Symmetry constraints on many-body localization

Andrew C. Potter\(^1\) and Romain Vasseur\(^{1,2}\)

\(^1\)Department of Physics, University of California, Berkeley, CA 94720, USA and
\(^2\)Materials Science Division, Lawrence Berkeley National Laboratories, Berkeley, CA 94720

We derive general constraints on the existence of many-body localized (MBL) phases in the presence of global symmetries, and show that MBL is not possible with symmetry groups that protect multiplets (e.g. all non-Abelian symmetry groups). Based on simple representation theoretic considerations, we derive general Mermin-Wagner-type principles governing the possible alternative fates of non-equilibrium dynamics in isolated, strongly disordered quantum systems. Our results rule out the existence of MBL symmetry protected topological phases with non-Abelian symmetry groups, as well as time-reversal symmetry protected electronic topological insulators, and in fact all fermion topological insulators and superconductors in the 10-fold way classification. Moreover, extending our arguments to systems with intrinsic topological order, we rule out MBL phases with non-Abelian anyons as well as certain classes of symmetry enriched topological orders.

The concept of symmetry plays a crucial role in our understanding of phases of matter. The interplay of symmetry and dimensionality leads to very general constraints on possible types of symmetry breaking phases and phase transitions, such as the Peierls or Mermin-Wagner theorems. Going beyond Landau’s theory of phases and phase transitions in terms of spontaneous symmetry breaking, it was recently understood that symmetries can protect topological distinctions among short-range entangled phases of matter — leading to the concept of symmetry protected topological (SPT) phases\(^{1-6}\) exemplified by the celebrated electronic topological insulators (TI)\(^{7-12}\) — and that symmetry can also enrich the possibilities of quantum phases with long-range entanglement and intrinsic topological order\(^{13-17}\).

Whereas traditionally, the existence of phases and phase transitions is considered within the framework of equilibrium statistical mechanics, a sharp notion of quantum phases can be extended to a certain class of far-from-equilibrium quantum systems that fail to self-thermalize in isolation. Highly excited eigenstates in such many-body localized (MBL) systems\(^{18,23}\) have properties akin to quantum groundstates\(^{23}\), leading to the prospect of quantum coherent phenomena and universal dynamics\(^{24-30}\), and symmetry-breaking, topological or SPT quantum orders at high energy density\(^{23,28,31-35}\).

Given the fundamental role of symmetry in our understanding of equilibrium phases of matter, it is natural to expect very general symmetry principles to play a crucial role in MBL systems. While certain examples of symmetry based constraints on localization have been identified\(^{26,38}\), a full and systematic understanding remains to be obtained. In this paper, we argue that the local conserved quantities that define MBL systems transform independently under the global symmetry, leading to an extensive number of local degeneracies in the presence of non-Abelian symmetries. This local action leads to very general constraints that dictate the fate of the excited state dynamics of strongly disordered systems on symmetry grounds alone. Namely, we show that symmetry preserving MBL phases are not possible with non-Abelian symmetries, either discrete or continuous. This eliminates the possibility of SPT and symmetry enriched topological (SET) order protected by non-Abelian symmetries, and for instance means that it is impossible to localize spin-1/2 electrons with time-reversal symmetry, ruling out the realization of electronic TIs in MBL settings. Moreover, based on the representation theory of the symmetry group, we derive very general Mermin-Wagner type principles governing the stability and possible fates of strongly disordered systems with symmetry. For example, systems with continuous non-Abelian symmetries inevitably thermalize even at strong disorder, whereas systems with discrete non-Abelian symmetries may yield non-ergodic phases that are either well localized but with spontaneous symmetry breaking or are delocalized and quantum-critical. Our results also constrain the emergence of non-equilibrium quantum phases with intrinsic topological order and anyonic excitations, with the notion of anyonic fusion algebras replacing the group representation theory. For example, we rule out the possibility of MBL with non-Abelian anyonic excitations.

I. MANY-BODY LOCALIZATION AND GLOBAL SYMMETRIES

A. Many-body localization and local conserved quantities

We begin by fixing a formal definition of MBL in terms of a complete set of quasi-local conservation laws\(^{26,27,39,40}\), which has been widely adopted, and from which all of the known phenomenology of MBL systems such as area-law entanglement\(^{22}\), absence of transport, slow dephasing and entanglement growth\(^{24,25,41}\) directly follow. Specifically, we define MBL in terms of the existence of a complete set of conserved quantities, \(\{n_{\alpha}\}\) \((\alpha = 1 \ldots N)\), each of which takes values from a set \(\{1, \ldots, d_\alpha\}\), with the associated quasi-local conserved
projectors called l-bits \[^{26,27}\].

\[
\Pi^{\alpha}_{\alpha} = \sum_{n_{\beta} \neq \alpha = 1}^{d_{\beta}} \langle n_1 \ldots n_\alpha \ldots n_N | n_1 \ldots n_\alpha \ldots n_N \rangle,
\]

that are exponentially well-localized within a localization length \(\xi\) of position \(r_\alpha\) and that commute with the Hamiltonian \(\Pi^{\alpha}_{\alpha}, H = 0\). We further assume that the quasi-local conserved quantities are related to local operators (with finite support) by dressing them with a quasi-local unitary transformation, which produces their exponential tails. This structure is obtained in all known examples of MBL, and in fact can be taken as the definition of the l-bits being quasi-local.

Note that the number of conserved quantities \(N\) can in general be different from the number of physical sites \(L\). We will restrict our attention the case where all of the energy eigenstates (or quasi-energy eigenstates for Floquet systems \[^{22,43}\]) are MBL, and neglect the theoretically more delicate case of a partially localized spectrum. This is equivalent to having each set \(\{n_\alpha\}\) are MBL, and neglect the theoretical argument for an extensive number of l-bits (i.e. with support only on a finite number of sites). For this case, it is clear that if \(r_\alpha\) and \(r_\beta\) are sufficiently far apart so that the corresponding supports do not overlap, then the action of the symmetry factorizes on the two l-bits \(\alpha\) and \(\beta\): \(\mathcal{V}_{n_\alpha n_\beta} = \mathcal{V}_{n_\alpha} \otimes \mathcal{V}_{n_\beta}\). Repeating the argument for an extensive number of l-bits \(\alpha_1, \alpha_2, \ldots, \alpha_p\) with \(p = O(N) \sim O(L)\) sufficiently far apart so that their support do not overlap, we find that the symmetry action factorizes on the local l-bits \(\mathcal{V}_{n_\alpha_1 \ldots n_\alpha_p} = \mathcal{V}_{n_{\alpha_1}} \otimes \cdots \otimes \mathcal{V}_{n_{\alpha_p}}\).

**B. Local symmetry action**

Having defined MBL we now derive some general constraints on many-body localization in the presence of symmetry. To begin, we start by showing that symmetry acts locally on the l-bits.

Consider a lattice of sites \(i\) containing quantum degrees of freedom that transform under a (possibly reducible) representation, \(\mathcal{V}\), of a symmetry group \(G\) – for example a chain of spins-\(\frac{1}{2}\) (\(\mathcal{V}\)), with spin-rotation symmetry \(G = SU(2)\). The Hilbert space of this system decomposes into a tensor product of on-site Hilbert spaces, \(\mathcal{H} = \mathcal{V}^{\otimes L}\). In order that different symmetry operations have non-trivial action on the physical degrees of freedom, we will demand that \(\mathcal{V}\) is a faithful representation of the symmetry \(G\), and that we cannot merely group degrees of freedom into larger clumps that transform under a simpler symmetry \(G'\). The former condition implies that all irreducible representations (irreps) of \(G\) are contained in the tensor product \(\mathcal{V}^{\otimes n}\) for sufficiently large \(n\).

We define a symmetry preserving MBL phase as one in which the local conserved quantities labelling is consistent with the symmetry \([\Pi^{\alpha}_{\alpha}, \prod_i g_i] = 0\), where \(g_i \in \mathcal{V}\) is the representation of the symmetry generator \(g \in G\) on the site \(i\). This is equivalent to having each set of conserved quantities \(\{n_\alpha\}\) label a multiplet of states \(\mathcal{V}_{n_1, n_2, \ldots, n_N}\) that form a representation of \(G\). To construct the local action of the symmetry, let us proceed as follows. Let us assume that at least one of the eigenstates labelled by a given set \(\{n_\alpha\}\) is non-degenerate and transforms trivially under the symmetry, so that the corresponding representation \(\mathcal{V}_{n_\alpha} = \mathcal{V}_{n_0}\) is the trivial representation (or singlet) with dimension 1. We then create a local excitation by changing the label \(n_\alpha\) to \(n_\beta\) for a given l-bit \(\alpha\): the resulting eigenstate(s) then transform in a different representation \(\mathcal{V}_{n_\beta} = \mathcal{V}_{n_0} = \cdots = \mathcal{V}_{n_{\hat{\alpha}}}\) which will generically be irreducible (otherwise, one may add generic local perturbations to reduce it). Because the change \(n_\alpha \rightarrow n_\beta\) is local, all the eigenstates \(|n_0, \ldots, n_a, \ldots, n_N\rangle\) with \(p = 1, \ldots, \dim \mathcal{V}_{n_0}\) in this representation and the singlet eigenstate \(\{|n_0\rangle\}\) should differ only locally around \(r_\alpha\). Let us now repeat the process to excite a different l-bit, \(\beta\), and let \(\mathcal{V}_{n_\alpha n_\beta}\) be the representation corresponding to the configuration where we changed the labels \(n_\beta \rightarrow n_\alpha\) and \(n_\beta \rightarrow n_\beta\) on two different locations \(\alpha\) and \(\beta\).

**C. Examples of local symmetry action**

This local factorization of the symmetry on the l-bits is particularly obvious for models of MBL paramagnets, such as the Ising paramagnet in arbitrary dimension with \(G = \mathbb{Z}_2\)

\[
H = -\sum_{i=1}^{L} h_i \sigma_i^z + \ldots
\]

where the dots represent small (but arbitrary) symmetry-preserving perturbations. The eigenstates of Eq. \(^{2}\) are related to product states of definite \(\sigma_i^z = \pm 1\) by a finite depth (quasi-local) unitary transformation, \(U\), such that the local conserved quantities of this MBL systems are the dressed projectors \(\Pi^{\alpha}_{\alpha} = 0, 1 = U^\dagger \frac{1 + \sigma_i^z}{2} U\). In this case, the local action of the symmetry is simply given by

\[
\tilde{g}_{\alpha_i} = U^\dagger \sigma_i^z U,
\]

with \([\tilde{g}_{\alpha_i}, H] = 0\), since \(g_{\alpha_i}\) commutes with all the conserved quantities \(\Pi^{\alpha}_{\alpha_j}\), readily verifying that the global symmetry \([\prod_i g_i, H] = 0\) with \(g_i = \sigma_i^z\) is promoted to a local symmetry \([\tilde{g}_{\alpha_i}, H] = 0\) for the MBL system.
This construction can be readily generalized to the generic MBL paramagnet Hamiltonian \( H_{\text{para}} = -\sum_i \sum_n h^n P^n_i + \ldots \) where the \( P^n_i \)'s are projection operators onto the different irreps (“channels”) in the decomposition \( \mathcal{V} = \oplus_n \mathcal{V}_n (\sum_n P^n_i = 1) \) of the on-site representation \( \mathcal{V} \) of \( G \), and the dots represent generic weak perturbations. As in the Ising example, the local conserved quantities of this MBL system are the dressed projectors \( \Pi^a_n = U^\dagger P^n_i U \) where \( U \) is a finite depth (quasilocal) unitary transformation, and the local action of the symmetry is simply given by \( \hat{g}_\alpha = U^\dagger g_i U \), where \( g_i \) in the representation of the group element \( g \) on site \( i \).

A less straightforward example are SPT phases, which cannot be continuously connected to a trivial paramagnet while preserving symmetry. However, the SPT eigenstates are non-continuously deformable, via a finite-depth unitary transformation \( U_{\text{SPT}} \) that preserves the symmetry everywhere in the bulk, to a trivial paramagnet (see e.g.46 and Appendix A for a specific example). We can then utilize the construction for paramagnets to identify, the local action of symmetry as being generated by \( \hat{g}_\alpha = U^\dagger U_{\text{SPT}} g_i U_{\text{SPT}} U^\dagger \), where \( U \) is the quasi-local unitary that dresses the l-bits. These generators form an ordinary local representation of symmetry in the bulk, but act non-trivially (e.g. projectively in 1D) at the edges of the system.

II. MBL AND NON-ABELIAN SYMMETRY

The local factorization of the symmetry on the l-bits has important consequences when \( G \) is non-Abelian, for which some irreps are necessarily multidimensional. Intuitively, this signals an obstacle to localization, since a generic MBL state will contain many of these multidimensional excitations, each with local degrees of freedom that cost no energy to excite, and can therefore freely inter-resonate with each other leading to a breakdown of localization. This rules out the existence of MBL paramagnets with Potts (permutation group \( G = S_n \)) or non-chiral clock (Diherdral group \( G = D_n = \mathbb{Z}_n \times \mathbb{Z}_2 \)) symmetry for instance. We emphasize that while our argument relied on the local integrability picture of MBL systems, we expect the main idea to be fairly general so that it would also rule out tentative MBL phases without an l-bit description (see discussion below).

More formally, if we were to have an MBL system with non-Abelian symmetry \( G \), then the local conserved quantities would transform as irreps of the symmetry group \( G \) so that each l-bit, \( n_\alpha \), labels an irrep \( \mathcal{V}_{n_\alpha} \) of \( G \). The Hilbert space therefore has a symmetry preserving tensor structure in the l-bit space \( \mathcal{H} = \otimes_\alpha \mathcal{V}_\alpha \), where the representation \( \mathcal{V}_\alpha \) is reducible and can be decomposed as \( \mathcal{V}_\alpha = \oplus_{n_\alpha} \mathcal{V}_{n_\alpha} \). Since the physical degrees of freedom transform in a faithful representation \( \mathcal{V} \) of the symmetry, at least a finite density of the \( \mathcal{V}_\alpha \)'s should be faithful representations of \( G \) as well. If \( G \) is non-Abelian, this immediately implies that some irreps \( \mathcal{V}_{n_\alpha} \) should have dimension larger than 1 so that the quantum numbers \( n_\alpha \) must be supplemented with an additional number \( p_\alpha = 1, \ldots, D_{n_\alpha} = \dim \mathcal{V}_{n_\alpha} \) to label uniquely an eigenstate. This finite density of local multidimensional irreps leads to an exponential degeneracy of the eigenstates of \( H \) since the energy cannot depend on the extra labels \( p_\alpha \). In other words, the global symmetry \( G \) is promoted to a local symmetry
\[
[g_{n_\alpha}, \Pi^a_{n_\alpha}] = [g_{n_\alpha}, H] = 0, \tag{4}
\]
because of the many-body localized structure of the eigenstates, with \( g_{n_\alpha} \) being the representation of \( g \in G \) in \( \mathcal{V}_{n_\alpha} \), acting locally around position \( r_\alpha \). This extended local symmetry leads to local degeneracies if there are multidimensional irreps, leading in turn to a massive exponential-in-system-size degeneracy of all eigenstates. Such degenerate eigenstates are inherently unstable, even to infinitesimally small, perturbations. However, the crucial point is that there is no local and symmetry-preserving way to resolve this degeneracy. Hence, either the symmetry or the localization must break down. Which of these fates may occur depends on the group structure, and below, we will identify some simple governing principles based on the number and dimensions of the irreps of the symmetry group.

Note that whereas the above discussion assumed a full factorization of the symmetry on the l-bits for simplicity (so that the Hilbert space factorizes as \( \mathcal{H} = \otimes_\alpha \mathcal{V}_\alpha \)), the existence of exponentially-degenerate eigenstates only requires a partial factorization of the symmetry for a general excitation involving an extensive number of l-bits far-enough apart \( \mathcal{V}_{n_{\alpha_1} \ldots n_{\alpha_p}} = \mathcal{V}_{n_{\alpha_1}} \otimes \cdots \otimes \mathcal{V}_{n_{\alpha_p}} \), which we showed in Sec. III.

III. GENERAL SYMMETRY PRINCIPLES

Above, we have shown that non-Abelian symmetries are not consistent with MBL phases, and we now seek some general insight into the possible fates of an isolated non-equilibrium system with non-Abelian symmetries. If disorder is too weak, we expect that the putative local degenerate excitations will strongly overlap and inter-resonate, driving the system into a thermalizing phase. Hence, we will subsequently focus on the regime of strong disorder, considering various classes of non-Abelian symmetry groups in turn.

When the non-Abelian group \( G \) has irreps of bounded dimension, with either infinitely many irreps (as for the group \( G = U(1) \times \mathbb{Z}_2 \) where all irreps have dimension \( \leq 2 \)) or a finite number of them (as for any finite non-Abelian symmetry), there are a few options to lift the degeneracies at strong disorder. One possible outcome is that the system forms an MBL state in which symmetry is spontaneously broken down to an Abelian subgroup by choosing a particular set of the numbers \( p_\alpha \). This spontaneous symmetry breaking (SSB) scenario was previously demonstrated for the particular example of
If on the other hand $G$ is Abelian ($|V|=1$), then a many-body localized phase is possible at strong disorder. If the other hand $G$ is non-Abelian, a symmetry-preserving MBL phase is not allowed, giving rise to either thermalization, MBL with the symmetry spontaneously broken (SSB) to an Abelian subgroup, or to non-trivial quantum critical glasses (QCG) depending on the properties of the symmetry group (see text).

A random XXZ spin chain, equivalent to fermions with particle-hole symmetric disorder with symmetry group $G = U(1) \rtimes \mathbb{Z}_2$ using renormalization group techniques and numerical simulations. Another possible option for finite groups would be for the system to form a symmetry preserving “quantum critical glass” (QCG) which is neither thermal nor exponentially localized, and that cannot be described in terms of independent conserved quantities (examples of such phases have been uncovered in analytically solvable random anyonic chains). It would be very interesting to find a concrete example of such a QCG phase in a random spin chain with non-Abelian symmetry. Note however that our argument also rules out marginal or quantum critical MBL states where independent l-bits exist, just with algebraic rather than exponential tails (as for the critical random transverse field Ising chain for example).

If the non-Abelian symmetry group $G$ is continuous, e.g. $G = SU(2)$, then it will possess infinitely many irreps with arbitrarily large dimension. E.g., for $G = SU(2)$, irreps are labelled by spin size $S$ and have dimension $2S + 1$ where $S$ can be arbitrarily large. Then following Ref. 46, in a large many-body system, one will encounter excitations with arbitrarily large local degeneracy, $D$ (large “spins”), whose quantum fluctuations are suppressed as $1/D$, leading to effectively classical dynamics, and resulting in thermalization, even for arbitrarily strong disorder. Note that it is furthermore not possible to realize an MBL phase by spontaneously breaking the continuous non-Abelian symmetry, as this would produce a delocalized Goldstone mode that would act as a bath, so that thermalization is the only possible scenario.

The only scenario that permits stable MBL phases with symmetry are Abelian groups, whose irreps all have dimension 1, avoiding the pitfalls of the above examples. These different scenarios are summarized in Tab. I.

| $|V|$ | # irreps $< \infty$ | # irreps $= \infty$ |
|------|----------------|-----------------|
| 1    | MBL $\checkmark$ Ex: $\mathbb{Z}_n$ | MBL $\checkmark$ Ex: $U(1)$ |
| $1 < |V| < \infty$ | MBL $\times$ Ex: $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ | MBL $\times$ Ex: $U(1) \rtimes \mathbb{Z}_2$ |
| $|V| = \infty$ | N/A | MBL $\times$ Ex: $SU(2)$ |

$\rightarrow$ MBL+SSB (or QCG?) $\rightarrow$ MBL+SSB $\rightarrow$ Thermalization only

**TABLE I. Symmetry constraints on MBL:** Possible phases of an isolated interacting system at strong disorder in terms of the representation theory of its symmetry group $G$. The relevant parameters are the number of irreps and the dimension of the largest irrep $|V| = \sup_k (\text{dim} V_k)$. If $G$ is Abelian ($|V|=1$), then a many-body localized phase is possible at strong disorder. If the other hand $G$ is non-Abelian, a symmetry-preserving MBL phase is not allowed, giving rise to either thermalization, MBL with the symmetry spontaneously broken (SSB) to an Abelian subgroup, or to non-trivial quantum critical glasses (QCG) depending on the properties of the symmetry group (see text).

| $|V|$ | # irreps $< \infty$ | # irreps $= \infty$ |
|------|----------------|-----------------|
| 1    | MBL $\checkmark$ Ex: $\mathbb{Z}_n$ | MBL $\checkmark$ Ex: $U(1)$ |
| $1 < |V| < \infty$ | MBL $\times$ Ex: $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ | MBL $\times$ Ex: $U(1) \rtimes \mathbb{Z}_2$ |
| $|V| = \infty$ | N/A | MBL $\times$ Ex: $SU(2)$ |

$\rightarrow$ MBL+SSB (or QCG?) $\rightarrow$ MBL+SSB $\rightarrow$ Thermalization only

**IV. CONSEQUENCES FOR NON-EQUILIBRIUM TOPOLOGICAL PHASES**

**A. Consequences for SPT order**

The above-identified obstruction to MBL rules out the possibility of localization stabilized SPT order (or Floquet SPT order$^{35,54-57}$) with non-Abelian symmetry groups such as the Haldane chains with continuous $SO(3)$ symmetry — as these phases require both symmetry and MBL to occur at high energy density. This further constrains the many-body localizability of SPT phases$^{34,35}$.

We remark that these results apply also to anti-unitary symmetries such as time reversal symmetry (TRS). The notion of local action of TRS is in general somewhat subtle, due to the nominally global action of complex conjugation. However, for MBL states, which by definition permit a tensor product state description, one may readily construct a well-defined local action of time-reversal$^{16,59}$. A notable case, is that of spin-1/2 electrons with time-reversal symmetry. In a putative TRS MBL state of such particles, electronic excitations would exhibit a local two-fold Kramers degeneracy, spoiling the stability of the localized phase. In particular, this rules out the possibility of 2D and 3D TRS electron topological insulators$^{42}$ in MBL systems.

In fact, this and related obstructions rule out the possibility of physically realizing any fermionic topological insulator in physically accessible dimensions ($d \leq 3$) in the 10-fold way classification$^{60,61}$, for the following reasons. First, any of the topological superconducting classes require a pair condensate, which in ultra-cold atomic systems in which MBL may be realized$^{62-65}$, implies the existence of a superfluid Goldstone mode which will lead to thermalization$^{38,51-53}$. Next, any non-superconducting TI class has either Kramers doublet fermions ($T^2 = -1$), a particle hole symmetry (leading to non-Abelian group structure)$^{37}$, or chiral edge states$^{34,48}$ — any of which prevent symmetry-preserving MBL. Whether any fermion SPT outside the 10-fold way is suitable for MBL protec-
tion can be examined on a case-by-case basis using the above criteria.

B. Localization of Anyons

Symmetry can also lead to new topologically ordered phases and our results immediately imply that such symmetry enriched topological (SET) phases with non-Abelian symmetry cannot be many-body localized. Moreover, our arguments also rule out MBL protection of classes of SET order in which the global symmetry group is Abelian, but where the local symmetry action on fractionalized anyons is projective (requiring multidimensional local degeneracy) and hence acts like a non-Abelian symmetry. Examples of this class of phases include discrete gauge theories in which the electric domain walls are decorated by one-dimensional SPTs, such that the electric charge excitations transform as the ends of 1D SPTs, and hence have symmetry protected degeneracy that prevents symmetry preserving MBL of generic excited states in which such excitations are present at finite density at random locations.

Even without any additional global symmetry, our argument can be naturally generalized to topologically ordered systems in 2+1 dimensions with non-Abelian anyonic excitations. If such systems could be many-body localized, the finite density of exponentially localized non-Abelian anyons in generic eigenstates would lead to an exponential degeneracy of eigenstates (the quantum dimension of the anyons playing the role of the dimension of the irreps in our previous discussion). This forbids area-law entangled MBL phases with non-Abelian topological order, and simply reflects the fact that the topological Hilbert space of non-Abelian anyons does not have a local tensor product structure and that the notion of topological charge cannot be made local. In general, we expect interacting anyons to either thermalize or to form more exotic non-ergodic states that cannot be described in terms of independent 1-bits, such as the QCG phase in 1D.

C. Localizability of anyonic edge modes

The constraints on the localization of anyons discussed above also have consequences for one-dimensional “trenches” of non-Abelian anyons $\psi$ with quantum dimension $d_\psi$, such as the 1D chain of Majorana bound states ($d_\psi = \sqrt{2}$) that emerges from gapping out the edge of a 2D TI or fractional TI by proximity to alternating ferromagnetic and superconducting regions. Focusing on the topological low-energy (in-gap) sector, we can ask whether the 1D topological phase with anyonic edge modes obtained by dimerizing the couplings can be protected to finite energy density (within the topological sector) using MBL. In the perfectly dimerized limit, the eigenstates of such a system consist of two dangling anyonic edge modes and of anyonic excitations resulting from the fusion $\psi \times \psi$ on the bulk dimerized bonds. Our discussion implies that an MBL phase away from the perfectly dimerized limit can exist if and only if the anyons appearing when fusing $\psi$ with itself all have dimension one, which can occur only if $d_\psi^2 = p$ is an integer — corresponding to the so-called parafermionic zero modes that generalize Majorana fermions ($p = 2$).

Only such parafermionic edge modes can be protected by MBL, which while interesting for topological quantum computing applications, are not enough to realize a set of universal quantum gates. One-dimensional chains of anyons whose braiding would provide a universal gate set (such as Fibonacci anyons for example) cannot be many-body localized even by strongly dimerizing the couplings, and instead generically thermalize (at weak disorder), or form a non-ergodic QCG phase (at strong disorder) consistent with recent real-space renormalization group results.

V. DISCUSSION AND GENERALIZATIONS

In this paper, we showed that MBL is not possible with symmetry groups that protect degeneracies (i.e. that have multidimensional irreps). This “no-go theorem” relies on a specific definition of MBL in terms of local integrability, and the existence of a complete set of local conserved quantities (“1-bits”) so that all eigenstates are smoothly connected by a quasi-local unitary transformation to a zero correlation length limit. This 1-bit picture has become central to our current understanding of area-law entangled MBL phases, and underlies all of the phenomenology of MBL systems (absence of transport, logarithmic dephasing etc.). The existence of MBL phases beyond the 1-bit picture is controversial, and could include systems with many-body mobility edges for instance. We emphasize here that our argument can be naturally extended beyond the local integrability picture, and would rule out tentative MBL phases that would escape the 1-bit description. The key point is that our argument relies only on the existence of local excitations over a symmetric eigenstate (say, the groundstate) that transform nicely under the symmetry. Even if the 1-bit picture breaks down, excitations of an MBL system should be local, and we expect that they should naturally transform under irreps of the symmetry group. A finite density of such local excitations transforming according to irreps of dimension larger than one would immediately lead to exponentially degenerate eigenstates, which are inherently unstable. While this argument can be made essentially rigorous within the 1-bit picture, we expect the main idea to be fairly general so that it would rule out MBL phases without an 1-bit description as well (if such phases do exist).

Acknowledgments. — We thank T. Morimoto, S. Parameswaran and A. Vishwanath for insightful discussions. This work was supported by the Gordon and Betty
Moore Foundation’s EPiQS Initiative through Grant GBMF4307 (ACP) and the Quantum Materials Program at LBNL (RV).

1. Z.-C. Gu and X.-G. Wen, Phys. Rev. B 80, 155131 (2009).
2. X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 84, 235128 (2011).
3. A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B 83, 075102 (2011).
4. F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B 85, 075125 (2012).
5. L. Fidkowski and A. Kitaev, Phys. Rev. B 83, 075103 (2011).
6. X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science 338, 1604 (2012).
7. C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005).
8. B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science 314, 1756 (2006).
9. L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. 98, 106803 (2007).
10. J. E. Moore and L. Balents, Phys. Rev. B 75, 121306 (2007).
11. R. Roy, Phys. Rev. B 79, 195522 (2009).
12. M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
13. X.-G. Wen, Phys. Rev. B 65, 165113 (2002).
14. J. Maciejko, X.-L. Qi, A. Karch, and S.-C. Zhang, Phys. Rev. Lett. 105, 246809 (2010).
15. B. Swingle, M. Barkeshli, J. McGreevy, and T. Senthil, Phys. Rev. B 83, 195139 (2011).
16. M. Levin and A. Stern, Phys. Rev. B 86, 115131 (2012).
17. A. Mesaros and Y. Ran, Phys. Rev. B 87, 155115 (2013).
18. L. Fleishman and P. Anderson, Phys. Rev. B 21, 2366 (1980).
19. I. Gornyi, A. Mirlin, and D. Polyakov, Phys. Rev. Lett. 95, 206603 (2005).
20. D. Basko, I. Aleiner, and B. Altshuler, Annals of Physics 321, 1126 (2006).
21. V. Oganesyan and D. A. Huse, Phys. Rev. B 75, 155111 (2007).
22. A. Pal and D. A. Huse, Phys. Rev. B 82, 174411 (2010).
23. B. Swingle, Phys. Rev. B 89, 115104 (2014).
24. C. Nandy, D. A. Huse, and A. Vishwanath, Phys. Rev. X 4, 041004 (2014).
25. M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. 110, 266601 (2013).
26. M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. 111, 127201 (2013).
27. D. A. Huse, R. Nandkishore, and V. Oganesyan, Phys. Rev. B 90, 174204 (2014).
28. Y. Bahri, R. Vosk, E. Altman, and A. Vishwanath, Nat Commun 5 (2014).
29. M. Serbyn, M. Nknap, Gopalakrishnan, Z. Papić, N. Y. Yao, C. R. Laumann, D. A. Abanin, M. D. Lukin, and E. A. Demler, Phys. Rev. Lett. 113, 147204 (2014).
30. E. M. Demler, Int. J. Mod. Phys. B 30, 014206 (2014).
31. D. A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S. L. Sondhi, Phys. Rev. B 88, 041404 (2013).
32. D. Bekker, G. Refael, E. Altman, E. Demler, and V. Oganesyan, Phys. Rev. X 4, 011052 (2014).
33. A. Chandran, V. Khemani, C. R. Laumann, and S. L. Sondhi, Phys. Rev. B 89, 144201 (2014).
34. A. C. Potter and A. Vishwanath, ArXiv e-prints (2015).
35. K. Slagle, Z. Bi, Y.-Z. You, and C. Xu, ArXiv e-prints (2015).
36. A. C. Potter, A. J. Friedman, S. A. Parameswaran, and S. L. Sondhi, Phys. Rev. B 91, 075125 (2015).
37. R. Vasseur, A. C. Potter, and S. A. Parameswaran, Phys. Rev. Lett. 114, 217201 (2015).
38. A. C. Potter, T. Morimoto, and A. Vishwanath, ArXiv e-prints (2016), arXiv:1602.05194 [cond-mat.str-el].
39. R. Vosk and E. Altman, Phys. Rev. Lett. 110, 067204 (2013).
40. J. Z. Imbrie, ArXiv e-prints (2014), arXiv:1403.7837 [math-ph].
41. M. Znidaric, T. Pronen, and P. Prelovsek, Phys. Rev. B 77, 064426 (2008).
42. A. Lazarides, A. Das, and R. Moessner, Phys. Rev. Lett. 115, 030402 (2015).
43. P. Ponte, Z. Papić, F. m. c. Hunevee, and D. A. Abanin, Phys. Rev. Lett. 114, 140401 (2015).
44. W. G. De Roeck, F. Hunevee, M. Müller, and M. Schiulaz, Phys. Rev. B 93, 014203 (2016).
45. S. D. Geraedts, R. N. Bhatt, and R. Nandkishore, ArXiv e-prints (2016), arXiv:1608.01328 [cond-mat.stat-mech].
46. X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 87, 155114 (2013).
47. R. Vosk and E. Altman, Phys. Rev. Lett. 112, 217204 (2014).
48. R. Nandkishore and A. C. Potter, Phys. Rev. B 90, 195115 (2014).
49. Y.-Z. You, X.-L. Qi, and C. Xu, Phys. Rev. B 93, 104205 (2016).
50. V. Oganesyan, A. Pal, and D. A. Huse, Phys. Rev. B 80, 115104 (2009).
51. S. John, H. Sonpolsky, and M. J. Stephen, Phys. Rev. B 27, 5592 (1983).
52. V. Gurarie and J. T. Chalker, Phys. Rev. B 68, 134207 (2003).
53. S. Banerjee, Phys. Rev. Lett. 116, 116601 (2016).
54. T. Iadecola, L. H. Santos, and C. Chamon, Phys. Rev. B 92, 121507 (2015).
55. C. W. von Keyserlingk and S. L. Sondhi, ArXiv e-prints (2016), arXiv:1602.02157 [cond-mat.str-el].
56. C. W. von Keyserlingk and S. L. Sondhi, ArXiv e-prints (2016), arXiv:1602.06949 [cond-mat.str-el].
57. D. V. Else and C. Nayak, ArXiv e-prints (2016), arXiv:1602.04804 [cond-mat.str-el].
58. I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. 59, 799 (1987).
59. X. Chen and A. Vishwanath, Phys. Rev. X 5, 041034 (2015).
60. A. Kitaev, AIP Conference Proceedings 1134, 22 (2009).
Appendix A: Finite-depth unitary mapping from SPT to paramagnet – an example

In this section, we explicitly construct an example of a finite depth unitary transformation that converts the SPT to a trivial paramagnet. We consider a discrete version of the Haldane spin chain with symmetry group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). The chain contains spin-1/2 degrees of freedom, and has two-sublattices (even and odd sites). The zero-correlation length paramagnet reads:

\[
H_{PM} = - \sum_{i=1}^{2L} h_i \sigma^x_i.
\]  

(A1)

The symmetry generators are \( g_e = \prod_i \sigma^z_i \), \( g_o = \prod_i \sigma^z_i \sigma^z_{i+1} \), which flip the spins about the z-axis on the even and odd sublattice respectively.

The SPT phase is described by zero-correlation length Hamiltonian:

\[
H_{SPT} = - \sum_{i=2}^{2L-1} J_i \sigma^z_{i-1} \sigma^z_i \sigma^z_{i+1}.
\]  

(A2)

Since \( H_{SPT} \) is a non-trivial SPT phase, then by definition, we cannot continuously deform the eigenstates of \( H_{SPT} \) to those of the trivial paramagnet \( H_{PM} \). Namely, there is no continuous family of symmetry-preserving unitary operators \( U(\lambda) \) for \( \lambda \in [0, 1] \), where \( U(0) = 1 \) and \( U(1) = U_{SPT} \), where \( U_{SPT} \) maps the SPT states to trivial ones.

However, if we sacrifice the continuity, we can write down the end result, \( U(\lambda = 1) \), as a finite-depth unitary circuit, which is all that is required to establish the local representation of symmetry. An explicit construction that does the job is

\[
U_{SPT} = e^{-i\pi/4 \sum_i \sigma^x_i \sigma^x_{i+1}},
\]  

(A3)

which takes \( \sigma^x_i \rightarrow \sigma^z_i \sigma^x_i \sigma^z_{i+1} \) everywhere in the bulk of the chain (see eq. [81]). The unitary operator \( U_{SPT} \) is finite depth, and preserves the form of the symmetry generators \( g_e \) and \( g_o \), except at the boundaries of the system.

While we have constructed an explicit example of the desired finite-depth unitary \( U_{SPT} \) for this particular symmetry class, similar constructions can be made for any general SPT class.\(^{46}\)