FACTORIZATION OF CONTRACTIONS

B. KRISHNA DAS, JAYDEB SARKAR, AND SRIJAN SARKAR

Dedicated to Professor Rajendra Bhatia on the occasion of his 65th birthday

Abstract. The celebrated Sz.-Nagy and Foias theorem asserts that every pure contraction is unitarily equivalent to an operator of the form \( P \mathcal{Q} M_z |_Q \) where \( \mathcal{Q} \) is a \( M^*_z \)-invariant subspace of a \( D \)-valued Hardy space \( H^2_D(\mathbb{D}) \), for some Hilbert space \( D \).

On the other hand, the celebrated theorem of Berger, Coburn and Lebow on pairs of commuting isometries can be formulated as follows: a pure isometry \( V \) on a Hilbert space \( H \) is a product of two commuting isometries \( V_1 \) and \( V_2 \) in \( \mathcal{B}(H) \) if and only if there exist a Hilbert space \( E \), a unitary \( U \) in \( \mathcal{B}(E) \) and an orthogonal projection \( P \) in \( \mathcal{B}(E) \) such that \( (V, V_1, V_2) \) and \( (M_z |_Q, M_\Phi |_Q, M_\Psi |_Q) \) on \( H^2_E(D) \) are unitarily equivalent, where \( \Phi(z) = (P + zP^\perp)U^* \) and \( \Psi(z) = U(P^\perp + zP) \) \((z \in \mathbb{D})\).

In this context, it is natural to ask whether similar factorization results hold true for pure contractions. The purpose of this paper is to answer this question. More particularly, let \( T \) be a pure contraction on a Hilbert space \( H \) and let \( \mathcal{Q} \) be the Sz.-Nagy and Foias representation of \( T \) for some canonical \( \mathcal{Q} \subseteq H^2_D(\mathbb{D}) \). Then \( T = T_1 T_2 \), for some commuting contractions \( T_1 \) and \( T_2 \) on \( H \), if and only if there exist \( \mathcal{B}(D) \)-valued polynomials \( \varphi \) and \( \psi \) of degree \( \leq 1 \) such that \( \mathcal{Q} \) is a joint \( (M^*_\varphi, M_\psi) \)-invariant subspace, \( P \mathcal{Q} M_z |_Q = P \mathcal{Q} M_\varphi |_Q = P \mathcal{Q} M_\psi |_Q \) and \( (T_1, T_2) \cong (P \mathcal{Q} M_\varphi |_Q, P \mathcal{Q} M_\psi |_Q) \).

Moreover, there exist a Hilbert space \( \mathcal{E} \) and an isometry \( V \in \mathcal{B}(D; \mathcal{E}) \) such that \( \varphi(z) = V^* \Phi(z)V \) and \( \psi(z) = V^* \Psi(z)V \) \((z \in \mathbb{D})\), where the pair \( (\Phi, \Psi) \), as defined above, is the Berger, Coburn and Lebow representation of a pure pair of commuting isometries on \( H^2_E(\mathbb{D}) \). As an application, we obtain a sharper von Neumann inequality for commuting pairs of contractions.

1. Introduction

Let \( \mathcal{H} \) be a Hilbert space and \( V \) be an isometry on \( \mathcal{H} \). It is a classical result, due to von Neumann and Wold (cf. \cite{13}), that \( V \) is unitarily equivalent to \( M_z \oplus U \) where \( M_z \) is the shift operator on an \( \mathcal{E} \)-valued Hardy space \( H^2_\mathcal{E}(\mathbb{D}) \), for some Hilbert space \( \mathcal{E} \), and \( U \) is a unitary operator on \( \mathcal{H}_u \), where

\[
\mathcal{H}_u = \bigcap_{m=0}^{\infty} V^m \mathcal{H}.
\]

We say that \( V \) is pure if \( \mathcal{H}_u = \{0\} \), or, equivalently, if \( V^m \rightarrow 0 \) in the strong operator topology (that is, \( \|V^m h\| \rightarrow 0 \) as \( m \rightarrow \infty \) for all \( h \in \mathcal{H} \)). Pure isometry, that is, shift operators on

\begin{footnotesize}
\textit{2010 Mathematics Subject Classification.} 47A13, 47A20, 47A56, 47A68, 47B38, 46E20, 30H10.

\textit{Key words and phrases.} pair of commuting contractions, pair of commuting isometries, Hardy space, factorizations, von Neumann inequality.
\end{footnotesize}
vector-valued Hardy spaces play an important role in the study of general operators that stems from the following result (see [13, 14]):

**Theorem 1.1. (Sz.-Nagy and Foias)** Let $T$ be a pure contraction on a Hilbert space $\mathcal{H}$. Then $T$ and $P_QM_z|_Q$ are unitarily equivalent, where $Q$ is a closed $M_z^*$-invariant subspace of a vector-valued Hardy space $H^2_D(D)$.

Here the $D$-valued Hardy space over $\mathbb{D}$, denoted by $H^2_D(D)$, is defined by

$$H^2_D(D) := \{ f = \sum_{k \in \mathbb{N}} \eta_k z^k \in \mathcal{O}(\mathbb{D}, D) : \eta_j \in D, j \in \mathbb{N}, \| f \| := \sum_{k \in \mathbb{N}} \| \eta_k \|^2 < \infty \}.$$  

Recall that a contraction $T$ on a Hilbert space $\mathcal{H}$ is pure (cf. [16]) if $T^m \to 0$ as $m \to \infty$ in the strong operator topology. Also note that, in the above theorem, one can choose the coefficient Hilbert space $D$ as $\text{ran} (I - TT^*)$ (see [13]).

In contrast with the von-Neumann and Wold decomposition theorem for isometries, the structure of commuting $n$-tuples of isometries, $n \geq 2$, is much more complicated and very little, in general, is known (see [3, 4, 5, 6, 7, 11, 12, 17, 18, 15]). However, for pure pairs of commuting isometries, the problem is more tractable.

A pair of commuting isometries $(V_1, V_2)$ on a Hilbert space $\mathcal{H}$ is said to be pure if $V_1V_2$ is a pure isometry, that is,

$$\bigcap_{m=0}^{\infty} V_1^m V_2^m \mathcal{H} = \{0\}.$$  

With this as motivation, a pair of commuting contractions $(T_1, T_2)$ is said to be pure if $T_1T_2$ is a pure contraction.

The concept of pure pair of commuting isometries introduced by Berger, Coburn and Lebow [6] is an important development in the study of representation and Fredholm theory for $C^*$-algebras generated by commuting isometries. They showed that a pair of commuting isometries $(V_1, V_2)$ on a Hilbert space $\mathcal{H}$ is pure if and only if there exist a Hilbert space $\mathcal{E}$, a unitary $U$ in $\mathcal{B}(\mathcal{E})$ and an orthogonal projection in $\mathcal{B}(\mathcal{E})$ such that $(V_1, V_2)$ on $\mathcal{H}$ and $(M_\Phi, M_\Psi)$ on $H^2_D(D)$ are jointly unitarily equivalent, where

$$(1.1) \quad \Phi(z) = (P + zP^\perp)U^* \quad \text{and} \quad \Psi(z) = U(P^\perp + zP) \quad (z \in \mathbb{D}).$$

Moreover, it follows that $M_\Phi M_\Psi = M_\Psi M_\Phi = M_z$, and $V_1V_2$ on $\mathcal{H}$ and $M_z$ on $H^2_D(D)$ are unitarily equivalent (see also [5, 10]). More precisely, if $\Pi : \mathcal{H} \to H^2_D(D)$ denotes the unitary map, implemented by the Wold and von Neumann decomposition of the pure isometry $V_1V_2$ with $\mathcal{E} = \text{ran} (I - V_1V_2V_1^*V_2^*)$ (cf. [13]), then

$$\Pi V_1 = M_\Phi \Pi, \quad \text{and} \quad \Pi V_2 = M_\Psi \Pi.$$  

In what follows, for a triple $(\mathcal{E}, U, P)$ as above we let $\Phi, \Psi \in H^\infty_{\mathcal{B}^\perp(\mathcal{E})}(\mathbb{D})$ denote the isometric multipliers as defined in (1.1). We call $(M_\Phi, M_\Psi)$ the pair of isometries associated with the triple $(\mathcal{E}, U, P)$.

Our work is motivated by the following equivalent interpretations of the Berger, Coburn and Lebow's characterizations of pure pairs of commuting isometries:
(I) Let \((V_1, V_2)\) be a pure pair of commuting isometries and let \(M_z\) on \(H_\infty^2(\mathbb{D})\) be the von Neumann and Wold decomposition representation of \(V_1 V_2\). Then there exist a unitary \(U\) and an orthogonal projection \(P\) in \(\mathcal{B}(\mathcal{E})\) such that the representations of \(V_1\) and \(V_2\) in \(\mathcal{B}(H_\infty^2(\mathbb{D}))\) are given by \(M_\Phi\) and \(M_\Psi\), respectively.

(II) Let \((X, Y)\) be a pair of commuting isometries in \(\mathcal{B}(H_\infty^2(\mathbb{D}))\). Moreover, let \(X\) and \(Y\) are Toeplitz operators \([13]\) with analytic symbols from \(H_\infty^2(\mathbb{D}))\). Then \(M_z = X Y\) if and only if there exist a unitary \(U\) and an orthogonal projection \(P\) in \(\mathcal{B}(\mathcal{E})\) such that \((X, Y) = (M_\Phi, M_\Psi)\).

In this paper we shall obtain similar results for pure pairs of commuting contractions acting on Hilbert spaces. More specifically, summarizing Theorems 3.1, 3.2, and 4.1 and Corollary 4.2 we have the following: Let \(T\) be a pure contraction and \((T_1, T_2)\) be a pair of commuting contractions on a Hilbert space \(\mathcal{H}\). Let \(Q\) be the Sz.-Nagy and Foias representation of \(T\), that is, \(Q\) is a \(M_z^*\)-invariant subspace of a vector-valued Hardy space \(H_\infty^2(\mathbb{D})\) and \(T\) and \(P_Q M_z\) \(Q\) are unitarily equivalent (see Theorem 1.1 and Section 2) where \(D = \text{ran}(I - TT^*)\). Then the following are equivalent:

(i) \(T = T_1 T_2\).

(ii) There exist a triple \((\mathcal{E}, U, P)\) and a joint \((M_\Phi, M_\Psi, M_\psi^*)\)-invariant subspace \(Q\) of \(H_\infty^2(\mathbb{D})\) such that

\[
T_1 \cong P_Q M_\Phi |_Q, T_2 \cong P_Q M_\Psi |_Q, T \cong P_Q M_z |_Q \quad \text{and} \quad M_\Phi M_\Psi = M_\Psi M_\Phi = M_z.
\]

In other words, \((T_1, T_2, T)\) on \(\mathcal{H}\) dilates to \((M_\Phi, M_\Psi, M_z)\) on \(H_\infty^2(\mathbb{D})\).

(iii) There exist \(\mathcal{B}(\mathcal{D})\)-valued polynomials \(\varphi\) and \(\psi\) of degree \(\leq 1\) such that \(Q\) is a joint \((M_\varphi^*, M_\psi^*)\)-invariant subspace,

\[
P_Q M_z \mid_Q = P_Q M_{\varphi^*} |_Q = P_Q M_{\psi^*} |_Q,
\]

and

\[
(T_1, T_2) \cong (P_Q M_{\varphi^*} |_Q, P_Q M_{\psi^*} |_Q).
\]

In particular, if \(T = T_1 T_2\) is pure then the Sz.-Nagy and Foias representations of \(T_1\) and \(T_2\) on \(Q\) are given by \(P_Q M_{\varphi} |_Q\) and \(P_Q M_{\psi} |_Q\), respectively. Moreover, it turns out that the pair \((M_{\varphi^*}, M_{\psi^*})\) can be chosen as

\[
\varphi(z) = V^* \Phi(z)V \quad \text{and} \quad \psi(z) = V^* \Psi(z)V \quad (z \in \mathbb{D}),
\]

where \(V \in \mathcal{B}(\mathcal{D} ; \mathcal{E})\) is an isometry and \(\Phi, \Psi\) are the isometric multipliers associated with the triple \((\mathcal{E}, U, P)\) from condition (ii). As an application of our results we give a sharper von Neumann inequality for commuting pairs of contractions: Let \((T_1, T_2)\) be a commuting pair of contractions on \(\mathcal{H}\). Also assume that \(T_1 T_2\) is a pure contraction and \(\text{rank} \ (I_H - T_i T_i^*) < \infty, \ i = 1, 2\). Then there exists a variety \(V\) in \(\mathbb{D}^2\) such that

\[
\|p(T_1, T_2)\| \leq \sup_{(z_1, z_2) \in V} \ |p(z_1, z_2)| \quad (p \in \mathbb{C}[z_1, z_2]).
\]

The plan of the paper is the following. Section 2 contains some preliminaries and a key dilation result. In Section 3 we prove that a pure pair of commuting contractions always dilates to a pure pair of commuting isometries. Our construction is more explicit for pairs of contractions with finite dimensional defect spaces. In Section 4, we obtain explicit representations of commuting and contractive factors of a pure contraction in its corresponding
Sz.-Nagy and Foias space. In the last section, we consider von Neumann inequality for pure pair of commuting contractions.

2. Preliminaries

In this section, we set notation and definitions and discuss some preliminaries. Also we prove a basic dilation result in Theorem 2.1. This result will play a fundamental role throughout the remainder of the paper.

Let $T$ be a contraction on a Hilbert space $H$ (that is, $\|Tf\| \leq \|f\|$ for all $f \in H$ or, equivalently, if $I_H - TT^* \geq 0$) and let $E$ be a Hilbert space. Then $M_z$ on $H^2_\mathbb{C}(\mathbb{D})$ is called an isometric dilation of $T$ (cf. [16]) if there exists an isometry $\Gamma : H \rightarrow H^2_\mathbb{C}(\mathbb{D})$ such that

$$\Gamma T^* = M_z \Gamma.$$ 

Similarly, a pair of commuting operators $(U_1, U_2)$ on $K$ is said to be a dilation of a commuting pair of operators $(T_1, T_2)$ on $H$ if there exists an isometry $\Gamma : H \rightarrow K$ such that

$$\Gamma T^*_j = U_j^* \Gamma \quad (j = 1, 2).$$

Note that, in this case, $Q := \text{ran} \Gamma$ is a joint $(U_1^*, U_2^*)$-invariant subspace of $K$ and $T_j \simeq P_Q U_j |_Q \quad (j = 1, 2)$.

Now let $T$ be a contraction on a Hilbert space $H$. Set

$$D_T = \text{ran} \left( I_H - TT^* \right), \quad D_T^\perp = \left( I_H - TT^* \right)^\perp.$$ 

If in addition, $T$ is pure then $M_z$ on $H^2_\mathbb{C}(\mathbb{D})$, induced by the isometry $\Pi : H \rightarrow H^2_\mathbb{C}(\mathbb{D})$, is an isometric dilation of $T$ (cf. [16]), where

$$(\Pi h)(z) = D_T (I_H - zT^*)^{-1} h \quad (z \in \mathbb{D}, h \in H).$$

In particular, this yields a proof of Theorem 1.1 that every pure contraction is unitarily equivalent to the compression of $M_z$ to an $M_z^*$-invariant closed subspace of a vector-valued Hardy space.

It is also important to note that the above dilation is minimal, that is,

$$H^2_{\mathbb{D}_T}(\mathbb{D}) = \text{span} \{ z^m \Pi f : m \in \mathbb{N}, f \in H \},$$

and hence unique in an appropriate sense (see [13]).

Our considerations will also rely on the techniques of transfer functions (cf. [8]). Let $H_1$ and $H_2$ be two Hilbert spaces, and

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(H_1 \oplus H_2),$$

be a unitary operator. Then the $\mathcal{B}(H_1)$-valued analytic function $\tau_U$ on $\mathbb{D}$ defined by

$$\tau_U(z) := A + zB(I - zD)^{-1}C \quad (z \in \mathbb{D}),$$

is called the transfer function of $U$. Using $U^* U = I$, a standard and well known computation yields (cf. [8])

$$I - \tau_U(z)^* \tau_U(z) = (1 - |z|^2)C^*(I - zD^*)^{-1}(I - zD)^{-1}C \quad (z \in \mathbb{D}).$$
However, in this paper, we will mostly deal with transfer functions corresponding to unitary matrices of the form \( U = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \). In this case, it follows (see (2.3)) from the identity
\[
I - \tau_U(z)^* \tau_U(z) = (1 - |z|^2) C^* C \quad (z \in \mathbb{D})
\]
that \( \tau_U \) is a \( B(\mathcal{H}_1) \)-valued inner function \([13]\).

Now let \((T_1, T_2)\) be a pair of commuting contractions. Since
\[
(I_{\mathcal{H}} - T_1 T_1^*) + T_1 (I_{\mathcal{H}} - T_2 T_2^*) T_1^* = T_2 (I_{\mathcal{H}} - T_1 T_1^*) T_2^* + (I_{\mathcal{H}} - T_2 T_2^*),
\]
it follows that
\[
\|D_{T_1} h\|^2 + \|D_{T_2} T_1^* h\|^2 = \|D_{T_1} T_2^* h\|^2 + \|D_{T_2} h\|^2 \quad (h \in \mathcal{H}).
\]
Thus
\[
U : \{ D_{T_1} h \oplus D_{T_2} T_1^* h : h \in \mathcal{H} \} \to \{ D_{T_1} T_2^* h \oplus D_{T_2} h : h \in \mathcal{H} \}
\]
defined by
\[
(2.4) \quad U (D_{T_1} h, D_{T_2} T_1^* h) = (D_{T_1} T_2^* h, D_{T_2} h) \quad (h \in \mathcal{H}),
\]
is an isometry. This operator will play a very important role in the sequel.

We now formulate the main theorem of this section, a result which will play a very important part in our considerations later on. Here the proof is similar in spirit to the main dilation result of \([8]\).

Let \( \mathcal{H} \) and \( \mathcal{E} \) be Hilbert spaces and let \((S,T)\) be a pair of commuting contractions on \( \mathcal{H} \). Let \( T \) be pure and \( V \in \mathcal{B}(\mathcal{D}_T; \mathcal{E}) \) be an isometry. Then the isometric dilation of \( T \), \( \Pi : \mathcal{H} \to H_{\mathcal{D}_T}^2(\mathbb{D}) \) as defined in (2.1), allows us to define an isometry \( \Pi_V \in \mathcal{B}(\mathcal{H}; H_{\mathcal{E}}^2(\mathbb{D})) \) by setting
\[
\Pi_V := (I_{H_{\mathcal{D}_T}} \otimes V) \Pi.
\]
It is easy to check that
\[
\Pi_V T^* = (M_x^* \otimes I_\mathcal{E}) \Pi_V,
\]
and hence we conclude that \( M_x \) on \( H_{\mathcal{E}}^2(\mathbb{D}) \) is an isometric dilation of \( T \). In particular, \( Q = \Pi_V \mathcal{H} \) is a \( M_x^* \)-invariant subspace of \( H_{\mathcal{E}}^2(\mathbb{D}) \) and \( T \cong P_Q M_x |_Q \).

**Theorem 2.1.** With the notations as above, let
\[
U = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} : \mathcal{E} \oplus \mathcal{D}_S \to \mathcal{E} \oplus \mathcal{D}_S,
\]
be a unitary operator such that
\[
U (VD_{T} h, D_{S} T^* h) = (VD_{T} S^* h, D_{S} h) \quad (h \in \mathcal{H}).
\]
We denote by \( \Phi(z) = A^* + zC^* B^* \) the transfer function of \( U^* \). Then \( \Phi \) is a \( \mathcal{B}(\mathcal{E}) \)-valued inner function and
\[
\Pi_V S^* = M_x^\Phi \Pi_V.
\]
In particular, \( Q = \Pi_V \mathcal{H} \) is a joint \( (M_x^*, M_\Phi^*) \)-invariant subspace of \( H_{\mathcal{E}}^2(\mathbb{D}) \) and
\[
T^* \cong M_x^* |_Q \quad \text{and} \quad S^* \cong M_\Phi^* |_Q.
Proof. We only need to prove that $\Pi V S^* = M_\Phi^* \Pi V$. Now for each $h \in H$ we have the equality
\[
\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} V D_T h \\ D_S T^* h \end{bmatrix} = \begin{bmatrix} V D_T S^* h \\ D_S h \end{bmatrix},
\]
that is,
\[
V D_T S^* h = AV D_T h + BD_S T^* h, \quad \text{and} \quad D_S h = CV D_T h.
\]
This implies
\[
V D_T S^* = AV D_T + BCV D_T T^*.
\]
Now if $n \geq 1$, $h \in H$ and $\eta \in \mathcal{E}$, then
\[
\langle M_\Phi^* \Pi V h, z^n \eta \rangle = \langle (I \otimes V) D_T (I - z T^*)^{-1} h, (A^* + z C^* B^*) (z^n \eta) \rangle
\]
\[
= \langle V D_T T^{*n} h, A^* \eta \rangle + \langle V D_T T^{*(n+1)} h, C^* B^* \eta \rangle
\]
\[
= \langle (AV D_T + BCV D_T T^*) (T^{*n} h), \eta \rangle
\]
\[
= \langle V D_T S^* (T^{*n} h), \eta \rangle.
\]
On the other hand, since
\[
\langle \Pi V S^* h, z^n \eta \rangle = \langle V D_T (I - z T^*)^{-1} S^* h, z^n \eta \rangle = \langle (V D_T S^*) (T^{*n} h), \eta \rangle,
\]
we get $\Pi V S^* = M_\Phi^* \Pi V$. This completes the proof. \hfill \Box

3. Dilating to pure isometries

In this section we prove that a pure pair of commuting contractions dilates to a pure pair of commuting isometries. We describe the construction of dilations more explicitly in the case of finite dimensional defect spaces.

**Theorem 3.1.** Let $(T_1, T_2)$ be a pure pair of commuting contractions on $H$ and $\dim \mathcal{D}_{T_j} < \infty$, $j = 1, 2$. Then $(T_1, T_2)$ dilates to a pure pair of commuting isometries.

**Proof.** Set $\mathcal{E} := \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}$ and $T := T_1 T_2$. Let $\Pi : H \to H^2_{\mathcal{D}_T}(\mathbb{D})$ be the isometric dilation of $T$ as defined in [2.1]. Now observe that the equality
\[
I - TT^* = I - T_1 T_2 T^* T_2^* = (I - T_1 T_1^*) + T_1 (I - T_2 T_2^*) T_1^*,
\]
implies that the operator $V \in \mathcal{B}(\mathcal{D}_T; \mathcal{E})$ defined by
\[
V(D_T h) = (D_{T_1} h, D_{T_2} T_1^* h) \quad (h \in H),
\]
is an isometry. Consequently,
\[
(3.1) \quad \Pi V := (I_{H^2_{\mathbb{D}}} \otimes V) \Pi : H \to H^2_{\mathcal{E}}(\mathbb{D})
\]
is an isometric dilation of $T$, and hence $T \cong P_Q M_\Phi |_Q$ where $Q = \Pi V H$ is a $M_\Phi^*$-invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ (see the proof of Theorem [2.1]). Let $\iota_j : \mathcal{D}_{T_j} \to \mathcal{E}$, $j = 1, 2$, be the inclusion maps, defined by
\[
\iota_1(h_1) = (h_1, 0) \quad \text{and} \quad \iota_2(h_2) = (0, h_2) \quad (h_1 \in \mathcal{D}_{T_1}, h_2 \in \mathcal{D}_{T_2}).
\]
Then $P := \iota_2 \iota_2^* \in \mathcal{B}(\mathcal{E})$ is the orthogonal projection onto $\mathcal{D}_{T_2}$, that is,

$$P(h_1, h_2) = (0, h_2) \quad ((h_1, h_2) \in \mathcal{E}).$$

Thus, $\iota_1 \iota_1^* = P^\perp$ is the orthogonal projection onto $\mathcal{D}_{T_1}$, and so

$$\begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} : \mathcal{E} \oplus \mathcal{D}_{T_1} \to \mathcal{E} \oplus \mathcal{D}_{T_1}$$

is a unitary. Now since $\dim \mathcal{E} < \infty$, it follows that the isometry $U$, as defined in (2.4), extends to a unitary, denoted again by $U$, on $\mathcal{E}$. In particular, there exists a unitary operator $U$ on $\mathcal{E}$ such that

$$U(D_{T_1}T_2^*h, D_{T_2}h) = (D_{T_1}h, D_{T_2}T_1^*h) \quad (h \in \mathcal{H}).$$

Then

$$U_1 = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} = \begin{bmatrix} UP & U\iota_1 \\ \iota_1^* & 0 \end{bmatrix},$$

is a unitary operator in $\mathcal{B}(\mathcal{E} \oplus \mathcal{D}_{T_1})$. Moreover, for all $h \in \mathcal{H}$, we have

$$U_1(V(D_T h), D_{T_1}T_1^*h) = U_1(D_{T_1}h, D_{T_2}T_1^*h, D_{T_1}T_1^*T_2^*h)$$

$$= (U(D_{T_1}T_1^*T_2^*h, D_{T_1}T_1^*h), D_{T_1}h)$$

$$= (D_{T_1}T_1^*h, D_{T_2}(T_1^*h), D_{T_1}h)$$

$$= (V(D_{T_1}T_1^*h), D_{T_1}h).$$

Consequently, by Theorem 2.1 we have

$$\Pi_V T_1^* = M_\Phi^* \Pi_V,$$

where

$$\Phi(z) = PU^* + z \iota_1 \iota_1^* U^* = (P + z P^\perp) U^* \quad (z \in \mathbb{D}),$$

is the transfer function of the unitary operator $U_1^*$. Similarly, if we define a unitary $U_2 \in \mathcal{B}(\mathcal{E} \oplus \mathcal{D}_{T_2})$ by

$$U_2 = \begin{bmatrix} P^\perp & \iota_2 \\ \iota_2^* & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} P^\perp U^* & \iota_2 \\ \iota_2^* U^* & 0 \end{bmatrix},$$

then

$$U_2(V(D_T h), D_{T_2}T_2^*h) = (V(D_{T_1}T_2^*h), D_{T_2}h) \quad (h \in \mathcal{H}),$$

and hence by Theorem 2.1 we have

$$\Pi_V T_2^* = M_\Psi^* \Pi_V,$$

where

$$\Psi(z) = U P^\perp + z \iota_2 \iota_2^* U = U(P^\perp + z P),$$

is the transfer function for the unitary operator $U_2^*$. This completes the proof that the pure pair of commuting isometries $(M_\Phi, M_\Psi)$ on $H^2_\mathcal{E}(\mathbb{D})$ corresponding to the triple $(\mathcal{E}, U, P)$ dilates $(T_1, T_2)$.

We will now go on to give a proof of the general result. The proof is essentially the same as the previous theorem except the constructions of unitary operators and inclusion maps.
Theorem 3.2. Let \((T_1, T_2)\) be a pure pair of commuting contractions on \(H\). Then \((T_1, T_2)\) dilates to a pure pair of commuting isometries.

Proof. Let \(\dim D_{T_1} = \infty\), or \(\dim D_{T_2} = \infty\) and \(D\) be an infinite dimensional Hilbert space. Set \(\mathcal{E} := (D) \oplus D_{T_1} \oplus D_{T_2}\). We now define inclusion maps \(\iota_1 : D \oplus D_{T_1} \rightarrow \mathcal{E}\) and \(\iota_2 : D_{T_2} \rightarrow \mathcal{E}\) by

\[
\iota_1(h, h_1) = (h, h_1, 0) \quad \text{and} \quad \iota_2(h_2) = (0, 0, h_2), \quad (h \in D, h_1 \in D_{T_1}, h_2 \in D_{T_2})
\]

respectively, and an isometric embedding \(V \in \mathcal{B}(D_T; \mathcal{E})\) by

\[
VD_T h = (0, D_{T_1} h, D_{T_2} T_1^* h) \quad (h \in H).
\]

We also define the orthogonal projection \(P\) by \(P = \iota_2 \iota_2^*\). Therefore

\[
P(h_1, h_2, h_3) = (0, 0, h_3) \quad ((h_1, h_2, h_3) \in \mathcal{E}).
\]

Finally, since

\[
U_D (0_D, D_{T_1} h, D_{T_2} T_1^* h) = (0_D, D_{T_1} T_2^* h, D_{T_2} h) \quad (h \in H),
\]

defines an isometry from \(\{0_D\} \oplus \{D_{T_1} h \oplus D_{T_2} T_1^* h : h \in H\}\) to \(\{0_D\} \oplus \{D_{T_1} T_2^* h \oplus D_{T_2} h : h \in H\}\), we can therefore extend \(U_D\) to a unitary, denoted again by \(U_D\), acting on \(\mathcal{E}\). With these notations we define unitary operators

\[
U_1 = \begin{bmatrix} U_D P & U_D \iota_1 \\ \iota_1^* & 0 \end{bmatrix} \in \mathcal{B}(D \oplus (D \oplus D_{T_1})) \quad \text{and} \quad U_2 = \begin{bmatrix} P^\perp U_D^* & \iota_2 \\ \iota_2^* U_D & 0 \end{bmatrix} \in \mathcal{B}(E \oplus D_{T_2}).
\]

The rest of the proof proceeds in the same way as in Theorem 3.1. This completes the proof. \(\square\)

The main inconvenience of our approach seems to be the nonuniqueness of the triple \((\mathcal{E}, U, P)\). This issue is closely related to the nonuniqueness of Ando dilation [2] and solutions of commutant lifting theorem [9].

It is also important to note that Theorem 3.2 is a sharper version of Ando dilation [2] for pure pairs of commuting contractions. More precisely, one can dilate a pure pair of commuting contractions to a pure pair of commuting isometries, in the sense of Berger, Coburn and Lebow. In the context of concrete isometric dilations for commuting pairs of pure contractions, see [11] and [8].

4. Factorizations

Let \((T_1, T_2)\) be a pair of commuting contractions on \(H\) and \(T = T_1 T_2\) be a pure contraction. Then by Theorem 1.1 we can realize \(T\) as \(P_\mathcal{Q} M_s|_\mathcal{Q}\) where \(Q = \text{ran} \Pi = \Pi H\) is the Sz.-Nagy and Foias model space and \(\Pi : H \rightarrow H^2_{D_T}(\mathbb{D})\) is the minimal isometric dilation of \(T\) (see (2.2)).

In this section we will show that \(T_1\) and \(T_2\) can be realized as compressions of two \(\mathcal{B}(D_T)\)-valued polynomials of degree \(\leq 1\) in the Sz.-Nagy and Foias model space \(\mathcal{Q}\) of the pure contraction \(T\).

Let \(\Pi_V : H \rightarrow H^2_\mathcal{Q}(\mathbb{D})\) be the isometric dilation as in Theorems 3.1 and 3.2, that is,

\[
\Pi_V T_1^* = M^*_\psi \Pi_V \quad \text{and} \quad \Pi_V T_2^* = M^*_\psi \Pi_V.
\]
Then it follows from 3.1 that
\[ \Pi T^*_1 = (I \otimes V^*)M_\varphi^*(I \otimes V)\Pi = M_\varphi^*\Pi, \]
where
\[ \varphi(z) = V^*\Phi(z)V \quad (z \in \mathbb{D}), \]
and \( V \in \mathcal{B}(\mathcal{D}_T; \mathcal{E}) \) is an isometry. Similarly, we derive
\[ \Pi T^*_2 = (I \otimes V^*)M_\psi^*(I \otimes V)\Pi = M_\psi^*\Pi, \]
where
\[ \psi(z) = V^*\Psi(z)V \quad (z \in \mathbb{D}). \]
In particular, \( \text{ran} \, \Pi \) is a joint \((M_\varphi^*, M_\psi^*)\)-invariant subspace and by construction of \( \Pi \) it follows that
\[ \Pi T^* = M_z^*\Pi, \]
and \( \text{ran} \, \Pi \) is a also a \( M_z^* \)-invariant subspace of \( H^2_{\mathcal{D}_T}(\mathbb{D}) \). We have thus proved the following theorem.

**Theorem 4.1.** Let \( T \) be a pure contraction on a Hilbert space \( \mathcal{H} \) and let \( T \cong P_Q M_z|_Q \) be the Sz.-Nagy and Foias representation of \( T \) as in Theorem 1.1. If \( T = T_1 T_2 \), for some commuting pair of contractions \( (T_1, T_2) \) on \( \mathcal{H} \), then there exists \( \mathcal{B}(\mathcal{D}_T) \)-valued polynomials \( \varphi \) and \( \psi \) of degree \( \leq 1 \) such that \( Q \) is a joint \((M_\varphi^*, M_\psi^*)\)-invariant subspace and
\[ (T_1, T_2) \cong (P_Q M_\varphi|_Q, P_Q M_\varphi|_Q). \]
In particular,
\[ P_Q M_z|_Q = P_Q M_\varphi|_Q = P_Q M_\psi|_Q. \]
It is important to note that \( P_Q M_\varphi|_Q = P_Q M_\psi|_Q \), even though, in general
\[ \varphi \psi \neq \psi \varphi. \]
A reformulation of Theorem 4.1 is the following:

**Corollary 4.2.** Let \( T \) be a pure contraction on a Hilbert space \( \mathcal{H} \) and let \( T \cong P_Q M_z|_Q \) be the Sz.-Nagy and Foias representation of \( T \) as in Theorem 1.1. Then \( T = T_1 T_2 \), for some commuting pair of contractions \( (T_1, T_2) \) on \( \mathcal{H} \), if and only if there exist \( \mathcal{B}(\mathcal{D}_T) \)-valued polynomials \( \varphi \) and \( \psi \) of degree \( \leq 1 \) such that \( Q \) is a joint \((M_\varphi^*, M_\psi^*)\)-invariant subspace,
\[ P_Q M_z|_Q = P_Q M_\varphi|_Q = P_Q M_\psi|_Q, \]
and
\[ (T_1, T_2) \cong (P_Q M_\varphi|_Q, P_Q M_\psi|_Q). \]
Moreover, there exists a triple \((\mathcal{E}, U, P)\) and an isometry \( V \in \mathcal{B}(\mathcal{D}_T; \mathcal{E}) \) such that
\[ \varphi(z) = V^*\Phi(z)V \quad \text{and} \quad \psi(z) = V^*\Psi(z)V \quad (z \in \mathbb{D}). \]
5. von Neumann Inequality

In this section we consider the von Neumann inequality for pure pair of commuting contractions with finite dimensional defect spaces. We show that for such a pair there exists a variety in the bidisc where the von Neumann inequality holds.

**Theorem 5.1.** Let \((T_1, T_2)\) be a pure pair of commuting contractions on \(\mathcal{H}\) and \(\dim \mathcal{D}_{T_i} < \infty\), \(i = 1, 2\). Then there exists an algebraic variety \(V\) in \(\overline{D}^2\) such that

\[
\|p(T_1, T_2)\| \leq \sup_{(z_1, z_2) \in V} |p(z_1, z_2)| \quad (p \in \mathbb{C}[z_1, z_2]).
\]

Moreover, if \(m = \dim(\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2})\), then there exists a pure pair of commuting isometries \((M_\Phi, M_\Psi)\) on \(H^2_C(\mathbb{D})\) such that

\[
V = \{(z_1, z_2) \in \overline{D}^2 : \det(\Phi(z_1 z_2) - z_1 I) = 0 \text{ and } \det(\Psi(z_1 z_2) - z_2 I) = 0\}.
\]

**Proof.** Let \(\mathcal{E} = \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}\). Then by Theorem 3.1 there exists a pure pair of commuting isometries \((M_\Phi, M_\Psi)\) on \(H^2_C(\mathbb{D})\) and a joint \((M_\Phi^*, M_\Psi^*)\)-invariant subspace \(\mathcal{Q}\) of \(H^2_C(\mathbb{D})\) such that \(T_1 \cong P_\mathcal{Q} M_\Phi \big|_{\mathcal{Q}}\) and \(T_2 \cong P_\mathcal{Q} M_\Psi \big|_{\mathcal{Q}}\). Then for each \(p \in \mathbb{C}[z_1, z_2]\), we have

\[
\|p(T_1, T_2)\|_{\mathcal{B}(\mathcal{H})} = \|P_\mathcal{Q} p(M_\Phi, M_\Psi)\|_{\mathcal{B}(\mathcal{Q})} \\
\leq \|p(M_\Phi, M_\Psi)\|_{\mathcal{B}(H^2_C(\mathbb{D}))} \\
= \|M_p(\Phi, \Psi)\|_{\mathcal{B}(H^2_C(\mathbb{D}))} \\
\leq \sup_{z \in \mathcal{T}} \|p(\Phi(z), \Psi(z))\|_{\mathcal{B}(\mathcal{E})} \\
\leq \sup_{\{p(\lambda_1, \lambda_2) : (\lambda_1, \lambda_2) \in \sigma(\Phi(z), \Psi(z)), z \in \mathbb{T}\}} \|p(\lambda_1, \lambda_2)\|_{\mathcal{B}(\mathcal{H})}.
\]

where we denote \(\sigma(\Phi(z), \Psi(z))\) by the joint spectrum of the commuting pair of unitary matrices \((\Phi(z), \Psi(z)), z \in \mathbb{T}\). Now observe that if \((\lambda_1, \lambda_2) \in \sigma(\Phi(z), \Psi(z))\) for some \(z \in \mathbb{T}\), then there exists a non-zero \(h \in \mathcal{E}\) such that \(\Phi(z)h = \lambda_1 h\) and \(\Psi(z)h = \lambda_2 h\). Then \(zh = \Phi(z)\Psi(z)h = \lambda_1 \lambda_2 h\) and hence \(z = \lambda_1 \lambda_2\). With this observation we have

\[
\{(\lambda_1, \lambda_2) \in \sigma(\Phi(z), \Psi(z)) : z \in \mathbb{T}\} \subset \partial V,
\]

where

\[
V_1 = \{(\lambda_1, \lambda_2) \in \overline{D}^2 : \det(\Phi(\lambda_1 \lambda_2) - \lambda_1 I) = 0\},
\]

\[
V_2 = \{(\lambda_1, \lambda_2) \in \overline{D}^2 : \det(\Psi(\lambda_1 \lambda_2) - \lambda_2 I) = 0\},
\]

and

\[
V = V_1 \cap V_2.
\]

Now since \(\Phi\) and \(\Psi\) are matrix valued polynomial, the variety \(V\) is an algebraic variety in \(\overline{D}^2\). Note also that equation (5.1) implies that

\[
\|p(T_1, T_2)\| \leq \sup_{(z_1, z_2) \in \partial V} |p(z_1, z_2)|,
\]
and hence
\[ \|p(T_1, T_2)\| \leq \sup_{(z_1, z_2) \in V} |p(z_1, z_2)|, \]
for all \( p \in \mathbb{C}[z_1, z_2] \). This completes the proof.

With the hypotheses of Theorem 5.1, let \((\mathcal{E}, U, P)\) be the triple corresponding to the pure isometric pair \((M_\Phi, M_\Psi)\). Let also assume that \(P \overline{U}^* \) and \(U \overline{P}^\perp\) be completely non-unitary. It now follows from [8, Proposition 4.1] that \(\Phi(z)\) and \(\Psi(z)\), \(z \in \mathbb{D}\), does not have any unimodular eigenvalue. Therefore
\[ V \cap \{(D \times T) \cup (T \times D)\} = \emptyset, \]
and hence
\[ V \cap \partial \mathbb{D}^2 = V \cap T^2. \]
This allows one to replace the algebraic variety \(V\) in Theorem 5.1 by an algebraic distinguished variety (see [1])
\[ \tilde{V} = \tilde{V}_1 \cap \tilde{V}_2, \]
in \(\mathbb{D}^2\), where
\[ \tilde{V}_1 = \{(\lambda_1, \lambda_2) \in \mathbb{D}^2 : \det(\Phi(\lambda_1 \lambda_2) - \lambda_1 I) = 0\}, \]
and
\[ \tilde{V}_2 = \{(\lambda_1, \lambda_2) \in \mathbb{D}^2 : \det(\Psi(\lambda_1 \lambda_2) - \lambda_2 I) = 0\}. \]

Acknowledgement: The authors are grateful to the referee for pointing out an error in an earlier version and for numerous suggestions. The first author’s research work is supported by DST-INSPIRE Faculty Fellowship No. DST/INSPIRE/04/2015/001094. The research of the second author is supported in part by NBHM (National Board of Higher Mathematics, India) Research Grant NBHM/R.P.64/2014.

References

[1] J. Agler and J. McCarthy, Distinguished Varieties Acta. Math. 194 (2005), 133-153.
[2] T. Ando, On a pair of commutative contractions, Acta Sci. Math. (Szeged) 24 (1963), 88-90.
[3] H. Bercovici, R.G. Douglas and C. Foias, Canonical models for bi-isometries, A panorama of modern operator theory and related topics, 177-205, Oper. Theory Adv. Appl., 218, Birkhauser/Springer Basel AG, Basel, 2012.
[4] H. Bercovici, R.G. Douglas and C. Foias, Bi-isometries and commutant lifting, Characteristic functions, scattering functions and transfer functions, 51-76, Oper. Theory Adv. Appl., 197, Birkhauser Verlag, Basel, 2010.
[5] H. Bercovici, R.G. Douglas and C. Foias, On the classification of multi-isometries, Acta Sci. Math. (Szeged) 72 (2006), 639-661.
[6] C. A. Berger, L. A. Coburn and A. Lebow, Representation and index theory for \(C^*\)-algebras generated by commuting isometries, J. Funct. Anal. 27 (1978), no. 1, 51–99.
[7] K. Bickel and G. Knese, Canonical Agler decompositions and transfer function realizations, Trans. Amer. Math. Soc. 368 (2016), 6293-6324.
[8] B. K. Das and J. Sarkar, Ando dilations, von Neumann inequality, and distinguished varieties, J. Funct. Anal. 272 (2017), 2114-2131.
[9] C. Foias and A. E. Frazho, *The commutant lifting approach to interpolation problems*, Operator Theory: Advances and Applications, 44. Birkhauser Verlag, Basel, 1990.

[10] D. Gaspar and P. Gaspar, *Wold decompositions and the unitary model for bisomtries*, Integral Equations Operator Theory 49 (2004), 419-433.

[11] K. Guo and R. Yang, *The core function of submodules over the bidisk*, Indiana Univ. Math. J. 53 (2004), 205-222.

[12] W. He, Y. Qin and R. Yang, *Numerical invariants for commuting isometric pairs*, Indiana Univ. Math. J. 64 (2015), 1–19.

[13] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*. North-Holland, Amsterdam-London, 1970.

[14] B. Sz.-Nagy, *Sur les contractions de l’espace de Hilbert*, Acta Sci. Math. (Szeged) 15 (1953), 87–92.

[15] W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York, 1969.

[16] J. Sarkar, *An Introduction to Hilbert module approach to multivariable operator theory*, Operator Theory, (2015) 969–1033 , Springer.

[17] R. Yang, *The core operator and congruent submodules*, J. Funct. Anal. 228 (2005), 469-489.

[18] R. Yang, *Hilbert-Schmidt submodules and issues of unitary equivalence*, J. Operator Theory 53 (2005), 169-184.

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, 400076, India

E-mail address: dasb@math.iitb.ac.in, bata436@gmail.com

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India

E-mail address: jay@isibang.ac.in, jaydeb@gmail.com

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India

E-mail address: srijan_rs@isibang.ac.in, srijansarkar@gmail.com