Analysis on image recovery for digital Fresnel hologram with aliased fringe generated from self-similarity of point spread function

BYUNG GYU CHAE

Biomedical Imaging Group, Electronics and Telecommunications Research Institute, 218 Gajeong-ro, Yuseong-gu, Daejeon 305-700, South Korea

*bgchae@etri.re.kr

Abstract: We investigate the aliasing phenomenon for digital Fresnel hologram with an enhanced numerical aperture (NA). The enhanced-NA digital Fresnel hologram acquired computationally or optically at a closer distance from the object has an aliased fringe generated by undersampling process of the Fresnel prefactor called as a point spread function. The point spread function shows a self-similar envelope of concentric Fresnel zone when being sampled, which becomes a crucial mechanism in making this type of aliasing fringe of hologram. We describe that as the enhanced-NA hologram involves already the Fresnel factor with complementary aliasing error that might come up in the reconstruction process, the robust recovery of object image can be realized. Numerical simulation reveals that the original image is completely recovered, and the subsequently propagated diffraction wave from the image is well described even at a severe aliasing environment. This result makes it possible to obtain the high-resolution holographic image by using digital sensor with a finite pixel size.

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1. Introduction

Digital holography is an imaging process which retrieves an original image from the digital hologram recording an object wave [1,2]. The digital hologram has a finite spatial-bandwidth product due to its sampling on a digitized device [3]. The sampling condition on the basis of Fourier optics has been widely studied especially for the Fresnel diffraction fields [4-6]. The present digital sensor has a finite size and consists of pixel array with several micrometer pixel pitch. Thus, there exist some restrictions about an object size and resolving power of restored image in the imaging procedure to avoid an aliasing effect. Particularly, the resolution enhancement is very essential in acquiring restored image of a high quality [7,8]. Several researches [9,10] such as a synthetic aperture imaging have been performed in the manner of the increase of hologram aperture size. Furthermore, various algorithms for adjustable change of output resolution have been developed to overcome a drawback having a finite resolution in the reconstruction process [11-13], where the intrinsic resolution would be still restricted by a numerical aperture (NA).

In general, the spatial resolution of restored image is determined by the NA of hologram, which is proportional to its lateral size and in inverse proportion to a reconstruction distance. We can expect that the high numerical aperture is accomplished by acquiring the hologram at a distance close to the object. In this case, the aliasing error of hologram fringe could be produced, and hence, it is important to comprehend the effect of aliasing error on the hologram acquisition and image reconstruction processes. The analysis of aliasing phenomenon for the digital hologram is rather complicated despite an abundant literature [4-6,14-16], because it is the sampling of a propagating diffraction field at a proper distance other than a simple Fourier analysis of image intensity.

In-line holographic system has a distinct advantage to acquire the digital hologram for the object with a large field size and high resolution rather than off-axis technology. Phase-
shifting holography enables us to directly acquire the complex hologram to remove the annoying effects such as the zeroth-order term and twin image in the reconstruction process [17,18]. Here, the analysis of sampling condition for complex hologram is to investigate the sampled diffraction field itself. The diffracted Fresnel field is described as the convolution of the object field and point spread function known as optical kernel. The point spread function $h(x,y)$ has a quadratic phase term showing a rapid oscillation with lateral space [19,20]:

$$h(x,y) = \frac{e^{ikz}}{i\lambda z} \exp \left[ i \frac{\lambda}{2z} \left( x^2 + y^2 \right) \right].$$  \hspace{1cm} (1)

where $k$ is wavenumber of $\lambda/2\pi$ with wavelength $\lambda$, and $z$ is a distance between object plane and hologram plane. Since the Fresnel diffraction involves the Fourier transform of a product of object field and quadratic phase term, the sampling theorem based on the Fourier analysis is applicable by itself. Typical approach to interpret an aliasing effect for diffracted field is to investigate the sampling condition about the quadratic phase term. In this analysis [14-16], the sampling condition is well defined for the geometrical configuration such as the object field size and propagation distance.

In practical applications, since digital hologram is loaded on the digitized device with a finite extent, an aliased fringe of hologram could be generated owing to a finite pixel pitch of device although it obeys the sampling condition based on the Fourier analysis with respect to object plane. This type of aliasing error can be analyzed by using a Fresnel prefactor in the hologram plane [15], where the sampling rate of diffracted field should fulfill the Nyquist criterion as well. When the pixel pitch of digital hologram is larger than that of sampling rate of diffracted field, the undersampling of hologram fringe arises. The digital hologram generated at a close distance will reveal a severe undersampling phenomenon because of the dense fringe pattern of diffraction field. The undersampling leads to an aliasing error of hologram fringe, which could be assumed to act on the restored image deterioration.

Recently, it has been known that for the object with a finite size, the hologram periodically undersampled can restore the original object image completely by filtering the modulated replica images [5,6]. This undersampling is principally different from that of above hologram fringe where only the Fresnel prefactor except for the spectrum of diffraction field is undersampled, but there seems to have a robust recovery property of image at lower sampling of optical kernel function. Thus, the analysis of hologram fringe itself with respect to undersampling is necessary for detailed understanding.

In this study, we analyze minutely an aliasing phenomenon of hologram fringe generated by undersampling process of the Fresnel prefactor, and then investigate the effect of aliasing error of hologram fringe on image recovery. It is elucidated that the point spread function has an internal structure of envelope curve when being sampled, which shows a self-similarity depending on a sampling pixel pitch. The fractal-like structure of optical kernel function becomes a key factor inducing an aliasing fringe of hologram. We carry out numerical simulation for the hologram synthesized at a closer distance from the object, where the hologram having the enhanced numerical aperture is highly undersampled by a finite pixel. The property of hologram fringe and image recovery is studied especially for the digital hologram suffering an overly aliased error.

2. Analysis on aliasing effect for digital Fresnel hologram with enhanced numerical aperture

2.1 Aliasing effect of digital Fresnel hologram

The convolution form of the Fresnel diffraction formula describing the Fresnel field $U(x,y)$ diffracted from the object field $U_0(\xi,\eta)$ can be written by
\[ U(x,y) = \frac{e^{ikz}}{i\lambda z} \left[ \int \int U_0(\xi,\eta) \exp \left[ \frac{i\pi}{\lambda z} (x-\xi)^2 + (y-\eta)^2 \right] d\xi d\eta \right]. \]  

(2)

In order to investigate aliasing effect for diffraction field sampling, we expand above formula as follows,

\[ U(x,y) = \frac{e^{ikz}}{i\lambda z} \exp \left[ \frac{i\pi}{\lambda z} (x^2 + y^2) \right] \text{FT} \left[ \frac{U_0(\xi,\eta) \exp \left[ \frac{i\pi}{\lambda z} (\xi^2 + \eta^2) \right]}{} \right]. \]  

(3)

This equation is represented as a multiplication of two separate parts: a preceding point spread function and the Fourier transform \( \text{FT} \) of a product of object field and embedded point spread function. The spectrum analysis of propagating diffraction field is carried out by using the Fourier transform term.

For convenience, we choose an in-line holographic system using a complex hologram. This approach allows us to interpret the aliasing effect using the propagated diffraction field itself. In addition, one-dimensional description for both fields discretized into \( N \times N \) pixels will be used hereafter. According to the Fourier analysis, the spatial frequency is \( \frac{\lambda}{z} \) with lateral coordinate, and thus the pixel resolution \( \Delta x \) of hologram field is defined in terms of resolution \( \Delta \xi \) of the object field [1]:

\[ \Delta x = \frac{\lambda z}{N \Delta \xi}. \]  

(4)

This value is explained by the NA of digital hologram, which in free space, is geometrically given by

\[ \text{NA} = \sin \Theta = \frac{N \Delta x}{2z}. \]  

(5)

Based on the Abbe criterion, the resolution limit \( R_{\text{Abbe}} = \Delta \xi \) of restored image has the following relation [21],

Fig. 1. Schematic diagram for the configuration of digital hologram and object images in in-line holographic system. The object resolution increases at a closer distance from digital hologram. Red and blue lines indicate the diffraction zone with respect to pixel pitch of hologram and object, respectively.

For convenience, we choose an in-line holographic system using a complex hologram.
\[ R_{\text{Abbe}} = \frac{\lambda}{2 \text{NA}}. \]  

(6)

Figure 1 depicts the configuration of digital hologram and object images in the hologram synthesis or acquisition system. Considering a finite device with constant pixel pitch, the object size at a closer distance is reduced due to \( \Delta \xi \) decrement in Eq. (4). This means that to realize the object image with a higher resolution, the digital hologram should be acquired at a closer distance from the object. Here, we set \( z_0 \) as the distance that both pixel sizes of object plane and hologram plane are the same, \( z_0 = N \Delta \xi^2 / \lambda \). The digital hologram to be captured at a distance lower than \( z_0 \) can get the image resolution higher than the pixel pitch of digital sensor, and hence, for convenience we define this type of hologram as an enhanced-NA hologram. The diffraction angle \( \Omega \) with respect to the pixel resolution of object field increases with decreasing a distance. Thus, the diffraction extent is subject to the diffraction angle,

\[ \Omega = 2 \sin^{-1} \left( \frac{\lambda}{2 \Delta \xi} \right). \]  

(7)

Furthermore, the object field-views in Fig. 1 are confined in the diffraction zone by a diffraction angle \( \theta \) of hologram pixel pitch to evade the high-order diffraction errors,

\[ \theta = 2 \sin^{-1} \left( \frac{\lambda}{2 \Delta \chi} \right), \]  

(8)

which will be studied in Section 4.

As described in previous Section, the aliasing phenomenon of digital hologram is examined in the manner of searching the sampling condition for the quadratic phase term in parenthesis of \( \text{FT} \) [14-16]. It is assumed that the object field is a slowly varying function compared to the quadratic phase term. To avoid an aliasing effect in the digital hologram synthesis, the sampling rate \( f_s \) of object field should be larger than two times the maximum spatial frequency \( f_{\xi, \max} \):

\[ f_s \geq 2 f_{\xi, \max} = \frac{2 \xi_{\max}}{\lambda z}, \]  

(9)

where the maximum frequency \( f_{\xi, \max} \) is estimated from the embedded quadratic phase \( \phi(\xi, \eta) \) of Eq. (3):

\[ f_{\xi, \max} = \frac{1}{2\pi} \left| \frac{\partial \phi(\xi, \eta)}{\partial \xi} \right|_{\max} = \frac{\xi_{\max}}{\lambda z}, \]  

(10)

whose value depends on both a distance and object field size. If a sampling rate in the object plane is put to be \( \Delta z \) and the relation of object field size that \( \text{NA} \xi = 2|\xi_{\max}| \) is used, well-sampling condition with respect to a synthesis distance is given by

\[ z \geq \frac{\text{NA} \xi^2}{\lambda}. \]  

(11)

On the other hand, as shown in Fig. 1, if the pixel pitch \( \Delta \chi \) of hologram field is larger than \( \Delta \xi \) of the object field at a closer distance, the hologram fringe is undersampled by its finite pixel pitch of hologram plane. When the hologram is synthesized at a distance lower than \( z_0 \), the undersampling is inevitable although the diffracted field propagates in the well-defined area from Eq. (4) and it follows the sampling rule of Eq. (11). This undersampling
process will form an aliased hologram fringe, which is unusual behavior compared to that of previous spectral analysis, i.e. it is a kind of phenomenon that there arises a spatially aliased 2D image when being undersampled. This phenomenon is characterized by using a quadratic phase factor in front of \( \text{FT} \) operation in Eq. (3). Through similar derivation to Eqs. (9)-(11), we can obtain the following sampling condition in the hologram plane:

\[
z \geq \frac{N \Delta x^2}{\lambda}. \tag{12}\]

By incorporating Eq. (4) to Eq. (12), the sampling operation without this type of aliased fringe is possible only at a constant distance \([14,15]\),

\[
z_c = \frac{N \Delta x^2}{\lambda}. \tag{13}\]

From this, we know that in the hologram with enhanced numerical aperture, an aliasing error of hologram fringe could be generated from the undersampling of oscillating phase of a preceding Fresnel factor. Here, the diffraction spectrum related to \( \text{FT} \) term is well sampled because it places within the diffraction area by object pixel pitch in Fig. 1. Surely, the Riemann integral method can avoid this aliased error because the pixel resolution of hologram plane can be arbitrary controlled. However, it is not realistic because actual device has a fixed pixel specification, as illustrated in Fig. 1. Either the convolutional method or angular spectrum method having a double Fourier transform can also evade this type aliased fringe, where the low-pass filtering takes place to prevent the aliasing error [22]. This approach is not under consideration, because it also deviates our research direction trying to find the method for obtaining a high-resolution image.

2.2 Stable recovery of holographic image under severe aliased hologram fringe

The reconstruction process is a backward propagation from digital hologram to object plane, and the diffraction equation is calculated through a reverse transform of Eq. (3):

\[
U_0(\xi, \eta) = \frac{ie^{-ikz}}{\lambda z} \exp \left[ -i \frac{\pi}{\lambda z} (\xi^2 + \eta^2) \right] \text{IFT} \left[ U(x, y) \exp \left[ -i \frac{\pi}{\lambda z} (x^2 + y^2) \right] \right]. \tag{14}\]

Here, we note that both quadratic phase terms change places. The Fresnel prefactor is conversely embedded in the parenthesis of inverse Fourier transform \( \text{IFT} \). The sampling condition derived from the quadratic phase term in the \( \text{IFT} \) operation appears to be the same as Eq. (12). The minimum distance to avoid an aliasing error appears to be \( z_0 \), and thus in the reconstruction process, the digital holograms synthesized at a distance lower than \( z_0 \) violate the sampling rule although these holograms are synthesized on the basis of well-defined sampling condition of Eq. (11). This means that considering its reconstruction process together, the well-sampling range seems to be limited to a constant distance \( z_c \).

In the digital hologram synthesized at a closer distance, the hologram fringe in the parenthesis of \( \text{IFT} \) has already complementary Fresnel factor with the aliasing error that might come up in the reconstruction process. The fringe error arises due to the increase of the maximum spatial frequency, where an error extent of fringe might be estimated by a ratio of the maximum frequencies. When the extent of error occurred in the hologram synthesis at an arbitrary distance \( z_0 \) below \( z_0 \) is represented as

\[
\text{Error} \left( \begin{array}{c} z_0 \\ z_0 \end{array} \right), \tag{15}\]
the error amount that happens in the reconstruction process will be the same as this value. Thus, even severe aliasing error generated from the hologram synthesis will be accurately compensated. From this, we find that the original image can be stably reconstructed despite of this type of aliasing error of hologram fringe.

2.3 Matrix formalism for image recovery of digital Fresnel hologram

We explain the property of image recovery through the matrix formalism. In Eq. (3), the column vector $X$ of the hologram with respect to object vector $\xi$ is represented as follows:

$$ X = \Psi_F \xi. \quad (16) $$

The Fresnel matrix $\Psi_F$ is constructed by using the Fresnel factor and Fourier matrix. The component of Fresnel matrix is given by

$$ \Psi_{mp} = H_x^m \omega^{mp} H_x^p, \quad (17) $$

where $\omega = e^{-2\pi i/N}$ and the component of Fresnel factor $H_x^m$ is

$$ \exp\left(\frac{i\pi}{\lambda z} m^2 \Delta x^2\right). \quad (18) $$

The Fresnel matrix has a unitary property because the transpose conjugate matrix is directly the inverse matrix [23]:

$$ \sum_{p=0}^{N-1} \Psi_{mp}^* \Psi_{pm} = \delta_{mnl}. \quad (19) $$

The original object can be completely restored by inverse Fresnel transform when there is no aliasing effect:

$$ \xi = \Psi_F^{-1} X. \quad (20) $$

Furthermore, it is not hard to see that the Fresnel matrix periodically undersampled is unitary, and thus this undersampled hologram can recover original image, but modulated replica image appears due to the limited diffraction area by sampling pitch [5,6].

As previously stated, a conventional approach to extract the sampling condition is to deal with the embedded Fresnel factor $H_x$. Whereas, in case of digital hologram synthesized at a closer distance lower than $z_0$ in Fig. 1, the undersampling of hologram fringe occurs only for the Fresnel prefactor $H_x$, where the spectrum term $\omega H_x$ related to the Fourier transform in Eq. (3) is correctly sampled. If the aliased Fresnel prefactor is put to be $H_{x,alias}$, the matrix form of Eq. (16) becomes

$$ X = H_{x,alias} \omega H_x \xi. \quad (21) $$

Since the aliased Fresnel factor that happens during inverse transform is like $H_{x,alias}^{-1}$, the inverse Fresnel transform is written by

$$ \xi = H_x^{-1} \omega^{-1} H_{x,alias}^{-1} X. \quad (22) $$

Substituting the hologram field of Eq. (21) into above equation, we can extract below identity with respect to Fresnel factor:

$$ I = H_{x,alias}^{-1} H_{x,alias}. \quad (23) $$
This type of aliased error from undersampling of Fresnel prefactor is canceled out, and hence, the original image is restored even at overly aliased hologram fringe.

Meanwhile, the fact that the original image is completely recovered at a proper distance does not imply the well-description of diffraction field propagation. This scheme should be able to obviously describe the propagating behavior of diffracted wave from the restored image. We can suppose that the error amount that required at a distance \( z_n \) away from the imaging plane is smaller than the quantity arisen at a proper distance \( z_n \):

\[
\text{Error} \left( \frac{z_0}{z} \right) \leq \text{Error} \left( \frac{z_0}{z_n} \right)
\]  

(24)

Referring to Eq. (23), a final error amount from the product form becomes larger than the quantity required in the reconstruction process, and thus, even the propagated diffraction field away from the restored image can be well explained within above scheme. In Section 4, we report that the well diffracted fringe is observed as a function of a propagation distance away from the imaging plane when the diffraction field is simulated by using the enhanced-NA digital hologram.

In the reconstruction process, the resolution limit of restored image is decided by Eq. (6), whose relation is related to the NA of hologram. Above mathematical property also preserves the aperture size of digital hologram. That is, the property of Fresnel matrix is consistent with the conservation of hologram numerical aperture. The simple operation that truncates the hologram surely deteriorates image resolution [10]. We confirmed a deteriorated image even in the truncated hologram with an overly aliased fringe. In addition, the hologram non-periodically undersampled worsens the image. This property means that the spatial resolution of restored image is not governed by the maximum frequency of hologram fringe, but strongly depends on the NA of digital hologram.

3. Self-similarity of hologram fringe for undersampled digital Fresnel hologram

3.1 Replication of concentric Fresnel zone by undersampling of digital Fresnel hologram

For numerical simulation, the diffraction field can be written by the discrete form using the discrete Fourier transform (DFT) [1]:

\[
U(m\Delta x, n\Delta y) = \frac{e^{ikz}}{i\Lambda z} \exp \left[ i \frac{\pi}{\Lambda z} \left( m^2 \Delta x^2 + n^2 \Delta y^2 \right) \right] 
\]

\[
\times \text{DFT} \left[ U_0(p\Delta \xi, q\Delta \eta) \exp \left[ i \frac{\pi}{\Delta \xi} \left( p^2 \Delta \xi^2 + q^2 \Delta \eta^2 \right) \right] \right],
\]  

(25)

where the fields are digitized with steps \( \Delta x \) and \( \Delta y \) in the hologram plane, and \( \Delta \xi \) and \( \Delta \eta \) in the object plane.

To analyze an aliasing effect through synthetic hologram, we calculate digital hologram about on-axis point sources located at various distances. The object and hologram has 256×256 pixels, where all the holograms have same pixel pitch of 8 \( \mu \text{m} \), in Fig. 1. The coherent plane wave has 532-nm wavelength, and in this condition, the distance \( z_0 \) having 8\( \mu \text{m} \) pixel pitch of the object is 30.8 mm. The relation of object and hologram resolutions with respect to a distance obeys Eq. (4). For convenience, we choose three points located at distances of a half and a quarter of \( z_0 \) as well as a distance of \( z_0 \). If the object field is put to be delta function \( \delta(x, y) \) located on axis, DFT term becomes constant, and the hologram fringe is formed from the sampling of preceding quadratic phase term, which is known as the Fresnel zone plate [24]. The hologram fringe shape will be adaptively determined by the sampling interval, where the minimum distance to get correct sampling, \( z_{\text{min}} = N\Delta x^2 / \lambda \).
Fig. 2. Quadratic sinusoids of point spread function for a point source located at 30.8-mm distance. The curves are drawn along vertical line at a horizontal center of hologram and sampled by a sampling pitch of (a) 8 μm, (b) 16 μm, and (c) 32 μm.

Fig. 3. Quadratic sinusoids of point spread function for point sources located at (a) 15.4-mm and (b) 7.7-mm distances.

In Fig. 2(a), the hologram synthesized using point source placed at $Z_0$ spot of $z_0$ distance shows the point spread function of a quadratic sinusoidal shape. Here, the function is drawn in one-dimensional $x$-coordinate. We know that as the resolution of hologram in the synthesis process is 8 μm, a complete sampling is expected, but we will show later that even this
sampling is not sufficient. Especially, we find that the similar patterns of quadratic sinusoid are generated when the function is undersampled. The shape of replica patterns does not exactly coincide with each other. The number of the replicas grows in proportion to the increment of sampling pitch, where the curves of Figs. 2(b) and 2(c) are undersampled by pixel intervals of 16 μm and 32 μm, respectively.

We can observe naturally this behavior when synthesizing hologram by using point sources at closer spots, B and A, as depicted in Fig. 3. Since the maximum spatial frequency of digital hologram is decided from the Fresnel factor preceding the Fourier transform in Eq. (3), the value for the hologram synthesized at a distance 15.4 mm of $z_1$ is double in comparison to that for the hologram at $z_0$ distance, where 4-μm pixel sampling is required. Here, as the sampling pixel pitch is still 8 μm, the hologram fringe should be undersampled. We note that this geometrical situation fulfills well the Nyquist sampling criterion for the object field, based on Eq. (11), but the aliasing error of hologram fringe due to undersampling is created. Likewise, the hologram at a distance 7.7 mm of $z_2$ is overly undersampled owing to the requirement of 2-μm sampling interval. The density of quadratic sinusoids in Fig. 2(a), Fig. 3(a), and Fig. 3(b) are different because of difference in corresponding maximum frequencies. The diffraction angles $\Omega_0$, $\Omega_1$, and $\Omega_2$ with respect to the pixel resolution of object images are calculated to be about 3.8°, 7.6°, and 15.3°, respectively.

Figure 4 illustrates 2D hologram fringe of point sources, representing the sinusoidal Fresnel zone plate [22]. We can see a replication of similar zone pattern apparently. The number of replicas of zone plate increases by a square of scale. The four-fold undersampling forms zone plates of 16, in Fig. 4(c). Figure 4(d) is the hologram fringe for off-axis point source located at $E$ spot in Fig. 1, where this spot is still placed within a diffraction area. Nevertheless, the replica zone plate by undersampling takes place, which is resulted from that a zone center placed at a boundary has two times the maximum frequency.
3.2 Fractal-like structure of point spread function

Particularly, we can observe that the peripheral zones between replicas come out blurry. To examine this phenomenon carefully, 2D hologram with 512×512 pixels is drawn in Fig. 5. The zoom-out operation of hologram in a digital display changes its shape due to undersampling. For clarity, the corresponding graph of quadratic sinusoid graph is drawn on the hologram fringe, as shown in inset of Fig. 5. It is certain that the sampled quadratic sinusoid has its similar pattern inherently. Namely, a self-similar pattern emerges in the form of envelope curve in accordance with a scale. This is a fractal-like character having a self-similarity. We find that the sampling operation to the point spread function induces its fractal structure, which plays a role in making an aliasing fringe in digital hologram when being undersampled.

This fractal-like structure shows an extraordinary behavior. That is to say, as a conventional fractal image continues to be magnified by a scaling parameter, its similar pattern shows up constantly [25]. The envelope shape of zone pattern showing a fractal-like structure seems to correspond to this condition. However, an apparent zone plate is generated from undersampling process, where the image shrinks with an increasing pixel interval, which can be viewed as the zoom-out process. This property is irrelevant to the negative Fractal dimension [26], because as confirmed in Fig. 2, physical size of similar patterns is constant.

Recently, the several fractals have been found in optical system [27,28]. The fractal dimension for hologram fringe could be calculated by using the box-counting method. However, because the self-similar curves are in the form of the envelope, the box-counting technique is not applicable directly, and a further study is required for detailed analysis.

![Fig. 5. Digital hologram for on-axis point source at 30.8-mm distance. The hologram is synthesized with object of 512×512 pixels. When the hologram is zoomed out in a digital display, a similar pattern of Fresnel zone is appeared constantly due to undersampling. The inset graph reveals a self-similar envelope pattern of quadratic sinusoid.](image-url)
4. Numerical simulation of aliasing phenomenon and image recovery for digital Fresnel hologram with enhanced numerical aperture

4.1 Aliasing phenomenon of digital Fresnel hologram synthesized with finite object

We confirmed that even digital holograms suffering a severe aliased fringe such as Fig. 4(c) recover the original point object through inverse Fresnel transform. This aliased hologram could play a role in multifocusing Fresnel lens, i.e. the replica zones could produce the high-order point images in the imaging plane [29,30]. The point object is only mathematical concept because physical object should have a finite extent. Physical objects can be considered as an aggregation of point objects. Thus, the object having a finite size will reveal a size effect for optical diffraction. For the further study, we simulate the hologram synthesis and its image reconstruction by using circular objects with a finite size and ‘HOLO’ letter object. The simulation specifications are put to be that of four-fold undersampling in Fig. 4(c). Figure 6 shows the hologram fringe property for synthesized holograms.

Fig. 6. Digital holograms synthesized using circular objects with a radius of (a) 4 μm, (b) 8 μm, and (c) 16 μm, and (d) ‘HOLO’ letter object. Below graphs indicate one-dimensional hologram phase and intensity profiles in the horizontal direction at the center of image.
We write the Fresnel diffraction from the object with a finite extent \( l \) in the convolution form of the object field and point spread function:

\[
U(x, y) = U_0(x, y) \text{rect} \left( \frac{x}{l}, \frac{y}{l} \right) * h(x, y).
\]  

(26)

Above equation is rewritten by

\[
U(x, y) = \frac{e^{ikz}}{i\lambda z} \exp \left[ i\pi \left( \frac{x^2 + y^2}{\lambda z} \right) \right] \text{FT} \left\{ U_0(x, y) h(x, y) \right\} \text{sinc} \left( \frac{\pi x}{\lambda z} \right) \text{sinc} \left( \frac{\pi y}{\lambda z} \right).
\]  

(27)

The Fourier transform term of a rectangular function becomes a modulating sinc function. As the object size increases, the maximum peak width \( \frac{\lambda z}{l} \) of sinc function decreases. As shown in Fig. 6, the size effect of diffraction becomes stronger with increasing a radius of circular object. That is to say, the diffracted wave from object with a large extent propagates more straight forward. This character does not mean the change in the width of propagated diffraction spectrum, because the whole range of diffracted field is fundamentally determined by the object resolution according to Eqs. (4) and (7).

The digital holograms synthesized by using circular objects with a radius of 4 \( \mu \)m, 8 \( \mu \)m, and 16 \( \mu \)m are displayed in Figs. 6(a)-6(c). We find that as with hologram fringe of point source, the replica of concentric fringe is formed. However, when object size is large, the modulus of replica fringe is considerably suppressed by a modulating sinc function. Likewise, the hologram fringe of letter object reveals the suppressed replicas of interferogram positioned at a center, in Fig. 6(d). The suppressed replicas of interferograms can not generate replica images other than the replica hologram zones arising from point object. Meanwhile, the whole area of digital hologram contributes to the retrieval of object image. From this property, we find that the digital hologram synthesized at a closer distance than \( z_0 \) have an enhanced numerical aperture, which results in the recovery of object image with higher resolution.

![Fig. 7. Restored images from real-valued digital holograms for (a) circular object with a radius of 8 \( \mu \)m and (b) letter object. Below figures display the restored images from complex holograms.](image-url)
4.2 Numerical simulation of image recovery for digital Fresnel hologram

Figure 7(a) is the restored image of circular object with a radius of 8 μm by using a real-valued hologram. Since the resolution of restored image is 2 μm, the total image size becomes 512 μm. We can clearly distinguish the object image from its conjugate term. We know that in in-line holographic system, only the real-valued or imaginary-valued hologram makes it difficult to analyze the aliasing effect owing to an overlap of noise terms. Below image is the restored image from complex hologram, which shows a most complete retrieval. The restored image for letter object is displayed in Fig. 7(b). The real-valued hologram reconstructs the discriminated original image clearly.

The diffraction fringes propagating away from the restored image are displayed in Fig. 8. For clarity, the letter object is expanded to 512×512 pixels by zero-padding, where z₁ distance is estimated to be 30.8 mm. The diffraction fringe reconstructed from complex hologram is displayed in the logarithmic scale, which enables us to investigate diffraction behavior in detail. We observe the well-behaved diffraction fringes according to the propagation distance. In the digital hologram with an aliased error from the sampling condition violation with respect to Fourier spectrum in Eq. (11), the rupture of diffraction fringe is observed. Especially, we can notice that the active diffraction fringe emerged from the letter object propagates at a diffraction angle larger than that of diffraction zone by a pixel pitch in Eq. (8). In our previous work [22], we elucidate that the active diffraction angle is closely related to the hologram numerical aperture, and from Eqs. (4)-(7) this diffraction angle Ω₁ is written by

\[ \Omega_1 = 2 \sin^{-1} \left( \frac{N \Delta x}{2 z} \right). \]  

This property allows us to develop the holographic display with a wide-viewing angle. Present spatial light modulator to load the digital hologram has a pixel pitch with several micrometer scale, and thus, the viewing-angle of holographic image is limited to several degree, which is known as the critical obstacle for realizing the holographic display. According to our scheme, the computed generated hologram with the enhanced-NA loaded on the spatial light modulator can reconstruct optically the image with a wide angle compared to that by the modulator pixel pitch. In Fig. 8, the diffraction angle θ with respect to hologram pixel pitch 8 μm is calculated to be about 3.8°, while the angle value angle Ω₁ becomes 7.6°.
Fig. 9. Digital hologram synthesized by using extended object to 512×512 pixels. Red box indicates original letter of 256×256 pixels.

Fig. 10. Simulation results of image reconstruction for the hologram synthesized using extended object: (a) Restored image from original hologram, and (b) restored image from two-fold upsampled hologram.

We also studied the digital hologram for the object being extended to the deviating area from a diffraction zone by pixel pitch. The extended object added by a rectangular image outside of original letter image is located at the \( z_1 \) distance of 15.4 mm in Fig. 1. The extended object with 512×512 pixels of a 4-\( \mu \)m resolution has the same size as 2048-\( \mu \)m of the hologram. The angle value \( \angle \Omega_1 \) with respect to object pixel pitch 4 \( \mu \)m is 7.6°, and thus, the calculated diffraction field places in the diffraction zone. The digital hologram is calculated from the Riemann integral of the Rayleigh-Sommerfeld diffraction formula to control arbitrary the input and output sizes. To fulfil the Nyquist sampling criterion enough, the digital hologram is calculated through a two-fold upsampling of object field, in Fig. 9. The aliasing error due to a self-similarity of point spread function will be inhibited in hologram fringe although it is not apparent to the naked eye. The reconstructed image by using complex hologram in Fig. 10(a) reveals the overlapping of high-order aliasing images because of a finite diffraction zone of 8-\( \mu \)m pixel pitch, where the diffraction area of 1024 \( \mu \)m is a half of the object field-view of 2048 \( \mu \)m. This high-order aliasing can be largely suppressed by a diffraction zone enlargement from two times upsampling of hologram fringe, as displayed in Fig. 10(b).
In the hologram acquisition system, the digital hologram could be adaptively acquired in accordance with a pixel pitch of digitized sensor. Since a real object can be regarded as a continuous signal, it does not have the limitation of diffraction angle. Conversely, the spectral window of captured diffraction field defines the image resolution depending on a numerical aperture. If we consider the real object as a collection of the point object appeared as a delta function, the spatial information is encoded in the shape of fringe pattern. The fringe pattern will depend on the lateral size of sensor and acquisition distance. The hologram fringe is undersampled by a finite pixel pitch when it is captured at a closer distance than \( \varepsilon_0 = \frac{NAx^2}{\lambda} \), which leads to an aliasing error of hologram fringe. Our result reveals that even digital hologram suffering a severe aliasing error can restore the original almost completely. In this acquisition system the object reduction is inevitable, but the limitation of object size can be also overcome by enlargement of diffraction zone for digital hologram. After all, our work shows the possibility for retrieving the high resolution image with a sufficient extent irrespective of the degree of aliasing error of hologram fringe.

5. Conclusions

We elucidate that optical kernel function reveals a fractal-like behavior when being sampled. A self-similarity of concentric Fresnel zone becomes a crucial mechanism in inducing an aliasing fringe for digital hologram with the enhanced numerical aperture. The enhanced-NA digital hologram acquired at a closer distance from the object could have an aliased fringe due to a finite pixel size of digitized sensor. The original image can be almost completely retrieved even under this severe aliasing phenomenon. We find that the spatial resolution of restored image is determined by a numerical aperture of digital hologram other than the maximum frequency of hologram fringe. This result provides a method for acquiring the digital hologram with a high numerical aperture and for restoring the high-resolution image with a sufficient extent.

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