Effect of particle fluctuation on isoscaling and isobaric yield ratio of nuclear multifragmentation

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I. INTRODUCTION

The study of the nuclear equation of state is an important area of research in intermediate energy heavy ion reactions [1, 2] and results from nuclear multifragmentation reactions are extensively used for such study. The statistical models are extremely powerful and widely used tools for study of the multifragmentation reactions. In models of statistical disassembly of a nuclear system formed by the collision of two heavy ions at intermediate energy one assumes that the hot and compressed nuclear system expands and subsequently fragments into composites of different masses depending on the initial conditions. The fragmentation of the nucleus into available channels (depends on phase space) can be solved in different statistical ensembles (microcanonical, canonical and grand canonical). For finite nuclei, in general the results for different observables differ in different ensembles and they are found to converge under certain conditions [6].

Isoscaling [7–12] and isobaric yield ratio [12–15] are two well known methods which are used to study the nuclear EOS and to extract liquid drop model parameters (symmetry energy coefficient for example) from multifragmentation reactions. The isoscaling parameters are related to the difference between the chemical potentials of the two fragmenting isotopes and hence provide insight into their symmetry energy. The isobaric yield ratio method is used to extract the liquid drop model parameters and also the difference in chemical potential between neutrons and protons. The formulae for both these methods are derived using the framework of the grand canonical model. Hence results from this ensemble agree exactly with those from these equations. The canonical model is better suited compared to grand canonical model for describing intermediate energy nuclear reactions where baryon and charge numbers are conserved. On the contrary, results from the canonical ensemble differ from the formulae based on grand canonical ensemble. The nature of this deviation depends on the size of the fragmenting systems as well as on the neutron to proton ratio (asymmetry) of them.

It is seen that results from both the ensembles differ when the source size is small and asymmetry is large and they come closer when one increases the source size or decrease its asymmetry. The results of the isoscaling as well as the isobaric yield ratio equations depend on the statistical ensemble used for the calculation. Since the equations for extracting the relevant parameters are derived from the grand canonical framework, hence the results from the canonical model should be carefully analyzed for extraction of different parameters from these methods.

The letter is structured as follows. In section II, we give a brief introduction to the canonical and the grand canonical models where as section III contains the theoretical framework of isoscaling and isobaric yield ratio methods from grand canonical ensemble. The results will be presented in Section IV. Finally we shall summarize and conclude in section V.

II. THE CANONICAL AND THE GRAND CANONICAL MODEL

In this section we describe briefly the canonical and the grand canonical models of nuclear multifragmentation. The basic output from canonical [3] or grand canonical model [16] is multiplicity of the fragments. After calculating the multiplicities, isoscaling and isobaric yield ratio parameters can be obtained. By multiplicity we mean that the average number of fragments produced for each proton number \( Z \) and neutron number \( N \). Assuming that the system with \( A_0 \) nucleons and \( Z_0 \) protons at temperature \( T \), has expanded to a volume higher than normal nuclear volume, the partitioning into different composites can be calculated according to the rules of equilibrium statistical mechanics.

In a canonical model [3], the partitioning is done such that all partitions have the correct \( A_0, Z_0 \) (equivalently \( N_0, Z_0 \)). The canonical partition function is given by

\[
Q_{N_0,Z_0} = \sum \prod \frac{\omega_{N,Z}^{n_{N,Z}}}{n_{N,Z}!} \tag{1}
\]
where the sum is over all possible channels of break-up (the number of such channels is enormous) satisfying \( N_0 = \sum N \times n_{N,Z} \) and \( Z_0 = \sum Z \times n_{N,Z} \); \( \omega_{N,Z} \) is the partition function of the composite with \( N \) neutrons and \( Z \) protons and \( n_{N,Z} \) is its multiplicity. The partition function \( Q_{N_0,Z_0} \) is calculated using a recursion relation \([3]\).

From Eq. (1), the average number of composites is given by \([3]\)

\[
\langle n_{N,Z} \rangle_c = \frac{\omega_{N,Z} Q_{N_0-N,Z_0-Z}}{Q_{N_0,Z_0}} \tag{2}
\]

It is necessary to specify which nuclei are included in computing \( Q_{N_0,Z_0} \). For \( N, Z \) we include a ridge along the line of stability. The liquid-drop formula gives neutron and proton drip lines and the results shown here include all nuclei within the boundaries.

In the grand canonical model \([10]\), if the neutron chemical potential is \( \mu_n \) and the proton chemical potential is \( \mu_p \), then statistical equilibrium implies \([17]\) that the chemical potential of a composite with \( N \) neutrons and \( Z \) protons is \( \mu_{N,Z} = \mu_n N + \mu_p Z \). The average number of composites with \( N \) neutrons and \( Z \) protons is given by \([10]\)

\[
\langle n_{N,Z} \rangle_{gc} = e^{\beta \mu_{N,Z} Z} \omega_{N,Z} \tag{3}
\]

The chemical potentials \( \mu_n \) and \( \mu_p \) are determined by solving two equations \( N_0 = \sum \omega_{N,Z} \) and \( Z_0 = \sum \omega_{N,Z} \). This amounts to solving for an infinite system but we emphasize that this infinite system can break up into only certain kinds of species as are included in the above two equations. We can look upon the sum on \( N \) and \( Z \) as a sum over \( A (= N + Z) \) and a sum over \( Z \). In principle \( A \) goes from 1 to \( \infty \) and for a given \( A, Z \) can go from 0 to \( A \). Here for a given \( A \) we restrict \( Z \) by the same drip lines used for canonical model.

In both the models, the partition function of a composite having \( N \) neutrons and \( Z \) protons is a product of two parts: one is due to the the translational motion and the other is the intrinsic partition function of the composite:

\[
\omega_{N,Z} = \frac{V}{\hbar^3} (2\pi mT)^{3/2} A^{3/2} \times z_{N,Z}(\text{int}) \tag{4}
\]

where \( V \) is the volume available for translational motion. Note that \( V \) will be less than \( V_f \), the volume to which the system has expanded at break up (freeze-out volume). We use \( V = V_f - V_0 \), where \( V_0 \) is the normal volume of nucleus with \( Z_0 \) protons and \( N_0 \) neutrons. In this work the temperature and freeze-out volume are kept constant at 5 MeV and 3\( V_0 \) respectively.

We list now the properties of the composites used in this work. The proton and the neutron are fundamental building blocks thus \( z_{1,0}(\text{int}) = z_{0,1}(\text{int}) = 2 \) where 2 takes care of the spin degeneracy. For deuterons, triton, \(^3\)He and \(^4\)He we use \( z_{N,Z}(\text{int}) = (2s_{N,Z} + 1) \exp(-\beta E_{N,Z}(\text{gr})) \) where \( \beta = 1/T, E_{N,Z}(\text{gr}) \) is the ground state energy of the composite and \( (2s_{N,Z} + 1) \) is the experimental spin degeneracy of the ground state. Excited states for these very low mass nuclei are not included. For mass number \( A \geq 5 \) we use the liquid-drop formula. For nuclei in isolation, this reads

\[
z_{N,Z}(\text{int}) = \exp \left[ \frac{1}{T} \left[ W_0 A - \sigma(T) A^{2/3} - a_c^* \frac{Z^2}{A^{1/3}} \right] - C_{\text{sym}} \frac{(N - Z)^2}{A} + \frac{T^2 A}{\epsilon_0} \right] \tag{5}
\]

The expression includes the volume energy \( W_0 = 15.8 \text{ MeV} \), the temperature dependent surface energy \( \sigma(T) = \sigma_0 \left( \frac{T_c^2 - T^2}{T_c^2 + T^2} \right)^{5/4} \) with \( \sigma_0 = 18.0 \text{ MeV} \) and \( T_c = 18.0 \text{ MeV} \), the Coulomb energy with Wigner-Seitz approximation \( a_c^* = a_c \{1 - (V_0/V_f)^{1/3}\} \) with \( a_c = 0.72 \text{ MeV} \) and the symmetry energy \( C_{\text{sym}} = 23.5 \text{ MeV} \). The term \( T^2 A / \epsilon_0 \) \( (\epsilon_0 = 16.0 \text{ MeV}) \) represents contribution from excited states since the composites are at a non-zero temperature.

In canonical ensemble though chemical potential does not come into picture directly, but we can define them as \( \mu = -\frac{\partial F}{\partial N} \). So in our case,

\[
\mu_p = F_1(N_0, Z_0 - 1) - F_1(N_0, Z_0) \tag{6}
\]

\[
\mu_n = F_1(N_0 - 1, Z_0) - F_1(N_0, Z_0) \tag{7}
\]

where \( F_1(N_0, Z_0) = -T \ln Q_{N_0,Z_0} \) is the total free energy.

### III. THEORETICAL FRAMEWORK OF ISOSCALING AND ISOBARIC YIELD RATIO METHOD

Substituting \( \omega_{N,Z} \) from Eq. (4) and Eq. (5) in Eq. (3), the average multiplicity from grand canonical ensemble can be written as,

\[
\langle n_{N,Z} \rangle_{gc} = \frac{V}{\hbar^3} (2\pi mT)^{3/2} A^{3/2} \times \exp \left[ - \frac{F(N,Z) - \mu_n N - \mu_p Z}{T} \right] \tag{8}
\]

where \( F(N,Z) = -T \ln z_{N,Z}(\text{int}) \), whereas \( z_{N,Z}(\text{int}) \) is given by Eq. 5. In isobaric yield ratio method, ratio of yields of two different types of fragments having same mass number \( A \) but different isospin asymmetry \( I = N - Z \) and \( I' = N' - Z' \) originating from same source \([14]\) is given by,

\[
R[I, I', A] = \frac{\langle n_{I,A} \rangle_{gc}}{\langle n_{I',A} \rangle_{gc}} \tag{9}
\]

With the choice of \( I = 1 \) and \( I' = -1 \) the ratio will be

\[
\ln R[1, -1, A] = \frac{\mu_n - \mu_p}{T} + \frac{a_c^*}{T} A^{2/3} \tag{10}
\]
And isoscaling is the ratio of yields of the same type of fragment \((N, Z)\) originating from two sources having different mass number \(A_1\) and \(A_2\) (\(A_2 > A_1\)) but same charge \(Z_1 = Z_2 = Z\). From (6),

\[
R_{21} = \frac{\langle n_{2N,Z} \rangle_{gc}}{\langle n_{1N,Z} \rangle_{gc}} = C \exp\left(\frac{\mu_{1Z} - \mu_{2Z}}{T} N + \frac{\mu_{1n} - \mu_{2n}}{T} Z\right) = C \exp(\alpha N + \beta Z) \tag{11}
\]

In canonical and grand canonical model calculations, \(R_{21}\) is calculated from the ratio of \(\langle n_{2N,Z} \rangle\) and \(\langle n_{1N,Z} \rangle\) and then isoscaling parameters \(\alpha\) and \(\beta\) are obtained by linear fitting of \(\ln R_{21}\) with \(N\) (at constant \(Z\)) and \(Z\) (at constant \(N\)) respectively.

Since the above formalisms are valid for equilibrium condition \([12]\) and secondary decay affects the equilibrium scenario \([12, 18]\), hence in the entire theoretical calculation secondary decay is not included.

### IV. RESULTS AND DISCUSSION

We have studied the dependence of neutron and proton chemical potentials on both source size and source asymmetry and the results are displayed in Fig. 1. In Fig. 1(a), the variation of both \(\mu_p\) and \(\mu_n\) with source proton number \(Z_0\) is shown for two values of the asymmetry parameter \(y = (N_0 - Z_0)/(N_0 + Z_0) = 0.11\) and \(0.27\). The source size \(A_0\) is varied from 44 to 264 (\(Z_0\) from 16 to 96) and it is seen that both the neutron and the proton chemical potentials remains almost constant as one increases the source size irrespective of the value of \(y\). In the same figure results are shown from both the canonical and the grand canonical models and it is seen that the chemical potentials calculated from both the ensembles are almost equal except for the very small sources where they are slightly different. In Fig. 1(b), the variation of chemical potentials with the asymmetry parameter \(y\) is shown where \(y\) is varied from 0 to 0.33. The source size \(A_0\) is kept fixed at 60 whereas \(Z_0\) varies from 20 (\(y=0.33\)) to 30 (\(y=0\)). The change of \(\mu_n\) and \(\mu_p\) with \(y\) is almost linear for both the ensembles and also the results from the canonical and the grand canonical ensembles are more or less the same except for higher \(y\) values where they are slightly different.

Since it is seen from Fig. 1(a) & 1(b) that the chemical potentials are almost same for both models for the entire range of source size and source asymmetry, hence for all other results to be presented for these systems, the chemical potentials obtained from grandcanonical model are being used.

The isoscaling parameters \(\alpha\) and \(\beta\) (Eq. 11) depend only on the difference in chemical potentials of more neutron-rich and less neutron rich fragmenting systems and on the temperature at freeze-out. For isoscaling studies a pair of sources with same \(Z_0\) value is required. We have kept the asymmetry value of the more neutron-rich source to be 0.27 and that of the less neutron-rich one to be 0.11. The study was done for different source sizes ranging from \(Z_0 = 16\) to \(Z_0 = 96\), the \(y\) values being the same for each pair. Since for a particular pair of reaction the difference in chemical potentials are the same, therefore it is expected that the isoscaling parameters \(\alpha\) (or \(\beta\)) would remain constant throughout the entire \(Z\) (or \(N\)) regime of the fragments. It can also be concluded from Fig. 1(a) that at constant temperature and same \(y\) value of the fragmenting sources, \(\mu_p\) and \(\mu_n\) are almost independent of the source size. On the contrary, from theoretical calculation by the canonical model it is observed from Fig 2(a) & 2(b) that \(\alpha\) increases with the increase of fragment proton number \(Z\) and \(\beta\) decreases with increase of fragment neutron number \(N\) and the change is more for smaller fragmenting sources. On the other hand, \(\alpha\) and \(\beta\) values calculated from the grand canonical model (Fig 2(c) & 2(d)) are independent of the fragment size. The dependence on source size is also very small as compared to the canonical results (Fig 2(a) & 2(b)). Therefore if we take the average of \(\alpha\)(or \(\beta\)) in the range of \(Z = 1\) to \(8\)(or \(N = 1\) to \(8\)) for each source and compare those with that calculated from the formula \(\alpha = (\mu_{2Z} - \mu_{1Z})/T\) or \(\beta = (\mu_{2N} - \mu_{1N})/T\), then it is observed that the average values obtained from canonical and grand canonical model are different for the smaller fragmenting sources and the difference decreases substantially as one increases the source size as seen in Fig 2(e) & 2(f). The values of the isoscaling parameter \(\alpha\) and \(\beta\) calculated from the slopes of the ratio \(R_{21}\) of the grand canonical model coincides exactly with those calculated from the formula. This is seen from the dotted lines and the stars in Fig 2(e) & 2(f). This is what is expected since the formulae connecting the isoscaling parameters with the difference in chemical potentials is deduced from the grand canonical ensemble.
and hence results from the later exactly coincide with those calculated from the formulae.

In Fig. 3(a) & 3(b) we show the variation of the isobaric yield ratio $\ln R(1, -1, A)$ with the fragment sizes where each line represents a particular source having different size but same isospin asymmetry 0.27. In the Isobaric yield ratio method, it is observed from Eq. 10 that the quantity $\ln R(1, -1, A)$ calculated for odd $A$ nuclei (since $N - Z = 1or -1$), varies linearly with $A^{2/3}$ by an equation like $y = mx + c$ with $m = a_\alpha^*/T$ & $c = \Delta \mu/T$ (where $a_\alpha^* = a_\alpha(1 - (V_0/V_1))^{1/3}$) and $\Delta \mu = \mu_m - \mu_p$). The difference $\mu_m - \mu_p$ is almost independent of the source size (Fig. 1(a)); hence $c$'s should be almost equal and $m$'s are exactly equal for all the fragmenting sources. Therefore the plot of the ratio $\ln R(1, -1, A)$ originating from all the fragmenting sources (as used in Fig. 1(a)) should almost coincide. But from the canonical model, the variation of $\ln R(1, -1, A)$ with $A^{2/3}$ is different for different sources as shown in Fig. 3(a). The slopes of the lines from different sources vary with the source size although the slope should be exactly equal according to the formula. This deviation arises because from Eq. 10

the slope $m = a_\alpha^*/T$ is derived from the grand canonical model and the same may not hold true for the canonical results. The results from the grand canonical model are shown in Fig. 3(b). Here it is seen that the slopes are exactly equal irrespective of the source size. The calculated values of the slope $m$ and the y-intercept $c$ obtained from linear fitting of the lines from canonical models (Fig. 3(a)) and those calculated from formula $c = \Delta \mu/T$ and $m = a_\alpha^*/T$ are not same for smaller fragmenting sources, but are close for the larger sources. This is shown in Fig. 3(c) & 3(d). The reason for this deviation is that the formulae are derived using the grand canonical ensemble and hence they are in general not true for the canonical model results. For larger sources the fragmentation is more, therefore the particle number fluctuation in grand canonical model is very less. In canonical model, particle number is strictly conserved and there is no such fluctuation. Hence the isoscaling parameters and isobaric yield ratios obtained from canonical and grand canonical model become closer compared to that from the smaller fragmenting sources. The values of $c$ and $m$ obtained by fitting the lines from the grand canonical model (Fig. 3(b)) coincide exactly with the values given by the formula. This is shown by the dashed line and the symbols in Fig. 3(c) and Fig. 3(d).

In Fig. 4 we show the effect of variation of the source

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**FIG. 2:** (Color online) Variation of isoscaling parameter $\alpha$ ($\beta$) with fragment proton number $Z$ ($N$) for each pair of sources having same isospin asymmetry 0.27 and 0.11 but different charges $Z_0 = 16$ (black squares joined by solid lines), 24 (red circles joined by dotted lines), 32 (green circles joined by dashed lines) and 96 (magenta pentagons joined by short dotted lines) from canonical [2(a), 2(b)] and grand canonical [2(c), 2(d)] model. $2(e)$ and $2(f)$ shows the variation of the isoscaling parameters $\alpha$ and $\beta$ respectively with source charge ($Z_0$) obtained from canonical model (red solid lines), grand canonical model (black dotted lines) and that calculated from the formulæ $\alpha = (\mu_{n2} - \mu_{n1})/T$ and $\beta = (\mu_{p2} - \mu_{p1})/T$ (blue stars).

**FIG. 3:** (Color online) Variation of isobaric yield ratio $\ln R(1, -1, A)$ with $A^{2/3}$ for sources having same isospin asymmetry 0.27 but different charge $Z_0 = 16$ (black squares), 24 (red circles), 32 (green circles) 48 (blue stars) and 96 (magenta pentagons) from canonical (a) and grand canonical (b) model. Here the lines connecting the points represent the linear fittings. (c) and (d) shows variation of the isobaric yield ratio parameters $c$ and $m$ respectively with source charge ($Z_0$) obtained from canonical model (red solid lines), grand canonical model (black dotted lines) and that calculated from the formulæ $c = \Delta \mu/T$ and $m = a_\alpha^*/T$ (blue stars).
Fig. 1(b). It is seen from Fig. 4(a) that the slopes are different for different sources from the canonical model calculation, the deviation being more for the source which is more asymmetric. For the grand canonical model(Fig. 4(b)), the slopes are exactly equal for each source as expected from the formulae. The values of the parameters $c$ and $m$ are plotted in Fig. 4(c) & 4(d) respectively. It is seen that results from the canonical model differs from that of the formulae for higher values of $y$ and they become close as $y$ value approaches 0 or in other words the source becomes symmetric. The results from the grand canonical ensemble coincide exactly with that from the formulae. It has been already studied that results from the canonical and grand canonical models converge more as the fragmenting system becomes more symmetric as the particle fluctuation in grand canonical model becomes less in such cases. Similar effect is obtained earlier for mass distribution [7].

V. SUMMARY AND CONCLUSION

Isobaric yield ratio method as well as the isoscaling equations connect the liquid drop model parameters to the ratio of fragment yields from break-up of hot nuclei. These relations are deduced using the grand canonical ensemble for calculating the fragment yields. Hence results from this ensemble agree exactly with those obtained from the equations of isobaric yield ratio and isoscaling. On the contrary, results from the canonical ensemble (which is more suitable for describing finite nuclei with conserved mass and charge) deviate from these equations and this deviation is more pronounced for smaller source size and more asymmetric nuclei. This is very much expected since results from both the ensembles differ in general for finite nuclei and are found to converge when fragmentation of the nucleus is more and it happens for the sources having larger mass or less asymmetry [8].