Approximating $n$-th Differentiable Functions of Two Variables and Mid-Point Formula

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Abstract. In this work, approximations for real two variables function $f$ which has continuous partial $(n-1)$-derivatives ($n \geq 1$) and has the $n$-th partial derivative of bounded bivariation or absolutely continuous are established. Explicit bounds for this representation are given. An approximation of a function $f$ by its mid-point formula with its error is established.

1. Introduction

The Trapezoid formula

$$\int_a^b f(x)dx \approx (b-a)f(a) + f(b)$$

is one of the most popular rule that approximate the definite integral $\int_a^b f(x)dx$ of a real function $f$ over an interval $[a, b]$.

In the last forty years, a great attention and valuable efforts were devoted to investigate this formula in several ways and for various type of functions. Among others, the generalized trapezoid formula (GFT)

$$\frac{1}{b-a}[(x-a)f(a) + (b-x)f(b)]$$

was expanded by Cerone et al. in [7], and eligible to be considered as one of the important works in the latest presented literature. So that instead of (1.1) one may use (1.2), i.e.,

$$\int_a^b f(t)dt \approx \frac{1}{b-a}[(x-a)f(a) + (b-x)f(b)], \quad \forall x \in [a, b].$$

In another approach, if $f : [a, b] \to \mathbb{R}$ is assumed to be bounded on $[a, b]$. The chord that connects its end points $A = (a, f(a))$ and $B = (b, f(b))$ has the equation

$$d_f : [a, b] \to \mathbb{R}, \quad d_f(x) = \frac{1}{b-a}[f(a)(b-x) + f(b)(x-a)].$$

Some error approximations for the value of the function $f(x)$ by the functional

$$\Phi_f(x) := \frac{b-x}{b-a} \cdot f(a) + \frac{x-a}{b-a} \cdot f(b) - f(x).$$

were introduced by Dragomir in [8], e.g., the following bounds for $\Phi_f(x)$ are hold:
Theorem 2. Let \( f : [a, b] \to \mathbb{R} \) be a function of bounded variation, then

\[
|\Phi_f(x)| \leq \left( \frac{b - x}{b - a} \right)^{\frac{1}{q}} \int_a^b \frac{1}{x} \int_a^x |f(y)| \, dy \, dx + \left( \frac{x - a}{b - a} \right)^{\frac{1}{p}} \left\{ \left[ \left( \frac{b - a}{b - a} \right)^p + \left( \frac{a}{b - a} \right)^p \right]^\frac{1}{p} \left[ \left( \int_a^x |f(y)|^q \right)^{\frac{1}{q}} \right] \right. \\
\left. \left. \leq \left( \frac{b - x}{b - a} \right)^{\frac{1}{q}} \left( \int_a^b f^p(y) \, dy \right)^{\frac{1}{p}} \int_a^x \left[ \left( \int_a^y f^q(z) \, dz \right)^{\frac{1}{q}} \right] \, dy \right\}^{\frac{1}{p}} \right.
\]

The first inequality in (1.4) is sharp. The constant \( \frac{1}{q} \) is best possible in the first and third branches.

In 2008, Dragomir [9] provided an approximation for the function \( f \) which possesses continuous derivatives up to the order \( n - 1 \) \((n \geq 1)\) and has the \( n \)-th derivative of bounded variation, in terms of the chord that connects its end points \( A = (a, f(a)) \) and \( B = (b, f(b)) \) and some more terms which depend on the values of the \( k \) derivatives of the function taken at the end points \( a \) and \( b \), where \( k \) is between 1 and \( n \).

The following representation of \( f \) was presented by Dragomir in [9], as follows:

Theorem 1. If \( f : [a, b] \to \mathbb{R} \) is of bounded variation, then

\[
|\Phi_f(x)| \leq \left( \frac{b - x}{b - a} \right)^{\frac{1}{q}} \int_a^b \frac{1}{x} \int_a^x |f(y)| \, dy \, dx + \left( \frac{x - a}{b - a} \right)^{\frac{1}{p}} \left\{ \left[ \left( \frac{b - a}{b - a} \right)^p + \left( \frac{a}{b - a} \right)^p \right]^\frac{1}{p} \left[ \left( \int_a^x |f(y)|^q \right)^{\frac{1}{q}} \right] \right. \\
\left. \left. \leq \left( \frac{b - x}{b - a} \right)^{\frac{1}{q}} \left( \int_a^b f^p(y) \, dy \right)^{\frac{1}{p}} \int_a^x \left[ \left( \int_a^y f^q(z) \, dz \right)^{\frac{1}{q}} \right] \, dy \right\}^{\frac{1}{p}} \right.
\]

The first inequality in (1.4) is sharp. The constant \( \frac{1}{q} \) is best possible in the first and third branches.

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The following representation of \( f \) was presented by Dragomir in [9], as follows:

Theorem 2. Let \( I \) be a closed subinterval on \( \mathbb{R} \), let \( a, b \in I \) with \( a < b \) and let \( n \) be a nonnegative integer. If \( f : I \to \mathbb{R} \) is such that the \( n \)-th derivative \( f^{(n)}(x) \) is of bounded variation on the interval \([a, b]\), then for any \( x \in [a, b] \) we have the representation

\[
f(x) = \frac{1}{b - a} \left[ (b - x) f(a) + (x - a) f(b) \right] + \frac{(b - x)(x - a)}{b - a} \cdot \sum_{k=1}^n \frac{1}{k!} \left[ (x - a)^{k-1} f^{(k)}(a) + (-1)^k (b - x)^{k-1} f^{(k)}(b) \right]
\]

where the kernel \( S_n : [a, b]^2 \to \mathbb{R} \) is given by

\[
S_n(x, t) = \begin{cases} 
(x - t)^n (b - x), & a \leq t \leq x \\
(-1)^{n+1} (t - x)^n (x - a), & a \leq t \leq x 
\end{cases}
\]

and the integral in the remainder is taken in the Riemann–Stieltjes sense.

On utilizing the notations

\[
D_n(f; x, a, b) := \frac{1}{b - a} \left[ (b - x) f(a) + (x - a) f(b) \right] + \frac{(b - x)(x - a)}{b - a} \cdot \sum_{k=1}^n \frac{1}{k!} \left[ (x - a)^{k-1} f^{(k)}(a) + (-1)^k (b - x)^{k-1} f^{(k)}(b) \right]
\]

with

\[
E_n(f; x, a, b) := \frac{1}{b - a} \int_a^b S_n(x, t) d \left( f^{(n)}(t) \right),
\]
and under the assumptions of Theorem \[2\] Dragomir provided an error approximation of the function \(f\) using the formula
\[
f(x) - D_n(f; x, a, b) = E_n(f; x, a, b)
\]
for any \(x \in [a, b]\).

The concept of Riemann–Stieltjes (\(\mathcal{RS}\)) double integral
\[
\int_a^b \int_a^c f(x, y) \, dx \, dy \alpha(x, y)
\]
was characterized by Fréchet in \[13\], and investigated later on by Clarkson in \[5\]. Such \(\mathcal{RS}\)-integral plays an important role in Mathematics with multiple applications in several subfields including Probability Theory & Statistics, Complex Analysis, Functional Analysis, Operator Theory and others.

For \(a, b, c, d \in \mathbb{R}\), we consider the subset \(Q := \{(x, y) : a \leq x \leq b, c \leq y \leq d\}\) of \(\mathbb{R}^2\). If \(P := \{(x_i, y_j) : x_{i-1} \leq x_i, y_{j-1} \leq y_j, \ i = 1, \ldots, n; \ j = 1, \ldots, m\}\) is a partition of \(Q\), write
\[
\Delta_{11} f(x_i, y_j) = f(x_{i-1}, y_{j-1}) - f(x_{i-1}, y_j) - f(x_i, y_{j-1}) + f(x_i, y_j)
\]
for \(i = 1, 2, \ldots, n\) and \(j = 1, 2, \ldots, m\). The function \(f(x, y)\) is said to be of bounded variation in the Vitali sense (or simply bounded bivariation \[14\]) if there exists a positive quantity \(M\) such that for every partition on \(Q\) we have
\[
\sum_{i=1}^n \sum_{j=1}^m |\Delta_{11} f(x_i, y_j)| \leq M
\]
(see \[6\]). Consequently, the total bivariation of a function of two variables, over bidimensional interval \(Q\), is defined to be the number
\[
\nabf(f) := \bigvee_{c}^{d} \bigvee_{a}^{b} (f) := \sup \left\{ \sum P : P \in \mathcal{P}(Q) \right\},
\]
where \(\sum (P)\) denote the sum \(\sum_{i=1}^n \sum_{j=1}^m |\Delta_{11} f(x_i, y_j)|\) corresponding to the partition \(P\) of \(Q\). For further properties of mappings of bounded bivariation we refer the interested reader to the book recent \[14\] and the monograph \[1\].

In 2009, Alomari \[1\] has studied the approximation problem of the Riemann–Stieltjes double integrals \(\int_a^b \int_a^c f(t, s) \, ds \, d\alpha(t, s)\) for various kind of the integrand \(f\) and the integrator \(\alpha\) via inequalities approach. Among others, we need the following two basic results from \[1\]:

**Lemma 1.** \((Integration\ by\ parts\ [1])\) If \(f \in \mathcal{RS}(\alpha)\) on \(Q\), then \(\alpha \in \mathcal{RS}(f)\) on \(Q\), and we have
\[
(1.8) \quad \int_a^b f(t, s) \, dt \, ds \, \alpha(t, s) + \int_a^b \int_a^c \alpha(t, s) \, dt \, ds \, f(t, s) = f(b, d) \alpha(b, d) - f(c, d) \alpha(c, d) - f(a, d) \alpha(a, b) + f(a, c) \alpha(a, c).
\]

**Lemma 2.** \((1)\) Assume that \(g \in \mathcal{RS}(\alpha)\) on \(Q\) and \(\alpha\) is of bounded bivariation on \(Q\), then the inequality
\[
(1.9) \quad \left| \int_a^b \int_a^c g(x, y) \, dx \, dy \, \alpha(x, y) \right| \leq \sup_{(x, y) \in Q} |g(x, y)| \cdot \bigvee_{Q} (\alpha),
\]
holds.

A small attention and a few works have been considered for mappings of two variables; particularly, the approximation problem of \(\mathcal{RS}\)-double integral \(\int_a^b \int_a^c f(t, s) \, ds \, d\alpha(t, s)\)
in terms of RS-double sums. Although the important applications in approximations, numerical integration in two or more variables is still much less developed area than its one-dimensional counterpart, which has been worked on intensively. For recent inequalities involving functions of two variables the reader may refer to [11–13], [10–12] and [15–18].

In this paper, we extend Cerone et al [7] and Dragomir [8, 9] results (mainly Theorem 2) by giving bidimensional representation for functions whose n-th partial derivatives are of bounded bivariation or absolutely continuous. Explicit bounds for this representation are given. An approximation of a function \( f \) by its mid-point formula with its error is established.

2. The Result

For a bounded function \( f : Q \to \mathbb{R} \) the chord plane that connects its end points may be represented as:

\[
C_f (x, y) := \frac{1}{(b-a)(d-c)} \left[ (b-x)(d-y) f(a, c) + (b-x)(y-c) f(a, d) \\
+ (x-a)(d-y) f(b, c) + (x-a)(y-c) f(b, d) \right],
\]

which is the generalized Trapezoid formula for functions of two variables. Consequently, the difference between the function \( f \) and its chord plane may considered by the functional:

\[
\psi (f, C) = C_f (x, y) - f(x, y).
\]

One may study this difference for various type of functions. On the other hand, this representation can be extended to n-th differentiable functions, as follows:

**Theorem 3.** Let \( f : Q \to \mathbb{R} \) be a real valued function which has continuous \((n-1)\) partial derivatives \((n \geq 1)\). If the n-th partial derivatives \( D^n f \) \((n \geq 1)\) are of bounded bivariation on \( Q \), then for any \((x, y) \in Q\) we have the representation

\[
f(x, y) = \frac{1}{(b-a)(d-c)} \left[ \frac{1}{n!} \sum_{j=1}^{n} \left\{ (b-x) (x-a)^n-j (y-c)^{j-1} D^n f(a, c) + (-1)^j (d-y)^{j-1} D^n f(a, d) \right\} \\
+ (x-a) (b-x)^{n-j} (-1)^{j-1} D^n f(b, c) + (d-y)^{j-1} D^n f(b, d) \right] \\
\times \int_a^b \int_c^d S_n(x, t, y, s) \, dt \, ds \left( D^n f(t, s) \right)
\]

where, \( D^n f(t, s) = \frac{\partial^n f}{\partial t^n \partial s^j}(t, s) \) \((j = 0, 1, \ldots, n)\) and

\[
S_n(x, t; y, s) = \frac{1}{n!} \left\{ (x-t)^n (b-x) (y-s)^n (d-y), \quad a \leq t \leq x, \quad c \leq s \leq y \\
- (t-x)^n (t-x)^n (x-a) (y-s)^n (d-y), \quad x < t \leq b, \quad c \leq s \leq y \\
- (t-x)^n (t-x)^n (x-a) (y-s)^n (y-c), \quad a \leq t \leq x, \quad y < s \leq d \\
- (t-x)^n (t-x)^n (x-a) (s-y)^n (y-c), \quad x < t \leq b, \quad y < s \leq d
\]
Proof. We utilize the following Taylor’s representation formula for functions $f : Q \subset \mathbb{R}^2 \to \mathbb{R}$ such that the $n$-th partial derivatives $D^nf$ are of locally bounded bivariation on $Q$,

\begin{equation}
(2.2) \quad f(x, y) = P_n(x, y) + R_n(x, y)
\end{equation}

such that,

\begin{equation}
(2.3) \quad P_n(x, y) = \sum_{j=0}^{n} \frac{1}{j!} \binom{n}{j} (x-x_0)^{n-j} (y-y_0)^j D^n f(x_0, y_0)
\end{equation}

and

\begin{equation}
(2.4) \quad R_n(x, y) = \frac{1}{n!} \int_{0}^{d} \int_{a}^{y} (x-t)^n (y-s)^n dtds \, (D^n f(t,s))
\end{equation}

where, $D^n f(t,s) = \frac{\partial^n f}{\partial t^n \partial s^n}(t,s)$, and $(x, y), (x_0, y_0)$ are in $Q$ and the double integral in the remainder is taken in the Riemann-Stieltjes sense.

Choosing $x_0 = a, y_0 = c$ and then $x_0 = b, y_0 = d$ in (2.1) we can write that

\begin{equation}
(2.5) \quad f(x, y) = \sum_{j=0}^{n} \frac{1}{j!} \binom{n}{j} (x-a)^{n-j} (y-c)^j D^n f(a, c)
\end{equation}

\begin{equation}
+ \frac{1}{n!} \int_{0}^{d} \int_{a}^{y} (x-t)^n (y-s)^n dtds \, (D^n f(t,s)),
\end{equation}

\begin{equation}
(2.6) \quad f(x, y) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n}{j} (x-a)^{n-j} (d-y)^j D^n f(a, d)
\end{equation}

\begin{equation}
+ \frac{(-1)^{n+1}}{n!} \int_{0}^{d} \int_{a}^{y} (x-t)^n (s-y)^n dtds \, (D^n f(t,s)),
\end{equation}

\begin{equation}
(2.7) \quad f(x, y) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n}{j} (b-x)^{n-j} (y-c)^j D^n f(b, c)
\end{equation}

\begin{equation}
+ \frac{(-1)^{n+1}}{n!} \int_{0}^{d} \int_{a}^{y} (t-x)^n (y-s)^n dtds \, (D^n f(t,s))
\end{equation}

\begin{equation}
(2.8) \quad f(x, y) = \sum_{j=0}^{n} \frac{1}{j!} \binom{n}{j} (b-x)^{n-j} (d-y)^j D^n f(b, d)
\end{equation}

\begin{equation}
+ \frac{1}{n!} \int_{y}^{d} \int_{x}^{b} (t-x)^n (s-y)^n dtds \, (D^n f(t,s))
\end{equation}

for $(x, y) \in Q$. 
Now, by multiplying (2.5) with \((b - x)(d - y)\), (2.6) with \((b - x)(y - c)\), (2.7) with \((x - a)(d - y)\), (2.8) with \((x - a)(y - c)\), we get

\[
\frac{1}{n!} (b - x)(d - y) f(x, y) = \frac{1}{n!} (b - x)(d - y) f(a, c) + \frac{1}{n!} \sum_{j=1}^{n} \binom{n}{j} (x - a)^{n-j} (y - c)^j D^n f(a, c)
\]

\[
+ \frac{1}{n!} (b - x)(d - y) \int_{a}^{b} \int_{a}^{x} (x - t)^n (y - s)^n dtds (D^n f(t, s)),
\]

\[
\frac{1}{n!} (b - x)(y - c) f(x, y) = \frac{1}{n!} (b - x)(y - c) f(a, d) + \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (x - a)^{n-j} (d - y)^j D^n f(a, d)
\]

\[
+ \frac{1}{n!} (b - x)(y - c) \int_{a}^{b} \int_{a}^{x} (x - t)^n (s - y)^n dtds (D^n f(t, s)),
\]

\[
\frac{1}{n!} (x - a)(d - y) f(x, y) = \frac{1}{n!} (x - a)(d - y) f(b, c) + \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (b - x)^{n-j} (y - c)^j D^n f(b, c)
\]

\[
+ \frac{1}{n!} (x - a)(d - y) \int_{a}^{b} \int_{a}^{x} (t - x)^n (y - s)^n dtds (D^n f(t, s))
\]

\[
\frac{1}{n!} (x - a)(y - c) f(x, y) = \frac{1}{n!} (x - a)(y - c) f(b, d) + \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (b - x)^{n-j} (d - y)^j D^n f(b, d)
\]

\[
+ \frac{1}{n!} (x - a)(y - c) \int_{a}^{b} \int_{a}^{x} (t - x)^n (s - y)^n dtds (D^n f(t, s))
\]

respectively, for ny \((x, y) \in Q\).
Finally, by adding the equalities (2.9)–(2.12) and dividing the sum with \((b - a)(d - c)\), we obtain

\[
f(x, y) = \frac{1}{(b - a)(d - c)} \left\{ \frac{1}{n!} \binom{n}{j} \left\{ (b - x)(x - a)^{n-j} D^n f(a, c) + (b - y)(y - a)^{n-j} D^n f(b, c) \right\} \right.
\]

\[
+ \frac{1}{n!} \frac{1}{(b - a)(d - c)} \int_c^y \int_a^x (x - t)^n (y - s)^n dtds (D^n f(t, s)) \right.
\]

\[
+ \frac{(-1)^{n+1}}{n!} \frac{1}{(b - a)(d - c)} \int_a^x \int_c^y (t - x)^n (y - s)^n dtds (D^n f(t, s)) \right.
\]

\[
+ \frac{1}{n!} \frac{1}{(b - a)(d - c)} \int_a^b \int_c^y (t - x)^n (s - y)^n dtds (D^n f(t, s)) \right.
\]

which gives the desired representation (2.11).

Remark 1. The case \(n = 0\) provides the representation

\[
\text{(2.13)} \quad f(x, y) = \frac{1}{(b - a)(d - c)} \left\{ (b - x)(d - y) f(a, c) + (b - x)(y - c) f(a, d) \right.
\]

\[
+ (x - a)(d - y) f(b, c) + (x - a)(y - c) f(b, d) \right.
\]

\[
+ \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b S_0(x; t; y, s) dtds (f(t, s)) \right.
\]

for any \(x \in Q\), where

\[
S_0(x; t; y, s) = \begin{cases} 
    (b - x)(d - y), & a \leq t \leq x, \quad c \leq s \leq y \\
    (x - a)(d - y), & x < t \leq b, \quad c \leq s \leq y \\
    (b - x)(y - c), & a \leq t \leq x, \quad y < s \leq d \\
    (x - a)(y - c), & x < t \leq b, \quad y < s \leq d 
\end{cases}
\]

and \(f\) is of bounded variation on \(Q\).

The case when \(n = 0\), provides an integral formula which compare any value of \(f(x, y), (x, y) \in Q\) with the values of the function and its derivatives at the rectangle end points (the corners of the rectangle generated by the end points). More specific, the Rectangular Mid-point value of \(f\) can be represented in the following corollary:
Corollary 1. With the assumptions of Theorem 3 for \( f \) and \( Q \), we have the identity

\[
(2.14) \quad f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) = \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \\
+ \frac{1}{2n+2} \sum_{j=1}^{n} \frac{1}{j!} \binom{n}{j} (b-a)^{n-j} (d-c)^j \left\{ D^n f (a, c) + (-1)^j D^n f (a, d) \right\} \\
+ (-1)^j D^n f (b, c) + D^n f (b, d) \right\} \\
+ \frac{1}{(b-a) (d-c)} \int_c^b \int_a^d M_n (t, s) \, dt \, ds \left( D^n f (t, s) \right)
\]

where, \( D^n f (t, s) = \frac{\partial^n f}{\partial t^n \partial s^n} (t, s) \) and

\[
M_n (t, s) = \frac{(b-a)(d-c)}{4n!} \left\{ \begin{array}{ll}
\left( \frac{a+b}{2} - t \right)^n \left( \frac{c+d}{2} - s \right)^n, & a \leq t \leq \frac{a+b}{2}, \quad c \leq s \leq \frac{c+d}{2} \\
\left( -1 \right)^n \left( t - \frac{a+b}{2} \right)^n \left( \frac{c+d}{2} - s \right)^n, & \frac{a+b}{2} < t \leq b, \quad c \leq s \leq \frac{c+d}{2} \\
\left( -1 \right)^n \left( t - \frac{a+b}{2} \right)^n \left( s - \frac{c+d}{2} \right)^n, & a \leq t \leq \frac{a+b}{2}, \quad \frac{c+d}{2} < s \leq d \\
\left( t - \frac{a+b}{2} \right)^n \left( s - \frac{c+d}{2} \right)^n, & \frac{a+b}{2} < t \leq b, \quad \frac{c+d}{2} < s \leq d
\end{array} \right.
\]

On utilizing the following notations

\[
(2.15) \quad A (f, Q) = \frac{1}{(b-a)(d-c)} \left[ (b-x)(d-y)f(a, c) + (b-x)(y-c)f(a, d) \right] \\
+ (x-a)(d-y)f(b, c) + (x-a)(y-c)f(b, d) \right] \\
\times \sum_{j=1}^{n} \frac{1}{j!} \binom{n}{j} \left\{ (b-x) (x-a)^{n-j} [ (y-c)^j-1 D^n f (a, c) + (-1)^j (d-y)^{j-1} D^n f (a, d) ] \\
+ (x-a) (b-x)^{n-j} [ (-1)^j (y-c)^{j-1} D^n f (b, c) + (d-y)^{j-1} D^n f (b, d) ] \right\}
\]

and

\[
(2.16) \quad B_n (f, Q) := \frac{1}{(b-a)(d-c)} \int_c^b \int_a^d S_n (x, t; y, s) \, dt \, ds \left( D^n f (t, s) \right)
\]

under the assumptions of Theorem 3, we can approximate the function \( f \) utilizing the polynomials \( A_n (f, Q) \) with the error \( B_n (f, Q) \). In other words, we have

\[
f(x, y) = A_n (f, Q) + B_n (f, Q)
\]

for any \((x, y) \in Q\).

The error \( B_n (f, Q) \) of a bounded bivariation function \( f(x, y) \) satisfies the following bounds.
Theorem 4. With the assumptions of Theorem 3 for \( f \) and \( Q \), we have

\[
|B_n(f, Q)| \leq \frac{1}{n!(b-a)(d-c)} \cdot \max \{(y-c)^n (d-y), (y-c) (d-y)^n\} \\
\times \max \{(x-a)^n (b-x), (x-a) (b-x)^n\} \cdot \max \{D^n f\} \\
\leq \frac{(b-a)^n (d-c)^n}{2^{2n+2(n)!}} \cdot \max \{D^n f\}
\]

Proof. Using the inequality for the Riemann–Stieltjes integral of continuous integrands and bounded bivariation integrators, we have

\[
|B_n(f, Q)| = \left| \frac{1}{n!(b-a)(d-c)} \left[ \int_{c}^{y} \int_{a}^{x} (x-t)^n (b-x) (y-s)^n (d-y) d_t d_s \left( D^n f(t,s) \right) \right. \right. \\
\left. \left. + \int_{y}^{d} \int_{a}^{x} (-1)^{n+1} (x-t)^n (b-x) (s-y)^n (y-c) d_t d_s \left( D^n f(t,s) \right) \right. \right. \\
\left. \left. + \int_{c}^{y} \int_{x}^{b} (-1)^{n+1} (t-x)^n (x-a) (y-s)^n (d-y) d_t d_s \left( D^n f(t,s) \right) \right. \right. \\
\left. \left. + \int_{y}^{d} \int_{x}^{b} (t-x)^n (x-a) (y-s)^n (d-y) d_t d_s \left( D^n f(t,s) \right) \right] \right| \\
\leq \frac{1}{n!(b-a)(d-c)} \left[ \int_{c}^{y} \int_{a}^{x} (x-t)^n (b-x) (y-s)^n (d-y) d_t d_s \left( D^n f(t,s) \right) \right. \right. \\
\left. \left. + \int_{y}^{d} \int_{a}^{x} (-1)^{n+1} (x-t)^n (b-x) (s-y)^n (y-c) d_t d_s \left( D^n f(t,s) \right) \right. \right. \\
\left. \left. + \int_{c}^{y} \int_{x}^{b} (-1)^{n+1} (t-x)^n (x-a) (y-s)^n (d-y) d_t d_s \left( D^n f(t,s) \right) \right. \right. \\
\left. \left. + \int_{y}^{d} \int_{x}^{b} (t-x)^n (x-a) (y-s)^n (d-y) d_t d_s \left( D^n f(t,s) \right) \right. \right. \\
\left. \left. \left. \right. \right| \right|
\]
\[
\frac{1}{n!(b-a)(d-c)} \left[ (b-x) (d-y) \max_{t \in [a,x]} \left\{ (x-t)^n (y-s)^n \right\} \cdot \bigvee_{c \leq a}^y D^n f \\
+ (b-x) (y-c) \max_{t \in [a,x]} \left\{ (x-t)^n (s-y)^n \right\} \cdot \bigvee_{y \leq a}^d D^n f \\
+ (x-a) (d-y) \max_{t \in [x,b]} \left\{ (t-x)^n (y-s)^n \right\} \cdot \bigvee_{c \leq x}^y D^n f \\
+ (x-a) (y-c) \max_{t \in [x,b]} \left\{ (t-x)^n (s-y)^n \right\} \cdot \bigvee_{y \leq x}^d D^n f \right]
\leq \frac{1}{n!(b-a)(d-c)} \left[ (x-a)^n (b-x) (y-c)^n (d-y) \cdot \bigvee_{c \leq a}^y D^n f \\
+ (x-a)^n (b-x) (d-y)^n (y-c) \cdot \bigvee_{y \leq a}^d D^n f \\
+ (b-x)^n (x-a) (y-c)^n (d-y) \cdot \bigvee_{c \leq x}^y D^n f \\
+ (b-x)^n (x-a) (d-y)^n (y-c) \cdot \bigvee_{y \leq x}^d D^n f \right]
\leq \frac{1}{n!(b-a)(d-c)} \left[ (x-a)^n (b-x) \max \left\{ (y-c)^n (d-y), (y-c) (d-y)^n \right\} \cdot \bigvee_{c \leq a}^y D^n f \\
+ (x-a) (b-x)^n \max \left\{ (y-c)^n (d-y), (y-c) (d-y)^n \right\} \cdot \bigvee_{c \leq x}^y D^n f \right]
\leq \frac{1}{n!(b-a)(d-c)} \cdot \max \left\{ (y-c)^n (d-y), (y-c) (d-y)^n \right\} \\
\times \max \left\{ (x-a)^n (b-x), (x-a) (b-x)^n \right\} \cdot \bigvee_{c \leq a}^d D^n f
\leq \frac{(b-a)^n (d-c)^n}{2^{2n+2}n!} \cdot \bigvee_{c \leq a}^d D^n f,
\]
and the first inequality in (2.17) is proved. \qed
Remark 2. Under the assumptions of Theorem [4], for all \((x, y) \in Q\), the case \(n = 0\) provides the following inequality:

\[
|B_0 (f, Q)| \leq \left[ \frac{1}{2} + \frac{|x - \frac{a+c}{2}|}{b-a} \right] \left[ \frac{1}{2} + \frac{|y - \frac{c+d}{2}|}{d-c} \right] \cdot \mathcal{V} \mathcal{V} (f) \\
\leq \frac{1}{4} \cdot \mathcal{V} \mathcal{V} (f).
\]

- **Midpoint formula:** Now, if we denote

\[
E^n_M (f; Q) = \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{2n+2} \sum_{j=1}^{n} \frac{1}{j!} \binom{n}{j} (b-a)^{n-j} (d-c)^j \left\{ D^n f (a,c) + (-1)^j D^n f (a,d) \\
+ (-1)^j D^n f (b,c) + D^n f (b,d) \right\}
\]

and

\[
F^n_M (f; Q) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d M_n (t,s) dtds (D^n f (t, s))
\]

where,

\[
M_n (t,s) = \frac{(b-a)(d-c)}{4n!} \times \begin{cases} 
\left( \frac{a+b}{2} - t \right)^n \left( \frac{c+d}{2} - s \right)^n, & a \leq t < \frac{a+b}{2}, \quad \frac{c+d}{2} < s \leq \frac{c+d}{2} \\
\left( -1 \right)^{n+1} \left( t - \frac{a+b}{2} \right)^n \left( \frac{c+d}{2} - s \right)^n, & \frac{a+b}{2} < t \leq b, \quad \frac{c+d}{2} < s \leq \frac{c+d}{2} \\
\left( -1 \right)^{n+1} \left( \frac{a+b}{2} - s \right)^n \left( s - \frac{c+d}{2} \right)^n, & a \leq t < \frac{a+b}{2}, \quad s < \frac{a+b}{2} \\
\left( t - \frac{a+b}{2} \right)^n \left( s - \frac{c+d}{2} \right)^n, & \frac{a+b}{2} < t \leq b, \quad \frac{a+b}{2} < s \leq d \\
\end{cases}
\]

then we can approximate the value of a function at its midpoint in terms of values of the function and its partial derivatives taken at the end points with error \(F_M (f; Q)\). Namely, we have the representation formula

\[
f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) = E^n_M (f; Q) + F^n_M (f; Q)
\]

The absolute value of the error can be bounded as follows:

**Corollary 2.** With the assumptions of Theorem [4] for \(f, Q\) and \(n\), we have the inequality

\[
|F^n_M (f; Q)| \leq \frac{(b-a)^n (d-c)^n}{2^{2n+2}n!} \cdot \mathcal{V} \mathcal{V} (f)
\]

For another assumptions on \(f\), the case when \(D^n f\) are absolutely continuous on \(Q\) for all \((n \geq 0)\), we give the following estimates for the remainder \(B_n (f, Q)\):

**Theorem 5.** Let \(f : Q \to \mathbb{R}\) be a real valued function which has continuous \((n-1)\) partial derivatives \((n \geq 1)\). If the \(n\)-th partial derivatives \(D^n f \ (n \geq 1)\) are...
absolutely continuous on $Q$, then for any $(x, y) \in Q$ we have

\begin{equation}
(2.22) \quad \frac{1}{(a+1)(b-a)(d-c)} \left[ (b-x)(x-a)^{n+1} + (x-a)(b-x)^{n+1} \right] \times \left[ (d-y)(y-c)^{n+1} + (y-c)(d-y)^{n+1} \right] \cdot \|D^{n+1}f\|_{Q,\infty}, \quad \text{if } D^{n+1}f \in L_{\infty}(Q);
\end{equation}

\begin{equation}
|B_n(f, Q)| \leq \frac{1}{(n+1)(b-a)(d-c)} \left[ (b-x)(x-a)^{n+1/q} + (x-a)(b-x)^{n+1/q} \right] \times \left[ (d-y)(y-c)^{n+1/q} + (y-c)(d-y)^{n+1/q} \right] \cdot \|D^{n+1}f\|_{Q,\infty}, \quad \text{if } D^{n+1}f \in L_{p}(Q), \quad p > 1; \quad \frac{1}{p} + \frac{1}{q} = 1;
\end{equation}

**Proof.** Since $D^n f$ is absolutely continuous on $Q$ then for any $(x, y) \in Q$ we have the representation

\begin{equation}
(2.23) \quad f(x, y) = \frac{1}{(b-a)(d-c)} \left[ (b-x)(d-y) + (b-x)(y-c)f(a, d) \right] \times \sum_{j=1}^{n} \frac{1}{j!} \left\{ (b-x)(x-a)^{n-j} \left[ (y-c)^{j-1} D^n f(a, c) + (-1)^j (d-y)^{j-1} D^n f(a, d) \right] \right.
\end{equation}

\begin{equation}
+ (x-a)(b-x)^{n-j} \left[ (-1)^j (y-c)^{j-1} D^n f(b, c) + (d-y)^{j-1} D^n f(b, d) \right] \right\}
\end{equation}

\begin{equation}
+ \frac{1}{(b-a)(d-c)} \int_{c}^{d} \int_{a}^{b} S_n(x, t; y, s) D^{n+1}f(t, s) \, dt \, ds,
\end{equation}

where the integral is considered in the Lebesgue sense and the kernel $S_n(x, t; y, s)$ is given in Theorem 3. Utilizing the properties of the Stieltjes integral, we have

\begin{equation}
|B_n(f; Q)| = \frac{1}{(b-a)(d-c)} \left| \int_{c}^{d} \int_{a}^{b} S_n(x, t; y, s) D^{n+1}f(t, s) \, dt \, ds \right|
\end{equation}

\begin{equation}
= \left| \frac{1}{n!} \int_{c}^{d} \int_{a}^{b} S_n(x, t; y, s) D^{n+1}f(t, s) \, dt \, ds \right|
\end{equation}

\begin{equation}
+ \int_{y}^{d} \int_{a}^{x} (-t)^{n+1} (b-x)^{n}(y-s)^{n} (d-y)^{n+1} f(t, s) \, dt \, ds
\end{equation}

\begin{equation}
+ \int_{c}^{y} \int_{x}^{b} (-1)^{n+1} (t-x)^{n} (x-a) (y-s)^{n} (d-y)^{n+1} f(t, s) \, dt \, ds
\end{equation}

\begin{equation}
+ \int_{y}^{d} \int_{x}^{b} (t-x)^{n} (x-a) (y-s)^{n} (d-y)^{n+1} f(t, s) \, dt \, ds
\end{equation}
\[
\leq \frac{1}{n!(b-a)(d-c)} \left[ \int_c^d \int_a^x (x-t)^n (b-x) (y-s)^n (d-y) D^{n+1} f(t,s) \, dt \, ds \right] \\
+ \int_y^d \int_a^x (-1)^{n+1} (x-t)^n (b-x) (s-y)^n (y-c) D^{n+1} f(t,s) \, dt \, ds \\
+ \int_y^d \int_x^b (-1)^{n+1} (t-x)^n (x-a) (y-s)^n (d-y) D^{n+1} f(t,s) \, dt \, ds \\
+ \int_y^d \int_x^b (t-x)^n (x-a) (y-s)^n (d-y) D^{n+1} f(t,s) \, dt \, ds \right] 
\]

Utilizing properties of integration together with Hölder integral inequality for the Lebesgue integral we have

\[
(2.24) \int_c^d \int_a^x (x-t)^n (y-s)^n |D^{n+1} f(t,s)| \, dt \, ds \\
= \begin{cases} 
(x-a)^{n+1+1/y} (y-c)^{n+1/y} (n+1)^y 
\cdot \|D^{n+1} f\|_{[a,x] \times [y,d], \infty}, & D^{n+1} f \in L_\infty [a,x] \times [c,y]; \\
(x-a)^{n+1/q} (y-c)^{n+1/q} (n+1)^{1/y} 
\cdot \|D^{n+1} f\|_{[a,x] \times [c,y],p}, & D^{n+1} f \in L_p [a,x] \times [c,y], \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; 
\end{cases}
\]

\[
(2.25) \int_y^d \int_a^x (s-y)^n |D^{n+1} f(t,s)| \, dt \, ds \\
= \begin{cases} 
(x-a)^{n+1+1/y} (d-y)^{n+1/y} (n+1)^y 
\cdot \|D^{n+1} f\|_{[a,x] \times [y,d], \infty}, & D^{n+1} f \in L_\infty [a,x] \times [y,d]; \\
(x-a)^{n+1/q} (d-y)^{n+1/q} (n+1)^{1/y} 
\cdot \|D^{n+1} f\|_{[a,x] \times [y,d],p}, & D^{n+1} f \in L_p [a,x] \times [y,d], \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; 
\end{cases}
\]

\[
(2.26) \int_c^d \int_x^b (t-x)^n (y-s)^n |D^{n+1} f(t,s)| \, dt \, ds \\
= \begin{cases} 
(b-x)^{n+1+1/y} (c-y)^{n+1/y} (n+1)^y 
\cdot \|D^{n+1} f\|_{[x,b] \times [y,d], \infty}, & D^{n+1} f \in L_\infty [x,b] \times [c,y]; \\
(b-x)^{n+1/q} (c-y)^{n+1/q} (n+1)^{1/y} 
\cdot \|D^{n+1} f\|_{[x,b] \times [c,y],p}, & D^{n+1} f \in L_p [x,b] \times [c,y], \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; 
\end{cases}
\]
and

\[
(2.27) \quad (b-x)^n (d-y)^n |D^{n+1} f(t,s)| \ dt \, ds
\]

\[
\begin{cases}
(b-x)^n (d-y)^n \frac{1}{(n+1)\gamma} : \|D^{n+1} f\|_{[x,b]\times[y,d],\infty}, & D^{n+1} f \in L_\infty [x,b] \times [y,d];
\end{cases}
\]

\[
\begin{cases}
(b-x)^n (d-y)^n \frac{1}{(nq+1)^{1/q}} : \|D^{n+1} f\|_{[x,b]\times[y,d],p}, & D^{n+1} f \in L_p [x,b] \times [y,d],
p > 1, \frac{1}{p} + \frac{1}{q} = 1.
\end{cases}
\]

Adding (2.24) – (2.27) we deduce that

\[
|B_n (f, Q)|
\]

\[
\leq \frac{(b-x) (d-y)}{n! (b-a) (d-c)} \times \begin{cases}
\frac{\|D^{n+1} f\|_{[a,x]\times[c,y],\infty}}{(n+1)^2}, & D^{n+1} f \in L_\infty [a,x] \times [c,y];
\end{cases}
\]

\[
\leq \frac{(b-x) (y-c)}{n! (b-a) (d-c)} \times \begin{cases}
\frac{\|D^{n+1} f\|_{[a,x]\times[y,d],\infty}}{(n+1)^2}, & D^{n+1} f \in L_\infty [a,x] \times [y,d];
\end{cases}
\]

\[
\leq \frac{(x-a) (d-y)}{n! (b-a) (d-c)} \times \begin{cases}
\frac{\|D^{n+1} f\|_{[x,b]\times[c,y],\infty}}{(n+1)^2}, & D^{n+1} f \in L_\infty [x,b] \times [c,y];
\end{cases}
\]

\[
\leq \frac{(x-a) (y-c)}{n! (b-a) (d-c)} \times \begin{cases}
\frac{\|D^{n+1} f\|_{[x,b]\times[y,d],\infty}}{(n+1)^2}, & D^{n+1} f \in L_\infty [x,b] \times [y,d];
\end{cases}
\]

\[
\leq \frac{1}{n!(n+1)^2(b-a)(d-c)} \times \begin{cases}
\frac{\|D^{n+1} f\|_{Q_\infty}}{Q_\infty}, & D^{n+1} f \in L_\infty (Q_{x,c});
\end{cases}
\]

\[
\leq \frac{1}{n!(n+1)^2(b-a)(d-c)} \times \begin{cases}
\frac{\|D^{n+1} f\|_{Q_p}}{Q_p}, & D^{n+1} f \in L_p (Q_{x,c});
\end{cases}
\]

\[
p > 1, \frac{1}{p} + \frac{1}{q} = 1;
\]
Thus, the value of the function $f$ can be approximated at its midpoint in terms of values of $f$ and its partial derivatives taken at the end points with error $F_M(f; Q)$, i.e.,

$$f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) = E_M(f; Q) + F_M(f; Q),$$

such that the error $F_M(f; Q)$ satisfies the following bounds:

**Corollary 3.** With the assumptions of Theorem 3 for $f$, $Q$ and $n$, we have the inequality

$$|F_M^n(f; Q)| \leq \frac{1}{n!} \times \begin{cases} \frac{(b-a)^{n+1}(d-c)^{n+1}}{2^{n+1}(n+1)!} \cdot \|D^{n+1}f\|_{Q,\infty}, & \text{if } D^{n+1}f \in L_\infty(Q); \\
\frac{(b-a)^{n+1/q}(d-c)^{n+1/q}}{2^{n+2/q}(nq+1)!} \cdot \|D^{n+1}f\|_{Q,p}, & \text{if } D^{n+1}f \in L_p(Q), \\
p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$

In particular, for $n = 0$, we have

$$|F_M^0(f; Q)| \leq \begin{cases} \frac{(b-a)}{4} \cdot \|Df\|_{Q,\infty}, & \text{if } Df \in L_\infty(Q); \\
\frac{(b-a)^{1/q}(d-c)^{1/q}}{4^{1/q}} \cdot \|Df\|_{Q,p}, & \text{if } Df \in L_p(Q), \\
p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$

**Remark 3.** Different kinds of Hölder continuous functions of two variable have been defined in literature, for example a function $f : Q \to \mathbb{R}$ is to be of $(\beta_1, \beta_2)$–Hölder type mapping on the co-ordinate, if for all $(t_1, s_1), (t_1, s_1) \in Q$, there exist $H_1, H_2 > 0$ and $\beta_1, \beta_2 > 0$ such that

$$|f(t_1, s_1) - f(t_2, s_2)| \leq H_1 |t_1 - t_2|^\beta_1 + H_2 |s_1 - s_2|^\beta_2.$$

If $\beta_1 = \beta_2 = 1$, then $f$ is called $(L_1, L_2)$–Lipschitz on the co-ordinate, i.e.,

$$|f(t_1, s_1) - f(t_2, s_2)| \leq L_1 |t_1 - t_2| + L_2 |s_1 - s_2|.$$

Another definition of Hölder continuous mapping can be stated as:

$$|f(t_1, s_1) - f(t_2, s_2)| \leq M_{\beta_1, \beta_2}(t_1, t_2; s_1, s_2),$$

where, $M_{\beta_1, \beta_2}(t_1, t_2; s_1, s_2)$ is any real norm. In particular, if $\beta_1 = \beta_2 = 1$, the usual Euclidean norm is well known as:

$$M_{1,1}(t_1, t_2; s_1, s_2) := \sqrt{(t_1 - s_1)^2 + (t_2 - s_2)^2}.$$

Under any type of Hölder continuity, one may has several bounds for the error bounds $E_n(f, Q)$ and $F_M^n(f, Q)$ of the function $f$. We leave this to the interested reader (see [1]).

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