Dynamics of Interfaces in Superconductors

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Abstract

The dynamics of an interface between the normal and superconducting phases under nonstationary external conditions is studied within the framework of the time-dependent Ginzburg-Landau equations of superconductivity, modified to include thermal fluctuations. An equation of motion for the interface is derived in two steps. First, the method of matched asymptotic expansions is used to derive a diffusion equation for the magnetic field in the normal phase, with nonlinear boundary conditions at the interface. These boundary conditions are a continuity equation which relates the gradient of the field at the interface to the normal velocity of the interface, and a modified Gibbs-Thomson boundary condition for the field at the interface. Second, the boundary integral method is used to integrate out the magnetic field in favor of an equation of motion for the interface. This equation of motion, which is highly nonlinear and nonlocal, exhibits a diffusive instability (the Mullins-Sekerka instability) when the superconducting phase expands into the normal phase (i.e., when the external field is reduced below the critical field). In the limit of infinite diffusion constant the equation of motion becomes local in time, and can be derived variationally from a static energy functional which includes the bulk free energy difference between the two phases, the interfacial energy, and a long range self-interaction of the interface of the Biot-Savart form. In this limit the dynamics is identical to the interfacial dynamics of ferrofluid
domains recently proposed by S. A. Langer et al. [Phys. Rev. A 46, 4894 (1992)]. As shown by these authors, the Biot-Savart interaction leads to mechanical instabilities of the interface, resulting in highly branched labyrinthine patterns. The application of these ideas to the study of labyrinthine patterns in the intermediate state of type-I superconductors is briefly discussed.

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I. INTRODUCTION

Over the last decade there has evolved a fairly complete understanding of the physics of several types of nonequilibrium growth patterns, such as the dendritic growth of solidifying systems (e.g., “snowflakes”) or the fingered growth which occurs at the interface of driven immiscible fluids (e.g., viscous fingers in Hele-Shaw cells); for reviews of the theoretical situation, see Kessler et al. [1] and Langer [2]. The similar patterns which grow in these ostensibly different physical systems are the consequence of a competition between a dynamic instability (the Mullins-Sekerka instability [3] for dendritic growth) which promotes the growth of a highly ramified interface, and surface tension, which favors a smooth interface. One of the important theoretical insights which has emerged from this work is that surface tension anisotropy plays a crucial role in determining the morphology of the pattern [1-2].

It has recently been shown that the process of magnetic flux expulsion in type-I superconductors subjected to a magnetic field quench share many features with these other pattern forming systems [3-5]. In particular, as a superconducting nucleus grows, the expelled flux generates eddy currents in the normal phase; the magnetic field \( \mathbf{h} \) in the normal phase therefore satisfies a diffusion equation,

\[
\partial_t \mathbf{h} = D \nabla^2 \mathbf{h}, \tag{1.1}
\]

where the magnetic diffusion constant \( D = 1/4\pi\sigma \), with \( \sigma \) the normal state conductivity (we will set \( c = 1 \)). By applying Maxwell’s equations to the interface itself, and noting that the electric and magnetic fields both vanish in the superconducting phase, we arrive at a continuity equation for the field at the interface [3-5],

\[
(\nabla \times \mathbf{h}) \times \hat{n}_i = -Dv_n \mathbf{h}_i, \quad \mathbf{h} \cdot \hat{n}_i = 0, \tag{1.2}
\]

where \( \hat{n} \) is the unit normal at the interface, directed toward the normal phase, and \( v_n \) is the normal velocity at the interface. Finally, the field at the interface should equal the superconducting critical field \( H_c \), with curvature corrections:
\[ |h|_i = H_c \left\{ 1 - \frac{4\pi}{H_c^2} \left[ \sigma_{\text{ns}}(\theta) + \sigma_{\text{ns}}''(\theta) \right] K \right\}, \]  

(1.3)

where \( \sigma_{\text{ns}}(\theta) \) is the surface tension of the normal/superconducting interface (not to be confused with the conductivity), which depends on the angle \( \theta \) with respect to the crystal axes (it would depend on two angles \( \theta_1, \theta_2 \) in three dimensions), and \( K \) is the curvature of the interface (or the sum of the principle curvatures in three dimensions) \[4\]. Similar equations (without the surface tension) were used by Pippard \[6\] and Lifshitz \[7\] to discuss the growth of the normal phase into the superconducting phase, which is a dynamically stable process. The analogy with the solidification problem is apparent when the magnetic field is identified with the temperature of the liquid, and the magnetic diffusion constant with the thermal diffusivity; the continuity condition, Eq. (1.2), is replaced with the continuity condition for the heat generated at the solidifying interface, and Eq. (1.3) is a modified form of the Gibbs-Thomson boundary condition \[4\]. Due to this formal similarity between the equations describing solidification and those describing the kinetics of the normal/superconducting transition, it was predicted that the growth of a superconducting nucleus into the “supercooled” normal state would be dynamically unstable \[4\]. Such instabilities were observed in numerical studies of the time-dependent Ginzburg-Landau (TDGL) equations of superconductivity for propagating interfaces; see e.g., Fig. 1 of Ref. \[4\].

In this paper we will connect the TDGL equations to the “sharp-interface” equations discussed above. In Sec. II the TDGL equations, including fluctuations, are presented. In Sec. III we use the method of matched asymptotic expansions to reduce these equations to a set of sharp interface equations, generalized to include fluctuations. In Sec. IV we formally reduce the sharp interface equations to an equation of motion for the interface itself, and discuss some of the features of this equation of motion. In particular, we will show that in the limit that the diffusion constant \( D \to \infty \), the interface equation of motion becomes local, and can be derived from an interfacial energy functional. This functional contains three terms: the bulk free energy difference between the normal and superconducting phases, the interfacial surface energy, and a Biot-Savart interaction of the interface with itself. When
written in this form, the equation of motion is identical to a phenomenological equation of
motion for the dynamics of two dimensional domains of ferrofluids, recently proposed by
Langer et al. \cite{10}. Appendix A reviews some properties of the surface tension, along with
a new result on the behavior of the surface tension near the critical value of $\kappa = 1/\sqrt{2}$.
Appendix B includes some details on the calculation of the kinetic coefficient which appears
in the equation of motion. In Appendix C the noise correlations are calculated and shown to
satisfy the fluctuation-dissipation theorem. Some useful definitions and results concerning
the differential geometry of curves in two dimensions are provided in Appendix D.

II. THE TIME-DEPENDENT GINZBURG-LANDAU MODEL

The time-dependent Ginzburg-Landau (TDGL) model consists of equations of motion
for the complex superconducting order parameter $\psi$, the magnetic vector potential $a$, and
the scalar potential $\phi$ \cite{11}. The origin and validity of the equations have been extensively
discussed elsewhere \cite{12,13}. In conventional units, these equations are

$$\hbar\gamma(\partial_t + \frac{e^*}{\hbar}\phi)\psi = \frac{\hbar^2}{2m}(\nabla - \frac{e^*}{\hbar}a)^2\psi + |a|\psi - b|\psi|^2\psi + \theta, \quad (2.1)$$

$$\nabla \times \nabla \times a = 4\pi(J_n + J_s + \tilde{J}), \quad (2.2)$$

where the normal current $J_n$ is given by

$$J_n = \sigma e = \sigma(-\nabla \phi - \partial_t a), \quad (2.3)$$

and where the supercurrent $J_s$ is given by

$$J_s = \frac{\hbar e^*}{2m i}(\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{(e^*)^2}{m}|\psi|^2 a. \quad (2.4)$$

In these equations $\gamma$ is a dimensionless order parameter relaxation time; $e^*$ and $m$ are the
charge and mass of a Cooper pair; $|a| = a_0(1 - T/T_{c0})$, with $T_{c0}$ the mean-field transition
temperature; $\sigma$ is the conductivity of the normal phase. We have also included two noise
terms, \( \theta \) and \( \tilde{J} \), which are chosen so as to generate the correct Boltzmann weights in equilibrium \([14]\). We then have \( \langle \theta \rangle = \langle \tilde{J} \rangle = 0 \), and

\[
\langle \theta(x, t)\theta(x', t') \rangle = 2\hbar \gamma k_B T \delta^{(d)}(x - x') \delta(t - t'),
\]

(2.5)

\[
\langle \tilde{J}_i(x, t)\tilde{J}_j(x', t') \rangle = 2\sigma k_B T \delta^{(d)}(x - x') \delta(t - t') \delta_{ij},
\]

(2.6)

with all cross correlations zero (the brackets denote an average with respect to the noise distribution). In terms of the parameters in the TDGL equations, the correlation length \( \xi = \hbar/(2m|a|)^{1/2} \), the penetration depth \( \lambda = [mb/4\pi(e^*)^2|a|]^{1/2} \), the Ginzburg-Landau parameter \( \kappa = \lambda/\xi \), and the thermodynamic critical field \( H_c = (4\pi|a|^2/b)^{1/2} \).

In deriving a set of sharp interface equations it will be useful to work with a judiciously chosen set of dimensionless variables. Sharp interfaces between the superconducting and normal phases will be produced when the coherence length \( \xi \) is small; i.e., when \(|a|\) is large (this implies that we are far from the normal/superconducting phase boundary). We will therefore introduce a small parameter \( \epsilon \) such that \(|a| = \bar{a}/\epsilon^2\), with \( \bar{a} \) a fixed constant. To recast the TDGL equations into dimensionless form we introduce the following primed dimensionless variables,

\[
x = \bar{\lambda} x', \quad t = (\hbar \gamma / \bar{a}) t', \quad \psi = (|a|/b)^{1/2} \psi',
\]

(2.7)

\[
a = \sqrt{2} \bar{H}_c \bar{\lambda} a', \quad \phi = (\bar{a}/e^* \gamma) \phi',
\]

along with dimensionless conductivity and temperature variables

\[
\bar{\sigma} = 4\pi \kappa^2 (\hbar/2m\gamma) \sigma, \quad T = k_B T/[(H_c^2/4\pi) \bar{\lambda}^d].
\]

(2.8)

Here we have defined \( \bar{\lambda} = [mb/4\pi(e^*)^2\bar{a}]^{1/2} \), \( \bar{\xi} = \hbar/(2\bar{m}\bar{a})^{1/2} \), and \( \bar{H}_c = (4\pi\bar{a}^2/b)^{1/2} \), so that \( \xi = \epsilon \bar{\xi} \) and \( \lambda = \epsilon \bar{\lambda} \). Therefore the \( \epsilon \to 0 \) limit is equivalent to taking \( \xi, \lambda \to 0 \) while holding \( \kappa \) fixed. In these units, the TDGL equations become (we will henceforth drop the primes)

\[
\epsilon^2 (\partial_t + i\phi) \psi = \epsilon^2 \left( \frac{\nabla}{\kappa} - i a \right)^2 \psi + \psi - |\psi|^2 \psi + \epsilon^3 \theta,
\]

(2.9)

\[
\epsilon^2 \nabla \times \nabla \times a = \epsilon^2 \bar{\sigma} \left( -\frac{1}{\kappa} \nabla \phi - \partial_t a \right) + \frac{1}{2\kappa i} (\psi^* \nabla \psi - \psi \nabla \psi^*) - |\psi|^2 a + \epsilon^2 \tilde{J},
\]

(2.10)
\begin{equation}
\langle \theta^* (x, t) \theta (x', t') \rangle = 2 \bar{T} \delta^{(d)} (x - x') \delta (t - t'),
\end{equation}

(2.11)

\begin{equation}
\langle \tilde{J}_i (x, t) \tilde{J}_j (x', t') \rangle = \bar{\sigma} \bar{T} \delta^{(d)} (x - x') \delta (t - t') \delta_{ij}.
\end{equation}

(2.12)

To further simplify, we rewrite the order parameter in terms of an amplitude and a phase, 
\( \psi = f \exp(i \chi) \); in terms of the gauge-invariant quantities 
\( q = a - \nabla \chi / \kappa \) and 
\( p = \phi + \partial_t \chi \),
the magnetic and electric fields are

\begin{equation}
h = \nabla \times q,
\end{equation}

(2.13)

\begin{equation}
e = -\frac{1}{\kappa} \nabla p - \partial_\tau q.
\end{equation}

(2.14)

Equating the real and imaginary parts of Eq. (2.9), and noting that the total current has zero divergence \[13\], we arrive at the final form of the TDGL equations:

\begin{equation}
\epsilon^2 \partial_t f = \epsilon^2 \left( \frac{1}{\kappa^2} \nabla^2 f - q^2 f \right) + f - f^3 + \epsilon^3 \zeta,
\end{equation}

(2.15)

\begin{equation}
\epsilon^2 \nabla \times \nabla \times q = \epsilon^2 \bar{\sigma} \left( -\frac{1}{\kappa} \nabla p - \partial_\tau q \right) - f^2 q + \epsilon^2 \tilde{J},
\end{equation}

(2.16)

\begin{equation}
\epsilon^2 \bar{\sigma} \nabla \cdot \left( \frac{1}{\kappa} \nabla p + \partial_\tau q \right) - f^2 p = \frac{\epsilon^2}{\kappa} \nabla \cdot \tilde{J} - \frac{\epsilon}{\kappa} \zeta,
\end{equation}

(2.17)

where the noise term \( \zeta \) has zero mean and correlations

\begin{equation}
\langle \zeta (x, t) \zeta (x', t') \rangle = \bar{T} \delta (x - x') \delta (t - t'),
\end{equation}

(2.18)

with the correlations of the current noise \( \tilde{J} \) given by Eq. (2.12) above. Note that in these scaled units there are three dimensionless parameters: the Ginzburg-Landau parameter \( \kappa \), which measures the coupling between the order parameter and the gauge field, the dimensionless normal state conductivity \( \bar{\sigma} \), which determines the rate at which flux diffuses in the normal state, and the dimensionless temperature \( \bar{T} \), which measures the relative strength of thermal fluctuations. The remainder of this paper is concerned with solving Eqs. (2.13), (2.16), and (2.17) for a moving interface.
III. THE SHARP-INTERFACE LIMIT

In this section we will derive a sharp interface equation from the TDGL equations, in the limit that $\epsilon \to 0$ while $\kappa$, $\bar{\sigma}$, and $\bar{T}$ are held fixed. In order to simplify the analysis we will assume that all quantities are translationally invariant along the direction of the applied magnetic field; in particular, the magnetic field is $h(\mathbf{x}, t) = h(\mathbf{x}, t)\mathbf{z}$, with $\mathbf{x} = (x, y)$. Extending the results to the more general three dimensional situation significantly complicates the calculation without offering any new physical insights. The derivation in this section is in the same spirit as derivations of the sharp interface equations of solidification from the “phase field equations” in certain distinguished limits [16–18]. The idea is that far from the interface the magnetic field in the normal phase satisfies a diffusion equation; this is the “outer region.” Near the interface, in the “inner region,” we solve the full nonlinear TDGL equations perturbatively in the velocity and curvature of the interface. By matching the solutions in an appropriate overlap region, we will see that the inner solution provides the boundary conditions for the outer region. This allows us to effectively “integrate out” the order parameter field in favor of a diffusion equation for the magnetic field with nonlinear boundary conditions. While this considerably reduces the complexity of the problem, the remaining “modified Stefan problem” is very challenging in its own right. There has been considerable progress in recent years in understanding the properties of this class of models within the context of dendritic growth; see Refs. [1,2] for reviews.

A. The outer solution

In the outer region we assume an expansion of the form

$$f(\mathbf{x}, t; \epsilon) = f_0(\mathbf{x}, t) + \epsilon f_1(\mathbf{x}, t) + \epsilon^2 f_2(\mathbf{x}, t) + \ldots, \quad (3.1)$$

$$q(\mathbf{x}, t; \epsilon) = q_0(\mathbf{x}, t) + \epsilon q_1(\mathbf{x}, t) + \epsilon^2 q_2(\mathbf{x}, t) + \ldots, \quad (3.2)$$

$$p(\mathbf{x}, t; \epsilon) = p_0(\mathbf{x}, t) + \epsilon p_1(\mathbf{x}, t) + \epsilon^2 p_2(\mathbf{x}, t) + \ldots. \quad (3.3)$$
Substituting these expansions into Eqs. (2.15), (2.16), and (2.17), and equating terms of $O(1)$, $O(\epsilon)$, and $O(\epsilon^2)$, we obtain the following two sets of solutions:

**Superconducting solution.** This solution corresponds to $f_0 = 1$, $f_1 = f_2 = 0$, $q_0 = q_1 = 0$, $q_2 = \tilde{J}$, $p_0 = 0$, $p_1 = -\zeta/\kappa$, $p_2 = -\nabla \cdot \tilde{J}/\kappa$.

**Normal solution.** This solution corresponds to $f_0 = f_1 = f_2 = 0$,

$$\nabla \times \nabla \times q_0 = \bar{\sigma} \left( -\frac{1}{\kappa} \nabla p_0 - \partial_t q_0 \right) + \tilde{J},$$

(3.4)

$$\nabla \cdot \left[ \bar{\sigma} \left( -\frac{1}{\kappa} \nabla p_0 - \partial_t q_0 \right) + \tilde{J} \right] = 0,$$

(3.5)

with $q_1, q_2, p_1, p_2$ undetermined (note that the second equation can be obtained by taking the divergence of the first equation). Taking the curl of Eq. (3.4) and using $\nabla \cdot h_0 = 0$, we see that the magnetic field in the normal phase satisfies the diffusion equation

$$\bar{\sigma} \partial_t h_0 = \nabla^2 h_0 + \nabla \times \tilde{J}.$$  

(3.6)

This expansion may be carried to higher order in $\epsilon$; order by order, the magnetic field satisfies the diffusion equation (without the noise term). Therefore, we find that quite generally the magnetic field in the normal phase satisfies

$$\bar{\sigma} \partial_t h = \nabla^2 h + \nabla \times \tilde{J},$$

(3.7)

the corrections to which are exponentially small.

**B. The inner solution**

To solve the TDGL equations in the inner region, we first affix a set of local coordinates $(r, s)$ to the interface, where $r$ measures the distance from the interface and $s$ measures the arclength along the interface. In this coordinate system, the Laplacian becomes

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \kappa \frac{\partial}{\partial r} + (\nabla s)^2 \frac{\partial^2}{\partial s^2} + \nabla^2_s \frac{\partial}{\partial s},$$

(3.8)
where $K$ is the curvature of the interface \[16\]. Since the coordinates now evolve in time, the time derivatives become

$$\partial_t \to \partial_t + \dot{r} \frac{\partial}{\partial r} + \dot{s} \frac{\partial}{\partial s}.$$  \hspace{1cm} (3.9)$$

We now “stretch out” the dimension normal to the interface by introducing the scaled variable $R = r/\epsilon$. Then, keeping the lowest order terms, we have

$$\nabla^2 = \frac{1}{\epsilon^2} \frac{\partial^2}{\partial R^2} + \frac{1}{\epsilon} K \frac{\partial}{\partial R} + O(1),$$  \hspace{1cm} (3.10)$$

$$\partial_t = -\frac{1}{\epsilon} v_n \frac{\partial}{\partial R} + O(1),$$  \hspace{1cm} (3.11)$$

where $v_n = -\dot{r}$ is the velocity normal to the interface. To ensure gauge invariance it is necessary to rescale the vector potential so that $Q = q/\epsilon$. The vector potential will be parallel to the interface, so that $Q = Q(R, s, t) \hat{t}$, with $\hat{t}$ the unit vector tangent to the interface. Finally, since we want to treat the noise terms as first order perturbations about the equilibrium solution, we will introduce rescaled noise terms of the form

$$\tilde{\zeta}(R, s, t) = \epsilon^2 \zeta(x, t), \hspace{1cm} \tilde{J}(R, s, t) = \epsilon \tilde{J}(x, t),$$  \hspace{1cm} (3.12)$$

with correlations

$$\langle \tilde{\zeta}(R, s, t) \tilde{\zeta}(R', s', t') \rangle = \epsilon^3 \tilde{T} \delta(R - R') \delta(s - s') \delta(t - t'),$$  \hspace{1cm} (3.13)$$

$$\langle \tilde{J}_i(R, s, t) \tilde{J}_j(R', s', t') \rangle = \epsilon^2 \sigma \tilde{T} \delta(R - R') \delta(s - s') \delta(t - t') \delta_{ij}.$$  \hspace{1cm} (3.14)$$

Then in terms of these scaled variables, the order parameter amplitude $F(R, s, t; \epsilon) = f(r, s, t; \epsilon)$ and the vector potential $Q(R, s, t; \epsilon) = q(r, s, t; \epsilon)/\epsilon$ satisfy

$$\frac{1}{\kappa^2} F'' - Q^2 F + F - F^3 = -\epsilon \left( v_n + \frac{1}{\kappa} K \right) F' - \epsilon \zeta + O(\epsilon^2),$$  \hspace{1cm} (3.15)$$

$$Q'' - F^2 Q = -\epsilon (\sigma v_n + K) Q' - \epsilon \tilde{J}_t + O(\epsilon^2),$$  \hspace{1cm} (3.16)$$
where \( \bar{J}_t = \bar{J} \cdot \hat{t} \) is the component of the current noise parallel to the interface, and where the primes denote derivatives with respect to \( R \). As before, we expand \( F, Q, \) and \( H = \nabla \times Q = \epsilon_{ij} \partial_i Q_j \) in powers of \( \epsilon \):

\[
F(R, s, t; \epsilon) = F_0(R, s, t) + \epsilon F_1(R, s, t) + \ldots, \tag{3.17}
\]

\[
Q(R, s, t; \epsilon) = Q_0(R, s, t) + \epsilon Q_1(R, s, t) + \ldots, \tag{3.18}
\]

\[
H(R, s, t; \epsilon) = H_0(R, s, t) + \epsilon H_1(R, s, t) + \ldots. \tag{3.19}
\]

Substituting these expansions into Eqs. (3.15) and (3.16), at \( O(1) \) we find the equilibrium Ginzburg-Landau equations,

\[
\frac{1}{\kappa^2} F''_0 - Q_0^2 F_0 + F_0 - F_0^3 = 0, \tag{3.20}
\]

\[
Q''_0 - F_0^2 Q_0 = 0, \tag{3.21}
\]

with \( H_0 = Q'_0 \). The solutions corresponding to an interface between the normal and superconducting phases have \( F_0 = 1, Q_0 = 0, \) and \( H_0 = 0, \) for \( R \to -\infty \) (the superconducting phase), and \( F_0 = 0, Q_0 \sim R/\sqrt{2}, \) and \( H_0 = 1/\sqrt{2} \) for \( R \to \infty \) (the normal phase).

At \( O(\epsilon) \), we have

\[
\frac{1}{\kappa^2} F''_1 - Q_0^2 F_1 - 2Q_0 Q_1 F_0 + F_1 - 3F_0^2 F_1 = - \left( v_n + \frac{1}{\kappa^2} \mathcal{K} \right) F'_0 - \bar{\zeta}, \tag{3.22}
\]

\[
Q''_1 - F_0^2 Q_1 - 2F_0 F_1 Q_0 = - (\bar{\sigma} v_n + \mathcal{K}) Q'_0 - \bar{J}_t, \tag{3.23}
\]

\[
H_1 = Q'_1 + \mathcal{K} Q_0. \tag{3.24}
\]

The \( O(\epsilon) \) perturbations satisfy a set of linear inhomogeneous differential equations. The homogeneous versions of these equations have the solution \( F_1 = F'_0, Q_1 = Q'_0, \) which is easily seen by direct substitution (this is a consequence of the translational invariance of
the equilibrium Ginzburg-Landau equations). This allows us to determine the value of $Q_1$ far from the interface without explicitly solving this set of coupled equations. To see this, multiply Eq. (3.22) by $F'_0$, Eq. (3.23) by $Q'_0$, add, and integrate from $-\infty$ to $R$. We then integrate the derivative terms by parts twice, using the boundary conditions $F'_1(R) = F_1(R) = Q''_0(R) = 0$ for $R \to -\infty$, and using the fact that $F'_0$ and $Q'_0$ satisfy the homogeneous forms of Eqs. (3.22) and (3.23). After some rearranging, we finally arrive at

$$H_0 H_1 + \mathcal{K} Q_0 \left( \frac{1}{\sqrt{2}} - H_0 \right) - Q_1 H'_0 + \frac{1}{\kappa^2} (F'_1 F'_0 - F_1 F''_0)$$

$$= -\mathcal{K} \int_{-\infty}^{R} dx \left[ \frac{1}{\kappa^2} (F'_0)^2 + (Q'_0)^2 - \frac{1}{\sqrt{2}} Q'_0 \right]$$

$$-v_n \int_{-\infty}^{R} dx \left\{ (F'_0)^2 + \tilde{\sigma} \left[ (Q'_0)^2 - \frac{1}{\sqrt{2}} Q'_0 \right] \right\}$$

$$-\frac{1}{\sqrt{2}} \tilde{\sigma} Q_0 v_n - Q_0 \tilde{J}_t + \int_{-\infty}^{R} dx Q_0 \tilde{J}'_t - \int_{-\infty}^{R} dx F'_0 \tilde{\zeta}. \quad (3.25)$$

C. Asymptotic matching

We are now in a position to match the inner and outer solutions. First, rewrite the outer expansion in terms of the inner variables, and take $\epsilon \to 0$ while holding $R$ fixed:

$$h(\epsilon R, s, t) = h_0(\epsilon R, s, t) + \epsilon h_1(\epsilon R, s, t) + \ldots$$

$$= h_0(0, s, t) + \epsilon \left[ R \frac{\partial h_0(r, s, t)}{\partial r} \bigg|_{r=0} + h_1(0, s, t) \right] + O(\epsilon^2). \quad (3.26)$$

This expansion must match order by order onto the inner solution in the limit $R \to \infty$. We then have the matching conditions

$$\lim_{r \to \pm 0} h_0(r, s, t) = \lim_{R \to \pm \infty} H_0(R, s, t), \quad (3.27)$$

$$\lim_{r \to \pm 0} \frac{\partial h_0(r, s, t)}{\partial r} = \lim_{R \to \pm \infty} \frac{\partial H_1(R, s, t)}{\partial R}, \quad (3.28)$$

$$\lim_{r \to \pm 0} h_1(r, s, t) = \lim_{R \to \pm \infty} \left[ H_1(R, s, t) - R \frac{\partial H_1(R, s, t)}{\partial R} \right]. \quad (3.29)$$
Using Eqs. (3.23) and (3.25), we find that at the interface the magnetic field in the outer region is
\[ h_0(0^+, s, t) + e h_1(0^+, s, t) = \frac{1}{\sqrt{2}} \left[ 1 - \epsilon \left( \tilde{\sigma}_{ns} \mathcal{K} + \tilde{\Gamma}^{-1} v_n - \bar{\eta} \right) \right], \] (3.30)
where \( \tilde{\sigma}_{ns} \) and \( \tilde{\Gamma}^{-1} \) are the dimensionless surface tension and kinetic coefficient for the normal/superconducting interface, given by
\[ \tilde{\sigma}_{ns} = 2 \int_{-\infty}^{\infty} dx \left[ \frac{1}{\kappa^2} (F_0')^2 + (Q_0')^2 - \frac{1}{\sqrt{2}} Q_0' \right], \] (3.31)
and
\[ \tilde{\Gamma}^{-1} = 2 \tilde{\sigma} \int_{-\infty}^{\infty} dx \left[ \frac{1}{\tilde{\sigma}} (F_0')^2 + (Q_0')^2 - \frac{1}{\sqrt{2}} Q_0' \right]. \] (3.32)
Some properties of the surface tension are discussed in Appendix A, and the kinetic coefficient is discussed in Appendix B. The noise term \( \bar{\eta} \) is the projection of the current and order parameter noise onto the interface,
\[ \bar{\eta}(s, t) = -2 \int_{-\infty}^{\infty} dx F_0'(x) \tilde{\zeta}(x, s, t) + 2 \int_{-\infty}^{\infty} dx Q_0(x) \tilde{J}_t(x, s, t). \] (3.33)
The average of the noise is easily seen to be zero; as shown in Appendix C, the noise correlations are
\[ \langle \bar{\eta}(s, t) \bar{\eta}(s', t') \rangle = 2 \bar{T} \tilde{\Gamma}^{-1} \delta(s - s') \delta(t - t'), \] (3.34)
so that the fluctuation-dissipation theorem is satisfied [14]. The boundary condition at the interface for the derivative of the magnetic field in the normal phase is
\[ \frac{\partial h_0(r, s, t)}{\partial r} \bigg|_{r=0^+} = -\frac{1}{\sqrt{2}} \sigma v_n - \bar{J}_i(r = 0^+, s, t). \] (3.35)
Finally, setting \( \epsilon = 1 \) and returning to conventional units, we have the diffusion equation for the magnetic field,
\[ \partial_t h = D \nabla^2 h + 4\pi D \nabla \times \bar{J}, \] (3.36)
the boundary condition for the field at the interface (denoted by the subscript \( i \)),
\[ h_i = H_c \left[ 1 - \frac{4\pi}{H_c^2} \left( \sigma_{ns} \mathcal{K} + \Gamma^{-1} v_n - \eta \right) \right], \]  

(3.37)

and a conservation condition for the field at the interface,

\[- \hat{n} \cdot \nabla h_i = D^{-1} H_c v_n + 4\pi \bar{J}_i |. \]  

(3.38)

In conventional units the surface tension and kinetic coefficient are

\[ \sigma_{ns} = \left( \frac{H_c^2 \lambda}{4\pi} \right) \bar{\sigma}_{ns}, \quad \Gamma^{-1} = \left( \frac{H_c^2 \lambda}{4\pi} \right) \left( \frac{2m\gamma}{\kappa^2 h} \right) \bar{\Gamma}^{-1}. \]  

(3.39)

Having now derived a set of sharp interface equations from the TDGL equations, it is instructive to compare our results with the heuristic set of sharp interface equations, Eqs. (1.1)-(1.3), which were discussed in the Introduction, and to compare our results with some previous work. (1) We have chosen to focus here on the two dimensional limit, in order to reduce the algebraic complexity of the derivation. In three dimensions we would reinstate the vector character of \( h \), and the curvature \( \mathcal{K} \) would be replaced by the sum of the principal curvatures of the interface; the expressions for the surface tension and kinetic coefficient would be unchanged. (2) We have assumed that the material parameters of the superconductor are isotropic, leading to an isotropic surface tension. One way of introducing anisotropy is to assume that there are different effective masses along the \( x \) and \( y \) axes. The analysis above could then be easily modified along the lines of Ref. [19], resulting in the more general anisotropic boundary condition in Eq. (1.2). (3) In the conservation condition, Eq. (3.38), the critical field \( H_c \) appears on the right hand side multiplying \( v_n \), whereas in Eq. (1.2) the magnetic field at the interface, \( h_i \), appears. This difference is of higher order in \( \epsilon \), and has therefore been neglected in our calculation. (4) The boundary condition derived above, Eq. (3.37), depends upon the interfacial velocity, in contrast to Eq. (1.2). Such velocity dependent corrections also arise in derivations of the sharp interface equations of solidification from phase-field models [16-18]. As discussed below, this term will have a natural interpretation as a local viscous damping term in the equation of motion for the interface. (5) We have explicitly included the effects of thermal fluctuations in deriving
the sharp interface equations, in contrast to Eqs. (1.1)-(1.3), which neglect fluctuations entirely. In fact, our Eqs. (3.36)-(3.38) bear a striking resemblance to a sharp interface model of solidification which incorporates fluctuations, recently proposed by Karma [20]. Nechiporenko [21] used the TDGL equations to study the nucleation of the superconducting phase, and essentially derived the inner solution in the limit of zero conductivity, and without noise. This results in curvature driven dynamics for the interface, without the interesting diffusion driven instabilities which occur when the inner solution is matched onto the outer solution.

IV. THE INTERFACE EQUATION OF MOTION

Having now reduced the TDGL equations to a sharp interface problem, we will carry the analysis one step further and obtain an explicit equation of motion for the interface itself by integrating out the magnetic field. This type of analysis has been used extensively in the study of the solidification problem [1].

A. The boundary integral method

We will consider a simply-connected superconducting domain which is expanding into the surrounding supercooled normal phase. The normal/superconducting interface is a closed curve $C$, specified by the position vector $r(s, t)$. As the magnetic field in the normal phase satisfies the diffusion equation, we start by introducing the Green’s function $G(x, t|x', t')$ for the diffusion equation,

$$(-D^{-1}\partial_t - \nabla^2)G(x, t|x', t') = \delta^{(d)}(x-x')\delta(t-t'),$$

the solution of which is (in $d$-dimensions)

$$G(x, t|x', t') = D(4\pi D|t-t'|)^{-d/2}\exp(-|x-x'|^2/4D|t-t'|)\theta(t-t').$$

Next, we (i) multiply Eq. (4.1) by $h(x, t)$, and integrate over $(x', t')$; (ii) integrate the term involving the time derivative of the Green’s function by parts, discarding the transient term;
(iii) use the diffusion equation for \( h(\mathbf{x}, t) \) to eliminate the time derivatives. The final result is

\[
h(\mathbf{x}, t) = \int_{-\infty}^{t^+} dt' \int_N d^2 \mathbf{x}' \left[ G(\mathbf{x}, t|\mathbf{x}', t') \nabla'^2 h(\mathbf{x}', t') - h(\mathbf{x}', t') \nabla'^2 G(\mathbf{x}, t|\mathbf{x}', t') \right] \\
+ 4\pi \int_{-\infty}^{t^+} dt' \int_N d^2 \mathbf{x}' G(\mathbf{x}, t|\mathbf{x}', t') \nabla' \times \mathbf{\tilde{J}}(\mathbf{x}', t'),
\]

(4.3)

where \( t^+ \equiv t + \delta \), with \( \delta \) is arbitrarily small, and where the subscript \( N \) on the area integrals denotes an integral over the normal phase. Using Green’s theorem, the first integral may be written as a contour integral along the curve \( C \); the second integral may be integrated by parts with the final result

\[
h(\mathbf{x}, t) = H_0 - \int_{-\infty}^{t^+} dt' \oint_C ds' \hat{n}' \cdot [G(\mathbf{x}, t|\mathbf{r}', t') \nabla' h(\mathbf{r}', t') - h(\mathbf{r}', t') \nabla' G(\mathbf{x}, t|\mathbf{r}', t')] \\
+ 4\pi \int_{-\infty}^{t^+} dt' \oint_C ds' \mathbf{\tilde{J}}(\mathbf{r}', t') G(\mathbf{x}, t|\mathbf{r}', t') \\
- 4\pi \int_{-\infty}^{t^+} dt' \int_N d^2 \mathbf{x}' \nabla' G(\mathbf{x}, t|\mathbf{x}', t') \times \mathbf{\tilde{J}}(\mathbf{x}', t'),
\]

(4.4)

where \( H_0 \) is the external magnetic field at the boundaries of the sample (the minus sign on the first integral is due to the definition of the normal vector as pointing outward from the superconducting phase). Using the boundary conditions at the interface, Eqs. (3.37), (3.38), inside the integrals, we obtain

\[
\frac{H_c}{4\pi} h(\mathbf{x}, t) = \frac{H_c H_0}{4\pi} + \frac{H_c^2}{4\pi D} \int_{-\infty}^{t^+} dt' \oint_C ds' G(\mathbf{x}, t|\mathbf{r}', t') v_n' \\
+ \frac{H_c^2}{4\pi} \int_{-\infty}^{t^+} dt' \oint_C ds' \left[ 1 - \frac{4\pi}{H_c^2} (\sigma_{ns} K' + \Gamma^{-1} v_n' - \eta') \right] \hat{n}' \cdot \nabla' G(\mathbf{x}, t|\mathbf{r}', t') \\
- H_c \int_{-\infty}^{t^+} dt' \int_N d^2 \mathbf{x}' \nabla' G(\mathbf{x}, t|\mathbf{x}', t') \times \mathbf{\tilde{J}}(\mathbf{x}', t'),
\]

(4.5)

where \( v_n' \equiv v_n(s', t') \), etc. This equation determines the magnetic field in the normal phase once the shape and velocity of the interface are specified. This equation must also hold as \( \mathbf{x} \) approaches the interface itself from the normal phase. Then evaluating Eq. (4.3) on the interface \( \mathbf{r}(s, t) \), and using the modified Gibbs-Thomson boundary condition, Eq. (3.37), we obtain after some rearranging
\[
\Gamma^{-1}v_n + \frac{H_c^2}{4\pi D} \int_{-\infty}^{t^+} dt' \oint_C ds' G(\mathbf{r}, t|\mathbf{r}', t') v_n' = \frac{H_c^2 - H_cH_0}{4\pi} - \sigma_{ns} K

- \frac{H_c^2}{4\pi} \int_{-\infty}^{t^+} dt' \oint_C ds' \left[ 1 - \frac{4\pi}{H_c^2} \left( \sigma_{ns} K' + \Gamma^{-1}v_n' - \eta' \right) \right] \mathbf{n}' \cdot \nabla' G(\mathbf{r}, t|\mathbf{r}', t')

+ \eta + \tilde{F},
\]

where the noise term \( \tilde{F} \) is

\[
\tilde{F}(\mathbf{r}, t) = H_c \int_{-\infty}^{t^+} dt' \int_{N} d^2x' \nabla' G(\mathbf{r}, t|x', t') \times \tilde{J}(x', t').
\]

The correlations of \( \tilde{F} \) are discussed in Appendix C.

Our final interface equation of motion, Eq. (4.6) is highly nonlinear and nonlocal. In order to understand the physics contained in this equation, it is helpful to dissect it and discuss the different terms separately. On the left hand side, the first term provides local viscous damping of the interface. The second term may be viewed as a nonlocal viscous damping term due to the eddy currents produced in the normal phase by the moving interface. It is this term which is responsible for the diffusive Mullins-Sekerka instability [1–3]; the propagation of the superconducting phase into the normal phase is dynamically unstable at long wavelengths. On the right hand side, the first term can be derived from the free energy difference between the two phases, and is analogous to the “undercooling” in the solidification problem [1,2]. The second term arises from the surface free energy of the interface. The third term is a consequence of the discontinuity of \( h \) across the interface, which results in an effective surface current density localized at the interface. This produces a retarded self-interaction of the interface; in the limit of infinite diffusion constant (see below) this term becomes local in time and takes the form of a Biot-Savart interaction. The last two terms contain the thermal fluctuations of the interface. The equation of motion looks like a Langevin equation with nonlocal damping and colored noise. However, the retarded self-interaction term on the right hand side cannot be derived from an energy functional, so strictly speaking our equation is not of the Langevin form. As shown below, the reduction to a Langevin equation is complete in the infinite diffusion constant limit.
B. The $D \to \infty$ limit

The interface equation of motion simplifies considerably in the limit that the diffusion constant becomes large. In this limit the second term on the left hand side of Eq. (4.6), which is responsible for the Mullins-Sekerka instability, vanishes, as does the noise term $\tilde{F}$. In addition, in this limit the magnetic field satisfies Laplace's equation rather than the diffusion equation, and the diffusion Green's functions may be replaced by a delta function in time multiplied by the Green's function for Laplace's equation:

$$G(r,t|\mathbf{r}',t') = -\frac{1}{2\pi} \ln(R/R_0) \delta(t-t'), \quad (4.8)$$

where $R = |\mathbf{r}(s) - \mathbf{r}(s')|$, and $R_0$ is some long distance cutoff. The normal gradient of $G$ is

$$\hat{n}(s') \cdot \nabla G(r,t|\mathbf{r}',t') = \frac{[\mathbf{r}(s) - \mathbf{r}(s')] \times \hat{t}(s')}{2\pi R^2} \delta(t-t'). \quad (4.9)$$

In this limit, the interface equation of motion becomes local in time:

$$\Gamma^{-1} n_n(s,t) = \frac{H_c^2 - H_c H_0}{4\pi} - \sigma_{ns} K(s,t) - \frac{H_c^2}{8\pi^2} \oint_C ds' \left\{ 1 - \frac{4\pi}{H_c^2} \left[ \sigma_{ns} K(s',t) \right. \right.$$

$$\left. + \Gamma^{-1} n_n(s',t) - \eta(s',t) \right\} \frac{[\mathbf{r}(s,t) - \mathbf{r}(s',t)] \times \hat{t}(s',t)}{R^2} + \eta(s,t), \quad (4.10)$$

where $\hat{t} = -\hat{n} \times \hat{z}$ has been used. This last term on the right hand side describes the self-interaction of the interface, in the form of a Biot-Savart interaction of the current which is flowing parallel to the interface. In general this current is not constant, but rather is a function of the arclength $s$. For the moment we will ignore this complication and assume that the current is constant. In addition, the in our asymptotic expansion the velocity and curvature corrections were treated as small perturbations; this is equivalent to assuming that the external field $H_0 \approx H_c$. With this in mind, we see that the first term on the right hand of Eq. (4.10) is approximately $(H_c^2 - H_0^2)/8\pi$. With these simplifications, the equation of motion becomes (using $v_n = \hat{n} \cdot \partial_t \mathbf{r}$)

$$\Gamma^{-1} \mathbf{n} \cdot \partial_t \mathbf{r} = \frac{H_c^2 - H_0^2}{8\pi} - \sigma_{ns} K - \frac{H_c^2}{8\pi^2} \oint_C ds' \frac{[\mathbf{r}(s,t) - \mathbf{r}(s',t)] \times \hat{t}(s')}{R^2} + \eta. \quad (4.11)$$
This equation may be written in the variational form

\[ \Gamma^{-1}\partial_t r = -\frac{1}{\sqrt{g}} \frac{\delta H_{\text{eff}}}{\delta r} + \eta, \]  

(4.12)

where \( g \) is the metric for the interface (see Appendix D), and where the effective interface Hamiltonian is

\[ H_{\text{eff}}[r] = -\frac{H_c^2 - H_0^2}{8\pi} A + \frac{H_c^2}{16\pi^2} \oint_C ds \oint_C ds' \hat{t}(s) \cdot \hat{t}(s') \ln(R/R_0). \]  

(4.13)

Here \( A \) is the area enclosed by the curve \( C \) and \( L \) is the perimeter of \( C \) (the necessary functional derivatives are carried out in Appendix D). The first term is the free energy difference between the two phases, the second term is the free energy of the interface, and the third term is the self-interaction of the interface. As mentioned above, this last term is a type of Biot-Savart interaction of a current sheet, with a current per unit length of \( H_c/4\pi \). The integral is therefore one-half of the self-inductance of the current sheet.

The local equation of motion, Eq. (4.12), is identical to an equation of motion for the interface of two dimensional ferrofluid domains recently proposed by Langer et al. [10] (for an overview, see [22]). As shown by these authors, the repulsive Biot-Savart interaction tends to favor an extended interface, resulting in the highly branched labyrinthine patterns which are observed when a ferrofluid droplet is subjected to an applied magnetic field. Labyrinthine structures are also observed in the intermediate state of type-I superconductors [23], and the long range Biot-Savart interaction may play an important role in understanding the development of these patterns [22].

V. DISCUSSION AND SUMMARY

In summary, we have derived an equation of motion for the normal/superconducting interface by starting from a fluctuating version of the TDGL equations. In the limit of infinite diffusion constant (zero normal state conductivity), these equations become local and can be derived variationally from an energy functional. For finite diffusion constant the equation...
of motion becomes nonlocal, and contains an additional term responsible for the Mullins-Sekerka instability of the moving interface. What remains to be studied is the competition between the self-interaction of the interface and the Mullins-Sekerka instability, and the implications for pattern formation in the intermediate state of type-I superconductors [24].

After this work was completed, results similar to those obtained in Sec. III have been reported by Chapman et al. [25]. I would like to thank Dr. Weinan E for bringing this reference to my attention.

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APPENDIX A: THE SURFACE TENSION

In this Appendix we will review some properties of the surface tension. In addition, we will calculate the first order corrections to surface tension near the critical value of $\kappa = 1/\sqrt{2}$, leading to Eq. (A18). This result has not appeared in the literature previously, and is the principle new result in this Appendix.

The canonical form of the dimensionless surface tension of the normal/superconducting interface is [20]

$$\bar{\sigma}_{ns} = \int_{-\infty}^{\infty} dx \left[ -F_0^2 + \frac{1}{2} F_0^4 + \frac{1}{\kappa^2} (F_0')^2 + Q_0^2 F_0^2 + \left( Q_0' - \frac{1}{\sqrt{2}} \right)^2 \right]. \quad (A1)$$

To show that this is equivalent to Eq. (3.31), we can use the following identity:

$$- F_0^2 + \frac{1}{2} F_0^4 + F_0^2 Q_0^2 = \frac{1}{\kappa^2} (F_0')^2 + (Q_0')^2 - \frac{1}{2}. \quad (A2)$$

To obtain this result, multiply the first equilibrium Ginzburg-Landau equation, Eq. (3.20), by $F_0'$, multiply the second Ginzburg-Landau equation, Eq. (3.21), by $Q_0'$, add the two
equations, and note that the resulting equation is an exact differential; the constant of integration is determined by the boundary conditions on $F_0$ and $Q_0$. Using this identity in conjunction with Eq. (A1) produces our result, Eq. (3.31).

An explicit calculation of the surface tension requires a numerical solution of the equilibrium Ginzburg-Landau equations. Lacking such numerical solutions, it is still possible to make some qualitative observations about the surface tension. The second set of terms in Eq. (3.31) which involve the vector potential produce a negative contribution to the surface tension; for sufficiently large $\kappa$ these terms will dominate and the surface tension will become negative. Detailed analytic arguments [27] show that for $\kappa \ll 1/\sqrt{2}$, $\bar{\sigma}_{ns} \approx 2\sqrt{2}/3\kappa$, so that in conventional units $\sigma_{ns} \approx 1.89\xi(H_c^2/8\pi)$, while for $\kappa \gg 1/\sqrt{2}$, $\bar{\sigma} = 4(\sqrt{2} - 1)/3$, so that in conventional units $\sigma_{ns} \approx 1.10\lambda(H_c^2/8\pi)$.

As first shown in the landmark paper by by Ginzburg and Landau [28], the surface tension vanishes at $\kappa = 1/\sqrt{2}$. Here we will extend these results somewhat, by expanding the surface tension about the $\kappa = 1/\sqrt{2}$ limit. We will introduce a small parameter $\tilde{\epsilon}$ such that $1/(2\kappa^2) = 1 + \tilde{\epsilon}$. We then expand the equilibrium order parameter $F_0$ and vector potential $Q_0$ in powers of $\tilde{\epsilon}$,

\begin{align*}
F_0(x; \tilde{\epsilon}) &= F_{00}(x) + \tilde{\epsilon}F_{01}(x) + \ldots, \quad (A3) \\
Q_0(x; \tilde{\epsilon}) &= Q_{00}(x) + \tilde{\epsilon}Q_{01}(x) + \ldots, \quad (A4)
\end{align*}

where $F_{01}$ and $Q_{01}$ and all of their derivatives vanish at $\pm\infty$. Substituting these expansions into the surface tension, and noting that the $O(1)$ term vanishes, we obtain

\[ \bar{\sigma}_{ns} = 2\tilde{\epsilon} \int_{-\infty}^{\infty} dx \left[ 2(F'_{00})^2 + 4F_{00}F_{01} + 2Q'_{00}Q'_{01} \right]. \quad (A5) \]

Below we will solve the equilibrium Ginzburg-Landau equations for $F'_{00}$, and derive an identity for the second and third integrals above.

The $O(1)$ terms satisfy the equilibrium Ginzburg-Landau equations with $1/\kappa^2 = 2$:

\[ 2F''_{00} - Q^2_{00}F_{00} + F_{00} - F^3_{00} = 0, \quad (A6) \]
\[ Q'''_{00} - F'^2_{00} Q_{00} = 0. \]  

(A7)

Since the surface tension vanishes at \( O(1) \), we have [27, 28]

\[ F'_{00} = -\frac{1}{\sqrt{2}} F_{00} Q_{00}. \]  

(A8)

This equation may be used to eliminate \( Q_{00} \) in Eq. (A6), so that at \( \kappa = 1/\sqrt{2} \) the order parameter amplitude satisfies

\[ 2 F''_{00} - 2 \frac{(F'_{00})^2}{F_{00}} + F_{00} - F^3_{00} = 0. \]  

(A9)

As \( F_{00} > 0 \), we can introduce a new function \( u(x) \) through \( F_{00}(x) = \exp[-u(x)] \), so that \( u \) is the solution to

\[ 2u'' + e^{-2u} - 1 = 0, \]  

(A10)

with \( u, \ u' \to 0 \) as \( x \to -\infty \). This equation may be integrated once; using the boundary conditions, we obtain

\[ u' = \left( u + \frac{1}{2} e^{-2u} - \frac{1}{2} \right)^{1/2}. \]  

(A11)

This equation could be integrated again to obtain \( u(x) \), and therefore \( F_{00}(x) \), but this will not be necessary for our purposes.

At \( O(\tilde{\epsilon}) \), we find

\[ 2F''_{01} - Q^2_{00} F_{01} - 2Q_{00} Q_{01} F_{00} + F_{01} - 3F_{00}^2 F_{01} = -2F''_{00}, \]  

(A12)

\[ Q''_{01} - F'^2_{00} Q_{01} - 2F_{00} F_{01} Q_{00} = 0. \]  

(A13)

As in Sec. III.B, we see that the \( O(\tilde{\epsilon}) \) perturbations satisfy a set of linear, inhomogeneous differential equations. The method of solution is identical: we multiply Eq. (A12) by \( F'_{00} \), Eq. (A13) by \( Q'_{00} \), add, and integrate from \(-\infty \) to \( x \). We then integrate the derivative terms by parts twice, and use the fact that \( F'_{00} \) and \( Q'_{00} \) solve the homogeneous versions of Eqs. (A12) and (A13). The final result is
\[ 2(F_{00}'F_{01}' - F_{00}''F_{01}) + Q_{00}'Q_{01}' - Q_{00}''Q_{01} = -(F_{00}')^2. \] (A14)

By substituting this result into Eq. (A13), and performing a few integrations by parts, we find that the surface tension is

\[ \bar{\sigma}_{ns} = \tilde{\epsilon} I_1 \] (A15)

where the integral \( I_1 \) is given by

\[ I_1 = 2 \int_{-\infty}^{\infty} dx (F_{00}')^2. \] (A16)

Since we have now expressed the \( O(\tilde{\epsilon}) \) correction to the surface tension entirely in terms of \( F_{00}' \), we can now use our results above to calculate the integral. By changing variables to \( u \), \( I_1 \) may be written as

\[
I_1 = 2 \int_{-\infty}^{\infty} dx (u')^2 e^{-2u} \\
= 2 \int_{-\infty}^{\infty} du u' e^{-2u} \\
= 2 \int_{0}^{\infty} du \left( u + \frac{1}{2} e^{-2u} - \frac{1}{2} \right)^{1/2} e^{-2u}.
\] (A17)

The integral is rapidly convergent, and a numerical evaluation produces \( I_1 = 0.388 \). We therefore have our final result for the surface tension near \( \kappa = 1/\sqrt{2} \):

\[ \bar{\sigma}_{ns} \approx 0.388 \left( \frac{1}{2\kappa^2} - 1 \right) \left( \frac{1}{2\kappa^2} \approx 1 \right). \] (A18)

**APPENDIX B: THE KINETIC COEFFICIENT**

As with the surface tension, an explicit evaluation of the kinetic coefficient \( \Gamma^{-1} \) defined in Eq. (3.32) requires a numerical solution of the equilibrium Ginzburg-Landau equations. In this Appendix we will make a few simple qualitative observations, and then evaluate the kinetic coefficient in some limiting cases. First, note that when \( \bar{\sigma} = \kappa^2 \), which in conventional units implies that \( 2\pi\hbar\sigma/m\gamma = 1 \) [see Eq. (2.8)], the kinetic coefficient is proportional to the surface tension:
\[ \Gamma^{-1} = \kappa^2 \bar{\sigma}_{ns} \quad (\bar{\sigma} = \kappa^2). \]  

(B1)

This is a consequence of a type of “duality” in the TDGL equations which has been previously noted in the context of vortex motion in superconductors [15]. However, there is generally no simple relation between the surface tension and the kinetic coefficient. Second, as with the surface tension, the terms in Eq. (3.32) involving the vector potential yield a negative contribution to the kinetic coefficient. It is therefore possible for the kinetic coefficient to become negative for sufficiently large conductivity \( \bar{\sigma} \). By balancing the two sets of terms in the kinetic coefficient, we see that it will become negative when \( \bar{\sigma}/\kappa \sim 1 \); in conventional units we have \( (2\pi \hbar \bar{\sigma}/m\gamma) \sim 1/\kappa \).

In order to calculate the kinetic coefficient in the large and small \( \kappa \) limits, we can use the results of Ref. [24]. In the limit of small \( \kappa \) the terms involving the gradient of the order parameter amplitude dominate, and we have

\[ \Gamma^{-1} = \frac{2\sqrt{2}}{3} \kappa \quad (\kappa \ll 1). \]  

(B2)

In the limit of large \( \kappa \) the terms involving the magnetic field dominate, with the result

\[ \Gamma^{-1} = -\frac{4}{3} (\sqrt{2} - 1) \kappa^2 \left( \frac{2\pi \hbar}{m\gamma} \right) \sigma \quad (\kappa \gg 1). \]  

(B3)

Finally, we can also calculate the kinetic coefficient exactly at \( \kappa = 1/\sqrt{2} \) by using the results derived in Appendix A above. The result is

\[ \Gamma^{-1} = 0.388 \left( 1 - \frac{2\pi \hbar}{m\gamma} \sigma \right) \quad (\kappa = 1/\sqrt{2}). \]  

(B4)

At \( \kappa = 1/\sqrt{2} \), the kinetic coefficient vanishes when \( 2\pi \hbar \sigma/m\gamma = 1 \). This is to be expected, for at this value of the conductivity the kinetic coefficient is proportional to the surface tension, which also vanishes at \( \kappa = 1/\sqrt{2} \).

**APPENDIX C: NOISE CORRELATIONS**

In order to demonstrate that the noise term \( \tilde{\eta} \) defined in Eq. (3.33) satisfies the fluctuation-dissipation, we will first work with
\[ \bar{\eta}(R, s, t) = -2 \int_{-\infty}^{R} dx F_0'(\zeta) + 2 \int_{-\infty}^{R} dx Q_0 \bar{J}_t, \] (C1)

and take the limit as \( R \to \infty \) at the end of the calculation. The average of \( \bar{\eta} \) is zero. The two point correlation is given by

\[ \langle \bar{\eta}(R, s, t)\bar{\eta}(R', s', t') \rangle = 4 \int_{-\infty}^{R} dx \int_{-\infty}^{R'} dx' F_0'(x) F_0'(x') \langle \bar{\zeta}(x, s, t)\bar{\zeta}(x', s', t') \rangle + 2 \int_{-\infty}^{R} dx \int_{-\infty}^{R'} dx' Q_0(x)Q_0(x') \frac{d^2}{dxdx'} \langle \bar{J}_t(x, s, t)\bar{J}_t(x', s', t') \rangle. \] (C2)

Substituting the two point correlations from Eqs. (3.12) and (3.13), and integrating the second term in Eq. (C2) by parts twice, we obtain

\[ \langle \bar{\eta}(R, s, t)\bar{\eta}(R', s', t') \rangle = 2T\delta(s - s')\delta(t - t') \left\{ 2 \int_{-\infty}^{R} dx (F_0')^2 \theta(R' - x) + 2\bar{\sigma} \left[ \int_{-\infty}^{R} dx (Q_0')^2 \theta(R' - x) - Q_0(R')Q_0(R')\theta(R - R') - Q_0(R)Q_0(R')\delta(R - R') \right] \right\}, \] (C3)

where \( \theta \) is the step function. Now take the limit \( R, R' \to \infty \) (\( R \neq R' \)):

\[ \langle \bar{\eta}(s, t)\bar{\eta}(s', t') \rangle = \lim_{R, R' \to \infty} \langle \bar{\eta}(R, s, t)\bar{\eta}(R', s', t') \rangle = 2T\delta(s - s')\delta(t - t') \left\{ 2 \int_{-\infty}^{R} dx (F_0')^2 + 2\bar{\sigma} \lim_{R \to \infty} \left[ \int_{-\infty}^{R} dx (Q_0')^2 - Q_0(R)Q_0'(R) \right] \right\}. \] (C4)

Since \( \lim_{R \to \infty} Q_0'(R) = 1/\sqrt{2} \), we may combine the last two terms into a single integral, and finally arrive at Eq. (3.34), with the kinetic coefficient \( \bar{\Gamma}^{-1} \) defined in Eq. (3.32). We thus see that the projected noise \( \bar{\eta} \) satisfies the fluctuation-dissipation theorem.

Next, we will calculate the correlations of the noise term \( \bar{F} \) defined in Eq. (4.7). Again, the average of \( \bar{F} \) is zero, while the two point correlation is

\[ \langle \bar{F}(r, t)\bar{F}(r', t') \rangle = H_c^2 \int_{-\infty}^{t_1} dt_1 \int_{-\infty}^{t_2} dt_2 \int_{N} d^2x_1 \int_{N} d^2x_2 \times \epsilon_{ij} \epsilon_{kl} \partial_i G(r, t|\mathbf{x}_1, t_1) \partial_k G(r', t'|\mathbf{x}_2, t_2) \langle J_j(\mathbf{x}_1, t_1)J_i(\mathbf{x}_2, t_2) \rangle = 2\sigma k_BTH_c^2 \int_{-\infty}^{t_1} dt_1 \int_{N} d^2x_1 \nabla G(r, t|\mathbf{x}_1, t_1) \cdot \nabla G(r', t'|\mathbf{x}_1, t_1), \] (C5)
where Eq. (2.6) has been used for the correlations of the current noise. By using several vector identities, along with the heat equation, we find that this may be written as

\[
\langle \tilde{F}(r,t)\tilde{F}(r',t') \rangle = \frac{H^2}{4\pi D}k_B T[G(r,t|r',t') + G(r',t'|r,t)]
\]

\[- \frac{H^2}{4\pi} 2k_B T \int_{-\infty}^{t'} dt_1 \oint_C ds_1 \hat{n}_1 \cdot \nabla_1 [G(r,t|\tau_1,t_1)G(r',t'|\tau_1,t_1)]. \tag{C6}
\]

The first term is what would be expected for colored noise, given the kernel in the second term on the left hand side of the equation of motion, Eq. (4.6). The physics of the second term is unclear. It may cancel the cross-correlation \(\langle \eta \tilde{F} \rangle\), although I have been unable to demonstrate this convincingly.

**APPENDIX D: DIFFERENTIAL GEOMETRY OF CURVES IN TWO DIMENSIONS**

Here we will review a few facts regarding the differential geometry of closed curves in two dimensions, much of which can be found in Refs. [10] and [29].

Let \(r(\alpha)\) trace out a closed curve \(C\), with \(0 \leq \alpha \leq 1\). This parameterization is connected to the arclength parameterization \(s(\alpha)\) by \(ds = \sqrt{g} d\alpha\), where \(g = |\partial r/\partial \alpha|^2\) is the metric for the curve. The counterclockwise tangent vector is \(\tau(\alpha) = \partial r/\partial \alpha\), and the unit tangent is \(\hat{t} = \tau/\sqrt{g}\). In the arclength parameterization, \(\partial \hat{t}/\partial s = -\mathcal{K}\hat{n}\), where \(\mathcal{K}\) is the curvature and \(\hat{n}\) is the outward unit normal, and \(\partial \hat{n}/\partial s = \mathcal{K}\hat{t}\). The perimeter \(L\) of the curve \(C\) is a functional of \(r(\alpha)\), and is given by

\[
L[r] = \oint_C ds = \int_0^1 d\alpha \sqrt{g}, \tag{D1}
\]

and the area \(A\) enclosed by \(C\) is

\[
A[r] = \frac{1}{2} \oint_C ds \mathbf{r} \times \hat{t} = \frac{1}{2} \int_0^1 d\alpha \mathbf{r}(\alpha) \times \tau(\alpha). \tag{D2}
\]
Functional derivatives of these quantities are obtained by using the Euler-Lagrange equations in the $\alpha$-parameterization, with the results

$$\frac{\delta L}{\delta r} = \sqrt{g}K\hat{n}, \quad \frac{\delta A}{\delta r} = \sqrt{g}\hat{n}. \quad (D3)$$

The Biot-Savart contribution to the energy is of the general form

$$I[r] = \frac{1}{2} \oint_C ds \oint_C ds' \hat{t}(s) \cdot \hat{t}(s')\Phi(R)$$

$$= \frac{1}{2} \int_0^1 d\alpha \int_0^1 d\alpha' \tau(\alpha) \cdot \tau(\alpha')\Phi(R), \quad (D4)$$

where $R = |r(s) - r(s')|$. The functional derivative of this quantity is

$$\frac{\delta I}{\delta r} = \sqrt{g}\hat{n} \oint_C ds' \left[ r(s) - r(s') \right] \times \hat{t}(s') \frac{\Phi'(R)}{R}, \quad (D5)$$

where $\Phi'(R) = \partial \Phi/\partial R$. 
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