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ABSTRACT
Multiple states emerge in the fluid flow with certain or sudden external interference. In this paper, multiple solutions of the velocity and temperature distributions of power-law fluids are found when the fluid is flowing through a viscous sheet with suction or blowing. By coupling the sheet dynamics with the surrounding fluid equations by the stress balance, the investigation results shed some light on many engineering applications of flow on various soft surfaces, for example, the synthetic plastics. The present research considers the nonuniformity of the mass suction or blowing effects, which is connected with both the stretching velocity and the thickness of the sheet. A similar transformation is adopted to transform the boundary layer equations into a series of ordinary differential equations, and multiple solutions are obtained. The existence of similar solutions has some limitations. The multiple solutions of the friction coefficient and the generalized Nusselt number affected by the power-law index $n$ and the suction/blowing rate $f_w$ are shown, and we also discuss the influence of the generalized Prandtl number $N_{Pr}$ and the Eckert number $Ec$ on the velocity and temperature distributions.

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NOMENCLATURE

| Symbol | Description                          |
|--------|--------------------------------------|
| $c_p$  | fluid specific heat                  |
| $Ec$   | Eckert number                        |
| $f_w$  | the dimensionless mass suction or blowing parameter |
| $g$    | dimensionless function               |
| $h_0$  | half-thickness of the sheet          |
| $h$    | sheet thickness                      |
| $n$    | power-law index                      |
| $N_{Pr}$ | generalized Prandtl number            |
| $T_{sh}$ | temperature at the sheet surface     |
| $T_{\infty}$ | temperature at infinity              |
| $\Delta T$ | temperature difference              |
| $u, v$ | velocity components along the x and y directions |
| $U$    | horizontal velocity of the sheet     |
| $U_0$  | velocity of the sheet at $x = 0$     |
| $U_L$  | velocity of the sheet at $x = L$     |
| $V_w$  | constant mass suction or blowing parameter |
| $x, y$ | coordinates along the sheet and normal to it, respectively |

Greek symbols

| Symbol | Description                          |
|--------|--------------------------------------|
| $\alpha$ | thermal diffusivity                  |
| $\gamma$ | constant parameter                   |
| $\mu_{sh}$ | viscosity of the sheet               |
| $\mu$  | viscosity of the Newtonian fluid     |
| $\mu$  | consistency of the power-law fluid   |
| $\rho$ | density of the fluid                 |
| $\rho_s$ | density of the sheet                 |
| $\tau_{xy}$ | shear stress of the fluid            |
| $\omega$ | constant parameter                   |

I. INTRODUCTION

Non-Newtonian fluids are common in nature. It is extremely important to study a type of non-Newtonian fluid, i.e., the power-law fluid, which is widely used in industrial productions. In recent years, research on power-law fluids has emerged in an endless stream. So far, people have analyzed the fluid flow and heat transfer of the non-Newtonian power-law fluid under different kinds...
of conditions. For instance, Ahmed et al.\textsuperscript{7} considered the axisymmetric flow and heat transfer of a boundary layer flow of the non-Newtonian power-law fluid on a stretching plate. Xun et al.\textsuperscript{34} analyzed the flow field and heat transfer of the Ostwald-de Waele fluid with a reduction in the power-law index on a rotary disk with changeable thickness. Shokouhmand and Soleimani\textsuperscript{1} discussed the influence of viscous dissipation on temperature distributions in the case of a power-law fluid on a moving plate with suction or injection. Sherer et al.\textsuperscript{25} researched a mathematical model about the natural convection of the power-law fluid with a constant energy source in a closed square cavity. Abdel-Gaied and Eid\textsuperscript{27} studied the heat and mass transfer of the power-law fluid in a porous medium on an axisymmetric, two-dimensional, arbitrary shape. Bahraei et al.\textsuperscript{29} finished the computational fluid dynamics (CFD) simulation on the power-law nanofluid of laminar flow.

When fluids flow over a soft material, the combination of the kinetics of the sheet and the fluid dynamics has an influence on the flow field and heat transfer. Many scholars have done a lot of research on this aspect. Al-Housseiny et al.\textsuperscript{15} analyzed the boundary layer flow with the coupling of a fluid to a thin plate. Ahmad et al.\textsuperscript{13} conducted a study on the magnetohydrodynamics of Newtonian fluids, which achieved good results when the fluid equations were coupled with those describing the sheet dynamics. Ahmad\textsuperscript{30} also studied the nanofluid flow in the case of coupling a nanofluid with a flat plate. In recent years, Liu et al.\textsuperscript{36} has made progress in the direction of coupling the power-law fluid and the dynamics of the thin plate. In the present research, we investigate the power-law fluid flowing over a viscous sheet, and the equation describing the sheet dynamics is combined with the governing equations of the surrounding fluid.

In predecessors’ work, Fourier’s law has been generalized to simulate the heat transfer of some certain non-Newtonian fluids. For instance, Pop et al.\textsuperscript{11} assumed that the thermal conductivity of certain non-Newtonian fluids varied with the velocity gradient. According to this model, Subba et al.\textsuperscript{11} made a comparison of the numerical and analytical solutions of the power-law fluid boundary layer problem in the cubic critical point. Grosan and Ece et al.\textsuperscript{13,14} discussed some natural convection problems of the power-law fluid in the boundary layer, and they gave similar solutions to the classical plume wall problem. Later, considering the impact of the power-law kinematic viscosity on the heat transfer, Zheng et al.\textsuperscript{15} put forward that the thermal conductivity would be a power-law function of the temperature gradient. They analyzed the fully developed power-law natural convective heat transfer in a circular tube to verify this model. Zheng’s model has been adopted in the present research.

Nonuniform suction and injection are very common in engineering, such as in filter systems or over membrane components. Roy and Saikrishnan\textsuperscript{16,17} studied the nonuniform injection (suction) impact on the incompressible boundary layer flow when temperature-dependent viscosity was taken into account. Ganapathirao and Ravindran\textsuperscript{18} assessed the influence of nonuniform suction/blowing stabilization on a vertical wedge with chemical reactions in convective magnetohydrodynamic (MHD) boundary layer flows. Ganapathirao and Ravindran\textsuperscript{18} also investigated the combined effects of Soret Dufour and chemical reactions on mixed convective with nonuniform mass injection/suction. Furthermore, Babu et al.\textsuperscript{19} focused on the creeping transport of viscous fluids through suction and injection channels. Wang and Hayat\textsuperscript{21} solved the wave flow of Maxwell fluid by considering a permeable plate with blowing or suction. Ravindran et al.\textsuperscript{25} showed that nonuniform single or double suction/injection had a certain effect on unsteady mixed convective MHD flow. Ganji et al.\textsuperscript{25} investigated the properties of the ferrofluid in the case of suction or injection. Ellahi et al.\textsuperscript{33} analyzed the heat transfer of a viscoelastic fluid with suction conditions on a stretched sheet.

Nonuniform suction or blowing can affect the fluid flow and heat transfer considerably, while multiple solutions of the governing equations appear. The emergence of multiple solutions, an interesting discovery when people investigate the fluid flow and heat transfer problems with certain or sudden disturbance, has become the direction of many investigations. Li et al.\textsuperscript{34} analyzed the multiple states of the laminar flow in the channel by adding a transverse magnetic field. Turkylmazoglu et al.\textsuperscript{35} assessed the multiplicity of the fluid and heat transfer of MHD flow in viscoelastic fluids through a stretched surface. Alloui and Vasseur\textsuperscript{36} analyzed Marangoni convection and multiple solutions of power-law fluids in layers occurring in gravity-free surroundings. Thibault et al.\textsuperscript{37} studied multiple water retention solutions of the stratified flow in oblique channels. Raza et al.\textsuperscript{38} analyzed the rheology of wall channel micropolar fluids with various solutions. Picchi and Poesio\textsuperscript{39} had reviewed the stability of multiple solutions of the stratified flow of inclined gas/shear thinning fluids. Hashim\textsuperscript{40} devoted to the multiplicity of the critical values of the flow pattern of the Magnetocarreau fluid over a cylinder. Khan et al.\textsuperscript{41} discussed a variety of solutions for non-Newtonian Carreau fluids on inclined shrinkage plates. Hashim and Khan\textsuperscript{42} analyzed the multiple solutions of the transient flow of the Williamson nanofluid magnetic fluid with convective heat transfer. Pop et al.\textsuperscript{43} focused on the multisolutions of the power-law boundary layer fluid flow on a moving wedge.

Inspired by previous studies, this paper investigates the fluid flow and heat transfer of the power-law fluid by taking the coupling effect of the fluid and the viscous sheet into account. The thermal conductivity is modeled by a power-law expression related to the temperature gradient. Multiple solutions of momentum and energy equations exist when nonuniform suction or blowing occurs in the boundary. Similar solutions are found in this research with certain limitations. This article is organized as follows: in Sec. II, the problem of fluid flow in the boundary layer and the governing equations in the case of coupling the surrounding fluid with the viscous sheet are described. Section II A shows the momentum equation of the viscous plate, and the control equations of the boundary layer are shown in Sec. II B. Section II C discusses the ordinary differential equations (ODEs) and corresponding boundary conditions of self-similar boundary layer flow on a stretched viscous plate. Section III obtains similar solutions, and the physical interpretations of the results are clarified in detail. Sections III A and III B display the multiple results of the momentum equation and the energy equation, respectively. Section IV is the conclusion.

II. PROBLEM FORMULATION AND GOVERNING EQUATIONS

Consider a viscous and permeable sheet immersed in an incompressible power-law fluid. It is a steady system that does not change...
with time. \( \rho \) and \( \rho_i \) are the density of the outer fluid and sheet, respectively. The horizontal velocity of the viscous sheet is \( U(x) \); \( U_0 \) at \( x = 0 \), while \( U_L \) at \( x = L \). The sheet has an initial half-thickness \( h_0 \), which is supposed to be much smaller than the lateral width \( W \) of the sheet, i.e., \( h_0 \ll W \). Half of the sheet thickness is a function of \( x \), namely, \( h(x) \), and with an increase in \( x \), it changes slowly \([h'(x) \leq 1]\). A nonuniform suction or blowing velocity is introduced into the system. In order to ensure similar solutions, the nonuniform mass suction/blowing has an equivalent form, \( V_w + U(x)h'(x) \). \( V_w + U(x)h'(x) < 0 \) means suction, while \( V_w + U(x)h'(x) > 0 \) means blowing. The temperature boundary layer is also investigated, by assuming that the temperature of the viscous sheet is \( T_{sh} \). As shown in Fig. 1, \( x \) represents the coordinates along the direction of the viscous sheet; \( y \) represents the coordinates perpendicular to it; and \( u \) and \( v \) represent velocities of fluids in the \( x \) and \( y \) directions, respectively.

A. Momentum equation of the viscous sheet

According to Ref. 7, the following equation can be obtained:

\[
2h(x)U(x) = 2h_0U_0. \tag{1}
\]

In the \( x \) direction, the momentum equation on the sheet is written as

\[
\frac{\partial \tau_{xx}}{\partial y} + \rho_i U(x) \frac{dU}{dx} = \frac{\partial U}{\partial y} \frac{\partial h}{\partial y} + \rho_i U(x) \frac{dU}{dx}. \tag{2}
\]

\( \sigma_{xx} \) and \( \tau_{yx} \) are the stress components of the sheet. With \( \frac{dh}{dx} \ll 1 \), \( \sigma_{xx} \) is just a function of \( x \). By calculating the integral of Eq. (2) from 0 to \( h(x) \), it can be obtained as

\[
\frac{d}{dx} \left( h(x) \sigma_{xx} \right) + \tau_{yx} = \rho_i h(x) U(x) \frac{dU}{dx}, \tag{3}
\]

By inserting Eq. (1) into Eq. (3), the momentum equation of the viscous sheet in the \( x \) direction can be obtained. Assuming that the fluid around the sheet is a Newtonian fluid, the controlling equation is written as in Ref. 7,

\[
\frac{d}{dx} \left( \mu \frac{dU}{dx} \right) - \rho_i U(x) + \frac{\mu}{U_0h_0} \frac{\partial U}{\partial y} \bigg|_y = 0. \tag{4}
\]

However, when the fluid is replaced with a power-law fluid, like in the current research, the functions of \( \tau_{yx} = \rho_i \left( \frac{\partial U}{\partial y} \right)^n \) are adopted\(^\dagger\) such that the controlling equation of the sheet will be of the form

\[
\frac{d}{dx} \left( \mu_i \left( \frac{dU}{dx} \right)^n \right) - \rho_i U(x) + \frac{\mu}{U_0h_0} \frac{\partial U}{\partial y} \bigg|_y = 0. \tag{5}
\]

B. The equations of the surrounding fluid

The velocity vector of the fluid around the sheet is \( u = (u(x, y), v(x, y)) \). It is assumed that the velocity is 0 when it is away from the sheet surface. For steady incompressible power-law fluids, the continuity, momentum, and energy equations are, respectively,

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6}
\]

\[
u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = \frac{\partial }{\partial y} \left( \rho \frac{\partial u}{\partial y} \right)^{n-1} = \frac{\partial }{\partial y} \left( \rho \frac{\partial u}{\partial y} \right). \tag{7}
\]

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\partial }{\partial y} \left( \rho \frac{\partial T}{\partial y} \right)^{n-1} = \frac{\partial }{\partial y} \left( \rho \frac{\partial T}{\partial y} \right), \tag{8}
\]

where \( \nu \frac{\partial T}{\partial y} \) is the kinematic viscosity. The generalized Fourier’s law is adopted, and \( \alpha = \nu \frac{\partial T}{\partial y} \) is the thermal diffusivity.\(^\dagger\) The symmetry of the system allows us to consider only the upper half \( y > 0 \). Moving boundary conditions and no slip make the velocity of the fluid on the surface consistent with the speed of the viscous sheet \( U(x) \). As the viscous sheet becomes thinner, the boundary condition of the nonuniform motion of velocity \( v \) in the vertical direction is established. The boundary conditions of velocity and temperature are

\[
u(x, h(x)) = U(x), v(x, h(x)) = V_w + U(x)h'(x), T(x, h(x)) = T_{sh}. \tag{9}
\]

\[
u \to \infty, u(x, y) = 0, T(x, y) \to T_\infty. \tag{10}
\]
The following similarity transformations are used:

\[ u(x, y) = U(x)f' (\eta), \eta = \frac{y - h(x)}{h_0(x)}, \]

\[ v(x, y) = -h_0 \frac{d(g(x)U(x))}{dx} f(\eta) + U(x) \left( \frac{dh}{dx} + h_0 \frac{dg(x)}{dx} \right) f'(\eta), \]

\[ T = T_\infty + \Delta T \theta(\eta), \]

where \( \Delta T = T_\text{th} - T_\infty \), and the velocity \( U(x) \) and the dimensionless function \( g(x) \) will be coupled by obtaining the surrounding fluid with the sheet dynamics in Sec. II C. Inserting (11) into Eqs. (7) and (8), the original momentum and energy equations become

\[ \left( \frac{f''(\eta)}{N_{\text{pr}}} \right)^{n-1} f''(\eta) + \frac{h_0 g(x)}{U(x)} \frac{dU}{dx} f'(\eta) \right) + \frac{1}{N_{\text{pr}}} \left( \left| f'(\eta) \right|^{n-1} f'(\eta) \right) \right)\right)^{n-1} f''(\eta) \right) \right)\right)^2 = 0, \]

\[ \left( \left| f'(\eta) \right|^{n-1} f'(\eta) \right)\right)^2 = 0. \]

In Eq. (13), \( N_{\text{pr}} = \frac{\left| \frac{U_0(x)}{x} \right|^{n-1}}{\frac{\mu}{\alpha}} \), \( \eta = \frac{c}{U_0(x) \Delta x} \). Regarding the general stretching viscous sheet when \( U'(x) = 0 \), the following equations are obtained for the guaranteed existence of the solutions:

\[ \left( \left| f'(\eta) \right|^{n-1} f'(\eta) \right)\right)^2 = 0. \]

On solving Eq. (18), the following expression can be reached:

\[ h_0^{n+1} A^{n+1} U^{n+1} \frac{dU}{dx} = ydx. \]

By integrating the two sides of Eq. (19) at the same time, the expression of \( U(x) \) is obtained as

\[ U(x) = \left( \frac{cn + c - n + 1 + 1}{h_0^{n+1} A^{n+1}} \gamma(x + B) \right)^{\frac{1}{n+1}}. \]

By comparing Eqs. (15) and (20), the following relation between \( \alpha \) and \( c \) could be obtained:

\[ \alpha = \frac{1}{cn + c - n + 2}. \]

Substituting Eq. (15) into Eq. (14), the parameters \( Re \) and \( m \) in Eq. (15) have the following relation:

\[ Re \left( \frac{h_0^n U_0^{n-1} \mu}{yn} \right), \]

where \( m \) can be determined when the speed of the sheet at \( x = L \), i.e., \( U_L \), is substituted into Eq. (15).

\[ m = 1 - \left( \frac{U_L}{U_0} \right)^{\frac{1}{n}}. \]

Inserting Eq. (14) into Eqs. (12) and (13), the corresponding equations are obtained as

\[ \left( \left| f''(\eta) \right|^{n-1} f''(\eta) \right)\right)^2 + (1 + c) \left| f''(\eta) \right|^{n-1} f''(\eta) \right)\right)^2 + (1 + c) f(\eta) \theta'(\eta) = 0. \]

C. The combination of fluid and sheet equations

By substituting Eqs. (1) and (11) into Eq. (5), the equation of the viscous sheet immersed in a power-law fluid becomes

\[ \left( \frac{d}{dx} \left( \frac{dU}{dx} \right) - \rho U \right) + \frac{U(x)}{h_0 g(x)} \right)\right)^{n-1} f''(0) = 0. \]

Substituting Eq. (15) into Eq. (26),

\[ \mu U_0^{n-1} \left( -m \frac{U}{x} + 1 \right)^{-\alpha} \right)\right)^{n-1} f''(0) = 0. \]

Substituting Eq. (15) into Eq. (26),

\[ \mu U_0^{n-1} \left( -m \frac{U}{x} + 1 \right)^{-\alpha} \right)\right)^{n-1} f''(0) = 0. \]

\[ \left( \frac{d}{dx} \left( \frac{dU}{dx} \right) - \rho U \right) + \frac{U(x)}{h_0 g(x)} \right)\right)^{n-1} f''(0) = 0. \]
By inserting the expression of \( \alpha \) into Eq. (27), it is obtained that
\[
\mu k U_0 n^{-1} \left( \frac{n-1}{cn + c - n + 2} L \right)^n \left( \frac{n-1}{cn + c - n + 2} - n \right) \times \left( \frac{m x}{L} + 1 \right)^{\frac{n-1}{(n-1)}} \frac{1}{\partial \chi \cdot \frac{1}{h_0 \Re^n} \left( - \frac{n-1}{n-2} \right)^{\frac{n-1}{(n-1)}}} \frac{1}{\partial \chi \cdot \frac{1}{h_0 \Re^n} \left( - \frac{n-1}{n-2} \right)^{\frac{n-1}{(n-1)}}} \times \frac{m x}{L} + 1 \right)^{\frac{n-1}{(n-1)}} = \left( \frac{m x}{L} + 1 \right)^{\frac{n-1}{(n-1)}} = \left( \frac{m x}{L} + 1 \right)^{\frac{n-1}{(n-1)}}.
\]
From Eqs. (29) and (30), it is obtained that when \( c = \frac{n-2}{n} \), the following expressions are established:
\[
c = \frac{n-2}{n}, \quad c = \frac{n^2 - n - 3}{n^2 + n + 1}.
\]
From the above equation, it could be seen that the two expressions of \( c \) have been obtained by solving Eqs. (29) and (30). We draw the curves of these two expressions to find their compatibility in Fig. 2. As it can be seen from the graph, when \( n \) is bigger than 0.8, these two expressions are almost coincident. Therefore, we consider the range of \( n \) between 0.8 and 1.6 in the following calculation, when both expressions in Eq. (31) can be satisfied.

Take the expression of \( c \) as
\[
c = \frac{n-2}{n}.
\]

The governing equations of fluids are as follows:
\[
\left( |f''(\eta)|^{n-1} f''(\eta) \right)' + \left( 1 + \frac{n-2}{n} \right) f''(\eta) - \left( f'_{\eta} \right)^2 = 0,
\]
\[
\frac{1}{N_T} \left( |f''(\eta)|^{n-1} f''(\eta) \right)' + Ec |f''(\eta)|^{n-1} f''(\eta) = 0.
\]
The boundary conditions become
\[
f(0) = \frac{-f_{w}}{1 + c}, \quad f_{\infty} = V_w \frac{1}{y} \log(x) \left| h_{g}(x) \right|^{n-1},
\]
\[
f'(0) = 1, \quad f'(\infty) = 0,
\]
\[
\theta(0) = 1, \quad \theta(\infty) = 0,
\]
where \( f_{w} \) is the suction/blowing rate, \( f_{w} < 0 \) and \( f_{w} > 0 \) denote suction and blowing, respectively. The friction coefficient \( C_{f_{k}} \) and Nusselt number \( N_{T_{u}} \), are two significant physical quantities investigated in Sec. III, and their expressions are
\[
C_{f_{k}} = \frac{2x_{w}}{\rho U^{2}(x_{1})}, \quad N_{T_{u}} = \frac{x_{w}q_{w}}{k_{f}(T_{w} - T_{\infty})},
\]
where the shear stress \( \tau_{w} \) and the heat flux \( q_{w} \) of the surface at \( y = h(x) \) are
\[
\tau_{w} = \mu \frac{\partial u}{\partial y} |^{n-1} \frac{\partial u}{\partial y}, \quad q_{w} = -k \frac{\partial T}{\partial y} |^{n-1} \frac{\partial T}{\partial y}.
\]

By inserting Eq. (39) into Eq. (38), the following relations can be obtained:
\[
C_{f_{k}} \Re^{\frac{n}{1}} \sim |f''(0)|^{n-1} f''(0),
\]
\[
N_{T_{u}} \Re^{\frac{n}{2}} \sim |\theta'(0)|^{n-1} \theta'(0).
\]
\[|f''(0)|^{n-1} f''(0) \] and \[|\theta'(0)|^{n-1} \theta'(0) \] will be calculated, and they share similar physical meanings with \( C_{f_{k}} \Re^{\frac{n}{1}} \) and \( N_{T_{u}} \Re^{\frac{n}{2}} \). Thus, the friction coefficient and generalized Nusselt number refer to \[|f''(0)|^{n-1} f''(0) \] and \[|\theta'(0)|^{n-1} \theta'(0) \] in Sec. III.

### III. RESULTS AND DISCUSSIONS

For the sake of proving the feasibility of the numerical approach employed in the current research for which we have used the package of BVP4C in Matlab, some comparisons are made with former studies, as shown in Tables I and II. Table I demonstrates the comparison between our numerical results of the skin friction coefficient and the previous solutions of Ahmad et al. by using RK45. It displays the effect of the suction/blowing rate \( f_{w} \) on the friction coefficient at different \( M \) (we use the same symbols as in Ref. 8). Table II shows a comparison between the numerical solution and the exact solution of the friction coefficient for the stretched sheet (\( c = 0 \)) and the elastic sheet (\( c = -3 \)) with varying suction/blowing rates \( f_{w} \). The numerical results of the skin friction coefficient...
TABLE I. Comparison of the previous and current results of the skin friction coefficient with the stretching sheet \((c = 0)\) at different \(M\).

| \(f_w\) | Ahmad et al. | Current calculation | Difference | Ahmad et al. | Current calculation | Difference |
|------|--------------|---------------------|------------|--------------|---------------------|------------|
| \(c = 0\) | \(M = 0.25\) | \(M = 0.25\) | (%) | \(M = 4\) | \(M = 4\) | (%) |
| 3 | 3.3706 | 3.3708 | 0.02 | 4.1924 | 4.1926 | 0.02 |
| 2 | 2.4999 | 2.5000 | 0.01 | 3.4492 | 3.4495 | 0.03 |
| 1 | 1.7244 | 1.7248 | 0.04 | 2.7910 | 2.7913 | 0.03 |
| 0 | 1.1178 | 1.1185 | 0.07 | 2.2359 | 2.2361 | 0.02 |
| −1 | 0.7245 | 0.7252 | 0.07 | 1.7910 | 1.7913 | 0.03 |
| −2 | 0.5000 | 0.5001 | 0.01 | 1.4492 | 1.4495 | 0.03 |
| −3 | 0.3706 | 0.3708 | 0.02 | 1.1926 | 1.1926 | 0.00 |

TABLE II. Comparison of the exact and the current numerical solutions of the skin friction coefficient with varying mass suction/blowing rates \(f_w\) with the stretching sheet \((c = 0)\) and the elastic sheet \((c = −3)\).

| \(f_w\) | Exact | Current calculation | Difference | Exact | Current calculation | Difference |
|------|-------|---------------------|------------|-------|---------------------|------------|
| \(c = 0\) | \(M = 0.1\) | \(M = 0.1\) | (%) | \(c = −3\) | \(M = 0.1\) | (%) |
| −0.5 | 1.32819 | 1.3287 | 0.051 | 0.79835 | 0.7984 | 0.005 |
| −0.4 | 1.26770 | 1.2683 | 0.060 | 0.76646 | 0.7665 | 0.004 |
| −0.3 | 1.20948 | 1.2101 | 0.062 | 0.73646 | 0.7365 | 0.004 |
| −0.2 | 1.15356 | 1.1543 | 0.074 | 0.70823 | 0.7082 | 0.003 |
| −0.1 | 1.10000 | 1.1008 | 0.080 | 0.68162 | 0.6816 | 0.002 |
| 0 | 1.04880 | 1.0497 | 0.090 | 0.65653 | 0.6565 | 0.003 |
| 0.1 | 1.00000 | 1.0001 | 0.010 | 0.63283 | 0.6328 | 0.003 |
| 0.2 | 0.95356 | 0.9546 | 0.104 | 0.61044 | 0.6104 | 0.004 |
| 0.3 | 0.90948 | 0.9105 | 0.102 | 0.58925 | 0.5893 | 0.005 |
| 0.4 | 0.86770 | 0.8688 | 0.110 | 0.56920 | 0.5692 | 0.000 |
| 0.5 | 0.82819 | 0.8292 | 0.101 | 0.55021 | 0.5502 | 0.001 |

obtained by our numerical method are in good agreement with the exact ones.

In Secs. III A and III B, the multisolutions of the friction coefficient \(|f''(0)|^{n-1}f''(0)\) and the generalized Nusselt number \(|\theta'(0)|^{n-1}\theta'(0)\) are analyzed under the influence of parameters, such as the power-law index \(n\), the suction/blowing rate \(f_w\), the generalized Prandtl number \(N_p\), and the Eckert number \(Ec\).

A. Multiple solutions of the velocity distribution and friction coefficient

In Fig. 3, \(|f''(0)|^{n-1}f''(0)\) represents the friction coefficient, as is explained in the end of Sec. II. For a Newtonian fluid \((n = 1)\), \(c = −1\) can be obtained according to Eq. (32). In this case, the denominator in Eq. (35) is 0, which makes the boundary condition meaningless. As a result, the case of \(n = 1\) is not discussed in our calculation. As \(n\) becomes larger, the viscosity is enhanced and the interaction between the fluid molecules becomes closer. A little disturbance from the outside will bring more possibilities of the fluid state. Thus, when the power-law index \(n\) is greater than 0.9,
the third solution of the corresponding friction coefficient emerges. In the case of three sets of solutions, as the power-law index \( n \) increases, the friction coefficient decreases. Since the friction coefficient is proportional to the surface shear stress, the surface shear stress will also be reduced as \( n \) increases.

Figure 4 depicts the effect of the suction/blowing rate \( f_w \) on the friction coefficient. As can be found out, the friction coefficient increases with an increase in \( f_w \). For pseudoplastic fluids \( (n < 1) \), when \( f_w \) is greater than 1.4, the third set of friction coefficient appears. For dilatant fluids \( (n > 1) \), the third solution occurs when \( f_w \) is greater than about 1.6. The fluid will have a triple solution when blowing \( (f_w > 0) \) occurs, and two solutions will occur when the fluid is under suction \( (f_w < 0) \). As can be seen from the figure, when the fluid is blown, the fluid flow is more unstable than when the fluid is under suction. The blowing tends to be more intense as \( f_w \) increases, and the fluid flow has more possibilities to be in a new state. The third solution is larger than the former two sets of solutions, but the increasing trend is the same as the previous two sets of solutions.

Figure 5 depicts the velocity distribution in the case of multiple solutions. \( f'(\eta) \) represents the velocity. It can be seen that whether it is a pseudoplastic fluid or a dilatant fluid, the trend of the velocity distribution is basically consistent: in the case of multiple solutions, the results all decrease with an increase in \( \eta \). For pseudoplastic fluids \( (n < 1) \), there are two sets of solutions that are infinitely close. However, for dilatant fluids \( (n > 1) \), the three states of fluid are totally different from each other.

B. Multisolutions of temperature distribution and generalized Nusselt number

\[ \theta'(0) \] shares the same physical meanings with the generalized Nusselt number. Figure 6 reflects the change in the generalized Nusselt number of the power-law fluid with the power-law index \( n \). As Fig. 5 indicates, the Newtonian fluid \( (n = 1) \) does not conform to the above figure, so it is represented by an empty point and is not analyzed. According to the expression \( \text{Nu}_x = \frac{Nu}{(1 - \theta'_{\infty})} \), it can be clearly seen that \( Nu \) and \( q_w \) have a direct relationship. Through
careful observation, the variation trend of the three sets of solutions is consistent: $|\theta'(0)|^{n-1} \theta'(0)$ is reduced as the power-law index $n$ increases. It is known from Eq. (39) that the heat flux decreases as $n$ increases. According to Fig. 6, the three states of power-law fluids have large differences, and when a set of solutions is determined, the convective heat transfer of the surface of the viscous sheet will be decided.

Figure 7 depicts the effect of the mass suction/blowing rate $f_w$ on the generalized Nusselt number. When the fluid is blown or under suction with different strengths, the influence on the generalized Nusselt number is different. The change in boundary conditions has an effect on the thermal diffusivity, resulting in more possibilities for the generalized Nusselt numbers. Obviously, whether it is blown or under suction, the generalized Nusselt coefficient of the three sets of solutions decreases as $f_w$ increases. For pseudoplastic fluids, the changes in the curves are not linear. For dilatant fluids, all three solutions are linearly decreasing. Compared with Fig. 4, an interesting phenomenon is that three sets of solutions of generalized Nusselt numbers can be found, while only two sets of solutions of friction coefficients are obtained when $f_w$ is less than 1.4 ($n < 1$) or 1.6 ($n > 1$). It means that with some value of mass suction/blowing rate $f_w$, the velocity distribution of the fluid has two possibilities, while the temperature has three.

In Figs. 8 and 9, the influences of the generalized Prandtl number $N_{pr}$ and Eckert number $Ec$ in the energy equation on the generalized Nusselt number under three different sets of solutions are discussed. When $N_{pr}$ and $Ec$ increase, $|\theta'(0)|^{n-1} \theta'(0)$ decrease. It can be seen that all these three solutions, the generalized Nusselt number is decreasing, and thus, the heat transfer is decreasing. The expression $N_{pr} = \left[ \frac{U(x)}{\Delta T} \right]^{n-1} \frac{n^2}{\gamma \omega}$ characterizes the relationship between kinematic viscosity and thermal diffusivity. It can be concluded from the above figure that as $N_{pr}$ increases, the kinematic viscosity becomes much bigger than the thermal diffusivity. By the expression $Ec = \frac{c_p U(x)^2}{\Delta T}$, the fluid specific heat $c_p$ will decrease as $Ec$ increases. All the generalized Nusselt numbers in three states decrease with an
increase in the Eckert number $Ec$, which is the same as the curve trends with the generalized Prandtl number $N_{Pr}$ in Fig. 8.

Figure 10 depicts the temperature distributions in the presence of multiple solutions. $\theta(\eta)$ represents the temperature. In both the pseudoplastic fluid and the dilatant fluid, as the fluid flows, changes in the intermolecular forces and external disturbances cause more variations in temperature. Whether it is a pseudoplastic fluid or a dilatant fluid, the three solutions have the same change trend, and when the independent variable $\eta$ increases, the temperature decreases, but the slopes of these curves are different.

IV. CONCLUSIONS

In this paper, the multistate of the boundary layer flow coupling the viscous sheet is studied. It can be seen from the research that the existence of similar solutions has certain limitations. Multiple solutions of the friction coefficient and the generalized Nusselt number can be obtained under the impacts of the power-law index $n$, the suction/blowing rate $f_w$, the generalized Prandtl number $N_{Pr}$, and the Eckert number $Ec$. The following conclusions can be obtained:

I. Similar solutions exist with the limitation that $n$ is greater than 0.8.

II. In all the multiple states, when the power-law index $n$ increases, the friction coefficient and the generalized Nusselt number decrease.

III. In the case of multiple solutions of the friction coefficient, for a pseudoplastic fluid, the third set of solutions will occur when $f_w$ is greater than 1.4, while the third set of solutions for expansive fluids will occur when $f_w$ is greater than 1.6.

IV. Three sets of solutions of generalized Nusselt numbers can be found, while only two sets of solutions of the friction coefficient are obtained when $f_w$ is less than 1.4 ($n < 1$) or 1.6 ($n > 1$).

V. When the generalized Prandtl number $N_{Pr}$ and the Eckert number $Ec$ increase, the generalized Nusselt number decreases.

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