Abstract

We introduce a family of parameterised counting problems on graphs, \( p\text{-}\#\text{INDUCED SUBGRAPH WITH PROPERTY}(\Phi) \), which generalises a number of problems which have previously been studied. This paper focusses on the case in which \( \Phi \) defines a family of graphs whose edge-minimal elements all have bounded treewidth; this includes the special case in which \( \Phi \) describes the property of being connected. We show that exactly counting the number of connected induced \( k \)-vertex subgraphs in an \( n \)-vertex graph is \#W[1]-hard, but on the other hand there exists an FPTRAS for the problem; more generally, we show that there exists an FPTRAS for \( p\text{-}\#\text{INDUCED SUBGRAPH WITH PROPERTY}(\Phi) \) whenever \( \Phi \) is monotone and all the minimal graphs satisfying \( \Phi \) have bounded treewidth. We then apply these results to a counting version of the \textsc{Graph Motif} problem.

1 Introduction

Parameterised counting problems were introduced by Flum and Grohe in \cite{FlumGrohe10}. In this paper we focus on problems of the following form:

*Research supported by EPSRC grant “Computational Counting”*
**Input:** An $n$-vertex graph $G = (V, E)$, and $k \in \mathbb{N}$.

**Parameter:** $k$.

**Question:** How many (labelled) $k$-vertex subsets of $V$ induce graphs with a given property?

It should be noted that, while the statement of this problem is concerned with induced subgraphs, it also encompasses problems more often formulated in terms of counting subgraphs that are not necessarily induced. For example, to count the number of $k$-vertex paths in $G$ (not necessarily induced), we would consider the labelled subgraph induced by $v_1, \ldots, v_k$ to have the desired property if and only if $v_iv_{i+1}$ is an edge for $1 \leq i \leq k - 1$, regardless of what other edges may be present.

Many problems of this form are known to be #W[1]-hard (see Section 1.2 for definitions of concepts from parameterised complexity), and thus are unlikely to be solvable exactly in time $f(k) n^{O(1)}$ for any function $f$; examples of #W[1]-hard problems of this form include counting the number of $k$-vertex cliques ($p$-#CLIQUE $[10]$), paths ($p$-#PATH $[10]$), cycles ($p$-#CYCLE $[10]$) and matchings ($p$-#MATCHING $[3]$).

A natural question, therefore, is whether such counting problems can be efficiently approximated. It is shown in $[2]$ that there exists an efficient approximation scheme for $p$-#EMB($\mathcal{H}$) whenever $\mathcal{H} = \{H_1, H_2, \ldots\}$ (where $|H_i| = i$) is a class of graphs of bounded treewidth; informally, this is the problem of counting embeddings of graphs from $\mathcal{H}$ in an instance graph.

In Section 1.3.1 below, we introduce a family of parameterised counting problems which includes all the specific problems discussed above. This family also includes the problem of counting the number of $k$-vertex connected induced subgraphs, a problem which we show to be #W[1]-hard in Section 2. In Section 3 we give a positive approximation result, showing that there exists an FPTRAS for the more general problem of counting the number of (labelled) $k$-vertex subsets of a graph satisfying a monotone property $\Phi$, provided that the edge-minimal graphs satisfying $\Phi$ all have bounded treewidth. Examples of problems in this class for which there exists an FPTRAS include those of counting the number of $k$-vertex induced subgraphs that are connected, the number of $k$-vertex induced subgraphs that are Hamiltonian, and the number of $k$-vertex induced subgraphs that are not bipartite. This last example contrasts with the result of Khot and Raman $[15]$ that deciding whether a graph contains an induced $k$-vertex subgraph that is bipartite is W[1]-hard.
Finally, in Section 4, we apply some of these results to a counting version of the problem \textsc{Graph Motif}, introduced by Lacroix, Fernandes and Sagot \cite{16} in the context of metabolic networks. The problem takes as input an \(n\)-vertex coloured graph, together with a motif or multiset of colours \(M\), and a solution is a subset \(U\) of \(|M|\) vertices such that the subgraph induced by \(U\) is connected and the colour-(multi)set of \(U\) is exactly \(M\). A counting version of this problem was studied by Guillemot and Sikora \cite{12}; we define and analyse a different natural counting version of \textsc{Graph Motif}, which is a more direct translation of the standard decision version into the counting world.

In the remainder of this section, we first introduce some notation in Section 1.1, then introduce some key concepts in the study of parameterised counting complexity in Section 1.2, before giving formal definitions of the problems we consider in Section 1.3.

1.1 Notation

Given a graph \(G = (V, E)\), and a subset \(U \subset V\), we write \(G[U]\) for the subgraph of \(G\) induced by the vertices of \(U\). If \(v \in V\), then \(\Gamma(v)\) denotes the set of neighbours of \(v\) in \(G\). For any \(k \in \mathbb{N}\), we write \([k]\) as shorthand for \(\{1, \ldots, k\}\), and \(V^{(k)}\) for the set of all subsets of \(V\) of size exactly \(k\). If \(G\) is coloured by some colouring \(\omega : V \to [k]\), we say that a subset \(U \subset V\) is \emph{colourful} (under \(\omega\)) if, for every \(i \in [k]\), there exists a unique vertex \(u \in U\) such that \(\omega(u) = i\); note that can only be achieved if \(U \in V^{(k)}\). We write \(\omega|_U\) for the restriction of \(\omega\) to the set \(U\); if \(U\) is colourful under \(\omega\) then \(\omega|_U\) is a bijection.

We say that \((T, \mathcal{D})\) is a \textit{tree decomposition} of \(G\) if \(T\) is a tree and \(\mathcal{D} = \{D(t) : t \in V(T)\}\) is a collection of non-empty subsets of \(V(G)\) (or \textit{bags}), indexed by the nodes of \(T\), satisfying:

1. \(V(G) = \bigcup_{t \in V(T)} D(t)\),
2. for every \(e = uv \in E(G)\), there exists \(t \in V(T)\) such that \(u, v \in D(t)\),
3. for every \(v \in V(G)\), if \(T(v)\) is defined to be the subgraph of \(T\) induced by nodes \(t\) with \(v \in D(t)\), then \(T(v)\) is connected.

The width of the tree decomposition \((T, \mathcal{D})\) is defined to be \(\max_{t \in V(T)} |D(t)| - 1\), and the treewidth of \(G\), written \(\text{tw}(G)\), is the minimum width over all tree decompositions of \(G\).
Finally, a boolean formula $\phi : \{0,1\}^m \to \{0,1\}$ is monotone if, whenever $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \{0,1\}^m$ satisfy $x_i \leq y_i$ for all $1 \leq i \leq m$, 

$$\phi(x) = 1 \implies \phi(y) = 1.$$  

### 1.2 Parameterised counting complexity

In this section, we introduce key notions from parameterised counting complexity, which we will use in the rest of the paper.

To understand the complexity of parameterised counting problems, Flum and Grohe [10] introduce two kinds of reductions between such problems.

**Definition.** Let $(\Pi, \kappa)$ and $(\Pi', \kappa')$ be parameterised counting problems.

1. An fpt parsimonious reduction from $(\Pi, \kappa)$ to $(\Pi', \kappa')$ is an algorithm that computes, for every instance $I$ of $\Pi$, an instance $I'$ of $\Pi'$ in time $f(k) \cdot |I|^c$ such that $\kappa'(I') \leq g(\kappa(I))$ and 

$$\Pi(I) = \Pi'(I')$$

(for computable functions $f, g : \mathbb{N} \to \mathbb{N}$ and a constant $c \in \mathbb{N}$). In this case we write $(\Pi, \kappa) \leq_{\text{fpt pars}}^{\text{pars}} (\Pi', \kappa').$

2. An fpt Turing reduction from $(\Pi, \kappa)$ to $(\Pi', \kappa')$ is an algorithm $A$ with an oracle to $\Pi'$ such that 

(a) $A$ computes $\Pi$,

(b) $A$ is an fpt-algorithm with respect to $\kappa$, and

(c) there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for all oracle queries “$\Pi'(y) =?\)” posed by $A$ on input $x$ we have $\kappa'(y) \leq g(\kappa(x))$.

In this case we write $(\Pi, \kappa) \leq_{\text{fpt}}^{\text{T}} (\Pi', \kappa').$

Using these notions, Flum and Grohe introduce a hierarchy of parameterised counting complexity classes, $\#W[t]$, for $t \geq 1$; this is the analogue of the W-hierarchy for parameterised decision problems. In order to define this hierarchy, we need some more notions related to satisfiability problems.

The definition of levels of the hierarchy uses the following problem, where $\psi$ is a first-order formula with a free relation variable of arity $s$. 
If $\Psi$ is a class of first-order formulas, then $p$-$\#$-WD-$\Psi$ is the class of all problems $p$-$\#$-WD-$\psi$ where $\psi \in \Psi$. The classes of first-order formulas $\Sigma_t$ and $\Pi_t$, for $t \geq 0$, are defined inductively. Both $\Sigma_0$ and $\Pi_0$ denote the class of quantifier-free formulas, while, for $t \geq 1$, $\Sigma_t$ is the class of formulas

$$\exists x_1 \ldots \exists x_k \psi,$$

where $\psi \in \Pi_{t-1}$, and $\Pi_t$ is the class of formulas

$$\forall x_1 \ldots \forall x_k \psi,$$

where $\psi \in \Sigma_{t-1}$. We are now ready to define the classes $\#W[t]$, for $t \geq 1$.

**Definition ([10] [11]).** For $t \geq 1$, $\#W[t]$ is the class of all parameterised counting problems that are fpt parsimonious reducible to $p$-$\#$-WD-$\Pi_t$.

Just as it is considered to be very unlikely that $W[1] = \text{FPT}$, it is very unlikely that there exists an algorithm running in time $f(k)n^{O(1)}$ for any problem that is hard for the class $\#W[1]$; hardness of a problem can be shown using either of the forms of reductions defined above.

In [11], Flum and Grohe also define a counting version of the A-hierarchy for parameterised problems; this turns out to be easier to use for some of our purposes. The definition is in terms of the following model-checking problem, where $C$ is a class of structures and $\Psi$ a class of formulas.

$p$-$\#$-MC($C, \Psi$)

*Input:* A structure $A \in C$ and a formula $\psi \in \Psi$.

*Parameter:* $|\psi|$.

*Question:* What is $|\psi(A)|$?

Here, $\psi(A)$ is the set of tuples $(a_1, \ldots, a_k) \in A^k$ such that $\psi(a_1, \ldots, a_k)$ is true in $A$, where $k$ is the number of free variables in $\psi$, and $A$ the universe of $A$. If $C$ is the class of all structures, we write simply $p$-$\#$-MC($\Psi$). The counting analogue of the A-hierarchy is then defined as follows.
Definition (I). For all $t \geq 1$, $\#A[t]$ is the class of all parameterised counting problems reducible to $p\text{-}MC(\Pi_{t-1})$ by an fpt parsimonious reduction.

It is known that the first levels of these two hierarchies for parameterised counting problems coincide:

Theorem 1.1 (I). $\#W[1] = \#A[1]$.

Thus, to prove that a problem belongs to $\#W[1]$ ($= \#A[1]$) it suffices to show that it is reducible, under fpt parsimonious reductions, to $p\text{-}MC(\Pi_0)$.

When considering approximation algorithms for parameterised counting problems, an “efficient” approximation scheme is an FPTRAS, as introduced by Arvind and Raman [2]; this is the analogue of a FPRAS (fully polynomial randomised approximation scheme) in the parameterised setting.

Definition. An FPTRAS for a parameterised counting problem $\Pi$ with parameter $k$ is a randomised approximation scheme that takes an instance $I$ of $\Pi$ (with $|I| = n$), and real numbers $\epsilon > 0$ and $0 < \delta < 1$, and in time $f(k) \cdot g(n, 1/\epsilon, \log(1/\delta))$ (where $f$ is any computable function, and $g$ is a polynomial in $n$, $1/\epsilon$ and $\log(1/\delta)$) outputs a rational number $z$ such that

$$\mathbb{P}[(1 - \epsilon)\Pi(I) \leq z \leq (1 + \epsilon)\Pi(I)] \geq 1 - \delta.$$ 

1.3 Problems considered

In this section we begin by introducing a general family of parameterised counting problems on graphs, in which the goal is to count $k$-tuples of vertices that induce subgraphs with particular properties. We then give formal definitions of the problems we will consider in Sections 2 and 3. Our model can be regarded as a generalisation of many problems that involve counting labelled subgraphs, including $p\text{-}\#\text{Cycle}$ [10], $p\text{-}\#\text{Path}$ [10], $p\text{-}\#\text{STREMB}(C)$ [4] and $\#k\text{-}\text{MATCHING}$ [5]. Induced subgraph problems that are invariant under relabelling of vertices have also been studied in the literature on parameterised counting, including $p\text{-}\#\text{CLIQUE}$ [10]; such problems can also be considered as instances of our model subject to appropriate rescaling of the result.

1.3.1 The model

Let $\Phi$ be a family $(\phi_1, \phi_2, \ldots)$ of functions $\phi_k : \{0, 1\}^{(k)} \to \{0, 1\}$, such that the function mapping $k \mapsto \phi_k$ is computable. Let $(i_1, \ldots, i_{(k)})$ be a fixed
ordering of all pairs in $[k]^{(2)}$. For any labelled graph, that is a graph $H$ on $k$ vertices together with a bijective labelling of its vertices $\pi : [k] \to V(H)$, we define

$$\phi_k(H, \pi) = \phi_k(e_{i_1}^{H, \pi}, e_{i_2}^{H, \pi}, \ldots, e_{(2)}^{H, \pi}),$$

where $e_{ij}^{H, \pi} \in \{0, 1\}$ and $e_{ij}^{H, \pi} = 1$ if and only if $\pi(j) \pi(l) \in E(H)$. Given a graph $G = (V, E)$ and a $k$-tuple of vertices $(v_1, \ldots, v_k)$, let $G[v_1, \ldots, v_k]$ denote the pair $(H, \pi)$ where $H = G[\{v_1, \ldots, v_k\}]$ and $\pi(i) = v_i$ for each $i \in [k]$. We write $(H, \pi) \subseteq (H', \pi')$ if, for all $e = uv \in E(H)$, $\pi'(\pi^{-1}(u))\pi'(\pi^{-1}(v)) \in E(H')$.

We then define the following problem.

\textbf{$p$-\#INDUCED SUBGRAPH WITH PROPERTY($\Phi$)}

\textbf{Input:} A graph $G = (V, E)$ and $k \in \mathbb{N}$.

\textbf{Parameter:} $k$.

\textbf{Question:} What is the cardinality of the set \{(v_1, \ldots, v_k) \in V^k : \phi_k(G[v_1, \ldots, v_k]) = 1\}?

We say that $\Phi$ is a \textit{symmetric} property if the value of $\phi_k(H, \pi)$ depends only on the graph $H$ and not on the labelling of the vertices; this corresponds to “unlabelled” graph problems, such as $p$-\#CLIQUE. We can define a related problem for symmetric properties:

\textbf{$p$-\#INDUCED UNLABELLED SUBGRAPH WITH PROPERTY($\Phi$)}

\textbf{Input:} A graph $G = (V, E)$ and $k \in \mathbb{N}$.

\textbf{Parameter:} $k$.

\textbf{Question:} What is the cardinality of the set \{\{v_1, \ldots, v_k\} \in V^{(k)} : \phi_k(G[v_1, \ldots, v_k]) = 1\}?

For any symmetric property $\Phi$, the output of $p$-\#INDUCED SUBGRAPH WITH PROPERTY($\Phi$) is exactly $k!$ times the output of $p$-\#INDUCED UNLABELLED SUBGRAPH WITH PROPERTY($\Phi$).

Note that, for any $\phi_k \in \Phi$, the problem of determining the cardinality of the set \{(v_1, \ldots, v_k) \in V^k : \phi_k(G[v_1, \ldots, v_k]) = 1\} can easily be expressed as an instance of $p$-\#MC($\Pi_0$); thus, by Theorem 1.1 we obtain the following result:

\textbf{Proposition 1.2.} For any $\Phi$, the problem $p$-\#INDUCED SUBGRAPH WITH
Property (Φ) belongs to $\#W[1]$. If Φ is symmetric, then the same is true for $p\text{-}\#\text{Induced Unlabelled Subgraph With Property}(\Phi)$.

In order to give an fpt parsimonious reduction from $p\text{-}\#\text{Induced Unlabelled Subgraph With Property}(\Phi)$ to $p\text{-}\#\text{WD-Π}_0$, we can introduce an additional relation in our structure which imposes an order on the elements, and thus ensure that each unlabelled subset is counted exactly once.

### 1.3.2 Problem definitions

In Section 2 we consider the following problem.

$p\text{-}\#\text{Connected Induced Subgraph}$

*Input:* A graph $G = (V,E)$ and $k \in \mathbb{N}$.

*Parameter:* $k$.

*Question:* For how many subsets $U \in V^{(k)}$ is $G[U]$ connected?

This problem can be regarded, subject to rescaling, as a particular case of the general problem $p\text{-}\#\text{Induced Subgraph With Property}(\Phi)$ introduced above. Let $\mathcal{T}_k$ be the set of all trees on $k$ vertices with vertices labelled $1,\ldots,k$, and then set $\Phi_{\text{conn}} = (\phi_{\text{conn}}^1, \phi_{\text{conn}}^2, \ldots)$ where

$$
\phi_{\text{conn}}^k(e_{i_1}, \ldots, e_{i_{\binom{k}{2}}}) = \bigvee_{T \in \mathcal{T}_k} \bigwedge_{\{j,l\} \in E(T)} e_{\{j,l\}}.
$$

Then those tuples $(v_1, \ldots, v_k) \in V^k$ such that $\phi_k(G[v_1, \ldots, v_k]) = 1$ are exactly the tuples such that $G[\{v_1, \ldots, v_k\}]$ is connected. Note that in this case $\phi_k$ is invariant under relabelling of the vertices, so we then need to divide the output of $p\text{-}\#\text{Induced Subgraph With Property}(\Phi_{\text{conn}})$ by $k!$ to obtain the correct value for $p\text{-}\#\text{Connected Induced Subgraph}$.

In Section 3 we consider the more general problem of solving $p\text{-}\#\text{Induced Subgraph With Property}(\Phi)$, whenever $\Phi = (\phi_1, \phi_2, \ldots)$ is a monotone property and there exists a positive integer $t$ such that, for each $\phi_k$, all edge-minimal labelled $k$-vertex graphs $(H,\pi)$ such that $\phi_k(H,\pi) = 1$ satisfy $\text{tw}(H) \leq t$. Note that $p\text{-}\#\text{Connected Induced Subgraph}$ is a special case of this more general problem: the set of edge-minimal labelled $k$-vertex graphs $(H,\pi)$ such that $\phi_k^{\text{conn}}(H,\pi) = 1$ is in fact precisely the set of labelled trees on $k$ vertices.
2 \textbf{p-\#Connected Induced Subgraph is \#W[1]-complete}

In this section, we prove the following result.

\textbf{Theorem 2.1.} p-\#Connected Induced Subgraph is \#W[1]-complete under fpt Turing reductions.

We begin in Section 2.1 by noting some background results we will need for the proof, before demonstrating \#W[1]-hardness with a series of fpt Turing reductions in Section 2.2.

\subsection*{2.1 Lattices and M"obius functions}

In Section 2.2 we will need to consider the lattice formed by partitionsof a \( k \)-element set, with a partial order given by the refinement relation: the partition \( P' \) refines the partition \( P \) if every block of \( P' \) is contained in some block of \( P \). In this section we note some existing results about lattices on arbitrary posets and also more specifically about partition lattices, which we will use in the proof of Lemma 2.5.

In any lattice, we will denote by \( \hat{1} \) and \( \hat{0} \) the top and bottom elements of the lattice respectively. Given two elements \( x \) and \( y \) of a lattice, the \textit{meet} of \( x \) and \( y \), \( x \land y \), is defined to be the unique element \( z \) of the lattice such that

1. \( z \leq x \) and \( z \leq y \), and
2. for any \( w \) such that \( w \leq x \) and \( w \leq y \), we have \( w \leq z \).

We will also need to make use of the M"obius function \( \mu \) on a poset, which is defined inductively by

\[
\mu(x, y) = \begin{cases} 
1 & \text{if } x = y \\
- \sum_{z : x \leq z < y} \mu(x, z) & \text{for } x < y \\
0 & \text{otherwise}.
\end{cases}
\]

In Lemma 2.5, we will consider a so-called meet-matrix on a partition lattice, and make use of the following lemma (which follows immediately from a result of Haukkanen \[13, \text{Cor. 2}\]).
Lemma 2.2. Let $x_1, \ldots, x_n$ be the elements of a finite lattice $(P, \leq)$, let $f : P \to \mathbb{C}$ be a function, and let $A = (a_{ij})_{1 \leq i, j \leq n}$ be the matrix given by $a_{ij} = f(x_i \land x_j)$. Then

$$\det(A) = \prod_{i=1}^{n} \sum_{x_k \leq x_i} f(x_k) \mu(x_k, x_i),$$

where $\mu$ is the Möbius function for $P$.

To make use of this result in Section 2.2, we will need to be able to calculate certain values of the Möbius function for the special case in which $P$ is the partition lattice; in fact, due to our choice of $f$ below, it will suffice to be able to compute $\mu(x, \hat{1})$ for each partition $x$. To do this, we will use the following lemma, which is an immediate consequence of a result of Rota [17, Section 4, Prop. 3].

Lemma 2.3. Let $L_n$ be the lattice of partitions of a set with $n$ elements, where $\pi \leq \sigma$ if and only if $\pi$ refines $\sigma$. If $\pi \in L_n$ is of rank $r$, then

$$\mu(\pi, \hat{1}) = (-1)^{n-r-1}(n-r-1)!$$

where the rank of $\pi$ is equal to $n$ minus the number of blocks of $\pi$.

Since the rank of any element lies in the range $[0, n-1]$, it follows that $\mu(\pi, \hat{1})$ is non-zero for all $\pi \in L_n$.

In Lemma 2.4 below, we will in fact be considering the lattice of partitions of a set with the dual order, i.e. $\pi \leq \sigma$ if and only if $\sigma$ refines $\pi$. To make use of Lemma 2.3 in this situation, we will need another result from [17].

Lemma 2.4 ([17, Section 3, Prop. 3]). Let $P^*$ be the partially ordered set obtained by inverting the order of a locally finite partially ordered set $P$, and let $\mu^*$ and $\mu$ be the Möbius functions of $P^*$ and $P$. Then $\mu^*(x, y) = \mu(y, x)$.

We finish this section with one further piece of notation: in any lattice, we will denote by $\hat{1}$ and $\hat{0}$ the top and bottom elements of the lattice respectively.

2.2 The reduction

In this section, we prove Theorem 2.1. The main work in this proof is to give an fpt Turing reduction from $p$-#$\text{MULTICOLOUR INDEPENDENT SET}$
to \texttt{p-\#Multicolour Connected Induced Subgraph}, where these two problems are defined as follows.

\textbf{p-\#Multicolour Independent Set}

\textit{Input:} A \( k \)-coloured graph \( G = (V, E) \) and an integer \( k \).

\textit{Parameter:} \( k \).

\textit{Question:} For how many colourful subsets \( U \in V^{(k)} \) is \( U \) an independent set in \( G \)?

\textbf{p-\#Multicolour Connected Induced Subgraph}

\textit{Input:} A \( k \)-coloured graph \( G = (V, E) \) and an integer \( k \).

\textit{Parameter:} \( k \).

\textit{Question:} For how many colourful subsets \( U \in V^{(k)} \) is \( G[U] \) connected?

In this reduction we will set up a system of equations, argue that, with an oracle to \texttt{p-\#Multicolour Connected Induced Subgraph}, we can compute the entries, and show that the system can be solved to give the number of colourful independent sets in our graph. Throughout, we will need to switch between considering colourful subsets of vertices and partitions of \([k]\). Let \( \mathcal{P}_k \) be the set of all partitions of the set \([k]\); thus the cardinality of \( \mathcal{P}_k \) is precisely the \( k \)th \textit{Bell number}, \( B_k \). We consider these partitions to be partially ordered by the refinement relation, so \( P_i \leq P_j \) if \( P_j \) refines \( P_i \) (note that \( \mathcal{P}_k \) gives the dual lattice of \( L_k \), as defined in Lemma 2.3). Set \( P^1 = 0 = \{[k]\} \) and \( P_{B_k} = 1 = \{\{1\}, \ldots, \{k\}\} \).

Suppose that \( G = (V, E) \) is the \( k \)-coloured graph in an instance of \texttt{p-\#Multicolour Independent Set}. Given a multicolour subset \( U \in V^{(k)} \), we set \( P(U) \) to be the partition of \([k]\) in which \( i, j \in [k] \) belong to the same set of the partition if and only if the vertices of \( U \) with colours \( i \) and \( j \) belong to the same connected component in \( G[U] \).

We define a function \( f : \mathcal{P}_k \to \{0, 1\} \) such that, for any partition \( P \in \mathcal{P}_k \),

\[
    f(P) = \begin{cases} 
    1 & \text{if } P = \{[k]\} \\
    0 & \text{otherwise}.
    \end{cases}
\]

In the following lemma we set up our system of equations, and use results from Section 2.1 to demonstrate that the system can be solved to determine the number of colourful independent sets in our graph.
Lemma 2.5. Given all values of $\sum_{U \in V^{(k)}} f(P(U) \land P')$ for $P' \in P_k$, we can compute the number of colourful independent sets in $G$ in time $h(k)$, where $h$ is some computable function.

Proof. For $1 \leq i \leq B_k$, let $N_i$ be the number of subsets $U \in V^{(k)}$ such that $P(U) = P_i$. Our goal is therefore to calculate $N_{B_k}$.

Let $A = (a_{ij})_{0 \leq i, j \leq B_k}$ be the matrix given by $a_{ij} = f(P_i \land P_j)$. We first claim that $A \cdot N = z$ where

$$N = (N_0, \ldots, N_{B_k})^T,$$

and

$$z = (z_1, \ldots, z_{B_k})^T$$

with $z_i = \sum_{U \in V^{(k)}} f(P(U) \land P_i)$.

To see that this is true, observe that the $i^{th}$ element of $A \cdot N$ is

$$\sum_{j=1}^{B_k} a_{ij} N_j = \sum_{j=1}^{B_k} f(P_i \land P_j) N_j$$

$$= \sum_{j=1}^{B_k} \sum_{U \in V^{(k)}} f(P_i \land P_j)$$

$$= \sum_{U \in V^{(k)}} f(P_i \land P(U))$$

$$= z_i,$$

as required. Thus it suffices to prove that the matrix $A$ is nonsingular, as then we can compute $N_{B_k}$ as the last element of $A^{-1} \cdot z$.

To see that this is indeed the case, first note that, by Lemma 2.2 ([13, Cor. 2]),

$$\det(A) = \prod_{j=1}^{B_k} \sum_{P_i \leq P_j} f(P_i) \mu(P_i, P_j)$$

$$= \prod_{j=1}^{B_k} \mu(\hat{0}, P_j).$$
Thus it suffices to verify that all values of \( \mu(\hat{0}, P_j) \) for \( P_j \in \mathcal{P} \) are non-zero. However, by Lemma 2.4 ([17, Section 3, Prop. 3]), if \( P^* \) is the lattice obtained by inverting the order of elements of \( \mathcal{P} \), and \( \mu^* \) is the Möbius function of \( P^* \), we have
\[
\mu(\hat{0}, P_j) = \mu^*(P_j, \hat{1}).
\]
Moreover, by Lemma 2.3 ([17, Prop. 3]), we have
\[
\mu^*(P_j, \hat{1}) = (-1)^r (r - 1)!
\]
for some integer \( r \geq 1 \), and so certainly we have \( \mu(\hat{0}, P_j) \neq 0 \) for every \( P_j \). Hence \( \det(A) \neq 0 \) and \( A \) is nonsingular, as required.

Now we show that, with an oracle to \( p\text{-}\#\text{Multicolour Connected Induced Subgraph} \), we can compute the values required to set up the equations in the previous lemma, completing the reduction from \( p\text{-}\#\text{Multicolour Independent Set} \) to \( p\text{-}\#\text{Multicolour Connected Induced Subgraph} \).

**Lemma 2.6.** There exists a computable function \( g \) such that, with an oracle to \( p\text{-}\#\text{Multicolour Connected Induced Subset} \), the value of \( \sum_{U \in V(k)} f(P(U) \land P_i) \) can be computed, for every \( P_i \in \mathcal{P}_k \), in time \( g(k) \cdot n^{O(1)} \). Moreover, for every oracle call, the parameter value is at most \( 2^k \).

**Proof.** We begin by considering how to compute all values of \( \sum_{U \in V(k)} f(P(U) \land P_i) \) for a single \( P_i \in \mathcal{P}_k \). Suppose \( P_i = \{X_1, \ldots, X_\ell\} \), where each \( X_j \subset [k] \).

We construct a new coloured graph \( G' \), with vertex set \( V(G') = V(G) \cup \{x_1, \ldots, x_\ell\} \), and where the colouring \( c \) of \( V(G) \) is extended to \( V(G') \) by setting \( c(x_j) = k + j \) for \( 1 \leq j \leq \ell \). \( G' \) has edge-set
\[
E(G') = E(G) \cup \bigcup_{1 \leq j \leq \ell} \{x_jv : v \text{ has colour } d \text{ for some } d \in X_j\}.
\]
Suppose that \( W \subset V(G') \) is a multicoloured subset of \( V(G') \) (so \( |W| = k + \ell \), and all vertices in \( W \) have distinct colours), and set \( U = W \cap V(G) \). Note that, in order for \( W \) to be colourful, we must have \( W \setminus U = V(G') \setminus V(G) \).

We make the following claim.

**Claim 1.** \( G'[W] \) is connected if and only if \( f(P(U) \land P_i) = 1 \).

Suppose first that \( G'[W] \) is connected. It suffices to prove that, for any \( u_1, u_2 \in U \), \( c(u_1) \) and \( c(u_2) \) belong to the same block of \( P(U) \land P_i \). Note
that, by connectedness of $G'[W]$, there must exist a path in $G'[W]$ from $u_1$ to $u_2$. We now proceed by induction on the length of a shortest $u_1$-$u_2$ path in $G'[W]$.

For the base case, suppose that there is at most one internal vertex on such a path. In this case, either $u_1$ and $u_2$ belong to the same connected component of $G[U]$ (in which case we are done, since by definition $c(u_1)$ and $c(u_2)$ then belong to the same block of $P(U)$ and hence $P(U) \cap P_i$), or else there is a single internal vertex $x_j \in W \setminus U$ lying on this path. Thus $u_1, u_2 \in \Gamma(x_j)$, implying by the construction of $G'$ that $c(u_1), c(u_2) \in X_j$, so $c(u_1)$ and $c(u_2)$ belong to the same block of $P_i$ and hence of $P(U) \cap P_i$. This completes the proof of the base case.

We may now assume that there are at least two internal vertices on a shortest $u_1$-$u_2$ path, and that the result holds for any $u'_1, u'_2$ that are connected by a shorter path in $G[W']$. Since there are at least two internal vertices on the shortest $u_1$-$u_2$ path in $G'[W]$, and no two vertices in $W \setminus U$ are adjacent, there must be some vertex $u_3 \in U \setminus \{u_1, u_2\}$ that lies on this path. But then there exists a shorter $u_1$-$u_3$ path in $G'[W]$, implying by the inductive hypothesis that $c(u_1)$ and $c(u_3)$ belong to the same block of $P(U) \cap P_i$. Similarly, we see that $c(u_2)$ and $c(u_3)$ belong to the same block of $P(U) \cap P_i$, and hence it must be that $c(u_1)$ and $c(u_2)$ belong to the same block of $P(U) \cap P_i$, as required.

We now consider the reverse implication. Suppose that $G'[W]$ is not connected, so this graph has connected components with vertex sets $W_1, \ldots, W_r$, where $r \geq 2$. We claim that the partition $P_W = \{c(W_1 \cap U), \ldots, c(W_r \cap U)\}$ of $[k]$ is refined by both $P(U)$ and $P_i$; hence

$$P(U) \cap P_i \geq P_W > \hat{0},$$

so $P(U) \cap P_i \neq \hat{0}$, as required. To see that this claim holds, it suffices to check that, for every block $X$ of $P(U)$ or $P_i$, the vertices having colours from $X$ belong to the same component of $G'[W]$. If $X \in P(U)$ then this follows immediately, since by definition blocks of $P(U)$ are sets of colours that appear in the same connected component of $G[U]$, and so must certainly belong to the same connected component of $G'[W]$. Suppose therefore that $X \in P_i$. But then $X = X_j$ for some $1 \leq j \leq \ell$, and so all vertices of $U$ with colours from $X$ are adjacent to $x_j$, and hence belong to the same component of $G'[W]$. This completes the proof of the claim.

Thus we see that $G'[W]$ is connected if and only if $P(U) \cap P_i = \hat{0}$, completing the proof of Claim \[\square\]
Thus, by Claim 1, \( \sum_{U \in V^{(k)}} f(P(U) \land P_i) \) is exactly equal to the number of colourful connected subsets in \( G' \). Thus, with \( B_k < k^k \) calls to an oracle to \( p-\#\text{Multicolour Connected Induced Subgraph} \), we can compute the value of \( \sum_{U \in V^{(k)}} f(P(U) \land P_i) \) for every \( P_i \in \mathcal{P}_k \); for each call, the parameter value \( k + \ell \) is at most \( 2k \).

Using Lemmas 2.5 and 2.6, it is now straightforward to prove the main result of this section.

**Proof of Theorem 2.1.** The fact that \( p-\#\text{Connected Induced Subgraph} \) \( \in \#W[1] \) follows immediately from Proposition 1.2, so it suffices to prove that the problem is \( \#W[1] \)-hard. To do this, we give a sequence of fpt Turing reductions from \( p-\#\text{Clique} \), shown to be \( \#W[1] \)-complete in [10].

\[ p-\#\text{Clique} \leq_{\text{fpt}} p-\#\text{Multicolour Independent Set} \]

For this reduction, we mimic the proof of Fellows et. al. [8] that \( \text{Multicolour Clique} \) is \( \#W[1] \)-complete. Let \( G = (V, E) \) be the graph in an instance of \( p-\#\text{Clique} \), with parameter \( k \). Now define \( G' \) to be the cartesian product \( G \times K_k \), in which each vertex in the complement of \( G \) is “blown up” to a \( k \)-clique; the vertices of each such \( k \)-clique are given distinct colours \( \{1, \ldots, k\} \). It is straightforward to check that if \( \alpha \) is the number of multicolour independent sets in \( G' \), then the number of \( k \)-cliques in \( G \) is exactly equal to \( \alpha/k! \).

\[ p-\#\text{Multicolour Independent Set} \leq_{\text{fpt}} p-\#\text{Multicolour Connected Subgraph} \]

The reduction follows immediately from Lemmas 2.5 and 2.6.

\[ p-\#\text{Multicolour Connected Subgraph} \leq_{\text{fpt}} p-\#\text{Connected Induced Subgraph} \]

The number of multicoloured connected induced subgraphs in a graph \( G \) can be computed, from the numbers of connected induced subgraphs in the \( 2^k \) subgraphs of \( G \) induced by different combinations of colour-classes, by inclusion-exclusion. Suppose the graph \( G \) is coloured with colours \( [k] \), and for any \( C \subseteq [k] \) let \( G_C \) be the subgraph of \( G \) induced by the vertices with colours belonging to \( C \). Then, if \( N_k(H) \) denotes the number of connected induced \( k \)-vertex subgraphs in \( H \), the number of colourful connected induced subgraphs in \( G \) is exactly

\[ \sum_{\emptyset \neq C \subseteq [k]} (-1)^{k-|C|} N_k(G_C). \]
Combining these reductions, we have \( p\#\text{Clique} \leq_{\text{fpt}}^T p\#\text{Connected Induced Subgraph} \), and so \( p\#\text{Connected Induced Subgraph} \) is \#W[1]-hard under fpt Turing reductions, as required.

3 Approximating \( p\#\text{Induced Subgraph} \) With Property(\( \Phi \))

In contrast to the hardness result of the previous section, we now give a positive result about the approximability of a class of parameterised counting problems that includes \( p\#\text{Connected Induced Subgraph} \), as well as, for example, the problems of counting the number of induced \( k \)-vertex Hamiltonian subgraphs, and that of counting the number of induced \( k \)-vertex non-bipartite subgraphs.

**Theorem 3.1.** Let \( \Phi = (\phi_1, \phi_2, \ldots) \) be a monotone property, and suppose there exists a positive integer \( t \) such that, for each \( \phi_k \), all edge-minimal labelled \( k \)-vertex graphs \((H, \pi)\) such that \( \phi_k(H) = 1 \) satisfy \( \text{tw}(H) \leq t \). Then there is an FPTRAS for \( p\#\text{Induced Subgraph} \) With Property(\( \Phi \)).

Recall from Section 1.3.2 that \( p\#\text{Connected Induced Subgraph} \) is a special case of this problem, so we obtain the following immediate corollary to Theorem 3.1.

**Corollary 3.2.** There is an FPTRAS for \( p\#\text{Connected Induced Subgraph} \).

The proof of Theorem 3.1 is adapted from the proof of Arvind and Raman \([2]\) that there is an FPTRAS for \( p\#\text{Emb}(\mathcal{H}) \) whenever \( \mathcal{H} \) is a class of graphs of bounded treewidth. We begin in Section 3.1 by summarising the existing results we will use, and then in Section 3.2 give a proof of the existence of an FPTRAS in the setting of Theorem 3.1.

3.1 Background

The algorithm we describe in the next section uses random sampling to count approximately, and relies heavily on the parameterised version of the Karp-Luby result \([14]\) on this subject, given by Arvind and Raman.
Theorem 3.3 ([2] Thm. 1)]. For every positive integer $n$, and for every integer $0 \leq k \leq n$, let $U_{n,k}$ be a finite universe, whose elements are binary strings of length $n^{O(1)}$. Let $\mathcal{A}_{n,k} = \{A_1, \ldots, A_m\} \subseteq U_{n,k}$ be a collection of $m = m_{n,k}$ given sets, let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a function and let $d > 0$ be a constant with the following conditions:

1. There is an algorithm that computes $|A_i|$ in time $g(k)n^d$, for each $i$, and every $\mathcal{A}_{n,k}$.

2. There is an algorithm that samples uniformly at random from $A_i$ in time $g(k)n^d$, for each $i$, and every $\mathcal{A}_{n,k}$.

3. There is an algorithm that takes $x \in U_{n,k}$ as input and determines whether $x \in A_i$ in time $g(k)n^d$, for each $i$, and every $\mathcal{A}_{n,k}$.

Then there is an FPTRAS for estimating the size of $A_i = A_1 \cup \cdots \cup A_m$ whenever $m_{n,k}$ is $l(k)n^{O(1)}$ for some function $l$. In particular, for $\epsilon = 1/g(k)$, and $\delta = 1/2^{n^{O(1)}}$, the running time of the FPTRAS algorithm is $(g(k))^{O(1)}n^{O(1)}$.

In proving that there exists an FPTRAS for the problem $\text{p-\#Emb}(C)$, Arvind and Raman prove two further results which we will use in Section 3.2. Firstly, they give an algorithm to compute the number of colourful copies of a $k$-vertex graph $H$ (of bounded treewidth) in a $k$-coloured graph $G$.

Lemma 3.4 ([2] Lemma 1]). Let $G = (V, E)$ be a graph on $n$ vertices that is $k$-coloured by some colouring $f : V(G) \rightarrow [k]$, and let $H$ be a $k$-vertex graph of treewidth $t$ that is $k$-coloured by some colouring $\pi$ such that $H$ is colourful. Then there is an algorithm taking time $O(c^tkt + n^{t+2t^2/2})$ time to exactly compute the cardinality of the set $\{K : K$ is a colourful $k$-vertex subgraph of $G$ and $K$ is colour-preserving isomorphic to $H$ coloured by $\pi\}$, where $c > 0$ is some constant.

The graph $H_1$ with colouring $\omega_1$ is said to be colour-preserving isomorphic to $H_2$ with colouring $\omega_2$ if there exists an isomorphism $\theta$ from $H_1$ to $H_2$ such that, for all $u \in V(H)$, $\omega_1(u) = \omega_2(\theta(u))$.

Secondly, they describe an algorithm to sample uniformly at random from the set of colourful copies of $H$ in $G$.

Lemma 3.5 ([2] Lemma 2]). Let $G = (V, E)$ be a graph on $n$ vertices that is $k$-coloured by some colouring $f : V(G) \rightarrow [k]$, and let $H$ be a $k$-vertex
graph of treewidth $t$ that is $k$-coloured by some colouring $\pi$ such that $H$ is colourful. Then there is an algorithm taking time $O(c^3 k + n^{t+O(1)}2^t)$ time to sample uniformly at random from the set \{ $K : K$ is a colourful $k$-vertex subgraph of $G$ under the colouring $f$ and $K$ is colour-preserving isomorphic to $H$ coloured by $\pi$ \}.

The proofs of Lemmas 3.4 and 3.5 use the concept of $k$-perfect families of hash functions. A family $\mathcal{F}$ of hash functions from $[n]$ to $[k]$ is said to be $k$-perfect if, for every subset $A \subset [n]$ of size $k$, there exists $f \in \mathcal{F}$ such that the restriction of $f$ to $A$ is injective. In the following section, we will use the following bound on the size of such a family of hash functions, proved in [1].

**Theorem 3.6.** For all $n, k \in \mathbb{N}$ there is a $k$-perfect family $\mathcal{F}_{n,k}$ of hash functions from $[n]$ to $[k]$ of cardinality $2^{O(k)} \cdot \log n$. Furthermore, given $n$ and $k$, the family $\mathcal{F}_{n,k}$ can be computed in time $2^{O(k)} \cdot n \log n$.

### 3.2 An FPTRAS for $p$-$\#$Induced Subgraph With Property($\Phi$)

In this section we use Theorem 3.3 ([2, Thm. 1]) to give a proof of Theorem 3.1, that is we show that there exists an FPTRAS for $p$-$\#$Induced Subgraph With Property($\Phi$) whenever $\Phi = (\phi_1, \phi_2, \ldots)$ is a monotone property and there exists a positive integer $t$ such that, for each $\phi_k$, all edge-minimal labelled $k$-vertex graphs $(H, \pi)$ such that $\phi_k(H) = 1$ satisfy $\text{tw}(H) \leq t$.

When considering this problem, we will take $U_{n,k}$ to be the set of all $k$-tuples of $[n]$; thus an element of $U_{n,k}$ can be regarded as a choice of a $k$-tuple of vertices in an $n$-vertex graph. Our goal is to approximate the cardinality of the set $A$, where

$$A = \{(v_1, \ldots, v_k) \in V^k : \phi_k(G[v_1, \ldots, v_k]) = 1\}.$$ 

Thus, in order to make use of Theorem 3.3 to prove the existence of an FPTRAS for $p$-$\#$Induced Subgraph With Property($\Phi$), we need to express $A$ as a union of sets $A_1, \ldots, A_{m_{n,k}}$ which satisfy the conditions of the theorem.

For each $k$, let $\mathcal{H}_k = \{(H_j, \sigma_j) : j \in J\}$ (for some indexing set $J$, with $|J| \leq 2^{O(k)}$) be the set of edge-minimal labelled graphs on $k$ vertices satisfying $\phi_k(H_j, \sigma_j) = 1$ (since $\Phi$ is monotone, this set of minimal elements uniquely
specifies $\Phi$); thus by assumption every $H \in \mathcal{H}_k$ has treewidth at most $t$. It is clear that $A$ can be expressed as a union of sets indexed by elements of $\mathcal{H}_k$, setting
\[ A = \bigcup_{(H, \sigma_H) \in \mathcal{H}_k} A_{(H, \sigma_H)}, \]
where
\[ A_{(H, \sigma_H)} = \{(v_1, \ldots, v_k) \in V^k : G[v_1, \ldots, v_k] \supseteq (H, \sigma_H)\}. \]
We now consider how to express each $A_{(H, \sigma)}$ as a union of sets satisfying the three conditions of Theorem 3.3.

Note that, rather than considering a tuple $(v_1, \ldots, v_k)$ of vertices, we can instead consider an unordered set $U \in V^{(k)}$, together with a bijective labelling $\sigma_U : [k] \to U$, giving
\[ A_{(H, \sigma_H)} = \{(U, \sigma_U) : U \in V^{(k)}, \sigma_U : [k] \to U \text{ is bijective}, (G[U], \sigma_U) \supseteq (H, \sigma_H)\}. \]
Recall from Theorem 3.6 that there exist families of $k$-perfect hash functions from $[n]$ to $[k]$ of size $2^{O(k) \log n}$; fix such a family $F$. With the goal of expressing $A$ as a union of sets whose sizes can be efficiently computed, it will be helpful to express each $A_{(H, \sigma)}$ as a union over elements of $F$. For any $U \in V^{(k)}$ there exists, by definition of a $k$-perfect family, some $f \in F$ such that $U$ is colourful under $f$. Thus it is clear that
\[ A_{(H, \sigma_H)} = \bigcup_{f \in F} A_{(H, \sigma_H, f)}, \]
where
\[ A_{(H, \sigma_H, f)} = \{(U, \sigma_U) : U \in V^{(k)}, \sigma_U : [k] \to U \text{ is bijective, } U \text{ is colourful under } f, \text{ and } (G[U], \sigma_U) \supseteq (H, \sigma_H)\}. \]
Using the language of Lemmas 3.4 and 3.5 we can equivalently write this as
\[ A_{(H, \sigma_H, f)} = \{(U, \sigma_U) : U \in V^{(k)}, \sigma_U : [k] \to U \text{ is bijective, } U \text{ is colourful under } f, \text{ and there exists } K \subseteq G[U] \text{ such that } K \text{ with colouring } \sigma_U^{-1} \text{ is colour-preserving isomorphic to } H \text{ with colouring } \sigma_H^{-1}\}. \]
If we now set
\[ \mathcal{I}_{n,k} = \{(H, \sigma_H, f) : (H, \sigma_H) \in \mathcal{H}_k, f \in F\}, \]
it is clear that
\[ A = \bigcup_{i \in I_{n,k}} A_i. \]  
(3.1)

Note that \(|I_{n,k}| \leq 2^k \cdot 2^{O(k)} \log n = 2^{O(k^2)} \log n\), and so if we set \(A_{n,k} = \{A_i : i \in I_{n,k}\}\) in Theorem 3.3, we certainly satisfy the requirement that \(m_{n,k}\) is \(l(k)n^{O(1)}\) for some function \(l\).

In order to show that the sets \(A_{n,k}\) satisfy the other conditions of Theorem 3.3, we will in fact consider a different collection of sets. For any \(f \in F\) and any \(H\) such that \((H,\sigma_H) \in H_k\) for some \(\sigma_H\), let
\[
A'(H,f) = \{(K,\pi) : K \text{ is a colourful } k\text{-vertex subgraph of } G \text{ under } f, \\
\pi : V(H) \to [k] \text{ is colourful, and } K \text{ with colouring } f \text{ is colour-preserving isomorphic to } H \text{ with colouring } \pi\}.
\]

In the following proposition, we will show that, for any \((H,\sigma_H,f) \in I_{n,k}\), the set \(A_{(H,\sigma_H,f)}\) is in one to one correspondence with \(A'_{(H,f)}\).

**Proposition 3.7.** For any \(i = (H,\sigma_H,f) \in I_{n,k}\), there exists a bijection \(\psi_i\) from \(A'_{(H,f)}\) to \(A_{(H,\sigma_H,f)}\). Moreover, for any \((K,\pi) \in A'_{(H,f)}\), the value of \(\psi_i((K,\pi))\) can be computed in time \(h(k)\), where \(h\) is some computable function.

**Proof.** Fix \(i = (H,\sigma_H,f) \in I_{n,k}\). For any \((K,\pi) \in A'_{(H,f)}\), we define
\[
\psi_i((K,\pi)) = (V(K), f|_{V(K)}^{-1} \circ \pi \circ \sigma_H).
\]

Note that the value of \(\psi_i((K,\pi))\) can clearly be computed in time \(h(k)\) for some computable function \(h\), so it will suffice to prove that \(\psi_i\) does indeed define a bijection from \(A'_{(H,f)}\) to \(A_{(H,\sigma_H,f)}\).

We begin by showing that \(\psi_i\) is a mapping from \(A'_{(H,f)}\) to \(A_{(H,\sigma_H,f)}\). In order to do this, we need to check that, for any \((K,\pi) \in A'_{(H,f)}\), the following conditions hold:

1. \(V(K) \in V(k)\),
2. \(f|_{V(K)}^{-1} \circ \pi \circ \sigma_H : [k] \to V(K)\) is bijective,
3. \(V(K)\) is colourful under \(f\), and
4. there exists \( K' \subseteq G[V(K)] \) such that \( K' \) with colouring \((f|_{V(K)})^{-1} \circ \pi \circ \sigma^{-1}_H \) is colour-preserving isomorphic to \( H \) with colouring \( \sigma^{-1}_H \).

Conditions 1 and 3 follow immediately from the fact that \( K \) is a colourful \( k \)-vertex subgraph of \( G \). For condition 2, first note that we have a mapping from \([k] \) to \( U \): \( \sigma_H \) maps elements of \([k] \) to \( V(H) \), \( \pi \) maps \( V(H) \) to \([k] \), and \( f|_{V(K)}^{-1} \) maps elements of \([k] \) to \( V(K) \). The fact that \((f|_{V(K)})^{-1} \circ \pi \circ \sigma_H \) is bijective follows from the fact that each of \( f|_{V(K)} \), \( \pi \) and \( \sigma_H \) is a bijection. Thus it remains only to check the final condition. Setting \( K' = K \), it follows immediately that \( K \subseteq G[V(K)] \). We know that \( K \) with colouring \( f \) is colour-preserving isomorphic to \( H \) with colouring \( \pi \), so the mapping \( \theta : V(K) \to V(H) \) given by \( \theta(v) = \pi^{-1}(f|_{V(K)}(v)) \) (that is, the mapping from \( V(K) \) to \( V(H) \) that preserves colours, when \( K \) is coloured by \( f \) and \( H \) is coloured by \( \pi \)) is an isomorphism. We claim that in fact \( \theta \) is a colour-preserving isomorphism from \( K \) with colouring \((f|_{V(K)})^{-1} \circ \pi \circ \sigma_H^{-1} \) to \( H \) with colouring \( \sigma_H^{-1} \). To see that this is true, observe that

\[
\sigma_H^{-1}(\theta(v)) = \sigma_H^{-1}(\pi^{-1}(f|_{V(K)}(v))) = (f|_{V(K)}^{-1} \circ \pi \circ \sigma_H^{-1})(v),
\]

as required, so \( \psi_i \) does indeed give a mapping from \( A'_{(H,f)} \) to \( A_{(H,\sigma_H,f)} \).

Next we will show that \( \psi_i \) is injective. Suppose we have \((K, \pi), (K', \pi') \in A'_{(H,f)} \) such that \( \psi_i((K, \pi)) = \psi_i((K', \pi')) \). By definition of \( \psi_i \), we must have \( V(K) = V(K') \). Moreover,

\[
f|_{V(K)}^{-1} \circ \pi \circ \sigma_H \equiv f|_{V(K')}^{-1} \circ \pi' \circ \sigma_H,
\]

and so, as \( V(K) = V(K') \), and \( f|_{V(K)} \) and \( \sigma_H \) are bijective, we must have \( \pi = \pi' \). Thus both \( K \) and \( K' \), when coloured by \( f \), are colour-preserving isomorphic to \( H \) with colouring \( \pi \). Clearly there can be at most one graph on a given vertex set which, when coloured by \( f \), is colour-preserving isomorphic to \( H \) with colouring \( \pi \), and so it follows that \( K' = K \). Hence \((K, \pi) = (K', \pi') \), and we see that \( \psi_i \) is indeed injective.

Finally we show that \( \psi_i \) is surjective. Let \((U, \sigma_U) \in A_{(H,\sigma_H,f)} \). Our goal is to show that there exists \((K, \pi) \in A'_{(H,f)} \) such that \( \psi_i((K, \pi)) = (U, \sigma_U) \). Note that, by definition of \( A_{(H,\sigma_H,f)} \), there exists \( K \subseteq G[U] \) such that \( K \) with colouring \( \sigma_U^{-1} \) is colour-preserving isomorphic to \( H \) with colouring \( \sigma_H^{-1} \). We claim that \((K, f|_{V(K)} \circ \sigma_U \circ \sigma_H^{-1}) \) is as required.
First we observe that \((K, f|_{V(K)} \circ \sigma_U \circ \sigma_H^{-1}) \in A'_{(H, f)}\). It is clear that \(K\) is a colourful \(k\)-vertex subgraph under \(f\), so it remains only to check that \(K\) with colouring \(f\) is colour-preserving isomorphic to \(H\) with colouring \(f|_{V(K)} \circ \sigma_U \circ \sigma_H^{-1}\). We know that \(K\) with colouring \(\sigma_H^{-1}\) is colour-preserving isomorphic to \(H\) with colouring \(\sigma_H^{-1}\); thus the colour-preserving mapping \(\theta : V(K) \to V(H)\) given by \(\theta(v) = \sigma_H(\sigma_U^{-1})(v)\) is an isomorphism. But then
\[
(f|_{V(K)} \circ \sigma_U \circ \sigma_H^{-1}) \circ \theta(v) = f|_{V(K)} \circ \sigma_U \circ \sigma_H^{-1} \circ \sigma_U \circ \sigma_H^{-1}(v)
\]
and so \(\theta\) is in fact a colour-preserving isomorphism from \(K\) with colouring \(f\) to \(H\) with colouring \(f|_{V(K)} \circ \sigma_U \circ \sigma_H^{-1}\). So we do indeed have \((K, f|_{V(K)} \circ \sigma_U \circ \sigma_H^{-1}) \in A'_{(H, f)}\).

We now need to check that \(\psi_i((K, f|_{V(K)} \circ \sigma_U \circ \sigma_H^{-1})) = (U, \sigma_U)\). It is clear from our choice of \(K\) that \(V(K) = U\). Moreover,
\[
f|_{V(K)}^{-1} \circ (f|_{V(K)} \circ \sigma_U \circ \sigma_H^{-1}) \circ \sigma_H = \sigma_U,
\]
so have \(\psi_i((K, f|_{V(K)} \circ \sigma_U \circ \sigma_H^{-1})) = (U, \sigma_U)\), as required.

Thus \(\psi_i\) does indeed define a bijection from \(A'_{(H, f)}\) to \(A_{(H, \sigma_H, f)}\). \(\square\)

We are now ready to show that the sets \(A_{n,k}\) defined above do satisfy the three conditions of Theorem 3.3. The first two conditions will follow easily from results proved in [2].

Lemma 3.8. Let \(A_{n,k} = \{A_1, \ldots, A_m\}\) be defined as above. Then there exists an algorithm that computes \(|A_i|\) in time \(g_1(k)n^{d_1}\), where \(d_1\) is an integer and \(g_1 : \mathbb{N} \to \mathbb{N}\) is a computable function, for each \(i\) and every \(A_{n,k}\).

Proof. Fix \(i = (H, \sigma_H, f) \in \mathcal{I}_{n,k}\). Recall from Proposition 3.7 that \(|A_{(H, \sigma_H, f)}| = |A'_{(H, f)}|\), and so it suffices to compute the cardinality of the set \(A'_{(H, f)}\). Observe that we can write
\[
A'_{(H, f)} = \{(K, \pi) : K \in A'_{(H, f, \pi)}, \pi : V(H) \to [k] \text{ is colourful}\},
\]
where
\[
A'_{(H, f, \pi)} = \{K : K \text{ is a colourful } k\text{-vertex subgraph of } G \text{ under } f, \text{ and } K \text{ with colouring } f \text{ is colour-preserving isomorphic to } H \text{ with colouring } \pi\}.
\]

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This implies that

\[ |A'(H,f)| = \sum_{\pi:V(H)\rightarrow [k]} |A'(H,f,\pi)|. \]

By Lemma 3.4 ([2, Lemma 1]), we can compute the cardinality of each set \(A'(H,f,\pi)\) in time \(O(c^3k + n^{t+2}2^{t^2/2})\). Thus we can clearly compute the sum of the cardinalities of these sets (over the \(k!\) possible values for \(\pi\)) in the specified time.

We now show that the second condition is satisfied.

**Lemma 3.9.** Let \(A_{n,k} = \{A_1, \ldots, A_m\}\) be defined as above. Then there exists an algorithm that samples uniformly at random from \(A_i\) in time \(g_2(k)n^{d_2}\), where \(d_2\) is an integer and \(g_2 : \mathbb{N} \rightarrow \mathbb{N}\) is a computable function, for each \(i\) and every \(A_{n,k}\).

**Proof.** Again, we start by fixing \(i = (H, \sigma_H, f) \in \mathcal{I}_{n,k}\). Since, by Proposition 3.7 there exists a bijection from \(A'(H,f)\) to \(A'(H,\sigma_H,f)\), it is sufficient to show that we can sample uniformly at random from \(A'(H,f)\); as we can then simply apply \(\phi_i\) to the result (taking additional time only \(h(k)\)).

Recall from Lemma 3.8 that

\[ A'(H,f) = \{(K, \pi) : \pi:V(H)\rightarrow [k] \text{ is colourful, } K \in A'(H,f,\pi)\}, \]

and that we can compute the cardinality of each set \(A'(H,f,\pi)\) in time \(g_1(k)n^{d_1}\). Thus, in time \(k! \cdot g_1(k)n^{d_1}\), we can compute the cardinality of all the sets \(A'(H,f,\pi)\) such that \(\pi\) is a colourful \(k\)-colouring of the vertices of \(H\); we can then select a colourful \(k\)-colouring of \(H\) at random, where each such colouring \(\pi\) is selected with probability \(|A'(H,f,\pi)|/|A'(H,f)|\). Having chosen a random colouring \(\pi\) in this way, we then apply the algorithm from Lemma 3.6 ([2, Lemma 2]) to sample a colourful subgraph \(K\) uniformly at random from the set \(A'(H,f,\pi)\) in time \(O(c^3k + n^{t+O(1)}2^{t^2/2})\). We then return \((K, \pi) \in A'(H,f)\).

Observe that any element \((K, \pi) \in A'(H,f)\) will be returned with probability exactly

\[ \frac{1}{|A'(H,f,\pi)|} \cdot \frac{|A'(H,f)|}{|A'(H,f)|} = \frac{1}{|A'(H,f)|}. \]

Thus we can indeed sample uniformly at random from \(A'(H,f)\), and hence from \(A(H,\sigma_H,f)\) in the required time. \(\square\)
Thus, in order to apply Theorem 3.3 it remains to check that our sets satisfy the third condition; we demonstrate this in the following lemma.

**Lemma 3.10.** There is an algorithm that takes \( x \in U_{n,k} \) as input and determines whether \( x \in A_i \) in time \( g_3(k)n^{d_3} \), where \( d_3 \) is an integer and \( g_3 : \mathbb{N} \rightarrow \mathbb{N} \) is a computable function, for each \( i \) and every \( A_{n,k} \).

**Proof.** For any \( x = (x_1, \ldots, x_k) \in U_{n,k} \), in order to determine whether \( x \in A_i \) for given \( i = (H, \sigma_H, f) \in \mathcal{I} \), it suffices to check whether both the following conditions are satisfied:

1. \((x_1, \ldots, x_k)\) is colourful with respect to the colouring \( f \), and
2. \( G[x_1, \ldots, x_k] \supseteq (H, \sigma_H) \).

The first of these two conditions can clearly be verified in time \( O(k) \). For the second condition we need to check, for every edge \( e = uv \in E(H) \), whether \( x_{\sigma_H^{-1}(u)}x_{\sigma_H^{-1}(v)} \in E(G) \); this can be done in time \( O(k^2) \). Thus for any \( X \in U_{n,k} \), and each \( i \in \mathcal{I} \), we can determine whether \( x \in A_i \) in time \( O(k^2) \), for every \( A_{n,k} \).

With these three lemmas, we can prove Theorem 3.2.

**Proof.** We wish to approximate the cardinality of a set \( A \) which, by (3.1), can be written as a union of sets \( A_1, \ldots, A_m \) (where \( m_{n,k} = l(k)n^{O(1)} \) for some function \( l \)). By Lemmas 3.8, 3.9 and 3.10 if we set \( g(k) = \max\{g_1(k), g_2(k), g_3(k)\} \) and \( d = \max\{d_1, d_2, d_3\} \), these sets satisfy the three conditions of Theorem 3.3 so it follows immediately that there exists a FPTRAS for estimating the size of \( A \), in other words there exists a FPTRAS for \( p \#\text{INDUCED SUBGRAPH WITH PROPERTY}(\Phi) \).

\[ \square \]

## 4 Application to \#GRAPH MOTIF

The **Graph Motif** problem was first introduced by Lacroix, Fernandes and Sagot [16] in the context of metabolic network analysis, and is defined as follows.

**Graph Motif**

**Input:** A vertex-coloured graph \( G \) and a multiset of colours \( M \).
**Question:** Does $G$ have a connected subset of vertices whose multiset of colours equals $M$?

This decision problem, and a number of variations, have since been studied extensively ([3, 6, 7, 9, 12]). The problem is known to be NP-complete in general [16], and remains NP-complete even if the input is restricted so that $G$ is a tree of maximum degree three and $M$ is a set rather than a multiset [9]. However, the decision problem is fixed parameter tractable when parameterised by the motif size $|M|$.

It is natural to consider counting versions of the **Graph Motif** problem, and counting the number of occurrences of a given motif in a graph has applications in determining whether a motif is over- or under-represented in a biological network with respect to the null hypothesis [18]. In [12], Guille-mot and Sikora consider the following parameterised counting version of the problem.

$p$-$\#XMGM$

**Input:** A graph $G = (V, E)$, a colouring $c$ of $V$, and a multiset of colours $M$.

**Parameter:** $k = |M|$.

**Question:** How many $k$-vertex trees in $G$ have a multiset of colours equal to $M$?

The authors prove that this problem is $\#W[1]$-hard in the case that $M$ is a multiset, but is fixed parameter tractable when $M$ is in fact a set ($\#XCGM$).

In $p$-$\#XMGM$, the output is the number of connected induced subgraphs of $G$ having colour-set exactly equal to $M$, where each such subgraph is weighted by its number of spanning trees. In this section we consider a more direct translation of **Graph Motif** into the counting world, in which the goal is to compute simply the total number of connected induced subgraphs having the desired colour-set.

$p$-$\#Graph Motif$

**Input:** A graph $G = (V, E)$, a colouring $c$ of $V$, and a multiset of colours $M$.

**Parameter:** $k = |M|$.

**Question:** How many subsets $U \subset V^{(k)}$ are such that $G[U]$ is connected and the multiset of colours assigned to $U$ is exactly $M$?
We adapt results from Sections 2 and 3 to show that

- \( p\text{-#GRAPH MOTIF} \) is \#W[1]-hard, even in the case that \( M \) is a set,

- there exists an FPTRAS for \( p\text{-#GRAPH MOTIF} \).

First, we give our hardness result.

**Theorem 4.1.** \( p\text{-#GRAPH MOTIF} \) is \#W[1]-hard, even when \( M \) is a set.

**Proof.** We give a simple reduction from \( p\text{-COLOURFUL CONNECTED INDUCED SUBGRAPH} \), shown to be \#W[1]-hard in Section 2.1. Let \( G \) with \( k \)-colouring \( c \) be the input in an instance of \( p\text{-COLOURFUL CONNECTED INDUCED SUBGRAPH} \). We then define an instance of \( p\text{-#GRAPH MOTIF} \) with input graph \( G \), colouring \( c \) and motif \( M = [k] \). Then occurrences of \( M \) as a motif in \( G \) are exactly the colourful connected induced subgraphs of \( G \), so this gives an fpt-parsimonious reduction.

Now we show that it is possible to approximate \( p\text{-#GRAPH MOTIF} \), for any input \((G,M)\).

**Theorem 4.2.** There exists an FPTRAS for \( p\text{-#GRAPH MOTIF} \).

**Proof.** Once again, we use Theorem 3.3 to demonstrate the existence of an FPTRAS for this problem. We will in fact prove that there exists an FPTRAS for the labelled version of the problem, so an approximation to the version defined above can be obtained by dividing the output by \( k! \). As before, we will take \( U_{n,k} \) to be the set of \( k \)-element subsets of \( n \). To make use of Theorem 3.3 we need to express

\[
B = \{(v_1, \ldots, v_k) \in V^k : \ G[v_1, \ldots, v_k] \text{ is connected and the colour-set of } \{v_1, \ldots, v_k\} \text{ under } c \text{ is } M\},
\]

as a union of sets \( B_{n,k} = \{B_1, \ldots, B_{m_{n,k}}\} \) that satisfy the conditions of the theorem. Applying the same reasoning as in the proof of Theorem 3.1 and defining \( I \) as before (note that in this case \( H_k \) consists of all labelled trees on \( k \) vertices), we see that

\[
B = \bigcup_{i \in I} B_i,
\]

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where

\[ B_{(H,\sigma_H,f)} = \{(U,\sigma_U) : U \in V^k, \sigma_U : [k] \to U \text{ is bijective, } U \text{ is colourful under } f, (G[U],\sigma_U) \supseteq (H,\sigma_H), \text{ and the colour-set of } U \text{ under } c \text{ is } M\}. \]

Now, if we set \( D \) to be the set of all bijective mappings \( d : M \to [k] \), we can write

\[ B_{(H,\sigma_H,f)} = \bigcup_{d \in D} B_{(H,\sigma_H,f,d)}, \]

where

\[ B_{(H,\sigma_H,f,d)} = \{(U,\sigma_U) : U \in V^k, \sigma_U : [k] \to U \text{ is bijective, } U \text{ is colourful under } f, (G[U],\sigma_U) \supseteq (H,\sigma_H), \text{ and for each } u \in U, c(u) = d^{-1}f(u)\}. \]

So if we set

\[ J = \{(H,\sigma_H,f,d) : (H,\sigma_H) \in \mathcal{H}_k, f \in \mathcal{F}, d \in D\}, \]

we have

\[ B = \bigcup_{j \in J} B_j. \]

Note that \( |J| \leq k^k \cdot |\mathcal{I}| \), and so we clearly have that \( m_{n,k} = |J| \) is \( l(k)n^{O(1)} \) for some function \( l \). It remains to check that these sets \( B_j \) satisfy the conditions of Theorem 3.3.

To see that the first two conditions hold, we make use of Lemmas 3.8 and 3.9. Observe that the set \( B_{(H,\sigma_H,f,d)} \), calculated with respect to the graph \( G \), is precisely equal to the set \( A_{(H,\sigma_H,f)} \), as defined in the proof of Theorem 3.1, if instead of considering the graph \( G \) we consider the graph

\[ G_{f,d} = G[\{v \in V : c(v) = d^{-1}(f(v))\}]. \]

Note that, given \( f \) and \( d \), the graph \( G_{f,d} \) can be computed from \( G \) in time \( O(n) \), so it follows from Lemmas 3.8 and 3.9 that

- there is an algorithm that computes \( |B_j| \) in time \( g_1(k)n^{d_1+1} \), for each \( j \), and every \( B_{n,k} \), and
there is an algorithm that samples uniformly at random from $B_j$ in time $g_2(k)n^{d_2+1}$, for each $j$, and every $\mathcal{B}_{n,k}$.

Thus it remains only to check that the third condition holds. Given a subset a labelled $k$-subset $(U,\sigma_U)$ of $V$ and $(H,\sigma_H,f) \in \mathcal{I}$, we know by Lemma 3.10 that we can check in time $g_3(k)n^{d_3}$ whether $U$ is colourful under $f$, and whether there exists $K \subseteq G[U]$ such that $K$ with colouring $\sigma_U^{-1}$ is colour-preserving isomorphic to $H$ with colouring $\sigma_H^{-1}$. If both these conditions do hold, then to determine whether in fact $U \in B_j$, we also need to check, for each $v \in U$, whether $c(v) = d^{-1}(f(v))$, but the time required to do this clearly depends only on $k$. Hence there exists a function $g_4 : \mathbb{N} \rightarrow \mathbb{N}$, and an algorithm that takes $x \in U_{n,k}$ as input and determines whether $x \in B_j$ in time $g_4(k)n^{d_3}$, for each $j$, and every $\mathcal{B}_{n,k}$.

Hence, setting $g = \max\{g_1(k),g_2(k),g_4(k)\}$ and $d = \max\{d_1, d_2, d_3\}$, all the conditions of Theorem 3.3 are satisfied, and therefore there exists an FPTRAS for the labelled version of $p\text{-}\#\text{GRAPH MOTIF}$. To obtain an approximation to the unlabelled version, we simply divide the result by $k!$, so this implies that there exists an FPTRAS for $p\text{-}\#\text{GRAPH MOTIF}$, as required.

5 Conclusions and Open Problems

We have shown that the problem $p\text{-}\text{CONNECTED INDUCED SUBGRAPH}$ is #W[1]-hard, but that on the other hand there exists an FPTRAS for a more general problem $p\text{-}\text{INDUCED SUBGRAPH WITH PROPERTY } (\Phi)$, where $\Phi$ is a monotone property such that the edge-minimal graphs satisfying $\Phi$ all have bounded treewidth. We then adapted these results to show that a natural counting version of the problem $\text{GRAPH MOTIF}$ is #W[1]-hard, but has an FPTRAS.

We finish with two natural related open questions.

1. Are all (non-trivial) special cases of the class of problems covered by Theorem 3.1 #W[1]-hard?

2. Does there exist an FPTRAS for any problems $p\text{-}\text{INDUCED SUBGRAPH WITH PROPERTY } (\Phi)$ where $\Phi$ is a monotone property but at the edge-minimal graphs that satisfy $\Phi$ do not all have bounded treewidth?
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