PARAMETRIZED SPECTRA, MULTIPLICATIVE THOM SPECTRA, AND THE TWISTED UMKEHR MAP

MATTHEW ANDO, ANDREW J. BLUMBERG, AND DAVID GEPNER

ABSTRACT. We introduce and study a general theory of objects parametrized by spaces, in the setting of ∞-categories. This framework specializes to give an ∞-categorical model of parametrized spectra, and we apply these foundations to study the multiplicative properties of the generalized Thom spectrum functor. As part of this work we study the Picard space of a presentable monoidal ∞-category. We sharpen classical results due to Lewis about the multiplicative properties of the Thom isomorphism. Our main application is the construction of twisted Umkehr maps on twistings of generalized cohomology theories.

CONTENTS

1. Introduction 1
2. The base-change functors: $f_*$ and its adjoints $f_!$ and $f_*$ 6
3. Interlude on ∞-operads 9
4. The closed monoidal structure 12
5. Wirthmüller contexts 14
6. The proper pushforward $p_*$ and its right adjoint $p_!$ 15
7. Parametrized spaces, spectra, and modules 17
8. Picard ∞-groupoids of presentable monoidal ∞-categories 23
9. Multiplicative structures on generalized Thom spectra 27
10. Twisted cohomology theories and the twisted Umkehr map 28
References 37

1. INTRODUCTION

In recent work with Hopkins and Rezk [1], we introduced an ∞-categorical approach to parametrized spaces and spectra and showed that it provides a useful context in which to study Thom spectra and orientations. If $S$ is a Kan complex and $S$ is the ∞-category of spectra, then our model for the ∞-category of spectra parametrized by $S$ is just the ∞-category $\text{Fun}(S^{op}, S)$ of $S$-valued presheaves on $S$. Conceptually this approach exhibits the ∞-category $S$ as the “classifying space” for bundles of spectra.

In this paper we develop these ideas to give a complete theory of parametrized objects of a general ∞-category $\mathcal{C}$. We frame this construction in terms of a functor which associates to any space $S$ the ∞-category of objects of $\mathcal{C}$ parametrized over $S$. We show that this functor satisfies descent and therefore we think of it as a “stack

Ando was supported in part by NSF grant DMS-1104746. A. J. Blumberg was supported in part by NSF grant DMS-0906105.
of $\infty$-categories on spaces”. This geometric perspective allows us to give a general treatment of base-change. Following the seminal treatment of parametrized spectra in May-Sigurdsson [27] and motivated by Grothendieck’s “six functor” formalism, we regard the study of base-change as the cornerstone of any satisfactory theory of parametrized objects.

More precisely, we show in Section 2 that $C$ determines a limit preserving functor

$$C_{/(-)} : \mathcal{T}^{\text{op}} \longrightarrow \mathcal{C} \mathbf{at}_\infty$$

from the (opposite of the) $\infty$-category of spaces to the $\infty$-category of (not necessarily small) $\infty$-categories. The value $C_{/S}$ at the space $S$ is equivalent to the $\infty$-category of $C$-valued presheaves $\text{Fun}(S^{\text{op}}, C)$ on (the $\infty$-groupoid of) $S$. In particular, associated to maps of spaces $f : S \rightarrow T$ we have base-change functors

$$f^* : C_{/T} \longrightarrow C_{/S},$$

which collectively assemble into a contravariant functor from the $\infty$-category of spaces to the $\infty$-category of $\infty$-categories. Moreover, we show that this functor satisfies descent, which in this context means merely that it preserves limits. Although we do not assume that $C_{/S}$ is an $\infty$-groupoid, because $C_{/(-)}$ satisfies descent we refer to it as a stack on the $\infty$-category $\mathcal{T}$ of spaces.

In practice it is important to know when each $f^*$ has a left adjoint $f_!$ and a right adjoint $f^*$, giving a sort of “three functor formalism”: this is easily seen to be the case if $C$ is a presentable $\infty$-category. Let $\text{Pr}^{L,R}$ denote the subcategory of $\mathcal{C} \mathbf{at}_\infty$ consisting of the presentable $\infty$-categories and the functors which are both left and right adjoints.

**Theorem 1.1.** A presentable $\infty$-category $C$ uniquely determines a stack of presentable $\infty$-categories and left and right adjoint functors

$$C_{/(-)} : \mathcal{T}^{\text{op}} \longrightarrow \text{Pr}^{L,R}$$

on $\mathcal{T}$ whose value at the space $S \in \mathcal{T}$ is equivalent to the $\infty$-category $\text{Fun}(S^{\text{op}}, C)$ of $C$-valued presheaves on $S$.

Second, we develop the multiplicative theory of parametrized objects. Among other things this leads us in Section 4 to a “five functor formalism”. Let $\mathcal{O}^{\otimes}$ be an $\infty$-operad and let $C^{\otimes}$ be a presentable $\mathcal{O}$-monoidal $\infty$-category. Then there is an $\mathcal{O}$-monoidal $\infty$-category $C^{\otimes}_{/S}$ with underlying $\infty$-category $C_{/S}$.

**Theorem 1.2.** There is a unique stack of presentable $\mathcal{O}$-monoidal $\infty$-categories $C^{\otimes}_{/(-)}$ on $\mathcal{T}$ whose value on the space $S \in \mathcal{T}$ is the $\mathcal{O}$-monoidal $\infty$-category $C^{\otimes}_{/S}$ of $C^{\otimes}$-valued presheaves on $S$.

If we assume that $\mathcal{O}^{\otimes}$ comes equipped with a fixed map $E^{\otimes}_1 \rightarrow \mathcal{O}^{\otimes}$, then any $\mathcal{O}^{\otimes}$-monoidal $\infty$-category $C^{\otimes}$ has a distinguished tensor structure $\otimes$. We write $\otimes_S$ for the resulting tensor structure on $C^{\otimes}_{/S}$. For each space $S$ and each object $X \in C_{/S}$, the “left multiplication by $X$” functor

$$X \otimes_S (-) : C_{/S} \longrightarrow C_{/S}$$

admits a right adjoint

$$F_S(X, -) : C_{/S} \longrightarrow C_{/S},$$

giving us two additional families of functors.
These five functors interact in standard ways. In the case of a symmetric monoidal \(\infty\)-category, the induced structure is called a “Wirthmüller Context” in [13]. Suppose that the map \(E \otimes f \to O \otimes f\) factors through \(E \otimes \infty\), which is to say that \(\mathcal{E} \otimes\) is a symmetric monoidal presentable \(\infty\)-category. Let 
\[ \text{CAlg}(P_r^L)^R \subset \text{CAlg}(P_r^L) \]
denote the subcategory of \(\text{CAlg}(P_r^L)^R\) spanned by all the objects (the symmetric monoidal presentable \(\infty\)-categories) but only those symmetric monoidal functors which are also right adjoints. Our generalization of a Wirthmüller context is a stack on \(T\) with values in \(\text{CAlg}(P_r^L)^R\). We discuss the relationship in Section 5.

We also address an analogue of the algebro-geometric notion of “proper pushforward” which occurs in versions of Grothendieck’s “six functor formalism”. Indeed, by analogy, a map \(f: S \to T\) in the \(\infty\)-category of spaces is proper if its (homotopy) fibers are (homotopically) compact, in which case, again by analogy, one might expect that the pushforward functor \(f_*: \mathcal{C}/S \to \mathcal{C}/T\) admits a right adjoint \(f^!\). In Section 6 we show this to be the case when \(\mathcal{C} \otimes\) is the symmetric monoidal \(\infty\)-category of \(R\)-modules for an \(E_\infty\)-ring spectrum \(R\), and so we also obtain a sort of “six functor formalism”.

**Theorem 1.3.** Let \(R\) be an \(\mathbb{E}_\infty\)-ring spectrum, let \(\mathcal{C} = \text{Mod}_R\) denote the \(\infty\)-category of \(R\)-modules, and let \(p: S \to T\) be a proper map. Then the pushforward \(p_*: \mathcal{C}/S \to \mathcal{C}/T\) admits a right adjoint \(p^!: \mathcal{C}/T \to \mathcal{C}/S\).

The remainder of the paper is concerned with a number of applications of the general theory. First, we show in Section 7 that when \(\mathcal{C} \otimes\) is the symmetric monoidal \(\infty\)-category of (based) spaces or spectra, the resulting stacks \(\mathcal{C}[\dash] \otimes\) recover the Wirthmüller context of May-Sigurdsson [27].

**Theorem 1.4.** Let \(\mathcal{C} \otimes\) be the symmetric monoidal model category of spaces, based spaces, or spectra. Then for each topological space \(S\), there exists an equivalence 
\[ N(\mathcal{C})[W^{-1}\mathcal{C}]_{\Pi_{\infty}} / S \simeq N(\mathcal{C}_S)[W^{-1}\mathcal{C}_S] \]
where \(\Pi_{\infty}\) denotes the \(\infty\)-groupoid (singular complex) functor, \(N(-)[W^{-1}\mathcal{C}]\) Lurie’s symmetric monoidal version of the Dwyer-Kan simplicial localization, and \(\mathcal{C}_S\) the May-Sigurdsson symmetric monoidal model category of objects of \(\mathcal{C}\) parametrized over \(S\). Moreover, the equivalences above are compatible with all of the base-change and tensor functors; i.e., the respective Wirthmüller contexts are equivalent.

We next give several applications of our theory to Thom spectra. Let \(R\) be an \(E_n\)-ring spectrum \((n > 1)\) and let \(\text{Mod}_R\) be the \(\infty\)-category of right \(R\)-modules. Within \(\text{Mod}_R\) is the full subgroupoid spanned by the invertible \(R\)-modules, \(\text{Pic}_R\). Given a space (Kan complex) \(X\) and a map \(f: X \to \text{Pic}_R\) (which we think of as classifying a bundle of invertible \(R\)-modules over \(X\)), we defined in [1, 2] the Thom spectrum of \(f\) to be the colimit \(Mf\) of the composite map 
\[ X \xrightarrow{f} \text{Pic}_R \to \text{Mod}_R. \]
Here it is necessary to take this colimit in \(\text{Mod}_R\), since \(\text{Pic}_R\) is not closed under colimits unless \(R\) is trivial. The Thom spectrum \(Mf\) is equivalent to the pushforward along the projection \(p: X \to \ast\) of the bundle of invertible \(R\)-modules over \(X\) classified by \(f\).
Remark 1.5. We proved in [1] that when restricted to the full subgroupoid $BGL_1 \mathcal{S}$ of $\text{Pic}_\mathcal{S}$ spanned by the (unshifted) sphere spectrum $\mathcal{S}$, this definition agrees with the classical Lewis-May construction of the Thom spectrum over $BF$ [22, 19].

To understand the multiplicative properties of this definition, in Section 8 we give a general treatment of the Picard space associated to a stable monoidal $\infty$-category. Suppose given a map of $E_1^\otimes \to O^\otimes$, and let $\mathcal{R}^\otimes$ be an $O$-monoidal $\infty$-category. We define $\text{Pic}(\mathcal{R})$ to be the maximal grouplike $\infty$-groupoid in the $O$-monoidal $\infty$-category of invertible objects of $\mathcal{R}^\otimes$. We then have the following results.

Theorem 1.6. For any $\infty$-operad $O^\otimes$ equipped with a map from $E_1^\otimes$, the Picard $\infty$-groupoid defines a functor

$$\text{Pic}: \text{Alg}_O^\otimes(\text{Pr}^L) \to \text{Alg}_O^\otimes(\mathcal{T})$$

that is right adjoint to the free functor

$$\mathcal{T}[-]: \text{Alg}_O^\otimes(\mathcal{T}) \to \text{Alg}_O^\otimes(\text{Pr}^L).$$

Here $\text{Alg}_O^\otimes(-)$ denotes the $\infty$-category of $O$-algebra objects, $\text{Alg}_O^\otimes(\mathcal{T})$ its full subcategory of grouplike algebra objects, and $\mathcal{T}[-]$ is the “free $\mathcal{T}$-module functor”, i.e. the covariant functor induced from $\mathcal{T}(\mathcal{T})$ and the left adjoints of the restrictions.

Remark 1.7. The generality in which we work permits a categorification of Pic. Let $R$ be an $E_n^\otimes$-ring spectrum. The Brauer spectrum of $R$ is the delooping of the grouplike $E_n^\infty$-space

$$\text{Br}_R = \text{Pic}(\text{Mod}_{\text{Mod}_R^L}(\text{Pr}^L)),$$

the Picard $\infty$-groupoid of the symmetric monoidal $\infty$-category $\text{Mod}_{\text{Mod}_R^L}(\text{Pr}^L)$ of (compactly generated) $\text{Mod}_R$-modules in $\text{Pr}^L$. This construction has been studied in detail by the third author [3, 14]. As an indication of the usefulness of this definition, we sketch in section 8.1 how the theory of Thom spectra in this setting recovers the definitions and main results of Douglas’ work on “twisted parametrized spectra”, defined in terms of sections of sections of rank-one bundles of stable categories [12].

Lurie has recently written down a proof of a conjecture of Mandell [21, 6.3.5.17] which in part shows that $E_n$-algebras admit $E_{n-1}$-monoidal module categories (with the latter notion defined in terms of the symmetric monoidal structure on presentable $\infty$-categories). Thus we can apply the Theorem in the case that $\mathcal{R}^\otimes = \text{Mod}_R^\otimes$ for an $E_n$-ring spectrum $R$ ($n > 1$). In Section 9 we use this idea to investigate the multiplicative properties of Thom spectra. The key point is the following.

Theorem 1.8. The resulting functor of $E_{n-1}$-monoidal presentable $\infty$-categories

$$\mathcal{T}/\text{Pic}_R \to \text{Mod}_R$$

arising from the counit of the adjunction of Theorem 1.6 is the generalized Thom spectrum functor.

As a corollary, we generalize Lewis’ theorem about Thom spectra that are ring spectra from the case of $S$ to an arbitrary $E_n$-ring spectrum.

Corollary 1.9. Let $R$ be an $E_n$-ring spectrum, with $n > 1$. Then $\text{Pic}_R$ is an $E_{n-1}$-space, and if $f: X \to \text{Pic}_R$ is $E_m$-monoidal for some $m < n$, then the Thom spectrum $Mf$ is an $E_m$-ring spectrum.
We also study the multiplicative properties of the Thom isomorphism. Lewis showed that an $E_n$-orientation gives rise to an $E_n$ Thom isomorphism “up to homotopy” [19, 7.4], and various authors have observed that in the $E_\infty$ setting the Thom isomorphism is a map of commutative ring spectra (e.g., [6]). We sharpen Lewis’ result as follows:

**Theorem 1.10.** Let $R$ be an $E_n$-ring spectrum, $n > 1$, and let $f: X \to \text{Pic}_R$ be an $E_m$-monoidal map for some $m < n$. Suppose that $Mf$ admits an $E_m$-orientation over a spectrum $R$, i.e., an $E_m$-algebra map $Mf \to R$. Then the composite

$$Mf \to \Sigma^\infty_+ X \wedge Mf \to \Sigma^\infty_+ X \wedge R$$

is an equivalence of $E_m$-ring spectra.

In Section 10, we use our treatment of parametrized spectra and our multiplicative theory of Thom spectra to study twisted cohomology theories and particularly twisted Umkehr maps. Recall from [2, 1] that given a map $\alpha: X \to \text{Pic}_R$, regarding such a map $\alpha$ as classifying a twisted form of the trivial $R$-line bundle over $X$, we can consider the associated $R$-module Thom spectrum $M\alpha$ to be the $\alpha$-twisted and $R$-stable homotopy type of $X$.

**Definition 1.11.** Let $R$ be an $E_n$-ring spectrum, $n > 0$. The $\alpha$-twisted $R$-homology and $R$-cohomology groups of $X$ are given by

$$R_\alpha^n(X) = \pi_0 \text{map}_R(M\alpha, \Sigma^n R)$$

$$R_n(\alpha) = \pi_0 \text{map}_R(\Sigma^n R, M\alpha) \cong \pi_n M\alpha.$$

Here $\text{map}_R(-,-)$ denotes the mapping space in the $\infty$-category $\text{Mod}_R$ of right $R$-modules.

**Remark 1.12.** We discuss the relationship of this definition to that of Atiyah and Segal in Section 10.

We now apply this in geometric contexts. For convenience, we switch to using exponential notation for Thom spectra; e.g., given a twist $\alpha: X \to \text{Pic}_R$ the Thom spectrum will be written $X^{\alpha}$. Now let $X$ be a compact manifold with tangent bundle $T$. The Pontryagin-Thom construction gives a stable map

$$\text{PT}(X): S \to X^{-T}$$

which realizes geometrically the stable map

$$S \to DX = F(\Sigma^\infty_+ X, S)$$

dual to the map $X \to \ast$. If $f: E \to B$ is a fiber bundle of compact manifolds with tangent bundle along the fibers $T_f$, then the Pontryagin-Thom construction gives rise to a stable map

$$\text{PT}(f): B_+ \to E^{-T_f}.$$  

If $R$ is a ring spectrum, then we get a map

$$R^*(E^{-T_f}) \to R^*(B_+);$$

composing with a Thom isomorphism $R^{+d}(E_+) \cong R^* E^{-T_f}$, we obtain an Umkehr map

$$R^{+d}(E_+) \to R^*(B_+).$$
Recently it has become important to consider the following generalization of these constructions (see for example [17, 29, 9]). A twist $\alpha : B \to \text{Pic}_R$ gives rise to a twist

$$E \xrightarrow{\alpha f} \text{Pic}_R,$$

and using the multiplicative structure of $\text{Pic}_R$ we can form the generalized $R$-module Thom spectrum $E^{-T_f+\alpha f}$. We construct a twisted Pontryagin-Thom map

$$(1.15) \quad \text{PT}(f, \alpha) : B^\alpha \to E^{-T_f+\alpha f}.$$

Given an orientation $R^* + d(E) \cong R^*(E^{-T_f+\alpha f})$, we then obtain a twisted Umkehr map

$$R^* + d(E) \to R^*(B)^\alpha.$$

The key idea is to show that the Pontryagin-Thom map $\text{PT}(f)$ in the form (1.14) arises from a two-step process: first apply the Pontryagin-Thom construction (1.13) fiberwise to obtain a map of spectra over $B$

$$\text{PT}(f/B) : S_B \to D(f/B)$$

from the sphere spectrum over $B$ to the duals of the fibers of the map $f$, then apply the push forward $p_!$ along the map $p : B \to \ast$ to obtain the map $\text{PT}(f)$.

Then given a map $\alpha : B \to \text{Pic}_R$, we can twist by $\alpha$ to get the map (of $R$-module spectra over $B$)

$$\text{PT}(f/B) \wedge_B \alpha : \alpha \to D(f/B) \wedge_B \alpha.$$

Applying the pushforward $p_!$ associated to $p : B \to \ast$ yields the map $\text{PT}(f, \alpha)$.

**Remark 1.16.** The idea that the Pontryagin-Thom map (1.14) arises from a fiberwise construction goes all the way back to the origins of the Umkehr map, participating as it does in the “families” index theorems of Atiyah and Singer [5]. It is also explicit in Becker and Gottlieb’s classic paper [7], for example. May and Sigurdsson have a beautiful exposition of a fiberwise construction in the setting of parametrized spectra in [27].

### 1.1. Acknowledgments.

Our debt to Peter May and Johann Sigurdsson should be obvious. We thank them also for many useful conversations and correspondence. We thank David Ben-Zvi, Dan Freed, Jacob Lurie, and Mike Mandell for helpful conversations and encouragement. Finally, we wish to thank our collaborators Mike Hopkins and Charles Rezk, without whom this project would not exist.

### 2. The base-change functors: $f^*$ and its adjoints $f_!$ and $f_*$

We begin by summarizing the central desiderata for a theory of “objects of $\mathcal{C}$ parametrized over $S$”, where $\mathcal{C}$ is an $\infty$-category and $S$ is a space or, equivalently (as we are only concerned with the weak homotopy type of $S$), a Kan complex. Specifically, we develop an analogue of what is known in algebraic geometry as a “six functor formalism” and in equivariant stable homotopy theory as a “Wirthmüller context”. The full scope of these formalisms, however, require substantial assumptions on the $\infty$-category $\mathcal{C}$, and it is perhaps instructive to begin with a significantly more general and less structured situation.

Given an $\infty$-category $\mathcal{C}$ and a Kan complex $S$ (which we think of as corresponding to a space via the singular complex), we wish to define an $\infty$-category $\mathcal{C}_S$ of objects
of \( \mathcal{C} \) parametrized over \( S \). Of course, we should not think of \( S \) as being fixed; rather, we require restriction (a.k.a. pullback or base-change) functors

\[ f^* : \mathcal{C}/T \rightarrow \mathcal{C}/S \]

for each map of Kan complexes \( f : S \rightarrow T \).

Moreover, these must be compatible with composition. Given a 2-simplex which exhibits \( g \circ f : S \rightarrow U \) as a composite of \( f : S \rightarrow T \) followed by \( g : T \rightarrow U \), we require a natural 2-simplex exhibiting \((g \circ f)^* : \mathcal{C}/U \rightarrow \mathcal{C}/S\) as a composite of \( g^* : \mathcal{C}/U \rightarrow \mathcal{C}/T \) followed by \( f^* : \mathcal{C}/T \rightarrow \mathcal{C}/S \), and so on for all higher dimensional simplices. In other words, we require a functor

\[ \mathcal{T}^{\text{op}} \rightarrow \hat{\text{Cat}}_{\infty} \]

from the \( \infty \)-category \( \mathcal{T}^{\text{op}} \) of spaces to the \( \infty \)-category \( \hat{\text{Cat}}_{\infty} \) of (not necessarily small) \( \infty \)-categories.

Lastly, this functor must satisfy the usual sort of descent conditions. Specifically, given maps \( f : S \rightarrow T \) and \( g : S \rightarrow U \), the restriction functors \( f^* : \mathcal{C}/T \rightarrow \mathcal{C}/S \) and \( g^* : \mathcal{C}/U \rightarrow \mathcal{C}/S \) induce a pullback square

\[
\begin{array}{ccc}
\mathcal{C}/T \coprod_S U & \rightarrow & \mathcal{C}/U \\
\downarrow & & \downarrow \\
\mathcal{C}/T & \rightarrow & \mathcal{C}/S
\end{array}
\]

from the \( \infty \)-category of objects parametrized over the pushout \( T \coprod_S U \) to the pullback of the \( \infty \)-categories of parametrized objects, and we require that this map is an equivalence. Similarly, given a family of spaces \( S_\lambda \) indexed by some (possibly infinite) set \( \Lambda \) with coproduct \( S = \coprod_\Lambda S_\lambda \), the inclusions \( i_\lambda : S_\lambda \rightarrow S \) induce a map

\[ \mathcal{C}/S \rightarrow \coprod_\lambda \mathcal{C}/S_\lambda \]

which also must be an equivalence. Together, these two conditions amount to saying that the \( \infty \)-category \( \mathcal{C}/S \) of objects of \( \mathcal{C} \) parametrized over \( S \) is local on the base space \( S \). Finally, we have the obvious normalization condition: if \( S = * \) is the terminal space, then we must have an equivalence \( \mathcal{C}/* \simeq \mathcal{C} \).

We can make this precise using the language of stacks. For us, a stack will mean a contravariant functor from an \( \infty \)-topos to the \( \infty \)-category \( \hat{\text{Cat}}_{\infty} \) of (not necessarily small) \( \infty \)-categories which satisfies descent in the sense that it preserves limits. In the present paper we only consider the \( \infty \)-topos \( \mathcal{T}^{\text{op}} \) of spaces, though many interesting generalizations are possible, e.g. the \( \infty \)-topos of \( G \)-spaces for a topological group \( G \). We plan to pursue some of these generalizations in future work.

**Proposition 2.1.** Evaluation at the point induces an equivalence between \( \hat{\text{Cat}}_{\infty} \) and the full subcategory of the (very large) \( \infty \)-category \( \text{Fun}(\mathcal{T}^{\text{op}}, \hat{\text{Cat}}_{\infty}) \) spanned by the stacks (i.e. the limit-preserving functors).

**Proof.** The \( \infty \)-category \( \mathcal{T}^{\text{op}} \) is freely generated under limits by the initial object \( * \) and the \( \infty \)-category \( \hat{\text{Cat}}_{\infty} \) admits all small limits. \( \square \)

**Definition 2.2.** Let \( \mathcal{C} \) be a (possibly large) \( \infty \)-category and let \( S \) be a Kan complex. The \( \infty \)-category \( \mathcal{C}/S \) of objects of \( \mathcal{C} \) parametrized over \( S \) is the \( \infty \)-category \( \text{Fun}(S^{\text{op}}, \mathcal{C}) \) of \( \mathcal{C} \)-valued presheaves on \( S \).
Proposition 2.3. Let $\mathcal{C}$ be an $\infty$-category. Then there exists a unique stack of (not necessarily small) $\infty$-categories

$\mathcal{C}_{(-)} : \mathcal{T}^{\text{op}} \to \hat{\text{Cat}}_{\infty}$

on $\mathcal{T}$ whose value at $S \in \mathcal{T}$ is equivalent to the $\infty$-category $\mathcal{C}/S$ of $\mathcal{C}$-valued presheaves on $S$.

Proof. By proposition 2.1, to specify a limit-preserving functor $F : \mathcal{T}^{\text{op}} \to \text{Cat}_{\infty}$, it is enough to specify the image of the initial object $*$, which we take to be $\mathcal{C}$. For a given space $S$, we have that $\text{Fun}(S^{\text{op}}, \mathcal{C}) \simeq \lim_S \mathcal{C}$, which shows that the value of $F$ on $S$ is equivalent to $\mathcal{C}/S$. □

In practice, we require more structure on $\mathcal{C}$, namely, that $\mathcal{C}$ is presentable, which is to say that $\mathcal{C}$ is the freely generated under $\kappa$-filtered colimits by the small subcategory $\mathcal{C}_{\kappa}$ of $\kappa$-compact objects (which itself admits $\kappa$-small colimits). Moreover, maps between presentable $\infty$-categories are typically taken to be colimit preserving, in which case it is formal that they admit right adjoints. Thus, when $\mathcal{C}$ is presentable, we require that the base-change functors $f_* : \mathcal{C}/T \to \mathcal{C}/S$ admit right adjoints $f^* : \mathcal{C}/S \to \mathcal{C}/T$.

Proposition 2.4. Let $\mathcal{C}$ be a presentable $\infty$-category. Then there exists a unique stack of presentable $\infty$-categories and left-adjoint functors

$\mathcal{C}_{/(-)} : \mathcal{T}^{\text{op}} \to \text{PrL}_{\mathcal{T}}$

on $\mathcal{T}$ whose value at the space $S \in \mathcal{T}$ is equivalent to the $\infty$-category $\mathcal{C}/S$ of $\mathcal{C}$-valued presheaves on $S$.

Proof. We must lift $\mathcal{C}_{/(-)}$ through the projection $\text{PrL} \to \hat{\text{Cat}}_{\infty}$. By the adjoint functor theorem, it is enough to show that the base-change functor $f^* : \mathcal{C}/T \to \mathcal{C}/S$ preserves colimits. Since colimits in functor $\infty$-categories are computed objectwise, this is immediate. □

From this we also see that the pullback functors $f^*$ are also left adjoints. Let $\text{PrL}^{\text{LR}}$ denote the subcategory of $\text{PrL}$ consisting of those left-adjoint functors $f : \mathcal{C} \to \mathcal{D}$ which are also right adjoints (equivalently, those $f : \mathcal{C} \to \mathcal{D}$ which preserve all small limits and colimits).

Lemma 2.5. The inclusion $\text{PrL}^{\text{LR}} \subset \text{PrL}$ preserves all small limits.

Proof. We must show that the limit of a small diagram of limit and colimit preserving functors preserves limits. This follows from [20, Theorem 5.5.3.18]. □

Theorem 2.6. Let $\mathcal{C}$ be a presentable $\infty$-category. Then the stack $\mathcal{C}_{/(-)}$ of presentable $\infty$-categories on $\mathcal{T}$ factors through the subcategory $\text{PrL}^{\text{LR}} \subset \text{PrL}$. In particular, there exists a unique stack of presentable $\infty$-categories and left and right adjoint functors

$\mathcal{C}_{/(-)} : \mathcal{T}^{\text{op}} \to \text{PrL}^{\text{LR}}$

on $\mathcal{T}$ whose value at the space $S \in \mathcal{T}$ is equivalent to the $\infty$-category $\mathcal{C}/S$ of $\mathcal{C}$-valued presheaves on $S$.

Proof. This follows from lemma 2.5 and the proof of proposition 2.3. □

Thus far, we have constructed, for each presentable $\infty$-category $\mathcal{C}$, a “three functor formalism” for the parametrized theory associated to $\mathcal{C}$. 
3. Interlude on $\infty$-operads

In this section, we quickly review the aspects of the framework of $\infty$-categories and $\infty$-operads from [20] and [21] that we need.

3.1. The passage from categories with weak equivalences to $\infty$-categories. There are now several ways to pass from the data of a model category $\mathcal{C}$ to an associated $\infty$-category. When $\mathcal{C}$ is a simplicial model category, restricting to the full subcategory of cofibrant-fibrant objects $\mathcal{C}^{cf}$ yields a fibrant simplicial category, and then the simplicial nerve [20, 1.1.5.5]

$$N(\mathcal{C}^{cf})$$

is an $\infty$-category. Although any combinatorial model category is Quillen equivalent to a simplicial model category [15], this replacement process can be inconvenient. Furthermore, very few functors preserve cofibrant-fibrant objects; this is a particular problem when studying (symmetric) monoidal model categories.

More recently, [21, §1.3.3] provides an analogue of the Dwyer-Kan simplicial localization. Starting with a (not necessarily simplicial) model category $\mathcal{C}$, one passes to an $\infty$-category via the ordinary nerve applied to the full subcategory of cofibrant objects and subsequently inverts the weak equivalences:

$$N(\mathcal{C}^{c})[W^{-1}].$$

Given a simplicial model category $\mathcal{C}$, there is an equivalence of $\infty$-categories

$$N(\mathcal{C}^{cf}) \simeq N(\mathcal{C}^{c})[W^{-1}],$$

which implies that we can apply either process as needed [21, 1.3.3.7].

3.2. $\infty$-operads and symmetric monoidal $\infty$-categories. We now review the theory of $\infty$-operads as we will apply it in the body of the paper, following [21, §2]. Let $\Gamma$ denote the category with objects the pointed sets $\{\ast, 1, 2, \ldots, n\}$ for each natural number $n \in \mathbb{N}$ and morphisms the pointed maps of sets. An $\infty$-operad is then specified by an $\infty$-category $\mathcal{O}^\otimes$ and a functor

$$p: \mathcal{O}^\otimes \rightarrow N(\Gamma)$$

satisfying certain conditions [21, 2.1.1.10].

**Remark 3.1.** This is the generalization of the notion of a multicategory (colored operad); to obtain the generalization of an operad we restrict to $\infty$-operads equipped with an essentially surjective functor $\Delta^0 \rightarrow p^{-1}(\{\ast, 1\})$. To make sense of this, note that $p^{-1}(\{\ast, 1\})$ should be thought of as the "underlying" $\infty$-category associated to $\mathcal{O}^\otimes$, which we’d want to contain only a single (equivalence class of) object if we’re interested studying the $\infty$-version of an ordinary operad.

The identity map $N(\Gamma) \rightarrow N(\Gamma)$ is an $\infty$-operad; this is the analogue of the $E_\infty$ operad. More generally, we can define a topological category $\mathcal{E}(k)$ [21, 5.1.0.2] such that there is a natural functor $N(\mathcal{E}(k)) \rightarrow N(\Gamma)$ which is an $\infty$-operad. We refer to the resulting $\infty$-operads as the $E_k$ operads.

**Remark 3.2.** This uses a general correspondence result which associates to a simplicial multicategory an operadic nerve such that the operadic nerve is an $\infty$-operad provided that each morphism simplicial set of the multicategory is a Kan complex [21, 2.1.1.27].
A symmetric monoidal ∞-category is then an ∞-category \( \mathcal{C}^\otimes \) equipped with a coCartesian fibration of \( \infty \)-operads \([21, 2.1.2.18]\)

\[ p: \mathcal{C}^\otimes \to \text{N}(\Gamma). \]

The “underlying” ∞-category is obtained as the fiber \( \mathcal{C} = p^{-1}(\{\ast, 1\}) \). In abuse of terminology, we will say that an \( \infty \)-category \( \mathcal{C} \) is a symmetric monoidal ∞-category if it is equivalent to \( p^{-1}(\{\ast, 1\}) \) for some symmetric monoidal ∞-category \( \mathcal{C}^\otimes \). More generally, if \( \mathcal{O}^\otimes \) is an \( \infty \)-operad and \( \mathcal{C}^\otimes \to \mathcal{O}^\otimes \) is a coCartesian fibration of \( \infty \)-operads such that the composite

\[ \mathcal{C}^\otimes \to \mathcal{O}^\otimes \to \text{N}(\Gamma) \]

exhibits \( \mathcal{C}^\otimes \) as an \( \infty \)-operad \([21, 2.1.2.13]\), then \( \mathcal{C} \) as an \( \mathcal{O} \)-monoidal ∞-category.

We can use the results of the previous section to produce symmetric monoidal model ∞-categories: Given a symmetric monoidal model category \( \mathcal{C} \), we can associated a symmetric monoidal ∞-category \( \text{N}(\mathcal{C}|W^{-1}|^{\otimes}) \) with underlying ∞-category \( \text{N}(\mathcal{C}|W^{-1}|) \)[21, 4.1.3.6].

Given symmetric monoidal ∞-categories \( \mathcal{C}^\otimes \) and \( \mathcal{D}^\otimes \), we have two associated categories of functors between them:

1. The ∞-category of \( \infty \)-operad maps \( \text{Alg}_{\mathcal{C}}(\mathcal{D}) \), which should be thought of as the analogue of lax symmetric monoidal functors \([21, 2.1.2.7]\),
2. and the ∞-category \( \text{Fun}^\otimes(\mathcal{C}, \mathcal{D}) \) of symmetric monoidal functors, which should be regarded as strong symmetric monoidal functors \([21, 2.1.3.7]\).

For a fibration \( q: \mathcal{C}^\otimes \to \mathcal{O}^\otimes \) of \( \infty \)-operads and a map of \( \infty \)-operads \( \alpha: \mathcal{O}^\otimes \to \mathcal{O}^\otimes \), we define an \( \mathcal{O}' \)-algebra object of \( \mathcal{C} \) over \( \mathcal{O} \) to be a map of \( \infty \)-operads \( A: \mathcal{O}^\otimes \to \mathcal{C}^\otimes \) over \( \mathcal{O}' \) such that \( q \circ A \) is \( \alpha \)[21, 2.1.3.1]. The ∞-category of \( \mathcal{O}' \)-algebra objects in \( \mathcal{C} \) over \( \mathcal{O} \), denoted \( \text{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \), is the full subcategory of the functor category \( \text{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}^\otimes, \mathcal{C}^\otimes) \) spanned by the maps of \( \infty \)-operads. When \( \alpha \) is the identity map we denote this by \( \text{Alg}_{/\mathcal{O}}(\mathcal{C}) \).

A particularly interesting class of symmetric monoidal structures on ∞-categories come from cartesian monoidal structures. Any ∞-category with finite products admits a unique cartesian symmetric monoidal structure; the monoidal product is given by the categorical product \([21, \S 2.4.1]\). When \( \mathcal{C} \) is a cartesian symmetric monoidal ∞-category and \( \mathcal{O} \) is an \( \infty \)-operad, we often write \( \text{Mon}_\mathcal{O}(\mathcal{C}) \) in place of \( \text{Alg}_\mathcal{O}(\mathcal{C}) \)[21, \S 2.4.2].

Finally, we will primarily be interested in algebras over \( \infty \)-operads which are unital and coherent. A unital ∞-operad \([21, 2.3.1.1]\) has an essentially unique “nullary” operation. For instance, the \( \mathbb{E}_n \) operads are unital for any \( n \). As one might expect, algebras over unital operads are equipped with well-behaved unit maps. Coherent \( \infty \)-operads \( \mathcal{O}^\otimes \)[21, \S 3.3.1] satisfy conditions so that the categories of modules over \( \mathcal{O} \)-algebras have reasonable properties.

### 3.3. The ∞-category of presentable ∞-categories

Recall from [21, \S 6.3.1] that the ∞-category \( \text{Pr}^L \) of presentable ∞-categories and left adjoint functors is complete and cocomplete. Furthermore, \( \text{Pr}^L \) is tensored over spaces.

**Lemma 3.3.** Given a presentable ∞-category \( \mathcal{C} \) of and a space \( S \), the presentable ∞-category \( S \otimes \mathcal{C} \) is naturally equivalent to the presentable ∞-category \( \text{Fun}(S^{op}, \mathcal{C}) \) of \( \mathcal{C} \)-valued presheaves on \( S \).
Proof. We must have $\ast \otimes \mathcal{C} \simeq \mathcal{C} \simeq \text{Fun}(\ast^{\text{op}}, \mathcal{C})$. Since $(-) \otimes \mathcal{C}$ commutes with colimits of spaces, we have that $S \otimes \mathcal{C} \simeq \text{colim}_{S^{op}} \mathcal{C}$. Finally, since colimits in $\mathcal{P}^L$ correspond to limits (indexed by the opposite diagram) in $\mathcal{P}^R$ [20, 5.5.3.4], and limits in $\mathcal{P}^R$ can be calculated on the level of underlying $\infty$-categories, we have that $S \otimes \mathcal{C} \simeq \text{colim}_{S^{op}} \mathcal{C} \simeq \text{Fun}(\text{colim}_{S^{op}} \ast, \mathcal{C}) \simeq \text{Fun}(S^{op}, \mathcal{C})$.

□

Recall that $\mathcal{P}^L$ is a symmetric monoidal $\infty$-category with unit $\mathcal{T}$, the $\infty$-category of spaces. The $\infty$-category $\mathcal{T}$ can be explicitly modeled as the coherent nerve applied to the cofibrant-fibrant objects in the standard model category $\text{Top}$ of topological spaces (or, equivalently, simplicial sets).

The fact that $\mathcal{T}$ is the unit of the symmetric monoidal structure on $\mathcal{P}^L$ implies that it is canonically a commutative algebra object and that the forgetful functor $\text{Mod}^\text{Comm}(\mathcal{P}^L) \to \mathcal{P}^L$ is an equivalence. Since we will always be working in the ambient symmetric monoidal $\infty$-category $\mathcal{P}^L$ of presentable $\infty$-categories, and $\mathcal{T}$ is the trivial commutative algebra object of $\mathcal{P}^L$, we will simply write $\text{Mod}^\mathcal{T}$ in place of $\text{Mod}^\text{Comm}(\mathcal{P}^L)$.

Remark 3.4. The above construction of $\mathcal{P}^L$ as an $\infty$-category tensored over $\mathcal{T}$ generalizes the construction of the free $\mathcal{T}$-module associated to a space. Strictly speaking, the free $\mathcal{T}$-module functor $\mathcal{T} \to \text{Mod}^\mathcal{T}$ is not the left adjoint of the forgetful functor $\text{Mod}^\mathcal{T}((\mathcal{T}, -)) : \text{Mod}^\mathcal{T} \to \mathcal{T}$, since the underlying $\infty$-groupoid of a presentable $\infty$-category is not generally small. It is possible to remedy this by restricting to the compact objects of compactly generated $\mathcal{T}$-modules, but we avoid adding this extra level of complexity since we will not explicitly need this adjunction.

Similarly, recall that the $\infty$-category $\mathcal{P}^L_{\text{St}}$ of stable presentable $\infty$-categories admits a symmetric monoidal structure with unit the $\infty$-category $\mathcal{S}$ of spectra [21, §6.3.1]. There is a map of commutative algebra objects $\mathcal{T} \to \mathcal{T}_* \to \mathcal{S}$ of $\mathcal{P}^L$, where $\mathcal{T}_*$ is the $\infty$-category of pointed spaces. Both of these maps are “localizations” in the sense that the endofunctors $(-) \otimes_{\mathcal{T}} \mathcal{T}_*$ and $(-) \otimes_{\mathcal{T}} \mathcal{S}$ of $\text{Mod}^\mathcal{T}$ are idempotent: clearly $\mathcal{T}_* \otimes_{\mathcal{T}} \mathcal{T}_* \simeq \mathcal{T}_*$, and the same is true for $\mathcal{S}$ since $\mathcal{S} \simeq \mathcal{T}_*[\Sigma^{-1}]$.

Remark 3.5. The inclusion of the subcategory $\mathcal{P}^L \subset \text{Cat}_\infty$ is lax symmetric monoidal, and it induces subcategories $\text{Alg}(\mathcal{P}^L) \subset \text{Alg}(\text{Cat}_\infty)$ and $\text{CAlg}(\mathcal{P}^L) \subset \text{CAlg}(\text{Cat}_\infty)$.

A (commutative) algebra object of $\text{Cat}_\infty$ is a (commutative) algebra object of $\mathcal{P}^L$ if and only if the underlying $\infty$-category is presentable and the tensor product commutes with colimits in each variable. Given (symmetric) monoidal presentable $\infty$-categories $\mathcal{E}^\otimes$ and $\mathcal{D}^\otimes$, a (symmetric) monoidal functor $f : \mathcal{E}^\otimes \to \mathcal{D}^\otimes$ is a morphism of (commutative) algebra objects in $\mathcal{P}^L$ if and only if, on the level of underlying $\infty$-categories, it is a left adjoint (equivalently, by virtue of presentability, it preserves colimits) [21, §6.3.2].
Given an arbitrary $\infty$-operad $\mathcal{O}$, a (not necessarily small) $\mathcal{O}$-monoidal $\infty$-category is an object of $\text{Alg}_{\mathcal{O}}(\text{Cat}_{\infty})$, and an $\mathcal{O}$-monoidal presentable $\infty$-category is an object of $\text{Alg}_{\mathcal{O}}(\text{Pr}^L)$. In particular, associated to an $E_n$ algebra $R$ in a symmetric monoidal $\infty$-category $\mathcal{C}$, there is an $\infty$-category $\text{Mod}_R$ of right $R$-modules. A central theorem in the subject is that an $E_n$-algebra $R$ induces an $E_{n-1}$-monoidal structure on $\text{Mod}_R$ with unit $R$ such that the tensor product commutes with colimits in each variable; moreover, the functor from $E_n$-algebras to $E_{n-1}$-monoidal $\infty$-categories is fully-faithful [21, 6.3.5.17]. Furthermore, in an $E_n$-monoidal $\infty$-category $\mathcal{C}$, for $m \leq n$ the map of $\infty$-operads $E_m \rightarrow E_n$ implies that we have an $E_{n-m}$ monoidal $\infty$-category $\text{Alg}_{E_m}(\mathcal{C})$ of $E_m$-algebra objects.

4. The closed monoidal structure

We fix an $\infty$-operad $\mathcal{O}$ and suppose that $\mathcal{O}$ is an $\mathcal{O}$-monoidal presentable $\infty$-category. The goal now is to construct, for each Kan complex $S$, an $\mathcal{O}$-monoidal $\infty$-category $\mathcal{C}^{\mathcal{O}}_{/S}$ with underlying $\infty$-category $\mathcal{C}_{/S}$ such that the restriction functor $f^* : \mathcal{C}_{/T} \rightarrow \mathcal{C}_{/S}$ induced by a map of Kan complexes $f : S \rightarrow T$ is $\mathcal{O}$-monoidal.

Let $\text{Alg}_{\mathcal{O}}(\text{Pr}^L)^R$ denote the pullback diagram

\[
\begin{array}{ccc}
\text{Alg}_{\mathcal{O}}(\text{Pr}^L)^R & \rightarrow & \text{Alg}_{\mathcal{O}}(\text{Pr}^L) \\
\downarrow & & \downarrow \\
\text{Pr}^L & \rightarrow & \text{Pr}^L,
\end{array}
\]

that is, the subcategory of $\text{Alg}_{\mathcal{O}}(\text{Pr}^L)$ consisting of those $\mathcal{O}$-monoidal functors which are also right adjoints. To analyze the behavior of limits in $\text{Alg}_{\mathcal{O}}(\text{Pr}^L)^R$ we require the following technical lemma.

**Lemma 4.1.** The subcategory $\hat{\text{Cat}}^R_{\infty} \subset \hat{\text{Cat}}_{\infty}$ spanned by the complete $\infty$-categories and the limit preserving functors is stable under pullbacks.

**Proof.** Suppose given a pullback diagram

\[
\begin{array}{ccc}
\mathcal{A} & \rightarrow & \mathcal{B} \\
\downarrow^g & & \downarrow^h \\
\mathcal{C} & \rightarrow & \mathcal{D}
\end{array}
\]

in $\hat{\text{Cat}}_{\infty}$ such that $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are complete $\infty$-categories and $h, i$ are limit preserving functors. We first show that $\mathcal{A}$ is complete, which amounts to showing that the constant diagram functor $\mathcal{A} \rightarrow \text{Fun}(K, \mathcal{A})$ admits a right adjoint $\text{lim} : \text{Fun}(K, \mathcal{A}) \rightarrow \mathcal{A}$. Since the corresponding result holds for $\mathcal{B}, \mathcal{C}, \mathcal{D}$ by assumption, we obtain a map

\[
\text{Fun}(K, \mathcal{A}) \simeq \text{Fun}(K, \mathcal{B}) \times_{\text{Fun}(K, \mathcal{D})} \text{Fun}(K, \mathcal{C}) \rightarrow \mathcal{B} \times \mathcal{D} \simeq \mathcal{A}
\]

which is easily seen to be right adjoint to $\mathcal{A} \rightarrow \text{Fun}(K, \mathcal{A})$ since mapping spaces in a limit of $\infty$-category are computed as the limit of the mapping spaces. To see that the projections $f : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{A} \rightarrow \mathcal{C}$ preserve limits, we note that, by construction, $\text{lim} : \text{Fun}(K, \mathcal{A}) \rightarrow \mathcal{A}$ is the pullback of the diagram of maps $\text{lim} : \text{Fun}(K, \mathcal{B}) \rightarrow \mathcal{B}$ and $\text{lim} : \text{Fun}(K, \mathcal{C}) \rightarrow \mathcal{C}$ over $\text{lim} : \text{Fun}(K, \mathcal{D}) \rightarrow \mathcal{D}$, so this
is clear. Finally, given a commutative diagram

\[
\begin{array}{ccc}
\mathcal{A}' & \xrightarrow{f} & \mathcal{B} \\
\downarrow{g} & & \downarrow{h} \\
\mathcal{C} & \xrightarrow{i} & \mathcal{D}
\end{array}
\]

in $\widehat{\text{Cat}}^R_\infty$, we must show that the $\infty$-groupoid of limit preserving functors from $\mathcal{A}'$ to $\mathcal{A}$ over $\mathcal{B} \to \mathcal{D} \leftarrow \mathcal{C}$ is contractible. This follows from the fact that these functors form a full $\infty$-subgroupoid of the $\infty$-groupoid of all functors from $\mathcal{A}'$ to $\mathcal{A}$ over $\mathcal{B} \to \mathcal{D} \leftarrow \mathcal{C}$ coupled with the observations that this latter $\infty$-groupoid is contractible and that the unique such functor preserves limits.

\[\square\]

**Remark 4.2.** A similar argument shows that the subcategory $\widehat{\text{Cat}}^R_\infty \subset \widehat{\text{Cat}}_\infty$ is stable under all small limits.

**Corollary 4.3.** The $\infty$-category $\text{Alg}_O(\text{Pr}^L)^R_\infty$ admits all small limits and the inclusion of the subcategory $\text{Alg}_O(\text{Pr}^L)^R_\infty \subset \text{Alg}_O(\text{Pr}^L)^R_\infty$ preserves them.

**Proof.** This is immediate from lemma 4.1 above, as $\text{Alg}_O(\text{Pr}^L)^R_\infty$ is the pullback of a diagram of complete $\infty$-categories and limit preserving functors.

Our main foundational theorem is the following result:

**Theorem 4.4.** There exists a unique stack of presentable $\mathcal{O}$-monoidal $\infty$-categories $\mathcal{C}^\mathcal{O}_/(-)$ on $\mathcal{T}$ whose value at the space $S$ is the $\mathcal{O}$-monoidal $\infty$-category $\mathcal{C}^\mathcal{O}S$ of $\mathcal{C}^\mathcal{O}$-valued presheaves on $S$.

**Proof.** This follows from corollary 4.3 and the proof of proposition 2.3. □

**Remark 4.5.** In the symmetric monoidal context, one expects that the right adjoint $f_*$ is lax $\mathcal{O}$-monoidal and that the left adjoint $f_!$ is oplax $\mathcal{O}$-monoidal. Such a result can be shown by consideration of point-set models for the universal example of parametrized spaces; since we find this inelegant and do not need these results, we defer a treatment to a future point when the relevant foundational results on adjoint functors in the $\infty$-categories are available in the literature.

We now further suppose that $\mathcal{O}^\otimes$ comes equipped with a fixed map $E^\otimes_1 \to \mathcal{O}^\otimes$. This implies (by restriction along this map) that any $\mathcal{O}$-monoidal $\infty$-category $\mathcal{C}^\otimes$ is equipped with a distinguished monoidal structure $\otimes$.

**Lemma 4.6.** Let $\mathcal{C}^\otimes$ be a monoidal presentable $\infty$-category. Then $\mathcal{C}^\otimes$ is closed in the sense that, for each object $X$ of $\mathcal{C}$, the left and right multiplication functors $X \otimes (-) : \mathcal{C} \to \mathcal{C}$ and $(-) \otimes X : \mathcal{C} \to \mathcal{C}$ admit right adjoints.

**Proof.** Because the tensor product preserves colimits on the left and on the right, this is immediate from the adjoint functor theorem. □

Writing $\otimes$ for the monoidal product obtained by restriction along the map $E^\otimes_1 \to \mathcal{O}^\otimes$, then, for each object $X \in \mathcal{C}$, we write

\[F(X, -) : \mathcal{C} \to \mathcal{C}\]

for the right adjoint of the right multiplication functor $(-) \otimes X : \mathcal{C} \to \mathcal{C}$. As $S$ varies over all spaces, the base-change functors and closed tensor structures collectively give rise to a (non necessarily symmetric) sort of “Wirthmüller context” [13].
5. Wirthmüller contexts

In the context of parametrized spaces, \( f^* \) is a symmetric monoidal functor. The situation of a symmetric monoidal functor with left and right adjoints gives rise to a series of compatibility formulas (e.g., the projection formula). In this section, following [13], we abstract this relationship into what we will refer to as a Wirthmüller context.

We continue to fix an \( \infty \)-operad \( O \otimes \) over \( E_1 \) and an \( O \)-monoidal \( \infty \)-category \( C \otimes \).

To say more, we now suppose that the map \( E_1 \otimes \rightarrow O \otimes \) factors through \( E_1 \otimes \), which is to say that \( C \otimes \) is a symmetric monoidal presentable \( \infty \)-category [20, Proposition 4.1.1.20].

Specializing the definition of \( \text{Alg}_O(Pr^L)^R \), we let \( \text{CAlg}(Pr^L)^R \) denote the pullback

\[
\begin{array}{ccc}
\text{CAlg}(Pr^L)^R & \longrightarrow & \text{CAlg}(Pr^L) \\
\downarrow & & \downarrow \\
Pr^L \otimes & \longrightarrow & Pr^L,
\end{array}
\]

e.i., the subcategory of \( \text{CAlg}(Pr^L) \) consisting of those symmetric monoidal functors which are also right adjoints.

**Definition 5.1.** A Wirthmüller context is a \( \text{CAlg}(Pr^L)^R \)-valued stack on \( T \); that is, a limit preserving functor

\[
T^{op} \longrightarrow \text{CAlg}(Pr^L)^R.
\]

The \( \infty \)-category of Wirthmüller contexts is the full subcategory of the \( \infty \)-category \( \text{Fun}(T^{op}, \text{CAlg}(Pr^L)^R) \) spanned by the stacks.

In this setting, our basic existence result is the following corollary of theorem 4.4.

**Corollary 5.2.** A Wirthmüller context determines, and is determined by, a symmetric monoidal presentable \( \infty \)-category.

**Proof.** Since a Wirthmüller context is a \( \text{CAlg}(Pr^L)^R \)-valued stack on \( T \), this follows immediately. \( \square \)

Some of the useful consequences of the existence of a Wirthmüller context are summarized in the following standard proposition.

**Proposition 5.3.** Let \( \mathcal{C}^\otimes \) be a symmetric monoidal presentable \( \infty \)-category, let \( f: S \rightarrow T \) be a map of Kan complexes, and let \( X \) and \( Y \) be objects of \( \mathcal{C}[T]^\otimes \). Then:

1. \( f^*(X \otimes_T Y) \simeq f^*X \otimes_S f^*Y \)
2. \( F_T(X, f_*Y) \simeq f_*F_S(f^*X, Y) \)
3. \( f^*F_T(X, Y) \simeq F_S(f^*X, f^*Y) \)
4. \( f_!(f^*X \otimes_S Y) \simeq X \otimes_T f_!Y \)
5. \( F_S(f_!X, Y) \simeq f_!F_T(X, f^*Y) \)

**Proof.** The construction of the maps follows [13], and to check the equivalences it suffices to work on the level of the homotopy categories. See also [27, 2.2.2.2.5.8] for a verification in the particular case of parametrized spaces and spectra. \( \square \)

We can also consider this situation when \( \mathcal{C}^\otimes \) is an \( E_n \)-monoidal presentable \( \infty \)-category. In this case, provided that \( n > 2 \), the theory is the same because the equivalences of proposition 5.3 arise as isomorphisms in the homotopy category.
and for \( n > 2 \) an \( \mathbb{E}_n \)-monoidal presentable \( \infty \)-category has a closed symmetric monoidal homotopy category. (We also use the fact that \( \mathbb{E}_n \)-monoidal functors induce symmetric monoidal functors on the homotopy category in this case.) We suspect that analogous formulas hold for the braided monoidal case \( n = 2 \) and even the monoidal case \( n = 1 \).

**Remark 5.4.** We note that this definition implicitly specifies the 2-categorical coherences which are a necessary part of a complete description of base-change phenomena (see [28] for a treatment of these issues in a somewhat different context).

6. The proper pushforward \( p_* \) and its right adjoint \( p! \)

Under more restrictive hypotheses on \( C \), it turns out that for proper maps \( p : S \to T \) in the \( \infty \)-category of spaces (i.e., maps \( p \) such that the fibers of \( p \) are compact), the pushforward functor \( p_* : \mathcal{C}/S \to \mathcal{C}/T \), right adjoint to the restriction functor \( p^* : \mathcal{C}/T \to \mathcal{C}/S \), itself admits a right adjoint \( p^! : \mathcal{C}/T \to \mathcal{C}/S \). We will not attempt to pursue the most general version of this theorem, but instead will suppose (as in the case of greatest interest) that \( C \simeq \text{Mod}_R \) for some \( \mathbb{E}_\infty \)-ring spectrum \( R \).

In this case, \( C \) is the underlying \( \infty \)-category of the symmetric monoidal stable presentable \( \infty \)-category \( C \otimes = \text{Mod}_R^\otimes \) of \( R \)-modules. The unit of the symmetric monoidal structure on \( \text{Mod}_R^\otimes \) is \( R \) itself, which also acts as a compact generator of the underlying stable \( \infty \)-category \( \text{Mod}_R \).

**Definition 6.1.** A map of \( p : S \to T \) in the \( \infty \)-category \( T \) of space is proper if, for each vertex \( t : \Delta^0 \to T \) of \( T \), the fiber \( S_t = S \times_T \Delta^0 \) is a compact object of \( T \). (Recall that an object is compact if maps out of it commute with filtered colimits.)

Given a proper map \( p : S \to T \), we’ll see that the pushforward \( p_* \) admits a right adjoint \( p^! \). This gives a series of adjunctions \( p_!, p^*, p_*, p^! \), each of which is right adjoint to the functor on its left. First, we deal with the case in which \( T \) is contractible.

Let \( S \) be a Kan complex and form the symmetric monoidal stable presentable \( \infty \)-category \( \mathcal{C}/S \) of \( \mathcal{C} \)-valued presheaves on \( S \). Using the free \( R \)-module functor \( R[-] : \mathcal{T} \to \mathcal{C} \), an object \( s \) of \( S \) determines an object \( R[s] \) of \( \mathcal{C} \); here \( s \) denotes the representable presheaf

\[
\Delta^0 \to S \to \text{Fun}(S^{\text{op}}, \mathcal{T}) \simeq \mathcal{T}/S
\]

in which the middle map is the Yoneda embedding, and \( R[s] \) is obtained by composing with the map \( \mathcal{T}/S \to \mathcal{C}/S \) induced by the free \( R \)-module functor.

**Proposition 6.2.** For any Kan complex \( S \), the colimit of the Yoneda embedding

\[
S \to \text{Fun}(S^{\text{op}}, \mathcal{T})
\]

is a terminal object of \( \text{Fun}(S^{\text{op}}, \mathcal{T}) \).

**Proof.** We may immediately reduce to the case in which \( S \) is connected. Choosing an arbitrary basepoint of \( S \) and letting \( G \simeq \Omega S \) be the loop group of \( S \) at this point, we obtain an equivalence \( BG \simeq S \). Since the mapping space

\[
\text{map}_{BG}(-,-) : BG^{\text{op}} \times BG \to \mathcal{T}
\]

classifies \( G \) as a left and right \( G \)-space, the Yoneda embedding \( BG \to \text{Fun}(BG^{\text{op}}, \mathcal{T}) \) classifies the left action of \( G \) on itself as an object of \( \infty \)-category \( \text{Fun}(BG^{\text{op}}, \mathcal{T}) \) of
right $G$-spaces. Finally, the quotient of $G$ by the left action of $G$ in the $\infty$-category $\text{Fun}(BG^{op}, \mathcal{T})$ of right $G$-spaces is a terminal object of $\text{Fun}(BG^{op}, \mathcal{T})$.

**Proposition 6.3.** For any Kan complex $S$, the colimit of the stable Yoneda embedding
\[ S \to \text{Fun}(S^{op}, \mathcal{T}) \xrightarrow{\Sigma^\infty_i} \text{Fun}(S^{op}, S) \]
is an equivalence in $S/S$.

**Proof.** $\Sigma^\infty_i$ commutes with colimits. \qed

**Proposition 6.4.** Let $\kappa$ be an infinite regular cardinal, let $S$ be a $\kappa$-compact $\infty$-groupoid, and write $q : S \to *$ for the projection to the terminal $\infty$-groupoid. Then
\[ q^* : \mathcal{C} \simeq \mathcal{C}_{/*} \to \mathcal{C}_{/*} \]
preserves $\kappa$-compact objects.

**Proof.** We reduce to checking that $RS \simeq q^* R$ is $\kappa$-compact using the following facts: any $\kappa$-compact $R$-module is equivalent to a (retract of a) colimit of a $\kappa$-small diagram with value $R$, $\kappa$-compact objects are closed under $\kappa$-small colimits, and $q^*$ preserves colimits. Now $S$ is equivalent to (a retract of) the colimit of the trivial functor $I \to \mathcal{T}$, where $I$ is a $\kappa$-small simplicial set [20, 5.4.1.5,5.4.1.6]. It follows from this that (up to retract) there exists a weak equivalence $f : I \to S$, and thus we deduce that
\[ RS \simeq \text{colim}_{i \in I} R[\hat{f}_i] \]
is a $\kappa$-small colimit of representables and therefore $\kappa$-compact. \qed

**Proposition 6.5.** Let $\kappa$ be an infinite regular cardinal, let $S$ be a $\kappa$-compact $\infty$-groupoid, and write $q : S \to *$ for the projection to the terminal object. Then $q_* : \mathcal{C}_{/S} \to \mathcal{C}_{/*} \simeq \mathcal{C}$ preserves $\kappa$-filtered colimits.

**Proof.** Let $N_\alpha$ be a $\kappa$-filtered diagram of objects of $\mathcal{C}_{/S}$. We must show that the map
\[ \text{colim} q_* N_\alpha \to q_* \text{colim} N_\alpha \]
is an equivalence of $R$-modules. Since $R$ is compact, and hence $\kappa$-compact, and $q^*$ preserves $\kappa$-compact objects, we have that
\[ \text{map}(\Sigma^n R, q_* \text{colim} N_\alpha) \simeq \text{map}(q^* \Sigma^n R, \text{colim} N_\alpha) \]
\[ \simeq \text{colim} \text{map}(q^* \Sigma^n R, N_\alpha) \]
\[ \simeq \text{colim} \text{map}(\Sigma^n R, q_* N_\alpha) \simeq \text{map}(\Sigma^n R, \text{colim} q_* N_\alpha). \]
Since the $\Sigma^n R$, $n \in \mathbb{Z}$, generate $\mathcal{C}$, the result follows. \qed

**Proposition 6.6.** Let $p : S \to T$ be a proper map in $\mathcal{T}$. Then the pushforward
\[ p_* : \mathcal{C}_{/S} \to \mathcal{C}_{/T}, \]
preserves filtered colimits.
Proof. As equivalences are detected fiberwise, it is clear that $p_*$ preserves filtered colimits if and only if, for all objects $t$ of $T$, the pushforward $q_*$ along the projection $q: S_t \to \Delta^0$ defined by the pullback

\begin{center}
\begin{tikzcd}
S_t \arrow{r}{s} \arrow{d}{q} & S \arrow{d}{p} \\
\Delta^0 \arrow{r}{t} & T
\end{tikzcd}
\end{center}

preserves filtered colimits. But $S_t$ is compact since $p$ is proper, so this follows from Proposition 6.5. \qed

**Corollary 6.7.** Let $p: S \to T$ be a proper map in $\mathcal{T}$. Then $p_*$ admits a right adjoint $p^!: \mathcal{C}_T \to \mathcal{C}_S$.

Proof. Since $p_*: \mathcal{C}_S \to \mathcal{C}_T$ is a functor of stable presentable $\infty$-categories, $p_*$ admits a right adjoint if (and only if) $p_*$ preserves colimits. But $p_*$ is exact, so $p_*$ preserves colimits if (and only if) $p_*$ preserves filtered colimits, and this is the content of Proposition 6.6. \qed

7. Parametrized spaces, spectra, and modules

We now specialize the constructions of the preceding sections to obtain $\infty$-categorical models of parametrized spaces, spectra, and $R$-modules for an $\mathcal{E}_n$-ring spectrum $R$. Our main goal in this section is to compare these $\infty$-categories to those that result from the approach of [27]. Some of this material was previously discussed (in a more specialized form) in [1] and [2], but we intend that the systematic treatment given below will provide a more convenient and complete reference.

7.1. Parametrized spaces. We begin by comparing our model of parametrized spaces to the May-Sigurdsson notion of parametrized spaces [27]. In the context of $\infty$-categories, we think of spaces as $\infty$-groupoids, and model them with Kan complexes (as in the previous section). To reduce confusion, we will use the term “topological space” when we are referring to a bona-fide topological space, and we will write $\text{Top}$ for the category of compactly-generated weak Hausdorff topological spaces. The following is the specialization of theorem 4.4.

**Definition 7.1.** Let $S$ be a Kan complex. Define the $\infty$-category of “spaces over $S$” as the $\infty$-category

$$\mathcal{C}/S = \text{Fun}(S^{op}, \mathcal{T})$$

of presheaves of spaces on $S$ and the $\infty$-category of “pointed spaces over $S$” (or ex-spaces) as the $\infty$-category

$$\mathcal{(T}/S)_* = \text{Fun}(S^{op}, \mathcal{T})_* \simeq \text{Fun}(S^{op}, \mathcal{T}_*) = (\mathcal{T}_*)/S$$

of presheaves of pointed spaces on $S$.

**Remark 7.2.** Pointed objects of the category of spaces over $X$ are often referred to in the literature as “ex-spaces”, as they are spaces equipped with a map to $X$ together with a “cross-section” of that map.
Recall that $T/S$ can be described via the straightening and unstraightening correspondence as the $\infty$-category associated to the model category $\text{Set}_{\Delta/\Pi_\infty S}$, and $T^*/S$ can analogously be described as the $\infty$-category of pointed objects in $\text{Set}_{\Delta/\Pi_\infty S}$. This provides a comparison to the $\infty$-categories associated to the May-Sigurdsson categories of parametrized spaces $\text{Top}_B$ and $(\text{Top}_B)_*$ over a space $B$.

**Proposition 7.3.** Let $B$ be a space. There are equivalences of symmetric monoidal $\infty$-categories

$$\mathcal{T}_{\Pi_\infty B} \simeq \mathcal{N}(\text{Set}_{\Delta/\Pi_\infty B})[W^{-1}]^{\otimes} \simeq \mathcal{N}(\text{Top}_B)[W^{-1}]^{\otimes}$$

and

$$(\mathcal{T}_{\Pi_\infty B})^* \simeq \mathcal{N}((\text{Set}_{\Delta/\Pi_\infty B})_*)[W^{-1}]^{\otimes} \simeq \mathcal{N}((\text{Top}_B)_*)[W^{-1}]^{\otimes}.$$

**Proof.** For a space $B$, the projective model structure on $\text{Top}_B$ is Quillen equivalent to the corresponding simplicial model category structure on simplicial sets over $\Pi_\infty B$, which in turn is Quillen equivalent to the simplicial model category of simplicial presheaves on the simplicial category $\mathcal{C}[\Pi_\infty B]$ (with the projective model structure) [20, 2.2.1.2]. (Here $\mathcal{C}$ denotes the left adjoint to the simplicial nerve; it associates a simplicial category to a simplicial set [20, 1.1.5].)

Next, we have a comparison

$$\text{St}: \text{N Set}_{\Delta/\Pi_\infty B} \longrightarrow \text{Fun}(\Pi_\infty B^{\text{op}}, \text{N Set}_{\Delta});$$

the map, called the _straightening_ functor, rigidifies a fibration over $\Pi_\infty B$ into a presheaf of $\infty$-groupoids on $\Pi_\infty B$ whose value at the point $b$ is equivalent to the fiber over $b$ [20, 3.2.1].

Finally, the symmetric monoidal structure on $\mathcal{T}_{/S}$ is Cartesian and therefore unique [21, 2.4.1.9]. Thus, we can promote this equivalence to an equivalence of symmetric monoidal $\infty$-categories. The result for pointed objects follows. $\square$

To complete the comparison to the model of May-Sigurdsson, we need to study the base-change functors. Almost all of the subtlety and difficulty of the foundational portion of their work arises from the complexities of topological spaces (which they must contend with in order to handle the equivariant setting) and the fact that it is impossible to have a model structure in which the pairs $(f_!, f^*)$ and $(f^*, f_*)$ are simultaneously Quillen adjunctions.

Although the point-set category $(\text{Top}_B)_*$ of ex-spaces has a model structure induced by the standard model structure on $\text{Top}$ (which they refer to as the $q$-model structure), one of the key insights of May and Sigurdsson is that for the purposes of stable parametrized homotopy theory it is essential to work with the (Quillen equivalent) $qf$-model structure [27, 6.2.6].

The situation is easier in the simplicial setting: For a map $f: A \rightarrow B$, we can obtain point-set models of the functors $f^*$, $f_*$, and $f_!$ by considering model categories of simplicial presheaves. We must still confront the fact that

$$f^* : \text{Fun}(\mathcal{C}[\Pi_\infty B^{\text{op}}], \text{Set}_\Delta) \longrightarrow \text{Fun}(\mathcal{C}[\Pi_\infty A^{\text{op}}], \text{Set}_\Delta)$$

is a _right_ Quillen functor for the _projective_ model structure, with left adjoint $f_!$, and a _left_ Quillen functor for the _injective_ model structure, with right adjoint $f_*$, on the above categories of simplicial presheaves. Nonetheless, this suffices to produce the desired adjoint pairs on the level of $\infty$-categories.
Theorem 7.4. The Wirthmüller context we construct in corollary 5.2 on $(\mathcal{I}/S)_*$ is compatible with that of May-Sigurdsson.

Proof. To see this, observe that it suffices to check this for $f^*$; compatibility then follows formally for the adjoints $f_*$ and $f^!$. Thus, we need to check that the right derived functor of $f^*: (\text{Top}/B)_* \to (\text{Top}/A)_*$ in the $qf$-model structure is compatible with the right derived functor of

$$f^*: \text{Fun}(\mathcal{C}[\Pi_\infty B^{\text{op}}], \text{Set}_\Delta) \to \text{Fun}(\mathcal{C}[\Pi_\infty A^{\text{op}}], \text{Set}_\Delta)$$

in the projective model structure. By the work of [27, 9.3], it suffices to check the compatibility for $f^*$ in the $q$-model structure. Since both versions of $f^*$ that arise here are Quillen right adjoints, this amounts to the verification that the diagram

$$\begin{array}{ccc}
\text{Fun}(\mathcal{C}[\Pi_\infty B^{\text{op}}], \text{Set}_\Delta) & \xrightarrow{f^*} & \text{Fun}(\mathcal{C}[\Pi_\infty A^{\text{op}}], \text{Set}_\Delta) \\
\downarrow \text{Un} & & \downarrow \text{Un} \\
\text{Set}_{\Delta/B} & \xrightarrow{f^*} & \text{Set}_{\Delta/A}
\end{array}$$

commutes when applied to fibrant objects, where here $\text{Un}$ denotes the unstraightening functor (which is the right adjoint of the Quillen equivalence). Finally, this follows from [20, 2.2.1.1].

The promotion of this comparison to the symmetric monoidal structure is a consequence of the fact that $f^*$ preserves products and the fact that the Cartesian symmetric monoidal structure is unique. \hfill \Box

7.2. Parametrized spectra. We now move on to compare parametrized spectra with the May-Sigurdsson definition of parametrized spectra [27]. Specializing theorem 4.4 once again, we have the following description of parametrized spectra.

Definition 7.5. Let $S$ be a Kan complex. The $\infty$-category of “spectra over $S$” is defined to be the $\infty$-category

$$S/S = \text{Fun}(S^{\text{op}}, S)$$

of presheaves of spectra on $S$.

The proof of theorem 4.4 leads us to expect that the $\infty$-category $S[S]$ of spectra parametrized over $S$ can be understood as the stabilization of the $\infty$-category $\mathcal{I}/S$ (or $(\mathcal{I}/S)_*$) of (pointed) spaces parametrized over $S$. More generally, we have the following proposition.

Proposition 7.6. Let $S$ be a space and let $\mathcal{C}$ be a presentable $\infty$-category. Then we have a natural equivalence

$$\text{Stab}(\mathcal{C}/S) \simeq \text{Stab}(\mathcal{C})/S$$

of stable presentable $\infty$-categories. Moreover, if $\mathcal{C}$ is an $O$-monoidal presentable $\infty$-category for some $\infty$-operad $O$, then this equivalence extends to an equivalence of $O$-monoidal stable presentable $\infty$-categories.

Proof. First recall that the functor $\text{Stab}(-) \simeq (-) \otimes S$ is a symmetric monoidal localization of $\text{Pr}^L$ whose essential image is precisely the full subcategory of stable presentable $\infty$-categories. Thus

$$\text{Stab}(\mathcal{C}/S) \simeq (S \otimes \mathcal{C}) \otimes S \simeq S \otimes (\mathcal{C} \otimes S) \simeq \text{Stab}(\mathcal{C})/S.$$
The final assertion follows from the fact that $(-)/S$ is also symmetric monoidal, which we prove as proposition 7.13 below.

Therefore, in order to compare our model of parametrized spectra over $B$ to the May-Sigurdsson model, we will work with the corresponding formal stabilization of model categories. Specifically, given a left proper cellular model category $\mathcal{C}$ and an endofunctor of $\mathcal{C}$, Hovey constructs a cellular model category $\text{Sp}^N\mathcal{C}$ of spectra [18]. When the $\mathcal{C}$ is additionally a simplicial symmetric monoidal model category, the endofunctor given by the tensor with $S^1$ yields a simplicial symmetric monoidal model category of symmetric spectra $\text{Sp}^\Sigma\mathcal{C}$ (in addition to the simplicial model category $\text{Sp}^N\mathcal{C}$ of spectra). These models of the stabilization are functorial in left Quillen functors which are suitably compatible with the respective endofunctors (see [18, 5.2]).

**Proposition 7.7.** Let $\mathcal{C}$ be a left proper cellular simplicial model category and write $\text{Sp}^N\mathcal{C}$ for the cellular simplicial model category of spectra generated by the tensor with $S^1$. Then there is an equivalence of $\infty$-categories

$$N((\text{Sp}^N\mathcal{C})^c)[W^{-1}] \simeq \text{Stab}(N(\mathcal{C})[W^{-1}]).$$

When $\mathcal{C}$ is a simplicial symmetric monoidal model category, this equivalence extends to an equivalence

$$N((\text{Sp}^\Sigma\mathcal{C})^c)[W^{-1}]^\otimes \simeq \text{Stab}(N(\mathcal{C})[W^{-1}]^\otimes)$$

of symmetric monoidal $\infty$-categories.

**Proof.** The functors $E_n : \text{Sp}^N\mathcal{C} \to \mathcal{C}$ which associate to a spectrum its $n$th-space $A_n$, induce a functor

$$f : N((\text{Sp}^N\mathcal{C})^c)[W^{-1}] \longrightarrow \lim\{ \cdots \xrightarrow{\Omega} N(\mathcal{C})[W^{-1}] \xrightarrow{\Omega} N(\mathcal{C})[W^{-1}] \}$$

which is evidently essentially surjective. To see that it is fully faithful, it suffices to check that for cofibrant-fibrant spectrum objects $A$ and $B$ in $\text{Sp}^N\mathcal{C}$, there is an equivalence of mapping spaces

$$\text{map}(A, B) \simeq \text{holim}\{ \cdots \xrightarrow{\Omega} \text{map}(A_1, B_1) \xrightarrow{\Omega} \text{map}(A_0, B_0) \},$$

where $\Omega : \text{map}(A_{n+1}, B_{n+1}) \to \text{map}(A_n, B_n)$ acts as

$$A_{n+1} \to B_{n+1} \mapsto A_n \simeq \Omega A_{n+1} \mapsto \Omega B_{n+1} \simeq B_n.$$

Since any cofibrant $A$ is a retract of a cellular object, inductively we can reduce to the case in which $A = F_m X$, i.e., the shifted suspension spectrum on a cofibrant object $X$ of $\mathcal{C}$. Then $\text{map}(A, B) \simeq \text{map}(X, B_m)$ by adjunction. The latter is in turn equivalent to $\text{map}(\Sigma^{n-m} X, B_n)$, where we interpret $\Sigma^{n-m} X = *$ for $m > n$, in which case the homotopy limit is equivalent to that of the homotopically constant (above degree $n$) tower whose $n$th term is $\text{map}(\Sigma^{n-m} X, B_n)$.

In the symmetric monoidal setting, the fact that $\text{Stab}\mathcal{C}$ is the initial stable symmetric monoidal $\infty$-category which accepts a symmetric monoidal $\infty$-functor from $\mathcal{C}$ coupled with the equivalence between prespectra and symmetric spectra implies the desired comparison. □
May and Sigurdsson construct a symmetric monoidal stable model structure on the category $\mathcal{S}_B$ of orthogonal spectra in $(\text{Top}/B)_*$. This model structure is based on the $gf$-model structure on $\text{ex}$-spaces, leveraging the diagrammatic viewpoint of [25, 24]. Similarly, they construct a stable model structure on the category $\mathcal{P}_B$ of prespectra in $(\text{Top}/B)_*$. The forgetful functor $\mathcal{S}_B \to \mathcal{P}_B$ induces a Quillen equivalence [27, 12.3.10].

**Theorem 7.8.** Let $B$ be a topological space. There is an equivalence of symmetric monoidal $\infty$-categories between the $\infty$-category associated to the model category of orthogonal spectra and the $\infty$-category of parametrized spectra:

$$N(\mathcal{S}_B)[W^{-1}]^{\otimes} \simeq \text{Fun}(\Pi_\infty B^{op}, S)^{\otimes}.$$ 

**Proof.** This is essentially an immediate consequence of proposition 7.7, using the standard comparison between orthogonal spectra and symmetric spectra [25]. □

**7.3. Parametrized module spectra.** Finally, we describe the specialization of theorem 4.4 to the case of parametrized module spectra. An advantage of our framework is that we can handle both categories of $R$-module spectra parametrized over a space $S$ (for an $E_1$ ring spectrum $R$) and categories of module spectra over an $E_1$ ring object in $\text{Fun}(S^{op}, S)$.

**Definition 7.9.** Let $R$ be an $E_1$-ring spectrum, let $S$ be a space, and let $\text{Mod}_R$ denote the stable presentable $\infty$-category of right $R$-modules. The $\infty$-category of parametrized $R$-module spectra (i.e., bundles of $R$-modules) over $S$ is the stable presentable $\infty$-category $\text{(Mod}_R\text{)}_S$.

In practice, however, $R$ is often more than an $E_1$-ring spectrum; rather, it may be an $E_n$-ring spectrum for some $1 \leq n \leq \infty$. In this case, the category of parametrized module spectra inherits a multiplicative structure.

**Proposition 7.10.** Let $R$ be an $E_n$-ring spectrum, $n > 0$, and let $S$ be a space. Then the $\infty$-category $(\text{Mod}_R)_S$ of parametrized $R$-module spectra over $S$ is the underlying $\infty$-category of an $E_{n-1}$-monoidal stable presentable $\infty$-category $(\text{Mod}_R)^{\otimes}_S$.

**Proof.** Let $\text{Mod}_R$ denote the $\infty$-category of right $R$-modules, which is an $E_{n-1}$-algebra object of $\text{Pr}^L$ and in particular an $E_{n-1}$-monoidal $\infty$-category. Then

$$\text{Fun}(S^{op}, \text{Mod}_R)^{\otimes} = \text{Fun}(S^{op}, \text{Mod}_R)^{\otimes} \times_{\text{Fun}(S^{op}, N(\Gamma))} N(\Gamma)$$

is an $E_{n-1}$-monoidal $\infty$-category such that the underlying $\infty$-category

$$(\text{Mod}_R)_S = \text{Fun}(S^{op}, \text{Mod}_R)$$

is stable and presentable. By construction, the operations in $\text{Fun}(S^{op}, \text{Mod}_R)^{\otimes}$ are computed pointwise, so that the tensor product commutes with colimits in each variable. Hence $(\text{Mod}_R)^{\otimes}_S = \text{Fun}(S^{op}, \text{Mod}_R)^{\otimes}$ is an $E_{n-1}$-monoidal stable presentable $\infty$-category. □

**Remark 7.11.** More generally, our framework allows us to resolve a problem which was not handled in [27]. Due to difficulties in maintaining homotopical control over the product, categories of modules over general ring objects in the parametrized categories were not constructed.
7.4. Parametrized objects over $\mathbb{E}_n$ spaces. In this section, we study multiplicative structures that arise on $\infty$-categories of parametrized objects over $\mathbb{E}_n$ spaces. In the previous sections, the multiplicative structure on $\mathbb{C}/S$ was obtained pointwise, or, equivalently, from the evident external product via pullback along the diagonal $\Delta: S \to S \times S$ of the base space. Here, the multiplicative structures arise from an actual product on $S$ itself.

Recall that the $\infty$-categorical Day convolution product \cite[§6.3]{Day} is a consequence of the existence of a symmetric monoidal functor from spaces to presentable $\infty$-categories. The relevant functor on objects agrees with our functor $T/(-)$: $T \to \text{Pr}_L$ on objects, but it is the covariant version which comes from using the left adjoint $f_: T/S \to T/T$ in place of the right adjoint $f^*: T/T \to T/S$ for $f: S \to T$ a map of Kan complexes.

**Proposition 7.12.** There is a unique colimit-preserving functor

$$\mathcal{T}[-]: \mathcal{T} \to \text{Pr}^L$$

whose value at the space $S$ is the $\infty$-topos $\mathcal{T}/S$ of presheaves of spaces on $S$.

**Proof.** The $\infty$-category of spaces $\mathcal{T}$ is freely generated under colimits by the one-point space \cite[5.1.5.8]{Lurie}. Since any space $S$ is equivalent to the $S$-indexed colimit of the constant diagram on the point, it follows that

$$\mathcal{T}[S] \simeq S \otimes \mathcal{T} \simeq \text{Fun}(S^{op}, \mathcal{T}).$$

$\square$

**Proposition 7.13.** The functor $\mathcal{T}[-]: \mathcal{T} \to \text{Pr}^L$ extends to a symmetric monoidal functor

$$\mathcal{T}[-]: \mathcal{T}^\otimes \to (\text{Pr}^L)^\otimes.$$  

**Proof.** This is a consequence of the properties of the $\infty$-categorical Day convolution project \cite[6.3.1.2]{Lurie}. $\square$

An immediate consequence of proposition 7.13 is that the functor $\mathcal{T}[-]$ preserves multiplicative structures:

**Corollary 7.14.** Let $\mathcal{O}$ be an $\infty$-operad and let $X$ be an $\mathcal{O}$-algebra object of $\mathcal{T}$. Then $\mathcal{T}[X]$ is an $\mathcal{O}$-algebra object of $\text{Mod}_\mathcal{T}$.

We now assume that $\mathcal{O}$ is a unital and coherent $\infty$-operad. Since $\text{Mod}_\mathcal{T}$ is a symmetric monoidal $\infty$-category, we can consider $\mathcal{O}$-algebra objects in $\text{Mod}_\mathcal{T}$. We require $\mathcal{O}$ to be unital since we will need to consider the unit map $\eta: \mathcal{T} \to \mathcal{R}$ of an $\mathcal{O}$-algebra object $\mathcal{R}$ of $\text{Mod}_\mathcal{T}$.

Recall that if $\mathcal{R}$ is an object of $\text{Alg}_{/\mathcal{O}}(\text{Mod}_\mathcal{T})$ then $\text{Mod}^\mathcal{O}_{\mathcal{R}} = \text{Mod}^\mathcal{O}_{\mathcal{R}}(\text{Mod}_\mathcal{T})$ is the $\infty$-category of $\mathcal{R}$-module objects in $\text{Mod}_\mathcal{T}$ \cite[§3.3.3]{Lurie}.

**Proposition 7.15.** Let $\mathcal{O}$ be a coherent and unital $\infty$-operad and let $\mathcal{R}$ be an $\mathcal{O}$-algebra object of $\text{Mod}_\mathcal{T}$. Then there exists a unique colimit-preserving functor

$$\mathcal{R}[-]: \mathcal{T} \to \text{Mod}^\mathcal{O}_{\mathcal{R}}$$

whose value on the one-point space is the “free rank one” $\mathcal{R}$-module $\mathcal{R}$. 

Proof. The functor
\[ \eta^*: (-) \otimes \mathcal{R}: \text{Mod}_O^\mathcal{R} \to \text{Mod}_R^O \]
preserves colimits, as it is left adjoint to the restriction
\[ \eta_*: \text{Mod}_R^O \to \text{Mod}_O^\mathcal{R} \]
along \( \eta: \mathcal{R} \to \mathcal{R} \). Thus, the composite
\[ \mathcal{R}[-] \simeq \eta_* \circ \mathcal{T}[-]: \mathcal{T} \to \text{Mod}_O^\mathcal{R} \to \text{Mod}_R^O \]
preserves colimits, and sends the one-point space \( * \) to \( R \simeq \mathcal{T} \otimes \mathcal{R} \).

The main example of this phenomenon which will be of interest to us is the case of an \( O \)-algebra object \( R \) of \( \text{Mod}_S \), where \( O \) is a coherent \( \infty \)-operad under \( E_1 \). Then \( R \) is in particular an associative algebra object of \( \text{Mod}_S \), and so it has a Picard \( \infty \)-groupoid \( \text{Pic}(R) \), the full subgroupoid of \( R \) spanned by the invertible objects.

8. Picard \( \infty \)-groupoids of presentable monoidal \( \infty \)-categories

In this section we define and study the Picard \( \infty \)-groupoid of an \( O \)-monoidal stable presentable \( \infty \)-category \( R \) (for suitable \( \infty \)-operads \( O^\odot \)) and the categories of parametrized objects over Picard \( \infty \)-groupoids. Roughly speaking, we define the Picard \( \infty \)-groupoid of \( R \) as the space of invertible objects. Much of the work of the section is to keep track of the multiplicative structure. Note that contrary to the standard convention, our Picard \( \infty \)-groupoids will be grouplike \( O \)-spaces, not necessarily grouplike \( E_\infty \)-spaces.

We begin by recalling some details concerning grouplike \( E_1 \)-spaces. In any \( \infty \)-topos (in particular, such as the \( \infty \)-category of spaces), there is a notion of a grouplike \( E_1 \)-space \([21, 5.1.3.2]\). Specifically, we have the following characterization \([21, 5.1.3.5]\).

**Definition 8.1.** An \( E_1 \)-space \( X \) is said to be *grouplike* if the monoid \( \pi_0 X \) is a group. Given a map \( \eta: E_1 \to O \) of coherent \( \infty \)-operads, we say that an \( O \)-monoidal space \( X \) is *grouplike* if \( \eta^* X \) is a grouplike \( E_1 \)-space.

Given any \( O \)-monoidal space \( X \), we can restrict to the maximal grouplike subspace of \( X \).

**Lemma 8.2.** For an \( O \)-monoidal space \( X \), there is a maximal grouplike subspace \( GL_1 X \). That is, the inclusion
\[ \text{Mon}_{O^\odot}^\mathcal{R}(\mathcal{T}) \to \text{Mon}_O(\mathcal{T}) \]
of grouplike \( O \)-monoidal spaces into \( O \)-monoidal spaces has a right adjoint \( GL_1 \) given by passage to the maximal grouplike \( O \)-monoidal space.

**Proof.** The inclusion functor preserves colimits \([21, 5.1.3.5]\) and therefore the adjoint functor theorem implies that there exists a right adjoint \( GL_1 \). We can explicitly identify this as follows: Given an \( O \)-monoidal space \( X \), \( \pi_0(X) \) is a monoid. The maximal grouplike space \( GL_1 X \) is the full subgroupoid obtained by passage to the invertible elements of \( \pi_0(X) \) (i.e., the maximal group contained in \( \pi_0(X) \)). Since any product of invertible objects in \( \pi_0(X) \) is invertible, the criterion of \([21, 2.2.1.1]\) implies that this space is itself \( O \)-monoidal. Because \( GL_1 X \) is a full subgroupoid of \( X \), it is clear that any map from a grouplike \( O \)-monoidal space uniquely factors through it. \( \square \)
More generally, given any $O$-monoidal $\infty$-category $\mathcal{R}$, we can pass to the full subcategory of invertible objects in $\mathcal{R}$, which we will denote by $\mathcal{R}^\times$. Explicitly, this can be built as the pullback

\begin{equation}
(\mathcal{R}^\otimes)^\times \rightarrow \mathcal{R}^\otimes \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{Ho}(\mathcal{R}^\otimes)^\times \rightarrow \text{Ho}(\mathcal{R}^\otimes),
\end{equation}

where $\text{Ho}(\mathcal{R}^\otimes)^\times$ denotes the full monoidal subcategory of the (ordinary) monoidal category $\text{Ho}(\mathcal{R}^\otimes)$ spanned by the invertible objects. This is the $O$-monoidal $\infty$-category of invertible objects. The same argument as in the proof of lemma 8.2 proves the following lemma.

**Lemma 8.4.** For an $O$-monoidal $\infty$-category $\mathcal{R}$, the full $\infty$-subcategory $\mathcal{R}^\times$ of invertible objects is an $O$-monoidal $\infty$-category.

However, we want the Picard object to be a space. Recall that the inclusion of $\infty$-groupoids into $\infty$-categories preserves products and has a right adjoint; explicitly, if $\mathcal{C}$ is an $\infty$-category, then $\mathcal{C}_{\text{iso}}$ is the subcategory of $\mathcal{C}$ consisting of the invertible morphisms.

We can now define the Picard $\infty$-groupoid of an $E_1$ object in $\text{Pr}^L$.

**Definition 8.5.** Let $\mathcal{R}$ be an $S$-algebra in $\text{Pr}^L$. Then $\text{Pic}(\mathcal{R})$ is the maximal grouplike $\infty$-groupoid inside of the monoidal $\infty$-category $\mathcal{R}^\times$. When $\mathcal{R} = \text{Mod}_R$ for an $E_n$-ring spectrum $R$, $n > 1$, we typically write $\text{Pic}_R$ in place of $\text{Pic}(\mathcal{R})$.

**Remark 8.6.** In fact, we can perform this construction in either order. First, given $\mathcal{R}$, pass to the full $\infty$-subcategory $\mathcal{R}^\times$ of invertible objects in $\mathcal{R}$, and then take the maximal $\infty$-groupoid in $\mathcal{R}^\times$. Equivalently, given $\mathcal{R}$, pass to the maximal $\infty$-groupoid $\mathcal{R}_{\text{iso}}$ contained in $\mathcal{R}$, then pass to the largest grouplike object inside $\mathcal{R}_{\text{iso}}$.

Furthermore, if $\mathcal{R}$ is a closed symmetric monoidal stable $\infty$-category, we can characterize $\text{Pic}(\mathcal{R})$ as a subspace of the subcategory of dualizable objects in $\mathcal{R}$. (See for example [26, §2] for an excellent discussion of this perspective on the level of homotopy categories.)

**Lemma 8.7.** When $\mathcal{R}$ is a closed symmetric monoidal stable $\infty$-category with unit 1, the inverse of $X \in \text{Pic}(\mathcal{R})$ is the functional dual $F_{\mathcal{R}}(X,1)$.

**Proof.** The equivalences witnessing the invertibility of $X$ are duality data; this follows from [26, 2.9] since $\infty$-categorical duality can be detected on the homotopy category. \qed

**Remark 8.8.** It is not difficult to extend the description of equation 8.3 and lemma 8.7 to the situation when $\mathcal{R}$ has weaker monoidal structures, but to state the results requires a discussion of duality in these settings which we do not wish to pursue herein.

In order to obtain a multiplicative structure, we would like to describe $\text{Pic}(\mathcal{R})$ more explicitly as part of an adjunction. To make this precise, we first need the following result which allows us to control the size of the Picard group.
Theorem 8.9. Let $A$ be a monoidal presentable $\infty$-category. Then there exists a regular cardinal $\kappa$ such that the inclusion

$$\mathrm{Pic}(A^\kappa) \subseteq \mathrm{Pic}(A)$$

is an equivalence of $\infty$-groupoids. In particular, $\mathrm{Pic}(A)$ is essentially small.

Proof. By [21, Lemma 6.3.7.12], there exists a regular cardinal $\kappa$ such that $A$ is $\kappa$-presentable, the unit $1_A$ of $A$ is $\kappa$-compact, and the full subcategory $A^\kappa \subset A$ consisting of the $\kappa$-compact objects is a monoidal subcategory. Let $A \in \mathrm{Pic}(A)$ be an invertible object of $A$. Since $A \simeq \text{Ind}_\kappa(A^\kappa)$, $A = \text{colim}_I A_i$ for some $\kappa$-filtered diagram of $\kappa$-compact objects of $A$. Since $A$ has an inverse $B$, $1 \simeq A \otimes B \simeq \text{colim}_I (A_i \otimes B)$, and since $1$ is $\kappa$-compact, the equivalence $1 \to \text{colim}_I (A_i \otimes B)$ factors through a $\kappa$-small stage $J \subset I$. But then $1 \simeq \text{colim}_J (A_j \otimes B)$ implies that $\text{colim}_J A_j \simeq B^{-1} \simeq \text{colim}_I A_i$, so that $A$ is a $\kappa$-small colimit of $\kappa$-compact objects and hence itself is $\kappa$-compact. □

This now permits us to give the following characterization of Pic.

Theorem 8.10. Let $O$ be a coherent $\infty$-operad equipped with a map $A_\infty \to O$. Then

$$\mathrm{Pic} : \text{Alg}_O(\text{Pr}_L) \to \text{Alg}_O(\text{gp}(T))$$

is right adjoint to the free presentable $\infty$-category functor $T[-] : \text{Alg}_O(\text{gp}(T)) \to \text{Alg}_O(\text{Pr}_L)$.

Proof. Let $G$ be a grouplike $O$-monoidal space and $R$ a presentable $O$-monoidal $\infty$-category. Restriction along the Yoneda embedding $G \to T[G]$ gives maps

$$\text{map}_{\text{Alg}_O(\text{Pr}_L)}(T[G], R) \to \text{map}_{\widehat{\text{Cat}}_{\infty}}(T[G], U(R)) \to \text{map}_{\text{Alg}_O(\text{gp}(T))}(G, \text{Pic}(R)),$$

where $U : \text{Pr}_L \to \widehat{\text{Cat}}_{\infty}$ denotes the “underlying monoidal $\infty$-category” functor, such that the composite is an equivalence. □

Remark 8.11. The unit of the adjunction is the Yoneda embedding $G \to R[G]$. The counit of the adjunction is the map $T / \text{Pic}(R) \to R$ adjunct to the identity map $\text{Pic}(R) \to \text{Pic}(R)$. As a functor between presentable $\infty$-categories, this map preserves colimits and is uniquely determined by the image of $\text{Pic}(R)$ in $R$.

Remark 8.12. When $O$ is a model for the $E_n$ operad, we have the following specialization: The functor

$$\text{Pic} : \text{Alg}_{E_n}(\text{Pr}_L) \to \text{Alg}_{E_n}(\text{T})$$

is corepresented by the $E_n$-algebra $T[\Omega^n \Sigma^n_+ *]$. This is because $\Omega^n \Sigma^n_+ *$ is the free grouplike $E_n$-space on a single generator $*$. Passing to the stable setting, we obtain the following generalization:

Proposition 8.13. Let $A_\infty \to O$ be a map of coherent $\infty$-operads and let $R$ be a stable presentable $O$-monoidal $\infty$-category. Then the canonical map $S / \text{Pic}(R) \to R$ is a map of stable presentable $O$-monoidal $\infty$-categories.
Proof. This follows from the argument for theorem 8.10. □

We close with a remark about the way in which the work of this section is a categorification of the classical theory of the space of units of a ring spectrum. The multiplicative structure on $\text{Pic}(\mathcal{R})$ is such that the canonical map

$$S/\text{Pic}(\mathcal{R}) \rightarrow \mathcal{R},$$

adjoint to the inclusion $\text{Pic}(\mathcal{R}) \rightarrow \mathcal{R}$ of the invertible objects, is an $\mathcal{O}$-algebra map. Conceptually, this is a categorification of the adjunction which defines $GL_1$. Just as the underlying infinite loop space functor $\Omega^\infty: S \rightarrow \mathcal{T}$ is right adjoint to the symmetric monoidal suspension spectrum functor $\Sigma^\infty: \mathcal{T} \rightarrow S$, the forgetful functor

$$\text{map}_S: (S, -): \text{Pr}^L_{\text{St}} \rightarrow \mathcal{T}$$

is right adjoint to the symmetric monoidal functor

$$S[-]: \mathcal{T} \rightarrow \text{Pr}^L_{\text{St}}.$$

8.1. **The Brauer group and twisted parametrized spectra.** When applied to the category of modules over a commutative ring spectrum $R$, definition 8.5 recovers the usual construction of the Picard group. But we can also categorify the definition, as follows:

**Definition 8.14.** Let $R$ be an $E_\infty$-ring spectrum. The Brauer group of $R$ is

$$\text{Br}_R = \text{Pic}(\text{Mod}_{\text{Mod}_R}(\text{Pr}^L)^{cg}),$$

the Picard $\infty$-groupoid of the symmetric monoidal $\infty$-category $\text{Mod}_R^{cg}$ of compactly generated $\text{Mod}_R$-modules in $\text{Pr}^L$, the $\infty$-category of presentable $\infty$-categories.

It is straightforward to check that the Brauer group of $R$ provides a delooping of the Picard group.

**Lemma 8.15.** Let $R$ be an $E_\infty$-ring spectrum. There is a natural equivalence

$$\text{Pic}_R \simeq \Omega \text{Br}_R.$$

The connections to the classical definitions of the Brauer group have been studied by the third author with various collaborators [3, 14]. We do not go into detail about any of the applications of this here, other than to observe that definition 8.14 allows us to situate the work of Douglas on “twisted parametrized spectra” [12].

**Definition 8.16** (Haunts and specters). For a commutative $S$-algebra $R$, the $\infty$-category of $R$-haunts over a space $X$ is given by the $\infty$-category (actually, $\infty$-groupoid) $(\text{Br}_R)_{/X} = (\text{Pic}_{\text{Mod}_R})_{/X}$ of $\text{Mod}_R$-torsors over $X$. For a given haunt $\mathcal{K}$ on a space $X$, the $\infty$-category of specters is the limit of the composite

$$X \rightarrow \text{Br}_R \rightarrow \text{Mod}_{\text{Mod}_R}.$$

This can also be interpreted as the $\text{Mod}_R$-module of sections of an actual bundle of $\text{Mod}_R$-modules over $X$.

In light of this, it is straightforward to recover the various formal characterizations that Douglas obtains. In particular, we can expand Douglas’ sketch proof of the following illuminating comparison result:
Theorem 8.17. Let \( R \) be an \( \mathbb{E}_\infty \)-ring spectrum and let \( \mathcal{H} \) be a haunt classified by a pointed map \( f : X \rightarrow \text{Br}_R \) for a pointed connected space \( X \). Then the \( \infty \)-category of specters associated to \( \mathcal{H} \) is equivalent to the \( \infty \)-category of modules over the Thom spectrum associated to
\[
\Omega f : \Omega X \longrightarrow \Omega \text{Br}_R \cong \text{Pic}_R.
\]

9. Multiplicative structures on generalized Thom spectra

Proposition 8.13 has the following immediate consequence, which proves Theorem 1.8; this is a generalization of Lewis’ theorem.

Corollary 9.1. Let \( A_\infty \rightarrow \mathcal{O} \) be a map of \( \infty \)-operads and let \( \mathcal{R} \) be a stable presentable \( \mathcal{O} \)-monoidal \( \infty \)-category. The composite functor
\[
\mathcal{T}/\text{Pic}(\mathcal{R}) \longrightarrow \mathcal{S}/\text{Pic}(\mathcal{R}) \longrightarrow \mathcal{R}
\]
is a map of presentable \( \mathcal{O} \)-monoidal \( \infty \)-categories. Moreover, if \( \mathcal{R} = \text{Mod}_R \) for an \( \mathbb{E}_n \)-algebra object \( R \) of \( \mathcal{S} \), \( n > 1 \), then this composite is equivalent to the generalized Thom spectrum functor.

As an application, we use this to provide sharpenings of Lewis’ results about the multiplicative properties of the Thom isomorphism theorem. Recall that classically (working with topological operads), one can weaken the notion of an \( \mathbb{E}_\infty \) operad by requiring that the operadic structure maps need only commute in the homotopy category; this leads to the notion of an \( \mathbb{H}_\infty \) operad. More generally, for any operad \( \mathcal{V} \) there is a corresponding “operad up to homotopy” \( \overline{\mathcal{h}}\mathcal{V} \) obtained in this fashion.

Lewis proved [19, §IX.7.4] that given an \( \mathbb{E}_n \) classifying map \( f : X \rightarrow BGL_1S \) such that \( Mf \) admits an \( \mathbb{E}_n \) orientation over \( R \) (i.e., an \( \mathbb{E}_n \) map \( Mf \rightarrow R \)), then the map inducing the Thom isomorphism is an \( \overline{\mathcal{h}}\mathcal{h} \mathbb{E}_n \) map. In the \( \mathbb{E}_\infty \) setting, various authors (e.g., [6]) have observed that this equivalence can be promoted to an equivalence of \( \mathbb{E}_\infty \) ring spectra. We now prove analogous results for the case of any \( \mathbb{E}_n \) \( \infty \)-operad.

Assume that \( R \) is an \( \mathbb{E}_{n+1} \) ring spectrum and \( f \) is an object of the \( \infty \)-category \( \mathcal{Alg}_{/\mathbb{E}_n}(\mathcal{T}/BGL_1R) \), i.e., an \( \mathbb{E}_n \) map of spaces
\[
f : X \longrightarrow BGL_1R.
\]

One of the main theorems of our previous work on Thom spectra and units [1] shows that an orientation of the Thom spectrum \( Mf \) is specified by a map \( P \rightarrow GL_1R \) in \( \text{Mod}_{GL_1R} \), where here \( \text{Mod}_{GL_1R} \) is the \( \infty \)-category of \( GL_1R \)-modules in spaces. This suggests the following generalization of an orientation to the setting of \( \mathbb{E}_n \) maps.

Definition 9.2. Assume that \( R \) is an \( \mathbb{E}_{n+1} \)-ring spectrum. Let \( P \) be an object in \( \mathcal{Alg}_{/\mathbb{E}_n}(\text{Mod}_{GL_1R}) \). Then the space of \( \mathbb{E}_n \) orientations of \( P \) is the space of \( \mathbb{E}_n \)-algebra maps \( P \rightarrow GL_1R \) in \( \mathcal{Alg}_{/\mathbb{E}_n}(\text{Mod}_{GL_1R}) \).

It is convenient to view the Thom spectrum functor in this light, making use of the following lemma:

Lemma 9.3. There is an equivalence of \( \mathbb{E}_n \)-monoidal \( \infty \)-categories
\[
\mathcal{T}/BGL_1R \cong \text{Mod}_{GL_1R},
\]
and hence an equivalence of \( \infty \)-categories
\[
\mathcal{Alg}_{/\mathbb{E}_n}(\mathcal{T}/BGL_1R) \cong \mathcal{Alg}_{/\mathbb{E}_n}(\text{Mod}_{GL_1R}).
\]
As a consequence, the Thom spectrum functor can be written as the composite:

$$\text{Alg}_{/E_n}(\mathcal{F}_{/BGL_1 R}) \rightarrow \text{Alg}_{/E_n}(\mathcal{S}_{/BGL_1 R}) \rightarrow \text{Alg}_{/E_n}(\mathcal{S}),$$

where the first map is the stabilization.

Now given any object $P$ in $\text{Alg}_{/E_n}(\text{Mod}_{GL_1 R})$, we have the following version of the Thom diagonal, given by the $E_n$ map

$$\Delta: P \xrightarrow{id \times \ast} P \times (GL_1 R \times X)$$

where here $GL_1 R \times X$ is the free $GL_1 R$-module on the space $X$. We use the fact that both $X$ and $GL_1 R$ are based spaces.

Applying the Thom spectrum functor now yields a map

$$Mf \rightarrow Mf \wedge (R \wedge X_+),$$

of $E_n$-ring spectra.

On the other hand, given an orientation $P \rightarrow GL_1 R$, applying the Thom spectrum functor produces a map

$$Mf \rightarrow R$$

of $E_n$-ring spectra. Putting these together, we get the composite

$$Mf \rightarrow Mf \wedge (R \wedge \Sigma^\infty_+ X) \rightarrow R \wedge (R \wedge \Sigma^\infty_+ X) \rightarrow R \wedge \Sigma^\infty_+ X$$

which is a map of $E_n$-ring spectra realizing the Thom isomorphism:

**Theorem 9.4.** An $E_n$ orientation $P \rightarrow GL_1 R$ in $\text{Alg}_{E_n}(\text{Mod}_{GL_1 R})$ gives rise to a map of $E_n$-ring spectra

$$Mf \rightarrow R \wedge \Sigma^\infty_+ X$$

which is an equivalence and realizes the Thom isomorphism.

10. **Twisted cohomology theories and the twisted Umkehr map**

A basic application of Thom spectra is the construction of Umkehr or “wrong-way” maps in generalized cohomology theories. Roughly speaking, such maps arise from the composite of the Pontryagin-Thom map and the Thom isomorphism. In the absence of an orientation, one often wants to twist the Pontryagin-Thom map (and the cohomology theories) by a vector bundle, or more generally an arbitrary spherical fibration. These maps are closely related to the classical theory of transfer maps (e.g., the Becker-Gottlieb transfer [7]).

The modern viewpoint on such maps is that they arise from maps of parametrized spectra, for instance via a fiberwise Pontryagin-Thom map. Arguably, this perspective has driven much of the early development of parametrized homotopy theory (see for instance [11, 16]). In this section we develop this theory in the setting of our models of parametrized spectra and Thom spectra, using the framework of the Thom-theoretic model of twisted cohomology theories we introduced in [2]. The work of [10] implies that our Umkehr maps agree with all other possible definitions.

We take the perspective that Umkehr maps or transfer maps arise for an $E_n$-ring spectrum $R$ from maps

$$R_B \rightarrow X$$

of parametrized $R$-modules over a base space $B$ by applying the pushforward $p_!$ along the projection $p: B \rightarrow \ast$. The advantage of the parametrized viewpoint is that given “twisting data”

$$\alpha: B^{\text{op}} \rightarrow \text{Pic}_R \rightarrow \text{Mod}_R$$
we can consider the map induced by the fiberwise smash product
\[ \alpha \simeq R_B \land_R B \alpha \to X \land_R B \alpha. \]
We write \( \land_R B \) for the smash product of \( R \)-modules over \( B \); we can also think of this as smashing over the unit \( R_B \) of the monoidal structure.

Applying the pushforward \( p \) now yields the “twisted wrong-way map”
\[ p(R_B \land_R B \alpha) \to p(X \land_R B \alpha), \]
and passing to cohomology yields the twisted Umkehr map with domain the \( \alpha \)-twisted \( R \)-cohomology of \( B \). In settings of interest, the geometry of the situation allows us to identify \( p(X \land_R B \alpha) \), as we shall see in the discussion that follows.

10.1. The Becker-Gottlieb transfer. We begin by recalling the construction of the classical transfer map in the setting of parametrized homotopy theory; this story is beautifully explained in [27, §15.3], and our discussion in this section is simply the expression of that work in our setting. The transfer map arises from the categorical trace associated to the diagonal map of a stably dualizable space \( X \). Specifically, we have the composite
\[ S \xrightarrow{\eta} X \land DX \xrightarrow{DX \land X} DX \land X \xrightarrow{\text{id} \land \Delta} DX \land X \land X \xrightarrow{\epsilon \land \text{id}} S \land X \cong X, \]
where \( \eta \) and \( \epsilon \) are the unit and counit of the duality. These transfer maps satisfy a series of compatibility relations, see [27, 15.2.4]; the required conditions on the triangulation of the homotopy category hold here, either by comparison or as can be shown directly.

The key observation about duality in the parametrized setting is that we can characterize dualizability fiberwise:

**Lemma 10.1.** Let \( B \) be a space and \( X \in \text{Fun}(B^{op}, S) \) a parametrized spectrum. Then \( X \) is dualizable if and only if for each \( b \in B \), the value \( X(b) \) of \( X \) at \( b \) is a dualizable spectrum.

**Proof.** This follows from the fact that the smash product is computed pointwise. \( \square \)

In particular, given a map \( f : E \to B \) with stably dualizable (homotopy) fibers, such as a proper fibration, by adjoining a disjoint basepoint we get a diagonal map on \( \Sigma^\infty_B E_+ \) and so a transfer map
\[ S_B \to \Sigma^\infty_B E_+. \]
Pushing forward along the map \( B \to * \) now yields the classical transfer map
\[ \Sigma^\infty_+ B \to \Sigma^\infty_+ E. \]

Note that we can easily recover Dwyer’s generalization of the Becker-Gottlieb transfer [16]. Specifically, let \( R \) be an \( \mathbb{E}_\infty \)-ring spectrum and suppose that \( f : E \to B \) has homotopy fiber \( F \) such that \( R \land \Sigma^\infty_+ F \) is a dualizable object in the category of \( R \)-modules. Then the construction of the transfer in this setting gives rise to an \( R \)-module transfer map
\[ R \land \Sigma^\infty_+ B \to R \land \Sigma^\infty_+ E. \]

Summarizing, we obtain the following result of Dwyer.
Proposition 10.2. Let $R$ be an $\mathcal{E}_\infty$-ring spectrum and let $f: E \to B$ be a map of spaces with homotopy fiber $F$ such that $R \wedge \Sigma^\infty F$ is a dualizable object in $\text{Mod}_R$. Then the diagonal map on $E$ gives rise to a map of $R$-modules over $B$

$$R_B \to R \wedge \Sigma^\infty E_+$$

such that the pushforward along $B \to \ast$ is the $R$-module transfer

$$R \wedge \Sigma^\infty B \to R \wedge \Sigma^\infty E.$$ 

10.2. Duality and the Pontryagin-Thom map. The transfer map of the previous section is essentially a purely homotopy-theoretic construction. We now begin to study another kind of wrong-way map arising from duality that ultimately require geometry for their interpretation.

As above, recall that $\mathcal{I}$ denotes the $\infty$-category of spaces and $\mathcal{S}$ denote the $\infty$-category of spectra. We write $\mathcal{S}$ denote the sphere spectrum and, for a space $X$, we write $DX$ for the Spanier-Whitehead dual

$$DX = F(\Sigma^\infty X, \mathcal{S})$$

of the space $X$. We may regard $D$ as a sheaf of spectra $\mathcal{I}^{op} \to \mathcal{S}$ on $\mathcal{I}$.

Recall that applying $D$ to the unique map of spaces $X \to \ast$ yields a model of the (stable) Pontryagin-Thom map. This is the restriction to objects of an actual natural transformation of functors $\mathcal{I}^{op} \to \mathcal{S}$ from the constant functor on the sphere spectrum to $D$. Moreover, we can perform the Pontryagin-Thom map fiberwise over an arbitrary base space $B$. We write $(\mathcal{S}/B)_{\mathcal{S}/B}$ for the $\infty$-category of spectra over $B$ equipped with a map from the $B$-sphere $\mathcal{S}/B$.

Definition 10.3. The fiberwise Pontryagin-Thom map of spaces over $B$ is the functor

$$\text{PT}_B: (\mathcal{I}/B)^{op} \simeq ((\mathcal{I}/B)/_{B}^{op})^{op} (\Sigma^\infty B)_{\mathcal{S}/B}^{op} \xrightarrow{DB} (\mathcal{S}/B)_{\mathcal{S}/B}^{op},$$

where $\ast_B$ denotes the terminal object in $\mathcal{I}/B$. There is a spectrum-level map

$$p_! \circ \text{PT}_B: \mathcal{I}[B]^{op} \simeq ((\mathcal{I}/B)/_{B}^{op})^{op} (\Sigma^\infty B)_{\mathcal{S}/B}^{op} \xrightarrow{DB} \mathcal{S}[B]_{\mathcal{S}/B}^{op} \to \mathcal{S}^{p_{!}\mathcal{S}/B},$$

where $p: B \to \ast$ is the projection. In abuse of notation, we will denote the value of this functor on a space $X \to B$ over $B$ by $\text{PT}(X)$.

By our convention, $\text{PT}(X)$ is an object of spectra under $p_!\mathcal{S}/B$ and so we can regard it as a map

$$p_!\mathcal{S}/B \to p_!DB X.$$ 

As we have noted previously, the source is readily identified:

$$p_!\mathcal{S}/B \simeq \Sigma^\infty B.$$ 

To identify $p_!DB X$, one usually takes $f: X \to B$ to be some sort of family of compact manifolds, and finds that $p_!DB X$ can be identified with $B^{-Tf}$, the Thom spectrum of (minus the) bundle of tangents along the fiber of the family $f$. We now proceed to review the relevant geometry.
Remark 10.4. Using the symmetric monoidal structure $S \otimes_B$ on spectra over $B$, the fact that any space over $B$ is canonically a commutative coalgebra for the cartesian symmetric monoidal structure on $T/B$, and the fact that $PT_B$ is a composite of limit preserving functors, we may regard the fiberwise Pontryagin-Thom map as an $\text{CAlg}(\delta/B)$-valued sheaf on $T$.

10.3. Atiyah-Milnor-Spanier duality and Umkehr maps. Let $X$ be a compact manifold with tangent bundle $TX \to X$. Take an embedding $X \to \mathbb{R}^N$ with normal bundle $\nu_X$, and form the Pontryagin-Thom construction (collapse to a point the complement of a tubular neighborhood of $X$ in $\mathbb{R}^N$) to give a map

$$S^N \to X^{\nu_X}.$$ 

Desuspending $N$ times gives a stable map, the geometric Pontryagin-Thom map,

$$S \to X^{-TX}.$$

Atiyah-Milnor-Spanier duality compares the abstract and geometric Pontryagin-Thom maps, showing that $\mathbb{S} \to X^{-TX}$ is equivalent to the map $\mathbb{S} \to DX$.

More generally, let $j: W \to X$ be an embedding with normal bundle $\nu = \nu_{W,X}$. Up to a suspension, the geometric Pontryagin-Thom construction

$$X_+ \to W^\nu$$

is a model for $DX \to DW$. More precisely, suppose we have an embedding $k: X \to \mathbb{R}^N$, with normal bundle $\nu_X$. Then $kj: W \to \mathbb{R}^N$ is an embedding with normal bundle $\nu_W \cong \nu_{W,X} \oplus j^*\nu_X$. We have the composite

$$S^N \to X^{\nu_X} \to W^{\nu_W}$$

and desuspending $N$ times gives

$$S \to X^{-TX} \to W^{-TW}.$$

Summarizing, we have the following classical result.

Proposition 10.5. The geometric Pontryagin-Thom map

$$S \to X^{-TX}$$

realizes the map $PT_*(X)$ of definition 10.3 (here $B = \ast$), in the sense that there is an equivalence $X^{-TX} \to DX$ which makes the induced diagram commute up to homotopy.

If $j: W \to X$ is an embedding, then the geometric Pontryagin-Thom construction gives maps

$$S \to X^{-TX} \to W^{-TW}$$

realizing the associated maps

$$S \to DX \to DW$$

under a choice of equivalences $X^{-TX} \to DX$ and $W^{-TW} \to DW$. 

10.4. Parametrized manifolds and fiberwise Atiyah-Milnor-Spanier duality. Returning to the situation of section 10.2, we’d like to consider a family of compact manifolds over $B$, apply Atiyah duality as in section 10.3 fiberwise, and so identify the fiberwise dual. It’s not hard to construct families of manifolds, but the usual construction of the Atiyah duality equivalence

$$X^{-T X} \to DX$$

does not have attractive functoriality and naturality properties, and so it is not straightforward to assemble the fiberwise maps.

To handle this, we restrict to a fairly rigid geometric situation. Our motivating example is when $f : Y \to B$ is a proper smooth submersion which is also a fibration. The bundle of tangents along the fiber is the kernel in the sequence

$$T f \to TY \xrightarrow{df} f^*TB.$$ 

At each $b \in B$, the fiber of $f$ at $b$ is a compact manifold $Y_b$, with tangent bundle $T f|_{Y_b}$. If we let $Y \to \text{Pic}_S$ classify the (reduced) negative bundle of tangents along the fiber $-T f$, we expect to identify the Pontryagin-Thom map as

$$\text{PT}(f) : \Sigma_\infty^\infty B \to Y^{-T f}.$$ 

Let $M$ denote the $\infty$-category of smooth compact manifolds and diffeomorphisms. Specifically, $M$ is obtained as the nerve of the ordinary category of smooth compact manifolds. Associated to an object $M \in M$, we have the parametrized spectrum

$$S^{-T M}_M : M^{\text{op}} \to \mathcal{S}$$

associated to the negative tangent bundle. The value of $S^{-T M}_M$ at $m \in M$ is the dual of the Thom space of the tangent space at $m$.

We now adapt this construction to the setting of bundles of compact manifolds. An object $X \in M/B$ is by definition a map $B^{\text{op}} \to M$. Since there is a forgetful functor $M \to \mathcal{T}$, we may also regard $X$ as an object of $\mathcal{T}/B$, and hence a map $f : X \to B$. We obtain a spectrum $S^{-T f}_X$ over $X$ by gluing together the spectra $S^{-T X_b}_X$ defined by the compatible family of manifolds $X_b$. We are allowed to do this because the functor $S(-) : \mathcal{T}^{\text{op}} \to \text{Pr}^L$ is a stack and hence satisfies descent. Specifically,

$$X \simeq \colim_{b \in B^{\text{op}}} X_b,$$

so that

$$S/X \simeq \lim_{b \in B} S/X_b.$$ 

Finally, observe that each of the fibers of $S^{-T f}_X$ is invertible, and so the classifying map for the spectrum $S^{-T f}_X$ factors as

$$S^{-T f}_X : X \to \text{Pic}_S \to \text{Mod}_S.$$ 

That is, we have the following.

**Lemma 10.6.** Let $X$ be an object of $M/B$, regarded as a space $f : X \to B$ over $B$. The fiberwise spectra $S^{-T X_b}_X$ assemble into a family of invertible spectra $S^{-T f}_X$ over $X$. Viewing these fiberwise over $B$, we may regard $S^{-T f}_X$ as classified by a map

$$B^{\text{op}} \xrightarrow{S^{-T f}_X} \mathcal{T}/\text{Pic}_S.$$
For conciseness, we will often expose this data by writing
\[
\text{Pic}_\mathbb{S} \xrightarrow{-Tf} X \xrightarrow{f} B
\]
and refer to this situation as a “family of compact manifolds”.

By applying the functor \( D \) to the composite
\[
B^{\text{op}} \xrightarrow{X} M \xrightarrow{T} \Sigma^\infty_+ \xrightarrow{\Sigma} \mathbb{S},
\]
we obtain a parametrized spectrum \( D_B X \). On the other hand, pushing forward the spectrum \( \mathbb{S}^{-Tf}_X \) along the map \( f: X \to B \) yields a parametrized spectrum \( \text{Th}_B(-Tf) \).

**Definition 10.7.** A fiberwise Atiyah-Milnor-Spanier duality for the family of manifolds \( X \to B \) is an equivalence of spectra over \( B \)
\[
(10.8) \quad D_B X \simeq \text{Th}_B(-Tf).
\]

As we have defined it, any family of compact manifolds admits a fiberwise Atiyah-Milnor-Spanier duality. Specifically, since the fibers are diffeomorphic, we can choose a single Atiyah-Milnor-Spanier equivalence and assemble this map to a fiberwise Atiyah-Milnor-Spanier duality.

Using this, we can study the Umkehr map. First, recall from definition 10.3 that we have a map of spectra over \( B \)
\[
\text{PT}_B(X): \mathbb{S}_B \to D_B X.
\]
An equivalence (10.8) allows us to view this as a map
\[
\mathbb{S}_B \to \text{Th}_B(-Tf).
\]
Pushing forward along the projection \( p: B \to * \) we obtain a geometric Pontryagin-Thom map of spectra
\[
\Sigma^\infty_+ B \simeq p! \mathbb{S}_B^0 \to p! \text{Th}_B(-Tf).
\]

We can identify \( p! \text{Th}_B(-Tf) \) using the following lemma.

**Lemma 10.9.** We have an equivalence of spectra over \( B \)
\[
(10.10) \quad \text{Th}_B(-Tf) \simeq f_!(Tf),
\]
and an equivalence of spectra
\[
p_! \text{Th}_B(-Tf) \simeq X^{-Tf}.
\]

**Proof.** For the first assertion, observe that it suffices to check the claim at each fiber. Choose \( b \in B \) and let \( X_b \) be the fiber of \( f: X \to B \) over \( B \). Then the square is a pull-back diagram in
\[
\begin{array}{ccc}
X_b & \xrightarrow{j'} & X & \xrightarrow{-Tf} & \text{Pic}_\mathbb{S} \xrightarrow{f} \mathbb{S} \\
\{b\} & \xrightarrow{j} & \{b\} & \xrightarrow{f} & B
\end{array}
\]
and so we have
\[
j^* f_!(Tf) \simeq (j')_!(j')^*(-Tf)
\]
in \( \mathbb{S} \). The left side is the fiber of \( f_!(Tf) \) at \( b \). The right side is the Thom spectrum of \( -Tf \) at \( b \).
The second assertion is simply a statement about colimits commuting:

\[ p_! \Th_B(-Tf) \simeq \colim(B^{\text{op}} \xrightarrow{\mathcal{X}} \mathcal{X}/\text{Pic}_S \xrightarrow{\text{Th}} S) \]

\[ \simeq \Th(\colim(B^{\text{op}} \xrightarrow{\mathcal{X}} \mathcal{X}/\text{Pic}_S)) \]

\[ \simeq \Th(X \rightarrow \text{Pic}_S) = X^{-Tf}. \]

□

We summarize the situation in the following.

**Proposition 10.11.** Let

\[ \text{Pic}_S \xleftarrow{\mathcal{X}} X \xrightarrow{f} B \]

be a family of compact manifolds over \( B \), equipped with a parametrized Atiyah-Minor-Spanier duality

\[ D_B X \simeq \Th_B(-Tf). \]

Let \( p: B \rightarrow * \) be the projection. Then there are natural equivalences

\[ D_B X \simeq \Th_B(X) \]

\[ p_! S_B \simeq \Sigma^\infty B \]

\[ p_! \Th_B(X) \simeq X^{-Tf}, \]

and so the fiberwise Pontryagin-Thom map

\[ \text{PT}_B(X): S_B \rightarrow D_B X \simeq \Th_B(-Tf) \]

gives rise to the geometric Pontryagin-Thom map

\[ \Sigma^\infty \rightarrow \Sigma^\infty W^\nu \]

Becker and Gottlieb explain how to connect the Umkehr map with the transfer map [7]. In our framework, their argument shows that the composite of the Umkehr map of Proposition 10.11 and the cup product with the Euler class is equivalent to the transfer map constructed in section 10.1.

**Remark 10.12.** We also expect to have Umkehr maps arising from embeddings of manifolds. Let \( j: W \rightarrow M \) be an embedding of manifolds with normal bundle \( \nu \), and let \( p: M \rightarrow * \) be the map to a point. To obtain a similar view of the Umkehr map \( j \), we need to realize the geometric Pontryagin-Thom map

\[ \Sigma^\infty \rightarrow \Sigma^\infty \rightarrow \Sigma^\infty W^\nu \]

as \( p_! \alpha \), where \( t \) is a map of spectra over \( M \). Now if \( S^\nu \) is the parametrized space associated to \( \nu \), then

\[ W^\nu \simeq p_! \alpha : S^\nu \]

and this suggests that the map we seek is a suspension of a map of the form

\[ \alpha: S^0_M \rightarrow j_! S^\nu. \]

The requisite map is constructed by May and Sigurdsson [27, 18.6.3, 18.6.5].
10.5. Twists. Finally, we use the monoidal structures on our \( \infty \)-categories of parametrized spectra to twist the various Pontryagin-Thom maps we’ve produced. The basic idea is that the description of the Pontryagin-Thom map as the pushforward of a map of parametrized spectra allows us to twist. To explain the strategy precisely, it is useful to abstract the underlying phenomenon at work in our construction of the parametrized Umkehr maps.

Suppose that we have a map of spaces \( f: X \to Y \), a parametrized spectrum \( M \) over \( S \) and a parametrized spectrum \( N \) over \( Y \), and a map \( \varphi: N \to f_! M \) of parametrized spectra over \( Y \). As long as \( N \in S_Y \) is dualizable, we may smash over \( Y \) with the dual \( F_T(N, S_Y) \) of \( N \) to obtain a parametrized Pontryagin-Thom construction.

**Definition 10.13.** Let \( f: X \to Y \) be a map of spaces, let \( M \in S_X \) and \( N \in S_Y \) be parametrized spectra such that \( N \) is dualizable, and let \( \varphi: N \to f_! M \) be a map. The parametrized Pontryagin-Thom map is the composite

\[
S_Y \longrightarrow F_Y(N, S_Y) \land Y N \xrightarrow{\text{id} \land Y \varphi} F_Y(N, S_Y) \land Y f_! M \simeq f_!(f^* F_Y(N, S_Y) \land_X M) \\
\simeq f_!(f_X(f^* N, S_X) \land_X M) \\
\simeq f_! f_X(f^* N, M).
\]

**Remark 10.14.** Definition 10.13 makes sense in the category of parametrized \( R \)-modules for any \( E_n \)-ring spectrum \( R \) provided one has a projection formula relating \( f_! \) and \( f^* \). As such, there is no issue if \( n > 3 \) and it seems likely that this works for any \( n > 1 \).

Now recall the notion of a twist (see also [2]).

**Definition 10.15.** Let \( R \) be an \( E_{n+1} \)-ring spectrum. A twist is a map

\[
\alpha: B \longrightarrow \text{Pic}_R \longrightarrow \text{Mod}_R
\]

classifying a bundle of invertible \( R \)-modules over \( B \). Notice that the \( \infty \)-category \( \mathcal{T}_{/\text{Pic}_R} \) of twists is \( E_n \)-monoidal.

Given a twist \( \alpha: B \to \text{Pic}_R \to \text{Mod}_R \) and a generalized wrong-way map

\[
R_B \longrightarrow f_! (\mathcal{T}_f) = f_! f_X(f^* N, M),
\]

we can twist the map by fiberwise smash (i.e., using the monoidal structure on \((\text{Mod}_R)_{/Y}\)) to obtain

\[
R_B \land_{R_B} \alpha \longrightarrow f_! f_X(f^* N, M) \land_{R_B} \alpha.
\]

We can interpret this using our definition of twisted cohomology. As explained in [1, 2], \( M\alpha = p_!(R_B \land_{R_B} \alpha) \) is the generalized Thom spectrum of the map \( \alpha \); by definition, its \( R \)-cohomology is the \( \alpha \)-twisted \( R \)-cohomology of \( B \).

\[
R^k(B)^\alpha = \pi_0 \text{map}_R(M\alpha, \Sigma^k R).
\]

**Remark 10.16.** When \( R \) is the K-theory spectrum (real or complex), the question arises of comparing our version of twisted \( K \)-theory to the Atiyah-Segal construction of twisted \( K \)-theory in terms of Fredholm operators. In [2, §5], we interpret the Atiyah-Segal construction as associating to a twist \( f: X \to BGL_1 A \) the spectrum

\[
\Gamma_X(f) \simeq S_X(S_X, f),
\]
i.e., the spectrum of sections of $f$. Further, we explain how this spectrum is equivalent to the Thom spectrum functor applied to the twist $-f$ (the image under the involution $-1: BGL_1A \to BGL_1A$).

One might worry however that the geometric aspects of the Atiyah-Segal construction associate to a twist $X \to K(\mathbb{Z}/2, 2)$ a composite other than the that induced by the inclusion $K(\mathbb{Z}/2, 2) \to BGL_1KO$, and so there could be a potential discrepancy. But in [4], the third author (along with Antieau and Gomez) prove that up to homotopy any map $j: K(\mathbb{Z}/2, 2) \to BGL_1KO$ is either trivial or the canonical inclusion (and similarly for $KU$).

The monoidal structure on the category of twists gives rise to a product in twisted cohomology:

**Theorem 10.17.** Let $R$ be an $E_{n+1}$ ring spectrum. For any space $X$, the $E_n$ monoidal structure on $T_{/Pic_R}$ gives rise to a product map

$$R^m(X)^\alpha \otimes R^n(X)^\beta \to R^{m+n}(X)^{\alpha + \beta}.$$ 

We now specialize to the geometric example considered in section 10.4. For convenience, we use exponential notation for Thom spectra; e.g., $M\alpha$ will be written as $B\alpha$. We continue to let $Pic_S \xleftarrow{T_f} X \xrightarrow{f} B$ denote a family of compact manifolds over a space $B$, and we write $p$ for the map $B \to \ast$. Given a twist $\alpha$, we can then form the map of $R$-modules over $B$

$$PT_B(X) \otimes R_B \alpha \to Th_B(-Tf) \otimes R_B \alpha.$$ 

Applying the pushforward $p_!: (\text{Mod}_R)_B \to \text{Mod}_R$ gives rise to the twisted Pontryagin-Thom map.

**Definition 10.18.** The twisted Pontryagin-Thom map is defined as:

$$\text{PT}(X, \alpha) = p_!(PT_B(X) \otimes R_B \alpha \to Th_B(-Tf) \otimes R_B \alpha).$$

As for $Th_B(-Tf) \otimes R_B \alpha$, the projection formula yields

$$Th_B(-Tf) \otimes R_B \alpha \simeq f_!(R \otimes S_X^{Tf}) \otimes R_B \alpha$$

$$\simeq f_!(((R \otimes S_X^{Tf}) \otimes R_X f^* \alpha),$$

and so

$$p_!(Th_B(-Tf) \otimes R_B \alpha) \simeq p_! f_!(((R \otimes S_X^{Tf}) \otimes R_X f^* \alpha)$$

is the $R$-module Thom spectrum whose $R$-module cohomology is the cohomology of $X$, twisted by the sum of

$$X \xrightarrow{Tf} Pic_S \to Pic_R \to \text{Mod}_R$$

and

$$X \xrightarrow{f} B \xrightarrow{\alpha} Pic_R \to \text{Mod}_R.$$ 

That is, we have the following.

**Proposition 10.19.** Let

$$Pic_S \xleftarrow{Tf} X \xrightarrow{f} B$$
be a family of compact manifolds over $B$, equipped with a parametrized Pontryagin-Thom equivalence, and let $\alpha: B \to \text{Pic}_R \to \text{Mod}_R$ be a parametrized invertible $R$-module over $B$. Then we have an equivalence of $R$-modules

$$X^{-T_f + f\alpha} \simeq p_!(\text{Th}_B(-T_f) \wedge_R \alpha),$$

and so the twisted Pontryagin-Thom map $\text{PT}(X, \alpha)$ may be viewed as a map of $R$-modules

$$\text{PT}(X, \alpha): B^\alpha \to X^{-TX + f\alpha}.$$ 

This leads to the following definition.

**Definition 10.20.** In the situation of proposition 10.19, the $\alpha$-twisted cohomology of the Thom spectrum $X^{-T_f}$ is

$$R^k(X^{-T_f})^\alpha = \pi_0 \text{map}_R(X^{-T_f + \alpha f}, \Sigma^k R).$$

Passing to $R$-cohomology, we obtain the twisted Umkehr map

$$R^*(X^{-T_f})^\alpha \to R^*(B)^\alpha.$$ 

We close with an example of interest.

**Example 10.21.** Suppose that $\alpha$ makes the diagram

$$\begin{array}{ccc}
X & \xrightarrow{-T_f} & \text{Pic}_S \\
\downarrow f & & \downarrow \\
B & \xrightarrow{\alpha} & \text{Pic}_R
\end{array}$$

commute in the homotopy category. A choice of homotopy between the two compositions determines an equivalence of $R$-modules

$$X^{-T_f + \alpha f} \simeq \Sigma^\infty_+ X \wedge R,$$

so the twisted Pontryagin-Thom map takes the form

$$\text{PT}(X, \alpha): B^\alpha \to \Sigma^\infty_+ X \wedge R,$$

and passing to $R$-cohomology yields a twisted Umkehr map

$$R^*(X) \to R^*(B)^\alpha.$$ 

**References**

[1] M. Ando, A. J. Blumberg, D. Gepner, M. Hopkins, and C. Rezk. Units of ring spectra and Thom spectra. Preprint, arxiv:0810.4535v3, 2008.

[2] M. Ando, A. J. Blumberg, and D. Gepner. Twists of TMF and $K$-theory. Preprint, 2009.

[3] B. Antieau and D. Gepner. A cohomological characterization of the Brauer group of a ring spectrum. Preprint, 2011.

[4] B. Antieau, D. Gepner, and J. M. Gomez. Actions of Eilenberg-Mac Lane spaces on $K$-theory spectra and uniqueness of twisted $K$-theory. Preprint, arxiv:1106.5099v2, 2011.

[5] M. F. Atiyah and I. M. Singer. The index of elliptic operators. IV. Ann. of Math. (2), 93:119–138, 1971.

[6] M. Basterra and M. A. Mandell. Homology and cohomology of $E_\infty$ ring spectra. Math. Z., 249(4):903–944, 2005.

[7] J. C. Becker and J. H. Gottlieb. Transfer maps for fibrations and duality. Compositio Math., 33(2):107–133, 1976.

[8] A. J. Blumberg, R. Cohen, and C. Schlichtkrull. Topological Hochschild homology of Thom spectra and the free loop space. Geometry and Topology, 2010.
[9] Alan L. Carey and Bai-Ling Wang. Thom isomorphism and push-forward map in twisted $K$-theory, arxiv:math/0507414.

[10] R. Cohen and J. Klein. Unkern maps. Homology Homotopy Appl., 11(1):17-33, 2009.

[11] M. Crabb and I. James. Fiberwise homotopy theory. Springer-Verlag, Berlin, 1998.

[12] C. L. Douglas. Twisted parametrized spectra. Preprint, arXiv:0508070, 2005.

[13] H. Fausk, P. Hu, and J. P. May. Isomorphisms between left and right adjoints. Theory and Appl. of Categories, 11:107-131, 2003.

[14] D. Gepner and T. Lawson. Computing the relative Brauer group of a ring spectrum. Preprint, 2011.

[15] D. Dugger. Combinatorial model categories have presentations. Adv. Math. 164(1) (2001), 177–201.

[16] W. G. Dwyer. Transfer maps for fibrations. Math. Proc. Camb. Phil. Soc. 120:221-235, 1996.

[17] Daniel S. Freed and Edward Witten. Anomalies in string theory with D-branes. Asian J. Math., 3(4):819–851, 1999.

[18] M. Hovey. Spectra and symmetric spectra in general model categories. J. Pure Appl. Algebra, 165(1):63–127, 2001.

[19] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. Equivariant stable homotopy theory, volume 1213 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[20] J. Lurie, Higher topos theory. Annals of Mathematics Studies, 170, Princeton University Press, 2009.

[21] J. Lurie, Higher Algebra. Preprint, available at www.math.harvard.edu/lurie.

[22] M. Mahowald. Ring spectra which are Thom complexes. Duke Math. J., 46:549–559, 1979.

[23] M. A. Mandell. The smash product for derived categories in stable homotopy theory. Preprint, arxiv:1004.0006, 2010.

[24] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. Mem. Amer. Math. Soc., 159(755), 2002.

[25] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. Proc. London Math. Soc. (3), 82(2):441–512, 2001.

[26] J. P. May. Picard groups, Grothendieck rings, and Burnside rings of categories. Advances in Math., 163(1):1–16, 2001.

[27] J. P. May and J. Sigurdsson. Parametrized homotopy theory, volume 132 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2006.

[28] V. Voevodsky. Lectures on cross functors. Preprint, 2001.

[29] Bai-Ling Wang. Geometric cycles, index theory and twisted $K$-homology. J. Noncommut. Geom., 2(4):497–552, 2008.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA IL 61801, USA
E-mail address: mando@math.uiuc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TX 78703
E-mail address: blumberg@math.utexas.edu

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT REGENSBURG, 93040 REGENSBURG, GERMANY
E-mail address: djgepner@gmail.com