Sufficient criteria and sharp geometric conditions for observability in Banach spaces

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Abstract

Let $X, Y$ be Banach spaces, $(S_t)_{t \geq 0}$ a $C_0$-semigroup on $X$, $-A$ the corresponding infinitesimal generator on $X$, $C$ a bounded linear operator from $X$ to $Y$, and $T > 0$. We consider systems of the form

\[ \dot{x}(t) = -Ax(t), \quad y(t) = Cx(t) \quad t \in (0, T], \quad x(0) = x_0 \in X. \]

We provide sufficient conditions such that this system satisfies a final state observability estimate in $L^r((0,T); Y)$, $r \in [1, \infty]$. These sufficient conditions are given by an uncertainty relation and a dissipation estimate. Our approach unifies and generalizes the respective advantages from earlier results obtained in the context of Hilbert spaces. As an application we consider the example where $A$ is an elliptic operator in $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, and where $C = 1_{\omega}$ is the restriction onto a thick set $\omega \subset \mathbb{R}^d$. In this case, we show that the above system satisfies a final state observability estimate if and only if $\omega \subset \mathbb{R}^d$ is a thick set. Finally, we make use of the well-known relation between observability and null-controllability of the predual system, and investigate bounds on the corresponding control costs.

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1 Introduction

Let $X, Y$ be Banach spaces, $(S_t)_{t \geq 0}$ a $C_0$-semigroup on $X$, $-A$ the corresponding infinitesimal generator on $X$, and $C$ be a bounded operator from $X$ to $Y$. We consider systems of the form

\[ \dot{x}(t) = -Ax(t), \quad t \in (0, T], \quad x(0) = x_0 \in X, \]
\[ y(t) = Cx(t), \quad t \in [0, T], \tag{1} \]

where $T > 0$ can be thought of as a final time for the system. One interpretation of the second equation in (1) is that we cannot measure the state $x(t)$ at time $t$ directly, but just some $y(t) = Cx(t)$ from the range of $C$. The focus of this paper is laid to the question whether
the system (1) satisfies a final state observability estimate in $L_r((0, T); Y)$ with $r \in [1, \infty]$, that is, there exists $C_{\text{obs}} > 0$ such that for all $x_0 \in X$ we have $\|x(T)\|_X \leq C_{\text{obs}}\|y\|_{L_r((0, T); Y)}$. A final state observability estimate thus allows one to recover information on the final state $x(T)$ from suitable measurements $y(t)$ for $t \in (0, T)$.

The most studied example of the system (1) is the heat equation with heat generation term in $L_2(\Omega)$ with $\Omega \subset \mathbb{R}^d$ open, and some non-empty observability set, i.e. $A = \Delta - V$ is a self-adjoint Schrödinger operator in $L_2(\Omega)$ with bounded potential $V$, and $C = 1_\omega$ is the projection onto some non-empty measurable set $\omega \subset \Omega$. For bounded domains $\Omega \subset \mathbb{R}^d$ the observability problem for the heat equation is well understood since the seminal works by Lebeau and Robbiano [LR95], and Fursikov and Imanuvilov [FI96]. For unbounded domains this problem has been studied, e.g., in [Mil05a, Mil05b, Gd07, Bar14]. While for bounded domains it is sufficient that $\omega$ is open and non-empty, or even measurable with positive Lebesgue measure [AEWZ14, EMZ15], this is of course not true for unbounded domains. On unbounded domains a sufficient geometric condition for observability is given in [LM16]. In addition to that the papers [EV18, WWZZ19] show that the free heat equation in $L_2(\mathbb{R}^d)$ satisfies a final state observability estimate if and only if $\omega$ is a thick set.

Since the observability constant $C_{\text{obs}}$ can be interpreted (by duality) as the cost for the corresponding null-controllability problem, the problem of obtaining explicit bounds on $C_{\text{obs}}$ attached particular attention in the literature. The (optimal) dependence of $C_{\text{obs}}$ on the model parameter $T$ is investigated in [Güi85, FZ00, Mil04b, Phu04, Mil06a, TT07, Mil10, LL12, BPS18], while [Mil04b, TT11, EZ11, NTTV18, EV18, Phu18, Egi, LL, NTTVa] also study the dependence on the geometry of the control set $\omega$. Moreover, [Güi85, Mil06a, TT07, Lis12, Lis15, DE19] concern one-dimensional problems and boundary control.

One possible approach to show an observability estimate has been described in the papers [LR95, LZ98, JL99], that is to prove a quantitative uncertainty relation for spectral projectors. This is an inequality of the type

$$\forall \lambda > 0 \forall \psi \in L_2(\Omega): \quad \|P(\lambda)\psi\|_{L_2(\Omega)} \leq d_0 e^{d_1 \lambda^\gamma} \|1_\omega P(\lambda)\psi\|_{L_2(\omega)},$$

where $\gamma \in (0, 1)$, $d_0, d_1 > 0$, and where $P(\lambda)$ denotes the projector to the spectral subspace of $-\Delta + V$ below $\lambda$. Subsequently, this strategy is generalized to (contraction) semigroups in abstract Hilbert with (possibly self-adjoint) generators $-A$, to name those which are closest related to our result, see [Mil10, TT11, BPS18, NTTVa]. In particular, the papers [Mil10, BPS18] allow for the $P(\lambda)$ to be arbitrary projectors (onto semigroup invariant subspaces) by assuming additionally a so-called dissipation estimate. Since the constants appearing in the uncertainty relation transfer into the observability constant $C_{\text{obs}}$, it is important to achieve its dependence on $d_0$, $d_1$ and $\gamma$, on the set $\omega$, and on the coefficients of the operator $A$ as explicit as possible. Uncertainty relations with an explicit dependence on the geometry of $\omega$ are provided by the Logvinenko-Sereda theorem for the free heat equation observed on thick sets [LS74, Kov00, Kov01, EV]. For Schrödinger operators such uncertainty relations have, for instance, been proven in [NTTV18, NTTVb] for a certain class of equidistributed observation sets and bounded potentials, and in [LM] for thick observation sets and analytic potentials.

So far, the discussion was restricted to Hilbert spaces only. However, a natural setup to ask for observability estimates is the context of Banach spaces and $C_0$-semigroups, since there are various applications of the above concepts in this situation. In this paper, we extend (some
of the above mentioned results to the Banach space setting. In particular, in Section 2 we show in the general framework of Banach spaces that an uncertainty relation together with a dissipation estimate implies that the system (1) satisfies a final state observability estimate. Our observability constant $C_{\text{obs}}$ is given explicitly with respect to the parameters coming from the uncertainty relation and the dissipation estimate, and in addition is sharp in the dependence on $T$. Let us stress that, besides the fact that this result holds in its natural Banach space setting, our approach unifies and generalizes the respective advantages from earlier results even in the context of Hilbert spaces, cf. Remark 2.2 for more details. In Section 3 we verify these sufficient conditions in $L^p$-spaces for a class of elliptic operators $A$ and observation operators $C = 1_\omega$. This way we obtain an observability estimate with an explicit dependence on the coefficients of the elliptic operator $A$, the final time $T$, and the geometry of the thick set $\omega$. Furthermore, we show that this result is sharp in the sense that the system (1) satisfies a final state observability estimate if and only if $\omega$ is a thick set. Finally, in Section 4 we make use of the well-known relation between observability and null-controllability of the predual system to (1), and investigate bounds on the corresponding control costs.

2 Sufficient criteria for observability in Banach spaces

For normed spaces $V$ and $W$ we denote by $\mathcal{L}(V,W)$ the space of bounded linear operators from $V$ to $W$. Let $X,Y$ be Banach spaces, $(S_t)_{t \geq 0}$ a $C_0$-semigroup on $X$, $-A$ the corresponding infinitesimal generator on $X$ with domain $\mathcal{D}(-A)$, and $C \in \mathcal{L}(X,Y)$. For $T > 0$ we consider the system

\begin{align*}
\dot{x}(t) &= -Ax(t) \quad t \in (0,T], \quad x(0) = x_0 \in X, \\
y(t) &= Cx(t) \quad t \in [0,T].
\end{align*}

The mild solution of (2) is given by

$$
x(t) = S_t x_0, \quad y(t) = C S_t x_0, \quad t \in [0,T].
$$

In particular, if $x_0 \in \mathcal{D}(A)$ we may differentiate $x(\cdot) = S(\cdot) x_0$ to obtain (2). Let $r \in [1, \infty]$. We say that the system (2) satisfies a final state observability estimate in $L^r((0,T);Y)$, if there exists $C_{\text{obs}} > 0$ such that for all $x_0 \in X$ we have $\|x(T)\|_X \leq C_{\text{obs}} \|y\|_{L^r((0,T);Y)}$, or equivalently, if for all $x_0 \in X$ we have

$$
\|S_T x_0\|_X \leq C_{\text{obs}} \left( \int_0^T \|CS_{\tau} x_0\|_Y \, d\tau \right)^{1/r} \quad \text{if } 1 \leq r < \infty, \text{ or}
$$

$$
\|S_T x_0\|_X \leq C_{\text{obs}} \underset{\tau \in [0,T]}{\text{ess sup}} \|CS_{\tau} x_0\|_Y \quad \text{if } r = \infty.
$$

One motivation to study final state observability estimates is its relation to null-controllability of the predual system to (2) and its control cost. This is discussed in more detail in Section 4.

The following theorem provides sufficient conditions such that the system (2) satisfies a final state observability estimate.
**Theorem 2.1.** Let $X$ and $Y$ be Banach spaces, $C \in \mathcal{L}(X,Y)$, $(S_t)_{t \geq 0}$ be a $C_0$-semigroup on $X$, $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S_t\| \leq M e^{\omega t}$ for all $t \geq 0$, $\lambda^* > 0$, $(P_\lambda)_{\lambda > \lambda^*}$ be a family of bounded linear operators in $X$, $r \in [1, \infty]$, $d_0, d_1, d_3, \gamma_1, \gamma_2, \gamma_3, T > 0$ with $\gamma_1 < \gamma_2$, and $d_2 \geq 1$. Assume further that

$$\forall x \in X \ \forall \lambda > \lambda^*: \quad \|P_\lambda x\|_X \leq d_0 e^{d_3 \lambda^* t} \|CP_\lambda x\|_Y, \quad (3)$$

and

$$\forall x \in X \ \forall \lambda > \lambda^* \ \forall t \in (0,T/2]: \quad \|(\text{Id} - P_\lambda)S_t x\|_X \leq d_2 e^{-d_3 \lambda^* t \gamma_3} \|x\|_X. \quad (4)$$

Then we have for all $x \in X$

$$\|S_T x\|_X \leq C_{\text{obs}} \|CS^r x\|_L_r((0,T);Y) \quad \text{with} \quad C_{\text{obs}} = \frac{C_1}{T^{1/r}} \exp \left( \frac{C_2}{T^{1/2 - \gamma_1}} + C_3 T \right),$$

where $T^{1/r} = 1$ if $r = \infty$, and

$$C_1 = (4Md_0) \max \left\{ \left( \left( (4d_2^2 M^2 (d_0 \|C\|_{\mathcal{L}(X,Y)} + 1) \right)^{8/(e \ln 2)}, e^{4d_1 (2 \gamma_3)^{3/2}} \right) \right\},$$

$$C_2 = 4 \left( 2^{\gamma_1} (2 \cdot 4^{\gamma_3})^{1/2 - \gamma_1} d_1^{\gamma_1} d_3^{\gamma_3} \right)^{1/2 - \gamma_1},$$

$$C_3 = \max \{ \omega, 0 \} \left( 1 + 10/(e \ln 2) \right).$$

**Remark 2.2.** (a) Assumption (3) is called **uncertainty relation**, since a state $P_\lambda x \neq 0$ in the range of $P_\lambda$ cannot be in the kernel of $C$. In particular, if $X = Y$ is a Hilbert space and $P_\lambda$ and $C$ are orthogonal projections, Assumption (3) can be rewritten as

$$P_\lambda \leq d_0 e^{d_3 \lambda^* t} P_\lambda CP_\lambda, \quad (5)$$

where the inequality is understood in the quadratic form sense. If $X = Y = L_2(\Omega)$ with $\Omega \subset \mathbb{R}^d$ open, $A$ is a Schrödinger operator, $C = 1_\omega: X \to Y$ is the restriction operator (i.e. the multiplication operator with $1_\omega$) on some measurable set $\omega \subset \Omega$, and if $P_\lambda$ is the spectral projector of a self-adjoint operator onto the interval $(-\infty, \lambda]$, then the spectral projector corresponds to a restriction in momentum-space and enforces delocalization in direct space, i.e. an uncertainty relation. Inequality (5) is sometimes also called **gain of positive definiteness**, since the restriction $P_\lambda CP_\lambda$ of $C$ is strictly positive on the subspace Ran $P_\lambda$. In control theory inequalities of the type (3) are often called **spectral inequality**. We omit this notation, since the operators $P_\lambda$ are in our setting not necessarily spectral projectors of some self-adjoint operator in a Hilbert space.

Assumption (4) is called **dissipation estimate**, as it assumes an exponential decay of $(\text{Id} - P_\lambda)S_t$ with respect to $\lambda$ and $t$. In particular, it implies that $P_\lambda \to \text{Id}$ strongly as $\lambda \to \infty$.

(b) The dependence of $C_{\text{obs}}$ on $T$ is optimal for large and small $T$. In [Sei84] Seidman showed for one-dimensional controlled heat systems that $C_{\text{obs}}$ blows up at most exponentially for small $T$. This result was extended to arbitrary dimension by Fursikov and Imanuvilov in [FI96]. That the exponential blow-up has to occur for small $T$ has first been shown by Güichal [Güi85] for one-dimensional systems, and by Miller [Mil04a] in arbitrary dimension. It is folklore that in the large time regime, the decay rate $T^{-1/r}$ is optimal; for a proof see, e.g., [NTTVa, Theorem 2.13].
Let us discuss the novel aspects of Theorem 2.1 compared to earlier results in the literature. We restrict our discussion to the case where $X$ and $Y$ are Hilbert spaces and $r = 2$, since to the best of our knowledge, sufficient conditions for observability in Banach spaces as in Theorem 2.1 have not been obtained before.

That uncertainty relations imply observability estimates has been first shown in the seminal papers [LR95, LZ98, JL99]. Subsequently, there is a huge amount of literature concerning abstract theorems which turn uncertainty relations into observability estimates in Hilbert spaces, to name a few, see [Mil10, TT11, BPS18, NTTVa]. The paper [Mil10] considered general $C_0$-semigroups and the operators $(P_\lambda)_{\lambda > 0}$ are projections onto a non-decreasing family of semigroup invariant subspaces. The obtained observability constant $C_{\text{obs}}$ is of the form $C \exp(C/T^{\gamma_1\gamma_3/(\gamma_2-\gamma_1)})$ and hence misses the factor $T^{-1/2}$. A similar result has been obtained in [BPS18] for contraction semigroup and orthogonal projections $(P_\lambda)_{\lambda > 0}$ onto semigroup invariant subspaces. The papers [TT11, NTTVa] considered non-negative and self-adjoint operators $A$, and the operators $(P_\lambda)_{\lambda > 0}$ are assumed to be spectral projections of $A$ onto the interval $[0, \lambda)$. In this setting, the dissipation estimate is automatically satisfied with $\gamma_2 = \gamma_3 = 1$. While both papers obtain the “optimal” bound $CT^{-1/2} \exp(C/T^{\gamma_1/(1-\gamma_1)})$ (including the factor $T^{-1/2}$), the paper [TT11] assumed additionally that $A$ has purely discrete spectrum with an orthogonal basis of eigenvectors. Moreover, [NTTVa] slightly improved the dependence of $C_{\text{obs}}$ on the parameters $d_0$ and $d_1$ which was essential for their application to certain homogenization regimes. Let us emphasize that our result recovers this dependence on $d_0$ and $d_1$ as well, and hence allows also for homogenization.

To conclude, our result extends the earlier mentioned results into three directions.

1. We allow for an arbitrary family $(P_\lambda)_{\lambda > \lambda^*}$ of bounded linear operators, and obtain at the same time the factor $T^{-1/2}$ in $C_{\text{obs}}$. In particular, we do not require that $P_\lambda$ is an orthogonal projection.

2. We allow for general $C_0$-semigroups, possibly with exponential growth. We do not require contraction (or quasi-contraction, or bounded) semigroups.

Thus, even in the Hilbert space setting, our result is new and combines the respective advantages from earlier results, e.g. the factor $T^{-1/2}$ in $C_{\text{obs}}$, general $C_0$-semigroups, and arbitrary family $(P_\lambda)_{\lambda > \lambda^*}$ of bounded linear operators at the same time.

3. We consider Banach spaces $X$ and $Y$ instead of Hilbert spaces, and $r \in [1, \infty)$ instead of $r = 2$.

Indeed, since the theory of strongly continuous semigroups essentially is a Banach space theory, our Theorem 2.1 now formulates the link from uncertainty relations and dissipation estimates to observability estimates in its natural setup. Let us stress that, in contrast to the above mentioned references, we have no spectral calculus in the general framework of Banach spaces.

4. Suppose we have a discrete sequence $(P_k)_{k \in \mathbb{N}}$ of bounded linear operators in $X$ which satisfies the following discrete version of conditions (3) and (4):

$$\forall x \in X \forall k \in \mathbb{N}: \|P_k x\|_X \leq \tilde{d}_0 e^{d_1 k\gamma_1} \|CP_k x\|_Y,$$

and

$$\forall x \in X \forall k \in \mathbb{N} \forall t \in (0, T/2]: \|(1 - P_k)S_t x\|_X \leq \tilde{d}_2 e^{-\tilde{d}_3 k\gamma_2 t\gamma_3} \|x\|_X.$$
for constants $\tilde{d}_0, \tilde{d}_1, \tilde{d}_2 > 0$ and $\tilde{d}_2 \geq 1$, similar as in [BPS18]. Then we can apply Theorem 2.1 in the following way. Let $(P_\lambda)_{\lambda > 0}$ be defined by $P_\lambda = P_k$ for $\lambda \in (k - 1, k]$, $k \in \mathbb{N}$. Then $(P_\lambda)_{\lambda > 0}$ fulfills the assumptions (3) and (4) of Theorem 2.1 with

$$d_0 := \tilde{d}_0 e^{\tilde{d}_1}, \quad d_1 := 2^{\gamma_3} \tilde{d}_1, \quad d_2 := \tilde{d}_2, \quad d_3 := \tilde{d}_3, \quad \text{and} \quad \lambda^* = 0.$$  

**Proof of Theorem 2.1.** Assume we have shown the statement of the theorem in the case $r = 1$, i.e. for all $x \in X$ we have

$$\|S_T x\|_X \leq C_{\text{obs}} \|CS(\cdot)x\|_{L_1((0,T);Y)}, \quad \text{where} \quad C_{\text{obs}} = \frac{C_1}{T} \exp \left( \frac{C_2}{T^{\gamma_3}} + C_3 T \right).$$

Then, by Hölder’s inequality we obtain for all $r \in [1, \infty]$ and all $x \in X$

$$\|S_T x\|_X \leq C_{\text{obs}} T^{1/r'}\|CS(\cdot)x\|_{L_r((0,T);Y)},$$

where $r' \in [1, \infty]$ is such that $1/r + 1/r' = 1$. Since $T^{-1} T^{1/r'} = T^{-1/r}$ the statement of the theorem follows. Thus, it is sufficient to prove the theorem in the case $r = 1$.

For the first part of the proof, we adapt the strategy in [TT11, NTTVa] with a slight modification in order to deal with general $P_\lambda$’s instead of spectral projectors. Fix $x \in X$ arbitrary, and introduce for $t > 0$ and $\lambda > \lambda^*$ the notation

$$F(t) = \|S_t x\|_X, \quad F_\lambda(t) = \|P_\lambda S_t x\|_X, \quad F_\lambda^+(t) = \|(\text{Id} - P_\lambda) S_t x\|_X, \quad G(t) = \|CS_t x\|_{Y'}, \quad G_\lambda(t) = \|C P_\lambda S_t x\|_{Y'}, \quad G_\lambda^+(t) = \|C(\text{Id} - P_\lambda) S_t x\|_{Y'}.$$

Then for $0 \leq \tau \leq t$ we obtain

$$F(t) = \|S_t x\|_X = \|S_{t-\tau} S_\tau x\|_X \leq M e^{\omega_+ t} \|S_\tau x\|_X = M e^{\omega_+ t} F(\tau),$$

where $\omega_+ = \max \{\omega, 0\}$. Integrating this inequality, we obtain

$$F(t) \leq M e^{\omega_+ t} \frac{2}{t} \int_{t/2}^t F(\tau) d\tau.$$

We now use (3) to obtain for all $t > 0$ and $\lambda > \lambda^*$

$$F(t) \leq M e^{\omega_+ t} \frac{2}{t} \int_{t/2}^t F(\tau) d\tau \leq M e^{\omega_+ t} \frac{2}{t} \int_{t/2}^t \left( F_\lambda(\tau) + F_\lambda^+(\tau) \right) d\tau \leq M e^{\omega_+ t} \frac{2}{t} \int_{t/2}^t \left( d_0 e^{d_1 \gamma_3} G_\lambda(\tau) + F_\lambda^+(\tau) \right) d\tau.$$

By the semigroup property and assumption (4) we have for all $\tau \in (0,T]$ the estimate

$$F_\lambda^+(\tau) = \|(\text{Id} - P_\lambda) S_{\tau/2} S_{\tau/2} x\|_X \leq d_2 e^{-d_1 \gamma_3 (\tau/2)\gamma_3} F(\tau/2). \quad (6)$$

Since $F(\tau/2) \leq M e^{\omega_+ t/4} F(t/4)$ for $t > 0$ and $\tau \in [t/2, t]$, we obtain for all $t \in (0,T] \text{ and } \lambda > \lambda^*$

$$F(t) \leq \frac{2 M e^{\omega_+ t} d_0 e^{d_1 \gamma_3}}{t} \int_{t/2}^t G_\lambda(\tau) d\tau + d_2 M^2 e^{5 \omega_+ t/4} e^{-d_1 \gamma_3 (t/4)\gamma_3} F(t/4). \quad (7)$$

\[6\]
Using $G_\lambda(\tau) \leq G(\tau) + G_\lambda^l(\tau) \leq G(\tau) + \|C\|_{\mathcal{L}(X,Y)} F_\lambda^l(\tau)$ and (6) again, we obtain for all $t \in (0, T]$ and $\lambda > \lambda^*$

$$\int_{t/2}^t G_\lambda(\tau)d\tau \leq \int_{t/2}^t G(\tau)d\tau + \|C\|_{\mathcal{L}(X,Y)} d_2 e^{-d_3 \lambda^{72}(t/4)^{\gamma_3}} \int_{t/2}^t F(\tau/2)d\tau. \quad (8)$$

Since $F(\tau/2) \leq M e^{e^{1/4} \lambda^{71}} F(t/4)$ for $t > 0$ and $\tau \in [t/2, t]$ and $1 \leq e^{d_1 \lambda^{71}}$, we conclude from (7) and (8) for all $t \in (0, T]$ and $\lambda > \lambda^*$

$$F(t) \leq \frac{2 M e^{e^{1/4} \lambda^{71}} T}{t} \int_{t/2}^t G(\tau)d\tau + \frac{d_2 M^2 e^{5e^{1/4} \lambda^{71}}}{e^{d_3 \lambda^{72}(t/4)^{\gamma_3}}} (d_0 \|C\|_{\mathcal{L}(X,Y)} + 1) F(t/4).$$

With the short hand notation

$$D_1(t, \lambda) = \frac{2 M e^{e^{1/4} \lambda^{71}} T}{t} \int_{t/2}^t G(\tau)d\tau, \quad D_2(t, \lambda) = K_1 e^{d_1 \lambda^{71} - d_3 \lambda^{72}(t/4)^{\gamma_3}},$$

where $K_1 = (d_0 \|C\|_{\mathcal{L}(X,Y)} + 1) d_2 M^2 e^{5e^{1/4} \lambda^{71}}$, this can be rewritten as

$$F(t) \leq D_1(t, \lambda) + D_2(t, \lambda) F(t/4). \quad (9)$$

This inequality can be iterated. Let $(\lambda_k)_{k \in \mathbb{N}_0}$ be a sequence with $\lambda_k > \lambda^*$ for $k \in \mathbb{N}_0$. First we apply Ineq. (9) with $t = T$ and $\lambda = \lambda_0$. The term $F(4^{-1}T)$ on the right hand side is then estimated by Ineq. (9) with $t = 4^{-1}T$ and $\lambda = \lambda_1$. This way, we obtain after two steps

$$F(T) \leq D_1(T, \lambda_0) + D_2(T, \lambda_0) (D_1(4^{-1}T, \lambda_1) + D_2(4^{-1}T, \lambda_1) F(4^{-2}T))$$

$$= D_1(T, \lambda_0) + D_1(4^{-1}T, \lambda_1) D_2(T, \lambda_0) + D_2(T, \lambda_0) D_2(4^{-1}T, \lambda_1) F(4^{-2}T).$$

After $N + 1$ steps of this type we obtain

$$F(T) \leq D_1(T, \lambda_0) + \sum_{k=1}^N D_1(4^{-k}T, \lambda_k) \prod_{l=0}^{k-1} D_2(4^{-l}T, \lambda_l) + F(4^{-N-1}T) \prod_{k=0}^{N} D_2(4^{-k}T, \lambda_k). \quad (10)$$

We now choose the sequence $(\lambda_k)_{k \in \mathbb{N}_0}$ given by $\lambda_k = \nu \alpha^k$ with

$$\alpha = \begin{cases} \alpha_0, & \text{if } T \leq T_0, \\ \alpha_0 \left( \frac{T}{T_0} \right)^{\frac{3}{2}}, & \text{if } T > T_0, \end{cases} \quad \text{and} \quad \nu = \begin{cases} \nu_0 \left( \frac{T}{T} \right)^{\frac{3}{2} - \gamma_1}, & \text{if } T \leq T_0, \\ \nu_0 \left( \frac{T}{T} \right)^{\frac{3}{2} - \gamma_1}, & \text{if } T > T_0. \end{cases}$$

where

$$\alpha_0 := (2 \cdot 4^{73})^{\frac{1}{72 - \gamma_1}}, \quad \nu_0 := \max \left\{ \left( \frac{2 \ln(4K_1)}{e \ln(2)d_1} \right)^{\frac{1}{3}}, 2\lambda^* \right\}, \quad \text{and} \quad T_0 := \left( \frac{2d_1 \alpha_0^{72}}{d_3 \nu_0^{72 - \gamma_1}} \right)^{\frac{1}{73}}.$$

With this notation we have (in both cases $T \leq T_0$ and $T > T_0$) the equality

$$d_3 T^{73} \nu^{72 - \gamma_1} = 2d_1 \alpha^{72}, \quad (11)$$
which we will use frequently in the following. Moreover, the choice of $\alpha$ and $\nu$ ensures that the constants
\[
K_2 := d_3 \left( \frac{T}{4} \right)^{\gamma_3} \nu^{\gamma_2} - d_1 \nu^{\gamma_1} \quad \text{and} \quad K_3 := \frac{K_2}{\alpha^{\gamma_2/4\gamma_3} - 1} - d_1 \nu^{\gamma_1}
\]
are positive. Indeed, using (11) we find
\[
K_3 = \frac{d_3 (T/4)^{\gamma_3} \nu^{\gamma_2} - \alpha^{\gamma_2} d_1 \nu^{\gamma_1}/4^{\gamma_3}}{\alpha^{\gamma_2/4\gamma_3} - 1} = \nu^{\gamma_1} \frac{d_1 \alpha^{\gamma_2/4\gamma_3}}{\alpha^{\gamma_2/4\gamma_3} - 1} - 1.
\]
Since $\alpha^{\gamma_2} > 2 \cdot 4^{\gamma_3}$ we conclude that $K_3$ is positive. Note that $K_2 > K_3$, hence $K_2$ is positive as well. Let us now show that the right hand side in (10) converges for $N \to \infty$. Since $\alpha^{\gamma_1} \leq \alpha^{\gamma_2/4\gamma_3}$ we have
\[
\prod_{k=0}^{N} D_2(4^{-k}T, \lambda_k) = \prod_{k=0}^{N} K_1 \exp \left( d_1 \nu^{\gamma_1} \alpha^{\gamma_1 k} - d_3 \left( \frac{T}{4} \right)^{\gamma_3} \nu^{\gamma_2} \left( \frac{\alpha^{\gamma_2}}{4^{\gamma_3}} \right)^k \right)
\]
\[
\leq K_1^{N+1} \prod_{k=0}^{N} \exp \left( -K_2 \left( \frac{\alpha^{\gamma_2}}{4^{\gamma_3}} \right)^k \right).
\]
(12)
Since $K_1, K_2 > 0$ and $\alpha^{\gamma_2/4\gamma_3} > 1$ this tends to zero as $N$ tends to infinity. Moreover, using (12) and $\alpha^{\gamma_1} \leq \alpha^{\gamma_2/4\gamma_3}$, we infer that the middle term of the right hand side of (10) satisfies
\[
\sum_{k=1}^{N} D_1(4^{-k}T, \lambda_k) \prod_{\ell=0}^{k-1} D_2(4^{-\ell}T, \lambda_\ell)
\]
\[
\leq 2M e^\omega T d_0 \frac{1}{T} \int_0^T G(\tau)d\tau \sum_{k=1}^{N} (4K_1)^k \exp \left( -K_2 \left( \frac{\alpha^{\gamma_2/4\gamma_3}}{\alpha^{\gamma_2/4\gamma_3} - 1} + d_1 \nu^{\gamma_1} \alpha^{\gamma_1 k} \right) \right)
\]
\[
\leq 2M e^\omega T d_0 \frac{1}{T} \int_0^T G(\tau)d\tau \exp \left( \frac{K_2}{\alpha^{\gamma_2/4\gamma_3} - 1} \right) \sum_{k=1}^{N} (4K_1)^k \exp \left( -K_3 \left( \frac{\alpha^{\gamma_2}}{4^{\gamma_3}} \right)^k \right).
\]
Since $K_3 > 0$ the right hand side converges as $N$ tends to infinity, and we obtain from (10) that
\[
\|S_t x\|_X \leq \tilde{C}_{\text{obs}} \int_0^T \|CS_t y\|_Y dt,
\]
where
\[
\tilde{C}_{\text{obs}} = \frac{2M e^\omega T d_0}{T} \left( e^{d_1 \nu^{\gamma_1}} + \exp \left( \frac{K_2}{\alpha^{\gamma_2/4\gamma_3} - 1} \right) \sum_{k=1}^{\infty} (4K_1)^k \exp \left( -K_3 \left( \frac{\alpha^{\gamma_2}}{4^{\gamma_3}} \right)^k \right) \right).
\]
It remains to show the upper bound $\tilde{C}_{\text{obs}} \leq C_{\text{obs}}$ with $C_{\text{obs}}$ as in the theorem. To this end, we note that for all $A > 1$ and $B > 0$ we have
\[
\sum_{k=1}^{\infty} A^k e^{-B^2 k} \leq \sup_{x \geq 1} A^x e^{-\frac{B}{2^x}} \sum_{k=1}^{\infty} e^{-\frac{B}{2^k}} = \left( \frac{2 \ln(A)}{B \ln(2)} \right)^{\frac{\ln(A)}{\ln(2)}} \sum_{k=1}^{\infty} e^{-\frac{B}{2^k}}.
\]
where the last identity follows from elementary calculus. Using $2^k \geq 2k$ and $e^B - 1 \geq B$ we further estimate

$$\sum_{k=1}^{\infty} e^{-\frac{B}{2}2^k} \leq \sum_{k=1}^{\infty} e^{-kB} = \frac{e^{-B}}{1 - e^{-B}} \leq \frac{1}{B}.$$  

Hence, we find

$$\sum_{k=1}^{\infty} A^k e^{-B2^k} \leq \left( \frac{2 \ln(A)}{Be \ln(2)} \right)^{\ln(A)} \frac{1}{B}.$$  

(13)

We now apply inequality (13) with $A = 4K_1$ and $B = K_3$ and obtain by using $\alpha^{7/2}/4^{3/2} \geq 2$

$$\tilde{C}_{\text{obs}} \leq \frac{2Me^{\nu+T}d_0}{T} \left(e^{d_1\nu_1} + \exp \left( \frac{K_2}{\alpha^{7/2}/4^{3/2} - 1} \right) \right) \leq \frac{4Md_0}{Te^{-\omega+T}} \exp \left( \frac{K_2}{\alpha^{7/2}/4^{3/2} - 1} \right).$$  

(15)

For $K_3$ we have the lower bound

$$K_3 = \nu^{\gamma_1} \frac{d_1\alpha^{7/2}/4^{3/2}}{\alpha^{7/2}/4^{3/2} - 1} \geq \nu^{\gamma_1} d_1 \geq \nu_0^{\gamma_1} d_1 = \frac{2\ln(4K_1)}{e \ln(2)}.$$  

(14)

Since $Me^{\omega+T}, d_2 \geq 1$ we have $K_1 \geq 1$, hence $\ln(4K_1) \geq \ln 4$ and $K_3 > 1$. From this, Ineq. (14), and $d_1\nu_1 \leq d_1\nu_1 + K_3 = K_2/(\alpha^{7/2}/4^{3/2} - 1)$, we conclude

$$\tilde{C}_{\text{obs}} \leq \frac{2Md_0}{Te^{-\omega+T}} \left(e^{d_1\nu_1} + \exp \left( \frac{K_2}{\alpha^{7/2}/4^{3/2} - 1} \right) \right) \leq \frac{4Md_0}{Te^{-\omega+T}} \exp \left( \frac{K_2}{\alpha^{7/2}/4^{3/2} - 1} \right).$$  

(15)

For the constant $K_2$ we calculate, using (11) and $\alpha^{7/2}/4^{3/2} - 1 \geq (1/2)\alpha^{7/2}/4^{3/2}$,

$$\frac{K_2}{\alpha^{7/2}/4^{3/2} - 1} = \frac{d_3(T/4)^{3/2} \nu^{\gamma_2} - d_1\nu_1^{\gamma_2}}{\alpha^{7/2}/4^{3/2} - 1} \leq \frac{d_3(T/4)^{3/2} \nu^{\gamma_2}}{\alpha^{7/2}/4^{3/2} - 1} = \nu^{\gamma_1} d_1\alpha^{7/2}/4^{3/2} \alpha^{7/2}/4^{3/2} - 1 \leq 4d_1\nu^{\gamma_1}.$$

By our choice of $\nu$ we have

$$\nu^{\gamma_1} = \begin{cases} (2d_1\alpha_0^{7/2}d_3^{1/2})^{\gamma_1/(7/2-\gamma_1)} \left( \frac{1}{T} \right)^{\gamma_1/\gamma_2 - \gamma_1} & \text{if } T \leq T_0, \\
\max \left\{ \frac{2\ln(4K_1)}{e \ln(2)} (2\lambda^*)^{\gamma_1} \right\} & \text{if } T > T_0, \end{cases}$$

and hence

$$\frac{K_2}{\alpha^{7/2}/4^{3/2} - 1} \leq 4 \left( \frac{2^{7/2}\alpha_0^{7/2}d_1^{1/2}/d_3^{3/2}}{T^{7/2}4^{3/2}} \right)^{\gamma_1/\gamma_2 - \gamma_1} + \max \left\{ \frac{8\ln(4K_1)}{e \ln(2)}, 4d_1 (2\lambda^*)^{\gamma_1} \right\}.$$  

(16)

From Ineqs. (15) and (16) we conclude

$$\tilde{C}_{\text{obs}} \leq \frac{4Md_0}{Te^{-\omega+T}} \exp \left( 4 \left( \frac{2^{7/2}\alpha_0^{7/2}d_1^{1/2}/d_3^{3/2}}{T^{7/2}4^{3/2}} \right)^{\gamma_1/\gamma_2 - \gamma_1} \right) \max \left\{ (4K_1)^{8/(e \ln 2)}, e^{4d_1 (2\lambda^*)^{\gamma_1}} \right\}.$$ 

Finally, we insert the values of $\alpha_0$ and $K_1$ and factor out $e^{10\omega+T/(e \ln 2)}$ from the maximum to obtain the assertion. \(\square\)
3 Sharp geometric conditions for observability of elliptic operators in $L_p(\mathbb{R}^d)$

In this section we consider the case where $X = L_p(\mathbb{R}^d)$, $Y = L_p(\omega)$ with $1 < p < \infty$, $A_p$ is an elliptic operator in $L_p(\mathbb{R}^d)$ associated with a strongly elliptic polynomial in $\mathbb{R}^d$ of degree $m \geq 2$, $(S_t)_{t \geq 0}$ the $C_0$-semigroup on $L_p(\mathbb{R}^d)$ generated by $-A_p$, and where $C = 1_\omega$ is the restriction operator of a function in $L_p(\mathbb{R}^d)$ to some measurable subset $\omega \subset \mathbb{R}^d$, i.e. $1_\omega \in \mathcal{L}(L_p(\mathbb{R}^d), L_p(\omega))$ and $1_\omega f = f$ on $\omega$. Let $T > 0$. Our goal is to show that the system

$$
\begin{align*}
\dot{x}(t) &= -A_p x(t), \quad t \in (0, T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d), \\
y(t) &= 1_\omega x(t), \quad t \in [0, T],
\end{align*}
$$

(17)

satisfies a final state observability estimate in $L_r((0, T); L_p(\mathbb{R}^d))$, $r \in [1, \infty]$ if and only if $\omega$ is a so-called thick set, cf. Definition 3.2. In particular, if $\omega$ is a thick set, we conclude an observability estimate with an explicit dependence of $C_{obs}$ on $T$, the order $m$ of the operator $A_p$, and the geometry of the set $\omega$.

We start by recalling the class of elliptic operators $A_p$ which we consider. We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions, which is dense in $L_p(\mathbb{R}^d)$ for all $1 < p < \infty$. For $f \in \mathcal{S}(\mathbb{R}^d)$ let $\mathcal{F} f : \mathbb{R}^d \to \mathbb{C}$ be the Fourier transform of $f$ defined by

$$
\mathcal{F} f(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} \, dx.
$$

Then $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is bijective, continuous and has a continuous inverse, given by

$$
\mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\xi) e^{i\xi \cdot x} \, d\xi
$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$. Let $a : \mathbb{R}^d \to \mathbb{C}$ be a homogeneous strongly elliptic polynomial of degree $m \geq 2$, that is, $a$ is of the form

$$
a(\xi) = \sum_{|\alpha| = m} a_\alpha i^{|\alpha|} \xi^\alpha
$$

for given $a_\alpha \in \mathbb{C}$, and there is $c > 0$ such that for all $\xi \in \mathbb{R}^d$ we have

$$
\Re a(\xi) \geq c|\xi|^m.
$$

Note that this implies that $m$ is even. For $f \in \mathcal{S}(\mathbb{R}^d)$ define $A f \in \mathcal{S}(\mathbb{R}^d)$ by

$$
A f := \sum_{|\alpha| = m} a_\alpha \partial^\alpha f = \mathcal{F}^{-1}(a \mathcal{F} f).
$$

Then, for every $1 < p < \infty$, $A$ is closable in $L_p(\mathbb{R}^d)$, and its closure $A_p$ is a sectorial operator of angle $\omega_a < \pi/2$. As a consequence, $-A_p$ generates a bounded $C_0$-semigroup $(S_t)_{t \geq 0}$ on $L_p(\mathbb{R}^d)$. We call $A_p$ the elliptic operator associated with $a$. For details we refer, e.g., to the book [Haa06].
**Example 3.1.** Let $1 < p < \infty$ and $a: \mathbb{R}^d \to \mathbb{R}$ defined by $a(\xi) = |\xi|^2$. Then $a$ is a homogeneous strongly elliptic polynomial of degree $m = 2$ and $A_p = -\Delta$ is the negative Laplacian in $L_p(\mathbb{R}^d)$.

More generally, let $(a_{i,j}) \in \mathbb{R}^{d \times d}$ be a symmetric and negative definite matrix, and define $a: \mathbb{R}^d \to \mathbb{R}$ by $a(\xi) = \xi^\top (a_{i,j}) \xi$ for all $\xi \in \mathbb{R}^d$. Then $a$ is a homogeneous strongly elliptic polynomial of degree $m = 2$ and $A_p = -\text{div} (a_{i,j}) \text{grad}$ is the corresponding elliptic operator in $L_p(\mathbb{R}^d)$.

The following definition characterizes the class of subsets $\omega \subset \mathbb{R}^d$ which we consider.

**Definition 3.2.** Let $\rho \in (0, 1]$ and $L \in (0, \infty)^d$. A set $\omega \subset \mathbb{R}^d$ is called $(\rho, L)$-thick if $\omega$ is measurable and for all $x \in \mathbb{R}^d$ we have
\[
|\omega \cap \left( \prod_{i=1}^d (0, L_i) + x \right) | \geq \rho \prod_{i=1}^d L_i.
\]

Here, $|\cdot|$ denotes Lebesgue measure in $\mathbb{R}^d$. Moreover, $\omega \subset \mathbb{R}^d$ is called thick if there are $\rho \in (0, 1)$ and $L \in (0, \infty)^d$ such that $\omega$ is $(\rho, L)$-thick.

We are now in position to state our main theorems of this section.

**Theorem 3.3.** Let $1 < p < \infty$, $r \in [1, \infty]$, $a: \mathbb{R}^d \to \mathbb{C}$ a homogeneous strongly elliptic polynomial in $\mathbb{R}^d$ of degree $m \geq 2$, $A_p$ the associated elliptic operator in $L_p(\mathbb{R}^d)$, $(S_t)_{t \geq 0}$ the bounded $C_0$-semigroup on $L_p(\mathbb{R}^d)$ generated by $-A_p$, $\omega \subset \mathbb{R}^d$ a $(\rho, L)$-thick set, and $T > 0$. Then the system (17) satisfies a final state observability estimate in $L_r((0, T); L_p(\mathbb{R}^d))$. In particular, we have for all $x_0 \in L_p(\mathbb{R}^d)$
\[
\|S_T x_0\|_{L_p(\mathbb{R}^d)} \leq C_{\text{obs}} \|1_{\omega} S_t(\cdot) x_0\|_{L_r((0, T); L_p(\omega))}
\]

with
\[
C_{\text{obs}} = \frac{D_1 M^{16}}{T^{1/r}} \left( \frac{K^d}{\rho} \right)^{D_2} \exp \left( \frac{D_3 (|L|_1 \ln (K^d/\rho))^{m/(m-1)}}{(cT)^{m-1}} \right),
\]

where $K \geq 1$ is a universal constant, $D_1, D_2 \geq 1$ depending on $d$, $D_3 \geq 1$ depending on $d$ and $p$, $M = \sup_{t \geq 0} \|S_t\|$, and where $c > 0$ is such that $\Re a(\xi) \geq c|\xi|^m$ for all $\xi \in \mathbb{R}^d$.

The universal constant $K$ in Theorem 3.3 can be chosen to be the same as the constant $K$ in the Logvinenko-Sereda Theorem 3.5. Theorem 3.3 shows that the system (17) satisfies a final state observability estimate if $\omega$ is a thick set. The following theorem shows converse:

If the system (17) satisfies a final state observability estimate, then the set $\omega$ is necessarily a thick set.

**Theorem 3.4.** Let $1 < p < \infty$, $r \in [1, \infty]$, $a: \mathbb{R}^d \to \mathbb{C}$ a homogeneous strongly elliptic polynomial in $\mathbb{R}^d$ of degree $m \geq 2$, $A_p$ the associated elliptic operator in $L_p(\mathbb{R}^d)$, $(S_t)_{t \geq 0}$ the bounded $C_0$-semigroup on $L_p(\mathbb{R}^d)$ generated by $-A_p$, $\omega \subset \mathbb{R}^d$ measurable, $T > 0$, and assume there exists $C_{\text{obs}} > 0$ such that for all $x_0 \in L_p(\mathbb{R}^d)$ we have
\[
\|S_T x_0\|_{L_p(\mathbb{R}^d)} \leq C_{\text{obs}} \|1_{\omega} S_t(\cdot) x_0\|_{L_r((0, T); L_p(\omega))}.
\]

Then $\omega$ is a thick set.
In order to prove Theorem 3.3 we apply Theorem 2.1 in the case where \( X = L_p(\mathbb{R}^d), Y = L_p(\omega), C \in \mathcal{L}(X,Y) \) is the restriction operator of functions from \( \mathbb{R}^d \) to \( \omega \), and \( A = A_p \). To this end we define a family \( (P_\lambda)_{\lambda > 0} \) of operators in \( L_p(\mathbb{R}^d) \) such that the assumptions of Theorem 2.1, i.e. the uncertainty relation (3) and the dissipation estimate (4), are satisfied. Concerning the dissipation estimate we first consider the case \( p = 2 \) and then apply the Riesz-Thorin interpolation theorem. For the uncertainty relation we shall need a so-called Logvinenko-Sereda theorem. It has originally been proven by Logvinenko and Sereda in [LS74], and significantly improved by Kovrijkine in [Kov00, Kov01]. Recently, it has been adapted to functions on the torus instead of \( \mathbb{R}^d \), see [EV]. We quote a special case of Theorem 1 from [Kov01].

**Theorem 3.5** (Logvinenko-Sereda theorem). There exists \( K \geq 1 \) such that for all \( p \in [1, \infty] \), all \( \lambda > 0 \), all \( \rho \in (0,1] \), all \( L \in (0, \infty)^d \), and all \( \rho, L \)-thick sets \( \omega \subset \mathbb{R}^d \), and all \( f \in \mathcal{S}(\mathbb{R}^d) \) satisfying \( \text{supp} \mathcal{F}f \subset [-\lambda, \lambda]^d \) we have

\[
\|f\|_{L_p(\mathbb{R}^d)} \leq (\frac{p}{K^d})^{-K(d+2\lambda|L|_1)} \|f\|_{L_p(\omega)}.
\]

We now proceed with the proofs of Theorems 3.3 and 3.4.

**Proof of Theorem 3.3.** We apply Theorem 2.1 in the case where \( X = L_p(\mathbb{R}^d), Y = L_p(\omega), C \in \mathcal{L}(X,Y) \) is the restriction operator of functions from \( \mathbb{R}^d \) to \( \omega \), and \( A = A_p \). For this purpose we define a family \( (P_\lambda)_{\lambda > 0} \) of operators in \( L_p(\mathbb{R}^d) \) such that the assumptions of Theorem 2.1 are satisfied. Let \( \eta \in \mathcal{C}_c^\infty([0, \infty)) \) with \( 0 \leq \eta \leq 1 \) such that \( \eta(r) = 1 \) for \( r \in [0, 1/2] \) and \( \eta(r) = 0 \) for \( r \geq 1 \). For \( \lambda > 0 \) we define \( \chi_\lambda : \mathbb{R}^d \to \mathbb{R} \) by \( \chi_\lambda(\xi) = \eta(|\xi|/\lambda) \). Since \( \chi_\lambda \in \mathcal{S}(\mathbb{R}^d) \) for all \( \lambda > 0 \) we have \( \mathcal{F}^{-1}\chi_\lambda \in \mathcal{S}(\mathbb{R}^d) \). For \( \lambda > 0 \) we define \( P_\lambda : L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d) \) by \( P_\lambda f = (2\pi)^{-d/2}(\mathcal{F}^{-1}\chi_\lambda) * f \). By Young’s inequality we have for all \( f \in L_p(\mathbb{R}^d) \)

\[
\|P_\lambda f\|_{L_p(\mathbb{R}^d)} = \| (2\pi)^{-d/2}(\mathcal{F}^{-1}\chi_\lambda) * f \|_{L_p(\mathbb{R}^d)} \leq (2\pi)^{-d/2}\|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)}\|f\|_{L_p(\mathbb{R}^d)}.
\]

Moreover, the norm \( \|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)} \) is independent of \( \lambda > 0 \). Indeed, by the scaling property of the Fourier transform and by change of variables we have for all \( \lambda > 0 \)

\[
\|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)} = |\lambda|^d\|(\mathcal{F}^{-1}\chi_\lambda)(\lambda\cdot)\|_{L_1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)}.
\]

Hence, for all \( \lambda > 0 \) the operator \( P_\lambda \) is a bounded linear operator and the family \( (P_\lambda)_{\lambda > 0} \) is uniformly bounded by \( C_d := (2\pi)^{-d/2}\|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)} \). For all \( f \in \mathcal{S}(\mathbb{R}^d) \) we have by construction \( P_\lambda f \in \mathcal{S}(\mathbb{R}^d), \mathcal{F}P_\lambda f = \chi_\lambda \mathcal{F}f \in \mathcal{S}(\mathbb{R}^d) \) and \( \text{supp} \mathcal{F}P_\lambda f \subset \{ y \in \mathbb{R}^d : |y| \leq \lambda \} \subset [-\lambda, \lambda]^d \). By Theorem 3.5 we obtain for all \( \lambda > 0 \) and all \( f \in \mathcal{S}(\mathbb{R}^d) \)

\[
\|P_\lambda f\|_{L_p(\mathbb{R}^d)} \leq d_0 e^{d_1 \lambda} \|P_\lambda f\|_{L_p(\omega)},
\]

where

\[
d_0 = e^{-Kd \ln(\rho/K^d)} \quad \text{and} \quad d_1 = -2K|L|_1 \ln \left( \frac{\rho}{K^d} \right).
\]

Since \( \mathcal{S}(\mathbb{R}^d) \) is dense in \( L_p(\mathbb{R}^d) \) and \( P_\lambda \) is bounded, Ineq. (18) holds for all \( f \in L_p(\mathbb{R}^d) \). Thus, the uncertainty relation (3) of Theorem 2.1 is satisfied with \( d_0 \) and \( d_1 \) as in (18), \( \gamma_1 = 1 \), and \( \lambda^* = 0 \).
It remains to verify the dissipation estimate. Since $\mathcal{F}S_t\mathcal{F}^{-1}f = e^{-\lambda t}f$ for $f \in \mathcal{S}(\mathbb{R}^d)$ by functional calculus arguments, see e.g. Section 8 in [Haa06], and since the Fourier transform is an isometry in $L_2(\mathbb{R}^d)$, we obtain for all $f \in \mathcal{S}(\mathbb{R}^d)$ and all $\lambda > 0$

$$\| (1 - \Lambda)S_t f \|_{L_2(\mathbb{R}^d)} = \| \mathcal{F}^{-1}(1 - \chi)e^{-\lambda t}\mathcal{F}f \|_{L_2(\mathbb{R}^d)} \leq \| (1 - \chi)e^{-\lambda t}\|_{L_\infty(\mathbb{R}^d)} \| f \|_{L_2(\mathbb{R}^d)}.$$  

Since $1_{B(0,\lambda/2)} \leq \lambda \leq 1$, $\text{Re}(\xi) \geq c|\xi|_m$ for all $\xi \in \mathbb{R}^d$, and since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d)$ this yields for all $f \in L_2(\mathbb{R}^d)$

$$\| (1 - \Lambda)S_t f \|_{L_2(\mathbb{R}^d)} \leq e^{-c\lambda(\lambda/2)^m} \| f \|_{L_2(\mathbb{R}^d)}. \tag{19}$$

This shows that the dissipation estimate (4) of Theorem 2.1 is satisfied if $p = 2$. In order to treat the case $p \neq 2$ we apply the Riesz-Thorin interpolation theorem. Let

$$p_0 := \begin{cases} p^2 - 2p + 2 & \text{if } p \in (1, 2), \\ 2p & \text{if } p \in (2, \infty), \end{cases} \quad \text{and} \quad \theta := \begin{cases} -2p^2 + 6p - 1 & \text{if } p \in (1, 2), \\ 1 & \text{if } p \in (2, \infty). \end{cases}$$

If $p = 2$ we set $p_0 := 2$ and $\theta := 1$ for convenience. Then $p_0 \in (1, \infty)$, $\theta \in (0, 1]$ and

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{2}.$$  

Since the family $(P_\lambda)_{\lambda > 0}$ is uniformly bounded by $C_d$, we have for all $f \in L_{p_0}(\mathbb{R}^d)$ and all $\lambda > 0$

$$\| (1 - \Lambda)S_t f \|_{L_{p_0}(\mathbb{R}^d)} \leq (1 + C_d) M \| f \|_{L_{p_0}(\mathbb{R}^d)},$$

where $M = \sup_{t \geq 0} \| S_t \|$. Interpolation between $L_2(\mathbb{R}^d)$ and $L_{p_0}(\mathbb{R}^d)$ if $p \neq 2$, and Ineq. (19) if $p = 2$ now yields for all $f \in L_p(\mathbb{R}^d)$

$$\| (1 - \Lambda)S_t f \|_{L_p(\mathbb{R}^d)} \leq d_2 e^{-d_3 t \lambda^m} \| f \|_{L_p(\mathbb{R}^d)}, \tag{20a}$$

where

$$d_2 = (1 + C_d)^{1 - \theta} M_1^{1 - \theta} \quad \text{and} \quad d_3 = c\theta/2^m. \tag{20b}$$

Thus, the dissipation estimate (3) of Theorem 2.1 is satisfied with $d_2$ and $d_3$ as in (20), $\gamma_2 = m$, $\gamma_3 = 1$, and $\lambda^* = 0$. Since $\gamma_2 = m > 1 = \gamma_1$, we conclude from the uncertainty relation (18), the dissipation estimate (20), and Theorem 2.1, that the statement of the theorem holds with

$$C_{\text{obs}} = \frac{4Md_0(4M_1)^{\frac{3}{T^{1/r}}}}{T^{1/r}} \exp \left( \frac{4(2 \cdot 8 \frac{m}{T \cdot 1} d_0/d_3)^{\frac{m}{T \cdot 1}}}{T^{m-1}} \right),$$

where $K_1 = (d_0 + 1)M_2 d_2$, and $T^{1/r} = 1$ if $r = \infty$. From the definitions of $d_i$, $i \in \{0, 1, 2, 3\}$, and straightforward estimates we obtain $C_{\text{obs}} \leq C_{\text{obs}}$ with $C_{\text{obs}}$ as in the theorem. 

**Remark 3.6.** As the proof shows we can obtain an explicit dependence of $C_{\text{obs}}$ on $p$. Then, it turns out that $C_{\text{obs}} \to \infty$ as $p \to 1$ and as $p \to \infty$. This shows that our method of proof is only valid for $p \in (1, \infty)$. 

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\textbf{Proof of Theorem 3.4.} We improve the strategy developed in [EV18]. We show the contraposition. Assume that \( \omega \) is not thick. Then there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) in \( \mathbb{R}^d \) such that for all \( n \in \mathbb{N} \) we have
\[
|\omega \cap B(x_n, n)| < \frac{1}{n}.
\] (21)

Note that \( e^{-ta} \in \mathcal{S}(\mathbb{R}^d) \) for \( t > 0 \). Thus, for \( t > 0 \), the operator \( S_t \) is given as a convolution operator with convolution kernel \( p_t := (1/(2\pi)^{d/2})F^{-1}e^{-ta} \in \mathcal{S}(\mathbb{R}^d) \) for \( t > 0 \). Indeed, for \( f \in \mathcal{S}(\mathbb{R}^d) \) and \( t > 0 \) we have \( e^{-ta}Ff \in \mathcal{S}(\mathbb{R}^d) \) and
\[
S_t f = F^{-1}e^{-ta}F f = (2\pi)^{d/2}F^{-1}(Fp_t Ff) = p_t * f,
\]
and the claim follows by density. For \( n \in \mathbb{N} \) we define \( f_n := p_t(\cdot - x_n) \). As a consequence, we observe for all \( t > 0 \) and \( n \in \mathbb{N} \)
\[
S_t f_n = p_t * f_n = p_t * p_t(\cdot - x_n) = p_{t+1}(\cdot - x_n),
\] (22)
and hence by translation invariance of the Lebesgue measure
\[
\|S_T f_n\|_{L^p(\mathbb{R}^d)} = \|p_{T+1}(\cdot - x_n)\|_{L^p(\mathbb{R}^d)} = \|p_{T+1}\|_{L^p(\mathbb{R}^d)}.
\] (23)

For \( n \in \mathbb{N} \) we now shift the set \( \omega \) by \( x_n \) and consider the set \( \omega - x_n = \{y \in \mathbb{R}^d : y + x_n \in \omega\} \). Note that (21) is equivalent to \( |(\omega - x_n) \cap B(0, n)| < 1/n \) for all \( n \in \mathbb{N} \). From the latter fact, Eq. (22) and substitution we obtain for all \( t > 0 \) and \( n \in \mathbb{N} \)
\[
\|1_{\omega} S_t f_n\|_{L^p(\mathbb{R}^d)}^p = \|1_{(\omega-x_n) \cap B(0,n)} p_{t+1}\|_{L^p(\mathbb{R}^d)}^p + \|1_{(\omega-x_n)} (1 - 1_{B(0,n)}) p_{t+1}\|_{L^p(\mathbb{R}^d)}^p
\leq \|p_{t+1}\|_{L^p(\mathbb{R}^d)}^p |(\omega - x_n) \cap B(0, n)| + \|1_{B(0,n)} p_{t+1}\|_{L^p(\mathbb{R}^d)}^p
\leq \frac{1}{n} \|p_{t+1}\|_{L^p(\mathbb{R}^d)}^p + \|1_{B(0,n)} p_{t+1}\|_{L^p(\mathbb{R}^d)}^p.
\] (24)

Since \( a \) is a homogeneous polynomial, we have by substitution \( p_t(x) = t^{-d/m} p_1(x/t^{1/m}) \). Hence, we find for all \( t > 0 \)
\[
\|p_{t+1}\|_{L^\infty(\mathbb{R}^d)} = \frac{1}{(t+1)^{d/m}} \|p_1\|_{L^\infty(\mathbb{R}^d)} \leq \|p_1\|_{L^\infty(\mathbb{R}^d)}.
\] (25)

Moreover, it follows for all \( t \in (0, T] \) that
\[
\|(1 - 1_{B(0,n)}) p_{t+1}\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} (1 - 1_{B(0,n)}(x)) \left| \frac{p_1(x/(t+1)^{1/m})}{(t+1)^{d/m}} \right|^p dx
\leq \int_{\mathbb{R}^d} (1 - 1_{B(0,n/(t+1)^{1/m})}(x)) \left| p_1(x) \right|^p \frac{1}{(t+1)^{(p-1)d/m}} dx
\leq \int_{\mathbb{R}^d} (1 - 1_{B(0,n/(T+1)^{1/m})}(x)) \left| p_1(x) \right|^p dx.
\] (26)

From (24), (25) and (26) (and since \( p_1 \) is a Schwartz function and hence integrable), we obtain that
\[
\|1_{\omega} S_t f_n\|_{L^p((0,T);L^p(\mathbb{R}^d))} \rightarrow 0
\] (27)
as \( n \) tends to infinity. From (23) and (27) we conclude that for all \( C_{\text{obs}} > 0 \) there exists \( x_0 \in L^p(\mathbb{R}^d) \) such that
\[
\|S_T x_0\|_{L^p(\mathbb{R}^d)} > C_{\text{obs}} \|1_{\omega} S_T \|_{L^p((0,T);L^p(\mathbb{R}^d))}.
\]
This proves the contraposition of the theorem. \( \square \)
4 Null-controllability and control costs

Let $X$ and $U$ be Banach spaces, $(S_t)_{t \geq 0}$ be a $C_0$-semigroup on $X$, $-A$ the corresponding infinitesimal generator on $X$, $B \in \mathcal{L}(U,X)$, and $T > 0$. We consider the linear control system

$$x'(t) = -Ax(t) + Bu(t), \quad t \in (0,T], \quad x(0) = x_0 \in X,$$  \hspace{1cm} (28)

where $u \in L_r((0,T);U)$ with $1 \leq r \leq \infty$. The function $x$ is called state function and $u$ is called control function. The unique mild solution of (28) is given by Duhamel’s formula

$$x(t) = S_tx_0 + \int_0^t S_{t-\tau}Bu(\tau)d\tau, \quad t \in [0,T].$$

We say that the system (28) is null-controllable in time $T$ via $L_r((0,T);U)$ if for all $x_0 \in X$ there exists $u \in L_r((0,T);U)$ such that $x(T) = 0$. The controllability map is given by $B^T: L_r((0,T);U) \rightarrow X$,

$$B^T f := \int_0^T S_{T-\tau}Bu(\tau)d\tau.$$  \hspace{1cm} (29)

Note that we suppress the dependence of $B^T$ on $r$. The system (28) is null-controllable in time $T$ via $L_r((0,T);U)$ if and only if $\text{Ran } B^T \supset \text{Ran } S_T$. This gives an alternative definition of null-controllability.

Denote by $A'$ in $X'$ the dual operator of $A$, and by $B' \in \mathcal{L}(X',U')$ the dual operator of $B$. It is well known that null-controllability of the system (28) is in certain situations equivalent to final state observability of its adjoint or dual system

$$\dot{\varphi}(t) = -A'\varphi(t), \quad \varphi(0) = \varphi_0 \in X',$$

$$\psi(t) = B'\varphi(t), \quad t \in [0,T].$$  \hspace{1cm} (30)

Recall that the system (30) satisfies a final state observability estimate in $L_{r'}((0,T);U')$, $r' \in [1,\infty]$, if there exists $C_{\text{obs}} > 0$ such that for all $\varphi_0 \in X'$ we have $\|\varphi(T)\|_{X'} \leq C_{\text{obs}}\|\varphi\|_{L_{r'}((0,T);U')}$. This equivalence can be described in an abstract form due to Douglas [Dou66] and Dolecki and Russell [DR77], see in particular Theorem 2.5 and Section 5 in [DR77].

Lemma 4.1 ([Dou66, DR77]). Let $V,W,Z$ be reflexive Banach spaces and $F \in \mathcal{L}(V,Z)$, $G \in \mathcal{L}(W,Z)$. Then the following are equivalent:

(a) $\text{Ran } F \subset \text{Ran } G$.

(b) There exists $c_1 > 0$ such that $\|F'z'\|_V \leq c_1\|G'z'\|_{W'}$ for all $z' \in Z'$.

(c) There exists $H: \text{Ran } G' \rightarrow V'$ and $c_2 > 0$ such that $HG' = F'$ and $\|Hw'\|_{V'} \leq c_2\|w'\|_{W'}$ for all $w' \in \text{Ran } G'$.

Moreover, in (b) and (c) we can choose $c_1 = c_2$. 

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In particular, if \( V = Z = X, W = L_r((0,T); U) \), \( F = S_T \) and \( G = B^T \), statement (a) of Lemma 4.1 is equivalent to the fact that the system (28) is null-controllable in time \( T > 0 \) via \( L_r((0,T); U) \), while statement (b) of Lemma 4.1 is equivalent to the fact that the system (30) satisfies a final state observability estimate in \( L_r((0,T); U') \), where \( 1/r + 1/r' = 1 \) provided \( r \in (1, \infty) \). Thus, if \( X \) and \( U \) are reflexive and \( r \in (1, \infty) \), Lemma 4.1 implies that null-controllability of the system (28) is equivalent to final state observability of the adjoint or dual system (30). More recently, in [Vie05] and [YLC06] it is shown that this equivalence holds true even if \( X \) is a general Banach space, \( U \) a reflexive Banach space and \( r \in (1, \infty) \).

**Theorem 4.2** ([Vie05, YLC06]). Let \( X \) and \( U \) be Banach spaces, \( U \) reflexive, \((S_t)_{t \geq 0}\) be a \( C_0 \)-semigroup on \( X \), \( -A \) the corresponding infinitesimal generator on \( X \), \( B \in \mathcal{L}(U, X) \), \( r \in (1, \infty) \), and \( r' \in [1, \infty) \); with \( 1/r + 1/r' = 1 \). Then the system (28) is null-controllable in time \( T > 0 \) via \( L_r((0,T); U) \) if and only if there exists \( C_{obs} > 0 \) such that

\[
\forall x' \in X': \quad \|S'_T x'\|_{X'} \leq C_{obs} \|B' S'_t x'\|_{L_r((0,T);U')}.
\]

Note that \(-A'\) in general does not generate a \( C_0\)-semigroup on \( X' \) but merely the weak* generator of the weak*-continuous semigroup \( S' \) on \( X' \) given by \( S'_t := (S_t)' \) for all \( t \geq 0 \). However, if \( X \) is reflexive then \( S' \) is strongly continuous and \(-A'\) is the infinitesimal generator of \( S' \). If we assume that \( S' \) is strongly continuous, we can combine Theorem 2.1 and Theorem 4.2 and obtain sufficient conditions for null-controllability of the system (28).

**Theorem 4.3.** Let \( X, U \) be Banach spaces, \( U \) reflexive, \((S_t)_{t \geq 0}\) be a \( C_0 \)-semigroup on \( X \), \( -A \) the corresponding infinitesimal generator on \( X \), \( B \in \mathcal{L}(U, X) \), and assume that \((S_t)_{t \geq 0}\) is strongly continuous. Let further \( \lambda^s \geq 0 \) and \((P'_t)_{t \geq 0}\) be a family of bounded linear operators in \( X' \), \( r \in (1, \infty) \), \( d_0, d_1, d_3, \gamma_1, \gamma_2, \gamma_3, T > 0 \) with \( \gamma_1 < \gamma_2, d_3 \geq 1 \) and assume that

\[
\forall x' \in X' \quad \forall \lambda > \lambda^s: \quad \|P'_t x'\|_{X'} \leq d_0 e^{d_1 t \lambda^{\gamma_1}} \|B' P'_t x'\|_{U'},
\]

and

\[
\forall x' \in X' \quad \forall \lambda > \lambda^s \forall t \in (0, T/2]: \quad \|(\text{Id} - P'_t) S'_t x'\|_{X'} \leq d_2 e^{-d_3 t \lambda^{\gamma_2 + \gamma_3}} \|x'\|_{X'}.
\]

Then the system (28) is null-controllable in time \( T \) via \( L_r((0,T); U) \).

Combining Theorem 4.2 with Theorems 3.3 and 3.4 we obtain a sharp geometric condition on null-controllability for linear systems governed by strongly elliptic operators with interior control. For \( \omega \subset \mathbb{R}^d \) measurable we denote by \( 1'_\omega \in \mathcal{L}(L_p(\omega), L_p(\mathbb{R}^d)) \) the canonical embedding, i.e. \( 1'_\omega f = f \) on \( \omega \) and \( 1'_\omega f = 0 \) on \( \mathbb{R}^d \setminus \omega \).

**Theorem 4.4.** Let \( 1 < p < \infty, \ a: \mathbb{R}^d \to \mathbb{C} \) a homogeneous strongly elliptic polynomial in \( \mathbb{R}^d \) of degree \( m \geq 2 \), \( A_p \) the associated elliptic operator in \( L_p(\mathbb{R}^d) \), \((S_t)_{t \geq 0}\) the bounded \( C_0 \)-semigroup on \( L_p(\mathbb{R}^d) \) generated by \(-A_p \), \( \omega \subset \mathbb{R}^d \) measurable, \( r \in (1, \infty) \), and \( T > 0 \). Then the system

\[
x'(t) = -A_p x(t) + 1'_\omega u(t), \quad t \in (0, T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d),
\]

is null-controllable in time \( T \) via \( L_r((0,T); L_p(\mathbb{R}^d)) \) if and only if \( \omega \) is a thick set.
Proof. Let $1 < p' < \infty$ be such that $1/p + 1/p' = 1$. Note that $L_p(\mathbb{R}^d)' \cong L_{p'}(\mathbb{R}^d)$, $(A_p)' = A_{p'}$ (note that $m$ is even) and $A_{p'}$ is the generator of the bounded $C_0$-semigroup $(S_t)_{t \geq 0}$. If we set $B = 1_{\omega} \in \mathcal{L}(L_p(\omega), L_p(\mathbb{R}^d))$ we have for the dual operator $B' = 1_{\omega} \in \mathcal{L}(L_{p'}(\mathbb{R}^d), L_{p'}(\omega))$, i.e. $B'$ is the restriction operator of a function $x \in L_{p'}(\mathbb{R}^d)$ on $\omega$. Hence, combining Theorem 4.2 with Theorems 3.3 and 3.4 for the dual system we obtain the assertion.

We now turn to the discussion of the control costs. For $T > 0$ we call the quantity

$$C_T := \sup_{x_0 \in X} \inf_{\|x_0\|_X = 1}\left\{\|u\|_{L_r((0,T);U)} : u \in L_r((0,T);U), \; S_T x_0 + B^T u = 0\right\}$$

the control cost in time $T$ via $L_r((0,T);U)$ of the system (28). If $X$ and $U$ are Hilbert spaces and $r = 2$ it well known that the control cost $C_T$ equals the smallest constant $C_{obs}$ such that the system (30) satisfies a final state observability estimate. This fact is a direct consequence of Lemma 4.1.

If $U$ is not a Hilbert space, or $r \neq 2$, the construction above does not apply directly since it is not clear how to extend the operator $H$ to the whole space by keeping its relevant properties. It is an open question if control costs can be estimated by the observability constant in the general setting. Under some additional assumption we can formulate the following lemma.

Lemma 4.5. Let $X$ and $U$ be Banach spaces, $(S_t)_{t \geq 0}$ be a $C_0$-semigroup on $X$, $-A$ the corresponding infinitesimal generator on $X$, $B \in \mathcal{L}(U,X)$, $T > 0$, $r, r' \in (1, \infty)$ with $1/r' + 1/r = 1$, and $C_{obs} > 0$. Assume that the system (30) satisfies the final state observability estimate

$$\forall x' \in X' : \|S_T^r x'\|_{X'} \leq C_{obs}\|B' S_T^r x'\|_{L_r((0,T);U')}.$$  

Then the system (28) is null-controllable in time $T$ via $L_r((0,T);U)$. Moreover, there exists

$$H : \text{Ran} B'^T \to X' \quad \text{such that} \quad H B'^T = S_T'^r \quad \text{and} \quad \|H u'\|_{X'} \leq C_{obs}\|u'\|_{L_r((0,T);U')}$$

for all $u' \in \text{Ran} B'^T$, where $B'^T : L_r((0,T);U) \to X'$ is as in (29). Suppose further that there is an extension $\tilde{H} : L_{r'}((0,T);U') \to X'$ of $H$ with $\|\tilde{H} u'\| \leq C_{obs}\|u'\|_{L_{r'}((0,T);U')}$ for all $u' \in L_{r'}((0,T);U')$. Then the control cost in time $T$ via $L_r((0,T);U)$ of the system (28) satisfies $C_T \leq C_{obs}$.

Proof of Lemma 4.5. Equation (31) is equivalent to statement (b) of Lemma 4.1 with $V = X = W = L_r((0,T);U)$, $F = S_T$, and $G = B'^T \in \mathcal{L}(W,X)$ with $B'^T$ as in (29). The implication (b) $\Rightarrow$ (a) of Lemma 4.1 implies that $\text{Ran}(S_T') \subset \text{Ran}(B'^T)$, i.e. null-controllability of the system (28). The implication (b) $\Rightarrow$ (c) of Lemma 4.1 ensures the existence of the operator $H$ with the desired properties, which proves the first assertion.

The dual operator of $B'^T$ is given by $(B'^T x')(t) = B' S_t' x'$ for $x' \in X'$. For an arbitrary initial state $x_0 \in X$ we choose the control function $u \in L_r((0,T);U)$, $u(t) = (-\tilde{H} x_0)(T-t)$, with $\tilde{H}$ as in the hypothesis of the lemma. Since $H B'^T = S_T'$ by assumption, we obtain for all $x' \in X'$

$$\langle S_T x_0, x'\rangle_{X,X'} = \langle \tilde{H} x_0, B'^T x'\rangle_{L_r((0,T);U),L_r((0,T);U')} = -\int_0^T \langle u(T-t), B' S_t' x'\rangle_{U,U'} \, dt.$$
Thus, the solution of (28) satisfies $x(T) = S_T x_0 + B^T u = 0$. For the norm of the control function we have by assumption on $\tilde{H}$

$$
\|u\|_{L_r(0,T); U} \leq \|\tilde{H}\|_r \|x_0\|_X \leq C_{\text{obs}} \|x_0\|_X.
$$

This shows that the control cost in time $T$ via $L_r((0,T); U)$ of the system (28) satisfies $C_T \leq C_{\text{obs}}$. \hfill \Box

From Lemma 4.5 and Theorem 3.3 we obtain the following corollary. It complements Theorem 4.4 and provides an explicit upper bound on the control cost for elliptic operators $A$ and interior control on thick sets.

**Corollary 4.6.** Let $p, p', r, r' \in (1, \infty)$ such that $1/p' + 1/p = 1$ and $1/r' + 1/r = 1$, $a: \mathbb{R}^d \to \mathbb{C}$ a homogeneous strongly elliptic polynomial in $\mathbb{R}^d$ of degree $m \geq 2$, $A_p$ the associated elliptic operator in $L_p(\mathbb{R}^d)$, $(S_t)_{t \geq 0}$ the bounded $C_0$-semigroup on $L_p(\mathbb{R}^d)$ generated by $-A_p$, $\omega \subset \mathbb{R}^d$ a $(\rho, L)$-thick set, and $T > 0$. Then the system

$$
\dot{x}(t) = -A_p x(t) + 1'_\omega u(t), \quad t \in (0, T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d),
$$

is null-controllable in time $T$ via $L_r((0,T); L_p(\omega))$. Moreover, there exists $H: \text{Ran } B^T \to L_{p'}(\mathbb{R}^d)$ such that $H B^T = S'_T$

and $\|H u'\|_{L_{p'}(\mathbb{R}^d)} \leq C_{\text{obs}} \|u'\|_{L_r((0,T); L_{p'}(\omega))}$ for all $u' \in \text{Ran } B^T$, where as in (29) we have $B^T: L_r((0,T); L_p(\omega)) \to L_p(\mathbb{R}^d)$ with $B = 1'_\omega$,

$$
C_{\text{obs}} = \frac{D_1 M^{16}}{T^{1/r'}} \left( \frac{K^d}{\rho} \right)^{D_2} \exp \left( \frac{D_3 (|L_1 \ln(K^d/\rho))^{m/(m-1)}}{(cT)^{m-1}} \right),
$$

where $K \geq 1$ is a universal constant, $D_1, D_2 \geq 1$ depending on $d$, $D_3 \geq 1$ depending on $d$ and $p'$, $M = \sup_{t \geq 0} \|S_t\|$, and where $c > 0$ is such that $\text{Re } a(\xi) \geq c |\xi|^m$ for all $\xi \in \mathbb{R}^d$. Suppose further that there is an extension $\tilde{H}: L_{p'}((0,T); L_{p'}(\omega)) \to L_{p'}(\mathbb{R}^d)$ of $H$ with $\|\tilde{H} u'\|_{L_{p'}(\mathbb{R}^d)} \leq C_{\text{obs}} \|u'\|_{L_{p'}((0,T); L_{p'}(\omega))}$ for all $u' \in L_{p'}((0,T); L_{p'}(\omega))$. Then the control cost in time $T$ via $L_r((0,T); L_p(\omega))$ of the system (32) satisfy $C_T \leq C_{\text{obs}}$.

**Remark 4.7.** In general, it may be difficult to show the existence of an extension $\tilde{H}$ of $H$ as in Lemma 4.5 and Corollary 4.6. However, if $U$ is a Hilbert space (or $p = 2$) and $r = 2$, the existence is trivial since we can choose $\tilde{H} = H P$ with a suitable orthogonal projection $P$.

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