EFFICIENT DERIVATIVE-BASED SIMPSON’S 1/3-TYPE SCHEME USING CENTROIDAL MEAN FOR RIEMANN-STIELTJES INTEGRAL

Kashif Memon¹, Muhammad Mujtaba Shaikh², Kamran Malik³, Muhammad Saleem Chandio⁴, Abdul Wasim Shaikh⁵

¹,³,⁴,⁵ Institute of Mathematics and Computer Sciences, University of Sindh, Jamshoro, Pakistan
² Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Pakistan
³ Department of Mathematics, Government College University, Hyderabad, Pakistan

¹memonkashif.84@gmail.com, ²mujtaba.shaikh@faculty.muet.edu.pk,
³kamranmk99@gmail.com, ⁴saleem.chandio@usindh.edu.pk,
⁵wasim.shaikh@usindh.edu.pk

Corresponding Author: Muhammad Mujtaba Shaikh

https://doi.org/10.26782/jmcms.2021.03.00006

Abstract

In this paper, a new efficient derivative-based quadrature scheme of Simpson’s 1/3-type is proposed using the centroidal mean for the approximation of Riemann-Stieltjes integral (RS-integral). Theorems are proved related to the basic form, composite form, local and global errors of the new scheme for the RS-integral. The reduction of the new proposed scheme is verified using $g(t) = t$ for Riemann integral. The theoretical results of new proposed scheme have been proved by experimental work using programming in MATLAB against existing schemes. The order of accuracy, computational cost and average CPU time (in seconds) of the new proposed scheme are determined. The results obtained show the effectiveness of the proposed scheme compared to the existing schemes.

Keywords: Quadrature rule, Riemann-Stieltjes integral, Centroidal Mean, Simpson’s 1/3 rule, Composite form, Local error, Global error, Cost-effectiveness, Time-efficiency

I. Introduction

From science and engineering, the nonlinear models arise quite frequently. Such models demand numerical solution due to complexity of equations [XIX],[XX], [VII],[XXII]. The numerical computation of a definite integral is the important problem in numerical integration and this numerical value is known as the area under the curve which is applied in many engineering applications. Definite integral $I(f) = \int_{a}^{b} f(x)dx$

Kashif Memon et al
cannot be integrated analytically for integrand \( f(x) = e^{x^2} \) or \( \sin x^2 \). These integrals can be integrated numerically by numerical methods. The methods of numerical integration is known as quadrature methods. The Riemann-Stieltjes RS-integral is a variation of the Riemann integral. Let \( f(x) \) and \( \alpha(x) \) be two bounded functions on \([a, b]\) and \( \alpha(x) \) is monotonically increasing on \([a, b]\), the RS-integral is defined [IV] as:

\[
RS(f(x): \alpha; a, b) = \int_a^b f(x) d\alpha(x)
\]

(1)

where \( f(x) \) is integrand and \( \alpha(x) \) is an integrator.

RS-integrals have been used in various areas of mathematics. For instance, Statistics and theory of Probability, Complex analysis, Functional analysis, the theory of the Operator, etc.

Several works have been focused on the improvement of quadrature rules for the Riemann integral in the literature as in [II], [XVIII]. Moreover, it is extended for the approximation of integral equations in [XXI], and used to improve the reluctance motors in [X]. However, few works have been indicated on the approximation of quadrature rules for the RS-integral as in [XI] Hadamard inequality has been derived for the RS-integral in Trapezoid-type. Also in [XII], the relative convexity concept has been used to derive some inequalities for the approximation of RS-integral using the midpoint and Simpson rules. Authors in [XXIII] used Midpoint derivative approach in order to improve the closed Newton-Cotes quadrature schemes for the approximation of Riemann integral.

Many strategies have been applied to improve Newton-Cotes formulas numerically.

Authors in [XIV],[XV],[XVI],[XVII] used different means at the functional derivative for the improvement of closed Newton-Cotes quadrature schemes. In [V],[VI] cubature schemes using derivatives were proposed.

Initially, the midpoint derivative-based quadrature scheme of trapezoid-type was introduced for the RS-integral in [XXV] without numerical verification. Later, the composite form of trapezoidal rule has been presented for the RS-integral in [XXIV] without verification of theoretical results. Recently, the new efficient derivative-based and derivative-free quadrature schemes have been presented in [VIII],[IX] for the RS-integral with verification of theoretical results.

In this research paper, a new efficient derivative-based Simpson’s 1/3-type scheme is presented using the centroidal mean for the RS-integral. The basic and composite forms with error terms are described in theorems. The theoretical results of the new scheme have been verified by experimental work which shows cost efficiency, time efficiency and rapid convergence.

### II. Existing Trapezoid-type Schemes for the RS-Integral

Some basic existing schemes of Trapezoid-type can be presented in T [XI], ZT[XXV], MZT [VIII], for the RS-integral and defined in (2)-(4) as:

\[
T \approx \left( \frac{1}{b-a} \int_a^b g(t) dt - g(a) \right) f(a) + \left( g(b) - \frac{1}{b-a} \int_a^b g(t) dt \right) f(b)
\]

(2)
\[ ZT \approx \left( \frac{1}{b-a} \int_a^b g(t) dt - g(a) \right) f(a) + \left( g(b) - \frac{1}{b-a} \int_a^b g(t) dt \right) f(b) + \left( \int_a^b f(x) dx \right) dt - b-a \int_a^b g(t) dt \right) f''(c_{ZT}) \]  

where, \( c_{ZT} = \frac{(-2b^2+a^2-ab) \int_a^b g(t) dt + 6b \int_a^b f(x) dx dt - 6 \int_a^b f(x) dx} {6 \int_a^b f(x) dx dt - 3(b-a) \int_a^b g(t) dt} \)

\[ MZT \approx \left( \frac{1}{b-a} \int_a^b g(t) dt - g(a) \right) f(a) + \left( g(b) - \frac{1}{b-a} \int_a^b g(t) dt \right) f(b) + \left( \int_a^b f(x) dx \right) dt - b-a \int_a^b g(t) dt \right) f''(c_{MZT}) \]  

where, \( c_{MZT} = \frac{(-2b^2+a^2+ab) \int_a^b g(t) dt + 6b \int_a^b f(x) dx dt - 6 \int_a^b f(x) dx} {6 \int_a^b f(x) dx dt - 3(b-a) \int_a^b g(t) dt} \)

The composite forms of the CT, ZCT and MZCT schemes are described in (5)-(7) as:

\[ CT \approx \left[ \frac{n}{b-a} \int_a^b g(t) dt - g(a) \right] f(a) + \left[ g(b) - \frac{n}{b-a} \int_a^b g(t) dt \right] f(b) \]

\[ ZCT \approx \left[ \frac{n}{b-a} \int_a^b g(t) dt - g(a) \right] f(a) + \left[ g(b) - \frac{n}{b-a} \int_a^b g(t) dt \right] f(b) + \sum_{k=1}^{n-1} \left[ \int_{x_{k-1}}^{x_k} g(x) dx \right] dt - b-a \int_{x_{n-1}}^{x_n} g(t) dt \right) f''(c_{ZT,k}) \]  

\[ MZT \approx \left[ \frac{n}{b-a} \int_a^b g(t) dt - g(a) \right] f(a) + \left[ g(b) - \frac{n}{b-a} \int_a^b g(t) dt \right] f(b) + \sum_{k=1}^{n-1} \left[ \int_{x_{k-1}}^{x_k} g(x) dx \right] dt - b-a \int_{x_{n-1}}^{x_n} g(t) dt \right) f''(c_k) \]  

Where,

\[ c_{ZT,k} = \frac{(-2x_k^2+x_{k-1}^2-x_{k-1}x_k) \int_{x_{k-1}}^{x_k} g(t) dt + 6b \int_{x_{k-1}}^{x_k} f(x) dx dt - 6 \int_{x_{k-1}}^{x_k} f(x) dy g(x) dx dt} {6 \int_{x_{k-1}}^{x_k} f(x) dx dt - 3(b-a) \int_{x_{k-1}}^{x_k} g(t) dt} \]

\[ c_{MZT,k} = \frac{(-2x_k^2+x_{k-1}^2+x_{k-1}x_k) \int_{x_{k-1}}^{x_k} g(t) dt + 6b \int_{x_{k-1}}^{x_k} f(x) dx dt - 6 \int_{x_{k-1}}^{x_k} f(x) dy g(x) dx dt} {6 \int_{x_{k-1}}^{x_k} f(x) dx dt - 3(b-a) \int_{x_{k-1}}^{x_k} g(t) dt} \]

III. Proposed Scheme using Centroidal Mean for the Riemann-Stieltjes Integral

The basic centroidal mean derivative-based Simpson’s 1/3-type (CMS13) rule for the Riemann integral is defined in (8) as:

\[ Kashif Memon et al \]

71
The proposed scheme i.e. CMS13 is described in its basic form for the RS-integral on the basis of (8) in Theorem 1.

**Theorem 1.** Let \( \Gamma(t) \) and \( g(t) \) be continuous on \([a, b]\) and \( g(t) \) be increasing there.

Then the proposed derivative-based Simpson’s 1/3 scheme using the centroidal mean for the RS-integral with local error term \( R_{CMS13}[f] \) can be described as:

\[
\int_a^b f(t) dg = CMS13 + R_{CMS13}[f] = \left( \frac{4}{(b-a)^2} \int_a^b f g(x) dx dt \right) f(a)
+ \left( \frac{1}{b-a} \int_a^b g(t) dt - g(a) \right) f'(a)
+ \left( g(b) - \frac{3}{b-a} \int_a^b g(t) dt + \frac{4}{(b-a)^2} \int_a^b f g(x) dx dt \right) f(b)
+ \left( \frac{(b-a)^2(3a+b)}{2(b-a)2} \int_a^b g(t) dt + \frac{17b^2-10ab-7a^2}{48} \int_a^b f g(x) dx dt \right) f(\eta)
+ \left( \frac{(b-a)^2(39a^3+91a^2b+61ab^2+49b^3)}{2880(a+b)} \right)
\left( \int_a^b g(t) dt \right)
+ \left( \frac{b(4a^2+ab+b^2)}{6(a+b)} \right)
\left( \int_a^b f t g(x) dx dy dt \right)
+ \left( \frac{b^2+ab-2a^2}{3(a+b)} \right)
\left( \int_a^b f t^2 g(x) dx dy dz dt \right)
+ \left( \int_a^b f t^3 g(x) dx dy dz dt \right)
+ \left( \int_a^b f t^4 g(x) dx dy dz dw dt \right)
\]

where \( \xi, \eta \in (a, b) \).

**Proof of Theorem 1.**

To derive the centroidal mean derivative-based Simpson’s 1/3 scheme for the RS-integral, we search numbers: \( a_0, b_0, c_0, d_0 \) so that:

\[
\int_a^b f(t) dg = a_0 f(a) + b_0 f(\frac{a+b}{2}) + c_0 f(b) + d_0 f'\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)
\]

is exact for \( f(t) = 1, t, t^2, t^3, t^4 \). That is,

\[
\int_a^b 1 dg = a_0 + b_0 + c_0
\]
\[
\int_a^b t dg = a_0 a + b_0 \left(\frac{a+b}{2}\right) + c_0 b
\]
\[
\int_a^b t^2 dg = a_0 a^2 + b_0 \left(\frac{a+b}{2}\right)^2 + c_0 b^2
\]

Kashif Memon et al
\[ \int_a^b t^3 dg = a_0 a^3 + b_0 \left( \frac{a + b}{2} \right)^3 + c_0 b^3 \]
\[ \int_a^b t^4 dg = a_0 a^4 + b_0 \left( \frac{a + b}{2} \right)^4 + c_0 b^4 + 24d_0 \]

By using integration by parts of the RS-integral, as in [XXV], we have the following system of equations (11)-(15).

\[ a_0 + b_0 + c_0 = g(b) - g(a) \tag{11} \]
\[ a_0 a + b_0 \left( \frac{a + b}{2} \right) + c_0 b = bg(b) - ag(a) - \int_a^b g(t)dt \tag{12} \]
\[ a_0 a^2 + b_0 \left( \frac{a + b}{2} \right)^2 + c_0 b^2 = b^2 g(b) - a^2 g(a) \]
\[ -2b \int_a^b g(t)dt + 2 \int_a^b \int_a^t g(x)dx dt \tag{13} \]
\[ a_0 a^3 + b_0 \left( \frac{a + b}{2} \right)^3 + c_0 b^3 = b^3 g(b) - a^3 g(a) - 3b^2 \int_a^b g(t)dt \]
\[ +6b \int_a^b \int_a^t g(x)dx dt - 6 \int_a^b \int_a^t \int_a^y g(x)dx dy dt \tag{14} \]
\[ a_0 a^4 + b_0 \left( \frac{a + b}{2} \right)^4 + c_0 b^4 + 24d_0 = b^4 g(b) - a^4 g(a) - 4b^3 \int_a^b g(t)dt \]
\[ +12b^2 \int_a^b \int_a^t g(x)dx dt - 24b \int_a^b \int_a^t \int_a^y g(x)dx dy dt \]
\[ +24 \int_a^b \int_a^t \int_a^y \int_a^z g(x)dx dy dz dt \tag{15} \]

The system of linear equations (11)-(15) can be described with the coefficient matrix as:

\[
M = \begin{bmatrix}
1 & 1 & 1 & 0 \\
a & \frac{a + b}{2} & b & 0 \\
a^2 & \left( \frac{a + b}{2} \right)^2 & b^2 & 0 \\
a^3 & \left( \frac{a + b}{2} \right)^3 & b^3 & 0 \\
a^4 & \left( \frac{a + b}{2} \right)^4 & b^4 & 24 \\
\end{bmatrix}
\]

The reduced row echelon form of \( M \) is:

\[
M \cong M_R = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Kashif Memon et al
As in \( M_a \), rank(\( M \)) = 4. To check the linearly independent rows, take \( a = -1 \) and \( b = 1 \) in matrix \( M \) then

\[
M = \begin{bmatrix}
1 & 1 & 1 & 0 \\
-1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
-1 & 1 & 0 & 1 \\
1 & 0 & 1 & 24
\end{bmatrix}
\]

This shows that the first three and fifth are linearly independent rows whereas fourth row is linearly dependent. To find the coefficients \( a_0, b_0, c_0 \) and \( d_0 \), we solve equations (11), (12), (13) and (15) simultaneously, to have:

\[
a_0 = \frac{4}{(b-a)^2} \int_a^b g(x)dx - \frac{1}{b-a} \int_a^b g(t)dt - g(a),
\]
\[
b_0 = \frac{4}{(b-a)^2} \int_a^b g(t)dt - \frac{8}{(b-a)^2} \int_a^b g(x)dxdt,
\]
\[
c_0 = g(b) - \frac{3}{b-a} \int_a^b g(t)dt + \frac{4}{(b-a)^2} \int_a^b g(x)dxdt,
\]
\[
d_0 = - \frac{(a-b)^2(3a+b)}{96} \int_a^b g(t)dt + \frac{(17b^2-10ab-7a^2)}{48} \int_a^b g(x)dxdt
\]

Putting the values of coefficients \( a_0, b_0, c_0 \) and \( d_0 \) in (10), we have:

\[
\int_a^b f(t)dg \approx CMS13 = \left(\frac{4}{(b-a)^2} \int_a^b g(x)dx \right) f(a)
\]
\[
+ \left(\frac{4}{(b-a)^2} \int_a^b g(t)dt - \frac{8}{(b-a)^2} \int_a^b g(x)dxdt\right) f\left(\frac{x+b}{2}\right)
\]
\[
+ \left(g(b) - \frac{3}{b-a} \int_a^b g(t)dt + \frac{4}{(b-a)^2} \int_a^b g(x)dxdt\right) f\left(\frac{a+b}{2}\right)
\]
\[
+ \left(-\frac{(a-b)^2(3a+b)}{96} \int_a^b g(t)dt + \frac{(17b^2-10ab-7a^2)}{48} \int_a^b g(x)dxdt\right) f\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)
\]

Now we derive the local error term of the proposed CMS13 scheme.

Since the precision of the proposed CMS13 scheme is 4, we take \( f(t) = \frac{t^5}{5!} \) to find the leading error term defined as:

\[
R_{CMS13}\left[f\right] = \frac{1}{5!} \int_a^b t^5 dg - CMS13(t^5; g; a, b)
\]

We learn, from [XXV], that:

\[
\frac{1}{5!} \int_a^b t^5 dg = \frac{1}{120} (b^5 g(b) - a^5 g(a)) - \frac{b^4}{24} \int_a^b g(t)dt + \frac{b^3}{6} \int_a^b g(x)dxdt
\]
\[
- \frac{b^2}{2} \int_a^b \int_a^x \frac{\partial}{\partial t} g(x)dxdt + b \int_a^b \int_a^x \int_a^y g(x)dxdt + \frac{1}{4} \int_a^b \int_a^x \int_a^y \int_a^z g(x)dxdt
\]

Kashif Memon et al
By Theorem 1 and scheme (9), we have:

\[
CMS13(t^5; g; a, b) = \left( \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt - \frac{1}{b-a} \int_a^b g(t)dt - g(a) \right) \frac{a^5}{5!}
\]

\[
+ \left( \frac{4}{b-a} \int_a^b g(t)dt - \frac{8}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt \right) \frac{(a+b)^5}{252!}
\]

\[
+ (g(b) - \frac{3}{b-a} \int_a^b g(t)dt + \frac{4}{(b-a)^2} \int_a^b \int_a^t g(x)dxdt) \frac{b^5}{5!}
\]

\[
+ \left( \frac{-(a-b)^2(3a^2+5b)}{2} \right) \int_a^b g(t)dt + \frac{17b^2-10ab-7a^2}{48} \int_a^b \int_a^t \int_a^x g(x)dxdt \right) \frac{\frac{2(a^2+ab+b^2)}{3}}{(a+b)}
\]

\[
- b \int_a^b \int_a^t \int_a^x g(x)dxdt + \frac{b^2+ab-2a^2}{3(a+b)} \int_a^b \int_a^t \int_a^x g(x)dxdt
\]

\[
-f \left( 6 \int_a^b \int_a^x \int_a^y g(x)dx dy dz dt \right) \int_a^x \int_a^y \int_a^z g(x)dx dy dz dt
\]

\[
\int_a^x \int_a^y \int_a^z g(x)dx dy dz dt
\]

which is the same as (9), and the precision of this scheme is 4.

**Theorem 2.** When \( g(t) = t \), the proposed CMS13 scheme with the error term (9) for the RS-integrals is reduced to the corresponding CMS13 scheme [XVII], i.e. (8) for the Riemann integrals.

**Proof of Theorem 2.**

By Theorem 1, we have:

\[
\int_a^b f(t)dt = \int_a^b f(t)dt = \left( \frac{4}{(b-a)^2} \int_a^b \int_a^t x dxdt - \frac{1}{b-a} \int_a^b t dt - g(a) \right) f(a)
\]

\[
+ \left( \frac{4}{b-a} \int_a^b t dt - \frac{8}{(b-a)^2} \int_a^b \int_a^t x dxdt \right) f \left( \frac{a+b}{2} \right)
\]

\[
+ g(b) - \frac{3}{b-a} \int_a^b t dt + \frac{4}{(b-a)^2} \int_a^b \int_a^t x dxdt \right) f(b)
\]

\[
+ \left( \frac{-(a-b)^2(3a^2+5b)}{2} \right) \int_a^b t dt + \frac{17b^2-10ab-7a^2}{48} \int_a^b \int_a^t \int_a^x g(x)dxdt \right) f \left( \frac{2(a^2+ab+b^2)}{3} \right)
\]

\[
- b \int_a^b \int_a^t \int_a^x g(x)dxdydt + \frac{b^2+ab-2a^2}{3(a+b)} \int_a^b \int_a^t \int_a^x g(x)dxdydz dt
\]

\[
-f \left( 6 \int_a^b \int_a^x \int_a^y g(x)dx dy dz dt \right) \int_a^x \int_a^y \int_a^z g(x)dx dy dz dt
\]

\[
\int_a^x \int_a^y \int_a^z g(x)dx dy dz dt
\]

It is obvious to get:

\[
\int_a^b t dt = b^2 - a^2
\]

\[
\int_a^b x dx = \frac{b^3}{6} - \frac{a^2 b}{2} + \frac{a^3}{3}
\]

Kashif Memon et al
\[
\begin{align*}
\int_a^b \int_a^x y \, dx \, dy \, dt &= \frac{b^4}{24} - \frac{a^3 b^2}{4} + \frac{a^3 b}{3} + \frac{b^6}{6} + \frac{a^2 b^3}{12} + \frac{a^5}{30} \\
\int_a^b \int_a^y x \, dx \, dy \, dz &= \frac{b^6}{720} + \frac{a^5 b^2}{30} - \frac{a^4 b}{16} + \frac{a^2 b^3}{48} - \frac{a^6}{144} \\
\end{align*}
\]

And, finally using these in (20) we get:
\[
\int_a^b f(t) \, dt = \frac{b-a}{6} \left[ f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{(b-a)^5}{2880} f(4) \left( \frac{2(a^2 + ab + b^2)}{3(a+b)} \right) + \frac{(b-a)^7}{17280(a+b)} f(6) (\xi), \quad (21)
\]
where $\xi \in (a, b)$.

This illustrates that the proposed CMS13 rule is reducible to the classical Riemann integral form (8) in terms of the centroidal mean.

Now, the proposed composite centroidal mean derivative-based Simpson’s 1/3 scheme for the RS-integral is derived by dividing the interval into small subintervals and applying integration rule to each subinterval, and the results are showcased in Theorem 3.

**Theorem 3.** Let $f(t)$ and $g(t)$ be continuous on $[a, b]$ and $g(t)$ be increasing there. Let the interval $[a, b]$ be subdivided into $2n$ subintervals $[x_k, x_{k+1}]$ with width $h = \frac{b-a}{n}$ by using the equally spaced nodes $x_k = a + kh$, where $k = 0, 1, \ldots, n$. The composite centroidal mean Simpson’s 1/3 scheme to $2n$ subintervals for the RS-integral can be described as

\[
\begin{align*}
\int_a^b f(t) \, dg &\approx CMS13 = \left[ \frac{4n^2}{(b-a)^2} \int_a^t \int_a^x g(x) \, dx \, dt - \frac{n}{b-a} \int_a^x g(t) \, dt - g(a) \right] f(a) \\
&+ \frac{4n}{b-a} \sum_{k=1}^{n} \left[ \int_{x_{k-1}}^{x_k} g(t) \, dt - \frac{2n}{b-a} \int_{x_{k-1}}^{x_k} g(x) \, dx \, dt \right] f\left( \frac{1}{2} (x_{k-1} + x_k) \right) \\
&+ \frac{n}{b-a} \sum_{k=1}^{n-1} \left[ \int_{x_{k-1}}^{x_k} g(t) \, dt - \int_{x_{k-1}}^{x_k} g(x) \, dx \, dt \right] f\left( \frac{1}{2} (x_{k-1} + x_k) \right) \\
&+ \sum_{k=1}^{n} \left[ \int_{x_{k-1}}^{x_k} g(t) \, dt - \int_{x_{k-1}}^{x_k} g(x) \, dx \, dt \right] f\left( \frac{1}{2} (x_{k-1}^2 + x_{k-1} x_k + x_k^2) \right) \\
&+ \left[ g(b) - \frac{3n}{b-a} \int_{x_{n-1}}^{b} g(t) \, dt + \frac{4n^2}{(b-a)^2} \int_{x_{n-1}}^{b} g(x) \, dx \, dt \right] f(b) \quad (22)
\end{align*}
\]

**Proof of Theorem 3.**

The proposed basic form CMS13 scheme for the RS-integral is given in (9). Applying the proposed CMS13 rule over each subinterval, we have:

\[
\int_a^b f(t) \, dg \approx \left[ \frac{4}{b-a} \int_a^t \int_a^x g(x) \, dx \, dt - \frac{1}{b-a} \int_a^x g(t) \, dt - g(a) \right] f(a)
\]

Kashif Memon et al
\[ \begin{align*}
  &\left[ \frac{4}{(b-a)^2} \int_x^t g(t)dt - \frac{8}{(b-a)^2} \int_x^t \int_a^t g(x)dxdt \right] f \left( \frac{a+x}{2} \right) \\
  + &\left[ g(x_1) - \frac{3}{b-a} \int_a^x g(t)dt + \frac{4}{(b-a)^2} \int_a^t \int_a^t g(x)dxdt \right] f(x_1) \\
  + &\left( \frac{(b-a)^2}{96} \int_x^t x \int_a^x g(t)dt + \frac{17x_1^2 - 10ax_1 - 7a^2}{48} \int_x^t \int_a^t g(x)dxdt \right) f(x_1) \\
  + &\left( \frac{(b-a)^2}{96} \int_x^t x \int_a^x g(t)dt + \frac{17x_1^2 - 10ax_1 - 7a^2}{48} \int_x^t \int_a^t g(x)dxdt \right) f(x_1) \\
  + &\left( \frac{8}{(b-a)^2} \int_x^t x \int_a^x g(t)dt - \frac{1}{b-a} \int_x^t g(t)dt - g(x_1) \right) f(x_1) \\
  + &\left[ g(x_2) - \frac{3}{b-a} \int_a^x g(t)dt + \frac{4}{(b-a)^2} \int_a^t \int_a^t g(x)dxdt \right] f(x_2) \\
  + &\left( \frac{(b-a)^2}{96} \int_x^t x \int_a^x g(t)dt + \frac{17x_1^2 - 10ax_1 - 7a^2}{48} \int_x^t \int_a^t g(x)dxdt \right) f(x_2) \\
  + &\left( \frac{8}{(b-a)^2} \int_x^t x \int_a^x g(t)dt - \frac{1}{b-a} \int_x^t g(t)dt - g(x_1) \right) f(x_1) \\
  + &\left[ g(x_2) - \frac{3}{b-a} \int_a^x g(t)dt + \frac{4}{(b-a)^2} \int_a^t \int_a^t g(x)dxdt \right] f(x_2) \\
  + &\left( \frac{(b-a)^2}{96} \int_x^t x \int_a^x g(t)dt + \frac{17x_1^2 - 10ax_1 - 7a^2}{48} \int_x^t \int_a^t g(x)dxdt \right) f(x_2) \\
  + &\left( \frac{8}{(b-a)^2} \int_x^t x \int_a^x g(t)dt - \frac{1}{b-a} \int_x^t g(t)dt - g(x_1) \right) f(x_1) \\
  + &\left[ g(x_2) - \frac{3}{b-a} \int_a^x g(t)dt + \frac{4}{(b-a)^2} \int_a^t \int_a^t g(x)dxdt \right] f(x_2) \\
  + &\left( \frac{(b-a)^2}{96} \int_x^t x \int_a^x g(t)dt + \frac{17x_1^2 - 10ax_1 - 7a^2}{48} \int_x^t \int_a^t g(x)dxdt \right) f(x_2) \\
  + &\left( \frac{8}{(b-a)^2} \int_x^t x \int_a^x g(t)dt - \frac{1}{b-a} \int_x^t g(t)dt - g(x_1) \right) f(x_1) \\
  + &\left[ g(x_2) - \frac{3}{b-a} \int_a^x g(t)dt + \frac{4}{(b-a)^2} \int_a^t \int_a^t g(x)dxdt \right] f(x_2) \\
  + &\left( \frac{(b-a)^2}{96} \int_x^t x \int_a^x g(t)dt + \frac{17x_1^2 - 10ax_1 - 7a^2}{48} \int_x^t \int_a^t g(x)dxdt \right) f(x_2) \\
  + &\left( \frac{8}{(b-a)^2} \int_x^t x \int_a^x g(t)dt - \frac{1}{b-a} \int_x^t g(t)dt - g(x_1) \right) f(x_1) \\

\text{Kashif Memon et al}
\end{align*} \]
\[\begin{align*}
&= +\left[\frac{4}{n} \int_{x_{n-1}}^b f(x) dx dt - \frac{1}{n} \int_{x_{n-1}}^b f(x) dx dt - g(x_{n-1})\right] f(x_{n-1}) \\
&+ \left[\frac{4}{n} \int_{x_{n-1}}^b g(x) dx dt - \frac{8}{(b-a)^2} \int_{x_{n-1}}^t f(x) dx dt\right] f(x_{n-1} + \frac{b}{2}) \\
&+ \left[\int_{x_{n-1}}^b g(x) dx dt + \frac{4}{(b-a)^2} \int_{x_{n-1}}^t f(x) dx dt\right] f(b) \\
&+ \left[\frac{(b-a)^2}{96} \int_{x_{n-1}}^b g(x) dx dt + \frac{17b^2-10x_{n-1} b^2 + 7x_{n-1}^2}{48} \int_{x_{n-1}}^b f(x) dx dt\right] f(4) \left(\frac{2(x_{n-1} + x_{n-1} b + b^2)}{3(x_{n-1} + b)}\right) \\
&+ \left[\frac{4n^2}{(b-a)^2} \int_a^b x g(x) dx dt - \frac{n}{b-a} \int_a^b x g(x) dt - g(a)\right] f(a) \\
&+ \left[\frac{4n}{b-a} \int_a^b x g(x) dx dt - \frac{8n^2}{(b-a)^2} \int_a^b x g(x) dx dt\right] f(x_{n-1} + \frac{x_{n-1} + x_{n-1}}{2}) \\
&+ \left[\frac{4n^2}{(b-a)^2} \int_a^b x g(x) dx dt - \frac{8n^2}{(b-a)^2} \int_a^b x g(x) dx dt\right] f(\frac{x_{n-1} + x_{n-1}}{2}) \\
&+ \left[\frac{4n^2}{(b-a)^2} \int_a^b x g(x) dx dt + \frac{4n^2}{(b-a)^2} \int_a^b x g(x) dx dt\right] f(x_{n-1} + b) \\
&+ \left[\frac{4n^2}{(b-a)^2} \int_a^b x g(x) dx dt + \frac{4n^2}{(b-a)^2} \int_a^b x g(x) dx dt\right] f(x_{n-1} + b) \\
&+ \left[\frac{4n^2}{(b-a)^2} \int_a^b x g(x) dx dt - \frac{3n}{b-a} \left(3 \int_a^b x g(x) dt + \int_a^b x g(x) dx dt\right)\right] f(x_{n-1}) \\
&+ \left[\frac{4n^2}{(b-a)^2} \int_a^b x g(x) dx dt - \frac{3n}{b-a} \left(3 \int_a^b x g(x) dt + \int_a^b x g(x) dx dt\right)\right] f(b) \\
&+ \left[\frac{-h^2}{96} \int_{x_{n-1}}^b g(x) dx dt + \frac{17b^2-10x_{n-1} b^2 + 7x_{n-1}^2}{48} \int_{x_{n-1}}^b f(x) dx dt\right] f(4) \left(\frac{2(x_{n-1} + x_{n-1} b + b^2)}{3(x_{n-1} + b)}\right) \\
&= \left[\frac{4n^2}{(b-a)^2} \int_a^b x g(x) dx dt - \frac{n}{b-a} \int_a^b x g(x) dt - g(a)\right] f(a) \\
&+ \left[\frac{4n}{b-a} \int_a^b x g(x) dx dt - \frac{2n}{b-a} \int_a^b x g(x) dx dt\right] f(\frac{x_{k-1} + x_{k-1}}{2}) \\
&+ \left[\int_a^b x g(x) dx dt - \frac{n}{b-a} \int_a^b x g(x) dx dt\right] f(x_k) \\
&+ \left[\frac{4n}{b-a} \int_a^b x g(x) dx dt + \frac{4n}{b-a} \int_a^b x g(x) dx dt\right] f(x_{k+1}) \\
&+ \left[\frac{4n}{b-a} \int_a^b x g(x) dx dt - \frac{3n}{b-a} \left(3 \int_a^b x g(x) dt + \int_a^b x g(x) dx dt\right)\right] f(x_{k-1}) \\
&+ \left[\frac{4n}{b-a} \int_a^b x g(x) dx dt - \frac{3n}{b-a} \left(3 \int_a^b x g(x) dt + \int_a^b x g(x) dx dt\right)\right] f(x_{k+1}) \\
\end{align*}\]

Kashif Memon et al
\[
+ \left[ g(b) - \frac{3n}{b-a} \int_{x_{n-1}}^{b} g(t) dt + \frac{4n^2}{(b-a)^3} \int_{x_{n-1}}^{b} \int_{x_{n-1}}^{t} g(x) dx dt \right] f(b) \\
+ \sum_{k=1}^{n} \left[ \frac{-\kappa^2(x_{k-1}+5x_k)}{96} \int_{x_{k-1}}^{x_k} g(t) dt + \frac{17\kappa^2-10x_{k-1}x_k-7x_{k-1}^2}{48} \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{t} g(x) dx dy dt + \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{t} \int_{x_{k-1}}^{x_k} g(x) dx dy dx dt \right] f(4) \left( \frac{2(x_{n-1}+x_{n+1}+b+b^2)}{3(x_{n-1}+b)} \right)
\]

So the proposed composite derivative-based Simpson’s 1/3 scheme using the centroidal mean for the RS-integral is proved.

It is noted that the global error terms of the proposed CMCS13 scheme cannot be defined in classical form.

IV.

Results and Discussion

Theoretical results did not verify in [XI],[XII],[XXV], by experimental works on quadrature schemes for RS-integral. In this research paper, experimental works have been performed on quadrature schemes for RS-integral to verify the theoretical results. Three different numerical problems have been tested using MATLAB software for each scheme taken from [VIII], [IX], [III], [XIII], [I] etc. All the results are observed in Intel (R) Core (TM) Laptop with RAM 8.00GB and a processing speed of 1.00GHz-1.61GHz.. Double-precision arithmetic is used for numerical results.

Example 1. \( \int_{3.5}^{4.5} \sin 5 \ x \ dx \) \((cos \ x) = 0.227676016130689\)

Example 2. \( \int_{6}^{6} \sin x \ \ dx(x^3) = -59.655908136641912\)

Example 3. \( \int_{6}^{6} e^x d \ sin \ x = 187.4269314248657\)

The absolute error and computational order of accuracy (COC) formulae have been obtained from [IX].

In Table 1, the error plots of the proposed CMCS13 scheme have been matched with other existing schemes: CT, ZCT and MZCT under similar conditions, and it is noted that the proposed CMCS13 scheme has the smallest error for all examples. When the number of strips is increased, it is noticed from Figs. 1-3 through the line plots that errors in the proposed scheme are reduced rapidly as compared to other schemes.

The observed COC has been verified for all discussed methods for the RS-integral except the ZCT scheme due to errors highlighted in [VIII], the order oscillates and did not converge to 4, as in Tables 2-4.

In Table 5, the total evaluation of functions, derivatives and integrations are used per strip are mentioned for the discussed methods, which are required to compute the computational costs. Figs. 4-9 represent the total computational cost and the average CPU time in seconds for three different integrals using CT, ZCT, MZCT and CMCS13 schemes. The numerical results show that the proposed scheme computed less cost and smaller average CPU time to achieve the error \( 10^{-5} \) as compared to existing schemes for all test problems.
Table 1: Absolute error comparison by CMCS13 and others for Examples 1-3.

| Quadrature variants | Example 1 (m=20) | Example 2 (m=100) | Example 3 (m=20) |
|---------------------|------------------|-------------------|------------------|
| CT                  | 1.1862E-03       | 4.9713E-04        | 3.9042E-02       |
| ZCT                 | 1.6698E-03       | 5.5959E-04        | 3.9042E-02       |
| MZCT                | 1.8552E-06       | 1.2428E-09        | 2.4399E-06       |
| CMCS13              | 1.5474e-07       | 8.3806e-11        | 3.8453e-07       |

Fig 1. Comparison of error drops by all methods for Example 1

Fig 2. Comparison of error drops by all methods for Example 2

Fig 3. Comparison of error drops by all methods for Example 3

Kashif Memon et al
### Table 2: Comparison of COC in all methods for Example 1

| Number of strips (m) | CT    | ZCT    | MZCT   | CMCS13 |
|---------------------|-------|--------|--------|--------|
| 1                   | NA    | NA     | NA     | NA     |
| 2                   | 2.5728| 3.3788 | 5.0724 | 4.2210 |
| 4                   | 2.0420| 4.3106 | 4.1474 | 4.0528 |
| 8                   | 2.0091| -0.8154| 4.0348 | 4.0131 |
| 16                  | 2.0022| 2.1713 | 4.0086 | 4.0032 |
| 32                  | 2.0006| 3.2118 | 4.0021 | 4.0008 |

### Table 3: Comparison of COC in all methods for Example 2

| Number of strips (m) | CT    | ZCT    | MZCT   | CMCS13 |
|---------------------|-------|--------|--------|--------|
| 5                   | NA    | NA     | NA     | NA     |
| 10                  | 2.0018| 1.9243 | 4.0025 | 4.0176 |
| 20                  | 2.0004| 2.4487 | 4.0007 | 4.0044 |
| 40                  | 2.0001| 2.3985 | 4.0001 | 4.0011 |
| 80                  | 2.0001| 2.3985 | 4.0000 | 4.0003 |

### Table 4: Comparison of COC in all methods for Example 3

| Number of strips (m) | CT    | ZCT    | MZCT   | CMCS13 |
|---------------------|-------|--------|--------|--------|
| 1                   | NA    | NA     | NA     | NA     |
| 2                   | 1.9319| 1.9319 | 3.8984 | 4.0266 |
| 4                   | 1.9844| 1.9844 | 3.9761 | 4.0069 |
| 8                   | 1.9962| 1.9962 | 3.9941 | 4.0018 |
| 16                  | 1.9990| 1.9990 | 3.9985 | 4.0004 |
| 32                  | 1.9998| 1.9998 | 3.9985 | 4.0001 |

### Table 5: Computational cost in quadrature variants for m strips.

| Quadrature Variants | Total evaluations |
|---------------------|-------------------|
| CT                  | 2m+3 [XI]         |
| ZCT                 | 5m+3 [XI]         |
| MZCT                | 5m+3 [XI]         |
| Proposed CMCS13     | 7m+3              |

Kashif Memon et al
Figs. 4-6 represent a computational cost to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

**Fig 4.** Total computational cost by quadrature variants for Example 1

**Fig 5.** Total computational cost by quadrature variants for Example 2

**Fig 6.** Total computational cost by quadrature variants for Example 3

Figs. 7-9 represent average CPU time to achieve at most 1E-05 absolute error in quadrature variants for Examples 1-3.

**Fig 7.** Average CPU usage time by quadrature variants for Example 1

*Kashif Memon et al*
V. Conclusion

A new efficient derivative-based quadrature scheme of Simpson’s 1/3-type was derived using the centroidal mean for the RS-integral. The local and global error terms were derived in theorems. Three different numerical problems were analyzed from the literature to check the performance of the proposed scheme against few other existing schemes. The error drops, observed orders of accuracy and computations performance in terms of evaluations and CPU usage show the dominance of the proposed scheme over other discussed schemes for the evaluation for the RS-integral numerically.

Conflict of Interest:

The authors declare that there is no conflict of interest regarding this article

References

I. Bartle, R.G. and Bartle, R.G., The elements of real analysis, (Vol. 2). John Wiley and Sons, 1964.

Kashif Memon et al
II. Bhatti AA, Chandio MS, Memon RA and Shaikh MM, A Modified Algorithm for Reduction of Error in Combined Numerical Integration, Sindh University Research Journal-SURJ (Science Series) 51.4, (2019): 745-750.

III. Burden, R.L., Faires, J.D., Numerical Analysis, Brooks/Cole, Boston, Mass, USA, 9th edition, 2011.

IV. Dragomir, S.S., and Abelman S., Approximating the Riemann-Stieltjes integral of smooth integrands and of bounded variation integrators, Journal of Inequalities and Applications, 2013.1 (2013), 154.

V. Malik K., Shaikh, M. M., Chandio, M. S. and Shaikh, A. W. Error analysis of closed Newton-Cotes cubature schemes for double integrals: J. Mech. Cont. & Math. Sci., 15 (11): 95-107, 2020.

VI. Malik K., Shaikh, M. M., Chandio, M. S. and Shaikh, A. W. Some new and efficient derivative-based schemes for numerical cubature. : J. Mech. Cont. & Math. Sci., 15 (10): 67-78, 2020.

VII. Mastoi, Adnan Ali, Muhammad Mujtaba Shaikh, and Abdul Wasim Shaikh. A new third-order derivative-based iterative method for nonlinear equations. : J. Mech. Cont. & Math. Sci., 15 (10): 110-123, 2020.

VIII. Memon K, Shaikh MM, Chandio MS and Shaikh AW, A Modified Derivative-Based Scheme for the Riemann-Stieltjes Integral, Sindh University Research Journal-SURJ (Science Series) 52.1, (2020): 37-40.

IX. Memon K., Shaikh, M. M., Chandio, M. S. and Shaikh, A. W. A new and efficient Simpson’s 1/3-type quadrature rule for Riemann-Stieltjes integral. : J. Mech. Cont. & Math. Sci., 15 (11): 132-148, 2020.

X. Memon, A. A., Shaikh, M. M., Bukhari, S. S. H., & Ro, J. S. Look-up Data Tables-Based Modeling of Switched Reluctance Machine and Experimental Validation of the Static Torque with Statistical Analysis. Journal of Magnetics, 25(2): 233-244, 2020.

XI. Mercer, P.R., Hadamard's inequality and Trapezoid rules for the Riemann-Stieltjes integral, Journal of Mathematica Analysis and Applications, 344 (2008), 921-926.

XII. Mercer, P.R., Relative convexity and quadrature rules for the Riemann-Stieltjes integral, Journal of Mathematica inequality, 6 (2012), 65-68.

XIII. Protter, M.H. and Morrey, C.B., A First Course in Real Analysis . Springer, New York, NY, 1977.

XIV. Ramachandran Thiagarajan, Udayakumar.D, Parimala .R, Geometric mean derivative-based closed Newton-Cotes quadrature, International Journal of Pure & Engineering Mathematics, 4, 107-116, April 2016.

XV. Ramachandran Thiagarajan, Udayakumar.D, Parimala .R, Harmonic mean derivative-based closed Newton-Cotes quadrature, IOSR-Journal of Mathematics, 12, 36-41, May-June 2016.

XVI. Ramachandran Thiagarajan, Udayakumar.D, Parimala .R, Heronian mean derivative- based closed Newton cotes quadrature, International Journal of Mathematical Archive, 7, 53-58, July 2016.
XVII. Ramachandran Thiagarajan, Parimala R, Centroidal mean derivative–based closed Newton cotes quadrature, International Journal of Science and Research, 5, 338-343, August 2016.

XVIII. Shaikh, MM., MS Chandio and AS Soomro, A Modified Four-point Closed Mid-point Derivative Based Quadrature Rule for Numerical Integration, Sindh University Research Journal-SURJ (Science Series), 48.2, 2016.

XIX. Shaikh, Muhammad Mujtaba, Shafiq-ur-Rehman Massan, and Asim Imdad Wagan. "A new explicit approximation to Colebrook’s friction factor in roughpipes under highly turbulent cases." International Journal of Heat and Mass Transfer 88 (2015): 538-543.

XX. Shaikh, Muhammad Mujtaba, Shafiq-ur-Rehman Massan, and Asim Imdad Wagan. "A sixteen decimal places' accurate Darcy friction factor database using non-linear Colebrook's equation with a million nodes: A way forward to the soft computing techniques." Data in brief 27 (2019): 104733.

XXI. Shaikh, Muhammad Mujtaba. "Analysis of Polynomial Collocation and Uniformly Spaced Quadrature Methods for Second Kind Linear Fredholm Integral Equations–A Comparison." Turkish Journal of Analysis and Number Theory 7.4 (2019): 91-97

XXII. Umar, Sehrish, Muhammad Mujtaba Shaikh, and Abdul Wasim Shaikh. A new quadrature-based iterative method for scalar nonlinear equations. : J. Mech. Cont. & Math. Sci., 15 (10): 79-93, 2020.

XXIII. Zhao, W., and H. Li, Midpoint Derivative-Based Closed Newton-Cotes Quadrature, Abstract And Applied Analysis, Article ID 492507, (2013).

XXIV. Zhao, W., Z. Zhang, and Z. Ye, Composite Trapezoid rule for the Riemann-Stieltjes Integral and its Richardson Extrapolation Formula, Italian Journal of Pure and Applied Mathematics, 35 (2015), 311-318.

XXV. Zhao, W., Z. Zhang, and Z. Ye, Midpoint Derivative-Based Trapezoid Rule for the Riemann-Stieltjes Integral, Italian Journal of Pure and Applied Mathematics, 33, (2014), 369-376.