 CYLINDERS OVER AFFINE SURFACES

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For an affine variety $S$ we consider the ring $AK(S)$, which is the intersection of the rings of constants of all locally-nilpotent derivations of the ring $\mathcal{O}(S)$. We show that $AK(S \times \mathbb{C}^n) = AK(S)$ for a smooth affine surface $S$ with $H^2(S, \mathbb{Z}) = \{0\}$.

Introduction.

In this paper we are trying to understand better a ring invariant, which was introduced in [ML1], (see also [KML]). This invariant was used in order to show that non of the Dimca-Koras-Russell threefolds, which are related to the linearizing question, is isomorphic to $\mathbb{C}^3$ ([ML2], [KML], [KKMLR]). It also helped to describe the automorphisms of a surface $x^n y = f(z)$ (see [ML3]) and to give a new proof of the theorem of S. Abhyankar, P. Eakin, W. Heinzer [AEH] on cancelations for curves (in characteristic zero case) and some generalizations of this theorem ([ML4]).

Let us start with necessary algebraic notions in the generality which corresponds to our intended setting.

Let $R$ be an algebra over a field $\mathbb{C}$. Then a $\mathbb{C}$-homomorphism $\partial$ of $R$ is called a derivation of $R$ if it satisfies the Leibniz rule: $\partial(ab) = \partial(a)b + a\partial(b)$.

Any derivation $\partial$ determines two subalgebras of $R$. One is the kernel of $\partial$, which is usually denoted by $R^\partial$ and is called the ring of $\partial$-constants, by analogy with the ordinary derivative. The other is $nil(\partial)$, the ring of nilpotency of $\partial$. It is
determined by $\text{nil}(\partial) = \{ a \in R | \partial^n(a) = 0, n >> 1 \}$. In other words $a \in \text{nil}(\partial)$ if for a sufficiently large natural number $n$ we have $\partial^n(a) = 0$.

Both $R^\partial$ and $\text{nil}(\partial)$ are subalgebras of $R$ because of the Leibniz rule.

Let us call a derivation \textit{locally nilpotent} if $\text{nil}(\partial) = R$. Let us denote by $\text{ln}(R)$ the set of all locally nilpotent derivations.

The best examples of locally nilpotent derivations are the partial derivatives on the rings of polynomials $\mathbb{C}[x_1, \ldots, x_n]$.

The intersection of the rings of constants of all locally nilpotent derivations of $R$ is the invariant mentioned above. It will be called the \textit{ring of absolute constants} and denoted by $AK(R)$.

If $V$ is a complex affine variety and $\mathcal{O}(V)$ is the ring of regular functions on $V$, let us denote $AK(\mathcal{O}(V))$ by $AK(V)$. A locally nilpotent derivation of $\mathcal{O}(V)$ corresponds via exponentiation to a $\mathbb{C}$ action on $V$ (see [S]). It follows, that if $AK(V) \neq \mathcal{O}(V)$, variety $V$, as we see later, is ruled or “cylinderlike,” which means that it contains an open subset which is a product of affine variety and a complex line $\mathbb{C}$. It seems that the invariant $AK(V)$ is especially helpful when one tries to compare a variety with $\mathbb{C}^n$. E.g. M. Miyanishi ([Mi1]) showed that an affine surface $V$ with factorial ring $\mathcal{O}(V) \neq AK(V)$, is isomorphic to $\mathbb{C}^2$, provided $\mathcal{O}(V) = \mathbb{C}$. So, we think it is rather important to learn to compute this invariant.

One of the approaches to this is to find a connection between $AK(V \times W)$, $AK(V)$, and $AK(W)$ where $V$ and $W$ are affine varieties. E.g. it is known (see [ML4]), that if $V$ is a curve which is not an affine line, then $AK(V \times W) = AK(V) \otimes_{\mathbb{C}} AK(W)$. It is also known (see [ML1]) that if $AK(V) = \mathcal{O}(V)$, then $AK(V \times \mathbb{C}^1) = AK(V) \otimes_{\mathbb{C}} \mathbb{C} = AK(V)$. Any locally nilpotent derivation $\partial$ of $\mathcal{O}(V)$ or of $\mathcal{O}(W)$ can be extended to a locally nilpotent derivation of $\mathcal{O}(V \times W)$ by $\partial(f) = 0$ for any $f \in \mathcal{O}(W)$ (of $f \in \mathcal{O}(V)$ correspondingly). So it is clear that $AK(V \times W) \subset AK(V) \otimes AK(W)$. In geometric terms it is the following obvious observation. If $f \in \mathcal{O}(V \times W)$ is invariant under all $\mathbb{C}$-actions it is also invariant under all $\mathbb{C}$ actions which are “lifted” from the components.
Unfortunately, it is not true in general even when \( W = \mathbb{C}^1 \), that \( AK(V \times W) = AK(V) \otimes AK(W) \), which is demonstrated by the following example.

Let surfaces \( S_1 \) and \( S_2 \) be defined in \( \mathbb{C}^3 \) by equations \( xy = z^2 - 1 \) and \( x^2y = z^2 - 1 \). Danielewski ([D], [K]) showed that the cylinders over these surfaces are isomorphic and Fieseler ([F]) proved that the surfaces themselves are not isomorphic.

These surfaces were suggested by Danielewski as a counterexample to the generalized Zariski cancelation conjecture.

In our setting they provide an example of a situation when \( AK(R) \subsetneq AK(R[x]) \).

Let \( R_1 \) and \( R_2 \) be the rings of regular functions on \( S_1 \) and \( S_2 \) correspondingly. It is easy to find two locally nilpotent derivations on \( R_1 \) such that the intersection of their kernels is just \( \mathbb{C} \). Say, take \( \partial_1 \) defined by \( \partial_1(x) = 0, \partial_1(y) = 2z, \partial_1(z) = x \), and \( \partial_2 \) defined by \( \partial_2(x) = 2z, \partial_2(y) = 0, \partial_2(z) = y \).

On the other hand it is possible to show that any locally nilpotent derivation of \( R_2 \) has \( x \) in the kernel and that \( AK(R_2) = \mathbb{C}[x] ([ML3]) \). Thus \( AK(R_2) = \mathbb{C}[x] \neq AK(R_2[x]) = AK(R_1[x]) = \mathbb{C} \).

So it seems rather natural to find conditions on a variety which ensure the equality \( AK(V) = AK(V \times \mathbb{C}^n) \).

The goal of this paper is to show that if \( S \) is a smooth surface and \( H^2(S, \mathbb{Z}) = 0 \), then \( AK(S \times \mathbb{C}^n) = AK(S) \).

We also would like to state the following

**Conjecture.** \( AK(V) = AK(V \times \mathbb{C}^n) \) if \( O(V) \) is a factorial ring.

If this, indeed, is true it will advance rather substantially an understanding of Zariski cancelation conjecture and the linearizing question for \( \mathbb{C}^* - \) action on \( \mathbb{C}^n \).

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1. Auxiliary Facts
Assume that a group $G$, possibly infinite dimensional, is generated by a finite number of $C$-actions $\{\varphi_i\}$, which act algebraically on a $n$-dimensional irreducible reduced affine variety $X$. This means that for any $i$ there is a regular rational map $\varphi_i : C \times X \to X$ such that

a) $\varphi_i(z_0, x) = \varphi_i^{z_0}(x)$ is an algebraic regular automorphism of $X$;

b) $\varphi_i^{z_0 + z_1}(x) = \varphi_i^{z_0} \circ \varphi_i^{z_1}(x)$.

If the group $G$ is algebraic, then, due to the Rozenlicht Theorem (see, for example, [P-V]), there are two possibilities: either the a general orbit is Zariski dense in $X$, or there exists a $G$-invariant rational function on $X$.

The following proposition is a generalization of this fact for a non-algebraic group.

**Proposition 1.1.** If the $G$-orbit $G_{x_0} = \{y : y = g(x_0), g \in G\}$ of a general point $x_0$ is not dense in $X$, then there exists a $G$-invariant rational map $\pi : X \to X_G$ of the variety $X$ into an irreducible algebraic variety $X_G$ with $\dim X_G < \dim X$.

We need two Lemmas to prove the Proposition.

**Lemma 1.2.** The graph $D_G = \{(x, y) \in X \times X : y = g(x), g \in G\}$ is a dense subset of a closed algebraic subset of $X \times X$. Moreover, the orbit of a general point $x_0$ is a dense subset of an algebraic subset of $X$.

**Proof of Lemma 1.2.** Let word $I = \{i_1 \ldots i_s\}$ be a word of length $s$, where $i_k$ are natural numbers. Let us define a regular algebraic map $F_I : C^s \times X \to X$ by

$$F_I(z, x) = \varphi_{i_1}^{z_1} \circ \cdots \circ \varphi_{i_s}^{z_s}(x),$$

where $z = (z_1 \ldots z_s) \in C^s$ and $x \in X$.

For a multiindex $I = \{i_1, \ldots i_s\}$ the graph of this map $\Gamma_I = \{(z, x, y) : y = F_I(z, x)\}$ is a closed subset of $C^s \times X \times X$. Since this graph is isomorphic to $C^s \times X$, it is irreducible.

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1Further on we shall say that a set $W$ is dense in the algebraic set $V$, if $W$ contains a Zariski open subset of $V$. 
Denote by $\Gamma_I$ its closure in the product $\mathbb{P}^s \times \overline{X} \times \overline{X}$ of closures $\mathbb{P}^s, \overline{X}$ of $\mathbb{C}^s$ and $X$ respectively. It is irreducible because $\Gamma_I$ is irreducible. Hence its projection $Z_I \subset \overline{X} \times \overline{X}$ into $\overline{X} \times \overline{X}$ is an irreducible closed subset of $\overline{X} \times \overline{X}$, containing the projection $W_I$ of $\Gamma_I$ as a dense subset (since $\Gamma_I$ is dense in $\overline{\Gamma}_I$). Let $Z_I = \overline{Z}_I \cap (X \times X)$. Then $W_I \subset Z_I \subset \overline{Z}_I$, where $Z_I$ is an irreducible closed subset of $X \times X$ and $W_I$ is its dense subset. For any two words $I$ and $J$, such that $J$ contains $I$, $W_I \subset W_J$ and $\overline{Z}_I \subset \overline{Z}_J$. Now from the irreducibility of both $\overline{Z}_I$ and $\overline{Z}_J$ follows that either $\overline{Z}_J = \overline{Z}_I$ or $\dim \overline{Z}_J > \dim \overline{Z}_I$. Since $\dim Z_I \leq 2n$, it is possible to chose a word $I$ such that $k = \dim \overline{Z}_I$ is maximal among all indices $I$.

Consider two cases.

1. Let $k = 2n$. Then $Z_I = \overline{X}$ and for a general point $x_0 \in X$ we have $\dim Z_I \cap \{x_0 \times X\} = n$. Thus its projection together with projection of $W_I$ into second factor $X$ is dense in $X$. But the projection of $W_I$ is precisely the orbit of a point $x_0$, which shows that in this case the general orbit is dense in $X$.

2. Let $k$ be less then $2n$. Since $\overline{Z}_J \subset \overline{Z}_I \cup E$ and by the choice of $I$, $\overline{Z}_J \subset \overline{Z}_I$ for any word $J$. So, $W_J \subset \overline{Z}_I$, for any such word $J$, and $D_G = \cup W_J$ is contained in $\overline{Z}_I$ as well. On the other hand, $D_G$ contains $W_I$, and the last is dense in $\overline{Z}_I$. Moreover by definition $D_G \subset X \times X$. Thus, $D_G$ is a dense subset of a closed subset $Z_I$ of the product $X \times X$. The orbit of a general point $x_0$ is a projection of $D_G \cap (x_0 \times X)$ into $X$, which is a dense subset of the intersection of the projection of $\overline{Z}_I \cap (x_0 \times \overline{X})$ into $\overline{X}$ and $X$.

Lemma 1.3. If the general $G$-orbit $G_{x_0}$ is not dense in $X$, then there are rational functions on the variety $X$, invariant under the action of the group $G$.

Proof of Lemma 1.3.

Let $K$ be an ideal of the functions in, which are equal to zero on the closure $\overline{D}_G \subset X \times X$ in $X \times X$ of the set $D_G$. Any such function $f(x, y)$ has the form

$$f(x, y) = \sum_{i=1}^{n_f} p_i(x)q_i(y),$$

where $(x, y)$ are the points of $X \times X$ and $q_i(y) \in \mathcal{O}(X)$, $p_i(x) \in \mathcal{O}(X)$. If for all functions $f(x, y) \in K$ all $p_i(x) = 0$, then any pair $(x, y) \in X \times X$ belongs to $\overline{D}_G$. 

\[\square\]
i.e. $D_G$ is dense in $X \times X$, and the general orbit $D_G \cap \{x_0 \times X\}$ is dense in $X$ for a general point $x_0 \in X$, which contradicts the assumptions of the Lemma. Thus, $K$ contains non-zero functions.

Let $f_0$ be a function in $K \setminus \{0\}$, such that $\nu = n_{f_0} = \min \{n_f | f \in K \setminus \{0\} \}$:

$$f_0 = \sum_{i=1}^{\nu} p_i(x)q_i(y).$$

Once more two cases are possible.

1) $\nu = 1$ and $f_0(x, y) = p_1(x)q_1(y)$.

Then $D_G$ is contained in $\{(X \times N) \cup (M \times X)\}$, where $N = \{y | q_1(y) = 0\}$ and $M = \{x | p_1(x) = 0\}$. For any point $x_0$ the pair $(x_0, x_0) \in D_G \subset (X \times N) \cup (M \times X)$ which is impossible, if $x_0 \notin N \cap M$. Thus, $\nu \geq 2$.

2) $f_0(x, y) = p_1(x)q_1(y) + \sum_{i=2}^{\nu} p_i(x)q_i(y)$, where $p_1(x) \neq 0$. Then for $(x, y) \in D_G$ and $g \in G$ two equalities hold:

$$p_1(x)q_1(y) + \sum_{i=2}^{\nu} p_i(x)q_i(y) = f_0(x, y) = 0.$$

(2)

$$p_1(gx)q_1(y) + \sum_{i=2}^{\nu} p_i(gx)q_i(y) = f_0(gx, y) = 0.$$

Therefore

$$p_1(gx)f_0(x, y) - p_1(x)f_0(gx, y) = \sum_{i=2}^{\nu} (p_1(gx)p_i(x) - p_1(x)p_i(gx))q_i(y) \in K$$

and is “shorter” then $f_0$. Since $\nu$ was minimal by the choice of $f_0$, it means that

$$p_1(gx)p_i(x) - p_1(x)p_i(gx) \equiv 0 \text{ for } i = 2, ..., \nu.$$

Thus $\frac{p_i(x)}{p_1(x)}$ are $G$–invariant rational functions. They cannot be constant: if, say, $\frac{p_2(x)}{p_1(x)} = c$, then $f_0$ could have been written in a “shorter “ way:

$$f_0 = p_1(x)(q_1(y) + cq_2(y)) + \sum_{i=3}^{\nu} p_i(x)q_i(y).$$

Hence $\frac{p_i(x)}{p_1(x)}$ are the needed $G$-invariant rational functions. □
Proof of the Proposition 1.1.

Let \( f_0(x, y) = p_1(x)q_1(y) + \sum_{i=2}^{\nu} p_i(x)q_i(y) \). Then the map

\[
\pi : x \mapsto (p_1(x) : p_2(x) : \cdots : p_\nu(x))
\]

is a rational map of \( \overline{X} \) into a projective space \( \mathbb{P}^{\nu-1} \). Since there always exists a resolution \( X' \) of the map \( \pi \), (i.e. an irreducible projective variety birationally equivalent to \( \overline{X} \), such that the induced map \( \pi' : X' \to \pi(\overline{X}) \) is regular), the set \( \pi(\overline{X}) \subseteq \mathbb{P}^{\nu-1} \) is an image of an irreducible projective set under a regular map. Thus, \( \pi(\overline{X}) \) is projective and irreducible ([Shi2], 5.2) and contains \( \pi(X) \) as a dense subset. Since the general orbit \( G_x \) is dense in a closed subset \( \overline{G}_x \) of \( X \), \( \dim \pi(\overline{X}) \leq \dim(\overline{X}) - \dim \overline{G}_x < \dim(\overline{X}) \), which completes the proof. \( \square \)

The next Lemma is a particular case of Lemma 2.2 in a paper of M. Miyanishi [Mi2].

**Lemma 1.4.** (see [Mi2]) Let \( R \) be a finitely generated ring. Then it has a non-zero locally nilpotent derivation if and only if there exists an element \( t \in R \), such that \( R[t^{-1}] \) is isomorphic to a polynomial ring \( S[x] \).

**Proof of Lemma 1.4.** See [Mi2]. \( \square \)

**Corollary 1.5.** Let \( X \) be an affine normal variety and \( \pi : X \to Y \) be a regular map into a normal affine variety \( Y \). Let \( t \in \mathcal{O}(Y) \) and let \( D \) be divisor of its zeros. Assume that \( V = X \setminus \pi^{-1}(D) \cong (Y \setminus D) \times \mathbb{C} \). Then there is a \( \mathbb{C} \)-action on \( X \), such that its general orbit is a fiber of the map \( \pi \).

**Proof of Corollary 1.5.** Let \( S = \mathcal{O}(Y \setminus D) \). Then \( \mathcal{O}(V) = \mathcal{O}((Y \setminus D) \times \mathbb{C}) = S[x] \). On the other hand \( \mathcal{O}(V) = \mathcal{O}(X)[t^{-1}] \), since for any function \( r \in \mathcal{O}(V) \) there is such positive integer \( k \) that \( t^kr \in \mathcal{O}(X) \). So, according to Lemma 1.4 there is a locally nilpotent derivation \( \partial \) on \( R \), such that \( S \subset R^\partial \). Since variety \( Y \) is affine, the functions \( s \in S \) divide points in \( Y \). That means that a general fiber of \( \pi \) may be described as \( s_1 = \text{const}, \ldots, s_n = \text{const} \) for some \( s_1, \ldots, s_n \in S \). Since all \( s \) are \( \partial \)-constants, it means that the general fiber is invariant under \( \mathbb{C} \)-action.
corresponding to $\partial$. On the other hand, a general fiber is isomorphic to $\mathbb{C}$ and consequently is an orbit. $\square$

2. Invariant of product.

In this section we prove the following

**Theorem 2.1.** Let $S$ be a smooth surface with $H^2(S, \mathbb{Z}) = \{0\}$. Then $AK(S \times \mathbb{C}^n) = AK(S)$.

**Remark.** The condition $H^2(S, \mathbb{Z}) = \{0\}$ is essential. In the introduction an example is given of a surface with $H^2(S, \mathbb{Z}) \neq \{0\}$, and such that $AK(S \times \mathbb{C}^n) \neq AK(S)$.

Let $S$ be a smooth affine surface, $X = S \times \mathbb{C}^n$ and $\pi : X \to S$ the natural projection. Assume that there is a $\mathbb{C}$–action $\varphi_\lambda$ on $X$ such that the orbit $\Gamma_{x_0} = \{\varphi_\lambda(x_0), \lambda \in \mathbb{C}\}$ of a general point $x_0$ is not contained in $\pi^{-1}(\pi(x_0))$. Denote by $\psi_{1,\lambda}, ..., \psi_{n,\lambda}$ the standard actions acting along the fibers of a projection $\pi$ and by $G$ the automorphisms group of $X$ generated by $\varphi_\lambda, \psi_{1,\lambda}, ..., \psi_{n,\lambda}$. In Lemma 1.2 we proved that the orbit $G_{x_0}$ of a general point $x_0$ is a dense subset of a closed subset $\overline{G_{x_0}}$ of $X$.

If $\dim \overline{G_{x_0}} = n+1$ then by Proposition 1.1 there exists a dominant $G$–invariant rational map $p : X \to \mathbb{P}^1$.

The fibers of this map contain $G$–orbits.

Since the map $p$ is $G$–invariant, it induces the map $p_1 : S \to \mathbb{P}^1$, such that the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & \mathbb{P}^1 \\
\downarrow \pi & & \\
S & \xrightarrow{p_1} & \mathbb{P}^1
\end{array}
\]

We consider two different cases.
Case 1. The map \( p_1 \) is regular (i.e. it is everywhere defined).

Case 2. For any \( G \)-invariant rational map \( p \) the map \( p_1 : S \to \mathbb{P}^1 \) is not regular.

**Lemma 2.2.** In case 1 there is a \( \mathbb{C} \)-action \( \theta_\lambda \) on the surface \( S \), such that a general orbit \( \gamma_{s_0} = \{ \theta_\lambda(s_0), \lambda \in \mathbb{C} \} = \pi(G_{x_0}) \), where \( \pi(x_0) = s_0 \).

**Proof of Lemma 2.2.**

In this case both maps \( p_1 \) and \( p \) are regular.

Choose a closure \( \tilde{S} \) of \( S \) in such a way that the map \( p_1 \) may be extended to a regular map \( \tilde{p}_1 : \tilde{S} \to \mathbb{P}^1 \).

Let \( \tilde{C} \) be the normalization of the Stein factorization of a map \( \tilde{p}_1 \). By definition (see, e.g, [B], p.66) that means, that \( \tilde{C} \) is a smooth curve included into the following commutative diagram:

\[
\begin{array}{c}
\tilde{S} & \xrightarrow{\tilde{p}_1} & \mathbb{P}^1 \\
 \downarrow h & & \downarrow q \\
 \tilde{C}
\end{array}
\]

where all the maps are regular, \( h \) has connected fibers and \( q \) is finite. Let \( F_c = h^{-1}(c) \subset \tilde{S} \) be a fiber over a general point \( c \in \tilde{C} \). Take points \( s \in F_c \cap S \) and \( x \in \pi^{-1}(s) \). The orbit \( \Gamma_x = \{ \varphi_\lambda(x), \lambda \in \mathbb{C} \} \subset X \) is a rational curve with a single puncture, because it is an image of a complex plane. It follows, that \( \pi(\Gamma_x) \subset S \) is a rational curve with a single puncture as well. Since \( F_c \) is a closed connected curve, containing \( \pi(\Gamma_x) \), \( F_c \) has to be the closure of \( \pi(\Gamma_x) \). Moreover, it has to be smooth (for general \( c \)), since it is a fiber of a regular map of a smooth surface onto a smooth curve (see, for example, [Sh1], §4). Thus, the restriction of \( h \) onto \( S \) has a general fiber \( h^{-1}(c) \cap S = \pi(\Gamma_x) \), which is isomorphic to the complex plane, and a general fiber \( h^{-1}(c) \) is an irreducible smooth rational curve. Hence, \( \tilde{S} \) is a ruled surface, (see [Sh1], Theorem 2, chapter 4). Moreover, the divisor \( D = \tilde{S} \setminus S \) has precisely one irreducible component \( D_0 \), which is mapped by \( h \) isomorphically.
onto $\tilde{C}$, since a general fiber of $h|_S$ is isomorphic to $\mathbb{C}^1$. All other components of the divisor $D$ do not intersect with a general fiber of $h$, which is irreducible. Consider the reducible fibers $F_i, i = 1, \ldots, s$. Since intersection $(F_i, D_0) = 1$, they have the following structure: $F_i = C_i + \sum_{j=1}^{n_i} \alpha_j E_{ij}$, where $C_i, E_{ij}$ are irreducible components, $(C_i, D_0) = 1, (E_{ij}, D_0) = 0$. According to [Mi3], 4.4.1, every reducible fiber contains at least one exceptional curve of the first type.

Since $H^2(S) = \{0\}$, the inclusion $D \to \tilde{S}$ induces the epimorphism $H_2(D) \to H_2(\tilde{S})$.

The group $H_2(\tilde{S}, \mathbb{Z})$ may be described as follows. Let $S_0$ be a surface obtained by blowing down all the exceptional curves of the first type in all the fibers of the map $h$. We may repeat this procedure till we obtain the surface $S_n$, the map $t : \tilde{S} \to S_n$, and the regular map $h_n : S_n \to \tilde{C}$ with irreducible fibers. Thus, it will be geometrically ruled surface. The group $H_2(S_n, \mathbb{Z})$ is a direct sum $d_0.\mathbb{Z} \oplus f.\mathbb{Z}$, where $d_0$ and $f$ are the homology classes of $D_0$ and a general fiber respectively ([B],III.18). Let $E_{ij}$ be all the irreducible curves in $\tilde{S}$, which are contracted by $t$, and $e_{ij}$ their homology classes (there is precisely $n_i$ of such curves in reducible fiber $F_i$). Then $H_2(\tilde{S}, \mathbb{Z}) = H_2(S_n) \oplus \sum_i \sum_j e_{ij}.\mathbb{Z}$. Thus, $H_2(D) \to H(\tilde{S})$ may be an epimorphism only in case when $D$ contains at least one fiber $F$ and other fibers have only one irreducible component in $S$.

There are two important consequences of this fact.

1) $p_1(S)$ is an affine subset of $\tilde{C}$. Indeed, a point $c = h(F) \in (\tilde{C} \setminus h(S))$.

2) $S$ has a “cylinderlike” subset. Indeed, since $\tilde{S}$ is ruled, by taking away the finite number of points $c_1, c_2, \ldots, c_N$ from $\tilde{C}$ we obtain the Zariski open subset $\tilde{U} \subset \tilde{S}$, which is isomorphic to $(\tilde{C} \setminus \{c_1, \ldots, c_N\}) \times \mathbb{P}^1$, ([B], p. 26) and $\tilde{U} \cap S$ is isomorphic to $(\tilde{C} \setminus \{c_1, \ldots, c_N\}) \times \mathbb{C}$.

Adding, if needed, some other points, we may assume that $c_1 + \ldots + c_N$ is the zero-divisor of a regular function on $\tilde{C} \setminus h^{-1}(c)$.

By Corollary 1.5 there is a $\mathbb{C}^*$-action $\theta(c)$ on $S$ such that an orbit $\gamma_c = \{\theta_t(c) : t \in \mathbb{C}^*\}$ is a cylinder in $\tilde{U} \cap S$. 

\{\theta_\lambda(s), \lambda \in \mathbb{C}\} of a general point \(s\) coincides with \(h^{-1}(h(s)) = \pi(\Gamma_x) = \pi(G_x)\) for a point \(x \in X\), such that \(\pi(x) = s\). □

**Lemma 2.3.** In Case 2 all the units in the ring \(\mathcal{O}(S)\) are constants.

*Proof of Lemma 2.3.* Assume that the Lemma is not true. Let \(t \in \mathcal{O}(S)\) be a non-constant unit in \(\mathcal{O}(S)\) and \(t^*\) be its lift into \(\mathcal{O}(X)\). Since there exists a dominant regular map \(F_i\) (see Lemma 1.2) of \(\mathbb{C}^*\) into any \(G\) orbit, \(t^*\) has to be constant along any \(G\)-orbit, hence, it has to be \(G\)-invariant and provides a \(G\)-invariant regular function \(p\), which is constant over every point \(s \in S\). Hence, it generates a regular function \(p_1\) on \(S\), which is impossible in Case 2. □

**Lemma 2.4.** In case 2 \(\mathcal{O}(S)\) is factorial.

*Proof of Lemma 2.4.* It is enough to show that any effective divisor \(A\) in \(S\) may be defined as \(f = 0\) for some \(f \in \mathcal{O}(S)\). Consider once more diagrams (3) and (4), where \(\tilde{S}\) is any compactification of \(S\). In case 2 map \(p_1\) is not regular and \(\tilde{p}_1\) is a rational non-regular map of \(\tilde{S}\) onto \(\mathbb{P}^1\).

At first we are going to prove that \(\tilde{S}\) is rational.

Let \((S', \alpha)\) a resolution of \(\tilde{S}\), such that the lift \(p_1' : S' \to \mathbb{P}^1\) of \(\tilde{p}_1\) onto \(S'\) is regular. Let \(B'\) be a normal Stein factorization of \(p_1'\). We obtain the following commutative diagram.

\[
\begin{array}{c}
S' \xrightarrow{q} B' \\
\downarrow \alpha \quad \downarrow \tau \\
\tilde{S} \xrightarrow{\tilde{p}_1} \mathbb{P}^1
\end{array}
\]

In this diagram the maps \(p_1', q, \alpha, \tau\) are regular, \(\tau\) is finite and \(\alpha\) is a blowing down of finite number of exceptional curves ([Sh1],[Sh2]).

The map \(p_1'\) is not constant on the exceptional divisor of the map \(\alpha\). That means that there is an irreducible component \(E\) of this divisor, which is mapped
by $p'_1$ onto $\mathbb{P}^1$, and, hence, $q(E) = B'$. Since $E$ is exceptional, it has to be rational, and $B'$ is rational as well.

Consider now the fiber $F_{b'} = q^{-1}(b')$ over a general point $b' \in B'$. We have:

$$\alpha(F_{b'}) \cap S = \pi(\Gamma_x),$$

where $\Gamma_x$ is an orbit of any point $x \in X$, such that $\pi(x) \in \alpha(F_{b'})$. Since $\Gamma_x \cong \mathbb{C}$ and $\alpha|_{F_{b'}}$ is birational, it follows that $F_{b'}$ is a rational curve.

We obtained that $S'$ is a ruled surface with rational base. Therefore, $S'$ is a rational surface and $H^1(S', \mathbb{Z}) = 0$. Since the group $H^1$ is invariant under the blowing-downs, it means that $H^1(\tilde{S}, \mathbb{Z}) = 0$.

Let $D = \sum D_i = \tilde{S} \setminus S$. Let $A$ be any irreducible curve in $S$, and $\tilde{A}$ be its closure in $\tilde{S}$. Since $H^2(S, \mathbb{Z}) = 0$, the map $H_2(D, \mathbb{Z}) \to H_2(\tilde{S}, \mathbb{Z})$ is an epimorphism, hence $\tilde{A} = \sum a_iD_i$ in $H_2(\tilde{S}, \mathbb{Z})$. But since $H^1(\tilde{S}, \mathbb{Z}) = 0$, from the topological equivalence follows the linear equivalence (see, e.g. [B], p. 7), hence $\tilde{A} = \sum a_iD_i$ as a divisor, and there exist a function in $\mathcal{O}(S)$, such that $A$ is its zero divisor. □

Thus, in Case 2, $S$ is a smooth affine surface with factorial $\mathcal{O}(S)$ without non-constant units, and $H^2(S, \mathbb{Z}) = \{0\}$. Moreover, the logarithmic Kodaira dimension $k(S) = k(\tilde{S} \setminus D) = k(S' \setminus \alpha^{-1}(D)) = -\infty$, since the fiber of the restriction of $q$ onto $S' \setminus \alpha^{-1}(D)$ is isomorphic to $\mathbb{C}$. By the Miyanishi-Sugie Theorem ([M-S], [Su]), $S$ is isomorphic to $\mathbb{C}^2$.

**Lemma 2.5.** If $\dim \overline{G_x} = n + 2$ for a general point $x \in X$, then $S$ is isomorphic to $\mathbb{C}^2$.

**Proof of Lemma 2.5.** By virtue of Proposition 1.1 in this case the general orbit contains an image of $\mathbb{C}^k$ under a regular rational map $F_I$ for some $k \geq n + 2$ as a dense subset. Let $\overline{S, X} = \overline{S} \times \mathbb{P}^n$ be the closures of $S, X$ respectively, and $\overline{F_I} : \mathbb{P}^k \to \overline{X}$ be an extension of $F_I$. Then $\overline{\pi} \cdot \overline{F_I}$ will be a rational map of $\mathbb{P}^k$ onto $\overline{S}$. Since any unirational surface is rational ([Sh2], ch 3), $\overline{S}$ has to be rational. That means that, as in Lemma 2.4, from $H^2(S, \mathbb{Z}) = \{0\}$ follows that $\mathcal{O}(S)$ is factorial.
Moreover, \( \mathcal{O}(S) \) has no non constant units and \( k(S) = -\infty \), since \( S \) is dominated by \( \mathbb{C}^k \). By the Miyanishi-Sugie Theorem ([M-S], [Su]), \( S \) is a plane. □

**Proof of Theorem 2.1.**

Let function \( f \in \mathcal{O}(S) \) be invariant under all \( \mathbb{C}^- \) actions on \( S \). We have to prove that its lift \( f^* \) onto the product \( X \) is invariant under any \( \mathbb{C}^- \) action \( X \). If a general \( G^- \) orbit is contained in a fiber of projection \( \pi \), (lies over one point of \( S \)) then it is obviously true. If \( \dim G_x = n + 2 \) for a general point \( x \in X \), by Lemma 2.5 \( AK(S) = \mathbb{C} \) and the statement of Theorem is valid as well.

Thus, we may assume, that \( \dim G_x = n + 1 \) for an orbit \( G_x \) of a general point \( x \in X \). Let \( f \in AK(S), f \not= const \), and let \( f^* \) be its lift into \( \mathcal{O}(X) \). Let \( x_1, x_2 \) be two points in \( X \) belonging to the same \( G^- \) orbit, then \( \pi(x_1), \pi(x_2) \), by Lemma 2.2, belong to the same orbit of a \( \mathbb{C}^- \) action on \( S \), thus \( f(\pi(x_1)) = f(\pi(x_2)) \). But the lift is invariant under the actions along the fibers of \( \pi \), hence \( f^*(x_1) = f^*(x_2) \). □.

**Remark.** In the proof of the Theorem 2.1 the fiber \( \mathbb{C}^n \) of the product \( X = S \times \mathbb{C}^n \) may be replaced by any other affine variety such that the group of its \( \mathbb{C}^- \) actions has a dense orbit. This is the only property of the fiber used in the proof.

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