Direct diagonalization of Fock Space for an exact solution of pairing model

An Min Wang
Quantum Theory Group, Department of Modern Physics
University of Science and Technology of China, Hefei 230026, People’s Republic of China

We investigate the exact solution of BCS pairing model using direct diagonalization of Fock space. By the data analysis and numerical calculation, we verify the symmetry between energy spectrum of Fock subspaces, obtain the common structure features of energy gaps and energy bands in Hamiltonian spectrum of reduced model, propose the formula to estimate the lowest energy levels in all of the subspaces of reduced model, and suggest a scheme to estimate the respective energy spectrum which can reveal the structure of energy spectrum of pairing model.

PACS numbers: 74.20.Fg, 03.65.-w, 03.67.Lx

Recently, the Bardeen, Cooper and Schrieffer (BCS) model for superconductivity has been connected with the problems in different areas of physics such as superconductivity, nuclear physics, physics of ultrasmall metallic grains and color superconductivity in quantum chromodynamics. Since Richardson’s works from 60’s to now, the exact solution of the reduced (constant) BCS pairing model has been well known. However, for the more general cases in which the coupling coefficients among different pairs are different, the exact solution of BCS pairing model is on studying. In fact, the exactly solvable models have been proven to be very useful tools to understanding the physics of strongly correlated many-body quantum systems. There are a serial of the important and interesting works about the pairing model being published that have not cited here (see the references in [2, 3]).

Let us consider the BCS pairing Hamiltonian:

\[
H_{\text{BCS}} = \sum_{m=1}^{L} \left( \frac{\epsilon_m - \epsilon_F}{2} (n_m + n_{-m}) - \sum_{m,l=1}^{L} V_{ml} c_m^\dagger c_{-l} c_{l} \right)
\]

where \( n_{\pm m} \equiv c_{\pm m}^\dagger c_{\pm m} \) are the electron number operators, \( c_{m}^\dagger (c_{m}) \) is the fermionic creation (annihilation) operator, and the coupling coefficients are real and symmetric, that is \( V_{ml} = V_{lm} \). Based on the Refs. [2, 3, 9], one is able to study equivalently its spin-analogy form as the following

\[
H_p^{(L)} = \frac{1}{2} \sum_{m=1}^{L} \epsilon_m \sigma_{z}^{(m)} - \frac{1}{2} \sum_{m<l}^{L} V_{ml} \left( \sigma_{x}^{(m)} \sigma_{x}^{(l)} + \sigma_{y}^{(m)} \sigma_{y}^{(l)} \right)
\]

where \( \epsilon_m = (\epsilon_m - \epsilon_F) - V_{mm} \), and the constant term \( \sum_{m} \epsilon_m/2 \) has been ignored, which vanishes anyway since we cut off symmetrically above and below \( \epsilon_F \). Note that \( L \) is the number of the pairs or qubits here.

Quantum theory tell us, it is natural to exactly solve the pairing model by the direct diagonalization in Fock space. Several works have been published in this aspect, for example [10, 11]. However, its feasibility still has to be further considered under no any approximation. Clear calculations and interesting analysis of the results need to further study. Obviously, a difficulty of exact solution of spin-analogy of pairing model is, in the large-\( L \) limit, the obsession of exponentially complicated problem to the direct diagonalization. Fortunately, this difficulty can be partially avoided by making use of decomposition of Fock space. At least, for some given subspaces of Fock space, the direct diagonalization becomes a polynomial problem. On the other hand, the direct diagonalization needs that one knows the general forms of the Hamiltonian in the given subspaces. This implies that a key problem is to seek for their explicit forms. As soon as they are found and expressed, it is indeed feasible to exactly solve the pairing model by the direct diagonalization of Fock space.

Actually, our motivation is arose originally by our study on quantum simulation of pairing model on a quantum computer [4, 12, 13]. Because we did not know how to implement the relevant approximations in the simulating solution of pairing model in a quantum computer, and we realize that the direct diagonalization will be feasible and direct at least on a quantum computer, and so we would like to develop it for the quantum simulation of pairing model in the near future.

*Electronic address: anmwang@ustc.edu.cn
Now we first recall the technology of the decomposition of Fock space and give out a strict and simple proof about it. In fact, the spin space $S_{\text{spin}}(L)$ in a $L$-pair (qubit) system can be divided into the different subspaces which correspond to the different numbers of spin-up states, that is $S_{\text{spin}}(L) = S_{\text{spin}}(0) \oplus S_{\text{spin}}(1) \oplus S_{\text{spin}}(2) \oplus \cdots \oplus S_{\text{spin}}(L)$, where the subspace $n$, i.e. $S_{\text{spin}}(n)$, is a subspace with $n$ spin-up states $|0\rangle$ (corresponding to “occupation”). Thus, we can use $i_1, i_2, \ldots, i_n$ to indicate the values of positions appearing 0 in the bit-string $\alpha_1 \alpha_2 \cdots \alpha_L$ ($\alpha_i = 0$ or 1), and then the bases of $S_{\text{spin}}(n)$ can be denoted by $s^{(L)}_{i_1, i_2, \ldots, i_n}$ ($n \neq 0$) and $s^{(L)}_0 \in S_{\text{spin}}(0)$. Obviously, the dimension of $S_{\text{spin}}(n)$ is 
\[
\frac{L!}{n!(L-n)!}
\]

From \(\sigma_x^{(m)} \sigma_x^{(l)} + \sigma_y^{(m)} \sigma_y^{(l)}\) = \(\sigma_x^{(m)} + i \sigma_y^{(m)}\) = \(\sigma_x^{(+)}\) = \(\sigma_x^{(-)}\), \((m \neq l)\) in eq.(2), it follows that for the arbitrary basis $s^{(L)}_{i_1, \ldots, i_n}$ belonging to $S_{\text{spin}}(n)$, \(H \cdot s^{(L)}_{i_1, \ldots, i_n}\) also belongs to $S_{\text{spin}}(n)$ because that $\sigma^+$ and $\sigma^-$ appear in pairs or do not appear in the various $H_p$. It implies that \(\langle s^{(L)}_{i_1, \ldots, i_n} | H_p \cdot s^{(L)}_{i_1, \ldots, i_n} \rangle = 0\). (If $m \neq n; n, n = 1, 2, \ldots, L$) and \(\langle s^{(L)}_0 | H_p \cdot s^{(L)}_{i_1, \ldots, i_n} \rangle = 0\). Therefore, we have proved that the pairing model Hamiltonian is able to be decomposed into the direct sum of submatrices in Fock subspaces, that is
\[
H_p^{(L)} = H_{\text{sub}0} + H_{\text{sub}1} + H_{\text{sub}2} + \cdots + H_{\text{sub}L}
\]
(3)
It is clear that two the simplest eigenvectors of $H_p^{(L)}$ are in the one dimension $S_{\text{spin}}(0)$ and $S_{\text{spin}}(L)$ respectively with the eigenvalues $\pm \sum_{m=1}^{L} \epsilon_m/2$. Thus, the concerned subspaces or submatrices of Hamiltonian in our method only includes those from 1 to $L-1$.

Furthermore, we can derive out
\[
H_p^{(L)}|s^{(L)}_{i_1, \ldots, i_n}\rangle = \left(-\frac{1}{2} \sum_{m=1}^{L} \epsilon_m + \sum_{a=1}^{n} \epsilon_{i_a}\right) s^{(L)}_{i_1, \ldots, i_n} + \sum_{b=1}^{n} \sum_{m \neq i_b, a \neq b} V_{i_a i_b} s^{(L)}_{p_{i_1, \ldots, i_n}^{+}, \ldots, i_b^{+}, m} (i_a, i_b \in \{i_1, i_2, \ldots, i_n\})
\]
(4)
where $P_{i_1, \ldots, i_n}^{+}$ is such a permutation that $i_c$ is dropped, $r$ is added, and \{etc\} is rearranged according to their values from small to large. In terms of the orthogonality of bases, we obtain immediately the general and explicit forms of Hamiltonian submatrices of pairing model in the Fock space. Consequently, we can carry out the direct diagonalization of them within the limitation of the computer's power.

It must point out that in the half filling case i.e. $n = [L/2]$, the dimension of its submatrix $H_{\text{sub}hf}$ is the highest and arrives at $L!/([L/2]!([L+1]/2)!)$, where “$[M]$” means taking the integer part of $M$. Note the fact that $H_{\text{sub}hf}$ is a sparse matrix, for example, the dimension of $H_{\text{sub}hf}$ is 12870, but the number of its nonzero elements is only 65 x 65, one is able to diagonalize it to some L. Nevertheless, with $L$ increasing, the dimension of half filling submatrix will exceeds quickly the limits of large scale diagonalization in a classical computer. Perhaps, quantum simulation can solve this difficulty [12][12][12]. Here, we focus on the problem how obtain the more knowledge of Hamiltonian spectrum for the moderate $L$, which is helpful for understanding many body quantum theory, as well as for providing a precision comparison with the possible result of quantum simulating in the near future. Actually, we note that
\[
H_p^{(L)}|s^{(L)}_1\rangle = \left(-\frac{1}{2} \sum_{m=1}^{L} \epsilon_m + \epsilon_{i_1}\right) s^{(L)}_1 - \sum_{m=1}^{n-1} V_{mi} s^{(L)}_m - \sum_{l=i+1}^{L} V_{li} s^{(L)}_l
\]
(5)
It implies the submatrix one of paring Hamiltonian with a simple construction
\[
H_{\text{sub}1}[i, i] = -\frac{1}{2} \sum_{m=1}^{L} \epsilon_m + \epsilon_{i_1}, \quad H_{\text{sub}1}[i, j]|_{i \neq j} = -V_{ij} (i, j = 1, 2, \ldots, L)
\]
Obviously, it is easy to diagonalize the $H_{\text{sub}1}$ for $L = 100 \sim 180$ in an ordinary PC [14]. By making use of eq.4 we strictly obtain the general and explicit form of $H_{\text{sub}1}$ and the arbitrary subspace $H_{\text{sub}2}$. In this letter, we consider the typical three kinds of models, that is, the reduced (constant) model in which $V_{ij} = V$ (constant) for any $i$ and $j$, the nearest neighbor model in which $V_{ij} = \delta_{ij}$ (only there is the coupling in the nearest neighbor two pairs) and the third model which takes $V_{ij} = \beta V/|j - i|$ (the coupling coefficient decreases pro rate with the “distance”). They are denoted respectively by the subscripts RE, NB and RA. Moreover, we take $V = 2 \times 10^{-6}$, $\epsilon_m = mV/\lambda$, $\beta = 10^{-1}$ and $\lambda$ is a running parameter which can be taken as 10, 20 and etc in the following numerical calculation. Fig.1 shows the energy spectrum of subspace 1 of $H_p$ which is obtained by the direct diagonalization.
Fig. 1 and the other figures in this letter are all so-called energy level figures, in which, every line segment represents an energy level (or several energy levels whose number is equal to its degeneration degree), that is, an eigenvalue of Hamiltonian, and $y$-axis indicates the values of energy levels. In fact, for saving space, we always put a set of energy level figures together, where every sub-figure occupies one column and on its top a name of the considered models (Fig. 1) or an ordinal (not including the trivial ones of level figures together, where every sub-figure occupies one column and on its top a name of the considered models (Fig. 1) or an ordinal (not including the trivial ones of $S_0^L$ and $S_L^L$) of subspace (the others) are denoted. It must be emphasized that in our energy level figures, there are some energy levels that have not been distinguished clearly since the limitation of scale of $y$-axis and resolution from printer or monitor. Actually, it is just the effect we want to display, because those dense energy levels (large number of energy levels within the unit scale of energy) form so-called “energy bands” (the line segment with some width), two near but obviously separate energy levels (the difference of two near energy levels has the larger value) forms so-called “energy gaps” (two parallel line segment with an obviously larger interval). For example, it is easy to see that in the subspace 1 of reduced model has an energy gap standing at between the energy level 1 and energy level 2, and an energy band including all energy levels except for the energy level 1.

By the numerical calculating and fitting, analytical continuation and then theoretical deduce, we can obtain several important and interesting conclusions.

1) **Symmetry in the energy spectrum**: Denoting $E_{RE, NB, RA}^{(L)}(i, n)$ as the $i$-th energy level of $H_p^{(L)}$ in the subspace $n$ respectively for the models RE, NB and RA and arranging as $E^{(L)}(i, n) \leq E^{(L)}(j, n)$ for $i < j$, our data analysis and numerical fitting indicate that there is the symmetry between $E^{(L)}(i, n)$ and $E^{(L)}(i, L - n)$. That is

\[
E^{(L)}(i, n) = E^{(L)}(i, L - n) - (L + 1) \left( \frac{L}{2} - n \right) \frac{V}{\lambda}
\]  

Moreover, this symmetry is independent of the considered models here and so the subscript RE or NB or RA are omitted. We think that it can be an exact theoretical formula since its precision $\leq 10^{-36}$.

In fact, a physical symmetry should be a strict result. Newly, we have finished a strict analytical proof about this property.

2) **Structure of energy gaps and bands**: From the Fig. 2 it is easy to see the obvious structure of energy gaps and bands. Actually, we find that it is a common feature of energy spectrum in the reduced model. Here, the energy gap(s) means such a (some) difference(s) between two nearest neighbor energy levels that it is much larger than the other ones, and the energy band means such a set of energy levels with very small differences among the nearest neighbor energy levels, even appearing approximate degeneration. From the data of energy levels we can conclude that the number of energy gaps is equal to $n$ (if $n \leq \lfloor L/2 \rfloor$) or $L - n$ (if $n > \lfloor L/2 \rfloor$) for a given subspace $n$ of $H_p^{(L)}$, and so is the number of the energy bands since it does not contain a lowest energy level in this given subspace. The positions of energy gaps $E_{RE}^{(L)}(k, n)$ appear between $E_{RE}^{(L)}(L/((k - 1)!(L - k + 1)!)) + 1, n)$ and $E_{RE}^{(L)}(L/((k - 1)!(L - k + 1)!), n)$,
where \( k = 1, 2, \ldots, n \) (if \( n \leq [L/2] \)) or \( k = 1, 2, \ldots, L - n \) (if \( n > [L/2] \)). We have verified, in numerical, the following variation rules for \( N \leq 10 \): the energy gaps enlarge with \( \lambda \) or \( n \leq [L/2] \) or \( L \) increasing, and lessen with \( k \) increasing; the energy bands widen with \( k \) or \( L \) increasing, and narrow with \( \lambda \) or \( n \leq [L/2] \) increasing. When \( n > [L/2] \), the variation of energy gaps and bands can be known by the symmetry in energy spectrum. However, Fig. 3 shows that the nearest neighbor model destroys the structure of energy gaps and energy bands, and so almost does the third model except for the first energy gap. The destruction strength of the third model for the first energy gap depends on variety strength of the coupling coefficients with the distance \( "|j - i|^" \), the more rapidly the coupling coefficients decrease with the distance, the more largely the first energy gap is destroyed. Fig. 3 gives out an example displaying these features.

\[
E_{\text{RE}}^{(L)} \left(1, \left[ \frac{L}{2} \right] \right) \approx - \left[ \frac{L}{2} \right] \left[ \frac{L + 1}{2} \right] V - 2 \log \left( \left[ \frac{L - 1}{2} \right] \right) \left( \frac{\sin^2 \left( \frac{L \pi}{2} \right)}{\lambda} + \cos^2 \left( \frac{L \pi}{2} \right) \frac{V}{\lambda^2} \right)
\]

(7)

From the fact that \( E_{\text{RE}}^{(L)}(1, n) \approx E_{\text{RE}}^{(L)}(1, n-1) + (E_{\text{RE}}^{(L)}(1, [L/2]) - E_{\text{RE}}^{(L)}(1, 1))/([L/2] - 1) - ([L/2] - 2(n-1))V \) obtained by the data analysis and numerical fitting, it follows that

\[
E_{\text{RE}}^{(L)}(1, n) \approx \left(1 - \frac{n - 1}{[L/2] - 1}\right) E_{\text{RE}}^{(L)}(1, 1) - \frac{(n - 1)}{[L/2] - 1} \left( \left[ \frac{L}{2} \right] \left[ \frac{L + 1}{2} \right] + ([L/2] - 1)([L/2] - n) \right) V
- \frac{2(n-1) \log([[(L-1)/2]])}{[L/2] - 1} \left( \frac{\sin^2 \left( \frac{L \pi}{2} \right)}{\lambda} + \cos^2 \left( \frac{L \pi}{2} \right) \frac{V}{\lambda^2} \right)
\]

(8)

It implies that

\[
E_{\text{RE}}^{(L)}(1, n) \approx \left(1 - \frac{n - 1}{[L/2] - 1}\right) E_{\text{RE}}^{(L)}(1, 1) + (n - (L - 1)) \frac{[L/2]}{[L/2] - 1} V
- \frac{2(n-1) \log([[(L-1)/2]])}{[L/2] - 1} \left( \frac{\sin^2 \left( \frac{L \pi}{2} \right)}{\lambda} + \cos^2 \left( \frac{L \pi}{2} \right) \frac{V}{\lambda^2} \right)
\]
which can be used to estimate the LEL $E^{(L)}_{\text{RE}}(1, n)$ in a given subspace $n \leq [L/2]$. For the other subspaces ($n > [L/2]$), we can estimate their LEL by eq. (6). When $L = 10, \lambda = 20$, the absolute errors of the estimated values for all of subspaces from 1 to $[L/2]$ are $\leq 10^{-9}$ and their relative errors are $\leq 10^{-5}$.

(4) The representative energy levels and energy spectrum: Since the typical structure of energy gaps and energy bands as well as the large density of energy levels for a given band in the reduced model, we can introduce a so-called representative energy level (REL) which is defined by an average of all of energy levels in a given energy band, that is

$$E^{(L)}_{r}(k, n) = \frac{1}{L!/(k!(L-k)!) - L!/((k-1)!(L-k+1)!)} \sum_{i=L!/(k!(L-k)!)}^{L!/(k!(L-k)!)} E^{(L)}_{\text{RE}}(i, n)$$

(9)

where $k = 1, 2, \cdots, n$ (if $n \leq [L/2]$) or $k = 1, 2, \cdots, L - n$ (if $n > [L/2]$)). This REL can be used to indicated the corresponding energy band, in particular, in the interesting half filling case since it with narrower energy bands. Obviously, in a given subspace $n$, the structure features of energy spectrum of pairing model can be represented by the simple representative energy spectrum which consists of a LEL and $n \leq [L/2]$ or $L - n > [L/2]$ RELs. In particular, based on the data analysis and numerical fitting for the RELs, we find that the widths of representative energy gaps, that is, $\Delta^{(L)}_{r}(k, n) = E^{(L)}_{r}(k, n) - E^{(L)}_{r}(k-1, n)$ (where $E^{(L)}_{r}(0, 1) = E^{(L)}_{\text{RE}}(1, 1)$) from low to high decrease near a constant $2V$, which means that $\Delta^{(N)}_{r}(k, n) \approx \Delta^{(N)}_{r}(k-1, n) - 2V$, and $\Delta^{(L)}_{r}(k, n) \approx \Delta^{(L)}_{r}(k, n+1)$. From them we have

$$E^{(L)}_{r}(k, n) \approx \Delta^{(L)}_{r}(k, n) + E^{(L)}_{r}(k-1, n) + (\text{high order approximation})$$

$$= \sum_{i=1}^{k} \Delta^{(L)}_{r}(i, n) + E^{(L)}_{r}(0, n) + (\text{high order approximation})$$

$$= \sum_{i=1}^{k} \Delta^{(L)}_{r}(i, n) + E^{(L)}_{\text{RE}}(1, n) + (\text{high order approximation})$$

$$= k\Delta^{(L)}_{r}(1, n) - k(k-1)V + E^{(L)}_{r}(1, n) + (\text{high order approximation})$$

$$= k\Delta^{(L)}_{r}(1, 1) - k(k-1)V + E^{(L)}_{r}(1, n) + (\text{high order approximation})$$

Adding two additional compensatory terms (high approximation) to eliminate the error since the many times recursions

$$(\text{high order approximation}) \approx \frac{2 \delta_{k,L/2}}{[L/2] - 1} \frac{V}{\lambda^2} - \sin^2 \left( \frac{(n-k)\pi}{2} \right) \frac{V}{\lambda^2}$$

Of course, the form of this two additional terms is given by the numerical fitting. Therefore, we obtain the formula to estimate the RELs in the subspace $n \leq [L/2]$ by

$$E^{(L)}_{r}(k, n) \approx k\Delta^{(L)}_{r}(1, 1) + \left( 1 - \frac{n - 1}{[L/2] - 1} \right) E^{(L)}_{\text{RE}}(1, 1)$$

$$+ \left( (n-1)n - (n-1)(L-1) \frac{[L/2]}{[L/2] - 1} - k(k-1) \right) V$$

$$- \frac{2(n-1)(L-1)}{[L/2] - 1} \left( \sin^2 \left( \frac{L\pi}{2} \lambda \right) \frac{V}{\lambda^2} + \cos^2 \left( \frac{L\pi}{2} \lambda \right) \frac{V}{\lambda^2} \right)$$

$$+ \frac{2 \delta_{k,L/2}}{[L/2] - 1} \frac{V}{\lambda^2} - \sin^2 \left( \frac{(n-k)\pi}{2} \right) \frac{V}{\lambda^2}$$

(10)

where $\Delta^{(L)}_{r}(1, 1) = E^{(L)}_{r}(1, 1) - E^{(L)}_{\text{RE}}(1, 1)$, which can be calculated by the directly diagonalizing the subspace 1 of $H_{\gamma}$. Note that we, based on the data analysis and numerical fitting, add two little addition terms in order to include the contribution from high order approximations among the representative energy gaps. When $L = 10, \lambda = 20$, the absolute error of this estimation is $\sim 10^{-9}$ and its relative error is $\sim 10^{-3}$. Then, based on the symmetry in the energy spectrum, we can obtain the values of representative energy spectrum in the other subspaces. Fig[4] is an example which is estimated by our above scheme. The representative energy spectrum in subspaces 11 to 19 has not been
FIG. 4: The estimated the lowest energy level and representative energy spectrum \((\lambda = 20)\) for the subspaces from \(n = 1\) to \(n = 10\)

picted since their shapes can be known by the symmetry of energy spectrum. For the subspace 2, we can calculate out the absolute errors of one estimated the lowest energy level and two estimated representative energy levels are respectively \(5.80231 \times 10^{-9}\), \(2.63859 \times 10^{-9}\) and \(6.47206 \times 10^{-9}\), their relative errors are respectively \(0.000721591\), \(0.000653016\) and \(0.00147345\). Therefore, the lowest energy levels and the representative energy levels can have the better and usable precision in numerical calculation as well as estimation by our method.

It is clear that we have solved the lowest energy levels of all of the subspaces and representative energy spectrum of the pairing model in numerical by our direct diagonalization of Fock space. Moreover, we show the structure of the energy levels and energy gaps in the pairing model for the reduced model. After finishing the proof of symmetry between the energy spectrum of pairing model, we would like to show the origin of leading term of the LEL of \(H_p\) in the near. In principle, our method should be able to extend to some similar spin Hamiltonian systems. Of course, because our estimated formula are obtained by the numerical fitting and analytical continuation, it is still a problem how large \(N\) our scheme is suitable to. In addition, for the width of energy bands we have not found a good estimated method yet. More knowledge about the other models needs to explore.

We specially acknowledge interesting discussions with Feng Xu and valuable suggestions by her. We are grateful to Ningbo Zhao, Xiaodong Yang, Xiaosan Ma, Hao You, Wanqing Niu, Rengui Zhu and Xiaofang Su for our cooperations in the Quantum Theory Group, the Institute for Theoretical Physics of University of Science and Technology of China. This work was funded by the National Fundamental Research Program of China under No. 2001CB309310, and partially supported by the National Natural Science Foundation of China under Grant No. 60573008.

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer Phys. Rev. 108(1957) 1175
[2] von Delft, J. and D.C. Ralph, Phys. Rep. 345 (2001) 61 and its refs.
[3] J. Dukelsky and G. Sierra, Mod. Phys. Rev. 76 (2004) 643 and its refs.
[4] R. W. Richardson., Phys. Lett. 3 (1963) 277; Phys. Lett. 5 (1963)82
[5] J. Dukelsky and G. Sierra Phys. Rev. Lett. 83 (1999)172
[6] L. Amico and A. Osterloh Phys. Rev. Lett. 88 (2002)127003
[7] L. Amico, A. Di Lorenzo, and A. Osterloh Phys. Rev. Lett. 86 (2001)5759
[8] A. Volya, B. A. Brown, and V. Zelevinsky, Phys.Lett. B 509 (2001)37
[9] L.-A. Wu, M. S. Byrd, and D. A. Lidar, Phys. Rev. Lett. 89 (2002) 057904, Phys. Rev. Lett., 90 (2003) 249804; J. Dukelsky, J. M. Román, and G. Sierra, Phys Rev Lett., 90 (2003) 249803
[10] O.Burglin and N.Rowley, Nucl. Phys. A 602(1996)21
[11] H. Molique and J. Dudek, Phys. Rev. C 56(1997)1795
[12] An Min Wang and Xiaodong Yang, Phys. Lett. A, 352(2006) 304
[13] Xiaodong Yang, An Min Wang, Feng Xu, and Jiangfeng Du. Chemical Physics Letters, 422 (2006) 20
[14] Xu Feng, An Min Wang, Xiaodong Yang, Xiaoan Ma, and Hao You, Commun. Theor. Phys. 44 (2005) 171
[15] An Min Wang and Ren Gui Zhu, Chinese Phys. Lett. 23 (2006) 2542
[16] We verify the symmetry using a program in Mathematica (© Copyright 1988-2005 Wolfram Research, Inc.) and we do not consider the error that might be brought by the algorithm in Mathematica software.