Bohr Phenomenon for Locally Univalent Functions and Logarithmic Power Series

Bappaditya Bhowmik · Nilanjan Das

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Abstract
In this article we prove Bohr inequalities for sense-preserving $K$-quasiconformal harmonic mappings defined in the unit disk $\mathbb{D}$ and obtain the corresponding results for sense-preserving harmonic mappings. In addition, Bohr inequalities are established for uniformly locally univalent holomorphic functions, and for $\log(f(z)/z)$ where $f$ is univalent or inverse of a univalent function.

Keywords Bohr radius · Locally univalent functions · Logarithmic coefficients

Mathematics Subject Classification 30B10 · 30C45 · 30C50 · 30C62 · 31A05

1 Introduction
The origin of the Bohr phenomenon lies in the seminal work by Harald Bohr [10], which included the following (improved) result.

Theorem A Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic in the open unit disk $\mathbb{D}$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$, then

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1 \quad (1.1)$$

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Bappaditya Bhowmik
bappaditya@maths.iitkgp.ac.in

Nilanjan Das
nilanjan@iitkgp.ac.in

1 Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India
for all \( z \in \mathbb{D} \) with \(|z| = r \leq 1/3\).

Inequalities of similar nature are being extensively investigated nowadays in different frameworks, and have become famous by the name Bohr inequalities. To have a glimpse of the ongoing current research in the Bohr radius problem, the reader is referred to some recent articles, e.g. [4,9,11,12,23,26] and the references therein. Now we concentrate on a generalized treatment of the Bohr radius problem introduced in [1], using the concept of subordination. For two holomorphic functions \( f \) and \( g \) in \( \mathbb{D} \), we say \( g \) is subordinate to \( f \) if there exists a function \( \phi \), holomorphic in \( \mathbb{D} \) with \( \phi(0) = 0 \) and \(|\phi(z)| < 1 \), satisfying \( g = f \circ \phi \). Throughout this article we denote \( g \) is subordinate to \( f \) by \( g \prec f \). Also the class of functions \( g \) subordinate to a fixed function \( f \) will be denoted by \( S(f) \). Now according to [1] we say that \( S(f) \) has the Bohr phenomenon if for any \( g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f) \), there is a \( r_0 \in (0, 1] \) such that

\[
\sum_{n=1}^{\infty} |b_n|r^n \leq d(f(0), \partial f(\mathbb{D}))
\]

(1.2)

for \(|z| = r < r_0\). Here \( d(f(0), \partial f(\mathbb{D})) \) denotes the Euclidean distance between \( f(0) \) and the boundary of domain \( f(\mathbb{D}) \). It is seen that whenever a holomorphic function \( g \) maps \( \mathbb{D} \) into a domain \( \Omega \) other than \( \mathbb{D} \), then in a general sense the Bohr inequality (1.2) can be established if \( g \) can be recognized as a member of \( S(f) \), \( f \) being the covering map from \( \mathbb{D} \) onto \( \Omega \) satisfying \( f(0) = g(0) \). In particular, if we take \( \Omega = \mathbb{D} \), then for any holomorphic \( g : \mathbb{D} \to \mathbb{D} \) there exists a disk automorphism \( f \) such that \( g(0) = f(0) \) and \( g \in S(f) \). In this case \( d(f(0), \partial \mathbb{D}) = 1 - |f(0)| \), and hence (1.2) reduces to (1.1). The Bohr phenomenon has been explored using the above definition in a number of papers, e.g. [1–3,6,8]. One of the goals of the present article is to extend the Bohr inequalities of type (1.2) for certain harmonic functions in a suitable fashion. A complex valued function \( f(z) = u(x, y) + iv(x, y) \) of \( z = x + iy \in \mathbb{D} \) is called harmonic if both \( u \) and \( v \) satisfy the Laplace’s equation

\[
\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0,
\]

where \( H(x, y) \) is a real valued function. It is well known that under the assumption \( g(0) = 0 \), \( f \) has a unique canonical representation \( f = h + \overline{g} \), where \( h \) and \( g \) are holomorphic in \( \mathbb{D} \). In view of this representation, \( f \) is locally univalent and sense-preserving whenever the Jacobian \( J_f(z) := |h'(z)|^2 - |g'(z)|^2 > 0 \) for all \( z \in \mathbb{D} \). A sense-preserving homeomorphism defined in \( \mathbb{D} \) which is also harmonic is called K-quasiconformal, \( K \in [1, \infty] \) if the (second complex) dilatation \( w_f := g'/h' \) satisfies \(|w_f(z)| \leq k, k = (K - 1)/(K + 1) \in [0, 1]\). Now it is easy to see that the aforesaid definitions and notation for subordination of holomorphic functions can be adopted for harmonic functions without any change (cf. [28]). In the present day theory of harmonic mappings, investigations are often carried out to explore the connections between the holomorphic part, or some suitable holomorphic counterpart of a given harmonic mapping and the map itself (see for instance [15,22]). Motivated by this perspective, in this article we prove Bohr inequalities similar to (1.2) for \( S(f) \) under...
the assumption that \( f \) is a sense-preserving \( K \)-quasiconformal harmonic mapping defined in \( \mathbb{D} \), where the holomorphic part \( h \) is univalent or convex univalent. Further, as another application of the technique used in proving this theorem, we establish the sharpened version of [16, Thm. 3.1]. We mention here that a number of Bohr inequalities for sense-preserving \( K \)-quasiconformal harmonic mappings have been obtained in [16], most of them bearing the classical flavor of the Bohr radius problem. Besides, the recent articles [18,19] contain several new results on the Bohr phenomenon for certain types of harmonic mappings.

We now turn our attention to the class \( \mathcal{H} \) of complex valued holomorphic functions \( f \) defined in \( \mathbb{D} \). An interesting subfamily of \( \mathcal{H} \) is the class of uniformly locally univalent functions (see [17,24,29,30]). Here we clarify that \( f \in \mathcal{H} \) is said to be uniformly locally univalent if there exists \( a > 0 \) such that \( f \) is univalent on each hyperbolic disk \( \mathbb{D}^h_{a}(z_0) := \{z \in \mathbb{D} : |(z - z_0)/(1 - \overline{z_0}z)| < \tanh a\} \) with center \( z_0 \in \mathbb{D} \) and radius \( a \). It is well known that (cf. [17,30]) a function \( f \in \mathcal{H} \) is uniformly locally univalent if and only if the pre-Schwarzian norm

\[
\|P_f\| := \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right) \left|\frac{f''(z)}{f'(z)}\right| < \infty.
\]

Now let \( \mathcal{A} := \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\} \). Since \( f''/f' \) remains invariant under the post-composition by a non-constant linear function, in view of the above characterization it is quite natural to consider the class \( \mathcal{B}(\lambda) := \{f \in \mathcal{A} : \|P_f\| \leq 2\lambda\} \) for any \( \lambda \in [0, \infty) \) (compare [17]). In this paper we derive a Bohr inequality of type (1.2) for the functions in \( \mathcal{B}(\lambda) \). As \( \mathcal{B}(0) = \{z\} \), we consider \( \lambda \in (0, \infty) \) only to prove the result.

Before we proceed further we need to introduce the following subfamilies of \( \mathcal{A} \) to facilitate our discussion. Let the subclass of univalent functions in \( \mathcal{A} \) be denoted by \( S \). Two well known subclasses of \( S \) are \( S^* \) and \( C \) which consist of starlike and convex univalent functions respectively. We now consider the logarithmic coefficients of any \( f \in S \), which are defined by

\[
\log \left(\frac{f(z)}{z}\right) = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}
\]

(see, for instance [13, p. 151]). The importance of logarithmic coefficients in univalent function theory is already well recognised due to the substantial role played by them in the proof of the Bieberbach conjecture. We know that obtaining inequalities involving the \( |\gamma_n| \) is considered to be a challenging problem till date (see for instance [21,27] and references therein) due to the unavailability of sharp bounds on \( |\gamma_n| \) for \( n \geq 3 \), where \( f \in S \). Inspired by this fact, in this article we have considered the problem of establishing Bohr inequalities similar to the inequality (1.1) for \( \log( f(z)/z) \). More precisely, we will say that \( \log(f(z)/z) \) has Bohr radius \( r_0 \in (0, 1] \) if

\[
2 \sum_{n=1}^{\infty} |\gamma_n| r^n \leq 1
\]
for $|z| = r < r_0$. We comment here that the quantity $d(\log(f(z)/z), \partial \Omega)$, where $\Omega$ is the image of $\mathbb{D}$ under the function $\log(f(z)/z)$, can be an arbitrarily small positive number for $f \in S$. One can easily see this by choosing the univalent polynomials $f_n(z) = z + (z^2/n)$, $z \in \mathbb{D}$ for each $n \geq 2$ (cf. [13, p. 267]), and observing that the image of $\log(f_n(z)/z)$ does not include the point $\log(1 + (1/n))$. This fact confirms our choice of defining the Bohr phenomenon for $\log(f(z)/z)$ in the classical manner instead of using an inequality of the type (1.2). We derive Bohr inequalities in the form (1.4) while $f$ is a member of $S$, $S^*$, $\mathcal{L}$. Moreover, for $f \in S(\text{or } S^*)$, $f^{-1}(w)$ is defined in a neighborhood of the origin, which in particular can be chosen to be $\mathbb{D}_{1/4} := \{w \in \mathbb{C} : |w| < 1/4\}$, as we know that any $f \in S(\text{or } S^*)$ covers $\mathbb{D}_{1/4}$.

Therefore it is possible to define the logarithmic coefficients of $f^{-1}$ for $f \in S(\text{or } S^*)$ by the following expression:

$$\log \left( \frac{f^{-1}(w)}{w} \right) = 2 \sum_{n=1}^{\infty} \gamma_n w^n, \quad w \in \mathbb{D}_{1/4}$$  (1.5)

(compare [25]). We also compute the Bohr radius for $\log \left( \frac{f^{-1}(w)}{w} \right)$ with respect to the inequality (1.4), where $r = |w|$ and $f \in S(\text{or } S^*)$. Another important class $\mathcal{U}(\lambda)$ is being extensively studied by many authors (cf. [20,21] and the references therein) which is defined by $\mathcal{U}(\lambda) = \{f \in \mathcal{A} : |U_f(z)| < \lambda\}$ where $0 < \lambda \leq 1$, and

$$U_f(z) := \left( \frac{z}{f(z)} \right)^2 \frac{f'(z)}{f(z)} - 1, \quad z \in \mathbb{D}.$$  

It is well known that $\mathcal{U}(\lambda) \subseteq S$, and also that $\mathcal{U}(\lambda)$ neither contains $S^*$ nor is contained in it. Since the coefficient problem for $f$ or $\log(f(z)/z)$, $f \in \mathcal{U}(\lambda)$ has not yet been fully solved, the Bohr radius problem for $\log(f(z)/z)$ becomes quite appealing whenever $f \in \mathcal{U}(\lambda)$. Therefore, we end this article with a Bohr inequality for $\log(f(z)/z)$, $f \in \mathcal{U}(\lambda)$. Note that the power series described in (1.3) and (1.5) will be called logarithmic power series in this article.

## 2 Bohr Phenomenon for Locally Univalent Functions

We state the following lemma which will be required to establish two theorems in this section. It should be noted that this lemma is a direct consequence of [7, Cor. 2.3].

**Lemma 1** Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two holomorphic functions defined in $\mathbb{D}$ such that $g(z) = M\phi(z)h(z)$ for some $M > 0$, $\phi : \mathbb{D} \to \mathbb{D}$ being a holomorphic function with an expansion $\phi(z) = \sum_{n=0}^{\infty} c_n z^n$. Then

$$\sum_{n=0}^{\infty} |b_n| r^n \leq M \sum_{n=0}^{\infty} |a_n| r^n$$  (2.1)

for $|z| = r \leq 1/3$.  

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We are now ready to prove the first theorem of this section, which includes sharp Bohr radius for the subordinating family of a sense-preserving $K$-quasiconformal harmonic mapping with univalent holomorphic part.

**Theorem 1** Let $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ be a sense-preserving $K$-quasiconformal harmonic mapping defined in $\mathbb{D}$ such that $h$ is univalent in $\mathbb{D}$, and let $f_1(z) = h_1(z) + g_1(z) = \sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{\infty} d_n z^n \in S(f)$. Then

$$\sum_{n=1}^{\infty} |c_n|r^n + \sum_{n=1}^{\infty} |d_n|r^n \leq d(h(0), \partial h(\mathbb{D}))$$

(2.2)

for $|z| = r \leq r_0 = (5K + 1 - \sqrt{8K(3K + 1)})/(K + 1)$. This result is sharp for the function $p(z) = z/(1 - z)^2 + k\bar{z}/(1 - z)^2$, where $k = (K - 1)/(K + 1)$. Moreover if we take $h$ to be convex univalent then the inequality (2.2) holds for $r \leq r_0 = (K + 1)/(5K + 1)$. This result is again sharp for the function $q(z) = z/(1 - z) + k\bar{z}/(1 - z)$.

**Proof** From the definition of sense-preserving $K$-quasiconformal harmonic mappings, $h'(z) \neq 0$ for all $z \in \mathbb{D}$, and the dilatation $w_f = g'/h'$ satisfies $|w_f(z)| \leq k < 1$, $z \in \mathbb{D}$, where $k = (K - 1)/(K + 1)$. From maximum modulus principle, if $w_f$ is non-constant then $|w_f(z)| < k$ for all $z \in \mathbb{D}$. Therefore assuming $w_f$ non-constant, we see that there exists a holomorphic function $\phi : \mathbb{D} \to \mathbb{D}$ such that $g'(z) = k\phi(z)h'(z)$, $z \in \mathbb{D}$. An application of Lemma 1 readily gives

$$\sum_{n=1}^{\infty} n|b_n|r^{n-1} \leq k \sum_{n=1}^{\infty} n|a_n|r^{n-1}$$

for $r \leq 1/3$, which, upon integration from 0 to $r$ gives

$$\sum_{n=1}^{\infty} |b_n|r^n \leq k \sum_{n=1}^{\infty} |a_n|r^n$$

(2.3)

for $r \leq 1/3$. Now it is well known that since $h$ is univalent, $|a_1| \leq 4d(h(0), \partial h(\mathbb{D}))$ (see for instance [1, Lem. 1]), and the famous de Branges’s theorem asserts that $|a_n| \leq n|a_1|$ for $n \geq 1$. Consequently $|a_n| \leq 4nd(h(0), \partial h(\mathbb{D}))$ for all $n \geq 1$. Therefore from (2.3) we get

$$\sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n \leq (1 + k) \sum_{n=1}^{\infty} |a_n|r^n \leq \frac{4(1 + k)r}{(1 - r)^2} d(h(0), \partial h(\mathbb{D}))$$

(2.4)

for $r \leq 1/3$. From a direct computation we obtain that the right hand side of the inequality (2.4) is less or equal to $d(h(0), \partial h(\mathbb{D}))$ if $r^2 - (6 + 4k)r + 1 \geq 0$, or equivalently if $r \leq r_0 = (5K + 1 - \sqrt{8K(3K + 1)})/(K + 1)$. Again by straightforward
calculations one can verify that $r_0 < 1/3$. Therefore to prove our first assertion in
the theorem, it suffices to show that for $r \leq 1/3$
\[ \sum_{n=1}^{\infty} |c_n|r^n + \sum_{n=1}^{\infty} |d_n|r^n \leq \sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n, \]
which is indeed true, because $f_1 < f$ implies $h_1 < h$ and $g_1 < g$ (cf. [28, p. 164, Sec. 2]), and therefore from [8, Lem. 1], $\sum_{n=1}^{\infty} |c_n|r^n \leq \sum_{n=1}^{\infty} |a_n|r^n$ and $\sum_{n=1}^{\infty} |d_n|r^n \leq \sum_{n=1}^{\infty} |b_n|r^n$ respectively for $r \leq 1/3$. Now if $w_f$ is constant, then $w_f = ck$ for some $|c| = 1$, i.e. $g'(z) = ckh'(z)$. As a result equality occurs in (2.3) for all $r < 1$, and hence this case can be settled by following lines of reasoning similar to those we have already used. The sharpness part for the function $p$ can be verified from direct calculations.

If $h$ is taken to be convex univalent, we only need to note that $|a_n| \leq |a_1|$ for $n \geq 1$ and $|a_1| \leq 2d(h(0), \partial h(\mathbb{D}))$ (see, for example [1, Lem. 2]). Rest of the proof can be completed by following similar lines of argument presented above. \( \Box \)

**Remark** In connection with the above theorem the following interesting observations are made.

1. Theorem 1 and Remark 1 in [1] are special instances of the above Theorem 1, obtained by setting $K = 1$.
2. One can observe that the proof of Theorem 1 could be worked out by allowing $k$ to be 1 (with some small changes) to get that (2.2) holds for $r \leq r_0 = 5 - 2\sqrt{6}$, where $f$ is a sense-preserving harmonic mapping defined in $\mathbb{D}$ with $h$ univalent, and for $r \leq r_0 = 1/5$ with $h$ convex univalent. Both of these radii are the best possible.

In the next theorem we prove a sharp Bohr inequality for a sense-preserving $K$-quasiconformal harmonic mapping $f$ with the canonical representation $f = h + \overline{g}$, under the additional assumptions that $h$ is bounded and $g'(0) = 0$.

**Theorem 2** Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n$ be a sense-preserving $K$-quasiconformal harmonic mapping defined in $\mathbb{D}$, where $h$ is bounded on $\mathbb{D}$. Then
\[ \sum_{n=0}^{\infty} |a_n|r^n + \sum_{n=2}^{\infty} |b_n|r^n \leq \|h\|_\infty \] (2.5)
for $|z| = r \leq r_0$, where $r_0$ is the only root in $(0, 1)$ of the equation
\[ \frac{4Kr}{(K+1)(1-r)} + \frac{2(K-1)\log(1-r)}{K+1} = 1. \] (2.6)
This $r_0$ is the best possible.

**Proof** Without loss of generality we can consider $\|h\|_\infty = 1$. Also we observe that the case $K = 1$ follows from Theorem A. Hence it is enough to consider $K > 1$. As in

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the proof of Theorem 1, it is easily seen that if \( w_f \neq 0 \), then \( w_f/k \) is a holomorphic self mapping of \( \mathbb{D} \) with \( w_f(0) = 0 \). From the Schwarz lemma we can conclude that \( \phi(z) := w_f(z)/kz \) is again a holomorphic self mapping of \( \mathbb{D} \) if \( w_f(z) \neq kc, \ z \in \mathbb{D} \) for some \( |c| = 1 \). Therefore assuming \( w_f(z) \neq kc \), a use of Lemma 1 on \( g'(z) = k\phi(z)h'(z) \) yields

\[
\sum_{n=2}^{\infty} n|b_n| r^{n-1} \leq k \sum_{n=1}^{\infty} n|a_n| r^n,
\]

for \( |z| = r \leq 1/3 \), or equivalently

\[
\sum_{n=1}^{\infty} (n+1)|b_{n+1}| r^n \leq k \sum_{n=1}^{\infty} n|a_n| r^n
\]

for \( r \leq 1/3 \). Integrating both sides of the above inequality from 0 to \( r \) we have, for \( r \leq 1/3 \):

\[
\sum_{n=1}^{\infty} |b_{n+1}| r^{n+1} \leq k \sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right) |a_n| r^{n+1},
\]

which is same as saying

\[
\sum_{n=2}^{\infty} |b_n| r^n \leq k \sum_{n=2}^{\infty} \left( \frac{n-1}{n} \right) |a_{n-1}| r^n. \tag{2.7}
\]

Therefore using (2.7) and the well known estimates \( |a_n| \leq 1 - |a_0|^2 \) for \( n \geq 1 \), we have, for \( r \leq 1/3 \):

\[
\sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n \leq |a_0| + (1 - |a_0|^2) \sum_{n=1}^{\infty} r^n + k(1 - |a_0|^2) \sum_{n=2}^{\infty} \left( \frac{n-1}{n} \right) r^n. \tag{2.8}
\]

We here mention that for the case \( w_f(z) = kc, \ |c| = 1 \), the inequality (2.8) can be obtained from direct calculation, i.e. without any use of Lemma 1, and will hold for all \( r < 1 \). A little computation will now reveal that the right hand side of (2.8) is equal to \( |a_0| + (1 - |a_0|^2) ((1 + k)(r/(1 - r)) + k \log(1 - r)) \), which is less or equal to 1 if

\[
(1 + |a_0|) \left( \frac{2Kr}{(K+1)(1-r)} + \frac{(K-1) \log(1-r)}{K+1} \right) \leq 1,
\]

\( K \geq 1.23 \).
which is again true if
\[
\psi(r) := \frac{4Kr}{(K + 1)(1 - r)} + \frac{2(K - 1)\log(1 - r)}{K + 1} - 1 \leq 0.
\]

To prove the theorem it is sufficient to show that \(\psi(r)\) has exactly one root \(r_0\) in \((0, 1)\), \(r_0 < 1/3\) and \(\psi(r) \leq 0\) if and only if \(r \leq r_0\). We observe that \(\psi(0) = -1 < 0\) and \(\psi(1/3) = (K - 1)(1 + \log 4 - \log 9)/(K + 1) > 0\). By the intermediate value property of continuous functions there exists \(r_0 \in (0, 1/3)\) such that \(\psi(r_0) = 0\). Moreover, we observe that for all \(r \in (0, 1)\), \(\psi'(r) > 0\), which implies \(\psi\) is strictly increasing in \((0, 1)\). This asserts that \(r_0\) is the only root of \(\psi\) in \((0, 1)\), and that \(\psi(r) \leq 0\) if and only if \(r \leq r_0\). To see that \(r_0\) is the best possible we consider the functions (see also [16, p. 1763]) \(f_a(z) = h_a(z) + g_a(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n, z \in \mathbb{D}\), where \(a \in [0, 1)\) and
\[
h_a(z) = \frac{a - z}{1 - az}, \quad g_a'(z) = kzh'_a(z).
\]

It should be noted that each \(f_a\) satisfies the hypotheses of the theorem. A little computation reveals that for any fixed \(a \in [0, 1)\),
\[
\sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n = a + (1 - a^2) \left(\sum_{n=1}^{\infty} a^{n-1} r^n + k \sum_{n=2}^{\infty} \left(\frac{n - 1}{n}\right) a^{n-2} r^n\right) \leq 1
\]
if and only if
\[
M(a, r) := (1 + a) \left(\sum_{n=1}^{\infty} a^{n-1} r^n + k \sum_{n=2}^{\infty} \left(\frac{n - 1}{n}\right) a^{n-2} r^n\right) - 1 \leq 0,
\]
which holds if and only if \(r \leq r_0(a)\), \(r_0(a)\) being a non-negative real number. Now we observe that for each fixed \(a\), \(M(a, r)\) is strictly increasing in \(r \in (0, 1)\), and that \(M(a, 0) = -1 < 0\), \(M(a, 1) = 2a/(1 - a) + k(1 + a) \sum_{n=2}^{\infty} (n - 1)/n a^{n-2} > 0\). Hence the number \(r_0(a)\) is the only real root of \(M(a, r) = 0\) in \((0, 1)\). Again, we observe that \(M(a, r)\) is strictly increasing in \(a \in [0, 1)\) for each fixed \(r\), and therefore for any \(a_1, a_2 \in [0, 1)\), \(r_0(a_1) < r_0(a_2)\) whenever \(a_1 > a_2\). As a consequence, \(M(a, r) \leq 0\) for all \(a \in [0, 1)\) if and only if \(r \leq \inf_{a \in [0, 1)} r_0(a)\), and \(\inf_{a \in [0, 1)} r_0(a) = \lim_{a \to 1^-} r_0(a)\) is the root of \(\lim_{a \to 1^-} M(a, r) = 0\). Noting that
\[
M(a, r) = (1 + a) \left(\frac{(a + k)r}{a(1 - ar)} + k \frac{\log(1 - ar)}{a^2}\right) - 1,
\]
\(\lim_{a \to 1^-} M(a, r) = 0\) turns out to be the equation (2.6). Hence \(r_0 = \inf_{a \in [0, 1)} r_0(a)\), i.e. the inequality (2.5) holds for all \(f_a, a \in [0, 1)\) if and only if \(r \leq r_0\). This completes the proof. \(\square\)
Corollary 1  Let \( f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n \) be a sense-preserving harmonic mapping defined in \( \mathbb{D} \), where \( h \) is bounded on \( \mathbb{D} \). Then inequality (2.5) holds for \( |z| = r \leq r_0 = 0.299 \ldots \), where \( r_0 \) is the only root in \( (0, 1) \) of the equation
\[
\frac{4r}{1 - r} + 2 \log(1 - r) = 1.
\] (2.9)

This \( r_0 \) is the best possible.

Proof  We only need to observe that for \( f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n \) being a sense-preserving harmonic mapping defined in \( \mathbb{D} \), either \( g'(z) = czh'(z) \) for some \( |c| = 1 \), or \( g'(z) = z\phi(z)h'(z) \) for some holomorphic self mapping \( \phi \) of \( \mathbb{D} \); and hence the proof of this corollary can be completed by following similar lines of argument as in the proof of Theorem 2. Again, to see that \( r_0 \) is the best possible, one can consider the functions \( f_a(z) = h_a(z) + g_a(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n \), \( z \in \mathbb{D} \), where \( a \in (0, 1) \) and
\[
h_a(z) = \frac{a - z}{1 - az}, \quad g'_a(z) = zh'_a(z),
\]
and then proceed like the proof of the corresponding part in Theorem 2. To be more specific, a careful examination of the proof of Theorem 2 reveals that to prove this corollary, one has to reproduce the entire argument as in the proof of Theorem 2 by allowing \( k \) to be 1, with minor modifications only.

Remark

(1) Theorem 2 (resp. Corollary 1) is the refined version of [16, Thm. 3.1] (resp. [16, Cor. 3.2]). It may be observed that our proof uses a substantially different method compared to [16, Thm. 3.1]. Also Corollary 1 can be taken as a generalization of Proposition 1 from [14, p. 211], provided we note that the conclusion of Corollary 1 remains unchanged if the hypothesis “sense-preserving” is replaced by \( |g'(z)| \leq |zh'(z)| \), \( z \in \mathbb{D} \).

(2) In Corollary 1, if we take \( h \) such that \( h(z) \prec \tau(z) := (h(0) - z)/(1 - \overline{h(0)}z) \), \( z \in \mathbb{D} \), then \( \|h\|_{\infty} \leq 1 \), and hence from the inequality (2.5) we have
\[
\sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=2}^{\infty} |b_n|r^n \leq 1 - |a_0| = d(\tau(0), \partial \tau(\mathbb{D})). \tag{2.10}
\]

Therefore, Corollary 1 settles Conjecture 1(a) from [19, p. 13] for the special case \( h \prec \tau \). On the other hand, as we can assume \( \|h\|_{\infty} = 1 \) in Theorem 2 without loss of generality, we have \( h \prec \tau \), and the inequality (2.5) can be rewritten as (2.10). Hence [19, Thm. 2] can be regarded as a (non-sharp) generalization of Theorem 2 in the present article.

(3) It is interesting to note that Theorems 1.1 and 1.3 (and hence Corollary 1.4 and Corollary 1.5) from the paper [16] can also be established using the Lemma 1. One should, however, note that the part of Corollary 1.4 which remarks on the cases
that $a_0 = 0$ or $|a_0|$ being replaced by $|a_0|^2$ would produce a better Bohr radius $1/3$ instead of $1/5$, has to be proved separately, as [16, Thm. 1.2] cannot be derived from the Lemma 1.

We now establish a (possibly non-sharp) Bohr inequality for normalized uniformly locally univalent holomorphic functions with bounded pre-Schwarzian norm.

**Theorem 3** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{B}(\lambda)$ for some $\lambda \in (0, \infty)$. Then the inequality

$$r + \sum_{n=2}^{\infty} |a_n|r^n \leq d(f(0), \partial f(D))$$

holds for $|z| = r \leq r_0$, where $r_0$ is the only root in $(0, 1)$ of the equation

$$r + r\sqrt{\frac{4\lambda^2r^2}{1-r^2}} - 1 - \sqrt{\frac{2}{6} - 1} = -F_\lambda(-1).$$

The function $F_\lambda$ is given by

$$F_\lambda(z) := \int_0^z \left(\frac{1+t}{1-t}\right)^{\lambda} dt.$$

**Proof** From the definition of $\mathcal{B}(\lambda)$ we see, for any $|z| = r$

$$|(\log f'(z))'| \leq \frac{2\lambda}{1-r^2},$$

where $\log f'(z) = \sum_{n=1}^{\infty} c_n z^n, z \in \mathbb{D}$. Now the above inequality implies

$$\sum_{n=1}^{\infty} n|c_n|^2r^{2n} = \frac{1}{\pi} \int_{|\xi|<r} |(\log f'(|\xi|))'|^2 R\ dR \ d\theta \leq \frac{4\lambda^2}{\pi} \int_{|\xi|<r} \frac{R}{(1-R^2)^2} dR \ d\theta = \frac{4\lambda^2r^2}{1-r^2}.$$  

We clarify that the dummy variable $\xi$ inside the integration is taken to be $\xi = Re^{i\theta}$, $0 \leq R \leq r$ and $0 \leq \theta < 2\pi$. Using a minor variant of first Lebedev–Milin inequality (see [13, pp. 143–144]) we obtain

$$\sum_{n=1}^{\infty} n^2|a_n|^2r^{2n-2} \leq \exp\left(\sum_{n=1}^{\infty} n|c_n|^2r^{2n}\right) \leq \exp\left(\frac{4\lambda^2r^2}{1-r^2}\right).$$
In other words
\[ \sum_{n=2}^{\infty} n^2 |a_n|^2 r^{2n} \leq r^2 \left( \exp \left( \frac{4 \lambda^2 r^2}{1 - r^2} \right) - 1 \right), \]
which, along with an application of Cauchy–Schwarz inequality yields
\[
\sum_{n=1}^{\infty} |a_n| r^n \leq r + r \sqrt{\exp \left( \frac{4 \lambda^2 r^2}{1 - r^2} \right) - 1} \left( \frac{\pi^2}{6} - 1 \right) + F_\lambda(-1).
\]
From [17, Cor. 2.4] we observe that
\[ d(f(0), \partial f(\mathbb{D})) \geq -F_\lambda(-1). \] The inequality (2.11) now holds whenever \( r \leq r_0 \) for some \( r_0 \), if we can show
\[ \phi(r) := r + r \sqrt{\exp \left( \frac{4 \lambda^2 r^2}{1 - r^2} \right) - 1} \left( \frac{\pi^2}{6} - 1 \right) + F_\lambda(-1) \]
has one and only one root \( r_0 \) in \((0, 1)\) and \( \phi(r) \leq 0 \) if and only if \( r \leq r_0 \). It is easy to observe that \( \phi(0) = F_\lambda(-1) < 0 \) and \( \lim_{r \to 1^-} \phi(r) = \infty \) which together, by a use of intermediate value property for continuous functions, ensure the existence of one root \( r_0 \in (0, 1) \) of \( \phi(r) \). Again observing that \( \phi'(r) > 0 \) for all \( r \in (0, 1) \), we conclude that \( \phi \) is strictly increasing in \((0, 1)\). This proves that \( r_0 \) is the only root of \( \phi \) in \((0, 1)\), and that \( \phi(r) \leq 0 \iff r \leq r_0 \).

**Remark** We end this section with a few words on another kind of Bohr inequality for \( B(\lambda) \) which arises from the study of a larger function class containing \( B(\lambda) \). From [18, Thm. 3.2] we have that \( B(\lambda) \subset B_j(\lambda) \) for all \( \lambda > 0 \), where \( B_j(\lambda) \) is the space of holomorphic functions \( f \) in \( \mathbb{D} \) satisfying \( \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\lambda} |f(z)| < \infty \), and equipped with the norm
\[
\| f \|_{B_j(\lambda)} := |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\lambda} |f'(z)|
\]
(compare [18, p. 2]). From [17, Thm. 2.3] it is immediate that for any \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B(\lambda), \| f \|_{B_j(\lambda)} \leq 2^{2\lambda} \), and therefore, [18, Thm. 5.1] asserts that the inequality
\[
r + \sum_{n=2}^{\infty} |a_n| r^n \leq 2^{2\lambda}
\]
holds for \( r \leq \tilde{r}_c(\lambda) \), \( \tilde{r}_c(\lambda) \) being the Bohr radius obtained in [18, Thm. 5.1] for the larger class \( B_j(\lambda) \). The Bohr inequality in [18, Thm. 5.1] is similar to the classical one.
As a result it does not seem possible to draw a direct comparison between the r₀ in Theorem 3. However, following the computations in the proof of Theorem 3, we get that for any \( f \in B(\lambda) \) the inequality (2.13) holds for \( r \leq r_\alpha(\lambda) \), \( r_\alpha(\lambda) \) being the only root in \((0,1)\) of the equation

\[
\xi(r) := r + r\sqrt{\exp\left(\frac{4\lambda^2r^2}{1-r^2}\right) - 1} \sqrt{\frac{\pi^2}{6} - 1} - 2^{2\lambda} = 0.
\]

As \( B(\lambda) \) is a subclass of \( B_1(\lambda) \), it is natural to ask if \( r_\alpha(\lambda) \) improves over \( \tilde{r}_\alpha(\lambda) \). This is, however, not always the case, as using Mathematica we get: \( r_\alpha(1) \approx 0.688601 > \tilde{r}_\alpha(1) \approx 0.553567, r_\alpha(2) \approx 0.554248 > \tilde{r}_\alpha(2) \approx 0.492552 \) (see the Table 1 in [18, p. 16] for values of \( \tilde{r}_\alpha(\lambda), \lambda \in (0,3] \)), but \( r_\alpha(6) \approx 0.343103 < \tilde{r}_\alpha(6) \approx 0.352208 \). Further, it could be interesting to obtain a harmonic analogue of Theorem 3, and the Bohr inequalities analogous to [18, Thm. 5.2, 5.3] for uniformly locally univalent holomorphic or harmonic functions.

### 3 Bohr Phenomenon for Logarithmic Power Series

In the first theorem of this section, we compute sharp Bohr radii for \( \log(f(z)/z) \), \( z \in \mathbb{D} \) and \( \log(f^{-1}(w)/w) \), \( w \in \mathbb{D}_{1/4} \) where \( f \in S(\text{or} S^*) \). Besides, the sharp Bohr radius for \( \log(f(z)/z) \), \( f \in \mathcal{C} \) has been recorded in a subsequent remark.

**Theorem 4** Let \( f \in S(\text{or} S^*) \) with \( \log(f(z)/z) \) having Taylor expansion (1.3). Then the inequality (1.4) holds for \( |z| = r \leq r_0 = 1 - (1/\sqrt{e}) = 0.393 \ldots \) Moreover if the logarithmic coefficients of \( f^{-1} \) are given by (1.5) \( f \in S(\text{or} S^*) \), then inequality (1.4) is satisfied for \( |w| = r \leq r_0 = (1/e)(\sqrt{e} - 1) = 0.238 \ldots \) Both results are sharp for the function \( k_1(z) = z/(1+z)^2 \).

**Proof** For \( f \in \mathcal{S} \), the following inequality is well known (see for instance [5, p. 722]):

\[
\sum_{n=1}^{\infty} n|\gamma_n|^2 r^n \leq \log \left(\frac{1}{1-r}\right)
\]

where \( |z| = r \). Therefore using the Cauchy–Schwarz inequality gives

\[
2 \sum_{n=1}^{\infty} |\gamma_n|r^n \leq \sqrt{\sum_{n=1}^{\infty} n|\gamma_n|^2 r^n \sum_{n=1}^{\infty} r^n \sum_{n=1}^{\infty} \frac{r^n}{n} \leq 2 \log \left(\frac{1}{1-r}\right)}
\]

which is less or equal to 1 whenever \( r \leq 1 - (1/\sqrt{e}) \). Now to prove the second part of this theorem, we note that using the recent result [25, Thm. 1] gives

\[
2 \sum_{n=1}^{\infty} |\gamma_n|r^n \leq \sum_{n=1}^{\infty} \frac{2n}{n} \left(\frac{2n}{n}\right) r^n.
\]
It can be observed that for \( r < 1/4 \),

\[
\sum_{n=1}^{\infty} \binom{2n}{n} r^n = \frac{1}{\sqrt{1-4r}} - 1,
\]
or equivalently

\[
\sum_{n=1}^{\infty} \binom{2n}{n} r^{n-1} = \frac{4}{\sqrt{1-4r} (1 + \sqrt{1-4r})}.
\]

Integrating both sides of the above equation from 0 to \( r \) we get

\[
\sum_{n=1}^{\infty} \frac{1}{n} \binom{2n}{n} r^n = \int_{0}^{r} \frac{4}{\sqrt{1-4x} (1 + \sqrt{1-4x})} \, dx =: I.
\]

Setting \( 1 - 4x = t^2 \), a little calculation reveals that

\[
I = 2 \int_{\sqrt{1-4r}}^{1} \frac{dt}{1 + t} = 2 \log \frac{2}{1 + \sqrt{1-4r}}.
\]

Therefore from (3.1) it is seen that inequality (1.4) will be satisfied whenever

\[
2 \log \left( \frac{2}{1 + \sqrt{1-4r}} \right) \leq 1,
\]
or, in other words, whenever \( r \leq (1/e) (\sqrt{e} - 1) \).

Observing the fact that the function \( k_1(z) \in S^* \), the sharpness of both the results for the classes \( S \) and \( S^* \) can be shown from direct computations.

**Remark** For \( f \in C \) with \( \log(f(z)/z) \) having Taylor expansion (1.3), it is easy to prove the bounds \( |\gamma_n| \leq 1/2n \) for \( n \geq 1 \). As a result the inequality (1.4) holds for \( |z| = r \leq r_0 = 1 - (1/e) = 0.632 \ldots \). This result is sharp for the function \( l(z) = z/(1-z) \).

The next result includes the Bohr phenomenon for \( \log(f(z)/z) \) where \( f \in U(\lambda) \).

**Theorem 5** Suppose \( 0 < \lambda \leq 1 \) and \( f \in U(\lambda) \) with \( \log(f(z)/z) \) having Taylor expansion (1.3). Then the inequality (1.4) holds for

\[
|z| = r \leq r_0 = \begin{cases} 
\frac{1}{2\lambda} \left( (1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda \left( 1 - \frac{1}{e} \right)} \right) & \text{if } \lambda \geq \lambda_0, \\
\frac{1 + \lambda^2}{2(1+\lambda)} & \text{if } \lambda < \lambda_0,
\end{cases}
\]

where \( \lambda_0 \approx 0.750792 \) is the only root in \((0, 1)\) of the equation

\[
\lambda^5 - 2\lambda^4 - 2\lambda^3 - \frac{4}{e}\lambda^2 + \left( 5 - \frac{8}{e} \right) \lambda + \left( 2 - \frac{4}{e} \right) = 0. \tag{3.2}
\]

When \( \lambda \geq \lambda_0 \), this result is sharp for the function \( k_{\lambda}(z) = z/(1+z)(1+\lambda z) \).
In [20, Thm. 4], it was shown that for \( f \in \mathcal{U}(\lambda) \)

\[
\frac{f(z)}{z} < \frac{1}{(1 + z)(1 + \lambda z)}, \quad z \in D,
\]

which yields

\[
\log \left( \frac{f(z)}{z} \right) < - \log(1 - z) - \log(1 - \lambda z),
\]

or equivalently, in terms of Taylor expansions:

\[
2 \sum_{n=1}^{\infty} \gamma_n z^n < \sum_{n=1}^{\infty} \left( \frac{1 + \lambda^n}{n} \right) z^n.
\]

An application of [13, Thm. 6.3] on (3.3) gives

\[
4 \sum_{n=1}^{\infty} \left( \frac{n}{1 + \lambda^n} \right) |\gamma_n|^2 r^n \leq \sum_{n=1}^{\infty} \left( \frac{n}{1 + \lambda^n} \right) \left( \frac{1 + \lambda^n}{n} \right)^2 r^n = \sum_{n=1}^{\infty} \left( \frac{1 + \lambda^n}{n} \right) r^n,
\]

whenever \((n/(1 + \lambda^n))r^n \geq ((n + 1)/(1 + \lambda^{n+1}))r^{n+1}\) for all \(n \geq 1\), i.e. whenever \(r \leq ((1 + \lambda^{n+1})/(1 + \lambda^n)(n/(n + 1))\) for all \(n \geq 1\). We now observe that

\[
u_n := \frac{1 + \lambda^{n+1}}{1 + \lambda^n} \quad \text{and} \quad \gamma_n := \frac{n}{n + 1}\]

both are monotonically increasing sequences, which imply \(u_n \geq u_1 = (1 + \lambda^2)/(1 + \lambda)\) and \(v_n \geq v_1 = 1/2, \quad n \in \mathbb{N}\). Therefore the inequality (3.4) remains valid for \(r \leq (1 + \lambda^2)/2(1 + \lambda)\). Now a use of Cauchy–Schwarz inequality gives

\[
2 \sum_{n=1}^{\infty} |\gamma_n| r^n \leq \sqrt{4 \sum_{n=1}^{\infty} \left( \frac{n}{1 + \lambda^n} \right) |\gamma_n|^2 r^n} \sqrt{\sum_{n=1}^{\infty} \left( \frac{1 + \lambda^n}{n} \right) r^n}.
\]

Using (3.4) on the right hand side of the above inequality we obtain

\[
2 \sum_{n=1}^{\infty} |\gamma_n| r^n \leq \sum_{n=1}^{\infty} \left( \frac{1 + \lambda^n}{n} \right) r^n = - \log(1 - r) - \log(1 - \lambda r)
\]

for \(r \leq (1 + \lambda^2)/2(1 + \lambda)\). It is readily seen that the inequality (1.4) would be satisfied if \(- \log(1 - r) - \log(1 - \lambda r) \leq 1\), i.e. if \(r \leq r_0 = \min\{r_b, (1 + \lambda^2)/2(1 + \lambda)\} \) where \(r_b\) be the smallest positive root of

\[
\n
\]
\[ h(\lambda) := r^2 - \left(1 + \frac{1}{\lambda}\right)r + \frac{1}{\lambda}\left(1 - \frac{1}{e}\right) = 0. \] (3.6)

It is clear that \( r_b = ((1+\lambda)-\sqrt{(1+\lambda)^2 - 4\lambda(1-(1/e)))}/2\lambda \). We observe that \( h(0) = (1/\lambda)(1-(1/e)) > 0 \), \( h(1) = -1/\lambda e < 0 \), which by the intermediate value property implies that \( h \) has at least one root in \((0, 1)\). Also since \( h'(r) = 2r - (1+(1/\lambda)) < 0 \), \( h \) is strictly decreasing in \((0, 1)\), and therefore \( h \) has exactly one root in \((0, 1)\) which is clearly \( r_b \). Now it needs to be proved that \( r_b = r_b \) precisely when \( \lambda \geq \lambda_0 \). In fact we have to show that \( r_b \leq (1+\lambda^2)/2(1+\lambda) \) if and only if \( \lambda \geq \lambda_0 \). In other words we need to establish that \( h((1+\lambda^2)/2(1+\lambda)) \leq 0 \) if and only if \( \lambda \geq \lambda_0 \), or equivalently, after a little calculation:

\[ \lambda \geq \lambda_0 \iff g(\lambda) := \lambda^5 - 2\lambda^4 - 2\lambda^3 - \frac{4}{e}\lambda^2 + \left(5 - \frac{8}{e}\right)\lambda + \left(2 - \frac{4}{e}\right) \leq 0. \] (3.7)

Now \( g(0) = 2 - (4/e) > 0 \), and \( g(1) = 4 - (16/e) < 0 \) assert that \( g \) has at least one root in \((0, 1)\), which we choose to be \( \lambda_0 \). Now it is sufficient to show that \( g(\lambda) > 0 \) for \( \lambda < \lambda_0 \) and \( g(\lambda) \leq 0 \) for \( \lambda \geq \lambda_0 \). We see that

\[ g'(\lambda) = 5\lambda^4 - 8\lambda^3 - 6\lambda^2 - \frac{8}{e}\lambda + \left(5 - \frac{8}{e}\right), \]

and therefore \( g'(0) = 5 - (8/e) > 0 \) and \( g'(1) = -4 - (16/e) < 0 \), which ensure that \( g' \) has at least one root \( \mu_0 \in (0, 1) \). Again we observe that \( g''(\lambda) = 20\lambda^3 - 24\lambda^2 - 12\lambda - (8/e), \) and therefore \( g''(0) = -8/e. \) Since for \( \lambda \in (0, 1) \)

\[ g''(\lambda) = 60\lambda^2 - 48\lambda - 12 = 12(\lambda - 1)(5\lambda + 1) < 0, \]

\( g'' \) is strictly decreasing in \((0, 1)\) and hence \( g''(\lambda) < g''(0) < 0 \). This now asserts that \( g' \) is strictly decreasing in \((0, 1)\). Therefore \( \mu_0 \) is the only root of \( g' \) in \((0, 1)\), \( g'(\lambda) > 0 \) for \( \lambda < \mu_0 \) and \( g'(\lambda) < 0 \) for \( \lambda > \mu_0 \). Now let if possible, \( \mu_0 > \lambda_0 \). Then \( g \) is strictly increasing in \((0, (\lambda_0 + \mu_0)/2)\), and as a result \( g(\lambda_0) > g(0) > 0 \) which is contrary to our assumption. Therefore \( \mu_0 \leq \lambda_0 \), which shows that \( g \) is strictly increasing in \((0, \mu_0)\) and strictly decreasing in \((\mu_0, \lambda_0) \cup [\lambda_0, 1)\), \( \mu_0 \) being a local maximum of \( g \). Clearly for any \( \lambda \in (0, \mu_0) \), \( g(\lambda) > g(0) > 0 \); for \( \lambda \in [\mu_0, \lambda_0) \), \( g(\lambda) > g(\lambda_0) = 0 \); and for \( \lambda \in [\lambda_0, 1) \), \( g(\lambda) \leq g(\lambda_0) = 0 \). This validates our assertion (3.7). Using Mathematica it can be computed that \( \lambda_0 \) is approximately 0.750792. The sharpness part for \( \lambda \geq \lambda_0 \) is immediate from our computation.

\[ \square \]

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