \]

Fig. 7. Ray \( \overrightarrow{cq} \) does not intersect the Hausdorff boundary of \( c \). It intersects the boundary \( K \) of \( \text{freg}_{\hat{C}^f}(c) \) at point \( y \) not closest to \( C \) among clusters in \( F^f \). The mixed vertex \( x \) is closer to \( y \) on curve \( K \) and is equidistant to \( C \) and \( Q \). \( D^f \) is shown in gray.

Now, walk on \( K \) from \( y \) until you reach the Hausdorff boundary of \( c \), say at point \( x \). Then, \( x \) is a mixed vertex of \( C \) and thus it is equidistant from \( c \), \( c^* \), and the neighboring cluster \( Q \). Consider the closed disk \( D_x = D(x, |cx|) \). Cluster \( \hat{Q}^f \) is limiting w.r.t. \( \hat{C}^f \) circle \( K_x \) (see Definition 3). Assume without loss of generality that it is forward limiting. Then, \( \hat{Q}^f \subset D^f_x \subset \text{conv } \hat{C}^f \), where \( D^f_x = \text{conv } K_x \). Since the cluster \( C \) is not closest to \( y \) among clusters in \( F^f \), we have \( D^f_x \subset D_y \subseteq D_q \) (see Fig. 7, where \( D^f_y \) is shown with gray color). Except \( c \) and \( c^* \) every point in \( D^f_x \cup \text{conv } \hat{C}^f \) is contained in the interior of \( D_q \). Thus, \( \hat{Q}^f \) (which contains neither \( c \) nor \( c^* \)) is contained in the interior of \( D_q \) and this implies \( d^f(q, Q) < d^f(q, C) \leq d^f(q, C) \).

The point \( c \) is one of two tangents from \( q \) to \( \text{CH} (\hat{C}^f) \):

**Lemma 12.** Let \( C \) be a cluster at level \( \ell \) and \( b \) an inactive point of cluster \( C \) at level \( \ell \). Then, \( \text{freg}_{\hat{C}^f}(b) \subset \text{freg}_{\hat{C}^f}(c_1) \cup \text{freg}_{\hat{C}^f}(c_2) \), where \( c_1, c_2 \in \hat{C}^f \) lie on the two tangents from \( b \) to \( \text{conv } \hat{C}^f \).
Proof. Set \( C' = \hat{C}(l) \cup \{b\} \). Since \( C' \subseteq C \), we have \( \text{freg}_{C'}(b) \subseteq \text{freg}_C(b) \) and thus it is enough to prove that \( \text{freg}_{C'}(b) \subseteq \text{freg}_C(c_1) \cup \text{freg}_C(c_2) \).

First, it is not difficult to see that \( c_1 \) and \( c_2 \) are consecutive points in \( \text{CH}(\hat{C}(l)) \). Assume for the sake of contradiction that \( \text{freg}_{C'}(b) \cap \text{freg}_C(c_3) \neq \emptyset \) for some \( c_3 \in \hat{C}(l) \) such that \( c_3 \) is different from \( c_1 \) and \( c_2 \). Since both \( T(\hat{C}(l)) \) and \( T(C') \) have a tree-like structure and \( \text{freg}_{C'}(b) \) includes elements from at least three regions of \( \text{FVD}(\hat{C}(l)) \), \( \text{freg}_{C'}(b) \) has to contain at least one vertex \( v \) of \( \text{FVD}(\hat{C}(l)) \). Since \( v \in \text{freg}_{C'}(b) \), we have \( d(v, b) < d(v, c) \) for every \( c \in \hat{C}(l) \). But \( v \) also belongs to \( T(C) \cap \text{hreg}_{\mathcal{F}(l)}(C) \), which implies \( d(v, c) > d(v, b) \) for some \( c \in \hat{C}(l) \), since \( b \) is not active at level \( l \); a contradiction.

Fig. 8. For cluster \( C = \{b, c_1, \ldots, c_4\} \in \mathcal{F}(l) \), point \( b \notin \hat{C}(l) \). Say \( c_1 \) and \( c_2 \) lie on the two tangents from \( b \) to \( \hat{C}(l) \). In subfigure (a), the statement of Lemma 12 is true and we have \( v \notin \text{freg}_{C'(l)}(b) \). In subfigure (b), \( v \in \text{freg}_{C'(l)}(b) \), which gives a contradiction. Region \( \text{hreg}_{\mathcal{F}(l)}(C) \) is shown as a gray area and a minimum enclosing circle centered at \( v \) is shown dotted.

8 Discussion and Open Problems

We have provided improved complexity algorithms for constructing the Hausdorff Voronoi diagram of a family of non-crossing point clusters based on randomized incremental construction and point location. There is still a gap in the complexity of constructing the Hausdorff Voronoi diagram between our best \( O(n \log^2 n) \) expected time algorithm and the well-known \( \Omega(n \log n) \) time lower bound. An open problem is to close or reduce this gap. It is interesting that in the \( L_\infty \) metric, a simple \( O(n \log n) \)-time \( O(n) \)-space algorithm, based on plane sweep, is known [22]. In parallel, we are considering the application of the randomized incremental construction paradigm through history graphs. In future research we plan to consider families of arbitrary point clusters that may be crossing. In this case, the size of the diagram can vary from linear to quadratic, and therefore, an output-sensitive algorithm is most desirable. Another direction for research is to study the problem for clusters of segments, clusters of convex polygons or other shapes, rather than clusters of points.
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A Details of Growing the New Region

Given the diagram $\text{HVD}(F_{i-1})$, the cluster $C_i$, and a point $t \in \text{hreg}_{F_i}(C_i)$, we shall grow $\text{hreg}_{F_i}(C_i)$ around $t$ and update $\text{HVD}(F_{i-1})$ to get $\text{HVD}(F_i)$.

Visibility-based decomposition of the diagram. It will be beneficial to have a refinement of $\text{HVD}(F_i)$ such that the work performed when updating its face $f$ is proportional only to the deleted vertices and edges of $f$. To achieve this, we maintain the visibility-based decomposition [21] of $\text{HVD}(F_i)$, denoted by $\text{HVD}^*_i(F_i)$ and defined as follows. For every point $c \in C_i$, and every Voronoi vertex $x$ on the Hausdorff boundary of $c$, add to the diagram the segment $s = cx \cap \text{hreg}(c)$. See Fig. 9. The combinatorial complexity of $\text{HVD}^*_i(F_i)$ is $O(n)$ [19]. Each face $f$ of $\text{HVD}^*_i(F_i)$ is convex. The boundary of $f$ that is common with the farthest skeleton of some cluster is called the $T$-chain of $f$.

![Fig. 9. Visibility decomposition of a Hausdorff region of a point](image-url)

Computing vertices of $T(C_i)$ closest to $C_i$. For every vertex $v \in T(C_i)$, we do a point location of $v$ in $\text{HVD}^*_i(F_{i-1})$ and we check whether $v$ is closer to $C_i$ or another cluster $C \in F_{i-1}$. Remember that we also have a point $t \in \text{hreg}_{F_i}(C_i)$. If $t$ is not a vertex of $T(C_i)$, we also add it to this computed set. By Property 1, these vertices are contained in a subtree of $T(C_i)$.

Edges of $T(C_i)$ through which $\text{bd}(\text{hreg}_{F_i}(C_i))$ passes. If $|C_i| > 1$, we call an edge $vu$ of $T(C_i)$ a switch edge if $v$ is closer to $C_i$ than to any $C \in F_{i-1}$ and $u$ not closer to $C_i$ than to some $C \in F_{i-1}$. We compute the set of all switch edges of $T(C_i)$. Say their cyclic order is $e_0, \ldots, e_{\kappa-1}$. Then, $\text{bd}(\text{hreg}_{F_i}(C_i))$ consists of $\kappa$ curves, each one connecting two points in consecutive edges in the above cyclic order. These points $p_0, \ldots, p_{\kappa-1}$ correspond to $C_i$-mixed vertices in $\text{HVD}(F_i)$ [19].

Computing $p_0 \in e_0$. For switch edge $e_0 = uv$, with $u \in \text{hreg}_{F_i}(C_i)$ and $v \notin \text{hreg}_{F_i}(C_i)$, the method we use is as follows: We trace the movement of a point $y$ on $e_0$ starting from $u$ and going towards $v$, until $y$ reaches an element (vertex, edge, or face) of $\text{HVD}^*_i(F_{i-1})$ which intersects $\text{bd}(\text{hreg}_{F_i}(C_i))$. Then, we take the intersection of $yu$ with a bisector between some specific point of $C_i$ and another specific point of a cluster in $F_{i-1}$.

Tracing is done by following elements of $\text{HVD}^*_i(F_{i-1})$ along the switch edge $e_0$. A face $f$ can have a $T$-chain of non-constant size. When we trace on $e_0$ along
such a face \( f \), it is crucial to only visit a number of elements adjacent to \( f \), proportional to the number of updated elements adjacent to \( f \).

In particular, assume that \( y \in \text{cl} f \), and \( v \notin \text{cl} f \). Moreover, \( yv \cap f \neq \emptyset \). Since \( f \) is convex, there is a single point \( z \) of \( yv \) which is different from \( y \) and lies on the boundary of \( f \) (see Fig. 10). If \( p_0 \in \text{cl} f \), this can be easily detected without computing \( z \), by computing the intersection of \( yv \) with a bisector between some specific point of \( C_i \) and another specific point of a cluster in \( F_{i-1} \) (corresponding to face \( f \)). Otherwise, we have to compute \( z \), as follows.

We first check if \( yv \) intersects any of the edges not on the \( T \)-chain of \( f \) (there are at most three of these edges). If there is such an intersection, we found our point \( z \). Otherwise, \( yv \) must intersect with the \( T \)-chain of \( f \). In that case, consider the two extreme vertices \( v^- \) and \( v^+ \) of the \( T \)-chain (see Fig. 10). By Property 1, at least one of \( v^- \) and \( v^+ \) is in \( \text{hreg}_{F_i}(C_i) \). By comparing distances of \( v^- \), \( v^+ \) from \( C_i \) and the closest clusters in \( F_{i-1} \), we find (at least) one of \( v^- \) and \( v^+ \) that is closest to \( C_i \) than any cluster in \( F_{i-1} \). Let it be \( v^- \). Then, by Property 1, the part of the \( T \)-chain from \( v^- \) to \( z \) is closest to \( C_i \) and has to be updated. Therefore, we can search for the intersection point \( z \) by following the \( T \)-chain, starting from \( v^- \) (see Fig. 10).

**Computing the boundary of \( \text{hreg}_{F_i}(C_i) \).** We now have a point \( p_0 \in e_0 \) on the boundary of \( \text{hreg}_{F_i}(C_i) \). We start from \( p_0 \) and we trace \( \text{bd}(\text{hreg}_{F_i}(C_i)) \) counterclockwise until we reach (the yet unknown) \( p_1 \) on \( e_1 \). This tracing is done by following parts of bisectors of \( \text{HVD}^*(F_{i-1}) \) similarly to the tracing to find \( p_0 \) described above. See Fig. 11. As we move along different faces, we might have to use a different halfline \( h \), having the direction of the relevant bisector in the current face of \( \text{HVD}^*(F_{i-1}) \). Finally, we reach point \( p_1 \) on \( e_1 \) having traced the portion of \( \text{bd}(\text{hreg}_{F_i}(C_i)) \) between \( p_0 \) and \( p_1 \). We continue and trace the remaining \( \kappa - 1 \) portions until we get back to \( p_0 \), having traced completely \( \text{bd}(\text{hreg}_{F_i}(C_i)) \).

**Updating the diagram.** Now that we have computed the boundary of the region of the new cluster, we superimpose it on the existing diagram \( \text{HVD}^*(F_{i-1}) \), delete everything from \( \text{HVD}^*(F_{i-1}) \) that is in the interior of the new cluster’s region,
add $\text{hreg}_F(C_i) \cap T(C_i)$ and the visibility decomposition segments in order to get the full $\text{HVD}^*(F_i)$. We note that, with some care, the amount of elements of the arrangement of $\text{HVD}^*(F_{i-1})$ that we visit is of the order of the number of changes that have to be done in $\text{HVD}^*(F_{i-1})$, in order to get $\text{HVD}^*(F_i)$. 