Minimal Distortion Bending and Morphing of Compact Manifolds

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Abstract

Let $M$ and $N$ be compact smooth oriented Riemannian $n$-manifolds without boundary embedded in $\mathbb{R}^{n+1}$. Several problems about minimal distortion bending and morphing of $M$ to $N$ are posed. Cost functionals that measure distortion due to stretching or bending produced by a diffeomorphism $h: M \rightarrow N$ are defined, and new results on the existence of minima of these cost functionals are presented. In addition, the definition of a morph between two manifolds $M$ and $N$ is given, and the theory of minimal distortion morphing of compact manifolds is reviewed.

1 Introduction

Two diffeomorphic compact embedded hypersurfaces admit infinitely many diffeomorphisms between them, which we view as prescriptions for bending one hypersurface into the other. We ask which diffeomorphic bendings have minimal distortion with respect to some natural bending energy functionals. More precisely, let $M$ and $N$ be diffeomorphic compact and connected

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smooth oriented \( n \)-manifolds without boundary embedded in \( \mathbb{R}^{n+1} \). The manifolds \( M \) and \( N \) inherit Riemannian metrics \( g_M \) and \( g_N \) and corresponding volume forms from the usual metric and orientation of \( \mathbb{R}^{n+1} \). Although we will use this structure here, only the existence of the metrics on \( M \) and \( N \) is essential. We pose the problem of bending \( M \) into \( N \) via a diffeomorphism \( h : M \to N \) so that the distortion produced by \( h \) is minimal with respect to some functional that measures bending or stretching (cf. the problem of optimal development of surfaces [16, 7]).

The problem of minimal distortion bending of manifolds may be considered a special case of the problem of minimal morphing. A morph is defined to be a transformation between two shapes through a set of intermediate shapes. A minimal morph is such a transformation that minimizes distortion. There are important applications of minimal morphing in manufacturing [7, 16], computer graphics [14, 15], movie making [11], and mesh construction [5, 6]. We will formulate and solve a problem about the existence of minimal morphs with respect to stretching for \( n \)-dimensional manifolds.

2 Distortion Minimal Bending

Let \((M, g_M)\) and \((N, g_N)\) be smooth compact Riemannian \( n \)-manifolds without boundary with all the additional properties stated in the introduction. By \( \text{Vol}(M) \) we denote the volume of \( M \). Also, let \( \text{Diff}(M, N) \) denote the set of all diffeomorphisms between \( M \) and \( N \).

Our first functional measures distortion due to stretching. For a point \( p \in M \), we define the distortion due to stretching at \( p \) as the infinitesimal relative change of volume. More precisely, let \( \{A_k\}_{k=1}^\infty \subset M \) be a sequence of open neighborhoods of the point \( p \) that shrink to the point as \( k \to \infty \). For example, one can choose \( A_k = B(p, \frac{1}{k}) \cap M \), where \( B(p, R) \) is the open ball of radius \( R \) in \( \mathbb{R}^{n+1} \) centered at \( p \in M \subset \mathbb{R}^{n+1} \).

**Definition 2.1.** The distortion due to stretching produced by a diffeomorphism \( h \in \text{Diff}(M, N) \) at a point \( p \in M \) is defined to be

\[
\xi(p) = \lim_{k \to \infty} \frac{\int_{h(A_k)} \omega_N - \int_{A_k} \omega_M}{\int_{A_k} \omega_M} = |J(h)(p)| - 1,
\]

where \( J(h) \) is the Jacobian of \( h \) with respect to the (Riemannian) volume forms \( \omega_M \) and \( \omega_N \). The functional \( \Phi_1 : \text{Diff}(M, N) \to \mathbb{R}_+ \) is defined by

\[
\Phi_1(h) = \int_M \left( |J(h)| - 1 \right)^2 \omega_M.
\]
The following results are proved in [3].

**Lemma 2.2.** A diffeomorphism \( h \in \text{Diff}(M, N) \) is a critical point of \( \Phi_1 \) if and only if \( J(h)(m) = \frac{\text{Vol}(N)}{\text{Vol}(M)} \) for all \( m \in M \).

**Theorem 2.3** (Existence of minimizers for \( \Phi_1 \)). If \((M, g_M)\) and \((N, g_N)\) are diffeomorphic compact connected oriented Riemannian \(n\)-manifolds without boundary, then there exists a minimizer of the functional \( \Phi_1 \) over the class \( \text{Diff}(M, N) \) and the minimum value of \( \Phi_1 \) is

\[
\Phi_{1\text{min}} = \left( \frac{\text{Vol}(M) - \text{Vol}(N)}{\text{Vol}(M)} \right)^2.
\]

The functional \( \Phi_1 \) is invariant with respect to compositions with volume preserving maps: \( \Phi_1(h \circ k) = \Phi_1(h) \) provided that \( k \in \text{Diff}(M) \) is volume preserving (it has the Jacobian \( J(k) = 1 \)). Therefore, the minimizer of \( \Phi_1 \) is not unique.

For a vector bundle \( V \) over \( M \), we denote by \( \Gamma(V) \) the space of all sections of \( V \); \( \Gamma^r(V) \) denotes the space of all \( C^r \) sections of \( V \). Let \( T^{(0,2)}(M) \) denote the vector bundle of covariant order-two tensors over \( M \) (see [1]). In order to measure distortion with respect to bending, we introduce the strain tensor field \( S : \text{Diff}(M, N) \rightarrow \Gamma(T^{(0,2)}(M)) \) by \( S(h) = h^*g_N - g_M \), where \( h^*g_N \) is the pull-back of the metric \( g_N \) by \( h \).

**Definition 2.4.** The deformation energy functional \( \Phi_2 : \text{Diff}(M, N) \rightarrow \mathbb{R}_+ \) is given by

\[
\Phi_2(h) = \int_M \| h^*g_N - g_M \|^2 \omega_M,
\]

where the fiber norm \( \| \cdot \| \) on the bundle \( T^{(0,2)}(M) \) is induced by the fiber metric \( G := g_M^* \otimes g_M^* \) (see [12]).

Let \( \Gamma^\infty(TM) \) denote the space of \( C^\infty \) sections of the tangent bundle of \( M \). Fix a diffeomorphism \( h \in \text{Diff}(M, N) \). In order to derive the Euler-Lagrange equation, we consider all variations of the diffeomorphism \( h \) of the form \( h \circ \phi_t \), where \( \phi_t \) is the flow of a vector field \( Y \in \Gamma^\infty(TM) \). Because the tangent space \( T_h \text{Diff}(M, N) \) can be identified with the space of all sections \( \Gamma^\infty(h^{-1}TN) \) of the pull back bundle \( h^{-1}TN \) over \( M \), every smooth variation of the diffeomorphism \( h \) can be represented in the form \( h \circ \phi_t \) (see [2] for a more detailed description).

The diffeomorphism \( h \in \text{Diff}(M, N) \) is a critical point of \( \Phi_2 \) if

\[
\frac{d}{dt} \Phi_2(h \circ \phi_t)|_{t=0} = D\Phi_2(h)h_*Y = 2 \int_M G(h^*g_N - g_M, L_Y h^*g_N) = 0 \quad (2)
\]
for all $Y \in \Gamma^{\infty}(TM)$, where $h_* Y$ is the push forward of $Y$ by $h$, and $L_Y$ is the Lie derivative in the direction of $Y$ (see [1]).

Let $\nabla$ be the Riemannian connection on $M$ generated by the Riemannian metric $g_M$ with Christoffel symbols $\Gamma^k_{ij}$ (see [10]). The connection $\nabla$ with Christoffel symbols $\tilde{\Gamma}^k_{ij}$ is the Riemannian connection of the metric $h^* g_N$ on $M$.

**Definition 2.5.** Define $B(h) = (h^* g_N - g_M)$. That is, for each $p \in M$, the tensor $B(h)(p)$ of type $(2, 0)$ is defined as the tensor $(h^* g_N - g_M)(p)$ with its indices raised.

**Definition 2.6.** Define the bilinear form $A(h)$ to be

$$A(h)(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$$

for $X, Y \in \Gamma^{\infty}(TM)$ (see [12]).

It is easy to prove that $A(h)$ is a tensor field of type $(1, 2)$ on $M$ with components

$$A(h)^m_{kp} = \tilde{\Gamma}^m_{kp} - \Gamma^m_{kp}.$$

**Lemma 2.7.** The first variation of the functional $\Phi_2$ in the direction $Y \in \Gamma^{\infty}(TM)$ is given by

$$D\Phi_2(h)(h_* Y) = -4 \int_M g_M(\text{div} B(h) + A(h) : B(h), Y) \omega_M,$$

where $B(h) : A(h)$ is the contraction of the tensor fields $B(h)$ and $A(h)$ (see [8]). Moreover, $h$ is a critical point of the functional $\Phi_2$ if and only if

$$\text{div} B(h) + A(h) : B(h) = 0.$$  

The Euler-Lagrange equation for the functional $\Phi_2$ is the system of nonlinear partial differential equations [5].

Let $h_R : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be the radial map given by $h_R(x) = Rx$ for some number $R > 0$ and for all $x \in \mathbb{R}^{n+1}$. It is easy to check that if $N = RM$ is a rescaling of the manifold $M$, then the map $h = h_R \circ f$ satisfies the Euler Lagrange equation [5], whenever $f \in \text{Diff}(M)$ is an isometry on $M$.

The following results on minimizing $\Phi_2$ in the one-dimensional case are proved in [2].

**Proposition 2.8.** (i) Suppose that $M$ and $N$ are smooth simple closed curves in $\mathbb{R}^2$ with arc lengths $L(M)$ and $L(N)$ and base points $p \in M$ and
\( q \in N; \gamma \) and \( \xi \) are the arc length parametrizations of \( M \) and \( N \) with \( \gamma(0) = p \) and \( \xi(0) = q \) that induce positive orientations; and, the functions \( v \) and \( w \) are defined by \( v(t) = L(N)/L(M)t \) and \( w(t) = -L(N)/L(M)t + L(N) \) for all \( t \in [0, L(M)] \). If \( L(N) \geq L(M) \), then the functional \( \Phi_2 \) has exactly two minimizers in the admissible set

\[
\mathcal{A} = \{ h \in \text{Diff}(M, N) : h(p) = q \} :
\]

the orientation preserving minimizer

\[
h_1 = \xi \circ v \circ \gamma^{-1}
\]

and the orientation reversing minimizer

\[
h_2 = \xi \circ w \circ \gamma^{-1}
\]

(\text{where we consider } \gamma \text{ as a function defined on } [0, L(M)] \text{ so that } \gamma^{-1}(p) = 0). \]

Moreover, the minimum value of the functional \( \Phi_2 \) is

\[
\Phi_2^{\text{min}} = \frac{(L(N)^2 - L(M)^2)^2}{L(M)^3}.
\]

**Proposition 2.9.** Assume the notation of the proposition 2.8.

(i) If \( L(N) < L(M) \), then the functional \( \Phi_2 \) has no minimum in the admissible set

\[
\mathcal{Q} = \{ h \in C^\infty(M, N) : h \text{ is orientation preserving and } h(p) = q \}.
\]

(ii) If \( \frac{L(N)}{L(M)} < \frac{1}{\sqrt{3}} \), then the functional \( \Phi_2 \) has no minimum in the admissible set \( \mathcal{A} = \{ h \in \text{Diff}(M, N) : h(p) = q \} \).

The main ingredients for a proof of (i) in proposition 2.9 are simply illustrated. The curve \( M \) is wrapped around the curve \( N \) without stretching and the excess is removed. This wrapping function can be expressed in the form \( h = \xi \circ u \circ \gamma^{-1} \) (in the notation of proposition 2.8), where \( u : [0, L(M)] \to \mathbb{R} \) is a discontinuous piecewise linear function. The function \( h \) is not smooth; but, it is possible to approximate it by a minimizing sequence \( \{ h^k \}_{k=1}^\infty \subset \mathcal{Q} \) whose deformation energies \( \Phi_2(h^k) \) converge to \( \Phi_2(h) = 0 \). On the other hand, \( \Phi_2(f) > 0 \) for all \( f \in \mathcal{Q} \). The proof of (ii) uses the second variation of \( \Phi_2 \).

By proposition 2.9, we see that even in the one-dimensional case the functional \( \Phi_2 \) exhibits nontrivial behavior: the minimum does not always exist, and the existence depends on properties of the curves \( M \) and \( N \).
The general problem of the existence of minimizers for the functional $\Phi_2$ is open. On the other hand, we have solved the problem for the case where $M$ and $N$ are Riemann spheres or compact Riemann surfaces of genus greater than one. Let $H(M, N) = \{ h \in \text{Diff}(M, N) : h \text{ is a holomorphic map} \}$.

**Theorem 2.10.** (i) Let $h_R : \mathbb{R}^3 \to \mathbb{R}^3$ be the radial map given by $h_R(p) = R p$ for some number $R > 0$. If $M = S^2 \subset \mathbb{R}^3$ and $N = h_R(M)$, then $h := f \circ h_R$ is a global minimum of the functional $\Phi_2$, restricted to the admissible set $H(M, N)$, whenever $f$ is an isometry of $N$.

(ii) Let $M$ and $N$ be compact Riemann surfaces. If $H(M, N)$ is not empty and the genus of $M$ is at least two, then there exists a minimizer of the functional $\Phi_2$ in $H(M, N)$.

The general problem of minimization of the functional $\Phi_2$ seems to be very difficult because the admissible set is an infinite-dimensional manifold $\text{Diff}(M, N)$ whose structure is not completely understood. In theorem 2.10, the admissible set is a finite-dimensional homogeneous space in case $M$ and $N$ are two-spheres and a finite group in case $M$ is a compact Riemann manifold of genus greater than one. A natural idea is to reformulate the problem of minimal distortion bending in such a way that the admissible set is a linear space.

Fix a diffeomorphism $f \in \text{Diff}(M, N)$. Every diffeomorphism $h : M \to N$ can be represented in the form $h = f \circ \phi$, where $\phi \in \text{Diff}(M)$. For simplicity of notation, let $g_2 := f^* g_N$ and $g_1 := g_M$. To measure the deformation produced by $h = f \circ \phi \in \text{Diff}(M, N)$, we use the strain tensor field $S(\phi) = \phi^* g_2 - g_1$. In other words, the problem reduces to minimization of the deformation energy produced by some class of diffeomorphisms $\phi : (M, g_1) \to (M, g_2)$ in $\text{Diff}(M)$.

The tangent bundle $TM$ is equipped with the Riemannian metric $g_1$. Let $W^{k,2}(TM)$ be the $(k, 2)$-Sobolev space of sections of the tangent bundle $TM$ (see [13]). We choose the number of generalized derivatives $k \in \mathbb{N}$ large enough so that the Sobolev space $W^{k,2}(TM)$ is embedded into the space $C^2(TM)$ of all $C^2$ sections of $TM$ and some additional estimates hold. Consider the space $H = L^2([0, 1]; W^{k,2}(TM))$ of time dependent vector fields $v : M \times [0, 1] \to \Gamma(TM)$. The space $H$ is a Hilbert space equipped with the norm

$$\langle v, w \rangle_H = \int_0^1 \langle v(\cdot, t), w(\cdot, t) \rangle_{W^{k,2}(TM)} dt.$$ 

Every vector field $v \in H$ generates a diffeomorphism on $M$ in the fol-
following sense. The nonautonomous ordinary differential equation

$$\frac{dq}{dt} = v(q, t),$$  \hspace{1cm} (7)

is solved (on the compact manifold $M$) by an evolution operator $\eta^v(t; s, p)$ that satisfies the Chapman-Kolmogorov conditions (see [4]) and is such that $\eta^v(s; s, p) = p$ for every $p \in M$. The function $\phi^v : M \to M$ given by $\phi^v(p) = \eta^v(1; 0, p)$ is called the time-one map of the evolution operator $\eta^v$; it is a diffeomorphism on the manifold $M$.

We define the distortion energy functional $E : H \to \mathbb{R}_+$ to be

$$E(v) = \|v\|_H^2 + \int_M \|(\phi^v)^*g_2 - g_1\|^2 \omega_M + \int_M \|(\phi^v)^*\Pi_2 - \Pi_1\|^2 \omega_M, \hspace{1cm} (8)$$

where $\Pi_i$ is the second fundamental form on $M$ associated with $g_i$, $i = 1, 2$ (see [10]). This functional incorporates strain (which is intrinsic to the manifold $M$) and bending (which is extrinsic).

**Theorem 2.11** (Existence of minimizers for $E$). There exists a minimum of the functional $E$ in the space $H$.

The proof uses the direct method of the calculus of variations as well as convergence properties of the evolution operators $\eta^v(t; s, x)$ generated by weakly convergent sequences $\{v^l\}_{l=1}^\infty$ of time dependent vector fields in $H$.

### 3 Distortion Minimal Morphing

**Definition 3.1.** Let $M$ and $N$ be compact connected oriented $n$-dimensional smooth manifolds without boundary embedded in $\mathbb{R}^{n+1}$. A $C^1$ function $F : [0, 1] \times M \to \mathbb{R}^{n+1}$ is a morph from $M$ to $N$ if the following conditions hold:

(i) $p \mapsto F(t, p)$ is a diffeomorphism onto its image for each $t \in I = [0, 1]$;

(ii) the image $M^t = F(t, M)$ is an $n$-dimensional manifold possessing all the properties of $M$ and $N$ mentioned above;

(iii) $p \mapsto F(0, p)$ is a diffeomorphism of $M$;

(iv) the image of the map $p \mapsto F(1, p)$ is $N$.

We denote the set of all $C^2$ morphs between the manifolds $M$ and $N$ by $\mathcal{M}(M, N)$. 

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For simplicity, we will only consider morphs $F$ such that $p \mapsto F(0, p)$ is the identity map. We assume that each manifold $M^t = F(t, M)$ (with $M^0 = M$ and $M^1 = N$) is equipped with the volume form $\omega_t = i_{\eta_t} \Omega$, where $\Omega = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}$ is the standard volume form on $\mathbb{R}^{n+1}$ and $\eta_t : M^t \to \mathbb{R}^{n+1}$ is the outer unit normal vector field on $M^t$ with respect to the usual metric on $\mathbb{R}^{n+1}$. Also, as a convenient notation, we use $f^t = F(t, \cdot) : M \to M^t$.

**Definition 3.2.** The functional $\Phi_{s,t}^{s,t} : \text{Diff}(M^s, M^t) \to \mathbb{R}_+$ is defined by formula (1), were $M$ and $N$ are replaced with $M^s$ and $M^t$ respectively. A morph $F$ is distortion pairwise minimal (or, for brevity, pairwise minimal) if $f_{s,t}^t = f_{s,t}^t \circ (f_{s,t}^s)^{-1} : M^s \to M^t$ minimizes the functional $\Phi_{s,t}^{s,t}$ for every $s, t \in [0, 1]$. We denote the set of all $C^2$ distortion pairwise minimal morphs between manifolds $M$ and $N$ by $\mathcal{P}M(M, N)$.

Using lemma 2.2, it is easy to derive a necessary and sufficient condition for pairwise minimality.

**Proposition 3.3.** Let $M = M^0$ and $N = M^1$ be $n$-dimensional manifolds as in definition 3.1 equipped with the (respective) volume forms $\omega_0$ and $\omega_1$. A morph $F$ between $M$ and $N$ is distortion pairwise minimal if and only if

$$J(f_{s,t}^t)(m) \frac{\text{Vol}(M^t)}{\text{Vol}(M)} = 1$$

for all $t \in [0, 1]$ and $m \in M$, where $J(f_{s,t}^t)$ is the Jacobian of $f_{s,t}^t$ with respect to the volume forms $\omega_0$ and $\omega_1$.

The following proposition states the existence of pairwise minimal morphs. It can be proved by rescaling morphs between $M$ and $N$, which are not necessarily pairwise minimal, to conform to property (9). Moser’s theorem on volume forms (see [9]) plays a crucial role in the proof.

**Proposition 3.4.** Let $M$ and $N$ be $n$-dimensional manifolds as in definition 3.1. If $M$ and $N$ are connected by a $C^2$ morph, then there is a distortion pairwise minimal morph between them.

Having the preliminary study of pairwise minimal morphs at hand, we define minimal morphs.

**Definition 3.5.** The infinitesimal distortion of a $C^2$ morph $F$ from $M$ to $N$ at $t \in [0, 1]$ is

$$\varepsilon_F^t = \lim_{s \to t} \frac{E_{s,t}^t}{(s-t)^2} = \int_M \frac{(dJ(f_{s,t}^t))^2}{J(f_{s,t}^t)} \omega_M,$$
where $E^{s,t} = \Phi^{s,t}(f^{s,t})$ is the distortion energy of the transition map $f^{s,t}$. The total distortion functional $\Psi$ defined on such morphs is given by

$$\Psi(F) = \int_0^1 \varepsilon F(t) \, dt = \int_0^1 \left( \int_M \frac{d}{dt} J(f^t) \right)^2 J(f^t) \omega_M \, dt. \quad (10)$$

The following proposition implies that it suffices to minimize the functional $\Psi$ over the class $\mathcal{P}M(M, N)$ of all pairwise minimal morphs instead of the class $\mathcal{M}(M, N)$ of all $C^2$ morphs.

**Proposition 3.6.** (i) The following inequality holds:

$$\inf_{G \in \mathcal{P}M(M, N)} \Psi(G) \leq \inf_{P \in \mathcal{M}(M, N)} \Psi(P). \quad (11)$$

(ii) If there exists a minimum $F$ of the total distortion functional $\Psi$ over the class $\mathcal{P}M(M, N)$, then $F$ minimizes the functional $\Psi$ over the class $\mathcal{M}(M, N)$ as well; in fact,

$$\Psi(F) = \inf_{G \in \mathcal{P}M(M, N)} \Psi(G) = \inf_{P \in \mathcal{M}(M, N)} \Psi(P). \quad (12)$$

Using proposition 3.3, it is easy to recast the functional $\Psi$ into a simpler form:

**Lemma 3.7.** The total distortion of a $C^2$ pairwise minimal morph $F$ from $M$ to $N$ is

$$\Psi(F) = \int_0^1 \frac{\left( \frac{d}{dt} \text{Vol}(M^t) \right)^2}{\text{Vol}(M^t)} \, dt. \quad (13)$$

The latter form of the functional $\Psi$ and proposition 3.6 allow us to solve the problem of minimization of the total distortion functional $\Psi$ over the class of all $C^2$ morphs $\mathcal{M}(M, N)$. In order to solve the problem, we minimize the auxiliary functional

$$\Xi(\phi) = \int_0^1 \frac{\dot{\phi}^2}{\phi} \, dt \quad (14)$$

over the admissible set

$$Q = \{ \phi \in C^1([0,1]; \mathbb{R}_+) : \phi(0) = \text{Vol}(M), \phi(1) = \text{Vol}(N) \}.$$ 

The following theorem—our main result on distortion minimal morphing—is proved using proposition 3.6 and lemma 3.7.

**Theorem 3.8.** Let $M$ and $N$ be two $n$-dimensional manifolds satisfying the assumptions of definition 3.1. If $M$ and $N$ are connected by a $C^2$ morph, then they are connected by a minimal morph. The minimal value of $\Psi$ is

$$\min_{F \in \mathcal{M}(M, N)} \Psi(F) = 4 \left( \sqrt{\text{Vol}(N)} - \sqrt{\text{Vol}(M)} \right)^2. \quad (15)$$
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