An Application of the Poincare-Hopf Index Theorem: A Mathematical Model of Earthquake

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Abstract. In this paper, we will present a differential equation system which describes the stress (force) state on the earth's crust before and during the earthquake occurs by using the famous Poincare-Hopf index theorem which claims that any continuous vector field on a sphere has a positive index singularity.

1. Introduction
An earthquake (also known as a quake, tremor or temblor) is the result of a sudden release of energy in the earth's crust that creates seismic waves. One of the basic theories about earth-quakes is the Elastic Rebound Model (it might be the only one satisfactory theory about earthquake so far) created by Harry Fielding Reid[6], which states that at a geological fault between two moving plates, stress occurs and deforms the rocks. This occurs in four main steps of rock deformation, the original position, build up of strain, slippage, and strain release. If the fault creeps, it will produce frequent micro earth-quakes; if it binds together and then slips, it will produce large earthquakes. Stress will then quickly be released; sides of the fault will become offset; rocks will rebound to their initial state of stress.

In the literature there is no mathematical formula or equation system to describe the earthquake phenomenon. From the Elastic Re-bound model, it is possible for us to define a continuous (stress) vector field on the Earth and then apply some mathematical theories to describing the earthquake phenomenon. In this paper, we will present a differential equation system to describe the stress (force) state on the earth's crust before and during the earthquake occurs by using the famous Poincare-Hopf index theorem which claims that any continuous vector field on a sphere has a positive index singularity.

2. 2-Sphere and Its Representations
In this section, we will review the 2-sphere and its representations. Generally, there are two different ways (external way and intrinsic way) to describe a 2-sphere. Externally, a 2-sphere (with a radius $\rho > 0$) is defined as a subset $S^2$ of $R^3$ (the Euclidean space with dimension 3 or the 3-Euclidean space) which consists of all points $(x, y, z) \in R^3$ satisfying the equation $x^2 + y^2 + z^2 = \rho^2$.

Intrinsically, a 2-sphere is looked to be a topological manifold which can be locally parameterized by two variables (we will expound it in detail in the following).

Definition 1 A topological manifold of dimension $n$ is a topological space $M$ which is locally homeomorphic to an open subset of the n-Euclidean space $R^n$. To be precisely, there is an open covering $\{U_i\}_{i \in I}$ of $M$ and for each open set $U_i$ there is a homeomorphism embedding $\phi_i: U_i \rightarrow R^n$. such a pair $(U_i, \phi_i)$ is called a local coordinate system or a chart on $M$. On the other hand, the inverse...
mapping \( \phi_i^{-1} \) of \( \phi_i \) is called a local parameterization of \( M \). The collection \( A = \{ U_i, \phi_i \}_{i \in I} \) of charts is called an atlas on \( M \).

**Remark 1** The topological manifold \( M \) is generally required to be an Hausdorff space having a countable basis. In differential geometry the atlas \( A = \{ U_i, \phi_i \}_{i \in I} \) on \( M \) is additionally required to be smooth (such an \( M \) is called a differential manifold) i.e. for each pair of charts \( ( U, \theta ) \) and \( ( V, \varphi ) \) in \( A \), the mapping (called a change of coordinate system) \( F = \varphi \circ \theta^{-1} : \theta ( U \cap V ) \to \varphi ( U \cap V ) \) is a smooth mapping between open sets in \( n \)-Euclidean space \( \mathbb{R}^n \) (see Figure 1). One natural way to represent the 2-sphere as a topological manifold is using the polar coordinate charts.

**Definition 2** The polar coordinate of the 2-sphere (with a radius \( \rho > 0 \)) is a mapping \( F : D \to \mathbb{R}^3 \) defined by \( F ( \theta, \varphi ) = ( x, y, z ) \), where \( x = \rho \sin \theta \cos \varphi \), \( y = \rho \sin \theta \sin \varphi \), \( z = \rho \cos \theta \) \(( \forall ( \theta, \varphi ) \in D \) is the standard Cartesian coordinate, and \( D = [0, \pi] \times [0, 2\pi] \) (a rectangle).

**Remark 2** Obviously the image or range of the above mapping is a 2-sphere with a radius \( \rho > 0 \) (i.e. we can look the 2-sphere to be the image of a closed rectangle \( D = [0, \pi] \times [0, 2\pi] \) under the polar coordinate mapping). Notice that the image of \([0, \pi] \times [0, 2\pi] \) is a 2-sphere (with a radius \( \rho > 0 \)) punching the south point. To see this, let us first collapse the vertical edge \((0,0) \times [0,2\pi] \) to a point, and then glue the two horizontal edges \((0, \pi) \times [0, \pi] \) and \((0, \pi) \times [2\pi] \) using the equivalence relation \(~\) defined by \((a, 0) \sim (b, 2\pi) \) if \( a = b \) (i.e. the two points \((\theta, 0) \) and \((\theta, 2\pi) \) on the two horizontal edges are equivalence but the other points are equivalence to themselves). Then the result space \( \tilde{D} \) can be regarded as a quotient space \( \tilde{D} = ( D / \{0\} \times [0, 2\pi] ) / ( (\theta, 0) \sim (\theta, 2\pi) ) \) of \( D \).

Since the mapping \( F \) in Definition 2 takes constant value \((0,0,1)\) on the vertical edge \([0] \times [0, 2\pi] \) (i.e. \( F ([0] \times [0, 2\pi]) = \{(0,1)\} \)) and satisfies \( F (\theta, 0) = F (\theta, 2\pi) \) \((\forall \theta \in (0, \pi)) \). It can be lifted to a mapping \( \tilde{F} : \tilde{D} \to \mathbb{R}^2 \) \( \sim \{0, 0, -1\} \) = \( U_1 \) (\( \tilde{F} \) is actually a homeomorphism, where \( \tilde{D} \) has the quotient topology). It can be seen that the interior \( D^0 = (0, \pi) \times (0, 2\pi) \) of the rectangle \( D \) is homeomorphic with the 2-sphere cut out the one latitude (we denote this set by \( U_1 \)), and from this we can define a chart \((U_1, \phi_1)\) on the 2-sphere, where \( \phi_1 : U_1 \to D^0 = (0, \pi) \times (0, 2\pi) \) (see Figure 2). In this way we obtain an atlas \( A = \{(U_1, \phi_1), (U_1, \phi_2), (U_1, \phi_3), (U_1, \phi_4)\} \) on the 2-sphere \( S^2 \) which consists of only four charts (it shows a rectangle gluing the two horizontal edges and collapsing the thick vertical edge to the origin in \((0,0)\) and a sphere punching the south point \((0,0, -1)\)).

![Figure 1. Change of coordinate system.](image1)

![Figure 2. A chart on the 2-sphere.](image2)

Another natural way to represent the 2-sphere as a topological manifold is using the stereographic projection. In fact, as a punched 2-sphere is homeomorphic to the 2-Euclidean space \( \mathbb{R}^2 \), the 2-sphere can be looked to be a topological manifold which has an atlas consisting of just two natural charts.

**Definition 3** The stereographic projection parameterization of the punched 2-sphere \( S^2 - \{(0,0,1)\} \) (with a radius \( \rho = 1 \)) is a mapping \( G : \mathbb{R}^2 \to \mathbb{R}^3 \) defined by \( G (u,v) = (x,y,z) \), whose standard Cartesian coordinate is as follows:

\[
\begin{align*}
    x &= \frac{2u}{1+u^2+v^2},
    y &= \frac{2v}{1+u^2+v^2},
    z &= \frac{1+u^2+v^2}{1+u^2+v^2} \quad (\forall (u,v) \in \mathbb{R}^2)
\end{align*}
\]  

(1)

**Remark 3** (1) Let \( U_N = S^2 - \{(0,0,1)\} \). To see how the inverse \( G^{-1} : U_N \to \mathbb{R}^2 \) of the mapping \( G \) in Definition 3 works (the inverse \( G^{-1} \) of \( G \) is also written as \( \pi_N : U_N \to \mathbb{R}^2 \), where \( N = (0,0,1) \) is called the north point). Let \( P = (x,y,z) \in U_N \). Then the equation of the line \( l_P \) through points \( P \) and \( N \) in 3-
Euclidean space $\mathbb{R}^3$ can be given by $\frac{x}{y} = \frac{y}{z} = \frac{z-1}{x-1}$, so the preimage of $P$ under the projection $G$ (i.e. the image of $P$ under $G^{-1} = \pi_N$) is exactly the intersection point of $l_{PN}$ with the plane $Z = 0$. By a simple calculation, we have $G^{-1}(P) = \pi_N(P) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

If we use $u, v$ to represent the coordinates of plane $Z = 0$, then (1) is the formula of the stereographic projection parameterization.

(2) Clearly the stereographic projection parameterization, defined as (1), is a homeomorphism from the 2-Euclidean plane to a punctured 2-sphere. The equator is mapped by $\pi_N$ to a unit circle, the centre of this circle is just the south pole, the interior of the circle is the northern hemisphere, the exterior of the circle is the southern hemisphere, and a latitude is a line through the origin (see Figure 3, where the black point is on the sphere, the white point is the corresponding point on the plane, and the gray point is punched). Apparently, $(\pi_N, U_N)$ is a chart on the 2-sphere. We can define another chart $(\pi_S, U_S)$ on the 2-sphere, and in this way we obtain an atlas $A = \{(\pi_N, U_N), (\pi_S, U_S)\}$ on the 2-sphere $S^2$ which consists of only two charts.

3. Vector Fields and the Poincare-Hopf Index Theorem

In mathematics a vector field on a subset $M \subseteq \mathbb{R}^n$ is a mapping $F: M \to \mathbb{R}^n$ in standard Cartesian coordinates $(x_1, \ldots, x_n)$, where $x_i = p_i \circ F(s)$ $(s \in S)$ and $p_i: \mathbb{R}^n \to \mathbb{R}$ is the $i$-th projection $i = 1, 2, \ldots, n$. In physics vector fields are often used to model the speed and direction of a moving fluid throughout space, or the strength and direction of some force (such as the magnetic or gravitational force) as it changes from point to point. In the rigorous mathematical treatment, (tangent) vector fields are defined on manifolds as sections of a manifold’s tangent bundle. They are one kind of tensor field on the manifold. If $S$ is an open set, then $F$ is a continuous function provided that each component of $F$ is continuous. More generally $F$ is said to be a $C^k$ vector field if each component or coordinate of $F$ is $k$ times continuously differentiable.

**Definition 4** Let $F$ be a smooth tangent vector field on the manifold $M$ of dimension 2 with $p_0 \in M$ is an isolate zero point of $M$ (i.e. there exist an open subset $U \subseteq M$, $p_0 \in U$ is the only zero point of the vector field $F$). The local index $\text{index}(p_0)$ of $F$ at $p_0$ is defined by

$$\text{index}(p_0) = \frac{1}{2\pi} \oint \frac{dF(y(t))}{\|dF(y(t))\|} dt$$

where $y$ is a loop on the manifold $M$ given by parameterization $y: [0, 1] \to M$ with $y(0) = y(1)$, and we associated a direction of $y$ when $t \in [0, 1]$ increase the loop $y$ rotate anticlockwise. The composition mapping $F \circ y: [0, 1] \to \mathbb{R}^2$ is called Gaussian mapping, which describes how the vector field rotates along a loop. The local index is just the number of circle which the vector rotates along the loop we choose and it is positive when the vector rotate is anticlockwise, and negative if not.

Let $M$ be a compactorientable differentiable manifold, and $F$ be a vector field on $M$ with isolated zeroes. If $M$ has boundary, then we insist that $F$ be pointing in the outward normal direction along the boundary. Then the following theorem holds:

**Theorem 1** $\sum_i \text{index}(x_i) = \chi(M)$, where the sum of the indices is over all the isolated zeroes of $F$, and $\chi(M)$ is the Euler characteristic of $M$.

This theorem is called the Poincare-Hopf index theorem, which is named after the famous French mathematician Henri Poincaré and the Swiss mathematician Heinz Hopf (see [1, Theorem 11.25]). Henri Poincare first showed the existence of singularities of vector fields on a compact surface in order to study the qualitative theory of ordinary differential equations. Later, Heinz Hopf proved the full version of the theorem. Because the Euler characteristic of a compact surface is 2, the original Poincaré result becomes a special case of the Poincare-Hopf index theorem. The Poincare-Hopf index theorem is very important in modern differential topology and still attracts extensive investigations.

Several generalizations or new proofs are presented in recent years (see [5,7,10]). However, we only need the original Poincare theorem in this paper, and the readers only need to know some necessary mathematical notions appeared in the original Poincare theorem.
Example 1 (1) If we look the 2-sphere in the external way, the vector field on it is just the restriction of a vector field defined on the 3-Euclidean space. Here are two examples.

The simplest vector field on the 3-Euclidean space \( R^3 \) is given by the identity mapping \( F: R^3 \rightarrow R^3 \), i.e. \( F(x, y, z) = (y, -x, 0) \). The restriction \( F|S^2 \) of \( F \) to \( S^2 \) gives the first vector field (which is actually a smooth vector field taking values in \( R^3 \)) on \( S^2 \) and there is no zero point on the sphere. It is a section of the normal bundle on the sphere since each vector perpendicular to the tangent plane of point where the vector defined (see Figure 4).

Another vector field on the 3-Euclidean space \( R^3 \) is given by \( F(x, y, z) = (y, x, 0) \). The restriction \( F|S^2 \) of \( F \) to \( S^2 \) gives the second vector field (which is actually a smooth vector field taking values in \( R^3 \)). It is perpendicular to the normal vector at each point of the sphere, so it is a tangent vector field. As the value of this vector field is independent to \( z \), it is constant along a latitude. The pole point is the only zero point of this vector field, and the index of zero are \( \pm 1 \), the sum of the index is 0 (see Figure 5).

(2) If we consider the sphere \( S^2 \) as a two dimension manifold, then we can define the vector field on \( S^2 \) intrinsically. As \( S^2 \) is locally homeomorphic to the 2-Euclidean space, we first define two vector fields \( F_N: \pi_N(U_N) \rightarrow R^2 \) and \( F_S: \pi_S(U_S) \rightarrow R^2 \), then using the gluing lemma for open subsets we can get a vector field \( G: S^2 \rightarrow R^2 \) on \( S^2 \):

\[
G(x, y, z) = \begin{cases} 
F_N \circ \pi_N(x, y, z), (x, y, z) \in U_N, \\
F_S \circ \pi_S(x, y, z), (x, y, z) \in U_S - U_N,
\end{cases}
\]

(3)

\[\pi_N(x, y, z) = \begin{pmatrix} x \\ \frac{y}{1-\frac{1}{1-z}} \end{pmatrix} \quad (\forall (x, y, z) \in U_N) \quad \text{and} \quad \pi_S(x, y, z) = \begin{pmatrix} x \\ \frac{y}{1+\frac{1}{1+z}} \end{pmatrix} \quad (\forall (x, y, z) \in U_S) \]

(4)

Remark 3.1 (1) Notice that \( F_N \) and \( F_S \) must satisfy the following condition (under which the gluing lemma holds): \( F_N \circ \pi_N(x, y, z) = F_S \circ \pi_S(x, y, z) \) \( (\forall (x, y, z) \in U_N \cap U_S = S^2 - \{(0,0,1), (0,0,-1)\}) \)

This is equivalent to the following equality (since both \( \pi_N \) and \( \pi_S \) and homeomorphisms): \( F_N = F_S \circ \pi_S \circ \pi_N^{-1}(u, v) \) \( (\forall (u, v) \in R^2) \). This is also equivalent to the equality: \( F_S = F_N \circ \pi_N \circ \pi_S^{-1}(u, v) \) \( (\forall (u', v') \in R^2) \). In other words, if we have defined \( F_N \) (resp. \( F_S \)), then we can obtain \( F_S \) (resp. \( F_N \)) by these equality. We expound it by the following example. Suppose that \( F_S(u, v) = (u, v) \) is known (see the left of Figure 6). From \( \pi_N(x, y, z) = \begin{pmatrix} x \\ \frac{y}{1-z} \end{pmatrix} \) \( (\forall (x, y, z) \in U_N) \) we know \( \pi_S(U_N) \in R^2 \)

\[
\pi_S(x, y, z) = \begin{pmatrix} x \\ \frac{y}{1+z} \end{pmatrix} \quad (\forall (x, y, z) \in U_S) \quad \text{as} \quad \pi_N^{-1}(u, v) = \begin{pmatrix} 2u \\ 2v \\ -1+u^2+v^2 \end{pmatrix} \quad (u, v) \in R^2 \quad (5)
\]

\[
\pi_S \left( \frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right) = \begin{pmatrix} u \\ v \end{pmatrix} \quad (\forall (u, v) \in R^2) \quad (6)
\]

Thus \( F_N(u, v) = F_S \circ \pi_S \circ \pi_N^{-1}(u, v) = F_S \left( \frac{u}{1+u^2+v^2}, \frac{v}{1+u^2+v^2} \right) = \begin{pmatrix} u \\ v \end{pmatrix} \quad (\forall (u, v) \in R^2) \quad (7)
\]

This gives the third vector field (taking values in \( R^2 \)) on \( S^2 \).
The Euler characteristic is a topological invariant, i.e. a number that describes a topological space's shape or structure regardless of the way it is bent. It is commonly denoted by $\chi$ (a Greek letter). The Euler characteristic was originally defined for polyhedral land used to prove various theorems about them (including the classification of the Platonic solids). Leonhard Euler, for whom the concept is named, was responsible for much of this early work. In modern mathematics, the Euler characteristic arises from homology and its connections with many other invariants. The Euler characteristic of a closed orientable surface can be calculated from its genus $g$ (the number of tori in a connected sum decomposition of the surface; intuitively, the number of "handles") as $\chi = 2 - 2g$.

Example 2 The Euler characteristic of a 2-sphere, torus and double torus is 2, 0 and -2 respectively. To understand this paper, the readers only need to memorize the fact that the Euler characteristic of a 2-sphere is 2.

Example 3 Intuitively speaking, hairs on the head of a person can be regarded as a vector field on a sphere. The Poincare theorem says that everybody has at least one swirling hair, but nobody has more than two swirling hairs.

Theorem 2 Any continuous vector field on a sphere has a positive index singularity (i.e. the index of the vector field is positive integer).

Proof Suppose Theorem 2 is not true, then the sum of the indices of a vector field is zero or negative by the Poincare-Hopf index theorem. This is a contradiction because the Euler characteristic of the 2-sphere is positive.

4. A Mathematical Description of Earth-Quake

According to Harry Fielding Reid’s elastic rebound theory [6], we can define a vector field $V$ on the surface of the earth as a function which takes a value in a point as the sum stress (force) of all tectonic forces acting on this point. Because the surface of the earth is topologically homeomorphic to a sphere, we can regard $V$ as a vector field on a sphere. By Theorem 2, we see that the vector field $V$ has a singularity. Moreover, the sphere is locally homeomorphic to Euclidean plan. Consider the open neighbourhood of the singularity which is homeomorphic to an open subset of Euclidean plan. Because we are interested only in the local property of the singularity, without loss of generality, we suppose that the vector field is locally defined on an open subset $U$ of Euclidean plan. Since the process of the earthquake is variation with time $t$, it is naturally to study a family of vector fields with parameter $t$. Let $V: U \times [0, \infty) \times \mathbb{R}^2$ be the vector fields with parameter $t \in I = [0, \infty)$, $V_x = p_x \circ V$ and $V_y = p_y \circ V$ (where $p_x$ and $p_y$ are the projections). We assume that $V$ is a continuous map (i.e. the vector field is variation continuously with the time $t \in I$), thus for a fixed time $t_0$ and a point $p_0 = (x_0, y_0)$ on $U$ the value $V(p_0, t_0)$ represent the force at point $p_0$ in time $t_0$.

Definition 5 A curve on a manifold $M$ is a continue mapping $[0,1] \rightarrow M$. It can be represented by a vector value function locally: $\phi_t \circ \gamma: [t_1, t_2] \rightarrow \mathbb{R}^2$ compose with a coordinate chart $(U_1, \phi_t)$ when $\gamma([t_1, t_2]) \subseteq U_1$. Let $p = \gamma(t)$ be a point on $M$ for some $t \in [t_1, t_2]$. A tangent vector at $p$ of $y$ is a vector given by

$$
\frac{d(\phi_t \circ \gamma)}{dt} \bigg|_{t=t_0} = \left( \frac{d(x(t))}{dt} \bigg|_{t=t_0}, \frac{d(y(t))}{dt} \bigg|_{t=t_0} \right)
$$

where $x(t), y(t)$ are the coordinates of the curve $y$ in the chart $(U_1, \phi_t)$.
Consider a vector field $V$ on an open set $U \subseteq \mathbb{R}^2$ and a point $p = (x_0, y_0)$ in $U$. A curve $\gamma$ through $p$ and whose tangent vector at any $q \in \gamma$ is equal to $V(q)$ is called a integral curve in physics. If the vector field $V$ is a stress force field, then an integral curve of this vector field is called a energy flow.

Hence we have a 2-dimensional autonomous system of differential equations:

$$\frac{dx(t)}{dt} = V_1(x, y, \lambda), \quad \frac{dy(t)}{dt} = V_2(x, y, \lambda) \quad \text{with} \quad \frac{dx(t)}{dt} \bigg|_{t=t_0} = V_1(p_0), \quad \frac{dy(t)}{dt} \bigg|_{t=t_0} = V_2(p_0) \quad (9)$$

where $V_1$, $V_2$ are the components of the vector field $V$, and $\lambda(t)$ is an energy function of $t$ which is independent of $x$ and $y$. $(x_0, y_0) \in U$ is a given point. A point $(x, y) \in U$ is called a singularity point or an equilibrium point if it satisfies $V_1(x, y, \lambda(t)) = 0$ and $V_2(x, y, \lambda(t)) = 0$.

The right-hand side of the above equation is a family of vector fields with parameter $\lambda$. The equilibrium points are important for the analysis on the earthquake, and there are several cases that we will discuss in the following.

**Example 4** A way to get the stress force field is considering the potential energy function (a binary continuous function) of the tectonic forces. The gradient of the energy function gives a vector field on the earth plane. As an example, here we give a function which has two peaks and two valleys:

$$E(x, y) = (x - \lambda_1)(x - \lambda_2) \exp(-x^2 - y^2) - (y - \lambda_3)(y - \lambda_4) \exp(-x^2 - y^2), \quad \text{where} \quad \lambda_i \quad (i = 1, 2, 3, 4)$$

is a function of time $t$, depending on the speed of the absorption and releasing energy which can trigger earthquake. The vector fields mentioned above can be given by the gradient field of the energy function (10):

$$\begin{align*}
\dot{x} &= (2x - (\lambda_1 + \lambda_2) - 4x\sqrt{x^2 + y^2})(x - \lambda_1)(x - \lambda_2) - (y - \lambda_3)(y - \lambda_4)) \exp(-x^2 - y^2), \\
\dot{y} &= (-2y + (\lambda_1 + \lambda_2) - 4y\sqrt{x^2 + y^2})(x - \lambda_1)(x - \lambda_2) - (y - \lambda_3)(y - \lambda_4)) \exp(-x^2 - y^2),
\end{align*} \quad (10)$$

Setting $\lambda_1 = -0.2, \lambda_2 = 0.4, \lambda_3 = -0.3, \lambda_4 = -0.3$, we obtain the picture of the vector field (which has two absorption points and two releasing points):

![Figure 7. vector field](image)

Since the Euler characteristic of a sphere is 2, the singularity of a sphere has positive index by the Poincare-Hopf index theorem. Because the earthquake has no fixed tangent direction and releases a great deal of energy, we assume that the singularity is unstable. However before earthquake, it can be regarded as a stable equilibrium point. Suppose that the Hopf bifurcation is $\lambda(t_0) = \lambda_0$, that is, the equation system has a stable equilibrium point when $t \leq t_0$, and it has an unstable equilibrium point when $t > t_0$. Recall that the constant matrix of the given differential equations is as follows:

$$A = \begin{pmatrix}
\frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} \\
\frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y}
\end{pmatrix} \quad \left(\frac{\partial V_x}{\partial x} \text{ is shorten of } \frac{\partial V_x(x,y,\lambda)}{\partial x} \bigg|_{x=x_0,y=y_0}\right) \quad (11)$$

the first partial derivative with respect to $x$ at $(x_0, y_0)$; the similar explanations are suitable for the other three cases). Before the earthquake or between two earth-quakes, the equilibrium is at rest and
receives energy. When the energy is enough, the earthquake occurs and the earth starts to release energy. Recall that the characteristic polynomial of the constant matrix \( A \) is \( x^2 - Tx + D \), where \( T \) is the trace of the matrix (i.e. the sum of the diagonal elements), and \( D \) is the determinant of the matrix. Without loss of generality, we assume that \( A \) has the Jordan canonical form. Write \( \Delta = T^2 - 4D \). Then we get its eigenvalues \( x = 0.5(T \pm \sqrt{\Delta}) \).

From the qualitative theory of ordinary differential equations (see [2]) we know
(a) If the real parts of the eigenvalues are all negative, then the equilibrium is asymptotic stable (and thus stable).
(b) If there is one eigenvalue whose real part is positive, then the equilibrium is unstable.
Thus we have the following

**Theorem 3** At the first (or rest) stage, the constant matrix \( A \) has the following cases:
1. It has exactly two real eigenvalues, and both are negative;
2. It has no real eigenvalue, and all the eigenvalues have negative real parts;
3. \( T \) is negative and \( D \) is positive;
The phase orbit looks like Figure 8 (first row).
At the second (release) stage, the constant matrix has the following cases:
4. It has exactly two real eigenvalues, and at least one of them is positive;
5. It has no real eigenvalue, and both eigenvalues have positive real parts;
6. Both \( T \) and \( D \) are uncertain;
The phase orbit looks like Figure 8 (second row).

**Proof** (1) and (2) come from property (a), and (5) and (6) come from property (b). (3) can be show easily if we notice the following facts: the elements on the diagonal of the Jordan canonical form are just the real part of the eigenvalues, the determinant \( D \) is the product of all elements on the diagonal and the matrix is a 2×2 matrix (Fig.8). Note that we exclude the following singularity because the index is negative 1(Fig. 9):

![Figure 8](Image)

**Figure 8.** Charts of orbits

To study the earthquake phenomenon, the properties of the close orbits around the singularity may be useful because the sphere is a compact manifold and any solution function of the equation system can be defined on the whole real axis.

Without loss of generality, let \( U \) be the open neighbourhood of the singularity such that the closure of \( U \) is compact (remember that we are now working in an Euclidean space).

For each \( p = (x_0, y_0) \in U \), let \( \Phi_p(t) \) be the (unique) solution function satisfying \( \Phi_p(0) = (x_0, y_0) \), \( \Phi: U \times (-\infty, +\infty) \to U \) be the function defined by \( \Phi(p, t) = \Phi_p(t) \) (\( \forall (p, t) \in U \times (-\infty, +\infty) \)), and \( |p| \) be the norm of the vector \( p \).

For each \( p = (x, y) \in U \), let \( f_V(p) = \Phi(p, |p|) \), then we have a function on \( U.p = (x, y) \in U \) is called a fixed point if \( f_V(p) = p \). If we define the norm of the equilibrium to be zero, then each equilibrium point is a fixed point (but the converse is not true). Furthermore, we have the following general result:

**Theorem 4** If a fixed point is not equilibrium, then the solution function passing the fixed point is a closed orbit.
**Proof** Let \( p \) be a fixed point and \( \phi_p(t) \) be the solution curve satisfying \( \phi_p(0) = p \). Since \( p \) is not an equilibrium point, \(|p|\) is not zero. But we have \( \phi_p(0) = p = \phi_p(|p|) \) and thus the solution curve is a closed orbit. The existence of a fixed point is guaranteed by the famous Brouwer fixed point theorem. For the relationships between the fixed point and the singularity, please refer to [9, Corollary 1, p.87]) which says that any closed orbit contains an equilibrium point.

5. **Conclusions and Suggestions on Earthquake Research**

To get some information about the earthquake, we need to know the vector fields at different time. Then we can study the evolution of the singular points of these vector fields according to time. The earthquake happens at the time that a stable equilibrium point became unstable. The evolution of differential equations on a manifold is another topic in mathematic. Earthquake prediction is a very difficult task. In March, 1997, R. J. Geller, D. D. Jackson, Y. Y. Kagan, and F. Mulargia wrote an article [3] in which they claimed that “earthquake can’t be predicted”. Very soon, Max Wyss in Geophysical Institute, University of Alaska, presented an opposite point of view in [8] at the same journal. In this paper, we claim that the real and deep cause of the earthquake comes from our Earth’s topological property, not only from the faults, which might make the difficult task more complicated. However our mathematical description may also provide a new method to study this complicated earthquake phenomenon. For this purpose, we need detailed crustal parameters measured which can furnish constraints for our models. We also need some new method of fractional order systems control(see [4]) in our subsequent work.

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7. **References**

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