Embedding simply connected
2-complexes in 3-space
II. Rotation systems

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May 14, 2018

Abstract
We prove that 2-dimensional simplicial complexes whose first homology group is trivial have topological embeddings in 3-space if and only if there are embeddings of their link graphs in the plane that are compatible at the edges and they are simply connected.

1 Introduction
This is the second paper in a series of five papers. In the first paper [2] of this series we give an overview about this series as a whole. In this paper we give combinatorial characterisations for when certain simplicial complexes embed in 3-space. This completes the proof of a 3-dimensional analogue of Kuratowski’s characterisation of planarity for graphs, started in [2].

A (2-dimensional) simplicial complex has a topological embedding in 3-space if and only if it has a piece-wise linear embedding if and only if it has a differential embedding [1, 5, 9]. Perelman proved that every compact simply connected 3-dimensional manifold is isomorphic to the 3-sphere $S^3$ [11, 12, 13]. In this paper we use Perelman’s theorem to prove a combinatorial characterisation of which simply connected simplicial complexes can be topologically embedded into $S^3$ as follows.

The link graph at a vertex $v$ of a simplicial complex is the graph whose vertices are the edges incident with $v$ and whose edges are the faces incident with $v$ and the incidence relation is as in $C$, see Figure 1. Roughly, a planar rotation system of a simplicial complex $C$ consists of cyclic orientations $\sigma(e)$
Figure 1: The link graph at the vertex $v$ is indicated in grey. The edge $e$ projects down to a vertex in the link graph. The faces incident with $e$ project down to edges.

of the faces incident with each edge $e$ of $C$ such that there are embeddings in the plane of the link graphs such that at vertices $e$ the cyclic orientations of the incident edges agree with the cyclic orientations $\sigma(e)$. It is easy to see that if a simplicial complex $C$ has a topological embedding into some 3-dimensional manifold, then it has a planar rotation system. Conversely, for simply connected simplicial complexes the existence of planar rotation systems is enough to characterise embeddability into $S^3$:

**Theorem 1.1.** Let $C$ be a simply connected simplicial complex. Then $C$ has a topological embedding into $S^3$ if and only if $C$ has a planar rotation system.

The main result of this paper is the following extension of Theorem 1.1.

**Theorem 1.2.** Let $C$ be a simplicial complex such that the first homology group $H_1(C, \mathbb{F}_p)$ is trivial for some prime $p$. Then $C$ has a topological embedding into $S^3$ if and only if $C$ is simply connected and it has a planar rotation system.

This implies characterisations of topological embeddability into $S^3$ for the classes of simplicial complexes with abelian fundamental group and simplicial complexes in general, see Section 7 for details.

The paper is organised as follows. After reviewing some elementary definitions in Section 2, in Section 3 we introduce rotation systems, related concepts and prove basic properties of them. In Sections 4 and 5 we prove Theorem 1.1. The proof of Theorem 1.2 in Section 6 makes use of Theorem 1.1. Further extensions are derived in Section 7.
2 Basic definitions

In this short section we recall some elementary definitions that are important for this paper.

A closed trail in a graph is a cyclically ordered sequence \((e_n|n \in \mathbb{Z}_k)\) of distinct edges \(e_n\) such that the starting vertex of \(e_n\) is equal to the endvertex of \(e_{n-1}\). An (abstract) (2-dimensional) complex is a graph \(G\) together with a family of closed trails in \(G\), called the faces of the complex. We denote complexes \(C\) by triples \(C = (V,E,F)\), where \(V\) is the set of vertices, \(E\) the set of edges and \(F\) the set of faces. We assume furthermore that every vertex of a complex is incident with an edge and every edge is incident with a face. The 1-skeleton of a complex \(C = (V,E,F)\) is the graph \((V,E)\). A directed complex is a complex together with a choice of direction at each of its edges and a choice of orientation at each of its faces. For an edge \(e\), we denote the direction chosen at \(e\) by \(\vec{e}\). For a face \(f\), we denote the orientation chosen at \(f\) by \(\vec{f}\).

Examples of complexes are (abstract) (2-dimensional) simplicial complexes. In this paper all simplicial complexes are directed – although we will not always say it explicitly. A (topological) embedding of a simplicial complex \(C\) into a topological space \(X\) is an injective continuous map from (the geometric realisation of) \(C\) into \(X\). In our notation we suppress the embedding map and for example write \({S^3 \setminus C}\) for the topological space obtained from \(S^3\) by removing all points in the image of the embedding of \(C\).

In this paper, a surface is a compact 2-dimensional manifold (without boundary)[2]. Given an embedding of a graph in an oriented surface, the rotation system at a vertex \(v\) is the cyclic orientation[3] of the edges incident with \(v\) given by ‘walking around’ \(v\) in the surface in a small circle in the direction of the orientation. Conversely, a choice of rotation system at each vertex of a graph \(G\) defines an embedding of \(G\) in an oriented surface as explained in [2].

A cell complex is a graph \(G\) together with a set of directed walks such that each direction of an edge of \(G\) is in precisely one of these directed walks. These directed walks are called the cells. The geometric realisation of a cell complex is obtained from (the geometric realisation of) its graph by gluing discs so that the cells are the boundaries of these discs. The geometric realisation is always an oriented surface. Note that cell complexes need not be complexes as cells are allowed to contain both directions of an edge. The

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1 In this paper graphs are allowed to have parallel edges and loops.
2 We allow surfaces to be disconnected.
3 A cyclic orientation is a choice of one of the two orientations of a cyclic ordering.
rotation system of a cell complex $C$ is the rotation system of the graph of $C$ in the embedding in the oriented surface given by $C$.

3 Rotation systems

In this section we introduce rotation systems of complexes and some related concepts.

The link graph of a simplicial complex $C$ at a vertex $v$ is the graph whose vertices are the edges incident with $v$. The edges are the faces incident with $v$. The two endvertices of a face $f$ are those vertices corresponding to the two edges of $C$ incident with $f$ and $v$. We denote the link graph at $v$ by $L(v)$.

A rotation system of a directed complex $C$ consists of for each edge $e$ of $C$ a cyclic orientation $\sigma(e)$ of the faces incident with $e$.

Important examples of rotation systems are those induced by topological embeddings of complexes $C$ into $S^3$ (or more generally in some 3-manifold); here for an edge $e$ of $C$, the cyclic orientation $\sigma(e)$ of the faces incident with $e$ is the ordering in which we see the faces when walking around some midpoint of $e$ in a circle of small radius – in the direction of the orientation of $S^3$. It can be shown that $\sigma(e)$ is independent of the chosen circle if small enough and of the chosen midpoint.

Such rotation systems have an additional property: let $\Sigma = (\sigma(e)|e \in E(C))$ be a rotation system of a simplicial complex $C$ induced by a topological embedding of $C$ in the 3-sphere. Consider a ball of small radius around a vertex $v$. We may assume that each edge of $C$ intersects the boundary of that ball in at most one point and that each face intersects it in an interval or not at all. The intersection of the boundary of the ball and $C$ is a graph: the link graph at $v$. Hence link graphs of complexes with induced rotation systems must always be planar. And even more: the cyclic orientations $\sigma(e)$ at the edges of $C$ form – when projected down to a link graph to rotators at the vertices of the link graph – a rotation system at the link graph, see Figure 2.

Next we shall define ‘planar rotation systems’ which roughly are rotation systems satisfying such an additional property. The cyclic orientation $\sigma(e)$ at the edge $e$ of a rotation system defines a rotation system $r(e, v, \Sigma)$ at

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4A face is incident with a vertex if there is an edge incident with both of them.

5If the edge $e$ is only incident with a single face, then $\sigma(e)$ is empty.

6Formally this means that the circle intersects each face in a single point and that it can be contracted onto the chosen midpoint of $e$ in such a way that the image of one such contraction map intersects each face in an interval.
Figure 2: On the left we depicted a cyclic orientation $\sigma(e)$ at an edge $e$. On the right we depicted the link graph $L$ at an endvertex of $e$. The edge $e$ is a vertex of $L$ and $\sigma(e)$ is a rotator at $L$.

Each vertex $e$ of a link graph $L(v)$: if the directed edge $\vec{e}$ is directed towards $v$ we take $r(e, v, \Sigma)$ to be $\sigma(e)$. Otherwise we take the inverse of $\sigma(e)$. As explained in Section 2, this defines an embedding of the link graph into an oriented surface. The link complex for $(C, \Sigma)$ at the vertex $v$ is the cell complex obtained from the link graph $L(v)$ by adding the faces of the above embedding of $L(v)$ into the oriented surface. By definition, the geometric realisation of the link complex is always a surface. To shortcut notation, we will not distinguish between the link complex and its geometric realisation and just say things like: ‘the link complex is a sphere’. A planar rotation system of a directed simplicial complex $C$ is a rotation system such that for each vertex $v$ all link graphs are a disjoint union of spheres. The paragraph above shows the following.

**Observation 3.1.** Rotation systems induced by topological embeddings of locally connected simplicial complexes in 3-manifolds are planar. 

Next we will define the local surfaces of a topological embedding of a simplicial complex $C$ into $S^3$. The local surface at a connected component of $S^3 \setminus C$ is the following. Pour concrete into this connected component. The surface of the concrete is a 2-dimensional manifold. The local surface is the simplicial complex drawn at the surface by the vertices, edges and faces of $C$. Note that if an edge $e$ of $G$ is incident with more than two faces that are on the surface, then the surface will contain at least two clones of the edge $e$, see Figure 3.

Now we will define local surfaces for a pair $(C, \Sigma)$ consisting of a complex $C$ and one of its rotation systems $\Sigma$. **Lemma 3.4** below says that under

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7 **Observation 3.1** is also true without the assumption of ‘local connectedness’. In that case however the link complex is disconnected. Hence it is no longer directly given by the drawing of the link graph on a ball of small radius as above.
fairly general circumstances the local surfaces of a topological embedding are the local surfaces of the rotation system induced by that topological embedding. The set of faces of a local surface will be an equivalence class of the set of orientations of faces of $C$. The local-surface-equivalence relation is the symmetric transitive closure of the following relation. An orientation $\vec{f}$ of a face $f$ is *locally related* via an edge $e$ of $C$ to an orientation $\vec{g}$ of a face $g$ if $f$ is just before $g$ in $\sigma(e)$ and $e$ is traversed positively by $\vec{f}$ and negatively by $\vec{g}$ and in $\sigma(e)$ the faces $f$ and $g$ are adjacent. Here we follow the convention that if the edge $e$ is only incident with a single face, then the two orientations of that face are related via $e$. Given an equivalence class of the local-surface-equivalence relation, the local surface at that equivalence class is the following complex whose set of faces is (in bijection with) the set of orientations in that equivalence class. We obtain the complex from the disjoint union of the faces of these orientations by gluing together two of these faces $f_1$ and $f_2$ along two of their edges if these edges are copies of the same edge $e$ of $C$ and $f_1$ and $f_2$ are related via $e$. Of course, we glue these two edges in a way that endvertices are identified only with copies of the same vertex of $C$. Hence each edge of a local surface is incident with precisely two faces. Hence its geometric realisation is always a is a surface. Similarly as for link complexes, we shall just say things like ‘the local surface is a sphere’.

**Observation 3.2.** Local surfaces of planar rotation systems are always connected. □

A *(2-dimensional) orientation* of a complex $C$ such that each edge is in precisely two faces is a choice of orientation of each face of $C$ such that each
edge is traversed in opposite directions by the chosen orientation of the two incident faces. Note that a complex whose geometric realisation is a surface has an orientation if and only its geometric realisation is orientable.

**Observation 3.3.** The set of orientations in a local-surface-equivalence class defines an orientation of its local surface.

In particular, local surfaces are cell complexes.

We will not use the following lemma in our proof of Theorem 1.1. However, we think that it gives a better intuitive understanding of local surfaces. We say that a simplicial complex $C$ is locally connected if all link graphs are connected.

**Lemma 3.4.** Let $C$ be a connected and locally connected complex embedded into $S^3$ and let $\Sigma$ be the induced planar rotation system. Then the local surfaces of the topological embedding are equal to the local surfaces for $(C, \Sigma)$.

There is the following relation between vertices of local surfaces and faces of link complexes.

**Lemma 3.5.** Let $\Sigma$ be a rotation system of a simplicial complex $C$. There is a bijection $\iota$ between the set of vertices of local surfaces for $(C, \Sigma)$ and the set of faces of link complexes for $(C, \Sigma)$, which maps each vertex $v'$ of a local surface cloned from the vertex $v$ of $C$ to a face $f$ of the link complex at $v$ such that the rotation system at $v'$ is an orientation of $f$.

**Proof.** The set of faces of the link complex at $v$ is in bijection with the set of $v$-equivalence classes; here the $v$-equivalence relation on the set of orientations of faces of $C$ incident with $v$ is the symmetric transitive closure of the relation ‘locally related’. Since we work in a subset of the orientations, every $v$-equivalence class is contained in a local-surface-equivalence class. On the other hand the set of all clones of a vertex $v$ of $C$ contained in a local surface $S$ is in bijection with the set of $v$-equivalence classes contained in the local-surface-equivalence class of $S$. This defines a bijection $\iota$ between the set of vertices of local surfaces for $(C, \Sigma)$ and the set of faces of link complexes for $(C, \Sigma)$.

It is straightforward to check that $\iota$ has all the properties claimed in the lemma.

**Corollary 3.6.** Given a local surface of a simplicial complex $C$ and one of its vertices $v'$ cloned from a vertex $v$ of $C$, there is a homeomorphism from
a neighbourhood around $v'$ in the local surface to the cone with top $v'$ over the face boundary of $\iota(v')$ that fixes $v'$ and the edges and faces incident with $v'$ in a neighbourhood around $v'$.

The definitions of link graphs and link complexes can be generalised from simplicial complexes to complexes as follows. The *link graph* of a complex $C$ at a vertex $v$ is the graph whose vertices are the edges incident with $v$. For any traversal of a face of the vertex $v$, we add an edge between the two vertices that when considered as edges of $C$ are in the face just before and just after that traversal of $v$. We stress that we allow parallel edges and loops.

Given a complex $C$, any rotation system $\Sigma$ of $C$ defines rotation systems at each link graph of $C$. Hence the definition of link complex extends.

## 4 Constructing piece-wise linear embeddings

In this section we prove [Theorem 4.4](#) below, which is used in the proof of [Theorem 1.1](#).

Throughout this section we fix a connected and locally connected simplicial complex $C$ with a rotation system $\Sigma$. An associated topological space $T(C, \Sigma)$ is defined as follows. For each local surface $S$ of $(C, \Sigma)$ we take an embedding into $S^3$. Each local surface is oriented and we denote by $\hat{S}$ the topological space obtained from $S^3$ by deleting all points on the outside of $S$. We obtain $T(C, \Sigma)$ from the simplicial complex $C$ by gluing onto each local surface $S$ the topological space $\hat{S}$ along $S$.

We remark that associated topological spaces may depend on the chosen embeddings of the local surfaces $S$ into $S^3$. However, if all local surfaces are spheres, then any two associated topological spaces are isomorphic and in this case we shall talk about ‘the’ associated topological space.

Clearly, associated topological spaces $T(C, \Sigma)$ are compact and connected as $C$ is connected.

**Lemma 4.1.** *The rotation system $\Sigma$ is planar if and only if the associated topological space $T(C, \Sigma)$ is a 3-dimensional manifold.*

**Proof.** [Observation 3.1](#) implies that if $T(C, \Sigma)$ is a 3-dimensional manifold, then $\Sigma$ is planar. Conversely, now assume that $\Sigma$ is a planar rotation system. We have to show that there is a neighbourhood around any point $x$ of $T(C, \Sigma)$ that is isomorphic to the closed 3-dimensional ball $B_3$.

If $x$ is a point not in $C$, this is clear. If $x$ is an interior point of a face $f$, we obtain a neighbourhood of $x$ by gluing together neighbourhoods of copies of $x$ in the local surfaces that contain an orientation of $f$. Each orientation
of $f$ is contained in local surfaces exactly once. Hence we glue together the two orientations of $f$ and clearly $x$ has a neighbourhood isomorphic to $B_3$.

Next we assume that $x$ is an interior point of an edge $e$. Some open neighbourhood of $x$ is isomorphic to the topological space obtained from gluing together for each copy of $e$ in a local surface, a neighbourhood around a copy $x'$ of $x$ on those edges. A neighbourhood around $x'$ has the shape of a piece of a cake, see Figure 4.

![Figure 4: A piece of a cake. This space is obtained by taking the product of a triangle with the unit interval. The edge $e$ is mapped to the set of points corresponding to some vertex of the triangle.](image)

First we consider the case that $x$ has several copies. As $\sigma(e)$ is a cyclic orientation, these pieces of a cake are glued together in a cyclic way along faces. Since each cyclic orientation of a face appears exactly once in local surfaces, we identify in each gluing step the two cyclic orientations of a face. Informally, the overall gluing will be a ‘cake’ with $x$ as an interior point. Hence a neighbourhood of $x$ is isomorphic to $B_3$. If there is only one copy of $x'$, then the copy of $e$ containing $x'$ is incident with the two orientations of a single face. Then we obtain a neighbourhood of $x$ by identifying these two orientations. Hence there is a neighbourhood of $x$ isomorphic to $B_3$.

It remains to consider the case where $x$ is a vertex of $C$. We obtain a neighbourhood of $x$ by gluing together neighbourhoods of copies of $x$ in local surfaces. We shall show that we have one such copy for every face of the link complex for $(C, \Sigma)$ and a neighbourhood of $x$ in such a copy is given by the cone over that face with $x$ being the top of the cone, see Figure 5. We shall show that the glued together neighbourhood is the cone over the link complex with $x$ at the top. Since $\Sigma$ is planar and $C$ is locally connected, the link complex is isomorphic to the 2-sphere. Since the cone over the 2-sphere is a 3-ball, the neighbourhood of $x$ has the desired type.

Now we examine this plan in detail. By Lemma 3.5 and Corollary 3.6, the copies are mapped by the bijection $\iota$ to the faces of the link complex at $x$ and a neighbourhood around such a copy $x'$ is isomorphic to the cone with top $x'$ over the face $\iota(x')$. We glue these cones over the faces $\iota(x')$ on
Figure 5: In this example the link complex of $x$ is a tetrahedron. The three faces visible in our drawing are highlighted in red, gold and grey. On the left we see how the four cones over the faces of the link complex are pasted together to form the cone over the link complex depicted on the right.

their faces that are obtained from edges of $\iota(x')$ by adding the top $x'$.

The glued together complex is isomorphic to the cone over the complex $S$ obtained by gluing together the faces $\iota(x')$ along edges, where we always glue the edge the way round so that copies of the same vertex of the local incidence graph are identified. Hence the vertex-edge-incidence relation and the edge-face-incidence relation of $S$ are the same as for the link complex at $x$. The same is true for the cyclic orderings of edges on faces. So $S$ is equal to the link complex at $x$.

Hence a neighbourhood of $x$ is isomorphic to a cone with top $x$ over the link complex at $x$. Since $\Sigma$ is a planar rotation system, the link complex is a disjoint union of spheres. As $C$ is locally connected, it is a sphere. Thus its cone is isomorphic to $B_3$.

Lemma 4.2. If $C$ is simply connected, then so is any associated topological space $T(C, \Sigma)$.

Proof. This is a consequence of Van Kampen’s Theorem [6, Theorem 1.20]. Indeed, we obtain $T$ from $T(C, \Sigma)$ by deleting all interior points of the sets $\hat{S}$ for local surfaces $S$ that are not in a small open neighbourhood of $C$. This can be done in such a way that $T$ has a deformation retract to $C$, and thus is simply connected. Now we recursively glue the spaces $\hat{S}$ back onto $T$. In each step we glue a single space $\hat{S}$. Call the space obtained after $n$ gluings $T_n$.

The fundamental group of $\hat{S}$ is a quotient of the fundamental group of the intersection of $T_n$ and $\hat{S}$. And the fundamental group of $T_n$ is trivial
by induction. So we can apply Van Kampen’s Theorem to deduce that the gluing space $T_{n+1}$ has trivial fundamental group. Hence the final gluing space $T(C, \Sigma)$ has trivial fundamental group. So it is simply connected.

The converse of Lemma 4.2 is true if all local surfaces for $(C, \Sigma)$ are spheres.

**Lemma 4.3.** If all local surfaces for $(C, \Sigma)$ are spheres and the associated topological space $T(C, \Sigma)$ is simply connected, then so is $C$.

**Proof.** Let $\varphi$ an image of $S^1$ in $C$. Since $T(C, \Sigma)$ is simply connected, there is a homotopy from $\varphi$ to a point of $C$ in $T(C, \Sigma)$. We can change the homotopy so that it avoids an interior point of each local surface of the embedding. Since each local surface is a sphere, for each local surface without the chosen point there is a continuous projection to its boundary. Since these projections are continuous, the concatenation of them with the homotopy is continuous. Since this concatenation is constant on $C$ this defines a homotopy of $\varphi$ inside $C$. Hence $C$ is simply connected.

We conclude this section with the following special case of Theorem 1.1.

**Theorem 4.4.** A locally connected simplicial complex $C$ has a planar rotation system $\Sigma$ if and only if $T(C, \Sigma)$ is a 3-manifold. And if $C$ is simply connected, then $T(C, \Sigma)$ must be the 3-sphere.

**Proof.** By treating different connected components separately, we may assume that $C$ is connected. The first part follows from Lemma 4.1. The second part follows from Lemma 4.2 and Perelman’s theorem [11, 13, 12] that any compact simply connected 3-manifold is isomorphic to the 3-sphere.

**Remark 4.5.** We used Perelman’s theorem in the proof of Theorem 4.4. On the other hand it together with Moise’s theorem [8] that every compact 3-dimensional manifold has a triangulation implies Perelman’s theorem: let $M$ be a simply connected 3-dimensional compact manifold. Let $T$ be a triangulation of $M$. And let $C$ be the simplicial complex obtained from $T$ by deleting the 3-dimensional cells. Let $\Sigma$ be the rotation system given by the embedding of $C$ into $T$. It is clear from that construction that $T$ is equal to the triangulation given by the embedding of $C$ into $T(C, \Sigma)$. Hence we can apply Lemma 4.3 to deduce that $C$ is simply connected. Hence by Theorem 4.4 the topological space $T(C, \Sigma)$, into which $C$ embeds, is isomorphic to the 3-sphere. Since $T(C, \Sigma)$ is isomorphic to $M$, we deduce that $M$ is isomorphic to the 3-sphere.
5 Cut vertices

In this section we deduce Theorem 1.1 from Theorem 4.4 proved in the last section. Given a prime \( p \), a simplicial complex \( C \) is \( p \)-nullhomologous if every directed cycle of \( C \) is generated over \( \mathbb{F}_p \) by the boundaries of faces of \( C \). Note that a simplicial complex \( C \) is \( p \)-nullhomologous if and only if the first homology group \( H_1(C, \mathbb{F}_p) \) is trivial. Clearly, every simply connected simplicial complex is \( p \)-nullhomologous.

A vertex \( v \) in a connected complex \( C \) is a cut vertex if the 1-skeleton of \( C \) without \( v \) is a disconnected graph\(^8\). A vertex \( v \) in an arbitrary, not necessarily connected, complex \( C \) is a cut vertex if it is a cut vertex in a connected component of \( C \).

**Lemma 5.1.** Every \( p \)-nullhomologous simplicial complex without a cut vertex is locally connected.

**Proof.** We construct for any vertex \( v \) of an arbitrary simplicial complex \( C \) such that the link graph \( L(v) \) at \( v \) is not connected and \( v \) is not a cut vertex a cycle containing \( v \) that is not generated by the face boundaries of \( C \).

Let \( e \) and \( g \) be two vertices in different components of \( L(v) \). These are edges of \( C \) and let \( w \) and \( u \) be their endvertices different from \( v \). Since \( v \) is not a cut vertex, there is a path in \( C \) between \( u \) and \( w \) that avoids \( v \). This path together with the edges \( e \) and \( g \) is a cycle \( o \) in \( C \) that contains \( v \).

Our aim is to show that \( o \) is not generated by the boundaries of faces of \( C \). Suppose for a contradiction that \( o \) is generated. Let \( F \) be a family of faces whose boundaries sum up to \( o \). Let \( F_v \) be the subfamily of faces of \( F \) that are incident with \( v \). Each face in \( F_v \) is an edge of \( L(v) \) and each vertex of \( L(v) \) is incident with an even number (counted with multiplicities) of these edges except for \( e \) and \( g \) that are incident with an odd number of these faces. Let \( X \) be the connected component of the graph \( L(v) \) restricted to the edge set \( F_v \) that contains the vertex \( e \). We obtain \( X' \) from \( X \) by adding \( k - 1 \) parallel edges to each edge that appears \( k \) times in \( F_v \). Since \( X' \) has an even number of vertices of odd degree also \( g \) must be in \( X \). This is a contradiction to the assumption that \( e \) and \( g \) are in different components of \( L(v) \). Hence \( o \) is not generated by the boundaries of faces of \( C \). This completes the proof. \( \square \)

\( ^8 \)We define this in terms of the 1-skeleton instead of directly in terms of \( C \) for a technical reason: The object obtained from a simplicial complex by deleting a vertex may have edges not incident with faces. So it would not be a 2-dimensional simplicial complex in the terminology of this paper.
Given a connected complex $C$ with a cut vertex $v$ and a connected component $K$ of the 1-skeleton of $C$ with $v$ deleted, the complex attached at $v$ centered at $K$ has vertex set $K + v$ and its edges and faces are those of $C$ all of whose incident vertices are in $K + v$.

**Lemma 5.2.** A connected simplicial complex $C$ with a cut vertex $v$ has a piece-wise linear embedding into $S^3$ if and only if all complexes attached at $v$ have a piece-wise linear embedding into $S^3$.

**Proof.** If $C$ has an embedding into $S^3$, then clearly all complexes attached at $v$ have an embedding. Conversely suppose that all complexes attached at $v$ have an embedding into $S^3$. Pick one of these complexes arbitrarily, call it $X$ and fix an embedding of it into $S^3$. In that embedding pick for each component of $C$ remove $v$ except that for $X$ a closed ball contained in $S^3$ that intersects $X$ precisely in $v$ such that all these closed balls intersect pairwise only at $v$. Each complex attached at $v$, has a piece-wise linear embedding into the 3-dimensional unit ball as they have embeddings into $S^3$ such that some open set is disjoint from the complex. Now we attach these embeddings into the balls of the embedding of $X$ inside the reserved balls by identifying the copies of $v$. This defines an embedding of $C$.

Recall that in order to prove Theorem 1.1 it suffices to show that any simply connected simplicial complex $C$ has a piece-wise linear embedding into $S^3$ if and only if $C$ has a planar rotation system.

**Proof of Theorem 1.1.** Clearly if a simplicial complex is embeddable into $S^3$, then it has a planar rotation system. For the other implication, let $C$ be a simply connected simplicial complex and $\Sigma$ be a planar rotation system. We prove the theorem by induction on the number of cut vertices of $C$. If $C$ has no cut vertex, it is locally connected by Lemma 5.1. Thus it has a piece-wise linear embedding into $S^3$ by Theorem 4.4.

Hence we may assume that $C$ has a cut vertex $v$. As $C$ is simply connected, every complex attached at $v$ is simply connected. Hence by the induction hypothesis each of these complexes has a piece-wise linear embedding into $S^3$. Thus $C$ has a piece-wise linear embedding into $S^3$ by Lemma 5.2.

6 local surfaces of planar rotation systems

The aim of this section is to prove Theorem 1.2. A shorter proof is sketched in Remark 6.10 using algebraic topology. As a first step in that direction,
we first prove the following.

**Theorem 6.1.** Let $C$ be a locally connected $p$-nullhomologous simplicial complex that has a planar rotation system. Then all local surfaces of the planar rotation system are spheres.

Before we can prove Theorem 6.1 we need some preparation. The complex dual to a simplicial $C$ with a rotation system $\Sigma$ has as its set of vertices the set of local surfaces of $\Sigma$. Its set of edges is the set of faces of $C$, and an edge is incident with a vertex if the corresponding face is in the corresponding local surface. The faces of the dual are the edges of $C$. Their cyclic ordering is as given by $\Sigma$. In particular, the edge-face-incidence-relation of the dual is the same as that of $C$ but with the roles of edges and faces interchanged.

Moreover, an orientation $\vec{f}$ of a face $f$ of $C$ corresponds to the direction of $f$ when considered as an edge of the dual complex $D$ that points towards the vertex of $D$ whose local-surface-equivalence class contains $\vec{f}$. Hence the direction of the dual complex $C$ induces a direction of the complex $D$. By $\Sigma_C = (\sigma_C(f) | f \in E(D))$ we denote the following rotation system for $D$: for $\sigma_C(f)$ we take the orientation $\vec{f}$ of $f$ in the directed complex $C$.

In this paper we follow the convention that for edges of $C$ we use the letter $e$ (with possibly some subscripts) while for faces of $C$ we use the letter $f$. In return, we use the letter $f$ for the edges of a dual complex of $C$ and $e$ for its faces.

**Lemma 6.2.** Let $C$ be a connected and locally connected simplicial complex. Then for any rotation system, the dual complex $D$ is connected.

**Proof.** Two edges of $C$ are $C$-related if there is a face of $C$ incident with both of them. And they are $C$-equivalent if they are in the transitive closure of the symmetric relation ‘$C$-related’. Clearly, any two $C$-equivalent edges of $C$ are in the same connected component. If $C$ however is locally connected, also the converse is true: any two edges in the same connected component are $C$-equivalent. Indeed, take a path containing these two edges. Any two edges incident with a common vertex are $C$-equivalent as $C$ is locally connected. Hence any two edges on the path are $C$-equivalent.

We define $D$-equivalent like ‘$C$-equivalent’ with ‘$D$’ in place of ‘$C$’. Now let $f$ and $f'$ be two edges of $D$. Let $e$ and $e'$ be edges of $C$ incident with $f$ and $f'$, respectively. Since $C$ is connected and locally connected the edges $e$ and $e'$ are $C$-equivalent. As $C$ and $D$ have the same edge/facce incidence relation, the edges $f$ and $f'$ of $D$ are $D$-equivalent. So any two edges of $D$ are $D$-equivalent. Hence $D$ is connected. $\blacksquare$
First, we prove the following, which is reminiscent of Euler’s formula.

**Lemma 6.3.** Let $C$ be a locally connected $p$-nullhomologous simplicial complex with a planar rotation system and $D$ the dual complex. Then

$$|V(C)| - |E| + |F| - |V(D)| \geq 0$$

Moreover, we have equality if and only if $D$ is $p$-nullhomologous.

**Proof.** Let $Z_C$ be the dimension over $\mathbb{F}_p$ of the cycle space of $C$. Similarly we define $Z_D$. Let $r$ be the rank of the edge-face-incidence matrix over $\mathbb{F}_p$. Note that $r \leq Z_D$ and that $r = Z_C$ as $H_1(C, \mathbb{F}_p) = 0$. So $Z_D - Z_C \geq 0$. Hence it suffices to prove the following.

**Sublemma 6.4.**

$$|V(C)| - |E| + |F| - |V(D)| = Z_D - Z_C$$

**Proof.** Let $k_C$ be the number of connected components of $C$ and $k_D$ be the number of connected components of $D$. Recall that the space orthogonal to the cycle space (over $\mathbb{F}_p$) in a graph $G$ has dimension $|V(G)|$ minus the number of connected components of $G$. Hence $Z_C = |E| - |V(C)| + k_C$ and $Z_D = |F| - |V(D)| + k_D$. Subtracting the first equation from the second yields:

$$|V(C)| - |E| + |F| - |V(D)| + (k_D - k_C) = Z_D - Z_C$$

Since the dual complex of the disjoint union of two simplicial complexes (with planar rotation systems) is the disjoint union of their dual complexes, $k_C \leq k_D$. By **Lemma 6.2** $k_C = k_D$. Plugging this into the equation before, proves the sublemma. \qed

This completes the proof of the inequality. We have equality if and only if $r = Z_D$. So the ‘Moreover’-part follows. \qed

Our next goal is to prove the following, which is also reminiscent of Euler’s formula but here the inequality goes the other way round.

**Lemma 6.5.** Let $C$ be a locally connected simplicial complex with a planar rotation system $\Sigma$ and $D$ the dual complex. Then:

$$|V(C)| - |E| + |F| - |V(D)| \leq 0$$

with equality if and only if all link complexes for $(D, \Sigma_C)$ are spheres.
Before we can prove this, we need some preparation. By \( a \) we denote the sum of the faces of link complexes for \((C, \Sigma)\). By \( a' \) we denote the sum over the faces of link complexes for \((D, \Sigma_C)\). Before proving that \( a \) is equal to \( a' \) we prove that it is useful by showing the following.

**Claim 6.6.** Lemma 6.5 is true if \( a = a' \) and all link complexes for \((D, \Sigma_C)\) are connected.

**Proof.** Given a face \( f \) of \( C \), we denote the number of edges incident with \( f \) by \( \text{deg}(f) \). Our first aim is to prove that

\[
2|V(C)| = 2|E| - \sum_{f \in F} \text{deg}(f) + a
\]

To prove this equation, we apply Euler’s formula \([4]\) in the link complexes for \((C, \Sigma)\). Then we take the sum of all these equations over all \( v \in V(C) \). Since \( \Sigma \) is a planar rotation system, all link complexes are a disjoint union of spheres. Since \( C \) is locally connected, all link complexes are connected and hence are spheres. So they have euler characteristic two. Thus we get the term \( 2|V(C)| \) on the left hand side. By definition, \( a \) is the sum of the faces of link complexes for \((C, \Sigma)\).

The term \( 2|E| \) is the sum over all vertices of link complexes for \((C, \Sigma)\). Indeed, each edge of \( C \) between the two vertices \( v \) and \( w \) of \( C \) is a vertex of precisely the two link complexes for \( v \) and \( w \).

The term \( \sum_{f \in F} \text{deg}(f) \) is the sum over all edges of link complexes for \((C, \Sigma)\). Indeed, each face \( f \) of \( C \) is in precisely those link complexes for vertices on the boundary of \( f \). This completes the proof of \([1]\).

Secondly, we prove the following inequality using a similar argument. Given an edge \( e \) of \( C \), we denote the number of faces incident with \( e \) by \( \text{deg}(e) \).

\[
2|V(D)| \geq 2|F| - \sum_{e \in E} \text{deg}(e) + a'
\]

To prove this, we apply Euler’s formula in link complexes for \((D, \Sigma_C)\), and take the sum over all \( v \in V(D) \). Here we have ‘\( \geq \)’ instead of ‘\( = \)’ as we just know by assumption that the link complexes are connected but they may not be a sphere. So we have \( 2|V(D)| \) on the left and \( a' \) is the sum over the faces of link complexes for \((D, \Sigma_C)\).

The term \( 2|F| \) is the sum over all vertices of link complexes for \((D, \Sigma_C)\). Indeed, each edge of \( D \) between the two different vertices \( v \) and \( w \) of \( D \) is a vertex of precisely the two link complexes for \( v \) and \( w \). A loop gives rise to two vertices in the link graph at the vertex it is attached to.
The term $\sum_{e \in E} \deg(e)$ is the sum over all edges of link complexes for $(D, \Sigma_C)$. Indeed, each face $e$ of $D$ is in the link complex at $v$ with multiplicity equal to the number of times it traverses $v$. This completes the proof of (2).

By assumption, $a = a'$. The sums $\sum_{f \in F} \deg(f)$ and $\sum_{e \in E} \deg(e)$ both count the number of nonzero entries of $A$, so they are equal. Subtracting (2) from (1), rearranging and dividing by 2 yields:

$$|V(C)| - |E| + |F| - |V(D)| \leq 0$$

with equality if and only if all link complexes for $(D, \Sigma_C)$ are spheres. □

Hence our next aim is to prove that $a$ is equal to $a'$. First we need some preparation.

Two cell complexes $C$ and $D$ are (abstract) surface duals if the set of vertices of $C$ is (in bijection with) the set of faces of $D$, the set of edges of $C$ is the set of edges of $D$ and the set of faces of $C$ is the set of vertices of $D$. And these three bijections preserve incidences.

**Lemma 6.7.** Let $C$ be a simplicial complex and $\Sigma$ be a rotation system and let $D$ be the dual. The surface dual of a local surface $S$ for $(C, \Sigma)$ is equal to the link complex for $(D, \Sigma_C)$ at the vertex $\ell$ of $D$ that corresponds to $S$.

**Proof.** It is immediate from the definitions that the vertices of the link complex $\bar{L}$ at $\ell$ are the faces of $S$. The edges of $S$ are triples $(e, \vec{f}, \vec{g})$, where $e$ is an edge of $C$ and $\vec{f}$ and $\vec{g}$ are orientations of faces of $C$ that are related via $e$ and are in the local-surface-equivalence class for $S$. Hence the local-surface-equivalence class for $S$ is the triple $(e, \vec{f}, \vec{g})$ such that $\vec{f}$ and $\vec{g}$ are directions of edges that point towards $\ell$ and $f$ and $g$ are adjacent in the cyclic ordering of the face $e$. This is precisely the edges of the link graph $\bar{L}(\ell)$. Hence the link graph $\bar{L}(\ell)$ is the dual graph of the cell complex $S$.

Now we will use the Edmonds-Hefter-Ringel rotation principle, see [7, Theorem 3.2.4], to deduce that the link complex $\bar{L}$ at $\ell$ is the surface dual of $S$. We denote the unique cell complex that is a surface dual of $S$ by $S^\ast$. Above we have shown that $\bar{L}$ and $S^\ast$ have the same 1-skeleton. Moreover, the rotation systems at the vertices of the link complex $\bar{L}$ are given by the cyclic orientations in the local-surface-equivalence class for $S$. By Observation 3.3 these local-surface-equivalence classes define an orientation of $S$. So $\bar{L}$ and

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The dual graph of a cell complex $C$ is the graph $G$ whose set of vertices is (in bijection with) the set of faces of $C$ and whose set of edges is the set of edges of $C$. And the incidence relation between the vertices and edges of $G$ is the same as the incidence relation between the faces and edges of $C$.

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$S^*$ have the same rotation systems. Hence by the Edmonds-Hefter-Ringel rotation principle $\bar{L}$ and $S^*$ have to be isomorphic. So $\bar{L}$ is a surface dual of $S$.

Proof of Lemma 6.5. Let $C$ be a locally connected simplicial complex and $\Sigma$ be a rotation system and let $D$ be the dual. Let $\Sigma_C$ be as defined above. By Observation 3.2 and Lemma 6.7 every link complex for $(D, \Sigma_C)$ is connected. By Claim 6.6 it suffices to show that the sum over all faces of link complexes of $C$ with respect to $\Sigma$ is equal to the sum over all faces of link complexes for $D$ with respect to $\Sigma_C$. By Lemma 6.7 the second sum is equal to the sum over all vertices of local surfaces for $(C, \Sigma)$. This completes the proof by Lemma 3.5.

Proof of Theorem 6.1. Let $C$ be a $p$-nullhomologous locally connected simplicial complex that has a planar rotation system $\Sigma$. Let $D$ be the dual complex. Then by Lemma 6.5 and Lemma 6.3 $C$ and $D$ satisfy Euler’s formula, that is:

$$|V(C)| - |E| + |F| - |V(D)| = 0$$

Hence by Lemma 6.5 all link complexes for $(D, \Sigma_C)$ are spheres. By Lemma 6.7 these are dual to the local surfaces for $(C, \Sigma)$. Hence all local surfaces for $(C, \Sigma)$ are spheres.

The following theorem gives three equivalent characterisations of the class of locally connected simply connected simplicial complexes embeddable in $S^3$.

**Theorem 6.8.** Let $C$ be a locally connected simplicial complex embedded into $S^3$. The following are equivalent.

1. $C$ is simply connected;
2. $C$ is $p$-nullhomologous for some prime $p$;
3. all local surfaces of the planar rotation system induced by the topological embedding are spheres.

Proof. Clearly, 1 implies 2. To see that 2 implies 3, we assume that $C$ is $p$-nullhomologous. Let $\Sigma$ be the planar rotation system induced by the topological embedding of $C$ into $S^3$. By Theorem 6.1 all local surfaces for $(C, \Sigma)$ are spheres.

It remains to prove that 3 implies 1. So assume that $C$ has an embedding into $S^3$ such that all local surfaces of the planar rotation system induced
by the topological embedding are spheres. By treating different connected
components separately, we may assume that \( C \) is connected. By Lemma 3.4
all local surfaces of the topological embedding are spheres. Thus 3 implies
1 by Lemma 4.3.

Remark 6.9. Our proof actually proves the strengthening of Theorem 6.8
with ‘embedded into \( S^3 \)’ replaced by ‘embedded into a simply connected
3-dimensional compact manifold.’ However this strengthening is equivalent
to Theorem 6.8 by Perelman’s theorem.

Recall that in order to prove Theorem 1.2 it suffices to show that every
\( p \)-nullhomologous simplicial complex \( C \) has a piece-wise linear embedding
into \( S^3 \) if and only if it is simply connected and \( C \) has a planar rotation
system.

Proof of Theorem 1.2. Using an induction argument on the number of cut
vertices as in the proof of Theorem 1.1 we may assume that \( C \) is locally
connected. If \( C \) has a piece-wise linear embedding into \( S^3 \), then it has a
planar rotation system and it is simply connected by Theorem 6.8. The
other direction follows from Theorem 1.1.

Remark 6.10. One step in proving Theorem 1.2 was showing that if a
simplicial complex whose first homology group is trivial embeds in \( S^3 \), then
it must be simply connected. In this section we have given a proof that only
uses elementary topology. We use these methods again in [3].

However there is a shorter proof of this fact, which we shall sketch in
the following. Let \( C \) be a simplicial complex embedded in \( S^3 \) such that one
local surface of the embedding is not a sphere. Our aim is to show that the
first homology group of \( C \) cannot be trivial.

We will rely on the fact that the first homology group of \( X = S^3 \setminus S^1 \) is
not trivial. It suffices to show that the homology group of \( X \) is a quotient of
the homology group of \( C \). Since here by Hurewicz’s theorem, the homology
group is the abelisation of the fundamental group, it suffices to show that
the fundamental group \( \pi_1(X) \) of \( X \) is a quotient of the fundamental group
\( \pi_1(C) \).

We let \( C_1 \) be a small open neighbourhood of \( C \) in the embedding of \( C \)
in \( S^3 \). Since \( C_1 \) has a deformation retract onto \( C \), it has the same fundamental
group. We obtain \( C_2 \) from \( C_1 \) by attaching the interiors of all local surfaces
of the embedding except for one – which is not a sphere. This can be done
by attaching finitely many 3-balls. Similar as in the proof of Lemma 4.2.
one can use Van Kampen’s theorem to show that the fundamental group of $C_2$ is a quotient of the fundamental group of $C_1$. By adding finitely many spheres if necessary and arguing as above one may assume that remaining local surface is a torus. Hence $C_2$ has the same fundamental group as $X$. This completes the sketch.

7 Embedding general simplicial complexes

There are three classes of simplicial complexes that naturally include the simply connected simplicial complexes: the $p$-nullhomologous ones that are included in those with abelian fundamental group that in turn are included in general simplicial complexes. Theorem 1.2 characterises embeddability of $p$-nullhomologous complexes. In this section we prove embedding results for the later two classes. The bigger the class gets, the stronger assumptions we will require in order to guarantee topological embeddings into $S^3$.

A curve system of a surface $S$ of genus $g$ is a choice of at most $g$ genus reducing curves in $S$ that are disjoint. An extension of a rotation system $\Sigma$ is a choice of curve system at every local surface of $\Sigma$. An extension of a rotation system of a complex $C$ is simply connected if the topological space obtained from $C$ by gluing a disc at each curve of the extension is simply connected. The definition of a $p$-nullhomologous extension is the same with ‘$p$-nullhomologous’ in place of ‘simply connected’.

Theorem 7.1. Let $C$ be a connected and locally connected simplicial complex with a rotation system $\Sigma$. The following are equivalent.

1. $\Sigma$ is induced by a topological embedding of $C$ into $S^3$.
2. $\Sigma$ is a planar rotation system that has a simply connected extension.
3. We can subdivide edges of $C$, do baricentric subdivision of faces and add new faces such that the resulting simplicial complex is simply connected and has a topological embedding into $S^3$ whose induced planar rotation system $\Sigma'$ ‘induces’ $\Sigma$.

Here we define that ‘$\Sigma'$ induces $\Sigma$’ in the obvious way as follows. Let $C$ be a simplicial complex obtained from a simplicial complex $C'$ by deleting faces. A rotation system $\Sigma = (\sigma(e)|e \in E(C))$ of $C$ is induced by a rotation system $\Sigma' = (\sigma'(e)|e \in E(C))$ of $C'$ if $\sigma(e)$ is the restriction of $\sigma'(e)$ to the

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10 We stress that the curves need not go through edges of $C$. ‘Gluing’ here is on the level of topological spaces not of complexes.
faces incident with $e$. If $C$ is obtained from contracting edges of $C'$ instead, a rotation system $\Sigma$ of $C$ is *induced* by a rotation system $\Sigma'$ of $C'$ if $\Sigma$ is the restriction of $\Sigma'$ to those edges that are in $C$. If $C'$ is obtained from $C$ by a baricentric subdivision of a face $f$ we take the same definition of ‘induced’, where we make the identification between the face $f$ of $C$ and all faces of $C'$ obtained by subdividing $f$. Now in the situation of Theorem 7.1 we say that $\Sigma'$ *induces* $\Sigma$ if there is a chain of planar rotation systems each inducing the next one starting with $\Sigma'$ and ending with $\Sigma$.

Before we can prove Theorem 7.1, we need some preparation. The following is a consequence of the Loop Theorem [10, 5].

**Lemma 7.2.** Let $X$ be an orientable surface of genus $g \geq 1$ embedded topologically into $\mathbb{R}^3$, then there is a genus reducing circle $\gamma$ of $X$ and a disc $D$ with boundary $\gamma$ and all interior points of $D$ are contained in the interior of $X$.

**Corollary 7.3.** Let $X$ be an orientable surface of genus $g \geq 1$ embedded topologically into $\mathbb{R}^3$, then there are genus reducing circles $\gamma_1, \ldots, \gamma_g$ of $X$ and closed discs $D_i$ with boundary $\gamma_i$ such that the $D_i$ are disjoint and the interior points of the discs $D_i$ are contained in the interior of $X$.

**Proof.** We prove this by induction on $g$. In the induction step we cut out the current surface along $D$. Then we apply Lemma 7.2 to that new surface. \qed

**Proof of Theorem 7.1.** 1 is immediately implied by 3.

Next assume that $\Sigma$ is induced by a topological embedding of $C$ into $\mathbb{S}^3$. Then $\Sigma$ is clearly a planar rotation system. It has a simply connected extension by Corollary 7.3. Hence 1 implies 2.

Next assume that $\Sigma$ is a planar rotation system that has a simply connected extension. We can clearly subdivide edges and do baricentric subdivision and change the curves of the curve system of the simply connected extension such that in the resulting simplicial complex $C'$ all the curves of the simply connected extension closed are walks in the 1-skeleton of $C'$. We define a planar rotation system $\Sigma'$ of $C'$ that induces $\Sigma$ as follows. If we subdivide an edge, we assign to both copies the cyclic orientation of the original edge. If we do a baricentric subdivision, we assign to all new edges the unique cyclic orientation of size two. Iterating this during the construction of $C'$ defines $\Sigma' = (\sigma'(e)|e \in E(C'))$, which clearly is a planar rotation.

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11 A circle is a topological space homeomorphic to $\mathbb{S}^1$. 

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system that induces $\Sigma$. By construction $\Sigma'$ has a simply connected extension such that all its curves are walks in the 1-skeleton of $C'$. Informally, we obtain $C''$ from $C'$ by attaching a disc at the boundary of each curve of the simply connected extension. Formally, we obtain $C''$ from $C'$ by first adding a face for each curve $\gamma$ in the simply connected extension whose boundary is the closed walk $\gamma$. Then we do a baricentric subdivision to all these newly added faces. This ensures that $C''$ is a simplicial complex. Since $C$ is locally connected, also $C''$ is locally connected. Since the geometric realisation of $C''$ is equal to the geometric realisation of $C$, which is simply connected, the simplicial complex $C''$ is simply connected.

Each newly added face $f$ corresponds to a traversal of a curve $\gamma$ of some edge $e$ of $C'$. This traversal is a unique edge of the local surface $S$ to whose curve system $\gamma$ belongs. For later reference we denote that copy of $e$ by $e_f$.

We define a rotation system $\Sigma'' = (\sigma''(e)|e \in E(C'))$ of $C''$ as follows. All edges of $C''$ that are not edges of $C'$ are incident with precisely two faces. We take the unique cyclic ordering of size two there.

Next we define $\sigma''(e)$ at edges $e$ of $C'$ that are incident with newly added faces. If $e$ is only incident with a single face of $C'$, then $e$ is only in a single local surface and it only has one copy in that local surface. Since the curves at that local surface are disjoint. We could have only added a single face incident with $e$. We take for $\sigma''(e)$ the unique cyclic orientation of size two at $e$.

So from now assume that $e$ is incident with at least two faces of $C'$. In order to define $\sigma''(e)$, we start with $\sigma'(e)$ and define in the following for each newly added face in between which two cyclic orientations of faces adjacent in $\sigma'(e)$ we put it. We shall ensure that between any two orientations we put at most one new face. Recall that two cyclic orientations $\vec{f}_1$ and $\vec{f}_2$ of faces $f_1$ and $f_2$, respectively, are adjacent in $\sigma'(e)$ if and only if there is a clone $e'$ of $e$ in a local surface $S$ for $(C', \Sigma')$ containing $\vec{f}_1$ and $\vec{f}_2$ such that $e'$ is incident with $\vec{f}_1$ and $\vec{f}_2$ in $S$. Let $f$ be a face newly added to $C''$ at $e$. Let $\gamma_f$ be the curve from which $f$ is build and let $S_f$ be the local surface that has $\gamma_f$ in its curve system. Let $e_f$ be the copy of $e$ in $S_f$ that corresponds to $f$ as defined above. when we consider $f$ has a face obtained from the disc glued at $\gamma_f$. We add $f$ to $\sigma'(e)$ in between the two cyclic orientations that are incident with $e_f$ in $S_f$. This completes the definition of $\Sigma''$. Since the copies $e_f$ are distinct for different faces $f$, the rotation system $\Sigma''$ is well-defined. By construction $\Sigma''$ induces $\Sigma$. We prove the following.

**Sublemma 7.4.** $\Sigma''$ is a planar rotation system of $C''$. 


Proof. Let \( v \) be a vertex of \( C'' \). If \( v \) is not a vertex of \( C' \), then the link graph at \( v \) is a cycle. Hence the link complex at \( v \) is clearly a sphere. Hence we may assume that \( v \) is a vertex of \( C' \).

Our strategy to show that the link complex \( S'' \) at \( v \) for \( (C'',\Sigma'') \) is a sphere will be to show that it is obtained from the link complex \( S' \) for \( (C',\Sigma') \) by adding edges in such a way that each newly added edge traverses a face of \( S' \) and two newly added edges traverse different face of \( S' \).

So let \( f \) be a newly added face incident with \( v \) of \( C' \). Let \( x \) and \( y \) be the two edges of \( f \) incident with \( v \). We make use of the notations \( \gamma_f, S_f, x_f \) and \( y_f \) defined above. Let \( v_f \) be the unique vertex of \( S_f \) traversed by \( \gamma_f \) in between \( x_f \) and \( y_f \). By Lemma 3.5 there is a unique face \( z_f \) of \( S' \) mapped by the map \( \iota \) of that lemma to \( v_f \). And \( x \) and \( y \) are vertices in the boundary of \( z_f \). The edges on the boundary of \( z_f \) incident with \( x \) and \( y \) are the cyclic orientations of the faces that are incident with \( x_f \) and \( y_f \) in \( S_f \). Hence in \( S'' \) the edge \( f \) traverses the face \( z_f \).

It remains to show that the faces \( z_f \) of \( S' \) are distinct for different newly added faces \( f \) of \( C'' \). For that it suffices by Lemma 3.5 to show that the vertices \( v_f \) are distinct. This is true as curves for \( S_f \) traverse a vertex of \( S_f \) at most once and different curves for \( S_f \) are disjoint.

Since \( \Sigma'' \) is a planar rotation system of the locally connected simplicial complex \( C'' \) and \( C'' \) is simply connected, \( \Sigma'' \) is induced by a topological embedding of \( C'' \) into \( S^3 \) by Theorem 4.4. Hence 2 implies 3.

A natural weakening of the property that \( C \) is simply connected is that the fundamental group of \( C \) is abelian. Note that this is equivalent to the condition that every chain that is \( p \)-nullhomologous is simply connected.

**Theorem 7.5.** Let \( C \) be a connected and locally connected simplicial complex with abelian fundamental group. Then \( C \) has a topological embedding into \( S^3 \) if and only if it has a planar rotation system \( \Sigma \) that has a \( p \)-nullhomologous extension.

In order to prove Theorem 7.5 we prove the following.

**Lemma 7.6.** A \( p \)-nullhomologous extension of a planar rotation system of a simplicial complex \( C \) with abelian fundamental group is a simply connected extension.

*Proof.* Let \( C' \) be the topological space obtained from \( C \) by gluing discs along the curves of the \( p \)-nullhomologous extension. The fundamental group \( \pi' \)
of $C'$ is a quotient of the fundamental group $\pi$ of $C$, see for example [6, Proposition 1.26]. Since $\pi$ is abelian by assumption, also $\pi'$ is abelian. That is, it is equal to its abelisation, which is trivial by assumption. Hence $C'$ is simply connected.

**Proof of Theorem 7.5.** If $C$ has a topological embedding into $S^3$, then by Theorem 7.1 it has a planar rotation system that has a $p$-nullhomologous extension. If $C$ has a planar rotation system that has a $p$-nullhomologous extension, then that extension is simply connected by Lemma 7.6. Hence $C$ has a topological embedding into $S^3$ by the other implication of Theorem 7.1.

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