Noncommutative sedeons and their application in field theory

Victor L. Mironov* and Sergey V. Mironov

Institute for physics of microstructures RAS, Nizhniy Novgorod, Russia

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Abstract

We present sixteen-component values "sedeons", generating associative noncommutative space-time algebra. The generalized second-order and first-order equations of relativistic quantum mechanics based on sedeonic wave function and sedeonic space-time operators are proposed. We also discuss the description of fields with massive quantum on the basis of second-order and first-order equations for sedeonic potentials.

Introduction

The application of multicomponent hypercomplex numbers and multivectors in classical and quantum physics has a long history. In particular, the simplest generalization of electrodynamics and quantum mechanics was developed on the basis of quaternions [1]-[6]. The structure of quaternions with four components (scalar and vector) corresponds to the relativistic four-vector approach that allows one to reformulate field equations in terms of quaternionic algebra. However, the essential imperfection of the quaternionic algebra is that the quaternions do not include pseudoscalar and pseudovector components. The consideration of full space symmetry with respect to spatial inversion leads to the eight-component structures enclosing scalar, pseudoscalar, vector and pseudovector. There is a lot of works on application of different eight-component values such as biquaternions and octonions in classical electrodynamics and relativistic quantum mechanics [7]-[20]. However, a consistent relativistic approach implies equally the space and time symmetries that requires the consideration of the extended sixteen-component space-time algebras.

There are a few approaches in the development of theory on the basis of sixteen-component structures. One of them is the application of hypercomplex numbers sedenions, which are obtained from octonions by Cayley-Dickson extension procedure [21]-[25]. But as in the case of octonions the essential imperfection of sedenions is their nonassociativity. Another approach is based on the application of hypercomplex multivectors generating associative space-time Clifford algebras. The basic idea of such multivectors is an introduction of additional noncommutative time unit vector, which is orthogonal to the space unit vectors [26, 27]. However, the application of such multivectors in quantum mechanics is considered in general as one of abstract algebraic scheme enabling the reformulation of Klein-Gordon and Dirac equations for the multicomponent wave functions but does not touch the physical entity of these equations.

Recently we have developed an alternative approach based on our scalar-vector concept [28]-[30] realized in sixteen-component sedeons. In present paper we demonstrate the application of the sedeons to the reformulation of relativistic quantum mechanics and massive field equations.

*E-mail: mironov@ipmras.ru

1
1 Sedeonic space-time algebra

The sedeonic algebra encloses four groups of values, which are differed with respect to spatial and time inversion.

1. Absolute scalars ($V$) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.

2. Time scalars ($V_t^i$) and time vectors ($\vec{V}_t^i$) are changed (in sign) under time inversion but are not transformed under spatial inversion.

3. Space scalars ($V_s^i$) and space vectors ($\vec{V}_s^i$) are changed under spatial inversion but are not transformed under time inversion.

4. Space-time scalars ($V_{ts}^i$) and space-time vectors ($\vec{V}_{ts}^i$) are changed under spatial and time inversion.

Here indexes $t$ and $r$ indicate the transformations ($t$ for time inversion and $r$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\vec{V}$, which is defined by the following expression:

$$\vec{V} = V + \vec{V} + V_t + \vec{V}_t + V_s + \vec{V}_s + V_{ts} + \vec{V}_{ts}.$$ (1)

Let us introduce scalar-vector basis $a_0$, $a_1$, $a_2$, $a_3$, where the value $a_0 \equiv 1$ is absolute scalar unit and the values $a_1$, $a_2$, $a_3$ are absolute unit vectors generating the right Cartesian basis. We introduce also four space-time scalar units $e_0$, $e_1$, $e_2$, $e_3$, where value $e_0 \equiv 1$ is a absolute scalar unit; $e_1 \equiv e_t$ is a time scalar unit; $e_2 \equiv e_s$ is a space scalar unit; $e_3 \equiv e_{ts}$ is a space-time scalar unit. Using space-time scalar units $e_j$ ($j = 0, 1, 2, 3$) and scalar-vector basis $a_k$ ($k = 0, 1, 2, 3$) we can introduce unified sedeonic components $V_{jk}$ in accordance with the following relations:

$$V = e_0 V_{00} a_0,$$
$$\vec{V} = e_0 (V_{01} a_1 + V_{02} a_2 + V_{03} a_3),$$
$$V_t = e_1 V_{10} a_0,$$
$$\vec{V}_t = e_1 (V_{11} a_1 + V_{12} a_2 + V_{13} a_3),$$
$$V_s = e_2 V_{20} a_0,$$
$$\vec{V}_s = e_2 (V_{21} a_1 + V_{22} a_2 + V_{23} a_3),$$
$$V_{ts} = e_3 V_{30} a_0,$$
$$\vec{V}_{ts} = e_3 (V_{31} a_1 + V_{32} a_2 + V_{33} a_3).$$ (2)

Then the sedeon [1] can be written in the following expanded form:

$$\vec{V} = e_0 (V_{00} a_0 + V_{01} a_1 + V_{02} a_2 + V_{03} a_3)
+ e_1 (V_{10} a_0 + V_{11} a_1 + V_{12} a_2 + V_{13} a_3)
+ e_2 (V_{20} a_0 + V_{21} a_1 + V_{22} a_2 + V_{23} a_3)
+ e_3 (V_{30} a_0 + V_{31} a_1 + V_{32} a_2 + V_{33} a_3).$$ (3)

The sedeonic components $V_{jk}$ are numbers (complex in general). Further we will use symbol 1 instead of units $a_0$ and $e_0$ for simplicity.

The multiplication and commutation rules for sedeonic absolute unit vectors $a_1$, $a_2$, $a_3$ and space-time units $e_1$, $e_2$, $e_3$ are presented in tables 1 and 2 respectively.

In the tables and further the value $i$ is the imaginary unit ($i^2 = -1$). Note that sedeonic units $e_1$, $e_2$, $e_3$ and unit vectors $a_1$, $a_2$, $a_3$ generate the anticommutative algebras:

$$a_n a_m = -a_m a_n,$$
$$e_n e_m = -e_m e_n.$$ (4)

for $n$ and $m = 1, 2, 3$ ($n \neq m$), but $e_1$, $e_2$, $e_3$ commute with $a_1$, $a_2$, $a_3$:

$$a_n e_m = e_m a_n.$$ (5)
Table 1:

|   | \( a_1 \) | \( a_2 \) | \( a_3 \) |
|---|---|---|---|
| \( a_1 \) | 1 | \( ia_3 \) | \( -ia_2 \) |
| \( a_2 \) | \( -ia_3 \) | 1 | \( ia_1 \) |
| \( a_3 \) | \( ia_2 \) | \( -ia_1 \) | 1 |

Table 2:

|   | \( e_1 \) | \( e_2 \) | \( e_3 \) |
|---|---|---|---|
| \( e_1 \) | 1 | \( ie_3 \) | \( -ie_2 \) |
| \( e_2 \) | \( -ie_3 \) | 1 | \( ie_1 \) |
| \( e_3 \) | \( ie_2 \) | \( -ie_1 \) | 1 |

for any \( n \) and \( m \).

Thus the sedeon \( \tilde{V} \) is the complicated space-time object consisting of absolute scalar, time scalar, space scalar, space-time scalar, absolute vector, time vector, space vector and space-time vector.

Introducing the designations of scalar-vector values

\[
\begin{align*}
\nabla_0 &= V_{00} + V_{01} a_1 + V_{02} a_2 + V_{03} a_3, \\
\nabla_1 &= V_{10} + V_{11} a_1 + V_{12} a_2 + V_{13} a_3, \\
\nabla_2 &= V_{20} + V_{21} a_1 + V_{22} a_2 + V_{23} a_3, \\
\nabla_3 &= V_{30} + V_{31} a_1 + V_{32} a_2 + V_{33} a_3,
\end{align*}
\]

we can write the sedeon (3) in the compact form

\[
\tilde{V} = \nabla_0 + e_1 \nabla_1 + e_2 \nabla_2 + e_3 \nabla_3.
\]

On the other hand, introducing the designations of space-time sedeon-scalars

\[
\begin{align*}
V_0 &= V_{00} + e_1 V_{01} + e_2 V_{20} + e_3 V_{30}, \\
V_1 &= V_{01} + e_1 V_{11} + e_2 V_{21} + e_3 V_{31}, \\
V_2 &= V_{02} + e_1 V_{12} + e_2 V_{22} + e_3 V_{32}, \\
V_3 &= V_{03} + e_1 V_{13} + e_2 V_{23} + e_3 V_{33},
\end{align*}
\]

we can write the sedeon (3) in another form

\[
\tilde{V} = V_0 + V_1 a_1 + V_2 a_2 + V_3 a_3.
\]

or introducing the sedeon-vector

\[
\tilde{V} = \tilde{V} + \tilde{V}_t + \tilde{V}_r + \tilde{V}_{tr} = V_1 a_1 + V_2 a_2 + V_3 a_3.
\]

it can be represented in following compact form:

\[
\tilde{V} = V_0 + \tilde{V}.
\]

Further we will indicate the sedeon-scalars and the sedeon-vectors with the bold capital letters.

Let us consider the sedeonic multiplication in detail. The sedeonic product of two sedeons \( \tilde{A} \) and \( \tilde{B} \) can be presented in the following form:

\[
\tilde{A} \tilde{B} = \left( A_0 + \tilde{A} \right) \left( B_0 + \tilde{B} \right)
= A_0 B_0 + A_0 \tilde{B} + A \tilde{B}_0 + \left( \tilde{A} \cdot \tilde{B} \right) + [\tilde{A} \times \tilde{B}].
\]
Here we denote the sedeonic scalar multiplication of two sedeon-vectors (internal product) by symbol “•” and round brackets

\[
(\hat{A} \cdot \hat{B}) = A_1B_1 + A_2B_2 + A_3B_3,
\]

(13)

and sedeonic vector multiplication (external product) by symbol “×” and square brackets

\[
\begin{align*}
\hat{A} \times \hat{B} &= i (A_2B_3 - A_3B_2) a_1 + i (A_3B_1 - A_1B_3) a_2 \\
&+ i (A_1B_2 - A_2B_1) a_3.
\end{align*}
\]

(14)

In (13) and (14) the multiplication of sedeonic components is performed in accordance with (8) and table 2. Note that in sedeonic algebra the expression for the vector product has some differences from analogous expression in Gibbs vector algebra. Let us consider three absolute vectors \( \vec{A} \), \( \vec{B} \) and \( \vec{C} \). Then the formula for the vector triple product in sedeonic algebra has the following form:

\[
\left[ \vec{A} \times \left[ \vec{B} \times \vec{C} \right] \right] = -\hat{\vec{B}} \left( \vec{A} \cdot \vec{C} \right) + \hat{\vec{C}} \left( \vec{A} \cdot \hat{\vec{B}} \right).
\]

(15)

Thus, the sedeonic product

\[
\hat{F} = \hat{\vec{A}} \hat{\vec{B}} = F_0 + \hat{\vec{F}}
\]

(16)

has the following components:

\[
\begin{align*}
F_0 &= A_0B_0 + A_1B_1 + A_2B_2 + A_3B_3, \\
F_1 &= A_1B_0 + A_0B_1 + iA_2B_3 - iA_3B_2, \\
F_2 &= A_2B_0 + A_0B_2 + iA_3B_1 - iA_1B_3, \\
F_3 &= A_3B_0 + A_0B_3 + iA_1B_2 - iA_2B_1.
\end{align*}
\]

(17)

2 Sedeonic spatial rotation and space-time conjugation

The rotation of the sedeon \( \hat{V} \) on the angle \( \theta \) around the absolute unit vector \( \vec{n} \) is realized by uncompleted sedeon

\[
\hat{U} = \cos(\theta/2) + i\vec{n}\sin(\theta/2)
\]

(18)

and by complex conjugated sedeon

\[
\hat{U}^* = \cos(\theta/2) - i\vec{n}\sin(\theta/2),
\]

(19)

which satisfy the relation

\[
\hat{U}^*\hat{U} = \hat{U}\hat{U}^* = 1.
\]

(20)

The transformed sedeon \( \hat{V}' \) is defined as the sedeonic product

\[
\hat{V}' = \hat{U}^*\hat{V}\hat{U}.
\]

(21)

Thus the transformed sedeon \( \hat{V}' \) can be written in the following expanded form:

\[
\hat{V}' = (\cos(\theta/2) - i\vec{n}\sin(\theta/2)) (V_0 + \hat{\vec{V}})(\cos(\theta/2) + i\vec{n}\sin(\theta/2))
\]

\[
= V_0 + \hat{\vec{V}} \cos \theta + (1 - \cos \theta)(\vec{n} \cdot \hat{\vec{V}})\vec{n} - i \sin \theta[\vec{n} \times \hat{\vec{V}}].
\]

(22)

It is clearly seen that rotation does not transform the sedeon-scalar part, but sedeonic vector \( \hat{V} \) is rotated on the angle \( \theta \) around \( \vec{n} \).

The operations of time conjugation (\( \hat{R}_t \)), space conjugation (\( \hat{R}_r \)) and space-time conjugation (\( \hat{R}_{tr} \)) are connected with transformations in \( e_1, e_2, e_3 \) basis and can be presented as

\[
\begin{align*}
\hat{R}_t\hat{V} &= e_2\hat{V}e_2 = \hat{\vec{V}}_0 - e_1\hat{\vec{V}}_1 + e_2\hat{\vec{V}}_2 - e_3\hat{\vec{V}}_3, \\
\hat{R}_r\hat{V} &= e_1\hat{V}e_1 = \hat{\vec{V}}_0 + e_1\hat{\vec{V}}_1 - e_2\hat{\vec{V}}_2 - e_3\hat{\vec{V}}_3, \\
\hat{R}_{tr}\hat{V} &= e_3\hat{V}e_3 = \hat{\vec{V}}_0 - e_1\hat{\vec{V}}_1 - e_2\hat{\vec{V}}_2 + e_3\hat{\vec{V}}_3.
\end{align*}
\]

(23)
3 Sedeonic Lorentz transformations

The relativistic event four-vector can be represented in the following sedeon form:

\[ \mathbf{S} = i\mathbf{e}_1 ct + \mathbf{e}_2 \mathbf{r}, \]

(24)

where \( c \) is the velocity of light, \( t \) is the absolute scalar of time and \( \mathbf{r} \) is the absolute radius-vector. The square of this value is the Lorentz invariant

\[ \mathbf{SS} = -c^2 t^2 + x^2 + y^2 + z^2. \]

(25)

The Lorentz transformation of event four-vector is realized by sedeons

\[ \mathbf{L} = \cosh \vartheta - \mathbf{e}_3 \mathbf{m} \sinh \vartheta, \]

(26)

\[ \mathbf{L}' = \cosh \vartheta + \mathbf{e}_3 \mathbf{m} \sinh \vartheta, \]

(27)

where \( \tanh 2\vartheta = v/c; v \) is velocity of motion along the absolute unit vector \( \mathbf{m} \). Note that

\[ \mathbf{L} \cdot \mathbf{L}' = \mathbf{L}' \cdot \mathbf{L}^* = 1. \]

(28)

The transformed event four-vector \( \mathbf{S}' \) is written as

\[ \mathbf{S}' = \mathbf{L}^* \mathbf{S} \mathbf{L} = (\cosh \vartheta + \mathbf{e}_3 \mathbf{m} \sinh \vartheta)(i\mathbf{e}_1 ct + \mathbf{e}_2 \mathbf{r})(\cosh \vartheta - \mathbf{e}_3 \mathbf{m} \sinh \vartheta) \]

(29)

\[ = i\mathbf{e}_1 ct \cosh 2\vartheta - i\mathbf{e}_1 (\mathbf{m} \cdot \mathbf{r}) \sinh 2\vartheta + \mathbf{e}_2 \mathbf{r} \cosh^2 \vartheta - \mathbf{e}_2 ct \mathbf{m} \sinh 2\vartheta + \mathbf{e}_2 (\mathbf{m} \cdot \mathbf{r}) \mathbf{m} \sinh^2 \vartheta + \mathbf{e}_2 [\mathbf{m} \times \mathbf{r}] \times \mathbf{m} \sinh^2 \vartheta. \]

Separating the values with \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) we get the well-known expressions for the time and coordinates transformations [31]:

\[ t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}, \quad x' = \frac{x - tv}{\sqrt{1 - v^2/c^2}}, \quad y' = y, \quad z' = z, \]

(30)

where \( x \) is the coordinate along the \( \mathbf{m} \) vector.

Let us also consider the Lorentz transformation of the full sedeon \( \mathbf{V} \). The transformed sedeon \( \mathbf{V}' \) can be written as sedeonic product

\[ \mathbf{V}' = \mathbf{L} \cdot \mathbf{V} \mathbf{L}. \]

(31)

In expanded form:

\[ \mathbf{V}' = \mathbf{V}_0 \cosh^2 \vartheta - \mathbf{e}_{\mathbf{e}_1} \mathbf{V}_0 \mathbf{e}_{\mathbf{e}_1} \sinh^2 \vartheta + (\mathbf{e}_{\mathbf{e}_1} \mathbf{V}_0 - \mathbf{V}_0 \mathbf{e}_{\mathbf{e}_1}) \mathbf{m} \cosh \vartheta \sinh \vartheta \]

\[ + \mathbf{V} \cosh^2 \vartheta - \mathbf{e}_{\mathbf{e}_1} \mathbf{V}_0 \mathbf{e}_{\mathbf{e}_1} \sinh^2 \vartheta + (\mathbf{e}_{\mathbf{e}_1} \mathbf{V}_0 - \mathbf{V}_0 \mathbf{e}_{\mathbf{e}_1}) \mathbf{m} \cosh \vartheta \sinh \vartheta. \]

(32)

Rewriting the expression [32] with scalar [33] and vector [34] products we get

\[ \mathbf{V}' = \mathbf{V}_0 \cosh^2 \vartheta - \mathbf{e}_{\mathbf{e}_1} \mathbf{V}_0 \mathbf{e}_{\mathbf{e}_1} \sinh^2 \vartheta + (\mathbf{e}_{\mathbf{e}_1} \mathbf{V}_0 - \mathbf{V}_0 \mathbf{e}_{\mathbf{e}_1}) \mathbf{m} \cosh \vartheta \sinh \vartheta \]

\[ + \mathbf{V} \cosh^2 \vartheta - \mathbf{e}_{\mathbf{e}_1} \mathbf{V}_0 \mathbf{e}_{\mathbf{e}_1} \sinh^2 \vartheta + (\mathbf{e}_{\mathbf{e}_1} \mathbf{V}_0 - \mathbf{V}_0 \mathbf{e}_{\mathbf{e}_1}) \mathbf{m} \cosh \vartheta \sinh \vartheta \]

\[ + (\mathbf{e}_{\mathbf{e}_1} [\mathbf{m} \times \mathbf{V}] - [\mathbf{V} \times \mathbf{m}] \mathbf{e}_{\mathbf{e}_1}) \cosh \vartheta \sinh \vartheta. \]

(33)

Thus, the transformed sedeon have the following components:

\[ V'_0 = V_0, \]

\[ V'_{\mathbf{e}_1} = V_{\mathbf{e}_1}, \]

\[ V'_{\mathbf{e}_2} = V_{\mathbf{e}_2} \cosh 2\vartheta + \mathbf{e}_{\mathbf{e}_1} (\mathbf{m} \cdot \mathbf{V}) \sinh 2\vartheta, \]

\[ V'_{\mathbf{e}_3} = V_{\mathbf{e}_3} \cosh 2\vartheta + \mathbf{e}_{\mathbf{e}_1} (\mathbf{m} \cdot \mathbf{V}) \sinh 2\vartheta, \]

\[ V'_x = V_x \cosh 2\vartheta - 2(\mathbf{m} \cdot \mathbf{V}) \mathbf{m} \sinh^2 \vartheta + \mathbf{e}_{\mathbf{e}_1} [\mathbf{m} \times \mathbf{V}] \sinh 2\vartheta, \]

\[ V'_y = V_y \cosh 2\vartheta - 2(\mathbf{m} \cdot \mathbf{V}) \mathbf{m} \sinh^2 \vartheta + \mathbf{e}_{\mathbf{e}_1} [\mathbf{m} \times \mathbf{V}] \sinh 2\vartheta, \]

(34)

\[ V'_z = V_z + 2(\mathbf{m} \cdot \mathbf{V}) \mathbf{m} \sinh^2 \vartheta + \mathbf{e}_{\mathbf{e}_1} V \mathbf{m} \sinh 2\vartheta, \]

\[ V'_t = V_t + 2(\mathbf{m} \cdot \mathbf{V}) \mathbf{m} \sinh^2 \vartheta + \mathbf{e}_{\mathbf{e}_1} V \mathbf{m} \sinh 2\vartheta. \]
4 Sedeonic generalization of Klein-Gordon equation

The wave function of relativistic particle satisfies an equation, which is obtained from the Einstein relation between energy and momentum

$$E^2 - cp^2 = m_0^2 c^4$$  \hspace{1cm} (35)

by means of changing classical energy $E$ and momentum $\vec{p}$ on corresponding quantum-mechanical operators:

$$\hat{E} = i\hbar \frac{\partial}{\partial t} \quad \text{and} \quad \hat{\vec{p}} = -i\hbar \vec{\nabla},$$  \hspace{1cm} (36)

where $c$ is the speed of light, $m_0$ is the mass of particle, $\hbar$ is the Planck constant. The absolute vector of gradient has the following form:

$$\vec{\nabla} = \frac{\partial}{\partial x} a_1 + \frac{\partial}{\partial y} a_2 + \frac{\partial}{\partial z} a_3.$$  \hspace{1cm} (37)

In sedeonic algebra the Einstein relation (35) can be written as

$$(ie_t E + e_r c\vec{p} + e_{tr} m_0 c^2)(ie_t E + e_r c\vec{p} + e_{tr} m_0 c^2) = 0.$$  \hspace{1cm} (38)

Let us consider the wave function in the form of space-time sedeon

$$\tilde{V}(\vec{r}, t) = V_0(\vec{r}, t) + \vec{V}(\vec{r}, t).$$  \hspace{1cm} (39)

Then the generalized sedeonic wave equation for sedeonic wave function is written in the following form

$$\left( ie_t \frac{1}{c} \frac{\partial}{\partial t} - e_r \vec{\nabla} - ie_{tr} \frac{m_0 c}{\hbar} \right) \left( ie_t \frac{1}{c} \frac{\partial}{\partial t} - e_r \vec{\nabla} - ie_{tr} \frac{m_0 c}{\hbar} \right) \tilde{V} = 0.$$  \hspace{1cm} (40)

In this equation the basis elements $e_t$, $e_r$, $e_{tr}$ and $a_1$, $a_2$, $a_3$ play the role of the space-time operators, which transform the sedeonic wave function $\tilde{V}$ by means of component permutation. In fact the equation (40) is the system of 16 scalar equations for each component of wave function.

Redefining the operators

$$\partial_t = e_t \frac{1}{c} \frac{\partial}{\partial t},$$
$$\nabla_r = e_r \vec{\nabla},$$
$$m_{tr} = e_{tr} \frac{m_0 c}{\hbar},$$  \hspace{1cm} (41)

we can rewrite the equation (40) in compact form:

$$\left( i\partial_t - \nabla_r - im_{tr} \right) \left( i\partial_t - \nabla_r - im_{tr} \right) \tilde{V} = 0.$$  \hspace{1cm} (42)

Formally, the sedeonic equation (42) can be represented in the form of the system of Maxwell-like first-order equations. Let us consider the sequential action of operators. After the action of the first operator in the left part of equation (42) we obtain

$$\left( i\partial_t - \nabla_r - im_{tr} \right) \tilde{V} = i\partial_t V_0 + i\partial_t \vec{V}$$
$$-\nabla_r V_0 - \left( \nabla_r \cdot \vec{V} \right) - \left[ \nabla_r \times \vec{V} \right] - im_{tr} V_0 - im_{tr} \vec{V}.$$  \hspace{1cm} (43)

Introducing the scalar and vector values

$$E_0 = i\partial_t V_0 - \left( \nabla_r \cdot \vec{V} \right) - im_{tr} V_0,$$  \hspace{1cm} (44)
$$\vec{E} = i\partial_t \vec{V} - \nabla_r \vec{V}_0 - \left[ \nabla_r \times \vec{V} \right] - im_{tr} \vec{V},$$  \hspace{1cm} (45)
the relation (45) is presented as
\[ (i\partial_t - \vec{\nabla}_r - im_{tr}) \tilde{V} = E_0 + \vec{E}. \]

Then the wave equation (42) can be rewritten in the following form:
\[ (i\partial_t - \vec{\nabla}_r - im_{tr}) \left( E_0 + \vec{E} \right) = 0. \]

Applying the operator \( (i\partial_t - \vec{\nabla}_r - im_{tr}) \) to both parts of equation (47) and separating sedeon-scalar and sedeon-vector parts we get the wave equations for the values \( E_0 \) and \( \vec{E} \):
\[ (i\partial_t - \vec{\nabla}_r - im_{tr}) (i\partial_t - \vec{\nabla}_r - im_{tr}) E_0 = 0, \]
\[ (i\partial_t - \vec{\nabla}_r - im_{tr}) (i\partial_t - \vec{\nabla}_r - im_{tr}) \vec{E} = 0. \]

On the other hand, performing sedeonic multiplication in expression (47) and separating sedeon-scalar and sedeon-vector parts we obtain the Maxwell-like system of first-order equations:
\[ i\partial_t E_0 - \left( \vec{\nabla} \cdot \vec{E} \right) - im_{tr} E_0 = 0, \]
\[ i\partial_t \vec{E} - \left[ \vec{\nabla} \times \vec{E} \right] - im_{tr} \vec{E} - \vec{\nabla}_r E_0 = 0. \]

As one can see the values \( E_0 \) and \( \vec{E} \) can be interpreted as the quantum field intensities. These fields are defined on the whole space and carry information about the kinematic properties of the particle.

The equation (40) can be generalized for a particle in an external electromagnetic field. Let us consider the charged particle with electrical charge \( e \). In this case we have to change operators in (40) by
\[ \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{e}{\hbar} \phi, \quad \vec{\nabla} \to \vec{\nabla} - i \frac{e}{\hbar c} \vec{A}, \]
where \( \phi \) is scalar potential and \( \vec{A} \) is vector potential of electromagnetic field. Then we obtain the following wave equation
\[ \left( i e \frac{1}{c} \frac{\partial}{\partial t} - e \frac{1}{h c} \phi - e_r \vec{\nabla} + i e_r \frac{e}{h c} \vec{A} - i e_{tr} \frac{m_0 c}{h} \right) \tilde{V} = 0. \]

This equation describes the charged particle with spin 1/2 in an external electromagnetic field.

5 Sedeonic generalization of Dirac equation

The sedeonic algebra enables the reformulation of the first-order Dirac equation as a wave equation for the sedeonic wave function:
\[ \left( i e \frac{1}{c} \frac{\partial}{\partial t} - e_r \frac{e}{h c} \phi - e_r \vec{\nabla} - i e_{tr} \frac{m_0 c}{h} \right) \tilde{V} = 0. \]

In this equation the basis elements \( e_t, e_r, e_{tr} \) and \( a_1, a_2, a_3 \) play the role of the space-time operators, which transform the sedeonic wave function \( \tilde{V} \) by means of component permutation. In fact, equation (53) describes the special quantum field with zero field intensities \( E_0 \) and \( \vec{E} \) (see expression (46)).

The equations (53) can be generalized for a particle in an external electromagnetic field. In this case we have
\[ \left( i e \frac{1}{c} \frac{\partial}{\partial t} - e_t \frac{e}{h c} \phi - e_r \vec{\nabla} + i e_r \frac{e}{h c} \vec{A} - i e_{tr} \frac{m_0 c}{h} \right) \tilde{V} = 0. \]

This equation describes the particle with spin 1/2 in an external electromagnetic field.
6 Sedeonic second-order equation for massive field

6.1 Homogeneous equation

The Einstein relation between energy and momentum (35) allows another field interpretation. In this case \( E, \vec{p} \) and \( m_0 \) can be interpreted as energy, momentum and mass of a quantum of field. Then the equation

\[
\left(ie_1 \frac{1}{c} \frac{\partial}{\partial t} - e_2 \vec{\nabla} - ie_3 \frac{m_0 c}{\hbar}\right) \left(ie_1 \frac{1}{c} \frac{\partial}{\partial t} - e_2 \vec{\nabla} - ie_3 \frac{m_0 c}{\hbar}\right) \tilde{W} = 0
\]  

(55)

is the wave equation for the field potential \( \tilde{W} \) and relation

\[
E^2 - c^2 \vec{p}^2 = m_0^2 c^4
\]  

(56)

can be considered as the dispersion relation for the free wave of massive field.

Let us introduce new operators

\[
\partial = \frac{1}{c} \frac{\partial}{\partial t},
\]

\[
m = \frac{m_0 c}{\hbar}.
\]  

(57)

Then we can rewrite the equation (55) in compact form:

\[
\left(ie_1 \partial - e_2 \vec{\nabla} - ie_3 m\right) \left(ie_1 \partial - e_2 \vec{\nabla} - ie_3 m\right) \tilde{W} = 0.
\]  

(58)

Let us choose the potential in the following form:

\[
\tilde{W} = a + ie_1 b - ie_2 c - ie_3 d + i\vec{A} + e_1 \vec{B} + e_2 \vec{C} - e_3 \vec{D},
\]  

(59)

where the components \( a, b, c, d, \vec{A}, \vec{B}, \vec{C} \) and \( \vec{D} \) are the functions of spatial coordinates and time.

Introducing the scalar and vector fields strengths according to the following definitions:

\[
e = \partial b + (\vec{\nabla} \cdot \vec{C}) + md,
\]

\[
f = \partial a + (\vec{\nabla} \cdot \vec{D}) + mc,
\]

\[
g = \partial d + (\vec{\nabla} \cdot \vec{A}) - mb,
\]

\[
h = \partial c + (\vec{\nabla} \cdot \vec{B}) - ma,
\]

\[
\vec{E} = -\partial \vec{B} - \vec{\nabla} c - i[\vec{\nabla} \times \vec{C}] - m\vec{D},
\]

\[
\vec{F} = -\partial \vec{A} - \vec{\nabla} d + i[\vec{\nabla} \times \vec{D}] - m\vec{C},
\]

\[
\vec{G} = -\partial \vec{B} - \vec{\nabla} a - i[\vec{\nabla} \times \vec{A}] + m\vec{B},
\]

\[
\vec{H} = -\partial \vec{C} - \vec{\nabla} b + i[\vec{\nabla} \times \vec{B}] + m\vec{A},
\]  

(60)

we get

\[
\left(ie_1 \partial - e_2 \vec{\nabla} - ie_3 m\right) \left(a + ie_1 b - ie_2 c - ie_3 d + i\vec{A} + e_1 \vec{B} + e_2 \vec{C} - e_3 \vec{D}\right) 
\]

\[
= -e + ie_1 f - ie_2 g + ie_3 h - i\vec{E} + e_1 \vec{F} + e_2 \vec{G} + e_3 \vec{H}
\]  

(61)

and the wave equation (58) takes the form

\[
\left(ie_1 \partial - e_2 \vec{\nabla} - ie_3 m\right) \left(-e + ie_1 f - ie_2 g + ie_3 h - i\vec{E} + e_1 \vec{F} + e_2 \vec{G} + e_3 \vec{H}\right) = 0.
\]  

(62)

Performing the action of operator in the left part of the equation (62), and separating the terms with different space-time properties, we obtain the system of equations for the field’s strengths,
similar to the system of Maxwell’s equations in electrodynamics:

\[
\begin{align*}
\partial f + (\nabla \cdot \mathbf{G}) - mh &= 0, \\
\partial e + (\nabla \cdot \mathbf{H}) - mg &= 0, \\
\partial h + (\nabla \cdot \mathbf{E}) + mf &= 0, \\
\partial g + (\nabla \cdot \mathbf{F}) + me &= 0, \\
\partial \mathbf{F} + \nabla g + i(\nabla \times \mathbf{G}) - m\mathbf{H} &= 0, \\
\partial \mathbf{E} + \nabla h - i(\nabla \times \mathbf{H}) - m\mathbf{G} &= 0, \\
\partial \mathbf{H} + \nabla e + i(\nabla \times \mathbf{E}) + m\mathbf{F} &= 0, \\
\partial \mathbf{G} + \nabla f - i(\nabla \times \mathbf{F}) + m\mathbf{E} &= 0.
\end{align*}
\]  
\(63\)

The proposed equations for massive field possess a specific gauge invariance. It is easy to see that fields strengths \(60\) and equations \(63\) are not changed under the following substitutions for potentials:

\[
\begin{align*}
a &\Rightarrow a + \partial \varepsilon_a - m\varepsilon_c, \\
b &\Rightarrow b + \partial \varepsilon_b - m\varepsilon_d, \\
c &\Rightarrow c + \partial \varepsilon_c + m\varepsilon_a, \\
d &\Rightarrow d + \partial \varepsilon_d + m\varepsilon_b, \\
\mathbf{A} &\Rightarrow \mathbf{A} - \nabla \varepsilon_d, \\
\mathbf{B} &\Rightarrow \mathbf{B} - \nabla \varepsilon_c, \\
\mathbf{C} &\Rightarrow \mathbf{C} - \nabla \varepsilon_b, \\
\mathbf{D} &\Rightarrow \mathbf{D} - \nabla \varepsilon_a,
\end{align*}
\]  
\(64\)

where \(\varepsilon_a, \varepsilon_b, \varepsilon_c, \varepsilon_d\), are arbitrary scalar functions, which satisfy homogeneous Klein-Gordon equation. These gauge conditions are different from those taken in electrodynamics \(34\).

Multiplying each of the equations \(63\) to the corresponding field strength and adding these equations to each other, we obtain:

\[
\begin{align*}
\frac{1}{2} \partial \left( f^2 + e^2 + h^2 + g^2 + \mathbf{F}^2 + \mathbf{E}^2 + \mathbf{H}^2 + \mathbf{G}^2 \right) \\
+ f \left( \nabla \cdot \mathbf{G} \right) + e \left( \nabla \cdot \mathbf{H} \right) + h \left( \nabla \cdot \mathbf{E} \right) + g \left( \nabla \cdot \mathbf{F} \right) \\
+ \left( \mathbf{F} \cdot \nabla g \right) + \left( \mathbf{E} \cdot \nabla h \right) + \left( \mathbf{H} \cdot \nabla e \right) + \left( \mathbf{G} \cdot \nabla f \right) \\
+ i \left( \mathbf{F} \cdot \left( \nabla \times \mathbf{G} \right) \right) - i \left( \mathbf{E} \cdot \left( \nabla \times \mathbf{H} \right) \right) \\
+ i \left( \mathbf{H} \cdot \left( \nabla \times \mathbf{E} \right) \right) - i \left( \mathbf{G} \cdot \left( \nabla \times \mathbf{F} \right) \right) &= 0.
\end{align*}
\]  
\(65\)

Let us introduce the following notations:

\[
\begin{align*}
w &= -\frac{1}{8\pi} \left( f^2 + e^2 + h^2 + g^2 + \mathbf{F}^2 + \mathbf{E}^2 + \mathbf{H}^2 + \mathbf{G}^2 \right), \\
\mathbf{P} &= -\frac{c}{4\pi} \left( e\mathbf{H} + f\mathbf{G} + g\mathbf{F} + h\mathbf{E} + \left[ \mathbf{E} \times \mathbf{H} \right] + i \left[ \mathbf{G} \times \mathbf{F} \right] \right).
\end{align*}
\]  
\(66\), \(67\)

Then the equation \(65\) can be written as:

\[
\frac{1}{c} \frac{\partial w}{\partial t} + \left( \nabla \cdot \mathbf{P} \right) = 0.
\]  
\(68\)

This expression is an analog of the Poynting theorem for massive field. The value \(w\) plays the role of the field energy density and \(\mathbf{P}\) is a vector of energy flux density. The minus sign in expressions \(66\) and \(67\) are chosen with respect to the attractive character of charge interaction (see further Section 6.2.).
6.2 Nonhomogeneous equation

Let us consider the sedeonic nonhomogeneous equation for massive field

\[
\left(\imath e_1 \partial - e_2 \vec{\nabla} - \imath e_3 m\right) \left(\imath e_1 \partial - e_2 \vec{\nabla} - \imath e_3 m\right) \hat{W} = \hat{J},
\]

(69)

where \( \hat{J} \) is the source of massive field. By analogy with electrodynamics we consider the source in the following form \[29\]

\[
\hat{J} = -\imath e_1 4\pi \rho_B - e_2 \frac{4\pi}{c} \vec{j}_B,
\]

(70)

where \( \rho_B \) is a volume density of charge and \( \vec{j}_B \) is density of current. In this case we can describe the field by sedeonic potential \( \hat{W} \) written in the following form

\[
\hat{W} = \imath e_1 b + e_2 \vec{C},
\]

(71)

where \( b(\vec{r}, t) \) is a scalar part and \( \vec{C}(\vec{r}, t) \) is a vector part of field potential. In this case we have only the following nonzero field’s strengths

\[
\begin{align*}
e & = \partial b + (\vec{\nabla} \cdot \vec{C}), \\
g & = -mb, \\
\vec{E} & = -i \left[\vec{\nabla} \times \vec{C}\right], \\
\vec{F} & = -m\vec{C}, \\
\vec{H} & = -\partial \vec{C} - \vec{\nabla} b,
\end{align*}
\]

(72)

and the equation \[39\] can be rewritten as

\[
\left(\imath e_1 \partial - e_2 \vec{\nabla} - \imath e_3 m\right) \left(-e - \imath e_2 g - i\vec{E} + e_1 \vec{F} + e_3 \vec{H}\right) = -\imath e_1 4\pi \rho_B - e_2 \frac{4\pi}{c} \vec{j}_B.
\]

(73)

Then we obtain the following equations for the field strengths:

\[
\begin{align*}
\partial e + (\vec{\nabla} \cdot \vec{H}) - mg & = 4\pi \rho_B, \\
(\vec{\nabla} \cdot \vec{E}) & = 0, \\
\partial g + (\vec{\nabla} \cdot \vec{F}) + me & = 0, \\
\partial \vec{F} + \vec{\nabla} g - m\vec{H} & = 0, \\
\partial \vec{E} - i[\vec{\nabla} \times \vec{H}] & = 0, \\
\partial \vec{H} + \vec{\nabla} e + i[\vec{\nabla} \times \vec{E}] + m\vec{F} & = -\frac{4\pi}{c} \vec{j}_B, \\
i[\vec{\nabla} \times \vec{F}] - m\vec{E} & = 0.
\end{align*}
\]

(74)

On the other hand, applying the operator \( (\imath e_1 \partial - e_2 \vec{\nabla} - \imath e_3 m) \) to the equation \[73\] we obtain the following wave equations for the field strengths:

\[
\begin{align*}
(\partial^2 - \Delta + m^2)e & = 4\pi(\partial \rho_B + \frac{1}{c}(\vec{\nabla} \cdot \vec{j}_B)), \\
(\partial^2 - \Delta + m^2)g & = -4\pi m\rho_B, \\
(\partial^2 - \Delta + m^2)\vec{F} & = -\frac{4\pi}{c} m\vec{j}_B, \\
(\partial^2 - \Delta + m^2)\vec{E} & = -i\frac{4\pi}{c} [\vec{\nabla} \times \vec{j}_B], \\
(\partial^2 - \Delta + m^2)\vec{H} & = -4\pi(\frac{1}{c} \partial \vec{j}_B + \vec{\nabla} \rho_B).
\end{align*}
\]

(75)
Assuming the charge conservation

\[ \partial \rho_B + \frac{1}{e} (\vec{\nabla} \cdot \vec{j}_B) = 0 \]  

(76)

we can choose the field strength \( e \) equal to zero. This is equivalent to the following gauge condition (see (72)):

\[ \partial b + (\vec{\nabla} \cdot \vec{C}) = 0 \]  

(77)

similar to the Lorentz gauge.

Let us consider the simplest case of stationary field of point scalar source. In the stationary case \( \vec{j}_B = 0 \) and field potential can be chosen in a scalar form

\[ \tilde{\mathbf{W}} = ie_1 b(\vec{r}). \]  

(78)

Then we have only two nonzero field components

\[ g = -mb, \]
\[ \vec{H} = -\vec{\nabla} b \]  

(79)

and the following field equations:

\[ (\vec{\nabla} \cdot \vec{H}) - mg = 4\pi \rho_B, \]
\[ \vec{\nabla} g - m\vec{H} = 0, \]
\[ [\vec{\nabla} \times \vec{H}] = 0. \]  

(80)

Let us calculate the field produced by a scalar stationary point source

\[ \mathbf{J} = -4\pi q_B \delta(\vec{r}), \]  

(81)

where \( q_B \) is the point charge and \( \delta(\vec{r}) \) is delta function. Then stationary wave equation can be written in spherical coordinates as

\[ \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{m_0^2 c^2}{\hbar^2} \right) b(r) = -4\pi q_B \delta(\vec{r}). \]  

(82)

The partial solution of the equation (82), which decays at \( r \to \infty \), is

\[ b = \frac{q_B}{r} \exp \left( -\frac{m_0 c}{\hbar} r \right). \]  

(83)

Thus in this case the stationary field has scalar and vector components

\[ g = \frac{m_0 c q_B}{\hbar} \frac{q_B}{r} \exp \left( -\frac{m_0 c}{\hbar} r \right), \]  

(84)

\[ \vec{H} = \left( \frac{1}{r} + \frac{m_0 c}{\hbar} \right) \frac{q_B}{r} \exp \left( -\frac{m_0 c}{\hbar} r \right) \vec{r}_0, \]  

(85)

where \( \vec{r}_0 \) is a unit radial vector.

Two point charges interact due to the overlap of their fields. Taking into account that the field in this case is the sum of the two fields \( g = g_1 + g_2 \) and \( \vec{H} = \vec{H}_1 + \vec{H}_2 \) the energy of interaction is equal (see (66))

\[ W_{BB} = -\frac{1}{4\pi} \int \{g_1 g_2 + (H_1 \cdot H_2)\} dV, \]  

(86)

where the integral is over all space. This expression can be derived analytically:

\[ W_{BB} = -\frac{q_B q_B}{R} \exp \left( -\frac{m_0 c}{\hbar} R \right), \]  

(87)

where \( R \) is the distance between the point charges. This expression coincides with a well-known law of interaction between two baryons, which is described by Yukawa potential [35], therefore \( q_B \) can be interpreted as a baryon charge.
7 Sedeonic first-order equation for massive field

7.1 Homogeneous equation

Let us consider a special massive field that is described by sedeonic first-order equation. In sedeonic algebra the homogeneous first-order Dirac-like equation corresponding to the equation (55) is written as

\[
(i \mathbf{e}_1 \partial - \mathbf{e}_2 \vec{\nabla} - i \mathbf{e}_3 m) \tilde{W} = 0. \tag{88}
\]

Choosing potential in the form (59) we find that sedeonic equation (88) is equivalent to the following system

\[
\begin{align*}
\partial a + (\vec{\nabla} \cdot \vec{D}) + mc &= 0, \\
\partial b + (\vec{\nabla} \cdot \vec{C}) + md &= 0, \\
\partial c + (\vec{\nabla} \cdot \vec{B}) - ma &= 0, \\
\partial d + (\vec{\nabla} \cdot \vec{A}) - mb &= 0, \\
\partial \vec{A} + \vec{\nabla} d - i(\vec{\nabla} \times \vec{D}) + m \vec{C} &= 0, \\
\partial \vec{B} + \vec{\nabla} c + i(\vec{\nabla} \times \vec{C}) + m \vec{D} &= 0, \\
\partial \vec{C} + \vec{\nabla} b - i(\vec{\nabla} \times \vec{B}) - m \vec{A} &= 0, \\
\partial \vec{D} + \vec{\nabla} a + i(\vec{\nabla} \times \vec{A}) - m \vec{B} &= 0.
\end{align*}
\]  

(89)

In fact, these equations describe the special field with zero field strengths (see for comparison the expressions (60)).

Let us consider the plane wave solution of equation (88) in detail. In this case the potential can be written as

\[
\tilde{W} = \tilde{U} \exp \left\{ -i\omega t + i \left( \vec{k} \cdot \vec{r} \right) \right\}, \tag{90}
\]

where \( \omega \) is a frequency and \( \vec{k} \) is an absolute wave vector; the amplitude of the wave \( U \) does not depend on the coordinates and time. In this case, the dependence of frequency on the wave vector has two branches:

\[
\omega_{\pm} = \pm \sqrt{c^2 k^2 + \frac{m_0^2 c^4}{\hbar^2}}. \tag{91}
\]

Let us consider the amplitude of the wave function in the form of (59):

\[
\tilde{U} = a + i \mathbf{e}_1 b - i \mathbf{e}_2 c - i \mathbf{e}_3 d + i \vec{A} + \mathbf{e}_1 \vec{B} + \mathbf{e}_2 \vec{C} - \mathbf{e}_3 \vec{D}, \tag{92}
\]

where \( a, b, c, d, \vec{A}, \vec{B}, \vec{C} \) and \( \vec{D} \) are arbitrary constants. Then the solution can be written as

\[
\tilde{W} = \left( a + i \mathbf{e}_1 b - i \mathbf{e}_2 c - i \mathbf{e}_3 d + i \vec{A} + \mathbf{e}_1 \vec{B} + \mathbf{e}_2 \vec{C} - \mathbf{e}_3 \vec{D} \right) \exp \left\{ -i\omega_{\pm} t + i \left( \vec{k} \cdot \vec{r} \right) \right\}. \tag{93}
\]

Substituting this expression in the original equation (54) we get:

\[
\left( \mathbf{e}_1 \frac{\omega_{\pm}}{c} - i \mathbf{e}_2 \vec{k} - i \mathbf{e}_3 \frac{mac}{\hbar} \right) \left( a + i \mathbf{e}_1 b - i \mathbf{e}_2 c - i \mathbf{e}_3 d + i \vec{A} + \mathbf{e}_1 \vec{B} + \mathbf{e}_2 \vec{C} - \mathbf{e}_3 \vec{D} \right) = 0. \tag{94}
\]

For convenience we introduce the following notation:

\[
\omega' = \frac{\omega_{\pm}}{c}, \quad m = \frac{m_0 c}{\hbar}, \tag{95}
\]

then equation (94) can be rewritten as

\[
\left( \mathbf{e}_1 \omega' - i \mathbf{e}_2 \vec{k} - i \mathbf{e}_3 m \right) \left( a + i \mathbf{e}_1 b - i \mathbf{e}_2 c - i \mathbf{e}_3 d + i \vec{A} + \mathbf{e}_1 \vec{B} + \mathbf{e}_2 \vec{C} - \mathbf{e}_3 \vec{D} \right) = 0. \tag{96}
\]
For fixed \( \vec{k} \) let us represent the vector constants in (92) in the form

\[
\vec{A} = \vec{A}_\parallel + \vec{A}_\perp,
\]
\[
\vec{B} = \vec{B}_\parallel + \vec{B}_\perp,
\]
\[
\vec{C} = \vec{C}_\parallel + \vec{C}_\perp,
\]
\[
\vec{D} = \vec{D}_\parallel + \vec{D}_\perp,
\]

(97)

where the vectors \( \vec{A}_\parallel, \vec{B}_\parallel, \vec{C}_\parallel \) and \( \vec{D}_\parallel \) are parallel to the vector \( \vec{k} \) while the vectors \( \vec{A}_\perp, \vec{B}_\perp, \vec{C}_\perp \) and \( \vec{D}_\perp \) are perpendicular to \( \vec{k} \). Then performing the multiplication in (95), we obtain the following system of algebraic equations:

\[
i\omega' b - ikC_\parallel - md = 0,
\]
\[
\omega' a - kD_\parallel + imc = 0,
\]
\[
-\omega'd + kA_\parallel + imb = 0,
\]
\[
\omega' c - kB_\parallel - ima = 0,
\]
\[
\omega'B_\parallel - kc + imD_\parallel = 0,
\]
\[
i\omega'A_\parallel - ikd - mC_\parallel = 0,
\]
\[
i\omega'D_\parallel - ika + mB_\parallel = 0,
\]
\[
i\omega'C_\parallel - ikb + mA_\parallel = 0,
\]

(98)

(99)

(100)

(101)

(102)

(103)

(104)

(105)

where the values \( A_\parallel, B_\parallel, C_\parallel \) and \( D_\parallel \) are the projections of the vectors \( \vec{A}_\parallel, \vec{B}_\parallel, \vec{C}_\parallel \) and \( \vec{D}_\parallel \) on the vector \( \vec{k} \).

Let us solve this system of equations. From (108) and (109) we find

\[
\vec{D}_\perp = \frac{im}{\omega'} \vec{B}_\perp + \frac{i}{\omega'} \left[ \vec{k} \times \vec{A}_\perp \right],
\]

(110)

\[
\vec{C}_\perp = \frac{im}{\omega'} \vec{A}_\perp - \frac{i}{\omega'} \left[ \vec{k} \times \vec{B}_\perp \right].
\]

(111)

Using (111) one can easily check that for arbitrary vector constants \( A_\perp \) and \( B_\perp \) equations (106) and (107) are fulfilled.

As a next step from equations (98)-(101) we obtain:

\[
C_\parallel = \frac{\omega'}{k} b + \frac{m}{k} d,
\]

(112)

\[
D_\parallel = \frac{\omega'}{k} a + \frac{m}{k} c,
\]

(113)

\[
A_\parallel = \frac{\omega'}{k} d - \frac{m}{k} b,
\]

(114)

\[
B_\parallel = \frac{\omega'}{k} c - \frac{m}{k} a.
\]

(115)
One can check that these solution fulfill the equations \((102) - (105)\). Thus the sedeon \(\tilde{U}\) has the form

\[
\tilde{U} = a + ie_1 b - ie_2 c - ie_3 d + \{i\omega'd + mb + e_1\omega'c - ie_1 ma + e_2\omega'b + ie_2 md - e_3\omega'a - ie_3 mc\} \frac{\bar{k}}{k^2} + i\bar{A}_1 + e_1 \bar{B}_1 + ie_2 \frac{m}{\omega'} \bar{A}_1 - ie_3 \frac{m}{\omega'} \bar{B}_1 - ie_3 \frac{1}{\omega} \left[\bar{k} \times \bar{A}_1\right] - ie_2 \frac{1}{\omega} \left[\bar{k} \times \bar{B}_1\right].
\]

\((116)\)

Note that this expression can be rewritten in the following form:

\[
\tilde{U} = \left( e_1 \omega' - ie_2 \bar{k} - ie_3 m \right) \left( e_1 \omega' - ie_2 \bar{k} - ie_3 m \right) \equiv 0.
\]

\((118)\)

In general, the plane wave solution for the equation \((88)\) can be written in the following sedonic form:

\[
\tilde{W} = \left( e_1 \omega' - ie_2 \bar{k} - ie_3 m \right) \tilde{M} \exp \left\{ -i\omega t + i \left( \bar{k} \cdot \bar{r} \right) \right\},
\]

\((119)\)

where \(\tilde{M}\) is an arbitrary sedeon with constant components. In this case after performing multiplication in \((119)\) we obtain that the components of the resulting sedeon are defined only by 8 independent combinations of the sedeon \(\tilde{M}\) components. Note that the internal structure of this wave is changed under space and time inversion.

In massless case the dispersion relation is

\[
\omega = \pm ck
\]

\((120)\)

and plane wave solution can be written as

\[
\tilde{W} = \left( e_1 \frac{\omega}{c} - ie_r \bar{k} \right) \tilde{M} \exp \left\{ -i\omega t + i \left( \bar{k} \cdot \bar{r} \right) \right\}.
\]

\((121)\)

Let us analyze the structure of the plane wave \((121)\) in detail. We suppose that wave vector is directed along z axis. Then the first-order equation \((88)\) can be rewritten in the following equivalent form:

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + e_r a_3 \frac{\partial}{\partial z} \right) \tilde{W}' = 0,
\]

\((122)\)

where \(\tilde{W}' = ie_1 \tilde{W}\). Using \((120)\) and \((121)\) we can write solution of \((122)\) in the following form:

\[
\tilde{W}'_+ = - \left( 1 + e_r a_3 \right) k \tilde{M} \exp \left\{ -i\omega t + i k z \right\},
\]

\((123)\)

and

\[
\tilde{W}'_- = \left( 1 - e_r a_3 \right) k \tilde{M} \exp \left\{ -i\omega t + i k z \right\}.
\]

\((124)\)

Note that the wave function \(\tilde{W}'_+\) describes the positive branch of dispersion law \((120)\) that corresponds, for example, to the "antiparticle", while \(\tilde{W}'_-\) describes the negative branch that corresponds to the "particle" state. Besides, as it is seen the wave functions \((123)\) and \((124)\) are the eigenfunctions of spin operator \(\hat{S}_z\):

\[
\hat{S}_z = \frac{1}{2} e_r a_3.
\]

\((125)\)
Indeed it is simple to check that
\[ \hat{S}_z \tilde{W}' = S_z \hat{W}', \] (126)
where eigenvalue \( S_z = \pm 1/2 \). It is seen that plane waves \((123)\) and \((124)\) correspond to the different eigenvalues \( S_z \). Thus \( \tilde{W}'_+ \) describes "antiparticle" state with spirality \( S_z = +1/2 \), while \( \tilde{W}'_- \) describes "particle" state with spirality \( S_z = -1/2 \). However in the case of massive field the plane wave \((119)\) has more complicated space-time structure.

### 7.2 Nonhomogeneous equation

Let us consider the nonhomogeneous equation corresponding to the equation \((88)\)
\[ \left( i e_1 \partial - e_2 \vec{\nabla} - i e_3 m \right) \hat{W} = \hat{I}. \] (127)
Here \( \hat{I} \) is the field source. Choosing the potential \( \hat{W} \) in the form \((59)\), we obtain the following equation for the field strengths:
\[ -e + i e_1 f - i e_2 g - e_1 \vec{E} + e_2 \vec{G} + e_3 \vec{H} = I_0 + \hat{I}. \] (128)
This equation means that the strengths of this field are nonzero only in the region of field source.

Let us consider the sedeonic source in the following form:
\[ \hat{I} = -i e_2 4\pi \rho_L + e_1 \frac{4\pi}{c} \vec{j}_L. \] (129)
where \( \rho_L \) is a volume density of charge and \( \vec{j}_L \) is volume density of current. In this case the equation \((128)\) is rewritten as
\[ -i e_2 g + e_1 \vec{F} = -i e_2 4\pi \rho_L + e_1 \frac{4\pi}{c} \vec{j}_L. \] (130)
Applying the operator \( (i e_1 \partial - e_2 \vec{\nabla} - i e_3 m) \) to the equation \((130)\) and separating the values with different space-time properties we obtain the following equations for the field strengths:
\[ g = 4\pi \rho_L, \]
\[ \vec{F} = \frac{4\pi}{c} \vec{j}_L, \]
\[ \partial g + (\vec{\nabla} \cdot \vec{F}) = 4\pi [\partial \rho_L + \frac{1}{c} (\vec{\nabla} \cdot \vec{j}_L)], \]
\[ \vec{\nabla} \times \vec{F} = \frac{4\pi}{c} [\vec{\nabla} \times \vec{j}_L], \]
\[ \partial \vec{F} + \vec{\nabla} g = 4\pi [\frac{1}{c} \partial \vec{j}_L + \vec{\nabla} \rho_L]. \] (131)
Assuming charge conservation
\[ \partial \rho_L + \frac{1}{c} (\vec{\nabla} \cdot \vec{j}_L) = 0 \] (132)
we have the following gauge condition:
\[ \partial g + (\vec{\nabla} \cdot \vec{F}) = 0 \] (133)
which is similar to the Lorentz gauge, but for the field strengths.

Let us consider a stationary field generated by a scalar point source
\[ I_0 = -i e_2 4\pi q_L \delta(\vec{r}), \] (134)
where \( q_L \) is the point charge. Then the intensity of the scalar field is
\[ g_L(\vec{r}) = 4\pi q_L \delta(\vec{r}). \] (135)
This field is non-zero only in the region of source. It indicates that two point charges interact only if they are at the same point of space. The interaction energy for two point charges $q_{L1}$ and $q_{L2}$ is equal

$$W_{LL} = -\frac{1}{4\pi} \int g_{L1} g_{L2} dV = -4\pi q_{L1} q_{L2} \delta(\vec{R}),$$  \hspace{1cm}(136)$$

where $\vec{R}$ is the vector of distance between point charges. Such a law of interaction is typical for leptons involved in a weak interaction. So the $q_L$ can be interpreted as a lepton charge.

Moreover one can suppose the interaction between $q_B$ and $q_L$ charges due to the overlap of scalar fields $g_B$ and $g_L$. In the case of point $q_B$ and $q_L$ the fields are determined by the expressions (84) and (135), so that the interaction energy is equal to:

$$W_{BL} = -\frac{1}{4\pi} \int g_B g_L dV.$$  \hspace{1cm}(137)$$

As a result, we get:

$$W_{BL} = -\frac{m_0 c}{\hbar} q_B q_L \exp \left(-\frac{m_0 c}{\hbar} R \right),$$  \hspace{1cm}(138)$$

where $R$ is the distance between $q_B$ and $q_L$ charges.

## 8 Concluding remarks

Thus, in this paper we have presented the sixteen-component sedeons generating associative non-commutative space-time algebra. This algebra can be considered as the scalar-vector variant of complexified Clifford algebra with specific commutation and multiplication rules. The sedeonic basis elements $a_1, a_1, a_3$ are responsible for the spatial rotation, while the elements $e_t, e_r$ and $e_{tr}$ are responsible for the space-time inversions. Mathematically, these two bases are equivalent, and the different physical properties attributed to them are an important physical essence of our sedeonic hypothesis.

In contrast to the Gibbs-Heaviside vector algebra the multiplication rules for vector basis in sedeonic algebra contain the imaginary unit (see Table 1). It enables the realization of scalar-vector algebra whith Clifford algebra with specific commutation and multiplication rules. The sedeonic basis elements $a_1, a_1, a_3$ are responsible for the spatial rotation, while the elements $e_t, e_r$ and $e_{tr}$ are responsible for the space-time inversions. Mathematically, these two bases are equivalent, and the different physical properties attributed to them are an important physical essence of our sedeonic hypothesis.

The important point is that the sedeonic basis elements simultaneously play a role of the operators and space-time basis of the wave function. From a physical point of view, this allows us to reformulate the Klein-Gordon equation of relativistic quantum mechanics as the wave equation for special scalar-vector field that carries information about the kinematic properties of quantum particles. This sedeonic Klein-Gordon equation can be reformulated as Maxwell-like equations for the field intensities. At the same time the sedeonic first-order Dirac wave equation can be interpreted as the equation describing special field with zero field intensities.

On the other hand the sedeonic Klein-Gordon equation allows another interpretation as the wave equation for the potentials of a force massive field. In this case the Einstein relation between energy and momentum can be interpreted as the relation for the energy, momentum and mass of a quantum of force field. The sources of this field are corresponding charges $q_B$ and currents $\vec{j}_B$. At the same time the sedeonic first-order wave equation describes the special force field with zero field strengths. The sources of this field are corresponding charges $q_L$ and currents $\vec{j}_L$. We defined the concept of energy and energy flux for the force massive field and derived an expression that describes the energy conservation for a massive field, similar to the Poynting theorem in electrodynamics. Based on this concept, we have considered the interaction of point charges due to the overlap of scalar and vector fields.

16
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References

[1] S.L. Adler, Quaternionic quantum mechanics and quantum fields, (New York: Oxford University Press, 1995).

[2] V. Majernik – Quaternionic formulation of the classical fields, Advances in Applied Clifford Algebras, 9(1), 119 (1999).

[3] K. Imaeda – A new formulation of classical electrodynamics, Nuovo Cimento B, 32(1), 138 (1976).

[4] A.J. Davies – Quaternionic Dirac equation, Physical Review D, 41, 2628 (1990).

[5] S. De Leo, P. Rotelli – Quaternion scalar field, Physical Review D, 45(2), 575 (1992).

[6] C. Schwartz – Relativistic quaternionic wave equation, J. Mathematical Physics, 47, 122301 (2006).

[7] R. Penney – Octonions and Dirac equation, American Journal of Physics, 36, 871 (1968).

[8] M. Gogberashvili – Octonionic electrodynamics, J. Physics A: Mathematics in General, 39, 7099 (2006).

[9] A. Gamba – Maxwell’s equations in octonion form, Nuovo Cimento A, 111(3), 293 (1998).

[10] T. Tolan, K. Özdas and M. Tanisli – Reformulation of electromagnetism with octonions, Nuovo Cimento B, 121(1), 43 (2006).

[11] S. Ulrych – The Poincare mass operator in terms of a hyperbolic algebra, Physics Letters B, 612(1-2), 89 (2005).

[12] N. Candemir, M. Tanisli, K. Ozdas and S. Demir – Hyperbolic octonionic Proca-Maxwell equations, Zeitschrift für Naturforschung A, 63a, 15-18, (2008).

[13] S. Demir and M. Tanisli – A compact biquaternionic formulation of massive field equations in gravi-electromagnetism, European Physical Journal - Plus, 126, 115 1-12 (2011).

[14] B.C. Chanyal, P.S. Bisht and O.P.S. Negi – Generalized octonion electrodynamics, Int. J. Theoretical Physics, 49(6), 1333 (2010).

[15] V. Dzhumushaliev – Nonassociativity, supersymmetry, and hidden variables, Journal of Mathematical Physics, 49, 042108 (2008).

[16] M. Gogberashvili – Octonionic version of Dirac equations, Int. J. Modern Physics A, 21(17), 3513 (2006).

[17] S. De Leo and K. Abdel-Khalek – Octonionic Dirac equation, Progress in Theoretical Physics, 96, 833 (1996).

[18] V.L. Mironov and S.V. Mironov – Octonic representation of electromagnetic field equations, Journal of Mathematical Physics, 50, 012901 (2009).

[19] V.L. Mironov and S.V. Mironov – Octonic second-order equations of relativistic quantum mechanics, Journal of Mathematical Physics, 50, 012302 1-13 (2009).
[20] V.L. Mironov and S.V. Mironov – Octonionic first-order equations of relativistic quantum mechanics, International Journal of Modern Physics A, 24(22), 4157 (2009).

[21] K. Imaeda and M. Imaeda – Sedenions: algebra and analysis, Applied Mathematics and Computations, 115, 77 (2000).

[22] K. Carmody – Circular and hyperbolic quaternions, octonions, and sedenions, Applied Mathematics and Computations, 28, 47 (1988).

[23] K. Carmody – Circular and hyperbolic quaternions, octonions, and sedenions - further results, Applied Mathematics and Computations, 84, 27 (1997).

[24] J. Köpplinger – Dirac equation on hyperbolic octonions, Applied Mathematics and Computations, 182, 443 (2006).

[25] S. Demir and M. Tanisli – Sedenionic formulation for generalized fields of dyons, Int. J. Theoretical Physics, 51(4), 1239 (2012).

[26] W.P. Joyce – Dirac theory in spacetime algebra: I. The generalized bivector Dirac equation, J. Physics A: Mathematical and General, 34, 1991 (2001).

[27] C. Cafaro and S.A. Ali – The spacetime algebra approach to massive classical electrodynamics with magnetic monopoles, Advances in Applied Clifford Algebras, 17, 23 (2006).

[28] V.L. Mironov and S.V. Mironov – Reformulation of relativistic quantum mechanics equations with non-commutative sedenions, Applied mathematics, 4(10C), 53 (2013).

[29] V.L. Mironov and S.V. Mironov – Sedenionic equations of gravitoelectromagnetism, Journal of Modern Physics, 5(10), 917 (2014).

[30] S.V. Mironov and V.L. Mironov – Sedenionic equations of massive fields, International Journal of Theoretical Physics, 54(1), 153 (2015).

[31] L.D. Landau and E.M. Lifshits, Classical Theory of Fields (New York: Pergamon Press, 1975).

[32] V.L. Mironov and S.V. Mironov, Space-time sedenions and their application in relativistic quantum mechanics and field theory (Institute for physics of microstructures RAS, Nizhniy Novgorod, 2014).

[33] P.A.M. Dirac, The Principles of Quantum Mechanics (Clarendon Press, Oxford, 1958).

[34] J.D. Jackson and L.B. Okun – Historical roots of gauge invariance, Review of Modern Physics, 73, 663 (2001).

[35] H. Yukawa – On the interaction of elementary particles I, Proceedings of the Physico-Mathematical Society of Japan, 17, 48 (1935).

[36] A. Macfarlane, – Hyperbolic quaternions, Proc. R. Soc. Edinburgh, 1899-1900 session, pp. 169181.

[37] W. Pauli – Zur quantummechanik des magnetischen elektrons, Zeitschrift fur Physik, 43, 601 (1927).

[38] P.A.M. Dirac – The quanrum theory of electron, Proceedings of Royal Society at London, Ser. A, 117, 610 (1928).