Structure of the most singular vortices in fully developed turbulence

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Using high Reynolds number experimental data, we search for most dissipative, most intense vortices. These structures possess a scaling predicted by log-Poisson model for the dissipation field $\varepsilon_r$. These new experimental data suggest that the most intense structures have co-dimension less than 2. The log-Poisson statistics is compared with log-binomial which follows from the random $\beta$-model.

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It is known, at least from numerical simulations, that the large-amplitude dissipation occurs around vortex tubes in turbulence. We thus expect some structure to exist in a signal that characterizes the large values of the dissipation field. There are some statistics, although very incomplete ones, on the distance between the vortex tubes, the size of vortex tubes, etc. The largest value of the dissipation is also important in determining the resolution of DNS [1]. It would be therefore of interest to provide a direct experimental study of the dissipative field extremal values. On the other hand, very large values of the dissipation field correspond to intermittency. Traditionally, the latter is expressed through so-called intermittency corrections to the exponents for the structure functions, $\langle|u(x+r)−u(x)|^p\rangle \sim r^{\zeta_p}$, where $u$ is the longitudinal velocity, and $\zeta_p = p/3$ [2]. Thus, these corrections result in $\zeta_p = p/3 + \tau(p/3)$. A theory, incorporating the intermittency, the refined similarity hypothesis [3], links the statistic of these corrections with the statistic of the dissipation field $\varepsilon_r$, the energy dissipation averaged over a ball of size $r$. Namely, $\langle \varepsilon_r^p \rangle \sim r^{\tau(p)}$. Many models have been proposed to explain intermittency. It was originally suggested that the statistics of $\varepsilon_r$ is log-normal [3]. More recently, She and Lévêque [4] (hereafter SL, see also [5], [6], and recent study [7]) have proposed log-Poisson statistics for the dissipation field, with agreement with the experimentally found $\zeta_p$ in [8], [9]. These experimental exponents are obtained in Extended Self-Similarity approach, which is useful because of extended scaling range.

The simplest idea to study large values of dissipation is to measure the maxima. However, for many distributions a maximum of a big array can be “anything”, or arbitrary large. This is true, for example, for Gaussian statistics. The same is true for log-normal distribution. To see this, recall that, first, $\tau(p) = -d_p(p−1)$, $d_p = D − D_p$, where $D_p$ are so-called generalized dimensions [10]. Second, studying maxima is in a way equivalent to measuring asymptotically high moments, $d_{\infty} = \lim_{p\to\infty} (-\tau(p)/(p−1))$. For the log-normal distribution, $\tau(p) = -\mu/2p(p−1)$, and therefore asymptotically, $d_p = (\mu/2)p \to \infty$. Remarkably, the Poisson statistics provide some distinctive maximum. To see this, recall that for the Poisson distribution (see, e.g., [11]),

$$P(\alpha, \xi) = e^{-\xi^a}/a!,$$

and $a < 0$, the maximal value is defined through $b = \ln \max \xi_r$. In order to specify $b$, we calculate the moments, $\langle \xi_r^p \rangle$. Noting that $\langle \xi_r^1 \rangle = 1$, we get $b = \xi(1−e^a)$, where $\xi = C \ln (\ell/r)$, $\ell$ being external scale, and $C$ is a constant. As a result (of calculation of the moments), we get,

$$\tau(p) = C[1−(1−\gamma/C)^p] − p\gamma,$$  \hspace{1cm} (2)

where $\gamma = C(1−e^a)$ [12]. Using (2), it is easy to show that this time $d_{\infty} = \gamma$, which is a finite number.

SL is recovered from (2) if $C = 2$, $\gamma = 2/3$, so that $a = \ln (2/3)$, $C$ being the co-dimension of most dissipative structures, and $\gamma$ is defined by the dissipation rate, i.e., inverse time-scale, $1/\tau_r \sim r^{−2/3}$ [4].

The meaning of $C$ becomes even more clear directly from (1): the most intense fluctuations correspond to $\alpha = 0$ (as $a < 0$), so that the probability $P(\alpha = 0) = e^{-\xi} = (r/\ell)^C = (r/\ell)^{(D−H_0)}$, $D$ is dimension of space ($= 3$). Thus the Hausdorff dimension for most dissipative structures in SL theory, $H_0 = 1$, i.e., the structures are filaments. On the other hand, using expressions for $b$, $\xi$ and $\gamma$, we now rewrite (1) as follows,

$$\varepsilon_r = e^{\alpha a} \max \varepsilon_r = e^{\alpha a} \left(\frac{r}{\ell}\right)^{−\gamma}.$$

Putting $\alpha = 0$ in (3), we can see that the most intense structures are expected to scale $\sim r^{−\gamma}$. Thus, for the the log-Poisson statistics, the maxima of $\varepsilon_r(x)$ are not “anything”, and they are supposed to be self-similar.

This scaling is proved to be possible to verify experimentally. We used 10 million points of atmospheric data, with an estimated Taylor microscale Reynolds number 9540, (experiment A) and 40 million points for both longitudinal and transfer velocities (experiment B). The data are treated in spirit of Taylor hypothesis, that is,
the time series is treated as one-dimensional cut of the process. The dissipation rate can be written as

$$\varepsilon(x) = \nu (\partial_x v_x \partial_y v_y + \partial_y v_y \partial_x v_x), \tag{4}$$

(summation over repeating induces). The second term on the rhs vanishes after averaging, for homogeneous incompressible turbulence. The first term consists of 3 longitudinal and 6 transverse components. Therefore, it is natural to present the dissipation as

$$\varepsilon_l(x) = \nu \left[ 3 (\partial_x v_x(x))^2 + 6 (\partial_x v_y(x))^2 \right]. \tag{5}$$

For isotropic turbulence, $\langle (\partial_x v_y(x))^2 \rangle = 2 \langle (\partial_x v_x(x))^2 \rangle$, and therefore, following [13], we may consider three types of dissipation, longitudinal, transverse,

$$\varepsilon_l(x) = 15 \nu (\partial_x v_x(x))^2, \quad \varepsilon_t(x) = (15/2) \nu (\partial_x v_y(x))^2, \tag{6}$$

and combined, (4).

We will deal with coarse-grain dimensionless dissipation,

$$\varepsilon_r = \frac{1}{r} \int_{x-r/2}^{x+r/2} \frac{\varepsilon(x') \, dx'}{\langle \varepsilon \rangle}, \tag{7}$$

and maxima of $\varepsilon_r$ can be measured. Note that this measurement is meaningful because (7) contains some average. Figure 1 shows longitudinal scaling for the experiment A, which holds for 45 decades. The deviation from SL is small, and we recall that SL suggest that there is no anomalous scaling for $t_r$. This small deviation in Fig. 1 can be interpreted as anomalous persistence of the eddies, which is indeed observed [15], see also discussion in [16]. The value of $\gamma$ is 0.61 ± 0.01, only slightly smaller than 2/3. In order to compare with a “regular” random process we generated a Gaussian process $\omega_g$ with correlation function coinciding with experimental, i.e., $\langle \omega_g(x+r) \omega_g(x) \rangle = \langle \partial_x v_x(x+r) \partial_x v_x(x) \rangle$. Then, the “dissipation” $\varepsilon_r^{(g)} = \omega_g^2$, and $\varepsilon_r^{(g)} = 1/r \int_{x-r/2}^{x+r/2} \varepsilon_r^{(g)}(x') \, dx'$. Corresponding calculation for the maxima are reported in Fig. 1. If any scaling can be extracted from the Gaussian process, it would be at large asymptotic distances, and the scaling is trivial, $\gamma = 0$, meaning no singularity.

Figure 2 presents the scaling for experiment B. This time, the scaling holds for almost 6 decades. The scaling exponent for longitudinal dissipation is again 0.61 ± 0.01, while for the combined dissipation (which is quite close to the transversal dissipation) the exponent is 0.57 ± 0.01. In both Figs. 1 and 2 there is also $\lambda$-scale. Note that there is characteristic transfer region at $r/\lambda \approx 1$. We may interpret it as a transition to the inertial range [14], which is formed due to the fact that the vortices are expected to have scales between Kolmogorov microscale $\eta$ and Taylor microscale $\lambda$.

![FIG. 1. Scaling for most intense structures. The power law fitting of the experimental data (solid thick line) has been extended to reach unity (solid line), where it is supposed to match with SL scaling. The distances are given in terms of Kolmogorov micro-scale $\eta$, and in units of $\lambda$.](image1)

![FIG. 2. Experiment B scaling for longitudinal and transversal dissipation, including combined dissipation.](image2)

Although the experimental $\gamma$ is not that different from $2/3$, the value of the other parameter $C$ is quite sensitive to that difference. In order to find $C$ we substitute $\gamma$ from our measurements into (2), and use computer routines to find a best fit for these data with free parameter $C$ and the exponents $\zeta_p^{(ESS)}$ from experiment [8], [9]. We start with the longitudinal dissipation (6) from experiments A and B (recall that $\gamma$ is the same for them). As a result, we find $C = 1.67$ and $a = -0.45$ (cf. ln \{2/3\} = -0.41). With these parameters, the deviation of these calculated exponents $\zeta_p^{(c)}$ from the experimental exponents, $\sqrt{\langle (\zeta_p^{(c)} - \zeta_p^{(ESS)})^2 \rangle} = 0.0063$. To compare: for SL, $\sqrt{\langle (\zeta_p^{(SL)} - \zeta_p^{(ESS)})^2 \rangle} = 0.0078$. The $\zeta_p^{(c)}$ exponents seem to be “better” than $\zeta_p^{(SL)}$, but considering that the experimental exponents have errors of about ±1% [9], we conclude that these exponents are similar.
Note that if we substitute in (2) the value of \( \gamma = 0.61 \) and put \( C = 2 \), then, for the obtained exponents, \( \zeta_p^{(C=2)} \), we have, \( \sqrt{(\zeta_p^{(C=2)} - \zeta_p^{(ESS)})^2} = 0.050 \), much too high.

Consider now the combined dissipation, defined in (5). The best fitting with this parameter fixed results in \( C = 1.43 \), and \( a = -0.50 \). This time, the deviation of the computer generated spectrum from the experimental is \( 0.0071 \). Figure 3 shows \( \tau(p/3) \) from experiment, and for different theories. It can be seen that all the curves collapse into one, corresponding to the experiment, except that one with \( \gamma \) from our measurements, and \( C = 2 \). This illustrates that the data are indeed sensitive to the measured \( \gamma \), that is to its (small) difference from 2/3.

The codimension \( C = 1.43 \) corresponds to \( H_0 = 1.57 \). This value of \( H_0 > 1 \) seems to be consistent with the distinction between persistent vortical filaments and the dissipative structures associated with regions of strong strain [18]. That means that the most dissipative structures consist not only of filaments, but in part of sheets, or filaments convoluted into complex structures, covering more than 1 dimension. According to the intersection theorem [19], that \( D - H_0 = D^{(m)} - H_0^{(m)} \), where \( D^{(m)} \) is the dimension of the measurements (in our case \( D^{(m)} = 1 \)), and \( H_0^{(m)} \) - corresponding measured Hausdorff dimension. It is clear from this formula that, if \( H_0 < 2 \) then \( H_0^{(m)} < 0 \). This actually means that the dimension \( H_0 < 2 \) cannot be detected in 1D measurements directly, and therefore our conclusion is inevitably indirect. Indeed, it is obtained from spectrum (2), really formed in 3D, but projected into 1D assuming isotropy. Therefore, it would be important to measure the Hausdorff dimension in 3D simulations directly. Another reason for that is the surrogacy issue [20].

These statements about the dimensions of the most intense structures can also be formulated for log-binomial distribution, and, it is known that the Poisson process is a limit of the binomial distribution for "rare events". In particular, the Poisson distribution can be obtained from the random \( \beta \)-model [17] by a suitable limiting process [5], [21]. Let \( \beta \) take two values, \( W = \beta_1 \) with probability \( x \), and \( W = \beta_2 \) with probability \( 1 - x \), and \( \beta_1 x + \beta_2 (1 - x) = 1 \) (in order to have \( \tau(1) = 0 \)). Let also \( \beta_1 \leq 1 \leq \beta_2 \). Then, on the \( n \)-th level, the distribution is binomial, that is, \( W_n = \varepsilon_n = \beta_1^n \beta_2^{-m} \) with probability \( \binom{n}{m} x^m (1 - x)^{n-m} \). Hence, \( \langle p^n \rangle = [x \beta_1^m + (1 - x) \beta_2^m] \). Taking into account that \( n = \ln(r) / \ln \Gamma \), \( \Gamma \) being the ratio of successive scales, we obtain,

\[
\tau(p) = \frac{\ln [x \beta_1^p + (1 - x) \beta_2^p]}{\ln \Gamma}.
\]

In [5] and [21], \( \Gamma \) was treated as a free parameter. It was shown that, if \( \Gamma = 1 - x/C \), \( \beta_1 = 1 - \gamma/C \) and \( x \to 0 \), then \( \beta_2 \approx 1 + x \gamma/C \), and (8) reduces to (2).

The most intense structures on \( n \)-th level, \( \beta_2^n = \frac{\Gamma}{\tau} \ln \beta_2 / \ln \Gamma = \left(\frac{\Gamma}{\tau}\right)^{-\gamma_\beta} \), cf. (3). On the other hand, the probability of these maxima,

\[
P = (1 - x)^n = \left(\frac{\Gamma}{\tau}\right)^{(1-x)/\ln \Gamma} = \left(\frac{\Gamma}{\tau}\right)^{C_\beta}.
\]

In particular, if \( \Gamma = 1 - x/2 \) and \( x \to 0 \), then \( C_\beta = 2 \) [22].

If we do not treat \( \Gamma \) as a free parameter, and consider that it is a fixed number, then the log-binomial distribution generally cannot be reduced to the log-Poisson PDF: in particular, if \( x \) is small, then, \( \tau(p) \sim x \to 0 \), and thus the intermittency is negligible. As in our case the division level \( n \gg 1 \), the log-binomial distribution becomes essentially log-normal with maximum at \( m = xn \) [11], [25]. However, the log-normal distribution has many shortcomings, and it has been repeatedly criticized when used to explain anomalous spectrum [26], [27]. Nevertheless, the spectrum (8) does not even look like log-normal (for which \( \tau(p) = -\delta(2p)(p - 1) \)) and rather behaves like log-Poisson for \( p \gg 1 \). Indeed, according to (8), for \( p \gg 1 \),

\[
\tau(p) = C_\beta + C_1 \beta^p - p \gamma_\beta, \quad C_1 = \frac{x}{(1 - x) \ln \Gamma} < 0, \quad (11)
\]

\( \beta = \beta_1/\beta_2 \). This spectrum resembles (2); and the constants in (11) happen to be numerically close to corresponding numbers in (2). The reason for such a dramatic difference with log-normal distribution is as follows. For binomial distribution,

\[
\langle p^n \rangle = \beta_2^n \sum_{m=0}^n \binom{n}{m} x^m (1 - x)^{n-m} \delta_m, \quad (12)
\]

\( \delta = \beta^p \ll 1 \) for large \( p \). Then \( \delta_m \) decreases dramatically with increasing \( m \), and therefore the terms of the sum (12) of maximal probability, at \( m \sim xn \), where normal distribution if formed, do not contribute substantially. In contrast, only the first few terms of this sum (responsible for "rare" and very intense events) really contribute. Thus, effectively, the distribution works like a Poisson distribution. To see this explicitly, consider a probability distribution \( \binom{n}{m} x^m (1 - x)^{n-m} \delta_0^m / A \), where \( A \) is a normalization constant, \( A = (x \delta_0 + 1 - x)^n \), and \( \delta_0 = \delta_1/\beta_2 \), \( p_0 \gg 1 \). Then, for large \( n \) we express the factorials entering the binomial coefficients through Stirling formula (except for \( m! \), because \( m \) is not necessarily large), to get,

\[
P_1(m) = \frac{1}{A} \binom{n}{m} x^m (1 - x)^{n-m} \delta_m \approx e^{-\xi^m / mn}, \quad (13)
\]

where \( \xi_0 = n \delta_0 x / (1 - x) \). For \( p \geq p_0 \), the sum (12) can be written as \( A \beta_2^p \sum_{m=0}^\infty P_1(m) \delta_m(\beta_{p-p_0}) \), and thus the
distribution effectively corresponds to the Poisson distribution.

Let us take the random $\beta$ model “for real”, that is, consider the Poisson distribution as an approximation to the binomial, as in (13). Then, we may consider that the model is realized as follows. Denote the number of divisions of each volume by $N$. Then $\Gamma = 1/N^{1/D}$. We now multiply the values of $m$ divisions by $\beta_1$ and multiply the remaining $N - m$ divisions by $\beta_2$. This actually means that the probability $x = m/N$, and $1 - x = (N - m)/N$.

A particular case of $N = 2$ and $D = 1$, i.e., $x = 1/2$, corresponds to the model proposed in [23]. Then, according to (10), $P = (r/\ell)^D$, i.e., the Hausdorff dimension $H_0$ is 0, while $\gamma_\beta$, defined from (9), $= D \ln \beta_2/\ln 2$. The case $\beta_1 = 0$ returns us to the $\beta$-model [24]. In that case, $\beta_2 = 1/(1 - x)$, and $\gamma_\beta = C_\beta = D \ln (1 - x)/\ln (1/2)$.

In conclusion, one of the predictions of SL theory about intensity structures geometry in fully developed turbulence. The PDF’s of the exponents of the dissipation field are compared with the log-Poisson distribution to show a good agreement with the theory. The log-Poisson statistic can be considered as a limiting case for the log-binomial distribution appearing in random $\beta$-model. We estimated the parameters of the log-binomial distribution with $\gamma$ found in our measurements, and to fit the exponents for the structure functions found elsewhere. We conclude that the estimated Hausdorff co-dimension of the most intense structures is less than 2.

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![FIG. 3. Intermittency corrections from experiment [9], and from other theoretical models.](image)

In general case, we may take $D = 3$, and we are dealing with (8) with $\gamma_\beta$ given from our measurements (so that $\beta_2$ is defined according to (9)). We thus are left with two free parameters, $N$, an integer, and $x = m/N$, where $m$ is also an integer. These numbers can be found with help of computer search to fit experimental data [8], [9] in optimal way. As a result of this search, we get: For $\gamma = 0.57$ (combined longitudinal and transversal dissipation), $N = 11$, $x = 7/11$. With these parameters, the deviation of the spectrum from experimental is 0.0098, quite satisfactory. Indeed, the corresponding $\tau(p/3)$ depicted in Fig. 3 is indistinguishable from other approximations which collapse to the experimental data. For $\gamma = 0.61$ (longitudinal dissipation), $N = 4$, $x = 2/4 = 1/2$, $C_\beta = 1.5$, the deviation is 0.0115, still okay. As mentioned, at $p \gg 1$, the log-binomial spectrum (8) is essentially reduced to the log-Poisson spectrum (2), and therefore we would prefer to consider the log-binomial distribution to be more general.

In conclusion, one of the predictions of SL theory about the scaling of maxima $\sim r^{-\gamma}$ is experimentally confirmed. This makes it possible to make a better estimate of the

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