Large deviations of the ballistic Lévy walk model

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(Dated: July 7, 2020)

We study the ballistic Lévy walk and obtain the far-tail of the distribution for the walker’s position. When the position is of the order of the observation time, its distribution is described by the well-known Lamperti-arcsine law. However this law blows up at the far-tail which is nonphysical, in the sense that any finite time observation will never diverge. We claim that one can find two laws for the position of the particle, the first one is the mentioned Lamperti-arcsine law describing the central part of the distribution and the second is an infinite density illustrating the far tail of the position. We identify the relationship between the largest position and the longest waiting time describing the single big jump principle. From the renewal theory we find that the distribution of rare events of the position is related to the derivative of the average of the number of renewals at a small ‘time’ using a rate formalism.

I. INTRODUCTION

In recent years there is a growing interest in the characterisation of rare events in systems governed by the fat tailed distribution. It is therefore natural to investigate such a topic in the context of the widely applicable Lévy walk. The Lévy walk model [1–7] is a stochastic model of anomalous super-diffusion. It has many applications, ranging from blinking quantum dots [8], to foraging, dynamics of cold atoms, Lévy glasses, and dynamics generated by deterministic processes, like the Lorentz gas [9]. For the rare fluctuations of the Lévy walk model two techniques were promoted: the big jump principle, that characterises the process with a single event [10–12], and a moment generating function approach [13]. These however mainly focused on the case where the mean of the time between flights is finite, the variance diverges. Now we wish to extend these studies to the case when the mean flight time is infinite. This case presents new challenges. The spreading of packets of particles is ballistic [1, 14] and the moments of the process scale with time as $\langle x|^q \rangle \sim t^q$, a behavior called mono-scaling. In contrast when the mean is finite, we get a bi-fractal description of the moments, also known as strong anomalous diffusion [15, 16]. This implies that based on moments alone, and in the ballistic limit investigated here, we cannot get the desired information for the characterisation of the rare events, and the moment generating function approach in [13] is not useful (in some sense the moments are described by the typical fluctuations for the ballistic motion, while for enhanced sub-ballistic transport, the moments are sensitive to the far tail of the density of spreading particles). More importantly, in this field standard large deviation principle approach [17, 18] is not applicable, both for the ballistic case considered here and for cases considered previously [11], hence new tools must be developed for the description of the rare events.

In our analysis, we use the power law distribution of the times between flights capturing a heavy tail [19–22]

$$\phi(\xi) \sim \tau^{-\alpha-1}$$

with $0 < \alpha < 1$ for large $\tau$. Recall that in the velocity model of Lévy walk, the particle travels at a constant speed $v_0$ for time duration $\tau$ drawn from the probability density function (PDF) Eq. (1), then the process is renewed, with a velocity either $+v_0$ or $-v_0$ with equal probability (see details below). For $0 < \alpha < 1$, the average of the waiting time diverges, which leads to ballistic-diffusion [1], namely $\langle x^2(t) \rangle \sim (v_0)^2(1-\alpha)t^2$. In [23], the typical fluctuations were discussed in detail, i.e., the position $x$ is of the order of $t$. When $\alpha = 1/2$, the distribution of the position follows the arcsine law [23, 24]

$$P_\xi(\xi) \sim \frac{1}{\pi \sqrt{1-\xi^2}}$$

with $\xi = x/v_0 t$. Clearly, the arcsine law works very well for the central part of the distribution of the position; see the red solid line in Fig. 1 (a). While, when $|\xi| \to 1$ or $|x| \to \pm v_0 t$, the typical fluctuations Eq. (2) blow up at the far tail which is nonphysical at least for a finite time $t$. This drawback of the arcsine law, i.e., the nonphysical divergence at the far tail, is circumvented in this paper when a second type of scaling of the density is considered. See the data circled in red on the bottom panel of Fig. 1.

It implies that under certain conditions the density of the position is characterized by two scaling laws. The first one is the mentioned normalized arcsine law Eq. (2), the second corresponds to the non-normalized state, which is described by infinite densities [13, 25–30].

Another interesting problem is the relation between the far tail of the distribution of the longest time interval of a renewal process and the far tail of the distribution of the random walker. This problem is related to extreme value statistics [31–40]. Extreme events are natural phenomena and play an important role in our life. Thus, it is important to study how these rare events are related to...
II. MODEL

A. Renewal process and Lévy walk model

We first outline the main ingredients of the renewal process [28, 43–45] and Lévy walk model. The former is defined as follows: Events happen at the random epochs of time $t_1, t_2, \cdots, t_N, \cdots,$ from some time origin $t = 0$. Here we suppose time intervals $\tau_1 = t_1, \tau_2 = t_2 - t_1, \cdots, \tau_N = t_N - t_{N-1}, \cdots$, are IID random variables with a common PDF $\phi(\tau)$. Thus, the considered process is a renewal process. Given that the number of renewals during $(0, t)$ is $N$, i.e., $N = \max\{N, t_N \leq t\}$, the corresponding observation time $t$ is

$$t = \sum_{j=1}^{N} \tau_j + B_t. \quad (4)$$

Here $B_t$, defined by $t - t_N$, is the time interval between the time $t$ and the last event before $t$. When $t$ is fixed, our $N$ is a random variable.

We further consider the Lévy walk model in which the directions of each step are introduced. The particles move continuously with a constant velocity $\pm v_0$ for a random time $\tau_1$ drawn from a PDF $\phi(\tau)$. Here the directions of particles, i.e., $+ \text{ or } -$ are chosen randomly with equal probability. The corresponding displacement is $x_1 = -v_0 \tau_1$ ($x_1 = v_0 \tau_1$) on condition that the direction of the first step is negative (positive). We further generate another waiting time $\tau_2$ from $\phi(\tau)$ and the direction of the particle. Then the process is renewed. Here as mentioned $\tau_j$ are IID random variables with a common PDF $\phi(\tau)$. We are interested in the position of the particle at time $t$

$$x(t) = \sum_{j=1}^{N} \chi_j + v_{j+1}B_t, \quad (5)$$

where $\chi_j = \pm v_0 \tau_j$ ($j = 1, 2, \cdots, N$) are the displacement of $j$ step and $v_{j+1}B_t = \pm v_0 B_t$ is the last displacement. Below we will show how to derive the distribution of $x(t)$.

B. Three types of distributions of waiting times

Motivated by applications and also mathematical convenience, we consider three types of waiting time PDFs with the same heavy-tails. In Laplace space, from the Tauberian theorem [43] and Eq. (1) we have

$$\hat{\phi}(s) \sim 1 - b_\alpha s^\alpha, s \to 0 \quad (6)$$

with $0 < \alpha < 1$. Here $b_\alpha$ is a constant determined by the details of $\phi(\tau)$. In this paper we denote $\hat{\phi}(s)$ as the Laplace transform of $\phi(\tau)$ and $s$ is conjugate to $\tau$. We have $\hat{\phi}(0) = 1$, since $\phi(\tau)$ is a normalized density.
1. Pareto distribution

Our first example is called the Pareto distribution [46]. It is defined as follows

\[
\phi(\tau) = \begin{cases} 
0, & \tau \leq \tau_0; \\
\frac{\tau_0^\alpha}{\tau^{1+\alpha}}, & \tau > \tau_0.
\end{cases} \tag{7}
\]

When \(0 < \alpha < 1\), the first moment of \(\tau\) is divergent. Note that for Eq. (7), we have \(b_\alpha = \tau_0^\alpha |\Gamma(1-\alpha)|\) according to Tauberian theorem.

2. One-sided Lévy distribution

We further introduce the one-sided Lévy PDF \(\ell_\alpha(\tau)\). In Laplace space, \(\ell_\alpha(\tau)\) has a simple form

\[
\int_0^\infty \exp(-s\tau)\ell_\alpha(\tau)d\tau = \exp(-s^\alpha) \tag{8}
\]

and the small \(s\) expansion is \(\tilde{\phi}(s) \sim 1-s^\alpha\) with \(0 < \alpha < 1\). Let us first consider the special case of \(\alpha = 1/2\), i.e.,

\[
\ell_{1/2}(\tau) = \frac{1}{2\sqrt{\pi}}\tau^{-3/2} \exp\left(-\frac{1}{4\tau}\right), \quad \text{for} \quad \tau > 0. \tag{9}
\]

We see from Eq. (9) that \(\ell_{1/2}(\tau) \to 0\) for \(\tau \to 0\). However, for any small and finite \(\tau\), \(\ell_\alpha(\tau) \neq 0\), which is obviously different from Eq. (7).

3. Mittag-Leffler distribution

Another density of the waiting time is the Mittag-Leffler PDF [47, 48], i.e.,

\[
\phi(\tau) = \tau^{\alpha-1}E_{\alpha,\alpha}(-\tau^\alpha), \quad 0 < \alpha < 1 \tag{10}
\]

with \(E_{\alpha,\alpha}()\) being the Mittag-Leffler function defined by

\[
E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(an + \beta)}. \tag{11}
\]

In Laplace space, \(\tilde{\phi}(s)\) has the specific form

\[
\tilde{\phi}(s) = \frac{1}{1 + s^\alpha}. \tag{12}
\]

The Mittag-Leffler distribution is a geometric stable distribution [48]. When \(\tau \to 0\), we have \(\phi(\tau) \propto \tau^{\alpha-1} \to \infty\).

C. Propagator of Lévy walk

Let us briefly recap the basic equations of the model considered in this paper. For the velocity model under study, the particle moves continuously with a constant velocity and changes directions at random times [4, 49]. Mathematically, the joint probability of the step’s length \(\chi\) and duration time \(\tau\) is

\[
\phi(\chi, \tau) = \frac{1}{2}\delta(\tau)\delta(\chi - v_0\tau) + \delta(\chi + v_0\tau). \tag{13}
\]

The above equation describes the probability to move a distance \(\chi\) in time \(\tau\) with a single event and \(\delta(|\chi| - v_0\tau)\) accounts for the space-time correlation. The PDF of the particle’s position at time \(t\) is governed by [50]

\[
Q(x,t) = \delta(t)\delta(x) + \int_0^t \int_{-\infty}^{\infty} Q(y,t')\phi(x-y,t-t')dydt'. \tag{14}
\]

and the PDF of the particle’s position reads

\[
P(x,t) = \int_{-\infty}^{\infty} \int_0^t Q(y,t')\Phi(x-y,t-t')dt'dy, \tag{15}\]

where

\[
\Phi(x,t) = \frac{1}{2}\delta(x - v_0t) + \delta(x + v_0t) \int_0^\infty \phi(\tau)d\tau
\]

is the probability of moving a distance \(x\) in time \(t\) in a single motion during the last uncompleted step, and \(Q(x,t)\) is probability of just arriving at \(x\) at time \(t\) after completing a step. In Eqs. (14, 15) we identify the convolution both in time and in space, hence the analysis proceeds with Laplace-Fourier transforms. Combining Eqs. (14) and (15) yields [4, 49]

\[
\tilde{P}(k,s) = \frac{\tilde{\Phi}(s + ikv_0) + \tilde{\Phi}(s - ikv_0)}{2 - (\tilde{\phi}(s + ikv_0) + \tilde{\phi}(s - ikv_0))}, \tag{16}
\]

where \(\tilde{P}(k,s)\) is the Fourier \(x \to k\) and Laplace \(t \to s\) transforms of \(P(x,t)\). Such equations are known as Montroll-Weiss equations, they are not generally easy to invert, and hence later we turn to the asymptotic analysis.

The exact details of waiting time PDFs, namely, Eqs. (7, 8) and (10), are not vitally important for the typical fluctuations on condition that they have the same heavy-tails governed by the index \(\alpha\), for example, see Eq. (2). In contrast, here our aim is to find the statistics of rare fluctuations \(|x| \approx v_0t\), and then the detailed structure of the waiting time PDF is of importance.

III. RESULTS FOR LÉVY WALK

A. Bulk fluctuations

First, we focus on the typical fluctuations, namely the case \(|x| \sim v_0t\) and both are large, implying that \(s\) and \(k\) are small and comparable. Inserting Eq. (6) into Eq. (16) and taking the inverse Fourier-Laplace transform yield
the description of what we call bulk or typical fluctuations [5, 14, 23]

\[ P_\xi(\xi) \sim \frac{\sin(\pi \alpha)}{\pi} \times \frac{|1 - \xi^\alpha|}{|1 - \xi|^{\alpha + 1} + |1 + \xi|^{\alpha - 1}} \]

with the scaling form \( \xi = x/(v_0 t) \). Here as usual, the subscript \( \xi \) means that \( P(\cdot) \) is the corresponding PDF of \( \xi \). The propagator Eq. (17) is called the Lamperti distribution [24]. Here \(-1 \leq \xi \leq 1\) since \(-v_0 t \leq x \leq v_0 t\), namely there exists a finite “light cone” in which we may find the particle. The second moment of the position is \( \langle x^2(t) \rangle \sim (1 - \alpha)(v_0 t)^2 \), which corresponds to a ballistic behavior [1, 23]. When \( \alpha = 1/2 \), Eq. (17) reduces to Eq. (2) which is plotted in Fig. 1, and as expected describes well the central part of the packet of spreading particles. A related expression is

\[ P_\epsilon(\epsilon, t) \sim \frac{1}{\pi \sqrt{\epsilon(2v_0 t)}} \]

with \( \epsilon = v_0 t - x \) which is plotted by the dashed line in Fig. 2. There when \( \epsilon \) is small, we identify the deviations from the arcsine law as discussed in the introduction.

B. Rare fluctuations

We consider the case of \( x \rightarrow v_0 t \) using the random variable \( \epsilon = v_0 t - x \) where \( \epsilon \) is small. In Fourier–Laplace spaces, the density of \( \epsilon \) becomes

\[ \tilde{P}_\epsilon(k, s) = \tilde{P}(-k, s - ik_v) \]

Here \( k_v \) is the Fourier pair of the shifted position \( \epsilon \). Utilizing Eqs. (16) and (19), we get

\[ \tilde{P}_\epsilon(k, s) = \frac{\tilde{\Phi}(s - 2ik_v) + \tilde{\Phi}(s)}{2 - |\tilde{\Phi}(s - 2ik_v) + \tilde{\Phi}(s)|} \]

We are interested in analyzing the behavior of the position in the long time regime \( (s \rightarrow 0) \), where \( \epsilon \) and \( t \) are sufficient small and large, respectively. Using Eq. (6), Eq. (20) reduces to a simple expression

\[ \tilde{P}_\epsilon(k, s) \sim \frac{\tilde{\Phi}(-2ik_v) + b a s^{\alpha - 1}}{1 - \tilde{\Phi}(-2ik_v)} \]

Note that the inverse Laplace transform of \( \tilde{\Phi}(-2ik_v)/(1 - \tilde{\Phi}(-2ik_v)) \) gives a delta function \( \delta(t) \) which is ignored and not related to our long time behavior. Taking the inverse Laplace-Fourier transform of Eq. (21) gives the main result of this section

\[ \epsilon^{\alpha - 1} \Gamma(1 - \alpha) b a P_\epsilon(\epsilon, t) \sim \mathcal{I}(\epsilon) \]

with

\[ \mathcal{I}(\epsilon) = \mathcal{F}_\epsilon^{-1} \left[ \frac{1}{1 - \hat{\phi}(-2ik_v)} \right] \]  

(23)

Here we used the fact that \( s^{\alpha - 1} \) and \( t^{-\alpha}/\Gamma(1 - \alpha) \) are Laplace pairs. Eq. (22) with \( \phi(\tau) \) being the one-sided Lévy distribution is plotted in Fig. 2 by using the numerical inverse Fourier transform. The comparison to numerical simulation is excellent, while the arcsine law completely fails to describe the observed behavior. It can be seen that \( \mathcal{I}(\epsilon) \) given in Eq. (23) is an infinite density since \( \tilde{P}_\epsilon(k, s = 0) \neq 1/s \), namely the \( \mathcal{I}(\epsilon) \) is not normalised, which is hardly surprising since it is obtained from a normalised density multiplied by \( t^\alpha \) hence the area under the left hand side of Eq. (22) is obviously diverging.

Let us consider three examples:

i) For the Pareto distribution, we can not invert Eq. (22) exactly, however we may invert it numerically. While, there is a simply way by considering the limit \( \epsilon \rightarrow 0 \). Using \( 1/(1 - \tilde{\Phi}(-2ik_v)) \sim 1 + \tilde{\Phi}(-2ik_v) + \tilde{\Phi}(-2ik_v)^2 \) and taking the inverse Fourier transform gives

\[ \mathcal{I}(\epsilon) \sim \delta(\epsilon) + \frac{1}{2v_0} \phi \left( \frac{\epsilon}{2v_0} \right) + \int_0^{\epsilon/2v_0} \phi(y) \phi \left( \frac{\epsilon}{2v_0} - y \right) dy \]

(24)

If we are only interested in the behavior of \( x \rightarrow v_0 t \), the second term works perfectly and Eq. (24) reduces to

\[ \mathcal{I}(\epsilon) \sim \frac{1}{2v_0} \phi \left( \frac{\epsilon}{2v_0} \right) \]

(25)

with \( \epsilon \neq 0 \). Note that Eq. (25) in the limit \( \epsilon \rightarrow 0 \) is valid for a large range of PDFs, for example the mentioned Mittag-Leffler and the one-sided Lévy distributions.

ii) For the one-sided Lévy distribution, we use the geometric series \( 1/(1 - \tilde{\Phi}(-2ik_v)) = \sum_{n=0}^\infty \tilde{\Phi}^n(-2ik_v) \) and get by inversion \( k_v \rightarrow \epsilon \)

\[ \mathcal{I}(\epsilon) = \delta(\epsilon) + \sum_{n=0}^\infty \frac{1}{(2v_0)^{1/\alpha}} L_\alpha \left( \frac{\epsilon}{(2v_0)^{1/\alpha}} \right) \]

(26)

where \( L_\alpha(x) \) is the Lévy PDF [20], defined by

\[ L_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) \exp(|ik|^\alpha) dk. \]

(27)

When \( \epsilon = 0 \) or \( x = v_0 t \), only the function \( \delta(\epsilon) \) is of importance. As expected, Eq. (22) reduces to the survival probability, describing the probability of moving in the same direction for the whole observation time \( t \). On the contrary, the function \( \delta(\epsilon) \) loses its role for \( \epsilon \neq 0 \) or \( x \neq v_0 t \).

iii) For the Mittag-Leffler distribution, using Eq. (12), it is easy to show

\[ \mathcal{I}(\epsilon) = \left( \delta(\epsilon) + \frac{\epsilon^{\alpha - 1}}{(2v_0)^{\alpha} \Gamma(\alpha)} \right) \]

(28)
Utilizing Eqs. (28) and (22), as $t \to \infty$, we have

$$P_\xi(\xi, t) \sim \frac{b_{\alpha}}{t^{2\alpha} \Gamma(1-\alpha)} \delta(1-\xi) + \frac{\sin(\pi\alpha)}{2^\alpha \pi} (1-\xi)^{\alpha-1}$$

(29)

according to the relationship $t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha) \sim -1/(\Gamma(-\alpha)t^{\alpha+1})$. It indicates that Eq. (29) agrees with the far tail of the Lamperti distribution Eq. (17) in the limit of $\xi \to 1$. Namely, Eq. (29) exhibits a unique behavior that the rare events are described by the same theory as the typical fluctuations; see Fig. 2. Besides, an interesting feature of $P_\xi(\xi, t)$ is exclusively exhibited by the rare events analysis, i.e., a discrete probability describing the survival probability of the particles $\Phi(t) \sim t^{-\alpha}$ is found.

In Fig. 2, we show the propagator corresponding to Eq. (22) in the scaling form. For $\alpha = 1/2$, the calculated $P_\xi(\xi)$, Eq. (17), follows reasonably the arcsine law and this is only valid for the central part of the distribution of the position, namely $x \ll \nu_0 t$ but $x \neq \nu_0 t$. As the figure shows, it is difficult to find the difference between the typical fluctuations and the theoretical result with the Mittag-Leffler waiting time statistics, while, if $x \to \nu_0 t$ and the waiting time follows the Pareto or the one-sided Lévy distributions, deviations from Eq. (18) are clearly presented. When $x \to -\nu_0 t$, the rare events of $x$ can be obtained by using the symmetry property of the density and here we did not discuss this in detail. In Fig. 3, the scaling form $I(\varepsilon)$ is exhibited with different observation time $t$ to show the properties of the infinite density. Clearly, for small $\varepsilon$, $I(\varepsilon)$ is independent of the observation time $t$ and its shape does not change.

FIG. 2: The behavior of $P_\xi(\xi, t)$ for small $\varepsilon$ with $\varepsilon = \nu_0 t - x$. The full line [Eq. (22)], the dash-dotted one [Eq. (25)], and the dotted one [Eq. (28)] describing the rare events are the theoretical predictions with different waiting time distributions showing different behaviors of rare fluctuations. The dashed line Eq. (17) is the Lamperti distribution, which illustrates the PDF when both $x$ and $t$ are of the same order and comparable. The symbols are the simulation results obtained by averaging $10^7$ trajectories of the particles with $\alpha = 1/2$. Here 'ML' denotes the Mittag-Leffler distribution Eq. (10).

FIG. 3: The PDF of $\varepsilon = \nu_0 t - x$ multiplied by $t^{\alpha} \Gamma(1-\alpha)$ versus $\varepsilon$ for a model where the travel time PDF $\phi(\tau)$ is the one-sided Lévy distribution. The solid line is the analytical solutions $I(\varepsilon)$ [Eq. (23)] obtained by the numerical inverse Fourier transform and the symbols are simulations with different $t$, namely, $t = 10^3, 10^4$, and $t = 10$. Other parameters are the same as in Fig. 2.

IV. RELATION BETWEEN THE POSITION AND THE LONGEST WAITING TIME

The aim of this section is to study the rare events of the position from a new point of view by establishing a direct contact between the position and the longest waiting time in the renewal process. As mentioned in the introduction such relations come under the title: big jump principle [11, 12]. For the well known IID case, $N$ in Eq. (3) is fixed. In our model, $N$ is a random variable since $\tau_1 + \tau_2 + \cdots + \tau_N + B_1 = t$; see Fig. 4. As mentioned in the introduction this constraint also implies that we have correlations in the process, though they stem from a renewal process, and hence still can be analysed. Here we must distinguish between two types of rare events. We will focus on $x$ being large but strictly $x < \nu_0 t$. Then in Sec. VI we will treat $x = \nu_0 t$. The latter gives a delta function making a contribution to $P(x, t)$.

Here we first define

$$\tau_{\text{max}} = \max\{\tau_1, \tau_2, \cdots, \tau_N, B_1\}.$$

The limiting law of typical fluctuations of $\tau_{\text{max}}$ has been studied by C. Godrèche et al. in Ref. [45]. Here we recently showed that another law will be found when the second scaling is introduced [40]. If $\tau_{\text{max}} \to t$, the density of $\tau_{\text{max}}$ can be deduced from the following inverse Fourier transform

$$f_\eta(\eta, t) \sim \frac{b_{\alpha}}{(t-\eta)^{2\alpha} \Gamma(1-\alpha)} \frac{1}{1 - \phi(-ik_\eta)}$$

(30)

with $\eta = t - \tau_{\text{max}}$; see Eq. (A4) in Appendix A. We focus on the case where $\tau_{\text{max}}$ is large and $t - \tau_{\text{max}} \to 0$. The limit $k_\eta \to 0$ corresponds to the large 'time' $\eta$. Note
that here we must consider the full form of the density, namely \( \hat{\phi}(-i\kappa) \) is important, while for the typical fluctuations only the small \( k_b \) behavior of Eq. 6 is important. Rewriting Eq. (22), for the variable \( \epsilon = v_0 t - x \) we have

\[
P_\epsilon(\epsilon, t) \sim \frac{b_\alpha}{t^{\alpha} \Gamma(1 - \alpha)} F_\epsilon^{-1} \left[ \frac{1}{1 - \hat{\phi}(-2i\kappa,v_0)} \right]
\]

(31)

Combining Eqs. (30) and (31), we get the main result of this section

\[
t - \tau_{\text{max}} \approx \frac{d}{2v_0}(v_0 t - x), \tag{32}
\]

where \( \tau_{\text{max}} \) is large and \( d \) means that the distributions of the random variables on both sides of Eq. (32), i.e., \( v_0 t - x \) and \( 2v_0(t - \tau_{\text{max}}) \), are the same. As shown in Fig. 5, the density of \( t - \tau_{\text{max}} \) is consistent with that of \( (t - x)/(2v_0) \) for large \( x \) and \( \tau_{\text{max}} \). Rewriting Eq. (32) yields

\[
x = v_0 \tau_{\text{max}} - v_0(t - \tau_{\text{max}}). \tag{33}
\]

This behavior can be tested based on a correlation plot. As can be seen in Fig. 6, the strong relation between \( t - \tau_{\text{max}} \) and \( (v_0 t - x)/(2v_0) \) is illustrated [53]. As expected, we find that \( t - \tau_{\text{max}} \) grows linearly with \( (v_0 t - x)/(2v_0) \) for a small \( t - \tau_{\text{max}} \).

This relation stressed here is different from the case of \( \alpha > 1 \) discussed in [11] in which the relation is \( x \approx v_0 t \). The reason is as follows: For \( \alpha > 1 \), the length of the displacement made in \((0,t - \tau_{\text{max}})\), which is the time interval free of the longest waiting time, follows \( x(t) \propto (t - \tau_{\text{max}})^{1/\alpha} \ll t \). While, for \( \alpha < 1 \), the situation is changed since \( t - \tau_{\text{max}} \propto t \) and the term \( v_0(t - \tau_{\text{max}}) \) comes into play. See further details in Appendix B.

We further treat Eq. (32) heuristically to explain its meaning. Now the total observation time \( t \) is divided into two parts: One is the sum of waiting times denoted by \( t^{\text{positive}} \) when directions of particles are positive and the other one is \( t - t^{\text{positive}} \). We assume that \( x \approx v_0 t \), the particles arrive there by a mechanism of large jump. This means that we have \( t^{\text{positive}} = \tau_{\text{max}} \) and \( \tau_{\text{max}} \approx t \). More specifically, we consider two random variables \( \tau_{\text{max}} \) and the remaining time \( t - \tau_{\text{max}} \). The corresponding position of the particle is \( x(t) \approx v_0 \tau_{\text{max}} \pm v_0(t - \tau_{\text{max}}) \). We further suppose that the particle moves with velocity \( +v_0 \) in \((0,\tau_{\text{max}})\) and \(-v_0 \) in the remaining time, then the position of the particle at time \( t \) is

\[
x \approx v_0 \tau_{\text{max}} - v_0(t - \tau_{\text{max}}). \tag{34}
\]

If the particle does not change its direction in the mentioned two time intervals, we have \( x(t) \approx v_0 t \) (a delta function).

To conclude we see that rare events are obtained by a particle moving only in one direction from Eq. (34) (for \( \tau_{\text{max}} \)) and reversing direction in the remaining time, we do not see this as an intuitive result, but when \( \tau_{\text{max}} \) is really big the particle is left with little time to reverse, hence the ballistic motion with the reverse direction is plausible (and if it continues in the same direction, we are on the horizon of the walk, which is not considered here). Note that in principle the number of renewals related to Eq. (32) can be a large number and is not limited to one. This will be discussed rigorously in the following section.

![Graphs showing the relationship between rare events of the position and the average of renewals](image)

FIG. 4: Step size \( \chi_i \) for the velocity model when the PDF of travel times is the one-sided Lévy distribution and \( \alpha = 0.5 \). The observation time \( t \) is 1000 and \( i = 1, 2, \ldots \), correspond to the first, second, \ldots, waiting time of the Lévy walk model, respectively. The step length of each step denoted by \( \chi_i \) is \( \chi_i = \pm v_0 \tau_{\text{max}} \) with \( \tau_{\text{max}} \) being the sojourn time of the \( i \)-th time interval. Note that the directions of the particles are either + or − chosen randomly with equal probability. We see that one displacement is dominating the land-scope, this is the biggest jump. In this section we mainly consider the relation between the largest position and the longest waiting time.

V. RELATION BETWEEN RARE EVENTS OF THE POSITION AND THE AVERAGE OF RENEWALS (\( N \))

Now the aim is to investigate the relation between \( x \) and the number of renewals. Note that the rare events of the position are governed by \( \hat{\phi}^n(-i\kappa) \) with \( n \) being a positive integer according to Eqs. (26) and (24). It indicates that the rare events of the position have a strong relationship with the number of renewals. Based on the renewal theory [44], in Laplace space \( (t \rightarrow s) \), the probability of the number of renewals during time interval 0 and \( t \) is [44]

\[
\hat{P}_N(s) = \hat{\phi}^N(s) \frac{1 - \hat{\phi}(s)}{s}. \tag{35}
\]

We can check that \( P_N(t) \) is normalized by using \( \sum_{N=0}^\infty P_N(s) = 1/s \). According to Eq. (35), the mean
Rewriting Eq. (24), we get

\[ s(\widehat{N}(s)) = \frac{\widehat{\phi}(s)}{1 - \phi(s)} = \frac{1}{1 - \phi(s)} - 1 \]  

Taking the inverse Laplace transform yields

\[ \frac{d\langle N(t) \rangle}{dt} = \mathcal{L}^{-1}_{\epsilon} \left[ \frac{1}{1 - \phi(s)} \right] \]  

with \( t > 0 \). Note that Eq. (38) is the exact result for \( t > 0 \) and \( \mathcal{L}^{-1}_t[1/(1 - \phi(s))] \) corresponds to the rate of the number of renewals [51]. Combining the second line of the right hand side of Eqs. (31) and (38), the relation between the behavior of \( P_\epsilon(\epsilon, t) \) and \( \langle N(t) \rangle \) is found

\[ P_\epsilon(\epsilon, t) \sim \frac{b_\epsilon\Gamma(1 - \alpha)^{-1}}{2v_0t^\alpha} \left( \frac{d\langle N(z) \rangle}{dz} \right)_{z = \frac{\epsilon}{2v_0}}, \]  

which is illustrated in Fig. 7. Here the right-hand side of Eq. (39) is obtained by averaging \( 10^7 \) realizations. It can be seen that \( P_\epsilon(\epsilon, t) \) is connected to the average of renewals at a small ‘time’ \( z = \epsilon/(2v_0) \). The simulations for the mean number of renewals were made only up to time \( t = 10 \), still they predict the rare events with simulations made for time \( t = 1000 \). Thus, we just observe a very short time to get the data of the number of renewals and then we can map this to describe the rare fluctuations of positions. This is particularly important for real experiments saving a lot of time and expense.

VI. PROPAGATOR FOR THE PARETO TRAVELLING TIME PDF

Now we deal with a general observation time \( t \) instead of the long time limit considered in previous sections, then use a different method to explain the rare fluctuations again. Here we focus on the case of the waiting time following the Pareto distribution Eq. (7). Recall that \( \tau_0 \) is the cutoff for this distribution. If \( x \in (v_0t - \tau_0, v_0t) \), from Eq. (25) we get \( P(x, t) = 0 \). Indeed in Fig. 2 we see this effect rather easily, however this is an approximation valid in the long time limit only, for finite times this rule is not strictly valid as the probability of finding the particle in this interval is not identically zero. Intuitively, the large position is related to the large waiting time and
the large waiting time is determined by the far tail of the waiting time. So it would be interesting to consider the relationship between the far tail of the waiting time and the large position.

Using Taylor’s expansion on Eq. (16), we get
\[
\tilde{P}(k, s) = \left(\tilde{\Phi}(s + ikv_0) + \tilde{\Phi}(s - ikv_0)\right) / 2 \\
\times \sum_{n=0}^{\infty} \left(\frac{\tilde{\Phi}(s + ikv_0) + \tilde{\Phi}(s - ikv_0)}{2}\right)^n.
\] (40)

Here the summation over \(n\) is a sum over the number of renewals, which as mentioned is random. We focus on the case of \(x \in (v_0 t - 2v_0 \tau_0, v_0 t)\). If \(n = 0\), then clearly the particle is not in the interval under study. In order to obtain \(P(x, t)\) in the mentioned spatial interval, we need to consider the propagator Eq. (40). Utilizing the definition of \(\phi(x, t)\) and \(\Phi(x, t)\), for \(x \in (v_0 t - 2v_0 \tau_0, v_0 t)\) the above equation reduces to
\[
\tilde{P}(k, s) = \frac{1}{2} \tilde{\Phi}(s + ikv_0) \sum_{n=1}^{\infty} \left(\frac{\tilde{\Phi}(s - ikv_0)}{2}\right)^n.
\] (41)

The infinite terms on the right-hand side of Eq. (41) describe the probability of moving in positive direction all the time but the direction of the last step is negative. This is the only way the particles can reach \((v_0 t - 2v_0 \tau_0, v_0 t)\) at time \(t\). Taking the inverse Laplace-Fourier transform, we obtain
\[
P(x, t) = \frac{1}{4v_0} \int_{-\infty}^{\infty} \phi(y) dy \frac{L_{\frac{1}{1 + s/v_0}}^{-1}}{L_{\frac{1}{1 + s/v_0}}} \left[\frac{\tilde{\Phi}(s)}{2 - \tilde{\Phi}(s)}\right],
\] (42)

which reduces to
\[
P(x, t) = \frac{1}{4v_0} \left(L_{\frac{1}{1 + s/v_0}}^{-1} \left[\frac{\tilde{\Phi}(s)}{2 - \tilde{\Phi}(s)}\right]\right),
\] (43)

with \(x \in (v_0 t - 2v_0 \tau_0, v_0 t)\). When \(t\) is large, i.e., \(\tilde{\Phi}(s) \sim 1 - b_\alpha s^\alpha\), we have
\[
P(x, t) = \frac{1}{4v_0} \left(L_{\frac{1}{1 + s/v_0}}^{-1} \left[\frac{1 - b_\alpha s^\alpha}{1 + b_\alpha s^\alpha}\right]\right)
\] \(\propto -L_{\frac{1}{1 + s/v_0}}^{-1} [s^\alpha].\) (44)

Thus,
\[
P(x, t) \propto \left(\frac{t}{2} + \frac{x}{2v_0}\right)^{-\alpha-1}.
\] (45)

Clearly, the far tail of the position decays as a power law. Contrary to Eq. (25) with \(t \to \infty\), the behavior of \(x \in (v_0 t - 2\tau_0 v_0, v_0 t)\) is governed by the far tail of the PDF for a finite time; see Fig. 8. With the increasing of observation time \(t\), Eq. (43) goes to zero and approaches Eq. (25) since the probability of reaching \((v_0 t - 2\tau_0 v_0, v_0 t)\) becomes smaller and smaller.

We proceed with the discussion of \(x = v_0 t\). Recall that for Eq. (32), it is meaningless for \(\tau_{\text{max}} = t\). When all directions of the particles are the same, this contributes to the probability of \(x = v_0 t\). For this maximum point, we have
\[
\tilde{P}(k, s) = \frac{1}{2} \tilde{\Phi}(s - ikv_0) \sum_{n=0}^{\infty} \left(\frac{\tilde{\Phi}(s - ikv_0)}{2}\right)^n
\] (46)

with \(x = v_0 t\). The inversion of Eq. (46) is
\[
\tilde{P}(x, t) = \frac{1}{2} \delta(x - v_0 t) L_{\frac{1}{1 + s/v_0}}^{-1} \left[\frac{\tilde{\Phi}(s)}{1 - \tilde{\Phi}(s)/2}\right].
\] (47)

In the long time limit, using \(1 - \tilde{\Phi}(s)/2 \sim 1/2\), Eq. (47) reduces to
\[
\tilde{P}(x, t) \sim \delta(x - v_0 t) \Phi(t).
\] (48)

It can be seen that Eq. (48) depends on the initial position of the particle and is related to the survival probability. In reality, Eq. (48) is related to the single big jump principle, namely, the particles go in the one direction for the first step and continue in the same way for the rest steps.

One may wonder how can we understand the mechanism of rare fluctuations discussed in previous section. Motivated by the finite time limit Eq. (45), we consider the case that the particle just has only one negative velocity (the rest epochs are positive) to study the behavior.
of \(x \to v_0 t\). Similar to Eq. (41), we obtain from Eq. (40)
\[
\tilde{P}(k, s) \sim \frac{\tilde{\phi}(s + ikv_0)}{2} \frac{\tilde{\phi}(s - ikv_0)}{2} \left( \sum_{n=0}^{\infty} \left( \frac{\tilde{\phi}(s - ikv_0)}{2} \right)^n \right)^2
+ \frac{\tilde{\phi}(s + ikv_0)}{2} \left( \sum_{n=1}^{\infty} \left( \frac{\tilde{\phi}(s - ikv_0)}{2} \right)^n \right)^2.
\]
(49)

Here the second line of the right hand side of Eq. (49) corresponds to case where the negative velocity is only in the last step and the first line is when sole negative direction is not the last one. The sum of the infinite geometric series yields
\[
\tilde{P}(k, s) \sim \frac{\tilde{\phi}(s + ikv_0)}{2} \frac{\tilde{\phi}(s - ikv_0)}{2} \frac{1}{1 - \frac{\tilde{\phi}(s - ikv_0)}{2}}
+ \frac{\tilde{\phi}(s + ikv_0)}{2} \frac{1}{1 - \frac{\tilde{\phi}(s - ikv_0)}{2}}.
\]
(50)

Using relation Eq. (19) again, we have
\[
\tilde{P}_c(k, s) \sim \frac{\tilde{\phi}(s - 2ik\epsilon v_0)}{2} \frac{\tilde{\phi}(s)}{2} \frac{1}{1 - \frac{\tilde{\phi}(s)}{2}}
+ \frac{\tilde{\phi}(s - 2ik\epsilon v_0)}{2} \frac{1}{1 - \frac{\tilde{\phi}(s)}{2}}.
\]
(51)

The inverse Laplace transform of the above equation gives
\[
P_c(\epsilon, s) \sim \frac{\tilde{\phi}(s - v_0)}{2v_0} \frac{b_0}{\Gamma(1 - \alpha) t^\alpha} + \frac{1}{4v_0} \phi \left( t - \frac{\epsilon}{2v_0} \right).
\]
(52)

Based on Eq. (52), the leading term of Eq. (26) is obtained again in the limit \(t \to \infty\). It can be seen that Eq. (51) or (52) is an exact solution with Eq. (7) when \(x \in (v_0 t - 3\gamma v_0, v_0 t - 2\gamma v_0)\). Besides, it is easy to find that the rare events of the position are closely contact to the number of the negative directions. This is also corresponding to the expanded terms of Eq. (23) in powers of \(\phi(-ik\epsilon)\).

VII. CONCLUSION

The main focus of this manuscript has been on the rare fluctuations of the ballistic Lévy walk model in one dimension. We have shown that the rare events of the position have a strong relationship with the full shape of the distribution of waiting times, unlike the typical fluctuations which are described by the Lamperti-arcsine law. To highlight the rare fluctuations, we use a second non-ballistic scaling. Namely, we multiply the PDF with \(t^n\) and obtain the infinite density \(I(\epsilon)\). The integral of \(I(\epsilon)\) diverges at large \(\epsilon = v_0 t - x\). The infinite density and the normalized Lamperti-arcsine are complementary with the former describing the far tail of the position. Certainly, our infinite density is different from the case of \(1 < \alpha < 2\) [26]. We gave the relation between the infinite density of the waiting times and the infinite density of the position \(x\). This yields a different single big jump principle for the ballistic Lévy walk, where the mentioned principle is different from the one of the enhanced Lévy walk model [11].

We have carried out an investigation on the moving forward particles, leading to a delta function contribution at \(x = v_0 t\); see Eq. (48). This is clearly a description of a rare event, hence the theory is developed in two stages, \(x\) large and comparable to \(v_0 t\) but strictly smaller, and \(x = v_0 t\). Note that when \(x = v_0 t\), the particle did not change its direction, but the renewal process may have many collision events. If the velocity distribution is not \(+v_0\) and \(-v_0\) with equal probability we might obtain different behaviors than what we presented here, however this is hardly surprising as it is also true for the bulk [52].

We have developed a rate formalism to the rare events, see Eq. (39). Recently, Akimoto et al. considered a related rate formalism for a different observable: the velocity [51], while here we consider the position, we believe that the rate approach is a valuable tool for the calculations of large fluctuations, at least for renewal processes. See also the related work [11, 12]. In real experiments, the observation of the rare fluctuations is a challenge, since we need many samples and a long observation time \(t\). Here we investigate the rare fluctuations of the position from a new and different point of view. Utilizing the renewal theory, we find that the rare events of the Lévy walk is related to the mean number of renewals in the observation time \(0, (x - v_0 t)/(2v_0))\). This is to say, if we are interested in the rare events of the position of the velocity model, we just need the data of the renewals at some ‘time’ \((x - v_0 t)/(2v_0))\); see Eq. (39). Thus, our results give an effective way to measure the rare events of the position from extremely short time dynamics.

Acknowledgments

E. B. acknowledges the Israel Science Foundation for support through Grant No. 1898/17. M.H. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)–436344834. W.W. thanks Felix Thiel for the discussions. W.W. was supported by Bar-Ilan University together with the Planning and Budgeting Committee fellowship program.

Appendix A: Calculation of \(\tau_{\text{max}}\)

Let us give a brief account of the statistic of \(\tau_{\text{max}} = \max\{\tau_1, \tau_2, \cdots, \tau_N, B_1\}\). In Laplace space, the density of
\( \tau_{\text{max}} \) satisfies \([45]\)

\[
\int_0^\infty \tilde{f}_{\tau_{\text{max}}}(z, s)dz = \frac{1}{s} \frac{1}{1 + \tilde{g}(M, s)} \tag{A1}
\]

with

\[
\tilde{g}(M, s) = \frac{s \exp(s M) \int_0^M \Phi(z) \exp(-sz)dz}{\int_0^\infty \Phi(z)dz}. \tag{A2}
\]

From Eq. (A1), the density of \( \tau_{\text{max}} \) follows

\[
\tilde{f}_{\tau_{\text{max}}}(M, t) \sim \frac{\exp(-sM)}{1 - \Phi(s)} \left( \Phi(M) + \frac{\phi(M)}{s} \right). \tag{A3}
\]

In the limit \( t \to \infty \) and \( M \to t \), the leading term of Eq. (A3) is

\[
\tilde{f}_{\tau_{\text{max}}}(M, s) \sim \frac{\Phi(M) \exp(-sM)}{1 - \phi(s)}. \tag{A4}
\]

Notice that Eq. (A4) can be further simplified. Substituting Eq. (6) into Eq. (A4) and taking the inverse Laplace transform give

\[
f_{\tau_{\text{max}}}(M, t) \sim \frac{\sin(\pi \alpha)}{\pi} (t - M)^{\alpha - 1} t^{-\alpha}. \tag{A5}
\]

Rewriting Eq. (A5) yields the scaling form of \( f_{\tau_{\text{max}}}(M, t) \)

\[
f_{\tau_{\text{max}}/t}(x) \sim \frac{\sin(\pi \alpha)}{\pi} (1 - x)^{\alpha - 1}; \tag{A6}
\]

see also \([45]\). Let us now take the integral over \( x \) from 0 to 1 of Eq. (A6), we get

\[
\int_0^1 f_{\tau_{\text{max}}/t}(x)dx \sim \frac{\sin(\pi \alpha)}{\pi \alpha}. \tag{A7}
\]

It indicates that only \( \alpha \to 0 \), the density is normalized otherwise not. Note that Eq. (A5) is valid under condition that \( t \to \infty \) and \( t - \tau_{\text{max}} \) is large. While, when another scaling is introduced, i.e., \( \tau_{\text{max}} \to t \), the statistics of \( \tau_{\text{max}} \) will be changed. Using the definition of inverse Laplace transform on Eq. (A4) gives

\[
f_{\tau_{\text{max}}}(M, t) \sim \frac{\Phi(M)}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \exp(s(t - M))/\left(1 - \tilde{\phi}(s)\right)ds. \tag{A8}
\]

When \( s \to 0 \), the random variable \( t - M \) tends to infinity and in this limit Eq. (A8) corresponds to the statistics of large \( t - M \). While, for \( t - M \to 0 \) we need the information of large \( s \) rather than \( s \to 0 \). In other words, the dynamic of \( M \to t \) is governed by the full form of \( \phi(\tau) \). Here we just use the density of \( \tau_{\text{max}} \) to build the relationship between \( \tau_{\text{max}} \) and \( x \), and discussion of Eq. (A3) or (A4) will be shown by another paper \([40]\).

**Appendix B: The relation between the largest position and the longest traveling time with \( 0 < \alpha < 2 \)**

Here we make a brief comparison for the rare events of position and the waiting time. Contrary to \( 1 < \alpha < 2 \), the rare event of the position is not only determined by the longest waiting times but also by another fluctuant term. Though the micro term is small, we can not ignore it; see Eq. (33). This relation can also be found for the case of \( \alpha > 1 \); see Figs. 9 and 10 from the view of the distribution and the correlation plot. This means that in some special region near \( \eta \), the rare fluctuations are determined by the full form of the waiting time even for \( 1 < \alpha < 2 \). While, with the decrease of \( \tau_{\text{max}} \), the behavior is governed by the asymptotic behavior of the distribution of the waiting time showing the behavior when \( \epsilon \tau_{\text{max}} \) is not too large, and the statistics change. As expected, the single big jump principle Eq. (32) under study, will vanish due to \( \tau_{\text{max}} \to 0 \) with the increase of \( t \) and \( 1 < \alpha < 2 \). With the help of our work, one can get a better understanding about the rare events, the relation between \( 0 < \alpha < 1 \) and \( 1 < \alpha < 2 \), and the infinite densities.

![FIG. 9: The PDFs of random variables \( \eta = 2\epsilon(t - \tau_{\text{max}}) \) and \( \epsilon = \epsilon(t - \tau_{\text{max}}) \) for a Lévy walk model. We use the Pareto PDF for the travel times, and choose \( t = 1000 \), \( \epsilon = 1 \), \( \tau_{\text{max}} = 10 \), and \( N = 6 \times 10^6 \). When \( \epsilon \) and \( \eta \) are small, clearly the distribution of \( \eta \) is the same as that of \( \epsilon \); see the inset.](image-url)
FIG. 10: Correlation plot between $\eta = 2v_0(t - \tau_{\text{max}})$ and $\epsilon = v_0t - x$ shown in Eq. (32) for a Lévy walk model with different observation time $t$. When the observation time $t$ is not very large, for example $t = 10^2$, the statistics of the single big jump Eq. (32) shown by the dashed red line are found. On the other hand, if $t$ is enough large, the distribution of the biggest position and the largest waiting times agree with each other; see the black solid line. As mentioned this is a diversion, since $\alpha > 1$.

Note that the slope of the red dashed and the black solid lines are one and half, respectively. The parameters are the same as in Fig. 9.

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