A variational principle for asymptotically Randall-Sundrum black holes

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We prove the following variational principle for asymptotically Randall-Sundrum (RS) black holes, based on the first law of black hole mechanics: Instantaneously static initial data that extremizes the mass yields a static black hole, for variations at fixed apparent horizon area, AdS curvature length, cosmological constant, brane tensions, and RS brane warp factors. This variational principle is valid with either two branes (RS1) or one brane (RS2), and is applicable to variational trial solutions. This paper is the second in a series on asymptotically RS black holes.

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I. INTRODUCTION

The first law for a static or stationary black hole expresses conservation of energy, and takes the general form \( \delta M = (\kappa/8\pi G)\delta A + \sum_i p_i \delta Q_i \). This relates the variations of mass \( M \), horizon area \( A \), and other physical quantities \( Q_i \). Thus, if a black hole is static or stationary, it extremizes \( M \) under variations that hold constant the remaining variables \( (A, Q_i) \). The converse of this statement motivates a variational principle: If a black hole’s spatial geometry is initially static (or initially stationary) and extremizes the mass with other physical variables held fixed, then the black hole is static (or stationary).

In this variational principle, the specific variables to hold fixed depend on the first law. The appropriate area to hold fixed is that of the apparent horizon, which is determined by the spatial geometry alone (unlike the event horizon, which is a global spacetime property). The apparent horizon generally lies inside the event horizon, and coincides with it for a static or stationary spacetime.

A version of the above variational principle was proved by Hawking for stationary black holes [1], and was extended to Einstein-Yang-Mills theory by Sudarsky and Wald [2]. These results are for asymptotically flat black holes in four spacetime dimensions. There is also much ongoing interest in extra dimensions, including the Randall-Sundrum (RS) braneworld models [3, 4]. In this paper, we prove a version of the above variational principle for asymptotically RS black holes.

The RS models are phenomenologically interesting, and have holographic interpretations [5] in the AdS/CFT correspondence. In the RS models, our universe is a brane surrounded by an AdS bulk. The bulk is warped by a negative cosmological constant. The RS1 model [3] has two branes of opposite tension, with our universe on the negative-tension brane. Tuning the interbrane distance appropriately predicts the production of small black holes at TeV-scale collider energies [6], and LHC experiments [7] continue to test this hypothesis.

In the RS2 model [4], our universe resides on the positive-tension brane, with the negative-tension brane removed to infinite distance. Perturbations of RS2 reproduce Newtonian gravity at large distance on the brane, while in RS1 this requires a mechanism to stabilize the interbrane distance [8]. In RS2, solutions for static black holes on the brane have been found numerically, for both small black holes [9] and large black holes [10], compared to the AdS curvature length. The only known exact analytical black hole solutions are the static and stationary solutions [11] in a lower-dimensional version of RS2.

This paper is the second in a series [12, 13]. In [12], we proved a first law for a static asymptotically RS black hole, including variations of the AdS curvature length, cosmological constant, brane tensions, and RS brane warp factors. This first law motivates the following variational principle, which we prove in this paper.

Variational principle for asymptotically RS black holes: Instantaneously static initial data that extremizes the mass \( M \) is initial data for a static black hole, for variations that leave fixed the apparent horizon area \( A \), the AdS curvature length \( \ell \), cosmological constant \( \Lambda \), brane tensions \( \lambda_i \), and RS values (at spatial infinity) of the warp factors \( \Omega_i \) on each brane.

This paper is organized as follows. We review the RS spacetimes in section [11] and prove the variational principle in section [11]. We demonstrate an explicit application of the variational principle in section [14] and conclude in section [15]. Throughout this paper, we use two branes, so our results apply to either RS1 or RS2 in the appropriate limit. We work on the orbifold region (between the branes) and use \( D \) spacetime dimensions. A timelike surface has metric \( \gamma_{ab} \), extrinsic curvature \( K_{ab} = \gamma_a^{c} \nabla_c n_b \), and outward unit normal \( n_b \). A spatial hypersurface \( \Sigma \) has unit normal \( n_a \), metric \( h_{ab} \), and covariant derivative \( D_a \). Each boundary \( B \) of \( \Sigma \) has metric \( \sigma_{ab} \), extrinsic curvature \( k_{ab} = h_a^{c} D_c n_b \), and outward unit normal \( n_b \). The boundaries \( B \) of \( \Sigma \) are: \( B_1 \), \( B_2 \) (the branes), \( B_\infty \) (spatial infinity), and \( B_H \) (the apparent horizon).
II. THE RANDALL-SUNDRUM SPACETIMES

The RS spacetimes \[3, 4\] are portions of an anti-de Sitter (AdS) spacetime, with metric
\[
ds^2_{\text{RS}} = \Omega(Z)^2 \left(-dt^2 + d\rho^2 + \rho^2 d\omega^2_{D-3} + dZ^2\right).
\]
Here \(d\omega^2_{D-3}\) denotes the unit \((D-3)\)-sphere. The warp factor is \(\Omega(Z) = \ell/Z\), with values \(\Omega\) on each brane. Here \(\ell\) is the AdS curvature length, related to the bulk cosmological constant \(\Lambda < 0\) given below. The RS1 model \[3\] contains two branes, which are the surfaces \(Z = Z_i\) with brane tensions \(\lambda_i\), where \(i = 1, 2\). The brane tensions \(\lambda_i\) and bulk cosmological constant \(\Lambda\) are
\[
\lambda_1 = -\lambda_2 = \frac{2(D-2)}{8\pi G_D \ell}, \quad \Lambda = -\frac{(D-1)(D-2)}{2\ell^2}.
\]
The dimension \(Z\) is compactified on the orbifold \(S^1/Z_2\) and the branes have orbifold mirror symmetry: in the covering space, symmetric points across a brane are identified. There is a discontinuity in the extrinsic curvature \(K_{ab}\) across each brane given by the Israel condition \[14\]. Using orbifold symmetry, the Israel condition requires the extrinsic curvature at each brane to satisfy
\[
2K_{ab} = \frac{\varepsilon}{\ell} \gamma_{ab}, \quad 2k_{ab} = \frac{\varepsilon}{\ell} \sigma_{ab},
\]
where \(\varepsilon = \pm 1\) is the sign of each brane tension. The RS2 spacetime \[4\] is obtained from RS1 by removing the negative-tension brane (now a regulator) to infinite distance \((Z_2 \to \infty)\) and the orbifold region has \(Z \geq Z_1\).

III. PROOF OF THE VARIATIONAL PRINCIPLE

Our main step, which we carry out in section III.A, reduces the proof to two auxiliary boundary value problems, which are the topics of section III.B.

Our setup is general: it applies to a black hole localized on a brane, or isolated in the bulk (away from either brane), and also applies to the asymptotically RS black string \[15\]. Our key assumptions will be the following. First, we assume our initial data \(h_{ab}\) is instantaneously static. We will also assume the variations \(\delta h_{ab}\) extremize the mass \((\delta M = 0)\), while holding fixed the apparent horizon area \(A\) and the quantities \((\ell, \Lambda, \lambda_i, \Omega_i)\).

A. Main proof

Our method closely follows our proof of the first law for static asymptotically RS black holes \[12\], which is based on the Hamiltonian formulation of general relativity. The full Hamiltonian contains a bulk term and surface terms. The bulk term is defined on an initial data surface \(\Sigma\),
\[
H_\Sigma = \int_\Sigma d^{D-1}x \left( N_{\dot{c}} + N^a C_a \right).
\]
Here \(N\) and \(N^a\) are the lapse and shift functions in the standard decomposition of the metric. Our focus is the initial data \((h_{ab}, p^{ab})\) where \(h_{ab}\) is the spatial metric and \(p^{ab}\) is its canonically conjugate momentum,
\[
16\pi G_D p^{ab} = \sqrt{h} K^{ab} - \mathcal{K}_{ab}, \quad \mathcal{K}_{ab} = h_a \nabla_c u_b.
\]
Initial data must satisfy constraints, \(C_0 = 0\) and \(C_a = 0\), which we henceforth assume, where
\[
C_0 = \frac{\sqrt{h}(2\Lambda - \mathcal{R})}{16\pi G_D} + \frac{16\pi G_D}{\sqrt{h}} \left(p^{ab} p_{ab} - \frac{\rho^2}{D-2}\right), \quad C_a = -2D_b p_a b.
\]
Here \(\mathcal{R}\) and \(D_a\) are the Ricci scalar and covariant derivative associated with \(h_{ab}\). We now consider the change \(\delta H_\Sigma\) under variations \((\delta h_{ab}, \delta p^{ab})\). One finds \(\delta C_0\) and \(\delta C_a\) involve derivatives \((D_c \delta h_{ab}, D_c \delta p^{ab})\). Integrating by parts to remove these derivatives yields surface terms \(I_B\),
\[
\delta H_\Sigma = \int_\Sigma d^{D-1}x \left( \mathcal{P}^{ab} \delta h_{ab} + \mathcal{H}_{ab} \delta p^{ab} \right) + \sum_B I_B.
\]
The quantities \((\mathcal{P}^{ab}, \mathcal{H}_{ab})\) appear in the time evolution equations, which involve the Lie derivative (denoted by an overdot) along the vector \(v^a = Nu^a + N^a\),
\[
\dot{h}_{ab} = \mathcal{H}_{ab}, \quad \dot{p}^{ab} = -\mathcal{P}^{ab}.
\]
We will not need the most general forms of \(\mathcal{P}^{ab}, \mathcal{H}_{ab}\), and \(I_B\). Will give their simplified forms below, after implementing some of our key assumptions. We now assume our variations take one solution of the constraints to another solution of the constraints, so we take \(\delta C_0 = 0\) and \(\delta C_a = 0\). Then the variation of \((\mathcal{P}^{ab}, \mathcal{H}_{ab})\) immediately gives
\[
\delta H_\Sigma = 0.
\]
We henceforth assume the initial data is instantaneously static, for which \(\rho^{ab} = \delta \rho^{ab} = 0\) and we take \(N^a = 0\). One then finds \(\mathcal{H}_{ab} = 0\), so \((7)\) and \((9)\) give
\[
\int_\Sigma d^{D-1}x \mathcal{P}^{ab} \delta h_{ab} + \sum_B I_B = 0,
\]
In what follows, \((10)\) will be our primary equation, where
\[
\mathcal{P}^{ab} = \frac{\sqrt{h}}{16\pi G_D} \left( \mathcal{R}^{ab} + h^{ab} D_c D^c - D^a D^b \right) N.
\]
For instantaneously static initial data, the constraint \(C_0 = 0\) simplifies to \(\mathcal{R} = 2\Lambda\) and \(\delta C_a\) vanishes identically. The linearized constraint \((\delta C_0 = 0)\) simplifies to
\[
(\mathcal{R}^{ab} + h^{ab} D_c D^c - D^a D^b) \delta h_{ab} = 0.
\]
We now evaluate the terms \(I_B\) in \((10)\). We found the values of \(I_{B_1}\) (at each brane) and \(I_{B_{\infty}}\) (at spatial infinity) in \[12\], including the variations of quantities \((\ell, \Lambda, \lambda_i, \Omega_i)\) held constant here by assumption. In this case, \((12)\) gives
\[
I_{B_1} = 0, \quad I_{B_{\infty}} = -\delta M.
\]
Additionally, we have $\delta M = 0$, by our assumption of a mass extremum, so $I_{B,\infty} = 0$. At the apparent horizon,
\begin{equation}
I_{B_H} = \int d^{D-2} x \sqrt{\sigma} A^{abcd} [(D_b N) \delta h_{cd} - N D_b \delta h_{cd}] ,
\end{equation}
where
\begin{equation}
16\pi G_D A^{abcd} = n_a (h^{ac} h^{bd} - h^{ab} h^{cd}) .
\end{equation}
The boundary condition on the lapse is $N = 0$, whence $D_a N = - f n_a$, where $f^2 = (D^a N) (D_b N)$. Then (14) is
\begin{equation}
I_{B_H} = \frac{1}{8\pi G_D} \int d^{D-2} x f \delta \sqrt{\sigma} ,
\end{equation}
using $\sigma_{ab} = h_{ab} - n_a n_b$ and $\delta \sqrt{\sigma} = \sqrt{\sigma} \delta \sigma_{ab} / 2$. For convenience, we now choose to set $I_{B_H} = 0$ using the following gauge transformation,
\begin{equation}
\delta \sigma_{ab} \rightarrow \delta \sigma_{ab} + 2 D_a (\xi_b) , \quad \sigma_{ab} \delta \sigma_{ab} \rightarrow 0 ,
\end{equation}
where $D_a$ is the covariant derivative associated with $\sigma_{ab}$. If we let $\xi_a = D_a F$, then $\sigma_{ab} \delta \sigma_{ab} \rightarrow 0$ requires
\begin{equation}
- \sqrt{\sigma} D^a D_a F = \delta \sqrt{\sigma} .
\end{equation}
Note the apparent horizon is a closed surface (which is most clearly seen in the covering space, if the apparent horizon intersects a brane). A solution $F$ to (18) on a closed surface is well known to exist if and only if the integral of the right-hand side of (18) vanishes. This integral is simply $\delta A$, which indeed vanishes since we hold $A$ constant. Thus a solution $\xi_a$ exists to achieve (17), and we henceforth set $I_{B_H} = 0$. Since $I_{B_i} = I_{B_H} = 0$ and we set $I_{B_H} = 0$, our primary equation (10) simplifies to
\begin{equation}
\int_\Sigma d^{D-1} x P^{ab} \delta h_{ab} = 0 .
\end{equation}

Our goal is to conclude that the initial geometry $h_{ab}$ evolves to a static spacetime. The well known condition for this is that $P^{ab} = 0$ on $\Sigma$. We cannot, however, immediately conclude that $P^{ab} = 0$ from (19), because not all of the variations $\delta h_{ab}$ are arbitrary: the linearized constraint (12) removes one degree of freedom, which can be taken as $h^{ab} \delta h_{ab}$ or as the variation $\delta h$ of the determinant $h$. These are related by $h^{ab} \delta h_{ab} = \delta h / h$. As an identity, we may decompose $\delta h_{ab}$ into a trace-free (TF) part and a part proportional to $h$:
\begin{equation}
\delta h_{ab} = (\delta h_{ab})^{TF} + \frac{1}{D-1} \left( \frac{\delta h}{h} \right) h_{ab} .
\end{equation}
Using (20), our primary equation (19) then becomes
\begin{equation}
\int_\Sigma d^{D-1} x \left[ (P^{ab})^{TF} (\delta h_{ab})^{TF} + \frac{P}{D-1} \frac{\delta h}{h} \right] = 0 ,
\end{equation}
where
\begin{equation}
P^{ab} = (P^{ab})^{TF} + \frac{P}{D-1} h_{ab} ,
\end{equation}
\begin{equation}
P = h_{ab} P^{ab} .
\end{equation}
The arbitrary variations are $(\delta h_{ab})^{TF}$, subject to smoothness at the apparent horizon, $(\delta h_{ab})^{TF} \rightarrow 0$ at $B_\infty$, and boundary conditions at the branches that we will specify in the next section. As a completeness check, the arbitrary variations $(\delta h_{ab})^{TF}$ alone should determine the dependent quantity $\delta h$, which we verify below by showing the linearized constraint (12) is a well posed boundary value problem for $\delta h$.

Our proof then reduces to showing $P = 0$, which allows us to conclude from (21) that $(P^{ab})^{TF} = 0$, since $(\delta h_{ab})^{TF}$ are arbitrary variations. It then follows from (22) that $P^{ab} = 0$, which is the desired result. The statement $P = 0$ is a boundary value problem for $N$ that we demonstrate is solvable in the following section, which completes our proof of the variational principle.

**B. Auxiliary boundary value problems**

The boundary value problem for $N$ is
\begin{equation}
D^a D_a N - \frac{(D-1)}{\ell^2} N = 0 , \quad (24a)
\end{equation}
\begin{equation}
\left( n^a D_a N - \frac{\varepsilon}{\ell} N \right) |_{B_i} = 0 , \quad (24b)
\end{equation}
\begin{equation}
N |_{B_H} = 0 , \quad (24c)
\end{equation}
\begin{equation}
N |_{B_\infty} \rightarrow \Omega . \quad (24d)
\end{equation}
Here, $\Omega$ is the warp factor of the asymptotic RS solution [1] and $\varepsilon = \pm 1$ is the sign of each brane tension. The result (24a) follows from setting $P = h_{ab} P^{ab} = 0$, using (11) and $R = 2\Lambda$. The boundary conditions (24c) and (24d) are straightforward. Our main concern is the brane boundary condition (24b), which results from using
\begin{equation}
2K_{ab} = n^c \partial_c \gamma_{ab} + \gamma_{ac} \partial_c n^c + \gamma_{bc} \partial_a n^c .
\end{equation}
Now $\gamma_{tt} = -N^2$ and $\gamma_{ta} = 0$ gives $2K_{tt} = n^c \partial_c (-N^2)$, and the Israel condition $K_{tt} = (\varepsilon / \ell) \gamma_{tt}$ then gives (24b).

As shown in [16], the following approach can put a Robin boundary condition (24b) in a standard form while keeping its associated elliptic equation (24a) in a divergence form. Let $v_a$ be any vector field and define $V_a N = (D_a - v_a) N$. Then (24a) and (24b) become
\begin{equation}
D^a V_a N + v^a D_a N + \left[ D_a v^a - \frac{(D-1)}{\ell^2} \right] N = 0 \quad (26)
\end{equation}
and
\begin{equation}
\left[ n^a V_a N + \left( n^a v_a - \frac{\varepsilon}{\ell} \right) N \right] |_{B_i} = 0 . \quad (27)
\end{equation}
As in [16], we now choose $v_a$ so $(n^a v_a - \varepsilon / \ell) \geq 0$ at $B_1$, which is the usual prerequisite for applying an existence theorem to a boundary value problem of the form (26-27). For example, we choose $v_a = -\tilde{n}_a / \ell$, where $\tilde{n}_a$ is any vector field, pointing from $B_1$ to $B_2$, that interpolates from the inward unit normal $(-n_a)$ of $B_1$ to
the outward unit normal \( n_a \) of \( B_2 \). Then \( n^a \tilde{n}_a = -\varepsilon \) at each brane \( B_i \), and \( n^a \tilde{n}_a / \ell \) gives \( (n^a \nu_a - \varepsilon / \ell) = 0 \) in \( \Sigma \). With the brane boundary conditions in standard form, and the remaining standard (Dirichlet) boundary conditions, (24a) and (24b), we then readily infer that the boundary value problem (24) for \( N \) is solvable.

We now turn to the boundary value problem for \( \delta h \), which we will state in terms of a scalar quantity \( (\delta h / h) \),

\[
D^a D_a (\delta h / h) - \frac{(D - 1)}{\ell^2} (\delta h / h) = f_\Sigma ,
\]

(28a)

\[
\left[ n^a D_a (\delta h / h) - \frac{\varepsilon}{\ell} (\delta h / h) \right]_{B_i} = f_i ,
\]

(28b)

\[
D^a D_a (\delta h / h) \bigg|_{B_H} = f_H ,
\]

(28c)

\[
(\delta h / h) \bigg|_{B_\infty} \rightarrow 0 .
\]

(28d)

As above, \( \varepsilon = \pm 1 \) is the sign of each brane tension. The result (28a) follows from substituting (20) into the linearized constraint (12) with \( R = 2 \Delta \). We will give the source terms and derive the boundary conditions below.

The key point is that (28) is a well posed boundary value problem. The elliptic equation (28a) and the boundary conditions (28b) are similar in form to (24a) and (24b) in the previous boundary value problem (24). The remaining boundary conditions, (28c) and (28d), are well known types: Neumann and Dirichlet, respectively.

In the remainder of this section, we provide the details of the source terms and the boundary conditions in (28). The source terms in (28) are

\[
f_\Sigma = \frac{D - 1}{D - 2} \left[ D^a D^b (\delta h_{ab})^{TF} - R^{ab} (\delta h_{ab})^{TF} \right] ,
\]

\[
-f_i = \frac{D - 1}{D - 2} \left[ \sigma^{ab} n^c D_c (\delta h_{ab})^{TF} + \varepsilon D^a n^b (\delta h_{ab})^{TF} \right] ,
\]

\[
f_H = \frac{1}{D - 2} \left[ 2k^{ab} (\delta h_{ab})^{TF} - \sigma^{ab} n^c D_c (\delta h_{ab})^{TF} \right] .
\]

The boundary conditions on \( \delta h \) are given by varying those on \( h_{ab} \), which at the apparent horizon and the branes involve the extrinsic curvature \( k_a = \sigma_a \epsilon D_c n_b \),

\[
k \bigg|_{B_H} = 0 , \quad k_{ab} \bigg|_{B_i} = \frac{\varepsilon}{\ell} \sigma_{ab} .
\]

(29)

By varying these, we obtain

\[
\delta k \bigg|_{B_H} = 0 , \quad \delta k \bigg|_{B_i} = 0 , \quad \delta k_{ab} \bigg|_{B_i} = \frac{\varepsilon}{\ell} \delta \sigma_{ab} .
\]

(30)

To evaluate these, we use the general results

\[
2k h_{ab} = (\sigma^c \eta^d \delta h_{cd}) k_{ab} - \sigma_a \epsilon \sigma_b n^d J_{abdf} ,
\]

(31a)

\[
-2\delta k = 2k^{ab} \delta \sigma_{ab} - \sigma a \epsilon \sigma b n^d \delta h_{ab} + \sigma a \epsilon \sigma b \sigma_{abc} ,
\]

(31b)

\[
J_{abc} = D_a \delta h_{bc} + D_b \delta h_{ac} - D_c \delta h_{ab} .
\]

(31c)

The boundary conditions at the branes (28b) result from evaluating \( \delta k = 0 \) using (20), (29), and (31b). The last relation in (30) expresses brane boundary conditions for \( (\delta h_{ab})^{TF} \), since it reduces to a form independent of \( \delta h \) after substituting (3), (20), (28b), and (31a).

IV. A DIRECT APPLICATION OF THE VARIATIONAL PRINCIPLE

Here we demonstrate the utility of the variational principle, by applying it to a trial solution and reproducing the static asymptotically RS black string [15], which is the only known exact solution for an asymptotically RS black object in 5-dimensional spacetime. We first specify a trial geometry for an initially static black string. After evaluating the apparent horizon area \( A \) and mass \( M \), we then apply the variational principle.

A black string is a set of lower dimensional black holes stacked in an extra dimension \( Z \), which is how we will construct the trial geometry. We take

\[
ds^2 = \Omega(Z)^2 \left[ \Psi^4(x, Z) \, dx^2 + dZ^2 \right] ,
\]

(32)

where \( \Omega = \ell / Z \) and the branes are the surfaces \( Z = Z_1 \) and \( Z = Z_2 \). We will take

\[
\Psi = 1 + \frac{\rho_0}{\rho} \left( \frac{1}{|x + x_0|} + \frac{1}{|x - x_0|} \right) .
\]

(33)

Here \( \rho_0 > 0 \) is a constant and \( x_0 = (0, 0, d) \), using Cartesian coordinates \( x = (x, y, z) \) with origin at \( x = 0 \). We choose a function \( d(Z) \) as follows. The constraint, \( R = -12 / \ell^2 \), after linearizing in \( d \) and its derivatives, has the solution \( d(Z) = d_0 + c_0 Z^2 \), where \( c_0 \) and \( d_0 \) are constants. For the case \( c_0 = 0 \), (33) is an exact solution and the RS1 limit (\( \Omega \to 0 \)) is easily taken. We will work in RS2 and take \( c_0 \) as a small nonzero parameter.

On each slice \( Z=\)constant, we now transform to spherical coordinates centered at \( x = 0 \), with \( z = \rho \cos \theta \), and expand \( \Psi \) in Legendre polynomials \( P_n(\cos \theta) \) for \( \rho > d \),

\[
\Psi = 1 + \frac{\rho_0}{\rho} + \frac{\rho_0}{\rho} \sum_{j=1}^{\infty} \left( \frac{d}{\rho} \right)^{2j} P_{2j}(\cos \theta) .
\]

(34)

On each slice \( Z=\)constant, we will take \( \rho_0 \gg d \), which describes a 3-dimensional black hole [17], and \( d \) parametrizes the 2-dimensional apparent horizon's distortion from the sphere \( \rho = \rho_0 \). The full 3-dimensional apparent horizon (the union of the 2-dimensional apparent horizons) therefore describes a distorted black string.

On each slice \( Z=\)constant, as in [17], the surface \( \rho(\theta) \) of the 2-dimensional apparent horizon can be found as the sum of Legendre polynomials that minimizes the area \( A_2 \),

\[
A_2 = 2\pi \int_0^{\pi} d\theta \rho^4 \sqrt{\rho^2 + \left( \frac{d}{\rho} \right)^2} .
\]

(35)

To lowest order in \( (d/\rho_0) \ll 1 \), we find the apparent horizon on each slice \( Z=\)constant is

\[
\rho(\theta) = \rho_0 \left[ 1 + \frac{5}{4} \left( \frac{d}{\rho_0} \right)^2 P_2(\cos \theta) \right] .
\]

(36)

This agrees with the numerical results of [17], and gives, to lowest order in \( (d/\rho_0) \),

\[
A_2(Z) = 64\pi \rho_0^2 \left[ 1 - \frac{5}{4} \left( \frac{d}{\rho_0} \right)^4 \right] .
\]

(37)
Integrating in $Z$ gives the 3-dimensional area of the full apparent horizon in the black string geometry,

$$A = \int_{Z_1}^{Z_2} dZ \Omega^3 A_2 = 32\pi w \left( \rho_0^2 - \frac{5Q}{4\rho_0} \right), \quad (38)$$

where, with $\Omega_i$ the warp factor at each brane,

$$w = \ell \left( \Omega_1^2 - \Omega_2^2 \right), \quad Q = d_0 + \frac{8c_0 d_0^4 \ell^4}{\Omega_1 \Omega_2(\Omega_1 + \Omega_2)}. \quad (39)$$

We defined the mass $M$ in [12], which for (34) gives

$$G_5 M = w\rho_0. \quad (40)$$

To lowest order in $Q$, combining (38)–(40) gives

$$\left( \frac{G_5 M}{w} \right)^2 = \left( \frac{A}{32\pi w} \right) + \frac{5}{7} \left( \frac{32\pi w}{A} \right) Q. \quad (41)$$

We now apply our variational principle: we extremize $M$ in [11] at fixed $A$, $\ell$, and $\Omega_i$. This yields the conditions $c_0 = 0$ and $d_0 = 0$, which we conclude describes a static black string. We can verify this directly, since $d = 0$ in [34] gives $\Psi = 1 + \rho_0/\rho$ and the apparent horizon [30] is located at $\rho = \rho_0$. This is indeed the initial geometry of the static black string [15] in isotropic coordinates.

We can also deduce that the evolution of the distorted black string, with $d \neq 0$, will not be static, since each slice $Z=\text{constant}$ is the initial geometry for an attracting two-body problem [17], and the 2-dimensional apparent horizon considered above describes a black hole formed by, and surrounding, two closely separated smaller black holes (with a small minimal surface surrounding each point $x = \pm x_0$). From the perspective in each slice $Z=\text{constant}$, as in [17], the two small interior black holes will coalesce as the initial data evolves, due to mutual gravitational attraction. This results in a time-dependent geometry on each slice $Z=\text{constant}$, and results in a time-dependent black string geometry in the bulk perspective.

### V. DISCUSSION

The variational principle developed in this paper states that for an asymptotically RS black hole initially at rest, initial data that extremizes the mass yields a static black hole, for variations at fixed values of the apparent horizon area and the remaining physical variables in the first law. It would be interesting to investigate the consequences of holding fewer variables fixed. An example of this in 4-dimensional spacetime is Hawking’s proof [1] that the static (Schwarzschild) black hole is an extremum of mass at fixed apparent horizon area but arbitrary angular momentum.

Our example application of the variational principle to a trial solution serves as a prelude to the approach we will take in the next paper [13] in this series. In [13], we will conclude that solutions exist for small static black holes in RS2, both on and off the brane, as special members of a general family of initially static black holes. This family of black hole initial data will also indicate that a small black hole on an orbifold-symmetric braneworld model, and is an important result for future collider experiments.

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