Numerical solution of the complex modified Korteweg-de Vries equation by DQM

Ali Başhan¹, Yusuf Uçar², N. Murat Yağmurlu², Alaattin Esen²

¹Department of Mathematics, Faculty of Science and Art, Bulent Ecevit University, Zonguldak, 67100, Turkey
²Department of Mathematics, Faculty of Science and Art, Inonu University, Malatya, 44200, Turkey
E-mail: alibashan@gmail.com

Abstract. In this paper, a method based on the differential quadrature method with quintic B-spline has been applied to simulate the solitary wave solution of the complex modified Korteweg-de Vries equation (CMKdV). Three test problems, namely single solitary wave, interaction of two solitary waves and interaction of three solitary waves have been investigated. The efficiency and accuracy of the method have been measured by calculating maximum error norm \( L_\infty \) for single solitary waves having analytical solutions. Also, the three lowest conserved quantities and obtained numerical results have been compared with some of the published numerical results.

1. Introduction
In nature, various problems are modeled by partial differential equations. Being one of the well-known natural phenomena it is also a model for nonlinear evolution of plasma waves [1], propagation of transverse waves in a molecular chain model [2], and in a generalized elastic solid [3, 4]. Because of its importance, many researchers have dealt with the Complex Modified Korteweg-de Vries (CMKdV) equation given in the following form

\[
\frac{\partial W(x,t)}{\partial t} + \alpha \left( |W(x,t)|^2 W(x,t) \right) \frac{\partial W(x,t)}{\partial x} + \frac{\partial^3 W(x,t)}{\partial x^3} = 0, \quad -\infty < x < \infty, \quad t > 0, \quad (1)
\]

where \( W \) is a complex valued function of the spatial coordinate \( x \) and the time \( t \), \( \alpha \) is a constant parameter. To avoid complex computation, we need to transformation of CMKdV equation (1) into a nonlinear coupled system by decomposing \( W(x,t) \) into its real and imaginary parts

\[
W(x,t) = U(x,t) + iV(x,t), \quad i = \sqrt{-1}
\]

and obtain the real valued-modified Korteweg-de Vries equation system,

\[
U_t + \alpha \left[ 3U^2 U_x + V^2 U_x + 2UU V_x \right] + U_{3x} = 0, \quad (2)
\]

\[
V_t + \alpha \left[ 3V^2 V_x + U^2 V_x + 2UV U_x \right] + V_{3x} = 0, \quad (3)
\]

where \( U(x,t) \) and \( V(x,t) \) are real functions.
2. Quintic B-spline DQM

Bellman et al.[8] first introduced DQM in 1972 where partial derivative of a function with respect to a coordinate direction is expressed as a linear weighted sum of all the functional values at all mesh points along that direction[9]. Let’s take the grid distribution \( a = x_1 < x_2 < \cdots < x_N = b \) of a finite interval \([a, b]\) into consideration. Provided that any given function \( U(x) \) is smooth enough over the domain, its derivatives with respect to \( x \) at a grid points \( x_i \) can be approximated by a linear summation of all the functional values in the domain, namely,

\[
U_x^{(r)}(x_i) = \frac{d^r U}{dx^r}(x_i) = \sum_{j=1}^{N} w_{ij}^{(r)} U(x_j), \quad i = 1, 2, \ldots, N, \quad r = 1, 2, \ldots, N - 1
\]

where \( r \) denotes the order of derivative, \( w_{ij}^{(r)} \) represent the weighting coefficients of the \( r \)-th order derivative approximation, and \( N \) denotes the number of grid points in the solution domain. Here, the index \( j \) represents the fact that \( w_{ij}^{(r)} \) is the corresponding weighting coefficient of the functional value \( U(x_j) \). The quintic B-splines used as a base functions which are defined as given in [10].

2.1. Weighting coefficients of the first order derivative

From Eq.(4) with value of \( r = 1 \), and using quintic B-splines as test functions we have obtained the following equations

\[
Q_k(x_i) = \sum_{j=k-2}^{k+2} w_{ij}^{(1)} Q_k(x_j), \quad (5)
\]

For example, for the first grid point \( x_1 \) (5), we get the following equation

\[
Q_1'(x_1) = \sum_{j=0}^{N+2} w_{1,j}^{(1)} Q_1(x_j), \quad (6)
\]

By substituting the values of quintic basis functions into Eq.(6) and using four additional equations obtained from the derivative of Eq.(6) at four different B-spline \( Q_k \) \((k = -1, 0, N + 1, N + 2)\) and eliminating four unknown terms from the system of equations, we obtain the following system of equations

\[
\begin{bmatrix}
37 & 82 & 21 \\
8 & 33 & 18 & 1 \\
1 & 26 & 66 & 26 & 1 \\
1 & 26 & 66 & 26 & 1 \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 26 & 66 & 26 & 1 \\
1 & 18 & 33 & 8 \\
21 & 82 & 37 \\
\end{bmatrix}
\begin{bmatrix}
w_{1,-1}^{(1)} \\
w_{1,0}^{(1)} \\
w_{1,1}^{(1)} \\
w_{1,2}^{(1)} \\
w_{1,3}^{(1)} \\
w_{1,4}^{(1)} \\
\vdots \\
w_{1,N+1}^{(1)} \\
w_{1,N+2}^{(1)} \\
\end{bmatrix} = 
\begin{bmatrix}
-109 \\
-29 \\
0 \\
50 \\
26 \\
0 \\
\vdots \\
0 \\
0 \\
\end{bmatrix}
\]

(7)

Similarly, using the value of quintic basis functions at \( x_i \), \((2 \leq i \leq N)\) grid points, respectively, the equation systems is obtained which are used to determine the weighting coefficients.

So, weighting coefficients \( w_{ij}^{(1)} \) which are related to the \( x_i \), \((i = 1, 2, \ldots, N)\) are found quite easily by solving the obtained equation systems with Thomas algorithm. Determining of the weighting coefficients of the third order derivatives have same process.
3. Numerical discretization

We discretize the equations (2) – (3) separately by using forward finite difference and Crank-Nicolson.

\begin{equation}
2U^{n+1} + \Delta t \left[ U_{3x}^{n+1} + \alpha \left( 3 \left( U^2 U_x \right)^{n+1} + \left( V^2 U_x \right)^{n+1} + 2 \left( UV V_x \right)^{n+1} \right) \right]
= 2U^n + \Delta t \left[ -U_{3x}^n + \alpha \left( 3 \left( U^2 U_x \right)^n + \left( V^2 U_x \right)^n + 2 \left( UV V_x \right)^n \right) \right]
\end{equation}

Then, Rubin and Graves linearization technique\cite{11} is used at the left side of the Eq. (8) to linearize the nonlinear terms so we obtained

\begin{equation}
2U^{n+1} + \Delta t \left[ U_{3x}^{n+1} + 3 \alpha \left( U^2 U_x U_{n+1}^n + 2 U^n U_{x}^n \right) \right] + 3 \alpha \left( 2 U_x U_{n+1}^n + 3 \alpha \left( 2 \right)^n \right)
= 2U^n + \Delta t \left[ -U_{3x}^n + 3 \alpha \left( 2 U_x U_n \right) + \alpha \left( 2 \right)^n \right]
\end{equation}

Let us define some terms to use in Eq. (9) as,

\begin{equation}
A_i^n = \sum_{j=1}^{N} w_{ij}^{(1)} U_j^n = U_{x_i}, \quad B_i^n = \sum_{j=1}^{N} w_{ij}^{(3)} U_j^n = U_{3x_i}, \quad C_i^n = \sum_{j=1}^{N} w_{ij}^{(1)} V_j^n = V_{x_i}, \quad D_i^n = \sum_{j=1}^{N} w_{ij}^{(3)} V_j^n = V_{3x_i}
\end{equation}

where \( A_i^n \) and \( B_i^n \) are the first and third-order derivative approximations of \( U \) functions at the n-th time level on \( x_i \) points, respectively. And \( C_i^n \) and \( D_i^n \) are the first and third order derivative approximations of \( V \) function at the \( n - th \) time level on \( x_i \) points, respectively. By the substitution of definition (10) at Eq. (9) we obtained

\begin{equation}
2U_i^{n+1} + \Delta t \left[ \sum_{j=1}^{N} w_{ij}^{(3)} U_j^{n+1} + 3 \alpha \left( U_i^n \right) \sum_{j=1}^{N} w_{ij}^{(1)} U_j^{n+1} + 2 U_i^n \right]
+ 3 \alpha \left( 2 U_x U_i^n \right)
= 2U_i^n + \Delta t \left[ -B_i^n + 3 \alpha \left( 3 \left( U_i^n \right)^2 A_i^n + \left( V_i^n \right)^2 A_i^n + 2 U_i^n V_i^n C_i^n \right) \right]
\end{equation}

we reorganised Eq. (11) for each grid points as,

\begin{equation}
2 + \Delta t \left( w_{ii}^{(3)} + \alpha \left( 3 \left( U_i^n \right)^2 w_{ii}^{(1)} + 6 U_i^n A_i^n + \left( V_i^n \right)^2 w_{ii}^{(1)} + 2 V_i^n C_i^n \right) \right)
\left[ U_i^{n+1} \right]
+ \sum_{j=1,i \neq j}^{N} \Delta t \left( w_{ij}^{(3)} + \alpha \left( 3 \left( U_i^n \right)^2 w_{ij}^{(1)} + \left( V_i^n \right)^2 w_{ij}^{(1)} \right) \right) U_j^{n+1}

+ \left[ 2 \alpha \left( V_i^n A_i^n + U_i^n V_i^n w_{ii}^{(1)} \right) \right] V_i^{n+1}
+ \sum_{j=1,i \neq j}^{N} \left( 2 \alpha \left( U_i^n V_i^n w_{ij}^{(1)} \right) \right) V_j^{n+1} = f_i^n
\end{equation}

By the same process, the Eq. (3) is discretized, linearized and organised. Then, boundary conditions have been applied to system of equations and the first and last equations are eliminated from each systems and solved by Gauss elimination method easily.
4. Numerical examples

The accuracy of the numerical method is checked by using the error norm $L_\infty$ and three lowest invariants:

$$L_\infty \simeq \max \left| U_j^{\text{exact}} - (U_N)_j \right|, \quad I_1 = \int_{-\infty}^{\infty} w dx,$$

$$I_2 \simeq \sum_{j=1}^{N} h_j |w_j^n|^2, \quad I_3 \simeq \sum_{j=1}^{N} h_j \left[ \frac{\alpha}{2} |w_j^n|^4 - (w_{x,j}^n)^2 \right]$$

4.1. Single soliton

The analytically solution of complex mKdV equation is given in [5] as:

$$W(x, t) = \sqrt{\frac{2c}{\alpha}} \text{sech} \left[ \sqrt{c}(x - x_0 - ct) \right] \exp (i\theta)$$

where soliton standing at $x_0$ position initially and moving to the right hand with constant $c$ velocity and satisfies the boundary conditions $W \to 0$ as $x \to \pm \infty$. We first take $\alpha = 2$, $\theta = 0$, $c = 1$, $x_0 = 0$ in $[-20, 40]$ and at $t = 0$, we obtain the initial condition.

Figure 1. Single soliton

Figure 2. Absolute error

| $t$     | $L_\infty$ Present, $N = 336$ | $I_1$ | $I_2$ | $I_3$ | $L_\infty$ Present, $N = 600$ | $I_1$ | $I_2$ | $I_3$ |
|---------|-------------------------------|-------|-------|-------|-------------------------------|-------|-------|-------|
| 0       | 0.000000          3.141590 | 2.000000 | 0.666667 |       | 0.000000          3.141590 | 2.000000 | 0.669765 |
| 5       | 0.000065          3.141578 | 1.999999 | 0.666666 |       | 0.000057          3.141592 | 2.000000 | 0.669764 |
| 10      | 0.000070          3.141606 | 1.999999 | 0.666666 |       | 0.000108          3.141592 | 1.999999 | 0.669764 |
| 15      | 0.000068          3.141588 | 1.999999 | 0.666666 |       | 0.000163          3.141592 | 1.999999 | 0.669763 |
| 20      | 0.000066          3.141572 | 2.000000 | 0.666667 |       | 0.000218          3.141592 | 1.999999 | 0.669763 |

As it is seen from the Figure 1, with the time run up to the $t = 20$ the amplitude and velocity of wave do not change as a result of properties of solitons. As it is seen straightforward from Table 1 and 2 the present error norms $L_\infty$ are smaller than earlier works [6, 7]. It is seen from Figure 2 that the maximum absolute error at time $t = 20$ is found $4.02 \times 10^{-5}$ at $x = 19.94$.

4.2. Interaction of two solitary waves

The sum of two solitary waves which initial condition is given in [7] as:

$$W(x, 0) = \sqrt{\frac{2c_1}{\alpha}} \text{sech} \left[ \sqrt{c_1}(x - x_1) \right] \exp (i\theta_1) + \sqrt{\frac{2c_2}{\alpha}} \text{sech} \left[ \sqrt{c_2}(x - x_2) \right] \exp (i\theta_2)$$
Table 2. $L_\infty$ error norms and invariants for $\theta = \pi/4$ and $\Delta t = 0.01$.

| Time | Present, $N = 336$ | Coll. FEM [7], $N = 600$ |
|------|---------------------|--------------------------|
|      | $L_\infty$ | $L_2$ | $L_3$ | $L_\infty$ | $L_2$ | $L_3$ |
| 0.0  | 0.000000 | 4.000000 | 1.333344 | 0.000000 | 4.000000 | 1.339529 |
| 5.0  | 0.000049 | 4.000000 | 1.333344 | 0.000078 | 3.999999 | 1.339529 |
| 10.0 | 0.000051 | 3.999999 | 1.333332 | 0.001540 | 3.999998 | 1.339528 |
| 15.0 | 0.000050 | 3.999998 | 1.333333 | 0.002310 | 3.999998 | 1.339528 |
| 20.0 | 0.000049 | 3.999999 | 1.333334 | 0.003080 | 3.999997 | 1.339526 |

where $x_1 = 25$ and $x_2 = 50$ are initial positions of two solitary waves, respectively in [0, 100].

We investigated the interaction of two orthogonally polarized solitary waves which are interact with $y$-polarized ($\theta_1 = 0$) and $z$-polarized ($\theta_2 = \pi/2$) then the interaction of two $y$-polarized solitary waves which are interact with $y$-polarized ($\theta_1 = \theta_2 = 0$). We used fix value of $\alpha = 2$, $c_1 = 2$, $c_2 = 0.5$, and both simulations time run up to $t = 30$. As it seen clearly from Figure 3 and Figure 4 that at simulation of two orthogonally polarized solitary waves after the interaction a tail appeared behind the shorter wave and in opposition to two orthogonally polarized solitary waves there is not any tail appeared after the interaction of two $y$-polarized solitary waves. The obtained invariants given in Table 3 is acceptable good.

Table 3. Invariants of two orthogonally and two $y$-polarized solitons for $\Delta t = 0.01$.

| Time | two orthogonally polarized solitary waves | two $y$-polarized solitary waves |
|------|----------------------------------------|----------------------------------|
|      | $I_1(\Re a)$ | $I_1(\Im a)$ | $I_2$ | $I_3$ | $I_2$ | $I_3$ |
| 0    | 3.141593 | 3.141591 | 4.242640 | 2.121332 | 4.242640 | 2.121333 |
| 5    | 3.141630 | 3.141593 | 4.242590 | 2.121228 | 4.242587 | 2.121222 |
| 10   | 3.141667 | 3.141593 | 4.242537 | 2.121124 | 4.242529 | 2.121112 |
| 15   | 3.141534 | 3.141590 | 4.242371 | 2.120882 | 4.242330 | 2.120804 |
| 20   | 3.141566 | 3.141598 | 4.242489 | 2.120443 | 4.242473 | 2.121001 |
| 25   | 3.141812 | 3.142181 | 4.242424 | 2.120901 | 4.242431 | 2.120914 |
| 30   | 3.138532 | 3.135895 | 4.242476 | 2.120740 | 4.242377 | 2.120808 |

4.3. Interaction of three solitons

Our third test problem is interaction of three solitons which is given in [7] as follows

$$W(x, 0) = \sqrt{\frac{2c_1}{\alpha}} \sech [\sqrt{c_1} (x - x_1)] \exp (i\theta_1) + \sqrt{\frac{2c_2}{\alpha}} \sech [\sqrt{c_2} (x - x_2)] \exp (i\theta_2)$$
where $x_1 = 10$, $x_2 = 30$ and $x_3 = 50$ are initial positions of three single solitons, respectively. We investigated the interaction of three solitons which are interact with $y$-polarized ($\theta_1 = \theta_2 = \theta_3 = 0$) by using $\alpha = 2, c_1 = 1, c_2 = 0.5$ and $c_3 = 0.3$ time up to $t = 80$.

![Figure 5. Three y-polarized solitons](image)

Table 4. Invariants of three $y$-polarized solitons for $\Delta t = 0.01$.  

| $t$ | $I_2$ | $I_3$ | $I_2$ | $I_3$ |
|-----|-------|-------|-------|-------|
| 0   | 4.510004 | 1.012099 | 4.510006 | 1.015897 |
| 20  | 4.510002 | 1.012098 | 4.510005 | 1.015783 |
| 40  | 4.509997 | 1.012103 | 4.510006 | 1.015031 |
| 60  | 4.510002 | 1.012097 | 4.510008 | 1.015660 |
| 80  | 4.510003 | 1.012096 | 4.510005 | 1.015752 |

5. Conclusion

In this work, we have implemented DQM based on quintic B-splines for numerical solution of complex mKdV equation. One of the main characteristics of the present method is to be able to obtain good results by using less number of grid points. As can be observed by the comparison between the obtained results of present method and earlier works, QBDQM results are acceptable good. The obtained results show that QBDQM can be used to produce reasonable accurate numerical solutions of the complex mKdV equation. So, QBDQM is a reliable one for getting the numerical solutions of some physically important nonlinear problems.

6. References

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