Numerical determination of the contours of the area watering from the fluid source

E A Mikishanina
Department of Actuarial and Financial Mathematics, Chuvash State University,
Moscow avenue, 15, Cheboksary, 428015, Russia
E-mail: evaeva_84@mail.ru

Abstract. The paper presents the results of numerical determination of water-flooding contours (or as they are called, depression curves) of a flat porous region from a source with a smooth convex boundary. Based on the Green integral formula, formulas are obtained for determining the values of the harmonic function of the fluid pressure at each point in the region. Areas with constant permeability and variable are considered. Based on the obtained integral identity, a numerical algorithm for determining filtration rates for these and other areas are constructed using the boundary element method is constructed, and depression curves are constructed for sources with a round and elliptical boundary. The eccentricities of the region's water-flooding conditions are determined. To compare the numerical and analytical results, calculations were made for the test example. You can control the accuracy of calculations by changing the number of nodes on the boundary of the simulated area.

1. Introduction
There is often a problem of determining the shape of the watering areas of the medium from the source of the liquid in matters of irrigation, subsurface irrigation, and watering of oil wells and coal seams. Determining the contours of such areas (depression curves) with subsequent management of watering processes makes it possible to ensure high efficiency of reclamation and good indicators of oil and coal production.

Many works of various authors are devoted directly to the study of stationary filtration in an elastic medium [1-4]. Among other things, the author's works are devoted to numerical modeling of the filtering process itself [5-7]. Based on the results obtained earlier, a numerical algorithm will be presented below for determining the contours of the area being watered from the source.

Areas with a constant filtering coefficient and a variable will be discussed below. As a rule, to simplify calculations, the filter coefficient in the area is considered constant. However, when a hydraulic pressure occurs at the boundary of the region, the stresses in the region change, which also leads to a change in the filtration coefficient. This was shown in the article [6].

Within the framework of Darcy's law, we consider a flat porous region with a constant filtration coefficient or a variable filtration coefficient, in which a liquid source with a convex smooth closed boundary is placed. The determination of filtration rates and depression curves is analytically possible only for some sources with the simplest border shape, for example, round ones. Modeling for arbitrary-form sources is analytically difficult, but modeling can be performed numerically using the boundary
element method. We consider a number of aspects of numerical modeling of the stationary filtration process.

2. Numerical simulation of the stationary filtration process in a region with a constant filtration coefficient

The pressure of a liquid within the framework of Darcy's law in a flat region with a closed boundary is described by a harmonic function:

$$\Delta P = 0,$$  \hspace{1cm} (1)

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ - Laplace operator.

The boundary conditions are determined depending on the simulated scenario. In the case of a permeable boundary for a doubly connected region, they can have the form

$$P\bigg|_{(\partial \Omega)_1} = p_1, \quad P\bigg|_{(\partial \Omega)_2} = p_2.$$  \hspace{1cm} (2)

According to Green's integral formula, the pressure at each point in the region can be determined from the identity, \[8,9\]

$$\varepsilon P(x, y) = \iint_{\partial \Omega} P(\tau) G(\tau) d\sigma - \iint_{\partial \Omega} P(\tau) G_n(\tau) d\sigma,$$  \hspace{1cm} (3)

where for a point on the border $\varepsilon = 0.5$ and for an internal point $\varepsilon = 1$.

Let's approximate the border with an inscribed polygon with vertices at points \((x^k, y^k)\) \[10\]. If you determine the values of the pressure \(P^k\) and its normal derivative \(P^k_n\) at the average (control) points \((x^j, y^j, z^j)\), the formulas for the components of the filtration rate \(u, v\) at an arbitrary point \((x, y)\) will take the form

$$u(x, y) = -\frac{k}{\rho g} \sum_{j=1}^{N} \left[ P^k_i B_i^j - P^k_n A_i^j \right], \quad v(x, y) = -\frac{k}{\rho g} \sum_{j=1}^{N} \left[ P^k_n B_i^j - P^k A_i^j \right].$$  \hspace{1cm} (4)

where \(k\) - filtration coefficient, \(\rho\) - fluid density, \(g\) - acceleration of gravity and

$$A_i^j(x, y) = \frac{\partial}{\partial x} \left( \int_{s_{i-1}}^{s_i} G_n(s, x, y) ds \right), \quad A_i^j(x, y) = \frac{\partial}{\partial y} \left( \int_{s_{i-1}}^{s_i} G_n(s, x, y) ds \right),$$  \hspace{1cm} (5)

$$B_i^j(x, y) = \frac{\partial}{\partial x} \left( \int_{s_{i-1}}^{s_i} G(s, x, y) ds \right), \quad B_i^j(x, y) = \frac{\partial}{\partial y} \left( \int_{s_{i-1}}^{s_i} G(s, x, y) ds \right).$$

If the point belongs to the integration segment \((s_{j-1}, s_j)\), then

$$B_i^j = 0, \quad A_i^j = -\frac{2}{\pi} \frac{y^j - y^{j-1}}{h_j^4}, \quad A_i^j = \frac{2}{\pi} \frac{x^j - x^{j-1}}{h_j^2},$$

where \(h_j\) - length of the \(j\)-th integration segment.
3. Numerical simulation of the stationary filtration process in a region with a variable filtration coefficient

When filtering a homogeneous incompressible fluid in a flat area with a variable filtration coefficient \(k(x, y)\), the filtration equation is made its way to the form

\[ \nabla \cdot k(x, y) \nabla P = 0. \]  

(6)

The boundary conditions are determined depending on the simulated scenario. In the case of a permeable boundary for a doubly connected region, they can have the form (2)

Enter a new variable [11]

\[ U = \sqrt{k} \cdot P. \]  

(7)

If the condition \(\Delta \sqrt{k} = 0\) is met, the solution of the stationary filtration problem in a region with a variable filtration coefficient will be reduced to solving the boundary value problem for the harmonic equation [11]

\[ \Delta U = 0. \]  

(8)

The boundary conditions will be obtained from the boundary conditions for equation (6). According to Green's integral formula, the value of the desired function \(U\) at any point can be obtained using a formula similar to (3)

Given that the pressure in an area with a variable filtration coefficient is determined by the formula

\[ P(x, y) = \frac{U(x, y)}{\sqrt{k(x, y)}}, \]  

(9)

the components of the filtration rate will be defined as

\[ u = -\frac{1}{\rho g} \left( \frac{\partial U}{\partial x} \sqrt{k} - \frac{\partial k}{\partial x} \frac{U}{2\sqrt{k}} \right), \]

\[ v = -\frac{1}{\rho g} \left( \frac{\partial U}{\partial y} \sqrt{k} - \frac{\partial k}{\partial y} \frac{U}{2\sqrt{k}} \right), \]  

(10)

where \(U_n\) - normal derivative of function \(U\), and the values of partial derivatives are found by the formulas

\[ \frac{\partial U}{\partial x}(x, y) = \sum_{j=1}^{N} \left[ U_j B_1^j - U_1^j A_1^j \right], \]

\[ \frac{\partial U}{\partial y}(x, y) = \sum_{j=1}^{N} \left[ U_j B_2^j - U_2^j A_2^j \right]. \]  

(11)

Values \(A_1^j, A_2^j, B_1^j, B_2^j\) are determined by the formulas (5).

4. Comparison of numerical simulation results with an analytical solution

To compare the numerical and analytical results, the stationary filtration rates in a ring with internal and external radiuses \(R_1, R_2\) and pressures \(p_1\) and \(p_2\) are determined (figure 1). Analytical formulas for determining the speed components have the form

\[ u = -\frac{k}{\rho g} \left( \frac{p_1 - p_2}{x^2 + y^2} \right) \ln \left( \frac{R_2}{R_1} \right), \]

\[ v = -\frac{k}{\rho g} \left( \frac{p_1 - p_2}{x^2 + y^2} \right) \ln \left( \frac{R_2}{R_1} \right). \]
Graphs of the velocity components are built on a circle with a radius of \((R_1 + R_2)/2\): 
\[ R_1 = 0.75, \quad R_2 = 1, \quad p_1 = 10^4, \quad p_2 = 0, \quad s \in [0, \pi]. \]
The number of boundary elements \(N = 60\), (figure 2). Analytically found functions are represented by a solid line, numerically defined functions are represented by points.

![Figure 1. Stationary filtering through the area in the form of a circular ring.](image)

![Figure 2. The graphics component of the velocity of steady filtration.](image)

5. **Numerical algorithm for determining the contours of the area watering**

From a source with a smooth convex closed boundary \(C_0 = (x_0(s), y_0(s))\), \(s\) – an arc coordinate, and the liquid pressure \(p_0\) at this boundary, placed in an area \(T\) where atmospheric pressure acts \(p_a\), the process of unsteady filtration begins. It is necessary to define the contours of the gradual watering of the area.

Since the movement of the liquid is directed opposite to the normal to the source contour, the initial velocity can be defined as [12]

\[
v = \frac{k(p_0 - p_a)}{\rho g} \cdot \mathbf{n},
\]

where \(\mathbf{n}\) – normal vector to the source contour.

Thus, after an arbitrarily small time interval \(\Delta t\) of the filtering process, the depression curve \(C_1 = (x_1(s), y_1(s))\) is determined from the equalities:

\[
x_i(s) = x_0(s) + u\Delta t, \quad y_i(s) = y_0(s) + v\Delta t,
\]

Next, the problem is reduced to solving the harmonic equation (1) in a region bounded by curves \(C_0, C_1\) with boundary conditions.
and defined by formula (3), the velocity component is already on the contour \( C_1 \). Each new contour of the waterline area will be iterated out of equalities

\[
x_i(s) = x_{i-1}(s) + u_{i-1}\Delta t, \quad y_i(s) = y_{i-1}(s) + v_{i-1}\Delta t,
\]

in which components \( u_{i-1}, v_{i-1} \) are defined on the \((i-1)\)th contour.

6. Numerical example

6.1. Determining depression curves in a region with a circular source

Non-stationary filtering occurs from a source with a boundary \( C : x = a \cos s, y = -a \sin s, \ s \in [0, 2\pi) \). On the source circuit, the pressure is \( p \). In accordance with the described numerical algorithm, for \( a = 1 \), \( p = 10^5 \), \( k\Delta t/\rho g = 10^{-6} \) and the number of nodes \( N = 80 \), the contours of the gradual watering of the area are obtained (figure 3). At each step, the eccentricities of the curves are determined (table 1).

![Figure 3](image-url)  
**Figure 3.** The curves of the depression.

| Step number, \( i \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|----------------------|----|----|----|----|----|----|----|
| Eccentricity, \( \varepsilon \) | 0  | 0.002 | 0.0015 | 0.0019 | 0.0022 | 0.0022 | 0.0021 |

6.2. Determining depression curves in an area with an elliptical source

Non-stationary filtration occurs from a source with a boundary \( C : x = a \cos s, y = -b \sin s, \ a > b, \ s \in [0, 2\pi) \) and pressure \( p \). For \( a = 1, b = 0.75 \), \( p = 10^5 \), \( k\Delta t/\rho g = 10^{-6} \) and the number of nodes \( N = 80 \), the contours of the gradual watering of the area and the dynamics of the absolute value of the filtration rate on each contour \( i = 1, 7 \), \( s \in [0, \pi] \) are obtained (figure 4). Also, at each step, the eccentricities of the curves are determined to control the shape of the depression curves (table 2).

| Step number, \( i \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|----------------------|----|----|----|----|----|----|----|
| Eccentricity, \( \varepsilon \) | 0.661 | 0.635 | 0.609 | 0.596 | 0.587 | 0.578 | 0.571 |
Figure 4. The curves of the depression and the dynamics of the speed of fluid motion.

7. Conclusion
As can be seen from the examples discussed above, the contours of the water area from the fluid source with increasing linear dimensions of the water area approach the circle. Numerical calculations performed for other forms of the liquid source show that the above conclusions are valid for any smooth convex boundary of the source.

In this paper, we have developed a General algorithm for determining the contours of a gradual watering area from a source with a convex smooth border. Cases with both constant and variable filtration coefficients were considered. We have numerically constructed depression curves for sources with a circular and elliptical boundary. You can control the accuracy of calculations by changing the number of nodes on the boundary of the simulated area.

References
[1] Vabishchevich P and Vasilieva M 2012 Math. Modelling and Analysis 19 532-48
[2] Landau L D and Livshits E M 2006 Hydrodynamics (Moscow: Fizmatlit) p 736
[3] Toro E F 2009 Riemann Solvers and Numerical Methods for Fluid Dynamics: A Practical Introduction (USA: Springer) p 724
[4] Lighthill J 1981 Waves in liquids (Moscow: Mir) 603
[5] Mikishanina E A and Terentiev A G 2017 Scientific notes of Kazan University. Series of physical and mathematical sciences 159 204-15
[6] Mikishanina E A 2019 Bulletin of Udmurt University. Mathematics. Mechanics. Computer science 29 396-407
[7] Mikishanina E A 2016 Information Technologies for Intelligent Decision Making Support (Ufa: Ufa State Aviation Technical University Press) 2 138-41
[8] Terentiev A G and Petrova T N 1996 News of NANI CR 56-73
[9] Kazakova A O, Mikishanina E A, Terentiev A G 2017 Mathematical models and their applications 27-37
[10] Mikishanina E A 2019 Journal of Physics: Conf. series 1158 032033
[11] Golubev G V and Tumashev G G 1972 Incompressible fluid filtration in a non-uniform porous medium (Kazan: Kazan University Publ.) p 195
[12] Terentiev A G, Kirschner I N and Uhlman J S 2011 The Hydrodynamics of Cavitating Flows (USA: Backbone Publ. Company) p 598