Whishart–Pickrell distributions and closures of group actions

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Consider probabilistic distributions on the space of infinite Hermitian matrices \( \text{Herm}(\infty) \) invariant with respect to the unitary group \( U(\infty) \). We describe the closure of \( U(\infty) \) in the space of spreading maps (polymorphisms) of \( \text{Herm}(\infty) \), this closure is a semigroup isomorphic to the semigroup of all contractive operators.

1 The statement

1.1. Notation. Denote by \( \text{Herm}_\infty \) the space of all infinite Hermitian matrices. By \( \text{Herm}_0^\infty \) we denote the space of all infinite Hermitian matrices having a finite number of non-zero matrix elements.

By \( U(\infty) \) we denote the group of infinite unitary matrices \( g \) such that \( g^{-1} \) has only finite number of nonzero matrix elements. By \( \overline{U}(\infty) \) we denote the complete unitary group in \( \ell^2 \), we equip it with the weak operator topology.

By \( B(\infty) \) we denote the semigroup of all linear operators in \( \ell^2 \) with norm \( \leq 1 \).

1.2. Whishart–Pickrell distributions. For any probabilistic measure \( \mu \) on \( \text{Herm}_\infty \) we assign the characteristic function on \( \text{Herm}_0^\infty \) by

\[
\chi(\mu|A) = \int_{\text{Herm}_\infty} e^{i \text{tr} AX} d\mu(X).
\]

Obviously, such a function determines a measure in a unique way.

Consider the group \( U(\infty) \) of infinite unitary matrices \( g \) such that \( g^{-1} \) has only finite number of nonzero matrix elements. This group acts on \( \text{Herm}_\infty \) by conjugations,

\[
U : X \mapsto U^{-1}XU.
\]

There is the following theorem of Pickrell [11] (see also another proof with additional details in [9]) in the spirit of the de Finetti theorem:

Any \( U(\infty) \)-invariant measure on \( \text{Herm}_\infty \) can be decomposed into ergodic measures. An ergodic \( U(\infty) \)-invariant measure has a characteristic function of the form

\[
\chi_{\gamma_1, \gamma_2, \lambda}(A) = e^{-\frac{1}{2} \text{tr} A^2 + i \gamma_2 \text{tr} A} \prod_{k=1}^{\infty} \left( \det \frac{e^{-i\lambda_k A}}{1 - i\lambda_k A} \right),
\]

(1.1)

where \( \gamma_1 \geq 0 \), \( \gamma_2 \), and \( \lambda_1, \lambda_2, \ldots \) are real numbers, and \( \sum \lambda_k^2 < \infty \).

Denote this measure by \( \mu_{\gamma_1, \gamma_2, \lambda} \).

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A characteristic function is a product and the corresponding measure can be decomposed as an (infinite) convolution. The factor $e^{-\frac{1}{2} \text{tr} A^2 + i\gamma_2 \text{tr} A}$ corresponds to a Gaussian measure on $\text{Herm}_\infty$. Let us explain a meaning of factors $\text{det}(1 - i\lambda_k A)^{-1}$. Consider the space $\mathbb{C}$ equipped with the Gaussian measure $\pi_\infty^{-1} e^{-|u|^2} d\text{Re} u d\text{Im} u$. Consider the space $\mathbb{C}_\infty$ equipped with a product-measure $\nu$. Consider the map $\mathbb{C}_\infty \to \text{Herm}_\infty$ given by $u \mapsto \lambda_k u^* u$. Consider the image $\mu$ of $\nu$ under this map. The characteristic function of $\mu$ is

$$\int_{\text{Herm}_\infty} e^{i\text{tr} AX} d\mu(X) = \int_{\mathbb{C}_\infty} e^{i\text{tr} \lambda_k Au^* u} d\nu(u) = \int_{\mathbb{C}_\infty} e^{i\lambda_k Au^* u} d\nu(u) = \text{det}(1 - i\lambda_k A)^{-1}.$$

If $\sum |\lambda_k| < \infty$, then we can transform the expression (1.1) to

$$\chi_{\gamma_1, \gamma_2, \lambda}(A) = e^{-\frac{1}{2} \text{tr} A^2 + i(\gamma_2 - \sum \lambda_k) \text{tr} A} \prod_{k=1}^{\infty} \text{det}(1 - i\lambda_k A)^{-1}.$$

If the series $\sum \lambda_k$ diverges, we get a divergent series in the exponent and a divergent product.

1.3. Polymorphisms. See [3], [10], [5], Section VIII.4. Consider a Lebesgue measure space $M$ with a non-atomic probabilistic measure $\mu$. Denote by $\text{Ams}(M)$ the group of measure preserving bijective a.s. transformations of $M$.

A polymorphism of $M$ is a measure $\pi$ on $M \times M$, whose pushforwards to $M$ with respect to both projections $M \times M \to M$ coincide with $\mu$. Denote by $\text{Pol}(M)$ the set of all polymorphisms of $M$. We say that a sequence $\pi_j \in \text{Pol}(M)$ converges to $\pi$ if for any measurable sets $A, B \subset M$ we have convergence $\pi_j(A \times B) \to \pi(A \times B)$. The space $\text{Pol}(M)$ is compact and the group $\text{Ams}(M)$ is dense in $\text{Pol}(M)$.

Polymorphisms can be regarded as maps, spreading points of $M$ to probabilistic measures on $M$. Namely, for $\pi \in \text{Pol}(M)$ consider a system of conditional measures $\pi_m$ on sets $m \times M \subset M \times M$, where $m$ ranges in $M$. We declare that the 'map' $\pi$ send each point $m$ to the measure $\pi_m$. If $\pi, \pi_n \in \text{Pol}(M)$, then the product $\rho = \pi \circ \pi_n$ is defined from the condition

$$\rho_m = \int_M \pi_n d\pi_m(n).$$

We get a semigroup with separately continuous product.

For any $g \in \text{Ams}(M)$ consider the map from $M$ to $M \times M$ defined by $m \mapsto (m, g(m))$ and take the pushforward of the measure $\mu$. In this way, we get a polymorphism supported by the graph of $g$. The group $\text{Ams}(M)$ is dense in $\text{Pol}(M)$.
A Markov operator $R$ in $L^2(M)$ is a bounded operator satisfying 3 properties:

– for any function $f \geq 0$ we have $Rf \geq 0$;
– $R \cdot 1 = 1$, $R^* \cdot 1 = 1$.

Recall that automatically $\|R\| = 1$. There is a one-to-one correspondence between the set of Markov operators $\text{Mar}(M)$ and $\text{Pol}(M)$. Namely, let $R$ be a Markov operator, then we define a polymorphism $\pi$ by

$$\pi(A \times B) = \langle RI_A, I_B\rangle_{L^2(M)},$$

where $A, B \subset M$ are measurable sets, and $I_A, I_B$ are their indicator functions. The weak convergence in $\text{Mar}(M)$ corresponds to the convergence in $\text{Pol}(M)$, the product of Markov operators corresponds to the product of polymorphisms.

1.4. Closures of actions. Let a group $G$ act on $M$ by measure-preserving transformations, i.e., we have a homomorphism $G \to \text{Ams}(M)$, to be definite assume that this is an embedding. Then the closure of $G$ in $\text{Pol}(M)$ is a compact semigroup $\Delta \supset G$. A description of the closure is not too interesting for connected Lie groups (for instance, for semisimple linear Lie groups we get a one-point compactification, this follows from [2], Theory 5.3). However, a description of such closure is interesting for infinite-dimensional groups.

The first result of this type was obtained by Nelson [4], 1973. He showed that the standard action of the infinite-dimensional orthogonal group on the space with Gaussian measure admits an extension to an action of the semigroup of all contractive operators by polymorphisms and obtained formulas for such measures. Now, it is known a big collection of actions infinite-dimensional groups on measure spaces, however the closures are evaluated in few cases [7], [6]. Here we describe a new (relatively simple) example.

1.5. The statement. For any polymorphism $\pi$ on $\text{Herm}_\infty$ we write a characteristic function on $\text{Herm}_\infty^0 \times \text{Herm}_\infty^0$ by

$$F(\pi|A, B) := \int_{\text{Herm}_\infty \times \text{Herm}_\infty} e^{i \text{tr} AX + BY} d\pi(X, Y).$$

Denote by $\mathcal{B}(\infty)$ the semigroup of all operators in $\ell_2$ with norm $\leq 1$. Our purpose is the following statement

**Theorem 1.1** For any ergodic measure on $\text{Herm}_\infty$, the closure of $U(\infty)$ in $\text{Pol}(\text{Herm}_\infty)$ is isomorphic to $\mathcal{B}(\infty)$. If $S \in \mathcal{B}(\infty)$, then the characteristic function of the corresponding polymorphism $\pi_S$ is given by

$$F(\pi_S|A, B) = \exp\left\{ -\frac{\gamma_1}{2} \text{tr} \left[ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & S \\ S^* & 1 \end{pmatrix} \right]^2 + i\gamma_2(\text{tr} A + \text{tr} B) \right\} \times \prod_k \frac{e^{-i\lambda_k(\text{tr} A + \text{tr} B)}}{\det \left[ 1 - i\lambda_k \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & S \\ S^* & 1 \end{pmatrix} \right]}.$$

(1.2)
Since the characteristic function is a product, the measure $\pi_S$ can be decomposed as a convolution of measures. The exponential factor corresponds to a Gaussian measure, let us explain meaning of factors in the product. Consider a measure $\nu_S$ on $C^\infty \times C^\infty$ defined in the following way in terms of its characteristic function

$$\int_{C^\infty \times C^\infty} e^{i \text{Re} \tau + i \text{Re} \tau} d\nu_S := \exp\left\{ -\frac{1}{2} \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} S & 1 \\ S^* & 1 \end{pmatrix} \begin{pmatrix} u^* \\ v^* \end{pmatrix} \right\}.$$ 

Consider a map $C^\infty \times C^\infty \to \text{Herm}_\infty \times \text{Herm}_\infty$ given by

$$(u, v) \mapsto (\lambda u^* u, \lambda v^* v).$$

Then the image of $\nu_S$ under this map is a measure whose characteristic function is

$$\det\left[ 1 - i\lambda_k \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 \\ S^* \end{pmatrix} \right]^{-1}.$$ 

2 Proof

2.1. A priori remarks.

**Theorem 2.1** Let the complete unitary group $\overline{U}(\infty)$ act by measure preserving transformation\(^2\) on a Lebesgue space $M$ with a probability measure. Then the closure of $\overline{U}(\infty)$ in $\text{Pol}(M)$ is isomorphic to $\mathcal{B}(\infty)$.

**Proof.** Let $\rho$ be a unitary representation of $\overline{U}(\infty)$ in a Hilbert space $H$. The closure of the group $\rho(\overline{U}(\infty))$ in the group of all bounded operators in $H$ with respect to the weak operator topology is a semigroup isomorphic to $\mathcal{B}(\infty)$ (this follows from Kirillov–Olshanski classification of representations of $\overline{U}(\infty)$, see [8], Theorem 1.2).

We apply this to the action of $\overline{U}(\infty)$ in $L^2(M)$. The group $\overline{U}(\infty)$ acts by Markov operators, and weak limits of Markov operators are Markov operators. Therefore the semigroup $\mathcal{B}(\infty)$ also acts by Markov operators. \hfill \Box

**Lemma 2.2** Any ergodic action of $U(\infty)$ on $\text{Herm}_\infty$ admits a continuous extension to an action of $\overline{U}(\infty)$.

**Proof.** According [9], Corollary 2.14, the representation of $U(\infty)$ in the space $L^2(\text{Herm}_\infty, \mu)$ has a continuous extension to $\overline{U}(\infty)$. By the continuity arguments, the group $\overline{U}(\infty)$ acts by Markov operators $\rho(g)$. We have $\|\rho(g)\| \leq 1$, $\|\rho(g)^{-1}\| \leq 1$. Therefore $\rho(g)$ is a unitary operator. Hence it corresponds to a measure preserving transformation. \hfill \Box

\(^2\)Such an action can not be point-wise; the transformations are defined a.s., and products also are defined a.s., see [1].
2.2. Calculation. Consider a measure $\mu$ with a characteristic function \( \hat{\mu} \). For $g \in U(\infty)$ consider the corresponding polymorphism $\pi_g$ of $\text{Herm}_\infty$.

The characteristic function of $\pi_g$ is

$$F(\pi_g|A,B) = \exp \left\{ -\frac{\gamma_1}{2} \text{tr}(A + UB)^2 + i\gamma_2(\text{tr} A + \text{tr} UB^{-1}) \right\} \times \prod_{k=1}^{\infty} \frac{e^{-i\lambda_k \text{tr}(A + UB)}}{\det[1 - i\lambda_k(A + UB^{-1})]}.$$  \(2.1\)

By Theorem 2.1, for any $R \in \mathcal{B}(\infty)$, we have a polymorphism $\pi_R$ in the space $L^2(\text{Herm}_\infty, \mu)$. If a sequence $R_j \in \mathcal{B}(\infty)$ weakly converges to $R$, then we have the weak convergence of the corresponding polymorphisms, $\pi_{R_j} \to \pi_R$. This is equivalent to a point-wise convergence of characteristic functions. Now for $R \in \mathcal{B}(\infty)$ and a sequence $g_j \in U(\infty)$ weakly converging to $R$ we can find $F(\pi_{R_j}|A,B)$ as a point-wise limit of $F(\pi_{g_j}|A,B)$.

Let $S$ be a finitary operator with norm $\leq 1$, let actually $S$ be an $\alpha + \infty$-block matrix of the form

$$S = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}.$$  

Then we can build $u$ as a block of a unitary $\alpha + \alpha$-block matrix $\begin{pmatrix} u & v \\ w & z \end{pmatrix}$ (see, e.g., [3], Theorem VIII.3.2). Choose $U = U_m$ as a block unitary $\alpha + m + m + \alpha + \infty$-matrix of the form

$$U_m = \begin{pmatrix} \begin{pmatrix} u & 0 & 0 & v & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ w & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix}.$$  

Clearly, we have a weak convergence

$$U_m \to S,$$

we wish to watch the convergence of characteristic functions $F(\pi_{U_m}|A,B)$. In fact, we will show that this sequence is eventually constant for any fixed $A, B$.

Fix $A, B \in \text{Herm}_\infty^0$. Let actually $A, B \in \text{Herm}_{\alpha + \beta}$. Let $m$ be sufficiently large (in fact, $m \geq \beta$). We represent the matrices $A, B$ as block matrices of size $\alpha + \beta + (m - \beta) + \beta + (m - \beta) + \infty$:

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & \ldots & 0 \\ a_{21} & a_{22} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & 0 & \ldots & 0 \\ b_{21} & b_{22} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$  


We wish to evaluate
\[
det[1 - i\lambda_k(A + U_mB^{-1}U_m^*)] \quad \text{and} \quad \text{tr}(A + U_mB^{-1}U_m^*)^2.
\]
A straightforward calculation gives
\[
A + U_mB^{-1}U_m^* = \begin{pmatrix}
a_{11} + ub_{11}u^* & a_{12} & ub_{12} & ub_{11}w^* & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
b_{21}u^* & 0 & 0 & b_{22} & b_{21}w^* & 0 & 0 \\
w_{b_{11}}u^* & 0 & w_{b_{12}} & w_{b_{11}}w^* & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Clearly, we can remove zero columns and zero rows from this matrix. Formally,
\[
\det(1 - i\lambda_k(A + U_mB^{-1}U_m^*)) = \det(1 - i\lambda_kH),
\]
where
\[
H = \begin{pmatrix}
a_{11} + ub_{11}u^* & a_{12} & ub_{12} & ub_{11}w^* \\
a_{21} & a_{22} & 0 & 0 \\
b_{21}u^* & 0 & b_{22} & b_{21}w^* \\
w_{b_{11}}u^* & 0 & w_{b_{12}} & w_{b_{11}}w^*
\end{pmatrix}.
\]
Denote by \(\Delta\) the block diagonal matrix with blocks 1, 1, 1, \(w\). Represent \(H\) as \(H = \Delta Z\) (where the expression for \(Z\) is clear). We apply the formula
\[
\det(1 - i\lambda_k\Delta Z) = \det(1 - i\lambda_kZ\Delta).
\]
Denote \(H' := Z\Delta\),
\[
H' = \begin{pmatrix}
a_{11} + ub_{11}u^* & a_{12} & ub_{12} & ub_{11}w^* w \\
a_{21} & a_{22} & 0 & 0 \\
b_{21}u^* & 0 & b_{22} & b_{21}w^* w \\
b_{11}u^* & 0 & b_{12} & b_{11}w^* w
\end{pmatrix}.
\]
(2.2)
Since the matrix \(\begin{pmatrix} u & v \\ w & z \end{pmatrix}\) is unitary, we have \(w^*w = 1 - uu^*\). We substitute this to the 4-th column of (2.2), keeping a result in mind we continue our calculations. Denote
\[
T = \begin{pmatrix}
1 & 0 & 0 & u \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Then
\[
H' = \begin{pmatrix}
a_{11} + ub_{11}u^* & a_{12} & a_{11}u + ub_{12} & ub_{11} \\
a_{21} & a_{22} & 0 & a_{21}u \\
b_{21}u^* & 0 & b_{22} & b_{21} \\
b_{11}u^* & 0 & b_{12} & b_{11}
\end{pmatrix} T^{-1} =
\]
\[
T \begin{pmatrix}
a_{11} & a_{12} & 0 & a_{11}u \\
a_{21} & a_{22} & 0 & a_{21}u \\
b_{21}u^* & 0 & b_{22} & b_{21} \\
b_{11}u^* & 0 & b_{12} & b_{11}
\end{pmatrix} T^{-1}.
\]

Hence
\[
\det (1 - i\lambda_k H') = \det \left[ 1 - i\lambda_k \begin{pmatrix}
a_{11} & a_{12} & 0 & a_{11}u \\
a_{21} & a_{22} & 0 & a_{21}u \\
b_{21}u^* & 0 & b_{22} & b_{21} \\
b_{11}u^* & 0 & b_{12} & b_{11}
\end{pmatrix} \right]
\]
\[
= \det \left[ 1 - i\lambda \begin{pmatrix}
a_{11} & a_{12} & a_{11}u & 0 \\
a_{21} & a_{22} & a_{21}u & 0 \\
b_{21}u^* & 0 & b_{22} & b_{21} \\
b_{11}u^* & 0 & b_{12} & b_{11}
\end{pmatrix} \right]
\]
\[
= \det \left[ 1 - i\lambda \begin{pmatrix}
A & 0 & S \end{pmatrix} \begin{pmatrix}
1 & \end{pmatrix} \begin{pmatrix}
S^* & 1
\end{pmatrix} \right] \quad (2.3)
\]

and we get the desired expression.

Next,
\[
\text{tr}(A + U_mB^{-1}U_m^*)^2 = \text{tr}A^2 + \text{tr}B^2 + 2 \text{tr}AU_mB^{-1}U_m^*.
\]

Multiplying matrices we observe that \(AU_mB^{-1}U_m^*\) has a unique nonzero diagonal block, \(a_{11}ub_{11}u^*\). Thus,
\[
\text{tr}AU_mB^{-1}U_m^* = \text{tr}a_{11}ub_{11}u^* = \text{tr}ASBS^*,
\]
and this implies
\[
\text{tr}(A + U_mB^{-1}U_m^*)^2 = \text{tr} \left[ \begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix} \begin{pmatrix}
1 & \\
S^* & 1
\end{pmatrix} \right]^2.
\]

Thus, for \(m \geq \beta\) the value of \(F(\pi_{U_m}|A, B)\) is given by formula (1.2).

So, the theorem holds for finitary matrices \(S\). Consider an arbitrary \(S \in \mathcal{B}(\infty)\). Denote by \(S[m]\) the left upper corner of \(S\). Denote
\[
S_m := \begin{pmatrix}
S[m] & 0 \\
0 & 0
\end{pmatrix}.
\]

We have a weak convergence \(S_m \to S\). On the other hand, we have the pointwise convergence
\[
F(\pi_{S_m}|A, B) \to F(\pi_S|A, B)
\]
(in fact, this sequence is eventually constant for any fixed \(A, B\)).

This completes the proof of the theorem.
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