Universal prethermal dynamics and self-similar relaxation in the two-dimensional Heisenberg model

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We characterize the universal far-from-equilibrium dynamics of the isolated two-dimensional quantum Heisenberg model. For a broad range of initial conditions, we find a long-lived universal prethermal regime characterized by self-similar behavior of spin-spin correlations. We analytically derive the spatial-temporal scaling exponents and find excellent agreement with numerics using phase space methods. The scaling exponents are insensitive to the choice of initial conditions, which include coherent and incoherent spin states as well as values of magnetization and energy in a wide range. Compared to previously studied self-similar dynamics in non-equilibrium $O(n)$ field theories and Bose gases, we find qualitatively distinct scaling behavior originating from the presence of spin modes which remain gapless at long times and which are protected by the global SU(2) symmetry. Our predictions, which suggest a new non-equilibrium universality class, are readily testable in ultra-cold atoms simulators of Heisenberg magnets.

Introduction.—Self-similarity is a distinguishing feature that signals the emergence of universality in complex systems. Close to equilibrium, diverse and seemingly distinct models can be classified using symmetry, dimensionality, and conserved charges into universality classes sharing the same self-similar scaling [1]. Systems far from equilibrium, however, break a symmetry concomitant with detailed balance [2], and can therefore exhibit self-similar dynamics beyond conventional equilibrium statistical mechanics. Celebrated examples in classical physics comprise turbulence [3, 4], ageing [5], phase-ordering kinetics [6], KPZ scaling [7], reaction-diffusion models, and percolation [8].

Universality and self-similar relaxation typically arise in open systems with external friction and noisy forces that drive the system to or across a critical point [1, 8–12]. However, self-similar scaling has recently been shown to occur also in isolated systems where the system acts as its own bath. Prominent examples of dynamical scaling in isolated quantum many-body systems include pre-thermal critical states [5, 13–23] and non-thermal fixed points in scalar and gauge theories [24–40]. Research into these phenomena is further fueled by a surge of experimental evidence for universal dynamics in cold atoms [41–47], and dynamical phase transitions in trapped ions [48] and cavity QED [49].

Motivated by recent experiments on non-equilibrium spin dynamics in cold atomic gases [50–52], here we study the long-lived universal scaling dynamics and related critical exponents of the two-dimensional Heisenberg model. Specifically, starting from an initial textured state with a characteristic wavevector $q$, and by using the truncated Wigner approximation (TWA) for spins [53, 54], we find that the equal-time spin-spin correlation functions exhibit self-similar scaling at late times,

$$\sum_{a=x,y,z} \langle \hat{S}_a^\alpha(t)\hat{S}_k^\beta(t) \rangle = t^{\alpha \Phi(q,|k|)} \quad (1)$$

for a broad range of initial conditions and arbitrary spin number $S$. The spatial-temporal scaling exponents $(\alpha, \beta)$ are independent of the details of the initial condition, whereas the universal function $\Phi$ is only sensitive to a combination of $q$ and the global magnetization of the initial state. In a loose sense, the initial length-scale $1/q$ defines a dynamical renormalization group integra-
tion scale that governs the temporal scaling of correlation functions. Analytical considerations solely based on symmetries and the structure of the equations of motion, combined with a quasi-condensate picture describing coherent magnetization fluctuations, allow us to derive analytically the scaling exponents for a Gaussian and an interacting non-thermal fixed point. Using TWA we show that spin dynamics undergoes a crossover between these two scaling regimes. Remarkably, the late time dynamics exhibits critical exponents resulting from the existence of a slow spin mode whose gapless nature is symmetry-protected in the presence of SU(2) symmetry, as we numerically demonstrate by computing unequal-time correlations [55–59].

In the spirit of the Halperin-Hohenberg classification [1], the values of (α, β) combined with the presence of this additional slow mode with quadratic dispersion suggest that the dynamical scaling exponents of the Heisenberg model belong to a different non-equilibrium universality class than Bose gases [29–32, 55, 56], dissipative O(n) models [60–64], and other non-integrable high dimensional spin systems [65–74], as we discuss below.

**Microscopic model.**—We consider the two-dimensional isotropic Heisenberg model on a square \( L \times L \) lattice with lattice constant \( \ell \) and total number of sites \( N = L^2 \):

\[
\hat{H} = -J \sum_{\langle i,j \rangle} \left( \hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y + \hat{S}_i^z \hat{S}_j^z \right),
\]

where \( \langle i,j \rangle \) denotes summation over nearest neighbours. Each site contains a spin \( S \) degree of freedom and periodic boundary conditions are assumed in each spatial direction. This model has an SU(2) symmetry with conserved global spin magnetization \( S^a = \sum_i \langle \hat{S}^a_i \rangle \) in all directions \( a \in \{x, y, z\} \), where \( \langle \cdot \rangle \) denotes the quantum expectation value. A linear Zeeman field, which is present in many experimental Material (SM). Equation (3) defines a character-

\[
\langle \hat{S}_i^0 \rangle = S \sin \theta e^{i q \cdot r_i}, \quad \langle \hat{S}_i^z \rangle = S \cos \theta,
\]

with \( \hat{S}_i^\pm = \hat{S}_i^x \pm i \hat{S}_i^y \) and \( r_i \) the position of site \( i \). Other classes of initial conditions are considered in the Supplemental Material (SM). Equation (3) defines a characteristic timescale \( 1/\tau_c = J S^2 \sin^2 \theta [2 - \cos(q_x \ell) - \cos(q_y \ell)] \) associated to the energy density of the initial state.

**Spatial-temporal scaling via phase space methods.**—We begin by computing the real time dynamics using TWA [75]. This method incorporates quantum fluctuations present in the initial state by considering classical spins \( \mathbf{S}_i = (\hat{S}_i^x, \hat{S}_i^y, \hat{S}_i^z) \) with quantum noise at \( t = 0 \) and evolving them with the classical Landau-Lifshitz equations of motion. Defining \( \hat{S}_i^\pm \) as the transverse magnetization to \( \langle \hat{S}_i^0 \rangle \) for the initial condition (3), we assume initial Gaussian fluctuations of \( \hat{S}_i^\pm \) given by \( \langle \hat{S}_i^\pm \rangle_{cl} = 0 \) and \( \langle \hat{S}_i^\pm : \hat{S}_i^\pm \rangle_{cl} = S \), where \( \langle \cdot \rangle_{cl} \) denotes average over classical trajectories.

The subsequent dynamics follows three relaxation stages as illustrated in Fig. 1: (I) quantum fluctuations trigger a dynamical instability which leads to (II) a depletion of the macroscopically occupied state \( q \) and is followed by (III) the formation of a quasi-condensate with a time-dependent correlation length \( \xi(t) \sim t^\beta \) and self-similar evolution following Eq. (1).

Figure 2(a) shows the evolution of the spin-spin correlation function \( \sum_k (\hat{S}_k^a \hat{S}_k^a) \) after a quench from an initial spin spiral. Shown with solid and dotted lines are the initial distribution and the process of depletion of the initial \( q \) mode (stages I-II), respectively. We show with colored dashed lines the spin-spin correlation in the self-similar regime (stage III). Dotted lines are plotted for the range \( 15 < t/\tau_c < 40 \), with lighter colors for increasing \( t \). (b) Collapsed datapoints during stage III, c.f. Eq. (1), with scaling exponents \( \alpha = \frac{2}{3} \) and \( \beta = \frac{1}{3} \), (c) Fourier transform of the unequal time spin-spin correlation function \( C_{\alpha x}(k, \omega) = \int dt e^{i \omega t} \langle \hat{S}_k^x(t) \hat{S}_k^x(0) \rangle \) for \( t_0 = 20 \tau_c \), see SM. Different values of \( t_0 \) do not affect the qualitative features of the plot. Simulations parameters: \( L = 400 \), \( q_x = 0.2 \), \( \theta = \pi / 2 \), \( S = 5 \).
proximately $4\tau_s$ [67] and leads to a quick redistribution of fluctuations into other $k$ modes [stages I and II in Fig. 1; dotted lines in Fig. 2(a)]. At later times [stage III in Fig. 1; blue dashed lines in Fig. 2(a)] the system exhibits self-similarity as demonstrated in Fig. 2(b) by the excellent collapse of the rescaled curves with

$$\alpha = 0.63 \pm 0.05, \quad \beta = 0.34 \pm 0.03, \quad (4)$$

which also agree with our analytical estimates below. The procedure for fitting the exponents is discussed in the SM.

The SU(2) symmetry of the Heisenberg model precludes the opening of a gap during the dynamics. To find the relevant excitations, we evaluate in Fig. 2(c) the energy conservation in the Heisenberg model, $\partial_\tau \mathcal{S}_0 = \sum_{j,b,c} c_{abc} \mathcal{S}_0^b \mathcal{S}_j^c$, which yields

$$\partial_\tau \langle \mathcal{S}_0^a \mathcal{S}_\alpha^{-k} \rangle = 2 \sum_{p,b,c} (\gamma_0 - \gamma_p) \text{Re} \left[ \epsilon_{abc} \langle \mathcal{S}_\alpha^{-k} \mathcal{S}_p^b \mathcal{S}_\alpha^{-p} \mathcal{S}_c^c \rangle \right], \quad (7)$$

see details in the SM. The central assumption in our derivation of the scaling exponents $(\alpha, \beta)$.—We now analytically estimate the scaling exponents assuming for simplicity that the system has no net magnetization (non-zero magnetization does not affect the argument in any essential way). In this case, spin-spin fluctuations eventually become isotropic both in real and spin space. We find this to occur after a short transient timescale $\approx 5\tau_s$ (c.f. Fig. 1 and SM) and, therefore, the three components of the spin-spin correlation function exhibit the same scaling. In addition, we find that the mean-field components $2\sum_{k,\alpha} \langle \hat{S}_\alpha^{-k} \hat{S}_\alpha^k \rangle$ vanish on average at the onset of stage III. As a result, we use the full and connected component of $\langle \hat{S}_\alpha^{-k} \hat{S}_\alpha^k \rangle$ indistinguishably.

The first relation between $\alpha$ and $\beta$ is obtained from the local constraint of spin operators $\hat{S}_+ \hat{S}_- = S(S + 1)$. In momentum space, this relation is written as

$$\frac{1}{N} \sum_{k,\alpha} \hat{S}_\alpha^{-k} \hat{S}_\alpha^k = S(S + 1). \quad (5)$$

Equation (5) is an exact relation that is independent of the state of the system. If the spin-spin correlation function satisfies Eq. (1), then Eq. (5) implies that $\rho_{\alpha\beta} \propto \mathcal{S}^{\alpha \beta} \Phi(|x|)$ is a constant or, equivalently, that $\alpha$ and $\beta$ are related through

$$\alpha = d\beta. \quad (6)$$

In comparison to the universal dynamics of a Bose gas [29], Eq. (6) is equivalent to particle number conservation in the self-similar range. Interestingly, Eq. (6) also implies energy conservation in the Heisenberg model, $E = J \sum_{k,t} e^{i\ell} \hat{S}_\alpha^{-k} \hat{S}_\alpha^k$, if $\xi \gg \ell$, where $\sum_{\ell}$ denotes a sum over all nearest-neighbor vectors. This occurs because energy is quadratic in spin operators, unlike the Bose gas where energy has quartic contributions. If the SU(2) symmetry is broken (e.g., in the XXZ model), then Eq. (6) does not imply energy conservation and, instead, two self-similar regimes can coexist [68].

The second relation between $\alpha$ and $\beta$ is obtained from the dynamics. The time dependence of the spin-spin correlation function $\langle \hat{S}_\alpha^{-k} \hat{S}_\alpha^k \rangle$ can be straightforwardly obtained from the microscopic equations of motion of the spin operators, $\partial_\tau \mathcal{S}_0 = \sum_{j,b,c} c_{abc} \mathcal{S}_0^b \mathcal{S}_j^c$, which yields

$$\partial_\tau \langle \mathcal{S}_0^a \mathcal{S}_\alpha^{-k} \mathcal{S}_\alpha^{-p} \mathcal{S}_c^c \rangle = 2 \sum_{p,b,c} (\gamma_0 - \gamma_p) \text{Re} \left[ \epsilon_{abc} \langle \mathcal{S}_\alpha^{-k} \mathcal{S}_p^b \mathcal{S}_\alpha^{-p} \mathcal{S}_c^c \rangle \right], \quad (7)$$

Combining Eq. (8) and Eq. (6) in $d = 2$ yields $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$, consistent with the numerical results in Eq. (4) and with the scaling found numerically within kinetic theory close to the ferromagnetic ground state ($\theta \approx 0$) [66]. Remarkably, however, kinetic theory failed to analytically yield the correct exponents from scaling arguments alone.

The universal scaling function.—So far, we considered the dynamics of a single dimensionless parameter $\xi/\ell$ that governs the self-similar scaling. In addition, we find that the scaling function is sensitive to the global magnetization of the system. Empirically, we find that the dimensionless ratio

$$\Theta = \frac{\tan \theta}{q\ell} \quad (9)$$

alone governs the scaling function $\Phi$, as shown in Fig. 3. If $\theta = \pi/2$, then $\Theta \to \infty$ and the scaling function becomes independent of $q$, as we already noted above. For other values of $\theta$ and $q$, we find an excellent collapse of the data points.

In the presence of finite magnetization $S^z$, this second dimensionless quantity $\Theta$ can be related to the ratio
between magnetization fluctuations and the global magnetization as follows. The average magnetization is given by \( S^2/N = S \cos \theta \). The average amplitude of magnetization fluctuations at the onset of stage III is obtained from the identity (5) after removing the disconnected component associated to the global magnetization, which scales as \( S^2 \cos^2 \theta \), and assuming that fluctuations are equally distributed in a phase space region of size \( |k| \lesssim q \). For large \( S \), this results in \( \sum_a \langle S^a_k S^a_k \rangle_c \sim (S \sin \theta q \ell)^2 \), where \( 'c' \) stands for connected. The ratio between these two quantities gives Eq. (9).

**Dynamical crossovers.**—The self-similar scaling regime described above cannot be captured by an effective Gaussian description. This can be readily checked using a self-consistent Holstein-Primakoff approximation in the low spin-wave density expansion [69, 78], which would instead yield \( \alpha = 1 \) and \( \beta = 1/2 \) for the \( d = 2 \) Heisenberg model (see SM). Interestingly, these Gaussian exponents can be observed at early times before crossing over into the non-thermal fixed point characterised by different values of \( \alpha \) and \( \beta \) in the low spin-wave density expansion [69, 78], which was originally derived within spin-wave hydrodynamics [69].

This could shed light on models with degrees of freedom different from spins but exhibiting the same universal dynamics. A recent experiment simulated the Heisenberg model.

**Discussion.**—The self-similar exponents and scaling functions obtained here are distinct from those found in previous works on non-equilibrium dynamics in classical and quantum \( O(n) \) field theories and \( U(n) \) bosonic models [29–32, 55, 56]. Compared to these previous works, the central difference of our results is that the dynamics of the Heisenberg model is dominated by gapless spin excitations at all times. This is in sharp contrast to typical nonthermal fixed points in bosonic theories, where an effective mass (or chemical potential) has been observed to be dynamically generated by fluctuations [20, 29, 55, 56]. It also differs from scaling dynamics of \( O(n) \) theories quenched to (or across) a critical point [13, 15, 18, 20], where a dynamically generated gap only vanishes asymptotically in time.

To rationalize such differences, we note that the leading non-linearity in the low-density Holstein-Primakoff spin-wave expansion of the Heisenberg magnet is a ‘soft’ vertex [69, 78], i.e. it vanishes at low momenta (see also SM). In Keldysh non-equilibrium field theory [81], this appears as a quartic term \( \propto \int \nabla^2 \varphi^2 \tilde{\varphi}^2 \), while for \( O(n) \) models we have the ‘hard’ vertex \( \propto g \int \varphi^2 \tilde{\varphi}^2 \). Here, the fields \( \varphi \) and \( \tilde{\varphi} \) incorporate classical and quantum fluctuations [75, 81, 82]. The Laplacian \( \nabla^2 \) is responsible for the soft vertex proportional to momentum squared in Eq. (7), and it was originally derived within spin-wave hydrodynamics [69, 78]. At the level of canonical power counting, the two vertices scale respectively as \( \lambda \sim \xi^{-(2-d)} \) and \( g \sim \xi^{-(4-d)} \) in a regime dominated by classical-statistical fluctuations, which is justified by the high energy of the initial states. As a result, a different power counting is obtained for higher order non-linearities: they become less relevant with higher powers of \( \varphi \) for \( O(n) \) models in \( d = 2 \), while they are all marginal in the Heisenberg magnet. Therefore, the effective field theory for \( O(n) \) models and the Heisenberg magnet differ already at the level of canonical power counting, and this in turn dictates a different set of exponents for dynamical scaling whenever diagrammatic corrections are included.

**Conclusions.**—Our results on self-similar relaxation in the Heisenberg model extend the paradigm of scaling close to non-thermal fixed points to spin models [24–40]. In this regard, we plan to develop in the near future a non-equilibrium renormalization group theory to inspect the nature of the universality class unveiled in this work. This could shed light on models with degrees of freedom different from spins but exhibiting the same universal dynamics.

A recent experiment simulated the Heisenberg model...
with ultra-cold lithium atoms [50, 51] and it offers the opportunity to directly test our predictions. A natural next step would consist in considering long-range spin interactions $\propto 1/r^n$, with the perspective to investigate how dynamical scaling exponents change with $n$ (similarly to equilibrium criticality in long-range interacting systems [83]). A tunable profile of the interactions with long-range spatial character is achievable in photonic waveguides or trapped ions using, for instance, Raman sidebands engineering [84, 85]. Finally, the extension of our self-similar scaling to non-thermal fixed points with $SU(n)$ symmetry, as realised in ultra-cold alkaline-earth Fermi gases [86], appears as another natural step for a systematic classification in terms of universality classes.

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[76] In TWA, correlation functions of classical variables correspond to quantum expectation values of symmetrized operators [75]. In particular, we have \( \langle \frac{1}{2} \{ S_{A}^{a}(t), S_{B}^{b}(t') \} \rangle \approx \langle S_{A}^{a}(t) S_{B}^{b}(t') \rangle_{cl} \), where \( \{ A, B \} \equiv AB + BA \). Note that for equal times, \( \langle \frac{1}{2} \{ S_{A}^{a}(t), S_{B}^{b}(t) \} \rangle = \langle S_{A}^{a}(t), S_{B}^{b}(t) \rangle \).

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SUPPLEMENTAL MATERIAL

Universal prethermal dynamics and self-similar relaxation in the two-dimensional Heisenberg model

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The outline of the Supplemental Material is as follows. In Sec. I, we numerically demonstrate the insensitivity of the scaling exponents ($\alpha, \beta$) to variations in the initial conditions by considering incoherent initial conditions, which are different from the spin spiral initial conditions used in the main text. In Sec. II, we study the time evolution of the spin-spin correlation functions for the transverse and longitudinal components of magnetization and show that they become isotropic even when the initial condition is not. In Sec. III, we show the fitting scheme used to numerically estimate the scaling exponents ($\alpha, \beta$). In Sec. IV, we discuss the derivation of Eq. (7) in the main text. In Sec. V, we derive the canonical power counting used to estimate the scaling exponents of the gaussian fixed point.

I. INDEPENDENCE OF INITIAL CONDITIONS

To illustrate the universal character of our results, we repeat our calculations for incoherent initial conditions and show that the system exhibits the same scaling behavior as the one observed in the main text. Here we use initial conditions of the form

$$\langle S^x_i \rangle = S \cos \phi_i \sin \theta, \quad \langle S^y_i \rangle = S \sin \phi_i \sin \theta, \quad \langle S^z_i \rangle = S \sin \theta, \quad \phi_i = \arg \left( \sum_k f_k e^{i k \cdot r_i} \right), \quad (S1)$$

where $f_k$ is a Gaussian-distributed complex function $f_k = e^{i \theta_k - (|k| - q)^2/\sigma^2} q$ that satisfies $f_{-k} = f^*_k$, with $\sigma = 0.1 q$ and $\theta_k$ a random phase uniformly sampled $[0, 2\pi)$. Figure S1(a) shows the self-similar scaling for initial conditions with zero net magnetization and with finite magnetization computed through the Truncated Wigner Approximation (TWA). In both cases, we rescaled the datapoints using $(\alpha, \beta) = (2/3, 1/3)$. We find excellent agreement with the results reported in the main text using spin spiral initial conditions. In addition, the scaling function also matches remarkably well with the ones found in the main text, both with zero magnetization (Fig. 2 of main text) and non-zero magnetization (Fig. 3 of main text).

II. ISOTROPIC SPIN-SPIN CORRELATIONS

In the analytical derivation of the scaling exponents in the main text, we assumed that the spin-spin correlation function becomes isotropic in spin space. Although this is a plausible assumption, we numerically check that this is indeed the case. Figures S1(b-c) show the evolution of the (b) $xx$ and (c) $zz$ spin-spin correlation function at short times computed using TWA. We find that in a short timescale $\approx 5 \tau_s$, the spin-spin correlations become isotropic. In addition, both the $xx$ and $zz$ correlations exhibit the same scaling behavior. We recall that $\tau_s$ is defined from the initial conditions as $1/\tau_s = JS^2 \sin^2 \theta [2 - \cos(qx\ell) - \cos(qy\ell)]$, with $(qx, qy)$ the characteristic wavevector of the initial state, and $\theta$ defines the magnetization of the initial states, $S^z = NS \cos \theta$.

III. FITTING OF THE SCALING EXPONENTS

To find the scaling exponents $(\alpha, \beta)$ that best fit the numerical data, we minimize an error function that quantifies the accuracy of the datapoint collapse, as we describe next. We denote as $C(|k|, t) = \langle S_{-k}(t) S_k(t) \rangle_c$ the connected component of the spin-spin correlation function. The values of $|k| = k_i$ take discrete values on a lattice, and we choose...
FIG. S1. (a) Insensitivity of the scaling function and scaling exponents to the initial conditions. We show the rescaled spin-spin correlator $\sum_a \langle \hat{S}^a_i(t) \hat{S}^a_k(t) \rangle$ using incoherent initial conditions [see Eq. (S1)] with no net magnetization (blue circles) and with average magnetization $\langle S^z_i \rangle = S/2$ (orange triangles). Shown are datapoints in the time range $15\tau_\ast < t < 40\tau_\ast$, with decreasing shades of color indicating increasing time. We rescaled the data with scaling exponents $(\alpha, \beta) = (2/3, 1/3)$. (b-c) Evolution of the spin-spin correlation function $\langle \hat{S}^a_i(t) \hat{S}^a_k(t) \rangle$ for (b) $a = x$ and (c) $a = z$. Shown with black lines is $\langle \hat{S}^a_i(0) \hat{S}^a_k(0) \rangle$, and with blue lines is $\langle \hat{S}^a_i(t) \hat{S}^a_k(t) \rangle$ for $t = 2\tau_\ast$ to $t = 10\tau_\ast$ in steps of $2\tau_\ast$ (color shade decreases with time). The plots show that, after a transient time $\sim 5\tau_\ast$, correlations become isotropic in spin space.

IV. EQUATIONS OF MOTION

In the main text, we used the equations of motion of spin operators to analytically derive the dynamical scaling exponents. This approach is far more general than kinetic theory which may not be legitimate if, for example, magnetization fluctuations are large (as in our case). The microscopic equations of motion (in units of $J$) are given by:

$$\partial_t \hat{S}^a_i = \epsilon_{abc} \sum_j \hat{S}^b_j \hat{S}^c_j,$$

(S4)
where we sum over repeated indices only if the appear both as subscripts and superscripts. Going to momentum space, we obtain

\[ \partial_t \hat{S}^a_k = (\gamma_0 - \gamma_p) \epsilon_{abc} \hat{x}^b_{k+p} \hat{x}^c_p + i\gamma_0 \hat{S}^a_k, \]  

(S5)

with \( \gamma_k = \sum_\ell \epsilon^{i \ell \cdot k} \) (\( \ell \) are unit cell vectors). Multiplying on the left with the operator \( \hat{S}^a_{-k} \), summing with the complex conjugate of \( \hat{S}^a_{-k} \partial_t \hat{S}^a_k \), and taking expectation value, results in

\[ \partial_t \langle \hat{S}^a_{-k} \hat{S}^a_k \rangle = 2 \sum_p (\gamma_0 - \gamma_p) \text{Re} \left[ \epsilon_{abc} \langle \hat{S}^a_{-k} \hat{x}^b_{k-p} \hat{x}^c_p \rangle \right]. \]  

(S6)

We emphasize that correlations of the form \( \langle \hat{S}^a_{-k} \hat{S}^a_{k} \rangle \) appearing on the right-hand side of Eq. (S6) are not necessarily zero because the three components of magnetization are not independent. Equation (S6) is used in the main text to derive the scaling exponents under the assumption that a single lengthscale \( \xi \) (the quasi-condensate correlation length) describes collective dynamics and correlation functions at intermediate timescales.

V. EFFECTIVE HAMILTONIAN AND GAUSSIAN FIXED POINT

At short times, we observe a scaling regime with exponents \( (\alpha, \beta) \approx (1, 1/2) \), which we attribute to a Gaussian fixed point. In addition, we argue in the main text that non-linearities in the Heisenberg model are marginal in \( d = 2 \). Both observations can be rationalized from a Holstein-Primakoff expansion of the Heisenberg model close to the ferromagnetic ground state. Assuming small deviations from the ferromagnetic ground state \( |F\rangle = \hat{F} |0\rangle \), to quartic order in the bosonic operators \( \hat{\psi}_j \) leads to the long-wavelength Hamiltonian

\[ \hat{H} = J S a^2 \int_x \left( \nabla \hat{\psi}^\dagger \nabla \hat{\psi}_x + \frac{1}{4a^2} \hat{\psi}_x^\dagger \hat{\psi}_x^\dagger \nabla \hat{\psi}_x \nabla \hat{\psi}_x + h.c. \right). \]  

(S7)

Unlike the usual Bose gas with hard core collisions, here the collision amplitude of two quasiparticles with momentum \( k \) and \( p \) is \( - (k \cdot p) \). This reflects the SU(2) symmetry of the Hamiltonian: collisions become negligible at small momenta because a \( k \to 0 \) magnon state, \( \hat{\psi}_k |F\rangle \approx \hat{S}_k |F\rangle \), is effectively a global rotation of \( |F\rangle \) that would not affect the dynamics of a second incoming magnon.

The Gaussian exponents discussed in the main text can be derived by dropping non-linearities from (S7), and scaling the Fourier transform of the field with \( \xi \) as in Eq. (7), following \( \psi_k \sim \xi^{\alpha/(2\beta)} \) and \( \xi \sim t^\beta \) of the main text. We can now take the action associated to the free version of (S7) (equivalent to considering just the kinetic term), and require that this is invariant under a running scale, \( \xi \). It will yield \( \xi^{1/\beta - d - 2 + \alpha/\beta} \sim \xi^0 \) or, in other words, \( (\alpha + 1)/\beta = d + 2 \). Combining it with relation (6) in the main text, \( \alpha = \beta d \), related to total spin conservation, we find \( \alpha = 1 \) and \( \beta = 1/2 \) in \( d = 2 \). Such exponents are observed at short times in Fig. S3. Note that one can also derive the same Gaussian...
exponents in a similar way as for the nonthermal fixed point in Eq. (7) by deriving the equations of motion associated to Eq. (S7) without non-linearities, \( \partial_t \hat{\psi}_k \sim |k|^2 \hat{\psi}_k \), and rescaling both sides, which directly yields \( \beta = 1/2 \).