Singularity of optimal control in the problem of stabilizing a nonlinear inverted spherical pendulum

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Abstract. We study an optimal control problem for a nonlinear spherical inverted pendulum on a movable base. As the cost functional, the mean-squared deviation of the pendulum from the upper equilibrium is considered, so optimal controls stabilize the pendulum at the unstable upper position. We show that the problem under consideration posses a singular point of the second order and there are spiral-similar solution which attains the singular point in finite time.

1. Introduction
Models of inverted pendulum systems are widely used to study the dynamics of different real complex nonlinear objects. There are many papers in which computer modeling and simulation are used for controlling and stabilizing the inverted pendulum systems. In [1, 2] for a spherical inverted pendulum a stabilizing controller is designed that brings the pendulum from any initial condition in the upper hemisphere to the preferred upright position. In [3] the controller stabilizing a spherical inverted pendulum-cart system can bring the pendulum to the unstable upright equilibrium point with the position of the cart at a desirable target position. In [4] it was proposed a controller for the tracking of elliptical paths. Many methods to control pendulum systems use an optimal control technique: a stabilizing controller is derived from minimization of a quadratic cost functional (LQR controllers) [5]–[7]. But applied to real physical systems, such controllers produced a relatively large deviations of pendulum systems from the upper equilibrium position. To improve this, there are different approaches, i.e., in [5] it was proposed to combine optimal and neural network techniques. In present work we study an optimal synthesis in the minimization problem of the mean square deviation of a spherical inverted pendulum from the upper equilibrium position over an infinite time interval. For the linearized model optimal solutions were found in the form of logarithmic spirals [11,13] that hit the origin in a finite time $T$. The corresponding optimal controls perform an infinite number of rotations along the circle $S^1$. We generalize the results obtained for the linear model to the nonlinear case.
2. Mathematical model of a spherical inverted pendulum

We consider a mathematical model of a spherical inverted pendulum with a moving support point (Fig.1). The pendulum consists of the point mass $B$ on the end of the rigid massless rod of length $l$. The rod is attached by a hinge to a moving support point $S$. We suppose that there is no friction in the hinge. The support $S$ can move in the horizontal plane under the action of an external planar control force $u = (u_1, u_2) \in \mathbb{R}^2$.

Let $O\xi\eta\zeta$ be a fixed coordinate system. The position of $S$ is described by $(\xi, \eta, 0)$. The position of the pendulum $B$ is described by $(x_1, x_2)$. Using the Euler-Lagrange equations we obtain nonlinear equations of motion of the spherical inverted pendulum system [11]:

\[
\begin{align*}
(M + m)\ddot{\xi} + ml\dot{x}_1 \cos x_1 - ml\dot{x}_1^2 \sin x_1 &= u_1 \\
(M + m)\ddot{\eta} + ml\dot{x}_2 \cos x_2 - ml\dot{x}_2^2 \sin x_2 &= u_2 \\
\ddot{\xi}b_{11} + \dot{x}_1 b_{13} + \dot{x}_2 b_{14} + \dot{x}_1^2 a_{11} + \dot{x}_2^2 a_{12} + \dot{x}_1 \dot{x}_2 a_{13} + c_1 &= 0 \\
\ddot{\eta}b_{22} + \dot{x}_1 b_{23} + \dot{x}_2 b_{24} + \dot{x}_1^2 a_{21} + \dot{x}_2^2 a_{22} + \dot{x}_1 \dot{x}_2 a_{23} + c_2 &= 0
\end{align*}
\]

where $m$ – mass of $B$, $M$ – mass of $S$, $b_{ij}, a_{ij}, c_i$ are some functions of $\cos x_k$ and $\sin x_k$. Eliminating the variables $\xi$ and $\eta$ we can rewrite (1) in the following form

\[
\begin{align*}
\dot{x}_1 &= y_1, \quad \dot{x}_2 = y_2 \\
\dot{y}_1 &= \alpha_{11} u_1 + \alpha_{21} u_2 + \beta_1, \\
\dot{y}_2 &= \alpha_{12} u_1 + \alpha_{22} u_2 + \beta_2
\end{align*}
\]

Here $\alpha_{ij}, \beta_k$ are functions of $x_i, y_i$. We will study solutions of the following optimal control problem for nonlinear spherical inverted pendulum: to minimize the mean square deviation of the pendulum from the upper unstable equilibrium position over an infinite time interval.
3. Optimal Control problem
Denote \( x(t) = (x_1(t), x_2(t)) \), \( y(t) = (y_1(t), y_2(t)) \). Assume that initial states of the system \( (x(0), y(0)) \) are in a sufficiently small neighbourhood of the upper unstable equilibrium position \( x = y = 0 \). Our goal is to find the bounded control function \( u(t) \)
\[
 u(t) \in U = \{ u_1^2 + u_2^2 \leq 1 \} \tag{3}
\]
and the corresponding state function \( (x(t), y(t)) \) which satisfies (2) and minimizes
\[
 \int_0^\infty (x_1^2(t) + x_2^2(t)) \, dt \tag{4}
\]
Following the Pontryagin maximum principle we define the Hamiltonian
\[
 H(x, y, \phi, \psi, u) = -\frac{1}{2} \langle x, x \rangle + \langle y, \phi \rangle + \langle \beta, \psi \rangle + u_1 \langle \alpha_1, \psi \rangle + u_2 \langle \alpha_2, \psi \rangle = H_0 + u_1 H_1 + u_2 H_2 \tag{5}
\]
where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( \mathbb{R}^2 \), \( \phi, \psi \) are adjoint variables that satisfy the adjoint equation
\[
 \dot{\phi} = -\frac{\partial H}{\partial x}, \quad \dot{\psi} = -\frac{\partial H}{\partial y}, \tag{6}
\]
Assume that \( (x^*(t), y^*(t), u^*(t)) \) is an optimal solution for (2)–(4). The optimal control is determined from the maximum condition:
\[
 u^*(t) = \arg\max_{u \in U} H(x^*(t), y^*(t), \phi(t), \psi(t), u) \tag{7}
\]
(3), (5) and (7) imply that \( u_1^* = H_i / (H_1^2 + H_2^2)^{1/2}, \ H_1^2 + H_2^2 \neq 0 \). If \( H_1^2 + H_2^2 = 0 \) over some interval \( (\theta_1, \theta_2) \) than any admissible control meets the maximum condition. In this case there is a singular arc. We prove that in the problem (2)–(4) there is a second-order singular point. We show that there is a family of solutions that approach the singular point \( (x^0, y^0) \) in finite time \( T \). If \( t \to T \) then \( (x^*(t), y^*(t)) \to (x^0, y^0) \) and switches to the singular mode. The corresponding control \( u^*(t) \) has a singularity of the second type at \( T \) and makes countably many rotations along the circle \( S^1 \) in finite time as \( t \to T \).

4. Singular solution
Denote by \( q = (x, y), \ p = (\phi, \psi) \). Let \( (\text{ad}A)F \) be the Poisson bracket of functions \( A \) and \( F \):
\[
 (\text{ad}A)F = \{ F, A \}. \tag{8}
\]

**Definition.** Point \( (q_0, p_0) \in \mathbb{R}^8 \) is the **singular point of second order**, if the following conditions are satisfied:
1) Functions
\[
 H_i, \ (\text{ad}H_k)H_i, \ (\text{ad}H_l)H_i, \ (\text{ad}H_t)(\text{ad}H_k)(\text{ad}H_l)H_i, \ (i = 1, 2, j, k, l = 0, 1, 2) \tag{9}
\]
are equal to zero at the point \((q_0, p_0)\). The set of their differentials at the point \((q_0, p_0)\) has a constant rank.
2) The bilinear form \( B = (B_{ij})_{i,j=1,2} : B_{ij} = \left. \text{ad}H_i(\text{ad}H_0)^3H_j \right|_{(q_0, p_0)}, \ i, j = 1, 2, \) has rank 2 and is symmetric and negative definite.
3) All other the Poisson brackets of the fifth order (independent of the listed) from the functions \( H_j \) \((j = 0, 1, 2)\) are equal to zero at the point \((q_0, p_0)\).

**Theorem 1.** Point \( (q_0, p_0) = (0, 0) \) is a singular point of the second order of the control Hamiltonian system (2), (5)–(7).
Direct calculations show that the point satisfies the conditions 1-3 of the definition with \( B = -E \), where \( E \) is the identity matrix.

To study the behaviour of the solutions of (2), (5)–(7) in the neighborhood of the singular point \((q_0, p_0)\) we use the construction of the descending system of Poisson brackets of the functions \( H_i \). We call the brackets \( H_i, (ad H_0) H_i, (ad H_0)^2 H_i, (ad H_0)^3 H_i \) \((i = 1, 2)\) primary brackets. The other brackets of order not exceeding 4 are called secondary.

Denote \( z_1 = H_1, z_{2i} = (ad H_0) H_i, z_{3i} = (ad H_0)^2 H_i, z_{4i} = (ad H_0)^3 H_i \) \((i = 1, 2)\). Introduce new local coordinates \( z_{ij}, (i = 1, \ldots, 4, j = 1, 2)\) in the neighborhood of the point \((q_0, p_0)\). In new coordinates the Hamiltonian system (2), (5)–(6) has the following form:

\[
\begin{align*}
\frac{d}{dt} z_{1i} &= z_{2i} + f_{1i1}(z) u_1 + f_{1i2}(z) u_2 \\
\frac{d}{dt} z_{2i} &= z_{3i} + f_{2i1}(z) u_1 + f_{2i2}(z) u_2 \\
\frac{d}{dt} z_{3i} &= z_{4i} + f_{3i1}(z) u_1 + f_{3i2}(z) u_2 \\
\frac{d}{dt} z_{4i} &= (ad H_0) z_{4i} + (ad H_1) z_{4i} u_1 + (ad H_2) z_{4i} u_2
\end{align*}
\]

Here \( f_{ijk}(z), (i = 1, \ldots, 4, j = 1, 2, k = 1, 2)\) are the secondary brackets. It was proved [8] that the first order approximation of the behavior of the Hamiltonian system (8) in the neighborhood of the singular extremal is fully determined by the primary Poisson brackets. The first-order approximation of the behavior of the primary Poisson brackets coincides with the optimal synthesis in the model problem provided below. Under first approximation we understand an approximation of the order \( o(\lambda) \) with respect to the homothety \( g_\lambda \) which is used for blowing up the singularity.

5. Model Problem

\[
\int_0^\infty \|x(t)\|^2 dt \rightarrow \inf
\]

\[
\begin{align*}
\dot{x} &= y, \quad \dot{y} = u, \quad \|u(t)\| \leq 1, \quad x(0) = x^0, \quad y(0) = y^0
\end{align*}
\]

Here \( x, y, u \in \mathbb{R}^2, \|\cdot\| \) means the standard Euclidean norm on \( \mathbb{R}^2 \). The Hamiltonian system of the maximum principle for the problem is given by

\[
\begin{align*}
\dot{\phi} &= x, \quad \dot{\psi} = -\phi, \quad \dot{x} = y, \quad \dot{y} = \dot{u}
\end{align*}
\]

\[
\dot{u}(t) = \arg\max_{\|u\| \leq 1} H(x(t), y(t), \phi(t), \psi(t), u)
\]

where

\[
H(x, y, \phi, \psi) = \frac{1}{2} \langle x, x \rangle + \langle y, \phi \rangle + \langle u, \psi \rangle
\]

The system (11) is homogeneous with respect to the action of an one-parameter group \( G = \{g_\lambda\}, \lambda \in \mathbb{R}_+ \)

\[
g_\lambda \left( y, x, \phi, \psi \right) \overset{def}{=} \left( \lambda y, \lambda^2 x, \lambda^3 \phi, \lambda^4 \psi \right)
\]

This property was used [9, 10] to find solutions of (11)–(13) in the form of logarithmic spirals

\[
\begin{align*}
x(t) &= A_2 (T - t)^2 e^{i\alpha t[T-T]} \\
y(t) &= A_3 (T - t) e^{i\alpha t[T-T]} \\
u(t) &= e^{i\alpha t[T-T]} \\
phi(t) &= A_1 (T - t)^3 e^{i\alpha t[T-T]} \\
n(t) &= A_0 (T - t)^4 e^{i\alpha t[T-T]} \\
A_0 &= 1/126, \quad \alpha = \pm \sqrt{5}, \quad A_{j+1} = -A_j (4 - j + i\alpha), \quad j = 1, 2, 3.
\end{align*}
\]
where \( T \) is a time at which solution hits the origin (the hitting time). Here we use the complex notation for vectors in \( \mathbb{R}^2 \): \( r e^{i\varphi} = (r \cos \varphi, r \sin \varphi) \).

6. Main Result
The system (2), (5)–(7) does not obey homogeneity. However, we show that in this case there are similar logarithmic spirals. To prove this we use for the Hamiltonian systems (11)–(13) and (8) the procedure of resolution of singularity that was proposed in [9], and then improved in [8]. We use the scheme as in [8], [12, 14] and the similar change of coordinates.

Consider the blowing up the singularity for (11)–(13) at the origin by the map

\[
B : (x, y, \phi, \psi) \mapsto (\mu, \tilde{x}, \tilde{y}, \bar{\phi}, \bar{\psi})
\]

\[
\mu = \left( |\tilde{y}|^{24} + |x|^{12} + |\phi|^8 + |\psi|^6 \right)^{\frac{1}{24}},
\]

\[
\left( \tilde{y}, \tilde{x}, \bar{\phi}, \bar{\psi} \right) = g_{1/\mu} (y, x, \phi, \psi),
\]

Here \( \mu \in \mathbb{R}_+ \), \( A_0 = 1/126 \), \( A_{j+1} = -A_j (4 - j + i\alpha) \), \( \alpha = \sqrt{5} \), and \( \tilde{x}, \tilde{y}, \bar{\phi}, \bar{\psi} \in \mathbb{R}^2 \) lie on the manifold

\[
S = \left\{ |\tilde{y}|^{24} + |\tilde{x}|^{12} + |\bar{\phi}|^8 + |\bar{\psi}|^6 = 1 \right\}.
\]

In the coordinates (\( \mu, \tilde{x}, \tilde{y}, \bar{\phi}, \bar{\psi} \)) and a new time parameter \( s \) defined by \( ds = \frac{1}{\mu} dt \) the system (11) has the form:

\[
\begin{align*}
\mu' &= \mu \mathcal{M}_0 (\tilde{x}, \tilde{y}, \bar{\phi}, \bar{\psi}), \\
\bar{\psi}' &= -\bar{\phi} - 4\psi \mathcal{M}_0, \\
\bar{\phi}' &= \bar{x} - 3\tilde{y} \mathcal{M}_0, \\
\bar{\xi}' &= -\tilde{y} - 2\tilde{x} \mathcal{M}_0, \\
u' &= \bar{\psi} / |\bar{\psi}|.
\end{align*}
\]

where \( \mathcal{M}_0 = \frac{1}{24} \left( 24 |\tilde{y}|^{22} \langle \tilde{y}, \psi \rangle + 12 |\tilde{x}|^{10} \langle \tilde{x}, \tilde{y} \rangle + 8 \bar{\phi}^6 (\phi, x) - 6 |\bar{\psi}|^4 (\psi, \phi) \right) \).

Using similar techniques we apply the blowup for (8) at \((q^0, p^0)\). We get that the blowup for the principal part of the descending system (8) coincides with the system (15). The spirals (14) represent hyperbolic periodic trajectories (circles) in the space \((\tilde{x}, \tilde{y}, \bar{\phi}, \bar{\psi})\) and exist in both the Hamiltonian systems. By the invariant manifold theorem for a limit cycle [15] we have the following result.

**Theorem 2.** In a sufficiently small neighborhood of the origin there exist solutions of (2)–(4) that attain the upper equilibrium position and have the form of logarithmic spirals

\[
x(t) = C_x (T - t)^2 e^{i \varphi_0 (T - t)} (1 + g_x (T - t)),
\]

\[
y(t) = C_y (T - t) e^{i \varphi_0 (T - t)} (1 + g_y (T - t)),
\]

\[
u(t) = -C_a e^{i \varphi_0 (T - t)} (1 + g_u (T - t)),
\]

Here \( 0 < T < \infty \) is a time at which solution hits the origin (the hitting time), \( g_{x,y,u} (T - t) \to 0 \) as \( t \to T \), \( q > 0 \), \( C_{x,y,u} \in \mathbb{C} \).

7. Conclusion
We consider a mathematical model of a spherical inverted pendulum on a movable cart. We generalize the results obtained for the linear model to the nonlinear case. We show that this problem posses a singular point of the second order and there are spiral-similar solution which attains the singular point in finite time. We hope that obtained analytical results can be useful in developing effective algorithms for computer modeling and construction of a stabilizing controller.
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