AFFINE INTERVAL EXCHANGE TRANSFORMATIONS
WITH FLIPS AND WANDERING INTERVALS

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Abstract. There exist uniquely ergodic affine interval exchange transformations of \([0,1]\) with flips which have wandering intervals and are such that the support of the invariant measure is a Cantor set.

1. Introduction

Let \(N\) be a compact subinterval of either \(\mathbb{R}\) or the circle \(S^1\), and let \(f : N \to N\) be piecewise continuous. We say that a subinterval \(J \subset N\) is a wandering interval of the map \(f\) if the forward iterates \(f^n(J), n = 0, 1, 2, \ldots\), are pairwise disjoint intervals, each not reduced to a point, and the \(\omega\)-limit set of \(J\) is an infinite set.

A great deal of information about the topological dynamics of a map \(f : N \to N\) is revealed when one knows whether \(f\) has wandering intervals. This turns out to be a subtle question whose answer depends on both the topological and regularity properties of the map \(f\).

The question of the existence of wandering intervals first arose in the case where \(f\) is a diffeomorphism of the circle \(S^1\). The Denjoy counterexample shows that even a \(C^1\) diffeomorphism \(f : S^1 \to S^1\) may have wandering intervals. This behaviour is ruled out when \(f\) is smoother. More specifically, if \(f\) is a \(C^1\) diffeomorphism of the circle such that the logarithm of its derivative has bounded variation, then \(f\) has no wandering intervals [6]. In this case the topological dynamics of \(f\) is simple: if \(f\) has no periodic points, then \(f\) is topologically conjugate to a rotation.

The first results ensuring the absence of wandering intervals on continuous maps satisfying some smoothness conditions were provided by Guckenheimer [8], Yoccoz [19], and Blokh and Lyubich [2]. Later on, de Melo et al. [13] generalised these results, proving that if \(N\) is compact and \(f : N \to N\) is a \(C^2\) map with non-flat critical points, then \(f\) has no wandering intervals. Concerning discontinuous maps, Berry and Mestel [4] found a condition which excludes wandering intervals in Lorenz maps — interval maps with a single discontinuity. Of course, conservative maps and, in particular, interval exchange transformations admit no wandering intervals.

We consider the following generalisation of interval exchange transformations.

Let \(0 \leq a < b\) and let \(\{a, b\} \subset D \subset [a, b]\) be a discrete set consisting of \(n + 1\) points. We say that an injective, continuously differentiable map \(T : [a, b] \to [a, b]\)
defined on $D(T) = [a, b] \setminus D$ is an affine interval exchange transformation of $n$ subintervals, or $n$-AIET for short, if $|DT|$ is a positive, locally constant function such that $T([a, b] \setminus D)$ is all of $[a, b]$ except for finitely many points. We also assume that the points in $D \setminus \{a, b\}$ are non-removable discontinuities of $T$. We say that an AIET is oriented if $DT > 0$; otherwise we say that $T$ has flips. An isometric IET of $n$ subintervals, or $n$-IET for short, is an $n$-AIET satisfying $|DT| = 1$ everywhere.

Levitt [11] found an example of a non-uniquely ergodic oriented AIET with wandering intervals. Gutierrez and Camelier [4] constructed an AIET with wandering intervals that is semiconjugate to a self-similar IET. The regularity of conjugacies between AIETs and self-similar IETs was examined by Cobo [5] and by Liouss and Marzougui [12]. Recently, Bressaud, Hubert and Maass [3] provided sufficient conditions for a self-similar IET to have an AIET with a wandering interval semi-conjugate to it.

In this paper we present an example of a self-similar IET with flips having the particular property that we can apply the main result of the work [3] to obtain a 5-AIET with flips that is semiconjugate to the IET and has densely distributed wandering intervals. The AIET so obtained is uniquely ergodic [17] (see [14, 18]), and the support of the invariant measure is a Cantor set.

A few remarks are due in order to place this example in context. The existence of minimal non-uniquely ergodic AIETs with flips and wandering intervals would follow by the argument of Levitt [11], provided we know a minimal non-uniquely ergodic IET with flips. However, no example of a minimal non-uniquely ergodic IET with flips is known, although it is possible to insert flips in the example of Keane [10] (for oriented IETs) to get a transitive non-uniquely IET with flips having saddle-connections. Computational evaluations indicate that it is impossible to obtain, via Rauzy induction, examples of self-similar 4-IETs with flips that meet the hypotheses of [3], despite this being possible in the case of oriented 4-IETs (see [4, 5]). Thus, the example we present here is the simplest possible in the sense that wandering intervals do not occur for AIETs with flips that are semiconjugate to a self-similar IET, obtained via Rauzy induction, defined on a smaller number of intervals.

2. Self-similar interval exchange transformations

Let $T : [a, b] \to [a, b]$ be an $n$-AIET defined on $[a, b] \setminus D$, where $D = \{x_0, \ldots, x_n\}$ and $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. Let $\beta_i \neq 0$ be the derivative of $T$ on $(x_{i-1}, x_i)$, $i = 1, 2, \ldots, n$. We shall refer to

$$x = (x_0, x_1, \ldots, x_n)$$

as the $D$-vector of $T$ (i.e. the domain-of-definition-vector of $T$). The vectors

$$\gamma = (\log |\beta_1|, \log |\beta_2|, \ldots, \log |\beta_n|)$$

and

$$\tau = \left( \frac{\beta_1}{|\beta_1|}, \frac{\beta_2}{|\beta_2|}, \ldots, \frac{\beta_n}{|\beta_n|} \right)$$

will be called the log-slope-vector and the flips-vector of $T$, respectively. Notice that $T$ has flips if and only if some coordinate of $\tau$ is equal to $-1$. Let

$$\{z_1, \ldots, z_n\} = \left\{ T \left( \frac{x_0 + x_1}{2} \right), T \left( \frac{x_1 + x_2}{2} \right), \ldots, T \left( \frac{x_{n-1} + x_n}{2} \right) \right\}$$
be such that $0 < z_1 < z_2 < \cdots < z_n < 1$; we define the permutation $\pi$ associated to $T$ as the one that takes $i \in \{1, 2, \ldots, n\}$ to $\pi(i) = j$ if and only if $z_j = T((x_{i-1} + x_i)/2)$.

It should be remarked that an AIET $T : [a, b] \to [a, b]$ with flips-vector $\tau \in \{-1, 1\}^n$ which has the zero vector as the log-slope-vector is an IET (with flips-vector $\tau$), and conversely. Let $J = [c, d]$ be a proper subinterval of $[a, b]$. We say that the IET $E$ is self-similar (on $J$) if there exists an orientation preserving affine map $L : \mathbb{R} \to \mathbb{R}$ such that $L(J) = [a, b]$ and $L \circ E = E \circ L$, where $E : J \to J$ denotes the IET induced by $E$ and $L(D(E)) \subset D(E)$. A self-similar IET $E : [a, b] \to [a, b]$ on a proper subinterval $J \subset [a, b]$ will be denoted by $(E, J)$.

Given an AIET $T : [a, b] \to [a, b]$, the orbit of $p \in [a, b]$ is the set

$$O(p) = \{T^n(p) \mid n \in \mathbb{Z} \text{ and } p \in D(T)\}.$$ 

The AIET $T$ is called transitive if there exists an orbit of $T$ that is dense in $[a, b]$. We say that the orbit of $p \in [a, b]$ is finite if $\#(O(p)) < \infty$. In this way, a point $p \in [a, b] - (D(T) \cup D(T^{-1}))$ has a finite orbit $O(p) = \{p\}$. A transitive AIET is minimal if it has no finite orbits.

Let $E : [a, b] \to [a, b]$ be an IET with D-vector $(x_0, x_1, \ldots, x_n)$. Denote by $J = [c, d]$ a proper subinterval of $[a, b]$. Suppose that $E$ is self-similar (on $J$); then there exists an IET $\tilde{E} : J \to J$ such that $L(J) = [a, b]$ and $L \circ \tilde{E} = \tilde{E} \circ L$. For $i = 0, 1, \ldots, n$, let $y_i = L^{-1}(x_i)$. Thus, the sequence of discontinuities of $\tilde{E}$ is $\{y_1, \ldots, y_{n-1}\}$.

We say that a non-negative matrix is eventually positive if some power of it is a positive matrix. A non-negative matrix is eventually positive if and only if it is both irreducible and aperiodic. Let $A$ be an $n \times n$ non-negative matrix whose entries are:

$$A_{ji} = \#\{0 \leq k \leq N_i : E^k((y_{i-1}, y_i)) \subset (x_{j-1}, x_j)\},$$

where $N_i$ is the smallest non-negative integer such that for some $y \in (y_{i-1}, y_i)$ (and therefore for all $y \in (y_{i-1}, y_i)$), $E^{N_i+1}(y) \in J$. We shall refer to $A$ as the matrix associated to $(E, J)$. Being self-similar, $E$ is also transitive, which implies that $A$ is eventually positive. Hence, by the Perron-Frobenius Theorem [7], $A$ possesses exactly one probability right eigenvector $\alpha \in \Lambda_n$, where

$$\Lambda_n = \{\lambda = (\lambda_1, \ldots, \lambda_n) \mid \lambda_i > 0 \forall i\}.$$ 

Moreover, the eigenvalue $\mu$ corresponding to $\alpha$ is simple, real and greater than 1; also, all other eigenvalues of $A$ have absolute value less than $\mu$. It was proved by Veech [17] (see also [14, 18]) that every self-similar IET is minimal and uniquely ergodic. Furthermore, following Rauzy [16], we conclude that

$$\alpha = (x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}).$$

3. The theorem of Bressaud, Hubert and Maass

Let $A \in SL_n(\mathbb{Z})$ and let $\mathbb{Q}[t]$ be the ring of polynomials with rational coefficients in one variable. We say that two real eigenvalues $\theta_1$ and $\theta_2$ of $A$ are conjugate if there exists an irreducible polynomial $f \in \mathbb{Q}[t]$ such that $f(\theta_1) = f(\theta_2) = 0$. We say that an AIET $T$ of $[0, 1]$ is semiconjugate (resp. conjugate) to an IET $E$ of $[0, 1]$ if there exists a non-decreasing (resp. bijective) continuous map $h : [0, 1] \to [0, 1]$ such that $h(D(T)) \subset D(E)$ and $E \circ h = h \circ T$.
Theorem 1 (Bressaud, Hubert and Maass, 2007). Let $J$ be a proper subinterval of $[0,1]$, let $E : [0,1] \to [0,1]$ be an interval exchange transformation which is self-similar on $J$, and let $A$ be the matrix associated to $(E,J)$. Let $\theta_1$ be the Perron-Frobenius eigenvalue of $A$. Assume that $A$ has a real eigenvalue $\theta_2$ such that

1. $1 < \theta_2 < \theta_1$;
2. $\theta_1$ and $\theta_2$ are conjugate.

Then there exists an affine interval exchange transformation $T$ of $[0,1]$ with wandering intervals that is semiconjugate to $E$.

Proof. This theorem was proved in [3] for oriented IETs. The same proof holds word for word for IETs with flips. In this case, the AIET $T$ inherits its flips from the IET $E$ through the previously constructed semiconjugacy. □

4. The interval exchange transformation $E$

In this section we shall present the IET we shall use to construct the AIET with flips and wandering intervals. We shall need the Rauzy induction [16, 15] to obtain a minimal, self-similar IET whose associated matrix satisfies all the hypotheses of Theorem 1.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \Lambda_5$ be the probability Perron-Frobenius right eigenvector of the matrix

$$A = \begin{pmatrix} 2 & 4 & 6 & 5 & 2 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 & 0 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}.$$ 

The eigenvalues $\theta_1, \theta_2, \rho_1, \rho_2, \rho_3$ of $A$ are real and have approximate values

$$\theta_1 = 7.829, \quad \theta_2 = 1.588, \quad \rho_1 = 1, \quad \rho_2 = 0.358, \quad \rho_3 = 0.225,$$

and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, the probability right eigenvector associated to $\theta_1$, has approximate value

$$\alpha = (0.380, 0.091, 0.070, 0.170, 0.289).$$

In what follows we represent a permutation $\pi$ of the set $\{1,2,\ldots,n\}$ by the $n$-tuple $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$.

We consider the IET $E : [0,1] \to [0,1]$, which is determined by the following conditions:

1. $E$ has the D-vector $x = (x_0, x_1, x_2, x_3, x_4, x_5)$, where
   $$x_0 = 0, \quad x_i = \sum_{k=1}^{i} \alpha_k$$
   for $i = 1, \ldots, 5$;

2. $E$ has associated permutation $(5, 3, 2, 1, 4)$;
3. $E$ has flips-vector $(-1, -1, 1, 1, -1)$.

Lemma 2. The map $E$ is self-similar on the interval $J = [0,1/\theta_1]$, and $A$ is precisely the matrix associated to $(E,J)$. 

Proof: We apply the Rauzy algorithm to the IET $E$. We represent $E : I \rightarrow I$ by the pair $E^{(0)} = (\alpha^{(0)}, p^{(0)})$, where $\alpha^{(0)} = \alpha$ is its length vector and $p^{(0)} = (-5, -3, 2, 1, -4)$ is its signed permutation obtained by elementwise multiplication of the permutation $(5, 3, 2, 1, 4)$ with the flips-vector $(-1, -1, 1, 1, -1)$. We shall apply the Rauzy procedure fourteen times, obtaining IETs $E^{(k)} = (\alpha^{(k)}, p^{(k)})$, $k = 0, \ldots, 14$, with D-vector $\alpha^{(k)}$ given by $x^{(k)}_0 = 0$ and $x^{(k)}_i = \sum_{j=1}^{i} \alpha^{(k)}_j$ for $i = 1, 2, \ldots, 5$.

| $k$ | $p^{(k)}$ | $t^{(k)}$ |
|-----|-----------|-----------|
| 0   | -5 -3 2 1 -4 | 1         |
| 1   | -5 -3 2 1 0  | 0         |
| 2   | -4 -2 3 1 1  | 1         |
| 3   | -4 -2 3 -2 1 | 0         |
| 4   | -3 -4 2 5 1  | 1         |
| 5   | -3 -4 2 5 0  | 0         |
| 6   | -2 -5 1 -3 1 | 1         |
| 7   | -2 3 -5 4 1  | 0         |
| 8   | -3 4 -2 5 1  | 1         |
| 9   | -3 4 -2 5 1  | 1         |
| 10  | -3 4 -2 5 1  | 0         |
| 11  | -4 -5 3 2 1  | 1         |
| 12  | -4 -5 1 3 2  | 1         |
| 13  | -4 -5 2 1 -3 | 0         |
| 14  | -5 -3 2 1 -4 | 1         |

Table 1. Rauzy cycle with associated matrix $A$.

Given an IET $E^{(k)}$, defined on an interval $[0, L^{(k)}]$ and represented by the pair $(\alpha^{(k)}, p^{(k)})$, the IET $E^{(k+1)}$ is defined to be the map induced on the interval $[0, L^{(k+1)}]$ by $E^{(k)}$, where $L^{(k+1)} = L^{(k)} - \min \{\alpha^{(k)}_5, \alpha^{(k)}_s\}$ and $s$ is such that $|p^{(k)}_n(s)| = 5$. We say that the type $t^{(k)}$ of $E^{(k)}$ is 0 if $\alpha^{(k)}_5 > \alpha^{(k)}_s$ and 1 if $\alpha^{(k)}_5 < \alpha^{(k)}_s$. Notice that $\sum_{i=1}^{5} \alpha^{(k)}_i = L^{(k)}$.

The new signed permutations $p^{(k)}$ obtained by this procedure are given in Table 1 along with the type $t^{(k)}$ of $E^{(k)}$. The length vector $\alpha^{(k+1)}$ is obtained from $\alpha^{(k)}$ by the equation $\alpha^{(k)} = M(p^{(k)}, t^{(k)}) \cdot \alpha^{(k+1)}$, where $M(p^{(k)}, t^{(k)}) \in SL_n(\mathbb{Z})$ is a certain elementary matrix (see [9]). Moreover, we have that

$$M(p^{(0)}, t^{(0)}) \cdots M(p^{(13)}, t^{(13)}) = A.$$ 

Thus $\alpha^{(14)} = A^{-1} \cdot \alpha^{(0)} / \theta_1$, and $J = [0, L^{(14)}]$. Notice that $p^{(14)} = p^{(0)}$, so we have a Rauzy cycle: $E^{(14)}$ and $E^{(0)}$ have the same flips-vector and permutation. Hence $\tilde{E} = E^{(14)}$ is a $1/\theta_1$-scaled copy of $E = E^{(0)}$, and so $E$ is self-similar on the interval $J$.

As remarked earlier, since $E$ self-similar, we have that the matrix associated to $(E, J)$ is eventually positive. In fact, we have that $A$ is the matrix associated to $(E, J)$. To see this, for $i \in \{0, \ldots, 5\}$ let $y_i = x_i / \theta_1$ be the points of discontinuity for $\tilde{E}$. Table 2 shows the itinerary $I(i) = \{I(i)_k\}_{k=1}^{N_i}$ of each interval.
\[(y_{i-1},y_i), \text{ where } N_i = \min \{ n > 1 : E^{n+1}((y_{i-1},y_i)) \subset J \} \text{ and } I(i)_k = r \text{ if and only if } E^k((y_{i-1},y_i)) \subset (x_{r-1},x_r).\]

| i | N_i | I(i) |
|---|-----|-----|
| 1 | 4   | 1514|
| 2 | 11  | 15214152154|
| 3 | 17  | 15214153153153154|
| 4 | 14  | 15214153153154|
| 5 | 6   | 152154|

Table 2. Itineraries \(I(i), i \in \{1, \ldots, 5\}\).

The number of times that \(j\) occurs in \(I(i)\), for \(i,j \in \{1, \ldots, 5\}\), is precisely \(A_{ji}\); thus \(A\) is the matrix associated to the pair \((E,J)\), as required. \(\square\)

**Theorem A.** There exists a uniquely ergodic affine interval exchange transformation of \([0,1]\) with flips that has wandering intervals and is such that the support of the invariant measure is a Cantor set.

**Proof.** By construction, the matrix \(A\) associated to \((E,J)\) satisfies hypothesis (1) of Theorem 1. The characteristic polynomial \(p(t)\) of \(A\) can be written as the product of two irreducible polynomials over \(\mathbb{Q}[t]\):

\[p(t) = (1-t)(1-8t+18t^2-10t^3+t^4).\]

Thus the eigenvalues \(\theta_1\) and \(\theta_2\) are zeros of the same irreducible polynomial of degree four and so are conjugate. Hence, \(A\) also satisfies hypothesis (2) of Theorem 1, which finishes the proof. \(\square\)

Note that for an AIET \(T\), the forward and backward iterates of a wandering interval \(J\) form a pairwise disjoint collection of intervals. Moreover, when \(T\) is semi-conjugate to a transitive IET, as is the case in Theorem A, the \(\alpha\)-limit set and \(\omega\)-limit set of \(J\) coincide.

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