Some Examples of K3 Surfaces with Infinite Automorphism Group which Preserves an Elliptic Pencil

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Abstract—We give more detail to our examples in [1] of K3 surfaces over C which have an infinite automorphism group that preserves some elliptic pencil of the K3 surface.

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Dedicated to the memory of Érnest Borisovich Vinberg

1. INTRODUCTION

In this paper, we intend to give more detail to cases of [1] in which we considered K3 surfaces $X$ which have an infinite automorphism group $\text{Aut} \ X$, but which preserves some elliptic pencil of the K3 surface $X$. In Sec. 2, we recall some results of [1] related to these cases. In the remaining sections, we give more detail to these cases considered in [1]. A preliminary version of this paper was published as preprint [2].

2. SOME RESULTS OF [1] WHEN THE AUTOMORPHISM GROUP OF A K3 SURFACE IS INFINITE BUT PRESERVES SOME ELLIPTIC PENCIL

We remind the reader (see [3], [4]) that a K3 surface is a Kählerian compact complex surface $X$ such that its canonical class $K_X$ is zero (i.e. there exists a holomorphic 2-dimensional differential form $\omega_X$ of $X$ with zero divisor) and $X$ has no nonzero 1-dimensional holomorphic differential forms (equivalently, $H^1(X, \mathcal{O}_X) = 0$). In this paper, we shall consider algebraic K3 surfaces.

In [1, Sec. 4], in particular, we gave examples of algebraic K3 surfaces whose automorphism group preserves some elliptic pencil of the K3 surface.

K3 surfaces $X$ of [1, Sec. 4] have 2-elementary Picard lattices

$$S = S_X = \{ x \in H^2(X, \mathbb{Z}) \mid x \cdot \omega_X = 0 \}.$$ 

Here the word “lattice” means that $S$ is a free $\mathbb{Z}$-module of a finite rank with an integer symmetric bilinear form given, in our case, by the intersection pairing. The lattice $S$ is 2-elementary if for $S^* = \text{Hom}(S, \mathbb{Z})$, the discriminant group

$$A_S = S^*/S$$

of the lattice $S$ is 2-elementary, that is, $A_S \cong (\mathbb{Z}/2\mathbb{Z})^a$, where $a \geq 0$ is an integer.

By the global Torelli theorem for K3 surfaces [4], such K3-surfaces $X$ possess a unique involution $\theta$ which acts trivially (identically) on the Picard lattice $S$, but acts as $-1$ on the transcendental lattice
$T = T_X = (S_X)_{H^2(X, \mathbb{Z})}$ of the K3 surface $X$. In particular, $\theta(\omega_X) = -\omega_X$ for a 2-dimensional holomorphic differential form $\omega_X \in T_X \otimes \mathbb{C}$ of $X$. Thus, $\theta$ is a canonical nonsymplectic involution of $X$. In particular, all automorphisms of $X$ commute with $\theta$.

The geometry of such K3 surfaces $X$ was studied in [1, Sec. 4], and their classification was obtained.

For an algebraic K3 surface $X$, the Picard lattice $S = S_X$ is hyperbolic and has the signature $(t_+ = 1, t_-)$, where $t_+$ and $t_-$ are the numbers of positive and negative squares, respectively, of the real symmetric bilinear form $S \otimes \mathbb{R}$. Then $\text{rk} S = r = 1 + t_-$. Moreover, $S$ is even, that is, $x^2 = x \cdot x \equiv 0 \mod 2$ is even for any $x \in S$. There is a third important invariant $\delta$ of any 2-elementary lattice $S = S_X$ which is equal to 0 or 1. The invariant $\delta$ is 0 if $(x^2)^2 \in \mathbb{Z}$ for any $x \in S$; otherwise, $\delta = 1$.

The invariants $(r, a, \delta)$ define the isomorphism class of a 2-elementary Picard lattice $S = S_X$ of a K3 surface $X$. See [5].

All possible $(r, a, \delta)$ are described in the table of Sec. 5 of [1, paragraph 4]: we have

\begin{align*}
    r + a &\equiv 0 \mod 2, \quad r \geq 1, \quad 0 \leq a \leq r \leq 20, \quad 1 \leq r + a \leq 22, \quad (1) \\
    a &\equiv 0 \mod 2 \text{ if } \delta = 0, \quad r \equiv 2 \mod 4 \text{ if } \delta = 0, \quad (2) \\
    r &\equiv 2 \mod 8 \text{ if } a = 0, \quad r \equiv 2 \pm 1 \mod 8 \text{ if } a = 1, \quad (3) \\
    \delta &\equiv 1 \text{ if } r = a = 6, \quad \delta = 1 \text{ if } (r, a) = (14, 8). \quad (4)
\end{align*}

The set $X^\theta$ of fixed points of $\theta$ on $X$ was described. The set $X^\theta$ is a nonsingular algebraic curve of $X$ which may have several irreducible components that do not intersect each other. If $(r, a, \delta) \neq (10, 8, 0), (10, 10, 0)$, then

$$X^\theta = C^{(g)} + \sum_{i=1}^{k} E_i, \quad g = 11 - (r + a)/2, \quad k = (r - a)/2,$$

where $C^{(g)}$ is a nonsingular irreducible curve of genus $g$ and $E_i$, $1 \leq i \leq k$, are nonsingular irreducible rational curves (or $(-2)$-curves). We have

$$X^\theta = C_1^{(1)} + C_2^{(1)} \text{ if } (r, a, \delta) = (10, 8, 0),$$

where $C_1^{(1)}$ and $C_2^{(1)}$ are different nonsingular irreducible curves of genus 1. Further,

$$X^\theta = \emptyset \text{ if } (r, a, \delta) = (10, 10, 0).$$

We remind the reader that, for a K3 surface $X$, for an irreducible curve $D$, we have

$$p_a(D) = \frac{D^2}{2} + 1$$

and $E^2 = -2$, and the linear system $|E|$ is zero-dimensional for a nonsingular irreducible rational curve $E$, and $C^2 = 0$ and the linear system $|C|$ is one-dimensional for a nonsingular irreducible curve $C$ of genus 1 (i.e., $C$ is an elliptic curve). Thus, $|C|$ defines an elliptic pencil.

Since the involution $\theta$ is canonical for $X$, it follows from (5) — (7) that the automorphism group $\text{Aut} X$ preserves the elliptic pencil $\pi : X \to \mathbb{P}^1 = |C^{(1)}|$ (and the elliptic curve $C^{(1)}$) in case (5) if $r + a = 20$, and the elliptic pencil $\pi : X \to \mathbb{P}^1 = |C_1^{(1)}| = |C_2^{(1)}|$ in case (6) when $(r, a, \delta) = (10, 8, 0)$. It is these interesting cases where the automorphism group $\text{Aut} X$ of the K3 surface $X$ preserves an elliptic pencil $\pi : X \to \mathbb{P}^1$ that we wish to consider in more detail in this paper. We note that if $\text{Aut} X$ is infinite, then such an invariant elliptic pencil is unique. More generally, if a subgroup $H \subset \text{Aut} X$ preserves two different linear systems of elliptic curves $|C_1|$ and $|C_2|$, then $H$ is finite. Indeed, then $H$ preserves the divisor class $h = e(C_1 + C_2)$, which has a positive square $h^2$. From the geometry of K3 surfaces, it follows that $H$ is finite.

In [1, Sec. 4], it was shown by considering degenerate fibers and the Mordell–Weil group of the elliptic pencil $\pi : X \to \mathbb{P}^1$ that this pencil has no reducible fibers in case (5) when $k = 0$ or, equivalently, when $(r, a, \delta) = (10, 10, 1)$. In the remaining cases, the canonical involution $\theta$ has 2 fixed points on $\mathbb{P}^1$. 

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In case (6) when \((r, a, \delta) = (10, 8, 0)\), the curves \(C_1^{(1)}\) and \(C_2^{(1)}\) give fibers of \(\pi\) over these fixed points, and in case (5) when \(r + a = 20\) and \((r, a, \delta) \neq (10, 10, 1)\), the fiber of \(\pi\) over one fixed point is the elliptic curve \(C^{(1)}\), and the fiber of \(\pi\) over another fixed point contains the curves \(E_i, 1 \leq i \leq k\), and is a reducible elliptic curve \(\mathcal{E}\) of Dynkin type, where

\[
\mathcal{E} = \begin{cases} 
\tilde{E}_6 & \text{if } k = 4, \delta = 0, \\
\tilde{A}_{2k-1} & \text{in the other cases.}
\end{cases}
\]  

(9)

We recall that \(k = (r - a)/2\).

In the next sections, we shall give more detail in each of these cases \((r, a, \delta)\) related to the elliptic curves. In particular, we exactly calculate the classes in the Picard lattice of the curves \(C^{(1)}, C_1^{(1)}, C_2^{(1)}, \) and \(E_1, \ldots, E_k\) in (5), (6). We will give more details to the proof of (9) and exactly calculate the classes in the Picard lattice of the components of \(\mathcal{E}\) in (9).

3. THE CASE \((r, a, \delta) = (10, 10, 1)\)

We shall use the standard notation for lattices. By \(\langle A \rangle\), we denote a lattice with integer symmetric matrix \(A\). By \(M(n)\) we denote the lattice which is obtained from the lattice \(M\) by multiplication of the symmetric bilinear form of \(M\) by the integer \(n\). By \(nM\) we denote the orthogonal sum of \(n\) copies of a lattice \(M\), where \(n \geq 0\) is an integer. By \(M_1 \oplus M_2\) we denote the orthogonal sum of lattices \(M_1\) and \(M_2\). By \(A_n, n \geq 1, D_m, m \geq 4, E_k, k = 6, 7, 8\), we denote standard negative definite root lattices corresponding to the Dynkin diagrams \(A_n, D_m, E_k\), respectively, with roots whose square is \((-2)\). See [6].

The even hyperbolic 2-elementary lattice with invariants \((r = 10, a = 10, \delta = 1)\) is isomorphic to the lattice

\[
S_{10,10,1} = \bigoplus \left( \begin{array}{cc} 0 & 2 \\ 2 & -2 \end{array} \right) \oplus E_8(2)
\]

with basis \(\{c, d, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}\), where \(\{c, d\}\) have the matrix \(\begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}\) and \(e_1, \ldots, e_8\) give the standard basis of \(E_8\) with diagram \(E_8\) (see [6]).

We consider K3 surfaces \(X\) with Picard lattice \(S_{10,10,1}\), elliptic pencil \(|C|\) with the \(c|C| = c\) and nonsingular irreducible rational curve \(D\) of class \(d\). Such surfaces \(X\) exist, since \(E_8(2)\) has no elements with square \(-2\) and \(c, d\) define the NEF cone of \(X\).

Since \((c)_{S} = \mathbb{Z}c \oplus E_8(2)\) has no elements with square \(-2\), it follows that the Mordell—Weil group of this elliptic pencil \(|c|\) (see [3]) contains the subgroup \(\mathbb{Z}^4, 8 = \text{rk} E_8(2)\), which gives a subgroup of finite index of the automorphism group of the elliptic pencil \(|C|\). Since this group is infinite, \(c = c|C^{(1)}|\) and \(k = 0\) in (5). The elliptic pencil \(|c|\) is the unique elliptic pencil of \(X\) with infinite automorphism group and \(\text{Aut} X\) is the automorphism group of this elliptic pencil. We note that \(C^{(1)} \cap D\) gives two fixed points of the canonical involution \(\theta\) on the nonsingular rational curve \(D\).

4. THE CASE \((r, a, \delta) = (10, 8, 0)\)

An even hyperbolic 2-elementary lattice with invariants \((r = 10, a = 8, \delta = 0)\) is isomorphic to

\[
S_{10,8,0} = U \oplus E_8(2), \quad \text{where} \quad U = \bigoplus \left( \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right)
\]

with basis \(\{c, d, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}\), where \(\{c, d\}\) have the matrix \(U\) and \(e_1, \ldots, e_8\) correspond to the standard basis of \(E_8\) with diagram \(E_8\) (see [6]).
We consider K3 surfaces $X$ with Picard lattice $S_{10,8,0}$, elliptic fibration $|C|$ with $cl(C) = c$ and nonsingular irreducible rational curve $D$ of class $d$. We write fibration instead of pencil, because for this case $D$ gives a section of this fibration: we have $c \cdot d = 1$ and the linear system $|c|$ can be identified with $D$.

Such surfaces $X$ exist, since $E_8(2)$ has no elements with square $-2$ and $c$, $d$ define the NEF cone of $X$.

Since $(c)\frac{1}{2} = Zc \oplus E_8(2)$ has no elements with square $-2$, it follows that the Mordell—Weil group of this elliptic fibration $|c|$ (see [3]) contains the subgroup $\mathbb{Z}^8$, $8 = \text{rk} E_8(2)$, which gives a subgroup of finite index of the automorphism group of the elliptic fibration $|C|$. Since this group is infinite, $c = cl(C^{(1)}) = cl(C^{(2)})$ in (6). The elliptic fibration $|c|$ is the unique elliptic pencil of $X$ with infinite automorphism group and $\text{Aut} X$ is the automorphism group of this elliptic fibration. We note that $C^{(1)} \cap D$ and $C^{(2)} \cap D$ give two fixed points of the canonical involution $\theta$ on the nonsingular rational curve $D$.

This lattice $S_{10,8,0} = U \oplus E_8(2)$ was missed in Theorem 5.12 of the preprint [7].

5. THE CASE $(r, a, \delta) = (11, 9, 1)$

An even hyperbolic 2-elementary lattice with invariants $(r = 11, a = 9, \delta = 1)$ is isomorphic to

$$S_{11,9,1} = U \oplus E_8(2) \oplus A_1,$$

where $U = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \right\rangle$

with basis $\{c, d, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, f_1\}$, where $\{c, d\}$ have the matrix $U$, elements $e_1, ..., e_8$ correspond to the standard basis of $E_8$ with diagram $E_8$ (see [6]) and $f_1$ gives the basis of $A_1$, $f_1^2 = -2$.

We consider K3 surfaces $X$ with Picard lattice $S_{11,9,1}$, elliptic fibration $|C|$ with $cl(C) = c$ and nonsingular irreducible rational curve $D$ of class $d$ which gives the section of this fibration, and nonsingular irreducible rational curve $F_1$ of class $f_1$. Non-singular irreducible rational curves with classes $f_1$ and $f_2 = c - f_1$ define a reducible fiber of the type $\tilde{A}_1$ of this fibration with nonsingular irreducible rational curves $F_1$ with $cl(F_1) = f_1$ and $F_2$ with $cl(F_2) = f_2 = c - f_1$.

Such surfaces $X$ exist, since $E_8(2)$ has no elements with square $-2$ and $c$, $d$, $f_1$, $f_2 = c - f_1$ define the NEF cone of $X$.

Since $(c, f_1, f_2)\frac{1}{2} = Zc \oplus E_8(2)$ has no elements with square $-2$, it follows that the Mordell—Weil group of this elliptic fibration $|c|$ (see [3]) contains the subgroup $\mathbb{Z}^8$, $8 = \text{rk} E_8(2)$, which gives a subgroup of finite index of the automorphism group of the elliptic fibration $|C|$. Since this group is infinite, $c = cl(C^{(1)})$ in (5). The elliptic fibration $|c|$ is the unique elliptic pencil of $X$ with infinite automorphism group and $\text{Aut} X$ is the automorphism group of this elliptic fibration. We note that $C^{(1)} \cap D$ and $C^{(2)} \cap D$ give two fixed points of the canonical involution $\theta$ on the nonsingular rational curve $D$. In case (5), we have $k = 1$ and $E_1$ has the class $f_2 = c - f_1$ and $E_1 = F_2$. The curve $E$ in (9) has type $\tilde{A}_1$ with components with classes $f_1$ and $f_2 = c - f_1$.

6. CASES $(r, a = 20 - r, \delta = 1), r = 12, ..., 17, \text{AND } (r = 18, a = 2, \delta = 0)$

These are the cases where

$$S = U \oplus D_{16-2t} \oplus tA_1, \quad t = 0, 1, 2, 3, 4, 5, 6.$$

Then $r = 18 - t, a = 2 + t$ and $\delta = 1$ if $t > 0$, and $\delta = 0$ if $t = 0$.

The lattice $S$ has the basis $\{e, d, f_1, f_2, f_3, ..., f_{16-2t}, g_1, ..., g_t\}$, where $\{e, d\}$ give the standard basis of $U$, $f_1, f_2, f_3, ..., f_{16-2t}$ is the standard basis for the root lattice $D_{16-2t}$ and $\{g_1, ..., g_t\}$ give the standard basis for the root lattice $tA_1$. See [6].

We consider K3 surfaces $X$ with Picard lattice $S$, the elliptic fibration $|e|$ with section $D$, which is a nonsingular irreducible rational curve of class $d$, the degenerate fiber of type $\tilde{D}_{16-2t}$ with divisor...
where $k$ denotes by $t$ curves and should be $\{0\}$, because they are defined in the Picard lattice. We say that such a curve has type $+$ if $\theta$ is identical on this curve (or it is one of the curves $E_1, \ldots, E_k$ in (5)), and it has type $-$ if not. Then $\theta$ has two fixed points on this curve. If a nonsingular irreducible rational curve of type $+$ intersects transversally another nonsingular irreducible rational curve, then the second curve has type $-$, because of the action of $\theta$ by $(-1)$ on the canonical form $\omega_X$. It follows that the curves $D$ of class $d$ and $F_2, F_4, \ldots, F_{14-2t}$ with classes $f_2, f_4, f_6, \ldots, f_{14-2t}$, respectively, have type $+$. Thus, they give the curves $E_1, E_2, \ldots, E_k$, where $k = (r - a)/2 = 8 - t$ in (5). Thus, they have classes

$$d, f_2, f_4, f_6, \ldots, f_{14-2t}. \quad (10)$$

All other nonsingular irreducible rational curves on $X$ have type $-$. Let $C = C^{(1)}$ in (5). Two fixed points of $\theta$ on a nonsingular irreducible rational curve $M$ on $X$ of type $-$ come from the intersections of $M$ with $D, F_2, F_4, \ldots, F_{14-2t}$ and $C$. Using this information, we see that the class $c$ of $C$ is equal to

$$c = 3e^* + f_1^* + f_{15-2t}^* + f_{16-2t}^* + 2g_1^* + 2g_2^* + \cdots + 2g_t^* = 6e + 3d - 2f_1 - 3f_2 - 4f_3 - 5f_4 - 6f_5 - \cdots - (15 - 2t)f_{14-2t} - (8 - t)f_{15-2t} - (8 - t)f_{16-2t} - g_1 - \cdots - g_t, \quad (11)$$

where we denote by $\{e^*, d^*, f_1^*, \ldots, f_t^*, g_1^*, \ldots, g_t^*\}$ the corresponding dual basis from $S^*$ for our basis in the lattice $S$. One can check that $c^2 = 0$.

Our considerations show that the curve $D$ is the unique section of the elliptic fibration $|c|$, because we showed that this section is one of the $k$ curves $E_1$, $E_2$, $\ldots$, $E_k$ from (5), but the other $k - 1$ curves have classes $f_2, f_4, \ldots, f_{14-2t}$ of the degenerate fiber of type $\tilde{A}_{16-2t}$ of $|c|$, distinguished in the Dynkin diagram $\tilde{D}_{16-2t}$.

The elliptic pencil $|c|$ has the section $F_1$, because $f_1 \cdot c = 1$. Since the involution $\theta$ has exactly two fixed points in $F_1$, one of these points belongs to the fiber $C$ and the other fixed point belongs to the fiber of $|c|$ with curves $D, F_0, F_2, F_3, \ldots, F_{14-2t}$ with classes $d, f_0, f_2, f_3, \ldots, f_{14-2t}$, respectively. The class

$$\alpha = c - d - f_0 - f_2 - \cdots - f_{14-t}$$

has $\alpha^2 = -2$ and it has a nonnegative intersection with all classes of nonsingular irreducible rational curves $M$ of $X$ which have $e \cdot M = 0$ or $e \cdot M = 1$. We have $e \cdot \alpha = 2$. It follows that $\alpha$ corresponds to an irreducible nonsingular rational curve on $X$ by Vinberg’s criterion [8]. It follows that

$$d, f_0, f_2, f_3, \ldots, f_{14-2t}, \alpha \quad (12)$$

give all the classes of irreducible components of the degenerate fiber of type $\tilde{A}_{15-2t}$ of the elliptic fibration $|c|$. The elliptic fibration $|c|$ has no other reducible fibers, because they contain nonsingular rational curves and should be fibers over other fixed points of $\theta$ on the section $F_1$ of $|c|$. But $\theta$ has only 2 fixed points on $F_1$.

Thus, from the theory of elliptic pencils on K3, we see that the subgroup of finite index of the automorphism group of the elliptic fibration $|c|$ is a subgroup of finite index of the group $MW^*$, where

$$MW = (c, f_1, f_3, f_4, \ldots, f_{14-2t}, f_0, d, \alpha)^{\frac{1}{2}} \cdot \frac{1}{s},$$

which has rank $t + 1 = a - 1$. The lattice $MW$ has no elements with square $(-2)$.

Thus, we obtain the following statement.

**Theorem 1.** The automorphism group Aut $X$ of a K3 surface over $\mathbb{C}$ with the 2-elementary Picard lattice $S$ with invariants $(r, a = 20 - r, \delta = 1)$, where $r = 12, ..., 17$, and $(r = 18, a = 2, \delta = 0)$, equivalently, $S = U \oplus D_{16-2t} \oplus tA_1$, where $t = a - 2$ is equal to the automorphism group of the elliptic fibration $|c|$ of $X$, where $c$ is given in (11). Classes of the curves $E_1, \ldots, E_k$ from (5) are
given in (10), where \( k = (r - a)/2 = 8 - t \). The elliptic fibration \(|e|\) has the degenerate fiber of type \( \tilde{\mathbb{A}}_{15 - 2t} \) with classes (12) of components and no other reducible fibers. Then \( \text{Aut} X \) is isomorphic to \( \mathbb{Z}^{a - 1} = \mathbb{Z}^{t + 1} \) up to finite index.

See details in the considerations above.

7. THE CASE \((r, a, \delta) = (18, 2, 1)\)

In this case, \( S = S_{18,2,1} = U \oplus E_8 \oplus E_7 \oplus A_1 \).

The lattice \( S \) has the basis \( \{e, d, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, g_1, g_2, g_3, g_4, g_5, g_6, g_7, h_1\} \), where \( \{e, d\} \) is the standard basis of \( U \), \( f_1, \ldots, f_8 \) is the standard basis of \( E_8 \), \( g_1, \ldots, g_7 \) is the standard basis of \( E_7 \) and \( h_1 \) is the standard basis of \( A_1 \). See [6].

We consider K3 surfaces \( X \) with Picard lattice \( S \), the elliptic fibration \(|e|\) with the section \( D \) which is a nonsingular irreducible rational curve of class \( d \), the degenerate fiber of type \( \bar{E}_8 \) with divisor

\[
2F_1 + 3F_2 + 4F_3 + 6F_4 + 5F_5 + 4F_6 + 3F_7 + 2F_8 + F_0,
\]

where the nonsingular irreducible rational curves \( F_i \), \( i = 1, \ldots, 8 \), have classes \( f_i \), respectively, and \( F_0 \) has the class

\[
f_0 = e - 2f_1 - 3f_2 - 4f_3 - 6f_4 - 5f_5 - 4f_6 - 3f_7 - 2f_8,
\]

the degenerate fiber of type \( \bar{E}_7 \) with divisor

\[
G_0 + 2G_1 + 2G_2 + 3G_3 + 4G_4 + 3G_5 + 2G_6 + G_7,
\]

where \( G_i \), \( i = 1, \ldots, 7 \), give the standard basis of \( E_7 \), and \( G_0 \) has the class

\[
g_0 = e - 2g_1 - 2g_2 - 3g_3 - 4g_4 - 3g_5 - 2g_6 - g_7,
\]

the degenerate fiber of type \( \tilde{\mathbb{A}}_3 \) has the divisors \( H_1 + H'_1 \), where \( H_1 \) has the class \( h_1 \), and \( h'_1 \) has the class \( e - h_1 \). The curves \( H_1 \) and \( H'_1 \) are also nonsingular irreducible rational curves.

By considering the canonical involution \( \theta \), as in the previous case, we see that the curves \( F_1, F_4, F_6, F_8, D, G_1, G_4, G_6 \) with classes

\[
f_1, f_4, f_6, f_8, d, g_1, g_4, g_6,
\]

respectively, have type +. Thus, they give the curves \( E_1, \ldots, E_k \), where \( k = (r - a)/2 = 8 \) in (5). All other nonsingular irreducible rational curves on \( X \) have type -. Let \( C = C^{(1)} \) in (5). As in the previous case, using this information, we see that the class \( c \) of \( C \) is equal to

\[
c = 3e^* + f_2^* + g_2^* + g_7^* + 2h_1^* = 6e + 3d - 5f_1 - 8f_2 - 10f_3 - 15f_4
\]
\[
- 12f_5 - 9f_6 - 6f_7 - 3f_8 - 3g_1 - 5g_2 - 6g_3 - 9g_4 - 7g_5 - 5g_6 - 3g_7 - h_1,
\]

where we denote by \( \{e^*, d^*, f_1^*, \ldots, f_8^*, g_1^*, \ldots, g_7^*, h_1^*\} \) the corresponding dual basis from \( S^* \) for our basis in the lattice \( S \). One can check that \( c^2 \). 

Our considerations show that the curve \( D \) is the unique section of the elliptic fibration \(|e|\), because we showed that this section is one of \( k \) curves \( E_1, \ldots, E_k \), \( k = 8 \), from (5), but the other 7 curves have classes \( f_1, f_4, f_6, f_8, g_1, g_4, g_6 \) of the degenerate fibers of types \( \bar{E}_8 \) and \( \bar{E}_7 \) of \(|e|\), distinguished in their Dynkin diagrams.

The elliptic pencil \(|e|\) has the section \( G_7 \), because \( g_7 \cdot c = 1 \). Since the involution \( \theta \) has exactly two fixed points in \( G_7 \), one of these points belongs to the fiber \( C \) and another fixed point belongs to the fiber of \(|e|\) with curves \( D, F_1, F_3, F_4, F_5, F_6, F_7, F_8, F_0, G_0, G_1, G_3, G_4, G_5, G_6 \) with classes \( d, f_1, f_3, f_4, f_5, f_6, f_7, f_8, f_0, g_1, g_3, g_4, g_5, g_6 \), respectively. The class

\[
\alpha = c - d - f_1 - f_3 - f_4 - f_5 - f_6 - f_7 - f_8 - f_0 - g_0 - g_1 - g_3 - g_4 - g_5 - g_6
\]
has $\alpha^2 = -2$ and it has a nonnegative intersection with all classes of nonsingular irreducible rational curves $M$ of $X$ for which $e \cdot M = 0$ or $e \cdot M = 1$. We have $e \cdot \alpha = 2$. It follows that $\alpha$ corresponds to an irreducible nonsingular rational curve on $X$ by Vinberg’s criterion [8]. Therefore,

$$d, f_1, f_3, f_4, f_5, f_6, f_7, f_8, f_9, g_1, g_3, g_4, g_5, g_6, \alpha$$

(16)

give the classes of all irreducible components of the degenerate fiber of type $\tilde{h}_{15}$ of the elliptic fibration $|c|$. The elliptic fibration $|c|$ has no other reducible fibers, because they contain nonsingular rational curves and should be fibers over other fixed points of $\theta$ on the section $G_7$ of $|c|$. But $\theta$ has only 2 fixed points on $G_7$.

Thus, from the theory of elliptic pencils on K3, we see that the subgroup of finite index of the automorphism group of the elliptic fibration $|c|$ is a subgroup of finite index of the group $MW^*$, where

$$MW = (c, g_7, d, f_1, f_3, f_4, f_5, f_6, f_7, f_8, f_9, g_1, g_3, g_4, g_5, g_6, \alpha)$$

which has rank 1. The lattice $MW$ has no elements with square $(-2)$.

Thus, we obtain the following statement.

**Theorem 2.** The automorphism group $\text{Aut} \ X$ of a K3 surface $X$ over $\mathbb{C}$ with the 2-elementary Picard lattice $S$ with invariants $(r = 18, a = 2, \delta = 1)$, equivalently, $S = U \oplus E_8 \oplus E_7 \oplus A_1$, is equal to the automorphism group of the elliptic fibration $|c|$ of $X$, where $c$ is given in (15). Classes of the curves $E_1, \ldots, E_9$ from (5) are given in (13), where $k = 8$. The elliptic fibration $|c|$ has a degenerate fiber of type $\tilde{h}_{15}$ with components with classes (16) and no other reducible fibers. Then $\text{Aut} \ X$ is isomorphic to $\mathbb{Z}$ up to finite index.

See details in the considerations above.

8. THE CASE $(r, a, \delta) = (14, 6, 0)$

In this case,

$$S = S_{14,6,0} = U \oplus D_4 \oplus D_4 \oplus D_4.$$  

(17)

The lattice $S$ has the basis \{e, d, f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4\}, where \{e, d\} is the standard basis of $U$, $f_1, \ldots, f_4, g_1, \ldots, g_4$ and $h_1, \ldots, h_4$ give the standard basis of $D_4$. See [6].

We consider K3 surfaces $X$ with Picard lattice $S$, the elliptic fibration $|c|$ with section $D$ which is a nonsingular irreducible rational curve of class $d$, three degenerate fiber of type $\tilde{D}_4$ with divisors

$$F_0 + F_1 + 2F_2 + F_3 + F_4, \quad G_0 + G_1 + 2G_2 + G_3 + G_4, \quad H_0 + H_1 + 2H_2 + H_3 + H_4,$$

where nonsingular irreducible rational curves $F_i, G_i, H_i, i = 1, \ldots, 4$, have classes $f_i, g_i, h_i$, respectively, and $F_0, G_0, H_0$ have classes

$$f_0 = e - f_1 - 2f_2 - f_3 - f_4, \quad g_0 = e - g_1 - 2g_2 - g_3 - g_4, \quad h_0 = e - h_1 - 2h_2 - h_3 - h_4,$$

respectively. They are also nonsingular irreducible rational curves.

By considering the canonical involution $\theta$ on $X$, as in the previous cases, we see that the curves $F_2, G_2, H_2, D$ with classes

$$f_2, g_2, h_2, d,$$

(18)

respectively, have type $\pm$. Thus, they give the curves $E_1, \ldots, E_6$, where $k = (r - a)/2 = 4$ in (5). All other nonsingular irreducible rational curves on $X$ have type $-$. Let $C = C^{(1)}$ in (5). As in the previous cases, using this information, we see that the class $c$ of $C$ is equal to

$$c = 3e^* + f_1^* + f_2^* + f_4^* + g_1^* + g_2^* + g_3^* + h_1^* + h_2^* + h_4^*$$

$$= 6e + 3d - 2f_1 - 3f_2 - 2f_3 - 2f_4 - 2g_1 - 3g_2 - 2g_3 - 2g_4 - 2h_1 - 3h_2 - 2h_3 - 2h_4,$$

(19)

where we denote by \{e^*, d^*, f_1^*, \ldots, f_4^*, g_1^*, \ldots, g_4^*, h_1^*, \ldots, h_4^*\} the corresponding dual basis from $S^*$ for our basis in the lattice $S$. One can check that $c^2 = 0$.  

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We see that the nonsingular irreducible rational curves $D, F_0, F_2, G_0, G_2, H_0, H_2$ (or the corresponding classes $d, f_0, f_2, g_0, g_2, h_0, h_2$) have Dynkin diagrams of type $\tilde{E}_6$, where $D$ is the central element. Moreover, $3d + 2f_0 + f_2 + 2g_0 + g_2 + 2h_0 + h_2 = c$ in (19), where $c$ is the class of $C = C^{(1)}$ in (5). Thus, the elliptic pencil $|c|$ has the nonsingular fiber $C = C^{(1)}$ in (5) and the degenerate fiber $3D + 2F_0 + F_2 + 2G_0 + G_2 + 2H_0 + H_2$

of type $\tilde{E}_6$ with components with classes

$$d, f_0, f_2, g_0, g_2, h_0, h_2$$

(20)

that contain the curves $E_1, \ldots, E_k$, $k = 4$, in (5). The elliptic pencil $|c|$ has the section $F_1$, because $f_1 \cdot c = 1$, and it has type $-\tilde{c}$. Since the canonical involution $\theta$ has only two fixed points in $F_1$ corresponding to the two fibers $C^{(1)}$ and to the degenerate fiber $3D + 2F_0 + F_2 + 2G_0 + G_2 + 2H_0 + H_2$, the elliptic fibration $|c|$ does not have other reducible fibers.

Thus, from the theory of elliptic pencils on K3, we see that the subgroup of finite index of the automorphism group of the elliptic fibration $|c|$ is a subgroup of finite index of the group $MW^*$, where

$$MW = (c, f_1, g_0, h_2, h_0, d)$$

which has rank $14 - 2 - 6 = 6$. The lattice $MW$ has no elements with square $(-2)$.

Thus, we obtain the following statement.

**Theorem 3.** The automorphism group $\text{Aut} \ X$ of a K3 surface $X$ over $\mathbb{C}$ with the 2-elementary Picard lattice $S$ with invariants $(\tau = 14, a = 6, \delta = 0)$ or, equivalently, $S = U \oplus 3D_4$, is the automorphism group of the elliptic fibration $|c|$ of $X$, where $c$ is given in (19). The classes of the curves $E_1, \ldots, E_k$ from (5) are given in (18), where $k = 4$. The elliptic fibration $|c|$ has a degenerate fiber of type $\tilde{E}_6$ with components with classes (20) and no other reducible fibers. Then $\text{Aut} \ X$ is isomorphic to $\mathbb{Z}^6$ up to finite index.

See details in the considerations above.

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