1. Introduction and notation. In this paper the word "local" is used in at least three different meanings. Our aim is to study local Banach spaces of Fréchet or other locally convex spaces, and it turns out that it is convenient to use the local theory of Banach spaces for this purpose. Recall that given a locally convex space $E$ and a continuous seminorm $p$ on $E$ the completion of the normed space $(E/\ker(p), p)$ is a local Banach space of $E$, and it is denoted by $E_p$. If $(p_\alpha)_{\alpha \in A}$ is a system of seminorms defining the topology of $E$, then we call $(E_{p_\alpha})_{\alpha \in A}$ a system of local Banach spaces.

Valdivia proved in [V] that every infinite dimensional nuclear locally convex space $E$ has for all separable infinite dimensional Banach spaces $X$ a system of local Banach spaces isomorphic to $X$. In this paper we study Schwartz spaces and, consequently, compact linear operators in Banach spaces. We consider "dense" factorizations of operators in Banach spaces, i.e., factorizations $T = T^{(2)}T^{(1)}$, $T \in L(X,Y)$, through $Z$ such that $T^{(1)}(X) \subset Z$ is dense. Assuming density is enough to guarantee a direct application to the study of local Banach spaces of locally convex spaces.

In Section 2, we introduce a way to measure the distance of nonisomorphic Banach spaces, the so called local distance function. We prove general theorems (Theorems 2.8 and 2.14) which show that a compact operator $T$ on a separable reflexive Banach space $X$ factors through a Banach space $Y$ in a strong sense (the image of $X$ is dense in $Y$; both factors of the given operator are compact), if the distance of $X^*$ and $Y^*$ is small enough and some technical assumptions are satisfied. For example we show that an arbitrary compact operator on a separable $L_p$–space $X$, $1 < p < \infty$, factors through any separable $L_p$–space $Y$ in the above strong sense (Theorem 2.10). Consequently, if $X$ and $Y$ are such spaces, we see that a Schwartz space having a system of local Banach space isomorphic to $X$ also has a system isomorphic to $Y$ (Corollary 2.16).

The concept of local distance function is also analyzed from the Banach space theoretical point of view. We show (Proposition 2.4) that there exist separable, reflexive Banach
spaces which are in a sense very close to $\ell_2$ but which do not have a Schauder basis. This result is a consequence of a construction of Szarek, [S].

Section 3 contains some more remarks on dense factorizations. The considerations in this section are technically easy, compared to Section 2, and the results are not so deep from the Banach–space theoretical point of view. But the consequences to the locally convex space theory are so strong that it is worthwhile to present the results in detail. In Theorem 3.2 we show that if $X$ is a separable Banach space with a complemented unconditional basic sequence, if $Y$ and $Z$ are Banach spaces, $Z$ separable, and if $T \in L(X,Y)$ is compact, then $T$ factors through $X \times Z$ such that the image of $X$ is dense in $X \times Z$. Combining this observation with the results of Section 2 we see that a compact $T \in L(X)$ factors densely through $Y \times Z$ for an arbitrary separable Banach space $Z$, if the (uniform, see later) local distance of $X^*$ and $Y^*$ is small enough and if some technical assumptions are satisfied.

In the second part of the paper, Section 4, we study some duality problems for local Banach spaces. The relations of the local Banach spaces of a locally convex space $E$ and of the strong dual $E'$ are not yet well understood in general. For example, it is an open problem whether, given a Fréchet or a $(DF)$–space $E$ with a system of local Banach spaces isomorphic to some Banach space $X$, $E$ also has a fundamental system of Banach discs $(B_\lambda)_{\lambda \in \Lambda}$ such that the corresponding Banach spaces $E_{B_\lambda}$ are isomorphic to $X$. We are not able to solve this problem here, but in Proposition 4.1 we generalize the known partial positive results, and in Proposition 4.6 we construct an example which can be considered as a partial negative solution. This construction uses an estimate (Lemma 4.2) of the absolute projection constant of a Banach space. The method used in the proof of this lemma may be new.

We use mainly the terminology of [K1], [LT1] and [TJ]. Let us however mention some notations and definitions. $\mathbb{N}$ stands for the set $\{1,2,3,\ldots\}$. The closure of a set $A$ is denoted by $\overline{A}$. A subset $A$ of a vector space is absolutely convex, if $\sum_{i=1}^n \lambda_i x_i \in A$ for all sequences $(x_i)_{i=1}^n \subset A$ and for all scalar sequences $(\lambda_i)$ satisfying $\sum_{i=1}^n |\lambda_i| \leq 1$. A closed, bounded and absolutely convex subset of a Fréchet space is called a Banach disc. The vector spaces are over the real or complex scalar field unless otherwise stated. By a subspace we mean a linear subspace, by an operator a continuous linear operator, and by an isomorphism a linear homeomorphism. Two Banach spaces $X$ and $Y$ are $C$–isomorphic, $C \geq 1$, if the Banach–Mazur distance $d(X,Y) \leq C$, where $d(X,Y) := \inf \|\psi\|\|\psi^{-1}\|$, $\psi : X \to Y$ isomorphism. A subspace $X$ of a Banach space $Y$ is $C$–complemented, if there exists a projection $P$ from $Y$ onto $X$ with $\|P\| \leq C$; the projection constant $\lambda(X,Y)$ is the number $\inf \{\|P\| | P$ is a projection from $X$ onto $Y\}$. The absolute projection constant $\lambda(X)$ of a Banach space $X$ is the supremum of $\lambda(Z,Y)$ over all Banach spaces $Y$ containing a subspace $Z$ isometric to $X$.

For the definition of the bounded approximation property, (Schauder) basis and basis constant we refer to [LT1]. The definition and properties of $L_p$–spaces can be found in [LT], Chapter 5, or in [TJ], §2. Type, cotype and the corresponding constants are defined in [TJ], §4. Some elementary facts on and definitions of tensor products, especially the projective tensor product, are mentioned in [TJ], §5 (or in [T]; for more details, see [J]),
and the real interpolation method is also presented in the same book, §3.

Most of these definitions occur also in [J].

If $X$ and $Y$ are Banach spaces we denote by $L(X,Y)$ the space of bounded linear operators $X \to Y$. If $T \in L(X,Y)$ is compact, we define the $n$:th approximation number $a_n(T), \ n \in \mathbb{N}$, by

$$a_n(T) := \inf \{ \| T - T_n \| \mid T_n \in L(X,Y), \ \text{rank } T_n < n \}. \quad (1.1)$$

Recall that a locally convex space $E$ is a Schwartz space, if for every continuous seminorm $p$ there exists a continuous seminorm $q$, $q \geq p$, such that the canonical mapping $E_q \to E_p$ induced by the identity operator on $E$ is compact. For other definitions concerning locally convex spaces we refer to [K1] and [J].

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2. Local distance function of Banach spaces with applications to compact operators and Schwartz spaces. The Banach–Mazur distance provides a natural way to measure differences of two isomorphic Banach spaces, but for nonisomorphic spaces the Banach–Mazur distance is not a finite number. In this section we introduce a way to measure the distance of nonisomorphic spaces, the so called local distance function. The local distance function is not a metric, since the ”local distance” of two Banach spaces is a function, not a number. Some basic properties of local distance function are given in Remarks 1.–8.

2.1. Definition. Let $X$ and $Y$ be Banach spaces and $C \geq 1$. The $C$–local distance function of $X$ and $Y$ is the function $(C)–ld(X,Y) : \mathbb{N} \to \mathbb{R}^+ \cup \{ \infty \}$, the value of which at $n \in \mathbb{N}$ is the infimum of $\infty$ and all numbers $f(n) \in \mathbb{R}^+$ satisfying the following: If $M \subset X$ and $M_Y \subset Y$ are at most $n$–dimensional subspaces, $n \in \mathbb{N}$, then there exist $C$–complemented finite dimensional subspaces $N \subset X$ and $N_Y \subset Y$, such that $M \subset N, \ M_Y \subset N_Y$, $\dim (N) = \dim (N_Y)$ and

$$d(N, N_Y) \leq f(n).$$

Let $X, Y$ and $C$ be as above and let $K : \mathbb{N} \to \mathbb{N}$ be a non–decreasing function. The $K$–uniform $C$–local distance function of $X$ and $Y$ is the function $(K,C)–ld(X,Y) : \mathbb{N} \to \mathbb{R}^+ \cup \{ \infty \}$ the value of which at the point $n \in \mathbb{N}$ is the infimum of $\infty$ and all numbers $f(n) \in \mathbb{R}^+$ satisfying the following condition: If $M \subset X$ and $M_Y \subset Y$ are subspaces with dimension not greater than $n$, then there exist $C$–complemented subspaces $N \subset X$ and $N_Y \subset Y$ containing $M$ and $M_Y$, respectively, such that $\dim (N) = \dim (N_Y) \leq K(n)$ and $d(N, N_Y) \leq f(n)$.

We call the $C$–local distance function just the local distance function, if $C$ is clear from context or does not need to become specified. In the same way we speak about the uniform local distance function.
We say that the local distance (respectively, uniform local distance) of $X$ and $Y$ is bounded, if $(C)−ld(X,Y)$ is a bounded function for some $C$ (resp. $(K,C)−ld(X,Y)$ is a bounded function for some $C$ and $K$).

We say that the local distance of $X$ and $Y$ is finite, if, for some $C ≥ 1$, $(C)−ld(X,Y)(n)$ is finite for all $n$.

In the same way we define what means that the uniform local distance of $X$ and $Y$ is finite.

2.2. Remarks. 1. It may happen that the local distance of $X$ from itself is not finite. This is the case for the Banach space $X$ constructed by Pisier, [Ps1]: it has the property that all $n$–dimensional subspaces are not better than $C\sqrt{n}$–complemented in $X$, where $C$ is a constant depending on the space $X$ only. In fact, the property that the local distance of $X$ from itself is finite is equivalent to $X$ being a $\pi$–space in the sense of [JRZ]. This property is stronger than the bounded approximation property. We refer to [JRZ], Proposition 1.1 and 1.2.

2. It would also be possible to define another local distance function roughly speaking as the infimum of all functions $f(n)$ satisfying the following condition $(X, Y$ and $C$ as in Definition 2.1): Given finite dimensional subspaces $M \subset X, M_Y \subset Y$ there exist $C$–complemented subspaces $N$ and $N_Y$ containing $M$ and $M_Y$, respectively, such that $\dim(N) = \dim(N_Y)$ and such that $d(N, N_Y) ≤ f(\dim(N))$. We would get a distance function which would be easier to estimate for example in the case of $L_p$–spaces (cf. Remark 6.). However, this function seems not to be so useful in considerations like Theorem 2.8.

Omitting the space $M_Y$ in Definition 2.1 would lead to another concept of distance function. This distance function would not be symmetric with respect to $X$ and $Y$.

3. If $X$ and $Y$ are $L_p$–spaces, then the uniform local distance of $X$ and $Y$ is bounded. This statement follows immediately from definitions and [PR], Corollary 2.1. We do not know, if the converse statement (“if $X$ is a $L_p$–space and the local distance of $X$ and $Y$ is bounded, then $Y$ is a $L_p$–space”) also holds.

4. Let $X$ (resp. $Y$) be a Banach space of type $p$ (resp. $p'$). If the local distance of $X$ and $Y$ is bounded, then $p = p'$. Suppose by antithesis that $1 ≤ p < p' ≤ 2$. By the Maurey–Pisier theorem, [TJ], Theorem 7.6, $X$ contains a subspace $M$ which is $2$–isomorphic to $\ell^k_p, \infty > k > (4C'D)^{pp'/(p-p')}$, where $D = \sup_n(C)−ld(X,Y)(n)$ and $C' = T_{p'}(Y)$, the type $p'$–constant of $Y$.

Let $N \subset Y$ be any subspace which is $D$–isomorphic to a subspace of $X$ containing $M$. Then we have for the type $p'$–constant of $N$ (see [TJ], (4.5), p. 15),

$$T_{p'}(N) ≥ D^{-1}T_{p'}(M) ≥ D^{-1}k^{1/p'-1/p}/2 ≥ 2C',$$

which contradicts $T_{p'}(Y) = C'$.

In the same way one shows that if the local distance of $X$ and $Y$ is bounded, then the spaces $X$ and $Y$ have the same cotype.

5. It is a direct consequence of definitions that if $(C)−ld(X,Y)$ is a bounded function for some $C$, then $X$ (resp. $Y$) is crudely finitely representable (and even strongly representable) in $Y$ (resp. $X$). For the definition of these concepts, see [MS], 11.6 and [BDG].
6. We show that if \( X \) (resp. \( Y \)) is an \( L_p \)-space (resp. \( L_{p'} \)-space), \( 1 \leq p \leq p' \leq \infty \), then, for all \( \varepsilon > 0 \), for all \( n \in \mathbb{N} \),

\[
(K, C) - \mathrm{ld}(X, Y)(n) \leq (1 + \varepsilon)(2(n + 1)^2/C(\varepsilon))^{n/p-n/p'},
\]

where \( C = 1 + \varepsilon \) and \( K(n) = (2(n + 1)^2/C(\varepsilon))^n \) and \( C(\varepsilon) \) is a constant depending on \( \varepsilon \). Pełczyński, Rosenthal and Kwapien have shown ([PR], Theorem 2.1) that, given an \( n \)-dimensional subspace \( M \) of an \( L_q \)-space, there exists a \( 1 + \varepsilon \)-complemented subspace \( N \supset M \) such that \( \dim(N) \leq (2(n + 1)^2/C(\varepsilon))^n \) and \( d(N, \ell_{q}^{\dim(N)}) \leq 1 + \varepsilon/3 \).

So, given \( n \)-dimensional subspaces \( M \subset X \) and \( M_Y \subset Y \), we choose \( k \)-dimensional \( 1 + \varepsilon \)-complemented subspaces \( N \supset M \), \( N_Y \supset M_Y \) such that \( k \leq (2(n + 1)^2/C(\varepsilon))^n \), \( d(N, \ell_{p}^{k}) \leq 1 + \varepsilon/3 \), \( d(N_Y, \ell_{p}^{k}) \leq 1 + \varepsilon/3 \). Then we have

\[
d(N, N_Y) \leq (1 + \varepsilon)k^{1/p-1/p'} \leq (1 + \varepsilon)(2(n + 1)^2/C(\varepsilon))^{n/p-n/p'}.
\]

We do not know how sharp this estimate is.

7. We do not know what is the relation of the local distance function of Banach spaces \( X \) and \( Y \) and on the other hand of the local distance function of the duals \( X^* \) and \( Y^* \). This question should be compared with [LT], II.5.7 and 8, and [M]. The local distance function of \( X \) and \( Y \) gives direct information only on some quotients of \( X^* \) and \( Y^* \).

8. Let \( X \) be a weak Hilbert space in the sense of [Ps2]. If \( M \subset X \) is an \( n \)-dimensional subspace, then it is known that for a constant \( c \) depending on the space \( X \) only we have \( d(M, \ell_2^n) < c \log(n + 1) \); see [Ps2], Corollary 2.5. According to some yet unpublished information Maurey has improved the result of Johnson and Pisier, [JP], showing that weak Hilbert spaces have the ”linear uniform projection property”. This means that given \( \varepsilon > 0 \) and a finite-dimensional subspace \( M \) of \( X \) we can find a \( 1 + \varepsilon \)-complemented subspace \( N \) containing \( M \) such that \( \dim(N) \leq c_1 \dim(M) \), where \( c_1 > 0 \) depends on the space \( X \) and \( \varepsilon \) only.

Summing up the above statements we get for every \( \varepsilon > 0 \)

\[
(K, C) - \mathrm{ld}(X, \ell_2)(n) \leq c_0 \log(n + 1),
\]

where \( C = 1 + \varepsilon \), \( K: \mathbb{N} \to \mathbb{N} \) is an asymptotically linear function depending on \( \varepsilon \) and \( X \), and the constant \( c_0 \) depends on the space \( X \) and \( \varepsilon \) only.

Proposition 2.4 below shows that the properties of Banach spaces with a ”small” (but not bounded) local distance may be quite different. For this result we need to use a construction of Szarek. We begin with

**2.3. Lemma.** For all \( n \in \mathbb{N} \) and all \( q, 2 < q < \infty \), there exists an \( n \)-dimensional subspace \( Y_q^n \) of the \( \mathbb{R} \)-Banach space \( L_q \) such that

\[
bc(Y_q^n \bigoplus \ell_2 F) \geq cn^{(1/2)(1/2-1/q)}D^{-1/2},
\]

(2.2)
for an absolute constant $c$ and for all normed spaces $F$ satisfying the following: if $M \subset F$ is an $n$–dimensional subspace, then $d(M, \ell^2_n) \leq D$.

Here $bc(X)$ denotes the basis constant of the Banach–space $X$. Recall that $X$ has a basis if and only if $bc(X)$ is finite.

This lemma is an important result of Szarek, see [S], Proposition 3.1. Now we can prove

2.4. Proposition. Let $f : \mathbb{N} \to \mathbb{R}^+$ be a non–decreasing, unbounded function, $f(1) \geq 3$. For all $\varepsilon$, $0 < \varepsilon < 1$, there exists a separable reflexive $\mathbb{R}$–Banach space $X$ having no Schauder basis such that, for all $n \in \mathbb{N},$

$$(K, C) - ld(\ell_2, X)(n) \leq f(n), \quad (2.3)$$

where $C = 1 + \varepsilon$ and $K$ is some function $\mathbb{N} \to \mathbb{N}$.

The function $K$ is specified in (2.13) below.

Since $X$ is reflexive, also the dual $X^*$ does not have a basis.

Proof. We choose the sequences $(n_k)_{k=1}^\infty$, $n_k \in \mathbb{N}$, and $(q_k)_{k=1}^\infty$, $2 < q_k < \infty$, as follows: Let $q_0 = 4$, $n_0 = 1$, and assume that $k \in \mathbb{N}$ and that $q_t, n_t$ are chosen for $t < k$. We define $q'_{k}, 2 < q'_k < q_{k-1}$, such that

$$n_{k-1}^{1/2 - 1/q'_k} \leq 1 + \varepsilon/3 \quad (2.4)$$

and then $n_k > 2n_{k-1}$ such that

$$f(n_k) \geq kf(n_{k-1}), \quad n_k^{1/2 - 1/q'_k} \geq f(n_{k-1})/2 \quad (2.5)$$

and, finally, $q_k$, $2 < q_k < q'_k$ such that

$$n_k^{1/2 - 1/q_k} = f(n_{k-1})/2. \quad (2.6)$$

A useful refinement of this choice is described in Remark 2.5 below.

We define

$$X = \bigoplus_{k=1}^\infty Y_k \ell_2, \quad (2.7)$$

where $Y_k := Y_{n_k}^{q_k}$ and the spaces $Y_{n_k}^{q_k}$ are as in Lemma 2.3. We first show that (2.3) holds. So let $n \in \mathbb{N}$ and let $M \subset X$, $\dim(M) \leq n$. Let $k \in \mathbb{N}$ be such that $n_{k-1} \leq n \leq n_k$. We define the finite dimensional subspace $N$ by

$$N := \bigoplus_{t \leq k} Y_t \bigoplus Q(M), \quad (2.8)$$

6
where \( Q \) is the natural projection from \( X \) onto \( \bigoplus_{t > k} Y_t \). Clearly, \( M \subset N \). We show that \( N \) is \( 1 + \varepsilon \)-complemented in \( X \). Let \( Q_t \) be the natural projection from \( X \) onto \( Y_t \). Each space \( Q_t(M) \) is at most \( n_k \)-dimensional. Since every \( Y_t \) is, by definition, a subspace of \( L_q(0, 1) \), we can use a result of Lewis, [L], Corollary 4, to find for each \( t > k \) a projection \( P_t \) from \( Y_t \) onto \( Q_t(M) \) such that
\[
\|P_t\| \leq n_k^{1/2 - 1/q_t} \leq 1 + \varepsilon/3. \tag{2.9}
\]
Hence, there exists a projection \( \tilde{Q} \) from \( \bigoplus_{t > k} Y_t \subset X \) onto \( \bigoplus_{t > k} Q_t(M) \) such that \( \|\tilde{Q}\| \leq 1 + \varepsilon/3 \). On the other hand, since \( \dim (Q_t(M)) \leq n_k \) for all \( t \), we get, for \( t > k \),
\[
d(Q_t(M), \ell_2^{\dim(Q_t(M))}) \leq n_k^{1/2 - 1/q_t} \leq 1 + \varepsilon/3, \tag{2.10}
\]
see [L], Corollary 5. This means that
\[
d(\bigoplus_{t > k} Q_t(M), \ell_2^m) \leq 1 + \varepsilon/3 \tag{2.11}
\]
where \( m = \dim(\bigoplus_{t > k} Q_t(M)) \in \mathbb{N} \cup \{\infty\} \). Using the orthonormal projection in Hilbert space we thus find a projection \( \hat{Q} \) from \( \bigoplus_{t > k} Q_t(M) \) onto \( Q(M) \) with \( \|\hat{Q}\| \leq 1 + \varepsilon/3 \). Now \( \hat{Q}\tilde{Q} \) is a projection from \( \bigoplus_{t > k} Y_t \) onto \( Q(M) \) satisfying \( \|\hat{Q}\tilde{Q}\| \leq 1 + \varepsilon \), and this means that there exists a projection \( R \) from \( X \) onto \( N \) with \( \|R\| \leq 1 + \varepsilon \).

We have
\[
d((\bigoplus_{t \leq k} Y_t)_{\ell_2}, \ell_2^{(n_1 + \ldots + n_k)}) \leq \sup_{t \leq k} d(Y_t, \ell_2^{n_t}) \leq \sup_{t \leq k} n_t^{1/2 - 1/q_t}
\leq f(n_k - 1)/2 \leq f(n), \tag{2.12}
\]
see (2.6). So, (2.11) and (2.12) imply \( d(N, \ell_2^{\dim(N)}) \leq f(n) \). This proves the statement
\[
(C) -ld(X, \ell_2)(n) \leq f(n)
\]
for all \( n \).

From (2.8) we see that \( n_k \leq \dim(N) \leq 3n_k \); so, this upper estimate for \( \dim(N) \) depends only on \( \dim(M) \) (see the choice of \( k \)) and we get
\[
n_k \leq K(n) \leq 3n_k \tag{2.13}
\]
for \( n_k - 1 \leq n \leq n_k \). This proves the uniform version of the estimate (2.3).

The proof of [S], Proposition 4.1, with small modifications, shows that \( X \) does not have even a ”local basis structure”. In fact, to prove that \( X \) does not have a basis, it is sufficient to show that for all \( t \in \mathbb{N} \) large enough, the \( n_t \)-dimensional subspaces \( Z \) of
\[( \bigoplus_{k \neq t} Y_k \bigoplus \ell_2 \dim(Z) ) \leq f(n_{t-2})/2 \] (see Lemma 2.3); then we have, by (2.6) and (2.5)

\[ bc(X) = bc(Y_t \oplus ( \bigoplus_{k \neq t} Y_k ) \ell_2 ) \geq \sqrt{2}c n_t^{(1/2)(1/2-1/q_t)} f(n_{t-2})^{-1/2} \]

\[ = cf(n_{t-1})^{1/2} f(n_{t-2})^{-1/2} \geq c t^{1/2} \rightarrow \infty. \]

Given \( M \subset ( \bigoplus_{k \neq t} Y_k ) \ell_2 \), \( \dim(M) = n_t \), we have \( M \subset ( \bigoplus Q_k(M) ) \ell_2 \). We then get, analogously to (2.10) and (2.12),

\[ d(Q_k(M), \ell_2 \dim(Q_k(M)) \leq 1 + \varepsilon \]

for \( k > t \), and

\[ d(Q_k(M), \ell_2 \dim(Q_k(M)) \leq n_t^{1/2-1/q_{t-1}} = f(n_{t-2})/2 \]

for \( k < t \). This implies

\[ d(M, \ell_2 \dim(M)) \leq d( \bigoplus_{k \neq t} Q_k(M), \ell_2 \dim ) \leq f(n_{t-2})/2, \]

where \( m = \dim( \bigoplus_{k \neq t} Q_k(M) ) \). \( \Box \)

We refer also to Remark 2.11 below.

**2.5. Remark.** We show that given any non-decreasing unbounded functions \( g : \mathbb{N} \rightarrow \mathbb{R}^+ \) and \( h : \mathbb{N} \rightarrow \mathbb{R}^+ \), \( h(1) \geq 3 \), the function \( f \) and the sequence \( (n_k) \) in the construction of the space \( X \) above can be chosen such that both of the following are satisfied:

1° \[ n_k \geq g(k)n_{k-1} \text{ for all } k \in \mathbb{N}, \]

2° \[ f(n_k) \leq h(n_{k-2}) \text{ for all } k \in \mathbb{N}, k \geq 2. \]

Indeed, we set \( q_0 = 4, n_0 = 1 \) and \( f(1) = 3 \). We assume that \( k \in \mathbb{N} \) and that \( q_t, n_t \) are chosen for \( t < k \) and \( f(n) \) is defined for \( n \leq n_{k-1} \). We define \( q'_k, 2 < q'_k < q_{k-1} \), such that

\[ n_k^{1/2-1/q'_k} \leq 1 + \varepsilon/3 \quad (2.4a) \]

and then \( n_k \) such that \( n_k > \max\{2, g(k)\}n_{k-1} \) and such that

\[ h(n_k) \geq (k+2)h(n_{k-1}), \quad (2.5a) \]

\[ n_k^{1/2-1/q'_k} \geq f(n_{k-1})/2. \quad (2.5b) \]

Then we define \( f(n) = 3 \) for \( n_0 < n \leq n_1 \), if \( k = 1 \)

\[ f(n) = h(n_{k-2}) \quad (2.5c) \]
for $n_{k-1} < n \leq n_k$ and, finally, $q_k$, $2 < q_k \leq q_k'$ such that
\[ n_k^{1/2-1/q_k} = f(n_{k-1})/2. \] (2.6a)

Note that (2.5a,c) imply
\[ f(n_k) \geq kf(n_{k-1}) \] (2.7a)
for all $k \geq 3$.

So, in view of (2.4a), (2.5b), (2.6a) and (2.7a), none of the inequalities (2.4)–(2.6) is changed because of these extra requirements (except (2.5) for $k \leq 2$, but this does not matter), and hence Proposition 2.4 holds also for the function $f$ and the space $X$ satisfying $1^\circ$ and $2^\circ$.

Our main application of local distance function is a result which says that a compact operator $T \in L(X)$ factors in a "strong" sense through $Y$ if $(K, C)–ld(X*, Y^*)$ is small enough. We begin with a lemma which is in principle known; compare [LT], Proposition II.5.10.

2.6. Lemma. Let $X$ and $Y$ be reflexive Banach spaces such that $(K, C)–ld(X^*, Y^*)$ is a finite function for some $C \geq 1$ and some $K : \mathbb{N} \to \mathbb{N}$. Let $C_1 > C \geq 1$. Given $C_1$–complemented subspaces $M \subset X$, $M_Y \subset Y$, $0 \leq \dim (M) = \dim (M_Y) = n < \infty$, projections $P$ (resp. $P_Y$) from $X$ onto $M$, $\|P\| \leq C_1$ (resp. from $Y$ onto $M_Y$, $\|P_Y\| \leq C_1$) and $m$–codimensional, $m \in \mathbb{N}$, closed subspaces $X_0 \subset X$ and $Y_0 \subset Y$, there exist $C$–complemented subspaces $N \subset X$, $N_Y \subset Y$ satisfying the following

1\(^\circ\) \quad \dim (N) = \dim (N_Y) \leq K(n + m),
2\(^\circ\) \quad M \subset N, M_Y \subset N_Y,
3\(^\circ\) \quad there exist projections $Q$ from $X$ onto $N$ and $Q_Y$ from $Y$ onto $N_Y$ such that $PQ = P$, $P_YQ_Y = P_Y$,
\[(\text{id}_X - Q)(X) \subset X_0, \quad (\text{id}_Y - Q_Y)(Y) \subset Y_0,\]
\[\text{and } \|Q\| \leq C_1(C + 2), \quad \|Q_Y\| \leq C_1(C + 2)\]
4\(^\circ\) \quad d(N, N_Y) \leq C_1^2((C + 2)^2(K, C)–ld(X^*, Y^*)(n + m)).

Proof. Let $\tilde{M} := P^*(X^*) \subset X^*$, $\tilde{M}_Y := P_Y^*(Y^*) \subset Y^*$, $\tilde{M} := X_0^{\perp} \subset X^*$ and $\tilde{M}_Y := Y_0^{\perp} \subset Y^*$. We have $\dim (\tilde{M}) = \dim (\tilde{M}_Y) = n$, $\dim (\tilde{M}) = \dim (\tilde{M}_Y) = m$. By the assumption on the local distance function of $X^*$ and $Y^*$ we find $C$–complemented subspaces $\tilde{N} \subset X^*$ and $\tilde{N}_Y \subset Y^*$ such that
\[\tilde{M} + \tilde{M} \subset \tilde{N}, \quad \tilde{M}_Y + \tilde{M}_Y \subset \tilde{N}_Y,\] (2.14)
\[\dim (\tilde{N}) = \dim (\tilde{N}_Y) \leq K(n + m) \text{ and } d(\tilde{N}, \tilde{N}_Y) \leq (1 + \varepsilon)(K, C)–ld(X^*, Y^*)(n + m),\]
where $\varepsilon > 0$ is such that $(1 + \varepsilon)(C_1 + C_1 + C)^2 \leq C_1^2(C + 2)^2$. Let $\hat{Q}$ and $\hat{Q}_Y$ be projections from $X^*$ onto $\tilde{N}$ and $Y^*$ onto $\tilde{N}_Y$, respectively with norm not greater than $C$. We define
\[\hat{Q} := \hat{Q} + P^* - P^*\hat{Q}, \quad \hat{Q}_Y := \hat{Q}_Y + P_Y^* - P_Y^*\hat{Q}_Y.\] (2.15)
Then it is elementary to see, using (2.14) that, \( \hat{Q} \) is a projection from \( X^* \) onto \( \tilde{N} \) which commutes with \( P^* \) and satisfies \( \| \hat{Q} \| \leq CC_1 + C_1 + C \leq C_1(C + 2) \). The same holds for \( \hat{Q}_Y \) with respect to \( Y^*, \tilde{N}_Y \) and \( P_Y^* \).

Finally, we define using reflexivity the projections \( Q := \hat{Q}^* \), \( Q_Y := \hat{Q}_Y^* \) and the subspaces \( N := Q(X) \) and \( N_Y := Q_Y(Y) \). To see that \( M \subset N \) let \( x \in M \) and \( y \in X^* \). Then, by (2.14) and the commutativity of \( \hat{Q} \) and \( \hat{Q}_Y \),

\[
\langle Qx, y \rangle = \langle QPx, y \rangle = \langle x, P^*Q^*y \rangle \\
= \langle x, Q^*P^*y \rangle = \langle x, P^*y \rangle = \langle Px, y \rangle \\
= \langle x, y \rangle.
\]

Hence, \( Qx = x \) for all \( x \in M \), which means that \( M \subset N \). Concerning the property 3\(^o\), the relation \( PQ = P \) follows from definitions. Let us prove that \( (id_X - Q)(X) \subset X_0 \). If \( x \in X \) and \( y \in X_0 \), then \( \hat{Q} \) is a projection onto \( \tilde{N} \). This implies the statement. The norm estimate in 3\(^o\) follows from (2.15).

Clearly, the statements for \( N_Y \) etc. are proved in the same way.

The distance estimate 4\(^o\) follows from \( d(N^*, \tilde{N}) \leq CC_1 + C_1 + C \), \( d(N_Y^*, \tilde{N}_Y) \leq CC_1 + C_1 + C \) and \( d(\tilde{N}, \tilde{N}_Y) \leq (1 + \varepsilon)(K, C) - id(X^*, Y^*)(n + m) \) and from the choice of \( \varepsilon \). □

The main difficulty in applying our method is the following. We have finite dimensional subspaces \( M, M_1, M_1 \subset M \) and \( N, N_1, N_1 \subset N \), of the Banach spaces \( X \) and \( Y \), respectively, such that \( d(M, N) \) and \( d(M_1, N_1) \) are quite small and such that all the subspaces are quite well complemented. However, we do not in general know if \( d(M_2, N_2) \) is small enough for all ”good” complements \( M_2 \) and \( N_2 \) of \( M_1 \) and \( N_1 \) in \( M \) and \( N \), respectively. So, we give our theorem in two versions. In the first one we make assumptions on the spaces \( X \) and \( Y \) which enable us to avoid the difficulty mentioned above. We will assume that given a Banach space \( X \) the following property is satisfied for a suitable non-decreasing function \( f_X : N \to \mathbb{R}^+ \) which will be given later:

\( \text{(D) There exist constants } d(X) \geq 0 \text{ and } D(X) > 0 \text{ such that for each } n \text{-dimensional } c \text{-complemented subspace } M \subset X \text{ and projection } P \text{ from } X \text{ onto } M \text{ with } ||P|| \leq c \text{ there exist an } n \text{-dimensional subspace } N \text{ and a projection } Q \text{ from } X \text{ onto } N \text{ such that} \)

1\(^o\) \( M \cap N = \{0\} \),
2\(^o\) \( ||Q|| \leq D(X)e^{d(X)} \),
3\(^o\) \( PQ = QP = 0 \),
4\(^o\) \( d(M, N) \leq f_X(n) \).

It is not difficult to see that for example the reflexive separable \( L_p \)-spaces satisfy this property with a constant function \( f_X \) and \( d(X) = 2 \). Namely, assume that \( X \) is a separable \( L_{p, \lambda} \)-space, \( 1 < p < \infty, \lambda \geq 1 \), and that \( M \subset X \) is finite dimensional and that \( P \) is a projection from \( X \) onto \( M \). There exists a subspace \( M_1 \supseteq M \) such that \( d(M_1, \ell_p) \leq \lambda \),
where \(m = \dim(M_1)\); let \(\phi : M_1 \to \ell_p^m\) be an isomorphism with \(\|\phi\|\|\phi^{-1}\| \leq \lambda\). It is known that \(X\) is \(\lambda\)–isomorphic to a subspace of \(L_p(0,1)\) (see [LP], Corollary 7.2). On the other hand, every infinite dimensional subspace of \(L_p(0,1)\) which is not isomorphic to \(\ell_2\) contains, say, a 3–complemented subspace 2–isomorphic to \(\ell_p\) (see [KP], the proof of Theorem 2 and Theorem 3, and [LT1], Proposition 1.a.9). So, in view of these remarks \(\ker(P)\) contains a subspace \(Y\) which is 3\(\lambda\)–complemented in \(X\) and 2\(\lambda\)–isomorphic to \(\ell_p\).

Let \(\psi : Y \to \ell_p\) be an isomorphism with \(\|\psi\|\|\psi^{-1}\| \leq 2\lambda\). Let \(I_0\) be the natural embedding of \(\ell_p^m\) into \(\ell_p\) and let \(R_0\) be the canonical projection from \(\ell_p\) onto \(I_0(\ell_p^m)\), \(\|I_0\| = \|R_0\| = 1\).

Now \(N = \psi^{-1}I_0\phi(M)\) is a subspace of \(Y \subset X\) satisfying \(M \cap N = \{0\}\), \(d(M,N) \leq 2\lambda^2\), and the projection

\[Q = \psi^{-1}I_0\phi P \phi^{-1}I_0^{-1}R_0\psi R(id_X - P),\]

where \(R\) is a projection from \(X\) onto \(Y\) with \(\|R\| \leq 3\lambda\), satisfies \(Q(X) = N\), \(PQ = QP = 0\), \(\|Q\| \leq 12\|P\|^2\lambda^3\).

Moreover, if the space \(X\) is such that for a constant \(C \geq 1\), given a finite codimensional closed subspace \(Y\) of \(X\) there exists a \(C\)–complemented subspace \(Z\) which is \(C\)–isomorphic to \(Z\), then (D) is satisfied: given \(M \subset X\) and the projection \(P\) from \(X\) onto \(N\) we take \(Y = (id_X - P)(X)\) and apply the isomorphy of \(Z\) and \(X\) to find \(N\) analogously to the case of \(L_p\)–spaces. Furthermore, in Theorem 2.8 it is often sufficient that a priori only one of the given Banach–spaces satisfy (D) in full strength. This fits well to the philosophy that Theorem 2.8 is applicable to ”small perturbations” of ”regular” Banach–spaces; see Remark 2.11 below.

2.8. Theorem. Let \(X\) and \(Y\) be separable, reflexive Banach spaces such that \((K,C)\)–ld\((X^*,Y^*)\) is finite for some \(C \geq 1\) and \(K : \mathbb{N} \to \mathbb{N}\), and let \(T \in L(X)\) be compact. Choose the sequence \((m_k)_{k=0}^\infty\) such that \(m_0 = 2\), \(m_k \geq K(4m_{k-1} + 2) + 1\) for \(k \in \mathbb{N}\). Assume that for some constants \(0 \leq \alpha < 1/4, 0 \leq \beta < 1/4\) and \(C' > 0\) the spaces \(X\) and \(Y\) have property (D) with \(f_X, f_Y\) such that for all \(k \in \mathbb{N}\)

\[f_X(n) \leq C'(a_{m_k-1}(T))^{-\alpha}, \text{ for } n \leq m_{k+1}\]

\[f_Y(n) \leq C'(a_{m_k-1}(T))^{-\beta}, \text{ for } n \leq m_{k+1},\]

and assume that for all \(k \in \mathbb{N}\) the inequality

\[(K,C)\)–ld\((X^*,Y^*)(4m_{k+1}) \leq C'k^{-3/4}C(k + 2)^{-5}a_{m_k}(T)^{(-1+4\max\{\alpha,\beta\})/4},\]

(2.17)

where \(C(k) := (4D(C + 3))^{(k+1)d^k}\) and \(d = \max\{d(X),d(Y),1\}\), \(D = \max\{D(X),D(Y)\}\) (see property (D)), holds. Then \(T = T^{(2)}T^{(1)}\), where \(T^{(1)} \in L(X,Y), T^{(2)} \in L(Y,X)\) and \(T^{(1)}(X)\) is dense in \(Y\). Moreover, both \(T^{(1)}\) and \(T^{(2)}\) are compact.

Note that the assumptions on \(X\) and \(Y\) are symmetric so that the factorization applies as well to compact operators in \(Y\).

Proof. It is not a restriction to assume that \(a_n(T) \leq 1\) for all \(n\).

Let for each \(n \in \mathbb{N}, n > 1\), the operator \(T_n \in L(X)\) be such that \(\text{rank} (T_n) < n\) and

\[\|T - T_n\| \leq 2a_n(T).\]

(2.18)
Let $(Y_k)_{k=0}^\infty$ be a decreasing sequence of closed, finite codimensional subspaces of $Y$ satisfying
\[
\text{codim } (Y_k) = \text{codim } \left( \bigcap_{t=0}^k \ker (T_{m_t}) \right).
\]
Let $(x_n)_{n=1}^\infty \subset X$ and $(y_n)_{n=1}^\infty \subset Y$ be sequences of non zero elements such that $\text{sp } (x_n) \subset X$ and $\text{sp } (y_n) \subset Y$ are dense.

We choose the sequences $(M_k)_{k=1}^\infty$, $(\tilde{M}_k)_{k=1}^\infty$, $(N_k)_{k=1}^\infty$, $(\tilde{N}_k)_{k=1}^\infty$ of finite dimensional subspaces of $X$ or $Y$ by an inductive method as follows. We set $M_0 = \tilde{M}_0 = N_0 = \tilde{N}_0 = \{0\}$, $P_0 = \tilde{P}_0 = Q_0 = \tilde{Q}_0 = 0$. Assume that $k \in \mathbb{N}$ and that $M_n \subset X$, $\tilde{M}_n \subset X$, $N_n \subset Y$, $\tilde{N}_n \subset Y$ and the projections $P_n$ from $X$ onto $M_n$ (respectively, $\tilde{P}_n : X \to \tilde{M}_n$, $Q_n : Y \to N_n$, $\tilde{Q}_n : Y \to \tilde{N}_n$) are chosen for $0 \leq n < k$ such that the following holds:

1. $M_{n-1} + M_n \subset M_n$, $N_{n-1} + \tilde{N}_n \subset N_n$ for all $1 \leq n < k$,
2. $\|P_n\| < C(n)$, $\|\tilde{P}_n\| < C(n)$, $\|Q_n\| < C(n)$, $\|\tilde{Q}_n\| < C(n)$, for all $n < k$,
3. $P_n$ commutes with $P_{n-1}$ and $\tilde{P}_{n-1}$, and $Q_n$ commutes with $Q_{n-1}$ and $\tilde{Q}_{n-1}$ for all $1 \leq n < k$,
4. $(\text{id}_X - P_n)(X) \subset \bigcap_{t=0}^{n-1} \ker (T_{m_t})$, $(\text{id}_Y - Q_n)(Y) \subset Y_{n-1}$ for all $1 \leq n < k$,
5. $d(M_n, \tilde{M}_n) \leq C' a_{m_{\max(n-2,0)}}(T)^{-\alpha}$,
6. $d(N_n, \tilde{N}_n) \leq C' a_{m_{\max(n-2,0)}}(T)^{-\beta}$,
7. $d(M_n, N_n) \leq 16C(n)^2(K, C) - l d(X^*, Y^*)(4m_{n-1})$ for $1 \leq n < k$,
8. $P_n \tilde{P}_n = \tilde{P}_n P_n = Q_n \tilde{Q}_n = \tilde{Q}_n Q_n = 0$ for all $n < k$,
9. $x_n \in M_n$, $y_n \in N_n$ for all $1 \leq n < k$,
10. $\dim(M_n) = \dim(\tilde{M}_n) = \dim(N_n) = \dim(\tilde{N}_n) \leq m_n$ for all $n$.

An application of Lemma 2.6 yields the same for $n \leq k$; more specifically, we take $M = M_{k-1} + \tilde{M}_{k-1}$, $M_Y = N_{k-1} + \tilde{N}_{k-1}$, $X_0 = \bigcap_{t=0}^{k-1} \ker (T_{m_t})$, $Y_0 = Y_{k-1}$, and for the projections $P$ and $P_Y$ we take $P_{k-1} + \tilde{P}_{k-1}$ and $Q_{k-1} + \tilde{Q}_{k-1}$, respectively. Then we use Lemma 2.6 and first get the spaces $\tilde{M}_k := N$, $\tilde{N}_k := N_Y$ and the projections $\tilde{P}_k = Q$ and $\tilde{Q}_k = Q_Y$ with the properties $1^\circ - 4^\circ$ of the lemma (e.g. $\|\tilde{P}_k\| < 2C(k-1)(C+2)$ by $2^\circ$ above). Note that, by Lemma 2.6, $\tilde{P}_k$ and $P_{k-1} + \tilde{P}_{k-1}$ commute. Since $P_{k-1} \tilde{P}_{k-1} = \tilde{P}_{k-1} P_{k-1} = 0$, we see that
\[
P_{k-1} \dot{P}_k = P_{k-1}(P_{k-1} + \tilde{P}_{k-1}) \dot{P}_k = P_{k-1} \dot{P}_k (P_{k-1} + \tilde{P}_{k-1})
\]
\[
= P_{k-1}(P_{k-1} + \tilde{P}_{k-1}) = P_{k-1}
\]
which means that also $P_{k-1}$ and $\dot{P}_k$, and, similarly, $\dot{P}_{k-1}$ and $\dot{P}_k$, $Q_{k-1}$ and $\dot{Q}_k$, as well as, $\tilde{Q}_{k-1}$ and $\dot{Q}_k$, commute.

We choose the smallest $n$ (resp. $m$) such that $x_n \notin \tilde{M}_k$ (resp. $y_m \notin \tilde{N}_k$) and denote $x^{(k)} := x_n$ (resp. $y^{(k)} := y_m$). We have $(\text{id}_X - \tilde{P}_k)x^{(k)} \neq 0$ and $(\text{id}_Y - \tilde{Q}_k)y^{(k)} \neq 0$.\[\text{(2.19)}\]
Let $R_X$ and $R_Y$ be projections from $(id_X - \hat{P}_k)(X)$ onto $sp\left((id_X - \hat{P}_k)x^{(k)}\right)$ and from $(id_Y - \hat{Q}_k)(Y)$ onto $sp\left((id_Y - \hat{Q}_k)y^{(k)}\right)$, respectively, with norm one. We define

$$P_k = \hat{P}_k + R_X(id_X - \hat{P}_k), \quad Q_k = \hat{Q}_k + R_Y(id_Y - \hat{Q}_k)$$

and $M_k = P_k(X), N_k = Q_k(Y)$. Then clearly

i) $P_k$ commutes with $\hat{P}_k$ and $Q_k$ commutes with $\hat{Q}_k$,

ii) $\|P_k\| \leq 2\|\hat{P}_k\| + 1$, $\|Q_k\| \leq 2\|\hat{Q}_k\| + 1$.

Now 1° above holds for $n \leq k$. The property 2° for $P_k$ and $Q_k$ follows from Lemma 2.6:

$$\|P_k\| \leq 2\|\hat{P}_k\| + 1 \leq 4C(k - 1)(C + 2) + 1. \tag{2.20}$$

To verify 3° we have, by (2.19) and i)

$$P_{k-1}P_k = P_{k-1}\hat{P}_kP_k = P_{k-1}\hat{P}_k = P_{k-1} = P_kP_{k-1}$$

The same reasoning proves the other cases, too. Concerning 4°, we have for $x \in X$

$$(id_X - \hat{P}_k)(id_X - P_k)x = x - \hat{P}_kx - P_kx + \hat{P}_kP_kx = (id_X - P_k)x,$$

so that $(id_X - P_k)x \in (id_X - \hat{P}_k)(X) \subset \bigcap_{t=0}^{k-1} ker(T_{m_t})$, by the choice of $\hat{P}_k$. Similar proofs in $Y$ show that 3° and 4° hold.

The distance estimate for $d(M_k, N_k)$ in 5° follows from Lemma 2.6 and the definitions above; note that the Hahn–Banach–theorem implies the existence of decompositions $M_k = \hat{M}_k \oplus M^{(k)}$, $N_k = \hat{N}_k \oplus N^{(k)}$, where $\dim(M^{(k)}) = \dim(N^{(k)}) = 1$, such that the norms of the corresponding projections do not exceed 2. This again implies $d(M_k, N_k) \leq 16d(\hat{M}_k, \hat{N}_k)$, and 4° of Lemma 2.6 can be applied to estimate $d(\hat{M}_k, \hat{N}_k)$.

We use the assumption on the property (D) of the spaces $X$ and $Y$ and (2.20) to find spaces $\hat{M}_k$ and $\hat{N}_k$, and projections $\hat{P}_k$ and $\hat{Q}_k$ satisfying the properties 2°, 5° and 6°. It is clear that also 7° is satisfied. The property 8° follows from definitions.

We denote $P_k^+ = P_k + \hat{P}_k$ and $Q_k^+ = Q_k + \hat{Q}_k$ for all $k$. It is easy to see that $P^+_{k-1}P_k^+ = P_k^+P_{k-1}^+ = P_{k-1}^+$, and similarly for $Q_k^+$, hold.

For all $k$ we denote by $\psi_k : M_k \rightarrow N_k$ an isomorphism satisfying

$$\|\psi_k\| \leq 2d(M_k, N_k)^{1/2}, \quad \|\psi_k^{-1}\| \leq d(M_k, N_k)^{1/2}$$

and by $\alpha_k : M_k \rightarrow \hat{M}_k$ and $\beta_k : N_k \rightarrow \hat{N}_k$ isomorphisms satisfying

$$\|\alpha_k\| \leq 2d(M_k, \hat{M}_k)^{1/2}, \quad \|\alpha_k^{-1}\| \leq d(M_k, \hat{M}_k)^{1/2},$$

$$\|\beta_k\| \leq 2d(N_k, \hat{N}_k)^{1/2}, \quad \|\beta_k^{-1}\| \leq d(N_k, \hat{N}_k)^{1/2}.$$ We define for all $k \in \mathbb{N}$ the isomorphisms $\Phi_k : M_k \oplus \hat{M}_k \rightarrow N_k \oplus \hat{N}_k,$

$$\Phi_k : x + y \mapsto \beta_k \psi_k x + \psi_k \alpha_k^{-1} y,$$
where \( x \in M_k, y \in \tilde{M}_k \). We have
\[
\Phi_k^{-1}(x + y) = \alpha_k \psi_k^{-1} x + \psi_k^{-1} \beta_k^{-1} y,
\]
where \( x \in N_k, y \in \tilde{N}_k \), and
\[
\|\Phi_k\| \leq 4C(k) d(M_k, N_k)^{1/2} (d(M_k, \tilde{M}_k)^{1/2} + d(N_k, \tilde{N}_k)^{1/2}),
\]
\[
\|\Phi_k^{-1}\| \leq 2C(k) d(M_k, N_k)^{1/2} (d(M_k, \tilde{M}_k)^{1/2} + d(N_k, \tilde{N}_k)^{1/2}).
\]
Let \( k \geq 2 \). Note that \( \Phi_k(M_{k-1} + \tilde{M}_{k-1}) \subset \tilde{N}_k \) and that there exists a projection \( R_k \) from \( N_k \oplus \tilde{N}_k \) onto \( \Phi_k(M_{k-1} \oplus \tilde{M}_{k-1}) \) such that \( \|R_k\| \leq 16C(k)^3 d(M_k, N_k)(d(M_k, \tilde{M}_k)^{1/2} + d(N_k, \tilde{N}_k)^{1/2})^2 \); we can take
\[
R_k = \Phi_k P_{k-1}^+ \Phi_k^{-1}. \tag{2.21}
\]
Moreover, for \( k \geq 2 \) there exists an isomorphism \( \gamma_k \) from \( N_{k-1} \oplus \tilde{N}_{k-1} \) onto \( \Phi_k(M_{k-1} \oplus \tilde{M}_{k-1}) \) such that both \( \|\gamma_k\| \) and \( \|\gamma_k^{-1}\| \) are not greater than
\[
8C(k)^2 d(M_k, N_k)^{1/2} (d(M_k, \tilde{M}_k)^{1/2} + d(N_k, \tilde{N}_k)^{1/2}) d(M_{k-1}, N_{k-1})^{1/2}
\times (d(M_{k-1}, \tilde{M}_{k-1})^{1/2} + d(N_{k-1}, \tilde{N}_{k-1})^{1/2});
\]
we can take
\[
\gamma_k = \Phi_k \Phi_k^{-1} \tag{2.22}
\]
We define
\[
T^{(1)}x := \sum_{k=1}^{\infty} c(k)((\text{id}_Y - Q_{k-1}^+) + \gamma_k Q_{k-1}^+) \Phi_k (P_k^+ - P_{k-1}^+) x, \tag{2.23}
\]
where \( P_0^+ = 0, \gamma_1 = 0, x \in X \) and \( c(1) = 1, c(k) = k^{-3/2} \|\gamma_k\|^{-1} \|\Phi_k\|^{-1} C(k)^{-2} \) for \( k > 1 \), and
\[
T^{(2)}x := \sum_{k=1}^{\infty} c(k)^{-1} T \Phi_k^{-1} ((\text{id}_Y - R_k) + \gamma_k^{-1} R_k)(Q_k^+ - Q_{k-1}^+) x, \tag{2.24}
\]
where \( x \in Y, Q_0^+ = 0 \) and \( R_1 = 0 \). It is a direct consequence of the choice of \( c(k) \) that (2.23) converges absolutely in \( Y \) for all \( x \), and that \( T^{(1)} \) is a bounded operator. The convergence of the series determined by the right-hand side of (2.23) is even absolute in \( L(X, Y) \) so that \( T^{(1)} \) is compact.

We prove the same for \( T^{(2)} \). Let \( k \geq 2 \). We have for all \( x \in Y \)
\[
R_k Q_{k-1}^+ x = \Phi_k P_{k-1}^+ \Phi_k^{-1} Q_{k-1}^+ x = 0, \tag{2.25}
\]
since \( \Phi_k^{-1} Q_{k-1}^+ x \in \tilde{M}_k \) (see also 6° and 3°). Moreover, for \( x \in N_k \oplus \tilde{N}_k \),
\[
Q_{k-1}^+ R_k x = 0, \tag{2.26}
\]
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since \( R_k x \in \tilde{N}_k \). So, denoting \( y(x) := ((\text{id}_Y - R_k) + \gamma_k^{-1} R_k)(Q_k^+ - Q_{k-1}^+)x \) for \( x \in Y \), we get

\[
P_{k-1}^{-1} \Phi_k^{-1} y(x) = \Phi_k^{-1} \Phi_k P_{k-1}^{-1} \Phi_k^{-1} y(x) = \Phi_k^{-1} R_k y(x)
\]

\[
= \Phi_k^{-1} R_k \gamma_k^{-1} R_k (Q_k^+ - Q_{k-1}^+)x
\]

\[
= \Phi_k^{-1} R_k Q_{k-1}^+ \gamma_k^{-1} R_k (Q_k^+ - Q_{k-1}^+)x = 0,
\]

by (2.25), since \( \gamma_k^{-1} z \in N_{k-1} \oplus \tilde{N}_k \) for all \( z \in R_k(N_k \oplus \tilde{N}_k) \). Clearly, (2.27) implies

\[
\Phi_k^{-1}((\text{id}_Y - R_k) + \gamma_k^{-1} R_k)(Q_k^+ - Q_{k-1}^+)(Y) \subset (P_k^+ - P_{k-1}^+)(X)
\]

for all \( k \geq 2 \). On the other hand, \((P_k^+ - P_{k-1}^+)(X) \subset (\text{id}_X - P_{k-1})(X) \subset \bigcap_{t=0}^{k-2} \ker(T_{m_{t-1}})\), by the properties 3°, 4° and 6° above. Hence, for all \( x \in Y \),

\[
z_k(x) := \Phi_k^{-1}((\text{id}_Y - R_k) + \gamma_k^{-1} R_k)(Q_k^+ - Q_{k-1}^+)x \in \ker(T_{m_{k-2}}).
\]

So, we have by 5° for all \( k \geq 2 \)

\[
c(k)^{-1}\|T z_k(x)\| = c(k)^{-1}\|(T - T_{m_{k-2}}) z_k(x)\|
\]

\[
\leq 2k^{3/2}C(k)^2\|T - T_{m_{k-2}}\|\|\Phi_k\|\|\Phi_k^{-1}\|\|R_k\|\|\gamma_k\|\|\gamma_k^{-1}\|\|Q_k^+ - Q_{k-1}^+\|\|x\|
\]

\[
\leq 2^{15}k^{3/2}C(k)^4a_{m_{k-2}}(T)d(M_k, N_k)^4d(M_k, \tilde{M}_k)^{1/2} + d(N_k, \tilde{N}_k)^{1/2}8\|x\|
\]

\[
\leq 2^{100}k^{3/2}C'4C(k)^2a_{m_{k-2}}(T)^{1-4\max\{,\}}((K, C) - ld(X^*, Y^*)(4m_{k-1}))^4\|x\|;
\]

note that we have normalized \( a_n(T) \leq 1 \) for all \( n \). The inequality (2.17), combined with (2.30) now implies

\[
\|c(k)^{-1} T z_k(x)\| \leq 2^{100}k^{-3/2}C'8\|x\|
\]

for \( k \geq 2 \). The constants here do not depend on \( k \), so we see that (2.30) converges for all \( x \) and defines a bounded operator \( T \). The same inequality even shows that the convergence is absolute in \( L(Y, X) \). Hence, also \( T^{(2)} \) is compact.

We show that \( T = T^{(2)} T^{(1)} \). A direct calculation using \( \gamma_k Q_{k-1}^+ \subset (\text{id}_Y - Q_{k-1}^+)(Y) \) shows that for all \( k, t \in \mathbb{N} \) and for all \( x \in X \)

\[
(Q_t^+ - Q_{t-1}^+)((\text{id}_Y - Q_{k-1}^+) + \gamma_k Q_{k-1}^+)(P_k^+ - P_{k-1}^+)x
\]

\[
= \delta_{kt}((\text{id}_Y - Q_{k-1}^+) + \gamma_k Q_{k-1}^+)\Phi_k(P_k^+ - P_{k-1}^+)x.
\]

Hence, for \( x \in X \),

\[
T^{(2)} T^{(1)} x = \sum_{k=1}^{\infty} c(k)^{-1} c(k) T \Phi_k^{-1}((\text{id}_Y - R_k)
\]

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\[+ \gamma_k^{-1} R_k((\text{id}_Y - Q_{k-1}^+) + \gamma_k Q_{k-1}^+) \Phi_k(P_k^+ - P_{k-1}^+) x\]
\[= \sum_{k=1}^{\infty} T \Phi_k^{-1}(\text{id}_Y - R_k - Q_{k-1}^+)\]
\[+ R_k Q_{k-1}^+ + \gamma_k^{-1} R_k(\text{id}_Y - Q_{k-1}^+) + (\text{id}_Y - R_k) \gamma_k Q_{k-1}^+\]
\[+ \gamma_k^{-1} R_k \gamma_k Q_{k-1}^+ \Phi_k(P_k^+ - P_{k-1}^+) x.\] (2.32)

In view of (2.25) this is equal to
\[\sum_{k=1}^{\infty} T \Phi_k^{-1}(\text{id}_Y - R_k - Q_{k-1}^+ + \gamma_k^{-1} R_k + \gamma_k Q_{k-1}^+)\]
\[= \sum_{k=1}^{\infty} T \Phi_k^{-1}(\text{id}_Y - R_k + \gamma_k^{-1} R_k) \Phi_k(P_k^+ - P_{k-1}^+) x,\] (2.33)

But we have
\[R_k \Phi_k(P_k^+ - P_{k-1}^+) x = \Phi_k P_{k-1}^+ \Phi_k^{-1} \Phi_k(P_k^+ - P_{k-1}^+) x = \Phi_k P_{k-1}^+(P_k^+ - P_{k-1}^+) x = 0,\]

so that we finally get, using (2.32) and (2.33)
\[T^{(2)}T^{(1)} x = \sum_{k=1}^{\infty} T (P_k^+ - P_{k-1}^+) x,\] (2.34)

where \(P_0^+ = 0\). Note that, as a consequence of the absolute convergence of the series (2.23) and (2.24), we have absolutely convergent series in our calculations (2.32), (2.33) and (2.34). So, (2.34) implies \(T^{(2)}T^{(1)} x = T x\) for \(x\) in arbitrary \(M_k\); now \(\bigcup_k M_k = X\) implies this for all \(x\).

Finally, we show that \(T^{(1)}(X) \subset Y\) is dense. To this end it is enough to show that \(N_k \subset T^{(1)}(X)\); see the choice of \(N_k\) and \((y_k)_{k=1}^{\infty}\) above. So, assume that \(y \in (Q_k^+ - Q_{k-1}^+)\). We define
\[x := \Phi_k^{-1}((\text{id}_Y - R_k)y + \gamma_k^{-1} R_k y),\]

We have clearly \(x \in P_k^+(X)\), and, moreover,
\[P_{k-1}^+ x = \Phi_k^{-1}(\Phi_k P_{k-1}^+ \Phi_k^{-1} y - \Phi_k P_{k-1}^+ \Phi_k^{-1} R_k y) + P_{k-1}^+ \Phi_k^{-1} \gamma_k^{-1} R_k y = 0,\] (2.35)

since \(\Phi_k P_{k-1}^+ \Phi_k^{-1} = R_k\) and since
\[\Phi_k^{-1} \gamma_k^{-1} R_k(y) \in M_k.\]
So, \( c(k)^{-1}x \in (P^+_k - P^+_{k-1})(X) \), and we get, by (2.23), (2.26), by an analogue of (2.31) and by the assumption \( Q^+_{k-1}y = 0 \),

\[
T^{(1)}c(k)^{-1}x = ((\text{id} - Q^+_{k-1}) + \gamma_k Q^+_{k-1})((\text{id} - R_k)y + \gamma_k^{-1}R_ky)
\]

\[
= y - Q^+_{k-1}y - R_ky + Q^+_{k-1}R_ky + \gamma_k Q^+_{k-1}y
\]

\[
- \gamma_k Q^+_{k-1}R_ky + \gamma_k^{-1}R_ky - Q^+_{k-1}\gamma_k^{-1}R_ky + \gamma_k Q^+_{k-1}\gamma_k^{-1}R_ky
\]

\[
= y - R_ky + \gamma_k^{-1}R_ky - \gamma_k^{-1}R_ky + R_ky = y.
\]

This completes the proof for the density of \( T^{(1)}(X) \) and hence, for Theorem 2.8. □

The formulation of Theorem 2.8 looks quite technical, so it is useful to consider some simple numerical examples.

1°. First note that to get an increasing sequence

\[
\left( k^{-3/4}C(k + 2)^{-5}a_m(T)^{-1}(1 + 4\max\{\alpha, \beta\})/4 \right)_{k=1}^\infty
\]

in (2.17) it is always necessary to choose the sequence \((m_k)\) such that, roughly speaking, at least

\[
(a_{m_{k+1}}(T))^{-1} > (a_{m_k}(T))^{-d}
\]

holds for large \( k \). To be more exact, if for example \((a_{m_{k+1}}(T))^{-1} \leq (a_{m_k}(T))^{-d}\) for all \( k \), we have

\[
C(k + 2)^{-1}a_m(T)^{-1} \leq 4^{-(k+1)d^k}a_m(T)^{-d^k} \to 0,
\]

as \( k \to \infty \), and in this case (2.17) cannot be satisfied.

2°. Let \( T \in L(X) \) be compact and \( a_n(T) = n^{-c} \) for some \( c > 0 \). Now (2.17) holds, provided for some \( \varepsilon > 0 \) and \( \gamma > 0 \)

\[
(K, C) - ld(X^*, Y^*)(n) \leq (\log(n))^{c(1 - 4\max\{\alpha, \beta\})/4 - \varepsilon}
\]

and \( K(n) \leq e^{\gamma n} \) for all \( n \). (We choose \( m_{k+1} = e^{\gamma(4m_k + 2) + 1} \).)

3°. If \( d = 1 \), \( a_n(T) = (\log(n))^{-1} \), then (2.17) holds provided

\[
(K, C) - ld(X^*, Y^*)(n) \leq (\log(n))^{(1 - 4\max\{\alpha, \beta\})/8}
\]

for some \( c > 0 \) and \( K(n) \leq n^c \); we choose \( m_{k+1} = (7m_k)^{c'} \), where

\[
c' = (c + 4D(C + 3))^{40/(1 - 4\max\{\alpha, \beta\})}.
\]

We get

\[
(K, C) - ld(X^*, Y^*)(4m_{k+1}) \leq (\log(4m_{k+1}))^{(1 - 4\max\{\alpha, \beta\})/8}
\]

\[
\leq (28c' \log(m_k))^{(1 - 4\max\{\alpha, \beta\})/8}
\]

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so that
\[
((K, C) - ld(X^*, Y^*)(4m_{k+1}))^{-1} a_{m_k}(T)^{(-1+4\max\{\alpha, \beta\})/4} \geq c_0(\log(m_k))^{(1-4\max\{\alpha, \beta\})/8}
\]

(2.36)

for a constant \( c_0 > 0 \). Using \( m_k > m_{k-1}^C \) recursively we get \( m_k > m_{k-1}^{C^k} \), and combining this with (2.36) implies
\[
2((K, C) - ld(X^*, Y^*)(4m_{k+1}))^{-1} a_{m_k}(T)^{(-1+4\max\{\alpha, \beta\})/4} \geq c_0 c^k(1-4\max\{\alpha, \beta\})/8 \geq c_0(c + 4D(C + 3))^5k.
\]

Hence, in view of the definition of \( C(k) \) and \( d = 1 \) we see that (2.17) holds.

Clearly, Theorem 2.8 is in some cases far from being a necessary condition for factorization. For example in the case of a diagonal operator \( T \) on \( \ell_p \), \( Te_n = \lambda_n e_n \), where \((e_n)_{n=1}^{\infty}\) is the canonical basis and \((\lambda_n)_{n=1}^{\infty}\) tends to 0, the operators \( T_n \) connected with the numbers \( a_n(T) \) can be chosen such that the spaces \( T_n(\ell_p) \) are 1–complemented \( \ell_p \)–spaces, and consequently, the uniformity function \( K \) is not always needed. On the other hand in the case of general operators we do not know how to cope without the uniformity function.

2.9. Lemma. Let \( X \) and \( Y \) be separable reflexive Banach spaces. Assume that both of them satisfy property (D) with constant functions \( f_X \) and \( f_Y \), and that the uniform local distance of \( X^* \) and \( Y^* \) is bounded. Then \( X \) and \( Y \) satisfy the assumptions of Theorem 2.8 for an arbitrary compact \( T \in L(X) \).

To prove this we only need to choose the sequence \((m_k)\) in Theorem 2.8 so rapidly increasing that the right hand side of (2.17) is for all \( k \in \mathbb{N} \) larger than a constant (=the uniform local distance of \( X^* \) and \( Y^* \))

2.10. Theorem. Let \( X \) and \( Y \) be separable \( L_p \)–spaces, \( 1 < p < \infty \). Then an arbitrary compact \( T \in L(X) \) factors through \( Y \) as \( T = T^{(2)} T^{(1)} \) such that the image of \( X \) in \( Y \) is dense and such that both \( T^{(1)} \) and \( T^{(2)} \) are compact.

Proof. We refer to Lemma 2.9 and the considerations after the definition of property (D) and Remark 2.2,3°.

2.11. Remark. We show that if \( T \in L(\ell_2) \) is compact, then \( \ell_2 \) and the dual \( X^* \) of a suitably chosen space \( X \) of Proposition 2.4 (with \( \varepsilon = 1/2 \)) satisfy the assumptions of Theorem 2.8 with \( d(X^*) = 1 \), \( D(X^*) = 4 \), \( \alpha = 0 \), \( \beta = 1/8 \) and \( m_{k+1} = K(4m_k + 2) + 1 \). The consideration below is just to show that Theorem 2.8 is applicable to various situations. The conclusion, Corollary 2.12 itself has also a simpler proof, see Corollary 3.10. However, see also the remark after Corollary 2.12.

We first show that there exists a function \( f : \mathbb{N} \to \mathbb{N} \) and a choice of numbers \( n_k \) in the construction of the space \( X \) of Proposition 2.4 such that \((K, C) - ld(\ell_2, X^{**}) = (K, C) - ld(\ell_2, X) \) satisfies (2.17). Our aim is to use Remark 2.5. We first choose the
function \( g : \mathbb{N} \to \mathbb{N} \) such that \( g(k) \geq g(k - 1) \) for all \( k > 1 \), \( g(1) \geq 18 \) and such that the sequence
\[
(k^{-3/4}C(k + 2)^{-5}a_{g(k)}(T)^{-1/8})_{k=1}^\infty,
\]
where \( C(k) \) is as in Theorem 2.8 (with \( d = 1 \), \( D = 4 \) and \( C = 3/2 \), is increasing and unbounded and the first element is larger than 3. We define \( h(n) = 3 \) for \( n \leq g(1) \) and
\[
h(n) = k^{-3/4}C(k + 2)^{-5}a_{g(k)}(T)^{-1/8}
\]
for \( g(k) \leq n < g(k + 1) \). Then both \( h \) and \( g \) are non-decreasing unbounded functions and by Remark 2.5 we can choose the sequence \((n_k)\) and the function \( f \) such that \( 1^\circ \) and \( 2^\circ \) of Remark 2.5 hold and construct the space \( X \) satisfying Proposition 2.4 with \( \varepsilon = 1/2 \). By (2.13) we have
\[
n_k \leq K(n) \leq 3n_k,
\]
if \( n_{k-1} \leq n \leq n_k \). This implies that
\[
n_k \leq m_k \leq 3n_k + 1.
\]
Namely, if for some \( k \in \mathbb{N} \)
\[
n_{k-1} \leq m_{k-1} \leq 3n_{k-1} + 1,
\]
we have, by the choice of \( g(k) \) above and by the choice \( n_k \), Remark 2.5, \( 1^\circ \),
\[
4m_{k-1} + 2 \leq 12n_{k-1} + 6 \leq n_k.
\]
Hence, by (2.40) and (2.38) we have \( n_k \leq K(4m_{k-1} + 2) \leq 3n_k \). Since \( K(4m_{k-1} + 2) + 1 = m_k \), we get (2.40) for \( k \), and, by induction, this holds for all \( k \). (We have \( n_0 = 1 \), \( m_0 = 2 \).) Hence, for all \( k \in \mathbb{N} \), by Proposition 2.4 and by \( 2^\circ \), Remark 2.5,
\[
(K, C) - ld(\ell_2, X)(4m_{k+1}) \leq f(4m_{k+1}) \leq f(12n_{k+1} + 4) \leq f(n_{k+2}) \leq h(n_k)
\]
\[
\leq h(g(t + 1) - 1) = t^{-3/4}C(t + 2)^{-5}a_{g(t)}(T)^{-1/8} \leq k^{-3/4}C(k + 2)^{-5}a_{n_k}(T)^{-1/8}
\]
\[
\leq k^{-3/4}C(k + 2)^{-5}a_{m_k}(T)^{-1/8},
\]
where \( t \) is such that \( g(t) \leq n_k < g(t + 1) \) (note that \( n_k \geq g(k) \) by Remark 2.5, \( 1^\circ \) so that \( t \geq k \) holds). Hence, (2.17) holds.

Let us then consider property (D). For \( \ell_2 \) there is nothing to prove: it satisfies property (D) with \( f_{\ell_2} = d(\ell_2) = 1 \), \( D(\ell_2) \leq 1 \), so that we can take \( \alpha = 0 \). We note that
\[
X^* = (\bigoplus_{k=1}^\infty Y_k^*)_{\ell_2},
\]
where \( Y_k \) is as in the proof of Proposition 2.4. Since \( Y_k \) is for all \( k \) isometric to a subspace of \( L_{q_k} \) we see that \( Y_k^* \) is a quotient of \( L_{p_k} \), where \( 1/p_k + 1/q_k = 1 \). So, again by [L], Corollary 5, if \( Z \) is an \( n \)-dimensional subspace of \( Y_k^* \), then
\[
d(Z, \ell_2^{\dim(N)}) \leq n^{1/p_k-1/2} = n^{1/2-1/q_k}
\]
(2.41)
and there exist a projection from $Y_k^*$, and hence a projection $R$ from $X^*$, onto $Z$ with

$$
\|R\| \leq n^{1/2-1/q_k}.
$$

(2.42)

Let $M \subset X^*$ be such that $\dim(M) \leq m_k \leq 3n_k + 1$, $k > 2$. By (2.41) and considerations simpler than in (2.9)–(2.12) we see that $d(M, \ell_2^{\dim(M)}) \leq 3n_k^{1/2-1/q_k} + 1$. By (2.6), 2° of Remark 2.5 and (2.36) we thus get

$$
d(M, \ell_2^{\dim(M)}) \leq 2f(n_k-1) \leq 2h(n_k-2) \leq 2h(g(t+1)-1)
$$

$$
\leq a_g(t)(T)^{-1/8} \leq a_m(t)^{-1/8} \leq a_m(t)^{-1/8},
$$

(2.43)

where $t$ is such that $g(t) \leq n_k-2 < g(t+1)$.

On the other hand, let $P$ be a projection from $X^*$ onto $M$. The subspace $N_0 = \ker(P) \cap Y_{k+2}^*$ has dimension not smaller than $n_{k+2} - m_k$. Let $N$ be any $\dim(M)$-dimensional subspace of $N_0$. We have $d(N_0, \ell_2^{\dim(N_0)}) \leq (3n_k+1)^{1/2-1/q_k+2} \leq 1 + \varepsilon = 3/2$, see (2.41) and (2.4). By (2.41), (2.42) and (2.4) we find a projection $R$ from $X^*$ onto $N$ with $\|R\| \leq 2$. Now (2.16) follows from (2.43), and $Q = R(id_X - P)$ is a projection satisfying the conditions of property (D) with $d(X^*) = 1$, $D(X^*) = 4$.

In fact, we have proved

2.12. Corollary. Every compact operator $T$ on the real Hilbert space $\ell_2$ factors through a separable reflexive Banach space $Y$ (= $X^*$ above) without basis such that $T = T_1T_2$, where $T_1 \in L(\ell_2, Y), T_2 \in L(Y, \ell_2)$ and $T_1(\ell_2) \subset Y$ is dense.

Remark. The preceding consideration also yields a method to factorize compact operators on some $\ell_2$-sums of finite dimensional Banach spaces through the separable Hilbert space. Note that our factorization result remains valid, if we replace the spaces $Y_k$ of (2.7) by arbitrary $n_k$-dimensional subspaces of $L_{q_k}$. It is also clear that Theorem 3.2 or similar methods do not work in this case since all closed subspaces of the Hilbert space are Hilbert spaces.

In the second version of our main result we simply give a condition for factorization in terms of the solution of problem mentioned after Lemma 2.6. To be more exact, we give the following

2.13. Definition. Let $X$ and $Y$ be Banach spaces and let $f : \mathbb{N} \to \mathbb{R}^+$, $h : \mathbb{N} \to \mathbb{R}^+$ and $g : \mathbb{N} \to \mathbb{N}$, $g(n) \leq n$ for all $n$, be positive, non-decreasing. We define (if possible for these $X$, $Y$, $f$, $g$, and $h$) the function $\Phi(f, g, h) : \mathbb{N} \to \mathbb{R}$ as follows. If $n \in \mathbb{N}$, then $\Phi(f, g, h)(n)$ is the supremum over all pairs $(M, N)$ of at most $n$-dimensional subspaces $M \subset X$, $N \subset Y$, $\dim(M) = \dim(N)$, $d(M, N) \leq f(n)$, and subspaces $M_1 \subset M$, $N_1 \subset N$ satisfying $\dim(M_1) = \dim(N_1) \leq g(n)$ and $d(M_1, N_1) \leq f(g(n))$, of the number

$$
\sup \{d(M_2, N_2)\},
$$

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where the supremum is taken over all complements $M_2$, $M_2 \oplus M_1 = M$, and $N_2$, $N_2 \oplus N_1 = N$ such that the projections corresponding to these direct sums have norms smaller than $h(g(n))$.

In general, sufficiently good estimates for $\Phi$ are unknown. The philosophy is that, if for our spaces for example
\[
\Phi(f, g, h)(n) \leq cf(n)^\alpha
\]
holds for some constants $c, \alpha > 0$ and for some relatively slowly increasing functions $g$ and $h$, we get reasonable results.

The motivation of Definition 2.13 is the fact that knowing a good estimate for $\Phi$ makes the proof of our main result quite straightforward; compare the proofs of Theorems 2.8 and 2.14.

2.14. Theorem. Let $X$ and $Y$ be separable, reflexive Banach spaces such that $(K, C) – ld(X^*, Y^*)$ is finite for some $C \geq 1$ and $K : \mathbb{N} \to \mathbb{N}$, and let $T \in L(X)$ be compact. Let $m_0 = 2$, $m_k \geq K(3m_{k-1} + 2) + 1$ for all $k \in \mathbb{N}$. Let the functions $f : \mathbb{N} \to \mathbb{R}^+$ and $h : \mathbb{N} \to \mathbb{R}^+$ be defined by $f(n) = h(n) = 1$ for $n \leq 2$, and
\[
f(n) = 16C(k(n))^2(K, C) - ld(X^*, Y^*)(3m_{k(n)} - 1),
\]
\[h(n) = C(k(n)) + 1, \quad \text{for } n > 2
\]
where $k(n) \in \mathbb{N}$ is such that $m_{k(n)} - 1 < n \leq m_{k(n)}$ and $C(k) := (2(C + 3))^{k+1}$, and let $g : \mathbb{N} \to \mathbb{N}$ be the largest non-decreasing function satisfying $g(k) \leq k$ and $g(m_{k+1}) = m_k$ for all $k \in \mathbb{N}$.

Assume that for a constant $C'$ and for all $k \in \mathbb{N}$ the inequality
\[
\Phi(f, g, h)(n) \leq (k + 1)^{-3}C'C(k + 1)^{-1}a_{m_{k-1}}^{-1}(T), \quad \text{for all } n \leq m_{k+1},
\]
holds (\Phi with respect to the spaces $X$ and $Y$). Then $T = T^{(2)}T^{(1)}$, where $T^{(1)} \in L(X, Y)$, $T^{(2)} \in L(Y, X)$ and $T^{(1)}(X)$ is dense in $Y$. Moreover, the operators $T^{(1)}$ and $T^{(2)}$ are compact.

Note that if the estimate (2.44) holds, then the condition
\[
(K, C) - ld(X^*, Y^*)(3m_k) \leq (k + 1)^{-3/\alpha}C(k + 1)^{-2-1/\alpha}a_{m_{k-1}}^{-1}(T)^{-1/\alpha}
\]
implies (2.47), and hence, the operator $T$ factors. (Combine (2.44), (2.45) and (2.48).) On the other hand, (2.48) resembles (2.17) so we get numerical examples like those after the proof of Theorem 2.8, but with a bit better $C(k)$. Of course, applying Theorem 2.14 one does not need to take care of property (D).

Proof. For each $n \in \mathbb{N}$, $n > 1$, we choose an operator $T_n \in L(X)$ satisfying rank $(T_n) < n$ and
\[
\|T - T_n\| \leq 2a_n(T).
\]
Let \((Y_n)_{n=0}^{\infty}\) be a sequence of closed subspaces of \(Y\) such that \(Y_n \supset Y_{n+1}\) and \(\text{codim} (Y_n) = \text{codim} \left( \bigcap_{k=0}^{n} \ker (T_{m_k}) \right)\).

Let \((x_n)_{n=1}^{\infty} \subset X\) and \((y_n)_{n=1}^{\infty} \subset Y\) be sequences of non zero elements such that \(\text{sp} (x_n) \subset X\) and \(\text{sp} (y_n) \subset Y\) are dense.

We apply Lemma 2.6 inductively as follows. We set \(M_0 = N_0 = \{0\},\ P_0 = Q_0 = 0\). Assume that \(k \geq 1\) and that for \(0 \leq n < k\) the projections \(P_n \in L(X)\) and \(Q_n \in L(Y)\) are defined such that for all \(0 < n < k\):

1° \[\|P_n\| \leq C(n), \|Q_n\| \leq C(n),\]

2° \[P_n(X) \supset P_{n-1}(X),\ Q_n(Y) \supset Q_{n-1}(Y),\]

3° \[(\text{id}_X - P_n)(X) \subset (\text{id}_X - P_{n-1})(X) \cap \bigcap_{t=0}^{n-1} \ker (T_{m_t}),\]

4° \[\dim (M_n) = \dim (N_n) \leq m_n,\]

5° \[d(M_n, N_n) \leq 16C(n)^2KC\ell df(X^*, Y^*)(3m_{n-1})\]

6° \[x_n \in M_n,\ y_n \in N_n,\]

where \(M_n := P_n(X)\) and \(N_n := Q_n(Y)\). Then applying Lemma 2.6 with \(X, M_{k-1},\ k-1 \bigcap_{t=0}^{N_{k-1}} \ker (T_{m_t}), Y, N_{k-1}\) and \(Y_{k-1}\) as \(X, M, X_0, Y, M_Y\) and \(Y_0,\) and \(P_{k-1}\) and \(Q_{k-1}\) as \(P\) and \(P_Y\) we get projections \(Q = \hat{P}_k\) and \(Q_Y = \hat{Q}_k\) onto the subspaces \(M_k = N \subset X\) and \(N_k = N_Y \subset Y\), respectively, satisfying the properties mentioned in Lemma 2.6. So, we have \(\|\hat{P}_k\| \leq 2^k(C + 3)^k(C + 2)\) and \(\|\hat{Q}_k\| \leq 2^k(C + 3)^k(C + 2)\).

We choose the smallest \(n\) (resp. \(m\)) such that \(x_n \notin \hat{M}_k\) (resp. \(y_m \notin \hat{N}_k\)) and denote \(x^{(k)} := x_n\) (resp. \(y^{(k)} := y_m\)). We have \((\text{id}_X - \hat{P}_k)x^{(k)} \neq 0\) and \((\text{id}_Y - \hat{Q}_k)y^{(k)} \neq 0\). Let \(R_X\) and \(R_Y\) be projections from \((\text{id}_X - \hat{P}_k)(X)\) onto \(\text{sp} ((\text{id}_X - \hat{P}_k)x^{(k)})\) and from \((\text{id}_Y - \hat{Q}_k)(Y)\) onto \(\text{sp} ((\text{id}_Y - \hat{Q}_k)y^{(k)})\), respectively, with norm one. We define

\[P_k = \hat{P}_k + R_X(\text{id}_X - \hat{P}_k),\ Q_k = \hat{Q}_k + R_Y(\text{id}_Y - \hat{Q}_k)\]

and \(M_k = P_k(X), N_k = Q_k(Y)\). Then clearly

i) \(P_k\) commutes with \(\hat{P}_k\) and \(Q_k\) commutes with \(\hat{Q}_k\),

ii) \(\|P_k\| \leq 2\|\hat{P}_k\| + 1,\ \|Q_k\| \leq 2\|\hat{Q}_k\| + 1.\)

Now it is straightforward to see using Lemma 2.6 that properties 1°-6° are satisfied for \(k\) instead of \(k - 1\). We just remark that \((\text{id}_X - P_k)(X) \subset (\text{id}_X - P_{k-1})(X)\) follows from the commutativity of \(P_k\) and \(P_{k-1}\), which again is a consequence of Lemma 2.6 and i). Note also that the Hahn–Banach-theorem implies the existence of decompositions \(M_k = \hat{M}_k + M^{(k)}\), \(N_k = \hat{N}_k + N^{(k)}\), where \(\dim (M^{(k)}) = \dim (N^{(k)}) = 1\), such that the norms of the corresponding projections do not exceed 2. This again implies \(d(M_k, N_k) \leq 16d(\hat{M}_k, \hat{N}_k)\). Lemma 2.6 yields an estimate for \(d(\hat{M}_k, \hat{N}_k)\), so we get 5°.

Using Definition 2.13, (2.45), (2.46) and the properties 1° and 4° above we define for all \(k \in \mathbb{N}, k > 1\) the isomorphism \(\psi_k : (P_k - P_{k-1})(X) \to (Q_k - Q_{k-1})(Y)\) such that

\[\|\psi_k\|\|\psi_k^{-1}\| \leq \Phi(f, g, h)(m_k).\]  

(2.50)
(We take \( n = m_k \) and the spaces \( M_k, M_{k-1}, (P_k - P_{k-1})(M_k) \) correspond to \( M, M_1 \) and \( M_2 \) in Definition 2.13; similarly for \( N_k \) etc.) Let \( \psi_1 : M_1 \to N_1 \) be an isomorphism such that \( \|\psi_1\|\|\psi_1^{-1}\| = d(M_1, N_1) \).

We now define

\[
T^{(1)} x = \sum_{k=1}^{\infty} k^{-3/2} C(k)^{-1} \|\psi_k\|^{-1} \psi_k (P_k - P_{k-1}) x, \tag{2.51}
\]

where \( P_0 = 0 \) and \( x \in X \),

\[
T^{(2)} x = \sum_{k=1}^{\infty} k^{3/2} C(k) \|\psi_k\| T\psi_k^{-1} (Q_k - Q_{k-1}) x, \tag{2.52}
\]

where \( Q_0 = 0 \) and \( x \in Y \). The facts that \( T^{(1)} \in L(X, Y) \) and that \( T^{(1)} \) is compact are consequences of \( \|P_k\| \leq C(k), \ k \in \mathbb{N} \). Concerning \( T^{(2)} \), we have by \( 2^o \) for \( k \leq 2 \) and for all \( x \in Y \)

\[
\psi_k^{-1}(Q_k - Q_{k-1}) x \in (P_k - P_{k-1})(X) = (\text{id}_X - P_{k-1}) P_k(X) \subset \ker (T_{m_{k-2}}).
\]

Hence, by (2.49), for \( k \geq 2 \)

\[
\|T\psi_k^{-1}(Q_k - Q_{k-1}) x\| = \|(T - T_{m_{k-2}})\psi_k^{-1}(Q_k - Q_{k-1}) x\|
\leq \|T - T_{m_{k-2}}\| \|\psi_k^{-1}\| \|Q_k - Q_{k-1}\| \|x\|
\leq 4C(k) a_{m_{k-2}}(T) \|\psi_k^{-1}\| \|x\|.
\]

Now the choice of \( \psi_k \), (2.50) and (2.47) imply for \( k \geq 2 \)

\[
\|C(k)\|\|\psi_k\| T\psi_k^{-1}(Q_k - Q_{k-1}) x\|
\leq 4C(k) a_{m_{k-2}}(T) \|\psi_k\| \|\psi_k^{-1}\| \|x\|
\leq 4C(k) a_{m_{k-2}}(T) \Phi(f, g, h)(m_k)
\leq 4k^{-3} C' \|x\|. \tag{2.53}
\]

So (2.53) shows that (2.52) converges absolutely for all \( x, T^{(2)} \in L(Y, X) \) and \( T^{(2)} \) is even compact.

We have

\[
T^{(2)} T^{(1)} X = \sum_{k=1}^{\infty} k^{3/2} C(k) \|\psi_k\| T\psi_k^{-1} (Q_k - Q_{k-1}) \left( \sum_{t=1}^{\infty} t^{-3/2} C(t)^{-1} \|\psi_t\|^{-1} \psi_t (P_t - P_{t-1})(x) \right)
\]

\[
= \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} k^{3/2} t^{-3/2} C(k) C(t)^{-1} \|\psi_k\| \|\psi_t\|^{-1} T\psi_k^{-1} (Q_k - Q_{k-1}) \psi_t (P_t - P_{t-1})(x). \tag{2.54}
\]
Here
\[ T\psi_k^{-1}(Q_k - Q_{k-1})\psi_t(P_t - P_{t-1})(x) \]
\[ = \delta_{kt}T(P_t - P_{t-1})(x) \]
so that (2.24) is equal to
\[ \sum_{t=1}^{\infty} T(P_t - P_{t-1})(x). \]

So \( T^{(2)}T^{(1)}x = Tx \) for \( x \in M_k \) for all \( k \in \mathbb{N} \), and hence also for all
\[ x \in \bigcup_{k=1}^{\infty} M_k = X. \]

The density of \( T^{(1)}(X) \subset Y \) follows from the choice of the sequence \( (y_k) \) and the fact that \( y_k \in T(M_{k+1}) \) for all \( k \in \mathbb{N} \).

**2.15. Corollary.** 1°. Let \( T \in L(\ell_2) \) be compact, \( a_n(T) \leq (\log n)^{-1-\varepsilon} \) for some \( 0 < \varepsilon < 1 \). Then \( T \) factors as \( T = T_2T_1 \) through every weak Hilbert space \( X \) such that \( T_1(\ell_2) \subset X \) is dense.

2°. If \( X \) is a weak Hilbert space, \( T \in L(X) \) is compact and \( a_n(T) \leq (\log n)^{-1-\varepsilon} \), then \( T \) factors through a Hilbert space.

Note that to prove this we need to use the unpublished results mentioned in Remark 2.2.8°.

**Proof.** Let \( X \) be weak Hilbert, let \( C = 2 \) and \( K : \mathbb{N} \to \mathbb{N}, K(n) = c'n, \) where \( c' \geq 1 \) is such that \((K,C)-ld(X,\ell_2)(n) \leq c_0 \log(n+1) \) as in Remark 2.2.8°. Let \( c > 0 \) be such that if \( M \subset X \) is an \( n \)-dimensional subspace, then \( d(M,\ell_2^{\dim(M)}) \leq c \log(n+1) \), see Remark 2.2.8°. We define \( \alpha = 10^{2/\varepsilon} \) and \( m_k = \max\{e^{\alpha k}, 7c'm_{k-1}\} \) for all \( k \in \mathbb{N} \); for large \( k \) we thus have \( m_k = e^{\alpha k} \).

Assume now that \( k \in \mathbb{N} \) and \( n \leq m_{k+1} \). By the choice of \( c \) and the definition of \( \Phi \) we have the following trivial estimate for \( \Phi(f,g,h)(n) \) (with \( X = X, Y = \ell_2 \)) which in fact does not depend on \( f, g \) and \( h \):
\[ \Phi(f,g,h)(n) \leq c \log(m_{k+1} + 1). \]

On the other hand \((a_{m_{k-1}}(T))^{-1} \geq (\log m_{k-1})^{1+\varepsilon} \) so that we get for some constants \( c_i > 0, i = 1,2,3 \), for large \( k \) and \( n \leq m_{k+1} \)
\[ (\Phi(f,g,h)(n))^{-1}a_{m_{k-1}}^{-1}(T) \geq c_1^{-1}\alpha^{-k-2}\alpha^{(k-1)(1+\varepsilon)} \]
\[ \geq c_2\alpha^{k\varepsilon} = c_210^{2k} \geq c_310^{k}C(k + 1), \]
see the choice of \( C \) and \( C(k) \). This shows that (2.47) holds. So, Theorem 2.14 applies to prove our Corollary in both cases.
We now turn to a study of locally convex spaces. Factorization theorems in Banach spaces can often be used to find new systems of local Banach spaces of locally convex spaces. For example, given a Schwartz space $E$ as a projective limit of nonreflexive spaces we can use the factorization result of [DFJP] which says that a weakly compact operator always factors through a reflexive space, to find a system of reflexive local Banach spaces on $E$. However, not all factorization results are useful in this respect. To get a trivial counterexample, let us consider a Banach space $(X, \| \cdot \|)$ as a Fréchet space with the system $(p_k)_{k=1}^\infty$, $p_k = k \cdot \| \cdot \|$, of seminorms. Given any Banach space $Y$ the linking map (the identity operator on $X$) between the local Banach spaces $X_{p_{k+1}}$ and $X_{p_k}$ factors trivially through $X \times Y$. However, it is not possible that $X$ can have a system of local Banach spaces isomorphic to $X \times Y$ unless $X$ is isomorphic to $X \times Y$.

The situation is different, if the factorization is dense. To see this, assume that $(E, (p_\alpha)_{\alpha \in A})$ is a locally convex space. Let $\alpha, \beta \in A$, $p_\alpha \geq p_\beta$, and let $T_{\alpha,\beta} : E_{p_\alpha} \to E_{p_\beta}$ be the canonical mapping induced by the identity operator on $E$. If $T_{\alpha,\beta}$ factors as $T_{\alpha,\beta} = T^{(2)}T^{(1)}$ through some Banach spaces $Y$ such that $T^{(1)}(E_\alpha) \subset Y$ is dense, then there exists a continuous seminorm $q$ on $E$ such that $p_\alpha \leq q \leq p_\beta$ and $E_q \cong Y$: we can take $q(x) = \|T^{(1)}\psi x\|_Y$, where $\psi$ is the quotient mapping from $E$ onto $E/\ker(p_\alpha)$. Of course, this does not necessarily hold without the assumption on the density of $T^{(1)}(E_\alpha)$.

Recall that in the case of Schwartz spaces we can find for all $\alpha \in A$ an index $\beta$ such that $T_{\alpha,\beta}$ is compact. Assume that there exists a separable, reflexive Banach space $X$ such that for all $\alpha \in A$, $E_{p_\alpha} \cong X$. In view of the preceding remarks it is now clear that, given a Schwartz space as above, we can use Theorems 2.8 and 2.14 to find new systems of seminorms $(q_\beta)_{\beta \in B}$ on $E$ such that the local Banach spaces of this systems are not necessarily isomorphic to $X$.

We could formalize this statement as a corollary but we do not want to repeat the many assumptions of Theorems 2.8 and 2.14. We just give some special cases.

**2.16. Corollary.** Let $E$ be a Schwartz space as above such that $X$ is isomorphic to a $\mathcal{L}_p$–space, $1 < p < \infty$. Given any $\mathcal{L}_p$–space $Y$ the space $E$ has a system of local Banach spaces isomorphic to $Y$.

To prove this we use Theorem 2.10.

**2.17. Corollary.** Let $E$ be a hilbertizable Fréchet–Schwartz space over $\mathbb{R}$, i.e., a real Fréchet–Schwartz space having a system of local Banach spaces isomorphic to $\ell_2$. Then there exists a system of local Banach spaces $(E_{p_k})_{k=1}^\infty$ such that none of the spaces $E_{p_k}$ has a basis.

This follows from Corollary 2.12.

I think it should be possible to choose the spaces $E_{p_k}$ isomorphic to each other.

**3. Remarks on dense factorizations.** In this section we present some relatively simple, but efficient, remarks on dense factorizations of operators in Banach spaces. The
results can be combined for example with the considerations in the previous section to get much more new examples of systems of local Banach spaces in locally convex spaces.

We recall the trivial fact (see also the remark above Corollary 2.16) that given a Banach Space $X$ the identity operator on $X$ factors through $X \times Y$ for any Banach space $Y$, but that this factorization can in general not be done such that the image of $X$ in $X \times Y$ is dense, even if $Y$ is separable. We shall see in this section that the situation changes dramatically, if we consider for example a compact operator on $X$.

We begin with a lemma in the spirit of [V], Theorem 1: to prove the lemma we use the existence of a total, bounded biorthogonal system in a separable Banach space, as in [V].

**3.1. Lemma.** Let $X$ be a Banach space with a normalized basis $(e_n)_{n=1}^{\infty}$, let $Y$ and $Z$ be Banach spaces, $Z$ separable, and let $T \in L(X,Y)$ satisfy for some $\varepsilon > 0$

$$\|Te_n\| \leq n^{-2-\varepsilon} \quad (3.1)$$

for all $n$. Then $T$ factors as $T = T^{(2)}T^{(1)}$ through $Z$ such that $T^{(1)}(X) \subset Z$ is dense and such that $T^{(1)}$ and $T^{(2)}$ are compact.

**Proof.** Let $((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty})$ be a total, fundamental, biorthogonal system in $Z$, as in [LT1], Theorem 1.f.4. Recall that $((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty})$ satisfies the following properties:

$(x_n)_{n=1}^{\infty} \subset Z$ is total, $\langle x_n, y_m \rangle = \delta_{nm}$ for $n, m \in \mathbb{N}$, and $\sup_{n \in \mathbb{N}} \|x_n\|\|y_n\| \leq 20$, so that we may assume $\|x_n\| \leq 1$, $\|y_n\| \leq 20$ for all $n$.

Let $(e_n^*)_{n=1}^{\infty} \subset X^*$ be the sequence of the coefficient functionals of $(e_n)_{n=1}^{\infty}$; we have $\sup_n \|e_n^*\| < C < \infty$ for a constant $C > 0$.

We define

$$T^{(1)}x = \sum_{n=1}^{\infty} n^{-1-\varepsilon/2} \langle x, e_n^* \rangle x_n \quad (3.2)$$

for $x \in X$, and

$$T^{(2)}x = \sum_{n=1}^{\infty} n^{1+\varepsilon/2} \langle x, y_n \rangle Te_n$$

for $x \in Z$. The boundedness of the sequences $(e_n^*)_{n=1}^{\infty}$ and $(x_n)_{n=1}^{\infty}$ implies

$$\|T^{(1)}x\| \leq \sum_{n=1}^{\infty} n^{-1-\varepsilon/2} \|x\| \|e_n^*\| \|x_n\| \leq C' \|x\|$$

for a constant $C' > 0$, so that $T^{(1)}$ is a bounded and even a compact operator. Moreover, $T^{(1)}(X)$ is dense in $Z$ since $(x_n)_{n=1}^{\infty}$ is total in $Z$. Similarly, by (3.1), for $x \in Z$,

$$\|T^{(2)}x\| \leq \sum_{n=1}^{\infty} n^{1+\varepsilon/2} \|x\| \|y_n\| \|Te_n\| \leq \sum_{n=1}^{\infty} 20n^{-1-\varepsilon/2} \|x\|.$$
Hence, also $T^{(2)}$ is bounded and compact.

We have for all $x \in X$
\[
T^{(2)}T^{(1)}x = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n^{1+\epsilon/2} m^{-1-\epsilon/2} \langle x, e^*_m \rangle \langle x_m, y_n \rangle T e_n = \sum_{n=1}^{\infty} n^{1+\epsilon/2} n^{-1-\epsilon/2} \langle x, e^*_n \rangle T e_n = Tx,
\]
since $\langle x_m, y_n \rangle = \delta_{m,n}$. □

The following result provides a trick to make dense factorizations. We again refer to [LT1] for the terminology of bases and basic sequences in Banach spaces.

**3.2. Theorem.** Let $X$ and $Y$ be Banach spaces, let $T \in L(X,Y)$ be compact and assume that $X$ has a complemented unconditional basic sequence. If $Z$ is an arbitrary separable Banach space, then $T$ factors as $T = T^{(2)}T^{(1)}$ through $X \times Z$ such that $T^{(1)}(X) \subset X \times Z$ is dense and $T^{(2)}$ is compact.

**Proof.** Let $(e_n)_{n=1}^{\infty} \subset X$ be a normalized complemented unconditional basic sequence. By compactness, $(T e_n)_{n=1}^{\infty}$ has a convergent subsequence $(T e_{n_k})_{k=1}^{\infty}$, and, hence, the sequence
\[
(T e_{n_{2k}} - T e_{n_{2k+1}})_{k=1}^{\infty}
\]
converges to 0 in $Y$. Let $(m_k)_{k=1}^{\infty}$ be a subsequence of $(n_k)$ such that for all $k \in \mathbb{N}$
\[
\|T e_{m_{2k}} - T e_{m_{2k+1}}\| \leq k^{-3},
\]
and let us denote $f_k = e_{m_{2k}} - e_{m_{2k+1}}$ for $k \in \mathbb{N}$. Since $(e_n)_{n=1}^{\infty}$ is an unconditional, complemented basic sequence we see that $(f_n)_{n=1}^{\infty}$ is a complemented block basic sequence of $(e_n)$ satisfying
\[
\|T f_n\| \leq n^{-3} \tag{3.3}
\]
for all $n \in \mathbb{N}$. Moreover, we have
\[
2 \geq \|f_k\| = \|e_{m_{2k}} - e_{m_{2k+1}}\| \geq C^{-1}\|e_{m_{2k}}\| = C^{-1}
\]
for all $k$, where $C > 0$ is the basis constant of $(e_n)_{n=1}^{\infty}$.

By Lemma 3.1 and (3.3), the restriction of $T$ to the complemented subspace $X_1 := \text{sp} \{f_n \mid n \in \mathbb{N}\}$ of $X$ factors densely through $X_1 \times Z$. Let us denote this factorization by $T|_{X_1} = R^{(2)}R^{(1)}$, where the operators $R^{(1)} \in L(X_1, X_1 \times Z)$ and $R^{(2)} \in L(X_1 \times Z, X)$ are compact. We define
\[
T^{(1)} = R^{(1)} \oplus \text{id}_{X_2}, \quad T^{(2)} = R^{(2)}P + TQ, \tag{3.4}
\]
with $P = 0$ and $Q = \text{id}_Z$. □
where $X_2$ is a closed subspace of $X$ such that $X_1 \oplus X_2 = X$, and $P$ (respectively, $Q$) is the natural projection from $(X_1 \oplus X_2) \times Z$ onto $X_1 \times Z$ (respectively, onto $X_2$). Clearly, $T^{(2)}$ is compact as a sum of two compact operators, see Lemma 3.1. □

Examples. 1° The straightforward application of Theorem 3.2 to the Schwartz space $(E, (p_\alpha)_{\alpha \in A})$ yields the result that if the spaces $E_{p_\alpha}$ have complemented unconditional basic sequences, then we can find a system of local Banach spaces $(E_{q_\alpha})_{\alpha \in A}$ where each $E_{q_\alpha}$ is an arbitrary separable Banach space containing $E_{p_\alpha}$ as a complemented subspace. This result is quite natural in view of the facts that each Schwartz space contains a nuclear subspace and that in nuclear spaces the geometry of local Banach spaces may be chosen arbitrarily, [V]. However, these known facts do not imply our results.

2° Theorem 3.2 also leads to many examples of the following type: Let $(E_n)_{n=1}^\infty$ be a family of Banach spaces and let $\ell_p((E_n)_{n=1}^\infty)$ be the Banach space of $E_n$–valued, $\ell_p$–summable sequences, where $1 \leq p < \infty$. The space $\ell_p((E_n))$ contains a complemented copy of $\ell_p$ (fix one vector $0 \neq e_n \in E_n$ for all $n$ and use the Hahn–Banach theorem to find for all $n \in \mathbb{N}$ a projection from $E_n$ onto $sp(e_n)$). By Theorem 3.2, every compact operator on $\ell_p$ factors densely through $\ell_p((E_n))$. Generalizations and applications to Schwartz spaces are left to the reader.

3.3. Corollary. Every compact operator $T : \ell_1 \to \ell_1$ factors through an arbitrary separable $L_1$–space $Y$ such that the image of $\ell_1$ in $Y$ is dense.

Proof. Follows immediately from Theorem 3.2, since each $L_1$–space contains a complemented copy of $\ell_1$, [LP], Proposition 7.3. □

Corollary 3.3 complements Theorem 2.10.

In the case of $\ell_2$ a much stronger result holds.

3.4. Theorem. Let $T \in L(\ell_2)$ be compact and let $X$ be an arbitrary separable $L_p$–space, where $1 < p < \infty$. The operator $T$ factors as $T = T^{(2)} T^{(1)}$ through $X$ such that $T^{(2)} \subset X$ is dense and both $T^{(1)} \subset L(\ell_2, X)$ and $T^{(2)} \subset L(X, \ell_2)$ are compact.

Proof. Using for example the polar decomposition we can find compact operators $S^{(i)} \in L(\ell_2)$, $i = 1, 2$, such that $T = S^{(2)} S^{(1)}$ and such that each $S^{(i)}$ has a dense range. The space $L_p(0, 1)$ has a complemented subspace $Y$ isomorphic to $\ell_2$ (see, e.g. [LT], p.215). By Theorem 3.2, both $S^{(1)}$ and $S^{(2)}$ factor through $L_p(0, 1)$,

$$S^{(1)} = S^{(1,2)} S^{(1,1)}, \quad S^{(2)} = S^{(2,2)} S^{(2,1)},$$

such that $S^{(i,1)} \subset L_p(0, 1)$ is dense and $S^{(i,2)}$ is compact for $i = 1, 2$. By Theorem 2.10, $S := S^{(2,1)} S^{(1,2)} \in L(L_p(0, 1)$ factors as $S = R^{(2)} R^{(1)}$ through $X$ with a dense range such that each $R^{(i)}$ is compact. Now

$$T^{(1)} = R^{(1)} S^{(1,1)}, \quad T^{(2)} = S^{(2,2)} R^{(2)}$$

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3.5. Corollary. Every hilbertizable Fréchet–Schwartz space has for all $1 < p < \infty$ and for all separable $L_p$–spaces $Y$ a system of local Banach spaces isomorphic to $Y$.

Let us still mention the following interesting case.

3.6. Corollary. Let $(E, (p_\alpha)_{\alpha \in A})$ be a locally convex space such that every local Banach space $E_\alpha := E_{p_\alpha}$ has the bounded approximation property and such that for every $\alpha$ there exists $\beta, p_\alpha \leq p_\beta$, and a complemented unconditional basic sequence $(e_n(\alpha, \beta))_{n=1}^\infty \subset \text{sp}(e_n(\alpha, \beta) \mid n \in \mathbb{N})$ is compact. Then $E$ has a system of local Banach spaces $(E_{q_\alpha})_{\alpha \in A}$ so that each $E_{q_\alpha}$ has a basis.

Proof. It is easy to see, using Theorem 3.2, that each linking map $T_{\alpha\beta}$ as above factors densely through $E_\beta \times Z_{\alpha\beta}$ for an arbitrary separable Banach space $Z_{\alpha\beta}$. Since $E_\beta$ has the bounded approximation property, we can choose $Z_{\alpha\beta}$ such that $E_\beta \times Z_{\alpha\beta}$ has a basis, see [P]. We then define the system of seminorms as above Corollary 2.16.

Examples. 1° Every Schwartz space having a system of local Banach spaces with the bounded approximation property and a complemented unconditional basic sequence satisfies the conditions of Corollary 3.6.

2°. Assume that $E$ is a Fréchet space of Moscatelli type (for definition, see [BD], Definition 1.3) with respect to $X$, $(Y_n)_{n=1}^\infty$, $(Z_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$, where $X$ is a normal Banach sequence space, the Banach spaces $Y_n$ and $Z_n$ have the bounded approximation property for all $n$, the spaces $Y_n$ have for all $n$ unconditional complemented basic sequences, and the linear maps $f_n : Y_n \to Z_n$ are compact embeddings. The conditions of Corollary 3.6 are satisfied also in this case.

Remark. The result that a Banach–space with an unconditional basis is isomorphic to a complemented subspace of Banach space with a symmetric basis (see [LT1], Theorem 3.b.1) also yields an application similar to Corollary 3.6.

The following remark yields another method to combine the results of Section 2 with the present considerations.

3.7. Lemma. Let $X, Y, Z$ and $W$ be Banach spaces, $Z$ separable, let $T \in L(X, Y)$ be compact and assume that every compact operator $S \in L(X, Y)$ factors densely through $W$. If $X$ has a complemented unconditional basic sequence $(e_n)_{n=1}^\infty$, then $T$ factors densely through $W \times Z$.

Proof. According to Theorem 3.2, $T$ factors densely through $X \times Z$ as $T = S^{(2)}S^{(1)}$. Moreover, the factorization is such that $S^{(2)}$ is compact. By assumption, the restriction of $S^{(2)}$ to $X$ factors densely through $W$, $S^{(2)}|_X = R^{(2)}R^{(1)}$, where $R^{(1)} \in L(X, W)$, $R^{(2)} \in L(W, X)$. So, writing

\[
T^{(1)} = (R^{(1)}P_X + P_Z)S^{(1)},
\]

\[T^{(2)} = R^{(2)}P_W + S^{(2)}P_Z,
\]

(3.5)
where \( P_X \) denotes the natural projection onto \( X \) etc. and \( T^{(1)} \in L(X, W \times Z) \), \( T^{(2)} \in L(W \times Z, X) \), we get the desired factorization \( T = T^{(2)} T^{(1)} \).

**Example.** Applying the results of Section 2 and Lemma 3.5 we see that a compact operator on a separable reflexive Banach space \( X \) factors densely through a separable Banach space \( W \times Z \), where \( W \) is reflexive, if merely the uniform local distance of \( X^\ast \) and \( W^\ast \) is small enough and if some technical regularity assumptions are satisfied. To be more exact, the assumptions of Lemma 3.5 are satisfied, if \( X = Y \) and both \( X \) and \( W \) are reflexive, separable and have property \( (D) \) with constant functions \( f_X \) and \( f_W \), and if the uniform local distance of \( X^\ast \) and \( W^\ast \) is bounded (Lemma 2.9). For example, every compact operator on a separable \( \ell_p \)-space, \( 1 < p < \infty \), factors densely through \( \ell_p(E) \), the space of \( p \)-summable, \( E \) valued sequences, where \( E \) is an arbitrary separable Banach space; see the example after Theorem 3.2.

Johnson constructed in [Jo] a family of separable Banach spaces \( C_p \), \( 1 \leq p \leq \infty \), with the strong property that every compact operator on a separable Banach space \( X \) with the approximation property factors through an \( C_p \). The spaces \( C_p \) are \( \ell_p \)-sums of some sequences of finite dimensional Banach spaces. Having a look at the proof of [Jo] one finds that the factorization is not dense. It would be interesting to know, if the factorization could be done in a dense way for example in the case \( X \) has the bounded approximation property. Unfortunately, Theorems 2.8 or 2.14, combined with Lemma 3.5, do not work here, since the local distance of a space \( X \) and an \( \ell_p \)-sum of finite dimensional subspaces of \( X \) is usually not bounded. The following result implies a positive answer to this problem in a restricted case. Note that because of the polar decomposition, the case of compact operators in Hilbert space is included there. For the properties of symmetric bases we refer to [LT1], p. 113.

**3.8. Proposition.** Let \( X \) be a Banach space with a normalized symmetric basis \( (e_n)_{n=1}^\infty \) and let \( T \in L(X) \) be a compact diagonal operator, \( T e_n = \lambda e_n \) for a sequence \( (\lambda_n)_{n=1}^\infty \) of scalars satisfying \( \lim_{n \to \infty} |\lambda_n| = 0 \). Let \( Y \) be a Banach space which has an unconditional finite dimensional decomposition, \( (M_{n_k})_{k=1}^\infty \), where \( M_n = \text{sp} \{ e_k | k \leq n \} \subset X \), for some increasing sequence \( (n_k)_{k=1}^\infty \). If \( Z \) is an arbitrary separable Banach space, then the operator \( T \) factors as \( T = T^{(2)} T^{(1)} \) through \( Y \times Z \) such that \( T^{(1)}(X) \) is dense in \( Y \times Z \) and such that both \( T^{(1)} \) and \( T^{(2)} \) are compact.

**Proof.** We may assume that \( |\lambda_n| \leq 1 \) for all \( n \). We choose the sequence \( (m_k)_{k=1}^\infty \), \( m_k \in \mathbb{N} \), such that

\[
|\lambda_n| \leq k^{-3}
\]

for all \( n \geq m_k \), for all \( k \in \mathbb{N} \). Let

\[
X_1 = \text{sp} \{ e_n | n \neq m_k \text{ for all } k \} \subset X,
\]

\[
X_2 = \text{sp} \{ e_{m_k} | k \in \mathbb{N} \} \subset X.
\]
Let \((n_k)_{k=1}^\infty\) be an increasing subsequence of \((\nu_k)_{k=1}^\infty\) such that \(n_k \geq m_k\) for all \(k\). We define
\[
Y_1 = \bigoplus_{k=1}^{\infty} M_{\nu_k},
\]
\[
Y_2 = \bigoplus_{k \in J} M_{n_k}
\]
where \(J = \{k \in \mathbb{N} | n_k \neq \nu_k\text{ for all }t \in \mathbb{N}\}\).

The restriction of \(T\) to \(X_2\) factors densely through \(Y_2 \times Z\) by (3.6) and Lemma 3.1. Moreover, the factorization is such that both factors are compact. It is thus sufficient to prove that the restriction of \(T\) to \(X_1\) factors densely through \(Y_1\) such that both factors are compact. We define the sets \(N_k \subset \mathbb{N}\), \(k = 1, 2, \ldots\), inductively as follows: \(N_k\) consists of the smallest \(\nu_k\) natural numbers \(m\) such that \(m \neq m_t\) for all \(t \in \mathbb{N}\) and \(m \notin N_t\) for \(t < k\). Clearly, we have
\[
m \geq \nu_{k-1} \geq m_{k-1}
\]
for all \(m \in N_k\). Moreover,
\[
X_1 = \bigoplus_{k=1}^{\infty} N_k,
\]
where \(N_k = \text{sp}\{e_n | n \in N_k\}\). Let \(\varphi_k : \{1, \ldots, \nu_k\} \to \mathbb{N}_k\) be a bijection. We define for all \(k \in \mathbb{N}\) the operator \(R_k^{(1)} \in L(N_k, M_{\nu_k})\) by
\[
R_k^{(1)} e_{\varphi_k(n)} = |\lambda_{\varphi_k(n)}|^{1/2} e_n,
\]
where \(1 \leq n \leq \nu_k\), and \(R_k^{(2)} \in L(M_{\nu_k}, N_k)\) by
\[
R_k^{(2)} e_n = \lambda_{\varphi_k(n)} |\lambda_{\varphi_k(n)}|^{-1/2} e_{\varphi_k(n)}
\]
for \(1 \leq n \leq \nu_k\). We have \(|\lambda_{\varphi_k(n)}| \leq (k-1)^{-3}\) for all \(k > 1\) and for \(1 \leq n \leq \nu_k\), see the choice of \(\varphi_k\), (3.9) and (3.6). Since the basis \((e_n)_{n=1}^\infty\) is symmetric, we get
\[
\|R_k^{(1)}\| \leq Ck^{-3/2}, \quad \|R_k^{(2)}\| \leq Ck^{-3/2}
\]
for a constant \(C\) (depending only on the properties of the basis \((e_n)\); see [LT1], p.113) and for all \(k\). Denoting by \((P_k)_{k=1}^\infty\) (resp. \((Q_k)_{k=1}^\infty\)) the uniformly bounded family of natural projections, \(P_k : X_1 \to N_k\) (resp. \(Q_k : Y_1 \to M_{\nu_k}\)) we define
\[
R^{(1)} = \sum_{k=1}^\infty R_k^{(1)} P_k, \quad R^{(2)} = \sum_{k=1}^\infty R_k^{(2)} Q_k.
\]
These operators are bounded and compact because of (3.13) and the properties of \((P_k)_{k=1}^\infty\) and \((Q_k)_{k=1}^\infty\). Moreover, (3.11) and (3.12) imply that \(R^{(2)} R^{(1)} e_n = \lambda_n e_n\) for all \(n \in \bigcup_k \mathbb{N}_k\).
so that \( R^{(2)}R^{(1)}x = Rx \) for \( x \in X_1 \), see (3.10) and (3.7). The density of \( R^{(1)}(X_1) \) in \( Y_1 \) also follows from (3.11). □

3.9. Corollary. Let \( X \) and \( T \) be as in Proposition 3.6. For all \( p, 1 \leq p \leq \infty \), the operator \( T \) factors through the Johnson space \( C_p \) such that the image of \( X \) in \( C_p \) is dense. As a consequence, each \( \ell_q \)–Köthe sequence space, \( 1 \leq q \leq \infty \), which is a Schwartz space has a system of local Banach spaces isomorphic to \( C_p \).

Proof. Given any sequence \( (M_n)_{n=1}^{\infty} \) of finite dimensional Banach spaces the Johnson space contains a complemented subspace \( Y \) isomorphic to \( (C_p)_{\ell_p} \). So, Proposition 3.8 implies the desired factorization of \( T \). The statement concerning Köthe spaces follows from the fact that the linking maps between their natural local Banach spaces satisfy the assumptions of Proposition 3.8. □

We finally present a consequence or Theorem 3.2 containing Corollary 2.12.

3.10 Corollary. Let \( Y \) be a Banach space. Every compact operator \( T \in L(\ell_2, Y) \) factors densely through a separable reflexive Banach space \( X \) without basis.

Proof. Let \( X \) be the Szarek space as in Proposition 2.4 with e.g. \( f(n) = 3n \). It is easy to see, using considerations like those in the proof of Proposition 2.4, that for all \( n \in \mathbb{N} \) we can find \( k \) such that \( Y_k \) contains a 2–complemented subspace 2–isomorphic to \( \ell_2^k \). Since \( X \) is the \( \ell_2 \)–sum of the spaces \( Y_k \), we see that \( X \) in fact contains a complemented copy of \( \ell_2 \). Our result now follows directly from Theorem 3.2. □

4. Duality problems for local Banach spaces of Fréchet spaces. We consider the following problems on the local Banach spaces of, say, a Fréchet or a \((DF)\)–space \( E \) and its strong dual \( E'_b \).

(L1) Assume that \( E \) has a system of local Banach spaces isomorphic to a Banach space \( X \). Does \( E'_b \) have a system of local Banach spaces isomorphic to \( X^* \)?

(L2) Is it possible to construct an example of a Fréchet or a \((DF)\)–space \( E \) such that, given large enough continuous seminorms \( p \) and \( q \) on \( E \) and \( E'_b \), respectively, we have \( (E_p)^* \not\cong (E'_q)^* \) for all continuous seminorms \( \rho \geq p \) and \( \gamma \geq q \) on \( E \) and \( E'_b \)?

There exists a simple counterexample to (L1). Let \( E \) be any nuclear \( \ell_1 \)–Köthe sequence space with a continuous norm. Then \( E \) is a Fréchet–space having by definition a system of local Banach spaces \( (E_{p_k})_{k=1}^{\infty} \) isometric to \( \ell_1 \). We have, for all \( k \in \mathbb{N} \), \( (E_{p_k})' \cong \ell_\infty \), which is not a separable space. On the other hand \( E'_b \) is separable so that all the local Banach spaces of \( E'_b \) are also separable. So (L1) has a negative answer in this case.

The preceding counterexample is a separability argument, and it does not give any information on the local structure of the Banach spaces involved. In fact it is known that \( E'_b \) has a system of local Banach spaces isomorphic to \( c_0 \) (see [V]!). Both \( c_0 \) and \( \ell_\infty \) are \( L_\infty \)–spaces so that at least in this sense they are still quite similar. I do not think the above counterexample is yet a satisfactory answer to problem (L1).
We can also ask the following natural question:

(B1) Assume that $E$ has a system of local Banach spaces isomorphic to the Banach space $X$. Does $E$ also have a family of Banach discs $(B_\alpha)_{\alpha \in A}$ such that $E_{B_\alpha} \cong X$ and such that every bounded set of $E$ is contained in some $B_\alpha$?

Clearly, questions (L1) and (B1) are related. To be more exact, every weakly closed (with respect to the dual pair $<E,E'_b>$) absolutely convex neighbourhood of zero $U \subset E'_b$ is the polar of a Banach disc $B \subset E$, and, moreover, $(E')_U$ is isometric to a subspace of $(E_B)'$. The counterexample above does not solve (B1).

We are not able to solve (B1) here. I conjecture that the answer is negative. The difficulty in proving this is that one has to be able to deal with a large class of Banach spaces on which one has only little information and which may be very pathological (cf. Corollary 2.17!).

To give some reference we mention the book [Ju] which contains a study of related factorization problems. The above problems are also connected with the problem of projective descriptions of inductive limits. There exist a lot of papers on this topic by K.D.Bierstedt, J.Bonet, R.Meise and others. We refer to the survey articles [BM] and [BB]. In this context problem (B1) was solved in the positive for $\ell_p$–Köthe sequence spaces in [BMS], Proposition 2.5.

The result can easily be generalized to $X$–Köthe sequence spaces in the sense of Bellenot, [Be], where $X$ is a Banach space with an unconditional basis $(e_n)_{n=1}^\infty$. There is no difficulty to give even a vector valued version. Given $X$ as above and a Köthe matrix $(a_{kn})_{k,n=1}^\infty$ (i.e. a matrix consisting of non negative numbers $a_{kn}$ such that $a_{k+1,n} \geq a_{kn}$ for all $k, n$ and such that for all $n$ there exists $k$ with $a_{kn} > 0$) and a sequence $(E_n,q_n)_{n=1}^\infty$ of Banach spaces, a vector valued $X$–Köthe sequence space $E$ is defined by

$$E := \{x = (x_n)_{n=1}^\infty \mid x_n \in E_n, \ p_k(x) := \| \sum_{n=1}^\infty a_{kn}q_n(x_n)e_n\|_X < \infty\}. $$

Clearly, $E$ is a Fréchet space. We assume now that $a_{kn} > 0$ for all $k$ and $n$. In this case all local Banach spaces $E_{p_k}$ are isometric. Using the same method as [BMS], Proposition 2.5, we can now prove

4.1. Proposition. An arbitrary bounded set $B \subset E$ is contained in a bounded set of the form

$$B_0 = \{x \mid \| \sum_{n=1}^\infty b_nq_n(x_n)e_n\|_X \leq 1\}$$

for some positive sequence $(b_n)_{n=1}^\infty$. Moreover, $E_{B_0}$ is isometric to the spaces $E_{p_k}$.

Proof. Let the numbers $r_k$, $k \in \mathbb{N}$, be such that $cB \subset r_kU_k$ and $r_k < r_{k+1}$ for all $k \in \mathbb{N}$, where $U_k := \{x \in E \mid p_k(x) \leq 1\}$ and $c$ is the unconditionality constant of $(e_n)$. We define for all $n \in \mathbb{N}$

$$b_n = \max_{1 \leq k \leq n} 2^{-k}r_k^{-1}a_{kn}.$$
Then the set $B_0$, defined as above, is easily seen to be bounded. We denote for all $k \in \mathbb{N}$

$$K_k := \{n \in \mathbb{N}, n \geq k \mid b_n = 2^{-k}r_k^{-1}a_{kn}, b_n \neq 2^{-t}r_t^{-1}a_{tn} \text{ for all } t, k < t \leq n\}.$$ 

Clearly, each $n \in \mathbb{N}$ belongs to exactly one $K_k$ so that $\mathbb{N}$ is a disjoint union of the sets $K_k$. We have for all $x = (x_n) \in \bigcap_k r_k U_k$

$$\| \sum_{n=1}^{\infty} b_n q_n(x_n) e_n \|_X$$

$$= \| \sum_{k=1}^{\infty} \sum_{n \in K_k} 2^{-k}r_k^{-1}a_{kn}q_n(x_n)e_n \|_X$$

$$\leq \sum_{k=1}^{\infty} 2^{-k}r_k^{-1} \| \sum_{n \in K_k} a_{kn}q_n(x_n)e_n \|_X$$

$$\leq c \sup_{k \in \mathbb{N}} \{ r_k^{-1} \| \sum_{n=1}^{\infty} a_{kn}q_n(x_n)e_n \|_X \}.$$ 

This means that $\bigcap_{k=1}^{\infty} r_k U_k \subset cB_0$.

The last statement in our Proposition is clear from definitions. \( \Box \)

So, in the case of vector valued $X$–Köthe sequence spaces with a continuous norm the answer to question (B1) is positive. These spaces are special examples of $T$–spaces defined in $[BD]$ or $(FG)$–spaces studied in $[BDT]$. It is an open problem if the answer to (B1) is positive also in these more general classes.

Below we make an attempt to a negative solution of (B1). This construction shows the obstructions one has when trying to solve (B1) in a positive direction.

We first concentrate on the following phenomenon. Let $A$ and $B$ be closed absolutely convex sets in $\mathbb{R}^n$ such that $\text{sp} (A) = \text{sp} (B) = \text{sp} (A \cap B) = \mathbb{R}^n$, and such that, say, $(\mathbb{R}^n, A)$ and $(\mathbb{R}^n, B)$ are isometric. Then many isometric invariants occurring in Banach space theory (like projection constants) may be very different for $(\mathbb{R}^n, A)$ and $(\mathbb{R}^n, A \cap B)$. We give an example which will be used to analyze problem (B1).

Note that if $\| \cdot \|_A$ and $\| \cdot \|_B$ are the Minkowski functionals associated with $A$ and $B$, then the Minkowski functional of $A \cap B$ equals $x \mapsto \max \{ \| x \|_A, \| x \|_B \}, x \in \mathbb{R}^n$.

We shall use tensor products and projection constants. Let $n \in \mathbb{N}$, $n = 2^k$ for some $k \in \mathbb{N}$, and let $(e_i)_{i=1}^{n}$ be the canonical basis of $\ell_2^n$. Let for all $i, j, 1 \leq i, j \leq n$, the numbers $\varepsilon_{ij} = 1$ or $-1$ be such that the matrix $n^{-1/2}(\varepsilon_{ij})_{i,j=1}^{n}$ is symmetric and orthogonal. (See, for example, [K1], 31.3.(5), p.429). Let $A : \ell_2^n \to \ell_2^n$ be an operator such that the matrix of $A$ with respect to the basis $(e_i)$ equals $n^{-1/2}(\varepsilon_{ij})$. Then also $(Ae_i)_{i=1}^{n}$ is an orthonormal basis of $\ell_2^n$.
We denote by $M$ the $n$–dimensional subspace of $\ell_2^{2n}$, spanned by the vectors $f_i := (e_i, Ae_i)$, $i = 1, \ldots, n$. Note that then also the vectors $g_i := (Ae_i, e_i)$ belong to $M$, since

$$g_i = n^{-1/2} \sum_j \varepsilon_{ji} f_j$$

(4.1)

because of the choice of the matrix $n^{-1/2}(\varepsilon_{ij})$.

In the following we endow $M$ with the norm of $\ell_\infty^n$, which we denote by $\| \cdot \|_\infty$. The norm of $(M, \| \cdot \|_\infty)^*$ is denoted by $\| \cdot \|_*$.

The following result is contained in [KTJ], Lemma 6. We give a different proof.

4.2. Lemma. The absolute projection constant of $(M, \| \cdot \|_\infty)$ is at least $\sqrt{n}/2$.

Proof. Let $(f_i^*)_{i=1}^n$ (resp. $(g_i^*)_{i=1}^n$) be the dual basis of $(f_i)$ (resp. $(g_i)$). We consider the tensor

$$z = \sum_{i=1}^n f_i^* \otimes f_i \in M^* \otimes M \subset M^* \otimes \ell_2^{2n}.$$ 

If $z = \sum a_i \otimes b_i$, $a_i \in M^*$, $b_i \in M$ for all $i$, is an arbitrary finite representation, we have, by [Pi], B.1.4,

$$\sum_i \|a_i\|_* \|b_i\|_\infty \geq \sum_i \langle b_i, a_i \rangle = \sum_{i=1}^n \langle f_i, f_i^* \rangle = n.$$ 

(4.2)

Hence, the $M^* \otimes \pi M$–norm of $z$ is at least $n$.

We claim that $z$ has also the representation

$$z = \sum_{i=1}^n f_i^* \otimes (0, Ae_i) + \sum_{i=1}^n g_i^* \otimes (Ae_i, 0)$$ 

(4.3)

in $M^* \otimes \ell_\infty^{2n}$. Indeed, the first sum in (4.3) can be written as

$$\frac{1}{2} \sum_i f_i^* \otimes ((e_i, Ae_i) + (-e_i, Ae_i))$$ 

(4.4)

and the second one as

$$\frac{1}{2} \sum_i g_i^* \otimes ((Ae_i, e_i) + (Ae_i, -e_i))$$

$$= \frac{1}{2} \sum_i \left( \frac{1}{\sqrt{n}} \sum_j \varepsilon_{ji} f_j^* \right) \otimes \left( \frac{1}{\sqrt{n}} \sum_j \varepsilon_{ji} (e_j, Ae_j) \right)$$

$$+ \frac{1}{\sqrt{n}} \sum_j \varepsilon_{ji} (e_j, -Ae_j))$$

$$= \frac{1}{2} \sum_i f_i^* \otimes (e_i, A_i) + \sum_i f_i^* \otimes (e_i, -Ae_i)$$ 

(4.5)
Note that here $g_i^* = n^{-1/2} \sum_j \varepsilon_{ji} f_j^*$ follows from (4.1); moreover, for an orthonormal, symmetric $k \times k$–matrix $(\sigma_{ij})_{i,j=1}^k$ we always have
\[
\sum_{i=1}^k a_i \otimes b_i = \sum_{i=1}^k \left( \sum_{j=1}^k \sigma_{ji} a_j \right) \otimes \left( \sum_{j=1}^k \sigma_{ji} b_j \right)
\]
for vectors $a_i, b_i$ in an arbitrary vector space. So (4.3) follows from (4.4) and (4.5).

We show that $\|f_i^*\| , \|g_i^*\| \leq 1$ for all $i$. Let $x \in M, \|x\|_\infty \leq 1$. Then $x$ has the representations $x = \sum_{i=1}^n a_i f_i = \sum_{i=1}^n b_i g_i,$ and it follows from the definitions of $f_i$ and $g_i$ that
\[
\sup |a_i| \leq 1, \quad \sup |b_i| \leq 1.
\]
Hence, $|\langle x, f_i^* \rangle| = |a_i| \leq 1,$
and, similarly, $|\langle x, g_i^* \rangle| \leq 1$. This proves that $\|f_i^*\|, \|g_i^*\| \leq 1$.

Using (4.3) we now get the estimate
\[
\|z\|_{M' \otimes \ell^2_\infty} \leq \sum_{i=1}^n \|f_i^*\||(0, Ae_i)\|
\]
\[
+ \sum_{i=1}^n \|g_i^*\||(Ae_i, 0)\| \leq 2n n^{-1/2} = 2n^{1/2}.
\]
This, combined with (4.2) implies $\lambda(M) \geq n^{1/2}/2$ (see for example [T], Lemma 4.1).  

We now denote by $\| \cdot \|_f$ and $\| \cdot \|_g$ the $\ell^n_\infty$–norms in $M$ with respect to the bases $(f_i)_{i=1}^n$ and $(g_i)_{i=1}^n$, respectively. For $1/\sqrt{n} < \alpha < \sqrt{n}$ we define the norm
\[
\|x\|^{(\alpha)} := \max \{ \|x\|_f, \alpha \|x\|_g \}
\]
in $M$.

4.3. Remark. The equality
\[
\|x\|^{(1)} = \|x\|_\infty
\]
holds for all $x \in M$.

Indeed, if $x = \sum_{i=1}^n x_i f_i,$ then we also have
\[
x = \sum_{i=1}^n x_i \sum_{j=1}^n n^{-1/2} \varepsilon_{ji} g_j
\]
\[
= \sum_{j=1}^n (\sum_{i=1}^n n^{-1/2} x_i \varepsilon_{ji}) g_j,
\]

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so that

$$
\|x\|^{(1)} = \max_{i,j} \{|x_i|, \sum_{k=1}^n n^{-1/2} x_k \varepsilon_{jk}\}. \quad (4.9)
$$

On the other hand, if we consider $x$ as an element of $\ell_\infty^{2n}$,

$$
x = \sum_{i=1}^n x_i (e_i, Ae_i) = \sum_{i=1}^n x_i (e_i, \sum_{j=1}^n n^{-\frac{1}{2}} \varepsilon_{ji} e_j)
$$

$$
= (x_1, \ldots, x_n, \sum_{i=1}^n n^{-1/2} x_i \varepsilon_{1i}, \ldots, \sum_{i=1}^n n^{-\frac{1}{2}} x_i \varepsilon_{ni}).
$$

This, together with (4.9), implies (4.7).

**4.4. Remarks.** We recall that every finite dimensional Banach space $M$ is isometric to a subspace of $c_0$. Moreover, if $X \subset c_0$ is a finite dimensional subspace, then the absolute projection constant of $\lambda(X)$ satisfies

$$
\lambda(X) = \lambda(X, c_0); \quad (4.10)
$$

see for example [TJ], Propositions 32.1 and 13.3. We also remark that if $X$ and $Y$ are finite dimensional Banach spaces such that $\lambda(X) > C$ and $d(X, Y) \leq D$, then

$$
\lambda(Y) > C/D. \quad (4.11)
$$

One can also easily prove that if $Y$ is a Banach space isomorphic to $c_0$, $d(Y, c_0) < D \geq 1$ and $X$ is a finite dimensional $C$–complemented subspace of $Y$, then

$$
\lambda(X) \leq CD^2. \quad (4.12)
$$

The following lemma is essentially known; see for example [Ju], Proposition 6.5.6 for the case $\alpha = 2$. For the sake of completeness we give the proof.

**4.5. Lemma.** Let $(E, (p_k)_{k=1}^\infty)$ be a Fréchet space. If $(r_k)_{k=1}^\infty$ is a sequence of positive numbers and $1 \leq \alpha < \infty$, then the set

$$
B := \{x \in E \mid \left(\sum_{k=1}^\infty (r_k p_k(x))^\alpha\right)^{1/\alpha} \leq 1\}
$$

is a Banach disc.

**Proof.** It is easy to see that $B$ is bounded and absolutely convex. That $B$ is closed can be seen as follows. Let $\psi : E \to E_{p_k}$ be the canonical mapping induced by the identity
operator on $E$. Let us identify $E$ in the canonical way with a subspace of $E_0 = \prod_{k=1}^{\infty} E_{p_k}$.

Then the set $B$ is identified with

$$B_0 = \{ x = (x_k)_{k=1}^{\infty} \mid \|x\| := \left( \sum_{k=1}^{\infty} (r_k p_k(x_k))^\alpha \right)^{1/\alpha} \leq 1, \exists y \in E \text{ such that } x_k = \psi_k y \text{ for all } k \}. $$

Assume that $y = (y_k) \in \overline{B_0} \subset E \subset E_0$ and that $(x^{(n)})_{n=1}^{\infty} \subset B_0$ is a sequence converging to $y$ in the topology of $E_0$. Since we have the product topology on $E_0$, the sequence $(x^{(n)})$ converges coordinatewise to $y$. Given $\varepsilon > 0$ and $m \in \mathbb{N}$ we thus find $n \in \mathbb{N}$ such that

$$\sum_{k=1}^{m} (r_k p_k(x_k^{(n)} - y_k))^\alpha \leq \varepsilon^\alpha, $$

where $x^{(n)} = (x_k^{(n)})_k$. Since $\|x^{(n)}\| \leq 1$, this implies $(\sum_{k=1}^{m} (r_k p_k(y_k))^\alpha)^{1/\alpha} < 1 + \varepsilon$, and since $m$ is arbitrary, $\|y\| \leq 1 + \varepsilon$. Hence, $y \in B_0$. □

Using the remarks above we now construct an example of a Fréchet space $E$ which is a projective limit of Banach spaces isomorphic to $c_0$ such that $E_B$ is not isomorphic to $c_0$ for ”many” bounded Banach discs $B$.

4.6. Proposition. There exists a separable Fréchet space $(E, (U_k)_{k=1}^{\infty}, (p_k)_{k=1}^{\infty})$ such that $E_{p_k} \cong c_0$ for all $k$, and such that $E_B \not\cong c_0$, if $B$ is a Banach disc satisfying the following:

$(\ast)$ $p_B$ is formed using the real interpolation method from the norms $p^{(\alpha)} := (\sum_{k=1}^{\infty} (r_k^{-1} p_k)^{\alpha})^{1/\alpha}$ and $p^{(\beta)} := (\sum_{k=1}^{\infty} (s_k^{-1} p_k)^{\beta})^{1/\beta}$ where $1 \leq \alpha \leq \beta \leq \infty$, $(r_k)$ and $(s_k)$ are arbitrary positive increasing sequences, $r_k \leq s_k$ for all $k$, and $\sum_{k=1}^{\infty} r_k^{-\alpha} \leq 1$.

To be more exact, we have $p_{\alpha} \geq p_{\beta}$ and thus the Banach spaces $E_\alpha$ and $E_\beta$ corresponding to the Banach discs $\{ x \mid p^{(\alpha)}(x) \leq 1 \}$ and $\{ x \mid p^{(\beta)}(x) \leq 1 \}$, respectively, satisfy $E_\alpha \subset E_\beta$ with a continuous embedding, and, hence, they form a Banach interpolation couple (see [TJ], §3). So, we can use the real interpolation method to produce norms $p_B$ defined on subspaces of $E_\alpha + E_\beta = E_\beta$. We always have $p_B \geq p_\beta$ so that the closed unit ball of $p_B$ is in fact a bounded disc. However, we do not claim that every $B$ produced in such a way is closed in $E$.

Clearly, Proposition 4.6 contains as a special case the Banach discs

$$B = \{ x \mid p(x) = \left( \sum_{k=1}^{\infty} (r_k^{-1} p_k(x_k))^q \right)^{1/q} \leq 1 \}, $$

where $1 \leq q \leq \infty$; take $\alpha = \beta = q$ and $(s_k) = (r_k)$ in Proposition 4.6. Note that if $q = \infty$, we have $B = \bigcap_{k=1}^{\infty} r_k U_k$. 

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In fact, the only thing we need to assume on \( B \) is that certain projections on \( E_B \) are well enough bounded. Unfortunately, it is hard to describe this condition exactly before constructing the space \( E \). This is why we use the interpolation method to present our result. We refer to the proof.

**Proof.** We denote \( \mathbb{N}_d = \{2^k | k \in \mathbb{N}\} \). For all \( n \in \mathbb{N}_d \), we denote by \( M_n \) the \( n \)-dimensional vector space \( M \) constructed above in this section, and by \( (f_i^{(n)})_{i=1}^n \) and \( (g_i^{(n)})_{i=1}^n \) the bases \( (f_i)_{i=1}^n \) and \( (g_i)_{i=1}^n \) of \( M \). We also denote by \( \| \cdot \|_{f,n} \) and \( \| \cdot \|_{g,n} \) the norms \( \| \cdot \|_f \) and \( n^{1/2} \cdot \| \cdot \|_g \) on \( M_n \). Note that then \( \| x \|_{f,n} \leq \| x \|_{g,n} \) for all \( x \in M_n \), see (4.1) and the definition above (4.6).

Next we choose a bijection \( \phi: \mathbb{N}_d \to \mathbb{N} \times \mathbb{N} \) and define for all \( k \in \mathbb{N} \) the norms

\[
p_k((x_n)_{n \in \mathbb{N}_d}) = \sup_{n \in \mathbb{N}_d} \{p_{k,n}(x_n)\} \tag{4.13}
\]

where \( (x_n) \in \bigoplus_{n \in \mathbb{N}_d} M_n \) and

\[
p_{k,n}(x_n) := \begin{cases} \| x_n \|_{f,n}, & \text{if } \pi_1(\phi(n)) \geq k \\ \| x_n \|_{g,n}, & \text{if } \pi_1(\phi(n)) < k; \end{cases} \tag{4.14}
\]

here \( \pi_i, i = 1, 2 \), denote the canonical projections of \( \mathbb{N} \times \mathbb{N} \) onto the first and second coordinate spaces, respectively. Taking the completion of \( \bigoplus_{n \in \mathbb{N}_d} M_n \) with respect to the topology determined by the norms \( p_k \) we get a separable Fréchet space \((E, (U_k)_{k=1}^\infty, (p_k)_{k=1}^\infty)\). Clearly, each local Banach space \( E_{p_k} \) is isometric to \( c_0 \).

Assume now that \( B \subset E \) is a Banach disc satisfying (*) for some \( \alpha \) and \( \beta \) and for some sequences \((r_k)\) and \((s_k)\). We show that \( E_B \) has an infinite family \( (X_m)_{m=1}^\infty \) of finite dimensional 1–complemented subspaces such that

\[
\sup_{m \in \mathbb{N}} \{\lambda(X_m)\} = \infty. \tag{4.15}
\]

We choose for all \( m \in \mathbb{N} \) the number \( K \) such that \( (\sum_{k=K}^\infty r_k^{-\alpha})^{-1/\alpha} > 4ms_1 \), and then \( N \in \mathbb{N}_d \) such that \( \pi_1(\phi(N)) = K - 1 \) and \( N^{1/2} > (\sum_{k=K}^\infty r_k^{-\alpha})^{-1/\alpha} \) and \( N > s_1^{-2} s_2^2 \). Let \( P \) be the natural projection from \( \bigoplus_n M_n \) onto \( M_N \). Now \( P \) is continuous with the operator norm equal to 1 when \( \bigoplus_n M_n \) is endowed with either of the norms \( p^{(\alpha)} \) or \( p^{(\beta)} \). Since \( p_B \) is formed by interpolation, we see that \( \| P \| = 1 \) also as an operator in \( E_B \) (cf. [TJ], §3), so that \( M_N \) is a 1–complemented subspace of \( E_B \). (This is the only point where we need to use some extra assumption on \( B \).)

Using the properties of the sequence \((r_k)\) and the definition of the norms \( p_k \) we get

\[
p^{(\alpha)}|_{M_N} = (\sum_{k=1}^{K-1} (r_k^{-1} p_k|_{M_N})^{\alpha} + \sum_{k=K}^{\infty} (r_k^{-1} p_k|_{M_N})^{\alpha})^{1/\alpha}
\]

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\[
\leq \left( \sum_{k=1}^{K-1} (r_k^{-1} \cdot \|f, N\|^\alpha) + \sum_{k=K}^\infty (r_k^{-1} \cdot \|g, N\|^\alpha) \right)^{1/\alpha}
\]
\[
\leq \left( R_1^\alpha \|f, N\|^\alpha + R_K^\alpha \|g, N\|^\alpha \right)^{1/\alpha}
\]
\[
\leq 2 \max \left\{ R_1 \|f, N\|, R_K \|g, N\| \right\},
\]

where \( R_k' = \left( \sum_{k=k'}^\infty r_k^{-\alpha} \right)^{1/\alpha} \). On the other hand, obviously

\[
\max \left\{ s_1^{-1} \|f, N\|, s_K^{-1} \|g, N\| \right\} \leq p^{(\beta)} |_{MN}.
\]

By the properties of interpolation, we have \( p_\alpha |_{MN} \geq p_B |_{MN} \geq p_\beta |_{MN} \). Combining this with (4.16) and (4.17) and the assumption \( R_1 \leq 1 \) (see (*)) yields

\[
2 \max \left\{ \|x\|_{f, N}, R_K \|x\|_{g, N} \right\}
\]
\[
\geq p_B(x) \geq \max \left\{ s_1^{-1} \|x\|_{f, N}, s_K^{-1} \|x\|_{g, N} \right\}
\]

for \( x \in M_N \), and, using the notation (4.6) and the definitions of \( \cdot \cdot f, N \) and \( \cdot \cdot g, N \), we get

\[
2R_K N^{1/2} \|x\|^{(1)} \geq 2 \max \left\{ \|x\|_{f, N}, R_K \|x\|_{g, N} \right\} \geq p_B(x)
\]
\[
\geq s_1^{-1} \|x\|^{(N^{1/2} s_1 s_K^{-1})} \geq s_1^{-1} \|x\|^{(1)}
\]

for all \( x \in M_N \), by the choice of \( N \). Hence,

\[
d\left( (M_N, p_B), (M_N, \| \cdot \|^{(1)}) \right) \leq 2s_1 R_K N^{1/2}.
\]

On the other hand, by Lemma 4.2 and Remark 4.3, \( \lambda((M_N, \| \cdot \|^{(1)})) \geq \sqrt{N}/2 \). So, by the remark (4.11) and the choice of \( K \),

\[
\lambda((M_N, p_B)) \geq s_1^{-1} R_K^{-1}/4 \geq m,
\]

which proves (4.15).

By the Remark 4.4 it is now clear that \( E_B \) cannot be isomorphic to \( c_0 \), since \( E_B \) contains 1–complemented subspaces, the absolute projection constants of which can be chosen arbitrarily large.

4.7. Remark. The last conclusion in the proof of Proposition 4.6 even shows that the local distance of \( c_0 \) and \( E_B \) is not bounded. Hence, the counterexample given in Proposition 4.6. is much stronger than the trivial counterexample in the beginning of this section.
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