The generating function of amplitudes with $N$ twisted and $M$ untwisted states

Igor Pesando$^1$

$^1$Dipartimento di Fisica Teorica, Università di Torino
and I.N.F.N. - sezione di Torino
Via P. Giuria 1, I-10125 Torino, Italy
ipesando@to.infn.it

Abstract: We show that the generating function of all amplitudes with $N$ twisted and $M$ untwisted states, i.e. the Reggeon vertex for magnetized branes on $\mathbb{R}^2$ can be computed once the correlator of $N$ non excited twisted states and the corresponding Green function are known and we give an explicit expression as a functional of the these objects.

Keywords: D-branes, Conformal Field Theory.
1. Introduction and conclusions

In the late 80s a lot of work was done in computing the generating functions of all amplitudes for the bosonic string and superstring. Many methods were (further) developed such as the sewing method ([1]), the group theoretic method ([2]) and conserved charges method ([3]). Following the main idea of ([4],[5]) in this paper we would like to compute the generating function for $N$ generic excited twisted states and $M$ generic untwisted states on $R^2$ for the open string in presence of magnetic fields in the upper half plane using the path integral approach. Much work has been already done in computing non excited twisted states correlation functions, especially on $T^2$ (see for example [6], [7], [8] and [9]) but not so much on the computation of correlators involving excited twisted fields ([15], [16] see for earlier work) which remain quite mysterious.

In this paper we want to show that there is a quite simple way of labeling excited twisted states which is deeply connected with the operator-state map and that few ingredients are actually needed for computing all correlators involving excited twisted state and arbitrary untwisted ones on $R^2$. To obtain any correlator is only necessary the knowledge of the full (i.e. classical and quantum) $N$ non excited twist correlator on the disk$^1$

$$C(x_1, \ldots, x_N) = \langle \sigma_{\epsilon_1}(x_1, \bar{x}_1) \ldots \sigma_{\epsilon_N}(x_N, \bar{x}_N) \rangle_{\text{disk, full}} \quad x_t \in \mathbb{R}$$  \hspace{1cm} (1.1)

$^1$The twist fields in this and the following correlators are actually $\sigma_{\epsilon, \kappa=0}(x, \bar{x})$, see the in the main text.
and the boundary Green function in presence of such operators
\[ G_{\text{bou}}^{ij}(x; y; \{x_t\}_t=1\ldots N) = G_{\text{bou}}^{ij}(y; x; \{x_t\}_t=1\ldots N) = G^{ij}(x, \bar{x}; y, \bar{y}; \{x_t\}_t=1\ldots N) \] (1.2)
which can be derived from
\[ G^{ij}(z, \bar{z}; w, \bar{w}; \{x_t\}_t=1\ldots N) = \frac{\langle X^i(z, \bar{z})X^j(w, \bar{w})\sigma_{x1}(x_1, \bar{x}_1)\ldots \sigma_{xN}(x_N, \bar{x}_N) \rangle_{\text{disk}}}{\langle \sigma_{x1}(x_1, \bar{x}_1)\ldots \sigma_{xN}(x_N, \bar{x}_N) \rangle_{\text{disk}}} \] (1.3)
by setting \( z = x, \bar{w} = y \in \mathbb{R} \).

The main result of the paper is the generating function for the above mentioned amplitudes given in eq.s (1.29) and (1.30). These two expressions have exactly the same contain but the latter is written in a more usual way, i.e. using auxiliary expansion variables while the former has an expression like those used in the previous literature ([1]). Let us now explain the building blocks of this last version of the main formula (1.29).

- To any (excited) twisted operator inserted at \( x_t \) (\( t = 1 \ldots N \)) in the amplitude we associate an auxiliary Hilbert space \( \mathcal{H}_t \). On \( \mathcal{H}_t \) act the quantum fields \( X^i_t(z, \bar{z})^2 \) \( (i = 1, 2 \) or \( i = z, \bar{z} \))

\[ Z_{(t)}(z, \bar{z}) = X^i(z, \bar{z}) = \frac{X^1(t) + iX^2(t)}{\sqrt{2}} = \frac{1}{2} \left( Z_{L(t)}(z) + Z_{R(t)}(\bar{z}) \right), \]
\[ Z_{(t)}(z, \bar{z}) = X^i(z, \bar{z}) = \frac{X^1(t) - iX^2(t)}{\sqrt{2}} = \frac{1}{2} \left( Z_{L(t)}(z) + Z_{R(t)}(\bar{z}) \right) \] (1.4)
which have expansions
\[ Z_{(t)L}(z) = z(t)_0 + i\sqrt{2} \alpha e^{-i\gamma} \sum_{n=0}^{\infty} \left[ \frac{\alpha(t)n+1-\epsilon_t}{n+1-\epsilon_t} z^{-(n+1-\epsilon_t)} - \frac{\alpha^\dagger(t)n+\epsilon_t}{n+\epsilon_t} z^{n+\epsilon_t} \right] \]
\[ Z_{(t)R}(\bar{z}) = z(t)_0 + i\sqrt{2} \alpha e^{-i\gamma} \sum_{n=0}^{\infty} \left[ \frac{\alpha^\dagger(t)n+1-\epsilon_t}{n+1-\epsilon_t} \bar{z}^{-(n+1-\epsilon_t)} - \frac{\alpha(t)n+\epsilon_t}{n+\epsilon_t} \bar{z}^{n+\epsilon_t} \right] \] (1.5)

and
\[ \bar{Z}_{(t)L}(z) = \bar{z}(t)_0 + i\sqrt{2} \alpha e^{i\gamma} \sum_{n=0}^{\infty} \left[ \frac{\alpha(t)n+1-\epsilon_t}{n+1-\epsilon_t} \bar{z}^{-(n+1-\epsilon_t)} + \frac{\alpha^\dagger(t)n+\epsilon_t}{n+\epsilon_t} \bar{z}^{n+\epsilon_t} \right] \]
\[ \bar{Z}_{(t)R}(\bar{z}) = \bar{z}(t)_0 + i\sqrt{2} \alpha e^{i\gamma} \sum_{n=0}^{\infty} \left[ \frac{\alpha^\dagger(t)n+1-\epsilon_t}{n+1-\epsilon_t} \bar{z}^{-(n+1-\epsilon_t)} + \frac{\alpha(t)n+\epsilon_t}{n+\epsilon_t} \bar{z}^{n+\epsilon_t} \right] \] (1.6)

The previous fields satisfy the boundary conditions\(^3\)
\[ e^{-i\gamma} \partial Z_{(t)L}|x = e^{-i\gamma} \partial Z_{(t)R}|x \quad x \in \mathbb{R}^+ \] (1.7)
\[ e^{i\gamma-1} \partial Z_{(t)L}|y = e^{-i\gamma-1} \partial Z_{(t)R}|y \quad y = |y| e^{i\pi} \in \mathbb{R}^- \] (1.8)

\(^2\)In the following quantum fields have attached the label of the Hilbert space they act on, e.g. \( X^i_t(z, \bar{z}) \) while classical fields in path integral have no label, i.e. \( X^i(z, \bar{z}) \).

\(^3\)These can also be written as \( e^{-i\gamma} \partial Z_{(t)L}(x) = \frac{1}{\cos \gamma \epsilon} \partial Z_{(t)}(x, x) \) when \( x > 0 \) and \( e^{i\gamma-1} \partial Z_{(t)L}(y) = \frac{1}{\cos \gamma \epsilon} \partial Z_{(t)}(y, \bar{y}) \) when \( y < 0 \). These expressions are those used to connect the open string operators when naturally expressed as function of \( X(x, \bar{x}) \) to their expressions as functional of \( X_L(x) \).
where we have defined the phases $(-\frac{\pi}{2} < \gamma_t < \frac{\pi}{2})$

\[
e^{i\gamma_t} = \frac{1 + i B_t}{\sqrt{1 + B_t^2}} \rightarrow B_t = \tan \gamma_t = 2\pi \alpha' q_{(0)} F_{12(0)}
\]

\[
e^{i\gamma_{t-1}} = \frac{1 + i B_{t-1}}{\sqrt{1 + B_{t-1}^2}} \rightarrow B_{t-1} = \tan \gamma_{t-1} = 2\pi \alpha' q_{(\pi)} F_{12(\pi)}
\] (1.9)

where $B_{t-1} = 2\pi \alpha' q_{(\pi)} F_{12(\pi)}$ and $B_t = 2\pi \alpha' q_{(0)} F_{12(0)}$ are the adimensional magnetic fields which are on the $x < 0$ ($\sigma = \pi$) and $x > 0$ ($\sigma = 0$) boundaries. In the field expansion the shift $\epsilon_t$ is given by

\[
\epsilon_t = \begin{cases} 
\frac{1}{\pi} (\gamma_t - \gamma_{t-1}) & \gamma_t > \gamma_{t-1} \\
1 + \frac{1}{\pi} (\gamma_t - \gamma_{t-1}) & \gamma_t < \gamma_{t-1}
\end{cases} \quad 0 \leq \epsilon_t < 1
\] (1.10)

The previous operators act on the $H_t$ twisted ground state defined by

\[
\tilde{\alpha}_{(t)n+1-\epsilon_t}[T_t] = \alpha_{(t)m+\epsilon_t}[T_t] = x_0^2[T_t] = 0
\] (1.11)

and have the non vanishing commutation relations

\[
[\tilde{\alpha}_{(t)n+1-\epsilon_t}, \tilde{\alpha}_{(t)m+1-\epsilon_t}] = (n + 1 - \epsilon_t)\delta_{n,m} \quad n, m \geq 0
\]

\[
[\alpha_{(t)n+\epsilon_t}, \alpha_{(t)m+\epsilon_t}] = (n + \epsilon_t)\delta_{n,m} \quad n, m \geq 0
\]

\[
[z_{(t)0}, \tilde{z}_{(t)0}] = \frac{2\pi \alpha'}{B_t - B_{t-1}}
\] (1.12)

Notice that the choice of the definition of the zero modes vacuum is somewhat arbitrary since they do not change the energy, our choice is dictated by our gauge choice for the background magnetic field $A = B x^1 dx^2$ which implies the translational invariance $X^2 \rightarrow X^2 + \epsilon$ and by the observation that is almost the proper choice in toroidal compactifications. The existence of the zero modes imply that the vacuum is degenerate since all the states $|T_t, \kappa_t\rangle = e^{i\kappa x_{(t)0}}[T_t]$ have exactly the same energy of the vacuum and therefore there exists a one parameter family of twist fields ([12]) $\sigma_{\epsilon_t, \kappa_t}(x, \bar{x})$.

\footnote{Since the annihilator and creator operators have flat indexes this holds independently of our choice of the taking the metric diagonal; in particular from definition of the complex fields we have $(dX^1)^2 + (dX^2)^2 = 2dZd\bar{Z}$, i.e. $G_{z\bar{z}} = 1$.}
Given the previous vacuum definition we have the following twisted Green functions

\[ G_{T(t)}^{zz}(z, \bar{z}; w, \bar{w}) = [Z^{(+)}(z, \bar{z}), Z^{(-)}(w, \bar{w})]_{\text{an.cont}} = \frac{\pi \alpha'}{B_t - B_{t-1}} \]

\[ G_{T(t)}^{\bar{z}z}(z, \bar{z}; w, \bar{w}) = [\bar{Z}^{(+)}(z, \bar{z}), \bar{Z}^{(-)}(w, \bar{w})]_{\text{an.cont}} = -\frac{\pi \alpha'}{B_t - B_{t-1}} \]

\[ G_{T(t)}^{z\bar{z}}(z, \bar{z}; w, \bar{w}) = [Z^{(+)}(z, \bar{z}), \bar{Z}^{(-)}(w, \bar{w})]_{\text{an.cont}} = \frac{\pi \alpha'}{B_t - B_{t-1}} \]

\[ G_{T(t)}^{\bar{z}\bar{z}}(z, \bar{z}; w, \bar{w}) = [\bar{Z}^{(+)}(z, \bar{z}), Z^{(-)}(w, \bar{w})]_{\text{an.cont}} = -\frac{\pi \alpha'}{B_t - B_{t-1}} \]

\[ -\frac{\alpha'}{2} \left[ g_{\epsilon t} \left( \frac{w}{z} \right) + g_{\epsilon t} \left( \frac{\bar{w}}{\bar{z}} \right) + e^{-2i\gamma t} g_{\epsilon t} \left( \frac{\bar{w}}{\bar{z}} \right) + e^{2i\gamma t} g_{\epsilon t} \left( \frac{w}{z} \right) \right] \]

(1.13)

which can be obtained by analytically continuing their operatorial expression from \(|z| > |w|\) to the whole upper plane in such a way to preserve the symmetry \(G_{ij}(z, \bar{z}; w, \bar{w}) = G_{ji}(w, \bar{w}; z, \bar{z})\). In the previous expressions we have defined \(g_\nu(z)\) as the analytic continuation of

\[ g_{\nu,s}(z) = -\sum_{n-\nu>0} \frac{1}{n-\nu} z^{n-\nu} \quad |z| < 1, \quad -\pi + 2\pi s < \phi = \arg(z) \leq \pi + 2\pi s. \quad (1.14) \]

in the properly chosen sheet \(s\). Notice that the symmetry of the Green function \(G_{ij}(z, \bar{z}; w, \bar{w}) = G_{ji}(w, \bar{w}; z, \bar{z})\) is not obvious in the zero modes sector, i.e. for the constant terms but it holds due to the \(g\) transformation property ([12])

\[ g_{\nu,s}(z) = C_{\nu,s}(\phi) + g_{1-\nu,-s} \left( \frac{1}{z} \right), \quad C_{\nu,s}(\phi) = \begin{cases} \frac{\pi e^{-i\pi\nu}}{\sin \pi \nu} e^{-i2\pi s} & 2\pi s < \phi < \pi + 2\pi s \\ \frac{\pi e^{i\pi\nu}}{\sin \pi \nu} e^{-i2\pi s} & -\pi + 2\pi s < \phi < 2\pi s \end{cases} \]

(1.15)

This fact implies that we cannot really completely separate the zero modes and non zero modes also for the twisted sector as it already happens for the untwisted one.
For $x > 0 > y$ and $|y/x| < 1$ the previous Green functions become on the boundary\(^5\)

\[
G_{T(t)}^{\ast \ast} \text{ bou}(x; y) = \frac{\pi \alpha'}{B_t - B_{t-1}}
\]

\[
G_{T(t)}^{\ast \ast} \text{ bou}(x; y) = -\frac{\pi \alpha'}{B_t - B_{t-1}}
\]

\[
G_{T(t)}^{\ast \ast} \text{ bou}(x; y) = \frac{\pi \alpha'}{B_t - B_{t-1}} - 2\alpha' \cos \gamma_t \cos \gamma_{t-1} e^{i\gamma_t - i\gamma_{t-1}} g_{\epsilon t} \left( \frac{y}{x} \right)
\]

\[
G_{T(t)}^{\ast \ast} \text{ bou}(x; y) = -\frac{\pi \alpha'}{B_t - B_{t-1}} - 2\alpha' \cos \gamma_t \cos \gamma_{t-1} e^{-i\gamma_t + i\gamma_{t-1}} g_{1-\epsilon_t} \left( \frac{y}{x} \right)
\]

The other cases can be obtained with the substitution rule $x > 0 \cos \gamma_t e^{i\gamma_t} \leftrightarrow x < 0 \cos \gamma_{t-1} e^{-i\gamma_{t-1}}$ and the same for $y$ in the $G_{T(t)}^{\ast \ast}$ propagator. For the $G_{T(t)}^{\ast \ast}$ propagator one takes the complex conjugate of the previous substitution rule.

- In a similar way to any untwisted operator we insert in the amplitude we associate an auxiliary Hilbert space $H_{a,t_a}$. This Hilbert space as well as the position where the untwisted vertex is inserted $x_{a,t_a}$ are better labeled by both a counting label $a = 1 \ldots M$ and a further label $t_a \in \{1, \ldots, N\}$ which specify which is the magnetic field felt by the untwisted state (dipole string). This could seem irrelevant but it is important in defining the regularized Green functions (1.27) and in computing the non commutative phases. In the following we will use a lighter notation as $x_{a,t_a} \rightarrow x_a$ when there is not possibility of confusion.

On the auxiliary Hilbert space $H_{a,t_a}$ act the quantum fields

\[
Z_{(a,t_a)}(z, \bar{z}) = \frac{1}{\sqrt{2}} \left( X_{(a,t_a)}^1(z, \bar{z}) + iX_{(a,t_a)}^2(z, \bar{z}) \right) = \frac{1}{2} (Z_{(a,t_a)L}(z) + Z_{(a,t_a)R}(\bar{z}))
\]

which have expansions

\[
Z_{(a,t_a)L} = e^{-i\gamma_t a} \left( z_{(a,t_a)0} - 2\alpha' \bar{p}_{(a,t_a)} \ i \ln(z) + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{\bar{\alpha}(a,t_a)n}{n} z^{-n} - \frac{\alpha^\dagger(a,t_a)n}{n} z^n \right)
\]

\[
Z_{(a,t_a)R} = e^{+i\gamma_t a} \left( z_{(a,t_a)0} - 2\alpha' \bar{p}_{(a,t_a)} \ i \ln(\bar{z}) + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{\bar{\alpha}(a,t_a)n}{n} \bar{z}^{-n} - \frac{\alpha^\dagger(a,t_a)n}{n} \bar{z}^n \right)
\]

\(^5\)When $|y/x| > 1$ we must be more careful since we want to evaluate the $g$ on a cut; for example when $0 < x, y$ the expression which is valid for all ranges is $G_{T(t)}^{\ast \ast} \text{ bou}(x; y) = \frac{\pi \alpha'}{B_t - B_{t-1}} - \alpha^\prime \cos \gamma_t e^{i\gamma_t} g_{\epsilon t} \left( \frac{x}{y} \right) + e^{i\gamma_t} g_{\epsilon t} \left( \frac{y}{x} \right)$. In any case we can always use the symmetry property for the Green functions to reduce the computation in the range where we can apply the given expressions.
and
\[
\hat{Z}_{(a,t_a)L} = e^{+i \gamma_{t_a}} \left( \hat{z}_{(a,t_a)0} - 2\alpha' \hat{p}_{(a,t_a)} \right) \iota \ln(z) + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\bar{a}_{(a,t_a)n}^{\dagger} z^n + \alpha_{(a,t_a)n} \bar{z}^{-n}}{n}
\]
\[
\hat{Z}_{(a,t_a)R} = e^{-i \gamma_{t_a}} \left( \hat{z}_{(a,t_a)0} - 2\alpha' \hat{p}_{(a,t_a)} \right) \iota \ln(\bar{z}) + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\bar{a}_{(a,t_a)n}^{\dagger} \bar{z}^n + \alpha_{(a,t_a)n} z^{-n}}{n}
\]

The previous quantum fields satisfy the boundary conditions
\[
e^{+i \gamma_{t_a} \partial Z_{(a,t_a)}|_x} = e^{-i \gamma_{t_a} \bar{\partial} Z_{(a,t_a)}|_x} \quad x \in \mathbb{R}^+
\]
\[
e^{+i \gamma_{t_a} \partial Z_{(a,t_a)}|_y} = e^{-i \gamma_{t_a} \bar{\partial} Z_{(a,t_a)}|_y} \quad y = |y|e^{i\pi} \in \mathbb{R}^- \]

where we have defined the angle \( \gamma_{t_a} \), in a similar way for the twisted scalar (charged string), as
\[
e^{i \gamma_{t_a}} = \frac{1 + iB_{t_a}}{\sqrt{1 + B_{t_a}}} \Rightarrow B_{t_a} = \tan \gamma_{t_a}, \quad -\frac{\pi}{2} < \gamma_{t_a} < \frac{\pi}{2}
\]

The creation and destruction operators act on the dipole ground state defined by
\[
\tilde{a}_{(a,t_a)0}|0_{(a,t_a)}\rangle = \alpha_{(a,t_a)0}|0_{(a,t_a)}\rangle = \bar{p}_{(a,t_a)}|0_{(a,t_a)}\rangle = p_{(a,t_a)}|0_{(a,t_a)}\rangle = 0
\]
and have non trivial commutation relations
\[
\begin{align*}
[z_{(a,t_a)0}, \bar{z}_{(a,t_a)0}] &= 2\pi \alpha' B_{t_a} \\
[z_{(a,t_a)0}, \hat{p}_{(a,t_a)}] &= i \\
[\alpha_{(a,t_a)n}, \alpha_{(a,t_a)m}^{\dagger}] &= n\delta_{m,n} \\
[\tilde{a}_{(a,t_a)n}, \tilde{a}_{(a,t_a)m}^{\dagger}] &= n\delta_{m,n}
\end{align*}
\]

The normal ordering is the usual one but it worth noticing that in the zero modes sector is defined as
\[
:e^{i(kZ_{(a)z \pm} + k\bar{Z}_{(a)z \pm})}(x,\bar{x}) := \left\{ \begin{array}{ll}
e^{i\cos \gamma(kz_{(a)0} + k\bar{z}_{(a)0})} e^{2\alpha' \ln(|x|) \cos \gamma(kp_{(a)0} + kp_{(a)0})} x > 0 \\
e^{i\cos \gamma(k\bar{z}_{(a)0} + k\bar{z}_{(a)0})} e^{2\alpha' \ln(|x|) \cos \gamma(k\bar{p}_{(a)0} + k\bar{p}_{(a)0})} x < 0
\end{array} \right. .
\]

with \( \hat{z}_{(a)0} = z_{(a)0} - i2\pi \alpha' \tan \gamma \bar{p} \) which have the property that their commutation relations are the opposite of the \( z_{(a)0} \) ones. Finally the untwisted Green functions in a magnetic background \( B_{t_a} \) are given by \( 0 < arg(z - \bar{w}) < \pi \)
\[
G^z_{U(t_a)}(z, \bar{z}, w, \bar{w}) = G^{\bar{z}}_{U(t_a)}(z, \bar{z}, w, \bar{w}) = 0
\]
\[
G^{zz}_{U(t_a)}(z, \bar{z}, w, \bar{w}) = [Z^+(z, \bar{z}), \bar{Z}^-(w, \bar{w})]|_{an.cont}
\]
\[
= + \frac{1}{2} \pi \alpha' \sin(2\gamma_{t_a}) + \alpha' [\ln |z - w| + \cos(2\gamma_{t_a}) \ln |z - \bar{w}| + \sin(2\gamma_{t_a}) \arg(z - \bar{w})]
\]
\[
G^{\bar{z}\bar{z}}_{U(t_a)}(z, \bar{z}, w, \bar{w}) = [\bar{Z}^+(z, \bar{z}), Z^-(w, \bar{w})]|_{an.cont}
\]
\[
= - \frac{1}{2} \pi \alpha' \sin(2\gamma_{t_a}) - \alpha' [\ln |z - w| + \cos(2\gamma_{t_a}) \ln |z - \bar{w}| - \sin(2\gamma_{t_a}) \arg(z - \bar{w})]
\]
The constant terms can be obtained by rewriting

\[ z_{(a)0} = z_{(a)00} + i\pi\alpha' \tan \gamma_t \tilde{p} \]

so that \([z_{(a)00}, \tilde{z}_{(a)00}] = 0\) an considering the additional term proportional to \(\tilde{p}\) coming from this rewriting as belonging to \(Z^{(+)}(z, \tilde{z})\). Notice however once again that the constant terms are needed to ensure the symmetry \(G^{ij}(z, \tilde{z}; w, \tilde{w}) = G^{ji}(w, \tilde{w}; z, \tilde{z})\).

The previous Green functions become on the boundary \(z = x, w = y \in \mathbb{R}\)

\[
G^z_{U(t_a), \text{bou}}(x; y) = \frac{1}{2} \pi\alpha' \sin(2\gamma_{t_a}) - 2\alpha' \left[ \cos^2(\gamma_{t_a}) \ln |x - y| + \frac{1}{2} \sin(2\gamma_{t_a}) \arg(x - y) \right]
\]

\[
G^{\bar{z}}_{U(t_a), \text{bou}}(x; y) = -\frac{1}{2} \pi\alpha' \sin(2\gamma_{t_a}) - 2\alpha' \left[ \cos^2(\gamma_{t_a}) \ln |x - y| - \frac{1}{2} \sin(2\gamma_{t_a}) \arg(x - y) \right]
\]

(1.26)

From these expressions we can read the open string metric \(G^{z\bar{z}} = \cos^2(\gamma_t)\) and the non commutativity parameter \(\Theta^{z\bar{z}} = \frac{1}{2} \sin(2\gamma_t)\), we can also read the \(\mathbb{R}^2\) vielbein \(V^{z\bar{z}} = \frac{1}{2} \cos(\gamma_t)\) where \(z, \bar{z}\) are the flat indexes which are also implicit in the creation and destruction operators.

- We define the boundary Green function regularized by the untwisted Green function for a background \(B_t\) as

\[
G^{ij}_{\text{bou}, \text{reg} U(t_a)}(x; y; \{x_v\}) = G^{ij}_{\text{bou}}(x; y; \{x_v\}) - G^{ij}_{U(t_a), \text{bou}}(x; y) \quad x, y \in \mathbb{R}
\]

(1.27)

where \(G^{ij}_{U(t_a), \text{bou}}(x, y)\) are defined in eq.s (1.26). The choice of the background \(B_t\) in the regularization would seem arbitrary but it is not since these regularized Green functions (and their derivatives) enter only where an untwisted dipole state is emitted and this is on a well defined interval of the boundary.

We also define the analogous twisted boundary Green function regularized by the twisted Green function at the twist insertion point \(t\) as\(^7\)

\[
G^{ij}_{\text{bou}, \text{reg} T(t)}(x, y; \{x_v\}) = G^{ij}_{\text{bou}}(x, y; \{x_v\}) - G^{ij}_{T(t), \text{bou}}(x, y; \{x_v\} = \{x_0 = x_t, x_\infty = \infty\})
\]

(1.28)

where \(G^{ij}_{T(t), \text{bou}}\) are given in eq.s (1.16) with the substitution \(\frac{y - x}{x - x_t} \rightarrow \frac{x - y}{x - x_t}\).

\(^6\)When using these Green functions in eq. (3.14) in absence of twist fields we recover the results from the operatorial formalism.

\(^7\)The symmetrization is because we a symmetric function in \(x \leftrightarrow y\), i.e. independent on the way we take the limit \(x > y\) or \(y > x\).
Given the previous building blocks the main formula is given by \((x_t \neq x_a \ \forall t, a)\)

\[
\langle V_{N+M}(\{x_t\}_{t=1,..N}; \{x_a, t_a\}_{a=1,..M}) \rangle = C(x_1, \ldots, x_N)
\]

\[
\prod_{a=1}^{M} \left( \prod_{i=1}^{N} \langle T(t), x(t) \rangle = 0 \right) \prod_{t}^{N} \left( \langle T(t), x(t) \rangle = 0 \right) \delta(i \sum_{a}^{M} (\alpha_{(a)} - \alpha_{(a)}') + i \sum_{t} (z(t) - \bar{z}(t)))
\]

\[
\prod_{a} \exp \left\{ - \frac{1}{4\alpha'} \alpha_{(a)}^2 \gamma_{(a)}^2 \gamma_{(a)} \bar{z} \gamma_{(a)} \bar{z} \right\} G_{x}^{zz} \text{reg U}(t_a)(x; \{x_v\}) - \frac{1}{4\alpha'} \alpha_{(a)}^2 \gamma_{(a)}^2 \gamma_{(a)} \bar{z} \gamma_{(a)} \bar{z} \right\} G_{x}^{zz} \text{reg U}(t_a)(x; \{x_v\})
\]

\[
\prod_{t} \exp \left\{ \frac{1}{2} \left( \frac{\tan \gamma_{(a)} - \tan \gamma_{(a-1)}}{2\pi \alpha'} \right)^2 \gamma_{(a)}^{zz} \gamma_{(a)} \right\} \left( G_{x}^{zz} \gamma_{(a)} \gamma_{(a)} \bar{z} \gamma_{(a)} \bar{z} \right) \gamma_{(a)}^{zz} \gamma_{(a)} \gamma_{(a)} \bar{z} \gamma_{(a)} \bar{z} \right\} \gamma_{(a)}^{zz} \gamma_{(a)} \gamma_{(a)} \bar{z} \gamma_{(a)} \bar{z} \right\}
\]

\[
\prod_{a \neq b} \exp \left\{ - \frac{1}{2\alpha'} \alpha_{(a)} \alpha_{(b)} \gamma_{(a)} \gamma_{(b)} \bar{z} \gamma_{(b)} \bar{z} \right\} G_{x}^{zz} \text{reg U}(t_a)(x; \{x_v\}) - \frac{1}{2\alpha'} \alpha_{(a)} \alpha_{(b)} \gamma_{(a)} \gamma_{(b)} \bar{z} \gamma_{(b)} \bar{z} \right\} G_{x}^{zz} \text{reg U}(t_b)(x; \{x_v\})
\]

\[
\pi_{t<\pi} \exp \left\{ \left( \frac{\tan \gamma_{(a)} - \tan \gamma_{(a-1)}}{2\pi \alpha'} \right)^2 \gamma_{(a)}^{zz} \gamma_{(a)} \right\} \left( G_{x}^{zz} \gamma_{(a)} \gamma_{(a)} \bar{z} \gamma_{(a)} \bar{z} \right) \gamma_{(a)}^{zz} \gamma_{(a)} \gamma_{(a)} \bar{z} \gamma_{(a)} \bar{z} \right\}
\]
\[
\prod \exp \left\{ -\frac{1}{2\alpha'} \left( \tan \gamma_{t} - \tan \gamma_{t-1} \frac{x(t)_{0} \cdot i = 2}{\pi} \right) \frac{\alpha_{(a)} m V(t)}{2\alpha'} \right\} \sum_{m=0}^{\infty} \alpha_{(a)} m V(t) \frac{\partial_{m}^{y}}{m!} G_{\text{bo}}^{zz}(x, y; \{x_v\}) \right) \\
\prod \exp \left\{ -\frac{1}{2\alpha'} \left( \tan \gamma_{t} - \tan \gamma_{t-1} \frac{x(t)_{0} \cdot i = 2}{\pi} \right) \frac{\alpha_{(a)} m V(t)}{2\alpha'} \right\} \sum_{m=0}^{\infty} \alpha_{(a)} m V(t) \frac{\partial_{m}^{y}}{m!} G_{\text{bo}}^{zz}(x, y; \{x_v\}) \right) \\
\prod \exp \left\{ -\frac{1}{2\alpha'} \sum_{n=1, m=0}^{\infty} \frac{\alpha_{(1)} n \alpha_{(a)} m V(t)}{n - \epsilon_{t}} \frac{\partial_{n}^{x} \partial_{m}^{y}}{(n - 1)!} \frac{\partial_{m}^{y}}{m!} \left[ (x - x_t)^{n-1} \partial_{x} G_{\text{bo}}^{zz}(x, y; \{x_v\}) \right] \right) \\
\prod \exp \left\{ -\frac{1}{2\alpha'} \sum_{n=1, m=0}^{\infty} \frac{\alpha_{(1)} n \alpha_{(a)} m V(t)}{n - \epsilon_{t}} \frac{\partial_{n}^{x} \partial_{m}^{y}}{(n - 1)!} \frac{\partial_{m}^{y}}{m!} \left[ (x - x_t)^{n-1} \partial_{x} G_{\text{bo}}^{zz}(x, y; \{x_v\}) \right] \right) \bigg|_{x=x_{t}, y=x_{a}}
\right)
\]

where the operator indexes are raised an lowered using the flat metric while Green function indexes are raised an lowered using $\mathbb{R}^{2}$ metric. The previous expression can also be written without using the auxiliary operators as a more conventional generating function. In order to do so we introduce the auxiliary parameters $d(t)_{a}$, $d(t)_{b}$ and $c(a)_{n}$ and $c(a)_{n}$ which roughly correspond to $\alpha(t)n+1-\epsilon$, $\alpha(t)n-\epsilon$ and $\alpha(a)_{n}$, $\alpha(a)_{n}$ (see eqs (4.16) and (3.18) for a precise mapping) of the previous expression. Then we can write the generating function as

\[
\mathcal{V}_{N+M}(\{d(t)_{a}\}_{t=1,...,N}; \{c(a)_{n}\}_{a=1,...,M}; \{x(t)_{a}\}_{a=1,...,M}; \{x_{a,t}a\}_{a=1,...,M} = \delta \left( \text{Re} \left( \sum_{t} d(t)_{0} + \sum_{c(a)_{0}} \right) \right) C(x_{1}, \ldots x_{N})
\]

\[
\prod_{a} \exp \left\{ \frac{1}{2} \frac{\partial_{y}}{y} G_{\text{bo}}^{zz}(x, y; \{x_v\}) + \frac{1}{2} d(t)_{0} \right\} \sum_{n=0}^{\infty} c(a)_{n} \bar{c}(a)_{m} \partial_{x} \partial_{y} G_{\text{bo}}^{zz}(x, y; \{x_v\}) \right) \bigg|_{x=x_{a}}
\]

\[
\prod_{t} \exp \left\{ \frac{1}{2} \frac{\partial_{y}}{y} G_{\text{bo}}^{zz}(x, y; \{x_v\}) + \frac{1}{2} d(t)_{0} \right\} \sum_{n=1, m=0}^{\infty} \frac{d(t)_{n} \partial_{n}^{x} \partial_{m}^{y}}{(n - 1)!} (x - x_t)^{n-1-\epsilon} \partial_{x} G_{\text{bo}}^{zz}(x, y; \{x_v\}) \right) \bigg|_{x=x_{t}}
\]

\[
\prod_{a < b} \exp \left\{ \bar{c}(a)_{n} \partial_{x} \partial_{y} G_{\text{bo}}^{zz}(x, y; \{x_v\}) + c(a)_{n} \partial_{x} C_{\text{bo}}^{zz}(x, y; \{x_v\}) \right) \right) \bigg|_{x=x_{a}, y=x_{b}}
\]

\[
\right)
\]

\[
\right)
\]

\[
\right)
\]

\[
\right)
\]
\[
\prod_{t<u} \exp \left\{ \sum_{m=0}^{\infty} d(t)_0 \ c(a)_m \ \partial^m_y \ G^zz_{bou}(x; y; \{x_v\}) + \tilde{d}(t)_0 \ c(a)_m \ \partial^m_y \ G^zz_{bou}(x; y; \{x_v\}) \right\} + \sum_{n,m=1}^{\infty} d(t)_n \ \tilde{d}(u)_m \ G^zz_{bou}(x_t; x_u; \{x_v\}) + \sum_{n,m=1}^{\infty} \tilde{d}(t)_n \ d(u)_m \ \partial_x^{n-1} \partial_y^{m-1} \ \left[ (x - x_t)^{1-\epsilon_t} (y - x_u)^{1-\epsilon_u} \partial_x \partial_y G^zz_{bou}(x; y; \{x_v\}) \right] \bigg|_{x = x_t, y = x_v}
\]

\[
\prod_{t,a} \exp \left\{ \sum_{m=0}^{\infty} d(t)_0 \ c(a)_m \ \partial^m_y \ G^zz_{bou}(x; y; \{x_v\}) + \tilde{d}(t)_0 \ c(a)_m \ \partial^m_y \ G^zz_{bou}(x; y; \{x_v\}) \right\} + \sum_{n=1}^{\infty} d(t)_n \ \tilde{d}(a)_m \ \partial_x^{n-1} \partial_y^{m-1} \ (x - x_t)^{1-\epsilon_t} \partial_x G^zz_{bou}(x; y; \{x_v\}) \bigg|_{x = x_t, y = x_v} \quad (1.30)
\]

Notice that all the previous expressions are meaningful because of the behavior of the Green functions

\[
G^zz_{bou}(x; y; \{x_v\}) = const
\]
\[
G^zz_{bou}(x; y; \{x_v\}) = const
\]
\[
G^zz_{bou}(x; y; \{x_v\}) = G^zz_{bou}(y; x; \{x_v\}) \sim_{x \to x_t} const \ + \ (x - x_t)^{1-\epsilon_t} [g^zz_0(y; \{x_v\}) + O(x - x_t)]
\]
\[
G^zz_{bou}(x; y; \{x_v\}) = G^zz_{bou}(y; x; \{x_v\}) \sim_{y \to x_u} const \ + \ (y - x_u)^{1-\epsilon_u} [g^zz_0(x; \{x_v\}) + O(y - x_u)]
\]
\[
G^zz_{bou}(x; y; \{x_v\}) = G^zz_{bou}(y; x; \{x_v\}) \sim_{x \to y; x \in (x_t, x_{t+1})} const \ - \ 2\alpha' \cos^2 \gamma_t \text{log} |x - y| + O(x - y)
\]

where \(g^zz_0\) and \(g^zz_0\) are some functions of the given variables and the last line is strictly speaking true when \(x \to y\) but not at the same time when \(x \to x_t\) and \(y \to x_t\). It is anyhow true that \((x - x_t)^{1-\epsilon_t} (y - x_t)^{1-\epsilon_t} \partial_x \partial_y G^zz_{bou, reg T(t)}(x; y; \{x_v\})\) is well defined for \(x = y = x_t\) as discussed in the appendix B.

The rest of the paper is organized in the following way: in the next section we make some examples of the use of the previous formulae and we clarify the operator to state mapping we use in the twisted sector. In section 3 we derive the previous formulae for the case with non excited twisted matter and finally in section 4 we consider excited twisted matter.

2. Examples

We want now apply the main formulae stated in the previous section to some examples while elucidating the nature of excited twisted states.

We start from the simplest example and then move to some more complex ones while in appendix A we check the \(N = 2\) not excited states and \(M\) tachyons amplitude against the result found in ([12]).
2.1 Example 1: $N$ not excited twisted states

From the operator to auxiliary state map

$$
\sigma_{\epsilon_1, \kappa_1}(x_t, \bar{x}_t) \leftrightarrow |T_t, \kappa_t\rangle = \lim_{x \to 0} \sigma_{\epsilon_1, \kappa_1}(x, \bar{x})|0_{SL}\rangle
$$

we deduce that

$$
\langle \sigma_{\epsilon_1, \kappa_1}(x_1, \bar{x}_1) \ldots \sigma_{\epsilon_N, \kappa_N}(x_N, \bar{x}_N) \rangle = \langle V_{N+0}(\{x_t\}_{t=1, \ldots, N}) \prod_t |T_t, \kappa_t\rangle
$$

$$
= \delta(\sum_t \kappa_t) e^{-\frac{1}{2} \sum_t \kappa_t^2 G_{bou, reg T(t)}^{22}(x_t; x_t)} e^{-\frac{1}{2} \sum_{u,t} \kappa_u \kappa_{u,t}} G_{bou}^{22}(x; x) C(x_1, \ldots, x_N)
$$

where the phases proportional to $\kappa$ probably vanish as they do in the $N = 2$ case but this must be checked with an explicit computation of the Green function which can, in principle, be extracted from (6) after T-dualizing. The same computation can be performed using the more conventional generating function as

$$
\langle \sigma_{\epsilon_1, \kappa_1}(x_1, \bar{x}_1) \ldots \sigma_{\epsilon_N, \kappa_N}(x_N, \bar{x}_N) \rangle = V_{N+M} \prod_t e^{i \kappa_t \frac{2}{\delta(2)}} |c=0; d=0\rangle.
$$

2.2 Example 2: $N$ not excited twisted and 2 untwisted states

Similarly to what done in the previous example from the maps

$$
\sigma_{\epsilon_t}(x_t, \bar{x}_t) \leftrightarrow |T_t\rangle = \lim_{x \to 0} \sigma_{\epsilon_t}(x, \bar{x})|0_{SL}\rangle
$$

$$
\partial X^a(y_a, \bar{y}_a) \leftrightarrow -i \sqrt{2 \alpha^2} \cos \gamma_{a1} \alpha_{(a)}^\dagger|0_{(a)}\rangle = \lim_{y \to 0^+} \partial Z_{(a)}(y, \bar{y})|0_{SL}\rangle
$$

where we have made the choice $x_{a1} < y_a < x_{a1} + 1$ (which fixes the magnetic field felt by the untwisted state), we deduce

$$
\langle \sigma_{\epsilon_1, \kappa_1}(x_1, \bar{x}_1) \ldots \sigma_{\epsilon_N, \kappa_N}(x_N, \bar{x}_N) \rangle_{\partial y_1 Z(y_1, \bar{y}_1) \partial y_2 Z(y_2, \bar{y}_2)}
$$

$$
= \langle V_{N+2}(\{x_t\}_{t=1, \ldots, N}, \{y_a\}_{a=1, \ldots, 2}) \otimes |T_t\rangle \otimes (-i \sqrt{2 \alpha^2} \cos \gamma_{t1} \alpha_{(1)}^\dagger|0_{(1)}\rangle \otimes (-i \sqrt{2 \alpha^2} \cos \gamma_{t2} \alpha_{(2)}^\dagger|0_{(2)}\rangle
$$

$$
= \delta(\sum_t \kappa_t) e^{-\frac{1}{2} \sum_t \kappa_t^2 G_{bou, reg T(t)}^{22}(x_t; x_t)} e^{-\frac{1}{2} \sum_{u,t} \kappa_u \kappa_{u,t}} G_{bou}^{22}(x; x) C(x_1, \ldots, x_N) \partial y_1 \partial y_2 G_{bou}^{22}(y_1; y_2; \{x_t\})
$$

where $\bar{y}$ is a function of $y$ as in eq. (3.2). The previous result is an immediate consequence of the the definition of $G$ given in (1.3) but it can also be interpreted as a “proof” of eq. (1.3) since the Green function entering in the previous formula is the Green function obtained from the path integral.
2.3 Example 3: \(N-1\) not excited twisted, 1 excited twisted and 2 untwisted states

We can now discuss the excited twisted states. The easiest way to denote an excited twisted state is by writing from which untwisted state it can be obtained by OPE, for example \(y\) as a normal ordered operator in a twisted auxiliary Hilbert space where

\[\lim_{y \to 0^+} : [y^{1-\epsilon} \partial_y Z(y, y) \partial_y y^{1-\epsilon} \partial_y Z(y, y)] : |T, \kappa\rangle = (-i \sqrt{2 \alpha'} \cos \gamma)^2 \alpha^\dagger_1 \alpha^\dagger_{1+e} |T, \kappa\rangle \] (2.6)

Similarly we can consider the map

\[(\partial \bar{Z} \sigma_{\epsilon,t,\kappa_0})(x_1, x_1) \leftrightarrow -i \sqrt{2 \alpha'} \cos \gamma \alpha^\dagger_{1-\epsilon} |T_1, \kappa_1\rangle \] (2.7)

and compute the correlator

\[
\langle (\partial \bar{Z} \sigma_{\epsilon,t,\kappa_0})(x_1, x_1) \ldots \sigma_{\epsilon,N,\kappa_N}(x_N, x_N) \partial_{y_1} Z(y_1, y_1) \rangle = \langle V_{N+1}(\{x_t\}_{t=1,\ldots,N}, \{x_0\}_{a=1,2}(-i \sqrt{2 \alpha'} \cos \gamma \alpha^\dagger_{(1)} |T_1, \kappa_1\rangle \otimes (1)_{T_1, \kappa_1} \otimes (1)_{T_1, \kappa_1} |0(1)\rangle
\]

The same computation can be performed using the generating function as

\[
\langle (\partial \bar{Z} \sigma_{\epsilon,t,\kappa_0})(x_1, x_1) \ldots \sigma_{\epsilon,N,\kappa_N}(x_N, x_N) \partial_{y_1} Z(y_1, y_1) \rangle = V_{N+M} \frac{\partial}{\partial c(1)_{1+e}} \prod_{t>1} e^{i \kappa_t \frac{\partial}{\partial c_{(1)_{1+e}}} \frac{\partial}{\partial c_t}} |c=0, d=0\rangle
\] (2.10)

3. Derivation for untwisted matter

The starting point is very similar to ([4],[5]) where it was recognized that the generator for all closed (super)string amplitudes is a quadratic path integrals. The idea in the previous papers is that the appropriate boundary condition for R and/or NS sector can be obtained simply by inserting linear sources with the desired boundary conditions. Because of this assumption the quantum fluctuations are the same for all the amplitudes: from the purely NS to the mixed ones. It was later realized that this prescription misses a proper treatment of the quantum fluctuations ([10]) and that when this part is considered the amplitudes factorize correctly ([11]).

Here we consider open strings and we realize the proper twisted boundary conditions by quadratic boundary terms which are nothing else but the coupling of the string to the
magnetic field background. Therefore a non excited twist field is realized by a discontinuity in the magnetic field.

The $N$ non excited twist field amplitudes with Euclidean worldsheet metric is then computed by the quadratic path integral:

\[
C(x_1, \ldots, x_N)\delta(0) = \mathcal{N} \int \mathcal{D}X \ e^{-\frac{1}{2\pi\alpha'} \left[ \int_H d^2 z \ \frac{1}{2} G_{ij} \partial_x X^i \partial_x X^j - i \int_{\partial H} dx \ B(x) X^i(x, \bar{x}) \partial_x X^2(x, \bar{x}) \right]} \tag{3.1}
\]

where $dx \partial_x X^2(x, \bar{x})$ must be interpreted as the pullback on the boundary of $dX^2$ in such a way that $\bar{x}$ depends on $x$ as

\[
\bar{x} = \begin{cases} 
  x & x = |x| > 0 \\
  xe^{-i2\pi} & x = |x|e^{i\pi} < 0
\end{cases}
\tag{3.2}
\]

$H$ is the superior half plane and the adimensional magnetic field $B(x)$ is given by

\[
B(x) = 2\pi\alpha' \ qF_{12}(x) = \sum_{t=1}^{N} B_t \ \theta(x - x_t) \ \theta(x_{t+1} - x)
\tag{3.3}
\]

so that the dicharged string at $x = x_t$ feels a magnetic field $B_{t-1}$ on the left, i.e. $\sigma = \pi$ and $B_t$ on the right, i.e. $\sigma = 0$. Here we have set $B_{-1} = B_{N+1}$ and $x_{-1} = -\infty$, $x_{N+1} = +\infty$.

In the previous equation (3.1) we have chosen the gauge

\[
A_1 = 0, \quad A_2 = Bx
\tag{3.4}
\]

in order to make clear that we have only one zero mode which is associated with the shift $X^2 \rightarrow X^2 + \epsilon$ and therefore there is only one conserved momentum as it is the case in the Landau levels problem on $\mathbb{R}^2$.

Now we can add the untwisted states vertexes. This can be done by considering the generating function of all untwisted vertexes at $x = x_a \in \mathbb{R}$ as

\[
\mathcal{S}(c(a)) = \exp\left\{ \sum_{n=0}^{\infty} c(a)n_i \partial_{x_a}^n |_{x=x_a} X^i(x, \bar{x}) \right\} = \exp\left\{ \int_{\partial H} dx \ J_i(x; x_a) X^i(x, \bar{x}) \right\}
\tag{3.5}
\]

where $c$ are arbitrary complex numbers (or functions as we use in the next section). To understand how the previous generating vertex work let us take the example from ([12]).

Consider the vertex which describes the fluctuations of the gauge vector around the dipole

---

8 And in particular the transition from an eigenstate $|\alpha\rangle$ in magnetic field $B_t$ to an eigenstate $|\beta\rangle$ in magnetic $B_{t+1}$ at worldsheet time $\tau$, can be computed as in usual quantum mechanics as $\langle \beta|\alpha\rangle = \int \mathcal{D}X \ \langle X(\sigma), \tau_i^- |\alpha\rangle \langle X(\sigma), \tau_i^+ |\beta\rangle$.

9 We define $d^2 z = 2dz \ d\bar{z}$ and $z = e^{r+i\sigma} \in H$.

10 This is the path integral corresponding to all twist fields with zero “momentum” and therefore it is proportional to a $\delta(0)$ which arises from $X^2$ zero mode. The general case is treated in the next section.

11 This formulation involving the full field on the boundary is only right for NN boundary condition. For DD boundary condition one should use a slightly different one ([13]).
string background we can derive it from a generating functional for the dipole string as\textsuperscript{12}

\[ V(x_a; \epsilon, k) = \epsilon_i \partial_x X^i(x, \bar{x}) e^{ik, X^i(x, \bar{x})} \bigg|_{x=x_a} = S(c(a), x_a) \epsilon_i \frac{\partial}{\partial c(a)_i} e^{ik, \gamma_{c(a)j}} \bigg|_{c(a)=0} \] (3.6)

As a matter of facts the previous vertex gives an indefinite result when inserted in the path integral even when \( B = 0 \). We must therefore regularize it and consider

\[ S_{\text{reg}}(c(a)) = N(a)(x_a) \exp \left\{ \sum_{n=0}^{\infty} c(a)_n \partial^n_x \big|_{x=x_a} \langle X^i(x, \bar{x}) \rangle \right\} \]

\[ = N(a)(x_a) \exp \left\{ \int_{\partial H} dx J_{i,\text{reg}}(x; x_a) X^i(x, \bar{x}) \right\} \] (3.7)

where the regularized current is given by

\[ J_{i,\text{reg}}(x; x_a) = \sum_{n=0}^{\infty} c(a)_n \partial^n_x \delta_{\text{reg}}(x - x_a), \] (3.8)

the averaged field by

\[ \langle X^i(x, x) \rangle = \int_{\partial H} dy \delta_{\text{reg}}(x - y)X(y, y) \] (3.9)

and the normalization factor is

\[ N(a)(x_a) = \exp \left\{ -\frac{1}{2} \sum_{n,m=0}^{\infty} c(a)_{ni} c(a)_{mj} \int_{\partial H} dx \int_{\partial H} dy \partial^n_x \delta_{\text{reg}}(x - x_a) \partial^m_y \delta_{\text{reg}}(y - x_a) G^{ij}_{U(t_a) , \text{bou}}(x; y) \big|_{x=x_a, y=x_a} \right\} \] (3.10)

with \( G^{ij}_{U(t_a)}, \text{bou}(x; y) \) the boundary Green functions in the dipole case with magnetic field \( B_{t_a} \) given in eq.s (1.26). There are two reasons why we have introduced the previous definitions. The first is that it works in reproducing the amplitudes for \( N = 2 \) as discussed in appendix A. The second is connected to the way the regularization terms is suggested from the operatorial formalism. In operatorial formalism the simplest approach is to consider a point splitting, i.e. \[ \exp \left( c(a)i\bar{i}X^{i\bar{i}}(x_a, x_a) \right) \big|_{\text{p.s.}} = \exp \left( c(a)i[X^i|x_a e^{-\eta}, x_a e^{-\eta} + X^i|x_a(x_a, x_a)] \right) \]

\textsuperscript{12}Notice that here we are talking about the abstract path integral representation of the vertex and not of the operatorial representation. The operatorial representation of the vertex can be realized in an auxiliary space with both twisted and untwisted boundary conditions. The untwisted auxiliary Hilbert space representation is the usual operatorial representation while the twisted one is the one derived in ([12]). Just because of these different realizations this auxiliary Hilbert space must not be confused with \( H_{(a,t_a)} \) introduced before which is a way of representing the \( c(a)_{ni} \), see (3.18).

\textsuperscript{13}It is worth noticing how vertexes for dipole strings have the same functional form independently on the magnetic backgrounds \( B_{t_a} \) nevertheless they differ because different conditions for physical states, in the previous example we have \( k_\mu \eta^{\mu\nu} k_\nu + k_i \bar{G}^{ij}(B_{t_a}) k_j = \epsilon_\mu \eta^{\mu\nu} k_\nu + \epsilon_i \bar{G}^{ij}(B_{t_a}) k_j = 0 \) where \( \mu, \nu \neq 1, 2 \).
which implies a regularization factor \( N(a)(x_a) = \exp \left( -\frac{1}{2} c(a)i c(a)j G^{ij}_{\text{bou}}(x; y) \right) \bigg|_{x=x_a, y=x_a e^{-\eta}} \). When we smooth the fields the previous regularization factor becomes

\[
N(a)(x_a) = \exp \left( -\frac{1}{2} c(a)i c(a)j \int_{x>y} dx \ dy \ 2 \ \delta_{\text{reg}}(x - x_a) \delta_{\text{reg}}(y - x_a) G^{ij}_{\text{bou}}(x; y) \right)
\]

\[
= \exp \left( -c(a)i c(a)j \int dx \ dy \ \delta_{\text{reg}}(x - x_a) \delta_{\text{reg}}(y - x_a) \frac{G^{ij}_{\text{bou}}(x; y) \ \theta(x - y) + G^{ji}_{\text{bou}}(y; x) \ \theta(y - x)}{2} \right)
\]

\[
= \exp \left( -\frac{1}{2} c(a)i c(a)j \int dx \ dy \ \delta_{\text{reg}}(x - x_a) \delta_{\text{reg}}(y - x_a) G^{ij}_{\text{bou}}(x; y) \right)
\]

\[ (3.11) \]

where the factor 2 in the first line is due to the fact we are using one of the two \( \delta \) just one half because of the constraint \( x > y \) as it can be directly verified by using a step function regularization of the delta. In the last step we have used the property \( G^{ij}_{\text{bou}}(y; x) = G^{ij}_{\text{bou}}(x; y) \).

In conclusion the path integral we want to compute in order to get the generating function for all the \( M \) untwisted correlators in presence of \( N \) twists is

\[
Z(\{x_l\}; \{x_a\}) = N \int \mathcal{D}X \ e^{-\frac{1}{2\alpha} \left[ J_H + \frac{\epsilon}{2} G_{ij} \partial_x X^i \partial_x X^j + i \int \partial_H \ dx \ B(x) X^i(x, \bar{x}) \partial_x X^2(x, \bar{x}) \right]}
\]

\[
\times \prod_{a=1}^{M} N(a)(x_a) e^{\int \partial_H \ dx \ J_i, \text{reg}(x; x_a) X^i(x, \bar{x})}
\]

\[ (3.12) \]

This path integral can then be performed to get

\[
Z(\{x_l\}; \{x_a\}) = C(x_1, \ldots x_N) \ \delta(i \sum_a c(a)i = 2)
\]

\[
\prod_a \exp \left\{ \frac{1}{2} \int \partial_H \ dx \int \partial_H \ dy J_i, \text{reg}(x; x_a) \ J_j, \text{reg}(y; x_a) \left[ G^{ij}_{\text{bou}}(x; y; \{x_v\}) - G^{ij}_{U(t_a), \text{bou}}(x; y) \right] \right\}
\]

\[
\prod_{a,b;a \neq b} \exp \left\{ \frac{1}{2} \int \partial_H \ dx \int \partial_H \ dy J_i, \text{reg}(x; x_a) \ J_j, \text{reg}(y; x_b) \ G^{ij}_{\text{bou}}(x; y; \{x_v\}) \right\}
\]

\[ (3.13) \]

where \( G^{ij}_{\text{bou}}(x; y; \{x_v\}) \) is the Green function of the quadratic operator which turns out to be the one defined in (1.2). Notice that the quadratic operator is not even hermitian exactly as it happens for Landau levels in the plain quantum mechanics. The previous
result can be also rewritten after the regularization has been removed as

\[
Z(\{x_i\}; \{x_a\}) = C(x_1, \ldots, x_N) \delta(i \sum_a c_{(a)0} \bar{c}_{(a)0}) \\
\prod_a \exp \left\{ \frac{1}{2} \sum_{n,m=0}^{\infty} c_{(a)ni} c_{(a)mj} \partial^n_x |_{x=x_a} \partial^m_y |_{y=x_a} G_{bou \ reg \ U(t_a)}^{ij}(x; y; \{x_v\}) \right\} \\
\prod_{a,b(a \neq b)} \exp \left\{ \frac{1}{2} \sum_{n,m=0}^{\infty} c_{(a)ni} c_{(b)mj} \partial^n_x |_{x=x_a} \partial^m_y |_{y=x_b} G_{bou \ reg \ U(t_a)}^{ij}(x; y; \{x_v\}) \right\}
\]

(3.14)

where we have defined the boundary Green function regularized by the untwisted Green function for a background \(B_{t_a}\)

\[
G_{bou \ reg \ U(t_a)}^{zz}(x; y; \{x_v\}) = \left[ G_{bou}^{zz}(x; y; \{x_v\}) - G_{bou (t_a), bou}(x; y) \right]
\]

(3.15)

The previous expression can be simplified a little using the symmetry \(G_{bou}^{ij}(x; y) = G_{bou}^{ji}(y; x)\) and by rewriting it in the complex basis as

\[
Z(\{x_i\}; \{x_a\}) = C(x_1, \ldots, x_N) \delta \left( \sum_a (c_{(a)0} - \bar{c}_{(a)0}) \right) \\
\prod_a \exp \left\{ \frac{1}{2} c_{(a)0}^2 G_{bou \ reg \ U(t_a)}^{zz}(x; y; \{x_v\}) + \frac{1}{2} \bar{c}_{(a)0}^2 G_{bou \ reg \ U(t_a)}^{zz}(x; y; \{x_v\}) \right\} \\
\sum_{n,m=0}^{\infty} \bar{c}_{(a)n} c_{(a)m} \partial^n_x |_{x=x_a} \partial^m_y |_{y=x_a} G_{bou \ reg \ U(t_a)}^{zz}(x; y; \{x_v\}) \\
\prod_{a < b} \exp \left\{ \bar{c}_{(a)n} \bar{c}_{(b)m} G_{bou(x_a;x_b)}^{zz}(x_a; x_b, \{x_v\}) + c_{(a)n} c_{(b)m} G_{bou(x_a;x_b)}^{zz}(x_a; x_b, \{x_v\}) \right\} \\
\sum_{n,m=0}^{\infty} \bar{c}_{(a)n} \bar{c}_{(b)m} \partial^n_x |_{x=x_a} \partial^m_y |_{y=x_b} G_{bou(x_a;x_b)}^{zz}(x_a; x_b, \{x_v\}) \\
\sum_{n,m=0}^{\infty} c_{(a)n} \bar{c}_{(b)m} \partial^n_x |_{x=x_a} \partial^m_y |_{y=x_b} G_{bou(x_a;x_b)}^{zz}(x_a; x_b, \{x_v\}) \right\}
\]

(3.16)

with \(\bar{c}_{(a)n} = c_{(a)n\bar{z}}\) and \(c_{(a)n} = c_{(a)n\bar{z}}\).

We can now give a different formulation of the previous result if we realize the algebra

\[
[c_{(a)ni}, \frac{\partial}{\partial c_{(b)mj}}] = \delta^i_j \delta_{n,m} \delta_{a,b}
\]

(3.17)

on the untwisted scalar (dipole string) auxiliary Hilbert spaces \(\mathcal{H}_{a,t_a}\) with backgrounds \(B_{t_a}\).
introduced before as

\[ 1 \rightarrow \langle z(a)_{00} = \bar{z}(a)_{00} = 0 | 0(a) \rangle \]

\[ \bar{c}(a)_n \rightarrow \frac{i}{\sqrt{2\alpha'} \cos \gamma_{ta}} \frac{\alpha(a)_n}{n!}, \quad c(a)_n \rightarrow \frac{i}{\sqrt{2\alpha'} \cos \gamma_{ta}} \frac{\bar{\alpha}(a)_n}{n!} \quad n \geq 0 \]

\[ \frac{\partial}{\partial \bar{c}(a)_m} \rightarrow -i\sqrt{2\alpha'} (m-1)! \cos \gamma_{ta} \bar{\alpha}(a)_m, \quad \frac{\partial}{\partial c(a)_m} \rightarrow -i\sqrt{2\alpha'} (m-1)! \cos \gamma_{ta} \alpha(a)_m \quad m > 0 \]

where the “strange” choice of the normalization is due to the last expressions which arise from the desire of identifying

\[ -i\sqrt{2\alpha'} (m-1)! \cos \gamma_{ta} \alpha(a)_m \sim \partial^m X^{(-)}_{(a)}(x, \bar{x})|_{x=0}. \quad (3.19) \]

Using these auxiliary Hilbert spaces we can now rewrite the previous expression for the M untwisted correlators (3.16) as

\[ Z(\{x_t\}; \{x_a\}) = C(x_1, \ldots x_N) \delta \left( i \sum_a (\alpha(a)_0 - \bar{\alpha}(a)_0) \right) \prod_{a=1}^{M} \langle z(a)_{00} = \bar{z}(a)_{00} = 0 | 0(a) \rangle \]

\[ \prod_a \exp \left\{ \frac{1}{4\alpha'} \alpha^2(a)_0 \mathcal{V}_{(ta)} \right\} G_{\text{bou, reg} U(ta)}(x; y; \{x_v\}) \]

\[ - \frac{1}{4\alpha'} \bar{\alpha}^2(a)_0 \mathcal{V}_{(ta)} \right\} G_{\text{bou, reg} U(ta)}(x; y; \{x_v\}) \]

\[ - \frac{1}{2\alpha'} \sum_{n,m=0}^{\infty} \alpha(a)_n \bar{\alpha}(a)_m \mathcal{V}_{(ta)} \right\} G_{\text{bou, reg} U(ta)}(x; y; \{x_v\}) \]

\[ \prod_{a \leq b} \exp \left\{ - \frac{1}{2\alpha'} \sum_{n,m=0}^{\infty} \bar{\alpha}(b)_m \alpha(a)_n \mathcal{V}_{(ta)} \right\} G_{\text{bou} \left( x; y; \{x_v\} \right)} \right\}_{x=x_a, y=x_b} \]

where \( \mathcal{V} \) are the \( \mathbb{R}^2 \) vielbein which connect the \( \alpha \) flat index with the Green function \( G \) curved index.

4. Derivation for twisted matter

The strategy we are going to follow is to consider the amplitude derived in previous section with \( N + M \) untwisted states at the positions \( \{x_a\}_{a=1 \ldots M}, \{x_f\}_{f=1 \ldots N} \) and unexcited twists at positions \( \{x_t\}_{t=1 \ldots N} \). Then we choose \( N \) of untwisted states at the positions \( \{x_f\}_{f=1 \ldots N} \) for which we take the limit \( x_f \rightarrow x_t \). In order to get the desired amplitude with \( M \) untwisted and \( N \) excited twisted states we must choose in a proper way the \( c(f)_{ni} \). This amounts not only to choose \( c(f)_{ni} \) in (3.7) as a function of \( x_f \) as in eq. (4.9) but to introduce
a further normalization $\mathcal{R}(x_f)$ as in eq. (4.10) in such a way that we can “undo” the OPE and get a result which is a generating function for the twisted states

$$\exp \left\{ d_t^{(t_0)} x_0 = 2(a_{x,t}) - i \sqrt{2} \omega \cos \gamma_t \sum_{n=1}^{\infty} d_t(n)(n-1) ! \alpha_{n+1}^{(x,t)} + \bar{d}_t(n-1) ! \alpha_{n+1}^{(x,t)} \right\} |T_{(aux,t)}\rangle$$

(4.1)

when expressed in a chart where $x_t = 0$. The “strange” normalization is chosen because it is the easiest map from operators to states, f.x. the twisted excited state which can be obtained by subtracting the divergences of the limit $y \to x_t^+$ of $[\partial_0 Z(y,y)]^2 \sigma_{x_t}(x_t, t)$ gives the state

$$\lim_{y \to 0^+} \left[ \partial_y^{[2]} (y,y) \right] |T_{(aux,t)}\rangle = \left( -i \sqrt{2} \omega \cos \gamma_t \right)^2 \left| T_{(aux,t)} \right\rangle$$

(4.2)

thus making contact between eq. (4.1) and eq. (4.5).

Let us start studying the OPE $\mathcal{S}(c,x_f) \sigma_t(0,0)$. This can be studied in an auxiliary Hilbert space $\mathcal{H}_{aux,t}$ (not to be confused the the Hilbert space $\mathcal{H}_t$ which we introduced in the first section and which is associated with coefficients $d_t(n)$) where $\sigma_t(0,0)$ is represented by the twisted vacuum $|T_{(aux,t)}\rangle$ and the generating function $\mathcal{S}(c,x_f)$ as

$$\mathcal{S}_{(aux,t)}(c_f, x_f) = e^{-\frac{1}{2} \sum_{m,n=0} G^{zz}_{aux, reg} U(t_a) x_f(x_f;} \{x=0, x=\infty\}) - \frac{1}{2} \sum_{m,n=0} G^{zz}_{aux, reg} U(t_a) x_f(x_f;} \{x=0, x=\infty\})$$

$$\mathcal{S}_{(aux,t)}(c_f, x_f) = e^{-\frac{1}{2} \sum_{m,n=0} G^{zz}_{aux, reg} U(t_a) x_f(x_f;} \{x=0, x=\infty\})$$

(4.3)

In the previous equation the normal ordering is performed with respect to the operators entering the expansion of the quantum fields $Z_{(aux,t)}(z, \bar{z})$ and $\bar{Z}_{(aux,t)}(z, \bar{z})$ which act on $\mathcal{H}_{aux,t}$. Then the OPE can be computed as

$$\mathcal{S}(c,x_f) \sigma_t(0,0) \leftrightarrow e^{-\frac{1}{2} \sum_{m,n=0} G^{zz}_{aux, reg} U(t_a) x_f(x_f;} \{x=0, x=\infty\}) - \frac{1}{2} \sum_{m,n=0} G^{zz}_{aux, reg} U(t_a) x_f(x_f;} \{x=0, x=\infty\})$$

$$\mathcal{S}(c,x_f) \sigma_t(0,0) \leftrightarrow e^{-\frac{1}{2} \sum_{m,n=0} G^{zz}_{aux, reg} U(t_a) x_f(x_f;} \{x=0, x=\infty\})$$

(4.4)

which is similar to a rewriting of eq. (4.1) as

$$\lim_{x_f \to 0^+} e^{d(t_0)} \left[ Z_{(aux,t)}^{(-)}(x_f, x_f) \right] + d(t_0) \left[ Z_{(aux,t)}^{(-)}(x_f, x_f) \right]$$

$$\lim_{x_f \to 0^+} e^{d(t_0)} \left[ Z_{(aux,t)}^{(-)}(x_f, x_f) \right] + d(t_0) \left[ Z_{(aux,t)}^{(-)}(x_f, x_f) \right]$$

(4.5)

where in the second line we have written $\partial Z$ since we want to get rid of zero modes and in the limit it is necessary to write $x_f \to 0^+$ since the the behavior of $\partial Z$ changes by an overall normalization when $x < 0$. 


Comparison between the two previous expressions suggests to consider then the operator acting on the Hilbert space $\mathcal{H}_{(\text{aux } t)}$

\[
\mathcal{T}_{(\text{aux } t)}(d(t), x_f) = \mathcal{N}(d(t), x_f, x_f) e^{\tilde{d}(t)0} \left[ Z_{(\text{aux } t,\text{reg})}(x_f, x_f) \right]^{+} \right] + d(t)0 \left[ Z_{(\text{aux } t,\text{reg})}(x_f, x_f) \right]^{+}
\]

where $Z_{(\text{aux } t,\text{reg})}(x_f, x_f)$ is point split regularized of $Z_{(\text{aux } t)}(x_f, x_f)$ defined as

\[
Z_{(\text{aux } t,\text{reg})}(x_f, x_f) = Z_{(\text{aux } t)}^{(-)}(x_f e^{-\eta}, x_f e^{-\eta}) + Z_{(\text{aux } t)}^{(+)}(x_f, x_f),
\]

no normal ordering is performed and the normalization factor is given by

\[
\mathcal{N}^{-1}(d(t), x_f, x_f) = \left\{ \begin{array}{l}
\frac{1}{2} \sum_{n=0}^{\infty} d(t)n \frac{\partial^{n-1} \left[ x^{1-ct} \partial_x Z_{(\text{aux } t,\text{reg})}(x, x) \right]^{+} + d(t)n \frac{\partial^{n-1} \left[ x^{1-ct} \partial_x Z_{(\text{aux } t,\text{reg})}(x, x) \right]^{+}}{\partial_{x}^{n}}(x_f)  \\
\frac{1}{2} \sum_{n=0}^{\infty} d(t)n \frac{\partial^{n-1} \left[ x^{1-ct} \partial_x Z_{(\text{aux } t,\text{reg})}(x, x) \right]^{+} + d(t)n \frac{\partial^{n-1} \left[ x^{1-ct} \partial_x Z_{(\text{aux } t,\text{reg})}(x, x) \right]^{+}}{\partial_{y}^{n}}(y_f)  \\
\frac{1}{2} \sum_{n=0}^{\infty} d(t)n \frac{\partial^{n-1} \left[ x^{1-ct} \partial_x Z_{(\text{aux } t,\text{reg})}(x, x) \right]^{+} + d(t)n \frac{\partial^{n-1} \left[ x^{1-ct} \partial_x Z_{(\text{aux } t,\text{reg})}(y, y) \right]^{+}}{\partial_{y}^{n}}(y_f)
\end{array} \right\}
\]

\[
\mathcal{T}_{(\text{aux } t)}(d(t), x_f) \mathcal{S}_{(\text{aux } t)}(c(f)(d(t), x_f), x_f)
\]

where $G_{(\text{aux } t,\text{reg})}(y;x) = G_{(\text{aux } t,\text{reg})}(x,y) = G_{\text{reg}}^{ij}(y;x;\{x_1 = 0, x_2 = \infty\})$ are the (analytic continuation of the) boundary Green functions defined in eqs. (1.16).

The reason why we have written the previous expression in a non normal ordered way is to understand the expression of the regularization factor of the corresponding “classical” vertex (4.11) which we want to insert in the path integral.

The previous operator can also be written in a way to make its connection with the idea of undoing the OPE clearer as

\[
\mathcal{T}_{(\text{aux } t)}(d(t), x_f)
\]

where

\[
\begin{align*}
\tilde{c}_{(f)0}(d(t), x_f) &= d(t)0 \\
\tilde{c}_{(f)n}(d(t), x_f) &= \sum_{k=0}^{n} \frac{k-1}{n} d(t)k \partial^{k-n} x_f^{(1-ct)} \\
\tilde{c}_{(f)n}(d(t), x_f) &= \sum_{k=0}^{n} \frac{k-1}{n} \tilde{d}(t)k \partial^{k-n} x_f^{(1-ct)}
\end{align*}
\]
and the normalization factor is
\[
\mathcal{R}(d(t), x_f) = e^{-\sum_{n=0}^{\infty} d_{(2n)} \sum_{n=0}^{\infty} \partial_y^{n-1} \partial_x^{n-1} \left[ x^{1-\gamma} y^{1-\gamma} \partial_x \partial_y G_{\text{bou}}^{\bar{z}, z}(x,y;x=0,y=\infty) \right]} \bigg|_{x=y=x_f} \tag{4.10}
\]
in order to undo the OPE and get the desired result as in eq. (4.5).

We shall now translate the previous operator (4.6) into an abstract operator we can insert in the path integral at an arbitrary point \(x_t\), therefore we move it from \(x_t = 0\) to a generic \(x_t\) and we consider a generating vertex as
\[
\mathcal{T}(d(t), x_t) = \lim_{x \to x_t^+} \mathcal{N}_T(d(t), x, x_t) \ e^{d(t) \langle Z(x,x) \rangle + d(t) \langle \bar{Z}(x,x) \rangle}
\]
\[
e^{\sum_{n=0}^{\infty} \bar{G}_{(2n)} \sum_{n=0}^{\infty} \partial_x^{n-1} \partial_y^{n-1} \left\langle (x-x_t)^{1-\gamma} \partial_x Z(x,x) + d_{(2n)} \partial_x^{n-1} \partial_y \bar{Z}(x,x) \right\rangle} \tag{4.11}
\]
where \(x \to x_t\) has to be understood as taking the limit after the path integral has been computed. We have defined the averaged fields such as
\[
\langle (x-x_t)^{1-\gamma} \partial_x Z(x,x) \rangle = \int_{\partial H} dy \ \delta_{\text{reg}}(x-y)(y-x_t)^{1-\gamma} \partial_y Z(y,y) \tag{4.12}
\]
because we want a well defined regulated expression after performing the path integral and introduced the normalization factor
\[
\mathcal{N}_T(d(t), x, x_t) = e^{-\frac{1}{2} \sum_{n=0}^{\infty} \bar{G}_{(2n)} \big\langle (x-x_t)^{1-\gamma} \partial_x \partial_y G_{\text{bou}}^{\bar{z}, z}(x,y) \big\rangle} \tag{4.13}
\]
where the doubly regularized Green functions are defined such as
\[
\big\langle (x-x_t)^{1-\gamma} \partial_x \partial_y G_{\text{bou}}^{\bar{z}, z}(x,y) \big\rangle = \int dy_1 \int dy_2 \ \delta_{\text{reg}}(x-y_1) \delta_{\text{reg}}(y-y_2) \ G_{\text{bou}}^{\bar{z}, z}(y_1; y_2, \{x_1 = x_t, x_2 = \infty\})
\]
\[
\big\langle (y - x_t)^{1-\gamma} \partial_x \partial_y G_{\text{bou}}^{\bar{z}, z}(x,y) \big\rangle = \int dy_1 \int dy_2 \ \delta_{\text{reg}}(x-y_1) \delta_{\text{reg}}(y-y_2) \ (y_2 - x_t)^{1-\gamma} \partial_1 \partial_2 G_{\text{bou}}^{\bar{z}, z}(y_1; y_2, \{x_1 = x_t, x_2 = \infty\}) \tag{4.14}
\]
Since the previous expression is in nuce the same as for the untwisted matter, we can immediately deduce from (3.16) the result of inserting and integrating over the \(X\) to be eq. (1.30).

In analogy with what done for the untwisted states we can realize the algebra
\[
\left[ d_{(2n)}, \frac{\bar{\partial}}{\partial d_{(2n)}} \right] = \left[ \hat{d}_{(2n)}, \frac{\bar{\partial}}{\partial \hat{d}_{(2n)}} \right] = \delta_{m,n} \delta_{u,t} \tag{4.15}
\]
with operators acting on the twisted scalar (dicharged string) auxiliary Hilbert spaces $\mathcal{H}_t$ as

$$1 \rightarrow \langle T_\epsilon, x_1^t \rangle_0 = 0 |$$

$$\tilde{d}(t)_n \rightarrow \frac{i}{\sqrt{2\alpha'}} \cos \gamma \frac{\alpha(t)_n+1}{(n-1)! (n-1+\epsilon_t)}, \quad d(t)_n \rightarrow \frac{i}{\sqrt{2\alpha'} \cos \gamma} \frac{\alpha(t)_n-\epsilon_t}{(n-1)! (n-\epsilon_t)} \quad n > 0$$

$$\frac{\partial}{\partial \tilde{d}(t)_m} \rightarrow -i\sqrt{2\alpha'} (m-1)! \cos \gamma \frac{\alpha(t)_m-1}{(m-1+\epsilon_t)}, \quad \frac{\partial}{\partial d(t)_m} \rightarrow -i\sqrt{2\alpha'} (m-1)! \cos \gamma \frac{\alpha(t)_m}{(m-\epsilon_t)} \quad m > 0$$

which gives eq. (1.29) when substituted into eq. (1.30).

Acknowledgments

We would like to thank P. Di Vecchia and F. Pezzella for discussions. The author thanks the Nordita for hospitality during different stages of this work.

A. Check of the $N = 2$ amplitudes

We would now check that the operatorial amplitudes with $N = 2$ and the path integral approach give the same result, phases included. Let us consider the tachyonic amplitude

$$\langle \sigma_{-\epsilon, \lambda}(x_\infty, x_\infty) \sigma_{\epsilon, \kappa}(x_0, x_0) \mathcal{V}_T(x_1; k(1)) \ldots \mathcal{V}_T(x_M; k(M)) \rangle$$

(A.1)

with $x_t=1 = x_0$, $x_t=2 = x_\infty$ and $\gamma_0$ and $\gamma_1$ arbitrary but $\gamma_2 = \gamma_0$ so that $\pi \epsilon_1 = -\pi \epsilon_2$. Using the results from ([12]) we can compute it in the limit $x_0 \rightarrow 0$ and $x_\infty \rightarrow \infty$, for $x_1 > \ldots x_M > 0$ and when multiplied by the appropriate power of $x_\infty$ as

$$\langle T_\epsilon, -\lambda | e^{-\frac{1}{2} R^2(\epsilon)} \Delta(k(1)) \mathcal{A}_1 e^{i(\bar{k}(0)+k(1))_0} e^{i \cos \gamma \bar{k}(1) Z_{nzm} (x_1, x_1)+k(1) \bar{Z}_{nzm} (x_1, x_1)} \ldots | T_\epsilon, \kappa \rangle$$

$$= \delta(\kappa + \kappa + \sum_a k(a)2)$$

$$\prod_a e^{\frac{1}{2} \pi \alpha' \tan \frac{1}{\tan \gamma_1 - \tan \gamma_0} (k(a) - \bar{k}(a))}$$

$$\prod_a \left[ e^{-\frac{1}{2} R^2(\epsilon)} \Delta(k(a))_x \Delta(k(a)) \right]$$

$$\prod_a e^{-\pi \alpha' \tan \frac{1}{\tan \gamma_1 - \tan \gamma_0} \sqrt{2} (k(a) - \bar{k}(a))}$$

$$\prod_{a < b} \left[ e^{\pi \alpha' \tan \frac{1}{\tan \gamma_1 - \tan \gamma_0} (k(a) - \bar{k}(a))(k(b) + \bar{k}(b))} e^{\alpha' \cos^2 \gamma (k(a) + k(b)) g_{1-\epsilon} \left( \frac{x_a}{x_b} \right) + k(a) k(b) g_{\epsilon} \left( \frac{x_a}{x_b} \right)} \right]$$

(A.2)

where we have used the commutation relations (1.22), $R^2(\epsilon) = \lim_{u \rightarrow 1^-} [g_\epsilon(u) + g_{1-\epsilon}(u) - 2 \log(1-u)] = - (\psi(\epsilon) + \psi(1-\epsilon) - 2 \psi(1))$ and $\Delta(k(a)) = 2 \alpha' \cos^2 \gamma k(a) \bar{k}(a)$ is the conformal dimension of the tachyonic vertex.

We can now compare with the general expression (1.30) with the identifications $c(a)_0 \rightarrow i k(a)_0$, $\bar{c}(a)_0 \rightarrow i \bar{k}(a)_0$, $d_1(0) \rightarrow i \frac{x_a}{\sqrt{2}}$, $\bar{d}_1(0) \rightarrow i \frac{x_a}{\sqrt{2}}$, $d_2(0) \rightarrow i \frac{x_a}{\sqrt{2}}$ and $\bar{d}_2(0) \rightarrow i \frac{x_a}{\sqrt{2}}$. We can also compare with the expression (1.29) upon the product with the state...
in order to understand where the different terms come from the path integral point of view. In matching these terms is important to be careful in rewriting all the Green functions \(G(x; y)\) in such a way that \(x > y\) by using the symmetry (1.2) since this is the natural way they appear from the operatorial formalism. We recognize that

- the factor \(\exp\left\{ \frac{1}{2} \pi \alpha' \tan \gamma_1 - \tan \gamma_0 \left( k^2 - \bar{k}^2 \right) \right\} \) come from the \(G_{\text{bou reg}}^{zz} U(t_a)\) and \(G_{\text{bou reg}}^{zz} U(t_a)\) terms.

- The factors \(\prod_a \left[ e^{-\frac{1}{2} R^2(\epsilon) \Delta(k(a))} \right] \) come from the \(G_{\text{bou reg}}^{zz} U(t_a)\) terms. In particular the result follows from the following steps

\[
G_{\text{bou reg}}^{zz} U(t_a)(x_a^+, x_a) = \frac{\pi \alpha'}{\tan \gamma_1 - \tan \gamma_0} - \pi \alpha' \sin \gamma_1 \cos \gamma_1 + 2 \alpha' \cos^2 \gamma_1 \ln |x_a| - 2 \alpha' \cos^2 \gamma_1 \left( g_\epsilon \left( \frac{x_a}{x_a^+} \right) - \log \left( 1 - \frac{x_a}{x_a^+} \right) \right)
\]

\[
= \frac{\pi \alpha'}{\tan \gamma_1 - \tan \gamma_0} \cos^2 \gamma_1 + 2 \alpha' \cos^2 \gamma_1 \ln |x_a| - 2 \alpha' \cos^2 \gamma_1 (\psi(1 - \epsilon) - \psi(1))
\]

\[
= 2 \alpha' \cos^2 \gamma_1 \ln |x_a| - \alpha' \cos^2 \gamma_1 (\psi(\epsilon) + \psi(1 - \epsilon) - 2\psi(1))
\]

where we have used the expression for the Green function given in eq. (1.16) since we have chosen \(x = x_a^+\) and in the last line we have used the digamma property \(\psi(1 - \epsilon) = \psi(\epsilon) + \pi \cot(\pi \epsilon)\). It is also worth stressing that the result is independent on setting \(x_a^+\) in the first argument since the function \(G_{\text{bou reg}}^{zz} U(t_a)\) is continued analytically at \(x = y = x_a\) in such a way that \(G^{ij}(x; y) = G^{ji}(y; x)\) so that we have chosen the first argument to be \(x_a\) we would have got the same result computing \(G_{\text{bou reg}}^{zz} U(t_a)(x_a, x_a^-)\).

- The terms \(e^{-\pi \alpha' \tan \gamma_1 - \tan \gamma_0 \sqrt{2} (k(a) - \bar{k}(a))}\) arise from the \(\prod_{t,a}\) terms. While rewriting the Green functions \(G(x; y)\) in such a way that \(x > y\) we see that the terms with \(t = 2\) (at \(x_\infty\)) cancel while those from \(t = 1\) (at \(x_0\)) do not and reproduce the operatorial result.

- The terms \(\prod_{a < b}\) come trivially from the corresponding ones in eq. (1.30).

- The terms \(\prod_{t\in (1,30)}\) give a trivial result in a non trivial way. It is immediate to find that \(G_{\text{bou, reg}}^{zz} (t) = G_{\text{bou, reg}}^{zz} (t) = 0\). On the other side we get for \(0 < \frac{y - x_0}{x_0 - x_\infty} < 1\) and \(0 \leq \omega = \frac{y - x_0}{x_0 - y - x_\infty} < 1\)

\[
G_{\text{bou, reg}}^{zz}(t)(x; y, \{x_0, x_\infty\}) = -2 \alpha' \cos^2 \gamma \left[ g_\epsilon(\omega) - g_\epsilon(\frac{y - x_0}{x_\infty - x_0}) \right]
\]

Now we can write \(x = x_0 + \alpha, y = x_0 + (1 - \delta)\) with \(\alpha > 0\) and \(0 < \delta < 1\) and expand in \(\delta\) to get

\[
G_{\text{bou, reg}}^{zz}(t)(x; y, \{x_0, x_\infty\}) = -2 \alpha' \cos^2 \gamma \left[ -\log \left( 1 + \frac{\alpha}{x_\infty - x_0} \right) + \frac{(\epsilon - 1)\alpha}{x_\infty - x_0 + \alpha} \delta + O(\delta^2) \right]
\]
which vanishes when $\alpha = \delta = 0$. In a similar way can be treated the case when $x, y \to x_\infty$.

- The terms $\prod_{t<\omega}$ in (1.30) give also a trivial result in a not completely trivial fashion. We have to evaluate the Green functions for $\omega = \frac{y-x_0}{x-x_0} - \frac{y-x_\infty}{x-x_\infty}$ when $x = x_0$ and $y = x_\infty$ so $\omega = 0$ and we are left with only the constant terms, as we expect from the general asymptotic (1.31) hence $\prod_{t<\omega} = \prod_{\omega \to x_0} \exp \left\{ \tan \gamma_1 - \tan \gamma_0 \right\} \left[ -d_{(1)} d_{(2)} + d_{(1)} d_{(2)} - d_{(1)} d_{(2)} + d_{(1)} d_{(2)} \right]$ which vanishes when evaluated with the previously stated substitutions for which $d_{(t)} = \bar{d}_{(t)}$.

**B. Behavior of the Green function when $x, y \to x_t$**

In this section we follow and adapt the computation done in ([6]). We start considering the derivative of Green function of the left moving part defined as

$$\partial_z \partial_w \mathcal{G}_{LL}^{zz}(z; w; \{x_t\}_{t=1\ldots N}) = \frac{(\partial_z Z_L(z) \partial_w \bar{Z}_L(w) \sigma_{\epsilon_1, \kappa_1} (x_1, \bar{x}_1) \ldots \sigma_{\epsilon_N, \kappa_N} (x_N, \bar{x}_N))_{\text{disk}}}{\langle \sigma_{\epsilon_1, \kappa_1} (x_1, \bar{x}_1) \ldots \sigma_{\epsilon_N, \kappa_N} (x_N, \bar{x}_N) \rangle_{\text{disk}}} \quad (B.1)$$

which has asymptotics

$$-\frac{1}{2\alpha'} \partial_z \partial_w \mathcal{G}_{LL}^{zz}(z; w; \{x_t\}_{t=1\ldots N}) \sim_{z \to w} \frac{1}{(z-w)^2} + O(1)$$

$$\sim_{z \to x_t} (z-x_t)^{\alpha_t-1}$$

$$\sim_{w \to x_t} (w-x_t)^{-\alpha_t} \quad (B.2)$$

then we can write

$$-\frac{1}{2\alpha'} \partial_z \partial_w \mathcal{G}_{LL}^{zz}(z; w; \{x_t\}_{t=1\ldots N}) = \prod_{u} \left( \frac{w-x_u}{z-x_u} \right)^{1-\epsilon_u} \left[ \frac{1}{(z-w)^2} \sum_{u<v} a_{uv}(z-x_u)(z-x_v) \right]$$

$$+ \sum_{u_1<u_2<u_3<u_4} \frac{b_{u_1 u_2 u_3 u_4}}{(w-x_{u_1})(w-x_{u_2})(w-x_{u_3})(w-x_{u_4})} \quad (B.3)$$

Now we can study the behavior $x, y \to x_1$ by setting

$$z = x_1 + \alpha, \quad w = x_1 + \alpha(1-\delta) \quad (B.4)$$

and letting $\alpha, \delta \to 0^+$. A simple computation gives

$$-\frac{1}{2\alpha'} \partial_z \partial_w \mathcal{G}_{LL}^{zz}(z; w; \{x_t\}_{t=1\ldots N}) \sim_{z \to w} \frac{1}{\delta^2} \sum_{u<v} a_{uv} + \frac{1}{\alpha 1-\delta} \sum_{1<u_2<u_3<u_4} \frac{b_{u_2 u_3 u_4}}{(x_1-x_{u_2})(x_1-x_{u_3})(x_1-x_{u_4})} + O(1) \quad (B.5)$$

from which we deduce the constraint

$$\sum_{u<v} a_{uv} = 1. \quad (B.6)$$
Then the regularized Green function which is obtained by subtracting the corresponding Green function with only two twist, one of which in $x_1$ and the other at $\infty$ is of the form

$$
\partial_z \partial_w G_{LL, \text{reg}}^{zz}(z; w; \{x_t\}_{t=1 \ldots N}) \sim \frac{B_1(x_u)}{\alpha} \frac{1}{1 - \delta} + O(1) \tag{B.7}
$$

hence the terms in $\prod_t$ are well defined since

$$(z - x_1)^{1-\epsilon_1} (w - x_1)^{\epsilon_1} \partial_z \partial_w G_{LL, \text{reg}}^{zz}(z; w; \{x_t\}_{t=1 \ldots N}) \sim \alpha (1 - \delta)^{\epsilon_1} \left[ \frac{B_1(x_u)}{\alpha} \frac{1}{1 - \delta} + O(1) \right]. \tag{B.8}$$

References

[1] P. Di Vecchia, R. Nakayama, J. L. Petersen, J. Sidenius, S. Sciuto, “Brst Invariant N Reggeon Vertex,” Phys. Lett. B182 (1986) 164.

P. Di Vecchia, M. Frau, A. Lerda and S. Sciuto, “A simple expression for the multiloop amplitude in the bosonic string,” Phys. Lett. B 199 (1987) 49.

P. Di Vecchia, K. Hornfeck, M. Frau, A. Lerda and S. Sciuto, “N string, g loop vertex for the bosonic string,” Phys. Lett. B 206 (1988) 643.

P. Di Vecchia, F. Pezzella, M. Frau, K. Hornfeck, A. Lerda, A. Sciuto, “N point g loop vertex for a free bosonic string with vacuum charge Q,” Nucl. Phys. B322 (1989) 317.

P. Di Vecchia, K. Hornfeck, M. Frau, A. Lerda and S. Sciuto, “N string, g loop vertex for the fermionic string,” Phys. Lett. B 211 (1988) 301.

P. Di Vecchia, F. Pezzella, M. Frau, K. Hornfeck, A. Lerda and A. Sciuto, “N point g loop vertex for a free bosonic string with vacuum charge Q,” Nucl. Phys. B 322 (1989) 317.

P. Di Vecchia, F. Pezzella, M. Frau, K. Hornfeck, A. Lerda and S. Sciuto, “N point g loop vertex for a free fermionic string with arbitrary spin,” Nucl. Phys. B 333 (1990) 635.

[2] A. Neveu, P. C. West, “Cycling, Twisting And Sewing In The Group Theoretic Approach To Strings,” Commun. Math. Phys. 119 (1988) 585.

A. Neveu, P. C. West, “Group Theoretic Approach To The Perturbative String S Matrix,” Phys. Lett. B193 (1987) 187.

[3] P. Di Vecchia, K. Hornfeck, M. Yu, “Infinite Number Of Conserved Charges And The N String Vertex,” Phys. Lett. B195 (1987) 557.

B. Gato Riveira, “Construction Of Vertex Operators Using Operator Formalism Techniques,” Nucl. Phys. B322 (1989) 555.

[4] P. Di Vecchia, R. Nakayama, J. L. Petersen, J. R. Sidenius, S. Sciuto, “Covariant N String Amplitude,” Nucl. Phys. B287 (1987) 621.

[5] J. L. Petersen, J. R. Sidenius, A. K. Tollsten, “Covariant Loop Calculus For The Neveu-schwarz String,” Phys. Lett. B213 (1988) 30.

J. L. Petersen, J. R. Sidenius, A. K. Tollsten, “Covariant Superreggeon Calculus For Superstrings,” Nucl. Phys. B317 (1989) 109.

J. L. Petersen, J. R. Sidenius, “Covariant Loop Calculus For The Closed Bosonic String,” Nucl. Phys. B301 (1988) 247.

J. L. Petersen, J. R. Sidenius, A. K. Tollsten, “A Note On Multiloop Vertices In String Theory,” Phys. Lett. B214 (1988) 533.
[6] S. A. Abel, A. W. Owen, “N point amplitudes in intersecting brane models,” Nucl. Phys. B682 (2004) 183-216. [hep-th/0310257].
J. J. Atick, L. J. Dixon, P. A. Griffin, D. Nemeschansky, “Multiloop Twist Field Correlation Functions For Z(n) Orbifolds,” Nucl. Phys. B298 (1988) 1-35.

[7] M. Cvetic and I. Papadimitriou, “Conformal field theory couplings for intersecting D-branes on orientifolds,” Phys. Rev. D 68 (2003) 046001 [Erratum-ibid. D 70 (2004) 029903] [arXiv:hep-th/0303083].

[8] M. Bertolini, M. Billo, A. Lerda, J. F. Morales and R. Russo, “Brane world effective actions for D-branes with fluxes,” Nucl. Phys. B 743 (2006) 1 [arXiv:hep-th/0512067].

[9] D. Duo, R. Russo, S. Sciuto, “New twist field couplings from the partition function for multiply wrapped D-branes,” JHEP 0712 (2007) 042. [arXiv:0709.1805 [hep-th]].
R. Russo, S. Sciuto, “The Twisted open string partition function and Yukawa couplings,” JHEP 0704 (2007) 030. [hep-th/0701292].

[10] P. Di Vecchia, R. Madsen, K. Hornfeck, K. O. Roland, “A Vertex Including Emission Of Spin Fields,” Phys. Lett. B235 (1990) 63.
P. Di Vecchia, R. A. Madsen, K. Roland, “A vertex including emission of spin fields for an arbitrary BC system,” Nucl. Phys. B354 (1991) 154-190.

[11] N. Di Bartolomeo, P. Di Vecchia, R. Guatieri, “General properties of vertices with two Ramond or twisted states,” Nucl. Phys. B347 (1990) 651-686.

[12] I. Pesando, “Strings in an arbitrary constant magnetic field with arbitrary constant metric and stringy form factors,” JHEP 1106 (2011) 138. [arXiv:1101.5898 [hep-th]].
I. Pesando, “Open and Closed String Vertices for branes with magnetic field and T-duality,” JHEP 1002 (2010) 064 [arXiv:0910.2576 [hep-th]].

[13] I. Pesando, “Multibranes boundary states with open string interactions,” Nucl. Phys. B 793 (2008) 211 [arXiv:hep-th/0310027].
I. Pesando, “On the effective potential of the Dp Dp-bar system in type II theories,” Mod. Phys. Lett. A 14 (1999) 1545 [arXiv:hep-th/9902181].
I. Pesando, “A comment on discrete Kalb-Ramond field on orientifold and rank reduction,” arXiv:0804.3931 [hep-th].

[14] S. Sciuto, “The General Vertex Function In Dual Resonance Models,” Lett. Nuovo Cim. 2 (1969) 411.
A. Della Selva and S. Saito, Lett. Nuovo Cim. 4 (1970) 689.

[15] E. Corrigan, D. B. Fairlie, “Off-Shell States in Dual Resonance Theory,” Nucl. Phys. B 91 (1975) 527.
J. H. Schwarz, C. C. Wu, “Off Mass Shell Dual Amplitudes. 2.,” Nucl. Phys. B 72 (1974) 397.
J. H. Schwarz, “Off-mass-shell dual amplitudes without ghosts,” Nucl. Phys. B 65 (1973) 131-140.

[16] P. Hermansson, B. E. W. Nilsson, A. K. Töllsten, A. Watterstam, “Derivation Of The Brink-olive Correction Factor Using The Dual Ramond Superghost Vertex,” Phys. Lett. B 244 (1990) 209-214.
B. E. W. Nilsson, A. K. Töllsten, “General NSR String Reggeon Vertices From A Dual Ramond Vertex,” Phys. Lett. B 240 (1990) 96.
N. Engberg, B. E. W. Nilsson, A. Westerberg, “The Twisted string vertex algorithm applied to the Z(2) twisted scalar string four vertex,” Nucl. Phys. B 435 (1995) 277-294. [hep-th/9405159].
[17] N. Engberg, B. E. W. Nilsson, P. Sundell, “An Algorithm for computing four Ramond vertices at arbitrary level,” Nucl. Phys. B404 (1993) 187-214. [hep-th/9301107].
N. Engberg, B. E. W. Nilsson, P. Sundell, “On the use of dual Reggeon vertices for untwisted and twisted scalar fields,” Int. J. Mod. Phys. A7 (1992) 4559-4583.