Eigenfunctions of Galactic Phase Space Spirals from Dynamic Mode Decomposition

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ABSTRACT

We introduce the method of Dynamic Mode Decomposition (DMD) to the field of galactic dynamics. In DMD the dynamics of non-linear systems is studied by determining the dominant modes of an approximate linear model for the evolution. Typically, the DMD algorithm is applied to simulation data. In this paper, we consider the evolution of a plane-symmetric collisionless system of massive particles (sheets) that evolve through their mutual gravitational interactions and through their interactions with an external potential. The system has been used extensively as a toy model for the vertical structure of the Galactic disc, which has been the subject of intense interest since the discovery of number count asymmetries, bulk motions, and phase spirals in the Galactic $z-v_z$ plane. We show that the DMD analysis allows us to write the distribution function as a linear combination of an equilibrium distribution, bending and breathing modes, and spiral modes. These DMD modes are essentially the eigenfunctions of a linear operator that is constructed from a sequence of simulation snapshots. The associated eigenvalues determine the lifetime and frequency of the modes. We outline how to apply the method to full three-dimensional simulations.

Key words: Galaxy: kinematics and dynamics – Galaxy: structure – Galaxy: disc

1 INTRODUCTION

Astrometric and radial velocity surveys of the Milky Way such as SEGUE (Yanny et al. 2009), RAVE (Steinmetz et al. 2006), LAMOST Cui et al. (2012), and Gaia Data Release 2 (GDR2) (Gaia Collaboration et al. 2018a,b), have revealed a panoply of phase space structures in the Galactic stellar disc. In the vicinity of the Sun, these structures include vertical asymmetries in the local stellar number density (Widrow et al. 2012; Yanny & Gardner 2013; Bennett & Bovy 2018), vertical bulk motions of disc stars (Widrow et al. 2012; Williams et al. 2013; Carlin et al. 2013; Quillen et al. 2018; Gaia Collaboration et al. 2018b) and phase spirals in the $z-v_z$ plane. The spirals have now been studied as a function of position within the disc, action variables, and stellar properties (Khanna et al. 2019; Laporte et al. 2019; Li & Shen 2019).

A heuristic explanation of the spirals is that a local bend in the disc phase mixes due to the anharmonic nature of the vertical potential (Antoja et al. 2018), while coupling of the vertical and in-plane motions then leads to the $v_R$ and $v_\phi$ spirals (Schönrich & Binney 2018; Darling & Widrow 2018). In the simplest implementation of this picture, one treats stars as test particles in a fixed potential. This model seems to capture the basic features of the spirals and allows one to estimate the time at which the initial perturbation that gave rise to them took place (Antoja et al. 2018). However, it leaves several questions unanswered, which include the following: What perturbed the disc? Can we point to a singular event that drove the disc from equilibrium or is the disc in a perpetual state of disequilibrium? What is the underlying distribution function of the perturbed disc? The phase spirals are likely the deprojection of the number count asymmetry structures were the phase spirals discovered by Antoja et al. (2018) in the radial velocity subsample of GDR2. They were first seen by selecting stars in an arc of about 8 degrees in Galactic azimuth and 200 pc in Galactocentric radius centered on the Sun and plotting the number density, mean $v_\phi$ or mean $v_R$ across the $z-v_z$ plane. The spirals have now been studied as a function of position within the disc, action variables, and stellar properties (Khanna et al. 2019; Laporte et al. 2019; Li & Shen 2019).

Perhaps the most intriguing of the aforementioned structures were the phase spirals discovered by Antoja et al. (2018) in the radial velocity subsample of GDR2. They were first seen by selecting stars in an arc of about 8 degrees in Galactic azimuth and 200 pc in Galactocentric radius centered on the Sun and plotting the number density, mean $v_\phi$ or mean $v_R$ across the $z-v_z$ plane. The spirals have now been studied as a function of position within the disc, action variables, and stellar properties (Khanna et al. 2019; Laporte et al. 2019; Li & Shen 2019).

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along the $z$ axis mentioned above to the $z - \nu_z$ plane. Can we come to understand the spirals as the projection of some higher dimensional structure?

A number of candidates have been proposed as the agent of disequilibrium. For example, the disc may have been perturbed by a passing satellite galaxy or dark matter subhalo with the Sagittarius dwarf a prime suspect (Laporte et al. 2019; Khanna et al. 2019). On the other hand, the buckling of the stellar bar has been shown to generate phase spirals in simulations of a Milky Way-like galaxy (Khoperskov et al. 2019).

The more challenging problem is to discern the structure of the underlying phase space distribution function. Tremaine (1999) stressed the idea that the dimensionality of the distribution function can change via phase mixing. For example, a satellite galaxy that is being tidally disrupted by the gravitational potential of its host galaxy changes from a six-dimensional structure to a three-dimensional stream. Tremaine (1999) showed that one can relate changes in the phase space structure of a system to the eigenvalues of the Hessian matrix for the Hamiltonian. In principle, the method could be applied to phase mixing in a Galactic disc.

The main drawback of the phase mixing arguments is that they ignore the self-gravity of the perturbation, which is clearly important for the development of both bending and density waves in the disc. In short, when a local region of the disc is displaced from the midplane, it exerts a perturbing force on the unperturbed disc that, at least in linear theory, is the same order as the restoring force pulling it back into the midplane from the unperturbed disc (Hunter & Toomre 1969). A striking example of the importance of self-gravity can be found in the toy model simulations of bending waves in Darling & Widrow (2018) (Figure 7). Of course, self-gravity is built into the simulations of Laporte et al. (2019); Khanna et al. (2019) and Khoperskov et al. (2019).

These considerations suggest that phase mixing and self-gravity are competing effects. A particularly simple toy model in which this competition plays out is the one-dimensional slab. This system can be thought of as an idealized disc comprising concentric, rotating, razor-thin rings. This is similar in aim to the methods in Tremaine (1999) terms studying structure and its dimensionality. However since DMD is data driven, the inclusion of self gravity is trivial compared to the process of obtaining an appropriate action space Hamiltonian. The layout of the paper is as follows. In Section 2 we present a summary of the DMD method and how it connects to the theory of small oscillations. In Section 3 we apply this method to simulation data from a simple one-dimensional model for the vertical structure of a Galactic disc. The simulations are constructed so that we can adjust the relative importance of self-gravity and an external potential. In Section 4 we suggest a path forward for using DMD with full six-dimensional simulation data. Finally, we conclude with a summary and discussion in Section 5.

2 CHARACTERISTIC OSCILLATIONS

2.1 Small Oscillations

Consider a classical system with $n$ degrees of freedom described by the generalized phase space coordinates $\mathbf{x} = (\mathbf{q}, \mathbf{p})^T$ where $\mathbf{q}, \mathbf{p} \in \mathbb{R}^n$ and the Hamiltonian $H(\mathbf{q}, \mathbf{p}) = U(\mathbf{q}) + K(\mathbf{p})$. In the neighborhood where oscillations of the
system are small, we consider a linearized system such that both the potential energy $U(q)$ and the kinetic energy $K(p)$ are quadratic forms (Arnold 1989):

$$U(q) = \frac{1}{2} q^T B q, \quad K(p) = \frac{1}{2} p^T C p. \quad (1)$$

In this case, the equations of motion can be written as a linear operator equation,

$$\frac{dX}{dt} = AX, \quad (2)$$

where

$$A = \begin{pmatrix} 0 & C \\ -B & 0 \end{pmatrix}. \quad (3)$$

The solution is then given in terms of the eigendecomposition of $A$,

$$x(t) = \sum_j b_j \phi_j e^{\omega_j t}, \quad (4)$$

where $\phi_j$ and $\omega_j$ are the eigenvectors and eigenvalues of $A$, that is, $A \phi_j = \omega_j \phi_j$.

### 2.2 Dynamic Mode Decomposition

In this section, we provide a compact overview of DMD, which draws from the introductory chapters of Kutz et al. (2016). We consider a nonlinear dynamical system that is not necessarily belong to a Hamiltonian system. The goal of DMD is to determine the best fit linear model that approximates the non-linear dynamics of the system. The key idea is that over a sufficiently short time interval $\Delta t$, the dynamics of the system can be approximately described by a linear system of equations of the form given in equation 2 with a solution given by equation 4.

With these considerations in mind, we draw $m$ discrete time samples $x_t$ from the system with a sampling period of $\Delta t$. We then construct the equivalent discretized system of equations

$$x_{j+1} = Ax_j, \quad (5)$$

where the discrete-time map is given by $A = e^{A \Delta t}$. We emphasize here that the states $x_t$ need not be coordinates of the system, but can be any set of observables. For example, in the one-dimensional example presented in Section 3 we use phase space density as the observable.

In general, the operator $A$ is not known but is approximated from the data. To do so, we construct the data matrix

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \end{pmatrix}, \quad (6)$$

and the time shifted data matrix

$$X' = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_m \end{pmatrix}. \quad (7)$$

Our system of equations can then be approximated with the matrix equation

$$X' \approx AX. \quad (8)$$

From this, $A$ is estimated by minimizing the matrix norm, $\|X' - AX\|$, which yields the result

$$A = X' X^+. \quad (9)$$

Here, $X^+$ denotes the Moore-Penrose pseudo-inverse of $X$, which can be computed via singular value decomposition (SVD). As in Press et al. (2002) the SVD of $X$ is defined by the relation $X \approx U \Sigma V^T$ where $V^T$ denotes the Hermitian conjugate transpose, $\Sigma$ holds the singular values along its diagonal, and $U$ and $V$ are comprised of left and right orthonormal vectors respectively. Using the SVD, we have that $X^+ = V \Sigma^{-1} U^T$, with which the discrete-time map becomes

$$A \approx X' V \Sigma^{-1} U^T. \quad (10)$$

Our next goal is to obtain the eigendecomposition of $A$ as we did with $A$ in Section 2.1 to facilitate understanding the time evolution of the system in terms of dominant modes. In the spirit of principal component analysis, we assume that the dominant structure of the system may be described by $r < m$ modes. Recall that in principle component analysis, when a data matrix $X$ possesses low dimensional structure, it may be reasonably approximated in a basis spanned by the $r$ column vectors in $U$ of its SVD corresponding to the $r$ largest singular values. We therefore work with the projection of $A$ into this $r$-dimensional subspace,

$$\tilde{A} = U^T A U = U^T X' V \Sigma^{-1}. \quad (11)$$

By working with this projection, we drastically reduce the dimension of the discrete-time map, making its eigendecomposition computationally tractable despite the typically large data matrices. Doing so also improves the numerical stability of the pseudo-inverse of $X$.

We now determine the eigenvalues and eigenvectors of $\tilde{A}$. That is, we solve the equation $\tilde{A} \Xi = \Xi \Lambda$ where $\Lambda$ is a diagonal matrix whose elements $\lambda_j$, $j = 1, \ldots, r$ are the eigenvalues of $\tilde{A}$, and $\Xi$ is a matrix whose columns are the corresponding eigenvectors. To a good approximation, the $r$ most dominant eigenvalues of $A$ are the $\lambda_j$ while the corresponding eigenvectors, which are often referred to as the DMD modes, are given by

$$\Phi = X' V \Sigma^{-1} \Xi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_r \end{pmatrix}. \quad (12)$$

(For a detailed explanation and proof, see Tu et al. (2014).)

We now have all of ingredients necessary to write a series solution for the state of the system. Defining a set of frequencies $\omega_j = \ln(\lambda_j)/\Delta t$, the solution takes the form of equation 4, where $b_j$ are the initial amplitudes of the modes, given by $b = \Phi^* x_1 = (b_1 \ldots b_r)^T$.

In general, $\lambda_j$ can be real, imaginary, or complex. The case $\lambda_j = 1$ ($\omega_j = 0$) corresponds to a time-independent mode and arises, for example, when one has a system that
is oscillating about some equilibrium configuration. Imaginary \( \lambda_j \) indicates a pure oscillating mode, what one would usually refer to as a normal or true mode of the system. Real \( \lambda_j \) correspond to pure growing or decaying modes while complex \( \lambda_j \) correspond to pure damped or growing oscillations. It is often convenient to split the modal decomposition into terms with real and complex eigenvalues. In particular, for a system described by some real function, the DMD eigenvalues come in complex conjugate pairs and the associated pairs of modes combine to yield real contributions in the modal decomposition.

We conclude this Section with a few remarks that relate DMD back to our earlier discussion of small oscillations in Section 2.1. When DMD is applied to a system that is solvable by the method of characteristic oscillations, say, a linearized system with \( n \) degrees of freedom, it is natural to construct the data matrices from measurements of the \( n \) generalized coordinates. One can then use the full discrete-time map and expect to obtain the \( n \) characteristic or normal modes of the system. The power of DMD becomes manifest when we consider complex, nonlinear systems where simple analytic methods fail. In such cases, we can construct the data matrices using some convenient set of observables, where the \( r \) modes that one obtains are data rather than model driven.

3 SELF-GRAVITY AND PHASE MIXING

In this section, we apply DMD to a simple 1D system of massive particles that interact with each other through mutual gravitational interactions and with an external gravitational field. As discussed below, the observable in this case is the phase space distribution function.

We begin with the isothermal plane model, first developed by Spitzer (1942) and Camm (1950) and used as an approximation to the vertical structure of a stellar disc by Freeman (1978) and van der Kruit & Searle (1981). In this model, the equilibrium distribution function and density are given by

\[
f_{\text{eq}}(z, v_z) = \frac{\rho_0}{(2\pi\sigma^2)^{1/2}} e^{-E_z/2\sigma_z^2}
\]

and

\[
\rho_{\text{eq}}(z) = \rho_0 e^{-\psi(z)/2\sigma_z^2},
\]

where \( E_z = v_z^2/2 + \psi(z) \) is the vertical energy, \( \psi(z) \) is the potential, \( \rho_0 \) is the central density, and \( \sigma_z \), the velocity dispersion. For an isolated, self-gravitating system, the density and potential must satisfy the Poisson equation and we have

\[
\psi(z) = 2\sigma_z^2 \ln \cosh (z/\zeta_0),
\]

where \( \zeta_0 = \sigma_z^2/2\pi G \rho_0 \).

In the present discussion, we split the gravitational force into two parts: a time-independent part, which one might think of as coming from masses external to the disc, say the dark halo, and a live part coming from the disc itself. That is, we write the potential as

\[
\psi(z, t) = \psi_{\text{ext}}(z) + \psi_{\text{live}}(z, t),
\]

where \( \psi_{\text{ext}} = (1 - \alpha) \psi_{\text{eq}} \) and \( \psi_{\text{live}} \) comes from the disc with masses reduced by a factor of \( \alpha \). In equilibrium, the total potential is just \( \psi_{\text{eq}} \), but once the system is perturbed, \( \psi_{\text{live}} \), \( \rho \), and \( f \) all depend on time. For definiteness, we use \( \sigma_z = 20 \text{ km s}^{-1} \) and \( \zeta_0 = 500 \text{ pc} \), which yields a surface density of \( \Sigma = 2\rho_0 \rho_0 = 60 M_\odot \text{ pc}^{-2} \).

We sample \( N = 10^6 \) particles from \( f_{\text{eq}} \) and then impose a simple bending wave perturbation by shifting the velocities \( 10 \text{ km s}^{-1} \). This form of perturbation has been shown to yield spiral arms in the \( (z, v_z) \) phase space similar to that observed in Gaia DR2 (Antoja et al. 2018; Darling & Widrow 2018; Schönrich & Binney 2018). We then evolve the distribution for four orbital periods, or approximately 450 Myr. The time evolution of the phase space density for the cases \( \alpha = 0.2, 0.5, 0.8 \) are shown in Figure 1.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Snapshots of \( \log_{10}(f(z, v_z)) \) for \( \alpha = 0.8, 0.5, 0.2 \) from top to bottom. Time increases in intervals of 110 Myr from left to right. The bin sizes are \( \Delta z = 0.079 \text{ kpc} \) and \( \Delta v_z = 0.032 \text{ km s}^{-1} \).}
\end{figure}

3.1 Modes in the Distribution Function

We now apply the DMD algorithm to our simulation data. The state vectors \( \mathbf{x}_t \) are composed of the phase space density \( f(z, v_z, t) \) evaluated on a grid in \( (z, v_z) \). The grid of densities at each time is reshaped into a single column vector and placed into the data matrices \( \mathbf{X} \) and \( \mathbf{X}' \). The DMD solution is then computed up to a rank of \( r = 15 \).

In keeping with the discussion of the previous section, we split the series solution into separate sums over modes with real and complex eigenvalues:

\[
f(z, v_z, t) = \sum_{j=1}^{20} b_j \phi_j(z, v_z)e^{\omega_j t} + \sum_{k=-2}^{3} b_k \phi_k(z, v_z)e^{\omega_k t},
\]

where the \( q \) complex modes satisfy \( \phi_k^* = \phi_{-k} \) and the complex eigenvalues satisfy \( \lambda_k^* = \lambda_{-k} \). For each conjugate pair, we can calculate the density by integrating over velocities, and the potential by solving the Poisson equation. We can then construct approximations to the total distribution function and to the contribution to the potential from the disk. In principle, this distribution function and poten-
Figure 2. Five most dominant modes for the case $\alpha = 0.8$ where most of the gravitational force is provided by the particles themselves. From left to right: real component, imaginary component, magnitude, phase, and time dependent amplitude. Red is positive; blue is negative. In the right most column, the solid and dashed curves correspond to the real and imaginary components of the amplitudes respectively. The modes are evaluated on the same grid used for the histograms in Figure 1.

potential should provide an approximate solution to the time-dependent collisionless Boltzmann equation.

In Figures 2, 3, and 4 we show the first five dominant modes of the distribution function and their time dependent coefficients for the live fractions $\alpha = 0.8, 0.5, 0.2$. For the complex modes, we show only one member of each conjugate pair. We include the real and imaginary components as well as the magnitude and complex phase. The real and imaginary components give one a sense of what the mode looks like during different phases of its oscillations. The magnitude allows one to see the general structure of the mode while phase space spirals are most pronounced in the plot of the complex phase. In general we obtain a combination of damped and undamped modes. The undamped modes are likely related be the true modes found by Mathur (1990) and Weinberg (1991). Strongly damped modes correspond to transient responses such as Landau damping (Binney & Tremaine 2008). Most of the dominant modes are weakly damped and persist on long time scales compared to the dynamical time. These modes, which contain phase space spirals even when self-gravity is moderate to strong ($\alpha = 0.5$ and $\alpha = 0.8$), may describe a situation where self-gravity allows spirals to persist for times much longer than expected from pure phase mixing arguments.

In Figure 6, we show the eigenvalue spectra for each choice of $\alpha$. These plots provide an indication of the dimensionality of the structure, as they show how many modes will persist through the time evolution of the system either as stationary or oscillatory modes. Recalling that the mode frequencies are proportional to the natural logarithm of the eigenvalues, and considering the eigenvalues in polar form, $\lambda = |\lambda|e^{i\phi}$, one has that

$$\omega = \frac{1}{\Delta t} \left( \ln |\lambda| + i\phi \right).$$

(18)
From equation 17, we see that $|\lambda|$ determines the growth or decay rate of each mode, while $\theta_\lambda$ sets the mode oscillation frequency. It is also convenient to define the mode lifetime

$$\tau = \frac{\Delta t}{\ln |\lambda|}, \quad (19)$$

which we can compare to the timescale of the orbital period $T$.

We next turn to the eigenvalues. Recall that the frequencies are related to the eigenvalues via $\omega_j = \ln \lambda_j / \Delta t$. To gain some intuition into how the eigenvalues are to be interpreted we show in Figure 5 lines of constant mode lifetime and oscillatory frequency in the complex $\lambda$ plane.

The eigenvalues themselves are shown in Figure 6. Also shown in the figure is the unit circle and circles corresponding to mode lifetimes of $\tau = sT$, $s = 1, 2, 3, 4$. Modes with $|\lambda| \approx 1$ will persist as true modes while modes with $|\lambda| < 1$ will damp on the timescales indicated by the dashed lifetime curves. Thus, the long term behaviour of the system may be approximated by a sum over modes with eigenvalues near the unit circle. Although we do not expect growing modes in a stable system like the one we are studying here, the DMD algorithm can find very slowly growing modes. This seems to be the situation in our $\alpha = 0.2$ case though the growth rate is very slow compared to the timescale of the simulation. It is important to note then that the DMD solution in equation 17 is only valid for times close to the range of the simulation, and should not be used for the late times where the slowly growing modes begin to dominate.

3.1.1 Bending and Breathing Modes

In the case of $\alpha = 0.8$, where self-gravity dominates the evolution over the external potential, the dominant modes in rows two and three of Figure 2 share their morphology with the low-order breathing and bending modes derived in Mathur (1990) and Weinberg (1991) and further studied in Widrow et al. (2014) and Widrow & Bonner (2015). These modes are not damped, and have integer multiple frequencies. The sum of the conjugate pairs for each of these modes rotate clockwise in the $(z, v_z)$ plane with a pattern speed set by the mode frequency, which is also consistent with theoretical models. Apart from the stationary zero-frequency mode, there are only three eigenfunctions that persist through the observed evolution of the system. Although not all modes are shown in Figure 2, one can see in the left panel of Figure 6 that there are three conjugate pairs, and one real eigenvalue that lie outside of the $\tau = 4T$ circle. We then say that the structure is four dimensional in the DMD basis, as we may approximate the time evolution of the system with four independent DMD modes.

3.1.2 Phase Space Spirals

In the cases with moderate or weak self-gravity ($\alpha = 0.5$ and $\alpha = 0.2$, respectively) phase mixing becomes important and the DMD analysis readily finds modes with spiral structure. As these spiral modes capture the structure in the distribution function we expect from phase mixing, one might think they are strictly a transient response. Although this is the case for many of the modes, there are undamped persisting spiral modes, such as the mode in row two of Figure 3. Additionally, the mode in row three of Figure 3, which is qualitatively similar to a bending mode, has a weak left-handed spiral structure apparent in both the real and imaginary components, and the plot of the phase. Although we do not claim that there are true spiral modes, it is evident that the inclusion of self-gravity in a phase mixing system allows...
phase space spirals to persist on much longer time scales than expected from pure kinematic phase mixing (Darling & Widrow 2018).

Following the discussion in this section, one can link both the dimensionality of the phase space structure (number of persisting DMD modes) and morphology of the modes to the dominance of self gravity. A more quantitative understanding of this relation may help in estimating the timing of the GDR2 spiral perturbation.

4 EXTENSIONS

Given the promise of the DMD framework for studying phase space structure in the one-dimensional system studied here, a logical next step is to apply it to full three-dimensional simulations. The DMD algorithm itself is robust to very large data matrices. The challenge is to find an appropriate set of observables for each snapshot. In the one-dimensional case, we had the luxury of high particle resolution in the \((z,v_z)\) phase space, and consequently could use a high grid resolution when evaluating \(f(z,v_z)\). In the case of full three-dimensional simulations, a simple binning procedure in 6D phase space is unfeasible for anything beyond the coarsest grid.

Suppose one is interested in using DMD to study the phase spirals found by Antoja et al. (2018) across the disc. One might imagine the following strategy: First sort particles in \(N_R\) radial bins. For each bin, one can construct a Fourier series in Galactic azimuth, keeping only \(N_\phi\) terms. Then, for each azimuthal mode number \(m\) one can bin particles in an \(N_z \times N_v\) grid across the \(z-v_z\) plane. Finally, for each of the \(N_R N_\phi N_z N_v\) cells, one can compute the first \(m_{\text{max}}\) moments of \(v_R\) and \(v_\phi\). For example, with \(m_{\text{max}} = 2\) there are six moments. For \(N_R = 20, N_\phi = 9, N_z = N_v = 40\), and \(m_{\text{max}} = 2\) we have approximately 1.7M cells, which would require a simulation with several hundred million particles, a large amount but still feasible.

On the other hand, if one is interested in spiral structure and warps, then the Fourier methods introduced by Sellwood & Athanassoula (1986) and extended to bending waves by Chequers et al. (2018) provide an attractive alternative. In this case, one computes the surface density and vertical moments of the distribution function, such as the mean midplane displacement and mean vertical velocity as functions of \(R\) and \(\phi\) across the disc. One might also include moments of the radial and azimuthal velocities.

5 CONCLUSION

We have showed that DMD facilitates an analysis analogous to eigenmode decomposition with great generality in how it may be applied to problems in galactic dynamics. When applied to time series measurements of phase space density, DMD yields a finite series solution for the distribution function. The dominant terms in this series are typically undamped or very weakly damped oscillations that can persist on time scales of several orbital periods. Moreover, the analysis captures the physics of both phase mixing and self-gravitating oscillations. By computing DMD modes, one can describe and study time-dependent phase space structure throughout its evolution in terms of a few stationary eigenfunctions and time-dependent coefficients. This provides a much richer look at the evolution of structure than typical Fourier methods and allows analysis in terms of very few quantities compared to the full data set yielded by N-body simulations. The method should be even more powerful in the case of simulations of the complete 6D phase space.

We have observed how the competing effects of self gravity and phase mixing manifest in DMD modes. In the presence of both effects, persisting spiral modes arise in addition to the standard bending and breathing modes. In the DMD solution, the spiral modes are responsible for the apparent phase mixing in the full distribution function. In principle, the eigenvalues associated with these modes should yield insight into the timescale of the phase space spirals and the initial perturbations. Additionally, noting that each mode must satisfy the Poisson equation, it may be possible to learn about perturbative contributions to the potential associated with non-equilibrium behavior like persisting phase spirals that cannot be described by simple phase mixing.

The observational evidence for a Galaxy in disequilibrium has led to a keen interest in the distribution function for the stellar disc and specifically the form and timescale of the perturbations. Since this is inherently a time-dependent problem, numerical studies are a key component in understanding the complete picture. DMD has the potential to provide valuable insight into stellar dynamics just as it has in the field of fluid dynamics.

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