How is the learning trajectory of polynomial division topic?

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Abstract. The objective of this research was to describe the learning trajectory for topic of polynomial division. The research was carried out through design research. This research used the approach of Gravemeijer & Cobb that consists of: 1) Preparing for the experiment covering the preparation of Hypothetical Learning Trajectory (HLT) comparing three components, i.e. learning objectives, learning activities, and students’ hypothetical learning trajectory; 2) Experimental design was conducted in class of 53 students of XI MIPA 3 and XI MIPA 4; 3) Retrospective analysis, comparing HLT to actual learning practice. The result of this research indicated that the learning trajectory for topic of polynomial division was began with polynomial division by divisor \((x - h)\) first, then \((ax + b)\), and was divided by \((ax^2 + bx + c)\). Next, the methods in solving for polynomial division were long division, Horner, algebraic operation, and Horner-Kino, respectively. The next topic was the remainder theorem, consisting 3 subtopics, i.e. the remainder theorem with divisor \((x - h)\), \((ax - b)\) and \((x - a)(x - b)\). The last topic was factor theorem.

1. Introduction

In mathematics learning, students are required to have an understanding of the material and able to use their prior knowledge to construct new knowledge [1]. Mathematics learning in the classroom requires students to think, reason, and organize their prior knowledge to solve the problem. As the result from a lot of abilities which need to be owned by students in solving problems, the assumption arises that mathematics is a difficult subject and not desirable by students.

According to Liu and Ke [2], problem solving is intensive mastery which consists of the activities of gaining relevant knowledge to identify the main causes of the problem, develop solutions, and determine the best steps to solve the problem. A problem is a situation faced by a person or group that needs a solution, but they do not have a procedure to directly determine the solution [3]. Problem solving is a part of the whole of mathematics learning and should not be separated from it [1]. Problem solving is oftentimes one of challenges for students as they do not understand how the process or steps to solve the problems are [4].

One of the ways to find out students’ problem solving skills is by considering PISA’s scores since the aspects assessed in the Programme for International Students Assessment (PISA) are problem solving skills, reasoning ability, and communication skills [5]. Based on PISA 2015 results [6], Indonesian’s mathematics score reached 386, which increased 11 points compared to 2012 which only
reached 375. However this achievement is still less than the international mean which reached 490. In mathematical ability, Indonesia was ranked 63 out of 69 countries evaluated. PISA 2015 showed that 42.3% of Indonesian students who took the PISA test entered at level 2. At this level, students are able to interpret the relevant information from one source and can use basic algorithms, formulas, procedures or rules of problem solving in integers [6]. This result indicated that students still have not mastered in understanding of concepts, analytical skills and mathematical skills to solve problems. In learning mathematics, students in Indonesia are more accustomed with routine problems level 1 and level 2 at PISA [7]. Routine problems are problems which have predictable patterns and can be solved correctly without reading the entire problem carefully, whereas non-routine problems have more than one step solutions, and the problems given need to be read carefully [8]. In consequence, non-routine problems are more complex and difficult than routine problems [9]. Usually, the problems given in class more focus on problems which have a low level of difficulty [10].

Problem solving skills is the skill owned by students through the learning process in the class, the teacher as a facilitator should develop an effective learning design in order to improve students’ problem solving skills. In developing of learning design, teachers need to observe learning trajectory [11]. Clement and Sarama [12] defined mathematical learning trajectory as a description of how children think and learn in a specific mathematical domain, connect, and also guess the trajectory through a series of learning tasks which have been designed. It intends to appear children’s mental processes or suspected behaviour that will be carried out by children through the development of their thinking level to reach certain learning objective. Teacher as a facilitator is expected to be able to create an effective learning trajectory, so that the material, which is learned by students, can be understood properly.

Learning trajectory can be illustrated as a flow to reach the expected goals. To reach the goal, steps are needed to be done. The steps to achieve the goal require an ability to organize a variety of ways which are appropriate and structured. In learning trajectory, students organize the right trajectory to achieve the learning objectives, such as what concept that needs to be understood first in order to achieve the learning objectives. Learning trajectory describes how the level of thinking of children [12]. It is arranged systematically, so that students can solve the problems given in accordance with the rules of a good problem solving process.

One of the mathematics materials that has a high of element of interactivity in terms of learning trajectory and the process of solving problems is polynomial in class XI. This is supported by the absorption data of National Examination in year 2016 [13] which shows that the percentage of national completeness on indicator determines the quotient of polynomial having a degree of 3 with one of the coefficients being a variable if divided by \((ax + b)\) just reaches 47.24%. It means that there are 52.76% of students who have not been able to solve problems about degree 3 polynomial. By considering this result, it is assumed that the learning trajectory usage in the topic of polynomial division is not systematic yet.

Solving a mathematical problem is a continuous step. On the issue of polynomial division, the result of observations in class XI MIPA 1 SMAN 1 Pengasih found that some students have not understand yet how to determine the remainder of polynomial division by divisor \((ax + b)\) using the Horner method. Whereas, the prior knowledge is important to learn the next topic [14]. So after being given next problem related to polynomial division by divisor\((ax^2 + bx + c)\), students cannot solve the problem. This indicates that the prior knowledge of student in polynomial division was still low.

Learning trajectory of polynomial division has not been used in SMAN 1 Pengasih. Based on observation, we found that a teacher only gives formulas to students without giving them a chance to activate their prior knowledge concerning how the element of interactivity of polynomial division with the other topic is. It is one of the cases that occurred in the learning of polynomial division in the class, so we need a solution to increase the percentage of completeness of polynomial division. Therefore, this study creates a learning trajectory in polynomial division.
2. Method

2.1. Type of Study

Design research was used in this study. Design research which requires a design process is an important part [15]. The design process intended in this study is the design of learning trajectory on polynomial division. There are several design research models, in this study researchers used a Gravemeijer and Cobb model, i.e. preparing for the experiment/preparation and design phase, design experiment and Retrospective analysis (review analysis).

The research design process is guided by an instrument called hypothetical learning trajectory (HLT) which was developed by Fruedenthal. According to Simon [15] hypothetical learning trajectory is defined as having three components namely, learning objectives that define the direction of learning objectives, learning activities, and learning hypotheses to predict how students’ thoughts and understanding will develop in the context of learning activities.

2.2. Setting of Study

The study was conducted at SMAN 1 Pengasih, Kulon Progo, Daerah Istimewa Yogyakarta. The participants of this study were 23 students of the class XI MIPA 3 and 24 students of the class XI MIPA 4. Researchers conducted research on the teaching and learning process in the polynomial division topic.

2.3. Instrumen

The instruments used in this study were an instrument of hypothetical Learning Trajectory (HLT) and student worksheet.

- Hypothetical Learning Trajectory (HLT)

Hypothetical Learning Trajectory (HLT), the main instrument in describing learning trajectory, consists of 3 component, i.e. learning objective, learning activities, and students’ hypothetical learning trajectory.

- Student Worksheet

Student worksheet contains tasks that are designed, so it can develop problem solving skills. This worksheet is arranged by estimating the students’ obstacles in understanding polynomial division in order after carrying out these tasks, it is hoped that these obstacles will not arise. Student work on this worksheet is analysed at retrospective analysis phase.

2.4. Data Analysis

This phase uses qualitative analysis. In the retrospective analysis phase, HLT that has been prepared in preparing for the experiment phase is compared with learning trajectory that occurs at the design experiment phase. It aims to determine whether the students’ hypothetical learning trajectory is accordance with what happens in class or not. Descriptive method is used in this study and describes the information that occurs at design experiment phase with the information contained in the results of students’ work at worksheet, so that it obtained the learning trajectory of polynomial division in class XI.

3. Result and Discussion

3.1. First meeting

3.1.1. Preparing for the experiment

In the preparation for the experiment phase at this first meeting, the researchers composed hypothetical learning trajectory (HLT) on polynomial division topic. It consists of learning objectives, learning activities, and students’ hypothetical learning trajectory. Learning objective was prepared by considering the material of polynomial division which refers to the curriculum 2013 used in SMAN 1 Pengasih. Learning objectives at this first meeting was to determine the quotient and the remainder of polynomial division by divisor \((x - h)\) and \((ax + b)\). Next designed the learning activities. This meeting used Worked-Example model.

Students’ hypothetical learning trajectory was made based on student’s responses which appear during learning process. Furthermore, Students’ hypothetical learning trajectory which corresponded
to learning objectives and learning activities in this first meeting were students: (1) determine the quotient and the remainder of polynomial division from contextual problem by divisor \((x - h)\) using algebraic operations, long division, and Horner method respectively; (2) determine the quotient and the remainder of polynomial division by divisor \((ax + b)\) using Horner method; (3) determine the polynomial coefficient on polynomial division by divisor \((x - h)\) and its remainder using Horner method; (4) determine the coefficient of polynomial on the polynomial division which is divisible by divisor \((x - h)\) and \((ax + b)\) respectively; (5) analyse the same remainder from polynomial division with different divisors; (6) analyse the problem of polynomial division by divisor \((x - h)\) and \((ax + b)\) and (7) determine the coefficient of polynomial \(g(x)\), if \(f(x)\) divided by \((x - h)\) has the same remainder with \(g(x)\) divided by \((x - h)\).

3.1.2. Design experiment In this phase, the lesson plan and HLT that were prepared at preparing for the experiment phase tested on the experimental class or commonly called teaching for experiment. Every data obtained from this meeting was going to be an improvement for the next meeting.

In this first meeting, there was an error in the scheme of learning trajectory in providing material of method to solve polynomial division. The scheme of settlement method polynomial division in student worksheet began from algebraic operation, then long division, and then Horner method. Based on level of difficulty, algebraic method was more difficult than the others, and long division was more difficult than Horner method. So, by using a scheme like HLT arranged, that was providing algebraic operation first, from the beginning the students had lost their motivation because algebraic operation was a complex and difficult method. Therefore, there were students who began to judge the problems contained in worksheet were very difficult.

According to the learning trajectory’s error, it is required to repair the scheme of polynomial division method, learning process began with long division first, then Horner method. Long division method was done as a basic for abstraction process to the Horner method. It does not matter if Horner method is studied before algebraic operation, because students have studied Horner when determining the value of polynomial. Algebraic operation method has a complex step and needs a high of accuracy.

3.1.3. Retrospective analysis According to the student work at the worksheet and by referring to the students’ hypothetical learning trajectory which researchers made in HLT, students’ hypothetical learning trajectory corresponded with the student work at the worksheet. As the model of learning used worked-example, students could solve the problem according to the example in worksheet. Therefore, learning trajectory of students was done in worksheet matching with the students’ hypothetical learning trajectory in HLT.

Firstly, given the problem of determining a quotient and remainder from a contextual problem about polynomial division using algebraic operation method. In this case, students had not understand yet how to make a supposition on the quotient and remainder because they had not understand yet about the maximum degree of quotient and remainder, so teacher guided students to the example in worksheet. They started to understand if the degree of polynomial \(n\) and the degree of divisor was 1 then the maximum degree of its quotient was \((n - 1)\), and also the degree of remainder was 0. Most of them already understand how to find the quotient and the remainder using long division method. Next, students learned about Horner method. In using Horner method, almost all students understood how to solve the problem yet constrained on accuracy of students in calculating.

3.2. Second meeting

3.2.1. Preparing for the experiment Learning objectives in this second meeting were students can determine the quotient and the remainder of polynomial division by divisor \((ax^2 + bx + c)\) correctly. This meeting used Worked-example model. Firstly, students learned about polynomial division by
divisor \((ax^2 + bx + c)\) which identified the divisor. Students determined whether the divisor has rational or irrational roots, then found the best way to solve the problem.

The students’ hypothetical learning trajectory in this second meeting which was suitable to the learning objectives and learning activities were students: (1) determine the quotient and the remainder in polynomial division using Horner method and Algebraic operation by divisor \((ax^2 + bx + c)\) which has rational roots; (2) determine the quotient and the remainder in polynomial division using Horner-Kino method by divisor \((ax^2 + bx + c)\) which has irrational roots; (3) determine the quotient and the remainder in polynomial division using long division method by divisor which has more than 2 degrees; (4) determine the coefficient of polynomial in polynomial division which is divisible by \((ax^2 + bx + c)\) using Horner-Kino method; (5) determine the difference between the quotient and the remainder in polynomial division by divisor \((ax^2 + bx + c)\) using Horner method; (6) determine the coefficient of polynomial from polynomial equation; (7) determine the coefficient of polynomial in polynomial division by divisor \((ax^2 + bx + c)\) which has irrational roots using Horner-Kino method.

3.2.2. Design experiment Based on the error on the first meeting about the lack of students motivation in solving problem as a result from the difficult problem at the beginning of learning, so this meeting was held for rotating the presentation of method for solving polynomial division. First, students presented the steps of solution using Horner method, since the divisor easily factored. In giving easy material first, students become more motivated to solve the next problem. Next were algebraic operation method and long division, and the last is Horner-Kino method.

3.2.3. Retrospective analysis In this second meeting, there was students’ hypothetical learning trajectory which was not suitable with what students did in worksheet. Students’ hypothetical learning trajectory in determining the quotient and the remainder of polynomial division using algebraic operation by divisor \((ax^2 + bx + c)\) which has irrational roots, students solved these problem using long division. They said that based on the material which was learned in the first meeting, algebraic operation method was a difficult method. When they saw the example and steps of solutions in student worksheet, this method had a long and complex steps. So that there were students who solved these problems using long division.

The solutions due to students work were suitable with learning trajectory which was made by researchers in HLT, asking students about which steps from algebraic operation are considered difficult. It turns out that students did not understand yet how to make a supposition on the quotient and the remainder. Then researchers showed the explanation in student worksheet and helped them to understand how the strategy to make a supposition on the quotient and the remainder was. In Horner-Kino method and long division method, most of them could use this method to solve the problem correctly.

3.3. Third meeting
3.3.1. Preparing for the experiment Learning objectives at this third meeting were to determine the remainder using the remainder theorem by divisor\((x - h), (ax - b)\), and \((x - a)(x - b)\). This meeting used Discovery Learning model. Students’ hypothetical learning trajectory which was suitable to learning objectives and learning activities were students: (1) determine the remainder of polynomial division using Horner method, algebraic operation, and long division; (2) determine the value of polynomial; (3) analyse the correlation between a remainder and a value of polynomial; (4) mention the maximum degree of polynomial if there is polynomial which is divided by \((x - a)(x - b)\); (5) Make a supposition on the remainder from polynomial division by \((x - a)(x - b)\), then doing the substitution-elimination operation to find the concept of the remainder theorem; (6) make a conclusion about the remainder theorem.
3.3.2. Design experiment In the third meeting, learning about the remainder theorem using Discovery Learning model, which consisted of 6 steps, i.e. stimulation, problem statement, data collection, processing, verification, and generalization. In this meeting, learning trajectory was implemented successfully, because the material was already sorted.

3.3.3. Retrospective analysis There was students’ hypothetical learning trajectory which was not suitable to learning trajectory in HLT such as in analyzing the correlation between a remainder and a value of polynomial. The solution was asking emphasize questions to the students. i.e. “What is the quotient in polynomial form? What is the remainder in polynomial form? And what is the value of polynomial? From these three results, investigate which forms have the same result?” After that, students realized that the remainder and the value of polynomial had the same value.

Furthermore, students had a difficulty in generalizing the concept of the remainder theorem. The solutions were that researchers reminded students about the general form of polynomial division, i.e. $P(x) \equiv Q(x)H(x) + S(x)$, then guided students into the instruction in worksheet, that was “substitute the value of x with $x = h$ in general form of polynomial division”.

3.4. Fourth meeting

3.4.1. Preparing for the experiment Learning objective in this fourth meeting was students can determine the factors of polynomial appropriately. This meeting used the scientific approach. Students’ hypothetical learning trajectory in this meeting were students: (1) observe the problem about the remainder theorem and the factor theorem; (2) mention the result from observation process; (3) investigate the validity of factor theorem; (4) determine the remainder of polynomial division using the remainder theorem or polynomial division method; (5) determine the factors of polynomial; (6) determine the other factors from the polynomial which one factor is given; (7) determine the coefficient of polynomial using the factor theorem.

3.4.2. Design experiment Scientific approach consists of 5 steps, i.e. observation, ask a question, collecting the information, analyze, and communication. Learning trajectory in this meeting constrained on how students proof the factor theorem and should explore on collecting information.

3.4.3. Retrospective analysis Based on students’ hypothetical learning trajectory in HLT, there was a distinct mismatch with what students do in the actual learning practice. It lied in mention the quotient on observation. Students did not understand how to write the results which they got in the observation process. The solution was asking them about an important thing contained in the observation process. Then students could mention it, that was if $(x - h)$ is a factor of $P(x)$ then $P(h) = 0$, it means the remainder is 0.

Based on the result in retrospective analysis and design experiment phase, learning objective on HLT after design experiment do not change. However, the scheme of methods of solving polynomial division needs to be changed. After the three phases of design research has been carried out, learning trajectory in polynomial division can be summarized as follows:
Figure 1. Learning Trajectory of polynomial division after the Design Research phase

The difference between the results from this study and the Indonesia curriculum is lies on the method to find the quotient and the remainder of polynomial division by divisor \((ax^2 + bx + c)\). The innovation carried out in this study is giving Horner-Kino method to find the quotient and the remainder of polynomial division.

Based on this study, a learning trajectory has been created and can be used by teachers for teaching and learning activities in class. The systematic learning trajectory helps teacher to teach, so the question given to students can be responded well. It corresponds with the result of research which was conducted by Wilson, Sztajn, Edgington, and Confrey [16] that learning trajectory helps to improve teacher’s teaching skills and supports the teacher in learning mathematics.

4. Conclusion

Based on the result of this study, it is obtained a learning trajectory of polynomial division. Learning begins with recognizing the general form of polynomial division equation, next learns about polynomial division by divisor \((x - h)\). In this phase, there are several methods to solve polynomial division, it starts from the easier method that is long division, then Horner method, and the last is algebraic operation method. Students should learn with easier method first because it affects student
motivation in learning. Next is polynomial division by divisor \((ax + b)\). The learning principle used is same as the polynomial division by divisor \((x - h)\).

Next, students learn about polynomial division by divisor \((ax^2 + bx + c)\). In this material, the main thing is identifying divisor of polynomial, whether the divisor has rational or irrational roots. For divisor which have rational roots, in solving polynomial division can use long division, Horner method, and algebraic operation. While for divisor which have irrational roots, there is an additional method, the Horner-Kino method. Horner-Kino method is a method with a low error, so it is an easy method to solve polynomial division by divisor \((ax^2 + bx + c)\). According to the result of this study, Horner-Kino method on polynomial division by divisor \((ax^2 + bx + c)\) does not need to identify the divisor which has rational or irrational roots, so it is categorized as an easy problem solving method.

After students recognize the forms of divisors and method of solving polynomial division, students learn the remainder theorem. The remainder theorem is classified into 3 different divisors, i.e. the remainder theorem by divisor \((x - h)\), \((ax + b)\) and \((x - a)(x - b)\). The last is the factor theorem. The systematic learning trajectory helps teacher to teach. It corresponds with the result of research which was conducted by Wilson, Sztajn, Edgington, and Confrey [16] that learning trajectory helps in improving their teaching skills as well as supports them mathematics learning.

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