Rate-optimal sparse approximation of compact break-of-scale embeddings

Glenn Byrenheid∗ Janina Hübner† Markus Weimar‡
March 21, 2022

Abstract

The paper is concerned with the sparse approximation of functions having hybrid regularity borrowed from the theory of solutions to electronic Schrödinger equations due to Yserentant [43]. We use hyperbolic wavelets to introduce corresponding new spaces of Besov- and Triebel-Lizorkin-type to particularly cover the energy norm approximation of functions with dominating mixed smoothness. Explicit (non-)adaptive algorithms are derived that yield sharp dimension-independent rates of convergence.

Keywords: hyperbolic wavelets, tensor-product structures, best $m$-term approximation, linear approximation, function spaces, dominating mixed smoothness, energy norm

2010 Mathematics Subject Classification: 42C40, 41A25, 46E35, 41A45, 41A46

1 Motivation and main result

The electronic Schrödinger equation describes the motion of a huge system of electrons under Coulomb interaction forces in a field of clamped nuclei. It forms the basis of modern quantum chemistry. Solutions to this equation, so-called wave functions, depend on $d = 3N$ variables (three spatial dimensions for each of the $N \gg 1$ electrons) and thus are hard to approximate.

∗Friedrich-Schiller-University Jena, Institute of Mathematics, Ernst-Abbe-Platz 2, 07737 Jena. Email: glenn.byrenheid@uni-jena.de

†Corresponding author. Ruhr University Bochum, Faculty of Mathematics, Research Group Numerics, Universitätsstraße 150, 44801 Bochum. Email: janina.huebner@rub.de

‡Ruhr University Bochum, Faculty of Mathematics, Research Group Numerics, Universitätsstraße 150, 44801 Bochum, Germany. Email: markus.weimar@rub.de
numerically in general. Nonetheless, in a series of articles Yserentant and co-authors proved that physically relevant solutions (those which respect the so-called Pauli principle) possess a special type of smoothness that connects classical (isotropic) Sobolev regularity with square integrable mixed weak derivatives of order up to $N+1$; see [42, 43]. As we will show, this kind of hybrid smoothness can help to reduce the numerical effort of the high-dimensional problem at hand drastically. This is the initial motivation for us to study the approximation problem in spaces $H^{r,s}_p X$, where $X \in \{B, F\}$, of multivariate functions with hybrid regularity of Besov- or Triebel-Lizorkin-type which particularly cover standard $L_p$-Sobolev spaces $H^s_p$ and $S^s_p H$ of isotropic and dominating mixed smoothness [37], respectively, as special cases. The central question considered in this paper is the optimal worst-case (non-)linear approximability of functions w.r.t. the norms in $H^{r,s}_p F$ or $H^{r,s}_p B$, respectively. Our interest is founded by the analysis of Galerkin discretizations of elliptic partial differential equations (PDEs). In this context, Céa’s lemma allows to bound the norm of the resulting error in the respective energy space $H$ by the best approximation error w.r.t. the underlying Galerkin subspace. In the simplest case of the Dirichlet problem for Poisson’s equation on a bounded domain $\Omega \subset \mathbb{R}^d$, we have $H = H^1_0(\Omega)$ and hence

$$\|u - u_m\|_{H^1(\Omega)} \lesssim \inf_{v \in \mathbb{V}_m} \|u - v\|_{H^1(\Omega)},$$

where $u_m \in \mathbb{V}_m \subset H^1_0(\Omega)$ denotes the Galerkin approximation to the solution $u$ with $m$ degrees of freedom. Taking the supremum over all $u$ in the unit ball of a corresponding function space and the infimum over all linear subspaces $\mathbb{V}_m$ with $\dim(\mathbb{V}_m) \leq m$, yields the so-called Kolmogorov $m$-width $d_m$; see [33, Chapter 11]. The rate of convergence of this quantity, as $m$ tends to infinity, is governed by the regularity of the function class under consideration as well as by the target norm $\| \cdot \|_{H^1(\Omega)}$. In case of Hilbert target spaces, $d_m$ serves as a benchmark for the performance of optimal linear algorithms [33, Proposition 11.6.2]. An appropriate class of source spaces is given by Sobolev-Hilbert spaces $H^2_{\text{mix}}(\Omega)$ with bounded mixed derivatives up to second order (which coincides with our spaces $H^2_{2,2} B(\Omega)$ and $H^2_{2,2} F(\Omega)$, see Section 2 below). In this context, a first result based on hierarchic bases in combination with the introduction of so-called energy sparse grids implies that

$$d_m(\operatorname{Id}: H^2_{\text{mix}}((0,1)^d) \rightarrow H^1((0,1)^d)) \sim m^{-1},$$

see [1, 20] for details. Note that, in particular, there is no $d$-dependent logarithmic term as it is known for approximation w.r.t. $L_2((0,1)^d)$ or more generally $L_p((0,1)^d)$. A wide overview of this classical $L_p$-situation including

$$d_m(\operatorname{Id}: H^2_{\text{mix}}((0,1)^d) \rightarrow L_2((0,1)^d)) \sim (m^{-1} \log^{d-1} m)^2$$
is provided in the survey [8] and the references therein. In connection with measuring errors in the energy norm let us further mention [3] and [7], where the problem of energy norm-based sampling recovery is considered. Additionally, we want to mention [9], where $d_m$ is considered in the periodic Hilbert case with special interest to the $d$-dependence of the corresponding constants.

In the realm of PDEs or integral equations on non-smooth domains or manifolds, solutions typically contain singularities caused by irregular points of the underlying geometry [21]. In order to resolve these singular parts numerically, usually iterative schemes based on adaptive refinement strategies are employed [6, 12]. That is, the next Galerkin subspace is chosen during the run time of the algorithm, depending on the concrete (unknown) solution $u$ of interest, rather than being fixed in advance as for linear schemes based on uniform refinement. Therefore, the rate of convergence of non-linear quantities such as best $m$-term widths $\sigma_m$ (cf. Definition 2.5) yields a much better benchmark for such adaptive methods than $d_m$ discussed above. Again these rates are closely related to the regularity of the underlying function spaces [19]. While the smoothness of solutions with singular parts is known to be quite limited in the scale of Sobolev-Hilbert spaces [5], regularity theory shows that such functions admit higher order smoothness when derivatives are measured w.r.t. Lebesgue-norms weaker than $L_2(\Omega)$; see, e.g., [4, 10, 11, 14, 15, 22, 25]. This finally leads to the observation that best $m$-term widths decay faster than corresponding linear quantities, as this additional regularity can be exploited by adaptive algorithms, but not by linear ones. However, in the case of isotropic source and target spaces, usually the optimal rate of convergence is given by the difference in smoothness divided by the dimension $d$ which is commonly referred to as the curse of dimensionality [40]. If both spaces solely possess dominating mixed smoothness, this dimensional dependence in the main rate can be avoided, but still additional $d$-dependent logarithmic factors appear. For details and typical results we refer to [13, 23, 24]. Anyhow, except of [26, 27, 31] which consider the periodic setting and [17, 32, 35] dealing with Hilbert target spaces only, to the best of our knowledge, not much is known for general break-of-scale embeddings (such as, e.g., the setting in Theorem 1.1 below) and/or hybrid-type smoothness spaces.

Since wavelets are known to be a powerful tool in signal processing and numerical analysis [16, 18], in this paper we shall focus on algorithms based on a system of hyperbolic wavelets. In contrast to classical isotropic wavelets, their tensor product structure is perfectly suited to resolve anisotropies which naturally arise in various applications, e.g., in physics, engineering, or medical image processing; see [34] and the references therein. On the other hand, these wavelets can be employed to characterize function spaces measuring dominating mixed smoothness [38] as well as spaces of isotropic regularity [34]. In order to ensure a fair comparison of the performance of linear and non-linear methods, we restrict the corresponding widths $d_m$ and $\sigma_m$ to a dictionary $\Psi$ consisting of such hyperbolic wavelets;
see Definition 2.5 below for details.

The most important special cases of our break-of-scale main result (see Theorem 4.1) read as follows:

**Theorem 1.1.** For \( d \in \mathbb{N} \) let \( \Omega \subset \mathbb{R}^d \) be a bounded domain, \( 0 < p_0 \leq \infty \), and \( 1 < p_1 < \infty \) as well as \( r, s \in \mathbb{R} \) such that

\[
r - \left( \frac{1}{p_0} - \frac{1}{p_1} \right)_+ > s > 0.
\]

(i.) If \( 1 < p_0 < \infty \), then the embedding \( \Id_1 : S^r_{p_0} H(\Omega) \rightarrow H^s_{p_1}(\Omega) \) is compact and there holds

\[
d_m(\Id_1; \Psi) \sim m^{-[r-s-(1/p_0-1/p_1)_+]}, \quad \text{as well as} \quad \sigma_m(\Id_1; \Psi) \sim m^{-(r-s)}.
\]

(ii.) If further \( 0 < q_0 \leq \infty \), then the embedding \( \Id_2 : S^r_{p_0,q_0} B(\Omega) \rightarrow H^s_{p_1}(\Omega) \) is compact, where

\[
d_m(\Id_2; \Psi) \sim m^{-[r-s-(1/p_0-1/p_1)_+]}, \quad \text{and} \quad \sigma_m(\Id_2; \Psi) \sim m^{-(r-s)}.
\]

This theorem reveals several important effects simultaneously. First of all, our convergence rates for the energy norm neither contain a perturbating \( d \)-dependent logarithm, nor a dimensionally deteriorating main rate. Second, similar to the \( L_p \)-setting studied in [23, 24], best \( m \)-term approximation is not affected by different integrabilities between source and target space. While the latter observation resembles a typical feature of non-linear approximation methods, the first one heavily relies on the break-of-scale structure of the embeddings under consideration which is expressed by condition (1): We give up dominating mixed smoothness \( r \) and gain isotropic regularity \( s \).

The paper is organized as follows. In Section 2 we introduce our new function spaces of hybrid smoothness based on hyperbolic wavelets via a characterization by suitably chosen sequence spaces. There we also collect basic properties and recall the definition of the considered approximation widths. Afterwards, in Section 3 we derive sharp asymptotic approximation rates at the level of sequence spaces. For the upper bounds explicit (non-) linear algorithms are constructed. Finally, Section 4 contains the main result on break-of-scale embeddings of hybrid-type function spaces.

**Notation:** By \( \mathbb{N}_0 \) we denote the set of integers \( n \in \mathbb{Z} \) that are larger than or equal to zero and \( \mathbb{R} \) denotes the real numbers. For \( x \in \mathbb{R} \) we further write \( x_+ := \max\{x, 0\} \). Given two quasi-Banach spaces \( X \) and \( Y \), we write \( X \hookrightarrow Y \) if they are continuously embedded, i.e., \( X \subseteq Y \) and \( \Id \in \mathcal{L}(X, Y) \). We will write \( A \lesssim B \) if there exists a constant \( c > 0 \), such that
$A \leq c \cdot B$. With $A \sim B$ we mean that $A \lesssim B \lesssim A$. Further, $|D|$ denotes the cardinality of a discrete set $D$. For $d \in \mathbb{N}$, we use $\mathcal{S}'(\mathbb{R}^d)$ to denote the space of tempered distributions, the topological dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions, and $\mathcal{D}'(\Omega)$ is the dual of the space $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ of test functions with compact support in some open set $\Omega \subset \mathbb{R}^d$. Finally, the restriction of $g \in \mathcal{S}'(\mathbb{R}^d)$ to $\Omega$ is given by $g|_{\Omega} \in \mathcal{D}'(\Omega)$, where $(g|_{\Omega})(\varphi) := g(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$.

2 Preliminaries

We start with collecting all basic requirements needed later on. To do so, we first give a brief introduction to hyperbolic wavelets and define our hybrid function spaces through suitably chosen sequence spaces. In Proposition 2.3 we shall see that in this way we cover several well-known function spaces of interest as special cases. Afterwards, we formally introduce the widths measuring the performance of optimal (non-)adaptive algorithms and show how their behaviour at the level of function spaces can be reduced to the much simpler sequence space setting.

2.1 Hyperbolic wavelets and function spaces of hybrid smoothness

Let us recap some basics about hyperbolic wavelets as described in some more detail in [34, Section 4]. Let $\phi$ be a univariate scaling function and $\psi$ the corresponding wavelet which fulfill the following conditions for some $K \in \mathbb{N}_0$:

(i.) $\phi, \psi \in C^K(\mathbb{R})$ with compact support,
(ii.) $\|\phi\|_{L_2(\mathbb{R})} = \|\psi\|_{L_2(\mathbb{R})} = 1$, and
(iii.) $\psi$ has at least $K$ vanishing moments, i.e.

$$\int_{\mathbb{R}} \psi(x) x^b \, dx = 0, \quad b = 0, \ldots, K - 1. \quad \text{(In case of } K = 0, \text{ this condition is void.)}$$

Based on this we define the univariate wavelets via

$$\psi_{j,k} := 2^{-1/2} \psi(2^{j-1} \cdot -k) \quad \text{and} \quad \psi_{0,k} := \phi(\cdot - k), \quad j \in \mathbb{N}, \ k \in \mathbb{Z},$$

such that every $f \in L_2(\mathbb{R})$ has the wavelet expansion

$$\sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}} 2^j \langle f, \psi_{j,k} \rangle_{L_2} \psi_{j,k}.$$
In particular, all required properties are fulfilled by the classical Daubechies wavelets.

For the multivariate case, we let $\mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{N}_0^d$ as well as $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, $d \in \mathbb{N}$ and apply the usual tensor product ansatz to obtain the hyperbolic wavelet functions
\[
\psi^{\mathbf{j},\mathbf{k}}(\mathbf{x}) := \psi_{j_1,k_1}(x_1) \cdots \psi_{j_d,k_d}(x_d), \quad \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d,
\]
that form a basis in $L_2(\mathbb{R}^d)$. Moreover, let $\chi$ denote the characteristic function of $[0,1]$ and
\[
\chi_{j_i,k_i} := \chi(2^{j_i} - k_i), \quad i = 1, \ldots, d,
\]
the characteristic functions of the dyadic intervals $I_{j_i,k_i} := [2^{-j_i}k_i, 2^{-j_i}(k_i + 1)]$. Finally, let
\[
I^{\mathbf{j},\mathbf{k}} := I_{j_1,k_1} \times \cdots \times I_{j_d,k_d} \quad \text{and} \quad \chi^{\mathbf{j},\mathbf{k}}(\mathbf{x}) := \chi_{j_1,k_1}(x_1) \cdots \chi_{j_d,k_d}(x_d).
\]
Then $\text{supp}(\psi^{\mathbf{j},\mathbf{k}}) \subset c \text{supp}(\chi^{\mathbf{j},\mathbf{k}}) = c I^{\mathbf{j},\mathbf{k}}$ with some $c > 0$ independent of $\mathbf{j}$ and $\mathbf{k}$.

For several decades it is well-known that wavelets can be used to describe smoothness and approximation properties of functions and, more general, distributions. Usually, the point of departure is a Fourier-analytic definition of a class of function spaces such as, e.g., the classical Besov or Triebel-Lizorkin spaces $B^{s}_{p,q}(\mathbb{R}^d)$ and $F^{s}_{p,q}(\mathbb{R}^d)$ of isotropic smoothness $s$, respectively, which contains familiar scales like Bessel potential Sobolev spaces $H^{s}_p(\mathbb{R}^d)$ and Hölder-Zygmund spaces $C^{s}_d(\mathbb{R}^d)$ as special cases. We refer to [36] for a detailed discussion. Then a wavelet representation of this class is derived which characterizes the membership of a function in those scales in terms of decay properties of its wavelet coefficients (typically described in terms of sequence spaces). For hyperbolic wavelets and Besov/Triebel-Lizorkin spaces $S^{r}_{p,q}X(\mathbb{R}^d)$ of dominating mixed smoothness $r$, this has been done in [38]. Quite recently, it was found in [34] that exactly the same wavelets can be used to characterize also other anisotropic spaces $\widetilde{X}^{s}_{p,q}(\mathbb{R}^d)$ of Besov- and Triebel-Lizorkin-type which in some cases coincide with the classical (isotropic!) spaces $B^{s}_{p,q}(\mathbb{R}^d)$ and $F^{s}_{p,q}(\mathbb{R}^d)$, respectively. This is surprising, as previously only isotropic wavelets were employed to describe isotropic spaces.

The structural similarity of the representations of $S^{r}_{p,q}X(\mathbb{R}^d)$ and $\widetilde{X}^{s}_{p,q}(\mathbb{R}^d)$ in terms of hyperbolic wavelets inspires the following wavelet-based definition of Besov and Triebel-Lizorkin spaces of hybrid smoothness.

**Definition 2.1.** For $d \in \mathbb{N}$ let $X \in \{B,F\}$, $0 < p, q \leq \infty$ (with $p < \infty$ if $X = F$), and $r,s \in \mathbb{R}$. Further let $\{\psi^{\mathbf{j},\mathbf{k}} \mid \mathbf{j} \in \mathbb{N}_0^d, \mathbf{k} \in \mathbb{Z}^d\}$ be a hyperbolic wavelet system as described above, where $K \in \mathbb{N}_0$ is chosen sufficiently large.

(i.) $H^{r,s}_{p,q}X(\mathbb{R}^d)$ denotes the set of all $f \in S'(\mathbb{R}^d)$ such that
\[
f = \sum_{(\mathbf{j},\mathbf{k}) \in \mathbb{N}_0^d \times \mathbb{Z}^d} a_{\mathbf{j},\mathbf{k}} \psi^{\mathbf{j},\mathbf{k}} \quad \text{(convergence in } S'(\mathbb{R}^d)\text{)}
\]
with (unique) coefficients in

\[ h_{p,q}^{r,s} := \{ a = (a_{j,k})_{j \in \mathbb{N}_0, k \in \mathbb{Z}^d} \subset \mathbb{C} \mid \| a \|_{h_{p,q}^{r,s}} < \infty \}, \]

where

\[
\| f \|_{H_{p,q}^{r,s}(\mathbb{R}^d)} := \| a \|_{h_{p,q}^{r,s}} \\
:= \left\{ \begin{array}{l}
\sum_{j \in \mathbb{N}_0} 2^{q(r-1/p)j_1 + s|j|_{\infty}} \left( \sum_{k \in \mathbb{Z}^d} |a_{j,k}|^p \right)^{q/p}^{1/q}, & x = b, \\
\left( \sum_{j \in \mathbb{N}_0} 2^{q(r|j_1 + s|j|_{\infty})} \left( \sum_{k \in \mathbb{Z}^d} a_{j,k} \chi_{j,k}(\cdot) \right)^q \right)^{1/q} L_p(\mathbb{R}^d), & x = f
\end{array} \right.
\]

(usual modification if \( \max\{p,q\} = \infty \)).

(ii.) Let \( \Omega \subset \mathbb{R}^d \) be open. We then define \( H_{p,q}^{r,s}(\Omega) \) via restrictions, i.e.

\[ H_{p,q}^{r,s}(\Omega) := \{ f \in \mathcal{D}'(\Omega) \mid f = g|_{\Omega} \text{ for some } g \in H_{p,q}^{r,s}(\mathbb{R}^d) \} \]

where

\[
\| f \|_{H_{p,q}^{r,s}(\Omega)} := \inf_{g \in H_{p,q}^{r,s}(\mathbb{R}^d), f=g|_{\Omega}} \| g \|_{H_{p,q}^{r,s}(\mathbb{R}^d)}.
\]

Remark 2.2. Some comments are in order:

(i.) As usual, \( p \) indicates the integrability and \( q \) is a fine index. Moreover, we shall see that, roughly speaking, \( r \) describes the minimal degree of dominating mixed smoothness, while \( s \) measures the minimal isotropic regularity of the functions under consideration.

(ii.) Standard arguments show that the introduced spaces are complete w.r.t. the given quasi-norms.

(iii.) Since we are only interested in approximation properties and algorithms based on a given, fixed system of hyperbolic wavelets, we follow the route taken in [15], avoid the usual fourier-analytic detour, and take the expected outcome of a wavelet characterization as a definition. The drawback of this approach is that the spaces \( H_{p,q}^{r,s}(\mathbb{R}^d) \) formally depend on the concrete choice of the underlying hyperbolic wavelet system. However, Proposition 2.3 below and the analysis in [41] indicate that systems with similar properties most likely will lead to the same spaces (up to equivalent quasi-norms). To keep this paper as short as possible, we leave this point as well as the Littlewood–Paley analysis of \( H_{p,q}^{r,s}(\mathbb{R}^d) \) for further research.
The following important special cases can be identified. Therein, \( H^s_p(\mathbb{R}^d) \) and \( S^r_pH(\mathbb{R}^d) \) denote the classical \( L_p \)-Bessel potential spaces (isotropic Sobolev spaces) and \( L_p \)-Sobolev spaces of dominating mixed smoothness, respectively; see, e.g., [37, (1.17) and (1.41)].

**Proposition 2.3.** Let \( d \in \mathbb{N} \) and \( 0 < p_0, p_1, q_0, q_1 \leq \infty \) (with \( p_0, p_1 < \infty \) for Triebel-Lizorkin spaces), as well as \( r_0, r_1, s_0, s_1 \in \mathbb{R} \). Then we have

(i.) \( H^0_{p_0, q_0} F(\mathbb{R}^d) = F^{s_1}_{p_1, q_1}(\mathbb{R}^d) \) \iff \( p_0 = p_1, \ q_0 = q_1 = 2, \ r_0 = 0, \ \text{and} \ s_0 = s_1. \)

(ii.) \( H^{r_0, s_0} B(\mathbb{R}^d) = B^{s_1}_{p_1, q_1}(\mathbb{R}^d) \) \iff \( p_0 = p_1 = q_0 = q_1 = 2, \ r_0 = 0, \ \text{and} \ s_0 = s_1. \)

(iii.) \( H^{r_0, s_0} F(\mathbb{R}^d) = S^{r_1}_{p_1, q_1} F(\mathbb{R}^d) \) \iff \( p_0 = p_1, \ q_0 = q_1, \ r_0 = r_1, \ \text{and} \ s_0 = 0. \)

(iv.) \( H^{r_0, s_0} B(\mathbb{R}^d) = S^{r_1}_{p_1, q_1} B(\mathbb{R}^d) \) \iff \( p_0 = p_1, \ q_0 = q_1, \ r_0 = r_1, \ \text{and} \ s_0 = 0. \)

Especially, there holds

\[ H^s_p(\mathbb{R}^d) = H^{0, s}_{p, 2} F(\mathbb{R}^d) \] \and \[ S^r_p H(\mathbb{R}^d) = H^{r, 0}_{p, 2} F(\mathbb{R}^d), \quad r, s \in \mathbb{R}, \ 1 < p < \infty. \]

Moreover, all statements remain valid if \( \mathbb{R}^d \) is replaced by some domain \( \Omega \).

**Proof.** Per definition \( h_{p,q}^{r,s} x \) discretizes \( H^{r,s}_{p,q} X(\mathbb{R}^d) \). On the other hand, according to [34, Theorem 4.6 and Remark 7.2] as well as [38, Theorem 2.12], they also describe the classical function spaces mentioned in (i.)–(iv.) provided that the stated conditions are fulfilled. Furthermore, it is well-known that these spaces are different for different parameters. \( \blacksquare \)

For our subsequent analysis we will need sequence spaces associated to hybrid smoothness spaces \( H^{r,s}_{p,q} X(\Omega) \) on domains \( \Omega \subset \mathbb{R}^d \). Therefore, given \( j \in \mathbb{N}_0^d \), we let

\[ \mathfrak{D}_j := \{ \mathbf{k} \in \mathbb{Z}^d \mid \text{supp}(\psi_j^d \mathbf{k}) \cap \Omega \neq \emptyset \} \quad \text{and} \quad \nabla := \{ \lambda = (j, \mathbf{k}) \in \mathbb{N}_0^d \times \mathbb{Z}^d \mid \mathbf{k} \in \mathfrak{D}_j \}. \]

If \( \Omega \) is bounded and contains (a scaled and shifted version of) the unit cube \([0, 1]^d\), we obviously have

\[ |\mathfrak{D}_j| \sim 2^{j_1}, \quad j \in \mathbb{N}_0^d. \]

Based on this assumption we modify the above Definition 2.1 in the following way:

**Definition 2.4.** For \( x \in \{ b, f \}, \ 0 < p, q \leq \infty \) (with \( p < \infty \) if \( x = f \)), and \( r, s \in \mathbb{R} \), we define hybrid sequence spaces \( h_{p,q}^{r,s} x(\nabla) \) as in Definition 2.1 with \( \mathbb{Z}^d \) being replaced by \( \mathfrak{D}_j \).

### 2.2 Quantities of interest

In the course of this paper we shall study the asymptotic rate of convergence of the following three quantities as the number \( m \) of degrees of freedom tends to infinity.
Definition 2.5. Let $A, B$ be quasi-Banach spaces, $I \in \mathcal{L}(A, B)$, and $D := \{ b^\lambda \in B \mid \lambda \in \Lambda \}$ be a dictionary indexed by $\lambda \in \Lambda$. For $m \in \mathbb{N}_0$ we define

(i.) the best $m$-term approximation width

$$
\sigma_m(I; D) := \sigma_m(I: A \to B; D) := \sup_{\|a\| \leq 1} \inf_{|\Lambda_m| \leq m} \inf_{\lambda \in \Lambda_m} \sup_{c_{\lambda} \in \mathbb{C}, c_{\lambda} \in \mathbb{C}} \| Ia - \sum_{\lambda \in \Lambda_m} c_{\lambda} b^\lambda \| B,
$$

(ii.) the $m$-th Kolmogorov dictionary width

$$
d_m(I; D) := d_m(I: A \to B; D) := \inf_{|\Lambda_m| \leq m} \sup_{\|a\| \leq 1} \inf_{c_{\lambda} \in \mathbb{C}, c_{\lambda} \in \mathbb{C}} \| Ia - \sum_{\lambda \in \Lambda_m} c_{\lambda} b^\lambda \| B,
$$

(iii.) the $m$-th non-adaptive algorithm width

$$
\zeta_m(I; D) := \zeta_m(I: A \to B; D) := \inf_{|\Lambda_m| \leq m} \sup_{\|a\| \leq 1} \inf_{\lambda \in \Lambda_m} \| Ia - \sum_{\lambda \in \Lambda_m} c_{\lambda}(a) b^\lambda \| B,
$$

w.r.t. the dictionary $D$.

Since for best $m$-term approximation $\Lambda_m$ and $c_{\lambda}$ may depend on the input $a$ in an arbitrary way, $\sigma_m(I; D)$ reflects how well each individual $Ia$ can be approximated using an optimal linear combination of at most $m$ dictionary elements. Clearly, the collection of all such approximants forms a highly non-linear manifold in the target space $B$. Hence, $\sigma_m(I; D)$ serves as a benchmark for the performance of optimal adaptive algorithms based on $D$. In contrast, the Kolmogorov dictionary widths $d_m(I; D)$ measure the worst case error of approximation within non-adaptively chosen optimal linear subspaces in $B$ spanned by at most $m$ dictionary elements. Finally, $\zeta_m(I; D)$ describes the performance of optimal algorithms that are allowed to evaluate $m$ optimal non-adaptively chosen functionals on the input and compose these pieces of information in a linear way with a fixed collection of no more than $m$ dictionary elements to form an output. Note that these functionals neither have to be linear nor continuous.

Remark 2.6. Let us add some further comments which will be useful later on:

(i.) From Definition 2.5 it is obvious that all three quantities are monotonically non-increasing in $m$ and satisfy

$$
\sigma_m(I; D) \leq d_m(I; D) \leq \zeta_m(I; D), \quad m \in \mathbb{N}_0.
$$

9
(ii.) All quantities defined above are based on a dictionary $D$ which has to be fixed in advance. For the application we have in mind this allows for a fair comparison of adaptive and non-adaptive wavelet algorithms by choosing $D := \Psi$ later on.

(iii.) Let us stress that $d_m(I; D)$ upper bounds the classical Kolmogorov $m$-widths
\[ d_m(I) := d_m(I: A \to B) := \inf_{V \subset B \text{ linear}} \sup_{\|a\| \leq 1} \inf_{v \in V} \|Ia - v\| B, \quad m \in \mathbb{N}_0, \]
whose convergence to zero is known to characterize the compactness of $I \in \mathcal{L}(A; B)$.

(iv.) The sequence of best $m$-term widths yields a so-called pseudo-$s$-scale as introduced by Pietsch [33, Chapter 12] and hence satisfies the multiplicativity assertion
\[ \sigma_m(S \circ I \circ T; S(D)) \leq \|T\|_\mathcal{L}(Z,A) \sigma_m(I; D) \|S\|_\mathcal{L}(B,C), \quad m \in \mathbb{N}_0, \quad (3) \]
as well as the (pre)additivity
\[ \sigma_{m_1 + m_2}(I + J; D) \lesssim \sigma_{m_1}(I; D) + \sigma_{m_2}(J; D), \quad m_1, m_2 \in \mathbb{N}_0, \quad (4) \]
for all quasi-Banach spaces $A, B, C, Z$, as well as operators $I, J \in \mathcal{L}(A, B), T \in \mathcal{L}(Z, A)$, and $S \in \mathcal{L}(B, C)$, respectively; see, e.g., [2, Lemma 6.1] and the references therein.

### 2.3 Reduction to sequence spaces and their embeddings

One of the main tools in our analysis is given by the next Proposition 2.7 which allows to lift results for hybrid sequence spaces (see Definition 2.4) to the level of function spaces introduced in Definition 2.1. For the convenience of the reader, a detailed proof is given in Section A.1 below.

**Proposition 2.7.** For $d \in \mathbb{N}$ let $X, Y \in \{B, F\}$ and $0 < p_0, p_1, q_0, q_1 \leq \infty$ (with $p_0 < \infty$ if $X = F$ and $p_1 < \infty$ if $Y = F$, respectively), as well as $r_0, r_1, s_0, s_1 \in \mathbb{R}$. Then the embedding
\[ \text{Id}: H_{p_0; q_0}^{r_0, s_0} X(\Omega) \to H_{p_1; q_1}^{r_1, s_1} Y(\Omega) \]
is continuous if and only if the same holds true for $\text{id}: h_{p_0; q_0}^{r_0, s_0} x(\nabla) \to h_{p_1; q_1}^{r_1, s_1} y(\nabla)$. In this case,
\[ \sigma_m(\text{Id}; \Psi) \sim \sigma_m(\text{id}; E) \quad \text{and} \quad d_m(\text{Id}; \Psi) \sim \zeta_m(\text{Id}; \Psi) \sim \zeta_m(\text{id}; E) = d_m(\text{id}; E), \quad m \in \mathbb{N}_0, \]
with dictionaries $\Psi := \{\psi^\lambda \mid \lambda \in \nabla\}$ and $E := \{e^\lambda \mid \lambda \in \nabla\}$ consisting of hyperbolic wavelets and unit vectors, respectively.
At the level of sequence spaces, continuous embeddings can be easily proven by standard techniques. We omit details.

**Lemma 2.8** (Continuous embeddings). Let $d \in \mathbb{N}$, as well as $x,y \in \{b,f\}$. Further let $0 < p, p_0, p_1, q, q_0, q_1 \leq \infty$ (with finite integrability for $f$-spaces), $r, r_0, r_1, s, s_0, s_1 \in \mathbb{R}$ and

$$\alpha := r_0 - r_1 - \left( \frac{1}{p_0} - \frac{1}{p_1} \right)_+ \quad \text{and} \quad \beta := s_1 - s_0.\quad (5)$$

(i.) Change of fine parameter: There holds

$$h_{r_0,s_0}^{r,s} x(\nabla) \hookrightarrow h_{r_1,s_1}^{r,s} x(\nabla) \quad \text{if and only if} \quad q_0 \leq q_1.$$

(ii.) Change of type: If $p < \infty$, then

$$h_{r,s}^{r,s} \hookrightarrow h_{r,q}^{r,s} \hookrightarrow h_{r,s}^{r,s} \quad \text{if} \quad \alpha \geq 0 > \beta \quad \text{or} \quad \alpha > \beta \geq 0 \quad \text{or} \quad 0 > \alpha d > \beta.$$

(iii.) Change of integrability and/or smoothness I: We have

$$h_{p_0,q_0}^{r,s} x(\nabla) \hookrightarrow h_{p_1,q_1}^{r,s} y(\nabla) \quad \text{if} \quad \alpha \geq 0 > \beta \quad \text{or} \quad \alpha > \beta \geq 0 \quad \text{or} \quad 0 > \alpha d > \beta.$$

(iv.) Change of integrability and/or smoothness II: Let $\alpha = \beta \geq 0$ or $0 > \alpha d = \beta$. Then

$$h_{p_0,q_0}^{r,s} b(\nabla) \hookrightarrow h_{p_1,q_1}^{r,s} b(\nabla) \quad \text{if and only if} \quad q_0 \leq q_1.$$

We close this section with some final remarks.

**Remark 2.9.**

(i.) The proof of Lemma 2.8 shows that all stated embeddings remain valid if the spaces $h^{r,s}_{p,q} x(\nabla)$ are replaced by corresponding spaces $h^{r,s}_{p,q} x$ associated to function spaces on $\mathbb{R}^d$ provided that we additionally assume $p_0 \leq p_1$.

(ii.) Using Lemma 2.8(i) and (ii), simple examples show that

$$h_{p_0,q_0}^{r_0,s_0} x(\nabla) \hookrightarrow h_{p_1,q_1}^{r_1,s_1} y(\nabla) \quad \text{only if} \quad \begin{cases} \alpha \geq \beta & \text{if } \alpha > 0, \\ \alpha d \geq \beta & \text{else}. \end{cases}$$

In cases of equality further restrictions on the fine parameters might come into play; see, e.g., Lemma 2.8(iv). If, in addition, $p_0 \leq p_1$, these cases can be viewed as generalized Sobolev embeddings, since they particularly cover the classical statements for the ranges of purely isotropic and dominating mixed smoothness spaces, respectively.
(iii.) Finally, let us add some interpretation on the quantities $\alpha$ and $\beta$ in (5). Obviously, $\beta > 0$ is equivalent to a gain of isotropic smoothness. In contrast, $\alpha > 0$ refers to a loss of dominating mixed regularity, regardless of the integrability parameters involved. In combination, this break-of-scale trade-off is exactly the situation we are faced with in applications, where the PDE solutions we like to approximate are known to possess dominating mixed smoothness while errors have to be measured in isotropic energy spaces like $H^1(\Omega)$; cf. Theorem 1.1.

3 Approximation rates in hybrid sequence spaces

In this section, we investigate the asymptotic decay of best $m$-term and $m$-th Kolmogorov dictionary widths, respectively, of the embedding

$$h^r_{p_0,q_0} x(\nabla) \hookrightarrow h^{r_1,s_1}_{p_1,q_1} y(\nabla),$$

where $r_0 - r_1 - \left( \frac{1}{p_0} - \frac{1}{p_1} \right)_+ > s_1 - s_0 > 0$,

of hybrid sequence spaces w.r.t. the dictionary $E := \{ e^\lambda | \lambda \in \nabla \}$ consisting of unit vectors. Note that according to Lemma 2.8 there are more possibilities for continuous embeddings. However, our methods of proof seem to be limited to this most interesting situation; see Remark 2.9(iii). So, we leave the remaining cases open for further research.

3.1 Lower bounds

In order to derive lower bounds for our quantities of interest, we use two different arguments. For best $m$-term widths we employ a factorization technique that allows us to make use of results for embeddings of classical Besov sequence spaces in $d = 1$ stated in [13]. In contrast, for Kolmogorov dictionary widths we explicitly construct fooling sequences.

**Proposition 3.1** (Lower bound, non-linear). Let $d \in \mathbb{N}$ and $x, y \in \{b, f\}$. Further assume $0 < p_0, p_1, q_0, q_1 \leq \infty$ (with $p_0 < \infty$ if $x = f$ and $p_1 < \infty$ if $y = f$, respectively), and $r_0, r_1, s_0, s_1 \in \mathbb{R}$ such that

$$r_0 - r_1 - \left( \frac{1}{p_0} - \frac{1}{p_1} \right)_+ > s_1 - s_0 > 0. \tag{6}$$

Then for all $m \geq m_0$ there holds

$$\sigma_m(\text{id}: h^r_{p_0,q_0} x(\nabla) \rightarrow h^{r_1,s_1}_{p_1,q_1} y(\nabla); E) \gtrsim m^{-[(r_0 - r_1) - (s_1 - s_0)]}.$$
Let us consider the case \( x = y = b \) first. Let \( h_{p,q}^{r,s} b(\nabla) \) denote the subspace of all \( a \in h_{p,q}^{r,s} b(\nabla) \) such that \( a_{j,k} = 0 \) if \( j_i \neq 0 \) for some \( i = 2, \ldots, d \). Then, setting \( \gamma := r + s - 1/2 \), it is obvious that \( h_{p,q}^{r,s} b(\nabla) \) is isometrically isomorphic to the space \( \mathcal{E}_{p,q}^{\gamma} \) of complex sequences \( c = (c_{\nu,k})_{\nu \in \mathbb{N}_0, k \in M_\nu} \) quasi-normed by

\[
\| c \|_{\mathcal{E}_{p,q}^{\gamma}} := \left( \sum_{\nu \in \mathbb{N}_0} 2^{\nu(\gamma + 1/2 - 1/p)\nu} \left[ \sum_{k \in M_\nu} |c_{\nu,k}|^p \right]^{q/p} \right)^{1/q}
\]

with usual modifications for \( \max\{p,q\} = \infty \) and \( |M_\nu| \sim 2^\nu \). Therefore, there exist canonical universal linear restriction and extension operators,

\[
\text{re}: h_{p,q}^{r,s} b(\nabla) \to \mathcal{E}_{p,q}^{r+s-1/2} \quad \text{and} \quad \text{ex}: \mathcal{E}_{p,q}^{r+s-1/2} \to h_{p,q}^{r,s} b(\nabla),
\]

respectively, with norms bounded by one and \( \text{re}(E) \) being the set of unit vectors in \( \mathcal{E}_{p,q}^{\gamma} \). The spaces \( \mathcal{E}_{p,q}^{\gamma} \) arise in the context of (wavelet) discretizations of classical Besov spaces on bounded intervals and hence embeddings as well as corresponding approximation widths are known [13]. In particular, for \( \gamma_0, \gamma_1 \in \mathbb{R} \) and \( 0 < p_0, p_1, q_0, q_1 \leq \infty \) we have

\[
\mathcal{E}_{p_0,q_0}^{\gamma_0} \hookrightarrow \mathcal{E}_{p_1,q_1}^{\gamma_1} \quad \text{if} \quad \gamma_0 - \gamma_1 > \left( \frac{1}{p_0} - \frac{1}{p_1} \right)_{+}.
\]

If now \( \text{id}: h_{p_0,q_0}^{r_0,s_0} b(\nabla) \to h_{p_1,q_1}^{r_1,s_1} b(\nabla) \) denotes our embedding of interest with parameters satisfying (6) and we let \( \gamma_i := r_i + s_i - 1/2, \ i = 0, 1 \), then we obtain the factorization

\[
\text{re} \circ \text{id} \circ \text{ex} = \widetilde{\text{id}}: \mathcal{E}_{p_0,q_0}^{\gamma_0} \to \mathcal{E}_{p_1,q_1}^{\gamma_1},
\]

see Figure 1. Thus, the multiplicativity (3) yields that lower bounds for \( \sigma_m(\text{id}; E) \) are obtained from those for \( \sigma_m(\widetilde{\text{id}}; \text{re}(E)) \) stated in [13, Theorem 7] (with \( d = 1 \)). This proves the assertion for \( x = y = b \).
Note that the result we derived so far does not depend on the fine parameters \( q_i \). Hence, for the general case of embeddings \( \text{id} : h_{p_0,q_0}^r x(\nabla) \to h_{p_1,q_1}^{r_1} y(\nabla) \) we can choose suitable \( q_i \), as well as embeddings \( \text{emb}_i \) such that \( \text{emb}_1 \circ \text{id} \circ \text{emb}_0 = \text{id} \), see Lemma 2.8(i),(ii), and Figure 1 again. So, \( \sigma_m(\text{id}; E) \gtrsim \sigma_m(\text{id}; E) \) which completes the proof in view of our previous considerations.

\[ \begin{array}{c}
\text{Remark 3.2.} \text{ Some comments are in order:} \\
(i.) \text{ If } d = 1, \text{ the presented arguments can also be used to prove matching upper bounds, while for } d > 1 \text{ this approach fails in view of the lack of surjectivity of ex.} \\
(ii.) \text{ Although Proposition 3.1 holds true for } s_1 = s_0 \text{ as well, there are good reasons to assume that sharp lower bounds should contain additional log terms if } d > 1. \\
(iii.) \text{ In arbitrary dimensions our proof technique allows to derive lower bounds for all other pseudo-s-numbers of } \text{id} \text{ (such as, e.g., entropy numbers or Gelfand widths) from known results for } \hat{\text{id}}, \text{ too. In particular, for classical Kolmogorov } m\text{-widths as discussed in Remark 2.6, } [39, \text{Theorem 4.6}] \text{ implies}
\end{array} \]

\[ d_m(\text{id}) \gtrsim m^{-[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)_+]}, \quad m \geq m_0, \]

if, in addition to the assumptions of Proposition 3.1, there holds \( 0 < p_0 \leq p_1 \leq 2 \) or \( p_1 \leq p_0 \).

If we restrict ourselves to the larger Kolmogorov \emph{dictionary} widths \( d_m(\text{id}; E) \), we can get rid of additional assumptions on the relation of the integrability parameters.

\[ \begin{array}{c}
\text{Proposition 3.3 (Lower bound, linear). Let } d \in \mathbb{N} \text{ and } x,y \in \{b,f\}. \text{ Further assume } 0 < p_0,p_1,q_0,q_1 \leq \infty \text{ (with } p_0 < \infty \text{ if } x = f \text{ and } p_1 < \infty \text{ if } y = f, \text{ respectively), and } r_0,r_1,s_0,s_1 \in \mathbb{R} \text{ such that}
\end{array} \]

\[ r_0 - r_1 - \left( \frac{1}{p_0} - \frac{1}{p_1} \right)_+ > s_1 - s_0 > 0. \]

Then for all \( m \geq m_0 \) there holds

\[ d_m(\text{id} : h_{p_0,q_0}^r x(\nabla) \to h_{p_1,q_1}^{r_1} y(\nabla); E) \gtrsim m^{-[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)_+]} \].

\[ \begin{array}{c}
\text{Proof. If } p_1 \leq p_0 \text{ and hence } (1/p_0 - 1/p_1)_+ = 0, \text{ the assertion follows from Proposition 3.1 and (2). For the remaining case } p_0 < p_1 \leq \infty \text{ note that, as } d_m(\text{id}; E) \text{ is monotone in } m, \text{ it suffices to prove the claim for all } m := m(M) := \ceil{c2^M} \text{ with } M \in \mathbb{N} \text{ being large, where}
\end{array} \]
c > 0 is arbitrarily fixed. For each $M \in \mathbb{N}$ let $\mathbf{j}_M^* := (M, 0, \ldots, 0)$. Since by assumption there holds $|\mathcal{D}_j| \sim 2^{|j_1|}$ for every $j \in \mathbb{N}^d$, we can fix $c > 0$ such that

$$|\mathcal{D}_j| > m(M), \quad M \geq M_0.$$ Hence, for every given index collection $\Lambda_m \subset \nabla$ with $|\Lambda_m| \leq m$ we find $\mathbf{k}_M^* \in \mathcal{D}_j^*$ such that $\lambda^*_M := (\mathbf{j}_M^*, \mathbf{k}_M^*) \in \nabla \setminus \Lambda_m$. Then the fooling sequence

$$a^M := C e^{\lambda^*_M}$$

with $C := 2^{-(r_0 - 1/p_0 + s_0)M}$ satisfies

$$\|a^M h_{p_0,q_0}^0 x(\nabla)\| \lesssim \|a^M h_{p_0,q_0} b(\nabla)\| = 2^{(r_0 - 1/p_0)|j^*_M|_1 + s_0 |j^*_M|_\infty} C = 1,$$

where we used $q := \min\{p_0, q_0\} < \infty$ and Lemma 2.8. On the other hand, the so-called lattice property (see Lemma A.1 below) together with $\text{supp}(a^M) \cap \Lambda_m = \emptyset$ yields

$$\inf_{c_{\lambda} \in \mathbb{C}, \lambda \in \Lambda_m} \left\| a^M - \sum_{\lambda \in \Lambda_m} c_{\lambda} e^{\lambda} h_{p_1,1}^{r_1,s_1} y(\nabla) \right\| \gtrsim \inf_{c_{\lambda} \in \mathbb{C}, \lambda \in \Lambda_m} \left\| a^M - \sum_{\lambda \in \Lambda_m} c_{\lambda} e^{\lambda} h_{p_1,\infty}^{r_1,s_1} b(\nabla) \right\|$$

$$= \left\| a^M - \sum_{\lambda \in \Lambda_m} a_{\lambda}^M e^{\lambda} h_{p_1,\infty}^{r_1,s_1} b(\nabla) \right\|$$

$$= \left\| a^M h_{p_1,\infty}^{r_1,s_1} b(\nabla) \right\|$$

$$= C 2^{(r_1 - 1/p_1 + s_1)M}$$

$$\sim m^{-\left[(r_0 - r_1) - (s_1 - s_0) - (1/p_0 - 1/p_1)\right]}.$$ ■

Note that for $p_1 > p_0$ the proof does not require any restriction on the parameters. Further, we like to mention that more advanced fooling sequences can be used to give an alternative proof of Proposition 3.1 for certain parameter constellations.

### 3.2 Upper bounds

We complement the lower bounds obtained by abstract arguments in the previous subsection by the analysis of some explicitly constructed optimal approximation algorithms. In view of Proposition 3.1 and 3.3, there is some hope that optimal linear methods already show the maximal rate of convergence $(r_0 - r_1) - (s_1 - s_0)$. Therefore, we treat such algorithms first, but introduce some technicalities beforehand.
Definition 3.4. Let \( d \in \mathbb{N} \) and \( \mathcal{D}_j \subset \mathbb{Z}^d \) be as above, as well as \( \alpha, \beta \in \mathbb{R} \). We let
\[
\Delta_\mu := \{ j \in \mathbb{N}_0^d \mid \alpha |j|_1 - \beta |j|_\infty \leq \mu \}, \quad \mu \in \mathbb{N}_0.
\]
Further we set \( \mathcal{L}_0 := \Delta_0 \), as well as
\[
\mathcal{L}_\mu := \Delta_\mu \setminus \Delta_{\mu - 1} = \{ j \in \mathbb{N}_0^d \mid \mu - 1 < \alpha |j|_1 - \beta |j|_\infty \leq \mu \}, \quad \mu \in \mathbb{N}.
\]
Finally, let
\[
\nabla_\mu := \{ (j, k) \in \mathbb{N}_0^d \times \mathbb{Z}^d \mid j \in \mathcal{L}_\mu, \ k \in \mathcal{D}_j \}.
\]
We note in passing that \( \mathbb{N}_0^d = \bigcup_{\mu=0}^{\infty} \mathcal{L}_\mu \) and hence \( \nabla = \bigcup_{\mu=0}^{\infty} \nabla_\mu \) (disjoint unions). If we like to stress the dependence on \( \alpha \) and \( \beta \) in the notation, we write \( \Delta_\mu(\alpha, \beta) \), \( \mathcal{L}_\mu(\alpha, \beta) \), and \( \nabla_\mu(\alpha, \beta) \), respectively. Later on, we shall choose these parameters depending on the source and target spaces of the embedding under consideration.

A visualization of the just defined quantities can be found in Figure 2. Therein, the “kink” at the boundary of the shaded area \( \Delta_\mu \) is caused by the interplay of the different vector norms involved and the fact that \( \beta > 0 \) which distinguishes our setting of interest from the purely dominating mixed situation.

Some combinatorics related to the quantities from Definition 3.4 may be found in Section A.3 below. At this point, we shall restrict ourselves to the following observation.

Remark 3.5. If \( \alpha > \beta > 0 \), then \( |\mathcal{D}_j| \sim 2^{|j|_1} \) and Lemma A.6 applied for \( \delta := 1 \) yield
\[
|\nabla_\mu| \sim \sum_{j \in \Delta_\mu} 2^{|j|_1} - \sum_{j \in \Delta_{\mu-1}} 2^{|j|_1} \sim 2^{\mu/(\alpha-\beta)} - 2^{(\mu-1)/(\alpha-\beta)} \sim 2^{\mu/(\alpha-\beta)}
\]
for all \( \mu \geq \alpha - \beta \) (> 0) with constants independent of \( \mu \).
3.2.1 Linear approximation

Let us consider linear algorithms $\mathcal{A}_M$, $M \in \mathbb{N}_0$, of the form

$$\mathcal{A}_M a := \sum_{j \in \Delta_M} \sum_{k \in \mathbb{D}_j} a_{j,k} e^{j,k}, \quad a = \sum_{j \in \mathbb{N}_0^d} \sum_{k \in \mathbb{D}_j} a_{j,k} e^{j,k},$$

where $\Delta_M = \Delta_M(\alpha, \beta)$ is defined as above and $e^{j,k}$ denote the respective unit vectors. In this way, $\mathcal{A}_M$ takes into account the complete first $M + 1$ layers $\mathcal{L}_\mu$ of resolution vectors $\mathbf{j} \in \mathbb{N}_0^d$ which contribute most to the sequence space (quasi-)norm of the input $a$; see Figure 2.

**Proposition 3.6** (Upper bound, linear). For $d \in \mathbb{N}$ let $0 < p_0, p_1, q_1 \leq \infty$, as well as $r_0, r_1, s_0, s_1 \in \mathbb{R}$ such that

$$r_0 - r_1 - \left(\frac{1}{p_0} - \frac{1}{p_1}\right)^+ > s_1 - s_0 > 0.$$

Then there exist constants $M_0, c_1, c_2 > 0$ and a sequence $(\mathcal{A}_M)_{M \in \mathbb{N}}$ of linear algorithms such that for all $M \geq M_0$ and $m := m(M) := \left[ c_1 2^M/[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)+] \right]$ there holds that

$$\|a - \mathcal{A}_M a|_{h^{r_1,s_1}_{p_1,q_1} b(\nabla)} \| \leq c_2 m^{-(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)+} \|a|_{h^{r_0,s_0}_{p_0,\infty} b(\nabla)} \|, \quad a \in h^{r_0,s_0}_{p_0,\infty} b(\nabla),$$

and $\mathcal{A}_M$ uses at most $m$ degrees of freedom.

**Proof.** It suffices to consider the case $p_1 \geq p_0$ since otherwise Lemma 2.8(iv) allows to reduce to this situation as follows:

$$\left|a - \mathcal{A}_M a|_{h^{r_1,s_1}_{p_1,q_1} b(\nabla)} \right| \lesssim \left|a - \mathcal{A}_M a|_{h^{r_1,s_1}_{p_0,q_1} b(\nabla)} \right| \lesssim m^{-(r_0-r_1)-(s_1-s_0)} \|a|_{h^{r_0,s_0}_{p_0,\infty} b(\nabla)} \|$$

for all $a \in h^{r_0,s_0}_{p_0,\infty} b(\nabla)$.

So let $p_1 \geq p_0$. In addition, we may assume that $p_1, q_1 < \infty$ as the remaining cases can be obtained by obvious modifications. Choose $\varepsilon \in (0, s_1 - s_0)$ and define

$$\alpha := r_0 - r_1 - \left(\frac{1}{p_0} - \frac{1}{p_1}\right) - \varepsilon \quad \text{as well as} \quad \beta := s_1 - s_0 - \varepsilon$$

such that $\alpha > \beta > 0$. Based on this, we let $\Delta_M := \Delta_M(\alpha, \beta)$ and consider the linear algorithms $\mathcal{A}_M$, $M \in \mathbb{N}_0$, as defined in (7). If $M \geq M_0 := \lceil \alpha - \beta \rceil$, then $\mathcal{A}_M$ uses

$$\sum_{j \in \Delta_M} |\mathcal{D}_j| \sim \sum_{j \in \Delta_M} 2^{j_1} \sim 2^{M/[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)+]} = 2^{M/[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)+]}$$
degrees of freedom, where the implied constants are independent of $M$; see Lemma A.6. Given $a \in h_{p_0,\infty}^{r_0,s_0} b(\nabla)$, for its error $E_1(a) := a - \mathcal{A}_M a$ there holds

$$\| E_1(a) \| h_{p_1,q_1}^{r_1,s_1} b(\nabla) \|
= \left\| \sum_{j \in (\Delta M)^C} 2^{q_1 \left( (r_1-1/p_1) |j|_1 + s_1 |j|_\infty \right)} \left( \sum_{k \in \mathcal{D}_j} |a_{j,k}| \right)^{q_1/p_1} \right\|^{1/q_1}
= \left\| \sum_{\mu = M+1}^\infty \sum_{j \in \mathcal{E}_\mu} 2^{-q_1 \epsilon (|j|_1 - |j|_\infty)} 2^{q_1 \left( (r_0-1/p_0) |j|_1 + s_0 |j|_\infty \right)} \left( \sum_{k \in \mathcal{D}_j} |a_{j,k}| \right)^{q_1/p_1} \right\|^{1/q_1}
\lesssim \left\| \sum_{\mu = M+1}^\infty \sum_{j \in \mathcal{E}_\mu} 2^{-q_1 \epsilon (|j|_1 - |j|_\infty)} 2^{q_1 \left( (r_0-1/p_0) |j|_1 + s_0 |j|_\infty \right)} \left( \sum_{k \in \mathcal{D}_j} |a_{j,k}| \right)^{p_0} \right\|^{q_1/p_1} \left\| a \right\| h_{p_0,\infty}^{r_0,s_0} b(\nabla) \|
\leq \left\| a \right\| h_{p_0,\infty}^{r_0,s_0} b(\nabla) \| .$$

Moreover, Lemma A.7 yields

$$\sum_{\mu = M+1}^\infty 2^{-q_1 \epsilon (|j|_1 - |j|_\infty)} \lesssim \sum_{\mu = M+1}^\infty 2^{-q_1 \epsilon \mu} \sim 2^{-q_1 M}$$

such that finally

$$\| E_1(a) \| h_{p_1,q_1}^{r_1,s_1} b(\nabla) \| \lesssim 2^{-M} \| a \| h_{p_0,\infty}^{r_0,s_0} b(\nabla) \| \sim m^{-\left( (r_0-r_1) - (s_1-s_0) - (1/p_0 - 1/p_1) \right)} \| a \| h_{p_0,\infty}^{r_0,s_0} b(\nabla) \|. \quad (8)$$
Using the embeddings from Lemma 2.8, we can replace the source space \( h^{r_0, s_0}_{p_0, \infty} b(\nabla) \) by an arbitrary \( b \)- or \( f \)-space of same smoothness and integrability. The same applies for the target space \( h^{r_1, s_1}_{p_1, q_1} b(\nabla) \). Together with the monotonicity of the Kolmogorov dictionary widths this proves

**Corollary 3.7.** For \( d \in \mathbb{N} \) let \( x, y \in \{b, f\} \), \( 0 < p_0, p_1, q_0, q_1 \leq \infty \), and \( r_0, r_1, s_0, s_1 \in \mathbb{R} \) s.t.

\[
  r_0 - r_1 - \left( \frac{1}{p_0} - \frac{1}{p_1} \right)_+ > s_1 - s_0 > 0
\]

(with \( p_0 < \infty \) if \( x = f \) and \( p_1 < \infty \) if \( y = f \), respectively). Then

\[
d_m(\text{id}: h^{r_0, s_0}_{p_0, q_0} x(\nabla) \rightarrow h^{r_1, s_1}_{p_1, q_1} y(\nabla); E) \lesssim m^{-[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)_+]}, \quad m \geq m_0.
\]

In particular, the embedding \( h^{r_0, s_0}_{p_0, q_0} x(\nabla) \hookrightarrow h^{r_1, s_1}_{p_1, q_1} y(\nabla) \) is compact.

**Remark 3.8.** Corollary 3.7 implies several optimality statements.

(i.) From the lattice property (Lemma A.1) it follows that on the level of sequence spaces the Kolmogorov dictionary widths \( d_m(\text{id}; E) \) dominate classical approximation numbers

\[
a_m(\text{id}) := a_m(\text{id}: h^{r_0, s_0}_{p_0, q_0} x(\nabla) \rightarrow h^{r_1, s_1}_{p_1, q_1} y(\nabla))
\]

\[
:= \inf_{A_m \in \mathcal{A}} \sup_{h^{r_0, s_0}_{p_0, q_0} x(\nabla), h^{r_1, s_1}_{p_1, q_1} y(\nabla)} \sup_{\text{rank}(A_m) \leq m} \|a - A_m a\| h^{r_1, s_1}_{p_1, q_1} y(\nabla)\|
\]

which in turn are general upper bounds for usual Kolmogorov m-widths \( d_m(\text{id}) \); see [33, Chapter 11]. Combined with the lower bounds mentioned in Remark 3.2 we thus have

\[
a_m(\text{id}) \sim d_m(\text{id}) \sim d_m(\text{id}; E) \sim m^{-[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)_+]}, \quad m \geq m_0,
\]

if either \( 0 < p_0 \leq p_1 \leq 2 \) or \( p_1 \leq p_0 \).

(ii.) Note that in view of Proposition 3.3 the rate found in Corollary 3.7 is sharp.

(iii.) Due to (2) we can conclude from Corollary 3.7 that

\[
\sigma_m(\text{id}: h^{r_0, s_0}_{p_0, q_0} x(\nabla) \rightarrow h^{r_1, s_1}_{p_1, q_1} y(\nabla); E) \lesssim m^{-[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)_+]}, \quad m \geq m_0,
\]

which according to Proposition 3.1 is optimal if \( p_1 \leq p_0 \). That is, in this regime there is no need for non-linear algorithms, since the linear approximation by \( \mathcal{A}_L \) (as constructed in the proof of Proposition 3.6) is already best possible. However, this becomes false for \( p_0 < p_1 \) as we shall see below.
3.2.2 Non-linear approximation

In order to improve the speed of convergence if $p_0 < p_1$, we will approximate given sequences through non-linear algorithms $B_M$, $M \in \mathbb{N}_0$, of the form

$$B_M(a) := A_M a + \sum_{\mu=M+1}^{N_M} \sum_{\lambda \in \Lambda_{M,\mu}} a_\lambda e^\lambda$$

with some $N_M > M$ and subsets $\Lambda_{M,\mu} \subseteq \nabla_\mu = \{ \lambda = (j,k) \in \mathbb{N}_0^d \times \mathbb{Z}_o^d \mid j \in \mathcal{L}_\mu, k \in \mathcal{D}_j \}$ indicating the most important coefficients of $a$ at layer $\mathcal{L}_\mu$, $\mu = M + 1, \ldots, N_M$. Therein, $\Lambda_{M,\mu} := \{ \varphi_\mu(n) \mid n = 1, \ldots, m_{M,\mu} \}$ with some $m_{M,\mu} \in \mathbb{N}$ (to be specified later) and a bijection $\varphi_\mu : \{1,2,\ldots,|\nabla_\mu|\} \rightarrow \nabla_\mu$ which yields a non-increasing rearrangement of the weighted coefficients portion

$$\left(2^{-|j_1|}2^{(r_0-1)p_0}|j_1+s_0||a_j,1|\right)_{(j,k) \in \nabla_\mu}$$

with some $\varepsilon > 0$. That is,

$$2^{-|j_{\varphi_\mu(n)}|}2^{(r_0-1)p_0}|j_{\varphi_\mu(n)}+s_0|a_{\varphi_\mu(n)}| \geq 2^{-|j_{\varphi_\mu(n+1)}|}2^{(r_0-1)p_0}|j_{\varphi_\mu(n+1)}+s_0|a_{\varphi_\mu(n+1)}|,$$

where $j_{\varphi_\mu(n)} \in \mathcal{L}_\mu$ denotes the projection of $\varphi_\mu(n) = (j_{\varphi_\mu(n)}, k_{\varphi_\mu(n)}) \in \nabla_\mu$ to its first component. Hence, at first $B_M$ takes into account the first full $M+1$ layers of resolution (linear approximation) followed by a sparse non-linear correction based on information from some subsequent layers $\mathcal{L}_\mu$; see Figure 2 again.

Since for $p_0 \geq p_1$ the linear algorithm is already optimal, it suffices to consider $p_0 < p_1$.

Proposition 3.9 (Upper bound, non-linear). For $d \in \mathbb{N}$ let $0 < p_0, p_1, q_1 \leq \infty$ with $p_0 < p_1$, as well as $r_0, r_1, s_0, s_1 \in \mathbb{R}$ such that

$$r_0 - r_1 - \left(\frac{1}{p_0} - \frac{1}{p_1}\right) > s_1 - s_0 > 0.$$

Then there exist constants $M'_0$, $c'_1, c'_2 > 0$ and a sequence $(B_M)_{M \in \mathbb{N}}$ of non-linear algorithms such that for all $M \geq M'_0$ and $m := m(M) := \left[c'_1 2^{M/[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)]}\right]$ there holds

$$\|a - B_M(a)\|_{L^{r_1,s_1}(\nabla)} \leq c'_2 m^{-[(r_0-r_1)-(s_1-s_0)]} \|a\|_{L^{r_0,s_0}(\nabla)}, \quad a \in L^{r_0,s_0}(\nabla),$$

and $B_M$ uses at most $m$ degrees of freedom.
Proof. Once more, choose $\varepsilon \in (0, s_1 - s_0)$ and fix

$$\alpha := r_0 - r_1 - \left(\frac{1}{p_0} - \frac{1}{p_1}\right) - \varepsilon \quad \text{as well as} \quad \beta := s_1 - s_0 - \varepsilon$$

such that $\alpha > \beta > 0$. Then, as $p_0 < p_1$ and $M$ is supposed to be large, we have

$$N_M := \left\lfloor \frac{(r_0 - r_1) - (s_1 - s_0)}{\alpha - \beta} \right\rfloor > M$$

and we can choose $\kappa$ such that

$$\frac{1}{\alpha - \beta} < \kappa < \frac{1}{\alpha - \beta} + \frac{1}{1/p_0 - 1/p_1} = \frac{1}{\alpha - \beta} \left(1 + \frac{\alpha - \beta}{1/p_0 - 1/p_1}\right). \quad (10)$$

Finally, we let

$$m_{M, \mu} := \left\lceil C 2^{\kappa M + (1/(\alpha - \beta) - \kappa)\mu} \right\rceil, \quad \mu = M + 1, \ldots, N_M,$$

where $C > 0$ is chosen such that (with $\nabla_\mu = \nabla_\mu(\alpha, \beta)$ as defined above) there holds

$$m_{M, \mu} \leq C 2^{\kappa M + (1/(\alpha - \beta) - \kappa)\mu} + 1 < C 2^{\mu/(\alpha - \beta)} + 1 \leq |\nabla_\mu| \sim 2^{\mu/(\alpha - \beta)},$$

see Remark 3.5. Then the lower bound on $\kappa$ implies

$$\sum_{\mu = M+1}^{N_M} m_{M, \mu} \lesssim N_M + 2^{\kappa M} \sum_{\mu = M+1}^{N_M} 2^{(1/(\alpha - \beta) - \kappa)\mu} \sim N_M + 2^{\kappa M} 2^{(1/(\alpha - \beta) - \kappa)M} \lesssim 2^{M/(\alpha - \beta)}$$

which together with Proposition 3.6 proves that $B_M$ as defined in (9) uses not more than

$$m = \left\lceil c_1' 2^{M/(\alpha - \beta)} \right\rceil$$

coefficients of $a$.

We are left with bounding the error

$$a - B_M(a) = \sum_{j \in (\Delta N_M)^{\mathbb{C}}} \sum_{k \in \mathbb{D}_j} a_{j,k} e^{j,k} + \sum_{\mu = M+1}^{N_M} \sum_{(j,k) \in \nabla_\mu \setminus \Lambda_{M,\mu}} a_{j,k} e^{j,k}, \quad (11)$$

where $e^{j,k}$ again denote the respective unit vectors. For the tail, i.e. the first sum in (11), denoted by $E_1(a)$, we can employ (8) with $M$ replaced by $N_M$ to conclude

$$\left\| E_1(a) \right\|_{h_{r_0,s_0}^{r_1,s_1} b(\nabla)} \lesssim 2^{-N_M} \left\| a \right\|_{h_{p_0,0}^{r_0,s_0} b(\nabla)} \lesssim (2^{M/(\alpha - \beta)})^{-[(r_1 - r_0) - (s_1 - s_0)]} \left\| a \right\|_{h_{p_0,0}^{r_0,s_0} b(\nabla)}.$$
Hence, it remains to bound the second sum in (11) which we call $E_2(a)$. To do so, let us rewrite the norm in the target space for arbitrary sequences $c = (c_{j,k})_{(j,k) \in \mathbb{N}} \in h_{p_1,q_1}^{r_1,s_1} b(\mathbb{N})$ as follows. For the ease of presentation, w.l.o.g. we once more assume that $p_1, q_1 < \infty$.

$$
\| c \|_{h_{p_1,q_1}^{r_1,s_1} b(\mathbb{N})}^{q_1}
= \sum_{\mu=0}^{\infty} \sum_{j \in \Sigma_\mu} 2^{q_1 \left( (r_0-1/p_1)|j|_1 + s_1 |j|_\infty \right)} \left[ \sum_{k \in \mathbb{D}_j} |c_{j,k}|^{p_1} \right]^{q_1/p_1}
= \sum_{\mu=0}^{\infty} \sum_{j \in \Sigma_\mu} 2^{-q_1 \left( (r_0-1/p_1)-1/p_1 \right)|j|_1 - (s_1-\epsilon)|j|_\infty \right)} 2^{-q_1(|j|_1-|j|_\infty)\epsilon} \cdot 2^{q_1 \left( (r_0-1/p_0)|j|_1 + s_0 |j|_\infty \right)} \left[ \sum_{k \in \mathbb{D}_j} \left( 2^{-\left( |j|_1-|j|_\infty \right)\epsilon/2} 2^{(r_0-1/p_0)|j|_1 + s_0 |j|_\infty |c_{j,k}|} \right)^{p_1} \right]^{q_1/p_1}
$$

Setting $c := E_2(a)$, we can reduce the first sum to $\mu = M+1, \ldots, N_M$ and use Stechkin's Lemma A.4 in order to bound the most inner sum for every fixed $j \in \Sigma_\mu$ by

$$
\left[ \sum_{k \in \mathbb{D}_j} \left( 2^{-\left( |j|_1-|j|_\infty \right)\epsilon/2} 2^{(r_0-1/p_0)|j|_1 + s_0 |j|_\infty |c_{j,k}|} \right)^{p_1} \right]^{q_1/p_1}
\leq \left[ \sum_{(j,k) \in \mathbb{D}_j \setminus \Lambda_M,\mu} \left( 2^{-\left( |j|_1-|j|_\infty \right)\epsilon/2} 2^{(r_0-1/p_0)|j|_1 + s_0 |j|_\infty |a_{j,k}|} \right)^{p_1} \right]^{q_1/p_1}
\leq (|\Lambda_M,\mu| + 1)^{-q_1(1/p_0-1/p_1)} \left[ \sum_{(j,k) \in \mathbb{D}_j \setminus \Lambda_M,\mu} \left( 2^{-\left( |j|_1-|j|_\infty \right)\epsilon/2} 2^{(r_0-1/p_0)|j|_1 + s_0 |j|_\infty |a_{j,k}|} \right)^{p_0} \right]^{q_1/p_0}
\leq m^{-q_1(1/p_0-1/p_1)} \left[ \sum_{j \in \Sigma_\mu} 2^{-p_0 \left( |j|_1-|j|_\infty \right)\epsilon/2} \left( 2^{(r_0-1/p_0)|j|_1 + s_0 |j|_\infty} \left[ \sum_{k \in \mathbb{D}_j} |a_{j,k}|^{p_0} \right]^{1/p_0} \right)^{p_0} \right]^{q_1/p_0}.
$$
Now Lemma A.7 yields
\[
\left[ \sum_{k \in \mathcal{D}_j} \left( 2^{-(|j|_1 - |\bar{j}|_\infty) \varepsilon / 2} \sum_{j = \mathcal{L}_\mu} 2^{-|\bar{j}|_1} \right)^{p_1} \right]^{q_1 / p_1} 
\]
\[
\leq m_{M, \mu}^{-q_1(1/p_0 - 1/p_1)} \left[ \sum_{j \in \mathcal{L}_\mu} 2^{-|\bar{j}|_1} \right]^{q_1 / p_0} \| a \|_{H_{p_0, \infty}^0} \| h_{p_0, \infty}^{\beta_0, s_0} b(\nabla) \|^{q_1}
\]
\[
\lesssim m_{M, \mu}^{-q_1(1/p_0 - 1/p_1)} \| a \|_{H_{p_0, \infty}^0} \| h_{p_0, \infty}^{\beta_0, s_0} b(\nabla) \|^{q_1}, \quad j \in \mathcal{L}_\mu.
\]
Therefore, we can conclude
\[
\left\| E_2(a) \right\|_{H_{p_1, q_1}^{\beta_0, s_1} b(\nabla)}^{q_1} \lesssim \sum_{\mu = M + 1}^{N_M} 2^{-q_1 \mu} \sum_{j \in \mathcal{L}_\mu} 2^{-q_1(1/p_0 - 1/p_1)} m_{M, \mu}^{-q_1(1/p_0 - 1/p_1)} \| a \|_{H_{p_0, \infty}^0} \| h_{p_0, \infty}^{\beta_0, s_0} b(\nabla) \|^{q_1}
\]
\[
\lesssim \| a \|_{H_{p_0, \infty}^0} \| h_{p_0, \infty}^{\beta_0, s_0} b(\nabla) \|^{q_1} \sum_{\mu = M + 1}^{N_M} 2^{-q_1 \mu} m_{M, \mu}^{-q_1(1/p_0 - 1/p_1)},
\]
where we once again used Lemma A.7. Finally, in view of the definition of \( m_{M, \mu} \) and the upper bound on \( \kappa \) from (10), we see that the remaining sum can be estimated by
\[
\sum_{\mu = M + 1}^{N_M} 2^{-q_1 \mu} m_{M, \mu}^{-q_1(1/p_0 - 1/p_1)} \sim \sum_{\mu = M + 1}^{N_M} 2^{-q_1 \mu + \kappa M + (1/(\alpha - \beta) - \kappa) \mu} [-q_1(1/p_0 - 1/p_1)]
\]
\[
= 2^{-q_1 \kappa M(1/p_0 - 1/p_1)} \sum_{\mu = M + 1}^{N_M} 2^{-q_1 \mu \left(1 + \frac{1}{(\alpha - \beta) - \kappa} (1/p_0 - 1/p_1)\right)}
\]
\[
\sim 2^{-q_1 \kappa M(1/p_0 - 1/p_1)} 2^{-q_1 M \left(1 + \frac{1}{(\alpha - \beta) - \kappa} (1/p_0 - 1/p_1)\right)}
\]
\[
= 2^{-q_1 M \left(1 + \frac{1}{(\alpha - \beta)} (1/p_0 - 1/p_1)\right)}
\]
\[
= \left[2^{M/(\alpha - \beta)}\right]^{-[(r_0 - r_1) - (s_0 - s_1)]} q_1
\]
\[
\sim m^{-[(r_0 - r_1) - (s_0 - s_1)]} q_1.
\]
In conclusion, we have shown that
\[
\left\| E_2(a) \right\|_{H_{p_1, q_1}^{\beta_0, s_1} b(\nabla)} \lesssim m^{-[(r_0 - r_1) - (s_0 - s_1)]} \| a \|_{H_{p_0, \infty}^0} \| h_{p_0, \infty}^{\beta_0, s_0} b(\nabla) \|
\]
and the proof is complete. \(\blacksquare\)
Remark 3.10. The presentation of the proof above is a matter of taste. According to Remark 2.6(iv) it would also be possible to apply the (pre)additivity of $\sigma_m$ stated in (4) in connection with Maiorov’s discretization technique [29] on the level of pseudo-s-numbers, in order to conclude the proof with a more abstract presentation. In the context of Weyl and Bernstein numbers, this approach has been used, e.g., in [30].

Similar to Corollary 3.7 we derive the following Corollary 3.11 which is optimal in view of Proposition 3.1.

Corollary 3.11. For $d \in \mathbb{N}$ let $x, y \in \{b, f\}$, $0 < p_0, p_1, q_0, q_1 \leq \infty$, and $r_0, r_1, s_0, s_1 \in \mathbb{R}$ s.t.

$$r_0 - r_1 - \left(\frac{1}{p_0} - \frac{1}{p_1}\right)_+ > s_1 - s_0 > 0$$

(with $p_0 < \infty$ if $x = f$ and $p_1 < \infty$ if $y = f$, respectively). Then

$$\sigma_m(\text{id}; h^{r_0,s_0}_{p_0,q_0}x(\nabla) \to h^{r_1,s_1}_{p_1,q_1}y(\nabla); E) \lesssim m^{-[(r_0-r_1)-(s_1-s_0)]}, \quad m \geq m_0.$$ 

Proof. If $p_0 \geq p_1$, the assertion follows from Remark 3.8(iii). If otherwise $p_0 < p_1$, we employ Proposition 3.9 together with Lemma 2.8 and the monotonicity of $\sigma_m(\text{id}; E)$.

\[\blacksquare\]

4 Approximation rates in function spaces

We are now able to formulate our main result, i.e., transfer the assertions from Section 3 to the level of function spaces of hybrid smoothness.

Theorem 4.1. For $d \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $X, Y \in \{B, F\}$ and $0 < p_0, p_1, q_0, q_1 \leq \infty$ (with $p_0 < \infty$ if $X = F$ and $p_1 < \infty$ if $Y = F$, respectively), as well as $r_0, r_1, s_0, s_1 \in \mathbb{R}$ such that

$$r_0 - r_1 - \left(\frac{1}{p_0} - \frac{1}{p_1}\right)_+ > s_1 - s_0 > 0.$$

Then the embedding $\text{Id} : H^{r_0,s_0}_{p_0,q_0}X(\Omega) \to H^{r_1,s_1}_{p_1,q_1}Y(\Omega)$ is compact and for some $M_0, c, c' > 0$ there exist sequences of algorithms $(B_M)_{M \in \mathbb{N}}$ and $(A_M)_{M \in \mathbb{N}}$ such that

(i.) for all $M \geq M_0$ and $m := m(M) := \left\lceil c 2^{M/[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)_+]} \right\rceil$ there holds

$$\sigma_m(\text{id}; \Psi) \sim \sup_{\|f\|_{H^{r_0,s_0}_{p_0,q_0}X(\Omega)} \leq 1} \|f - B_M(f)\|_{H^{r_1,s_1}_{p_1,q_1}Y(\Omega)} \sim m^{-[(r_0-r_1)-(s_1-s_0)]}.$$

For each input, $B_M$ produces a linear combination of at most $m$ adaptively chosen elements from the (hyperbolic wavelet) dictionary $\Psi := \{\psi^\lambda|\lambda \in \nabla\}$.  

24
The proof of Proposition 2.7 given below is based on the so-called lattice property of our unit vectors at the level of sequence spaces and the discrete analogue of \( \text{id} \). Inserting the upper bound of \( \sigma \) in Proposition 2.7 it was shown that

\[
\zeta_m(\text{id}; \Psi) \sim d_m(\text{id}; \Psi) \sim \sup_{\|f\|_{H^{r_0,s_0}_{p_0,q_0}X(\Omega)}} \|f - \mathcal{A}_M(f)\|_{H^{r_1,s_1}_{p_1,q_1}Y(\Omega)} \sim m^{-[(r_0-r_1)-(s_1-s_0)-(1/p_0-1/p_1)_+]}.
\]

For each input, \( \mathcal{A}_M \) uses a linear combination of the same at most \( m \) elements from \( \Psi \).

Proof. In Proposition 2.7 it was shown that \( \sigma_m(\text{id}; \Psi) \sim \sigma_m(\text{id}; E) \), where \( E \) is the set of unit vectors at the level of sequence spaces and \( \text{id} : h^{r_0,s_0}_{p_0,q_0}x(\nabla) \to h^{r_1,s_1}_{p_1,q_1}y(\nabla) \) denotes the discrete analogue of \( \text{id} \). Inserting the the upper bound of \( \sigma_m(\text{id}; E) \) from Corollary 3.11 and the lower bound from Proposition 3.1 proves (i.). Assertion (ii.) is shown likewise using Corollary 3.7 and Proposition 3.3, respectively. Finally, compactness of \( \text{id} \) follows from Remark 2.6(iii).

Note that this assertion combined with Proposition 2.3 especially proves Theorem 1.1. The remarks given there apply likewise for the more general situation of Theorem 4.1.

Appendix A

A.1 Proof of Proposition 2.7

The proof of Proposition 2.7 given below is based on the so-called lattice property of our sequence spaces and a carefully chosen non-linear extension operator \( \mathcal{E}^* \) from \( \Omega \) to \( \mathbb{R}^d \) which is inspired by ideas in [23, Section 4.6.6].

Lemma A.1 (Lattice property). Let \( y \in \{b, f\}, 0 < p, q < \infty \) (with \( p < \infty \) if \( y = f \)) and \( r, s \in \mathbb{R} \). Then for all \( \Lambda \subset \nabla \)

\[
\inf_{c_\lambda \in \mathbb{C}, \lambda \in \Lambda} \left\| a - \sum_{\lambda \in \Lambda} c_\lambda e^\lambda \right\|_{h^{r,s}_{p,q}y(\nabla)} = \left\| a - \sum_{\lambda \in \Lambda} a_\lambda e^\lambda \right\|_{h^{r,s}_{p,q}y(\nabla)}, \quad a = (a_\lambda)_{\lambda \in \nabla} \in h^{r,s}_{p,q}y(\nabla).
\]

Proof. The non-trivial estimate follows from the lattice structure of the spaces \( s : = h^{r,s}_{p,q}y(\nabla) \):

\[
\left\| a - \sum_{\lambda \in \Lambda} c_\lambda e^\lambda \right\|_{J} = \left\| \left( a - \sum_{\lambda \in \Lambda} a_\lambda e^\lambda \right) + \left( \sum_{\lambda \in \Lambda} (a_\lambda - c_\lambda) e^\lambda \right) \right\|_{J} \geq \left( a - \sum_{\lambda \in \Lambda} a_\lambda e^\lambda \right),
\]

for every choice of \( c_\lambda, \lambda \in \Lambda \). \( \blacksquare \)
Let us first bound the quantities of interest on the level of function spaces by corresponding ones for hybrid sequence spaces.

**Lemma A.2.** For \( d \in \mathbb{N} \) let \( x, y \in \{ b, f \} \) and \( 0 < p_0, p_1, q_0, q_1 \leq \infty \) (with \( p_0 < \infty \) if \( x = f \) and \( p_1 < \infty \) if \( y = f \)), as well as \( r_0, r_1, s_0, s_1 \in \mathbb{R} \) be such that \( id: h^{r_0, s_0}_{p_0, q_0} x(\nabla) \hookrightarrow h^{r_1, s_1}_{p_1, q_1} y(\nabla) \). Then \( Id: H^{r_0, s_0}_{p_0, q_0} X(\Omega) \hookrightarrow H^{r_1, s_1}_{p_1, q_1} Y(\Omega) \) and

\[
\zeta_m(\text{Id}; \Psi) \lesssim d_m(\text{Id}; E) \quad \text{as well as} \quad \sigma_m(\text{Id}; \Psi) \lesssim \sigma_m(\text{Id}; E), \quad m \in \mathbb{N}_0.
\]

**Proof.** For every \( f \in H^{r,s}_{p,q} X(\Omega) \) there exists an extension \( F = \sum_{\lambda \in \mathbb{N}_0^d \times \mathbb{Z}^d} a_\lambda \psi^\lambda \in H^{r,s}_{p,q} X(\mathbb{R}^d) \) with \( \frac{1}{2} \| F \|_{H^{r,s}_{p,q} X(\mathbb{R}^d)} \leq \| f \|_{H^{r,s}_{p,q} X(\Omega)} \). Setting \( a_f := (a_\lambda)_{\lambda \in \nabla} \), we can thus define another, local, but possibly non-linear extension to \( f \),

\[
\delta^*(f) := \sum_{\lambda \in \nabla} a_\lambda \psi^\lambda \in H^{r,s}_{p,q} X(\mathbb{R}^d), \tag{12}
\]

such that with constants independent of \( f \) there holds

\[
\| f \|_{H^{r,s}_{p,q} X(\Omega)} \gtrsim \| \delta^*(f) \|_{H^{r,s}_{p,q} X(\mathbb{R}^d)} = \| a_f \|_{H^{r,s}_{p,q} X(\nabla)} . \tag{13}
\]

If \( id \in \mathcal{L}(h^{r_0, s_0}_{p_0, q_0} x(\nabla), h^{r_1, s_1}_{p_1, q_1} y(\nabla)) \) and \( f \in H^{r_0, s_0}_{p_0, q_0} X(\Omega) \), then \( \delta^*(f) \in \mathcal{S}'(\mathbb{R}^d) \) provides an extension to it for which

\[
\| \delta^*(f) \|_{H^{r_1, s_1}_{p_1, q_1} Y(\mathbb{R}^d)} = \| a_f \|_{h^{r_1, s_1}_{p_1, q_1} \langle \nabla \rangle} \lesssim \| a_f \|_{h^{r_0, s_0}_{p_0, q_0} X(\nabla)} \lesssim \| f \|_{H^{r_0, s_0}_{p_0, q_0} X(\Omega)}
\]

is finite. Hence, \( f \in H^{r_1, s_1}_{p_1, q_1} Y(\Omega) \) and \( \| f \|_{H^{r_1, s_1}_{p_1, q_1} Y(\Omega)} \lesssim \| f \|_{H^{r_0, s_0}_{p_0, q_0} X(\Omega)} \).

Now let \( \Lambda_m \subset \nabla \) with \( |\Lambda_m| \leq m \) be arbitrarily fixed. Further let us choose continuous linear functionals \( c_\lambda^* \in H^{r,s}_{p,q} X(\mathbb{R}^d)' \) such that \( c_\lambda^*(\psi^\varepsilon) = \delta_\lambda \varepsilon \) for all \( \lambda, \varepsilon \in \nabla \) and define \( \tilde{c}_\lambda := c_\lambda^* \circ \delta^* : H^{r,s}_{p,q} X(\Omega) \rightarrow \mathbb{C} \). Then for all \( f \in H^{r_0, s_0}_{p_0, q_0} X(\Omega) \) there holds

\[
\tilde{c}_\lambda(f) = c_\lambda^* \left( \sum_{\varepsilon \in \nabla} a_\varepsilon \psi^\varepsilon \right) = a_\lambda, \quad \lambda \in \nabla,
\]

with \( a_f = (a_\lambda)_{\lambda \in \nabla} \in h^{r_0, s_0}_{p_0, q_0} x(\nabla) \) as in (12) and hence

\[
\left\| f - \sum_{\lambda \in \Lambda_m} \tilde{c}_\lambda(f) \psi^\lambda \right\|_{H^{r_1, s_1}_{p_1, q_1} Y(\Omega)} \leq \left\| \delta^*(f) - \sum_{\lambda \in \Lambda_m} \tilde{c}_\lambda(f) \psi^\lambda \right\|_{H^{r_1, s_1}_{p_1, q_1} Y(\mathbb{R}^d)}
\]

\[
= \left\| a_f - \sum_{\lambda \in \Lambda_m} a_\lambda \varepsilon^\lambda \right\|_{h^{r_1, s_1}_{p_1, q_1} \langle \varepsilon \rangle} \lesssim \inf_{c_\lambda^* \in \mathbb{C}} \left\| a_f - \sum_{\lambda \in \Lambda_m} c_\lambda \varepsilon^\lambda \right\|_{h^{r_1, s_1}_{p_1, q_1} \langle \varepsilon \rangle}, \tag{14}
\]

26
due to Lemma A.1. So, (13) implies
\[
\sup_{\|f\|_{H^0_{p,q}(\Omega)}} \left\| \frac{f - \sum_{\lambda \in \Lambda_m} \tilde{c}_\lambda (f) \psi^\lambda}{c_{\lambda m}} \right\|_{L^1} H^{r_1,s_1}_{p_1,q_1} Y(\Omega)
\leq \sup_{\|f\|_{H^0_{p,q}(\Omega)}} \left\| \inf_{c_{\lambda m} \in \mathbb{C}}, \sum_{\lambda \in \Lambda_m} c_{\lambda m} e^\lambda \right\|_{L^1} H^{r_1,s_1}_{p_1,q_1} y(\nabla)
\leq \sup_{\|f\|_{H^0_{p,q}(\Omega)}} \left\| \inf_{c_{\lambda m} \in \mathbb{C}}, \sum_{\lambda \in \Lambda_m} c_{\lambda m} e^\lambda \right\|_{L^1} H^{r_1,s_1}_{p_1,q_1} y(\nabla)
\]
which (by taking the infimum w.r.t. \(\Lambda_m\)) yields \(\zeta_m(\text{id}; \Psi) \leq d_m(\text{id}; E)\).

For the best \(m\)-term w.r.t. \(\Lambda_m\) we can argue similarly. Indeed, (14) implies
\[
\inf_{\Lambda_m \subset V', c_{\lambda m} \in \mathbb{C}}, \sum_{\lambda \in \Lambda_m} c_{\lambda m} e^\lambda \left\| f - \sum_{\lambda \in \Lambda_m} c_{\lambda m} e^\lambda \right\|_{L^1} H^{r_1,s_1}_{p_1,q_1} y(\nabla)
\]
for all \(f \in H^0_{p,q}(\Omega)\), hence taking the sup w.r.t. \(\|f\|_{H^0_{p,q}(\Omega)} \leq 1\) proves the claim. ■

The converse to Lemma A.2 reads as follows:

**Lemma A.3.** For \(d \in \mathbb{N}\) let \(X,Y \in \{B,F\}\) and \(0 < p_0, p_1, q_0, q_1 \leq \infty\) (with \(p_0 < \infty\) if \(X = F\) and \(p_1 < \infty\) if \(Y = F\), respectively), as well as \(r_0, r_1, s_0, s_1 \in \mathbb{R}\) be such that \(\text{id} : H^0_{p_0,q_0}(\Omega) \to H^{r_1,s_1}_{p_1,q_1}(\Omega)\). Then \(\text{id} : H^0_{p_0,q_0}(\nabla) \to H^{r_1,s_1}_{p_1,q_1}(\nabla)\) and
\[
d_m(\text{id}; E) \leq d_m(\text{id}; \Psi) \quad \text{as well as} \quad \sigma_m(\text{id}; E) \leq \sigma_m(\text{id}; \Psi), \quad m \in \mathbb{N}_0.
\]

**Proof.** Set \(\nabla' := \{\lambda \in \nabla \mid \text{supp}(\psi^\lambda) \cap \partial \Omega = \emptyset\}\). Then every \(\mathbf{a} = (a_{\lambda})_{\lambda \in \nabla'} \in H^{r_1,s_1}_{p,q}(\nabla')\) defines a distribution \(f_{\mathbf{a}} := \sum_{\lambda \in \nabla'} a_{\lambda} \psi^\lambda \in H^{r_1,s_1}_{p_1,q_1}(\nabla', \mathbb{R})\) with \(\text{supp}(f_{\mathbf{a}}) \subset \Omega\). Hence, we actually have \(f_{\mathbf{a}} \in H^{r_1,s_1}_{p,q}(\Omega)\) and
\[
\|f_{\mathbf{a}} - H^{r_1,s_1}_{p_1,q_1}(\Omega)\|_{L^1} = \|f_{\mathbf{a}} - H^{r_1,s_1}_{p_1,q_1}(\mathbb{R})\|_{L^1} = \|\mathbf{a} - H^{r_1,s_1}_{p_1,q_1}(\nabla')\|_{L^1}.
\]

Therefore, \(\text{id} : H^{r_1,s_1}_{p_0,q_0}(\Omega) \to H^{r_1,s_1}_{p_1,q_1}(\Omega)\) and \(\mathbf{a} \in H^{r_1,s_1}_{p_0,q_0}(\nabla')\) yield that
\[
\|\mathbf{a} - H^{r_1,s_1}_{p_1,q_1}(\nabla')\| = \|f_{\mathbf{a}} - H^{r_1,s_1}_{p_1,q_1}(\Omega)\| \leq \|f_{\mathbf{a}} - H^{r_1,s_1}_{p_0,q_0}(\Omega)\| = \|\mathbf{a} - H^{r_1,s_1}_{p_0,q_0}(\nabla')\|,
\]
is finite and thus \(H^{r_1,s_1}_{p_0,q_0}(\nabla') \to H^{r_1,s_1}_{p_1,q_1}(\nabla')\).
Now let $\Lambda_m \subset \nabla'$ with $|\lambda_m| \leq m$ be arbitrarily fixed. Then for all $a \in h_{p_0,q_0}^{r_0,0} x(\nabla')$ we can select coefficients $\widehat{\alpha}_\lambda$, $\lambda \in \Lambda_m$, with

$$\inf_{c_\lambda \in \mathbb{C}, \lambda \in \Lambda_m} \left\| a - \sum_{\lambda \in \Lambda_m} c_\lambda e^\lambda \right\| h_{p_1,q_1}^{r_1,1} y(\nabla') \leq \left\| a - \sum_{\lambda \in \Lambda_m} \widehat{\alpha}_\lambda e^\lambda \right\| h_{p_1,q_1}^{r_1,1} y(\nabla')$$

$$= \left\| f_a - \sum_{\lambda \in \Lambda_m} \widehat{\alpha}_\lambda \psi^\lambda \right\| H_{p_1,q_1}^{r_1,1} Y(\Omega)$$

$$\leq 2 \inf_{c_\lambda \in \mathbb{C}, \lambda \in \Lambda_m} \left\| f_a - \sum_{\lambda \in \Lambda_m} c_\lambda \psi^\lambda \right\| H_{p_1,q_1}^{r_1,1} Y(\Omega).$$

Next, we take the sup over all $\left\| a \right\| h_{p_0,q_0}^{r_0,0} x(\nabla') \leq 1$ which at the right-hand side can be replaced by the sup over all $f_a \in H_{p_0,q_0}^{r_0,0} X(\Omega)$ with (quasi-)norm at most one. So, $d_m(id; \lambda \in \Lambda_m) \subset \nabla'$ can be replaced by the sup over all $f_a \in \nabla'$ and essentially the same arguments show that also $\sigma_m(id; \lambda \in \Lambda_m)$ scale like $2^{j_1}$. Therefore, in all proven assertions the sequence spaces on $\nabla'$ can be replaced by corresponding ones on $\nabla$ and thus the proof is complete.

Now we are well-prepared to prove the lifting assertion stated in Proposition 2.7.

Proof of Proposition 2.7. Combining Lemma A.2 and A.3 shows the continuity statement, as well as the assertion on best-$m$-term widths. Moreover, together with (2) they yield

$$d_m(id; E) \lesssim d_m(id; \Psi) \leq \zeta_m(id; \Psi) \lesssim d_m(id; E) \quad \text{and} \quad d_m(id; E) \leq \zeta_m(id; E).$$

To finish the proof, we note that Lemma A.1 implies that for all $m \in \mathbb{N}_0$

$$d_m(id; E) = \inf_{\Lambda_m \subset \nabla'} \| a \| h_{p_0,q_0}^{r_0,0} x(\nabla') \| \leq 1 \| a - \sum_{\lambda \in \Lambda_m} a_\lambda e^\lambda \| h_{p_1,q_1}^{r_1,1} y(\nabla') \| \geq \zeta_m(id; E).$$

A.2 Sparse approximation of sequences: Stechkin’s inequality

Our upper bounds are based on a result which is frequently attributed to Sergey Stechkin. For the convenience of the reader, we add its simple proof based on [28, Lemma 3.3].
Lemma A.4 (Stechkin). Let \( \mathcal{G} \neq \emptyset \) denote some countable index set, \( 0 < p_0 \leq p_1 \leq \infty \), and \( a = (a_i)_{i \in \mathcal{G}} \in \ell_{p_0}(\mathcal{G}) \) be some real or complex sequence. Then for all finite subsets \( \Lambda \subseteq \mathcal{G} \) with \( |a_\lambda| \geq |a_i| \) for all \( \lambda \in \Lambda \) and \( i \in \mathcal{G} \setminus \Lambda \) there holds

\[
\left( \sum_{i \in \mathcal{G} \setminus \Lambda} |a_i|^{p_1} \right)^{1/p_1} \leq (|\Lambda| + 1)^{-(1/p_0-1/p_1)} \left( \sum_{i \in \mathcal{G}} |a_i|^{p_0} \right)^{1/p_0}
\]

with the usual modifications if \( p_1 \) or \( p_0 \) equal infinity.

Note that Lemma A.4 implies

\[
\sigma_m(\text{id}: \ell_{p_0}(\mathcal{G}) \to \ell_{p_1}(\mathcal{G}) ; E) \lesssim m^{-(1/p_0 - 1/p_1)}
\]

with \( E \) denoting the unit vectors in the corresponding sequence spaces.

Proof. The cases \( \Lambda = \emptyset \), or \( \Lambda = \mathcal{G} \), or \( p_0 = p_1 \) are trivial. Hence, we can assume that \( |\mathcal{G}| > |\Lambda| \geq 1 \) and \( p_0 < p_1 \leq \infty \). Let \( (b_n)_{n=1}^{||\mathcal{G}||} \subset \mathbb{R} \) be any non-increasing rearrangement of \((|a_i|)_{i \in \mathcal{G}}\), i.e., \( b_n \geq b_{n+1} \) for all \( n \). Then for \( m := |\Lambda| + 1 \) there holds

\[
m b_m^{p_0} \leq b_1^{p_0} + \ldots + b_m^{p_0} \leq \sum_{n=1}^{||\mathcal{G}||} b_n^{p_0} \quad \text{and hence} \quad b_m^{1-p_0/p_1} \leq m^{-(1/p_0-1/p_1)} \left( \sum_{n=1}^{||\mathcal{G}||} b_n^{p_0} \right)^{1/p_0-1/p_1},
\]

since \( 1/p_0 > 1/p_1 \). If \( p_1 = \infty \), this implies the claim as follows:

\[
\max_{i \in \mathcal{G} \setminus \Lambda} |a_i| = \max_{n \geq m} b_n \leq b_m \leq m^{-1/p_0} \left( \sum_{n=1}^{||\mathcal{G}||} b_n^{p_0} \right)^{1/p_0} = (|\Lambda| + 1)^{-1/p_0} \left( \sum_{i \in \mathcal{G}} |a_i|^{p_0} \right)^{1/p_0}.
\]

On the other hand, if \( p_1 < \infty \), we can argue similarly and obtain

\[
\left( \sum_{i \in \mathcal{G} \setminus \Lambda} |a_i|^{p_1} \right)^{1/p_1} = \left[ \sum_{n=m}^{||\mathcal{G}||} b_n^{p_0} \right]^{1/p_1} \leq b_m^{1-p_0/p_1} \left( \sum_{n=m}^{||\mathcal{G}||} b_n^{p_0} \right)^{1/p_1} \leq m^{-(1/p_0-1/p_1)} \left( \sum_{n=1}^{||\mathcal{G}||} b_n^{p_0} \right)^{1/p_0}
\]

which finishes the proof. \( \blacksquare \)
A.3 Combinatorics

In this appendix, we collect estimates related to the sets $\Delta_{\mu} := \Delta_{\mu}(\alpha, \beta)$ and $\mathcal{L}_{\mu} := \mathcal{L}_{\mu}(\alpha, \beta)$ introduced in Definition 3.4. We start with bounding their cardinality.

**Lemma A.5.** Let $d \in \mathbb{N}$.

(i.) If $\alpha, \beta \geq 0$, then

$$|\Delta_{\mu}| \gtrsim \mu^d, \quad \mu \in \mathbb{N}.$$ 

(ii.) If $\alpha \geq 0$ and $\beta < \alpha$, then

$$|\Delta_{\mu}| \lesssim \mu^d, \quad \mu \in \mathbb{N}.$$ 

(iii.) If $\alpha > \beta \geq 0$, we have

$$|\mathcal{L}_{\mu}| \sim \mu^{d-1}, \quad \mu \in \mathbb{N} \setminus \{1\}.$$ 

**Proof.** The last statement follows from the previous ones since

$$|\mathcal{L}_{\mu+1}| = |\Delta_{\mu+1}| - |\Delta_{\mu}| \sim (\mu + 1)^d - \mu^d = \sum_{k=0}^{d-1} \binom{d}{k} \mu^k \sim \mu^{d-1} \sim (\mu + 1)^{d-1}.$$ 

For the lower bound we note that $\mathcal{L} := \{0, 1, \ldots, \left\lfloor \frac{\mu}{\alpha d} \right\rfloor \}^d \subset \Delta_{\mu}$, since every $j \in \mathcal{L}$ satisfies

$$\mu \geq \alpha d |j|_{\infty} \geq \alpha |j|_1 \geq \alpha |j|_1 - \beta |j|_{\infty},$$

i.e. $j \in \Delta_{\mu}$. So,

$$|\Delta_{\mu}| \geq |\mathcal{L}| = \begin{cases} \infty, & \alpha = 0, \\ (1 + \left\lfloor \frac{\mu}{\alpha d} \right\rfloor)^d \geq \left( \frac{\mu}{\alpha d} \right)^d, & \alpha > 0, \end{cases} \gtrsim \mu^d.$$ 

Similarly, every $j \in \Delta_{\mu}$ satisfies $\mu \geq \alpha |j|_1 - \beta |j|_{\infty} \geq (\alpha - \beta) |j|_{\infty}$ such that $j$ belongs to $\mathcal{U} := \left\{ 0, 1, \ldots, \left\lfloor \frac{\mu}{\alpha - \beta} \right\rfloor \right\}^d$. Therefore, $|\Delta_{\mu}| \leq |\mathcal{U}| \leq (1 + \frac{\mu}{\alpha - \beta})^d \leq (\frac{2\mu}{\alpha - \beta})^d \lesssim \mu^d$ if $\mu \geq \alpha - \beta$. Otherwise, $\Delta_{\mu} \lesssim 1 \lesssim \mu^d$. 

Further, we shall use the following sharp estimate which generalizes [3, Lemma 6.3].

**Lemma A.6.** Let $d \in \mathbb{N}$, $\alpha > \beta > 0$, and $\delta > 0$. Then

$$\sum_{j \in \Delta_{\mu}} 2^{\delta |j|_1} \sim 2^{\delta \mu/(\alpha - \beta)}, \quad \mu \geq \alpha - \beta.$$
Proof. Step 1 (Lower bound). We first show that
\[ j^* := \left\lfloor \frac{\mu}{\alpha - \beta} \right\rfloor (1, 0, \ldots, 0) \]
belongs to \( \Delta_\mu \). Indeed,
\[ j^* \in \Delta_\mu \iff \alpha |j^*|_1 - \beta |j^*|_{\infty} \leq \mu \iff (\alpha - \beta) \left\lfloor \frac{\mu}{\alpha - \beta} \right\rfloor \leq \mu, \]
which is obviously true. Therefore,
\[ \sum_{j \in \Delta_\mu} 2^\delta |j|_1 \geq 2^\delta |j^*|_1 = 2^\delta \lceil \mu/(\alpha - \beta) \rceil \geq 2^\delta \mu/(\alpha - \beta) - \delta \sim 2^\delta \mu/(\alpha - \beta). \]

Step 2 (Upper bound). If \( d = 1 \), we have \( j \in \Delta_\mu \) iff \( j \leq \mu/(\alpha - \beta) \) and thus
\[ \sum_{j \in \Delta_\mu} 2^\delta |j|_1 \leq \sum_{j = 0}^{\lceil \mu/(\alpha - \beta) \rceil} 2^\delta j \sim 2^\delta \mu/(\alpha - \beta), \]
and
\[ \sum_{j \in \Delta_\mu} 2^\delta |j|_1 \leq \sum_{j = 0}^{\lfloor \mu/(\alpha - \beta) \rfloor} 2^\delta \mu/(\alpha - \beta) \sim 2^\delta \mu/(\alpha - \beta). \]

Now let \( d \geq 2 \). For each \( j = (j_1, \ldots, j_d) \in \mathbb{N}_0^d \) set \( j' := (j_2, \ldots, j_d) \). Further, for \( i = 1, \ldots, d \) let \( \bar{j}_i := \{ j \in \mathbb{N}_0^d | j_i = |j|_\infty \} \). Due to symmetry, it suffices to estimate
\[ \sum_{j \in \Delta_\mu} 2^\delta |j|_1 \leq \sum_{i = 1}^{d} \sum_{j \in \bar{j}_i \cap \Delta_\mu} 2^\delta j \sim \sum_{i = 1}^{d} \sum_{j \in \bar{j}_i \cap \Delta_\mu} 2^\delta j \sim \sum_{i = 1}^{d} \sum_{j \in \bar{j}_i \cap \Delta_\mu} 2^\delta j \sum_{j \in \bar{j}_i \cap \Delta_\mu} 2^\delta j. \]
If \( j = (j_1, j') \in \bar{j}_1 \), then \( \alpha - \beta > 0 \) yields
\[ j \in \Delta_\mu \iff \alpha (|j'|_1 + j_1) - \beta j_1 \leq \mu \iff j_1 \leq \frac{\mu - \alpha |j'|_1}{\alpha - \beta}. \]
Therefore, \( j \in \bar{j}_1 \cap \Delta_\mu \) implies \( |j'|_{\infty} \leq |j|_{\infty} = j_1 \leq (\mu - \alpha |j'|_1)/(\alpha - \beta) \) and hence
\[ \alpha |j'|_1 + (\alpha - \beta) |j'|_{\infty} \leq \mu. \]
Combining these estimates we conclude
\[ \sum_{j \in \Delta_\mu} 2^\delta |j|_1 \leq \sum_{j \in \bar{j}_1 \cap \Delta_\mu} 2^\delta j \sum_{j_1 = |j'|_{\infty}} 2^\delta j_1. \]
Since \( \delta > 0 \), up to constants the inner geometric sum is upper bounded by
\[
2^{\delta \left( (\mu - \alpha |j'_{1}|)/(\alpha - \beta) \right)} \leq 2^{\delta \left( (\mu - \alpha |j'_{1}|)/\alpha \right)} \lesssim 2^{\delta \left( (\mu - \alpha |j'_{1}|)/(\alpha - \beta) \right)}
\]
such that
\[
\sum_{j \in \Delta_{\mu}} 2^{\delta |j'_{1}|} \lesssim \sum_{j' \in \mathbb{N}^{d-1}_{0}} 2^{\delta \left( |j'_{1}| + (\mu - \alpha |j'_{1}|)/(\alpha - \beta) \right)}
\]
\[
\leq 2^{\delta \mu/(\alpha - \beta)} \sum_{j' \in \mathbb{N}^{d-1}_{0}} 2^{\delta \left( 1 - \alpha/(\alpha - \beta) \right) |j'_{1}|}
\]
\[
= 2^{\delta \mu/(\alpha - \beta)} \prod_{i=1}^{d-1} \sum_{j_{i}=0}^{\infty} 2^{-\delta \beta/(\alpha - \beta) j_{i}}
\]
\[
\lesssim 2^{\delta \mu/(\alpha - \beta)},
\]
where we used that due to the assumption \( \alpha > \beta > 0 \) we have \( \delta \beta/(\alpha - \beta) > 0 \).}

Finally, our proofs of the upper bounds in Section 3.2 make use of

Lemma A.7. Let \( d \in \mathbb{N}, \alpha > \beta \geq 0, \) and \( \delta > 0 \). Then
\[
\sum_{j \in \mathcal{L}_{\mu}} 2^{-\delta (|j'_{1}| - |j|_{\infty})} \lesssim 1, \quad \mu \in \mathbb{N} \setminus \{1\}.
\]

Proof. If \( d = 1 \), Lemma A.5 yields
\[
\sum_{j \in \mathcal{L}_{\mu}} 2^{-\delta (|j|_{1} - |j|_{\infty})} = \sum_{j \in \mathcal{L}_{\mu}} 1 = |\mathcal{L}_{\mu}| \sim 1, \quad \mu \in \mathbb{N} \setminus \{1\}.
\]
So let \( d \geq 2 \) and define \( \mathcal{J}_{i} := \{ j = (j_{1}, \ldots, j_{d}) \in \mathbb{N}_{0}^{d} \mid j_{i} = |j|_{\infty} \} \) for \( i = 1, \ldots, d \). Then
\[
\sum_{j \in \mathcal{L}_{\mu}} 2^{-\delta (|j|_{1} - |j|_{\infty})} \leq \sum_{i=1}^{d} \sum_{j \in \mathcal{J}_{i} \cap \mathcal{L}_{\mu}} 2^{-\delta (|j|_{1} - |j|_{\infty})} = d \sum_{j = (j_{1}, j')} \sum_{j_{1} \in \mathcal{J}_{1} \cap \mathcal{L}_{\mu}} 2^{-\delta |j'_{1}|},
\]
where every \( j = (j_{1}, j') \in \mathcal{J}_{1} \cap \mathcal{L}_{\mu} \) satisfies \( \mu - 1 < \alpha(j_{1} + |j'|_{1}) - \beta j_{1} \leq \mu \), i.e.
\[
\frac{\mu - 1}{\alpha - \beta} - \frac{\alpha}{\alpha - \beta} |j'_{1}| < j_{1} \leq \frac{\mu}{\alpha - \beta} - \frac{\alpha}{\alpha - \beta} |j'_{1}|.
\]

32
Thus, independent of $\mu$ there are only constantly many different values for $j_1$ for fixed $j'$. So,

\[
\sum_{j \in \mathcal{L}_\mu} 2^{-\delta|j_1| - |j|_\infty} \lesssim \sum_{j = (j_1, j') \in \mathcal{J}_1 \cap \mathcal{L}_\mu} 2^{-\delta|j'|_1} \lesssim \sum_{j' \in \mathbb{N}_0^{d-1}} 2^{-\delta|j'|_1} = \prod_{i=1}^{d-1} \sum_{j_i=0}^{\infty} 2^{-\delta j_i} \lesssim 1. \quad \blacksquare
\]

References

[1] H.-J. Bungartz and M. Griebel. Sparse grids. Acta Numer., 13:147–269, 2004.

[2] G. Byrenheid. Sparse representation of multivariate functions based on discrete point evaluations. PhD thesis, University of Bonn, 2018.

[3] G. Byrenheid, D. Dung, W. Sickel, and T. Ullrich. Sampling on energy-norm based sparse grids for the optimal recovery of Sobolev type functions in $H^s$. J. Approx. Theory, 207:207–231, 2016.

[4] P. Cioica, S. Dahlke, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R. Schilling. Spatial Besov regularity for stochastic partial differential equations on Lipschitz domains. Studia Math., 207(3):197–234, 2011.

[5] P. A. Cioica-Licht and M. Weimar. On the limit regularity in Sobolev and Besov scales related to approximation theory. J. Fourier Anal. Appl., 26(1):Art. 10, 2020.

[6] A. Cohen, W. Dahmen, and R. A. DeVore. Adaptive wavelet methods for elliptic operator equations: Convergence rates. Math. Comp., 70:27–75, 2001.

[7] D. Dung. Sampling and cubature on sparse grids based on a B-spline quasi-interpolation. J. Found. Comput. Math., 16(5):1193–1240, 2016.

[8] D. Dung, V. Temlyakov, and T. Ullrich. Hyperbolic Cross Approximation. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer, Cham, 2018.

[9] D. Dung and T. Ullrich. N-widths and $\varepsilon$-dimensions for high-dimensional approximations. J. Found. Comput. Math., 13:965–1003, 2013.

[10] S. Dahlke and R. A. DeVore. Besov regularity for elliptic boundary value problems. Comm. Partial Differential Equations, 22(1-2):1–16, 1997.

[11] S. Dahlke, L. Diening, C. Hartmann, B. Scharf, and M. Weimar. Besov regularity of solutions to the $p$-Poisson equation. Nonlinear Anal., 130:298–329, 2016.
[12] S. Dahlke, H. Harbrecht, M. Utzinger, and M. Weimar. Adaptive wavelet BEM for boundary integral equations: Theory and numerical experiments. *Numer. Funct. Anal. Optim.*, 39(2): 208–232, 2018.

[13] S. Dahlke, E. Novak, and W. Sickel. Optimal approximation of elliptic problems by linear and nonlinear mappings II. *J. Complexity*, 22(4):549–603, 2006.

[14] S. Dahlke and C. Schneider. Regularity in Sobolev and Besov spaces for parabolic problems on domains of polyhedral type. *J. Geom. Anal.*, 31(12):11741–11779, 2021.

[15] S. Dahlke and M. Weimar. Besov regularity for operator equations on patchwise smooth manifolds. *J. Found. Comput. Math.*, 15(6):1533–1569, 2015.

[16] W. Dahmen. Wavelet and multiscale methods for operator equations. *Acta Numer.*, 6:55–228, 1997.

[17] M. Dauge and R. Stevenson. Sparse tensor product wavelet approximation of singular functions. *SIAM J. Math. Anal.*, 42(5):2203–2228, 2010.

[18] R. DeVore and A. Kunoth, editors. *Multiscale, Nonlinear and Adaptive Approximation*. Springer, Berlin/Heidelberg, 2009.

[19] R. A. DeVore. Nonlinear approximation. *Acta Numer.*, 7:51–150, 1998.

[20] M. Griebel and S. Knapek. Optimized tensor-product approximation spaces. *Constr. Approx.*, 16(4):525–540, 2000.

[21] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman, Boston, MA, 1985.

[22] M. Hansen. Nonlinear approximation rates and Besov regularity for elliptic PDEs on polyhedral domains. *J. Found. Comput. Math.*, 15(2):561–589, 2015.

[23] M. Hansen and W. Sickel. Best $m$-term approximation and Lizorkin-Triebel spaces. *J. Approx. Theory*, 163(8):923–954, 2011.

[24] M. Hansen and W. Sickel. Best $m$-term approximation and Sobolev-Besov spaces of dominating mixed smoothness—the case of compact embeddings. *Constr. Approx.*, 36(1):1–51, 2012.

[25] C. Hartmann and M. Weimar. Besov regularity of solutions to the $p$-poisson equation in the vicinity of a vertex of a polygonal domain. *Results Math.*, 73(1):Art. 41, 2018.

[26] L. Kämmerer, D. Potts, and T. Volkmer. Approximation of multivariate periodic functions by trigonometric polynomials based on rank-1 lattice sampling. *J. Complexity*, 31(4):543–576, 2015.

34
[27] Y. Kolomoitsev, T. Lomako, and S. Tikhonov. Sparse grid approximation in weighted Wiener spaces. Preprint, arXiv:2111.06335, 2021.

[28] D. Kressner and C. Tobler. Low-rank tensor Krylov subspace methods for parametrized linear systems. *SIAM J. Matrix Anal. Appl.*, 32(4):1288–1316, 2011.

[29] V. E. Ma˘ıorov. Discretization of the problem of diameters. *Uspehi Mat. Nauk*, 30(6(186)):179–180, 1975.

[30] V. K. Nguyen. Weyl and Bernstein numbers of embeddings of Sobolev spaces with dominating mixed smoothness. *J. Complexity*, 36:46–73, 2016.

[31] V. K. Nguyen and V. D. Nguyen. Best $n$-term approximation of diagonal operators and application to function spaces with mixed smoothness. Preprint, arXiv:2108.12974, 2021.

[32] P.-A. Nitsche. Best $N$ term approximation spaces for tensor product wavelet bases. *Constr. Approx.*, 24(1):49–70, 2006.

[33] A. Pietsch. *Operator Ideals*, volume 20 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam/New York, 1980.

[34] M. Schäfer, T. Ullrich, and B. Vedel. Hyperbolic wavelet analysis of classical isotropic and anisotropic Besov-Sobolev spaces. *J. Fourier Anal. Appl.*, 27(3-51):1–55, 2021.

[35] P. Siedlecki and M. Weimar. Notes on $(s,t)$-weak tractability: A refined classification of problems with (sub)exponential information complexity. *J. Approx. Theory*, 200:227–258, 2015.

[36] H. Triebel. *Theory of Function Spaces III*. Birkhäuser, Basel, 2006.

[37] H. Triebel. *Function Spaces with Dominating Mixed Smoothness*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2019.

[38] J. Vybíral. Function spaces with dominating mixed smoothness. *Dissertationes Math.*, 436:1–73, 2006.

[39] J. Vybíral. Widths of embeddings in function spaces. *J. Complexity*, 24(4):545–570, 2008.

[40] M. Weimar. Breaking the curse of dimensionality. *Dissertationes Math.*, 505:1–112, 2015.

[41] M. Weimar. Almost diagonal matrices and Besov-type spaces based on wavelet expansions. *J. Fourier Anal. Appl.*, 22(2):251–284, 2016.

[42] H. Yserentant. On the regularity of the electronic Schrödinger equation in Hilbert spaces of mixed derivatives. *Numer. Math.*, 98(4):731–759, 2004.

[43] H. Yserentant. *Regularity and Approximability of Electronic Wave Functions*, volume 2000 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010.