SMOOTH PROJECTIVE PLANES

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Abstract. Using symplectic topology and the Radon transform, we prove that smooth 4-dimensional projective planes are diffeomorphic to \( \mathbb{CP}^2 \). We define the notion of a plane curve in a smooth projective plane, show that plane curves in high dimensional regular planes are lines, prove that homeomorphisms preserving plane curves are smooth collineations, and prove a variety of results analogous to the theory of classical projective planes.

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1. INTRODUCTION

A smooth projective plane is an object of topological geometry, a manifold with a family of submanifolds, called lines, satisfying the axioms of projective plane geometry. A symplectic manifold is an object of symplectic topology, a manifold $M^{2n}$ with a closed 2-form $\Omega$ for which $\Omega^n \neq 0$. This article builds a bridge between symplectic topology and topological geometry and

1. constructs differential forms on smooth projective planes via the Radon transform
2. proves that these forms tame lines (in the sense of Gromov’s work on elliptic differential equations)
3. proves that smooth 4-dimensional projective planes are symplectomorphic to $\mathbb{CP}^2$. (Previously only homeomorphism with $\mathbb{CP}^2$ was known. Diffeomorphism of smooth projective planes of all other dimensions with $\mathbb{RP}^2$, $\mathbb{HP}^2$ or $\mathbb{OP}^2$ was recently proven by Kramer & Stolz.)
4. defines a concept of plane curve in a smooth projective plane
5. proves that the dual (i.e. set of tangent lines) of a plane curve is a plane curve
6. uncovers the lowest order differential invariant of a smooth projective plane (the tableau)
7. proves that the tableau determines the 2-jet of the smooth projective plane
8. proves that the tableau is a triality (in the sense of Cartan) just when the plane is regular (in the sense of Breitsprecher and Bödi)
9. proves that regularity is invariant under projective duality
10. determines when a smooth projective plane agrees to second order with a classical projective plane
11. proves that on regular projective planes of dimension 4 or more, the differential equations defining plane curves are elliptic (determined for 4-dimensional projective planes, overdetermined for higher dimensions)
12. proves that the only plane curves in higher dimensional regular projective planes are lines
13. connects the analysis of plane curves in regular 4-dimensional projective planes to symplectic topology
14. proves that every regular 4-dimensional projective plane can be deformed through a family of 4-dimensional projective planes into one which is isomorphic to $\mathbb{CP}^2$
15. proves that the differential system for plane curves in an irregular smooth 4-dimensional projective plane is a uniform limit of differential systems for plane curves in a regular projective plane.
16. proves that the differential system for plane curves in a regular projective plane determines the projective plane
17. characterizes the classical projective planes $\mathbb{RP}^2$ and $\mathbb{CP}^2$ in terms of local differential invariants
(18) proves a collection of results about smooth quadrics in regular 4-dimensional projective planes, showing that
(a) they have a connected 10-dimensional moduli space
(b) they are diffeomorphic to 2-spheres
(c) through any 5 points, with no 3 collinear, there is a unique smooth quadric
(d) the dual of a smooth quadric is a smooth quadric
(19) uncovers a dynamical system on the 2-torus which is the analogue of the elliptic curve found in the proof of Poncelet’s porism.

2. Definition of smooth projective planes

**Definition 1.** An incidence geometry is a triple \((P, \Lambda, F)\) where \(P\) is a set (whose elements are called the points of the geometry), \(\Lambda\) is a set (whose elements are called the lines of the incidence geometry) and \(F \subset P \times \Lambda\) (whose elements are called the pointed lines of the incidence geometry). \(F\) is called the correspondence space, incidence correspondence or flag space. We will say that a point \(p \in P\) is on a line \(\lambda \in \Lambda\) if \((p, \lambda) \in F\). The dual incidence geometry is \((P^*, \Lambda^*, F^*)\) where \(P^* = \Lambda\), \(\Lambda^* = P\) and \(F^* = \{(\lambda, p) \mid (p, \lambda) \in F\}\). An incidence geometry is called a projective plane if

1. any two distinct points \(p_1, p_2 \in P\) are on a unique line \(p_1p_2 \in \Lambda\)
2. any two distinct lines \(\lambda_1, \lambda_2 \in \Lambda\) have a unique common point \(\lambda_1\lambda_2 \in P\) on them
3. there are at least 4 points, no three of which are on the same line.

**Definition 2.** We call a projective plane topological if \(P\) and \(\Lambda\) are compact topological spaces, and the maps \(p_1, p_2 \mapsto p_1p_2\) and \(\lambda_1, \lambda_2 \mapsto \lambda_1\lambda_2\) are continuous.

**Definition 3.** We call a topological projective plane smooth if \(P\) and \(\Lambda\) are smooth manifolds, \(F \subset P \times \Lambda\) is a smooth embedded submanifold, and the maps \(p_1, p_2 \mapsto p_1p_2\) and \(\lambda_1, \lambda_2 \mapsto \lambda_1\lambda_2\) are smooth maps.

Topological projective planes are studied by certain German topologists; the standard reference is Salzmann et. al. [43].

3. State of the art on smooth projective planes

There is a useful characterization of smooth projective planes, due to Bődi and Immervoll:

**Definition 4.** Suppose that we have a projective plane, for which \(P\) and \(\Lambda\) are closed smooth manifolds, both of dimensions \(2n\) \((n \geq 0\) an integer), and suppose that \(F \subset P \times \Lambda\) is a \(3n\)-dimensional closed smoothly embedded submanifold, so that the canonical maps

\[
\begin{array}{ccc}
\pi_P & F & \pi_\Lambda \\
\downarrow & & \downarrow \\
P & \Lambda & \\
\end{array}
\]

given by \(\pi_P(p, \lambda) = p\) and \(\pi_\Lambda(p, \lambda) = \lambda\) are both submersions. Call \(P\) a smooth generalized plane. Consider the subsets \(\lambda = \pi_P\pi_\Lambda^{-1}(\lambda) \subset P\) and \(\bar{\lambda} = \pi_\Lambda\pi_P^{-1}(p) \subset \Lambda\) for \(p \in P\) and \(\lambda \in \Lambda\). We will identify \(\lambda \in \Lambda\) with \(\bar{\lambda}\), and still call it a line (\(\lambda\) is
also often called a point row in the incidence geometry literature), while \( \bar{p} \) will be called the pencil through \( p \). Lines are submanifolds of \( P \).

**Theorem 1** (Bödi & Immervoll [8]). A smooth generalized plane is a smooth projective plane just when any two lines are transverse, and the pencils of any two points are transverse. Conversely, every smooth projective plane is a smooth generalized plane.

**Corollary 1.** Every submanifold of \( P \times \Lambda \) which is \( C^2 \) close enough to \( F \) defines a smooth projective plane; hence smooth projective planes of dimension \( 2n \) depend on \( n \) functions of \( 3n \) variables.

**Remark 1.** Bödi [7] remarks that it is not known if there are real analytic smooth projective planes not isomorphic to the standard real projective plane. It is easy to construct lots of real analytic functions on products of projective planes, for example eigenfunctions of the Laplacian. Thereby one can easily construct lots of real analytic vector fields, and deform the flag space \( F \) of the real projective plane. The deformations are too many to be accounted for by the finite dimensional symmetry group. See Hilbert [27] for proof that all symmetries of the standard real projective plane are continuous, and that the group of symmetries is the projective general linear group. Therefore there are infinitely many real analytic smooth projective planes not isomorphic to one another.

**Theorem 2** (Freudenthal [19]). The dimension of a smooth projective plane is either 0, 2, 4, 8 or 16. For proof, see Salzmann et. al. [43] p.258. Ignoring 0, these happen to be the dimensions of the smooth projective planes \( \mathbb{R}P^2 \), \( \mathbb{C}P^2 \), \( \mathbb{H}P^2 \) and \( \mathbb{O}P^2 \) respectively (see Salzmann et. al. [43] for definitions); we call each of these four spaces the model of any smooth projective plane with the same dimension. Zero dimensional projective planes are discrete, and henceforth dimensions 0 and 2 will be largely ignored.

**Theorem 3** (Salzmann [44, 42], [43] 51.29, Löwen [36], LeBrun & Mason [35]). Two-dimensional smooth projective planes are diffeomorphic to the real projective plane.

A two-dimensional projective plane carries a projective structure (a quite general type of path geometry, corresponding in local coordinates to a second-order ordinary differential equation for lines, see Bryant, Griffiths & Hsu [12]). Generic smooth projective plane structures on the real projective plane are not projective connections (i.e. the lines are not geodesics of any connection on the tangent bundle). LeBrun & Mason were concerned with projective connections, although their proof of this theorem works for projective structures as well; most of their work in that paper does not appear to apply to projective structures.

**Theorem 4** (Kramer [31]). The space of points \( P \) of a smooth projective plane of positive dimension is homeomorphic to its model, as is the space of lines \( \Lambda \).

Kramer’s theorem uses some quite deep differential topology. It is much easier to show that each line is compact and connected, and that the point space and line space are compact and connected; see Salzmann et. al. [43] p. 225. It is also not difficult to show:
Theorem 5 (Breitsprecher [43] pp. 257,262). The lines of a positive dimension smooth projective plane are homeomorphic to spheres, and the cohomology class of any line is the generator of the cohomology ring of \( P \). The cohomology rings of \( P, \Lambda \) and \( F \), and their relations under pullback, are identical to those of the model.

Henceforth, if the dimension is greater than 2, we will use this identification of cohomology rings to orient \( P, \Lambda, F \) and to orient each line and the pencil of each point, to match with the model. In particular, a pair of lines will intersect at a unique point, by hypothesis, but moreover this intersection will be transverse by theorem 1 on the facing page, and in a projective plane of dimension 4 or more the intersection will be positive by matching of cohomology.

Definition 5. A morphism of projective planes \( P_0 \to P_1 \) is a pair of maps \( P_0 \to P_1 \) and \( \Lambda_0 \to \Lambda_1 \), taking the flag space \( F_0 \subset P_0 \times \Lambda_0 \) to the flag space \( F_1 \subset P_1 \times \Lambda_1 \). An isomorphism is often called a collineation.

Theorem 6 (Bödi & Kramer [9]). Every continuous isomorphism of smooth projective planes is smooth.

Note that the inverse is not assumed continuous, and it is not known whether the inverse must be continuous.

4. Affine charts

Consider a smooth projective plane \( P \). We will write down the well-known affine charts on \( P \). Pick any three points 0, \( X \) and \( Y \in P \). (Think of 0 as the origin of an affine plane, and \( X \) and \( Y \) as the points at infinity on the x and y axes.) Call \( \overline{0X}, \overline{0Y} \) and \( XY \subset P \) the axes associated to this choice of three points, and \( 0, \overline{XY} \) and \( \overline{Y} \subset \Lambda \) the dual axes. Define a map

\[
\alpha : p \in P \setminus XY \mapsto (p_X, p_Y) \in (\overline{0X} \setminus 0X) \times (\overline{0Y} \setminus 0Y)
\]

by

\[
p_X = (pY) (0X), \quad p_Y = (pX) (0Y).
\]

Define another map,

\[
\hat{\alpha} : \lambda \in \Lambda \setminus \overline{0} \mapsto (\lambda_X, \lambda_Y) \in (\overline{0X} \setminus 0X) \times (\overline{0Y} \setminus 0Y),
\]

by

\[
\lambda_X = \lambda (0X), \quad \lambda_Y = \lambda (0Y).
\]

Lemma 1. \( \alpha \) and \( \hat{\alpha} \) are diffeomorphisms.

Proof. It is easy to check that if

\[
\alpha(p) = (p_X, p_Y)
\]

then

\[
p = (p_X Y) (p_Y X).
\]

Therefore \( \alpha \) is smooth with smooth inverse, so a diffeomorphism. Similarly, if \( \hat{\alpha}(\lambda) = (\lambda_X, \lambda_Y) \), then \( \lambda = \lambda_X \lambda_Y \), so \( \hat{\alpha} \) is also a diffeomorphism. \( \Box \)
\( \hat{\alpha}^{-1}\alpha \) is a diffeomorphism between \( P \) with axes removed and \( \Lambda \) with dual axes removed.

A pointed line, i.e. an element \((p, \lambda) \in F\), can be mapped to \( 0\mathcal{X} \times 0\mathcal{Y} \times 0\mathcal{X} \) in two ways:

\[
\alpha(p, \lambda) = (p_X, p_Y, \lambda_X) \\
\hat{\alpha}(p, \lambda) = (p_X, \lambda_Y, \lambda_X).
\]

Given \((p_X, p_Y, \lambda_X)\), we can compute \( p = (p_X Y)(p_Y X) \) (as before), and \( \lambda = p\lambda_X \). It is easy to check that \( \alpha \) and \( \hat{\alpha} \) are diffeomorphisms

\[
\alpha : \{ (p, \lambda) \in F \mid p \not\in \mathcal{O}\mathcal{X} \cup \mathcal{X}\mathcal{Y} \text{ and } \lambda \not\in \mathcal{O}\mathcal{X} \} \to (0\mathcal{X}\setminus 0\mathcal{X}) \times (0\mathcal{Y}\setminus \{0, Y\}) \times 0\mathcal{X}.
\]

\[
\hat{\alpha} : \{ (p, \lambda) \in F \mid \lambda \not\in \hat{\mathcal{O}} \cup \hat{\mathcal{Y}} \} \to 0\mathcal{X} \times (0\mathcal{Y}\setminus \{0, Y\}) \times (0\mathcal{X}\setminus 0\mathcal{X}).
\]

Affine charts are defined on complements of submanifolds of the obvious codimensions. Moreover, as in the model, these charts are orientation preserving in projective planes of dimension 4 or more, since at the intersection points of the various lines, the maps are clearly orientation preserving, by positivity of intersection of lines.

**Lemma 2.** The lines and pencils of a smooth generalized plane are smooth embedded submanifolds. Lines meet transversely.

**Proof.** In the \( \hat{\alpha} \) chart on \( F \), this is immediate: we fix \( \lambda_X \) and \( \lambda_Y \) and vary only \( p_X \). We can cover \( F \) with three affine charts. An affine chart turns its axes into obviously transverse submanifolds. \( \square \)

### 5. Blowup

**Definition 6.** Given a point \( p_0 \in P \), define the blowup \( \text{Bl}_{p_0}(P) \) at \( p_0 \) to be the subset of \( F \) consisting of pairs \((p, \lambda) \in F \) so that \((p_0, \lambda) \in F \).

**Lemma 3.** The blowup of a smooth projective plane at a point is a smooth submanifold of the flag space. Moreover the map \((p, \lambda) \in \text{Bl}_{p_0}(P) \mapsto \lambda \in \tilde{p}_0 \) is a smooth fiber bundle. This fiber bundle admits the global section \( \lambda \in \tilde{p}_0 \mapsto (p_0, \lambda) \in \text{Bl}_{p_0}(P) \); the image of this section is called the exceptional divisor at \( p_0 \), and is also written \( \tilde{p}_0 \). The map \((p, \lambda) \in \text{Bl}_{p_0}(P) \mapsto p \in P \) is smooth, surjective, and a local diffeomorphism away from the exceptional divisor.

**Proof.** In an affine chart \( \hat{\alpha} \), setting \( X = p_0 \), the open subset of the blowup intersecting the domain of that chart is given by points \((p, \lambda) \) with \( p_Y = \lambda_Y \neq Y \) and \( \lambda_X = X \) and \( p_X \) an arbitrary point of \( \mathcal{O}\mathcal{X} \) other than \( X \). We obtain a bijection

\[
\hat{\alpha} : \{(p, \lambda) \in \text{Bl}_{p_0}(P) \mid \lambda \not\in 0\mathcal{X}, \mathcal{X}\mathcal{Y} \} \mapsto (p_X, p_Y) \in 0\mathcal{X} \times (0\mathcal{Y}\setminus \{0, Y\}),
\]

(The map \( \text{Bl}_{p_0}(P) \to \tilde{p}_0 \) is \((p_X, p_Y) \mapsto \lambda_Y = p_Y \), identifying an open set of the pencil \( \tilde{p}_0 \) with an open set of \( \mathcal{O}\mathcal{X} \).) This covers all of the blowup except for \( \lambda \in \mathcal{O} \cup Y \).

To cover those two pencils of lines, we switch coordinates, taking \( o = X, x = O, y = Y \). Now we can try to use the other coordinates, \( \alpha \) on \( F \). We then find that in terms of \((p_x, p_y, \lambda_x)\), the blowup \( \text{Bl}_o(P) \) is given by the equation \( \lambda_x = o \). This gives a map

\[
\alpha : (p, \lambda) \mapsto (p_x, p_y)
\]
from
\[ \{ \lambda \neq ox, xy \text{ and } p \neq 0 \text{ and } p \neq \overrightarrow{xy} \} \subset \text{Bl}_{p_0}(P) \]
to
\[ (\overrightarrow{o}x \setminus \{o, x\}) \times (\overrightarrow{oy} \setminus \{o, y\}) . \]
In this chart, the map \( \text{Bl}_{p_0}(P) \to \overline{p}_0 \) is expressed as \((p_x, p_y) \mapsto \lambda = po\) which we can map to the \(x\) or the \(y\) axis.

Where both \(\alpha\) and \(\hat{\alpha}\) are defined, we easily compute \((p_x, p_y)\) in terms of \((p, p_X)\) and vice versa, via diffeomorphisms. We have now covered all of the blowup except for the points \((o, oy), (y, oy)\) and the set of points of the form \((p, ox)\).

Swapping \(x\) and \(y\), so that the blowup is at \(p_0 = y\), and then checking the smoothness of all of the maps, covers all of the blowup except for \((o, oy), (y, oy), (o, ox), (x, ox)\).

Finally, choosing any other choice of affine charts, perhaps perturbing the \(x\) and \(y\) points slightly, covers the rest of the blowup. □

6. Hopf fibrations

**Lemma 4.** Pick a point \(p_0 \in P\) and a line \(\lambda_0 \in \Lambda\) with \(p_0\) not on \(\lambda_0\). The map
\[ f : p \in P\setminus p_0 \mapsto (pp_0) \lambda_0 \in \overline{\lambda}_0 \]
is a fiber bundle mapping, with fiber above \(q \in \overline{\lambda}_0\) the punctured line \(\overline{p_0q}\setminus p_0\).

*Proof.* As in Salzmann et. al. [43] p. 252; pick any \(q \in \overline{\lambda}_0\), and let \(U = \overline{\lambda}_0 \setminus q\), and let \(A\) be the pencil of lines through \(q\), with \(pq\) deleted. Define the map
\[ \phi : (p, \lambda) \in U \times A \mapsto (pp_0p) \lambda \in f^{-1}U. \]
These maps trivialize our fiber bundle. □

**Lemma 5.** For each point \(p_0 \in P\), the map
\[ f : p \in P\setminus p_0 \mapsto pp_0 \in \overline{p}_0 \]
is a fiber bundle map with fiber through \(\lambda\) being \(\overline{\lambda}_0 \setminus p_0\). Call this map \(f\) the Hopf fibration at \(p_0\).

*Proof.* This is essentially the same map. □

7. The infinitesimal Hopf fibration

We need to introduce an infinitesimal analogue of the Hopf fibration. Given a point \(p_0 \in P\), start by constructing the pullback vector bundle:

\[
\begin{array}{ccc}
\tau & \longrightarrow & \ker p'_\Lambda \\
\downarrow & & \downarrow \\
p_0 \times \overline{p}_0 & \longrightarrow & F \\
\downarrow & & \downarrow \\
\overline{p}_0 & \longrightarrow & \Lambda
\end{array}
\]
so that the fiber of \(\tau\) over a point \(\lambda \in \overline{p}_0\) is \(T_{p_0}\overline{\lambda}\). We can clearly map
\[ \tau \to T_{p_0}P \]
by inclusion.
Lemma 6. The map $\tau \to T_{p_0}P$ is a smooth bijection.

Proof. By transversality of lines, this map is an injection away from the zero section. Rescaling vectors by positive numbers gives an injective smooth map

$$S\tau \to ST_{p_0}P$$

from the bundle of spheres

$$S\tau = (\tau \setminus 0) / \mathbb{R}^+,$$

to the single sphere

$$ST_{p_0}P = (T_{p_0}P \setminus 0) / \mathbb{R}^+.$$  

Each fiber of the bundle $S\tau$ is the sphere of the tangent plane of a line, and is taken by the identity map to that same sphere of that same tangent plane. The map $\tau \to T_{p_0}P$ is just the derivative of the map $\text{Bl}_{p_0}(P) \to P$ restricted to the submanifold $\bar{p}_0 \subset \text{Bl}_{p_0}(P)$. We need to see why $\tau \to T_{p_0}P$ is onto. Let's suppose that it misses some open set. Then this open set, by rescaling, must contain an open cone. Picture taking local coordinates with origin at the point $p_0$. We can dilate these coordinates freely, zooming in on the origin. As we do, the lines through $p_0$ become flatter, with as many derivatives as we like, and we can therefore approximate them uniformly by their tangent planes at $p_0$. Therefore picking any point $p$ lying in our cone (in the given system of coordinates), the line $pp_0$ enters that open cone. So $\tau \to T_{p_0}P$ has dense image, and by rescaling $S\tau \to ST_{p_0}P$ must as well. By compactness of $S\tau$, the image must be closed. Therefore $S\tau \to ST_{p_0}P$ is a smooth bijection, and so $\tau \to T_{p_0}P$ is also a smooth bijection. By Sard's lemma, the map $S\tau \to ST_{p_0}P$ identifies the fundamental classes in cohomology. Again, by Sard's lemma, the inverse map $ST_{p_0}P \to S\tau$ is a diffeomorphism near a generic point. We call $ST_{p_0} \to S\tau \to \bar{p}_0$ the infinitesimal Hopf fibration.

Theorem 7 (Bödi [6]). The map $S\tau \to ST_{p_0}P$ is a smooth homeomorphism. In particular, the infinitesimal Hopf fibration is a topological sphere bundle, and a smooth submersion near a generic point.

Proof. Smoothness of $S\tau \to ST_{p_0}P$ is obvious, while homeomorphism is just the topological pigeonhole principle (see Salzmann et. al. [43] p. 251); Bödi's proof is different, employing the theory of microbundles.

Definition 7. We call the line $\lambda(\ell) \in \Lambda$ tangent to a given real line $\ell \subset T_pP$ the magnification of $\ell$.

Lemma 7. Magnification is continuous, and smooth near a generic point.

Proof. The magnification is the map taking a real line $\ell \subset T_{p_0}P$ through the homeomorphism $T_{p_0}P \to \tau$ and then through the vector bundle map $\tau \to \bar{p}_0$.  

\[1\]The term magnification is intended to remind the reader of complexification of a real line to a complex line in $\mathbb{C}^2$. 

We call $ST_{p_0} \to S\tau \to \bar{p}_0$ the infinitesimal Hopf fibration.
8. The tangential affine translation plane

We have a new approach to defining the tangent plane of a smooth projective plane.

Definition 8. An affine plane is a choice of sets \(P, \Lambda, F\) with map \(F \to P \times \Lambda\) so that, if we write \(\pi_P : F \to P, \pi_\Lambda : F \to \Lambda\), as usual, and call the sets \(\lambda = \pi_P \pi_\Lambda^{-1}(\lambda)\) lines then

1. any two points \(p_1, p_2\) lie on a unique line \(p_1p_2\) and
2. for any point \(p\) and line \(\lambda\), there is a unique line \(\lambda_p\) (called the parallel to \(\lambda\) through \(p\)) so that \(p\) lies on \(\lambda_p\) and either \(\lambda_p = \lambda\) or \(\lambda_p\) has no point in common with \(\lambda\) and
3. there are three points not contained in any line.

Following Bödi & Immervoll p. 66, we call an affine plane smooth (topological) if

1. the maps \(p_1, p_2 \mapsto p_1p_2\) and \(p, \lambda \mapsto \lambda_p\) are smooth (continuous) and
2. the intersection point of two lines is unique, if it exists, and depends smoothly (continuously) on the choice of the lines, and exists for an open set of lines, and
3. there are four points with no three on a common line.

A translation of an affine plane is a map taking points to points, lines to parallel lines, and preserving the flag space. We call an affine plane a (topological) smooth affine translation plane if the group of (homeomorphic) diffeomorphic translations acts transitively on points.

Definition 9. Given a smooth projective plane \(P\) and a point \(p_0 \in P\), let \(P_0 = T_{p_0}P\), let \(\tau \mapsto \tilde{p}_0\) be the vector bundle defined above, let \(E_0 = \tilde{p}_0 \times P_0 \to \tilde{p}_0\) be the trivial bundle, and let \(\Lambda_0\) be the quotient bundle \(E_0/\tau \to \tilde{p}_0\). Consider the quotient map \(Q : E_0 \to E_0/\tau\). Let \(F_0\) be the set of pairs \((p, l)\) in \(E_0 \oplus \tilde{p}_0, E_0/\tau\) for which \(Q(p) = l\).

Define maps \((p, l) \in F_0 \mapsto q \in \Lambda_0\) and \((p, l) \in F_0 \mapsto p \in P_0\).

Lemma 8. \((P_0, \Lambda_0, F_0)\) is a topological affine translation plane with choice of origin \(0 \in P_0\), called the tangent plane to \(P\) at \(p_0\). The translations are precisely the usual translations of \(P_0 = T_{p_0}P\), and are homeomorphisms.

Proof. Clearly \(Q\) is smooth and of constant rank, so that \(F_0\) is a smooth fiber subbundle of \(E_0 \oplus \tilde{p}_0, E_0/\tau\). The maps \(\pi_P\) and \(\pi_\Lambda\) are obviously smooth submersions. The rest is proven in Bödi [7].

Remark 2. For some smooth projective planes, the tangent plane is not necessarily a smooth translation plane. The trouble comes from the smoothness of the map \(p_1, p_2 \mapsto p_1p_2\). Indeed the map \(p_2 \mapsto 0p_2\) is the magnification map.

9. The Radon Transform

Definition 10. Let \(P\) be a smooth projective plane, with \(\Lambda\) its space of lines, and \(F\) its correspondence space. Take any top degree form \(\eta\) on \(\Lambda\), and let \(\tilde{\eta} = \pi_P^* \pi_\Lambda^* \eta\). We call \(\tilde{\eta}\) the Radon transform of \(\eta\). If the dimension of \(P\) is \(2n\), then \(\tilde{\eta}\) is an \(n\)-form.

Lemma 9. If \(\eta\) is a volume form on \(\Lambda\) (i.e. a nowhere vanishing top degree form), then \(\tilde{\eta}\) is a closed form on \(P\), \(\tilde{\eta}^2\) is a volume form on \(P\), and \(\phi^* \tilde{\eta}\) pulls back to a positive volume form on any line.
Proof. Suppose that $C\overline{\lambda}$ is a line through $p \in \mathcal{P}$, and $\hat{\eta} = 0$ at $T_p \overline{\lambda}$. Using the map $\hat{\alpha}$ on a dense open subset of

$$\overline{0X} \times \overline{0Y} \setminus (0 \times \overline{0Y} \cup \overline{0X} \times 0),$$

we write $\eta$ as

$$\eta = f \, d\lambda_X \wedge d\lambda_Y,$$

with $f > 0$ and $d\lambda_X$ and $d\lambda_Y$ any volume forms on the lines $\overline{0X}$ and $\overline{0Y}$ compatible with the orientations on those lines. Pulling back,

$$\pi^* \Lambda \eta = f \, d\lambda_X \wedge d\lambda_Y = f \, d\lambda_X \wedge (\frac{\partial \lambda_Y}{\partial p_X} \, dp_X + \frac{\partial \lambda_Y}{\partial p_Y} \, dp_Y).$$

Pushing down,

$$\hat{\eta} = \pi\pi^* \Lambda \eta = \left( \int f \, \frac{\partial \lambda_Y}{\partial p_X} \, d\lambda_X \right) \, dp_X + \left( \int f \, \frac{\partial \lambda_Y}{\partial p_Y} \, d\lambda_X \right) \, dp_Y.$$

Therefore

$$dp_X \wedge \hat{\eta} = \left( \int f \, \frac{\partial \lambda_Y}{\partial p_X} \, d\lambda_X \right) \, dp_X \wedge dp_Y.$$

We have to check signs: we need to ensure that we can consistently orient lines to keep $\frac{\partial \lambda_Y}{\partial p_Y} > 0$. The equations

$$\lambda_Y = ((p_X Y) (p_Y X) \lambda_X) (0Y)$$

$$p_Y = (0\lambda_Y) ((0Y Y \lambda_Y) (p_X Y)),$$

gives $\lambda_Y$ in terms of $p_Y$ and conversely, so these are diffeomorphically mapped to one another, and so $\frac{\partial \lambda_Y}{\partial p_Y} \neq 0$. By preservation of orientations under affine charts, $\frac{\partial \lambda_X}{\partial p_Y} > 0$. As a consequence, $\hat{\eta} > 0$ on $T_0 \overline{0X}$. We can pick the points 0 and $X$ to be anywhere we like, in particular pick 0 = $p$ and pick $X$ on the line $\overline{\lambda}$. So $\hat{\eta}$ is a volume form on each line.

Given any point $p \in \mathcal{P}$, take two distinct lines $\lambda_1, \lambda_2$ through $p$. They are transverse at $p$, and $\hat{\eta} \neq 0$ on each of their tangent planes at $p$, so $\hat{\eta}^2 \neq 0$ at $p$. □

Note that a volume form $\eta$ on $\Lambda$ exists just when the dimension of $\mathcal{P}$ is 0,4,8 or 16 (i.e. not 2), as is apparent from the cohomology.

Corollary 2. If the dimension of a projective plane is 4, then the Radon transforms $\hat{\eta}$ of positive volume forms $\eta$ are all symplectomorphic, up to rescaling.

Proof. Apply the Moser homotopy method. □

Lemma 10. If $\Sigma \subset \mathcal{P}$ is a compact oriented smooth submanifold (perhaps with boundary and corners) of dimension $n > 0$ in a dimension $2n$ projective plane, then

$$\int_{\Sigma} \hat{\eta} = \int_{\Lambda} \# (\Sigma \cap \overline{\lambda}) \, \eta$$

where $\# (\Sigma \cap \overline{\lambda})$ is the number of intersections of $\Sigma$ and $\overline{\lambda}$, defined on the full measure subset of $\lambda$ for which $\Sigma$ and $\overline{\lambda}$ only intersect transversely.
Remark 3. We count \( \# (\Sigma \cap \lambda) \) keeping track of signs for positivity or negativity of intersection. For a full measure set of \( \lambda \in \Lambda \), the intersection will be transverse, so we don’t need to worry about whether we are counting with multiplicity or not at points of nontransverse intersection.

**Proof.** First, if we cut \( \Sigma \) into two submanifolds \( \Sigma_1 \) and \( \Sigma_2 \), possibly with boundary and corners, which overlap only on their boundaries, then clearly

\[
\int_{\Sigma} \eta = \int_{\Sigma_1} \eta + \int_{\Sigma_2} \eta.
\]

On the other hand, the intersections of a line \( \lambda \) with \( \Sigma \) could occur either on \( \Sigma_1 \) or on \( \Sigma_2 \), but possibly on both. However, the intersection \( \Sigma' = \Sigma_1 \cap \Sigma_2 \) is a compact submanifold of dimension \( n - 1 \), perhaps with boundary and corners. Define \( F' \) to be the pullback bundle

\[
F' \longrightarrow F
\]

\[
\Sigma' \longrightarrow P.
\]

The lines striking \( \Sigma' \) are the image of \( F' \to F \to \Lambda \), so by Sard’s theorem, counting dimension, the lines striking \( \Sigma' \) form a measure zero set. Therefore

\[
\int_{\Lambda} \# (\Sigma \cap \lambda) \eta = \int_{\Lambda} \# (\Sigma_1 \cap \lambda) \eta + \int_{\Lambda} \# (\Sigma_2 \cap \lambda) \eta.
\]

Therefore we only need to prove the result for “small pieces” of \( \Sigma \). The result holds for lines (for which it reduces to a statement in cohomology), and therefore for any \( \Sigma \) built out of finitely many compact subsets of lines.

For any \( \Sigma \), after cutting into enough submanifolds, we can see that \( \pi^{-1}_p \Sigma \) is a smooth manifold with boundary and corners, and \( \pi^{-1}_p \Sigma \to \Sigma \) is a fiber bundle, with fiber over point \( p \) the pencil \( \overline{p} \). Our integral is:

\[
\int_{\Sigma} \eta = \int_{\pi^{-1}_p \Sigma} \pi^*_\Lambda \eta.
\]

The map \( \pi_\Lambda : \pi^{-1}_p \Sigma \to \Lambda \) has preimage at each point \( \lambda \) given by all of the pairs \( (p, \lambda) \) with \( p \in \Sigma \), i.e. \( \Sigma \cap \lambda \). For \( \Sigma \) a subset of a line \( \lambda_0 \), these are transverse positive intersections. For generic \( \Sigma \) and generic \( \lambda \) they are transverse intersections. The integrand vanishes except at the points where \( \pi_\Lambda \) takes \( \pi^{-1}_p \Sigma \) locally diffeomorphically to \( \Lambda \). Again by Sard’s theorem, the integral away from those points is just precisely the integral of \( \# (\Sigma \cap \lambda) \eta \).

\[ \square \]

**Corollary 3.** The same is true if \( \Sigma \) is a finite union of compact rectifiable submanifolds, possibly with boundaries and corners.

**Corollary 4.** If \( \Sigma \subset P \) is a compact \( n \)-dimensional rectifiable cycle (e.g. a line), then

\[
\int_{\Lambda} \# (\Sigma \cap \lambda) \eta = \int_{\Sigma} \eta = [\eta] [\Sigma]
\]

where the cohomology classes \( [\eta] \in H^{2n}(\Lambda, \mathbb{R}) = \mathbb{R} \) and \( [\Sigma] \in H_2(P, \mathbb{R}) = \mathbb{R} \) are thought of as numbers, using the standard basis for the cohomology.

**Corollary 5.** The Radon transform \( \eta \mapsto \hat{\eta} \) on projective planes of dimension \( 4 \) or more is injective.
Theorem 8. Every smooth projective plane of dimension 4 is diffeomorphic to the complex projective plane.

Proof. Such a projective plane is a compact 4-manifold with a symplectic structure. Each line is diffeomorphic to a sphere, because we know its cohomology is that of a sphere. Moreover, its self-intersection is nonnegative, because in the orientation coming from the symplectic form, it belongs to a 4-dimensional family of spheres, and any two distinct spheres from that family intersect positively. Lalonde & McDuff [34] prove that a compact symplectic 4-manifold containing a symplectic sphere with nonnegative self-intersection is symplectomorphic to the complex projective plane, or a blowup of the complex projective plane at a finite number of points, or $S^2 \times S^2$. By cohomology, all of these are ruled out except the complex projective plane. □

Remark 4. This idea is easy to generalize: Gromov [23] p. 336 suggests that a compact 4-manifold admitting a smooth incidence geometry of any reasonable type with compact embedded curves is diffeomorphic to $\mathbb{C}P^2$, although he gives no details. See Bődi & Immervoll [8] for the definition of smooth incidence geometries.

Corollary 6. Every smooth projective plane of positive dimension is diffeomorphic to its model.

Proof. We proved the result in dimension 4 above; for dimensions 8 and 16, the result was proven by Linus Kramer & Stephan Stolz [32]. □

10. IMMERSED PLANE CURVES

Definition 11. A plane curve in a smooth projective plane $P$ of dimension $2n$ is a smooth immersion of manifolds $C^n \to P$ which is tangent to a line at each point.

Note:

1. we do not ask the curve to be compact; e.g. in $\mathbb{C}P^2$ this allows transcendental (i.e. nonalgebraic) curves
2. we do not allow singularities; picturing complex curves in $\mathbb{C}P^2$, we would have to remove their singular points to fit this definition
3. in $\mathbb{R}P^2$ (or any smooth projective plane of dimension 2) this definition allows all immersed curves, and is therefore useless. A reasonable concept of plane curve in a 2-dimensional projective plane, generalizing the concept of algebraic curve in $\mathbb{R}P^2$, has never been formulated. There probably is one, in terms of Cartan’s normal projective connection (see Bryant, Griffiths & Hsu [12]).

Lemma 11. Suppose that $P$ is a smooth projective plane, with space of lines $\Lambda$. The Radon transform of any volume form on $\Lambda$ is a positive volume form on every plane curve.

Proof. Plane curves are tangent to lines, and the Radon transform of a volume form is positive on lines. □

Lemma 12. In a smooth projective plane of dimension 4 or more, the homology class of any closed plane curve is a positive multiple of the homology class of a line.
Proof. The homology is $H_n(P) = \mathbb{Z}$, generated by $[\bar{\lambda}]$; see theorem 5 on page 5. Therefore $[C] = d [\bar{\lambda}]$, for some integer $d$. But

$$0 < \int_C \bar{\eta} = d \int_\lambda \eta.$$

\[\square\]

Definition 12. The degree of a plane curve $C$ in a smooth projective plane of dimension 4 or more is the ratio

$$d = \frac{[C]}{[\bar{\lambda}]} \in \mathbb{Z}^+.$$

Lemma 13. If $C$ is an embedded plane curve, and a sequence of points $p_j \in C$ approaches a limit $q \in C$, then the secant lines $p_jq$ approach the tangent line to $C$ at $q$.

Proof. There is a tangent line $\lambda$ to $C$ by definition, and it is unique by transversality. The points $p_j$ lie on the submanifold $C$, and approach $q$. Therefore in any local coordinates with $q$ as origin, if $p_j$ is close enough to $q$, then $p_j$ lies near to $T_qC$. Dilating coordinates as needed, the line $p_jq$ is nearly a linear subspace in some tiny coordinate ball. Therefore $T_qp_jq \rightarrow T_qC$ in the Grassmann bundle of linear subspaces. The Gauss map $F \rightarrow \text{Gr}(n, TM)$ taking a line to its tangent plane is clearly smooth, and injective, and $F$ is a compact manifold. Therefore the Gauss map is a topological embedding. So the convergence of the tangent spaces $T_qp_jq \rightarrow T_qC$ implies convergence $p_jq \rightarrow \lambda$. \[\square\]

11. Polycontact system

Define a field $\Theta$ of $2n$-planes on the flag space $F$ by assigning to each point $(p, \lambda) \in F$ the $2n$-plane

$$\Theta = \ker \pi'_P \oplus \ker \pi'_\Lambda \subset TF.$$

Call $\Theta$ the polycontact plane field.

Lemma 14. $\Theta$ is a smooth plane field, invariant under projective duality, and

$$\Theta_{(p, \lambda)} = (\pi'_P)^{-1} T_p \bar{\lambda} \subset T_{(p, \lambda)} F.$$

Proof. Smoothness is obvious, as is invariance under projective duality. By definition, $\bar{\lambda} = \pi_P \left( \pi^-_\Lambda (\lambda) \right)$. Differentiating,

$$T_p \bar{\lambda} = T_p \pi_P \left( \pi^-_\Lambda (\lambda) \right)$$

$$= \pi'_P(p, \lambda) T_{(p, \lambda)} \pi^-_\Lambda (\lambda)$$

$$= \pi'_P(p, \lambda) \ker \pi'\Lambda(p, \lambda)$$

$$= \pi'_P(p, \lambda) \Theta_{(p, \lambda)},$$

so

$$\Theta_{(p, \lambda)} = \pi'_P(p, \lambda)^{-1} T_p \bar{\lambda}.$$

\[\square\]
Definition 13. Let $\pi : \tilde{\text{Gr}}(n, TP) \to P$ be the Grassmann bundle of oriented $n$-planes in the tangent spaces of $P$. Let $\tilde{\Theta} \subset \tilde{T\text{Gr}}(n, TP)$ be the field of tautological planes, 
\[ \tilde{\Theta}_\Pi = \pi'(\Pi)^{-1}\Pi. \]

Lemma 15. The Gauss map 
\[ g : (p, \lambda) \in F \to T_p\lambda \in \tilde{\text{Gr}}(n, TP) \]
is injective, and 
\[ \Theta_{(p, \lambda)} = g'(p, \lambda)^{-1}\tilde{\Theta}_{g(p, \lambda)}. \]
In particular, if the Gauss map is an immersion, then $\Theta$ is a subbundle of $g^*\tilde{\Theta}$.

The proof: unwind the definitions.

12. Plane curves

Definition 14. A generalized plane curve in a projective plane of dimension $2n$ is a Lipschitz map $\phi : C \to F$ from an $n$-dimensional manifold $C$, perhaps with boundary, so that every differential form $\vartheta$ (of any degree) on $F$, vanishing on $\Theta$, pulls back to $\phi^*\vartheta = 0$. A generalized plane curve is called basic if $\phi$ is injective on a dense open set and intersects each fiber of $\pi_P : F \to P$ on a discrete set of points.

Remark 5. We will further generalize the notion of plane curve below to allow singularities.

Lemma 16. A continuously differentiable immersed plane curve $\phi : C \to P$ lifts to a continuous map $\Phi : C \to F$ defined by $\Phi(c) = (\phi(c), \lambda(c))$, where $T\phi(c)\lambda(c) = \phi'(c)T_C$. If the Gauss map is an immersion, and $\phi$ is $C^{k+1}$, then $\Phi$ is a $C^k$ basic generalized curve.

Proof. By hypothesis that $\phi$ is a plane curve, $\Phi$ is defined; to see that $\Phi$ is continuous, take any real immersed curve $c(t)$ on $C$, and look at its tangent lines $\phi'(c(t))c'(t)$, and magnify them: $\lambda(c(t)) = \lambda(\phi'(c(t))c'(t))$. By theorem 7 on page 8, this map is continuous.

Suppose that $F \to \tilde{\text{Gr}}(n, TP)$ is an immersion. Then $\Phi$ maps to $\phi'$, so is continuously differentiable. To show that $\Phi : C \to F$ is a generalized plane curve, we will show that $\Phi'(c)T_C \subset \Theta$. To see this, first note that $\pi_P\Phi'(c) = \phi'(c)$. Next, if $p = \phi(c)$ and $\lambda = \lambda(c)$, then
\[
\Theta_{(p, \lambda)} = (\pi'_P)^{-1}T_p\lambda
= (\pi'_P)^{-1}\phi'(c)T_C
= (\pi'_P)^{-1}(\pi_P\Phi'(c))T_C
= (\pi'_P)^{-1}\pi_P\Phi'(c)T_C
\supset \Phi'(c)T_C.
\]

Lemma 17. A continuously differentiable generalized plane curve $\Phi : C \to F$ is the lift of a plane curve $\phi : C \to P$ just when it is basic.
Proof. Define $\phi = \pi_P \Phi$. Clearly $\phi : C \to P$ is an immersion because $\Phi$ is not transverse to the fibers of $\pi_P$. We need to show that $\phi$ is tangent to some line at every point. Write $\Phi(c) = (\phi(c), \lambda(c))$. Since $\Phi'(c) T_c C \subset \Theta_{\phi(c)}$, 

$$
\phi'(c) T_c C = \pi'_P \Phi'(c) T_c C \\
\subset \pi'_P \Theta_{\phi(c), \lambda(c)} \\
= \pi'_P (\pi_P)^{-1} T_{\phi(c)} \tilde{\lambda}(c) \\
= T_{\phi(c)} \tilde{\lambda}(c).
$$

\[\square\]

Definition 15. If $\phi : C \to P$ is an immersed projective curve, then its dual curve $\phi^* : C \to P^*$ is $\phi^* = \pi_P \Phi$, where $\Phi : C \to F$ is the lift of $\phi$.

Corollary 7. The dual curve $\phi^* : C \to P^*$ of an immersed plane curve $\phi : C \to P$ is an immersed plane curve in the dual plane, at every point $c \in C$ where $\phi$ does not have second-order tangency with a line.

Lemma 18. If $\eta$ is a positive volume form on $\Lambda$, and $\phi : C \to F$ is a plane curve, then $\phi^* \pi_P \eta$ is a positive volume form on $C$.

Lemma 19. Generalized plane curves are precisely the solutions of a smooth system of first order partial differential equations.

Proof. The differential equations for the lift $\Phi \subset F$ are just $\partial = 0$ for every $\partial$ satisfying $\Theta = 0$. In local coordinates, a local basis of such $\partial$ is easy to write down, since $\Theta$ is a vector bundle. \[\square\]

Lemma 20. Plane curves (not generalized) are precisely the solutions of a system of first order partial differential equations.

Proof. Consider the map $(p,\lambda) \in F \mapsto T_p \tilde{\lambda} \in \widetilde{\text{Gr}}(n, TP)$. The requirement that an immersed manifold be a plane curve is that its tangent space lie in the image of this map, a set of possible first derivatives in any system of local coordinates. \[\square\]

13. Regularity

Suppose that $\ell \subset T_{p_0} P$ is a real line. Let $\widetilde{\text{Gr}}(n, T_{p_0} P)$ be the set of oriented $n$-planes in $T_{p_0} P$ containing $\ell$. Our map $F \to \widetilde{\text{Gr}}(n, TP)$ maps $p_0 \to \widetilde{\text{Gr}}(n, T_{p_0} P)$. Transversality of lines (which holds for all smooth projective planes) is precisely the statement that $\tilde{p}_0$ can only intersect $\widetilde{\text{Gr}}(n, T_{p_0} P)$ in at most one point.

Definition 16. A smooth projective plane is regular if for any point $p_0 \in P$ and any real line $\ell \subset T_{p_0} P$, $\tilde{p}_0$ and $\widetilde{\text{Gr}}(n, T_{p_0} P)$ only have transverse intersections, if they have any intersections at all.

Example 1. The classical projective planes $\mathbb{R}P^2, \mathbb{C}P^2, \mathbb{H}P^2$ and $\mathbb{O}P^2$ are all regular. Indeed by homogeneity of their collineation groups on their projective tangent bundles, one need only check a single real line.

We will present several properties which are equivalent to regularity.

Definition 17. A linear map $t : U \to V^* \otimes W$ (written $u \mapsto t_u$) with $\dim U = \dim V = \dim W$ is called a trilinearity (following Cartan [14]) if all of the linear maps $t_u$ are invertible, except at $u = 0$. 


Example 2. If $A$ is an algebra, then the multiplication map $A \otimes A \to A$ determines a map $t : A \to A^* \otimes A$, by $t_u = R_u$ (right multiplication by $u \in A$). This map is a triality just when $A$ has no zero divisors. For $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$, we will call this a classical triality.

As is well known, the tangent planes to a Grassmannian $\widetilde{\text{Gr}} \left( n, \mathbb{R}^N \right)$ at a point $E$ are intrinsically identified with linear maps $E^* \otimes (\mathbb{R}^N / E)$ as follows. Consider the principal bundle
\[
\text{GL}(n, \mathbb{R}) \longrightarrow \text{Epi} \left( \mathbb{R}^N, \mathbb{R}^n \right) \longrightarrow \widetilde{\text{Gr}} \left( n, \mathbb{R}^N \right).
\]

The vertical map takes an epimorphism $T$ to its kernel. The tangent plane to $\text{Epi} \left( \mathbb{R}^N, \mathbb{R}^n \right)$ at any point is $\mathbb{R}^{N^*} \otimes \mathbb{R}^n$. Given a vector $\tilde{E}$ tangent to the Grassmannian at $E$, take any tangent vector $\tilde{T}$ to $\text{Epi} \left( \mathbb{R}^N, \mathbb{R}^n \right)$ at $T$, so that $\tilde{T}$ maps to $\tilde{E}$ under the derivative of the bundle map. Identifying $T^T \text{Epi} \left( \mathbb{R}^N, \mathbb{R}^n \right)$ with $\mathbb{R}^{N^*} \otimes \mathbb{R}^n$, consider $T^{-1} \tilde{T} \big|_E \in E^* \otimes (\mathbb{R}^N / E)$. The reader can easily check that $\tilde{E} \mapsto T^{-1} \tilde{T} \big|_E$ smoothly identifies the tangent vectors to the Grassmannian with linear maps. The tangent plane to a subGrassmannian $\widetilde{\text{Gr}}_\ell \left( n, \mathbb{R}^N \right)$ is precisely the space of linear maps $\xi \in E^* \otimes (\mathbb{R}^N / E)$ for which $\ell \subset \ker \xi$.

**Lemma 21.** Map $\bar{p}_0$ to $\widetilde{\text{Gr}} \left( n, T_{p_0}P \right)$ by mapping a line $\lambda$ to its tangent plane $T_{p_0}\lambda$. Regularity is just the requirement that this map is an immersion and that the induced maps on tangent planes of $\bar{p}_0$
\[
T_{\lambda} \bar{p}_0 \to E^* \otimes (T_{p_0}P / E).
\]
are trialities.

**Proof.** Transversality to all of these subGrassmannians is precisely the absence of kernel of all of the linear maps representing tangent vectors. \hfill \square

**Lemma 22** (Otte [41]). A smooth projective plane is regular just when the infinitesimal Hopf fibration is a smooth fiber bundle, i.e. the tangent planes $T_{p_0}\lambda$ of lines at a point $p_0$ divide the tangent plane $T_{p_0}P$ into a fiber bundle (except at the origin). This is equivalent to smoothness of the magnification map $\ell \mapsto \lambda(\ell)$ (the line $\lambda$ so that $\ell \in T_p\lambda$, for $\ell \subset T_{p_0}P$ a real line).

**Proof.** Assume regularity. The map $\tau \to T_{p_0}P$ above maps the tangent planes of lines injectively and smoothly into the tangent plane. We need to show that it is a diffeomorphism away from the 0 section, and then we will use the fiber bundle structure of $\tau \setminus 0 \to \bar{p}_0$ to induce such a structure on $T_{p_0}P \setminus 0$. If the map $\tau \to T_{p_0}P$ has a nonzero vector, say $v \in T_{p_0}$, in its image, then let $\ell$ be the span of $v$. Inside the Grassmannian, $\bar{p}_0$ strikes $\widetilde{\text{Gr}}_\ell \left( n, T_{p_0}P \right)$ transversely at some point $\lambda$ (which is identified with $T_{p_0}P \lambda$ in $\widetilde{\text{Gr}} \left( n, T_{p_0}P \right)$). Take $E \subset T_{p_0}P$ any linear subspace for which $E \cap T_{p_0}\lambda = \ell$. For each real line $\ell'$ in $E$ close to $\ell$, the submanifold $\widetilde{\text{Gr}}_\ell \left( n, T_{p_0}P \right)$ must intersect $\bar{p}_0$ transversely at a single point, say $\lambda'$. By compactness of $\bar{p}_0$, this must happen for all $\ell'$, not just those close to...
Moreover, \( \lambda \) clearly varies smoothly with \( \ell' \) and \( E \), by the implicit function theorem. Therefore \( \tau \rightarrow \tilde{p}_0 \) has a smooth inverse.

Conversely, assume that \( \tau \rightarrow \tilde{p}_0 \) is a diffeomorphism away from 0, use the map \( \tau \rightarrow \tilde{p}_0 \) to associate to each \( \ell \) a smooth choice of \( \lambda (\ell) \). We need to show that for any nonzero vector \( v \in T_{p_0} \tilde{\lambda} \), and any vector \( \xi \in T_{\tilde{p}_0} \tilde{\lambda} \), \( \xi (v) \neq 0 \), thinking of \( \xi \) as a linear map. Suppose that \( \xi (v) = 0 \) for some such \( \xi \neq 0 \) and \( v \neq 0 \). Write \( \xi \) as \( \xi = T^{-1} \tilde{T} \), so that \( \tilde{T} v = 0 \) and \( T v = 0 \) and \( v \neq 0 \). We must have \( \xi = T^{-1} \tilde{T} \) the velocity at \( t = 0 \) of a curve \( \lambda (t) \) in \( \tilde{p}_0 \), and we can produce a curve \( T(t) \) in \( \operatorname{Epi}(T_{p_0} P, \mathbb{R}^n) \) so that \( \lambda (t) \) is its image in the Grassmannian. By assumption on \( \xi \), \( T(0) v = T(0) \tilde{v} = 0 \). Since \( T_{p_0} P \) fibers over \( \tilde{p}_0 \) away from the origin, we can replace the single vector \( v \) with a curve \( v(t) \in T_{p_0} P \setminus 0 \) so that \( \lambda(t) \) is its image in \( \tilde{p}_0 \), \( \tilde{v}(0) \) is not in \( T_{p_0} \tilde{\lambda} \). Therefore \( v(t) \in T_{p_0} \tilde{\lambda}(t) \) with \( T_{p_0} \tilde{\lambda}(t) \) the kernel of \( T(t) \), i.e. \( T(t) v(t) = 0 \). Putting our equations together,

\[
T(t) v(t) = \tilde{T}(0) v(0) = 0.
\]

Plugging in Taylor expansions for \( T(t) \) and \( v(t) \) to these equations, we find

\[
T(0) v(0) = 0,
\]

so that \( v(0) \in T_{p_0} \tilde{\lambda}(0) \), contradicting our hypothesis. Therefore the tangent planes of \( \tilde{p}_0 \) are trialities.

**Corollary 8.** Regularity occurs just when for any point \( p_0 \in P \) and real line \( \ell \subset T_{p_0} P \), \( \tilde{p}_0 \) strikes \( \tilde{G}_\ell (n, T_{p_0} P) \) transversely at a single point, which depends smoothly on \( p_0 \) and \( \ell \).

**Corollary 9.** Regularity in the sense above is identical to regularity in the sense of Breitsprecher [10] and Bödi [6].

**Proof.** They define regularity as the property that the infinitesimal Hopf fibration is an isomorphism. \( \square \)

**Corollary 10.** The generic point of the flag space of a smooth projective plane is regular.

**Proof.** By Bödi’s theorem 7, the infinitesimal Hopf fibration is a smooth homeomorphism. By Sard’s lemma, the generic point of the target is a regular value. Therefore the generic point of the source is a regular point. \( \square \)

**Theorem 9.** Regularity is precisely smoothness of the tangent plane (i.e. the affine translation plane of lemma 8 on page 9).

### 14. Trialities

Recall from definition 17 on page 15 the concept of triality.

**Definition 18.** A bilinear map \( t : U \otimes V \rightarrow W \) is called a tableau (see Bryant et. al. [11]). Define the dual tableau to be \( t^* : V \otimes U \rightarrow W \), given by \( t^*(v, u) = t(u, v) \).

Consider a tableau \( t : U \otimes V \rightarrow W \), with the property that \( t(u, v) \neq 0 \) unless \( u = 0 \) or \( v = 0 \). This is identified with the triality \( t : U \rightarrow V^* \otimes W \) given by \( t_u (v) = t(u, v) \).

**Remark 6.** The concept of tableau does not require that \( U, V, W \) have the same dimension. However, the concept of triality does.
Definition 19. Given a triality \( t : U \to V^* \otimes W \), pick any elements \( \epsilon_U \in U \setminus 0 \) and \( \epsilon_V \in V \setminus 0 \). Define \( \epsilon_U : u \in U \mapsto t_u (\epsilon_V) \in V \) and \( \epsilon_V : v \in V \mapsto t_{\epsilon_U} (v) \in W \). Given \( v_0, v_1 \in V \) define
\[
v_0 v_1 = \epsilon_V^{-1} t (\epsilon_U^{-1} (v_0), v_1).
\]
This determines a real algebra with identity (not necessarily commutative or associative). We call this the triality algebra of these two points. We define the real part of any element \( v \) of the triality algebra to be
\[
\Re(v) = \frac{\tr(t_u)}{\dim U},
\]
where \( \epsilon_U(u) = v \). We can think of this as a real number, or as an element of the algebra. We define the imaginary part to be \( \Im(v) = v - \Re(v) \).

Remark 7. If \( U, V, W \) have the same dimension, then the obvious notion of mapping tableau under linear transformations, taking a tableau \( t \) and transformations \( (g_U, g_V, g_W) \) with \( g_U \in \text{GL}(U) \), etc., and defining
\[
(g_U, g_V, g_W) t(u, v) = g_W t(g_U^{-1} u, g_V^{-1} v),
\]
is called isotopy of algebras by algebraists (see Albert [1]), perhaps an unfortunate term. The relevant objects are really tableaux, not algebras.

Definition 20. Given an algebra \( A \), and element \( x \in A \), let \( L_x : A \to A \) be the operation of left multiplication by \( x \). An algebra with identity element is called classical if for any imaginary element \( x \in A \), the endomorphism \( L_x^2 \) is a multiple of the identity element.

Lemma 23. A triality algebra is classical just when the underlying triality is classical (i.e. isomorphic to precisely one of the trialities of right multiplication by real numbers, complex numbers, quaternions or octave numbers). Given a triality \( U \subset V^* \otimes W \), the triality algebra associated to a choice of two points \( \epsilon_U \in U \setminus 0 \) and \( \epsilon_V \in V \setminus 0 \) is classical just when it is classical at any pair of points.

Proof. Write \( L_x^2 = -Q(x) \in \mathbb{R} \). Identifying \( U \) and \( V \) via \( \epsilon_U \) and taking determinant
\[
\det(L_x^2) = Q(x)^n.
\]
Since \( x \neq 0 \), we must have \( \det(L_x^2) = \det(L_x)^2 \neq 0 \). Therefore \( Q \) is a definite quadratic form. If \( Q \) is not positive definite, then pick some \( x \) on which \( Q(x) = -1 \), and compute
\[
(1 - L_x)(1 + L_x) = 0.
\]
Because there are no zero divisors, \( x = \pm 1 \), is not imaginary. So \( Q \) is positive definite. Therefore we can take \( n \)-th roots above and get
\[
Q(x) = \det(L_x)^2/n.
\]
Let \( V' \subset V \) be the set of imaginary elements. Then
\[
V' \hookrightarrow \mathfrak{sl}(V)
\]
extends to a representation of the Clifford algebra \( \mathcal{C}^\ell (V', Q) \), since it satisfies \( L_x^2 = -Q(x) \). It is not a trivial representation, since it is not trivial on \( V' \). Therefore it is a sum of irreducible representations of \( \text{Spin}(n - 1) \) in dimension \( n \). Representation theory tells us that such a representation exists only for \( n = 1, 2, 4 \) or 8, and is determined up to isomorphism. In particular, it tells us that the algebra is isomorphic to \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \). \( \square \)
15. Regularity and the polycontact system

**Definition 21.** An adapted coframing on the flag space $F$ of a smooth projective plane is a choice of 1-forms $\vartheta^\mu, \omega^\mu, \pi^\mu$, $\mu = 1, \ldots, n$ on an open subset of $F$ so that $\vartheta$ and $\omega$ are semibasic for $F \to P$, $\vartheta$ and $\pi$ are semibasic for $F \to \Lambda$ and $\Theta = (\vartheta = 0)$.

**Proposition 1.** Pick any adapted coframing, and calculate $d\vartheta = -\varpi \wedge \omega$ modulo $\vartheta$, where $\varpi$ is a combination of $\pi$ and $\omega$ 1-forms. Then $\varpi$ has the form $\varpi^\mu = t_{\nu\sigma}^\mu \pi^\sigma$.

Call $t = (t^\mu_{\nu\sigma})$ the tableau of the adapted coframing. The expression $t^\mu(u,v) = t^\mu_{\nu\sigma}u^\nu v^\sigma$ is a triality $t : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}^n$ just where the projective plane is regular.

**Proof.** We can ensure that $d\vartheta$ has the stated form, because unabsorbable $\omega \wedge \omega$ terms would prevent existence of integral manifolds, but we have the lines as integral manifolds, and similarly dually there can be no unabsorbable $\pi \wedge \pi$ terms (see Bryant et. al. [11]).

Let us start off with a special choice of adapted coordinates. Take point $(p, \lambda) \in F$, and take coordinates $x, y$ on $P$ near $p$, so that $p$ is $(x,y) = (0,0)$, and so that $T_p \lambda$ is $dy = 0$. Up on $F$, we can let $\omega = dx$, and construct a suitable $\vartheta$ by demanding that $\vartheta = dy - z \, dx$, for some uniquely determined function $z$ defined on an open set. Take $\pi^\mu$ arbitrary 1-forms semibasic for $F \to \Lambda$ and independent of $\vartheta$.

I claim that for any vector $v \in T_{(p,\lambda)}F$ tangent to the fiber $\bar{p}$, under motion with velocity $v$,

$$\frac{dv^\mu}{dt} = t^\mu_{\nu\sigma}v^\nu,$$

where we write $v^\sigma$ for $v \wedge \pi^\sigma$. To prove the claim, extend $v$ to a vector field $X$ on an open subset of $F$ and tangent to the fibers of $F \to P$, and compute $\mathcal{L}_X \vartheta = X \vartheta - d(\mathcal{L}_X \vartheta)$. Note that since $X$ is tangent to the fibers, and $\vartheta$ is semibasic, $X \vartheta = 0$.

Pick $\ell \subset (dy = 0)$ any line, say spanned by a vector $u = u^\mu \frac{\partial}{\partial x^\mu}$. The open subset of the Grassmannian $\text{Gr}_\ell (n, T_p P)$ on which $dx^1 \wedge \cdots \wedge dx^n \neq 0$ consists of the planes $dy = z \, dx$ with $z^\sigma u^\nu = 0$. Therefore the tangent space of $\text{Gr}_\ell (n, T_p P)$ at $z = 0$ is $dz^\sigma u^\nu = 0$. Consequently, the transversality claimed is precisely expressed by the requirement that

$$t^\mu_{\nu\sigma}v^\nu u^\sigma \neq 0$$

for any $u$ and $v$.

Finally, if we change the choice of adapted coframing, say to

$$\left( \begin{array}{c} \vartheta' \\ \omega' \\ \pi' \end{array} \right) = \left( \begin{array}{ccc} a & 0 & 0 \\ b & c & 0 \\ d & 0 & e \end{array} \right) \left( \begin{array}{c} \vartheta \\ \omega \\ \pi \end{array} \right),$$

then we can compute how $t$ changes:

$$t'(u,v) = at^{-1} (e^{-1} u, c^{-1} v).$$

**Corollary 11.** The tableaux $t^\mu_{\nu\sigma}$ of the dual of a smooth projective plane are the dual tableaux of the original plane.

**Proof.** A adapted coframing $\vartheta, \omega, \pi$ with $d\vartheta = -t \pi \wedge \omega$ for a smooth projective plane determines a adapted coframing $\vartheta^* = \vartheta, \omega^* = \pi, \pi^* = -\omega$ with $d\vartheta^* = -t^* \pi^* \wedge \omega^*$, $t^*$ the dual triality.
Corollary 12. The dual plane is regular just when the original plane is regular.

16. Embedding the flag space into the Grassmannian bundles

Lemma 24 (Otte [41] 5.14). Write $\widetilde{Gr}(n, T_p P)$ for the Grassmannian of oriented $n$-planes in a tangent plane $T_p P$, and $\widetilde{Gr}(n, TP)$ for the fiber bundle

$$\begin{array}{c}
\widetilde{Gr}(n, T_p P) \\
\downarrow \\
\widetilde{Gr}(n, TP)
\end{array}$$

For a regular projective plane, the Gauss map

$$(p, \lambda) \in F \mapsto T_p \bar{\lambda} \in \widetilde{Gr}(n, TP)$$

is an embedding, and a fiber bundle mapping over $P$.

Remark 8. Projective duality gives an obvious dual to this lemma.

Remark 9. The reader might notice that $F \rightarrow \widetilde{Gr}(n, TP)$ could perhaps be an embedding without regularity of the projective plane.

Proof. By transversality of lines, the map is injective. To see that it is smooth, note that

$$T_p \bar{\lambda} = \ker \pi'_A (p, \lambda)$$

is a vector bundle over $F$. Transversality of the fibers $\bar{p}$ with the submanifolds $\widetilde{Gr}_\ell(n, T_p P)$ forces the $\bar{p}$ to be immersed submanifolds, because at each point $\lambda \in \bar{p}$, we can pick any $\ell \subset T_p \lambda$, and find $\bar{p}$ transverse to a submanifold of complementary dimension. Since the fibers of $F \rightarrow P$ are compact (they are the $\bar{p}$ submanifolds), and the map $F \rightarrow P$ is a submersion, it is a fiber bundle. Therefore since the fibers of $F \rightarrow P$ are embedded into $\widetilde{Gr}(n, TP)$, it is clear from the local triviality that $F \rightarrow \widetilde{Gr}(n, TP)$ is an embedding. \qed

17. Nondegenerate coordinates

Lemma 25. For any choice of point $p \in P$ in a smooth projective plane, and choice of line $\lambda \in \Lambda$, generic local coordinates $x, y : P \rightarrow \mathbb{R}^n$ on $P$ defined near $p$, and generic local coordinates $X, Y : \Lambda \rightarrow \mathbb{R}^n$ defined near $\lambda$, the submanifold $F \subset P \times \Lambda$ is given by equations in these coordinates, so that we can pick any three of $x, y, X, Y$, and the fourth will be a smooth function of those three.

Proof. Pick any coordinates on $P$ near $p_0$, and write them as $x, y$ (with $x$ and $y$ each valued in $\mathbb{R}^n$). Similarly write coordinates $X, Y$ on $\Lambda$. After possibly a change of coordinates (which need only be a linear change of coordinates at worst), we can suppose that any choice of three of $x, y, X, Y$ gives coordinates on $F$. In particular, there must be functions

$$x = x(y, X, Y)$$
$$y = y(x, X, Y)$$
$$X = X(x, y, Y)$$
$$Y = Y(x, y, X).$$
To check this, prove it first for linear functions $x, y, X, Y$ on vector spaces $T_{(p_0, \lambda)} F \subset T_{p_0} P \times T_{\lambda} \Lambda$, and then the result is clear by the implicit function theorem. Or just look at an affine chart. □

**Lemma 26.** A smooth projective plane is regular just when, in any nondegenerate coordinates, at any point $(x, y, X, Y) \in F$, if we write $h^i_j$ for the inverse matrix of

$$\frac{\partial y^i}{\partial Y^j},$$

and write

$$t^i_{jk} = \frac{\partial^2 y^i}{\partial x^j \partial X^k} - \frac{\partial^2 y^i}{\partial x^j \partial Y^k} h^m_l \frac{\partial y^m}{\partial X^k},$$

then for any nonzero vectors $\dot{x}$ and $\dot{X}$,

$$t^i_{jk} \dot{x}^j \dot{X}^k \neq 0,$$

i.e. $t^i_{jk}$ determines a triality.

**Proof.** The tangent space at a point $(x, y, X, Y)$ to a line is given by the equation

$$dy = \frac{\partial y}{\partial x} \bigg|_{(x, X, Y)} dx.$$

Parameterize the Grassmannian by associating to any plane $E \subset T_{p_0} P$ the matrix $M$ so that $dy = M dx$. Map $F \rightarrow \tilde{Gr}(n, TP)$ by

$$(x, X, Y) \mapsto \left(x, y, \frac{\partial y}{\partial x}\right).$$

If $\ell \subset T_{p_0} P$ is a real line contained in $T_{p_0} \lambda$, lets suppose that $\ell$ is spanned by a vector $a_i \frac{\partial}{\partial x} + b_i \frac{\partial}{\partial y}$. Without loss of generality, we can suppose that $a_1 = 1$. The subGrassmannian $\tilde{Gr}(n, T_{p_0} P)$ is the set of matrices $M$ so that $b = Ma$. To be transverse to this, we need that whenever $b = \frac{\partial a}{\partial x}$, for any curve $\lambda(t)$ in the fiber $\bar{p}_0$, which passes through the given point with nonzero velocity at $t = 0$,

$$\left.\frac{d}{dt}\right|_{t=0} \frac{\partial y}{\partial x} a \neq 0.$$

Since we have to stay inside the pencil $\bar{p}_0$, we need $\dot{x} = 0$ and $\dot{y} = 0$ on $\lambda(t)$. Therefore

$$0 = \dot{y} = \frac{\partial y}{\partial x} \dot{x} + \frac{\partial y}{\partial X} \dot{X} + \frac{\partial y}{\partial Y} \dot{Y},$$

from which we conclude that

$$\dot{Y} = -\frac{\partial Y}{\partial y} \frac{\partial y}{\partial X} \dot{X}.$$

Differentiating along such a curve gives precisely the stated condition. □

**Lemma 27.** The flag space of any smooth projective plane is embedded in the Grassmannian, i.e. the map $F \rightarrow \tilde{Gr}(n, TP)$ is an embedding, just when there is a vector $\dot{X}$ for which the matrix $X^k t^i_{jk}$ is not zero.
Proof. Immersion of the flag space in Grassmannian is expressed in nondegenerate coordinates by requiring that \((x,X,Y) \mapsto \frac{\partial Y}{\partial y}\) be an immersion. However, it suffices for the fibers \(\bar{p}\) to be immersed in the Grassmannians, which is just the requirement that every nonzero vector \((\dot{X},\dot{Y})\) which is tangent to the fiber \(\bar{p}\) must satisfy

\[
0 \neq \frac{\partial^2 y}{\partial x \partial X} \dot{X} + \frac{\partial^2 y}{\partial x \partial Y} \dot{Y}.
\]

Since we must fix the point \((x,y)\), we find the embeddedness of the flag space in the Grassmannian is precisely the condition that for every vector \(\dot{X}\), there is a vector \(\dot{x}\) for which \(t_{jk} \dot{x} ^{j} \dot{X} ^{k} \neq 0\). □

We see clearly how regularity strengthens the requirement of being an embedding. Note that \(t_{ji} \dot{x} ^{j} \dot{X} ^{i}\) determines a triality: to each \(\dot{x}\) we associate the linear map \(\dot{x} ^{k} t_{kj}\).

**Corollary 13.** The flag space \(F\) of any smooth projective plane embeds into \(\tilde{\text{Gr}} (n,TP)\) just when it embeds dually into \(\tilde{\text{Gr}} (n,T\Lambda)\).

18. ADAPTED COFRAMINGS AND NONDEGENERATE COORDINATES

**Lemma 28.** Given any point of \(F\), there is a system of nondegenerate coordinates in which at any chosen point we can arrange

\[
\frac{\partial Y}{\partial x} = 1, \quad \frac{\partial X}{\partial y} = 0, \quad \frac{\partial X}{\partial Y} = 0.
\]

The tableau \(t\) is expressed at that point as

\[
t^\mu _{\nu \sigma} = \frac{\partial^2 Y^\mu}{\partial x^\nu \partial X^\sigma},
\]

in any adapted coframing which satisfies \(\vartheta = dy, \omega = dx, \pi = dX\) at that point.

**Proof.** Start with any coordinates \(z^I\) on \(F\) \((I = 1, \ldots, 3n)\), and nondegenerate coordinates \(x,y\) on \(P\) and \(X,Y\) on \(\Lambda\). Use the fact that the maps \(z \mapsto (x,y)\) and \(z \mapsto (X,Y)\) have full rank, and the differentials of these maps have transverse kernels, to show that after perhaps a linear change of variables, we can arrange

\[
\frac{\partial}{\partial z} \begin{pmatrix} x \\ y \\ X \\ Y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Now change variable to \(z = (x,y,X)\), and you won’t affect these arrangements. So we have \(Y = Y(x,y,X)\) a function with the required derivatives.

Write an adapted coframing \(\vartheta, \omega, \pi\), with \(\vartheta = dy - p dx\) where

\[
p = - \left( \frac{\partial Y}{\partial y} \right)^{-1} \left( \frac{\partial Y}{\partial x} \right).
\]

We can arrange \(\omega = dx\) and \(\pi = dX\) at our distinguished point, by changes of adapted coframing, so computing \(d\vartheta\) gives

\[
t = \frac{\partial Y}{\partial x \partial X}
\]

as expected. Changing the coframing won’t affect \(t\), as long as it doesn’t change the coframe at that one point. □
Lemma 29. The tableau $t = t_{\nu\sigma}$ of a smooth projective plane at a point of $F$ determines, and is determined by the 2-jet of the map $F \to P \times \Lambda$ at that point.

Proof. The 1-jet of the map $F \to P \times \Lambda$ determines and is determined by the $G$-structure whose sections are the adapted coframings. The 1-jet of that $G$-structure is precisely determined by the torsion (see, e.g., McKay [37]), which in this case one easily computes to be the tableau. □

19. Ellipticity of the differential equations for immersed plane curves

Lemma 30. On a regular projective plane of dimension 4 or more, the system of differential equations for plane curves is elliptic.

Proof. Pick a basis of semibasic 1-forms $\vartheta^\mu, \omega^\mu$ for the map $F \to P$ so that $\Theta = (\vartheta = 0)$. Following Bryant et. al. [11], ellipticity is the absence of a vector $v \in \Theta_{(p,\lambda)}$ so that $v \cdot \varpi$ has rank 1. As in the proof of proposition 1 on page 19, $v \cdot \varpi$ has nonzero determinant or vanishes, and therefore cannot have rank 1. □

Elliptic regularity results are quite involved for projective planes of dimension 8 and 16, since the relevant equations are overdetermined elliptic. Nonetheless, with some effort one should be able to identify in local coordinates a determined subsystem, and prove elliptic regularity results for it. We are thereby encouraged when studying plane curves to assume that they are smooth.

In principle, it seems possible that the equations for immersed plane curves in some smooth irregular projective plane might be elliptic, because $v \cdot \varpi$ might never have rank 1, but might not still not have full rank. Examples of irregular projective planes would be valuable.

20. Cartan’s count

The reader who finds the material of this section mumbo-jumbo might consult Bryant et. al. [11] for assistance.

Lemma 31. The Cartan integers for the differential system of plane curves in a regular projective plane are $s_1 = n, s_2 = \cdots = s_n = 0$.

Proof. The $v \cdot \varpi$ matrices are all invertible, for $v \neq 0$, so they must all have a nonzero entry in the first column, and no linear combination of those entries vanish, so $\varpi$ has $n$ independent 1-forms in its first column. There are only $n$ independent directions available, modulo the independence condition (thinking of the differential system as a linear Pfaffian system), and therefore $s_2 = \cdots = s_n = 0$. □
Therefore formally,\(^2\) plane curves depend on at most \(n\) functions of 1 variable. However, we still have to check involutivity. Let us compute the prolongation. Modulo \(\omega = dx\), we can write
\[
\varpi^\mu_\nu = t^\mu_\nu \pi^\sigma,
\]
with the \(\pi^\sigma\) linearly independent. Regularity will ensure that \(t^\mu_\nu\) is a triality. Plugging in \(\pi = p\omega\) to find the prolongation gives the equations
\[
t^\mu_\nu p^\sigma_\tau = t^\mu_\tau p^\sigma_\nu.
\]
In terms of the algebra
\[
(xy)^\mu_\nu = t^\mu_\nu x^\nu y^\sigma,
\]
this says that the prolongation consists in the \(n \times n\) matrices \(p\) so that
\[
x(py) = y(px)
\]
for any \(x\) and \(y\).

**Definition 22.** A linear transformation \(p : A \rightarrow A\) of an algebra \(A\) is an integral element of the algebra if
\[
x(py) = y(px)
\]
for all \(x, y \in A\). More generally, an integral element of a tableau \(t : U \otimes V \rightarrow W\) is a linear map \(p : U \rightarrow V\) so that \(t(pu_0, u_1) = t(pu_1, u_0)\) for any \(u_0, u_1 \in U\) (see Bryant et. al. [11]).

A simple calculation gives:

**Lemma 32.** The integral elements of the algebra \(\mathbb{C}\) are multiplications by complex numbers. More generally, any triality on \(\mathbb{R}^2\), after suitable linear transformation, has the form
\[
t^\mu_\nu = \delta^\mu_\nu \delta^0_\sigma + a^\mu_\nu \delta^1_\sigma.
\]
The integral elements of the triality are the matrices \(p^\mu_\nu\) given by
\[
p^1_0 = a^1_0 p^0_1 - a^0_0 p^1_1
\]
\[
p^0_1 = a^1_1 p^0_0 - a^0_1 p^1_0.
\]
The set of such matrices is a 2-dimensional vector space.

**Corollary 14.** The exterior differential system for plane curves in a 4-dimensional regular projective space is involutive; plane curves depend formally on 2 functions of 1 variable. The system is elliptic. Formally (rigorously in the real analytic category), every connected real immersed curve sits in a unique maximal connected plane curve.

---

\(^2\)In stating that a system of smooth partial differential equations has formal solution dependings on \(s\) functions of \(d\) variables, we mean that the equations pass Cartan’s test, with last nonzero Cartan character \(s_d = s\). In the real analytic category, this will ensure that there is a well posed Cauchy problem for solutions of the partial differential equations, with this generality. In the smooth category, it tells us that while the corresponding Cauchy problem might not really be well posed anymore, it is at least possible, given \(s\) functions of \(d\) variables, to solve at a point in a formal Taylor expansion solution which solves the equations at all orders. Moreover, the formal solution will consist in Taylor coefficients determined algebraically by the Taylor coefficients of those \(s\) functions. This is often useful in trying to carry out approximation arguments which start with a formal Taylor expansion solution, so even outside the real analytic category, it is worthwhile to know the formal result of Cartan’s test. This is well but not widely known.
Proof. See Bryant et. al. [11] for the relevant theory of noncharacteristic submanifolds for elliptic equations.

Warning: a smooth but not analytic real immersed curve does not generally sit in any plane curve. For instance, if we consider \( \mathbb{CP}^2 \), take any real curve which is contained in a complex curve. Now perturb the real curve in some small open set, so that in that open set it no longer lies on that complex curve, but in some other open set it still does. Clearly there is no immersed complex curve containing the real curve.

Lemma 33. All integral elements vanish for all division algebras (i.e. triadical) on \( \mathbb{R}^n \) except in dimension \( n = 1, 2 \).

Proof. By regularity, we can arrange with a simple change of coordinates at any required point that our algebra have a left identity (replace multiplication \( uv \) with \( u * v = L_{e_0}^{-1}(uv) \) where \( L_{e_0} \) mean left multiplication by \( e_0 \); see Albert [1]). It follows that \( py = y(p) \). So if we let \( \epsilon = p1 \), we find that

\[
x(y\epsilon) = y(xe)
\]

for all \( x, y \). Let \( R_{\epsilon} \) be the operation of right multiplication by \( \epsilon \). Define a new multiplication operation \( * \) by

\[
x * y = R_{\epsilon}^{-1}(x(y\epsilon))
\]

The new multiplication is commutative and has no zero divisors, and has the same identity element as the old one. By a famous theorem of Heinz Hopf [29] (see Springer [47] for a beautiful proof, and also [29, 4]), the dimension of a commutative (not necessarily associative) real algebra without zero divisors must be 1 or 2. □

Theorem 10. Let \( P \) be a regular projective plane of dimension 8 or 16. The only basic plane curves in \( P \) are lines.

Proof. Our differential system prolongs to a holonomic plane field, since the prolongation has dimension 0, and therefore the space \( F \) is foliated by the integral manifolds, a unique one through each point. But \( F \) is already foliated by (lifts of) lines, which are integral manifolds. □

Theorem 11. For any smooth projective plane, \( (P, \Lambda, F) \), of dimension 8 or 16, and generic flag \( (p, \lambda) \in F \), there is no plane curve containing \( p \) tangent to \( \bar{\lambda} \), except \( \bar{\lambda} \).

Proof. See corollary 10 on page 17. □

Remark 10. Note that this theorem is purely local: there are no “little pieces” of plane curves, not asking plane curves to be compact. We can strengthen this slightly. Immersed plane curves are basic, but moreover every plane curve is basic on a dense open set unless it is a fiber of \( F \to P \), i.e. a \( \bar{p} \) pencil, which is a kind of degenerate plane curve, which we can just think of as a point. So roughly put, the only curves in high dimensional projective spaces are points and lines.

Remark 11. This theorem was previously unknown in any case, except for

(1) the quaternionic projective plane (a folk theorem) and

(2) the octave projective plane (Robert Bryant).
Robert Bryant proved this result (but did not publish it) for the octave projective plane using the canonical differential system on the space of 8-planes calibrated by the $F_4$ invariant 8-form on the octave projective plane. The same approach can be used (much more easily) on the quaternionic projective plane. (It is also easy to generalize to quaternionic projective spaces.) Generic smooth projective planes of any dimension do not bear differential forms calibrating their lines. Intuitively, the absence of calibrating forms is explained by closed differential $n$-forms on a manifold depending on $\binom{2n}{n}$ functions of $2n$ variables, while a smooth projective plane structure depends on $n$ functions of $3n$ variables. Small perturbations with compact support of a map $F \to P \times \Lambda$ which started out with all lines calibrated by a given $n$-form will not remain calibrated by that $n$-form, and it is easy to ensure that the lines will not be calibrated by anything.

**Remark 12.** It remains possible that a plane curve could exist on a smooth irregular projective plane of dimension 8 or 16, but the local nature of the proof would require the projective plane to have *irregularities* at all points of the plane curve. It should be possible to strengthen the results above to show that generic smooth projective planes of dimension 8 or 16 have no projective curves other than lines.

### 21. Foliating $\mathbb{R}^4$ by 2-planes after Gluck & Warner

Following Gluck & Warner [20], imagine a family of 2-planes in $\mathbb{R}^4$ which foliate $\mathbb{R}^4$ away from 0. They prove that these 2-planes can be oriented continuously, so that they all intersect positively, and then determine a surface $\Sigma \subset Gr(2, \mathbb{R}^4)$, in the space of oriented 2-planes. Pick a metric on $\mathbb{R}^4$ and write each 2-plane $\Pi \subset \mathbb{R}^4$ as $\xi \wedge \eta$ where $\xi$ and $\eta$ are perpendicular unit-length 1-forms vanishing on $\Pi$. Assume that $\Pi$ is oriented, so that the pair $\xi, \eta$ are well-defined up to rotation. A vector $v$ belongs to $\Pi$ just when $v \wedge \xi \wedge \eta = 0$. Given two 2-planes $\Pi, \Pi'$ we write them as $\xi \wedge \eta, \xi' \wedge \eta'$, and clearly $0 = \xi \wedge \eta \wedge \xi' \wedge \eta'$ just when $\Pi \cap \Pi' \neq 0$. Moreover, $\xi \wedge \eta \wedge \xi' \wedge \eta' > 0$ just when the 2-planes have positive intersection.

Split $\xi \wedge \eta = \sigma_+ + \sigma_-$, where $\sigma_+$ is a self-dual 2-form, and $\sigma_-$ is anti-self-dual. If we write out an orthonormal basis $\sigma_+^j$ for the self-dual 2-forms, and an orthonormal basis $\sigma_-^j$ for the anti-self-dual 2-forms, then $\sigma_+^j \wedge \sigma_-^j = \pm dV$ and $\sigma_+^j \wedge \sigma_-^j = 0$. Therefore writing

$$
\xi \wedge \eta = X_i \sigma_+^i + Y_i \sigma_-^i,
$$

we find

$$
1 = \frac{(\xi \wedge \eta)^2}{dV} = X_i^2 - Y_i^2,
$$

so that $\sigma_+$ and $\sigma_-$ belong to the unit spheres $S^+$ and $S^-$ of self-dual and anti-self-dual 2-forms. Gluck & Warner show that map

$$
(\sigma_+, \sigma_-) : \Sigma \to S^+ \times S^-
$$

of a surface to the Grassmannian satisfies $|\hat{\sigma}_-| > |\hat{\sigma}_+|$ just when the family of 2-planes it represents locally smoothly foliates some region in $\mathbb{R}^4$ with 2-planes. Consequently, if the entirety of $\mathbb{R}^4$ is foliated by 2-planes, then the image of this map $\Sigma \to S^+ \times S^-$ is the graph of a smooth strictly contracting map $S^- \to S^+$, and this strictly contracting map determines the foliation.

All of this applies immediately to the tangent planes to the lines through a point $p_0 \in P$ in a 4-dimensional smooth projective projective plane. These tangent planes foliate $\mathbb{R}^4 = T_{p_0} P$ just when $p_0$ is a regular point.
A self-dual 2-form in the formalism is represented by a choice of point $\omega_+ \in S^+$, and it is positive on a 2-plane $\xi \wedge \eta$ just when $\xi \wedge \eta \wedge \omega_+ > 0$, a positive volume form.

22. Regular 4-dimensional projective planes

McKay [38] develops the general theory of pseudocomplex structures. In local coordinates, these are determined systems of first order elliptic partial differential equations for two functions of two variables. Globally, they are a choice of such equations on each coordinate chart of a 4-manifold, and having the same local solutions on overlaps of coordinate charts.

**Theorem 12.** The category of regular 4-dimensional projective planes is isomorphic to the category of compact, symplectically tameable pseudocomplex 4-manifolds which contain a pseudoholomorphic sphere with nonnegative selfintersection, which is isomorphic to the category of pseudocomplex structures on $\mathbb{CP}^2$ tameable by a symplectic structure. In particular, smooth families from one category are smoothly equivalent to smooth families from the other. Every regular 4-dimensional smooth projective plane can be deformed through regular 4-dimensional projective planes tamed by a fixed symplectic form into a classical 4-dimensional projective plane.

**Remark 13.** We will be brief in our analysis, for which all details are worked out completely in Lalonde & McDuff [34] and McKay [38].

**Proof.** Given a regular 4-dimensional projective plane $P$, pick a volume form $\eta$ on $\Lambda$, in the cohomology class dual to $[\Lambda]$, and let $\tilde{\eta}$ be the Radon transform. Take the pseudocomplex structure to be $F \subset \tilde{\text{Gr}} (n, TP)$. Pseudoholomorphic curves are precisely plane curves. It follows by transversality of lines and intersection theory from [38] that $\Lambda$ is the moduli space of pseudoholomorphic spheres in the generating homology class. More specifically, if we have any compact pseudoholomorphic curve $\Sigma$ in the homology class of a line, and $p \in \Sigma$ is a smooth point of $\Sigma$, then take $\lambda$ the tangent line to $\Sigma$ at $p$. The surfaces $\bar{\lambda}$ and $\Sigma$ must have intersection number at least 2 at $p$ unless they are equal (see McKay [38] for proof). By cohomology calculations, $\Sigma = \lambda$. So the map of categories is defined and injective.

Conversely, suppose that $P$ is a compact 4-manifold, has a symplectically tame pseudocomplex structure, and contains a pseudoholomorphic sphere with nonnegative selfintersection. By results of McDuff (see Lalonde & McDuff [34]), this forces $P$ to be symplectomorphic to $\mathbb{CP}^2$. Let $\Lambda$ be the moduli space of pseudoholomorphic spheres in that same homology class as the given sphere (call these lines). By intersection theory arguments presented in [38] (see below for a little more detail), any two lines intersect transversely in a unique point, and (looking at the linearized equations) the moduli space $\Lambda$ is a smooth 4-manifold.

By results of Taubes (see Lalonde & McDuff [34]), there is a unique symplectic structure on $\mathbb{CP}^2$ up to symplectomorphism and rescaling. Therefore we can ensure (possibly by reorienting) that our symplectic structure is the usual one. As proven in Gluck & Warner [20] (and see McKay [38] for more details), the space of all pseudocomplex structures tamed by a given symplectic structure is contractible; simply put this is just the statement that the space of strictly contracting maps of a sphere to a hemisphere is contractible. Therefore we can produce a homotopy
through smooth pseudocomplex structures, which starts at the standard pseudocomplex structure, and ends at the given one. Fixing any two points, take the line between them in the standard complex structure.

As proven in Duistermaat [17] \(^3\) (and again see McKay [38] for more), cokernel computations proceed as in the standard structure to show that the problem of constructing a pseudoholomorphic sphere through two distinct points is a well-posed elliptic system whose linearization has vanishing kernel and cokernel, and that as we pass through the homotopy, we can deform the sphere to continue to pass through the points. To give a little more detail, \(T_\lambda \Lambda = H^0(\nu_\lambda)\) where \(\nu_\lambda\) is the normal bundle of \(\bar{\lambda}\), equipped with the Duistermaat complex structure (see Duistermaat [17]). But \(\mathbb{CP}^1\) has only one complex structure up to diffeomorphism, and the normal bundle is topologically determined to be \(O(1)\). To ask that a section of this bundle vanish at 2 points forces it to vanish everywhere. Moreover, the uniqueness is clear by intersection theory, and these must be lines. As above, distinct lines intersect at a unique point transversely. Therefore the 4-manifold is a projective plane.

Let \(F \subset P \times \Lambda\) be the incidence correspondence. We need to show that it is a smooth embedded submanifold of dimension 6, and that the maps \(F \to P\) and \(F \to \Lambda\) are submersions. By the same ellipticity argument that shows that \(\Lambda\) is a 4-manifold, we find that \(F\) is a 6-manifold, the moduli space of pointed pseudoholomorphic spheres in the given homology class. Suppose that \(E \to \text{Gr}(2, TP)\) is our pseudocomplex structure. We know by definition that \(E \to P\) is a fiber bundle with compact fibers. We can map \(F \to E\) by taking a pointed line to the tangent plane of that line at that point. This map is injective and smooth.

By the same homotopy argument we used above for pairs of points, adapted now to tangent planes, we can take any point of \(E\), i.e. a point of \(P\) and a potential tangent direction \(P \in \text{Gr}(2, P)\) for a plane curve, and find a unique line tangent to \(P\). Therefore \(F \to E\) is onto. The line depends smoothly on the choice of tangent plane \(P\), by bootstrapping, so \(F \to E\) is a diffeomorphism, and \(F \to P\) is a submersion. We still have to show that \(F \to \Lambda\) is a submersion and that \(F \to P \times \Lambda\) is an embedding. Smoothness of each of these maps follows from the previous remarks.

Thinking of \(F\) as a (smooth) moduli space of pointed lines, elliptic theory (again looking at the linearized equations) tells us that \(T_{(p, \lambda)} F\) is the set of pairs \((\dot{p}, \dot{\lambda})\) where \(\dot{p} \in T_p P\) and \(\dot{\lambda} \in H^0(\nu_\lambda)\), so that \(\dot{\lambda}(p) = \dot{p}\) modulo \(T_p \bar{\lambda}\) (there are no other obstructions, because those would live in \(H^1(\nu_\lambda) = 0\)). For the same reason, there are no base points for this line bundle, since it is just \(O(1)\), and so the tangent space to \(F\) has 6 real dimensions. Indeed it has an almost complex structure. Clearly \(F \to \Lambda\) is a fiber bundle.

\(^3\)As the reviewer points out, it is remarkable how much effort the pseudoholomorphic community could have saved had they been aware of Duistermaat’s beautiful work earlier. At this late stage, Duistermaat’s results have probably all been rediscovered by researchers in pseudoholomorphic curves, but there is no better place to read them than the original [17].
Finally, we need to prove that point pencils meet transversely in $\Lambda$. This follows immediately from section 6 of McKay [38], where we constructed the dual pseudocomplex structure, and from intersection theory presented in the same paper. Now we can apply theorem 1 on page 4.

In case we start with the manifold $P = \mathbb{CP}^2$ equipped with a pseudocomplex structure tamed by a symplectic structure, as above we can assume that the symplectic structure is the Fubini–Study symplectic structure, and find a homotopy through pseudocomplex structures to the standard complex structure, and homotope lines, to ensure that there are pseudoholomorphic spheres in the usual homology class.

\begin{remark}
Using the results of Gluck & Warner [20] (also see McKay [38] p. 258), we can identify every regular 4-dimensional projective plane with a smooth fiber bundle map $\mathbb{S}\Lambda^2^+ \to \mathbb{S}\Lambda^2^+$ (between the unit sphere bundles of the bundles of anti-self-dual and self-dual 2-forms) which are strictly contracting on each fiber, and have image contained in the same hemisphere of $\mathbb{S}\Lambda^2^+$ that contains the Fubini–Study symplectic form. This description gives the differential system explicitly, and makes the taming symplectic structure manifest, but leaves the lines as unknown solutions of a differential system. It is also not functorial, since families might have varying symplectic structures. The point of view of a 4-dimensional projective plane as a map $F \to P \times \Lambda$ makes the lines explicit, but leaves the symplectic structure hidden.

Either description parameterizes the regular 4-dimensional projective planes with 2 functions of 6 variables.

\begin{proposition}
Every elliptic system of 2 equations for 2 functions of 2 variables occurs locally as the differential system for plane curves on a regular 4-dimensional projective plane.
\end{proposition}

\begin{proof}
The system can be locally symplectically tamed (see McKay [38]). In local Darboux coordinates, take an open set of small volume, and paste it into $\mathbb{CP}^2$ matching up with the usual Fubini–Study symplectic structure on $\mathbb{CP}^2$. The picture of Gluck & Warner shows us that we can glue together, outside of some compact set, this elliptic equation with the usual one for complex curves in $\mathbb{CP}^2$.
\end{proof}

23. Smooth but irregular 4-dimensional projective planes

\begin{remark}
This last approach might provide a mechanism to construct irregular smooth projective planes. We might hope to take a regular projective plane, thought of in the Gluck & Warner picture as a strictly contracting bundle map $\mathbb{S}\Lambda^2^+ \to \mathbb{S}\Lambda^2^+$, and deform it while keeping it symplectically tamed into a map which is contracting but not strictly. One has to prove that the lines survive as smooth surfaces, in a smooth family, forming a smooth projective plane. The lines would remain symplectically tamed, so this \textit{a priori} estimate, together with some local analysis of degenerations of elliptic systems, might build examples of irregular smooth projective planes.

\begin{lemma}
Let $P$ be a smooth 4-dimensional projective plane. Identify $P = \mathbb{CP}^2$ by a symplectomorphism (whose existence is ensured by theorem 8 on page 12). Take the usual Fubini–Study metric on $\mathbb{CP}^2$. Note that this makes the symplectic form a self-dual 2-form. Identify 2-planes in tangent spaces of $T_p\mathbb{CP}^2$ with unit
2-forms in $\Lambda^2(T^*CP^2)$ by the Plücker embedding:

$$\text{Pl}: \tilde{\text{Gr}}(2, TCP^2) \rightarrow \Lambda^2(T^*CP^2).$$

Let $SA^2+, SA^2- \subset \Lambda^2(T^*CP^2)$ be the bundles of unit length self-dual and anti-self-dual 2-forms. Then under the Gauss map, followed by the Plücker embedding, the space $T$ is smoothly homeomorphically mapped to the graph of a fiber bundle morphism $SA^2- \rightarrow SA^2+$ which is contracting on each fiber. The image of each fiber lies inside the hemisphere containing the symplectic form. The map is strictly contracting just when the projective plane is regular, and is $C^{k-1}$ if the projective plane is $C^k$. Continuous isomorphisms of smooth projective planes determine continuous isomorphisms of the maps $SA^2- \rightarrow SA^2+$.

**Proof.** For regular 4-dimensional projective planes, see McKay [38], section 2.6, where it is shown that the map $SA^2- \rightarrow SA^2+$ is strictly contracting. Consider an irregular 4-dimensional projective plane. By corollary 10 on page 17, the generic point is regular. So locally the image of $\text{Pl}g$ is locally the graph of a strictly contracting map near regular points. By continuity (see theorem 7 on page 8), the contracting map near regular points. By continuity (see theorem 7 on page 8), the image of each fiber is strictly contracting on each fiber. The image of each fiber lies inside the hemisphere containing the symplectic form. The map is strictly contracting just when the projective plane is regular, and is $C^{k-1}$ if the projective plane is $C^k$. Continuous isomorphisms of smooth projective planes determine continuous isomorphisms of the maps $SA^2- \rightarrow SA^2+$.

In homology, let $A = (\sigma_+, \sigma_-)[p] \in H_2(S^+ \times S^-)$. The image of $\sigma_+$ is contained in the hemisphere around the symplectic form (this precisely expresses the condition that the symplectic form tames the smooth projective plane; see McKay [38] 2.4), so $A$ has no component in $H_2(S^+)$ and so $A = d[S^-]$, some integer $d$. Consider the intersection with the graph of an isometry $S^- \rightarrow S^+$. For example, the subGrassmannian $\text{Gr}_2(T_pP)$ is such a graph (see McKay [38] 2.4). If we pick a real line $\ell \subset T_pP$ for a regular $\lambda$, then the intersection point is unique, $\lambda = \lambda(\ell)$ the magnification. But moreover, the intersection is negative (see McKay [38] 2.4 again). The isometry in this case is orientation reversing, and its graph has homology $[S^+] - [S^-]$ (once again [38] 2.4). Therefore $d = 1$. So the map $\sigma_-$ has degree 1.

If $\sigma_-$ is not 1-1, then image of the map $\text{Pl}g$ inside $S^+ \times S^-$ must intersect some submanifold $S^+ \times s_-$ at 2 points at least. For every $s_-$, there must be an intersection, because $\sigma_-$ is onto. For a full measure set of choices of $s_-$, all of the intersection points will be regular values of $(\sigma_+, \sigma_-)$, and therefore will be points of positive intersection. But the intersection number is 1, so for a full measure open set of points $s_- \in S^-$, there will be a unique transverse point of intersection. The same will be true if we replace $S^+ \times s_-$ with the graph of any strictly contracting map $S^+ \rightarrow S^-$. This is as far as I can get with homology arguments, but it is satisfying to see that we can nearly prove the result this way.

Suppose that $\sigma_-(\lambda_1) = \sigma_-(\lambda_2)$, but $\sigma_+(\lambda_1) \neq \sigma_+(\lambda_2)$. Write the relevant 2-planes as $\sigma_+^1 + \sigma_-$ and $\sigma_+^2 + \sigma_-$. Since the Fubini-Study form $\omega_+$ tames both 2-planes, we must have both $\sigma_+^1$ lying in the same hemisphere as $\omega_+$, and therefore
$0 < \sigma_+^1 \land \sigma_+^2 < 1$. Let $1 - \epsilon = \sigma_+^1 \land \sigma_+^2$. Therefore

\[
(\sigma_+^1 + \sigma_-) \land (\sigma_+^2 + \sigma_-) = 1 - \epsilon - 1 = -\epsilon < 0.
\]

But the intersections are positively oriented. \qed

**Corollary 15.** The differential system for curves on any smooth 4-dimensional projective plane is a topological submanifold of the Grassmann bundle, and is the limit in $C^0$ topology of the differential system for curves in a regular 4-dimensional projective plane.

**Remark 16.** The purpose of this corollary is the give evidence for the conjecture that every $C^k$ smooth projective plane is a limit of $C^k$ regular projective planes, in the $C^k$ topology. Note the importance of the Radon transform.

**Remark 17.** Take $F_0 \to P \times \Lambda_0$ a smooth 4-dimensional projective plane, perhaps not regular, and assume $P = \mathbb{CP}^2$ bears the standard symplectic structure, positive on the lines of $F_0$. Take $\sigma_0 : S\Lambda^2_* \to S\Lambda^2_+$ the associated Gluck–Warner map. From this lemma, we can construct a smooth family $\sigma_t : S\Lambda^2_* \to S\Lambda^2_+$ of strictly contracting maps approaching $\sigma_0$. Let $F_t \subset \tilde{\text{Gr}}(2, TP)$ be the associated embedded submanifold (the inverse image under the Plücker map). Take $\Lambda_t$ the moduli space of lines (i.e. degree 1 maps $\mathbb{CP}^1 \to P$, $F_t$-holomorphic). Let $\tilde{\Lambda}$ be the set of pairs $(t, \lambda)$ where $t \geq 0$ and $\lambda$ is an $F_t$-line. Elliptic theory tells us that $\tilde{\Lambda}$ is a smooth cobordism away from the boundary $t = 0$. If we could show that $\tilde{\Lambda}$ is a smooth cobordism, then we would see that irregular 4-dimensional planes are all limits of regular planes.

**Remark 18.** Plane curve theory in irregular smooth projective planes is thus a special case of singular perturbation theory of first order determined elliptic PDE for two functions of two variables.

### 24. Maps Taking Curves to Curves

**Definition 23.** A curve morphism of smooth projective planes is a map taking plane curves (thought of as subsets) to plane curves. For instance, every diffeomorphic collineation is a curve morphism.

**Theorem 13.** A continuous [homeomorphic] curve isomorphism of regular projective planes of dimension 8 or 16 [4] is a smooth collineation. In particular, smooth isomorphisms of the differential system for plane curves are smooth collineations.

**Proof.** In 8 or 16 dimensional regular projective planes, curves are pieces of lines, so the isomorphism must take lines to lines, hence a collineation. In 4 dimensional regular projective planes similarly, any line must be taken to a pseudoholomorphic curve. Homeomorphism ensures that it is in the appropriate cohomology class. By the intersection theory of McKay [38], it must be a line. Therefore the map is a collineation. By theorem 6 on page 5, a continuous collineation is smooth. \qed

This strengthens the rather poor results of McKay [39]. It is surprising, since the differential system for plane curves on a 4 dimensional projective plane can be very flexible locally (see McKay [38] for examples, besides the classical $\mathbb{CP}^2$).
25. LOCAL CHARACTERIZATION OF CLASSICAL PROJECTIVE PLANES

**Proposition 3.** A smooth projective plane of dimension 2 or 4 has all of its tangent trialitys classical just when it is regular.

*Proof.* The proof in dimension 4 is a long calculation; see McKay [38]. □

**Theorem 14** (Tresse). A smooth projective plane of dimension 2 is classical just when the Tresse invariants vanish.

*Proof.* See Tresse [48], Arnol’d [2], Cartan [13, 15]. □

**Theorem 15.** A smooth projective plane of dimension 4 is isomorphic to $\mathbb{CP}^2$ just when it and its dual are regular, and the differential invariants $T_2$ and $T_3$ of pseudocomplex structures (presented in McKay [38]) vanish. Equivalently, a smooth projective plane of dimension 4 is isomorphic to $\mathbb{CP}^2$ just when it and its dual bear almost complex structures for which the line and pencils are pseudoholomorphic curves.

**Remark 19.** Note that it is not sufficient to check $T_3$ just for the smooth projective plane, since the smooth projective planes with $T_3 = 0$ are precisely the almost complex structures on $\mathbb{CP}^2$ tamed by the usual symplectic structure (an open condition), and these form an infinite dimensional family, depending on 8 functions of 4 variables. Vanishing of $T_2$ alone is not invariantly defined, since $T_2$ is a relative invariant, and arbitrary multiples of $T_3$ can be added to it.

*Proof.* As shown in McKay [38], a smooth projective plane is an almost complex manifold, and its curves pseudoholomorphic curves, just when $T_3 = 0$. In terms of that paper, this is just the condition that $\tau_1 \wedge \bar{\omega} = 0$, which is shown to be equivalent to $0 = \sigma = \tau_1 \wedge \bar{\omega} = \tau_3 \wedge \bar{\omega}$. If the dual is also almost complex, then this forces the same equations on the dual invariants. McKay [38] section 6 shows that $T_3$ of the dual plane is $T_2$ of the original plane. This leaves only the invariants $U_3$ and $V_2$. Writing out the structure equations of McKay [38] pg. 20, and differentiating once, reveals that these are also forced to vanish. This forces the differential system for curves to be isomorphic to the Cauchy–Riemann equations, so by theorem 13 on the preceding page, the smooth projective plane is isomorphic to the classical model $\mathbb{CP}^2$. □

**Remark 20.** This proof requires $C^5$ differentiability, to define and differentiate all of the required invariants.

**Remark 21.** A similar approach via the method of equivalence could certainly determine local invariants for smooth projective planes of dimension 8 and 16 whose vanishing is necessary and sufficient for isomorphism with the model projective planes.

26. CURVES WITH SINGULARITIES

It will be helpful to adopt a more general notion of plane curve in a regular 4-dimensional projective plane, as in McKay [38] p. 280; the reader will need a copy of that article in hand to follow our arguments from here on. Essentially the idea is that $F$ admits a canonical almost complex structure for which $\Theta$ is a complex subbundle of the tangent bundle, and for which the fibers of $F \rightarrow P$ and $F \rightarrow \Lambda$ are pseudoholomorphic curves, and for which generalized plane curves are
pseudoholomorphic. This almost complex structure is derived in McKay [38]. A plane curve is a pseudoholomorphic map \( \Phi : C \to F \) (not necessarily an immersion) which is tangent to \( \Theta \), i.e. for which all 1-forms \( \vartheta \) on \( F \) vanishing on \( \Theta \) pull back to \( \Phi^* \vartheta = 0 \). If such a map is Lipschitz, then it is smooth by elliptic regularity (see McKay [38]).

**Theorem 16** (Micallef & White). Every compact plane curve (perhaps with boundary) in a regular 4-dimensional projective plane is taken by a Lipschitz homeomorphism of a neighborhood of a point to a plane curve in the classical 4-dimensional projective plane \( \mathbb{CP}^2 \). If the tangents to the curve at that point all lie tangent to the same line (for example, not a nodal point), then we can make a smooth diffeomorphism instead of merely Lipschitz.

**Remark 22.** The crucial idea is that this result holds true at singular points, and at intersections, whether transverse or not. Thus the intersection theory of plane curves is isomorphic to the intersection theory of complex curves. For proof see Micallef & White [40] and also see Sikorav [46].

**Remark 23.** The reader might be able to generalize the work of Duval [18] or of Kharlamov & Kulikov [33] to plane curves in 4-dimensional smooth projective planes.

### 27. Quadrics

**Definition 24.** A quadric or conic is a closed immersed plane curve \( \phi : C \to P \), whose degree \( \phi_*[C]/[\bar{\lambda}] \) is 2, so that no path component of \( C \) is mapped to a point.

**Lemma 35.** Any quadric in a regular 4-dimensional projective plane is either a pair of lines, or is a smooth embedded Riemann surface and contains 5 points with no three of them on the same line.

**Proof.** A quadric is a map \( Q \to F \) from a compact Riemann surface \( Q \) which is pseudoholomorphic and tangent to \( \Theta \), and which maps \( Q \to F \to P \) to a degree 2 curve. Split \( Q \) into components \( Q = \bigcup Q_\alpha \). Each \( Q_\alpha \) must map to a point or have a positive degree. No components mapping to points are allowed by definition, so each component must have positive degree. Because the total degree is 2, either \( Q \) has 2 components of degree 1, which are therefore lines (by intersection theory with their tangent lines), or has one component of degree 2. The singularities of the map \( Q \to F \to P \) must be diffeomorphic to singularities of a plane curve in the classical projective plane \( \mathbb{CP}^2 \). The selfintersection number at each singularity must be at most 2, by the intersection theory of McKay [38] proposition 13, p.290. Moreover, by the Micallef & White theorem 16, the self-intersections must look like those of an algebraic curve, so to have selfintersection number at most 2 at a singularity, it must be a double point, i.e. \( Q \) is an immersed submanifold. But in that case, the local picture of two surfaces intersecting transversely, we can take the line tangent to one of those surfaces at the intersection point, and it will have intersection number 3 or more. This contradicts the degree being 2, so the tangent line must have infinite order intersection. By McKay [38] theorem 4, p. 282, this forces a line to be contained in \( Q \). Looking at the other tangent line, we find that \( Q \) is a union of two lines. Therefore either \( Q \) is smooth or a union of two lines. Suppose that \( Q \) is smooth. If \( Q \) contains a line, say \( \bar{\lambda} \), then take any two points \( p_0 \) and \( p_1 \) of \( Q \) not on \( \bar{\lambda} \), and
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draw the line $\mu = p_0 p_1$. The line $\mu$ strikes $Q$ at $p_0, p_1$ and a point of $\bar{\lambda}$. Therefore $\mu$ must also have infinite order intersection with $Q$, and again $Q$ is the union of $\bar{\mu}$ and $\bar{\lambda}$.

Therefore either $Q$ is smooth containing no lines, or is a union of two lines (possibly a double line, i.e. parameterizing a line twice over). If $Q$ is smooth, then no three points of $Q$ lie in a line, and therefore any 5 points of $Q$ will do. □

**Lemma 36.** A smooth quadric in a regular 4-dimensional projective plane is diffeomorphic to a 2-sphere.

**Proof.** Take a smooth quadric $Q$, a point $q \in Q$. Map $p \in P \setminus q \mapsto pq \in \bar{q}$, the Hopf fibration. Restricted to $Q \setminus q$, this map has smooth inverse $\lambda \mapsto p$ where $p$ lies on $\lambda$ and on $Q$, defined for all $\lambda \in \bar{q}$ with two distinct points of intersection with $Q$. But every line has either a pair of distinct points of intersection with $Q$, or is tangent to $Q$ (double intersection point), since higher intersections are forbidden by homology count. We can extend the map to take $q$ to the tangent line at $q$ and obtain a bijection $Q \to \bar{q}$. Away from $q$, this map identifies a smooth quadric minus a point with a line minus a point. By lemma 13 on page 13, if we extend the map to take $q$ to the tangent line to $Q$, then it extends to a homeomorphism. Because the 2-sphere has a unique smooth structure, $Q$ is diffeomorphic to the 2-sphere. □

**Remark 24.** If the tangent plane to $Q$ at $q$ is regular, then the map $Q \to \bar{q}$ is an immersion, because $Q$ is never tangent to second order to any of its tangent lines. Second order tangency implies intersection number 3, as shown in McKay [38]. If $Q$ has irregular tangent line at $q$, then it isn’t clear.

**Lemma 37.** In the canonical conformal structure (see McKay [38]), a smooth quadric is biholomorphic to $\mathbb{C}P^1$, and its normal bundle is diffeomorphic to $\mathcal{O}(4)$.

**Proof.** The uniqueness of conformal structure on the sphere is well-known. The self-intersection number of a smooth quadric must be 4, by its homology, giving the topology of the normal bundle. Apply the classification of rank 2 vector bundles on the sphere. □

**Lemma 38.** The space of smooth quadrics is an oriented smooth real manifold of dimension 10.

**Proof.** Consider the normal bundle $\nu$. It comes equipped with the linearization of the differential system for plane curves. Following Duistermaat [17] (or McKay [38]), we can write the linearized operator $Lu = \bar{\partial}u + b\bar{u}$, for sections $u$ of the normal bundle, turning the normal bundle into a complex line bundle, and the smooth quadric into a complex curve biholomorphic to $\mathbb{C}P^1$. The surjectivity of $L$ is proven in Duistermaat [17] p.238. The smoothness of the moduli space follows by standard elliptic theory. We can count the dimension of the moduli space as the dimension of the kernel of $L$, by following Gromov & Shubin [24]:

$$\dim \ker L = \text{ind } L + \dim \ker L^t,$$

(or just using the standard Riemann–Roch theorem, following Duistermaat, as the reviewer points out), and $L^t$ is the adjoint operator on the dual bundle $\kappa \nu^{-1}$ (where $\kappa$ is the canonical bundle). But this operator also has the form $L^t v = \bar{\partial}v + B\bar{v}$ (see Duistermaat [17]). By the Bers similarity theorem (see Bers [5]), if $v$ lies in the kernel of $L^t$, then $v = e^S V$ for $e^S$ a nowhere zero holomorphic section of a trivial
line bundle, and $V$ a holomorphic section of another line bundle, say $L$. Counting Chern classes, $c_1(\kappa v^{-1}) = -6$. So $V$ must be a holomorphic section of $\mathcal{O}(-6)$, and therefore vanishes, so $v$ does as well. By the Atiyah–Singer index theorem (or again just the Riemann–Roch theorem)

$$\dim \ker L = \text{ind } L = 2c_1(v) + 2 = 10$$

as a real vector space.

**Lemma 39.** Pick 5 points $p_1, \ldots, p_5$ in a regular 4-dimensional projective plane, no 3 of them lying in a line. There is a smooth 1-parameter family of regular projective plane structures $F \times \mathbb{R} \to P \times \Lambda$ (write $F_1$ for $F \times \{t\}$) and a smooth family of points $F_j : \mathbb{R}P, \ j = 1, \ldots, 5$, so that $F_0$ is isomorphic to the classical projective plane $\mathbb{CP}^2$, $F_1$ is the regular 4-dimensional plane of our hypothesis, all $F_i$ are tamed by the standard symplectic structure, and for each time $t$, no 3 of the points $P_1(t), \ldots, P_5(t)$ are collinear with respect to $F_i$.

**Proof.** Take any smooth deformation $F_i$ from the classical structure to the given structure, tamed by the usual symplectic structure, guaranteed to exist by theorem 12 on page 27. Let $M$ be the set of sextuples $(q_1, \ldots, q_5, t) \in \prod^5 P \times \mathbb{R}$ for which no three of the $q_j$ are colinear with respect to $F_i$. Clearly $M$ is an open subset of $\prod^5 P \times \mathbb{R}$. Indeed $M$ is just the leftover part when we remove the diagonal loci $q_i = q_j$ and the loci where $q_k \in \overline{q_i q_j}$. The diagonal loci are obviously submanifolds of codimension 4. The locus where $q_k \in \overline{q_i q_j}$ can be written as $(q_i, q_j q_k) \in F$. So this locus is the inverse image under the map

$$\begin{align*}
(q_i, q_j, q_k) \mapsto (q_i, q_j q_k) & \in P \times \Lambda
\end{align*}$$

of $F$.

**Lemma 40.** The map $(p_1, p_2) \in P \times P \setminus \Delta P \mapsto p_1 p_2 \in \Lambda$ has full rank at all points where it is defined (i.e. where $p_1 \neq p_2$).

**Proof.** Pick two points $p_1 \neq p_2$, and take two lines $\lambda_1, \lambda_2 \neq p_1 p_2$. Consider the smooth map $\lambda \mapsto (\lambda_1, \lambda_2)$ defined where $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$. This map provides a local section open $\subset \Lambda \to P \times P \setminus \Delta P$. Therefore the map has full rank.

Therefore, returning to our map $(q, q_1, q_2) \mapsto (q, q_1 q_2)$, this map has full rank, and therefore the inverse image of $F$ is a smooth submanifold of codimension 2.

Our manifold $M$ is the complement in $\prod^5 P \times \mathbb{R}$ of a finite set of codimension 2 and codimension 4 submanifolds, and is therefore connected.

**Lemma 41.** Given any 5 points in a regular 4-dimensional projective plane, no 3 of which lie on a line, there is a unique smooth quadric through them, and the quadric depends smoothly on the choice of the 5 points.

**Remark 25.** The proof is Gromov’s [23] 2.4.B’$''$, with some more details.

**Proof.** Start by deforming the 5 points $P_1(t), \ldots, P_5(t)$ and the projective plane structure $F_i$ from the classical projective plane at $F_0$, to the given projective plane at $F_1$. Ensure that the no three of the points $P_j(t)$ lie in a line, at any time $t$, and that all of the $F_i$ projective plane structures are tamed by the same symplectic
structure. Gromov compactness ensures that the set of smooth quadrics through the
given points is compact on any compact interval of $t$ values in $\mathbb{R}$. There is a unique
such quadric at $t = 0$, by classical plane algebraic geometry. To ensure the survival
of the smooth quadric on any open subset of $\mathbb{R}$, we employ the continuity method
of elliptic partial differential equations. This reduces to showing the surjectivity of the linear operator

$$Lu = \bar{\partial}u + b\bar{u}$$

where $\bar{\partial} + b$ is derived in Duistermaat [17], p. 237, $u$ a section of the normal bundle $\nu$, and we need surjectivity among sections $u$ with specified values $u(P_j(t))$ at the
5 points. By linearity, we can assume that those specified values are $u = 0$. (As
the reviewer points out, surjectivity here is proven in Ivashkovich & Shevchisin [30]
and Barraud [3].)

The surjectivity of this operator is equivalent to the injectivity of the adjoint
operator, which is

$$v \mapsto \left( (\bar{\partial} + b)^t, v(P_1(t)), \ldots, v(P_5(t)) \right),$$

$v$ a section of $\kappa\nu^{-1}$ and $\kappa$ the canonical bundle. Topologically, the first Chern
classes are

$$c_1(\kappa\nu^{-1}) = c_1(\kappa) - c_1(\nu)$$

$$= -2 - 4$$

$$= -6.$$

Consider the divisor $\mu = P_1(t) + \cdots + P_5(t)$. By the Gromov–Shubin–Riemann–
Roch theorem (see Gromov & Shubin [24, 25, 26]), the space of solutions of $L$ with
zeros on $\mu$ satisfies

$$\dim \ker(L, \mu) = \text{ind } L - \deg \mu + \dim \ker(L^t, -\mu).$$

The index of $L$, by the Atiyah–Singer index theorem, is $c_1(L) + 1$ (see Duistermaat
[17], p.238). The number $\deg \mu$ depends only on $\mu$, not on the operators involved
(see Gromov & Shubin [26], p.169), so we can calculate it for a smooth quadric in
the standard projective plane, and find that $\deg \mu = 5$. Therefore

$$\dim \ker(L, \mu) = \dim \ker(L^t, -\mu).$$

Applying the Bers similarity principle (see Bers [5]), any $u$ in $\ker(L, \mu)$ has the
form $u = e^s U$ where $e^s$ is a nowhere vanishing section of a (obviously trivial) line
bundle, and $U$ is a section of a line bundle $\nu'$ with the same topology as $\nu$. By
the Birkhoff–Grothendieck theorem, $\nu' = O(4)$. The section $U$ will have the same
zeros as $u$, and so will have 5 zeros (at the $P_j(t)$). This forces $U = 0$. Therefore
$\ker(L, \mu) = 0$, and so $\ker(L^t, -\mu) = 0$, ensuring that $L$ is surjective. □

Remark 26. We didn’t really need to use the full force of Bers’s similarity principle
for line bundles; it is enough to notice that $u$ has only positive intersections with
the zero section, which follows from the local Bers similarity principle.

Corollary 16. The space of smooth quadrics is connected.

Proof. The space of 5-tuples with no 3 points colinear is the complement of a
codimension 2 subset of the space of 5-tuples, and therefore is connected, and maps
smoothly onto the space of smooth quadrics. □
Lemma 42. Take $Q$ a smooth quadric in a regular 4-dimensional projective plane. The map $q \in Q \rightarrow T_qQ \in \Lambda$ takes $Q$ to a smooth quadric in $\Lambda$.

Proof. By corollary 7 on page 15, this is the map to the dual curve, so it is a plane curve (perhaps with singularities). By intersection theory of McKay [38], no line can be tangent to a quadric at two points, so the map is injective. The map has full rank, because no smooth quadric can be tangent to higher than first order with a line. By deforming to the classical case, we can compute the degree. Therefore the dual curve is a quadric. If the dual curve is not smooth, then it must be a pair of lines. But the dual of a line is a point, so the original curve would have been a pair of points. Therefore the dual curve is a smooth quadric. □

Lemma 43. There are precisely 4 lines simultaneously tangent to any pair of distinct smooth quadrics, counting multiplicities. We don’t have to count multiplicities unless there are one or two points of nontransverse intersection.

Proof. Lines tangent to $Q$ are points of the dual curve $Q^*$. Count intersections of the dual curves, which are smooth quadrics. □

Lemma 44. Given a smooth quadric, the smooth quadrics nowhere tangent to it form a dense open subset of the quadrics.

Proof. Pick 4 distinct points on the given quadric, and one point not on the given quadric, not on a line through any two of the 4 points. The quadric through those 5 points is nowhere tangent to the original quadric, because it has 4 distinct points of intersection, so by homology counting and positivity of intersection (see McKay [38]), the two quadrics are nowhere tangent. Given any quadric which is tangent, we can pick our 4 intersection points close to its intersection points, and our fifth point close to it. □

Lemma 45. Given any pair of nowhere tangent smooth quadrics, we can deform the projective plane structure $F \rightarrow P \times \Lambda$ into the classical one, and deform the quadrics so that they remain quadrics throughout the deformation, and keep them from every becoming tangent.

Proof. Given one smooth quadric $Q_1$, pick any 4 distinct points on it, $p_1, \ldots, p_4 \in Q_1$. Draw the tangent line $\lambda$ to $Q_1$ at $p_1$, and pick any point $p_5$ of $\lambda$ other than $p_1, \lambda(p_4p_1), i, j = 1, \ldots, 4$. Then the quadric through the points $p_1, \ldots, p_5$ is smooth and nowhere tangent to $Q_1$. The space of choices of the $p_j$ points is clearly a connected manifold, a 4-fold covering space of the space of ordered pairs of nowhere tangent quadrics, of dimension 20, and a fiber bundle over the space of smooth quadrics. We can easily add a parameter $t$ to the construction, and look at points $p_1(t), \ldots, p_5(t)$, and not lose the connectedness. □

Lemma 46. Take $Q$ a smooth quadric, $X$ the set of pairs $(p, \lambda)$ for which $p \in Q$ and $\lambda$ is a line through $p$. Map $\iota : (p, \lambda) \in X \rightarrow (p', \lambda) \in X$ where $p, p'$ are the points where $\lambda$ intersects $Q$, and take $p' \neq p$ if possible, i.e. unless $\lambda$ is tangent to $Q$ at $p$. The map $\iota : X \rightarrow X$ is a smooth diffeomorphism.

Proof. Clearly $X = \pi_p^{-1}Q$ is a smooth manifold. Where $\lambda$ is not tangent to $X$, we can ensure the result by transversality. For tangent $\lambda$, the result is immediate in Micallef–White coordinates (see theorem 16 on page 33). □
Remark 27. Pascal’s mystic hexagon apparently does not give a mechanism for drawing smooth quadrics; see Hofmann [28].

Remark 28. Gromov [23] p. 309 0.2.B suggests that the approach we have taken here to construct quadrics can construct plane curves of all genera. The details of the argument have never been provided; Gromov (p. 338 2.4.B′′) suggests that there are some subtleties.

28. Poncelet’s porism

Because regular 4-dimensional projective planes are symplectomorphic to $\mathbb{CP}^2$, they share the same Gromov–Witten invariants, so that a huge collection of enumerative problems about plane curves have the same solutions. Let's consider some plane geometry which is not enumerative. We will search for an analogue of the elliptic curve in the classical proof of Poncelet’s porism. For proof in the classical 4-dimensional projective plane, see Griffiths & Harris [21, 22] and Schwartz [45].

Definition 25. A polygon in a smooth projective plane is an ordered collection of distinct points $p_1, \ldots, p_n, p_{n+1} = p_1$. The lines $p_jp_{j+1}$ are called the edges of the polygon, while the points $p_j$ are called the vertices. A polygon is circumscribed about a quadric if every edge of the polygon is tangent to the quadric. A polygon is inscribed in a quadric if every vertex lies in the quadric. Given two quadrics, a polygon circumscribed about the first one, and inscribed in the second, is called a Poncelet polygon of those quadrics.

The classical theorem:

Theorem 17 (Poncelet). For a given pair of nowhere tangent quadrics in the classical projective plane of dimension 4 or more, every Poncelet polygon of those quadrics belongs to a smooth family of distinct Poncelet polygons of those same quadrics, with any one of the vertices [or edges] being drawn over the entire quadric in which the polygon remains inscribed [circumscribed].

Remark 29. For smooth projective planes of dimension 8 or 16, the result is a triviality, since quadrics are pairs of lines. Henceforth, consider a regular projective plane of dimension 4. It seems very unlikely that the Poncelet porism is true for generic regular 4-dimensional projective planes.

Consider a Poncelet polygon. Let $Q_E$ be the quadric that the polygon circumscribes, and $Q_V$ the quadric in which the polygon is inscribed.

Lemma 47. Let $T$ be the set of pairs $(p, \lambda)$ so that $p \in Q_V$ and $\lambda$ is a line containing $p$ and tangent to $Q_E$. Map $T \rightarrow Q_E$, by taking each line to its point of intersection with $Q_E$. This map is well-defined, smooth, and a double covering branched at 4 points.

Proof. Let $\delta : Q_E \rightarrow Q'_E$ be the duality map, taking each point to its tangent line. The map $(1, \delta) : Q_V \times Q_E \rightarrow Q_V \times Q'_E$ identifies $T$ with the set $T'$ of pairs $(p, q) \in Q_E \times Q_V$ so that either $p = q$ and $Q_E$ is tangent to $Q_V$ at $p$ or $p \neq q$ and $pq = \delta(q)$. Since the map $(p, q) \mapsto pq$ has full rank, $T'$ is a submanifold of codimension 2, except possibly at points of the form $(p, p)$, i.e. tangent points of the two smooth quadrics.

Deform to the classical case, deforming the two smooth quadrics $Q_E$ and $Q_V$, and forgetting about the Poncelet polygon for the moment, and you see that $T$
deforms into the elliptic curve of the classical case, so \( T \) is diffeomorphic to a 2-torus. The map \( T \to Q_E \) is just the composition \( T \to Q_E^* \to Q_E \) of the projection with the dual map, therefore a smooth map.

We have two involutions defined on \( Q_V \times Q_{E_i}^* \): \( \iota_P(p, \lambda) = (p', \lambda) \) where \( p,p' \) are the points of intersection of \( \lambda \) with \( Q_V \), and \( \iota_A(p, \lambda) = (p, \lambda') \), where \( \lambda, \lambda' \) are the points of \( Q_{E_i}^* \) which contain \( p \). These maps are well-defined, except where the intersections are double points, where we take \( p = p' (\lambda = \lambda' \) respectively).

**Lemma 48.** The maps \( \iota_P, \iota_A \) are smooth, and each has 4 fixed points.

*Proof.* The fixed points of \( \iota_P \) are obviously the points \((p, \lambda)\) where \( \lambda \) is tangent to both \( Q_V \) and \( Q_E \) (i.e. in \( Q_V \cap Q_{E_i}^* \)). Dually, the fixed points of \( \iota_A \) are the points \((p, \lambda)\) where \( p \) belongs to both \( Q_V \) and \( Q_{E_i}^* \). By transversality, both maps are smooth away from their fixed points. Lemma 46 on page 37 assures smoothness near tangent points.

**Lemma 49.** The map \( \varpi = \iota_A \circ \iota_P \) is a diffeomorphism isotopic to the identity. Neither \( \varpi \) nor \( \varpi \circ \varpi \) have any fixed points.

*Proof.* The fixed points of \( \varpi \) are the points \((p, \lambda)\) for which the line \( \lambda \) has double intersection with \( Q_E \) at \( p \), so \( \lambda \) is tangent to \( Q_E \) at \( p \), and for which there is only one line, \( \lambda \), tangent to \( Q_V \) passing through \( p \), so \( \bar{\lambda} \) has double intersection with \( Q_{V_i}^* \) at \( \lambda \), and therefore so \( \bar{\lambda} \) is tangent to \( Q_V \) at \( \lambda \). But the tangent line to \( Q_{V_i}^* \) at \( \lambda \) is \( \bar{\lambda} \) for \( q \in Q_V \) the corresponding point, so \( \bar{\lambda} = \bar{\lambda} \), i.e. \( p = q \). Therefore \( p \in Q_E \cap Q_V \).

So \( p \) is one of the four points of \( Q_V \cap Q_E \), and \( \lambda \) is tangent to both \( Q_V \) and \( Q_E \), necessarily at \( p \). Therefore \( Q_V \) and \( Q_E \) have a common tangent, contradicting our hypotheses.

Consider a fixed point of \( \varpi^2 \), i.e. \( \varpi(p, \lambda) = (p', \lambda') \) and \( \varpi(p', \lambda') = (p, \lambda) \). Then \( \lambda \) and \( \lambda' \) are two tangent lines to \( Q_E \) which both intersection \( Q_V \) at both points \( p \) and \( p' \). Therefore \( p = p' \) or \( \lambda = \lambda' \). If \( p = p' \), then \( \lambda \) and \( \lambda' \) are both tangent to both \( Q_V \) and \( Q_E \), and both must be tangent to \( Q_V \) at \( p = p' \), so must be equal. Dually, if \( \lambda = \lambda' \) then \( p = p' \). Therefore \( \varpi(p, \lambda) = (p, \lambda) \), contradicting the last paragraph. Under deformation to the classical projective plane, \( \varpi \) is taken by isotopy to a translation on an elliptic curve, and therefore is isotopic to the identity map.

**Lemma 50.** Poncelet polygons are precisely periodic orbits of \( \varpi \).

*Proof.* \( \varpi \) by definition takes a point of \( Q_V \) and tangent line to \( Q_E \) to another such point and line, with the next point also contained in the first line. Thus a periodic orbit of \( \varpi \) draws a closed Poncelet polygon.

**Remark 30.** Given any smooth quadric, and 5 tangent lines to it \( \lambda_1, \ldots, \lambda_5 \), their pairwise intersections \( \lambda_i \lambda_{i+1} \) lie on a quadric. For a generic choice of 5 tangent lines, one should be able to show that the resulting quadric is smooth, so that there should be Poncelet pentagons. In a generic smooth projective plane, these might be the only Poncelet polygons.

**Remark 31.** Take an orbit of \( \varpi \), and pick an oriented real surface \( \Sigma \), say a smooth \( CW \) complex. Then average its number of intersections with the first \( k \) tangent lines from that orbit. Let \( k \) get large. I imagine that either a symplectic form emerges, in the cohomology class of the usual Radon–Fubini–Study type symplectic forms, or a Poncelet polygon, but I have no proof.
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