DERIVED CATEGORIES OF BHK MIRRORS

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ABSTRACT. We prove a derived analogue to the results of Borisov, Clarke, Kelly, and Shoemaker on the birationality of Berglund-Hübsch-Krawitz mirrors. Heavily bootstrapping off work of Seidel and Sheridan, we obtain Homological Mirror Symmetry for Berglund-Hübsch-Krawitz mirror pencils to hypersurfaces in projective space.

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1. INTRODUCTION

In 1989, Candelas, Lynker, and Schimmrigk wrote a prophetic paper with computer-based evidence of a mathematical phenomenon predicted by string theorists. Their paper provides a list of Calabi-Yau hypersurfaces in weighted-projective 4-space which mostly partner off. Namely, if there is a Calabi-Yau threefold with Hodge numbers \((h^{1,1}, h^{2,1})\) on the list then there is often one with the Hodge numbers flipped: \((h^{2,1}, h^{1,1})\) [CLS90] - the so called mirror. Greene and Plesser followed with a physical construction of the mirror partners to Fermat hypersurfaces in weighted-projective spaces [GP90].

The next generalization was provided by Berglund and Hübsch [BH93]. The Berglund-Hübsch construction provides a mirror for quasismooth hypersurfaces in a weighted-projective space. One takes a polynomial

\[
F_A := \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}}
\]

associated to an invertible matrix \(A = (a_{ij})\) which defines a quasismooth hypersurface in weighted projective space \(\mathbb{P}(q_0, \ldots, q_n)\). Its mirror is roughly the hypersurface given by the...
transposed polynomial

\[ F_{AT} := \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}} \]

in another weighted projective space. More precisely, one takes additional quotients on both sides by finite groups which correspond to an exchange of the geometric and quantum symmetries of the polynomials \( F_A \) and \( F_{AT} \).

This proposal had its limitations. For example, it was unable to accommodate the latest theory seen in a paper of Candelas, de la Ossa, and Katz [CdK95]. Fortunately, a toric mirror construction due to Batyrev [Bat94] saved the day. Batyrev’s mirror construction was extended to Calabi-Yau complete intersections by Batyrev and Borisov the following year, providing a pivotal construction for future work on mirror symmetry.

In 2007, Berglund-Hübsch mirrors resurfaced in a series of articles after Fan, Jarvis, and Ruan used the Berglund-Hübsch construction to explain the self-duality of \( A_n \) and \( E_n \) singularities and study Landau-Ginzburg mirror symmetry [FJR13]. Soon afterward, Krawitz gave a well-defined version of Berglund-Hübsch mirror symmetry [Kr09] and Chiodo and Ruan [CR11] went on to prove that the Berglund-Hübsch-Krawitz (BHK) mirrors form a mirror pair on the level of Chen-Ruan orbifold cohomology [CR04] (and consequently stringy cohomology).

At this point, both Batyrev-Borisov mirrors and Berglund-Hübsch-Krawitz mirrors had evidence of being correct mirrors; however, given a Calabi-Yau hypersurface that has both a Batyrev-Borisov mirror and a BHK mirror, these mirrors may not be isomorphic. To make matters worse, varying certain choices involved in either construction can result in multiple mirrors. What to do?

As it turns out, this phenomenon is not so mysterious. In the physics literature, it is a well-studied story about different phases or energy limits of the mirror. Meanwhile in the math literature, we have a more specific ansatz: the paper of Clarke [Cla08] which unifies the constructions of Givental, Hori-Vafa, Berglund-Hübsch, and Batyrev-Borisov, together with Kontsevich’s Homological Mirror Symmetry Conjecture.

In light of Kontsevich’s Homological Mirror Symmetry Conjecture, a mirror pair of Calabi-Yau manifolds \( \mathcal{M} \) and \( \mathcal{W} \) should exchange symplectic and complex data at the level of categories. Namely, the Fukaya category of \( \mathcal{M} \) (the A-model) should be equivalent to the bounded derived category of coherent sheaves of its mirror \( \mathcal{W} \) (the B-model), i.e.,

\[ \text{Fuk}(\mathcal{M}) \cong D^b(\text{coh } \mathcal{W}) \quad \text{and} \quad \text{Fuk}(\mathcal{W}) \cong D^b(\text{coh } \mathcal{M}). \]

Consider a Calabi-Yau manifold \( \mathcal{M} \). As a consequence of the Homological Mirror Symmetry Conjecture, the derived category of its mirror should depend neither on the construction of the mirror nor on the complex structure of \( \mathcal{M} \). In summary, if we have multiple mirrors \( \mathcal{W}_1, ..., \mathcal{W}_r \) that arise from various choices of complex structure on \( \mathcal{M} \) or mirror constructions, then we expect that these mirrors have equivalent derived categories

\[ \text{Fuk}(\mathcal{M}) \cong D^b(\text{coh } \mathcal{W}_1) \cong ... \cong D^b(\text{coh } \mathcal{W}_r). \]

In this paper, we prove that this is precisely the case for Berglund-Hübsch-Krawitz mirrors in Gorenstein toric varieties. We will now provide a more precise mathematical explanation of our results.
1.1. **Precise Results.** Let us fix once and for all, $\kappa$, an algebraically closed field of characteristic 0. We work strictly over such a field.

The context of BHK mirror symmetry consists of taking a polynomial

$$F_A := \sum_{i=0}^{n} \prod_{j=0}^{n} x_i^{a_{ij}}$$

where the matrix $A := (a_{ij})$ is invertible and the polynomial $F_A$ cuts out a quasismooth Calabi-Yau hypersurface in some weighted-projective stack $\mathbb{P}(q_0, \ldots, q_n)$. Then one takes a group $G$ that is a subset of the group of diagonal automorphisms

$$\text{Aut}(F_A) = \{(\lambda_i) \in (\mathbb{G}_m)^{n+1} | F_A(\lambda_i x_i) = F_A(x_i)\}$$

so that $G$ acts trivially on holomorphic $(n, 0)$ forms of $Z(F_A)$. We take the quotient stack

$$Z_{A,G} = \left[ \frac{\{F_A = 0\}}{G \mathbb{G}_m} \right] \subseteq \left[ \frac{\mathbb{A}^{n+1} \setminus \{0\}}{G \mathbb{G}_m} \right] = \frac{\mathbb{P}(q_0, \ldots, q_n)}{G}$$

where $\bar{G}$ is the quotient of $G$ by the intersection of $G$ with the group $\mathbb{G}_m$ by which one quotients $\mathbb{A}^{n+1} \setminus \{0\}$ to obtain the weighted-projective stack $\mathbb{P}(q_0, \ldots, q_n)$. BHK mirror symmetry proposes a mirror that is associated to the transposed polynomial

$$F_{AT} := \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ji}}.$$

The polynomial $F_{AT}$ cuts out a quasismooth Calabi-Yau hypersurface in another weighted-projective stack $\mathbb{P}(r_0, \ldots, r_n)$. Krawitz [Kr09] identified the dual group $G^T$ (see Equation (2.6)) so that one can state the BHK mirror to be:

$$Z_{AT,GT} := \left[ \frac{\{F_{AT} = 0\}}{G^T \mathbb{G}_m} \right] \subseteq \left[ \frac{\mathbb{A}^{n+1} \setminus \{0\}}{G^T \mathbb{G}_m} \right] = \frac{\mathbb{P}(r_0, \ldots, r_n)}{G^T}.$$  

Chiodo and Ruan [CR11] proved that:

**Theorem 1.1** (Chiodo-Ruan). On the level of Chen-Ruan cohomology the Hodge diamonds for $Z_{A,G}$ and its BHK mirror $Z_{AT,GT}$ flip:

$$H^{p,q}_{CR}(Z_{A,G}, k) \cong H^{n-1-p,q}_{CR}(Z_{AT,GT}, k).$$

This is the analogous result to that of Batyrev and Borisov for their construction. One can ask how this construction compares to the mirror construction of Batyrev for hypersurfaces of Fano toric varieties. The answer is that the mirror construction matches if and only if the polynomial $F_A$ is a Fermat variety in a (necessarily Gorenstein) Fano toric variety. In fact, if one starts with a non-diagonal polynomial $F_A$ sitting in a (possibly Fano) toric variety, very often one gets a BHK mirror $Z_{AT,GT_A}$ that is in a non-Gorenstein (and consequently non-Fano) toric variety (see Example 2.4). Such a BHK mirror $Z_{AT,GT_A}$ does not have a mirror prescribed by Batyrev and Borisov, and consequently does not match up to the varieties prescribed to be the Batyrev mirror. This lead to the following question of Iritani:

**Question 1.2** (Iritani). Given two quotient stacks $Z_{A,G}$ and $Z_{A',G'}$ that sit in the same toric variety, are their BHK mirrors $Z_{AT,GT_A}$ and $Z_{(A')T,GT'_{A'}}$ birationally equivalent?
This question is answered affirmatively in many ways in the literature by Borisov [Bor13], Shoemaker [Sho14], Kelly [Kel13], and Clarke [Cla13]. In this paper, we prove that these mirrors are the same from the perspective of homological mirror symmetry.

**Theorem 1.3** (=Corollary 5.13). *Given two quotient stacks* $Z_{A,G}$ and $Z_{A',G}$ *that sit in the same Gorenstein toric variety, their BHK mirrors* $Z_{A^\vee,G^\vee}$ and $Z_{(A')^\vee,G'_{(A')^\vee}}$ *are derived equivalent.*

By joining this theorem with the main theorem of [FK14], we can say the following: given a Calabi-Yau complete intersection or hypersurface in a Gorenstein toric variety, there may be various distinct ways to construct its mirror using Berglund-Hübsch-Krawitz or Batyrev-Borisov mirrors, but all of these mirrors are derived equivalent.

Moreover, when proving Theorem 1.3 one gets derived equivalences amongst families of hypersurfaces in the different weighted-projective stacks. A priori, Berglund and Hübsch proposed their mirror duality to specific Calabi-Yau hypersurfaces, but we can explicitly match families of Calabi-Yau varieties to one another pointwise under derived equivalence via variations of GIT.

The most basic extension to families allows one to apply Polishchuk-Zaslow, Seidel, and Sheridan’s proof of Homological Mirror Symmetry for Calabi-Yau hypersurfaces in projective space [PZ98, Sei03, She14]. Since the Polishchuk-Zaslow result (dimension 1) is analogous but slightly different to state, we treat the cases of Seidel (dimension 2) and Sheridan (dimension $\geq 3$) which one can do simultaneously.

Namely, let $\Lambda$ be the universal Novikov field which contains $\mathbb{C}[r] \subseteq \Lambda$ so that $r$ is a formal parameter. Over the universal Novikov field, we define a *Berglund-Hübsch-Krawitz pencil* as

\[
Z_{A,G}^{\text{pencil}} := \left\{ x_0 \ldots x_n + rF_A = 0 \right\} \sslash \mathbb{G}_m \subseteq \left\{ x_0 \ldots x_n \right\} \sslash \mathbb{G}_m = \mathbb{P}(q_0, \ldots, q_n).
\]

For Berglund-Hübsch-Krawitz pencils we have the following.

**Theorem 1.4** (=Theorem 5.15). *Homological Mirror Symmetry holds for Berglund-Hübsch-Krawitz mirror pencils in projective space over the universal Novikov field.*

More precisely, if $F_A$ defines a smooth hypersurface in complex projective space $\mathbb{C}P^n$ (in particular $G = \mathbb{Z}_{n+1}$) with $n \geq 3$, there is an equivalence of triangulated categories,

\[
\text{Fuk } Z_{A,G} \cong D^b(\text{coh } Z_{A^\vee,G^\vee}^{\text{pencil}}).
\]

**1.2. Plan of the Paper.** Here is a brief summary of how the paper is organized.

In Section 2 we outline BHK mirror symmetry, give a toric reinterpretation due to Borisov and Shoemaker, and define the multiple mirrors that we will prove are derived equivalent.

In Section 3 we provide background on the category of singularities and in particular the theorems of Orlov, Isik, and Shipman which we will use.

In Section 4 we prove criteria for derived equivalences for complete intersections that are zero loci of sections of different vector bundles. This is placed in the context of equivalences of categories of singularities amongst various partial compactifications of vector bundles, and we show how the latter follows from some recent results on variations of GIT quotients.

In Section 5 we apply our framework to prove the derived analogue to the birationality result of Borisov, Clarke, Shoemaker, and the second-named author on BHK mirrors. We then discuss this in an explicit example.
Acknowledgments: We heartily thank Colin Diemer for suggesting that VGIT may relate to the BHK picture and give special thanks to Charles Doran for input on this project from start to finish. This project was also greatly aided by stimulating conversations and suggestions from many great mathematicians; Matthew Ballard, Ionut Ciocan-Fontanine, Ron Donagi, Daniel Halpern-Leistner, and Xenia de la Ossa. Furthermore, the first-named author is grateful to the Korean Institute for Advanced Study for their hospitality while this document was being prepared and especially to Bumsig Kim for insightful conversations on Clarke’s mirror construction. The second-named author thanks the Pacific Institute for the Mathematical Sciences for its hospitality in his visits as they expedited the progress of this work.

The first-named author is grateful to the Natural Sciences and Engineering Research Council of Canada for support provided by a Canada Research Chair and Discovery Grant. The second-named author acknowledges that this paper is based upon work supported by the National Science Foundation under Award No. DMS-1401446 and the Engineering and Physical Sciences Research Council under Grant EP/N004922/1.

2. Background

2.1. Berglund-Hübsch-Krawitz Mirror Symmetry. Let

\[ F_A = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}}, \quad a_{ij} \geq 0 \]

be a polynomial equation that is the sum of \( n + 1 \) monomials in \( n + 1 \) variables and set the matrix \( A := (a_{ij})_{i,j=0}^{n} \). We impose the following conditions:

Definition 2.1. The polynomial \( F_A \) above is a Kreuzer-Skarke polynomial if:

a) the matrix \( A \) is invertible;

b) there exists positive integers \( q_i \) so that the sum \( \sum_i q_i a_{ij} \) is constant for all \( i \); and

c) when viewed as a polynomial map, \( F_A : \mathbb{A}^{n+1} \to \mathbb{A} \) has exactly one critical point, namely at the origin.

Remark 2.2. These conditions are restrictive. Their classification is discussed in Section 5.1.

We then can look at the well-defined hypersurface in a weighted projective stack that is cut out by the polynomial \( F_A \),

\[ Z_A := \{ F_A = 0 \} \subseteq \mathbb{P}(q_0, \ldots, q_n) \]

Condition (b) implies that the hypersurface is well-defined in this weighted projective space and condition (c) implies that the hypersurface is quasismooth. We further impose the condition that \( Z_A \) is Calabi-Yau. This is equivalent to the condition that the degree of the polynomial \( F_A \) is the sum of the weights \( \sum_i q_i \). This is equivalent to the condition that the sum of the entries in the inverse matrix \( A^{-1} \) is one, i.e., \( \sum_{i,j} (A^{-1})_{ij} = 1 \). If we want that the hypersurface \( Z_A \) is just Fano Calabi-Yau, we merely desire that the sum of the entries of the inverse matrix \( A^{-1} \) sums to an integer.

These hypersurfaces are highly symmetric. If we take the the torus \((\mathbb{G}_m)^{n+1}\) acting coordinatewise on \( \mathbb{P}(q_0, \ldots, q_n) \), we can describe many subgroups of the torus that represent certain
symmetries of the polynomial $F_A$ and the hypersurface $Z_A$. Consider the group $\text{Aut}_{\text{diag}}(F_A)$ of diagonal symmetries rescaling the coordinates and preserving $F_A$:

$$\text{Aut}_{\text{diag}}(F_A) = \{ (\lambda_i) \in (\mathbb{G}_m)^{n+1} | F_A(\lambda_i x_i) = F_A(x_i) \} \quad (2.1)$$

This group is generated by the elements $\rho_j = (\exp(2\pi i a^{j0}), \ldots, \exp(2\pi i a^{jn}))$.

In the case where $Z_A$ is a Calabi-Yau variety, not all the elements in the group of diagonal symmetries leave the unique (up to scaling) holomorphic form invariant, hence we define a subgroup

$$\text{SL}(F_A) = \left\{ (\lambda_i) \in \text{Aut}_{\text{diag}}(F_A) \mid \prod_i \lambda_i = 1 \right\} \quad (2.2)$$

of elements that, when viewed a diagonal matrix acting on the coordinates $x_i$ has determinant one.

Some of these symmetries of $F_A$ act trivially on the hypersurface $Z_A$. In particular, one has the exponential grading operator subgroup

$$J_{F_A} = \langle \rho_0, \ldots, \rho_n \rangle \subseteq \text{Aut}_{\text{diag}}(F_A)$$

which acts trivially on the hypersurface $Z_A$. Take a group $G$ so that

$$J_{F_A} \subseteq G \subseteq \text{SL}(F_A) \quad (2.3)$$

and denote by $G/\overline{G}$ the quotient $G/J_{F_A}$. If we start with a Calabi-Yau hypersurface $Z_A$, when we quotient by $G/\overline{G}$ we get a Calabi-Yau orbifold $Z_{A,G} := [Z_A/\overline{G}]$. Alternatively, we may view this as a (smooth) Deligne-Mumford global quotient stack

$$Z_{A,G} = \left[ \frac{\{F_A = 0\}}{G \mathbb{G}_m} \right] \subseteq \left[ \frac{\mathbb{A}^{n+1} \setminus \{0\}}{G \mathbb{G}_m} \right] = \frac{\mathbb{P}(q_0, \ldots, q_n)}{G} \quad (2.4)$$

Berglund-Hübsch-Krawitz mirror symmetry provides a mirror for this orbifold in the following way. We define the transposed polynomial

$$F_{A^T} = \sum_{i=0}^n \prod_{j=0}^n x_j^{a_{ji}} \quad (2.5)$$

and the transposed group

$$G_{A^T} = \left\{ \prod_j (\rho_j^{T})^{x_j} \mid \prod_j x_j^{s_j} \text{ is } G \text{-invariant} \right\} \quad (2.6)$$

where $\rho_j^{T} := ((\exp(2\pi i a^{0j}), \ldots, \exp(2\pi i a^{nj})))$. Provided $F_A$ and $G$ above, we enjoy the following properties about their transposed counterparts:

i. $F_{A^T}$ is a Kreuzer-Skarke polynomial, but with possibly different weights $r_i$.
ii. If $J_{F_A} \subseteq G$, then $G_{A^T} \subseteq \text{SL}(F_{A^T})$.
iii. If $G \subseteq \text{SL}(F_A)$, then $J_{F_{A^T}} \subseteq G_{A^T}$.
iv. The hypersurface $Z_{A^T} := \{ F_{A^T} = 0 \} \subseteq \mathbb{P}(r_0, \ldots, r_n)$ is (Fano) Calabi-Yau if $Z_A$ is (Fano) Calabi-Yau.

Denote by $G_{A^T}/\overline{G}_{A^T}$ the quotient $G_{A^T}/J_{F_{A^T}}$. If we start with a Calabi-Yau hypersurface $Z_A$ and a group $G$ so that $J_{F_{A^T}} \subseteq G \subseteq \text{SL}(F_A)$, we obtain the quotient stack

$$Z_{A^T,G^T} = \left[ \frac{\{F_{A^T} = 0\}}{G_{A^T} \mathbb{G}_m} \right] \subseteq \left[ \frac{\mathbb{A}^{n+1} \setminus \{0\}}{G_{A^T} \mathbb{G}_m} \right] = \frac{\mathbb{P}(r_0, \ldots, r_n)}{G_{A^T}} \quad (2.7)$$
that is also a Calabi-Yau orbifold where
\[ \mathbb{P}(r_0, \ldots, r_n) := [\mathbb{A}^{n+1}\setminus 0 / G_m] \]
is a weighted projective stack.

**Example 2.3.** If one takes \( A \) to be the \( 5 \times 5 \) diagonal matrix, \( A = 5I_5 \), then one gets the Fermat polynomial \( F_{5I_5} = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 \) which carves out the Fermat hypersurface \( X_{5I_5} \subseteq \mathbb{P}^4 \). Take the group \( G \) to be the exponential grading operator \( J_{F_{5I_5}} \) so that we are looking at the Fermat quintic threefold \( Z_{A,G} = X_{5I_5} \). BHK mirror symmetry predicts the mirror \( Z_{(A,G)} = X_{5I_5}/(\mathbb{Z}_5)^3 \subseteq \mathbb{P}^4 / (\mathbb{Z}_5)^3 \) where the \((\mathbb{Z}_5)^3 \) acts coordinatewise by the generators \((\zeta, \zeta^{-1}, 1, 1, 1)\), \((\zeta, 1, \zeta^{-1}, 1, 1)\), and \((\zeta, 1, 1, \zeta^{-1}, 1)\) where \( \zeta \) is a primitive fifth root of unity. This is the same mirror hypersurface that is predicted by Greene-Plesser and Batyrev.

**Example 2.4.** Suppose one takes \( A' \) to be the matrix of exponents for the polynomial \( F_A = x_0^5x_1 + x^4_1x_2 + x_3^4x_4 + x_3^4x_4 + x_4^5 \), which carves out a quintic hypersurface \( Z_{A'} \subseteq \mathbb{P}^4 \). As before, take the group \( G \) to be the exponential grading operator. BHK mirror symmetry predicts the mirror \( Z_{(A,G),G} = Z(\mathbb{A}^5_{(\mathbb{F}_5)^4}) = Z(y_0^4 + y_0y_1^4 + y_1y_2 + y_2y_3 + y_3y_4) \subseteq \mathbb{P}^4(20, 15, 13, 20, 12) \). The hypersurface \( Z_{(A,G)} \) is not predicted by Greene-Plesser or Batyrev, rather, it does not sit in a Fano (or Gorenstein) toric variety. Hypersurfaces in non-Gorenstein toric varieties do not have mirror constructions due to any of the naturally toric mirror constructions created. For more examples of BHK mirrors to projective hypersurfaces, consult Tables 5.1-3 of [DG11].

The mirrors in Examples 2.3 and 2.4 do not have an obvious relation. Although Batyrev mirror symmetry would predict the same family of mirrors for two hypersurfaces \( Z_{A,G} \) and \( Z_{A',G} \) that sit in the same toric variety, the BHK mirror construction does not give the same prediction. This question of if \( Z_{(A,G)} \) and \( Z_{(A',G)} \) are birational has been well-studied recently by many approaches. The theorem below states a relevant amalgamation of these results (which is not described in full generality):

**Theorem 2.5 ([Bor13, Sho14, Kel13, Cla13]).** Take two polynomials \( F_A \) and \( F_{A'} \) as above so that the Calabi-Yau hypersurfaces \( Z_A \) and \( Z_{A'} \) are hypersurfaces in the same weighted projective space \( \mathbb{P}(q_0, \ldots, q_n) / G \) where \( J_{F_A} = J_{F_{A'}} \subseteq G \subseteq SL(F_A) \cap SL(F_{A'}) \). One then has two CY orbifolds \( Z_{A,G} \) and \( Z_{A',G} \) as hypersurfaces in the orbifold \( \mathbb{P}(q_0, \ldots, q_n) / (G / J_{F_A}) \). The BHK mirrors \( Z_{(A,G)} \) and \( Z_{(A',G)} \) are birational.

In the following sections, we mesh the many approaches to this question with variational geometric invariant theory in order to prove a result more in line with Kontsevich’s homological mirror symmetry—derived equivalence.

2.2. Toric reinterpretation of BHK mirrors. There have been a few toric reinterpretations of BHK mirror duality in the literature ([Bor13, Cla08, Sho14]). In this subsection, we will give a brief overview of the framework that we will use and introduce the relevant notation for the BHK mirror construction both in a Landau-Ginzburg and a Calabi-Yau setting.

We start with the setup of [Bor13]. Take two free abelian groups \( M_0 \) and \( N_0 \) with bases \( \{u_i\} \) and \( \{v_i\} \), respectively. Take the matrix \( A \) to be the defined as \( A := (a_{ij})_{i,j} \), where \( a_{ij} := \langle u_i, v_j \rangle \). We want to choose overlattices \( M \) and \( N \) so that \( M \) and \( N \) are dual to one
another and we have the following containments:

\[ N_0 \subseteq N \subseteq M; \quad M_0 \subseteq M \subseteq N_0^\vee. \]

We then have exact sequences

\[ 0 \rightarrow M \rightarrow N_0^\vee \rightarrow N_0^\vee / M \rightarrow 0; \]

\[ m \mapsto (m, -), \]  

\[ 0 \rightarrow N \rightarrow M_0^\vee \rightarrow M_0^\vee / N \rightarrow 0; \]

\[ n \mapsto ( -, n). \]  

The first map is the toric divisor map div for the toric variety \((k \otimes N_0)/(N_0^\vee / M)\) with ray generators \(v_i\), as it can be written

\[ m \mapsto \sum_i (m, v_i) v_i^\vee. \]  

The second map is the monomial map mon for the rational function \(\sum_i x^{u_i}\) as it can be written

\[ n \mapsto \sum_i (u_i, n) u_i^\vee. \]  

This gives us a pair consisting of a space and a function

\[ \left( (k \otimes N_0)/(N_0^\vee / M); \sum x_i^{u_i} \right), \]  

often referred to as a Landau-Ginzburg (LG) model.

Following Clarke [Cla08], the mirror LG model is given by swapping \(M\) and \(N\) and the maps mon and div. Hence, in this setting, the mirror is the pair

\[ (k \otimes M_0)/(M_0^\vee / N); \sum x_i^{u_i}. \]  

Notice that we have a \(\mathbb{Z}\)-basis for \(N_0\), namely \(\{v_i\}\), so we have natural functions on the semi-ring \(\kappa[N_0]\) given by the \(v_i^\vee\). We denote these functions by \(x_i\). In this basis, we write the monomial \(x_i^{u_i}\) as

\[ x_i^{u_i} = \prod_j x_j^{(u_i, v_j)} = \prod_j x_j^{a_{ij}}, \]  

hence

\[ \sum_i x_i^{u_i} = \sum_i \prod_j x_j^{a_{ij}} = F_A. \]  

Analogously, we take the natural functions on the semi-ring \(\kappa[M_0]\) given by the dual elements \(u_i^\vee\). We denote these functions by \(y_i\). In this basis, we write the monomial \(x_i^{v_i}\) as

\[ x_i^{v_i} = \prod_i y_i^{(u_i, v_j)} = \prod_i x_i^{a_{ij}}, \]  

hence

\[ \sum_j x_j^{v_j} = \sum_j \prod_i x_i^{a_{ij}} = F_A r. \]  

We have now checked that the polynomials in this toric interpretation match to the original construction. We also have the groups match:
**Proposition 2.6** (Proposition 2.3.1 of [Bor13]). The groups $N_0^\vee / M$ and $M_0^\vee / N$ are naturally isomorphic to the groups $G$ and $G^T_A$, respectively.

Note that we do not necessarily have yet that the polynomials $F_A$ and $F_{AT}$ are quasihomogeneous for positive weights. In order to have this, we take the elements $\deg \in N_0^\vee$ and $\deg^\vee \in M_0^\vee$ so that $\langle \deg, v_i \rangle = \langle u_i, \deg^\vee \rangle = 1$ for all $i$.

In order to have quasihomogeneity, we require $\deg$ and $\deg^\vee$ to be in the lattices $M$ and $N$ respectively. Note that, given a general choice of $\{u_i\}$ and $\{v_j\}$ as above, we do not necessarily have overlattices $M$ and $N$ so that these elements sit inside them.

**Proposition 2.7** (Proposition 2.3.4 of [Bor13]). There exists such dual lattices $M$ and $N$ if and only if $\sum_{i,j} (A^{-1})_{i,j} \in \mathbb{Z}_+$. If we have that the sum $\sum_{i,j} (A^{-1})_{i,j}$ is exactly one, then the way to produce a Calabi-Yau hypersurface is straightforward. Take the cones $C_M = \text{Cone}(u_i)$ and $C_N = \text{Cone}(v_j)$ and produce fans $\Sigma_M$ and $\Sigma_N$ by taking the collection of cones that are the proper faces of the cones $C_M$ and $C_N$. We star subdivide each fan by the ray generated by $\deg$ and $\deg^\vee$, respectively. We then have two new fans, call them $\Sigma_{M,\deg}$ and $\Sigma_{N,\deg^\vee}$. These fans correspond to toric varieties that are canonical bundles over quotients of weighted projective spaces where the polynomials $\sum_i x^{u_i}$ and $\sum_j x^{v_j}$ are zero-sections of the dual bundles. By taking the zero loci of these polynomials, we obtain the Calabi-Yau orbifolds:

$$Z_{A,G} \subseteq [\mathbb{P}(q_0, \ldots, q_n)/\bar{G}]; \quad Z_{AT,G^T_A} \subseteq [\mathbb{P}(r_0, \ldots, r_n)/\bar{G}^T_A].$$

To obtain this correspondence see Section 2 of [Sho14]. Namely, in the notation of loc. cit. Section 2, the fans $\Sigma$ and $\Sigma^\vee$ correspond to the projections of $\Sigma_{M,\deg}$ and $\Sigma_{N,\deg^\vee}$ under the maps $\pi_M : M \to M/((\deg))$ and $\pi_N : N \to N/((\deg^\vee))$ respectively.

As we view both Calabi-Yau orbifolds $Z_{A,G}$ and $Z_{AT,G^T_A}$ as smooth Deligne-Mumford stacks, we must treat the corresponding toric varieties as toric stacks. For a treatment of toric stacks that will be relevant to the proof of the derived equivalence of BHK mirrors presented here, we direct the reader to Section 5 of [FK14].

### 3. Categories of Singularities

In this section, we provide the necessary details on categories of singularities for global quotient stacks. We start by reminding the reader of the framework set up in Section 3 of [FK14], and then continue with an additional observation from Orlov’s original discussion of such categories [Orl04], which we require later.

Let $X$ be a variety and $G$ be an algebraic group acting on $X$.

**Definition 3.1.** An object of $D^b(\text{coh}[X/G])$ is called **perfect** if it is locally quasi-isomorphic to a bounded complex of vector bundles. We denote the full subcategory of perfect objects by $\text{Perf}([X/G])$. The Verdier quotient of $D^b(\text{coh}[X/G])$ by $\text{Perf}([X/G])$ is called the category of **singularities** and denoted by

$$D_{sg}([X/G]) := D^b(\text{coh}[X/G])/\text{Perf}([X/G]).$$

We now repeat Orlov’s observation that the category of singularities localizes about the singular locus (Proposition 1.14 in [Orl04]) in the presence of a group action.
Proposition 3.2 (Orlov). Assume that $\text{coh}[X/G]$ has enough locally-free sheaves. Let $i : U \to X$ be a $G$-equivariant open immersion such that the singular locus of $X$ is contained in $i(U)$. Then the restriction,
\[ i^* : D_{\text{sg}}([X/G]) \to D_{\text{sg}}([U/G]), \]
is an equivalence of categories.

Proof. The proof of Proposition 1.14 in [Orl04] works verbatim for equivariant sheaves. □

Our goal later on, will be to convert a problem on hypersurfaces in weighted projective space to a toric calculation. This is done using a theorem of Isik and Shipman which also uses to pass from studying a hypersurface to the (toric) total space of the line bundle defining it.

The setup is general and does not involve toric varieties. Namely, consider a variety $X$ with the action of an algebraic group $G$ and a vector bundle $E$ on $X$.

Take the section $s \in H^0(X, E)$ and consider the zero locus $Z$ of $s$ in $X$. The pairing with $s$ induces a global function on the total space of $E^\vee$. Let $Y$ be the zero locus of the pairing with $s$ and consider the fiberwise dilation action of $\mathbb{G}_m$ on $Y$.

Theorem 3.3 (Isik, Shipman, Hirano). Suppose the Koszul complex on $s$ is exact. Then there is an equivalence of categories
\[ D_{\text{sg}}([Y/(G \times \mathbb{G}_m)]) \cong D^b(\text{coh}[Z/G]). \]

Proof. The theorem is originally due to independently to Isik [Isi13] and Shipman [Shi12]. With the $G$-action, it is a special of Theorem 1.2 of [Hir16]. □

Corollary 3.4. Let $X$ be an algebraic variety with a $G \times \mathbb{G}_m$ action. Suppose there is an open subset $U \subseteq X$ such that $U$ is $G \times \mathbb{G}_m$ equivariantly isomorphic to $Y$ as above and that $U$ contains the singular locus of $X$. Then
\[ D_{\text{sg}}([X/(G \times \mathbb{G}_m)]) \cong D^b(\text{coh}[Z/G]). \]

Proof. We have
\[
D_{\text{sg}}([X/(G \times \mathbb{G}_m)]) \cong D_{\text{sg}}([U/(G \times \mathbb{G}_m)]) \\
\cong D^b(\text{coh}[Z/G])
\]
where the first line is Theorem 3.3 and the second line is Proposition 3.2. □

4. Torus Actions on Affine Space

In this section, we extend the setup of Section 4 of [FK14] to partial compactifications of vector bundles. Consider an affine space $X := \mathbb{A}^{n+t}$ with coordinates $x_i, u_j$ for $1 \leq i \leq n, 1 \leq j \leq t$. Let $T = \mathbb{G}_m^{n+t}$ be the open dense torus with the standard embedding and action on $X$. Take $S \subseteq T$ to be a subgroup and $\tilde{S}$ be the connected component of the identity.

The possible GIT quotients for the action of $\tilde{S}$ on $X$ [MFK94] have both an algebraic and toric description. The description in terms of GIT variations comes from varying linearization on trivial bundle (which is ample as $X$ is affine). The choice of linearization on the trivial bundle is the same thing as a choice of a a character of $\tilde{S}$. That is, given an
element $\chi \in \text{Hom}(\tilde{S}, \mathbb{G}_m)$, we can form the corresponding line bundle $\mathcal{O}_X$ by pulling back the representation of $\tilde{S}$ via the morphism of stacks $[X/\tilde{S}] \to [\text{pt}/\tilde{S}]$.

In studying GIT variations, it is often convenient to consider $\chi$ as an element of the vector space $\text{Hom}(\tilde{S}, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$ by rationalizing denominators in order to get an equivariant line bundle. Now, each linearization in Mumford’s GIT, or in our case, each choice of $\chi$, determines an open subset $U_\chi$ corresponding to the semi-stable locus of $X$ with respect to $\chi$.

Furthermore, if we think of the vector space $\text{Hom}(\tilde{S}, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$ as a parameter space for linearization, then it was shown in [GKZ94] that this parameter space has a natural fan-structure $\Sigma_{\text{GKZ}}$ called the GKZ-fan. The fan is defined by the following property, each $U_\chi$ is constant on the interior of each cone in the fan.

The maximal cones of this fan are called chambers and the codimension 1 cones are called walls. There are finitely many chambers $\sigma_1, \ldots, \sigma_r$ in the fan $\Sigma_{\text{GKZ}}$ which are in bijection with regular triangulations of the set $\{\nu_1(S), \ldots, \nu_{n+t}(S)\}$, described as follows:

Apply $\text{Hom}(-, \mathbb{G}_m)$ to the exact sequence

$$0 \longrightarrow S \xrightarrow{i_S} \mathbb{G}_m^{n+t} \xrightarrow{p} \text{Coker}(i_S) \longrightarrow 0$$

to get

$$\text{Hom}(\text{Coker}(i_S), \mathbb{G}_m) \xrightarrow{\hat{p}} \mathbb{Z}^{n+t} \xrightarrow{\hat{i}_S} \text{Hom}(S, \mathbb{G}_m) \longrightarrow 0.$$ 

Set $\nu_i(S)$ to be the element of $\text{Hom}(\text{Coker}(i_S), \mathbb{G}_m)^\vee$ given by the composition of $\hat{i}_S$ with the projection of $\mathbb{Z}^n$ onto its $i$th factor. Then, we define $\nu(S)$ as the following vector

$$\nu(S) := (\nu_1(S), \ldots, \nu_{n+t}(S)).$$

For any character $\chi_p$ in the interior of $\sigma_p$, we can consider the semi-stable points with respect to that character. This yields an open subset in $X$ which we denote by $U_p$. It also corresponds to a regular triangulation $T_p$ of the collection of points $\{\nu_1(S), \ldots, \nu_{n+t}(S)\}$.

**Definition 4.1.** Let $\times : \mathbb{G}_m^{n+t} \to \mathbb{G}_m$ be the multiplication map. We say that $S$ satisfies the quasi-Calabi-Yau condition if $\times|_{\tilde{S}} = 1$, i.e., the multiplication map restricted to $\tilde{S}$ is the trivial homomorphism.

**Definition 4.2.** Let $G$ be a group acting on a space $X$ and let $f$ be a global function on $X$. We say that $f$ is semi-invariant with respect to a character $\chi$ if, for any $g \in G$,

$$f(g \cdot x) = \chi(g)f(x).$$

Equivalently, this means that $f$ is a section of the equivariant line bundle $\mathcal{O}(\chi)$ on the global quotient stack $[X/G]$.

**Remark 4.3.** Each variable $x_i$ is semi-invariant with respect to a unique character of $S$ which we can denote by $\text{deg}(x_i)$. The quasi-Calabi-Yau condition is equivalent to

$$\sum \text{deg}(x_i) + \sum \text{deg}(u_j)$$

being torsion.

To apply Corollary 3.4, we will add an auxiliary $\mathbb{G}_m$-action and an $S$-invariant function which is $\mathbb{G}_m$-semi-invariant. This auxiliary $\mathbb{G}_m$-action acts with weight 0 on the $x_i$ for all $i$ and with weight 1 on the $u_j$ for all $j$. We refer to this auxiliary action as $R$-charge.
The action of $S$ on $\text{Spec} \kappa[u_j]$ gives a character $\gamma_j$ of $S$. Let $f_1, ..., f_t$ be $S$-semi-invariant functions in the $x_i$ with respect to the character $\gamma_j^{-1}$. The functions $f_i$ determine a complete intersection in $\mathbb{A}^n$ as their common zero-set. We can also use them to define a function

$$w := \sum_{j=1}^t u_j f_j.$$ 

we call the superpotential.

The superpotential $w$ is $S$-invariant and $\chi$-semi-invariant for the projection character $\chi : S \times \mathbb{G}_m \to \mathbb{G}_m$. This means that it is homogeneous of degree 0 for the $S$-action and homogeneous of degree 1 with respect to the $R$-charge.

Let $Z$ denote the zero-locus of $w$ in $X$ and

$$Z_p := Z \cap U_p.$$

**Theorem 4.4** (Herbst-Walcher). If $S$ satisfies the quasi-Calabi-Yau condition, there is an equivalence of categories,

$$D_{sg}([Z_p/S \times \mathbb{G}_m]) \cong D_{sg}([Z_q/S \times \mathbb{G}_m])$$

for all $1 \leq p, q \leq r$.

*Proof.* This is essentially Theorem 3 of [HW12] stated in geometric as opposed to algebraic language. For the geometric translation see Theorem 5.2.1 of [BFK12] (version 2 on arXiv) or [H-L12] Cor 4.8 and Prop 5.5. 

We now refocus our attention to describe explicitly the open sets $U_p \subseteq X$ corresponding to the semistable loci associated to the characters $\chi$ in $\text{Hom}(\tilde{S}, \mathbb{G}_m) \otimes \mathbb{Q}$. For $1 \leq p \leq r$, we can define the irrelevant ideal $\mathcal{I}_p$ that is associated to the character $\chi_p$ in the chamber $\sigma_p$ of the secondary fan:

$$\mathcal{I}_p := \left\langle \prod_{i \in I} x_i \prod_{j \notin J} u_j \left| \bigcap_{i \in I} F_{i, \chi_p} \cap \bigcap_{j \in J} F_{j, \chi_p} \neq \emptyset \right. \right\rangle$$

where $I \subseteq \{1, ..., n\}$, $J \subseteq \{1, ..., t\}$ and $F_{\chi_p}$ are the virtual facets of the polyhedron $P_{\chi_p}$ (see Sections 14.2 and 14.4 of [CLS11]).

Alternatively, $\mathcal{I}_p$ can be defined by $\mathcal{T}_p$, the corresponding triangulation of $\nu(S)$. Namely,

$$\mathcal{I}_p = \left\langle \prod_{i \in I} x_i \prod_{j \notin J} u_j \left| \text{Conv} \left( \bigcup_{i \in I} \nu_i(S) \cup \bigcup_{j \in J} \nu_{n+j}(S) \right) \in \mathcal{T}_p \right. \right\rangle. \quad (4.2)$$

The complement $U_p$ of the irrelevant ideal is the zero set of an ideal generated by monomials, i.e.,

$$U_p = X \setminus Z(\mathcal{I}_p).$$
We also consider a certain subideal of the irrelevant ideal given by taking all generators found by fixing $J = \{1, \ldots, t\}$:

$$J_p := \left\langle \prod_{i \notin I} x_i \left| \bigcap_{i \in I} F_{i,x_p} \cap \bigcap_{j=1}^t F_{j,x_p} \neq \emptyset \right. \right\rangle$$

$$= \left\langle \prod_{i \notin I} x_i \left| \text{Conv} \left( \bigcup_{i \in I} \nu_i(S) \cup \bigcup_{j \in J} \nu_{n+j}(S) \right) \in T_p \right. \right\rangle. \quad (4.3)$$

The complement of the zero-locus of $J_p$ gives a new open set

$$V_p := X \setminus Z(J_p) \subseteq U_p.$$ 

We may also view $J_p$ as an ideal in $\kappa[x_1, \ldots, x_n]$ in which case we denote it by $J_p^x$. Now, restrict the action of $S$ to $\mathbb{A}^n = \text{Spec} \kappa[x_1, \ldots, x_n]$ (considered as a plane in $\mathbb{A}^{n+t}$). This gives an open set of $\mathbb{A}^n$.

$$V_p^x := \mathbb{A}^n \setminus Z(J_p^x)$$

and a toric Deligne-Mumford stack

$$X_p := [V_p^x/S].$$

The inclusion of rings $\kappa[x_1, \ldots, x_n] \to \kappa[x_1, \ldots, x_n, u_1, \ldots, u_t]$ restricts to a $S$-equivariant morphism

$$[V_p/S] \to [V_p^x/S] = X_p.$$ 

**Proposition 4.5.** The morphism

$$[V_p/S] \to X_p.$$ 

realizes $[V_p/S]$ as the total space of a vector bundle

$$[V_p/S] \cong \text{tot} \bigoplus_{j=1}^t \mathcal{O}(\gamma_j).$$

Furthermore, the $R$-charge action of $\mathbb{G}_m$ is the dilation action along the fibers. Finally, for each $j$, the function $f_j$ gives a section of $\mathcal{O}(\gamma_j^{-1})$ and the superpotential $w = \sum u_j f_j$ restricts to the pairing with the section $\bigoplus f_j$.

**Proof.** Notice first that the open set $V_p$ decomposes as a product

$$V_p = V_p^x \times \text{Spec} \kappa[u_1, \ldots, u_t].$$

It is then a standard fact that the stack

$$[V_p/S] = [V_p^x \times \text{Spec} \kappa[u_1, \ldots, u_t]/S]$$

can be realized as the equivariant bundle on $[V_p^x/S]$ given by the representation of $S$ on $\text{Spec} \kappa[u_1, \ldots, u_t]$.

Now, the group $S$ acts on $\text{Spec} \kappa[u_1, \ldots, u_t]$ via the characters $\gamma_j$ and the representation is nothing more than the diagonal action of these characters. Hence, we get precisely the statement:

$$[V_p/S] \cong \text{tot} \bigoplus_{j=1}^t \mathcal{O}(\gamma_j). \quad (4.4)$$
By definition, the R-charge action of $\mathbb{G}_m$ acts with weight 0 on $V^x_p$ and weight 1 on $\text{Spec } \kappa[u_1, ..., u_t]$, i.e., by scaling on the second factor. Under the isomorphism (4.4), this $\mathbb{G}_m$ just acts with weight 1 along the fibers of the vector bundle, as desired.

Finally, by definition,

$$\text{tot } \bigoplus_{j=1}^{t} \mathcal{O}(\gamma_j) = \text{Spec} \left( \text{Sym} \left( \bigoplus_{j=1}^{t} \mathcal{O}(\gamma_j^{-1}) \right) \right)$$

with global functions identified as

$$H^0 \left( \text{Sym} \left( \bigoplus_{j=1}^{t} \mathcal{O}(\gamma_j^{-1}) \right) \right) = \bigoplus_{j=1, ..., t, r \in \mathbb{Z}} u_j^{r} H^0 \left( \bigoplus_{j=1}^{t} \mathcal{O}(\gamma_j^{-r}) \right)$$

so that $w = \sum u_j f_j$ is identified with $\bigoplus f_j \in H^0(\bigoplus_{j=1}^{t} \mathcal{O}(\gamma_j^{-1})) \subseteq H^0(\text{Sym}(\bigoplus_{j=1}^{t} \mathcal{O}(\gamma_j^{-1})))$ as desired.

From Proposition 4.5, we see that for all $p$, the zero set of $\bigoplus f_j$ as a section of $V_p$ defines a complete intersection

$$Z_p := Z(\bigoplus f_j) \subseteq X_p.$$ 

We can also consider the zero locus of $w|_{U_p}$, which we denote by

$$Y_p := Z(w) \cap U_p.$$ 

Let $\partial w$ be the Jacobian ideal, i.e., the ideal generated by the partial derivatives of $w$ with respect to the $x_i$ and the $u_j$.

**Proposition 4.6.** If $\mathcal{I}_p \subseteq \sqrt{\partial w}, \mathcal{J}_p$ then

$$\text{D}_{sg}(\mathcal{Y}_p / S \times \mathbb{G}_m) \cong \text{D}^b(\text{coh } Z_p).$$

**Proof.** Since, $\mathcal{I}_p \subseteq \sqrt{\partial w}, \mathcal{J}_p$ this implies that the singular locus of $w$ is contained in $V_p$. By Proposition 4.5 we may apply Corollary 3.4 with $X = Y_p$ and $U = Y_p \cap V_p$ to obtain the result. \qed

**Corollary 4.7.** If $\mathcal{I}_p \subseteq \sqrt{\partial w}, \mathcal{J}_p$ and $\mathcal{I}_q \subseteq \sqrt{\partial w}, \mathcal{J}_q$ for some $1 \leq p, q \leq r$ then

$$\text{D}^b(\text{coh } Z_p) \cong \text{D}^b(\text{coh } Z_q).$$

**Proof.** We have

$$\text{D}^b(\text{coh } Z_p) \cong \text{D}_{sg}(\mathcal{Y}_p / S \times \mathbb{G}_m)$$

$$\cong \text{D}_{sg}(\mathcal{Y}_q / S \times \mathbb{G}_m)$$

$$\cong \text{D}^b(\text{coh } Z_q)$$

where the first line is Proposition 4.6, the second line is Theorem 4.4, and the third line is Proposition 4.6 again. \qed

**Remark 4.8.** For each $p$, the condition that $\mathcal{I}_p \subseteq \sqrt{\partial w}, \mathcal{J}_p$ is a locally closed condition on the set of $t$-tuples $f_j$ of $S$-invariant functions. Hence, given two partial compactifications of vector bundles related by GIT, there is a locally-closed family of zero-sections of each bundle which are derived equivalent.
Remark 4.9. For a single wall-crossing in the GKZ fan of a toric variety, one can look at the corresponding wall crossing in the GKZ fan of the total space of the canonical bundle. The condition that $I_p \subseteq \sqrt{\partial w}, J_p$ and $I_q \subseteq \sqrt{\partial w}, J_q$ is then equivalent to the hypersurface $w$ being nonsingular on the contracting loci. These wall-crossings were first described independently by Dolgachev and Hu, and Thaddeus [DH98, Tha96] and by Gel’fand, Kapranov, and Zelevinsky in the toric setting [GKZ94]. For an explanation of terminology see [BFK12], especially Proposition 5.1.4 where the relevant contracting loci are described.

5. Derived Equivalence of Berglund-Hübsch-Krawitz Mirrors

Suppose one has polynomials $F_A$ and $F_{A'}$ so that they are quasihomogeneous with weights $q_i$ and there is a group $G \subseteq SL(F_A) \cap SL(F_{A'})$ as in Equation 2.3. Then, one can define the Calabi-Yau orbifolds $Z_{A,G}$ and $Z_{A',G}$ as well as the BHK mirrors $Z_{A,G}^{T,G}$ and $Z_{A',G}^{T,A'}$.

In this section, we prove the following theorem:

Theorem 5.1. Let $Z_{A,G}$ and $Z_{A',G}$ be hypersurfaces in $\mathbb{P}(q_0, \ldots, q_n)/\bar{G}$, where $J_{F_A} = J_{F_{A'}} \subseteq G \subseteq SL(F_A) \cap SL(F_{A'})$. If the coarse moduli space of $\mathbb{P}(q_0, \ldots, q_n)/\bar{G}$ is Gorenstein, then the BHK mirrors $Z_{A,G}^{T,G}$ and $Z_{A',G}^{T,A'}$ are derived equivalent.

This is the derived analogue to the result on the birationality of Berglund-Hübsch-Krawitz mirrors (Theorem 2.5). Theorem 5.1 is proven by decomposing the differences between the potentials $F_A$ and $F_{A'}$ into a sequence $F_{A_i}$ such that the BHK mirrors associated to consecutive elements of the sequence are derived equivalent.

5.1. Kreuzer-Skarke Cleaves. In this subsection, we explain the sequence $F_{A_i}$ that we use to prove Theorem 5.1. This uses the classification of Kreuzer-Skarke polynomials i.e. quasihomogeneous, quasismooth potentials in $n+1$ variables with $n+1$ monomials terms:

Theorem 5.2 (Kreuzer-Skarke Classification [KS92]). Up to relabelling, all Kreuzer-Skarke polynomials can be written as a sum of the following polynomials in separate variables:

i. Fermat: $W_{\text{fermat}} := x^{a}$;

ii. Loops of length $\ell > 2$: $W_{\text{loop}} := x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{\ell-1}^{a_{\ell-1}}x_\ell + x_\ell^{a_\ell}x_1$; and

iii. Chains of length $\ell > 2$: $W_{\text{chain}} := x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{\ell-1}^{a_{\ell-1}}x_\ell + x_\ell^{a_\ell}$.

The polynomials in the list above are called atomic types. In the original Kreuzer-Skarke paper, the diagrams for such atomic types are the following:

1. Fermat:

   \[ \bullet_a \]

2. Loop:

   \[ \bullet_a \quad \bullet_{a_1} \quad \bullet_{a_2} \quad \ldots \quad \bullet_{a_{\ell-1}} \quad \bullet_{a_\ell} \]

3. Chain:

   \[ \bullet_a \quad \bullet_{a_1} \quad \bullet_{a_2} \quad \ldots \quad \bullet_{a_{\ell-1}} \quad \bullet_{a_\ell} \]

To each point in such a diagram, one can associate a monomial $x_i^{a_i}$ or $x_i^{a_i}x_j$ where $a_i$ is the weight at the vertex corresponding to $x_i$ and the factor $x_j$ depends on if there’s an arrow pointing to the vertex corresponding to the variable $x_j$. One obtains the three atomic types
of polynomials by summing over vertices. Hence, all Kreuzer-Skarke polynomials can be visualized as disjoint unions of the three types above.

**Remark 5.3.** If one takes the Kreuzer-Skarke diagram of a polynomial $F_A$, the Kreuzer-Skarke diagram of the transposed polynomial $F_{A^T}$ is the dual diagram resulting from reversing the direction of all the arrows.

**Definition 5.4.** Take Kreuzer-Skarke polynomials $F_A$ and $F_{A'}$ so that they cut out hypersurfaces in $\mathbb{P}(q_0, \ldots, q_n)/\bar{G}$. Suppose that $F_A$ and $F_{A'}$ are related by deleting or adding a single arrow and changing the weight $a_i$ at the source of the arrow. In this case we say that the pair $(A, A')$ is a Kreuzer-Skarke cleave.

**Definition 5.5.** Given an element $b \in \kappa^l$ and a diagram as above, we define a generalized Kreuzer-Skarke polynomial as a polynomial the form

$$F^b_A = \sum_{i=1}^l b_i p_i$$

where $p_i = x_i^{a_i} x_j$ or $p_i = x_i^{a_i}$ according to the prescription above associated to the diagram.

**Remark 5.6.** Given a Kreuzer-Skarke cleave $(A, A')$, we may also compare $F^b_A$, $F^b_{A'}$ for fixed $b \in \kappa^l$.

**Proposition 5.7.** Fix $b \in \kappa^l$. Take $d = \sum_i q_i$. Suppose the coarse moduli space of $\mathbb{P}(q_0, \ldots, q_n)/\bar{G}$ is Gorenstein. Any $\bar{G}$-invariant generalized Kreuzer-Skarke polynomials of (weighted) degree $d$ with weights $q_0, \ldots, q_n$ is related to the generalized Fermat polynomial

$$\sum b_i x_i^{d_i}$$

by a sequence of Kreuzer-Skarke cleaves.

**Proof.** The coarse moduli space of $\mathbb{P}(q_0, \ldots, q_n)/\bar{G}$ is Gorenstein if and only if the coarse moduli space of $\mathbb{P}(q_0, \ldots, q_n)/\bar{G}$ is Fano. Hence, the anticanonical polytope $\Delta$ is reflexive. Consequently, the support of the fan for the canonical bundle on $\mathbb{P}(q_0, \ldots, q_n)/\bar{G}$ is the cone over $\Delta^\vee$. Therefore, the vertices of the anticanonical polytope pair to 0 against a facet of $\Delta^\vee$, meaning they correspond to Fermat polynomials. That is, there exists an $a'_i$ so that $x_i^{a'_i}$ is a $\bar{G}$-invariant polynomial in $\mathbb{P}(q_0, \ldots, q_n)$. Since the monomial $x_i^{a'_i}$ is a section of the anticanonical, it is of degree $d$. This is the monomial term corresponding to the vertex corresponding to $x_i$ in the Kreuzer-Skarke diagram with no outgoing arrow. This means that if we start with any $\bar{G}$-invariant polynomial, any Kreuzer-Skarke cleave which deletes an arrow will remain $\bar{G}$-invariant. Delete all arrows in any order to get a sequence of Kreuzer-Skarke cleaves that relate $F_A$ with a Fermat polynomial. \qed

We now will prove that if $F_A$ and $F_{A'}$ are related by a Kreuzer Skarke cleave, then their BHK mirrors are derived equivalent. A technical tool in the proof will be the use of the following triangulations.

First we introduce notation. Fix an ambient space $X_{\Sigma} := \mathbb{P}(q_0, \ldots, q_n)/\bar{G}$ given by a fan $\Sigma$ which is the normal fan to a simplex $\Delta$. We have a fan $\Sigma_K \subseteq (N \oplus \mathbb{Z})_\mathbb{R}$ corresponding to the canonical bundle of toric variety $X_{\Sigma}$ and by Lemma 5.17 of [FKLM], we have $|\Sigma_K|\cap \text{Cone}(\Delta, 1)$. Therefore, anticanonical sections of $\mathbb{P}(q_0, \ldots, q_n)/G$ are given by elements of $(\Delta, 1) \cap (M \oplus \mathbb{Z})$. 
Now consider \( n + 1 \) lattice points
\[
Ξ = \{m_0, \ldots, m_n\} \subseteq (\Delta, 1) \cap M \oplus \mathbb{Z}
\]
so that the polynomial
\[
F_A := \sum_{i=1}^{n} x^{m_i}.
\]
is a Kreuzer-Skarke polynomial. Take a new lattice element \( m'_k \in (\Delta, 1) \cap (M \oplus \mathbb{Z}) \) such that the polynomial
\[
F_{A'} = x^{m'_k} + \sum_{i \neq k} x^{m_i}
\]
is also a Kreuzer-Skarke polynomial and \((A, A')\) is a Kreuzer-Skarke cleave. Define
\[
Ξ' := (Ξ \setminus \{m_k\}) \cup \{m'_k\}.
\]
We may now consider the set
\[
ν := \{(0, 1), m_0, \ldots, m_n, m'_k\} = Ξ \cup Ξ' \cup \{(0, 1)\} \subseteq (\Delta, 1) \cap M \oplus \mathbb{Z}
\]
and define two triangulations of \( ν \) as follows.

Let \( T \) be the set of simplices generated by any proper face of the convex hull of \( n \) elements of the set \( Ξ \) together with the element \((0, 1)\). We also have the collection of simplices
\[
S := \{\text{Conv}\{\xi_{\in I}\} | I \subseteq Ξ, \text{Conv}\{\xi_{\in I}, m'\} \cap \text{int}(\text{Conv}(Ξ)) = \emptyset\}
\]
We define
\[
T := \begin{cases} 
C & \text{if } m' \in \text{Conv}(Ξ) \\
C \cup S & \text{otherwise}.
\end{cases}
\]

We now define another set of simplicies analogously. That is, we define \( C' \) to be the set of simplicies generated by less than \( n \) elements of the set \( Ξ' \) together with the element \((0, 1)\) and
\[
S' := \{\text{Conv}\{\xi_{\in I}, m\} | I \subseteq Ξ', \text{Conv}\{\xi_{\in I}, m\} \cap \text{int}(\text{Conv}(Ξ')) = \emptyset\}.
\]
We define
\[
T' := \begin{cases} 
C' & \text{if } m \in \text{Conv}(Ξ') \\
C' \cup S' & \text{otherwise}.
\end{cases}
\]

**Lemma 5.8.** Given a Kreuzer-Skarke cleave \((A, A')\) associated to anticanonical sections as above, the corresponding sets of simplicies \( T, T' \) are regular triangulations of \( ν \).

**Proof.** By Theorem 4 of [Lee90], all triangulations with at most \( n + 3 \) vertices of an \( n \)-dimensional polytope are regular. Hence, it is enough to show that \( T, T' \) are triangulations. Since \( T, T' \) are defined completely analogously, we only provide a proof for \( T \). Begin by observing that from the Kreuzer-Skarke classification that any subset of \( n + 1 \) elements in \( ν \) do not lie in a hyperplane.

Now to check that \( T \) is a triangulation, we check the conditions of the definition. First, in all cases, each simplex has codimension 1 in \( M_r \) by definition. Second, it is easy to check that the intersection of any two simplices in \( T \) is given by the convex hull of the terms in \( ν \) they have in common hence a face of both simplices. Third, we need to check that
\[
\bigcup_{t \in T} t = \text{Conv}(ν)
\]
or equivalently that \( \bigcup_{t \in T} t \) is convex. From the Kreuzer-Skarke classification, \((0, 1)\) is in the interior of the simplex \( \text{Conv}(\Xi) \) and hence
\[
\bigcup_{c \in \mathcal{C}} t = \text{Conv}(\Xi).
\]
Furthermore, \( \text{Conv}(\nu) \) is the union of all lines between points in \( \text{Conv}(\nu) \) and \( m'_k \). Let \( p \in \text{Conv}(\nu) \setminus \text{Conv}(\Xi) \) and \( q \) be the point where the line from \( p \) to \( m'_k \) intersects the boundary of \( \text{Conv}(\Xi) \). Consider any facet \( F \) of \( \text{Conv}(\Xi) \) which contains \( q \). The plane spanned by \( F \) separates the interior of \( \text{Conv}(\Xi) \) and \( p \). Therefore, \( p \) lies in the simplex \( \text{Conv}(F, m'_k) \) which lies in \( \mathcal{S} \).

5.2. Derived Equivalence of BHK Mirrors Related by a Kreuzer-Skarke Cleave.

In this section, we prove our main result. The method is partially toric and will use results from Section 5 of [FK14]. We refer the reader there for a connection between the algebraic and toric language.

Given a Kreuzer-Skarke polynomial \( A \), a group \( G \in \text{Aut}(F_A) \) and a vector \((c, b) \in k^{l+1}\) we can define a generalized BHK pencil by the formula
\[
Z_{c,b}^{c,b} = \left[ \left\{ F_A + c \prod_{i \in I} x_i = 0 \right\} \right]_{G/G_m} \subseteq \left[ \mathbb{A}^{n+1} \setminus \{0\} \right]_{G/G_m} = \mathbb{P}(q_0, \ldots, q_n) / G.
\]

Any Kreuzer-Skarke cleave \((A, A')\), by definition, removes an arrow from the diagram for \( F_A \) or \( F_{A'} \). The removal of an arrow always results in the formation of a new chain or Fermat diagram. This chain or Fermat diagram has its tail at the head of the removed arrow. Let \( I \) be the indexing set which records the \( a_i \) which this chain passes through.

**Theorem 5.9.** Suppose \((A, A')\) is a Kreuzer-Skarke cleave where \( F_A, F_{A'} \) define anticanonical hypersurfaces in \( \mathbb{P}(q_0, \ldots, q_n) / G \). If \( b_i \neq 0 \) for \( i \in I \), then the generalized BHK mirror pencils \( Z_{c,b}^{c,b} \) and \( Z_{(A')^T,G_{A'}}^{c,b} \) are derived equivalent, i.e.,
\[
D^b(\text{coh } Z_{c,b}^{c,b}) \cong D^b(\text{coh } Z_{(A')^T,G_{A'}}^{c,b}).
\]

**Proof.** Notationally, we let \( m_i \) be the vertex such that the monomial \( x^{m_i} \) corresponds to the vertex associated to \( x_i \) in the Kreuzer-Skarke diagram of \( F_A \) and \( F_{A'} \) and the variables are arranged in order according to atomic types. As in the previous section, we set
\[
F_A^b := \sum_{i=1}^{n} b_i x^{m_i} \text{ and } F_{A'}^b := b_k x^{m'_k} + \sum_{i \neq k} b_i x^{m_i}.
\]

The CY orbifolds \( Z_{c,b}^{c,b} \) and \( Z_{c,b}^{c,b} \) are hypersurfaces in the same toric variety \( \mathbb{P}(q_0, \ldots, q_n) \) and their BHK mirrors \( Z_{A^T,G_A}^{c,b} \) and \( Z_{(A')^T,G_{A'}}^{c,b} \) are hypersurfaces in quotients of weighted projective stacks, say \( \mathbb{P}(t_0, \ldots, t_n) / (G_A)^T \) and \( \mathbb{P}(r_0, \ldots, r_n) / (G_A')^T \).

Without loss of generality, the Kreuzer-Skarke cleave deletes an arrow, i.e., the monomial \( x^{m_k} \) is \( x_k^{a_k} \) and the monomial \( x^{m_k} \) is part of a loop or chain. Recall the set of \( n + 3 \) points \( \nu \) as above:
\[
\nu := \{(0, 1), m_0, \ldots, m_n, m'_k\} = \Xi \cup \Xi' \cup \{(0, 1)\} \subseteq (\Delta, 1) \cap M \oplus \mathbb{Z}
\]
and the two regular triangulations \( T, T' \) of \( \nu \) (see Lemma [5.8]). Furthermore, \( S_\nu \) satisfies the quasi-Calabi-Yau condition by Lemma 5.12 of [FK14] since \( \nu \) lies in the affine plane \((M, 1)\).
Since we are discussing mirror symmetry, we are flipping the usual roles of $M$ and $N$. Moreover, the lattice points in $\nu$ have two interpretations depending on which side you are on: (1) the monomial terms, e.g. $x^m_i$, have two separate interpretations as monomials - they are anticanonical sections of $\mathbb{P}(g_0, \ldots, g_n)/\bar{G}$ and (2) the rays of the fan of the dual toric variety and hence correspond to variables in the Cox construction.

Therefore, for notational purposes, we set $y_i$ to be the monomial associated to the ray $m_i$, $u$ to be the monomial associated to the ray $(0,1)$, and $y'_i$ to be the monomial associated to the ray $m'_i$. We get two irrelevant ideals $\mathcal{I}_p$ and $\mathcal{I}_q$ (as defined in Equation (12)) associated to the triangulations $\mathcal{T}$ and $\mathcal{T}'$ respectively. Both have subideals $\mathcal{J}_p \subseteq \mathcal{I}_p$ and $\mathcal{J}_q \subseteq \mathcal{I}_q$ as defined in Equation (4.3) that correspond to the simplices in $C$ and $C'$ that are of maximal dimension. Recall the open sets

$$U_p = \text{Spec}(\kappa[y_0, \ldots, y_n, y'_i, u]) \setminus Z(\mathcal{I}_p); \quad U_q = \text{Spec}(\kappa[y_0, \ldots, y_n, y'_i, u]) \setminus Z(\mathcal{I}_q)$$

We have subsets

$$V_p = \text{Spec}(\kappa[y_0, \ldots, y_n, y'_i, u]) \setminus Z(\mathcal{J}_p); \quad V_q = \text{Spec}(\kappa[y_0, \ldots, y_n, y'_i, u]) \setminus Z(\mathcal{J}_q).$$

The coarse moduli spaces associated to the stacks $[V_p/S_\nu]$ and $[V_q/S_\nu]$ are the toric varieties that correspond to the fans $\Sigma_p$ and $\Sigma_q$ which are the collections of cones obtained by coning over the set of simplices in $C$ and $C'$. These varieties are the canonical bundles of the quotients of the weighted projective spaces $\mathbb{P}(r_0, \ldots, r_n)/\mathbb{G}_A$ and $\mathbb{P}(r'_0, \ldots, r'_n)/\mathbb{G}_{A'}$, see Definition 2.7 and Proposition 2.9 of [Sho14].

Now introduce the potential

$$w := \sum_{i=0}^n b_i(y_{u_i}^{-1}) + cu \prod y_i,$$

and define

$$Z_p := Z(w) \subseteq X_p = \mathbb{P}(r_0, \ldots, r_n)/\mathbb{G}_A$$

and

$$Z_q := Z(w) \subseteq X_q = \mathbb{P}(r'_0, \ldots, r'_n)/\mathbb{G}_{A'}.$$

When we take these zero loci, the polynomial $w$ specializes to only having the variables that correspond to the elements in $\Xi$ and $\Xi'$, respectively. By Equations (2.16) and (2.17), it follows that $w$ specializes to $F_A^T$ and $F_{(A')^T}$ respectively. In summary, we have defined the two CY orbifolds

$$Z_{A^T,G_A^T} = Z_p; \quad Z_{(A')^T,G_{A'}^T} = Z_q.$$

The derived equivalence desired now follows if we can use Corollary 4.7. In Lemma 5.10 below, we prove that the hypotheses of Corollary 4.7 hold, finishing the proof. □

**Lemma 5.10.** Take the potential function associated to the sum of the monomials corresponding to the lattice points $u_{p_i}$ that are the minimal generators of the rays in the fan $\Sigma$:

$$w := \sum_{i=0}^n b_i(y_{u_i}^{-1}) + cu \prod y_i.$$ 

If $b_i \neq 0$ for all $i \in I$, then we have the following containment of ideals

$$\mathcal{I}_p \subseteq \sqrt{\partial w}, \mathcal{J}_p$$ 

and

$$\mathcal{I}_q \subseteq \sqrt{\partial w}, \mathcal{J}_q.$$
Proof. We use the notation in the previous proof. Take $F_A$ to be the sum of $\beta$ invertible polynomials of atomic types $F_{A_1}, \ldots, F_{A_{\beta}}$. Without loss of generality, we say that $m_k$ is in $F_{A_1}$. Due to the assumption that $F_{A'}$ corresponds to having a Fermat term for the variable $x_k$, we know that $F_{A_1}$ must be either a chain or a loop. We split our proof into these two cases as they give triangulations of a slightly different nature.

**Case 1: $F_{A_1}$ is a chain of length $\ell + 1$.**

Since by assumption, $x_k^{m_k}x_{k+1}$ is a summand of the atomic part $F_{A_1}$, we know that $k < \ell$. We now look at the polytope $(\Delta, 1) \subseteq M_R \times R$. We have two triangulations $\mathcal{T}$ and $\mathcal{T}'$ as above. These triangulations correspond to irrelevant ideals $I_p$ and $I_q$ for some maximal chambers of the secondary fan corresponding to some characters $\chi_p$ and $\chi_q$. The subideals of $I_p$ and $I_q$ generated by taking the monomials associated to the maximal simplices in the collections $\mathcal{C} \subseteq \mathcal{T}$ and $\mathcal{C}' \subseteq \mathcal{T}'$ yield the subideals $J_p$ and $J_q$ as in Equation (4.3), namely,

$$J_p = y_k(y_0, \ldots, y_n)$$

and

$$J_q = y_k(y_0, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_n)$$

The quotients $I_p/J_p$ and $I_q/J_q$ are generated by the monomials associated to the simplices in the collections $S$ and $S'$ that are of maximal dimension. While we need to prove that $I_p \subseteq \sqrt{\partial w}, J_p$ and $I_q \subseteq \sqrt{\partial w}, J_q$, we will instead prove something slightly stronger. Namely

$$I_p \subseteq \langle y_k(y_0, \ldots, y_n), u(y_{k+1}, \ldots, y_\ell) \rangle \subseteq \sqrt{\partial w}, J_p$$

and

$$I_q \subseteq \langle y_k(y_0, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_n), u(y_{k+1}, \ldots, y_\ell) \rangle \subseteq \sqrt{\partial w}, J_q.$$  

We first establish the containments,

$$I_p \subseteq \langle y_k(y_0, \ldots, y_n), u(y_{k+1}, \ldots, y_\ell) \rangle$$

and

$$I_q \subseteq \langle y_k(y_0, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_n), u(y_{k+1}, \ldots, y_\ell) \rangle,$$

from Equations (5.1) and (5.2). This is equivalent to showing that the simplices in $S, S'$ lie in the set of simplices which do not contain $(0, 1)$ and some $v \in \{m_{k+1}, \ldots, m_\ell\}$. It is clear that each simplex in $S, S'$ does not contain $(0, 1)$ and now must drop precisely one more element.

The key observation is that the variables $m'_{k}, m_{k}, \ldots, m_{\ell}$ all live on the same $\ell - k - 1$ dimensional face of the polytope $(\Delta, 1)$. In particular, this is the face defined by taking the intersection of $(\Delta, 1)$ with the half spaces corresponding to the elements $(u_{\rho_i}, 1)$ for $k \leq i \leq \ell$, i.e.,

$$m'_{k}, m_{k}, \ldots, m_{\ell} \in (\Delta, 1) \cap \bigcap_{i \in \{k, \ldots, n\}} H(u_{\rho_i}, 1).$$

This implies that one must drop an element from $\{m'_{k}, m_{k}, \ldots, m_{\ell}\}$. If one drops $m'_{k}$ you get $\text{Conv}(\Xi)$ and if one drops $m_{k}$ you get $\text{Conv}(\Xi')$. Neither of these is in $\mathcal{T}, \mathcal{T}'$. This implies the desired containment.

We now establish the containments,

$$\langle y_k(y_0, \ldots, y_n), u(y_{k+1}, \ldots, y_\ell) \rangle \subseteq \sqrt{\partial w}, J_p$$

and

$$\langle y_k(y_0, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_n), u(y_{k+1}, \ldots, y_\ell) \rangle \subseteq \sqrt{\partial w}, J_q.$$
from Equations (5.1) and (5.2).

It suffices to prove that the monomial $uy_j$ is in both ideals $\sqrt{\partial w, J_q}$ and $\sqrt{\partial w, J_p}$ for $k < j \leq \ell$.

First, one can describe all of the monomials of $w$ explicitly in terms of the matrix $A$:

$$y^{(u_{\rho_1},1)} = \begin{cases} y_0^{a_{00}} u & \text{if } k \neq 0 \text{ and } i = 0. \\ y_0^{a_{00}} (y_0')^b u & \text{if } k = i = 0. \\ y_{ij}^{a_{ij}} y_{i-1} u & \text{if } 0 < i \leq \ell, \ i \neq k \\ y_k^{a_{i1}} y_{k-1} (y'_k)^b u & \text{if } 0 < k = i \\ \prod_{j=\ell+1}^n y_{j,i}^{a_{j,i}} u & \text{if } i > \ell. \end{cases}$$

Note that $y_j$ does not divide the monomial $y^{(u_{\rho_1},1)}$ whenever $0 \leq j \leq \ell$ and $i > \ell$.

We now take the partial derivative of $w$ with respect to the variable $y_k$ and consider:

$$y_k \partial k w = b_ka_{kk}y_k^{a_{kk}} y_{k-1}(y'_k)^b u + b_{k+1}y_k y_{k+1}^{a_{(k+1)(k+1)}} u + cu \prod_{i \neq k} y_i.$$

The first and third summands are in the ideals $J_p, J_q$. Therefore $y_k y_{k+1} u$ is in the radical ideals $\sqrt{\partial w, J_p}$ and $\sqrt{\partial w, J_q}$ as $b_{k+1} \neq 0$ by assumption.

Inductively, we now show that, provided that $y_{j-1} y_j u$ for $k < j < \ell$ is in $\sqrt{\partial w, J_p}$ and $\sqrt{\partial w, J_q}$, the monomial $y_{j+1} u$ is as well. We take the partial derivative with respect to $y_j$ of the potential $w$:

$$\partial_j w = b_j y_{j-1} y_{j,k}^{a_{jj}} y_{j-1}(y'_j)^b u + b_{j+1} y_{j+1}^{a_{(j+1)(j+1)}} u + cu \prod_{i \neq j} y_i.$$

The first and third summands are in $\sqrt{\partial w, J_p}$ and $\sqrt{\partial w, J_q}$, consequently $y_{j+1} u$ is as well. Finally, return to the partial derivative

$$\partial_k w = b_k a_{kk} y_k^{a_{kk}} y_{k-1}(y'_k)^b u + b_{k+1} y_k y_{k+1}^{a_{(k+1)(k+1)}} u + cu \prod_{i \neq k} y_i.$$

The first and third summands are in $\sqrt{\partial w, J_p}$, $\sqrt{\partial w, J_q}$ therefore $y_{k+1} u$ is as well. This completes Case 1 as Equations (5.1) and (5.2) are satisfied.

Case 2: $F_{A_1}$ is a loop of length $\ell + 1$.

Similarly, we prove

$$\mathcal{I}_p \subseteq \langle y_0(y_0, \ldots, y_n), u(y_1, \ldots, y_\ell) \rangle \subseteq \sqrt{\partial w, J_p}$$

and

$$\mathcal{I}_q \subseteq \langle y_0(y_0, \ldots, y_n), u(y_1, \ldots, y_\ell) \rangle \subseteq \sqrt{\partial w, J_q}. (5.4)$$

As $F_{A_1}$ is a loop, without loss of generality we set $k = 0$. We apply a similar strategy to that of Case 1, but we have that $m_0, m_0, \ldots, m_\ell$ all sit in the same face of $(\Delta, 1)$, namely:

$$(\Delta, 1) \cap \bigcap_{j=\ell+1}^n H_{(u_{\rho_j}, 1)}.$$

The same argument gives the containments

$$\mathcal{I}_p \subseteq \langle y_0(y_0, \ldots, y_n), u(y_1, \ldots, y_\ell) \rangle$$

and

$$\mathcal{I}_q \subseteq \langle y_0(y_0, \ldots, y_n), u(y_1, \ldots, y_\ell) \rangle$$

and consider:
from Equations (5.3) and (5.4).

Again, one can explicitly describe the monomial terms of \( w \) in terms of the matrix \( A \):

\[
y^{(u_{pi},1)} = \begin{cases} 
y_0^{a_{00}} (y_0')^{a_0} y_\ell u & \text{if } i = 0, \\
y_{i-1}^{a_{ii}} y_{i}^{a_{ii}} u & \text{if } 0 < i \leq \ell \\
\prod_{j=\ell+1}^{n} y_j^{a_{ji}} u & \text{if } j > \ell
\end{cases}
\]

We now prove the containments,

\[
\langle y_0(y_0, \ldots, y_n), u(y_1, \ldots, y_\ell) \rangle \subseteq \sqrt{\partial w, J_p}
\]

and

\[
\langle y_0(y_0, \ldots, y_n), u(y_1, \ldots, y_\ell) \rangle \subseteq \sqrt{\partial w, J_q},
\]

from Equations (5.3) and (5.4).

First, take the partial derivative of \( w \) with respect to \( y_0 \):

\[
y_0 \partial_0 w = b_0 a_{00} y_0^{a_{00}} (y_0')^{b} y_\ell u + b_1 y_0 y_1^{a_{11}} u + cu \prod y_i.
\]

As the first and third summands are in both \( J_p \) and \( J_q \), we know that \( y_0 y_1 u \) is in both the radical ideals \( \sqrt{\partial w, J_p} \) and \( \sqrt{\partial w, J_q} \). We now can iterate the procedure.

Given that the monomial \( y_{j-1} y_j u \) is in both the ideals \( \sqrt{\partial w, J_p} \) and \( \sqrt{\partial w, J_q} \), we can prove that \( y_{j+1} u \) is as well for \( 0 < j < \ell \). Take the partial derivative with respect to \( y_j \):

\[
\partial_j w = b_j a_{jj} y_{j-1} y_{j} y_{j}^{a_{jj}} u + b_{j+1} a_{(j+1)(j+1)} y_{j+1}^{a_{(j+1)(j+1)}} u + cu \prod_{i \neq j} y_i
\]

as the first and third summands are in both ideals \( \sqrt{\partial w, J_p} \) and \( \sqrt{\partial w, J_q} \), we have that the second summand is as well, hence \( y_{j+1} u \) is in both the radical ideals \( \sqrt{\partial w, J_p} \) and \( \sqrt{\partial w, J_q} \).

Finally, return to the partial derivative at \( y_0 \):

\[
\partial_0 w = b_0 a_{00} y_0^{a_{00}-1} (y_0')^{b} y_\ell u + b_1 y_1^{a_{11}} u + cu \prod_{i \neq 0} y_i.
\]

The first and third summands are in \( \sqrt{\partial w, J_p}, \sqrt{\partial w, J_q} \) therefore \( y_1 u \) is as well. This completes Case 2 as Equations (5.3) and (5.4) are satisfied. \( \square \)

**Corollary 5.11.** Suppose \((A, A')\) is a Kreuzer-Skarke cleave where \( F_A, F_{A'} \) define hypersurfaces define anticanonical hypersurfaces in \( \mathbb{P}(q_0, \ldots, q_n)/\bar{G} \). Then the BHK mirrors \( Z_{A^T,G_A^T} \) and \( Z_{(A')^T,G_{A'}^T} \) are derived equivalent, i.e.,

\[
D^b(\text{coh} \ Z_{A^T,G_A^T}) \cong D^b(\text{coh} \ Z_{(A')^T,G_{A'}^T}).
\]

**Proof.** We set \( c = 0 \) and \( b_i = 1 \) in Theorem 5.9 \( \square \)

**Corollary 5.12.** Fix \( b \in (\kappa^*)^l, c \in \kappa \). Take two polynomials \( F_A \) and \( F_{A'} \) which define hypersurfaces in a quotient of a Gorenstein weighted projective stack \( \mathbb{P}(q_0, \ldots, q_n)/\bar{G} \). Then the generalized BHK mirror pencils \( Z_{A^T,G_A^T}^c \) and \( Z_{(A')^T,G_{A'}^T}^c \) are derived equivalent.

**Proof.** Since we assume \( b_i \not= 0 \) for all \( i \), this follows directly from iteratively using Theorem 5.9 to compare both \( F_A \) and \( F_{A'} \) through a sequence of Kreuzer-Skarke cleaves, which is guaranteed to exist by Proposition 5.7 \( \square \)
Corollary 5.13 (Theorem 5.1). Take two polynomials $F_A$ and $F_{A'}$ which define hypersurfaces $Z_{A,G}$ and $Z_{A',G}$ in a quotient of a Gorenstein weighted projective stack $\mathbb{P}(q_0, \ldots, q_n)/G$. Then the BHK mirrors $Z_{A^T,G_A}$ and $Z_{(A')^T,G_{A'}}$ are derived equivalent.

Proof. This is the special case of Corollary 5.12 where $b_i = 1, c = 0$. □

Remark 5.14. Since $Z_{c,b}^{A,G}$, $Z_{c,b}^{(A')^T,G_{A'}}$ are open substacks of the irreducible component of the critical locus of $w$ lying on $Z(u)$, it follows that they are birational. In the Gorenstein case, this immediately recovers Theorem 2.5 in the case of families.

We can now rephrase Seidel and Sheridan’s Homological Mirror Symmetry result for hypersurfaces in projective space [Sei03, She14] in the language of Berglund-Hübsch-Krawitz mirror symmetry. They define the universal Novikov field, to be the field whose elements are formal sums
\[
\sum_{j=0}^{\infty} c_j r^{\lambda_j}
\]
where $c_j \in \mathbb{C}$, and $\lambda_j \in \mathbb{R}$ is an increasing sequence of real numbers such that
\[
\lim_{j \to \infty} \lambda_j = \infty.
\]
The universal Novikov field is algebraically closed of characteristic zero.

Over the universal Novikov field, we define a Berglund-Hübsch-Krawitz pencil as
\[
Z_{A,G}^{\text{pencil}} := \left\{ x_0 \ldots x_n + rF_A = 0 \right\} \subseteq \left[ \mathbb{A}^{n+1} \setminus \{0\} \right] / \mathbb{G}_m.
\]
Since Sheridan and Seidel have proven Homological Mirror Symmetry when $A^T$ is a Fermat polynomial, we obtain the following.

Theorem 5.15. Homological Mirror Symmetry holds for Berglund-Hübsch-Krawitz mirror pencils in projective space over the universal Novikov field.

More precisely, if $F_A$ defines a smooth hypersurface in projective space $\mathbb{P}^n$ over the universal Novikov field (in particular $G = \mathbb{Z}_{n+1}$) and $n \geq 3$, there is an equivalence of triangulated categories,
\[
\text{Fuk } Z_{A,G} \cong \text{D}^b(\text{coh } Z_{A^T,G_A}^{\text{pencil}}).
\]

Proof. Set $A' = (n+1) \text{Id}$, $G = J_{A'} = \mathbb{Z}_{n+1}$ and $q_0 = \ldots = q_n = 1$. We have
\[
\text{Fuk } Z_{A,G} = \text{Fuk } Z_{A',G} = \text{D}^b(\text{coh } Z_{(A')^T,G_{A'}}^{\text{pencil}}) = \text{D}^b(\text{coh } Z_{A^T,G_A}^{\text{pencil}}).
\]
The first line follows from the fact that $Z_{A,G}$ is symplectomorphic to $Z_{A',G}$ by Moser’s theorem. The second line is Theorem 1.3 of [Sei03] in the case $n = 3$ and Theorem 1.2.7 of [She14] in the case $n \geq 4$. The third line is Corollary 5.12 in the special case $b_i = 1, c = r$, and $\kappa = \Lambda$. □

Remark 5.16. In the case of elliptic curves ($n = 2$), a variant of this theorem can be proven using work of Polishchuk and Zaslow [PZ98].
**Remark 5.17.** The category $\text{Fuk} Z_{A,G}$ is equipped with a $\Lambda$-linear structure and the equivalence is $\Lambda$-linear after changing the module structure of $D^b(\text{coh} Z_{A^T,G_A^T})$ by an automorphism of $\Lambda$. See [Sei03, She14] for details. It can then be extended to an equivalence of dg-categories using Theorem 9.8 of [LO10].

5.3. **An Example.** In the following example, we will see that our proof extends to families as well.

**Example 5.18.** Consider the polynomials $F_A = x_0^3 + x_1^2 x_2 + x_2^3$ and $F_{A'} = x_0^3 + x_1^3 + x_3^3$. Both carve out cubic hypersurfaces in $\mathbb{P}^2$. Let us take the fan of $\mathbb{P}^2$ which is the complete fan in $N_{\mathbb{R}} = (\mathbb{Z})^2 \otimes \mathbb{R}$ generated by rays $(1,0)$, $(0,1)$ and $(-1,-1)$ and enumerate these rays as $x_{(1,0)} =: x_0$, $x_{(0,1)} =: x_1$ and $x_{(-1,-1)} =: x_2$ respectively. The canonical bundle of $\mathbb{P}^2$ is the toric variety associated to the fan $\Sigma_K$ which is defined to be the fan with rays generated by $u_{p_0} = (1,0,1)$, $u_{p_1} = (0,1,1)$, $u_{p_2} = (-1,-1,1)$ and $u_{p_3} = (0,0,1)$ and is the star subdivision along $\rho_3$ of the fan generated by $p_0$, $p_1$, and $p_2$.

The dual cone to $|\Sigma_K|$ is generated by the elements $(2,-1,1)$, $(-1,2,1)$, and $(-1,-1,1)$. The polytope $\Delta$ that is associated to $\mathbb{P}^2$ is found by looking at the one slice $|\Sigma_K|_{(1)} = (\Delta, 1)$. Note that since each lattice point corresponds to a monomial we can look at which lattice points correspond to monomials that are nonzero in $F_A$ and $F_{A'}$.

![Figure 1](image-url)  
**Figure 1.** The polytope $\Delta$ with lattice points marked by sections of $\omega_{\mathbb{P}^2}$.

To consider the BHK mirrors, we set $\nu := \{v_{\tau_0}, v_{\tau_1}, v'_{\tau_1}, v_{\tau_2}, v_{\tau_3}\}$ where $v_{\tau_0} = (2,-1,1)$, $v_{\tau_1} = (-1,2,1)$, $v'_{\tau_1} = (-1,1,1)$, $v_{\tau_2} = (-1,-1,1)$ and $v_{\tau_3} = (0,0,1)$. We introduce variables for each ray: $y_i$ for $\tau_i$ where $i \in \{0,1,2\}$, $y'_{i}$ for $\tau'_i$ and $u$ for $\tau_3$. The triangulations $\mathcal{T}$, $\mathcal{T}'$ are pictured in Figure 2.

The corresponding irrelevant ideals are $\mathcal{I}_p = \langle y_1(y_0, y'_1, y_2), u y_2 \rangle$ and $\mathcal{I}_q = \langle y'_1(y_0, y_1, y_2) \rangle$ respectively.

There exists subideals $\mathcal{J}_p = \langle y_1(y_0, y'_1, y_2) \rangle$ and $\mathcal{J}_q = \mathcal{I}_q$ which correspond to the fans over the triangulations in Figure 3. The toric varieties associated to $\Xi$ and $\Xi'$ are $\text{tot}(\omega_{\mathbb{P}^2(3,2,1)})$ and $\text{tot}(\omega_{\mathbb{P}^2/\mathbb{Z}_2})$ respectively.

We now need to discuss the potential $w$ that is a function on the partial compactifications of these bundles. To do this, we must turn back to the dual cone to $\text{Cone}(v_{\tau_1}, v'_{\tau_1})$. In this
The dual cone is just $|\Sigma_K|$ (on a general Gorenstein quotient of weighted projective space, the dual cone contains $|\Sigma_K|$ with equality if and only if $F_A$ or $F_A'$ is a Fermat polynomial). We draw the support of the dual cone $|\Sigma_K|$ below along with the functions corresponding to the lattice points in Figure 4.

Now, let

$$w := c_0y_0^3u + c_1y_1^3(y_1')^2u + c_2y_1'y_2^3u + c_3y_0y_1y_1'y_2u$$

for some constants $c_i \in k$. We need to check that we have that $I_p \subseteq \sqrt{\partial w}$, $J_p$ in order to be able to use Corollary 4.7 (as $I_q = J_q$ this is automatic for the other triangulation). Here,
we compute the partial derivative of \( w \) with respect to \( y_1' \):

\[
\partial_{y_1'} w = 2c_1y_1^3(y_1')u + c_2y_2^2u + c_3y_0y_1y_2u.
\]

Here we can see that the first and third summands are both in \( \mathcal{J}_p \), hence \( y_2u \) is in \( \sqrt{\partial w, \mathcal{J}_p} \) as long as the constant \( c_2 \) is nonzero. In other words, one can apply Corollary 4.7 as long as \( c_2 \) is nonzero. Applying the framework outlined in Section 4, we get:

\[
\begin{align*}
U_p &:= \mathbb{A}^5 \setminus Z(\mathcal{I}_p); & U_q &:= \mathbb{A}^5 \setminus Z(\mathcal{I}_q); \\
V_p &:= \mathbb{A}^5 \setminus Z(\mathcal{J}_p); & V_q &:= \mathbb{A}^5 \setminus Z(\mathcal{J}_q); \\
V^x_p &:= \mathbb{A}^4 \setminus Z(\mathcal{J}^x_p); & V^x_q &:= \mathbb{A}^4 \setminus Z(\mathcal{J}^x_q); \\
[V_p/S] &:= \text{tot}(\omega_{\mathbb{P}(2,3,1)}); & [V_q/S] &:= \text{tot}(\omega_{\mathbb{P}^2/Z_3}); \\
X_p &:= [V^x_p/S] = \mathbb{P}(2,3,1); & X_q &:= [V^x_q/S] = \mathbb{P}^2/Z_3; \\
Z_p &:= Z(w_p); & Z_q &:= Z(w_q);
\end{align*}
\]

where

\[
\begin{align*}
w_p &:= c_0y_0^3 + c_1(y_1')^2 + c_2y_1'y_2^3 + c_3y_0y_1'y_2; \\
w_q &:= c_0y_0^3 + c_1y_1^3 + c_2y_2^3 + c_3y_0y_1y_2.
\end{align*}
\]

Then we have the equivalence of categories \( D^b(\text{coh } Z_p) \cong D^b(\text{coh } Z_q) \).

The special case \( c_0 = c_1 = c_2 = 1 \) and \( c_3 = 0 \) is \( w_p = F_{A\lambda} \) and \( w_q = F_{(A')\lambda} \), which gives us the BHK mirrors to \( Z_A \) and \( Z_{A'} \). If we take \( c_0 = c_1 = c_2 = 1 \) and \( c_3 = \lambda \), we have pencils. Also, we can take degenerate loci, for example, \( c_2 = 1 \) and \( c_0 = c_1 = c_3 = 0 \) so that \( w_p = y_1'y_2^3 \) and \( w_q = y_2^3 \). In general, we have locally-closed BHK mirror families that are pointwise derived equivalent to one another.

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