SEMISTAR-KRULL AND VALUATIVE DIMENSION OF
INTEGRAL DOMAINS

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ABSTRACT. Given a stable semistar operation of finite type $\star$ on an integral domain $D$, we show that it is possible to define in a canonical way a stable semistar operation of finite type $\star[X]$ on the polynomial ring $D[X]$, such that, if $n := \text{-dim}(D)$, then $n + 1 \leq \star[X]\text{-dim}(D[X]) \leq 2n + 1$. We also establish that if $D$ is a $\star$-Noetherian domain or is a Prüfer $\star$-multiplication domain, then $\star[X]\text{-dim}(D[X]) = \text{-dim}(D) + 1$. Moreover we define the semistar valuative dimension of the domain $D$, denoted by $\text{-dim}_v(D)$, to be the maximal rank of the $\star$-valuation overrings of $D$. We show that $\star\text{-dim}_v(D) = n$ if and only if $\star[X_1, \ldots, X_n]\text{-dim}_v(D[X_1, \ldots, X_n]) = 2n$, and that if $\star\text{-dim}_v(D) < \infty$ then $\star[X]\text{-dim}_v(D[X]) = \star\text{-dim}_v(D) + 1$. In general $\star\text{-dim}(D) \leq \star\text{-dim}_v(D)$ and equality holds if $D$ is a $\star$-Noetherian domain or is a Prüfer $\star$-multiplication domain. We define the $\star$-Jaffard domains as domains $D$ such that $\star\text{-dim}(D) < \infty$ and $\star\text{-dim}(D) = \star\text{-dim}_v(D)$. As an application, $\star$-quasi-Prüfer domains are characterized as domains $D$ such that each $(\star, \star)$-linked overring $T$ of $D$, is a $\star'$-Jaffard domain, where $\star'$ is a stable semistar operation of finite type on $T$. As a consequence of this result we obtain that a Krull domain $D$, must be a $\text{w}_D$-Jaffard domain.

1. INTRODUCTION

Throughout this paper, $D$ denotes a (commutative integral) domain with identity and $K$ denotes the quotient field of $D$. Let $X$ be an algebraically independent indeterminate over $D$. Seidenberg proved in [35] Theorem 2], that if $D$ has finite Krull dimension, then

$$\text{dim}(D) + 1 \leq \text{dim}(D[X]) \leq 2(\text{dim}(D)) + 1.$$ 

Moreover, Krull [27] has shown that if $D$ is any finite-dimensional Noetherian ring, then $\text{dim}(D[X]) = 1 + \text{dim}(D)$ (cf. also [35] Theorem 9]). Seidenberg subsequently proved the same equality in case $D$ is any finite-dimensional Prüfer domain. To unify and extend such results on Krull-dimension, Jaffard [23] introduced and studied the valuative dimension denoted by $\text{dim}_v(D)$, for a domain $D$. This is the maximum of the ranks of the valuation overrings of $D$. Jaffard proved in [23] Chapitre IV (see also Arnold [2]), that if $D$ has finite valuative dimension, then $\text{dim}_v(D[X]) = 1 + \text{dim}_v(D)$ and that if $D$ is a Noetherian or a Prüfer domain, then $\text{dim}(D) = \text{dim}_v(D)$. Also he showed that $\text{dim}_v(D) = n$ if and only if $\text{dim}(D[X_1, \ldots, X_n]) = 2n$, where $X_1, \ldots, X_n$ are indeterminates over $D$. In [1] the authors introduced the notion of Jaffard domains, as integral domains $D$ such that $\text{dim}(D) = \text{dim}_v(D)$. The class of Jaffard domains contains most of the

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well-known classes of finite dimensional rings involved in dimension theory of commutative rings, such as Noetherian domains, Prüfer domains, universally catenary domains \cite{41}, and stably strong S-domains \cite{28} \cite{24}. A good and available reference for the dimension theory of commutative rings is Gilmer \cite[Section 30]{17}.

For several decades, star operations, as described in \cite[Section 32]{17}, have proven to be an essential tool in multiplicative ideal theory, for studying various classes of domains. In \cite{30}, Okabe and Matsuda introduced the concept of a semistar operation to extend the notion of a star operation. Since then, semistar operations have been extensively studied and, because of a greater flexibility than star operations, have permitted a finer study and new classifications of special classes of integral domains. For instance, semistar-theoretic analogues of the classical notions of Krull dimension, Noetherian and Prüfer domains have been introduced: see \cite{10} and \cite{13} for the basics on \star-Krull dimension, \star-Noetherian domains and Prüfer \star-multiplication domains (for short P\star MD), respectively.

Now it is natural to ask:

**Question 1.1.** Given a semistar operation of finite type \star on \(D\), is it possible to define in a canonical way a semistar operation of finite type \(\star[X]\) on \(D[X]\), such that \(\star\cdot\dim(D) + 1 \leq \star[X]\cdot\dim(D[X]) \leq 2(\star\cdot\dim(D)) + 1\), and that if \(D\) is a \star-Noetherian domain or a P\star MD, then \(\star[X]\cdot\dim(D[X]) = \star\cdot\dim(D) + 1\)?

In this paper, we answer this question, in case that \(\star\) is a stable semistar operation of finite type on \(D\). More precisely, in Section 2, using the technique introduced by Chang and Fontana in \cite{6}, we define in a canonical way a semistar operation stable and of finite type \(\star[X]\) on \(D[X]\): see Theorem 2.1. In Section 3 we show among other things that this question has an affirmative answer: see Theorems 3.1, 3.2, and 3.3.

Let \(\star\) be a semistar operation on the integral domain \(D\) and let \(\tilde{\star}\) be the stable semistar operation of finite type canonically associated to \(\star\) (the definitions are recalled later in this section). We define in Section 4, what it means the semistar valuative dimension of \(D\), denoted by \(\star\cdot\dim_{\nu}(D)\). It extends the "classical" valuative dimension of P. Jaffard \cite{23}, denoted by \(\dim_{\nu}(D)\) to the setting of semistar operations. We show that the semistar valuative dimension of \(D\) has various nice properties, like the classical valuative dimension. For example we show that if \(\tilde{\star}\cdot\dim_{\nu}(D) < \infty\) then \(\star[X]\cdot\dim_{\nu}(D[X]) = \tilde{\star}\cdot\dim_{\nu}(D) + 1\): see Theorem 4.5. Also we established that \(\tilde{\star}\cdot\dim(D) \leq \tilde{\star}\cdot\dim_{\nu}(D)\), and equality holds if \(D\) is a \(\tilde{\star}\)-Noetherian domain or a P\star MD; see Corollaries 4.6 and 4.8. In relation with the \(\star\)-Nagata ring \(\text{Na}(D, \star)\), it is shown that \(\tilde{\star}\cdot\dim_{\nu}(D) = \dim_{\nu}(\text{Na}(D, \star))\): see Theorem 4.17. If \(\tilde{\star}\cdot\dim(D) < \infty\) and \(\tilde{\star}\cdot\dim(D) = \tilde{\star}\cdot\dim_{\nu}(D)\), we say that, \(D\) is a \(\tilde{\star}\)-Jaffard domain.

We establish that \(D\) is a \(\tilde{\star}\)-quasi-Prüfer domain if and only if each \((\star, \star')\)-linked overring \(T\) of \(D\) is a \(\tilde{\star}'\)-Jaffard domain, where \(\star'\) is a semistar operation on \(T\): see Theorem 4.14. As a consequence of this result we obtain that a Krull domain \(D\), must be a \(w_{D}\)-Jaffard domain.

To facilitate the reading of the introduction and of the paper, we first review some basic facts on semistar operations. Let \(\mathcal{F}(D)\) denote the set of all nonzero \(D\)-submodules of \(K\). Let \(\mathcal{F}(D)\) be the set of all nonzero fractional ideals of \(D\); i.e., \(E \in \mathcal{F}(D)\) if \(E \in \mathcal{F}(D)\) and there exists a nonzero element \(r \in D\) with \(rE \subseteq D\). Let \(f(D)\) be the set of all nonzero finitely generated fractional ideals of \(D\). Obviously, \(f(D) \subseteq \mathcal{F}(D) \subseteq \mathcal{F}(D)\). As in \cite{30}, a semistar operation on \(D\) is
a map \( \star : \mathcal{F}(D) \to \mathcal{F}(D) \), \( E \mapsto E^\star \), such that, for all \( x \in K, x \neq 0 \), and for all \( E, F \in \mathcal{F}(D) \), the following three properties hold:

\[
\begin{align*}
\star_1 : & \quad (xE)^\star = xE^\star; \\
\star_2 : & \quad E \subseteq F \text{ implies that } E^\star \subseteq F^\star; \\
\star_3 : & \quad E \subseteq E^\star \text{ and } E^{\star\star} := (E^\star)^\star = E^\star.
\end{align*}
\]

Recall from [30 Proposition 5] that if \( \star \) is a semistar operation on \( D \), then, for all \( E, F \in \mathcal{F}(D) \), the following basic formulas follow easily from the above axioms:

\[
\begin{align*}
(1) \quad (EF)^\star = (E^\star F^\star) = (E^\star F^\star)^\star; \\
(2) \quad (E + F)^\star = (E^\star + F^\star) = (E^\star + F^\star)^\star; \\
(3) \quad (E : F)^\star \subseteq (E^\star : F^\star) = (E^\star : F) = (E^\star : F)^\star, \text{ if } (E : F) \neq (0); \\
(4) \quad (E \cap F)^\star \subseteq E^\star \cap F^\star = (E^\star \cap F^\star)^\star \text{ if } (E \cap F) \neq (0).
\end{align*}
\]

It is convenient to say that a (semi)star operation on \( D \) is a semistar operation which, when restricted to \( \mathcal{F}(D) \), is a star operation (in the sense of [17 Section 32]). It is easy to see that a semistar operation \( \star \) on \( D \) is a (semi)star operation on \( D \) if and only if \( D^\star = D \).

Let \( \star \) be a semistar operation on the domain \( D \). For every \( E \in \mathcal{F}(D) \), put \( E^\star := \cup F^\star \), where the union is taken over all finitely generated \( F \in f(D) \) with \( F \subseteq E \). It is easy to see that \( \star \) is a semistar operation on \( D \), and \( \star \) is called the semistar operation of \( \star \). Note that \( (*f)_f = \star_f \). A semistar operation \( \star \) is said to be of \( \text{finite type} \) if \( \star \) is of \( \text{finite type} \) and, in particular \( \star_f \) is of \( \text{finite type} \).

We say that a nonzero ideal \( I \) of \( D \) is a \( \text{quasi-}\star\)-ideal of \( I \), if \( I \) is an \( \text{prime quasi-}\star\)-ideal of \( D \); and a \( \text{quasi-}\star\)-maximal ideal of \( D \), if \( I \) is maximal in the set of all proper \( \text{quasi-}\star\)-ideals of \( D \). Each \( \text{quasi-}\star\)-maximal ideal is a prime ideal. It was shown in [11 Lemma 4.20] that if \( D^\star \neq K \), then each proper \( \text{quasi-}\star_f\)-ideal of \( D \) is contained in a \( \text{quasi-}\star_f\)-maximal ideal of \( D \). We denote by \( \text{QMax}^\star(D) \) (resp., \( \text{QSpec}^\star(D) \)) the set of all \( \text{quasi-}\star\)-maximal ideals (resp., \( \text{quasi-}\star\)-prime ideals) of \( D \). When \( \star \) is a (semi)star operation, it is easy to see that the notion of \( \text{quasi-}\star\)-ideal is equivalent to the classical notion of \( \star\)-ideal (i.e., a nonzero ideal \( I \) of \( D \) such that \( I^\star = I \)).

If \( \star_1 \) and \( \star_2 \) are semistar operations on \( D \), one says that \( \star_1 \leq \star_2 \) if \( E^{\star_1} \subseteq E^{\star_2} \) for each \( E \in \mathcal{F}(D) \) (cf. [30 page 6]). This is equivalent to saying that \( (E^\star)^{\star_2} = E^{\star_2} = (E^{\star_1})^{\star_1} \) for each \( E \in \mathcal{F}(D) \) (cf. [30 Lemma 16]). Obviously, for each semistar operation \( \star \) on \( D \), we have \( \star_f \leq \star \). Let \( d_D \) (or, simply, \( d \)) denote the identity (semi)star operation on \( D \). Clearly, \( d \leq \star \) for all semistar operations \( \star \) on \( D \).

If \( \Delta \) is a set of prime ideals of a domain \( D \), then there is an associated semistar operation on \( D \), denoted by \( \star_\Delta \), defined as follows:

\[
E^{\star_\Delta} := \cap \{ED_P | P \in \Delta \}, \text{ for each } E \in \mathcal{F}(D).
\]

If \( \Delta = \emptyset \), let \( E^{\star_\Delta} := K \) for each \( E \in \mathcal{F}(D) \). Note that \( E^{\star_\Delta} D_P = ED_P \) for each \( E \in \mathcal{F}(D) \) and \( P \in \Delta \) by [11 Lemma 4.1 (2)]. One calls \( \star_\Delta \) the spectral semistar operation associated to \( \Delta \). A semistar operation \( \star \) on a domain \( D \) is called a spectral semistar operation if there exists a subset \( \Delta \) of the prime spectrum of \( D \), Spec\( (D) \), such that \( \star = \star_\Delta \). When \( \Delta := \text{QMax}^\star(D) \), we set \( \hat{\star} := \star_\Delta \); i.e.,

\[
E^{\hat{\star}} := \cap \{ED_P | P \in \text{QMax}^\star(D) \}, \text{ for each } E \in \mathcal{F}(D).
\]

It has become standard to say that a semistar operation \( \star \) is stable if \( (E \cap F)^\star = E^\star \cap F^\star \) for all \( E, F \in \mathcal{F}(D) \). (“Stable” has replaced the earlier usage, “quotient”, in
All spectral semistar operations are stable \[11\] Lemma 4.1(3)]. In particular, for any semistar operation \(*\), we have that \(\tilde{*}\) is a stable semistar operation of finite type \[11\] Corollary 3.9].

Let \(D\) be a domain, \(\star\) a semistar operation on \(D\), \(T\) an overring of \(D\), and \(\iota : D \rightarrow T\) the corresponding inclusion map. In a canonical way, one can define an associated semistar operation \(\star \iota\), on \(T\), by \(E \rightarrow E^\star \iota := E^\star\), for each \(E \in \mathcal{F}(T)(\subseteq \mathcal{F}(D))\).

The most widely studied (semi)star operations on \(D\) have been the identity \(d_D\) and \(v_D\), \(d_D := (v_D)_{\iota}\), and \(v_D := \tilde{v}_D\) operations, where \(E^{v_D} := (E^{-1})^{-1}\), with \(E^{-1} := (D : E) := \{x \in K | xE \subseteq D\}\).

Let \(D\) be a domain with quotient field \(K\), and let \(X\) be an indeterminate over \(K\). For each \(f \in K[X]\), we let \(c_D(f)\) denote the content of the polynomial \(f\), i.e., the (fractional) ideal of \(D\) generated by the coefficients of \(f\). Let \(\star\) be a semistar operation on \(D\). If \(N_\star := \{g \in D[X]| g \neq 0, \text{ and } c_D(g)^* = D^*\}\), then \(N_\star = D[X]\setminus \bigcup \{P[X]|P \in \text{QMax}^\star(D)\}\) is a saturated multiplicative subset of \(D[X]\). The ring of fractions \(D[X]_{N_\star}\) is called the \(\star\)-Nagata domain \((\text{of } D \text{ with respect to the semistar operation } \star)\). When \(\star = d\), the identity (semi)star operation on \(D\), then \(D(D, d)\) coincides with the classical Nagata domain \(D(X)\) (as in, for instance \[29\] page 18, \[17\] Section 33] and \[14\]).

2. Semistar operations on polynomial rings

In \[6\], Chang and Fontana introduced a new technique for defining new semistar operations on integral domains. Let \(D\) be an integral domain with quotient field \(K\), and let \(X\) be an indeterminate over \(K\). For a given multiplicative subset \(S\) of \(D[X]\), set

\[E^\bigcirc_S := E[X]_{S} \cap K, \text{ for all } E \in \mathcal{F}(D).\]

Then it is proved in \[6\] Theorem 2.1] among other things that, the mapping \(\bigcirc_S : \mathcal{F}(D) \rightarrow \mathcal{F}(D), E \mapsto E^\bigcirc_S\) is a stable semistar operation of finite type on \(D[X]\), i.e., \(\bigcirc_S = \bigcirc_S\), and \(\text{QMax}^{\bigcirc_S}(D) = \text{the set of maximal elements of } \Delta(S) := \{P \in \text{Spec}(D) | P[X] \cap S = \emptyset\}\).

Let \(D\) be an integral domain, and \(\star\) a semistar operation on \(D\). Using the technique discussed in the first paragraph, Chang and Fontana defined canonically a semistar operation denoted by \(\star\) on the polynomial ring \(D[X]\). More precisely suppose that \(X, Y\) are two indeterminates over \(D\), and set \(D_1 := D[X], K_1 := K(X)\). Take the following subset of \(\text{Spec}(D_1)\):

\[\Delta_1^\star := \{Q_1 \in \text{Spec}(D_1) | Q_1 \cap D = (0) \text{ or } Q_1 = (Q_1 \cap D)[X] \text{ and } (Q_1 \cap D)^* \subseteq D^*\}.

Set \(S_1^\star := S(\Delta_1^\star) := D_1[Y]\setminus \{Q_1[Y]|Q_1 \in \Delta_1^\star\}\) and \([\star] := \bigcirc_{S_1^\star}\), that is:

\[E^\star := E[Y]_{S_1^\star} \cap K_1, \text{ for all } E \in \mathcal{F}(D_1).\]

They proved answering their question \[71\] Question], that \(D\) is a \(\tilde{*}\)-quasi-Prüfer domain if and only if each upper to zero, is a quasi-[\(\star]\]-maximal ideal of \(D[X]\). Recall that \(D\) is said to be a \(\star\)-quasi-Prüfer domain, in case, if \(Q\) is a prime ideal in \(D[X]\), and \(Q \subseteq P[X]\), for some \(P \in \text{QSpec}^\star(D)\), then \(Q = (Q \cap D)[X]\). This notion is the semistar analogue of the classical notion of the quasi-Prüfer domains.
Section 6.5] (that is among other equivalent conditions, the domain \(D\) is said to be a quasi-Prüfer domain if it has Prüferian integral closure).

Now by the same technique, we define canonically a semistar operation denoted by \(\star[X]\) on the polynomial ring \(D[X]\), which has desired semistar (Krull) dimension theoretic properties.

**Theorem 2.1.** Let \(D\) be an integral domain with quotient field \(K\), let \(X, Y\) be two indeterminates over \(D\) and let \(\star\) be a semistar operation on \(D\). Set \(D_1 := D[\{X\}]\), \(K_1 := K(X)\) and take the following subset of \(\text{Spec}(D_1)\):

\[
\Theta_1^* := \{Q_1 \in \text{Spec}(D_1) \mid Q_1 \cap D = (0) \text{ or } (Q_1 \cap D)^{\star} \subseteq D^*\}
\]

Set \(\mathcal{S}_1^* := S(\Theta_1^*) := D_1[Y]\backslash \bigcup \{Q_1[Y] \mid Q_1 \in \Theta_1^*\}\) and:

\[
E^\circ \Theta_1^* := E[Y]_\Theta_1^* \cap K_1, \text{ for all } E \in \mathcal{F}(D_1).
\]

(a) The mapping \(\star[X] := \circ \Theta_1^* : \mathcal{F}(D_1) \to \mathcal{F}(D_1), E \mapsto E^\circ \Theta_1^*\) is a stable semistar operation of finite type on \(D[X]\), i.e., \(\star[X] = \star[X]\).

(b) \(\mathcal{F}[X] = \star f[X] = \star[X]\).

(c) \(\star[X] \subseteq [\star]\). In particular, if \(\star\) is a (semi)star operation on \(D\), then \(\star[X]\) is a (semi)star operation on \(D[X]\).

(d) \(d_D[X] = d_{D[X]}\).

**Proof.** Note that, if \(Q_1 \in \text{Spec}(D[X])\) is not an upper to zero and \((Q_1 \cap D)^{\star} \subseteq D^*\), then the prime ideal \(Q_1 \cap D\) is contained in a quasi-\(\star\)-f maximal ideal of \(D\). Moreover if \(Q_1 \cap D = (0)\) and \(c_D(Q_1)^{\star} \subseteq D^*\) then \(c_D(Q_1)^{\star}\) is contained in a quasi-\(\star\)-f prime ideal of \(D\) and hence \(Q_1 \subseteq P[X]\) with \(P^\star \subseteq D^*\). Set \(\Theta_1^* := \{Q_1 \in \text{Spec}(D_1) \mid Q_1 \cap D = (0)\) and \(\Theta_1^* = D^*\) or \(Q_1 \cap D \in \text{QMax}^\star(D)\}. \) Now we show that:

\[
\mathcal{S}_1^* := D_1[Y]\backslash \bigcup \{Q_1[Y] \mid Q_1 \in \Theta_1^*\} = D_1[Y]\backslash \bigcup \{Q_1[Y] \mid Q_1 \in \Theta_1^*\} = S(\Theta_1^*).
\]

Since \(\Theta_1^* \subseteq \Theta_1^*\) one has \(\mathcal{S}_1^* \subseteq S(\Theta_1^*).\) For the other inclusion suppose that \(f \in S(\Theta_1^*).\) So that \(f \notin Q_1[Y]\) for each \(Q_1 \in \Theta_1^*\). We want to show that \(f \notin \mathcal{S}_1^*\).

Suppose the contrary, hence \(f \in Q_1[Y]\) for some \(Q_1 \in \Theta_1^*.\) Therefore there are two cases to consider:

1) If \(Q_1 \cap D = (0)\), we have \(c_D(Q_1)^{\star} \neq D^*\) as \(Q_1 \notin \Theta_1^*\). Thus \(Q_1 \subseteq P[X]\) for some quasi-\(\star\)-f prime ideal \(P\) of \(D\). Choose \(M \in \text{QMax}^\star(D)\) such that \(P \subseteq M\). So that \(Q_1 \subseteq M[X]\) and hence \(f \in M[X][Y]\) while \(M[X] \in \Theta_1^*\), which is a contradiction.

2) If \((Q_1 \cap D)^{\star} \subseteq D^*\), then \(Q_1 \cap D \subseteq M\) for some quasi-\(\star\)-f maximal ideal of \(D\). We have \(Q_1 \cap D \neq M\), since otherwise \(Q_1 \in \Theta_1^*\) and \(f \notin \Theta_1^*\) which is a contradiction. Note that \((Q_1 + M[X]) \cap D = (M[X] + (X)) \cap D = M\) and \(Q_1 \subseteq Q_1 + M[X] \subseteq M[X] + (X)\). Therefore \(f \in (M[X] + (X))[Y]\) while \(M[X] + (X) \in \Theta_1^*\), which is again a contradiction.

So that we have \(f \notin Q_1[Y]\) for each \(Q_1 \in \Theta_1^*\). Thus \(f \in \mathcal{S}_1^*\), that is \(\mathcal{S}_1^* = S(\Theta_1^*).\)

(a) It follows from [\(\Theta_1^* \subseteq \Theta_1^*\), that \(\star[X]\) is a stable semistar operation of finite type on \(D[X]\).

(b) Since \(\text{QMax}^\star(D) = \text{QMax}^\star(D)\), the conclusion follows easily from the fact that \(\mathcal{S}_1^* = \mathcal{S}_1^* = \mathcal{S}_1^*\).

(c) It is easily seen that \(\mathcal{S}_1^* \subseteq \mathcal{S}_1^*\). Then

\[
E^\star[X] = E[Y]_\Theta_1^* \cap K_1 \subseteq (E[Y]_\Theta_1^* \cap K_1 \cap K_1 = E[Y]_{\Theta_1^*} \cap K_1 = E^\star[X].
\]
This means that $\star [X] \leq [\star]$ by definition. Now if $\star$ is a (semi)star operation on $D$, then $[\star]$ is a (semi)star operation on $D[X]$ by [6] Theorem 2.3 (a)]. So that $D_1 \subseteq D^\star_1[X] \subseteq D_1[\star] = D_1$, that is $D_1^\star[X] = D_1$. Hence $\star [X]$ is a (semi)star operation on $D[X]$. 

(d) Note that we have:

$$E_1^{d_D} = D_1[Y] \setminus \left( \bigcup \{Q_1[Y] | Q_1 \in \Theta_1^{d_D} \} \right) = D_1[Y] \setminus \left( \bigcup \{Q_1[Y] | Q_1 \in \text{Spec}(D_1) \text{ and } Q_1 \cap D \neq D \} \right) = D_1[Y] \setminus \left( \bigcup \{Q_1[Y] | Q_1 \in \text{Max}(D_1) \} \right).$$

So for an element $E \in \mathcal{F}(D_1)$ we have:

$$E = E^{d_D[X]} \subseteq E^{d_D} \cap K_1 = E^D_1(Y) \cap K_1 = E.$$

The last equality follows from [14] Proposition 3.4 (3)]. Thus $E^{d_D[X]} = E^{d_D} [X]$, that is $d_D[X] = d_D [X]$. \hfill \qedsymbol

A different approach to the semistar operations on polynomial rings is possible by using the notion of localizing system. Recall that a localizing system of ideals $\mathcal{F}$ of $D$ is a set of (integral) ideals of $D$ verifying the following conditions (a) if $I \in \mathcal{F}$ and if $I \subseteq J$, then $J \in \mathcal{F}$; (b) if $I \in \mathcal{F}$ and if $J$ is an ideal of $D$ such that $(J :_D iD) \in \mathcal{F}$, for each $i \in I$, then $J \in \mathcal{F}$. The relation between stable semistar operations and localizing systems has been deeply investigated by M. Fontana and J. Huckaba in [11] and by F. Halter-Koch in the context of module systems [25]. If $\star$ is a semistar operation on $D$, then $\mathcal{F} := \{I \text{ ideal of } D | I^\star = D^\star \}$ is a localizing system (called the localizing system associated to $\star$) of $D$. And if $\mathcal{F}$ is a localizing system of $D$, then the map $E \mapsto E^\mathcal{F} := \bigcup \{E : J | J \in \mathcal{F} \}$, for each $E \in \mathcal{F}(D)$, is a stable semistar operation on $D$. It is proved in [32] Proposition 3.1] that if $\mathcal{F}$ is a localizing system of $D$, then $\mathcal{F}[X] := \{A \text{ ideal of } D[X] | A \cap D \in \mathcal{F} \}$ is a localizing system of the polynomial ring $D[X]$. Now let $\star$ be a stable semistar operation on $D$ and let $\mathcal{F} \star$ be the localizing system of $D$ associated to $\star$. Consider the localizing system $\mathcal{F} \star [X]$ of $D[X]$. Then G. Picozza [32] Page 426] introduced a semistar operation denoted by $\star'$ on the polynomial ring $D[X]$ as $\star_{\mathcal{F} \star [X]}$. He used the semistar operation $\star'$ to provide the semistar version of the Hilbert basis Theorem 32 Theorem 3.3).

**Proposition 2.2.** If $\star$ is a stable semistar operation of finite type on $D$, that is if, $\star = \star'$ then $\star' = \star [X]$.\hfill \qedsymbol

**Proof.** Adapt the notation in the paragraph before the proposition. Recall from [6] Corollary 2.2] that if $\mathcal{F}$ is a localizing system of $D$, $Y$ is an indeterminate over $D$, and $S(\mathcal{F}) := D[Y] \setminus \bigcup \{Q[Y] | Q \in \text{Spec}(D) \text{ and } Q \notin \mathcal{F} \}$ which is a saturated multiplicatively closed subset of $D[Y]$, then $\star_{\mathcal{F}} = \cap_{S(\mathcal{F})}$. Now let $\mathcal{F} := \mathcal{F}^\star [X] = \cap_{S(\mathcal{F})}$.
A ideal of $D_1|(A \cap D)^* = D^*$}. Then
\[
S(F) = D_1[Y] \bigcup \{Q_1[Y] | Q_1 \in \text{Spec}(D_1) \text{ and } Q_1 \notin F \}
\]
\[
= D_1[Y] \bigcup \{Q_1[Y] | Q_1 \in \text{Spec}(D_1) \text{ s.t. } Q_1 \cap D = (0) \text{ or } (Q_1 \cap D)^* \subsetneq D^* \}
\]
\[
= D_1[Y] \bigcup \{Q_1[Y] | Q_1 \in \Theta_1^* \}
\]
\[
= \Theta_1^*.
\]
Consequently $\star' = \star_\mathcal{F} = \bigcirc_{S(F)} = \bigcirc_{\Theta_1} = \star[X]$ which ends the proof. \hfill \Box

Note that the semistar operation $\star$ has a main difference with $\star[X]$ and $\star'$. Indeed let $\star = d_D$. Then one has $d'_D = d_D[X] = d_{D[X]}$ by Theorem 2.1 (d) and Proposition 2.2. But $[d_D] \neq d_D[X]$. Since if $[d_D] = d_D[X]$; then [7, Corollary 2.5 (1)] implies that if $D$ is a Prüfer domain then $D[X]$ should be a Prüfer domain which is absurd.

Remark 2.3. Note that the set of quasi-$\star[X]$-prime ideals of $D[X]$, coincides with the set $\Theta_1^* \setminus \{0\}$. Indeed let $Q$ be an element of $\Theta_1^* \setminus \{0\}$. Then we have $Q[Y] \cap \Theta_1^* = 0$. Hence
\[
Q^{\star[X]} \cap D[X] = (Q[Y]_{\Theta_1} \cap K(X)) \cap D[X]
\]
\[
= (Q[Y]_{\Theta_1} \cap D[X,Y]) \cap D[X]
\]
\[
= Q[Y] \cap D[X] = Q.
\]
Therefore $Q$ is a quasi-$\star[X]$-prime ideal of $D[X]$; i.e., $\Theta_1^* \setminus \{0\} \subseteq \text{QSpec}^{\star[X]}(D[X])$. Since the other inclusion is trivial, we obtain that $Q\text{Spec}^{\star[X]}(D[X]) = \Theta_1^* \setminus \{0\}$.

In the rest of the paper for every semistar operation $\star$ on an integral domain $D$, we let $\star[X]$, to be the stable semistar operation of finite type on $D[X]$ canonically associated to $\star$ as in Theorem 2.1(a).

Let $\star$ be a semistar operation on a domain $D$. As in [13 and 9] (cf. also [20] for the case of a star operation), $D$ is called a Prüfer $\star$-multiplication domain (for short, a $\star\text{-MD}$) if each finitely generated ideal of $D$ is $\star_f$-invertible; i.e., if $(I \star_f^{-1})^{\star_f} = D^*$ for all $I \in \{f(D)$. When $\star = v$, we recover the classical notion of Prüfer domain. When $\star = d_D$, the identity (semi)star operation, we recover the notion of Prüfer domain.

Remark 2.4. Let $\star$ be a semistar operation on an integral domain $D$. Suppose that $D[X]$ is a $P\star[X]\text{MD}$ (resp. a $\star[X]$-quasi-Prüfer domain). Since $\star[X] \leq \star$ by Theorem 2.1(c), we obtain that $D[X]$ is a $P\star[X]\text{MD}$ by [13, resp. a $\star[X]$-quasi-Prüfer domain by [7, Corollary 2.4]). So that $D$ is a $P\star\text{MD}$ by [6, Corollary 2.5 (1)] (resp. a $\star$-quasi-Prüfer domain by [6, Corollary 2.4]).

In [10, Section 3], El Baghdadi, Fontana and Picozza defined and studied the semistar Noetherian domains, i.e., domains having the ascending chain condition on quasi-semistar-ideals.

Remark 2.5. (Cf. [32, Theorem 3.6]) Let $\star$ be a semistar operation on an integral domain $D$. Then $D$ is a $\star$-Noetherian domain if and only if $D[X]$ is a $\star[X]$-Noetherian domain. In fact if $D[X]$ is a $\star[X]$-Noetherian domain, since $\star[X] \leq \star$ by Theorem 2.1(c), we obtain that $D[X]$ is a $\star$-Noetherian domain. So that $D$ is
is a chain of $D$ ⋆ Remark 2.3 that, $Q(\dim(P))$ is defined to be the supremum of the lengths of the chains of quasi-⋆-prime ideals of $D$, between prime ideal (0) (included) and $P$. Obviously, if $\star = d_D$ is the identity (semi)star operation on $D$, then $\star$-height of $P = \operatorname{ht}(P)$, for each prime ideal $P$ of $D$. If the set of quasi-⋆-prime of $D$ is not empty, the $\star$-dimension of $D$ is defined as follows:

$$\star \dim(D) := \sup\{\operatorname{ht}(P) | P \text{ is a quasi-⋆-prime of } D\}.$$ 

If the set of quasi-⋆-primes of $D$ is empty, then pose $\star \dim(D) := 0$. Thus, if $\star = d_D$, then $\star \dim(D) = \dim(D)$, the usual (Krull) dimension of $D$.

Note that, the notions of $t$-dimension and of $w$-dimension have received a considerable interest by several authors (cf. for instance, [32, 33, 14]).

It is known (see [10 Lemma 2.11]) that

$$\tilde{\star} \dim(D) = \sup\{\operatorname{ht}(P) | P \text{ is a quasi-\tilde{\star}-prime ideal of } D\} = \sup\{\operatorname{ht}(P) | P \text{ is a quasi-\tilde{\star}-maximal ideal of } D\}.$$

We answer to the Question 1.1 in the results [31, 32] and [11]. The following result is the semistar version of the classical theorem of Seidenberg [35] Theorem 2).

\textbf{Theorem 3.1.} Let $\star$ be a semistar operation on an integral domain $D$. Suppose that $n := \tilde{\star} \dim(D)$. Then

$$n + 1 \leq \star[X] \dim(D[X]) \leq 2n + 1.$$ 

\textbf{Proof.} Consider a chain $P_1 \subseteq \cdots \subseteq P_n$ of quasi-\tilde{\star}-prime ideals of $D$. Let $Q := P_n[X] + (X)$. Since $Q \cap D = (P_n[X] + (X)) \cap D = P_n \in \mathcal{QSpec}(D)$, we have, using Remark 2.2 that, $Q$ is a quasi-⋆$[X]$-prime ideal of $D[X]$. Then

$$P_1[X] \subseteq \cdots \subseteq P_n[X] \subseteq P_n[X] + (X),$$

is a chain of $n + 1$ quasi-⋆$[X]$-prime ideals of $D[X]$. Hence $n + 1 \leq \star[X] \dim(D[X])$.

For the second inequality suppose that $Q \in \operatorname{QMax}^{\star[X]}(D[X])$ is such that

$$\operatorname{ht}_{D[X]} Q = \star[X] \dim(D[X]).$$

Hence by [20] Theorem 38 we obtain that $\operatorname{ht}_{D[X]} Q \leq 2(\operatorname{ht}_{D}(Q \cap D)) + 1 \leq 2n + 1$. Consequently we have $\star[X] \dim(D[X]) \leq 2n + 1$. \hfill $\square$

In [36, Theorem 3], Seidenberg showed that for any pair of positive integers $(n, m)$ with $n + 1 \leq m \leq 2n + 1$, there exists a domain $D$ such that $\dim(D) = d_D \dim(D) = n$ and $\dim(D[X]) = d_{D[X]} \dim(D[X]) = d_D[X] \dim(D[X]) = m$.

If $X_1, \cdots, X_r$ are indeterminates over $D$, for $r \geq 2$, we let

$$\star[X_1, \cdots, X_r] := \{\star[X_1, \cdots, X_{r-1}]|X_r]$$

where $\star[X_1, \cdots, X_{r-1}]$ is a stable semistar operation of finite type on $D[X_1, \cdots, X_{r-1}]$. 

3. Semistar-Krull dimension

Let $\star$ be a semistar operation on an integral domain $D$. In this section we make use of the semistar operation $\star[X]$ on $D[X]$, canonically associated to the given semistar operation $\star$ on $D$, to provide an answer to the question raised in the introduction. First we recall some definitions and properties of $\star$-dimension. For each quasi-⋆-prime $P$ of $D$, the $\star$-height of $P$ (for short, $\star$-height of $P$) is defined to be the supremum of the lengths of the chains of quasi-⋆-prime ideals of $D$, between prime ideal (0) (included) and $P$.Obviously, if $\star = d_D$ is the identity (semi)star operation on $D$, then $\star$-height of $P = \operatorname{ht}(P)$, for each prime ideal $P$ of $D$. If the set of quasi-⋆-prime of $D$ is not empty, the $\star$-dimension of $D$ is defined as follows:

$$\star \dim(D) := \sup\{\star \operatorname{ht}(P) | P \text{ is a quasi-⋆-prime of } D\}.$$
Theorem 3.2. Let $*$ be a semistar operation on an integral domain $D$. Suppose that $D$ is a $\kappa$-Noetherian domain of $\kappa$-Krull dimension $n$. Then
\[ *\{X_1, \cdots, X_m\} \cdot \dim(D[X_1, \cdots, X_m]) = n + m. \]

Proof. Since $D[X_1, \cdots, X_{m-1}]$ is $*[X_1, \cdots, X_{m-1}]$-Noetherian domain, it suffices to prove the theorem for the case $m = 1$. By Theorem 3.1, we have $n + 1 \leq *\{X\} \cdot \dim(D[X])$. Now let $M$ be an arbitrary quasi-$*[X]$-maximal ideal of $D[X]$. Then $M$ is either an upper to zero, or $P := M \cap D \in \text{QSpec}^*\{D\}$. Note that in either case $D_P$ is a Noetherian domain ([10 Proposition 3.8]). Hence:
\[ \text{ht}_{D[X]} M = \dim(D[X]_M) = \dim(D_P[X]_{M_D[X]}), \]
\[ \leq \dim(D_P[X]) = \dim(D_P) + 1 \]
\[ \leq n + 1. \]

The third equality holds since $D_P$ is a Noetherian domain and [17 Theorem 30.5], and the second inequality holds by [10 Lemma 2.11]. So that by [10 Lemma 2.11] we obtain that
\[ *\{X\} \cdot \dim(D[X]) = \sup\{\text{ht}_{D[X]} M | M \in \text{QMax}^{*\{X\}}(D[X])\} \leq n + 1, \]
which ends the proof.

Theorem 3.3. Let $*$ be a semistar operation on an integral domain $D$. Suppose that $D$ is a $P\text{-MD}$ of $\kappa$-Krull dimension $n$. Then $*\{X\} \cdot \dim(D[X]) = n + 1$.

Proof. Use the fact that if $D$ is a Prüfer domain, then $\dim(D[X]) = \dim(D) + 1$ [36 Corollary] and by the same argument as Theorem 3.2 the proof is complete.

In Corollary 1.11, we show that if $D$ is a $P\text{-MD}$ then
\[ *\{X_1, \cdots, X_m\} \cdot \dim(D[X_1, \cdots, X_m]) = \kappa \cdot \dim(D) + m. \]

One of the key concepts of Jaffard in [23], is that of a special chain, defined as follows. A chain $C = \{P_i\}_{i=0}^m$ of primes in a polynomial ring $D[X_1, \cdots, X_m]$ is called a special chain if, for each $P_i \in C$, the ideal $(P_i \cap D)[X_1, \cdots, X_m]$ is a member of $C$. Jaffard's special chain theorem asserts that, if $Q$ is a prime ideal of $D[X_1, \cdots, X_m]$ of finite height, then $\text{ht}(Q)$ can be realized as the length of a special chain of primes in $D[X_1, \cdots, X_m]$ with terminal element $Q$. In particular, if $D$ is a finite dimensional domain, then $\dim(D[X_1, \cdots, X_m])$ can be realized as the length of a special chain of prime ideals of $D[X_1, \cdots, X_m]$ (see [17 Corollary 30.19] for a simple proof). So we make the following remark.

Remark 3.4. Let $*$ be a semistar operation on an integral domain $D$. If $\kappa \text{-dim}(D)$ is finite, then $*\{X_1, \cdots, X_m\} \cdot \dim(D[X_1, \cdots, X_m])$ can be realized as the length of a special chain of quasi-$*\{X_1, \cdots, X_m\}$-prime ideals of $D[X_1, \cdots, X_m]$. In fact there exists a quasi-$*\{X_1, \cdots, X_m\}$-maximal ideal $Q$ of $D[X_1, \cdots, X_m]$ such that
\[ *\{X_1, \cdots, X_m\} \cdot \dim(D[X_1, \cdots, X_m]) = \text{ht}(Q). \]

Now by Jaffard's special chain theorem [17 Corollary 30.19], $\text{ht}(Q)$ can be realized as the length of a special chain $(0) = Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_n$ of prime ideals in $D[X_1, \cdots, X_m]$ with $Q_n = Q$. Since $Q_n$ is a quasi-$*\{X_1, \cdots, X_m\}$-prime ideal of $D[X_1, \cdots, X_m]$, then each of $Q_1, \cdots, Q_{n-1}$ is a quasi-$*\{X_1, \cdots, X_m\}$-prime ideal of $D[X_1, \cdots, X_m]$ by Theorem 2.7(a) and [11 Lemma 4.1, and Remark 4.5].
As an application of Theorem 3.1, is the following result, which is the semistar version of [35, Theorem 8].

**Theorem 3.5.** Let $\star$ be a semistar operation on an integral domain $D$. Suppose that $\star\text{-dim}(D) = 1$. Then $\star[X]\text{-dim}(D[X]) = 2$ if and only if $D$ is a $\star$-quasi-Prufer domain.

**Proof.** ($\Rightarrow$). Suppose the contrary. Hence by [7, Lemma 2.3], there exists an upper to zero $Q$ of $D[X]$ such that $c_D(Q)^{\star} \nsubseteq D^\star$. Then $c_D(Q)^{\star}$ is contained in a quasi-$\star$-prime ideal $P$ of $D$ and hence $Q \nsubseteq P[X]$. So that $2 \leq \text{ht}_{D[X]}(P[X]) \leq \star[X]\text{-dim}(D[X]) = 2$, that is $\text{ht}_{D[X]}(P[X]) = 2$. This means that $P[X]$ is a quasi-$\star[X]$-maximal ideal of $D[X]$. Therefore since $(P[X] + (X)) \cap D = P \in \text{QSpec}^\star(D)$, we obtain that $P[X] + (X) \in \text{QSpec}^{\star[X]}(D[X])$. Hence $P[X] = P[X] + (X)$ since $P[X]$ is a quasi-$\star[X]$-maximal ideal of $D[X]$. So that $(X) \subseteq P[X]$. Consequently $D = c_D((X)) \subseteq c_D(P[X]) \subseteq P$, which is a contradiction.

($\Leftarrow$). By Theorem 3.1, we have $2 \leq \star[X]\text{-dim}(D[X]) \leq 3$. If $\star[X]\text{-dim}(D[X]) = 3$, then $\text{ht}_{D[X]}(M) = 3$ for some $M \in \text{QMax}^{\star[X]}(D[X])$. By [17, Corollary 30.2], $M$ can not be an upper to zero. So that $P := M \cap D \in \text{QMax}^\star(D)$. From [7, Lemma 2.1] and the hypothesis, we obtain that $D_P$ is a quasi-Prufer domain of dimension 1. Hence $\text{dim}(D_P[X]) = 2$ by [17, Proposition 30.14]. So we have:

$$3 = \text{ht}_{D[X]}(M) = \text{dim}(D[X]_M) = \text{dim}(D_P[X]_{MD_P[X]}) \leq \text{dim}(D_P[X]) = 2,$$

which is a contradiction. Hence $\star[X]\text{-dim}(D[X]) = 2$. □

Recall that an integral domain $D$ is called a UMt-domain (UMt means “uppers to zero are maximal $t$-ideals”) if every upper to zero in $D[X]$ is a maximal $t$-ideal [21, Section 3]. It is observed in [7, Corollary 2.4 (b)] that $D$ is a $w$-quasi-Prufer domain if and only if $D$ is a UMt-domain.

**Corollary 3.6.** Let $D$ be an integral domain. Suppose that $w\text{-dim}(D) = 1$. Then $w[X]\text{-dim}(D[X]) = 2$ if and only if $D$ is a UMt domain.

**Corollary 3.7.** Let $\star$ be a semistar operation on an integral domain $D$. Suppose that $\star\text{-dim}(D) = 1$. The following statements are equivalent:

1. $D$ is a $P\star$MD.

2. $D^\star$ is integrally closed and $\star[X]\text{-dim}(D[X]) = 2$.

**Proof.** The equivalence follows easily from Theorem 3.1 and from the fact that $D$ is a $P\star$MD if and only if $D$ is a $\star$-quasi-Prufer domain and $D^\star$ is integrally closed, [7, Lemma 2.17]. □

In the following result we collect the semistar (Krull) dimension properties of $[\star]$.

**Proposition 3.8.** Let $\star$ be a semistar operation on an integral domain $D$. Suppose that $n := \star\text{-dim}(D)$. Then $n \leq [\star]\text{-dim}(D[X]) \leq 2n$. Moreover if $D$ is a $\star$-Noetherian domain or a $P\star$MD, then $[\star]\text{-dim}(D[X]) = \star\text{-dim}(D)$.

**Proof.** Consider a chain $P_1 \subseteq \cdots \subseteq P_n$ of quasi-$\star$-prime ideals of $D$. Since $P_1[X] \subseteq \cdots \subseteq P_n[X]$ is a chain of $n$ quasi-$\star$-prime ideals of $D[X]$, we have $n \leq [\star]\text{-dim}(D[X])$. For the second inequality suppose that $Q \in \text{QMax}^\star(D[X])$ is such that $\text{ht}_{D[X]}(Q) = [\star]\text{-dim}(D[X])$. Then...
If $Q$ is an upper to zero, then $\text{ht}_{D[X]} Q \leq 1 \leq 2n$. Otherwise by [6, Theorem 2.3 (e)], there exists a quasi-$\pi$-maximal ideal $P$ of $D$ such that $Q = P[X]$. Hence by [26, Theorem 38] we obtain that $\text{ht}_{D[X]} Q \leq 2(\text{ht}_D(P)) \leq 2n$. Consequently we have $[\ast]-\dim(D[X]) \leq 2n$.

Now suppose that $D$ is a $\pi$-Noetherian domain or a $P\ast$MD. We know that $\overline{\ast}$-$\dim(D) \leq [\ast]-\dim(D[X])$. Let $M$ be an arbitrary quasi-$[\ast]$-maximal ideal of $D[X]$. Then $M$ is either an upper to zero, or $M = P[X]$ for some $P \in \text{QMax}^\ast(D)$ by [6, Theorem 2.3 (e)]. Note that in either case $D_P$ is a Noetherian domain by [10, Proposition 3.8] (resp. a valuation domain by [13, Theorem 3.1]). Hence:

$$\text{ht}_{D[X]} P[X] = \dim(D[X]_{P[X]}) = \dim(D_P[X]_{PDP[X]})$$
$$\leq \dim(D_P[X]) - \dim((D_P/PDP)[X])$$
$$= \dim(D_P[X]) - \dim((D_P/PDP)[X])$$
$$= \dim(D_P) \leq \overline{\ast}$-$\dim(D).$$

The fourth equality holds since $D_P$ is a Noetherian domain and [17, Theorem 30.5] (resp. a valuation domain and [36, Theorem 4]) and the second inequality holds by [10, Lemma 2.11]. So that by [10, Lemma 2.11] we obtain that $[\ast]-\dim(D[X]) \leq \overline{\ast}$-$\dim(D)$, which ends the proof. $\square$

Analogous to Seidenberg, in [38, Theorem 2.10], Wang, showed that for any pair of positive integers $(n, m)$ with $1 \leq n \leq m \leq 2n$, there exists a domain $D$ such that $w_D$-$\dim(D) = n$ and $w_{D[X]}$-$\dim(D[X]) = m$. Note that $[w_D] = w_D[X]$ by [6, Theorem 2.3].

**Remark 3.9.** Let $D$ be an integral domain which is $w_D$-Noetherian and of $w_D$-dimension $n$. Then $[w_D]-\dim(D[X]) = w_{D[X]}-\dim(D[X]) = n$ by Proposition [38, 3.2] while $w_D[X]$-$\dim(D[X]) = n + 1$ by Theorem [2.3]. This means that $w_D[X] \neq w_{D[X]}([w_D])$. Actually noting Part (c) of Theorem [2.4] we have $w_D[X] \leq [w_D]$.

4. **Semistar-valuative dimension**

It is worth reminding the reader of the nice behavior of the valuative dimension with respect to polynomial rings, in the sense that $\dim_n(D[X_1, \ldots, X_n]) = \dim_n(D) + n$ for each positive integer $n$ and each ring $D$ ([23, Theorem 2]). In this section we define the *semistar-valuative dimension* of integral domains and derive its properties.

For this section we need to recall the notion of $\ast$-valuation overring (a notion due essentially to P. Jaffard [22, page 46]). For a domain $D$ and a semistar operation $\ast$ on $D$, we say that a valuation overring $V$ of $D$ is a $\ast$-valuation overring of $D$ provided $F^\ast \subseteq FV$, for each $F \in f(D)$. Note that, by definition, the $\ast$-valuation overrings coincide with the $\ast_f$-valuation overrings. By [14, Theorem 3.9], $V$ is a $\overline{\ast}$-valuation overring of $D$ if and only if $V$ is a valuation overring of $D_P$ for some quasi-$\ast_f$-maximal ideal $P$ of $D$. Also $V$ is a $\ast$-valuation overring of $D$ if and only if $V^{\ast_f} = V$, (cf. [10, Page 34]).

Let $R$ be a Bézout domain. Then each (nonzero) finitely generated ideal of $R$ is principal. So that if $J$ is a nonzero finitely generated ideal of $R$, then $J = J'$, and hence each nonzero ideal of $R$ is a $t$-ideal. This implies that the $d_R$-operation on $R$ is a unique (semi)star operation of finite type on $R$. Therefore every (semi)star
operation of finite type on a valuation domain, is the trivial identity operation. The following result is the key lemma in this section.

**Lemma 4.1.** Let $\ast$ be a semistar operation on an integral domain $D$. Suppose that $W$ is a valuation overring of $D[X]$. Then $W$ is a $\ast[X]$-valuation overring of $D[X]$, if and only if $W \cap K$ is a $\ast$-valuation overring of $D$.

*Proof.* ($\Rightarrow$). Suppose that $W$ is a $\ast[X]$-valuation overring of $D[X]$. Then by [14, Theorem 3.9], there exists a $Q \in \text{QMax}^{\ast[X]}(D[X])$, such that $D[X]_Q \subseteq W$. Put $P := Q \cap D$. Note that $D[X]_Q = D_P[X]_{QD_P[X]}$. Therefore $D_P[X] \subseteq W$, and hence $D_P \subseteq W \cap K$. If $P = 0$, then $K = W \cap K$, and hence clearly $W \cap K$ is a $\ast$-valuation overring of $D$. If $P \neq 0$, then $P^\ast = (Q \cap D)^\ast \subsetneq D^\ast$ by Remark 2.3. Hence $P \in \text{QSpec}^\ast(D)$. Choose a quasi-$\ast$-maximal ideal $M$ of $D$ containing $P$ by [14, Lemma 2.3 (1)]. So that $D_M \subseteq D_P \subseteq W \cap K$. Therefore $W \cap K$ is a $\ast$-valuation overring of $D$ by [14, Theorem 3.9].

($\Leftarrow$). Let $M$ be the maximal ideal of $W$, and set $Q := M \cap D[X]$. We need to show that $Q$ is a quasi-$\ast[X]$-prime ideal of $D[X]$. Note that $M \cap K$ is the maximal ideal of $W \cap K$ by [17, Theorem 19.16]. Since $W \cap K$ is a $\ast$-valuation overring of $D$, we have $(W \cap K)^\ast = W \cap K$ by [10, Page 34]. Thus $\ast_i$ is a (semi)star operation of finite type by [33, Proposition 3.4], on $W \cap K$, where $i$ is the canonical inclusion of $D$ to $W \cap K$. So that since $W \cap K$ is a valuation domain it is the identity operation. Put $P := Q \cap D = (M \cap K) \cap D$. If $P = 0$ then by construction of $\ast[X]$, $Q$ is a quasi-$\ast[X]$-prime ideal of $D[X]$. So assume that $P \neq 0$. Now we show that $P^\ast \neq D^\ast$. If not

$$D^\ast = P^\ast = ((M \cap K) \cap D)^\ast = (M \cap K)^\ast \cap D^\ast = (M \cap K) \cap D^\ast.$$ 

Hence $D^\ast \subseteq M \cap K$ and therefore, intersecting with $D$ we find that $D = M \cap D$, which is a contradiction. Now using Remark 2.3 we see that $Q$ is a quasi-$\ast[X]$-prime ideal of $D[X]$. Now choose a quasi-$\ast[X]$-maximal ideal $M$ of $D[X]$ containing $Q$. Thus we have $D[X]_M \subseteq D[X]_Q \subseteq W$. Consequently by [14, Theorem 3.9], we obtain that $W$ is a ($\ast[X] = \ast[X]$)-valuation overring of $D[X]$. \hfill $\Box$

The following theorem is one of the main results of this section, whose proof is based on that of [17, Theorem 30.8]. First, we need the following definition. Let $D$ be a domain and $T$ an overring of $D$. Let $\ast$ and $\ast'$ be semistar operations on $D$ and $T$, respectively. One says that $T$ is ($\ast, \ast'$)-linked to $D$ (or that $T$ is a ($\ast, \ast'$)-linked overring of $D$) if

$$F^{\ast} = D^{\ast} \Rightarrow (FT)^{\ast'} = T^{\ast'}$$

for each nonzero finitely generated ideal $F$ of $D$. It was proved in [8, Theorem 3.8] that $T$ is ($\ast, \ast'$)-linked to $D$ if and only if $\text{Na}(D, \ast) \subseteq \text{Na}(T, \ast')$.

**Theorem 4.2.** Let $\ast$ be a semistar operation on an integral domain $D$, and let $n$ be an integer. Then the following statements are equivalent:

1. Each ($\ast, \ast'$)-linked overring $T$ of $D$ has $\tilde{\ast}'$-dimension at most $n$, whenever $\ast'$ is a semistar operation on $T$.
2. Each ($\ast, w_T$)-linked overring $T$ of $D$ has $w_T$-dimension at most $n$.
3. Each overring $T$ of $D$ has $\tilde{\ast}_i$-dimension at most $n$, where $i : D \rightarrow T$ is the canonical inclusion.
4. Each $\tilde{\ast}$-valuation overring of $D$ has dimension at most $n$. 

For each finite subset \( \{ t_i \}_{i=1}^n \) of \( K \), \( \bar{\dim}_s(D[t_1, \ldots, t_n]) \leq n \), where \( \iota : D \to D[t_1, \ldots, t_n] \) is the canonical inclusion.

For each finite subset \( \{ t_i \}_{i=1}^n \) of \( K \), such that \( D[t_1, \ldots, t_n] \) is a \((*, *)\)-linked overring of \( D \), \( \bar{\dim}_s(D[t_1, \ldots, t_n]) \leq n \), whenever \( * \) is a semistar operation on \( D[t_1, \ldots, t_n] \).

\( \bar{\dim}_s(X_1, \ldots, X_n) \leq 2n \).

Proof. \((1) \Rightarrow (2), (1) \Rightarrow (3), (1) \Rightarrow (6), (3) \Rightarrow (5) \) and \((6) \Rightarrow (5) \) are trivial.

\((2) \Rightarrow (4) \). By [10, Lemma 2.7], \( V \) is a \( \bar{\pi} \)-valuation overring of \( D \) if and only if \( V \) is a \((\bar{\pi}, d_V)\)-linked valuation overring of \( D \). The assertion therefore follows since \( w_V = d_V \) for a valuation domain.

\((4) \Rightarrow (3) \). Suppose the contrary. So there exists an overring \( T \) of \( D \) containing \( P_0 \subset P_1 \subset \cdots \subset P_n \), of quasi-\( \pi \)-prime ideals of \( T \), where \( \iota : D \to T \) is the canonical inclusion. Actually one can choose \( P_n \) so that \( P_n \in \text{QMax}^{\pi} (T) \). Consider the chain \( P_0 T_p \subset P_1 T_p \subset \cdots \subset P_n T_p \) of distinct prime ideals of \( T_p \). Using [17, Corollary 19.7], there exists a valuation overring \( V \) of \( T_p \), such that \( V \) contains a chain \( M_0 \subset M_1 \subset \cdots \subset M_n \) of prime ideals of \( V \) and \( M_i \cap T_p = P_i T_p \).

Since \( P_n \in \text{QMax}^{\pi} (T) \) and \( V \) is an overring of \( T_p \), we obtain that \( V \) is a \( \pi \)-valuation overring of \( T \), by [13, Theorem 3.9]. So that \( V^\pi = V \), (see [10, Page 34]).

Hence \( V^\pi = V \). Therefore \( V \) is a \( \pi \)-valuation overring of \( D \) (see [10, Page 34]) and \( \text{dim}(V) > n \), which is impossible.

\((5) \Rightarrow (3) \). Suppose there exists an overring \( T \) of \( D \) containing a chain \( P_0 \subset P_1 \subset \cdots \subset P_n \) of quasi-\( \pi \)-prime ideals of \( T \), where \( \iota : D \to T \) is the canonical inclusion. Choose \( t_i \in P_i \setminus P_{i-1} \), for each \( i = 1, \cdots, n \). If \( D' = D[t_1, \ldots, t_n] \), then \( 0 \leq P_0 \cap D' \subset P_1 \cap D' \subset \cdots \subset P_n \cap D' \subset D' \).

And since \( T \) is an overring of \( D' \), \( P_0 \cap D' \neq 0 \). Indeed let \( r/s \in P_0 \), where \( r, s \in D' \setminus \{0\} \). Then \( r = s(r/s) \) is an element of \( P_0 \cap D' \). On the other hand each \( P_i \cap D' \) is a quasi-\( \pi \)-prime ideals of \( D' \), where \( \iota : D \to D' \) is the canonical inclusion.

More precisely

\[ (P_i \cap D')^\pi \cap D' = (P_i \cap D')^\pi \cap D' = P_i \cap D' = P_i \cap D' = P_i \cap D' = \]

\[ P_i \cap (T \cap D') = (P_i \cap T) \cap D' = (P_i \cap T) \cap D' = P_i \cap D'. \]

Therefore \( \bar{\dim}_s (D[t_1, \ldots, t_n]) > n \), which is a contradiction.

\((3) \Rightarrow (4) \). Let \( V \) be a \( \pi \)-valuation overring of \( D \). Hence we have \( V^\pi = V \) by [10, Page 34]. This means that \( \pi \) is a (semi)star operation on \( V \), where \( \iota : D \to V \) is the canonical inclusion. Note that since \( \pi \) is of finite type, then it is the identity operation on the valuation domain \( V \). Thus \( \text{dim}(V) = \pi \)-dim(\( V \)) \leq n.

\((4) \Rightarrow (1) \). Suppose the contrary. So there exists a \((*, *)\)-linked overring \( T \) of \( D \) containing a chain \( P_0 \subset P_1 \subset \cdots \subset P_n \) of quasi-\( \pi \)-prime ideals of \( T \). By the same reasoning as in the proof of \((4) \Rightarrow (3) \), there exists a \( \pi \)-valuation overring \( V \) of \( T \) with \( \text{dim}(V) > n \). Thus, by [10, Lemma 2.7], \( V \) is a \((\pi, d_V)\)-linked overring of \( T \). Since linked-ness is a transitive relation ([9, Theorem 3.8]), \( V \) is a \((\pi, d_V)\)-linked overring of \( D \). Consequently \( V \) is a \( \pi \)-valuation overring of \( D \), which is impossible.

So we showed that \((1) \sim (6) \) are equivalent.

\((4) \Rightarrow (7) \). To prove \( \pi (X_1, \ldots, X_n) \)-dim(\( D[X_1, \ldots, X_n] \)) \leq 2n \), it suffices in view of what we have just shown, to prove that each \( \pi (X_1, \ldots, X_n) \)-valuation overring \( W \) of \( D[X_1, \ldots, X_n] \) has dimension at most \( 2n \). Thus by Lemma [4.1] \( W \cap K \) is a
Lemma 30.7], shows that $h_t(\mathcal{W})$ is the dimension of $\mathcal{W}$. So that $\dim(\mathcal{W} \cap K) \leq n$. Then by [17, Theorem 20.7], we have $\dim(W) \leq 2n$.

(7) $\Rightarrow$ (5). We consider a subset $\{s_i\}_{i=1}^n$ of $K$. If $Q_0$ is the kernel of the $D$-homomorphism $\varphi : D[X_1, \cdots, X_n] \to D[s_1, \cdots, s_n]$, sending $X_i$ to $s_i$, then [17, Lemma 30.7], shows that $ht(Q_0) = n$. Note that $D[t_1, \cdots, t_n] \cong D[X_1, \cdots, X_n]/Q_0$.

Suppose that $\beta \in Q\mathrm{Spec}^*(D[t_1, \cdots, t_n])$ is such that $ht(\beta) = t_1.$ $\dim(D[t_1, \cdots, t_n]),$ where $\iota : D \to D[t_1, \cdots, t_n]$ is the canonical inclusion. There exists a prime ideal $Q$ of $D[X_1, \cdots, X_n]$, such that $\beta = \varphi(Q) \cong Q/Q_0$. We claim that $Q$ is a quasi-$\star[X_1, \cdots, X_n]$-prime ideal of $D[X_1, \cdots, X_n]$. To this end set $P := \beta \cap D$, which, by the same argument as in the proof of part (5) $\Rightarrow$ (3), is a quasi-$\star$-prime ideal of $D$. Note that $Q \cap D = \beta \cap D = P$. Therefore $(Q \cap D)^\star = P^\star \subseteq D^\star$. Then by repeated applications of Remark 3, we claim that $Q$ is a quasi-$\star[X_1, \cdots, X_n]$-prime ideal of $D[X_1, \cdots, X_n]$. This means that $ht(Q) \leq 2n$ by the hypothesis. Thus we have

$$t_1 - \dim(D[t_1, \cdots, t_n]) = ht(\beta) = ht(Q/Q_0) \leq ht(Q) - ht(Q_0) \leq 2n - n = n,$$

which ends the proof. \hfill \Box

In [23] Jaffard defines the valuative dimension, denoted $\mathrm{dim}_v(D)$, of the domain $D$ to be the maximal rank of the valuation overrings of $D$. Now we make the following definition.

**Definition 4.3.** Let $\star$ be a semistar operation on an integral domain $D$. We say that $D$ has $\star$-valuative dimension $n$, and we write $\star\mathrm{dim}_v(D) = n$, if each $\star$-valuation overring of $D$ has dimension at most $n$ and if there exists a $\star$-valuation overring of $D$ of dimension $n$. If no such integer exists, we say that the $\star$-valuative dimension of $D$ is infinite.

Note that $d_D\mathrm{dim}_v(D) = \mathrm{dim}_v(D)$. Since by definition, the $\star$-valuation overrings coincide with the $\star_f$-valuation overrings we have $\star_f\mathrm{dim}_v(D) = \star\mathrm{dim}_v(D)$. In particular $t_D\mathrm{dim}_v(D) = v_D\mathrm{dim}_v(D)$. Suppose that $\star_1$ and $\star_2$ are two semistar operations on an integral domain $D$, such that $\star_1 \leq \star_2$. If $V$ is a $\star_2$-valuation overring of $D$, then for each $F \in f(D)$ we have $F^{\star_1} \subseteq F^{\star_2} \subseteq FV$. Hence $V$ is a $\star_1$-valuation overring of $D$ by definition. So we have:

$$\star_2\mathrm{dim}_v(D) \leq \star_1\mathrm{dim}_v(D).$$

Using [17, Corollary 19.7] together with [14, Theorem 3.9], one can easily see that $\bar{\mathrm{dim}}(D) \leq \bar{\mathrm{dim}}(D)$. The following example shows that this inequality is not true in general.

**Example 4.4.** Let $(D, M)$ be a two dimensional local Noetherian domain and suppose that $0 \subseteq P \subseteq M$ be the corresponding chain of prime ideals. Let $(T_1, N_1)$ and $(T_2, N_2)$ be two rank one discrete valuation rings [8] dominating the local rings $D_P$ and $D$ respectively. Let $\star$ be a semistar operation on $D$ defined by $E^\star = ET_1 \cap ET_2$ for each $E \in \mathcal{F}(D)$. Then clearly $\star = \star_f$. We show that $P, M \in \mathrm{QSpec}(D)$. Indeed there exists a positive integer $k$ such that $PT_1 = N_1^k$. Hence $P \subseteq P^\star \cap D = PT_1 \cap PT_2 \cap D \subseteq PT_1 \cap D = N_1^k \cap D \subseteq N_1 \cap D = P$. Therefore $P^\star \cap D = P$. By the same way $M^\star \cap D = M$. Therefore we have $\star\mathrm{dim}(D) = 2$. Now we compute $\star_1\mathrm{dim}_v(D)$. Suppose that $V$ is a $\star$-valuation overring of $D$. Thus in particular we have $D^\star \subseteq DV$ that is $T_1 \cap T_2 \subseteq V$. Using [17, Theorem 26.1] we obtain that $T_1 \subseteq V$ or $T_2 \subseteq V$. Consequently $\dim V \leq 1$. This means that
$\star\text{-dim}_n(D) = 1$. Thus we have

$$2 = \star\text{-dim}(D) > \star\text{-dim}_n(D) = 1.$$  

Note that $\tilde{\star} = d_D$. So that we have $\tilde{\star} \leq \star$ and $1 = \star\text{-dim}_n(D) < \tilde{\star}\text{-dim}_n(D) = 2$.

By a slight modification of Theorem 4.2 we have:

**Theorem 4.5.** Let $\star$ be a semistar operation on an integral domain $D$, and let $n$ be an integer. Then the following statements are equivalent:

1. Each $(\star, \star')$-linked overring $T$ of $D$ has $\star'$-dimension at most $n$, and $n$ is minimal, whenever $\star'$ is a semistar operation on $T$.
2. Each $(\star, \text{wr})$-linked overring $T$ of $D$ has $\text{wr}$-dimension at most $n$, and $n$ is minimal.
3. Each overring $T$ of $D$ has $\tilde{\star}$-dimension at most $n$, and $n$ is minimal, where $\iota : D \rightarrow T$ is the canonical inclusion.
4. $\tilde{\star}\text{-dim}_n(D) = n$.
5. For each finite subset $\{t_i\}_{i=1}^n$ of $K$, $\tilde{\star}, \text{dim}(D[t_1, \ldots, t_n]) \leq n$, and $n$ is minimal, where $\iota : D \rightarrow D[t_1, \ldots, t_n]$ is the canonical inclusion.
6. For each finite subset $\{t_i\}_{i=1}^n$ of $K$, such that $D[t_1, \ldots, t_n]$ is a $(\star, \star')$-linked overring of $D$, $\tilde{\star}', \text{dim}(D[t_1, \ldots, t_n]) \leq n$, and $n$ is minimal, whenever $\star'$ is a semistar operation on $D[t_1, \ldots, t_n]$.
7. $\{X_1, \ldots, X_n\} \text{-dim}(D[X_1, \ldots, X_n]) = 2n$.

**Corollary 4.6.** Let $\star$ be a semistar operation on an integral domain $D$. If $D$ is a $\tilde{\star}$-Noetherian domain of $\tilde{\star}$-dimension $n$, then $\tilde{\star}\text{-dim}_n(D) = n$.

**Proof.** By Theorem 4.2 we know $\{X_1, \ldots, X_n\} \text{-dim}(D[X_1, \ldots, X_n]) = 2n$. Hence $\tilde{\star}\text{-dim}_n(D) = n$. $\square$

Let $D$ be a $\text{P}\times\text{MD}$. Since for each $M \in \text{QMax}^*(D)$, $D_M$ is a valuation domain by [13] Theorem 3.1, we have $\tilde{\star}\text{-dim}(D) = \tilde{\star}\text{-dim}_n(D)$. For an integer $r$, it is convenient to put $\star[r]$ to denote $\star[X_1, \ldots, X_r]$ and $D[r]$ to denote $D[X_1, \ldots, X_r]$, where $X_1, \ldots, X_r$ are indeterminates over $D$.

**Corollary 4.7.** Let $\star$ be a semistar operation on an integral domain $D$. Suppose that $\tilde{\star}\text{-dim}_n(D) = k$. Then $\star[r] \text{-dim}(D[r]) = \star[r] \text{-dim}_n(D[r])$, for each $r \geq k$.

**Proof.** Theorem 4.3 shows that $\star[k] \text{-dim}(D[k]) = 2k$. Since $D[r] = D[k][X_{k+1}, \ldots, X_r]$, it follows that:

$$\star[r] \text{-dim}(D[r]) \geq \text{dim}(D[k]) + r - k = 2k + r - k = r + k.$$  

If $V$ is a $\star[r]$-valuation overring of $D[r]$, then $V \cap K$ is a $\tilde{\star}$-valuation overring of $D$ by Lemma 4.4. So that by [17] Theorem 20.7, we have $\text{dim}(V) \leq \text{dim}(V \cap K) + r \leq k + r$. Consequently $\star[r] \text{-dim}_n(D[r]) \leq k + r$. Since $\star[r] \text{-dim}(D[r]) = \star[r] \text{-dim}_n(D[r]) = k + r = \tilde{\star} \text{-dim}_n(D) + r$. $\square$

**Theorem 4.8.** Let $\star$ be a semistar operation on an integral domain $D$. Then:

$$\star[m] \text{-dim}_n(D[m]) = \tilde{\star} \text{-dim}_n(D) + m.$$  

**Proof.** Put $n := \tilde{\star} \text{-dim}_n(D)$. If $W$ is a $\star[m]$-valuation overring of $D[m]$, then by Lemma 4.3 $W \cap K$ is a $\tilde{\star}$-valuation overring of $D$. So that $\text{dim}(W \cap K) \leq \tilde{\star} \text{-dim}_n(D) + m$. $\square$
Corollary 4.9. Let $\star$ be a semistar operation on an integral domain $D$. Suppose that $\tilde{\star}\text{-dim}(D) = \star\text{-dim}_v(D) < \infty$. If $n$ is a positive integer, then

$$\star[n]\text{-dim}(D[n]) = \star[n]\text{-dim}_v(D[n]) = n + \tilde{\star}\text{-dim}(D).$$

Let $\star$ be a semistar operation on an integral domain $D$. Recall that the $\star$-closure of $D$, defined by:

$$D^{cl\star} := \bigcup\{(F*: F*) | F \in f(D)\}$$

is an integrally closed overring of $D$ and, more precisely, $D^{cl\star} = \bigcap\{V | V \text{ is a } \star\text{-valuation overring of } D\}$. For more details on this subject and for the proof of the result recalled above, see [31], [18], [15] Proposition 3.2 and Corollary 3.6. Set $\tilde{D} := D^{cl\star}$ and $\tilde{\star} := \star\iota$, where $\iota: D \to \tilde{D}$ is the canonical embedding. Note that $\tilde{\star} = \star$ by [33] Proposition 3.1.

Proposition 4.10. Let $\star$ be a semistar operation on an integral domain $D$. Suppose that $D$ is a $\tilde{\star}$-quasi-Prüfer domain. Then

$$\tilde{\star}\text{-dim}(D) = \tilde{\star}\text{-dim}_v(D).$$

Proof. Recall from [7] Theorem 2.16 that, $D$ is a $\tilde{\star}$-quasi-Prüfer domain if and only if $Na(\tilde{D},\tilde{\star})$ is a Prüfer domain, that is $\tilde{D}$ is a P*MD by [13] Theorem 3.1. Since $\tilde{D}$ is a P*MD, we have $\star\text{-dim}(\tilde{D}) = \star\text{-dim}_v(\tilde{D})$. Also an easy application of [7] Lemma 2.15, yields us that $\tilde{\star}\text{-dim}(D) = \star\text{-dim}(\tilde{D})$. So

$$\tilde{\star}\text{-dim}(D) = \star\text{-dim}(\tilde{D}) = \star\text{-dim}_v(\tilde{D}) = \tilde{\star}\text{-dim}_v(D).$$

The last equality holds true since by [15] Corollary 3.6 a valuation domain is a $\tilde{\star}$-valuation overring of $D$ if and only if it is a $\star$-valuation overring of $\tilde{D}$ (see Remark 4.20 for another reasoning of this equality). \qed

Corollary 4.11. Let $\star$ be a semistar operation on an integral domain $D$. Suppose that $D$ is a $\tilde{\star}$-quasi-Prüfer domain (e.g., if $D$ is a P*MD). Then

$$\star[n]\text{-dim}(D[n]) = \star[n]\text{-dim}_v(D[n]) = n + \tilde{\star}\text{-dim}(D).$$

Combining Corollary 4.11 with Theorem 3.10 we obtain the following corollary. The special case of $\star = d_D$ is contained in [36].

Corollary 4.12. Let $\star$ be a semistar operation on an integral domain $D$. Suppose that $\tilde{\star}\text{-dim}(D) = 1$. The following statements are equivalent:

1. $\star[X]\text{-dim}(D[X]) = 2$.
2. $\star[m]\text{-dim}(D[m]) = m + 1$ for any integer $m$. 

n. Therefore [17] Theorem 20.7, shows that $\dim(W) \leq n + m$. Consequently $\star[m]\text{-dim}_v(D[m]) \leq n + m$.

But by assumption, there exists a $\tilde{\star}$-valuation overring $V$ of $D$ of rank $n$. So that by [17] Remark 20.4, $V$ has an extension to a valuation domain $W$ on $K(X_1, \cdots, X_m)$, with $\dim(W) = n + m$ and such that $\{X_1, \cdots, X_m\}$ is contained in the maximal ideal of $W$. Therefore $W$ is a valuation overring of $D[m]$ of dimension $n + m$. Since $V = W \cap K$ is a $\tilde{\star}$-valuation overring of $D$, Lemma 4.1 shows that $W$ is a $\star[m]$-valuation overring of $D[m]$. So that $\star[m]\text{-dim}_v(D[m]) \geq n + m$. Thus we have

$$\star[m]\text{-dim}_v(D[m]) = n + m,$$

which is the desired equality. \qed
In [1], to honor Jaffard, the authors defined a domain \( D \) to be a Jaffard domain, in case \( \dim(D) = \dim_v(D) < \infty \). The class of Jaffard domains contains most of the well-known classes of finite dimensional rings involved in dimension theory of commutative rings, such as Noetherian domains, Prüfer domains, universally catenarian domains [4], and stably strong S-domains [28, 24]. As the semistar analogue we define:

**Definition 4.13.** Let \( \star \) be a semistar operation on an integral domain \( D \). The domain \( D \) is said to be a \( \tilde{\star} \)-Jaffard domain, if \( \tilde{\star} \)-dim\( (D) < \infty \) and \( \tilde{\star} \)-dim\( (D) = \tilde{\star} \)-dim\( _v(D) \).

Note that the notion of \( d_P \)-Jaffard domain coincides with the “classical” notion of Jaffard domain. Note that \( D \) is \( \tilde{\star} \)-Jaffard domain if and only if \( \tilde{\star} \)-dim\( (D) < \infty \) and \( \star[r]\)-dim\( (D[r]) = r + \tilde{\star} \)-dim\( (D) \) for every \( r \in \mathbb{N} \). Indeed let \( k = \tilde{\star} \)-dim\( _v(D) \).

Then by Corollaries 4.8 and 4.7 respectively we have

\[
k + \tilde{\star} \)-dim\( _v(D) = \star[k]\)-dim\( _v(D[k]) = \star[k]\)-dim\( (D[k]) = k + \tilde{\star} \)-dim\( (D).
\]

Hence \( \tilde{\star} \)-dim\( (D) = \tilde{\star} \)-dim\( _v(D) \). The converse is true by Corollary 4.3.

Every \( \sim \)-Noetherian domain and every \( \sim \)-quasi-Prüfer domain (e.g., every \( P \star \)-MD) of finite \( \tilde{\star} \)-dimension is a \( \tilde{\star} \)-Jaffard domain. As Theorem 3.5 and Corollary 4.12 show, if \( \tilde{\star} \)-dimension is one, then \( \sim \)-quasi-Prüfer domains and \( \tilde{\star} \)-Jaffard domains coincide. For the general case, we have the following theorem. See also [34, Theorem 4.3] for several other characterizations of \( \tilde{\star} \)-quasi-Prüfer domains. The special case of \( \star = d_P \), of the following theorem is contained in [4].

**Theorem 4.14.** Let \( \star \) be a semistar operation on an integral domain \( D \). Suppose that \( \tilde{\star} \)-dim\( (D) \) is finite. Then the following statements are equivalent:

1. \( D \) is a \( \sim \)-quasi-Prüfer domain.
2. Each \( (\star, \star') \)-linked overring \( T \) of \( D \) is a \( \tilde{\star}' \)-quasi-Prüfer domain, where \( \star' \) is a semistar operation on \( T \).
3. Each \( (\star, \star') \)-linked overring \( T \) of \( D \) is a \( \tilde{\star}' \)-Jaffard domain, where \( \star' \) is a semistar operation on \( T \).
4. Each overring \( T \) of \( D \) is a \( \tilde{\star}, \tilde{\star}' \)-Jaffard domain, where \( \iota \) is the canonical embedding of \( D \) into \( T \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose that \( D \) is a \( \sim \)-quasi-Prüfer domain. Hence \( \text{Na}(D, \star) \) is a quasi-Prüfer domain by [7, Theorem 2.16]. If \( T \) is a \( (\star, \star') \)-linked overring of \( D \), where \( \star' \) is a semistar operation on \( T \), then by [9, Theorem 3.8], we have \( \text{Na}(D, \star) \subseteq \text{Na}(T, \star') \). Consequently \( \text{Na}(T, \star') \) is a quasi-Prüfer domain by [12, Corollary 6.5.14]. Therefore \( T \) is a \( \tilde{\star}' \)-quasi-Prüfer domain by [7, Theorem 2.16].

(2) \( \Rightarrow \) (3). and (3) \( \Rightarrow \) (4) are trivial.

(4) \( \Rightarrow \) (1). In order to show that \( D \) is a \( \tilde{\star} \)-quasi-Prüfer domain, it suffices by [7, Theorem 2.16], to show that \( D_P \) is a quasi-Prüfer domain for all \( P \in \text{QMax}(\tilde{\star}) \). And for this, it suffices to prove that each overring \( T \) of \( D_P \), is a Jaffard domain by [12, Theorem 6.7.4]. To this end let \( P \) be an arbitrary quasi-\( \tilde{\star} \)-maximal ideal of \( D \), and \( T \) be an overring of \( D_P \). Let \( V \) be a valuation overring of \( T \). Since \( D_P \subseteq V \), and \( P \) is a quasi-\( \tilde{\star} \)-maximal ideal of \( D \), we have \( V \) is a \( \tilde{\star} \)-valuation overring of \( D \) by [14, Theorem 3.9]. Thus \( V \tilde{\star} = V \) by [10, Page 34]. This means that \( V \) is a \( \tilde{\star} \)-valuation overring of \( T \) ([10, Page 34]), where \( \iota \) is the canonical embedding of \( D \) into \( T \). So we obtain that \( \dim_v(T) = \tilde{\star} \)-dim\( _v(T) \). Therefore by the hypothesis we
have:
\[
\dim(T) \leq \dim_\star(T) = \tilde{\dim}_\star(T) = \tilde{\dim}(T) \leq \dim(T).
\]
Thus \(\dim(T) = \dim_\star(T)\), that is \(T\) is a \(\star\)-Jaffard domain. Hence \(DP\) is a \(\star\)-Prüfer domain for all \(P \in \text{QMax}^\star(D)\), that is \(D\) is a \(\star\)-\(\star\)-Prüfer domain.

Recall that if \(D\) is a Krull domain then it is a \(\text{Pr} \cap \text{MD} \) (c.f. \cite{10} Remark 4.2)). Hence from the above theorem, it can be seen that a Krull domain is \(\omega\)-Jaffard.

There is an old question (see \cite{5}) asking if it possible to find a UFD (or a Krull domain) which is not Jaffard. So, the natural question is the following: is it possible to find a \(\omega\)-Jaffard non Jaffard domain?

Next, we wish to establish that, if \(D\) is a \(\star\)-Jaffard domain, then \(\text{Na}(D, \star)\) is a Jaffard domain. First we compute the Krull dimension of the \(\star\)-Nagata ring.

**Theorem 4.15.** Let \(\star\) be a semistar operation on an integral domain \(D\). Then \(\dim(\text{Na}(D, \star)) = [\star X] \cdot \dim(D[X]) - 1\). In particular if \(D\) is a \(\star\)-Jaffard domain, then \(\dim(\text{Na}(D, \star)) = \star \cdot \dim(D)\).

**Proof.** Note that if \(Q\) is an upper to zero, then, \(\text{ht}(Q) \leq 1\). Also if \(Q \in \text{Spec}(D[X])\), and \(P := Q \cap D\), such that \(P[X] \subseteq Q\), then \(\text{ht}(Q) = \text{ht}(P[X]) + 1\) by \cite{17} Lemma 30.17]. So we have:
\[
\star \cdot [\star X] \cdot \dim(D[X]) = \sup \{\text{ht}(Q) | Q \in \text{QMax}^\star(D[X])\} \\
= \sup \{\text{ht}(Q) | Q \cap D \in \text{QMax}^\star(D)\} \\
= \sup \{\text{ht}(P[X]) + 1 | P \in \text{QMax}^\star(D)\} \\
= \sup \{\text{ht}(P[X]) | P \in \text{QMax}^\star(D)\} + 1 \\
= \dim(\text{Na}(D, \star)) + 1.
\]

For the third equality note that if \(Q \in \text{QMax}^\star(D[X])\), and \(P := Q \cap D\), then \(P[X] \subseteq Q\). Otherwise \(Q = P[X]\). Note that \(P \in \text{QSpec}^\star(D)\) (or equal to zero). Due to the fact that \((P[X] + (X)) \cap D = P\), we obtain by Remark 2.3 that \(P[X] + (X) \in \text{QSpec}^\star(D[X])\). Since \(P[X] \in \text{QMax}^\star(D[X])\) and is contained in \(P[X] + (X)\), we have \(P[X] = P[X] + (X)\). Then \((X) \subseteq P[X]\) and therefore \(D = c_D((X)) \subseteq c_D(P[X]) \subseteq P\) which is a contradiction. For the last equality note that \(\text{Max}(\text{Na}(D, \star)) = \{P \text{Na}(D, \star) | P \in \text{QMax}^\star(D)\}\) \cite{14} Proposition 3.1 (3).

Next we compute the valuative dimension of the \(\star\)-Nagata ring. Before that, we need some observations and one lemma. Let \(D\) be an integral domain and \(\star\) a semistar operation on \(D\). One can consider the contraction map \(h : \text{Spec}(\text{Na}(D, \star)) \rightarrow \text{QSpec}^\star(D) \cup \{0\}\). Indeed if \(N\) is a prime ideal of \(\text{Na}(D, \star)\), then there exists a quasi-\(\star\)-maximal ideal \(M\) of \(D\), such that \(N \subseteq M \text{Na}(D, \star)\). So that
\[
h(N) = N \cap D \subseteq M \text{Na}(D, \star) \cap D = M \text{Na}(D, \star) \cap K \cap D = M^\star \cap D = M.
\]

The third equality holds by \cite{14} Proposition 3.4 (3)]. So that \(h(N) \in \text{QSpec}^\star(D) \cup \{0\}\), since it is contained in \(M\) and \cite{11} Lemma 4.1 and Remark 4.5]. Note that if \(P \in \text{QSpec}^\star(D)\), then
\[
h(P \text{Na}(D, \star)) = P \text{Na}(D, \star) \cap D = P \text{Na}(D, \star) \cap K \cap D = P^\star \cap D = P.
\]
Therefore $h(\text{Spec}(\mathcal{N}(D, \star))) = \text{QSpec} \tilde{\mathcal{N}}(D) \cup \{0\}$. In fact using [7, Theorem 2.16], the map $h$ is bijective if and only if $D$ is a $\star$-quasi-Prüfer domain.

**Lemma 4.16.** Let $\star$ be a semistar operation on an integral domain $D$. Then each valuation overring of $\mathcal{N}(D, \star)$ is a $\star[X]$-valuation overring of $D[X]$.

**Proof.** Let $W$ be a valuation overring of $\mathcal{N}(D, \star)$. Let $M$ be the maximal ideal of $W$. Set $\Omega := M \cap \mathcal{N}(D, \star)$ and $Q := M \cap [D][X]$. Since $\Omega \in \text{Spec}(\mathcal{N}(D, \star))$, we have $h(\Omega) = \Omega \cap D = Q \cap D \in \text{QSpec} \tilde{\mathcal{N}}(D) \cup \{0\}$. Thus by Remark 2.3, we obtain that $Q$ is a quasi-$\star[X]$-prime ideal of $[D][X]$. Now choose a quasi-$\star[X]$-maximal ideal $\mathcal{M}$ of $[D][X]$ containing $Q$. Thus we have $[D][X]_{\mathcal{M}} \subseteq [D][X]_Q \subseteq W$. Consequently by [14, Theorem 3.9], we obtain that $W$ is a $\star[X]$-valuation overring of $D[X]$.

Recall that for each domain $D$, $\text{dim}_v(D) = \sup\{\text{dim}_v(D_M)\mid M \in \text{Max}(D)\}$. In fact if $n = \text{dim}_v(D)$, then there exists a valuation overring $V$, with maximal ideal $N$, of $D$ such that $\text{dim}(V) = n$. Put $M := N \cap D$. So that $V$ is a valuation overring of $D_M$. Hence $\text{dim}_v(D) = n = \text{dim}_v(V) \leq \text{dim}_v(D_M) \leq \text{dim}_v(D) = n$. Actually one can assume that $M$ is a maximal ideal of $D$.

Let $\star$ be a semistar operation on an integral domain $D$. Recall from [10] that the Kronecker function ring of $D$ with respect to the semistar operation $\star$ is defined by:

$$\text{Kr}(D, \star) := \left\{ \frac{f}{g} \mid f, g \in [D][X], g \neq 0, \text{ and there exists } h \in [D][X] \setminus \{0\} \text{ with } (c(f)c(h))^* \subseteq (c(g)c(h))^* \right\}. $$

It is an overring of the $\star$-Nagata ring with quotient field $K(X)$, which is a Bézout domain [16]. From [15, Theorem 3.5], we have $V$ is a valuation overring of $D$ if and only if $V(X)$ is a valuation overring of $\text{Kr}(D, \star)$. Now we are ready to prove the following theorem.

**Theorem 4.17.** Let $\star$ be a semistar operation on an integral domain $D$. Then

$$\bar{\star}-\text{dim}_v(D) = \text{dim}_v(\mathcal{N}(D, \star)).$$

**Proof.** Consider the following inequalities:

$$\bar{\star}-\text{dim}_v(D) \leq \text{dim}_v(\text{Kr}(D, \bar{\star})) \leq \text{dim}_v(\mathcal{N}(D, \star)) \leq \star[X]-\text{dim}_v([D][X]) = \bar{\star}-\text{dim}_v(D) + 1.$$  

The first inequality follows from the fact that if $V$ is a $\bar{\star}$-valuation overring of $D$, then $V(X)$ is a valuation overring of $\text{Kr}(D, \bar{\star})$ and that $\text{dim}(V) = \text{dim}(V(X))$; second inequality follows from the fact that $\mathcal{N}(D, \star) \subseteq \text{Kr}(D, \bar{\star})$, while the third one uses the Lemma 4.16. So that we can assume that $\bar{\star}-\text{dim}_v(D)$ and $\text{dim}_v(\mathcal{N}(D, \star))$ are finite numbers. Now by observation before the theorem, choose a quasi-$\bar{\star}$-maximal ideal $P$ of $D$, such that the maximal ideal $M := PN\mathcal{N}(D, \star)$ has the property that $\text{dim}_v(\mathcal{N}(D, \star)) = \text{dim}_v(\mathcal{N}(D, \star)_M) = \text{dim}_v(D_P(X))$.

But since $P \in \text{QMax} \tilde{\mathcal{N}}(D)$, each valuation overring of $D_P$, is a $\bar{\star}$-valuation overring of $D$ [14, Theorem 3.9]. Hence we find the inequality $\text{dim}_v(D_P) \leq \bar{\star}-\text{dim}_v(D)$. Consequently we have

$$\bar{\star}-\text{dim}_v(D) \leq \text{dim}_v(\mathcal{N}(D, \star)) = \text{dim}_v(D_P(X)) = \text{dim}_v(D_P) \leq \bar{\star}-\text{dim}_v(D),$$
in which the second equality holds by [11 Proposition 1.22]. Thus we find the desired equality 
\[ \dim_v(D) = \dim_v(\text{Na}(D, \star)). \]

As an immediate corollary we have:

**Corollary 4.18.** Let \( \star \) be a semistar operation on an integral domain \( D \). Then:

(a) \( D[X] \) is a \( \star[X] \)-Jaffard domain, if and only if, \( \text{Na}(D, \star) \) is a Jaffard domain.

(b) \( D \) is a \( \bar{\star} \)-Jaffard domain if and only if \( \text{Na}(D, \star) \) is a Jaffard domain and
\[ \dim_v(D[X]) = \bar{\star} \cdot \dim(D) + 1. \]

**Proof.** Both statements are easy consequences of Theorems 4.15 and 4.17, and for
(a) use also Theorem 4.8. \( \square \)

**Remark 4.19.** By the proof of the above theorem, we have
\[ \dim_v(D) = \dim_v(\text{Kr}(D, \bar{\star})). \]
Since \( \text{Kr}(D, \bar{\star}) \) is a Bézout, and hence a Prüfer domain, we have
\[ \bar{\star} \cdot \dim_v(D) = \dim_v(\text{Kr}(D, \bar{\star})) = \dim(\text{Kr}(D, \bar{\star})). \]

**Remark 4.20.** Let \( D, \bar{D}, \star, \) and \( \star \) be as in the Proposition 4.10. Note that by the
proof of part (6.\( \bar{\star} \)) \( \Rightarrow \) (10.\( \star \)) of [7] Theorem 2.16] we have \( \text{Na}(\bar{D}, \star) = \text{Na}(D, \star). \) So
that by Theorem 4.17 we have
\[ \star \cdot \dim_v(\bar{D}) = \dim_v(\text{Na}(\bar{D}, \star)) = \dim_v(\text{Na}(D, \star)) = \bar{\star} \cdot \dim_v(D), \]
which is another reason for the last equality in the proof of Proposition 4.10.

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