Joint Equidistribution of CM Points

Ilya Khayutin
September 29, 2017
Theorem (Pila ’11 for general \( n \), André ’98 for \( n = 2 \). Conditionally on GRH: Edixhoven ’98 and Edixhoven ’05)

Let \( X = Y \times \ldots \times Y \) be the Cartesian power of a modular curve. Assume \( \{ x^i = (x^i_1, \ldots, x^i_n) \} \), is a sequence of special points in \( X \), i.e. each \( x^i_k \in Y_k \) is a CM point. If the intersection of this sequence with any proper special subvariety is finite then this sequence is Zariski dense in \( X \).
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**Definition**

We call a sequence of special points *generic* if it has finite intersection with every proper special subvariety.
Proper Special Subvarieties for $n = 2$

- A special point $(x_1, x_2)$,
- $\{x\} \times Y$ and $Y \times \{x\}$ for $x \in Y$ a CM point,
- image of a Hecke correspondence $T_n \hookrightarrow Y \times Y$, e.g. the diagonal embedding $Y \trianglerightarrow Y \times Y$. 
Equidistribution Conjecture

Conjecture

Let \( \{x_i\}_i \) be a generic sequence of special points in \( X \) – the Cartesian power of a modular curve. Let the probability measure \( \mu_i \) on \( X \) be the normalized counting measure on the Galois orbit of \( x_i \)

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\mu_i := \frac{1}{|\text{Orb}(x_i)|} \sum_{y \in \text{Orb}(x_i)} \delta_y
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Then \( \{\mu_i\}_i \) converges weak-* to the uniform measure

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m_X = m_Y \times \ldots m_Y. \quad \underbrace{m_Y \times \ldots m_Y}_{n}
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Asymptotic density of Galois orbits in the locally compact topology.
Previously Known Results

- $n = 1$ – Duke ’88, Iwaniec ’87... Michel ’04, SW Zhang ’05. Assuming a split prime: Linnik ’55.

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Let $X = Y \times \ldots Y$ be the Cartesian power of a modular curve.
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Let \( \{x_i\}_i \) be a generic sequence of special points such that all coordinates of \( x_i \) have CM by the same maximal order of discriminant \( D_i \) and CM field \( E_i/\mathbb{Q} \). Fix two primes \( p_1, p_2 \) and assume that for all \( i \in \mathbb{N} \)
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Then $\{\mu_i\}$ converges weak-* to the uniform measure.
Let $x \in Y$ be a CM point. The theory of complex multiplication implies that the Galois group of the field of definition of $x$ is isomorphic to $\text{Pic}(\Lambda)$ and

$$\text{Orb}_{\text{Galois}}(x) = \text{Orb}_{\text{Pic}(\Lambda)}(x)$$

\[\text{Galois Action} \xrightarrow{\text{reciprocity}} \text{Torus Action}\]
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Conjecture (Michel & Venkatesh)

Let $G$ be a form of $\text{PGL}_2$ over $\mathbb{Q}$ and set $Y := \Gamma \backslash \frac{G(\mathbb{R})}{K_\infty}$ for $\Gamma < G(\mathbb{R})$ a congruence lattice and $K_\infty < G(\mathbb{R})$ a compact torus.
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Denote by $\mu_i^{\text{joint}}$ the $\text{Pic}(\Lambda_i)^{\Delta}$-invariant probability measure supported on $\mathcal{H}_i^{\text{joint}}$. If

$$\min_{a \subset \Lambda_i \text{ invertible ideal} \atop a \in \sigma_i} \text{Nr } a \to i \to \infty \infty$$

Then $\mu_i^{\text{joint}} \to m_Y \times m_Y$. 
Measure Rigidity and Intermediate Measures

• Einsiedler-Lindenstrauss Joinings Theorem (’15): Splitting Condition + \( n = 1 \) case \( \iff \) Any limit measure is a convex combination of the uniform measure and translates of Hecke correspondences.

• Main Obstacle: Exclude intermediate measures, i.e. translates of Hecke correspondences.

• Relative Trace: Express cross-correlation between \( H_i \) and a Hecke correspondence via a \( G_\Delta \backslash G \times G / T_\Delta \) relative trace where \( T \) is the anisotropic torus \( / \mathbb{Q} \) associated to \( H_i \).

• Geometric Expansion: Transform relative trace into a short shifted convolution sum of ideal counting functions using the geometric expansion of the relative trace and fine arithmetic invariants (valued in \( \Lambda \)-ideals).

• Sieve: Upper bound on shifted convolution sum using a vector sieve. This is conditional on non-existence of exceptional zeros.
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The method of Ellenberg, Michel and Venkatesh applies with a fixed single split prime and when

$$\exists \eta > 0 \forall i \geq 1: \min_{\substack{a \subseteq \Lambda_i \text{ invertible ideal} \\ a \in \sigma_i}} \text{Nr} \ a \ll |D_i|^{1/2-\eta}$$

This covers approximately $\sim |D_i|^{-\eta} \# \text{Pic}(\Lambda)$ twists $\sigma_i$. 
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where $\theta$ is the best available exponent in Gauss’s circle problem for imaginary quadratic fields. The Van der Corput bound yields $\theta = 2/3 \Rightarrow \frac{2-\theta^{-1}}{3} = 1/6$. 
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