CURVATURE PROPERTIES OF GÖDEL METRIC

RYSZARD DESZCZ, MARIAN HOTLOŠ, JAN JELOWICKI, HARADHAN KUNDU AND ABSOS ALI SHAIKH

Abstract. The main aim of this article is to investigate the geometric structures admitting by the Gödel spacetime which produces a new class of semi-Riemannian manifolds (see Theorem 4.1 and Theorem 4.4). We also consider some extension of Gödel metric (see Example 4.1).

1. Introduction

In 1949 Gödel [1] obtained an exact solution of Einstein field equation with a non-zero cosmological constant corresponding to a universe in rotation and with an incoherent matter distribution. In that paper he described a metric nowadays called Gödel metric as exact and stationary solution of Einstein field equation, which describes a rotating, homogeneous but non-isotropic spacetime. Possessing a series of strange properties, it remains still today quite interesting mathematically and significant physically. For example, it contains rotating matter but have not singularity, and also it is cyclic Ricci parallel [2]. It is known that the Weyl conformal tensor of the Gödel solution has Petrov type D, and Gödel solution is, up to local isometry, the only perfect fluid solution of Einstein field equation admitting five dimensional Lie algebra of Killing vectors. Gödel spacetime is geodesically complete, its timelike curves are closed [3]. Also Gödel spacetime is not globally hyperbolic but diffeomorphic to $\mathbb{R}^4$ and is simply connected. Gödel metric is the Cartesian product of a factor $\mathbb{R}$ with a three dimensional Lorentzian manifold with signature $(-+++)$.

Gödel metric and its properties have been studied by various authors to describe the Gödel universe. Kundt [4] studied its geodesics in 1956, and Hawking and Ellis [5] emphasized on coordinates showing its rotational symmetry to draw a nice picture of its dynamics in their book in 1973. Malament [6] calculated the minimal energy of a closed timelike curve of Gödel spacetime. In 2001 Radojević [7] presented modification of Gödel metric in order to find out some other perfect fluid solutions. Induced matter theory and embedding of Gödel universe...
in five-dimensional Ricci flat space was studied by Fonseca-Neto et. al. [8] in 2005. Riemann extension of Gödel metric was considered by Dryuma [2] in 2005, and Dautcourt et. al. [9] studied light cone of Gödel universe. Lanczos spin tensor of Gödel geometry was studied by García-Olivo et. al. [10] in 2006. Gödel metric in various dimensions was studied by Gürses et. al. [11]. Generalized Gödel metric is given by Plaue et. al. [12] in 2008. The Gödel metric [1] is given by:

\[ ds^2 = g_{ij} dx^i dx^j = a^2 \left( -(dx^1)^2 + \frac{1}{2} e^{2x^1} (dx^2)^2 - (dx^3)^2 + (dx^4)^2 + 2 e^{x^1} dx^2 dx^4 \right), \]

where \(-∞ < x^i < ∞, i, j \in \{1, 2, 3, 4\}\) and \(a^2 = \frac{1}{ω^2}\), \(ω\) is a non-zero real constant, which turns out to be the angular velocity, as measured by any non-spinning observer located at any one of the dust grains.

The object of the paper is to present the curvature properties of Gödel metric. Section 2 deals with semi-Riemannian manifolds with cyclic parallel and Codazzi type Ricci tensor and we provide a metric whose Ricci tensor is of Codazzi type but not cyclic parallel (see Example 2.1). However, Gödel spacetime is a manifold with cyclic parallel Ricci tensor but the Ricci tensor is not of Codazzi type. Section 3 is concerned with rudiments of pseudosymmetry type manifolds, and in the last section we investigate the geometric structures admitting by Gödel metric. Among others, it is shown that Gödel spacetime is neither pseudosymmetric nor Ricci pseudosymmetric but it is quasi-Einstein and a special type of Ricci generalized pseudosymmetric (i.e., \(R \cdot R = Q(S, R)\)). Although, it is not conformally pseudosymmetric but its Weyl conformal curvature tensor is pseudosymmetric (i.e., \(C \cdot C = \frac{9}{6} Q(g, C)\)) (see Theorem 4.1). Hence Gödel spacetime induces a new class of semi-Riemannian manifolds which are quasi-Einstein with pseudosymmetric Weyl conformal curvature tensor satisfying \(R \cdot R = Q(S, R)\). Finally, we consider some extension of Gödel metric (see Example 4.1).

2. MANIFOLDS WITH CYCLIC PARALLEL AND CODAZZI TYPE RICCI TENSOR

Let \((M, g)\), \(\text{dim } M = n \geq 3\), be a connected paracompact manifold of class \(C^\infty\) with the metric \(g\) of signature \((s, n-s)\), \(0 \leq s \leq n\). The manifold \((M, g)\) will be called a semi(pseudo)-Riemannian manifold. Clearly, if \(s = 0\) or \(s = n\) then \((M, g)\) is a Riemannian manifold. If \(s = 1\) or \(s = n - 1\) then \((M, g)\) is a Lorentzian manifold. Further, let \(\nabla\), \(R\), \(S\) and \(κ\) be the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature of the semi-Riemannian manifold \((M, g)\), respectively.
The semi-Riemannian manifold \((M, g)\) is called locally symmetric if \(\nabla R = 0\) (locally \(R^h_{ijkl} = 0\)), which is equivalent to the fact that for each point \(x \in M\) the local geodesic symmetry is an isometry. For 2-dimensional manifolds being of locally symmetric and being of constant curvature are equivalent. But for \(n \geq 3\), the locally symmetric manifolds are a generalization of the manifolds of constant curvature. A full classification of locally symmetric manifolds is given by Cartan [13] for the Riemannian case, and Cahen and Parker ([14], [15]) for the non-Riemannian case.

The semi-Riemannian manifold \((M, g)\) is said to be Ricci symmetric if \(\nabla S = 0\) (locally \(S_{ij,k} = 0\)). Every locally symmetric semi-Riemannian manifold is Ricci symmetric but not conversely. However, the converse statement is true when \(n = 3\). For a compendium of natural symmetries of semi-Riemannian manifolds, we refer to [16] and [17]. We mention that Gray in [18], among other things, investigated various extensions of the class of Ricci symmetric manifolds. We denote by \(A\) the class of semi-Riemannian manifolds whose Ricci tensor \(S\) is cyclic parallel, i.e.,

\[
(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0
\]

for all vector fields \(X, Y, Z \in \chi(M)\), \(\chi(M)\) being the Lie algebra of all smooth vector fields on \(M\). The local expression of (2.1) is \(S_{ij,k} + S_{jk,i} + S_{ki,j} = 0\). A semi-Riemannian manifold satisfying (2.1) is said to be a manifold with cyclic parallel Ricci tensor. We mention that D’Atri and Nickerson [19] proposed to study some class of Riemannian manifolds whose curvature tensor satisfies certain conditions of which the first one is equivalent to (2.1).

Evidently, every Ricci symmetric semi-Riemannian manifold is a manifold with cyclic parallel Ricci tensor but not conversely. However, the converse statement is true if the Ricci tensor is a Codazzi tensor. We recall that an \((0, 2)\)-symmetric tensor \(B\) is said to be a Codazzi tensor if it satisfies the Codazzi equation, i.e. \((\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z)\). The local expression of the last equation is \(B_{ij,k} = B_{k,j,i}\), where \(B_{ij}\) are the local components of the tensor \(B\). A Codazzi tensor is trivial if it is a constant multiple of the metric tensor [20]. We denote by \(B\) the class of semi-Riemannian manifolds with Ricci tensor \(S\) as Codazzi tensor, i.e. \(S\) satisfies \((\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)\), for all vector fields \(X, Y, Z \in \chi(M)\). Every Ricci symmetric semi-Riemannian manifold is of class \(B\) but not conversely. Codazzi tensors are of great interest in the geometric literature and have been studied by several authors, as Berger and Ebin [21],
We note that every semi-Riemannian manifold of constant curvature and hence Einstein semi-Riemannian manifold is of class $\mathcal{A}$ as well as of $\mathcal{B}$. We note that the scalar curvature $\kappa$ of every semi-Riemannian manifold of the class $\mathcal{A}$ or $\mathcal{B}$ is constant. We note that Gödel spacetime is of class $\mathcal{A}$ but not of class $\mathcal{B}$.

It is known that Cartan hypersurfaces are Riemannian manifolds, with non-parallel Ricci tensor, satisfying the generalized Einstein metric condition \((2.1)\) [28, Theorem 4.1]. As it was noted in [29](p. 109), the Cartan hypersurfaces do not satisfy \((2.2)\)

\[
\nabla_Z \left( S(X,Y) - \frac{\kappa}{2(n-1)} g(X,Y) \right) = \nabla_Y \left( S(X,Z) - \frac{\kappa}{2(n-1)} g(X,Z) \right).
\]

We mention that \((2.2)\) is presented in the Table 1, pp. 432-433 of [27]. We also refer to [27] for results on Riemannian manifolds satisfying \((2.2)\). This was noted that Codazzi tensors occur naturally in the study of harmonic Riemannian manifolds. The Ricci tensor is a Codazzi tensor if and only if $\text{div } R = 0$ i.e., if and only if the manifold has harmonic curvature tensor [27].

We note that in a 3-dimensional Riemannian manifold \((M, g)\), the following conditions: (a) \((M, g)\) is locally symmetric, (b) \((M, g)\) is Ricci symmetric and (c) \((M, g)\) is a conformally flat manifold with cyclic parallel Ricci tensor, are equivalent [30]. Also for a Riemannian manifold \((M, g)\), $n \geq 4$, the following conditions: (a) \((M, g)\) is Ricci symmetric, (b) \((M, g)\) is a manifold with cyclic parallel Ricci tensor and harmonic conformal curvature tensor, and (c) \((M, g)\) is a manifold with cyclic parallel and Codazzi type Ricci tensor, are equivalent [30].

Example 2.1 (i) Let $M = \{(x^1, x^2, x^3, x^4) : x^i > 0, i = 1, 2, 3, 4\}$ be the subset of $\mathbb{R}^4$ endowed with the metric $g$ defined by $ds^2 = \varepsilon(dx^1)^2 + x^1((dx^2)^2 + (dx^3)^2 + (dx^4)^2)$, $\varepsilon = \pm 1$. It is easy to check that \((M, g)\) is a conformally flat quasi-Einstein Riemannian manifold, $\text{rank } (S - \frac{1}{4(x^1)^2} g) = 1$, whose scalar curvature $\kappa$ is equal to zero, and the Ricci tensor $S$ is of Codazzi type but not cyclic parallel. Moreover, we have $R \cdot R = Q(S, R)$ and $R \cdot R = L Q(g, R)$, $L = -\frac{1}{4\varepsilon(x^1)^2}$.

(ii) Let $M = \{(x^1, x^2, \ldots, x^5) : x^i > 0, i = 1, 2, \ldots, 5\}$ be the subset of $\mathbb{R}^5$ endowed with the metric $g$ defined by $ds^2 = \varepsilon(dx^1)^2 + x^1((dx^2)^2 + \ldots + (dx^5)^2)$, $\varepsilon = \pm 1$. It is easy to check that \((M, g)\) is a conformally flat quasi-Einstein Riemannian manifold, $\text{rank } (S - \frac{1}{2(x^1)^2} g) = 1$, whose scalar curvature $\kappa$ is non-zero, $\kappa = \frac{1}{\varepsilon(x^1)^2}$, and the Ricci tensor $S$ is not of Codazzi type and not cyclic parallel. Moreover, we have $R \cdot R = Q(S, R)$ and $R \cdot R = L Q(g, R)$, $L = -\frac{1}{4\varepsilon(x^1)^2}$.
3. Pseudosymmetry type curvature conditions

We define on a semi-Riemannian manifold \((M, g)\), \(n \geq 3\), the endomorphisms \(X \wedge_A Y\), \(\mathcal{R}(X, Y)\), \(\mathcal{C}(X, Y)\), \(\mathcal{K}(X, Y)\) and \(\text{conh}(\mathcal{R})\) by \((31), (32), (33), (34), (35)\)

\[
(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,
\]

\[
\mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,
\]

\[
\mathcal{C}(X, Y) = \mathcal{R}(X, Y) - \frac{1}{n-2}(X \wedge_g \mathcal{L}Y + \mathcal{L}X \wedge_g Y - \frac{\kappa}{n-1}X \wedge_g Y),
\]

\[
\mathcal{K}(X, Y) = \mathcal{R}(X, Y) - \frac{\kappa}{n(n-1)}X \wedge_g Y,
\]

\[
\text{conh}(\mathcal{R})(X, Y) = \mathcal{R}(X, Y) - \frac{1}{n-2}(X \wedge_g \mathcal{L}Y + \mathcal{L}X \wedge_g Y),
\]

respectively, where \(A\) is an \((0,2)\)-tensor on \(M\), \(X, Y, Z \in \chi(M)\). The Ricci operator \(\mathcal{L}\) is defined by \(g(X, \mathcal{L}Y) = S(X, Y)\), where \(S\) is the Ricci tensor and \(\kappa\) the scalar curvature of \((M, g)\), respectively. The tensor \(S^2\) is defined by \(S^2(X, Y) = S(X, \mathcal{L}Y)\). Further, we define the Gaussian curvature tensor \(G\), the Riemann-Christoffel curvature tensor \(R\), the Weyl conformal curvature tensor \(C\), concircular curvature tensor \(K\) and conharmonic curvature tensor \(\text{conh}(R)\) of \((M, g)\), by \((31), (32), (33), (34)\)

\[
G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4),
\]

\[
R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4),
\]

\[
C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),
\]

\[
K(X_1, X_2, X_3, X_4) = g(\mathcal{K}(X_1, X_2)X_3, X_4),
\]

\[
\text{conh}(R)(X_1, X_2, X_3, X_4) = g(\text{conh}(\mathcal{R})(X_1, X_2)X_3, X_4),
\]

respectively. For \((0,2)\)-tensors \(A\) and \(B\) we define their Kulkarni-Nomizu product \(A \wedge B\) by (see e.g. \(32), (33), (35)\)

\[
(A \wedge B)(X_1, X_2; X, Y) = A(X_1, Y)B(X_2, X) + A(X_2, X)B(X_1, Y)
\]

\[
- A(X_1, X)B(X_2, Y) - A(X_2, Y)B(X_1, X).
\]

We note that the Weyl conformal curvature tensor \(C\) can be presented in the following form

\[
(3.1)\quad C = R - \frac{1}{n-2}g \wedge S + \frac{\kappa}{(n-2)(n-1)}G.
\]
For an \((0, k)\)-tensor \(T\), \(k \geq 1\) and a symmetric \((0, 2)\)-tensor \(A\) we define the \((0, k)\)-tensor \(A \cdot T\) and the \((0, k + 2)\)-tensors \(R \cdot T\), \(C \cdot T\) and \(Q(A, T)\) by

\[
(A \cdot T)(X_1, \cdots, X_k) = -T(AX_1, X_2, \cdots, X_k) - \cdots - T(X_1, X_2, \cdots, AX_k),
\]

\[
(R \cdot T)(X_1, \cdots, X_k; X, Y) = (R(X, Y) \cdot T)(X_1, \cdots, X_k)
\]

\[
= -T(R(X, Y)X_1, X_2, \cdots, X_k) - \cdots - T(X_1, \cdots, X_{k-1}, R(X, Y)X_k),
\]

\[
(C \cdot T)(X_1, \cdots, X_k; X, Y) = (C(X, Y) \cdot T)(X_1, \cdots, X_k)
\]

\[
= -T(C(X, Y)X_1, X_2, \cdots, X_k) - \cdots - T(X_1, \cdots, X_{k-1}, C(X, Y)X_k),
\]

\[
Q(A, T)(X_1, \cdots, X_k; X, Y) = ((X \wedge_A Y) \cdot T)(X_1, \cdots, X_k)
\]

\[
= -T((X \wedge_A Y)X_1, X_2, \cdots, X_k) - \cdots - T(X_1, \cdots, X_{k-1}, (X \wedge_A Y)X_k),
\]

where \(A\) is the endomorphism of \(\chi(M)\) defined by \(g(AX, Y) = A(X, Y)\). Putting in the above formulas \(T = R, T = S, T = C\) or \(T = K\), \(A = g\) or \(A = S\), we obtain the tensors: \(R \cdot R, R \cdot S, R \cdot C, R \cdot K, C \cdot R, C \cdot S, C \cdot C, C \cdot K, Q(g, R), Q(g, S), Q(g, C), Q(g, K), Q(S, R), Q(S, C), Q(g, K), S \cdot R, S \cdot C, S \cdot K, \text{conh}(R) \cdot \text{conh}(R), \text{conh}(R) \cdot R, R \cdot \text{conh}(R), \text{conh}(R) \cdot S\), etc. The tensor \(Q(A, T)\) is called the Tachibana tensor of the tensors \(A\) and \(T\), or the Tachibana tensor for short (36). We like to point out that in some papers, \(Q(g, R)\) is called the Tachibana tensor (see e.g. 36, 37, 38, 39). We also have

**Proposition 3.1.** (cf. 10) For any semi-Riemannian manifold \((M, g)\), \(n \geq 4\), we have

\[
\text{conh}(R) \cdot S = C \cdot S - \frac{\kappa}{(n-2)(n-1)} Q(g, S),
\]

\[
R \cdot \text{conh}(R) = R \cdot C,
\]

\[
\text{conh}(R) \cdot R = C \cdot R - \frac{\kappa}{(n-2)(n-1)} Q(g, R),
\]

\[
\text{conh}(R) \cdot \text{conh}(R) = C \cdot C - \frac{\kappa}{(n-2)(n-1)} Q(g, C).
\]

(3.2)

A semi-Riemannian manifold \((M, g)\), \(n \geq 3\), satisfying the condition

\[
R \cdot R = 0
\]

(3.3)

is called semisymmetric (41). We mention that non-conformally flat and non-locally symmetric semi-Riemannian manifolds having parallel Weyl conformal curvature tensor are semisymmetric (42, Theorem 9), their scalar curvature is equal to zero (42, Theorem 7) and the Ricci tensor is
a Codazzi tensor ([42], eq. (6)). We refer to [43]-[46] for the recent results on semi-Riemannian manifolds with parallel Weyl conformal curvature tensor and, in particular, for classification results. Semi-Riemannian warped products having parallel Weyl conformal curvature tensor were investigated in [47]. We also mention that, recently, conformally semisymmetric manifolds and special semisymmetric Weyl conformal tensors are studied in [48]. Another important subclass of semisymmetric semi-Riemannian manifolds form manifolds satisfying

\[ \nabla \nabla R = 0. \]

(3.4)

We refer to [49] and [50] and references therein for results on manifolds satisfying (3.4).

A semi-Riemannian manifold \((M, g), n \geq 3\), is said to be pseudosymmetric [51] if the tensor \(R \cdot R\) and the Tachibana tensor \(Q(g, R)\) are linearly dependent at every point of \(M\). This is equivalent to

\[ R \cdot R = L_R Q(g, R) \]

(3.5)

on \(U_R = \{ x \in M : R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x \}\), where \(L_R\) is some function on this set. We refer to [52], [53], [16], [37] and [17] for surveys on such manifolds. In particular, a geometrical interpretation of pseudosymmetric manifolds, in the Riemannian case, is given in [37].

We note that [51] is the first publication, in which a semi-Riemannian manifold satisfying (3.5) was named the pseudosymmetric manifold. In [51] pseudosymmetric warped products with 1-dimensional base manifold and \((n-1)\)-dimensional fibre, \(n \geq 4\), which is not a semi-Riemannian space of constant curvature, were investigated. In [54] it was shown that hypersurfaces in spaces of constant curvature, with exactly two distinct principal curvatures at every point, are pseudosymmetric. Thus in particular, Cartan’s and Schouten’s investigations of quasi-umbilical hypersurfaces in spaces of constant curvature are closely related to pseudosymmetric manifolds (cf. [16]). It is clear that every semisymmetric manifold is pseudosymmetric. However, the converse statement is not true. For instance, the Schwarzschild spacetime, the Kottler spacetime and the Reissner-Nordström spacetime satisfy (3.5) with non-zero function \(L_R\) [55] (see also [56], [57]). We also mention that Friedmann-Lemaître-Robertson-Walker spacetimes are pseudosymmetric (cf. [16]). It is well-known that the Schwarzschild spacetime was discovered in 1916 by Schwarzschild, during his study on solutions of Einstein’s equations. It seems that the Schwarzschild spacetime is the first example of a non-semisymmetric, pseudosymmetric
warped product. Finally, we note that (3.5) is equivalent to (e.g. see [53])

\[(R - L_R G) \cdot (R - L_R G) = 0.\]

We also note that in [58] Chaki introduced another kind of pseudosymmetry. However, both notions of pseudosymmetry are not equivalent. Throughout the paper we will confine the pseudosymmetry related to (3.5).

A semi-Riemannian manifold \((M, g)\), \(n \geq 3\), is said to be Ricci-pseudosymmetric ([59], [60]) if the tensor \(R \cdot S\) and the Tachibana tensor \(Q(g, S)\) are linearly dependent at every point of \(M\). Thus the manifold \((M, g)\) is Ricci-pseudosymmetric if and only if

\[(3.7) \quad R \cdot S = L_S Q(g, S)\]

holds on \(U_S = \{ x \in M : S - \frac{r}{n} g \neq 0 \text{ at } x \}\), where \(L_S\) is some function on this set. We note that \(U_S \subset U_R\). It is easy to check that (3.7) is equivalent to

\[(3.8) \quad (R - L_S G) \cdot (S - L_S g) = 0.\]

We refer to [52], [32], [16], [29] and [37] for surveys and comments on such manifolds. A geometrical interpretation of Ricci-pseudosymmetric manifolds, in the Riemannian case, is given in [38]. It is clear that every pseudosymmetric semi-Riemannian manifold is Ricci-pseudosymmetric. However, the converse statement is not true. For instance, the Cartan hypersurfaces of dimension 6, 12 or 24 are non-quasi-Einstein and non-pseudosymmetric Ricci-pseudosymmetric manifolds ([61], see also [32], [29]). The 3-dimensional Cartan hypersurface is a quasi-Einstein pseudosymmetric manifold [54]. We mention that recently quasi-Einstein Ricci-pseudosymmetric hypersurfaces in semi-Riemannian spaces of constant curvature were investigated in [62].

A semi-Riemannian manifold \((M, g)\), \(n \geq 4\), is said to be conformally pseudosymmetric ([53], [63]) if the tensor \(R \cdot C\) and the Tachibana tensor \(Q(g, C)\) are linearly dependent at every point of \(M\). Again a semi-Riemannian manifold \((M, g)\), \(n \geq 4\), is said to be a manifold with pseudosymmetric Weyl conformal curvature tensor ([63], [53]) if the tensor \(C \cdot C\) and the Tachibana tensor \(Q(g, C)\) are linearly dependent at every point of \(M\). This is equivalent to

\[(3.9) \quad C \cdot C = L_C Q(g, C)\]
on $U_C = \{x \in M : C \neq 0 \text{ at } x\}$, where $L_C$ is some function on this set. We note that $U_C \subset U_R$.

It is easy to check that \((3.9)\) is equivalent to

\[(3.10) \quad (C - L_C G) \cdot (C - L_C G) = 0.\]

Using \((3.1)\), we also can check that \((3.6)\) and \((3.10)\) are equivalent on every Einstein manifold. As it was stated in \([63]\), any warped product $M_1 \times_F M_2$, with $\dim M_1 = \dim M_2 = 2$, satisfies \((3.9)\). Thus in particular, the Schwarzschild spacetime, the Kottler spacetime and the Reissner-Nordström spacetime satisfy \((3.9)\).

A semi-Riemannian manifold $(M, g)$, $n \geq 3$, is said to be Ricci-generalized pseudosymmetric \([64], [65]\) if at every point of $M$ the tensor $R \cdot R$ and the Tachibana tensor $Q(S, R)$ are linearly dependent. Hence $(M, g)$ is Ricci-generalized pseudosymmetric if and only if

\[(3.11) \quad R \cdot R = L Q(S, R)\]

holds on $U = \{x \in M : Q(S, R) \neq 0 \text{ at } x\}$, where $L$ is some function on this set. An important subclass of Ricci-generalized pseudosymmetric manifolds is formed by the manifolds realizing the condition \(([64], [66])\)

\[(3.12) \quad R \cdot R = Q(S, R).\]

At the end of this section we also present some other curvature conditions closely related to the above presented conditions. Namely, it was stated in \([67]\), on every hypersurface $M$ immersed isometrically in a semi-Riemannian space of constant curvature $N$, $\dim N = n + 1$, $n \geq 4$, we have

\[(3.13) \quad R \cdot R - Q(S, R) = -\frac{(n - 2)\tilde{\kappa}}{n(n + 1)} Q(g, C),\]

where $\tilde{\kappa}$ is the scalar curvature of the ambient space. It is clear, that if the ambient space is a semi-Euclidean space then \((3.13)\) reduces to \((3.12)\). We also note that any warped product $M_1 \times_F M_2$, with $\dim M_1 = 1$, $\dim M_2 = 3$, satisfies \([68]\)

\[(3.14) \quad R \cdot R - Q(S, R) = L Q(g, C),\]

for some function $L$ on $U_C$. Thus generalized Robertson-Walker spacetimes fulfills \((3.14)\). In particular, Friedmann-Lemaître-Robertson-Walker spacetimes satisfy \((3.12)\). We mention that the Vaidya spacetime also satisfies \((3.14)\) \([35]\), Example 5.2).
The conditions: (3.5), (3.7), (3.9), (3.11) and (3.14) or other conditions of this kind are called conditions of pseudosymmetry type. We refer to [52], [31], [53], [32] and [16] for surveys on semi-Riemannian manifolds satisfying such conditions. In particular, we refer to [69] for recent results on quasi-Einstein manifolds satisfying curvature conditions of this kind.

It is easy to check that on every Ricci-pseudosymmetric manifold the following condition is satisfied ([70], Lemma 3.3; [71], Proposition 3.1(iv))

\[(3.15) \quad R(\mathcal{L}X, Y, Z, W) + R(\mathcal{L}Z, Y, W, X) + R(\mathcal{L}W, Y, X, Z) = 0.\]

Semi-Riemannian manifolds \((M, g)\), \(n \geq 3\), satisfying (3.15) are called Riemann compatible ([72], [73]). We also note that (3.15) remains invariant under geodesic mappings. In [36] (Proposition 2.1) it was proved that (3.15) holds on every manifold satisfying (3.14). Thus in particular, manifolds satisfying (3.12) are also Riemann compatible ([64], Lemma 2.2(i)). We refer to [74], [75] and [76] for further results on Riemann compatible manifolds.

4. Gödel metric admitting geometric structures

Let on \(\mathbb{R}^4\) be given the Gödel metric \(g\) defined by (1.1) and let \(h, i, j, k, l, m \in \{1, 2, 3, 4\}\). From (1.1) the non-zero components \(\Gamma^{h}_{ij}\) of the Christoffel symbols of second kind of \(g\) are given by [1]:

\[
\Gamma^{4}_{12} = \frac{e^{x_1}}{2}, \quad \Gamma^{2}_{14} = -e^{-x_1}, \quad \Gamma^{4}_{14} = \Gamma^{1}_{22} = \frac{e^{2x_1}}{2}, \quad \Gamma^{1}_{24} = \frac{e^{x_1}}{2}.
\]

Further, the non-zero components \(R_{hijk}\) and \(S_{ij}\) of the Riemann-Christoffel curvature tensor \(R\) and the Ricci tensor \(S\), respectively, and the scalar curvature \(\kappa\) are given by [1]:

\[
R_{1212} = \frac{3}{4} a^2 e^{2x_1}, \quad R_{1214} = \frac{1}{2} a^2 e^{x_1}, \quad R_{1414} = \frac{a^2}{2}, \quad R_{2424} = \frac{1}{4} a^2 e^{2x_1},
\]

\[
S_{22} = e^{2x_1}, \quad S_{24} = e^{x_1}, \quad S_{44} = 1 \text{ and } \kappa = \frac{1}{a^2}.
\]

Again the non-zero components \(R_{hijk,l}\) and \(S_{ij,l}\) of the covariant derivatives of the Riemann-Christoffel curvature tensor \(\nabla R\) and the Ricci tensor \(\nabla S\), respectively, are given by:

\[
R_{1212,1} = a^2 e^{2x_1}, \quad R_{1214,1} = \frac{1}{2} a^2 e^{x_1}, \quad R_{1224,2} = \frac{1}{4} a^2 e^{3x_1},
\]

\[
S_{12,2} = -\frac{e^{2x_1}}{2}, \quad S_{14,2} = -\frac{e^{x_1}}{2}, \quad S_{22,1} = e^{2x_1}, \quad S_{24,1} = \frac{e^{x_1}}{2}.
\]
The non-zero components $C_{hijk}$ of the Weyl conformal curvature tensor $C$ are given below:

$$C_{1212} = \frac{1}{3} a^2 e^{2x^1}, \quad C_{1214} = \frac{1}{6} a^2 e^{x^1}, \quad C_{1414} = \frac{a^2}{6}, \quad C_{2424} = \frac{1}{12} a^2 e^{2x^1},$$

$$C_{1313} = -\frac{1}{6} a^2, \quad C_{2323} = -\frac{5}{12} a^2 e^{2x^1}, \quad C_{2334} = \frac{1}{3} a^2 e^{x^1}, \quad C_{3434} = -\frac{1}{3} a^2.$$

The non-zero components $(R \cdot R)_{hijklm}$ of the tensor $R \cdot R$ are given below:

$$(R \cdot R)_{122412} = \frac{1}{4} a^2 e^{3x^1}, \quad (R \cdot R)_{122414} = \frac{1}{4} a^2 e^{2x^1},$$

$$(R \cdot R)_{121224} = -\frac{1}{2} a^2 e^{2x^1}, \quad (R \cdot R)_{121424} = -\frac{1}{4} a^2 e^{2x^1}.$$

The non-zero components $Q(S, R)_{hijklm}$ of the Tachibana tensor $Q(S, R)$ are given below:

$$Q(S, R)_{122412} = \frac{1}{4} a^2 e^{3x^1}, \quad Q(S, R)_{122414} = \frac{1}{4} a^2 e^{2x^1},$$

$$Q(S, R)_{121224} = -\frac{1}{2} a^2 e^{2x^1}, \quad Q(S, R)_{121424} = -\frac{1}{4} a^2 e^{2x^1}.$$

The non-zero components $(C \cdot C)_{hijklm}$ of the tensor $C \cdot C$ are given below:

$$(C \cdot C)_{122412} = \frac{1}{24} a^2 e^{3x^1}, \quad (C \cdot C)_{132312} = \frac{1}{12} a^2 e^{2x^1}, \quad (C \cdot C)_{133412} = -\frac{1}{12} a^2 e^{x^1},$$

$$(C \cdot C)_{122313} = -\frac{1}{8} a^2 e^{2x^1}, \quad (C \cdot C)_{123413} = \frac{1}{12} a^2 e^{x^1}, \quad (C \cdot C)_{142313} = -\frac{1}{12} a^2 e^{x^1},$$

$$(C \cdot C)_{143413} = \frac{1}{12} a^2, \quad (C \cdot C)_{122414} = \frac{1}{24} a^2 e^{2x^1}, \quad (C \cdot C)_{132314} = \frac{1}{12} a^2 e^{x^1},$$

$$(C \cdot C)_{133414} = -\frac{1}{12} a^2, \quad (C \cdot C)_{121323} = \frac{1}{24} a^2 e^{2x^1}, \quad (C \cdot C)_{232423} = -\frac{1}{24} a^2 e^{x^1},$$

$$(C \cdot C)_{234323} = \frac{1}{24} a^2 e^{2x^1}, \quad (C \cdot C)_{121224} = -\frac{1}{12} a^2 e^{3x^1}, \quad (C \cdot C)_{121424} = -\frac{1}{24} a^2 e^{2x^1},$$

$$(C \cdot C)_{232324} = \frac{1}{12} a^2 e^{3x^1}, \quad (C \cdot C)_{233424} = -\frac{1}{24} a^2 e^{2x^1}.$$
The non-zero components \(Q(g, C)_{hijklm}\) of the Tachibana tensor \(Q(g, C)\) are given below:

\[
\begin{align*}
Q(g, C)_{122412} &= \frac{1}{4}a^4e^{3x^1}, & Q(g, C)_{132312} &= \frac{1}{2}a^4e^{2x^1}, & Q(g, C)_{133412} &= -\frac{1}{2}a^4e^{x^1}, \\
Q(g, C)_{122313} &= -\frac{3}{4}a^4e^{2x^1}, & Q(g, C)_{123413} &= \frac{1}{2}a^4e^{x^1}, & Q(g, C)_{142313} &= -\frac{1}{2}a^4e^{x^1}, \\
Q(g, C)_{143413} &= \frac{1}{2}a^4, & Q(g, C)_{122414} &= \frac{1}{4}a^4e^{2x^1}, & Q(g, C)_{132314} &= \frac{1}{2}a^4e^{x^1}, \\
Q(g, C)_{133414} &= -\frac{1}{2}a^4, & Q(g, C)_{121323} &= \frac{1}{4}a^4e^{2x^1}, & Q(g, C)_{232423} &= -\frac{1}{4}a^4e^{3x^1}, \\
Q(g, C)_{243423} &= \frac{1}{4}a^4e^{2x^1}, & Q(g, C)_{121224} &= -\frac{1}{2}a^4e^{3x^1}, & Q(g, C)_{121424} &= -\frac{1}{4}a^4e^{2x^1}, \\
Q(g, C)_{232324} &= \frac{1}{2}a^4e^{3x^1}, & Q(g, C)_{233424} &= -\frac{1}{4}a^4e^{2x^1}.
\end{align*}
\]

Thus we see that \(\mathbb{R}^4\) equipped with the Gödel metric \(g\) has the following curvature properties:

(i) The Ricci tensor is cyclic parallel \([2]\), the rank of the Ricci tensor \(S\) is 1 \([1]\), precisely,

\[
S = \kappa \omega \otimes \omega, \quad \kappa = \frac{1}{a^2}, \quad \omega = (\omega_1, \omega_2, \omega_3, \omega_4) = (0, ae^{x^1}, 0, a),
\]

and the vector field \(X\) corresponding to 1-form \(\omega\) is given by \(X = (0, 0, 0, \frac{1}{a})\),

(ii) \(R \cdot R = Q(S, R)\),

(iii) \(C \cdot C = \frac{5}{6}Q(g, C)\),

(iv) \(3R \cdot K - 2Q(S, K) = Q(S, C)\).

Gödel metric also realizes the following pseudosymmetric type conditions:

(v)

\[
(2a^2L_1 + \frac{2}{3}L_2)(R \cdot C + C \cdot R) = (\frac{2}{3}L_1 + \frac{1}{9a^2}L_2)
\left(Q(g, R) - 3Q(S, R)\right) + L_1Q(g, C) + L_2Q(S, C),
\]

where \(L_1\) and \(L_2\) are some functions. This condition implies that \(R \cdot C, C \cdot R, Q(g, R), Q(S, R), Q(g, C)\) and \(Q(S, C)\) are linearly dependent.

(vi)

\[
(L_1 + L_2)(C \cdot K + K \cdot C) = (\frac{1}{12a^2}L_1 + \frac{7}{12a^2}L_2)Q(g, C) + L_1Q(S, C)
- \frac{1}{2a^2}L_2Q(g, K) + L_2Q(S, K),
\]

where \(L_1\) and \(L_2\) are some functions. This condition implies that \(C \cdot K, K \cdot C, Q(g, C), Q(S, C), Q(g, K)\) and \(Q(S, K)\) are linearly dependent.
(vii) \[
\left(-\frac{2}{5}L_1 + \frac{12a^2}{5}L_2\right)(K \cdot \text{conh}(R) + \text{conh}(R) \cdot K) \\
= \left(\frac{1}{30a^2}L_1 - \frac{6}{5}L_2\right)Q(g, K) + L_1Q(S, K) + L_2Q(g, \text{conh}(R)) \\
+ \left(-\frac{7}{5}L_1 + \frac{12a^2}{5}L_2\right)Q(S, \text{conh}(R)),
\]
where $L_1$ and $L_2$ are some functions. This condition implies that $K \cdot \text{conh}(R)$, $\text{conh}(R) \cdot K$, $Q(g, K)$, $Q(S, K)$, $Q(g, \text{conh}(R))$ and $Q(S, \text{conh}(R))$ are linearly dependent.

(viii) \[
2a^2L_1(R \cdot \text{conh}(R) + \text{conh}(R) \cdot R) = -L_1\left(Q(g, R) - Q(g, \text{conh}(R))\right) \\
+ L_2Q(S, R) + (2a^2L_1 - L_2)Q(S, \text{conh}(R)),
\]
where $L_1$ and $L_2$ are some functions. This condition implies that $R \cdot \text{conh}(R)$, $\text{conh}(R) \cdot R$, $Q(g, R)$, $Q(S, R)$, $Q(g, \text{conh}(R))$ and $Q(S, \text{conh}(R))$ are linearly dependent.

We note that the condition $\text{rank } S = 1$ holds at a point of a semi-Riemannian manifold $(M, g)$, $n \geq 3$, if and only if $S \wedge S = 0$ at this point. Further, it is easy to check that (iv) is an immediate consequence of (ii) and $S \wedge S = 0$ and the definitions of the tensors $C$ and $K$.

We also note that the condition (3.12) holds at every point $x$ of a semi-Riemannian manifold $(M, g)$, $n \geq 3$, at which the condition
\[
\omega(X_1)\mathcal{R}(X_2, X_3) + \omega(X_2)\mathcal{R}(X_3, X_1) + \omega(X_3)\mathcal{R}(X_1, X_2) = 0
\]
is satisfied, where $\omega$ is a non-zero covector at $x$ ([60], Theorem 3.1, [64], p. 110). However, in case of Gödel metric (3.12) holds, but does not satisfy the above condition. Since the Gödel metric is a product metric of a 3-dimensional metric and an 1-dimensional metric, the property (3.12) also follows from Corollary 4.1 of [65]. As it was stated in Section 3, any semi-Riemannian manifold satisfying (3.12) is Riemann compatible. Thus the Gödel metric satisfies also (3.15). Now (3.15), by (4.1), turns into \[
\omega_t g^{rs}(\omega_h R_{sijk} + \omega_j R_{sikh} + \omega_k R_{sihj}) = 0,
\]where $R_{sijk}$ and $g^{rs}$ are the local components of the the Riemann-Christoffel curvature tensor $R$ and the tensor $g^{-1}$ of the Gödel metric $g$. We note that the 1-form $\omega$, with respect to Definition 3.1 of [77], is called $R$-compatible and hence Weyl compatible. Thus we have
Theorem 4.1. The Gödel spacetime \((M, g)\) is a cyclic Ricci parallel and Riemann compatible manifold satisfying: \(\text{rank} S = 1, \ R \cdot R = Q(S, R), \ \text{conh}(R) \cdot C = \text{conh}(R) \cdot \text{conh}(R) = 0, \ C \cdot C = C \cdot \text{conh}(R) = \frac{\kappa}{6} Q(g, C)\) and the 1-form \(\omega\), defined by (4.1), is \(R\)-compatible as well as Weyl compatible.

The above presented results lead to the following generalizations.

Let \((\bar{M} \times \bar{N}, g = \bar{g} \times \bar{g})\) be the product manifold of an \((n-1)\)-dimensional semi-Riemannian manifold \((\bar{M}, \bar{g})\), \(n \geq 4\), and an 1-dimensional manifold \((\bar{N}, \bar{g})\). Moreover, let \((\bar{M}, \bar{g})\) be a conformally flat manifold, provided that \(n \geq 5\). The local components \(C_{hijk}, h, i, j, k \in \{1, 2, \ldots, n\}\), of the Weyl conformal curvature tensor \(C\) of \((\bar{M} \times \bar{N}, g)\) which may not vanish identically are the following (cf. [78], eqs. (49)-(51))

\[
C_{abcd} = \frac{1}{(n-3)(n-2)} \left( \bar{g}_{ad}A_{bc} - \bar{g}_{ac}A_{bd} + \bar{g}_{bc}A_{ad} - \bar{g}_{bd}A_{ac} \right),
\]

\[
C_{nbcn} = -\frac{1}{n-2} \bar{g}_{nn}A_{bc},
\]

where \(A_{ab} = \bar{S}_{ab} - \frac{n}{n-1} \bar{g}_{ab}\), and \(\bar{g}_{ab}\) and \(\bar{S}_{ab}\) denote the local components of the metric tensor \(\bar{g}\) and the Ricci tensor \(\bar{S}\) of \((\bar{M}, \bar{g})\), respectively, \(a, b, c, d \in \{1, 2, \ldots, n-1\}\), and \(\kappa\) is the scalar curvature of \((\bar{M}, \bar{g})\). Further, we denote by \(U_C\) the set of all points of \((\bar{M} \times \bar{N}, g)\) at which the Weyl conformal curvature tensor \(C\) of \((\bar{M} \times \bar{N}, g)\) is non-zero. We note that the tensor \(C\) is non-zero at a point of \(U_C\) if and only if \(\bar{S} \neq \frac{n}{n-1} \bar{g}\) at this point.

As an immediate consequence of Proposition 2 of [78] we get the following equivalence: (3.9) holds on the set \(U_C\) of the defined above manifold \((\bar{M} \times \bar{N}, g)\) for some function \(L_C\) on this set, if and only if at every point of \(U_C\) we have

\[
g^{bc}A_{ab}A_{cd} = (n-3)(n-2)L_C A_{ad} + \lambda \bar{g}_{ad},
\]

\[
B_{ad}B_{bc} - B_{ac}B_{bd} = \left( (n-2)^2 L_C^2 - \frac{\lambda}{n-1} \right) \left( \bar{g}_{ad}\bar{g}_{bc} - \bar{g}_{ac}\bar{g}_{bd} \right),
\]

where \(B = A + (n-2)L_C \bar{g}\), and \(\lambda\) is a constant. Furthermore, in view of Lemma 3.1 of [79], at every point of \(U_C\) (4.5) is equivalent to \(\text{rank} B = 1\), i.e.

\[
\text{rank} \left( \bar{S} - \left( \frac{\kappa}{n-1} - (n-2)L_C \right) \bar{g} \right) = 1.
\]
Further, we denote by $S$ and $\kappa$ the Ricci tensor and the scalar curvature of $(\tilde{M} \times \tilde{N}, g = \bar{g} \times \bar{g})$, respectively. It is obvious that $S = \overline{S}$ and $\kappa = \overline{\kappa}$. Therefore (4.6) yields

$$(4.7) \quad \text{rank}\left(S - \left(\frac{\kappa}{n-1} - (n-2)L_C\right)g\right) = 1.$$ 

From the above presented considerations and Proposition 3.1 it follows

**Theorem 4.2.** Let $(\tilde{M} \times \tilde{N}, g = \bar{g} \times \bar{g})$ be the product manifold of an $(n-1)$-dimensional semi-Riemannian manifold $(\tilde{M}, \bar{g})$, $n \geq 4$, and an 1-dimensional manifold $(\tilde{N}, \tilde{g})$. Moreover, let $(\tilde{M}, \tilde{g})$ be a conformally flat manifold, provided that $n \geq 5$. If on $\tilde{M}$ we have $\text{rank}(\overline{S} - \rho \overline{g}) = 1$, for some function $\rho$, then $\text{rank}(S - \rho g) = 1$ and (3.9), i.e. $C \cdot C = L_C Q(g, C)$, with $L_C = \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \rho\right)$, hold on $\tilde{M} \times \tilde{N}$. In particular, if the rank of the Ricci tensor of $(\tilde{M}, \tilde{g})$ is one, then the rank of the Ricci tensor of $\tilde{M} \times \tilde{N}$ is also one and (3.9), with $L_C = \frac{\kappa}{(n-2)(n-1)}$, or equivalently, $\text{conh}(R) \cdot \text{conh}(R) = 0$ holds on this manifold.

We present now an application of the last theorem.

From Theorem 4.1 of [67] it follows that a hypersurface $M$ immersed isometrically in a semi-Riemannian space of constant curvature $N$, dim $N \geq 5$, is a quasi-umbilical hypersurface if and only if it is a conformally flat manifold. Furthermore, using the Gauss equation of $M$ in $N$, we can easily prove that if $M$ is quasi-umbilical hypersurface then it is also a quasi-Einstein manifold. These facts, together with Theorem 4.2, leads to the following

**Theorem 4.3.** Let $(\tilde{M}, \tilde{g})$ be a manifold which is isometric with a quasi-umbilical hypersurface immersed isometrically in a semi-Riemannian space of constant curvature $N$, dim $N \geq 5$. Let $(\tilde{N}, \tilde{g})$ be an 1-dimensional manifold. Then the manifold $(\tilde{M} \times \tilde{N}, g = \bar{g} \times \bar{g})$ is a quasi-Einstein manifold with pseudosymmetric Weyl conformal curvature tensor.

In this way we obtain a family of quasi-Einstein manifolds with pseudosymmetric Weyl conformal curvature tensor. We mention that quasi-Einstein hypersurfaces with pseudosymmetric Weyl conformal curvature tensor immersed isometrically in semi-Riemannian spaces of constant curvature were investigated in [33].

Theorem 4.3 together with Theorem 4.2 and Example 4.1 of [68], yields
Theorem 4.4. Let \((\overline{M}, \overline{g})\) be a manifold which is isometric with a quasi-umbilical hypersurface immersed isometrically in a semi-Euclidean space \(N\), \(\dim N \geq 5\). Let \((\tilde{N}, \tilde{g})\) be an 1-dimensional manifold. Then the manifold \((\overline{M} \times \tilde{N}, g = \overline{g} \times \tilde{g})\) is a quasi-Einstein manifold with pseudosymmetric Weyl conformal curvature tensor satisfying \(R \cdot R = Q(S, R)\).

Finally, we consider some extension of the Gödel metric.

Example 4.1. (i) We define the metric \(g\) on \(M = \{(t, r, \phi, z) : t > 0, r > 0\} \subset \mathbb{R}^4\) by (cf. [80], Section 1)

\[
ds^2 = (dt + H(r)\, d\phi)^2 - D^2(r)\, d\phi^2 - dr^2 - dz^2,
\]

where \(H\) and \(D\) are certain functions on \(M\). In the special case, if \(H(r) = \frac{2\sqrt{2}}{m} \sinh^2\left(\frac{mr^2}{2}\right)\) and \(D(r) = \frac{2}{m} \sinh\left(\frac{mr^2}{2}\right) \cosh\left(\frac{mr^2}{2}\right)\) then \(g\) is the Gödel metric (e.g. see [80], eq. (1.6)).

(ii) Since the metric \(g\) defined by (4.8) is the product metric of a 3-dimensional metric and a 1-dimensional metric (3.12) holds on \(M\). We can check that the Riemann-Christoffel curvature tensor \(R\) of \((M, g)\) is expressed by a linear combination of the Kulkarni-Nomizu products formed by \(S\) and \(S^2\), i.e. by the tensors \(S \wedge S\), \(S \wedge S^2\) and \(S^2 \wedge S^2\),

\[
R = \phi_1 S \wedge S + \phi_2 S \wedge S^2 + \phi_3 S^2 \wedge S^2,
\]

\[
\phi_1 = \frac{D^2}{\tau}(2D^2H'' - 4DD'H'H'' - 3H'^4 + 8DD'H'^2 + 2D^2H'^2 - 8D^2D'^2),
\]

\[
\phi_2 = \frac{2D^4}{\tau}(H'^2 - 4DD''),
\]

\[
\phi_3 = \frac{4D^6}{\tau}, \quad H' = \frac{dH}{dr}, \quad H'' = \frac{dH'}{dr},
\]

\[
\tau = (H'^2 - 2DD'')(D^2H'^2 - 2DD'H'H'' - H'^4 + 2DD'H'^2 + D'^2H'^2),
\]

provided that the function \(\tau\) is non-zero at every point of \(M\).

(iii) If \(H(r) = ar^2, a = \text{const.} \neq 0\) and \(D(r) = r\) then (4.8) turns into ([80], eq. (3.20))

\[
ds^2 = (dt + ar^2\, d\phi)^2 - r^2\, d\phi^2 - dr^2 - dz^2.
\]

The spacetime \((M, g)\) with the metric \(g\) defined by (4.9) is called the Som-Raychaudhuri solution of the Einstein field equations ([81]). For the metric (4.9) the function \(\tau\) is non-zero at every point of \(M\).

(iv) We refer to [32] and [82] for surveys on semi-Riemannian manifolds \((M, g), n \geq 4\), having
Riemann-Christoffel curvature tensor $R$ expressed by a linear combination of the Kulkarni-Nomizu products formed by $g$ and $S$, i.e. by the tensors $g \wedge g$, $g \wedge S$ and $S \wedge S$. In particular, we mention that in the class of the Reissner-Nordström-de Sitter spacetimes there are spacetimes having that property ([35], Example 5.3).

It may be mentioned that we have calculated the local components of various tensors using Wolfram Mathematica, as well as SymPy and Maxima packages for symbolic calculation.

**CONCLUSION:**

By considering the dust particles as galaxies, the Gödel spacetime can be taken as a cosmological model of rotating universe. Although Gödel spacetime is not a realistic model of the universe in which we live but it realized many peculiar properties. For example, the existence of closed timelike curves implies a form of time travel in an alternative universe described by the Gödel spacetime. Also Gödel spacetime is quasi-Einstein, Ricci tensor is cyclic parallel but not Codazzi type, which may be physically interpreted as the content of the spacetime is of rotating matter without singularity. It is neither pseudosymmetric nor Ricci pseudosymmetric but a special type of Ricci generalized pseudosymmetric, and it is not conformally pseudosymmetric but its Weyl conformal curvature tensor is pseudosymmetric (i.e., $C \cdot C = \frac{2}{6}Q(g, C)$) and also the spacetime is Riemann compatible as well as Weyl compatible (Theorem 4.1). Hence Gödel spacetime forced us to obtain a new class of semi-Riemannian manifolds which is quasi-Einstein with pseudosymmetric Weyl conformal curvature tensor and is a special type of Ricci generalized pseudosymmetric manifolds (Theorem 4.4).

**ACKNOWLEDGMENTS**

The first and third named authors are supported by a grant of the Wrocław University of Environmental and Life Sciences, Poland [WIKSiG/441/212/S]. The fourth and fifth named authors gratefully acknowledge the financial support of CSIR, New Delhi, India [File no: 09/025(0194)/2010-EMR-I, Project F. No. 25(0171)/09/EMR-II].

**REFERENCES**

[1] K. Gödel, An example of a new type of cosmological solutions of Einstein’s field equations of gravitation, *Rev. Modern Phys.* **21**(3) (1949), 447–450.
[2] V. Dryuma, On the Riemann extension of the Gödel space-time metric, *Bull. Acad. Stin. Republ. Mold. Math.* 49(3) (2005), 43–62.

[3] D. Malament, A note about closed timelike curves in Gödel space-time, *J. Math. Phys.* 28(10) (1987), 2427–2430.

[4] W. Kundt, Trägheitsbahnen in einem von Gödel angegebenen kosmologischen Modell, *Zeitschrift für Physik* 145 (1956), 611–620.

[5] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time* (Cambridge University Press, 1973).

[6] D. Malament, Minimal acceleration requirement for time travel in Gödel space-time, *J. Math. Phys.* 26(4) (1985), 774–777.

[7] D. Radojević, The modification of Gödel metric, *Facta Univ.* 3(11) (2001), 149–152.

[8] J.B. Fonseca-Neto, C. Romero and F. Dahia, Gödel’s Universe and induced-matter theory, *Brazilian J. Phys.* 35(4B) (2005), 1067–1069.

[9] G. Dautcourt and M. Abdel-Megied, Revisiting the light cone of the Gödel universe, *arXiv:gr-qc/0511015* 3 Nov. 2005, 1–23.

[10] R. García-Olivo, J. López-Bonilla and S. Vidal-Beltrán, Gödel’s geometry: embedding and Lanczos spin-tensor, *Elect. J. Theo. Phys.* 3(12) (2006), 55–58.

[11] M. Gürses, A. Karasu and S. Özdür, Gödel-type metrics in various dimensions, *Class. Quantum Gravit.* 22(2005), 1527–1543.

[12] M. Plaue, M. Scherfner and L.A.M. De Sousa Jr., On spacetimes with given kinematical invariants: construction and examples, arXiv: 0801.3364v2 [gr-qc], 24 Jan. 2008.

[13] É. Cartan, *Lecons sur la géométrie des espaces de Riemann*, (Gauthier-Villars, Paris, 1963).

[14] M. Cahen and M. Parker, Sur des classes d’espaces pseudo-riemanniens symmetriques, *Bull. Soc. Math. Belg.* 22 (1970), 339–354.

[15] M. Cahen and M. Parker, Pseudo-Riemannian symmetric spaces, *Mem. Amer. Math. Soc.* 24(229) (1980), 1–108.

[16] R. Deszcz, S. Haesen and L. Verstraelen, On natural symmetries, in: *Topics in Differential Geometry*, eds. A. Mihai, I. Mihai and R. Miron, (Editura Academiei Române, 2008).

[17] S. Haesen and L. Verstraelen, Natural Intrinsic Geometrical Symmetries, Symmetry, Integrability and Geometry, Methods and Applications SIGMA 5 (2009), 086, 15 pages.

[18] A. Gray, Einstein-like manifolds which are not Einstein, *Geom. Dedicata* 7 (1978), 259–280.

[19] J.E. D’Atri and H.K. Nickerson, Divergence preserving geodesic symmetries, *J. Diff. Geom.* 3 (1969), 467–476.

[20] A. Derdziński and C.L. Shen, Codazzi tensor fields, curvature and Pontryagin forms, *Proc. London Math. Soc.*, 47(3) (1983), 15–26.

[21] M. Berger and D. Ebin, Some characterizations of the space of symmetric tensors on a Riemannian manifold, *J. Diff. Geom.* 3 (1969), 379–392.
[22] J.P. Bourguignon, Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d’Einstein, *Invent. Math.* **63**(2) (1981), 263–286.

[23] A. Derdziński, Some remarks on the local structure of Codazzi tensors, *Glob. Diff. Geom. Glob. Ann.*, Lecture Notes **838** (1981), 251–255, Springer-Verlag.

[24] A. Derdziński, On compact Riemannian manifolds with harmonic curvature, *Math. Ann.* **259** (1982), 145–152.

[25] D. Ferus, A remark on Codazzi tensors on constant curvature space, *Glob. Diff. Geom. Glob. Ann.*, Lecture Notes, **838**, (Springer, Heidelberg, 1981).

[26] U. Simon, Codazzi tensors, *Glob. Diff. Geom. and Glob. Ann.*, Lecture Notes **838**, (Springer-Verlag, Heidelberg, 1981) pp. 289–296.

[27] A.L. Besse, *Einstein Manifolds* (Springer-Verlag, Berlin, Heidelberg, 1987).

[28] U-H. Ki and H. Nakagawa, A characterization of the Cartan hypersurfaces in a sphere, *Tohoku Math. J.* **39** (1987), 27–40.

[29] R. Deszcz, P. Verheyen and L. Verstraelen, On some generalized Einstein metric conditions, *Publ. Inst. Math. (Beograd) (N.S.)* **60**(74) (1996), 108–120.

[30] A.A. Shaikh and T.Q. Binh, On some class of Riemannian manifolds, *Bull. Transilvania Univ.* **15**(50) (2008), 351–362.

[31] F. Defever, R. Deszcz, M. Hotloś, M. Kucharski and Z. Şentürk, Generalisations of Robertson-Walker spaces, *Annales Univ. Sci. Budapest. Eötvös Sect. Math.* **43** (2000), 13–24.

[32] R. Deszcz, M. Głogowska, M. Hotloś and K. Sawicz, A Survey on Generalized Einstein Metric Conditions, in: *Advances in Lorentzian Geometry: Proceedings of the Lorentzian Geometry Conference in Berlin*, eds. M. Plaue, A.D. Rendall and M. Scherfner (AMS/IP Studies in Advanced Mathematics **49**, S.-T. Yau (series ed.), 2011) pp. 27–46.

[33] M. Głogowska, On quasi-Einstein Cartan type hypersurfaces, *J. Geom. Phys.* **58** (2008), 599–614.

[34] Y. Ishii, On conharmonic transformations, *Tensor (N.S.)* **7** (1957), 73–80.

[35] D. Kowalczyk, On the Reissner-Nordström-de Sitter type spacetimes, *Tsukuba J. Math.* **30** (2006), 263–281.

[36] R. Deszcz, M. Głogowska, M. Plaue, K. Sawicz and M. Scherfner, On hypersurfaces in space forms satisfying particular curvature conditions of Tachibana type, *Kragujevac J. Math.* **35** (2011), 223–247.

[37] S. Haesen and L. Verstraelen, Properties of a scalar curvature invariant depending on two planes, *Manuscripta Math.* **122** (2007), 59–72.

[38] B. Jahanara, S. Haesen, Z. Şentürk and L. Verstraelen, On the parallel transport of the Ricci curvatures, *J. Geom. Phys.* **57** (2007), 1771–1777.

[39] B. Jahanara, S. Haesen, M. Petrović-Torgašev and L. Verstraelen, On the Weyl curvature of Deszcz, *Publ. Math. Debrecen* **74** (2009), 417–431.

[40] R. Deszcz, M. Głogowska, and M. Hotloś, Some identities on hypersurfaces in conformally flat spaces, in: *Proceedings of the International Conference XVI Geometrical Seminar, Vrnjačka Banja, September, 20-25, 2010*, (Faculty of Science and Mathematics, University of Niš, Serbia, 2011) pp. 34–39.
[41] Z.I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R = 0$, I, The local version, *J. Diff. Geom.* 17 (1982), 531–582.

[42] A. Derdziński and W. Roter, Some theorems on conformally symmetric manifolds, *Tensor (N.S.)* 32 (1978), 11–23.

[43] A. Derdziński and W. Roter, Projectively flat surfaces, null parallel distributions, and conformally symmetric manifolds, *Tohoku Math. J.* 59 (2007), 565–602.

[44] A. Derdziński and W. Roter, On compact manifolds admitting indefinite metrics with parallel Weyl tensor, *J. Geom. Physics* 58 (2008), 1137–1147.

[45] A. Derdziński and W. Roter, The local structure of conformally symmetric manifolds, *Bull. Belg. Math. Soc. - Simon Stevin* 16 (2009), 117–128.

[46] A. Derdziński and W. Roter, Compact pseudo-Riemannian manifolds with parallel Weyl tensor, *Ann. Global Anal. Geom.* 37 (2010), 73–90.

[47] M. Hotloś, On conformally symmetric warped products, *Annales Academiae Paedagogicae Cracoviensis* 23 (2004), 75–85.

[48] S.B. Edgar and J.M.M. Senovilla, (Conformally) semisymmetric spaces and special semisymmetric Weyl tensor, *J. Phys.: Conference Series* 314 (2011) 012019, 1-4.

[49] O.F. Blanco, M. Sánchez and J.M.M. Senovilla, Structure of second-order symmetric Lorentzian manifolds, *J. Eur. Math. Soc.* 15 (2013), 595–634.

[50] J.M.M. Senovilla, Second-order symmetric Lorentzian manifolds: I. Characterization and general results, *Class. Quantum Grav.* 25 (2008), 245011 (25 pages).

[51] R. Deszcz and W. Grycak, On some class of warped product manifolds, *Bull. Inst. Math. Acad. Sinica* 15 (1987), 311–322.

[52] M. Belkhelfa, R. Deszcz, M. Glogowska, M. Hotloś, D. Kowalczyk and L. Verstraelen, On some type of curvature conditions, *Banach Center Publ.* 57, Inst. Math. Polish Acad. Sci., 2002, 179–194.

[53] R. Deszcz, On pseudosymmetric spaces, *Bull. Belg. Math. Soc., Ser. A* 44 (1992), 1–34.

[54] R. Deszcz, L. Verstraelen and Ş. Yaprak, Pseudosymmetric hypersurfaces in 4-dimensional spaces of constant curvature, *Bull. Inst. Math. Acad. Sinica* 22 (1994), 167–179.

[55] R. Deszcz, L. Verstraelen and L. Vrancken, The symmetry of warped product space-times, *Gen. Rel. Gravitation* 23, 1991, 671–681.

[56] R. Deszcz, S. Haesen and L. Verstraelen, Classification of space-times satisfying some pseudo-symmetry type conditions, *Soochow J. Math.* 23 (2004), 339–349 (Special issue in honor of Professor Bang-Yen Chen).

[57] S. Haesen and L. Verstraelen, Classification of the pseudosymmetric space-times, *J. Math. Phys.* 45 (2004), 2343–2346.

[58] M.C. Chaki, On pseudosymmetric manifolds, *An. Ştiinţ. Univ., ”Al.I. Cuza” Iaşi Sect., I a Mat.* 33 (1987), 53–58.

[59] R. Deszcz, On Ricci-pseudosymmetric warped products, *Demonstratio Math.* 22 (1989), 1053–1065.
[60] R. Deszcz and M. Hotloś, Remarks on Riemannian manifolds satisfying certain curvature condition imposed on the Ricci tensor, *Prace Nauk. Pol. Szczec.* 11 (1989), 23–34.

[61] R. Deszcz and S. Yaprak, Curvature properties of Cartan hypersurfaces, *Colloq. Math.* 67 (1994), 91–98.

[62] R. Deszcz, M. Hotloś and Z. Şentürk, On curvature properties of certain quasi-Einstein hypersurfaces, *Int. J. Math.*, 23 (2012), 1250073 (17 pages).

[63] R. Deszcz, On four-dimensional warped product manifolds satisfying certain pseudosymmetry curvature conditions, *Colloq. Math.* 62 (1991), 103–120.

[64] F. Defever and R. Deszcz, On semi-Riemannian manifolds satisfying the condition $R \cdot R = Q(S, R)$, in: *Geometry and Topology of Submanifolds, III*, (World Sci., River Edge, NJ, 1991), pp. 108–130.

[65] F. Defever and R. Deszcz, On warped product manifolds satisfying a certain curvature condition, *Atti. Acad. Peloritana Cl. Sci. Fis. Mat. Natur.* 69 (1991), 213–236.

[66] R. Deszcz and W. Grycak, On manifolds satisfying some curvature conditions, *Colloq. Math.* 58 (1998), 259–268.

[67] R. Deszcz and L. Verstraelen, Hypersurfaces of semi-Riemannian conformally flat manifolds, in: *Geometry and Topology of Submanifolds, III*, (World Sci., River Edge, NJ, 1991), pp. 131–147.

[68] F. Defever, R. Deszcz and M. Prvanović, On warped product manifolds satisfying some curvature condition of pseudosymmetry type, *Bull. Greek Math. Soc.* 36 (1994), 43–67.

[69] J. Chojnacka-Dulas, R. Deszcz, M. Głogowska and M. Prvanović, On warped products manifolds satisfying some curvature conditions, *J. Geom. Phys.* 74 (2013), 328–341.

[70] K. Arslan, Y. Çelik, R. Deszcz and R. Ezentas, On the equivalence of the Ricci-semisymmetry and semisymmetry, *Colloq. Math.* 76 (1998), 279–294.

[71] R. Deszcz, M. Głogowska, M. Hotloś and Z. Şentürk, On certain quasi-Einstein semisymmetric hypersurfaces, *Ann. Univ. Sci. Budap. Rolando Eötvös Sect. Math.* 41 (1998), 151–164.

[72] C.A. Mantica and L.G. Molinari, Extended Derdziński-Shen theorem for curvature tensors, *Colloq. Math.* 128 (2012), 1–6.

[73] C.A. Mantica and L.G. Molinari, Riemann compatible tensors, *Colloq. Math.* 128 (2012), 197–210.

[74] R. Deszcz, M. Głogowska, J. Jelowicki, M. Petrović-Torgašev and G. Zafindratafa, On Riemann and Weyl compatible tensors, to appear.

[75] C.A. Mantica and L.G. Molinari, A second order identity for the Riemann tensor and applications, *Colloq. Math.* 122 (2011), 69–82.

[76] C.A. Mantica and L.G. Molinari, Weakly Z-symmetric manifolds, *Acta Math. Hungar.* 35 (2012), 80–96.

[77] C.A. Mantica and L.G. Molinari, Weyl compatible tensors, arXiv:1212.1273v1 [math-ph] 6 Dec 2012.

[78] R. Deszcz, L. Verstraelen and Ş. Yaprak, Warped products realizing a certain condition of pseudosymmetry type imposed on the Weyl curvature tensor, *Chinese J. Math.* 22 (1994), 139–157.

[79] M. Głogowska, Semi-Riemannian manifolds whose Weyl tensor is a Kulkarni-Nomizu square, *Publ. Inst. Math. (Beograd) (N.S.)* 72(86) (2002), 95–106.
Ryszard Deszcz and Jan Jelowicki,
Department of Mathematics,
Wrocław University of Environmental and Life Sciences
Grunwaldzka 53, 50-357 Wrocław, Poland
E-mail address: ryszard.deszcz@up.wroc.pl jan.jelowicki@up.wroc.pl

Marian Hotloś
Institute of Mathematics and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
E-mail address: marian.hotlos@pwr.wroc.pl

Haradhan Kundu and Absos Ali Shaikh
Department of Mathematics,
University of Burdwan, Golapbag,
Burdwan-713104,
West Bengal, India
E-mail address: kundu.haradhan@gmail.com, aask2003@yahoo.co.in, aashaikh@mathburuniv.ac.in