Re-embedding a 1-Plane Graph into a Straight-line Drawing in Linear Time *

Seok-Hee Hong¹ and Hiroshi Nagamochi²

¹ University of Sydney, Australia
seokhee.hong@sydney.edu.au
² Kyoto University, Japan
nag@amp.i.kyoto-u.ac.jp

Abstract. Thomassen characterized some 1-plane embedding as the forbidden configuration such that a given 1-plane embedding of a graph is drawable in straight-lines if and only if it does not contain the configuration [C. Thomassen, Rectilinear drawings of graphs, J. Graph Theory, 10(3), 335-341, 1988].

In this paper, we characterize some 1-plane embedding as the forbidden configuration such that a given 1-plane embedding of a graph can be re-embedded into a straight-line drawable 1-plane embedding of the same graph if and only if it does not contain the configuration. Re-embedding of a 1-plane embedding preserves the same set of pairs of crossing edges. We give a linear-time algorithm for finding a straight-line drawable 1-plane re-embedding or the forbidden configuration.

1 Introduction

Since the 1930s, a number of researchers have investigated planar graphs. In particular, a beautiful and classical result, known as Fáry’s Theorem, asserts that every plane graph admits a straight-line drawing [5]. Indeed, a straight-line drawing is the most popular drawing convention in Graph Drawing.

More recently, researchers have investigated 1-planar graphs (i.e., graphs that can be embedded in the plane with at most one crossing per edge), introduced by Ringel [13]. Subsequently, the structure of 1-planar graphs has been investigated [4, 12]. In particular, Pach and Toth [12] proved that a 1-planar graph with n vertices has at most 4n − 8 edges, which is a tight upper bound. Unfortunately, testing the 1-planarity of a graph is NP-complete [6, 11], however linear-time algorithms are available for special subclasses of 1-planar graphs [1, 3, 7].

Thomassen [14] proved that every 1-plane graph (i.e., a 1-planar graph embedded with a given 1-plane embedding) admits a straight-line drawing if and only if it does not contain any of two special 1-plane graphs, called the B-configuration or W-configuration, see Fig. 1.

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Recently, Hong et al. [8] gave an alternative constructive proof, with a linear-time testing algorithm and a drawing algorithm. They also showed that some 1-planar graphs need an exponential area with straight-line drawing.

We call a 1-plane embedding straight-line drawable (SLD for short) if it admits a straight-line drawing, i.e., it does not contain a B- or W-configuration by Thomassen [14]. In this paper, we investigate a problem of “re-embedding” a given non-SLD 1-plane embedding \( \gamma \) into an SLD 1-plane embedding \( \gamma' \). For a given 1-plane embedding \( \gamma \) of a graph \( G \), we call another 1-plane embedding \( \gamma' \) of \( G \) a cross-preserving embedding of \( \gamma \) if exactly the same set of edge pairs make the same crossings in \( \gamma' \).

More specifically, we first characterize the forbidden configuration of 1-plane embeddings that cannot admit an SLD cross-preserving 1-plane embedding. Based on the characterization, we present a linear-time algorithm that either detects the forbidden configuration in \( \gamma \) or computes an SLD cross-preserving 1-plane embedding \( \gamma' \).

Formally, the main problem considered in this paper is defined as follows.

**Re-embedding a 1-Plane Graph into a Straight-line Drawing**

**Input:** A 1-planar graph \( G \) and a 1-plane embedding \( \gamma \) of \( G \).

**Output:** Test whether \( \gamma \) admits an SLD cross-preserving 1-plane embedding \( \gamma' \), and construct such an embedding \( \gamma' \) if one exists, or report the forbidden configuration.

To design a linear-time implementation of our algorithm in this paper, we introduce a rooted-forest representation of non-intersecting cycles and an efficient procedure of flipping subgraphs in a plane graph. Since these data structure and procedure can be easily implemented, it has advantage over the complicated decomposition of biconnected graphs into triconnected components [10] or the SPQR tree [2].
2 Plane Embeddings and Inclusion Forests

Let $U$ be a set of $n$ elements, and let $S$ be a family of subsets $S \subseteq U$. We say that two subsets $S, S' \subseteq U$ are intersecting if none of $S \cap S'$, $S - S'$ and $S' - S$ is empty. We call $S$ a laminar if no two subsets in $S$ are intersecting. For a laminar $S$, the inclusion-forest of $S$ is defined to be a forest $I = (S, E)$ of a disjoint union of rooted trees such that (i) the sets in $S$ are regarded as the vertices of $I$, and (ii) a set $S$ is an ancestor of a set $S'$ in $I$ if and only if $S' \subseteq S$.

Lemma 1. For a cyclic sequence $(u_1, u_2, \ldots, u_\delta)$ of $\delta \geq 2$ elements, define an interval $(i, j)$ to be the set of elements $u_k$ with $i \leq k \leq j$ if $i \leq j$ and $(i, j) = (i, \delta) \cup (1, j)$ if $i > j$. Let $S$ be a set of intervals. A pair of two intersecting intervals in $S$ (when $S$ is not a laminar) or the inclusion-forest of $S$ (when $S$ is a laminar) can be obtained in $O(\delta + |S|)$ time.

Throughout the paper, a graph $G = (V, E)$ stands for a simple undirected graph. The set of vertices and the set of edges of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v$, let $E(v)$ be the set of edges incident to $v$, $N(v)$ be the set of neighbors of $v$, and $\deg(v)$ denote the degree $|N(v)|$ of $v$. A simple path with end vertices $u$ and $v$ is called a $u, v$-path. For a subset $X \subseteq V$, let $G - X$ denote the graph obtained from $G$ by removing the vertices in $X$ together with the edges in $\cup_{v \in X} E(v)$.

A drawing $D$ of a graph $G$ is a geometric representation of the graph in the plane, such that each vertex of $G$ is mapped to a point in the plane, and each edge of $G$ is drawn as a curve. A drawing $D$ of a graph $G = (V, E)$ is called planar if there is no edge crossing. A planar drawing $D$ of a graph $G$ divides the plane into several connected regions, called faces, where a face enclosed by a closed walk of the graph is called an inner face and the face not enclosed by any closed walk is called the outer face.

A planar drawing $D$ induces a plane embedding $\gamma$ of $G$, which is defined to be a pair $(\rho, \varphi)$ of the rotation system (i.e., the circular ordering of edges for each vertex) $\rho$, and the outer face $\varphi$ whose facial cycle $C_\varphi$ gives the outer boundary of $D$. Let $\gamma = (\rho, \varphi)$ be a plane embedding of a graph $G = (V, E)$. We denote by $F(\gamma)$ the set of faces in $\gamma$, and by $C_f$ the facial cycle determined by a face $f \in F$, where we call a subpath of $C_f$ a boundary path of $f$. For a simple cycle $C$ of $G$, the plane is divided by $C$ in two regions, one containing only inner faces and the other containing the outer area, where we say that the former is enclosed by $C$ or the interior of $C$, while the latter is called the exterior of $C$. We denote by $F_{in}(C)$ the set of inner faces in the interior of $C$, by $E_{in}(C)$ the set of edges in $E(C_f)$ with $f \in F_{in}(C)$, and by $V_{in}(C)$ the set of end-vertices of edges in $E_{in}(C)$. Analogously define $F_{ex}(C)$, $E_{ex}(C)$ and $V_{ex}(C)$ in the exterior of $C$. Note that $E(C) = E_{in}(C) \cap E_{ex}(C)$ and $V(C) = V_{in}(C) \cap V_{ex}(C)$.

For a subgraph $H$ of $G$, we define the embedding $\gamma|_H$ of $\gamma$ induced by $H$ to be a sub-embedding of $\gamma$ obtained by removing the vertices/edges not in $H$, keeping the same rotation system around each of the remaining vertices/crossings and the same outer face.
2.1 Inclusion Forests of Inclusive Set of Cycles

In this and next subsections, let \((G, \gamma)\) stand for a plane embedding of \(\gamma = (\rho, \varphi)\) of a biconnected simple graph \(G = (V, E)\) with \(n = |V| \geq 3\).

Let \(C\) be a simple cycle in \(G\). We define the direction of \(C\) to be an ordered pair \((u, v)\) with \(uv \in E(C)\) such that the inner faces in \(\text{Fin}(C)\) appear on the right hand side when we traverse \(C\) in the order that we start \(u\) and next visit \(v\). For simplicity, we say that two simple cycles \(C\) and \(C'\) are intersecting if \(\text{Fin}(C)\) and \(\text{Fin}(C')\) are intersecting.

Let \(\mathcal{C}\) be a set of simple cycles in \(G\). We call \(\mathcal{C}\) inclusive if no two cycles in \(\mathcal{C}\) are intersecting, i.e., \(\{\text{Fin}(C) \mid C \in \mathcal{C}\}\) is a laminar. When \(\mathcal{C}\) is inclusive, the inclusion-forest of \(\mathcal{C}\) is defined to be a forest \(I = (\mathcal{C}, E)\) of a disjoint union of rooted trees such that:

(i) the cycles in \(\mathcal{C}\) are regarded as the vertices of \(I\), and
(ii) a cycle \(C\) is an ancestor of a cycle \(C'\) in \(I\) if and only if \(\text{Fin}(C') \subseteq \text{Fin}(C)\).

Let \(I(\mathcal{C})\) denote the inclusion-forest of \(\mathcal{C}\). For a vertex subset \(X \subseteq V\), let \(\mathcal{C}(X)\) denote the set of cycles \(C \in \mathcal{C}\) such that \(x \in V(C)\) for some vertex \(x \in X\), where we denote \(\mathcal{C}({v})\) by \(\mathcal{C}(v)\) for short.

**Lemma 2.** For \((G, \gamma)\), let \(\mathcal{C}\) be a set of simple cycles of \(G\). Then any of the following tasks can be executed in \(O(n + \sum_{C \in \mathcal{C}} |E(C)|)\) time.

(i) Decision of the directions of all cycles in \(\mathcal{C}\);
(ii) Detection of a pair of two intersecting cycles in \(\mathcal{C}\) when \(\mathcal{C}\) is not inclusive, and construction of the inclusion-forests \(I(\mathcal{C}(v))\) for all vertices \(v \in V\) when \(\mathcal{C}\) is inclusive; and
(iii) Construction of the inclusion-forest \(I(\mathcal{C})\) when \(\mathcal{C}\) is inclusive.

2.2 Flipping Spindles

A simple cycle \(C\) of \(G\) is called a spindle (or a \(u, v\)-spindle) of \(\gamma\) if there are two vertices \(u, v \in V(C)\) such that no vertex in \(V(C) - \{u, v\}\) is adjacent to any vertex in the exterior of \(C\), where we call vertices \(u\) and \(v\) the junctions of \(C\). Note that each of the two subpaths of \(C\) between \(u\) and \(v\) is a boundary path of some face in \(F(\gamma)\).

Given \((G, \gamma)\), we denote the rotation system around a vertex \(v \in V\) by \(\rho_v(v)\). For a spindle \(C\) in \(\gamma\), let \(J(C)\) denote the set of the two junctions of \(C\).

**Flipping a \(u, v\)-spindle \(C\)** means to modify the rotation system of vertices in \(V_{\text{in}}(C)\) as follows:

(i) For each vertex \(w \in V_{\text{in}}(C) - J(C)\), reverse the cyclic order of \(\rho_v(w)\); and
(ii) For each vertex \(u \in J(C)\), reverse the order of subsequence of \(\rho_v(u)\) that consists of vertices \(N(u) \cap V_{\text{in}}(C)\).

Every two distinct spindles \(C\) and \(C'\) in \(\gamma\) are non-intersecting, and they always satisfy one of \(E_{\text{in}}(C) \cap E_{\text{in}}(C') = \emptyset\), \(E_{\text{in}}(C) \subseteq E_{\text{in}}(C')\), and \(E_{\text{in}}(C') \subseteq E_{\text{in}}(C)\) Let \(\mathcal{C}\) be a set of spindles in \(\gamma\), which is always inclusive, and let \(I(\mathcal{C})\) denote the inclusion-forest of \(\mathcal{C}\).
When we modify the current embedding $\gamma$ by flipping each spindle in $\mathcal{C}$, the resulting embedding $\gamma_c$ is the same, independent from the ordering of the flipping operation to the spindles, since for two spindles $C$ and $C'$ which share a common junction vertex $u \in J(C) \cap J(C')$, the sets $N(u) \cap V_{u}(C)$ and $N(u) \cap V_{u}(C')$ do not intersect, i.e., they are disjoint or one is contained in the other.

Define the depth of a vertex $v \in V$ in $\mathcal{I}$ to be the number of spindles $C \in \mathcal{C}$ such that $v \in V_{\text{in}}(C) - J(C)$, and denote by $p(v)$ the parity of depth of vertex $v$, i.e., $p(v) = 1$ if the depth is odd and $p(v) = -1$ otherwise.

For a vertex $v \in V$, let $\mathcal{C}[v]$ denote the set of spindles $C \in \mathcal{C}$ such that $v \in J(C)$, and let $\gamma_{\mathcal{C}[v]}$ be the embedding obtained from $\gamma$ by flipping all spindles in $\mathcal{C}[v]$. Let $\text{rev}(\sigma)$ mean the reverse of a sequence $\sigma$. Then we see that $\rho_{\mathcal{C}[v]}(v) = \rho_{\gamma_{\mathcal{C}[v]}}(v)$ if $p(v) = 1$; and $\rho_{\mathcal{C}[v]}(v) = \text{rev}(\rho_{\gamma_{\mathcal{C}[v]}}(v))$ otherwise. To obtain the embedding $\gamma_c$ from the current embedding $\gamma$ by flipping each spindle in $\mathcal{C}$, it suffices to show how to compute each of $p(v)$ and $\rho_{\gamma_{\mathcal{C}[v]}}(v)$ for all vertices $v \in V$.

Lemma 3. Given $(G, \gamma)$, let $\mathcal{C}$ be a set of spindles of $\gamma$. Then any of the following tasks can be executed in $O(n + \sum_{C \in \mathcal{C}} |E(C)|)$ time.

(i) Decision of parity $p(v)$ of all vertices $v \in V$; and
(ii) Computation of $\rho_{\gamma_{\mathcal{C}[v]}}(v)$ for all vertices $v \in V$.

3 Re-embedding 1-plane Graph and Forbidden Configuration

A drawing $D$ of a graph $G = (V, E)$ is called a 1-planar drawing if each edge has at most one crossing. A 1-planar drawing $D$ of graph $G$ induces a 1-plane embedding $\gamma$ of $G$, which is defined to be a tuple $(\chi, \rho, \varphi)$ of the crossing system $\chi$ of $E$, the rotation system $\rho$ of $V$, and the outer face $\varphi$ of $D$. The planarization $\mathcal{G} (G, \gamma)$ of a 1-plane embedding $\gamma$ of graph $G$ is the plane embedding obtained from $\gamma$ by regarding crossings also as graph vertices, called crossing-vertices. The set of vertices in $\mathcal{G} (G, \gamma)$ is given by $V \cup \chi$. For a notational convenience, we refer to a subgraph/face of $\mathcal{G} (G, \gamma)$ as a subgraph/face in $\gamma$.

Let $\gamma = (\chi, \rho, \varphi)$ be a 1-plane embedding of graph $G$. We call another 1-plane embedding $\gamma' = (\chi', \rho', \varphi')$ of graph $G$ a cross-preserving 1-plane embedding of $\gamma$ when the same set of edge pairs makes crossings, i.e., $\chi = \chi'$. In other words, the planarization $\mathcal{G} (G, \gamma')$ is another plane embedding of $\mathcal{G} (G, \gamma)$ such that the alternating order of edges incident to each crossing-vertex $c \in \chi$ is preserved.

To eliminate the additional constraint on the rotation system on each crossing-vertex $c \in \chi$, we introduce “circular instances.” We call an instance $(G, \gamma, \gamma')$ of 1-plane embedding circular when for each crossing $c \in \chi$, the four end-vertices of the two crossing edges $u_1u_3$ and $u_2u_4$ that create $c$ (where $u_1, u_2, u_3$ and $u_4$ appear in the clockwise order around $c$) are contained in a cycle $Q_c = (u_1, w_1^i, u_2, w_2^i, u_3, w_3^i, u_4, w_4^i)$ of eight crossing-free edges for some vertices $w_i^j$, $i = 1, 2, 3, 4$ of degree 2, as shown in Fig. 1(c). By definition, $c$ and each $w_i^j$ not necessarily appear along the same facial cycle in the planarization $\mathcal{G} (G, \gamma)$. For example, path $(v, w, u)$ is part of such a cycle $Q_s$ for the crossing $s$ in the
circular instance in Fig. 2(a), but c and w are not on the same facial cycle in the planarization.

A given instance can be easily converted into a circular instance by augmenting the end-vertices of each pair of crossing edges as follows. In the plane graph, $\mathcal{G}(G, \gamma)$, for each crossing-vertex $c \in \chi$ and its neighbors $u_1, u_2, u_3$ and $u_4$ that appear in the clockwise order around $c$, we add a new vertex $w^c_i$, $i = 1, 2, 3, 4$ and eight new edges $u_iw^c_i$ and $w^c_iu_{i+1}$, $i = 1, 2, 3, 4$ (where $u_5$ means $u_1$) to form a cycle $Q_c$ of length 8 whose interior contains no other vertex than $c$.

Let $H$ be the resulting graph augmented from $G$, and let $\Gamma$ be the resulting 1-plane embedding of $H$ augmented from $\gamma$. Note that $|V(H)| \leq |V(G)| + 4|\chi|$ holds. We easily see that if $\gamma$ admits an SLD cross-preserving embedding $\gamma'$ then $\Gamma'$ admits an SLD cross-preserving embedding $\Gamma''$. This is because a straight-line drawing $D_{\gamma'}$ of $\gamma'$ can be changed into a straight-line drawing $D_{\Gamma''}$ of some cross-preserving embedding $\Gamma''$ of $\Gamma$ by placing the newly introduced vertices $w^c_i$ within the region sufficiently close to the position of $c$. We here see that cycle $Q_c$ can be drawn by straight-line segments without intersecting with other straight-line segments in $D_{\gamma'}$.

Note that the instance $(G, \gamma')$ remains circular for any cross-preserving embedding $\gamma'$ of $\gamma$. In the rest of paper, let $(G, \gamma)$ stand for a circular instance $(G = (V, E), \gamma = (\chi, \rho, \varphi))$ with $n \geq 3$ vertices and let $\mathcal{G}$ denote its planarization $\mathcal{G}(G, \gamma)$. Fig. 2 shows examples of circular instances $(G, \gamma)$, where the vertex-connectivity of $\mathcal{G}$ is 1.

As an important property of a circular instance, the subgraph $G_{(0)}$ with crossing-free edges is a spanning subgraph of $G$ and the four end-vertices of any two crossing edges are contained in the same block of the graph $G_{(0)}$. The biconnectivity is necessary to detect certain types of cycles by applying Lemma 2.

![Fig. 2. Circular instances $(G, \gamma)$ with a cut-vertex $u$ of $\mathcal{G}$, where the crossing edges are depicted by slightly thicker lines: (a) hard B-cycles $C = (u, c, v, s)$ and $C' = (u', c', v', s')$, (b) hard B-cycle $C = (u, c, v, s)$ and a nega-cycle $C' = (u', c', v', s')$ whose reversal is a hard B-cycle, where vertices $u, v, u', v' \in V$ and crossings $c, s, c', s' \in \chi$.](image-url)
3.1 Candidate Cycles, B/W Cycle, Posi/Nega Cycle, Hard/Soft Cycle

For a circular instance \((G, \gamma)\), finding a cross-preserving embedding of \(\gamma\) is effectively equivalent to finding another plane embedding of \(G\) so that all the current B- and W-configurations are eliminated and no new B- or W-configurations are introduced. To detect the cycles that can be the boundary of a B- or W-configuration in changing the plane embedding of \(G\), we categorize cycles containing crossing vertices in \(G\).

A candidate posi-cycle (resp., candidate nega-cycle) in \(G\) is defined to be a cycle \(C = (u, c, v)\) or \(C = (u, c, v, s)\) in \(G\) with \(u, v \in V\) and \(c, s \in \chi\) such that the interior (resp., exterior) of \(C\) does not contain a crossing-free edge \(uv \in E\) and any other crossing vertex \(c'\) adjacent to both \(u\) and \(v\).

Fig. 3. Candidate posi- and nega-cycles \(C = (u, c, v)\) and \(C = (u, c, v, s)\) in \(G\), where white circles represent vertices in \(V\) while black ones represent crossings in \(\chi\): (a) candidate posi-cycle of length 3, (b) candidate posi-cycle of length 4, (c) candidate nega-cycle of length 3, and (d) candidate nega-cycle of length 4.

Fig. 3(a)-(b) and (c)-(d) illustrate candidate posi-cycles and candidate nega-cycles, respectively. Let \(C^p\) and \(C^n\) be the sets of candidate posi-cycles and candidate nega-cycles, respectively. By definition we see that the set \(C^p \cup C^n \cup \{C_f \mid f \in F(\gamma)\}\) is inclusive, and hence \(|C^p \cup C^n \cup \{C_f \mid f \in F(\gamma)\}| = O(n)\).

A candidate posi-cycle \(C\) with \(C = (u, c, v)\) (resp., \(C = (u, c, v, s)\)) is called a B-cycle if

(a)-(B): the exterior of \(C\) contains no vertices in \(V - \{u, v\}\) adjacent to \(c\) (resp., contains exactly one vertex in \(V - \{u, v\}\) adjacent to \(c\) or \(s\)).

Note that \(uv \in E\) when \(C = (u, c, v)\) is a B-cycle, as shown in Fig. 4(a). Fig. 4(b) and (d) illustrate the other types of B-cycles.

A candidate posi-cycle \(C = (u, c, v, s)\) is called a W-cycle if

(a)-(W): the exterior of \(C\) contains no vertices in \(V - \{u, v\}\) adjacent to \(c\) or \(s\).

Fig. 4(c) and (e) illustrate W-cycles.

Let \(C_W\) (resp., \(C_B\)) be the set of W-cycles (resp., B-cycles) in \(\gamma\). Clearly a W-cycle (resp., B-cycle) gives rise to a W-configuration (resp., B-configuration).
Conversely, by choosing a W-configuration (resp., B-configuration) so that the interior is minimal, we obtain a W-cycle (resp., B-cycle). Hence we observe that the current embedding $\gamma$ admits a straight-line drawing if and only if $C_W = C_B = \emptyset$.

A W- or B-cycle $C$ is called hard if

(b): length of $C$ is 4, and the interior of $C = (u, c, v, s)$ contains no inner face $f$ whose facial cycle $C_f$ contains both vertices $u$ and $v$, i.e., some path connects $c$ and $s$ without passing through $u$ or $v$.

On the other hand, a W- or B-cycle $C = (u, c, v, s)$ of length 4 that does not satisfy condition (b) or a B-cycle of length 3 is called soft. We also call a hard B- or W-cycle a posi-cycle.

Fig. 4(d) and (e) illustrate a hard B-cycle and a hard W-cycles, respectively, whereas Fig. 4(a) and (b) (resp., (c)) illustrate soft B-cycles (resp., a soft W-cycle).

A cycle $C = (u, c, v, s)$ is called a nega-cycle if it becomes a posi-cycle when an inner face in the interior of $C$ is chosen as the outer face. In other words, a nega-cycle is a candidate nega-cycle $C = (u, c, v, s)$ of length 4 that satisfies the following conditions (a') and (b'), where (a') (resp., (b')) is obtained from the above conditions (a)-(B) and (a)-(W) (resp., (b)) by exchanging the roles of “interior” and “exterior”:

(a’): the interior of $C$ contains at most one vertex in $V - \{u, v\}$ adjacent to $c$ or $s$; and
Lemma 4. \( C \) is necessarily disjoint. Then:

\[ \gamma \]

embedding of \( C \) if and only if it has no forbidden cycle pair. Finding an SLD cross-preserving candidate nega-cycles that are not nega-cycles.

Let \( C^+ \) (resp., \( C^- \)) denote the set of posi-cycles (resp., nega-cycles) in \( \gamma \). By definition, it holds that \( C^+ \subseteq C_W \cup C_B \subseteq C^p \) and \( C^- \subseteq C^n \).

3.2 Forbidden Cycle Pairs

We define a forbidden configuration that characterizes 1-plane embeddings, which cannot be re-embedded into SLD ones. A forbidden cycle pair is defined to be a pair \( \{C, C'\} \) of a posi-cycle \( C = (u, c, v, s) \) and a posi- or nega-cycle \( C' = (u', c', v', s') \) in \( \gamma \) with \( u, v, u', v' \in V \) and \( c, s, c', s' \in \chi \) to which \( \gamma \) has a \( u, u' \)-path \( P_1 \) and a \( v, v' \)-path \( P_2 \) such that:

(i) when \( C' \in C^+ \), paths \( P_1 \) and \( P_2 \) are in the exterior of \( C \) and \( C' \), i.e., \( V(P_1) - \{u, u'\}, V(P_2) - \{v, v'\} \subseteq V_{\text{ex}}(C) \cap V_{\text{ex}}(C') \), where \( C \) and \( C' \) cannot have any common inner face; and

(ii) when \( C' \in C^- \), paths \( P_1 \) and \( P_2 \) are in the exterior of \( C \) and the interior of \( C' \), i.e., \( V(P_1) - \{u, u'\}, V(P_2) - \{v, v'\} \subseteq V_{\text{ex}}(C) \cap V_{\text{in}}(C') \), where \( C \) is enclosed by \( C' \).

In (i) and (ii), \( P_1 \) and \( P_2 \) are not necessary disjoint, and possibly one of them consists of a single vertex, i.e., \( u = u' \) or \( v = v' \).

The pair of cycles \( C \) and \( C' \) in Fig. 5(a) (resp., Fig. 5(b)) is a forbidden cycle pair, because there is a pair of a \( u, u' \)-path \( P_1 = (u, x, z, y, u') \) and a \( v, v' \)-path \( P_2 = (v, x', z, y', v') \) that satisfy the above conditions (i) (resp., (ii)). Note that the pair of cycles \( C \) and \( C' \) in Fig. 2(a)-(b) is not forbidden cycle pair, because there are no such paths.

Our main result of this paper is as follows.

**Theorem 1.** A circular instance \((G, \gamma)\) admits an SLD cross-preserving embedding if and only if it has no forbidden cycle pair. Finding an SLD cross-preserving embedding of \( \gamma \) or a forbidden cycle pair in \( \gamma \) can be computed in linear time.

**Proof of necessity:** The necessity of the theorem follows from the next lemma.

For a cycle \( C = (u, c, v, s) \in C^+ \) (resp., \( C^- \)) with \( u, v \in V \) and \( c, s \in \chi \) in \( \gamma \), we call a vertex \( z \in V \) an in-factor of \( C \) if the exterior of \( C \in C^+ \) (resp., the interior of \( C \in C^- \)) has a \( z, u \)-path \( P_{z,u} \) and a \( z, v \)-path \( P_{z,v} \), i.e., \( V(P_{z,u} - \{u\}) \cup V(P_{z,v} - \{v\}) \) is in \( V_{\text{ex}}(C) \) (resp., \( V_{\text{in}}(C) \)). Paths \( P_{z,u} \) and \( P_{z,v} \) are not necessarily disjoint.

**Lemma 4.** Given \( \gamma = \gamma(G, \gamma) \), let \( \gamma' \) be a cross-preserving embedding of \( \gamma \). Then:

(i) Let \( z \in V \) be an in-factor of a cycle \( C \in C^+ \cup C^- \) in \( \gamma \). Then cycle \( C \) is a posi-cycle (resp., a nega-cycle) in \( \gamma(G, \gamma') \) if and only if \( z \) is in the exterior (resp., interior) of \( C \) in \( \gamma' \);
(ii) For a forbidden cycle pair \( \{ C, C' \} \), one of \( C \) and \( C' \) is a posi-cycle in \( G, \gamma' \) (hence any cross-preserving embedding of \( \gamma \) contains a B- or W-configuration and \( (G, \gamma) \) admits no SLD cross-preserving embedding).

Proof of sufficiency: In the rest of paper, we prove the sufficiency of Theorem 1 by designing a linear-time algorithm that constructs an SLD cross-preserving embedding of an instance without a forbidden cycle pair.

4 Biconnected Case

In this section, \( (G, \gamma) \) stands for a circular instance such that the vertex-connectivity of the plane graph \( G \) is at least 2. In a biconnected graph \( G \), any two posi-cycles \( C = (u, c, v, s) \) and \( C' = (u', c', v', s') \in C^+ \) with \( u, v, u', v' \in V \) give a forbidden cycle pair if they do not share an inner face, because there is a pair of \( u, u' \)-path and \( v, v' \)-path in the exterior of \( C \) and \( C' \). Analogously any pair of a posi-cycle \( C \) and a nega-cycle \( C' \) such that \( C' \) encloses \( C \) is also a forbidden cycle pair in a biconnected graph \( G \).

To detect such a forbidden pair in \( G \) in linear time, we first compute the sets \( C_p, C_n, C_W, C_B \), \( C^+ \) and \( C^- \) in \( \gamma \) in linear time by using the inclusion-forest from Lemma 2.

Lemma 5. Given \((G, \gamma)\), the following in (i)-(iv) can be computed in \( O(n) \) time.

(i) The sets \( C_p, C_n \) and the inclusion-forest \( I \) of \( C_p \cup C_n \cup \{ C_f \mid f \in F(\gamma) \} \);
(ii) The sets \( C_W \) and \( C_B \);
(iii) The sets \( C^+, C^- \) and the inclusion-forest \( I^* \) of \( C^+ \cup C^- \); and
(iv) A set \( \{ f_C \mid C \in (C_W \cup C_B) - C^+ \} \) such that \( f_C \) is an inner face in the interior of a soft B- or W-cycle \( C \) with \( V(C_f) \supseteq V(C) \).
Given \((G, \gamma)\), a face \(f \in F(\gamma)\) is called admissible if all posi-cycles enclose \(f\) but no nega-cycle encloses \(f\). Let \(A(\gamma)\) denote the set of all admissible faces in \(F(\gamma)\).

**Lemma 6.** Given \((G, \gamma)\), it holds \(A(\gamma) \neq \emptyset\) if and only if no forbidden cycle pair exists in \(\gamma\). A forbidden cycle pair, if one exists, and \(A(\gamma)\) can be obtained in \(O(n)\) time.

By the lemma, if \((G, \gamma)\) has no forbidden cycle pair, i.e., \(A(\gamma) \neq \emptyset\), then any new embedding obtained from \(\gamma\) by changing the outer face with a face in \(A(\gamma)\) is a cross-preserving embedding of \(\gamma\) which has no hard B- or W-cycle.

### 4.1 Eliminating Soft B- and W-cycles

Suppose that we are given a circular instance \((G, \gamma)\) such that \(G\) is biconnected and \(C^+ = \emptyset\). We now show how to eliminate all soft B- and W-cycles in \(G\) in linear time using the inclusion-forest from Lemma 2 and the spindles from Lemma 3.

**Lemma 7.** Given \((G, \gamma)\) with \(C^+ = \emptyset\), there exists an SLD cross-preserving embedding \(\gamma' = (\chi, \rho', \varphi')\) of \(\gamma\) such that \(V(C_{\varphi'}) \supseteq V(C_\varphi)\) for the facial cycle \(C_\varphi\) (resp., \(C_{\varphi'}\)) of the outer face \(\varphi\) (resp., \(\varphi'\)), which can be constructed in \(O(n)\) time.

Given an instance \((G, \gamma)\) with a biconnected graph \(G\), we can test whether it has either a forbidden cycle pair or an admissible face by Lemmas 5 and 6. In the former, it cannot have an SLD cross-preserving embedding by Lemma 4. In the latter, we can eliminate all hard B- and W-cycles by choosing an admissible face as a new outer face, and then eliminate all soft B- and W-cycles by a flipping procedure based on Lemma 7. All the above can be done in linear time.

To treat the case where the vertex-connectivity of \(G\) is 1 in the next section, we now characterize 1-plane embeddings that can have an SLD cross-preserving embedding such that a specified vertex appears along the outer boundary. For a vertex \(z \in V\) in a graph \(G\), we call a 1-plane embedding \(\gamma\) of \(G\) \(z\)-exposed if vertex \(z\) appears along the outer boundary of \(\gamma\). We call \((G, \gamma)\) \(z\)-feasible if it admits a \(z\)-exposed SLD cross-preserving embedding \(\gamma'\) of \(\gamma\).

**Lemma 8.** Given \((G, \gamma)\) such that \(A(\gamma) \neq \emptyset\), let \(z\) be a vertex in \(V\). Then:

(i) The following conditions are equivalent:
   (a) \(\gamma\) admits no \(z\)-exposed SLD cross-preserving embedding;
   (b) \(A(\gamma)\) contains no face \(f\) with \(z \in V(C_f)\); and
   (c) \(G\) has a posi- or nega-cycle \(C\) to which \(z\) is an in-factor;

(ii) A \(z\)-exposed SLD cross-preserving embedding or a posi- or nega-cycle \(C\) to which \(z\) is an in-factor can be computed in \(O(n)\) time.
5 One-connected Case

In this section, we prove the sufficiency of Theorem 1 by designing a linear-time algorithm claimed in the theorem. Given a circular instance \((G, \gamma)\), where \(G\) may be disconnected, obviously we only need to test each connected component of \(G\) separately to find a forbidden cycle pair. Thus we first consider a circular instance \((G, \gamma)\) such that the vertex-connectivity of \(G\) is 1; i.e., \(G\) is connected and has some cut-vertices.

A block \(B\) of \(G\) is a maximal biconnected subgraph of \(G\). For a biconnected graph \(G\), we already know how to find a forbidden cycle pair or an SLD cross-preserving embedding from the previous section. For a trivial block \(B\) with \(|V(B)| = 2\), there is nothing to do. If some block \(B\) of \(G\) with \(|V(B)| \geq 3\) contains a forbidden cycle pair, then \((G, \gamma)\) cannot admit any SLD cross-preserving embedding by Lemma 4.

We now observe that \(G\) may contain a forbidden cycle pair even if no single block of \(G\) has a forbidden cycle pair.

**Lemma 9.** For a circular instance \((G, \gamma)\) such that the vertex-connectivity of \(G\) is 1, let \(B_1\) and \(B_2\) be blocks of \(G\) and let \(P_{1,2}\) be a \(z_1, z_2\)-path of \(G\) with the minimum number of edges, where \(V(B_i) \cap V(P_{1,2}) = \{z_i\}\) for each \(i = 1, 2\). If \(\gamma|_{B_i}\) has a posi- or nega-cycle \(C_i\) to which \(z_i\) is an in-factor for each \(i = 1, 2\), then \(\{C_1, C_2\}\) is a forbidden cycle pair in \(G\).

For a linear-time implementation, we do not apply the lemma for all pairs of blocks in \(B\). A block of \(G\) is called a leaf block if it contains only one cut-vertex of \(G\), where we denote the cut-vertex in a leaf block \(B\) by \(v_B\). Without directly searching for a forbidden cycle pair in \(G\), we use the next lemma to reduce a given embedding by repeatedly removing leaf blocks.

**Lemma 10.** For a circular instance \((G, \gamma)\) such that the vertex-connectivity of \(G = G(G, \gamma)\) is 1 and a leaf block \(B\) of \(G\) such that \(\gamma|_{B}\) is \(v_B\)-feasible, let \(H = G - (V(B) - \{v_B\})\) be the graph obtained by removing the vertices in \(V(B) - \{v_B\}\). Then

(i) The instance \((H, \gamma|_{H})\) is circular; and
(ii) If \((H, \gamma|_{H})\) admits an SLD cross-preserving embedding \(\gamma^*_H\), then an SLD cross-preserving embedding \(\gamma^*_B\) of \(\gamma|_{B}\) within a space next to the cut-vertex \(v_B\) in \(\gamma^*_H\).

Given a circular instance \((G, \gamma)\) such that \(G = G(G, \gamma)\) is connected, an algorithm **Algorithm Re-Embed-1-Plane** for Theorem 1 is designed by the following three steps.

The first step tests whether \(G\) has a block \(B\) such that \(\gamma|_{B}\) has a forbidden cycle pair, based on Lemma 8. If one exists, the algorithm outputs a forbidden cycle pair and halts.

After the first step, no block has a forbidden cycle pair. In the current circular instance \((G, \gamma)\), one of the following holds:
the number of blocks in $G$ is at least two and there is at most one leaf block $B$ such that $\gamma|_B$ is not $v_B$-feasible;
(ii) $G$ has two leaf blocks $B$ and $B'$ such that $\gamma|_B$ is not $v_B$-feasible and $\gamma|_{B'}$ is not $v_{B'}$-feasible; and
(iii) the number of blocks in $G$ is at most one.

In (ii), $v_B$ is an in-factor of a cycle $C$ in $\gamma|_B$ and $v_{B'}$ is an in-factor of a cycle $C'$ in $\gamma|_{B'}$ by Lemma 8, and we obtain a forbidden cycle pair $\{C, C'\}$ by Lemma 9. Otherwise if (i) holds, then we can remove all leaf blocks $B$ such that $\gamma|_B$ is not $v_B$-feasible by Lemma 10. The second step keeps removing all leaf blocks $B$ such that $\gamma|_B$ is not $v_B$-feasible until (ii) or (iii) holds to the resulting embedding. If (i) occurs, then the algorithm outputs a forbidden cycle pair and halts.

When all the blocks of $G$ can be removed successfully, say in an order of $B^1, B^2, \ldots, B^m$, the third step constructs an embedding with no B- or W-cycles by starting with such an SLD embedding of $B^m$ and by adding an SLD embedding of $B^i$ to the current embedding in the order of $i = m-1, m-2, \ldots, 1$. By Lemma 10, this results in an SLD cross-preserving embedding of the input instance $(G, \gamma)$.

Note that we can obtain an SLD cross-preserving embedding $\gamma^*_{H_1}$ of $\gamma$ in the third step when the first and second step did not find any forbidden cycle pair. Thus the algorithm finds either an SLD cross-preserving embedding of $\gamma$ or a forbidden cycle pair. This proves the sufficiency of Theorem 1.

By the time complexity result from Lemma 8, we see that the algorithm can be implemented in linear time.

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