Prescribed-Time Safety Design for a Chain of Integrators

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Abstract—Safety in dynamical systems is commonly pursued using control barrier functions (CBFs) which enforce safety-constraints over the entire duration of a system’s evolution. We propose a prescribed-time safety (PTSf) design which enforces safety only for a finite time of interest to the user. While traditional CBF designs would keep the system away from the barrier longer than necessary, our PTSf design lets the system reach the barrier by the prescribed time and obey the operator’s intent thereafter. To emphasize the capability of our design for safety constraints with high relative degrees, we focus our exposition on a chain of integrators where the safety condition is defined for the state furthest from the control input. In contrast to existing CBF-based methods for high-relative degree constraints, our approach involves choosing explicitly specified gains (instead of class $K$ functions), and, with the aid of backstepping, operates in the entirety of the original safe set with no additional restriction on the initial conditions. With Quadratic Programming (QP) being employed in the design, in addition to backstepping and CBFs with a PTSf property, we refer to our design as a QP-backstepping PT-CBF design. For illustration, we include a simulation for the double-integrator system.

I. INTRODUCTION

A. Hi-rel-deg CBFs

CBFs have become a popular tool for synthesizing safe controllers for dynamical systems and have been used in a wide range of problem domains: multi-agent robotics [10], [29], robust safety [15], [18], [41], automotive systems [2], [28], delay systems [1], [14], [22], and stochastic systems [6], [27], [30] to mention a few. First defined in [38] and later refined and popularized by the seminal papers [2], [3], CBFs are often employed in a “safety filter” framework where they’re used for generating safe control overrides for a potentially unsafe nominal controller. In essence, a nominal controller is designed to achieve a desired performance objective, and a CBF-based control override is used whenever the nominal controller is unsafe.

In its initial conception [2], [38], CBFs were specified for safety constraints of relative degree one i.e. constraints whose time derivative depend explicitly on the input. The extension of CBFs to constraints with high-relative degree (hi-rel-deg CBFs) was first studied independently in [13], [39] with much progress following in [5], [23], [40], [42]. In [39], the extension was limited to the relative degree two case, and in [13] where CBFs of arbitrarily high relative degree was introduced, its usage for a relative degree $r$ case involves choosing $r-1$ bounded, positive definite functions that satisfy additional derivative conditions [13, Eq. (26)] - a requirement that limits the utility of [13] for significantly high relative degree constraints. A similar limitation applies to more recent treatment [40] which requires choosing and tuning $r$ class-$K$ functions, whose choice determines the subset of the original safe set that is kept forward invariant. Building off [39], exponential CBFs were reported in [23] and allowed the use of simple linear control tools to design CBFs for high relative degree constraints.

B. The “non-overshooting control” roots of hi-rel-deg CBFs

A year before [38], eight years before [2], and ten years before [23], a design for stabilization to an equilibrium point at the barrier was introduced, under the name “non-overshooting control,” in the 2006 paper [19] (following its conference version in 2005). This design possesses all the attributes of a safety design with a CBF of a uniform and high relative degree (sans the CBF terminology) with only the QP step absent since, for stabilization at the barrier, QP is subsumed in the stabilization design (the nominal feedback and the safety-filtered feedback are the same).

The interest in non-overshooting control in the 1990s came from applications—spacecraft docking, aerial refueling, machining, etc., with no margin for error in downward setpoint regulation. The non-overshooting control problem for linear systems, albeit mostly for zero initial conditions and nonzero setpoints, was solved in [4], [25], [31].

The paper [19] introduced the following two ideas (translated to the current CBF terminology). First, for a system with a hi-rel-deg CBF, a transformation, by backstepping, into a particular target system in the form of a chain of first-order CBF subsystems (resulting in all real poles in the linear case) is performed. Second, in order to ensure that all the CBF “states” of this chain begin and remain positive, the positivity of their initial values is ensured by choosing the backstepping gains in accordance with the initial conditions so the entire CBF chain is initialized positively.

This chain structure and the gain selection of [19, Eq. (12), (13)], regarded through the lens of pole placement, were independently discovered in the 2016 paper [23, Cor. 2]. Likewise, the nonlinear damping choices in the CBF chain in [19, Eq. (53), (54)] was independently proposed in the 2019 paper [40]. Additionally pursued in [19], but not in [23], [40], was a form of input-to-state safety (ISSf) in the

1Uniformity of the CBF’s relative degree gives the equivalence of a general control-affine system with the strict-feedback class and the convertibility of the safe set, given by the CBF positivity constraint, into a semi-infinite interval constraint for the first state of the strict-feedback system.
presence of disturbances. This notion, though not explored for hi-rel-deg CBFs, is rigorously conceived in [18].

Inspired by non-overshooting control under disturbances in [19], (i.e. stabilization to an equilibrium at the barrier along with ISSf), mean-square stabilization of stochastic nonlinear systems to an equilibrium at the barrier, along with a guarantee of non-violation of the barrier in the mean sense, is solved in [20].

With a nearly negligible QP modification, the stabilizing feedbacks in [19] can be used in safety filters. Hence, backstepping generates a safety-filter with explicit tuning variables that dictate the exponential approach to the barrier.

C. Prescribed-time safety (PTSf)

Recent advances in prescribed-time stabilization (PTS) [32] have resulted in time-varying backstepping controllers that guarantee settling times independent of initial conditions. Extensions have been developed to stochastic nonlinear systems [21], infinite-dimensional systems [7], [35], [36], and even coupled systems with finite/infinite-dimensional subsystems [8], [33], [34]. PTS is a subset of both the finite-time [11] and fixed-time [26] notions (i.e., stronger than both).

The success in achieving stabilization in prescribed time, independent of initial conditions, raises the question of pursuing the safety counterpart of the same notion. The “translation” from stability to safety may be a bit counterintuitive: while PTS guarantees that the state reaches the equilibrium no later than a prescribed time \( T \), PTSf guarantees that the state cannot reach the barrier sooner than a prescribed time \( T \).

In what context is such a PTSf property useful? First, it should be noted that PTSf is “less safe” than exponential safety (ESf). Less safe is useful by not being needlessly conservative. If the desired operation is in the barrier’s proximity, and especially if the desired operation is beyond the barrier, where it is safe to be after time \( T \), PTSf offers obvious performance (or alertness) advantages over ESf. It is even known in the automotive area that “too safe” may mean very unsafe: a follower vehicle that keeps a large distance even known in the automotive area that “too safe” may mean

W. We distinguish our notion of PTSf from existing notions of limited duration safety [24], and periodic safety using fixed-time CBFs [9]. Specifically, [9] introduced the notion of periodic safety where the objective is to keep a system safe for all times while enforcing that it periodically (with time period \( T \)) visits a goal set inside the safe set. In [24] the notion of limited duration safety was studied, and like PTSf it implies that a system is kept safe only for a limited duration \( T \). While [24] restricts the set of initial conditions—a set that shrinks as \( T \) increases [24, Rk. 2]—to be a strict subset of the safe set [24, Eq. (3)], our notion of PTSf places no restriction on the initial conditions of the system.

Finally, two distinct features of the time-varying backstepping technique make it quite attractive for use in safety-filter design. The first is that, compared to ESf designs, the PTSf filters designed with time-varying backstepping do not exhibit large transients when the safety filter overrides the nominal controller. This is not the case for ESf filters with rapid decay rates: the so-called “peaking” phenomenon [16], [17], [37] is exhibited, which causes some of the states to become very large near the initialization time, before rapidly converging to the equilibrium. This behavior can cause large state-derivatives, which, e.g., is undesirable in vehicle systems where maneuvers causing large acceleration and “jerk” can be dangerous. PTSf safety-filter designs avoid peaking by using small gains near initialization time that only grow large as the state grows “small”.

The second feature making time-varying backstepping attractive is the behavior of the convergence it achieves near the terminal time. PTSf achieves convergence with “infinitely-soft” landing, that is, the state and all of its derivatives converge to the equilibrium by the terminal time. This feature occurring in finite time is unique to PTSf, and is desirable because it can ensure, e.g., “jerk-free” safety maneuvers by the terminal time.

II. PROBLEM DESCRIPTION AND PRELIMINARIES

A. Problem Description

We study systems in the chain-of-integrator form

\[
\begin{align*}
\dot{x}_i(t) &= x_{i+1}(t), \quad i = 1, \ldots, n-1, \\
\dot{x}_n(t) &= u(t), \\
y(t) &= x_1(t), \quad t \geq t_0,
\end{align*}
\]

with relative degree \( n \), where \( t_0 \geq 0 \) is the initialization time, \( x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n \) is the state with initial condition \( x(t_0) \), and \( u(t) \in \mathbb{R} \) is the control input. The control objective is to enforce “safety”, defined here as the non-positivity of the output \( y(t) \) over the finite time horizon \([t_0, t_0 + T]\), where \( T \) is a terminal time that can be a priori prescribed.

B. Preliminaries

We denote by \( \mathbb{N} \) the set of natural numbers excluding 0. Our PTSf designs will be generated by the following “blow-up” function:

\[
\mu_m(t - t_0, T) = \frac{1}{\nu^m(t - t_0, T)}, \quad t \in [t_0, t_0 + T]
\]

for \( m \in \mathbb{N}_{\geq 2} \) and the terminal time \( T > 0 \), where

\[
\nu(t - t_0, T) := \frac{T + t_0 - t}{T}
\]

decays linearly from one to zero by the terminal time. We denote by \( m^k \) the rising factorial for \( m, k \in \mathbb{N} \), that is,

\[
m^k := m(m + 1) \cdots (m + k - 1);
\]

decays linearly from one to zero by the terminal time. We denote by \( m^k \) the rising factorial for \( m, k \in \mathbb{N} \), that is,

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m^k := m(m + 1) \cdots (m + k - 1);
\]

decays linearly from one to zero by the terminal time. We denote by \( m^k \) the rising factorial for \( m, k \in \mathbb{N} \), that is,
with CBFs. For the control input, we use the safety-filter bounded provided that
where constraint $y$ and $\alpha$ for $t \geq t_0 + T$, this also holds true for the $x$-system states. See Section IV and in particular, (32) and (34) for the mathematical treatment of this “infinity-soft” convergence.

### III. Nth-Order Chain-of-Integrators Design

We begin by performing a time-varying backstepping transformation defined as follows, for $t \in [t_0, t_0 + T)$:

$$
\begin{align*}
\dot{h}_i &= -x_i + \alpha_i \dot{x}_{i-1}(t), \quad i = 1, \ldots, n, \\
\alpha_0(x, t) &= 0, \\
\alpha_i(x, t) &= c_i \mu_2 h_i + \frac{d}{dt} \alpha_i^{-1}(x_{i-1}, t), \quad i = 1, \ldots, n,
\end{align*}
$$

where $c_i, i = 1, \ldots, n$ are design parameters to be determined. We call the transformed states $h_i, i = 1, \ldots, n$ barrier functions to connote the desire to keep their values positive, provided that the initial values $h_i(t_0)$ are positive as is typical with CBFs. For the control input, we use the safety-filter

$$
\begin{align*}
\min \{ u_{\text{nom}}, \alpha_n(x, t) \}, & \quad \text{if } 0 \leq t < t_0 + T, \\
\alpha_n g(t, x_1(t_0 + T)), & \quad \text{if } t \geq t_0 + T,
\end{align*}
$$

where $u_{\text{nom}}$ is a potentially unsafe nominal controller and $g$ is the “ramp” function

$$
g(t, x_1(t_0 + T)) = \begin{cases} 1 - \nu^m(t), & \text{if } x_1(t_0 + T) = 0, \\
1, & \text{otherwise},
\end{cases}
$$

where $m \in \mathbb{N}$ and $\bar{T}$ are design parameters. The role of $g$ in the product $u_{\text{nom}} g(t, x_1(t_0 + T))$ in (9) is to ensure that the control law is continuous at $t = t_0 + T$, since we will show that the feedback law $\alpha_n(x, t_0 + T) = 0$ (cf. Section IV). As defined, the continuity of the controller (9) follows from the continuity of $u_{\text{nom}}$.

Strictly speaking, the safety-filter (9) during times $t_0 \leq t < t_0 + T$ is the solution of the QP problem

$$
u = \min_{v \in \mathbb{R}} |v - u_{\text{nom}}|^2
$$

subject to $v \leq \alpha_n

where constraint $v \leq \alpha_n$ is equivalent to $h_n + c_n \mu_2 h_n \geq 0$ under input $v$. Therefore, we refer to our design as a QP-backstepping PT-CSF design.

With the safety-filter (9), the CBFs (6) satisfy

$$
\dot{h}_i = -c_i \mu_2 h_i + h_{i+1}, \quad i = 1, \ldots, n - 1, \\
\dot{h}_n \geq -c_n \mu_2 h_n,
$$

for $t \in [t_0, t_0 + T)$. We can now state our main result.

**Theorem 1:** If the system (1) is initially safe, that is, $y(t_0) < 0$, then the controller (6)–(9) ensures that $y(t) < 0$ for all $t \in [t_0, t_0 + T)$ for the initial control gains

$$
c_i = \max \{ 0, \underline{c} \}, \quad i = 1, \ldots, n - 1,
$$

where

$$
\underline{c} = \frac{x_{i+1}(t) - d}{\alpha_i \alpha_{i-1}(x_{i-1}(t), t)}.
$$

and $c_n \geq 0$. Moreover, the control law (9) is uniformly bounded provided that $u_{\text{nom}}$ is continuous in the interval $[t_0, t_0 + T]$.

**Remark 1:** While not characterized in Theorem 1, if the safety filter overrides the nominal controller over the time interval $[t_0 + i, t_0 + T)$ for some $i < T$, then the convergence of the CBFs to zero with be “infinity-soft”: that is, all of the derivatives $d^k h_i(t), k \in \mathbb{N},$ will also converge to zero by the terminal time $t_0 + T$. This also holds true for the $x$-system states. See Section IV and in particular, (32) and (34) for the mathematical treatment of this “infinity-soft” convergence.

### IV. Proof of Theorem 1

The structure of our proof comes in two parts: one to establish non-positivity of $y(t)$ for $t \in [t_0, t_0 + T)$; and another to establish uniform boundedness of the control law which filters the nominal controller to enforce safety. To this end, we first present the following commutativity property of the “blow-up” function (2) which we will leverage to show controller boundedness. To simplify our presentation, we take $t_0 = 0$ henceforth.

**Lemma 2:** For $m \in \mathbb{N}_0$ and $0 \leq \bar{t} \leq T$, the “blow-up” function (2) satisfies

$$
\mu_m(t, T) = \mu_m(\bar{t}, T)\mu_m(t - \bar{t}, T - \bar{t})
$$

**Proof:** Omitted due to space limitation.

To demonstrate controller uniform boundedness, we must leverage the fact that the feedback law invokes PTsf whose convergence dominates the rate of divergence of the time-varying control gains in (6). To accomplish this, we characterize the following property of the closed-loop system.

**Lemma 3:** For $c > 0$, the $i$th derivative of the function

$$
\xi(t) := e^{-cT(\mu_1(t, T) - 1)}
$$

satisfies

$$
\lim_{t \to T} \frac{d\xi(t)}{dt} = \lim_{t \to T} P \frac{\xi(t)}{\mu_1(t, T)}
$$

**Proof:** Follows by induction and repeated application of l’Hôpital’s rule. Details to be included in a journal version.

We can now proceed with the proof of Theorem 1.

**Proof:** We first pursue non-positivity of $y(t)$ under (9), (14), (15). The system beginning from safety, that is, $y(t_0) = x_1(t_0) < 0$, implies that $h_1(t_0) > 0$. We proceed by induction: suppose $h_i(t_0) > 0$ for some $i = 1, \ldots, n - 1$; it follows from (12) and differentiating (6) along (1) that

$$
h_{i+1}(t_0) = c_i h_i(t_0) - x_{i+1}(t_0) + \frac{d}{dt} \alpha_{i-1}(x_{i-1}(t_0), t_0)
$$

The initial control gains (14), (15) are designed so that

$$
c_i h_i(t_0) - x_{i+1}(t_0) + \frac{d}{dt} \alpha_{i-1}(x_{i-1}(t_0), t_0) > 0,
$$

where we’ve used (6). With $h_i(t_0) > 0 \forall i = 1, \ldots, n$, it is easy to show by backwards strong induction and the variation of constants formula that $h_i(t) > 0$ for $t \in [t_0, t_0 + T)$. We now pursue uniform boundedness of the the control law (9). We partition the time horizon $[0, T)$ into intervals for which the system is either deemed safe or unsafe according to the safety filter (9) by defining

$$
k_t := \begin{cases} \min \{ t < T : u_{\text{nom}}(t) = \alpha_n(x_n, t) \}, \quad \text{if it exists,} \\
T, \quad \text{otherwise,}
\end{cases}
$$

and for all $t 

$$
\alpha_0(x, t) = 0, \\
\alpha_i(x, t) = c_i \mu_2 h_i + \frac{d}{dt} \alpha_i^{-1}(x_{i-1}, t), \quad i = 1, \ldots, n,
\end{align*}

where constraint $y$ and $\alpha$ for $t \geq t_0 + T$, this also holds true for the $x$-system states. See Section IV and in particular, (32) and (34) for the mathematical treatment of this “infinity-soft” convergence.
for \( k \in \mathbb{N} \) with \( t_0 = 0 \), where

\[
[0, T) = \bigcup_{k \in \mathbb{N} \setminus \{0\}} [t_k, t_{k+1}) .
\]

(21)

We have constructed this partition such that the control law (9) remains continuous at \( t_k \), precluding Zeno behavior of the closed-loop system. Since the system is initially safe, \( t_1 \neq T \) represents the first time that safety is enforced by (9). For \( t \in [t_{2k-1}, t_{2k+1}) \), \( k \in \mathbb{N} \cup \{0\} \) and \( t_{2k+1} \leq T \), we define

\[
h_i^{2k-1} := -x_i + \alpha_i^{2k-1}(x_i, t - t_{2k-1}),
\]

\[
\alpha_i^{2k-1}(x_i, t - t_{2k-1}) := \frac{d}{dt} \alpha_i^{2k-1}(x_i, t - t_{2k-1})
\]

(22)

\[
+ c_i^{2k-1} \mu_2(t - t_{2k-1}, T - t_{2k-1}) h_i^{2k-1}.
\]

(23)

for \( i = 1, \ldots, n \), with \( \alpha_0^{2k}(x_0, t - t_{2k-1}) \equiv 0 \). Similarly, for \( t \in [t_{2k-1}, t_{2k}) \), \( k \in \mathbb{N} \) and \( t_{2k} \leq T \), we define the CBFs

\[
h_i^{2k-1} := -x_i + \alpha_i^{2k-1}(x_i, t - t_{2k-1}),
\]

\[
\alpha_i^{2k-1}(x_i, t - t_{2k-1}) := \frac{d}{dt} \alpha_i^{2k-1}(x_i, t - t_{2k-1})
\]

(24)

\[
+ c_i^{2k-1} \mu_2(t - t_{2k-1}, T - t_{2k-1}) h_i^{2k-1}.
\]

(25)

for \( i = 1, \ldots, n \), with \( \alpha_0^{2k}(x_0, t - t_{2k-1}) \equiv 0 \). We select \( c_i^0 = c_i \) according to (14), (15), and we select

\[
c_i^k = c_i^{k-1} \mu_2(t_k - t_{k-1}, T - t_{k-1}), \quad k \in \mathbb{N}.
\]

(26)

Since \( \alpha_0^{2k}(x_0, t - t_{2k-1}) = \alpha_0^{2k-1}(x_0, t - t_{2k-1}) \equiv 0 \), it follows that \( h_i^{2k}(t_{2k-1}) = h_i^{2k-1}(t_{2k-1}) \) for \( k \in \mathbb{N} \). Furthermore, by applying the initial gain selection (26) to (24), (25) and comparing them to (22), (23) at \( t = t_{2k-1} \), we deduce that \( h_i^{2k}(t_{2k-1}) = h_i^{2k-1}(t_{2k-1}) \) for \( i = 2, \ldots, n \). The same treatment leads to the equalities \( h_i^{2k}(t_{2k}) = h_i^{2k-1}(t_{2k}) \) for \( i = 1, \ldots, n \). Hence, the initial gain selection (26) for each time partition in (21) ensures that the system dynamics remain continuous at every time. In fact, it simply tracks the growth of the “blow-up” function \( \mu_2 \) over the time intervals.

Furthermore, we can leverage Lemma 2 and the initial gain selection (26) to show that

\[
\prod_{k \in \mathbb{N}} \alpha_i^k \mu_2(t - t_k, T - t_k) = \frac{\mu_2(t, T)}{\mu_2(t_0, T_0)}.
\]

(27)

in other words, the CBF design over the partitioned set (21) is consistent with the design (6)-(15).

For \( t \in [t_{2k-1}, t_{2k+1}) \), \( k \in \mathbb{N} \cup \{0\} \) and \( t_{2k+1} \leq T \), the system is safe and the nominal control—which we assume to be uniformly bounded (continuous over a compact time interval)—is being applied. For \( t \in [t_{2k-1}, t_{2k}) \), \( k \in \mathbb{N} \) and \( t_{2k} \leq T \), we must estimate the size of the time-varying input to verify that it is bounded. It follows from (23) adapted as \( u = \alpha_0^{2k-1}(x_0, t - t_{2k-1}) \) that during these intervals, the CBFs satisfy

\[
h_i^{2k-1} = -c_i^{2k-1} \mu_2(t - t_{2k-1}, T - t_{2k-1}) h_i^{2k-1} + \mu_i(t - t_{2k-1}, T - t_{2k-1}) h_i^{2k-1},
\]

(28)

\[
h_n^{2k-1} = -c_n^{2k-1} \mu_2(t - t_{2k-1}, T - t_{2k-1}) h_n^{2k-1},
\]

(29)

for \( i = 1, \ldots, n - 1 \). To this end, we first study the stability of (28), (29). We can solve (29) explicitly to obtain

\[
h_i^{2k-1}(t) = e^{-c_i^{2k-1}(T-t_{2k-1})}(\mu_i(t-t_{2k-1}, T-t_{2k-1})-1) h_i^{2k-1}(t_{2k-1}),
\]

(30)

whereas for \( i = 1, \ldots, n - 1 \), we have the relationship

\[
h_i^{2k-1}(t) = e^{-c_i^{2k-1}(T-t_{2k-1})}(\mu_i(t-t_{2k-1}, T-t_{2k-1})-1) h_i^{2k-1}(t_{2k-1})
\]

\[
+ \int_t^{T} e^{-c_i^{2k-1}(T-t_{2k-1})}(\mu_i(t-t_{2k-1}, T-t_{2k-1})-1) h_i^{2k-1}(t_{2k-1})
\]

\[
+ \int_t^{T} e^{-c_i^{2k-1}(T-t_{2k-1})}(\mu_i(t-t_{2k-1}, T-t_{2k-1})-1) h_i^{2k-1}(t_{2k-1})
\]

(31)

We apply Lemma 3 to (30) to establish that successive derivatives of (30) will converge to zero by the terminal time:

\[
limit_{t \rightarrow T^-} \frac{d^r h_i^{2k-1}(t)}{dt^r} = 0, \quad t \in [t_{2k-1}, T^-],
\]

(32)

for \( r \in \mathbb{N} \cup \{0\} \). For \( i = 1, \ldots, n - 1 \), we compute

\[
\frac{d h_i^{2k-1}(t)}{dt} = h_i^{2k-1}(t) + h_i^{2k-1}(t_{2k-1}) \times
\]

\[
\frac{d}{dt} \left( e^{-c_i^{2k-1}(T-t_{2k-1})}(\mu_i(t-t_{2k-1}, T-t_{2k-1})-1) \right).
\]

(33)

By applying Lemma 3 to the second term within (33), and by backward strong induction on (32), we get for \( r \in \mathbb{N} \cup \{0\} \)

\[
limit_{t \rightarrow T^-} \frac{d^r h_i^{2k-1}(t)}{dt^r} = 0, \quad i = 1, \ldots, n - 1.
\]

(34)

Hence, we have shown that when the nominal controller is overridden by the safety filter to enforce safety during \( t \in [t_{2k-1}, t_{2k}) \), \( k \in \mathbb{N} \) and \( t_{2k} \leq T \), our time-varying backstepping design ensures that the CBF converge very smoothly to zero by the terminal time—indeed, all of their derivatives also converge to zero by the terminal time. By using (28) within the derivative term of (25) in an iterative fashion, we can verify by induction that (25) for \( i = n \) is equivalent to

\[
\alpha_n^{2k-1}(x_n, t - t_{2k-1}) = \sum_{r=1}^{n-1} \frac{d^r}{dt^r} \left( h_n^{2k-1} - h_n^{2k-1} \right)
\]

\[
+ c_n^{2k-1} \mu_2(t - t_{2k-1}, T - t_{2k-1}) h_n^{2k-1}.
\]

(35)

We conclude from (32), (34) and applying l’Hôpital’s rule to (35) as before that

\[
|\alpha_n^{2k-1}(x_n, t - t_{2k-1})| < +\infty, \quad \forall t \in [t_{2k-1}, t_{2k}),
\]

(36)

for \( k \in \mathbb{N} \) and \( t_{2k} \leq T \).

V. DOUBLE INTEGRATOR DESIGN INTERPRETATION

We consider the double integrator i.e. system (11) with \( n = 2 \). Suppose that the nominal control input \( u_{nom} \) is at risk of making the system unsafe, and we wish to design a time-invariant safety filter that overrides the nominal controller and takes the system to the origin. This problem was studied in [23, Sec. 3.B] for input-output linearized systems via pole-placement (which inherently relies on the backstepping
method—see [23, Rk. 5]), which achieves exponential convergence to the origin with arbitrary decay rate. We define the barrier functions \( h_1 := -x_1, h_2 := -x_2 + \rho h_1, \rho > 0, \) with the goal of keeping \( h_1 \geq 0 \) uniformly. Consider the following time-invariant safety filter designed as in [23]:

\[
u = \min \left\{ u_{\text{nom}}, -\left(2\rho^2 \ 3\rho \right) x \right\}, \text{ for } t_0 \leq t < \infty, \tag{37}\]

with \( \rho \geq \max \{0, -x_2(t)/x_1(t)\} \). Suppose the safety-filter overrides \( u_{\text{nom}} \) at \( t = t_0 + \tilde{t} < \infty \) and continues to enforce safety thereafter (i.e., \( u(t) = -\left(2\rho^2 \ 3\rho \right) x(t) \) for all \( t \geq t_0 + \tilde{t} \), placing the closed-loop poles for the \( x \)-system at \( (-\rho, -2\rho) \)). Then the closed-loop system is given by

\[
x(t) = e^{-\rho(t-t_0-\tilde{t})}
\times \left(\frac{2 - e^{-\rho(t-t_0-\tilde{t})}}{2\rho \left( e^{-\rho(t-t_0-\tilde{t})} - 1 \right)} \frac{1}{\rho} - \frac{e^{-\rho(t-t_0-\tilde{t})}}{2e^{-\rho(t-t_0-\tilde{t})} - 1} \right) x(t_0 + \tilde{t}). \tag{38}\]

If we wish to achieve large exponential decay when the system is unsafe, we can select \( \rho \gg \max \{0, -x_2(t)/x_1(t)\} \) as large as desired. However, for small \( t-t_0-\tilde{t} \), the righthand side of (38) can be very large depending on the size of \( \rho \) (in particular, \( x_2 \) grows with \( \rho \)). This illustrates the “peaking” phenomenon, which was studied for ODE control systems in [16], [17], [37]. We now compare these results graphically to demonstrate the advantages of time-varying backstepping. We perform numerical simulations for the double-integrator system under the nominal controller

\[
u_{\text{nom}} = -4\left[x_1 + \sin(\omega t) + 0.8\right] - 4\left[x_2 + \omega \cos(\omega t)\right] \tag{39}\]

with \( \omega = 2\pi/T \) where \( T = 4 \) is the prescribed time. For initial condition \( x(0) = (-4, 2)^T \), we use the time-varying PTSf safety-filter [23] with choice of gains \( c_2 = c_1 = \max \{0, -x_2(0)/x_1(0)\} + 0.1 = 0.6 \) and use ramp

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Fig. 1. System trajectories (left) and outputs (right) for double integrator under nominal controller (39) with terminal time \( T = 4 \), and initial condition \((x_1(0), x_2(0)) = (-4, 2)\). The PTSf safety-filter uses (39) with \( c_2 = c_1 = 0.6 \) while the ESf safety-filter uses (37) with \( \rho = 0.6 \) and \( \rho = 3.2 \) — the latter value tuned to make ESf react at the same instant as PTSf.

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Fig. 2. Left: Control signal. Right: Jerk during intervals when nominal command is overridden. When ESf is tuned (\( \rho = 3.2 \) case) to be less conservative like PTSf, the magnitude of the jerk increases significantly.
function \( \mu \) with \( m = 2 \) and \( \bar{T} = 0.5 \). For comparison, we use the time-invariant ESF safety-filter \( \rho \) with \( \rho = 0.6 \) and \( \bar{T} = 3.2 \). The choice \( \rho = 0.6 \) was made to allow a gain equivalent to the initial gains of PTSf, and the choice \( \rho = 3.2 \) was tuned to make ESF less conservative and to react at around the same instant as PTSf. For numerical stability near the origin, we clip the blow-up function \( \mu_2 \) at a maximum value \( \mu_2,\max = 1000 \) — which still allows the PTSf gains to grow to several orders of magnitudes larger than \( \rho = 3.2 \). The system trajectories under PTSf and ESF are shown in Fig. \( \text{[1]} \). There, we observe that while ESF can be tuned to be less conservative like PTSf by choosing larger gains, it comes at the expense of a significant jerk.

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