Phases of QCD with nonvanishing isospin density

Michael C. Birse\textsuperscript{a}, Thomas D. Cohen\textsuperscript{b} and Judith A. McGovern\textsuperscript{a}
\textit{a. Department of Physics and Astronomy, University of Manchester, Manchester M13 9PL, UK
b. Department of Physics, University of Maryland, College Park, 20742-4111}

The propagation of low-momentum baryons through QCD matter at low isospin density is studied using chiral perturbation theory. When the isospin chemical potential exceeds a critical value, the dispersion relation for the lowest nucleon mode becomes anomalous near zero momentum—increasing momentum yields decreasing energy—so that the momentum for the lowest energy state, $p_{\text{min}}$, becomes nonzero. This can be interpreted as a new phase of QCD, with $p_{\text{min}}$ serving as the order parameter and the kinetic mass (defined as the inverse of the second derivative of the energy with respect to the momentum at $p_{\text{min}}$) serving as the susceptibility. A lowest-order chiral-perturbation-theory calculation yields a critical isospin chemical potential of $\sim 285$ MeV. Since this is small compared to the chiral-symmetry-breaking scale, corrections to it are likely to be modest.

Recently, Son and Stephanov have considered QCD in a phase with nonzero isospin density \[3\]. Such a system is of interest for a number of reasons. It is an intrinsically nonperturbative system which is sensitive to much of the complex dynamics of QCD. In the limits of both very low and very high isospin density, there are analytical techniques which, given our present state of understanding of the theory, should be both valid and computationally tractable. At the same time the functional determinant for the system in Euclidean space is manifestly real and positive \[2\]. Accordingly, one can numerically simulate QCD at nonvanishing isospin density, $\rho_I$, (or more precisely nonvanishing isospin chemical potential, $\mu_I$) using standard lattice Monte Carlo techniques, unlike the case of QCD at finite baryon density \[8\]. Thus, even if it is hard to envision a situation in which one could study this regime experimentally, the prospect of lattice results means that it may serve as a useful testing ground for various theoretical approaches to QCD. The system may also provide insights into some of the properties of neutron matter—a system with both non-zero isospin density and baryon density. Moreover, the phase structure of the theory is interesting in its own right. At low, but non-zero, isospin density the medium is a superfluid—a pion condensate. The phase structure is, in fact, more rich. For example, if one generalizes isospin to three flavors as was recently considered by Kogut and Toublan \[4\] one finds a kaon condensed phase which is separated from the pion condensed phase by a first order phase transition. In this paper we show that even if one restricts one’s attention to two flavors and relatively low isospin density, the medium can be nontrivial. At sufficiently low $\rho_I$, both above and below the pion condensation transition, the nucleons have a normal dispersion relation—the energy increases with increasing momentum. However, when $|\mu_I|$ is
above a certain critical value, \( \mu_{ct} \), the situation is fundamentally different; for low values of the nucleon momenta the dispersion relation for the lowest lying nucleon excitation is anomalous in the sense of anomalous dispersion in optics—increasing the momentum leads to lower energy. This defines our new phase. It can be described by a simple order parameter,

\[ \Theta = \rho_{\text{baryon}}^\text{min}, \tag{1} \]

where \( \rho_{\text{baryon}}^\text{min} \) is the magnitude of the momentum of the lowest-lying excitation of nonzero baryon number. For \(|\mu_1| < \mu_{ct}, \Theta = 0\) while for \(|\mu_1| > \mu_{ct}, \Theta \neq 0\); thus \( \Theta \) serves as an order parameter in the traditional sense of distinguishing between two phases. As will be shown below, the critical chemical potential, \( \mu_{ct} \), can be estimated using \( \chi \)PT. At lowest order in the chiral expansion the critical chemical potential satisfies

\[ \frac{2M_Nm_\pi^2\mu_{ct}}{(1-g_\pi^2)m_\pi^2 + g_\pi^2\mu_{ct}^2} = 1, \tag{2} \]

which yields an estimate of \( \mu_{ct} \approx 285 \) MeV for the transition. Chiral corrections to this are of order \( Q^2/\Lambda^2 \), so that one might expect corrections to alter this value by \( \sim 20\% \).

We begin with a brief recapitulation of the arguments of ref. [1] for the low \( \mu_1 \) region. As all dynamical scales in the problem are much less than typical hadronic scales one can describe the system by an effective theory for interactions among pions, the only light degrees of freedom in the problem. The Lagrangian for the purely pionic part of the theory is simply that of the nonlinear sigma model. At leading order it is

\[ \mathcal{L} = \frac{1}{4}f_\pi^2 \text{Tr} [\nabla_\mu U \nabla^\mu U^\dagger + 2m_\pi^2 \text{Re}U], \tag{3} \]

where \( U \in SU(2) \) and can be parametrized by \( U \equiv \exp \left( i \frac{\pi^a}{f_\pi} \tilde{T}^a \right) \). The theory at this order contains two parameters: the pion mass, \( m_\pi \approx 139 \) MeV, and the pion decay constant \( f_\pi \approx 93 \) MeV; the covariant derivative \( \nabla \) incorporates couplings to gauge fields.

At the QCD level the isospin-chemical-potential term is simply \( \frac{1}{2} \mu_1 \tau_3 \cdot \tau q \), where \( \tau_q \) is a four velocity that describes the motion of the resulting medium; in its rest frame \( \tau_q = (1, 0, 0, 0) \). Thus, the isospin chemical potential couples directly to the conserved vector-isovector current, and at the level of the effective theory, the coupling of the isospin chemical potential must be the same as the coupling of a U(1) isovector gauge field. \( \[ \] \). Hence we use

\[ \nabla_\mu U = \partial_\mu U + i[U,V_\mu], \tag{4} \]

with \( V_\mu = \frac{1}{2} \mu_1 \tau_3 \delta_{\mu 3} = \frac{1}{2} \mu_1 \tau_3 \delta_{\mu 3} \). Using this lowest order Lagrangian one expects predictions of pionic observables and thermodynamic quantities to be accurate up to corrections of relative order \( Q^2/\Lambda^2 \) where \( Q \approx m_\pi, \mu_1, p \) (where \( p \) is the external momentum of a probe and \( \Lambda \sim 1 \) GeV is a characteristic hadronic scale).

The next step is to determine the \( U \) which minimizes the energy. The ground state is translationally invariant and static so one can simply use a constant \( U \) in the Lagrangian of eq. [3] and equate it to the negative of an effective potential,

\[ V_{\text{eff}}(U) = \frac{f_\pi^2 \mu_1^2}{8} \text{Tr}(\tau_3 U \tau_3 U^\dagger - 1) - \frac{f_\pi^2 m_\pi^2}{2} \text{Re} \text{Tr}(U). \tag{5} \]

It is clear that first term in eq. [3] lowers the energy if \( U \) aligns along the \( \tau_1 \) or \( \tau_2 \) directions while the last term favors the isoscalar direction. Accordingly, the minima can be found using an ansatz which allows for general rotation between the isoscalar and \( \tau_1 \) or \( \tau_2 \) directions,

\[ U = \cos \alpha + i(\tau_1 \cos \phi + \tau_2 \sin \phi) \sin \alpha, \tag{6} \]

for which the effective potential takes the form

\[ V_{\text{eff}}(\alpha) = \frac{f_\pi^2 \mu_1^2}{4} \cos 2\alpha - 1 - \frac{f_\pi^2 m_\pi^2}{2} \cos \alpha. \tag{7} \]

Note that \( V_{\text{eff}} \) depends on \( \alpha \) but not on \( \phi \). Minimizing with respect to \( \alpha \) one sees that for \( |\mu_1| < m_\pi \), the minimum occurs at \( \alpha = 0 \) (\( U = 1 \)) so that one is in the normal phase. For \( |\mu_1| > m_\pi \), the minimum occurs at \( [\] \)

\[ \cos \alpha = \frac{m_\pi^2}{\mu_1^2}. \tag{8} \]
which implies that the state has chirally rotated to a new pion-condensed state.

Now let us turn to the main subject of the present work, the propagation of nucleons through a pion-condensed phase. The interaction between nucleons and the chiral field can also be described using $\chi$PT. The lowest order Lagrangian describing the interaction of nucleons with pions in $\chi$PT is given by (3)

$$ \mathcal{L}_{\pi N} = \Psi \left( i D - M + \frac{g_A}{2} \gamma_5 u \mu \right) \Psi, \quad (9) $$

where $u^2 = U$ and

$$ D_\mu = \partial_\mu + \frac{i}{2} [u^\dagger, \partial_\mu u] - \frac{i}{2} i (u^\dagger V_\mu u + u V_\mu u^\dagger), $$

$$ u_\mu = i (u^\dagger \nabla_\mu u - u \nabla_\mu u^\dagger), $$

$$ \nabla_\mu u = \partial_\mu u + i [u, V_\mu]. \quad (10) $$

We note that the interaction between nucleons and pions considered by Son and Stephanov (1) was based on a nucleon field that transforms linearly under chiral transformations rather than one that transforms nonlinearly as considered here. In fact, it is straightforward to show that the two are equivalent—that is, they are related to each other by a change of variables—provided that one takes $\mu = 0$ is no longer the minimum for the lower branch. Expressing $\alpha$ in terms of $\mu_3$ through eq. (3), we find a critical value of $\mu_3$, already given in eq. (2), above which the minimum energy is at a non-zero value of $p$. Fig. 1 illustrates

![Fig. 1. The nucleon dispersion relation for values of $\mu_3$ below and above $\mu_{cr.}$](image-url)
the qualitative difference between the two regions. We plot $E_{\pm}$ for two choices of $\mu I$, one below the critical value and one above.

As noted earlier, $p_{\text{min}}$ functions as an order parameter. For $|\mu I| > \mu_{\text{cr}}$ it is given by

$$p_{\text{min}} = \frac{1}{2} \sqrt{\frac{(g_0^2 \mu_I^2 + (1 - g_0^2) m_N^2)^2 - 4M^2 m_N^2 \mu_I^2}{\mu_I^2 (g_0^2 \mu_I^2 + (1 - g_0^2) m_N^2)}}. \quad (14)$$

This is plotted in fig. 2.

An alternative definition for the mass to that considered in ref. 1 is the kinetic mass, which for any mode is defined by

$$M_{\text{kin}}^{-1} = \frac{\partial^2 E}{\partial p^2} |_{p_{\text{min}}}. \quad (15)$$

Clearly for the free-space dispersion relation the mass defined by the minimum-energy excitation and the kinetic mass are equal, but in medium this need not be so. The kinetic mass is important in that it is the inertial parameter which enters in an effective Schrödinger equation describing the lowest-energy excitations of this system. The kinetic mass for the lower of the two nucleon modes is

$$M_{\text{kin}} = \begin{cases} M m_N^2 \sqrt{(1 - g_0^2) m_N^2 + g_0^2 \mu_I^2} + 4M m_N^2 \mu_I + 4M^2 \mu_I^2 / 2M m_N^2 \mu_I - (1 - g_0^2) m_N^2 - g_0^2 \mu_I^2 \\ g_0 M \left((1 - g_0^2) m_N^2 + g_0^2 \mu_I^2\right)^{1/2} \sqrt{\mu_I^2 - m_N^2} / (g_0^2 \mu_I^2 + (1 - g_0^2) m_N^2)^2 - 4M^2 m_N^2 \mu_I^2 \end{cases} \quad |\mu I| < \mu_{\text{cr}},$$

$$M_{\text{kin}} = \left\{ \begin{array}{ll} M m_N^2 \sqrt{(1 - g_0^2) m_N^2 + g_0^2 \mu_I^2} + 4M m_N^2 \mu_I + 4M^2 \mu_I^2 / 2M m_N^2 \mu_I - (1 - g_0^2) m_N^2 - g_0^2 \mu_I^2 \\ g_0 M \left((1 - g_0^2) m_N^2 + g_0^2 \mu_I^2\right)^{1/2} \sqrt{\mu_I^2 - m_N^2} / (g_0^2 \mu_I^2 + (1 - g_0^2) m_N^2)^2 - 4M^2 m_N^2 \mu_I^2 \end{array} \right\} \quad |\mu I| > \mu_{\text{cr}}. \quad (16)$$

It is plotted in fig. 3. One striking feature is that $M_{\text{kin}}$ develops a pole, diverging at the critical value $\mu_{\text{cr}}$. This critical value occurs when the denominators of eq. (16) vanish.

Figures 2 and 3 look very much like a standard second order phase transition, with the kinetic mass diverging at the transition in the manner of a typical susceptibility and $p_{\text{min}}$ rising from zero as a fractional power in the manner of a typical order parameter. In a way, however, this phase transition is rather unusual. Generally at phase transitions there are discontinuities in quantities associated with bulk energetics such as specific heats. In the present case we have no evidence of such effects. The basic reason is very clear: the discontinuities associated with the phase transition reported here all involve baryon properties. To the order at which we have calculated, the pionic degrees of freedom are unaffected by the transition and the pions dominate the energetics. To the extent that there are discontinuities in quantities associated with energies they must involve changes in possible nucleon-antinucleon loop effects which might altered discontinuously when the nucleon dispersion relation is altered. However, any such possible effects are too small to be calculated in $\chi$PT.

We note that the value of the chemical potential from Eq. 2 is of order $(M_N m_N^2)^{1/3}$, and numerically equal to 285 MeV $\approx 2m_N$. Thus in practice $\mu_{\text{cr}}$ is comparable in size to $m_N$ and well below $M_N$, so we are in the usual regime of applicability of $\chi$PT. The leading corrections to our result should come from the inclusion of the higher-order terms in the pion-nucleon Lagrangian which are parameterised by the low-energy constants $c_i$. In fact to order $1/M_N$ the sole effect of these is to add a term $(4c_1 m_N^2 - (c_2 + c_3) \mu^2) \sin^2 \alpha$ to the nucleon energy. Not only is this small ($58 \pm 1$ MeV

![FIG. 2.](image)
at $\mu = \mu_{cr}$ using the second-order determinations of the $c_i$), it is also momentum-independent. Thus it cannot affect $\mu_{cr}$ at all, and corrections start only at third order.

Finally, we turn to a question of considerable significance in nuclear physics, namely neutron matter, which is QCD at finite baryon chemical potential and isospin chemical potential with the chemical potentials chosen so the that baryon density is twice the isospin density. It is unclear whether the $\chi$PT techniques used here can be generalized easily to deal with this problem since they are based on the assumption that the only light scale in the problem is the pion mass. On the other hand, the inverse of the nucleon-nucleon scattering length, $1/a$, defines a mass scale much lighter than $m_\pi$. This suggests that one should develop a systematic expansion treating both $1/a$ and $m_\pi$ is light. Formally, such expansions have been developed for the two nucleon problem, however, in that context there is clear evidence for problems with convergence of the chiral expansion. However, the effects of chiral symmetry may be very different in the case of neutron matter from those in the N-N scattering case and a viable expansion may prove possible. The possibility that at some density neutron matter exhibits pion condensation is interesting since the fact that the propagation of nucleons through a pion condensate changes characteristically above a certain isospin chemical potential could play a significant role.

With the above caveats, we have looked at the mean-field Fermi gas in the present framework. The possibility of pion condensation arises from the term $-\frac{1}{2}\mu_2^2 f_\pi^2 \sin^2 \alpha$ in the energy, which for sufficiently large $\mu$, can overcome the mass term $-m_\pi^2 f_\pi^2 \cos \alpha$; the question is whether a gas of fermions tends to stabilize or destabilize the phase with condensed pions. We see from eq. (13) that for a given value of $\mu_I$, the lowest single-nucleon energy is at $\alpha = 0$, the normal vacuum. Analytic calculation shows that for fixed $\mu_I$ and baryon density, $\alpha = 0$ is also a minimum of the full fermion energy, whereas $\alpha = \frac{1}{2}\pi$ is a maximum. Thus the fermions do not favor the pion-condensed phase. Numerical calculations were done for the case with the isospin density constrained to equal minus half the baryon density, as in neutron matter. In the chiral limit $\alpha = \frac{1}{2}\pi$ (which minimizes the pion energy) is a minimum of the full pion and fermion energy only for baryon densities below about a third of nuclear matter density ($\rho_0$). For higher densities the minimum moves round the chiral circle, reaching $\alpha = 0$ at about two-thirds of $\rho_0$. Thus even in the chiral limit the condensed phase is a low-density phenomenon. With the physical pion mass, pion condensation requires $\mu_I > m_\pi$ at zero baryon density; in neutron matter such a value is reached at around $4\rho_0$. Thus the Fermi gas model does not predict this form of $(s$-wave) pion condensation, much less the new phase discussed in this paper, in neutron stars.

TDC gratefully acknowledges the support of U.S. Department of Energy via grant DE-FG02-93ER-40762. MCB and JMcG are supported by the UK EPSRC, which also funded a visit by TDC to Manchester during which the current work was started. We would like to thank D. T. Son and M. A. Stephanov for their helpful comments.

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