THE SEVERI INEQUALITY

\[ K^2 \geq 4\chi \]

FOR SURFACES OF MAXIMAL ALBANESE DIMENSION

RITA PARDINI

ABSTRACT. We prove the so-called Severi inequality, stating that the invariants of a minimal smooth complex projective surface of maximal Albanese dimension satisfy:

\[ K^2_S \geq 4\chi(S). \]

2000 Mathematics Subject Classification: 14J29.

1. INTRODUCTION

In this paper we prove the so-called Severi inequality:

*If \( S \) is a minimal smooth complex projective surface of maximal Albanese dimension, then \( K^2_S \geq 4\chi(S) \).*

This inequality has a long history. Severi (Se) stated it as a theorem in 1932, but his proof was not correct (cf. Ca). Around the end of the 1970’s the inequality was posed as a conjecture by Reid (Re) and by Catanese (Ca). Motivated by this conjecture, Xiao wrote the foundational paper [Xi] on surfaces fibred over a curve, in which he proves the conjecture in the special case of a surface admitting a fibration over a curve of positive genus. Finally, at the end of the 1990’s, the conjecture was almost solved by Manetti (Ma), who proved it under the additional assumption that the surface have ample canonical bundle. His proof is based on a very fine study of the positivity properties of the tautological line bundle of \( \mathbb{P}(\Omega^1_S \otimes \omega_S) \) and it allows him also to classify the surfaces for which equality holds, still under the assumption that the canonical bundle be ample.

Our proof of the conjecture in the general case is completely different from Manetti’s. We derive the Severi inequality from a well known inequality for the slope of a fibered surface proven by Xiao in the same paper [Xi] and, independently, by Cornalba–Harris (CH) in the semistable case. In order to do this, we construct first an infinite
sequence of fibred surfaces \( f_d: Y_d \to \mathbb{P}^1 \) such that the slope of \( f_d \) converges to the ratio \( K_S^2/\chi(S) \) as \( d \) goes to infinity (cf. Prop. [23]). Then we obtain the Severi inequality by applying the slope inequality to the fibrations \( f_d \) and taking the limit for \( d \to \infty \).

**Acknowledgments.** This work was partially supported by P.R.I.N. 2002 “Geometria sulle varietà algebriche” of Italian M.I.U.R. I am indebted to Xiao Gang and János Kollár for suggesting simplifications of the original proof.

**Notation and conventions.** We work over the field of complex numbers; all varieties are projective. Linear equivalence is denoted by \( \equiv \) and numerical equivalence by \( \sim_{\text{num}} \). We say that a surface \( S \) has maximal Albanese dimension if the image of its Albanese map is a surface.

The remaining notation is standard in algebraic geometry. We just recall the numerical invariants associated to a surface \( S \): the self intersection \( K_S^2 \) of the canonical divisor \( K_S \), the geometric genus \( p_g(S) := h^0(S, K_S) \), the irregularity \( q(S) := h^0(S, \Omega_S^1) \) and the Euler–Poincaré characteristic \( \chi(S) := p_g(S) - q(S) + 1 \).

2. **The Severi inequality**

In this section we prove the following:

**Theorem 2.1** (Severi inequality). Let \( S \) be a smooth minimal complex surface of maximal Albanese dimension. Then:

\[
K_S^2 \geq 4\chi(S).
\]

We begin by recalling some facts about surfaces fibred over a curve.

Let \( Y \) be a smooth complex surface and let \( f: Y \to B \) be a fibration onto a smooth curve of genus \( b \). Assume that \( f \) is relatively minimal, i.e., that no fibre of \( f \) contains a \(-1\)-curve, and that a general fibre of \( f \) has genus \( g \geq 2 \). It is well known (cf. [Be]) that one has:

\[
\chi(Y) \geq (b - 1)(g - 1)
\]

with equality holding if and only if the fibration \( f \) is smooth and isotrivial. (Recall that a fibration is said to be isotrivial if all its smooth fibres are isomorphic).

If \( f \) is not smooth and isotrivial, then one defines (cf. [Xi]) the slope \( \lambda(f) \) of \( f \) as:

\[
\lambda(f) := \frac{K_Y^2 - 8(b - 1)(g - 1)}{\chi(Y) - (b - 1)(g - 1)}.
\]
The fundamental inequality for the slope of a fibration is the following:

**Theorem 2.2** (Xiao, Cornalba–Harris). Let \( f: Y \to B \) be a relatively minimal fibration, not smooth and isotrivial, with fibres of genus \( g \geq 2 \). Then one has the following inequalities for the slope \( \lambda(f) \) of \( f \):

\[
4(g - 1)/g \leq \lambda(f) \leq 12.
\]

**Proof.** See [Xi], Theorem 2. For the case of a semistable fibration, see also [CH]. \( \square \)

We are going to reduce the proof of the Severi inequality in the general case to Theorem 2.2 by means of the following:

**Proposition 2.3.** Let \( S \) be a smooth minimal complex surface of general type and of maximal Albanese dimension. There exists a sequence of smooth complex surfaces \( Y_d, d \in \mathbb{N}_{>0} \), and relatively minimal fibrations \( f_d: Y_d \to \mathbb{P}^1 \) such that:

(i) \( f_d \) is not isotrivial;

(ii) \( \lim_{d \to \infty} g_d = +\infty \), where \( g_d \) is the genus of a general fibre of \( f_d \);

(iii) \( \lim_{d \to \infty} \lambda(f_d) = K_S^2/\chi(S) \).

**Proof.** Let \( a: S \to A \) be the Albanese map and let \( q \geq 2 \) be the irregularity of \( S \). Let \( L \) be a very ample line bundle on \( A \), let \( H \) be the pull-back of \( L \) on \( S \) and set \( \alpha := H^2, \beta := K_SH \). Fix an integer \( d \geq 1 \), let \( \mu: A \to A \) be the multiplication by \( d \) and consider the following cartesian diagram:

\[
\begin{array}{ccc}
S' & \xrightarrow{p} & S \\
\alpha' \downarrow & & \downarrow a \\
A & \xrightarrow{\mu} & A
\end{array}
\]

The surface \( S' \) is smooth minimal of maximal Albanese dimension with invariants:

\[
K_{S'}^2 = d^2 K_S^2, \quad \chi(S') = d^2 \chi(S).
\]

By [LB] Ch. 2, Prop. 3.5, one has the following equivalence on \( A \):

\[
\mu^* L \sim_{\text{num}} d^2 L.
\]

Hence, denoting by \( H' \) the pull-back of \( L \) on \( S' \) via the map \( a' \), one has:

\[
p^* H \sim_{\text{num}} d^2 H'.
\]
Using this remark, one computes:

\[(2.3)\quad H'^2 = d^{2q-4}\alpha, \quad H'K_{S'} = d^{2q-2}\beta.\]

Now let \(D_1, D_2 \in [H']\) be general smooth curves. Set \(C_1 := D_1 + D_2\) and let \(C_2 \in [2H']\) be a general curve, so that \(C_2\) is smooth and \(C_1\) and \(C_2\) intersect transversely at \(4d^{2q-4}\alpha\) points. Let \(\varepsilon: Y_d \to S'\) be the blow up of the intersection points of \(C_1\) and \(C_2\). The invariants of \(Y_d\) are the following:

\[(2.4)\quad K_{Y_d}^2 = K_{S'}^2 - 4H'^2 = d^{2q}K_S^2 - 4d^{2q-4}\alpha\]

\[(2.5)\quad \chi(Y_d) = \chi(S') = d^{2q}\chi(S)\]

The pencil spanned by \(C_1\) and \(C_2\) defines a fibration \(f_d: Y_d \to \mathbb{P}^1\) which is relatively minimal by construction. Furthermore, \(f_d\) is not isotrivial, since the strict transform of \(C_1\) is a singular stable fibre. The genus \(g_d\) of a general fibre of \(f_d\) is equal to the genus of \(C_2\), namely one has:

\[(2.6)\quad g_d = 1 + K_{S'}H' + 2H'^2 = 1 + d^{2q-2}\beta + 2d^{2q-4}\alpha.\]

We have \(\beta = K_SH > 0\), since \(S\) is minimal of general type and \([H]\) is free, and \(q > 1\), since \(S\) has maximal Albanese dimension. By these remarks, statement (ii) follows from \(2.3\).

The slope of \(f_d\) is given by the following expression:

\[(2.7)\quad \lambda(f_d) = \frac{K_{Y_d}^2 + 8(g_d - 1)}{\chi(Y_d) + (g_d - 1)}\]

Hence statement (iii) follows by \(2.4\), \(2.5\) and \(2.6\). \(\Box\)

**Proof of Thm. 2.1** If \(S\) is not of general type, then the inequality follows from the classification of surfaces, hence we may assume that \(S\) is of general type.

Consider the surfaces \(Y_d\) of Proposition 2.3. By Theorem 2.2, we have:

\[(2.8)\quad \lambda(f_d) \geq 4(g_d - 1)/g_d \quad \forall d \geq 1.\]

By Proposition 2.3, taking the limit of \(2.8\) for \(d \to \infty\) we get:

\[K_S^2 \geq 4\chi(S).\]
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Rita Pardini
Dipartimento di Matematica, Università di Pisa
Via F. Buonarroti 2, 56127 Pisa (Italy)
Email: pardini@dm.unipi.it