Proof-Labeling Schemes: Broadcast, Unicast and In Between

Boaz Patt-Shamir and Mor Perry

School of Electrical Engineering, Tel Aviv University, Tel Aviv 6997801, Israel.

Abstract. We study the effect of limiting the number of different messages a node can transmit simultaneously on the verification complexity of proof-labeling schemes (PLS). In a PLS, each node is given a label, and the goal is to verify, by exchanging messages over each link in each direction, that a certain global predicate is satisfied by the system configuration. We consider a single parameter \( r \) that bounds the number of distinct messages that can be sent concurrently by any node: in the case \( r = 1 \), each node may only send the same message to all its neighbors (the broadcast model), in the case \( r \geq \Delta \), where \( \Delta \) is the largest node degree in the system, each neighbor may be sent a distinct message (the unicast model), and in general, for \( 1 \leq r \leq \Delta \), each of the \( r \) messages is destined to a subset of the neighbors.

We show that message compression linear in \( r \) is possible for verifying fundamental problems such as the agreement between edge endpoints on the edge state. Some problems, including verification of maximal matching, exhibit a large gap in complexity between \( r = 1 \) and \( r > 1 \). For some other important predicates, the verification complexity is insensitive to \( r \), e.g., the question whether a subset of edges constitutes a spanning-tree. We also consider the congested clique model. We show that the crossing technique [5] for proving lower bounds on the verification complexity can be applied in the case of congested clique only if \( r = 1 \). Together with a new upper bound, this allows us to determine the verification complexity of MST in the broadcast clique. Finally, we establish a general connection between the deterministic and randomized verification complexity for any given number \( r \).

Keywords: verification complexity, proof-labeling schemes, CONGEST model, congested clique

1 Introduction

Similarly to classical complexity theory, studying the verification complexity of various problems is one of the major approaches in the quest to understand the complexity of network tasks. The basic idea, proposed by Korman, Kutten and Peleg [22] under the name Proof-Labeling Schemes (PLS for short), is to assume that an oracle assigns a label to each node, so that by exchanging these labels, the nodes can collectively verify that a certain global predicate holds (see Sec. 2 for details). The verification complexity of a predicate \( \pi \) is defined to be the minimal label length which suffices to verify \( \pi \). This node-centric, space-based view was generalized in subsequent work, in which it was allowed for nodes to send different messages to different neighbors, rather than
the whole local label to all neighbors. Specifically, in [5] the verification complexity is defined to be the minimal message-length required to verify the given predicate.

The distinction between these two models is natural and appears in other contexts as well, like the broadcast and the unicast flavors of congested clique, proposed by Drucker et al. [9]: in the unicast flavor, a node may send a different message to each of its neighbors, while in the broadcast flavor, all neighbors receive the same message. Following up on this model, Becker et al. [6] proposed considering a spectrum of congested clique models, where a node may send up to $r$ distinct messages in a round, where $1 \leq r < n$ is a given parameter. This model, called henceforth MCAST($r$), can be motivated by observing that $r$ can be viewed as the number of network interfaces (NICs) a node possesses: Each interface may be connected to a subset of the neighbors, and it can send only a single message at a time.

**Our Results.** In this paper we present a few preliminary results concerning PLS in the MCAST($r$) model. Our main focus is on the tradeoff between the number $r$ of different messages a node can send in one round and the verification complexity (message length) $\kappa$. While there are problems whose verification complexity is independent of $r$, we prove that the verification complexity of some fundamental problems is highly dependent on $r$. First, we consider the problem of matching verification (MV), where every node has at most one incident edge marked, and the goal is to verify whether the set of marks implies a well defined matching, i.e., an edge is either marked in both endpoints or unmarked in both, and that this set is a matching. In [19], among other results, it is shown that maximal matching has verification complexity $\Theta(1)$, and that the verification complexity of maximum matching in bipartite graphs is also $\Theta(1)$. These results implicitly assume that the subset of edges is well defined; our results show that in fact, the main difficulty is in ensuring that both endpoints of an edge agree on its status. This motivates our next problem that focuses on consistency. Specifically, we define the primitive problem edge agreement (EA) as follows. Each node has a $b$-bit string for each incident edge, and a state is considered legal iff both endpoints of each edge agree on the string associated with that edge. It turns out that the arboricity of the graph, denoted $\alpha(G)$, plays an important role in the verification complexity of EA (and all problems that EA can locally be reduced to). In Theorem 2, we prove that $\kappa(EA) \cdot r \in \Theta(\alpha(G)b)$. Next, as a more sophisticated example, we consider the important problem of maximum flow (MF): In Theorem 3 we show that $\kappa(MF) \cdot r \in \Theta(\alpha(G) \log f_{\text{max}})$, where $f_{\text{max}}$ is the largest flow value over an edge. In [22], a scheme to verify that the maximum flow between a given pair of nodes $s$ and $t$ is exactly $k$ is given in the broadcast model, with complexity $O(k \log k + \log n))$. We prove, in Theorem 4, that the verification complexity of this problem in the broadcast model is $O(\min \{\alpha(G), k\} (\log k + \log \Delta))$, which is an exponential improvement in some cases. In addition, our upper bound scales linearly with $r$ in the MCAST($r$) model.

We also consider the congested clique model. To date, no lower bounds on the verification complexity in the congested clique were known. We show that the known technique of crossing [5] can be applied, but only in broadcast clique (i.e., MCAST(1)). We use this argument, along with a new scheme, to obtain a tight $\Theta(\log n + \log w_{\text{max}})$ bound for MST verification in broadcast cliques, where $w_{\text{max}}$ denotes the largest edge weight.
Finally, we show that all results translate to randomized PLS (RPLS) \cite{5}. Extending a result of \cite{5}, we show that if both PLS and RPLS are using the same number $r$, then an exponential difference in verification complexity holds in both directions, i.e., in the MCAST($r$) model, an RPLS with verification complexity $O(\log \kappa_d)$ can be constructed out of every PLS with verification complexity $\kappa_d$, and every RPLS with verification complexity $\kappa_r$ can be used to construct a PLS with verification complexity $O(2^{\kappa_r})$.

**Related Work.** Drucker et al. \cite{9} propose a local broadcast communication in the congested clique, where every node broadcasts a message to all other nodes in each round. Becker et al. \cite{6} proposed, still for congested cliques, a bounded number $r$ of different messages a node can send in each round.

Verification of a given property in decentralized systems finds applications in various domains, such as, checking the result obtained from the execution of a distributed program \cite{4,17}, establishing lower bounds on the time required for distributed approximation \cite{8}, estimating the complexity of logic required for distributed run-time verification \cite{18}, general distributed complexity theory \cite{16}, and self stabilizing algorithms \cite{7,21}.

The notion of distributed verification in a single round was introduced by Korman, Kutten, and Peleg in \cite{22}. The verification complexity of minimum spanning-trees (MST) was studied in \cite{20}. Constant-round schemes were studied in \cite{19}. Verification processes in which the global result is not restricted to be the logical conjunction of local outputs had been studied in \cite{2,3}. The role of unique node identifiers in local decision and verification was extensively studied in \cite{14,13,15}. Proof-labeling schemes in directed networks were studied in \cite{11}, where both one-way and two-way communication over directed edges is considered. Verification schemes for dynamic networks, where edges may appear or disappear after label assignment and before verification, are studied in \cite{12}. Recently, a hierarchy of local decision as an interaction between a prover and a disprover was presented in \cite{10}.

**Paper Organization.** The remainder of this paper is organized as follows. In Section 2 we formalize the model and recall some graph-theoretic concepts. In Section 3 we present two general techniques that apply to the MCAST($r$) model. In Section 4 we present results for verification of matching, edge agreement, and max-flow. In Section 5 we present our results for congested cliques. In Section 6 we analyze the relation between deterministic and randomized PLSs. We conclude in Section 7 with some open questions and directions for future work.

## 2 Model and Preliminaries

**Computational Framework and the MCAST Model.** Our model is derived from the CONGEST model \cite{27}. Briefly, a distributed network is modeled as a connected undirected graph $G = (V, E)$, where $V$ is the set of nodes, $E$ is the set of edges, and every node has a unique identifier. In each synchronous round every node performs a local computation, sends a message to each of its neighbors, and receives messages from all neighbors. We denote the number of nodes $|V|$ by $n$ and the number of edges $|E|$ by $m$. For every node $v \in V$, let $d(v)$ be the degree of $v$. We denote by $\Delta(G)$ the maximal
degree of a node in $G$. We assume that the edges incident to a node $v$ are numbered $1, \ldots, d(v)$.

The main difference between the model considered in this paper, called $\text{MCAST}(r)$, and $\text{CONGEST}$, is that in $\text{MCAST}(r)$ we are given a parameter $r \in \mathbb{N}$ such that a node may send at most $r$ distinct messages simultaneously. More precisely, we assume that prior to sending messages, the neighbors of a node are partitioned into $r$ disjoint subsets (some of which may be empty), such that $v$ sends the same message to all neighbors in a subset. We emphasize that in our model, for simplicity, $r$ is a uniform parameter for all nodes.

**Proof-Labeling Schemes in the MCAST model.** A configuration $G_s$ includes an underlying graph $G = (V, E)$ and a state assignment function $s : V \rightarrow S$, where $S$ is a (possibly infinite) state space. The state of a node $v$, denoted $s(v)$, includes all local input to $v$. In particular, the state usually includes a unique node identity $\text{ID}(v)$ and, in the case of weighted graphs, the weight $w(e)$ of each incident edge $e$. The state of $v$ typically include additional data whose integrity we would like to verify. For example, node state may contain a marking of incident edges, such that the set of marked edges constitutes a spanning tree.

Let $\mathcal{F}$ be a family of configurations, and let $\mathcal{P}$ be a boolean predicate over $\mathcal{F}$. A proof-labeling scheme consists of two conceptual components: a prover $p$, and a verifier $v$. The prover is an oracle which, given any configuration $G_s \in \mathcal{F}$ satisfying $\mathcal{P}$, assigns a bit string $\ell(v)$ to every node $v$, called the label of $v$. The verifier is a distributed algorithm running at every node. At each node $v$, the local verifier takes as input the state $s(v)$ of $v$, its label $\ell(v)$ and based on them sends messages to all neighbors. Then, using as input the messages received from the neighbors, the local state and the local label, the local verifier computes a boolean value. If the outputs are TRUE at all nodes, the global verifier $v$ is said to accept the configuration, and otherwise (i.e., at least one local verifier outputs FALSE), $v$ is said to reject the configuration. For correctness, a proof-labeling scheme $\Sigma = (p, v)$ for $(\mathcal{F}, \mathcal{P})$ must satisfy the following requirements, for every $G_s \in \mathcal{F}$:

- If $\mathcal{P}(G_s) = \text{TRUE}$ then, using the labels assigned by $p$, the verifier $v$ accepts $G_s$.
- If $\mathcal{P}(G_s) = \text{FALSE}$ then, for every label assignment, the verifier $v$ rejects $G_s$.

Given a configuration $G_s$, we denote by $c_\Sigma(G_s)$ the vector of length $|E|$ that contains the messages sent according to the scheme $\Sigma$, and we refer to this vector as the communication pattern of $\Sigma$ over $G_s$. For an underlying graph $G$, we denote by $L(G)$ the number of legal configurations of $G$, and by $W_\Sigma(G)$ the number of different communication patterns of $\Sigma$ in $G$, over all legal configurations. In our analysis, given an edge $(v, u) \in E$, we denote by $M_{v}(e)$ the message over $e$ from $v$ to $u$.

Our central measure for PLSs is its verification complexity, defined as follows.

**Definition 1.** The verification complexity of a proof labeling scheme $\Sigma = (p, v)$ for the predicate $\mathcal{P}$ over a family of configurations $\mathcal{F}$ is the maximal length of a message generated by the verifier $v$ based on the labels assigned to the nodes by the prover $p$ in a configuration $G_s$ for which $\mathcal{P}(G_s) = \text{TRUE}$.

In this paper we consider PLSs in the $\text{MCAST}(r)$ model, namely we impose the additional restriction that at most $r$ distinct messages may be sent by a node.
Arboricity, degeneracy and average degree. The average degree of a graph plays a central role in our study. However, graphs may have dense and sparse regions. We therefore use the following refined concepts.

**Definition 2.** The **arboricity** of a graph $G = (V, E)$, denoted by $\alpha(G)$, is defined as the minimum number of acyclic subsets of edges that cover $E$. The **degeneracy** of a graph $G$, denoted by $\delta(G)$, is defined as the smallest value $i$ such that the edges of $G$ can be oriented to form a directed acyclic graph with out-degree at most $i$.

The following properties are well known [25,26].

**Lemma 1.** For all graphs $G$, $\alpha(G) \leq \delta(G) < 2\alpha(G)$.

**Lemma 2.** For a given graph $G = (V, E)$, $\alpha(G) = \max \left\{ \left\lceil \frac{m_H}{n_H-1} \right\rceil \mid V_H \subseteq V, |V_H| \geq 2 \right\}$, where $m_H = |E_H|$ and $n_H = |V_H|$ over all induced subgraphs $H = (V_H, E_H)$ of $G$.\(^1\)

Note that by Lemmas 1 and 2, the minimal number of outgoing edges in the best orientation of a graph $G$ is proportional to the maximal average degree over all induced subgraphs of $G$.

## 3 Techniques for the MCAST Model

In this work, we consider problems expressible as a conjunction of edge predicates, where a node may have a different input for every edge. We present two techniques that can be used as building blocks in the design of efficient PLSs in the MCAST model.

The first technique, which we call minimizing orientation, reduces the number of incident edges a node sends its input on. We orient the edges such that the maximum out degree is minimized. Lemma 1 ensures that the maximum out degree is bounded by $2\alpha$. Using a minimizing orientation, we can prove the following lemma.

**Lemma 3.** Suppose that a verification problem $(\mathcal{F}, \mathcal{P})$ is expressible as a conjunction of edge predicates, each involving variables from a single pair of neighbors. Then there exists a PLS $\Sigma = (p, v)$ for $(\mathcal{F}, \mathcal{P})$ in the MCAST$(2\alpha)$ model with verification complexity $k$, where $k$ is the length of the largest local input to an edge predicate.

**Proof:** We describe the scheme $\Sigma = (p, v)$. The prover $p$ orients the edges such that the maximum out-degree of a node is minimized (this can be done in linear time, see, e.g., [24]). Then, every node sends its input only on outgoing edges.

The orientation can be verified simply by a local verification that a node receives a message from every incoming edge and does not receive a message on every outgoing edge. Note that the empty message is also counted in the number $r$ of different messages. By definition of degeneracy, the maximum out degree is $\delta$, and by Lemma 1 $\delta$ is strictly smaller than $2\alpha$. Therefore, in the MCAST$(2\alpha)$ model, in addition to sending a different message on every outgoing edge, we may use the empty message for

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\(^1\) Given a graph $G = (V, E)$, the induced subgraph $H = (V_H, E_H)$ over the set of nodes $V_H \subseteq V$ satisfies that $E_H = E \cap (V_H \times V_H)$. 
incoming edges. After verifying that the orientation is correct (i.e., consistent between neighbors), by definition of the scheme, the function of every edge is computed by one of its endpoints and the verification is completed.

**Color addressing.** In the unicast model, each node receives its own message. However, if we want to use a unicast PLS in the MCAST\((r)\) model with \(r < 2\alpha\), we may need to bundle together a few messages, and hence we need to somehow tag each part of the message with its intended recipient. Clearly this can be done by tagging each sub-message by the unique ID of recipient, but this adds \(\Theta(\log n)\) bits to each sub-message. The color addressing technique reduces this overhead to \(O(\log \Delta)\). The idea is that each node need only distinguish between its neighbors.\(^2\) We solve this difficulty by coloring the nodes so that no two neighbors of a node get the same color. Formally, color addressing is a PLS \(\Sigma_{COL} = (p, v)\) in the broadcast model, where the prover \(p\) first colors the nodes so that no two nodes at distance 1 or 2 receive the same color. This is possible using at most \(\Delta^2 + \Delta + 1 \in O(\Delta^2)\) colors, because every node has at most \(\Delta\) neighbors and \(\Delta^2\) nodes at distance 2 from it. Next, the prover assigns to every incident edge of a node the color of the neighbor at the other end of the edge. The verifier \(v\) at a node \(v\) broadcasts the color assigned to \(v\) by the prover. Every node verifies that every incident edge is assigned a different color and that the color received from every edge is the color assigned by the prover to this edge.

Clearly, \(\Sigma_{COL}\) guarantees a proper coloring as desired to use for addressing, and this coloring is locally verifiable. Moreover, since a color can be represented using \(O(\log \Delta)\) bits, we obtain local addressing with verification complexity \(O(\log \Delta)\) in the broadcast model. We summarize in the following lemma.

**Lemma 4.** \(\Sigma_{COL}\) is a PLS in the broadcast model, which assigns and verifies an \(O(\log \Delta)\)-bit coloring for proper addressing. The verification complexity of \(\Sigma_{COL}\) is \(O(\log \Delta)\).

## 4 Verification Complexity Trade-offs in the MCAST\((r)\) Model

In this section, we study the effect of \(r\) on the verification complexity of PLSs in the MCAST\((r)\) model. We start with the observation that for some problems, the asymptotic verification complexity is independent of \(r\). These problems include the deterministic verification of a spanning-tree and vertex bi-connectivity, and the randomized verification of an MST. For each of these problems, we provide a scheme for \(r = 1\) with verification complexity that matches the lower bound for \(r = \Delta\) [22,5]. In contrast, there are problems for which the verification complexity is sensitive to \(r\). Specifically, we present a tight bound for the matching verification problem in the broadcast model, which is reduced dramatically even for \(r = 2\). Finally, we show tight bounds for the primitive problem of edge agreement and the more sophisticated application of maximum flow, which scales linearly with \(r\).

\(^2\) We note that using simple port numbering requires agreement with the neighbors, which is costly, as we prove in Theorem 2.
4.1 Verification of Matchings

In the literature, in verification problems of the form “does a subset of edges satisfy a specified property,” it is usually assumed that the subset of edges is well defined, i.e., for every edge \( e = (u, v) \), the local state of \( v \) indicates that \( e \) is in the subset if and only if the local state of \( u \) indicates it. However, since edges do not have storage, an edge set is actually represented by the local state at the nodes, and hence consistency between neighbors is not always guaranteed.

In fact, there are problems for which the verification of consistency is the dominant factor of the verification complexity. In particular, consider matching problems: maximal matching, and maximum matching in bipartite graphs. Both problems are known to have constant verification complexity [19]. However, these results make the problematic assumption that the edge set in question is well defined. We consider the matching verification problem using the following definition.

**Definition 3 (Matching Verification(MV)).**

**Instance:** At each node \( v \), at most one edge is marked. We use \( I_v(e) \in \{\text{TRUE, FALSE}\} \) to denote whether \( e \) is marked in \( v \).

**Question:** Is the set \( M \) of marked edges well defined, i.e., \( I_v(e) = I_u(e) \) for every edge \( e = (u, v) \in E \), and \( M \) is a matching?

We argue that in the broadcast model, the verification complexity of this problem is \( \Theta(\log \Delta) \). Formally, we study the problem \((F_m, MV)\), where \( F_m \) is the family of connected configurations with edge indication at each node. We obtain the following result.

**Theorem 1.** The verification complexity of \((F_m, MV)\) in the broadcast model is \( \Theta(\log \Delta) \).

We start with proving the lower bound of the theorem as stated in the following lemma. The proof uses a variant of crossing arguments [5].

**Lemma 5.** The verification complexity of any PLS for \((F_m, MV)\) in the broadcast model is \( \Omega(\log \Delta) \).

**Proof:** By contradiction. Let \( n \) and \( 2 \leq \Delta \leq n/2 + 1 \) be given. We construct the following graph \( G = (V, E) \) with maximum degree \( \Delta \). The \( n \)-node graph \( G \) consists of two parts. One part is a complete bipartite graph over two sets of nodes \( A \) and \( B \) of size \( \Delta - 1 \) each: \( H = (A, B, E_H) = K_{\Delta-1, \Delta-1} \); the second part consists of an \( n - 2(\Delta - 1) \) path, connected by an edge to a node in \( A \).

Given a configuration \( G_s \) of \( G \), let \( I_v^G(e) \) denote \( I_v(e) \) (the mark of edge \( e \) as represented in the state of \( v \)). Given a configuration \( H_s \) of the nodes of \( H \), extend it to a configuration \( G(H_s) \) of \( G \) as follows. For every \( e = (u, v) \in E_H \) let \( I_{v}^{G(H_s)}(e) = I_{v}^{H_s}(e) \), and for every \( e = (u, v) \in E \setminus E_H \), let \( I_{v}^{G(H_s)}(e) = 0 \). Clearly, \( H_s \) is legal if and only if \( G(H_s) \) is legal. We note that the number of different matchings in \( H, L(H) \), is at least \((\Delta - 1)! \), because every permutation of \( \Delta - 1 \) elements represents a different matching in \( H \).
Now, let $\Sigma = (p, v)$ be a PLS for $(\mathcal{F}_m, \text{MV})$ in the broadcast model, and assume for contradiction that $\kappa(\Sigma) < \frac{1}{2} \log(\Delta - 1) - 1$. Recall that $W_\Sigma(H)$ is the number of different communication patterns of $\Sigma$ in $H$. Then

$$W_\Sigma(H) \leq 2^{(\Delta - 1)\kappa} \leq 2^{(\Delta - 1) \log \frac{\Delta - 1}{e}} \leq (\Delta - 1)! \leq L(H).$$

Inequality (1) is true since for every PLS in the broadcast model with verification complexity $\kappa$, every communication pattern in $H$ can be constructed by choosing a $\kappa$-bit message for each of the $2(\Delta - 1)$ nodes in $H$. Inequality (2) follows from our assumption that $\kappa < \frac{1}{2} \log(\Delta - 1) - 1$ and the fact that $\log e < 2$. Therefore, the number of communication patterns of $\Sigma$ in $H$ is strictly smaller than the number of legal configurations of $H$. Therefore, there must be two different legal configurations $G(H_s)$ and $G(H'_s)$ with the same communication pattern in $H$. Since $G(H_s)$ and $G(H'_s)$ differ only over $E_H$ edges, there must exist an edge $e^* = (v, u) \in E_H$ such that $I^G(H_s)(e^*) = I^G(H'_s)(e^*)$ and $I^G(H_s)(e^*) \neq I^G(H'_s)(e^*)$. Consider the configuration obtained from $G(H_s)$ with $I^G(H'_s)(e)$ replacing $I^G(H_s)(e)$ for every node $w \in B$ and every edge $e = (w, w') \in E_H$. Intuitively, in this configuration, the state of all nodes in $V \setminus B$ is as in $G(H_s)$, and the state of nodes in $B$ is as in $G(H'_s)$. Obviously, this configuration is illegal, because $I^G(H'_s)(e^*) \neq I^G(H'_s)(e^*)$, and $e \in A \times B$. However, since all nodes in $H$ send the same messages in $G(H_s)$ and in $G(H'_s)$ under $\Sigma$, we get the following. With the labels assigned by $p$ to the set of nodes in $B$ in $G(H'_s)$ and the labels assigned by $p$ to all $V \setminus B$ nodes in $G(H_s)$, since all edges connected to $B$ are in $E_H$, the local view of $B$ in verification is exactly as in $G(H'_s)$, and the local view of all other nodes in verification is exactly as in $G(H_s)$. Therefore, all nodes output $\text{TRUE}$ on an illegal configuration, which contradicts the correctness of $\Sigma$.

The following lemma shows a matching upper bound for this problem. This completes the proof of Theorem 1.

**Lemma 6.** There exists a PLS $\Sigma = (p, v)$ for $(\mathcal{F}_m, \text{MV})$ in the broadcast model with verification complexity $O(\log \Delta)$.

**Proof:** The constructed scheme $\Sigma = (p, v)$ uses color addressing. Let Col$_v$ be the color assigned to node $v$. The verifier $v$ at every node $v$ locally verifies that it has at most one marked incident edge. If none of the edges of $v$ is marked, then it sends the empty message, and if there is a marked edge $(v, u)$, then the verifier at node $v$ sends Col$_u$. Finally, locally verify consistency at every node $v$ as follows. The message received from edge $(v, u)$ is Col$_u$ if and only if edge $(v, u)$ is marked.

We now prove the correctness of $\Sigma$. According to the correctness of color addressing, every node reliably broadcasts the indication of its marked edge if any. If the marking is consistent and indicates a matching, then on every marked edge, every endpoint sends the color of the other endpoints, and all nodes output $\text{TRUE}$. If there exists a node with more than one marked edge, by definition of the scheme, this node outputs $\text{FALSE}$. Finally, if every node has at most one marked edge but the marking is inconsistent, then there exists an edge $e = (v', u')$ such that $M_{v'}(e) = 0$ and $M_{u'}(e) = 1$. By definition
of the scheme, \( u' \) broadcasts \( COL_{v'} \), and \( v' \) receives its color from an unmarked edge. Therefore, \( v' \) outputs \( \text{FALSE} \).

The result above says that in the broadcast model, the verification complexity of the maximal matching problem and the maximum matching in bipartite graphs is dominated by the consistency verification. Observe that in the \( \text{MCAST}(2) \) model, the verification complexity of \( (\mathcal{F}_m, \text{MV}) \) is \( O(1) \), by letting every node \( v \) send on every edge \( e = (v, u) \) the bit \( I_r(e) \): only two types of messages are needed!

We also note that for the problem of maximum matching in cycles, the asymptotic verification complexity is unchanged if we must verify consistency, since the verification complexity of this problem in the broadcast model is \( \Theta(\log n) \) [19].

### 4.2 The Edge Agreement Problem

Motivated by the results for matching verification, we now formalize and study the fundamental problem of consistency across edges.

**Definition 4 (\( b \)-bit Edge Agreement (EA\( b \))).**

**Instance:** Each node \( v \) holds in its state a \( b \)-bit string \( B_v(e) \) for each incident edge \( e \).

**Question:** Is \( B_v(e) = B_u(e) \) for every edge \( e = (u, v) \in E \)?

Let \( \mathcal{F} \) be the family of all configurations, and let \( \alpha \) denote the arboricity of the graph. Our first main result is the following tight trade-off between \( r \) (the number of different messages for a node) and verification complexity of EA\( b \).

**Theorem 2.** Let \( b \in \Omega(\log \Delta) \). For every \( 1 \leq r \leq \min \{ \Delta, 2^{b/4} \} \), the verification complexity of \( (\mathcal{F}, \text{EA}_b) \) in the \( \text{MCAST}(r) \) model is \( \Theta(\lceil \frac{n}{r} \rceil b) \).

This theorem states both an upper and a lower bound. We start with the lower bound.

**Lemma 7.** For every \( 1 \leq r \leq \min \{ \Delta, 2^{b/4} \} \), the verification complexity of any PLS for \( (\mathcal{F}, \text{EA}_b) \) in the \( \text{MCAST}(r) \) model is \( \Omega((\frac{n}{r} + 1)b) \).

To prove Lemma 7, we prove the following claim using ideas similar to those used in the proof of Lemma 5.

**Claim.** Let \( G = (V, E) \) be a graph, let \( 1 \leq r \leq \min \{ \Delta, 2^{b/4} \} \) and consider a PLS for \( (\mathcal{F}, \text{EA}_b) \) in the \( \text{MCAST}(r) \) model. For every induced subgraph \( H = (V_H, E_H) \) of \( G \), \( W_{\Sigma}(H) \geq L(H) \).

**Proof:** Given \( 1 \leq r \leq \min \{ \Delta, 2^{b/4} \} \), let \( \Sigma = (\mathbf{p}, \mathbf{v}) \) be a PLS for \( (\mathcal{F}, \text{EA}_b) \) in the \( \text{MCAST}(r) \) model. Given \( G = (V, E) \), let \( H = (V_H, E_H) \) be an induced subgraph of \( G \). Let \( B_G^v(e) \) be \( B_v(e) \) (the bit-string held by \( v \) for the edge \( e \)) in configuration \( G_H \). Given a configuration \( H_s \) of the nodes of \( H \), extend it to a configuration \( G(H_s) \) of \( G \) as follows: for every \( e = (u, v) \in E_H \) let \( B_H^v(H_s)(e) = B_{H_s}^v(e) \), and for every \( e = (u, v) \in E \setminus E_H \), let \( B_H^v(H_s)(e) = 0^b \). Clearly, \( H_s \) is legal if and only if \( G(H_s) \) is legal.
Now, to prove the claim, assume for contradiction that $W_{\Sigma}(H) < L(H)$. Then there must be two different legal configurations $G(H_s)$ and $G(H'_s)$ with the same communication pattern. There must exist an edge $e^* = (v, u) \in E_H$ such that $B_v^{G(H_s)}(e^*) = B_u^{G(H'_s)}(e^*) \neq B_v^{G(H'_s)}(e^*)$: This is because we assume $G(H_s) \neq G(H'_s)$, and by construction, the difference can be only in $E_H$ edges. Consider the configuration obtained from $G(H_s)$ with $B_v^{G(H'_s)}(e)$ replacing $B_v^{G(H_s)}(e)$ for every edge $e = (v, w) \in E$. This configuration is illegal, because $B_v^{G(H_s)}(e^*) \neq B_v^{G(H'_s)}(e^*)$.

However, since all nodes send the same messages in $G(H_s)$ and in $G(H'_s)$ under $\Sigma$, we get that with the label assigned by $p$ to $v$ in $G(H'_s)$ and the labels assigned by $p$ to all $V \setminus \{v\}$ nodes in $G(H_s)$, the local view of $v$ in verification is exactly as in $G(H'_s)$, and the local view of all other nodes in verification is exactly as in $G(H_s)$. Therefore, all nodes output $\text{true}$ on an illegal configuration, a contradiction.

**Proof of Lemma 7:** It is known that the non-deterministic two-party communication complexity of verifying the equality (EQ) of $b$-bit strings is $\Omega(b)$ [23, Example 2.5]. Simulating a verification scheme for $(\mathcal{F}, E_{A_b})$ on a network of one edge, is a correct non-deterministic two-party communication protocol for EQ. Therefore, $\Omega(b)$ is a lower bound for $(\mathcal{F}, E_{A_b})$.

We now prove that $\Omega\left(\frac{b}{2}\right)$ is also a lower bound for $(\mathcal{F}, E_{A_b})$. Let $G_s \in \mathcal{F}$ be a configuration with an underlying graph $G = (V, E)$, and let $H = (V_H, E_H)$ be the densest induced subgraph of $G$, i.e., $m_H/n_H \geq m_{H'}/n_{H'}$ for every $V'_H \subseteq V$. By Lemma 2, $\alpha = \lceil m_H/(n_H - 1) \rceil$. W.l.o.g., let $V_H = \{v_1, \ldots, v_{n_H}\}$, and let $d_H(v_i) = |\{(v_i, v_j) \in E_H\}|$ be the degree of node $v_i$ in $H$.

We now show that for $1 \leq r \leq \min\{\Delta, 2^{b/4}\}$ and any scheme $\Sigma$ for $(\mathcal{F}, E_{A_b})$ with verification complexity $\kappa < \frac{ab}{r^2} - 2$ in the $\text{MCAST}(r)$ model, it holds that $W_{\Sigma}(H) < L(H)$. Let $\Sigma$ be such a verification scheme. Then

$$W_{\Sigma}(H) \leq \prod_{i=1}^{n_H} \left(\frac{2^\kappa}{r}\right)^{r d_H(v_i)}$$

(1)

$$\leq \left(\frac{2^\kappa \cdot e^{r}}{r}\right)^{r m_H} \cdot r^{2m_H}$$

(2)

$$< 2^{\alpha b m_H / 4} \cdot r^{2m_H}$$

(3)

$$\leq 2^{\frac{b}{4} m_H} \cdot r^{2m_H}$$

(4)

$$\leq 2^{b m_H} = L(H).$$

(5)

Inequality (1) is true since for every PLS in the $\text{MCAST}(r)$ model with verification complexity $\kappa$, every communication pattern can be constructed by letting each node $v_i$ choose $r$ different messages of size $\kappa$ each, and for each of its $d_H(v_i)$ neighbors, let it choose one of the $r$ messages to send. Inequality (2) is due to the fact that $\left(\frac{x}{y}\right)^{y} \leq \left(\frac{x \cdot e^{x}}{y}\right)^{y}$ for $x, y \geq 0$. Inequality (3) follows from our assumption that $\kappa < \frac{ab}{r^2} - 2$. Inequality (4) follows from Lemma 2 which implies that $\alpha \leq 2m_H/n_H$, and Inequality (5) from our assumption that $r \leq 2^{b/4}$. 

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Therefore we may conclude that if \( \kappa < \frac{ab}{2r} - 2 \), then, by Claim 4.2, \( \Sigma \) is not a correct verification scheme for \((\mathcal{F}, \text{EA}_b)\). This concludes the proof of the lower bound.

Next, we turn to the upper bound. To this end we define a more general problem as follows.

**Definition 5 (b-bit Edge \( \psi (\text{EA}_b) \)).**

**Instance:** Each node \( v \) holds in its state a b-bit string \( B_v(e) \) for each incident edge \( e \).

**Question:** Is \( \psi (B_v(e), B_u(e)) = \text{TRUE} \) for every edge \( e = (u, v) \), where \( \psi_b \) is a given symmetric predicate of two b-bit strings, i.e., \( \psi_b : \{0, 1\}^b \times \{0, 1\}^b \rightarrow \{\text{TRUE}, \text{FALSE}\} \) and \( \psi(s, s') = \psi(s', s) \) for all \( s, s' \in \{0, 1\}^b \) ?

**Lemma 8.** For every \( 1 \leq r < 2\alpha \), there exists a PLS for \((\mathcal{F}, \text{EA}_b)\) in the MCAST(\( r \)) model with verification complexity \( O(\frac{\alpha}{r} (b + \log \Delta)) \), and for every \( 2\alpha \leq r \leq \Delta \), there exists a PLS for \((\mathcal{F}, \text{EA}_b)\) in the MCAST(\( r \)) model with verification complexity \( O(b) \).

**Proof:** Let \( 1 \leq r < 2\alpha \) be given. We construct a scheme \( \Sigma = (\mathbf{p}, \mathbf{v}) \) in the MCAST(\( r \)) model as follows. \( \Sigma \) uses color addressing and minimizing orientation. Let \( \text{Col}_v \) be the color assigned to node \( v \), and let \( d^v(v) \) denote the out-degree of node \( v \) under the orientation. The verifier \( \mathbf{v} \) at node \( v \) partitions the outgoing edges into \( r \) parts, each of size at most \( Q \overset{\text{def}}{=} \lceil d^v(v)/r \rceil \), such that all edges in a part are sent the same message as follows. Let \( \{e_1 = (v, w_1), \ldots, e_Q = (v, w_Q)\} \) be one part of outgoing edges of \( v \). The message \( v \) sends to all these edges is the list of \( Q \) pairs \((\text{Col}_{w_i}, B_v(e_i))\) for \( 1 \leq i \leq Q \). One of the messages that are sent over outgoing edges is sent over all incoming edges (in order to meet the limit of only \( r \) different messages.) Every node \( v \), upon receiving a message \( M_w(e) \) over an edge \( e = (v, w) \), verifies the following conditions.

1. If \( e \) is an outgoing edge, then there exists no pair \((\text{Col}, x)\) in \( M_w(e) \) such that \( \text{Col} = \text{Col}_v \).
2. If \( e \) is an incoming edge then:
   
   (a) There exists exactly one pair \((\text{Col}, x)\) in \( M_w(e) \) such that \( \text{Col} = \text{Col}_v \).
   
   (b) For the pair \((\text{Col}, x)\), it holds that \( \psi_B(x, B_v(e)) = \text{TRUE} \).

   Obviously, this is a PLS in the MCAST(\( r \)) model. We now prove its correctness. By Lemma 4, we can assume that the colors of neighbors of each node are different from each other and from the color of the node. If the configuration is legal and labels are assigned according to \( \mathbf{p} \), all nodes output TRUE. Suppose now that the configuration is illegal. Hence, there must be two neighbors, \( u \) and \( v \), such that for the edge \( e = (u, v) \) we have \( \psi_B(B_u(e), B_v(e)) = \text{FALSE} \). Since \( r < 2\alpha \), Lemma 3 does not imply a proper verification of the orientation. Therefore, our scheme should verify it. If both \( v \) and \( u \) consider \( e \) as an outgoing edge, then \( M_v(e) \) contains the pair \((\text{Col}_u, B_v(e))\). Therefore, \( u \) rejects condition (2) and outputs FALSE. If both \( v \) and \( u \) consider \( e \) as an incoming edge, then \( M_v(e) \) does not contain a pair \((\text{Col}, x)\) such that \( \text{Col} = \text{Col}_u \). Therefore, \( u \) rejects condition (3.a) and outputs FALSE. Assume now, w.l.o.g., that \( e \) is oriented from \( v \) to \( u \). By definition of the scheme, there exists exactly one pair \((\text{Col}_u, x)\) in \( M_v(e) \), and for this pair we know that \( x = B_u(e) \). Since \( \psi_B(B_u(e), B_v(e)) = \text{FALSE} \), \( u \) rejects condition (3.b) and outputs FALSE.
Regarding complexity, by definition of degeneracy, for every \( v \), it holds that \( \delta^0(v) \leq \delta(G) \). By Lemma 1, \( \delta(G) < 2\alpha(G) \). By Lemma 4, every Col can be represented using \( O(\log \Delta) \) bits, and overall, every message is of size \( O\left(\frac{\alpha}{r} (b + \log \Delta)\right) \).

For \( 2\alpha \leq r \leq \Delta \), by Lemma 3 and the fact that every scheme in the MCAST\( (r_1) \) model is in particular a scheme in the MCAST\( (r_2) \) model for \( r_2 \geq r_1 \), there exists a PLS \( \delta' = (p', \psi') \) for \((\mathcal{F}, \mathcal{EA}_{b})\) in the MCAST\( (r) \) model with verification complexity \( b \) \( \mathcal{EA}_{b} \) is a special case of \( \mathcal{E}\psi_{b} \), where \( \psi \) is the equality predicate. Therefore, Lemma 8 gives a tight upper bound for \((\mathcal{F}, \mathcal{EA}_{b})\) for the case \( b \in \Omega(\log \Delta) \). This concludes the proof of Theorem 2.

We note that Theorem 2, in conjunction with Theorem 7, gives the following corollary.

**Corollary 1.** Let \( b \in \Omega(\log \Delta) \). For every \( 1 \leq r \leq \min\{\Delta, 2^{\beta/4}\} \), the randomized verification complexity of \((\mathcal{F}, \mathcal{EA}_{b})\) in the MCAST\( (r) \) model is \( \Theta(\log(\lceil \frac{\alpha}{r} \rceil b)) \).

### 4.3 An Advanced Example: The Maximum Flow Problem

In this section we consider a more sophisticated problem, namely Maximum Flow in the context of the MCAST\( (r) \) model. The best previously known result [22] was for verification of “\( k \)-flow”: the goal is to verify that the maximum flow between a given pair of nodes is exactly \( k \). The verification complexity of the scheme of [22] is \( O(k(\log k + \log n)) \) in the broadcast model. In Theorem 4, we show an improvement of this result and a generalization to the MCAST\( (r) \) model.

First, we solve a slightly different problem, formalized as follows. Let \( \mathcal{F}_{st} \) be the family of configurations of graphs, where a graph in \( \mathcal{F}_{st} \) has two distinct nodes denoted \( s \) and \( t \) called source and sink, respectively, and a natural number \( c(e) \) called the capacity associated with each edge \( e \). The MF problem is defined over the family of configurations \( \mathcal{F}_{st} \) as follows.

**Definition 6 (Maximum Flow (MF)).**

**Instance:** A configuration \( G_s \in \mathcal{F}_{st} \), where each node \( v \) has an integer \( f(v, u) \) for every neighbor \( u \).

**Question:** Interpreting \( f(v, u) \) as the amount of flow from \( v \) to \( u \) \((f(v, u) < 0 \) means flow from \( u \) to \( v \)), is \( f \) a maximum flow from \( s \) to \( t \)?

Recall that \( f \) is a legal flow if it satisfies the following three conditions (see, e.g., [1]).

- Anti symmetry: for every \((v, u) \in E\), \( f(v, u) = -f(u, v) \).
- Capacity compliance: for every \((v, u) \in E\), \(|f(v, u)| \leq c(v, u)\).
- Flow conservation: for every node \( v \in V \setminus \{s, t\}\), \( \sum_{u \in V} f(v, u) = 0 \).

If all three conditions hold, then, by the max-flow min-cut theorem, \( f \) is maximum iff there is a saturated cut.

We denote by \( f_{\text{max}} \) the maximal flow amount over all edges of \( G \) (note that \( f_{\text{max}} \) need not be polynomial in \( n \)). Also, for a bit string \( x = x_0 x_1 \cdots x_k \), let \( \bar{x} = \sum_{i=0}^{k} x_i 2^i \).

**Theorem 3.** Let \( \log f_{\text{max}} \in \Omega(\log n) \). There exists a constant \( c > 1 \) such that for every \( 1 \leq r \leq \min\{\alpha/c, \sqrt{f_{\text{max}}}\} \), the verification complexity of \((\mathcal{F}_{st}, \text{MF})\) in the MCAST\( (r) \) model is \( \Theta(\log(f_{\text{max}})\alpha/r) \).
Again, we start with the lower bound. We note that the counting argument used for $\text{EAB}_s$ (Lemma 7) cannot be applied to this problem. To prove the lower bound for $\text{MF}$, we show a non-trivial reduction from a problem in $(\mathcal{F}, \text{EAB}_s)$ to a problem in $(\mathcal{F}_{st}, \text{MF})$.

**Lemma 9.** Let $\log f_{\text{max}} \in \Omega(\log n)$. There exists a constant $c > 1$ such that for every $1 \leq r \leq \min \{\alpha/c, \sqrt{f_{\text{max}}}\}$, the verification complexity of any PLS for $(\mathcal{F}_{st}, \text{MF})$ in the MCAST($r$) model is $\Omega(\log(f_{\text{max}})\alpha/r)$.

**Proof:** We reduce the problem $(\mathcal{F}, \text{EAB}_s)$ to the problem $(\mathcal{F}_{st}, \text{MF})$ with $f_{\text{max}} \leq m \cdot 2^t$. Let $\Sigma_f = (p_f, v_f)$ be a PLS for $(\mathcal{F}_{st}, \text{MF})$ in the MCAST($r$) model. We construct from $\Sigma_f$ a PLS $\Sigma = (p, v)$ for $(\mathcal{F}, \text{EAB}_s)$ in the MCAST($r$) model. Let $G_s \in \mathcal{F}$ be such that $\text{EAB}_s(G_s) = \text{TRUE}$ with an underlying graph $G = (V, E)$, whose arboricity is $\alpha$. Let $T = (V, E_T)$ be a breadth-first spanning tree of $G$ rooted at some node $v \in V$. We denote by $p(v)$ the parent of $v$ in $T$. Intuitively, the prover constructs from a legal $(\mathcal{F}, \text{EAB}_s)$ instance $G_s$, a legal $(\mathcal{F}_{st}, \text{MF})$ instance $G'_s$ by letting exactly one node to simulate nodes $s$ and $t$ which are connected to the rest of the network with edges of capacity 0. Assigning a capacity to every edge in $E$ results in $G'_s \in \mathcal{F}_{st}$. Then, the prover defines a cyclic legal flow according to the input strings $B_s(u)$. Think of an arbitrary orientation of the edges, and assume that for the oriented edge $(v, u)$ with $B_v(u) = B_u(v) = B$ we assign the flow $f(v, u) = -B$ and $f(u, v) = B$. Course, $f$ does not necessarily satisfy flow conservation. The actual flow we define would be the sum of $f$ with the convergecast on $T$ of excess flow from all nodes. The result is a legal flow of value 0 in a network where the minimum cut is 0. Therefore, $\text{MF}(G'_s) = \text{TRUE}$.

Formally, the prover orient $\Sigma_f$ all edges arbitrarily. Let $h_v(u)$ be the variable indicating to $v$ the orientation of edge $(v, u)$. If the prover orient $e = (v, u)$ from $v$ to $u$ then $h_v(u) = -1$ and $h_u(v) = 1$. We define $X(v)$ as the excess flow of node $v$ if the flow over every edge $e = (v, u)$ were $h_v(u) \cdot B_v(u)$, i.e., $X(v) = \sum_{(v, u) \in E} h_v(u) \cdot B_v(u)$. We denote by $T_v$ the set of nodes in the sub-tree rooted at $v$, and define $X(T_v)$ as the sum of all excess flow in this sub-tree, $X(T_v) = \sum_{u \in T_v} X(u)$. Let $G'_s \in \mathcal{F}_{st}$ be a configuration as follows. Add to $G$ nodes $s$ and $t$ connected to $w$ with edges $(s, w), (t, w)$ of capacity 0. The idea is that $w$ simulates in addition to the verification of $\text{MF}$ of itself, the verification of $\text{MF}$ of $s$ and $t$. For every edge $e \in E$, $c(e) = m \cdot 2^t$. For every node $v \in V$ and neighbor $u \in V$ the flow $f(v, u)$ in configuration $G'_s$ is defined as follows. If $u = p(v)$ then $f(v, u) = h_v(u) \cdot B_v(u) - X(T_v)$, if $v = p(u)$ then $f(v, u) = h_u(u) \cdot B_v(u) + X(T_u)$, and otherwise $f(v, u) = h_v(u) \cdot B_v(u)$. For completeness, we define $f(s, w) = f(u, s) = f(t, w) = f(w, t) = 0$.

We first prove that if $\text{EAB}_s(G_s) = \text{TRUE}$ then $\text{MF}(G'_s) = \text{TRUE}$. By construction, the value of flow going out of $s$ and into $t$ is 0, and the capacity of the cut $\langle \{s\} : V \cup \{t\} \rangle$ is 0. Therefore, if $f$ is a legal flow then it is a maximum flow. Symmetry holds by construction and the assumption that $B_v(u) = B_u(v)$ for every $u$ and $v$. Capacity constraints hold by definition of $X(T_v)$ and the fact that $B_v(u) \leq 2^t$. Finally, we prove that flow conservation holds. Let $\mathcal{C}(v) = \{u \mid p(u) = v\}$ be the set of children of $v$ in $T$, and let $\mathcal{N}(v) = \{u \mid (v, u) \in E, u \neq p(v), v \neq p(u)\}$ be the set of neighbors of $v$ which are neither the children of $v$ nor $v$’s parent in $T$. The flow at every node $v \neq w$ satisfies the following.
This concludes the proof that if $EA_{h}G_{X}(h) = h_{v}(u) \cdot B_{v}(u) - X(T_{v}) + \sum_{u \in C(v)}(h_{v}(u) \cdot B_{v}(u) + X(T_{u})) + \sum_{u \in C(v)}(h_{v}(u) \cdot B_{v}(u))$

Equality (6) is true by construction of $f$, and Equality (7) is true since by definition, $X(T_{v}) = X(v) + \sum_{u \in C(v)} X(T_{u})$.

For $w$, we know that $f(w, s) = f(w, t) = 0$. Therefore, we only need to prove flow conservation over the edges $(w, u) \in E$.

\[
\sum_{(w, u) \in E} f(w, u) = \sum_{u \in C(w)} (h_{w}(u) \cdot B_{w}(u) + X(T_{u}))
\]

Equality (8) is true since $T$ is a BFS rooted at $w$ and therefore, every neighbor of $w$ is a child of $w$ in $T$, and Equality (9) is true since by definition, $X(T_{v}) = X(v) + \sum_{u \in C(v)} X(T_{u})$ is the sum of all excess flow of nodes in $T$, and $T$ spans $G$. Equality (10) is true by assumption that for every $u$, $B_{v}(u) = B_{u}(v)$ and $h_{v}(u) \cdot h_{u}(v) = -1$. For $s$ and $t$, flow conservation holds immediately by construction. This concludes the proof that if $EA_{b}(G_{s}) = TRUE$ then $MF(G_{s}) = TRUE$.

We now describe the details of the scheme $\Sigma = (p, v)$ for $(F, EA_{b})$. Given a configuration $G_{s} \in F$ such that $EA_{b}(G_{s}) = TRUE$, the prover $p$ constructs the configuration $G_{s}^{p} \in F_{td}$ and assigns for every node $v \in V$ a label which is composed of seven parts:

- $\ell_{1, v} = ID_{w}$.
- $\ell_{2, v} = dist(w, v)$ is the distance between $w$ and $v$ in $T$.
- $\ell_{3, v} = ID_{p_{v}}$.
- $\ell_{4, v} = \{h_{v}(u) \mid (v, u) \in E\}$.
- $\ell_{5, v} = \{f(w, u) \mid (v, u) \in E\}$.
- $\ell_{6, v} = X(T_{v})$.
- $\ell_{7, v} = \ell_{f}(v)$ is the label assigned by $p_{f}$ to $v$ in $G_{s}^{p}$ (if $v = w$ then it receives also $\ell_{f}(s)$ and $\ell_{f}(t)$).
The verifier \( v \) at node \( v \) operates as follows. The message sent from node \( v \in V \) to its neighbor \( u \in V \) is \( \Sigma_r(u) = (\text{ID}_v, \ell_{1,v}, \ell_{2,v}, \ell_{3,v}, \ell_{6,v}, M^f(u)) \) where \( M^f(u) \) is the message \( v \) sends to \( u \) according to \( \Sigma_f(G_s') \). Upon receiving messages \( M_u(v) \) from all neighbors, \( v \) outputs the conjunction of the following.

1. If \( \ell_{2,v} = 0 \) then \( \ell_{1,v} = \text{ID}_v \).
2. \( \ell_{1,v} = \ell_{1,v} \) for every neighbor \( u \).
3. If \( \ell_{2,v} > 0 \) then there exists a neighbor \( u \) such that \( \ell_{2,u} = \ell_{2,v} - 1 \) and \( \ell_{3,v} = \text{ID}_u \).
4. For \( u \) such that \( \ell_{3,v} = \text{ID}_u \), \( f(v,u) = h_v(u) \cdot B_v(u) - \ell_{6,v} \).
5. For \( u \) such that \( \ell_{3,v} = \text{ID}_v \), \( f(v,u) = h_v(u) \cdot B_v(u) + \ell_{6,u} \).
6. For \( u \) such that neither \( \ell_{3,v} = \text{ID}_u \) nor \( \ell_{3,u} = \text{ID}_v \), \( f(v,u) = h_v(u) \cdot B_v(u) \).
7. If \( \ell_{2,v} > 0 \) then \( v \) simulates the output of \( v \) according to \( \Sigma_f \) with the set of flows \( \ell_{5,v} \), label \( \ell_{7,v} \), and the received message \( M^s(v) \) from every neighbor \( u \).
8. If \( \ell_{2,v} = 0 \) then \( v \) simulates the output of \( v \) according to \( \Sigma_f \) with the set of flows \( \ell_{5,v} \cup \{ f(v,s) = 0 \), \( f(v,t) = 0 \} \), label \( \ell_{7,v} \), and the received message \( M^s(v) \) from every neighbor \( u \) and messages \( M^f(v) \) and \( M^s(v) \) that are sent from \( s \) with label \( \ell_f(s) \) and from \( t \) with label \( \ell_f(t) \) according to \( \Sigma_f \) respectively. In addition, \( v \) simulates the output of \( s \) and \( t \) according to \( \Sigma_f \) with the flows \( f(s,v) = 0 \) and \( f(t,v) = 0 \), labels \( \ell_f(s) \) and \( \ell_f(t) \), and received messages \( M^s(v) \) and \( M^f(v) \) respectively. The result of this verification item is the conjunction of outputs of \( u, v \) and \( t \).

By construction, if \( M^f(u_1) = M^f(u_2) \) then \( M_v(u_1) = M_v(u_2) \). Therefore, if \( \Sigma_f \) is a PLS in the \text{mcast}(r) model then \( \Sigma \) is a PLS in the \text{mcast}(r) model. Let \( c^\ast \) be a constant such that every 1D is at most \( n^{c^\ast} \). If the verification complexity of \( \Sigma_f \) is \( \kappa \), then the verification complexity of \( \Sigma \) is \( \kappa + b + 3(c^\ast + 1) \log n \). This is true because \( X(T_r) \leq n^2 \cdot 2^b \). If \( \Sigma \) is a correct verification scheme for \((\mathcal{F}, EA_b)\), by the proof of \textbf{Lemma 7}, its verification complexity is greater than \( \frac{a}{r}\beta b - 2 \). If \( b \geq \log n \) and \( \frac{a}{r \beta} > 12c^\ast + 16 \) we get that \( \kappa \in \Omega(\frac{a}{r \beta}) \). Since by construction \( \ell_{max} \leq m \cdot 2^b \), it follows that \( \kappa \in \Omega(\frac{a}{r \beta} \log(\ell_{max}/n)) \), and if \( \log \ell_{max} \in \Omega(\log n) \) we get that \( \kappa \in \Omega(\frac{a}{r \beta} \log \ell_{max}) \).

We now prove the correctness of \( \Sigma \). If \( EA_b(G_s) = \text{true} \) and labels are assigned according to \( p \), then clearly, all nodes output \text{true}. Suppose now that \( EA_b(G_s) = \text{false} \). Then, there is at least one edge \((v,u) \in E \) such that \( B_v(u) \neq B_u(v) \). We assume that all nodes output \text{true} and show that it leads to a contradiction. If all nodes verify properties (1), (2) and (3) then there is exactly one node \( w \) which simulates \( s \) and \( t \). Therefore, the simulated configuration is in \( \hat{F}_{st} \). If all nodes verify properties (7) and (8) then the simulated configuration \( \hat{G}_s \) satisfies \( \text{mF}(\hat{G}_s) = \text{true} \). In particular, \( f(v,u) = -f(u,v) \). If \( u \) and \( v \) verify properties (4), (5) and (6) then the following holds. If \( \ell_{3,v} = \text{ID}_u \) then \( f(v,u) = h_v(u) \cdot B_v(u) + \ell_{6,u} \) and \( f(u,v) = h_u(v) \cdot B_u(v) - \ell_{6,u} \). If \( \ell_{3,v} = \text{ID}_u \) then \( f(v,u) = h_v(u) \cdot B_v(u) - \ell_{6,v} \) and \( f(u,v) = h_u(v) \cdot B_u(v) + \ell_{6,v} \). If neither \( \ell_{3,v} = \text{ID}_u \) nor \( \ell_{3,u} = \text{ID}_v \) then \( f(v,u) = h_v(u) \cdot B_v(u) \) and \( f(u,v) = h_u(v) \cdot B_u(v) \). In all three possible cases, since \( h_v(u) \) and \( h_u(v) \) have values either 1 or \(-1\), it holds that \( f(v,u) = -f(u,v) \) if and only if \( B_v(u) = B_u(v) \). Contradicting the assumption that \( B_v(u) \neq B_u(v) \). This concluded the proof.

\textbf{Lemma 10.} For every \( 1 \leq r < 2\alpha \), there exists a PLS for \((\mathcal{F}_{st}, \text{mF})\) in the \text{mcast}(r) model with verification complexity \( O(\frac{a}{r \beta} (\log \ell_{max} + \log \Delta)) \), and for every \( 2\alpha \leq r \leq \Delta \),
there exists a PLS for \((F_{st}, MF)\) in the MCAST(r) model with verification complexity \(O(\log f_{\text{max}})\).

**Proof:** Let \(\psi\) be the function of two input strings such that \(\psi(x, y) = \text{TRUE}\) iff \(x = \overline{y}\),

and let \(\Sigma_A = (p_A, v_A)\) be a PLS for \((F, c\psi_b)\) in the MCAST(r) model. We describe the details of a PLS \(\Sigma = (p, v)\) for \((F_{st}, MF)\) in the MCAST(r) model. Let \(G_s \in F_{st}\) be a configuration with an underlying graph \(G = (V, E)\) and \(\text{MF}(G_s) = \text{TRUE}\). Consider the configuration \(G'_s\) defined as follows. The underlying graph of \(G'_s\) is \(G = (V, E)\) and for every node \(v \in V\) and every edge \(e = (v, u)\), \(B_v(e) = f(v, u)\). Obviously, \(G'_s \in F\). In addition, \(\psi_b(G'_s) = \text{TRUE}\) since, in particular, the flow in \(G_s\) satisfies asymmetry on every edge. Let \(Z \subseteq V\) be such that \((Z; V \setminus Z)\) is a minimum \(s\)-\(t\) cut in \(G_s\) with \(s \in Z\) and \(t \notin Z\). The label assigned by \(p\) to every node \(v \in V\) is composed of two parts: \(\ell_{1, v} = z(v)\) where \(z(v)\) is a bit such that \(z(v) = 1 \iff v \in Z\), and \(\ell_{2, v} = \ell_A(v)\) is the label assigned by \(p_A\) to \(v\) in \(G'_s\).

The verifier \(v\) at node \(v\) operates as follows. The message sent from \(v\) over edge \(e = (v, u)\) is \(M_u(e) = (\ell_{1, v}; M^A_v(e))\) where \(M^A_v(e)\) is the message \(v\) sends over \(e\) in \(\Sigma_A(G'_s)\). Upon receiving a message \(M_u(e)\) over every edge \(e = (u, v)\), node \(v\) outputs the conjunction of the following:

1. The output of \(v_A\) upon receiving a message \(M^A_v(e)\) from every neighbor \(u\) of \(v\), where the label of \(v\) is \(\ell_{2, v}\).
2. For every edge \((v, u)\), \(|f(v, u)| \leq c(v, u)\).
3. If \(v \neq s, t\) then \(\sum_{u \in V} f(v, u) = 0\).
4. If \(v = s\) then \(\ell_{1, v} = 1\).
5. If \(v = t\) then \(\ell_{1, v} = 0\).
6. For every neighbor \(u\) of \(v\), if \(\ell_{1, u} \neq \ell_{1, v}\) then \(|f(v, u)| = c(v, u)|\).

If \(\Sigma_A\) is a PLS in the MCAST(r) model with verification complexity \(\kappa\), then \(\Sigma\) is a PLS in the MCAST(r) model with verification complexity \(\kappa + 1\). This is true because if \(M^A_v(e_1) = M^A_v(e_2)\) then \(M_v(e_1) = M_v(e_2)\) and \(\ell_{1, v}\) is one bit. \(\Sigma_A\) is a verification scheme for \((F, c\psi_b)\) where \(b = \log(f_{\text{max}})\). By Lemma 8, the upper bounds follow.

We now prove the correctness of \(\Sigma\). If \(\text{MF}(G_s) = \text{TRUE}\) then for every \(v\) and \(u\), \(f(v, u) = -f(u, v)\). From correctness of \(\Sigma_A\), the result of \(v\) in (1) is TRUE. The result of \(v\) in (2) and (3) is also TRUE since the flow is legal. If labels are assigned as described, then \((Z; V \setminus Z)\) is a minimum \(s\)-\(t\) cut. Since \(f\) is a maximum flow, we know that every minimum cut is saturated. Therefore, the result of \(v\) in (4),(5) and (6) is TRUE, and \(v\) outputs TRUE. If all nodes output TRUE then, by (1),(2) and (3), \(f\) is a legal flow. By (4) and (5), the \(z(v)\) bits indicate an \(s\)-\(t\) cut, and by (6) cut \((Z; V \setminus Z)\) is saturated, and therefore, \(f\) is a maximum flow, \(\text{MF}(G_s) = \text{TRUE}\).

For \(\log f_{\text{max}} \in \Omega(\log n)\), Lemma 10 gives a tight upper bound for \((F_{st}, MF)\) which concludes the proof of Theorem 3.

Consider now the \(k\)-MF problem as defined in [22] over the family of configurations \(F_{st}\).

**Definition 7** \((k\text{-Maximum Flow (k-MF))}\. 

**Instance:** A configuration \(G_s \in F_{st}\).

**Question:** Is the maximum flow between \(s\) and \(t\) in \(G_s\) is exactly \(k\)?
We give an upper bound for \((\mathcal{F}_{st}, k\text{-MF})\) in the MCAST\((r)\) model, which generalizes and improves the previous bound.

**Theorem 4.** For every \(1 \leq r < 2\alpha\), there exists a PLS for \((\mathcal{F}_{st}, k\text{-MF})\) in the MCAST\((r)\) model, with verification complexity \(O\left(\min\{\alpha, k\} r (\log k + \log \Delta)\right)\), and for every \(2\alpha \leq r \leq \Delta\), there exists a PLS for \((\mathcal{F}_{st}, k\text{-MF})\) in the MCAST\((r)\) model, with verification complexity \(O(\log k)\).

**Proof:** In a verification scheme for \((\mathcal{F}_{st}, k\text{-MF})\), the prover can assign the flow values \(f(v,u)\) for every edge \((v,u)\). W.l.o.g, assume that \(f\) does not contain cycles of positive flow. In this case, \(f_{\text{max}} \leq k\) and, since the flow value over each edge is an integer, the number of incident edges of every node \(v\) carrying non-zero flow is at most \(2k\). By Lemma 10, and the observation that it is sufficient that every node verifies the value of flow only on edges with \(f(v,u) \neq 0\), the upper bounds follow.

To be precise, the problem solved in [22] required in addition that every node holds the value \(k\) in its state. Verifying that all nodes hold the same value \(k\) is simply an additive \(\log k\) factor to message length – every node sends its value and verifies that all its neighbors have the same value. We argue in the following lemma, that \(\Omega(\log k)\) is a lower bound for \((\mathcal{F}_{st}, k\text{-MF})\) verification even if \(k\) is known to all nodes.

**Lemma 11.** For every \(1 \leq k \leq 2^{\Theta(n)}\), the verification complexity of any PLS for \((\mathcal{F}_{st}, k\text{-MF})\) is \(\Omega(\log k)\), even in the unicast model and for constant degree graphs.

**Proof:** Consider the following graph family \(\mathcal{F}'\). Node \(s\) is connected to two nodes \(s_0\) and \(s_1\), and node \(t\) is connected to two nodes \(t_0\) and \(t_1\) with edges of capacity \(2^n\). Each of the nodes \(s_0, s_1, t_0, t_1\) is connected to \(y\) additional different nodes with edges of capacity \(2^n\). For \(i \in \{0, 1\}\) consider the following structure. Let \(V^i = \{v^i_0, \ldots, v^i_{y-1}\}\) be the \(y\) nodes connected to \(s_i\) in addition to \(s\) and let \(U^i = \{u^i_0, \ldots, u^i_{y-1}\}\) be the \(y\)
nodes connected to \( t_i \) in addition to \( t \). For every \( 0 \leq j \leq y - 1 \) there is an edge \((v_j^i, u_j^i)\) with capacity either \( c(v_j^i, u_j^i) = 0 \) or \( c(v_j^i, u_j^i) = 2^j\). In particular, \( \sum_{j=0}^{y-1} c(v_j^i, u_j^i) \) can be every integer between 0 and \( 2^y - 1 \). We denote by \( G_{a,b} \) the configuration where the sum of capacities in the subgraph induced by the set of nodes \( V^0 \cup U^0 \) is \( a \), and the sum of capacities in the subgraph induced by the set of nodes \( V^1 \cup U^1 \) is \( b \) (see Figure 1). \( \mathcal{F}' = \{G_{a,b} \mid 0 \leq a, b \leq 2^y - 1\} \). Clearly, for every \( k \leq 2^y - 1 = 2^{\Theta(n)} \) and \( 0 \leq a \leq k \) it holds that \( G_{a,k-a} \in \mathcal{F}' \) and \( k\text{-MF}(G_{a,k-a}) = \text{TRUE} \).

Assume by contradiction that there is a unicast proof-labeling scheme \( \Sigma \) for the \((\mathcal{F}_st, k\text{-MF})\) problem with verification complexity less than \( \frac{\log k}{\alpha} \). Consider the collection of 4 messages sent over edges \((s, s_1)\) and \((t, t_1)\). By assumption, there are less than \( \log k \) bits in this sequence of messages. Hence, there are less than \( k \) different communication patterns over these edges. Therefore, there must be two configurations \( G_{a,k-a} \) and \( G'_{a',k-a'} \), where \( a \neq a' \), such that the communication pattern of \( \Sigma \) over edges \((s, s_1)\) and \((t, t_1)\) is the same for both configurations. Consider the configuration \( G_{a,k-a} \). Obviously, since \( a \neq a' \), \( k\text{-MF}(G_{a,k-a}) = \text{FALSE} \).

By construction, the state of every node \( v \in \mathcal{W} = \{s, t, s_0, t_0 \} \cup V^0 \cup U^0 \) in \( G_{a,k-a'} \) is the same as in \( G_{a,k-a} \), and the state of every node \( v \in \mathcal{W}' = \{s_1, t_1 \} \cup V^1 \cup U^1 \) in \( G_{a,k-a} \) is the same as in \( G'_{a',k-a'} \). Let \( \ell_a(v) \) (respectively \( \ell_{a'}(v) \)) be the label assigned to node \( v \) according to \( \Sigma(G_{a,k-a}) \) (respectively \( \Sigma(G'_{a',k-a'}) \)), and consider the following labeling \( \ell \) for \( G_{a,k-a'} \). For every \( v \in \mathcal{W} \) assign \( \ell(v) = \ell_a(v) \), and for every \( v \in \mathcal{W}' \) assign \( \ell(v) = \ell_{a'}(v) \). Since the state and label of every \( v \in \mathcal{W} \) (respectively \( v \in \mathcal{W}' \)) in \( G_{a,k-a'} \) are exactly as in \( \Sigma(G_{a,k-a}) \) (respectively \( \Sigma(G'_{a',k-a'}) \)), all messages these nodes send are as in \( \Sigma(G_{a,k-a}) \) (respectively \( \Sigma(G'_{a',k-a'}) \)).

By assumption on the communication patterns of \( \Sigma(G_{a,k-a}) \) and \( \Sigma(G'_{a',k-a'}) \), and the fact that \((s, s_1)\) and \((t, t_1)\) are the only edges in \( \mathcal{W} \times \mathcal{W}' \), all nodes in \( G_{a,k-a'} \) output \text{TRUE}, a contradiction to the correctness of \( \Sigma \). Therefore, the verification complexity of any proof-labeling scheme for \((\mathcal{F}', k\text{-MF})\) is \( \Omega(\log k) \). Since \( \mathcal{F}' \subset \mathcal{F}_st \), the lower bound holds for \((\mathcal{F}_st, k\text{-MF})\).

In order to show that this lower bound holds even for constant degree graphs, we change the construction so that every star structure induced by \( \{s_i\} \cup V^i \), for \( i \in \{0, 1\} \), is replaced by a binary tree rooted at \( s_i \) and its leaves are \( V^i \). In the same way, we replace every star structure induced by \( \{t_i\} \cup U^i \), for \( i \in \{0, 1\} \), by a binary tree. The maximum degree of the new graph family is \( O(1) \), and the lemma follows.

By Theorem 4, this lower bound is tight for \( 2\alpha \leq r \leq \Delta \), and the following theorem holds.

**Theorem 5.** For every \( 1 \leq k \leq 2^{\Theta(n)} \) and every \( 2\alpha \leq r \leq \Delta \), the verification complexity of \((\mathcal{F}_st, k\text{-MF})\) in the MCAST\((r)\) model is \( \Theta(\log k) \).

## 5 Verification in Congested Cliques

In the congested clique model, the communication network is a fully connected graph over \( n \) nodes (i.e., an \( n \)-clique). Given an input graph \( G = (V, E) \) with \( n = |V| \), the nodes of \( G \) are mapped 1-1 to the nodes of the clique, and the state of each node contains a bit for each port, indicating whether the edge to that port is in \( E \) or not, and, if the
edge is present and \( G \) is weighted, the weight of the edge. We assume that the part in the state that specifies whether the edge connected to this port is in \( E \) is reliable: since verification is done with respect to the given graph as input, there is no way to verify its authenticity, but only whether the combination of input and output satisfies the given predicate. Moreover, we assume that the input is consistent, in the sense that the state at node \( v \) indicates that \((v, u)\) is an edge in \( E \) (possibly with some weight \( w \)), if and only if so does the state of \( u \) (namely edge agreement on the input graph is guaranteed).

### 5.1 Crossing in Congested Cliques

In what follows, we say that an edge is oriented to indicate a specific order over its endpoints.

**Definition 8 (Independent Edges).** Let \( G = (V, E) \) be a graph and let \( e_1 = (v_1, u_1) \) and \( e_2 = (v_2, u_2) \) be two oriented edges of \( G \). The edges \( e_1 \) and \( e_2 \) are said to be independent if and only if \( v_1, u_1, v_2, u_2 \) are four distinct nodes and \((v_1, u_2), (v_2, u_1) \notin E \).

The following definition is illustrated in Figure 2.

**Definition 9 (Crossing [5]).** Let \( G = (V, E) \) be a graph, let \( e_1 = (v_1, u_1) \) and \( e_2 = (v_2, u_2) \) be two independent oriented edges of \( G \), and for \( i \in \{1, 2\} \), let \( p_i \) and \( q_i \) be the port numbers of \( e_i \) at \( v_i \) and \( u_i \) respectively. The crossing of \( e_1 \) and \( e_2 \) in \( G \), denoted by \( G(e_1, e_2) \), is the graph obtained from \( G \) by replacing \( e_1 \) and \( e_2 \) with the edges \( e'_1 = (v_1, u_2) \) and \( e'_2 = (v_2, u_1) \) so that \( e'_1 \) connects port \( p_1 \) at \( v_1 \) and port \( q_2 \) at \( u_2 \) and \( e'_2 \) connects port \( p_2 \) at \( v_2 \) and port \( q_1 \) at \( u_1 \).

Consider an input graph \( G = (V, E) \) in the clique, assume that \( e_1, e_2 \in E \) are independent edges and let \( G(e_1, e_2) = (V, E') \). Note that crossing a graph over a clique network does not result in a change of state: Due to the port preservation of the crossing operation, for every node \( v \in V \) and every port \( 0 \leq i \leq n - 1 \), the edge \((v, u)\) on port number \( i \) in \( G \) satisfies \((v, u) \in E \) if and only if the edge \((v, u')\) on port number \( i \) in \( G(e_1, e_2) \) satisfies \((v, u') \in E' \).

Whether we can prove a lower bound for verification in the congested clique for \( r > 1 \) is still an open question. However, for the broadcast clique model (i.e., \( r = 1 \)), it turns out that we can. The following lemma is the key to proving lower bounds for PLSs in the broadcast clique.

**Lemma 12.** Let \( \mathcal{F} \) be a family of configurations, let \( \mathcal{P} \) be a boolean predicate over \( \mathcal{F} \), and let \( \Sigma \) be a PLS for \((\mathcal{F}, \mathcal{P})\) in the broadcast clique model with verification complexity \( \kappa \). Suppose that there is a configuration \( G_x \in \mathcal{F} \) such that \( \mathcal{P}(G_x) = \text{TRUE} \) and \( G \) contains \( q \) pairwise independent oriented edges.
If $\kappa < \frac{\log q}{2}$, then there are $1 \leq i < j \leq q$ such that $G_s(e_i, e_j)$ is accepted by $\Sigma$.

**Proof:** Let $\Sigma = (p, v)$ be a PLS for $(F, P)$ in the broadcast clique model, with verification complexity $\kappa$, and let $G_s$ be a configuration as described in the statement. Assume that $\kappa < \frac{\log q}{2}$, and consider a collection of $q$ pairwise independent oriented edges $e_1 = (v_1, u_1), \ldots, e_q = (v_q, u_q)$. Let $\ell(v)$ be the label given by $p$ to $v$, let $M_v$ be the message sent by $v$ to all its neighbors according to $v$, and for every $i$, consider the bit-string $M^i = M_{v_i} \circ M_{u_i}$. We have $|M^i| < \log q$ for every $i$, and thus there are less than $q$ possible distinct $M^i$'s in total. Therefore, by the pigeonhole principle, there are $1 \leq i < j \leq q$ such that $M^i = M^j$. Consider the output of the verifier $v$ in $G_s$ and in $G_s(e_i, e_j)$.

By assumption, $G_s$ is accepted by $\Sigma$, i.e., with the labels provided by $p$, the verifier $v$ outputs $\text{TRUE}$ at all nodes of $G_s$. Therefore, clearly, all nodes other than $v_i, u_i, v_j, u_j$ output $\text{TRUE}$ in $G_s(e_i, e_j)$. Now, consider node $v_i$. Its neighbor $u_i$ in $G_s$ is replaced in $G_s(e_i, e_j)$ by the node $u_j$, and its communication edge $(v_i, u_j)$ in $G_s$ is replaced in $G_s(e_i, e_j)$ by communication edge $(v_i, u_i)$. Since $M_{u_i} = M_{u_j}$, the verifier acts the same at $v_i$ in both $G_s$ and $G_s(e_i, e_j)$. The same argument works for $u_i, v_j, u_j$, and therefore, the verifier also outputs $\text{TRUE}$ at all nodes in $G_s(e_i, e_j)$, which implies that $G_s(e_i, e_j)$ is accepted by $\Sigma$.

We use the following corollary of Lemma 12 to lower-bound verification complexity of broadcast clique PLSs.

**Corollary 2.** Let $F$ be a family of configurations, and let $P$ be a boolean predicate over $F$. If there is a configuration $G_s \in F$ satisfying that $P(G_s) = \text{TRUE}$ and $G$ contains $q$ pairwise independent oriented edges $e_1, \ldots, e_q$ such that for every $1 \leq i < j \leq q$ it holds that $P(G_s(e_i, e_j)) = \text{FALSE}$, then the verification complexity of any deterministic PLS for $(F, P)$ in the broadcast clique model is $\Omega(\log q)$.

Note that we essentially cross two pairs of edges in the crossing operation: one pair of edges in $E$, and one pair of edges in $\bar{E}$. These two pairs are uniquely associated with each other in a way that if we assume a PLS in the $\text{MCAST}(2)$ clique model, then we would not be able to apply the pigeonhole principle even with 1-bit messages. To see why this is true, consider any set of independent oriented edges $(v_1, u_1), \ldots, (v_q, u_q)$. For every $i \neq j$, both edges $(v_i, u_j), (v_j, u_i) \in \bar{E}$ are associated only with the pair of edges $(v_i, u_i), (v_j, u_j) \in E$. Therefore, with a PLS in the $\text{MCAST}(2)$ clique model, it is possible that $M_{v_i}(u_j) \neq M_{v_j}(u_i)$ for every $i \neq j$ independently of other pairs. Hence, the crossing of any two edges may change the local view of at least one node. Therefore, the crossing technique can not be applied for every $r > 1$ in the congested clique.

### 5.2 Minimum Spanning-Tree Verification

In this section we illustrate the use of Corollary 2 and prove tight bounds for the verification complexity of the Minimum Spanning-Tree (MST) problem. Recall that an MST of a weighted graph $G$ is a spanning tree of $G$ whose sum of all its edge-weights is
minimum among all spanning trees of $G$. In particular, in the clique, there is a fully connected communication network, a weighted input graph $G = (V, E, w)$ where $E$ is a subset of communication edges, $w : E \to \mathbb{N}$ is the edge weight assignment, and a subset $T \subseteq E$ is specified as the MST. It is important to notice that all specifications of edge subsets are local in the sense that every node $v \in V$ has $n - 1$ ports and in its state there is a specification for every edge $e_i$ on port number $i$ whether $e_i \in E$ and whether $e_j \in T$. According to our assumption on the clique model, the input graph $G$ is given in a reliable way, i.e., an edge $(v, u)$ is considered by $v$ to be in $E$ if and only if it is considered by $u$ to be in $E$. However, this consistency has to be verified for the edges of $T$. In addition, since the communication network is fully connected and does not depend on the input graph $G$, we also consider the case where $G$ is disconnected. In this case, we define the MST as the set of minimum spanning-trees of all connected components of $G$.

Let $F_{w_{\text{max}}}$ be the family of all weighted configurations (not necessarily connected) with maximum weight $w_{\text{max}}$. Formally, if $e$ is an edge of the underlying weighted graph of a configuration $G_s \in F_{w_{\text{max}}}$, then $w(e) \leq w_{\text{max}}$. Edge weights are assumed to be known at their endpoints.

**Theorem 6.** The verification complexity of $(F_{w_{\text{max}}}, \text{MST})$ in the broadcast clique model is $\Theta(\log n + \log w_{\text{max}})$.

**Proof:** We first use Corollary 2 to show $\Omega(\log n)$ lower bound for MST in the broadcast clique model. Consider the weighted graph $G = (V, E, w)$ where $V = \{v_0, \ldots, v_{n-1}\}$, $E = \{(v_i, v_{i+1}) | 0 \leq i \leq n - 2\}$, and $w(e) = 1$ for every $e \in E$. Intuitively, $G$ is a path of $n$ nodes with edges of weight 1. We define the configuration $G_s$ to be the graph $G$ where $T = E$. Obviously, MST($G_s$) = TRUE. Next, we define the set of $q = \left\lfloor \frac{n}{3} \right\rfloor - 1$ independent oriented edges $e_1, \ldots, e_q$ as follows. For $1 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1$, let $e_i = (v_{3i}, v_{3i+1})$. For every $1 \leq i < j \leq \left\lfloor \frac{n}{3} \right\rfloor - 1$, $G_s = G_s(e_i, e_j)$ is obtained from $G_s$ by removing edges $(v_{3i}, v_{3i+1})$ and $(v_{3j}, v_{3j+1})$ from $G_s$, and replacing them by $(v_{3i}, v_{3j+1})$ and $(v_{3j}, v_{3i+1})$. Thus, the crossing creates two connected components: the cycle $C = (v_{3i+1}, v_{3i+2}, \ldots, v_{3j-1}, v_{3j})$ and a path the contains the rest of the nodes, and therefore, MST($G_s'$) = FALSE. It follows from Corollary 2 that the verification complexity of any deterministic PLS for MST in the broadcast clique model is $\Omega(\log q) = \Omega(\log n)$.

For the $\Omega(\log W)$ lower bound, we show a variation of the proof in [22], which holds also for the broadcast clique model. Assume for contradiction that there exists a scheme $\Sigma$ for MST over $F_W$ in the broadcast clique model with verification complexity $\kappa < \frac{1}{4} \log \left(\frac{W-1}{2}\right)$. Let $G^i = (V, E, w_i)$ be a graph where $V = \{v_0, u_0, v_1, u_1\}$, $E = \{(v_0, u_0), (v_1, u_1), (v_0, v_1), (u_0, u_1)\}$, $w(v_0, v_1) = w(u_0, u_1) = 1$, $w(v_0, u_0) = 2i$ and $w(v_1, u_1) = 2i + 1$ (see Figure 3(a)). Let $G^i_s$ be the configuration over the graph $G^i$ with $T = E \setminus \{(v_1, u_1)\}$. Obviously, MST($G^i_s$) = TRUE for every $i \in \mathbb{N}$. In particular, for every $1 \leq i \leq \frac{W-1}{2}$ it holds that $G^i_s \in F_W$ and $\Sigma$ accepts $G^i_s$. Let $\ell_i(v)$ be the label assigned by $p$ to $v$ in $G^i_s$. Since $\kappa < \frac{1}{4} \log \left(\frac{W-1}{2}\right)$ and the fact that $\Sigma$ is a broadcast scheme, we get that there exist $1 \leq i < j \leq \frac{W-1}{2}$ such that $c_\Sigma(G^i_s) = c_\Sigma(G^j_s)$. Let $G_s$ be the same as $G^i_s$ except that $w(v_0, u_0) = 2j$ (see Figure 3(b)). Since $i < j$ it follows that $2i + 1 = w(v_1, u_1) < w(v_0, u_0) = 2j$. Therefore, since $T = E \setminus \{(v_1, u_1)\}$,
The configurations described in the proof of Theorem 6 for the $\Omega(\log W)$ lower bound. Dashed edges are communication edges, solid edges are in $E$ and thick solid edges are in $T$. (a) the configuration $G_i$ which satisfies MST($G_i$) = TRUE. (b) the configuration $G_s$ which satisfies MST($G_s$) = FALSE since $i < j$.

MST($G_s$) = FALSE. However, since $c_{\Sigma_i}(G_i) = c_{\Sigma_s}(G_s)$, with the labeling $\ell(v_0) = \ell_j(v_0)$, $\ell(u_0) = \ell_j(u_0)$, $\ell(v_1) = \ell_j(v_1)$ and $\ell(u_1) = \ell_j(u_1)$ for $G_s$ we get that nodes $v_0$ and $u_0$ act exactly as in $G_j$ and output TRUE, and nodes $v_1$ and $u_1$ act exactly as in $G_j$ and output TRUE, a contradiction to the correctness of $\Sigma$. This concludes the proof of the $\Omega(\log n + \log W)$ lower bound.

Finally, we show a PLS for MST in the broadcast clique model with verification complexity $O(\log n + \log W)$. Consider the following scheme $(p, v)$. Given a legal configuration $G_s$, i.e., the set of edges $T$ is consistent and is an MST, the prover $p$ roots $T$, and gives every node a pointer to its parent in $T$. The verifier $v$ uses one communication round in which every non-root node sends its identity, the identity of its parent and the weight of the edge connecting it to its parent; the root sends an indication that it is the root. When all messages are received, each node locally constructs $T'$ from the collection of all edges sent by all nodes. Finally, every node $v$ outputs TRUE if the following conditions are met.

1. For all incident edges $e = (v, u)$: $e \in T$ if and only if $e \in T'$.
2. $T'$ is a tree spanning all $n$ nodes.
3. For all $e = (v, u) \in E$: if $e \notin T$ then $w(e) \geq w(e')$ for every $e'$ in the unique path between $v$ and $u$ in $T'$.

We now prove the correctness of the scheme. Recall that by the “red rule” (cf. [28]), the heaviest edge of every cycle is not in the MST. Suppose MST($G_s$) = TRUE, i.e., the set of edges $T$ is an MST. With the labels assigned by $p$, since $T$ is an MST and hence satisfies the red rule, all nodes output TRUE. Assume now that all nodes output TRUE. Then by (1), $T$ must be consistent over all nodes, and by (2), we know that $T$ is a spanning tree. If (3) holds at all nodes, $T$ satisfies the red rule and therefore $T$ is an MST, i.e., MST($G_s$) = TRUE.

6 Randomized Proof-Labeling Schemes in the MCAST model

The concept of randomized proof labeling schemes was introduced in [5]. Briefly, the idea is that the messages generated by the verifier may depend not only on the local state and label, but also on local random bits, and the correctness requirement is that if $G_s$ satisfies $P$, then, using the labels assigned by the prover, all local verifiers accepts
An RPLS for a given family $F$ of configurations and a boolean predicate $P$ over $F$, in the $\text{MCAST}(r)$ model is defined in the same way deterministic PLS for $(F, P)$ is defined, with the exception that the messages sent by the verifier are a function of the local state, local label and a string of random bits. The restriction of the $\text{MCAST}(r)$ model means that the number of distinct messages (which are now random variables) that may be sent out by a node is at most $r$. In accordance with our concern about dynamic partitioning of the neighbors, we stress that in this case too, we assume that the partitioning of neighbors into same-message groups is done obliviously of the actual random bits. The correctness requirement are the same as in the standard requirements from a (one-sided) RPLS: A randomized scheme $(p, v)$ for $(F, P)$ must satisfy the following requirements, for every $G_s \in F$:

- If $P(G_s) = \text{TRUE}$ then, using the labels assigned by $p$, the verifier $v$ accepts $G_s$.
- If $P(G_s) = \text{FALSE}$ then, for every label assignment, $Pr[v \text{ rejects } G_s] \geq \frac{1}{2}$.

In this section, we extend the exponential relation between verification complexity of deterministic and randomized schemes (shown in [5] for broadcast deterministic schemes and unicast randomized schemes) to the $\text{MCAST}(r)$ model.

**Theorem 7.** Let $F$ be a family of configurations, let $P$ be a boolean predicate over $F$, and consider schemes for $(F, P)$ in the $\text{MCAST}(r)$ model. If there exists a (deterministic) PLS with verification complexity $\kappa_d$ then there exists an RPLS with verification complexity $O(\log \kappa_d)$, and if there exists an RPLS with verification complexity $\kappa_r$ then there exists a PLS with verification complexity $O(2^{\kappa_r})$.

**Proof:** In [5] it is shown that an RPLS (recall that this means a unicast RPLS) with verification complexity $O(\log \kappa)$ can be constructed from a (broadcast) PLS with verification complexity $\kappa$. The proof of this result can be easily adapted to show the following generalization.

**Lemma 13.** Let $F$ be a family of configurations and let $P$ be a boolean predicate over $F$. If there exists a PLS for $(F, P)$ in the $\text{MCAST}(r)$ model with verification complexity $\kappa$, then there exists an RPLS for $(F, P)$ in the $\text{MCAST}(r)$ model with verification complexity $O(\log \kappa)$.

The converse holds as well, as stated in the following lemma.

**Lemma 14.** Let $F$ be a family of configurations and let $P$ be a boolean predicate over $F$. If there exists a one-sided RPLS for $(F, P)$ in the $\text{MCAST}(r)$ model with verification complexity $\kappa$, then there exists a PLS for $(F, P)$ in the $\text{MCAST}(r)$ model with verification complexity $O(2^\kappa)$.

**Proof:** Let $(p, v)$ be a one-sided randomized proof-labeling scheme for $(F, P)$ in the $\text{MCAST}(r)$ model with verification complexity $\kappa$. Let $G_s \in F$ be a configuration satisfying predicate $P$. For every node $v$, let $\ell(v)$ be the label assigned to $v$ by $p$, let $u_1, \ldots, u_d$ be the $d = \deg(v)$ neighbors of $v$ and let $g_1, \ldots, g_r$ be the partition of $u_1, \ldots, u_d$ to $r$ groups used by $(p, v)$ (some groups may be empty). For every non-empty group of
neighbors \( g_i \), let \( \mathcal{C}(v, g_i) = \{ c_{1(v,g_i)}, \ldots, c_{v(g_i)} \} \) be the collection of all certificates with positive probability to be sent from \( v \) to its neighbors in group \( g_i \) according to \( v \) where labels are assigned by \( p \). By definition of MCAST\((r)\), all neighbors that belong to the same group receive the same certificate. We construct a deterministic proof-labeling scheme \((p', v')\) for \((\mathcal{F}, \mathcal{P})\) in the MCAST\((r)\) model as follows. For every \( v \), the label assigned by \( p' \) to \( v \) is \( \ell'(v) = \ell(v) \). For every neighbor \( u \) of \( v \), let \( g_u(v) \) be the group containing \( u \) in the partition of \( v \). The message sent from \( v \) to \( u \) according to \( v' \) is \( M'_u(v) = x_{1(v,u)}, \ldots, x_{v(u)} \) where \( x_{j(v,u)} \) is a bit whose value is 1 iff \( j \in \mathcal{C}(v, g_u(u)) \). Upon receiving \( M'_u(v), M'_{u_1}(v) \), \ldots, \( M'_{u_d}(v) \), the verifier at \( v \) outputs FALSE if and only if there exist \( j_1, \ldots, j_d \) such that \( v(v) = \text{FALSE} \) upon receiving \( j_1, \ldots, j_d \) from neighbors \( u_1, \ldots, u_d \) respectively, and for every \( 1 \leq i \leq d \) it holds that \( x_{j_i(v,u)} = 1 \). Intuitively, if for some combination of certificates in the support of \( v \) it holds that \( v(v) = \text{FALSE} \) then \( v'(v) = \text{FALSE} \), otherwise \( v'(v) = \text{TRUE} \). This concludes the construction of \((p', v')\).

The correctness of the scheme \((p', v')\) follows from the observation that by construction, every combination of messages \( j_1, \ldots, j_d \), such that for every \( 1 \leq i \leq d \) it holds that \( x_{j_i(v,u)} = 1 \), has positive probability to occur in \( v \). Therefore, since \((p, v)\) is a one-sided scheme, all combinations must lead to an output of TRUE. Regarding communication complexity, the length of the messages sent by \( v' \) is exactly the number of possible messages sent by \( v \). Since the verification complexity of \((p, v)\) is \( \kappa \), the number of bits in the messages of \( v' \) is \( 2^\kappa \).

7 Conclusion

In this paper we studied the MCAST\((r)\) model from the perspective of verification. This angle seems particularly convenient, because it involves a single round of message exchange. (If multiple rounds are allowed, one has to consider the possibility of reconfiguring the neighbor partitions: is it allowed to partition the neighbors anew in each round, and if so, at what cost?). We focus on the relation between the number of different messages of each node and the verification complexity of proof-labeling schemes. We gave tight bounds on the verification complexity of edge agreement and max flow in the MCAST\((r)\) model. We have shown that in the restrictive broadcast model, a well defined matching is harder to verify than the maximalitiy of a given matching, and that it is possible to obtain lower bounds on the verification complexity in congested cliques. Many interesting questions remain open. We list a few below.

- Develop a theory for a restricted number of interface cards (NICs). The number of NICs limits the number of messages that can be simultaneously transmitted. In this paper we looked only at a simple case of one round of communication. We believe that developing a tractable and realistic model in which the number of NICs is a parameter is an important challenge.
- As mentioned, in multiple round algorithms, dynamic reconfigurations can be exploited to convey information. It seems that an interesting challenge would be to account for dynamic reconfigurations.
- We considered a model in which a single parameter \( r \) is used to indicate the restriction of all nodes. What can be said about a model in which every node has its own restriction?
We have given examples of problems that have a linear improvement in verification complexity as a function $r$, and on the other hand, we have given examples of problems that are not sensitive at all to $r$. Can a characterization of problems be shown, according to their sensitivity of verification complexity to $r$?

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