Orthonormal bases of states in terms of labelling and Racah operators

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Abstract. The construction of orthonormal bases of eigenstates for the missing label problem is considered, with special emphasis on the special case of reductions with respect to the Cartan subalgebra of a reductive Lie algebra, leading to the so-called Racah operators. Some general properties of this type of labelling operators are analyzed. For the unitary case, it is shown that Racah operators constructed from Casimir operators generally do not suffice to construct orthonormal bases of states for generic irreducible representations. Various examples of explicit constructions of orthonormal bases of states are given.

1. Introduction
Possibly the most common application of group theoretical techniques to problems of physical interest is the internal labelling problem (also called missing label problem or MLP). Usually a physical problem not only determines the highest symmetry but also specific subgroup chains $G \supset G_1 \supset \cdots \supset G_n \cdots$ where, in general, the quantum numbers associated with irreducible representations (short IRs) of the involved subgroups do not suffice to completely label the multiplets of $G$, therefore requiring the construction of additional quantum number to separate the observed degeneracies. Classical examples of the MLP are given by the Elliott model $SU(3)$, the Wigner supermultiplet, the nuclear collective motion or the Interacting Boson Model [1].

A large number of other (internal) labelling problems has been studied in the literature, mainly focusing on the construction of complete sets of commuting operators. Many different methods have been used to this extent, from the direct brute force algebraic method to the reformulation of the problem in terms of projection operators, differential equations, the Vector Coherent State theory, the boson formalism or even contractions of Lie algebras [2, 3, 4, 5]. A recurring problem in these approaches is to find a structurally comprehensible way to interpret the multiplicity problem [6], and to keep the physical significance of the quantum numbers constructed. A second part of the MLP concerns the effective computation of these quantum numbers for IRs. Numerical results concerning the simultaneous diagonalization of missing label operators have also been considered for a number of reductions, although these results are generally limited to either low dimensional or multiplets of special type. To establish general formulae for the
eigenvalues of labelling operators, like their characteristic polynomial on a generic irreducible representation, has evaded a systematization, and still constitutes an open and difficult problem.

We review here some recent work on the internal labelling problem, notably a generic algorithm developed in [7] to construct orthonormal bases of states, based on the known results and some new criteria developed to simplify the computation of commutators in enveloping algebras. We reformulate these results in the analytical frame. Then we consider a special type of labelling problem, corresponding to reduce the irreducible representations of a (reductive) Lie algebra \( s \) with respect to a Cartan subalgebra \( h \). This special type of MLP has some particular features that distinguishes it from the usual cases. It is shown that there is a close connection to the MLP associated to maximal rank subalgebras, and the structure of the corresponding labelling operators (called Racah operators) is inspected.

For two examples the numerical analysis of the labelling operators is considered. In particular, we focus on the orthonormal bases of states of \( \mathfrak{sl}(3, \mathbb{R}) \) constructed with Racah operators. In spite of the simplicity of this example, it shows interesting features and indicates under which conditions the labelling operators obtained by decomposition of the Casimir operators are not sufficient to separate degeneracies.

2. Generalities on the labelling problem

Racah showed in [8] that for any semisimple Lie algebra \( s \) of rank \( l \) there exist \( \mathcal{N}(s) = l \) functionally independent Casimir operators, i.e., polynomials in the generators that commute with all elements of the algebra. Eigenvalues of these operators were used to label irreducible representations of \( s \), while the generators of the Cartan subalgebra \( h \) were used to identify states within each representation. Racah himself pointed out that in general these operators are not sufficient to completely characterize the states of IRs, and showed that the total number of internal labels required is given by:

\[
i = \frac{1}{2}(\dim s - \mathcal{N}(s)). \tag{1}\]

If we use a subalgebra \( s' \subset s \) to label the basis states of an IR \( \Gamma \) of \( s \), a similar lack of a complete set of labelling operators is observed for non multiplicity-free reductions. Sharp proved that in this situation, the subgroup provides \( \frac{1}{2}(\dim s' + \mathcal{N}(s')) - l_0 \) labels, where \( l_0 \) is the number of invariants of \( s \) that depend only on generators of the subalgebra \( s' \) [9]. Additional operators, called missing label operators or subgroup scalars, are needed to separate multiplicities of IRs of \( s' \) appearing in the decomposition of \( \Gamma \). It turns out that the total number of available operators is \( m = 2n \). For \( n > 1 \), labelling operators must moreover commute with each other. We recall that for a given reduction \( s \supset s' \), a degree \( p \) Casimir operator \( C_p \) of \( s \) can always be decomposed as

\[
C_p = \sum_{\alpha=0}^{m_p} \Theta^{[p-\alpha,\alpha]}, \tag{3}\]

i.e., as a sum of subgroup scalars \( \Theta^{[p-\alpha,\alpha]} \) having degree \( p - \alpha \) in the generators of \( s' \) and degree \( \alpha \) in the generators of the characteristic representation \( \Gamma \), where \( \text{ad}(s) = \text{ad}(s') \oplus \Gamma \) and \( m_p \leq p \).

We shall say that \( \Theta^{[p-\alpha,\alpha]} \) is an operator of bi-degree \( (p - \alpha, \alpha) \).

1 See [3, 7] for details.
Essentially there are two approaches to the internal labelling problem and the determination of invariants of Lie algebras: the direct algebraic way or the approach by means of differential operators. In many aspects they are equivalent approaches [7, 9]. Given a Lie algebra $\mathfrak{s}$ with generators $\mathbb{R}\{X_1, \ldots, X_n\}$ and commutators $[X_i, X_j] = C^k_{ij}X_k$, the $X_i$’s are realized in the space $C^\infty(\mathfrak{s}^*)$ by the differential operators:

$$\hat{X}_i = -C^k_{ij}x^i\frac{\partial}{\partial x^j},$$

where $\{x_1, \ldots, x_n\}$ are the coordinates of a covector in a dual basis of $\mathbb{R}\{X_1, \ldots, X_n\}$. The invariants of $\mathfrak{s}$ (in particular, the Casimir operators) are then solutions of the following system of partial differential equations:

$$\hat{X}_iF = 0, \quad 1 \leq i \leq n.$$  

The number $\mathcal{N}(\mathfrak{s})$ of functionally independent solutions of (5) follows from the usual analytic criteria:

$$\mathcal{N}(\mathfrak{s}) := \dim \mathfrak{s} - \sup_{x_1, \ldots, x_n} \text{rank} \left( C^k_{ij}x^k \right),$$

where $\left( C^k_{ij}x^k \right)$ is the matrix associated to the commutator table of $\mathfrak{s}$ over the given basis. For a polynomial solution of (5), the standard symmetrization defined by

$$\Lambda \left( x_{i_1} \cdots x_{i_p} \right) = \frac{1}{p!} \sum_{\sigma \in S_p} X_{\sigma(i_1)} \cdots X_{\sigma(i_p)}$$

provides the Casimir operators in their usual form as elements in the centre of the enveloping algebra $\mathcal{U}(\mathfrak{s})$. In particular, for any homogeneous polynomial $f = c^{k_1 \cdots k_p}x_{k_1} \cdots x_{k_p}$ we can define uniquely its symmetric representative as

$$\Lambda (f) = c^{k_1 \cdots k_p} \Lambda \left( x_{k_1} \cdots x_{k_p} \right).$$

Conversely, given $P = c^{k_1 \cdots k_p}x_{k_1} \cdots x_{k_p} \in \mathcal{U}(\mathfrak{s})$, the analytical counterpart is $\pi(P) = c^{k_1 \cdots k_p}x_{k_1} \cdots x_{k_p}$. In order to compare the usual bracket for elements in enveloping algebras and the Poisson bracket in the space $C^\infty(\mathfrak{s}^*)$, the notion of factorizable pairs was introduced in [10]: $P = X_{i_1} \cdots X_{i_p} \in \mathcal{U}(\mathfrak{s})$ and $Q = X_{j_1} \cdots X_{j_q} \in \mathcal{U}(\mathfrak{s})$ such that $\pi(P) \neq \pi(Q)$ are called factorizable if they can be written in the form

$$P = X_{i_1}^{a_1} \cdots X_{i_l}^{a_l}P_1 \in U(\mathfrak{s}), \quad Q = X_{i_1}^{a_1} \cdots X_{i_l}^{a_l}Q_1 \in U(\mathfrak{s}),$$

with $[P_1, Q_1] = 0$ and $[X_{i_1}, P_1] = [X_{i_1}, Q_1] = 0, \quad i = 1, \ldots, l$. Otherwise the pair $P, Q$ is called non-factorizable [10]. Using the projection $\pi$, it turns out that $P$ and $Q$ are factorizable exactly when $\pi(P)$ and $\pi(Q)$ share a common factor. This definition extends to polynomials: $F = c^{k_1 \cdots k_p}x_{k_1} \cdots x_{k_p}$ and $G = c^{j_1 \cdots j_q}x_{j_1} \cdots x_{j_q}$ are non-factorizable if for any pair $\{c^{k_1 \cdots k_p}, c^{j_1 \cdots j_q}\}$ the monomials $\{x_{k_1} \cdots x_{k_p}, x_{j_1} \cdots x_{j_q}\}$ are non-factorizable. Whenever there is no ambiguity, we will use the notion of non-factorizable pairs in both the algebraic and analytical frames. This notion was used in [10] to obtain the following criterion:

**Commutativity criterion:** Let $F, G$ be a non-factorizable pair of polynomials in the enveloping algebra $\mathcal{U}(\mathfrak{s})$ of $\mathfrak{s}$ such that $F = \Lambda(f)$, $G = \Lambda(g)$ for some homogeneous polynomials $f, g \in S(\mathfrak{s})$. Then $[F, G] = 0$ if and only if $\{f, g\} = 0$, i.e., if the functions $f, g$ are in involution with respect to the Berezin bracket

$$\{g, h\} = -C^k_{ij}x^k\frac{\partial g}{\partial x^i} \frac{\partial h}{\partial x^j}. \quad (10)$$
Recall that a Lie algebra $\mathfrak{s}$ is a reductive subalgebra of a semisimple Lie algebra. We recall here the two main criteria in their analytical form.

**Criterion A:** Let $C_p$ be a Casimir operators of $\mathfrak{s}$ of order $p$. If $\pi(C_p)$ decomposes as $\pi(C_p) = \lambda \pi(\Theta^{[\beta,\alpha]}) + \pi(\Theta^{[\beta-\beta,\alpha]}) + \pi(\Theta^{[\beta-2,\alpha]})$ with $|\beta - \alpha| \leq 2$ and $\lambda = 0, 1$, then

$$\{ \pi(\Theta^{[\beta-\beta,\alpha]}), \pi(\Theta^{[\beta,\alpha]}) \} = \{ \pi(\Theta^{[\beta-\beta,\alpha]}), \pi(\Theta^{[\beta-2,\alpha]}) \} = 0. \quad (11)$$

**Criterion B:** Let $\gamma - \alpha \geq 3$. If $\pi(C_p) = \pi(\Theta^{[\beta,\alpha]}) + \pi(\Theta^{[\beta,\alpha]}) + \pi(\Theta^{[\beta,\alpha]})$ (0 $\neq \alpha < \beta < \gamma$) and $\pi(\Theta^{[r,s]})$ is a subgroup scalar of $\mathfrak{s}' \subset \mathfrak{s}$ such that $\{ \pi(\Theta^{[r,s]}), \pi(\Theta^{[\beta,\alpha]}) \} = 0$, then $\{ \pi(\Theta^{[r,s]}), \pi(\Theta^{[\beta,\alpha]}) \} = 0$ and $\{ \pi(\Theta^{[r,s]}), \pi(\Theta^{[\beta,\alpha]}) \} = 0$.

3. **Racah operators of Lie algebras**

As commented, the labelling problem associated to the Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{s}$ is a special case, which however presents some different features to the usual MLP.

Since the number of internal labelling operators to classify states within an IR of a semisimple Lie algebra $\mathfrak{s}$ is given by $\frac{\dim \mathfrak{s} - \dim \mathfrak{h}}{2}$, the case where we use the reduction $\mathfrak{h} \subset \mathfrak{s}$ with respect to the Cartan subalgebra of $\mathfrak{s}$, we need exactly $f = \frac{\dim \mathfrak{s} - 3l}{2}$ labelling operators in addition to the Casimir operators of $\mathfrak{h}$, which coincide with the generators. Subgroup scalars for $\mathfrak{h}$ are usually called Racah operators.

Let $\{H_1, \cdots, H_l\}$ be a basis of $\mathfrak{h}$ and $\{H_1, \cdots, H_l, X_1, \cdots, X_{2r}\}$ be a basis of $\mathfrak{s}$, where $l = \text{rank } \mathfrak{s}$. The differential operators associated to the Cartan generators have the form

$$\hat{H}_i = -C_{ij}^{k} x_k \frac{\partial}{\partial x_j}, \quad i = 1, \cdots, l; \quad j, k = 1, \cdots, 2r. \quad (12)$$

This system has exactly $\dim \mathfrak{s} - l$ independent solutions, $l$ of them corresponding to $H_1, \cdots, H_l$. The remaining $\dim \mathfrak{s} - 2l$ solutions $\phi_{\alpha}$ to the system must satisfy the condition $\frac{\partial \phi_{\alpha}}{\partial x_j} \neq 0$ for some $j = 1, \cdots, 2r$. Further $l$ solutions are given by the Casimir operators of $\mathfrak{s}$, so that there are exactly $\dim \mathfrak{s} - 3l$ available Racah operators. As observed in [11], at most $\frac{1}{2} (\dim \mathfrak{s} - 3l)$ will commute with themselves.

Racah operators also appear naturally when considering the missing label problem for maximal rank reductive subalgebras of a semisimple Lie algebra. In fact, if $\mathfrak{r}$ is such a subalgebra, then it is known that $\mathfrak{r}$ can be written as

$$\mathfrak{r} = [\mathfrak{r}, \mathfrak{r}] \oplus Z(\mathfrak{r}), \quad (13)$$

where $Z(\mathfrak{r})$ denotes the centre. Now, if $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{r}$, then the direct sum $\mathfrak{h} = \mathfrak{h}' \oplus Z(\mathfrak{r})$ is a Cartan subalgebra of $\mathfrak{s}$ [12]. Therefore, any labelling operator for the labelling problem related to the chain $\mathfrak{s} \supseteq \mathfrak{r}$ also commutes with the generators of $\mathfrak{h}$, and hence defines a Racah operator of $\mathfrak{s}$, which is however of a particular type. The converse is easily seen to be not true: Consider for instance the simple Lie algebra $su(3)$ and the maximal rank subalgebra $\mathfrak{r} = su(2) \times u(1)$. For the reduction $su(3) \supset \mathfrak{r}$ we have $n = 0$, which means that it is multiplicity

2 Recall that a Lie algebra $\mathfrak{g}$ is called reductive if it decomposes as the direct sum $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}_0$ of its centre $Z(\mathfrak{g})$ and a semisimple ideal $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$. It is called of maximal rank if it contains a Cartan subalgebra of $\mathfrak{s}$. 


free, while we have exactly one Racah operator. This means that the only operators which commute with the generators of the reductive algebra are the Casimir operators of \( \mathfrak{su}(3) \).

As \( \mathfrak{h} \) is an Abelian algebra, any irreducible representation of it is one dimensional, labeled by the corresponding eigenvalues of the Cartan generators. Hence, if \( R \) is an IR of \( \mathfrak{s} \), its decomposition into representations of the Cartan subalgebra \( \mathfrak{h} \) will be simply the full character of \( R \), i.e., the weights occurring in the representation [12]. Degeneracy will be given whenever a weight has multiplicity greater than one. The Racah operators must separate these states having the same eigenvalues for \( \mathfrak{h} \). This means specifically that the labelling operators and the internal subgroup operators coincide, as follows from the identity \( l + \frac{1}{2} (\dim \mathfrak{s} - 3l) = \frac{1}{2} (\dim \mathfrak{s} - l) \). This double role of Racah operators constitutes the main structural difference of this labelling problem with respect to the classical MLP, and allows us to interpret Racah operators as a kind of "scalar" version of the usual labelling operators.

If \( \Theta^{[\alpha,\beta]} \) is a Racah operator having degree \( \alpha \) in the Cartan generators and degree \( \beta \) in the remaining generators of \( \mathfrak{s} \), the first question that arises is whether \( \alpha \) and \( \beta \) are subjected to some constraints. Obviously those of type \( \Theta^{[\alpha,0]} \) are simply functions in \( C^\infty (\mathfrak{h}^*) \) and of no interest for the labelling problem. An important restriction is given in the following result:

**Criterion on the bi-degree of Racah operators:** If \( \Theta^{[\alpha,\beta]} \) is a Racah operator of \( \mathfrak{s} \), then \( \beta \neq 1 \).

Without loss of generality, we can work with the complexification \( \hat{\mathfrak{s}} \) of the semisimple Lie algebra \( \mathfrak{s} \). As the Cartan operators commute, they can be simultaneously diagonalized in the adjoint representation, so that we can assume that over a basis \( \{ H_1, \ldots, H_l, Y_1, \ldots, Y_{2r} \} \) of \( \hat{\mathfrak{s}} \) we have \( [H_i, Y_j] = \lambda_{ij} Y_j \). Now let \( \Theta^{[p-1,1]} \) be a Racah operator of bi-degree \( [p-1,1] \) for some \( p > 2 \). Then its analytical counterpart \( O^{[p-1,1]} = \pi (\Theta^{[p-1,1]}) \) can be written in the form

\[
O^{[p-1,1]} = \sum_{\alpha=1}^{2r} \varphi^\alpha (h_1, \ldots, h_l) y_\alpha,
\]

where \( \{ h_1, \ldots, h_l, y_1, \ldots, y_{2r} \} \) are the coordinates in \( \hat{\mathfrak{s}}^* \). The function \( O^{[p-1,1]} \) satisfies the system (12):

\[
\hat{H}_i \left( O^{[p-1,1]} \right) = -\lambda_{i1} y_1 \varphi^1 - \cdots - \lambda_{i2r} y_{2r} \varphi^{2r} = 0, \quad 1 \leq i \leq l.
\]

As these identities must hold generically, we have that \( \varphi^k = 0 \) whenever there exists an index \( i_0 \) such that \( \lambda_{i_0 k} \neq 0 \). Now for any generator \( Y_j \) there exists some Cartan generator \( H_{j_0} \) such that \( [H_{j_0}, Y_j] \neq 0 \) (otherwise \( Y_j \) would belong to the Cartan subalgebra, which is excluded by hypothesis), so that we recursively obtain the identities \( \varphi^1 = \varphi^2 = \cdots = \varphi^{2r} = 0 \).

As a consequence, when decomposing the Casimir operators of \( \mathfrak{s} \) with respect to the scaling transformations on the Cartan subalgebra, we get decompositions of the type

\[
C_p = \Theta^{[p,0]} + \Theta^{[p-2,2]} + \cdots + \Theta^{[p-k_0,k_0]},
\]

where \( 2 \leq k_0 \leq p \). From equation (14) is straightforward to verify that a Casimir operator of degree \( p \) provides at most \( p - 2 \) independent Racah operators. It is however no ensured that the labelling operators obtained by this ansatz actually form a complete set of commuting operators.

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The invariance of the degrees of subgroup scalars follows from the Gruber-O’Raifeartaigh method [13].
4. Algorithm for the construction of orthonormal bases

Using the criteria A and B of Section 2 and the decomposition of Casimir operators, an algorithm for the MLP and the construction of orthonormal bases of states was proposed in [7]. We reproduce it here in its analytical form. We assume that \( \mathfrak{s} \supset \mathfrak{s}' \) is a chain of semisimple Lie algebras of rank \( l = \text{rank}(\mathfrak{s}) \) and \( l' = \text{rank}(\mathfrak{s}') \) respectively. With the symbols \( O^{[p-a, \alpha]} \) we denote the projected subgroup scalars \( \pi (\Theta^{[p-a, \alpha]}) \).

(I) Decompose the Casimir invariants of \( \mathfrak{s} \) of degree \( p \geq 3 \) following (3):
\[
\pi (C_p) = \sum_{\alpha=0}^{M_p} \pi (\Theta^{[p-a, \alpha]}) = \sum_{\alpha=0}^{M_p} O^{[p-a, \alpha]}.
\]

(II) Check if there are scalars satisfying criterion A.

(III) For the non-factorizable pairs \( (O^{[p-a, \alpha]}, O^{[p-a, \beta]}) \) not satisfying criterion A compute the Berezin bracket.

(IV) Determine if there are scalars satisfying criterion B.

(V) Let \( t \) be the total number of mutually involutive operators \( O^{[p-a, \alpha]} \).

(a) If \( t < n \), proceed to step (VI).
(b) If \( t \geq n \) mutually involutive operators \( O^{[p-a, \alpha]} \) are found, compute the rank of the system \( \mathcal{L} = \{ \pi (C_1), \ldots, \pi (C_1), \pi (C_1'), \ldots, \pi (C_l'), O^{[p-1, \alpha_1]}, \ldots, O^{[p-1, \alpha_l]} \} \).
   1. If \( \text{rank}(\mathcal{L}) \geq l + l' - l_0 + n \), the symmetrized representatives of the operators solve the missing labelling problem.
   2. If \( \text{rank}(\mathcal{L}) < l + l' - l_0 + n \), proceed to step (VI).

(VI) Compute additional \( t' \geq n - \text{rank}(\mathcal{L}) \) subgroup scalars \( \Phi_1, \ldots, \Phi_{t'} \) such that

(a) \( \{ \Phi_k, O^{[p-1, \alpha_1]}, \ldots, O^{[p-1, \alpha_l]} \} = 0 \) for \( k = 1, \ldots, t' \) and \( i = 1, \ldots, t \).
(b) \( \{ \Phi_k, \Phi_l \} = 0 \) for \( k, l = 1, \ldots, t' \).
(c) \( \text{rank}(\{ \pi (C_1), \ldots, \pi (C_l), \pi (C_1'), \ldots, \pi (C_l'), O^{[p-1, \alpha_1]}, \ldots, O^{[p-1, \alpha_l]}, \Phi_1, \ldots, \Phi_{t'} \}) \geq n \).

Extract \( n \) functionally independent operators from the previous system.

This first part of the algorithm systematizes the search and computation of labelling operators. The second part, referring to their simultaneous diagonalization, is carried out using the symmetric representatives (8) of the preceding labelling operators. The process, valid for normal operators \( T \) (i.e., satisfying the constraint \( TT^\dagger = T^\dagger T \)), is summarized in the following steps:

(VII) Decompose the IR \( R \) of \( \mathfrak{s} \) into IRs of \( \mathfrak{s}' : R = R_1^{n_1} \oplus \ldots \oplus R_q^{n_q} \).

(VIII) For each \( R_i^{n_i} \) \( (i = 1, \ldots, q) \) determine a basis of eigenvectors for the Casimir operators of \( \mathfrak{s}' \).

(IX) Find a basis of \( R_i^{n_i} \) that diagonalizes the labelling operators \( \Theta_k \ (k = 1, \ldots, n) \).

(X) Within any IR \( R_i \), diagonalize the internal subgroup operators \( J_1, \ldots, J_f \).

(XI) Apply Gram-Schmidt to orthonormalize the different eigenspaces.

For the special case of Racah operators, the steps (VIII) and (X) are redundant and can be skipped.

5. Construction of orthonormal bases

In this section we illustrate the method and point out the differences between orthonormal bases of eigenstates constructed with the usual MLP with respect to a subgroup and the labelling problem with respect to the Cartan subalgebra.

\(^{4}\) That is, considering the analytical counterpart of Casimir operators and subgroup scalars.
5.1. The chain $G_2 \supset \mathfrak{su}(3)$

The labelling problem $G_2 \supset \mathfrak{su}(3)$ with $n = 1$ missing label for generic irreducible representations was considered in [11, 14] using the algebraic approach. However, the diagonalization of the labelling operator was not considered. We start from the basis used in [15], consisting of the generators $g_k^l, a_k^\pm, a_k^- = (a_k^l)^\dagger$ with indices $1 \leq k, l \leq 3$ and the constraints $(g_k^l)^\dagger = g_k^l, \sum_{k=1}^3 g_k^l = 0$. The brackets of $G_2$ are:

$$
\begin{align*}
[g_k^l, g_m^m] &= \delta_{km} g_n^{nm} - \delta_{nm} g_k^m, \\
[a_k^l, a_m^-] &= g_k^m, \\
[a_k^l, a_m^+] &= \pm \delta_{nm} a_k^l \mp \frac{1}{3} \delta_k^l a_m^+, \\
a_k^+, a_k^- &= \mp \frac{2}{\sqrt{3}} klm a_m^-.
\end{align*}
$$

(15)

It is easy to verify that the generators $g_k^l$ span the $\mathfrak{su}(3)$ subalgebra. After some lengthy computations the sixth order Casimir operator of $G_2$ is seen to provide the subgroup scalars:

$$
C_6 = \Theta[6,0] + \Theta[4,2] + \Theta[3,3] + \Theta[2,4] + \Theta[0,6].
$$

(16)

Since $\Theta[0,6] = \frac{1}{144} \Theta[0,2]^3$, this operator will not be useful for the MLP. We may take $\Theta[2,4]$ as the missing label operator, and $C_2, C_6, C_2', C_3', \Theta[2,4]$ as independent set. We chose $g_1^1, g_2^1$ and $(g_1^1)^2 + (g_2^1)^2 + 2(g_1^1 g_2 + g_2^1 g_1 - g_1^1 g_2^1)$ as internal $\mathfrak{su}(3)$ operators, the latter being the Casimir operator of the $\mathfrak{so}(3)$ subalgebra of $\mathfrak{su}(3)$ generated by $g_1^2, g_2^1$ and $g_1^1 - g_2^2$. All these operators are Hermitean and commute, hence can be simultaneously diagonalized.

Table 1 specifies the eigenvalues of missing label operator $\Theta[2,4]$ for the IRs of $G_2$ having dimension $d \leq 100$. We see that [11] is the lowest dimensional IR having degeneracy [7].

| IR  |   | 01 | 01 | 00 |
|-----|---|----|----|----|
| 01  | 10 | 01 | 00 |
| 00  | 10 |
| 01  | 4  | 4  | 0  |
| 02  | 20 | -20 | 0 |
| K   | 1051 | 1051 | 0 |

| IR  |   | 11 | 12 | 02 | 01 | 00 |
|-----|---|----|----|----|----|----|
| 01  | 11 | 01 | 00 |
| 00  | 10 |
| 01  | 9  | 4  | 4  |
| 02  | 0  | 20 | -20 |
| K   | 40713 | 16327 | 16327 |

| IR  |   | 11 | 12 | 02 | 01 | 00 |
|-----|---|----|----|----|----|----|
| 01  | 11 | 01 | 00 |
| 00  | 10 |
| 01  | 9  | 10 | 10 |
| 02  | 0  | 70 | -70 |
| K   | 86281 | 37303 | 37303 |

| 01  | 10 | 01 | 00 |
| 00  | 10 |
| 01  | 10 |
| 02  | 0  | 70 | -70 |
| K   | 41779 | 51779 | 51779 |

| IR  |   | 03 | 03 | 02 | 01 | 00 |
|-----|---|----|----|----|----|----|
| 01  | 21 | 02 | 01 | 00 |
| 00  | 10 |
| 01  | 16 |
| 02  | 0  | 56 | -56 |
| K   | 168834 | 377284 | 377284 |

| IR  |   | 03 | 03 | 02 | 01 | 00 |
|-----|---|----|----|----|----|----|
| 01  | 21 | 02 | 01 | 00 |
| 00  | 10 |
| 01  | 16 |
| 02  | 0  | 56 | -56 |
| K   | 113230 | 113230 | 1506024 |

An orthonormal basis constructed with these operators would be given by $| \lambda_1, \lambda_2, \lambda_3, \varphi_1, \varphi_2, \varphi_3 \rangle$, where $\lambda_i$ denote the eigenvalues of the $\mathfrak{su}(3)$ Casimir operators, $K$ that of $\Theta[2,4]$ and the $\varphi_i$’s those of the internal labels.
5.2. Racah operators for $\mathfrak{so}(3, \mathbb{R})$

The Lie algebra $\mathfrak{so}(3, \mathbb{R})$ is the simplest case where the Racah operators obtained by decomposition of the Casimir operators does not suffice to construct an orthonormal basis that is valid for generic representations. This is the consequence of the structure of the Casimir operators, which will vanish identically for certain types of representations, therefore implying the nullity of the subgroup scalars. In this case, we will have to compute an even order labelling operator in order to determine a set of commuting operators valid to describe generic IRs.

We start from the following real matrices:

\[
L_0 = \begin{pmatrix} 1 & \vspace{0.5cm} 0 \\ 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1 & \vspace{0.5cm} 0 \\ 0 & -1 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 1 & \vspace{0.5cm} 0 \\ 0 & -1 \end{pmatrix}, \quad T_0 = \begin{pmatrix} 1 & \vspace{0.5cm} 0 \\ 0 & -2 \end{pmatrix},
\]

\[
T_2 = \begin{pmatrix} 1 & \vspace{0.5cm} 0 \\ 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & \vspace{0.5cm} 0 \\ 0 & -1 \end{pmatrix}, \quad T_{-1} = \begin{pmatrix} 1 & \vspace{0.5cm} 0 \\ 0 & -1 \end{pmatrix}, \quad T_{-2} = \begin{pmatrix} 1 & \vspace{0.5cm} 0 \\ 0 & -1 \end{pmatrix}.
\]

These matrices generate a simple Lie algebra isomorphic to $\mathfrak{so}(3, \mathbb{R})$.\(^5\) The Cartan subalgebra $\mathfrak{h}$ is generated by $L_0$ and $T_0$. Moreover, the matrices correspond to the fundamental representation $[10]$ of this algebra. Over this basis, the Casimir operators are given by

\[
C_2 = L_0^2 + \frac{1}{3} T_0^2 + L_1 L_{-1} + L_{-1} L_1 + 2 (T_2 T_{-2} + T_{-2} T_2) + T_1 T_{-1} + T_{-1} T_1,
\]

\[
C_3 = \Lambda \left( t_1^2 + 9 t_0 t_2 t_{-1} + 9 t_0^2 t_{-1} - 27 \Lambda \left( l_0 l_{-1} + l_0 l_{-1} t_1 + t_1^2 t_{-2} - t_2^2 t_{-2} \right) \right)
\]

where $\Lambda$ is the symmetrization map of (7). If we decompose $C_3$ we obtain the subgroup scalars

\[
C_3 = \Theta^{[3,0]} + \Theta^{[1,2]} + \Theta^{[0,3]},
\]

where $\Theta^{[3,0]} = T_0^3 - 9 T_0 L_0^2$, $\Theta^{[1,2]} = \Lambda \left( 9 t_0 \left( t_1 t_{-1} + t_1 l_{-1} + l_1 t_{-1} - l_1 l_{-1} t_1 - 36 t_2 t_{-2} t_{0} \right) \right)$ and $\Theta^{[0,3]} = \Lambda \left( -27 \left( (t_1^2 - t_2^2) t_{-2} + (t_1^2 - l_1^2) t_2 \right) \right)$. Either $\Theta^{[1,2]}$ or $\Theta^{[0,3]}$ can be chosen as Racah operator. Observe further that since $f = 1$, these operators will not commute in general. Using the diagonalization procedure of [7], we can construct the common eigenvectors of the sets $\{L_0, T_0, \Theta^{[1,2]}\}$ or $\{L_0, T_0, \Theta^{[0,3]}\}$. Now the question arises whether these sets can be used to build an orthonormal basis of states for an arbitrary IR $[\lambda \mu]$ of $\mathfrak{so}(3, \mathbb{R})$. The answer is however in the negative. Suppose that $[\lambda \mu]$ is a representation such that the eigenvalue of the cubic Casimir operator is zero. By (18), we obtain that for each state $|\varphi\rangle$ in the representation we have

\[
\Theta^{[1,2]} |\varphi\rangle = -\Theta^{[0,3]} |\varphi\rangle - \Theta^{[3,0]} |\varphi\rangle.
\]

Now we prove that $\Theta^{[1,2]}$ does not separate the states for which $L_0 |\varphi\rangle = T_0 |\varphi\rangle = 0$. If $|\varphi\rangle$ is such a state, then the action of the subgroup scalar $\Theta^{[1,2]}$ can be simplified (using the commutation relations of $\mathfrak{so}(3, \mathbb{R})$) and brought to the form:

\[
\Theta^{[1,2]} |\varphi\rangle = 9 T_1 T_{-1} T_0 |\varphi\rangle + 9 L_1 L_{-1} T_0 |\varphi\rangle - 27 L_1 T_{-1} L_0 |\varphi\rangle - 27 L_{-1} T_1 L_0 |\varphi\rangle - 36 T_2 T_{-2} T_0 |\varphi\rangle = 0.
\]

This means that both the operators $\Theta^{[1,2]}$ and $\Theta^{[0,3]}$ are zero on the eigenspace $V_{(0,0)}$ of $L_0$ and $T_0$ having eigenvalues $(0,0)$. Therefore these states cannot be distinguished by the subgroup scalars. It follows from the general theory of Lie algebras that for all representations $[\lambda \mu]$ of

\(^5\) In the literature, this basis has been used as a complex basis of $\mathfrak{su}(3)$. Computing the Killing form we obtain $\sigma = 2$, which means that as real Lie algebra, it is isomorphic to the normal form $\mathfrak{su}(3, \mathbb{R})$. 

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\( \mathfrak{sl}(3, \mathbb{R}) \) the eigenvalue of \( C_3 \) is zero, thus for these representations we cannot construct an orthonormal basis with the sets of commuting operators \( \{L_0, T_0, \Theta^{[1, 2]}\} \) or \( \{L_0, T_0, \Theta^{[0, 3]}\} \). In order to circumvent this difficulty, we consider the quadratic operator

\[
X^2 = T_1 L_{-1} + L_{-1} T_1 + T_{-1} L_1 + L_1 T_{-1} = 2 T_1 L_{-1} + 2 L_1 T_{-1},
\]

which does not arise from a decomposition of Casimir operators. It is straightforward to verify that it commutes with \( L_0 \) and \( T_0 \), thus constitutes a Racah operator. As an operator of even degree, it will not vanish at the self-adjoint representations \( [\lambda \lambda] \). The following table illustrates how the set \( \{L_0, T_0, X^2\} \) not only serves to construct an orthonormal basis for the \( [\lambda \lambda] \) IRs, but also for non-selfadjoint representations. It contains the eigenvalues for the basis states for the IRs \( \Gamma = [11], \Gamma = [21], \Gamma = [31] \) and \( \Gamma = [22] \) of dimensions 8, 15, 24 and 27 respectively.

| \([11]\) | \(L_0\) | \(T_0\) | \(X^2\) | \([31]\) | \(L_0\) | \(T_0\) | \(X^2\) | \([22]\) | \(L_0\) | \(T_0\) | \(X^2\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \(-1\) | \(-3\) | \(2\) | \(2\) | \(-2\) | \(-8 - 4\sqrt{6}\) | \(0\) | \(0\) | \(0\) |
| \(1\) | \(-3\) | \(-2\) | \(2\) | \(-2\) | \(-8 + 4\sqrt{6}\) | \(0\) | \(0\) | \(8\sqrt{7}\) |
| \(-1\) | \(3\) | \(-2\) | \(1\) | \(1\) | \(-8 - 2\sqrt{11}\) | \(0\) | \(0\) | \(-8\sqrt{7}\) |
| \(1\) | \(3\) | \(2\) | \(1\) | \(1\) | \(-8 + 2\sqrt{11}\) | \(-1\) | \(3\) | \(-4 - 2\sqrt{7}\) |
| \(-2\) | \(0\) | \(0\) | \(0\) | \(4\) | \(-4\sqrt{5}\) | \(-1\) | \(3\) | \(-4 + 2\sqrt{7}\) |
| \(0\) | \(0\) | \(4\sqrt{3}\) | \(0\) | \(-2\) | \(-4\sqrt{5}\) | \(-2\) | \(0\) | \(4\sqrt{5}\) |
| \(0\) | \(0\) | \(4\sqrt{3}\) | \(0\) | \(-2\) | \(-4\sqrt{5}\) | \(1\) | \(-3\) | \(-4 - 2\sqrt{7}\) |
| \([21]\) | \(L_0\) | \(T_0\) | \(X^2\) | \(-1\) | \(1\) | \(8 - 2\sqrt{11}\) | \(-1\) | \(-3\) | \(4 + 2\sqrt{7}\) |
| \(1\) | \(1\) | \(-4 - 2\sqrt{17}\) | \(-2\) | \(-2\) | \(8 + 4\sqrt{6}\) | \(-1\) | \(-3\) | \(-4 - 2\sqrt{7}\) |
| \(1\) | \(1\) | \(-4 + 2\sqrt{17}\) | \(-2\) | \(-2\) | \(8 - 4\sqrt{6}\) | \(1\) | \(3\) | \(4 + 2\sqrt{7}\) |
| \(0\) | \(2\) | \(-8\sqrt{2}\) | \(-1\) | \(7\) | \(-2\) | \(1\) | \(3\) | \(4 - 2\sqrt{7}\) |
| \(0\) | \(2\) | \(8\sqrt{2}\) | \(1\) | \(7\) | \(2\) | \(2\) | \(0\) | \(-4\sqrt{5}\) |
| \(-1\) | \(-1\) | \(4 + \sqrt{17}\) | \(-2\) | \(4\) | \(8\) | \(2\) | \(0\) | \(4\sqrt{5}\) |
| \(-1\) | \(-1\) | \(4 - \sqrt{17}\) | \(2\) | \(4\) | \(8\) | \(-2\) | \(6\) | \(-4\) |
| \(-1\) | \(5\) | \(-2\) | \(-3\) | \(1\) | \(10\) | \(2\) | \(-6\) | \(-4\) |
| \(1\) | \(5\) | \(2\) | \(3\) | \(1\) | \(10\) | \(-2\) | \(-6\) | \(4\) |
| \(-2\) | \(2\) | \(-4\) | \(-4\) | \(-2\) | \(-4\) | \(2\) | \(6\) | \(4\) |
| \(2\) | \(2\) | \(-4\) | \(4\) | \(-2\) | \(-4\) | \(-3\) | \(3\) | \(2\) |
| \(-3\) | \(-1\) | \(2\) | \(3\) | \(-5\) | \(-6\) | \(3\) | \(3\) | \(2\) |
| \(3\) | \(-1\) | \(-2\) | \(-3\) | \(-5\) | \(6\) | \(-3\) | \(-3\) | \(-2\) |
| \(2\) | \(-4\) | \(-4\) | \(1\) | \(-5\) | \(-2\) | \(3\) | \(-3\) | \(-2\) |
| \(-2\) | \(-4\) | \(4\) | \(-1\) | \(-5\) | \(2\) | \(-4\) | \(0\) | \(0\) |
| \(0\) | \(0\) | \(-4\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |

These bases of eigenstates have been constructed by means of symmetric and skew-symmetric tensor products of the fundamental representations \( R = [10] \) and \( R^* = [01] \), where the operators has the eigenvalues \((2, 0, -2)\). By the structure and symmetries \( \mathfrak{sl}(3, \mathbb{R}) \)-representations \([16]\), it follows that \( X^2 \) always has eigenvalues that are multiples of 2. Moreover, it will have the same eigenvalues for the representations \([\lambda \mu] \) and \([\mu \lambda] \). The distinction of states in this case will follow from the eigenvalues of the Cartan generators, which have opposite sign.

Table 3 contains the characteristic polynomial of the Racah operator \( X^2 \) for all IRs having dimension \( d < 65 \). All roots are real and provide the eigenvalues of \( X^2 \) on the states of these representations. Due to the symmetry, only the representations \([\lambda \mu] \) with \( \lambda > \mu \) are listed.
Table 3. Eigenvalues of $X^2$ for IRs $[\lambda \mu]$ of dimension $d \leq 65$.

| $\Gamma$ | $\dim \Gamma$ | Characteristic polynomial of $X^2$ |
|----------|----------------|----------------------------------|
| [00]     | 1              | $Y$                              |
| [10]     | 3              | $Y (Y + 2)$                      |
| [20]     | 6              | $Y^2 (Y + 6) (Y + 4)$            |
| [11]     | 8              | $Y^2 (Y + 2)^2 (Y^2 - 48)$       |
| [30]     | 10             | $Y^2 (Y + 12) (Y + 10) (Y + 6) (Y + 2)$ |
| [21]     | 15             | $Y (Y + 4)^2 (Y + 2)^2 (Y^2 - 128) (Y^2 + 8Y - 52)$ |
| [40]     | 15             | $Y^3 (Y + 20) (Y + 18) (Y + 6) (Y + 4)$ |
| [50]     | 21             | $Y^3 (Y + 30) (Y + 24) (Y + 18) (Y + 10) (Y + 6) (Y + 2)$ |
| [31]     | 24             | $(Y + 10) (Y + 6) (Y + 4) (Y + 2)^2 (Y^2 - 240) (Y^2 - 80) (Y^2 + 16X - 100) (Y^2 + 16Y - 32)$ |
| [22]     | 27             | $Y^5 (Y + 4)^2 (Y + 2)^2 (Y^2 - 448) (Y^2 - 80)^2 (Y^2 + 8Y - 148)^2$ |
| [60]     | 28             | $Y^4 (Y + 42) (Y + 40) (Y + 30) (Y + 22) (Y + 20) (Y + 18) (Y + 14) (Y + 12) (Y + 8) (Y + 6) (Y + 4)$ |
| [41]     | 35             | $Y (Y + 18) (Y + 16) (Y + 14) (Y + 12) (Y + 8) (Y + 6)^2 (Y + 4) (Y + 2) (Y^2 - 384) (Y^2 - 192) (Y^2 + 32Y + 48) \times (Y^2 + 24Y - 148) (Y^2 + 24Y + 12)$ |
| [70]     | 36             | $Y^4 (Y + 56) (Y + 54) (Y + 50) (Y + 44) (Y + 36) (Y + 30) (Y + 28) (Y + 24) (Y + 18) (Y + 14) (Y + 12) \times (Y + 10)^2 (Y + 6) (Y + 4)$ |
| [32]     | 42             | $Y^4 (Y + 14) (Y + 8) (Y + 6)^3 (Y + 4) (Y + 2)^2 (Y^2 - 896) (Y^2 - 192) (Y^2 + 16Y - 144) (Y^2 + 8Y - 276) (Y^2 + 8Y - 176) \times (Y^2 + 18Y - 452Y + 2248)$ |
| [80]     | 45             | $Y^5 (Y + 72) (Y + 70) (Y + 66) (Y + 60) (Y + 52) (Y + 42)^2 (Y + 40) (Y + 36) (Y + 30)^2 (Y + 22) (Y + 20) (Y + 18) (Y + 16) \times (Y + 14) (Y + 12) (Y + 8) (Y + 6) (Y + 4)$ |
| [51]     | 48             | $(Y + 28) (Y + 26) (Y + 22) (Y + 16) (Y + 10) (Y + 8) (Y + 6) (Y + 2)^2 (Y^2 - 560) (Y^2 - 336) (Y^2 - 112) (Y^2 + 48Y + 224) \times (Y^2 + 48Y + 316) (Y^2 + 32Y - 196) (Y^2 + 32Y + 80) (Y^2 + 16Y - 164) (Y^2 + 16Y - 64)$ |
| [90]     | 55             | $(Y + 90) (Y + 88) (Y + 84) (Y + 78) (Y + 70) (Y + 60) (Y + 56) (Y + 54) (Y + 50) (Y + 48) (Y + 44) (Y + 36) (Y + 34) (Y + 30) \times (Y + 28) (Y + 26) (Y + 24) (Y + 18)^2 (Y + 14) (Y + 12) (Y + 10)^2 (Y + 6) (Y + 2) Y^5$ |
| [42]     | 60             | $Y^4 (Y + 16) (Y + 14) (Y + 10) (Y + 8) (Y + 4)^3 (Y + 2) (Y^2 - 1472) (Y^2 - 640) (Y^2 + 24X - 116) (Y^2 + 24X - 84) \times (Y^2 + 16X - 272) (Y^2 + 16Y - 64) (Y^2 + 8Y - 436) (Y^2 + 8Y - 212) (Y^2 + 36X - 272X + 4762) (Y^2 + 30Y^2 - 772Y + 8568)$ |
| [61]     | 63             | $Y (Y + 40) (Y + 38) (Y + 34) (Y + 28) (Y + 20) (Y + 16) (Y + 10) (Y + 8) (Y + 4) (Y + 2) (Y^2 - 768) (Y^2 - 512) \times (Y^2 + 72X + 876) (Y^2 + 64Y + 496) (Y^2 + 64Y + 704) (Y^2 + 40Y - 244) (Y^2 + 40Y + 172) (Y^2 + 32Y - 16) (Y^2 + 24Y - 244) \times (Y^2 + 24Y - 20) (Y^2 + 8Y - 116)$ |
| [33]     | 64             | $(Y + 6)^4 (Y + 4)^2 (Y + 2)^2 (Y^2 - 640)^2 (Y^2 - 336)^2 (Y^2 - 228)^4 (Y^2 - 112)^2 (Y^2 + 16Y - 288)^2 (Y^3 + 18Y^2 - 964Y + 5064) \times (Y^4 - 2016Y^2 + 241920) Y^4$ |
6. Concluding remarks

We have reviewed some recent work concerning the construction of labelling operators and orthonormal bases of eigenstates for chains of reductive Lie algebras. Two types of labelling problems have been distinguished: the usual chain \( s \supset s' \) of reductive algebras and the reduction \( s \supset h \) with respect to the Cartan subalgebra. Some important structural differences between these types of labelling problems have been analyzed. Specifically, we have seen that some bi-degrees for Racah operators are forbidden by the structure of the Cartan subalgebra, in contrast to the general MLP for semisimple or reductive subgroups. Moreover, we pointed out the relation between Racah operators and labelling operators arising from the MLP associated to maximal rank subgroups. The results obtained for the Racah operators can be generalized in straightforward manner to reductions with respect to subalgebras of the Cartan subalgebra. Many structural observations remain valid, with the exception that in this case subgroup scalars of bi-degree \((p - 1, 1)\) are not excluded.

One of the main problems in constructing orthonormal bases of states using the labelling or Racah operators is to find explicit expressions of the eigenvalues. General formulae will only be possible for multiplicity-free reductions \( s \supset s' \) [17], as the eigenvalues of the labelling operators differ at the representations of \( s' \) appearing with degeneracy greater than one. Therefore the problem of obtaining formulae for the eigenvalues must be considered for each specific case of physical interest. For the case of Racah operators, the situation is rather similar, up to the difference that representations of the subalgebra are replaced by the states. However, exactly this pathology makes the analysis of interest, as Racah operators constitute an alternative to other constructions of orthonormal bases analyzed for semisimple Lie algebras. Actually they could be used as a tool to be compared with other (classical) constructions of orthonormal bases [18]. Further work in this direction is in progress.

However, even for the classical algebras, it has been pointed out that decomposition of Casimir operators does not necessarily provide the required number of Racah operators to construct an orthonormal basis of states. The analysis of such bases for \( \mathfrak{sl}(3, \mathbb{R}) \) has shown the limitations of the Racah operators when derived from Casimir operators. The same will happen for the compact form \( \mathfrak{su}(3) \), where the vanishing of the cubic operator on the representations of type \([\lambda \lambda] \) implies that some additional Racah operator must be computed to label the states. It is still an unsolved problem whether the characteristic polynomial of the labelling operator \( X^2 \) can be determined generically for the \([\lambda \mu] \) IRs. When considering the Racah operators for the \( \mathfrak{su}(n) \) series, it should also be expected that additional Racah operators to replace the subgroup scalars obtained from Casimir operators of odd degree are necessary. An open question is whether for higher rank, the replacing Racah operators can be taken as quadratic in the generators of the algebra. This would certainly simplify the numerical analysis of the eigenvalues for arbitrary representations. In addition, it is not guaranteed that in the case of even-degree invariants, there are sufficiently many independent ones to form a complete set of commuting operators. This refers specifically to degenerate representations, where some of the Racah operators may be dependent, as happens for the usual MLP [19, 20]. In this sense, new criteria and approaches must be developed to ensure the completeness of commuting sets of labelling operators, as well as to solve the difficulties intrinsic to the numerical part of the internal labelling problem in a satisfactory manner.

Acknowledgments

During the preparation of this work, the author was financially supported by the research project MTM2010-18556 of the Ministerio de Ciencia e Innovación.
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