Noncommutative Geometry and Spacetime Gauge Symmetries of String Theory

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Abstract

We illustrate the various ways in which the algebraic framework of noncommutative geometry naturally captures the short-distance spacetime properties of string theory. We describe the noncommutative spacetime constructed from a vertex operator algebra and show that its algebraic properties bear a striking resemblance to some structures appearing in M Theory, such as the noncommutative torus. We classify the inner automorphisms of the space and show how they naturally imply the conventional duality symmetries of the quantum geometry of spacetime. We examine the problem of constructing a universal gauge group which overlies all of the dynamical symmetries of the string spacetime. We also describe some aspects of toroidal compactifications with a light-like coordinate and show how certain generalized Kac-Moody symmetries, such as the Monster sporadic group, arise as gauge symmetries of the resulting spacetime and of superstring theories.

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1. Symmetries in Noncommutative Geometry

The description of spacetime and its symmetries at very short length scales is one of the main challenges facing theoretical physics at the end of this millennium. At distances much larger than the Planck length, spacetime is described by a differentiable manifold, whose symmetries are diffeomorphisms which describe the gravitational interactions. Additional interactions are described by gauge theories, associated with the symmetries of some internal structure living in the spacetime. The most promising candidate for a description of spacetime and its symmetries at the quantum level is string theory. Here the Planck length appears as the square root of the string tension, and the coordinates of spacetime are target space fields, defined on a two-dimensional world-sheet, which describe the surface swept out by the string in spacetime.

In the case of a toroidally compactified target space $T^d \cong \mathbb{R}^d / 2\pi \Gamma$, with $\Gamma$ a Euclidean rational lattice of rank $d$ with inner product $g_{\mu \nu}$, the string is described by the Fubini-Veneziano fields

$$X^\mu_\pm(z_\pm) = x^\mu_\pm + ig^{\mu \nu} p^\pm_\nu \log z_\pm + \sum_{k \neq 0} \frac{1}{ik} \alpha^{(\pm)\mu}_k z_\pm^{-k},$$  \hspace{1cm} (1.1)

where the zero-modes $x^\mu_\pm$ (the center of mass coordinates of the string) and the (center of mass) momenta $p^\pm_\mu = \frac{1}{\sqrt{2}} (p_\mu \pm d^\pm_\mu w^\nu)$ are canonically conjugate variables. Here $\{p_\mu\} \in \Gamma^*$ (the lattice dual to $\Gamma$) are the spacetime momenta and $\{w^\mu\} \in \Gamma$ are winding numbers representing the number of times that the string wraps around the cycles of the torus. The background matrices $d^\pm_{\mu \nu} = g_{\mu \nu} \pm \beta_{\mu \nu}$ are constructed from the spacetime metric $g_{\mu \nu}$ and an antisymmetric instanton tensor $\beta_{\mu \nu}$. The set of momenta $\{(p^+_\mu, p^-_\mu)\}$ along with the quadratic form

$$\langle p, q \rangle_\Lambda \equiv p^+_\mu g^{\mu \nu} q^+_\nu - p^-_\mu g^{\mu \nu} q^-_\nu = p^\mu v^\mu + q^\mu \bar{w}^\mu,$$  \hspace{1cm} (1.2)

where $q^\pm_\mu = \frac{1}{\sqrt{2}} (q_\mu \pm d^\pm_\mu w^\nu)$, form an even self-dual Lorentzian lattice $\Lambda = \Gamma^* \oplus \Gamma$ of rank $2d$ and signature $(d, d)$ called the Narain lattice [1]. The functions (1.1) define chiral multi-valued quantum fields. The oscillatory modes $\alpha^{(\pm)\mu}_k = (\alpha^{(\pm)\mu}_{-k})^*$ in (1.1) yield bosonic creation and annihilation operators (acting on some vacuum states $|0\rangle_\pm$) which generate the Heisenberg algebra

$$\left[\alpha^{(\pm)\mu}_k, \alpha^{(\pm)\nu}_m\right] = k \ g^{\mu \nu} \delta_{k+m,0}. \hspace{1cm} (1.3)$$

The fundamental continuous symmetries of these fields are target space reparametrization invariance and conformal invariance of the world-sheet. In addition there are a number of quantum symmetries, such as dualities, mirror symmetries, world-sheet parity, etc. which we shall discuss in section 4. The interactions of the strings are described by a vertex operator algebra, which we study in the next section. These interactions are crucial

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1In this paper we will consider only the simplest non-trivial case of closed strings with world-sheet which is an infinite cylinder with local coordinates $(\tau, \sigma) \in \mathbb{R} \times S^1$, and we set $z_\pm = e^{-i(\tau \pm \sigma)}$.\]
for the theory at short distances, where a description based on a cylindrical world-sheet breaks down. We want, however, to recover the usual classical spacetime in some low energy limit.

Although string theory is a dramatic generalization of quantum field theory, its tools still lie within the realm of traditional differential geometry, which utilizes sets of points, differentiable manifolds, and commuting or anticommuting fields. Recently, however, a much more radical approach to geometry has been pioneered by Connes and others which goes under the name of Noncommutative Geometry \[^2\]. The central idea of Noncommutative Geometry is that since all separable topological spaces (for example manifolds) are completely characterized by the commutative $C^*$-algebra of continuous complex-valued functions defined on it, it may be useful to regard this algebra as the algebra generated by the coordinates of the space. Conversely, given a commutative $C^*$-algebra, it is possible to construct, with purely algebraic methods, a topological space and recover, again with purely algebraic methods, all of the topological information about it. One can therefore substitute the study of the geometry and topology of a space in terms of the relations among sets of points with a purely algebraic description in terms of $C^*$-algebras. The terminology noncommutative geometry refers to the possibility of generalizing these concepts to the case where the algebra is noncommutative. This would be the case of some sort of space in which the coordinates do not commute, such as that predicted by the recent matrix model realizations of M Theory \[^3, 4\], but this is only one possibility (and as we will see a rather reductive one).

A noncommutative manifold is therefore a space described by a noncommutative $C^*$-algebra. Within the algebraic framework of noncommutative geometry, the symmetries of spacetime have a natural interpretation at the algebraic level \[^5\]. The algebra which describes the manifold (or its generalization) is unchanged as a whole under automorphisms, i.e. isomorphisms of the algebra into itself, as they just resuffle the elements without changing the basic algebraic relations. In fact, for a manifold $M$, the group of diffeomorphisms of the space is naturally isomorphic to the group of automorphisms of the commutative algebra $\mathcal{A} = C^\infty(M)$ of smooth functions defined on it,

$$f(\phi(x)) = \sigma_\phi(f)(x),$$  \hspace{1cm} (1.4)

where $\phi(x)$ is a diffeomorphism of $M$ and $\sigma_\phi(f) \equiv f \circ \phi$ is an (outer) automorphism of $\mathcal{A}$.

When the algebra $\mathcal{A}$ is noncommutative, a new class of automorphisms appears, the inner automorphisms, which act as conjugation of the elements of the algebra by a unitary element $u \in \mathcal{A}$,

$$\sigma_u(f) = uf u^{-1}.$$  \hspace{1cm} (1.5)

They represent internal fluctuations of the noncommutative geometry corresponding to rotations of the elements of $\mathcal{A}$. If, for example, we consider the algebra of smooth functions from a manifold $M$ into the space $M(N, \mathbb{C})$ of $N \times N$ complex-valued matrices, then the group of inner automorphisms coincides with the group of local gauge transformations of
a $U(N)$ gauge theory on $M$. In general, as is customary in theories with noncommutative fields, the algebra of observables is the tensor product of the infinite-dimensional algebra of functions on $M$ with a finite-dimensional algebra (typically a matrix algebra). In noncommutative geometry this algebra describes a manifold, represented by the infinite-dimensional algebra, in which the points have some sort of internal structure, described by the finite-dimensional algebra. This is the case in the example of the standard model [6], where the internal structure is described by an algebra whose unimodular group is the gauge group $SU(3) \times SU(2) \times U(1)$. The application of noncommutative geometry to the standard model has had some successes, but probably its most promising potential lies in considering more complicated structures than just the tensor product of a commutative algebra with a matrix algebra.

We see therefore that in noncommutative geometry diffeomorphisms and gauge transformations appear on a very similar footing. Both represent transformations of the algebra which leave it invariant, the difference being that the latter ones are automorphisms which are internally generated by elements of the algebra and appear as transformations of the internal space. In the following we will apply this formalism to the spacetime described by string theory. We will show how the geometry and topology relevant for general relativity must be embedded in a larger noncommutative structure in string theory.

2. Vertex Operator Algebras

At very high energies, strings appear as genuine extended objects, so that at small distance scales it may not be possible to localize the coordinates of spacetime with definite certainty. A long-standing idea in string theory is that, below the minimum distance determined by the finite size of the string (usually taken to be the Planck length), the classical concepts of spacetime geometry break down. It is therefore natural to assume that the spacetime implied by string theory is described by some noncommutative algebra. An explicit realization of this idea arises in D-brane field theory [3] and M Theory [4] where the noncommutative coordinates of spacetime are encoded in a finite-dimensional matrix algebra as described in the previous section. However, there is another approach which in essence describes the “space” of interacting strings by employing the vertex operator algebra of the underlying two-dimensional conformal quantum field theory [7]–[10]. Vertex operators describe the interactions of strings and act on the string Hilbert space as insertions on the world-sheet corresponding to the emission or absorption of string states. The idea behind this approach is that, to a large extent, the stringy spacetime is determined by its quantum symmetries.

The complicated algebraic properties of a vertex operator algebra lead to a very complex noncommutative spacetime. The underlying conformal field theory admits two mutually commuting chiral algebras $\mathcal{A}_\pm$ of observables, namely the operator product algebras of (anti-)holomorphic fields. Each chiral algebra contains a representation $\text{Vir}_\pm$ of the
operators in the present case are $L^\pm_k$ satisfy the commutation relations
\[
[L_k^+, L_m^+] = (k - m)L_{k+m}^\pm + \frac{c}{12} (k^3 - k) \delta_{k+m,0},
\] (2.1)

where $c$ is the conformal anomaly. The corresponding symmetry group represents the basic conformal invariance of the string theory. For the case of toroidally compactified bosonic strings, the vertex operator algebra $\mathcal{A} = \mathcal{A}^{+} \otimes_{\mathbb{C}(\Lambda)} \mathcal{A}^{-}$ is constructed using the operator-state correspondence of local quantum field theory (here $\mathbb{C}(\Lambda)$ denotes a twisting factor from the group algebra of the double cover of $\Lambda \otimes \mathbb{Z} \subset \mathbb{I}(\Lambda)$). The Hilbert space on which the quantum fields $\mathbb{I}$ act is
\[
\mathcal{H} = L^2(T^d)^\Gamma \otimes \mathcal{F}^+ \otimes \mathcal{F}^-,
\] (2.2)

where $L^2(T^d)^\Gamma = \bigoplus_{\omega^\mu \in \Gamma} L^2(T^d)$ is the $L^2$ space associated with the center of mass modes of the strings, whose connected components are labelled by the winding numbers and whose basis is denoted $|p^+; p^-\rangle = e^{-ip_x x^\mu}$ in each component. $\mathcal{F}^\pm$ are Fock spaces generated by the oscillatory modes $\alpha_k^{(\pm)\mu}$, and the unique vacuum state of $\mathcal{H}$ is $|\text{vac}\rangle \equiv |0; 0\rangle \otimes |0\rangle^+_\pm$. To a typical homogeneous state
\[
|\psi\rangle = |q^+; q^-\rangle \otimes \prod_j r^{(j)+}_\mu \alpha_{-\eta_j}^{(+)}|0\rangle_+ \otimes \prod_k r^{(k)-}_\nu \alpha_{-\eta_k}^{(-)}|0\rangle_+.
\] (2.3)

of $\mathcal{H}$, there corresponds the vertex operator
\[
V(\psi; z_+, z_-) = i V_{q^+q^-}(z_+, z_-) \prod_j r^{(j)+}_\mu \alpha_{-\eta_j}^{(+)}|0\rangle_+ \otimes \prod_k r^{(k)-}_\nu \alpha_{-\eta_k}^{(-)}|0\rangle_+.
\] (2.4)

where $(q^+, q^-), (r^+, r^-) \in \Lambda$, $\psi \in \mathcal{H}^*$ denotes the operator with $|\psi\rangle = \psi|\text{vac}\rangle$, and
\[
V_{q^+q^-}(z_+, z_-) \equiv V(e^{-i q^\mu x^\mu} - e^{-i q^\mu x^\mu} \otimes I; z_+, z_-) = (-1)^{q^\mu w^\mu} e^{-i q^\mu x^\mu}; (z_-) = (2.5)
\]
are the fundamental tachyon vertex operators. Then the smeared operators $V(\psi; f) = \int d^2 z \ V(\psi; z_+, z_-) f(z_+, z_-)$, with $f$ a Schwartz space test function, are well-defined and densely-defined on $\mathcal{H}$ with $|\psi\rangle = V(\psi; f)|\text{vac}\rangle$. They realize explicitly the non-locality property required of noncommuting spacetime coordinate fields.

The Hilbert space $\mathcal{H}$ can be graded by the conformal dimensions $\Delta^\pm_\eta$ which are the highest weights of the irreducible representations of the Virasoro algebras $\mathfrak{Vir}^\pm$. The Virasoro operators in the present case are $L^\pm_k = \frac{1}{2} \sum_{m \in \mathbb{Z}} g_{\mu\nu} : \alpha_m^{(\pm)\mu} \alpha_{k-m}^{(\pm)\nu} :$, with $\alpha_0^{(\pm)\mu} \equiv g^{\mu\nu} \rho^\pm_\nu$. They generate a representation of the Virasoro algebra $L^\pm_0$ of central charge $c = d$. The grading is defined on the subspaces $\mathcal{H}_{\Delta_\eta} \subset \mathcal{H}$ of states (2.3) which are highest weight vectors,
\[
L^\pm_k |\psi\rangle = \Delta^\pm_\eta |\psi\rangle, \quad L^\pm_0 |\psi\rangle = 0, \quad \forall k > 0,
\] (2.6)

where $\Delta^+_\eta = \frac{1}{2} g^{\mu\nu} q^\mu q^\nu + \sum_j n_j$ and $\Delta^-_\eta = \frac{1}{2} g^{\mu\nu} q^\mu q^\nu + \sum_k m_k$. The corresponding operator-valued distributions (2.4) are called primary fields.

The operators (2.5) generate the tachyon states $|q^+; q^-\rangle \otimes |0\rangle_+ \otimes |0\rangle_- \in L^2(T^d)$. The string oscillations vanish on this subspace of $\mathcal{H}$, and when $w^\mu = 0$ (equivalently $q^+ = q^-$)
the operators (2.5) projected onto this subspace coincide with the basis elements $e^{-iq_\mu x^\mu}$ of the commutative algebra $C^\infty(T^d)$. Thus the tachyon vertex operators represent the subspace of the full noncommutative string spacetime which corresponds to ordinary classical spacetime. This is the noncommutative geometry version of the appearance of classical general relativity in a low-energy limit (i.e. at distances much larger than the Planck length). In string theory a low-energy regime is one in which the vibrational modes of the string are negligible and no string modes wind around the spacetime. In this limit strings become effectively point particles which are well-described by ordinary quantum field theory.

The tachyon operators (2.5) can be thought of as sorts of “plane wave” basis fields on the space of conformal field configurations. The tachyon states are highest weight states of the level 2 $u(1)_+ \oplus u(1)_-$ current algebra (1.3), so that the entire Hilbert space can be built up from the actions of the $\alpha_k$’s for $k < 0$ on these states. This current algebra represents the target space reparametrization symmetry of the string theory. Thus the tachyon vertex operators in this sense span the noncommutative vertex operator algebra $\mathcal{A}$ of the string spacetime. Another important class of vertex operators for us will be the graviton operators

$$V_{q^+ q^-}^{\mu \nu} (z_+, z_-) = i V_{q^+ q^-} (z_+, z_-) \partial_{z_+} X_+^\mu \partial_{z_-} X_-^\nu : . \quad (2.7)$$

These operators represent the Fourier modes of the background matrices $d_{\mu \nu}^\pm$ of the toroidal target space, and they create the graviton state $|q^+; q^- \rangle \otimes \alpha_+^{(+)\mu} |0\rangle_+ \otimes \alpha_-^{(-)\nu} |0\rangle_-$ of polarization $(\mu \nu)$. The graviton states represent the lowest-lying states with non-trivial string oscillations, and as such they generate the smallest stringy excitations of the commutative spacetime determined by the tachyon sector of $\mathcal{A}$.

The algebraic relations of $\mathcal{A}$ which characterize the noncommutativity of the chiral algebras can be combined into a single relation known as the Jacobi identity of the vertex operator algebra [10, 11]. It can be regarded as a combination of the classical Jacobi identity for Lie algebras and the Cauchy residue formula for meromorphic functions. The operator product algebra can be encoded through the algebraic relations among the tachyon operators [10], for which we find the clock algebra

$$V_{q^+ q^-} (z_+, z_-) V_{r^+ r^-} (w_+, w_-) = e^{-\pi i (q_\mu r_\mu) \Lambda} V_{r^+ r^-} (w_+, w_-) V_{q^+ q^-} (z_+, z_-) \quad (2.8)$$

for $\pm \arg z_+ = \pm \arg w_\pm$. The algebra (2.8) is a generalization of one of the original examples of a noncommutative geometry, the noncommutative torus [2, 8, 12]. For $d = 2$ this algebra describes the quotient of the 2-torus $T^2$ by the orbit of a free particle whose velocity vector forms an angle $\theta$ with respect to the cycles of $T^2$. When $\theta$ is irrational the motion is ergodic and dense in the torus, and the resulting quotient is not a topological space in the usual sense. An equivalent way to visualize this is to consider the rotations of a circle. It is then possible to describe the space by the algebra of functions on a

\[ \text{See [2, 11] for a precise definition of this projection.} \]
circle together with the action of these rotations. Such an algebra $\mathcal{A}_\theta$ is generated by two elements $U$ and $V$ which obey
\[
UV = e^{-\pi i \theta} VU .
\]
(2.9)
This algebra has also appeared in the recent matrix model descriptions of M Theory \cite{4, 8} and in the noncommutative geometry of four-dimensional gauge theories \cite{13}. When $\theta = 2K/N$, with $K$ and $N$ relatively prime positive integers, the generators $U$ and $V$ can be represented by $N \times N$ matrices. Then the quotient $\mathcal{A}_\theta / \mathcal{I}_\theta$ by the ideal $\mathcal{I}_\theta$ of $\mathcal{A}_\theta$ generated by its center is isomorphic, as a $C^*$-algebra, to the full matrix algebra $M(N, \mathbb{C})$. When $\theta$ is irrational the algebra $\mathcal{A}_\theta$ is infinite-dimensional.

The clock algebra (2.8) (or rather its smeared version) thus resembles the algebra $\mathcal{A}_{\{\theta, r\}}$ of the noncommutative $d$-torus. However, because the vertex operator algebra $\mathcal{A}$ is determined from the quantum field algebra of the conformal field theory, it really represents some large, infinite-dimensional generalization of this structure. It can be regarded as an augmentation, in an appropriate limiting procedure $\mathcal{A} \sim M(\infty, \mathbb{C})$, of the matrix algebras which describe the symmetry group of the 11-dimensional supermembrane and the low-energy dynamics of M Theory determined by the large-$N$ limit of a supersymmetric matrix model \cite{4}. The matrix model for M Theory can be viewed as a particular truncation of a set of vertex operators to finite-dimensional $N \times N$ matrices such that the large-$N$ limit recovers aspects of the full non-perturbative dynamics. For $d = 2$, this limiting regime should be taken as a double scaling limit $K, N \to \infty$ with $\theta = 2K/N$ approaching an irrational number. In M Theory there are several conjectures about what the appropriate large-$N$ limit should be \cite{4, 14}, and from the above point of view such limiting regimes should result in a vertex operator algebra structure. It is in this way that the vertex operator algebra represents the noncommutative coordinates of string theory. Furthermore, this construction demonstrates the unity between gauge theories and string theories, an important ingredient of the unified framework of M Theory. It is also possible to prove that $\mathcal{A}_\theta \cong \mathcal{A}_{1/\theta} \cong \mathcal{A}_{\theta+1}$ \cite{12}. When translated into the language of the vertex operator algebra, this implies certain duality symmetries of the noncommutative spacetime \cite{9, 10, 13}.

3. Automorphisms of Vertex Operator Algebras

An automorphism of the vertex operator algebra $\mathcal{A}$ is a unitary isomorphism $\sigma : \mathcal{H} \to \mathcal{H}$ of vector spaces which preserves the operator-state correspondence,
\[
\sigma V(\psi; z_+, z_-) \sigma^{-1} = V(\sigma(\psi); z_+, z_-) .
\]
(3.1)
In other words, the mapping $|\psi\rangle \to V(\psi; z_+, z_-)$ on $\mathcal{H} \to \mathcal{A}$ is equivariant with respect to the (adjoint) actions of $\sigma$ on $\mathcal{H}$ and $\mathcal{A}$. The group of automorphisms (3.1) is denoted $\text{Aut}(\mathcal{A})$. The grading of $\mathcal{H}$ with respect to conformal dimension yields a decomposition of $\mathcal{A} = \mathcal{A}^+ \otimes \mathbb{C}\{A\} \mathcal{A}^-$ into the eigenspaces of the Virasoro operators $L_0^\pm$ according to
\( \mathcal{A}^\pm = \bigoplus_{\Delta^\pm} \mathcal{A}_{\Delta^\pm} \). The weight zero subspace of \( \mathcal{H} \) is one-dimensional and is spanned by the vacuum \( |\text{vac}\rangle \). The subgroup of \( \text{Aut}(\mathcal{A}) \) consisting of automorphisms that preserve this grading, i.e. \( \sigma \mathcal{A}_\Delta \sigma^{-1} = \mathcal{A}_\Delta \), is denoted \( \text{Aut}^{(0)}(\mathcal{A}) \). The normal subgroup of \( \text{Aut}(\mathcal{A}) \) consisting of inner automorphisms is denoted \( \text{Inn}(\mathcal{A}) \), while the remaining outer automorphisms are \( \text{Out}(\mathcal{A}) = \text{Aut}(\mathcal{A})/\text{Inn}(\mathcal{A}) \). The automorphism group is then the semi-direct product

\[
\text{Aut}(\mathcal{A}) = \text{Inn}(\mathcal{A}) \ltimes \text{Out}(\mathcal{A})
\]

of \( \text{Inn}(\mathcal{A}) \) by the natural action of \( \text{Out}(\mathcal{A}) \).

The classification of the full automorphism group \( (3.2) \) is a difficult problem in the general case. However, the structure of \( \text{Aut}^{(0)}(\mathcal{A}) \) can be determined to some extent. This group of automorphisms is represented as a group of unitary transformations on each of the homogeneous spaces \( \mathcal{H}_\Delta \). Consider the chiral algebras \( \mathcal{A}^\pm \) built on the compactification lattice \( \Gamma \) with vertex operators \( V(\psi_\pm; z_\pm) \), where \( \psi_\pm \in (\mathcal{H}^\pm)^* \). Let \( \psi_n \) be the component operators of the meromorphic function \( V(\psi; z_\pm) = \sum_{n \in \mathbb{Z}} \psi_n z_\pm^{-n-1} \). The condition \( (3.1) \) on \( \mathcal{A}^\pm \) is then equivalent to \( \sigma \psi_n \sigma^{-1} = (\sigma(\psi))_n \forall n \in \mathbb{Z} \). The automorphisms of \( \mathcal{A}^\pm \) which preserve their gradings are those which leave invariant the conformal vectors \( \omega^\pm \) defined by the stress-energy tensors \( T^\pm(z_\pm) = \sum_{k \in \mathbb{Z}} L_k z_\pm^{-k-2} = V(\omega^\pm; z_\pm) \) of the underlying conformal field theory. The operators \( \omega^\pm \) are given explicitly by

\[
\omega^\pm = \frac{1}{2} \mathbb{I} \otimes g_{\mu\nu}(\bar{e}^\mu)_{\lambda}(\bar{e}^\nu)^{\rho} \alpha^{(\pm)\lambda}_{-1} \alpha^{(\pm)\rho}_{-1},
\]

where \( \{\bar{e}^\mu\} \) is an arbitrary basis of the lattice \( \Gamma \), and they have conformal weight 2. The operations

\[
[[\psi, \varphi]] \equiv \psi_0 \varphi \quad , \quad \langle \langle \psi, \varphi \rangle \rangle \equiv \psi_1 \varphi
\]

define, respectively, a Lie bracket and an invariant bilinear form on the space

\[
\mathcal{L}_\Gamma^\pm = (\mathcal{H}_1^\pm)^*/(\mathcal{H}_1^\pm)^* \cap L_{-2}^\pm(\mathcal{H}_0^\pm)^*
\]

of primary states of weight one. Then the unitary group \( \exp i\mathcal{L}_\Gamma^\pm \) yields a Lie group of automorphisms acting on the chiral algebra \( \mathcal{A}^\pm \) by the adjoint representation\(^3\) and preserving its conformal grading. These define the continuous automorphisms of the vertex operator algebra, so that the inner automorphism subgroup of \( \text{Aut}^{(0)}(\mathcal{A}^\pm) \) is

\[
\text{Inn}^{(0)}(\mathcal{A}^\pm) = \text{Ad}_{\mathcal{A}^\pm} \exp i\mathcal{L}_\Gamma^\pm.
\]

Explicitly, \( \mathcal{L}_\Gamma^\pm \) is generated by the operators \( \varepsilon_q^\pm \equiv (-1)^{q_\mu q_\nu} e^{-iq^\pm x^\mu} \) (the generators of the twisted group algebra \( \mathbb{C}\{\Gamma\} \) of \( \Gamma \), where \( q^\pm \in \Gamma_2 \equiv \{ q \in \Gamma \mid q_{\mu}g^{\mu\nu}q_\nu = 2 \} \), and \( r^\pm_{\mu} \alpha_{-1}^{(\pm)\mu} \), where \( r^\pm \in \Gamma \). As a subspace of the noncommutative spacetime, the algebra \( (3.3) \) therefore contains the lowest-lying non-trivial oscillatory modes of the strings (i.e. \( \mathcal{A}^\pm = \bigoplus_{\Delta^\pm} \mathcal{A}_{\Delta^\pm} \). The *-conjugation on the vertex operator algebra is defined by \( V^*(\psi; z_\pm) = \sum_{n \in \mathbb{Z}} \psi_n^* z_\pm^{-n-1} = V(e^{\pm L_1} (-z_\pm^2) L_0^* \psi; z_\pm^{-1}) \). With this definition we have, in particular, that \( (L_0^*)^* = L_{-2}^\pm \).
the graviton operators \((\mathcal{L}_{1}^{+})\), and thus the smallest quantum perturbation of classical spacetime. With the Lie bracket in \((3.3)\) its commutation relations are

\[
\begin{align*}
[[\alpha_{-1}^{(\pm)\mu}, \alpha_{-1}^{(\pm)\nu}]] &= 0, \\
[[\alpha_{-1}^{(\pm)\mu}, \varepsilon_{q_{\pm}}]] &= g^{\mu\nu} q_{\nu} \varepsilon_{q_{\pm}}
\end{align*}
\]

Using the bilinear form in \((3.4)\) we have \(\langle \alpha_{-1}^{(\pm)\mu}, \alpha_{-1}^{(\pm)\nu} \rangle = g^{\mu\nu}, \langle \alpha_{-1}^{(\pm)\mu}, \varepsilon_{q_{\pm}} \rangle = 0\), and \(\langle \varepsilon_{q_{\pm}}, \varepsilon_{r_{\pm}} \rangle = 0\) if \(q_{\mu}^{\pm} g^{\mu\nu} r_{\nu}^{\pm} \geq 0\), \(1\) if \(q_{\mu}^{\pm} g^{\mu\nu} r_{\nu}^{\pm} = -1\). This determines a root space decomposition of \(\mathcal{L}_{1}^{\pm}\) such that its root lattice is precisely \(\Gamma\) and its set of roots is \(\Gamma_{2}\). Furthermore, we can use this bilinear form and the operator-state correspondence to construct a level 1 representation of the corresponding affineization \(\hat{\mathcal{L}}_{1}^{\pm}\), appropriate to the action of \((3.6)\) on the quantum fields of the string theory. One finds for the usual modes that \((\alpha_{-1}^{(\pm)\mu})_{n} = \alpha_{n}^{(\pm)\mu}\) and \((\varepsilon_{q_{\pm}})_{n}\) can be determined from the tachyon vertex operators \([\mathcal{L}_{1}^{\pm}]\), so that

\[
\begin{align*}
[[\alpha_{-1}^{(\pm)\mu} m, (\alpha_{-1}^{(\pm)\nu})_{n}]] &= m g^{\mu\nu} \delta_{m+n,0}, \\
[[\alpha_{-1}^{(\pm)\mu} m, (\varepsilon_{q_{\pm}})_{n}]] &= g^{\mu\nu} q_{\nu}^{\pm} (\varepsilon_{q_{\pm}})_{m+n}
\end{align*}
\]

(3.7)

The Kac-Moody algebras \(\hat{\mathcal{L}}_{1}^{\pm}\) always contain the affine \(u(1)_{d}^{d}\) gauge groups generated by the chiral Heisenberg fields

\[
\alpha_{\pm}^{\mu}(z_{\pm}) = -i \partial_{z_{\pm}} X_{\pm}^{\mu}(z_{\pm}) = \sum_{k=-\infty}^{\infty} \alpha_{k}^{(\pm)\mu} z_{\pm}^{-k-1} = -i V(1 \otimes \alpha_{-1}^{(\pm)\mu})_{\pm}
\]

(3.9)

which are the conserved currents associated with the target space reparametrization symmetry. The full algebraic structure of \(\mathcal{L}_{1}^{\pm}\) is, however, strongly dependent on the particular compactification lattice \(\Gamma\). For instance, when \(\Gamma = (2\mathbb{Z})^{d}\), for which \(\Gamma_{2} = \emptyset\), we have \(\mathcal{L}_{2}^{\pm} = u(1)_{d}^{d}\), while for \(\Gamma = \mathbb{Z}^{d}\) (corresponding to a complete factorization of the \(d\)-torus as \(T^{d} = (S^{1})^{d}\)), we find \(\mathcal{L}_{2}^{\pm} = su(2)_{d}^{d}\).

From \((3.3)\) it is also possible to immediately identify the outer automorphism subgroup of \(\text{Aut}(0)(\mathcal{A}_{\pm})\) as the (discrete) isometry group of the lattice \(\Gamma\),

\[
\text{Out}^{(0)}(\mathcal{A}_{\pm}) = \text{Aut}(\Gamma) = O(d; \mathbb{Z})
\]

(3.10)

which represents the symmetries of the Dynkin diagram of \(\mathcal{L}_{1}^{\pm}\). However, a complete classification of the full automorphism group, including the transformations which do not
preserve (3.3), is not as straightforward. For instance, the inner automorphism of \( A^\pm \) determined by the unitary element
\[
\sigma_{q^\pm} = \exp \left\{ 2i \varepsilon_{q^\pm} \otimes \mathbb{I} / (2 - q^\pm_{\mu} g^{\mu\nu} q^\pm_\nu) \right\}
\]
for \( q^\pm \in \Gamma - \Gamma_2 \) maps (3.3) to the conformal vector \( \omega_{q^\pm} = \sigma_{q^\pm} \omega^\pm \sigma_{q^\pm}^{-1} = \omega^\pm + L_{-1} (\varepsilon_{q^\pm} \otimes \mathbb{I}) \) of conformal dimension \( \frac{1}{2} q^\pm_{\mu} g^{\mu\nu} q^\pm_\nu + 1 \). Thus the full group of inner automorphisms in this case is quite large. The classification of the automorphism group for the full vertex operator algebra \( A = A^+ \otimes \mathbb{C}_* A^- \) is further complicated by the existence of transformations which do not factorize into chiral components and can mix between the two chiral algebras.

4. Quantum Geometry and Gauge Symmetries of String Theory

We shall now apply the formalism developed thus far to a systematic investigation of the symmetries of the noncommutative string spacetime, represented by the \(*\)-algebra \( A = A^+ \otimes \mathbb{C}_* A^- \). As discussed in section 1, the inner automorphisms of an algebra in Noncommutative Geometry represent gauge transformations. In simple cases (such as the standard model), the distinction between outer and inner automorphisms is, respectively, of diffeomorphisms of the spacetime and internal gauge symmetries. In the case at hand, however, the situation is far from being so simple, as the entire algebra is noncommutative and a manifold only emerges in a low energy limit.

The quantum geometry of \( T^d \) is determined by classifying its duality symmetries. The basic observation is that the entire algebraic structure of the vertex operator algebra is unchanged on the whole by redefining the Heisenberg fields (3.9) as \( \alpha_\pm^d (z_\pm) \rightarrow \pm \alpha_\pm^d (z_\pm) \). This transformation can be achieved via an automorphism of \( A \) in several ways. Such automorphisms define the duality transformations of the string theory and represent symmetries of the classical spacetime \( T^d \) when viewed as a (low-energy) subspace of the full noncommutative spacetime. The simplest example is the T-duality transformation \( d_\pm \rightarrow (d^\pm)^{-1} \), which corresponds essentially to an inversion of the spacetime metric \( g_{\mu\nu} \) and interchanges momentum and winding modes in the spectrum of the quantum string theory. It implies that, as subspaces of the noncommutative spacetime, the torus \( T^d \) is equivalent to its dual \( (T^d)^* = \mathbb{R}^d/2\pi \Gamma^* \). From the point of view of classical general relativity there is no reason for these two spacetimes to describe the same physics, and the noncommutative geometry naturally describes the stringy modification of classical spacetime geometry.

The augmentation of the generic \( u(1)^d_+ \oplus u(1)^d_- \) gauge symmetry of the string theory to \( \mathcal{L}_\Lambda \equiv \mathcal{L}_{\Gamma^*} \oplus \mathcal{L}_\Gamma \) is known as an ‘enhanced gauge symmetry’. For the full vertex operator algebra \( A \), it is due to the appearance of extra dimension \( (\Delta^+, \Delta^-) = (1, 0) \) and \( (0,1) \) operators in the theory (namely the appropriate tachyon fields). Such operators can be used to perturb the underlying conformal field theory to an isomorphic one with different target space properties. It turns out that to describe the full duality group of
toroidally compactified string theory, it suffices to examine the Lie algebra associated with
the unique fixed point \((d^\pm)^2 = I\) of the T-duality transformation, i.e. \(g_{\mu\nu} = \delta_{\mu\nu}, \beta_{\mu\nu} = 0\), or equivalently \(\Gamma = \mathbb{Z}^d\). As mentioned in the previous section, at this fixed point the generic \(U(1)^d_+ \times U(1)^d_-\) gauge symmetry is enhanced to a level 1 representation of the affine Lie group \(SU(2)^d_+ \times SU(2)^d_-\). To describe this structure explicitly, let \(k^{(i)}_{\mu}\), \(i = 1, \ldots, d\), be a suitable basis of (constant) Killing forms on \(T^d\). Then the vertex operators \(j^{(i)}_{\pm}(z_\eta) =: e^{\pm i k^{(i)}_{\mu}(z_\eta)} \mu, j^{(i)}_{3}(z_\eta) = ik^{(i)}_{\mu}(\alpha_\eta(z_\eta)), \) with \(\eta = \pm\), generate a level 1 \(su(2)^d_+ \oplus su(2)^d_-\) Kac-Moody algebra as described before. Associated with this gauge symmetry of the theory are the corresponding gauge group elements \(\sigma = e^{iG}\). The generators \(G\) that implement spacetime duality transformations of the string theory were originally constructed in [13] and are given as follows,

\[
G^{(\mu)}_{\pm}(z_+, z_-) = \frac{\tau}{2^\frac{d}{2}} : e^{i\sqrt{2}\pi k^{(\mu)}_{\mu}(X_{\pm}(z_+, z_-))} - e^{-i\sqrt{2}\pi k^{(\mu)}_{\mu}(X_{\pm}(z_+, z_-))} : \tag{4.1}
\]

\[
G^{(\mu)}_{\pm}(z_+, z_-) = \frac{\tau}{2^\frac{d}{2}} : e^{i\pi z_\pm} - e^{-i\pi z_\pm} : \tag{4.2}
\]

\[
G_{\lambda}(z_+, z_-) = \lambda_\mu(X)(z_+ \alpha^\mu_+(z_+) - z_- \alpha^\mu_-(z_-)) \tag{4.3}
\]

\[
G_{\xi}(z_+, z_-) = \xi_\mu(X)(z_+ \alpha^\mu_+(z_+) + z_- \alpha^\mu_-(z_-)) \tag{4.4}
\]

where \(X = \frac{1}{\sqrt{2}}(X_+ + X_-)\).

The operator (4.1) generates the \(\mu^\text{th}\) factorized duality map of the spacetime. For each \(\mu = 1, \ldots, d\) it is a generalization of the \(R \to 1/R\) circle duality in the \(X^\mu\) direction (of radius \(R\)) of \(T^d\). When \(d\) is even, it corresponds to mirror symmetry which interchanges the complex and Kähler structures of the target space and leads to the stringy phenomenon of spacetime topology change. The inner automorphism generated by \(G^{(\mu)}_{+} + G^{(\mu)}_{-}\) corresponds to the reflection \(X^\mu \to -X^\mu\) of the coordinates of \(T^d\). Thus factorized dualities and spacetime reflections are enhanced gauge symmetries, and hence intrinsic properties, of the noncommutative spacetime. The remaining two duality transformations are abelian gauge symmetries. The operator (4.3) generates local gauge transformations \(\beta \to \beta + d\lambda\) of the instanton two-form. Taking \(\lambda_\mu(X) = C_{\mu\nu}X^\nu\), with \(C_{\mu\nu}\) a constant antisymmetric matrix, gives the shift \(\beta_{\mu\nu} \to \beta_{\mu\nu} + C_{\mu\nu}\). Singlevaluedness of the corresponding gauge group element \(\sigma\) requires \(C_{\mu\nu} \in \mathbb{Z}\) yielding a large gauge transformation. Finally, the operator (4.4) generates a general spacetime coordinate transformation \(X \to \xi(X)\). Again for the large diffeomorphisms \(\xi_\mu(X) = T_{\mu\nu}X^\nu\) of \(T^d\), singlevaluedness requires \([T_{\mu\nu}] \in SL(d, \mathbb{Z})\). In particular, setting \(T_{\mu\nu} = \frac{\tau}{2} \text{sgn}(P)g_{P(\mu), \nu}\) yields a permutation \(P \in S_d\) of the coordinates of \(T^d\). Combining these with the factorized duality transformations yields T-duality in the form of an inner automorphism. As such, T-duality corresponds to the global gauge transformation in the Weyl subgroup \(\mathbb{Z}_2\) of \(SU(2)\). It can be shown that these discrete vertex operator inner automorphisms, corresponding to large gauge transformations of the internal string spacetime, generate the infinite discrete duality group \(O(d, d; \mathbb{Z})\) which is the isometry group of the Narain lattice \(\Lambda\). Explicit expressions for the actions of these operators on \(\mathcal{H}\) and \(\mathcal{A}\) can be found in [10]. We see therefore that, by viewing string theory as a noncommutative geometry, target space duality transformations, as well as generic diffeomorphisms, all appear naturally as elements of the group of gauge transformations.
It is also possible to write down the full outer automorphism group, representing the
diffeomorphisms (i.e. the gravitational symmetries) of the noncommutative spacetime, as
\[
\text{Out}(\mathcal{A}) = O(d, d; \mathbb{Z}) \otimes O(2). \quad (4.5)
\]
The \(O(2)\) part of (4.5) is a worldsheet symmetry group that acts by rotating the two
chiral sectors into each other. It arises from the spin structure of the string worldsheet
which yields a representation of \(\text{spin}(2)\) on the Hilbert space \(\mathcal{H}\) that implements the group
\(SO(2)\). In particular, its \(\mathbb{Z}_2\) subgroup generates the worldsheet parity transformation that
interchanges the left and right chiral algebras, \(A^+ \otimes A^- \to A^- \otimes A^+\). However, as discussed
earlier, it is not known what the general form of the inner automorphism group is. It is
an open problem to determine what symmetries are represented by inner automorphisms
such as (3.11) which correspond to deformations of the underlying conformal field theory
to an inequivalent two-dimensional quantum field theory.

All of these duality transformations are exact symmetries of the full noncommutative
spacetime described by the entire vertex operator algebra. The algebra \(C^\infty(T^d)\) which
describes the toroidal spacetime is contained in the (smeared) vertex operator algebra
as a subalgebra. The generators (4.1)–(4.3) act trivially on the low-energy sector, i.e.
\(e^{iG}|C^\infty(T^d)\rangle = \mathbb{I}\), as expected since the symmetries represented by the inner automorphisms
(3.6) live in the quantum perturbation of classical spacetime represented by the graviton
operators (2.7). The main feature is that the orthogonal projection \(A \to C^\infty(T^d)\) does
not commute with the duality maps, and therefore these symmetries change the geometry
and topology of the classical spacetime. Thus although duality is a gauge symmetry of the
full noncommutative geometry, the low energy sectors look dramatically different. Similar
statements also apply to the outer automorphisms (4.5) acting on \(C^\infty(T^d)\).

As for the inner automorphisms generated by (4.4), when projected onto the subalgebra
\(C^\infty(T^d) \subset \mathcal{A}\) they represent the generators of \(\text{Diff}(T^d)\) in terms of the canonically
conjugate center of mass variables \(x^\mu, p_\mu\). Thus the full group of inner automorphisms
of \(\mathcal{A}\), representing internal gauge symmetries of the string spacetime, projects onto the
group of outer automorphisms of \(C^\infty(T^d)\), representing the diffeomorphisms of the target
space. In this respect general covariance is represented as a gauge symmetry in the stringy
modification of general relativity, in that its transformations are generated by a unitary
group acting on a noncommutative geometry. This gives a remarkable interpretation of
the usual classical symmetries in quantum geometry, a dramatic example of which is that
gravity becomes a gauge theory.

5. Universal Gauge Symmetry

We have seen that Noncommutative Geometry gives a unifying description of spacetimes
that appear to be distinct at low energies, as well as of gauge transformations and
diffeomorphisms. It is natural then to envisage some sort of \textit{universal} gauge symmetry
which encompasses all of these features. The investigation of such a universal symmetry has been previously attempted in \cite{10}. There the Universal Gauge Group is defined as an ideal in the algebra of bounded operators acting on a Hilbert space, which contains all groups of gauge transformations as subgroups. These ideals are related to the Schatten ideals which play a central role in the description of infinitesimals in noncommutative geometry \cite{2}, and this construction can be carried out using the notion of a cycle in noncommutative geometry. This group is easier to deal with than the inductive large-$N$ limit $U(\infty)$ of the unitary groups $U(N)$, which also contains all compact gauge groups as subgroups (via appropriate unitary representations). Two Yang-Mills theories with different structure groups can have vastly different physical properties, so that this notion of a universal gauge theory overlies many different physical theories. In the present context we can make a construction similar in spirit to this.

We recall the dependence of the symmetry group (3.6) on the choice of compactification lattice $\Gamma$. Since nature cannot distinguish between different compactifications (as this part of the full string spacetime is unobservable in the physical spacetime in which we live), we seek a symmetry algebra which overlies every Lie algebra $\mathcal{L}_\Lambda$. This would ultimately lead to a unified framework for studying all of the dynamical symmetries of string theory. For this, we consider the unique 2d-dimensional even self-dual Lorentzian lattice $\Pi_{d,d}$, and we analytically continue in the spacetime momenta to allow for both purely real and purely imaginary momenta by extending it to a module over the Gaussian integers, $\Lambda(G) \equiv \Pi_{d,d} \otimes \mathbb{Z}[i]$. This extension will ensure that the roots of the appropriate Lie algebra lie inside the light-cone of the Lorentzian lattice $\Lambda(G)$. Now we carry out the constructions of sections 2 and 3 with the arbitrary lattice $\Lambda = \Gamma^* \oplus \Gamma$ replaced by $\Lambda(G)$. Then the corresponding Lie algebra of dimension 1 primary states

$$\mathcal{L}_U \equiv (\mathcal{H}_1^{(G)})^* / \bigcup_{k \geq 1} (\mathcal{H}_1^{(G)})^* \cap \left( L_+^k \otimes L_-^{-k} \right) \mathcal{H}^{(G)}$$

(5.1)

generates a universal gauge symmetry group, in the sense that for any compactification lattice $\Gamma$ there is a natural Lie subalgebra embedding $\mathcal{L}_\Lambda \hookrightarrow \mathcal{L}_U$ \cite{18}. In this way, noncommutative geometry yields a natural geometrical interpretation to the universal gauge group corresponding to (5.1). This construction, as well as those of \cite{10,17}, is based on the representation of gauge transformations as operators on an infinite-dimensional Hilbert space.

The universal gauge symmetry algebra (5.1) (as well as that of \cite{17}) is a generalization of the Lie algebra of vector fields on the circle to noncommutative geometry. These structures are all related to the symmetries of the noncommutative 2-torus with rational deformation parameter $\theta$, which in turn is the underlying noncommutative geometrical structure of M Theory. The Lie group $U(N)$ is just the unitary group of the $C^*$-algebra $M(N, \mathbb{C})$. Thus, given the relation between the vertex operator algebra and the large-$N$ limit of this noncommutative torus, the symmetry group Aut($A$) and the corresponding

\footnote{An extension to large-$N$ matrix models for open strings has also been recently given in \cite{17} involving some more familiar algebras such as $gl(\infty, \mathbb{C})$ and the Cuntz algebra.}
universal gauge group should be related in some way to those of the infinite-dimensional $C^*$-algebra $M(\infty, \mathbb{C})$. Thus the universal gauge symmetries described here could help in understanding the underlying dynamical symmetries of M Theory.

Another feature which arises in this context concerns the meaning of the gauge symmetries in $\text{Inn}(\mathcal{A}) - \text{Inn}^{(0)}(\mathcal{A})$. Such transformations represent higher-order quantum perturbations of classical spacetime and can dramatically alter the underlying world-sheet theory. The symmetry group $\text{Aut}(\mathcal{A})$ then overlies a rather large set of two-dimensional quantum field theories which can be physically quite different. But from the point of view of noncommutative geometry, these theories are all embedded into the same universal structure and are just different corners of some big model whose symmetry group is $\text{Aut}(\mathcal{A})$. The different corners are related to one another by gauge transformations, yielding duality maps between inequivalent physical theories. This is the earmark of M Theory, which overlies the five consistent superstring theories in ten dimensions by relating them to each other via duality isomorphisms. In this sense, the noncommutative geometry of string spacetimes is naturally suited to the unifying framework of M Theory. The focal point of the ease in which these interpretations arise is at the very heart of the techniques of noncommutative geometry.

6. Time-like Compactifications and Generalized Kac-Moody Symmetries

The structures we have described thus far change rather dramatically when the toroidal target space contains a time-like coordinate, i.e. $T^{1,d-1} \cong S^1_\bot \times T^{d-1}$, generated by a compactification lattice $\Gamma$ of Minkowski signature. It was shown in [18] that the action of the duality group $\text{Aut}(\Lambda)$ on the background parameters of the spacetime is ergodic and dense in this case. This means that the quantum moduli space of time-like toroidal compactifications does not exist as a manifold and only makes sense within the framework of noncommutative geometry. This was precisely the case of the noncommutative torus with irrational deformation parameter $\theta$, the duality action being regarded there as the orbit of a free particle in the 2-torus. Noncommutative geometry is therefore an important ingredient in the description of the space of time-dependent string backgrounds.

The structure of the vertex operator algebra also changes when the lattice $\Gamma$ is no longer Euclidean. For example, consider the unique 26-dimensional even self-dual Lorentzian lattice $\Gamma = \Pi_1,25$. Then $\Lambda_* = \Pi_1,25 \oplus \Pi_1,25$ is the unique point in the quantum moduli space of time-like toroidal compactifications at which the vertex operator algebra $\mathcal{A}$ completely factorizes between its left and right chiral sectors,

$$\mathcal{H}(\Lambda_*)^* = \mathcal{C}^+ \otimes \mathcal{C}^- \quad (6.1)$$

where

$$\mathcal{C}^\pm = \mathbb{C}\{\Pi_1,25\} \otimes \mathcal{F}^\pm(\Pi_1,25)^* \quad (6.2)$$
and we have denoted the explicit dependence of the Heisenberg algebras built on $\Pi_{1,25}$. In essence this factorizes the closed string into two open strings. The distinguished lattice $\Lambda$ is an enhanced symmetry point in the moduli space. The corresponding Lie algebra $\mathcal{L}_\Lambda = \mathcal{B} \oplus \mathcal{B}$ is a maximal symmetry algebra, in the sense that it contains all gauge symmetry algebras [18]. It is not, however, universal because the gauge symmetries are not necessarily embedded into it as Lie subalgebras. $\mathcal{L}_\Lambda$ contains many novel infinite-dimensional symmetry algebras, such as algebras of area-preserving and volume-preserving diffeomorphisms. The existence of these large symmetries is intimately connected to the unusual nature of time in string theory.

The chiral algebra $\mathcal{B}$ obtained from the complete factorization above is given explicitly by

$$\mathcal{B} = \mathcal{H}^\pm_1(\Pi_{1,25})^*/\ker\langle\cdot,\cdot\rangle$$

(6.3)

where in the present case we have to divide out by the kernel of the bilinear form in (6.4) because in Minkowskian signature additional null physical states besides the spurious states of $(\mathcal{H}^\pm_1)^* \cap L^\pm_1(\mathcal{H}^\pm_0)^*$ appear. $\mathcal{B}$ is called the fake Monster Lie algebra [11, 19]. It is not a Kac-Moody algebra because it contains light-like simple Weyl roots. This leads to the notion of a Borcherds or generalized Kac-Moody algebra [19] which resembles an ordinary affine Lie algebra in most respects, except for the existence of imaginary (non-positive norm) simple roots. The positive norm simple roots of the root lattice $\Pi_{1,25}$ of $\mathcal{B}$ lie in the Leech lattice $\Gamma_L$, the unique 24-dimensional even self-dual Euclidean lattice with no vectors of length $\sqrt{2}$.

The fake Monster Lie algebra is the starting point for the construction of another generalized Kac-Moody algebra, the Monster Lie algebra [17]. The Monster module $\mathcal{M}$ is built from the Leech lattice $\Gamma_L$ and the order 2 automorphism $\sigma$ of the corresponding lattice vertex operator algebra induced by the reflection isometry of $\Gamma_L$. $\mathcal{M}$ is the vertex operator algebra associated with the (unique) chiral $c = 24$ $\sigma$-twisted orbifold conformal field theory [11]. Since $\Pi_{1,25} \cong \Gamma_L \oplus \Pi_{1,1}$, with $\Pi_{1,1}$ the unique two-dimensional even self-dual Lorentzian lattice, $\mathcal{M}$ is naturally obtained from $\mathcal{B}$. The most startling aspect of $\mathcal{M}$ is that it contains no dimension 1 operators. Its symmetries (which are all discrete) are now classified according to the space of dimension 2 primary states. On this space there is a binary operation

$$\psi \star \varphi \equiv \psi_1 \varphi$$

(6.4)

which turns it into a commutative non-associative algebra. This algebra is the 196884-dimensional Griess algebra for which the Monster group $F_1$ is the full automorphism group. $F_1$ is the largest finitely-generated simple sporadic group.

The monster module $\mathcal{M}$ is naturally associated with a twisted heterotic string theory [20]. It leads to a rather exotic noncommutative spacetime which possesses only discrete outer automorphisms as its (diffeomorphism) symmetries, a subgroup of which is the Monster group. This spacetime is “topological” in the sense that it does not contain the usual (low-energy) diffeomorphism symmetries. It can also be shown that string
compactifications with Monster symmetry are not dense \cite{18}.

It has been recently argued \cite{21} that various superstring dualities are isomorphic to the elementary group structures occurring above in the construction of $F_1$. In particular, string-string duality appears to be identical to the reflection isometry of the Leech lattice, and an explicit $SU(\infty)$ gauge symmetry (the symmetry group of the 11-dimensional supermembrane) arises from this formulation whose fundamental quantum degrees of freedom can be identified with the elements of the Griess algebra. The emergence of matter-like states is thus an example of enhanced gauge symmetry. Thus the Monster sporadic group appears to be a hidden symmetry group of superstring theory\cite{21}. The Monster module has also been related recently to D-particle dynamics in type IIA superstring theory \cite{22}. In \cite{23} the relations between string duality and generalized Kac-Moody algebras are also discussed. The space of perturbative BPS string states of toroidally compactified heterotic string theories forms a generalized Kac-Moody algebra. The possibility of relating superstring theory to properties of exotic algebras, such as the Leech lattice and the Monster group, is quite appealing since many of the associated symmetry groups are unique.

Given these relations and the fact that the fake Monster Lie algebra \cite{23} is a maximal symmetry algebra of the string theory, it is becoming increasingly evident that Borcherds algebras, when interpreted as generalized symmetry algebras of the noncommutative geometry, are relevant to the construction of a universal symmetry of string theory. Being natural generalizations of affine Lie algebras, they may emerge as new symmetry algebras for string spacetimes. The appearance of exotic gauge symmetries such as Monster symmetry could have significant implications as the role of a dynamical Lie algebra representing the unified symmetries of the noncommutative string spacetime. The remarkable fact is that the irreducible representations of the algebras described above match exactly the Fock space of the bosonic string field theory with the underlying Kac-Moody algebra as spectrum-generating algebra for the one-string Hilbert space. Within these dynamical algebras then one would expect to find all quantum states in a single representation, with the underlying Lie algebra determining the maximal symmetry algebra of the string spacetime.

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References

[1] K.S. Narain, Phys. Lett. B169 (1986) 41; K.S. Narain, M.H. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369.

[2] A. Connes, Noncommutative Geometry (Academic Press, 1994).

\footnote{In this paper we have discussed only the simplest case of bosonic strings, but, as is discussed in \cite{21}, the natural arena for the noncommutative geometry of quantum field theories is appropriate supersymmetric extensions of these models.}
[3] E. Witten, Nucl. Phys. B460 (1996) 335; P.-M. Ho and Y.-S. Wu, Phys. Lett. B398 (1997) 52; F. Lizzi, N.E. Mavromatos and R.J. Szabo, hep-th/9711012.

[4] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, Phys. Rev. D55 (1997) 5112.

[5] A. Connes, Commun. Math. Phys. 182 (1996) 155; A.H. Chamseddine and A. Connes, Phys. Rev. Lett. 77 (1996) 4868; Commun. Math. Phys. 186 (1997) 731.

[6] A. Connes and J. Lott, Nucl. Phys. B (Proc. Suppl.) 18B (1990) 29; A. Connes, J. Math. Phys. 36 (1995) 619; For a recent review see C.P. Martín, J.M. Gracia-Bondía and J.C. Várilly, [hep-th/9605001], to appear in Phys. Rep.

[7] J. Fröhlich and K. Gawędzki, CRM Proc. Lecture Notes 7 (1994) 57; A.H. Chamseddine, Phys. Rev. D56 (1997) 5112.

[8] J. Fröhlich, O. Grandjean and A. Recknagel, in: Quantum Symmetries, Proc. LXIV Les Houches Session, eds. A. Connes and K. Gawędzki, to appear, [hep-th/9706132].

[9] F. Lizzi and R.J. Szabo, Phys. Rev. Lett. 79 (1997) 3581.

[10] F. Lizzi and R.J. Szabo, [hep-th/9707202].

[11] I.B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure Appl. Math. 134 (Academic Press, New York, 1988); R.W. Gebert, Intern. J. Mod. Phys. A8 (1993) 5441.

[12] M. Rieffel, Pac. J. Math. 93 (1981) 415.

[13] F. Lizzi and R.J. Szabo, [hep-th/9709198], Phys. Lett. B (in press).

[14] T. Banks, N. Seiberg and S.H. Shenker, Nucl. Phys. B490 (1997) 91.

[15] M. Evans and I. Giannakis, Nucl. Phys. B472 (1996) 139; I. Giannakis, Phys. Lett. B388 (1996) 543.

[16] S.G. Rajeev, Phys. Rev. D42 (1990) 2779; D44 (1991) 1836.

[17] C.-W.H. Lee and S.G. Rajeev, [hep-th/9712090].

[18] G. Moore, [hep-th/9305139].

[19] R.E. Borcherds, J. Algebra 115 (1988) 501; Invent. Math. 109 (1992) 405.

[20] J.A. Harvey, in: Unified String Theories, eds. M.B. Green and D.J. Gross (World Scientific, Singapore, 1985) 704.

[21] G. Chapline, [hep-th/9609162].

[22] M.B. Green and D. Kutasov, [hep-th/9712146].

[23] G. Moore, [hep-th/9710198].