First law of black hole mechanics with fermions

P B Aneesh1, Sumanta Chakraborty2,3,∗, Sk Jahanur Hoque1,4 and Amitabh Virmani1

1 Chennai Mathematical Institute, H1 SIPCOT IT Park, Kelambakkam, Tamil Nadu 603103, India
2 School of Mathematical and Computational Sciences, Indian Association for Cultivation of Science, Kolkata 70032, India
3 School of Physical Sciences, Indian Association for Cultivation of Science, Kolkata 70032, India
4 Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičká 2, 180 00 Prague 8, Czech Republic

E-mail: aneeshpb@cmi.ac.in, tpsc@iacs.res.in, skjhoque@cmi.ac.in and avirmani@cmi.ac.in

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Abstract
In the last few years, there has been significant interest in understanding the stationary comparison version of the first law of black hole mechanics in the vielbein formulation of gravity. Several authors have pointed out that to discuss the first law in the vielbein formulation one must extend the Iyer–Wald Noether charge formalism appropriately. Jacobson and Mohd (2015 Phys. Rev. D 92 124010) and Prabhu (2017 Class. Quantum Grav. 34 035011) formulated such a generalisation for symmetry under combined spacetime diffeomorphisms and local Lorentz transformations. In this paper, we apply and appropriately adapt their formalism to four-dimensional gravity coupled to a Majorana field and to a Rarita–Schwinger field. We explore the first law of black hole mechanics and the construction of the Lorentz-diffeomorphism Noether charges in the presence of fermionic fields, relevant for simple supergravity.

Keywords: first law of black hole thermodynamics, vielbein formalism of gravity, Rarita–Schwinger field, AdS supergravity

1. Introduction

Iyer and Wald [1] gave a derivation of the stationary comparison version of the first law of black hole mechanics for arbitrary perturbations around a stationary axisymmetric black hole in any diffeomorphism covariant theory of gravity. The black hole entropy was identified with...
the integral over the bifurcation surface of the diffeomorphism Noether charge for the horizon generating Killing field. The results of Iyer and Wald have found numerous applications over the years [2–4].

However, as emphasised in detail in references [5, 6], there are situations of physical interest where Iyer–Wald analysis cannot be applied directly. An assumption that goes into their analysis is that matter fields, if present, are smooth tensor fields on the spacetime. Often in gauge theories, one cannot always make a gauge choice such that the gauge fields are smooth tensor fields. A similar situation arises for gravity theories written in vielbein formulation [5], where the spin-connection might not be smooth in some chosen gauge (i.e., for a choice of vielbeins $e^\mu_\nu$). Since the coupling of fermions to gravity is typically through the vielbeins, Iyer–Wald analysis cannot be directly applied to gravity coupled to fermions. In the last few years, there has been interest in understanding the stationary comparison version of the first law of black hole mechanics in the vielbein formulation and renewed interest in the theory of Noether charges in the vielbein formulation of gravity, see e.g., [7–11].

In order to discuss the first law in the vielbein formulation, one must extend the Iyer–Wald’s Noether charge formalism appropriately. Prabhu [6] formulated a generalisation for symmetry under combined diffeomorphisms and internal gauge transformations in terms of fields living on a principal bundle over spacetime. Using this formalism he presented a derivation of the first law of black hole mechanics in the vielbein formulation of gravity, for gravity coupled to Yang–Mills field and for gravity coupled to a Dirac fermion. Interestingly, he found that the contribution due to the Dirac field to the Lorentz-diffeomorphism $^5$ Noether charge vanishes on-shell for the horizon generating Killing field at the bifurcation surface. Thus, we are led to the question: does the contribution to the Lorentz-diffeomorphism Noether charge vanish generically for fermions at the bifurcation surface?

The situation should be compared to bosonic fields. For a minimally coupled scalar too the contribution to the diffeomorphism Noether charge due to the scalar vanishes. Though, this is not the case for a vector field (see e.g., [12]). Thus, it is interesting to explore the case of the Rarita–Schwinger field, which besides being fermionic also carries a spacetime index. In this paper, we apply and appropriately adapt the formalism of [5, 6] to four-dimensional gravity coupled to a Majorana field and to a Rarita–Schwinger field. We explore the first law of black hole mechanics and the construction of Lorentz-diffeomorphism Noether charges. With the Rarita–Schwinger field one can in principle write down a few different diffeomorphism covariant Lagrangians. Perhaps the most natural set-up to consider is the case of $N = 1, D = 4$ supergravity, often called simple supergravity.

At this stage, it is important to note that local supersymmetry is not an internal gauge symmetry; it is a spacetime symmetry. Prabhu’s formalism [6], although quite general, is restricted to the cases of internal gauge symmetry. Thus, local supersymmetry cannot be properly taken into account within the framework of references [5, 6]. For this reason, we ‘switch off’ supersymmetry in our study of the Lorentz-diffeomorphism Noether charges. Perhaps an independent formulation of the first law can be achieved by considering supergravity as a geometric theory on superspace, where the fermionic gauge symmetry is properly taken into account. We leave this for future work.

$^5$ A Lorentz-diffeomorphism is a diffeomorphism accompanied by a Lorentz transformation, i.e., a frame rotation. The particular Lorentz transformation of the frames, given a diffeomorphism, is given by Kosmann derivative introduced in equation (2.1).
The rest of the paper is organised as follows. In section 2 we summarise the Lorentz diffeomorphism Noether charge formalism. For simple supergravity, the Rarita–Schwinger field is a Majorana vector–spinor. The Majorana condition brings in new elements in the computation. It is instructive to study the simpler spin-1/2 Majorana field first, before diving into the case of the simple supergravity. Therefore, in section 3 we study the Lorentz-diffeomorphism Noether charge for the simpler case of spin-1/2 Majorana field. This section connects to the case of the Dirac field studied in reference [6]. It also serves as a warm-up for simple supergravity considered in section 4. The key result of section 4 is an expression for the contribution to the Lorentz-diffeomorphism Noether charge due to the Rarita–Schwinger field. Using this Noether charge we formulate a stationary comparison version of the first law in section 5. We close with a brief discussion in section 6.

2. A summary of the Lorentz-diffeomorphism Noether charge formalism

For theories with internal gauge symmetries there is no natural action of spacetime diffeomorphisms on dynamical fields. We only have a notion of spacetime diffeomorphisms up to gauge transformations. Without such a separation between spacetime diffeomorphisms and gauge transformations, the diffeomorphism Noether charge is not an adequate concept to work with. This issue has been discussed in the context of the first law of black hole mechanics over the years [12–16] and, from our point of view, is satisfactorily addressed in [5, 6]. A different perspective on these issues is presented in [10].

Prabhu [6] formulates a given gravity theory in terms of fields living on a principal bundle over spacetime. Then he considers the full group of transformations, diffeomorphisms together with gauge transformations, viewed as automorphisms of the bundle. For a general such automorphism he defines a notion of Noether charge and uses it to obtain a first law of black hole mechanics as a variational identity. A key idea in this construction is the fact that the variation of the fields under a general automorphism $X$ is simply given by the standard Lie derivative with respect to vector field $X$ on the bundle.

Jacobson and Mohd in reference [5] take a more pragmatic approach. They propose a generalisation of the Iyer–Wald diffeomorphism Noether charge for a given spacetime diffeomorphism to what they call Lorentz-diffeomorphism Noether charge. The two approaches have some similarities and some differences. In general there is no unique way to associate an automorphism of the bundle to a given spacetime diffeomorphism. However, the requirement that the vielbeins (co-frames) be preserved by the corresponding Killing vector of a given spacetime metric, uniquely determines the infinitesimal bundle automorphism. A proof of this statement with references to the original literature can be found in [5, 6]. This uplift defines a Lorentz–Lie derivative. We use the notation $\mathcal{K}$ following [5] where the Lorentz–Lie derivative was defined for arbitrary spacetime vector fields by the same formula as for the Killing vector fields. We call it the Kosmann derivative. On co-frames, for arbitrary $\xi$,

$$K_\xi e^a_\mu = \mathcal{L}_\xi e^a_\mu + (E^\nu{}_b \mathcal{L}_\xi e^b_\nu) e^a_\mu, \quad (2.1)$$

where $\mathcal{L}_\xi$ is the standard Lie derivative computed by ignoring the internal indices, and $E^a_\mu$ are the inverse vielbeins. Since in the later parts of the paper we suppress spacetime indices and work in the form notation, a separate notation is required for the vielbeins and inverse vielbeins. The vielbeins and inverse vielbeins satisfy the usual properties, $E_a \cdot e^b := E^a_\mu e^b_\mu = \delta^b_a$, and under arbitrary variation $\delta E_a \cdot e^b = -E_a \cdot \delta e^b$. 

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In order to generalise the concept of Kosmann derivative to an arbitrary tensor–spinor object carrying both spacetime and/or Lorentz/spinor indices, we first rewrite equation (2.1) as:

\[
K_\xi e^a_\mu = L_\xi e^a_\mu + \frac{1}{2} \left( \xi \cdot \hat{\omega}_{cd} + E^c_\mu \partial_\delta b_\xi \right) \left( 2 \gamma^{\mu \nu} \delta^b_\nu \right) e^b_\mu, \tag{2.2}
\]

\[
= L_\xi e^a_\mu + \frac{1}{2} \left( \xi \cdot \hat{\omega}_{cd} + D_\mu \xi_{\delta} \right) \left( 2 \gamma^{\mu \nu} \delta^b_\nu \right) e^b_\mu, \tag{2.3}
\]

where \(\hat{\omega}_{cd}\) is the torsionless part of the spin-connection and \(D_\mu\) is the torsionless Lorentz covariant derivative. This rewriting makes the vector representation of the Lorentz group in equation (2.1) manifest, and allows us to propose a Kosmann derivative for an arbitrary tensor–spinor object\[17\]. For an arbitrary spacetime vector field \(\xi\), we define,

\[
K_\xi T_{\mu_1 \ldots \mu_m \nu_1 \ldots \nu_n} \equiv L_\xi T_{\mu_1 \ldots \mu_m \nu_1 \ldots \nu_n} + \frac{1}{2} \left( \xi \cdot \hat{\omega}_{ab} + D_\mu \xi_{\delta} \right) \Gamma_r(M_{ab}) T_{\mu_1 \ldots \mu_m \nu_1 \ldots \nu_n}, \tag{2.4}
\]

where the Lorentz/spinor indices are suppressed and the Lie derivative is the usual Lie derivative that only sees the spacetime indices, and where \(\Gamma_r(M_{ab})\) are the representation matrices for the Lorentz generators \(M_{ab}\) in the representation \(r\) of the Lorentz tensor–spinor \(T\). For vector representation,

\[
\Gamma_{\text{vec}}(M_{ab})^c_d = 2\eta^c[a \delta b], \tag{2.5}
\]

and for the four-dimensional spinor representation,

\[
\Gamma_{\text{spinor}}(M_{ab}) = \frac{1}{2} \gamma^{ab} = \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a). \tag{2.6}
\]

For spinors, gamma matrices, Majorana condition, Majorana flip conventions we exclusively follow reference\[18\].

It then follows that for spinors, the Kosmann derivative takes the form,

\[
K_\xi \Psi = \xi^\mu \hat{D}_\mu \Psi + \frac{1}{4} \partial_\mu \xi_{\delta} \gamma^{\mu \nu} \Psi, \tag{2.7}
\]

where \(\hat{D}_\mu\) is the torsionless spinor covariant derivative,

\[
\hat{D}_\mu \Psi(x) = \left( \partial_\mu + \frac{1}{4} \hat{\omega}_{ab} \gamma^{ab} \right) \Psi(x), \tag{2.8}
\]

and where \(\hat{\omega}_{ab}\) and the Christoffel symbol \(\hat{\Gamma}_\mu^{\alpha \beta}\) are related by the vielbein postulate,

\[
\partial_\mu e^a_\nu + \hat{\omega}_{ab} \gamma^{ab} - \Gamma^{\alpha \beta}_{\mu \nu} \gamma^\alpha = 0. \tag{2.9}
\]

Along identical lines, it also follows that the Kosmann derivative for the Rarita–Schwinger field, which is a vector–spinor field, takes the form,

\[
K_\xi \psi_\mu = L_\xi \psi_\mu + \frac{1}{4} \left( \xi \cdot \hat{\omega}_{ab} + D_\mu \xi_{\delta} \right) \gamma^{ab} \psi_\mu
\]

\[
= \xi^\alpha \partial_\alpha \psi_\mu + \psi_\alpha \partial_\mu \xi^\alpha + \frac{1}{4} \left( \xi \cdot \hat{\omega}_{ab} + D_\mu \xi_{\delta} \right) \gamma^{ab} \psi_\mu
\]

\[
= \xi^\alpha \left( \partial_\alpha \psi_\mu - \partial_\mu \psi_\alpha + \frac{1}{4} \hat{\omega}_{ab} \gamma^{ab} \psi_\mu - \frac{1}{4} \hat{\omega}_{ab} \gamma^{ab} \psi_\alpha \right) + \partial_\mu (\xi^\alpha \psi_\alpha) \tag{2.10}
\]
\[ \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} (\xi^\alpha \psi_\alpha) + \frac{1}{4} D_\mu \xi^\alpha \gamma^{ab} \psi_\mu = (D_\mu \psi_\mu - D_\mu \psi_\alpha) + \frac{1}{4} D_\mu \xi^\alpha \gamma^{ab} \psi_\mu, \]  

(2.10)

where in the third line we have added and subtracted appropriate terms to bring it in the final form. Later we will convert this final form in the index-free form notation.

Jacobson and Mohd [5] proposed to use the Kosmann derivative to define a Lorentz-diffeomorphism associated to \( \xi \). Although, the association of the (projection of the) bundle automorphism \( K_\xi \) to spacetime diffeomorphism \( \mathcal{L}_\xi \) is different from the spirit of [6], when restricted to Killing vectors the conclusions from the two formalisms are identical. In particular, one gets the same first law for stationary spacetimes using either of the two prescriptions. We make use of this fact and borrow convenient notation from both these references to discuss the first law with the Majorana and Rarita–Schwinger fields.

A comment about notation is in order: Prabhu [6] uses underline to distinguish between fields on the spacetime (base space) from those on the principal bundle. We do not use that notation. For us all quantities are on the spacetime. The main technical point we use from [5, 6] is the modification of the Lie derivative to the Kosmann derivative when evaluating the Noether currents.

Let \( L \) denote the Lagrangian \( d \)-form. Spacetime differential forms are denoted with boldface letters and center dot is used to denote the interior derivative: for a \( p \)-form \( A \), \( \xi \cdot A \). We assume that the Lagrangian is diffeomorphism covariant and is a Lorentz scalar, and its variations are the same when we vary the fields with the Kosmann derivative or with the Lie derivative, i.e.,

\[ K_\xi L = \mathcal{L}_\xi L = d(\xi \cdot L). \]  

(2.11)

Let us denote the dynamical field collectively as \( \varphi^\alpha \) (and also simply as \( \varphi \)), where \( \alpha \) denotes any internal indices that the field may carry. The variation of the Lagrangian \( d \)-form induced by a field variation \( \delta \varphi^\alpha \) can be written as,

\[ \delta L = \mathcal{E}_\alpha(\varphi) \wedge \delta \varphi^\alpha + d\theta(\varphi, \delta \varphi). \]  

(2.12)

The quantity \( \mathcal{E}_\alpha(\varphi) \) defines the equations of motion, \( \mathcal{E}_\alpha(\varphi) = 0 \), and the \((d-1)\)-form \( \theta \) is constructed out of the dynamical fields \( \varphi \) and their first variations \( \delta \varphi \). Now let us consider the variation of the Lagrangian induced by an arbitrary vector field \( \xi \), with field variations,

\[ \delta_\xi \varphi^\alpha = K_\xi \varphi^\alpha. \]  

(2.13)

To such a variation we can associate an \((d-1)\)-form called the Lorentz-diffeomorphism Noether current,

\[ J_\xi = \theta(\varphi, K_\xi \varphi) - \xi \cdot L. \]  

(2.14)

For all vector fields \( \xi \), the Noether current is closed on-shell. This implies [5, 6, 19] that on-shell \( J_\xi \) is an exact form \( J_\xi = dQ_\xi \). In this work, we are concerned with the construction of the Lorentz-diffeomorphism Noether charge \( Q_\xi \) for the theories of interest.

The Lagrangians considered in this paper are of the form,

\[ L = L_{\text{grav}} + L_{\text{matter}}, \]  

(2.15)

where \( L_{\text{grav}} \) depends on the vielbein one-forms (also called frame-field one forms or co-frames) \( e^\mu = e^\mu_\rho dx^\rho \) and the spin-connection one-forms \( \omega^{ab} = \omega^{ab}_\mu dx^\mu \). We exclusively work in the
first order formalism, i.e., we treat $e^a$ and $\omega^{ab}$ as independent fields. We write the equations of motion terms in the variation with respect to the co-frames $e^a$ and the spin-connection $\omega^{ab}$ as,

$$
(\delta L_{\text{grav}})_{\text{cem}} = E_a \wedge \delta e^a, \quad (\delta L_{\text{matter}})_{\text{cem}} = -T_a \wedge \delta e^a, \quad E_a - T_a = 0,
$$

(2.16)

$$
(\delta L_{\text{grav}})_{\text{cem}} = E_{ab} \wedge \delta \omega^{ab}, \quad (\delta L_{\text{matter}})_{\text{cem}} = -S_{ab} \wedge \delta \omega^{ab}, \quad E_{ab} - S_{ab} = 0.
$$

(2.17)

The above equations define the symbols $E_{ab}$, $T_a$, $E_{ab}$ and $S_{ab}$.

For the first order formulation of gravity with a cosmological constant we use the following Lagrangian,

$$
L_{\text{grav}} = \frac{(R - 2\Lambda)}{16\pi G} e_4,
$$

(2.18)

often called the Einstein–Hilbert–Palatini Lagrangian. This theory was considered in detail in references [5, 6]. For this Lagrangian we have a symplectic potential boundary term, which takes the form,

$$
\theta = \frac{1}{32\pi G} \varepsilon_{abcd} e^a \wedge e^b \wedge \delta \omega^{cd}.
$$

(2.19)

Now, in order to compute the Lorentz-diffeomorphism Noether current equation (2.14) we need the Kosmann derivative of the spin-connection. This can be obtained in a number of ways. We follow the following logic: we recall that infinitesimal variation of the spin-connection $\delta \omega^{ab}$ transforms covariantly under local Lorentz transformations. The first Cartan structure equation $(d e^a + \omega^{ab}_b \wedge e^b = T^a)$, where $T^a$ is the torsion two-form allows us to write $\delta \omega^{ab}$ in terms of $\delta e^a$ and $\delta T^a$. Since $T^a$ is a proper Lorentz-tensor, its Kosmann derivative is uniquely defined by the above considerations, cf equation (2.4). Therefore, taking $\delta \omega^{ab} = K_{\xi} \omega^{ab}$, $K_{\xi} \omega^{ab}$ can be related to $K_{\xi} e^a$ and $K_{\xi} T^a$. A calculation gives,

$$
K_{\xi} \omega^{ab} = L_{\xi} \omega^{ab} = D \left( e^{a[\xi} L_{\xi} e^{b]} \right),
$$

(2.20)

where $D$ is the standard local Lorentz covariant derivatives. When restricted to the torsionless case, this equation is same as the one given in [5].

The Lorentz-diffeomorphism Noether current for the gravity sector Lagrangian equation (2.18) takes the form

$$
J_{\xi} = \frac{1}{32\pi G} \varepsilon_{abcd} e^a \wedge e^b \wedge \left( K_{\xi} \omega^{cd} \right) - \left( \xi \cdot L_{\text{grav}} \right)
$$

$$
= dQ_{\xi} + E_a (\xi \cdot e^a) + E_{ab} \left\{ \frac{1}{2} E_a \cdot E_b \cdot [d \xi - T^c (E_c \cdot \xi)] - (E_c \cdot \xi) (E_{[a} \cdot E^{c]} \cdot T_b)] \right\},
$$

(2.21)

6 Sometimes the $d - 1$ form $\theta(\varphi, \delta \varphi)$ is called the presymplectic potential. Using this we construct the presymplectic current $\omega(\varphi, \delta \varphi, \delta^2 \varphi) = \delta(\theta(\varphi, \delta \varphi) - \delta^2 \theta(\varphi, \delta \varphi))$. Integrating $\omega$ on a Cauchy surface gives a presymplectic two-form on the configuration space $F$. The presymplectic two-form is not a true symplectic two-form as it is not non-degenerate. It is possible to construct a phase space with a non-degenerate symplectic form [20]. However, for our purposes the presymplectic forms will be sufficient. Thus, to simplify terminology we use the names symplectic potential for $\theta$ and symplectic current for $\omega$. 

6
where
\[ E_a = -\frac{1}{16\pi G} \varepsilon_{abcd} e^b \wedge R^{cd} + \frac{\Lambda}{8\pi G} (\ast e_a), \tag{2.22} \]
\[ E_{ab} = \frac{1}{16\pi G} \varepsilon_{abcd} e^c \wedge T^d, \tag{2.23} \]
and the Noether charge \( Q_\xi \) is
\[ Q_\xi = \frac{1}{32\pi G} \varepsilon_{abcd} e^a \wedge e^b \left\{ \frac{1}{2} E_c \cdot E_d \cdot [d\xi - T^* (E_c \cdot \xi)] - (E_c \cdot \xi) (E_e \cdot E^* \cdot T_d) \right\}. \tag{2.24} \]

In equation (2.22), \( \ast \) denotes the four-dimensional Hodge dual. Expression (2.24) can be written more conveniently using the contorsion tensor (see below) and can be compared with (the projection of) equation (5.8) of reference [6]. In the torsionless case, the Noether charge \( Q_\xi \) takes the familiar form,
\[ Q_\xi = -\frac{1}{16\pi G} (\ast d\xi). \tag{2.25} \]

To get the first law for a stationary, axisymmetric, black hole solution we use the horizon generating Killing field \( k \) in place of \( \xi \). Since \( k \) vanishes at the bifurcation surface, the integration of the Noether charge equation (2.24) over the bifurcation two-sphere \( B \) gives
\[ \int_B Q_k = \frac{\kappa_B}{8\pi} \text{area}(B), \tag{2.26} \]
where \( \kappa_B \) is surface gravity of the black hole horizon.

3. Lorentz-diffeomorphism Noether charge for Majorana field

In this section we study the Lorentz-diffeomorphism Noether charge for the spin-1/2 Majorana field. The aim of this section is to first compute the symplectic potential \((d-1)\)-form \( \theta \) and then the corresponding Lorentz-diffeomorphism Noether current \( J_\xi \).

The Lagrangian four-form for a massive Majorana field takes the form,
\[ L_{\text{matter}} = e_4 \left( -\frac{1}{2} \bar{\Psi} \not{\partial} \Psi + \frac{1}{2} m \bar{\Psi} \Psi \right), \tag{3.1} \]
where \( \bar{\Psi} \) is the Majorana conjugate and the derivative \( \not{\partial} \) is defined below. For the benefit of the reader we review the definition of Dirac, Majorana, and charge conjugate following [18]. Let \( C \) be the charge conjugation matrix and let \( \Gamma^{(r)} \) be the rank \( r \) product of antisymmetrised \( \gamma \) matrices
\[ \Gamma^{(r)} : \gamma^{\mu_1 \cdots \mu_r} = \gamma^{[\mu_1} \cdots \gamma^{\mu_r]}. \tag{3.2} \]
The conventions reference [18] follows for four-dimensions (and Lorentzian signature) are
\[ (CT^{(r)})^T = -t_r CT^{(r)}, \tag{3.3} \]
with \( t_0 = +1, t_1 = -1, t_2 = -1, t_3 = +1, \) and \( t_{r+4} = t_r \). The Majorana conjugate is defined as,
\[ \bar{\Psi} = \Psi^T C. \tag{3.4} \]
The Majorana field by definition satisfies the Majorana condition, \( \Psi^C = \Psi \), where \( \Psi^C \) is the charge conjugate defined via \( \Psi^* = i t_0 C^0 \Psi^C \). These definitions imply that the Dirac conjugate
\( \bar{\Psi} = i\Psi^\dagger \gamma^0 \) is same as the Majorana conjugate for a Majorana spinor. Due to this equivalence, \( \Psi \) in equation (3.1) can be taken to be the Dirac conjugate for the Majorana spinor \( \Psi \). For explicit computations it is easier to work with the Dirac conjugate.

The covariant derivative \( \overleftrightarrow{D} \) used in the above Lagrangian is defined as

\[
\overleftrightarrow{D} \Psi = \gamma^\mu D_\mu \Psi = \gamma^a E^\mu_a D_\mu \Psi = \gamma^a (E_a \cdot D_\Psi) = \gamma^\mu \left( \partial_\mu + \frac{1}{4} \omega^a_{\mu \lambda} \gamma_{ab} \right) \Psi ,
\]

where the Latin indices are Lorentz frame indices and the Greek indices are spacetime indices. It is convenient to define the derivative operator acting on \( \bar{\Psi} \) as well via

\[
\overrightarrow{D} \bar{\Psi} = -\frac{1}{2} \bar{\Psi} \gamma^\mu \Psi = - \left( \partial_\mu \bar{\Psi} - \frac{1}{4} \bar{\Psi} \omega^a_{\mu \lambda} \gamma_{ab} \right) \gamma^\mu \Psi .
\]

Unlike the case of the Dirac field, where the reality condition on the Lagrangian requires us to work with the symmetrised derivative \( (\bar{\Psi} \overleftrightarrow{D} \Psi + \overrightarrow{D} \bar{\Psi}) \), Lagrangian equation (3.1) is automatically real due to the Majorana condition. To see this we note that,

\[
(\bar{\Psi} \overleftrightarrow{D} \Psi) \dagger = (\overleftrightarrow{D} \Psi \Psi)(i \Psi^\dagger \gamma^0) \dagger \\
= i \left( \partial_\mu \Psi^\dagger + \frac{1}{4} \omega^a_{\mu \lambda} \Psi^\dagger \gamma^0 \gamma^{ab} \gamma^0 \right) (\gamma^0 \gamma^\mu \gamma^0 \Psi) \\
= - \left( \partial_\mu \Psi^\dagger - \frac{1}{4} \omega^a_{\mu \lambda} \Psi^\dagger \gamma^{ab} \gamma \right) \gamma^\mu \Psi \\
= \bar{\Psi} \gamma^\mu \partial_\mu \Psi + \frac{1}{4} \omega^a_{\mu \lambda} \bar{\Psi} \gamma^\mu \gamma^{ab} \Psi \\
= \bar{\Psi} \overrightarrow{D} \Psi ,
\]

where we have repeatedly used the relation \( \bar{\Psi} \gamma^\mu \Psi = 0 \) for a Majorana spinor. Note that this property is not available for the Dirac field.

The Lagrangian presented in equation (3.1) depends on three fields: the vielbein \( e^a \), the spin-connection \( \omega^{ab} \) and the Majorana field \( \Psi \). The variation of the Lagrangian with respect to the vielbein takes the form,

\[
\delta_{\epsilon_a} L = (\delta \epsilon_a) \left( -\frac{1}{2} \bar{\Psi} \overleftrightarrow{D} \Psi + \frac{1}{2} m \bar{\Psi} \Psi \right) - \frac{1}{2} \epsilon_4 \delta (\bar{\Psi} \overleftrightarrow{D} \Psi) \\
= (\delta \epsilon_a) \left( -\frac{1}{2} \bar{\Psi} \overleftrightarrow{D} \Psi + \frac{1}{2} m \bar{\Psi} \Psi \right) - \frac{1}{2} \epsilon_4 \bar{\Psi} \gamma^a (E_a \cdot D_\Psi) \\
= (\delta \epsilon_4) \left( -\frac{1}{2} \bar{\Psi} \overleftrightarrow{D} \Psi + \frac{1}{2} m \bar{\Psi} \Psi \right) + \frac{1}{2} \epsilon_4 \bar{\Psi} \gamma^a (E_a \cdot D_\Psi) (E_a \cdot \delta e^b) \\
= (\epsilon_a) \left( -\frac{1}{2} \bar{\Psi} \overleftrightarrow{D} \Psi + \frac{1}{2} m \bar{\Psi} \Psi \right) \wedge \delta e^a - \frac{1}{2} (E_b \cdot \epsilon_4) \bar{\Psi} \gamma^b (E_a \cdot D_\Psi) \wedge \delta e^a \\
= (\epsilon_a) \left( -\frac{1}{2} \bar{\Psi} \overleftrightarrow{D} \Psi + \frac{1}{2} m \bar{\Psi} \Psi \right) \wedge \delta e^a - \frac{1}{2} \epsilon_b \bar{\Psi} \gamma^b (E_a \cdot D_\Psi) \wedge \delta e^a \\
:= - \mathcal{T}_a \wedge \delta e^a .
\]
The above manipulations have been performed as follows. In going from the second to the third step we have related \( \delta E_a \) to \( \delta e^b \). In going from the third to the fourth step we have used the result, \( (\delta e^b) = - (\star e^b) \wedge |(V \cdot B) | \wedge (V \cdot e^b) \),

\[
V \cdot (A \wedge B) = (V \cdot A) \wedge B + (-1)^p A \wedge (V \cdot B),
\]

where, \( V \) is a vector field, \( A \) is a \( p \)-form and \( B \) is \( q \)-form. In the present context we have used identity equation (3.9) with \( A = e_4, B = \delta e^a \) and \( V = E_a \), such that, \( A \wedge B = 0 \). In going from the fourth to the fifth step we have used \( (E_b \cdot e_4) = \star e_b \). Finally, the last line of equation (3.8) defines the energy–momentum three-form \( T_a \),

\[
T_a = (\star e_a) \left( - \frac{1}{2} \bar{\Psi} \gamma^\mu E_a \cdot \left( \frac{1}{2} m \bar{\Psi} \gamma^\mu \right) + \frac{1}{2} (\star e_b) \bar{\Psi} \gamma^b E_a \cdot D \Psi \right).
\]

Similarly, the variation with respect to the spin connection gives,

\[
\delta_\omega L = - \frac{1}{2} e_d \bar{\Psi} \gamma^\mu E_a \cdot \left( \frac{1}{4} \delta \omega_{cd} \gamma^\mu \right) \Psi
\]

\[
= - \frac{1}{8} e_d \bar{\Psi} \gamma^\mu \gamma_{cd} (E_a \cdot \delta \omega_{cd}) = \frac{1}{8} (E_a \cdot e_4) \bar{\Psi} \gamma^\mu \gamma_{cd} \Psi \wedge \delta \omega_{cd}
\]

\[
= - S^{cd} \wedge \delta \omega_{cd},
\]

where we have again used identity equation (3.9) with \( A = e_4, B = \delta \omega_{cd}, \) and \( V = E_a \) such that \( A \wedge B = 0 \). The last line of the above equation defines the spin-current three-form \( S^{cd} \),

\[
S^{cd} = - \frac{1}{8} (E_a \cdot e_4) \bar{\Psi} \gamma^\mu \gamma_{cd} \Psi.
\]

Finally, we consider the variation of the Lagrangian with respect to \( \Psi \). Unlike the case of Dirac field, where variations of \( \Psi \) and \( \bar{\Psi} \) are treated as independent, here the field variations must also satisfy the Majorana condition so that \( \delta \bar{\Psi} \) is related to \( \delta \Psi \). In practice this condition is implemented through the Majorana flip relations. Let \( \lambda_1 \) and \( \lambda_2 \) be two arbitrary Majorana spinors (possibly with other spacetime indices), then,

\[
\tilde{\lambda}_1 \Gamma^{(i)} \lambda_2 = t_i \hat{\lambda}_1 \Gamma^{(i)} \lambda_1,
\]

which for our applications read,

\[
\delta \bar{\Psi} \gamma^\mu \Psi = \bar{\Psi} \gamma^\mu \delta \Psi,
\]

\[
\delta \bar{\Psi} \gamma_{cd} \Psi = - \bar{\Psi} \gamma_{cd} \delta \Psi,
\]

\[
\delta \bar{\Psi} \gamma^\mu \partial_\mu \Psi = - \bar{\Psi} \gamma^\mu \partial_\mu \delta \Psi,
\]

\[
\delta \bar{\Psi} \gamma^\mu \gamma_{cd} \Psi = - \bar{\Psi} \gamma^\mu \gamma_{cd} \delta \Psi.
\]

These relations imply,

\[
\delta \bar{\Psi} \gamma^\mu \Psi = \delta \bar{\Psi} \gamma^\mu \left( \partial_\mu + \frac{1}{4} \omega_\mu \gamma_{ab} \right) \Psi
\]

\[
= - \partial_\mu \bar{\Psi} \gamma^\mu \delta \Psi + \frac{1}{4} \bar{\Psi} \omega_\mu \gamma_{ab} \gamma^\mu \delta \Psi
\]

\[
= \bar{\Psi} \nabla \cdot E_a \gamma^a \delta \Psi = - \bar{\Psi} \nabla \cdot \delta \Psi.
\]
Using these relations, the variation of the Lagrangian with respect to the Majorana field yields,

\[
\delta \Psi L = \frac{1}{2} \epsilon_4 \left( -\delta \bar{\Psi} D \Psi - \bar{\Psi} D \delta \Psi + m \bar{\Psi} \delta \Psi + m \delta \bar{\Psi} \right) \\
= \frac{1}{2} \epsilon_4 \bar{\Psi} \delta \Psi + \frac{1}{2} (E_a \cdot \epsilon_4) \bar{\Psi} \gamma^a \wedge D \delta \Psi + \epsilon_4 (m \bar{\Psi} \delta \Psi) \\
= \epsilon_4 \left( \bar{\Psi} (\delta \Psi + m) - \frac{1}{3!} T_{ba} \bar{\Psi} \gamma^a \right) \delta \Psi - d \left[ \frac{1}{2} (E_a \cdot \epsilon_4) \bar{\Psi} \gamma^a \delta \Psi \right],
\]

(3.18)

where \( T_{ab} \) are the frame components of the torsion \( T_{ab} = E_b \cdot E_a \cdot T^c \). In going from the first to the second step we have used once again identity equation (3.9) with \( A = \epsilon_4, B = D \delta \Psi, \) and \( V = E_a \) such that \( A \wedge B = 0 \). In going from the second to the third line we have used integration by parts and manipulations similar to the ones performed in [6] for the Dirac field.

The contribution to the symplectic potential \( \theta \) from the Majorana field can be read off from equation (3.18),

\[
\theta(\Psi, \delta \Psi) = -\frac{1}{2} (E_a \cdot \epsilon_4) \bar{\Psi} \gamma^a \delta \Psi.
\]

(3.19)

Given the symplectic potential, the Lorentz-diffeomorphism Noether current can be obtained in a straightforward manner. It takes the form,

\[
J_\xi := \theta(\Psi, K_\xi \Psi) - \xi \cdot L \\
= -\frac{1}{2} (E_a \cdot \epsilon_4) \bar{\Psi} \gamma^a (K_\xi \Psi) - (\xi \cdot \epsilon_4) \left( -\frac{1}{2} \bar{\Psi} D \Psi + \frac{1}{2} m \bar{\Psi} \right).
\]

(3.20)

From equation (2.7) we have,

\[
K_\xi \Psi = \xi \cdot d \Psi + \frac{1}{4} \xi \cdot \omega^a \gamma^b \gamma^{ab} \Psi + \frac{1}{8} (E_b \cdot E_a \cdot d \xi) \gamma^{ab} \Psi.
\]

(3.21)

Inserting this expression in equation (3.20) and using the identity,

\[
V \cdot A = (V \cdot e_a)(E^a \cdot A),
\]

(3.22)

we get

\[
J_\xi = -T_a (\xi \cdot e^a) - S^{ab} \left( \frac{1}{2} E_a \cdot E_b \cdot (d \xi - T^c (E_c \cdot \xi)) - (E_c \cdot \xi) (E_{[a} \cdot E^b \cdot T_{b]}) \right),
\]

(3.23)

where \( T_a \) and \( S_{ab} \) are defined in equations (3.10) and (3.12) respectively.

Now, the full Lorentz-diffeomorphism Noether current of the gravity plus the matter takes the form,

\[
J_\xi = J^\text{grav}_\xi + J^\text{matter}_\xi \\
= dQ_\xi + (\mathcal{E}_a - T_a) (\xi \cdot e^a) + (\mathcal{E}^{ab} - S^{ab}) \\
\times \left( \frac{1}{2} E_a \cdot E_b \cdot (d \xi - T^c (E_c \cdot \xi)) - (E_c \cdot \xi) (E_{[a} \cdot E^b \cdot T_{b]}) \right).
\]

(3.24)

(3.25)
Therefore, on-shell we have
\[ J_\xi = \frac{1}{32\pi G} \epsilon_{abcd} e^a \wedge e^b \left( \frac{1}{2} E_c \cdot E_d \cdot (d\xi - T^e(E_e \cdot \xi)) - (E_e \cdot \xi)(E_e \cdot E_c \cdot T_d) \right), \]
(3.26)
and the torsion two-form is fixed by the equations of motion \( \mathcal{E}^{ab} - \mathcal{S}^{ab} = 0 \). At the bifurcation surface, for the horizon generating Killing field \( \xi^\mu = k^\mu = (\partial_t)^\mu + \Omega(\partial_\phi)^\mu \), \( \xi^\mu \) vanishes. As a result, the Noether charge
\[ Q_k = \frac{1}{32\pi G} \epsilon_{abcd} e^a \wedge e^b \left( \frac{1}{2} E_c \cdot E_d \cdot dk \right) = -\frac{1}{16\pi G} \star dk. \]
(3.27)
Thus, we see that the Majorana fermion does not contribute to the full Noether charge for the horizon generating Killing field at the bifurcation surface.

4. Lorentz-diffeomorphism Noether charge for simple (AdS) supergravity

In this section we study the Lorentz-diffeomorphism Noether charge for the spin-3/2 Rarita–Schwinger field in four spacetime dimensions. The most natural set-up to consider is the case of \( N = 1, D = 4 \) supergravity, often called simple supergravity. With very little effort, this discussion can be generalised to \( N = 1, D = 4 \) AdS supergravity. This is the set-up we work with. The Lagrangian four-form for simple AdS supergravity is,
\[ L_{\text{sugra}} = \frac{1}{4\kappa^2} \epsilon_{abcd} R^{ab} \gamma \wedge e^c \wedge e^d + i \frac{2}{\kappa^2} \bar{\psi} \gamma \gamma_5 \gamma \wedge \hat{D}\psi - \frac{1}{\kappa^2} \Lambda \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d, \]
(4.1)
where the Rarita–Schwinger field \( \psi \) is a Majorana spinor valued one-form, \( \gamma \) is a matrix-valued one-form
\[ \gamma = \gamma^a e_a, \]
(4.2)
\[ \gamma_5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3, \]
and the derivative operator on the Rarita–Schwinger field is defined as,
\[ \hat{D}\psi = D\psi - \frac{1}{2L} \gamma \wedge \psi = \left( \partial_\mu \psi^\nu + \frac{i}{4} \epsilon_{abcd} \gamma_\mu^c \partial_\nu \psi^a - \frac{1}{2L} \gamma^a \gamma_\mu \psi^a \right) dx^\mu \wedge dx^\nu. \]
(4.3)
Furthermore, \( \kappa^2 = 8\pi G \) and the cosmological constant \( \Lambda \) is related to the AdS radius \( L \) through \( \Lambda = -\frac{3}{L^2} \). Lagrangian equation (4.1) is presented in a slightly different form compared to standard references such as [18]. Since we use the form notation, the above structure of the Lagrangian is computationally simpler to work with.

The gravity sector of the Lagrangian presented in equation (4.1) is the same as in the previous sections. Therefore, it is enough to consider the matter part of the Lagrangian equation (4.1),
\[ L = \frac{i}{2\kappa^2} \bar{\psi} \wedge \gamma_5 \wedge \hat{D}\psi. \]
(4.4)
The variation with respect to \( e^a \) yields the energy-momentum three-form \( T_{\alpha\beta} \), and the variation with respect to \( \omega^{ab} \) yields the spin-current three-form \( S_{ab} \),

\[
(\delta L)_{\text{hom}} = -T_{\alpha\beta} \wedge \delta e^\alpha, \quad (\delta \omega)_{\text{hom}} = -S_{ab} \wedge \delta \omega^{ab}.
\]

(4.5)

It turns out that the variations \( \delta e_L \) and \( \delta S_L \) do not give total derivative terms. The variation with respect to \( \psi \) yields the equations of motion for the Rarita–Schwinger field and a total derivative term from where we read off the contribution to the symplectic potential three-form \( \theta \).

We have,

\[
\delta L = -\frac{i}{2\hbar^2} \left( \delta \bar{\psi} \bar{\gamma}_5 \gamma^\alpha \lambda \bar{\gamma}^\alpha \gamma^\beta \psi \right) \bar{D}_\beta \bar{\psi} + \frac{i}{2\hbar^2} \bar{\psi} \gamma^\alpha \gamma^\beta \gamma^\gamma \lambda \bar{\psi} \gamma^\beta \psi \delta e^\alpha + \frac{i}{2\hbar^2} \bar{\psi} \gamma^\alpha \lambda \bar{\psi} \gamma^\beta \psi \delta \omega^{ab} + \frac{i}{2\hbar^2} \bar{\psi} \gamma^\alpha \gamma^\beta \gamma^\gamma \lambda \bar{\psi} \gamma^\beta \psi \delta \omega^{ab},
\]

The variation of \( \bar{\psi} \) is related to the variation of \( \psi \) through the Majorana flip relations. The following more general version of the Majorana flip relation is very useful in actual computations:

\[
\bar{\lambda}_1 \Gamma^{(r)} \Gamma^{(r)} \lambda_2 = t_{r_1} \bar{\lambda}_1 \Gamma^{(r)} \lambda_2 = t_{r_1} t_{r_2} \bar{\lambda}_1 \Gamma^{(r)} \Gamma^{(r)} \lambda_2.
\]

(4.7)

Using \( \gamma_e = i\gamma_0 \gamma_1 \gamma_2 \gamma_3 \), together with equations (3.13) and (4.7), we have for the first term in the variation presented in equation (4.6),

\[
(\delta \bar{\psi}) \bar{\gamma}_5 \gamma_a e^a \wedge \bar{D}_\alpha \psi = \left( \delta \bar{\psi} \bar{\gamma}_5 \gamma_a \bar{D}_\alpha \psi \right) dx^\alpha \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho
\]

\[
= \left( \bar{D}_\alpha \psi \gamma_a \delta \bar{\psi} \right) dx^\alpha \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho
\]

\[
= \left\{ \partial_{\mu} \bar{\psi} + \frac{1}{4} \bar{\psi} \wedge \omega_{\mu} \gamma^{cd} - \frac{1}{2\hbar} \bar{\psi} \gamma^\mu \gamma^\rho \delta \psi \right\} \wedge \bar{\psi} \psi \gamma_a \gamma^a \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho
\]

\[
= -\bar{\psi} \bar{D} \lambda \gamma_a \gamma_a e^a \wedge \delta \psi,
\]

(4.8)

where the operation \( \bar{D} \) is defined as,

\[
\bar{D} \equiv \partial_{\mu} \bar{\psi} + \frac{1}{4} \bar{\psi} \wedge \omega_{\mu} \gamma^{cd} - \frac{1}{2\hbar} \bar{\psi} \gamma^\mu \gamma^\rho
\]

\[
= \left( \partial_{\mu} \bar{\psi} + \frac{1}{4} \omega_{\mu} \gamma^a \gamma^{cd} + \frac{1}{2\hbar} \bar{\psi} \gamma^\mu \gamma^\rho \right) dx^\mu \wedge dx^\nu \wedge dx^\rho
\]

(4.9)

Additionally, to simplify the second term in the variation equation (4.6), we note that
To compute this we need the Kosmann derivative concisely in the form notation as follows,

\[ \psi \wedge \gamma_a \wedge D \delta \psi = \psi \wedge \gamma_a \wedge d \delta \psi + \frac{1}{4} \psi \wedge \gamma_a \gamma_b \wedge \omega_{cd} \gamma^{cd} \wedge \delta \psi - \frac{1}{2L} \bar{\psi} \wedge \gamma_a \gamma_b \wedge \gamma_c \gamma_d \wedge \delta \psi \]

\[ = d \left( \psi \wedge \gamma_a \gamma_b \wedge \delta \psi \right) - d \left( \bar{\psi} \wedge \gamma_a \gamma_b \gamma_c \gamma_d \wedge \delta \psi \right) - \frac{1}{4} \bar{\psi} \wedge \omega_{cd} \gamma_b \wedge \gamma_c \wedge \gamma_d \wedge \delta \psi + \frac{1}{2L} \bar{\psi} \wedge \gamma_a \gamma_b \gamma_c \gamma_d \wedge \delta \psi \]

\[ = d \left( \psi \wedge \gamma_a \gamma_b \wedge \delta \psi \right) + \psi \wedge \gamma_a \gamma_b \gamma_c \gamma_d \wedge \delta \psi + \frac{1}{4} \bar{\psi} \wedge \omega_{cd} \wedge \gamma_a \gamma_b \gamma_c \gamma_d \wedge \delta \psi \]

\[ = d \left( \psi \wedge \gamma_a \gamma_b \wedge \delta \psi \right) + \psi \wedge \gamma_a \gamma_b \gamma_c \gamma_d \wedge \delta \psi - \psi \Psi \wedge d \gamma_a \gamma_b \gamma_c \gamma_d \wedge \delta \psi, \quad (4.10) \]

where we have separated out the total derivative term.

Using both equations (4.8) and (4.10), the variation of the Rarita–Schwinger Lagrangian simplifies to,

\[ \delta L = d \theta - T_a \wedge \delta e^a - S_{cd} \wedge \delta \omega^{cd} - E_{\psi} \wedge \delta \psi, \quad (4.11) \]

with

\[ \theta = \frac{i}{2 \kappa^2} \bar{\psi} \wedge \gamma_a \gamma_b \wedge \delta \psi, \quad (4.12) \]

\[ T_a = - \frac{i}{2 \kappa^2} \left[ \bar{\psi} \wedge \gamma_a \gamma_b \gamma_c \gamma_d \wedge \psi \right], \quad (4.13) \]

\[ S_{cd} = \frac{i}{8 \kappa^2} \bar{\psi} \wedge \gamma_a \gamma_b \gamma_c \gamma_d \wedge \psi, \quad (4.14) \]

\[ E_{\psi} = \frac{i}{\kappa^2} \left[ \bar{\psi} \wedge \gamma_a \gamma_b \gamma_c \gamma_d \wedge \psi \right]. \quad (4.15) \]

Having obtained an expression for \( \theta \), we can now obtain the Lorentz-diffeomorphism Noether current via our general formula,

\[ J_\zeta = \theta(\psi, \zeta \psi) - \xi \cdot L. \quad (4.16) \]

To compute this we need the Kosmann derivative \( \zeta \psi \) of the Rarita–Schwinger field. An expression for \( \zeta \psi \) was given in equation (2.10). Expression (2.10) can be written more concisely in the form notation as follows,

\[ \zeta \psi = \xi \cdot D \psi + \tilde{D} (\xi \cdot \psi) + \frac{1}{4} \left( - \frac{1}{2} E_a \cdot E_b \cdot d \xi \right) \gamma^{ab} \psi, \quad (4.17) \]

or equivalently,
\[ K_\xi \psi = \xi \cdot \left( D \psi - \frac{1}{4} K_{ab} \gamma^{ab} \wedge \psi \right) + D (\xi \cdot \psi) - \frac{1}{4} K_{ab} \gamma^{ab} (\xi \cdot \psi) \\
+ \frac{1}{4} \left( -\frac{1}{2} E_a \cdot E_b \cdot d \xi \right) \gamma^{ab} \psi \\
= \xi \cdot (D \psi) + D (\xi \cdot \psi) + \frac{1}{4} \left( -\frac{1}{2} E_a \cdot E_b \cdot d \xi - \xi \cdot K_{ab} \right) \gamma^{ab} \psi, \] (4.18)

where we have replaced \( \circ D \) with \( D \). In the new form the spinor covariant derivative includes torsion. The contorsion tensor \( K_{ab} \) is defined via \( \omega_{ab} = \tilde{\omega}_{ab} + K_{ab} \), and can be expressed in terms of the torsion tensor as

\[ K_{\alpha\mu\nu} = -\frac{1}{2} \left( T_{\alpha\mu\nu} - T_{\mu\nu\alpha} + T_{\nu\alpha\mu} \right). \] (4.19)

The contorsion tensor is antisymmetric in the last two indices, while torsion tensor is anti-symmetric in the first two indices. We can write contorsion tensor term \( \xi \cdot K_{ab} \) in terms of the torsion two form. We have,

\[ \xi \cdot K_{ab} = \xi^\alpha K_{\alpha\mu\nu} E^\mu_a E^\nu_b = -\frac{1}{2} \xi^\alpha \left( T_{\alpha\mu\nu} - T_{\mu\nu\alpha} + T_{\nu\alpha\mu} \right) E^\mu_a E^\nu_b = -\xi^\alpha T_{\mu\nu\alpha} E^\mu_a E^\nu_b - \frac{1}{2} \xi^\alpha \left( T_{\alpha\mu\nu} - T_{\alpha\nu\mu} \right) E^\mu_a E^\nu_b = -\frac{1}{2} (\xi \cdot e^\nu) [E_a \cdot (E_b \cdot T_{\nu\mu})] - E^\mu_a \cdot (\xi \cdot T_{\nu\mu}) \] (4.20)

where \( T_{\alpha\mu\nu} = (1/2) T_{\mu\nu\alpha} \) \( dx^{\alpha} \wedge dx^{\nu} \). Thus, the Kosmann derivative of the spinor one-form can be written as:

\[ K_\xi \psi = \xi \cdot (D \psi) + D (\xi \cdot \psi) + \frac{1}{4} \left[ \frac{1}{2} E_a \cdot (E_b \cdot d \xi) \right. \\
+ \frac{1}{2} \left( \xi \cdot e^\nu \right) [E_a \cdot (E_b \cdot T_{\nu\mu})] + E^\mu_a \cdot \left( \xi \cdot T_{\nu\mu} \right) \right] \gamma^{ab} \psi. \] (4.21)

It is slightly easier to work with expression (4.18), as it is less cumbersome. However, to compare with some expressions in [6] it is better to use the form (4.21). We continue to use expression (4.18) and use (4.20) to convert contorsion tensor into the torsion tensor when needed.

Now, from equations (4.12) and (4.15), it follows that

\[ \theta (\psi, K_\xi \psi) = \frac{i}{2\kappa^2} \bar{\psi} \wedge \gamma \gamma \wedge \xi \cdot D \psi + \frac{i}{2\kappa^2} \bar{\psi} \wedge \gamma \gamma \wedge D (\xi \cdot \psi) \\
- S^{cd} \left( \frac{1}{2} E_c \cdot E_d \cdot d \xi + \xi \cdot K_{cd} \right), \] (4.22)

The \( \xi \cdot L \) term in \( J_\xi = \theta (\psi, K_\xi \psi) - \xi \cdot L \) is more tedious. We find,
\[ \xi \cdot L = \frac{i}{2\kappa^2} (\xi \cdot \psi) \gamma_\alpha \gamma_a D \psi - \frac{i}{2\kappa^2} \bar{\psi} \gamma_a \gamma_\alpha \bar{D} \psi (\xi \cdot e^a) + \frac{i}{2\kappa^2} \bar{\psi} \gamma_\alpha \gamma_a \bar{D} \psi (\xi \cdot e^a) \]

\[ - \frac{i}{2\kappa^2} \left( \frac{1}{2L} \right) \bar{\psi} \gamma_\alpha \gamma_a \bar{\psi} (\xi \cdot e^a) + \frac{i}{2\kappa^2} \left( \frac{1}{2L} \right) \bar{\psi} \gamma_\alpha \gamma_a \bar{\psi} (\xi \cdot e^a) \]

\[ = \frac{i}{2\kappa^2} (\xi \cdot \psi) \gamma_\alpha \gamma_a D \psi + \frac{i}{2\kappa^2} \bar{\psi} \gamma_\alpha \gamma_a \bar{D} \psi (\xi \cdot e^a) \left[ \bar{\psi} \gamma_\alpha \gamma_a D \psi \right] \]

\[ - \frac{1}{2L} \bar{\psi} \gamma_\alpha \gamma_a \bar{\psi} (\xi \cdot e^a) \left[ \psi \gamma_\alpha \gamma_a D \psi \right] \]

\[ + \frac{i}{2\kappa^2} \left( \frac{1}{2L} \right) \bar{\psi} \gamma_\alpha \gamma_a \bar{\psi} (\xi \cdot \psi) \].

(4.23)

The above manipulations are as follows. In the first step we have expanded \( \xi \cdot \bar{D} \psi \) in \( \xi \cdot D \psi \) and the remaining terms. In the second step we have regrouped various terms to factor out \((\xi \cdot e^a)\). In the third step we have regrouped terms such that the coefficient of \((\xi \cdot e^a)\) is the energy-momentum three-form \( T_a \) introduced in equation (4.13).

Combining equations (4.22) and (4.23), the Lorentz-diffeomorphism Noether current for the Rarita–Schwinger part of the Lagrangian can be expressed as,

\[ J^{RS}_\xi = \theta (\psi, K_\xi \bar{\psi}) - \xi \cdot L \]

\[ = \frac{i}{2\kappa^2} \bar{\psi} \gamma_\alpha \gamma_a \bar{D} \psi (\xi \cdot \psi) - S^{cd} \left( \frac{1}{2} E_c \cdot E_d \cdot d\xi + \xi \cdot K_{cd} \right) \]

\[ - T_a (\xi \cdot e^a) + \frac{i}{2\kappa^2} \left( \xi \cdot \psi \right) \bar{\gamma}_\alpha \gamma_a D \psi. \]

(4.24)

The first term in the above equation can be rewritten as a sum of a total derivative and other terms (cf equation (4.10)),

\[ \frac{i}{2\kappa^2} \bar{\psi} \gamma_\alpha \gamma_a \bar{D} \psi (\xi \cdot \psi) = d \left( \frac{i}{2\kappa^2} \bar{\psi} \gamma_\alpha \gamma_a (\xi \cdot \psi) \right) - \frac{i}{2\kappa^2} \bar{\psi} \bar{D} \gamma_\alpha \gamma_a (\xi \cdot \psi) \]

\[ + \frac{i}{2\kappa^2} \bar{\psi} \gamma_\alpha \gamma_a T^a (\xi \cdot \psi), \]

(4.25)

while the last term in equation (4.24) can be Majorana flipped (cf equation (4.7)) to get,

\[ \frac{i}{2\kappa^2} \left( \xi \cdot \psi \right) \bar{\gamma}_\alpha \gamma_a D \psi = - \frac{i}{2\kappa^2} \bar{\psi} \bar{D} \gamma_\alpha \gamma_a (\xi \cdot \psi). \]

(4.26)

We finally have,

\[ J^{RS}_\xi = d \left[ \frac{i}{2\kappa^2} \bar{\psi} \gamma_\alpha \gamma_a (\xi \cdot \psi) \right] - T_a (\xi \cdot e^a) - E_b (\xi \cdot \psi) \]

\[ - S^{cd} \left( \frac{1}{2} E_c \cdot E_d \cdot d\xi + \xi \cdot K_{cd} \right). \]

(4.27)

The full Lorentz-diffeomorphism Noether current is therefore,
\[ J_\xi = J_\xi^{\text{grav}} + J_\xi^{\text{RS}} \]
\[ = dQ_\xi^{\text{grav}} + d \left[ \frac{i}{2}\bar{\psi} \gamma_\gamma (\xi \cdot \psi) \right] + (\mathcal{E}_a - \mathcal{T}_a)(\xi \cdot e^a) + \mathcal{E}_\psi (\xi \cdot \psi) \]
\[ + (\mathcal{E}^{cd} - S^{cd}) \left( \frac{1}{2} E_c \cdot E_d \cdot d\xi + \xi \cdot K_{cd} \right) . \] (4.28)

On-shell we have \( J_\xi = dQ_\xi \) where the total Noether charge \( Q_\xi \) is
\[ Q_\xi = Q_\xi^{\text{grav}} + Q_\xi^{\text{RS}} , \] (4.29)
with
\[ Q_\xi^{\text{RS}} = \frac{i}{2\kappa} \bar{\psi} \gamma_\gamma a e^a (\xi \cdot \psi) , \] (4.30)
and where we recall that,
\[ Q_\xi^{\text{grav}} = \frac{1}{32\pi} \varepsilon^{a b c d} e^a \wedge e^b \left( \frac{1}{2} E_c \cdot E_d \cdot d\xi + \xi \cdot K_{cd} \right) \]
\[ = \frac{1}{32\pi} \varepsilon^{a b c d} e^a \wedge e^b \left( \frac{1}{2} E_c \cdot E_d \cdot (d\xi - T^c(E_c \cdot \xi)) - (E_c \cdot \xi) \mathcal{P}_{[c} \cdot E^d \cdot T_{d]} \right) \] (4.31)
\[ \Rightarrow \frac{1}{2} \left( 1 + \frac{r^2}{L^2} \right) dr^2 + \left( 1 + \frac{r^2}{L^2} \right)^{-1} \left( \Omega^2 + \sin^2 \theta \Omega^2 \right) . \] (4.32)

Expressions (4.12) and (4.13), and the expression for the Noether charge (4.30) are the main results of this section.

5. First law for supergravity

We now have all the ingredients necessary to formulate a first law for simple AdS supergravity. We recall that a first law is an identity relating the perturbed Hamiltonians for the horizon generating Killing field evaluated at the bifurcation surface and at spatial infinity. The first variations of the Hamiltonians are constructed out of the Noether charge two-form \( Q_\xi \) and symplectic potential three-form \( \theta \) as,
\[ \delta H_\xi = \int_{\Sigma} \omega(\varphi, \delta_1 \varphi, K_\xi \varphi) = \int_{\partial\Sigma} (\delta Q_\xi - \xi \cdot \theta) . \] (5.1)

In this expression \( \Sigma \) is a Cauchy surface in the spacetime and \( \partial \Sigma \) is its boundary, \( \omega \) is the symplectic current
\[ \omega(\varphi, \delta_1 \varphi, \delta_2 \varphi) = \delta_1 \theta(\varphi, \delta_2 \varphi) - \delta_2 \theta(\varphi, \delta_1 \varphi) . \] (5.2)

The Noether charge two-form \( Q_\xi \) and symplectic potential three-form \( \theta \) for both the gravitational Lagrangian and the Rarita–Schwinger Lagrangian have been obtained in the previous sections.

A first step to discuss the first law is to specify the boundary conditions for the gravitational and Rarita–Schwinger fields. Asymptotically, we demand the metric to behave as global AdS,
\[ ds^2_{\text{AdS}} = - \left( 1 + \frac{r^2}{L^2} \right) dt^2 + \left( 1 + \frac{r^2}{L^2} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) . \] (5.3)
A set of co-frames that capture the above metric is simply,

\[ e^0_{\text{AdS}} = \left(1 + \frac{r^2}{L^2}\right)^{1/2} \, dr, \quad e^1_{\text{AdS}} = \left(1 + \frac{r^2}{L^2}\right)^{-1/2} \, dr, \]

\[ e^2_{\text{AdS}} = r \, d\theta, \quad e^3_{\text{AdS}} = r \sin \theta \, d\phi. \] (5.4)

The ‘Dirichlet’ boundary conditions that define asymptotically AdS spacetimes in simple supergravity were worked out in [21]. They are,

\[ e^0 = e^0_{\text{AdS}} + \mathcal{O}(r^{-5}) \, dr + \mathcal{O}(r^{-2}) \, dr + \mathcal{O}(r^{-2}) \, d\theta + \mathcal{O}(r^{-2}) \, d\phi, \] (5.6)

\[ e^1 = e^1_{\text{AdS}} + \mathcal{O}(r^{-4}) \, dr + \mathcal{O}(r^{-3}) \, dr + \mathcal{O}(r^{-3}) \, d\theta + \mathcal{O}(r^{-3}) \, d\phi, \] (5.7)

\[ e^2 = e^2_{\text{AdS}} + \mathcal{O}(r^{-2}) \, dr + \mathcal{O}(r^{-2}) \, dr + \mathcal{O}(r^{-2}) \, d\theta + \mathcal{O}(r^{-2}) \, d\phi, \] (5.8)

\[ e^3 = e^3_{\text{AdS}} + \mathcal{O}(r^{-2}) \, dr + \mathcal{O}(r^{-2}) \, dr + \mathcal{O}(r^{-2}) \, d\theta + \mathcal{O}(r^{-2}) \, d\phi. \] (5.9)

These boundary conditions ensure that the symplectic current is finite at the boundary and that the symplectic flux through the boundary vanishes [22]. The corresponding boundary conditions in the asymptotically flat setting were first discussed in [23, 24].

Let us assume that there is a stationary axisymmetric black hole solution to the field equations. Let the black hole horizon be a bifurcate Killing horizon generated by the Killing field \( k^\mu \). From general results on the bifurcate Killing horizon it follows that the Killing field is a linear combination of time translation \( t^\mu = (\partial/\partial t)^\mu \) and rotation \( \phi^\mu = (\partial/\partial \phi)^\mu \),

\[ k^\mu = t^\mu + \Omega_H \phi^\mu, \] (5.12)

where \( \Omega_H \) is a constant representing horizon angular velocity and that \( k^\mu \) vanishes at the bifurcation two-sphere.

Let us also assume that the Rarita–Schwinger field is smooth on the relevant part of the spacetime, i.e., in the neighbourhood of the future and past horizons, at the bifurcation two-sphere, and in the spacetime region outside the horizon all the way to infinity. Moreover, let us assume that the Rarita–Schwinger field is stationary and axisymmetric, i.e.,

\[ K_t \psi = 0 \quad \text{and} \quad K_\phi \psi = 0 \quad \Rightarrow \quad K_k \psi = 0. \] (5.13)

These conditions ensure stationarity and axi-symmetry of the solution to the field equations. Since the vector \( k^\mu \) vanishes at the bifurcation two-sphere, the contribution from the \( k \cdot \theta \) term in the perturbed Hamiltonian

\[ \delta H_k = \int_B \left( \delta Q_k - k \cdot \theta \right), \] (5.14)

is zero. Our boundary conditions at infinity are such that there exists [22] a two-form \( \Theta \), such that

\[ \int_{\partial \Sigma_{\infty}} k \cdot \theta = \delta \int_{\partial \Sigma_{\infty}} k \cdot \Theta. \] (5.15)
Hence, boundary Hamiltonians exist.

For the gravity sector, at the bifurcation two-sphere, the contribution to the perturbed Hamiltonian $\delta H_k$ becomes $T_H \delta S$. Here $T_H = (\kappa_B/2\pi)$ and $S = (A/4)$, where $\kappa_B$ is the surface gravity and $A$ is the area of the bifurcation two-sphere. This is because with $k^\mu = 0$ on the bifurcation surface and $\delta k^\mu = 0$ everywhere, one can argue that the variation of the temperature term is zero [1, 6].

Since the Rarita–Schwinger field is assumed to be smooth on the horizon, the contribution of the Rarita–Schwinger field to the Noether charge equation (4.30) at the bifurcation two-sphere is zero: since $k^\mu = 0$ at the bifurcation two-sphere, $k \cdot \psi$ vanishes. Hence on the bifurcation surface, the contribution from simple AdS supergravity is simply $T_H \delta S$.

At infinity, as is well known the contribution to the Hamiltonian from the gravitational field yields the ADM mass $M_{\text{ADM}}$ and angular momentum $J_{\text{ADM}}$. Thus the variation of the gravitational Hamiltonian at infinity yields the variation of the ADM mass and angular momentum. While depending on the nature of the solution, the Hamiltonian for the Rarita–Schwinger field may or may not contribute at infinity. The boundary conditions we mentioned above are such that the supercharges are finite. With these boundary conditions contributions to the energy and angular momentum from the Rarita–Schwinger field (which depends quadratically on the spinor field) vanish. Combining these elements, the stationary comparison version of the first law for black holes with bifurcate Killing horizons in simple AdS supergravity takes the form,

$$T_H \delta S = \delta M_{\text{ADM}} - \Omega H J_{\text{ADM}}.$$ (5.16)

In summary, we found that smooth, stationary, axisymmetric Rarita–Schwinger field does not explicitly contribute to the black hole entropy. The extra term in the Noether charge vanishes at the bifurcation surface. Near infinity, Rarita–Schwinger field falls-off sufficiently fast that it does not contribute to the integrals for the energy and angular momentum. Thus, the first law of black hole mechanics in simple supergravity retains the same form as in pure general relativity.

6. Conclusions

In this work we have applied and appropriately adapted the Lorentz-diffeomorphism Noether charge formalism of references [5, 6] to four-dimensional gravity coupled to a Majorana field and to a Rarita–Schwinger field. In section 3 we studied the Lorentz-diffeomorphism Noether charge for a spin-1/2 Majorana field. The Majorana condition brings in certain new elements in the computation. It served as a warm-up for the Rarita–Schwinger field in the context of simple supergravity considered in section 4. As we saw in that section the Majorana nature of the Rarita–Schwinger field played an important role in the computations. A key result of our work is expression (4.30) for the contribution to the Lorentz-diffeomorphism Noether charge due to the Rarita–Schwinger field. Using this Noether charge we formulated a stationary comparison version of the first law in section 5.

In our analysis of the first law with the Rarita–Schwinger field we made two important assumptions: (i) the Rarita–Schwinger field is smooth everywhere in the region of interest, (ii) the Rarita–Schwinger field is annihilated by the Kosmann derivative with respect the horizon generating Killing field. Using these assumptions, we concluded that the Rarita–Schwinger field does not contribute to the first law at the bifurcation surface. Perhaps these assumptions are too restrictive. This situation should be compared to the analysis of the Yang–Mills field by Sudarsky and Wald [13, 14]. Under similar assumptions, namely (i) a smooth Yang–Mills field can be chosen on the spacetime, and (ii) it is annihilated by the Lie derivative with respect
the horizon generating Killing field, they also concluded that the Yang-Mills field does not contribute to the first law at the bifurcation surface. Over the years, this conclusion has been refined. In 2003 Gao [12] argued that the Yang–Mills field does contribute to the first law at the horizon, but he was not able to write the contribution as a potential times the perturbed charge without making additional assumptions. In 2015 Prabhu [6] by formulating the problem in terms of the principal bundle gave a satisfactory discussion of the first law for gravity coupled to a Yang–Mills field. He showed that the Yang–Mills field contributes to the first law both at the bifurcation surface and at infinity. The contributions are of the form potential times the perturbed charge, and generically it is not possible to write the two terms as the ‘difference in the potential between infinity and the bifurcation surface’ times the perturbed charge.

It is natural to speculate that similar refinements are to be found with the Rarita–Schwinger field. A reason our analysis is ill-equipped to address this question is that we have ignored the fermionic gauge symmetry of the Rarita–Schwinger field. The fermionic gauge symmetry is supersymmetry—a spacetime symmetry. It changes the frame field as well. The principal bundle formalism of Prabhu [6], though quite general, is not equipped to handle supersymmetry. Perhaps a formulation of the first law is possible using the superspace formalism of supergravity. The superspace is discussed at length in the mathematical physics literature [25–28]. In such a formulation, we expect that the above mentioned shortcomings can be addressed and that supercharges may feature in the first law. Such a discussion would shed further light on black holes in supergravity. We leave this for future work.

For a class of supersymmetric black holes it is known that smooth (at least at the future horizon) normalisable linearised fermionic hair modes exist [29, 30]. It will be very interesting to understand how these modes appear in the first law for the corresponding black holes. We hope to return to these questions in our future work.

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ORCID iDs

P B Aneesh https://orcid.org/0000-0002-8723-0380
Sumanta Chakraborty https://orcid.org/0000-0003-3343-3227
Sk Jahanur Hoque https://orcid.org/0000-0003-2921-9586
Amitabh Virmani https://orcid.org/0000-0001-7701-4839

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