Dynamics of the Time Horizon Minority Game

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Abstract

We present exact analytic results for a new version of the Minority Game (MG) in which strategy performance is recorded over a finite time horizon. The dynamics of this Time Horizon Minority Game (THMG) exhibit many distinct features from the MG and depend strongly on whether the participants are fed real, or random, history strings. The THMG equations are equivalent to a Markov Chain, and yield exact analytic results for the volatility given a specific realization for the quenched strategy disorder.

1 Introduction

Agent-based models of complex adaptive systems are attracting significant interest across many disciplines[1]. Typically each agent has access to a limited set of recent global outcomes of the system; she then combines this information with her limited strategy set chosen randomly at the start of the game (i.e. quenched disorder) in order to decide an action at a given timestep. These decisions by the $N$ agents feed back to generate the fluctuations in the system’s output. The Minority Game (MG) introduced by Challet and Zhang[2, 3] offers possibly the simplest paradigm for such a complex, adaptive system and has therefore been the subject of intense research activity [1]-[9]. Most studies of the MG have focussed on a calculation of both time and configuration (i.e. quenched disorder) averaged quantities, in particular the ‘volatility’ $\sigma$ where $\sigma$ is the standard deviation of the fluctuations. Our own work has shown that the variations of this averaged $\sigma$ with memory size $m$ can be quantitatively explained in terms of ‘crowd-anticrowd’ collective behavior [5, 6]. This crowd-anticrowd theory, which implicitly assumes both time-averaging and configuration-averaging, is simple and intuitive yet it yields useful analytic expressions [6, 7]. In terms of more detailed microscopic theories, two complementary spin-glass approaches have been shown to be remarkably powerful [2, 9].

In this paper, we wish to focus on the dynamics of the multi-agent game for a given realization of the quenched disorder of initially picked strategies. We
present exact analytic results for a finite time horizon version of the Minority Game, the Time Horizon Minority Game (THMG), in which strategy performance is only recorded over the previous $\tau$ timesteps. This seemingly trivial modification of the basic MG yields a dynamical system with surprisingly rich dynamics. These dynamics depend strongly on whether the participants are fed real (as opposed to random) history strings, and on the nature of the quenched disorder corresponding to initial conditions. We present a set of equations describing the THMG - these equations are equivalent to a Markov Chain. This Markov Chain is used to generate accurate analytic results for the resulting volatility in the THMG. Throughout the paper, similarities and differences between the THMG and MG are pointed out where appropriate. Section 2 provides a brief introduction to the MG and provides exact analytic expressions for the volatility $\sigma$ for a given configuration of quenched disorder. Section 3 discusses the THMG and provides corresponding formulae for this game. Section 4 compares the analytic and numerical results for the THMG. Section 5 provides the conclusion.

2 The basic Minority Game (MG)

The MG [1, 2] comprises an odd number of agents $N$ (e.g. traders or drivers) choosing repeatedly between option 0 (e.g. buy or choose route 0) and option 1 (e.g. sell or choose route 1). The winners are those in the minority group; e.g. sellers win if there is an excess of buyers, drivers choosing route 0 encounter less traffic if most other drivers choose route 1. The outcome at each timestep represents the winning decision, 0 or 1. A common bit-string of the $m$ most recent outcomes is made available to the agents at each timestep [8]. The agents randomly pick $s$ strategies at the beginning of the game, with repetitions allowed (quenched disorder). Each strategy is a bit-string of length $2^m$ which predicts the next outcome for each of the $2^m$ possible histories. After each turn, the agent assigns one (virtual) point to each of her strategies which would have predicted the correct outcome, and penalizes a strategy by one (virtual) point if it incorrectly predicts the outcome. At each turn of the game, the agent uses the most successful strategy, i.e. the one with the most virtual points, among her $s$ strategies.

The number of agents holding a particular combination of strategies can be written as a $D \times D \times \ldots$ ($s$ terms) dimensional tensor $\Omega$, where $D$ is the total number of available strategies. For $s = 2$, this is simply a $D \times D$ matrix where the entry $(i, j)$ represents the number of agents who picked strategy $i$ and then $j$. The strategy labels are given by the decimal representation of the strategy plus unity, for example the strategy 0101 for $m = 2$ has strategy label 5+1=6. This quenched disorder $\Omega$ is fixed at the beginning of the game and can be written using the full or reduced strategy spaces [2]. The value of $a^R_{\mu}$ describes the prediction of strategy $R$ given the history $\mu$, where $\mu$ is the decimal number corresponding to the $m$-bit binary history string. Hence $a^R_{\mu} = -1$ denotes a prediction of choice ‘0’ while $a^R_{\mu} = 1$ denotes a prediction
of choice ‘1’). The approach of our (time and configuration-averaged) crowd-anticrowd theory was to partition the \( N \) agents into \( g \) groups in such a way that the groups themselves were uncorrelated. This was achieved by considering the reduced strategy space, which produces essentially identical results to the full strategy space for the volatility. Specifically, each group \( g \) contains a crowd of \( n^C_g \) agents using highly correlated strategies, and an anticrowd \( n^A_g \) of agents using strategies which are highly anti-correlated to the crowd. This leaves an effective super-agent of size \( n_g = n^C_g - n^A_g \) representing each group. Since the strong correlations have now been removed from the problem, the resulting groups are essentially uncorrelated and can be considered as executing random walks with respect to each other in terms of their decisions. As shown in Refs. [5, 6], summing the resulting variances yields excellent agreement with the numerical values for the volatility \( \langle \langle \sigma \rangle \rangle_\Omega \), where \( \langle \langle \ldots \rangle \rangle_\Omega \) denotes averaging over all initial configurations of quenched disorder \( \{\Omega\} \).

Here we are interested instead in the detailed dynamics of the game for a given choice of initial quenched disorder \( \Omega \), hence we follow a more microscopic approach. This is particularly relevant if the intended application of such games is to understand financial markets, since such markets should each correspond to just one realization of the game given an initial \( \Omega \). Hence we imagine that a particular \( \Omega \) has already been chosen. Since the game involves a coin-toss to break ties in strategy scores, this stochasticity also means that different runs for a given \( \Omega \) will also differ - we return to this point below. The number of traders making decision 1 (the ‘instantaneous crowd’) minus the number of traders making decision 0 (the ‘instantaneous anticrowd’) defines the net ‘attendance’ \( A[t] \) at a given timestep \( t \) of the game. This attendance \( A[t] \) is made up of two groups of traders at any one timestep: there are \( A_D[t] \) traders who act in a ‘decided’ way, i.e. they do not require the toss of a coin to decide which choice to make - this is because they have one strategy that is better than their others, or because their highest-scoring strategies are tied but give the same response as each other to the history \( \mu_t \) at that turn of the game. In addition, there are \( A_U[t] \) traders who act in an ‘undecided’ way, i.e. they require the toss of a coin to decide which choice to make - this is because they have two (or more) highest-scoring tied strategies and these give different responses to the history \( \mu_t \) at that turn of the game. Hence the outcome at timestep \( t \) is given by

\[
A[t] = A_D[t] + A_U[t] .
\] (1)

The state of the game at the beginning of timestep \( t \) depends on the strategy score vector \( s_t \) and a history \( \mu_t \) at that moment. Henceforth we will drop the variable \( t \) from the notation, but note that it remains an implicit variable through the time-dependence of \( s \) and \( \mu \). We also focus on \( s = 2 \) strategies per agent, although the formalism can be generalized in a straightforward way. At timestep \( t \), \( A_D \) is given exactly by

\[
A_D(s, \mu) = \sum_{R, R'} a^R(1 + \text{Sgn}[s_R - s_{R'}])\Psi_{R, R'} (2)
\]
where the symmetrized matrix $\Psi = \frac{1}{2}(\Omega + \Omega^T)$ with $\Omega$ representing the quenched disorder. The element $\Omega_{RR'}$ gives the number of agents picking strategy $R$ and then $R'$. The number of undecided traders $N_U$ is given exactly by

$$N_U(s, \mu) = \sum_{R, R'} \delta(s_R - s_{R'})[1 - \delta(a_R^\mu - a_{R'}^\mu)]\Omega_{RR'}$$ (3)

and hence the attendance of undecided traders $A_U$ is distributed binomially in the following way:

$$A_U(s, \mu) \equiv 2 \text{Bin}[N_U(s, \mu), \frac{1}{2}] - N_U(s, \mu)$$ (4)

where the term ‘Bin’ denotes a binomial distribution with $N_U(s, \mu)$ trials and probability $1/2$.

The so-called ‘volatility’ is used in finance to describe some statistical characteristic of the fluctuations in the market. It does not have a unique definition in the finance literature but is typically taken as some form of ‘root-mean-square’ fluctuation - however this definition leaves open the question of which mean should be computed. In the present context, it makes sense to define the volatility in terms of the time-average of a particular realization $k$ of the random process $A[t]$, given an initial quenched disorder $\Omega$. As mentioned above, our crowd-anticrowd theory was limited to consideration of a time-averaged volatility $\langle \langle \sigma \rangle \rangle_\Omega$ which had also been averaged over all configurations of initial quenched disorder $\{\Omega\}$. The present work goes beyond this limitation to consideration of a particular choice of quenched disorder $\Omega$. Consider any stochastic process $z(t)$ produced by a particular realization $k$ of the game, given an initial quenched disorder $\Omega$. This quantity $z(t)$ could represent the attendance $A[t]$ at timestep $t$, or any time-dependent quantity derived from it such as the mean attendance at timestep $t$ calculated over the past $n$ timesteps, or the volatility defined as the root-mean-square attendance over the past $n$ timesteps. The finite time average of the $k$'th realization of this process is given by

$$[^{(k)}z(t)]_T = \frac{1}{T} \int_{t-T/2}^{t+T/2} [^{(k)}z(t')] dt' .$$ (5)

If $T$ is finite, then $[^{(k)}z(t)]_T$ is itself a random process. In real financial markets, the volatility defined by such a finite-time average does indeed fluctuate producing ‘volatility clustering’. Here we instead wish to focus on the time-average which is defined in the $T \to \infty$ limit:

$$\overline{^{(k)}z} \equiv \lim_{T \to \infty} \frac{1}{T} \int_{t-T/2}^{t+T/2} [^{(k)}z(t')] dt' ,$$ (6)

which no longer depends on $t$ or $T$ but in general does depend on the particular realization $k$ of the ensemble that we have chosen in addition to the dependence on the initial quenched disorder $\Omega$. In the absence of stochastic tie-breaks,
the attendance $A[t]$ would be deterministic hence producing a deterministic Minority Game for a given initial quenched disorder $\Omega$. However in the presence of stochastic tie-breaks, which is the case of interest here, the game should self-average in the following sense: for a given quenched disorder $\Omega$, the time-average of $A[t]$ or any of its higher order correlation functions (e.g. volatility) for a given realization $k$ should be equivalent to the ensemble average value taken over all realizations $k$ at a given time $t$. We stress that this discussion is for one particular (fixed) quenched disorder $\Omega$ - we are not averaging over this quenched disorder. Henceforth we will therefore assume that this ergodic principle holds given a particular $\Omega$, i.e. we assume that the time-average and ensemble-average of both the attendance $A[t]$ and the volatility are equivalent for fixed quenched disorder $\Omega$. We will denote this average attendance as $\overline{A}$ and the associated average volatility as $\sigma$, noting that both have an implicit dependence on the initial quenched disorder $\Omega$. Hence for a given $\Omega$, the square of the volatility is given exactly by the expectation value of the mean-square attendance:

$$\sigma^2 = \sum_A (A - \overline{A})^2 P(A)$$

where $P(A)$ is the probability that the attendance is given by $A$. From Eq. (1) the attendance $A = A_D(s, \mu) + A_U(s, \mu, x)$ where we have included the stochastic variable $x$ to denote the coin-toss process. Because this coin-toss process is unbiased, we have $\overline{A_U(s, \mu, x)} = 0$ and hence $\overline{A} = \overline{A_D(s, \mu)}$. The probability $P(A)$ is exactly equivalent to the probability of obtaining a given $s, \mu$ and $x$. Since $x$ is an independent variable, we have

$$P(A) = P(\mu | s) P(s) P(x)$$

with $P(x)$ given by the binomial expression $\binom{N_U}{x} \left( \frac{1}{2} \right)^{N_U}$. Here $P(s)$ is the probability that the strategy score vector is $s$, while $P(\mu | s)$ is the probability that the game is in a state where the history is $\mu$ given that the strategy score vector is $s$. Hence Eq. (7) can be rewritten exactly as

$$\sigma^2 = \sum_s \left[ \sum_\mu \left( \sum_{x=0}^{N_U} \binom{N_U}{x} \left( \frac{1}{2} \right)^{N_U} (A_D + 2x - N_U - \overline{A})^2 P(\mu | s) \right) P(s) \right]$$

with the mean attendance being given exactly by

$$\overline{A} = \sum_s \left[ \sum_\mu \left( A_D P(\mu | s) \right) P(s) \right]$$

where in Eqs. (9) and (10) $A_D = A_D(s, \mu)$ and $N_U = N_U(s, \mu)$. The difficulty in evaluating this expression for the volatility in the MG, for a given quenched disorder $\Omega$, now lies in the complexity of $P(s)$ and $P(\mu | s)$. As an example of a specific realization of the initial quenched disorder, we shall take $\Omega$ throughout

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this paper to be the following randomly-chosen matrix in the reduced strategy space:

\[
\Omega = \begin{pmatrix}
1 & 2 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 \\
2 & 1 & 0 & 3 & 1 & 0 & 2 & 1 \\
2 & 2 & 0 & 3 & 3 & 2 & 2 & 2 \\
1 & 1 & 0 & 2 & 3 & 3 & 2 & 0 \\
2 & 3 & 2 & 0 & 1 & 5 & 1 & 1 \\
0 & 1 & 4 & 7 & 2 & 1 & 0 & 0 \\
3 & 2 & 2 & 0 & 2 & 2 & 2 & 4
\end{pmatrix}.
\]  \hspace{1cm} (11)

Figure 1 shows the resulting \( P(\mathbf{s}) \) for \( N = 101 \), \( m = 2 \), \( s = 2 \) and \( \Omega \) as shown above. The strategy scores themselves are written out explicitly for the dominant score vectors. As can be seen, the probability distribution \( P(\mathbf{s}) \) is uneven and has non-trivial structure.

### 3 Time Horizon Minority Game (THMG)

In the basic MG, strategy scores are kept since the beginning of the game. One might ask whether this is realistic in a ‘real-world’ situation, given the finite time-horizon under which most ‘agents’ (e.g. traders) tend to operate. Hence we will make a small modification to this rule - we introduce the Time Horizon Minority Game (THMG) in which strategy scores are only kept over the last \( \tau \) turns of the game. Hence agents (e.g. traders) are limited to assessing their strategies’ performance over the last \( \tau \) turns of the game, in addition to the basic MG rule of viewing just the last \( m \) steps of the history. We focus on the low \( m \) regime because of the richer dynamics, however our formalism applies for all \( m \).

Figure 2 shows numerical results for the variation in volatility as a function of \( \tau \), given different realizations of the initial quenched disorder \( \Omega \). Results are shown for \( m = 3 \) using the full strategy space, and for both real and random histories of the game. For real histories it can be seen that the volatility is essentially periodic in \( 2.2^m \). This value corresponds to the number of different paths in a De Bruijn graph linking all \( P = 2^m \) histories. Such a graph is necessarily Eulerian since all the vertex-degrees of a De Bruijn graph are even. Games that differ in \( \tau \) by multiples of \( 2.2^m \) show similar dynamics for \( t > \tau \). The dynamics of the THMG for \( t < \tau \) are that of the basic MG, as can be seen in Figs. 3a and 3b for small \( t \).

The peaks in Fig. 2 which arise at \( \tau = 2.2^m \lambda - 1 \) for real histories, where \( \lambda \geq 1 \) and is an integer, correspond to realizations of the game that are purely deterministic. If the game performs a path around the de Bruijn graph that is of a length \( \tau \) and satisfies the condition that all edges have been visited equally during the course of this path, then \( \mathbf{s} \) and \( \mu \) at the start of the cycle are identical to \( \mathbf{s} \) and \( \mu \) at the end of the cycle, i.e. \( \Delta s_{\text{cycle}} = 0 \) and \( \Delta \mu_{\text{cycle}} = 0 \). Note that even for large \( \lambda \) this is very likely to happen due to the nature of the minority
game in the efficient regime \([10]\). In the THMG it is observed to occur very soon after \(t\) becomes larger than \(\tau\). Once this type of path occurs for integer \(\lambda\), the game evolves such that \(a^\mu\) is equal to \(\pm a^\mu\) at any subsequent time-step of the game. Note that for integer \(\lambda\) the strategies are scored for one time-step less than the period of this special cycle; this fact together with \(\Delta s_{\text{cycle}} = 0\) and \(\Delta \mu_{\text{cycle}} = 0\) imply that \(a^\mu = \pm a^\mu\). When \(s^\mu = \pm a^\mu\), we are left with the unique situation where all tied strategies that have a particular score at history \(\mu\) give the same game decision, i.e. \(+1\) or \(-1\). When this happens there are no longer any traders that have tied strategies telling them to make opposing decisions. None of the game dynamics for this turn of the game are therefore decided by the tossing of coins - this turn is hence purely deterministic. Once the game has found such a deterministic state, it never leaves it and the game henceforth evolves such that \(s^\mu = \pm a^\mu\). Figure 3a shows the finite time-average standard deviation (over 100 turns) of the attendance, together with the actual number of traders making a given decision as a function of time. The case shown corresponds to \(\lambda = 100\) and hence \(\tau = 1599\). For time-steps 0 to 1599 the system is equivalent to that of the basic MG, whereas from \(t = 1600\) onwards the effect of the time horizon on strategy scores becomes apparent. The system only takes about 40 time-steps to become locked into the deterministic state described above. We have observed at low \(m\) and for several randomly selected \(\Omega\), that once \(t\) becomes larger than \(\tau\) then the game rapidly finds the deterministic state described above. The game dynamics hence become deterministic and periodic with period \(\tau + 1\). It is possible to construct specially chosen \(\Omega\) matrices such that the above special cycle is not found during a run of the corresponding game, however we are here interested in characterising the dynamics for a ‘typical’ (i.e. randomly chosen) \(\Omega\). Figure 3b shows the corresponding results for \(\tau = 1600\).

We now present an analytic description of the THMG. Consider \(\tau + 1\) consecutive histories in the game and call this a \(\mu\)-path, denoted as \(\mu_{\text{path}}\). We define \(G\) as the matrix telling us which transitions between histories are allowed in the game - the matrix \(G\) is the \(P \times P\) ‘adjacency’ matrix of a de Bruijn graph of order \(m\). The element \(G_{\mu,\nu}\) has value 1 if history \(\nu\) can follow history \(\mu\) in the game, but has value 0 if the transition is disallowed. The adjacency matrix is hence given by

\[
G_{\mu,\nu} = \delta(2\mu\%P - \nu) + \delta(2\mu\%P + 1 - \nu)
\]

where \(A\%B\) is the remainder when \(A\) is divided by \(B\). Here \(\mu\) and \(\nu\) denote the nodes (i.e. histories) in the de Bruijn graph where \(\mu, \nu = 0, 1, \ldots\). The matrix \(G\) quantifies which paths around history space are allowed and hence we can write down a rule for determining which \(\mu_{\text{path}}\) transitions are permitted in the game. Let \(\mu_{\text{path}}(t-1) = \mu_{t-\tau-1} \rightarrow \mu_{t-\tau} \rightarrow \ldots \mu_{t-1}\). Whether \(\mu_{\text{path}}(t-1)\) actually arises in the game, and whether the transition to \(\mu_{\text{path}}(t) = \mu_{t-\tau} \rightarrow \mu_{t-\tau+1} \rightarrow \ldots \mu_{t}\) is allowed, depends on whether all the transitions are allowed between histories in \(\mu_{\text{path}}(t-1)\) and the corresponding history in \(\mu_{\text{path}}(t)\). Figure 4 shows an
example of the allowed transitions between \( \mu \)-paths for \( m = 2, \tau = 2 \). Let us define a scalar quantity \( \Gamma \) given by

\[
\Gamma = \prod_{i=0}^{\tau} G_{\mu_{t-1-i}, \mu_{t-1}} .
\] (13)

The transition between \( \mu \)-path \((t-1)\) and \( \mu \)-path \((t)\) is allowed if \( \Gamma = 1 \), but is disallowed if \( \Gamma = 0 \). The increment in score vector when passing from one history at time-step \( t-1 \) to the next history at time-step \( t \) is given by

\[
\delta s_{\mu_{t-1} \rightarrow \mu_t} = (2a^{\mu_{t-1}} - 1)(2(\mu_t \% 2) - 1)
\] (14)

It follows that

\[
s_{\mu \text{path}} = \sum_{i=0}^{\tau-1} \delta s_{\mu_{t-1-i} \rightarrow \mu_t} .
\] (15)

We can evaluate the exact number of decided and undecided traders for a given \( \mu \)-path, and we can also identity which \( \mu \)-paths are allowed in the game; hence we can find the transition matrix giving the probability that a particular \( \mu \)-path at an arbitrary timestep evolves to the next allowed \( \mu \)-path with final history \( \mu_t \).

This transition matrix is given as follows:

\[
T_{\mu \text{path}(t-1), \mu \text{path}(t)} = \sum_{x=0}^{N_U} \left[ N_U C_x \left( \frac{1}{2} \right)^{N_U} \delta \left[ \text{Sgn} (A_D + 2x - N_U) \right] + (2(\mu_t \% 2) - 1) \right]
\] (16)

where \( A_D = A_D(\mu \text{path}(t-1)) = A_D(s_{\mu \text{path}(t-1)}, \mu_{t-1}) \) and \( N_U = N_U(\mu \text{path}(t-1)) = N_U(s_{\mu \text{path}(t-1)}, \mu_{t-1}) \). The size of \( T \) is given by \( \phi = 2^{(m+\tau)} \). For the basic MG \( \phi \) would grow indefinitely with time \( t \), however for the THMG it is of fixed size. Having obtained this transition matrix for the THMG then allows us to calculate various macroscopic quantities, in particular \( P(\mu \text{path}) \). The vector \( P(\mu \text{path}) \) satisfies \( P(\mu \text{path}) = P(\mu \text{path}) T \) which is the transpose of an eigenvector-eigenvalue equation. The vector \( P(\mu \text{path}) \) is also a stationary distribution of the Markov Chain:

\[
P(\mu \text{path}) = [\phi^{-1} \mathbf{1}^T T^\infty]_{\mu \text{path}} .
\] (17)

The expression given for \( P(\mu \text{path}) \) represents an average over all games with different \( \mu \)-paths defining the initial conditions of the game. This is achieved by taking an average over all \( \phi \) rows of the matrix \( T^\infty \). This method assumes that, for all possible states in the set \( \{ \mu \text{path} \} \), if a state is visited then the state can be re-visited during a run of the game. We note that if there are closed irreducible subsets in the set of \( \mu \)-paths \( \{ \mu \text{path} \} \) which are visited during the time evolution of the THMG, the game can then lock into a deterministic state. This situation
of deterministic states arises for $\tau = 2.2^m \lambda - 1$ as discussed earlier. In this case, the present method for calculating $P(\mu_{\text{path}})$ could be improved to account more fully for the deterministic dynamics since in general the system is not ergodic. We do not consider such a calculation here, but note that the values of $\sigma$ obtained using the present method may still be very accurate (see later Fig. 6 for $\tau = 7$ for example).

Figure 5a shows the measured $P(\mu_{\text{path}})$ for $N = 101$, $m = 2$, $s = 2$, $\Omega$ as in Eq. (11) and $\tau = 2$. Figure 5b shows the corresponding calculated $P(\mu_{\text{path}})$ using Eq. (13). The agreement is excellent, demonstrating that our expression for $P(\mu_{\text{path}})$ is exact.

4 Results

We now show how to calculate the volatility of the THMG exactly using Eq. (9). Without loss of generality, we replace both $P(s)$ and $P(\mu|s)$ by the probability $P(\mu_{\text{path}})$ that the game has just passed through one particular $\mu_{\text{path}}$ at a timestep $t$. This reduces the problem of calculating the volatility $\sigma$ in the THMG to that of studying a Markov Chain whose states are given by the set $\{\mu_{\text{path}}\}$. Note that whilst the THMG is homogeneous, the basic MG is not since the size of the set $\{\mu_{\text{path}}\}$ grows as $t$ increases. Replacing both $P(s)$ and $P(\mu|s)$ by the probability $P(\mu_{\text{path}})$ in Eq. (9), we have an equivalent exact expression for $\sigma$ given by:

$$\sigma^2 = \sum_{\{\mu_{\text{path}}\}} \sum_{x=0}^{N_U} \left[ N_U C_x \left( \frac{1}{2} \right)^N_U (A_U + 2x - N_U - \bar{A}) \right]^2 P(\mu_{\text{path}}) .$$

In order to calculate $A_U$ and $N_U$ exactly for a given $\mu_{\text{path}}$, the score vector to be used is that which would be obtained if the game went through a path of histories corresponding to $\mu_{\text{path}}$ as in Eq. (15); the history to be used is the last history in the path $\mu_{\text{path}}$. Denoting this last history as $\mu$, we have $A_U = A_U(\mu_{\text{path}}) = A_U(s_{\text{path}}, \mu)$ and $N_U = N_U(\mu_{\text{path}}) = N_U(s_{\text{path}}, \mu)$. We note that the exact analytic expression given for $\sigma$ in Eq. (18) applies to the limiting case where the number of realizations used to numerically determine $\sigma$ tends to infinity (i.e. ensemble average). We also note that using the formalism described, we could alternatively have obtained a finite time-average volatility $\sigma$ between $t = t_1$ and $t = t_2$ given a specific $\mu_{\text{path}}$ at $t = t_1$.

Using Eq. (17) in Eq. (18), together with Eqs. (2), (3) and (10), we obtain analytic values for $\sigma$ in the THMG given the initial quenched disorder $\Omega$. Figure 6 shows a comparison between the analytic values for the volatility $\sigma$ and the numerical values taken from the game as a function of the time-horizon $\tau$. Here $N = 101$, $m = 2$, $s = 2$, and $\Omega$ is given in Eq. (11). We only show $\tau = 1 \rightarrow 2.2^m + 1$, i.e. $\lambda = 1$; however similar results can also be obtained for $\lambda > 1$. The agreement is excellent, with the numerical and analytic lines essentially coincident. This demonstrates the power and accuracy of the present Markov Chain formalism as developed for the THMG.
5 Conclusion

In summary, we have introduced and studied a finite time horizon version of the Minority Game (THMG). We have presented exact analytic expressions for the volatility in both the THMG and the basic MG, for a given configuration of initial quenched disorder $\Omega$. We have presented an analytic theory to describe the dynamics of the THMG by obtaining an analytic expression for the transition matrix in terms of the set $\{\mu_{\text{path}}\}$. As an example of what can be achieved analytically, we obtained excellent agreement between analytic and numerical values for the THMG volatility, given knowledge of the initial quenched disorder $\Omega$. Finally we would like to stress that our theoretical approach and results avoid having to keep track of the labels of individual agents - our results are obtained for a specific initial quenched disorder $\Omega$, however this $\Omega$ describes many possible arrangements of individual agents. In this sense, $\Omega$ defines a macrostate for which there are many possible microstates corresponding to different initial strategy choices by the $N$ individual agents.

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References

[1] See http://www.unifr.ch/econophysics for a detailed account of previous work on agent-based games such as the Minority Game.

[2] D. Challet and Y.C. Zhang, Physica A 246, 407 (1997); ibid. 256, 514 (1998); ibid. 269, 30 (1999); D. Challet and M. Marsili, Phys. Rev. E 60, R6271 (1999); D. Challet, M. Marsili, and R. Zecchina, Phys. Rev. Lett. 84, 1824 (2000); M Marsili, D. Challet and R. Zecchina cond-mat/9908481; M Marsili and D. Challet cond-mat/0102257.

[3] R. Savit, R. Manuca and R. Riolo, Phys. Rev. Lett. 82, 2203 (1999). See also Physica A 276, 234 (2000) and 265 (2000).

[4] R. D’Hulst and G.J. Rodgers, Physica A 270, 514 (1999).

[5] M. Hart, P. Jefferies, N.F. Johnson and P.M. Hui, cond-mat/0003486; Phys. Rev. E 63, 017102 (2001); cond-mat/0005152; cond-mat/0008385 (to appear in Eur. J. Phys. B 2001); P. Jefferies, M. Hart, N.F. Johnson and P.M. Hui, J. Phys. A: Math. Gen. 33 L409 (2000).

[6] N.F. Johnson, P.M. Hui, D. Zheng and M. Hart, J. Phys. A: Math. Gen. 32 L427 (1999); N.F. Johnson, M. Hart and P.M. Hui, Physica A 269, 1 (1999).

[7] A. Cavagna, J.P. Garrahan, I. Giardina and D. Sherrington, Phys. Rev. Lett. 83, 4429 (1999); J.P. Garrahan, E. Moro and D. Sherrington, cond-mat/0012269; A. Cavagna, Phys. Rev. E 59, R3783 (1999).

[8] See D. Challet and M. Marsili, cond-mat/0004196, for discussions of the relevance of the actual memory in the MG, and diffusion around de Bruijn graphs.

[9] J.A.F. Heimel and A.C.C. Coolen, cond-mat/0012045.

[10] P. Jefferies, M.L. Hart and N.F. Johnson (in preparation).
FIG. 1. The probability of occurrence $P(s)$ of the strategy score vector obtained numerically from the Minority Game (MG). Strategy score vectors $s$ are listed in an arbitrary order along the x-axis. Unless otherwise stated, the game parameters in Figs. 1-6 are as follows: $N = 101, m = 2, s = 2$ with the initial quenched disorder matrix $\Omega$ taken from Eq. (11).

FIG. 2. Standard deviation (volatility) $\sigma$ as a function of time horizon $\tau$ for the Time Horizon Minority Game (THMG) with $m = 3$. Results are obtained using the full strategy space. Numerical data is collected using real histories (black circles) and random histories (grey circles) with randomly selected quenched disorder matrices $\Omega$. The lower short-dashed line shows the value of the volatility for the basic MG in the high $m$ limit. The upper long-dashed line shows the configuration-average volatility (i.e. average over quenched disorder $\Omega$) for the basic MG with $m = 3$.

FIG. 3. The finite time-average standard deviation (average taken over 100 turns) of the attendance of traders (black line) together with the number of traders choosing ‘1’ (grey circles) as a function of time $t$. (a) $\lambda = 100$, i.e. $\tau = 1599$. (b) $\tau = 1600$.

FIG. 4. Schematic diagram showing the allowed transitions between $\mu$-paths in the THMG for the following example: $\mu_{\text{path}}(t-1) = 2 \rightarrow 0 \rightarrow 1$ and $m = 2, \tau = 2$.

FIG. 5. (a) Numerical and (b) analytic results for the probability distribution $P(\mu_{\text{path}})$ as a function of $\mu_{\text{path}}$. $\mu$-path are listed in order of increasing decimal representation along the x-axis. Game parameters are $N = 101, m = 2, s = 2, \Omega$ as in Eq. (11), and $\tau = 2$.

FIG. 6. Standard deviation (volatility) $\sigma$ for the THMG as a function of time horizon $\tau$, for a given initial quenched disorder $\Omega$. Exact analytic values are grey diamonds joined by a thick grey dashed line, numerical values taken from the game simulation are black circles joined by a thick black line. These lines are essentially coincident thereby demonstrating the excellent agreement between the analytic theory and the numerical results. Here $N = 101, m = 2, s = 2, \Omega$ is from Eq. (11). Lower short-dashed line shows the value of the volatility for the basic MG in the high $m$ limit. Upper long-dashed line shows the configuration-average volatility (i.e. average over quenched disorder $\Omega$) for the basic MG with $m = 2$. 

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Fig. 1

Probability of occurrence

Strategy score vector:

- (0,0,0,0,0,0,0,0)
- (0,0,2,-2,-2,2,-2,0)
- (1,-1,1,-1,1,-1,1,-1)
- (1,1,-1,-1,1,1,-1,-1)
- (2,0,0,-2,2,0,0,-2)
- (3,-1,-1,-1,1,1,1,-3)
Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6