Delay reductions of the two-dimensional Toda lattice equation

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\textbf{Abstract}

Integrable delay analogues of the two-dimensional Toda lattice equation are presented and their multi-soliton solutions are constructed by applying the delay reduction to the Gram determinant solution.

\textit{Keywords:} Integrable systems, Delay-differential equations, Delay reduction, Multi-soliton solutions

\section{1. Introduction}

Delay-differential equations have been used as mathematical models in various fields of science and engineering, such as traffic flow and infectious diseases. Studies on integrability and exact solutions of delay-differential equations have been carried out from mathematical and applied point of view\cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}. Integrable delay-differential equations have been obtained by similarity reductions or symmetry reductions in previous studies, and it was confirmed that these integrable delay-differential equations exhibit singularity confinement type behavior which is one of important properties of discrete integrable systems\cite{15}. Villarroel and Ablowitz considered a delay analogue of the two-dimensional Toda lattice (2DTL) equation and found the Lax pair and established the inverse scattering transform\cite{7}. However multi-soliton

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solutions have not been known until now.

In this article, we present integrable delay analogues of the 2DTL equation and construct their multi-soliton solutions by a delay reduction. As far as we know, this is the first time that multi-soliton solutions of delay-differential equations has been obtained.

2. Delay analogues of the 2DTL equation and their multi-soliton solutions

We consider the 2DTL equation

$$\frac{\partial^2}{\partial x \partial y} \log(1 + V_n) = V_{n+1} - 2V_n + V_{n-1}$$

which can be written as

$$\frac{\partial^2 r_n}{\partial x \partial y} = e^{r_{n+1}} - 2e^{r_n} + e^{r_{n-1}}$$

or

$$\frac{\partial^2 u_n}{\partial x \partial y} = e^{u_{n-1}} - u_n - e^{u_{n+1}}$$

via the dependent variable transformation $r_n = u_n - u_{n+1} = \log(1 + V_n)$. The 2DTL equation is transformed into the bilinear equation

$$D_x D_y \tau_n \cdot \tau_n = 2(\tau_{n+1} \tau_{n-1} - \tau_n^2)$$

via the dependent variable transformation

$$V_n(x, y) = \frac{\partial^2}{\partial x \partial y} \log \tau_n = \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2} - 1 .$$

Here $D_x$ and $D_y$ are Hirota’s $D$-operator which is defined as

$$D_x^n D_y^m f \cdot g = (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n f(x, y) g(x', y')|_{x=x', y=y'} .$$
The Gram determinant form of the $N$-soliton solution to the 2DTL equation (1) is given as follows [16]:

\[ V_n(x,y) = \frac{\partial^2}{\partial x \partial y} \log \tau_n = \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2} - 1, \]

\[ \tau_n = \det \left( \delta_{ij} + \frac{\phi_i \psi_j}{p_i - q_j} \right)_{1 \leq i,j \leq N} \]

\[ = \prod_{i=1}^{N} \phi_i \det \left( \delta_{ij} + \frac{\phi_i \phi_j}{p_i - q_j} \right)_{1 \leq i,j \leq N} \]

\[ = \det \left( \delta_{ij} + \frac{\phi_i \psi_j}{p_i - q_j} \right)_{1 \leq i,j \leq N} \]

\[ \phi_i = p^n e^{p_i x - p_i^{-1} y + \phi_i^{(0)}}, \quad \psi_i = q^{-n} e^{-q_i x + q_i^{-1} y + \psi_i^{(0)}}, \]

\[ (i = 1, 2, \cdots, N), \]

where $p_i, q_i$ are positive constants and $\phi_i^{(0)}, \psi_i^{(0)}$ are real constants.

Applying the delay reduction

\[ V_n(x,y) = w(z,y), \quad z = x + hn, \]

to the 2DTL equation (1), where $h$ is a nonzero real constant, we obtain the delay 2DTL equation

\[ \frac{\partial^2}{\partial z \partial y} \log(1 + w(z,y)) = w(z + h,y) - 2w(z,y) + w(z - h,y), \]

which can be written as

\[ \frac{\partial^2 r(z,y)}{\partial z \partial y} = e^{r(z+h,y)} - 2e^{r(z,y)} + e^{r(z-h,y)} \]

or

\[ \frac{\partial^2 u(z,y)}{\partial z \partial y} = e^{u(z+h,y)-u(z,y)} - e^{u(z,y)-u(z+h,y)} \]

via the dependent variable transformation $r(z,y) = u(z,y) - u(z + h, y) = \log(1+w(z,y))$. Note that this delay 2DTL equation is a delay partial differential equation. The delay 2DTL equation (9) is transformed into the delay bilinear equation

\[ D_z D_y f(z,y) \cdot f(z,y) = 2(f(z + h,y)f(z - h,y) - f(z,y)^2) \]
via the dependent variable transformation
\[ w(z, y) = \frac{\partial^2}{\partial z \partial y} \log f(z, y) = \frac{f(z + h, y)f(z - h, y)}{f(z, y)^2} - 1. \quad (13) \]

We impose the reduction condition
\[ \log p_i - \log q_i = h(p_i - q_i) \quad (14) \]
to the parameters \( p_i, q_i \) (\( i = 1, 2, \cdots, N \)) in the N-soliton solution (7). Note that the reduction condition (14) includes the delay parameter \( h \). Then, by setting \( z = x + hn \), we have

\[
\phi_j \psi_j = p_j^n e^{p_j x - p_j^{-1} y + \phi_j^{(0)}} q_j^{-n} e^{-q_j x + q_j^{-1} y + \psi_j^{(0)}} \\
= e^{(p_j - q_j)x + n \log p_j + \phi_j^{(0)}} e^{-q_j x + q_j^{-1} y - n \log q_j + \psi_j^{(0)}} \\
= e^{(p_j - q_j)x + n}(p_j - q_j) - (p_j^{-1} - q_j^{-1})y + \phi_j^{(0)} + \psi_j^{(0)} \\
= e^{(p_j - q_j)x + n}(p_j - q_j) - (p_j^{-1} - q_j^{-1})y + \phi_j^{(0)} + \psi_j^{(0)} \\
= e^{(p_j - q_j)x + n}(p_j - q_j) - (p_j^{-1} - q_j^{-1})y + \phi_j^{(0)} + \psi_j^{(0)}.
\]

(15)

Thus, by imposing the reduction condition (14) and setting \( z = x + hn, f(z, y) = \tau_n(x, y) \), we obtain the following N-soliton solution of the delay 2DTL equation (9):

\[
w(z, y) = \frac{\partial^2}{\partial z \partial y} \log f(z, y) \\
= \frac{f(z + h, y)f(z - h, y)}{f(z, y)^2} - 1, \quad (16)
\]

\[
f(z, y) = \det \left( \delta_{ij} + \frac{\Phi_j(z, y)}{p_i - q_j} \right)_{1 \leq i, j \leq N},
\]

\[
\Phi_j(z, y) = e^{(p_j - q_j)x - (p_j^{-1} - q_j^{-1})y + \phi_j^{(0)}}.
\]

where \( p_i, q_i \) must satisfy
\[
\log p_i - \log q_i = h(p_i - q_i), \quad (i = 1, 2, \cdots, N)
\]

(17)

and \( \Phi_j^{(0)} \) are real constants.
Next, we consider the delay reduction of the 2DTL equation in a different coordinate and its one-dimensional reduction. By the independent variable transformation
\[ x = \frac{t + s}{2}, \quad y = \frac{t - s}{2}, \tag{18} \]
the 2DTL equation (1) leads to
\[ \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} \right) \log(1 + V_n(t, s)) = V_{n+1}(t, s) - 2V_n(t, s) + V_{n-1}(t, s), \tag{19} \]
which can be written as
\[ \frac{\partial^2 r_n}{\partial t^2} - \frac{\partial^2 r_n}{\partial y^2} = e^{r_{n+1}} - 2e^{r_n} + e^{r_{n-1}}, \tag{20} \]
or
\[ \frac{\partial^2 u_n}{\partial t^2} - \frac{\partial^2 u_n}{\partial y^2} = e^{u_{n+1} - u_n} - e^{u_{n-1} - u_n}. \tag{21} \]
via the dependent variable transformation \( r_n = u_n - u_{n+1} = \log(1 + V_n) \). The \( N \)-soliton solution of the 2DTL equation (19) is given as follows:
\[ V_n(t, s) = \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} \right) \log \tau_n \]
\[ = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} - 1, \tag{22} \]
\[ \tau_n = \det \left( \delta_{ij} + \frac{\phi_i \psi_j}{p_i - q_j} \right)_{1 \leq i, j \leq N}, \]
\[ = \det \left( \delta_{ij} + \frac{\phi_j \psi_j}{p_i - q_j} \right)_{1 \leq i, j \leq N}, \]
\[ \phi_i = p_i^n e^{\frac{1}{2}(p_i - p_{i-1})t + \frac{1}{2}(p_i + p_{i-1})s + \phi^{(0)}_i}, \]
\[ \psi_i = q_i^{-n} e^{-\frac{1}{2}(q_i - q_{i-1})t - \frac{1}{2}(q_i + q_{i-1})s + \psi^{(0)}_i}, \]
\[ (i = 1, 2, \ldots, N), \]
where \( p_i, q_i \) are positive constants and \( \phi^{(0)}_i, \psi^{(0)}_i \) are real constants.

Applying the delay reduction \( V_n(t, s) = w(z, s), \quad z = t + hn \), to the 2DTL equation (19), we obtain the delay 2DTL equation
\[ \left( \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial s^2} \right) \log(1 + w(z, s)) = w(z + h, s) - 2w(z, s) + w(z - h, s). \tag{24} \]
which can be written as
\[
\frac{\partial^2 r(z, s)}{\partial z^2} - \frac{\partial^2 r(z, s)}{\partial s^2} = e^{r(z+h,s)} - 2e^{r(z,s)} + e^{r(z-h,s)} \tag{25}
\]
or
\[
\frac{\partial^2 u(z, s)}{\partial z^2} - \frac{\partial^2 u(z, s)}{\partial s^2} = e^{u(z-h,s)-u(z,s)} - e^{u(z,s)-u(z+h,s)} \tag{26}
\]
via the dependent variable transformation \( r(z, s) = u(z, s) - u(z + h, s) = \log(1 + w(z, s)) \). Note that this delay 2DTL equation was considered by Villarroel and Ablowitz, and they found the Lax pair and established the inverse scattering transform \([7]\). The delay 2DTL equation \((24)\) is transformed into the delay bilinear equation
\[
(D^2_2 - D^2_1)f(z, s) \cdot f(z, s) = 2(f(z + h, s)f(z - h, s) - f(z, s)^2) \tag{27}
\]
via the dependent variable transformation
\[
w(z, s) = \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial s^2}\right) \log f(z, s) = \frac{f(z + h, s)f(z - h, s)}{f(z, s)^2} - 1. \tag{28}
\]
We impose the reduction condition
\[
\log p_i - \log q_i = \frac{h}{2}(p_i - p_i^{-1} - q_i + q_i^{-1}) \tag{29}
\]
to the parameters \( p_i, q_i \) (\( i = 1, 2, \cdots, N \)) in the \( N \)-soliton solution \((22)\). Then, by setting \( z = x + hn \), we have
\[
\phi_j \psi_j = p^n_j e^{\frac{1}{2}(p_j - p_j^{-1})t + \frac{n}{2}(p_j + p_j^{-1})s + \phi_j^{(0)}} q^{-n}_j e^{-\frac{1}{2}(q_j - q_j^{-1})t - \frac{n}{2}(q_j + q_j^{-1})s + \psi_j^{(0)}}
\]
\[
= e^{\frac{1}{2}(p_j - p_j^{-1} - q_j + q_j^{-1})t + n(\log p_j - \log q_j) + \frac{n}{2}(p_j + p_j^{-1} - q_j - q_j^{-1})s + \phi_j^{(0)} + \psi_j^{(0)}}
\]
\[
= e^{\frac{1}{2}(p_j - p_j^{-1} - q_j + q_j^{-1})(t + hn) + \frac{n}{2}(p_j + p_j^{-1} - q_j - q_j^{-1})s + \phi_j^{(0)} + \psi_j^{(0)}}
\]
\[
= e^{\frac{1}{2}(p_j - p_j^{-1} - q_j + q_j^{-1})t + \frac{n}{2}(p_j + p_j^{-1} - q_j - q_j^{-1})s + \phi_j^{(0)} + \psi_j^{(0)}}. \tag{30}
\]
Thus, by imposing the reduction condition \((29)\) and setting \( z = t + hn \),
\[
f(z, s) = \tau_n(t, s),
\]
we obtain the following \( N \)-soliton solution of the delay 2DTL
equation (24):

\[
    w(z, s) = \left( \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial s^2} \right) \log f(z, s)
    = \frac{f(z + h, s)f(z - h, s)}{f(z, s)^2} - 1,
\]

(31)

\[
    f(z, s) = \det \left( \delta_{ij} + \frac{\Phi_j(z, s)}{p_i - q_j} \right)_{1 \leq i, j \leq N},
\]

\[
    \Phi_j(z, s) = e^{\frac{1}{2}(p_j - p_{j-1} - q_j + q_{j-1})z + \frac{1}{2}(p_j + p_{j-1} - q_j - q_{j-1})s + \Phi_j^{(0)}},
\]

where \( p_i, q_i \) must satisfy (29) and \( \Phi_i^{(0)} \) are real constants.

By imposing another delay reduction

\[
    V_n(t, s) = w(z_1, z_2), \quad z_1 = t + h_1 n, \quad z_2 = s + h_2 n,
\]

(32)

to the 2DTL equation (24), where \( h_1, h_2 \) are nonzero real constants, we can also consider another delay 2DTL equation

\[
    \left( \frac{\partial^2}{\partial z_1^2} - \frac{\partial^2}{\partial z_2^2} \right) \log(1 + w(z_1, z_2)) = w(z_1 + h_1, z_2 + h_2)
    - 2w(z_1, z_2) + w(z_1 - h_1, z_2 - h_2),
\]

(33)

which can be written as

\[
    \frac{\partial^2 r(z_1, z_2)}{\partial z_1^2} - \frac{\partial^2 r(z_1, z_2)}{\partial z_2^2} = e^{r(z_1 + h_1, z_2 + h_2)} - 2e^{r(z_1, z_2)} + e^{r(z_1 - h_1, z_2 - h_2)}
\]

(34)

or

\[
    \frac{\partial^2 u(z_1, z_2)}{\partial z_1^2} - \frac{\partial^2 u(z_1, z_2)}{\partial z_2^2}
    = e^{u(z_1 - h_1, z_2 - h_2) - u(z_1, z_2)} - e^{u(z_1, z_2) - u(z_1 + h_1, z_2 + h_2)}
\]

(35)

via the dependent variable transformation \( r(z_1, z_2) = u(z_1, z_2) - u(z_1 + h_1, z_2 + h_2) = \log(1 + w(z_1, z_2)) \). In this case, we can obtain the \( N \)-soliton solution by imposing the reduction condition

\[
    \log p_i - \log q_i = \frac{h_1}{2}(p_i - p_{i-1} - q_i + q_{i-1}) + \frac{h_2}{2}(p_i + p_{i-1} - q_i - q_{i-1})
\]

(36)

to the parameters \( p_i, q_i \) \( (i = 1, 2, \cdots, N) \) in the \( N \)-soliton solution [22]. The
\( N \)-soliton solution of the delay 2DTL equation \( \text{(33)} \) is given as follows:

\[
w(z_1, z_2) = \left( \frac{\partial^2}{\partial z_1^2} - \frac{\partial^2}{\partial z_2^2} \right) \log f(z_1, z_2)
= \frac{f(z_1 + h_1, z_2 + h_2)f(z_1 - h_1, z_2 - h_2)}{f(z_1, z_2)^2} - 1,
\]

\( f(z_1, z_2) = \text{det} \left( \delta_{ij} + \Phi_j(z_1, z_2) \right)_{1 \leq i, j \leq N}, \)

\[
\Phi_j(z_1, z_2) = e^{\frac{1}{2}(p_j - p_{j-1} - q_j + q_{j-1})z_1 + \frac{1}{2}(p_j + p_{j-1} - q_j - q_{j-1})z_2 + \Phi_j(0)},
\]

where \( p_i, q_i \) must satisfy \( \text{(36)} \) and \( \Phi_j(0) \) are real constants. In the case of \( h_2 = 0 \), this delay 2DTL equation \( \text{(33)} \) is reduced to the delay 2DTL equation \( \text{(24)} \), but it becomes another delay 2DTL equation in the case of \( h_1 = 0 \).

Now we consider the reduction to the one-dimensional Toda lattice equation \( \text{(4)} \)

\[
\frac{d^2}{dz^2} \log(1 + w(z)) = w(z + h) - 2w(z) + w(z - h),
\]

which can be written as

\[
\frac{d^2 r(z)}{dz^2} = e^{r(z+h)} - 2e^{r(z)} + e^{r(z-h)}.
\]

or

\[
\frac{d^2 u(z)}{dz^2} = e^{u(z-h) - u(z)} - e^{u(z) - u(z+h)}.
\]

by the transformation \( r(z) = u(z) - u(z + h) = \log(1 + w(z)) \). The delay 1DTL equation \( \text{(33)} \) is transformed into the delay bilinear equation

\[
D^2 f(z) \cdot f(z) = 2(f(z + h)f(z - h) - f(z)^2)
\]

via the dependent variable transformation

\[
w(z) = \frac{d^2}{dz^2} \log f(z) = \frac{f(z + h)f(z - h)}{f(z)^2} - 1.
\]

We note that only 1-soliton solution survives under this reduction. Applying

\[
q_1 = \frac{1}{p_1}
\]
to the 1-soliton solution of the delay 2DTL equation (24), we obtain

\begin{align}
  w(z) &= \frac{d^2}{dz^2} \log f(z) = \frac{f(z + h)f(z - h)}{f(z)^2} - 1 , \\
  f(z) &= 1 + e^{(p_1 - p_1^{-1})z + \Phi(0)} ,
\end{align}

(44)

where \( p_1 \) must satisfy

\[ 2 \log p_1 = h(p_1 - p_1^{-1}) . \]

(45)

3. Conclusions

We have considered delay reductions of the 2DTL equation and obtained the \( N \)-soliton solution of the delay analogues of the 2DTL equation. To the best of our knowledge, this is the first time that multi-soliton solutions of delay-differential equations has been obtained. We believe that our result is useful to study delay-differential equations. In our forthcoming paper, we will propose a systematic method to generate delay soliton equations having multi-soliton solutions[17].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

The authors thank Prof. Willox for stimulating discussion. This work was partially supported by JSPS KAKENHI Grant Numbers 18K03435 and JST/CREST.
References

[1] K. Hasebe, A. Nakayama, Y. Sugiyama, Phys. Lett. A 259 (1999) 135.

[2] Y. Tutiya and M. Kanai, J. Phys. Soc. Jpn. 76 (2007) 083002.

[3] G. R. W. Quispel, H. W. Capel, R. Sahadevan, Phys. Lett. A 170 (1992) 379.

[4] D. Levi, P. Winternitz, J. Math. Phys. 34 (1993) 3713.

[5] B. Grammaticos, A. Ramani, I. C. Moreira, Physica A 196 (1993) 574.

[6] A. Ramani, B. Grammaticos, K. M. Tamizhmani, J. Phys. A: Math. Gen. 26 (1993) L53.

[7] J. Villarroel, M. J. Ablowitz, Phys. Lett. A 180 (1993) 413.

[8] N. Joshi, J. Phys. A: Math. Theor. 42 (2009) 022001.

[9] N. Joshi, P. E. Spicer, J. Phy. Soc. Jpn. 78 (2009) 094006.

[10] A. S. Carstea, J. Phy. A: Math. Theor. 44 (2011) 105202.

[11] C-M. Viallet, arXiv:1408.6161 (2014).

[12] R. Halburd, R. Korhonen, Proc. Amer. Math. Soc. 145 (2017) 2513.

[13] B. K. Berntson, SIGMA 14 (2018) 020.

[14] A. Stokes, J. Phys. A: Math. Theor. 53 (2020) 435201.

[15] B. Grammaticos, A. Ramani, V. G. Papageorgiou, Phys. Rev. Lett. 67 (1991) 1825.

[16] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, 2004.

[17] K. Nakata and K. Maruno, in preparation.