On the Construction of Generalised Bobillier Formula

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Abstract

In this paper, one-parameter planar motion in generalised complex plane (or \( p \)-complex plane) \( \mathbb{C}_p = \{x + iy : x, y \in \mathbb{R}, \ i^2 = p\} \) which is defined as a system of generalised complex numbers is studied. Firstly, generalised Bobillier formula is obtained by using the geometric interpretation of generalised Euler-Savary formula in the \( p \)-complex plane. Moreover, it is shown that the Bobillier formula may be obtained by an alternative method without the use of Euler-Savary formula in the generalised complex plane. Thus, this formula generalises the complex, hyperbolic and Galilean cases.

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1 Introduction

A mechanism is a system of rigid elements (linkages) arranged and connected to transmit motion or is a combination of links which can transform a determined motion. In kinematics, a mechanism is a means of transmitting, controlling, or constraining relative movement. A planar mechanism is a mechanical system that is constrained so the trajectories of points in all the bodies of the system lie on planes parallel to a grand plane. The rotational axes of hinged joints that connect the bodies in the system are perpendicular to this ground plane.\[1\]

Kinematics concerned with the characteristics of movement without considering the concepts of mass and force is sub-branch of mechanics and analyses the displacement of one point or a point system (object) with respect to time. Robot kinematics refers the analytical study of the motion of a robot manipulator. For a robot mechanism is very crucial for analysing the behaviour of industrial manipulators.

Robot kinematics studies the relationship between the dimensions and connectivity of kinematic chains and the position, velocity and acceleration of each of the links in the robotic system, in order to plan and control movement and
to compute actuator forces and torques. The relationship between mass and inertia properties, motion, and the associated forces and torques is studied as part of robot dynamics, [2, 3].

The Euler-Savary formula which is one of the important kinematics formulas and has an important place in many fields such as mathematics, engineering and astronomy is found by Euler in 1765 and Savary in 1845, [19]. This formula giving the relationship between the curvatures of trajectory curves drawn by the points of the moving plane in the fixed plane is studied by [5, 6, 7, 11, 12] in two and three dimensional Euclidean spaces. Moreover, the Euler-Savary formula has too many applications in kinematics including in mechanical engineering of robotics in science of machines and mechanisms.

In 1988, Fayet defined a formula giving the relation of the curvatures of second order of one-parameter planar motion and generalizing the Euler-Savary formula. Since this formula analytically solves the problem that the Bobillier’s construction solved graphically it is called the Bobillier formula, [13].

In [14], it was demonstrated that the Bobillier formula can be obtained without the use of the Euler-Savary formula by Fayet. In addition, in [15] the Bobillier formula was shown by conventional procedures.

Ersoy and Bayrak studied the Bobillier formula for the one-parameter planar motion in the complex plane, [8]. Also, they investigated the Bobillier formula in Lorentzian sense and saw that the same results can be achieved without the use of the Euler-Savary formula for the Bobillier formula. In doing so, they considered that the cases of polar curves to be timelike or spacelike, separately.

Erişir et al. studied the one-parameter planar motion in the $p-$complex plane. So, they defined a one-parameter planar motion in the $p-$complex plane by transformation $x' = (x - u)e^{i\theta_p}$ and obtained the relation between the velocities of any point in the moving $p-$complex plane as

$$V_a^p = V_f^p + V_r^p$$  \hspace{1cm} (1.1)

where $V_a^p$, $V_f^p$ and $V_r^p$ are the absolute, the sliding and the relative velocity vectors, respectively, [20]. Moreover, in [21], the generalised Euler-Savary formula giving the relation between the curvatures of the trajectory curves of one-parameter planar motion in the $p-$complex plane was given.

In this study, we have obtained the generalised Bobillier formula by the geometric interpretation of the generalised Euler-Savary formula given in [21] for $p-$complex plane. In addition, we have expressed another presentation of the generalised Bobillier formula without using the Euler-Savary formula. The $p-$complex plane corresponds to complex, Galilean and hyperbolic plane for the special cases of $p = -1, 0, +1$, respectively. Since the Bobillier formulas in Galilean and hyperbolic plane have not existed according to our best knowledges, firstly we obtained them. Then we proved that the generalised Bobillier formula in $p-$complex plane satisfies the complex, Galilean and hyperbolic cases.
2 Preliminaries

The generalised complex numbers or binary numbers are introduced as follows
\[ z = x + iy \quad (x, y \in \mathbb{R}), \quad i^2 = iq + p \quad (q, p \in \mathbb{R}). \]

The double, dual and ordinary numbers are the particular members of two parameter family of complex number systems. Moreover, the generalised complex number systems are isomorphic to the double, ordinary and dual complex numbers when \( p + q^2/4 \) is positive, negative and zero, respectively. [4]

Unless otherwise stated we assume that \( i^2 = p \) and \( q = 0 \) \( (p \in \mathbb{R}) \). This complex number system is denoted by
\[ \mathbb{C}_p = \{ x + iy : x, y \in \mathbb{R}, \quad i^2 = p \}. \]

The set \( \mathbb{C}_p \) is called the \( p \)-complex plane. For \( z_1 = (x_1 + iy_1), z_2 = (x_2 + iy_2) \in \mathbb{C}_p \) the addition, subtraction and product are defined by
\[ z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = x_1 \pm x_2 + i(y_1 \pm y_2). \]

and
\[ MP(z_1, z_2) = (x_1x_2 + py_1y_2) + i(x_1y_2 + x_2y_1), \]
respectively. The product definition yields the ordinary, Study and Clifford products as \( p \) is equal to \(-1, 0 \) and \( 1 \)
\[ M^{-1}(z_1, z_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \quad \text{for} \quad p = -1, \]
\[ M^{0}(z_1, z_2) = (x_1x_2) + i(x_1y_2 + x_2y_1) \quad \text{for} \quad p = 0, \]
\[ M^{1}(z_1, z_2) = (x_1x_2 + y_1y_2) + i(x_1y_2 + x_2y_1) \quad \text{for} \quad p = 1, \]
respectively. [4]. For \( z_1 = (x_1, y_1), \quad z_2 = (x_2, y_2) \in \mathbb{C}_p \), the scalar product is as follows
\[ MP(z_1, \overline{z_2}) = MP(\overline{z_1}, z_2) = x_1x_2 - py_1y_2 \quad (2.1) \]
and the magnitude of \( z = (x, y) \in \mathbb{C}_p \) is given by the non-negative real number
\[ ||z||_p = \sqrt{|MP(z, \overline{z})|} = \sqrt{|x^2 - py^2|} \quad (2.2) \]
where the over bar denotes the complex conjugation. The equalities
\[ |MP(z_1, z_2)| = ||z_1||_p||z_2||_p \cos p\theta_p, \quad (2.3) \]
\[ \|z_1 \wedge_p z_2\|_p = \sqrt{|-p||z_1||_p||z_2||_p \sin p\theta_p} \quad (2.4) \]
are valid in \( p \)-complex plane where \( \theta_p \) is the \( p \)-rotation angle between the vectors \( z_1 \) and \( z_2 \).

In the Euclidean plane, a circle is defined in two ways. The first definition is the geometric locus of points which are at a fixed distance from a point. The second is the set of points from which a segment \( AB \) is seen at a constant directed angle \( \theta_p \). Using these definitions, since a unit circle is the geometric locus of points \( z \) satisfying \( \|z\|_p = 1 \), the unit circles in the \( p \)-complex planes are drawn as in Figure 1.
Figure 1. Unit circles in $C_p$.

Let us interpret Figure 1. If we take $p < 0$, then the unit circle corresponds to unit ellipse of the form $x^2 + |p|y^2 = 1$ and the complex number system $C_p (p < 0)$ corresponds to elliptical complex number system. In this case, if $p = -1$, the unit circle in the $p$-complex plane corresponds to a standard unit circle defined by $x^2 + y^2 = 1$ and the $p$-complex plane corresponds to the complex plane. If we consider $p = 0$, the above circle definitions yield different sets of points in Galilean geometry. According to the first definition, the equation $\|z\|_0^2 = x^2$ is hold and the unit circle is as $x = \pm 1$, (Figure 1) and for the first circle we cannot talk of curvature. According to the second definition, the set of points is called a cycle. Figure 2 illustrates the cycle (infill cycle, red lines) in Galilean plane and this cycle is the (Euclidean) parabola. Therefore, the space $C_0$ is the parabolic complex number system and the $p$-complex plane corresponds to the Galilean plane. Moreover, as can be seen from Figure 1 the parabolic complex plane is divided into two area by the imaginary axis. Finally, when $p > 0$, unit circles in $C_p (p > 0)$ are the hyperbolas of the form $|x^2 - py^2| = 1$ whose asymptotes are $y = \pm x/\sqrt{|p|}$ (red dashed line in Figure 1). Thus, the space $C_p (p > 0)$ corresponds to hyperbolic complex number system. Especially, if we take $p = 1$, the $p$-complex plane is referred to as a well-known hyperbolic plane and the asymptotes of the unit circles separate the hyperbolic planes into four regions.

$p$-trigonometric functions ( $p$-cosine $(\cos p)$, $p$-sine $(\sin p)$ and $p$-tangent $(\tan p)$) are, respectively, defined by

\[
\begin{align*}
\cos p\theta_p &= \begin{cases} 
\cos \left(\theta_p\sqrt{|p|}\right), & p < 0 \\
1, & p = 0 \ (\text{branch I}) \\
\cosh \left(\theta_p\sqrt{|p|}\right), & p > 0 \ (\text{branch I})
\end{cases} \\
\sin p\theta_p &= \begin{cases} 
\frac{1}{\sqrt{|p|}} \sin \left(\theta_p\sqrt{|p|}\right), & p < 0 \\
\theta_p, & p = 0 \ (\text{branch I}) \\
\frac{1}{\sqrt{|p|}} \sinh \left(\theta_p\sqrt{|p|}\right), & p > 0 \ (\text{branch I})
\end{cases} \\
\tan p\theta_p &= \frac{\sin p\theta_p}{\cos p\theta_p}.
\end{align*}
\]
Additionally, the derivatives of $p-$trigonometric functions are as follows,

$$
\frac{d}{d\theta_p}(\cos p\theta_p) = p \sin p\theta_p, \quad \frac{d}{d\theta_p}(\sin p\theta_p) = \cos p\theta_p \quad (2.6)
$$

and the generalised Eulers formula is as follows

$$
e^{i\theta_p} = \cos p\theta_p + i \sin p\theta_p. \quad (2.7)
$$

The polar and exponential forms of any $p-$complex number $z$ are

$$
z = r_p (\cos p\theta_p + i \sin p\theta_p) = r_p e^{i\theta_p}
$$

where $r_p = \|z\|$ and $\theta_p$ are the $p-$magnitude and argument of $z$, respectively, [10].

### 3 The Generalised Euler-Savary Formula in $p$-Complex Plane

Let $T$ and $T'$ be the moving and fixed $p-$complex planes, respectively. $\{O; t_1, t_2\}$ and $\{O'; t'_1, t'_2\}$ are the perpendicular coordinate system of these planes, respectively. Let $N^1_p, N^2_p$ and $N^3_p$ be the fixed points in the moving $p-$complex plane $T$ and $\gamma^1_p, \gamma^2_p$ and $\gamma^3_p$ be centres of curvature of trajectory drawn by these fixed points in the fixed $p-$complex plane $T'$.

The normals of trajectories drawn by these points pass from the instantaneous rotation centre $I$ and there is a $p-$rotation pole at each $t$ moment. The geometric loci of the pole points in the fixed $p-$complex plane $T'$ and the moving $p-$complex plane $T$ are called fixed pole curve ($Q'$) and moving pole curve ($Q$), respectively, [20].

The pole curves roll upon each other without sliding during the motion of $p-$complex planes $T, T'$. So, the pole curves ($Q$), ($Q'$) are tangent to each other and have the same velocity at each $t$ moment, [20]. Thus, the real axis is the common tangent and the imaginary axis is the common normal for the pole curves.

If $\theta_p$ is the $p-$rotation angle of motion of the $p-$complex plane $T$ with respect to $T'$ at each $t$ moment, then each point $N^k_p$ makes a rotation motion with $\dot{\theta}_p$ angular velocity at the instantaneous centre $I$.

Let

$$
X^1_p = \frac{IN^1_p}{\|IN^1_p\|}, \quad X^2_p = \frac{IN^2_p}{\|IN^2_p\|}, \quad X^3_p = \frac{IN^3_p}{\|IN^3_p\|} \quad (3.1)
$$

be the unit vectors in the direction of the pole rays $IN^1_p, IN^2_p$ and $IN^3_p$, respectively, (Figure 2). So the exponential forms of these vectors are given by

$$
X^1_p = i e^{i\theta^1_p}, \quad X^2_p = i e^{i\theta^2_p}, \quad X^3_p = i e^{i\theta^3_p} \quad (3.2)
$$

where $\theta^1_p, \theta^2_p, \theta^3_p$ are $p-$rotation angles of motion in the $p-$complex plane.
The $p$-distances of the points $N_k^p$ and $\gamma_k^p$ from the origin $I$ are $\rho_k$ and $\rho'_k$ ($k = 1, 2, 3$), respectively then from the equations (2.2) and (2.3) we can write

\[ |M^p (IN_1^p, \bar{X}_p^p)| = \rho_1, \quad |M^p (I\gamma_1^p, \bar{X}_p^p)| = \rho'_1. \]  

(3.3)

Similarly

\[ |M^p (IN_2^p, \bar{X}_p^p)| = \rho_2, \quad |M^p (I\gamma_2^p, \bar{X}_p^p)| = \rho'_2 \]

and

\[ |M^p (IN_3^p, \bar{X}_p^p)| = \rho_3, \quad |M^p (I\gamma_3^p, \bar{X}_p^p)| = \rho'_3. \]

(3.4)

An inflection point may be defined to be a point whose trajectory momentarily has an infinite radius of curvature. Such points also have zero acceleration normal to their trajectory [7, 18, 22]. Let the inflection points be $N_1^{*p}$, $N_2^{*p}$ and $N_3^{*p}$, (Figure 2). The locus of such points is a circle in the moving $p$-complex plane $T$ called as an inflection circle. So, from the equation (2.2) and (2.3) we can write

\[ |M^p (IN_1^{*p}, \bar{X}_p^p)| = \rho_1^*, \quad |M^p (IN_2^{*p}, \bar{X}_p^p)| = \rho_2^*, \quad |M^p (IN_3^{*p}, \bar{X}_p^p)| = \rho_3^* \]

(3.4)

for the abscissae of the inflection points.
There is a relationship between $h$ which is diameter of the inflection circle and $\rho^*_1$ (the $p-$distance from the instantaneous centre $I$ to inflection point $N^*_p$). Namely; the sliding velocity vector of the inflection point $N^*_p$ is perpendicular to the vector which connects the centre to this point. Moreover, the sliding velocity vector is perpendicular to the angular velocity vector. So, we can write

$$V_P^f = w_p \rho^*_1 e^{i\theta^*_p}$$

(3.5)

where $w_p = w_p z_p$ is angular velocity vector of motion in the $p-$complex plane and $z_p$ is the unit vector in the direction of the angular velocity vector.

On the other hand the points move around an trajectory whose instantaneous centre is $I$ during the motion $T/T'$ in the $p-$complex plane. So, the amount of displacement of the point $I$ is equal to the product of the diameter $h$ with the amount of angular displacement. Thus, we can write

$$V_P^f = hw_p.$$  

(3.6)

Considering the equations (3.5) and (3.6), the relation between $\rho^*_1$ and $h$ is

$$h = \rho^*_1 e^{i\theta^*_p}.$$  

(3.7)

During one-parameter planar motion $T/T'$ in the $p-$complex plane, the point $N^*_p$ in the moving $p-$complex plane $T$ draws a trajectory which has curvature centre $\gamma^*_p$ in the fixed $p-$complex plane $T'$. Conversely, the point $\gamma^*_p$ in $T'$ draws a trajectory with curvature centre $N^*_p$ in $T$. This relation between the points $N^*_p$ and $\gamma^*_p$ is given by

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho'_1}\right) e^{-i\alpha_p} = p \left(\frac{1}{R'} - \frac{1}{R}\right)$$

(3.8)

where $R$ and $R'$ are the radii of curvature of the pole curves ($Q$) and ($Q'$), [21]. Considering the equation (3.6), (3.7) and (3.8) the following theorem can be given.

**Theorem:** Let $h$ be the diameter of the inflection circle in $p-$complex plane and $w_p$ be the angular velocity of motion in the $p-$complex plane. The first form and second form of the generalised Euler-Savary formula are

$$\frac{1}{R} = p \left(\frac{1}{R'} - \frac{1}{R}\right)$$

(3.9)

and

$$\frac{1}{h} = \frac{w_p}{V_P^f},$$

(3.10)

respectively.

This theorem can be interpreted as the following special cases;
Case 1. We consider \( p = -1 \). In that case we reduce that the generalised Euler-Savary formula given in the equation (3.8) to \( \left( \frac{1}{\rho_1} - \frac{1}{\rho_1'} \right) e^{-i\alpha} = \frac{i}{R} - \frac{j}{W} \)
where \( i \) is the complex imaginary unit satisfying \( i^2 = -1 \). So, the generalised Euler-Savary formula coincides with the Euler-Savary formula in the complex plane, [11]. Moreover, from the equations (3.9) and (3.10) it is seen that the first form and second form are \( i \frac{1}{h} = \frac{1}{R} - \frac{1}{W} \) and \( \frac{1}{h} = \frac{w}{v} \) as given in [16].

Case 2. Let \( p = 0 \). From the equation (3.8), we obtain \( \left( \frac{1}{\rho_1} - \frac{1}{\rho_1'} \right) e^{-i\alpha} = \varepsilon \left( \frac{1}{R} - \frac{1}{W} \right) \), where \( \varepsilon \) is the dual imaginary unit satisfying \( \varepsilon^2 = 0 \) which corresponds to the Euler-Savary formula in Galilean plane, [9]. Moreover, considering the equations (3.9) and (3.10) the first form and second form are \( \frac{1}{h} = \varepsilon \left( \frac{1}{R} - \frac{1}{W} \right) \) and \( \frac{1}{h} = \frac{w}{v} \). In addition, we calculated the first form and second form of Shear motion in Galilean plane on aspect of [9] and we saw that the above equations are the same with calculated equations by us.

Case 3. If we take \( p = 1 \), then we obtain that the Euler-Savary formula is as \( \left( \frac{1}{\rho_1} - \frac{1}{\rho_1'} \right) e^{-i\alpha} = \frac{1}{R} - \frac{1}{W} \) where \( j \) is the hyperbolic imaginary unit satisfying \( j^2 = 1 \). This formula corresponds to the Euler-Savary formula under the planar motion in the hyperbolic plane, [8]. Also, from the equations (3.9) and (3.10), the first form and second form are equal to \( \frac{1}{h} = \varepsilon \left( \frac{1}{R} - \frac{1}{W} \right) \) and \( \frac{1}{h} = \frac{w}{v} \). Similar to special case 2, considering the reference [8], we obtained that the first form and second form for the motion in the hyperbolic plane and proved that they are coincident with the above equations.

4 Generalised Bobillier Formula obtained by Generalised Euler-Savary Formula

We consider that the inflection points \( N^1_p, N^2_p \) and \( N^3_p \) are on the direction of \((I, X^1_p), (I, X^2_p)\) and \((I, X^3_p)\), respectively. So, the images of these inflection points are \( Q^1_p, Q^2_p \) and \( Q^3_p \) where \( \text{IQ}^k_p = \frac{1}{\rho^k_p} X^k_p \), \( 1 \leq k \leq 3 \), (Figure 3,4,5).

Then, from the equation (2.2) and (2.3) we have

\[
\left| M^p \left( \text{IQ}^1_p, \overline{X^1_p} \right) \right| = \frac{1}{\rho_1}, \left| M^p \left( \text{IQ}^2_p, \overline{X^2_p} \right) \right| = \frac{1}{\rho_2}, \left| M^p \left( \text{IQ}^3_p, \overline{X^3_p} \right) \right| = \frac{1}{\rho_3} \quad (4.1)
\]

So, from the equation (3.7) the following equations hold;

\[
\begin{align*}
\text{IQ}^1_p e^{-i\theta^1_p} &= \frac{1}{\rho^1_p} X^1_p e^{-i\theta^1_p} = \frac{1}{h} X^1_p, \\
\text{IQ}^2_p e^{-i\theta^2_p} &= \frac{1}{\rho^2_p} X^2_p e^{-i\theta^2_p} = \frac{1}{h} X^2_p, \\
\text{IQ}^3_p e^{-i\theta^3_p} &= \frac{1}{\rho^3_p} X^3_p e^{-i\theta^3_p} = \frac{1}{h} X^3_p.
\end{align*} \quad (4.2)
\]

The last three equations show that

\[
\left| M^p \left( \text{IQ}^k_p, \overline{X^k_p} \right) \right| e^{-i\theta^k_p} = \left| M^p \left( \text{IQ}^k_p, \overline{X^k_p} \right) \right| e^{-i\theta^k_p} = \left| M^p \left( \text{IQ}^k_p, \overline{X^k_p} \right) \right| e^{-i\theta^k_p} = \frac{1}{h}. \quad (4.3)
\]
So, the set of the points $Q_p$ is a straight line $D$ parallel to axis $x$. Thus, the line $D$ is an image of the inflection circle of the $p-$complex plane by means of an inversion at the rotation centre $I$, (Figure 3, 4, 5).

Since the points $Q_p^1, Q_p^2, Q_p^3$ are linear, the vectors $(IQ_p^1 - IQ_p^2)$ and $(IQ_p^2 - IQ_p^3)$ are linearly dependent. So, the $p-$cross product of these vectors is

\[(IQ_p^1 - IQ_p^2) \wedge_p (IQ_p^2 - IQ_p^3) = 0.\]

Thus, we can write

\[(IQ_p^1 \wedge_p IQ_p^2) + (IQ_p^3 \wedge_p IQ_p^1) + (IQ_p^2 \wedge_p IQ_p^3) = 0.\]

If we consider that $\rho^*_1 \rho^*_2 \rho^*_3 \neq 0$, since $IQ_p^1 = \frac{1}{\rho^*_1}ie^{-i\theta^*_1}$, $IQ_p^2 = \frac{1}{\rho^*_2}ie^{-i\theta^*_2}$ and $IQ_p^3 = \frac{1}{\rho^*_3}ie^{-i\theta^*_3}$ we can easily write

\[\rho^*_1 \left(ie^{-i\theta^*_1} \wedge_p ie^{-i\theta^*_2}\right) + \rho^*_2 \left(ie^{-i\theta^*_2} \wedge_p ie^{-i\theta^*_3}\right) + \rho^*_3 \left(ie^{-i\theta^*_3} \wedge_p ie^{-i\theta^*_1}\right) = 0.\]

Considering the last equation and the equation (2.4) the following theorem can be given.

**Theorem:** In one parameter motion $T/T'$ in the $p-$complex plane the relation between the centres of curvatures is

\[\rho^*_k \sin p\theta^{23} + \rho^*_l \sin p\theta^{31} + \rho^*_m \sin p\theta^{12} = 0\] \hspace{1cm} (4.4)

where $\frac{1}{\rho^*_k} = \frac{1}{\rho^*_k} - \frac{1}{\rho^*_k}$ and $\theta^{lm}$ are the $p-$rotation angles between $X^l_p$ and $X^m_p$ for the permutations of the indices $k, l, m = 1, 2, 3 : 2, 3, 1 ; 3, 1, 2$.

The formula (4.4) is called the generalised Bobillier formula which is generalisation the generalised Euler-Savary formula for all $p \in \mathbb{R}$. The special cases of the generalised Bobillier formula with respect to the sign of real number $p$ are as follows;

**Case 1.** If we take $p < 0$, the $p-$complex number system $\mathbb{C}_p$ corresponds to the elliptical complex number system. From the equation (6), since $\sin p\theta_p = \frac{1}{|p|} \sin \left(\theta_p \sqrt{|p|}\right)$ the generalised Bobillier formula is equal to

\[\rho^*_k \sin p \left(\theta^{23}_p \sqrt{|p|}\right) + \rho^*_l \sin p \left(\theta^{31}_p \sqrt{|p|}\right) + \rho^*_m \sin p \left(\theta^{12}_p \sqrt{|p|}\right) = 0.\]

Especially, if we take $p = -1$, this formula becomes

\[\rho^*_k \sin \theta^{23} + \rho^*_l \sin \theta^{31} + \rho^*_m \sin \theta^{12} = 0.\]

Under the circumstances, the generalised Bobillier formula is reduced to the Bobillier formula at the complex plane. Figure 3 represents the illustration of the Bobillier construction in the complex plane. [16].
Case 2. If we consider $p = 0$, the $p$–complex number system $\mathbb{C}_p$ is equal to the parabolic complex number system. By considering the equation (2.5) if we substitute $\sin p \theta_p = \theta_p$ into the equation (4.4), we get

$$\rho_1^p \theta_2^3 + \rho_2^p \theta_3^1 + \rho_3^p \theta_1^2 = 0.$$ 

This formula is the Bobillier formula for Shear motion in Galilean plane. The inflection cycle for Shear motion is illustrated as

Case 3. When $p > 0$, the $p$–complex number system $\mathbb{C}_p$ is referred to as the hyperbolic complex number system. In addition, considering the equation
(2.5), we have \( p \theta_p = \frac{1}{\sqrt{p}} \sin \left( \theta_p \sqrt{p} \right) \). Thus, we obtain that the generalised Bobillier formula is reduced to

\[
\rho^*_1 \sinh \left( \theta_p^{23} \sqrt{p} \right) + \rho^*_2 \sinh \left( \theta_p^{31} \sqrt{p} \right) + \rho^*_3 \sinh \left( \theta_p^{12} \sqrt{p} \right) = 0.
\]

Especially, if we take \( p = 1 \), we obtain that the Bobillier formula as

\[
\rho^*_1 \sin \theta_p^{23} + \rho^*_2 \sin \theta_p^{31} + \rho^*_3 \sin \theta_p^{12} = 0.
\]

Thus, this formula is equal to the Bobillier formula at the hyperbolic plane. The figure of the (hyperbolic) inflection circle for \( p = 1 \) is as follows

![Figure 5. Inflection Circle in Hyperbolic Plane.](image)

5 An Alternative Way for the Generalised Bobillier Formula

In this section, we will obtain the generalised Bobillier formula in the \( p \)-complex plane without the use of the generalised Euler-Savary formula. So, the generalised Bobillier formula will be obtained considering the velocities and accelerations of planar motion in the \( p \)-complex plane.

Firstly, we calculate the trajectory velocities and accelerations of the points in the moving \( p \)-complex plane \( T \). Let \( V_p^a \left( N^1_p \right) \) and \( J_p^a \left( N^1_p \right) \) be absolute velocity and acceleration vectors of the point \( N^1_p \), respectively. Moreover, if \( \omega_p \) is the angular velocity of motion \( T/T' \) in the \( p \)-complex plane, \( \omega_p = \frac{d \theta_p^1}{dt} \) where \( \theta_p^1 \) is \( p \)-rotation angle. So, considering the equation (3.5) the sliding velocity vector of the point \( N^1_p \) is

\[
V_p^f \left( N^1_p \right) = w_p \rho_1 e^{i \theta_p^1}. \tag{5.1}
\]
In addition, from the equation (1.1) there is the relation between the velocity vectors
\[ \mathbf{V}_p^a (N^1_p) = \mathbf{V}_p^r (I_p) + \mathbf{V}_p^f (N^1_p) \]  
(5.2)
where $\mathbf{V}_p^a$, $\mathbf{V}_p^f$ and $\mathbf{V}_p^r$ are the absolute, sliding and relative velocity vectors, respectively. If we consider the equations (5.1) and (5.2) it is obtained that
\[ \mathbf{V}_p^a (N^1_p) = \mathbf{V}_p^r (I_p) + w_p \rho_1 e^{i \theta_1^p}. \]  
(5.3)
Differentiating the equation (5.3) with respect to $t$, we get
\[ \mathbf{J}_p^a (N^1_p) \wedge_p \mathbf{J}_p^a (N^1_p) = \mathbf{J}_p^r (I_p) + \rho_1 \dot{w}_p e^{i \theta_1^p} + \rho_1 w_p^2 i e^{i \theta_1^p}. \]  
(5.4)
where the first term is the path wise tangential acceleration component, the second term is the centripetal component and the third term can be shown to be a pure imaginary component. Considering this analysis, the absolute velocity vector of the inflection point is linearly dependent with the absolute acceleration vector of the inflection point since the normal component of acceleration is zero. So we can write
\[ \mathbf{V}_p^a (N^1_p) \wedge_p \mathbf{J}_p^a (N^1_p) = 0. \]  
(5.5)
Here considering $\mathbf{V}_p^r = 0$ for the inflection point $N^1_p$ and using the equations (5.3) and (5.4) we can easily find
\[ w_p \rho_1 \left( e^{i \theta_1^p} \wedge_p \mathbf{J}_p^r (I_p) \right) + \rho_1 \dot{w}_p \left( e^{i \theta_1^p} \wedge_p e^{i \theta_1^p} \right) + \rho_1 w_p^2 \left( e^{i \theta_1^p} \wedge_p e^{i \theta_1^p} \right) = 0 \]
If we consider the equation (2.4) and make necessary arrangements we obtain that
\[ w_p \rho_1 \left( e^{i \theta_1^p} \wedge_p \mathbf{J}_p^r (I_p) \right) + \rho_1 w_p^2 \dot{z}_p = 0 \]
and finally it is said that
\[ \rho_1 \dot{z}_p = -\frac{e^{i \theta_1^p} \wedge_p \mathbf{J}_p^r (I_p)}{w_p^2}. \]  
(5.6)
With similar process for the inflection points $N^2_p$ and $N^3_p$ the following equations hold:
\[ \rho_2 \dot{z}_p = -\frac{e^{i \theta_2^p} \wedge_p \mathbf{J}_p^r (I_p)}{w_p^2} \]  
(5.7)
and
\[ \rho_3 \dot{z}_p = -\frac{e^{i \theta_3^p} \wedge_p \mathbf{J}_p^r (I_p)}{w_p^2}. \]  
(5.8)
On the other hand the linear connection between $X_p^1$, $X_p^2$ and $X_p^3$ may be written as follows
\[ \lambda_1 e^{i \theta_1^p} + \lambda_2 e^{i \theta_2^p} + \lambda_3 e^{i \theta_3^p} = 0 \]
where $\lambda_1, \lambda_3, \lambda_3 \in \mathbb{R}$. From the last equation by successive $p-$cross product with $e^{i\theta_1^p}$, $e^{i\theta_2^p}$ and $e^{i\theta_3^p}$, we get

$$
\lambda_1 = \sin p\theta_{23}^p, \quad \lambda_2 = \sin p\theta_{31}^p, \quad \lambda_3 = \sin p\theta_{12}^p
$$

where $\theta_{23}^p = \theta_3^p - \theta_2^p$, $\theta_{31}^p = \theta_1^p - \theta_3^p$ and $\theta_{12}^p = \theta_2^p - \theta_1^p$. So, we obtain that

$$
\sin p\theta_{23}^p e^{i\theta_1^p} + \sin p\theta_{31}^p e^{i\theta_2^p} + \sin p\theta_{12}^p e^{i\theta_3^p} = 0
$$

Finally, if the equations (5.6), (5.7) and (5.8) are substituted into the last equation, the following formula is hold;

$$
\rho_1^s \sin p\theta_{23}^p + \rho_2^s \sin p\theta_{31}^p + \rho_3^s \sin p\theta_{12}^p = 0.
$$

(5.9)

This is the formula given in (4.4). So, this direct way gives us the generalised Bobillier formula without using the generalised Euler-Savary formula in the $p-$complex plane.

6 Conclusion

The angle between the tangent of pole curve at the instantaneous pole centre of coupler with respect to the base of a four bar linkage and of the cranks is equal to the angle between the other crank and the collineation axis. This expression is can be verified by the Bobilliers construction by graphically. This has been a major interest to physicists, mathematicians and engineers. Also this problem can be solved by an analytical method called the Bobillier formula which is more practical to use. Various geometric and analytical methods have been developed for the Bobillier formula of Euclidean and Lorentzian planar motion. In literature, the Bobillier formula in Galilean and hyperbolic planes has not been obtained yet. Then the following questions can be asked. "Can be this formula obtained in Galilean and hyperbolic planes?" and "Is it possible to give a generalised formula for the Bobillier construction in a $p-$complex plane including all planes?" Thus, we find out the Bobillier formula for $p = 0$ (Galilean) and $p = 1$ (hyperbolic). Moreover, since the $p-$complex plane correspond to complex, Galilean and hyperbolic planes for the special cases of $p (-1, 0, +1)$, respectively, we obtain that the generalised Bobillier formula in $p-$complex plane. We give the special cases of the generalised Bobillier formula for the cases of $p$. So, we think that this study would be important to the sciences of mathematics, engineering and astronomy. There is no doubt that this study will add innovation to science.
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