Cavity evolution in relativistic self-gravitating fluids

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Abstract
We consider the evolution of cavities within spherically symmetric relativistic fluids, under the assumption that the proper radial distance between neighboring fluid elements remains constant during their evolution (purely areal evolution condition). The general formalism is deployed and solutions are presented. Some of them satisfy Darmois conditions whereas others present shells and must satisfy Israel conditions, on either one or both boundary surfaces. Prospective applications of these results to some astrophysical scenarios are suggested.

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1. Introduction

Many years ago, Skripkin [1] addressed the very interesting problem of the evolution of a spherically symmetric fluid distribution following a central explosion. As a result of the conditions imposed by Skripkin a Minkowskian cavity should surround the center of the fluid distribution.

Recently [2], this problem was studied in detail, proving that under Skripkin conditions (isotropic fluid with constant energy density distribution) the scalar expansion vanishes. It was further shown that the assumption of vanishing expansion scalar requires the existence of a cavity within the fluid distribution (of any kind). Next, it was shown in [3] that the Skripkin model is incompatible with Darmois junction conditions [4]. Also, the inhomogeneous expansion-free dust models presented in [3] are deprived of physical interest since they imply negative energy density distributions.

For the reasons above, we turn in this paper to another kinematical condition particularly suitable for describing the evolution of a fluid distribution with a cavity surrounding the center. This consists in assuming the vanishing of variation of proper radial distance between any
two infinitesimally close fluid elements per unit of proper time. We shall explore here the consequences derived from this condition (hereafter referred to as the purely areal evolution condition). In particular, we are interested in analytical models which even if they are relatively simple to analyze, still contain some of the essential features of a realistic situation. It should be emphasized that we are not interested in the dynamics and the conditions of the creation of the cavity itself, but only in its evolution once it is already formed.

We have two hypersurfaces delimiting the fluid. The external one separating the fluid distribution from a Schwarzschild or Vaidya spacetime (depending on whether we assume the evolution to be adiabatic or dissipative) and the internal one, delimiting the cavity within which we have Minkowski spacetime. It should be mentioned that for cavities with sizes of the order of 20 Mpc or smaller, the assumption of a spherically symmetric spacetime outside the cavity is quite reasonable, since the observed universe cannot be considered homogeneous on scales less than 150–300 Mpc. However, for larger cavities it should be more appropriate to consider their embedding in an expanding Lemaître–Friedmann–Robertson–Walker spacetime (for the specific case of void modeling in expanding universes see [5, 6] and references therein).

Thus, we have to consider junction conditions on both hypersurfaces. Depending on whether we impose Darmois conditions [4] or allow for the existence of thin shells [7], different kinds of models are obtained. In this paper we shall focus mainly on models satisfying Darmois conditions, although some models presenting thin shells will be briefly described too.

For the sake of generality we shall start our discussion by considering an anisotropic dissipative fluid (arguments to justify such kind of fluid distributions may be found in [8–10] and references therein). A detailed description of this kind of distribution, as well as definitions of kinematical and other important variables, is given in section 2. The Darmois junction conditions on both the inner and the outer boundary surface are briefly discussed in section 3.

In order to better understand the physical meaning of the purely areal evolution condition, we shall discuss two different definitions of radial velocity of a fluid element, in terms of which both the expansion and the shear can be expressed, and which renders intelligibly the origin of the term we use to denote such a condition in section 4.

We shall next deploy all the equations required for cavity modeling under the purely areal evolution condition in section 5. The specific case of cavities satisfying Darmois conditions, on both hypersurfaces, is treated in section 6, whereas cavities presenting shells are considered in section 7.

Finally, a brief summary of the results is presented in the last section and prospective applications of these results are briefly mentioned.

2. Fluid distributions and kinematical variables

We consider a spherically symmetric distribution of fluid, bounded by a spherical surface $\Sigma^{(e)}$. The fluid is assumed to be locally anisotropic, with principal stresses unequal, and undergoing dissipation in the form of heat flow (diffusion approximation).

Choosing comoving coordinates inside $\Sigma^{(e)}$, the general interior metric can be written as

$$ds_2^2 = -A^2 dt^2 + B^2 dr^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $A$, $B$ and $R$ are the functions of $t$ and $r$ and are assumed positive. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$. Observe that $A$ and $B$ are dimensionless, whereas $R$ has the same dimension as $r$.2
From (1) we have that inside $\Sigma^\epsilon$ any spherical surface has its proper radius given by $\int B \, dr$ and its areal radius by $R$.

The matter energy–momentum $T_{\alpha\beta}^-$ inside $\Sigma^\epsilon$ has the form

$$T_{\alpha\beta}^- = (\mu + P_\perp) V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp) \chi_\alpha \chi_\beta + q_\alpha V_\beta + V_\alpha q_\beta,$$

where $\mu$ is the energy density, $P_r$ is the radial pressure, $P_\perp$ is the tangential pressure, $q^\alpha$ is the heat flux, $V^\alpha$ is the four-velocity of the fluid and $\chi^\alpha$ is a unit four-vector along the radial direction. These quantities satisfy

$$V^\alpha V_\alpha = -1, \quad V^\alpha q_\alpha = 0, \quad \chi^\alpha \chi_\alpha = 1, \quad \chi^\alpha V_\alpha = 0.$$

The four-acceleration $a_\alpha$ and the expansion $\Theta$ of the fluid are given by

$$a_\alpha = V_\alpha \dot{V}^\beta, \quad \Theta = V^\alpha \dot{V}_\alpha, \quad (4)$$

and its shear $\sigma_{\alpha\beta}$ by

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha} V_{\beta)} - \frac{1}{3} \Theta h_{\alpha\beta}, \quad (5)$$

where

$$h_{\alpha\beta} = g_{\alpha\beta} + V_\alpha V_\beta. \quad (6)$$

Since we assumed metric (1) comoving, then

$$V^\alpha = A^{-1} \delta^\alpha_0, \quad q^\alpha = q B^{-1} \delta^\alpha_0, \quad \chi^\alpha = B^{-1} \delta^\alpha_1, \quad (7)$$

where $q$ is a function of $t$ and $r$. From (4) with (7) we have the non-zero component for the four-acceleration and its scalar:

$$a_1 = \frac{A'}{A}, \quad a = (a^\alpha a_\alpha)^{1/2} = \frac{A'}{AB}, \quad (8)$$

and for the expansion

$$\Theta = \frac{1}{A} \left( \frac{\dot{B}}{B} + 2 \frac{\dot{R}}{R} \right), \quad (9)$$

where the prime stands for $r$ differentiation and the dot stands for differentiation with respect to $t$. With (7) we obtain for shear (5) its non-zero components

$$\sigma_{11} = \frac{2}{3} B^2 \sigma, \quad \sigma_{22} = \sigma_{33} = -\frac{1}{3} R^2 \sigma, \quad (10)$$

and its scalar

$$\sigma_{\alpha\beta} \sigma^{\alpha\beta} = \frac{2}{3} \sigma^2, \quad (11)$$

where

$$\sigma = \frac{1}{A} \left( \frac{B}{B} - \frac{R}{R} \right). \quad (12)$$

Also observe that the shear tensor may be written as

$$\sigma_{\alpha\beta} = \sigma \left( \chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta} \right). \quad (13)$$

Sometimes it could be convenient [11, 12] to express the energy–momentum tensor (2) in the form

$$T_{\alpha\beta}^- = \mu V_\alpha V_\beta + \hat{P} h_{\alpha\beta} + \Pi_{\alpha\beta} + q (V_\alpha \chi_\beta + \chi_\alpha V_\beta) \quad (14)$$
with
\[ \dot{P} = \frac{1}{3} h_{\alpha \beta} T^{\alpha \beta} = \frac{P_r + 2 P_\perp}{3}, \]
\[ \Pi^{\alpha \delta} = \left( h^{\alpha \beta} h_{\delta \gamma} - \frac{1}{3} h^{\alpha \delta} h_{\gamma \beta} \right) T^{\gamma \delta} = \Pi \left( \chi^\alpha \chi^\beta - \frac{1}{3} h^{\alpha \beta} \right). \]
\[ \Pi = P_r - P_\perp. \]

Next, the mass function \( m(t, r) \) introduced by Misner and Sharp [13] (see also [14]) reads
\[ m = \frac{R^3}{2} R_{23}^{23} = \frac{R}{2} \left( \frac{\dot{R}}{A} \right)^2 - \left( \frac{R'}{B} \right)^2 + 1. \] (15)

To study the dynamical properties of the system, let us introduce, following Misner and Sharp [13], the proper time derivative \( D_T \) given by
\[ D_T = \frac{1}{A} \frac{\partial}{\partial t}. \] (16)

Using (16) we can define the velocity \( U \) of the collapsing fluid (for another definition of velocity see section 4) as the variation of the areal radius with respect to proper time, i.e.
\[ U = D_T R < 0 \] (in the case of collapse). (17)

Then (15), by using (17), can be rewritten as
\[ E \equiv \frac{R'}{B} = \left( 1 + U^2 - \frac{2m}{R} \right)^{1/2}. \] (18)

Using field equations (see [2] for details) with (16) and (17) we obtain from (15)
\[ m' = 4\pi \left( \mu + \frac{U}{E} \right) R' R^2, \] (19)
which implies
\[ m = 4\pi \int_0^r \left( \mu + \frac{U}{E} \right) R^2 R' \, dr, \] (20)
where we assumed a regular center to the distribution, so \( m(0) = 0 \).

It will be useful to introduce the Weyl tensor. Thus, let \( E_{\alpha \beta} \) denote the ‘electric’ part of the Weyl tensor (in the spherically symmetric case the ‘magnetic’ part of the Weyl tensor vanishes, \( H_{\alpha \beta} = 0 \)) defined by
\[ E_{\alpha \beta} = C_{\alpha \mu \beta \nu} V^\mu V^\nu, \] (21)
which may be written as
\[ E_{\alpha \beta} = \mathcal{E} \left( \chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha \beta} \right), \] (22)
where
\[ \mathcal{E} = \frac{1}{2A^2} \left[ \frac{\dot{R}}{R} - \frac{\dot{B}}{B} - \left( \frac{\dot{R}}{R} - \frac{\dot{B}}{B} \right) \left( \frac{\dot{A}}{A} + \frac{\dot{R}}{R} \right) \right] + \frac{1}{2B^2} \left[ \frac{A''}{A} - \frac{R''}{R} + \left( \frac{B'}{B} + \frac{R'}{R} \right) \left( \frac{R'}{R} - \frac{A'}{A} \right) \right] - \frac{1}{2R^2}. \] (23)

Using field equations (see [2] for details) and the definition of mass function (15) we may write \( \mathcal{E} \) as
\[ \mathcal{E} = 4\pi \left( \mu - P_r + P_\perp \right) - \frac{3m}{R^3}. \] (24)
3. The exterior spacetime and junction conditions

Outside $\Sigma^{(e)}$ we assume that we have the Vaidya spacetime (or Schwarzschild in the dissipationless case), i.e. we assume that all outgoing radiation is massless, described by

$$\text{d}s^2 = - \left(1 - \frac{2M(v)}{r}\right) \text{d}v^2 - 2 \frac{dr}{d\nu} dv + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2),$$  \hspace{1cm} (25)

where $M(v)$ denotes the total mass, and $v$ is the retarded time. The matching of the non-adiabatic sphere to the Vaidya spacetime, on the surface $r = r_{\Sigma^{(e)}} = \text{constant}$, in the absence of thin shells, where Darmois conditions hold, is discussed in [9, 15–17]. This requires the continuity of the first and the second fundamental forms through the matching hypersurface, producing

$$m(t, r) \big|_{\Sigma^{(e)}} = M(v),$$  \hspace{1cm} (26)

and

$$2 \left( \frac{R'}{R} - \frac{B R'}{R A} - \frac{R A'}{R A} \right) \big|_{\Sigma^{(e)}} = - \frac{B}{A} \left[ \frac{2}{A} - \frac{R'}{R} \right]$$

$$+ \frac{A}{B} \left[ \frac{2}{A} + \frac{R'}{R} \right] - \left( \frac{B}{R} \right)^2,$$  \hspace{1cm} (27)

and

$$q \big|_{\Sigma^{(e)}} = \frac{L}{4\pi r},$$  \hspace{1cm} (28)

where $\Sigma^{(e)}$ means that both sides of the equation are evaluated on $\Sigma^{(e)}$ and $L$ denotes the total luminosity of the sphere as measured on its surface and is given by

$$L = L_{\infty} \left(1 - \frac{2m}{r} + 2 \frac{dr}{d\nu}\right)^{-1},$$  \hspace{1cm} (29)

and where

$$L_{\infty} = \frac{dM}{d\nu}$$  \hspace{1cm} (30)

is the total luminosity measured by an observer at rest at infinity.

From (27) and field equations one obtains

$$q \big|_{\Sigma^{(e)}} = P_r.$$  \hspace{1cm} (31)

In the case when a cavity forms, then we also have to match the solution to the Minkowsky spacetime on the boundary surface delimiting the cavity. If we call $\Sigma^{(i)}$ the boundary surface between the cavity and the fluid, then the matching of the Minkowski spacetime within the cavity to the fluid distribution implies

$$m(t, r) \big|_{\Sigma^{(i)}} = 0,$$  \hspace{1cm} (32)

$$q \big|_{\Sigma^{(i)}} = P_r.$$  \hspace{1cm} (33)

However, since we are assuming our cavity to be empty, then $L \big|_{\Sigma^{(i)}} = 0$, which implies

$$q \big|_{\Sigma^{(i)}} = P_r \big|_{\Sigma^{(i)}} = 0.$$  \hspace{1cm} (34)

If we allow for the presence of thin shells on $\Sigma^{(i)}$ and/or $\Sigma^{(e)}$, then we have to relax the above conditions and allow for discontinuities in the mass function [7].

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4. Two definitions of radial velocity and the purely areal evolution condition

In section 2 we introduced the variable $U$ which, as mentioned before, measures the variation of the areal radius $R$ per unit proper time.

Another possible definition of ‘velocity’ may be introduced as the variation of the infinitesimal proper radial distance between two neighboring points ($\delta l$) per unit of proper time, i.e. $D_T(\delta l)$. Then, it can be shown that (see [2] for details)

$$\frac{D_T(\delta l)}{\delta l} = \frac{1}{3} (2\sigma + \Theta),$$

or, by using (9) and (12),

$$\frac{D_T(\delta l)}{\delta l} = \frac{B}{AB}.$$  

Then with (9), (12), (17) and (36) we can write

$$\sigma = \frac{D_T(\delta l)}{\delta l} - \frac{D_T R}{R} = \frac{D_T(\delta l)}{\delta l} - \frac{U}{R},$$

and

$$\Theta = \frac{D_T(\delta l)}{\delta l} + \frac{2D_T R}{R} = \frac{D_T(\delta l)}{\delta l} + \frac{2U}{R}.$$  

Thus, the ‘circumferential’ (or ‘areal’) velocity $U$ is related to the change of the areal radius $R$ of a layer of matter, whereas $D_T(\delta l)$ has also the meaning of ‘velocity’, being the relative velocity between neighboring layers of matter, and can be in general different from $U$.

In [2] it was shown that the condition $\Theta = 0$ requires the existence of a cavity surrounding the center of the fluid distribution.

Let us now consider the condition $D_T(\delta l) = 0$, but $U \neq 0$. From the comments above it is evident why we shall refer to it as the purely areal evolution condition.

Now, if $D_T(\delta l) = 0$, then it follows from (37) and (38) that $\Theta = -2\sigma$; feeding this back into the (01) component of the Einstein field equations (see [2]) for details) we obtain

$$\sigma' + \frac{\sigma R'}{R} = -\frac{4\pi q R'}{E},$$

whose integration with respect to $r$ yields

$$\sigma = \frac{\xi(t)}{R} - \frac{4\pi}{R} \int_0^r \frac{q R'}{E} \, dr,$$

where $\xi$ is an integration function of $t$. It should be observed that in the case where the fluid fills all the sphere, including the center ($r = 0$), we should impose the regularity condition $\xi = 0$. However, since we consider the possibility of a cavity surrounding the center, such a condition is not required.

On the other hand, (40) with (37) implies

$$U = -\xi + 4\pi \int_0^r \frac{q R'}{E} \, dr.$$  

Thus, in the non-dissipative case the purely areal evolution condition implies that $U = U(t)$. This condition is clearly incompatible with a regular symmetry center, unless $U = 0$. Therefore, if we want the purely areal evolution condition to be compatible with a time-dependent situation ($U \neq 0$), we must assume that either
• the fluid has no symmetry center,
  or
• the center is surrounded by a compact spherical section of another spacetime, suitably
  matched to the rest of the fluid.

Here we shall discard the first possibility since we are particularly interested in describing
localized objects without the unusual topology of a spherical fluid without a center. Also,
within the context of the second alternative we have chosen an inner vacuum Minkowski
spherical vacuole.

Let us now consider the dissipative case. Then assuming the purely areal evolution
condition, if the fluid fills the whole sphere (no cavity surrounding the center), and we have a
symmetry center, we have to put $\zeta = 0$ and (41) becomes

$$U = 4\pi \int_0^r q \frac{R'}{E} \, dr,$$

(42)

which is not incompatible with a regular symmetry center. In this case we shall assume in
an ad hoc manner that a cavity surrounds the center. However, this assumption is somehow
suggested by the following qualitative argument.

In the case of an outwardly directed flux vector ($q > 0$), all terms within the integral
are positive and we obtain from (38) and (42) that $\Theta > 0$ and $U > 0$. Now, during the
Kelvin–Helmholtz phase of evolution \[18\], when all the dissipated energy comes from the
gravitational energy, we should expect a contraction, not expansion, to be associated with an
outgoing dissipative flux. Inversely, an inwardly directed flux ($q < 0$) (during that phase)
would produce an overall expansion, not a contraction as it follows from (42).

Thus, we have seen that the purely areal evolution condition appears to be particularly
suitable to describe the evolution of a fluid distribution with a cavity surrounding the center.

Finally observe that using (37) and (38) in (13), the purely areal evolution condition can
be expressed in a covariant form as

$$\sigma_{\alpha\beta} = -\frac{\Theta}{2} \left( \chi_{\alpha} \chi_{\beta} - \frac{1}{3} h_{\alpha\beta} \right).$$

(43)

In the next section we shall consider some models.

5. Models of cavities

We shall now study the general properties of models satisfying the purely areal evolution
condition.

The general picture is similar to that proposed by Skripkin [1], namely an explosion at
the center initiates an overall expansion of the fluid, creating a cavity surrounding the center.
The difference here is that we shall not assume $\Theta = 0$ but instead, $D_T (\delta l) = 0$. Thus, we
have because of (36) $B = 0$ (but $R \neq 0$) which means that $B = B(r)$ and it can be chosen

$$B = 1,$$

(44)

with no loss of generality. As mentioned before, the physical appeal of this kind of models
stems from the fact that the condition $\dot{B} = 0$ requires for consistency, the existence of a cavity
surrounding the center.
Then the Einstein field equations become

$$8\pi \mu = \frac{1}{A^2} \left( \frac{\dot{R}}{R} \right)^2 - 2\frac{R''}{R} - \left( \frac{R'}{R} \right)^2 + \frac{1}{R^2},$$  \hspace{1cm} (45)

$$8\pi q = \frac{2}{A} \left( \frac{R'}{R} - \frac{\dot{R} A'}{AR} \right),$$  \hspace{1cm} (46)

$$8\pi P_r = -\frac{1}{A^2} \left[ 2\frac{\dot{R}}{R} - \left( \frac{2\dot{A}}{A} - \frac{\dot{R}}{R} \right) \frac{\dot{R}}{R} \right] + \left( \frac{2A'}{A} + \frac{R'}{R} \right) R' - \frac{1}{R^2},$$  \hspace{1cm} (47)

$$8\pi P_\perp = -\frac{1}{A^2} \left( \frac{\dot{R}}{R} - \frac{\dot{A}}{AR} \right) + \frac{A''}{A} + \frac{R''}{R} + \frac{A'}{A} R' - \frac{1}{R^2},$$  \hspace{1cm} (48)

and the non-trivial components of the Bianchi identities, $T_{\alpha\beta}^{\gamma} = 0$, become

$$\frac{1}{A} \left[ \mu + 2(\mu + P_\perp) \frac{R}{R} \right] + q' + 2q \frac{(AR)'}{AR} = 0,$$  \hspace{1cm} (49)

$$\frac{1}{A} \left( \dot{q} + 2q \frac{\dot{R}}{R} \right) + P' + \mu + P r \frac{A'}{A} + 2(P_r - P_\perp) \frac{R'}{R} = 0.$$  \hspace{1cm} (50)

In the particular geodesic case we have $A' = 0 \rightarrow A = 1$. Then the field equations (45)–(48) read

$$8\pi \mu = \left( \frac{\dot{R}}{R} \right)^2 - 2\frac{R''}{R} - \left( \frac{R'}{R} \right)^2 + \frac{1}{R^2},$$  \hspace{1cm} (51)

$$8\pi q = \frac{2R'}{R},$$  \hspace{1cm} (52)

$$8\pi P_r = \left[ 2\frac{\dot{R}}{R} + \left( \frac{\dot{R}}{R} \right)^2 \right] + \left( \frac{R'}{R} \right)^2 - \frac{1}{R^2},$$  \hspace{1cm} (53)

$$8\pi P_\perp = -\frac{R}{R} + \frac{R''}{R},$$  \hspace{1cm} (54)

and it follows with (15)

$$2\pi \mu = \frac{m}{R^3} + 2\pi (P_r - 2P_\perp),$$  \hspace{1cm} (55)

or, using (24),

$$2\pi \mu = 2\pi (P_r - 4P_\perp) - \mathcal{E}. $$  \hspace{1cm} (56)

Hence, from (55) and (56) we have for conformally flat spacetime $\mathcal{E} = 0$, and geodesic fluids with isotropic pressures $P_r = P_\perp = P$,

$$\frac{m}{R^3} + 4\pi P = 0,$$  \hspace{1cm} (57)

implying that such models, if they satisfy Darmois conditions, must be dissipative (otherwise $M = 0$), and absorbing energy ($q < 0$, otherwise $m < 0$).
6. Models satisfying Darmois conditions

Let us now consider some simple cases in order to find analytical models which do not present thin shells on either \( \Sigma^{(e)} \) or \( \Sigma^{(i)} \), but holding Darmois conditions.

The simplest models of this kind we have found are non-dissipative. Thus, considering \( q = 0 \) then from (46) after integration we have

\[
A = \frac{R}{h_1},
\]

(58)

where \( h_1(t) \) is an arbitrary function of \( t \). Reparametrizing \( t \), we may choose without loss of generality

\[
h_1 = \frac{\dot{R}}{\Sigma^{(i)}},
\]

(59)

which amounts to choosing

\[
A_{\Sigma^{(i)}} = 1.
\]

(60)

Observe that for all these models the velocity \( U \) is the same for all fluid elements, since, as it follows from (17) and (58) that

\[
U = h_1 = \frac{\dot{R}}{\Sigma^{(i)}}.
\]

(61)

This fact was already brought out in the previous section from (41).

Substituting (58) into (45), (47) and (48) we obtain, using (16), (58) and (59),

\[
8\pi \mu = -\frac{1}{R^2} \left( 2RR'' + R^2 - \dot{R}^2_{\Sigma^{(i)}} - 1 \right),
\]

(62)

\[
8\pi P_r = \frac{1}{R^2 R_{\Sigma^{(i)}}} D_T \left[ R (R^2 - \dot{R}^2_{\Sigma^{(i)}} - 1) \right],
\]

(63)

\[
8\pi P_{\perp} = \frac{1}{2R R_{\Sigma^{(i)}}} D_T \left( 2RR'' + R^2 - \dot{R}^2_{\Sigma^{(i)}} - 1 \right).
\]

(64)

From (62) and (64) it is clear that

\[
P_{\perp} = -\frac{D_T (\mu R^2)}{2RR_{\Sigma^{(i)}}}.
\]

(65)

Calculating the mass function (15) with (44) and (58) it yields

\[
m = -\frac{R}{2} \left( R^2 - \dot{R}^2_{\Sigma^{(i)}} - 1 \right),
\]

(66)

which allows us to re-express (63) like

\[
4\pi P_r = -\frac{m}{R^2 R}.
\]

(67)

The boundary conditions \( P_r^{\Sigma^{(i)}} = 0 \) and \( P_r^{\Sigma^{(e)}} = 0 \) are automatically satisfied with \( m = M = \) constant and \( m \equiv 0 \).

With (58) we obtain for (23)

\[
\mathcal{E} = \frac{R}{4R_{\Sigma^{(i)}}} D_T \left[ \frac{1}{R^2} \left( 2RR'' + R^2 - \dot{R}^2_{\Sigma^{(i)}} + 1 \right) \right].
\]

(68)

We shall next specialize to some particular cases.
6.1. Conformally flat models

If we assume the spacetime between \( r = r_{\Sigma(0)} \) and \( r = r_{\Sigma(0)} \) to be conformally flat \( \mathcal{E} = 0 \), then it follows from (68) that

\[
2RR'' + R^2 - f_1 R^2 + \dot{R}_{\Sigma(0)}^2 + 1 = 0, \tag{69}
\]

where \( f_1(r) \) is an arbitrary function of \( r \), and integrating again we obtain

\[
R^2 = R \left( \int f_1 \, dR + h_2 \right) + \dot{R}_{\Sigma(0)}^2 + 1, \tag{70}
\]

where \( h_2(t) \) is an arbitrary function of \( t \). Comparing (66) with (70) it follows that

\[
m = -\frac{R^2}{2} \left( \int f_1 \, dR + h_2 \right); \tag{71}
\]

therefore, \( h_2 \) may be obtained from the junction condition (32), producing

\[
h_2 \Sigma(0) = -\int f_1 \, dR. \tag{72}
\]

Thus, all models of this kind are defined by a single function \( f_1(r) \) which should be chosen so as to satisfy the remaining Darmois conditions.

Before proceeding further, the following remark is in order: all spherically symmetric conformally flat spacetimes (without dissipation) and isotropic fluids are shear-free (see equation (78) in [10]), but this is no longer true for anisotropic fluids (see equation (84) in [10]). Therefore, the models to be considered here are necessarily anisotropic.

6.2. Models with vanishing tangential stresses

Assuming \( P_\perp = 0 \) we have from (65) after integration

\[
\mu = \frac{f_2}{R^2}, \tag{73}
\]

where \( f_2(r) \) is an arbitrary function of \( r \). Substituting (73) into (62) it yields

\[
2RR'' + R^2 + 8\pi f_2 - \dot{R}_{\Sigma(0)}^2 - 1 = 0, \tag{74}
\]

and with (66) it becomes

\[
m' = 4\pi f_2 R'. \tag{75}
\]

In order to obtain models we have to assume a specific form of the mass function or the energy density distribution. As an example let us assume

\[
f_2 = c_1 = \text{constant} > 0, \tag{76}
\]

producing, due to (20) and (73)

\[
m = 4\pi c_1 (R - R_{\Sigma(0)}), \tag{77}
\]

and implying

\[
M = 4\pi c_1 (R_{\Sigma(0)} - R_{\Sigma(0)}), \tag{78}
\]

\[
\dot{R}_{\Sigma(0)} = \dot{R}_{\Sigma(0)}, \quad A_{\Sigma(0)} = A_{\Sigma(0)} = 1. \tag{79}
\]

Also, from (67), (73) and (77) it follows that

\[
P_r = \mu \left( \frac{R_{\Sigma(0)}}{R} - 1 \right). \tag{80}
\]
Next, substituting (77) into (66) we have
\[ RR' = \alpha R + \beta, \] (81)
where
\[ \alpha(t) = \dot{R}_0^2 + 1 - 8\pi c_1, \quad \beta(t) = 8\pi c_1 R_0, \] (82)
and after integration
\[ [\alpha R(\alpha R + \beta)]^{1/2} - \beta \ln[(\alpha R)^{1/2} + (\alpha R + \beta)^{1/2}] = \alpha^{3/2} [r - r_0], \] (83)
where \( r_0(t) \) is an arbitrary function of \( t \). Evaluating (83) on \( \Sigma_{(i)} \) we obtain
\[ \left[ [\alpha (\dot{R}_0^2 + 1)]^{1/2} - 8\pi c_1 \ln[\alpha^{1/2} + (\dot{R}_0^2 + 1)^{1/2}] - 4\pi c_1 \ln R \right] R \equiv a^{3/2} (r - r_0). \] (84)
This is a first-order differential equation for \( R/\Sigma_{(i)} \) which may be solved for any function \( r_0(t) \).

The result of this integration, together with (83), provides all the information required to obtain the \( t \) and \( r \) dependence of all physical and metric variables. Observe that the energy density is always positive and regular everywhere within the fluid distribution. Also, choosing \( r_0(t) \) such that \( 0 < \dot{h}_3 - f_3^2 < 1 \) we assure that the pressure is positive and smaller than the energy density.

7. Models with thin shells

As it is apparent from the discussion above, the fulfillment of Darmois conditions on both \( \Sigma_{(e)} \) and \( \Sigma_{(i)} \) severely restricts the possible models of cavities. Therefore, it might be pertinent to relax Darmois conditions and work within the thin wall approximation, which allows for the existence of discontinuities of the mass function across \( \Sigma_{(e)} \) and/or \( \Sigma_{(i)} \) (for models of voids within the thin wall approximation see [19–21] and references therein).

7.1. Non-dissipative geodesic model

The simplest model of this kind (under the purely areal evolution condition) corresponds to a Lemaître–Tolman–Bondi (LTB) spacetime [22–24], whose general line element is given by
\[ ds^2 = -A^2 \, dt^2 + \frac{R^2}{1 - K} \, dr^2 + R^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \] (85)
where \( K \) is a function of \( r \). Then, the purely areal evolution condition applied to (85) produces
\[ R' = (1 - K)^{1/2}, \] (86)
and since all LTB spacetimes are geodesic and dissipationless, (52) implies
\[ R = h_3 + f_3, \] (87)
where \( h_3(t) \) and \( f_3(r) \) are the arbitrary functions of \( t \) and \( r \) respectively. Then for the mass function (15) we obtain
\[ m = R \left( \frac{h_3}{2} - f_3^2 + 1 \right). \] (88)

Imposing Darmois conditions on \( \Sigma_{(i)} \) we obtain from (32) with (86)–(88),
\[ h_3 = (-K_{\Sigma_{(i)}})^{1/2} t + c_2, \] (89)
where \( c_2 \) is an arbitrary constant and we have to assume the condition \( K_{\Sigma_{(i)}} < 0 \). Once we have imposed Darmois conditions on \( \Sigma_{(i)} \), it follows that we have to assume the presence of a shell on \( \Sigma_{(i)} \). Indeed, the continuity of the mass function \( m \) on \( \Sigma_{(i)} \) implies from (26)
\[ R_{\Sigma_{(i)}} = \frac{2M}{h_3^2 - f_3^2 + 1}. \] (90)
which, because of (89) and the fact that $M$ is constant, produces

$$
\dot{R} \Sigma^{(i)} = 0,
$$
(91)
implying, because of (87),

$$
\dot{h}_3 = 0 \rightarrow \dot{R} = 0,
$$
(92)
thereby invalidating the continuity of mass function across $\Sigma^{(i)}$. Thus, all models of this kind should admit a thin shell on $\Sigma^{(i)}$. Alternatively, we may assume discontinuities of the mass function on $\Sigma^{(i)}$ or on both boundaries.

### 7.2. Geodesic radiating dust models

In an increasing order of complexity, let us consider the next simplest possible situation, namely geodesic $a = 0$, dust $P_r = P_\perp = 0$ with dissipation $q \neq 0$. It should be stressed that in the dissipative case, the pure dust condition, $P_r = P_\perp = 0$, does not imply vanishing four-acceleration $a$, as it can be seen from (50).

From (54) it follows at once

$$
\ddot{R} = R'',
$$
(93)
whose general solution is of the form

$$
R = c_3 \Psi(t + r) + c_4 \Phi(t - r),
$$
(94)
where $c_3$ and $c_4$ are the constants. Considering (93) together with (51) and (53) produces

$$
2\pi \mu = -\frac{R''}{R} = -\frac{\dot{R}}{R},
$$
(95)
and (50) in our case reduces to

$$
\dot{q} + 2q \frac{\dot{R}}{R} = 0,
$$
(96)
implying

$$
q = f_4 \frac{R}{R^2},
$$
(97)
where $f_4(r)$ is an arbitrary function of $r$.

We shall now assume that our model satisfies Darmois conditions on $\Sigma^{(i)}$; then from (34) it follows that

$$
f_4 \Sigma^{(i)} = 0.
$$
(98)
Next, from (8), (17) and (46) we have

$$
\dot{R} = -\frac{m}{R^2},
$$
(99)
and evaluating it at the cavity boundary by using (32) we obtain

$$
R \Sigma^{(i)} = c_5 t + c_6
$$
(100)
where $c_5$ and $c_6$ are the constants. Observe that this is consistent with (94). Also it follows from (95) that

$$
\mu \Sigma^{(i)} = 0.
$$
(101)
A further restriction on $f_4$ may be obtained from (49), which in our case reads

$$
\dot{\mu} + 2\mu \frac{\dot{R}}{R} + q' + 2q \frac{R'}{R} = 0.
$$
(102)
Evaluating (102) on the boundary of the cavity and using (98) and (101), we have
\[ f_4'_{\Sigma(e)} = 0. \tag{103} \]

Once we have assumed that Darmois conditions are satisfied on \( \Sigma^{(i)} \), then it follows that they are violated on \( \Sigma^{(e)} \). Indeed, if we assume continuity of the mass function across \( \Sigma^{(e)} \), then evaluating (99) on \( \Sigma^{(e)} \) and using (26) we obtain
\[ R'_{\Sigma^{(e)}} = \frac{M}{R^2}, \tag{104} \]

which can be integrated to obtain
\[ R_{\Sigma^{(e)}} = \pm \left( \frac{2M}{R} + c_7 \right)^{1/2}, \tag{105} \]

where \( c_7 \) is a constant. Combining (97) and (102) we obtain
\[ \mu R^2 = f_5 - tf_4', \tag{106} \]

where \( f_5(r) \) is an arbitrary function of \( r \). Then using (95) and (104) in (106) and evaluating at \( \Sigma^{(e)} \) it follows that
\[ R_{\Sigma^{(e)}} \equiv \frac{M}{2\pi(f_5 - tf_4')}, \tag{107} \]

and by using (105) becomes
\[ R_{\Sigma^{(e)}} = \text{constant}, \tag{108} \]

thereby ruling out the possibility of the continuity of the mass function across \( \Sigma^{(e)} \). Observe that in this particular case, there should be always a shell on \( \Sigma^{(e)} \).

The only constraints imposed on the functions \( f_4 \), \( \Psi_1 \) and \( \Phi_1 \) are that they must be regular and positive, so that the regularity and positiveness of \( \mu \) is assured for all values of \( t \) and \( r \).

### 7.3. Non-geodesic models

So far all presented models within the thin wall approximation have been geodesic; therefore, it would be instructive to present a non-geodesic model. For that purpose, we shall invoke an ansatz which proved to be very useful for describing dissipative collapse [16].

Thus, let us assume
\[ A = A_0(r), \quad R = R_0(r)g(t), \tag{109} \]

where we take \( A_0 \) and \( R_0 \) to describe a static anisotropic perfect fluid whose energy density \( \mu_0 \) and anisotropic pressures \( P_{\perp0} \) and \( P_{\parallel0} \) are given by
\[ 8\pi \mu_0 = -2 \frac{R_{\parallel0}^0}{R_0^0} - \left( \frac{R_{\parallel0}^0}{R_0^0} \right)^2 + \frac{1}{R_0^0}, \tag{110} \]
\[ 8\pi P_{\parallel0} = \left( \frac{2A_0^0}{A_0} + \frac{R_{\parallel0}^0}{R_0} \right) \frac{R_{\parallel0}^0}{R_0} - \frac{1}{R_0^2}, \tag{111} \]
\[ 8\pi P_{\perp0} = \frac{A_0^0}{A_0} + \frac{R_{\perp0}^0}{R_0} + \frac{A_0^0}{A_0} \frac{R_{\perp0}^0}{R_0}. \tag{112} \]

With (109)–(112) we can rewrite (45)–(48) like
\[ 8\pi \mu = \kappa \mu_0 + \frac{1}{A_0^2} \left( \frac{\dot{g}}{g} \right)^2 + \frac{1}{R_0^2} \left( \frac{1}{g^2} - 1 \right), \]  \hspace{1cm} (113)

\[ 8\pi q = \frac{2}{A_0} \left( \frac{R_0}{A_0} - \frac{A_0'}{A_0} \right) \frac{\dot{g}}{g}, \]  \hspace{1cm} (114)

\[ 8\pi P_r = \kappa P_{r0} - \frac{1}{A_0^2} \left[ \frac{\ddot{g}}{g} + \left( \frac{\dot{g}}{g} \right)^2 \right] - \frac{1}{R_0^2} \left( \frac{1}{g^2} - 1 \right), \]  \hspace{1cm} (115)

\[ 8\pi P_{\perp} = 8\pi P_{\perp0} - \frac{1}{A_0^2} \frac{\ddot{g}}{g}. \]  \hspace{1cm} (116)

All models of this kind present shells in either \( \Sigma^{(i)} \) or \( \Sigma^{(e)} \). Indeed, evaluating \( (115) \) on \( \Sigma^{(i)} \) and assuming Darmois conditions there, we obtain
\[ 2g\ddot{g} + \dot{g}^2 - c_0^2 (g^2 - 1) = 0, \]  \hspace{1cm} (117)

where
\[ c_0^{\Sigma^{(i)}} \equiv \frac{A_0}{R_0}. \]  \hspace{1cm} (118)

Then integration of \( (117) \) produces
\[ \dot{g}^2 = c_0^2 \left( \frac{g^2}{3} - 1 \right) + \frac{c_8}{g}, \]  \hspace{1cm} (119)

where \( c_8 \) is a constant. Finally, evaluating the mass function \( (15) \) on \( \Sigma^{(i)} \), and using \( (119) \) it follows that
\[ g(t) = \text{constant} \]  \hspace{1cm} (120)

implying the necessary violation of Darmois conditions on \( \Sigma^{(i)} \).

Choosing a physical meaningful static (‘seed’) solution, it is not difficult to choose a function \( g \) such that standard energy conditions are satisfied. Thus, for example if we demand \( g^2 < 1 \) and \( \frac{\dot{g}}{g} > 0 \), we assure those conditions for the models.

8. Conclusions

We have studied in detail the consequences emerging from the purely areal evolution condition. It has been shown that such a condition is particularly suitable for describing the evolution of a fluid distribution endowed with a cavity surrounding the center. All equations governing the dynamics under such a condition have been written down and some models have been presented. Some of them satisfy Darmois conditions on both hypersurfaces, \( \Sigma^{(i)} \) and \( \Sigma^{(e)} \), precluding thereby the appearance of shells on either of these hypersurfaces. More simple models result from relaxing Darmois conditions and adopting Israel junction conditions across shells.

One possible application of the presented results which comes to our minds is the modeling of evolution of cosmic voids. Indeed, the cavity associated with the purely areal evolution condition might be considered as a void precursor. Voids are, roughly speaking, underdensity regions in the large-scale matter distribution in the universe (see \cite{25–33} and references therein). Their relevance stems from the fact that, as stressed in \cite{34}, the actual universe has a spongelike structure, dominated by voids. Indeed, observations suggest \cite{35} that some 40–50% of the present volume of the universe is in voids of a characteristic scale 30 h\(^{-1}\) Mpc,
where \( h \) is the dimensionless Hubble parameter, \( H_0 = 100 \ h \ km \ s^{-1} \ \text{Mpc}^{-1} \). However, voids of very different scales may be found, from minivoids [36] to supervoids [37]. It should be emphasized that in general voids are neither empty nor spherical, either in simulations or in deep redshift surveys. However, for simplicity they are usually described as vacuum spherical cavities surrounded by a fluid.

However we are aware of the fact that cold dark matter at scales of the order of tens of Mpc is non-collisional, so that pressure and heat flux terms are negligible. Therefore, excluding the LTB case, it is not likely that our solutions could be used as toy models for cosmic voids.

Possibly our solutions could be applied as toy models of localized systems such as supernova explosion models. It is worth mentioning that for these scenarios, the Kelvin–Helmholtz phase is of the greatest relevance [38].

At any rate our purpose here has not been to generate specific models of any observed void, but rather to call the attention to the potential of the purely areal evolution condition for such a modeling, providing all necessary equations for their description.

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