Energy conditions bounds on $f(T)$ gravity

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In the standard approach to cosmological modeling in the framework of general relativity, the energy conditions play an important role in the understanding of several properties of the Universe, including singularity theorems, the current accelerating expansion phase, and the possible existence of the so-called phantom fields. Recently, the $f(T)$ gravity has been invoked as an alternative approach for explaining the observed acceleration expansion of the Universe. If gravity is described by a $f(T)$ theory instead of general relativity, there are a number of issues that ought to be reexamined in the framework of $f(T)$ theories. In this work, to proceed further with the current investigation of the limits and potentialities of the $f(T)$ gravity theories, we derive and discuss the bounds imposed by the energy conditions on a general $f(T)$ functional form. The null and strong energy conditions in the framework of $f(T)$ gravity are derived from first principles, namely the purely geometric Raychaudhuri equation along with the requirement that gravity is attractive. The weak and dominant energy conditions are then obtained in a direct approach via an effective energy-momentum tensor for $f(T)$ gravity. Although similar, the energy condition inequalities are different from those of general relativity, but in the limit $f(T) = T$, the standard forms for the energy conditions in general relativity are recovered. As a concrete application of the derived energy conditions to locally homogeneous and isotropic $f(T)$ cosmology, we use the recent estimated values of the Hubble and the deceleration parameters to set bounds from the weak energy condition on the parameters of two specific families of $f(T)$ gravity theories.

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I. INTRODUCTION

A diverse set of cosmological observations coming from different sources, including the supernovae-type Ia (SNe Ia)\cite{1}, the cosmic microwave background radiation (CMBR)\cite{2}, and the large-scale structure (LSS)\cite{3} clearly indicate that the Universe is currently expanding with an accelerating rate. A number of alternative models and different frameworks have been proposed to account for this observed late-time accelerated expansion of the Universe. These approaches can be classified into two broad groups. In the first, the framework of general relativity is kept unchanged and an unknown form of matter sources, the so-called dark energy, is invoked. In this regard, the simplest way to describe the accelerated expanding Universe is by introducing a cosmological constant into the general relativity field equations. Although this is entirely consistent with the available observational data, it faces difficulties, including the microphysical origin and the order of magnitude of the cosmological constant. In the second group, modifications of Einstein’s gravitation theory are assumed as an alternative for describing the accelerated expansion.\cite{4}

Examples of the latter group include generalized theories of gravity based upon modifications of the Einstein-Hilbert action by taking nonlinear functions $f(R)$ of the Ricci scalar $R$ or other curvature invariants (for reviews see Ref.\cite{5}).

An alternative modification of general relativity, known as $f(T)$ gravity, has been examined recently as a possible way of describing the current acceleration of the Universe\cite{6,7}. The origin of $f(T)$ gravity theory goes back to 1928 with Einstein’s attempt to unify gravity and electromagnetism through the introduction of a tetrad (vierbein) field along with the concept of absolute parallelism or teleparallelism\cite{8}. In the teleparallel gravity (TG) theories the dynamical object is not the metric $g_{\mu\nu}$ but a set of tetrad fields $e_a(x^\mu)$, and rather than the familiar torsionless Levi-Civita connection of general relativity, a Weitzenböck connection (which has no curvature but only torsion) is used to define the covariant derivative. The gravitational field equation of TG is then described in terms of the torsion instead of the curvature\cite{10,11}. In formal analogy with the $f(R)$, the $f(T)$ gravity theory was suggested by extending the Lagrangian of teleparallel gravity to a function $f(T)$ of a...
torsion scalar $T$. In comparison with $f(R)$ gravity in the metric formalism, whose field equations are of the fourth order, $f(T)$ gravity has the advantage that the dynamics are governed by second-order field equations.

The fact that $f(T)$ theories can potentially be used to explain the observed accelerating expansion along with the relative simplicity of their field equations has given birth to a number of papers on these gravity theories, in which several features of $f(T)$ gravity have been discussed, including observational cosmological constraints, solar system constraints, cosmological perturbations, dynamical behavior, spherical symmetric solutions, the existence of relativistic stars, the possibility of quantum divide crossing, cosmographic constraints, and the lack of local Lorentz invariance, which may give rise to undesirable outcomes from $f(T)$ gravity, although suitable tetrad fields can be chosen. For some further references on several aspects of $f(T)$ gravity theories we refer the readers to Ref. [32].

In the framework of general relativity the so-called energy conditions have been used to derive remarkable results in a number of contexts. For example, the famous Hawking-Penrose singularity theorems invoke the strong energy condition (SEC) [33], whose violation allows for the observed accelerating expansion, and the proof of the second law of black hole thermodynamics requires null energy conditions (NEC) [34, 35].

On macroscopic scales relevant for cosmology, the confrontation of the energy conditions predictions with observational data is another important issue that has been considered in a number of recent articles. In this regard, since the pioneering works by Visser [36], a number of articles have been published concerning this confrontation by using model-independent energy-conditions bounds on the cosmological observable quantities, such as the distance modulus, lookback time, and deceleration and curvature parameters.

Owing to their role in several important issues in general relativity and cosmology, the energy conditions have also been investigated in several frameworks of modified gravity theories, including $f(R)$ gravity [37, 38], gravity without minimal coupling between curvature and matter [39], Gauss-Bonnet gravity [40], modified $f(G)$ gravity [41], and Brans-Dicke theories [42] (see also the related Refs. [43, 44]).

In this article, to proceed further with these investigations on the potentialities, difficulties, and limitations of $f(T)$ gravity theories, we derive the energy conditions for the general functional form of $f(T)$ and discuss some concrete examples of these bounds by using observational constraints on the Hubble and the deceleration parameters. The null and strong energy conditions (NEC and SEC) are derived in the framework of $f(T)$ from first principles, i.e., from the purely geometric Raychaudhuri equation along with the requirement that gravity is attractive. We find that the NEC and the SEC in general $f(T)$ gravity, although similar, are different from those of Einstein’s gravity and $f(R)$ gravity, but in the limiting case $f(T) = T$, the standard general relativity forms for these energy conditions are recovered. The resulting inequalities for the SEC and NEC in the $f(T)$ gravity framework are then compared with what would be obtained by translating these energy conditions in terms of an effective energy-momentum tensor for $f(T)$ gravity. There emerges from this comparison a natural formulation for the weak and dominant energy conditions (WEC and DEC) in the context of $f(T)$ gravity, which also reduce to the standard GR forms for these conditions in the limit $f(T) = T$. As a concrete application of the energy conditions for spatially homogeneous and isotropic $f(T)$ cosmology, we use recent estimated values of the Hubble and the deceleration parameters to set bounds from the WEC on the parameters of two specific families of $f(T)$ gravity theories.

Our paper is organized as follows. In Sec. II we give a brief review on the $f(T)$ theories and derive the field equations. In Sec. III using the purely geometric Raychaudhuri equations for timelike and null congruences of curves, we derive the SEC and NEC from first principles, and the WEC and DEC through an effective energy-momentum tensor. In Sec. IV we use the constraints on present-day values of cosmographic parameters to set constraints on exponential as well as on the Born-Infeld $f(T)$ gravity from the WEC. Finally, conclusions and final remarks are presented in Sec. V.

II. $f(T)$ GRAVITY THEORY

In this section, we briefly introduce the teleparallel gravity and its generalization known as $f(T)$ gravity. We begin by recalling that the dynamical variables in teleparallel gravity are the vielbein or tetrad fields, $e_a(x^\mu)$, which is a set of four ($a = 0, \cdots, 3$) vectors defining a local orthonormal frame at every point $x^\mu$ of the space-time manifold. The tetrad vectors field $e_a(x^\mu)$ are vectors in the tangent space and can be expressed in terms of a coordinate basis as $e_a(x^\mu) = e_a^\mu \partial_\mu$. The spacetime metric tensor and the tetrads are related by

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2 Throughout this paper we use Greek letters to denote space-time coordinate indices, which are lowered and raised, respectively, with $g_{\mu\nu}$ and $g^{\mu\nu}$, and vary from 0 to 3, whereas firsts alphabetic latin lower case letters ($a$ and $b$) are tetrad indices, which are lowered and raised with the Minkowski tensor $\eta_{ab} = \text{diag} (1, -1, -1, -1)$ and $\eta^{ab}$, respectively. We denote the spatial components $(1, 2, 3)$ by using the middle alphabetic latin lower case letters $i$ and $j$. \[ \text{\footnotesize{\cite{3, 4}}} \]
where $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric of the tangent space at $x^\mu$. It follows that the relation between frame components, $e^a$, and coframe components, $e^a_\mu$, are given by

$$e^a e^b_\nu = \delta^a_\mu \quad \text{and} \quad e^a e^b_\nu = \delta^b_a.$$  

In general relativity one uses the Levi-Civita connection

$$\text{\Gamma}^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) ,$$

which leads to nonzero spacetime curvature but zero torsion.

In teleparallel gravity, instead of the Levi-Civita connection, one uses the Weitzenböck connection which is given by

$$\tilde{\text{\Gamma}}^\lambda_{\mu\nu} = e^a_\mu \partial_\nu e^a_\lambda - e^a_\nu \partial_\mu e^a_\lambda .$$

An immediate consequence of this definition is that the covariant derivative, $D_\mu$, of the tetrad fields

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu - \tilde{\text{\Gamma}}^a_{\mu\nu} = 0 ,$$

vanishes identically. This equation leads to a zero curvature but nonzero torsion.

To clarify the interrelations between Weitzenböck and Levi-Civita connections, one needs to introduce the torsion and contorsion tensors, which are given, respectively, by

$$T^\rho_{\mu\nu} = \tilde{\text{\Gamma}}^\rho_{\mu\nu} - \text{\Gamma}^\rho_{\mu\nu} = e^a_\mu (\partial_\nu e^a_\lambda - \partial_\lambda e^a_\nu) ,$$

$$K^\rho_{\mu\nu} = \tilde{\text{\Gamma}}^\rho_{\mu\nu} - \text{\Gamma}^\rho_{\mu\nu} = \frac{1}{2} (T^\rho_{\mu\nu} + T^\rho_{\nu\mu} - T^\rho_{\mu\nu}) ,$$

where above $\tilde{\text{\Gamma}}^\rho_{\mu\nu}$ is the Levi-Civita connection.

Now, if one further defines the so-called super-potential

$$S^\mu_{\sigma} = K^{\mu\nu} - \delta^\mu_{\sigma} T^{\xi\nu} - \delta^\mu_{\nu} T^{\xi\sigma} ,$$

one obtains the torsion scalar

$$T = \frac{1}{2} S^\mu_{\sigma} T^{\sigma\mu} = \frac{1}{4} T^{\xi\nu} T_{\xi\mu} + \frac{1}{2} T^{\xi\mu} T_{\nu\xi} - T^{\xi\nu} T_{\xi\mu} ,$$

which is used as the Lagrangian density in formulation of the teleparallel gravity theory, which is given by

$$\mathcal{L}_T = \frac{e T}{2 \kappa^2} ,$$

where $e = \text{det}(e^a_\mu) = \sqrt{-g}$, $\kappa^2 = 8\pi G$, and $G$ is the gravitational constant. Now, by taking an arbitrary function $f$ of the torsion scalar $T$, one obtains the Lagrangian density of $f(T)$ gravity theory, that is

$$\mathcal{L}_T \rightarrow \mathcal{L}_{f(T)} = \frac{e f(T)}{2 \kappa^2} .$$

Now, by adding a matter Lagrangian density $\mathcal{L}_M$ to Eq. (11) and varying the resultant action with respect to the vierbein, one obtains the following field equation for $f(T)$ gravity:

$$\partial_\xi (e e^a_\rho S^\xi_{\sigma} f_T) - e e^2 \left( S e^a_\rho S^\xi_{\sigma} T_{\rho\xi\lambda} f_T + \frac{1}{2} e^a_\rho f(T) \right) = \partial_\xi (e e^a_\rho S^\xi_{\sigma}) - e e^2 \left( S e^a_\rho S^\xi_{\sigma} T_{\rho\xi\lambda} f_T + e^a_\rho (\partial_\xi T) S^\rho_{\sigma} f_T T \right)$$

$$+ \frac{1}{2} e^a_\rho f(T) = e \Theta^a_\sigma ,$$

where $f_T = df(T)/dT$, $f_{TT} = d^2 f(T)/dT^2$, and $\Theta^a_\sigma$ is the energy-momentum tensor of the matter fields. Here and in what follows we have chosen units such that $\kappa^2 = c = 1$.

To bring the field equations (12) to a form suitable for our purpose in the next section. To this end, we first note that if one multiply $e^{-1} g_{\mu\nu} e^a_\mu$, both sides of (12), the resultant equation is such that the coefficient of that the term $f_T$ takes the form

$$e e^a_\mu \partial_\xi (e e^a_\rho S^\xi_{\sigma}) - S e^a_\rho S^\xi_{\sigma} T_{\rho\xi\nu}$$

$$= \partial_\xi (e e^a_\rho S^\xi_{\sigma} - \tilde{\text{\Gamma}}^\rho_{\nu\xi} S^\xi_{\sigma} + \tilde{\text{\Gamma}}^\rho_{\xi\xi} S^\xi_{\rho} - S e^2 T_{\rho\xi\nu})$$

$$= -\nabla^\xi S^\xi_{\rho} - S e^2 K_{\xi\rho\nu} ,$$

where the relation

$$S_{(\mu\nu)} = T^{\rho}_{(\mu\nu)} = S_{\mu\nu} = 0$$

has been used.

On the other hand, from the relation between Weitzenböck and Levi-Civita connection in the form

$$R^\rho_{\mu\lambda\nu} = \partial_\lambda \tilde{\text{\Gamma}}^\rho_{\mu\nu} - \partial_\nu \tilde{\text{\Gamma}}^\rho_{\mu\lambda} + \tilde{\text{\Gamma}}^\rho_{\lambda\nu} \tilde{\text{\Gamma}}^\rho_{\mu\lambda} - \tilde{\text{\Gamma}}^\rho_{\mu\lambda} \tilde{\text{\Gamma}}^\rho_{\lambda\nu} ,$$

whose associated Ricci tensor can then be written as

$$R^\rho_{\mu\nu} = \partial_\nu K^\rho_{\mu\lambda} - \partial_\mu K^\rho_{\nu\lambda} + K^\rho_{\nu\lambda} K^\sigma_{\mu\lambda} - K^\rho_{\mu\lambda} K^\sigma_{\nu\lambda} ,$$

Now, by using $K^\rho_{\mu\nu}$ given by Eq. (8) along with the relations (13) and considering that $S^\mu_{\rho\nu} = 2 K^\mu_{\rho\nu} = -2 T^\mu_{\rho\nu}$ one has

$$R^\rho_{\mu\nu} = -\nabla^\rho S^\nu_{\mu\rho} - g_{\mu\nu} \nabla^\sigma T^\rho_{\sigma\rho} - S e^\rho_{\mu} K_{\rho\sigma\nu} ,$$

$$R = -T - 2 \nabla^\nu T^\nu_{\mu} ,$$

and thus obtain

$$G_{\mu\nu} = \frac{1}{2} g_{\mu\nu} T = -\nabla^\rho S^\nu_{\mu\rho} - S e^\rho_{\mu} K_{\rho\sigma\nu} ,$$
where \( G_{\mu\nu} = R_{\mu\nu} - (1/2) g_{\mu\nu} R \) is the Einstein tensor.

Finally, combining Eq. (13) and Eq. (18), the field equations for \( f(T) \) gravity Eq. (12) can be rewritten in the form

\[
A_{\mu\nu} f_T + B_{\mu\nu} f_{TT} + \frac{1}{2} g_{\mu\nu} f(T) = \Theta_{\mu\nu},
\]

where

\[
A_{\mu\nu} = g_{\mu\nu} e^a \partial_a \left[ e^{-c} S^\mu_{\rho} S^\rho_{\xi} - e^\lambda S_\xi^\sigma S_\rho^\mu \right]
= -\nabla_\sigma S^\mu_{\nu\sigma} - S_{\rho\lambda}^\mu K_{\lambda\rho\nu} = G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T,
B_{\mu\nu} = S_{\nu\mu} \nabla_\sigma T.
\]

To close this section, we note that since \( A_{\mu\nu} = -(R + 2T) \), the trace of Eq. (19), which can be used as an independent relation to simplify the field equation, can be expressed as

\[
-(R + 2T) f_T + B f_{TT} + 2f(T) = \Theta,
\]

where \( B = B_{\mu}^\mu \) and \( \Theta = \Theta_\mu^\mu \).

III. ENERGY CONDITIONS

A. Strong and null energy conditions

The ultimate origin of strong and null energy conditions is the Raychaudhuri equation together with the requirement that gravity is attractive. The Raychaudhuri equation gives temporal variation of the expansion \( \theta \) of congruence of geodesics (for a review article see Ref. [57]). For a congruence of timelike geodesics whose tangent vector field is \( u^\mu \) Raychaudhuri equation reads

\[
\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} u^\mu u^\nu,
\]

where \( \theta \), \( \sigma_{\mu\nu} \) and \( \omega_{\mu\nu} \) are, respectively, the expansion, shear, and rotation associated with the congruence defined by the vector field \( u^\mu \), and \( R_{\mu\nu} \) is the Ricci tensor.

The evolution equation for the expansion of a congruence of null geodesics defined by a null vector field \( k^\mu \) has a similar form as the Raychaudhuri equation (22), but with a factor 1/2 rather than 1/3, and \( -R_{\mu\nu} k^\mu k^\nu \) instead of \( -R_{\mu\nu} u^\mu u^\nu \) as the last term (see Ref. [58] for more details). Thus, its reads

\[
\frac{d\theta}{d\tau} = -\frac{1}{2} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} k^\mu k^\nu,
\]

where the kinematical quantities \( \theta \), \( \sigma_{\mu\nu} \) and \( \omega_{\mu\nu} \) are now clearly associated with the congruence of null geodesics.

An important point to be emphasized is that Raychaudhuri Eqs. (22) and (23) are purely geometric statements, and as such they make no reference to any theory of gravitation.

Now, since the shear is a "spatial" tensor, i.e., \( \sigma^2 = \sigma_{\mu\nu} \sigma^{\mu\nu} \geq 0 \), from Eqs. (22) and (23), one has that for any hypersurface of orthogonal congruences (\( \omega_{\mu\nu} = 0 \)), the conditions for gravity to remain attractive (\( d\theta/d\tau < 0 \)) are given by

\[
R_{\mu\nu} u^\mu u^\nu \geq 0,
\]

\[
R_{\mu\nu} k^\mu k^\nu \geq 0.
\]

Thus, as long as one can use the field equations of any given gravity theory to relate \( R_{\mu\nu} \) to the energy-momentum tensor \( T_{\mu\nu} \), the above Raychaudhuri Eqs. (22) and (23), along with the requirement that gravity is attractive, lead to Eqs. (24) and (25), which can be employed to restrict the energy-momentum tensors in the framework of the gravity theory one is concerned with.

Equations (24) and (25) are ultimately the SEC and DEC stated in a coordinate-invariant way for an unfixed geometrical theory of gravitation. Hence, for example, in the framework of general relativity, they take, respectively, the forms\(^3\)

\[
R_{\mu\nu} u^\mu u^\nu \geq \left( T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right) u^\mu u^\nu \geq 0,
\]

\[
R_{\mu\nu} k^\mu k^\nu \geq T_{\mu\nu} k^\mu k^\nu \geq 0,
\]

which, for example, for a perfect fluid of density \( \rho \) and pressure \( p \), i.e., for \( T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu} \), reduce to the well-known forms of the SEC and NEC in general relativity

\[
\rho + 3p \geq 0 \quad \text{and} \quad \rho + p \geq 0.
\]

B. Energy conditions in \( f(T) \) gravity

According to the previous section the Raychaudhuri equations together with the attractive character of the gravitational interaction give rise to Eqs. (24) and (25), which hold for any geometrical theory of gravitation. In what follows, we maintain this approach to derive the SEC and NEC in the \( f(T) \) gravity context. To this end, we first rewrite the \( f(T) \) field equation (19) in the form

\[
G_{\mu\nu} = \frac{1}{f_T} \left[ \Theta_{\mu\nu} + \frac{1}{2} (f_T f - f) g_{\mu\nu} - B_{\mu\nu} f_{TT} \right].
\]

Here, \( \Theta_{\mu\nu} \) and \( \Theta \) denote, respectively, the energy momentum tensor and its trace.

\(^3\) Clearly, here \( T \) is not the torsion scalar, but the trace of the energy momentum tensor \( T = T_\mu^\mu \).
From Eq. (20) and by taking into account the trace equation (21), we have

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T,$$

(30)

where

$$T_{\mu\nu} = \frac{1}{f_T} (\Theta_{\mu\nu} - f f_{\mu\nu}) ,$$

(31)

$$T = \frac{1}{f_T} (\Theta + T f_T - f - B f_{TT}).$$

(32)

Now, for the homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) metric with scale factor $a(t)$, i.e., $g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2)$, from Eqs. (6) through (9) along with Eq. (20), we have

$$T = -6H^2,$$

(33)

$$A_{00} = 6H^2, \quad A_{ij} = -2a^2 (3H^2 + \dot{H}) \delta_{ij},$$

(34)

$$B_{ij} = 24a^2 H^2 \delta_{ij}, \quad B = -72H^2 \dot{H},$$

(35)

where a dot denotes derivative with respect to time, $H = \dot{a}/a$ is the Hubble parameter, and the simplest and suitable tetrad basis was used [31].

Now, for a perfect fluid of density $\rho$ and pressure $p$, namely for

$$\Theta_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - p g_{\mu\nu} \quad \text{with} \quad u_{\mu} = (1, 0, 0, 0),$$

(36)

taking $k_{\mu} = (1, a, 0, 0)$, we obtain the $T_{\mu\nu}$ and its trace $T$, namely

$$\Theta_{00} = \frac{1}{f_T} \rho, \quad \Theta_{ij} = \frac{a^2}{f_T} (p - 24H^2 \dot{H} f_{TT}) \delta_{ij},$$

(37)

and

$$T = \frac{1}{f_T} (\rho - 3p + T f_T - f + 72H^2 \dot{H} f_{TT}).$$

(38)

Thus, from equations (24) and (25) for a general $f(T)$ gravity, the strong energy condition (SEC) and the null energy condition (NEC) can be, respectively, written as

$$\text{SEC:} \quad \frac{1}{2f_T} (\rho + 3p + f - T f_T - 72H^2 \dot{H} f_{TT}) \geq 0, \quad (39)$$

and

$$\text{NEC:} \quad \frac{1}{f_T} (\rho + p - 24H^2 \dot{H} f_{TT}) \geq 0. \quad (40)$$

We note that the well-known forms for the SEC ($\rho + 3p \geq 0$) and NEC ($\rho + p \geq 0$) in the framework of general relativity can be recovered as a particular case of the above SEC and DEC in the context of $f(T)$ gravity for the special case $f(T) = T$, as one would expect.

To derive the weak and dominant energy conditions (WEC and DEC) in $f(T)$ gravity, it is important to realize that the above SEC and NEC inequalities [Eqs. (39) and (40)] can also be recast as an extension of the SEC and NEC conditions in the context of general relativity by defining suitably an effective energy-momentum tensor in the context of $f(T)$ gravity. In fact, in $f(T)$ gravity theories one can define an effective energy-momentum tensor as

$$\Theta_{\mu\nu}^{\text{eff}} = \frac{1}{f_T} [\Theta_{\mu\nu} + \frac{1}{2} (T f_T - f) g_{\mu\nu} - f f_{TT}],$$

(41)

from which one defines the effective energy density and the effective pressure in the FLRW by

$$\rho^{\text{eff}} = -g^{00} \Theta_{00}^{\text{eff}} = \frac{1}{f_T} [\rho + \frac{1}{2} (T f_T - f)],$$

(42)

$$p^{\text{eff}} = \frac{1}{3} g^{ij} \Theta_{ij}^{\text{eff}} = \frac{1}{f_T} [p - \frac{1}{2} (T f_T - f) - 24H^2 \dot{H} f_{TT}],$$

(43)

which in turn make apparent that the SEC and NEC given by Eqs. (39) and (40) can be obtained from the corresponding general relativity expressions [Eq. (28)] by using the above effective matter components. Thus, using the effective energy-momentum tensor approach, the weak energy condition (WEC) in $f(T)$ gravity ($\rho^{\text{eff}} \geq 0$) reduce to

$$\text{WEC:} \quad \frac{1}{f_T} [\rho + \frac{1}{2} (T f_T - f)] \geq 0.$$

(44)

Similarly, the dominant energy condition (DEC) in $f(T)$ gravity ($\rho^{\text{eff}} \geq |p|$) can be written in the form

$$\text{DEC:} \quad \frac{1}{f_T} [\rho - p + (T f_T - f) + 24H^2 \dot{H} f_{TT}] \geq 0.$$

(45)

IV. CONSTRAINING $f(T)$ GRAVITY THEORIES

The energy conditions [39, 40, 44, and 45] can be used to place bounds on a given $f(T)$ in the context of FLRW models. To investigate such bounds, we first note that to ensure the positivity of the effective Newton gravity constant, one has $f_T > 0$ [29]. Thus, after some algebra, in terms of present-day values for the cosmological parameters, the energy conditions [39, 40, 44,]

4 A comparison with the effective energy-momentum tensor of Ref. [31] makes clear that the one used in the present work includes the whole matter.
and (45) can be, respectively, rewritten as

SEC:
\[ \rho_0 + 3p_0 + f_0 + 6H_0^2 f_{T_0} + 72(1 + q_0)H_0^4 f_{T_0}T_0 \geq 0; \]  

(46)

NEC:
\[ \rho_0 + p_0 + 24(1 + q_0)H_0^4 f_{T_0}T_0 \geq 0; \]  

(47)

WEC:
\[ 2\rho_0 - f_0 - 6H_0^2 f_{T_0} \geq 0; \]  

(48)

DEC:
\[ \rho_0 - p_0 - f_0 - 6[H_0^2 f_{T_0} + 4(1 + q_0)H_0^4 f_{T_0}T_0] \geq 0, \]  

(49)

where \( q = -(\ddot{a}/a)H^{-2} \) is the deceleration parameter, and a subscript 0 indicates the present-day value of the corresponding parameter.

To make concrete applications of the above conditions to set bounds on \( f(T) \), we first note that apart from the WEC [Eq. (48)], all the above conditions depend on the current value of the pressure \( p_0 \). Therefore, for simplicity in what follows we shall focus on the observational WEC constraints on \( f(T) \) gravity. Furthermore, we will also take the best fit value \( H_0 = 0.718 \) as determined by Capozziello et al. [24].

A. Exponential \( f(T) \) gravity

As a first concrete example, we shall examine the WEC bounds on the parameter \( \beta \) of the following exponential family of \( f(T) \) gravity theories [15, 59]:
\[ f(T) = T + \alpha T(1 - e^{\beta T_0/T}) \]  

(50)

with
\[ \alpha = -\frac{1 - \Omega_{m0}}{1 - (1 - 2\beta)} \]  

(51)

where the limit \( \beta = 0 \) corresponds to \( \Lambda \)CDM model, \( \Omega_{m0} \) is the dimensionless matter density parameter, and \( T_0 = T(z = 0) \) is the current value for the torsion scalar.

By using \( T_0 = -6H_0^2 \), one finds from (18) the following WEC constraint
\[ \alpha \beta T_0 e^{\beta} \geq 0. \]  

(52)

Now we take \( \Omega_{m0} = 0.272^{+0.036}_{-0.034} \) —which arises from the combination of 557 Type Ia Supernovae (SNe Ia) Union 2 set, baryonic acoustic oscillation (BAO), and the cosmic microwave background (CMB) radiation at 95% confidence level —along with the above observational value of \( H_0 \). These values lead to \( \beta > -1.256 \) for the relation (52) to be satisfied. Reciprocally, the inequality (52) is always fulfilled for all values \( \beta \) such that \( \beta > -1.256 \). This makes explicit the constraint on parameter \( \beta \) of the exponential \( f(T) \) gravity [Eq. (50)] for the WEC fulfillment.

B. Born-Infeld \( f(T) \) gravity

As the second concrete example, we consider the Born-Infeld (BI) \( f(T) \) gravity given by [53]
\[ f(T) = \lambda \left[ 1 - \epsilon + \frac{2T}{\lambda} \right]^{1/2} - 1 , \]  

(53)

where \( \epsilon = 4\Lambda/\lambda \) is a dimensionless parameter, \( \Lambda \) is the cosmological constant, and \( \lambda \) is a Born-Infeld-like constant. This gravity theory has been considered in several cosmological contexts, which include the avoidance of singularity in the standard model [53], as a way to an inflationary scenario without inflaton [54], and also to bound the dynamics of the Hubble parameter [54]. Clearly, the BI \( f(T) \) gravity [53] reduces to the standard TG (often referred to as TEGR) when \( \lambda \rightarrow \infty \). Here, we focus on the case \( \lambda > 0 \) [54]. In this case, the WEC takes the form
\[ \epsilon - \frac{T_0}{\lambda} + \left(1 - \epsilon + \frac{2T_0}{\lambda}\right)^{1/2} - 1 > 0 \]  

(54)

This inequality holds for
\[ 0 < \epsilon < 1 \quad \text{and} \quad \lambda > \frac{T_0}{\sqrt{\epsilon(1 - \sqrt{\epsilon})}} , \]  

(55)

which makes apparent that the range of \( \epsilon \) in which the WEC is fulfilled coincides with that of an expanding universe where the cosmological constant is positive (type II of Ref. [53]). Furthermore, by using \( T_0 = -6H_0^2 \), one finds from inequations (55) the WEC lower bound on the parameter \( \lambda \) in the BI teleparallel gravity, namely \( \lambda > 12.36 \).

V. FINAL REMARKS

Motivated by the attempts to explain the observed accelerating expansion of the Universe with a modifying teleparallel gravitational theory, there have been many recent papers on \( f(T) \) gravity. Despite the arbitrariness in the choice of different functional forms of \( f(T) \), which call for ways of constraining the possible \( f(T) \) gravity theories on physical grounds, several features of \( f(T) \) gravity have been discussed in a number of recent articles.

In this paper we have proceeded further with the investigations on the potentialities, difficulties, and limitations of \( f(T) \) gravity theories by deriving the classical energy conditions in the \( f(T) \) gravity context. Starting from the Raychaudhuri equation along with the requirement
that gravity is attractive, we have derived the null and strong energy conditions in the framework of \( f(T) \) gravity and shown that, although similar, they differ from NEC and SEC of general relativity, but in the limiting case \( f(T) = T \), they reduce to well-known NEC and SEC of Einstein’s gravitational theory. The comparison of the NEC and SEC inequalities [Eqs. (39) and (40)] with those which would be obtained by translating these energy conditions in terms of an effective energy-momentum tensor for \( f(T) \) gravity, enabled us to obtain the general expressions for the weak and dominant energy conditions [Eqs. (44) and (45)], which also reduce to the known corresponding energy conditions in general relativity in the limit \( f(T) = T \).

As concrete examples of how these energy conditions requirements may constrain \( f(T) \) gravity theories, we have discussed the WEC bounds on two different \( f(T) \) families of theories, namely the exponential and Born-Infeld \( f(T) \) gravity theories (Secs. IV A and IV B). To this end, we have used the current observational bounds on \( H_0 \) and \( \Omega_{mb} \) to show that the WEC are fulfilled for \( \beta > -1.256 \) in the exponential \( f(T) \) gravity, whereas for Born-Infeld \( f(T) \) gravity the WEC fulfillment is guaranteed for any \( \lambda > 12.36 \) such that \( 0 < \epsilon < 1 \) holds.

Finally, we emphasize that although the energy conditions in \( f(T) \) gravity discussed in this paper have well-motivated physical grounds (the attractive character of gravity together with the Raychaudhuri equation), the question as to whether they should be employed to any solution of \( f(T) \) gravity theories is an open question, which is ultimately related to the confrontation between theory and observations. We recall that in the context of Einstein’s gravitational theory, this confrontation indicates that all energy conditions seem to have been violated in the recent past of cosmic evolution [37, 44].

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