AN INDUCED MAP BETWEEN RATIONALIZED CLASSIFYING SPACES FOR FIBRATIONS

TOSHIHIRO YAMAGUCHI

Abstract. Let $Baut_1 X$ be the Dold-Lashof classifying space of orientable fibrations with fiber $X$. We give a condition that there exists a map $(Baut_1 X)_0 \to (Baut_1 Y)_0$, where $S_0$ is the rationalization of a space $S$. Then we observe the map between the differential graded Lie algebras of derivations of Sullivan models for a rationally weakly trivial map $f : X \to Y$. Furthermore we consider some conditions that the maps admit sections.

1. Introduction

Let $X$ (and also $Y$) be a connected and simply connected CW complex with $\dim \pi_\ast(X)_Q < \infty$ ($G_Q = G \otimes Q$) and $Baut X$ be the Dold-Lashof classifying space of orientable fibrations [2]. Here $\text{aut}_1 X = \text{map}(X, X; \text{id}_X)$ is the identity component of the space $\text{aut} X$ of self-equivalences of $X$. Then any orientable fibration $\xi$ with fibre $X$ over a base space $B$ is the pull-back of a universal fibration $X \to E_\infty \to Baut X$ by a map $h : B \to Baut X$ and equivalence classes of $\xi$ are classified by their homotopy classes [2], [17], [1]. The Sullivan minimal model $M(X)$ ([18]) determines the rational homotopy type of $X$. The differential graded Lie algebra (DGL) $\text{Der} M(X)$, the negative derivations of $M(X)$ (see §2), gives rise to a Quillen model for $Baut X$ due to Sullivan [18] (cf. [19], [5]), i.e., the spatial realization $\text{||} \text{Der} M(X) \text{||}$ is $(Baut X)_0$. Therefore we obtain a map $(Baut_1 X)_0 \to (Baut_1 Y)_0$ if there is a DGL-map $\text{Der} M(X) \to \text{Der} M(Y)$. However it does not exist in general.

Let $f : X \to Y$ be a map whose homotopy fibration $\xi_f : F_f \to X \to Y$ is given by the relative model (Koszul-Sullivan extension)

$$M(Y) = (\Lambda V, d) \to (\Lambda V \otimes \Lambda W, D) \simeq M(X)$$

for a certain differential $D$ with $D|_{\Lambda V} = d$, where $M(F_f) \simeq (\Lambda W, \overline{D})$ for the homotopy fiber $F_f$ of $f$ [4]. In this paper, we say a map is rationally weakly trivial (abbr., Q-w.t.) if $\xi_f$ is rationally weakly trivial; i.e., $\pi_\ast(X)_Q = \pi_\ast(F_f)_Q \oplus \pi_\ast(Y)_Q$. Then $(\Lambda V \otimes \Lambda W, \overline{D})$ is the minimal model $M(X)$ of $X$.

Definition 1.1. We say that a Q-w.t. map $f : X \to Y$ strictly induces the map

$$a_f : (Baut_1 X)_0 \to (Baut_1 Y)_0$$

if there exists the DGL-map

$$b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \to \text{Der}(\Lambda V, d)$$

given by $b_f(\sigma) = \text{proj}_V \circ \sigma$ with $||b_f|| = a_f$. Here $\text{proj}_V : \Lambda V \otimes \Lambda W \to \Lambda V$ is the algebra map with $\text{proj}_V(w) = 0$ for $w \in W$ and $\text{proj}_V|_{\Lambda V} = \text{id}_{\Lambda V}$.
Notice that there is an isomorphism $\phi : \text{Der}(\Lambda V \otimes \Lambda W; D_1) \cong \text{Der}(\Lambda V \otimes \Lambda W; D_2)$ induced by a suitable basis-change $(\Lambda V \otimes \Lambda W; D_1) \cong (\Lambda V \otimes \Lambda W; D_2)$ if $\xi_{f_1}$ and $\xi_{f_2}$ are equivalent \cite{11} for two maps $f_1, f_2 : X \to Y$ and then $b_{f_1} \circ \phi = b_{f_2}$.

Let $\min \pi_i(S)_Q := \min \{i > 0 \mid \pi_i(S)_Q \neq 0\}$ and $\max \pi_i(S)_Q := \max \{i \geq 0 \mid \pi_i(S)_Q \neq 0\}$ for a space $S$. In particular, $\min \pi_i(S)_Q := \infty$ when $S$ is a one-point.

**Definition 1.2.** A fibration $\xi_f : F_f \to X \xrightarrow{f} Y$ or a map $f : X \to Y$ with homotopy fiber $F_f$ is said to be $\pi_Q$-separable if $\min \pi_i(F_f)_Q \geq \max \pi_i(Y)_Q$.

If a map $f : X \to Y$ is $\pi_Q$-separable, it is $\mathbb{Q}$-w.t. and the $\pi_Q$-separable condition is equivalent to that $\min W = \min \{i > 0 \mid W^i \neq 0\} \geq \max V = \max \{i > 0 \mid V^1 \neq 0\}$ in the relative minimal model $M(Y) = (\Lambda V, d) \to (\Lambda V \otimes \Lambda W, D)$ of $\xi_f$.

Recall a question related to Gottlieb\cite{6, §5]: Which map $f : X \to Y$ can be extended to a map between fibrations over a fixed base space $B$, that is, for any fibrations $\xi : X \to E \to B$, does there exist a fibration $\eta : Y \to E' \to B$ and a map $f' : E \to E'$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & E \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xrightarrow{\eta} & E'
\end{array}
$$

homotopically commutes? \cite[Example 3.8]{21}. If a map $f : X \to Y$ is $\pi_Q$-separable, due to Sullivan minimal model theory, it is obvious that there are a fibration $\eta$ after rationalization and a map $f' : E_0 \to E'_0$ as above. In particular, let $f : X \to X(n)$ be the rationalized Postnikov $n$-stage map of $X$, where $\pi_{>n}(X(n)) = 0$. Then $\eta$ is rationally fibre-trivial, i.e.,

$$
\begin{array}{ccc}
X & \xrightarrow{f} & E \\
\downarrow{f} & & \downarrow{f'} \\
X(n) & \xrightarrow{\psi_f} & X(n) \times B_0
\end{array}
$$

homotopically commutes when the classifying map $B \to (B\text{aut}_1 X)_0 \xrightarrow{a_f} (B\text{aut}_1 Y)_0$ is homotopic to the constant map for a sufficiently small $n$.

**Proposition 1.3.** A $\mathbb{Q}$-w.t.map $f : X \to Y$ strictly induces $a_f : (B\text{aut}_1 X)_0 \to (B\text{aut}_1 Y)_0$ if and only if $f$ is $\pi_Q$-separable.

Let $f : X \to Y$ be a map with a section $s$, i.e., there is a map $s : Y \to X$ with $f \circ s \simeq i_Y$.

Then there is a map $\psi_f : \text{aut}_1 X \to \text{aut}_1 Y$ with $\psi_f(g) := f \circ g \circ s$ for $g \in \text{aut}_1 X$.

In general, this does not preserve the monoid structures.

**Theorem 1.4.** If a $\pi_Q$-separable map $f$ admits a section, $\Omega a_f \simeq (\psi_f)_0$.

In §2, we give the proofs under some preparations of models of \cite{11} and \cite{19}.

In this paper, we consider only $\mathbb{Q}$-w.t.maps. For example, we don’t consider the inclusion map $i_X : X \to X \times Y$, which is not $\mathbb{Q}$-w.t. However $i_X$ induces the monoid map $\psi : \text{aut}_1 X \to \text{aut}_1 (X \times Y)$ by $\psi(g) = g \times 1_Y$ and therefore there exists the induced map $B\psi : B\text{aut}_1 X \to B\text{aut}_1 (X \times Y)$ without rationalization. The DGL model is given by the natural inclusion $\text{Der} M(X) \to \text{Der} (M(X) \otimes M(Y))$, which is a DGL-map.

Let Sep$_\mathbb{Q}$ be the category that the objects are simply connected CW-complexes of finite dimensional rational homotopy groups and morphisms are $\mathbb{Q}$-separable maps.
When \( f : X \to Y \) and \( g : Y \to Z \) are \( \pi_Q \)-separable maps, \( g \circ f : X \to Z \) is also a \( \pi_Q \)-separable map. Then

\[
a_{g \circ f} = a_g \circ a_f : (\text{Baut}_1 X)_0 \to (\text{Baut}_1 Z)_0
\]

by our construction. In particular \( f = \text{id}_X : X \to X \) is \( \pi_Q \)-separable and then \( a_f \) is of course the identity map of \((\text{Baut}_1 X)_0 \). Thus we obtain

**Theorem 1.5.** \((\text{Baut}_1 (\_))_0 \) is a functor from \( \text{Sep}_Q \) to \( \text{ho}(\mathbb{Q}-\text{CW}_1) \).

Here \( \text{ho}(\mathbb{Q}-\text{CW}_1) \) is the homotopy category of rational simply connected CW-complexes of finite dimensional rational homotopy groups. This functor is not essentially surjective on objects, i.e., there are rational spaces that cannot be realized as \((\text{Baut}_1 X)_0 \) for any \( X \) due to Lupton-Smith [11] (16). Now we propose

**Question 1.6.** Which map \((\text{Baut}_1 X)_0 \to (\text{Baut}_1 Y)_0 \) can be strictly induced by a \( \pi_Q \)-separable map \( f : X \to Y \) ?

We can easily find in Example 2.5 there exists a map that cannot be realized as \( a_f \) for any \( \pi_Q \)-separable map \( f \). Furthermore we consider a (rational) homotopy property of the map \( a_f \):

**Question 1.7.** When does (is) the strictly induced map \( a_f : (\text{Baut}_1 X)_0 \to (\text{Baut}_1 Y)_0 \) admit a section (fibre-trivial as a fibration) ? Conversely, when is it homotopic to the constant map ?

In §3, we give some such conditions. The essential proofs of them are induced from Proposition 3.1 that the DGL-model of the homotopy fibration \( F_{a_f} : (\text{Baut}_1 X)_0 \to (\text{Baut}_1 Y)_0 \) is given by

\[
\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \to \text{Der}(\Lambda V \otimes \Lambda W) \xrightarrow{\text{ho}} \text{Der}(\Lambda V).
\]

In particular we note in Theorems 3.10 and 3.11 some relations with a famous Halperin’s conjecture for fibrations [1] §39 due to Meier [12].

In §4, we observe the obstruction for the lifting \( h \) for a map \( h : B \to (\text{Baut}_1 Y)_0 : \)

\[
\begin{array}{ccc}
B & \xrightarrow{h} & (\text{Baut}_1 Y)_0 \\
\uparrow & \text{a_f} & \\
(\text{Baut}_1 X)_0 & \downarrow & \end{array}
\]

2. SULLIVAN MODELS, DERIVATIONS AND QUILLEN MODELS

Let \( M(X) = (\Lambda V, d) \) be the Sullivan minimal model of simply connected CW complex \( X \) of finite type [18]. It is a free \( \mathbb{Q} \)-commutative differential graded algebra (DGA) with a \( \mathbb{Q} \)-graded vector space \( V = \bigoplus_{i \geq 1} V^i \) where \( \dim V^i < \infty \) and a decomposable differential, i.e., \( d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1} \) and \( d \circ d = 0 \). Here \( \Lambda^+ V \) is the ideal of \( \Lambda V \) generated by elements of positive degree. The degree of a homogeneous element \( x \) of a graded algebra is denoted as \( |x| \). Then \( xy = (-1)^{|x||y|} y x \) and \( d(xy) = d(x)y + (-1)^{|x|} x d(y) \). Note that \( M(X) \) determines the rational homotopy type of \( X \), namely the spatial realization is given as \( \|M(X)\| \simeq X_0 \). In particular,

\[
V^n \cong \text{Hom}(\pi_n(X), \mathbb{Q}) \quad \text{and} \quad H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q}).
\]
Here the second is an isomorphism as graded algebras. Refer to [4] for detail.

Let $\text{Der}_i M(X)$ be the set of $\mathbb{Q}$-derivations of $M(X)$ decreasing the degree by $i$ with $\sigma(xy) = \sigma(x)y + (-1)^i x\sigma(y)$ for $x, y \in M(X)$. The boundary operator $\partial : \text{Der}_i M(X) \to \text{Der}_{i-1} M(X)$ is defined by

$$\partial(\sigma) = d \circ \sigma - (-1)^i \sigma \circ d$$

for $\sigma \in \text{Der}_i M(X)$. We denote $\oplus_{i \geq 0} \text{Der}_i M(X)$ by $\text{Der} M(X)$ in which $\text{Der}_1 M(X)$ is $\partial$-cycles. Then $\text{Der} M(X)$ is a (non-free) DGL by the Lie bracket

$$[\sigma, \tau] = \sigma \circ \tau - (-1)^{||\sigma||} \tau \circ \sigma.$$

Note that $H_*(\text{Der} M) = H_*(\text{Der} N)$ when free DGAs $M$ and $N$ are quasi-isomorphic [14]. Furthermore, recall the definition of D.Tanrê [19, p.25]: Let $(L, \partial)$ be a DGL of finite type with positive degree. Then $\text{C}^*(L, \partial) = (\mathbb{A}^{-\infty}_L L, D = d_1 + d_2)$ with

$$\langle d_1 s^{-1} z; sx \rangle = -\langle z; \partial x \rangle \text{ and } \langle d_2 s^{-1} z; sx_1, sx_2 \rangle = (-1)^{|x_1|} \langle z; [x_1, x_2] \rangle,$$

where $\langle s^{-1} z; sx \rangle = (-1)^{|x|} \langle z; x \rangle$ and $\mathbb{A}_L$ is the dual space of $L$.

**Theorem 2.1.** [18, §11],[19,5] For a Sullivan model $M(X)$ of $X$, $\text{Der} M(X)$ is a DGL-model of $\text{Baut}_1 X$. In particular, there is an isomorphism of graded Lie algebras $H_*(\text{Der} M(X)) \cong \pi_*(\Omega \text{Baut}_1 X)_\mathbb{Q}$ where the right-hand has the Samelson bracket. Furthermore $\text{C}^*(\text{Der} M(X))$ is a DGA-model of $\text{Baut}_1 X$.

Let $L(X) = (LU, \partial)$ be the Quillen model of $X$ [19 III.3], [4 §24]. It is a free $\mathbb{Q}$-commutative differential graded Lie algebra (DGL) with a $\mathbb{Q}$-graded vector space $U = \bigoplus_{i \geq 1} U_i$ where dim $U_i < \infty$ and $\partial(U_i) \subset (LU)_{i-1}$, which is the space of elements of $LU$ with degree $i - 1$. Note that $[x, y, z] = (-1)^{|x||y|} [y, x, z]$ and Jacobi identity:

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|} [y, [x, z]]$$

for $x, y, z \in LU$ and Leibniz rule:

$$\partial[x, y] = [\partial x, y] + (-1)^{|x|} [x, \partial y].$$

Note that $L(X)$ determines the rational homotopy type of $X$, namely $||L(X)|| \simeq X_0$. In particular, there are isomorphisms

$$\tilde{H}_n(X; \mathbb{Q}) \cong H_{n-1}(U, \partial_1) \text{ and } \pi_*(\Omega X)_\mathbb{Q} \cong H_*(LU, \partial),$$

where $\partial_1 : U \to U$ is the linear part of $\partial$. Here the second is an isomorphism as graded Lie algebras. Refer to [4] for detail.

**Convention.** For a DGA-model $(\Lambda V, d)$ the symbol $(v, f)$ means the *elementary derivation* that takes a generator $v$ of $V$ to an element $f$ of $\Lambda V$ and the other generators to 0. Note that $||(v, f)|| = |v| - |f|$.

**Proof of Proposition 1.2.** Let $M(Y) = (\Lambda V, d) \to (\Lambda V \otimes \Lambda W, D)$ be the model of $f$.

(i) When $\min W \geq \max V$, there is a decomposition of vector spaces

$$\text{Der}(\Lambda V \otimes \Lambda W) = \text{Der}(\Lambda V) \oplus \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$$

from degree arguments. Then there is a DGL-map $b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \to \text{Der}(\Lambda V, D)$ by $b_f(\sigma_1) = \sigma_1$ and $b_f(\sigma_2) = 0$ for $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1 \in \text{Der}(\Lambda V)$ and $\sigma_2 \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$. In particular, it preserves the differential since $b_f(\tau(\sigma_1)) = 0$ when $\partial_X(\sigma_1) = \partial_Y(\sigma_1) + \tau(\sigma_1)$ for $\tau(\sigma_1) \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$. 
(only if) Suppose that \( \min W < \max V \). There are elements \( w \in W \) and \( v \in V \) with \( |w| < |v| \). Then \( b_f \) is not a DGL-map since
\[ 0 \neq b_f(v, 1) = b_f([(w, 1), (v, w)]) = [b_f(w, 1), b_f(v, w)] = [0, 0] = 0 \]
from the definition of \( b_f \).

**Example 2.2.** Let \( f : S^{2n} \to K(\mathbb{Z}, 2n) \) be the natural inclusion. Then the homotopy fibre is \( S^{4n-1} \) and therefore \( f \) is \( \pi_Q \)-separable. Thus \( b_f : \text{Der}(\Lambda(x, y), D) \to \text{Der}(\Lambda(x, 0)) \) with \( |x| = 2n, |y| = 4n - 1, Dx = 0 \) and \( Dy = x^2 \) is given by \( b_f((x, 1)) = (x, 1) \) and \( b_f((y, 1)) = b_f((y, x)) = 0 \). Refer Theorem 3.10

**Example 2.3.** Consider the case that \( f \) is not \( \pi_Q \)-separable (not \( \mathbb{Q} \)-w.t.). Let \( f : S^7 \to S^4 \) be the Hopf map. Then the model is given by
\[ M(S^4) = (\Lambda(x, y), d) \to (\Lambda(x, y, z), D) \cong M(S^7) \]
with \( |x| = 4, |y| = 7, |z| = 3, dx = Dy = x^2, Dz = x, Dy = x^2 \). Then the bases of derivations are given as
\[
\begin{array}{c|c}
 n & \text{Der}_n(\Lambda(x, y, z), D) \\
7 & (y, 1) \\
4 & (x, 1) (y, z) \\
3 & (y, x) (z, 1) \\
1 & (x, z) \\
\end{array}
\text{ and } H_*(\text{Der}(\Lambda(x, y))) = \mathbb{Q}\{\{y, 1\}\}. \text{ By degree reason, any DGL-map }
\psi : (\text{Der}(\Lambda(x, y, z), D) \to (\text{Der}(\Lambda(x, y), d))
\]
is given by \( \psi(y, 1) = a_1(y, 1), \psi(x, 1) = a_2(x, 1), \psi(y, z) = a_3(x, 1), \psi(y, x) = a_4(y, x), \psi(z, 1) = a_5(y, x) \) and \( \psi(x, z) = 0 \) for some \( a_i \in \mathbb{Q} \).

From \( (x, 1) = [(z, 1), (x, z)] \) and \( (y, z) = [(z, 1), (y, x)] \) we have \( a_2 = 0 \) and \( a_3 = 0 \), respectively. Then from \( 2(y, 1) = [(z, 1), (y, z)] + [(x, 1), (y, x)] \), we obtain \( a_1 = 0 \). Thus \( ||\psi|| \) is homotopic to the constant map.

**Proof of Theorem 1.7** The map \( \pi_n(\psi_f) : \pi_n(\text{aut}_1 X) \to \pi_n(\text{aut}_1 Y) \) is given by \( \pi_n(\psi_f)([\sigma]) = [\tau] := [f \circ \sigma \circ (s \times 1_{S^n})] \) in the following homotopy commutative diagram:

\[
\begin{array}{c}
X \times S^n & \overset{s \times \text{id}_{S^n}}{\longrightarrow} & Y \times S^n \\
\downarrow \sigma & & \downarrow i_Y \\
X & \overset{f \circ \sigma \circ (s \times 1_{S^n})}{\longrightarrow} & Y \\
\end{array}
\]

from adjointness. That is the pointed homotopy classes of maps \( S^n \to \text{aut}_1 X = \text{map}(X, X; \text{id}_X) \) are in bijection with the homotopy classes of those maps \( X \times S^n \to X \) that composed with the inclusion \( i_X : X \to X \times S^n \) yield the identity \([\text{Id}_X] \). Let \( M(Y) = (\text{Av}, d) \to (\text{Av} \otimes \text{Av}, D) \) be the model of \( f \). There is a chain map
\[ c_f : \text{Der}(\text{Av} \otimes \text{Av}, D) \to \text{Der}(\text{Av}, d) \]

\[ 5 \]
given by \( cf(\sigma) = \text{proj}_V \circ \sigma \). It is well-defined, i.e., \( \partial_Y \circ cf = cf \circ \partial_X \), from \( DW \subset AV \otimes \Lambda^+ W \) since it admits a section. Notice that \( \pi_n(\psi f)_Q(\sigma_0) G \) group map induces the map between their Gottlieb groups \([7] \subset DW \)

The DGL-map

\[
\pi_n(\psi f)_Q : \pi_n(\text{aut}_1 X)_Q \cong \pi_n(\Omega \text{Baut}_1 X)_Q \xrightarrow{H_n(b_f)} \pi_n(\Omega \text{Baut}_1 Y)_Q \cong \pi_n(\text{aut}_1 Y)_Q
\]

for any \( n \). Thus we obtain \( \Omega a_f \cong (\psi f)_0 \).

Claim 2.4. For any \( \pi_Q \)-separable map \( f : X \to Y \), we have \( b_f(C) = 0 \) and \( b_f |_{\text{Der}(AV)} = \text{id}_{\text{Der}(AV)} \) for \( \text{Der}(AV \otimes AW) = C \oplus \text{Der}(AV) \).

Example 2.5. Let \( X = K(Q, n) \times K(Q, 2n) \) and \( Y = K(Q, n) \). Then \( M(X) = \Lambda(x, y), 0 \) and \( M(Y) = \Lambda(z), 0 \) with \( |x| = |z| = n \) and \( |y| = 2n \). The DGL-map \( \psi : \text{Der}\Lambda(x, y) \to \text{Der}\Lambda(z) \) such that \( \psi(y, x) = \psi(x, 1) = (z, 1) \) is not DGL-homotopic to \( b_f \) from Claim 2.4. Because the homotopy fibration of any \( \pi_Q \)-separable map is given by \( \Lambda(z), 0 \to \Lambda(z, y), 0 \cong \Lambda(x, y), 0 \) from the degree reason.

Recall that a map \( f : X \to Y \) is said to be a Gottlieb map \([21]\) if its homotopy group map induces the map between their Gottlieb groups \( f : G_*(X) \to G_*(Y) \). For example, if a map \( f \) admits a section, it is a Gottlieb map. Since elements of the rational Gottlieb group \( G_*(X)_Q \) is described by certain derivations of \( M(X) \) \([3]\), we note

Corollary 2.6. If a map \( f : X \to Y \) is \( \pi_Q \)-separable, it is a rational Gottlieb map.

3. When does it admit a section?

Let \( f : X \to Y \) be a \( \pi_Q \)-separable map with homotopy fiber \( F_f \) and \( \text{Der}(AW, AV \otimes AW) \) the sub-DGL of \( \text{Der}(AV \otimes AW) \) restricted to derivations out of \( AW \).

Proposition 3.1. Let \( F_{af} \) be the homotopy fiber of \( a_f \). Then the DGL-model of the fibration \( X_f : F_{af} \xrightarrow{\partial} (\text{Baut}_1 X)_0 \xrightarrow{\alpha} (\text{Baut}_1 Y)_0 \) is given by

\[
\text{Der}(AW, AV \otimes AW) \xrightarrow{\text{incl}} \text{Der}(AV \otimes AW) \xrightarrow{b_f} \text{Der}(AV).
\]

Proof. Since \( b_f \) is surjective and \( \text{Der}(AW, AV \otimes AW) \) is Ker \( b_f \), it follows from \([19]\) VI .1.(3) Proposition.

Let \( L(F) = \oplus_{i>0} L(F)_i \) be the degree decomposition of a DGL-model of a space \( F \).

Theorem 3.2. \( L(F_{af})_n = \oplus_{i-j=n} \text{Der}_i(AW) \otimes H^j(AV) \).

Proof. The DGL-map \( \rho : \text{Der}_i(AW) \otimes H^j(AV) \to \text{Der}_i(AW, AW \otimes AV)^j \) induced by an inclusion \( H^j(AV) \to (AV)^j \) is quasi-isomorphic since there is a decomposition \( \text{Der}(AW, AV \otimes AV) = (\text{Der}(AW) \otimes H^*(AV)) \oplus C \) for a complex \( C \) of derivations with \( H_*(C) = 0 \).

Corollary 3.3. \( H^*(F_{af}; Q) \) is free as a graded algebra if and only if \( H^*(\text{Baut}_1 F_f; Q) \) is so.
The rational homotopy exact sequence of the strictly induced fibration:

\[ \cdots \xrightarrow{\delta_f} \pi_{n+2}(\text{Baut}_1 Y)_Q \xrightarrow{a_f} \pi_{n+2}(\text{Baut}_1 X)_Q \xrightarrow{\delta_f} \cdots \]

is equivalent to the homology exact sequence:

\[ \cdots \rightarrow H_{n+1}(\text{Der}(\Lambda V \otimes \Lambda W)) \xrightarrow{b_f} H_{n+1}(\text{Der}(\Lambda V)) \xrightarrow{\delta_f} \cdots \]

Claim 3.4. The connecting map \( \delta_f \) is given by \( \delta_f([\sigma]) = [\tau] \) when \( \partial_X(\sigma) = \tau \) for a \( \partial_Y \)-cycle \( \sigma \) of \( \text{Der}(\Lambda V) \) and a \( \partial_X \)-cycle \( \tau \) of \( \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \).

Recall that the following implications hold for a general fibration \( \chi : F \to E \xrightarrow{p} B \) of simply connected spaces:

\( \chi \) is fibre-trivial \( \Rightarrow p \) admits a section \( \Rightarrow \chi \) is weakly trivial \( \Leftrightarrow \delta = 0 \). (\(*\))

Here \( \delta \) is the connecting map of the homotopy exact sequence for \( \chi \). The following may be a characteristic phenomenon in our case.

Proposition 3.5. \( a_f \) admits a section if and only if \( \delta_f = 0 \).

Proof. (if) Let the DGA-model of the fibration \( \chi_f \) be given as the commutative diagram:

\[
\begin{array}{ccc}
\Lambda U, d & \xrightarrow{\sim} & \Lambda U \otimes \Lambda Z, D_2 \\
\rho_2|_{\Lambda U} & \sim & \rho_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
C^*(\text{Der}\Lambda V) & \xrightarrow{\sim} & C^*(\text{Der}\Lambda V) \otimes \Lambda Z, D_1 \\
\rho_1 & \sim & \rho_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
C^*(\text{Der}\Lambda V) & \xrightarrow{\sim} & C^*(\text{Der}(\Lambda V \otimes \Lambda W)), D & \xrightarrow{\sim} & C^*(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)), D' \\
\end{array}
\]

where \( M(\text{Baut}_1 Y) \cong (\Lambda U, d) \) with \( U^{n+1} = H_n(\text{Der}(\Lambda V)) \) and \( M(\text{F}_a) \cong (\Lambda Z, \overline{D}_2) \) with \( Z^{n+1} = H_n(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)) \). From the assumption \( \chi_f \) is weakly equivalent, i.e., \( M(\text{Baut}_1 X) \cong (\Lambda U \otimes \Lambda Z, D_2) \). Let \( D = d_1 + d_2 \) in \( \S2 \). Then

\[ H^{n+1}(C^*(\text{Der}(\Lambda V \otimes \Lambda W)), d_1) \cong H_n(\text{Der}(\Lambda V \otimes \Lambda W)) = U^{n+1} \oplus Z^{n+1}. \]

Notice that \( (w, h) \notin [\text{Der}(\Lambda V), \text{Der}(\Lambda V)] \) for any \( w \in W \) and \( h \in \Lambda V \otimes \Lambda W \). That means

\[ d_2(s^{-1}(w, h^*)) \in I(C^*(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W))) \]

Here \( I(S) \) is the ideal generated by a basis of a vector space \( S \). Let \( \sigma \) be a \( \partial_X \)-cycle of \( \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \). Then \( [s^{-1}\sigma^*] \in H_c(C^*(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)), d_1') \cong \overline{Z} \) for \( D' = d_1' + d_2' \) in \( \S2 \). Since

\[ \rho_1(D_1([s^{-1}\sigma^*])) \sim d_3(s^{-1}\sigma^*) \text{ ; } D\text{-cohomologous}, \]

we have \( D_1([s^{-1}\sigma^*]) \in I(Z) \) by \( \rho_1 \), i.e., \( D_1(Z) \subset C^*(\text{Der}\Lambda V) \otimes \Lambda^+Z \). By \( \rho_2 \), \( D_2(Z) \subset \Lambda U \otimes \Lambda^+Z \). Then we have done since \( a_f \) admits a section if and only if \( D_2Z \subset \Lambda U \otimes \Lambda^+Z \) \([20]\).

(only if) It holds from the above implications \((\ast)\). \qed
Theorem 3.6. If a $\pi_Q$-separable map $f : X \to Y$ is rationally fibre-trivial (i.e., $X_0 \sim (F_f)_{Q_0} \times Y_0$), $a_f$ admits a section.

Proof. From the assumption and Claim 3.4, we have $\delta_f = 0$. Then it holds from Proposition 3.5. □

Refer [13] page 292] for related topics. Conversely, when $Y$ is an odd-sphere,

Theorem 3.7. If a $\pi_Q$-separable map $f : X \to Y = S^{2n+1}$ is not rationally fibre-trivial, $a_f$ does not admit a section. Furthermore $a_f \sim \ast$ (the constant map).

Proof. Let $M(S^{2n+1}) = (\Lambda v, 0)$. Since there exists an element $w \in W$ such that $Dw \in \Lambda v \otimes \Lambda^+ W$ from the assumption, $\partial_X(v, 1) = \pm(w, \partial Dw/\partial v) + \cdots \neq 0$ in $\text{Der}(AW)$. From Claim 3.4, $\delta_f$ is injective since $\delta_f([[v, 1]]) = [[w, v, 1]] \neq 0$. Then the former holds from Proposition 3.5. Furthermore, from the homotopy exact sequence, we have $a_{f_{\delta}} = 0$. Thus the latter holds. □

Example 3.8. (1) Let $S^0 \times S^7 \to X \to Y = S^3$ be a non-(fibre-)trivial $\pi_Q$-separable fibration given by the model

$$(\Lambda(v_1, 0)) \to (\Lambda(v_1, v_2, w), D) \to (\Lambda(w_1, w_2, 0), 0)$$

with $|v_1| = 3$, $|v_2| = 5$, $|w_2| = 7$, $Dv_1 = 0$ and $Dv_2 = v_1 w_1$. Then $a_f$ does not admit a section from Theorem 3.7. Indeed $\delta_f : H_3(\text{Der}(\Lambda v)) \to H_2(\text{Der}(\Lambda(v_1, v_2, w), \Lambda(v_1, v_2, w)))$ is non-trivial from $\delta_f([[v_1, 1]]) = [[w_2, v_1]] \neq 0$.

(2) Let $S^5 \times S^7 \to X' \to Y'$ be a non-(fibre-)trivial $\pi_Q$-separable fibration given by the model

$$(\Lambda(v_1, v_2, v_3, d_Y)) \to (\Lambda(v_1, v_2, v_3, w, w_2, 2), D') \to (\Lambda(w_1, w_2, 0), 0)$$

with $|v_1| = |v_2| = 3$, $|v_3| = 5$, $|v_1| = 7$, $|w_2| = 9$, $d_Y(v_1) = d_Y(v_2) = 0$, $d_Y(v_3) = v_1 v_2$, $Dw'w_1 = 0$ and $Dw_2 = v_1 w_1$. Then $a_f$ admits a section from Proposition 3.7. Since $\delta_f([[v_3, 1]]) = 0$ for $H_5(\text{Der}(\Lambda(v_1, v_2, v_3))) = \mathbb{Q}[[v_3, 1]]$. However $\chi_f$ is not trivial from $[[v_3, 1], (w_2, v_2 v_3)] = (w_2, v_2)$. Indeed, then

$$D(s^{-1}(w_2, v_2)^*) = d_2(s^{-1}(w_2, v_2)^*) = s^{-1}(v_3, 1)^* \cdot s^{-1}(w_2, v_2 v_3)^*$$

for $(C^*(\text{Der}(\Lambda(v_1, v_2, v_3, v_1, w_2)), D))$ with $D = d_1 + d_2$. Refer the proof of Proposition 3.2.

A simply connected CW complex $X$ is said to be elliptic if the dimensions of the rational cohomology algebra and homotopy group are both finite [4]. An elliptic space $X$ is said to be pure if $dM(X)^{even} = 0$ and $dM(X)^{odd} \subset M(X)^{even}$. Furthermore a pure space is said to be an $F_0$-space (or positively elliptic) if dim $\pi_{even}(X) \otimes \mathbb{Q} = \dim \pi_{odd}(X) \otimes \mathbb{Q}$ and $H^{odd}(X; \mathbb{Q}) = 0$. Then it is equivalent to $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$, in which $|x_i|$, the degree of $x_i$, are even and $f_1, \ldots, f_n$ forms a regular sequence in the $\mathbb{Q}$-polynomial algebra $\mathbb{Q}[x_1, \ldots, x_n]$, where $M(X) = \{\mathbb{Q}[x_1, \ldots, x_n] \otimes \Lambda(y_1, \ldots, y_n), d\}$ with $dx_i = 0$ and $dy_i = f_i$. In 1976, S. Halperin [8] conjectured that the Serre spectral sequences of all fibrations $X \to E \to B$ of simply connected CW complexes collapse at the $E_2$-terms for any $F_0$-space $X$ [4]. For compact connected Lie groups $G$ and $H$ where $H$ is a subgroup of $G$, when rank $G = \text{rank} H$, the homogeneous space $G/H$ satisfies the Halperin conjecture [15]. Also the Halperin conjecture is true when $n \leq 3$ [10]. From Claim 2.4, we obtain

8
Lemma 3.9. Let \( X \) be a pure space with \( M(X) = (Q[x_1, \ldots, x_m] \otimes \Lambda(y_1, \ldots, y_n), d) \) with \( \max\{|x_1|, \ldots, |x_m|\} < \min\{|y_1|, \ldots, |y_n|\} \). If \( f : X \to \Pi_{i=1}^n K(Q, [x_i]) \) is the rational principal fibration given by \( M(f) := (Q[x_1, \ldots, x_m], 0) \to M(X) \) with \( M(f)(x_i) = x_i \) for all \( i \). Then \( a_f \sim \ast \) if and only if \( f \) is a fibration whose relative model is given as \( \Lambda(y_1, \ldots, y_n) \otimes \Lambda(y_1, \ldots, y_n), d) \) with suitable degree and some \( \theta \in \operatorname{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \). Then we have \( \deg \theta = 0 \) from Claim 3.4 since \( H_{even}(\operatorname{Der}(M(Y))) = 0 \) \cite{12}. Furthermore the Lie bracket decomposition of an element of \( \operatorname{Der}(\Lambda V, \Lambda W) \) does not have an element of \( \operatorname{Der}(\Lambda V) \) as a factor from Theorem 3.2 since \( H_{even}(\operatorname{Der}(M(Y))) = 0 \) \cite{12} again. Thus we have \( D_2 = d \otimes 1 + 1 \otimes D_2 \) for the Sullivan minimal model \( (\Lambda U, d) \to (\Lambda U \otimes \Lambda Z, D_2) \to (\Lambda Z, D_2) \) of \( \operatorname{Der}(\Lambda V, \Lambda W) \) (in the proof of Proposition 3.3). (only if) Suppose there is a non-zero element \( \sum_i (x_i, h_i) \in H^{2m}(\operatorname{Der}(M(Y))) \) for \( h_i \in Q[x_1, \ldots, x_n], g_j \in \Lambda V \) and some \( m \). Then there is a rational fibration \( S^a \times S^b \to X \to Y \) of the model:

\[
(AV, d) \to (AV \otimes \Lambda(w_1, w_2), D) \to (\Lambda(w_1, w_2), 0)
\]

where \( |w_1| = a \) and \( |w_2| = b \) are odd with \( b - a = |x_k| - 1 \) for some \( k, h_k \) is not \( d_Y \)-exact, \( Dw_1 = 0 \) and \( Dw_2 = x_k w_1 \). Then

\[
\delta f([\sum_i (x_i, h_i) + \sum_j (y_j, g_j)]) = [(w_2, h_k w_1)] \neq 0
\]

for \( \delta f : H_{2m}(\operatorname{Der}(AV)) \to H_{2m-1}(\operatorname{Der}(\Lambda(w_1, w_2), AV \otimes \Lambda(w_1, w_2))) \). In particular, \( \chi_f \) is not trivial. \( \square \)

Example 3.12. Let \( Y \) be the homogeneous space \( SU(6)/SU(3) \times SU(3) \). Then \( Y \) is a pure space but not an \( F_0 \)-space since \( \operatorname{rank} SU(6) = 5 > 4 = \operatorname{rank}(SU(3) \times SU(3)) \). Let \( \xi : S^{11} \times S^{23} \to X \to Y \) be a fibration whose relative model is given as

\[
(\Lambda(x_1, x_2, y_1, y_2, y_3), d_Y) \to (\Lambda(x_1, x_2, y_1, y_2, y_3) \otimes \Lambda(w_1, w_2), D) \to (\Lambda(w_1, w_2), 0)
\]

where \( |x_1| = 4, |x_2| = 6, |y_1| = 7, |y_2| = 9, |y_3| = 11, |w_1| = 11, |w_2| = 23, d_Y y_1 = x_1^2, d_Y y_2 = x_1 x_2, d_Y y_3 = x_2^2, Dw_1 = 0 \) and \( Dw_2 = (x_1 y_2 - x_2 y_1)w_1 \). Then...
\[ \partial_X((y_1, 1)) = (w_2, x_2 w_1), \text{ i.e., } \delta_f([y_1, 1]) = [(w_2, x_2 w_1)] \neq 0. \text{ In particular } \chi_f \text{ is not trivial. Refer [13, Example 1.14(2)] for the Sullivan minimal model of } Y. \]

In this section finally we mention about a heredity property for a pull-back:

**Theorem 3.13.** Let \( f' : X' \to Y' \) be the pull-back of a map \( f : X \to Y \) by a map \( g : Y' \to Y \) with a rational section. Suppose that both \( f \) and \( f' \) are \( \pi_Q \)-separable. If \( f \) admits a section, then \( f' \) does so.

**Proof.** From [4, Proposition 15.8], the Sullivan model of the pull-back diagram:

\[
\begin{array}{ccc}
F_{f'} & \xrightarrow{i'} & F_f \\
\downarrow i' & & \downarrow i \\
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y \\
\end{array}
\]

is given as

\[
\Lambda W, D' \xrightarrow{\delta_f'} \Lambda W, D \\
\Lambda V \otimes \Lambda U \otimes \Lambda W, D' \xrightarrow{\delta_f} \Lambda V \otimes \Lambda W, D \\
\Lambda V \otimes \Lambda U, d' \xrightarrow{\delta_f''} \Lambda V, d
\]

where \( D'W = DW \subset \Lambda V \otimes \Lambda W \). Then, from Claim 3.4, the following is commutative:

\[
\begin{array}{ccc}
H_*(\text{Der}(\Lambda V)) & \xrightarrow{\delta_f} & H_{*-1}(\text{Der}(\Lambda W, \Lambda W \otimes H^*(Y; \mathbb{Q}))) \\
\downarrow h_*(c_g) & & \downarrow h_{*-1}(\text{Der}(g^*)) \\
H_*(\text{Der}(\Lambda V \otimes \Lambda U)) & \xrightarrow{\delta_f'} & H_{*-1}(\text{Der}(\Lambda W, \Lambda W \otimes H^*(Y'; \mathbb{Q}))),
\end{array}
\]

where \( c_g \) is same as \( b_g \) as a chain map (see the proof of Theorem 1.3). Here \( \text{Der}(g^*) : \text{Der}(\Lambda W, \Lambda W \otimes H^*(Y; \mathbb{Q})) \to \text{Der}(\Lambda W, \Lambda W \otimes H^*(Y'; \mathbb{Q})) \) is given by \( \text{Der}(g^*)((w, w' \otimes y)) = (w, w' \otimes g^*(y)) \). Then \( \delta_f' = 0 \) if \( \delta_f = 0 \). Thus it follows from Proposition 3.5.

Remark that \( \chi_{f'} \) is not a pull-back of \( \chi_f \). Moreover the converse of Theorem 3.13 is false in general. Indeed, we see in Example 3.8 that the fibration of (2) is the pull-back of (1) by a map \( g : Y' \to Y = S^3 \) with \( M(g)(v_1) = v_1 \).

4. The Obstruction Class for a Lifting

Let \( L(B) = (L(B), \partial_B) \) be the Quillen model of a simply connected CW complex \( B \) of finite type. Then \( L(B \cup e^N) \) is given by \( L(B) \coprod L(u), \partial_\alpha \) where \( |u| = N - 1 \), \( \partial_\alpha|_{L(B)} = \partial_B \) and \( \partial_\alpha(u) \in L(B) \) [19, Proposition III.3.(6)].
Theorem 4.1. For a $\pi_Q$-separable map $f : X \to Y$, let

$$
\begin{array}{c}
B \xrightarrow{h_X} (\text{Baut}_1 X)_0 \\
\downarrow i \\
B \cup_\alpha e^N \xrightarrow{h_Y} (\text{Baut}_1 Y)_0
\end{array}
$$

be a commutative diagram. Then there is a lift $h$ such that

$$
\begin{array}{c}
B \xrightarrow{h_X} (\text{Baut}_1 X)_0 \\
\downarrow h \\
B \cup_\alpha e^N \xrightarrow{h_Y} (\text{Baut}_1 Y)_0
\end{array}
$$

is commutative if and only if

$$\mathcal{O}_\alpha(h_X, h_Y) := \left[ \tau(h_Y(u)) - h_X''(\partial_\alpha(u)) \right] = 0$$

in $H_{N-2}(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)) = \pi_{N-1}(F_\alpha)_Q$ for the DGL-commutative diagram

$$
\begin{array}{c}
L(B) \xrightarrow{h_X} \text{Der}(\Lambda V \otimes \Lambda W), \partial_X \\
\downarrow i \\
L(B) \coprod \mathbb{L}(u), \partial_\alpha \xrightarrow{h_Y} \text{Der}(\Lambda V), \partial_Y
\end{array}
$$

with

- $\partial X |_V = \partial Y + \tau$ and $\partial X |_W = \tau$ where $\tau(\sigma) \in \text{Der}(\Lambda V, \Lambda V \otimes \Lambda W)$ for $\sigma \in \text{Der}(\Lambda V \otimes \Lambda W)$ and
- $h_X = h_X' + h_X''$ where $h_X'(b) \in \text{Der}(\Lambda V)$ and $h_X''(b) \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ for $b \in L(B)$.

Proof. Since $b_f \circ h_X = h_Y \circ i$ and $h_Y$ is a DGL-map,

$$h_X' \partial_\alpha(u) = h_Y \partial_\alpha(u) = \partial Y h_Y(u) \quad (1)$$

in $\text{Der}(\Lambda V)$. Notice that the obstruction element $\partial X(h_Y(u)) - h_X(\partial_\alpha(u))$ is a $\partial X$-cycle in $\text{Der}(\Lambda V \otimes \Lambda W)$. Therefore $\tau(h_Y(u)) - h_X''(\partial_\alpha(u))$ is a $\partial X$-cycle in $\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ from (1).

(if) Suppose that $\mathcal{O}_\alpha(h_X, h_Y) = 0$. Then there is an element $q \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$ such that

$$\partial X(q) = \tau(h_Y(u)) - h_X''(\partial_\alpha(u)). \quad (2)$$

Let

$$h \mid_{L(B)} := h_X \quad \text{and} \quad h(u) := h_Y(u) - q$$

Then $h$ is a DGL-map since

$$\partial X(h(u)) = \partial X(h_Y(u)) - \partial X(q)$$

$$= (h_X' \partial_\alpha(u) + \tau h_Y(u)) - (\tau h_Y(u) - h_X'' \partial_\alpha(u))$$

$$= h_X' \partial_\alpha(u) + h_X'' \partial_\alpha(u) = h_X(\partial_\alpha(u)) = h(\partial_\alpha(u))$$
from (1) and (2). Furthermore

\[
\begin{array}{ccc}
L(B) & \xrightarrow{h_X} & \text{Der}(\Lambda V \otimes \Lambda W), \partial_X \\
\text{i} & \downarrow & \downarrow \\
L(B) \bigotimes \text{L}(u), \partial_\alpha & \xrightarrow{h_Y} & \text{Der}(\Lambda V), \partial_Y
\end{array}
\]

is commutative since \(b_f(q) = 0\). Thus the (if)-part holds from the special realization of (\(*\)).

(only if) Suppose that there exists a map \(h\) such that (\(*\)) is commutative. Since \(h\) is a DGL-map,

\[
\partial_X(h(u)) = h(\partial_\alpha(u))
\]

in \(\text{Der}(\Lambda V \otimes \Lambda W)\) and

\[
h''_X \partial_\alpha(u) = \tau h(u)
\]

from (1) and (3). Furthermore

\[
\tau h(u) \sim \tau h_Y(u)
\]

in \(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)\). Here \(\sim\) means “homologous”. Indeed, (5) follows since

\[
h(u) = h_Y(u) + x
\]

for some element \(x \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)\) from \(b_f \circ h = h_Y\) and then since

\[
\tau h(u) = \tau(h_Y(u) + x) = \tau h_Y(u) + \partial_X(x).
\]

Thus we obtain that \(O_\alpha(h_X, h_Y) = \{\tau(h_Y(u)) - h''_X(\partial_\alpha(u))\} = 0\) from (4) and (5).

\[\square\]

From Theorem 3.2, we have

**Corollary 4.2.** If \(\pi_{N-1}(\text{Baut}_1 F_f) \cong 0\) for the homotopy fiber \(F_f\) of \(f\), there exists a lift \(h\) for the pair \((h_X, h_Y)\) of above.

**Example 4.3.** Let \(B = S^2 = \mathbb{C}P^1\). Let \(S^3 \times S^5 \to X \xrightarrow{f} Y = S^3\) be the fibration given by the model

\[
(\Lambda(v), 0) \to (\Lambda(v, w_1, w_2), D) \to (\Lambda(w_1, w_2), 0)
\]

with \(|v| = |w_1| = 3, |w_2| = 5, Dw_1 = 0\) and \(Dw_2 = vw_1\). Let \(L(\mathbb{C}P^2) = L(B \cup_a e^4) = (\text{L}(u_1, u_2), \partial)\) with \(|u_1| = 1, |u_2| = 3, \partial u_1 = 0\) and \(\partial u_2 = [u_1, u_1]\) [19]. Let

\[
S^2 \xrightarrow{h_X} (\text{Baut}_1 X)_0
\]

\[
S^2 \cup_a e^4 \xrightarrow{h_Y} (\text{Baut}_1 Y)_0
\]

be a commutative diagram given by the DGL-model

\[
\begin{array}{ccc}
\text{L}(u_1) & \xrightarrow{h_X} & \text{Der}(\Lambda(v, w_1, w_2), \partial_X) \\
\text{i} & \downarrow & \downarrow \\
\text{L}(u_1, u_2), \partial & \xrightarrow{h_Y} & \text{Der}(\Lambda v), 0
\end{array}
\]
by \( h_X(u_1) = h_Y(u_1) = 0 \) and \( h_Y(u_2) = (v, 1) \). Then \( \mathcal{O}_\alpha(h_X, h_Y) \neq 0 \) since
\[
\tau h_Y(u_2) = \partial_X(v, 1) = (w_2, w_1) \neq 0 = h''_X([u_1, u_1]) = h''_X(\partial_\alpha(u_2)).
\]
Thus \( h_Y : \mathbb{C}P^2 \to (\text{Baut}_1Y)_0 \) cannot lift to \( h : \mathbb{C}P^2 \to (\text{Baut}_1X)_0 \). Note that \( h_Y \) is extended to \( \mathbb{C}P^\infty \to (\text{Baut}_1Y)_0 \). Since \( BS^1 = \mathbb{C}P^\infty \), we obtain that any free \( S^1 \)-action on \( Y \) cannot lift to \( X \).

**References**

[1] G. Allaud, *On the classification of fiber spaces*, Math. Z. **92** (1966) 110-125

[2] A. Dold and R. Lashof, *Principal quasi-fibrations and fibre homotopy equivalence of bundles*, Illinois J. Math. **3** (1959), 285-305

[3] Y. Félix and S. Halperin, *Rational LS category and its applications*, Trans.A.M.S. **275** (1982) 1-38

[4] Y. Félix, S. Halperin and J. C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics **205**, Springer-Verlag, (2001).

[5] J. B. Gatsinzi, *LS-category of classifying spaces*, Bull. Belg. Math. Soc. Simon Stevin 2, no 2 (1995) 121-126

[6] D. H. Gottlieb, *On fibre spaces and the evaluation maps*, Ann. Math. **87** (1968) 42-55

[7] D. H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969) 729-756

[8] S. Halperin, *Finiteness in the minimal models of Sullivan*, Trans. A.M.S. **230** (1977) 173-199

[9] P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Math. Studies **15** (1975)

[10] G. Lupton, *Note on a Conjecture of Stephen Halperin*, Springer L.N.M. **1440** (1990) 148-163

[11] G. Lupton and B. S. Smith, *Realizing spaces as classifying spaces*, Proc. A.M.S. **144** (2016) 3619-3633

[12] W. Meier, *Rational universal fibrations and flag manifolds*, Math.Ann. **258** (1982) 329-340

[13] H. Nishinobu and T. Yamaguchi, *Sullivan minimal models of classifying spaces for non-formal spaces of small rank*, Topology and its Appl. **196** (2015) 290-307

[14] P. Salvatore, *Rational homotopy nilpotency of self-equivalences*, Topology and its Appl. **77** (1997) 37-50

[15] H. Shiga and M. Tezuka, *Rational fibrations, homogeneous spaces with positive Euler characteristics and Jacobians*, Ann. Inst. Fourier (Grenoble) **37** (1987) 81-106

[16] S. B. Smith, *The classifying spaces for fibrations and rational homotopy theory*, the 5th GeToPhyMa Summer School on “Rational Homotopy Theory and its Interactions” in Rabat (2016)

[17] J. Stasheff, *A classification theorem for fibre spaces*, Topology (1963) 239-246

[18] D. Sullivan, *Infinitesimal computations in topology*, I.H.E.S., **47** (1978) 269-331

[19] D. Tanrè, *Homotopie Rationnelle: Modèles de Chen, Quillen, Sullivan*, Lecture Note in Math. Springer **1025** (1983)

[20] J. C. Thomas, *Rational homotopy of Serre fibrations*, Ann. Inst. Fourier **331** (1981) 71-90

[21] T. Yamaguchi, *A rational obstruction to be a Gottlieb map*, J. Homotopy and Related Structures **5**(1). (2010) 97-111

**Kochi University, 2-5-1, Kochi, 780-8520, JAPAN**

**E-mail address:** tyamag@kochi-u.ac.jp