Linearization of two dimensional complex-linearizable systems of second order ordinary differential equations

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Abstract. Complex-linearization of a class of systems of second order ordinary differential equations (ODEs) has already been studied with complex symmetry analysis. Linearization of this class has been achieved earlier by complex method, however, linearization criteria and the most general linearizable form of such systems have not been derived yet. In this paper, it is shown that the general linearizable form of the complex-linearizable systems of two second order ODEs is (at most) quadratically semi-linear in the first order derivatives of the dependent variables. Further, linearization conditions are derived in terms of coefficients of system and their derivatives. These linearizable 2-dimensional complex-linearizable systems of second order ODEs are characterized here, by adopting both the real and complex procedures.

1. Introduction

Most of the algorithms constructed to solve differential equations (DEs) with symmetry analysis involve an invertible change of the dependent and/or independent (point transformations) variables. For solving nonlinear DEs, symmetry analysis uses a tool called linearization, which maps them to linear equations under invertible change of the variables. Linearization procedure requires; the most general forms of the DEs that could be candidates of linearization and linearization criteria that ensure existence of invertible transformations from nonlinear to linear equations. Though construction of point transformations and finally getting to an analytic solution of the concerned problem are also involved in linearization process, these
issues are of secondary nature as one needs to first investigate linearity of DEs. An explicit linearizable form and linearization criteria for the scalar second order ODEs have been derived by Sophus Lie (see, e.g., [1]). Similarly, linearization of higher order scalar ODEs and systems of these equations attracted a great deal of interest and studied comprehensively over the last decade (see, e.g., [2]-[7]).

Complex symmetry analysis has been employed to solve certain classes of systems of nonlinear ODEs and linear PDEs. Of particular interest here, is linearization of systems of second order ODEs (see, e.g., [8]-[11]) that is achieved by complex methods. These classes are obtained from linearizable scalar and systems of ODEs by considering their dependent variables as complex functions of a real independent variable, which when split into the real and imaginary parts give two dependent variables. In this way, a scalar ODE produces a system of two coupled equations, with Cauchy-Riemann (CR) structure on both the equations. These CR-equations appear as constraint equations that restrict the emerging systems of ODEs to special subclasses of the general class of such systems. These subclasses of 2-dimensional systems of second order ODEs may trivially be studied with real symmetry analysis, however, they appear to be nontrivial when viewed from complex approach. Complex-linearizable (c-linearizable) classes explored earlier [8]-[11] and studied in this paper provide us means to extend linearization procedure to \( m \)-dimensional systems (\( m \geq 3 \), of \( n^{th} \) order (\( n \geq 2 \)) ODEs. Though these classes are subcases of the general \( m \)-dimensional systems of \( n^{th} \) order ODEs, their linearization has not been achieved yet, with real symmetry analysis. Presently symmetry classification and solvability of higher dimensional systems of higher order ODEs seems to be exploitable only with complex symmetry analysis.

When linearizable scalar second order ODEs are considered complex by taking the dependent variable as a complex function of a real independent variable, they lead to c-linearization. The associated linearization criteria that consist of two equations (see, e.g., [1]) involving coefficients of the second order equations and their partial derivatives of (at most) order two, also yield four constraint equations for the corresponding system of two ODEs on splitting the complex functions involved, into the real and imaginary parts. These four equations constitute the c-linearization criteria [8], for the corresponding class of systems of two second order ODEs. The reason for calling them c-linearization instead of linearization criteria is that, in earlier works, explicit Lie procedure to obtain linearization conditions of this class of systems, was not performed after incorporating complex symmetry approach on scalar ODEs. The most general form of the c-linearizable 2-dimensional linearizable systems of second order ODEs is obtained here by real and complex methods. This derivation shows that the general linearizable forms (obtained by real and complex procedures) of 2-dimensional c-linearizable systems of second order ODEs are identical. Moreover, associated linearization criteria have
been derived, again by adopting both the real and complex symmetry methods. These linearization conditions are also shown to be similar whether derived from real Lie procedure developed for systems or by employing complex symmetry analysis on scalar ODE. The core result obtained here is refinement of the c-linearization conditions to linearization criteria for 2-dimensional systems of second order ODEs, obtainable from linearizable complex scalar second order ODEs.

The plan of the paper is as follows. The second section presents derivation of the linearizable form for the scalar second order ODEs and Lie procedure to obtain associated linearization criteria. The subsequent section is on the linearization of 2-dimensional c-lineariable systems of second order ODEs, by real and complex symmetry methods. The fourth section contains some illustrative examples. The last section concludes the paper.

2. A subclass of linearizable scalar second order ODEs

The following point transformations

\[ \tilde{x} = \phi(x, u), \quad \tilde{u} = \psi(x, u), \]  

(1)

where \( \phi \) and \( \psi \) are arbitrary functions of \( x \) and \( u \), yield the most general form of linearizable scalar second order ODEs

\[ u'' + \alpha(x, u)u^3 + \beta(x, u)u^2 + \gamma(x, u)u' + \delta(x, u) = 0, \]  

(2)

with four arbitrary coefficients, that is cubically semi-linear in the first order derivative of the dependent variable, for derivation see [1]. Restricting these transformations to

\[ \tilde{x} = \phi(x), \quad \tilde{u} = \psi(x, u), \]  

(3)

i.e., assuming \( \phi_u = 0 \), leads to a quadratically semi-linear scalar second order ODE that is derived here explicitly. Under transformations (3) the first and second order derivatives of \( \tilde{u}(\tilde{x}) \) with respect to \( \tilde{x} \) read as

\[ \tilde{u}' = \frac{D\psi(x, u)}{D\phi(x)} = \lambda(x, u, u'), \]  

(4)

and

\[ \tilde{u}'' = \frac{D\lambda(x, u, u')}{D\phi(x)} = \mu(x, u, u', u''), \]  

(5)
respectively. Here

\[ D = \frac{\partial}{\partial x} + u' \frac{\partial}{\partial u} + u'' \frac{\partial}{\partial u'} + \cdots, \]

(6)
is the total derivative operator. Inserting the total derivative operator in both the above equations leads us to the following

\[ \tilde{u}' = \frac{\psi_x + u' \psi_u}{\phi_x}, \]

(7)
and

\[ \tilde{u}'' = \frac{\phi_x (\psi_{xx} + 2u' \psi_{xu} + u'' \psi_{uu} + u'' \psi_u) - \phi_{xx} (\psi_x + u' \psi_u)}{\phi_x^3}, \]

(8)
respectively. Equating (8) to zero, i.e., considering \( \tilde{u}'' = 0 \), leaves a quadratically semi-linear ODE of the form

\[ u'' + a(x, u) u^2 + b(x, u) u' + c(x, u) = 0, \]

(9)
with the coefficients

\[ a(x, u) = \frac{\psi_{uu}}{\psi_u}, \quad b(x, u) = \frac{2 \phi_x \psi_{xu} - \psi_u \phi_{xx}}{\phi_x \psi_u}, \quad c(x, u) = \frac{\phi_x \psi_{xx} - \psi_u \phi_{xx}}{\phi_x \psi_u}. \]

(10)
The quadratic nonlinear (in the first derivative) equation (9) with three coefficients (10) is a subcase of the general linearizable (cubically semi-linear) second order ODE (2).

Now for the derivation of Lie linearization criteria of nonlinear equation (9), we start with a re-arrangement

\[ \psi_{uu} = a(x, u) \psi_u, \]
\[ 2 \psi_{xu} = \phi_x^{-1} \psi_u \phi_{xx} + b(x, u) \psi_u, \]
\[ \psi_{xx} = \phi_x^{-1} \psi_x \phi_{xx} + c(x, u) \psi_u. \]

(11)
of the relations (10). Equating the mixed derivatives of \( \psi \), such that \((\psi_{xu})_u = (\psi_{uu})_x \) and \((\psi_{xu})_x = (\psi_{xx})_u \), we find

\[ b_u - 2a_x = 0, \]

(12)
and

\[ \phi_x^{-2} (2 \phi_x \phi_{xx} - 3 \phi_{xx}^2) = 4(c_u + ac) - (2b_x + b^2). \]

(13)
As \( \phi_u = 0 \), differentiating (13) with respect to \( u \), simplifies it to

\[ c_{uu} - a_{xx} - a_x b + a_u c + c_u a = 0. \]

(14)
Equations (12) and (14) constitute the linearization criteria for the scalar second order quadratically semi-linear ODEs [5].
3. Linearizable two dimensional c-linearizable systems of second order ODEs

To obtain the most general form of a linearizable 2-dimensional c-linearizable system of second order ODEs we consider two approaches viz, by complex symmetry analysis and by real symmetry analysis.

3.1 Use of complex symmetry analysis

Suppose \( u(x) \) in (9) be complex function of a real variable \( x \) i.e., \( u(x) = y(x) + iz(x) \). Further assume that

\[
a(x, u) = a_1(x, y, z) + ia_2(x, y, z),
\]
\[
b(x, u) = b_1(x, y, z) + ib_2(x, y, z),
\]
\[
c(x, u) = c_1(x, y, z) + ic_2(x, y, z).
\] (15)

This converts the scalar ODE (9) to a system of two second order ODEs of the form

\[
y'' + a_1y' - b_1y' - c_1 + 0,
\]
\[
z'' + a_2y' + b_2y' + c_2 = 0,
\] (16)

with the coefficients \( a_j, b_j, c_j; (j = 1, 2) \), satisfying the CR-equations

\[
a_{1,y} = a_{2,z}, \quad a_{1,z} = -a_{2,y},
\]
\[
b_{1,y} = b_{2,z}, \quad b_{1,z} = -b_{2,y},
\]
\[
c_{1,y} = c_{2,z}, \quad c_{1,z} = -c_{2,y}.
\] (17)

Moreover, conditions (12) and (14) can now be converted into a set of four equations

\[
2a_{1,x} - b_{1,y} = 0,
\] (18)
\[
2a_{2,x} + b_{1,z} = 0,
\] (19)
\[
c_{1,zz} + a_{1,xx} + a_{1,x}b_1 - a_{2,xx}b_2 - (a_2c_1)_z - (a_1c_2)_z = 0,
\] (20)
\[
c_{2,yy} - a_{2,xx} - a_{2,x}b_1 - a_{1,xx}b_2 + (a_2c_1)_y + (a_1c_2)_y = 0,
\] (21)

by splitting the complex coefficients (17) into the real and imaginary parts.

As evident from [8], such a (complex) procedure leads us to c-linearization of systems of ODEs. Our claim here is that equations (18-21) are actually the linearization conditions
despite of being just the c-linearization conditions for system (16). In order to prove this fact, we now adopt real symmetry analysis in the next subsection to derive the lineariation conditions for system (16).

3.2 Use of real symmetry analysis

The previous work on c-linearizable [8, 11] and their linearizable subclass of systems [9, 10] of second order ODEs reveals that point transformations of the form

\[ \tilde{x} = \phi(x), \quad \tilde{y} = \psi_1(x, y, z), \quad \tilde{z} = \psi_2(x, y, z), \]

where

\[ \psi_{1,y} = \psi_{2,z}, \quad \psi_{2,y} = -\psi_{1,z}, \]

i.e., \( \psi_j \), for \( j = 1, 2 \), satisfy the CR-equations that involve derivatives with respect to both the dependent variables, linearizes the c-linearizable systems. Notice that (22) are obtainable from (3) that is a subclass of (1). These transformations map the first and second order derivatives as

\[ \tilde{y}' = \frac{D\psi_1}{D\phi} = \lambda_1(x, y, z, y', z'), \quad \tilde{z}' = \frac{D\psi_2}{D\phi} = \lambda_2(x, y, z, y', z'), \]

and

\[ \tilde{y}'' = \frac{D\lambda_1}{D\phi} = \mu_1(x, y, z, y', z', y'', z''), \quad \tilde{z}'' = \frac{D\lambda_2}{D\phi} = \mu_2(x, y, z, y', z', y'', z''), \]

where

\[ D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} + y'' \frac{\partial}{\partial y'} + z'' \frac{\partial}{\partial z'} + \cdots. \]

Inserting the total derivative operator in the above equations and simplifying, we arrive at the following 2-dimensional system

\[ y'' + \alpha_1 y'^2 - 2\alpha_2 y' z' + \alpha_3 z'^2 + \beta_1 y' - \beta_2 z' + \gamma_1 = 0, \]
\[ z'' + \alpha_4 y'^2 + 2\alpha_5 y' z' + \alpha_6 z'^2 + \beta_3 y' + \beta_4 z' + \gamma_2 = 0, \]

where

\[ \alpha_1 = \phi_x \Delta^{-1}(\psi_{2,z} \psi_{1,yy} - \psi_{1,z} \psi_{2,yy}), \quad \alpha_2 = \phi_x \Delta^{-1}(\psi_{1,z} \psi_{2,yz} - \psi_{2,z} \psi_{1,yz}), \]
\[ \alpha_3 = \phi_x \Delta^{-1}(\psi_{2,z} \psi_{1,zz} - \psi_{1,z} \psi_{2,zz}), \quad \alpha_4 = \phi_x \Delta^{-1}(\psi_{1,y} \psi_{2,yz} - \psi_{2,y} \psi_{1,yz}), \]
\[ \alpha_5 = \phi_x \Delta^{-1}(\psi_{1,y} \psi_{2,yy} - \psi_{2,y} \psi_{1,yy}), \quad \alpha_6 = \phi_x \Delta^{-1}(\psi_{1,y} \psi_{2,zz} - \psi_{2,y} \psi_{1,zz}), \]
\[ \beta_1 = 2\phi_x \Delta^{-1}(\psi_{2,z}\psi_{1,xy} - \psi_{1,z}\psi_{2,xy}) - \frac{\phi_{xx}}{\phi_x}, \quad \beta_2 = 2\phi_x \Delta^{-1}(\psi_{1,z}\psi_{2,xx} - \psi_{2,z}\psi_{1,xx}), \]
\[ \beta_3 = 2\phi_x \Delta^{-1}(\psi_{1,y}\psi_{2,xy} - \psi_{2,y}\psi_{1,xy}), \quad \beta_4 = 2\phi_x \Delta^{-1}(\psi_{1,y}\psi_{2,xx} - \psi_{2,y}\psi_{1,xx}) - \frac{\phi_{xx}}{\phi_x}, \]

and
\[ \gamma_1 = \Delta^{-1}(\phi_x \psi_{1,y}\psi_{1,xx} - \psi_{1,z} \phi_{xx} - \phi_{x}\psi_{1,z}\psi_{2,xx} + \psi_{1,z}\psi_{2,xx} \phi_{xx}), \]
\[ \gamma_2 = \Delta^{-1}(\phi_x \psi_{1,z}\psi_{1,xx} - \psi_{1,z} \phi_{xx} + \phi_{x}\psi_{1,y}\psi_{2,xx} + \psi_{1,y}\psi_{2,xx} \phi_{xx}), \]

(28)

where
\[ \Delta = \phi_x (\psi_{1,y}\psi_{2,z} - \psi_{1,z}\psi_{2,y}) \neq 0, \]

is the Jacobian of the transformation (22). The coefficients (10) of the scalar ODE (9) split into the coefficients of the corresponding 2-dimensional system of second order ODEs. This happens due to presence of the complex dependent function \( u \), in the coefficients (10). The restricted fibre preserving transformations (22) used to derive the linearizable form (27), are obtainable from the complex transformations (3) that are employed to deduce (9). Therefore, transformations (22) along with (23) appear to be the real and imaginary parts of complex transformation (3), they reveal the correspondence of the linearizable forms of 2-dimensional systems and scalar complex ODEs. The CR-equations are not yet incorporated in the linearizable form (27). Insertion of the CR-equations (23) and their derivatives
\[ \psi_{1,yy} = \psi_{2,yz} = -\psi_{1,zz}, \]
\[ \psi_{2,zz} = \psi_{1,yz} = -\psi_{2,yy}, \]

(30)
brings out the correspondence between the coefficients (10) of the complex linearizable ODEs (9) and coefficients (28) of the system (27). Employing (23) and (30) the coefficients (28) reduces to \textit{only six arbitrary} coefficients that read as
\[ \alpha_1 = -\alpha_3 = \alpha_5 = a_1, \quad \alpha_2 = \alpha_4 = -\alpha_6 = a_2, \]
\[ \beta_1 = \beta_4 = b_1, \quad \beta_2 = \beta_3 = b_2, \quad \gamma_1 = c_1, \quad \gamma_2 = c_2. \]

(31)

Here the coefficients \( a_j \), \( b_j \) and \( c_j \) are the real and imaginary parts of the complex coefficients (10). The linearizable form of systems derived in this section by real method appears to be the same as one obtains by splitting the corresponding form of the scalar complex equation (9). This analysis leads us to the following theorem.

**Theorem 1.** The most general form of the linearizable two dimensional c-linearizable systems of second order ODEs is quadratically semi-linear.
3.2.1 Sufficient conditions for the linearization of a c-linearizable system

Consider the most general form of the c-linearizable 2-dimensional systems of second order ODEs (16), with constraint equations (17). Rewriting the coefficients of the system (16) in the form

\[ a_1 = \Delta^{-1} \phi_x (\psi_{1,y} \psi_{1,yy} + \psi_{1,z} \psi_{1,yz}), \]
\[ a_2 = \Delta^{-1} \phi_x (\psi_{1,z} \psi_{1,yy} + \psi_{1,y} \psi_{1,yz}), \]
\[ b_1 = 2\Delta^{-1} \phi_x (\psi_{1,y} \psi_{1,xy} + \psi_{1,z} \psi_{1,xz}) - \frac{\phi_{xx}}{\phi_x}, \]
\[ b_2 = 2\Delta^{-1} \phi_x (\psi_{1,z} \psi_{1,xy} + \psi_{1,y} \psi_{1,xz}), \]
\[ c_1 = \Delta^{-1} (\phi_x \psi_{1,y} \psi_{1,xx} - \psi_{1,x} \psi_{1,y} \phi_{xx} - \phi_x \psi_{1,z} \psi_{1,xx} + \psi_{1,z} \psi_{2,x} \phi_{xx}), \]
\[ c_2 = \Delta^{-1} (\phi_x \psi_{1,z} \psi_{1,xx} - \psi_{1,x} \psi_{1,z} \phi_{xx} + \phi_x \psi_{1,y} \psi_{2,xx} + \psi_{1,y} \psi_{2,x} \phi_{xx}). \] (32)

For obtaining the sufficient linearizability conditions of (16), we have to solve compatibility problem, that has already been solved for the scalar equations earlier in this work, for the set of equations (32). It is an over determined system of partial differential equations for the functions \( \phi, \psi_1 \) and \( \psi_2 \) with known \( a_j, b_j, c_j \).

The system (32) gives us

\[ \psi_{1,yy} = \psi_{1,y}a_1 + \psi_{1,z}a_2, \]
\[ \psi_{1,zz} = \psi_{1,z}a_1 - \psi_{1,y}a_2, \]
\[ \psi_{1,xy} = \frac{1}{2}(\psi_{1,y}b_1 + \psi_{1,z}b_2 + \psi_{1,y} \phi_{xx}), \]
\[ \psi_{1,xz} = \frac{1}{2}(\psi_{1,z}b_1 - \psi_{1,y}b_2 + \psi_{1,z} \phi_{xx}), \]
\[ \psi_{1,xx} = \psi_{1,y}c_1 + \psi_{1,z}c_2 + \psi_{1,x} \phi_{xx}, \]
\[ \psi_{2,xx} = \psi_{1,y}c_2 - \psi_{1,z}c_1 + \psi_{2,x} \phi_{xx}. \]

The compatibility of the system (32) first requires to compute partial derivatives

\[ \Delta_x = 2\Delta \frac{\phi_{xx}}{\phi_x} + \Delta b_1, \]
\[ \Delta_y = 2\Delta a_1, \]
\[ \Delta_z = -2\Delta a_2, \]
of the Jacobian. Comparing the mixed derivatives \((\Delta_y)_z = (\Delta_z)_y, (\Delta_x)_y = (\Delta_y)_x\) and \((\Delta_x)_z = (\Delta_z)_x\), we obtain

\[
\begin{align*}
a_{1,z} + a_{2,y} &= 0, \\
2a_{1,x} - b_{1,y} &= 0, \\
2a_{2,x} + b_{1,z} &= 0,
\end{align*}
\]

respectively. Equating the mixed derivatives \((\psi_{1,yy})_z = (\psi_{1,yz})_y, (\psi_{1,yy})_x = (\psi_{1,xy})_y, (\psi_{1,xx})_y = (\psi_{1,xz})_x, (\psi_{1,xx})_z = (\psi_{1,xz})_y, (\psi_{2,xx})_y = (\psi_{2,xy})_y\) and \((\psi_{2,xx})_z = (\psi_{2,xz})_x\) gives us

\[
\begin{align*}
a_{1,y} - a_{2,z} &= 0, \\
b_{2,y} + b_{1,z} &= 0, \\
b_{2,z} - b_{1,y} &= 0, \\
c_{2,z} - c_{1,y} &= 0, \\
c_{2,y} + c_{1,z} &= 0, \\
c_{1,zz} + a_{1,xx} + a_{1,x}b_1 - a_{2,x}b_2 - (a_2c_1)_z - (a_1c_2)_z &= 0, \\
c_{2,yy} - a_{2,xx} - a_{2,x}b_1 - a_{1,z}b_2 + (a_1c_2)_y - (a_2c_1)_y &= 0.
\end{align*}
\]

Note that \((\psi_{1,yz})_x - (\psi_{1,xz})_y = 0\) and \((\psi_{1,xy})_z - (\psi_{1,yz})_x = 0\) are satisfied. Also \((33), (36)-(40)\) are CR-equations for the coefficients \(a_j, b_j, c_j\). Therefore, the solution of the compatibility problem of the system \((32)\), provides CR-constraints on the coefficients of \((16)\) and the linearization conditions.

**Theorem 2.** A two dimensional c-linearizable system of second order ODEs of the form \((16)\) is linearizable if and only if its coefficients satisfy the CR-equations and conditions \((34), (35), (41), (42)\).

These are the same conditions that are already obtained \((18)-(21)\), by employing complex analysis, i.e., splitting the linearization conditions associated with the base scalar equation \((9)\), into the real and imaginary parts.

**Corollary.** The c-linearization conditions for a two dimensional system of quadratically semi-linear second order ODEs are the linearization conditions.
4. Examples

We present some examples to illustrate our results.

1. The 2-dimensional system of second order ODEs

\[
y'' - \left(\frac{2y}{y^2 + z^2}\right)y'^2 - 2\left(\frac{2z}{y^2 + z^2}\right)y'z' + \left(\frac{2y}{y^2 + z^2}\right)z'^2 - \frac{2y}{x^2} - \frac{2y}{x^2} = 0, \\
z'' + \left(\frac{2z}{y^2 + z^2}\right)y'^2 - 2\left(\frac{2z}{y^2 + z^2}\right)y'z' - \left(\frac{2z}{y^2 + z^2}\right)z'^2 - \frac{2z}{x^2} - \frac{2z}{x^2} = 0.
\]

is of the same form as (16) with

\[
a_1 = -\frac{2y}{y^2 + z^2}, \quad a_2 = \frac{2z}{y^2 + z^2}, \quad b_1 = -\frac{2}{x}, \quad b_2 = 0, \quad c_1 = -\frac{2y}{x^2}, \quad c_2 = -\frac{2z}{x^2}.
\]

One can easily verify that (45) satisfy the conditions (34), (35), (41), (42) and CR-equations w.r.t \(y\) and \(z\). So the system of ODEs (44) is linearizable. The transformation

\[
t = x, \quad u = \frac{y}{x(y^2 + z^2)}, \quad v = \frac{-z}{x(y^2 + z^2)},
\]

reduces the nonlinear system (44) to the linear system \(u'' = 0, v'' = 0\).

2. Consider the following system of nonlinear ODEs

\[
y'' - \frac{1}{f(y, z)}(y'^2 \cos y \sin y - z'^2 \cos y \sin y - 2y'z' \cosh z \sinh z) + \frac{2y'}{x} = 0, \\
z'' - \frac{1}{f(y, z)}(y'^2 \cosh z \sinh z - z'^2 \cosh z \sinh z + 2y'z' \cosh y \sinh y) + \frac{2z'}{x} = 0,
\]

where \(f(y, z) = \sin^2 y \cosh^2 z + \cos^2 y \sinh^2 y\), and the coefficients satisfy the CR-constraint and linearization conditions (34), (35), (41), (42). Hence Theorem 2 guarantees that system (47) can be transformed to system of linear equations \(u'' = 0, v'' = 0\). The linearizing transformations in this case are

\[
t = x, \quad u = x \cos y \cosh z, \quad v = -x \sin y \sinh z.
\]

3. Consider the anisotropic oscillator system

\[
y'' + f(x)y = 0, \\
z'' + g(x)z = 0.
\]

In [6] it is shown that system (49) is reducible to the free particle system \((u'' = 0, v'' = 0)\) provided \(f = g\). Our c-linearization criteria also leads to the same condition, i.e. \(f = g\).
5. Conclusion

C-linearization of 2-dimensional systems of second order ODEs is achieved earlier by considering the scalar second order linearizable ODEs as complex. Their associated linearization criteria are separated into the real and imaginary parts due to complex functions involved. In this work, the c-linearization and linearization are shown to be two different criteria for a 2-dimensional systems of second order ODEs. Linearizable form of such c-linearizable systems has been derived and it is shown to be quadratically semi-linear in the first order derivatives. Moreover, complex linearization criteria have been refined to linearization criteria for such 2-dimensional systems that are linearizable due to their correspondence with the complex scalar ODEs.

Earlier in this work, c-linearizable classes of systems of ODEs are claimed to be non-trivial, when viewed from complex approach. The reason for calling them non-trivial is that the concept of c-linearization of systems of ODEs is extendable to $m$-dimensional systems of $n^{th}$ order ODEs. The simplest procedure that might lead us to linearization of $m$-dimensional system of second order ODEs, is to iteratively complexify a scalar second order linearizable ODE. Therefore, complex symmetry analysis needs to be extended to 2- and 3-dimensional systems of third and second order ODEs, respectively, in order to derive the general linearization results mentioned above. Likewise, complex symmetry analysis may lead us to algebraic classification of the higher dimensional systems of higher order ODEs.

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