A lower bound for $A_p$ exponents for some weighted weak-type inequalities

Carlos Pérez$^1$ · Israel P. Rivera-Ríos$^2$

Received: 2 October 2017 / Accepted: 30 November 2017 / Published online: 9 November 2018 © Forum D’Analystes, Chennai 2018

Abstract
We give a weak-type counterpart of the main result in Luque et al. (Math Res Lett 22(1):183–201, 2015) which allows to provide a lower bound for the exponent of the $A_p$ constant in terms of the behaviour of the unweighted inequalities when $p \to \infty$ and when $p \to 1^+$. We also provide some applications to classical operators.

Keywords $A_p$ weights · Calderón-Zygmund operators · Weighted estimates · Quantitative estimates · Weak type estimates

Mathematics Subject Classification 42B20 · 42B25

1 Introduction and main results

The purpose of this paper is to give a weak-type counterpart of the main result in [13]. If $T$ is an operator which satisfies a weak type bound like

$$\|T\|_{L^p(w)\to L^{p,\infty}(w)} \leq c \left[w\right]_{A_p}^\beta,$$  \hspace{1cm} (1.1)

with $\beta > 0$, then we will show in Theorem 1 that the optimal lower bound for $\beta$ is related to the asymptotic behaviour of the unweighted $L^p$ norm $\|T\|_{L^p(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)}$ as $p$ goes to 1 and $+\infty$. We recall that a weight $w$, namely a non-negative locally integrable function, belongs to the $A_p$ class of Muckenhoupt if

---

Carlos Pérez
carlos.perezmo@ehu.es

Israel P. Rivera-Ríos
petnapet@gmail.com

1 Departamento de Matemáticas, Universidad del País Vasco UPV/EHU, IKERBASQUE, Basque Foundation for Science, BCAM, Basque Center for Applied Mathematics, Bilbao, Spain

2 Departamento de Matemáticas, Universidad del País Vasco UPV/EHU and BCAM, Basque Center for Applied Mathematics, Bilbao, Spain
where $M$ stands for the Hardy–Littlewood maximal function, namely

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f|$$

and each $Q$ is a cube with its sides parallel to the axis.

The $A_p$ conditions characterize the weighted $L^p$ boundedness of the maximal function, namely $w \in A_p$ if and only if the corresponding estimate

$$\|Mf\|_{L^p(w)} \leq c_{w,n,p} \|f\|_{L^p(w)} \quad (1 < p < \infty)$$

holds, where $c_{w,n,p}$ is a constant that depends on the weight, on the dimension $n$ and on $p$. Since Muckenhoupt’s seminal work, many authors such as Wheeden, Hunt, Coifman or Fefferman, got involved in the study of weighted estimates, providing such interesting results for singular integrals as well.

In the last decade, one of the main problems in Harmonic Analysis has been the study of sharp norm inequalities for some of the classical operators on weighted Lebesgue spaces $L^p(w)$, $1 < p < \infty$. Some examples of those kind of results include the Hardy–Littlewood maximal operator, the Hilbert transform and more generally Calderón–Zygmund operators (C–Z operators). Given any of these operators $T$, the first part of this problem is to look for quantitative bounds of the norm $\|T\|_{L^p(w)}$ in terms of the $A_p$ constant of the weight, namely an estimate like (1.1). The following step is to establish the sharp dependence, typically with respect to the power of $[w]_{A_p}$, i.e. the optimality of $\beta$ in (1.1). In recent years, the answer to this last question has led a fruitful activity and development of new tools in Harmonic Analysis. Firstly, in the early 90s, Buckley [5] identified the sharp exponent in the case of the Hardy–Littlewood maximal function, i.e.,

$$\|Mf\|_{L^p(w)} \leq c_{n,p}[w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)} \quad (1 < p < \infty),$$

$$\|Mf\|_{L^{p,\infty}(w)} \leq c_1[w]_{A_p}^\frac{1}{2} \|f\|_{L^p(w)} \quad (1 \leq p < \infty).$$

However, Buckley’s work was not very influential initially. Quantitative estimates did not become an important topic until the work of Astala et al. [2] in which they proved that some self-improvement properties of the integrability of the solutions of the Beltrami equation relied upon the linear dependence on the $A_2$ constant of the Beurling transform, namely on the following estimate

$$\|Bf\|_{L^2(w)} \leq c_n [w]_{A_2} \|f\|_{L^2(w)}.$$
Shortly after, that estimate was proved to be true by Petermichl and Volberg [17], and can be considered the beginning of the “quantitative estimates era”. Several authors have made more than interesting contributions to this topic. Especially, the proof of the $A_2$ conjecture [9] (improved in [11]) and the quest for simpler proofs has led to developments such as the sparse domination theory that probably were inconceivable years ago.

In this work we provide a criterium to decide the sharp dependence of the $A_p$ constant for the weak-type $(p, p)$ estimate based on the behaviour of $\|T\|_{L^p(\mathbb{R}^n) \to L^{p, \infty}(\mathbb{R}^n)}$ when $p \to 1$ and $p \to \infty$. The main result is the following.

**Theorem 1** Given an operator $T$ such that for some $1 < p_0 < \infty$ and for any $w \in A_{p_0}$

$$\|T\|_{L^{p_0, \infty}(w)} \leq c[w]_{A_{p_0}}^\beta$$

then

$$\beta \geq \max \left\{ \gamma_T; \frac{\alpha_T}{p_0 - 1} \right\}$$

where

$$\alpha_T = \sup \left\{ \alpha \geq 0 : \forall \varepsilon > 0 \limsup_{p \to 1^+} (p - 1)^{\alpha - \varepsilon} \|T\|_{L^p \to L^{p, \infty}} = \infty \right\}$$

and

$$\gamma_T = \sup \left\{ \gamma \geq 0 : \forall \varepsilon > 0 \limsup_{p \to \infty} \frac{\|T\|_{L^p \to L^{p, \infty}}}{p^{\gamma - \varepsilon}} = \infty \right\}.$$ 

To apply the preceding result we need to provide sharp unweighted estimates in terms of $p$ and $p'$. We gather such estimates for some cases of interest in Lemma 1. Now we present those operators.

We say that $T$ is a Calderón–Zygmund operator if $T$ is bounded on $L^2$ and admits the following representation for $f \in C_c^\infty$

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy \quad x \not\in \text{supp} f$$

where $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\} \to \mathbb{R}$ is a kernel satisfying the following properties

1. $|K(x, y)| \leq \frac{C_K}{|x - y|^n}$

2. $|K(x, y) - K(x', y)| \leq C \left( \frac{|x - x'|}{|x - y|} \right)^\delta \frac{1}{|x - y|^n}$ where $|x - x'| \leq \frac{1}{2} |x - y|$ for some $\delta > 0$.

Relying upon the preceding definitions, we have that given $b \in BMO$ and $T$ a Calderón–Zygmund operator we define the commutator $[b, T]$ by
We recall that $b \in BMO$ if

$$\|b\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |b - b_Q| \, dx < \infty.$$  

At this point we are in the position to state the lemma that we announced before.

**Lemma 1** Let $1 < p < \infty$, $T$ a Calderón–Zygmund operator and $b \in BMO$. Then there exist constants $c_i > 0$ such that the following estimates hold

$$\|T\|_{L^p \to L^{p', \infty}} \leq c_1 p$$
$$\|[b, T]\|_{L^p \to L^{p', \infty}} \leq c_2 p' p^2$$
$$c_3 (p')^{k-1} \leq \|M^k\|_{L^p \to L^{p', \infty}} \leq c_4 (p')^{k-1}$$

where $M^k$ stands for $M^k \circ M$.

On the other hand, if $H$ is the Hilbert transform and $b(x) = \log |x|$, we also have that there exist constants $c_i > 0$ such that

$$c_5 p \leq \|H\|_{L^p \to L^{p', \infty}}$$
$$c_6 \max \{p', p^2\} \leq \|[b, H]\|_{L^p \to L^{p', \infty}}$$

Combining some known estimates in the literature and the preceding results we obtain the following result.

**Theorem 2** Let $1 < p < \infty$, $k$ a positive integer, $T$ a Calderón–Zygmund operator and $b \in BMO$. Then

1. $\|T\|_{L^p \to \max(w)} \leq c[w]_{A_p}$ and the exponent of the $A_p$ constant is sharp.
2. $\|[b, T]\|_{L^p \to \max(w)} \leq c[w]_{A_p}^{p'}$ where we have that $\max \left\{ 2, \frac{1}{p-1} \right\} \leq p' \leq \max \{ 2, p' \}$.
3. $\|M^k\|_{L^p \to \max(w)} \leq c[w]_{A_{p_0}}^\eta_{p_0}$ with $\frac{k-1}{p_0-1} \leq \eta_{p_0} \leq \frac{1}{p} + \frac{k-1}{p_0-1}$

At this point some remarks are in order. We observe that our method is completely satisfactory in the case of Calderón–Zygmund operators, contrary to what happens in the case of the commutator $[b, T]$ and for the maximal function $M$ and its iterations $M^k$. In the case of the maximal function, that fact is not a surprise. The information that the method provides comes from the relationship of the boundedness constant of the operator with the exponents of $p$ and $p'$ and in this case both exponents are zero, thus, the method cannot provide any kind of information. In the case of the commutator it is not clear whether the upper bound can be improved or the lower one should be larger.
A lower bound for exponents for some weighted weak-t…

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. Lemma 1 is established in Sect. 3. We end up this paper with the proof of Theorem 2 which is presented in Sect. 4.

2 Proof of Theorem 1

As we mentioned before, we will adapt here the main arguments from [13] which in turn is based on ideas from [6] and [7].

Firstly we prove that \( \frac{\alpha_T}{p_0-1} \leq \beta \). If \( \alpha_T = 0 \) there’s nothing to prove, so let us assume that \( \alpha_T > 0 \). We define then the following Rubio de Francia algorithm (see for instance [8])

\[
R_p f(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{2^k ||M||^k_{L^p}}.
\]

\( R \) satisfies the following properties

1. \( h \leq R_p h \)
2. \( ||R_p h||_{L^p} \leq 2 ||h||_{L^p} \)
3. \( R_p h \in A_1 \). Furthermore \( [R_p h]_{A_1} \leq 2 ||M||_{L^p} \).

Now, if we fix \( 1 < p < p_0 \)

\[
||Tf||_{L^{p,\infty}} = \sup_{\lambda > 0} \lambda \left( \int_{|Tf(x)| > \lambda} f(x)^{\frac{1}{p}} \right)^{\frac{1}{p}}
\]

\[
= \sup_{\lambda > 0} \lambda \left( \int_{|Tf(x)| > \lambda} (R_p f)^{-(p_0-p)}\frac{p_0-p}{p_0} f(x)^{\frac{p_0-p}{p_0}} \right)^{\frac{1}{p}}
\]

\[
\leq \sup_{\lambda > 0} \lambda \left( \int_{|Tf(x)| > \lambda} (R_p f)^{-(p_0-p)} \right)^{\frac{1}{p_0}}
\]

\[
\times \left( \int_{|Tf(x)| > \lambda} (R_p f)^{\frac{p_0-p}{p_0}} \right)^{\frac{p_0-p}{p_0}}
\]

\[
\leq \|Rf\|_{L^{p_0}(\mathbb{R}^n)} \sup_{\lambda > 0} \left( \int_{|Tf(x)| > \lambda} (R_p f)^{-(p_0-p)} \right)^{\frac{1}{p_0}}
\]

\[
= ||f||_{L^{p_0}(\mathbb{R}^n)} ||Tf||_{L^{p_0,\infty}} (R_p f)^{-(p_0-p)}.
\]
Applying the hypothesis (1.4) we have that

\[
\|f\|_{L^p(R^n)} \|Tf\|_{L^{p_0,\infty}(R^n)} \leq c \left[ (R_p f)^{-(p_0-p)} \right]^{\frac{p_0}{p}} \|f\|_{L^p(R^n)} \|f\|_{L^{p_0}(R^n)}^{-(p_0-p)} \leq c \left[ (R_p f)^{-(p_0-p)} \right]^{\frac{p_0}{p}} \|f\|_{L^p(R^n)}.
\]

Now, since it was established in [13] that 
\[
\left[ (R_p f)^{-(p_0-p)} \right] \leq c_n \|M\|_{L^p(R^n)}^{p_0-p},
\]
we have

\[
\|Tf\|_{L^p,\infty} \leq c \|M\|_{L^p(R^n)}^{\beta(p_0-p)} \|f\|_{L^p(R^n)} 1 < p < p_0.
\]

Recalling that

\[
\|M\|_{L^p(R^n)} \leq c \frac{1}{p-1}
\]

then for \( p \) close to 1 we get

\[
\|T\|_{L^p \rightarrow L^{p,\infty}} \leq c(p-1)^{-\beta(p_0-p)} \leq c(p-1)^{-\beta(p_0-1)}.
\]

Since \( \alpha_T > 0 \) if \( \alpha_T - \epsilon > 0 \), multiplying by \( (p-1)^{\alpha_T-\epsilon} \), and taking \( \lim sup \), by the definition of \( \alpha_T \)

\[
= \limsup_{p \rightarrow 1^+} \|T\|_{L^p,\infty} \rightarrow L^p \ (p-1)^{\alpha_T-\epsilon}
\]

\[
\leq \limsup_{p \rightarrow 1^+} c(p-1)^{-\beta(p_0-1)+\alpha_T-\epsilon}
\]

we have that

\[
-\beta(p_0-1) + \alpha_T - \epsilon < 0 \iff \frac{\alpha_T - \epsilon}{p_0-1} < \beta
\]

and taking \( \inf \) in \( \epsilon \),

\[
\frac{\alpha_T}{p_0-1} \leq \beta.
\]

Let us prove now that \( \gamma_T \leq \beta \). We follow the same extrapolation ideas, but now we use the dual space \( L^{p'}(\mathbb{R}^n) \). Fix \( p > p_0 \) and \( f \in L^p(\mathbb{R}^n) \). Firstly we observe that our hypothesis is equivalent to

\[
tw(\{ x \in \mathbb{R}^n : |Tf(x)| > t \}) \leq c[w] A_{p_0}^{\beta} \|f\|_{L^{p_0}} \quad t > 0.
\]

\( \text{ Springer} \)
Now
\[
\|Tf\|_{L^{p,w}(\mathbb{R}^n)} = \sup_{\lambda > 0} \lambda \left( \int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} \right)^{\frac{1}{p}} \\
= \sup_{\lambda > 0} \lambda \left\| \chi_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} \right\|_{L^p(\mathbb{R}^n)}
\]

By duality for each \( \lambda > 0 \) we can find \( h_\lambda \in L^{p'}(\mathbb{R}^n) \), \( h_\lambda \geq 0 \), \( \|h_\lambda\|_{L^{p'}(\mathbb{R}^n)} = 1 \) such that
\[
\lambda \left\| \chi_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} \right\|_{L^p(\mathbb{R}^n)} = \lambda \int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} h_\lambda \]
\[
\leq \lambda \int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} (R_{p'}h_\lambda)^{\frac{p-p_0}{p-1}} h_\lambda^{\frac{p(p_0-1)}{p}} \ dx
\]
\[
\leq \lambda \left( \int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} (R_{p'}h_\lambda)^{\frac{p-p_0}{p-1}} \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^n} h_\lambda^{p'} \ dx \right)^{\frac{1}{p'}}
\]
\[
= \lambda \left( \int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} (R_{p'}h_\lambda)^{\frac{p-p_0}{p-1}} \right)^{\frac{1}{p}}.
\]

Now we observe that \( w = (R_{p'}h_\lambda)^{\frac{p-p_0}{p-1}} \) is an \( A_{p_0} \) weight. Using hypothesis this yields,
\[
tw(\{x \in \mathbb{R}^n : |Tf(x)| > t\})^{\frac{1}{p_0}} \leq c[w]_{A_{p_0}}^\beta \|f\|_{L^{p_0}} \quad t > 0.
\]

In particular that inequality holds for \( t = \lambda \). Then we have that
\[
\lambda \left( \int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} (R_{p'}h_\lambda)^{\frac{p-p_0}{p-1}} \right)^{\frac{1}{p_0}} \leq c \left[ (R_{p'}h_\lambda)^{\frac{p-p_0}{p-1}} \right]_{A_{p_0}} \left( \int_{\mathbb{R}^n} |f|^{p_0} (R_{p'}h_\lambda)^{\frac{p-p_0}{p-1}} \ dx \right)^{\frac{1}{p_0}}
\]
Hölder
\[
\leq c \left[ (R_{p'}h_\lambda)^{\frac{p-p_0}{p-1}} \right]_{A_{p_0}} \left( \int_{\mathbb{R}^n} |f|^p \ dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^n} (R_{p'}h_\lambda)^{p'} \ dx \right)^{\frac{1}{p'}}
\]
Jensen
\[
\leq c \left[ R_{p'}h_\lambda \right]_{A_1} \left( \int_{\mathbb{R}^n} |f|^p \ dx \right)^{\frac{1}{p}}
\]
\[
\leq c \|M\|_{L^{p_0}(\mathbb{R}^n)} \left( \int_{\mathbb{R}^n} |f|^p \ dx \right)^{\frac{1}{p}}.
\]
Then we have that
\[
\lambda \left( \int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} \right)^\frac{1}{p} \leq c \|M\|^\frac{\beta p_0 - p}{p - 1}_L \left( \int_{\mathbb{R}^n} |f|^p \, dx \right)^\frac{1}{p}
\]
and consequently
\[
\|Tf\|_{L^{p,\infty}((\mathbb{R}^n))} \leq c \|M\|^\frac{\beta p_0 - p}{p - 1}_L \|f\|_{L^p((\mathbb{R}^n))}.
\]
To finish the proof we recall that, for large \( p >> p_0 \), we have that \( \|M\|_{L^p} \sim p \). Therefore, we have that
\[
\|Tf\|_{L^{p,\infty}((\mathbb{R}^n))} \leq cp^\frac{\beta p_0 - p}{p - 1} \leq cp^\beta.
\]
Since \( p >> p_0 \) we have that, dividing by \( p^{\gamma_T - \epsilon} \) and taking upper limits, we obtain
\[
\infty = \limsup_{p \to \infty} \frac{\|Tf\|_{L^{p,\infty}((\mathbb{R}^n))}}{p^{\gamma_T - \epsilon}} \leq c \limsup_{p \to \infty} p^{\beta - \gamma_T + \epsilon}.
\]
Consequently \( \beta \geq \gamma_T \) and we are done.

### 3 Proof of Lemma 1

#### 3.1 Lemmata

In order to prove the unweighted estimates we need some lemmas. We present first some of the of the main ingredients of those results. We recall that the sharp maximal function \( M^{\#}_s f \) is defined by
\[
M^{\#}_s f(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f - f_Q|^s \right)^{\frac{1}{s}} \quad 0 < s < \infty.
\]

In the case in which the supremum is taken only over dyadic cubes we write \( M^{\#}_s d \). We note that \( M^{\#}_s \) is comparable so replacing one by the other when dealing with norm estimates will not make a difference for us. Analogously we will denote \( M(f) = M(|f|^\frac{1}{2}) \frac{1}{2} \).

Given a measurable function \( f \) we define its non increasing rearrangement by
\[
f^*(t) = \inf \{ \lambda > 0 : d_f(\lambda) \leq t \}
\]
where \( d_f(\lambda) = \{ x \in \mathbb{R}^n : |f(x)| > \lambda \} \).
Lemma 2 Let $0 < \delta, \gamma < 1$. There exists a constant $c = c_{n,\gamma,\delta}$ such that for any measurable function

$$f^*(t) \leq c \left( M_{\delta}^\# f \right)^*(\gamma t) + f^*(2t) \quad t > 0. \quad (3.1)$$

These type of estimates in this context go back to the work of Bagby and Kurtz in the mid 80s (see [3, 4]). The proof of the Lemma can be found in [16] in the context of $A_p$ weights and in [15] in the context of $A_\infty$ weights. As a consequence we have the following.

Lemma 3 Let $1 \leq p < \infty$, and $0 < \delta < 1$. Then there exists a constant $c = c_{n,\delta}$ such that

$$\|f\|_{L^{p,\infty}} \leq c p \left\| M_{\delta}^\# f \right\|_{L^{p,\infty}}$$

for each function $f$ such that $|\{x : |f(x)| > t\}| < \infty$ for every $t > 0$.

Proof Iterating 3.1

$$f^*(t) \leq c \sum_{k=0}^{\infty} \left( M_{\delta}^\# f \right)^*(2^k \gamma t) + f^*(+\infty)$$

$$\leq c \frac{1}{\log 2} \int_{\frac{t}{2}}^{\infty} \left( M_{\delta}^\# f \right)^*(s) \frac{ds}{s}$$

using that $f^*(+\infty) = 0$ which follows since $|\{x : |f(x)| > t\}| < \infty$ for each $t > 0$. Now we recall that

$$Sf(x) = \int_{x}^{\infty} f(s) \frac{1}{s} ds$$

is the adjoint of Hardy operator. Then the preceding estimate can be restated as follows

$$f^*(t) \leq c \frac{1}{\log 2} S \left( \left( M_{\delta}^\# f \right)^* \right) \left( \frac{t \gamma}{2} \right). \quad (3.2)$$

Now we see that

$$\|S(g^*)\|_{L^{p,\infty}} \leq p \|g\|_{L^{p,\infty}}. \quad (3.3)$$

Indeed, since $Sg^*(x)$ is decreasing,
\[ \|S(g^\ast)\|_{L^p,\infty} = \sup_{t > 0} t^{\frac{1}{p}} S(g^\ast)(t) = \sup_{t > 0} t^{\frac{1}{p}} S(g^\ast)(t), \]
and the desired estimate follows from observing that for each \( t > 0 \) we have that
\[
\begin{align*}
&\frac{1}{t^p} S(g^\ast)(t) = t^{\frac{1}{p}} \int_t^\infty g^\ast(s) \frac{1}{s} \, ds = t^{\frac{1}{p}} \int_t^\infty s^{\frac{1}{p}} g^\ast(s) \frac{1}{s^{1 + \frac{1}{p}}} \, ds \\
&\leq \|g\|_{L^p,\infty} t^{\frac{1}{p}} \int_t^\infty \frac{1}{s^{1 + \frac{1}{p}}} \, ds = \|g\|_{L^p,\infty} t^{\frac{1}{p}} \int_t^\infty \frac{1}{s^{1 + \frac{1}{p}}} \, ds \\
&= \|g\|_{L^p,\infty} t^{\frac{1}{p}} \left[ \frac{1}{-s^{\frac{1}{p}}} \right]_t^\infty = p \|g\|_{L^p,\infty}.
\end{align*}
\]

Armed with (3.2) and (3.3) we can now establish the desired inequality:
\[
\|f\|_{L^p,\infty} = \sup_{t > 0} t^{\frac{1}{p}} f^\ast(t) \leq c \frac{1}{\log 2} \sup_{t > 0} t^{\frac{1}{p}} S\left( \left( M^\#_0 f \right)^\ast \right)(t^{\frac{1}{2}}) \\
= c \frac{1}{\log 2} \left( \frac{2}{\gamma} \right)^{\frac{1}{p}} \sup_{t > 0} \left( t^{\frac{1}{2}} \right)^{\frac{1}{p}} S\left( \left( M^\#_0 f \right)^\ast \right)(t^{\frac{1}{2}}) \\
= c \frac{1}{\log 2} \left( \frac{2}{\gamma} \right)^{\frac{1}{p}} \|S\left( \left( M^\#_0 f \right)^\ast \right)\|_{L^p,\infty} \\
\leq c \frac{1}{\log 2} \left( \frac{2}{\gamma} \right)^{\frac{1}{p}} p \|M^\#_0 f\|_{L^p,\infty}.
\]

Lemma 4 Let \( 1 \leq p < \infty \) and \( 0 < \epsilon \leq 1 \). Suppose that \( f \) is a function such that for every \( t > 0 \) \( \{x : |f(x)| > t\} < \infty \). Then there exists a constant \( c = c_{n,\epsilon} \) such that
\[
\left\| M^{\#}_\epsilon f \right\|_{L^p,\infty} \leq c p \left\| M^{\#,d}_\epsilon f \right\|_{L^p,\infty}.
\]

Proof We apply the preceding lemma with \( f \) replaced by \( M_\epsilon f \) and \( \delta = \epsilon_0 \) such that \( 0 < \epsilon_0 < \epsilon < 1 \). Then
\[
\|M_\epsilon f\|_{L^p,\infty} \leq c p \left\| M^{\#,d}_{\epsilon_0} \right( M_\epsilon f \right\|_{L^p,\infty}.
\]
We also know (see [14]) that

\[ M_{\epsilon_0}^{\#}(M_\epsilon f) \leq c M_\epsilon^{\#} f. \]

Consequently,

\[ \left\| M_{\epsilon_0}^{\#}(M_\epsilon f) \right\|_{L^{p,\infty}} \leq c \left\| M_\epsilon^{\#} f \right\|_{L^{p,\infty}}. \]

This concludes the proof of the lemma.

\[ \square \]

### 3.2 Proof of Lemma 1

Armed with the preceding results we are in a position to establish Lemma 1. We consider different cases.

#### 3.3 Calderón–Zygmund operators

Firstly we obtain the upper bound. Using Lemma 3

\[ \left\| Tf \right\|_{L^{p,\infty}} \leq c \left\| M_\delta^{\#}(Tf) \right\|_{L^{p,\infty}} \leq c \left\| Mf \right\|_{L^{p,\infty}} \leq c \left\| f \right\|_{L^p}, \]

since \( M_\delta^{\#}(Tf) \leq c_\delta Mf \), 0 < \( \delta \) < 1 as can be found [1]. Now we deal with the lower bound. It is well known that

\[ H(\chi_{[0,1]})(x) = \log \left( \frac{|x|}{|x-1|} \right). \]

Then

\[ \left\| H\chi_{(0,1]} \right\|_{L^{p,\infty}} = \sup_{t>0} t \left\{ x \in \left(0, \frac{1}{2}\right) : -\frac{1}{\pi} \log \left( \frac{x}{x-1} \right) > t \right\}^{\frac{1}{p}} \]

\[ = \sup_{t>0} t \left\{ x \in \left(0, \frac{1}{2}\right) : \Phi(x) > t \right\}^{\frac{1}{p}} \]

\[ \Phi \text{ decreasing} = \sup_{t>0} t \left\{ x \in \left(0, \frac{1}{2}\right) : \Phi^{-1}(t) > x \right\}^{\frac{1}{p}} = \sup_{t>0} t \Phi^{-1}(t)^{\frac{1}{p}}. \]

Using again the properties of \( \Phi \)

\[ \sup_{t>0} t \Phi^{-1}(t)^{\frac{1}{p}} = \sup_{0<x<\frac{1}{2}} \Phi(x)x^{\frac{1}{p}}. \]
Now we observe that for every \(0 < x < \frac{1}{2}\),
\[
\Phi(x)x^{\frac{1}{2}} \geq -\frac{1}{\pi} \log \left( \frac{x}{2} \right)x^{\frac{1}{2}} = -p2^{\frac{1}{2}} \frac{1}{\pi} \log \left( \left( \frac{x}{2} \right)^{\frac{1}{2}} \right) \left( \frac{x}{2} \right)^{\frac{1}{2}} \geq cp
\]
and we’re done.

### 3.4 Commutators

Firstly we obtain the upper bound. Suppose that \(\|b\|_{BMO} = 1\). Then using Lemma 3
\[
\| [b, T]f \|_{L^{p,\infty}} \leq cp \| M^\sharp_b (b, T) f \|_{L^{p,\infty}} \leq cp \| M^\sharp (Tf) \|_{L^{p,\infty}} + cp \| M^2 f \|_{L^{p,\infty}} = cp (L_1 + L_2).
\]

Now we observe that using Lemma 4 with \(0 < \varepsilon < 1\),
\[
L_1 = \| M^\varepsilon (Tf) \|_{L^{p,\infty}} \leq cp \| M^\sharp^\varepsilon (Tf) \|_{L^{p,\infty}} \leq cp \| Mf \|_{L^{p,\infty}} \leq cp \| f \|_{L^p}.
\]

For \(L_2\) we have
\[
L_2 = \| M^2 f \|_{L^{p,\infty}} \leq c \| Mf \|_{L^p} \leq cp' \| f \|_{L^p}.
\]

Consequently
\[
\| [b, T]f \|_{L^{p,\infty}} \leq cp^2 p' \| f \|_{L^p}.
\]

Let us focus on the lower bound. Consider the Hilbert transform
\[
Hf(x) = pv \int_\mathbb{R} \frac{f(y)}{x - y} dy,
\]
and consider the BMO function \(b(x) = \log |x|\). Let \(f = \chi_{(0,1)}\). If \(0 < x < 1\),
\[
[b, H]f(x) = \int_0^1 \frac{\log(x) - \log(y)}{x - y} dy = \int_0^1 \frac{\log \left( \frac{x}{y} \right)}{x - y} dy
\]
\[
= \int_0^{1/x} \frac{\log \left( \frac{1}{t} \right)}{1 - t} dt \int_0^1 \frac{\log \left( \frac{1}{t} \right)}{1 - t} dt = \int_0^1 \frac{\log \left( \frac{1}{t} \right)}{1 - t} dt + \int_1^{1/x} \frac{\log \left( \frac{1}{t} \right)}{1 - t} dt
\]
\[
\frac{\log \left( \frac{1}{t} \right)}{1 - t} \text{ is positive for } (0, 1) \cup (1, \infty) \text{ and then we have that for } 0 < x < 1
\]
\[
\| [b, H]f(x) \| > \int_1^{1/x} \frac{\log \left( \frac{1}{t} \right)}{1 - t} dt.
\]
But since
\[ \int_1^\infty \frac{\log t}{t-1} \, dt = \infty \]
and
\[ \lim_{L \to \infty} \frac{\int_1^L \frac{\log(t)}{1-t} \, dt}{(\log L)^2} = 1 \]
we have that for some \( x_0 < 1 \)
\[ ||[b, H]f(x)|| > c \left( \log \frac{1}{x} \right)^2 \quad 0 < x < x_0. \]
and then
\[
\begin{align*}
t \left\{ x \in \mathbb{R} : ||[b, H]f(x)|| > t \right\}^{\frac{1}{p}} & \geq t \left\{ x \in (0, x_0) : c \left( \log \frac{1}{x} \right)^2 > t \right\}^{\frac{1}{p}} \\
& \geq cte^{-\frac{\sqrt{t}}{p}} = cp^2 \frac{t}{p^2} e^{-\sqrt{\frac{t}{p^2}}}. 
\end{align*}
\]
Consequently
\[ \left\| [b, H]f \right\|_{L^p, \infty}(\mathbb{R}) = \sup_{t>0} t \left\{ x \in \mathbb{R} : ||[b, H]f(x)|| > t \right\}^{\frac{1}{p}} \]
\[ \geq \sup_{t>0} cp^2 \frac{t}{p^2} e^{-\sqrt{\frac{t}{p^2}}} \geq cp^2. \]
Let \( b = \log |x| \) and \( f(x) = \chi_{(0,1)}(x) \). If \( x > e \), we have that
\[ ||[b, H]f(x)|| = \int_0^1 \frac{\log (x) - \log (y)}{x-y} \, dy = \int_0^{\frac{1}{x}} \frac{\log \left( \frac{1}{t} \right)}{1-t} \, dt \]
\[ \geq \log(x) \int_0^{\frac{1}{x}} \frac{1}{1-t} \, dt = \frac{\log(x)}{x}. \]
Now we observe that
\[ \| [b, H]f \|_{L^p, \infty(\mathbb{R})} = \sup_{t > 0} \{ x \in \mathbb{R} : \| [b, H]f(x) \| > t \} \]^\frac{1}{p} \]
\[ \geq \sup_{t > 0} t \left\{ x > e : p^2 \frac{\log(x)}{x} > t \right\} \]^\frac{1}{p} \]
\[ = \sup_{t > 0} t \{ x > e : \Phi(x) > t \} \]^\frac{1}{p}. \]

As \( \Phi \) is decreasing
\[ \sup_{t > 0} t \{ x > e : \Phi(x) > t \} \]^\frac{1}{p} = \sup_{t > 0} t \{ x > e : \Phi^{-1}(t) > x \} \]^\frac{1}{p} \]
\[ = \sup_{t > 0} t (\Phi^{-1}(t) - e) \]^\frac{1}{p} = \sup_{x > e} \Phi(x)(x - e) \]^\frac{1}{p} \geq \sup_{x > 2e} \Phi(x)(x - e) \]^\frac{1}{p}. \]

Now we observe that
\[ \Phi(x)(x - e) \]^\frac{1}{p} \geq - \log \left( \frac{x - 1}{x} \right) \log(x) \left( x - \frac{x}{2} \right) \frac{1}{p} = \frac{\log(x)}{x^{\frac{1}{p}}} \left( \frac{1}{2} \right) \frac{1}{p} \]
\[ = p' \frac{\log \left( x^{\frac{1}{p'}} \right)}{x^{\frac{1}{p'}}} \left( \frac{1}{2} \right) \frac{1}{p'} \geq \frac{1}{2} p' \frac{\log \left( x^{\frac{1}{p'}} \right)}{x^{\frac{1}{p'}}} \]
and consequently
\[ \| [b, H]f \|_{L^p, \infty(\mathbb{R})} \geq \frac{1}{2} p'. \]

### 3.5 Maximal function

For the upper bound
\[ \| M^k f \|_{L^p, \infty} \leq c_n \| M^{k-1} f \|_{L^p} \leq c_n (p')^{k-1} \| f \|_{L^p}. \]

For the lower bound firstly we observe that \( M^k \chi_{(0,1)} \simeq \frac{(\log |x|)^{k-1}}{|x|} \). Hence
A lower bound for exponents for some weighted weak-t…

Now we see that

\[ \|M^k \chi_{(0,1)}\|_{L^p,\infty} \approx \sup_{t > 0} \left( \left\{ x \in \mathbb{R} : \frac{(\log |x|)^{k-1}}{|x|} > t \right\} \right)^{\frac{1}{p'}} \]

\[ \geq \sup_{t > 0} \left( \left\{ x > e : \frac{(\log x)^{k-1}}{x} > t \right\} \right)^{\frac{1}{p'}} \]

\[ = \sup_{t > 0} \left( \left\{ x > e : \Psi(x) > t \right\} \right)^{\frac{1}{p'}} = \sup_{t > 0} \left( \{ x > e : x < \Psi^{-1}(t) \} \right)^{\frac{1}{p'}} \]

\[ = \sup_{t > 0} (\Psi^{-1}(t) - e)^{\frac{1}{p'}} = \sup_{x > e} \frac{(\log x)^{k-1}}{x} (x - e)^{\frac{1}{p'}} \]

\[ \geq \sup_{x > 2e} \frac{(\log x)^{k-1}}{x} (x - e)^{\frac{1}{p'}}. \]

Now we see that

\[ \frac{(\log x)^{k-1}}{x} (x - e)^{\frac{1}{p'}} \geq \frac{(\log x)^{k-1}}{x} \left( x - \frac{e}{2} \right)^{\frac{1}{p'}} \geq \left( \frac{1}{2} \right)^{\frac{1}{p'}} (\log x)^{k-1} \]

\[ \geq \left( \frac{1}{2} \right)^{\frac{1}{p'}} \left( p' \right)^{k-1} \log (x p')^{k-1} \]

\[ \geq c \left( \frac{1}{2} \right)^{\frac{1}{p'}} (p')^{k-1}. \]

This gives the desired estimate.

4 Proof of Theorem 2

We consider each case separately.

4.1 Calderón–Zygmund operators

It was established in [10] that

\[ \|T\|_{L^p(w) \to L^{p,\infty}(w)} \leq c_n[w]_{A_p} \]

and the optimality of the exponent is a direct corollary of the combination of Theorem 1 and Lemma 1.
4.2 Commutators

The lower bound of the exponent is again a direct corollary of the combination of Theorem 1 and Lemma 1. For the upper exponent we are going to use a proof based on a sparse domination result obtained in [12].

We recall that a family of dyadic cubes $\mathcal{S}$ is $\eta$-sparse with $\eta \in (0, 1)$ if for each cube $Q \in \mathcal{S}$ there exists a measurable subset $E_Q \subset Q$ such that $E_Q$ are pairwise disjoint and $\eta |Q| \leq |E_Q|$.

The following result is well known

**Theorem 3** Let $1 < p < \infty$. Then if $w \in A_p$

$$\|A \|_{L^p(w) \to L^p(w)} \leq c_{n,p} [w]_{A_p}^{\max \left\{ 1, \frac{1}{p} \right\}},$$

$$\|A \|_{L^p(w) \to L^p(\infty(w))} \leq c_{n,p} [w]_{A_p}$$

where $A(f) = \sum_{Q \in \mathcal{S}} \frac{1}{|Q|} \int_Q f \chi_Q(x)$.

In [12] it was proved that commutators can be controlled by suitable sparse operators. The precise statement is the following.

**Theorem 4** Let $T$ a Calderón–Zygmund operator, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $f \in C^\infty$. There exist $3^n$ dyadic lattices $D_j$ and $3^n$ sparse families $S_j \subset D_j$ such that

$$|[b, T]f(x)| \leq c_n c_T \sum_{j=1}^{3^n} \left( T_{b,S_j}f(x) + T_{b,S_j}^*f(x) \right)$$

where $c_T = c_K + c_\delta + \|T\|_{L^2 \to L^2}$ and

$$T_{b,S_j}f(x) = \sum_{Q \in S} |b(x) - b_Q| \frac{1}{|Q|} \int_Q |f| \chi_Q(x),$$

$$T_{b,S_j}^*f(x) = \sum_{Q \in S} \frac{1}{|Q|} \int_Q |b - b_Q||f| \chi_Q(x).$$

Consequently it suffices to establish the weak-type $(p, p)$ for those sparse operators. Without loss of generality we may assume that $\|b\|_{\text{BMO}} = 1$.

We deal first with $T_{b,S_j}$. We observe that

$$\|T_{b,S_j}f\|_{L^p(\infty(w))} = \sup_{\|g\|_{L^{p',1} \to L^{1}}} \left| \int T_{b,S_j}f g w \right|.$$
Now
\[
\left| \int \mathcal{T}_{b,S}(f)gw \right| \leq \int \mathcal{T}_{b,S}(f)|g|w = \sum_{Q \in S} \frac{1}{|Q|} \int_Q |f| \int_Q |b - b_Q||g|w.
\]

Arguing as in the proof of [12, Theorem 1.4] we have that there exists a sparse family $\tilde{S} \supset S$ such that for every cube in $Q \in \tilde{S}$,
\[
\int_Q |b - b_Q||g|w \leq c_n \|b\|_{\text{BMO}} \int_Q A_{\tilde{S}}(|g|w).
\]

Then, taking also into account that $A_{\tilde{S}}$ is self-adjoint,
\[
\sum_{Q \in \tilde{S}} \frac{1}{|Q|} \int_Q |b - b_Q||g|w \leq c_n \|b\|_{\text{BMO}} \sum_{Q \in \tilde{S}} \frac{1}{|Q|} \int_Q |f| \int_Q A_{\tilde{S}}(|g|w)
\]
\[
= c_n \|b\|_{\text{BMO}} \int A_{\tilde{S}}(f) A_{\tilde{S}}(|g|w)
\]
\[
= c_n \|b\|_{\text{BMO}} \int (A_{\tilde{S}} \circ A_{\tilde{S}})(f)|g|w
\]

and we have that
\[
\| \mathcal{T}_{b,S} f \|_{L^{p,\infty}(w)} = \sup_{\|g\|_{L^{p',1}(w)}} \left| \int \mathcal{T}_{b,S}(f)gw \right| \leq \sup_{\|g\|_{L^{p',1}(w)}} \int (A_{\tilde{S}} \circ A_{\tilde{S}})(f)|g|w
\]
\[
\leq c_n \sup_{\|g\|_{L^{p',1}(w)}} \int (A_{\tilde{S}} \circ A_{\tilde{S}})(f)|g|w
\]
\[
\leq c_n \|(A_{\tilde{S}} \circ A_{\tilde{S}})f\|_{L^{p,\infty}(w)} \|g\|_{L^{p',1}(w)}.
\]

The desired inequality follows from applying twice Theorem 3.

For $\mathcal{T}_{b,S}^*$ we can argue analogously. Indeed, it is clear that we also have that
\[
\int_Q |b - b_Q||f||w \leq c_n \|b\|_{\text{BMO}} \int_Q A_{\tilde{S}}(|f||w)
\]

and this yields
\[
\mathcal{T}_{b,S}^*(f) \leq c_n \|b\|_{\text{BMO}} (A_{\tilde{S}} \circ A_{\tilde{S}})(|f|).
\]

Now it suffices to use Theorem 3 to end the proof.
4.3 Maximal operator

The lower bound of the exponent is a straightforward consequence of Theorem 1 and Lemma 1. For the upper bound it readily follows from (1.3) that

\[ \|M^k f\|_{L^{p,\infty}(w)} \leq c_n \|w\|_A^p \|M^{k-1} f\|_{L^p(w)} \leq c_n \|w\|_A^{p-k+1} \|f\|_{L^p(w)} \]

and we are done.

Acknowledgements Both authors are supported by the Basque Government through the BERC 2014-2017 program and by Spanish Ministry of Economy and Competitiveness MINECO through BCAM Severo Ochoa excellence accreditation SEV-2013-0323. and through the project MTM2014-53850-P. I.P.R-R. is also supported by Spanish Ministry of Economy and Competitiveness MINECO through the project MTM2012-30748.

References

1. Alvarez, J., and C. Pérez. 1994. Estimates with \( A_\infty \) weights for various singular integral operators, Bollettino U.M.I. (7) 8-A :123–133.
2. Astala, K., T. Iwaniec, and E. Saksman. 2001. Beltrami operators in the plane. Duke Mathematical Journal 107 (1): 27–56.
3. Bagby, R.J., and D.S. Kurtz. 1985. Covering lemmas and the sharp function. Proceedings of the American Mathematical Society 93: 291–296.
4. Bagby, R.J., and D.S. Kurtz. 1986. A rearranged good-\( \lambda \) inequality. Transactions of the American Mathematical Society 293: 71–81.
5. Buckley, S.M. 1993. Estimates for operator norms on weighted spaces and reverse Jensen inequalities. Transactions of the American Mathematical Society 340 (1): 253–272.
6. Cruz-Uribe, D., SFO, J.M. Martell and C. Pérez. 2011. Weights, Extrapolation and the Theory of Rubio de Francia, Operator Theory: Advances and Applications, vol. 215, Birkhäuser/Springer Basel AG, Basel.
7. Duoandikoetxea, J. 2011. Extrapolation of weights revisited: new proofs and sharp bounds. Journal of Functional Analysis 260 (6): 1886–1901.
8. García-Cuerva, J., and J.L. Rubio de Francia. 1985. Weighted Norm Inequalities and Related Topics, North Holland Math. Studies, vol. 116, North Holland, Amsterdam.
9. Hytönen, T.P. 2012. The sharp weighted bound for general Calderón–Zygmund operators. Annals of Mathematics (2) 175 (3): 1473–1506.
10. T.P. Hytönen, M.T. Lacey, H. Martikainen, T. Orponen, M.C. Reguera, E.T. Sawyer, I. Uriarte-Tuero. 2012. Weak and strong type estimates for maximal truncations of Calderón-Zygmund operators on \( A_p \) weighted spaces. Journal d’Analyse Mathématique 118 (1): 177–220.
11. Hytönen, T.P., and C. Pérez. 2013. Sharp weighted bounds involving \( A_\infty \). Analysis of PDE 6 (4): 777–818.
12. Lerner, A.K., S. Ombrosi, and I.P. Rivera-Rios. 2017. On pointwise and weighted estimates for commutators of Calderón–Zygmund operators, Advances in Mathematics 319: 153–181.
13. Luque, T., C. Pérez, and E. Rela. 2015. Optimal exponents in weighted estimates without examples. Mathematical Research Letters 22 (1): 183–201.
14. Ortiz-Caraballo, C. 2011. Quadratic \( A_1 \) bounds for commutators of singular integrals with BMO functions. Indiana University Mathematics Journal 60 (6): 2107–2129.
15. Ortiz-Caraballo, C., C. Pérez and E. Rela. 2013. Improving bounds for singular operators via Sharp Reverse Hölder Inequality for \( A_\infty \). In Operator Theory: Advances and Applications, Advances in Harmonic Analysis and Operator Theory, eds. A. Almeida, L. Castro, F. Speck, vol. 229, 303–321. Springer Basel.
16. Pérez, C. 2015. Singular integrals and weights. Harmonic and geometric analysis, Adv. Courses Math. CRM Barcelona, 91–143. Birkhäuser/Springer Basel AG, Basel.
17. Petermichl, S., and A. Volberg. 2002. Heating of the Ahlfors–Beurling operator: weakly quasiregular maps on the plane are quasiregular. Duke Mathematical Journal 112 (2): 281–305.