NONCOMMUTATIVE RATIONAL FUNCTIONS, THEIR
DIFFERENCE-DIFFERENTIAL CALCULUS AND
REALIZATIONS

DMITRY S. KALIUZHNYI-VERBOVETSKYI AND VICTOR VINNIKOV

Abstract. Noncommutative rational functions appeared in many contexts in system theory and control, from the theory of finite automata and formal languages to robust control and LMIs. We survey the construction of noncommutative rational functions, their realization theory and some of their applications. We also develop a difference-differential calculus as a tool for further analysis.

1. Introduction

Noncommutative rational functions first appeared in system theory in the context of recognizable formal power series in noncommuting indeterminates in the theory of formal languages and finite automata; see Kleene [36], Schützenberger [48, 49], and Fliess [23, 24, 25] (where the motivation comes also from applications to certain classes of nonlinear systems), and Berstel–Reutenauer [12] for a survey. In particular, noncommutative rational functions admit a good state space realization theory. More recently, state space realizations of rational expressions in Hilbert space operators (modelling structured possibly time varying uncertainty) have figured prominently in work on robust control of linear systems, see Beck [9], Beck–Doyle–Glover [10], Lu–Zhou–Doyle [40].

Another important application comes from the area of Linear Matrix Inequalities (LMIs); see, e.g., Nesterov–Nemirovski [42], Nemirovski [41], Skelton–Iwasaki–Grigoriadis [50]. As it turns out, most optimization problems appearing in systems and control are dimension-independent, i.e., the natural variables are matrices, and the problem involves rational expressions in these matrix variables which have therefore the same form independent of matrix sizes; see Helton [29], Helton–McCullough–Putinar–Vinnikov [30]. Realizations of rational functions in noncommuting indeterminates are exactly what is needed here to convert (numerically unmanageable) rational matrix inequalities into (highly manageable) linear matrix inequalities, see Helton–McCullough–Vinnikov [31].

Last but not least, in many situations one can establish a commutative result by “lifting” to the noncommutative setting, applying the noncommutative theory, and then “descending” again to the commutative situation. Some examples are:

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• The classical paper of Fornasini–Marchesini \cite{27} establishing a state space realization theorem for rational functions of several commuting variables.
• The results of Ball–Kaliuzhnyi-Verbovetskyi \cite{7} on conservative dilations of various classes of (commutative) multidimensional systems.
• The theorem of Kaliuzhnyi-Verbovetskyi–Vinnikov \cite{34} showing that the singularities of a matrix-valued rational function of several commuting variables which is regular at zero coincide with the singularities of the resolvent in any of its Fornasini–Marchesini realizations with the minimal possible state space dimension. This implies, in particular, the absence of zero-pole cancellations in a minimal factorization.

The goal of this paper is two-fold. First, we survey the basic concepts of the theory of noncommutative rational functions, and their realization theory. Second, we develop a difference-differential calculus for noncommutative rational functions. This is a new powerful tool for the needs of system theory and beyond. It is a special instance of the general theory of noncommutative functions which are defined as functions on tuples of matrices of all sizes satisfying certain compatibility conditions as we vary the size of matrices (they respect direct sums and simultaneous similarities); see Kaliuzhnyi-Verbovetskyi–Vinnikov \cite{35}.

It is important to notice that the NCAlgebra software, \url{http://www.math.ucsd.edu/~ncalg}, implements many symbolic algorithms in the noncommutative setting; see \cite{30} for examples, guidance, and detailed references.

2. Noncommutative rational functions

We first formally introduce noncommutative rational functions; this involves some non-trivial details since unlike the commutative case, a noncommutative rational function does not admit a canonical coprime fraction representation. We follow Kaliuzhnyi-Verbovetskyi–Vinnikov \cite{34}, to which we refer for both details and references to extensive algebraic literature; we only mention Amitsur \cite{3}, Bergmann \cite{11}, Cohn \cite{15,17} for some of the original constructions, and Rowen \cite{47,Chapter 8}, Cohn \cite{16,20} for good expositions.

We start with noncommutative polynomials in \(d\) noncommuting indeterminates \(z_1, \ldots, z_d\) over a field \(\mathbb{K}\). E.g., a noncommutative polynomial of total degree 2 in 2 indeterminates \(z_1, z_2\) is of the form

\[
p = \alpha + \beta z_1 + \gamma z_2 + \delta z_1^2 + \epsilon z_1 z_2 + \zeta z_2 z_1 + \eta z_2^2,
\]

where the coefficients \(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta \in \mathbb{K}\). The general form of a noncommutative polynomial is

\[
p = \sum_{w \in \mathbb{F}_d} p_w z^w.
\]

Here \(\mathbb{F}_d\) denotes the free semigroup with \(d\) generators (letters) \(g_1, \ldots, g_d\); elements of \(\mathbb{F}_d\) are arbitrary words \(w = g_{i_1} \cdots g_{i_\ell}\), and the semigroup operation is concatenation; the neutral element is the empty word \(\emptyset\), and \(|w| = \ell\) is the length of the word \(w\). We use noncommutative multipowers \(z^w = z_{i_\ell} \cdots z_{i_1}\). In the example above,

\[
p\emptyset = \alpha, \; p g_1 = \beta, \; p g_2 = \gamma, \; p g_1 g_2 = \delta, \; p g_1 g_2 g_1 = \epsilon, \; p g_2 g_1 = \zeta, \; p g_2^2 = \eta.
\]

Noncommutative polynomials form an algebra \(\mathbb{K}(z_1, \ldots, z_d)\) over \(\mathbb{K}\), often called the free associative algebra on \(d\) generators \(z_1, \ldots, z_d\). Notice that we can evaluate
a noncommutative polynomial \( p \in \mathbb{K}(z_1, \ldots, z_d) \) on a \( d \)-tuple \( Z = (Z_1, \ldots, Z_d) \) of \( n \times n \) matrices over \( \mathbb{K} \), for any \( n \), yielding a \( n \times n \) matrix \( p(Z) \).

A non-zero polynomial can vanish on tuples of matrices of a certain size. E.g., \( p = z_1 z_2 - z_2 z_1 \) vanishes on pairs of \( 1 \times 1 \) matrices (scalars), and

\[
p = \sum_{\pi \in S_{n+1}} \text{sign}(\pi) x_1^{\pi(1)-1} x_2 \cdots x_1^{\pi(n+1)-1} x_2
\]

vanishes on pairs of \( n \times n \) matrices (here \( S_{n+1} \) is the symmetric group on \( n + 1 \) elements); see [47] Proposition 1.1.37 and Exercise 1.4.11 on page 104 and [20] Theorem 7. However, if \( p(Z) = 0 \) for all \( d \)-tuples \( Z \) of square matrices of all sizes, then necessarily \( p \) is the zero polynomial. More precisely, if \( p(Z) = 0 \) for all \( d \)-tuples \( Z \) of \( n \times n \) matrices, then \( \deg p \geq 2n \); this follows by applying to \( p \) a multilinearization process to reduce to the case of a polynomial of degree 1 in each indeterminate, and then evaluating the resulting multilinear polynomial on a staircase of matrix units, see [47] page 6 and Lemma 1.4.3. We notice that a much stronger statement appears in [1]; in particular, the intersection of the kernels of the matrix evaluations \( p(Z) \) for all \( d \)-tuples of \( n \times n \) matrices is zero for \( n \) large enough compared to the degree of \( p \).

We next define (scalar) noncommutative rational expressions by starting with noncommutative polynomials and then applying successive arithmetic operations — addition, multiplication, and inversion. We emphasize that an expression includes noncommutative polynomials and then applying successive arithmetic operations — addition, multiplication, and inversion. We define a noncommutative rational expression \( r \) can be evaluated on a \( d \)-tuple \( Z \) of \( n \times n \) matrices in its domain of regularity, \( \text{dom} \, r \), which is defined as the set of all \( d \)-tuples of square matrices of all sizes such that all the inverses involved in the calculation of \( r(Z) \) exist. E.g., if \( r = (z_1 z_2 - z_2 z_1)^{-1} \) then \( \text{dom} \, r = \{ Z = (Z_1, Z_2) : \det(Z_1 Z_2 - Z_2 Z_1) \neq 0 \} \). We assume that \( \text{dom} \, r \neq \emptyset \), in other words, when forming noncommutative rational expressions we never invert an expression that is nowhere invertible.

Two noncommutative rational expressions \( r_1 \) and \( r_2 \) are called equivalent if \( \text{dom} \, r_1 \cap \text{dom} \, r_2 \neq \emptyset \) and \( r_1(Z) = r_2(Z) \) for all \( d \)-tuples \( Z \in \text{dom} \, r_1 \cap \text{dom} \, r_2 \). E.g., the three different noncommuting rational expressions in the paragraph above are equivalent. For another example, easy matrix algebra shows that \( r_1 = z_1 z_2 (z_1 z_2 - z_2 z_1)^{-1} \) and \( r_2 = 1 + z_2 z_1 (z_1 z_2 - z_2 z_1)^{-1} \) are equivalent.

We define a noncommutative rational function to be an equivalence class of noncommutative rational expressions. We usually denote noncommutative rational functions by German (Fraktur) letters.

Notice that, unlike in the commutative case, the “minimal complexity” of a noncommutative rational expression defining a given noncommutative rational function can be arbitrarily high; there is nothing similar to a coprime fraction representation.

\footnote{This requirement is in fact superfluous. For any noncommutative rational expression \( r \), it turns out [24] Remark 2.3 that the evaluation of \( r \) on \( d \)-tuples of \( n \times n \) generic matrices — see the discussion following Proposition 2.1 below — is defined for all sufficiently large \( n \). In the case where \( \mathbb{K} \) is an infinite field it follows that \( \text{dom}_n r \) is Zariski dense for all sufficiently large \( n \); therefore for any two noncommutative rational expressions \( r_1 \) and \( r_2 \), \( \text{dom} \, r_1 \cap \text{dom} \, r_2 \neq \emptyset \). The case where the field \( \mathbb{K} \) is finite can be handled as in the proof of Proposition 2.1.}
It turns out that any nonzero noncommutative rational function is invertible. This follows from the following result which is essentially well known and is non-trivial already in the case where \( r \) in the statement is a noncommutative polynomial.

**Proposition 2.1.** If \( r \) is a noncommutative rational expression and \( \det r(Z) = 0 \) for all \( Z \in \text{dom} \, r \) then \( r \) is equivalent to zero.

Proposition 2.1 means that noncommutative rational functions form a skew field — a skew field of fractions of the ring of noncommutative polynomials. We remind the reader that a skew field, also called a division ring, is a ring with identity in which every nonzero element has a multiplicative inverse; it is therefore similar to a field, except that the multiplication is not assumed to be commutative. A skew field of fractions of a given ring is a skew field containing the ring and generated by it in the sense that no proper skew subfield contains the ring. If a noncommutative integral domain \( R \) satisfies the so called right Ore condition,

\[
\forall a, b \in R, b \neq 0 \exists c, d \in R, d \neq 0: ad = bc,
\]

then one can construct a skew field of fractions analogously to the commutative case as the ring of right quotients, i.e., of formal fractions \( cd^{-1} \), \( d \neq 0 \). In general, a skew field of fractions of a noncommutative integral domain might or might not exist. We refer to [16, Sections 0.5–0.6 and Chapter 7], [20], [38, Chapter 4], [47, Section 1.7 and pages 105, 107–108] for more material on skew fields of fractions and noncommutative localization. If a skew field of fractions exists, it might not be unique; we will discuss this in more detail later in this section.

Before proceeding to the proof of Proposition 2.1 we notice that for a given matrix size \( n \), we can view a noncommutative polynomial or a noncommutative rational expression in \( d \) noncommuting indeterminates as a \( n \times n \) matrix-valued polynomial or rational function in commuting matrix entries. More formally, let \( T = (T_1, \ldots, T_d) \) be a \( d \)-tuple of \( n \times n \) matrices whose entries \( (T_i)_{jk} \) are \( dn^2 \) commuting indeterminates; \( T_1, \ldots, T_d \) are often called generic matrices. For a noncommutative polynomial \( p \) or for a noncommutative rational expression \( r \), we define

\[
p_n = p(T) \in \mathbb{K}^{n \times n}[(T_i)_{jk}: i = 1, \ldots, d; j, k = 1, \ldots, n]
\]

and

\[
r_n = r(T) \in \mathbb{K}^{n \times n}((T_i)_{jk}: i = 1, \ldots, d; j, k = 1, \ldots, n).
\]

Note that \( r_n \) is defined only for \( n \) in a subset \( \mathcal{N}_r \subseteq \mathbb{N} \) such that all the inversions involved in the calculation of the rational matrix-valued function \( r(T) \) exist. E.g., if \( r = (z_1 z_2 - z_2 z_1)^{-1} \) then \( \mathcal{N}_r = \mathbb{N} \setminus \{1\} \).

We make two useful remarks. First, if \( \mathbb{K} \) is an infinite field then \( n \in \mathcal{N}_r \) if and only if \( \text{dom}_n \, r := \text{dom} \, r \cap (\mathbb{K}^{n \times n})^d \) is nonempty. This may fail in the case of a finite field \( \mathbb{K} \). E.g., if \( \mathbb{K} = \mathbb{Z}/p\mathbb{Z} \) and \( r(z_1) = (z_1^p - z_1)^{-1} \) then \( 1 \in \mathcal{N}_r \), however \( \text{dom}_1 \, r = \emptyset \). The problem is, of course, in that there are nonzero polynomials over \( \mathbb{K} \) which vanish identically.

Second, we define the extended domain of regularity, \( \text{edom} \, r \), of a noncommutative rational expression \( r \) as follows: for each matrix size \( n \), \( \text{edom}_n \, r := \text{edom} \, r \cap (\mathbb{K}^{n \times n})^d \) is the domain of regularity of the rational matrix-valued function \( r_n \). (We set \( \text{edom}_n \, r = \emptyset \) if \( n \notin \mathcal{N}_r \).) Here the domain of regularity of a scalar rational function is the complement of the zero set of its denominator in the
coprime fraction representation. In general, dom $r \subseteq \text{edom } r$. As a silly example, take $r = z_1 z_2^{-1}$. Then dom $r = \{ Z_1 : \text{det } Z_1 \neq 0 \}$, however $N_r = \mathbb{N}$, $r_n = I_n$ for each $n$, so that edom $r$ consists of all square matrices $Z_1$ over $K$ of all sizes. For a more conceptual example, see the end of this section.

**Proof of Proposition 2.7** We consider a subring $G_n = \{ p_n : p \in \mathbb{K}(z_1, \ldots, z_d) \}$ of $\mathbb{K}^{n \times n}((T_i)_{jk})$, which is often called the ring of generic matrices. Let $D_n$ be the ring of central quotients of $G_n$, i.e., the ring of formal fractions $PQ^{-1}$ with $P, Q \in G_n$ and $Q$ a regular central element (see [47] Section 1.7 for details). By a theorem of Amitsur [47] Theorem 3.2.6, $D_n$ is a skew field. Assuming $d > 1$, any central element $Q \in G_n$ is central in the whole ring $\mathbb{K}^{n \times n}((T_i)_{jk})$ (see [47] Exercise 2.4.2 on page 149). Hence, any such nonzero $Q$ is a scalar $n \times n$ matrix of polynomials in $(T_i)_{jk}$; in particular, $Q$ is invertible in $\mathbb{K}^{n \times n}((T_i)_{jk})$. It follows that the skew field $D_n$ is a subring of $\mathbb{K}^{n \times n}((T_i)_{jk})$ (clearly, this is also true in the case of $d = 1$, where $G_n$ is commutative, every element is central, and every element is invertible in $\mathbb{K}^{n \times n}((T_i)_{jk})$). Therefore, $r_n \in D_n$ for any noncommutative rational expression $r$ with $n \in N_r$. If det $r_n = 0$ then $r_n$ is not invertible in $\mathbb{K}^{n \times n}((T_i)_{jk})$. On the other hand, since $D_n$ is a skew field, this can happen only if $r_n = 0$.

Assume now that det $r(Z) = 0$ for all $Z \in \text{dom } r$. We claim that det $r_n = 0$ for each matrix size $n \in N_r$; by the preceding paragraph, this will imply the conclusion of the proposition: $r(Z) = 0$ for all $Z \in \text{dom }_r n \in N_r$, i.e., $r$ is equivalent to zero. If the field $K$ is infinite then the assumption det $r(Z) = 0$ for all $Z \in \text{dom }_r n \in N_r$ is simply equivalent to det $r_n = 0$ (as a rational function of $(T_i)_{jk}$).

Let now $K$ be a finite field, and assume that det $r_n \neq 0$. Then there exists a finite extension $\overline{K}$ of $K$, say of degree $m$, and $\overline{Z} \in (\overline{K}^{n \times n})^d$ such that $\overline{Z} \in \text{dom } r$ (over $\overline{K}$) and det $r(\overline{Z}) \neq 0$. Since $K$ is finite, it is perfect, therefore every finite extension is separable, and thus admits a primitive element $\alpha$, i.e., $\overline{K} = \mathbb{K}(\alpha)$ (see [39] Theorem 7.6.1 and Corollary 7.7.8). Then $1, \alpha, \ldots, \alpha^{m-1}$ is a basis for $\overline{K}$ over $K$, and we can define a ring homomorphism $\phi : \overline{K} \rightarrow \mathbb{K}^{m \times m}$ by setting $\phi(\alpha) = A$ where $A$ is a $m \times m$ matrix over $K$ whose minimal polynomial coincides with that of $\alpha$. $\phi$ induces a ring homomorphism $\phi_n = \text{id}_{\mathbb{K}^{n \times n}} \otimes_{\mathbb{K}} \phi$ from $\mathbb{K}^{n \times n} \cong \mathbb{K}^{n \times n} \otimes_{\mathbb{K}} \overline{K}$ to $\mathbb{K}^{n \times n} \otimes_{\mathbb{K}} \mathbb{K}^{m \times m} \cong \mathbb{K}^{n m \times n m}$. The fact that $\phi_n$ is a homomorphism implies that $(\phi_n(\overline{Z}_1), \ldots, \phi_n(\overline{Z}_d)) \in \text{dom } r$ and

$$r(\phi_n(\overline{Z}_1), \ldots, \phi_n(\overline{Z}_d)) = \phi_n(r(\overline{Z}_1, \ldots, \overline{Z}_d)).$$

For any $\overline{X} = [\overline{x}_{ij}] \in \overline{K}^{n \times n}$, we claim that

$$\det \phi(\det \overline{X}) = \det \phi_n(\overline{X}),$$

which boils down to

$$\det \left( \sum_{\pi \in S_n} \text{sign}(\pi) \phi(\overline{x}_{1\pi(1)}) \cdots \phi(\overline{x}_{n\pi(n)}) \right) = \det [\phi(\overline{x}_{ij})]_{i,j=1,\ldots,n}.$$ 

Since the matrices $\phi(\overline{x}_{ij})$ commute, the last formula follows from a well known identity for the determinant of a block matrix with commuting blocks; see, e.g.,
Finally, 
\[
\det r \left( \phi_n(\overline{Z}_1), \ldots, \phi_n(\overline{Z}_d) \right) = \det \phi_n \left( r(\overline{Z}_1, \ldots, \overline{Z}_d) \right) = \det \phi \left( \det r(\overline{Z}_1, \ldots, \overline{Z}_d) \right) .
\]
Since \( \det r(\overline{Z}) \neq 0 \), and \( \phi \) is a ring homomorphism whose domain is a field, \( \phi \left( \det r(\overline{Z}) \right) \) is invertible. Therefore, 
\[
\det r \left( \phi_n(\overline{Z}_1), \ldots, \phi_n(\overline{Z}_d) \right) = \det \phi \left( \det r(\overline{Z}) \right) \neq 0,
\]
which contradicts the assumption that \( \det r(Z) = 0 \) for each \( Z \in \text{dom } r \). \( \square \)

The proof of Proposition 21 implies two interesting facts. First, while the “minimal complexity” of a noncommutative rational function can be arbitrarily high, its restriction to \( n \times n \) matrices, for every matrix size \( n \), is of a simple form: it is equal to the restriction of \( pq^{-1} \), where \( p \) and \( q \) are noncommutative polynomials, \( q \) being a central polynomial for \( n \times n \) matrices.

Second, we could have defined noncommutative rational expressions and their equivalence using evaluation on generic matrices or on matrices over the algebraic closure of \( K \) rather than evaluation on matrices over \( K \) as we did. However, we would obtain the same noncommutative rational expressions and the same equivalence relation.

We denote the skew field of noncommutative rational functions in \( z_1, \ldots, z_d \) by \( K(z_1, \ldots, z_d) \); it is often called the free skew field.

Unlike in the commutative case, skew fields of fractions are in general not unique. Here is an example of infinitely many embeddings of \( K(z_1, z_2) \) into skew fields (see [22] and [16, Exercise 7.2.10 on page 258]). Consider the polynomial ring in one indeterminate \( K[t] \) with the endomorphism \( \alpha_n \) \((n = 2, 3, \ldots)\) induced by \( \alpha_n(t) = t^n \) and let \( K[t][x; \alpha_n] \) be the skew polynomial ring over \( K[t] \) determined by \( \alpha_n \). The elements of \( K[t][x; \alpha_n] \) are polynomials over \( K \) in \( t \) and \( x \) with the indeterminates \( t \) and \( x \) satisfying the commutation relation \( tx = xt^n \). Like any skew polynomial ring, \( K[t][x; \alpha_n] \) is a right Ore ring, i.e., satisfies (2.1), and can be embedded into its skew field of right quotients, \( K(t)(x; \alpha_n) \). The elements of \( K(t)(x; \alpha_n) \) are rational functions over \( K \) in \( t \) and \( x \) with \( tx = xt^n \). See [16] Section 0.8 for details on skew polynomial rings. Let \( y = xt \). It turns out that for any noncommutative polynomial \( p \) in two indeterminates, \( p(x, y) \neq 0 \). This can be verified directly by showing that distinct noncommutative monomials in \( x \) and \( y \) yield distinct monomials of the form \( x^k t^l \); it can also be deduced from the result of [23], since the two left ideals in \( K[t][x; \alpha_n] \) generated by \( x \) and by \( y \) have a trivial intersection. We thus obtain an embedding \( p \mapsto p(x, y) \) of \( K(z_1, z_2) \) into \( K(t)(x; \alpha_n) \) and, therefore, into \( K(t)(x; \alpha_n) \). It is not hard to see that these embeddings are not isomorphic for distinct values of \( n \).

It is instructive to observe that in the embedding \( K(z_1, z_2) \hookrightarrow K(t)(x; \alpha_n) \) the images \( x \) and \( y \) of \( z_1 \) and \( z_2 \) satisfy the rational identity \( x^{-1}yx = y(x^{-1}y)^{n-1} \) (since \( tx = xt^n \) and \( y = xt \), hence \( t = x^{-1}y \)). This is in contrast to the free skew field \( K(z_1, z_2) \), where \( z_1, z_2 \) satisfy no nontrivial rational identities.

The non-uniqueness issue leads us to the notion of the universal skew field of fractions. A skew field of fractions \( K \) of a ring \( R \) is called a universal skew field
of fractions if for every homomorphism \( \phi: \mathcal{R} \to \mathcal{L} \) to a skew field \( \mathcal{L} \) there exists a subring \( \mathcal{K}_0 \subseteq \mathcal{K} \) containing \( \mathcal{R} \) and a homomorphism \( \theta: \mathcal{K}_0 \to \mathcal{L} \) extending \( \phi \) such that the following holds:

\[
(2.2) \quad \text{for every } x \neq 0 \text{ in } \mathcal{K}_0 \text{ its inverse } x^{-1} \text{ belongs to } \mathcal{K}_0 \text{ if and only if } \theta(x) \neq 0.
\]

Furthermore, the extension \( \theta \) is unique in the following local sense. Let \( \theta': \mathcal{K}'_0 \to \mathcal{L} \) be another extension satisfying (2.2). Then there exists a subring \( \mathcal{K}_0 \subseteq \mathcal{K}_0 \cap \mathcal{K}'_0 \) containing \( \mathcal{R} \) such that \( \theta \) and \( \theta' \) agree on \( \mathcal{K}_0 \), and the extension \( \bar{\theta}: \mathcal{K}_0 \to \mathcal{L} \) defined by \( \bar{\theta} := \theta|_{\mathcal{K}_0} = \theta'|_{\mathcal{K}_0} \) satisfies (2.2). It is straightforward to see that a universal skew field of fractions is unique (when it exists) up to a unique isomorphism.

The following result is essentially the first fundamental theorem of Amitsur on rational identities.

**Proposition 2.2.** \( \mathbb{K}\langle z_1, \ldots, z_d \rangle \) is the universal skew field of fractions of the ring \( \mathbb{K}\langle z_1, \ldots, z_d \rangle \).

**Proof.** Let \( \phi: \mathbb{K}\langle z_1, \ldots, z_d \rangle \to \mathcal{L} \) be a homomorphism to a skew field \( \mathcal{L} \). We notice that for any skew field \( \mathcal{D} \) over \( \mathbb{K} \), a rational expression \( r \) can be evaluated on a \( d \)-tuple \( x = (x_1, \ldots, x_d) \) of elements of \( \mathcal{D} \) provided all the inverses involved in the calculation of \( r(x) \) exist, i.e., all the elements to be inverted are nonzero. Therefore, we can define \( \theta(r) \) as the evaluation \( r(\phi(z_1), \ldots, \phi(z_d)) \) whenever this is possible.

We claim that if \( r \) is equivalent to 0 then \( r(\phi(z_1), \ldots, \phi(z_d)) \) is either 0 or undefined. Indeed, \( r \) being equivalent to 0 means that for every matrix size \( n \), \( r \) is either 0 or undefined on the skew field of fractions \( D_n \) of the ring of generic matrices \( G_n \), see the proof of Proposition 2.1. In other words, \( r \) is a rational identity for \( D_n \), \( n = 1, 2, \ldots \), hence (see [47] Theorem 8.3.3 and Corollary 8.2.16]) \( r \) is a rational identity for any skew field over \( \mathbb{K} \). In a little bit more details, the fact that \( r \) is a rational identity for \( D_n \), \( n = 1, 2, \ldots \) (or just for a sequence \( D_{n_j}, n_j \to \infty \)), implies by a simple ultraproduct construction in the proof of Corollary 8.2.16 in [47] that \( r \) is a rational identity for a skew field \( \mathcal{D} \) that is infinite dimensional over an infinite center; hence by the first fundamental theorem of Amitsur [47] Theorem 8.2.15], \( r \) is a rational identity for any skew field over \( \mathbb{K} \). At any rate, \( r \) is a rational identity for \( \mathcal{L} \), hence, \( r(\phi(z_1), \ldots, \phi(z_d)) \) is either 0 or undefined.

We can now define \( \mathcal{K}_0 \) to consist of all noncommutative rational functions \( r \) that can be represented by noncommutative rational expressions \( r \) such that \( \theta(r) \) is defined, and we set \( \theta(t) := \theta(r) \). It is clear that \( \mathcal{K}_0 \) is a subring of \( \mathbb{K}\langle z_1, \ldots, z_d \rangle \) containing \( \mathbb{K}\langle z_1, \ldots, z_d \rangle \), that \( \theta: \mathcal{K}_0 \to \mathcal{L} \) is a homomorphism extending \( \phi \) and satisfying (2.2). Furthermore, \( \theta \) is the only extension of \( \phi \) to \( \mathcal{K}_0 \). Hence, if \( \theta': \mathcal{K}_0 \to \mathcal{L} \) is another extension of \( \phi \) satisfying (2.2), then \( \theta \) coincides with \( \theta' \) on \( \mathcal{K}_0 = \mathcal{K}_0 \cap \mathcal{K}_0 \), and it is obvious that the extension \( \bar{\theta}: \mathcal{K}_0 \to \mathcal{L} \) defined by \( \bar{\theta} := \theta|_{\mathcal{K}_0} = \theta'|_{\mathcal{K}_0} \) satisfies (2.2). \( \square \)

Finally, we introduce matrix-valued noncommutative rational expressions and matrix-valued noncommutative rational functions. We start with matrix-valued noncommutative polynomials (having matrix rather than scalar coefficients) and use tensor substitutions for evaluations on tuples of matrices. E.g., if

\[
P = P_0 + P_{g_1} z_1 + P_{g_2} z_2 + P_{g_1^2} z_1^2 + P_{g_1 g_2} z_1 z_2 + P_{g_2 g_1} z_2 z_1 + P_{g_2^2} z_2^2
\]
is a matrix-valued noncommutative polynomial of total degree 2 with coefficients $P_w \in \mathbb{K}^{p \times q}$, $|w| \leq 2$, then for a $d$-tuple $Z = (Z_1, \ldots, Z_d)$ of $n \times n$ matrices over $\mathbb{K}$,

$$P(Z) = P_0 \otimes I_n + P_0 \otimes Z_1 + P_{g_1} \otimes Z_2 + P_{g_1} \otimes Z_1^2 + P_{g_2} \otimes Z_1 Z_2 + P_{g_2} \otimes Z_2 Z_1 + P_{g_2} \otimes Z_2^2 \in \mathbb{K}^{p \times q} \otimes \mathbb{K}^{n \times n}.$$  

We will often use the canonical identification of $\mathbb{K}^{p \times q} \otimes \mathbb{K}^{n \times n}$ with $\mathbb{K}^{pn \times qn}$, i.e., with $p \times q$ block matrices with $n \times n$ block entries. Thus we will often view $P(Z)$ above as a $pn \times qn$ matrix. We define matrix-valued noncommutative rational expressions by starting with matrix-valued noncommutative polynomials and applying successive matrix arithmetic operations — addition, multiplication, and inversion, and forming block matrices: a $p_1 \times q_1$ matrix of $p_2 \times q_2$ matrix-valued noncommutative rational expressions is a $p_1 p_2 \times q_1 q_2$ matrix-valued noncommutative rational expression. The domain of a matrix-valued noncommutative rational expression $R$, $\text{dom } R$, consists of all $d$-tuples $Z$ of square matrices of all sizes such that all the inverses involved in the calculation of $R(Z)$ exist. E.g., consider the $1 \times 1$ matrix-valued rational expression

$$R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -z_1 & -z_2 \\ -z_2 & 1 - z_1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

with

$$\text{dom } R_1 = \left\{ (Z_1, Z_2) : \det \begin{bmatrix} I - Z_1 & -Z_2 \\ -Z_2 & I - Z_1 \end{bmatrix} \neq 0 \right\}.$$  

Notice that a $1 \times 1$ matrix-valued noncommutative rational expression is not necessarily the same as a scalar noncommutative rational expression, since it may involve, as in this example, intermediate matrix operations.

Equivalence of matrix-valued noncommutative rational expressions, and matrix-valued noncommutative rational functions as equivalence classes, are defined as in the scalar case. E.g., using a standard Schur complement calculation, we can observe that the $1 \times 1$ matrix-valued noncommutative rational expression $R_1$ above is equivalent to the following two scalar noncommutative rational expressions,

$$r_2 = (1 - z_1 - z_2 (1 - z_1)^{-1} z_2)^{-1}$$

and

$$r_3 = -z_2^{-1} (1 - z_1) (z_2 - (1 - z_1) z_2^{-1} (1 - z_1))^{-1},$$

with

$$\text{dom } r_2 = \left\{ (Z_1, Z_2) : \det (I - Z_1) \neq 0, \det (I - Z_1 - Z_2 (I - Z_1)^{-1} Z_2) \neq 0 \right\}$$

and

$$\text{dom } r_3 = \left\{ (Z_1, Z_2) : \det (Z_2) \neq 0, \det (Z_2 - (I - Z_1) Z_2^{-1} (I - Z_1)) \neq 0 \right\}.$$  

It is not $a \text{ priori}$ clear whether a $p \times q$ matrix-valued noncommutative rational function is the same thing as a $p \times q$ matrix of (scalar) noncommutative rational functions; the question is whether any $p \times q$ matrix-valued noncommutative rational function can be represented by a $p \times q$ matrix of scalar noncommutative rational expressions. It turns out that this is true, because noncommutative rational functions form a skew field; see [34, Remarks 2.16 and 2.11] for details.
We define the domain of regularity of a matrix-valued noncommutative rational function $R$ as the union of the domains of regularity of all matrix-valued noncommutative rational expressions representing this function, i.e.,

$$\text{dom} R = \bigcup_{R \in \mathcal{R}} \text{dom} R.$$

We emphasize that even for the case of a (scalar) noncommutative rational function, we define its domain using all $1 \times 1$ matrix-valued noncommutative rational expressions representing the function, not just the scalar ones. E.g., in the examples above, it is easily seen that $\text{dom} r_2$ and $\text{dom} r_3$ are both properly contained in $\text{dom} R_1$: so, if $r$ is the corresponding noncommutative rational function, then $\text{dom} r \supseteq \text{dom} R_1$. In fact, the result on the singularities of minimal realization (to be discussed in Section 3) implies that $\text{dom} r = \text{dom} R_1$. See [34, Remark 2.11] for additional discussion and references.

We can also evaluate a matrix-valued noncommutative rational expression $R$ on generic matrices as in the discussion preceding the proof of Proposition 2.1, and introduce a subset $N_R \subseteq \mathbb{N}$ where the evaluation is defined and the extended domain $\text{edom} R$. We then define $N_R$ and $\text{edom} R$, the extended domain of regularity of a matrix-valued noncommutative rational function $R$, by

$$N_R = \bigcup_{R \in \mathcal{R}} N_R, \quad \text{edom} R = \bigcup_{R \in \mathcal{R}} \text{edom} R.$$

We notice that while in general, for $R \in \mathcal{R}$, $\text{dom} R \subset \text{dom} R$, it is always the case that $\text{edom} R = \text{edom} R$ provided that $N_R = N_R$; see [34, Section 2] for additional discussion.

3. Realization theory for noncommutative rational functions

It is a bitter experience that the constellation of foundational facts underlying the classical Kalman realization theory for 1D systems collapses for rational functions of several commuting variables and commutative multidimensional systems. It is all the more amazing that these facts do hold, with obvious modifications, in the noncommutative setting. Noncommutative systems of the form (3.1) below were first studied by Ball–Vinnikov [8] in the conservative setting, in the context of operator model theory for row contractions (Popescu [43, 44, 45, 46]) and of representation theory of the Cuntz algebra (Bratelli–Jorgensen [13] and Davidson–Pitts [21]). On the other hand, noncommutative realizations very similar to (3.2) were considered much earlier in the theory of formal languages and finite automata in the work of Kleene, Schützenberger and Fliess [36, 48, 24]. A comprehensive study of noncommutative realization theory appears in Ball–Groenewald–Malakorn [4, 6, 5]; these papers give a unified framework of structured noncommutative multidimensional linear systems for different kinds of realization formulae. We also mention the paper by Ball–Kaliuzhnyi-Verbovetskyi [7] where an even more general class of noncommutative systems (given though in a frequency domain) was described and the corresponding dilation theory was developed.

A noncommutative multidimensional system is a system with evolution along the free semigroup $F_d$ on $d$ letters $g_1, \ldots, g_d$ rather than along the multidimensional integer lattice $\mathbb{Z}^d$. An example of system equations with evolution along $F_d$ is given by a noncommutative Fornasini–Marchesini system (see [27] for the original
commutative version):

\[
\sum_{FM}^{\Sigma} : \begin{cases} 
  x(g_1 w) = A_1 x(w) + B_1 u(w), \\
  \vdots \\
  x(g_d w) = A_d x(w) + B_d u(w), \\
  y(w) = C x(w) + D u(w),
\end{cases} \quad (w \in F_d).
\]

Applying to the system equations (3.1) an appropriately defined formal noncommutative $z$-transform and under the assumption that the state of the system is initialized at 0 (so that $x(\emptyset) = 0$), we arrive at the input-output relation

\[\hat{y}(z) = T_{\Sigma FM}(z)\hat{u}(z)\]

where the transfer function is given by

\[(3.2) \quad T_{\Sigma FM}(z) = D + C(I_m - A_1 z_1 - \cdots - A_d z_d)^{-1}(B_1 z_1 + \cdots + B_d z_d).
\]

Here $m$ is the dimension of the state space $K^m$ where vectors $x(w)$ live. We see that the transfer function is a matrix-valued noncommutative rational function in noncommuting indeterminates $z_1, \ldots, z_d$ which is regular at zero, i.e., zero belongs to its domain of regularity (a little more precisely, the transfer function is the matrix-valued noncommutative rational function defined by the matrix-valued noncommutative rational expression (3.2)).

The system (3.1) is called controllable (resp., observable) if

\[\text{span}_{w \in F_d} \text{ran}\{A^w B_j\} = K^m, \quad (\text{resp., } \bigcap_{w \in F_d} \ker\{CA^w\} = \{0\}).\]

The following facts are fundamental for the noncommutative realization theory:

1. Every matrix-valued noncommutative rational function which is regular at zero admits a state space realization (3.2).
2. An arbitrary realization (3.2) of a given matrix-valued noncommutative rational function can be reduced via an analogue of the Kalman decomposition to a controllable and observable realization.
3. A realization (3.2) is controllable and observable if and only if it is minimal, i.e., it has the smallest possible state space dimension, and a minimal realization is unique up to a unique similarity.
4. A minimal realization (3.2) can be constructed canonically and explicitly from a matrix-valued noncommutative rational function by means of the corresponding Hankel operator; this ties in with the fact that the Hankel operator corresponding to a matrix-valued noncommutative formal power series has finite rank if and only if the power series represents a rational function (an analogue of Kronecker’s Theorem).
5. In a minimal realization (3.2), the singularities of the transfer function coincide with the singularities of the resolvent; more precisely, the domain of regularity \(2^2\) of the transfer function (3.2) is exactly

\[\{(Z_1, \ldots, Z_d) : \det(I - A_1 \otimes Z_1 - \cdots - A_d \otimes Z_d) \neq 0\}.\]

For the proofs of items (1)–(4), including missing details and exact references to the earlier literature, we refer to [4] where these facts are established in a more general setting of structured noncommutative multidimensional systems.

\(^{2}\)In fact, this is also the extended domain of regularity of the transfer function.
As for item (5), it is amazingly difficult to prove “by hands”; the usual proofs for $d = 1$ use the Hautus test for controllability / observability, but this is no longer available. A proof appears in \[34\] using noncommutative backward shifts which are a particular instance of the difference-differential calculus for noncommutative rational functions. This is a special case of the difference-differential calculus for general noncommutative functions, which are functions on tuples of square matrices of all sizes which respect direct sums and simultaneous similarities. The forthcoming basic reference is \[35\]. The difference-differential calculus for noncommutative rational functions can be developed in a more straightforward manner than in the general case, and we will do this later in Section 4.

Another important example of a structured noncommutative multidimensional system is a noncommutative Givone–Roesser system (for the original commutative version of these systems, see \[28\]):

\[
\Sigma^{GR}: \begin{cases}
  x_1(g_1 w) = A_{11} x_1(w) + \cdots + A_{1d} x_d(w) + B_1 u(w), \\
  \vdots \\
  x_d(g_d w) = A_{d1} x_1(w) + \cdots + A_{dd} x_d + B_d u(w), \\
  y(w) = C_1 x_1(w) + \cdots + C_d x_d(w) + Du(w),
\end{cases} \quad (w \in F_d).
\]

Here $x_j \in K^{m_j}$, i.e., the state space has $d$ components: $K^m = K^{m_1} \oplus \cdots \oplus K^{m_d}$. The transfer function of the noncommutative Givone–Roesser system is given by

\[
T_{\Sigma^{GR}}(z) = D + C (I_m - \Delta(z) A)^{-1} \Delta(z) B,
\]

where $A$ is a $d \times d$ block matrix with blocks $A_{ij} \in K^{m_i \times m_j}$, and $\Delta(z)$ is a $d \times d$ block diagonal matrix, with matrix-valued noncommutative monomials $I_{m_j}, z_j, \ldots, I_{m_d}, z_d$ on the diagonal.

The system \((3.3)\) is called controllable (resp., observable) if

\[
\text{span ran} \{P_j A^w B\} = K^{m_j}, \quad (\text{resp., } \bigcap_{w \in F_d} \ker \{C A^w|_{K^{m_j}}\} = \{0\}, \quad j = 1, \ldots, d.
\]

Here $P_j$ is the orthogonal projection of the state space $K^m$ onto its $j$-th component $K^{m_j}$.

As we already mentioned, items (1)–(4) above hold for arbitrary structured noncommutative multidimensional system realizations, in particular for the noncommutative Givone–Roesser realization \((3.3)\). The result in \[34\], i.e., item (5), has been proved for a much more general class of realizations than \((3.2)\); however, this class does not cover all structured noncommutative multidimensional system realizations, and we conjecture that the result might fail for noncommutative Givone–Roesser realizations.

On the other hand, the symmetry appearing in Givone–Roesser system equation\(\text{4}\) makes Givone–Roesser realizations more suitable for problems where this symmetry is essential. For general structured noncommutative multidimensional systems, basic arithmetic operations on transfer functions (sum, product, inversion) correspond to certain operations on systems, in the same manner as it occurs in the classical 1D case. For noncommutative Givone–Roesser systems, we have

\[\text{4In the case of } K = \mathbb{C}, \text{ the adjoint system } \Sigma^{GR} \text{ has the same form as } \Sigma^{GR} \text{, but with switching the input and output spaces and replacing the coefficient block matrix } \begin{bmatrix} [A_{ij}] & \text{col}[B_j] \\ \text{row}[C_i] & D \end{bmatrix} \text{ by its adjoint.}\]
that, in addition, the adjoint of the transfer function is the transfer function of the
adjoint system. Exploiting these correspondences, one can study noncommutative
rational functions with certain symmetries in terms of their realizations.

In the paper by Alpay–Kaliuzhnyi-Verbovetskyi [2], classes of matrix-valued non-
commutative rational functions with various symmetries were studied in terms of
their Givone–Roesser realizations. A sample result from [2] is a version of the so-
called lossless bounded real lemma (cf. [5] for the general bounded real lemma in
the noncommutative setting). Let $F$ be a $q \times q$ matrix-valued noncommutative ra-
tional function over the field $\mathbb{C}$ which is regular at zero. Let $J = J^{-1} = J^* \in \mathbb{C}^{q \times q}$. Then $F$ is called matrix-$J$-unitary on the set $J_d$ of $d$-tuples of skew-Hermitian $n \times n$ matrices, $n = 1, 2, \ldots$ (which is a noncommutative analogue of the imaginary axis
of the complex plain) if

$$F(Z)(J \otimes I_n)F(Z)^* = J \otimes I_n \quad (Z \in J_d)$$

at all points $Z \in J_d \cap \text{dom } F$. Suppose that $F$ is a $q \times q$ matrix-valued noncommutative rational function over $\mathbb{C}$ which is regular at zero, and let (3.3)–(3.4) be its
minimal noncommutative Givone–Roesser system realization. Then $F$ is matrix-$J$-
unitary on $J_d$ if and only if

(a) $D$ is $J$-unitary, i.e., $DJD^* = J$;

(b) there exists an invertible Hermitian solution $H = \text{diag}(H_1, \ldots, H_d)$, with $H_j \in \mathbb{C}^{m_j \times m_j}$, of the Lyapunov equation

$$A^*H + HA = -C^*JC,$$

and

$$B = -H^{-1}C^*JD.$$ 

This matrix $H$ is uniquely determined by a minimal realization (3.3)–(3.4), and for
this realization it is called the associated structured Hermitian matrix. Moreover,
$F$ is matrix-$J$-inner, i.e., in addition to (3.5), $F$ is $J$-contractive on the set of all
$d$-tuples of $n \times n$ matrices $Z_j$ such that $Z_j + Z_j^* > 0$, $n = 1, 2, \ldots$ (this set is
a noncommutative analogue of the right half-plane), if and only if the associated
structured Hermitian matrix is positive definite.

4. DIFFERENCE-DIFFERENTIAL CALCULUS

In this section we develop the difference-differential calculus for noncommutative
rational functions and discuss various special cases and applications: directional
derivatives, backward shifts, finite difference formulae, higher order difference-
differential operators, and connections with formal power series.

The difference-differential calculus for noncommutative rational functions is based
on difference-differential operators,

$$\Delta_j : K \to K \otimes K, \quad j = 1, \ldots, d,$$

which are noncommutative counterparts of both partial finite difference and partial
differential operators: here $K = \mathbb{K}\langle z_1, \ldots, z_d \rangle$. We extend $\Delta_j$ to matrix-valued
noncommutative rational functions by applying these operators entrywise: we re-
mind the reader that a matrix-valued noncommutative rational function is the same
as a matrix of (scalar) noncommutative rational functions. We thus have

$$\Delta_j : K^{p \times q} \to (K \otimes K)^{p \times q}, \quad j = 1, \ldots, d.$$
Our strategy will be to define $\Delta_j$ on matrix-valued noncommutative rational expressions recursively, starting with matrix-valued noncommutative polynomials, and postulating linearity and an appropriate version of the Leibniz rule. We then check that equivalence is preserved, thus we can define $\Delta_j$ on matrix-valued noncommutative rational functions.

To define $\Delta_j$ on matrix-valued noncommutative rational expressions, we will need to introduce matrix-valued noncommutative rational expressions in two tuples of noncommuting indeterminates, $z_1, \ldots, z_d$ and $z'_1, \ldots, z'_d$. They are obtained by applying successive matrix arithmetic operations to tensor products of matrix-valued noncommutative rational expressions in $z_1, \ldots, z_d$ and in $z'_1, \ldots, z'_d$ and forming block matrices. More precisely,

Definition 4.1.  

(1) If $R$ and $R'$ are $p \times q$ and $p' \times q'$ matrix-valued noncommutative rational expressions in $z_1, \ldots, z_d$ and in $z'_1, \ldots, z'_d$, respectively, then $R \otimes R'$ is a $pp' \times qq'$ matrix-valued noncommutative rational expression in $z_1, \ldots, z_d$ and $z'_1, \ldots, z'_d$, with $\text{dom}(R \otimes R') = \text{dom} R \times \text{dom} R'$, and the evaluation is defined by

$$ (R \otimes R')(Z, Z') = R(Z) \otimes R'(Z'). $$

Here for $Z \in (K^{n \times n})^d$ and $Z' \in (K^{n' \times n'})^d$ we have $R(Z) \in K^{p \times q} \otimes K^{n \times n}$, $R'(Z') \in K^{p' \times q'} \otimes K^{n' \times n'}$, and $R(Z) \otimes R'(Z') \in K^{pp' \times qq'} \otimes K^{n \times n} \otimes K^{n' \times n'}$ via the canonical identification of $K^{p \times q} \otimes K^{p' \times q'}$ with $K^{pp' \times qq'}$.

(2) If $R_1$ and $R_2$ are $p \times q$ matrix-valued noncommutative rational expressions in two tuples of indeterminates, then so is $R_1 + R_2$, $\text{dom}(R_1 + R_2) = \text{dom} R_1 \cap \text{dom} R_2$, and the evaluation is given by

$$ (R_1 + R_2)(Z, Z') = R_1(Z, Z') + R_2(Z, Z'). $$

(3) If $R_1$ and $R_2$ are $p \times q$ and $q \times r$ matrix-valued noncommutative rational expressions in two tuples of indeterminates, then $R_1 R_2$ is $p \times r$ matrix-valued, $\text{dom}(R_1 R_2) = \text{dom} R_1 \cap \text{dom} R_2$, and the evaluation is given by

$$ (R_1 R_2)(Z, Z') = R_1(Z, Z') R_2(Z, Z'). $$

(4) If $R$ is a $p \times p$ matrix-valued noncommutative rational expression in two tuples of indeterminates, and $\text{det} R(Z, Z')$ does not vanish identically on $\text{dom} R$, then so is $R^{-1}$, $\text{dom} R^{-1} = \{(Z, Z') \in \text{dom} R : \text{det} R(Z, Z') \neq 0\}$, and

$$ R^{-1}(Z, Z') = R(Z, Z')^{-1}. $$

(5) If $R_{ab}$, $a = 1, \ldots, p_2$, $b = 1, \ldots, q_2$, are $p_1 \times q_1$ matrix-valued noncommutative rational expressions in two tuples of indeterminates, then $R = [R_{ab}]_{a=1,\ldots,p_2;\,b=1,\ldots,q_2}$ is $p_1 p_2 \times q_1 q_2$ matrix-valued,

$$ \text{dom} R = \bigcap_{a=1,\ldots,p_2;\,b=1,\ldots,q_2} \text{dom} R_{ab} $$

and

$$ R(Z, Z') = [R_{ab}(Z, Z')]_{a=1,\ldots,p_2;\,b=1,\ldots,q_2}. $$
We notice that for a $p \times q$ matrix-valued noncommutative rational expression $R$ in two tuples of indeterminates and for $Z \in (K^{n \times n})^d$ and $Z' \in (K^{n' \times n'})^d$ with $(Z, Z') \in \text{dom } R$, the evaluation $R(Z, Z') \in K^{p \times q} \otimes K^{n \times n} \otimes K^{n' \times n'}$. We will often use the canonical identification of $K^{p \times q} \otimes K^{n \times n} \otimes K^{n' \times n'} = K^{p \times q} \otimes (K^{n \times n} \otimes K^{n' \times n'})$ with $K^{pnn' \times qnn'}$ (cf. page 8). Thus we will often view $R(Z, Z')$ as a $pmn' \times qmn'$ matrix. (An alternative interpretation of the values $R(Z, Z')$ as linear mappings will be considered later — see the discussion preceding Theorem 4.8.)

Two $p \times q$ matrix-valued noncommutative rational expressions, $R_1$ and $R_2$, in two tuples of indeterminates are called equivalent if $\text{dom } R_1 \cap \text{dom } R_2 \neq \emptyset$ and $R_1(Z, Z') = R_2(Z, Z')$ for all pairs of $d$-tuples $(Z, Z')$ in $\text{dom } R_1 \cap \text{dom } R_2$. It would be natural to define matrix-valued noncommutative rational functions in two tuples of noncommuting indeterminates as the corresponding equivalence classes. Doing this in a meaningful way requires analogues of Propositions 2.1 and 2.2; see [18, 19] for related issues. Here we restrict ourselves to a relatively simple situation.

**Theorem 4.2.** Equivalence classes of $p \times q$ matrix-valued noncommutative rational expressions in two tuples of indeterminates, which are formed by using only the rules (1), (2), (3), and (5) in Definition 4.1, are in a natural one-to-one correspondence with $p \times q$ matrices over $K \otimes K$.

**Proof.** Since a matrix-valued noncommutative rational expression is equivalent to a matrix of scalar noncommutative rational expressions, it is clear that any matrix-valued noncommutative rational expression in two tuples of indeterminates which is formed by using only the rules (1), (2), (3), and (5), is equivalent to a matrix whose entries are sums of tensor products of noncommutative rational expressions. It only remains to show that the corresponding elements of $K \otimes K$ are uniquely determined. Let $r_1, \ldots, r_\ell$ and $r'_1, \ldots, r'_\ell$ be noncommutative rational functions in $z_1, \ldots, z_d$ and in $z'_1, \ldots, z'_d$ represented by noncommutative rational expressions $r_1, \ldots, r_\ell$ and $r'_1, \ldots, r'_\ell$, respectively. We have to show that if $r_1 \otimes r'_1 + \cdots + r_\ell \otimes r'_\ell$ is equivalent to zero then $r_1 \otimes r'_1 + \cdots + r_\ell \otimes r'_\ell = 0$ in $K \otimes K$. We may assume that $r_1, \ldots, r_\ell$ are linearly independent over $K$, otherwise the number of terms in the tensor combination can be reduced by one.

We may assume that $r'_1, \ldots, r'_\ell$ are not all equivalent to zero, since otherwise there is nothing to prove. Take $Z' \in \text{dom } r'_1 \cap \cdots \cap \text{dom } r'_\ell$ such that $r'_i(Z'), \ldots, r'_\ell(Z')$ are not all zero. (The existence of such a $Z'$ is established analogously to the reasoning in the footnote on page 9) This implies that the matrix elements $(r'_i(Z'))_{ij}, \ldots, (r'_\ell(Z'))_{ij}$ are not all zero for some $i$ and $j$. For an arbitrary $Z \in \text{dom } r_1 \cap \cdots \cap \text{dom } r_\ell$, we have $r_1(Z) \otimes r'_1(Z') + \cdots + r_\ell(Z) \otimes r'_\ell(Z') = 0$, and therefore

$$(r'_i(Z'))_{ij}r_1(Z) + \cdots + (r'_\ell(Z'))_{ij}r_\ell(Z) = 0$$

is a nontrivial linear dependence relation for matrices $r_1(Z), \ldots, r_\ell(Z)$. Therefore $r_1, \ldots, r_\ell$ are linearly dependent, a contradiction. \hfill \Box

We define the domain of regularity, $\text{dom } \mathcal{R}$, of a matrix $\mathcal{R}$ over $K \otimes K$ as the union of the domains of regularity of all matrix-valued noncommutative rational expressions in two tuples of indeterminates representing $\mathcal{R}$.

We can also evaluate a matrix-valued noncommutative rational expression $R$ in two tuples of indeterminates on generic matrices $T_1, \ldots, T_d$ and $T'_1, \ldots, T'_d$, as in the proof of Proposition 2.1 and introduce a subset $N_R \subseteq \mathbb{N}$ where the evaluation
is defined and the extended domain, edom $R$. We then define $N_R$ and edom $\mathfrak{R}$, the extended domain of regularity of a matrix $\mathfrak{R}$ over $K \otimes K$, by taking the union over all matrix-valued noncommutative rational expressions in two tuples of indeterminates representing $\mathfrak{R}$.

**Remark 4.3.** We can also introduce matrix-valued noncommutative rational expressions in $\ell$ tuples of noncommuting indeterminates $z_1^{(j)}, \ldots, z_d^{(j)}$, $j = 1, \ldots, \ell$, analogously to Definition 4.1 except that $R$ and $R'$ in rule (1) are now matrix-valued noncommutative rational expressions in $t$ tuples and in $s$ tuples of indeterminates respectively, with $t + s = \ell$. Namely, if $R$ and $R'$ are $p \times q$ and $p' \times q'$ matrix-valued noncommutative rational expressions in $z_1^{(j)}, \ldots, z_d^{(j)}$, $j = 1, \ldots, t$, and in $z_1^{(j)}, \ldots, z_d^{(j)}$, $j = t + 1, \ldots, \ell$, respectively, then $R \otimes R'$ is a $pp' \times qq'$ matrix-valued noncommutative rational expression in $z_1^{(j)}, \ldots, z_d^{(j)}$, $j = 1, \ldots, \ell$, with $\text{dom}(R \otimes R') = \text{dom} R \times \text{dom} R'$, and the evaluation is defined by

$$(R \otimes R')(Z^{(1)}, \ldots, Z^{(\ell)}) = R(Z^{(1)}, \ldots, Z^{(t)}) \otimes R'(Z^{(t+1)}, \ldots, Z^{(\ell)}).$$

We notice that for a $p \times q$ matrix-valued noncommutative rational expression $R$ in $\ell$ tuples of indeterminates and for $Z^{(j)} \in (K^{n_j \times n_j})^d$, $j = 1, \ldots, \ell$, with $(Z^{(1)}, \ldots, Z^{(\ell)}) \in \text{dom} R$, the evaluation

$$R(Z^{(1)}, \ldots, Z^{(\ell)}) \in K^{p \times q} \otimes K^{n_1 \times n_1} \otimes \cdots \otimes K^{n_\ell \times n_\ell}.$$

We will often use the canonical identification of

$$K^{p \times q} \otimes K^{n_1 \times n_1} \otimes \cdots \otimes K^{n_{\ell-1} \times n_{\ell-1}} \otimes K^{n_\ell \times n_\ell}
\cong K^{p \times q} \otimes \left(\left(K^{n_1 \times n_1} \otimes \cdots \otimes \left(K^{n_{\ell-1} \times n_{\ell-1}} \otimes K^{n_\ell \times n_\ell}\right)\right)\right)$$

with $K^{pm_1 \cdots n_\ell \times qn_1 \cdots n_\ell}$. Thus we will often view $R(Z^{(1)}, \ldots, Z^{(\ell)})$ as a $pm_1 \cdots n_\ell \times qn_1 \cdots n_\ell$ matrix. We then define the equivalence of matrix-valued noncommutative rational expressions in $\ell$ tuples of indeterminates and show, as in Theorem 4.2, that equivalence classes of $p \times q$ matrix-valued noncommutative rational expressions in $\ell$ tuples of indeterminates, which are formed by using only the analogues of the rules (1), (2), (3), and (5) in Definition 4.1, are in a natural one-to-one correspondence with $p \times q$ matrices over $K^{\otimes \ell}$. We can now define the domain of regularity, dom $\mathfrak{R}$, of a matrix $\mathfrak{R}$ over $K^{\otimes \ell}$ as the union of the domains of regularity of all matrix-valued noncommutative rational expressions in $\ell$ tuples of indeterminates representing $\mathfrak{R}$. We can also introduce a subset $N_R \subseteq \mathbb{N}$ where the evaluation on generic matrices of a matrix-valued noncommutative rational expression $R$ in $\ell$ tuples of indeterminates is defined and the extended domain, edom $R$; we then define $N_R$ and edom $\mathfrak{R}$, the extended domain of regularity of a matrix $\mathfrak{R}$ over $K^{\otimes \ell}$.

We proceed now with the definition of difference-differential operators $\Delta_j$. For a $p \times q$ matrix-valued noncommutative rational expression $R$, $\Delta_j(R)$ is a $p \times q$ matrix-valued noncommutative rational expression in two tuples of indeterminates.

**Definition 4.4.** (1) For a matrix-valued noncommutative polynomial $P \in K^{p \times q}(z_1, \ldots, z_d)$, $P(z) = \sum_{w \in F_d} P_w z^w$, set

$$\Delta_j(P) = \sum_{w \in F_d} P_w \sum_{u, v : w = uv} z^u \otimes z^v \in (K(z_1, \ldots, z_d) \otimes K(z'_1, \ldots, z'_d))^{p \times q}.$$
(2) If $R_1$ and $R_2$ are $p \times q$ matrix-valued noncommutative rational expressions, then
\[ \Delta_j(R_1 + R_2) = \Delta_j(R_1) + \Delta_j(R_2). \]

(3) If $R_1$ is a $p \times q$ matrix-valued noncommutative rational expression and $R_2$ is a $q \times s$ matrix-valued noncommutative rational expression, then
\[ \Delta_j(R_1R_2) = \Delta_j(R_1)(1 \otimes R_2) + (R_1 \otimes 1)\Delta_j(R_2). \]

(4) If $R$ is a $p \times p$ matrix-valued noncommutative rational expression which is not identically singular, then
\[ \Delta_j(R^{-1}) = -(R^{-1} \otimes 1)\Delta_j(R)(1 \otimes R^{-1}). \]

(5) If $R_{ab}$, $a = 1, \ldots, p_2$, $b = 1, \ldots, q_2$, are $p_1 \times q_1$ matrix-valued noncommutative rational expressions and $R = [R_{ab}]_{a=1, \ldots, p_2; b=1, \ldots, q_2}$, then $\Delta_j(R) = [(\Delta_j(R))_{ab}]_{a=1, \ldots, p_2; b=1, \ldots, q_2}$.
\[ (\Delta_j(R))_{ab} = \Delta_j(R_{ab}). \]

It is clear that $\text{dom } \Delta_j(R) = \text{dom } R \times \text{dom } R$.

We give some examples to illustrate Definition 4.4.

**Example 4.5.** For a (scalar) noncommutative polynomial of total degree two in two indeterminates
\[ p = \alpha + \beta z_1 + \gamma z_2 + \delta z_1^2 + \epsilon z_1 z_2 + \zeta z_2 z_1 + \eta z_2^2, \]
we have
\[ \Delta_1(p) = \beta (1 \otimes 1) + \delta (1 \otimes z_1') + \delta (z_1 \otimes 1) + \epsilon (1 \otimes z_2') + \zeta (z_2 \otimes 1) \]
and
\[ \Delta_2(p) = \gamma (1 \otimes 1) + \epsilon (z_1 \otimes 1) + \zeta (1 \otimes z_1') + \eta (1 \otimes z_2') + \eta (z_2 \otimes 1). \]

**Example 4.6.** For the three equivalent $1 \times 1$ matrix-valued rational expressions $R_1$, $r_2$, and $r_3$ introduced on page 8 we have (up to trivial equivalences)
\begin{align*}
\Delta_1(R_1) & \sim ([1 \ 0] \otimes 1) \left( \begin{bmatrix} 1 - z_1 & -z_2 \\ -z_2 & 1 - z_1 \end{bmatrix}^{-1} \otimes 1 \right) \\
& \quad \cdot \left( 1 \otimes \begin{bmatrix} 1 - z_1' & -z_2' \\ -z_2' & 1 - z_1' \end{bmatrix}^{-1} \right) \left( 1 \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \\
\Delta_2(R_1) & \sim ([1 \ 0] \otimes 1) \left( \begin{bmatrix} 1 - z_1 & -z_2 \\ -z_2 & 1 - z_1 \end{bmatrix}^{-1} \otimes 1 \right) \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (1 \otimes 1) \\
& \quad \cdot \left( 1 \otimes \begin{bmatrix} 1 - z_1' & -z_2' \\ -z_2' & 1 - z_1' \end{bmatrix}^{-1} \right) \left( 1 \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \\
\Delta_1(r_2) & \sim ((1 - z_1 - z_2)(1 - z_1)^{-1}z_2)^{-1} \otimes 1 \\
& \quad \cdot \left( 1 \otimes 1 + (z_2(1 - z_1)^{-1} \otimes 1)(1 \otimes (1 - z_1')^{-1}z_2') ight) \\
& \quad \cdot \left( 1 \otimes (1 - z_1' - z_2'(1 - z_1')^{-1}z_2')^{-1} \right),
\end{align*}
\[ \Delta_2(r_2) \sim \left( (1 - z_1 - z_2 (1 - z_1)^{-1} z_2)^{-1} \otimes 1 \right) \cdot \left( 1 \otimes (1 - z'_1)^{-1} z'_2 + z_2 (1 - z_1)^{-1} \otimes 1 \right) \cdot \left( 1 \otimes (1 - z'_1 - z'_2 (1 - z'_1)^{-1} z'_2)^{-1} \right), \]

\[ \Delta_1(r_3) \sim (z_2^{-1} \otimes 1) \left( \left( \left( \left( (z_2 (1 - z_1) z_1^{-1} (1 - z_1)^{-1}) \right)^{-1} \right) \right) \cdot \left( (1 \otimes (1 - z'_1) (z_2 - (1 - z'_1) z'_2^{-1} (1 - z'_1))^{-1} \otimes 1 \right) \right) \cdot \left( 1 \otimes (1 - z'_1) + (1 - z_1) z_2^{-1} \otimes 1 \right) \cdot \left( 1 \otimes (z'_2 - (1 - z'_1) z'_2^{-1} (1 - z'_1))^{-1} \right), \]

\[ \Delta_2(r_3) \sim (z_2^{-1} \otimes 1) \left( 1 \otimes z'_2^{-1} \right) \left( (1 \otimes (1 - z'_1) (z_2 - (1 - z'_1) z'_2^{-1} (1 - z'_1))^{-1} \right) \right) + \left( z_2^{-1} (1 - z_1) \otimes 1 \right) \left( (z_2 - (1 - z_1) z_2^{-1} (1 - z_1))^{-1} \otimes 1 \right) \cdot \left( 1 + (1 - z_1) z_2^{-1} \otimes 1 \right) \left( (1 \otimes (1 - z'_1) z'_2^{-1} (1 - z'_1))^{-1} \right) \cdot \left( 1 \otimes (z'_2 - (1 - z'_1))^{-1} \right). \]

**Example 4.7.** Consider a \( p \times q \) matrix-valued noncommutative rational expression

\[ R = C(I_m - A_1 z_1 - \cdots - A_d z_d)^{-1} B, \]

where \( m \in \mathbb{N}, A_j \in \mathbb{K}^{m \times m} (j = 1, \ldots, d), B \in \mathbb{K}^{m \times q}, C \in \mathbb{K}^{p \times m}. \) (This is a recognizable series realization of a \( p \times q \) matrix-valued noncommutative rational function; see [33, 45, 21].) We have

\[ \Delta_j(R) = (C \otimes 1) \cdot \left( (I_m - A_1 z_1 - \cdots - A_d z_d)^{-1} \otimes 1 \right) \cdot A_j (1 \otimes 1) \cdot \left( 1 \otimes (I_m - A_1 z'_1 - \cdots - A_d z'_d)^{-1} \right) \cdot (1 \otimes B), \]

or up to trivial equivalences,

\[ \Delta_j(R) \sim (C(I_m - A_1 z_1 - \cdots - A_d z_d)^{-1} A_j \otimes 1) \cdot (1 \otimes (I_m - A_1 z'_1 - \cdots - A_d z'_d)^{-1} B) \]

\[ \sim \left( C(I_m - A_1 z_1 - \cdots - A_d z_d)^{-1} \otimes 1 \right) \cdot (1 \otimes A_j (I_m - A_1 z'_1 - \cdots - A_d z'_d)^{-1} B). \]

It is not *a priori* clear from Definition 4.4 that \( \Delta_j \) preserves the equivalence of matrix-valued noncommutative rational expressions and can be thus defined on matrix-valued noncommutative rational functions. This is a consequence of the following key theorem that relates the evaluation \( \Delta_j(R)(Z, Z') \) to the evaluation of \( R \) on \( d \)-tuples of \( 2 \times 2 \) block upper triangular matrices with \( Z_j \) and \( Z'_j \) on block diagonals. We will identify \( \mathbb{K}^{n \times n} \otimes \mathbb{K}^{n' \times n'} \) with hom(\( \mathbb{K}^{n \times n'}, \mathbb{K}^{n' \times n'} \)), so that \( \sum_i A_i \otimes B_i \) corresponds to the linear mapping \( H \mapsto \sum_i A_i H B_i \). This correspondence extends naturally to \( p \times q \) matrices: we identify \( \mathbb{K}^{p \times q} \otimes \mathbb{K}^{n \times n} \otimes \mathbb{K}^{n' \times n'} \cong (\mathbb{K}^{n' \times n'} \otimes \mathbb{K}^{n \times n'})^{p \times q} \) with hom \( (\mathbb{K}^{n \times n'}, (\mathbb{K}^{n \times n'})^{p \times q}) \), so that \( \sum_i A_i^{(j k)} \otimes B_i^{(j k)} \) corresponds to the linear mapping \( H \mapsto \left[ \sum_i A_i^{(j k)} H B_i^{(j k)} \right] \). We will use the \( pm \times pm \) permutation matrices \( P(p, n) = |E_{ij}^{T}|_{i=1,...,p; j=1,...,n} \) where each \( E_{ij} \in \mathbb{K}^{p \times n} \) has entry 1 in position \((i, j)\) and all other entries are zero. These
matrices allow us to change the order of factors in tensor products: \( A \otimes B = P(n, p)(B \otimes A)P(m, q)^\top \) for any \( A \in \mathbb{K}^{n \times m} \) and \( B \in \mathbb{K}^{p \times q} \). See [32] pages 259–261. We also use the notation \( \text{dom}_n R := \text{dom} R \cap (\mathbb{K}^{n \times n})^d \).

**Theorem 4.8.** Let \( R \) be a \( p \times q \) matrix-valued noncommutative rational expression. Let \( Z \in \text{dom}_n R \) and \( Z' \in \text{dom}_{n'} R \), and let \( W = (W_1, \ldots, W_d) \in (\mathbb{K}^{n \times n})^d \). Then

\[
\begin{bmatrix}
Z & W \\
0 & Z'
\end{bmatrix} = \begin{bmatrix}
Z_1 & W_1 \\
0 & Z'_1
\end{bmatrix}, \ldots, \begin{bmatrix}
Z_d & W_d \\
0 & Z'_d
\end{bmatrix} \in \text{dom}_{n+n'} R
\]

and

\[
P(n + n', p)R \left( \begin{bmatrix}
Z & W \\
0 & Z'
\end{bmatrix} \right) = \sum_{w \in F_d} \begin{bmatrix}
Z & W \\
0 & Z'
\end{bmatrix} w \otimes P_w
\]

\[
= \sum_{\ell, i_1, \ldots, i_d} \begin{bmatrix}
Z_{i_1} \cdots Z_{i_d} & \sum_{k=1}^{d} Z_{i_1} \cdots Z_{i_{k-1}} W_i Z'_{i_{k+1}} \cdots Z'_{i_d} \otimes P_{g_{i_1} \cdots g_{i_d}} \\
0 & Z'_{i_1} \cdots Z'_{i_d}
\end{bmatrix}
\]

\[
= \sum_{w} Z^w \otimes P_w \sum_{j=1}^{d} \sum_{u,v} Z^u W_j Z'^v \otimes P_w
\]

\[
= \left[ P(n, p)P(Z)P(n, q)^\top P(n, p)\left( \sum_{j=1}^{d} \Delta_j(P)(Z, Z')(W_j) \right) P(n', q)^\top \right].
\]

**Proof.** We establish (4.3) recursively; the reasoning will also imply that \( \begin{bmatrix}
Z & W \\
0 & Z'
\end{bmatrix} \in \text{dom}_{n+n'} R \).

We first verify that (4.3) holds when \( R \) is a matrix-valued noncommutative polynomial \( P = \sum_{w \in F_d} P_w z^w \).

\[
P(n + n', p)P \left( \begin{bmatrix}
Z & W \\
0 & Z'
\end{bmatrix} \right) = \sum_{w \in F_d} \begin{bmatrix}
Z & W \\
0 & Z'
\end{bmatrix} w \otimes P_w
\]

\[
= \sum_{\ell, i_1, \ldots, i_d} \begin{bmatrix}
Z_{i_1} \cdots Z_{i_d} & \sum_{k=1}^{d} Z_{i_1} \cdots Z_{i_{k-1}} W_i Z'_{i_{k+1}} \cdots Z'_{i_d} \otimes P_{g_{i_1} \cdots g_{i_d}} \\
0 & Z'_{i_1} \cdots Z'_{i_d}
\end{bmatrix}
\]

\[
= \sum_{w} Z^w \otimes P_w \sum_{j=1}^{d} \sum_{u,v} Z^u W_j Z'^v \otimes P_w
\]

\[
= \left[ P(n, p)P(Z)P(n, q)^\top P(n, p)\left( \sum_{j=1}^{d} \Delta_j(P)(Z, Z')(W_j) \right) P(n', q)^\top \right].
\]

If (4.3) is true for \( R_1 \) and for \( R_2 \) then it is clearly true for \( R = R_1 + R_2 \).

Now, assume that (4.3) is true for a \( p \times q \) matrix-valued noncommutative rational expression \( R_1 \) and for a \( q \times r \) matrix-valued noncommutative rational expression.
Thus, (4.3) is true for the matrix-valued noncommutative rational expression $R_2$. Then

$$P(n + n', p)(R_1R_2) \begin{bmatrix} Z & W \\ 0 & Z' \end{bmatrix} P(n + n', r)^\top$$

$$= P(n+n',p)R_1 \begin{bmatrix} Z & W \\ 0 & Z' \end{bmatrix} P(n+n',q)^\top P(n+n',q)R_2 \begin{bmatrix} Z & W \\ 0 & Z' \end{bmatrix} P(n+n',r)^\top$$

$$= \begin{bmatrix} P(n,p)R_1(Z)P(n,q)^\top & P(n,p) \left( \sum_{j=1}^{d} \Delta_j(R_1)(Z,Z')(W_j) \right) P(n',q)^\top \\ 0 & P(n',p)R_1(Z')P(n',q)^\top \end{bmatrix}$$

$$\cdot \begin{bmatrix} P(n,q)R_2(Z)P(n,r)^\top & P(n,q) \left( \sum_{j=1}^{d} \Delta_j(R_2)(Z,Z')(W_j) \right) P(n',r)^\top \\ 0 & P(n',q)R_2(Z')P(n',r)^\top \end{bmatrix}$$

$$= \begin{bmatrix} P(n,p)R_1(Z)P(n,r)^\top \\ 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} P(n,p) \left( \sum_{j=1}^{d} (R_1(Z)\Delta_j(R_2)(Z,Z')(W_j) + \Delta_j(R_1)(Z,Z')(W_j)R_2(Z')) \right) P(n',r)^\top \\ P(n',p)R_1(Z')P(n',r)^\top \end{bmatrix}$$

$$= \begin{bmatrix} P(n,p)(R_1R_2)(Z)P(n,r)^\top & P(n,p) \left( \sum_{j=1}^{d} \Delta_j(R_1R_2)(Z,Z')(W_j) \right) P(n',r)^\top \\ 0 & P(n',p)(R_1R_2)(Z')P(n',r)^\top \end{bmatrix}.$$  

Thus, (4.3) is true for the $p \times r$ matrix-valued noncommutative rational expression $R_1R_2$.

Next, assume that (4.3) is true for a $p \times p$ matrix-valued noncommutative rational expression $R$ which is not identically singular. Then

$$P(n + n', p)R^{-1} \begin{bmatrix} Z & W \\ 0 & Z' \end{bmatrix} P(n + n', p)^\top$$

$$= \begin{bmatrix} P(n,n',p)R \begin{bmatrix} Z & W \\ 0 & Z' \end{bmatrix} P(n,n',p)^\top \\ 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} P(n,p)R(Z)P(n,p)^\top & P(n,p) \left( \sum_{j=1}^{d} \Delta_j(R)(Z,Z')(W_j) \right) P(n',p)^\top \\ 0 & P(n',p)R(Z')P(n',p)^\top \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} P(n,p)R(Z)^{-1}P(n,p)^\top \\ 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} -P(n,p)R(Z)^{-1} \left( \sum_{j=1}^{d} \Delta_j(R)(Z,Z')(W_j) \right) R(Z')^{-1}P(n',p)^\top \\ P(n',p)R(Z')^{-1}P(n',p)^\top \end{bmatrix}$$

$$= \begin{bmatrix} P(n,p)R^{-1}(Z)P(n,p)^\top & P(n,p) \left( \sum_{j=1}^{d} \Delta_j(R^{-1})(Z,Z')(W_j) \right) P(n',p)^\top \\ 0 & P(n',p)R^{-1}(Z')P(n',p)^\top \end{bmatrix}.$$  

Thus, (4.3) is true for the matrix-valued noncommutative rational expression $R^{-1}$. 
Finally, we will show that if \((4.3)\) is true for \(R_{ab} \neq 0\), \(a = 1, \ldots, p_2, b = 1, \ldots, q_2\), then it is also true for \(R = [R_{ab}]\). Clearly, it suffices to prove \((4.3)\) for the case where only one block \(R_{ab}\) is nonzero, i.e., \(R = E_{ab} \otimes R_{ab}\), with some \(a, b\) and \(E_{ab}\) a \(p_2 \times q_2\) matrix with 1 at the \((a, b)\) position and 0 elsewhere. Since \((E_{ab} \otimes R_{ab})(X) = (E_{ab} \otimes I_{p_1})(I_{q_2} \otimes R_{ab})(X)\), this boils down to proving \((4.3)\) for \(I_{q_2} \otimes R_{ab}\). Simplifying the notation, we can state the problem as follows: show that if a \(p \times q\) matrix-valued noncommutative rational expression \(R\) satisfies \((4.3)\), then so does the \(sp \times sq\) matrix-valued noncommutative rational expression \(I_s \otimes R\) for every \(s \in \mathbb{N}\). We first observe that for any \(m, s \in \mathbb{N}\) and any \(X \in \text{dom}_m R\) one has \(X \otimes I_s \in \text{dom}_m R\) and \(R(X) \otimes I_s = R(X \otimes I_s)\) — this follows directly from the definition of matrix-valued noncommutative rational expressions. Second, using this observation and the identities of the form

\[ P(m, sp) = P(ms, p)P(mp, s) \]

(see [122], Problem 20, page 266], we obtain that

\[
P(m, sp)(I_s \otimes R)(X)P(m, sq)^\top = P(ms, p)(mp, s)(I_s \otimes R(X))P(ms, q)^\top P(ms, q)^\top
\]

Then we have

\[
P(n + n', sp)(I_s \otimes R)\begin{bmatrix} Z & W' \\ 0 & Z' \end{bmatrix} P(n + n', sq)^\top
\]

\[
= P((n + n')s, pr)R\begin{bmatrix} Z \otimes I_s & W \otimes I_s \\ 0 & Z' \otimes I_s \end{bmatrix} P((n + n')s, q)^\top
\]

\[
= \begin{bmatrix} P(ns, p)R(Z \otimes I_s)P(ns, q)^\top \\ 0 \\ P(ns, p)\left(\sum_{j=1}^d \Delta_j(R)(Z \otimes I_s, Z' \otimes I_s)(W_j \otimes I_s)\right) P(ns, q)^\top \\ 0 \\ P(ns, p)(I_s \otimes R)(Z')P(ns, q)^\top \end{bmatrix},
\]

i.e., \(I_s \otimes R\) satisfies \((4.3)\).

The proof is complete. \(\square\)

Remark 4.9. The equality \((4.3)\) is the accurate statement of the fact that matrix-valued noncommutative rational expressions respect a block triangular matrix structure, in particular direct sums — compare [24], formulae (2.3), (2.5), (2.12), (2.13), (2.19)], where the permutation matrices are missing. (This does not affect any subsequent arguments there.)

Corollary 4.10. For any matrix-valued noncommutative rational expression \(R\),

\[ \text{edom} \Delta_j(R) \supseteq \text{edom} R \times \text{edom} R. \]

The proof is obtained by substituting generic matrices \(T_1, \ldots, T_d\) and \(T_1', \ldots, T_d'\) into \((4.3)\) — the details are similar to [34], Corollary 2.12.]
Corollary 4.11. If $R_1$ and $R_2$ are two equivalent matrix-valued rational expressions then $\Delta_j(R_1)$ and $\Delta_j(R_2)$ are also equivalent.

The proof is immediate from (4.3).

Corollary 4.11 and Theorem 4.2 allow us to define $\Delta_j\mathcal{R}$ for a matrix-valued noncommutative rational function $\mathcal{R}$: $\Delta_j\mathcal{R}$ is the matrix over $K \otimes K$ corresponding to the equivalence class of $\Delta_j(R)$ for any $R \in \mathcal{R}$. We have

$$\text{dom } \Delta_j\mathcal{R} \supseteq \text{dom } \mathcal{R} \times \text{dom } \mathcal{R}, \quad \text{edom } \Delta_j\mathcal{R} \supseteq \text{edom } \mathcal{R} \times \text{edom } \mathcal{R},$$

where the second inclusion follows from Corollary 4.10.

Remark 4.12. We emphasize that while we always have equality $\text{dom } \Delta_j(R) = \text{dom } (R) \times \text{dom } (R)$ for a matrix-valued noncommutative rational expression $R$, we may have strict inclusion $\text{dom } \Delta_j\mathcal{R} \supset \text{dom } \mathcal{R} \times \text{dom } \mathcal{R}$. As an example, let $r = z^{-1}$. Then we have

$$\Delta_2r = -(z^{-1} \otimes 1) \cdot 0(1 \otimes 1) \cdot (1 \otimes z^{-1}) \sim 0,$$

and for the corresponding noncommutative rational function $r$ we have

$$\text{dom } \Delta_2r = \prod_{n=1}^{\infty} (K^n)^2,$$

while

$$\text{dom } r = \{(Z_1, Z_2) : \det Z_1 \neq 0\}.$$

This example also shows that we have a strict inclusion

$$\text{edom } (\Delta_2r) \supset \text{edom } (r) \times \text{edom } (r).$$

We proceed to describe some of the many facets of the noncommutative difference-differential operators.

4.1. Directional derivatives. Evaluating $\Delta_j(R)$, $j = 1, \ldots, d$, at $Z' = Z$ yields the differential or the directional derivatives of a matrix-valued noncommutative rational expression $R$ at $Z$:

$$\sum_{j=1}^{d} \Delta_j(R)(Z, Z)\langle W_j \rangle = \frac{d}{dt} R(Z + tW)\bigg|_{t=0}. \tag{4.4}$$

Here $Z \in \text{dom } R$ and we view $R(Z + tW)$ as a matrix-valued rational function in the indeterminate $t$. The formula (4.4) follows easily from the recursive definition of $\Delta_j$ and the corresponding properties of the derivative of a matrix-valued rational function in one indeterminate. Of course, one can also write the analogue of (4.4) for matrix-valued noncommutative rational functions.

4.2. Backward shifts. Let $R$ be a matrix-valued noncommutative rational expression which is regular at zero, i.e., $0 \in \text{dom}_1 R$. Then

$$\Delta_j(R)(Z, 0) = R_j R(Z), \quad \Delta_j(R)(0, Z) = L_j R(Z), \tag{4.5}$$

where $R_j$ and $L_j$ are the right and left backward shift operators introduced in [34]. The formula (4.5) follows easily from the recursive definitions of $\Delta_j$ and of the backward shifts. Of course, one can also write the analogue of (4.5) for matrix-valued noncommutative rational functions. As an illustration of the action of backward shifts, for the polynomial $p$ of Example 4.5 we have

$$R_1p = \beta + \delta z_1 + \zeta z_2, \quad L_1p = \beta + \delta z_1 + \epsilon z_2,$$
\[ R_2 p = \gamma + \epsilon z_1 + \eta z_2, \quad L_2 p = \gamma + \zeta z_1 + \eta z_2, \]

and for the recognizable series realization \(4.1\) of Example \(4.7\) we have
\[ R_j R = C(I_m - A_1 z_1 - \cdots - A_d z_d)\!^{-1} A_j B, \quad L_j R = C A_j (I_m - A_1 z_1 - \cdots - A_d z_d)\!^{-1} B. \]

4.3. Finite difference formulae. For a \(p \times q\) matrix-valued noncommutative rational expression \(R\), and for \(Z(0), Z \in \text{dom}_n R\), we have the noncommutative finite difference formula
\[ R(Z) - R\left(Z(0)\right) = \sum_{j=1}^{d} \Delta_j\left(R\right) \left(Z(0), Z\right) \left(Z_j - Z_j^{(0)}\right), \]

which can also be proved recursively, and extends naturally to matrix-valued noncommutative rational functions.

4.4. Higher order difference-differential operators. We can iterate the difference-differential operators \(\Delta_j\): we define a linear mapping
\[ \Delta_j: K^{\otimes \ell} \to K^{\otimes (\ell+1)} \]
by its action on pure tensors as
\[ \Delta_j(t_1 \otimes \cdots \otimes t_\ell) = t_1 \otimes \cdots \otimes t_{\ell-1} \otimes \Delta_j t_\ell. \]

It is easy to check that \(\Delta_j\) satisfies the following version of the Leibniz rule:
\[ \Delta_j(t_\ell t') = \Delta_j t \cdot \iota_\ell t' + (t \otimes 1) \cdot \Delta_j t', \]
for all \(t, t' \in K^{\otimes \ell}\), where the linear mapping \(\iota_\ell: K^{\otimes \ell} \to K^{\otimes \ell+1}\) (actually, a homomorphism of \(K\)-algebras) is defined on pure tensors by
\[ \iota_\ell(t_1 \otimes \cdots \otimes t_{\ell-1} \otimes t_\ell) = t_1 \otimes \cdots \otimes t_{\ell-1} \otimes 1 \otimes t_\ell. \]

Applying \(4.6\) to both sides of the identity \(tt^{-1} = 1\), we obtain that
\[ \Delta_j(t^{-1}) = -(t^{-1} \otimes 1) \cdot \Delta_j t \cdot \iota_\ell(t^{-1}). \]

We extend \(\Delta_j\) and \(\iota_\ell\) entrywise to matrices.

We can now define, for a word \(w = g_{i_1} \cdots g_{i_\ell}\) of length \(\ell\), the corresponding higher-order difference-differential operators
\[ \Delta^w := \Delta_{i_\ell} \cdots \Delta_{i_1}: K^{p \times q} \to \left(K^{\otimes (\ell+1)}\right)^{p \times q}. \]

Example 4.13. For a matrix-valued noncommutative rational function \(R\) defined by a matrix-valued noncommutative polynomial \(P = \sum_{v \in \mathcal{F}_d} P_v z^v\) and for a word \(w = g_{i_1} \cdots g_{i_\ell}\) of length \(\ell\) at most the total degree of \(P\), we have
\[ \Delta^w R = \sum_{v \in \mathcal{F}_d} P_v \sum_{u = u_1 g_{i_1} u_2 g_{i_2} \cdots g_{i_\ell} u_{\ell + 1}} (z(1))^{u_1} \otimes (z(2))^{u_2} \otimes \cdots \otimes (z(\ell+1))^{u_{\ell+1}}. \]

More precisely, every term in the second sum on the right-hand side is a tensor product of \(\ell + 1\) noncommutative rational functions defined by the corresponding noncommutative monomials; alternatively, the right-hand side (with some nesting of parentheses) is a matrix-valued noncommutative rational expression in \(\ell\) tuples of indeterminates defining \(\Delta^w R\).
Example 4.14. For the matrix-valued noncommutative rational function $\mathcal{R}$ defined by a recognizable series realization of the form (4.1), we have by iterating the first equality in (4.2) that for every $w = g_i \cdots g_{i_k}$, 

$$
\Delta^w \mathcal{R} = \left(C(I_m - A_1 z_1^{(1)} - \cdots - A_d z_d^{(1)})^{-1} A_{i_1} \otimes 1 \otimes \cdots \otimes 1 \right) \prod_{j=2}^\ell \left(1 \otimes \cdots \otimes 1 \otimes (I_m - A_1 z_1^{(j)} - \cdots - A_d z_d^{(j)})^{-1} A_{i_j} \otimes 1 \otimes \cdots \otimes 1 \right) \cdot 
\left(1 \otimes \cdots \otimes 1 \otimes (I_m - A_1 z_1^{(\ell+1)} - \cdots - A_d z_d^{(\ell+1)})^{-1} B \right).
$$

We identify $K^{n_1 \times n_1} \otimes \cdots \otimes K^{n_{\ell+1} \times n_{\ell+1}}$ with $\ell$-linear mappings

$$
K^{n_1 \times n_2} \times \cdots \times K^{n_\ell \times n_{\ell+1}} \rightarrow K^{n_1 \times n_{\ell+1}},
$$

so that $\sum_i \left(A_i^{(1)} \otimes \cdots \otimes A_i^{(\ell+1)}\right)$ corresponds to the $\ell$-linear mapping

$$(H_1, \ldots, H_\ell) \mapsto \sum_i A_i^{(1)} H_1 A_i^{(2)} \cdots A_i^{(\ell)} H_\ell A_i^{(\ell+1)}.
$$

This correspondence extends naturally to matrices. It follows that for a $p \times q$ matrix-valued noncommutative rational function $\mathcal{R}$ and for $Z^{(1)} \in K^{n_1 \times n_1}$, $\ldots$, $Z^{(\ell+1)} \in K^{n_{\ell+1} \times n_{\ell+1}}$ in appropriate domains, we have that

$$
\Delta^w \mathcal{R} \left(Z^{(1)}, \ldots, Z^{(\ell+1)}\right) \in K^{p \times q} \otimes K^{n_1 \times n_1} \otimes \cdots \otimes K^{n_{\ell+1} \times n_{\ell+1}} \cong \left(K^{n_1 \times n_1} \otimes \cdots \otimes K^{n_{\ell+1} \times n_{\ell+1}}\right)^{p \times q},
$$

and for $(H_1, \ldots, H_\ell) \in K^{n_1 \times n_2} \times \cdots \times K^{n_\ell \times n_{\ell+1}}$, we have that

$$
\Delta^w \mathcal{R} \left(Z^{(1)}, \ldots, Z^{(\ell+1)}\right) (H_1, \ldots, H_\ell) \in \left(K^{n_1 \times n_{\ell+1}}\right)^{p \times q}.
$$

In particular, we have

$$
\frac{d}{dt} \mathcal{R}(Z + tW) \bigg|_{t=0} = \sum_{i=1}^d \sum_{j=1}^d \Delta_i \Delta_j \mathcal{R}(Z, Z)(W_i, W_j) = \frac{d^2}{dt^2} \mathcal{R}(Z + tW) \bigg|_{t=0}.
$$

This is exactly the Hessian of $\mathcal{R}$ which plays a central role in the study of noncommutative convexity; see, e.g., [29, 31, 30]. Hessians and directional derivatives of matrix-valued noncommutative rational expressions were implemented in NCAlgebra to produce a convexity checking algorithm [14].

Remark 4.15. We can also define the difference-differential operators on matrices over tensor powers of $\mathbf{K}$ at the level of matrix-valued noncommutative rational expressions in several tuples of indeterminates. For a $p \times q$ matrix-valued noncommutative rational expression $\mathcal{R}$ in $\ell$ tuples of indeterminates (see Remark 4.3), we define $\Delta_j(\mathcal{R})$, a $p \times q$ matrix-valued noncommutative rational expression in $\ell + 1$ tuples of indeterminates, analogously to Definition 4.4, except that in rule (1) we consider “pure tensors” instead of polynomials, and we modify rules (3) and (4), cf. (4.6) and (4.7). Namely,
(1) If \( R \) and \( R' \) are matrix-valued noncommutative rational expressions in \( t \) tuples and in \( s \) tuples of indeterminates respectively, with \( t + s = \ell \), we set
\[
\Delta_j(R \otimes R') = R \otimes \Delta_j(R').
\]

(2) If \( R_1 \) and \( R_2 \) are matrix-valued noncommutative rational expressions of compatible sizes in \( \ell \) tuples of indeterminates, then
\[
\Delta_j(R_1 R_2) = \Delta_j(R_1) \iota_\ell(R_2) + (R_1 \otimes 1) \Delta_j(R_2).
\]

(3) If \( R_1 \) and \( R_2 \) are matrix-valued noncommutative rational expressions of the same size in \( \ell \) tuples of indeterminates, which is not identically singular, then
\[
\Delta_j(R^{-1}) = -(R^{-1} \otimes 1) \Delta_j(R \iota_\ell(R^{-1})).
\]

Here, for a matrix-valued noncommutative rational expression \( R \) in \( \ell \) tuples of indeterminates, \( \iota_\ell(R) \) is a matrix-valued noncommutative rational expression of the same size in \( \ell + 1 \) tuples of indeterminates defined by \( \iota_1(R) = 1 \otimes R \) and by the recursive relations
\[
\begin{align*}
(1) \ & \ i_{t+1}(R \otimes R') = R \otimes i_t(R'); \\
(2) \ & \ i_\ell(R_1 + R_2) = i_\ell(R_1) + i_\ell(R_2); \\
(3) \ & \ i_\ell(R_1 R_2) = i_\ell(R_1) i_\ell(R_2); \\
(4) \ & \ i_\ell(R^{-1}) = (i_\ell(R))^{-1}; \\
(5) \ & \ i_\ell([R_{ab}]) = [i_\ell(R_{ab})]
\end{align*}
\]

for \( \ell > 1 \). Notice that for \( d \)-tuples of matrices \( Z^j \in (K_{n \times n_j})^d, j = 1, \ldots, \ell + 1 \), the value \( \iota_\ell(R)(Z^{(1)}, \ldots, Z^{(\ell + 1)}) \) is the image of the value \( R(Z^{(1)}, \ldots, Z^{(\ell - 1)}, Z^{(\ell + 1)}) \) under the linear mapping
\[
K_{n_1 \times n_1} \otimes \cdots \otimes K_{n_{\ell - 1} \times n_{\ell - 1}} \otimes K_{n_{\ell} \times n_{\ell + 1}} \\
\longrightarrow K_{n_1 \times n_1} \otimes \cdots \otimes K_{n_{\ell - 1} \times n_{\ell - 1}} \otimes K_{n_\ell \times n_\ell} \otimes K_{n_{\ell + 1} \times n_{\ell + 1}},
\]
\[
A^{(1)} \otimes \cdots \otimes A^{(\ell)} \longrightarrow A^{(1)} \otimes \cdots \otimes A^{(\ell - 1)} \otimes I_{n_\ell} \otimes A^{(\ell)}
\]
extended naturally to the mapping of matrices
\[
(K_{n_1 \times n_1} \otimes \cdots \otimes K_{n_{\ell - 1} \times n_{\ell - 1}} \otimes K_{n_{\ell} \times n_{\ell + 1}})^{p \times q} \\
\longrightarrow (K_{n_1 \times n_1} \otimes \cdots \otimes K_{n_{\ell - 1} \times n_{\ell - 1}} \otimes K_{n_\ell \times n_\ell} \otimes K_{n_{\ell + 1} \times n_{\ell + 1}})^{p \times q}.
\]

We then have an analogue of Theorem 4.8 as follows. Let \( R \) be a \( p \times q \) matrix-valued noncommutative rational expression in \( \ell \) tuples of indeterminates, let \( Z^{(j)} \in (K_{n_j \times n_j})^d, j = 1, \ldots, \ell - 1 \), \( Z \in (K_{n \times n})^d, Z' \in (K_{n' \times n'})^d \), so that
\[
(Z^{(1)}, \ldots, Z^{(\ell - 1)}, Z), (Z^{(1)}, \ldots, Z^{(\ell - 1)}, Z') \in \text{dom } R,
\]
and let \( W \in (K_{n \times n'})^d \). Then
\[
\left( Z^{(1)}, \ldots, Z^{(\ell - 1)}, \begin{bmatrix} Z & W \\ 0 & Z' \end{bmatrix} \right) \in \text{dom } R.
\]
and

\[(4.8)\]
\[P(n + n', pm_1 \cdots n_{\ell-1})R \begin{pmatrix} Z^{(1)} & \cdots & Z^{(\ell-1)} \end{pmatrix}, \begin{bmatrix} Z & W \\ 0 & Z' \end{bmatrix}) P(n + n', qn_1 \cdots n_{\ell-1})^T = \begin{bmatrix} P(n, pm_1 \cdots n_{\ell-1})R(Z^{(1)}, \ldots, Z^{(\ell-1)}, Z) & 0 \\ P(n, pm_1 \cdots n_{\ell-1}) \sum_{j=1}^{d} \Delta_j(R)(Z^{(1)}, \ldots, Z^{(\ell-1)}, Z, Z')(W_{j}) & P(n', qn_1 \cdots n_{\ell-1})^T \end{bmatrix} \]

Here
\[\Delta_j(R)(Z^{(1)}, \ldots, Z^{(\ell-1)}, Z, Z') \in K^{p \times q} \otimes K^{n_1 \times n_1} \otimes \cdots \otimes K^{n_{\ell-1} \times n_{\ell-1}} \otimes K^{n \times n} \otimes K^{n' \times n'} \]
\[\cong (K^{n_1 \times n_1} \otimes \cdots \otimes K^{n_{\ell-1} \times n_{\ell-1}} \otimes K^{n \times n} \otimes K^{n' \times n'})^{p \times q} \]
and
\[\Delta_j(R)(Z^{(1)}, \ldots, Z^{(\ell-1)}, Z, Z')(W_{j}) \in (K^{n_1 \times n_1} \otimes \cdots \otimes K^{n_{\ell-1} \times n_{\ell-1}} \otimes K^{n \times n} \otimes K^{n' \times n'})^{p \times q}, \]
cf. the discussions preceding Theorem 4.8 and following Example 4.14. The proof is analogous to the proof of Theorem 4.8 except that instead of establishing (4.3) for polynomials we have to establish (4.8) for pure tensors. Namely, we have to show that if \( R \) and \( R' \) satisfy (4.8) then so does \( R \otimes R' \). This can be achieved using the identity

\[R(Z^{(1)}, \ldots, Z^{(\ell-1)}) \otimes R' \begin{pmatrix} Z & W \\ 0 & Z' \end{pmatrix} = (R(Z^{(1)}, \ldots, Z^{(\ell-1)}) \otimes I) \begin{bmatrix} Z & W \\ 0 & Z' \end{bmatrix}.\]

It follows from (4.8) that \( \Delta_j \) preserves the equivalence of matrix-valued noncommutative rational expressions in \( \ell \) tuples of indeterminates and can be thus defined on matrices over \( K^{\otimes \ell} \). It also follows that if
\[(Z^{(1)}, \ldots, Z^{(\ell-1)}, Z), (Z^{(1)}, \ldots, Z^{(\ell-1)}, Z') \in \text{edom } R, \]
then
\[(Z^{(1)}, \ldots, Z^{(\ell-1)}, Z, Z') \in \text{edom } \Delta_j(R).\]

Remark 4.16. As a special case of the previous remark, we see that for a matrix-valued noncommutative rational function \( R \) and for a word \( w \) of length \( \ell \), \( \Delta^w R \) is the matrix over \( K^{\otimes \ell} \) corresponding to the equivalence class of \( \Delta^w(R) \) for any \( R \in \mathcal{R} \). We further conclude that
\[\text{dom } \Delta^w \mathcal{R} \supseteq (\text{dom } \mathcal{R})^{\ell+1}, \quad \text{edom } \Delta^w \mathcal{R} \supseteq (\text{edom } \mathcal{R})^{\ell+1}.\]
The second inclusion follows from the last statement of Remark 4.15. The first inclusion follows from the equality \( \text{dom } \Delta^w R = (\text{dom } R)^{\ell+1} \) for a matrix-valued rational expression \( R \), which can be proved recursively. We leave the details to
the reader, noticing only that for the product we have the following higher order Leibniz rule:

\[(4.9) \quad \Delta^w(R_1 R_2) = \sum_{u,v \in \mathcal{F}_d \colon w=uv} (\Delta^u(R_1) \otimes 1 \otimes \cdots \otimes 1) \cdot (1 \otimes \cdots \otimes 1 \otimes \Delta^v(R_2)).\]

4.5. **Formal power series.** A matrix-valued noncommutative rational expression which is regular at zero determines a noncommutative formal power series with matrix coefficients. This correspondence is defined recursively by inverting formal power series with invertible constant term (the coefficient for \(z^0\)); see, e.g., [12]. Furthermore, \(R_1\) and \(R_2\) are equivalent if and only if the corresponding formal power series coincide, so that the noncommutative formal power series expansion of a matrix-valued noncommutative rational function which is regular at zero is well defined; see [34, Remark 2.14]. If \(\sum_{w \in \mathcal{F}_d} R_w z^w\) is the formal power series expansion of \(R\) then the formal power series expansion of \(\Delta^w R\) is given by

\[\sum_{u,v \in \mathcal{F}_d} R_{ugjv} z^u \otimes z^v.\]

For a proof, we use the recognizable series realization and represent \(R\) by a matrix-valued noncommutative rational expression of the form (4.1). Therefore, \(R_w = CA^w B\). On the other hand, it follows from Example 4.7 that the formal power series expansion of \(\Delta^w R\) is given by

\[\sum_{u,v \in \mathcal{F}_d} CA^u A^v Bz^u \otimes z^v.\]

So, we see that the coefficient for \(z^u \otimes z^v\) is exactly \(R_{ugjv}\).

A similar argument using Example 4.14 shows that the formal power series expansion of \(\Delta^w R\) for \(w = g_{i_1} \cdots g_{i_{\ell+1}}\) is given by

\[\sum_{u_1, \ldots, u_{\ell+1} \in \mathcal{F}_d} R_{u_1 g_{i_1} u_2 g_{i_2} \cdots u_{\ell+1} (z^{(1)})^{u_1} \otimes (z^{(2)})^{u_2} \otimes \cdots \otimes (z^{(\ell+1)})^{u_{\ell+1}}.\]

In particular, looking at the constant term of this expansion (i.e., for \(u_1 = \ldots = u_{\ell+1} = 0\)), we see that

\[\Delta^w R(0, \ldots, 0) = R_{w^0}.\]

5. **Conclusions**

Rational functions in noncommuting indeterminates occur in many areas of system theory: most control problems involve rational expressions in matrix parameters. In this paper we surveyed some aspects of the theory of noncommutative rational functions, and provided some pointers to a rapidly growing literature. We discussed in some details a construction of the skew field of noncommutative rational functions based on noncommutative rational expressions and their matrix evaluations. We explained its role as the universal field of fractions of the ring of noncommutative polynomials. We gave an outline of a noncommutative realization theory. Finally, we developed a difference-differential calculus for noncommutative rational functions.
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Department of Mathematics, Drexel University, 3141 Chestnut Str., Philadelphia, PA, 19104
E-mail address: dmitryk@math.drexel.edu

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, Israel, 84105
E-mail address: vinnikov@math.bgu.ac.il