KIRCHHOFF’S THEOREMS IN HIGHER DIMENSIONS
AND REIDEMEISTER TORSION

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Abstract. Using ideas from algebraic topology and statistical mechanics, we generalize Kirchhoff’s network and matrix-tree theorems to finite CW complexes of arbitrary dimension. As an application, we give a formula expressing Reidemeister torsion as an enumeration of higher dimensional spanning trees.

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1. Introduction

Gustav Kirchhoff’s results on electrical networks, which predate Maxwell’s theory of electromagnetism, are a product of the mid-19th century [Ki1], [Ki2]. Kirchhoff’s network theorem states that in any resistive network there is a unique current satisfying Ohm’s law and Kirchhoff’s current and voltage laws, and furthermore this current can be explicitly computed. The first complete treatment of the network theorem is attributed to Hermann Weyl [W] in 1923. By the mid-20th century, algebraic topology provided key ideas leading to a simple and
elegant proof [E],[R],[NS]. A companion result is Kirchhoff’s matrix-tree theorem which gives a formula for the number of spanning trees in a finite connected graph (see [Mo] for a history of this result). This paper is an outgrowth of our investigations on the interplay between algebraic topology and statistical mechanics [CKS1], [CKS2], [CKS3]. Our aim is to generalize Kirchhoff’s results to higher dimensions, as well as to connect these results to the theory of Reidemeister torsion.

A high dimensional network theorem. Suppose $X$ is a finite connected CW complex of dimension $d$. Let $C_j(X;\mathbb{R})$ denote the cellular chain complex of $X$ with real coefficients and the standard inner product $\langle , \rangle$ for which the set of $j$-cells, denoted $X_j$, is an orthonormal basis. In what follows we fix a function $r: X_d \rightarrow \mathbb{R}_+$; the value of $r$ at a $d$-cell $b$ is considered to be the resistance of $b$. Define a linear transformation $R: C_d(X;\mathbb{R}) \rightarrow C_d(X;\mathbb{R})$ by mapping a $d$-cell $b$ to $rb$ and extending linearly. Let $B_{d-1}(X;\mathbb{R}) \subset C_{d-1}(X;\mathbb{R})$ be the vector subspace of $(d-1)$-boundaries and let $Z_d(X;\mathbb{R}) \subset C_d(X;\mathbb{R})$ be the vector subspace of $d$-cycles.

Definition 1.1. A network problem for $X$ consists of a choice of $p \in B_{d-1}(X;\mathbb{R})$ and $q \in Z_d(X;\mathbb{R})$, respectively called $(d-1)$-boundary current and $d$-cycle voltage. A solution consists of $V, J \in C_d(X;\mathbb{R})$ such that

1. $V = RJ$, \hspace{3cm} (Ohm’s law)
2. $\partial J = p$, \hspace{3cm} (current law)
3. $\langle V, z \rangle = \langle q, z \rangle$, \hspace{3cm} for all $z \in Z_d(X)$. \hspace{2cm} (voltage law)

To see why a solution exists, define a modified inner product $\langle , \rangle_R$ on $C_d(X;\mathbb{R})$ by $\langle b, b' \rangle_R = \langle Rb, b' \rangle$ for $b, b' \in X_d$. Let

$\partial_R^*: C_{d-1}(X;\mathbb{R}) \rightarrow C_d(X;\mathbb{R})$

denote the formal adjoint to $\partial$ using the standard inner product on $C_{d-1}(X;\mathbb{R})$ and the modified inner product on $C_d(X;\mathbb{R})$. Let $B_R^d(X;\mathbb{R})$ be the image of $\partial_R^*$ and note that $B_R^d(X;\mathbb{R})$ is the orthogonal complement to $Z_d(X;\mathbb{R})$ in $C_d(X;\mathbb{R})$ with respect to the modified inner product. Elementary linear algebra implies $\partial: B_R^d(X;\mathbb{R}) \rightarrow B_{d-1}(X;\mathbb{R})$

\footnote{When $d = 1$, in the terminology of Roth [R], $p$ is a node current and $q$ is a mesh voltage, each arising from an external source. Bollobás [B, p. 41] only considers the case when $q = 0$ and $p$ is of the form $p_i i + p_j j$ for a pair of distinct vertices $i$ and $j.$}
is an isomorphism. Consequently, there is a unique $J_0 \in B^1_d(X; \mathbb{R})$ such that $\partial J_0 = p$. Set $V_0 = RJ_0$. Then $\langle V_0, z \rangle = \langle J_0, z \rangle_R = 0$ for all $z \in Z_d(X; \mathbb{R})$. Let $J_1$ be the orthogonal projection of $R^{-1}q$ onto $Z_d(X; \mathbb{R})$ in the modified inner product, and set $x = J_1 - R^{-1}q$. Then $\langle RJ_1 - q, z \rangle = \langle x, z \rangle_R = 0$ for all $z \in Z_d(X; \mathbb{R})$. Set $V_1 = RJ_1$. Then $J := J_0 + J_1$ and $V := V_0 + V_1$ solve the network problem. It is straightforward to show that this solution is unique.

The above solution to the network problem uses the orthogonal projection of $C_d(X; \mathbb{R})$ onto $Z_d(X; \mathbb{R})$ in the modified inner product. In the classical case $d = 1$, Kirchhoff gave a formula expressing the orthogonal projection as a weighted sum indexed over the set of spanning trees of $X$. To get an explicit formula in higher dimensions we will need a notion of spanning tree.

**Definition 1.2.** Assume as above that $X$ is a connected finite CW complex of dimension $d$. A spanning tree for $X$ is a subcomplex $T$ such that

- $H_d(T; \mathbb{Z}) = 0$,
- $\beta_{d-1}(T) = \beta_{d-1}(X)$, where $\beta_k(X)$ denotes the $k$-th Betti number,
- $X^{(d-1)} \subset T$, where $X^{(k)}$ is the $k$-skeleton of $X$.

**Remark 1.3.** We will show in the next section such spanning trees exist. The reader should be aware that the literature contains sundry notions of “high dimensional spanning tree.” \(^2\) Note that when $d = 1$, our definition reduces to the classical notion of spanning tree.

**Definition 1.4.** For a spanning tree $T$ of $X$, define a linear transformation

$$
\bar{T} : C_d(X; \mathbb{R}) \rightarrow Z_d(X; \mathbb{R})
$$

as follows: Let $b$ be a $d$-cell. If $b$ is contained in $T$ then we set $\bar{T}(b) = 0$. Otherwise, note that $H_d(T \cup b; \mathbb{Z}) = Z_d(T \cup b; \mathbb{Z})$ is infinite cyclic. Let $c$ be a generator. Set $t_b = \langle c, b \rangle$ (this is always non-zero). Then $\bar{T}(b) := c/t_b$, is a real $d$-cycle of $X$. It is easy to see that $\bar{T}(b)$ is independent of the choice of $c$.

\(^2\)For example, [DKM] replaces condition (2) with the requirement that the reduced Betti number $\bar{\beta}_{d-1}(T)$ is trivial. This implies $\bar{\beta}_{d-1}(X)$ is trivial as well, so the notion of spanning tree in [DKM] does not apply to a general finite complex $X$. See [P] for a detailed discussion of the various notions.
Let $\theta_T$ denote the order of the torsion subgroup of $H_{d-1}(T; \mathbb{Z})$ and define the weight of $T$ to be the positive real number

$$w_T := \theta_T^2 \prod_{b \in T_d} r_b^{-1}.$$ 

**Theorem A** (Higher Projection Formula). With respect to the modified inner product $\langle \ , \ \rangle_R$, the orthogonal projection $C_d(X; \mathbb{R}) \to Z_d(X; \mathbb{R})$ is given by

$$\frac{1}{\Delta} \sum_T w_T \bar{T},$$

where the sum is over all spanning trees, and $\Delta = \sum_T w_T$.

Let $\partial^* : C_{d-1}(X; \mathbb{R}) \to C_d(X; \mathbb{R})$ be the formal adjoint to the boundary operator with respect to the standard inner product. Define $B^d(X, \mathbb{R})$ to be the image of $\partial^*$. Then we have

**Addendum B** (Higher Network Theorem). Given a vector $V \in C_d(X; \mathbb{R})$, there is only one vector $z \in Z_d(X; \mathbb{R})$ such that $V - Rz \in B^d(X, \mathbb{R})$. Furthermore, for each $d$-cell $b$, we have

$$\langle z, b \rangle = \frac{1}{\Delta} \sum_T \frac{w_T}{r_b} \langle V, \bar{T}(b) \rangle.$$

*Remark 1.5.* In classical network terminology ($d = 1$), $\langle V, b \rangle$ is the voltage source on branch $b$ and $\langle z, b \rangle$ is the current resulting in branch $b$ (see [R],[NS]).

**A high dimensional matrix-tree theorem.** The classical matrix-tree theorem enumerates the number of spanning trees of a graph. In higher dimensions, the best we can achieve is an expression for $\sum_T \theta_T^2$.

Observe that $B_{d-1}(X; \mathbb{R})$ is an invariant subspace of the operator

$$\partial \partial^*_R : C_{d-1}(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R}).$$

Let

$$L^R : B_{d-1}(X; \mathbb{R}) \xrightarrow{\cong} B_{d-1}(X; \mathbb{R})$$

denote the associated restriction.

**Theorem C** (Higher Weighted Matrix-Tree Theorem). We have

$$\det L^R = \gamma_X \sum_T w_T,$$

where the sum is indexed over all spanning trees of $X$, and the normalizing factor is given by

$$\gamma_X = \frac{\mu_X}{\theta_X^2},$$
where $\mu_X \in \mathbb{N}$ is the square of the covolume of the lattice $B_{d-1}(X; \mathbb{Z}) \subset B_{d-1}(X; \mathbb{R})$ with respect to the restriction of the standard inner product of $C_{d-1}(X; \mathbb{R})$ and $\theta_X$ is the order of the torsion subgroup of $H_{d-1}(X; \mathbb{Z})$.

The unweighted case when $r: X_d \to \mathbb{R}_+$ is constant with value 1 is worth singling out, as it gives rise to the operator

$$\mathcal{L} = \partial \partial^*: B_{d-1}(X; \mathbb{R}) \xrightarrow{\sim} B_{d-1}(X; \mathbb{R}).$$

In this case $w_T = \theta_X^2$ and Theorem C becomes

**Corollary D (Higher Matrix-Tree Theorem).** For $\mathcal{L}$ as above, we have

$$\det \mathcal{L} = \gamma_X \sum_T \theta_T^2.$$

**Remark 1.6.** Variations of Corollary D have appeared in [Ka], [P], [DKM] and [L] (note: all but the last reference assume additional conditions on $X$, and each work utilizes its own notion of spanning tree). When $d = 1$, we have $\theta_T = 1 = \theta_X$ and $\mu_X$ is the number of vertices of $X$, so we obtain the classical Kirchhoff matrix-tree theorem. Theorem C is actually a special case of a more general result, Theorem 6.6 below.

Corollary D admits the following simpler reformulation (cf. Corollary 6.10 below).

**Addendum E.**

$$\det \mathcal{L} = \sum_T \mathcal{L}^T = \sum_T \mu_T,$$

where $\mathcal{L}^T = \partial \partial^*_T: B_{d-1}(T; \mathbb{R}) \xrightarrow{\sim} B_{d-1}(T; \mathbb{R})$.

**Reidemeister torsion counts spanning trees.** Milnor [M] introduced the notion of Reidemeister torsion $\tau(C_*)$ of a not necessarily acyclic finite chain complex $C_*$ over a field in which a preferred basis is chosen for $C_*$ as well as its homology. When $C_*$ is the real chain complex of a finite CW complex $X$, we will establish a connection between the torsion and the enumeration of spanning trees on the skeleta of $X$.

Suppose $X$ is a finite, connected CW complex. We give $C_*(X; \mathbb{R})$ the preferred basis given by its set of cells. We also choose a basis for $H_*(X; \mathbb{R})$ by selecting a basis for the torsion free part of each integral homology group $H_*(X; \mathbb{Z})$. Such a basis is called a combinatorial basis for the homology and we will denote it by $h$. The definition of Reidemeister torsion $\tau(X; h)$ is given in §7 below.

For $k \geq 0$, we define the following quantities:

- $T_k = \text{the set of spanning trees of } X^{(k)}$ (for this we require $k > 0$).
\( \mu_k \) = the square of the covolume of the lattice \( B_k(X; \mathbb{Z}) \subset B_k(X; \mathbb{R}) \) with respect to the inner product given by restricting the standard inner product on \( C_k(X; \mathbb{R}) \).

\( H_k(X; \mathbb{Z})_0 \) = the image of the evident homomorphism \( H_k(X; \mathbb{Z}) \to H_k(X; \mathbb{R}) \).

\( \eta_k \) = the square of the covolume of the lattice \( H_k(X; \mathbb{Z})_0 \subset H_k(X; \mathbb{R}) \), where we give \( H_k(X; \mathbb{R}) \) the inner product defined by identifying the latter with the orthogonal complement of \( B_k(X; \mathbb{R}) \subset Z_k(X; \mathbb{R}) \) using the inner product arising from the standard one on \( C_k(X; \mathbb{R}) \).

\( \theta_k \) = the order of the torsion subgroup of \( H_k(X; \mathbb{Z}) \).

With respect to the above, we set
\[ \delta_k := \frac{\eta_k \mu_k}{\theta_k^2}. \]

Then \( \delta_k \) is defined entirely in terms of \( X \).

**Theorem F** (Torsion-Tree Theorem). For a finite, connected CW complex \( X \), we have
\[ \tau^2(X; \mathfrak{h}) = \prod_{k \geq 0} (\delta_k \sum_{T \in \mathcal{T}_{k+1}} \theta^2_T (-1)^k), \]
where \( \theta^2_T \) denotes the order of the torsion subgroup of \( H_k(T; \mathbb{Z}) \) for \( T \in \mathcal{T}_{k+1} \).

**Conventions.** We assume the reader is familiar with basic linear algebra as well as a first year course on algebraic topology. The topological spaces of this paper are equipped with preferred CW structure and when we write \( H_*(X; A) \), we mean cellular homology with coefficients in an abelian group \( A \) (in practice, \( A \) is either \( \mathbb{Z} \) or \( \mathbb{R} \)). If \( X \) is a CW complex, we write \( X^{(k)} \) for its \( k \)-skeleton and \( X_k \) for its set of \( k \) cells. Thus
\[ X^{(k)} = X^{(k-1)} \cup (X_k \times D^k), \]
where the union is amalgamated along the attaching map \( X \times S^{k-1} \to X^{(k-1)} \). The \( k \)-th Betti number \( \beta_k(X) \) is defined to be the rank of the vector space \( H_k(X; \mathbb{R}) \). If \( A \) is a commutative ring, then the \( k \)-th real chain group \( C_k(X; A) \) is by definition the relative homology group \( H_k(X^{(k)}, X^{(k-1)}; A) \), which is just the free \( A \)-module having basis \( X_k \).

**Outline.** In §2 we develop basic results about higher dimensional spanning trees. In §3 we prove Theorem A and Addendum B. In §4 we prove Theorem C up to identification of the normalizing constant \( \gamma_X \). In §5 we introduce the low temperature limit and use it to show that for sufficiently well-behaved \( W \), the determinant of \( \mathcal{L} \) tends in the low
temperature limit to the determinant of $L^T$, where the latter is defined using a spanning tree $T$ in place of $X$. This result is employed in §6 to identify $\gamma_X$, thereby completing the proof of Theorem C; in so doing we generalize Theorem C to Theorem 6.6. Lastly, in §7, we outline Milnor’s definition of Reidemeister torsion and prove Theorem F. Also, in Theorem 7.11, we obtain a different expression for the Reidemeister torsion that is expressed in terms of both spanning tree and homology truncation data for $X$.

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2. Spanning Trees in higher dimensions

Definition 2.1. Let $X$ be a finite connected CW complex. A $k$-cell $b \in X_k$ is said to be essential if there exists a $k$-cycle $z \in Z_k(X; \mathbb{R})$ such that $\langle z, b \rangle \neq 0$.

Lemma 2.2. Assume in addition $X$ has dimension $d$. Then adding or removing an essential $d$-cell from $X$ increases or decreases $\beta_d(X)$ by one, respectively, and fixes $\beta_{d-1}(X)$.

Proof. Construct a decreasing filtration $Y^i$ on $X$ by removing the $d$-cells of $X$ one at a time, $X = Y^n \supset Y^{n-1} \supset \ldots \supset Y^0 = X_{d-1}$. Then for $1 \leq j \leq n$, we have an exact sequence in homology

$$0 \to H_d(Y^{j-1}) \to H_d(Y^j) \xrightarrow{\partial_*} \mathbb{Z} \to H_{d-1}(Y^{j-1}) \to H_{d-1}(Y^j) \to 0$$

The above factors into two short exact sequences

$$0 \to H_d(Y^{j-1}) \to H_d(Y^j) \xrightarrow{\im \partial_*} \mathbb{Z} \to \mathbb{Z}/\im \partial_* \to H_{d-1}(Y^{j-1}) \to H_{d-1}(Y^j) \to 0,$$

where $\im \partial_*$ is the image of $\partial_*$. If the attached cell is essential, then $\im \partial_*$ is a nontrivial subgroup of $\mathbb{Z}$. Therefore, the first sequence implies $\beta_d(Y^j) = \beta_d(Y^{j-1}) + 1$, while the second implies $\beta_{d-1}(Y^j) = \beta_{d-1}(Y^{j-1})$. We may view $Y^j$ as a complex with an additional essential cell, or $Y^{j-1}$ as a complex with an essential cell removed. \qed

Lemma 2.3. $X$ has a spanning tree.
Proof. If \( H_d(X; \mathbb{R}) = 0 \) then \( X \) is a spanning tree. If \( H_d(X; \mathbb{R}) \neq 0 \), then we can pick an essential \( d \)-cell and remove it, decreasing \( \beta_d(X) \) by one. Repeat this process until \( \beta_d \) is zero. Evidently, the resulting subcomplex \( T \) contains \( X_{d-1} \) and by Lemma 2.2, we have \( \beta_{d-1}(T) = \beta_{d-1}(X) \). Hence, \( T \) is a spanning tree. □

The following is straightforward, and its proof is left to the reader.

**Lemma 2.4.** Any spanning tree for \( X \) may be obtained by removing essential \( d \)-cells. Furthermore, if \( T \) is a spanning tree of \( X \), the number of essential \( d \)-cells withdrawn to construct \( T \) is equal to \( \beta_d(X) \).

**Lemma 2.5.** Let \( T \) be a spanning tree of \( X \) and let \( \tilde{T} = T \cup b \), where \( b \) is an essential cell in \( \tilde{T} \). If \( b' \) is an essential \( d \)-cell of \( \tilde{T} \) different from \( b \), then \( U := \tilde{T} \setminus b' \) is a spanning tree.

Proof. Since \( b' \) is essential, Lemma 2.2 implies \( H_d(U) \) has rank zero. This lemma also implies \( \beta_{d-1}(U) = \beta_{d-1}(\tilde{T}) = \beta_{d-1}(T) \). Since our construction leaves the \( d-1 \) skeleton fixed, \( U \) is a spanning tree. □

**Lemma 2.6.** Let \( T \) be a spanning tree of \( X \) and let \( b \in X_d \setminus T_d \). Then \( [\partial b] \) generates a torsion element of \( H_{d-1}(T; \mathbb{Z}) \).

Proof. Since \( T \) is a spanning tree, we have that \( b \) is attached to \( T \) along its attaching map \( \partial b \to T \). Hence, the homology class \([\partial b]\) lies in \( H_{d-1}(T; \mathbb{Z}) \). The isomorphism \( H_{d-1}(T; \mathbb{R}) \cong H_{d-1}(X; \mathbb{R}) \), along with the fact that \( \partial b \) bounds the cellular chain \( b \) in \( X \), implies \( \partial b \) is torsion in \( H_{d-1}(T; \mathbb{Z}) \). □

Recall the linear transformation \( \bar{T}: C_d(X; \mathbb{R}) \to Z_d(X; \mathbb{R}) \) defined in the introduction, which was defined on essential cells as
\[
\bar{T}(b) = \frac{c}{t_b},
\]
where \( c \) is a generator of \( H_d(T \cup b; \mathbb{Z}) \) and
\[
t_b = \langle c, b \rangle,
\]
where the inner product is taken in \( C_d(X; \mathbb{R}) \) (here we are using the inclusion \( H_d(T \cup b; \mathbb{R}) \subset C_d(X; \mathbb{R}) \) to make sense of the inner product). Observe that \( |t_b| \) is the order of \([\partial b] \in H_{d-1}(T; \mathbb{Z})\).

**Lemma 2.7.** Let \( T \) be a spanning tree of \( X \), let \( b_i \in X_d \setminus T_d \) and let \( b_j \) be an essential \( d \)-cell. Then
\[
\langle \bar{T}(b_i), b_j \rangle = \frac{t_{b_j}}{t_{b_i}}.
\]

Proof. \( \langle \bar{T}(b_i), b_j \rangle = \langle c/t_{b_i}, b_j \rangle = 1/t_{b_i} \langle c, b_j \rangle = t_{b_j}/t_{b_i} \). □
Corollary 2.8. If \( U \) is a spanning tree obtained by adding and then removing an essential cell from a spanning tree \( T \) as above, then
\[
\langle \bar{T}(b_i), b_j \rangle (b_i, U(b_j)) = 1. 
\]

Lemma 2.9. For an essential \( d \)-cell \( b \), the class \( [\partial b] \in H_{d-1}(T; \mathbb{Z}) \) is a torsion element of order \( |t_b| \). In particular, there is a short exact sequence
\[
0 \to \mathbb{Z}/t_b\mathbb{Z} \to H_{d-1}(T; \mathbb{Z}) \to H_{d-1}(T \cup b; \mathbb{Z}) \to 0. 
\]

Proof. By Lemma 2.6, \( [\partial b] \) is a torsion class. Let \( t \) be its order. By slight abuse of notation, we let \( \partial b \) denote the cycle representing \( [\partial b] \). Then \( t\partial b \) is also a cycle, which is also the boundary of a unique integral \( d \)-chain \( w \in C_d(T \cup b; \mathbb{Z}) \). It is straightforward to check that \( tb - w \) is a generator of \( H_d(T \cup b; \mathbb{Z}) = \mathbb{Z}_d(T \cup b; \mathbb{Z}) \). Then \( (tb - w, b) = t \). It follows that \( t = \pm t_b \). The short exact sequence is a direct consequence. \( \square \)

For a finite CW complex \( Y \) of dimension \( d \), let \( \theta_Y \) denote the order of the torsion subgroup of \( H_{d-1}(Y; \mathbb{Z}) \).

Corollary 2.10. For \( T, U, b_i \), and \( b_j \) as above,
\[
\theta_T^2 \langle \bar{T}(b_i), b_j \rangle = \theta_U^2 \langle b_i, U(b_j) \rangle. 
\]

Proof. Set \( t_i := t_{b_i} \) and let \( Y = T \cup b_i = U \cup b_j \). Then the exact sequence
\[
0 \to \mathbb{Z}/t_i\mathbb{Z} \to H_{d-1}(T; \mathbb{Z}) \to H_{d-1}(Y; \mathbb{Z}) \to 0 
\]
gives \( |t_i|\theta_Y = \theta_T \) and similarly \( |t_j|\theta_Y = \theta_U \). Consequently,
\[
\theta_T^2 \langle \bar{T}(b_i), b_j \rangle = \theta_U^2 \langle b_i, U(b_j) \rangle. \quad \square 
\]

3. Proof of Theorem A and Addendum B

The proof will proceed along the lines given in [NS] in the classical setting. Given a spanning tree \( T \), let \( \{b_1, \ldots, b_k\} \) elements of \( X_d \setminus T_d \).

Lemma 3.1. The collection \( \bar{T}(b_1), \ldots, \bar{T}(b_k) \) is a basis for \( Z_d(X; \mathbb{R}) \).

Proof. Recall that \( Z_d(X; \mathbb{R}) = H_d(X; \mathbb{R}) \). Let \( q_x \colon X \to X/T \) be the quotient map. Then the homomorphism \( q_x^* \colon H_d(X; \mathbb{R}) \to H_d(X/T; \mathbb{R}) \) is an isomorphism, and \( H_d(X/T; \mathbb{R}) \) is the vector space with basis \( b_1, \ldots, b_k \). It’s straightforward to check that \( q_x \circ \bar{T} \colon C_d(X; \mathbb{R}) \to H_d(X/T; \mathbb{R}) \) maps a \( d \)-cell \( b \) to itself when \( b \in X_d \setminus T_d \) and is zero otherwise. \( \square \)

Corollary 3.2. For any \( z \in Z_d(X; \mathbb{R}) \), we have \( \bar{T}(z) = z \).
Proof. Use the Lemma 3.1 to write $z = \sum_i s_i \bar{T}(b_i)$. Then
\[
\bar{T}(z) = \sum_i s_i \bar{T}(b_i) = \sum_i s_i \bar{T}(b_i) = z . \tag*{□}
\]

Lemma 3.3. For distinct $d$-cells $b_i, b_j \in X$, let $T_{ij}$ be the set of spanning trees such that $\langle \bar{T}(b_i), b_j \rangle \neq 0$. Then
\[
\sum_{T \in T_{ij}} w_T \langle \bar{T}(b_i), b_j \rangle_R = \sum_{U \in T_{ji}} w_U \langle \bar{b}_i, U(b_j) \rangle_R .
\]

Proof. From the definition of the weights, have $r_j w_T = r_i w_U$. Note that $\langle \bar{T}(b_i), b_j \rangle_R = r_j$. Using Corollary 2.10, we infer
\[
\theta^2_T \langle \bar{T}(b_i), b_j \rangle_R = \theta^2_U \langle \bar{b}_i, U(b_j) \rangle_R .
\]
Now sum up over all $T \in T_{ij}$. □

Proof of Theorem A. Consider the operator $F := \sum_T w_T \bar{T}$, where the sum is over all spanning trees of $X$. For any pair of $d$-cells $b_i$ and $b_j$ of $X$ we have
\[
\langle \sum_T w_T \bar{T}(b_i), b_j \rangle_R = \sum_{T \in T_{ij}} w_T \langle \bar{T}(b_i), b_j \rangle_R
\]
\[
\quad = \sum_{U \in T_{ji}} w_U \langle \bar{b}_i, U(b_j) \rangle_R \quad \text{by Lemma 3.3 ,}
\]
\[
\quad = \langle b_i, \sum_{U} w_U \bar{U}(b_j) \rangle_R
\]
\[
\quad = \langle b_i, \sum_T w_T \bar{T}(b_j) \rangle_R
\]
Hence $F$ is self-adjoint in the modified inner product.

If $z$ is a cycle, then $F(z) = (\sum_T w_T)z =: \Delta z$. Consequently, $(1/\Delta)F$ restricts to the identity on $Z_d(X; \mathbb{R})$. As $(1/\Delta)F$ is self-adjoint, it is the orthogonal projection in the modified inner product. □

Proof of Addendum B. Let $z$ be the orthogonal projection of $R^{-1}V$ in the modified inner product. Then $R^{-1}V - z \in B^d_R(X; \mathbb{R})$, i.e.,
\[
0 = \langle R^{-1}V - z, z' \rangle_R = \langle V - Rz, z' \rangle
\]
for all $z' \in Z_d(X; \mathbb{R})$. Hence, $V - Rz \in B^d(X; \mathbb{R})$. The uniqueness of $z$ is a consequence of the fact that $B^d(X; \mathbb{R})$ is the orthogonal complement to $Z_d(X; \mathbb{R})$ in the standard inner product.
The proof of the last part is given by direct calculation using the self-adjointness of the operator \( \sum_T w_T \bar{T} \):

\[
\langle z, b \rangle = \frac{1}{r_b} \langle z, b \rangle_R,
\]

\[
= \frac{1}{r_b} \langle \frac{1}{\Delta} \sum_T w_T R^{-1} \bar{V}, b \rangle_R,
\]

\[
= \frac{1}{\Delta} \sum_T \frac{w_T}{r_b} \langle R^{-1} \bar{V}, T(b) \rangle_R,
\]

\[
= \frac{1}{\Delta} \sum_T \frac{w_T}{r_b} \langle \bar{V}, \bar{T}(b) \rangle_R.
\]

\[\square\]

4. A weak form of Theorem C

The goal of this section is to show how Theorem A implies Theorem C up to the identification of the prefactor \( \gamma \). The prefactor will be computed in §6, where in addition we prove an enhanced version of Theorem C.

Recall the given function \( r: X_d \rightarrow \mathbb{R}_+ \) of §1. It is convenient to set \( W := \ln r: X_d \rightarrow \mathbb{R} \).

Then \( r_b = e^{W_b} \), and we may also write \( R = e^W: C_d(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R}) \).

Conversely, given any function \( W: X_d \rightarrow \mathbb{R} \), we set \( r := e^W: X_d \rightarrow \mathbb{R}_+ \).

It is convenient to think of \( W \) as lying in \( C_d(X; \mathbb{R}) \) by representing it as \( \sum_{b \in X_d} W_b b \).

Recall that to each spanning tree \( T \) we associated the weight \( w_T = \theta_T^2 \prod_{b \in T_d} r_b^{-1} \),

where \( \theta_T \) is the order of the torsion subgroup of \( H_{d-1}(T; \mathbb{Z}) \).

**Remark 4.1.** Let \( M \) be a smooth manifold and let \( V \) be a finite dimensional real vector space. Suppose \( f: M \rightarrow V \) is a smooth map. Then the directional derivative defines a \( V \)-valued, smooth, differential 1-form \( df \in \Omega^1(M; V) \). In the special case when \( M = U \) is a finite dimensional real vector space, then \( \Omega^1(M; V) \) can be identified with the space of smooth maps \( U \rightarrow \text{hom}(U, V) \).

Consider the linear operator

\[
\partial \partial_R^* = \partial e^{-W} \partial^*: C_{d-1}(X; \mathbb{R}) \rightarrow C_{d-1}(X; \mathbb{R}) .
\]

Since the image of \( \partial \partial_R^* \) is contained in \( B_{d-1}(X; \mathbb{R}) \), restriction of this operator to \( B_{d-1}(X; \mathbb{R}) \) gives an isomorphism

\[
\mathcal{L}(W): B_{d-1}(X; \mathbb{R}) \xrightarrow{\cong} B_{d-1}(X; \mathbb{R}) .
\]

For \( R = e^W \), \( \mathcal{L}(W) \) is the operator \( \mathcal{L}^R \) defined in §1.
We can regard $W \mapsto \mathcal{L}(W)$ as defining a smooth map
\begin{equation}
\mathcal{L} : C_d(X; \mathbb{R}) \rightarrow \text{end}(B_{d-1}(X; \mathbb{R})) ,
\end{equation}
which is a family of linear operators parametrized by $C_d(X; \mathbb{R})$. To avoid notational clutter, when $W$ is understood, we will often write $\mathcal{L}(W)$ without referring to its argument. Therefore, $\mathcal{L}$ can refer to either (5) or (4).

**Proposition 4.2.** Theorem A implies the identity
\begin{equation}
d\ln \det \mathcal{L} = d \ln \sum_T w_T.
\end{equation}

**Remark 4.3.** In keeping with our notational ambiguity, the left side of the display in Proposition 4.2 is to be interpreted as the value at $W$ of $d \ln \det \mathcal{L} \in \Omega^1(C_d(X; \mathbb{R}); \mathbb{R})$.

Proposition 4.2 is equivalent to the statement
\begin{equation}
\det \mathcal{L} = \gamma \sum_T w_T.
\end{equation}
for a suitable positive constant $\gamma$, as yet to be determined. This gives Theorem C modulo the determination of the prefactor $\gamma$.

**Proof of Proposition 4.2.** We take the differential of the natural logarithm of $\det \mathcal{L}$:
\begin{align}
d \ln \det \mathcal{L} &= d \text{tr} \ln \mathcal{L} \\
&= \text{tr} d(\ln \mathcal{L}) \\
&= \text{tr}(\mathcal{L}^{-1} d\mathcal{L}),
\end{align}
where $d\mathcal{L} = \partial d e^{-W} \partial^* = -\partial dW e^{-W} \partial^*$.

The cyclic property of the trace implies
\begin{equation}
\text{tr}(\mathcal{L}^{-1} d\mathcal{L}) = - \text{tr}(\partial dW e^{-W} \partial^* \mathcal{L}^{-1}).
\end{equation}
If we set $A := e^{-W} \partial^* \mathcal{L}^{-1} : B_{d-1}(X; \mathbb{R}) \rightarrow B^d_R(X; \mathbb{R})$, then $\text{tr}(\mathcal{L}^{-1} d\mathcal{L}) = - \text{tr}(\partial dWA) = - \text{tr}(dWA \partial)$. Consequently,
\begin{align}
d \text{tr} \ln \mathcal{L} &= - \text{tr}(dWA \partial) \\
&= - \sum_{b \in X_d} \langle b | dWA \partial | b \rangle \\
&= - \sum_{b \in X_d} \langle b | dWA | \partial b \rangle \\
&= - \sum_{b \in X_d} dW_b \langle b | A | \partial b \rangle ,
\end{align}
where \( dW_b \) denotes the \( b \)-coordinate function of \( dW \), i.e., \( dW_b(x) = dW(x)(b) = W(b) \), and \( \langle i|H|j \rangle \) stands for the inner product \( \langle i, H(j) \rangle \).

By definition, \( A \) is a left inverse to \( \partial : B^d_R(X; \mathbb{R}) \to B_{d-1}(X; \mathbb{R}) \), so the expression \( \langle b|A|\partial b \rangle \) is the same as \( \langle b, Pb \rangle \), where \( P : C^d(X; \mathbb{R}) \to B^d_R(X; \mathbb{R}) \) is the orthogonal projection in the modified inner product \( \langle \cdot, \rangle_R \). By Theorem A, we have

\[
P = I - \frac{1}{\Delta} \sum_T w_T T,
\]

where \( I \) is the identity operator. By inserting this expression into \( \langle b, Pb \rangle \) and doing some rewriting, we obtain

\[
\langle b|A|\partial b \rangle = \frac{1}{\Delta} \sum_{T,b} w_T,
\]

where \( \Delta = \sum_T w_T \) and the displayed sum is over all spanning trees \( T \) for which \( b \) lies in \( T \). This allows us to rewrite the expression appearing in the last line of Eq. (8) as

\[
\sum_{b \in X_d} dW_b \langle b|A|\partial b \rangle = \frac{1}{\Delta} \sum_T \sum_{b \in T_d} w_T dW_b.
\]

On the other hand, for any spanning tree \( T \) we have

\[
d \ln \sum_T w_T = \frac{1}{\Delta} \sum_T dw_T,
\]

where \( dw_T \) is given by

\[
dw_T = \theta_T^2 d \prod_{b \in T_d} e^{-W_b} = -\sum_{b \in T_d} dW_b w_T.
\]

Inserting Eq. (13) into Eq. (12) gives

\[
d \ln \sum_T w_T = -\frac{1}{\Delta} \sum_T \sum_{b \in T_d} w_T dW_b.
\]

Assembling equations (6), (8), (11), 12 and (14), we conclude

\[
d \ln \det \mathcal{L} = -\frac{1}{\Delta} \sum_T \sum_{b \in T_d} w_T dW_b = d \ln \sum_T w_T. \tag{15}
\]

5. The low temperature limit

Here we compute \( \det \mathcal{L} \) in the low temperature \( \beta \to \infty \) limit for a certain kind of \( W \). We set

\[
\mathcal{L} = \partial e^{-\beta W} \partial^*: B_{d-1}(X; \mathbb{R}) \to B_{d-1}(X; \mathbb{R})
\]

where \( W : X_d \to \mathbb{R} \), and \( \beta \in \mathbb{R}_+ \) represents inverse temperature.
Our freedom in choosing $W$ shows this determinant will tend to $\det \mathcal{L}^T$, where $\mathcal{L}^T$ is $\mathcal{L}$ restricted to a spanning tree.

**Definition 5.1.** Fix a spanning tree $T$ of $X$. A function $W: X_d \to \mathbb{R}$ is **good** if

$$W_\gamma > \sum_{\alpha \in T_d} W_\alpha - k \min_{\alpha \in T_d} W_\alpha$$

for any $\gamma \in X_d \setminus T_d$,

where $k$ is the number of $d$-cells of $X$.

**Proposition 5.2.** For good $W: X_d \to \mathbb{R}$, we have

$$\lim_{\beta \to \infty} \frac{\det \mathcal{L}^T}{\det \mathcal{L}} = 1.$$

Before commencing with the proof, recall the boundary of a $d$-cell $\alpha \in X_d$ is given by

$$\partial \alpha = \sum_{j \in X_{d-1}} b_{\alpha j} \langle \partial \alpha, j \rangle \neq 0$$

where $b_{\alpha j} := \langle \partial \alpha, j \rangle$ is the incidence number of $\alpha$ and $j$. With respect to the standard inner product, the adjoint operator $\partial^*$ on a $(d-1)$-cell $j$ is given by

$$\partial^* j = \sum_{\alpha \in X_d} b_{j \alpha}^* \alpha$$

where $b_{j \alpha}^* := b_{\alpha j}$. A straightforward computation of the matrix elements of $\mathcal{L}$ yields

$$\mathcal{L}_{ij} = \sum_{\alpha \in X_d} e^{-\beta W_\alpha} b_{\alpha i} b_{\alpha j}.$$

**Proof of Proposition 5.2.** Since $C_d(X; \mathbb{R})$ is a real vector space with basis spanned by the set of $d$-cells, $X_d$, we have an orthogonal projection $Q: C_d(X; \mathbb{R}) \to C_d(T; \mathbb{R})$. This allows us to write $\partial^* = \partial_T^* + \tilde{\partial}^*$, where $\partial_T^*$ is defined via the commutative diagram

$$B_{d-1}(X; \mathbb{R}) \xrightarrow{\partial^*} C_d(X; \mathbb{R}) \xrightarrow{\partial_T^*} C_d(T; \mathbb{R}) \xrightarrow{Q} C_d(T; \mathbb{R}).$$

It also enables us to write

$$\mathcal{L} = \mathcal{L}^T + \delta \mathcal{L},$$
where $\mathcal{L}^T = \partial \partial^T$. Together, these imply

$$\mathcal{L}_{ij}^T = \sum_{\alpha \in T_d} e^{-\beta W_\alpha} b_{\alpha i} b_{\alpha j}.$$  

Our choice of good $W$ implies any $e^{-\beta W_\gamma}$ appearing in the expansion of $\delta \mathcal{L}$ must be less than any $e^{-\beta W_\alpha}$ appearing in $\mathcal{L}^T$ and conversely. This also means the matrix elements of $\delta \mathcal{L}$ can be written as a similar sum; the only difference is we instead sum over $\alpha \in X_d \setminus T_d$.

To simplify taking the limit, we compute the quotient of $\det \mathcal{L}$ by $\det \mathcal{L}^T$ and let $\beta \to \infty$. Since $\det \mathcal{L}^T \neq 0$, we may write

$$\frac{\det (\mathcal{L}^T + \delta \mathcal{L})}{\det \mathcal{L}^T} = \frac{\det (I + (\mathcal{L}^T)^{-1} \delta \mathcal{L}) \det \mathcal{L}^T}{\det \mathcal{L}^T}.$$  

It suffices to prove that $(\mathcal{L}^T)^{-1} \delta \mathcal{L}$ tends to the zero operator as $\beta \to \infty$. Equivalently, it is enough to show that the matrix elements of $(\mathcal{L}^T)^{-1} \delta \mathcal{L}$ converge to zero. The first bound is of $\mathcal{L}^T_{ij}$:

$$|\mathcal{L}_{ij}^T| \leq \sum_{\alpha \in T_d} e^{-\beta W_\alpha} |b_{\alpha i} b_{\alpha j}| \leq e^{-\beta \min_\alpha W_\alpha} \sum_\alpha |b_{\alpha i} b_{\alpha j}|.$$  

This can be further bounded by defining $B^T = \max_{ij} \sum_\alpha |b_{\alpha i} b_{\alpha j}|$. Hence, we have

$$(15)$$  

$$|\mathcal{L}_{ij}^T| \leq e^{-\beta \min_{\alpha \in T_d} W_\alpha} B^T.$$  

The standard formula for the inverse of a matrix gives

$$(16)$$  

$$((\mathcal{L}^T)^{-1})_{ij} = \frac{\det \mathcal{L}_{ij}^T}{\det \mathcal{L}^T},$$  

where $\tilde{A}_{ij}$ is the $(i, j)$-th cofactor of $A$. Using the exact expression for the determinant of $\mathcal{L}^T$ appearing in Eq. (21) below, Eq. (20) below and the bound Eq. (15) in the case of the cofactor $\mathcal{L}_{ij}^T$, we obtain the estimate

$$((\mathcal{L}^T)^{-1})_{ij} \leq \frac{e^{-\beta \min_{\alpha \in T_d} W_\alpha}(n-1)! B^T}{e^{-\beta \sum_{\alpha \in T_d} W_\alpha} g_T},$$  

where $g_T = \det(\partial^T \partial T)$ depends only on $T$.

We bound the elements $\delta \mathcal{L}$ similarly by

$$|\delta \mathcal{L}_{jk}| \leq e^{-\beta \min_{\gamma \in X_d \setminus T_d} W_\gamma} B^T,$$

---

There is no circularity here; Eqs. (20) and (21) do not depend on the material in this section.
where $B^T$ is defined in the obvious fashion.

Finally, the matrix elements of $(\mathcal{L}^T)^{-1} \partial \mathcal{L}$ then satisfy the following inequality:

$$(\mathcal{L}^T)^{-1} \partial \mathcal{L})_{ik} \leq \frac{(n-1)!e^{-\beta \min_\alpha W_\alpha}(n-1)(B^T)^{n-1}ne^{-\beta \min_\gamma W_\gamma}B^T}{e^{-\beta \sum_{\alpha \in T_d} W_\alpha g_T}}.$$  

Collecting terms independent of $\beta$ into $N$, we see

$$(\mathcal{L}^T)^{-1} \partial \mathcal{L})_{ik} \leq e^{-\beta \left((n-1) \min_\alpha W_\alpha - \sum_\alpha W_\alpha + \min_\gamma W_\gamma\right)}N$$

where $\alpha \in T_d$ and $\gamma \in X_d \setminus T_d$. Our choice of $W$ forces the matrix elements to 0 as $\beta \to \infty$. Therefore,

$$\lim_{\beta \to \infty} \frac{\det \mathcal{L}}{\det \mathcal{L}^T} = \det I = 1,$$

completing the proof.  

\section{A generalized form of Theorem C}

In this section we will identify the prefactor $\gamma$ appearing in Theorem 6.6. We will also generalize Theorem C in a significant way.

\textbf{Covolume.} If $A$ is a finitely generated abelian group we let

$$A_\mathbb{R} := A \otimes_\mathbb{Z} \mathbb{R}$$

denote its \textit{realification}, and we let $\beta(A) = \dim_\mathbb{R} A_\mathbb{R}$ denote the rank of $A$. Let $t(A)$ be the order of the torsion subgroup of $A$.

For a homomorphism $\alpha : A \to B$ of abelian groups, we denote $\alpha_\mathbb{R} : A_\mathbb{R} \to B_\mathbb{R}$ be the induced homomorphism of real vector spaces.

\textbf{Definition 6.1.} A homomorphism $\alpha : A \to B$ of finitely generated abelian groups is called a \textit{real isomorphism} if the induced homomorphism $\alpha_\mathbb{R} : A_\mathbb{R} \to B_\mathbb{R}$ of real vector spaces is an isomorphism.

Clearly, $\alpha$ is a real isomorphism if and only if its kernel and its cokernel are finite. If $\alpha$ is a real isomorphism, then $\beta(A) = \beta(B)$, where we recall that $\beta(A)$ is the rank of $A$. We will henceforth assume that $A$ and $B$ are free abelian. In this case $\alpha$ is a real isomorphism if and only if $\alpha$ is a monomorphism with finite cokernel.

\textbf{Definition 6.2.} For $\alpha : A \to B$ a real isomorphism with $A$ and $B$ free abelian, we let

$$t(\alpha) \in \mathbb{N}$$

denote the order of the cokernel, i.e., $t(\alpha) := t(B/\alpha(A))$.  

An ordered basis for $A$ determines an ordered basis for $A_R$, and given any pair of ordered bases for $A$, the associated change of basis matrix for $A_R$ has determinant $\pm 1$. This defines an equivalence relation on ordered bases for $A$ with exactly two distinct equivalence classes. A choice of equivalence class is referred to as an orientation of $A$. Consequently, when orientations for $A$ and $B$ are chosen, and $\alpha: A \to B$ is a real isomorphism, then the determinant $\det \alpha \in \mathbb{R}$ is defined and depends only on the choice of orientations. Furthermore, its absolute value $|\det \alpha|$ is well defined and does not depend on the choice of orientations. The latter has the following interpretation: choose an ordered basis for $B$. This defines an inner product on $B_R$ making the ordered basis for $B$ into an orthonormal basis for $B_R$. Then $\alpha(A) \subset B_R$ is a lattice and $|\det \alpha|$ is its covolume, that is, the volume of the torus $B_R/\alpha(A)$ with respect to the induced Riemannian metric, or equivalently, the volume of a fundamental domain of the universal covering $B_R \to B_R/\alpha(A)$.

**Proposition 6.3.** For a real isomorphism $\alpha: A \to B$ of finitely generated free abelian groups we have $|\det \alpha| = t(\alpha)$.

**Proof.** Choose an ordered basis for $B$, and give $B_R$ the induced inner product.

Consider the inclusions $\alpha(A) \subset B \subset B_R$. Then we have a finite covering space
\[ B/\alpha(A) \to B_R/\alpha(A) \to B_R/B, \]
in which the covering projection $B_R/\alpha(A) \to B_R/B$ is a local isometry and $B_R/\alpha(A)$ is the fiber over the basepoint. This shows that the covolume of $\alpha(A)$ is the product of the covolume of $B \subset B_R$ with $|B/\alpha(A)| = t(\alpha)$. But the covolume of $B \subset B_R$ is 1.

**Generalization of Theorem 6.6.** Recall that for $W: X_{d} \to \mathbb{R}$, we have the operator
\[ \mathcal{L}(W) = \partial e^{-W} \partial^* : B_{d-1}(X; \mathbb{R}) \xrightarrow{\cong} B_{d-1}(X; \mathbb{R}) \]
which is just $\mathcal{L}^R = \partial \partial^*_R$, as defined in the introduction, with $R = e^W$. Again, we suppress the argument $W$ from the notation and refer to $\mathcal{L}(W)$ as $\mathcal{L}$.

As we showed earlier in Proposition 4.2, we have the following representation:

\[ \det \mathcal{L} = \gamma \sum_T w_T, \]

where the constant $\gamma$ is still to be determined.
Definition 6.4. Let $A \subset C_{d-1}(X; \mathbb{Z})$ be a subgroup. Define a natural number
$$\mu(A) \in \mathbb{N}$$
as follows: let $\{e_i\}$ be a basis for $A$. Consider the matrix $g$ whose $(i, j)$-entry is given by $g_{ij} = \langle e_i, e_j \rangle$, where the inner product is taken in $C_{d-1}(X; \mathbb{R})$. Set $\mu(A) := \det g$.

Since $e_i$ expressed in the standard basis for $C_{d-1}(X; \mathbb{R})$ has integer components, we infer that $g_{ij} \in \mathbb{Z}$, so $\mu(A)$ is an integer. Alternatively, one can define $\mu(A)$ as the square of the covolume of the lattice $A \subset A_{\mathbb{R}}$ given by restricting the standard inner product of $C_{d-1}(X; \mathbb{R})$ to $A_{\mathbb{R}}$.

The equivalence of the two definitions can be seen as follows: let $B$ be the matrix whose rows are the vectors $e_i$ expressed in an orthonormal basis for $C_{d-1}(X; \mathbb{R})$. Then $|\det B|$ is the covolume of $A \subset A_{\mathbb{R}}$. Furthermore, $g = BB^*$, so $\mu(A) = \det g = (\det B)^2 \in \mathbb{N}$.

For any abelian group $U$, we set
$$B^U_{d-1} := B_{d-1}(X; U),$$
that is, the image of the boundary operator $\partial: C_d(X; U) \to C_{d-1}(X; U)$ of the cellular chain complex of $X$ with $U$ coefficients.

Hypothesis 6.5. The inclusion $A \subset C_{d-1}(X; \mathbb{R})$ is such that the orthogonal projection $P_A: B_{d-1}^{\mathbb{R}} \to A_{\mathbb{R}}$ is induced by a real isomorphism $p_A: B_{d-1}^{\mathbb{Z}} \to A$, i.e., $P_A = (p_A)_{\mathbb{R}}$.

Consider the composite operator
$$\mathcal{L}_A : A_{\mathbb{R}} \xrightarrow{\cong} A_{\mathbb{R}},$$
defined by $\mathcal{L}_A = P_A \partial e^{-W} \partial^*|_{A_{\mathbb{R}}}$.

Theorem 6.6 (Generalized Higher Weighted Matrix-Tree Theorem). We have
$$\det \mathcal{L}_A = \gamma_A \sum_T w_T,$$
where the prefactor is given by
$$\gamma_A = \frac{\mu(A)t(p_A)^2}{\theta^2_X}.$$

Remark 6.7. The choice $A = B_{d-1}(X; \mathbb{Z})$ gives Theorem C.

Remark 6.8. If $A = A_S$ is the free abelian group generated by a judiciously chosen subset $S \subset X_{d-1}$, we will obtain $\mu(A_S) = 1$. Using this choice of $A$ as well as $W = 0$, Theorem 6.6 gives a generalization of the main result of [P] to CW complexes.
Proof of Theorem 6.6. As above, we have
\[ \mathcal{L} := \partial \partial_R^* = \partial e^{-W} \partial^*: B_{d-1}(X; \mathbb{R}) \xrightarrow{\sim} B_{d-1}(X; \mathbb{R}). \]

Then
\[ \mathcal{L}_A = P_A \mathcal{L} P_A^*, \]
which implies
\[ \det \mathcal{L}_A = \det(\mathcal{L}) \det(P_A P_A^*). \]

If we apply this to Eq. (17), we reproduce Eq. (18) with \( \gamma_A = \gamma \det(P_A P_A^*). \)

It suffices to identify the prefactor \( \gamma_A. \)

Consider the operator \( \mathcal{L}_T \) for some spanning tree. We have

\[ \det(\mathcal{L}_T) = \det(\partial_T e^{-W} \partial_T^*) = \det(\partial_T \partial_T^* e^{-W}) = \frac{w_T}{\theta_T} \det(\partial_T \partial_T^*) \]

Applying Eq. (18) in the case of good \( W \) and in the low temperature limit, the left hand side of that equation tends to the determinant of the operator \( \mathcal{L}_T^A \) for the spanning tree \( T \subset X \) of maximal weight, whereas the right hand side is dominated by the single contribution associated with the same spanning tree \( T \). Consequently, Eq. (21) implies

\[ \det(\partial_T \partial_T^*) \det(\mathcal{L}_T^A P_T^*(\mathcal{L}_T^A)^*) = \gamma_A \theta_T^2. \]

Since \( P_T^A = (p_T^A)_{R}, \) where the real isomorphism \( p_T^A : B_{d-1}(T; \mathbb{Z}) \to A \) is obtained by composing the real isomorphism \( p_A : B_{d-1}(X; \mathbb{Z}) \to A \) with the inclusion \( B_{d-1}(T; \mathbb{Z}) \subset B_{d-1}(X; \mathbb{Z}) \), we have
\[ \det(P_A P_A^*) = \mu(A)\mu(B_{d-1}(T; \mathbb{Z}))^{-1}(\det p_A^T)^2. \]

We further note that, since \( T \) is a spanning tree, the free abelian group \( B_{d-1}(T; \mathbb{Z}) \) has basis \( \{\partial e_1, \ldots, \partial e_k\} \), where \( e_1, \ldots, e_k \) are the \( k \)-cells of \( T \), so that we have a matrix \( g \) of inner products with the matrix elements \( g_{ij} = \langle \partial_T e_i, \partial_T e_j \rangle = \langle \partial_T e_i, e_j \rangle \), which implies \( \mu(B_{d-1}(T; \mathbb{Z})) = \det(\partial_T \partial_T^*) \). Then Eq. (22) assumes the form
\[ \mu(A)(\det p_A^T)^2 = \gamma_A \theta_T^2. \]

Combining this with Proposition 6.3 results in
\[ \gamma_A = \frac{\mu(A) t(p_A^T)^2}{\theta_T^2}. \]

The right side of Eq. (23) is written in terms of a particular spanning tree \( T \), however, it does not actually depend on this choice. An invariant expression that does not contain \( T \) is obtained by using the following relations:
\[ \frac{t(p_A^T)}{t(p_A)} = t(B_{d-1}(X; \mathbb{Z})/B_{d-1}(T; \mathbb{Z})) = \frac{\theta_T}{\theta_X}. \]
Substituting Eq. (24) into Eq. (23) results in an invariant expression for \( \gamma_A \), given by Eq. (19).

**Alternative forms of Theorem C.** In this subsection we deduce Addendum E as well as a generalization of it to the weighted case. Let us now return to the more general situation of Theorem 6.6.

**Theorem 6.9.** With \( A \subset C_{d-1}(X;\mathbb{Z}) \) as above, we have

\[
\det \mathcal{L}_A = \sum_T \det \mathcal{L}_A^T.
\]

**Proof.** Using Eq. (24) we infer that

\[
\gamma_A = \frac{\mu(A)t(P_A)^2}{\theta_X^2} = \frac{\mu(A)t(P_A^T)^2}{\theta_T^2}
\]

for any spanning tree \( T \). Combining this with Theorem 6.6 in the case of a spanning tree \( T \) we obtain

\[
\det \mathcal{L}_A^T = \gamma_A w_T.
\]

The conclusion now follows by summing over all \( T \).

In the special case when \( A = B_{d-1}(X;\mathbb{Z}) \), Theorem 6.9 reduces to

**Corollary 6.10.** \( \det \mathcal{L} = \sum_T \det \mathcal{L}^T = \sum_T \mu_T. \)

7. Reidemeister torsion and Theorem F

**Reidemeister torsion.** Milnor [M] defined the Reidemeister torsion of a not necessarily acyclic finite chain complex over a field equipped with the auxiliary structure of an ordered basis of its chains as well as a choice of ordered basis of its homology groups. In this section we restrict ourselves to the case of torsion for chain complexes defined over the real numbers.

Consider the case of a chain complex \( C_\ast \) of finite dimensional vector spaces over \( \mathbb{R} \) having non-trivial terms in degrees \( 0 \leq \ast \leq d \). Let \( \partial: C_k \to C_{k-1} \) be the boundary operator. Let \( Z_k \subset C_k \) be the subspace of \( k \)-cycles and let \( B_k \subset Z_k \) the subspace of \( k \)-boundaries. We also set \( H_k = Z_k/B_k \).

We then have short exact sequences

\[
0 \to Z_k \to C_k \to B_{k-1} \to 0 \quad \text{and} \quad 0 \to B_k \to Z_k \to H_k \to 0.
\]

If we choose splittings \( s_{k-1}: B_{k-1} \to C_k \) and \( t_k: H_k \to Z_k \), we are entitled to write \( C_k \cong Z_k \oplus B^k \cong B_k \oplus H_k \oplus B^k \).
Pick bases $b_k := \{b^k_i\}, c_k := \{c^k_i\}, h_k := \{h^k_i\}$ for $B_k, C_k,$ and $H_k,$ respectively. It follows that $\{s_{k-1}(b^k_{i-1}), t_k(h^k_i), b^k_k\}$ forms another basis for $C_k$. Let $\{b_k h_k b_{k-1}\}$ denote this basis and let

$$[b_k h_k b_{k-1}/c_k]$$

denote the change of basis matrix that expresses the basis $b_k h_k b_{k-1}$ in terms of the basis $c_k$. Let $c = \{c_k\}$ and $h = \{h_k\}$.

**Definition 7.1** (Milnor [M, p. 365]). The torsion of the pair $(C_*, h)$ is defined by

$$\tau(C_*) = \prod_{k \geq 0} \det[b_k h_k b_{k-1}/c_k]^{(-1)^k},$$

which is consistent with Milnor’s definition with respect to the identification of $K_1(\mathbb{R}) \cong \mathbb{R}^\times$ given by the determinant function.

Milnor shows that the definition is independent of the choice of $b$ as well as the splittings. Thus the torsion is really an invariant of the triple $(C_*, c, h)$.

In what follows, $C_* = C_*(X; \mathbb{R})$ is the cellular chain complex of a finite connected CW complex $X$ which has a preferred basis consisting of the set of cells. In this case, we think of the torsion as an invariant of the pair $(X, h)$ and we set

$$\tau(X; h) := \tau(C_*(X; \mathbb{R})),
$$

where we have indicated in the notation the dependence on the choice of homology basis. It will be useful to single out a specific kind of homology basis. Let $H_*(X; \mathbb{Z})_0 \subset H_*(X, \mathbb{R})$ be the lattice given by taking the image of the evident homomorphism $H_*(X; \mathbb{Z}) \to H_*(X; \mathbb{R})$. Note that $H_*(X; \mathbb{Z})_0$ has a preferred isomorphism to the torsion free part of $H_*(X; \mathbb{Z})$.

**Definition 7.2.** A combinatorial basis for $H_*(X; \mathbb{R})$ consists of a basis for $H_k(X; \mathbb{Z})_0$ for $k \geq 0$.

Henceforth we fix a combinatorial basis $h$. Let $r: \bigoplus_k X_k \to \mathbb{R}_+$ be a positive-valued function on the set of cells of $X$. As in previous sections, we write $R: C_*(X; \mathbb{R}) \to C_*(X; \mathbb{R})$ for the linear transformation determined by $b \mapsto r_b b$, and $R = e^W$. We have a modified inner product $\langle b, b' \rangle_R = \langle r_b b, b' \rangle$. We also have an operator

$$L_k(W) = \partial \partial_R^* := \partial e^{-W_{k+1}} \partial^* e^{W_k}: B_k(X; \mathbb{R}) \to B_k(X; \mathbb{R}),$$

where $\partial_R^*$ is the formal adjoint to $\partial: C_{k+1}(X; \mathbb{R}) \to C_k(X; \mathbb{R})$ in the modified inner product on both source and target. We define $H^R_k(X; \mathbb{R})$ to be the orthogonal compliment of $B_k(X; \mathbb{R})$ in $Z_k(X; \mathbb{R})$ with respect
to modified inner product on $C_k(X; \mathbb{R})$, and we then have a preferred identification $H^R_k(X; \mathbb{R}) \cong H_k(X; \mathbb{R})$ given by sending a cycle to its homology class. As in the introduction, we let $\eta_k$ be the square of the covolume of $H_k(X; \mathbb{Z})_0 \subset H^R_k(X; \mathbb{R})$, with respect to the basis $\mathfrak{h}_k$ for $H_k(X; \mathbb{Z})_0$ and the inner product on $H^R_k(X; \mathbb{R})$ obtained by restricting the modified inner product on $C_k(X; \mathbb{R})$.

**Theorem 7.3.** Let $X$ be a finite, connected CW complex. Then

$$\tau^2(X; \mathfrak{h}) = \prod_{k \text{ even}} \det \mathcal{L}_k(W) \prod_{k \text{ odd}, b \in X_k} e^{W_{kb}} \prod_{k \text{ even}, b \in X_k} \eta_k \prod_{k \text{ odd}, \gamma_k}. $$

**Remark 7.4.** If we take $W = 0$, then Theorem 7.3 immediately implies that $\tau^2(X; \mathfrak{h})$ is an invariant of the lattice $H_*(X; \mathbb{Z})_0 \subset H_*(X; \mathbb{R})$ rather than just an invariant of the specific choice of combinatorial basis $\mathfrak{h}$. Since this lattice doesn’t depend on any choices, we infer that $\tau^2(X; \mathfrak{h})$ depends only on the CW structure of $X$. In fact, the method of proof of [M, th. 7.2] shows that $\tau^2(X; \mathfrak{h})$ is invariant under subdivision.

**Proof of Theorem 7.3.** For the purpose of this proof we suppress $W$ and write $\mathcal{L} = \mathcal{L}(W)$. We also set $C_* := C_*(X; \mathbb{R})$. Define the splitting maps $s_{k-1} : B_{k-1} \to C_k$ by

$$s_{k-1}(b^i_{k-1}) = e^{-W_k} \partial^* e^{W_{k-1}} \mathcal{L}^{-1}_{k-1}(b^i) = \partial^*_{R} \mathcal{L}^{-1}_{k-1}(b^i_{k-1}).$$

Let $B_R^k(X; \mathbb{R})$ denote the image of $s_{k-1}$, and similarly we define $B_R^k(X; \mathbb{Z})$ to be $s_{k-1}(B_k(X; \mathbb{Z}))$. Note that $B_R^k(X; \mathbb{R})$ is the orthogonal complement to $Z_k$ in the modified inner product on $C_k$.

Let $\gamma^k$ denote the square of the covolume of $B_R^k(X; \mathbb{Z}) \subset B_R^k(X; \mathbb{R})$, using the inner product on $B_R^k(X; \mathbb{R})$ induced by the modified inner product on $C_k$. Similarly, let $\gamma_{k-1}$ denote the square of the covolume of $B_{k-1}(X; \mathbb{Z}) \subset B_{k-1}(X; \mathbb{R})$, where $B_{k-1}(X; \mathbb{R})$ is given the inner product by restricting the modified inner product on $C_{k-1}$. Using the isomorphism $B_k \oplus H_k \oplus B_{k-1} \xrightarrow{\cong} C_k$ determined by the splitting, we infer

$$\det[b_k, \mathfrak{h}_k, b_{k-1}/\mathfrak{c}_k]^2 = \prod_{b \in X_k} e^{W_{kb}} \gamma^k \eta_k \gamma_k,$$

so the square of the Reidemeister torsion is

$$\tau^2(X; \mathfrak{h}) = \prod_{k \text{ even}} \gamma^k \eta_k \gamma_k \prod_{k \text{ odd}, b \in X_k} e^{W_{kb}} \prod_{k \text{ even}, b \in X_k} \eta_k \prod_{k \text{ odd}, \gamma_k}.$$  

---

4In the introduction, $\eta_k$ was defined only in the case when $W = 0$; the current notation applies to an arbitrary $W$. 

Since \( s_k = \partial W \mathcal{L}_k^{-1} \), we have \( s_k^* = \mathcal{L}_k^{-1} \partial \) (since \( \mathcal{L} \) is self-adjoint). Therefore, \( s_k^* s_k = \mathcal{L}_k^{-1} \partial \partial^* \mathcal{L}_k^{-1} = \mathcal{L}_k^{-1} \). We use this fact to compute the quotient of \( \gamma^k / \gamma_{k-1} \). Recall that \( \gamma^k \) is given as the determinant of the inner product matrix, we compute

\[
\langle s_{k-1}(b^i_{k-1}), s_{k-1}(b^j_{k-1}) \rangle_R = \langle s_{k-1}^* s_{k-1}(b^i_{k-1}), (b^j_{k-1}) \rangle_R
\]

\[
= \langle \mathcal{L}_{k-1}^{-1}(b^i_{k-1}), (b^j_{k-1}) \rangle_R.
\]

The determinant of the matrix with these entries latter is, by definition,

\[
\left( \det U \right)^2 \det \mathcal{L}_{k-1},
\]

where \( U \) is the change of basis matrix expressing \( b_{k-1} \) in terms of an orthonormal basis for \( B_{k-1}(X; \mathbb{R}) \) in the modified inner product. A similar observation shows that the determinant of the matrix whose entries are \( \langle b^i_{k-1}, b^j_{k-1} \rangle_R \) is \( (\det U)^2 \), and this is just \( \gamma_{k-1} \).

Consequently, the quotient of these determinants is

\[
\frac{\gamma^k}{\gamma_{k-1}} = \frac{1}{\det \mathcal{L}_{k-1}}.
\]

Inserting Eqn. (27) into Eqn. (26) and performing the evident cancellations, we conclude

\[
\tau^2(X; \mathfrak{h}) = \prod_{k \text{ even}} \det \mathcal{L}_k(W) \cdot \frac{\prod_{k \text{ odd}, b \in X_k} e^{W_k}}{\prod_{k \text{ even}, b \in X_k} e^{W_k}} \cdot \prod_{k \text{ odd}} \eta_k.
\]

In the special case when \( W = 0 \), we can combine Theorem 7.3 with Corollary D. This immediately gives Theorem F:

**Corollary 7.5 (Torsion-Tree Theorem).** For a finite, connected CW complex \( X \), we have

\[
\tau^2(X; \mathfrak{h}) = \prod_{k \geq 0} (\delta_k \sum_{T \in T_{k+1}} \theta^2_T)^{(-1)^k}.
\]

**An alternative formula.** In this part we shall derive a different formula for the torsion in terms of a single spanning tree in each degree as well as a choice of auxiliary structure, namely, homology truncation data for \( X \).

**Hypothesis 7.6.** For each \( k \geq 1 \), we fix a spanning tree \( T_k \) for \( X^{(k)} \). Our convention is to set \( T^0 = \emptyset \).

**Definition 7.7.** A homology truncation of \( X \) in degree \( k \geq 0 \), subordinate to \( T_k \), is a subcomplex \( i: V^k \subset X^{(k)} \) such that \( T^k \subset V^k \) and \( i_*: H_*(V^k; \mathbb{R}) \to H_*(X; \mathbb{R}) \) are isomorphisms for \(* \leq k \).
In induction argument similar to the proof of Lemma 2.3 shows that homology truncations exist. Note that $V^0$ consists of a single vertex of $X$. We have a filtration

$$T^0 \subset V^0 \subset \cdots \subset X^{(k-1)} \subset T^k \subset V^k \subset X^{(k)} \subset \cdots$$

**Lemma 7.8.** The choice of spanning tree $T^k$ determines a splitting $B_{k-1}(X; \mathbb{R}) \rightarrow C_k(X; \mathbb{R})$. The choice of homology truncation $V^k$ subordinate to $T^k$ determines a splitting $H_k(X; \mathbb{R}) \rightarrow Z_k(X; \mathbb{R})$.

**Proof.** The first splitting is the composition

$$B_{k-1}(X; \mathbb{R}) = B_{k-1}(T^k; \mathbb{R}) \xrightarrow{\partial^{-1} \cong} B_k(T^k; \mathbb{R}) \rightarrow C_k(X; \mathbb{R})$$

and the second is given by

$$H_k(X; \mathbb{R}) \xrightarrow{i_k^{-1} \cong} H_k(V^k; \mathbb{R}) = Z_k(V^k; \mathbb{R}) \xrightarrow{i_k} Z_k(X; \mathbb{R}) . \quad \square$$

Define a basis for $B^k(X; \mathbb{Z})$, $b^k = \{b_i^k\}$, as given by the cells of $T^k$. Here we are using the preferred isomorphism $B^k(X; \mathbb{Z}) \cong C_k(T^k; \mathbb{Z})$. This defines a basis $b_{k-1}$ for $B_{k-1}(X; \mathbb{R})$ by $\{b_i^k = \partial b_i^k\}$. The basis for homology in degree $k$ is the combinatorial basis $h_k$ given as an input to the torsion. As always, the basis for $C_k(X; \mathbb{R})$ is given by the set of $k$-cells.

Before explicitly identifying the torsion, note that in each dimension $k$ there are essentially three types of cells:

$$X_k = (T^k_k) \cup (V^k_k \setminus T^k_k) \cup (X_k \setminus V^k_k) .$$

Roughly speaking, the first set of cells contributes to $B^k$, the second set contributes to $H_k$ and the last set contributes to $B_k$. This gives us a decomposition of the $k$-chains

$$(28) \quad C_k(X; \mathbb{R}) = C_k(T^k; \mathbb{R}) \oplus C_k(V^k/T^k; \mathbb{R}) \oplus C_k(X/V^k; \mathbb{R}) .$$

(when $k = 0$, we replace $C_0(V^0/T^0; \mathbb{R})$ with $C_0(V^0, T^0; \mathbb{R}) = \mathbb{R}$, etc.)

We first identify the homological contribution in degree $k$ to the torsion. With respect to the splitting Eq. (28), the combinatorial basis $h_k$ has image contained in the direct sum

$$C_k(T^k; \mathbb{R}) \oplus C_k(V^k/T^k; \mathbb{R}) = C_k(V^k; \mathbb{R}) .$$

Hence, its contribution to the torsion is left invariant if we project these elements onto $C_k(V^k/T^k; \mathbb{R}) = H_k(V^k/T^k; \mathbb{R}) = H_k(X; \mathbb{R})$ (since the other summand $C_k(T^k; \mathbb{R}) = B^k(T^k; \mathbb{R})$ maps to $B^k(X; \mathbb{R})$ and the relevant determinant remains unchanged if we project away from
\(B^k(X; \mathbb{R})\)). Consequently, the homological contribution to the torsion in degree \(k\) is given by the determinant of the composite

\[ H_k(X; \mathbb{R}) \xrightarrow{\sim} H_k(V^k; \mathbb{R}) \xrightarrow{p_*} H_k(V^k/T^k; \mathbb{R}), \]

where \(p: V^k \rightarrow V^k/T^k\) is the quotient map. So we wish to identify \(\det p_*/\det i_*\).

**Definition 7.9.** Let \(\chi_k \in \mathbb{N}\) denote the square of the determinant of \(i_*: H_k(V^k; \mathbb{R}) \rightarrow H_k(X; \mathbb{R})\), i.e., the square of the covolume of the lattice \(i_*(H_k(V^k; \mathbb{Z})) \subset H_k(X; \mathbb{R})\).

Applying Proposition 6.3 to the real isomorphism \(H_k(V^k; \mathbb{Z}) \rightarrow H_k(V^k/T^k; \mathbb{Z})\), we infer

**Lemma 7.10.** The determinant of \(p_*\) is the ratio \(\pm \theta_{T^k}/\theta_{V^k}\).

Consequently, up to sign, the contribution of \(h^k\) to the determinant defining the Reidemeister torsion is

\[ \chi_k \cdot \theta_{T^k}/\theta_{V^k}. \] (29)

We next identify the contribution in degree \(k\) to the torsion provided by the basis \(b^k\). As defined above this basis is given by the boundaries of the cells of \(T_{k+1}\). This leads us to consider the composite

\[ C_{k+1}(T^{k+1}; \mathbb{Z}) \xrightarrow{\partial} B_k(T^{k+1}; \mathbb{Z}) \xrightarrow{q_k} C_k(X/V^k; \mathbb{Z}), \] (30)

where \(q_k\) is induced by the quotient map \(T^{k+1} \rightarrow X/V^k\). The homomorphism \(\partial\) is an isomorphism and so it has determinant \(\pm 1\). The second homomorphism \(q_k\) is a real isomorphism and therefore the determinant of its realization, \(\det((q_k)_\mathbb{R})\), has value \(\pm t(q_k)\) by Proposition 6.3. Note that \((q_k)_\mathbb{R}\) is the restriction of the orthogonal projection \(C_k(X; \mathbb{R}) \rightarrow C_k(X/V^k; \mathbb{R})\) to the subspace \(B_k(T_{k+1}; \mathbb{R}) \subset C_k(X; \mathbb{R})\), and the projection of \(b^k\) onto this summand gives its contribution to the torsion. Hence, the determinant of the composition \((q_k)_\mathbb{R} \circ \partial\) is \(\pm t(q_k)\). So the contribution in degree \(k\) of \(b^k\) to the torsion is \(\pm t(q_k)\).

Lastly, the contribution to the torsion in degree \(k\) provided by the basis \(b^k_{k-1}\) is given by the standard basis of \(C_k(T_k; \mathbb{R})\) via the splitting Eq. (28). It is then evident that the contribution in degree \(k\) of \(b^k_{k-1}\) to the torsion is 1.

Assembling, we obtain

\[ \det[b^k b^k b^{k-1}] = \pm t(q_k) \cdot \frac{\theta_{T^k}}{\theta_{V^k} \sqrt{\chi_k}} \cdot 1. \] (31)
Forming the square of the Reidemeister torsion, we conclude

**Theorem 7.11.** For a connected, finite CW complex $X$ with combinatorial homology basis $\mathfrak{h}$, spanning tree data $\{T^k\}$ and homology truncation data $\{V^k\}$, we have

$$\tau^2(X; \mathfrak{h}) = \prod_{k \geq 0} \left( \frac{\theta^2_{T^k} t(q_k)^2}{\theta^2_{V^k} \chi_k} \right)^{(-1)^k},$$

where $q_k : B_k(T^{k+1}; \mathbb{Z}) \to C_k(X/V^k; \mathbb{Z})$ and $\chi_k \in \mathbb{N}$ are as above.

**Example 7.12.** If $X$ has dimension one, then all terms appearing in Theorem 7.11 are equal to one. Hence, $\tau^2(X; \mathfrak{h}) = 1$ whenever $X$ is a connected finite graph.

**Example 7.13.** Let $X = \mathbb{R}P^2$. Then we may choose $T^1 = * = V^1$ and $T^2 = \mathbb{R}P^2 = V^2$. In this instance, the only non-trivial term appearing in Theorem 7.11 is $t(q_1)^2 = 4$. Hence $\tau^2(\mathbb{R}P^2; \mathfrak{h}) = \frac{1}{4}$.

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