Regularity and uniqueness of the heat flow of biharmonic maps

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Abstract

In this paper, we first establish regularity of the heat flow of biharmonic maps into the unit sphere $S^L \subset \mathbb{R}^{L+1}$ under a smallness condition of renormalized total energy. For the class of such solutions to the heat flow of biharmonic maps, we prove the properties of uniqueness, convexity of hessian energy, and unique limit at $t = \infty$. We also establish both regularity and uniqueness for the class of weak solutions $u$ to the heat flow of biharmonic maps into any compact Riemannian manifold $N$ without boundary such that $\nabla^2 u \in L^q_t L^p_x$ for some $p > \frac{n}{2}$ and $q > 2$ satisfying (1.13).

1 Introduction

For $n \geq 4$ and $L \geq k \geq 1$, let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $N \subset \mathbb{R}^{L+1}$ be a $k$-dimensional compact Riemannian manifold without boundary. For $m \geq 1$, $p \geq 1$, the Sobolev space $W^{m,p}(\Omega, N)$ is defined by

$$W^{m,p}(\Omega, N) = \left\{ v \in W^{m,p}(\Omega, \mathbb{R}^{L+1}) : v(x) \in N \text{ for a.e. } x \in \Omega \right\}.$$ 

On $W^{2,2}(\Omega, N)$, there are two second order energy functionals:

$$E_2(u) = \int_\Omega |\Delta u|^2 \quad \text{and} \quad F_2(u) = \int_\Omega ||(\Delta u)^T|^2,$$

where $(\Delta u)^T$ is the tangential component of $\Delta u$ to $T_u N$ at $u$, which is also called the tension field of $u$ (see [6]). A map $u \in W^{2,2}(\Omega, N)$ is called an extrinsic (or intrinsic) biharmonic map, if $u$ is a critical point of $E_2(\cdot)$ (or $F_2(\cdot)$ respectively). It is well known that biharmonic maps are higher-order extensions of harmonic maps, which are critical points of the Dirichlet energy $E_1(u) = \int_\Omega |\nabla u|^2$ over $W^{1,2}(\Omega, N)$. Recall that the Euler-Lagrange equation of (extrinsic) biharmonic maps is (see [43] Lemma 2.1):

$$\Delta^2 u = N_{bh}[u] := \Delta(A(u)(\nabla u, \nabla u)) + 2\nabla \cdot \langle \Delta u, \nabla (P(u)) \rangle - \langle \Delta (P(u)), \Delta u \rangle \perp T_u N,$$ 

(1.1)
where \( P(y) : \mathbb{R}^{L+1} \to T_y N \) is the orthogonal projection for \( y \in N \), and \( A(y)(\cdot, \cdot) = \nabla P(y)(\cdot, \cdot) \) is the second fundamental form of \( N \) at \( y \in N \). Throughout this paper, we use \( \mathcal{N}_{bh}[u] \) to denote the nonlinearity in the right hand side of the biharmonic map equation (1.1).

Motivated by the regularity theory of harmonic maps by Schoen-Uhlenbeck \([41]\), Hélein \([13]\), Evans \([7]\), Bethuel \([2]\), Lin \([26]\), Rivièere \([32]\), and many others, the study of biharmonic maps has attracted considerable interest and prompted a large number of interesting works by analysts during the last several years. The regularity of biharmonic maps to \( N = S^L \) – the unit sphere in \( \mathbb{R}^{L+1} \) – was first studied by Chang-Wang-Yang \([4]\). Wang \([43, 44, 45]\) extended the main theorems of \([4]\) to any compact Riemannian manifold \( N \) without boundary. It asserts smoothness of biharmonic maps when the dimension \( n = 4 \), and the partial regularity of stationary biharmonic maps when \( n \geq 5 \). Here we mention in passing the interesting works on biharmonic maps by Angelsberg \([1]\), Strzelecki \([31]\), Hong-Wang \([17]\), Lamm-Rivière \([24]\), Struwe \([40]\), Ku \([20]\), Gastel-Scheven \([10]\), Scheven \([34, 35]\), Lamm-Wang \([25]\), Moser \([28, 29]\), Gastel-Zorn \([11]\), Hong-Yin \([18]\), and Gong-Lamm-Wang \([12]\).

Now we describe the initial and boundary value problem for the heat flow of biharmonic maps. For \( 0 < T \leq +\infty \), and \( u_0 \in W^{2,2}(\Omega, N) \), a map \( u \in W^{1,2}_\nu(\Omega \times [0,T], N) \), i.e. \( \partial_t u, \nabla^2 u \in L^2(\Omega \times [0,T]) \), is called a weak solution of the heat flow of biharmonic maps, if \( u \) satisfies in the sense of distributions

\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t u + \Delta^2 u = & \mathcal{N}_{bh}[u] \quad \text{in } \Omega \times (0,T) \\
\quad & u = u_0 \quad \text{on } \partial_p(\Omega \times [0,T]) \\
\frac{\partial u}{\partial \nu} = & \frac{\partial u_0}{\partial \nu} \quad \text{on } \partial \Omega \times [0,T),
\end{array} \right.
\end{align*}
\]

(1.2)

where \( \nu \) denotes the outward unit normal of \( \partial \Omega \). Throughout the paper, we denote the parabolic boundary of \( \Omega \times [0,T] \) by \( \partial_p(\Omega \times [0,T]) = (\Omega \times \{0\}) \cup (\partial \Omega \times (0,T)) \).

The formulation of heat flow of biharmonic maps (1.2) remains unchanged, if \( \Omega \) is replaced by a \( n \)-dimensional compact Riemannian manifold \( M \) with boundary \( \partial M \). On the other hand, if \( \Omega \) is replaced by a \( n \)-dimensional compact Riemannian manifold without boundary or a complete, non-compact Riemannian manifold without boundary \( M \), then the Cauchy problem of heat flow of biharmonic maps is considered. More precisely, if \( \partial M = \emptyset \), then (1.2) becomes

\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t u + \Delta^2 u = & \mathcal{N}_{bh}[u] \quad \text{in } M \times (0,T) \\
\quad & u = u_0 \quad \text{on } M \times \{0\},
\end{array} \right.
\end{align*}
\]

(1.3)

The Cauchy problem (1.3) was first studied by Lamm \([22, 23]\) for \( u_0 \in C^\infty(M, N) \) in dimension \( n = 4 \), where the existence of a unique, global smooth solution is established under the condition that \( \|u_0\|_{W^{2,2}(M)} \) is sufficiently small. For any \( u_0 \in W^{2,2}(M, N) \), the existence of a unique, global weak solution of (1.3), that is smooth away from finitely many times, has been independently proved by Gastel \([9]\) and Wang \([46]\). We would like to point out that with suitable modifications of their proofs, the existence theorem by \([9]\) and \([46]\) can be extended to (1.2) for any compact 4-dimensional Riemannian manifold \( M \) with boundary \( \partial M \), if, in additions, the trace of \( u_0 \) on \( \partial M \) for \( u_0 \in W^{2,2}(M, N) \) satisfies \( u_0|_{\partial M} \in W^{2,2}(\partial M, N) \) (see \([14]\)). Namely, there is a unique, global
weak solution $u \in W^{1,2}_2(M \times [0, \infty), N)$ of (1.2) such that (i) $E_2(u(t))$ is monotone decreasing for $t \geq 0$; and (ii) there exist $T_0 = 0 < T_1 < \ldots < T_k < T_{k+1} = +\infty$ such that

$$u \in \bigcap_{i=0}^k C^\infty(M \times (T_i, T_{i+1}), N) \quad \text{and} \quad \nabla u \in \bigcap_{i=0}^k C^\alpha(M \times (T_i, T_{i+1}), N), \ \forall \ \alpha \in (0,1).$$

For dimensions $n \geq 4$, Wang [47] established the well-posedness of (1.3) on $\mathbb{R}^n$ for any $u_0 : \mathbb{R}^n \to N$ that has sufficiently small BMO norm. Moser [30] showed the existence of global weak solutions $u \in W^{1,2}_2(\Omega \times [0, \infty), N)$ to (1.2) on any bounded smooth domain $\Omega \subset \mathbb{R}^n$ for $n \leq 8$ and $u_0 \in W^{2,2}(\Omega, N)$.

Because of the critical nonlinearity in the equation (1.2), the question of regularity and uniqueness for weak solutions of (1.2) is very challenging for dimensions $n \geq 4$. There has been very few works in this direction. This motivates us to study these issues for the equation (1.2) in this paper. Another motivation comes from our recent work [15] on the heat flow of harmonic maps. We obtain several interesting results concerning regularity, uniqueness, convexity, and unique limit at time infinity of the equation (1.2), under a smallness condition of renormalized total energy.

Before stating the main theorems, we introduce some notations.

**Notations:** For $1 \leq p, q \leq +\infty$, $0 < T \leq \infty$, define the Sobolev space

$$W^{1,2}_2(\Omega \times [0, T], N) = \left\{ v \in L^2([0, T], W^{2,2}(\Omega, N)) : \partial_t v \in L^2([0, T], L^2(\Omega)) \right\},$$

the $L^p_t L^r_x$-space

$$L^p_t L^r_x(\Omega \times [0, T], \mathbb{R}^{L+1}) = \left\{ f : \Omega \times [0, T] \to \mathbb{R}^{L+1} : f \in L^q([0, T], L^p(\Omega)) \right\},$$

and the Morrey space $M^{p,\lambda}_R$ for $0 \leq \lambda \leq n + 4$, $0 < R \leq \infty$, and $U = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}$:

$$M^{p,\lambda}_R(U) = \left\{ f \in L^p_{\text{loc}}(U) : \|f\|_{M^{p,\lambda}_R(U)} < +\infty \right\},$$

where

$$\|f\|_{M^{p,\lambda}_R(U)} = \left( \sup_{(x,t) \in U} \sup_{0 < r < \min\{R,d(x,\partial U_1)\} / \sqrt{t}} \int_{P_r(x,t)} |f|^p \right)^{1/p},$$

and

$$B_r(x) = \{ y \in \mathbb{R}^n : |y - x| \leq r \}, \quad P_r(x,t) = B_r(x) \times [t - r^4, t], \quad d(x, \partial U_1) = \inf_{y \in \partial U_1} |x - y|.$$  

Denote $B_r$ (or $P_r$) for $B_r(0)$ (or $P_r(0)$ respectively), and $M^{p,\lambda}_R(U) = M^{p,\lambda}_\infty(U)$ for $R = \infty$. We also define the weak Morrey space $W^{p,\lambda}_R(U)$, that is the set of functions $f$ on $U$ such that

$$\|f\|_{M^{p,\lambda}_R(U)} = \sup_{r > 0, (x,t) \in U} \left\{ r^{\lambda - (n+4)} \|f\|_{L^p\ast(P_r(x,t) \cap U)} \right\} < +\infty,$$

where $L^p\ast(P_r(x,t) \cap U)$ is the weak $L^p$-space, that is the collection of functions $v$ on $P_r(x,t) \cap U$ such that

$$\|v\|_{L^p\ast(P_r(x,t) \cap U)} = \sup_{a > 0} \left\{ a^p \{ z \in P_r(x,t) \cap U : |v(z)| > a \} \right\} < +\infty.$$
If $N = S^L := \{ y \in \mathbb{R}^{L+1} : |y| = 1 \}$, then direct calculations yield

$$\mathcal{N}_{bh}[u] = -(|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\langle \nabla u, \nabla \Delta u \rangle)u,$$

so that for the heat flow of biharmonic maps to $S$, $\text{Theorem 1.1}$ can be written into

$$\partial_t u + \Delta^2 u = -(|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\langle \nabla u, \nabla \Delta u \rangle)u. \quad (1.4)$$

The first theorem concerns the regularity of $(1.4)$. $\textbf{Theorem 1.1}$ For $\frac{3}{2} < p \leq 2$ and $0 < T < +\infty$, there exists $\epsilon_p > 0$ such that if $u \in W^{1,2}_2(\Omega \times [0,T], S^L)$ is a weak solution of $(1.4)$ and satisfies that, for $z_0 = (x_0, t_0) \in \Omega \times (0,T]$ and $0 < R_0 \leq \frac{1}{2} \min\{d(x_0, \partial \Omega), \sqrt{t_0}\}$,

$$\|\nabla^2 u\|_{M^{2p}_{R_0}(P_{R_0}(z_0))} + \|\partial_t u\|_{M^{4p}_{R_0}(P_{R_0}(z_0))} \leq \epsilon_p,$$

then $u \in C^\infty \left( P_{\frac{R_0}{4}}(z_0), S^L \right)$, and

$$\left| \nabla^m u(z_0) \right| \leq \frac{C \epsilon_p}{R_0^m}, \forall m \geq 1. \quad (1.6)$$

$\textbf{Remark 1.2}$ It is an open question whether Theorem $\text{1.1}$ holds for any compact Riemannian manifold $N$ without boundary (with $p = 2$).

Utilizing Theorem $\text{1.1}$ we obtain the following uniqueness theorem.

$\textbf{Theorem 1.3}$ For $n \geq 4$ and $\frac{3}{2} < p \leq 2$, there exist $\epsilon_0 = \epsilon_0(p, n) > 0$ and $R_0 = R_0(\Omega, \epsilon_0) > 0$ such that if $u_1, u_2 \in W^{1,2}_2(\Omega \times [0,T], S^L)$ are weak solutions of $(1.3)$, with the same initial and boundary value $u_0 \in W^{2,2}(\Omega, S^L)$, that satisfy

$$\max_{i=1,2} \left[ \|\nabla^2 u_i\|_{M^{2p}_{R_0}(\Omega \times (0,T))} + \|\partial_t u_i\|_{M^{4p}_{R_0}(\Omega \times (0,T))} \right] \leq \epsilon_0, \quad (1.7)$$

then $u_1 \equiv u_2$ on $\Omega \times [0,T]$.

There are two main ingredients in the proof of Theorem $\text{1.3}$

(i) The interior regularity of $u_i$ ($i = 1, 2$): $u_i \in C^\infty(\Omega \times (0,T), S^L)$ and

$$\max_{i=1,2} \left[ \|\nabla^m u_i\|_{L^1(\Omega \times (0,T))} \right] \lesssim \epsilon_0 \left( \frac{1}{R_0^m} + \frac{1}{d^m(x, \partial \Omega)} + \frac{1}{t^m} \right), \quad (1.8)$$

for any $(x,t) \in \Omega \times (0,T)$ and $m \geq 1$.

(ii) The energy method, with suitable applications of the Poincaré inequality and the second order Hardy inequality in Lemma $\text{3.1}$ below.

$\textbf{Remark 1.4}$ (i) We would like to point out that a novel feature of Theorem $\text{1.3}$ is that the solutions may have singularities at the parabolic boundary $\partial_p(\Omega \times [0,T])$ so that the standard argument to
prove uniqueness for classical solutions is not applicable.

(ii) For $\Omega = \mathbb{R}^n$, if the initial data $u_0 : \mathbb{R}^n \to N$ satisfies that for some $R_0 > 0$,

$$\sup \left\{ r^{4-n} \int_{B_r(x)} |\nabla^2 u_0|^2 : x \in \mathbb{R}^n, r \leq R_0 \right\} \leq \epsilon^2_0,$$

then by the local well-posedness theorem of Wang [47] there exists $0 < T_0( \approx R_0^4 )$ and a solution $u \in C^\infty(\mathbb{R}^n \times (0, T_0), N)$ of (1.3) that satisfies the condition (1.7).

Prompted by the ideas of proof of Theorem 1.3 we obtain the convexity property of the $E_2$-energy along the heat flow of biharmonic maps to $S^L$.

**Theorem 1.5** For $n \geq 4$, $\frac{2}{3} < p \leq 2$, and $1 \leq T \leq \infty$, there exist $\epsilon_0 = \epsilon_0(p, n) > 0$, $R_0 = R_0(\Omega, \epsilon_0) > 0$, and $0 < T_0 = T_0(\epsilon_0) < T$ such that if $u \in W^{1,2}_2(\Omega \times [0, T], S^L)$ is a weak solution of (1.3), with the initial and boundary value $u_0 \in W^{2,2}(\Omega, S^L)$, satisfying

$$\|\nabla^2 u\|_{M^{p-2p}_R(\Omega \times (0, T))} + \|\partial_t u\|_{M^{p-4p}_R(\Omega \times (0, T))} \leq \epsilon_0,$$  \hspace{1cm} (1.9)

then

(i) $E_2(u(t))$ is monotone decreasing for $t \geq T_0$; and

(ii) for any $t_2 \geq t_1 \geq T_0$,

$$\int_{\Omega} |\nabla^2 (u(t_1) - u(t_2))|^2 \leq C \left[ \int_{\Omega} |\Delta u(t_1)|^2 - \int_{\Omega} |\Delta u(t_2)|^2 \right]$$  \hspace{1cm} (1.10)

for some $C = C(n, \epsilon_0) > 0$.

A direct consequence of the convexity property of $E_2$-energy is the unique limit at $t = \infty$ of (1.2).

**Corollary 1.6** For $n \geq 4$ and $\frac{2}{3} < p \leq 2$, there exist $\epsilon_0 = \epsilon_0(p, n) > 0$, and $R_0 = R_0(\Omega, \epsilon_0) > 0$ such that if $u \in W^{1,2}_2(\Omega \times [0, \infty), S^L)$ is a weak solution of (1.3), with the initial and boundary value $u_0 \in W^{2,2}(\Omega, S^L)$, satisfying the condition (1.9), then there exists a biharmonic map $u_\infty \in C^\infty \cap W^{2,2}(\Omega, S^L)$, with \(\frac{\partial u_\infty}{\partial \nu} = (u_0, \frac{\partial u_0}{\partial \nu})\) on $\partial \Omega$, such that

$$\lim_{t \to \infty} \|u(t) - u_\infty\|_{W^{2,2}(\Omega)} = 0,$$  \hspace{1cm} (1.11)

and, for any compact subset $K \subset \subset \Omega$ and $m \geq 1$,

$$\lim_{t \to \infty} \|u(t) - u_\infty\|_{C^m(K)} = 0.$$  \hspace{1cm} (1.12)

**Remark 1.7** (i) We would like to remark that if Theorem 1.1 has been proved for any compact Riemannian manifold $N$ without boundary, then Theorem 1.3, Theorem 1.5, and Corollary 1.6 would be true for any compact Riemannian manifold $N$ without boundary.

(ii) With slight modifications of the proofs, Theorem 1.1, Theorem 1.3, Theorem 1.5, and Corollary 1.6 remain to be true, if $\Omega$ is replaced by a compact Riemannian manifold $M$ with boundary $\partial M$. 

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u consists of all weak solutions to the heat flow of biharmonic maps.

If (ii) \( u \in W^{2,2}(\Omega \times (0,T), N) \) for some \( p > n/2 \) and \( q < \infty \) satisfying
\[
\frac{n}{p} + \frac{4}{q} = 2. 
\]
(1.13)

We usually call (1.13) as Serrin’s condition (see [37]). In §5, we will prove that if \( u \) is a weak solution of (1.2) such that \( \nabla^2 u \in L^q_t L^p_x(\Omega \times [0,T]) \) for some \( p > \frac{n}{2} \) and \( q > 3 \) satisfying (1.13) and \( u_0 \in W^{2,r}(\Omega, N) \) for some \( r > \frac{n}{2} \), then \( u \) satisfies (1.3) for some \( p_0 > \frac{n}{2} \). Thus, for \( N = S^3 \), the regularity and uniqueness for such solutions of (1.2) follow from Theorem 1.1 and Theorem 1.3. However, for a compact Riemannian manifold \( N \) without boundary, the regularity and uniqueness for such a class of weak solutions of (1.2) require different arguments. More precisely, we have

**Theorem 1.8** For \( n \geq 4 \) and \( 0 < T \leq \infty \), let \( u_{1}, u_{2} \in W^{1,2}_{2}(\Omega \times [0,T], N) \) be weak solutions of (1.2), with the same initial and boundary value \( u_0 \in W^{2,2}(\Omega, N) \). If, in additions, \( \nabla^2 u_1, \nabla^2 u_2 \in L^q_t L^p_x(\Omega \times [0,T]) \) for some \( p > \frac{n}{2} \) and \( q < \infty \) satisfying (1.13), then \( u_{1}, u_{2} \in C^\infty(\Omega \times (0,T), N) \), and \( u_1 \equiv u_2 \) in \( \Omega \times [0,T] \).

**Remark 1.9** (i) It is a very interesting question to ask whether Theorem 1.8 holds for the end-point case \( p = \frac{n}{2} \) and \( q = \infty \).

(ii) If \( u_0 \in W^{2,r}(\Omega, N) \) for some \( r > \frac{n}{2} \), then the local existence of solutions \( u \) of (1.2) such that \( \nabla^2 u \in L^q_t L^p_x(\Omega \times [0,T]) \) for some \( p > \frac{n}{2} \) and \( q < \infty \) satisfying (1.13) can be shown by the fixed point argument similar to [3] §4. We leave it to interested readers.

For dimension \( n = 4 \), by applying Theorem 5.2 (with \( p = 2 \left(= \frac{n}{2} \right) \) and \( q = \infty \)) and the second half of the proof of Theorem 1.3, we obtain the following uniqueness result.

**Corollary 1.10** For \( n = 4 \) and \( 0 < T \leq \infty \), there exists \( \epsilon_1 > 0 \) such that if \( u_1 \) and \( u_2 \in W^{1,2}_{2}(\Omega \times [0,T], N) \) are weak solutions of (1.2), under the same initial and boundary value \( u_0 \in W^{2,2}(\Omega, N) \),
satisfying
\[ \limsup_{t \to t_0^+} E_2(u_i(t)) \leq E_2(u_i(t_0)) + \epsilon_1, \quad \forall \ t_0 \in [0, T), \]  
for \( i = 1, 2 \). Then \( u_1 \equiv u_2 \) in \( \Omega \times [0, T) \). In particular, the uniqueness holds among weak solutions of (1.2), whose \( E_2 \)-energy is monotone decreasing for \( t \geq 0 \).

We would like to point out that for the Cauchy problem (1.3) of heat flow of biharmonic maps on a compact 4-dimensional Riemannian manifold \( M \) without boundary, Corollary 1.10 has been recently proven by Rupflin [33] through a different argument.

Concerning the convexity and unique limit of (1.2) at \( t = \infty \) in dimension \( n = 4 \), we have

**Corollary 1.11** For \( n = 4 \), there exist \( \epsilon_2 > 0 \) and \( T_1 > 0 \) such that if \( u \in W_2^{1,2}(\Omega \times (0, +\infty), N) \) is a weak solution of (1.2), with the initial-boundary value \( u_0 \in W_2^{2,2}(\Omega, N) \), satisfying
\[ E_2(u(t)) \leq \epsilon_2^2, \quad \forall \ t \geq 0, \]  
then (i) \( E_2(u(t)) \) is monotone decreasing for \( t \geq T_1 \);
(ii) for \( t_2 \geq t_1 \geq T_2 \), it holds
\[ \int_\Omega |\nabla^2(u(t_1) - u(t_2))|^2 \leq C (E_2(u(t_1)) - E_2(u(t_2))) \]  
for some \( C = C(\epsilon_2) > 0 \); and
(iii) there exists a biharmonic map \( u_\infty \in C^\infty \cap W_2^{2,2}(\Omega, N) \), with \( (u_\infty, \frac{\partial u_\infty}{\partial \nu}) = (u_0, \frac{\partial u_0}{\partial \nu}) \) on \( \partial \Omega \), such that \( \lim_{t \to \infty} \|u(t) - u_\infty\|_{W_2^{2,2}(\Omega)} = 0 \), and for any \( m \geq 1 \), \( K \subset \subset \Omega \), \( \lim_{t \to \infty} \|u(t) - u_\infty\|_{C^m(K)} = 0 \).

It is easy to see that the condition (1.13) holds for any solution \( u \in W_2^{1,2}(\Omega \times [0, \infty), N) \) of (1.2) that satisfies \( E_2(u(t)) \leq E_2(u_0) \) for \( t \geq 0 \) (e.g., the solution by [9] and [16]) and \( E_2(u_0) \leq \epsilon_2^2 \).

The paper is written as follows. In §2, we will prove the \( \epsilon \)-regularity Theorem 1.1 for weak solutions of (1.2) under the assumption (1.5). In §3, we will show both convexity and uniqueness property for biharmonic maps with small \( E_2 \)-energy. In §4, we will prove the uniqueness Theorem 1.3, the convexity Theorem 1.5 and the unique limit Theorem 1.6. In §5, we will discuss weak solutions \( u \) of (1.2) such that \( \nabla^2 u \in L_p^q L_{t}^{p}(\Omega \times [0, T]) \) for some \( p \geq \frac{3}{2} \) and \( q \geq 2 \) satisfying (1.13), and prove Theorem 1.8 Corollary 1.10 and Corollary 1.11. In §6 Appendix, we will sketch a proof for higher-order regularities of the heat flow of biharmonic maps.

## 2 \( \epsilon \)-regularity

This section is devoted to the proof of Theorem 1.1 i.e., the regularity of heat flow of biharmonic maps to \( S^L \) under the smallness condition (1.5). The idea is motivated by [4] on the regularity of stationary biharmonic maps to \( S^L \).

The first step is to rewrite (1.4) into the form where nonlinear terms are of divergence structures, analogous to [4] on the equation of biharmonic maps to \( S^L \). As in [4], we divide the nonlinearities
It follows from (2.5) and (2.6) that
\[ T_{11}^{\alpha} = \left( u_j^\beta \Delta u^\beta (u^\beta - c^\beta) \right)_j \quad \text{or} \quad \left( u_j^\beta \Delta u^\alpha (u^\beta - c^\beta) \right)_j, \quad T_{12}^{\alpha} = \left( (u^\alpha - c^\alpha) u_i^\beta u_j^\alpha \right)_j, \]
\[ T_{21}^{\alpha} = \Delta \left( (u^\alpha - c^\alpha) |\nabla u|^2 \right), \quad T_{22} = \Delta \left( (u^\beta - c^\beta) \Delta u^\beta \right), \]
\[ T_{33} = \left( (u^\beta - c^\beta) u_j^\beta \right)_{jii}, \quad T_{41}^{\alpha} = \left( u^\alpha \partial_t u^\beta - u^\beta \partial_t u^\alpha \right) (u^\beta - c^\beta), \]
where the upper index $\alpha$, $\beta$ denotes a linear function of its arguments such that the coefficients can be bounded independent of $u$. 

**Lemma 2.1** The equation (1.7) is equivalent to
\[ \partial_t u^\alpha + \Delta^2 u^\alpha = F_\alpha (T_{11}^{\alpha}, T_{12}^{\alpha}, T_{21}^{\alpha}, T_{22}, T_{33}, T_{41}^{\alpha}), \]
where $F_\alpha$ denotes a linear function of its arguments such that the coefficients can be bounded independent of $u$.

**Proof.** We follow [4] Proposition 1.2 closely. First, by Lemma 1.3 of [4], we have that, for every fixed $\alpha$,
\[ c^\alpha \Delta (|\nabla u|^2) \quad \text{and} \quad (u_j^\beta |\nabla u|^2)_j \quad \text{are linear functions of} \quad T_{11}^{\alpha}, T_{12}^{\alpha}, T_{21}^{\alpha}, T_{22}, T_{33}, T_{41}^{\alpha}, \]
whose coefficients can be bounded independent of $u$. For $1 \leq \alpha \leq L + 1$, set
\[ S_1^\alpha = u^\alpha |\nabla u|^2, \quad S_2^\alpha = 2u^\alpha u_j^\beta \left( \Delta u_j^\beta \right)_j, \quad S_3^\alpha = u^\alpha \Delta (|\nabla u|^2). \]
Differentiation of $|u| = 1$ gives
\[ u^\alpha u_j^\alpha = 0, \quad u^\beta \Delta u^\gamma + |\nabla u|^2 = 0. \]
By the equation (1.2), we have
\[ u^\alpha \Delta^2 u^\beta + u^\alpha \partial_t u^\beta = u^\beta \Delta^2 u^\alpha + u^\beta \partial_t u^\alpha, \quad 1 \leq \alpha, \beta \leq L + 1. \]
It follows from (2.5) and (2.6) that
\[ \frac{S_2^\alpha}{2} = u^\alpha u_j^\beta (\Delta u_j^\beta)_j = u_j^\alpha \left( u^\alpha (\Delta u_j^\beta)_j - u^\beta (\Delta u_j^\alpha)_j \right) = u_j^\alpha \left( u^\alpha (\Delta u_j^\beta)_j - u^\beta (\Delta u_j^\alpha)_j - u_j^\beta \Delta u_j^\beta + u_j^\beta \Delta u_j^\alpha \right) + u_j^\beta \left( u_j^\alpha \Delta u_j^\alpha - u_j^\beta \Delta u_j^\alpha \right) = \left\{ \left( u^\beta - c^\beta \right) \left( u^\alpha (\Delta u_j^\beta)_j - u^\beta (\Delta u_j^\alpha)_j - u_j^\beta \Delta u_j^\beta + u_j^\beta \Delta u_j^\alpha \right) \right\}_j + \left\{ \left( u^\beta - c^\beta \right) \left( u^\alpha \partial_t u_j^\beta - u^\beta \partial_t u_j^\alpha \right) + u_j^\beta \left( u_j^\alpha \Delta u_j^\alpha - u_j^\beta \Delta u_j^\alpha \right) \right\}_j - 2 \left\{ \left( u^\beta - c^\beta \right) \left( u_j^\alpha \Delta u_j^\beta - u_j^\alpha \Delta u_j^\alpha \right) \right\}_j + u_j^\beta \left( u_j^\alpha \Delta u_j^\beta - u_j^\alpha \Delta u_j^\alpha \right) + T_{41}^{\alpha}. \]
where \( G_\alpha \) is a linear function of its arguments whose coefficients can be bounded independent of \( u \). By (2.3) and (2.5), we have
\[
S_3^\alpha = (u^\alpha - e^\alpha) \Delta (|\nabla u|^2) + e^\alpha \Delta (|\nabla u|^2)
\]
\[
= \Delta ((u^\alpha - e^\alpha) |\nabla u|^2 - 2 (u_j^\alpha)(|\nabla u|^2)_j - \Delta u^\alpha u^\beta \Delta u^\beta + e^\alpha \Delta (|\nabla u|^2) \hspace{1cm} (2.8)
\]
where \( H_\alpha \) is a linear function of its arguments whose coefficients can be bounded independent of \( u \). By (2.8), the definition of \( S_3^\alpha \), and (2.7), we have
\[
S_1^\alpha + S_3^\alpha = \left( u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha \right) \Delta u^\beta + H_\alpha(T_{11}^\alpha, T_{12}^\alpha, T_{22}^\alpha, T_{23}^\alpha, T_{33}^\alpha),
\]
\[
= \frac{1}{2} \left( u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha \right) u_j^\beta \frac{1}{j} - \frac{1}{2} \left( u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha \right) u_j^\beta
\]
\[
= - \frac{S_2^\alpha}{2} + S_2^\alpha + L_\alpha(T_{11}^\alpha, T_{12}^\alpha, T_{22}^\alpha, T_{23}^\alpha, T_{33}^\alpha),
\]
where \( L_\alpha \) is a linear function of its arguments whose coefficients can be bounded independent of \( u \). Therefore we obtain
\[
S_1^\alpha + S_2^\alpha + S_3^\alpha = L_\alpha(T_{11}^\alpha, T_{12}^\alpha, T_{22}^\alpha, T_{23}^\alpha, T_{33}^\alpha, T_{41}^\alpha).
\]
This completes the proof. \( \square \)

Next we recall some basic properties of the heat kernel for \( \Delta^2 \) in \( \mathbb{R}^n \), and the definition of Riesz potentials on \( \mathbb{R}^{n+1} \), and the definition of \textup{BMO} space and John-Nirenberg’s inequality (see [19]).

Let \( b(x, t) \) be the fundamental solution of
\[
(\partial_t + \Delta^2)u = 0 \text{ in } \mathbb{R}^{n+1}_+.
\]
Then we have (see [21] §2.2):
\[
b(x, t) = t^{-\frac{n}{2}} g \left( \frac{x}{t^{\frac{1}{2}}} \right), \text{ with } g(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi \eta - |\eta|^2} \text{, } \xi \in \mathbb{R}^n,
\]
and the estimate
\[
|\nabla^m b(x, t)| \leq C \left( |t|^{\frac{1}{2}} + |x| \right)^{-n-m}, \forall (x, t) \in \mathbb{R}^{n+1}_+, \forall m \geq 1. \hspace{1cm} (2.10)
\]
We equip \( \mathbb{R}^{n+1} \) with the parabolic distance \( \delta \):
\[
\delta((x, t), (y, s)) = |t - s|^{\frac{1}{2}} + |x - y|, (x, t), (y, s) \in \mathbb{R}^{n+1}.
\]
For \( 0 \leq \alpha \leq n + 4 \), define the Riesz potential of order \( \alpha \) on \( \mathbb{R}^{n+1} \) by
\[
I_\alpha(f)(x, t) = \int_{\mathbb{R}^{n+1}} \left( |t - s|^{\frac{1}{2}} + |x - y| \right)^{-\alpha - n - 4} |f|(y, s), (x, t) \in \mathbb{R}^{n+1}. \hspace{1cm} (2.11)
\]
For any open set $U \subset \mathbb{R}^{n+1}$, let $\text{BMO}(U)$ denote the space of functions of bounded mean oscillations: $f \in \text{BMO}(U)$ if

$$
[f]_{\text{BMO}(U)} := \sup \left\{ \int_{P_r(z)} |f - f_{P_r(z)}| : P_r(z) \subset U \right\} < +\infty,  \tag{2.12}
$$

where $\int_{P_r(z)} = \frac{1}{|P_r(z)|} \int_{P_r(z)} f$ denotes the average of $f$ over $P_r(z)$. By the celebrated John-Nirenberg inequality (see [19]), we have that if $f \in \text{BMO}(U)$, then for any $1 < q < +\infty$ it holds

$$
\sup \left\{ \left( \int_{P_r(z)} |f - f_{P_r(z)}|^q \right)^{\frac{1}{q}} : P_r(z) \subset U \right\} \leq C(q) [f]_{\text{BMO}(U)}. \tag{2.13}
$$

Now we are ready to prove the $\epsilon$-regularity for the heat flow of biharmonic maps to $\mathbb{S}^L$.

**Proposition 2.2** For any $\frac{3}{2} < p \leq 2$, there exists $\epsilon_p > 0$ such that if $u : P_\frac{3}{2} \to \mathbb{S}^L$ is a weak solution of (1.4) and satisfies

$$
\sup_{(x,t) \in P_5, 0 < r \leq 1} r^{2p - n - 4} \int_{P_r(x,t)} (|\nabla^2 u|^p + r^{2p} |\partial_t u|^p) \leq \epsilon_p, \tag{2.14}
$$

then $u \in C^\infty(P_\frac{1}{2}, \mathbb{S}^L)$, and

$$
\left\| \nabla^m u \right\|_{C^0(P_\frac{1}{2})} \leq C(p, n, m), \forall m \geq 1. \tag{2.15}
$$

**Proof.** We first establish Hölder continuity of $u$ in $P_\frac{3}{2}$. It is based on the decay estimate.

Claim. There exist $\epsilon_p > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that

$$
\left[ u \right]_{\text{BMO}(P_{\theta_0})} \leq \frac{1}{2} \left[ u \right]_{\text{BMO}(P_2)}. \tag{2.16}
$$

In order to establish (2.16), we first want to prove that there exists $q > 1$ such that

$$
\int_{P_{\theta_0}(z_0)} |u - u_{P_{\theta_0}(z_0)}| \leq C \left( \theta^{-(n+4)} \epsilon_p + \theta \right) \left( \int_{P_{\theta_0}(z_0)} |u - u_{P_{\theta_0}(z_0)}|^q \right)^{\frac{1}{q}}. \tag{2.17}
$$

holds for any $0 < \theta \leq \frac{1}{2}$, $z_0 \in P_1$, and $0 < r \leq 2$.

By translation and scaling, it suffices to show (2.17) for $z_0 = (0,0)$ and $r = 2$. First, we need to extend $u$ from $P_1$ to $\mathbb{R}^{n+1}$. Let the extension, still denoted by $u$, be such that

$$
|u| \leq 2 \text{ in } \mathbb{R}^{n+1}, \quad u = 0 \text{ outside } P_2,
$$

and

$$
\int_{\mathbb{R}^{n+1}} |\nabla^2 u|^p + |\partial_t u|^p \lesssim \int_{P_2} |\nabla^2 u|^p + |\partial_t u|^p.
$$

For $1 \leq \alpha \leq L + 1$, let $w_{ij}^\alpha : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be solutions of

$$
\partial_t w_{ij}^\alpha + \Delta^2 w_{ij}^\alpha = T_{ij}^\alpha \quad \text{in } \mathbb{R}^{n+1}_+, \quad w_{ij}^\alpha = 0 \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{2.18}
$$
for $ij \in \{11, 12, 21, 23, 41\}$, and and $w_{kk} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be solutions of

$$
\partial_t w_{kk} + \Delta^2 w_{kk} = T_{kk} \quad \text{in} \quad \mathbb{R}^{n+1}_+; \quad w_{kk} = 0 \quad \text{on} \quad \mathbb{R}^n \times \{0\}
$$

(2.19)

for $k \in \{2, 3\}$. Define $v : P_1 \to \mathbb{R}^{L+1}$ by letting

$$
v^\alpha = u^\alpha - \mathcal{F}_\alpha(u_{11}^\alpha, u_{12}^\alpha, w_{21}^\alpha, w_{22}^\alpha, w_{23}^\alpha, w_{41}^\alpha), \quad 1 \leq \alpha \leq L + 1.
$$

Here $\mathcal{F}_\alpha$ is the linear function given by Lemma 2.1. By (2.2), we have

$$
\partial_t v + \Delta^2 v = 0 \quad \text{in} \quad P_1.
$$

(2.20)

It follows from (2.19) and the Duhamel formula that for $1 \leq \alpha \leq L + 1$,

$$
\begin{cases}
  w_{ij}^\alpha(x, t) = \int_{\mathbb{R}^n \times [0, t]} b(x - y, t - s) T_{ij}^\alpha(y, s), \quad ij \in \{11, 12, 21, 23, 41\}, \\
  w_{kk}^\alpha(x, t) = \int_{\mathbb{R}^n \times [0, t]} b(x - y, t - s) T_{kk}^\alpha(y, s), \quad k \in \{2, 3\}.
\end{cases}
$$

(2.21)

Set $c^\alpha = u_{P_2}^\alpha$ in (2.1). Then it is easy to see $|c^\alpha| \leq 1$. Now we can estimate $w_{12}^\alpha$ by ($w_{11}^\alpha$ can be estimated similarly):

$$
|w_{12}^\alpha(x, t)| = \left| \int_{\mathbb{R}^n \times [0, t]} \nabla_j b(x - y, t - s)(u^\alpha - u_{P_2}^\alpha) u_i^\beta u_j^\beta(y, s) \right|
$$

$$
\lesssim \int_{\mathbb{R}^{n+1}} \left( |t - s|^{\frac{2}{\alpha}} + |x - y| \right)^{-n-1} |u - u_{P_2}| \|\nabla u\| \|\nabla^2 u\|(y, s)
$$

$$
\lesssim I_3 \left( \chi_{P_2} |u - u_{P_2}| \|\nabla u\| \|\nabla^2 u\| \right)(x, t),
$$

(2.22)

where $\chi_{P_2}$ is the characteristic function of $P_2$.

By the estimate of Riesz potentials in $L^q$-spaces (see also §5 below), we have that for any $f \in L^q$, $1 < q < +\infty$, $I_\alpha(f) \in L^q$, where $\frac{1}{q} = \frac{1}{q_1} = \frac{1}{q} - \frac{\alpha}{n+1}$. As $p > \frac{3}{2}$, we can check that for sufficiently large $q_1 > 1$, there exists $q_1 > 1$ such that

$$
\frac{1}{q_1} = \frac{1}{p} + \frac{1}{2p} + \frac{1}{q_1} - \frac{3}{n+4}.
$$

Hence we obtain

$$
\left\| w_{12}^\alpha \right\|_{L^{q_1}(P_2)} \leq C \left\| u - u_{P_2} \right\|_{L^{q_1}(P_2)} \left\| \nabla u \right\|_{L^{2p}(P_2)} \left\| \nabla^2 u \right\|_{L^p(P_2)} \leq C \epsilon_p \left\| u - u_{P_2} \right\|_{L^{q_1}(P_2)}.
$$

(2.23)

Next we can estimate $w_{21}^\alpha$ by ($w_{22}$ and $w_{23}^\alpha$ can be estimated similarly):

$$
|w_{21}^\alpha(x, t)| = \left| \int_{\mathbb{R}^n \times [0, t]} \Delta b(x - y, t - s)(u^\alpha - u_{P_2}^\alpha) \|\nabla u\|^2(y, s) \right|
$$

$$
\lesssim \int_{\mathbb{R}^{n+1}} \left( |t - s|^{\frac{2}{\alpha}} + |x - y| \right)^{-n-2} |u - u_{P_2}| \|\nabla u\|^2(y, s)
$$

$$
\lesssim I_2 \left( \chi_{P_2} |u - u_{P_2}| \|\nabla u\|^2 \right)(x, t).
$$

(2.24)
For $q_2 > 1$ sufficiently large, there exists $\tilde{q}_2 > 1$ be such that
$$\frac{1}{q_2} = \frac{1}{p} + \frac{1}{q_2} - \frac{2}{n+4}.$$ Hence we obtain
$$\left\| u_{21}^{q_2} \right\|_{L^{q_2}((P_2))} \leq C \left\| u - u_{P_2} \right\|_{L^{q_2}((P_2))} \left\| \nabla u \right\|_{L^p((P_2))} \leq C \epsilon_p \left\| u - u_{P_2} \right\|_{L^{q_2}((P_2))}. \quad (2.25)$$

For $w_{33}$, we have
$$|w_{33}(x,t)| = \left| \int_{\mathbb{R}^n \times [0,t]} \Delta b_j(x-y,t-s)(u^\beta - u_{P_2}^\beta)u_j^\beta(y,s) \right| \lesssim \int_{\mathbb{R}^n \times [1]} \left( |t-s|^{\frac{1}{4}} + |x-y| \right)^{-n-3} |u - u_{P_2}| \left\| \nabla u \right\|(y,s) \quad (2.26)$$
$$\lesssim I_1(\chi_{P_2}|u - u_{P_2}|\nabla u).$$

For $q_3 > 1$ sufficiently large, there exists $\tilde{q}_3 > 1$ such that
$$\frac{1}{q_3} = \frac{1}{2p} + \frac{1}{q_3} - \frac{1}{n+4}.$$ Hence we obtain
$$\left\| w_{33} \right\|_{L^{q_3}((P_2))} \leq C \left\| u - u_{P_2} \right\|_{L^{q_3}((P_2))} \left\| \nabla u \right\|_{L^{2p}((P_2))} \leq C \epsilon_p \left\| u - u_{P_2} \right\|_{L^{q_3}((P_2))}. \quad (2.27)$$

For $w_{41}^\alpha$, we have
$$\partial_tw_{41}^\alpha + \Delta^2 w_{41}^\alpha = \left( u^\alpha \partial_t u^\beta - u^\beta \partial_t u^\alpha \right) \left( u^\beta - u_{P_2}^\beta \right). \quad (2.28)$$

By the Duhamel formular, we have
$$w_{41}^\alpha(x,t) = \sum_{\beta} \int_0^t \int_{\mathbb{R}^n} b(x-y,t-s) \left( u^\alpha \partial_t u^\beta - u^\beta \partial_t u^\alpha \right) \left( u^\beta - u_{P_2}^\beta \right)(y,s),$$
so that by applying the Young inequality we obtain
$$\left\| w_{41} \right\|_{L^{q_4}((\mathbb{R}^n \times [0,2])} \lesssim \|b\|_{L^1((\mathbb{R}^n \times [0,2])} \left( \sum_{\alpha,\beta} \left\| \left( u^\alpha \partial_t u^\beta - u^\beta \partial_t u^\alpha \right)(u^\beta - u_{P_2}^\beta) \right\|_{L^{q_4}((\mathbb{R}^n \times [0,2])} \right) \lesssim \left\| \partial_t u \right\|_{L^{p}((P_2))} \left\| u - u_{P_2} \right\|_{L^{q_4}((P_2))}, \quad (2.29)$$

where $q_4 > \frac{p}{p-1}$ and $1 < \tilde{q}_4 < p$ satisfy
$$\frac{1}{q_4} = \frac{1}{p} + \frac{1}{q_4}.$$ Set
$$q = \max\{q_1, q_2, q_3, q_4\} > 1 \text{ and } \tilde{q} = \min\{\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4\} > 1.$$
By (2.23), (2.25), (2.27) and (2.29), we have

$$\sum_{ij=11,12,21,23,41} w_{ij}^2 \|L^i(P_2) + \sum_{k=2}^3 \|w_{kk}\|L^k(P_2) \leq C_\epsilon \|u - u_{P_2}\|_{L^\infty(P_2)}. \tag{2.30}$$

On the other hand, by the standard estimate on $v$, we have that for any $0 < \theta < 1$,

$$\left(\int_{P_0} |v - v_{P_0}|^q \right)^{\frac{1}{q}} \leq C \theta \left(\int_{P_1} |v - v_{P_1}|^q \right)^{\frac{1}{q}} \leq C \theta \|u - u_{P_2}\|_{L^\infty(P_2)} \tag{2.31}.$$

Adding (2.30) and (2.31) together and applying the Hölder inequality, we obtain

$$\int_{P_0} |u - u_{P_0}| \leq \left(\int_{P_0} |u - u_{P_0}|^q \right)^{\frac{1}{q}} \leq C \left(\theta^{-(n+4)} \epsilon_p + \theta \right) \left(\int_{P_2} |u - u_{P_2}|^q \right)^{\frac{1}{q}}. \tag{2.32}$$

This implies (2.17).

Now we indicate how (2.16) follows from (2.17). It follows from the Poincaré inequality and (2.14) that $u \in \text{BMO}(P_3)$, and hence by (2.13) we have

$$\int_{P_{2r}(z_0)} |u - u_{P_{2r}(z_0)}| \leq C \left(\theta^{-(n+4)} \epsilon_p + \theta \right) \left[ u \right]_{\text{BMO}(P_2)} \tag{2.33}$$

holds for any $0 < \theta \leq \frac{1}{2}$, $z_0 \in P_1$, and $0 < r \leq 1$. Taking supremum of (2.33) over all $z_0 \in P_0$ and $0 < r \leq 1$, we obtain

$$\left[ u \right]_{\text{BMO}(P_0)} \leq C \left(\theta^{-(n+4)} \epsilon_p + \theta \right) \left[ u \right]_{\text{BMO}(P_2)}. \tag{2.34}$$

If we choose $\theta = \theta_0 \in (0, \frac{1}{2})$ and $\epsilon_p$ small enough so that

$$C \left(\theta_0^{-(n+4)} \epsilon_p + \theta_0 \right) \leq \frac{1}{2},$$

then (2.34) implies (2.16).

It is standard that iterating (2.16) yields the Hölder continuity of $u$ by using the Campanato theory [3]. The higher-order regularity then follows from the hole-filling type argument and the bootstrap argument, which will be sketched in Proposition 6.1 of §6 Appendix. After this, we have that $u \in C^\infty(P_{\frac{3}{2}}, S^L)$ and the estimate (2.15) holds.

**Proof of Theorem 1.1** By the definition of Morrey spaces, for $z_0 = (x_0, t_0) \in \Omega \times (0, T)$ and $R_0 \leq \frac{1}{2} \min\{d(x_0, \partial \Omega), \sqrt{t_0}\}$, we have

$$\sup_{z \in P_{R_0}(z_0), r \leq R_0} r^{2p-(n+4)} \int_{P_r(z)} (\|\nabla^2 u\|^p + r^2 \|\partial_t u\|^p) \leq \epsilon_p. \tag{2.35}$$

Consider $v(x, t) = u(x + \frac{R_0}{R_0^4}, t_0 + (\frac{R_0}{R_0^4})^4 t) : P_4 \rightarrow S^L$. It is easy to check that $v$ is a weak solution of (1.4) and satisfies (2.14). Hence Proposition 2.2 implies that $v \in C^\infty(P_{\frac{3}{2}}, S^L)$ and satisfies (2.15). After rescaling, we see that $u \in C^\infty(P_{\frac{3}{2}}, S^L)$ and the estimate (2.16) holds.

Since biharmonic maps are steady solutions of the heat flow of biharmonic maps, as a direct consequence of Theorem 1.1 we have the following $\epsilon$-regularity for biharmonic maps to $S^L$.
Corollary 2.3  For $\frac{3}{2} < p \leq 2$, there exist $\epsilon_p > 0$ and $r_0 > 0$ such that if $u \in W^{2,p}(\Omega, S^L)$ is a weak solution of (1.1) and satisfies
\begin{equation}
\sup_{x \in \Omega} \sup_{0 < r \leq \min\{r_0, d(x, \partial \Omega)\}} r^{2p-n} \int_{B_r(x)} |\nabla^2 u|^p \leq \epsilon_p^p,
\end{equation}
then $u \in C^\infty(\Omega, S^L)$, and
\begin{equation}
|\nabla^m u(x)| \leq C\epsilon_p \left(\frac{1}{r_0^m} + \frac{1}{d^n(x, \partial \Omega)}\right), \quad \forall \ m \geq 1.
\end{equation}

Remark 2.4  For $p = 2$, Corollary 2.3 was first proved by Chang-Wang-Yang [4]. For biharmonic maps into any compact Riemannian manifold $N$ without boundary, Corollary 2.3 was proved by [43, 45] for $p = 2$.

3 Convexity and uniqueness of biharmonic maps

We will show the convexity and uniqueness properties for biharmonic maps with small energy, which are the second-order extensions of the theorems on harmonic maps with small energy by Struwe [39], Moser [27], and Huang-Wang [15].

Consider the Dirichlet problem for a biharmonic map $u \in W^{2,2}(\Omega, N)$:
\begin{equation}
\begin{cases}
\Delta^2 u = N_{bh}[u] & \text{in } \Omega \\
(u, \frac{\partial u}{\partial \nu}) = (u_0, \frac{\partial u_0}{\partial \nu}) & \text{on } \partial \Omega.
\end{cases}
\end{equation}

where $u_0 \in W^{2,2}(\Omega, N)$ is given.

We recall the second order Hardy inequality.

Lemma 3.1  There is $C > 0$ depending only on $n$ and $\Omega$ such that if $f \in W^{2,2}_0(\Omega)$, then
\begin{equation}
\int_{\Omega} \frac{|f(x)|^2}{d^4(x, \partial \Omega)} \leq C \int_{\Omega} |
abla^2 f(x)|^2.
\end{equation}

Proof.  For simplicity, we indicate a proof for the case $\Omega = B_1$ – the unit ball in $\mathbb{R}^n$. The readers can refer to [5] for a proof of general domains. By approximation, we may assume $f \in C^\infty_0(B_1)$. Writing the left hand side of (3.2) in spherical coordinates, integrating over $B_1$, and using the
Hölder inequality, we obtain
\[
\int_{B_1} \frac{|f(x)|^2}{(1-|x|)^4} = \int_0^1 \int_{S^{n-1}} \frac{|f|^2(r,\theta)}{(1-r)^4} r^{n-1} dH^{n-1}(\theta) dr \\
= - \int_0^1 \int_{S^{n-1}} \frac{1}{3(1-r)^3} (2ff_r r^{n-1} + |f|^2(n-1)r^{n-2}) dH^{n-1}(\theta) dr \\
\leq - \int_0^1 \int_{S^{n-1}} \frac{2}{3(1-r)^3} ff_r r^{n-1} dH^{n-1}(\theta) dr \\
\leq C \int_0^1 \int_{S^{n-1}} \frac{|f||f_r|^r n-1}{(1-r)^3} dH^{n-1}(\theta) dr \\
\leq C \left( \int_{B_1} \frac{|f(x)|^2}{(1-|x|)^4} \right)^{\frac{1}{2}} \left( \int_{B_1} \frac{\|
abla f(x)\|^2}{(1-|x|)^2} \right)^{\frac{1}{2}}.
\]

Thus, by using the first-order Hardy inequality, we obtain
\[
\int_{B_1} \frac{|f(x)|^2}{(1-|x|)^4} \leq C \int_{B_1} \frac{\|
abla f(x)\|^2}{(1-|x|)^2} \leq C \int_{B_1} |\nabla^2 f(x)|^2.
\]

This yields (3.2).

Now we introduce the Morrey spaces in \( \mathbb{R}^n \). For \( 1 \leq l < +\infty, 0 < \lambda \leq n, \) and \( 0 < R \leq +\infty \), \( f \in M_R^{l,\lambda}(\Omega) \) if \( f \in L_{\text{loc}}^l(\Omega) \) satisfies
\[
\|f\|_{M_R^{l,\lambda}(\Omega)} := \sup_{x \in \Omega} \sup_{0 < r \leq \min \{ R, d(x, \partial \Omega) \}} \left\{ r^{\lambda - n} \int_{B_r(x)} |f|^l \right\} < +\infty.
\]

We have the convexity property of biharmonic maps with small energy.

**Theorem 3.2** For \( n \geq 4, \delta \in (0,1), \) and \( \frac{3}{2} < p \leq 2, \) there exist \( \epsilon_p = \epsilon(p,\delta) > 0 \) and \( R_p = R(p,\delta) > 0 \) such that if \( u \in W^{2,2}(\Omega, N) \) is a biharmonic map satisfying either
(i) \( \|\nabla^2 u\|_{M_R^{2,2}(\Omega)} \leq \epsilon_2, \) when \( N \) is a compact Riemannian manifold without boundary, or
(ii) \( \|\nabla^2 u\|_{M_R^{p,2p}(\Omega)} \leq \epsilon_p, \) when \( N = S^n, \)
then
\[
\int_{\Omega} |\Delta u|^2 \geq \int_{\Omega} |\Delta u|^2 + (1-\delta) \int_{\Omega} |\nabla^2 (v-u)|^2
\]
holds for any \( v \in W^{2,2}(\Omega, N) \) with \( \left( v, \frac{\partial v}{\partial n} \right) = \left( u, \frac{\partial u}{\partial n} \right) \) on \( \partial \Omega. \)

**Proof.** First, it follows from Corollary 2.3 for \( N = S^n \) or Wang [45] that if \( \epsilon_p > 0 \) is sufficiently small then \( u \in C^\infty(\Omega, N), \) and
\[
|\nabla^m u(x)| \leq C \epsilon_p \left( \frac{1}{R_p^m} + \frac{1}{d^m(x, \partial \Omega)} \right), \quad \forall x \in \Omega, \ \forall m \geq 1.
\]
For \( y \in N \), let \( P^\perp(y) : \mathbb{R}^{L+1} \to (T_y N)^\perp \) denote the orthogonal projection from \( \mathbb{R}^{L+1} \) to the normal space of \( N \) at \( y \). Since \( N \) is compact, a simple geometric argument implies that there exists \( C > 0 \) depending on \( N \) such that

\[
|P^\perp(y)(z - y)| \leq C|z - y|^2, \quad \forall z \in N.
\] (3.7)

Since

\[
\mathcal{N}_{bh}[u] \perp T_u N,
\]

it follows from (3.7) that multiplying (1.1) by \((u - v)\) and integrating over \( \Omega \) yields

\[
\int_\Omega \Delta u \cdot \Delta (u - v) = \int_\Omega \mathcal{N}_{bh}[u] \cdot (u - v)
\]

\[
\leq \int_\Omega [|\nabla u|^2 \nabla^2 u| + |\nabla^2 u|^2 + |\nabla u| |\nabla^3 u|] (u - v)^2
\]

\[
\leq \epsilon_p^4 \int_\Omega \frac{|u - v|^2}{R_p^4} + \frac{|u - v|^2}{d^4(x, \partial \Omega)}
\]

\[
\leq \epsilon_p \int_\Omega |\nabla^2 (u - v)|^2,
\] (3.8)

where we choose \( R_p \geq \epsilon_p \), apply (3.6) and the Poincaré inequality and the Hardy inequality (3.2) during the last two steps.

It follows from (3.8) that

\[
\int_\Omega |\Delta v|^2 - \int_\Omega |\Delta u|^2 - \int_\Omega |\Delta u - \Delta v|^2 = 2 \int_\Omega \Delta u \cdot \Delta (v - u) \geq -C \epsilon_p \int_\Omega |\nabla^2 (u - v)|^2.
\] (3.9)

Since \((u - v) \in W^{2,2}_0(\Omega),\) we have that

\[
\int_\Omega |\Delta u - \Delta v|^2 = \int_\Omega |\nabla^2 (u - v)|^2,
\]

so that

\[
\int_\Omega |\Delta v|^2 - \int_\Omega |\Delta u|^2 \geq (1 - C \epsilon_p) \int_\Omega |\nabla^2 (u - v)|^2.
\]

This yields (3.5), if \( \epsilon_p > 0 \) is chosen so that \( C \epsilon_p \leq \delta \).

**Corollary 3.3** For \( n \geq 2 \) and \( \frac{2}{p} < p \leq 2 \), there exists \( \epsilon_p > 0 \) and \( R_p > 0 \) such that if \( u_1, u_2 \in W^{2,2}(\Omega, N) \) are biharmonic maps, with \( u_1 - u_2 \in W^{2,2}_0(\Omega, \mathbb{R}^{L+1}) \), satisfying either

(i) \( \max_{i=1,2} \|\nabla^2 u_i\|_{M^2_{R_2}(\Omega)} \leq \epsilon_2 \), when \( N \) is a compact Riemannian manifold without boundary, or

(ii) \( \max_{i=1,2} \|\nabla^2 u_i\|_{M^{p,2p}_{R_2}(\Omega)} \leq \epsilon_p \), when \( N = S^L \),

then \( u_1 \equiv u_2 \) in \( \Omega \).

**Proof.** Choose \( \delta = \frac{1}{2} \), apply Theorem 3.2 to \( u_1 \) and \( u_2 \) by choosing sufficiently small \( \epsilon_p > 0 \) and \( R_p > 0 \). We have

\[
\int_\Omega |\Delta u_2|^2 \geq \int_\Omega |\Delta u_1|^2 + \frac{1}{2} \int_\Omega |\nabla^2 (u_2 - u_1)|^2,
\]
\[
\int_\Omega |\Delta u_1|^2 \geq \int_\Omega |\Delta u_2|^2 + \frac{1}{2} \int_\Omega |\nabla^2(u_1 - u_2)|^2.
\]

Adding these two inequalities together yields \(\int_\Omega |\nabla^2(u_1 - u_2)|^2 = 0\). This, combined with \(u_1 - u_2 \in W^{2,2}_0(\Omega)\), implies \(u_1 \equiv u_2\) in \(\Omega\). \(\square\)

### 4 Uniqueness and convexity of heat flow of biharmonic maps

This section is devoted to the proof of uniqueness, convexity, and unique limit at \(t = \infty\) for (1.2) of the heat flow of biharmonic maps, i.e. Theorem 1.3, Theorem 1.5, and Corollary 1.6.

**Proof of Theorem 1.3** First, by Theorem 1.1, we have that for \(i = 1, 2, u_i \in C^\infty(\Omega \times (0, T), S^L)\), and

\[
|\nabla^m u_i(x, t)| \leq C_\varepsilon_p \left( \frac{1}{R_p^m} + \frac{1}{d_m(x, \partial \Omega)} + \frac{1}{t^m} \right), \forall (x, t) \in \Omega \times (0, T), \forall m \geq 1. \quad (4.1)
\]

Set \(w = u_1 - u_2\). Then \(w\) satisfies

\[
\begin{cases}
\partial_t w + \Delta^2 w = N_{bh}[u_1] - N_{bh}[u_2] & \text{in } \Omega \times (0, T) \\
w = 0 & \text{on } \partial_p(\Omega \times (0, T)) \\
\frac{\partial w}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T). 
\end{cases} \quad (4.2)
\]

Multiplying (4.2) by \(w\) and integrating over \(\Omega\), by (3.7), (4.1), the Poincaré inequality and the Hardy inequality (3.2), we obtain that

\[
\frac{d}{dt} \int_\Omega |w|^2 + 2 \int_\Omega |\nabla^2 w|^2 = 2 \int_\Omega (N_{bh}[u_1] - N_{bh}[u_2]) \cdot w
\]

\[
\lesssim \sum_{i=1}^2 \int_\Omega (|\nabla u_i|^2 |\nabla^2 u_i| + |\nabla^2 u_i|^2 + |\nabla u_i||\nabla^3 u_i|)|w|^2
\]

\[
\lesssim \epsilon_p^4 \int_\Omega \frac{|w(x, t)|^2}{R_p^4} + \frac{|w(x, t)|^2}{d^4(x, \partial \Omega)} + \frac{|w(x, t)|^2}{t}
\]

\[
\lesssim \epsilon_p \int_\Omega |\nabla^2 w|^2 + \frac{\epsilon_p}{t} \int_\Omega |w|^2.
\]

If we choose \(\epsilon_p > 0\) sufficiently small and \(R_p \geq \epsilon_p\), then it holds

\[
\frac{d}{dt} \int_\Omega |w|^2 \leq C \epsilon_p \int_\Omega |w|^2. \quad (4.3)
\]

It follows from (4.3) that

\[
\frac{d}{dt} \left(t^{-\frac{1}{2}} \int_\Omega |w|^2 \right) = t^{-\frac{1}{2}} \frac{d}{dt} \int_\Omega |w|^2 - \frac{1}{2} t^{-\frac{3}{2}} \int_\Omega |w|^2
\]

\[
\leq (C \epsilon - \frac{1}{2}) t^{-\frac{3}{2}} \int_\Omega |w|^2 \leq 0. \quad (4.4)
\]
Integrating this inequality from 0 to \( t \) yields
\[
\frac{t^{1/2}}{2} \int_{\Omega} |w|^2 \leq \lim_{t \downarrow 0^+} t^{1/2} \int_{\Omega} |w|^2. \tag{4.5}
\]
Since \( w(\cdot, 0) = 0 \), we have
\[
w(x, t) = \int_0^t w_t(x, \tau) d\tau, \text{ a.e. } x \in \Omega,
\]
so that, by the Hölder inequality,
\[
t^{1/2} \int_{\Omega} |w(x, t)|^2 \leq t^{1/2} \int_0^t \int_{\Omega} |w_t(x, \tau)|^2 \, dx \, d\tau \leq Ct^{1/2} \to 0, \text{ as } t \downarrow 0^+.
\]
This, combined with (4.5), implies \( w \equiv 0 \) in \( \Omega \times [0, T] \). The proof is complete.

Now we want to prove Theorem 1.5 and Corollary 1.6. To do so, we need

**Lemma 4.1** Under the same assumptions as in Theorem 1.5, there exists \( T_0 > 0 \) such that \( \int_{\Omega} |\partial_t u(t)|^2 \) is monotone decreasing for \( t \geq T_0 \):
\[
\int_{\Omega} |\partial_t u|^2(t_2) + C \int_{\Omega \times [t_1, t_2]} |\nabla^2 \partial_t u|^2 \leq \int_{\Omega} |\partial_t u|^2(t_1), \quad T_0 \leq t_1 \leq t_2 \leq T. \tag{4.6}
\]

**Proof.** For any sufficiently small \( h > 0 \), set
\[
u^h(x, t) = \frac{u(x, t + h) - u(x, t)}{h}, \quad (x, t) \in \Omega \times (0, T - h).
\]
Then \( u^h \in L^2([0, T - h], W_0^{2,2}(\Omega)) \), \( \partial_t u \in L^2(\Omega \times [0, T - h]) \), and \( \lim_{h \downarrow 0^+} \|u^h - \partial_t u\|_{L^2(\Omega \times [0, T - h])} = 0 \).

Since \( u \) satisfies (1.2), we obtain
\[
\partial_t u^h + \Delta^2 u^h = \frac{1}{h} (\mathcal{N}_{bh}[u(t + h)] - \mathcal{N}_{bh}[u(t)]). \tag{4.7}
\]

Multiplying (4.7) by \( u^h \), integrating over \( \Omega \), and applying (3.7) and (4.1), we have
\[
\frac{d}{dt} \int_{\Omega} |u^h|^2 + 2 \int_{\Omega} |\Delta u^h|^2 \lesssim \int_{\Omega} \left( |\mathcal{N}_{bh}[u(t + h)]| + |\mathcal{N}_{bh}[u(t)]| \right) |u^h|^2 \\
\lesssim \int_{\Omega} \left( |\nabla^2 u|^2 + |\nabla u| \nabla^3 u| + |\nabla u|^2 |\nabla^2 u| \right) (t + h) |u^h|^2 \\
+ \int_{\Omega} \left( |\nabla^2 u|^2 + |\nabla u| \nabla^3 u| + |\nabla u|^2 |\nabla^2 u| \right) |u^h|^2 \\
\lesssim \epsilon_p \frac{\int_{\Omega} |u^h|^2}{R_p^4} + \frac{|u^h|^2}{d^4(x, \partial \Omega)} + \frac{|u^h|^2}{T_0} \\
\lesssim \epsilon_p \int_{\Omega} |\nabla^2 u^h|^2
\]
provided that we choose \( R_p \geq \epsilon_p \) and \( T_0 \geq \epsilon_p \). Since
\[
\int_{\Omega} |\nabla^2 u^h|^2 = \int_{\Omega} |\Delta u^h|^2,
\]

Proof of Theorem 1.5 Thus (4.10) follows.

we obtain It suffices to show the right hand side of the above identity tends to 0 as 

\[ \int_\Omega |u^h|^2(t_2) + C \int_{t_1}^{t_2} \int_\Omega |\nabla^2 u^h|^2 \leq \int_\Omega |u^h|^2(t_1). \]  

(4.9)

Sending \( h \to 0 \), (4.9) yields (4.6). □

Now we can show the monotonicity of \( E_2 \)-energy for heat flow of biharmonic maps for \( t \geq T_0 \).

Lemma 4.2 Under the same assumptions as in Theorem 1.5, there is \( T_0 > 0 \) such that \( \int_\Omega |\Delta u(t)|^2 \) is monotone decreasing for \( t \geq T_0 \): 

\[ \int_\Omega |\Delta u(t_2)|^2 \eta_\delta^2 - \int_\Omega |\Delta u(t_1)|^2 \eta_\delta^2 + 2 \int_{t_1}^{t_2} \int_\Omega |\partial_t u|^2 \eta_\delta^2 \]

\[ = -4 \int_{t_1}^{t_2} \int_\Omega \Delta u \cdot \partial_t u \left( |\nabla \eta_\delta|^2 + \eta_\delta \Delta \eta_\delta \right) - 8 \int_{t_1}^{t_2} \int_\Omega \Delta u \cdot \nabla \partial_t u \eta_\delta \nabla \eta_\delta. \]

(4.11)

It suffices to show the right hand side of the above identity tends to 0 as \( \delta \to 0^+ \). By Lemma 4.1 we have that \( \partial_t u \in L^2([T_0, T], W^{2,2}_0(\Omega)) \) so that

\[ \int_{t_1}^{t_2} \int_\Omega |\nabla \partial_t u|^2 |\nabla \eta_\delta|^2 + |\partial_t u|^2 \left( |\nabla \eta_\delta|^4 + |\Delta \eta_\delta|^2 \right) \]

\[ \lesssim \delta^{-2} \int_{t_1}^{t_2} \int_\Omega |\nabla \partial_t u|^2 + \delta^{-2} |\partial_t u|^2 \]

\[ \lesssim \int_{t_1}^{t_2} \int_{\Omega_\delta} |\nabla^2 \partial_t u|^2 \to 0, \text{ as } \delta \to 0. \]

(4.12)

This, combined with the Hölder inequality, implies that for \( t_2 \geq t_1 \geq T_0 \),

\[ -4 \int_{t_1}^{t_2} \int_\Omega \Delta u \cdot \partial_t u \left( |\nabla \eta_\delta|^2 + \eta_\delta \Delta \eta_\delta \right) - 8 \int_{t_1}^{t_2} \int_\Omega \Delta u \cdot \nabla \partial_t u \eta_\delta \nabla \eta_\delta \to 0, \text{ as } \delta \to 0^+. \]

Thus (4.10) follows. □

Proof of Theorem 1.5 First, by Theorem 1.1 we have that \( u \in C^\infty(\Omega \times (0, T], S^L) \), and

\[ \left| \nabla^m u(x, t) \right| \leq C_{\epsilon_p} \left( \frac{1}{R_p^{m}} + \frac{1}{d^m(x, \partial \Omega)} + \frac{1}{t^{m}} \right), \quad \forall (x, t) \in \Omega \times (0, T), \forall m \geq 1. \]

(4.13)
For $t_2 > t_1 \geq T_0$, we have

\[
\int_\Omega |\Delta u(t_1)|^2 - \int_\Omega |\Delta u(t_2)|^2 - \int_\Omega |\Delta u(t_1) - \Delta u(t_2)|^2
= 2 \int_\Omega (\Delta u(t_1) - \Delta u(t_2)) \Delta u(t_2)
= -2 \int_\Omega (u(t_1) - u(t_2)) u_t(t_2)
+ \int_\Omega \mathcal{N}_{bh}[u(t_2)] \cdot (u(t_1) - u(t_2))
= I + II.
\]

For $II$, applying (3.7), we obtain

\[
|\mathcal{N}_{bh}[u(t_2)] \cdot (u(t_1) - u(t_2))| \lesssim |\mathcal{N}_{bh}[u(t_2)]||u(t_1) - u(t_2)|^2.
\]

Hence, by (4.13), the Hardy inequality and the Poincaré inequality, we have

\[
|II| \lesssim \epsilon_p^2 \int_\Omega \left( \frac{1}{R_p^2} + \frac{1}{d^2(x, \partial \Omega)} + \frac{1}{T_0} \right) |u(t_1) - u(t_2)|^2
\leq C \epsilon_p \int_\Omega |\nabla^2 (u(t_1) - u(t_2))|^2.
\]  

(4.15)

For $I$, by Lemma 4.1, we have

\[
\left\| \partial_t u(t_2) \right\|_{L^2(\Omega)}^2 \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \int_\Omega |\partial_t u|^2.
\]  

(4.16)

By the Hölder inequality and (4.10), this implies

\[
|I| \lesssim \int_\Omega |\partial_t u(t_2)||u(t_1) - u(t_2)|
\lesssim \left\| \partial_t u(t_2) \right\|_{L^2(\Omega)} \left\| u(t_1) - u(t_2) \right\|_{L^2(\Omega)}
\leq \sqrt{t_2 - t_1} \left\| \partial_t u(t_2) \right\|_{L^2(\Omega)} \left( \int_{\Omega \times [t_1, t_2]} |\partial_t u|^2 \right)^{\frac{1}{2}}
\leq \int_{\Omega \times [t_1, t_2]} |\partial_t u|^2 \leq \frac{1}{2} \left[ \int_\Omega |\Delta u(t_1)|^2 - \int_\Omega |\Delta u(t_2)|^2 \right].
\]  

(4.17)

Putting (4.17) and (4.15) into (4.14) implies (1.10). This completes the proof. \[ \square \]

**Proof of Corollary 1.6** It follows from Lemma 4.2 that $\int_\Omega |\Delta u(t)|^2$ is monotone decreasing for $t \geq T_0$. Hence

\[
c = \lim_{t \to \infty} \int_\Omega |\Delta u(t)|^2
\]

exists and is finite. Let $\{t_i\}$ be any increasing sequence such that $\lim_{i \to \infty} t_i = +\infty$. Then (1.10) implies that

\[
\int_\Omega \left| \nabla^2 (u(t_{i+j}) - u(t_i)) \right|^2 \leq C \left[ \int_\Omega |\Delta u(t_{i+j})|^2 - \int_\Omega |\Delta u(t_i)|^2 \right] \to 0, \text{ as } i \to \infty,
\]

as $i \to \infty$.\]
for all $j \geq 1$. Thus there exists $u_\infty \in W^{2,2}(\Omega, S^L)$, with $(u_\infty, \partial u_\infty) = (u_0, \partial u_0)$ on $\partial \Omega$, such that

$$\lim_{t \to \infty} \left\| u(t) - u_\infty \right\|_{W^{2,2}(\Omega)} = 0.$$  

Since (4.10) implies that there exists a sequence $t_i \to \infty$, such that

$$\lim_{i \to \infty} \left\| \partial_t u(t_i) \right\|_{W^{2,2}(\Omega)} = 0.$$  

Thus $u_\infty \in W^{2,2}(\Omega, S^L)$ is a biharmonic map. For any $m \geq 1$, and any compact subset $K \subset \subset \Omega$, since

$$\left\| u(t) \right\|_{C^m(K)} \leq C(n, m, K), \forall t \geq 1,$$

we conclude that

$$\lim_{t \to \infty} \left\| u(t) - u_\infty \right\|_{C^m(K)} = 0,$$

and $u_\infty \in C^\infty(\Omega, S^L)$. This completes the proof. \hfill \Box

5 Proof of Theorem 1.8

In this section, we will prove Theorem 1.8 on both smoothness and uniqueness for certain weak solutions of (1.2). First, we would like to verify

Proposition 5.1 For $n \geq 4$, $0 < T < +\infty$, suppose $u \in W^{1,2}_2(\Omega \times [0, T], N)$ is a weak solution of (1.2), with the initial and boundary value $u_0 \in W^{2, r}(\Omega, N)$ for some $\frac{n}{2} < r < +\infty$, such that $\nabla^2 u \in L^q_t L^p_x(M \times [0, T])$ for some $p > \frac{n}{2}$ and $q < \infty$ satisfying (1.13). Then

(i) $\partial_t u \in L^q_t L^p_x(\Omega \times [0, T])$; and

(ii) for any $\epsilon > 0$, there exists $R = R(u, \epsilon) > 0$ such that for any $1 < s < \min\{\frac{p}{2}, \frac{q}{2}\}$,

$$\sup \left\{ r^{2s-(n+4)} \int_{P_r(x,t) \cap (\Omega \times [0, T])} (|\nabla^2 u|^s + r^{2s}|\partial_t u|^s) \right\} < \epsilon. \quad (5.1)$$

Proof. For simplicity, we will sketch the proof for $\Omega = \mathbb{R}^n$. By the Duhamel formula, we have that $u(x, t) = u_1(x, t) + u_2(x, t)$, where

$$u_1(x, t) = \int_{\mathbb{R}^n} b(x - y, t) u_0(y), \quad (5.2)$$

$$u_2(x, t) = \int_0^t \int_{\mathbb{R}^n} b(x - y, t - s) \nabla \cdot [\nabla (A(u)(\nabla u, \nabla u)) + 2\Delta u \cdot \nabla (P(u)) - \Delta u \cdot \Delta (P(u))] (y, s). \quad (5.3)$$

We proceed with two claims.

Claim 1. $\nabla^3 u \in L^{\frac{2q}{4+q}}_t L^{\frac{2p}{4+p}}_x(\mathbb{R}^n \times [0, T])$. For $u_1$, we have

$$\nabla^3 u_1(x, t) = \int_{\mathbb{R}^n} \nabla b(x - y, t) \nabla^2 u_0(y). \quad (5.4)$$
Direct calculations, using the property of the kernel function \(b\), yield
\[
\left\| \nabla^3 u \right\|_{L^\frac{2p}{3} L^\frac{2q}{3} (\mathbb{R}^n \times [0,T])} \lesssim T^{\frac{2}{3} - \frac{2p}{3}} \left\| \nabla^2 u_0 \right\|_{L^r (\mathbb{R}^n)}.
\] (5.5)

For \(u_2\), we have
\[
\nabla^3 u_2(x,t) = \int_0^t \int_{\mathbb{R}^n} \nabla^4 b(x-y, t-s) \left[ \nabla (A(u)(\nabla u, \nabla u)) + 2\Delta u \cdot \nabla (P(u)) \right] \nonumber \\
- \int_0^t \int_{\mathbb{R}^n} \nabla^3 b(x-y, t-s) \Delta u \cdot \Delta (P(u))(y,s) \\
= M_1 + M_2.
\] (5.6)

By the Nirenberg interpolation inequality, we have \(\nabla u \in L^{2q}_{t} L^{2p}_{x} (\mathbb{R}^n \times [0,T])\). By the Hölder inequality, we then have \(\nabla (A(u)(\nabla u, \nabla u)) + 2\Delta u \cdot \nabla (P(u))) \in L^{2q}_{t} L^{2p}_{x} (\mathbb{R}^n \times [0,T])\). Hence, by the Calderon-Zygmund \(L^{\frac{q}{2}}_{t} L^{\frac{p}{2}}_{x}\)-theory, we have
\[
\left\| M_1 \right\|_{L^{\frac{2p}{3} L^{\frac{2q}{3}} (\mathbb{R}^n \times [0,T])}} \lesssim \left\| \nabla (A(u)(\nabla u, \nabla u)) + 2\Delta u \cdot \nabla (P(u)) \right\|_{L^{\frac{2p}{3} L^{\frac{2q}{3}} (\mathbb{R}^n \times [0,T])}} \\
\lesssim \left\| \nabla u \right\|_{L^{2p}_{t} L^{2q}_{x} (\mathbb{R}^n \times [0,T])} \left\| \nabla^2 u \right\|_{L^{2}_{t} L^{q}_{x} (\mathbb{R}^n \times [0,T])} \\
\lesssim 1 + \left\| \nabla^2 u \right\|_{L^{2}_{t} L^{q}_{x} (\mathbb{R}^n \times [0,T])}^2.
\] (5.7)

For \(M_2\), we have
\[
|M_2|(x,t) \lesssim I_1 \left( \left| \nabla^2 u \right|^2 + |\nabla u|^4 \right)(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,T].
\]

Recall the following estimate of \(I_1(\cdot)\) (see, for example, [8] §4):
\[
\left\| I_1(f) \right\|_{L^{s_2}_{t} L^{r_2}_{x} (\mathbb{R}^n \times [0,T])} \lesssim \left\| f \right\|_{L^{s_1}_{t} L^{r_1}_{x} (\mathbb{R}^n \times [0,T])},
\] (5.8)

where \(s_2 \geq s_1\) and \(r_2 \geq r_1\) satisfy
\[
\frac{n}{r_1} + \frac{4}{s_1} \leq \frac{n}{r_2} + \frac{4}{s_2} + 1.
\] (5.9)

Applying (5.8) to \(M_2\), we see that \(M_2 \in L^{\frac{2p}{3}}_{t} L^{\frac{2q}{3}}_{x} (\mathbb{R}^n \times [0,T])\), and
\[
\left\| M_2 \right\|_{L^{\frac{2p}{3} L^{\frac{2q}{3}} (\mathbb{R}^n \times [0,T])}} \lesssim 1 + \left\| \nabla^2 u \right\|_{L^{2}_{t} L^{q}_{x} (\mathbb{R}^n \times [0,T])}^2.
\] (5.10)

Combining these estimates of \(\nabla^3 u_1, M_1,\) and \(M_2\) yields Claim 1.

Claim 2. \(\nabla^4 u \in L^{\frac{2p}{7}}_{t} L^{\frac{2q}{7}}_{x} (\mathbb{R}^n \times [0,T])\). It follows from Claim 1 that
\[
\mathcal{N}_{bh}[u] = \left| \Delta (A(u)(\nabla u, \nabla u)) + 2\Delta u \cdot \nabla (P(u)) \right| - \Delta u \cdot \Delta (P(u)) \in L^{\frac{2p}{7}}_{t} L^{\frac{2q}{7}}_{x} (\mathbb{R}^n \times [0,T]).
\]

Since
\[
\nabla^4 u_2(x,t) = \int_0^t \int_{\mathbb{R}^n} \nabla^4 b(x-y, t-s)\mathcal{N}_{bh}[u](y,s),
\]
we can apply the Calderón-Zygmund $L^q_t L^p_x$-theory again to conclude that $\nabla^4 u_2 \in L^{q}_{t} W^{2,2}_{x}(\mathbb{R}^{n} \times [0, T])$. For $u_1$, we have

$$\nabla^4 u_1(x, t) = \int_{\mathbb{R}^{n}} \nabla^2 b(x - y, t) \nabla^2 u_0(y).$$

Hence, by direct calculations, we have

$$\left\| \nabla^4 u_1 \right\|_{L^{q}_{t} L^{p}_{x}(\mathbb{R}^{n} \times [0, T])} \lesssim T^{\frac{4}{q} - \frac{2}{p}} \left\| \nabla^2 u_0 \right\|_{L^{r}(\mathbb{R}^{n})}.$$

Combining these two estimates yields Claim 2.

By (1.2), it is easy to see that $\partial_t u \in L^{q}_{t} L^{p}_{x}(\mathbb{R}^{n} \times [0, T])$. In fact, we have

$$\left\| \partial_t u \right\|_{L^{q}_{t} L^{p}_{x}(\mathbb{R}^{n} \times [0, T])} \lesssim \left\| \nabla^2 u \right\|_{L^{r}_{t} L^{q}_{x}(\mathbb{R}^{n} \times [0, T])} + T^{\frac{4}{q} - \frac{2}{p}} \left\| \nabla^2 u_0 \right\|_{L^{r}(\mathbb{R}^{n})}. \tag{5.11}$$

This implies (i).

(ii) follows from (i) and the Hölder inequality. In fact, for any $1 < s < \min\{\frac{q}{2}, \frac{q}{2}\}$, it holds

$$\left( r^{2s-(n+4)} \int_{P_r(x, t) \cap (\Omega \times [0, T])} \left| \nabla^2 u \right|^s \right)^{\frac{1}{s}} \leq \left\| \nabla^2 u \right\|_{L^{r}_{t} L^{q}_{x}(P_r(x, t) \cap (\Omega \times [0, T]))},$$

and

$$\left( r^{4s-(n+4)} \int_{P_r(x, t) \cap (\Omega \times [0, T])} \left| \partial_t u \right|^s \right)^{\frac{1}{s}} \leq \left\| \partial_t u \right\|_{L^{q}_{t} L^{p}_{x}(P_r(x, t) \cap (\Omega \times [0, T]))}.$$ 

These two inequalities clearly imply (5.11), provided that $R = R(u, \epsilon) > 0$ is chosen sufficiently small.

Now we prove an $\epsilon$-regularity property for certain solutions of (1.2).

**Theorem 5.2** There exists $\epsilon_0 > 0$ such that if $u \in W^{1,2}_{x} W^{1,2}_{x}(P_1, N)$, with $\nabla^2 u \in L^{q}_{t} L^{p}_{x}(P_1)$ for some $q \geq \frac{n}{2}$ and $p \leq \infty$ satisfying (1.13), is a weak solution of (1.2) and satisfies

$$\left\| \nabla^2 u \right\|_{L^{q}_{t} L^{p}_{x}(P_1)} \leq \epsilon_0, \tag{5.12}$$

then $u \in C^{\epsilon}(P_{\frac{1}{2}}, N)$ and

$$\left\| \nabla^m u \right\|_{C^{0}(P_{\frac{1}{2}})} \leq C(m, p, q, n) \left\| \nabla^2 u \right\|_{L^{q}_{t} L^{p}_{x}(P_1)} \left( \int_0^T \left\| f \right\|_{L^{p}(\Omega)} \left\| g \right\|_{L^{2}(\Omega)} \right)^{\frac{1}{q}}, \tag{5.13}$$

Before proving this theorem, we recall the Serrin type inequalities (see [37]) and Adams' estimates of Riesz potential between Morrey spaces in $(\mathbb{R}^{n+1}, \delta)$.

**Lemma 5.3** Assume $p \geq \frac{q}{2}$ and $q \leq \infty$ satisfy (1.13). For any $f \in L^{q}_{t} L^{p}_{x}(\Omega \times [0, T])$, $g \in L^{2}_{t} W^{2,2}_{x}(\Omega \times [0, T])$, and $h \in L^{2}_{t} W^{1,2}_{x}(\Omega \times [0, T])$, we have

$$\int_{\Omega \times [0, T]} \left| f \right| \left| g \right| \left| h \right| \lesssim \left\| h \right\|_{L^{2}(\Omega \times [0, T])} \left\| g \right\|_{L^{q}_{t} W^{2,2}_{x}(\Omega \times [0, T])} \left( \int_0^T \left\| f \right\|_{L^{p}(\Omega)} \left\| g \right\|_{L^{2}(\Omega)} \right)^{\frac{1}{q}}. \tag{5.14}$$
Since 1
\[\int_{\Omega \times [0,T]} |f| \nabla g \cdot \nabla h |\Omega \times [0,T]| (|g| \nabla |\nabla h| \Omega \times [0,T]) \left( \int_0^T \|f\|^q_{L_p(\Omega)} \|g\|^q_{L_q(\Omega)} \right)^{\frac{1}{q}}. \tag{5.15}\]

**Proof.** For convenience, we sketch the proof here. By the Hölder inequality, we have
\[\int_{\Omega} |f| |g| |h| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \|h\|_{L^2(\Omega)}, \tag{5.16}\]
where \( \frac{1}{p} + \frac{1}{r} = \frac{1}{2} \). It follows from (1.13) that \( 2 \leq r \leq \frac{2n}{n-4} \). Hence by the Sobolev inequality we have
\[\|g\|_{L^p(\Omega)} \leq \|g\|_{L^2(\Omega)} \|g\|_{W^{2,2}(\Omega)} \lesssim \|g\|_{L^2(\Omega)} \|g\|_{W^{2,2}(\Omega)}. \tag{5.17}\]
Putting (5.17) into (5.16) yields
\[\int_{\Omega} |f| |g| |h| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^2(\Omega)} \|g\|_{W^{2,2}(\Omega)} \|h\|_{L^2(\Omega)}. \tag{5.18}\]
Since \( \frac{1}{q} + \frac{n}{4p} + \frac{1}{2} = 1 \), (5.14) follows by integrating over \([0,T]\) and the Hölder inequality.

To see (5.15), note that the Hölder inequality implies
\[\int_{\Omega} |f| |g| |h| \leq \|f\|_{L^p(\Omega)} \|\nabla g\|_{L^t(\Omega)} \|h\|_{L^s(\Omega)} \tag{5.19}\]
where \( \frac{1}{p} + \frac{1}{s} = \frac{n-2}{2n} \). Since
\[\frac{1}{s} = \frac{1}{n} + \frac{n}{2p} \left( \frac{1}{2} - \frac{2}{n} \right) + \left( 1 - \frac{n}{2p} \right) \frac{1}{2}, \]
the Nirenberg interpolation inequality implies
\[\|\nabla g\|_{L^t(\Omega)} \leq \|g\|_{L^2(\Omega)} \|g\|_{W^{2,2}(\Omega)}. \tag{5.20}\]
Putting (5.20) into (5.19) and using the Sobolev inequality, we obtain
\[\int_{\Omega} |f| |\nabla g| |h| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^2(\Omega)} \|g\|_{W^{2,2}(\Omega)} \|h\|_{W^{1,2}(\Omega)}. \tag{5.21}\]
Since \( \frac{1}{q} + \frac{n}{4p} + \frac{1}{2} = 1 \), (5.15) follows by integration on \([0,T]\) and the Hölder inequality. \( \square \)

Now we state Adams’ estimate for the Riesz potentials on \((\mathbb{R}^{n+1}, \delta)\). Since its proof is exactly the same argument as in Huang-Wang ([16] Theorem 3.1), we skip it here.

**Proposition 5.4** (i) For any \( \beta > 0 \), \( 0 < \lambda \leq n+4 \), \( 1 < p < \frac{n}{n-\beta} \), if \( f \in L^p(\mathbb{R}^{n+1}) \cap M^{p,\lambda}(\mathbb{R}^{n+1}) \), then \( I_\beta(f) \in L^{\tilde{p}}(\mathbb{R}^{n+1}) \cap M^{\tilde{p},\lambda}(\mathbb{R}^{n+1}) \), where \( \tilde{p} = \frac{p\lambda}{\lambda - p\beta} \). Moreover,
\[\|I_\beta(f)\|_{L^{\tilde{p}}(\mathbb{R}^{n+1})} \leq C\|f\|_{M^{p,\lambda}(\mathbb{R}^{n+1})} \|f\|_{L^p(\mathbb{R}^{n+1})}^{1-\frac{\beta}{n}}. \tag{5.22}\]
(ii) For any $0 < \beta < \lambda \leq n + 4$, if $f \in L^1(\mathbb{R}^{n+1}) \cap M^{1,\lambda}(\mathbb{R}^{n+1})$, then $f \in L^{\frac{\lambda}{\lambda - \beta}}(\mathbb{R}^{n+1}) \cap M^{\frac{\lambda - \beta}{\lambda}}(\mathbb{R}^{n+1})$. Moreover,

\[
\|I_\beta(f)\|_{L^{\frac{\lambda}{\lambda - \beta}}(\mathbb{R}^{n+1})} \leq C\|f\|_{M^{1,\lambda}(\mathbb{R}^{n+1})}^{\frac{\beta}{\lambda}} \|f\|_{L^1(\mathbb{R}^{n+1})}^{1 - \frac{\beta}{\lambda}}.
\]

(5.24)

\[
\|I_\beta(f)\|_{M^{\frac{\lambda - \beta}{\lambda}}(\mathbb{R}^{n+1})} \leq C\|f\|_{M^{1,\lambda}(\mathbb{R}^{n+1})}.
\]

(5.25)

**Proof of Theorem 5.2** The proof is based on three claims.

Claim 1. For any $0 < \alpha < 1$, we have that $\nabla^2 u \in M^{2,4-4\alpha}(P_{\frac{3}{4}})$, and

\[
\left\|\nabla^2 u\right\|_{M^{2,4-4\alpha}(P_{\frac{3}{4}})} \leq C\left\|\nabla^2 u\right\|_{L^1_t L^p_{\nu}(\mathbb{R}^n)}.
\]

(5.26)

For any $0 < r \leq \frac{T}{4}$ and $z_0 = (x_0, t_0) \in P_{\frac{3}{4}}$, by (5.12) we have

\[
\left\|\nabla^2 u\right\|_{L^1_t L^p_{\nu}(P_r(z_0))} \leq \epsilon.
\]

(5.27)

Let $v : P_r(z_0) \to \mathbb{R}^{L+1}$ solve

\[
\begin{cases}
\partial_t v + \Delta^2 v = 0 & \text{in } P_r(z_0) \\
v = u & \text{on } \partial P_r(z_0) \\
\partial v = \frac{\partial u}{\partial \nu} & \text{on } \partial B_r(x_0) \times (t_0 - r^4, t_0].
\end{cases}
\]

(5.28)

Set $w = u - v$. Multiplying (5.25) and (1.2) by $w$, subtracting the resulting equations and integrating over $P_r(z_0)$, we obtain

\[
\sup_{t_0 - r^4 \leq t \leq t_0} \int_{B_r(x_0)} |w|^2(t) + 2 \int_{P_r(z_0)} |\nabla^2 w|^2
\]

\[
= \int_{P_r(z_0)} \mathcal{N}_{bh}[u] : w
\]

\[
= \int_{P_r(z_0)} -\nabla(A(u)\nabla u, \nabla u))\nabla w - (\Delta u, \Delta(P(u))) w - 2 \langle \nabla u, \nabla(P(u)) \rangle \nabla w
\]

\[
\lesssim \int_{P_r(z_0)} |\nabla^2 u|^2 |w| + \int_{P_r(z_0)} |\nabla u||\nabla^2 u||\nabla w|
\]

\[
= I + II.
\]

(5.29)

For $I$, we can apply (5.14) to get

\[
|I| \lesssim \|\nabla^2 u\|_{L^2(P_r(z_0))} \|w\|_{L^2_t W^{2,2}_{\nu}(P_r(z_0))}^{\frac{p}{4}} \left(\int_{t_0 - r^4}^{t_0} \|\nabla^2 u\|_{L^{p}(B_r(x_0))}^2 \|w\|_{L^2(B_r(x_0))}^2\right)^{\frac{1}{p}}.
\]

(5.30)
For $I$, by (5.15), we have
\[
|II| \lesssim \|\nabla u\|_{L^2_w^1(B_r(z_0))}\|w\|_{L^p(B_r(z_0))}^{\frac{n}{mp}} \left(\int_{t_0-r^4}^{t_0} \|\nabla^2 u\|_{L^p(B_r(z_0))}^{q} \|w\|_{L^2(B_r(z_0))}^{2}\right)^{\frac{1}{q}}. \tag{5.31}
\]
Putting (5.30) and (5.31) into (5.29) and applying the Poincaré inequality, we obtain
\[
\sup_{t_0-r^4 \leq t \leq t_0} \int_{B_r(z_0)} |w|^2(t) + 2 \int_{P_r(z_0)} |\nabla^2 w|^2 \leq \begin{cases} \|\nabla u\|_{L^2_w^1(B_r(z_0))}\|\nabla^2 w\|_{L^p(B_r(z_0))}^{q} \|w\|_{L^2(B_r(z_0))}^{2}\right)^{\frac{1}{q}}, & q < \infty, \\
\|\nabla u\|_{L^2_w^1(B_r(z_0))}\|\nabla^2 w\|_{L^q(B_r(z_0))}^{q} \|w\|_{L^2(B_r(z_0))}^{2}\right)^{\frac{1}{q}}, & q = \infty. \end{cases} \tag{5.32}
\]
Since $\|\nabla^2 u\|_{L_t^q L_x^p(B_r(z_0))} \leq \epsilon$, we obtain, by the Young inequality,
\[
\sup_{t_0-r^4 \leq t \leq t_0} \int_{B_r(z_0)} |w|^2(t) + 2 \int_{P_r(z_0)} |\nabla^2 w|^2 \leq \begin{cases} \|\nabla u\|_{L^2_w^1(B_r(z_0))}^2 + \epsilon \|\nabla u\|_{L^2_w^1(B_r(z_0))}^2 + C\epsilon^\frac{2}{n} \sup_{t_0-r^4 \leq t \leq t_0} \|w\|_{L^2(B_r(z_0))}^{2}, & q < \infty, \\
\|\nabla u\|_{L^2_w^1(B_r(z_0))}^2 + C\|\nabla^2 u\|_{L_t^{\infty} L_x^q(B_r(z_0))}^2 \|w\|_{L^2(B_r(z_0))}^{2}\right)^{\frac{1}{q}}, & q = \infty. \end{cases} \tag{5.33}
\]
By choosing $\epsilon > 0$ sufficiently small, this implies
\[
\int_{P_r(z_0)} |\nabla^2 w|^2 \lesssim \epsilon \int_{P_r(z_0)} |\nabla u|^2 + |\nabla^2 u|^2. \tag{5.34}
\]
Since $N$ is compact and $u$ maps into $N$, $|u| \leq C_N$. Hence, by the Nirenberg interpolation inequality, we have
\[
\int_{P_r(z_0)} |\nabla u|^2 \lesssim \int_{P_r(z_0)} |\nabla^2 u|^2 + r^{n+4}. \tag{5.35}
\]
Combining (5.35) with (5.34), we have
\[
\int_{P_r(z_0)} |\nabla^2 w|^2 \lesssim \epsilon \int_{P_r(z_0)} |\nabla^2 u|^2 + \epsilon r^{n+4}. \tag{5.36}
\]
By the standard estimate on $v$, we have
\[
(\theta r)^{-n} \int_{P_{r}(z_0)} |\nabla^2 v|^2 \lesssim (\theta^4 + \theta^{-n}) \int_{P_{r}(z_0)} |\nabla^2 v|^2, \ \forall \ \theta \in (0, 1). \tag{5.37}
\]
Combining (5.36) with (5.37), we obtain
\[
(\theta r)^{-n} \int_{P_{r}(z_0)} |\nabla^2 u|^2 \leq C (\theta^4 + \theta^{-n}) r^{-n} \int_{P_{r}(z_0)} |\nabla^2 u|^2 + C\epsilon \theta^{-n} r^4, \ \forall \ \theta \in (0, 1). \tag{5.38}
\]
For any $0 < \alpha < 1$, choose $0 < \theta < 1$ and $\epsilon$ such that
\[
C\theta^4 \leq \frac{1}{2} \theta^{4\alpha} \text{ and } \epsilon \leq \min \left\{ \left( \frac{1}{2C} \right)^{\frac{2}{p}}, \frac{\theta^{4\alpha+n}}{2C} \right\}. \]
Therefore, for any \((z_0) \in P_{\frac{1}{4}}\) and \(0 < r \leq \frac{1}{4}\), it holds

\[
(\theta r)^{-n} \int_{P_{r}(x,t)} |\nabla^2 u|^2 \leq \theta^{4\alpha} r^{-n} \int_{P_{r}(x,t)} |\nabla^2 u|^2 + \theta^{4\alpha} r^4.
\]  

(5.39)

It is standard that iterating (5.39) implies

\[
\int_{P_{r}(x,t)} |\nabla^2 u|^2 \leq C r^{4\alpha} \left( \int_{P_{\frac{1}{4}}} |\nabla^2 u|^2 + 1 \right)
\]

(5.40)

for any \(z_0 \in P_{\frac{1}{4}}\) and \(0 < r \leq \frac{1}{4}\). (5.40) implies that \(\nabla^2 u \in M^{2,4-4\alpha}(P_{\frac{1}{4}})\), and the estimate (5.26) holds. This proves Claim 1.

Claim 2. For any \(1 < \beta < +\infty\), \(\nabla^2 u \in L^{\beta}(P_{\frac{1}{4}})\), and

\[
\left\| \nabla^2 u \right\|_{L^{\beta}(P_{\frac{1}{4}})} \lesssim \left\| \nabla^2 u \right\|_{L^2_xL^\beta_t(P_1)}^{1/2}.
\]

(5.41)

This can be proven by estimates of Riesz potentials between Morrey spaces. To do so, let \(\eta \in C_0^\infty(P_1)\) be such that

\[
0 \leq \eta \leq 1, \, \eta \equiv 1 \text{ in } P_{\frac{1}{8}}, \, \left| \eta \right| + \sum_{m=1}^{4} |\nabla^m \eta| \leq C.
\]

Let \(Q : \mathbb{R}^n \times [-1, \infty) \to \mathbb{R}^{L+1}\) solve

\[
\partial_t Q + \Delta^2 Q = \nabla \cdot \left( \eta^2 \nabla(A(u)(\nabla u, \nabla u)) + 2\eta^2 (\Delta u, \nabla (P(u))) \right) - \eta^2 (\Delta u, \Delta (P(u))) \quad (5.42)
\]

\[
Q \bigg|_{t=-1} = 0.
\]

Set

\[
J_1 = \nabla \cdot \left( \eta^2 \nabla(A(u)(\nabla u, \nabla u)) + 2\eta^2 (\Delta u, \nabla (P(u))) \right) \quad \text{and} \quad J_2 = -\eta^2 (\Delta u, \Delta (P(u))).
\]

By the Duhamel formula, we have, for \((x, t) \in \mathbb{R}^n \times (-1, \infty),

\[
\nabla^2 Q(x, t) = \int_{\mathbb{R}^n \times [-1, t]} \nabla^2 b(x - y, t - s) (J_1 + J_2) (y, s)
\]

\[
= \int_{\mathbb{R}^n \times [-1, t]} \nabla^3 b(x - y, t - s) \left( \eta^2 \nabla(A(u)(\nabla u, \nabla u)) + 2\eta^2 (\Delta u, \nabla (P(u))) \right) (y, s)
\]

\[
- \int_{\mathbb{R}^n \times [-1, t]} \nabla^2 b(x - y, t - s) \eta^2 (\Delta u, \Delta (P(u))) (y, s)
\]

\[
= K_1(x, t) + K_2(x, t).
\]

(5.43)

It is clear that for \((x, t) \in \mathbb{R}^n \times (-1, \infty),

\[
|K_1|(x, t) \lesssim I_1 \left( \eta^2 (|\nabla u|^3 + |\nabla u||\nabla^2 u|) \right)(x, t), \, |K_2|(x, t) \leq I_2 \left( \eta^2 (|\nabla^2 u|^2 + |\nabla u|^4) \right)(x, t).
\]

It follows from (5.26) and the Nirenberg interpolation inequality that \(\nabla u \in M^{4,4-4\alpha}(P_{\frac{1}{4}})\) and

\[
\left\| \nabla u \right\|_{M^{4,4-4\alpha}(P_{\frac{1}{4}})} \lesssim \left\| \nabla^2 u \right\|_{L^2_xL^\beta_t(P_1)}^{1/2}.
\]

(5.44)
Hence, by the Hölder inequality, we have that for any $0 < \alpha_1, \alpha_2 < 1$,
\[ \eta^2(|\nabla u|^3 + |\nabla u|\|\nabla^2 u\|) \in M^{4,4-\alpha_1}(\mathbb{R}^{n+1}) \text{ and } \eta^2(|\nabla^2 u|^2 + |\nabla u|^4) \in M^{1,4-\alpha_2}(\mathbb{R}^{n+1}), \]
and
\[ \left\| \eta^2(|\nabla u|^3 + |\nabla u|\|\nabla^2 u\|) \right\|_{M^{4,4-\alpha_1}(\mathbb{R}^{n+1})} \lesssim \left\| \nabla u \right\|_{M^{4,4-\alpha_1}(P,\mathbb{R})} \left\| \nabla^2 u \right\|_{M^{2,4-\alpha_1}(P,\mathbb{R})} \lesssim \left\| \nabla^2 u \right\|_{L^P_t L^6_x(P)}^2, \tag{5.45} \]
\[ \left\| \eta^2(|\nabla^2 u|^2 + |\nabla u|^4) \right\|_{M^{1,4-\alpha_2}(\mathbb{R}^{n+1})} \lesssim \left\| \nabla u \right\|_{M^{4,4-\alpha_2}(P,\mathbb{R})} + \left\| \nabla^2 u \right\|_{M^{2,4-\alpha_2}(P,\mathbb{R})} \lesssim \left\| \nabla^2 u \right\|_{L^P_t L^6_x(P)}^2. \tag{5.46} \]

Now applying Proposition 5.4, we conclude that
\[ K_1 \in M^{\frac{4-\alpha_1}{2}}(\mathbb{R}^{n+1}) \cap L^{\frac{4-\alpha_1}{2}}(\mathbb{R}^{n+1}), \quad K_2 \in M^{\frac{2+2\alpha_2}{2}}(\mathbb{R}^{n+1}) \cap L^{\frac{2+2\alpha_2}{2}}(\mathbb{R}^{n+1}), \]
and
\[ \left\| K_1 \right\|_{M^{\frac{4-\alpha_1}{2}}(\mathbb{R}^{n+1})} + \left\| K_2 \right\|_{M^{\frac{2+2\alpha_2}{2}}(\mathbb{R}^{n+1})} \lesssim \left\| \nabla^2 u \right\|_{L^P_t L^6_x(P)}^2. \tag{5.47} \]
Sending $\alpha_1 \uparrow \frac{2}{3}$ and $\alpha_2 \uparrow \frac{1}{2}$, we obtain that for any $1 < \beta < +\infty$, $K_1, K_2 \in L^\beta(\mathbb{R}^{n+1})$, and
\[ \left\| K_1 \right\|_{L^\beta(\mathbb{R}^{n+1})} + \left\| K_2 \right\|_{L^\beta(\mathbb{R}^{n+1})} \lesssim \left\| \nabla^2 u \right\|_{L^P_t L^6_x(P)}^2. \tag{5.48} \]
This implies that for any $1 < \beta < +\infty$, $\nabla^2 Q \in L^\beta(\mathbb{R}^{n+1})$, and
\[ \left\| \nabla^2 Q \right\|_{L^\beta(\mathbb{R}^{n+1})} \lesssim \left\| \nabla^2 u \right\|_{L^P_t L^6_x(P)}^2. \tag{5.49} \]
Since $(u - Q)$ solves
\[ (\partial_t + \Delta^2)(u - Q) = 0 \text{ in } P_\frac{x}{2}, \]
it follows that for any $1 < \beta < +\infty$, $\nabla^2 u \in L^\beta(P_\frac{x}{2})$, and
\[ \left\| \nabla^2 u \right\|_{L^\beta(P_\frac{x}{2})} \lesssim \left\| \nabla^2 u \right\|_{L^P_t L^6_x(P)}^2. \tag{5.50} \]
This implies (5.49). Hence Claim 2 is proven.

Claim 3. $u \in C^\infty(P_\frac{x}{2}, N)$ and (5.13) holds. It follows from (5.49) that for any $1 < \beta < +\infty$, there exist $f, g \in L^\beta(P_\frac{x}{2})$ such that (1.2) can be written as
\[ (\partial_t + \Delta^2)u = \nabla \cdot f + g. \]
Thus, by the $L^p$-theory of higher-order parabolic equations, we conclude that $\nabla^3 u \in L^\beta(P_\frac{x}{4})$. Applying the $L^p$-theory again, we would obtain that $\partial_t u, \nabla^4 u \in L^\beta(P_\frac{x}{16})$. Taking derivatives of
the equation (1.2) and repeating this argument, we can conclude that \( u \in C^\infty(P_{\frac{1}{2},t},N) \), and the estimate (5.13) holds. Putting together these three claims completes the proof. \( \Box \)

**Proof of Theorem 1.8** Let \( \epsilon_0 > 0 \) be given by Theorem 5.2. Since \( p > \frac{n}{2} \) and \( q < \infty \), there exists \( T_0 > 0 \) such that \[
\max_{i=1,2} \| \nabla^2 u_i \|_{L^q_t L^p_x(\Omega \times [0,T])} \leq \epsilon_0. \tag{5.51}
\]
This implies that for any \( x_0 \in \Omega \) and \( 0 < t_0 \leq T_0 \), if \( R_0 = \min \{ d(x_0, \partial \Omega), t_0^4 \} > 0 \), then
\[
\max_{i=1,2} \| \nabla^2 u_i \|_{L^q_t L^p_x(P_{R_0}(z_0))} \leq \epsilon_0. \tag{5.52}
\]
Hence by suitable scalings of the estimate of Theorem 5.2, we have that for \( i = 1,2 \), \( u_i \in C^\infty(P_{R_0}(z_0),N) \) and
\[
\left| \nabla^m u_i \right|_{(x_0,t_0)} \lesssim \epsilon_0 \left( \frac{1}{d^m(x_0, \partial \Omega)} + \frac{1}{t_0^m} \right). \tag{5.53}
\]
Using (5.53), the same proof of Theorem 1.3 implies that \( u_1 \equiv u_2 \) in \( \Omega \times [0,T_0] \). Repeating this argument on the interval \([T_0,T]\) yields \( u_1 \equiv u_2 \) in \( \Omega \times [0,T] \). \( \Box \)

**Proof of Corollary 1.10** Let \( \epsilon_0 > 0 \) be given by Theorem 5.2. Since \( u_0 \in W^{2,2}(\Omega,N) \), by the absolute continuity of \( \int |\nabla^2 u_0|^2 \) there exists \( r_0 > 0 \) such that
\[
\max_{x \in \Omega} \int_{\partial \Omega} |\nabla^2 u_0|^2 \leq \frac{\epsilon_0^2}{2}. \tag{5.54}
\]
Choosing \( \epsilon_1 \leq \frac{\epsilon_0^2}{2} \) and applying (1.14), we conclude that there exists \( 0 < t_0 \leq r_0^4 \) such that
\[
\max_{x \in \Omega, 0 \leq t \leq t_0} \int_{\partial \Omega} |\nabla^2 u_i(t)|^2 \leq \epsilon_0^2, \quad \text{for } i = 1,2. \tag{5.55}
\]
Set \( R_0 = \min \{ r_0, t_0^4 \} = t_0^4 > 0 \). Then (5.55) implies
\[
\max_{x \in \Omega, 0 \leq t \leq t_0} \int_{\partial \Omega} |\nabla^2 u_i(t)|^2 \leq \epsilon_0^2, \quad \text{for } i = 1,2. \tag{5.56}
\]
Hence \( u_1 \) and \( u_2 \) satisfy (5.12) of Theorem 5.2 (with \( p = 2 \) and \( q = \infty \)) on \( P_r(z) \), for any \( z \in \Omega \times [0,t_0] \) and \( r = \min \{ R_0, d(x, \partial \Omega), t_0^4 \} > 0 \). Hence by suitable scalings of the estimate of Theorem 5.2, we have
\[
\max_{1,2} \left| \nabla^m u_i(x,t) \right| \lesssim \epsilon_0 \left( \frac{1}{R_0^m} + \frac{1}{d^m(x, \partial \Omega)} + \frac{1}{t_0^m} \right) \lesssim \epsilon_0 \left( \frac{1}{d^m(x, \partial \Omega)} + \frac{1}{t_0^m} \right), \quad \forall m \geq 1, \tag{5.57}
\]
for any \((x,t) \in \Omega \times [0,t_0]\). Here we have used \( R_0 \geq t_0^4 \) in the last inequality. Applying (5.57) and the proof of Theorem 1.3 we can conclude that \( u_1 \equiv u_2 \) in \( \Omega \times [0,t_0] \). Continuing this argument on the interval \([t_0,T]\) shows \( u_1 \equiv u_2 \) in \( \Omega \times [0,T] \). \( \Box \)
Proof of Corollary 1.11. Let $\epsilon_2 > 0$ be given by Theorem 5.2. Then (1.15) yields
\[
\|\nabla^2 u\|_{L_t^\infty L_x^2(\Omega \times [0,\infty))} \leq \epsilon_2.
\] (5.58)
Hence by suitable scalings of the estimate of Theorem 5.2 we have $u \in C^\infty(\Omega \times (0,\infty), N)$ and there exists $T_1 > 0$ such that
\[
|\nabla^m u(x,t)| \leq \epsilon_2 \left(\frac{1}{\partial_m(x,\partial\Omega)} + \frac{1}{t^\alpha}\right), \forall m \geq 1,
\] (5.59)
holds for all $x \in \Omega$ and $t \geq T_1$. Now we can apply the same arguments as in the proof of Theorem 1.5 and Corollary 1.6 to prove the conclusions of Corollary 1.11.

6 Appendix: Higher order regularity

It is known, at least to experts, that higher order regularity holds for any Hölder continuous solution to (1.2) of the heat flow of biharmonic maps. However, we can’t find a proof in the literature. For the completeness, we will sketch a proof here.

Proposition 6.1 For $0 < \alpha < 1$, if $u \in W^{1,2}_2 \cap C^\alpha(P_2, N)$ is a weak solution of (1.2), then $u \in C^\infty(P_1, N)$, and
\[
\|\nabla^m u\|_{C^0(P_1)} \lesssim \left[ u \right]_{C^\alpha(P_2)} + \|u\|_{L_t^2 W^{2,2}_x(P_2)}, \forall m \geq 1.
\] (6.1)

Proof. By Claim 2 and Claim 3 in the proof of Theorem 5.2 it suffices to establish that $\nabla^2 u \in M^{2.4-4\delta}(P_2)$ for some $\frac{2}{3} < \tilde\alpha < 1$, and
\[
\|\nabla^2 u\|_{M^{2.4-4\delta}(P_2)} \lesssim \left[ u \right]_{C^\alpha(P_2)} + \|\nabla^2 u\|_{L^2(P_2)}.
\] (6.2)
This will be achieved by the hole-filling type argument. For any fixed $z_0 = (x_0, t_0) \in P_2$ and $0 < r \leq \frac{1}{4}$, let $\phi \in C^\infty_0(\mathbb{R}^n)$ be a cut-off function of $B_r(x_0)$, i.e.,
\[
0 \leq \phi \leq 1, \phi \equiv 1 \text{ in } B_r(x_0), \phi \equiv 0 \text{ outside } B_{2r}(x_0), |\nabla^m \phi| \leq Cr^{-m}, \forall m \geq 1.
\]
Set $c := \int_{B_{r}(z_0)} u \in \mathbb{R}^{L+1}$. Multiplying (1.2) by $(u - c)\phi^4$ and integrating over $\mathbb{R}^n$, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^n} |u - c|^2 \phi^4 + 2 \int_{\mathbb{R}^n} \Delta(u - c) \cdot \Delta((u - c)\phi^4) = 2 \int_{\mathbb{R}^n} N_{bh}[u] \cdot (u - c)\phi^4
\leq \int_{\mathbb{R}^n} |\nabla^2 u|^2|u - c|\phi^4 + \int_{\mathbb{R}^n} |\nabla u||\nabla^2 u|| \Delta((u - c)\phi^4)|.
\] (6.3)
For the second term in the left hand side of (6.3), we have
\[
2 \int_{\mathbb{R}^n} \Delta(u - c) \cdot \Delta((u - c)\phi^4) = 2 \int_{\mathbb{R}^n} \nabla^2 (u - c) \cdot \nabla^2 ((u - c)\phi^4)
\geq 2 \int_{B_{r}(z_0)} |\nabla^2 u|^2 - C \int_{\mathbb{R}^n} |u - c|^2(|\nabla^2 \phi|^2 + |\nabla \phi|^4) + \phi^2|\nabla \phi|^2|\nabla u|^2.
\] (6.4)
Substituting (6.4) into (6.3) and integrating over \( t \in [t_0 - r^4, t_0] \), we obtain
\[
\int_{P_r(z_0)} |\nabla^2 u|^2 \leq \int_{B_{2r}(z_0) \times \{t_0 - r^4\}} |u - c|^2 + \left( 2^{-(n+4)} + C_{\text{osc}P_{2r}(z_0)} u \right) \int_{P_{2r}(z_0)} |\nabla^2 u|^2 \\
+ Cr^n (\text{osc}P_{2r}(z_0)) u^2 + C \left[ 1 + (\text{osc}P_{2r}(z_0)) u^2 \right] r^{-2} \int_{P_{2r}(z_0)} \phi^2 |\nabla u|^2 \\
+C \int_{P_{2r}(z_0)} |\nabla u|^4 \phi^4
\] (6.5)
By integration by parts and the Hölder inequality, we have
\[
\int_{P_{2r}(z_0)} \phi^2 |\nabla u|^2 \leq Cr^{-2} (\text{osc}P_{2r}(z_0)) u \int_{P_{2r}(z_0)} |\nabla^2 u|^2 + Cr^n (\text{osc}P_{2r}(z_0)) u^2,
\]
and
\[
C \int_{P_{2r}(z_0)} \phi^4 |\nabla u|^4 \leq 2^{-(n+4)} \int_{P_{2r}(z_0)} |\nabla^2 u|^2 + Cr^n (\text{osc}P_{2r}(z_0)) u^2 + C (\text{osc}P_{2r}(z_0)) u^2 \int_{P_{2r}(z_0)} |\nabla^2 u|^2.
\]
Putting these two inequalities into (6.5) and using \( \text{osc}P_{2r}(z_0) u \leq Cr^\alpha \), we get
\[
\int_{P_r(z_0)} |\nabla^2 u|^2 \leq \left( 2^{-(n+3)} + Cr^\alpha \right) \int_{P_{2r}(z_0)} |\nabla^2 u|^2 + Cr^{n+2\alpha} + C(1 + r^{2\alpha}) r^{-2} \int_{P_{2r}(z_0)} |\nabla^2 u| \\
\leq \left( 2^{-(n+2)} + Cr^\alpha \right) \int_{P_{2r}(z_0)} |\nabla^2 u|^2 + Cr^{n+2\alpha},
\] (6.6)
where we have used the following inequality in the last step:
\[
C(1 + r^{2\alpha}) r^{-2} \int_{P_{2r}(z_0)} |\nabla^2 u| \leq 2^{-(n+3)} \int_{P_{2r}(z_0)} |\nabla^2 u|^2 + Cr^{n+2\alpha}.
\]
Choosing \( r > 0 \) so small that \( Cr^\alpha \leq 2^{-(n+3)} \), we see that (6.6) implies
\[
\int_{P_r(z_0)} |\nabla^2 u|^2 \leq \frac{1}{2} (2r)^{-n} \int_{P_{2r}(z_0)} |\nabla^2 u|^2 + Cr^{2\alpha}.
\] (6.7)
It is clear that iterating (6.7) implies that there is \( \alpha_0 \in (0, 1) \) such that \( \nabla^2 u \in M^{2,4-2\alpha_0}(P_{\frac{r}{2}}) \) and
\[
\left\| \nabla^2 u \right\|_{M^{2,4-2\alpha_0}(P_{\frac{r}{2}})} \lesssim \left[ u \right]_{C^\alpha(P_2)} + \left\| \nabla^2 u \right\|_{L^2(P_2)}.
\] (6.8)
We can apply the estimate (6.8) and repeat the above argument to show that \( \nabla^2 u \in M^{2,4-4\alpha_0}(P_{\frac{r}{4}}) \) and the estimate (6.8) holds with \( \alpha_0 \) replaced by \( 2\alpha_0 \). Repeating these argument again and again until there exists \( \bar{\alpha} \in (\frac{2}{3}, 1) \) such that \( \nabla^2 u \in M^{2,4-4\bar{\alpha}}(P_{\frac{r}{2}}) \) and the estimate (6.2) holds. The remaining parts of the proof can be done by following the same arguments as in Claim 2 and Claim 3 of the proof of Theorem 5.2. This completes the proof. \( \square \)

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