Abstract

It was recently observed in a lattice QCD measurement that the chiral condensate in the quenched approximation shows dramatically different behavior in the three $Z_3$-equivalent de-confined phases. We argue that this phenomenon can be understood qualitatively as an effect of $Z_3$ twists on fermionic fields. Quarks under these $Z_3$-twists become global anyons and hence display different thermodynamic properties. We further show that the lattice data can be roughly modeled by a Nambu-Jona-Lasinio type Lagrangian with a minimal coupling to a constant gauge field $A_0 = 2\pi nT/3$ (with $n = 0, \pm 1$), which arises naturally from the non-trivial phase of the Polyakov line.

Submitted to: Physics Letters B
I. INTRODUCTION

Understanding chiral symmetry breaking and its restoration at finite temperatures in QCD is of great interest. Many lattice simulations have been performed to study the expected phase transitions [1]. However, the underlying dynamics responsible for these phenomena are still unclear. Hence, it is sometimes useful to explore regions of QCD parameters beyond their physical domain. One such effort in lattice QCD is to study the fermionic Greens functions in the background dynamics of pure gauge fields. In such a study one ignores all the effects of internal quark loops. At low temperatures no essential difference is observed between such studies and those that take into account the effects of quark loops. However, at finite temperatures these effects can change the physics qualitatively. In the pure gauge theory there is a first order phase transition between a confined and a de-confined phase signaled by a non-vanishing expectation value of the Polyakov line. This is related to a global $Z_3$ symmetry breaking [2]. On the other hand, inclusion of massless quarks breaks this global $Z_3$ symmetry explicitly and chiral symmetry restoration becomes relevant, with the chiral condensate as the order parameter. The precise nature of the chiral transition depends on the number of flavors [3] appearing through internal quark loops.

Interestingly, in lattice QCD one can measure a quenched chiral condensate even after ignoring internal quark loops from the dynamics. Such a condensate has been measured in the past near the de-confining transition. A first order transition was observed in the quenched chiral condensate at the de-confining temperature [4]. However, no efforts were made to distinguish between the behavior of quarks in the three de-confined phases labeled by an element of $Z_3$. Recently, in an effort to understand the effects of internal quark loops near the chiral phase transition, this issue was revived by a careful and precise measurement of the quenched chiral condensate [5]. It was observed that, above the de-confining transition, the chiral condensate crucially depends on the $Z_3$ phase in which the gauge dynamics settles. Even the chiral symmetry restoration appears to occur at different temperatures, depending on the phase. This somewhat surprising result needs an explanation. In this article we present a qualitative understanding of the physics underlying these observations.

In a somewhat different context the effect of the boundary condition in the Gross-Neveu model, with one of the dimension compactified, was studied recently [6]. It was found there that the fermion, which obeys the usual antiperiodic boundary condition, becomes a global anyon when a non-trivial twist is introduced via a constant background gauge field. The anyon interpretation is supported by an explicit decomposition of the fermion propagator into a sum over winding numbers in the compactified direction. As a consequence of this statistics transmutation the thermodynamic properties of the Gross-Neveu model become sensitive to the value of the twist. We suggest that a similar physical setting is realized in quenched QCD in the various de-confined phases. The non-trivial phase of the Polyakov line provides a constant background gauge field in the temporal direction and turns quenched quarks into global anyons.

In section 2, we show that in QCD one can define a chiral condensate with a $Z_3$ twist in general. The usual chiral condensate is the one where the twist is zero. The other two are new quenched order parameters of chiral symmetry breaking. We then argue that the chiral condensates measured in [5] are related to these $Z_3$ twisted condensates in the quenched limit. In section 3, we elucidate the meaning of the $Z_3$ twisted chiral condensate by considering an
effective action which governs these quenched observables. In section 4, we show that the qualitative features of the lattice data can be obtained by the Nambu-Jona-Lasinio model in the large \( N_c \) limit, minimally coupled to the suggested constant gauge field. In section 5, we present some observations and conclusions.

II. \( Z_3 \) TWISTED CHIRAL CONDENSATES

The chiral condensate in QCD is the order parameter for chiral symmetry breaking and is defined formally as

\[
\langle \bar{\psi} \psi \rangle = \frac{1}{Z} \int [dA] \ tr \frac{1}{D + m_q} \ Det(D + m_q) \ exp(-S_g[A])
\]

where \( D(A) = \gamma^\mu (\partial_\mu - iA_\mu) \) is the Dirac operator and \( m_q \) is the quark mass and \( S_g \) is the gauge action. \( Z \) is the partition function. The above definition can be made more precise on the lattice.

We now introduce a chiral condensate with a \( Z_3 \) twist \( \theta \), given by

\[
\langle \bar{\zeta} \zeta \rangle_\theta = \frac{1}{Z} \int [dA] \ tr \frac{1}{D_\theta + m_\zeta} \ Det(D + m_q) \ exp(-S_g[A])
\]

where

\[
D(A)_\theta = \gamma^\mu (\partial_\mu - iA_\mu) + i\theta T \gamma^0
\]

with \( \theta = 0, \pm \frac{2\pi}{3} \) and \( T \) the temperature. The mass in the observable, \( m_\zeta \), is deliberately chosen to be distinct from the usual quark mass, \( m_q \). Thus \( \langle \bar{\zeta} \zeta \rangle \) is a quenched observable and can be measured on the lattice even in the presence of dynamical fermions. The physical content of this new observable will be elucidated shortly. Notice that \( D_\theta \) is simply the original Dirac operator with the temporal gauge field shifted by a phase of \( Z_3 \). Further note that \( \langle \bar{\zeta} \zeta \rangle_\theta = \langle \bar{\zeta} \zeta \rangle_{-\theta} \). We can show this by noting that \( D(A) \) and \( D(-A) \) have the same spectra due to a charge conjugation symmetry. However, in general we will have \( \langle \bar{\zeta} \zeta \rangle_\frac{2\pi}{3} \neq \langle \bar{\zeta} \zeta \rangle_0 \). This is related to the fact that fermions break the \( Z_3 \) symmetry.

In general, the dynamical quarks in eq. (2), will always pick the phase in which the gauge field distribution, \( \{A_\mu\} \), makes the expectation value of the Polyakov line \( P \) real and positive, i.e.

\[
\langle P(x) \rangle = \left\langle \frac{1}{3}Tr \left( P \int_0^T \exp(iA_0(x, t)) dt \right) \right\rangle = |\langle P \rangle| \quad (4)
\]

The gauge field distribution of a \( Z_3 \) rotated phase would be \( \{A_\mu\} + \theta T \delta_{\mu,0} \). Now the Polyakov line is given by \( |\langle P \rangle| e^{i\theta} \). The relevant Dirac operator for measuring the chiral condensate is \( D(A)_\theta \) defined in eq. (3). Thus measuring the usual chiral condensate in the various \( Z_3 \) phases is naturally equivalent to measuring \( \langle \bar{\zeta} \zeta \rangle_\theta \) in the quenched limit of the de-confined phase.

The effect of similar \( \theta \) terms on the thermodynamics has been studied in the past for simple local fermionic field theories \[3\]. There it is argued that the \( \theta \) term acts like a twist
in the boundary condition for the fermions and converts them into global anyons. This conversion then affects the critical temperature dramatically. We suggest that the same physics also happens in the present context. We anticipate that, if we study \( \langle \zeta \zeta \rangle \) as a function of the temperature, it is likely that the condensate with \( \theta = 0 \) and \( \theta = \pm \frac{2\pi}{3} \) will behave differently. In particular if there is a phase transition in the \( m_\zeta \to 0 \) limit it is quite likely that it will occur at different temperatures depending on \( \theta \).

The above discussion then suggests that it is likely that at some temperature in the deconfined phase \( \langle \zeta \zeta \rangle_0 = 0 \) whereas \( \langle \zeta \zeta \rangle_{\pm \frac{2\pi}{3}} \neq 0 \), as observed in \([5]\). However, in the confined phase the gauge dynamics is \( Z_3 \) symmetric in the quenched limit. The \( Z_3 \) twist in this phase ceases to be meaningful, since the gauge field ensemble is no longer clustered around any of the three \( \theta \) values there. It is of course still possible to study the \( Z_3 \) chiral condensates, as defined in eq. \([2]\), even in the presence a dynamical quark mass \( m_q \), because of the relative shift in the Dirac operator in the observable and in the internal quark determinant. In the presence of finite \( m_q \) the \( Z_3 \) symmetry is broken and thus it is possible to track the three phases into the confined phase and study their evolution towards the chiral limit. We do not know if such a study is physically relevant since the \( Z_3 \) condensates are in general some quenched order parameters. In the remainder of this article we will be interested only in the quenched limit.

## III. AN EFFECTIVE ACTION FOR THE \( Z_3 \) TWISTED CONDENSATES

In this section we suggest a local field theory which governs the behavior of the \( Z_3 \) twisted chiral condensates. Formally one can rewrite eq. \([2]\), introducing fermionic (\( \zeta \)) and bosonic (\( \phi \)) ghost fields in the spirit of \([7]\),

\[
\langle \zeta \zeta \rangle_\theta = \frac{1}{Z} \int_{\psi,\zeta,\phi,A} \zeta \zeta \exp \left[ -S_g[A] + \overline{\psi} (D + m_q) \psi + \overline{\zeta} (D_\theta + m_\zeta) \zeta + \phi^\dagger (D_\theta + m_\zeta) \phi \right] \tag{5}
\]

This enlarged theory contains another chiral symmetry in the limit \( m_\zeta \to 0 \) for arbitrary real quark mass \( m_q \). It is now obvious that the order parameter associated with this symmetry is \( \langle \zeta \zeta \rangle \). One can then formally integrate over the quark fields \( \psi \), the ghost fields \( \phi \) and the gauge fields \( A_\mu \) to generate an effective action for the \( \zeta \) fields. Thus

\[
\langle \zeta \zeta \rangle_\theta = \frac{1}{Z} \int_\zeta \overline{\zeta} \zeta \exp \left[ -S_{\text{eff}}^{\mu}[m_q,\theta,m_\zeta,T] (\overline{\zeta},\zeta) + \overline{\zeta} (\gamma^\mu \partial_\mu + i\theta T \gamma^0 + m_\zeta) \zeta \right] \tag{6}
\]

We have allowed for the possibility that the effective action could depend on the temperature through the couplings apart from the usual anti-periodic boundary conditions in the Euclidean time direction for the \( \zeta \) fields.

In general \( S_{\text{eff}}^{\mu}[m_q,\theta,m_\zeta,T] (\overline{\zeta},\zeta) \) would be arbitrarily complicated and possibly contains non-local interactions. It seems hopeless to understand the full structure of the effective action. However, it may be interesting to obtain some qualitative insight into the essential physics involved by studying a simplified model that is motivated phenomenologically and at the same time is tractable.

In the quenched limit, the chiral symmetry between fermionic Greens functions is governed only by the dynamics of the gauge fields. Non-perturbative gauge fields can in general
produce chiral symmetry breaking. One immediate consequence is that the quenched chiral condensate need not vanish. This feature, in principle, need not have any connection with confinement. Thus it is not unnatural to assume that, for some temperature range, chiral symmetry breaking happens even in the deconfined phase. Consequently we are led to study chiral symmetry breaking in the absence of confinement. To this end, we assume that the effective interaction generated after integrating out all other fields is dominated by the lowest dimensional term,

\[ S_{\text{eff}}^{\text{f}}[m_q, \theta, m, T, \zeta, \bar{\zeta}] = \frac{G(T)}{2N_c \Lambda^2} \left[ (\zeta \cdot \zeta)^2 - (\bar{\zeta} \cdot \gamma_5 \zeta)^2 \right] \tag{7} \]

with \( \Lambda \) being some cutoff scale specified later. Thus in this model the \( \theta \) dependence enters only through the quadratic term shown in eq. (6). We expect this term to produce the difference between the complex and the real phases as in \([6]\).

As mentioned earlier, in the confined phase there is no difference between the various \( Z_3 \) twisted condensates. This is due to the \( Z_3 \) symmetry of the dynamics. Thus the effective action must have no knowledge of \( \theta \). This is in accordance with the fact that the expectation value of the Polyakov line is zero in the confined phase, it is meaningless to include a \( \theta \) term in the confined phase. Here no attempt is made to construct an effective field theory that describes the confinement physics also in a natural way. However, as long as we are interested only in a model for the chiral condensate, it may be sufficient to drop the \( \theta \) term from the effective action in the confined phase. Additional effects of gauge dynamics are mimicked by a temperature dependence of the coupling \( G(T) \). Further since the deconfining transition is a first order transition, an abrupt jump is possible in \( G(T) \) at the de-confinement temperature.

IV. MODELING OF LATTICE DATA

To have a semi-quantitative understanding of the lattice data we start with the Lagrangian

\[ \mathcal{L}(\zeta, \bar{\zeta}) = \bar{\zeta} \cdot \beta \zeta + \bar{\zeta} \cdot \gamma^0 \zeta i \theta T + m \bar{\zeta} \zeta + \frac{G(T)}{2N_c \Lambda^2} \left[ (\zeta \cdot \zeta)^2 - (\bar{\zeta} \cdot \gamma_5 \zeta)^2 \right] \tag{8} \]

or equivalently

\[ \mathcal{L}(\zeta, \bar{\zeta}) = \bar{\zeta} \cdot \beta \zeta + \bar{\zeta} \cdot \gamma^0 \zeta i \theta T + m \bar{\zeta} \zeta + \bar{\zeta}(\sigma + i\gamma_5 \pi) \zeta + \frac{N_c \Lambda^2}{2G(T)}(\sigma^2 + \pi^2) \tag{9} \]

Since this model is non-renormalizable we consider a momentum cutoff in the spatial momentum \(|p| = \Lambda\). We will solve the model in the large \( N_c \) limit and fix the parameters in such a way that the model reproduces the lattice data obtained in \([3]\) qualitatively. The effective potential in the large \( N_c \) limit is given by

\[ V_{\text{eff}}(\sigma) = \frac{N_c \Lambda^2}{2G(T)} \sigma^2 - 2N_c \int_{-\Lambda}^{\Lambda} \frac{d^3p}{(2\pi)^3} T \sum_{n=-\infty}^{\infty} \ln \left[ \frac{\tilde{\omega}_n^2 + p^2 + (\sigma + m)^2}{\tilde{\omega}_n^2 + p^2 + (\sigma - m)^2} \right] \tag{10} \]

where
The Matsubara sum can be carried out by the standard contour integral technique, yielding
\[ \tilde{\omega}_n = [(2n - 1)\pi + \theta]T \]

The full effective potential can be obtained after integrating eq. (12),
\[
V_{\text{eff}}(\sigma) - V_{\text{eff}}(0) = \frac{N_c A^2}{G(T)} \left\{ \frac{\sigma^2}{2} - \frac{G(T)T}{\pi^2 A^2} \int_0^\Lambda p^2 dp \ln \left[ \frac{\cosh(\sqrt{p^2 + (\sigma + m)^2}/T) + \cos \theta}{\cosh(\sqrt{p^2 + m^2}/T) + \cos \theta} \right] \right\}
\]

We refer the reader to [6] for a detailed discussion of the effects of $\theta$. Here we only try to model the lattice data of [5].

The parameters in the NJL model are fixed in a conventional way [8]. Assuming the cutoff $\Lambda$ to be about 1 GeV, we set $\Lambda/T_c = 5.5$ and $G(T)/(2\pi^2) = 1.163$ for $T < T_c$ in order to reproduce the chiral condensate at low temperatures with $T_c$ around 200 MeV. The $\theta$ term is dropped in the confined phase for reasons already explained. The current quark mass on the lattice ($m_\zeta$), and in the effective action ($m$) can in general be different. Here we assume $m = Z m_\zeta$. Then $Z = 1.48$ can be cleanly determined by fitting the condensate ($\theta = 0$) in the chirally symmetric phase. However, in order to reproduce the lattice data quantitatively a non-trivial $T$-dependence in $G(T)$ needs to be introduced in the deconfined phase. The first order nature of the de-confining transition is incorporated by a jump in $G(T)$. Then a simple quadratic form $G(T)/(2\pi^2) = 1.385 - 0.385(T/T_c)^2$ is sufficient for $T > T_c$.

1In addition to contributions proportional to $m_\zeta$, $m$ could get other contributions, such as from the $U_A(1)$ anomalies. We will ignore such additive contributions here.
FIG. 1. This graph shows the lattice data. The behavior of the condensate in the large $N_c$ Nambu-Jona-Lasinio model described in the text, produces the solid lines. The lattice data corresponds to $m_\zeta a = 0.001$, where $a = \frac{1}{T N_t}$ with $N_t = 4$.

The lattice data corresponds to $N_t = 4$ and the lattice spacing is given by $a = \frac{1}{T N_t}$. Then the value of the condensate computed from the NJL model should be compared with the lattice data as

$$-\frac{1}{N_c} \langle \zeta \cdot \zeta \rangle = \frac{\sigma \Lambda^2}{G} = \frac{1}{G \Lambda} \left( \frac{\Lambda}{T_c} \right)^3 T_c^3 = \frac{\sigma}{\Lambda} \frac{5.5^3}{G} T_c^3$$

(15)

where $\sigma$ is obtained from the solution of the gap equation.

The data from lattice simulations and the model are given in figures 1 and 2. It is seen there that the NJL model, with its parameters properly tuned, is capable of describing the data semi-quantitatively. However, we would like to stress the fact that the final outcome of the NJL model sensitively depends on the precise temperature dependence. A small change in the numerical value of the coupling $G$ can change the evolution of the complex chiral condensate drastically. Thus in this regard we find the model somewhat unnatural, even though it is possible to capture the qualitative physics.

In figure 2 the current mass dependence of the condensate is compared between the model and the lattice data at $T/T_c = 1.067$. At this temperature the complex phase lattice

---

2 The actual data was given in terms of the lattice coupling. This has been converted to dimensionless units using the critical temperature as the scale. The bare lattice coupling $\beta = 5.692$ was used as the critical coupling for the de-confinement temperature.
data begins to show interesting power law dependence, and the real phase lattice data is essentially linear \([9]\). The real phase data at \(T = 1.067 T_c\) was used to fix the constant \(Z\) and hence will agree well with the model. However, it is obvious that the present model will not reproduce the power law behavior of the real phase lattice data at lower temperatures \([5]\). Similarly we do not expect the model to reproduce the power law behavior of the complex phase either. In fact we do not know if any universality arguments apply to these data. However, we do see the qualitative difference between the real and the complex phases.

\[
\begin{align*}
\langle \zeta \rangle^2 / T_c &< 0.01 \\
0.01 &< \langle \zeta \rangle^2 / T_c < 0.10 \\
0.10 &< \langle \zeta \rangle^2 / T_c < 1.00
\end{align*}
\]

**FIG. 2.** This graph compares the behavior of the condensate as a function of the current quark mass at \(T = 1.067 T_c\) in the large \(N_c\) Nambu-Jona-Lasinio model described in the text with the lattice data. This is approximately the temperature where a possible transition occurs in the complex phase. (r) refers to \(\theta = 0\) and (c) refers to \(\theta = \pm 2\pi/3\).

**V. DISCUSSION AND CONCLUSIONS**

In this work we have clarified the precise meaning of the chiral order parameter considered in a recent lattice study. The observed \(Z_3\) phase dependence of this order parameter in quenched QCD in the deconfined phase has been qualitatively explained. The relevant physical picture is that the non-trivial \(Z_3\) twists in the temporal boundary condition, due to the survival of expectation value of the Polyakov line, turn quarks into global anyons. The qualitative behavior of the lattice data can be modeled simply by a temperature dependent NJL model with a minimal coupling to a constant gauge field \(A_0 = \theta T\) with \(\theta = 0\) and \(\pm 2\pi/3\).

For \(m = 0\), the NJL model would predict a continuous phase transition when \(\theta = \pm 2\pi/3\), in addition to the jump associated with the first order de-confining phase transition. The present lattice data are consistent with this prediction. The second transition
temperature is estimated to be about 5 to 10 percent higher than the de-confining transition temperature.

As this work was being completed we received an article [10], which discusses the same issue along very similar lines using the NJL model. However, based on our calculation, we would like to point out that, on a quantitative level the model does not reflect one important feature of the gauge dynamics. It is clear from figure 1 that the lattice data for the complex phase evolves rapidly above $T_c$ towards a possible second transition at about $T = 1.07(3) \ T_c$. In the above model such a rapid evolution required very fine tuning of the coupling $G(T)$. Thus it is very likely that the gauge dynamics is not naturally captured in this model. In fact, the rapid evolution just mentioned is also seen in other physical quantities like the entropy density [11] in the same region. Thus an obvious direction to extend the present work, is to obtain a better model including the gauge dynamics. This may help in understanding the reason for the rapid evolution seen just above $T_c$.

As was suggested in section II, the $Z_3$ condensates as defined by eq. (2) can be studied even in the presence of small dynamical quark masses $m_q$. As quenched order parameters these may be measured in lattice simulations. However, more theoretical work would be necessary to see if they are interesting. Finally, in this article we have concentrated on the lattice data for $SU(3)$ gauge theories. A similar discussion in the case of $SU(2)$ leads to interesting possibilities since the de-confining transition is second order in this case. Extensions to $SU(N)$ with $N > 3$ may also be interesting and has been attempted in [10].

VI. ACKNOWLEDGMENTS

We would like to thank the Columbia Lattice group for allowing us to use the lattice data prior to their publication. These have been presented at the recent lattice conference [5] and the detailed results will be published shortly [9]. We would also like to thank U. -J. Wiese for helpful discussions.
REFERENCES

[1] C. DeTar, Quark Gluon Plasma in numerical simulations of lattice QCD, to appear in Quark Gluon Plasma II, R. Hwa ed., World Scientific 1995
[2] B. Svetitsky and L. Yaffe, Nucl. Phys. B210 (1982), 423-447.
[3] R. Pisarski and F. Wilczek, Phys. Rev. D29, (1984) 338.
[4] J. Kogut et al., Phys. Rev. Lett. 50, (1983) 393.
[5] S. Chandrasekharan and N. Christ, Contribution to Lattice 95, to appear in Nucl. Phys. B. Proc. Suppl. (1996), CU-TP-711, hep-lat/9509095
[6] S. Huang and B. Schreiber, Nucl. Phys. B426, (1994) 644-660
[7] M. Golterman, C. Bernard, Phys. Rev. D46, (1992) 853
[8] T. Hatsuda and T. Kunihiro, Phys. Rep. 247, (1994) 221
[9] Columbia University Preprint, in preparation., to be published
[10] P. N. Meisinger and M. C. Ogilvie, hep-lat/9512011
[11] F. R. Brown et al., Phys. Rev. Lett. 61, (1988) 2058