ON THE RANK OF $\pi_1(\text{Symp}_0)$

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Abstract. We show that for any positive integer $k$ there exists a closed symplectic 4-manifold $(M, \omega)$, such that $H^1_{\text{dR}}(M; \mathbb{R})$ is a $k$-dimensional real vector space and its Flux group is equal to $H^1_{\text{dR}}(M_1; \mathbb{Z})$.

1. Introduction

In symplectic topology it is a far reaching problem to determine the homotopy type of the groups of Hamiltonian and symplectic diffeomorphisms of a symplectic manifold. For the case of the group of Hamiltonian diffeomorphisms only a few cases have been fully worked out. See for instance [1, 2, 3, 4, 9, 10]. Notice that all of the examples outlined are 4-dimensional. The reason being that holomorphic curves in 4-dimensional symplectic manifolds provide more information than in higher dimensions. In [13], we proved that for any positive integer $k$ there exists a closed symplectic 4-manifold $(M, \omega)$ such that $\mathbb{Z}^k$ injects into $\pi_1(\text{Ham}(M, \omega))$. That is, it is possible to make the rank of $\pi_1(\text{Ham}(M, \omega))$ as large as possible by 4-dimensional manifolds. The techniques used in [13] are classical, they do not rely on holomorphic curves; it make use of the symplectic one-point blow up and Weinstein’s morphism.

The aim of this article is to prove an analogous result but for the group of symplectic diffeomorphisms. In this regard, recall that the inclusion map $\text{Ham}(M, \omega) \to \text{Symp}_0(M, \omega)$ induces an isomorphism of abelian groups

$$
\pi_j(\text{Ham}(M, \omega)) \simeq \pi_j(\text{Symp}_0(M, \omega))
$$

for $j \geq 2$. Here $\text{Symp}_0$ stands for the connected component of the group of symplectic diffeomorphisms that contains the identity diffeomorphism. And for the case when $j = 1$, the fundamental groups differ by the flux group $\Gamma_{(M, \omega)} \subset H^1_{\text{dR}}(M; \mathbb{R})$,

$$
0 \to \pi_1(\text{Ham}(M, \omega)) \to \pi_1(\text{Symp}_0(M, \omega)) \to \Gamma_{(M, \omega)} \to 0.
$$

The statements listed above about the group of symplectic diffeomorphisms can be found in [11] Sec. 10.2. It is known that for a closed symplectic manifold the flux group $\Gamma_{(M, \omega)}$ is discrete in $H^1_{\text{dR}}(M; \mathbb{R})$. See for instance the work of F. Lalonde, D. McDuff and L. Polterovich [3] for partial results in this direction;

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and K. Ono [12] for the proof in the general case. As for the case $j = 0$, recently D. Aroux and I. Smith [5] showed that the number of connected components of $\text{Symp}(M, \omega)$ can be made arbitrarily large by some 6-dimensional symplectic manifolds.

Returning back to the case of interest of this note, if we want to know if the rank of $\pi_1(\text{Symp}_0(M, \omega))$ can be made arbitrarily large by 4-dimensional symplectic manifolds we must be cautioned about the subgroup $\pi_1(\text{Ham}(M, \omega))$.

In that respect the correct statement is the following.

**Theorem 1.1.** Given $k \in \mathbb{N}$, there exists a closed connected symplectic 4-manifold $(M_k, \omega_k)$ such that

$$\Gamma_{(M_k, \omega_k)} = H^1_{dR}(M_k; \mathbb{Z}) \simeq \mathbb{Z}^k.$$

The construction of the symplectic manifold $(M_k, \omega_k)$ is inspired by the work of R. Gompf [6]. That is, using the symplectic fiber connected sum, it is possible to come up with a 4-dimensional symplectic manifold $(M_k, \omega_k)$ such that $H^1_{dR}(M; \mathbb{Z}) \simeq \mathbb{Z}^k$ for a given positive integer $k$. Theorem 1.1 says that it is in fact possible to find a 4-dimensional symplectic manifold $(M, \omega)$ such that its flux group is precisely $H^1_{dR}(M; \mathbb{Z})$.

The approach to prove Theorem 1.1 is the following. Using the same argument as in [6], we construct a closed symplectic 4-manifold $(M_1, \omega_1)$ such that $H^1_{dR}(M_1; \mathbb{R})$ is 1-dimensional. The manifold $(M_1, \omega_1)$ is obtained by attaching two ruled surfaces to the standard 4-torus $(T^4, \omega_{std})$, via the symplectic fiber connected sum procedure. Subsequently, by attaching together two copies of $(M_1, \omega_1)$ along a symplectic 2-torus we obtain the symplectic manifold $(M_2, \omega_2)$. Continuing with this process, $(M_k, \omega_k)$ is defined by the fiber connected sum of $(M_{k-1}, \omega_{k-1})$ and $(M_1, \omega_1)$ along a 2-torus. At last, the loops of symplectic diffeomorphisms on $(M_k, \omega_k)$ that are required to show $\Gamma_{(M_k, \omega_k)} = H^1_{dR}(M_k; \mathbb{Z})$ are induced from a loop of symplectic diffeomorphisms on $(T^4, \omega_{std})$ that is supported on a tubular neighbourhood of some non-contractible loop in $T^4$.

2. **The Loop of Symplectic Diffeomorphisms**

In this section we show how to build a loop of symplectic diffeomorphisms on the 4-torus that is supported in a tubular neighbourhood of some closed loop with non-zero flux. Later we will show that there is an open set in the 4-torus that contains the support of such loop of diffeomorphisms and maps by a symplectic embedding into $(M_1, \omega_1)$ or more generally into $(M_k, \omega_k)$. In this manner, for $k = 1$, we will have the desired loop in $\text{Symp}_0(M_1, \omega_1)$ whose flux generate $H^1_{dR}(M_1; \mathbb{Z})$. 
Consider the 4-torus, \((\mathbb{T}^4, \omega_{\text{std}})\), with the standard symplectic structure. That is,
\[
\omega_{\text{std}} := dx_1 \wedge dy_1 + dx_2 \wedge dy_2
\]
where \(x_j\) and \(y_j\) correspond to the standard periodic coordinates on the torus. Hence \(H^1_{\text{dR}}(\mathbb{T}^4; \mathbb{R})\) is the real vector space generated by \(dx_j\) and \(dy_j\) for \(j \in \{1, 2\}\). It is well known that the flux group of \((\mathbb{T}^4, \omega_{\text{std}})\) is \(\Gamma(\mathbb{T}^4, \omega_{\text{std}}) = H^1_{\text{dR}}(\mathbb{T}^4; \mathbb{Z}) = \mathbb{Z} \langle dx_1, \ldots, dy_2 \rangle\). For \(j \in \{1, 2\}\), fix simple and oriented closed curves \(\alpha_j\) and \(\beta_j\) in the torus such that its induced homology classes are dual to the 1-forms \(dx_j\) and \(dy_j\) respectively;
\[
\langle dx_i, \alpha_j \rangle = \delta_{i,j} \text{ and } \langle dy_i, \beta_j \rangle = \delta_{i,j}
\]
for \(i, j \in \{1, 2\}\).

Consider the loop \(\{\phi_t\}_{0 \leq t \leq 1}\) of symplectic diffeomorphisms on \((\mathbb{T}^4, \omega_{\text{std}})\) defined as
\[
\phi_t(x_1, y_1, x_2, y_2) := (x_1 + t, y_1, x_2, y_2),
\]
where the sum is mod 1. Note that, as a path of symplectic diffeomorphisms, \(\text{Flux}[\{\phi_t\}] = [dy_1]\). We claim that it is possible to obtain a loop of symplectic diffeomorphisms supported in a tubular neighborhood of \(\alpha_1\) with flux \([dy_1]\).

**Lemma 2.2.** There exists a loop \(\{\psi_t\}\) of symplectic diffeomorphisms of \((\mathbb{T}^4, \omega_{\text{std}})\) supported in a tubular neighborhood of the closed loop \(\alpha_1\) such that \(\text{Flux}[\{\psi_t\}] = [dy_1]\).

**Proof.** Let \(f : \mathbb{T}^4 \to \mathbb{R}\) be a nonzero smooth function supported on a disk neighborhood of the loop \(\alpha_1\). Then for \(0 \leq t \leq 1\),
\[
\phi_t^{(f)}(x_1, y_1, x_2, y_2) := (x_1 + tf(x_1, y_1, x_2, y_2), y_1, x_2, y_2)
\]
is a path of symplectic diffeomorphisms that starts at the identity and its flux, as a path, is \([f dy_1]\). Next consider the function \(g : \mathbb{T}^4 \to \mathbb{R}\) defined as \(g := f - 1\). By the same reasoning as before, the path
\[
\phi_t^{(g)}(x_1, y_1, x_2, y_2) := (x_1 + tg(x_1, y_1, x_2, y_2), y_1, x_2, y_2)
\]
is supported in a disk neighborhood of \(\alpha_1\), has flux \([gd y_1]\) and at \(t = 1\) we have that \(\phi_1^{(f)} = \phi_1^{(g)}\).

Henceforth, the loop \(\{\psi_t\}_{0 \leq t \leq 2}\) of symplectic diffeomorphisms defined as
\[
\psi_t := \begin{cases} 
\phi_t^{(f)} & 0 \leq t \leq 1 \\
\phi_2^{(g)} & 1 \leq t \leq 2
\end{cases}
\]
is supported in a disk neighborhood of the loop \(\alpha_1\) and \(\text{Flux}[\{\psi_t\}] = [dy_1]\). \(\square\)
3. The Construction of the Manifold \( (M_1, \omega_1) \)

For the sake of completeness of the article and in order to set notation, we review Gompf’s construction of a closed symplectic 4-manifold \( (M_1, \omega_1) \) such that \( \text{rank}(H^1_{\text{dR}}(M_1; \mathbb{R})) = 1 \). It is important to note that such symplectic manifold constitutes the main building block of the construction of the manifold \( (M_k, \omega_k) \) that will satisfy Theorem 1.1. Gompf’s construction is based on the symplectic fiber connected sum; that is, surgery along codimension-2 symplectic submanifolds. This surgery technique was proposed by Gromov in [7, Sec. 3.4.4.E’]. However, it was Gompf who really exploited this surgery in [6, Thm. 4.1] and [11, Thm. 7.2.8], where he constructed a closed symplectic 4-manifold having fundamental group isomorphic to a given finitely presented group.

Now we start explaining Gompf’s argument. Let \( (M, \omega) \) be a symplectic 4-manifold that contains a symplectic 2-torus \( (T_M, \omega|_{T_M}) \) with trivial normal bundle. Denote by \( \iota: (T_M, \omega|_{T_M}) \rightarrow (M, \omega) \) the symplectic embedding. If \( X \) denotes an elliptic surface, then the fibre \( T_X \) a is 2-torus with trivial normal bundle. Moreover, \( X \) admits a symplectic form \( \omega_X \) such that \( (T_M, \omega|_{T_M}) \) and \( (T_X, \omega_X|_{T_X}) \) have the same area. Let \( (M_0, \omega_0) \) be the symplectic manifold that is obtained by the symplectic fiber sum of \( (M, \omega) \) and \( (X, \omega_X) \) along the tori \( T_M \) and \( T_X \). It then follows that

\[
\pi_1(M_0) \cong \pi_1(M)/\iota_*(\pi_1(T_M)).
\]

That is, by attaching elliptic surfaces to a symplectic 4-manifold it is possible to eliminate some homology classes of loops. We refer to [11, Sec. 7.2] for a proof of this isomorphism between fundamental groups and the precise details of the symplectic fiber sum surgery. It is important to keep in mind that the closure of complement of some tubular of \( T_M \subset (M, \omega) \) symplectically embeds into \( (M_0, \omega_0) \).

The next result and proof is due to Gompf [6, Thm. 4.1]. We present it here in order to set the notation for the forthcoming discussion.

**Proposition 3.3** (Gompf [6]). There exists a closed symplectic 4-manifold \( (M_1, \omega_1) \) such that \( \text{rank}(H^1_{\text{dR}}(M_1; \mathbb{R})) = 1 \).

**Proof.** Start with the symplectic manifold \( \mathbb{T}^2 \times \mathbb{T}^2, \omega \oplus \omega \) with projection maps \( \pi_F \) and \( \pi_B \). On the base torus \( \mathbb{T}^2 \) consider simple, closed and oriented curves \( \alpha_B \) and \( \beta_B \) that induce a basis of \( H_1(\mathbb{T}^2; \mathbb{Z}) \). Proceed analogously on the fibre space \( \mathbb{T}^2 \). In this case we denote the generators by \( \alpha_F \) and \( \beta_F \). The goal is to wipe out the homology classes of \( \{pt\} \times \beta_B, \alpha_F \times \{pt\} \) and \( \beta_F \times \{pt\} \) in \( H_1(\mathbb{T}^2 \times \mathbb{T}^2; \mathbb{Z}) \) by performing symplectic fiber sum surgery along two disjoint 2-dimensional symplectic embedded tori with two copies of the elliptic surface \( X \). Out of this, it will follow that \( \pi_1(M_1) = \mathbb{Z} \langle \gamma_1 \rangle \) where the loop \( \gamma_1 \subset M_1 \) is induced by the loop \( \{pt\} \times \alpha_B \subset \mathbb{T}^2 \times \mathbb{T}^2 \).

To that end, we need to fix two symplectic 2-tori in \( \mathbb{T}^2 \times \mathbb{T}^2 \).
1) Set

\[ T_\Delta := \alpha_F \times \beta_B \]

in \( (T^2 \times T^2, \omega \oplus \omega) \). Then \( T_\Delta \) has trivial normal bundle and it is a Lagrangian embedded torus. On the base \( T^2 \) let \( \lambda_B \) be a closed 1-form such that when restricted to \( \beta_B \) is a volume form and it is supported in a tubular neighborhood of \( \beta_B \). Similarly let \( \lambda_F \) be a closed 1-form on the fibre \( T^2 \) that restricts to a volume form on \( \alpha_F \). Then, for small \( \epsilon > 0 \)

\[ \omega' := \omega \oplus \omega + \epsilon \pi_F^*(\lambda_F) \wedge \pi_B^*(\lambda_B) \]

is a symplectic form on \( T^2 \times T^2 \) and \( T_\Delta \) is a symplectic torus submanifold.

2) Subsequently, consider \( p_0 \) be a point on the base torus that does not lie on \( \alpha_B, \beta_B \) and the support of \( \lambda_B \). Since \( p_0 \) lies on the complement of the support of \( \lambda_B \) it follows that

\[ T_F := T^2 \times \{ p_0 \} \]

is also symplectic torus submanifold of \( (T^2 \times T^2, \omega') \).

Therefore \( T_\Delta \) and \( T_F \subset (T^2 \times T^2, \omega') \) are two disjoint symplectic tori. Next, consider two elliptic surfaces \( (X_1, \omega_{X_1}) \) and \( (X_2, \omega_{X_2}) \) with the symplectic forms such that the tori \( T_{X_1} \subset (X_1, \omega_{X_1}) \) and \( T_{X_2} \subset (X_2, \omega_{X_2}) \) have the same area as \( (T_\Delta, \omega'|_{T_\Delta}) \) and \( (T_F, \omega'|_{T_F}) \subset (T^2 \times T^2, \omega') \) respectively. In order to sum the elliptic surfaces to \( (T^2 \times T^2, \omega') \) consider two disjoint tubular neighborhoods of the tori \( T_\Delta \) and \( T_F \). Denote by \( (M_1, \omega_1) \) the symplectic manifold that is obtain by performing the symplectic fibers sum along the tori \( T_\Delta \) and \( T_F \) with \( (X_1, \omega_{X_1}) \) and \( (X_2, \omega_{X_2}) \).

Recall that the homology loops of \( T^2 \times T^2 \) that are in the image of the tori of the elliptic surfaces get canceled in the resulting manifold \( M_1 \). Henceforth, \( \pi(M_1) \cong \mathbb{Z} \). Furthermore, the loop \( \{ pt \} \times \alpha_B \subset T^2 \times T^2 \) induces the loop \( \gamma_1 \subset M_1 \) whose class is a generator of \( \pi(M_1) \). Thus \( H^1_{\text{dr}}(M_1; \mathbb{R}) \simeq \mathbb{R} \).

For the following computations it is essential to consider the standard coordinate system \( (x_1, y_1, x_2, y_2) \) on \( T^2 \times T^2 \). We will consider several open sets in the torus \( (T^2 \times T^2, \omega') \) that will symplectically embed into the symplectic manifold \( (M_1, \omega_1) \) defined above; and also into the symplectic manifold \( (M_k, \omega_k) \) that will be defined in the next section. In a way, the computations that will be performed on these open subsets of \( (T^2 \times T^2, \omega') \) will then be translated to the symplectic manifold \( (M_k, \omega_k) \), for \( k \geq 1 \), by symplectic embeddings.

Fix some points \( a_2, a_3, b_3, b_4 \in S^1 \) and put

\[ T_\Delta := \{(x_1, a_2, a_3, y_2) \mid y_1, x_2 \in S^1\} \quad \text{and} \quad T_F := \{(x_1, y_1, b_3, b_4) \mid x_1, y_1 \in S^1\}; \]
the tori used in the proof of Prop. 3.3. Now we take into account the symplectic structure. Fix tubular neighborhoods \( \mathcal{O}(T_\Delta) \) and \( \mathcal{O}(T_F) \) of the tori such that the symplectic surgery of \( (\mathbb{T}^2 \times \mathbb{T}^2, \omega') \) with the ruled surfaces is done in the interior of such neighborhoods. Thus on the complement of \( O \) is symplectomorphic to an open set in \( (\mathbb{T}, \omega) \). Fix tubular neighborhoods \( T_1 \) of the tori used in the proof of Prop. 3.3. Now we take into account the symplectic surgery of \( (\mathbb{T}^4, \omega) \) before surgery and \( U_1 \subset (M_1, \omega_1) \) after surgery.

**Lemma 3.4.** For the open set \( U_0 \), \( H^3_{\text{dr}}(U_0; \mathbb{R}) \) is two-dimensional with one generator being the class of \( dx_1 dy_1 dy_2 \). In particular, \( H^1_+(U_0; \mathbb{R}) \) has a generator given by the class of \( fdx_2 \) where the function \( f : U_0 \to \mathbb{R} \) is compactly supported.

**Proof.** The result follows from the Mayer-Vietoris sequence on \( \mathbb{T}^4 \) using the open covering \( \{\mathbb{T}^4 \setminus T_\Delta, \mathbb{T}^4 \setminus T_F\} \), and by noticing that \( U_0 \) is diffeomorphic to \( \mathbb{T}^4 \setminus (T_\Delta \cup T_F) \). \( \square \)

Below we will see how the class \([fdx_2] \in H^1_+(U_0; \mathbb{R})\) induces a class in \( H^1_{\text{dr}}(M_1; \mathbb{R}) \) that is dual to \([\gamma_1]\), where \( \gamma_1 \) is the loop described in Prop. 3.3

In the meantime, note that the open set \( U_0 \) in \( \mathbb{T}^4 \) can be described as

\[
U_0 = \{(x_1, y_1, x_2, y_2) \mid x_1, y_1, x_2, y_2 \in S^1, y_1 \neq a_2 + t, x_2 \neq a_3 + t, \text{ or } x_2 \neq b_3 + t, y_2 \neq b_4 + t \text{ for } -\epsilon < t < \epsilon\}
\]

for some fixed \( \epsilon > 0 \) and where the sum is mod 1. Next, fix parameters \( c_1, c_2, c_3, d_1, d_2, d_4 \in S^1 \) so that the loops

\[
\gamma(c)(y_2) := \{(c_1, c_2, c_3, y_2) \mid y_2 \in S^1\} \text{ and }\]

\[
\gamma(d)(x_2) := \{(d_1, d_2, x_2, d_4) \mid x_2 \in S^1\}
\]

lie inside of \( U_0 \).

For the proof of the next result, we remark that \( U_0 \) contains an open set described by constraining the \( x_2 \)-coordinate to some small interval:

\[
\{(x_1, y_1, x_2, y_2) \mid x_1, y_1, x_2, y_2 \in S^1, x_2 \neq a_3 + t \text{ and } x_2 \neq b_3 + t\}
\]

where the sum is mod 1 and \( t \in (-\epsilon', \epsilon') \) for some small \( \epsilon' \).

**Proposition 3.5.** Consider the open set \( U_0 \subset (\mathbb{T}^4, \omega \oplus \omega) \) and the loop \( \gamma(d)(x_2) \subset U_0 \) defined above. Then there exist a loop of symplectic diffeomorphisms \( \{\psi_t\} \) on \((U_0, \omega \oplus \omega)\) and compactly supported function \( f : U_0 \to \mathbb{R} \) such that

a) \( \psi_t \) is compactly supported for each \( t \),
b) on \((U_0, \omega \oplus \omega)\),

\[
\text{Flux}([\{\psi_t\}]) = [f \, dx_2] \in H^1_t(U_0; \mathbb{R}), \quad \text{and}
\]

c) \([f \, dx_2, [\gamma_{(d)}(x_2)]] = 1\).

**Proof.** According to Lemma 2.2 on \((\mathbb{T}^4, \omega \oplus \omega)\) there is a loop \(\{\psi_t\}\) of symplectic

diffeomorphisms supported in a tubular neighborhood of the loop \(\gamma_{(c)}(y_2)\) such that

\[
\text{Flux}([\{\psi_t\}]) = [dx_2].
\]

Since the loop \(\gamma_{(c)}(y_2)\) lies inside \(U_0\), from the proof of Lemma 2.2 it is possible to assure that

the support of each \(\psi_t\) lies inside \(U_0\).

Now let \(g : S^1 \to \mathbb{R}\) be any non zero function such that \(g(a_3) = g(b_3) = 0\).

Now define \(f : \mathbb{T}^4 \to \mathbb{R}\) as

\[
f(x_1, y_1, x_2, y_2) := g(x_2).
\]

As noted above, it is possible to choose \(g\) such that the the support of \(f\) lies in \(U_0\). In particular the support of \(f\) is compact. Thus, on \(H^1_{\text{dr}}(\mathbb{T}^4; \mathbb{R})\) we have

that \([dx_2] = [f \, dx_2]\). Notice that up to some constant we can assume that \(f\) is such that \([f \, dx_2, [\gamma_{(d)}(x_2)]] = 1\).

Henceforth, \(\{\psi_t\}\) is a loop in \(\text{Symp}_0(U_0, \omega \oplus \omega)\) such that

\[
\text{Flux}([\{\psi_t\}]) = [f \, dx_2] \in H^1_t(U_0; \mathbb{R}).
\]

The above result make use of the coordinate functions of the torus on \(U_0 \subset (\mathbb{T}^4, \omega \oplus \omega)\). Next, we rewrite the same result in a coordinate-free manner on the symplectic manifold \((M_1, \omega_1)\) Denote by \(j : U_1 \to M_1\) the inclusion map.

Recall that \((U_1, \omega_1)\) is symplectomorphic to \((U_0, \omega \oplus \omega)\). From the proof of Prop. 3.3 we know that \(\pi_1(M_1) = \mathbb{Z}\langle[\gamma_1]\rangle\) where the loop \(\gamma_1\) is induced by the

loop \(\{pt\} \times \alpha_B = \gamma_{(d)}(x_2)\) on \(\mathbb{T}^4\) for some values of \(d_1, d_2\) and \(d_4\). That is, the loop \(\gamma_{(d)}(x_2)\) can be considered inside \(U_1\) and under the map \(j_* : H_1(U_1; \mathbb{Z}) \to H_1(M_1; \mathbb{Z})\) we have that \(j_*([\gamma_{(d)}(x_2)]) = [\gamma_1]\).

Denote by \(\eta_1^{(1)}\) the generator of \(H^1(M_1; \mathbb{Z})\) such that \([\eta_1^{(1)}, [\gamma_1]] = 1\). Since \(U_1\) is diffeomorphic to \(U_0\), it follows from Prop. 3.3 and Lemma 3.4 that the map

\[
j_* : H^t_c(U_1; \mathbb{R}) \to H^1_{\text{dr}}(M_1; \mathbb{R})
\]

is surjective. Following the above result, denote by \([f \, dx_2] \in H^t_c(U_1; \mathbb{R})\) a class such that \(j_*[f \, dx_2] = \eta_1^{(1)}\). Thus we have the relation

\[
[\eta_1^{(1)}, [\gamma_1]] = ([f \, dx_2, [\gamma_{(d)}(x_2)]] = 1.
\]

Summarizing, on \((M_1, \omega_1)\) we have \(H_1(M_1; \mathbb{Z}) \simeq \mathbb{Z}\langle[\gamma_1]\rangle\) and \(H^1(M_1; \mathbb{Z}) \simeq \mathbb{Z}/\eta_1^{(1)}\) such that \([\eta_1^{(1)}, [\gamma_1]] = 1\).

In Lemma 2.2 we constructed a loop of symplectic diffeomorphisms supported in some open set inside \(\mathbb{T}^4\). The relation between the Flux on an open set of compactly supported symplectic diffeomorphisms and the Flux on the
ambient manifold it is given by the following result. Its proof follows directly from the definitions.

**Lemma 3.6.** Let \((M, \omega)\) be a closed symplectic manifold and \(U \subset (M, \omega)\) an open set. If \(j : U \to M\) and \(\tau : \text{Symp}_0(U, \omega) \to \text{Symp}_0(M, \omega)\) are the inclusion and extension maps respectively, then the diagram

\[
\begin{array}{ccc}
\pi_1(\text{Symp}_0(U, \omega)) & \xrightarrow{\tau_*} & \pi_1(\text{Symp}_0(M, \omega)) \\
\text{Flux}_U & & \text{Flux}_M \\
H^1_c(U; \mathbb{R}) & \xrightarrow{j_*} & H^1_{\text{dr}}(M; \mathbb{R})
\end{array}
\]

commutes, \(\text{Flux}_M \circ \tau_* = j_* \circ \text{Flux}_U\). In particular, \(j_*(\Gamma_U)\) is a subgroup of \(\Gamma_M\).

In light of the above results, we prove the main theorem for the case when \(k = 1\). The idea of the proof consists using Prop. 3.5 which is a statement on \((U_1, \omega_1)\), and then use Lemma 3.6.

**Proposition 3.7.** The closed symplectic 4-manifold \((M_1, \omega_1)\) constructed on Prop. 3.3 satisfies the following:

\[
\text{rank}(H^1_{\text{dr}}(M_1; \mathbb{R})) = 1 \quad \text{and} \quad \Gamma_{(M_1, \omega_1)} = H^1_{\text{dr}}(M_1; \mathbb{Z}).
\]

**Proof.** From Prop. 3.3 we know that \(\text{rank}(H^1_{\text{dr}}(M_1; \mathbb{R})) = 1\). It remains to compute its flux group.

We have that \(H^1_{\text{dr}}(M_1; \mathbb{Z}) = \mathbb{Z}\langle \eta_1 \rangle\) where the class is normalized by the condition \(\langle \eta_1, [\gamma_1] \rangle = 1\). Since \((U_1, \omega_1)\) is symplectomorphic to \((U_0, \omega \oplus \omega)\), it follows by Prop. 3.5 that there exists a loop \(\{\psi_t\}\) in \(\text{Symp}_0(U_1, \omega_1)\) such that is compactly supported, \(\text{Flux}_{(U_1, \omega_1)}(\{\psi_t\}) = \langle [f dx_2] \rangle\) and \(\langle [f dx_2], [\gamma_1(x_2)] \rangle = 1\) where \(f : U_1 \to \mathbb{R}\) has compact support. Recall that under the inclusion map \(j : U_1 \to M_1\) we have that \(j_*([\gamma_1(x_2)]) = [\gamma_1]\) and \(j_*([f dx_2]) = \eta_1\).

Therefore the loop \(\{\psi_t\}\) induces a loop in \(\text{Symp}(M_1, \omega_1)\), which we still denote by \(\{\psi_t\}\). Moreover by Lemma 3.6 \(\text{Flux}_{(M_1, \omega_1)}(\{\psi_t\}) = \eta_1\). Henceforth, \(\Gamma_{(M_1, \omega_1)} = H^1_{\text{dr}}(M_1; \mathbb{Z})\).

So far we have proved Theorem 1.1 for the special case when \(k = 1\). In some respect the above proof will carry word for word for the general case. In particular, the results concerning the open set \((U_0, \omega \oplus \omega)\) are the main building blocks of the above result and will also be used in the general case. That is, that in \(U_0\) we have two simple loops \(\gamma_c(y_2)\) and \(\gamma_d(x_2)\), where the loop of symplectic diffeomorphisms \(\psi_t\) is supported in a tubular neighborhood of \(\gamma_c(y_2)\) and its flux is equal to \([f dx_2]\) where \(f : U_0 \to \mathbb{R}\) has compact support and \(\langle [f dx_2], [\gamma_d(x_2)] \rangle = 1\).
Therefore the problem now reduces to construct a closed symplectic 4-manifold \((M, \omega)\), such that

- \(H^1_{\text{dR}}(M; \mathbb{R})\) is \(k\)-dimensional;
- it admits a symplectic embedding of \(k\) disjoint copies of \((U_0, \omega \oplus \omega)\),

\[
j : \bigsqcup_k U_0 \to M
\]

such that \(H^1_{\text{dR}}(M)\) is generated by the \(k\) classes

\[
j_*([f \, dx_2], 0, \ldots, 0), \ldots, j_*([f \, dx_2], 0, \ldots, 0), j_*([f \, dx_2], 0, \ldots, 0)
\]

The next section precisely takes care of this construction of the symplectic manifold \((M_k, \omega_k)\).

4. The construction of the manifold \((M_k, \omega_k)\) for \(k \geq 2\)

The construction of the symplectic manifold \((M_k, \omega_k)\) of Theorem 1.1 emulates the construction of the surface of genus \(k\); iteration of attaching \(k\) tori. By this, we mean that the symplectic manifold \((M_k, \omega_k)\) is built by considering \(k\) copies of \((M_1, \omega_1)\) and performing the fiber symplectic sum along some 2-tori.

To that end, recall from the proof of Prop. 3.3 that on \((\mathbb{T}^4, \omega')\) we considered the symplectic torus \(T_F = \mathbb{T}^2 \times \{p_0\}\) where \(p_0\) is a point in the base torus that does not lie on the support of the 1-form \(\lambda_B\), and on the loops \(\alpha_B\) and \(\beta_B\). Fix two other distinct points \(p_1\) and \(p_2\) on the base torus with the same properties as \(p_0\) such that the symplectic tori

\[
T_{F,1} := \mathbb{T}^2 \times \{p_1\} \quad \text{and} \quad T_{F,2} := \mathbb{T}^2 \times \{p_2\}
\]

lie inside the open set \(U_0 \subset (\mathbb{T}^4, \omega')\). Denote by \(T_{1,1}\) and \(T_{1,2}\) the corresponding tori in \(U_1 \subset (M_1, \omega_1)\). Notice that the tori \(T_F, T_{F,1}\) and \(T_{F,2}\) are homologous among them in \(\mathbb{T}^4\). And by Gompf’s result, Prop 3.3, \(T_{1,1}\) and \(T_{1,2}\) are null homologous in \(M_1\).

Since the points \(p_1\) and \(p_2\) on the base torus do not lie in the support of the 1-form \(\lambda_B\), the 2-tori \(T_{1,1}\) and \(T_{1,2}\) \((M_1, \omega_1)\) are symplectomorphic. Furthermore, since the tori have trivial normal bundle it is possible to fiber connected sum among two copies of \((M_1, \omega_1)\) along these tori. Finally, note that it is possible to take a smaller set \(U'_0 \subset U_0\), such that it is disjoint from the tori \(T_{F,1} := \text{and} \ T_{F,2}\), and is homeomorphic to \(U_0\). Let \(U'_1 \subset U_1\) be the corresponding open set in \((M_1, \omega_1)\).

Next we will define inductively the closed symplectic manifolds \((M_k, \omega_k)\) for \(k \geq 2\).
4.1. **The definition of the symplectic manifold** $(M_2, \omega_2)$. Define $(M_2, \omega_2)$ as the symplectic fiber connected sum of two copies of $(M_1, \omega_1)$ along $T_{1,1}$ and $T_{1,2}$ on each manifold;

\[ M_2 := M_1 \#_T M_1. \]

Moreover we assume that inside some neighborhood of each $T_{1,j} \subset M_1$ where the surgery is carried out is disjoint from the open set $U'_1$ on each copy of $(M_1, \omega_1)$. Therefore, the following result follows from this assumption.

**Lemma 4.8.** The symplectic manifold $(M_2, \omega_2)$ admits the symplectic embedding of the open set

\[ U'_1 \coprod U'_1 \]

where $U'_1 \subset (M_1, \omega_1)$.

From the above result, the symplectic 2-tori $T_{1,1} \subset U_1$ and $T_{1,2} \subset U_1$ that were not used in the surgery are symplectomorphic to two 2-tori in $(M_2, \omega_2)$. Denote such tori by $T_{2,1}, T_{2,2} \subset (M_2, \omega_2)$. As before, the tori are disjoint and have trivial normal bundle. Furthermore, note that all the tori $T_{1,i} \subset (M_1, \omega_1)$ and $T_{2,j} \subset (M_2, \omega_2)$ for $i,j \in \{1, 2\}$ are symplectomorphic.

4.2. **The definition of the symplectic manifold** $(M_k, \omega_k)$ for $k \geq 3$.

Next we define inductively the symplectic manifolds $(M_k, \omega_k)$ for $k \geq 3$. Let $(M_{k-1}, \omega_{k-1})$ be a closed symplectic 4-manifold with two disjoint symplectic 2-tori submanifolds $T_{k-1,1}$ and $T_{k-1,2}$. Furthermore, the tori have trivial normal bundle and are symplectomorphic to $T_{1,j}$ for $i,j \in \{1, 2\}$.

Define the symplectic manifold $(M_k, \omega_k)$ as the fiber connected sum of $(M_{k-1}, \omega_{k-1})$ and $(M_1, \omega_1)$ along the tori $T_{k-1,1}$ and $T_{1,2}$ respectively, that is $M_k := M_{k-1} \#_T M_1$.

Note that the symplectic manifold $(M_k, \omega_k)$ can also be viewed as the fiber connected sum of $k$ copies of $(M_1, \omega_1)$ along some tori;

\[ M_k = M_1 \#_T \cdots \#_T M_1. \]

Moreover, the symplectic manifold $(M_k, \omega_k)$ has two disjoint two disjoint symplectic 2-tori, $T_{k,1}$ and $T_{k,2}$. Such tori have trivial normal bundle and are symplectomorphic. We claim that the symplectic manifold $(M_k, \omega_k)$ satisfies the conclusion of the main theorem.

From the definition of the symplectic manifold $(M_k, \omega_k)$ and the fact that the tori $T_{1,1}$ and $T_{1,2}$ are disjoint from the open set $U'_1 \subset M_1$, we have the following result that is analogous to Lemma 4.8.

**Lemma 4.9.** The symplectic manifold $(M_k, \omega_k)$ admits the symplectic embedding of $k$ disjoint copies of the open set $U'_1 \subset (M_1, \omega_1)$,

\[ j : \coprod_k U'_1 \to M_k. \]
Next, we compute the group $H_1(M_k;\mathbb{Z})$. Let $M$ be a 4-manifold and $T \subset M$ a submanifold that is a 2-torus, has a trivial normal bundle $\nu$ and is null-homologous in $M$. Let $\nu_0$ be the normal bundle minus the zero-section; thus $\nu_0$ retracts to a 3-torus. Since $T$ is null-homologous in $M$, the induced map by the inclusion $\nu_0 \hookrightarrow M$,

$$H_1(\nu_0;\mathbb{R}) \to H_1(M \setminus T;\mathbb{R})$$

is the zero map. We will use this observation, where $M_k$ will play the role of $M$ and $T_{k,j} \subset M_k$ the role of $T$ for $j \in \{1, 2\}$.

So far we have seen in Prop. 3.3 that $T_{1,1}$ and $T_{1,2}$ are null-homologous in $M_1$. Since the fiber connected sum $M_k = M_{k-1} \#_T M_1$ is done along the tori $T_{k-1,1} \subset M_k$ and $T_{1,2} \subset M_1$ it follows that $T_{k,1}$ and $T_{k,2}$ are also null-homologous in $M_k$.

**Lemma 4.10.** For the symplectic manifold $(M_k, \omega_k)$ defined above, we have that

$$H_1(M_k;\mathbb{Z}) \simeq \mathbb{Z}^k.$$  

**Proof.** The case $k = 1$ follows by Gompf’s argument in Prop. 3.3 Next we proceed by induction on $k$, and compute the degree-one homology of $M_k = M_{k-1} \#_T M_1$ using the Mayer-Vietoris sequence relative the open cover $\{M_{k-1} \setminus \mathcal{O}(T_{k-1,1}), M_1 \setminus \mathcal{O}(T_{1,2})\}$.

Recall that $T_{1,2}$ is null homologous in $M_1$. Therefore $H_1(M_1 \setminus T_{1,2};\mathbb{Z}) \simeq \mathbb{Z}$. Similarly for $M_{k-1}$, that is $H_1(M_{k-1} \setminus T_{k-1,1};\mathbb{Z}) \simeq \mathbb{Z}^{k-1}$. Furthermore, if $\nu \subset M_{k-1}$ is a tubular neighbourhood of $T_{k-1,1}$ and $\nu_0 := \nu \setminus T_{k-1,1}$ then the inclusion map

$$H_1(\nu_0;\mathbb{Z}) \to H_1(M_{k-1} \setminus T_{k-1,1};\mathbb{Z})$$

is the zero map.

By exactness of the Mayer-Vietoris sequence relative to the cover $\{M_{k-1} \setminus \mathcal{O}(T_{k-1,1}), M_1 \setminus \mathcal{O}(T_{1,2})\}$ and by the previous remark we have that the map

$$H_1(M_{k-1} \setminus \mathcal{O}(T_{k-1,1});\mathbb{Z}) \oplus H_1(M_1 \setminus \mathcal{O}(T_{1,2});\mathbb{Z}) \to H_1(M_k;\mathbb{Z})$$

is an isomorphism and the claim follows. \hfill $\square$

Now we describe a basis for $H_1(M_k;\mathbb{Z})$ that is suitable for our needs. Recall that $H_1(M_1;\mathbb{Z}) = \mathbb{Z}\langle [\gamma_1] \rangle$ where the 1-cycle was induced from a closed simple loop in $U_1$ or even in smaller set $U'_1$. That is, the inclusion maps $U'_1 \hookrightarrow M_1 \setminus \mathcal{O}(T_{1,2}) \to M_1$ induce the isomorphisms $H_1(U'_1;\mathbb{Z}) \to H_1(M_1 \setminus \mathcal{O}(T_{1,2});\mathbb{Z}) \to H_1(M_1;\mathbb{Z})$. For this, recall that $U'_1$ and $U_1$ are homeomorphic.

In order to extend this idea to $H_1(M_k;\mathbb{Z})$, note that from the proof of Lemma 4.10

$$H_1(M_{k-1} \setminus \mathcal{O}(T_{k-1,1});\mathbb{Z}) \oplus H_1(M_1 \setminus \mathcal{O}(T_{1,2});\mathbb{Z}) \to H_1(M_k;\mathbb{Z})$$
is an isomorphism. But also \( H_1(M_{k-1} \setminus \mathcal{O}(T_{k-1,1}); \mathbb{Z}) \to H_1(M_{k-1}; \mathbb{Z}) \) is an isomorphism since \( T_{k-1,1} \subset M_{k-1} \) is null-homologous. Henceforth, by Lemma 4.9 obtain an isomorphisms of maps
\[
H_1(U'_1; \mathbb{Z}) \oplus k \to H_1(M_1 \setminus \mathcal{O}(T_{1,2}); \mathbb{Z}) \oplus k \to H_1(M_k; \mathbb{Z}).
\]

Summarizing, \( H_1(M_k; \mathbb{Z}) \) has a basis represented by 1-cycles \( \gamma_1^{(k)}, \ldots, \gamma_k^{(k)} \) such that each cycle \( \gamma_j^{(k)} \) is supported in a connected component of the image of the symplectic embedding \( j : \bigsqcup_k U'_1 \to M_k \) described in Lemma 4.9. Henceforth,
\[
H_1(M_k; \mathbb{Z}) = \mathbb{Z} \langle [\gamma_1^{(k)}], \ldots, [\gamma_k^{(k)}] \rangle.
\]
Next, we also used the symplectic embedding \( j : \bigsqcup_k U'_1 \to M_k \) of Lemma 4.9 to obtain suitable generators of \( H^1_{\text{DR}}(M_k) \). Let \([f \, dx_2] \in H^1(M_k; \mathbb{R})\) be the class described in Prop. 3.5. For \( 1 \leq i \leq k \), denote by \( \eta_i^{(k)} \in H^1_{\text{DR}}(M_k) \) the image of \([0, \ldots, [f \, dx_2] \ldots 0] \in \oplus_k H^1_{\text{DR}}(U'_1)\). Under the map \( j_* : \oplus_k H^1_{\text{DR}}(U'_1) \to H^1_{\text{DR}}(M_k) \) where \([f \, dx_2] \) appears in the \( i \)-entry. Again from Prop. 3.5 and the fact that we have a disjoint union of sets \( \bigsqcup_k j(U'_1) \subset M_k \) we have that for any \( 1 \leq i, r \leq k \)
\[
\langle \eta_i^{(k)}, [\gamma_r^{(k)}] \rangle = \delta_{i,r}.
\]
Hence it follows that \( H^1_{\text{DR}}(M_k; \mathbb{Z}) = \mathbb{Z} \langle \eta_1^{(k)}, \ldots, \eta_k^{(k)} \rangle \).

Proof of Theorem 1.1. For \( k \geq 2 \) consider the symplectic 4-manifold \((M_k, \omega_k)\) defined above. We claim that the flux group of \((M_k, \omega_k)\) is equal to \( H^1_{\text{DR}}(M_k; \mathbb{Z}) \).

For \( 1 \leq r \leq k \), consider the 1-form \( \eta_r^{(k)} \in H^1_{\text{DR}}(M_k; \mathbb{Z}) \). Recall that the support of the 1-form \( \eta_r^{(k)} \) lies inside some \( j(U'_1) \subset M_k \), where \( j \) is the symplectic embedding of Lemma 4.9. From Prop. 3.5 there is a loop of symplectic diffeomorphisms \( \{ \psi_t \} \) of \((M_k, \omega_k)\) such that the support of each \( \psi_t \) lies inside \( j(U'_1) \) and \( \text{Flux}_{(M_k, \omega_k)}([\{ \psi_t \}]) = \eta_r^{(k)} \). Therefore, \( \Gamma_{(M_k, \omega_k)} = H^1_{\text{DR}}(M_k; \mathbb{Z}) \). \( \square \)

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