BERRY-ESSEE TYPE ESTIMATES FOR NONCONVENTIONAL SUMS

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Abstract. We obtain Berry-Esseen type estimates for "nonconventional" expressions of the form
\[ \xi_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (F(X(q_1(n)), ..., X(q_r(n))) - \bar{F}) \]
where \( X(n) \) is a sufficiently fast mixing vector process with some moment conditions and stationarity properties, \( F \) is a continuous function with polynomial growth and certain regularity properties, \( \bar{F} = \int F d(\mu \times ... \times \mu) \), \( \mu \) is the distribution of \( X(0) \) and \( q_i(n) = in \) for \( 1 \leq i \leq k \) while for \( i > k \) they are positive functions taking integer values on integers with some growth conditions which are satisfies, for instance, when they are polynomials of increasing degrees. Our setup is similar to [14] where a nonconventional functional central limit theorem was obtained and the present paper provides estimates for the convergence speed. As a part of the study we provide answers for the crucial question on positivity of the limiting variance \( \lim_{N \to \infty} \text{Var}(\xi_N) \) which was not studied in [14]. Extensions to the continuous time case will be discussed as well. As in [14] our results are applicable to stationary processes generated by some classes of sufficiently well mixing Markov chains and dynamical systems.

1. Introduction

The classical Berry-Esseen theorem provides a uniform estimate of the error term in the central limit theorem for a sum of mean zero independent identically distributed (i.i.d.) random variables \( \{X(n)\}_{n=1}^{\infty} \). Namely, let \( F_n \) be the distribution function of \( \frac{1}{\sigma \sqrt{n}} \sum_{i=1}^{n} X(i) \) where \( \sigma = \sqrt{E(X(1))^2} > 0 \) and \( \Phi \) be the standard normal distribution function then
\[
\sup_{x \in \mathbb{R}} \left| F_n(x) - \Phi(x) \right| \leq \frac{C|X(1)|^3}{\sigma^3 \sqrt{n}}
\]
(see §6 of Ch. III in [19]) where \( C \) is an absolute constant which by efforts of many researchers was optimized by now to a number a bit less than \( 1/2 \).

Motivated partially by the research on nonconventional ergodic theorems (the name comes from [8]) the study of nonconventional limit theorems was initiated in [14]. More recently a functional central limit theorem was proved in [14] for
normalized nonconventional sums of the form

\[ \xi_N(t) = \frac{1}{\sqrt{N}} \sum_{Nt \geq n \geq 1} (F(X(q_1(n))), ..., X(q_\ell(n))) - \bar{F} \]

where \( \{X(n), n \geq 0\} \) is a sufficiently fast mixing vector valued process with some stationarity properties satisfying certain moment conditions, \( F \) is a continuous function with polynomial growth rate and certain regularity properties, \( \bar{F} = \int Fd(\mu \times \cdots \times \mu) \), \( \mu \) is the common distribution of \( X(n) \)'s and \( q_j(n) = jn \) for \( 1 \leq j \leq k \) while \( q_j(n) \) for \( k < j \leq \ell \) are positive functions taking integer values on integers and satisfying certain growth conditions. In this paper we derive Berry-Esseen type estimates for the convergence rate in such nonconventional limit theorems.

During last 50 years central limit theorems were extended to weakly dependent sequences of random variables and to martingale differences and corresponding Berry-Esseen type estimates of the speed of convergence were obtained, as well (see, for instance, [9], [17], [18], [6] and references there). We observe though that summands in nonconventional sums appearing in (1.2) are usually strongly long range dependent (even when \( X(n), n \geq 1 \) are independent) so the results for the weakly dependent case are not applicable here. Still, it was shown in [14] that under natural conditions nonconventional sums can be splitted into \( \ell \) subsums and each of the latter can be approximated by a martingale. We will show that, actually, in the arithmetic progression case \( q_j(n) = jn, j = 1, ..., \ell \) the whole nonconventional sum can be approximated by one martingale which will enable us to apply one of Berry-Esseen type results for martingales mentioned above. Still, in order to do so we will need to obtain appropriate asymptotic covariance estimates for nonconventional summands. We observe that when not all \( q_j(n) \)'s are linear but, say, \( q_j(n), j = k+1, ..., \ell \) grow faster, for instance, polynomially as in [14] then we have to deal with several martingales with respect to different filtrations which requires additional considerations described in the concluding Section 6.

As (1.1) and more advanced results show Berry-Esseen type estimates (with an absolute constant) depend crucially on variances of the corresponding sums which appear in some form in the denominators of corresponding bounds. In the standard (conventional) setup the conditions which ensure linear growth in the number of summands of these variances are well known for stationary sequences since [11]. On the other hand, the limiting behavior of the variance \( \xi_N \) in (1.2) was not studied in [14] in spite of the fact that a meaningful central limit theorem requires the limit of \( \text{Var}\xi_N \) as \( N \to \infty \) to be positive. Some partial results in this direction were obtained in [12] and [10]. Ensuring positivity of the limiting variance and obtaining appropriate lower bounds for it is especially important in Berry-Esseen type estimates and we provide here a rather complete answer concerning this question. Namely, we show that under appropriate mixing conditions the positivity question for the limiting variance of \( \xi_N \) can be reduced to the same question for the \( \ell \)-dimensional process constructed of independent copies of the process \( X(n), n \geq 0 \) which is plugged in the function \( F \). If \( X(n), n \geq 0 \) is stationary then (in the \( k = \ell \) case) this \( \ell \)-dimensional process is stationary, as well, and we can rely on the well known results concerning the latter (see [11] and the next section).

The structure of this paper is the following. In the next Section 2 we describe precisely our setup and formulate our main results. In Section 3 we derive some auxiliary estimates. In Sections 4 and 5 we prove our main theorems. In order
2. Preliminaries and main results

2.1. Setup and assumptions. Our setup consists of a \( \varphi \)-dimensional stochastic process \( \{X(n)\}_{n \geq 0} \) on a probability space \((\Omega, \mathcal{F}, P)\) and a nested family of \( \sigma \)–algebras \( \mathcal{F}_{k,l}, -\infty \leq k \leq l \leq \infty \) such that \( \mathcal{F}_{k,l} \subseteq \mathcal{F}_{k',l'} \) if \( k' \leq k \) and \( l' \geq l \). As usual (see [4]) the dependence between two sub \( \sigma \)–algebras \( \mathcal{G}, \mathcal{H} \subset \mathcal{F} \) will be measured by the expressions

\[
\varpi_{q,p}(\mathcal{G}, \mathcal{H}) = \sup\{||E(g|\mathcal{G}) - E g||_p : g \in L^q(\Omega, \mathcal{H}, P) \text{ and } ||g||_q \leq 1\}.
\]

and we refer the reader to [4] for relations between various dependence coefficients. Set also

\[
\varpi_{q,p}(n) = \sup_{k \geq 0}(\mathcal{F}_{-\infty,k}, \mathcal{F}_{k+n,\infty}).
\]

Our results below can be obtained assuming that \( X(n) \) is measurable with respect to \( \mathcal{F}_{n,n} \) without special assumptions on the function \( F \) beyond measurability similarly to [10]. Nevertheless, we prefer here the setup from [14] which allows applications to dynamical systems. Thus, we introduce approximation rate coefficients

\[
\beta(q,r) = \sup_{k \geq 0} ||X(k) - E(X(k) | \mathcal{F}_{k-r,k+r})||_q.
\]

We will not require stationarity of the process \( \{X(n) | n \geq 0\} \) assuming only that the distribution of \( X(n) \) does not depend on \( n \) and the joint distribution of \( (X(n), X(n')) \) depends only on \( n - n' \) which we write for further reference by

\[
X(n) \sim \mu \text{ and } (X(n), X(n')) \sim \mu_{n-n'}
\]

where \( Y \sim \mu \) means that \( Y \) has \( \mu \) for its distribution, denoted also \( \mu = L(Y) \).

Next, let \( F = F(x_1, \ldots, x_\ell) \), \( x_j \in \mathbb{R}^p \) be a function on \( \mathbb{R}^{\ell p} \) such that for some \( K, \ell > 0, \kappa \in (0,1] \) and all \( x_i, y_i \in \mathbb{R}^p \), \( i = 1, \ldots, \ell \),

\[
|F(x) - F(y)| \leq K(1 + \sum_{i=1}^{\ell}(|x_i|^\kappa + |y_i|^\kappa)) \sum_{i=1}^{\ell} |x_j - y_j|^\kappa
\]

and

\[
|F(x)| \leq K(1 + \sum_{i=1}^{\ell} |x_i|^\kappa)
\]

where \( x = (x_1, \ldots, x_\ell), y = (y_1, \ldots, y_\ell) \). To simplify the formulas we assume a centering condition

\[
\bar{F} = \int F(x_1, \ldots, x_\ell) d\mu(x_1) \ldots d\mu(x_\ell) = 0
\]

which is not really a restriction since we can always replace \( F \) by \( F - \bar{F} \). Our main goal is obtaining Berry-Esseen and covariance type estimates for \( \xi_N(t), t \in [0, T] \) defined in (1.2) (with \( \bar{F} = 0 \)). For each \( \theta > 0 \), set

\[
\gamma^\theta_\theta = ||X(n)||^\theta_\theta = \int |x|^\theta d\mu.
\]

Our results rely on the following assumptions (similar to [14]).
2.1. Assumption. With $d = (\ell - 1)p$ there exits $\infty > p, q \geq 1, b \geq 2, \alpha, \lambda \geq 0$ and $\delta, m > 0$ (these numbers will be called the initial parameters) with $\delta < \kappa - \frac{d}{p}$ satisfying
\begin{equation}
\theta(q, p, \alpha, 1) = \sum_{n \geq 1} n^{\alpha} \omega_{q,p}(n) < \infty,
\end{equation}
\begin{equation}
\Lambda(q, \delta, \lambda, 1) = \sum_{r=1}^{\infty} r^{\lambda} (\beta(q, r))^\delta < \infty,
\end{equation}
\begin{equation}
\gamma_m < \infty, \gamma_{bpq} < \infty; \text{ with } \frac{1}{b} \geq \frac{1}{p} + \frac{\ell + 2}{m} + \frac{\delta}{q},
\end{equation}
while conditions on $\alpha$ and $\lambda$ will be specified in the statements below.

As in [14] it will be useful to represent the function $F = F(x_1, \ldots, x_\ell)$ in the form
\begin{equation}
F = F_1(x_1) + \ldots + F_\ell(x_1, \ldots, x_\ell)
\end{equation}
where for $i < \ell$,
\begin{equation}
F_i(x_1, \ldots, x_i) = \int F(x_1, \ldots, x_* x_i) d\mu(x_i) \ldots d\mu(x_\ell) - \int F(x_1, \ldots, x_* x_i) d\mu(x_i) \ldots d\mu(x_\ell)
\end{equation}
and
\begin{equation}
F_\ell(x_1, \ldots, x_\ell) = F(x_1, \ldots, x_\ell) - \int F(x_1, \ldots, x_\ell) d\mu(x_\ell)
\end{equation}
which ensures that
\begin{equation}
\int F_i(x_1, \ldots, x_{i-1}, x_i) d\mu(x_i) = 0 \ \forall x_1, \ldots, x_{i-1}.
\end{equation}

Next, assume that $q_j(n) = jn$ for $j = 1, \ldots, k \leq \ell$ while when $\ell \geq j > k$ we have $q_j(n+1) - q_j(n) \to \infty$ and $q_j(\varepsilon n) - q_{j-1}(n) \to \infty$ as $n \to \infty$ for each $\varepsilon > 0$. Following [14] we will use the representation
\begin{equation}
\xi_N(t) = \sum_{i=1}^{k} \xi_i, N(t) + \sum_{i=k+1}^{\ell} \xi_i, N(t)
\end{equation}
where for $1 \leq i \leq k$,
\begin{equation}
\xi_i, N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{[N t]} F_i(X(n), \ldots, X(in))
\end{equation}
and for $i > k$
\begin{equation}
\xi_i, N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{[N t]} F_i(X(q_1(n)), \ldots, X(q_i(n)))
\end{equation}
The following result was proved in [14].

2.2. Theorem. Suppose that Assumption [2.7] holds true with $b = 2$ and $\alpha = \lambda = 0$. Then the $\ell$–dimensional process $\{\xi_i, N(t)\}_{i=1}^{\ell}$ converges in distribution as $N \to \infty$ to a vector Gaussian process $\{\eta_i(t)\}_{i=1}^{\ell}$ with stationary independent mean zero increments and covariances
\begin{equation}
E[\eta_i(s)\eta_j(t)] = \min(s, t) D_{i,j} = \lim_{N \to \infty} E[\xi_i, N(s)\xi_j, N(t)]
\end{equation}
Berry Esseen Theorem

where $D_{i,j} = 0$ if $i \neq j$ and $\max(i,j) > k$. This together with (2.13) yields that the limiting variance exists and has the form

$$
\lim_{N \to \infty} \text{Var}\xi_N(t) = \lim_{N \to \infty} E\xi_N^2(t) = t\sigma^2 = t(\sigma_0^2 + \sigma_1^2)
$$

where

$$
\sigma_0^2 = \lim_{N \to \infty} \left( \sum_{i=1}^{k} \xi_{i,N}(i) \right)^2 = \sum_{i=1}^{k} iD_{i,i} + 2 \sum_{1 \leq i < j \leq k} iD_{i,j}
$$

and

$$
\sigma_1^2 = \lim_{N \to \infty} \left( \sum_{i=k+1}^{\ell} \xi_{i,N}(1) \right)^2 = \sum_{i=k+1}^{\ell} D_{i,i}.
$$

Moreover, the process $\xi_N(\cdot)$ converges in distribution to the Gaussian process $\eta(\cdot)$ which can be represented in the form $\eta(t) = \sum_{i=1}^{k} \eta_i(it) + \sum_{i=k+1}^{\ell} \eta_i(t)$ which may have dependent increments.

2.2. Statement of main results. In order to make this paper more readable we will focus on the case $k = \ell$ and introduce several extensions in Section 6 (among them, results for $k < \ell$). In general, uniform Berry-Esseen type estimates can only be meaningful if the asymptotical variance $\sigma^2 = \lim_{N \to \infty} E\xi_N(1)$ is positive which can be seen already in (1.1). Some conditions for positivity of $\sigma^2$ were obtained in Theorem 2.3 from [10] but the following theorem provides a substantially stronger and more general result.

2.3. Theorem. Suppose that $k = \ell$ and that Assumption [2.7] holds true with $b = 2$ and $\alpha, \lambda \geq 1$. Let $\{X^{(i)}(n)\}_{n \geq 0}$, $i = 1, \ldots, \ell$ be $\ell$ independent copies of the process $\{X(n)\}_{n \geq 0}$ and set

$$
Z_n = F(X^{(1)}(n), X^{(2)}(2n), \ldots, X^{(\ell)}(\ell n)) \text{ and } \Sigma_N = \sum_{n=1}^{N} Z_n.
$$

Then the limit

$$
s^2 = \lim_{n \to \infty} \frac{1}{n} \text{Var}\Sigma_n
$$

exists. Moreover, $\sigma^2 > 0$ if and only if $s^2 > 0$ and the latter conditions holds true if and only if there exists no representation of the form

$$
Z_n = V_{n+1} - V_n, n = 0, 1, 2...
$$

where $\{V_n\}_{n=1}^{\infty}$ is a square integrable weakly (i.e. in the wide sense) stationary process. Furthermore, $s^2 = 0$ if and only if $\text{Var}\Sigma_N$ is bounded in $N$ and then for all $N \geq 2$, $\text{Var}\xi_N \leq CN^{-1} \ln^2 N$ for some $C > 0$ independent of $N$.

We observe that this theorem remains true with essentially the same proof also in the more general case $k < \ell$ described above. Actually, in this case $\sigma^2 > 0$ unless $F_j = 0$ for all $j = k + 1, k + 2, \ldots, \ell \mu \times \cdots \times \mu$-almost surely (a.s.) (see Section 6.2). The above result reduces the problem on positivity of the limiting variance for nonconventional sums to the corresponding much more studied question concerning sums of stationary in the wide sense processes. If $X(n), n \geq 0$ is a strictly stationary process then $(X^{(1)}(n), X^{(2)}(2n), \ldots, X^{(\ell)}(\ell n))_{n \geq 0}$ and


$$F(X^{(1)}(n), X^{(2)}(2n), \ldots, X^{(\ell)}(\ell n)), n \geq 0$$ are strictly stationary, as well, while under our condition (2.4) these processes are stationary in the wide sense. Limit theorems for sums of the latter were widely studied. We observe that it is not possible to give useful (i.e., computable) positive lower bounds for the limiting variance even in a general conventional situation of sums of stationary processes. In the nonconventional case the situation is more complicated and though some formulas for the limiting variances are given in [14] it is not possible to check directly when they are positive. Still, assuming that $$X(n), n \geq 0$$ are independent we provide in Section 6 some formulas for limiting variances which are easier to handle and to obtain estimates.

2.4. Remark. Similarly to [14] the results of this paper can be applied to some types of discrete time dynamical systems $$T: \Omega \circlearrowright$$ such as subshifts of finite type, expanding transformations and Axiom A diffeomorphisms considered with a Gibbs invariant measure $$\mu$$ (see, for instance, [3]). Such dynamical systems are exponentially fast $$\psi$$-mixing which is more than enough for our purposes. In this setup we should take $$X(n) = f \circ T^n$$ where, say, $$f$$ is a Hölder continuous (vector) function. Then Theorem 2.3 reduces the question on positivity of the limiting variance of $$N^{-1/2} \sum_{n=1}^{N} G(T^n \omega, T^{2n} \omega, \ldots, T^{\ell n} \omega)$$, where $$G(\omega_1, \ldots, \omega_\ell) = F(f(\omega_1), \ldots, f(\omega_\ell))$$, to the corresponding question for the product dynamical system $$T \times \cdots \times T: \Omega \times \cdots \times \Omega \circlearrowright$$, i.e., for $$N^{-1/2} \sum_{n=1}^{N} G(T^n \omega_1, T^{2n} \omega_2, \ldots, T^{\ell n} \omega_\ell)$$. Since $$T \times \cdots \times T$$ preserves the product measure $$\mu \times \cdots \times \mu$$ and also turns out to be an exponentially fast $$\psi$$-mixing dynamical system we arrive at a well studied problem. Furthermore, it is known since [5] that for a general measure preserving dynamical system $$T: \Omega \circlearrowright$$ and a bounded measurable function $$H$$ the sums $$\sum_{n=1}^{N} H(T^n \omega)$$ are almost surely uniformly bounded if and only if $$H$$ has a co-boundary representation $$H(\omega) = \varphi(T \omega) - \varphi(\omega)$$ for some other bounded measurable function $$\varphi$$. For nonconventional sums $$\sum_{n=1}^{N} G(T^n \omega_1, T^{2n} \omega_2, \ldots, T^{\ell n} \omega_\ell)$$ such result cannot hold true in this generality since the meaningful action here is only on the diagonal of $$\Omega \times \cdots \times \Omega$$, and so we can define $$G$$ to be a co-boundary for $$T \times \cdots \times T$$ on the diagonal which has zero product measure while defining $$G$$ arbitrarily outside of the diagonal still preserving measurability. Then the sum will be bounded but $$G$$ will not have necessarily a co-boundary representation on the whole product space. In the more restricted nonconventional situation of Theorems 2.2 and 2.3 the central limit theorem together with positivity of the limiting variance ensures that the sum $$\sum_{n=1}^{N} G(T^n \omega, \ldots, T^{\ell n} \omega)$$ is unbounded while if it is bounded then $$G$$ must have a co-boundary representation. It would still be interesting to understand whether boundedness of these sums in the nonconventional setup can be characterized in a more general situation. In clarifying some points discussed in this remark the second author benefited from several conversations with A. Katok at PennState University in September 2014.

Recall, that the Kolmogorov (uniform) metric is defined for each pair of distribution functions $$F, G$$ by

$$d_K(F, G) = \sup_{x \in \mathbb{R}} |F(x) - G(x)|$$

Now we can formulate our second main result.
2.5. Theorem. Suppose that \( k = \ell \) and that Assumption 2.7 holds true with \( b \geq 4, \alpha, \lambda > 1 \) and that \( \sigma^2 > 0 \). Then,
\[
d_K(\mathcal{L}(\xi_N(1)), \mathcal{N}(0, \sigma^2)) \leq CA(\sigma)N^{-\zeta(\alpha, \lambda)}
\]
where, \( \mathcal{N}(0, \sigma^2) \) is the zero mean normal distribution with the variance \( \sigma^2 > 0 \), the constant \( C > 0 \) depends only on the initial parameters and on the expressions \( (2.8) \) and \( (2.9) \). \( A(\sigma) = (1 + \frac{1}{\lambda}) \max(\sigma^{-\frac{1}{2}}, \sigma^{-\frac{1}{\lambda}}) \), and
\[
\zeta(\alpha, \lambda) = \frac{1}{10} \min(\min(\alpha, \lambda) - 1, \frac{\lambda}{\lambda + 8}).
\]
Moreover, if there exist \( c \in (0, 1) \) and \( r > 0 \) satisfying \( \varpi_{q,p}(n) + \beta(q, n) \leq rc^n \) then \( N^{-\zeta(\alpha, \lambda)} \) can be replaced by \( N^{-\frac{r}{4}} \ln N \).

In order to describe our method of the proof of Theorem 2.5 consider the simpler case when \( X(n), n \geq 0 \) is a sequence of independent identically distributed (i.i.d.) random variables and choose the \( \sigma \)-algebras \( \mathcal{F}_{n,m} = \sigma\{X(n), ..., X(m)\} \) for any \( n \leq m \). For each \( i = 1, 2, ..., \ell \) define
\[
M_{i,n} = \sum_{iN \leq m \leq n} F_i(X(m), X(2m), ..., X(im)) \quad \text{for} \quad n \leq iN
\]
and \( M_{i,n} = M_{i,iN} \) for \( n \geq iN \). Then \( M_{i,n}, n = 1, ..., \ell N \) is a martingale with respect to the filtration \( \{\mathcal{F}_{0,n}, n \geq 0\} \), and so \( M_n = \sum_{i=1}^{\ell} M_{i,n}, n = 1, 2, ..., \ell N \) is also a martingale or, more precisely, a martingale array since the construction depends on \( N \). Now observe that
\[
\xi_N(1) = \frac{1}{\sqrt{N}} M_{\ell N}
\]
and Theorem 2.5 will follow in this situation from estimates of rates of convergence in the martingale central limit theorem derived in [9]. Still, for this specific i.i.d. case we will give in Section 6 another more direct proof which yields better estimates. In the more general setup of the present paper we will need first a truncation procedure and then a martingale approximation similar but still somewhat different from [13]. Namely, as above in the i.i.d. case, we construct in the case \( k = \ell \) a martingale approximation of the whole sum \( \sqrt{N} \xi_N(1) \) and not only of its parts \( \sqrt{N} \xi_{i,N}(t) \) as in [13]. Some additional work, described in Section 6, is needed when \( \xi_N \) has the more general form \([12]\) with some of \( q_j(n) \)'s growing faster than linearly. In order to rely on [9] we will need also appropriate quadratic variation estimates which will be obtained in Section 6.

2.6. Remark. We construct a martingale array approximation (representation in the i.i.d. case described above) for the whole normalized sum \( \xi_N(1) \) and not only for its parts \( \xi_{i,N} \) as in [14]. This serves us well for the Berry-Esseen type estimates here and yields also the central limit theorem for \( \xi_N(1) \) from standard results for martingale arrays. Still, the functional central limit theorem for the whole process \( \xi_N(t), t \geq 0 \) cannot be obtained this way. Indeed, if we could approximate this process by a martingale array depending only on \( N \) but not on \( t \) then the limiting Gaussian process would have independent increments which is not the case in general (see [14]). Already in the above construction for the i.i.d. case we would have to define \( M_{i,n} = M_{i,[iN]} \) for \( n \geq [iN] \) obtaining martingales depending on \( t \) which would not enable us to employ standard theorems on martingale arrays.
3. Auxiliary estimates

We start with the following simple observation.

3.1. Lemma. Let $f : (\mathbb{R}^r)^d \to \mathbb{R}$ and $g : (\mathbb{R}^s)^p \to \mathbb{R}$ satisfy the conditions (2.5) and (2.6). Then the function $h : (\mathbb{R}^r)^{d+p} \to \mathbb{R}$ defined by $h(x,y) = f(x)g(y)$ satisfies these conditions with constants $2\epsilon, \kappa$ and $K = 2(1 + d + p)K^2$ in place of $\epsilon, \kappa$ and $K$, respectively.

Proof. The lemma follows from three simple inequalities $|ab| \leq \frac{1}{2}(a^2 + b^2)$, $|a| \leq 1 + a^2$, $|ab - a'b'| \leq |a(b - b')| + |b'(a - a')|$, the Hölder continuity of $f$ and $g$ and the concavity of the function $x \to x^q$ for $1 > a > 0$. □

Next, we will need

3.2. Lemma. Let $0 < \delta < \kappa \leq 1$ and $b \geq 1$ satisfy $\frac{1}{b} \geq \frac{1}{2} + \frac{\epsilon + 2}{m} + \frac{\delta}{q}$ for some $q, p \geq 1$ and $m, \epsilon > 0$. Then for any random variables $Y, X$,

$$||Y^\epsilon \cdot X^\kappa||_b \leq (1 + ||X||_{m})||Y||_m^\epsilon \cdot ||X||_q^\delta.$$  

Proof. First, clearly,

$$(3.1) \quad ||Y^\epsilon X^\kappa||_b \leq T_1 + T_2$$

where

$$T_1 = ||Y^\epsilon X^\kappa 1_{\{|X|>1\}}||_b \quad \text{and} \quad T_2 = ||Y^\epsilon X^\kappa 1_{\{|X|\leq 1\}}||_b.$$  

Observe that $T_1 = ||Y^\epsilon X^\kappa 1_{\{|X|>1\}}||_b$. Since $\frac{1}{b} > \frac{\epsilon + \kappa}{m} + \frac{\delta}{q}$ then Lemma 3.1 from [14] yields that

$$T_1 \leq ||Y^\epsilon X^\kappa||_m^\frac{\epsilon}{\epsilon + \kappa} \cdot ||1_{\{|X|>1\}}||_q^\delta.$$  

Since $||1_{\{|X|>1\}}||_q = (P\{|X|>1\})^\frac{1}{q} = (P\{|X|^q>1\})^\frac{1}{q}$ it follows by the Markov inequality that $||1_{\{|X|>1\}}||_q \leq (E|X|^q)^\frac{1}{q} = ||X||_q$. Moreover, since $(\frac{\epsilon + \kappa}{m})^{-1} = \frac{m}{\epsilon + \kappa}$, Lemma 3.1 from [14] yields $||Y^\epsilon X^\kappa||_m^\frac{\epsilon}{\epsilon + \kappa} \leq ||Y||_m^\epsilon \cdot ||X||^\kappa_{m}$, and so $T_1 \leq ||Y||_m^\epsilon \cdot ||X||^\kappa_{m} ||X||^\delta_{q}$. Next, set $Z = |X| 1_{\{|X|\leq 1\}}$. Clearly, $T_2 = ||Y^\epsilon Z^\kappa||_b$. Since $0 \leq Z \leq 1$ and $\delta < \kappa$ it follows that $T_2 \leq ||Y^\epsilon Z^\kappa||_b$. Since $\frac{1}{b} > \frac{\epsilon + \kappa}{m} + \frac{\delta}{q}$ we apply again Lemma 3.1 from [14] and use that $||Z||_q \leq ||X||_q$ in order to obtain $T_2 \leq ||Y||_m^\epsilon \cdot ||X||^\delta_{q}$. The lemma now follows from (3.1) and the above estimates. □

We will use also

3.3. Lemma. Let $X, Y$ and $Z$ be random variables and $\delta > 0$. Suppose that $X$ and $Y$ are defined on a common probability space and $Z$ has density bounded by $c > 0$. Then, for any $a \geq 1$,

$$d_K(Y, Z) \leq 3d_K(X, Z) + ||X - Y||_a^{\frac{1}{a}} (1 + 4c).$$  

Proof. Let $a, t \in \mathbb{R}$ and $\delta > 0$. Then,

$$(3.2) \quad |P\{Y \leq t\} - P\{Z \leq t\}| \leq d_k(X, Z) + |P\{X \leq t\} - P\{Y \leq t\}| \leq P\{|X - t| \leq \delta\} + P\{|X - Y| > \delta\}.$$  

By the definition of $d_k(X, Z)$ and the mean value theorem,

$$P\{|X - t| \leq \delta\} \leq P(t - 2\delta < X - t \leq t + 2\delta) = P(X \leq t + 2\delta) - P(X \leq t - 2\delta) \leq 2d_k(X, Z) + P(Z \leq t + 2\delta) - P(Z \leq t - 2\delta) \leq 2d_k(X, Z) + 4c\delta.$$  

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Therefore by the Markov inequality,

$$|P\{Y \leq t\} - P\{Z \leq t\}| \leq 3d_k(X, Z) + 4c\delta + \frac{E|X - Y|^a}{\delta^a}. $$

The lemma follows first by taking supremum over $t \in \mathbb{R}$ and then taking $\delta = \|X - Y\|_{\alpha}^{-\frac{\alpha}{a}}$. \hfill \Box

Next, we introduce notations which appeared in [14] and will be useful here, as well. Set

$$(3.3) \quad F_{i,n,r}(x_1, \ldots, x_{i-1}, \omega) = E(F_i(x_1, \ldots, x_{i-1}, X(n))|F_{n-r,n+r}),$$

and $Y_{i,q}(n) = F_i(X(q_1(n)), \ldots, X(q_i(n)))$

and $Y_{i,m} = 0$ if $m \not\in \{q_i(n)\}_{n=1}^\infty$, $X_r(n) = E(X(n)|F_{n-r,n+r})$,

$Y_{i,q}(n), r = F_i(X_r(q_1(n)), \ldots, X_r(q_{i-1}(n)), \omega)$ and $Y_{i,m} = 0$ if $m \not\in \{q_i(n)\}_{n=1}^\infty$.

We will rely on the following result obtained in Lemma 4.2 of [14] under Assumption 2.1 with $b = 2$. Set

$$b_{i,j}(n,l) = E(Y_{i,q_i(n)}Y_{j,q_j(n)})$$

and

$$s_{i,j}(n,l) = \min(q_i(n) - q_j,l, n) \quad \text{and} \quad s_{i,j}(n,l) = \max(s_{i,j}(n,l), s_{i,j}(l,n))$$

Then, there exits a nonincreasing sequence $h(m)$, satisfying

$$(3.4) \quad \sup_{n : i, s_{i,j}(n,l) \geq m} |b_{i,j}(n,l)| \leq h(m).$$

Moreover, for $m > L_1$ we can set $h(m) = C(\varpi_{q,p}(n) + \beta(q,n)\delta)$ with $n = \frac{1}{n} m$ and some positive constants $L_1$ and $C$ depending only on the initial parameters.

Next, we will obtain estimates of errors for approximating expectations of the form $E G(X(n_1), \ldots, X(n_s))$, where $n_1 < \ldots < n_s$, by corresponding expectations with respect to corresponding product measures. The result is similar to Lemma 4.3 from [14] but the latter does not provide specific estimates which we need here. First, we will recall the inequality (3.14) from Corollary 3.6 of [14]. Let $\mathcal{G}$ and $\mathcal{H}$ be sub-$\sigma$ algebras of a probability space $(\Omega, \mathcal{F}, P)$, $X$ be $d$-dimensional random vector and $f = f(x, \omega), x \in \mathbb{R}^d$ be a collection of random variables that are measurable with respect to $\mathcal{H}$ which satisfy

$$(3.5) \quad \|f(x, \omega) - f(y, \omega)\|_q \leq C_1(1 + |x|^\kappa + |y|^\kappa)|x - y|^\kappa \quad \text{and} \quad \|f(x, \omega)\|_q \leq C_2(1 + |x|^\kappa).$$

Then, assuming that $\frac{\kappa}{\alpha} \geq \frac{1}{p} + \frac{\kappa + 1}{q}$ and $1 \geq \kappa > \frac{\kappa}{d}$,

$$(3.6) \quad \|E(f(X, \cdot)|\mathcal{G}) - g(X)\|_a \leq c\varpi_{q,p}(\mathcal{G}, \mathcal{H})(C_1 + C_2)^{\frac{\kappa}{p}}C_2^{1 - \frac{\kappa}{p}}(1 + \|X\|_q^{\kappa + 1})$$

$$+ 2c(C_1 + C_2)(1 + 2\|X\|_q^{\kappa + 2})\|X - E(X|\mathcal{G})\|_q^2$$

where $c = c(\kappa, \varpi_{q,p}, p, q, a, \delta, \kappa, d) > 0$ depends only on parameters in brackets and $g(x) = Ef(x, \omega)$. Assuming that $a \geq 1$, taking expectation and using the H" older inequality together with (3.6) we obtain

$$(3.7) \quad |Ef(X, \cdot) - Eg(X)| \leq R$$

where $R$ is the right hand side of (3.6). As a conclusion of (3.7) we derive the following result.
3.4. **Lemma.** Suppose that Assumption 2.1 holds true. Let \( G : (\mathbb{R}^v)^{n_1 + ... + n_v} \rightarrow \mathbb{R} \) be a function satisfying conditions (2.3) and (2.4) with \( K', K \) and \( K' \leq b \). Suppose that the sets \( M_i = \{ i, i < ... < a_i, a_i \} \subset \mathbb{N} \) satisfy \( a_i, a_i < a_i+1, a_i+1 \) and set \( X(M_i) = (X(a_i,1), ..., X(a_i,n_i)) \) where \( i = 1, ..., v \) and \( r = \min_{1 \leq i \leq v} \{ a_{i+1,1} - a_{i,n_i} \} \). Let 
\[
\{ Y(M_i) \}_{i=1}^v \text{ be independent copies of } \{ X(M_i) \}_{i=1}^v.
\]
Assume that \( 1 \geq \frac{1}{p} + \frac{2 - \frac{q}{2}}{m} + \frac{4}{q} \).

Then (3.8) \[
|EG(X(M_1), ..., X(M_v)) - EG(Y(M_1), ..., Y(M_v))| \leq C((\beta(q, \frac{r}{4}))^\delta + \varpi_q,p_r(\frac{r}{4}))
\]
where \( C \) depends only on the initial parameters and on \( \max_i \{ n_i \} \), \( v \) and \( K' \).

**Proof.** For \( i = 1, ..., v \) set \( z_i = (x_{a_i,1}, ..., x_{a_i,n_i}) \),
\[
\hat{X}_i = (X(M_1), ..., X(M_i)) \text{ and } H^{(v)}(z_1, ..., z_v) = G(z_1, ..., z_v).
\]
Define recursively for \( j = v, v - 1, ..., 1, \)
\[
H^{(j-1)}(z_1, ..., z_j) = \int H^{(j)}(z_1, ..., z_j) d\mu(z_j) = EH^{(j)}(z_1, ..., z_j-1, X(M_j)).
\]
Notice that \( H^{(0)} = EH(Y_0) \). For any \( s > 0 \), set
\[
H^{(j-1)}(z_1, ..., z_j-1) = EH^{(j)}(z_1, ..., z_j-1, X_\hat{\mu}(M_j)).
\]
Observe that since \( m > b \), \( X(M_j) \) has a finite \( b' \) moment. Hence, \( H^{(j)} \) and \( H^{(j)}_v \) also satisfy conditions (2.3) and (2.5). Thus, by the contraction of conditional expectations and Lemma 3.2 we obtain,
\[
|EH^{(j)}(\hat{X}_j) - EH^{(j)}(\hat{X}_{j-1}, X_\hat{\mu}(M_j))| \leq C(\beta(q, \frac{r}{4}))^\delta
\]
and
\[
|EH^{(j-1)}(\hat{X}_{j-1}) - EH^{(j-1)}_v(\hat{X}_{j-1})| \leq C(\beta(q, \frac{r}{4}))^\delta.
\]
Let \( 1 \leq j \leq v \) and set \( f(y, \omega) = H^{(j)}(y, X_\hat{\mu}(M_j)). \) Observe that condition (3.3) is satisfied with constants which depend only on the initial parameters since \( X(M_j) \) has a finite \( bq' \) moment. Taking \( G = F_{(-\infty, a_{j-1,n_j-1}]} \) and applying (3.7) we obtain that
\[
|EH^{(j)}(\hat{X}_j - X_\hat{\mu}(M_j)) - EH^{(j-1)}_v(\hat{X}_{j-1})| \leq C'(|\varpi_q,p_r(\frac{r}{2})| + \beta(q, \frac{r}{4}))^\delta.
\]
and therefore,
\[
|EH^{(j)}(\hat{X}_j) - EH^{(j-1)}(\hat{X}_{j-1})| \leq C''(|\varpi_q,p_r(\frac{r}{2})| + \beta(q, \frac{r}{4}))^\delta.
\]
Finally, using the fact that
\[
H^{(v)}(\hat{X}_v) - H^{(0)} = \sum_{j=1}^v H^{(j)}(\hat{X}_j) - H^{(j-1)}(\hat{X}_{j-1})
\]
we obtain (3.8) completing the proof. \( \square \)

We will need the following general estimates which appeared as Lemmas 6.1 and 6.2 in earlier preprint versions of [14] (see arXiv:1012.2223v2) but not in its published version so for readers’ convenience we provide them here. Consider a
probability space \((\Omega, F, P)\) with a filtration of \(\sigma\)-fields \(G_j\). Suppose that random variables \(X_j\) are \(G_j\) measurable and for some \(2 \leq p < \infty\) satisfy

\[
\gamma_p = \sup_j \|X_j\|_p \leq \sup_j \|E[X_j|G_j]\|_p = A_p < \infty.
\]

We will explore the behavior of higher order moments for sums \(S_n = \sum_{i=1}^n X_i\) obtaining estimates of the form \(E[|S_n|^{2l}] \leq C_{2l} n^l\) with some control on dependence of constants \(C_{2l}\) on \(\gamma_{2l}\) and \(A_{2l}\).

3.5. Lemma. Suppose \(\{a_n\}\) is a sequence of nonnegative numbers such that for some integer \(l \geq 1\) and any integer \(n \geq 1\),

\[
a_{n+1} \leq c \sum_{r=2}^{2l} C^r a_j^{2l-r}.\]

Then

\[
a_n \leq A n^l
\]

with \(A = \max\{2^l c \cdot C^{2l}, C^{2l}, a_1\}\).

Proof. We derive the above inequality by induction. It is clearly valid for \(n = 1\). Assume it is valid for \(j = 1, 2, \ldots, n\). Then

\[
a_{n+1} \leq c \sum_{r=2}^{2l} C^r (A_j)^{2l-r} \leq c C^2 A^{1-\frac{1}{l}} \sum_{r=2}^{2l} C^r A^{-\frac{r}{2l}} \sum_{j=1}^n j^{l-1} \leq A^{l+1} n^l
\]

where

\[
A' = c C^2 A^{1-\frac{1}{l}} \sum_{r=0}^{2l-2} C^r A^{-\frac{r}{l}}
\]

and we need to pick \(A\) so that \(
\frac{A'}{A} \leq A.\) In particular, \(A = \max\{2^l c \cdot C^{2l}, C^{2l}, a_1\}\) will do because \(CA^{-\frac{1}{l}} \leq 1, 2cC^2A^{-\frac{1}{l}} \leq 1\) and

\[
c C^2 A^{1-\frac{1}{l}} \sum_{r=0}^{2l-2} C^r A^{-\frac{r}{l}} \leq c C^2 A^{1-\frac{1}{l}} (2l-1) \leq c C^2 A^{1-\frac{1}{l}} 2l \leq l A.
\]

3.6. Lemma. Let the sequence \(\{X_i\}\) of random variables satisfy (3.9) with \(p = 2l\) and some positive integer \(l\). Then there is a constant \(c_l\) depending only on \(l\) such that

\[
ES_{2l}^n \leq c_l A_{2l}^2 n^l.
\]

Proof. We begin by expanding \(S_{2l,j+1}^l = (S_j + X_{j+1})^{2l}\) by the binomial theorem,

\[
S_{2l,j+1}^l = S_{2l,j}^l + 2l S_{2l,j}^{l-1} X_{j+1} + \sum_{r=2}^{2l} \binom{2l}{r} S_{2l-r}^j X_{j+1}^r
\]

and expressing

\[
S_{2l-2}^j = \sum_{i=1}^{j} (S_{2l-1}^i - S_{2l-1}^{i-1}) = \sum_{1 \leq i \leq j} X_i \sum_{r=0}^{2l-2} S_{2l-r}^i S_{2l-2-r}^{i-1}.
\]
This enables us to rewrite
\[ S^2_{j+1} = S^2_j + 2l \sum_{1 \leq i \leq j} Z_i X_{j+1} + \sum_{r=2}^{2l} \binom{2l}{r} S^{2l-r}_j X^r_{j+1} \]
where \( Z_i = X_i \sum_{r=0}^{2l-2} S^{2l-2-r}_r \). Then,
\[
ES^2_{n+1} = EX^{2l}_i + 2l \sum_{1 \leq i \leq j \leq n} EZ_i X_{j+1} + \sum_{j=1}^{n} \sum_{r=2}^{2l} \binom{2l}{r} ES^{2l-r}_j X^r_{j+1}
\]
where \( W_i = \sum_{j=1}^{n} E(X_{j+1} | F_i) \). We note that \( ||X_i||_{2l} \leq \gamma_{2l} \leq A_{2l} \) and \( ||W_i||_{2l} \leq A_{2l} \). Hence,
\[
E[|Z_i W_i|] \leq ||S^2_{i} S^{2l-2-r}_i|| \gamma_{2l} ||X_i||_{2l} ||W_i||_{2l}
\]
\[
\leq c_1 A_{2l}^2 ((ES^{2l}_i)^{\frac{1}{2}} + (ES^{2l}_{i-1})^{\frac{1}{2}}).
\]
Next, for \( r \geq 2 \),
\[
||ES^{2l-r}_j X^r_{j+1}|| \leq ||S^2_j||^{2l-r} ||X_{j+1}||_{2l} \leq A_{2l}^r ||S^2_j||^{2l-r}.
\]
It follows that
\[
ES^{2l}_{n+1} \leq c_1 \left( \sum_{j=1}^{n} \left( \sum_{r=2}^{2l} A_{2l}^r ||S^2_j||^{2l-r} + A_{2l}^r ||S^2_j||^{2l-2} + A_{2l}^r ||S^2_{j-1}||^{2l-2} \right) \right) \]
\[
\leq c_1 \left( \sum_{j=1}^{n} \sum_{r=2}^{2l} A_{2l}^r ||S^2_j||^{2l-r} \right)
\]
where \( c_1 \) is an absolute constant which depends only on \( l \). The sequence \( a_n = E[S^2_n] \) satisfies the condition of Lemma 3.3 with \( c = c_1, C = A_{2l} \) and \( a_1 \leq \gamma_{2l} \) and the result follows.

4. Limiting Variance

In this section we will prove Theorem 2.3. For each \( i = 1, ..., \ell \) set
\[ Z_{i,n} = F_i \left( X^{(1)}(n), ..., X^{(\ell)}(\ell n) \right) \]
and \( \Sigma_{i,n} = \sum_{n=1}^{N} Z_{i,n} \) so that \( Z_n = \sum_{i=1}^{\ell} Z_{i,n} \) and \( \Sigma_N = \sum_{i=1}^{\ell} \Sigma_{i,n} \). Then, under the assumption (2.4) the processes \( \{Z_{i,n}\}_{n \geq 0}, i = 1, ..., \ell \) and \( \{Z_n\}_{n \geq 0} \) are (one sided) stationary in the wide sense. In view of (2.1),
\[
EZ_{i,n} Z_{j,m} = 0 \text{ if } i \neq j \text{ and so } \text{Var(}\Sigma_N\text{)} = \sum_{i=1}^{\ell} \text{Var(}\Sigma_{i,N}\text{)}.
\]

Hence, \( EZ_{i,n} Z_{0} = \sum_{i=1}^{\ell} EZ_{i,n} Z_{0,n} \). In the same way as Lemma 4.2 of [14] provides the estimate (3.3) with \( h(m) = C(\varepsilon q, p) + \beta(q, [q/m]^d) \) for some \( C > 0 \) and all
Given \( m \) large enough we obtain that for all \( n \) large enough and some \( C > 0 \) independent of \( n \),
\[
|EZ_{n,i}Z_{0,i}| \leq C(\varpi_{q,p}(\frac{1}{3}n)) + \beta(q,\frac{1}{3}n)^\delta.
\]
This together with Assumption 2.1 with \( \alpha, \lambda \geq 1 \) yields that
\[
(4.2) \quad \sum_{n=1}^{\infty} n|E(Z_nZ_0)| < \infty.
\]

By Proposition 8.3 and Theorem 8.6 from [4] (modified for a one sided process) if a stationary in the wide sense process satisfies (4.2) then if \( n \) is unbounded if and only if \( \text{Var}(\Sigma) \) is unbounded if and only if \( \text{Var}(\Sigma_{e,N}) \) is bounded for each \( i = 1, \ldots, \ell \).

Next, set
\[
S_N = \sum_{n=1}^{N} F(X(n), \ldots, X(\ell n)), S_{t,N} = \sum_{n=1}^{N} F_t(X(n), \ldots, X(\ell n)),
\]
\[
N_t = N_t^{(1)} = [N(1 - \frac{1}{2t})] + 1 \quad \text{and} \quad N_t^{(i)} = [N_t^{(i-1)}(1 - \frac{1}{2t})] + 1 \quad \text{for} \quad i = 1, 2, 3, \ldots
\]
\[
S_{t,N}^{(i-1)} = S_{t,N} = \sum_{n=1}^{N} F_t(X(n), \ldots, X(\ell n))
\]
and
\[
S_{t,N}^{(2i-1)} = \sum_{n=1}^{N} F_t(X(n), \ldots, X(\ell n)), S_{t,N}^{(2i)} = S_{t,N}^{(2i-3)} - S_{t,N}^{(2i-1)}, i = 1, 2, 3, \ldots.
\]

Set also \( \sigma_N^2 = \text{var}(S_N) \) and \( \sigma_N^2 = \text{var}(\Sigma_N) \).

Now we can write
\[
(4.3) \quad \sigma_N^2 = \text{Var}(\sum_{i=1}^{\ell-1} S_{i,N} + S_{t,N}^{(1)} + \text{Var}(S_{t,N}^{(2)})

+ 2\text{Cov}(\sum_{i=1}^{\ell-1} S_{i,N} + S_{t,N}^{(1)}, S_{t,N}^{(2)}).
\]

Observe that \( N_t \geq \frac{N}{N_t} \). Since \( \ell n - \ell \geq \frac{N}{N_t} \) whenever \( i < \ell, n \leq N \) and \( N_t \leq m \leq N \) then \( |b_{i,t}(n, m)| \leq h(\frac{N}{N_t}) \) by (4.4). Taking into account Assumption 2.1 with \( \alpha, \lambda \geq 1 \) and the choice of the nonincreasing function \( h \) we obtain that
\[
(4.4) \quad |\text{Cov}(\sum_{i=1}^{\ell-1} S_{i,N}, S_{t,N}^{(2)})| \leq \sum_{i=1}^{\ell-1} \sum_{n=1}^{N} \sum_{m=N_t}^{N} |b_{i,t}(n, m)|

\leq \ell N^2 h(\frac{N}{N_t}) \leq 4\ell \sum_{n=1}^{\infty} nh(n) < \infty.
\]

Furthermore, since \( |b_{i,t}(n, m)| \leq h(m-n) \) when \( n < m \) we obtain
\[
(4.5) |\text{Cov}(S_{t,N}^{(1)}, S_{t,N}^{(2)})| \leq \sum_{n=1}^{N_t-1} \sum_{m=N_t}^{N} h(m-n) = \sum_{n=1}^{N_t-1} \sum_{j=N_t-n}^{N-N_t} h(j) = \sum_{j=1}^{N_t-1} \sum_{n=\min(N-J, N_t)}^{\max(N-J, N_t)} h(j) = \sum_{j=1}^{N_t-1} \sum_{n=1}^{N-N_t} h(j) \leq \sum_{j=1}^{\infty} jh(j) < \infty.
\]
Next, define $\Sigma_{\ell,N}^{(j)}$ for $j = -1, 1, 2, 3, \ldots$ similarly to $S_{\ell,N}^{(j)}$ using $X^{(1)}(n), \ldots, X^{(\ell)}(n)$ in place of $X(n), \ldots, X(\ell n)$. Observe that $N_{\ell}^{(i-1)}(1 - \frac{1}{2\ell}) + 1 \geq N_{\ell}^{(i)}$ and so,

$$jN_{\ell}^{(i)} - (j-1)N_{\ell}^{(i-1)} \geq \frac{1}{2}N_{\ell}^{(i-1)}$$

and $i \geq N_{\ell}^{(i)} - N(1 - \frac{1}{2\ell}) \geq 0$.

Applying Lemma 3.4 for $\gamma$ where $\gamma(n) = \omega_{\ell,N}^{(i)}(n) + \beta^q(q,n)$ and $C > 0$ depends only on the initial parameters. Hence for all $i \geq 1$,

$$|\text{Var}(\Sigma_{\ell,N}^{(2i)}) - \text{Var}(\Sigma_{\ell,N}^{(2j)})| \leq C(N_{\ell}^{(i-1)} - N_{\ell}^{(i)})^2/(\gamma(n))$$

where $N_{\ell}^{(0)} = N$ and $c_1 > 0$ depends only on the initial parameters and the expressions (2.8) and (2.9).

Next, assume that $s_N^2$ is bounded. Then by (4.1) we see that $\text{Var}(\Sigma_{\ell,N})$ is bounded in $N$ for each $i = 1, \ldots, \ell$. Proving one direction of Theorem 2.3 we will derive from here by induction in $j$ that for each $j$ there exists $C_j > 0$ such that for all $N \geq 2$,

$$\text{Var}(\sum_{i=1}^{j} S_{\ell,N}) \leq C_j \ln^2 N.$$

When $j = 1$ we have $\text{Var}S_{1,N} = \text{Var}\Sigma_{1,N}$ which is bounded if $s_N^2$ is bounded. Now suppose that (4.8) holds true for all $j$ up to $\ell - 1$ and prove it for $j = \ell$. Recall that $Z_{\ell,n}, n \geq 0$ is a stationary in the wide sense process, and so

$$\text{Var}(\Sigma_{\ell,N}^{(2i)}) = \text{Var}(\sum_{n=N_{\ell}^{(i)}}^{N_{\ell}^{(i-1)}-1} Z_{\ell,n}) = \text{Var}\Sigma_{\ell,N}^{(i-1)} - N_{\ell}^{(i)}$$

and the latter expression is bounded in view of our assumption on $s_N^2$. This together with (4.7) yields

$$\text{Var}(S_{\ell,N}^{(2i)}) \leq \text{Var}(\Sigma_{\ell,N}^{(2i)}) + c_1 \leq c_2$$

for some $c_2 > 0$ independent of $N$. Now by (2.8)–(2.9), (4.9) and the induction hypothesis

$$\sigma_N^2 = \text{Var}(\sum_{i=1}^{\ell} S_{\ell,N}) \leq c_3 + 2C_{\ell-1} \ln^2 N + 2\text{Var}(S_{\ell,N}^{(1)})$$

for some $c_3 > 0$ independent of $N$. 
Next, applying the above definitions recursively for any $i$ such that $N_i^{(i)} \geq 2$ we can write

\begin{equation}
S_{\ell,N}^{(1)} = S_{\ell,N}^{(3)} + S_{\ell,N}^{(4)} = S_{\ell,N}^{(2i-1)} + \sum_{j=2}^{i} S_{\ell,N}^{(2j)}.
\end{equation}

Hence,

\[
\text{Var}S_{\ell,N}^{(1)} \leq 2\text{Var}S_{\ell,N}^{(2i-1)} + 2i \sum_{j=2}^{i} \text{Var}S_{\ell,N}^{(2j)}
\]

where we use that $(\sum_{j=1}^{m} a_j)^2 \leq m \sum_{j=1}^{m} a_j^2$. By (4.11) we can choose $i = M \ln N$ for some fixed $M > 0$ so that $2 \leq N_i^{(i)} \leq i + 4$. Then $\text{Var}S_{\ell,N}^{(2i-1)} \leq C'(1 + \ln^2(N))$ for some $C' > 0$ independent of $N$ and we obtain from (4.9) that $\text{Var}S_{\ell,N}^{(1)} \leq \tilde{C}(1 + \ln^2 N)$ for some $\tilde{C} > 0$ independent of $N$ which together with (4.10) yields (4.8) with $j = \ell$.

Next, we will prove Theorem 2.3 in the other direction assuming that $S_N^2$ is unbounded which, as explained above, is equivalent to the linear in $N$ growth of $s_N^2$ and to the fact that the corresponding limiting variance $s^2$ is positive. Our goal is to show that then

\begin{equation}
\sigma^2 = \lim_{N \to \infty} \frac{1}{N} \sigma_N^2 > 0.
\end{equation}

Recall, that the existence of the limit in (4.12) follows from (4.2) and only its positivity should be proved in our situation. The proof will proceed again by induction in $\ell$. For $\ell = 1$ we have $S_N = \Sigma_i$, and so $\text{Var}\Sigma_i$ grows linearly in $N$ then the same is true for $\text{Var}S_N$. Now suppose that we already established for each $j = 1, 2, ..., \ell - 1$ that if $\text{Var}(\sum_{i=1}^{j} \Sigma_i)$ grows linearly in $N$ then the same is true for $\text{Var}(\sum_{i=1}^{j} \Sigma_i)$ and now we will prove this for $j = \ell$. Indeed, assume that $s_N^2 = \text{Var}(\sum_{i=1}^{\ell} \Sigma_i)$ grows linearly in $N$. Then by (4.1) either $\text{Var}(\sum_{i=1}^{\ell-1} \Sigma_i)$ or $\text{Var}(\Sigma_{\ell,N})$ grow linearly in $N$. In the latter case we obtain also that $\text{Var}(\Sigma_{\ell,N})$ grows linearly in $N$ in view of stationarity in the wide sense of $Z_{\ell,n} \geq 0$. Then by (4.7) we see that $\text{Var}(S_{\ell,N}^{(2)})$ grows linearly in $N$. This together with (4.3)–(4.5) yields that $\sigma_N^2$ grows at least linearly in $N$ but since by (4.3) a (finite) limit $\lim_{N \to \infty} \frac{1}{N} \sigma_N^2$ exists we conclude that in this case $\sigma_N^2$ grows linearly in $N$ as required.

Now suppose that $\text{Var}(\Sigma_{\ell,N})$ is bounded while $\text{Var}(\sum_{i=1}^{\ell-1} \Sigma_i)$ grows linearly in $N$. Then $\text{Var}(\Sigma_{\ell,N}^{(2)})$ for all $i$ are also bounded by stationarity of $Z_{\ell,n}$, $n \geq 0$ in the wide sense which together with (4.7) yields that $\text{Var}(S_{\ell,N}^{(2)})$ are also bounded for all $i$. Using again the representation (4.1) we conclude as before that $\text{Var}S_{\ell,N}^{(1)} \leq \tilde{C}(1 + \ln^2 N)$ for some $\tilde{C} > 0$ independent of $N$. Since $\text{Var}(\sum_{i=1}^{\ell-1} \Sigma_i)$ grows linearly in $N$ then by the induction hypothesis $\text{Var}(\sum_{i=1}^{\ell-1} \Sigma_i)$ grows linearly in $N$, as well. It follows that

\begin{equation}
\sigma_N^2 = \text{Var}(\sum_{i=1}^{\ell} \Sigma_i) = \text{Var}(\sum_{i=1}^{\ell-1} \Sigma_i) + \text{Var}S_{\ell,N}
+ 2\text{Cov}(\sum_{i=1}^{\ell-1} \Sigma_i, S_{\ell,N})
\geq \text{Var}(\sum_{i=1}^{\ell-1} \Sigma_i)
- 2(\text{Var}(\sum_{i=1}^{\ell-1} \Sigma_i))^{1/2}(\text{Var}S_{\ell,N}^{(1)})^{1/2} + (\text{Var}S_{\ell,N}^{(2)})^{1/2}
\geq \text{Var}(\sum_{i=1}^{\ell-1} \Sigma_i)(1 - \tilde{C}N^{-1/2}(\ln N + 1))
\end{equation
for some $\hat{C} > 0$ independent of $N$. Hence, $\sigma_N^2$ grows at least linearly in $N$ but, again, since finite limit $\lim_{N \to \infty} \frac{1}{N} \sigma_N^2$ exists $\sigma_N^2$ grows, in fact, linearly in $N$ completing the proof of Theorem 2.3.

\section{Convergence estimates}

In this section we introduce martingale approximation technique which is similar but a bit different from [14]. Then we study the quadratic variation of the constructed martingale and use it to prove Theorem 2.5. The following representations from (5.2) in [14] will be useful here, as well.

\begin{equation}
Y_{i,n} = Y_{i,n,1} + \sum_{r=1}^{\infty} [Y_{i,n,2^r} - Y_{i,n,2^{r-1}}], \quad \zeta_{i,N,0}(t) = \frac{1}{\sqrt{N}} \sum_{1 \leq n \leq N} Y_{i,n,1},
\end{equation}

\begin{equation}
\zeta_{i,N,r}(t) = \frac{1}{\sqrt{N}} \sum_{1 \leq n \leq N} [Y_{i,n,2^r} - Y_{i,n,2^{r-1}}], \quad r \geq 1, \quad \zeta_{i,N}^{(u)}(t) = \sum_{r=1}^{u} \zeta_{i,N,r}(t)
\end{equation}

and

\begin{equation}
\xi_{i,N}(t) = \frac{1}{\sqrt{N}} \sum_{1 \leq n \leq N} Y_{i,n}.
\end{equation}

The convergence of series in (5.1) follows from [14]. Similarly to Proposition 5.8 of [14] we derive from Corollary 3.6(ii) there that for any $r \in \mathbb{N}, 1 \leq i \leq \ell$ and $l \leq n + r$,

\begin{equation}
|E(Y_{i,n,r} | \mathcal{F}_{-\infty,i})|_b \leq c_r(n - l)
\end{equation}

for some sequence $c_r$ satisfying

\begin{equation}
C_r = \sum_{m=0}^{\infty} c_r(m) \leq C_r
\end{equation}

where $b$ comes from Assumption 2.1 and, recall, that $b \geq 4$ in Theorem 2.5 while $C$ depends on the initial parameters and on the expressions (5.2) and (5.3).

Furthermore, similarly to the proof of Proposition 5.9 from [14] we obtain that

\begin{equation}
\sum_{n \geq r} \sup_{N \geq 1} \| \sup_{0 \leq t \leq T} |\zeta_{i,N,n}(t)| \|_2 \leq \frac{C T}{\lambda} \sum_{n=2^\delta}^{\infty} (\beta(q,n))^\delta
\end{equation}

where $C$ depends only on the initial parameters. Observe that since

\begin{equation}
\sum_{n=m}^{\infty} (\beta(q,n))^\delta \leq m^{-\lambda} \sum_{n=m}^{\infty} n^{\lambda}(\beta(q,n))^\delta
\end{equation}

then under Assumption 2.1 with $b \geq 2$ we obtain from (5.4) that

\begin{equation}
| \sup_{0 \leq t \leq T} |\xi_{i,N}(t) - \xi_{i,N}^{(u)}(t)\|_2 \leq \frac{C T 2^{-\gamma}}{\lambda}
\end{equation}

\subsection{Martingale approximations}

For any fixed $n, u \in \mathbb{N}$ and $1 \leq i \leq k$ set

\begin{equation}
W_{i,n,2^u} = Y_{i,n,2^u} + R_{i,n,u} - R_{i,n-1,u}
\end{equation}

where $R_{i,v,u} = \sum_{s \geq v+1} E(Y_{i,s,2^u} | \mathcal{F}_{-\infty,v+2^u})$. Clearly, $\{W_{i,n,2^u}\}_{n \geq 1}$ is a martingale difference sequence with respect to the filtration $\{\mathcal{F}_{-\infty,n+2^u}\}_{n \geq 1}$. For any $1 \leq i \leq \ell$ and $u(N) = \lceil \log_2(N) \rceil + 1$ define the truncated martingale differences $W_{i,n}^{(N)} = \mathbb{1}_{\{n \leq i\}} W_{i,n,2^u}(|N)$ and $W_{i,n}^{(N)} = \sum_{m=1}^{\ell} W_{i,m}^{(N)}$ where $\mathbb{1}_A = 1$ if an event $A$ occurs and $= 0$, otherwise. Set $M_{i,n}^{(N)} = \sum_{m=1}^{n} W_{i,m}^{(N)}$ and

\begin{equation}
M_{i,n}^{(N)} = \sum_{m=1}^{n} W_{i,m}^{(N)} = \sum_{i=1}^{\ell} M_{i,n}^{(N)}
\end{equation}
When $N$ is fixed the sequence $M_{N}^{(N)}$, $n \geq 1$ is a martingale with respect to the filtration $\{\mathcal{F}_{-\infty,n+2u(N)}\}_{n \geq 1}$ and when $N$ changes we have a martingale array. Taking into account that $\xi_{N}(t) = \sum_{i=1}^{\ell} \xi_{i,N}(t)$ we obtain by (5.3) that

$$\|\xi_{N}(1) - \frac{1}{\sqrt{N}} M_{N}^{(N)}\|_{2} \leq C N^{-\frac{1}{4} \min(\alpha, \lambda) - \frac{1}{4}}.$$

5.2. Quadratic variation estimates. Our goal in this subsection is to obtain the following result.

5.1. Proposition. Suppose that Assumption 2.1 holds true with $\alpha, \lambda > 1$. Let $1 \leq i, j \leq k$ and $1 \leq T \leq \ell$ and set

$$Z_{n} = Z_{n}^{(i,j,N)} = W_{i,n,2u(N)}W_{j,n,2u(N)}.$$

Then

$$\frac{1}{N} \sum_{n=1}^{NT} Z_{n} - TD_{i,j}\|_{2} \leq C N^{-\frac{1}{4} \min(\alpha, \lambda) - \frac{1}{4}}.$$ 

where $D_{i,j}$ was introduced in Theorem 2.2 and $C$ depends only on the initial parameters and the expressions (2.9) and (2.10).

We prove this proposition in several steps formulated as separate lemmas.

5.2. Lemma. Let $1 \leq i \leq \ell$. Suppose that $\{n_{k}\}_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers. Then, for any $u, m, k \in \mathbb{N}$ such that $k \leq \frac{b}{4}$,

$$\| \sum_{s=v}^{u+m-1} Y_{i,in,s,2^{s}}\|_{2k} \| \sum_{s=v}^{u+m-1} Y_{i,in,s}\|_{2k} \leq C \sqrt{m}$$

where $C$ depends only on the initial parameters and the expressions (2.9) and (2.10).

Proof. Let $s > s'$. Taking $l, n \in \mathbb{N}$ which satisfy $l \geq (i - 1)n$ and $in \geq l + 2s$ we can derive from Theorem 3.4 in [14] that

$$\|E(Y_{i,in,s} | \mathcal{F}_{-\infty,l+s}) - E(Y_{i,in,s} | \mathcal{F}_{-\infty,l+s})\|_{b} \leq C_{1} \varpi_{q,p}(in - l - 2s)(\beta(q, s'))^{\delta}$$

(see the proof of Lemma 3.11 in the early preprint version arXiv:1012.2223v2 of [14]). On the other hand, if $l < (i - 1)n$ and $2s < n$ then by the contraction of the conditional expectations similarly to the above,

$$\|E(Y_{i,in,s} - Y_{i,in,s'} | \mathcal{F}_{-\infty,l+s+2})\|_{b} \leq \|E(Y_{i,in,s} - Y_{i,in,s'} | \mathcal{F}_{-\infty,(i-1)n+2s})\|_{b} \leq C_{2} \varpi_{q,p}(n - 2s)(\beta(q, s'))^{\delta}.$$ 

Let $k, r, m \in \mathbb{N}$ and set $s = 2^{r+1}$, $s' = 2^{r}$, $l = in$, and $n = n_{k}$. Since $n_{k} - n_{k'} \geq k - k'$ if $k \geq k'$ there exist no more than $4s = 2^{r+3}$ natural numbers $k \geq m$ which do not satisfy either $in_{m} \geq (i - 1)n_{k}$ and $in_{k} \geq in_{m} + 2^{r+2}$ or $in_{m} < (i - 1)n_{k}$ and $n_{k} > 2^{r+2}$, i.e. for each $m$ we can use one of two inequalities above with such $s, s', l$ and $n = n_{k} \geq n_{m}$ except for at most $2^{r+3}$ of $k$’s. Using again Theorem 3.4 from [14] (or Lemma 3.12 from the above mentioned preprint), the contraction of the conditional expectations to bound those (at most) $2^{r+3}$ summands, the estimates above and the fact that $\sum_{n \in \mathbb{N}} \varpi_{q,p}(n) < \infty$ we obtain

$$\sup_{m} \sum_{k \geq m} \|E(Y_{i,in_{k},2^{r+1}} - Y_{i,in_{k},2^{r}} | \mathcal{F}_{in_{m}+2^{r+1}})\|_{b} \leq C_{3} 2^{r} (\beta(q, 2^{r}))^{\delta}.$$ 

Applying Lemma 5.6 with
\[ S_n = Y_{i, in_{s+n-1}, 2^r} - Y_{i, in_{s+n-1}, 2^{r+1}} \]
yields for \( k \leq \frac{b}{2} \) that
\[ \| \sum_{s=v}^{v+m-1} Y_{i, in_s, 2^r} - Y_{i, in_s, 2^{r+1}} \|_{2k} \leq C_4 \sqrt{m} 2^r (\beta(q, 2^r))^\delta. \]  
(5.8)

Since \( Y_{i, in_s, 2^r} = Y_{i, in_s} + \sum_{r'=u}^\infty Y_{i, in_s, 2^{r'} - 1} \), then we obtain for any \( u > 0 \) that
\[ \| \sum_{s=v}^{v+m-1} Y_{i, in_s, 2^r} \|_{2k} \leq C_4 \sqrt{m} \sum_{r'=u}^\infty 2^r (\beta(q, 2^r))^\delta + \| \sum_{s=v}^{v+m-1} Y_{i, in_s} \|_{2k}. \]  
(5.9)

Since \( Y_{i, in} = Y_{i, in_{1}} + \sum_{r'=1}^\infty (Y_{i, in_{2^r}} - Y_{i, in_{2^{r-1}}} - 1) \), almost surely then by (5.8), (5.12) applied with \( r = 1 \) and by Lemma 6.2 in the preprint of [14] applied with \( (Y_{i, in_{1}})_{s=v}^\infty \) we obtain \( \| \sum_{s=v}^{v+m-1} Y_{i, in_s} \|_{2k} \leq C \sqrt{m} \) and the lemma follows by (5.8).

5.3. Lemma. Suppose that Assumption 2.1 holds true with \( \alpha, \lambda > 1 \). Let \( 1 \leq i, j \leq \ell \). Let \( N \in \mathbb{N} \). Set \( l = l(N) = [N^{\frac{1}{4}}], N_l = \left[ \frac{N}{N_l} \right] \),
\[ V_{N,r} = V_r = \left( \sum_{n \in B_r'} \sum_{n \in B_r} \right) \]
and \( U_r = V_r - EV_{N,r} \), where \( B_r = \mathbb{N} \cap [l(r-1), lr] \). Then,
\[ \frac{1}{N} \sum_{r=1}^{N_l} N \left\| U_{N,r} \right\|_2 \leq C N^{-\frac{1}{8} \min\{\alpha, \lambda\} - 1} \frac{2\lambda}{\sqrt{m}}. \]
where \( C \) depends on the initial parameters and the expressions (2.8) and (2.9).

Proof. For any \( 1 \leq r \leq N_l \) set
\[ A_r = \{ 1 \leq r' \leq N_l : \min\{|sn - tn'| : 1 \leq s, t \leq \ell^2, n \in B_r, n' \in B_{r'} \} < l \}. \]

If for some \( 1 \leq t, s \leq \ell^2 \), \( sr' l \leq t(r-2)l \) or \( s(r'-1)l \geq t(r+1)l \) then for any \( n \in B_r \) and \( n' \in B_{r'} \), \( |tn - sn'| \geq l \). Hence, for any \( r' \in A_r \), there exist \( 1 \leq t, s \leq \ell^2 \) satisfying \( sr' l > t(r-2)l \) and \( s(r'-1)l < t(r+1)l \). Therefore, \( |r' - \frac{4\ell^3}{s} | < \max\{\frac{2\ell^3}{s} + 1, l\} \leq 2\ell \) and hence \( |A_r| \leq 2\ell^3 \). Next,
\[ \left( \frac{1}{N} \sum_{r=1}^{N_l} \left\| U_r \right\|_2 \right)^2 = \sum_{r=1}^{N_l} E( \sum_{r' \in A_r} \sum_{k=1}^{2} U_{r_k} ) + \sum_{r=1}^{N_l} E( \sum_{r' \notin A_r} \sum_{k=1}^{2} U_{r_k} ) = J_1 + J_2. \]

First, for any \( 1 \leq r_1, r_2 \leq N_l \), by Lemma 5.2 and the Cauchy-Schwarz inequality, \( |E(U_{r_1}U_{r_2})| \leq \left\| U_{r_1} \right\|_2 |U_{r_2} \right\|_2 \leq C N \) and hence,
\[ \left( \frac{1}{N} \sum_{r=1}^{N_l} |A_r| \right) \leq \frac{2C_3 C_0}{N} \sum_{r=1}^{N_l} \left| A_r \right| \leq \frac{4\ell^3 C_0}{\sqrt{N}} = C_1 \sqrt{N}. \]

Next, let \( 1 \leq r_1, r_2 \leq N_l \) and suppose that \( r_2 \notin A_{r_1} \). Since \( r_1 \notin A_{r_2} \), assume without loss of generality that \( r_1 < r_2 \). Next, we estimate \( |E(U_{r_1}U_{r_2})| \). For any
Hence, by the above, Lemma 5.2 and the Cauchy-Schwarz inequality,

\[ ||V_n - \hat{V}_n||_2 \leq ||\sum_{n \in B_r} Y_n||_4 ||\sum_{n \in B_r} (Y_{j,n} - Y_{j,n,2^v})||_4 \]

\[ \leq ||\sum_{n \in B_r} Y_{j,n,2^v}||_4 ||\sum_{n \in B_r} (Y_{i,n} - Y_{i,n,2^v})||_4 \leq C_1 l(N)2^{-u(N)\lambda}, \]

where

\[ \hat{V}_r = (\sum_{n \in B_r} Y_{i,n})(\sum_{n \in B_r} Y_{j,n}). \]

Hence, by the above, Lemma 5.2 and the Cauchy-Schwarz inequality,

\[ ||V_r V_2 - \hat{V}_r \hat{V}_2||_1 \leq ||V_r||_2 ||V_2 - \hat{V}_2||_2 + ||V_2||_2 ||V_r - \hat{V}_r||_2 \]

\[ \leq C_2 N^{-u(N)\lambda} \leq C_3 N^{1-\frac{1}{5\sigma}}. \]

In view of (5.11), it suffices to estimate \( \text{cov}(\hat{V}_r, \hat{V}_2) \). We show that Lemma 3.4 is applicable. For any \( n, m \in B_r, s = 1, 2 \) observe at \( Y_{i,n} Y_{j,m} \) and \( Y_{i,n} Y_{j,m} \). Then it vanishes unless \( i \) divides \( n \) and \( j \) divides \( m \) for \( s = 1, 2 \) and we can write

\[ n_1 = i n_1, n_2 = in_2, m_1 = jm_1, m_2 = jm_2. \]

Set

\[ \gamma_1 = (\{s m^t_1\}_{t=1}^s \cup \{s m^t_2\}_{t=1}^s) \quad \text{and} \quad \gamma_2 = (\{s m^t_1\}_{t=1}^s \cup \{s m^t_2\}_{t=1}^s). \]

By ordering the set \( \gamma_1 \cup \gamma_2 \) and considering the jump points from \( \gamma_1 \) to \( \gamma_2 \) (or vice versa) we can represent this set as a disjoint union of blocks with distances between them at least \( \frac{l}{k^\sigma} \) and which are contained in \( \gamma_1 \) or in \( \gamma_2 \). Applying Lemma 3.4 first with \( Y_{i,n} Y_{j,m} Y_{j,n} Y_{j,m} \) and then with \( Y_{i,n} Y_{j,m} \) and \( Y_{i,n} Y_{j,m} \) separately yields

\[ |E(Y_{i,n} Y_{j,m} Y_{j,n} Y_{j,m}) - E(Y_{i,n} Y_{j,m})E(Y_{i,n} Y_{j,m})| \leq C_4 \gamma(l_{\frac{1}{4}}) \]

where \( \gamma(n) = w_{q,p}(n) + (\beta(q,n))^6 \). Finally, by (5.11), the fact that \( l^2 \gamma(l_{\frac{1}{4}}) \leq cl^{-\min(\alpha,\lambda)-1} \) and the above inequality we see that

\[ \frac{1}{N^2}|J_2| \leq C \left( N^{-\frac{1}{5\sigma}} + N^{-\frac{\min(\alpha,\lambda)-1}{2}} \right) \]

and the lemma follows by (5.10)–(5.12).

5.4. Lemma. Suppose that Assumption 2.1 holds with \( \alpha, \lambda > 1 \). Let \( N \in \mathbb{N} \). Then

\[ \frac{1}{N} \sum_{n=1}^{N} (Z_n - E(Z_n))||_2 \leq CN^{-\frac{1}{2} \min(\min(\alpha,\lambda)-1, \frac{1}{5\sigma})} \]

where \( C \) depends only on the initial parameters.

Proof. Fix \( N \in \mathbb{N} \) and let \( l = [\sqrt{N}] \), \( u = u(N) \). For any \( r > 0 \) set \( \mathcal{G}_r = \mathcal{F}_{-\infty, r-l+2^v}, \)

\[ G_r = (\sum_{n \in B_r} W_{i,n,2^v})(\sum_{n \in B_r} W_{j,n,2^v}) \quad \text{and} \quad T_r = G_r - \sum_{n \in B_r} Z_n. \]

Clearly \( \{T_r\}_{r=1}^{\infty} \) is a martingale differences sequence with respect to the filtration \( \{\mathcal{G}_r\}_{r=1}^{\infty} \). Then by (5.2), the triangle inequality, the Cauchy-Schwarz inequality, Lemma 5.2 and using the fact that \( b \geq 4 \) and \( 2^u \leq \sqrt{t} \),

\[ ||G_r - V_r||_2 \leq ||(\sum_{n \in B_r} Y_{i,n,2^v})(\sum_{n \in B_r} Y_{j,n,2^v} - W_{j,n,2^v})||_2 \]

\[ + ||(\sum_{n \in B_r} W_{j,n,2^v})(\sum_{n \in B_r} Y_{i,n,2^v} - W_{i,n,2^v})||_2 \leq C_1 \sqrt{t} 2^u \leq C_2 N^{4+\frac{1}{10\sigma}} \]
and hence
\[ \| \sum_{r=1}^{N_t} (G_r - V_r) \|_2 \leq C_2 N^{\frac{3}{2} + \frac{1}{24}}. \]

By Lemma 5.2 and (5.13)
\[ \| G_r \|_2 \leq \| V_r \|_2 + C_3 \sqrt{2^u} \leq C_4 N^{\frac{3}{2}}. \]

By (5.13), \( \| Z_n \|_2 \leq C_5 2^u \). Thus, \( \| T_r \|_2 \leq 2^u C_6 N^{\frac{3}{2}} \). Therefore, by the martingale orthogonality property and since \( N - t N_t \leq l \),
\[ \| \sum_{n=1}^{N} Z_n - \sum_{r=1}^{N} G_r \|_2 \leq C_5 2^u + \| \sum_{r=1}^{N} T_r \|_2 \]
\[ = C_5 2^u + \sum_{r=1}^{N} \| T_r \|_2 \leq C_7 N^{\frac{3}{2} + \frac{1}{24}}. \]

The lemma follows by (5.13) and (5.14) writing \( \sum_{n=1}^{N} Z_n \) as sum of those two differences.

**Proof of Proposition 5.1** First write
\[ \sum_{n=1}^{[NT]} E(Z_n) = E\left( \sum_{n=1}^{[NT]} W_{i_1,n,2^u}\right)\left( \sum_{n=1}^{[NT]} W_{i_1,n,2^u}\right). \]

By the same reason as in (5.13), using the fact that \( T \geq 1 \) and \( \frac{2^u(N)}{\sqrt{N}} \leq 2 N^{-\frac{\lambda + 4}{\sqrt{N + \lambda}}} \leq 2 N^{-\frac{\lambda}{\sqrt{N + \lambda}}} \),
\[ \left\| \frac{1}{N} \sum_{n=1}^{[NT]} E(Z_n) - E\left( \xi_{i_1,N}^{(u)}(T)\xi_{i_1,N}^{(u)}(T)\right) \right\| \leq C_1 N^{-\frac{\lambda}{\sqrt{N + \lambda}}}, \quad u = u(N). \]

By (5.10), Lemma 5.2 and the Cauchy-Schwarz inequality,
\[ \left\| \xi_{i_1,N}^{(u)}(T)\xi_{i_1,N}^{(u)}(T) - \xi_{i_1,N}(T)\xi_{i_1,N}(T) \right\| \leq C_2 2^{-u\lambda} \leq C_3 N^{-\frac{\lambda}{\sqrt{N + \lambda}}} \].

As in (5.13), \( |E(\xi_{i_1,N}(T)\xi_{i_1,N}(T)) - TD_{i_1,j}| \leq C_4 \) for some constant which depends on the initial parameters and on the expressions (2.8) and (2.4). Thus,
\[ \left\| \frac{1}{N} \sum_{n=1}^{[NT]} Z_n - TD \right\| \leq \frac{1}{N} \left\| \sum_{n=1}^{[NT]} Z_n - E(Z_n) \right\| + C_5 N^{-\frac{\lambda}{\sqrt{N + \lambda}}}. \]

Now we estimate the second term on the right hand side of (5.13) by Lemma 5.3 and the lemma follows.

**5.3. Proof of Theorem 2.5** Let \( M_n = M_{n}^{(N)} \), \( n = 1, \ldots, \ell N \) and \( W_n = W_n^{(N)} \), \( n = 1, \ldots, \ell N \) be the martingale array and the corresponding martingale differences from Section 5.1. Let \( \sigma^2 > 0 \) be the limiting variance. We apply Theorem 2 from [9] with \( \delta = 1 \), and \( \varepsilon = \sigma^{-\frac{1}{2}} \), \(|||N-1\sum_{n=1}^{\ell N} W_n^2 - \sigma^2||_2^2\|\) and taking into account the Markov inequality and that always \( E\|Z\|_{\|Z\|_1} \leq EZ^2 \|Z\|_1 \leq EZ^4 \) we obtain
\[ d_K(\mathcal{L}(N^{-1/2}M_N), \mathcal{N}(0, \sigma^2)) = d_K(\mathcal{L}(N^{-1/2}\sigma^{-1}M_N), \mathcal{N}(0, 1)) \]
\[ \leq A(N^{-2/3}\sigma^{-1/3}V_{4,N}^2 + N^{-2/5}\sigma^{-1/5}V_{4,N}^2 + \sigma^{-\frac{1}{2}} \|\sum_{n=1}^{\ell N} W_n^2 - d^2\|_2^2) \]
where \( A > 0 \) is an absolute constant and \( V_{4,N} = \sum_{n=1}^{\ell N} EW_n^4 \).
Next, by (5.3) and the formulas for $W_{i,n,2^k}$, $W_{i,n}^{(N)}$ and $W_{i,n}^{(1)}$ we obtain that $||W_{i,n}^{(N)}||_b \leq C^2u^{(N)}$ for some constant $C > 0$ independent of $N$, and so $V_{i,n} \leq C^2u^{(N)} \leq CN$. Finally, (5.9) and (5.10) together with Proposition 6.1 and Lemma 6.3 yields the first assertion of Theorem 6.5. In order to prove the second assertion we take $u(N) = \frac{\log_{2}(\log_{2}(N))}{\log_{2}(N)}$ in place of $\frac{\log_{2}(\log_{2}(N))}{\log_{2}(N)}$ and repeat the proof of the first assertion, with appropriate modifications, using the fact that with $c_3 = C^{3}, n^2(\lambda_{q,p}(n) + \lambda_{q,p}(n)) \leq M\theta n$ for some $1 > \theta > c_3$ and $\sum_{n=2}^{\infty} (\beta(q,n))^4 \leq 8^{\log_{2}(C_{4})}$.

6. Special cases, extensions and concluding remarks

In this section we provide better estimates for the i.i.d. case, extend results to more general $q_{i}(n)$ functions and consider also the continuous time case.

6.1. Independent case. When $\{X(n)\}_{n \geq 1}$ are i.i.d. random variables we do not assume (2.4) and (2.5) but only that $F(X(1), X(2), ..., X(\ell))$ is a nonconstant random variable having third moment. The case $\ell = 1$ is the ”conventional” case, so we assume that $\ell > 1$ and set

$$S_N = \sum_{n=1}^{n} F(X(n), X(n), ..., X(n\ell)).$$

As in Section 2 from [15] $S_N$ can be splitted into sum of independent (blocks) random variables as follows. Set

$$\{l_1, l_2, ..., l_m\} = \{x \in \mathbb{N} : \text{is a prime number and } x \leq \ell\},$$

$$A_n = \{1 \leq a \leq n : \text{a is relatively prime with } l_1, l_2, ..., l_m\}$$

and

$$B_s(a) = \{b \leq s : b = a_{l_1}d_1 \cdot a_{l_2}d_2 \cdot ... \cdot a_{l_m}d_m \text{ for some nonnegative integers } d_1, d_2, ..., d_m\}.$$ 

For any $a \in A_N$ put

$$S_{N,a} = \sum_{b \in B_N(a)} F(X(b), X(2b), ..., X(\ell b)).$$

Then, the distribution of $S_{N,a}$ depends only on $|B_N(a)|$ where $|B|$ denotes here the cardinality of $B$. Observe that $\{S_{N,a}\}_{a \in A_N}$ are independent random variables and that $S_N = \sum_{n \in A_N} S_{N,n}$.

6.1. Assumption. Suppose that

$$EF(X(1), X(2), ..., X(\ell)) = 0 \text{ and } 0 < E|F^{3}(X(1), X(2), ..., X(\ell))| = b^{3} < \infty.$$

6.2. Theorem. Suppose that Assumption 6.1 holds true. Then,

$$\sigma^2 = \lim_{n \to \infty} \frac{1}{N} \text{Var}(S_N) > 0$$

exists and

$$d_{K}(S_N, \mathcal{N}(0, \sigma)) \leq C \frac{(1 + \ln N)^{3n}}{(N\sigma)^{n}}$$

where $C > 0$ depends only on $\ell$ and on $b^2$. 
Proof. The first statement of this theorem follows actually from Theorem 2.5 of [10] but we will give here a direct proof adapted to the i.i.d. case which will also provide some necessary estimates. We will use the construction and techniques from Section 4 in [15]. Put $Z_{N,a} = ES_{N,a}^2$, $Z_N = ES_N^2 = \sum_{a \in A_n} Z_{N,a}$,

$$D(\rho) = \{ n = (n_1, ..., n_m) \in \mathbb{Z}^m : n_i \geq 0, i = 1, ..., m \ and \ \sum_{i=1}^m n_i \ln (l_i) \leq \rho \}$$

and $R_l = Z_{N,a}$ if $l = |B_N(a)|$ where $|\Gamma|$ for a finite set $\Gamma$ denotes its cardinality. Observe that $R_1 = EF^2(X(1), ..., X(\ell)) > 0$ and that by the Cauchy-Schwartz inequality

$$(6.1) \quad R_l(F) \leq \ell^2 EF^2(X(1), ..., X(\ell)).$$

In Section 4 of [15] it was shown that the numbers

$$\rho_{\text{max}}(l) = \sup \{ \rho \geq 0 : |D(\rho)| = l \} \ and \ \rho_{\text{min}}(l) = \inf \{ \rho \geq 0 : |D(\rho)| = l \}$$

are well defined. Now, for any $l \in \mathbb{N}$,

$$(6.2) \quad (t^{\frac{1}{m}} - 1) \ln(2) < \rho_{\text{min}}(l) < \rho_{\text{max}}(l) < C_m l$$

where $C_m = 2 \sum_{i=1}^m \ln (l_i)$. Indeed, two inequalities on the left hand side were obtained already in [15], and so it remains to derive the last one. Observe $|D(lC_m)| > l$ and the right hand side of (6.2) follows since $|D(\rho)|$ is increasing.

Next, set

$$A_{N}^{(l)} = \{ a \in A_N : |B_N(a)| = l \}$$

and

$$\hat{A}_{N}^{(l)} = \{ a \in \mathbb{N} : Ne^{-\rho_{\text{max}}(l)} \leq a \leq Ne^{-\rho_{\text{min}}(l)} \}.$$

In (4.6) and (4.7) from [15] it was shown that

$$(6.3) \quad |A_{N}^{(l)}| \leq N2^{-l^{\frac{1}{m}} - 1} \ and \ \frac{1}{N} || A_{N}^{(l)} | - | \hat{A}_{N}^{(l)} || \leq \frac{1}{N}.$$

As in (4.10) from [15] with $|G_{N}^{(l)}(n)| = \left( \frac{N}{m} e^{-\rho_{\text{min}}(l)} - e^{-\rho_{\text{max}}(l)} \right)$,

$$|\hat{A}_{N}^{(l)}| = \sum_{k=1}^m (-1)^{k+1} \sum_{i_1 < i_2 < ... < i_k \leq m} [G_{N}^{(l)}(\prod_{s=1}^k i_s)]$$

and hence

$$\frac{1}{N} |\hat{A}_{N}^{(l)}| - c_{l}(e^{-\rho_{\text{min}}(l)} - e^{-\rho_{\text{max}}(l)}) \leq \frac{m^m}{N}$$

where

$$c_{l} = 1 - \prod_{k=1}^m \left( 1 - \frac{1}{l_k} \right) = \sum_{k=1}^m (-1)^{k+1} \sum_{i_1 < i_2 < ... < i_k \leq m} \prod_{s=1}^k \frac{1}{l_{i_s}}.$$

Therefore in view of (6.3),

$$(6.4) \quad \left| \frac{1}{N} |A_{N}^{(l)}| - c_{l}(e^{-\rho_{\text{min}}(l)} - e^{-\rho_{\text{max}}(l)}) \right| \leq \frac{m^m + 1}{N}.$$

It follows that

$$\frac{1}{N} Z_n = \frac{1}{N} \sum_{a \in A_N} Z_{N,a} = \frac{1}{N} \sum_{1 \leq l \leq \left( 1 + \frac{\ln N}{m} \right) m} |A_{N}^{(l)}| R_l \rightarrow c_{l} \sum_{l=1}^\infty (e^{-\rho_{\text{min}}(l)} - e^{-\rho_{\text{max}}(l)}) R_l = \sigma^2$$
where the last series converges absolutely in view of (6.1) and (6.2). Now, the first assertion of Theorem 6.2 follows since $R_1 > 0$ and $\rho_{\min}(l) < \rho_{\max}(l)$.

Next, in order to derive the second assertion we will need two following inequalities

\begin{equation}
\left| \frac{1}{N} \text{Var}(S_N) - \sigma^2 \right| \leq \frac{C}{N}(1 + \ln N)^{3m}, \tag{6.5}
\end{equation}

where $C > 0$ depends only on $b$ and $\ell$, and

\begin{equation}
\ell b^2 N \leq \text{Var}(S_N) \leq \ell^2 b^2 N \tag{6.6}
\end{equation}

where $b$ is from Assumption 6.1 and $c > 0$ is an absolute constant.

By (6.1)-(6.4) and the mean value theorem

\begin{align*}
\frac{1}{N} \sum_{1 \leq l \leq \lceil (1 + \frac{1}{m^2}) \ln N \rceil} |A_N^{(l)}| R_l - \sigma^2 | & \leq \\
C_1 \left( \sum_{l > \lceil (1 + \frac{1}{m^2}) \ln N \rceil + 1} I_l^2 \left( e^{-\rho_{\max}(l)} - e^{-\rho_{\min}(l)} \right) + \frac{1}{N} \sum_{1 \leq l \leq \lceil (1 + \frac{1}{m^2}) \ln N \rceil} I_l^2 \right) & \leq \\
C_2 \left( \sum_{l > \lceil (1 + \frac{1}{m^2}) \ln N \rceil + 1} e^{-\frac{l}{m}} - e^{-\frac{l - 1}{m}} \right) \ln(2) + \frac{(1 + \ln N)^{3m}}{N} & \leq \\
C_3 \left( \sum_{n = \lceil \ln m \rceil} \frac{n^3}{N} - \frac{2}{N^3} \right) + \frac{(1 + \ln N)^{3m}}{N} & \leq C_4 \frac{(1 + \ln N)^{3m}}{N}. 
\end{align*}

In order to obtain (6.6), observe that $|B_N(a)| = 1$ for any $a \in A_N$ with $a > \frac{N}{2}$, and so $\text{Var}(S_{N,a}) = b^2$. Thus,

$$\text{Var}(S_N) \geq \sum_{k \in A_N \cap \left( \frac{N}{2}, N \right]} b^2 = |A_N \cap \left( \frac{N}{2}, N \right]| b^2.$$ 

By the inclusion exclusion principle, $\lim_{N \to \infty} \frac{1}{N} A_N = c_\ell$ and the left hand side of (6.6) follows taking into account that $|A_N \cap \left( \frac{N}{2}, N \right]|$ is positive (since gcd$(n, n-1) = 1$ for any $n > 1$). Now, notice that for a given $n \in \mathbb{N}$ there are at most $\ell^2 m$’s such that

$$E(F(X(n), ..., X(\ell n))F(X(m), ..., X(\ell m)) \neq 0$$

while for any $n, m \in \mathbb{N}$ the Cauchy- Schwarz inequality implies that

$$\left| E(F(X(n), ..., X(\ell n))F(X(m), ..., X(\ell m)) \right| \leq b^2$$

and the right hand side of (6.6) follows.

Next, since $|x^{-\frac{1}{2}} - y^{-\frac{1}{2}}| = (xy)^{-\frac{1}{2}} (x^\frac{1}{2} + y^\frac{1}{2})^{-1} |x - y|$, (6.5) and (6.6) imply that

$$\left| \frac{1}{\sqrt{\text{Var}(S_N)}} - \frac{1}{N \sigma^2} \right| \leq \frac{1}{c b^2 \sigma} N^{-\frac{1}{2}} C (1 + \ln N)^{3m},$$

and so by the right hand side of (6.6)

\begin{equation}
\frac{S_N}{\sqrt{\text{Var}(S_N)}} - \frac{S_N}{\sqrt{\text{Var}(S_N)}} \leq \frac{1}{c b^2 \sigma} N^{-1} C \ell b (1 + \ln N)^{3m}. \tag{6.7}
\end{equation}

Now, in order to prove the last assertion of Theorem 6.2 first apply the assertion 4.1.1 from Chapter 4 of [1] which yields that with some absolute constant $A_1 > 0$,

$$d_K \left( \frac{S_N}{\sqrt{\text{Var}(S_N)}} , \mathcal{N}(0, 1) \right) \leq A_1 (\text{Var}(S_N))^{-\frac{1}{2}} \sum_{a \in A_n} E[|S_{N,a}|^3] \leq C b^3 \frac{(1 + \ln N)^{3m}}{e^2 N^{\frac{3}{2}}}.$$
In the last inequality above we used the fact that $|B_{N,k}| \leq C(1 + \ln(N))^m$ with some positive constant $C$ and the lower bound from (6.6). Now, Theorem 6.2 follows from the above estimate, Lemma 3.3 applied with $\alpha = 1$, (6.7) and the fact that
\[
d_K\left(\frac{S_N}{\sqrt{N}}, \mathcal{N}(0, \sigma^2)\right) = d_K\left(\frac{S_N}{\sigma \sqrt{N}}, \mathcal{N}(0, 1)\right).
\]

\[\square\]

6.2. Nonlinear functions $q_j$. Here we discuss the case $k < \ell$, where recall, $q_j(n) = jn$ for $j = 1, \ldots, k$ and $q_j(n+1) - q_j(n)$ and $q_j(\varepsilon n) - q_{j-1}(n)$ tend to $\infty$ as $n \to \infty$ whenever $\ell \geq j > k$ and $\varepsilon > 0$. First, observe that we can exclude the case when $F(x_1, \ldots, x_\ell) = G(x_1, \ldots, x_k)$ for some Borel function $G$ and $\mu^\ell = \mu \times \cdots \times \mu$ almost all $(x_1, \ldots, x_\ell)$ since then we arrive at the setup of Theorem 2.2. The above equality means that $F$ does not depend essentially on the variables $x_{k+1}, \ldots, x_\ell$ and this is equivalent to the condition that
\[
F_1 = 0 \mu^i - \text{almost surely (a.s.) for all } i = k + 1, \ldots, \ell.
\]

By Proposition 4.5 in [14], for any $i > k$,
\[
D_{i,i} = \int F_1^2(x_1, \ldots, x_i) d\mu^i(x_1, \ldots, x_i),
\]
and so if the above case is excluded then $D_{i,i} > 0$ for at least one $i > k$. This together with Theorem 2.2 yields that then $\sigma^2 > 0$ whence this question is settled here and it remains to deal only with Berry-Esseen type estimates in the present situation.

6.3. Theorem. Let $k < \ell$. Suppose that Assumption 2.2 is satisfied with some $\alpha, \lambda > 1$ and $b \geq 4$ and that there exists $1 > \gamma > 0$ such that $q_i([n^\gamma]) \geq q_{i-1}(n)$ and $q_i(n+1) - q_i(n) \geq n^\gamma$ for any $k < i \leq \ell$ and $n \in \mathbb{N}$. Assume that (6.8) does not hold true. Then for any $N \in \mathbb{N},$
\[
d_k(\mathcal{L}(\xi_N(1)), \mathcal{N}(0, \sigma^2)) \leq CRN^{-\frac{A}{2}(\gamma, \alpha, \lambda)}
\]
where $\theta(\gamma, \alpha, \lambda) = \min\left(\frac{1}{2}(1 - \gamma), \frac{\gamma \min(\alpha, \lambda) - 1}{4(\alpha + 2), \gamma \min(\alpha, \lambda)}, \frac{\lambda}{4(\lambda + 2)}\right)$, $D_{i,j}, 1 \leq i, j \leq \ell$ were introduced in Theorem 2.2, $D_{0,0} = \sigma_0 = \sum_{1 \leq i, j \leq k} \min(i, j) D_{i,j},$
\[
R = 1 + \max_{i \in \{0, k+1, \ldots, \ell\}: D_{i,i} > 0} D_{i,i}^{-1} \left(\max_{k < j < \ell: D_{j,j} > 0} \max(D_{j,j}^\uparrow, D_{j,j}^\downarrow)\right)
\]
and $C > 0$ depends only on the initial parameters and on the expressions (2.8)-(2.9).

As in Theorem 2.2 the main step in the proof of Theorem 6.3 is the construction of martingale approximations and their estimates. Still, unlike in the case $k = \ell$ we cannot provide here approximations of the whole process $\sqrt{n} \xi_n(1)$ by a single martingale. Thus, we will use separately the martingale approximation for $\sqrt{n} \xi_n(1), n \geq 1$ constructed in Section 5 and the martingale approximations of each $\sqrt{n} \xi_{i,n}, n \geq 1, i = k + 1, \ldots, \ell$ relying on Lemma 6.3 below.

For any $i = k + 1, \ldots, \ell$ and fixed $u, N \in \mathbb{N}$, we construct the martingales $M^u_{i,q_i}$, $n \geq 1$ with respect to the filtration $\mathcal{F}_{-\infty, q_i(n) + 2^n}$ and similarly to Section 3
\[
R_{i,q_i,v} = \sum_{s > v+1} E[Y_{i,q_i(s),2^n} | \mathcal{F}_{-\infty, q_i(v) + 2^n}]\]
\[ W_{i,q_i(n),2^n} = Y_{i,q_i(n),2^n} + R_{i,q_i(n),u} - R_{i,q_i(n-1),u}. \]

Let \( u(N) = \left[ \log_2(N) \right] + 1 \). Using techniques similar to Section 5 we obtain that for any \( N > L \) and \( i > k \),

\begin{align}
\frac{1}{\sqrt{N-L}} \left\| \sum_{n=L}^{N} Y_{i,q_i(n)} - ((M_i^u(N))_{N} - (M_i^u(N))_{L}) \right\|_b & \leq C\left( \frac{2^{u(N)}-1}{\sqrt{N-L}} + \sum_{r=2^{u(N)}}^{\infty} (\beta(q,r))^\delta \right) \\
& \leq C_1 \left( \frac{N-L}{\sqrt{N-L}} + N^{-\frac{\lambda}{2(q+r)}} \right)
\end{align}

where \( C_1 > 0 \) depends only on the initial parameters and on the expressions (2.8)–(2.9).

Next, observe that the proof of Lemma 6.2 also works for our setup and so,

\begin{equation}
\left\| \sum_{n=1}^{[N^*]} Y_{i,q_i(n)} \right\|_b \leq CN^{\frac{-4}{5}}
\end{equation}

where \( C > 0 \) depends only on the initial parameters and the expressions (2.8)–(2.9). Hence, applying Lemma 6.3 we can replace \( \sum_{n=1}^{N} Y_{i,q_i(n)} \) by \( \sum_{n=[N^*]}^{N} Y_{i,q_i(n)} \) with an error estimated by (6.10). Thus, for \( i > k \) and fixed \( N \) we consider the martingales \( (M_{i,N}) \) (with respect to the filtration \( \mathcal{F}_{-\infty,q_i(n+[N^*])}^{2^{u(N)})} \)) \( \Gamma \geq 1 \), where \( (M_{i,N})_{r} = (M_i^u(N))_{[N^*]+r} - (M_i^u(N))_{[N^*]} \) for \( N - [N^*] \geq r > 0 \). \( (M_{i,N})_{r} = M_{i,N-[N^*]} \) for \( r \geq N - [N^*] \). As in the proof of Theorem 4.3, quadratic variation estimates are crucial. Combining methods of Proposition 4.5 from [14] and Lemmas 5.3 and 5.4 above we obtain the following result.

6.4. **Lemma.** Suppose that Assumption 2.1 holds true with \( \alpha, \lambda > 1 \) and that there exists \( \gamma \) satisfying conditions of Theorem 6.3. Let \( k + 1 \leq i \leq \ell \) and \( N, u \in \mathbb{N} \) such that \( 2^u \leq N^{\frac{4}{5}} \). Set \( Z_n = Z_n^{i,u} = W_{i,q_i(n),2^n}^{2^n} \). Then

\[
\left\| \frac{1}{\sqrt{N}} \sum_{n=1}^{N} Z_n - D_{i,i} \right\|_2 \leq C\left(N^{-\tau(\alpha, \lambda, \gamma)} + 2^{\frac{u}{2}} + 2^u N^{-\frac{4}{5}} \right)
\]

where \( \tau(\alpha, \lambda, \gamma) = \frac{1}{2} \min(1 - \frac{\gamma}{2} \frac{\min(\alpha, \gamma - 1)}{2^{\gamma} \min(\alpha, \gamma)^{\frac{1}{2}}}) \) and \( C \) depends only on the initial parameters and on (2.8)–(2.9).

The use of approximations by several martingales as explained above works in the proof of Theorem 6.3 in view of the following result which is the main additional argument needed in comparison to the proof of Theorem 4.3. Let \( g_1(n) < \ldots < g_l(n) \) be positive and strictly increasing functions taking integer values. Set

\[
K_N = \max_{1 \leq j \leq l} \left( \min\left\{ 1 \leq m \leq n : g_j(m) > g_{j-1}(N) \right\} \right).
\]

6.5. **Lemma.** Let the \( \mathcal{F}_{n,i} \) be a nested family of \( \sigma \)-algebras (see Section 2). Let \( N \in \mathbb{N} \) and suppose that \( W^{(i)}, i = 1, \ldots, l \) is a martingale differences sequence with respect to the filtration \( \mathcal{F}_{-\infty,q_i(n+[K_{i,N}])} \) \( \Gamma \geq 1 \) where \( K_{1,N} = 0 \) and \( K_{i,N} = K_N \) if \( i > 1 \). Let \( M^{(i)} \) be the corresponding martingales. Suppose that \( \max\{ ||W^{(i)} n \leq N; i \leq l \} \leq C_1 2^n \) for some positive constant \( C_1 \) independent of \( N \) and \( u > 0 \) such that \( 2^u \leq N^{\frac{4}{5}} \) for some \( 0 < \zeta < \frac{1}{2} \). Let \( d_1, \ldots, d_l > 0 \) and assume that

\[
\frac{1}{\sqrt{N}} \max\{ ||M^{(i)} n \leq 2^n ; i = 1, \ldots, l \} \leq C_2
\]
and that \( \max\{A_{2,s} : s = 1, \ldots, l\} \leq C_3 N^{-\theta} \) for some \( 0 < \theta < 1 \), where

\[
A_{2,s} = \left\| \frac{1}{N} \sum_{n=1}^{N} \left( W_n^{(s)} \right)^2 - d_s^2 \right\|_2
\]

and \( C_2, C_3 > 0 \) are positive constants independent of \( N \). Let \( \eta_1, \ldots, \eta_l \) be independent and centered random variables which are normally distributed with variances \( d_i^2 > 0 \). Then

\[
d_K \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{l} M_N^{(i)}, \sum_{i=1}^{l} \eta_i \right) \leq A(1 + C_2)(1 + \left( \sum_{i=1}^{2} d_i^2 \right)^{-\frac{1}{4}}) DN^{-\frac{1}{8\theta} \min(56, 26(1 - 2\xi) - 8\phi)}
\]

where \( A > 0 \) is an absolute constant, \( D_s = \max(d_s^{-\frac{1}{4}}, d_s^{-\frac{1}{2}}) \) and \( D = \max\{D_s : 1 \leq s \leq l\} \).

**Proof.** First, observe that if \( Z_1, Z_2 \) and \( Z_3, Z_4 \) are pairs of independent random variables then

\[
(6.11) \quad d_K(Z_1 + Z_2, Z_3 + Z_4) \leq d_K(Z_1, Z_3) + d_K(Z_2, Z_4).
\]

By taking the product measure we can always assume that \( \{W_n^{(i)} , n \geq 1\} \) and \( \{\eta_j\}_{j=1}^{l} \) are defined on the same probability space and are independent from each other. For any \( s = 1, \ldots, l \) set

\[
Y^{(s)} = \frac{1}{\sqrt{N}} \sum_{i=1}^{s} M_N^{(i)} \quad \text{and} \quad \delta(s) = d_K(Y^{(s)}, \sum_{i=1}^{s} \eta_i).
\]

The main step of the proof is showing that for any \( 2 \leq s \leq l \) and \( U, L \in \mathbb{N} \)

\[
(6.12) \quad \delta(s) \leq C(1 + (\sum_{i=1}^{s} d_i^2)^{\frac{1}{2}})(\delta(s - 1) + AD_s((\frac{2^{uw}}{N})^{\frac{1}{8}} + (\frac{2^{uw}}{N})^{\frac{1}{4}}) + (A_{2,s})^{\frac{1}{2}}(UL)^{\frac{1}{2}} + (1 + \frac{1}{\sqrt{N}} \max_{1 \leq i \leq l} \| M_N^{(i)} \|_2)(\frac{1}{U} + \frac{1}{L})^{\frac{1}{2}})
\]

where \( C > 0 \) is an absolute constant.

Indeed, for any random variable \( Z \) and \( U, L \in \mathbb{N} \) set

\[
Z_{L,U} = \sum_{k=-LU}^{LU} \frac{k}{U^{\frac{1}{2}} \sqrt{\{L \leq k < L+U \text{ and } |Z| \leq L\}}}
\]

Observe that by the Hölder inequality, for any \( q > 1 \),

\[
(6.13) \quad \|Z - Z_{L,U}\|_1 \leq \frac{1}{U} + \|E[|Z|I_{\{|Z|>L\}}]\|_1 \leq \frac{1}{U} + \frac{E[|Z|^q]}{L^{q-1}}.
\]

In order to proceed we need some relations between probability metrics which can be found in [7]. Denote by \( d_P \) the Prokhorov metric on \( \mathbb{R} \) and by \( d_L \) the Levi metric on \( \mathbb{R} \) (see [7]). Then for any distribution functions \( F \) and \( G \),

\[
(6.14) \quad d_L(F, G) \leq d_K(F, G) \leq (1 + \sup_{x \in \mathbb{R}} |G'(x)|)d_L(F, G) \quad \text{and} \quad d_L(F, G) \leq d_P(F, G)
\]

where the right hand side of the first inequality holds true if \( G \) is differentiable.

Moreover, by the the Markov inequality and some standard estimates, one can show that for any random variables \( X \) and \( Y \) which are defined on the same probability space with distribution functions \( F \) and \( G \),

\[
(6.15) \quad d_P(F, G) = d_P(X, Y) \leq 2\|X - Y\|_1^{\frac{1}{2}}.
\]
Proceeding with the proof of (6.12) we observe that by (6.14),
\[\delta(s) = d_K(Y(s^{-1}) + \frac{1}{\sqrt{N}}M_N^{(s)}; \sum_{i=1}^{s} \eta_i) \leq (1 + (\sum_{i=1}^{s} d_i^2)^{-\frac{1}{2}})d_L(Y(s^{-1}) + \frac{1}{\sqrt{N}}M_N^{(s)}; \sum_{i=1}^{s} \eta_i).\]

By triangle inequality and then by (6.14) and (6.15),
\[d_L(Y(s^{-1}) + \frac{1}{\sqrt{N}}M_N^{(s)}; \sum_{i=1}^{s} \eta_i) \leq d_L(Y(s^{-1}) + \frac{1}{\sqrt{N}}M_N^{(s)}; Y_L,U) + d_L(Y(s^{-1}) + \frac{1}{\sqrt{N}}M_N^{(s)}; Y_L,U + \eta_s) + d_L(Y_L,U + \eta_s; Y(s^{-1}) + \eta_s) + \eta_s, \sum_{i=1}^{s} \eta_i \leq 4||Y(s^{-1}) - Y_L,U||_{L^2} + d_K(Y_L,U + \frac{1}{\sqrt{N}}M_N^{(s)}; Y_L,U + \eta_s) + d_K(Y(s^{-1}) + \eta_s, \sum_{i=1}^{s} \eta_i) = I_1 + I_2 + I_3.

By (6.11), since \(Y(s^{-1})\) and \(\eta_i, i = 1, \ldots, s\) are independent random variables, \(I_3 \leq \delta(s - 1)\). By (6.13) applied with \(q = 2\),
\[I_1 \leq 4(1 + \frac{1}{\sqrt{N}}) \max_{1 \leq i \leq l} ||M_N^{(j)}||_2(\frac{1}{U} + \frac{1}{L})^\frac{1}{2}.
\]

Next, for any measurable set \(A\) satisfying \(P(A) > 0\) let \(P_A = P(\cdot \mid A)\) be the corresponding conditional probability. For any probability measure \(\mu\) we denote the expectation with respect to it by \(E_\mu\). For any \(y \in \mathbb{R}\) set \(A_y = \{Y_L,U = y\}\) and \(1 = \{y : P(A_y) > 0\}\). Then, for any \(a \in \mathbb{R}\) taking into account that \(\Gamma\) is a finite set,
\[P(Y_L,U^{-1}) + \frac{1}{\sqrt{N}}M_N^{(s)} \leq a) = \sum_{y \in \Gamma} P(A_y)P_{A_y}(\frac{1}{\sqrt{N}}M_N^{(s)} \leq a - y).
\]
If \(A \in \mathcal{G}\) then \(E_{P_A}[Z \mid \mathcal{G}] = \mathbb{E}_A E_P[Z \mid \mathcal{G}]\) and hence \(\{M_{N}^{(s)}\}_{r \geq 1}\) is also a martingale with respect to the measure \(P_{A_y}\). Next, we apply (5.16) with \(\delta = 1, p = 2\) and use that \(E_{P_A}[Z] = \frac{1}{\mathbb{E}_{P_A}[Z \mid \mathcal{G}]}} E_P[Z \mid \mathcal{G}] \leq \frac{1}{\mathbb{E}_{P_A}[Z \mid \mathcal{G}]}} E_P[Z] \mid \mathcal{G})\) which yields
\[|P_{A_y}(\frac{1}{\sqrt{N}}M_N^{(s)} \leq a - y) - P_{\eta_s}(\eta_s \leq a - y)| \leq A'D_s(c^1 + c^1 + P_{A_y}A_{2,s,s}^2),
\]
where \(c = P^{-1}(A_y)2^{4u}N^{-2}\). Observe that cardinality(\(\Gamma\)) \(\leq 3L\). This together with (6.16) and (6.17) and the inequality \(\sum_{i=1}^{n} c_i^1 \leq n^{1-t}(\sum_{i=1}^{n} c_i)\) for any \(c_i \geq 0\) and \(0 \leq t \leq 1\) yields
\[I_2 \leq A'D_s(\frac{2^{4u}}{N^2}UL)^{\frac{1}{2}} + \frac{2^{4u}}{N^2}UL)^{\frac{1}{2}} + A_{2,s,s}(UL)^{\frac{1}{2}},
\]
where \(A' > 0\) is an absolute constant and (6.12) follows. Finally, applying (5.16) with the martingale \(M_N^{(1)}\), taking into consideration that \(2^u \leq \sqrt{N}\) we obtain
\[\delta(1) \leq A''D_1((\frac{2^{4u}}{N^2}UL)^{\frac{1}{2}} + A_{2,1}^2).
\]
for some absolute constant \(A'' > 0\). Making a repetitive use of (6.12) for \(s = 2, 3, \ldots, l\) yields
\[d_k(\frac{1}{\sqrt{N}}\sum_{i=1}^{l}M_N^{(i)}; \sum_{i=1}^{l} \eta_i) \leq A'D(1 + C)(1 + (\sum_{i=1}^{2} d_i^2)^{-\frac{1}{2}}((\frac{2^{4u}}{N^2}UL)^{\frac{1}{2}} + (\frac{2^{4u}}{N^2}UL)^{\frac{1}{2}} + (UL)^{\frac{1}{2}} + (UL)^{\frac{1}{2}})\sum_{i=1}^{d} (A_{2,i})^2 + (\frac{1}{U} + \frac{1}{L})^{\frac{1}{2}}).
\]
The lemma follows by taking $U = L = N^{\frac{1}{2}},$ recalling that $A_{2,n} \leq C' N^{-\theta}$ and $2^n \leq C'' N^\zeta$ (the power $\frac{1}{2}$ comes from taking $L = U = N^\nu$ and then comparing the powers of $N$ of the last two above summands). \hfill \square

In order to prove Theorem 6.3 we apply first Lemma 6.3 taking into consideration (6.9)-(6.10). Then we apply Lemmas 6.5 and 6.4 with the martingales $\tilde{M}_{i,N}$ for $i$ such that $D_{i,i} > 0$ (where $D_{0,0} = \sigma_0$) with $\zeta = \frac{1}{2(\lambda + 2)}$ and $\theta = \theta(\zeta, \gamma, \alpha, \lambda) < 2(1 - \zeta).

6.3. **Continuous time results.** Here we explain how to obtain convergence rates in the Levy-Prokhorov metric (\cite{2} Ch. 1, Sec. 6) in the case $k = \ell$ for the continuous time processes $\xi_N(t)$ defined in Section 2. Such results when $k < \ell$ will not be dealt with here since it is not clear how to adapt Lemma 6.5 for continuous time martingales, and so a different approach should be employed. It also possible to obtain such rates for the one dimensional processes $\xi_{i,N}(\cdot)$ for $i = k + 1, ..., \ell.$ First, relying on the H"{o}lder inequality for any random variables $\{X_i\}_{i \leq n},$

$$E(\max_{1 \leq i \leq n} |X_i|) \leq n^\frac{\zeta}{2} \max_{1 \leq i \leq n} \|X_i\|_q$$

we obtain by (5.10) that for any $1 \leq i \leq \ell$, the martingale approximation estimates in the form

$$E(\|M^{(n)}_{i,N}(t) - \xi_i,N(t)\|) \leq C\left(\frac{2^n}{\sqrt{N}}(M_i(NT))^{\frac{\delta}{2}} + T \sum_{n=2^{\alpha-1}}^\infty (\beta(q,n))^{\delta}\right)$$

where $\|f\| = \sup\{|f(t)| : t \in [0,T]\}$ and $M^{(n)}_{i,N}(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{[NT]} W_i \eta_i(R_{i,n}), 2^n$ with $R_{i,n}$ are the limits from Theorem 2.2. Next, we discuss how to obtain similar results for expressions of the form (6.8)-(6.9) and for $i \geq j$, $Z_n = W_{i,n,2^{k-1}}$, while for $i = j > k$, $Z_n = W_{i,q_i(n),2^n}$.

In order to prove Lemma 6.6 we have to improve somewhat Lemmas 5.3 and 5.4 obtaining similar results for expressions of the form $\|\sum_{n=1}^N (Z_n - E Z_n)\|_2$ with $z < N$ using the same technique and then applying Proposition 3 from \cite{16}. Finally, we can apply some Berry-Esseen type estimates (for instance, from \cite{9}) for continuous time martingales which will yield corresponding estimates in our setup.

6.4. **Integral type expressions.** Next, we discuss how to obtain similar results for expressions of the form

$$I_N(t) = \frac{1}{\sqrt{N}} \int_0^{NT} (F(X(q_1(n)), ..., X(q_k(n))) - \bar{F}) dt$$
where again \( q_i(n) = in \) for all \( i \). We introduce a reduction to the discrete time case where we can apply the technique used above for expressions (1.2). First, we represent again the function \( F \) in the form (2.11) and write

\[
\xi_{i,N}(t) = \frac{1}{\sqrt{N}} \int_0^{Nt/i} F_i(X(q_1(t)), ..., X(q_t(t))) dt.
\]

We will use below the same notations as in (3.3) with \( n \) replaced by \( t \) (see Section 6 in [14]). In order to apply our discrete time technique set \( \tilde{\xi}_{i,N}(t) = \sum_{n=1}^{[Nt/i]} J_i(n) \) where \( J_i(n) = \int_0^1 Y_i(q_i(n + s)) ds \). As in (6.2) from [14] applied with \( \delta = b - 2 \),

\[
P(\sup_{0 \leq t \leq T} |\xi_{i,N}(t) - \tilde{\xi}_{i,N}(t)| > \varepsilon) \leq \frac{C}{(\varepsilon \sqrt{N})^{b-2}}
\]

which by taking \( \varepsilon = \varepsilon_N = N^{-\left(\frac{b-4}{2} \frac{4}{b-2}\right)} \) bounds the Levi-Prokhorov and the Kolmogorov (uniform) distance between \( \tilde{\xi}_{i,N} \) and \( \xi_{i,N} \) by \( C \varepsilon_N \leq CN^{-\frac{b}{2}} \). We can approximate \( \tilde{\xi}_{i,N}(t) \) by

\[
\tilde{\xi}_{i,N,r}(t) = \sum_{n=1}^{[Nt/i]} J_i,r(n)
\]

where \( J_i,r(n) = \int_0^1 Y_{i,r}(q_i(n + s)) ds \) using an appropriate version of (5.5) (see Section 6 in [14]). As mentioned in [14] we will have an appropriate continuous time version of (5.5) with the expressions

\[
R_{i,r}(m) = \sum_{l=m+1}^{\infty} E(J_{i,r}(l)|\mathcal{F}_{-\infty,m+r})
\]

and the martingale differences \( Z_{i,r}(m) = J_{i,r}(m) + R_{i,r}(m) - R_{i,r}(m - 1) \). In order to extend the results of Section 5 to the present case we should have a continuous time version of Lemma 6.3. Such a version (adapted to our specific setup) follows directly from the observation that the bound from Lemma 6.3 depends only on the gaps between the sets \( M_i \) there and the initial parameters together with the facts that for \( T_j(s_j) = X(q_1(n_j + s_j)), ..., X(q_j(n_j + s_j)) \) and integrable functions \( G \) and \( G_i \), \( i = 1, ..., m \),

\[
E \int_{[0,1]^m} G(T_1(s_1), ..., T_k(s_k)) ds_1 ... ds_m = \int_{[0,1]^m} EG(T_1(s_1), ..., T_k(s_k)) ds_1 ... ds_m ,
\]

\[
E \prod_{j=1}^m \int_0^1 G_i(T_j(s_j)) ds_j = E \int_{[0,1]^m} \prod_{j=1}^m G_i(T_j(s_j)) ds_1 ... ds_m \quad \text{and}
\]

\[
\prod_{i=1}^m E[\int_0^1 X_i(s) ds] = E \prod_{i=1}^m \int_0^1 X_i(s) ds = E \int_{[0,1]^m} \prod_{i=1}^m X_i(s) ds_1 ... ds_m
\]

where \( X_1(\cdot), ..., X_k(\cdot) \) are independent random functions.

Using the technique from Section 5 with appropriate modifications we can prove a corresponding version of Proposition 6.1 which will yield some Berry-Esseen type convergence rates as above.

\[\square\]

**References**

[1] Z. Lin and Z. Bai, *Probability Inequalities*, Science Press and Springer-Verlag, Beijing and Heidelberg, 2010.

[2] P. Billingsley, *Convergence of Probability Measures*, 2nd ed. Wiley, New York, 1999.
[3] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Math. 470, Springer–Verlag, Berlin, 1975.
[4] R.C. Bradley, *Introduction to Strong Mixing Conditions*, Kendrick Press, Heber City, UT, 2007.
[5] F.E. Browder, *On the iteration of transformations in noncompact minimal dynamical systems*, Proc. Amer. Math. Soc. 9 (1958), 773-780.
[6] B. Courbot, *Rates of convergence in the functional CLT for multidimensional continuous time martingale*, Stoch. Proc. Appl. 91 (2001), 57-76.
[7] A. L. Gibbs and F. E. Su, *On choosing and bounding probability metrics*, Int. Stat. Rev., 70 (2002), 419–435.
[8] H. Furstenberg, *Nonconventional ergodic averages*, Proc. Symp. Pure Math. 50 (1990), 43-56.
[9] P. Hall and C.C. Heyde, *Rates of Convergence in the Martingale Central Limit Theorem*, Ann. Probab. 9 (1981), 395-404.
[10] Y. Hafouta and Y. Kifer, *A nonconventional local limit theorem*, arXiv: 1407.0143.
[11] I.A. Ibragimov and Y.V. Linnik, *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff, Groningen, 1971.
[12] Yu. Kifer, *Nonconventional limit theorems*, Probab. Th. Rel. Fields, 148 (2010), 71-106.
[13] Yu. Kifer, *Strong approximations for nonconventional sums and almost sure limit theorems*, Stoch. Proc. Appl. 123 (2013), 2286-2302.
[14] Yu. Kifer and S. R.S Varadhan, *Nonconventional limit theorems in discrete and continuous time via martingales*, Ann. Probab. 42 (2014), 649-688.
[15] Yu. Kifer and S. R.S Varadhan, *Nonconventional large deviations theorem*, Th. Rel. Fields, 158 (2014), 197-224.
[16] F. Merlevède, M. Peligrad and S. Utev, *Recent advances in invariance principles for stationary sequences*, Probability Surveys 3 (2006), 1-36.
[17] E. Rio, *Sur le théorème de Berry-Esseen pour les suites faiblement dépendantes*, Probab. Th. Relat. Fields 104 (1996), 255-282.
[18] Y. Rinott and V. Rotar, *Some bounds on the rate of convergence in the CLT for martingales*, Theory Probab. Appl. I, 43 (1998), 604-619; II, 44 (1999), 523-536.
[19] N. Shiryaev, *Probability*, Springer-Verlag, Berlin, 1995.

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