Chern–Simons–Antoniadis–Savvidy forms and standard supergravity

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Abstract

In the context of the so-called the Chern–Simons–Antoniadis–Savvidy (ChSAS) forms, we use the methods for FDA decomposition in 1-forms to construct a four-dimensional ChSAS supergravity action for the Maxwell superalgebra. On the other hand, we use the Extended Cartan Homotopy Formula to find a method that allows the separation of the ChSAS action into bulk and boundary contributions and permits the splitting of the bulk Lagrangian into pieces that reflect the particular subspace structure of the gauge algebra.
I. INTRODUCTION

In Refs. [1–4] Antoniadis, Konitopoulos and Savvidy introduced a procedure to construct gauge invariant, background-free gauge forms. The integrals of these forms over the corresponding space-time coordinates provides new topological actions that we have called Chern–Simons–Antoniadis–Savvidy (ChSAS) actions, which generalize the usual Chern–Simons theory. Of special interest are those which can be constructed in even dimensions.

Using calculation methods for Free Differential Algebra (FDA) that allow the decomposition of $p$-forms in 1-forms developed in Ref. [5] and applied in Refs. [6–8], we construct a four-dimensional ChSAS supergravity action for Maxwell superalgebra.

Following Ref. [9] it is found, in the context of the so called the ChSAS theory, a subspace separation method for the Lagrangian. The method is based on the iterative use of the generalized Extended Cartan Homotopy Formula, and allows one to separate the action in bulk and boundary contributions, and systematically split the Lagrangian in appropriate reflection of the subspace structure of the gauge algebra. In order to apply the method, one must regard ChSAS forms as a particular case of more general objects known as generalized transgression forms.

This work is organized as follows: in Section 2, we briefly review the principal aspects of transgression and Chern–Simons–Antoniadis–Savvidy forms. In section 3, we use the calculation methods for Free Differential Algebra (FDA) that allow the decomposition of $p$-forms in 1-forms developed in Ref. [5] and applied in Refs. [6–8] to construct a four-dimensional Chern–Simons–Antoniadis–Savvidy supergravity action for Maxwell superalgebra. Section 4 presents the generalized Extended Cartan Homotopy Formula and shows how a subspace separation method that allows for a deeper understanding of the ChSAS Lagrangian can be built upon it. We finish in Section 5 with conclusions and some considerations on future possible developments.

II. CHERN–SIMONS–ANTONIADIS–SAVVIDY FORMS IN $(2n + 2)$-DIMENSIONS

The idea of extending the Yang–Mills fields to higher rank tensor gauge fields was used in Ref. [1] in order to construct gauge invariant and metric independent forms in higher
dimensions. These forms are analogous to the Pontryagin–Chern forms in Yang–Mills gauge theory. These results were generalized in Refs. [2–4], where the authors found closed invariant forms similar to the Pontryagin–Chern forms in non-abelian tensor gauge field theory. These forms are based on non-abelian tensor gauge fields and are polynomials on the corresponding curvature forms.

A Lie algebra valued 1-form connection \( A \) can be written making more or less explicit the dependence on the Lie algebra generator basis \( T_a \) or the basis of 1-forms \( dx^\mu \),

\[
A = A_\mu \otimes dx^\mu = A^a_\mu T_a \otimes dx^\mu.
\]

The same is true for the 2-form \( B \) gauge potential \( B = \frac{1}{2} B_{\mu\nu} \otimes dx^\mu dx^\nu = \frac{1}{2} B^a_{\mu\nu} T_a \otimes dx^\mu dx^\nu \).

The corresponding 2-form and 3-form “curvatures” are given by \( F = \frac{1}{2} F_{\mu\nu} \otimes dx^\mu dx^\nu \) and \( H = \frac{1}{3} H_{\mu\nu\lambda} \otimes dx^\mu dx^\nu dx^\lambda \) respectively, where

\[
F = dA + A^2, \quad H = DB = dB + [A, B].
\]  

The curvatures \( F \) and \( H \) satisfy the Bianchi identities

\[
DF = 0, \quad DH + [B, F] = 0.
\]

The infinitesimal, non-abelian gauge transformations for the generalized gauge fields are given by

\[
\delta A = D\xi_0, \quad \delta B = D\xi_1 + [B, \xi_0],
\]

where \( \xi_0 \) and \( \xi_1 \) are a 0-form and a 1-form gauge parameters respectively [1]. Under these gauge transformations, the curvatures transform as [2],

\[
\delta F = D(\delta A) = [F, \xi_0],
\]

\[
\delta H = D(\delta B) + [\delta A, B].
\]

It may be of interest to note that we have used the definition of the commutator for differential forms given by

\[
[X, Y] = XY - (-1)^{pq} YX,
\]

where \( X \) is a \( p \)-form and \( Y \) is a \( q \)-form.

In Refs. [1, 2] there were found closed invariant forms similar to the Pontryagin–Chern forms in non-abelian tensor gauge field theory. In particular, it was found that there exists
a gauge invariant metric-independent invariant in \((2n + 3)\)-dimensional space-time

\[ \Gamma_{2n+3} = \langle F^n H \rangle. \]  \hfill (6)

**Chern–Weil theorem in the \((2n + 2)\)-dimensional case:** the theorem ingredients are:

(i) Two Lie-algebra valued, 1-forms connections \(A_0\) and \(A_1\). Their curvatures are given by, \(F_0 = dA_0 + A_0^2\) and \(F_1 = dA_1 + A_1^2\), respectively. (ii) Two Lie-algebra valued, generalized 2-forms gauge fields \(B_0\) and \(B_1\). Their generalized curvatures are given by \(H_0 = dB_0 + [A_0, B_0]\) and \(H_1 = dB_1 + [A_1, B_1]\) respectively. (iii) In terms of these fundamental ingredients, it is possible to define the differences \(\Theta = A_1 - A_0\) and \(\Phi = B_1 - B_0\), and the interpolating connections \(A_t = A_0 + t\Theta\) and \(B_t = B_0 + t\Phi\) with \(0 \leq t \leq 1\). Their corresponding curvatures are given by

\[ F_t = dA_t + A_t^2, \quad H_t = dB_t + [A_t, B_t], \]  \hfill (7)

which satisfy the conditions

\[ \frac{d}{dt} F_t = D_t \Theta, \quad \frac{d}{dt} H_t = D_t \Phi + [\Theta, B_t]. \]  \hfill (8)

**Theorem [8]:** Let \(A_0\) and \(A_1\) be two gauge connection 1-forms, and let \(F_0\) and \(F_1\) be their corresponding 2-forms curvature. Let \(B_0\) and \(B_1\) be two gauge connection 2-forms and let \(H_0\) and \(H_1\) be their corresponding curvature 3-forms. Then, the difference \(\Gamma_{2n+3}^{(1)} - \Gamma_{2n+3}^{(0)}\) is an exact form

\[ \Gamma_{2n+3}^{(1)} - \Gamma_{2n+3}^{(0)} = \langle F_1^n H_1 \rangle - \langle F_0^n H_0 \rangle = d\Sigma^{(2n+2)}(A_0, B_0; A_1, B_1), \]  \hfill (9)

where

\[ \Sigma^{(2n+2)}(A_0, B_0; A_1, B_1) = \int_0^1 dt \left( nF_0^{n-1}\Theta H_t + \langle F_1^n \Phi \rangle \right) \]  \hfill (10)

is what we call “Antoniadis–Savvidy transgression form”. A proof can be found in Ref. [8]. Following the same procedure followed in the case of the Chern–Simons forms, we define the \((2n+2)\)-Chern–Simons–Antoniadis–Savvidy form as

\[ \mathcal{C}_{\text{ChSAS}}^{(2n+2)}(A, B; 0, 0) = \int_0^1 dt \langle nAF_t^{n-1}H_t + BF_t^n \rangle. \]  \hfill (11)

This result agrees with the expression found by Antoniadis and Savvidy in Refs. [1, 2]. From eq. (11), we have for the \(n = 1\) case [2],

\[ \mathcal{C}_{\text{ChSAS}}^{(2)} = \int_0^1 dt \langle AH_t + F_t B \rangle = \langle FB \rangle - d \langle AB \rangle. \]  \hfill (12)
It is interesting to notice that transgression forms (both, standard ones and the above generalization) are defined globally on the spacetime basis manifold of the principal bundle, and are off-shell gauge invariant. Chern–Simons forms (both, standard ones and the Antoniadis–Savvidy generalization) are locally defined and are off-shell gauge invariant only up to boundary terms (i.e., quasi-invariants).

III. CHERN–SIMONS–ANTONIADIS–SAVVIDY FORM FOR MAXWELL SUPERALGEBRA

Now we will use this construction for the particular case of the Maxwell superalgebra, in order to show the connection between eq. (12) and supergravity in $D = 4$.

A. $sM_4$ Maxwell superalgebra

The minimal Maxwell superalgebra $sM_4$ in $D = 4$ is an algebra whose generators $\{P_a, J_{ab}, Z_{ab}, \bar{Z}_{ab}, Q_\alpha, \Sigma_\alpha\}$ satisfy the following commutation relation \[10, 11\]

$$
\begin{align*}
[J_{ab}, J_{cd}] &= \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{bd} J_{ac} - \eta_{ac} J_{bd}, \\
[J_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b, \quad [P_a, P_b] = Z_{ab}, \\
[J_{ab}, Z_{cd}] &= \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{bd} Z_{ac} - \eta_{ac} Z_{bd}, \\
[P_a, Q_\alpha] &= -\frac{1}{2} (\gamma_a \Sigma)_\alpha, \quad [J_{ab}, Q_\alpha] = -\frac{1}{2} (\gamma_{ab} Q)_\alpha, \\
[J_{ab}, \Sigma_\alpha] &= -\frac{1}{2} (\gamma_{ab} \Sigma)_\alpha, \quad [\bar{Z}_{ab}, Q_\alpha] = -\frac{1}{2} (\gamma_{ab} \Sigma)_\alpha, \\
\{Q_\alpha, Q_\beta\} &= -\frac{1}{2} \left[ (\gamma^{ab} C)_{\alpha\beta} \bar{Z}_{ab} - 2 (\gamma^a C)_{\alpha\beta} P_a \right], \\
\{Q_\alpha, \Sigma_\beta\} &= -\frac{1}{2} (\gamma^{ab} C)_{\alpha\beta} Z_{ab}, \\
[J_{ab}, \bar{Z}_{cd}] &= \eta_{bc} \bar{Z}_{ad} + \eta_{ad} \bar{Z}_{bc} - \eta_{bd} \bar{Z}_{ac} - \eta_{ac} \bar{Z}_{bd}, \\
[\bar{Z}_{ab}, \bar{Z}_{cd}] &= \eta_{bc} \bar{Z}_{ad} + \eta_{ad} \bar{Z}_{bc} - \eta_{bd} \bar{Z}_{ac} - \eta_{ac} \bar{Z}_{bd}, \\
\text{others} &= 0.
\end{align*}
$$

This algebra can be found by an S-expansion of $\mathfrak{osp}(4/1)$ superalgebra. \[10, 11\].

In order to write down a four dimensional ChSAS action, we start by expressing the gauge fields $A$ and $B$ at the base of Maxwell superalgebra.
To interpret the gauge field associated with a traslational generator $P_a$ as the vielbein, one is forced to introduce a length scale $\ell$ in the theory. Since one can always choose Lie algebra generators $T_A$ to be dimensionless as well, the one-form connection fields $A = A^a_T A dx^\mu$ must also be dimensionless. However, the vielbein $e^a = e^a_\mu dx^\mu$ must have dimensions of length if it is related to the spacetime metric $g_{\mu\nu}$ through the usual equation $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. This means that the “true” gauge field must be of the form $e^a/\ell$, with $\ell$ a length parameter. Therefore, following Refs. [12], [13], the one-form gauge field $A$ is given by

$$A = \frac{1}{\ell} e^a P_a + \frac{1}{2} \omega^{ab} J_{ab} + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{2} \bar{k}^{ab} \bar{Z}_{ab} + \frac{1}{\sqrt{\ell}} \psi^\alpha Q_\alpha + \frac{1}{\sqrt{\ell}} \xi^\alpha \Sigma_\alpha,$$

where $e^a$ is identified as the 1-form vierbein, $\omega^{ab}$ is the 1-form spin connection, $k^{ab}$ and $\bar{k}^{ab}$ are extra antisymmetric bosonic 1-form fields, and $\psi^\alpha, \xi^\alpha$ are fermionic 1-form fields. The corresponding 2-form curvature is given by

$$F = \frac{1}{\ell} \hat{T}^a P_a + \frac{1}{2} R^{ab} J_{ab} + \frac{1}{2} f^{ab} Z_{ab} + \frac{1}{2} \hat{f}^{ab} \bar{Z}_{ab} + \frac{1}{\sqrt{\ell}} \Psi^\alpha Q_\alpha + \frac{1}{\sqrt{\ell}} \Xi^\beta \Sigma_\beta,$$

with

$$\hat{T}^a = T^a + \frac{1}{2} \bar{\psi} \gamma^a \psi,$$

$$R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb},$$

$$f^{ab} = D k^{ab} + \frac{1}{\ell^2} \epsilon^{a b} e^c + \bar{k}^a_c \bar{k}^c b - \frac{1}{\ell} \bar{\psi} \gamma^{ab} \psi,$$

$$\hat{f}^{ab} = D \bar{k}^{ab} - \frac{1}{2\ell} \bar{\psi} \gamma^{ab} \psi,$$

$$\Psi^\alpha = D \psi^\alpha,$$

$$\Xi^\beta = D \xi^\beta - \frac{1}{4} \bar{k}^{ab} \psi^\alpha (\gamma_{ab})^\alpha_\beta - \frac{1}{2\ell} \epsilon^{a b} \psi^\alpha (\gamma_a)^\beta.$$

For the 2-form $B$, we can write

$$B = B^a P_a + \frac{1}{2} B^{ab} J_{ab} + \frac{1}{2} \beta^{ab} Z_{ab} + \frac{1}{2} \bar{\beta}^{ab} \bar{Z}_{ab} + \lambda^\alpha Q_\alpha + \chi^\alpha \Sigma_\alpha,$$

where $B^a, B^{ab}, \beta^{ab}, \bar{\beta}^{ab}, \lambda^\alpha, \chi^\alpha$ are 2-forms that we must determine. The corresponding 3-form curvature is given by

$$H = H^a P_a + \frac{1}{2} H^{ab} J_{ab} + \frac{1}{2} \Theta^{ab} Z_{ab} + \frac{1}{2} \bar{\Theta}^{ab} \bar{Z}_{ab} + \bar{H}^\alpha Q_\alpha + \bar{H}^\alpha \Sigma_\alpha,$$
with,

\[
H^a = DB^a - \frac{1}{\ell} B^a e^c + \frac{1}{\sqrt{\ell}} \tilde{\psi} \gamma^a \lambda, \tag{13}
\]

\[
H^{ab} = DB^{ab}, \tag{14}
\]

\[
\Theta^{ab} = D\beta^{ab} - B^{[a}\epsilon[B^{c]} - \frac{1}{\ell} B^{[a}\epsilon[B^{b]} - \frac{1}{\sqrt{\ell}} \tilde{\psi} \gamma^{ab} \chi + \frac{1}{\sqrt{\ell}} \tilde{\lambda} \gamma^{ab} \xi, \tag{15}
\]

\[
\tilde{H}^a = D\lambda^a + \frac{1}{4\sqrt{\ell}} B^{ab} \psi^\beta (\gamma_{ab})^\alpha, \tag{16}
\]

\[
\mathcal{H}^a = D\lambda^a + \frac{1}{4\sqrt{\ell}} B^{ab} \psi^\beta (\gamma_{ab})^\alpha - \frac{1}{2\ell} e^a \chi^\beta (\gamma_a)^\alpha + \frac{1}{2\sqrt{\ell}} B^a \psi^\beta (\gamma_a)^\alpha - \frac{1}{4\sqrt{\ell}} \tilde{\mathcal{H}}^a \psi^\beta (\gamma_{ab})^\alpha. \tag{17}
\]

The problem now is to express the form \( B \) defined by the equations (13-17) in terms of the one-forms \( \{e^a, \omega^a, k^a, \bar{k}^a, \psi^\alpha, \xi^a\} \) of the Maxwell superalgebra. To express the 2-forms \( \{B^a, B^{ab}, \beta^{ab}, \bar{\beta}^{ab}, \lambda, \chi\} \) as the wedge product of the 1-forms \( \{e^a, \omega^a, k^a, \bar{k}^a, \psi^\alpha, \xi^a\} \) we follow a procedure developed in Refs. 3, 6. Imposing the ansatz

\[
B^a = \frac{a_1}{2\ell} e^a e^b + \frac{a_2}{2\ell} k^a k^b + \frac{a_3}{2\ell} \bar{k}^a \bar{k}^b + \frac{a_4}{\ell} \tilde{\psi} \gamma^a \psi + \frac{a_5}{\ell} \tilde{\psi} \gamma^a \xi + \frac{a_6}{\ell} \tilde{\xi} \gamma^a \xi, \tag{18}
\]

\[
B^{ab} = \frac{b_1}{2\ell^2} e^a e^b + \frac{b_2}{\ell} \omega^a \epsilon[k^b] + \frac{b_3}{2} k^a k^b + \frac{b_4}{\ell} \omega^a \epsilon[\bar{k}^b] + \frac{b_5}{2} \bar{k}^a \bar{k}^b + \frac{b_6}{\ell} \tilde{\psi} \gamma^{ab} \psi + \frac{b_7}{\ell} \tilde{\psi} \gamma^{ab} \xi + \frac{b_8}{\ell} \tilde{\xi} \gamma^{ab} \xi, \tag{19}
\]

\[
\beta^{ab} = \frac{c_1}{2\ell} e^a e^b + \frac{c_2}{2\ell} \omega^a \epsilon[k^b] + \frac{c_3}{2} k^a k^b + \frac{c_4}{\ell} \omega^a \epsilon[\bar{k}^b] + \frac{c_5}{2} \bar{k}^a \bar{k}^b + \frac{c_6}{\ell} \tilde{\psi} \gamma^{ab} \psi + \frac{c_7}{\ell} \tilde{\psi} \gamma^{ab} \xi + \frac{c_8}{\ell} \tilde{\xi} \gamma^{ab} \xi, \tag{20}
\]

\[
\bar{\beta}^{ab} = \frac{d_1}{2\ell} e^a e^b + \frac{d_2}{\ell} \omega^a \epsilon[k^b] + \frac{d_3}{2} k^a k^b + \frac{d_4}{\ell} \omega^a \epsilon[\bar{k}^b] + \frac{d_5}{2} \bar{k}^a \bar{k}^b + \frac{d_6}{\ell} \tilde{\psi} \gamma^{ab} \psi + \frac{d_7}{\ell} \tilde{\psi} \gamma^{ab} \xi + \frac{d_8}{\ell} \tilde{\xi} \gamma^{ab} \xi, \tag{21}
\]

\[
\lambda_a = \frac{f_1}{\ell} e^a \gamma^a \psi + \frac{f_2}{\ell} \omega^a \epsilon[k^b] + \frac{f_3}{2} k^a \gamma^b \psi + \frac{f_4}{2} \omega^a \gamma^b \xi + \frac{f_5}{2} k^a \gamma^b \xi + \frac{f_6}{2} \omega^a \gamma^b \xi, \tag{22}
\]

\[
\chi_a = \frac{g_1}{\ell} e^a \gamma^a \psi + \frac{g_2}{\ell} \omega^a \epsilon[k^b] + \frac{g_3}{2} k^a \gamma^b \psi + \frac{g_4}{2} \omega^a \gamma^b \xi + \frac{g_5}{2} k^a \gamma^b \xi + \frac{g_6}{2} \omega^a \gamma^b \xi, \tag{23}
\]

where \( a_1, \ldots, a_6, b_1, \ldots, b_3, c_1, \ldots, c_9, d_1, \ldots, d_9, f_1, \ldots, f_8, g_1, \ldots, g_8 \) are arbitrary constants, and introducing eqs. (18-23) in eqs. (13-17), when \( H^a = H^{ab} = \Theta^{ab} = \tilde{H}^a = \mathcal{H}^a = 0, \)
we find

\[ B^a = \frac{a_4}{\ell} \bar{\psi} \gamma^a \psi + \frac{a_5}{\ell} \bar{\psi} \gamma^a \xi, \]  
(24)

\[ B^{ab} = 0, \]  
(25)

\[ \beta^{ab} = \frac{c_1}{2\ell^2} e^a e^b + \frac{c_5}{2} \bar{k}^a \bar{k}^b + \frac{c_8}{\ell} \bar{\psi} \gamma^{ab} \xi + \frac{c_9}{\ell} \bar{\xi} \gamma^{ab} \xi, \]  
(26)

\[ \tilde{\beta}^{ab} = \frac{d_7}{\ell} \bar{\psi} \gamma^{ab} \psi + \frac{d_8}{\ell} \bar{\psi} \gamma^{ab} \xi, \]  
(27)

\[ \lambda_\alpha = \frac{f_1}{\ell} e_a \gamma^a \psi_\alpha + \frac{f_2}{2} \bar{k}_{ab} \gamma^{ab} \psi_\alpha, \]  
(28)

\[ \chi_\alpha = \frac{g_1}{\ell} e_a \gamma^a \psi_\alpha + \frac{g_2}{\ell} e_a \gamma^a \xi_\alpha + \frac{g_7}{2} \bar{k}_{ab} \gamma^{ab} \psi_\alpha + \frac{g_8}{2} \bar{k}_{ab} \gamma^{ab} \xi_\alpha. \]  
(29)

The fields given by eqs. (24-29) represent the most general solution that can be built from the fields \{e^a, \omega^{ab}, k^{ab}, \bar{k}^{ab}, \psi^\alpha, \xi^\alpha\}. Any choice of the constants represent a solution to the FDA.

**B. Chern-Simons-Antoniadis-Savvidy form**

Using the invariant tensor found in Ref. [10]

\[ \langle J_{ab} J_{cd} \rangle = \alpha_0 \epsilon_{abcd}, \quad \langle J_{ab} \tilde{Z}_{cd} \rangle = \alpha_2 \epsilon_{abcd}, \] 
\[ \langle \tilde{Z}_{ab} Z_{cd} \rangle = \alpha_4 \epsilon_{abcd}, \quad \langle J_{ab} Z_{cd} \rangle = \alpha_4 \epsilon_{abcd}, \] 
\[ \langle Q_\alpha Q_\beta \rangle = 2\alpha_2 (\gamma_5)_{\alpha\beta}, \quad \langle Q_\alpha \Sigma_\beta \rangle = 2\alpha_4 (\gamma_5)_{\alpha\beta}, \]

being \( \alpha_0, \alpha_2 \) and \( \alpha_4 \) dimensionless arbitrary independent constants, the Chern–Simons–Antoniadis–Savvidy Lagrangian \( \mathcal{L}^{(4)}_{\text{ChSAS}} \equiv \mathcal{L}^{(4)}_{\text{CSAS}} \) is explicitly given by

\[
\mathcal{L}^{(4)}_{\text{ChSAS}} = \frac{1}{4} \epsilon_{abcd} \left( \alpha_0 R^{ab} B^{cd} + \alpha_4 \left( R^{ab} \beta^{cd} + f^{ab} B^{cd} \right) \right) \\
+ \frac{1}{4} \epsilon_{abcd} \alpha_2 \left( R^{ab} \tilde{\beta}^{cd} + \frac{1}{4} \tilde{f}^{ab} B^{cd} \right) + \alpha_4 \tilde{f}^{ab} \tilde{\beta}^{cd} \\
+ \frac{2\alpha_2}{\sqrt{\ell}} \psi^\alpha (\gamma_5)_\alpha^\beta \lambda_\beta + \frac{2\alpha_4}{\sqrt{\ell}} \psi^\alpha (\gamma_5)_\alpha^\beta \chi_\beta + \frac{2\alpha_4}{\sqrt{\ell}} \lambda^\alpha (\gamma_5)_\alpha^\beta \Xi_\beta. \]  
(30)
Using the FDA expansion given by eqs. [(24, 29)], the Chern–Simons–Antoniadis–Savvidy Lagrangian for the Maxwell algebra takes the form

\[
\mathcal{L}_{\text{ChSAS}}^{(4)} = \frac{1}{4} \epsilon_{abcd} \left( \alpha_4 R^{ab} \left( \frac{c_1}{\ell^2} e^c e^d + \frac{c_5}{2} \tilde{k}^c f \tilde{k}^d + \frac{c_8}{\ell} \tilde{\psi} \gamma^{cd} \xi + \frac{c_9}{\ell} \tilde{\xi} \gamma^{cd} \xi \right) \right) \\
+ \frac{1}{2} \epsilon_{abcd} \alpha_2 R^{ab} \left( \frac{d_7}{\ell} \tilde{\psi} \gamma^{cd} \psi + \frac{d_8}{\ell} \tilde{\psi} \gamma^{cd} \xi \right) + \alpha_1 \tilde{f}^{ab} \left( \frac{d_7}{\ell} \tilde{\psi} \gamma^{cd} \psi + \frac{d_8}{\ell} \tilde{\psi} \gamma^{cd} \xi \right)
\]

From eq. (31) we can see that if \( c_9 = d_8 = f_1 = f_7 = g_2 = g_8 = 0 \), which are conditions consistent with the equations (18-23) and (13-17), we have that \( \mathcal{L}_{\text{ChSAS}}^{(4)} \) is given by

\[
\mathcal{L}_{\text{ChSAS}}^{(4)} = \frac{\alpha_4 c_1}{8} \epsilon_{abcd} R^{ab} e^c e^d + 2 \alpha_4 g_1 \sqrt{\ell} \Psi \gamma_5 e_a \gamma^a \psi + \frac{\alpha_4 c_5}{8} \epsilon_{abcd} \tilde{k}^c f \tilde{k}^d \\
+ \frac{\alpha_4 c_3}{4} \epsilon_{abcd} R^{ab} \bar{\psi} \gamma^{cd} \x + \frac{\alpha_4 d_7}{4} \epsilon_{abcd} \tilde{f}^{ab} \bar{\psi} \gamma^{cd} \x
\]

Here it is necessary to notice that:

(a) The first two terms contains the Einstein–Hilbert and the Rarita-Schwinger terms given by \( \epsilon_{abcd} R^{ab} e^c e^d e^e \) and \( \Psi \gamma_5 e_a \gamma^a \psi \) respectively.

(b) The following terms could be interpreted as non-linear couplings between the bosonic and fermionic ”matter” fields \( \tilde{k}^{ab}, \x \), the Rarita-Schwinger field \( \psi \) and the curvature, where the parameter \( \ell \) can be considered as a kind of coupling constant.

From eq. (32), we can see that when \( \ell \ll 1 \), the Chern–Simons–Antoniadis–Savvidy Lagrangian for the Maxwell superalgebra is given by

\[
\mathcal{L}_{\text{ChSAS}}^{(4)} = \frac{\alpha_4 c_1}{8} \epsilon_{abcd} R^{ab} e^c e^d + 2 \alpha_4 g_1 \sqrt{\ell} \Psi \gamma_5 e_a \gamma^a \psi,
\]

where we can see that the Chern–Simons–Savvidy Lagrangian reproduces, except for numerical coefficients, the Lagrangian for standard supergravity.

It is perhaps interesting to note that the commutation relation \([P_a, P_b] = Z_{ab}\) depends on the \( Z_{ab} \) generators. The consequences on the Lagrangian of this non-zero bracket are related
to the gauge field $k_{ab}$ associated to $Z_{ab}$. If we do not consider the $k_{ab}$ dependence, or if we take a limit on the theory in which the gauge field $k_{ab}$ effects are not included, then it is not surprising that the curvature term in the above Lagrangian looks like the standard gravity Lagrangian.

IV. THE EXTENDED CARTAN HOMOTOPY FORMULA IN $(2n + 2)$-DIMENSIONS

The Extended Cartan Homotopy Formula (ECHF) reads [14]

$$\int_{\partial T_{r+1}} \frac{p!}{p!} \pi = \int_{T_{r+1}} \frac{p+1!}{(p+1)!} d\pi + (-1)^{p+q} d \int_{T_{r+1}} \frac{p+1!}{(p+1)!} \pi, \quad (34)$$

where, in this case, $\pi$ represents a polynomial in the forms $\{A_t, B_t, F_t, H_t, d_tA_t, d_tF_t\}$ which is also an $m$-form on $M$ and a $q$-form on $T_{r+1}$, with $m \geq p$ and $p + q = r$. The exterior derivatives on $M$ and $T_{r+1}$ are denoted respectively by $d$ and $d_t$. The operator $l_t$, called homotopy derivation, maps differential forms on $M$ and $T_{r+1}$ according to

$$l_t : \Omega^a (M) \times \Omega^b (T_{r+1}) \to \Omega^{a-1} (M) \times \Omega^{b+1} (T_{r+1}),$$

and it satisfies Leibniz’s rule as well as $d$ as $d_t$. In our case, we will consider the polynomial $\pi = \langle F^n H_t \rangle$. This choice has the three following properties: (i) $\pi$ is $M$-closed, i.e., $d\pi = 0$, (ii) $\pi$ is a 0-form on $T_{r+1}$, and (iii) $\pi$ is a $(2n + 3)$-form on $M$. The allowed values for $p$ are $p = 0, \ldots, 2n + 3$. The ECHF reduces in this case to

$$\int_{\partial T_{p+1}} \frac{p!}{p!} \pi = (-1)^{p+q} d \int_{T_{p+1}} \frac{p+1!}{(p+1)!} \pi. \quad (35)$$

Since the three operators $d$, $d_t$ and $l_t$ define a graded algebra given by [14]

$$d^2 = 0, \quad d_t^2 = 0, \quad \{d, d_t\} = 0, \quad (36)$$

$$[l_t, d] = d_t, \quad [l_t, d_t] = 0, \quad (37)$$

we have that the action of $l_t$ on $\{A_t, B_t, F_t, H_t, d_tA_t, d_tF_t\}$ reads [14]

$$l_t A_t = 0, \quad l_t F_t = l_t (dA_t + A_t A_t) = (dl_t + d_t) A_t = d_t A_t,$$

while the action of $l_t$ on $B_t$ and $H_t$ must be determined.
Particular cases of (35) with \( \pi \) given by (6) which we review below, reproduce both the Chern–Weil Theorem and the Triangle Formula. In fact, when \( p = 0 \), we find that Eq.(35) takes the form,

\[
\int_{\partial T_1} \pi = d \int_{T_1} l_t \pi, \tag{38}
\]

where \( \pi = \langle F^n_t H_t \rangle \) and \( A_t = A_0 + t \Theta, \, B_t = B_0 + t \Phi \). The left side of (38) is given by

\[
\int_{\partial T_1} \langle F^n_t, H_t \rangle = \int_0^1 dt \langle F^n_t, H_t \rangle = \langle F^n_1, H_1 \rangle - \langle F^n_0, H_0 \rangle,
\]

while for the right side we have

\[
d \int_{T_1} l_t \langle F^n_t, H_t \rangle = d \left\{ n \int_0^1 dt \langle F^{n-1}_t, \Theta, H_t \rangle + \int_{T_1} l_t \langle F^n_t, l_t H_t \rangle \right\},
\]

so that

\[
\langle F^n_1, H_1 \rangle - \langle F^n_0, H_0 \rangle = d \left\{ n \int_0^1 dt \langle F^{n-1}_t, \Theta, H_t \rangle + \int_{T_1} l_t \langle F^n_t, l_t H_t \rangle \right\}.
\]

From the Chern–Weil theorem and \( B_t = B_0 + t \Phi \), we see that

\[
\langle F^n_t, l_t H_t \rangle = \langle F^n_t, d_t B_t \rangle,
\]

so that \( l_t H_t = d_t B_t \). On the other hand, we have

\[
\langle F^n_t, l_t (dB_t + A_t B_t - B_t A_t) \rangle = \langle F^n_t, d_t B_t \rangle,
\]

\[
\langle F^n_t, ((d + A_t) (l_t B_t) + d_t B_t - (l_t B_t) A_t) \rangle = \langle F^n_t, d_t B_t \rangle,
\]

and therefore \( \langle F^n_t, D_t l_t B_t \rangle = d \langle F^n_t, l_t B_t \rangle = 0 \) and \( l_t B_t = 0 \). Summarizing, we can write \( l_t B_t = 0 \) and \( l_t H_t = d_t B_t \).

A. The subspace separation method

In this subsection we will show that the subspace separation method developed in Ref. [9] can be generalized to the case of Chern–Simons–Antoniadis–Savvidy formalism. This means that the so called triangle equation (46) splits the transgression form \( \mathcal{T}^{(2n+2)} (A_2, B_2; A_0, B_0) \) into the sum of two transgressions forms depending on an intermediate connection \( A_1, B_1 \) plus a exact form \( \mathcal{T}^{(2n+1)} (A_2, B_2; A_1, B_1; A_0, B_0) \) shown in Eq. (43).

When \( p = 1 \) we have that Eq. (35) is given by

\[
\int_{\partial T_2} l_t \langle F^n_t, H_t \rangle = -d \int_{T_2} \frac{l_t^2}{2} \langle F^n_t, H_t \rangle, \tag{39}
\]
where

\[ A_t = t^0 (A_0 - A_1) + t^2 (A_2 - A_1) + A_1, \]

\[ B_t = t^0 (B_0 - B_1) + t^2 (B_2 - B_1) + B_1. \]

The left side of (39) corresponds to an integral along the boundary of the simplex \( T_2 = (A_2, B_2; A_1, B_1; A_0, B_0) \):

\[
\int_{\partial T_2} l_t \langle F^n_t, H_t \rangle = \mathfrak{T}^{(2n+2)} (A_2, B_2; A_1, B_1) - \mathfrak{T}^{(2n+2)} (A_2, B_2; A_0, B_0) + \mathfrak{T}^{(2n+2)} (A_1, B_1; A_0, B_0). \tag{40}
\]

The right side of (39) is given by

\[
d \int_{T_2} \frac{t^2}{2} \langle F^n_t, H_t \rangle = d \mathfrak{T}^{(2n+1)} (A_2, B_2; A_1, B_1; A_0, B_0), \tag{41}
\]

where

\[
\mathfrak{T}^{(2n+1)} (A_2, B_2; A_1, B_1; A_0, B_0) = \int_0^1 dt \int_0^t ds \left\{ n (n - 1) \langle F^{n-1}_t, (A_2 - A_1), (A_1 - A_0), H_t \rangle + n \langle F^{n-1}_t, A_0, (B_2 - B_1) \rangle + n \langle F^{n-1}_t, A_1, (B_0 - B_2) \rangle + n \langle F^{n-1}_t, A_2, (B_1 - B_0) \rangle \right\}. \tag{42}
\]

In (43) we have introduced dummy parameters \( t = 1 - t^0 \) and \( s = t^2 \), in terms of which \( A_t \) reads

\[ A_t = A_0 + t(A_1 - A_0) + s(A_2 - A_1). \tag{44}\]

Thus we have that the triangle equation is given by

\[
\mathfrak{T}^{(2n+2)} (A_2, B_2; A_1, B_1) - \mathfrak{T}^{(2n+2)} (A_2, B_2; A_0, B_0) + \mathfrak{T}^{(2n+2)} (A_1, B_1; A_0, B_0) \]

\[ = -d \mathfrak{T}^{(2n+1)} (A_2, B_2; A_1, B_1; A_0, B_0), \tag{45}\]

or alternatively

\[
\mathfrak{T}^{(2n+2)} (A_2, B_2; A_0, B_0) = \mathfrak{T}^{(2n+2)} (A_2, B_2; A_1, B_1) + \mathfrak{T}^{(2n+2)} (A_1, B_1; A_0, B_0) + d \mathfrak{T}^{(2n+1)} (A_2, B_2; A_1, B_1; A_0, B_0). \tag{46}\]

We would like to stress that use of the Extended Cartan Homotopy Formula has allowed us to pinpoint the exact form of the boundary contribution \( \mathfrak{T}^{(2n+1)} (A_2, B_2; A_1, B_1; A_0, B_0) \).
Note that if we choose $A_0 = 0$ and $B_0 = 0$ we obtain an expression that relates the form Antoniadi–Savvidy transgression form to two Chern-Simons-Savvidy forms and a total derivative

$$\mathcal{S}^{(2n+2)}(A_2, B_2; A_1, B_1) = \mathcal{S}_{\text{ChSAS}}^{(2n+2)}(A_2, B_2) - \mathcal{S}_{\text{ChSAS}}^{(2n+2)}(A_1, B_1) - d\mathcal{S}^{(2n+1)}(A_2, B_2; A_1, B_1; 0, 0).$$

(47)

V. CONCLUDING REMARKS

In this Letter some features of the $(2n + 2)$-dimensional transgressions and Chern–Simons–Antoniadis–Savvidy forms used as Lagrangians for supergravity theories were briefly reviewed. The 2–form field $B$ can be decomposed in terms of components of the 1-form $A$. It is performed in a self–consistent way by considering the generalization of Maurer–Cartan approach to forms of higher order, i.e., free differential algebras, and by following the procedure used in Refs. [5–8]. The final result is a four-dimensional supergravity action, which is gauge quasi–invariant under the Maxwell superalgebra. These 4-dimensional results shown that an interesting problem is to extract physical information from the $(2n + 2)$-dimensional Lagrangian (11). A crucial step in this direction is the separation of the Lagrangian in a way that reflects the inner subspace structure of the gauge algebra. This is specially interesting in the case of higher-dimensional supergravity, where superalgebras come naturally split into distinct subspaces. Examples of the use of the method within the transgression/Chern–Simons–Antoniadis–Savvidy framework will be studied elsewhere.

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