ABSTRACT. It is proved that a convex polyhedral scatterer of impedance type can be uniquely determined by the electric far-field pattern of a single incident plane wave with fixed direction, polarization and wavenumber. Our proof relies on the reflection principle for Maxwell’s equations with the impedance (or Leontovich) boundary condition enforcing on a hyper-plane. We prove that it is impossible to analytically extend the total field across any vertex of the scatterer. This leads to a data-driven inversion scheme for imaging an arbitrary convex polyhedron.

Keywords: Uniqueness, inverse electromagnetic scattering, polyhedral scatterer, reflection principle, impedance boundary condition, single incident wave, data-driven scheme.

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1. Introduction and main result

The propagation of time-harmonic electromagnetic waves in a homogeneous isotropic medium in $\mathbb{R}^3$ is modelled by the Maxwell’s equations

$$\nabla \times E(x) - ikH(x) = 0, \quad \nabla \times H(x) + ikE(x) = 0 \quad \text{for } x \in \mathbb{R}^3, \quad (1.1)$$

where $E$ and $H$ represent the electric and magnetic fields respectively and $k > 0$ is known as the wave number. Let $E^{in}$ and $H^{in}$ satisfying Equation (1.1) denote the incident electric and magnetic fields respectively. Consider the scattering of given incoming waves $E^{in}$ and $H^{in}$ from a convex polyhedral scatterer $D \subset \mathbb{R}^3$ coated by a thin dielectric layer, which can be modeled by the impedance (or Leontovich) boundary value problem of the Maxwell equations (1.1) in $\mathbb{R}^3 \setminus D$. Then the total fields $E = E^{in} + E^{sc}$, $H = H^{in} + H^{sc}$, where $E^{sc}$ and $H^{sc}$ denote the scattered fields, are governed by the following set of Equations (1.2)-(1.5):

$$\nabla \times E - ikH = 0, \quad \nabla \times H + ikE = 0 \quad \text{in } \mathbb{R}^3 \setminus D, \quad (1.2)$$

$$E = E^{in} + E^{sc}, \quad H = H^{in} + H^{sc} \quad \text{in } \mathbb{R}^3 \setminus D, \quad (1.3)$$

$$\lim_{|x| \to \infty} (H^{sc} \times x - |x|E^{sc}) = 0, \quad (1.4)$$

$$\nu \times (\nabla \times E) + i\lambda \nu \times (\nu \times E) = 0 \quad \text{on } \partial D, \quad (1.5)$$

where $\nu$ denotes the outward unit normal to $\partial D$ and the impedance coefficient $\lambda > 0$ is supposed to be a constant. Equation (1.4) is known as the Silver-Müller radiation condition and is uniform in all directions $\hat{x} := x/|x|$. For the existence and uniqueness of the solution $(E, H)$ to the forward system (1.2)-(1.5), we refer to [9] when $\partial D$ is $C^2$-smooth and to [3,5] when $\partial D$ is Lipschitz with connected exterior. Moreover, the Silver-Müller radiation condition (1.4) ensures that scattered
fields $E^{sc}$ and $H^{sc}$ satisfy the following asymptotic behaviour (see [9])

$$E^{sc}(x) = \frac{e^{ik|x|}}{|x|} \left( E^{\infty}(\hat{x}) + O \left( \frac{1}{|x|} \right) \right), \quad \text{as } |x| \to \infty, \quad (1.6)$$

$$H^{sc}(x) = \frac{e^{ik|x|}}{|x|} \left( H^{\infty}(\hat{x}) + O \left( \frac{1}{|x|} \right) \right), \quad \text{as } |x| \to \infty \quad (1.7)$$

where the vector fields $E^{\infty}$ and $H^{\infty}$ defined on the unit sphere $\mathbb{S}^2$, are called the electric and magnetic far-field patterns of the scattered waves $E^s$ and $H^s$, respectively. It is well known that $E^{\infty}$ and $H^{\infty}$ are analytic functions with respect to the observation direction $\hat{x} \in \mathbb{S}^2$ and satisfy the following relations

$$H^{\infty} = \nu \times E^{\infty}, \quad \nu \cdot E^{\infty} = \nu \cdot H^{\infty} = 0, \quad (1.8)$$

where $\nu$ denotes the unit normal vector to the unit sphere $\mathbb{S}^2$.

Given the incoming wave $(E'^m, H'^m)$ and the scatterer $D \subset \mathbb{R}^3$, the direct problem arising from electromagnetic scattering is to find the scattered fields $(E^{sc}, H^{sc})$ and their far-field patterns. The inverse problem to be considered in this paper consists of determining the location and shape of $D$ from knowledge of the far-field patterns $(E^{\infty}, H^{\infty})$. We assume that the incident fields $E'^m$ and $H'^m$ are time-harmonic plane waves given by

$$E'^m(x, d, p) = pe^{ikx \cdot d} \quad H'^m(x, d, p) = (d \times p)e^{ikx \cdot d} \quad (1.9)$$

where $d \in \mathbb{S}^2$ is known as the incident direction and $p \in \mathbb{R}^3 \setminus \{0\}$ with $p \perp d$ is known as the polarization direction. Throughout this paper the wavenumber $k$, the polarization $p$ and the incident direction $d \in \mathbb{S}^2$ are all fixed.

The present article is concerned with a uniqueness result of determining the convex polyhedral scatterer $D$ appearing in the system of Equations (1.2)--(1.5) from a single electric far-field pattern $E^{\infty}$ over all observation directions $\hat{x} \in \mathbb{S}^2$. More precisely, we prove the following result:

**Theorem 1.1.** Let $D_1$ and $D_2$ be two convex polyhedral scatterers of impedance type. For fixed incident plane waves $(E'^m, H'^m)$, we denote by $E_j^{\infty}(\hat{x}; k, p, d)$ ($j = 1, 2$) the electric far-field patterns of the scattering problem (1.2)-(1.5) when $D = D_j$. Then the relation

$$E_1^{\infty}(\hat{x}; k, p, d) = E_2^{\infty}(\hat{x}; k, p, d) \quad \text{for all } \hat{x} \in \mathbb{S}^2 \quad (1.10)$$

implies that $D_1 = D_2$.

It is widely open how to uniquely determine the shape of a general impenetrable/penetrable scatterer using a single far-field pattern. As in the acoustic case [1][6][24], quite limited progress has also been made in inverse time-harmonic electromagnetic scattering. To the best of our knowledge, global uniqueness with a single measurement data is proved only for perfectly conducting obstacles with restrictive geometric shapes such like balls [23] and convex polyhedrons [9] Chapter 7.1. Without the convexity condition, it was shown in [26] that a general perfect polyhedral conductor (the closure of which may contain screens) can be uniquely determined by the far-field pattern for plane wave incidence with one direction and two polarizations. We shall prove Theorem 1.1 by using the reflection principle for Maxwell’s equations with the impedance boundary condition enforcing on a hyper-plane. It seems that such a reflection principle has not been studied in prior works, although the corresponding principle under the perfectly conducting boundary condition is well known in optics (see e.g. [26]). Theorem 1.1 carries over to perfectly conducting polyhedrons with a single far-field pattern (see Corollary 3.5), and thus improves the acoustic uniqueness result for impedance scatterers [7] where two incident directions were used. It is also worthy mentioning other works
in the literature related to reflection principles for the Helmholtz and Navier equations together with their applications to uniqueness in inverse acoustic and elastic scattering [1, 6, 8, 12, 14, 25]. The unique determination of non-convex polygons and polyhedrons of impedance type with a single far-field pattern still seems open. More remarks concerning our uniqueness proof will be concluded at the end of Section 3.

In the second part of this paper, we shall propose a novel non-iterative scheme for imaging an arbitrary convex-polyhedron from a single electric far-field pattern. The Linear Sampling Method in inverse electromagnetic scattering [2–5] was earlier studied with infinite number of plane waves at fixed energy. We are mostly motivated by the uniqueness proof of Theorem 1.1 (see also Corollary 3.4) and the one-wave factorization method in inverse elastic scattering [14]. The proposed scheme is essentially a domain-defined sampling approach, requiring no forward solvers. Promising features of our imaging scheme are summarised as follows.

(i) It requires lower computational cost and only a single measurement data. The proposed domain-defined indicator function involves only inner product calculations. Since the number of sampling variables is comparable with the original Linear Sampling Method and Factorization Method (2, 19, 21), the computational cost is not heavier than the aforementioned pointwise-defined sampling methods. (ii) It can be interpreted as a data-driven approach, because it relies on measurement data corresponding to a priori given scatterers (which are also called test domains in the literature or samples in the terminology of learning theory and data science). There is a variety of choices on the shape and physical properties of these samples, giving arise to quite “rich” a priori sample data in addition to the measurement data of the unknown target. In this paper, we choose perfectly conducting balls (or impedance balls) as test domains, because the spectra of the resulting far-field operator admit explicit representations. However, these test domains can also be chosen as any other convex penetrable and impenetrable scatterers, provided the classical factorization scheme for imaging this test domain can be verified using all incident and polarization directions. We refer to [20] for the Factorization Method applied to inverse electromagnetic medium scattering problems. (iii) It provides a necessary and sufficient criterion for imaging convex polyhedrons (see Theorem 4.2). We prove that the wave fields cannot be analytic around any vertex of \( D \) (see Corollary 3.4), excluding the possibility of analytical extension across a vertex. Some other domain-defined sampling approaches such as the range test approach [21, 22] and the one-wave no-response test [25, 29] usually pre-assume such extensions, leading to a sufficient condition for imaging general targets. Our approach is comparable with the one-wave enclosure method by Ikehata [17, 18] for capturing singular points of \( \partial D \). Detailed discussions on identifying singular points can be found in the monograph [32, Chapter 15]. If \( \partial D \) contains no singular points, only partial information of \( D \) can be numerically recovered; see [27] where the linear sampling method with a single far-field pattern was tested.

We organize the article as follows. In §2 we prove the reflection principle for Maxwell’s equations with the impedance boundary condition on a hyper-plane in \( \mathbb{R}^3 \) (see Theorem 2.1). Using this reflection principle we prove in §3 the main uniqueness Theorem 1.1. The reconstruction scheme will be described in §4.

2. Reflection Principle for Maxwell equations

Let \( \Omega \subseteq \mathbb{R}^3 \) be an open connected set which is symmetric with respect to a plane \( \Pi \) in \( \mathbb{R}^3 \) and we define by \( \gamma := \Omega \cap \Pi \). Denote by \( \Omega^+ \) and \( \Omega^- \) the two symmetric parts of \( \Omega \) which are divided by \( \Pi \) and by \( R_\Pi \) the reflection operator about \( \Pi \), that is, if \( x \in \Omega^\pm \) then \( R_\Pi x \in \Omega^\mp \) for \( x = (x_1, x_2, x_3) \in \Omega \). Throughout this article, \( \Omega \) will be assumed in such a way that any line segment
with end points in Ω and intersected with Π by the angle π/2 lies completely in Ω. In other words, the projection of any line segment in Ω onto the hyperplane Π is a subset of γ. This geometrical condition was also used in [11] where the reflection principle for the Helmholtz equation with the impedance boundary condition was verified. Now consider the time-harmonic Maxwell equations with the impedance boundary condition by

\[
\nabla \times E - ikH = 0, \quad \nabla \times H + ikE = 0, \quad \text{in} \quad \Omega^+,
\]

\[
\nu \times (\nabla \times E) + i\lambda \nu \times (\nu \times E) = 0, \quad \text{on} \quad \gamma \subset \Pi.
\] (2.2)

It is well known from (Theorem 6.4 in [9]) that a solution \((E, H)\) of Equation (2.1) satisfies the vectorial Helmholtz equations with the divergence-free condition:

\[
\Delta E + k^2 E = 0, \quad \Delta H + k^2 H = 0, \quad \nabla \cdot E = \nabla \cdot H = 0.
\] (2.3)

Since Equations (2.3) and (2.1) are rotational invariant, without loss of generality we can assume that the plane Π mentioned above coincides with the ox1x2-plane, i.e., Π = \(\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}\). Consequently, we have \(\nu = e_3 := (0,0,1)^T\) and \(R_\Pi x = (x_1, x_2, -x_3)\). In this section, we study the reflection principle for solutions to the Maxwell equation satisfying the impedance boundary condition on γ. Our aim is to extend the solution \((E, H)\) of Equation (2.1) from \(\Omega^+\) to \(\Omega^-\) by an analytical formula. The reflection principle is stated as follows.

**Theorem 2.1.** Let \(\Pi := \{(x', x_3) \in \mathbb{R}^3 : x_3 = 0\}\) with \(x' := (x_1, x_2)\), \(\gamma \subset \Omega \cap \Pi\) and \(\Omega^\pm := \{(x_1, x_2, x_3) \in \Omega : \pm x_3 > 0\}\). Assume that \((E, H)\) satisfies Equation (2.1) with the boundary condition (2.2). Then \((E, H)\) can be analytically extended to \(\Omega^-\) as a solution to (2.1). Moreover, the extended electric field \(\tilde{E} := (\tilde{E}_1, \tilde{E}_2, \tilde{E}_3)^T\) is given explicitly by \(\tilde{E} := E\) in \(\Omega^+ \cup \gamma\) and

\[
\tilde{E}_3(x) := E_3(x', x_3) - \frac{2k^2}{i\lambda} \int_0^{-x_3} e^{\frac{k^2}{2\lambda}(s+x_3)} E_3(x', s) ds,
\] (2.4)

\[
\tilde{E}_j(x', x_3) := E_j(x', x_3) + 2i\lambda \int_0^{-x_3} e^{-i\lambda(s+x_3)} E_j(x', s) ds,
\] (2.5)

for \(j = 1, 2\) in \(\Omega^-\).

Before going to the proof of Theorem 2.1, we first state and prove the reflection principle for the Helmholtz equation with an impedance boundary condition. The result in the following Lemma 2.2 has already been proved in [11], but since we have shortened and simplified the arguments of [11], we prefer to provide its proof here. Our proof of Theorem 2.1 is essentially motivated by proof of Lemma 2.2.

**Lemma 2.2.** [11] Let \(\Omega, \Pi, \gamma\) and \(\Omega^\pm\) be defined as in Theorem 2.1. If \(u\) is a solution to the boundary value problem of the Helmholtz equation

\[
\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega^+, \quad \partial_n u + i\lambda u = 0 \quad \text{on} \quad \gamma.
\] (2.6)
Then $u$ can be extended from $\Omega^+$ to $\Omega$ as a solution to the Helmholtz equation, with the extended solution $\tilde{u}$ given by the formula

$$
\tilde{u}(x) := u(x_1, x_2, -x_3) + 2i\lambda e^{-i\lambda x_3} \int_0^{-x_3} e^{-i\lambda s}u(x_1, x_2, s)ds \quad \text{in} \quad \Omega^-.
$$

(2.7)

Proof. Define the function $v := \partial_3 u + i\lambda u$ in $\Omega^+ \cup \gamma$. Then using (2.6), $v$ is a solution to

$$
\Delta v + k^2 v = 0 \quad \text{in} \quad \Omega^+, \quad v = 0 \quad \text{on} \quad \gamma.
$$

Now using the reflection principle (see [11]) for the Helmholtz equation with Dirichlet boundary condition, the solution $v$ can be extended to $\Omega$ as a solution to the Helmholtz equation by $\tilde{v} := v$ in $\Omega^+ \cup \gamma$ and by $\tilde{v}(x) := -v(x_1, x_2, -x_3)$ for $x \in \Omega^-$. Motivated by the relation between $v$ and $u$ in $\Omega^+$, we look for the extended solution $\tilde{u}$ of $u$ as the unique solution to the following ordinary equation

$$
\partial_3 \tilde{u} + i\lambda \tilde{u} = \tilde{v} \quad \text{in} \quad \Omega, \quad \tilde{u} = u \quad \text{on} \quad \gamma.
$$

Multiplying the above equation by $e^{i\lambda x_3}$ and integrating between 0 to $x_3$, we get

$$
\tilde{u}(x) = e^{-i\lambda x_3}u(x_1, x_2, 0) + e^{-i\lambda x_3} \int_0^{x_3} e^{i\lambda s} (\partial_s u(x_1, x_2, s) + i\lambda u(x_1, x_2, s)) ds.
$$

(2.8)

Since $\tilde{v} = v$ in $\Omega^+$, Equation (2.8) can be rewritten in $\Omega^+$ as

$$
\tilde{u}(x) = e^{-i\lambda x_3}u(x_1, x_2, 0) + e^{-i\lambda x_3} \int_0^{x_3} e^{i\lambda s} (\partial_s u(x_1, x_2, -s) + i\lambda u(x_1, x_2, -s)) ds,
$$

which together with the integration by parts yields $\tilde{u} = u$ in $\Omega^+$. For $x \in \Omega^-$, we can express the right hand side of (2.8) in terms of $u|_{\Omega^+}$ by

$$
\tilde{u}(x) = e^{-i\lambda x_3}u(x_1, x_2, 0) - e^{-i\lambda x_3} \int_0^{-x_3} e^{i\lambda s} (-\partial_s u(x_1, x_2, -s) + i\lambda u(x_1, x_2, -s)) ds
$$

$$
= e^{-i\lambda x_3}u(x_1, x_2, 0) + e^{-i\lambda x_3} \int_0^{-x_3} e^{-i\lambda t} (\partial_t u(x_1, x_2, t) + i\lambda u(x_1, x_2, t)) dt.
$$

After using the integration by parts, we get

$$
\tilde{u}(x) = u(x_1, x_2, -x_3) + 2i\lambda e^{-i\lambda x_3} \int_0^{-x_3} e^{-i\lambda s}u(x_1, x_2, s)ds, \quad x \in \Omega^-,
$$

which proves Equation (2.7). To check that $\tilde{u}$ is indeed the extended solution, one needs to verify $\Delta \tilde{u} + k^2 \tilde{u} = 0$ in $\Omega$. Simple calculations show that $\Delta \tilde{u} + k^2 \tilde{u} = 0$ in $\Omega^+ \cup \Omega^-$. For $x \in \Omega^-$, it is easy to see

$$
\partial_3 \tilde{u}(x) = -\partial_3 u(x_1, x_2, -x_3) + 2\lambda^2 e^{-i\lambda x_3} \int_0^{-x_3} e^{-i\lambda s}u(x_1, x_2, s)ds - 2i\lambda u(x_1, x_2, -x_3).
$$
Taking $x \to \gamma$, this implies
\[ \partial_x^3 \tilde{u}(x_1, x_2, 0) = -\partial_x^3 u(x_1, x_2, 0) - 2i\lambda u(x_1, x_2, 0). \]
Thus after using (2.6), we conclude that $\partial_x^3 \tilde{u} = \partial_x^3 \tilde{u}$ on $\gamma$. Since the Cauchy data of $\tilde{u}$ keep continuous on $\gamma$, we have $\Delta \tilde{u} + k^2 \tilde{u} = 0$ in $\Omega$.

### 2.1. Proof of Theorem 2.1

From (2.1), we deduce that $E$ is a solution to
\[ \nabla \times (\nabla \times E) + k^2 E = 0 \quad \text{in } \Omega^+, \quad e_3 \times (\nabla \times E) + i\lambda e_3 \times (e_3 \times E) = 0 \quad \text{on } \gamma \quad (2.9) \]
where
\[ e_3 \times (e_3 \times E) = (-E_1, E_2, 0)^T, \quad e_3 \times (\nabla \times E) = (\partial_1 E_3 - \partial_3 E_1, \partial_2 E_3 - \partial_3 E_2, 0)^T. \]

Analogously to the Helmholtz case, we define $F$ by
\[ F := e_3 \times (\nabla \times E) + i\lambda e_3 \times (e_3 \times E) =: (F_1, F_2, 0)^T \quad \text{in } \Omega^+, \]
\[ F_1 = \partial_1 E_3 - \partial_3 E_1 - i\lambda E_1, \quad F_2 = \partial_2 E_3 - \partial_3 E_2 - i\lambda E_2, \quad (2.10) \]
and define $V$ as
\[ V := \frac{\partial_1 F_1 + \partial_2 F_2}{i\lambda} = \frac{\partial_2 E_3 - \partial_3 E_1 - i\lambda \partial_1 E_1 + \partial_2 E_3 - \partial_3 E_2 - i\lambda \partial_2 E_2}{i\lambda}. \]

Now using the fact that $\Delta E_j + k^2 E_j = 0$ for $1 \leq j \leq 3$ and $\nabla \cdot E = 0$ in $\Omega^+$, we have in $\Omega^+$ that
\[ V = \frac{-\partial_3^2 E_3 - k^2 E_3 - \partial_3 (\partial_1 E_1 + \partial_2 E_2) - i\lambda (\partial_1 E_1 + \partial_2 E_2)}{i\lambda} = \partial_3 E_3 - \frac{k^2}{i\lambda} E_3. \]
Since $F_1 = F_2 = 0$ on $\gamma \subseteq \{ x \in \mathbb{R}^3 : \ x_3 = 0 \}$, we get $\partial_1 F_1 = \partial_2 F_2 = 0$ on $\gamma$ and thus
\[ \Delta E_3 + k^2 E_3 = 0 \quad \text{in } \Omega^+, \quad \partial_3 E_3 - k^2/(i\lambda) E_3 = 0 \quad \text{on } \gamma. \quad (2.11) \]

Applying Lemma 2.2, we can extend $E_3$ from $\Omega^+$ to $\Omega$ by $\tilde{E}_3 := E_3$ in $\Omega^+ \cup \gamma$ and
\[ \tilde{E}_3(x) := E_3(x', -x_3) - \frac{2k^2}{i\lambda} e^{i\lambda x_3} \int_0^{x_3} e^{i\lambda s} E_3(x', s) \, ds \quad \text{in } \Omega^-, \]
which gives the extension formula for $E_3$.

To find the extension formula for $E_j$ $(j = 1, 2)$, we observe that $F_j$ $(j = 1, 2)$ given by (2.10) satisfy the Helmholtz equation with the Dirichlet boundary condition,
\[ \Delta F_j + k^2 F_j = 0 \quad \text{in } \Omega^+, \quad F_j = 0 \quad \text{on } \gamma. \]
Applying the reflection principle with the Dirichlet boundary condition (see [11]), we can extend $F_j$ through $\tilde{F}_j := F_j$ in $\Omega^+ \cup \gamma$ and $\tilde{F}_j(x) := -F_j(x', -x_3)$ in $\Omega^-$. As done for the Helmholtz equation, we will derive the extension formula for $E_j$ for $j = 1, 2$ by considering the boundary value problem of the ordinary equation (cf. (2.10))
\[ \partial_3 \tilde{E}_j + i\lambda \tilde{E}_j - \partial_3 \tilde{E}_3 = -\tilde{F}_j \quad \text{in } \Omega, \quad \tilde{E}_j = E_j \quad \text{on } \gamma, \]
where $\tilde{E}_j$ $(j = 1, 2)$ denote the extended functions. Multiplying the above equation by $e^{i\lambda x_3}$ and integrating between 0 to $x_3$, we have
\[ \int_0^{x_3} e^{i\lambda s} \left( \partial_s \tilde{E}_j(x', s) + i\lambda \tilde{E}_j(x', s) \right) \, ds - \int_0^{x_3} e^{i\lambda s} \partial_j \tilde{E}_3(x', s) \, ds = -\int_0^{x_3} e^{i\lambda s} \tilde{F}_j(x', s) \, ds, \]
which gives us

$$\tilde{E}_j(x) = e^{-i\lambda x_3}E_j(x', 0) + \int_0^{x_3} e^{i\lambda(s-x_3)} \partial_j \tilde{E}_3(x', s) ds - \int_0^{x_3} e^{i\lambda(s-x_3)} \tilde{F}_j(x', s) ds. \quad (2.12)$$

Note that since $\tilde{F}_j = F_j$ and $\tilde{E}_3 = E_3$ in $\in \Omega^+$, Equation (2.12) can be rewritten as

$$\tilde{E}_j(x) = e^{-i\lambda x_3}E_j(x', 0) + e^{-i\lambda x_3} \int_0^{x_3} e^{i\lambda s} \partial_j E_3(x', s) ds$$

$$- e^{-i\lambda x_3} \int_0^{x_3} e^{i\lambda s} (\partial_j E_3(x', s) - \partial_s E_j(x', s) - i\lambda E_j(x', s)) ds$$

which can be proved to be identical with $E_j$ in $\Omega^+$ by applying the integration by parts. Next, we want to simplify the expression of $\tilde{E}_j(x)$ given by (2.12) in $\Omega^-$. Using the expression for $\tilde{F}_j$ and $\tilde{E}_3$ (see (2.4)), we obtain

$$\tilde{E}_j(x) = e^{-i\lambda x_3}E_j(x', 0) + \int_0^{x_3} e^{i\lambda(s-x_3)} (\partial_j E_3(x', -s) + \partial_s E_j(x', -s) - i\lambda E_j(x', -s)) ds$$

$$+ \int_0^{x_3} e^{i\lambda(s-x_3)} \partial_j E_3(x', -s) ds - \frac{2k^2}{i\lambda} \int_0^{x_3} e^{i\lambda(s-x_3)} \left( \int_0^{-s} e^{\frac{k^2}{i\lambda} (t-s)} \partial_j E_3(x', t) dt \right) ds.$$

This gives

$$\tilde{E}_j(x) = E_j(x', -x_3) + 2i\lambda \int_0^{-x_3} e^{-i\lambda(s+x_3)} E_j(x', s) ds - 2 \int_0^{-x_3} e^{-i\lambda(s+x_3)} \partial_j E_3(x', s) ds$$

$$+ \frac{2k^2}{i\lambda} \int_0^{x_3} e^{i\lambda(s-x_3)} \int_0^{-s} e^{\frac{-k^2}{i\lambda} (t-s)} \partial_j E_3(x', -t) dt ds.$$

Changing the order of integration in the last term of the previous equation, we obtain

$$\tilde{E}_j(x) = E_j(x', -x_3) + 2i\lambda \int_0^{-x_3} e^{-i\lambda(s+x_3)} E_j(x', s) ds - 2 \int_0^{-x_3} e^{-i\lambda(s+x_3)} \partial_j E_3(x', s) ds$$

$$+ \frac{2k^2}{i\lambda} \int_0^{x_3} e^{-i\lambda x_3} \frac{t=x_3}{t=0} e^{-\frac{k^2}{i\lambda} t} \partial_j E_3(x', -t) \int_0^{s=x_3} e^{\frac{-k^2}{i\lambda} s} ds dt,$$

$$= E_j(x', -x_3) + 2i\lambda \int_0^{-x_3} e^{-i\lambda(s+x_3)} E_j(x', s) ds - 2 \int_0^{-x_3} e^{-i\lambda(s+x_3)} \partial_j E_3(x', s) ds$$

$$+ \frac{2k^2}{k^2 - \lambda^2} \int_0^{x_3} e^{-\frac{k^2}{i\lambda} (s-x_3)} \partial_j E_3(x', -s) ds - \frac{2k^2}{k^2 - \lambda^2} \int_0^{x_3} e^{i\lambda(s-x_3)} \partial_j E_3(x', -s) ds.$$
After combining similar terms, we get

\[ \tilde{E}_j(x) = E_j(x', -x_3) + 2i\lambda \int_0^{-x_3} e^{-i\lambda(s+x_3)} E_j(x', s) ds \]

\[ + \frac{2\lambda^2}{k^2 - \lambda^2} \int_0^{-x_3} e^{-i\lambda(s+x_3)} \partial_j E_j(x', s) ds - \frac{2k^2}{k^2 - \lambda^2} \int_0^{-x_3} e^{i\lambda(s+x_3)} \partial_j E_3(x', s) ds. \]

This proves Equation (2.5). Remark that the right hand side of \( \tilde{E}_j \) (\( j = 1, 2 \)) depends on both \( E_j \) and \( E_3 \) in \( \Omega^+ \).

In order to show that \( \tilde{E}_j \) given by (2.5) and (2.4) are indeed the required extension formula for \( E_j \), we need to verify that \( \Delta \tilde{E}_j + k^2 \tilde{E}_j = 0 \) and \( \nabla \cdot \tilde{E} = 0 \) in \( \Omega \). For this purpose, we shall proceed with the following three steps.

Step 1. Prove that the Cauchy data of \( \tilde{E}_j \) taking from \( \Omega^\pm \) are identical on \( \gamma \). By Lemma 2.2, this is true for the third component \( \tilde{E}_3 \). On the other hand, it is clear from Equation (2.5) that \( E_j \) (\( j = 1, 2 \)) are continuous functions in \( \Omega \). Therefore, we only need to show that \( \partial_3^+ \tilde{E}_j = \partial_3^- \tilde{E}_j \) on \( \gamma \), \( j = 1, 2 \). Simple calculations show that

\[
\partial_3 \tilde{E}_j(x) =
\begin{cases}
\partial_3 E_j(x', x_3) & \text{in } \Omega^+,
-\partial_3 E_j(x', -x_3) + 2\lambda^2 e^{-i\lambda x_3} \int_0^{-x_3} e^{-i\lambda s} E_j(x', s) ds - 2i\lambda E_j(x', -x_3) \\
-\frac{2i\lambda^3}{k^2 - \lambda^2} e^{-i\lambda x_3} \int_0^{-x_3} e^{-i\lambda s} \partial_j E_j(x', s) ds - \frac{2\lambda^2}{k^2 - \lambda^2} \partial_3 E_3(x', -x_3) \\
-\frac{2k^4}{i\lambda(k^2 - \lambda^2)} e^{i\lambda x_3} \int_0^{-x_3} e^{i\lambda s} \partial_j E_3(x', s) ds + \frac{2k^2}{k^2 - \lambda^2} \partial_3 E_3(x', -x_3) & \text{in } \Omega^-,
\end{cases}
\]

from which it follows that

\[
\partial_3^- \tilde{E}_j(x', 0) = -\partial_3^+ E_j(x', 0) - 2i\lambda E_j(x', 0) + \frac{2(k^2 - \lambda^2)}{k^2 - \lambda^2} \partial_3 E_3(x', 0).
\]

Recalling the relation \( \partial_3 E_3(x', 0) - \partial_3^+ E_j(x', 0) - i\lambda E_j(x', 0) = 0 \) for \( j = 1, 2 \), we get from the previous equation that \( \partial_3^+ E_j = \partial_3^- \tilde{E}_j \) on \( \gamma \).

Step 2. Prove that \( \Delta \tilde{E}_j + k^2 \tilde{E}_j = 0 \) in \( \Omega \) for \( j = 1, 2, 3 \). In view of Step 1, it suffices to verify that \( \Delta \tilde{E}_j + k^2 \tilde{E}_j = 0 \) in \( \Omega^- \) for \( j = 1, 2 \). From Equation (2.5), we have

\[
\Delta \tilde{E}_j(x) = \Delta E_j(x', -x_3) + I_1 + I_2 + I_3, \quad x \in \Omega^-,
\]

(2.13)
where, for some fixed \( j = 1 \) or \( j = 2 \),

\[
I_1 := 2i\lambda\Delta \left( e^{-i\lambda x_3} \int_0^{x_3} e^{-i\lambda s} E_j(x', s) ds \right)
\]

\[
I_2 := \frac{2\lambda^2}{k^2 - \lambda^2} \Delta \left( e^{-i\lambda x_3} \int_0^{x_3} e^{-i\lambda s} \partial_j E_3(x', s) ds \right)
\]

\[
I_3 := -\frac{2k^2}{k^2 - \lambda^2} \Delta \left( e^{k^2 x_3} \int_0^{x_3} e^{k^2 s} \partial_j E_3(x', s) ds \right).
\]

Using \( \Delta E_j + k^2 E_j = 0 \) for \( j = 1, 2, 3 \) in \( \Omega^+ \) and applying integration by parts, the three terms \( I_j \) (\( j = 1, 2, 3 \)) can be calculated as follows:

\[
I_1 = -2i\lambda\partial_3 E_j(x', -x_3) + 2i\lambda e^{-i\lambda x_3} \partial_3 E_j(x', 0) + 2\lambda^2 E_j(x', -x_3) - 2\lambda^2 e^{-i\lambda x_3} E_j(x', 0) + 2i\lambda e^{-i\lambda x_3} \partial_3 E_j(x', 0) - 2i\lambda^3 \partial_3 E_j(x', -x_3)
\]

\[
I_2 = -\frac{2\lambda^2}{k^2 - \lambda^2} \partial_3 \partial_j E_3(x', -x_3) + \frac{2\lambda^2}{k^2 - \lambda^2} e^{-i\lambda x_3} \partial_3 \partial_j E_3(x', 0) - \frac{2\lambda^3}{k^2 - \lambda^2} \partial_3 \partial_j E_3(x', -x_3) + \frac{2\lambda^4}{k^2 - \lambda^2} e^{-i\lambda x_3} \partial_3 \partial_j E_3(x', 0) - \frac{2\lambda^4}{k^2 - \lambda^2} e^{-i\lambda x_3} \partial_3 \partial_j E_3(x', -x_3)
\]

\[
I_3 = -\frac{2k^2}{k^2 - \lambda^2} \partial_3 \partial_j E_3(x', -x_3) + \frac{2k^2}{k^2 - \lambda^2} e^{k^2 x_3} \partial_3 \partial_j E_3(x', 0) + \frac{2k^4}{k^2 - \lambda^2} \partial_3 \partial_j E_3(x', -x_3) - \frac{2k^4}{k^2 - \lambda^2} \partial_3 \partial_j E_3(x', 0) - \frac{2k^6}{k^2 - \lambda^2} e^{k^2 x_3} \partial_3 \partial_j E_3(x', s) ds
\]

\[
+ \frac{2k^4}{k^2 - \lambda^2} e^{k^2 x_3} \int_0^{x_3} e^{k^2 s} \partial_j E_3(x', s) ds + \frac{2k^6}{k^2 - \lambda^2} e^{k^2 x_3} \int_0^{x_3} e^{k^2 s} \partial_j E_3(x', s) ds
\]

\[
- \frac{2k^4}{k^2 - \lambda^2} \partial_3 \partial_j E_3(x', -x_3) - \frac{2k^2}{k^2 - \lambda^2} \partial_3 \partial_j E_3(x', -x_3).
\]
Using again the Helmholtz equation $\Delta E_j + k^2 E_j = 0$ in $\Omega^+$ and inserting expressions of $I_1$, $I_2$ and $I_3$ into Equation (2.13), we get

$$\Delta \tilde{E}_j(x) = -k^2 \left( E_j(x', -x_3) + 2i\lambda \int_0^{-x_3} e^{-i\lambda (s+x_3)} E_j(x', s) ds \right)$$

$$+ \frac{2\lambda^2}{k^2 - \lambda^2} \int_0^{-x_3} e^{-i\lambda (s+x_3)} \partial_j E_3(x', s) ds - \frac{2k^2}{k^2 - \lambda^2} \int_0^{-x_3} e^{i\lambda x_3} \partial_j E_3(x', s) ds$$

$$+ 2i\lambda e^{-i\lambda x_3} \left( \partial_3 E_3(x', 0) + i\lambda E_j(x', 0) \right) + \frac{2\lambda^2}{k^2 - \lambda^2} e^{-i\lambda x_3} \partial_j \left( \partial_3 E_3(x', 0) + i\lambda \partial_j E_j(x', 0) \right)$$

$$- \frac{2k^2}{k^2 - \lambda^2} e^{i\lambda x_3} \partial_j \left( \partial_3 E_3(x', 0) - \frac{k^2}{i\lambda} E_3(x', 0) \right).$$

This together with Equation (2.5) and the following boundary conditions

$$\partial_3 E_3(x', 0) - \frac{k^2}{i\lambda} E_3(x', 0) = 0, \quad \partial_j E_3(x', 0) - \partial_3 E_3(x', 0) - i\lambda E_j(x', 0) = 0$$

leads to the relation $\Delta \tilde{E}_j + k^2 \tilde{E}_j = 0$ in $\Omega^-$. 

Step 3. Prove that $\nabla \cdot \tilde{E} = 0$ in $\Omega$. It follows from Step 1 that $\tilde{E} \in C^4(\Omega)$. Hence, we only need to show the divergence-free condition in $\Omega^-$. For $x \in \Omega^-$, we see

$$\nabla \cdot \tilde{E}(x) = \partial_1 E_1(x', -x_3) + \partial_2 E_2(x', -x_3) - \partial_3 E_3(x', -x_3)$$

$$+ 2i\lambda \int_0^{-x_3} e^{-i\lambda (s+x_3)} \left( \partial_1 E_1 + \partial_2 E_2 \right)(x', s) ds$$

$$+ \frac{2\lambda^2}{k^2 - \lambda^2} \int_0^{-x_3} e^{-i\lambda (s+x_3)} \left( \partial_1^2 E_3 + \partial_2^2 E_3 \right)(x', s) ds$$

$$- \frac{2k^2}{k^2 - \lambda^2} \int_0^{-x_3} e^{i\lambda x_3} \left( \partial_1^2 E_3 + \partial_2^2 E_3 \right)(x', s) ds$$

$$+ 2k^4 \frac{1}{\lambda^2} \int_0^{-x_3} e^{i\lambda x_3} E_3(x', s) ds + \frac{2k^2}{i\lambda} E_3(x', -x_3).$$
Now using $\nabla \cdot E = 0$ and $\Delta E_j + k^2 E_j = 0$ for $1 \leq j \leq 3$ in $\Omega^+$, we have

$$\nabla \cdot \vec{E}(x) = -2\partial_3 E_3(x', -x_3) - 2i\lambda \int_0^{-x_3} e^{-i\lambda(s+x_3)} \partial_s E_3(x', s) ds$$

$$- \frac{2\lambda^2}{k^2 - \lambda^2} \int_0^{-x_3} e^{-i\lambda(s+x_3)} \partial_s^2 E_3(x', s) ds - \frac{2\lambda^2 k^2}{k^2 - \lambda^2} \int_0^{-x_3} e^{-i\lambda(s+x_3)} E_3(x', s) ds$$

$$+ \frac{2k^2}{k^2 - \lambda^2} \int_0^{-x_3} e^{\frac{k^2}{2\pi} (s+x_3)} \partial_s^2 E_3(x', s) ds + \frac{2k^4}{k^2 - \lambda^2} \int_0^{-x_3} e^{\frac{k^2}{2\pi} (s+x_3)} E_3(x', s) ds$$

$$+ \frac{2k^4}{\lambda^2} \int_0^{-x_3} e^{\frac{k^2}{2\pi} (s+x_3)} E_3(x', s) ds + \frac{2k^2}{i\lambda} E_3(x', -x_3).$$

Using integration by parts, we can rewrite $J_1$, $J_2$ and $J_3$ as

$$J_2 = \frac{2\lambda^2}{k^2 - \lambda^2} \partial_3 E_3(x', -x_3) - \frac{2\lambda^2}{k^2 - \lambda^2} e^{-i\lambda x_3} \partial_3 E_3(x', 0) + \frac{2i\lambda^3}{k^2 - \lambda^2} E_3(x', -x_3)$$

$$- \frac{2i\lambda^3}{k^2 - \lambda^2} e^{-i\lambda x_3} E_3(x', 0) - \frac{2\lambda^4}{k^2 - \lambda^2} e^{-i\lambda x_3} \int_0^{-x_3} e^{-i\lambda s} E_3(x', s) ds$$

$$+ \frac{2\lambda^2 k^2}{k^2 - \lambda^2} e^{-i\lambda x_3} \int_0^{-x_3} e^{-i\lambda s} E_3(x', s) ds,$$

$$J_1 = 2i\lambda E_3(x', -x_3) - 2i\lambda e^{-i\lambda x_3} E_3(x', 0) - 2\lambda^2 e^{-i\lambda x_3} \int_0^{-x_3} e^{-i\lambda s} E_3(x', s) ds,$$

$$J_3 = \frac{2k^2}{k^2 - \lambda^2} \partial_3 E_3(x', -x_3) - \frac{2k^2}{k^2 - \lambda^2} e^{\frac{i\lambda}{2\pi} x_3} \partial_3 E_3(x', 0) + \frac{2ik^4}{\lambda (k^2 - \lambda^2)} E_3(x', -x_3)$$

$$- \frac{2ik^4}{\lambda (k^2 - \lambda^2)} e^{\frac{i\lambda}{2\pi} x_3} E_3(x', 0) - \frac{2k^6}{\lambda (k^2 - \lambda^2)} e^{\frac{i\lambda}{2\pi} x_3} \int_0^{-x_3} e^{\frac{i\lambda}{2\pi} s} E_3(x', s) ds.$$

Inserting them into Equation (2.14), applying the integration by parts and rearranging terms, we get

$$\nabla \cdot \vec{E}(x) = 2e^{-i\lambda x_3} \left[ \left( i\lambda + \frac{i\lambda^3}{k^2 - \lambda^2} \right) E_3(x', 0) + \frac{\lambda^2}{k^2 - \lambda^2} \partial_3 E_3(x', 0) \right]$$

$$+ \frac{2k^2}{k^2 - \lambda^2} e^{\frac{k^2}{2\pi} x_3} \left( -\partial_3 E_3(x', 0) + \frac{k^2}{i\lambda} E_3(x', 0) \right).$$
Recalling \( \partial_3 E_3(x', 0) - \frac{k^2}{\lambda} E_3(x', 0) = 0 \), we finally get \( \nabla \cdot \tilde{E} = 0 \) in \( \Omega^- \).

By far we have proved that the function \( \tilde{E} \) with components given by Equations (2.5) and (2.4) is the extension of the solution \( E \) of the Maxwell equations.

3. Proof of Theorem 1.1

This section is devoted to proving the uniqueness result for recovering convex polyhedral scatterers of impedance type, which was stated in Theorem 1.1. Assuming two of such scatterers generate identical far-field patterns for a fixed electromagnetic plane wave, we shall prove via reflection principle that the scattered electric fields could be analytically extended into the whole space, which is impossible. Similar ideas were employed in [14, 15, 17] for proving uniqueness in inverse conductivity and elastic scattering problems. Later we shall remark why our approach cannot be applied to convex polyhedral scatterers and compare our arguments with the uniqueness proof of [7] in the Helmholtz case. The following results straightforwardly follow from Theorem 2.1.

**Corollary 3.1.** Let \( (E, H) \) be a solution to the Maxwell equations (2.1) in \( x_3 > 0 \) fulfilling the impedance boundary condition (2.2) on \( \Pi = \{ x \in \mathbb{R}^3 : x_3 = 0 \} \). Then \( (E, H) \) can be extended from the upper half-space \( x_3 \geq 0 \) to the whole space.

**Corollary 3.2.** Let \( \Omega = \Omega^+ \cup \gamma \cup \Omega^- \) be the domain defined in Theorem 2.1. Given a subset \( D \subset \Omega^- \), suppose that \( (E, H) \) is a solution to the Maxwell equations (2.1) in \( \Omega \setminus D \) fulfilling the impedance boundary condition (2.2) on \( \gamma \). Then \( (E, H) \) can be analytically extended onto \( \overline{D} \).

The above corollaries will be used in the proof of Theorem 1.1.

3.1. Proof of Theorem 1.1

Recall from Equation (1.10) that \( E^{\infty}_1(\hat{x}; k, p, d) = E^{\infty}_2(\hat{x}; k, p, d) \) for all \( \hat{x} \in S^2 \). Using the Rellich’s lemma (see [9]) we get

\[
E_1 = E_2 \quad \text{and} \quad H_1 = H_2 \quad \text{in} \quad \mathbb{R}^3 \setminus (D_1 \cup D_2).
\]

Assuming that \( D_1 \neq D_2 \), we shall prove the uniqueness by deriving a contradiction. By the convexity of \( D_1 \) and \( D_2 \), we may assume that there exists a vertex \( O \) of \( \partial D_1 \) and a neighborhood \( V_O \) of \( O \) such that \( V_O \cap \overline{D_2} = \emptyset \). Next using the impedance boundary condition of \( E_1 \) on \( \partial D_1 \) and \( E_1 = E_2 \) in \( \mathbb{R}^3 \setminus (D_1 \cup D_2) \), we have that \( \nu \times (\nabla \times E_2) + i \lambda \nu \times (\nu \times E_2) = 0 \) on \( V_O \cap \partial D_1 \). Since \( D_1 \) is a convex polyhedron, there exists a finite number of convex polygonal faces \( \Lambda_j (j \geq 3) \) of \( D_1 \) whose closure meet at \( O \); for example, see Figure 1 where \( j = 4 \) (left) and \( j = 3 \) (right). Denote by \( \Pi_j \) the maximum extension of \( \Lambda_j \) in \( \mathbb{R}^3 \setminus \overline{D_2} \). Then we get

\[
\nu \times (\nabla \times E_2) + i \lambda \nu \times (\nu \times E_2) = 0 \quad \text{on} \quad \Pi_j
\]
due to the analyticity of \( E_2 \) in the exterior of \( \overline{D_2} \).

Case (i): One of \( \Pi_j \) coincides with some hyper-plane \( \Pi \) in \( \mathbb{R}^3 \setminus \overline{D_2} \) (see Figure 1 right). Since \( D_2 \) is convex, it must lie completely on one side of the plane \( \Pi \). By Corollary 3.1, the electric field \( E_2 \) can be analytically extended to \( \mathbb{R}^3 \) as a solution to the Maxwell equation. This implies that \( E_2^{sc} \) is an entire radiating solution to the Maxwell equation. Consequently, we get \( E_2^{sc} \equiv 0 \) and thus the total field \( E_2 = pe^{ikx-d} \) satisfies the impedance boundary condition on \( \partial D_2 \).

Case (ii): None of \( \Pi_j \) coincides with an entire hyper-plane in \( \mathbb{R}^3 \setminus \overline{D_2} \) (see Figure 1 left). Denote by \( \tilde{\Pi}_j \supset \Pi_j \) the hyper-plane in \( \mathbb{R}^3 \) containing \( \Lambda_j \). We shall prove via reflection principle that \( E_2 \) satisfies the impedance boundary condition on each \( \tilde{\Pi}_j \), which again leads to the relation \( E_2 = pe^{ikx-d} \) by repeating the same arguments in case (i).
Without loss of generality we take \( j = 1 \) and consider the plane \( \Pi_1 \supset \Pi_2 \supset \Lambda \). Recall that \( \Lambda_1 \subset \mathbb{R}^2 \) is a convex polygonal face lying on the boundary of the polyhedron \( D_2 \) and that the total field \( E_2 \) is analytic near \( O \), a corner point of \( \partial \Lambda_1 \). It suffices to prove that \( E_2 \) is analytic on \( \partial \Lambda_1 \). Let \( O_1 \in \partial \Lambda_1 \) be a neighboring corner of \( O \), which is also a vertex of \( D_2 \). By the convexity of \( D_2 \), there exists at least one face \( \Lambda_j \) with \( j \neq 1 \) such that the finite line segment \( OO_1 \) lies completely on one side of the hyper-plane \( \Pi_j \) and the projection of \( OO_1 \) onto \( \Pi_j \), which we denote by \( L \), is a subset of \( \Pi_j \subset \Pi_1 \). We refer to Figure 2 for an illustration of the proof in two dimensions. Since \( D_2 \) does not intersect with \( \Pi_j \), one can always find a symmetric domain \( \Omega_0 \subset \mathbb{R}^3 \setminus D_2 \) with respect to \( \Pi_j \) such that \( OO_1 \subset \Omega \) and \( L \subset (\Omega_0 \cap \Pi_j) \). Recall that \( E_2 \) fulfills the impedance boundary on \( \Pi_j \). Now applying Corollary 3.3 (with \( D = OO_1 \) and \( \Omega = \Omega_0 \) to \( (E_2, H_2) \), we find that \( E_2 \) must also be analytic on \( OO_1 \), and in particular, \( E_2 \) is analytic near \( O_1 \). Analogously, one can prove the analyticity of \( E_2 \) at another neighboring corner point \( O_2 \) to \( O \in \partial \Lambda_1 \) and also the analyticity on the line segment \( OO_2 \subset \partial \Lambda_1 \). Applying the same arguments to \( O_1 \) and \( O_2 \) in place of \( O \), we can conclude that \( E_2 \) is analytic on the closure of \( \Lambda_1 \). This implies that \( E_2 \) satisfies the impedance boundary condition on \( \Pi_1 \supset \Pi_1 \) and thus \( E_2 = E^{in} \) in \( \mathbb{R}^3 \).

To proceed with the proof, we recall from cases (i) and (ii) that \( E^{in} \) fulfills the impedance boundary condition on \( \partial D_2 \). By Equation (1.9), it then follows that

\[
\text{i} k \nu \times (d \times p) + \text{i} \lambda \nu \times (\nu \times p) = 0,
\]

for any outward unit normal vector \( \nu \) to \( \partial D_2 \) and fixed vectors \( d, p \) such that \( d \bot p \). Applying the identity \( A \times (B \times C) = (A \cdot C) B - (A \cdot B) C \), we get

\[
\text{i} k (\nu \cdot p) d - \text{i} k (\nu \cdot d) p + \text{i} \lambda (\nu \cdot p) \nu - \text{i} \lambda (\nu \cdot \nu) p = 0.
\]

Taking the inner product with \( d \times p \) in (3.2) gives \( (\nu \cdot p) \{ \nu \cdot (d \times p) \} = 0 \) for any outward unit normal \( \nu \) to \( \partial D_2 \). This implies that the normal vector of \( \partial D_2 \) must be orthogonal to either \( p \) or \( d \times p \). Since \( d \cdot p = 0 \), the faces of \( D_2 \) should be parallel to two orthogonal hyper-planes, which is impossible for consisting the boundary of a convex polyhedron. Therefore we have \( D_1 = D_2 \). This proves Theorem 1.1. \( \square \)

Below we present several remark concerning the above uniqueness proof.
Remark 3.3. (i) Using the reflection principle for the Helmholtz equation (see Theorem 2.1 or [7]), one can prove uniqueness in recovering a convex polyhedral or polygonal scatterer of acoustically impedance-type with a single incoming wave; see Figure 2 for an illustration of the uniqueness proof in two dimensions. This improves the result of [7] with two incident directions. (ii) The uniqueness proof for convex sound-soft, sound-hard and perfectly conducting polyhedrons with a single incoming wave (see [7] and [9, Chapters 5.1 and 7.1]) cannot apply to the case of impedance condition. In fact, it is impossible to derive a contradiction from Equation (3.2) using only normal directions at faces $\Lambda_j$ ($j \geq 3$) meeting at the vertex $O \in \partial D_1$. For instance, suppose that $p = e_1$, $d \times p = e_2$ and $\eta = \lambda/k > 1$. Simple calculations show that Equation (3.1) for the unknown vector $\nu = c_1 e_1 + c_2 e_2 + c_3 e_3$ has three linearly independent solutions with the coefficients given by

$$c_1 = 0, c_2 = \sqrt{1 - \eta^{-2}}, c_3 = -1/\eta; \quad c_1 = \pm \sqrt{\eta^2 - 1}, c_2 = 0, c_3 = -\eta.$$

**Figure 2.** Illustration of two different convex polygonal scatterers: $D_2$ is a square and $D_1 := D_2 \cup G$ where $G$ denotes the gap domain between $D_1$ and $D_2$. There are two sides of $D_1$ around the corner $O$, both of them cannot be extended to a straight line in $\mathbb{R}^2 \setminus D_2$.

As a consequence of the proof of Theorem 1.1 we have the following corollaries.

**Corollary 3.4.** Let $D \subset \mathbb{R}^3$ be a convex polyhedron and let $E = E^{\text{in}} + E^{\text{sc}}$ be the solution to Equations (1.2)-(1.5). Then $E$ cannot be analytically extended from $\mathbb{R}^3 \setminus D$ to the interior of $D$ across a vertex of $\partial D$, or equivalently, $E$ cannot be analytic on the vertices of $D$.

**Corollary 3.5.** Let $D \subset \mathbb{R}^3$ be a perfectly conducting polyhedron such that $\mathbb{R}^3 \setminus D$ is connected. Suppose that $E = E^{\text{in}} + E^{\text{sc}}$ with $E^{\text{in}} = p e^{ikx \cdot d}$ is a solution to Equations (1.2)-(1.4) with the perfectly conducting boundary condition $\nu \times E = 0$ on $\partial D$. Then $\partial D$ can be uniquely determined by a single electric far-field pattern $E^{\infty}$ over all observation directions. Moreover, $E$ cannot be analytically extended from $\mathbb{R}^3 \setminus D$ to the interior of $D$ across a vertex of $\partial D$.

**Proof.** Suppose that two perfect polyhedral conductors $D_1$ and $D_2$ generate identical electric far-field patterns but $D_1 \neq D_2$. Combining the path arguments of [26] and the uniqueness proof in Theorem 1.1, one can always find a perfectly conducting hyperplane $\Pi \subset \mathbb{R}^3$ such that $D_j$ ($j = 1$ or $j = 2$) lies completely on one side of $\Pi$. This is possible, because the reflection principle with the perfectly conducting boundary condition is of ‘point-to-point’ type. This implies that the total electric field $E$ can be analytically extended into the whole space, leading to $E^{\text{sc}} \equiv 0$ in $\mathbb{R}^3$ and thus $\nu \times E^{\text{in}} = 0$ on $\partial D$. Hence, we get $\nu \times p = 0$ for any normal direction on $\partial D$, which is impossible. □
We remark that the perfectly conducting polyhedron in Corollary 3.5 is allowed to be non-convex, but cannot contain two-dimensional screens on its closure. Our arguments cannot be carried over to non-convex polyhedral scatterers of impedance type, because the reflection principle established in Theorem 2.1 is valid in a symmetric domain with a certain geometric assumption (see Section 2). For non-convex polyhedrons, one cannot find a vertex $O$ around which the total field is analytic under the assumption (1.10). In acoustics and optics, the unique determination of non-convex polyhedrons of impedance type with a single far-field pattern still seems open.

4. A DATA-DRIVEN IMAGING SCHEME

The 'singularity' of $E^{sc}$ at vertices (see Corollary 3.4) motivates us to locate all vertices of $D$ so that the position and shape of $D$ can be recovered from a single measurement data. The aim of this section is to establish a data-driven inversion scheme for imaging arbitrarily convex-polyhedral scatterers. Motivated by the one-wave factorization method in inverse elastic scattering [14], we shall propose a domain-defined indicator functional to characterize an inclusion relationship between a test domain and our target. Being different from other domain-defined sampling approaches ([21, 22, 28, 29]) arising from inverse scattering, our scheme will be interpreted as a data-driven method, because it also relies on measurement data corresponding to a priori given test domains. In this paper, we shall take for simplicity perfectly conducting balls with different centers and radii as test domains. Similar techniques were used in the Extended Linear Sampling Method [27] for extracting information of a sound-soft obstacle from a single far-field pattern.

Consider the scattering of an incident plane wave $E^{in} = ik (d \times p) \times de^{ikx \cdot d}$ by a ball $B_h(z) := \{ x \in \mathbb{R}^3 : |x - z| < a \}$ with $h > 0$, $z \in \mathbb{R}^3$, where $d \in S^2$ is the incident direction and $p \in \mathbb{R}^3$ is a polarization vector. Then the total field $E = E^{in} + E^{sc}$ satisfies

$$\begin{cases} \nabla \times (\nabla \times E) - k^2 E = 0 & \text{in } |x - z| > h, \\ (\nu \times E) \times \nu = 0 & \text{on } |x - z| = h, \\
\lim_{|x| \to \infty} (E^{sc} \times \hat{x} + \frac{1}{ik} \nabla \times E^{sc}) |x| = 0, & \hat{x} = \frac{\hat{z}}{r},
\end{cases}$$

where $\hat{z} = \frac{x}{r}$.

It is well known that (4.1) has a series solution $E(\hat{x}; d, p, h, z)$ for a given $E^{in}(x; d, p)$ ([31]). For notational convenience we will omit the dependance of solutions on $d$, $p$ and $k$ (all of them are fixed in our arguments) and only indicate the dependance on the center $z \in \mathbb{R}^3$ and radius $h > 0$ of the ball $B_h(z)$. Denote by $E^{\infty}(\hat{x}; h, z)$ the electric far-field pattern of the scattered electric field $E^{sc}$. We expand $E^{\infty}(\hat{x}; h, z)$ into a series by using vector spherical harmonics. For any orthonormal system $Y^m_n$, $m = -n, \ldots, n$ of spherical harmonics of order $n > 0$, the tangential fields defined on the unit sphere

$$U^m_n(\hat{x}) := \frac{1}{\sqrt{n(n+1)}} \text{Grad} Y^m_n(\hat{x}); \quad V^m_n(\hat{x}) := \hat{x} \times U^m_n(\hat{x})$$

are called vector spherical harmonics of order $n$. By coordinate translation, it is easy to check that $E^{\infty}$ can be expanded into the convergent series ([10, 31])

$$E^{\infty}(\hat{x}; h, z)e^{ikz \cdot \hat{x}} = 4\pi \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left( u_n^{(h)} [U^m_n(\hat{x}) \cdot p] U^m_n + v_n^{(h)} [V^m_n \cdot p] V^m_n \right)$$

where

$$u_n^{(h)} := \frac{u_n'(k h)}{(\zeta_n^{(1)})'(k h)} \in \mathbb{C}, \quad v_n^{(h)} := -\frac{v_n(k h)}{\zeta_n^{(1)}(k h)} \in \mathbb{C},$$

and $u_n^{(h)}$ and $v_n^{(h)}$ are the scattered field component expressed in the form of vector spherical harmonics.
with \( \psi_n(t) := t j_n(t) \) and \( \zeta_n^{(1)}(t) := th_n^{(1)}(t) \). Here \( j_n \) is the spherical Bessel function of order \( n \) and \( h_n^{(1)} \) is the spherical Hankel function of first kind of order \( n \). Denote the far-field operator \( F^{(z,h)} : T(S^2) \to T(S^2) \) by

\[
(F^{(z,h)}g)(\hat{x}) := \int_{S^2} E^\infty(\hat{x}, d, g(d), h, z)ds(d)
\]

where \( T(S^2) := \{ g \in L^2(S^2)^3 : g(\hat{x}) \cdot \hat{x} = 0 \text{ for all } \hat{x} \in S^2 \} \) denotes the tangential space defined on \( S^2 \). The expression (4.2) shows that \( F^{(z,h)} \) is diagonal in the basis

\[
\tilde{U}_{m,n}^{(z)}(\hat{x}) := e^{-ikz \hat{x}} U_{m,n}^z(\hat{x}), \quad \tilde{V}_{m,n}^{(z)}(\hat{x}) := e^{-ikz \hat{x}} V_{m,n}^z(\hat{x}).
\]

It can be verified that \( (4\pi u_n^{(h)}, 4\pi v_n^{(h)}) \) and \( (\tilde{U}_{m,n}^{(z)}, \tilde{V}_{m,n}^{(z)}) \) are eigenvalues and the associated eigenvectors of \( F^{(z,h)} \). Note that the eigenvalues depend on the radius \( h \) only and the eigenfunctions depend on the location \( z \) only. We refer to [10] for detailed analysis when the ball is located at the origin. The general case can be easily justified via coordinate translation.

To proceed, we suppose that \( w^\infty \in T(S^2) \) is the electric field pattern of some radiating electric field \( w^* \) in \( |x| > a \) for some \( a > 0 \) sufficiently large. Introduce the function

\[
I_{w^\infty}(z, h) := \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \frac{|\langle w^\infty, \tilde{U}_{m,n}^{(z)} \rangle|^2}{|u_n^{(h)}|^2} + \frac{|\langle w^\infty, \tilde{V}_{m,n}^{(z)} \rangle|^2}{|v_n^{(h)}|^2} \right),
\]

where \( z \in \mathbb{R}^3 \) and \( h > 0 \) will be referred to as sampling variables in this paper. Equation (4.4) can be regarded as a functional defined on the test domain \( B_h(z) \). If the above series is convergent, we shall prove below that the radiated electric field \( w^* \) can be analytically extended at least to the exterior of the test domain \( B_h(z) \). For simplicity we still denote by \( w^* \) the extended solution.

**Lemma 4.1.** Suppose that \( k^2 \) is not the Dirichlet eigenvalue (that is, the tangential component vanishes) of the operator \( \text{curl curl} \) over the ball \( B_h(z) \). We have \( I_{w^\infty}(z, h) < \infty \) if and only if \( w^\infty \) is the far-field pattern of the radiating field \( w^* \) which satisfies

\[
\nabla \times (\nabla \times w^*) - k^2 w^* = 0 \quad \text{in} \quad |x - z| > h, \quad \nu \times w^* \times \nu \in H^{-1/2}(\partial B_h(z)).
\]

Here \( H^{-1/2}(\partial D) \) denotes the trace space of \( H(\text{curl}, D) = \{ \phi \in L^2(D) : \nabla \times \phi \in L^2(D) \} \) of a bounded Lipschitz domain \( D \subset \mathbb{R}^3 \), given by

\[
H^{-1/2}(\partial D) := \{ u \in \left( H^{-1/2}(\partial D) \right)^3 : \nu \cdot u = 0 \text{ on } \partial D \text{ and } (\nabla \times u) \in H^{-1/2}(\partial D) \}.
\]

**Proof.** Without loss of generality we may assume that \( B_h(z) \) is located at the origin, so that \( \tilde{U}_{m,n}^{(z)} = U_{m,n}^z \) and \( \tilde{V}_{m,n}^{(z)} = V_{m,n}^z \). Since the assumption on the wavenumber \( k \) ensures that (see [33, Chapter 5] for related discussions)

\[
j_n(t) \neq 0 \quad \text{and} \quad j_n(t) + tj_n'(t) \neq 0 \quad \text{for} \quad t = kh, \ n = 1, 2, \ldots ,
\]

we have \( |u_n^{(h)}| \neq 0 \) and \( |v_n^{(h)}| \neq 0 \) for all \( n \). By [6, Equation 6.73] it follows that \( w^* \) can be expressed as

\[
w^*(x) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=-n}^{n} \left[ a_n^m q_n^m(x) + b_n^m \nabla \times q_n^m(x) \right] \quad \text{in} \quad |x| > a,
\]
with the coefficients \( a_n^m, b_n^m \in \mathbb{C} \) and \( q_n^m(x) := \nabla \times \{ x h_n^m(k|x|) Y_n^m(\hat{x}) \} \). Correspondingly, the far-field pattern \( w^\infty \) is given by (see Equation (6.74) on page 219 in [9]):

\[
w^\infty(\hat{x}) = \frac{i}{k} \sum_{n=1}^{\infty} \frac{1}{i^{n+1}} \sum_{m=-n}^{n} (ikb_n^m U_n^m(\hat{x}) - a_n^m V_n^m(\hat{x})).
\]

Inserting the above expression into (4.4), we get

\[
I_{w^\infty}(\theta, h) = \frac{1}{4\pi} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left( \frac{|b_n^m|^2}{|u_n^{(h)}|} + \frac{|a_n^m|^2}{|v_n^{(h)}|} \right), \quad \theta = (0, 0, 0).
\]

(4.6)

To analyze the convergence of the above series, we need the asymptotic behavior of \( u_n^{(h)} \) and \( v_n^{(h)} \) as \( n \to +\infty \). Using the asymptotics of special functions for large orders, it is easy to observe that

\[
\frac{1}{|u_n^{(h)}|} = \frac{\zeta_n^{(1)}}{\psi_n} \left| \frac{(th_n^{(1)}(t))'}{t=jn(t)} \right| \sim \frac{C_1}{n} \left| (h_n^{(1)})'(kh) \right|^2,
\]

\[
\frac{1}{|v_n^{(h)}|} = \frac{\zeta_n^{(1)}(kh)}{\psi_n(kh)} = \frac{h_n^{(1)}(kh)}{j_n(kh)} \sim C_2 |n| |h_n^{(1)}(kh)|^2,
\]

as \( n \to \infty \), where \( C_1, C_2 \in \mathbb{C} \) are fixed constants. Thus it follows from (4.6) that

\[
I_{w^\infty}(\theta, h) \sim \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left( \frac{C_1 b_n^m}{n} |h_n^{(1)}(kh)|^2 + C_2 a_n^m |n| |h_n^{(1)}(kh)|^2 \right)
\]

(4.7)

On the other hand, it is seen from the expression of \( w^s \) that on \( |x| = h \),

\[
(\hat{x} \times w^s \times \hat{x}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left( \frac{b_n^m}{h} (th_n^{(1)}(t))' \right|_{t=kh} U_n^m(\hat{x}) - a_n^m h_n^{(1)}(kh)V_n^m(\hat{x}) \right).
\]

By definition of the \( H^{-1/2}_{\text{curl}}(\partial B_h(z)) \) norm (see e.g. [33, Chapter 5] and [31, Chapter 9.3.3]) we obtain

\[
\| \hat{x} \times w^s \times \hat{x} \|_{H^{-1/2}_{\text{curl}}(\partial B_h(z))} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left( \frac{|b_n^m|^2}{|n|} \left[ (th_n^{(1)}(t))' \right|_{t=kh} \right]^2 + \sqrt{n(n+1)} |a_n^m|^2 |h_n^{(1)}(kh)|^2 \right)
\]

\[
\sim \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left( \frac{1}{n} |b_n^m|^2 |h_n^{(1)}(kh)|^2 + n |a_n^m|^2 |h_n^{(1)}(kh)|^2 \right).
\]

(4.8)

Obviously, (4.7) and (4.8) have the same convergence. In the same manner, one can prove that (4.7) has the same convergence with \( \| \nu \times w^s \|_{H^{-1/2}(\partial D)} \) where

\[
H^{-1/2}_{\text{div}}(\partial D) := \{ u \in (H^{-1/2}(\partial D))^3 : \nu \cdot u = 0 \text{ on } \partial D \text{ and } (\text{Div } u) \in H^{-1/2}(\partial D) \}.
\]

Using the relation

\[
\| \nu \times w^s \|_{L^2(\partial B_h(z))} + \| \nu \times w^s \times \nu \|_{L^2(\partial B_h(z))} \leq C \left( \| \nu \times w^s \|_{H^{-1/2}_{\text{div}}(\partial B_h(z))} + \| \nu \times w^s \times \nu \|_{H^{-1/2}_{\text{curl}}(\partial B_h(z))} \right),
\]

\[
\| \nu \times w^s \times \nu \|_{L^2(\partial B_h(z))} \leq C \left( \| \nu \times w^s \|_{H^{-1/2}_{\text{div}}(\partial B_h(z))} + \| \nu \times w^s \times \nu \|_{H^{-1/2}_{\text{curl}}(\partial B_h(z))} \right),
\]

\[
\| \nu \times w^s \|_{L^2(\partial B_h(z))} \leq C \left( \| \nu \times w^s \|_{H^{-1/2}_{\text{div}}(\partial B_h(z))} + \| \nu \times w^s \times \nu \|_{H^{-1/2}_{\text{curl}}(\partial B_h(z))} \right),
\]
we conclude that the tangential component of \( w^s \) on \( \partial B_h(z) \) is convergent in the \( L^2 \)-sense, if \( I_{w^s}(o, h) < \infty \). This together with \cite{9} Theorem 6.27 implies that \( w^s \) is a solution to the Maxwell equation in \(|x| > h\). The proof of Lemma 4.1 is thus complete. \qed

Combining Lemma 4.1 and Corollary 3.4, we may characterize an inclusion relation between \( D \) and \( B_h(z) \) through the measurement data \( E^\infty \) of our target and the spectra of the far-field operator \( F^{(z,h)} \) corresponding to the test ball.

**Theorem 4.2.** Let \( E^\infty \) be the electric far-field pattern of a convex-polyhedral scatterer \( D \) with a constant impedance coefficient. Suppose that \( k^2 \) is not the Dirichlet eigenvalue of the operator \( \text{curl curl} \) over the ball \( B_h(z) \). It holds that

\[
I_{E^\infty}(z, h) < \infty \quad \text{if and only if} \quad D \subset \overline{B_z(h)}.
\]

Hence we have

\[
D = \bigcap_{I_{E^\infty}(z, h) < \infty} B_h(z).
\]

**Proof.** If \( D \subset \overline{B_z(h)} \), the scattered electric field \( E^{sc} \) is well defined in \( \{x : |x - z| > h\} \) which lies in the exterior of \( D \). Hence, \( E^{sc} \) satisfies (4.5) and by Lemma 4.1 it holds that \( I_{E^\infty}(z, h) < \infty \). On the other hand, suppose that \( I_{E^\infty}(z, h) < \infty \) but the relation \( D \subset \overline{B_z(h)} \) does not hold. Since \( D \) is a convex polyhedron, there must exist at least one vertex \( O \) of \( \partial D \) such that \( |O - z| > h \). Again using Lemma 4.1, we conclude that \( E^{sc} \) can be extended from \( \mathbb{R}^3 \setminus D \) to the exterior of \( B_h(z) \). This implies that \( E^{sc} \) is analytic at \( O \), which contradicts Corollary 3.4. \qed

By Theorem 4.2, the function \( h \to I_{E^\infty}(z, h) \) for fixed \( |z| = R \) will blow up when \( h \geq \max_{y \in \partial D} |y - z| \), indicating a rough location of \( D \) with respect to \( z \in \mathbb{R}^3 \). In Table 4 we describe an inversion procedure for imaging an arbitrary convex-polyhedron \( D \) by taking both \( z \in \partial B_R \) and \( h \) as sampling variables. The mesh for discretizing \( h \in (0, 2R) \) should be finer than the mesh for \( z \in \partial B_R \). To avoid the fact that \( k^2 \) is a Dirichlet eigenvalue of \( \text{curl curl} \) over \( B_{hi}(z_j) \), one may use coated balls by a thin dielectric layer (which can be modeled by the impedance boundary condition) as test domains in place of our choice of perfectly conducting balls. We refer to \cite{10, Section 3.1} for description of the eigenvalues of the far-field operator corresponding to such coated balls centered at the origin. If \( \lambda > 0 \), \( k^2 \) cannot be an impedance eigenvalue of \( \text{curl curl} \) over any boundary domain. It should be remarked that the test domains can also be taken as penetrable balls under the assumption that \( k^2 \) is not an interior transmission eigenvalue. Both Theorem 4.2 and Lemma 4.1 can be carried over to these test domains. Finally, it is worthy mentioning that a regularization scheme should be employed to truncate the series (4.6), because the eigenvalues \( u_n^{(h)} \) and \( v_n^{(h)} \) decay very fast and the calculation of the inner product between \( E^\infty \) and the eigenfunctions \( \left( \tilde{U}_{n,m}^{(z)}, \tilde{V}_{n,m}^{(z)} \right) \) is usually polluted by data noise and numerical errors. We refer to \cite{30} for numerical examples in inverse acoustic scattering. Numerical tests for Maxwell’s equations will be reported in our forthcoming publications.
Table 1. Data-driven scheme for imaging convex polyhedral scatterers

**Step 1** Collect the measurement data \( E^\infty(\hat{x}) \) for all \( \hat{x} \in S^2 \) and suppose that \( D \subset B_R := \{ x : |x| < R \} \) for some large \( R > 0 \).

**Step 2** Choose sampling variables \( z_j \in \{ x : |x| = R \} \) and \( h_i \in (0, 2R) \) to get the spectra of the far-field operator \( F^{(z,h)} \) corresponding to testing balls \( B_{h_i}(z_j) \subset B_R \).

**Step 3** Calculate the domain-defined indicator function \( I_{E^\infty}(z_j, h_i) \) by (4.4) with \( w^\infty = E^\infty \). In particular, it follows from Theorem 4.2 that

\[
\begin{align*}
    h_i < \max_{y \in \partial D} |z_j - y| &\rightarrow I_{E^\infty}(z_j, h_i) = \infty, \\
    h_i \geq \max_{y \in \partial D} |z_j - y| &\rightarrow I_{E^\infty}(z_j, h_i) < \infty.
\end{align*}
\]

**Step 4** Image \( D \) as the intersection of all test balls \( B_{h_i}(z_j) \) such that \( I_{E^\infty}(z_j, h_i) < \infty \).

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