Bosonic string theory with constraints linear in the momenta

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(March 27, 2022)

The Hamiltonian analysis of Polyakov action is reviewed putting emphasis in two topics: Dirac observables and gauge conditions. In the case of the closed string it is computed the change of its action induced by the gauge transformation coming from the first class constraints. As expected, the Hamiltonian action is not gauge invariant due to the Hamiltonian constraint quadratic in the momenta. However, it is possible to add a boundary term to the original action to build a fully gauge-invariant action at first order. In addition, two relatives of string theory whose actions are fully gauge-invariant under the gauge symmetry involved when the spatial slice is closed are built. The first one is pure diffeomorphism in the sense it has no Hamiltonian constraint and thus bosonic string theory becomes a sub-sector of its space of solutions. The second one is associated with the tensionless bosonic string, its boundary term induces a canonical transformation and the fully gauge-invariant action written in terms of the new canonical variables becomes linear in the momenta.

I. CANONICAL ANALYSIS

Relativistic free strings propagating in an arbitrary D-dimensional fixed background spacetime with metric \( g_{\mu\nu}(X) dX^\mu dX^\nu; \mu, \nu = 0, 1, ..., D - 1, \) can be described, for instance, with the Polyakov action

\[
S[\gamma^{ab}, X^\mu] = \alpha \int_M d^2\xi \sqrt{-g} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X).
\]

The variation of \( S[\gamma^{ab}, X^\mu] \) with respect to the background coordinates \( X^\mu \) and the metric \( \gamma^{ab} \) yields the equations of motion

\[
\nabla^a \nabla_a X^\mu + \Gamma^\mu_{\theta\phi} \gamma^{bc} \partial_\theta X^b \partial_\phi X^c = 0,
\]

\[
T_{ab} := \frac{\alpha}{2} \gamma_{cd} \nabla^a X^\mu \nabla_b X^\nu g_{\mu\nu} - \alpha \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} = \frac{\alpha}{2} \gamma_{cd} h_{cd} - \alpha h_{ab} = 0,
\]

respectively. Here, \( h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \) is the induced metric on the world sheet, \( \nabla \) is the covariant derivative associated with the Levi-Civita connection of \( \gamma^{ab} \), \( \Gamma^\mu_{\theta\phi} \) are the Christoffel symbols associated with the background metric \( g_{\mu\nu} \). The Lagrangian formalism is more common than the Hamiltonian one for string people community. However, the Hamiltonian framework is a necessary step to perform the quantization of the theory using Dirac’s method. Also the Hamiltonian framework is the natural arena to analyze the issues of observables and gauge conditions for the theory which have relevance both in its classical and quantum dynamics. That is why here the canonical analysis is reviewed putting emphasis in these two topics. To go to the Hamiltonian formalism, it is mandatory to choose a time coordinate \( \xi^0 = \tau \) and a space coordinate \( \xi^1 = \sigma \) and assume that the world sheet \( M \) has the topology \( M = R \times \Sigma \). The metric \( \gamma_{ab} \) is put in the ADM form

\[
(\gamma_{ab}) = \begin{pmatrix} -N^2 + \lambda^2 \chi & \chi \lambda \\ \chi \lambda & \chi \end{pmatrix}, \quad (\gamma^{ab}) = \begin{pmatrix} -\frac{1}{\lambda^2} & \frac{1}{\lambda} \\ \frac{1}{\lambda} & \frac{1}{N^2} \end{pmatrix},
\]

and so \( \sqrt{-\gamma} := \sqrt{-\det \gamma_{ab}} = \epsilon N \sqrt{\chi} \) with \( \epsilon = +1 \) if \( N > 0 \) and \( \epsilon = -1 \) if \( N < 0 \). Due to the fact \( \tau \) is time-like and \( \sigma \) is space-like \( -N^2 + \lambda^2 \chi < 0 \) and \( \chi > 0 \). Taking into account \([3]\) \( S[\gamma^{ab}, X^\mu] \) acquires the form

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where the dependency of the phase space variables and Lagrange multipliers in terms of the Lagrangian variables is

$$\tilde{P}_\mu = -\frac{2\alpha\epsilon}{N} X^\nu g_{\mu\nu} + \frac{2\alpha\lambda}{N} X^{\prime\nu} g_{\mu\nu}, \quad \lambda_\epsilon = -\frac{N}{4\alpha\sqrt{X}},$$

with

$$\tilde{H} = \tilde{P}_\mu \tilde{P}^{\mu} + 4\alpha^2 X^{\prime\nu} X^{\prime\rho} g_{\mu\nu}, \quad \tilde{D} = X^{\prime\nu} \tilde{P}_\mu,$$

the Hamiltonian and diffeomorphism constraints, respectively. Here \(X^{\prime\nu} = \frac{\partial X^\nu}{\partial \sigma}\). The standard variational principle is formed with the action (4) and with the boundary conditions

$$X^{\nu}(\tau_1, \sigma) = x^{\nu}_i(\sigma), \quad i = 1, 2,$$

where \(x^{\nu}_i(\sigma)\) are the initial (at \(\tau_1\)) and final (at \(\tau_2\)) string configurations.

The variation of \(S[X^\mu, \tilde{P}_\mu, \lambda_\epsilon, \lambda_\lambda]\) under (3) and the condition \(\delta S = 0\) yield the equations of motion

$$\dot{X}^{\mu} = 2\lambda \tilde{P}_\nu g^{\nu\mu} \lambda X^{\mu}, \quad \ddot{\tilde{P}}_\mu = \lambda \tilde{Y}^{\rho\phi} \frac{\partial g^{\phi\nu}}{\partial X^{\rho}} + \left(\lambda^2 \alpha^2 X^{\prime\nu} g_{\mu\nu} + \lambda \tilde{P}_\mu\right),$$

and the constraints

$$\tilde{H} = 0, \quad \tilde{D} = 0,$$

where \(\tilde{Y} = \tilde{P}_\mu \tilde{P}^{\mu} g^{\rho\nu} - 4\alpha^2 X^{\rho\phi} X^{\rho\phi}\). To compute the algebra of constraints the Hamiltonian and diffeomorphism constraints are smeared with arbitrary fields \(\varepsilon(\tau, \sigma)\) and \(\varepsilon(\tau, \sigma)\)

$$H(\varepsilon) = \int d\sigma \varepsilon \tilde{H}, \quad D(\varepsilon) = \int d\sigma \varepsilon \tilde{D}. \quad (10)$$

Then a straightforward computation yields the Poisson brackets between the constraints

$$\{H(\varepsilon), H(\lambda)\} = D(\kappa) + B.T., \quad \{D(\varepsilon), H(\lambda)\} = H(L_\varepsilon \varepsilon) + B.T., \quad \{D(\varepsilon), D(\lambda)\} = D(L_\varepsilon \lambda) + B.T.,$$

where \(L_\varepsilon \lambda = \varepsilon X' - \lambda \varepsilon', L_\varepsilon \varepsilon = \varepsilon \varepsilon' - \varepsilon \varepsilon', \kappa = 16\alpha^2 (\varepsilon \lambda' - \lambda \varepsilon')\), and B.T. stands for boundary terms. Therefore, \(\tilde{H}\) and \(\tilde{D}\) are first class and the theory has \(D - 2\) physical degrees of freedom (\(2(D - 2)\) in the phase space) per space point.

**Geometric perspective.** Even though the Hamiltonian and diffeomorphism constraints have an evident meaning, it is interesting to see what the constraint surface means from the perspective of the geometry of the induced metric \(h_{\nu\rho}\). By using the equation of motion for \(X^{\mu}\) and the definitions of the induced metric components the constraints (3) become

$$\tilde{H} = \frac{1}{4\lambda^2} h_{\tau\tau} - \frac{\lambda}{2\lambda^2} h_{\tau\sigma} + \left(\frac{\lambda^2}{4\lambda^2} + 4\alpha^2\right) h_{\sigma\sigma}, \quad \tilde{D} = \frac{1}{2\lambda} h_{\tau\sigma} - \frac{\lambda}{2\lambda^2} h_{\sigma\sigma}.$$

Therefore, the constraint surface (4) just means

$$\frac{1}{4\lambda^2} h_{\tau\tau} - \frac{\lambda}{2\lambda^2} h_{\tau\sigma} + \left(\frac{\lambda^2}{4\lambda^2} + 4\alpha^2\right) h_{\sigma\sigma} = 0, \quad \frac{1}{2\lambda} h_{\tau\sigma} - \frac{\lambda}{2\lambda^2} h_{\sigma\sigma} = 0. \quad (13)$$

From the second equation \(h_{\tau\sigma}\) can be plugged into the first and then the constraint surface looks like

$$h_{\tau\tau} = \left(-16\lambda^2 + \lambda^2\right) h_{\sigma\sigma}, \quad h_{\tau\sigma} = \lambda h_{\sigma\sigma}. \quad (14)$$

These relationships among the world sheet metric components and the Lagrange multipliers just came as a consequence of describing string dynamics from a canonical perspective. Also it is possible to write down the induced metric \(h_{ab}\) in terms of the phase space variables and Lagrange multipliers
\[ h_{\tau\tau} = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} = 4 \lambda^2 \tilde{P}_\mu \tilde{P}_\nu g^{\mu\nu} + 4 \lambda \lambda \tilde{P}_\mu X'^\mu + \lambda^2 X'^\mu X'^\nu g_{\mu\nu}, \]
\[ h_{\tau\sigma} = \dot{X}^\mu X'^\nu g_{\mu\nu} = 2 \lambda \tilde{P}_\mu X'^\mu + \lambda X'^\mu X'^\nu g_{\mu\nu}, \]
\[ h_{\sigma\sigma} = X'^\mu X'^\nu g_{\mu\nu}, \] (15)

where the equation of motion for \( X^\mu \) was used. These expressions mean that in order to compute the induced metric on the world sheet is necessary 1) fix the gauge and thus to determine the Lagrange multipliers \( \lambda \) and \( \lambda \), 2) with each particular choice for the Lagrange multipliers solve the equations of motion, 3) plug these solutions into the constraints and into the gauge fixing conditions to drop the gauge freedom completely, finally, 4) insert the final expression for \( X^\mu \) together with the Lagrange multipliers into the RHS of the metric components.

By using (15) the metric components can be written in terms of the constraints, the Lagrange multipliers and just one metric component

\[ h_{\tau\tau} = 4 \lambda^2 \tilde{H} + 4 \lambda \lambda \tilde{D} + \left(-16 \lambda^2 \alpha^2 + \lambda^2\right) h_{\sigma\sigma}, \quad h_{\tau\sigma} = 2 \lambda \tilde{D} + \lambda h_{\sigma\sigma}. \] (16)

Thus, on the constraint surface

\[ h_{\tau\tau} = \left(-16 \lambda^2 \alpha^2 + \lambda^2\right) h_{\sigma\sigma}, \quad h_{\tau\sigma} = \lambda h_{\sigma\sigma}, \] (17)
as expected [see Eq. (14)].

In addition, it is interesting to compute the energy-momentum tensor (2) in terms of the phase space variables and the Lagrange multipliers. By doing this the components of \( T = T_{ab} \delta x^a \delta x^b \) become

\[ T_{\tau\tau} = -2\alpha \lambda^2 \left(1 + \frac{\lambda^2}{16 \alpha^2} \alpha^2 \tilde{H} - 4 \alpha \lambda \lambda \tilde{D}\right), \quad T_{\tau\sigma} = -\frac{\lambda}{8 \alpha} \tilde{H} - 2 \alpha \lambda \tilde{D}, \quad T_{\sigma\sigma} = -\frac{1}{8 \alpha} \tilde{H}, \] (18)

and thus the energy-momentum tensor vanishes on the constraint surface. However, the trace of the energy-momentum zero vanishes identically, \( T_{\alpha} = T_{ab} g^{ab} = 0 \tilde{H} + 0 \tilde{D} = 0 \), and not just on the constraint surface.

**Gauge transformations.** Before computing the gauge transformation on the phase space variables and Lagrange multipliers induced by the first class constraints, it is interesting to compute the finite transformation of these variables due to both Poincaré and Weyl invariance, i.e., it is assumed in this part of the paper that the background metric \( g_{\mu\nu} \) is the Minkowski one \( \eta_{\mu\nu} \).

i) **Poincaré invariance** is \( X^\mu (\tau, \sigma) = X^\mu (\tau', \sigma') + a^\mu, \quad \gamma_{ab} (\tau, \sigma) = \gamma_{ab} (\tau, \sigma) \) with \( a^\mu \) a Lorentz transformation and \( a^\mu \) a translation. Using the explicit form of \( \gamma_{ab} \) in (3) and the definition of the momentum (2) finite Poincaré invariance means, in the Hamiltonian framework, that the phase space variables and Lagrange multipliers transform as

\[ X^\mu (\tau, \sigma) = \Lambda^\mu\nu X^\nu (\tau, \sigma) + a^\mu, \quad \tilde{P}_\mu (\tau, \sigma) = \Lambda^\mu\nu \tilde{P}_\nu (\tau, \sigma), \quad \chi (\tau, \sigma) = \lambda (\tau, \sigma), \quad \lambda (\tau, \sigma) = \lambda (\tau, \sigma). \] (19)

ii) **Two-dimensional Weyl invariance** is \( X^\mu (\tau, \sigma) = X^\mu (\tau, \sigma) \) \( \gamma_{ab} (\tau, \sigma) = e^{2\omega (\tau, \sigma)} \gamma_{ab} (\tau, \sigma) \) for arbitrary \( \omega (\tau, \sigma) \). Using the explicit form of \( \gamma_{ab} \) in (3) and the definition of the momentum (2) finite two-dimensional Weyl invariance means, in the Hamiltonian framework, that the phase space variables and Lagrange multipliers transform as

\[ X^\mu (\tau, \sigma) = X^\nu (\tau, \sigma), \quad \tilde{P}_\mu (\tau, \sigma) = \frac{1}{\epsilon} \tilde{P}_\nu (\tau, \sigma), \quad \chi (\tau, \sigma) = \epsilon' \lambda (\tau, \sigma), \quad \lambda (\tau, \sigma) = \lambda (\tau, \sigma); \quad \epsilon' = \pm 1, \] (20)
due to the fact \( \chi \) and \( N \) transform as \( \chi (\tau, \sigma) = e^{2\omega (\tau, \sigma)} \chi (\tau, \sigma), \quad N (\tau, \sigma) = \epsilon' e^{\omega (\tau, \sigma)} N (\tau, \sigma) \). Nevertheless, only \( \epsilon' = 1 \) leaves action (4) invariant under the transformation (20).

Let us go back to general case, namely, when the background metric \( g_{\mu\nu} \) is left arbitrary. In this case, it is easy to compute the infinitesimal gauge transformation induced by the constraints (3) on the phase space variables (4)

\[ X^\mu (\tau, \sigma) = X^\mu (\tau, \sigma) + \{X^\mu (\tau, \sigma), H (\xi)\} + \{X^\mu (\tau, \sigma), D (\varepsilon)\}, \]
\[ = X^\mu (\tau, \sigma) + 2 \left(\xi \delta X^\mu \right) (\tau, \sigma) + \mathcal{L}_\varepsilon X^\mu (\tau, \sigma), \]
\[ \tilde{P}_\mu (\tau, \sigma) = \tilde{P}_\mu (\tau, \sigma) + \{\tilde{P}_\mu (\tau, \sigma), H (\xi)\} + \{\tilde{P}_\mu (\tau, \sigma), D (\varepsilon)\}, \]
\[ = \tilde{P}_\mu (\tau, \sigma) + \tilde{P}_\mu (\tau, \sigma) + \tilde{Y} \tilde{\theta} \frac{\partial \tilde{\theta}}{\partial X^\mu (\tau, \sigma)} - 8 \alpha^2 \tilde{\xi} (\tau, \sigma) \delta (\tau, \sigma) \gamma_{ab} (\tau, \sigma) g_{\mu\nu} (\tau, \sigma) g_{\nu\sigma} (\tau, \sigma) \quad \mathcal{L}_\varepsilon \tilde{P}_\mu (\tau, \sigma) \quad \mathcal{L}_\varepsilon \tilde{P}_\mu (\tau, \sigma). \] (21)
respectively, are observables; and thus the values of $P_\mu$ brackets with the Hamiltonian and diffeomorphism constraints. At infinitesimal level this means observables must be

\[ L \] invariance of the theory. For closed strings no boundary terms appear in the constraints algebra \[ (11) \] and the transformation law for the phase space variables simplifies accordingly

\[ \mathcal{X}^\mu(\tau, \sigma) = \mathcal{X}^\mu(\tau, \sigma) + 2 \left( \xi_{\lambda} g^{\mu\nu} \bar{P}_\nu \right)(\tau, \sigma) + L_\tau \mathcal{X}^\mu(\tau, \sigma), \]

\[ \bar{P}_\mu(\tau, \sigma) = \bar{P}_\mu(\tau, \sigma) + \xi \frac{\partial \theta \phi}{\partial \mathcal{X}^\mu}(\tau, \sigma) + (8\alpha^2 \xi \mathcal{X}^{\mu\nu} g_{\mu\nu})'(\tau, \sigma) + L_\tau \bar{P}_\mu(\tau, \sigma), \]  

while the Lagrange multipliers transform as

\[ \lambda'(\tau, \sigma) = \lambda(\tau, \sigma) + \xi + L_\tau \lambda - L_\tau \xi, \quad \lambda'(\tau, \sigma) = \lambda(\tau, \sigma) + \xi + L_\tau \lambda - \kappa. \]

Taking into account \[ (22) \] and \[ (23) \] the gauge transformation induces a transformation in the action for the closed string

\[ S[\mathcal{X}^\mu, \bar{P}_\mu, \lambda', \lambda'] = S[\mathcal{X}^\mu, \bar{P}_\mu, \lambda, \lambda] + \int_{\sigma_1}^{\sigma_2} d\sigma \left[ \left\{ \mathcal{X}^\mu(\tau, \sigma), H \right\} \bar{P}_\mu(\tau, \sigma) - \left( \tilde{\varepsilon} H + \varepsilon \bar{D} \right) \right]_{\tau = \tau_2}, \]

with

\[ H = \int_{\sigma_1}^{\sigma_2} d\sigma \left( \tilde{\varepsilon} H + \varepsilon \bar{D} \right) = H(\varepsilon) + D(\varepsilon). \]

After a direct computation

\[ S[\mathcal{X}^\mu, \bar{P}_\mu, \lambda', \lambda'] = S[\mathcal{X}^\mu, \bar{P}_\mu, \lambda, \lambda] + \int_{\sigma_1}^{\sigma_2} d\sigma \left[ \xi \left( \bar{P}_\mu \bar{P}_\nu g^{\mu\nu} - 4\alpha^2 \mathcal{X}^{\mu\nu} \mathcal{X}^{\mu\nu} g_{\mu\nu} \right) \right]_{\tau = \tau_2}, \]

\[ = S[\mathcal{X}^\mu, \bar{P}_\mu, \lambda, \lambda] + \int_{\sigma_1}^{\sigma_2} d\sigma \left( \tilde{\varepsilon} \mathcal{X}^{\mu\nu} g_{\mu\nu} \right)_{\tau = \tau_2}. \]

Therefore $S[\mathcal{X}^\mu, \bar{P}_\mu, \lambda, \lambda]$ is not gauge-invariant and behaves in the same way as the action for the relativistic free particle [cf. Ref. \[ 3 \]]. The reason why $S[\mathcal{X}^\mu, \bar{P}_\mu, \lambda, \lambda]$ fails to be gauge invariant is because the Hamiltonian constraint is quadratic in the momenta like in systems with finite degrees of freedom. \[ [14] \] In spite of this, fully gauge invariant actions under finite gauge symmetries for systems with finite degrees of freedom were built in Ref. \[ 3 \]. Now, those ideas are here extended to field theory. In the particular case when the background metric $g_{\mu\nu}$ is constant, for instance when $g_{\mu\nu}$ is the Minkowski metric $\eta_{\mu\nu}$, the action for the closed string

\[ S_{\text{inv}}[\mathcal{X}^\mu, \bar{P}_\mu, \lambda, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1}^{\sigma_2} d\sigma \left[ \mathcal{X}^\mu \bar{P}_\mu - \left( \lambda \bar{H} + \lambda \bar{D} \right) \right] - \frac{1}{2} \int_{\sigma_1}^{\sigma_2} d\sigma \left( \mathcal{X}^\mu \bar{P}_\mu \right)_{\tau = \tau_2}, \]

is, at first order, fully gauge-invariant.

**Observables.** Dirac observables or observables defined for short are functions defined on the reduced phase space of the theory. They are constant along the gauge orbits of the constraint surface and thus they have weakly vanishing Poisson brackets with the Hamiltonian and diffeomorphism constraints. At infinitesimal level this means observables must be gauge invariant under the gauge transformation \[ (22) \]. From the transformation law for the phase space variables \[ (22) \] it is clear that for closed strings propagating in a constant background $g_{\mu\nu}$ the linear and angular momentum

\[ P_\mu = \int_{\sigma_1}^{\sigma_2} d\sigma \bar{P}_\mu(\tau, \sigma), \quad M^{\mu\nu} = \int_{\sigma_1}^{\sigma_2} d\sigma \left( \mathcal{X}^\mu(\tau, \sigma) \bar{P}^\nu(\tau, \sigma) - \mathcal{X}^\nu(\tau, \sigma) \bar{P}^\mu(\tau, \sigma) \right), \]

respectively, are observables; and thus the values of $P_\mu$ and $M^{\mu\nu}$ are independent of any particular choice for the gauge conditions. From their own definitions apparently the linear momentum $P_\mu$ and the angular momentum $M^{\mu\nu}$ depend on $\tau$. However, by computing their derivative with respect to $\tau$ and using the equations of motion $\bar{P}_\mu = 0 = M^{\mu\nu}$. Therefore, $P_\mu$ and $M^{\mu\nu}$ are indeed independent of the time coordinate $\tau$ and, of course, of the space coordinate $\sigma$. Notice that two string configurations having same $P_\mu$ and $M^{\mu\nu}$ do not represent the same physical string configuration because $P_\mu$ and $M^{\mu\nu}$ do not label the full reduced phase space of string theory. Moreover, it is possible to build other observables from combinations of the previous ones as, for example, the square mass $M^2$ of the closed string $M^2 = -P_\mu P^\mu$. Up to here, it has been shown $P_\mu$ and $M^{\mu\nu}$ are observables because they are invariant.
under the gauge transformation generated by the first class constraints $\{22\}$. However, what about Poincaré and Weyl invariance? Notice that under Poincaré invariance $\{19\}$ the linear momentum $P_\mu$ and the angular momentum $M^{\mu\nu}$ transform as $P_\mu = \Lambda_\mu^\nu P_\nu$, $M^{\mu\nu} = \Lambda_\mu^\alpha \Lambda_\nu^\beta M^{\alpha\beta} + (a^\mu \Lambda_\nu^\alpha - a^\nu \Lambda_\mu^\alpha) \mathcal{P}^{\mu\nu}$ while under two-dimensional Weyl invariance $\{20\}$ they are fully gauge-invariant (taking $\epsilon' = 1$) $P_\mu = P_\mu$, $M^{\mu\nu} = M^{\mu\nu}$ because $\mathcal{P}_\mu (\tau, \sigma) = \tilde{P}_\mu (\tau, \sigma)$ and $X^\mu (\tau, \sigma) = X^\mu (\tau, \sigma)$. This will be very important in a moment. Due to the fact the ten observables $P_\mu$ and $M^{\mu\nu}$ in a Minkowski target are associated with its isometries (Killing vector fields), it is natural to expect that the analogous of $P_\mu$ and $M^{\mu\nu}$ in arbitrary backgrounds, where Poincaré invariance is lost, would be associated with their Killing vector fields too. In fact, if $v = v^\mu (X) \frac{\partial}{\partial X^\mu}$ is a Killing vector field of the background spacetime $g = g_{\mu\nu} (X) dX^\mu dX^\nu$, then a straightforward application of Noether’s theorem to $\{1\}$ implies

$$O_v = \int_{\sigma_1}^{\sigma_2} d\sigma \tilde{P}_\mu (\tau, \sigma) v^\mu (X (\tau, \sigma)),$$

are observables [see $\{3\}$ for an alternative approach in the case of $p$-branes]. But, what about if backgrounds had no isometries? This simply would mean that there would be no observables associated with isometries, however, still there would be observables, i.e., invariant entities under the transformation $\{22\}$ associated with the true physical degrees of freedom of strings.

Other quantities used in string theory are the ‘center of mass’ coordinates of the string

$$X^\mu (\tau) = \int_{\sigma_1}^{\sigma_2} d\sigma X^\mu (\tau, \sigma).$$

However, the ‘center of mass’ coordinates of a closed string are not observables under the transformation $\{22\}$. This might be source of confusion with intuition. Certainly, $X^\mu (\tau)$ are measurable quantities, but measurable quantities are not, in general, observables of the theory. In addition, under Poincaré invariance $\{19\}$, the ‘center of mass’ coordinates transform as $X^\mu (\tau) = \Lambda_\mu^\nu X^\nu (\tau) + a^\mu (\sigma_2 - \sigma_1)$ while under Weyl invariance $\{20\}$ they are fully gauge-invariant $X^\mu (\tau) = X^\mu (\tau)$ because $X^\mu (\tau, \sigma) = X^\mu (\tau, \sigma)$.

So far, it has been exhibited the transformation laws for $P_\mu$ and $M^{\mu\nu}$ under i) Poincaré invariance $\{19\}$, ii) two-dimensional Weyl invariance $\{20\}$, and iii) the transformation law associated with the first class constraints $\{22\}$. Let us compare with gravity. Here, gravity is not string gravity, rather, it is Einstein’s general relativity. In four dimensional general relativity is neither the symmetry of the kind associated with global Lorentz invariance $\{19\}$ nor the symmetry of the kind associated with two-dimensional Weyl invariance $\{20\}$ [this type of symmetry is also not present in the Dirac-Nambu-Goto action $\{3\}$], rather, the gauge symmetry present in general relativity is of the same kind that the one coming from the first class constraints $\{6\},\{22\}$. Indeed, from last computations important notions can been drawn which make shape to the meaning of observables in generally covariant theories, in particular, for general relativity. The first lesson from $P_\mu$ and $M^{\mu\nu}$ is that they are gauge-invariant under the gauge symmetry associated with the first class constraints $\{22\}$. The second lesson is that $P_\mu$ and $M^{\mu\nu}$ are independent of the time and space coordinates $\tau$ and $\sigma$; respectively, which label the points on the world sheet. Therefore, in any generally covariant theory having Hamiltonian and diffeomorphism constraints, as general relativity, must happen the same phenomenon: observables must be coordinate independent entities too. In string theory, on the other hand, fields have physical meaning because they are attached to the fixed background $\eta_{\mu\nu}$, and thus $P_\mu$ (or $M^{\mu\nu}$) can be measured in any ‘external’ Lorentz reference frame. The relationship between the values of the linear momentum $P_\mu$ measured from any two ‘external’ Lorentz observers is $P_\mu = \Lambda_\mu^\nu P_\nu$ with $\Lambda_\mu^\nu$ a matrix in the Lorentz group. However, the presence of ‘external’ observers placed in the background manifold is a peculiar fact of string theory and it is not a general property of generally covariant theories, for instance, in general relativity ‘external’ observers are not allowed; there is not a background manifold ‘outside’ of spacetime where ‘external’ observers sit to see how spacetime propagates, rather, dynamics of the gravitational field must be described from an ‘inside’ viewpoint. This is a key conceptual difference of general relativity with respect to string theory. Nevertheless, as already mentioned it is still true that in general relativity observables must be coordinate independent entities as well, and this fact implies a major problem in gravity. In general relativity spacetime coordinates are attached to ‘observers’ placed in some reference frame, so how can an ‘observer’ measure some observable, say in his (her) laboratory, if observables are independent of spacetime coordinates? In other words, ‘local’ observables in general relativity or in any other generally covariant theory are not allowed because of diffeomorphism invariance $\{3\}$.

Gauge fixing. In any gauge theory, determinism forces it to identify gauge related phase space variables as a single point in the reduced phase space of the gauge theory, and the total number of these orbits span its physical phase space. At classical level, good gauge conditions help to single out these physical degrees of freedom because they intersect just once the gauge orbits on the constraint surface. On the other hand, in quantum theory there are essentially two ways to proceed: i) reduce then quantize or ii) quantize then reduce. In i) the relevance of a good
gauge fixing is clear. Standard quantization of strings is of the kind (i) and so it is important to have good gauge conditions to do that. Before going to that point, some words about other unfortunate choice for the gauge conditions

\[ \lambda = 0, \quad \lambda = 1, \]  

(31)

usually found in the literature. Due to the fact \( \tau \) is time-like and \( \sigma \) is space-like \( \gamma_{\tau\tau} = -N^2 + \lambda^2 \chi < 0 \) and \( \gamma_{\sigma\sigma} = \chi > 0 \) must hold, which means \( \lambda^2 < 16\alpha^2\lambda^2 \). It is clear (31) does not satisfy this condition. Putting it in a different manner, the choice (31) breaks down the causal structure on the world sheet because with such a choice \( \tau \) becomes space-like and \( \sigma \) becomes time-like.

To fix consistently the gauge degrees of freedom in the action (1), the components of the inverse of the world sheet metric \( \gamma^{ab} \) will be considered as dynamical variables. In this case, there are three additional constraints, since the canonical momenta associated to \( \gamma^{ab} \) are weakly equal to zero

\[ \tilde{\pi}_{ab} \approx 0, \quad a, b = 1, 2. \]

(32)

In this approach, instead of (2), the canonical action is

\[ S[X^\mu, \gamma^{ab}, \tilde{P}_\mu, \tilde{\pi}_{ab}, \lambda_1, \lambda_1, \lambda^{ab}] = \int_R d\tau \int d\sigma \left[ \dot{X}^\mu \tilde{P}_\mu + \gamma^{ab} \tilde{\pi}_{ab} - \left( \lambda^{ab} \tilde{\pi}_{ab} + \lambda_1 \tilde{H} + \lambda_1 \tilde{D} \right) \right]. \]

(33)

This action becomes the action (1) when Weyl invariant variables are used, all the constraints being first class. So, to fix the gauge, five gauge conditions are needed. Notice that the Lagrange multipliers \( (\lambda_1, \lambda_1) \) are not exactly the same Lagrange multipliers of (1). Both sets are related by

\[ \lambda_1 = \lambda + \rho, \quad \tilde{\lambda}_1 = \tilde{\lambda} + \tilde{\rho}, \]

(34)

where the additional arbitrary parts \( (\rho, \tilde{\rho}) \) appear from the fact that the constraints (1) are secondary ones in this approach.

Now, the conformal fixing of the world sheet metric will be considered

\[ \gamma^{\tau\tau} = -1, \quad \gamma^{\tau\sigma} = 0, \quad \gamma^{\sigma\sigma} = 1. \]

(35)

These gauge conditions set the Lagrange multipliers \( \lambda^{ab} = 0 \), however, from the infinitesimal gauge transformation for the intrinsic metric

\[ \delta \gamma_{ab} = \eta^c \partial_c \gamma_{ab} + \partial_a \eta^c \gamma_{cb} + \partial_b \eta^c \gamma_{ca} + 2\omega \gamma_{ab}, \]

(36)

it follows that the conditions (35) do not fix completely the gauge freedom of the gauge parameters \( (\eta^a, \omega) \). These parameters are only restricted to satisfy the differential equations

\[ \dot{\eta}^a - \eta^a'' = 0, \quad \dot{\eta}^\sigma - \eta^\sigma'' = 0, \quad \dot{\eta}^\tau = \eta^{\sigma\sigma} = -\omega. \]

(37)

The remaining gauge freedom is associated to the conformal group in two dimensions, which is infinite-dimensional. The Lagrangian gauge parameters \( \eta^a \) and the Hamiltonian ones \( (\xi, \epsilon) \) are related by

\[ \eta^\tau = \frac{\xi}{\lambda_1}, \quad \eta^\sigma = -\frac{\xi}{\lambda_2} \lambda_1 + \epsilon. \]

(38)

Now to fix the additional gauge freedom associated with the constraints (1), the light-cone gauge conditions are chosen (in the case of the closed string propagating in the Minkowski spacetime)

\[ \tau = AX^+/P^+, \quad P^+ = \frac{B}{2} P^+ , \]

(39)

\( A, B \) constants, and where the light-cone coordinates are given by

\[ X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^{D-1}), \quad P^\pm = \frac{1}{\sqrt{2}} (P^0 \pm P^{D-1}). \]

(40)

These gauge conditions allow it to fix the Lagrange multiplier \( \lambda_1 = \frac{1}{A^2} \), however, they do not fix completely \( \lambda_1 \), rather, it is left as an arbitrary \( \tau \)-dependent function, \( \lambda_1 = \lambda_1(\tau) \). In addition, \( \xi = 0 \) and \( \epsilon = \epsilon(\tau) \). By plugging \( \xi = 0 \) and \( \epsilon = \epsilon(\tau) \) into (38), \( \eta^\tau = 0, \eta^\sigma = \epsilon(\tau) \). By inserting in (37), those equations set \( \omega = 0 \), and \( \eta^\sigma = \epsilon(\tau) = a_1 \tau + a_2 \), with \( a_1, a_2 \) constants. So, the system is still invariant under the \( \tau \)-dependent coordinate transformations

\[ \tau' = \tau, \quad \sigma' = \sigma + f(\tau). \]

(41)

This residual gauge invariance is important in the quantum theory of the string (1).
II. RELATIVES OF BOSONIC STRING THEORY

A. Pure diffeomorphism bosonic string theory

The algebra of constraints for string theory allows it to define a new theory that looks like the action for string theory except that it has no Hamiltonian constraint, being its dynamics attached to the diffeomorphism constraint only. This theory is defined by

\[ S[X^\mu, \tilde{P}_\mu, \lambda] = \int^{\tau_2}_{\tau_1} d\tau \int^{\sigma_2}_{\sigma_1} d\sigma \left[ \dot{X}^\mu \dot{P}_\mu - \lambda \tilde{D} \right], \]

(42)

with \( \tilde{D} = X'^\mu \tilde{P}_\mu, X'^\mu = \frac{\partial X^\nu}{\partial \sigma} \). The algebra of constraints closes and thus \( \tilde{D} \) is first class. The equations of motion are

\[ \dot{X}^\mu = \lambda X'^\mu, \quad \dot{\tilde{P}}_\mu = (\lambda \tilde{P}_\mu)' \]

(43)

The theory defined by (42) contains string theory (4) as a sub-sector of its space of solutions because the theory defined by (42) has one more physical degree of freedom than string theory (4). This is a general fact, always that a diffeomorphism constraint appears in the formalism of generally covariant theories it closes with itself, and thus it is possible to drop some of the other constraints involved in their algebra and thus to build larger theories which will contain the formers as sub-sectors, like the one defined by (42) which emerged from (44). It is pretty obvious that a similar construction holds for the bosonic p-branes where instead of having one single diffeomorphism constraint there will be a finite number of them. However, at first sight, a supersymmetric version of (42) might not be allowed. A more radical interpretation for the theory defined by (42) and its relationship with string theory is to see (42) as a kind of M-theory, and to consider different sectors of this M-theory as ones defined by different Hamiltonian constraints \( \tilde{H} \)'s. Notice that in the case when the spatial surface is closed, the action (42) is fully gauge-invariant under the gauge transformation generated by \( \tilde{D} \). Due to the fact \( \tilde{H} \) is missing in (42), a deep analysis of (42) can help to understand better the role that the Hamiltonian constraint \( \tilde{H} \) plays in string theory both classical and quantum mechanically. Finally, it is important to mention that the theory defined by (42) plays the same role with respect to string theory (4) as the Husain-Kuchar model plays with respect to self-dual gravity for self-dual gravity is a sub-sector of the space of solution of the Husain-Kuchar model (10). Actually, to have a better analogy it would be desirable to have a Lagrangian form for (42).

B. Tensionless bosonic string theory with constraints linear in the momenta

String action (4) has another relative in the case when the background metric \( g_{\mu\nu} \) is constant, say the Minkowski or Euclidean metric \( \eta_{\mu\nu} \). The later is defined by setting \( \alpha = 0 \) in the constraints, namely, it is defined by the action

\[ S[X^\mu, \tilde{P}_\mu, \chi, \lambda] = \int^{\tau_2}_{\tau_1} d\tau \int^{\sigma_2}_{\sigma_1} d\sigma \left[ \dot{X}^\mu \dot{P}_\mu - \left( \lambda \tilde{H} + \lambda \tilde{D} \right) \right], \]

(44)

with

\[ \tilde{H} = \tilde{P}_\mu \tilde{P}_\nu \eta^{\mu\nu}, \quad \tilde{D} = X'^\mu \tilde{P}_\mu, \]

(45)

where \( X'^\mu = \frac{\partial X^\nu}{\partial \sigma} \). Obviously, this action can not be obtained from the Polyakov action (4) because if \( \alpha \) were equal zero then the RHS of (4) would vanish too.

Let us focus in the case when the spatial slice of the ‘world sheet’ is closed. The algebra of constraints is

\[ \{ H(\varepsilon), H(\lambda) \} = 0, \quad \{ D(\varepsilon), H(\varepsilon) \} = H(\mathcal{L}_\varepsilon \lambda), \quad \{ D(\varepsilon), D(\lambda) \} = D(\mathcal{L}_\varepsilon \lambda). \]

(46)

Under the gauge symmetry generated by the constraints, the action changes, according to (27), as

\[ S[X^\mu, \tilde{P}_\mu, \chi', \lambda'] = S[X^\mu, \tilde{P}_\mu, \chi, \lambda] + \int^{\sigma_2}_{\sigma_1} d\sigma \left( \varepsilon \tilde{H} \right)_{\tau = \tau_2}. \]

(47)
Therefore, the boundary term is proportional to the Hamiltonian constraint, and thus the action is gauge invariant on the constraint surface. A similar situation appears in general relativity expressed in terms of Ashtekar variables [11]. In Ref. [11] it was not built the fully gauge-invariant action associated with the self-dual action, however, this could be carried out.

Let us come back to the action (44) and construct, following the steps of [3], its fully gauge-invariant action. This action is given by

\[ S_{\text{inv}}[X^\mu, \tilde{P}_\mu, \lambda_0, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1}^{\sigma_2} d\sigma \left[ \dot{X}^\mu \tilde{P}_\mu - \left( \lambda \tilde{H} + \lambda \tilde{D} \right) \right] - \frac{1}{2} \int_{\sigma_1}^{\sigma_2} d\sigma \left( X^\mu \tilde{P}_\mu \right)_{\tau=\tau_1}^{\tau=\tau_2}. \] (48)

A straightforward computation shows that, at first order, \( S_{\text{inv}}[X^\mu, \tilde{P}_\mu, \lambda_0, \lambda] \) is fully gauge-invariant. The boundary term in (48) induces the canonical transformation

\[ q^0 = \frac{1}{2} \ln \left( \frac{X^0}{\tilde{P}_0} \right), \quad p_0 = X^0 \tilde{P}_0, \]

\[ q^1 = \frac{1}{2} \ln \left( \frac{X^1}{\tilde{P}_1} \right), \quad p_1 = X^1 \tilde{P}_1, \]

\[ \ldots \]

\[ q^D = \frac{1}{2} \ln \left( \frac{X^D}{\tilde{P}_D} \right), \quad p_D = X^D \tilde{P}_D, \quad (\text{no sum over } D). \] (49)

In terms of the new phase space variables \( S_{\text{inv}} \) reads

\[ S_{\text{inv}}[q^\mu, p_\mu, \lambda_0, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1}^{\sigma_2} d\sigma \left[ \dot{q}^\mu p_\mu - \left( \lambda \tilde{H} + \lambda \tilde{D} \right) \right], \] (50)

with

\[ \tilde{H} = p_\mu e^{-2q^\nu \eta_{\mu\nu}}, \quad \tilde{D} = \frac{1}{2} \left( p_\mu l_\mu + 2p_\mu q'^\mu \right), \] (51)

with \( l^\nu = (1, 1, \ldots, 1) \) and it was assumed a diagonal background metric \( \eta_{\mu\nu} \). Notice that the constraints are linear and homogeneous in the momenta (and in their derivatives).

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