Remarks on factorization property of some stochastic integrals*

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June 22, 2013

Abstract. In the paper Sato (2006) there are introduced two families of improper random integrals and the corresponding two convolution semigroups of infinitely divisible laws on $\mathbb{R}^d$. Theorem 3.1 gives a relation (a factorization property) between those two integrals. Here, using the random integral mappings $I_{(a,b)}^{h,r}$ (cf. the survey article Jurek (2011)), we give a simpler proof that is also valid for measures on Banach spaces. Furthermore, using our technique we establish yet other relations between those two families of improper stochastic integrals.

Mathematics Subject Classifications (2010): Primary 60E07, 60H05, 60B11; Secondary 44A05, 60H05, 60B10.

Key words and phrases: Lévy process; infinite divisibility; random integral; tensor product; image measures; product measures; Banach space.

Abbreviated title: Factorization of some random integrals

*Research funded by Narodowe Centrum Nauki (NCN) Dec2011/01/B/ST1/01257
†Part of this work was done when Author was visiting Indiana University, Bloomington, Indiana, USA in 2013.
In the last few decades there have appeared many papers on random integral representations of convolution subsemigroups of the (master) semigroup, ID, of all infinitely probability distributions. Jurèk-Vervaat (1983) on the class, L, of selfdecomposable measures seems to be one of the first in that area. For more references cf. Jurek (2011), Sato (2006) and Maejima, Perez-Abreu and Sato (2012). Some of the subsemigroups were introduced via the random integrals while the others were described by transformations of the Lévy (spectral) measures of some infinitely divisible distributions. The latter approach was presented already in Jurek (1990) and the resulting measures were called there as $\lambda$-mixtures. Most of that research was done in Euclidean spaces but we have also techniques and proofs that are applicable in any infinite dimensional separable Banach space.

In this note using random integral technique we provide shorter and simpler proofs of the factorization property of the two transforms (integral operators) introduced in Sato (2006). It seems that the general random integral method is more useful than considerations of some specific cases.

1. For an interval $(a, b]$ in the positive half-line, two deterministic functions $h$ (space change) and $r$ (inner clock time change), and a Lévy process $Y_\nu(t)$, $t \geq 0$ on a real separable Banach space $E$, where $\nu \in ID$ is the law of random variable $Y_\nu(1)$, we consider the following mapping (or the operator):

$$\nu \mapsto I_{(a,b]}^{h,r}(\nu) := \mathcal{L} \left( \int_{(a,b]} h(t) \, dY_\nu(r(t)) \right) \quad (\ast)$$

and $\mathcal{L}$ denotes the probability distribution of the random (stochastic) integral. Random integrals $(\ast)$ are defined by formal integration by parts formula, i.e.,

$$\int_{(a,b]} h(t) dY_\nu(r(t)) := h(b)Y_\nu(r(b)) - h(a)Y_\nu(r(a)) - \int_{(a,b]} Y_\nu(r(t)-)dh(t) \in E,$$

cf. Jurek and Vervaat (1983) or Jurek and Mason (1993) for a discussion on the above random integrals.

Improper mappings $I_{(a,\infty]}^{h,r}$ are defined as limits as $b \to \infty$; similarly, as limits, are defined the improper random integrals $I_{(a,b]}^{h,r}$ cf. Jurek (2011) (invited Section Lecture at 10th Vilnius Conference on Probability in 2010) or Jurek (2012).

Recall here that the integral $I_{(a,b]}^{h,r}$ commute with each other, that is,

$$I_{(a_1,b_1]}^{h_1,r_1}(I_{(a_2,b_2]}^{h_2,r_2}(\mu)) = I_{(a_2,b_2]}^{h_2,r_2}(I_{(a_1,b_1]}^{h_1,r_1}(\mu)),$$
provided μ is in appropriate domains. It follows from the Lévy-Khintchine formula for characteristic functions of infinitely divisible distributions; cf. for details Jurek (2012).

2. For \(-\infty < \beta < \alpha < \infty\), let us define the following two families of time change clocks:

\[
r_{\alpha}(t) := \int_{t}^{\infty} u^{-\alpha-1} e^{-u} du, \quad \text{for } 0 < t < \infty; \quad \text{and}
\]

\[
r_{\beta,\alpha}(t) := (\Gamma(\alpha - \beta))^{-1} \int_{t}^{1} (1 - u)^{\alpha-\beta-1} u^{-\alpha-1} du, \quad \text{for } 0 < t < 1. \tag{1}
\]

Sato (2006) used the implicitly given inverse functions \(r_{\alpha}^{-1}\) and \(r_{\alpha,\beta}^{-1}\) to define two improper random integrals. In our notations these were random integral mappings \(I_{t, r_{\alpha}(t)}(0, \infty)\) and \(I_{s, r_{\alpha,\beta}(s)}(0, 1)\). One of the main result is the following factorizations of the above two mappings:

**PROPOSITION 1.** For \(-\infty < \beta < \alpha < \infty\) and infinitely divisible \(\nu\), on a real separable Banach space, such that the following integrals are well defined we have that

\[
I_{t, r_{\alpha}(t)}(0, \infty) (I_{s, r_{\alpha,\beta}(s)}(0, 1)) = I_{t, r_{\alpha}(t)}(0, \infty) (\nu) = I_{s, r_{\alpha,\beta}(s)}(0, 1) (\nu) \tag{2}
\]

**Remark 1.** (i) Above we keep the explicite form the inner clock time for an easy reference and comparison.

(ii) For general questions related to domains of the above random integrals we refer to Sato (2006) and Jurek (2012). However, from Jurek (2012), Corollary 10, we infer that in (2) for \(\nu\) we can take stable measures with the exponent \(p > \alpha\).

(iii) Also, the proof below is valid for any real separable infinite dimensional Banach space – not only for Euclidean space \(\mathbb{R}^d\) as it is in Sato (2006).

**Proof of Proposition 1.** As in Theorem 2, Section 4.2 in Jurek (2012), let us define Borel measures \(\rho_i\) using the inner clock time change from (1).

Namely, let

\[
\rho_1((c, d]) := \int_{[c,d]} u^{-\beta-1} e^{-u} du, \quad (c, d] \subset (0, \infty) \tag{3}
\]

and

\[
\rho_2((c, d]) := (\Gamma(\alpha - \beta))^{-1} \int_{[c,d]} (1 - u)^{\alpha-\beta-1} u^{-\alpha-1} du, \quad (c, d] \subset (0, 1) \tag{4}
\]
Furthermore, let define the space change functions as follows

\[ h_1(t) := t, \ t \in (0, \infty) \quad \text{and} \quad h_2(s) := s, \ s \in (0, 1) \quad (5) \]

Finally, let

\[ \rho := \rho_1 \times \rho_2 \quad \text{and} \quad h(t, s) := h_1(t)h_2(s) \quad (\text{tensor product}) \quad (6) \]

Now observe that for the image measure \( h\rho \) and \( u > 0 \) we have

\[
(h\rho)(x : x > u) = \int_0^\infty 1_{(x,x>u)}(v)h\rho(dv) = \int_0^\infty \int_0^1 1_{(x,x>u)}(s\cdot t)\rho_1(ds)\rho_2(dt) \\
= (\Gamma(\alpha-\beta))^{-1} \int_0^\infty \left( \int_0^1 1_{(x,x>u)}(s\cdot t)(1-s)^{\alpha-\beta-1}s^{-\alpha-1}ds \right) t^{-\beta-1} e^{-t} \, dt \\
= (\Gamma(\alpha-\beta))^{-1} \int_0^\infty \left( \int_0^1 1_{(x,x>u)}(w)(1-w)^{\alpha-\beta-1}w^{-\alpha-1}dw \right) t^{-\beta-1} e^{-t} \, dt \\
= (\Gamma(\alpha-\beta))^{-1} \int_0^\infty 1_{(x,x>u)}(w)(w-t)^{\alpha-\beta-1}w^{-\alpha-1}dw \, e^{-t} \, dt \quad (\text{changing order}) \\
= (\Gamma(\alpha-\beta))^{-1} \int_0^\infty 1_{(x,x>u)}(w)w^{-\alpha-1}(w-t)^{\alpha-\beta-1}e^{-t} \, dt \, dw \\
= (\Gamma(\alpha-\beta))^{-1} \int_0^\infty 1_{(x,x>u)}(w)w^{-\alpha-1}e^{-w} \left( \int_w^\infty (t-w)^{\alpha-\beta-1}e^{-(t-w)} \, dt \right) \, dw \\
= \int_0^\infty w^{-\alpha-1}e^{-w} \, dw.
\]

Hence and from Theorem 2 in Jurek (2012) we get the equality (2) which completes the proof.

**COROLLARY 1.** (a) For \(-\infty < \beta < \alpha < \infty\) and the inner clock changes \( r_\alpha \) and \( r_{\beta, \alpha} \) given in (1) we have a factorization

\[
I_{(0, \infty)}^{t, r_\beta(t)} \circ I_{(0,1)}^{s, r_{\beta, \alpha}(s)} = I_{(0, \infty)}^{t, r_\alpha(t)}
\]

(b) For \(-\infty < \alpha_k < \alpha_{k-1} < \alpha_{k-2} < \ldots < \alpha_2 < \alpha_1 < \infty\) we have

\[
I_{(0, \infty)}^{t, r_{\alpha_k}(t)} \circ I_{(0,1)}^{s, r_{\alpha_k, \alpha_{k-1}}(s)} \circ I_{(0,1)}^{s, r_{\alpha_{k-1}, \alpha_{k-2}}(s)} \circ \ldots \circ I_{(0,1)}^{s, r_{\alpha_2, \alpha_1}(s)} = I_{(0, \infty)}^{t, r_{\alpha_1}(t)},
\]

where \( \circ \) denotes the composition of the random integral mappings.

Proofs follows from Proposition 1 by mathematical induction argument.

3. The following factorization was predicted but not proved in Sato (2006) in Comment 2 on p. 86. Here it is phrased in the terms of our integral mappings \( I_{(a,b)}^{h,r} \)
PROPOSITION 2. For $-\infty < \gamma < \beta < \alpha < \infty$ and an infinitely divisible $\nu$, on a real separable Banach space, such that the following integral are well defined, we have that

$$I_{(0,1)}^{l,r,\alpha,\nu}(t) \circ I_{(0,1)}^{l,r,\beta,\nu}(s) = I_{(0,1)}^{l,r,\alpha,\nu}(s) \circ I_{(0,1)}^{l,r,\beta,\nu}(t) = I_{(0,1)}^{l,r,\alpha,\beta}(t)(\nu) \quad (7)$$

Proof of Proposition 2. For later use let recall the relation between the special functions beta and gamma. Namely, for $a > 0$, $b > 0$

$$B(a,b) := \int_0^1 (1-u)^{a-1}u^{b-1}du, \quad \text{and} \quad B(a,b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$$

As in proof of Proposition 1 we use Theorem 2 from Jurek (2012).

For the Lévy exponent of the ID measure on the right hand side in (7) we have

$$\Gamma(\alpha-\beta) \Gamma(\beta-\gamma) \int_0^1 \Phi(st, y) \left| dr_{\alpha,\beta}(t) \right| \left| dr_{\beta,\gamma}(t) \right|$$

$$= \int_0^1 \left( \int_0^1 \Phi(st, y)(1-t)^{\alpha-\beta-1}t^{\alpha-1}dt \right) (1-s)^{\beta-\gamma-1}s^{-\beta-1}ds \quad (\text{put } st =: w)$$

$$= \int_0^1 \left( \int_0^w \Phi(w, y)(1-w)^{\alpha-\beta-1}(w)^{-\alpha-1}d\frac{w}{s} \right) (1-s)^{\beta-\gamma-1}s^{-\beta-1}ds \quad (\text{change order})$$

$$= \int_0^1 \Phi(w, y)w^{-\alpha-1} \left( \int_0^1 (s-w)^{\alpha-\beta-1}(1-s)^{\beta-\gamma-1}ds \right) dw \quad (\text{put } 1-s =: z)$$

$$= \int_0^1 \Phi(w, y)w^{-\alpha-1} \left( \int_{1-w}^1 (1-w-z)^{\alpha-\beta-1}z^{\beta-\gamma-1}dz \right) dw \quad (\text{put } (1-w)^{-1}z =: x)$$

$$= \int_0^1 \Phi(w, y)w^{-\alpha-1}(1-w)^{\alpha-\gamma-1}dw \int_0^1 (1-x)^{\alpha-\beta-1}x^{\beta-\gamma-1}dx$$

$$= B(\alpha-\beta, \beta-\gamma) \Gamma(\alpha-\gamma) \int_0^1 \Phi(w, y) \left| dr_{\alpha,\gamma}(w) \right|$$

$$= \Gamma(\alpha-\beta) \Gamma(\beta-\gamma) \int_0^1 \Phi(w, y) \left| dr_{\alpha,\gamma}(w) \right|,$$

which proves identity (7) and Proposition 2.

COROLLARY 2. For positive integer $k \geq 2$ and reals $\alpha_i$, $i = 1, 2, ..., k$ such that $-\infty < \alpha_k < \alpha_{k-1} < ... < \alpha_2 < \alpha_1 < \infty$ we have

$$I_{(0,1)}^{l,r_{\alpha_2,\alpha_1}(t)} \circ I_{(0,1)}^{l,r_{\alpha_3,\alpha_2}(t)} \circ I_{(0,1)}^{l,r_{\alpha_4,\alpha_3}(t)} \circ ... \circ I_{(0,1)}^{l,r_{\alpha_k,\alpha_{k-1}}(t)} = I_{(0,1)}^{l,r_{\alpha_k,\alpha_1}(t)}$$

where $\circ$ denotes the composition of the random integral mappings.
Its proof follows from Proposition 2 via the induction argument.

Last but not least, from the few instances showed in this note, one may expect that the images of measures through tensor product will find more applications and may provide simpler proofs as well; cf. Jurek (2012).

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