Characteristic cycles and the conductor of direct image

Takeshi Saito

November 11, 2019

Abstract

We prove the functoriality for proper push-forward of the characteristic cycles of constructible complexes by morphisms of smooth projective schemes over a perfect field, under the assumption that the direct image of the singular support has the dimension at most that of the target of the morphism. The functoriality is deduced from a conductor formula which is a special case for morphisms to curves. The conductor formula in the constant coefficient case gives the geometric case of a formula conjectured by Bloch.

Let $k$ be a perfect field and $\Lambda$ be a finite field of characteristic invertible in $k$. For a constructible complex $F$ of $\Lambda$-modules on a smooth scheme $X$ over $k$, the characteristic cycle $CCF$ is defined in [17, Definition 5.10] as a cycle supported on the singular support $SSF$ defined by Beilinson in [2] as a closed conical subset of the cotangent bundle $T^*X$. We study the functoriality of characteristic cycles for proper push-forward.

Let $f: X \to Y$ be a morphism of smooth projective schemes over $k$. Then, we prove in Theorem 2.2.5 the equality

$$CCR_f F = f!CCF$$

conjectured in [18, Conjecture 1] under the assumption $\dim f_*SSF \leq \dim Y = m$ for the direct image $f_*SSF \subset T^*Y$. The precise definitions will be given in Subsection 2.1. We can slightly weaken the assumption, as is seen in Theorem 2.2.5.

A typical example where the assumption $\dim f_*SSF \leq \dim Y = m$ fails is the case where $f$ is the Frobenius. Without the assumption, the author does not know how to prove the equality (2.11) because the Milnor formula (1.8) on which the definition of characteristic cycles is based will not make sense. In the general case, the equality (2.11) implies an equality in the Chow group $\text{CH}_m (T^*Y) = \text{CH}_0 (Y)$. In the case where $Y = \text{Spec } k$ for a finite field $k$, the equality in $\text{CH}_0 (Y) = Z$ is proved in [21] under the assumption that $X$ is projective by a method different from that in the present article using the product formula for the constant term of the functional equation of $L$-function.

The formula (2.11) is an algebraic analogue of [13, Proposition 9.4.2] where functorial properties of characteristic cycles are studied in a transcendental context. In the case where $Y = \text{Spec } k$, the equality (2.11) is the index formula

$$\chi (X_k, F) = (CCF, T^*_X X)_{T^*X}$$

computing the Euler-Poincaré characteristic as an intersection number proved in [17, Theorem 7.13].
We deduce the functoriality (2.11) from the index formula (2.12) in Subsection 2.2 as follows. By taking a projective embedding of $Y$ and a good pencil, we reduce it to the case where $Y$ is a projective smooth curve. By the index formula (2.12) applied to a general fiber, the equality (2.11) is equivalent to a conductor formula

$$-a_y Rf_* F = (CC F, df)_{T^*X, X_y}$$

proved in Theorem 2.2.3, where the left hand side denotes the Artin conductor at a closed point $y \in Y$ of the direct image. In the case where $F$ is the constant sheaf $\Lambda$, the right hand side equals the localized self-intersection product defined in [4] and the formula (2.17) specializes to the geometric case, Corollary 2.2.4 of the conductor formula conjectured in [4] by Bloch. In [20], the authors announce an attempt to prove the conductor formula by a different approach.

Further the index formula implies that we have an equality (2.18) for the sums over $y \in Y$ of the both sides in (2.17). To deduce (2.17) from (2.18) for the sums, it suffices to show the existence of a covering of $Y$ étale at a fixed point $y$ killing the contributions of the other points.

For the vanishing of the left hand side, the local acyclicity of $f: X \to Y$ relative to $F$ is a sufficient condition. The $SSF$-transversality of $f: X \to Y$ defined in Definition 1.4.3 and studied in Subsection 1.4 after some preliminaries in Subsection 1.3 is a stronger condition and is a sufficient condition for the vanishing of the right hand side. Thus, the proof of (2.17) is reduced to showing variants of the stable reduction theorem on the existence of ramified covering $Y'$ of $Y$ and of a modification $F'$ of the pull-back of $F$ on $X' = X \times_Y Y'$ such that the base change $f': X' \to Y'$ is locally acyclic relatively to $F'$ and is $SSF'$-transversal.

We show that the existence of a modification of a perverse sheaf $F$ relatively to which $f: X \to Y$ is locally acyclic is equivalent to the condition that the inertia action on the nearby cycles complex $R\Psi F$ is trivial in Proposition 1.2.2.2. This is rather a direct consequence of the relation of the direct image by the open immersion of the generic fiber with the nearby cycles complex. As we work with torsion coefficients, the condition is satisfied over a ramified covering of $Y$.

Further, we show that the local acyclicity of $f: X \to Y$ relatively to $F$ implies the existence of a ramified covering $Y' \to Y$ such that the base change of $f: X \to Y$ is $SSF'$-transversal for the pull-back $F'$ of $F$ in Corollary 1.6.4 of Theorem 1.6.2. Theorem 1.6.2 is deduced from a weaker version Proposition 1.5.4 which is proved by using the alteration [5, Theorem 8.2]. In Proposition 1.5.4 the ramified covering may be inseparable, while it is generically étale in Theorem 1.6.2. This improvement is crucial because in the proof of Theorem 2.2.3 we need to find a covering of a curve $Y$ which is étale at a fixed point $y \in Y$. Theorem 1.6.2 is proved by an argument similar to that in the proof of [7, Proposition 3.2] by using a consequence of the stable reduction theorem [19, Theorem 1.5].

We also prove an index formula Proposition 2.3.3 for vanishing cycles complex.

The author thanks A. Beilinson for the remark that Theorem 2.2.3 implies the geometric case of the conductor formula conjectured by Bloch in [4] and for showing the proof of Lemma 2.2.6 in the characteristic zero case. The author thanks H. Haoyu for discussion on the subject of Subsection 2.3 and thanks H. Kato for pointing out an error in the proof of Proposition 2.3.3 in an earlier version. He also thanks an anonymous referee for careful reading, for pointing out gaps in the proofs of Lemma 1.4.9 and of Proposition 1.5.4 in an earlier version and proposing to improve the statement of Lemma 1.2.1.
The research was supported by JSPS Grants-in-Aid for Scientific Research (A) 26247002.

Contents

1 Local acyclicity and transversality .............................. 3
  1.1 Preliminaries on perverse sheaves .............................. 3
  1.2 Nearby cycles and local acyclicity .............................. 5
  1.3 C-transversality ................................................. 7
  1.4 SSF-transversality ............................................... 11
  1.5 Alteration and transversality ................................... 17
  1.6 Potential transversality ......................................... 23

2 Characteristic cycles and the direct image ....................... 28
  2.1 Direct image of a cycle ........................................... 28
  2.2 Characteristic cycle of the direct image ....................... 32
  2.3 Index formula for vanishing cycles ............................. 38

1 Local acyclicity and transversality

1.1 Preliminaries on perverse sheaves

We fix some conventions on perverse sheaves. Let $X$ be a noetherian scheme and let $\Lambda$ be a finite field of characteristic $\ell$ invertible on $X$. We say that a complex $F$ of $\Lambda$-modules on the étale site of $X$ is constructible if the cohomology sheaf $H^q F$ is constructible for every integer $q$ and $H^q F = 0$ except for finitely many $q$. Let $D^b_{\text{c}}(X, \Lambda)$ denote the category of constructible complexes of $\Lambda$-modules.

First we recall the case where $X$ is a scheme of finite type over a field $k$. Let $\Lambda$ be a finite field of characteristic $\ell$ invertible in $k$. Then, the $t$-structure ($pD^\leq 0$, $pD^\geq 0$) on $D^b_c(X, \Lambda)$ relative to the middle perversity is defined in [3, Théorème 2.2.10] and the perverse sheaves on $X$ form an abelian subcategory $\text{Perv}(X, \Lambda) = pD^\leq 0 \cap pD^\geq 0$.

Every object of $\text{Perv}(X, \Lambda)$ is of finite length by [3, Théorème 4.3.1 (i)]. Further by [3, Théorème 4.3.1 (ii)], simple objects are of the following form: Let $V \subset X$ be an irreducible locally closed subset of dimension $d$ such that the reduced part of the geometric fiber $V_k$ is smooth. Let $j: V \to X$ be the immersion and let $G$ be an irreducible locally constant sheaf on $V$. Then, $j_! G[\ell]$ is a simple perverse sheaf on $X$.

Let $j: U \to X$ be an open immersion and $F$ be a perverse sheaf on $X$. Let $G \subset F$ (resp. $H \subset F$) be the largest (resp. smallest) sub perverse sheaf supported (resp. such that $F/H$ is supported) on the complement $X - U$. Then, since $j_! j^* F$ is the unique extension of $j^* F$ without non-trivial sub or quotient perverse sheaf supported on $X - U$ by [3, Corollaire 1.4.25], there is a canonical isomorphism

\[
H/G \to j_* j^* F.
\]

Proposition 1.1.1. Let $h: W \to X$ be a finite and faithfully flat morphism of schemes of finite type over a field $k$ and let $F \in \text{Perv}(X, \Lambda)$ be a perverse sheaf on $X$. Then, the trace morphism $h_* h^* F \to F$ ([2, Théorème 6.2.3]) induces a surjection

\[
h_* p^h^0 h^* F \to F
\]
of perverse sheaves on $X$.

**Proof.** Since $h^*$ is right $t$-exact by [3, Proposition 2.2.5], and $h_*$ is $t$-exact by [3, Corollaire 4.1.3], for perverse sheaf $\mathcal{F}$ on $X$, we have $h_*h^*\mathcal{F} \in D^{\leq 0}$ and $pH^0 h_* h^* \mathcal{F} = h_* pH^0 h^* \mathcal{F}$ and the trace morphism induces (1.2).

First we show the surjectivity for a simple perverse sheaf $\mathcal{F}$ on $X$. Since $\mathcal{F}$ is simple, it suffices show that (1.2) is non trivial. By [3, Théorème 4.3.1 (ii)], we may assume that $\mathcal{F} = j_!* \mathcal{G}[d]$, for an irreducible locally closed subset $V \subset X$ of dimension $d$ such that the reduced part of the geometric fiber $V_k$ is smooth, the immersion $j: V \rightarrow X$ and an irreducible locally constant sheaf $\mathcal{G}$ on $V$. Let $h_V: V' \rightarrow V$ denote the base change of $h$. Shrinking $X$, we may assume that $V \subset X$ is a closed subset. Further shrinking $X$, we may assume that the reduced part of $V_k$ is smooth over $\bar{k}$ and that $h_V h^*_V \mathcal{G}$ is locally a constant sheaf on $V$. Then $h^* \mathcal{F}$ is a perverse sheaf on $W$ and the trace morphism $h_{V_*} h^*_V \mathcal{G} \rightarrow \mathcal{G}$ is a surjection. Hence the assertion follows in this case.

We show the general case. Since $h^*$ is right $t$-exact by [3, Proposition 2.2.5], and $h_*$ is $t$-exact by [3, Corollaire 4.1.3], for an exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ of perverse sheaves on $X$, we have a commutative diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
\mathcal{F}' \\
\downarrow \\
\mathcal{F} \\
\downarrow \\
\mathcal{F}'' \\
\downarrow \\
0
\end{array}
$$

(1.3)

of exact sequences of perverse sheaves on $X$, where the vertical arrows are induced by the trace morphisms. Hence by the induction on the length of $\mathcal{F}$, the assertion follows. \qed

**Corollary 1.1.2.** Let $h: W \rightarrow X$ be a proper morphism of schemes of finite type over a field $k$. Let $\mathcal{F}$ be a perverse sheaf on $X$ and $\mathcal{F}'$ be a perverse sheaf on $W$. Assume that there exists an open subset $U \subset X$ satisfying the following conditions: The base change $h_U: U' \rightarrow U$ of $h$ is finite and faithfully flat. The perverse sheaf $\mathcal{F}$ is isomorphic to $j_! \mathcal{F}_U$ for the open immersion $j: U \rightarrow X$ and $\mathcal{F}_U = j^* \mathcal{F}$. The perverse sheaf $\mathcal{F}_U = j_!^* \mathcal{F}'$, the assertion follows.

Then the perverse sheaf $\mathcal{F}$ on $X$ is isomorphic to a subquotient of $pH^0 R^0 h_* \mathcal{F}'$.

**Proof.** Since $h_{U*}$ is $t$-exact, we have $h_{U*}(\mathcal{F}'|_{U'}) = (pH^0 R^0 h_* \mathcal{F}')|_{U'}$. Hence $j_! h_{U*}(\mathcal{F}'|_{U'}) = j_! (pH^0 R^0 h_* \mathcal{F}')|_{U'}$ is isomorphic to a subquotient of $pH^0 R^0 h_* \mathcal{F}'$. Since $pH^0 h^*_U \mathcal{F}_U$ on $U'$ is isomorphic to a subquotient of the restriction $\mathcal{F}'|_{U'}$, the perverse sheaf $\mathcal{F}_U$ is a subquotient of $h_{U*} \mathcal{F}'|_{U'}$ by Proposition 1.1.1. Hence $\mathcal{F} = j_! \mathcal{F}_U$ is isomorphic to a subquotient of $j_! (h_{U*} \mathcal{F}'|_{U'})$ and the assertion follows. \qed

Next, we consider the case where $X$ is a scheme of finite type over the spectrum $S$ of a discrete valuation ring as in [12, 4.6]. Let $s$ and $\eta$ denote the closed point and the generic point of $S$ respectively and let $i: X_s \rightarrow X$ and $j: X_\eta \rightarrow X$ be the closed immersion and the open immersion of the fibers. Let $\Lambda$ be a finite field of characteristic $\ell$ invertible on $S$. Then, we consider the $t$-structure on $D_c^b(X, \Lambda)$ obtained by gluing ([3, 1.4.10]) the $t$-structure $(pD^{\leq 0}, pD^{\geq 0})$ on $D_c^b(X_s, \Lambda)$ and the $t$-structure $(pD^{\leq -1}, pD^{\geq -1})$ on $D_c^b(X_\eta, \Lambda)$. In particular, a constructible complex $\mathcal{F} \in D_c^b(X, \Lambda)$ is contained in $pD^{\leq 0}$ if and only if we have $i^* \mathcal{F} \in pD^{\leq 0}$ and $j^* \mathcal{F} \in pD^{\leq -1}$.

Note that if the $t$-structure on $D_c^b(X_\eta, \Lambda)$ where $X_\eta$ is regarded as a scheme over $\eta$ is $(pD^{\leq 0}, pD^{\geq 0})$, then that on $D_c^b(X_\eta, \Lambda)$ where $X_\eta$ is regarded as a scheme over $S$ is
\((\underline{p}D_\infty^{-1}, \underline{p}D_\infty^{-1})\). To distinguish them, we call the former the \(t\)-structure on \(X_\eta\) over \(\eta\) and the latter the \(t\)-structure on \(X_\eta\) over \(S\). We use the same terminology for perverse sheaves on \(X_\eta\).

By the same argument as [3, Théorème 4.3.1], we see that every object of \(\text{Perv}(X, \Lambda)\) is of finite length. Further, simple objects are of the following form: Let \(V \subset X_s\) (resp. \(V \subset X_\eta\)) be an irreducible locally closed subset of dimension \(d\) (resp. \(d - 1\)) such that the reduced part of the geometric fiber \(\overline{V}\) (resp. \(\overline{V}_\eta\)) is smooth. Let \(j: V \to X\) be the immersion and let \(G\) be an irreducible locally constant sheaf on \(V\). Then, \(j_!G[d]\) is a simple perverse sheaf on \(X\).

The functors \(j_!, Rj_*: D^b_c(X_\eta, \Lambda) \to D^b_c(X, \Lambda)\) are \(t\)-exact with respect to the \(t\)-structure on \(X_\eta\) over \(S\). This follows from [1, Théorème 3.1] by the argument in [12, 4.6 (a)]. Let \(F \in \text{Perv}(X_\eta, \Lambda)\) be a perverse sheaf on \(X_\eta\) over \(S\). Then the intermediate extension \(j_!F \in \text{Perv}(X, \Lambda)\) is defined as the image \(j_!F = \text{Im}(j_!F \to Rj_*F)\).

Similarly as [3 (4.11.1.1)], the distinguished triangle \(j_!F \to Rj_*F \to i_*i^*Rj_*F \to \) and the \(t\)-exactness of the functors \(j_!\) and \(Rj_*\) [3, Appendix Remark (i)], imply the vanishing \(\underline{p}H^0i^*Rj_*F = 0\) for \(q \neq 0, -1\) and define an exact sequence \(0 \to i_*\underline{p}H^{-1}i^*Rj_*F \to j_!F \to Rj_*F \to i_*\underline{p}H^0i^*Rj_*F \to 0\) of perverse sheaves on \(X\). This induces an isomorphism

\[
(1.4) \quad \underline{p}H^{-1}i^*Rj_*F \to i^*\text{Ker}(j_!F \to Rj_*F) = i^*\text{Ker}(j_!F \to j_!F) \leftarrow i^*j_!F[-1]
\]

of perverse sheaves on \(X_s\) similarly as [3 (4.11.2.1)].

### 1.2 Nearby cycles and local acyclicity

Assume that \(S = \text{Spec} \, \mathcal{O}_K\) is the spectrum of a strictly local discrete valuation ring. Let \(\bar{\eta}\) be a geometric point above \(\eta\). Let \(X\) be a scheme of finite type over \(S\), and let \(i: X_s \to X\) and \(j: X_\eta \to X\) denote the immersions and let \(\pi: X_\bar{\eta} \to X_\eta\) denote the canonical morphism. Then, the nearby cycles functor \(R\Psi = i^*R(j_!\pi)_*\pi^*: D^b_c(X_\eta, \Lambda) \to D^b_c(X_s, \Lambda)\) is \(t\)-exact with respect to the \(t\)-structure on \(X_\eta\) over \(\eta\) [12, Corollaire 4.5]. We have a canonical isomorphism \(F \to R\Gamma(I, \pi_*\pi^*F)\) and we identify \(i^*Rj_*F = R\Gamma(I, R\Psi F)\) by the induced isomorphism.

**Lemma 1.2.1.** Let \(S = \text{Spec} \, \mathcal{O}_K\) be the spectrum of a strictly local discrete valuation ring and let \(s\) and \(\eta\) denote the closed and the generic point of \(S\) respectively. Let \(\bar{\eta}\) be the spectrum of a separable closure \(K_s\) of \(K\) and \(I = \text{Gal}(K_s/K)\) be the inertia group. Let \(X\) be a scheme of finite type over \(S\), and let \(i: X_s \to X\) and \(j: X_\eta \to X\) denote the immersions. Let \(F\) be a perverse sheaf of \(\Lambda\)-modules on \(X_\eta\) over \(S\). Then the morphism \(i^*Rj_*F \to R\Psi F\) induces an isomorphism

\[
(1.5) \quad i^*j_!F[-1] \to (R\Psi F[-1])^I
\]

to the inertia fixed part as a perverse sheaf on \(X_s\).
Proof. Note that $\mathcal{F}[-1]$ is a perverse sheaf on $X_\eta$ over $\eta$ and hence $R\Psi\mathcal{F}[-1]$ is a perverse sheaf on $X_s$ by the $t$-exactness of $R\Psi$. Let $P \subset I$ denote the wild inertia subgroup. Then, since the functor taking the $P$-invariant parts is an exact functor, we have an isomorphism $i^*Rj_*\mathcal{F} \to R\Gamma(I/P, (R\Psi\mathcal{F}[-1])^P)$. Since the profinite group $I/P$ is cyclic, we have an isomorphism $[\sigma - 1: (R\Psi\mathcal{F}[-1])^P \to (R\Psi\mathcal{F}[-1])^P] \to R\Gamma(I/P, (R\Psi\mathcal{F}[-1])^P)$ for a topological generator $\sigma$ of $I/P$. Thus we obtain an isomorphism
\[
\mathcal{H}^{-1}i^*Rj_*\mathcal{F} \to (R\Psi\mathcal{F}[-1])^I
\]
of perverse sheaves on $X_s$. Hence the assertion follows from the isomorphism $[14]$. \qed

We study the local acyclicity of a morphism to the spectrum of a discrete valuation ring with respect to a perverse sheaf.

**Proposition 1.2.2.** Let $S = \text{Spec} \mathcal{O}_K$ be the spectrum of a discrete valuation ring and let $s$ and $\eta$ denote the closed and the generic point of $S$ respectively. Let $X$ be a scheme of finite type over $S$, and let $i: X_s \to X$ and $j: X_\eta \to X$ denote the immersions.

1. Let $\mathcal{G}$ be a perverse sheaf of $\Lambda$-modules on $X$. Assume that $X \to S$ is locally acyclic relatively to $\mathcal{G}$. Then $\mathcal{G}$ has no non-zero subquotient supported on the closed fiber and is isomorphic to $j^*\mathcal{G}$.

2. For a perverse sheaf $\mathcal{F}$ of $\Lambda$-modules on $X_\eta$ over $S$, the following conditions are equivalent:
   1. The morphism $X \to S$ is locally acyclic relatively to $j^*\mathcal{F}$.
   2. Let $\bar{s}$ be a geometric point above the closed point $s \in S$ and let $\bar{i}: X_{\bar{s}} \to X$ denote the canonical morphism. Then, the canonical morphism
   \[
   \bar{i}^*j^*\mathcal{F} \to R\Psi\mathcal{F}
   \]
is an isomorphism.
   3. The inertia group $I$ of $K$ acts trivially on the nearby cycles complex $R\Psi\mathcal{F}$.
   4. The formation of $j^*\mathcal{F}$ commutes with the pull-back by faithfully flat morphisms $S' \to S$ of the spectra of discrete valuation rings.

Proof. 1. We first show that $\mathcal{G}$ has no non-zero subquotient supported on the closed fiber. The local acyclicity is equivalent to the vanishing $R\Phi\mathcal{G} = 0$. Since the shifted vanishing cycles functor $R\Phi[-1]: D_c^b(X, \Lambda) \to D_c^b(X_s, \Lambda)$ is $t$-exact $[12, \text{Corollaire 4.6}]$, it is reduced to the case where $\mathcal{G}$ is a simple perverse sheaf by the induction on length of $\mathcal{G}$. If $\mathcal{G}$ is supported on the closed fiber, we have $R\Phi\mathcal{G}[-1] = \mathcal{G}$. Hence $\mathcal{G}$ has no non-zero subquotient supported on the closed fiber.

Since $j^*\mathcal{G}$ is the unique perverse sheaf extending $j^*\mathcal{G}$ without non-trivial sub or quotient perverse sheaf supported on the closed fiber by $[3, \text{Corollaire 1.4.25}]$, $\mathcal{G}$ is canonically isomorphic to $j^*\mathcal{G}$. \qed

2. (1)\(\Leftrightarrow\)(2): The condition (2) is equivalent to that for every geometric point $x$ of $X_s$, the canonical morphism $j^*\mathcal{F}_x \to R\Gamma(X(x) \times S_{\bar{s}}, \bar{\eta}, \mathcal{F})$ is an isomorphism.

(2)\(\Leftrightarrow\)(3): Clear from the isomorphism $[13]$.

(2)\(\Rightarrow\)(4): Since the formation of nearby cycles complex $R\Psi\mathcal{F}$ commutes with base change $[7, \text{Proposition 3.7}]$, the isomorphism $[17]$ implies the condition (4).

(4)\(\Rightarrow\)(2): There exists a finite extension $K'$ of $K$ such that the inertia action $I' \subset I$ on $R\Psi\mathcal{F}$ is trivial, since $\Lambda$ is a finite field. Let $j': X_{K'} \to X_{S'}$ denote the base change
of the open immersion $j$ by $S' = \text{Spec} \mathcal{O}_{K'} \to S$, let $i': X_{\bar{s}} \to X_{S'}$ denote the canonical morphism and let $\mathcal{F}'$ denote the pull-back of $\mathcal{F}$ on $X_{K'}$. We factorize the morphism (1.6) as the composition of $\tilde{i}' j_{s'}^* \mathcal{F} \to \tilde{i}' j_{s'}^* \mathcal{F}' \to R\Psi \mathcal{F}$. By (3)⇒(2) already proven, the second arrow is an isomorphism. The condition (4) implies that the first arrow is an isomorphism. Hence the composition (1.6) is an isomorphism.

Finally, we consider the case where $X$ is a scheme of finite type over a regular noetherian connected scheme $S$ of dimension 1. Let $\Lambda$ be a finite field of characteristic $\ell$ invertible on $S$. Then the $t$-structure $(pD^{\leq 0}, pD^{\geq 0})$ on $D^b_c(X, \Lambda)$ is defined as the intersection of the inverse images of the $t$-structures $(pD^{\leq 0}, pD^{\geq 0})$ on $D^b_c(X \times_S S_s, \Lambda)$ for the base changes by the localizations $S_s \to S$ at closed points $s \in S$. If $Y = S$ is a smooth curve over a field $k$ and if $f: X \to Y$ is a morphism of schemes of finite type over $k$, the $t$-structure on $D^b_c(X, \Lambda)$ defined above is the same as that defined by considering $X$ as a scheme of finite type over $k$.

**Corollary 1.2.3.** Let $S$ be a regular noetherian scheme of dimension 1. Let $X$ be a scheme of finite type over $S$ and $\mathcal{F}$ be a perverse sheaf of $\Lambda$-modules on $X$. Let $V \subset S$ be a dense open subscheme such that the base change $X_V \to V$ is universally locally acyclic relatively to the restriction $\mathcal{F}_V$ of $\mathcal{F}$.

Then, there exists a finite faithfully flat and generically étale morphism $S' \to S$ of regular schemes such that the base change $X' \to S'$ is locally acyclic relatively to $j_{i!} \mathcal{F}_{V'}$, where $\mathcal{F}_{V'}$ denotes the pull-back of $\mathcal{F}$ on $V' = V \times_S S'$ and $j': X'_{V'} \to X'$ denotes the base change.

**Proof.** By Proposition 1.2.2 (1)⇒(4) and Lemma 1.2.4 below, it suffices to consider locally on a neighborhood of each point of the complement $S - V$. Since the coefficient field $\Lambda$ is finite, the assertion follows from Proposition 1.2.2 (3)⇒(1).

**Lemma 1.2.4.** Let $S$ be a regular noetherian scheme of dimension 1 and let $s_1, \ldots, s_n$ be closed points of $Y$. Let $L_1, \ldots, L_n$ be finite separable extensions of the local fields $K_1, \ldots, K_n$ of $S$ at $s_1, \ldots, s_n$. Then, there exists a finite, faithfully flat and generically étale morphism $S' \to S$ such that $S' \times_S K_i$ is isomorphic to the disjoint union of finitely many copies of $\text{Spec} \, L_i$.

**Proof.** Let $m$ be a common multiple of the degrees $[L_i: K_i]$ and let $A_i$ be the product of copies of $L_i$ such that $\dim K_i A_i = m$. Then, by weak approximation, there exists a finite étale algebra $A$ over the fraction field $K$ of $S$ such that $A \otimes_K K_i = A_i$. Then, it suffices to take the normalization $S'$ of $S$ in $A$.

### 1.3 $C$-transversality

We introduce some terminology on proper intersection.

**Lemma 1.3.1.** Let $f: C \to X$ and $h: W \to X$ be morphisms of schemes of finite type over a field $k$. Assume that $C$ is irreducible of dimension $n$ and that $h$ is locally of complete intersection of relative virtual dimension $d$. Then every irreducible component of $h^*C = C \times_X W$ is of dimension $\geq n + d$.

**Proof.** Since the assertion is local on $W$, we may decompose $h = gi$ as the composition of a smooth morphism $g$ with a regular immersion of codimension $c$. Since the assertion is
clear for $g$, we may assume that $h = i$ is a regular immersion. Then, it follows from [9, Proposition (5.1.7)].

**Definition 1.3.2.** Let $f: C \to X$ and $h: W \to X$ be morphisms of schemes of finite type over a field $k$. Assume that every irreducible component of $C$ is of dimension $n$ and that $h$ is locally of complete intersection of relative virtual dimension $d$. We say that $h: W \to X$ meets $f: C \to X$ properly if $h^* C = C \times_X W$ is of dimension $n + d$.

By Lemma 1.3.1 the condition that $h^* C = C \times_X W$ is of dimension $n + d$ is equivalent to the condition that every irreducible component of $h^* C = C \times_X W$ is of dimension $n + d$.

**Lemma 1.3.3.** Let $f: C \to X$ be a morphism of schemes of finite type over a field $k$. Assume that $X$ is equidimensional of dimension $m$ and that $C$ is equidimensional of dimension $n \geq m$. We consider the following conditions:

1. Every morphism $h: W \to X$ locally of complete intersection meets $C$ properly.
2. For every closed point $x$ of $X$, the fiber $C \times_X x$ is of the dimension $n - m$.
3. Assume that $X = \mathbb{P}$ is a projective space and let $c$ be an integer. Then, the linear subspaces $V \subset \mathbb{P}$ of codimension $c$ such that the immersion $V \to \mathbb{P}$ meets $C$ properly form a dense open subset of the Grassmannian variety $G$.

**Proof.**

1. Assume that the condition (2) is satisfied and let $h: W \to X$ be a morphism locally of complete intersection of relative virtual dimension $d$. Then, we have $\dim C \times_X W \leq \dim W + n - m = n + d$. Hence, $C \times_X W$ is equidimensional of dimension $n + d$ by Lemma 1.3.1. The rest is clear.

2. If $X$ is regular and $x$ is a closed point, the closed immersion $i: x \to X$ is a regular immersion of codimension $m$ and hence the condition (1) implies that $\dim C \times_X x = n - m$.

3. Let $V \subset \mathbb{P} \times G$ be the universal family of linear subspaces of codimension $c$ and we consider the cartesian diagram

$$
\begin{array}{ccc}
G & \to & V \\
\downarrow & & \downarrow \\
C & \to & \mathbb{P},
\end{array}
$$

Then, since the projection $V \to \mathbb{P}$ is smooth of relative dimension $\dim \mathbb{P} - c$, we have $\dim C_V = \dim \mathbb{P} + n - c$. Hence the open subset of $G$ consisting of $V$ such that $\dim C \times_{\mathbb{P}} V \leq n - c$ is dense. □

Recall that a closed subset $C$ of a vector bundle $E$ on a scheme $X$ is said to be conical if it is stable under the action of the multiplicative group. For a closed conical subset $C \subset E$, the intersection $B = C \cap X$ with the 0-section identified with a closed subset of $X$ is called the base of $C$. We say that a morphism $f: X \to Y$ of noetherian schemes is finite (resp. proper) on a closed subset $Z \subset X$ if its restriction $Z \to Y$ is finite (resp. proper) with respect to a closed subscheme structure of $Z \subset X$. 

8
Definition 1.3.4. Let $f: X \to Y$ be a morphism of smooth schemes over a field $k$ and let $C \subset T^*X$ be a closed conical subset.

1. ([2, 1.2]) We say that $f: X \to Y$ is $C$-transversal if the inverse image of $C$ by the canonical morphism $X \times_Y T^*Y \to T^*X$ is a subset of the 0-section $X \times_Y T^*_Y Y \subset X \times_Y T^*Y$.

2. Assume that every irreducible component of $X$ is of dimension $n$ and that every irreducible component of $C$ is of dimension $n$. Assume that every irreducible component of $Y$ is of dimension $m \leq n$. We say that $f: X \to Y$ is properly $C$-transversal if $f: X \to Y$ is $C$-transversal and if for every closed point $y$ of $Y$, the fiber $C \times_Y y$ is of dimension $n - m$.

Definition 1.3.5. Let $h: W \to X$ be a morphism of smooth schemes over a field $k$ and let $C \subset T^*X$ be a closed conical subset. Let $K \subset W \times_X T^*X$ be the inverse image of the 0-section $T^*_W W \subset T^*W$ by the canonical morphism $W \times_X T^*X \to T^*W$.

1. ([2, 1.2]) We say that $h: W \to X$ is $C$-transversal if the intersection $(W \times_X C) \cap K \subset W \times_X T^*X$ is a subset of the 0-section $W \times_X T^*_X X$.

If $h: W \to X$ is $C$-transversal, a conical subset $h^\circ C \subset T^*W$ is defined to be the image of $h^\circ C = W \times_X C$ by $W \times_X T^*X \to T^*W$.

2. ([17, Definition 7.1]) Assume that every irreducible component of $X$ is of dimension $n$ and that every irreducible component of $C$ is of dimension $n$. Assume that every irreducible component of $W$ is of dimension $m$. We say that $h: W \to X$ is properly $C$-transversal if $h: W \to X$ is $C$-transversal and if $h: W \to X$ meets $C \to Y$ properly.

If $h: W \to X$ is $C$-transversal, the morphism $W \times_X T^*X \to T^*W$ is finite on $h^\circ C = W \times_X C$ and hence $h^\circ C \subset T^*W$ is a closed subset by [2, Lemma 1.2 (ii)]. For a morphism $r: X \to Y$ of smooth schemes proper on the base $B = C \cap T^*_X X \subset X$ of a closed conical subset $C \subset T^*X$, the closed conical subset $r_0 C \subset T^*Y$ is defined to be the image by the projection $X \times_Y T^*Y \to T^*Y$ of the inverse image of $C$ by the canonical morphism $X \times_Y T^*Y \to T^*X$.

Lemma 1.3.6. Let $f: X \to Y$ be a smooth morphism of smooth schemes over a field $k$ and let $C \subset T^*X$ be a closed conical subset. Let

$$
\begin{array}{ccc}
X & \xrightarrow{h} & W \\
\downarrow f & & \downarrow g \\
Y & \xleftarrow{i} & Z
\end{array}
$$

be a cartesian diagram of smooth schemes over $k$.

1. Assume that $f: X \to Y$ is $C$-transversal (resp. properly $C$-transversal). Then, $h: W \to X$ is $C$-transversal (resp. properly $C$-transversal) and $g: W \to Z$ is $h^\circ C$-transversal (resp. properly $h^\circ C$-transversal).

2. Assume that $f: X \to Y$ is proper on the base of $C$. Then, $i: Z \to Y$ is $f_0 C$-transversal if and only if $h: W \to X$ is $C$-transversal. If these equivalent conditions are satisfied, we have $i^\circ f_0 C = g_0 h^\circ C$.

Proof. 1. The assertion for the transversality is proved in [17, Lemma 3.9.2]. The proper transversality of $h: W \to X$ follows from the transversality and Lemma 1.3.3 applied to $C \to Y$ and $Z \to Y$. The proper $h^\circ C$-transversality of $g: W \to Z$ follows from that $h^\circ C \to h^\circ C$ is finite.
2. We consider the commutative diagram

\[
\begin{array}{ccccccccc}
T^*X & \leftarrow & W \times \mathcal{X} T^*X & \overset{dh}{\rightarrow} & T^*W \\
\uparrow & & \uparrow & & \uparrow \\
X \times \mathcal{Y} T^*Y & \leftarrow & W \times \mathcal{Y} T^*Y & \overset{g^*(di)}{\rightarrow} & W \times \mathcal{Z} T^*Z \\
\downarrow & & \downarrow & & \downarrow \\
T^*Y & \leftarrow & Z \times \mathcal{Y} T^*Y & \overset{di}{\rightarrow} & T^*Z \\
\end{array}
\]

with cartesian squares indicated by □. The upper vertical arrows are injections. Since \(dh\) induces an isomorphism \(W \times_X T^*X/Y \to T^*W/Z\) for the relative cotangent bundles and \(f: X \to Y\) is smooth, the upper right square is also cartesian.

Let \(K\) and \(K'\) be the inverse image of the 0-sections by \(dh: W \times_X T^*X \to T^*W\) and \(di: Z \times_Y T^*Y \to T^*Z\) respectively. Since the upper right square is cartesian, \(K\) is identified with the inverse image of the 0-section by \(g^*(di): W \times_Y T^*Y \to W \times_Z T^*Z\) which equals \(g^{-1}(K') \subset W \times_Y T^*Y\).

Since the lower left square is cartesian, the pull-back \(Z \times_Y f_0C\) is the image \(g_*(C')\) of \(C' = (W \times_X C) \cap (W \times_Y T^*Y)\). Hence the condition that \((Z \times_Y f_0C) \cap K' = g_*(C') \cap K' = g_*(C' \cap g^{-1}(K'))\) is a subset of the 0-section is equivalent to the condition that \((W \times_X C) \cap K = C' \cap g^{-1}(K')\) is a subset of the 0-section.

If these conditions are satisfied, the equality \(\overline{\mathcal{C}} f_0C = g_0 h_0 C\) follows from the cartesian diagram. \(\square\)

**Lemma 1.3.7.** Let \(\mathcal{P}\) be a projective space of dimension \(n\) and let \(C \subset T^*\mathcal{P}\) be a closed conical subset.

1. Let \(\mathcal{P}^\vee\) be the dual projective space, let \(Q \subset \mathcal{P} \times \mathcal{P}^\vee\) be the universal family of hyperplanes and let

\[
(1.7) \quad \mathcal{P} \leftarrow^p Q \overset{p^\vee}{\rightarrow} \mathcal{P}^\vee
\]

be the projections. Let \(C^\vee = p_0^\vee p_0 C\) be the Legendre transform. Let \(V \subset \mathcal{P}\) be a linear subspace and let \(V^\vee \subset \mathcal{P}^\vee\) be the dual subspace. Then the immersion \(V \to \mathcal{P}\) is \(C\)-transversal if and only if \(V^\vee \to \mathcal{P}^\vee\) is \(C^\vee\)-transversal.

2. Assume that every irreducible component of \(C\) is of dimension \(n = \dim \mathcal{P}\) and let \(0 \leq c \leq n\) be an integer. Then, the linear subspaces \(V \subset \mathcal{P}\) of codimension \(c\) such that the immersion \(V \to \mathcal{P}\) is properly \(C\)-transversal form a dense open subset of the Grassmannian variety \(\mathcal{G}\).

**Proof.**

1. The \(C\)-transversality of \(V \to \mathcal{P}\) means \(\mathcal{P}(T^*_V \mathcal{P}) \cap \mathcal{P}(C) = \emptyset\) and similarly for the \(C^\vee\)-transversality of \(V^\vee \to \mathcal{P}^\vee\). Then, the assertion follows from \(\mathcal{P}(T^*_V \mathcal{P}) = \mathcal{P}(T^*_V \mathcal{P}^\vee)\) and \(\mathcal{P}(C) = \mathcal{P}(C^\vee)\) under the identification \(\mathcal{P}(T^*_V \mathcal{P}) = Q = \mathcal{P}(T^*_V \mathcal{P}^\vee)\).

2. First, we show that the condition on \(V\) is an open condition. For the transversality, it suffices to apply a relative version of [17] Lemma 3.4.4 or equivalently [2] Lemma 1.2(i)] to the closed immersion of the universal family \(V \to \mathcal{P} \times \mathcal{G}\) since the projection \(V \to \mathcal{G}\) is proper. For the proper transversality, it follows from the semi-continuity of fiber dimension \([9\, \text{Théorème (13.1.3)}]\).

It suffices to show the existence of \(V\). By induction on \(c\), it is reduced to the case \(c = 1\). By 1, the hyperplanes \(H\) such that the immersion \(H \to \mathcal{P}\) is \(C\)-transversal is parametrized
by the complement of the image \( p^\vee(\mathcal{P}(C)) \subseteq \mathbb{P}^\vee \). Hence, the assertion follows from this and Lemma 1.3.3.3.

### 1.4 \( SSF \)-transversality

For the definitions and basic properties of the singular support of a constructible complex on a smooth scheme over a field, we refer to [2] and [17]. Let \( k \) be a field and let \( \Lambda \) be a finite field of characteristic \( \ell \) invertible in \( k \). Let \( X \) be a smooth scheme over \( k \) such that every irreducible component is of dimension \( n \) and let \( \mathcal{F} \) be a constructible complex on \( X \). The singular support \( SSF \) is defined in [2] as a closed conical subset of the cotangent bundle \( T^*X \). By [2, Theorem 1.3 (ii)], every irreducible component \( C_a \) of the singular support \( SSF = C = \bigcup a C_a \) is of dimension \( n = \dim X \).

Further if \( k \) is perfect, the characteristic cycle \( CC\mathcal{F} = \sum a m_a C_a \) is defined as a linear combination with \( \mathbb{Z} \)-coefficients in [17, Definition 5.10]. It is characterized by the Milnor formula

\[
- \dim \text{tot} \phi_u(\mathcal{F}, f) = (CC\mathcal{F}, df)_{T^*U,u}
\]

for morphisms \( f: U \to Y \) to smooth curves \( Y \) defined on an étale neighborhood \( U \) of an isolated characteristic point \( u \). For more detail on the notation, we refer to [17, Section 5.2].

**Lemma 1.4.1.** Let \( h: W \to X \) be a morphism of smooth schemes over a field \( k \). Let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules on \( X \) and let \( C \) denote the singular support \( SSF \). If \( h: W \to X \) is properly \( C \)-transversal, we have

\[
SSh^*\mathcal{F} = h^*SSF.
\]

**Proof.** By [2, Theorem 1.4 (iii)], we may assume that \( k \) is perfect. Suppose \( \dim W = \dim X + d \). If \( \mathcal{F} \) is a perverse sheaf on \( X \), then \( h^*\mathcal{F}[d] \) is a perverse sheaf on \( W \) by the assumption that \( h: W \to X \) is \( C \)-transversal and by [17, Lemma 8.6.5]. Hence by [2, Theorem 1.4 (ii)], we may assume that \( \mathcal{F} \) is a perverse sheaf. By [17, Proposition 5.14.2], we have \( CC\mathcal{F} \geq 0 \) and the support of \( CC\mathcal{F} \) equals the singular support \( SSF \). Also we have \( (-1)^dCCh^*\mathcal{F} \geq 0 \) and the support of \( CCh^*\mathcal{F} \) equals the singular support \( SSF \).

By the assumption that \( h: W \to X \) is properly \( C \)-transversal and by [17, Theorem 7.6], we have \( CC\mathcal{F} = h^*CC\mathcal{F} \). Hence by the positivity [8, Proposition 7.1 (a)], the singular support \( SSF \) equals the support \( h^*SSF \) of \( h^*CC\mathcal{F} \).

**Lemma 1.4.2.** Let \( k \) be a field and \( f: X \to Y \) be a morphism of schemes of finite type over \( k \). Assume that \( Y \) is smooth over \( k \). Let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules on \( X \). Let

\[
\begin{array}{ccc}
X & \xrightarrow{i} & P \\
in & & \downarrow g \\
P' & \xrightarrow{g'} & Y
\end{array}
\]
be a commutative diagram of schemes over $k$ such that $i$ and $i'$ are closed immersions and the schemes $P$ and $P'$ are smooth over $k$. Let $C = SS_i^*\mathcal{F} \subset X \times_P T^*P \subset T^*P$ and $C' = SS'_i\mathcal{F} \subset X \times_{P'} T^*P' \subset T^*P'$ denote the singular supports of the direct images. Then, $P \to Y$ is $C$-transversal (resp. properly $C$-transversal) if and only if $P' \to Y$ is $C'$-transversal (resp. properly $C'$-transversal).

**Proof.** By factorizing $P \to Y$ as the composition of the graph $P \to P \times Y$ and the projection $P \times Y$, we may assume that $P \to Y$ is smooth. Similarly, we may assume that $P' \to Y$ is smooth. By considering the morphism $(i, i') : X \to P \times Y P'$, we may assume that there exists a smooth morphism $P' \to P$ compatible with the immersions of $X$ and the morphisms to $Y$. Since the assertion is étale local on $P$, we may assume that there exists a section $s : P \to P'$ compatible with the immersions of $X$ and the morphisms to $Y$. Then, we have $C' = s^{-1}C$ and the assertion follows from [17, Lemma 3.8].

Lemma 1.4.2 allows us to make the following definition.

**Definition 1.4.3.** Let $k$ be a field and $f : X \to Y$ be a morphism of schemes of finite type over $k$. Assume that $Y$ is smooth over $k$ but we do not require $X$ to be smooth. Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$.

We say that $f : X \to Y$ is $SS\mathcal{F}$-transversal (resp. properly $SS\mathcal{F}$-transversal) if locally on $X$ there exist a closed immersion $i : X \to P$ to a smooth scheme $P$ over $k$ and a morphism $g : P \to Y$ over $k$ such that $f = g \circ i$ and that $g : P \to Y$ is $C$-transversal (resp. properly $C$-transversal) for $C = SS_i^*\mathcal{F}$.

In Definition 1.4.3 we obtain an equivalent condition by requiring that $g$ is smooth.

Let $f : X \to Y$ be a morphism of schemes of finite type over a field $k$ such that $Y$ is smooth over $k$ and let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$. For an open subset $U \subset X$, we say $f : X \to Y$ is $SS\mathcal{F}$-transversal (resp. properly $SS\mathcal{F}$-transversal) on $U$ if the restriction $U \to Y$ of $f$ is $SS\mathcal{F}_U$-transversal (resp. properly $SS\mathcal{F}_U$-transversal) for the restriction $\mathcal{F}_U$ of $\mathcal{F}$ on $U$. Similarly, for an open subset $V \subset Y$, we say $f : X \to Y$ is $SS\mathcal{F}$-transversal (resp. properly $SS\mathcal{F}$-transversal) on $V$ if the base change $X \times_Y V \to V$ of $f$ is $SS\mathcal{F}_{X \times_Y V}$-transversal (resp. properly $SS\mathcal{F}_{X \times_Y V}$-transversal) for the restriction $\mathcal{F}_{X \times_Y V}$ of $\mathcal{F}$ on $X \times_Y V$.

**Lemma 1.4.4.** Let $f : X \to Y$ be a morphism of schemes of finite type over a field $k$. Assume that $Y$ is smooth over $k$. Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$.

1. Assume that $Y$ is smooth over $k$ and that $\mathcal{F}$ is locally constant. Then, $f : X \to Y$ is properly $SS\mathcal{F}$-transversal.

2. Assume that $f : X \to Y$ is $SS\mathcal{F}$-transversal. Or more weakly, suppose that there exists a quasi-finite faithfully flat morphism $Y' \to Y$ of smooth schemes over $k$ such that the base change $f' : X' \to Y'$ is $SS\mathcal{F}$-transversal for the pull-back $\mathcal{F}'$ of $\mathcal{F}$ on $X' = X \times_Y Y'$. Then, $f : X \to Y$ is universally locally acyclic relatively to $\mathcal{F}$.

3. The following conditions are equivalent:

   (1) $f : X \to Y$ is $SS\mathcal{F}$-transversal (resp. properly $SS\mathcal{F}$-transversal).

   (2) For every integer $q$ and for every constituent $\mathcal{G}$ of the perverse sheaf $\Phi^q\mathcal{F}$, the morphism $f : X \to Y$ is $SS\mathcal{G}$-transversal (resp. properly $SS\mathcal{G}$-transversal).

4. Let $k'$ be an extension of $k$. Then $f : X \to Y$ is $SS\mathcal{F}$-transversal (resp. properly $SS\mathcal{F}$-transversal) if and only if the base change $f' : X' \to Y'$ by Spec $k' \to$ Spec $k$ is $SS\mathcal{F}'$-transversal (resp. properly $SS\mathcal{F}'$-transversal) for the pull-back $\mathcal{F}'$ on $X'$ of $\mathcal{F}$.
Proof. 1. If $\mathcal{F}$ is locally constant, then the singular support $\text{SS}\mathcal{F}$ is a subset of the 0-section $T^*_X X$. Hence the assertion follows.

Since the remaining assertions 2-4 are local on $X$, we may take a closed immersion $i: X \to P$ to a smooth scheme $P$ over $k$ such that $f$ is the composition of $i$ with a morphism $P \to Y$ over $k$. Replacing $X$ and $\mathcal{F}$ by $P$ and $i_* \mathcal{F}$, we may assume that $X$ is smooth over $k$. Set $C = \text{SS}\mathcal{F}$.

2. If $f: X \to Y$ is $C$-transversal, the morphism $f: X \to Y$ is universally locally acyclic relatively to $\mathcal{F}$ by the definition of singular support. Thus under the weaker assumption, the morphism $f': X' \to Y'$ is universally locally acyclic with respect to the pull-back $\mathcal{F}'$. Since $Y' \to Y$ is quasi-finite and faithfully flat and since the local acyclicity descends for faithfully flat morphisms, the morphism $f: X \to Y$ itself is universally locally acyclic with respect to $\mathcal{F}$.

3. By [2, Theorem 1.4 (ii)], the singular support $\text{SS}\mathcal{F}$ equals the union of $\text{SS}\mathcal{G}$ for the constituents $\mathcal{G}$ of the perverse sheaves $\mathcal{H}^q \mathcal{F}$ for integers $q$. Hence the assertion follows.

4. By [2, Theorem 1.4 (iii)], the construction of the singular support commutes with change of base fields. Hence the assertion follows. \hfill \Box

Lemma 1.4.5. Let $f: X \to Y$ be a morphism of schemes of finite type over a field $k$. Assume that $Y$ is smooth over $k$. Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$. Assume that $f: X \to Y$ is $\text{SS}\mathcal{F}$-transversal.

1. Assume that $\mathcal{F}$ is a perverse sheaf. Let $V \subset Y$ be a dense open subscheme and $j: X_V = X \times_Y V \to X$ be the open immersion. Then, there is a unique isomorphism $\mathcal{F} \to j_{!*}j^* \mathcal{F}$ such that the restriction on $X_V$ is the identity.

2. There exists a dense open subscheme $V \subset Y$ such that the base change $f: X_V \to V$ is properly $\text{SS}\mathcal{F}$-transversal on $V$.

Proof. 1. By [3, Corollaire 1.4.25], it suffices to show that for every constituent of $\mathcal{F}$, its restriction on $X_V$ is non-trivial. Let $\mathcal{G}$ be a constituent of $\mathcal{F}$. By Lemma 1.4.3 and 2, the morphism $f: X \to Y$ is locally acyclic relatively to $\mathcal{G}$. Let $x$ be a geometric point of $X$ such that $\mathcal{G}_x \neq 0$ and let $y \to f(x)$ be a specialization for a geometric point $y$ of $V$. Then, since the canonical morphism $\mathcal{G}_x \to R\Gamma(X(x) \times_Y (Y(y)), \mathcal{G})$ is an isomorphism, the restriction of $\mathcal{G}$ on $X_V$ is non-trivial. Thus the assertion is proved.

2. As in the proof of Lemma 1.4.4 we may assume that $X$ is smooth over $k$. Set $C = \text{SS}\mathcal{F}$. There exists a dense open subset $V \subset Y$ such that for every irreducible component $C_a$ with the reduced scheme structure of $C = \bigcup_a C_a$, the base change $C_a \times_Y V \to V$ is flat. \hfill \Box

Lemma 1.4.6. Let $f: X \to Y$ be a morphism of schemes of finite type over a field $k$. Assume that $Y$ is smooth over $k$. Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$. Let $Y' \to Y$ be a morphism of smooth schemes over $k$ and let

\[
\begin{array}{ccc}
X & \xleftarrow{h} & X' \\
\downarrow f & & \downarrow f' \\
Y & \xleftarrow{\text{id}} & Y',
\end{array}
\]

be a cartesian diagram. Let $\mathcal{F}'$ denote the pull-back of $\mathcal{F}$ on $X'$.

1. We consider the following conditions:
(1) $f: X \to Y$ is $\text{SS}\mathcal{F}$-transversal (resp. properly $\text{SS}\mathcal{F}$-transversal).
(2) \( f': X' \to Y' \) is \( SSF' \)-transversal (resp. properly \( SSF' \)-transversal).

Then, we have (1) \( \Rightarrow \) (2). Conversely, if \( Y' \to Y \) is \'{e}tale surjective, we have (2) \( \Rightarrow \) (1).

2. Assume that \( f: X \to Y \) is \( SSF \)-transversal, that \( F \) is a perverse sheaf on \( X \) and that \( \dim Y' = \dim Y + d \). Then \( F'[d] \) is a perverse sheaf on \( X' \).

3. Assume that \( f: X \to Y \) is smooth and is properly \( SSF \)-transversal. Then, we have \( SSF' = h^oSSF \). Further if \( k \) is perfect, we have \( CC' = h^oCCF \).

**Proof.** Since the assertions are local on \( X \), we may take a closed immersion \( i: X \to P \) to a smooth scheme \( P \) over \( Y \). As in the proof of Lemma \ref{lem:transversality-of-germs}, we may assume that \( f: X \to Y \) is smooth. Set \( C = SSF \).

1. Assume that \( f: X \to Y \) is \( C \)-transversal. The pair \((h, f')\) of morphisms is \( C \)-transversal by Lemma \ref{lem:transversality-of-germs}. Hence, \( F' = h^*F \) is micro-supported on \( h^oC \) by \cite{17} Lemma 4.2.4 and we have an inclusion \( SSF' \subset h^oC \) and \( f' \) is \( SSF' \)-transversal. Thus the implication \( (1) \Rightarrow (2) \) is proved for the \( C \)-transversality. The assertion on the proper \( C \)-transversality follows from this and Lemma \ref{lem:transversality-of-germs}.

Since the formation of singular support is \'{e}tale local, we have (2) \( \Rightarrow \) (1) if \( Y' \to Y \) is \'{e}tale surjective.

2. Since \( h: X' \to X \) is \( C \)-transversal by Lemma \ref{lem:transversality-of-germs}, the assertion follows from \cite{17} Lemma 8.6.5.

3. Since \( h: X' \to X \) is properly \( C \)-transversal by Lemma \ref{lem:transversality-of-germs}, the assertion for \( SSF' \) (resp. for \( CC'F' \)) follows from Lemma \ref{lem:transversality-of-germs} (resp. \cite{17} Theorem 7.6)]. \( \blacksquare \)

Lemma \ref{lem:transversality-of-germs} is closely related to the subject studied in \cite{10}.

**Lemma 1.4.7.** Let \( f: X \to Y \) be a morphism of schemes of finite type over a field \( k \). Assume that \( Y \) is smooth over \( k \). Let \( F \) be a constructible complex of \( \Lambda \)-modules on \( X \).

1. Let \( g: Y \to Z \) be a smooth morphism of smooth schemes over \( k \). If \( f: X \to Y \) is \( SSF \)-transversal (resp. properly \( SSF \)-transversal), then the composition \( gf: X \to Z \) is \( SSF \)-transversal (resp. properly \( SSF \)-transversal).

2. Let \( h: W \to X \) be a smooth morphism of schemes of finite type over \( k \). If \( f: X \to Y \) is \( SSF \)-transversal (resp. properly \( SSF \)-transversal), then the composition \( fh: W \to Y \) is \( SSh^*F \)-transversal (resp. properly \( SSh^*F \)-transversal).

3. Let

\[
\begin{array}{ccc}
X & \xrightarrow{r} & X' \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xrightarrow{\gamma} & Y \\
\end{array}
\]

be a commutative diagram of morphisms of schemes of finite type over \( k \). Assume that \( r: X \to X' \) is proper on the support of \( F \) and that \( f: X \to Y \) is quasi-projective. If \( f: X \to Y \) is \( SSF \)-transversal, then \( f': X' \to Y \) is \( SS Rr_*F \)-transversal.

**Proof.** 1. As in the proof of Lemma \ref{lem:transversality-of-germs} we may assume that \( X \) is smooth over \( k \). Set \( C = SSF \). Since \( g: Y \to Z \) is smooth, the \( C \)-transversality of \( f \) implies that of \( gf \) by \cite{17} Lemma 3.6.3. The assertion on the proper \( C \)-transversality follows from this and the smoothness of \( g \).
2. Since the question is étale local on $W$, we may assume that there exists a cartesian diagram

$$
\begin{array}{ccc}
W & \xrightarrow{h} & X \\
\downarrow & & \downarrow \circ \quad i \\
Q & \xrightarrow{p} & P
\end{array}
$$

of morphisms of schemes over $k$ such that the vertical arrows are closed immersions and the horizontal arrow $Q \rightarrow P$ is a smooth morphism of smooth schemes over $k$. By replacing $X$ and $\mathcal{F}$ by $P$ and $i_*\mathcal{F}$, we may assume that $X$ is smooth. Since $W \times_X T^*X \rightarrow T^*W$ is an injection and $SSh^*\mathcal{F} = h^*SS\mathcal{F}$ by Lemma 1.4.1, the assertion follows.

3. Since the assertion is local on $X'$, we may assume that $X'$ and $Y$ are affine and hence $X$ is quasi-projective. By taking a closed immersion $i': X' \rightarrow P'$ to an affine space and by factorizing $X' \rightarrow Y$ as the composition of the immersion $(i', f') : X' \rightarrow P' \times Y$ and the projection $P' \times Y \rightarrow Y$, we may assume that $X'$ is smooth. Similarly, we take an open subscheme $P$ of a projective space and a closed immersion $i : X \rightarrow P$. Then, by factorizing $X \rightarrow X'$ as the composition of the immersion $(i, r) : X \rightarrow P \times X'$ and the projection $P \times X' \rightarrow X'$, we may also assume that $X$ is smooth, by [17] Lemma 3.8 (2)⇒(1)]. By [2] Lemma 2.2 (ii)], we have $SS Rr_*\mathcal{F} \subset r_*SS\mathcal{F}$. Hence the assertion follows from [17] Lemma 3.8 (2)⇒(1)].

We give two methods to establish $SS\mathcal{F}$-transversality.

**Lemma 1.4.8.** Let $Y \rightarrow S$ be a smooth morphism of smooth schemes of finite type over a field $k$ and let $f : X \rightarrow Y$ be a morphism of schemes of finite type over a field $k$. Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$. Assume that the composition $X \rightarrow S$ is properly $SS\mathcal{F}$-transversal.

1. Assume that $k$ is perfect. Then, the following conditions are equivalent:
   1. $f : X \rightarrow Y$ is $SS\mathcal{F}$-transversal (resp. properly $SS\mathcal{F}$-transversal).
   2. For every closed point $s \in S$, the fiber $f_s : X_s \rightarrow Y_s$ is $SS\mathcal{F}_s$-transversal (resp. properly $SS\mathcal{F}_s$-transversal) for the pull-back $\mathcal{F}_s$ of $\mathcal{F}$ on $X_s = X \times_S s$.

2. Assume that $\mathcal{F}$ is a perverse sheaf on $X$ and that $f : X \rightarrow Y$ is locally acyclic relatively to $\mathcal{F}$. If there exists a closed subset $Z \subset X$ quasi-finite over $S$ such that $f : X \rightarrow Y$ is $SS\mathcal{F}$-transversal (resp. properly $SS\mathcal{F}$-transversal) on the complement of $Z$, then $f : X \rightarrow Y$ is $SS\mathcal{F}$-transversal (resp. properly $SS\mathcal{F}$-transversal) on $X$.

**Proof.** 1. The implication (1)⇒(2) is a special case of Lemma 1.4.6.1. We show (2)⇒(1). Since the question is local on $X$, we may assume that $f : X \rightarrow Y$ is smooth. Let $T^*X/S$ and $T^*Y/S$ denote the relative cotangent bundles and let $C = SS\mathcal{F}$. By the assumption that $X \rightarrow S$ is $C$-transversal, the canonical surjection $T^*X \rightarrow T^*X/S$ is finite on $C$ by [2] Lemma 1.2 (ii)]. Hence its image $\bar{C} \subset T^*X/S$ is a closed conical subset and $C \rightarrow \bar{C}$ is finite. The morphism $X \rightarrow Y$ is $C$-transversal if and only if the inverse image of $\bar{C} \subset T^*X/S$ by the canonical injection $X \times_Y T^*Y/S \rightarrow T^*X/S$ is a subset of the 0-section. This is equivalent to that for every closed point $s \in S$ and the closed immersion $i_s : X_s \rightarrow X$, the morphism $f_s : X_s \rightarrow Y_s$ is $i_s^*C$-transversal. Further, under the assumption that $f : X \rightarrow Y$ is $C$-transversal, this is properly $C$-transversal if and only if $f_s : X_s \rightarrow Y_s$ is properly $i_s^*C$-transversal for every closed point $s \in S$. 

15
By the assumption that $X \to S$ is properly $SSF$-transversal and by Lemma 1.4.6.3, we have $SSF_s = i_s^*SSF = i_s^*C$ for every closed point $s \in S$. Hence the assertion is proved.

2. By Lemma 1.4.4.4, we may assume that $k$ is perfect. By 1 and Lemma 1.4.6.2, we may assume that $S = \text{Spec } k$. As in the proof of Lemma 1.4.6 after replacing $X$ by a smooth scheme $P$ over $Y$ containing $X$ as a closed subscheme, we may assume that $f: X \to Y$ is smooth of relative dimension $d$. Let $u \in Z$. By replacing $X$ by a neighborhood of $u$, we may assume $Z = \{u\}$. Set $C = SSF$, $v = f(u) \in Y$ and regard $X \times_Y T^*Y$ as a closed subscheme of $T^*X$.

We show that $f: X \to Y$ is $C$-transversal, assuming that $f: X \to Y$ is locally acyclic relatively to $F$. Namely, we show that the intersection $C' = C \cap (X \times_Y T^*Y)$ is a subset of the 0-section $X \times_Y T^*_Y Y$. By the assumption that $f: X \to Y$ is $C$-transversal outside $u$, the intersection $C' = C \cap (X \times_Y T^*Y)$ is a subset of the union $(X \times_Y T^*_Y Y) \cup (u \times_Y T^*Y)$ with the fiber at $u$. Let $v = f(u)$ and $\omega \in u \times_Y T^*Y = v \times_Y T^*_Y Y$ be a non-zero element. After shrinking $Y$ to a neighborhood of $v$ if necessary, we take a smooth morphism $Y \to A^1 = \text{Spec } k[t]$ such that $dt(v) = \omega$. Then, by [17] Lemma 3.6.3], on a neighborhood of $u$, the composition $g: X \to Y \to A^1$ is $C$-transversal except at most at $u$. In other words, the point $u$ is at most an isolated $C$-characteristic point ([17] Definition 5.3.1) of $g: X \to A^1$.

Since $F$ is a perverse sheaf, the characteristic cycle $CCF$ is an effective cycle and its support equals $C = SSF$ by [17] Proposition 5.14. Let $dg$ denote the section of $X \times_Y T^*Y \subset T^*X$ defined by the function $g^*(t)$. Since the composition $X \to Y \to A^1$ is locally acyclic relatively to $F$ by [11] Corollaire 5.2.7, we have $(CCF, dg)T^*_X u = 0$ by the Milnor formula [10]. Therefore by the positivity [3] Proposition 7.1 (a)], the intersection $SSF \cap dg = C' \cap dg$ is empty and hence $\omega \notin C'$. Since $\omega$ is any non-zero element of $u \times_Y T^*Y$, we conclude that $C' \cap (u \times_Y T^*Y) \subset 0$ and that $f: X \to Y$ is $C$-transversal.

Assume further that $f: X \to Y$ is properly $C$-transversal outside $u$. Since $f: X \to Y$ is $C$-transversal, the morphism $T^*X \to T^*X/Y$ to the relative cotangent bundle is finite on $C$ by [2] Lemma 1.2 (ii)] and the image $\bar{C} \subset T^*X/Y$ of $C$ is a closed conical subset. It is sufficient to show that for every point $y \in Y$, the fiber $\bar{C} \times_Y y$ is of dimension $\leq d$. For $y \neq f(u)$, this follows from the assumption. Assume $y = f(u)$. Then, every irreducible component of $\bar{C} \times_Y y$ is either a closure of an irreducible component of $\bar{C} \times_Y y \cap (X \times_Y y = \{u\})$ or a subset of the fiber $T^*_u (X \times_Y y)$. Since $\dim T^*_u (X \times_Y y) = d$, the assertion is proved.

Lemma 1.4.9. Let

$$
\begin{array}{c}
W \xrightarrow{h} X \xrightarrow{f} Y \\
\uparrow \quad \uparrow j \\
U' \xrightarrow{h_U} U
\end{array}
$$

be a cartesian diagram of schemes of finite type over a field $k$. Assume that $Y$ is smooth over $k$ and that $j$ is an open immersion. Let $\mathcal{F}$ and $\mathcal{F}'$ be perverse sheaves of $\Lambda$-modules on $X$ and on $W$ respectively and let $\mathcal{F}_U$ and $\mathcal{F}'_U$ be the restrictions on $U$ and on $U'$ respectively. Assume that $\mathcal{F}$ is isomorphic to $j_* \mathcal{F}_U$ and that the perverse sheaf $HH^0(h_U^* \mathcal{F}_U)$ on $U'$ is isomorphic to a subquotient of $\mathcal{F}'_U$. If one of the following conditions (1) and (2) below is satisfied and if $f \circ h: W \to Y$ is $SSF'$-transversal, then $f: X \to Y$ is $SSF$-transversal.

1. The morphism $h: W \to X$ is proper, surjective and generically finite and the composition $W \to Y$ is quasi-projective. The reduced geometric fiber $U_k$ is smooth of
dimension $d$ over $k$ and there exists a locally constant sheaf $\mathcal{G}$ of $\Lambda$-modules on $U$ such that $\mathcal{F}_U = \mathcal{G}[d]$.

(2) The morphism $h$ is quasi-finite and faithfully flat and $U = X$.

Proof. Assume that (1) is satisfied. Replacing $X$ by the reduced closed subscheme, we may assume that $X$ is reduced. Shrinking $U$ if necessary, we may assume that $h_U: U' \to U$ is finite and faithfully flat. Then, by Corollary 1.1.2, $\mathcal{F}$ is isomorphic to a subquotient of $\mathcal{P}^0 \mathcal{R}h_* \mathcal{F}'$. Since $W \to Y$ is quasi-projective, the assertion follows from Lemma 1.4.7.3 and Lemma 1.4.4.3.

Assume that (2) is satisfied. Since the assertion is étale local on $X$ by Lemma 1.4.6.1, we may assume that $h: W \to X$ is finite and faithfully flat and that $W, X$ and $Y$ are affine. Then, by Corollary 1.1.2, $\mathcal{F}$ is isomorphic to a subquotient of the perverse sheaf $h_* \mathcal{F}'$. Since $W \to Y$ is affine and hence quasi-projective, the assertion also follows from Lemma 1.4.7.3 and Lemma 1.4.4.3. \qed

1.5 Alteration and transversality

Let $f: X \to Y$ be a morphism of smooth schemes over a field $k$ and let $D \subset Y$ be a divisor smooth over $k$. In this article, we say that $f: X \to Y$ is semi-stable relatively to $D$ if étale locally on $X$ and on $Y$, there exists a cartesian diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
A^n & \longrightarrow & A^1 \\
\end{array}
$$

where the lower left horizontal arrow $A^n = \text{Spec } k[t_1, \ldots, t_n] \to A^1 = \text{Spec } k[t]$ is defined by $t \mapsto t_1 \cdots t_n$ and the lower right horizontal arrow is the inclusion of the origin $0 \in A^1$. A semi-stable morphism $f: X \to Y$ is flat and the base change $f_V: X \times_Y V \to V = Y - D$ is smooth. We recall statements on the existence of alteration.

Lemma 1.5.1. Let $k$ be a perfect field and let $f: X \to Y$ be a dominant separated morphism of integral schemes of finite type over $k$.

1. There exists a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
W & \xrightarrow{q} & Y' \\
\end{array}
$$

of integral schemes of finite type over $k$ satisfying the following condition: The bottom horizontal arrow $Y' \to Y$ is dominant and is the composition $gh$ of an étale morphism $g$ and a finite flat radicial morphism $h$. The schemes $W$ and $Y'$ are smooth over $k$ and the morphism $q: W \to Y'$ is quasi-projective and smooth. The induced morphism $W \to X \times_Y Y'$ is proper surjective and generically finite.

2. Let $\xi \in Y$ be a point such that the local ring $\mathcal{O}_{Y, \xi}$ is a discrete valuation ring. Then, there exists a commutative diagram (1.9) of integral schemes of finite type over $k$ satisfying the following condition: The bottom horizontal arrow $Y' \to Y$ is quasi-finite and flat and its image is an open neighborhood of $\xi$. The schemes $W$ and $Y'$ are smooth over $k$, the
closure $D' \subset Y'$ of the inverse image of $\xi$ is a divisor smooth over $k$ and the morphism
$g: W \to Y'$ is quasi-projective and is semi-stable relatively to $D'$. The induced morphism
$W \to X \times_Y Y'$ is proper surjective and generically finite.

Proof. 1. Let $\eta$ be the generic point of $Y$. Then, it suffices to apply [5, Theorem 4.1] to
the generic fiber $X \times_Y \eta$.

2. Let $S = \text{Spec } \mathcal{O}_{Y, \xi}$ be the localization at $\xi$. Then, it suffices to apply [5, Theorem
8.2] to the base change $X \times_Y S \to S$. □

We prove an analogue of the generic local acyclicity theorem [7, Théorème 2.13].

Proposition 1.5.2. Let $f: X \to Y$ be a morphism of schemes of finite type over a perfect
field $k$. Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$. Then, there exists a cartesian
diagram

$$
\begin{array}{c}
X & \leftarrow & X' \\
\downarrow f & & \downarrow f' \\
Y & \leftarrow & Y'
\end{array}
$$

(1.10)

of schemes of finite type over $k$ satisfying the following conditions: The scheme $Y'$ is
smooth over $k$ and the morphism $Y' \to Y$ is dominant and is the composition $gh$ of an
étale morphism $g$ and a finite flat radicial morphism $h$. The morphism $f': X' \to Y'$ is
properly $\text{SSS}\mathcal{F}'$-transversal for the pull-back $\mathcal{F}'$ of $\mathcal{F}$ on $X'$.

Proof. We may assume that $\mathcal{F}$ is a simple perverse sheaf by Lemma 1.4.4.3 and Lemma
1.4.6. Hence, we may assume that there exist a locally closed immersion $j: Z \to X$ of
a smooth irreducible scheme of dimension $d$ and a simple locally constant sheaf $\mathcal{G}$ of $\Lambda$-
modules such that $j_! \mathcal{G}[d] = \mathcal{F}$ by [3, Théorème 4.3.1 (ii)]. By replacing $X$ by the closure
of $j(Z)$, we may assume that $j: Z \to X$ is an open immersion. It suffices to consider the
case where $Z \to Y$ is dominant since the assertion is clear if otherwise.

Let $Z_1 \to Z$ be a finite étale covering such that the pull-back of $\mathcal{G}$ is constant and let
$X_1$ be the normalization of $X$ in $Z_1$. Namely, $X_1$ is the scheme finite over $X$
\corresponding to the integral closure of $\mathcal{O}_X$ in the quasi-coherent $\mathcal{O}_X$-algebra defined as the direct
image of $\mathcal{O}_{Z_1}$. The inverse image of $Z \subset X$ in $X_1$ is canonically identified with $Z_1$. Applying
Lemma 1.5.1.1 to $X_1 \to Y$, we obtain a commutative diagram

$$
\begin{array}{c}
X & \leftarrow & W \\
\downarrow f & & \downarrow j' \\
Y & \leftarrow & Y'
\end{array}
$$

(1.11)

of schemes over $k$ satisfying the following conditions: The scheme $Y'$ is smooth and the
morphism $Y' \to Y$ is dominant and is the composition $gh$ of an étale morphism $g$ and
a finite flat radicial surjective morphism $h$. The morphism $W \to Y'$ is quasi-projective
and smooth. The induced morphism $r': W \to X' = X \times_Y Y'$ is proper surjective and
generically finite. The pull-back $\mathcal{G}'_W$ of $\mathcal{G}$ on $W \times_X Z$ is a constant sheaf.

We consider the cartesian diagram

$$
\begin{array}{c}
Z & \leftarrow & Z' \leftarrow W \times_X Z \\
j \downarrow & & \downarrow j' \downarrow jw \\
X & \leftarrow & X' \leftarrow W
\end{array}
$$
and let $G'$ be the pull-back of $G$ on $Z'$. After shrinking $Z$ if necessary, we may assume that the reduced part of $Z'$ is smooth over the perfect field $k$. Since the finite radical surjective morphism $h$ is universally a homeomorphism, we have $F' = j'_*G'[d]$.

Since $G'_W$ is a constant sheaf on $W \times_X Z$ and $W$ is smooth over $k$, the intermediate extension $j_{W^1*}G'_W[d]$ is constant. The smooth morphism $W \rightarrow Y'$ is properly $SSj_{W^1*}G'_W[d]$-transversal by Lemma 1.4.1. Since $W \rightarrow X'$ is proper and $W \rightarrow Y'$ is quasi-projective, the morphism $X' \rightarrow Y'$ is $SSF'$-transversal by the case (1) in Lemma 1.4.9. After shrinking $Y'$, the morphism $X' \rightarrow Y'$ is properly $SSF'$-transversal by Lemma 1.4.12.

**Corollary 1.5.3.** Let $f: X \rightarrow Y$ and $F$ be as in Proposition 1.5.2 and assume that $k$ is of characteristic $p > 0$. Then, there exist a dense open subscheme $V \subset Y$ smooth over $k$ and an iteration $\tilde{V} \rightarrow V$ of Frobenius such that the base change $X \times_Y \tilde{V} \rightarrow \tilde{V}$ is $SS\tilde{F}$-transversal for the pull-back $\tilde{F}$ on $\tilde{X}_V = X \times_Y \tilde{V}$.

**Proof.** After shrinking $Y''$ in the conclusion of Proposition 1.5.2, we may assume that $Y'' \rightarrow Y$ is the composition $jgh$ of an open immersion $j: V \rightarrow Y$, a finite surjective radical morphism $g$ and an étale surjective morphism $h$. By Lemma 1.4.11, we may assume that $Y'' \rightarrow Y$ is $jg$. Thus, the assertion follows.

We show an analogue of the stable reduction theorem.

**Proposition 1.5.4.** Let

$$
\begin{array}{ccc}
X & \xleftarrow{\gamma} & U \\
\downarrow f & & \downarrow \quad \downarrow f_V \\
Y & \xleftarrow{\gamma} & V
\end{array}
$$

be a cartesian diagram of schemes of finite type over a perfect field $k$. Assume that $Y$ is normal and that $V$ is a dense open subset of $Y$ smooth over $k$. Let $F_U$ be a perverse sheaf of $\Lambda$-modules on $U$ such that $f_V': U \rightarrow V$ is $SSF_U$-transversal.

Then, there exists a cartesian diagram

$$
\begin{array}{ccc}
X & \xleftarrow{j} & X' \xleftarrow{j'} & U' = U \times_X X' \\
\downarrow f & & \downarrow f' & \downarrow f_V \\
Y & \xleftarrow{g} & Y' \xleftarrow{\gamma} & V' = V \times_Y Y'
\end{array}
$$

of schemes of finite type over $k$ satisfying the following conditions: The scheme $Y'$ is smooth over $k$ and $V' \subset Y'$ is the complement of a divisor $D' \subset Y'$ smooth over $k$. The morphism $g: Y' \rightarrow Y$ is quasi-finite flat and the complement $Y - g(Y')$ is of codimension $\geq 2$ in $Y$. The pull-back $F'_U$ of $F_U$ is a perverse sheaf on $U'$ and for $F' = j'_*F'_U$, on $X'$, the morphism $f': X' \rightarrow Y'$ is $SSF'$-transversal.

First, we prove a basic case.

**Lemma 1.5.5.** Let $X$ and $Y$ be smooth schemes over a field $k$. Let $D \subset Y$ be a divisor smooth over $k$ and $V = Y - D$ be the complement. Let $f: X \rightarrow Y$ be a morphism over $k$ semi-stable relatively to $D$. Assume that $\dim X = n$. For a cartesian diagram 1.13 such that $Y' \rightarrow Y$ is a quasi-finite flat morphism of smooth schemes over $k$, let $F'$ be the perverse sheaf $F' = j'_*\Lambda_U[n]$ on $X'$.
1. Assume that \( \dim Y = 1 \). Let \( Y' \to Y \) be a flat morphism of smooth curves over \( k \) such that for every \( y' \in Y' \to V' \), the action of the inertia group \( I_y' \) on \( \mathcal{H}^n_{y'} \mathcal{L}_{y'} \) is trivial. Then the morphism \( X' \to Y' \) is properly \( SSF' \)-transversal.

2. There exists a quasi-finite faithfully flat morphism \( Y' \to Y \) of smooth schemes over \( k \) satisfying the following conditions: The open subscheme \( V' \subset Y' \) is the complement of a divisor \( D' \) smooth over \( k \) and the morphism \( X' \to Y' \) is properly \( SSF' \)-transversal.

Proof. 1. Since the question is étale local, we may assume that \( Y = A^1_k = \text{Spec } k[t] \), that \( X = X_n = A^n_k = \text{Spec } k[t_1, \ldots, t_n] \) and that the morphism \( f : X \to Y \) is defined by \( t \mapsto t_1 \cdots t_n \). We prove the assertion by induction on \( n \). If \( n = 1 \), then \( f : X \to Y \) is étale and \( F' \) is constant. Hence the assertion follows in this case by Lemma \[4.4\].

Assume \( n > 1 \). Outside the closed point \( u \in X \) defined by \( t_1 = \cdots = t_n = 0 \), locally there exists a smooth morphism \( X = X_n \to X_{n-1} \) over \( Y \). Hence, the induction hypothesis implies the assertion on the complement \( X - \{ u \} \) by Lemma \[4.7\]. Thus, the morphism \( f' : X' \to Y' \) is properly \( SSF' \)-transversal outside the inverse image of \( u \). By Proposition \[2.2\] (3) \( \Rightarrow \) (1), the morphism \( f' : X' \to Y' \) is locally acyclic relatively to \( F' \). Hence by Lemma \[4.8\], the morphism \( f' : X' \to Y' \) is properly \( SSF' \)-transversal on \( X' \).

2. It follows from 1 by Lemma \[4.6\] and Lemma \[4.5\].

Proof of Proposition \[5.4\]. The proof is similar to that of Proposition \[5.2\]. By Lemma \[4.6\] and Lemma \[4.5\] it suffices to show the assertion on a neighborhood of each point \( \xi \in Y \) of codimension 1 not contained in \( V \). Thus, we may assume that \( Y \) is smooth over \( k \), the closure \( D \) of \( \xi \) is a divisor smooth over \( k \) and that \( V = Y - D \).

We prove the assertion by the induction on the dimension \( d \) of the support of \( F'_U \). If \( d = 0 \), it suffices to take \( Y' = Y \). To prove the induction step, we show the following.

Claim. 1. Let \( Y'' \to Y' \) be a morphism of quasi-finite flat schemes over \( Y \) such that \( Y' \) and \( Y'' \) are smooth over \( k \) and that \( V' = V \times_Y Y' \) and \( V'' = V \times_Y Y'' \) are complements of divisors \( D' \) and \( D'' \) smooth over \( k \). Assume that \( F' = j''_! F''_{U'} \) satisfies the conclusion of Proposition \[5.4\]. Then its pull-back \( F'' \) on \( X'' = X \times_Y Y'' \) also satisfies the same condition.

2. Let \( 0 \to G_U \to F_U \to H_U \to 0 \) be an exact sequence of perverse sheaves on \( U \). Then, if the assertion of Proposition \[5.4\] holds for \( G_U \) and for \( H_U \), it also holds for \( F_U \).

Proof of Claim. 1. By Lemma \[4.6\] the \( SSF' \)-transversality of \( X' \to Y' \) implies that \( F'' \) is a perverse sheaf on \( X'' \) and that \( X'' \to Y'' \) is \( SSF'' \)-transversal. By Lemma \[4.5\], we have \( F'' = j''_! F''_{U''} \).

2. Let \( Y'_1 \) and \( Y'_2 \) be quasi-finite flat schemes over \( Y \) satisfying the conditions in Proposition \[5.4\] for \( G_U \) and for \( H_U \) are satisfied respectively. By Claim 1, replacing \( Y'_1 \) and \( Y'_2 \) by the normalization \( Y'_3 \) of the fiber product \( Y'_1 \times_Y Y'_2 \) and shrinking \( Y'_3 \) if necessary, we may assume \( Y'_1 = Y'_2 \). Let \( S \to Y'_1 \) be the strict localization at a geometric point \( \xi'_1 \in \xi'_1 \) above the generic point \( \xi \in D \). Then, by Proposition \[2.2\], there exists a finite ramified extension \( S' \to S \) such that the pull-back \( \mathcal{F}_{X''_y} \) of \( F \) to the generic fiber \( X_{y'} \subset X \times_Y S' \) satisfies the equivalent conditions loc. cit. Hence by Claim 1, further replacing \( Y'_1 \), we may assume that \( S' = S \).

Let \( j'_1 : U'_1 = U \times_X Y'_1 \to Y'_1 = X \times_Y Y'_1 \) be the open immersion, \( F'_{U'_1} \) be the pull-back of \( F_U \) on \( U'_1 \) and let \( A \subset X'_1 \) be the union of the supports of constituents of \( j'_1^! F'_{U'_1} \) that do not meet \( U'_1 = X'_1 \times_Y Y'_1 \). Since \( \mathcal{F}_{X''_y} \) satisfies the condition (1) in Proposition \[2.2\],
the intersection of $A$ with the fiber $X'_1 \times_{Y'_1} \xi'_1$ is empty. Hence by replacing $Y'_1$ by an open neighborhood $Y'$ of $\xi'_1$, we may assume that $A$ itself is empty. Namely, $j'_!* F'_{U'}$, has no non-trivial subquotient perverse sheaf supported on $X' \times_{Y'} D'$.

Then, we have an exact sequence $0 \to j'_!* G'_{U'} \to j'_!* F'_{U'} \to j'_!* H'_{U'} \to 0$ and the assertion follows by Lemma 1.4.13.

By Claim and by induction on the dimension of support, similarly as in the proof of Proposition 1.5.2, we may assume that there exist a dense open immersion $j : Z \to U$ of a smooth irreducible scheme of dimension $d$ and a simple locally constant sheaf $G$ of $\Lambda$-modules such that $F_U = j_* G[d]$. Further, we may assume that $Z \to Y$ is dominant.

Taking a finite étale covering trivializing $G$ and applying Lemma 1.5.12 as in the proof of Proposition 1.5.2 we obtain a commutative diagram

$\begin{array}{ccc}
X & \xleftarrow{r} & W_1 \\
\downarrow f & & \downarrow \\
Y & \xleftarrow{g} & Y_1
\end{array}$

(1.14)

of schemes over $k$ satisfying the following conditions: The scheme $Y_1$ is smooth over $k$, the morphism $g_1 : Y_1 \to Y$ is quasi-finite and flat and $Y \to g_1(Y_1)$ is of codimension $\geq 2$ in $Y$. The inverse image $V \times_Y Y_1$ is the complement of a divisor $D_1$ smooth over $k$ and the morphism $W_1 \to Y_1$ is quasi-projective and is semi-stable relatively to $D_1$. The induced morphism $r_1 : W_1 \to X_1 = X \times_Y Y_1$ is proper surjective and generically finite. The pull-back $G'_1$ of $G$ on $W_1 \times_X Z$ is a constant sheaf.

By Lemma 1.5.12 applied to the semi-stable morphism $W_1 \to Y_1$, we obtain a quasi-finite faithfully flat morphism $Y' \to Y_1$ of smooth schemes satisfying the condition loc. cit. We consider the cartesian diagram

$\begin{array}{ccc}
Z & \xleftarrow{j_2} & Z' \\
\downarrow j_Z & & \downarrow j_Z' \\
X & \xleftarrow{r'} & W'
\end{array}
\begin{array}{ccc}
& & = W_1 \times_{Y_1} Y' \\
& & = X \times_Y Y'
\end{array}$

$W' \times_X Z$

and let $G'$ and $G'_{W'}$ denote the pull-backs of $G$ on $Z'$ and on $W' \times_X Z$ respectively. After shrinking $Z$ if necessary, we may assume that the reduced part of $Z'$ is smooth over $k$. Since $G'_{W'}$ is a constant sheaf on $W' \times_X Z$, the morphism $W' \to Y'$ is $SS j_{W_1} G'_{W'}[d]$-transversal by Lemma 1.5.2.

The pull-back $F'_U$ is a perverse sheaf by Lemma 1.4.6. Let $\mathcal{H} \subset F'_U$ (resp. $\mathcal{H}' \subset F'_U$) be the largest (resp. smallest) sub perverse sheaf supported (resp. such that $F'_U / \mathcal{H}'$ is supported) on the complement $U' \to Z'$ and let $u : Z' \to U'$ be the open immersion. Since the restriction of $F'_U$ on $Z'$ is identified with $G'[d]$, the subquotient $\mathcal{H}' / \mathcal{H}$ is canonically identified with $u_* G'[d]$ by Corollaire 1.4.25. Since $r' : W' \to X'$ is proper surjective and generically finite and since $r' : W' \to Y'$ is quasi-projective, the morphism $X' \to Y'$ is $SS F'_1$-transversal for $F'_1 = j_{Z_1} G'[d] = j'_{!*} (\mathcal{H}' / \mathcal{H})$ by the case (1) in Lemma 1.4.9. Since $\dim(U' - Z') < d$, the assertion follows from the hypothesis of induction and Claim. □

Corollary 1.5.6. Let the cartesian diagram (1.12) and a perverse sheaf $F_U$ on $U = X \times_Y V$ be as in Proposition 1.5.2. Assume further that $Y$ is smooth over $k$ and that $V$ is the
complement of a divisor $D \subset Y$ smooth over $k$. Then, there exists a cartesian diagram \[ (1.13) \] satisfying the following conditions: The scheme $Y'$ is smooth over $k$ and $V' \subset Y'$ is the complement of a divisor $D' \subset Y'$ smooth over $k$. The morphism $g: Y' \to Y$ is quasi-finite flat, the morphism $D' \to D$ is dominant and the morphism $V' \to V$ is étale. The pull-back $\mathcal{F}_{U'}$, of $\mathcal{F}_U$ is a perverse sheaf on $U'$ and the morphism $f': X' \to Y'$ is universally locally acyclic relatively to $\mathcal{F}' = j'_* \mathcal{F}_{U'}$.

Proof. Let $V' \subset Y'$ be as in the conclusion of Proposition 1.5.4. Let $\bar{Y}'$ be the normalization of $Y$ in the separable closure of $k(Y)$ in $k(Y')$. Then, there exists a dense open subset $Y'' \subset \bar{Y}'$ smooth over $k$ of the image of $Y' \to \bar{Y}'$ such that $g': Y'' \to Y$ is flat, that $V'' = V \times_Y Y''$ is the complement of a divisor $D''$ smooth over $k$, that $D'' \to D$ is dominant, and that $V'' \to V$ is étale. Since $Y' \times_{\bar{Y}'} Y'' \to Y''$ is finite surjective radicial, the cartesian diagram \[ (1.13) \] defined by $Y'' \to Y$ in place of $Y' \to Y$ satisfies the conditions.

Corollary 1.5.7. Let $X \to Y$ be a morphism of schemes of finite type over a field $k$ and assume that $Y$ is smooth of dimension $1$. Then, for a constructible complex $\mathcal{F}$ of $\Lambda$-modules on $X$, the following conditions are equivalent:

1. $X \to Y$ is locally acyclic relatively to $\mathcal{F}$.
2. $X \to Y$ is universally locally acyclic relatively to $\mathcal{F}$.
3. There exists a finite faithfully flat morphism $Y' \to Y$ of smooth curves over $k$ such that the base change $X' \to Y'$ is $SS\mathcal{F}'$-transversal for the base change $\mathcal{F}'$ of $\mathcal{F}$ on $X'$.

The equivalence $(1) \Leftrightarrow (2)$ is proved in [16].

Proof. We show $(1) \Rightarrow (3)$. Since the nearby cycles functor is $t$-exact, we may assume that $\mathcal{F}$ is a perverse sheaf. Then, the assertion follows from Propositions 1.5.2, 1.5.4 and 1.2.2. The implication $(3) \Rightarrow (2)$ is proved in Lemma 1.4.4.2. The implication $(2) \Rightarrow (1)$ is trivial.

The following example shows that taking a covering $Y' \to Y$ in condition $(3)$ is necessary.

Example 1.5.8. Let $k$ be a field of characteristic $p \geq 2$. Let $X = \mathbb{A}^1 \times \mathbb{P}^1$ and $j: U = \mathbb{A}^1 \times \mathbb{A}^1 = \text{Spec } k[x, y] \to X$ be the open immersion. Let $W \to X$ be the Artin-Schreier covering defined by $t^p - t = xy$ ramified along the divisor $X = U$ and let $\mathcal{G}$ be the locally constant sheaf of $\Lambda$-modules of rank $1$ on $U$ trivialized by $W_X U$ and defined by a non-trivial character $F_p \to \Lambda^\times$. Let $\mathcal{F} = j_* \mathcal{G}$.

Let $W \to X \to Y = \mathbb{P}^1$ be the composition with the second projection and let $Y' \to Y = \mathbb{P}^1$ be a finite flat morphism of smooth curves over $k$ such that the ramification index at $\infty$ is divisible by $p$. Then, the normalization $W' \to W \times_Y Y'$ of the base change is smooth over $Y'$ and $W' \to X'$ is finite and faithfully flat. Hence by Lemma 1.4.9.2, for the pull-back $\mathcal{F}'$ of $\mathcal{F}$ on $X' = X \times_Y Y'$, the morphism $X' \to Y'$ is $SS\mathcal{F}'$-transversal. By Corollary 1.5.7, $(3) \Rightarrow (2)$, $X \to Y$ is local acyclicity relatively to $\mathcal{F}$ (cf. [15] Théorème 2.4.4).

On the other hand, on the complement of the origin $(0, \infty) \in X$, the singular support $C = SS\mathcal{F}$ is the union of the zero-section $T_X X$ and the conormal bundles $T_X X$ of the fiber $X_\infty = pr_2^{-1}(\infty)$. Hence the projection $pr_2: X \to Y = \mathbb{P}^1$ is not $C$-transversal.
1.6 Potential transversality

We prove a refinement of the analogue of the stable reduction theorem, using the following consequence of the stable reduction theorem for curves.

Lemma 1.6.1. Let

\[
\begin{array}{c}
U \\ f_V
\end{array} \longrightarrow \begin{array}{c}
X \\ f
\end{array}
\]

\[
\begin{array}{c}
V \\ g
\end{array} \longrightarrow \begin{array}{c}
Y \\ S
\end{array}
\]

be a cartesian diagram of morphisms of smooth schemes of finite type over a perfect field \( k \) satisfying the following conditions: The morphism \( f: X \rightarrow Y \) is flat and the morphisms \( g: Y \rightarrow S \) and \( f_V: U \rightarrow V \) are smooth of relative dimension 1. The horizontal arrows are open immersions and the open subset \( V \subset Y \) is the complement of a divisor \( D \subset Y \) smooth over \( k \) and quasi-finite and flat over \( S \).

Then, there exists a commutative diagram

\[
\begin{array}{c}
X' \\ f'
\end{array} \longrightarrow \begin{array}{c}
Y' \\ g'
\end{array} \longrightarrow \begin{array}{c}
S'
\end{array}
\]

\[
\begin{array}{c}
X \\ f
\end{array} \longrightarrow \begin{array}{c}
Y \\ g
\end{array} \longrightarrow \begin{array}{c}
S
\end{array}
\]

of smooth schemes over \( k \) satisfying the following conditions: The morphisms \( S' \rightarrow S \), \( Y' \rightarrow Y \times_S S' \) and \( X' \rightarrow X \times_Y Y' \) are quasi-finite flat and dominant. The morphisms \( g': Y' \rightarrow S' \) and \( f': X' \rightarrow Y' \) are smooth of relative dimension 1 and that \( V' = V \times_Y Y' \subset Y' \) is the complement of a divisor \( D' \subset Y' \) smooth over \( k \) and quasi-finite and flat over \( S' \). The morphism \( V' \rightarrow V \times_S S' \) is etale and the morphism \( U' = X' \times_Y V' \rightarrow U \times_V V' \) is an isomorphism. The quasi-finite morphisms \( D' \rightarrow D \) and \( X' \times_Y D' \rightarrow X \times_Y D \) are dominant.

Proof. Let \( \bar{\eta} \) be a geometric point of \( S \) defined by an algebraic closure of the function field of an irreducible component. Then, it suffices to apply [19, Theorem 1.5] to the base change of \( X \rightarrow Y \rightarrow S \) by \( \bar{\eta} \rightarrow S \).

\[ \qed \]

Theorem 1.6.2. Let

\[
\begin{array}{c}
U \\ f_V
\end{array} \longrightarrow \begin{array}{c}
X \\ f
\end{array}
\]

\[
\begin{array}{c}
V \\ g
\end{array} \longrightarrow \begin{array}{c}
Y \\ S
\end{array}
\]

be a cartesian diagram of morphisms of schemes of finite type over a perfect field \( k \). Assume that \( Y \) and \( S \) are smooth over \( k \), that \( Y \rightarrow S \) is smooth of relative dimension 1 and that \( V \subset Y \) is the complement of a divisor \( D \) smooth over \( k \) and quasi-finite and flat over \( S \). Let \( \mathcal{F}_U \) be a perverse sheaf of \( \Lambda \)-modules on \( U = X \times_Y V \) such that \( U \rightarrow V \) is \( SS\mathcal{F}_U \)-transversal.

Then, there exists a commutative diagram

\[
\begin{array}{c}
V' \\ f'
\end{array} \longrightarrow \begin{array}{c}
Y' \\ g'
\end{array} \longrightarrow \begin{array}{c}
S'
\end{array}
\]

\[
\begin{array}{c}
V \\ f
\end{array} \longrightarrow \begin{array}{c}
Y \\ g
\end{array} \longrightarrow \begin{array}{c}
S
\end{array}
\]
of smooth schemes over \( k \) satisfying the following conditions (1) and (2):

1. The morphisms \( S' \to S \) and \( Y' \to Y \times_S S' \) are quasi-finite flat and dominant. The horizontal arrow \( Y' \to S' \) is smooth of relative dimension 1. The left square is cartesian and \( V' \subset Y' \) is the complement of a divisor \( D' \subset Y' \) smooth over \( k \) and quasi-finite and flat over \( S' \). The induced morphism \( V' \to V \times_S S' \) is \( \text{étale} \) and \( D' \to D \) is dominant.
2. Let

\[
\begin{array}{ccc}
U' & \xrightarrow{j'} & X' \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow & \downarrow \\
U & \xrightarrow{f} & X & \xrightarrow{i} & Y
\end{array}
\]  

be a cartesian diagram and let \( F_U' \) denote the pull-back of \( F_U \) on \( U' \). Then the morphism \( f': X' \to Y' \) is \( SSF' \)-transversal for \( F' = j'_*F_U' \).

Proof. Since the assertion is local on \( X \), we may assume that there exists a closed immersion \( i: X \to \mathbb{A}_Y^n \) for an integer \( n \geq 0 \). By replacing \( X \) and \( F_U \) by \( \mathbb{A}_Y^n \) and \( i|_{U*}F_U \) on \( \mathbb{A}_Y^n \), we may assume that \( X \) is an open subscheme of \( \mathbb{A}_Y^n \). We prove the assertion by induction on \( n \).

Assume \( n = 0 \) and hence \( X \to Y \) is an open immersion. Since the open immersion \( U \to V \) is \( SSF_U \)-transversal, the singular support \( SSF_U \) is a subset of the 0-section \( T^0_U \) by Lemma \[17\] Lemma 3.6.3]. Hence \( F_U \) is locally constant by \([2\] Lemma 2.1(iii)]. Let \( U_1 \to U \) be a finite étale covering such that the pull-back of \( F_U \) is constant. Let \( Y_1 \) be the normalization of \( Y \) in \( U_1 \). There exists a quasi-finite flat and dominant morphism \( S' \to S \) of smooth scheme such that the normalization \( Y' \to Y_1 \times_S S' \) is smooth over \( U_1 \) and that \( V' \subset Y' \) is the complement of a divisor \( D' \) over \( S' \). After shrinking \( S' \), we may assume that \( Y' \to Y \times_S S' \) is flat. After shrinking \( Y' \) keeping \( D' \) dominant over \( D \), we may assume that \( V' \to V \times_S S' \) is \( \text{étale} \). Then, the condition (1) is satisfied. Since \( F' \) on \( X' \subset Y' \) is constant, the condition (2) is also satisfied by Lemma \[14.4.1\].

Assume that \( n \geq 1 \) and that the assertion holds for \( n-1 \). For the proof of the induction step, we first show the following weaker assertion.

Claim. Let \( X \subset \mathbb{A}_Y^n \to \mathbb{A}_Y^1 \) be a projection and assume that its restriction \( U \subset \mathbb{A}_Y^n \to \mathbb{A}_Y^1 \) is \( SSF_U \)-transversal. Then, there exist a commutative diagram \[1.15\] satisfying the condition (1) and an open subset \( W' \subset \mathbb{A}_Y^1 \), satisfying the following condition:

(2’) The intersection \( W' \cap \mathbb{A}_Y^1 \) is dense in \( \mathbb{A}_D^1 \). For the cartesian diagram \[1.16\] and for the pull-back \( F_U' \) of \( F \) on \( U' \) and \( F' = j'_*F_U' \) on \( X' \), the morphism \( f': X' \to Y' \) is \( SSF' \)-transversal on the inverse image \( X' \times_{\mathbb{A}_Y^1} W' \subset X' \).

Proof of Claim. By the induction hypothesis applied to \( X \subset \mathbb{A}_Y^n \to \mathbb{A}_Y^1 \to \mathbb{A}_S^1 \), there exists a commutative diagram

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{\iota} & S_1 \\
\downarrow & & \downarrow \\
\mathbb{A}_Y^1 & \xrightarrow{\iota} & \mathbb{A}_S^1
\end{array}
\]
satisfying the conditions (1) and (2) in Theorem \textbf{1.6.2}. We consider the cartesian diagram

\[
\begin{array}{c}
X_1 \quad \xrightarrow{j_1} \quad X_0 \quad \xrightarrow{\square} \quad Y_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
X \quad \xrightarrow{\square} \quad A^1_{\mathbb{Q}}
\end{array}
\]

and let $\mathcal{F}_{U_1}$ be the pull-back of $\mathcal{F}_U$. Then, for $\mathcal{F}_1 = j_{1!} \mathcal{F}_{U_1}$ on $X_1$, the morphism $X_1 \to Y_1$ is $SS\mathcal{F}_1$-transversal. The inverse image $V_1 = V \times_S Y_1 \subset Y_1$ is the complement of a divisor $D_1 \subset Y_1$ smooth over $k$ and quasi-finite and flat over $S_1$. The quasi-finite morphism $V_1 \to V \times_S S_1$ is étale and the quasi-finite morphism $D_1 \to A^1_{\mathbb{Q}}$ is dominant.

Since the morphism $S_1 \to A^1_{\mathbb{Q}}$ is quasi-finite and flat, there exists a quasi-finite, flat and dominant morphism $S' \to S$ of smooth schemes over $k$ such that the normalization $S'_1$ of $S_1 \times_S S'$ is smooth over $S'$ and that the induced morphism $S'_1 \to S_1$ is also quasi-finite, flat and dominant. After shrinking $S'_1$ if necessary, we may assume that the morphism $Y_1 \times_S S'_1 \to A^1_{\mathbb{Q}} \times A^1_{\mathbb{Q}}$ is smooth curves over $S'_1$ is flat. Hence, by replacing $S, Y, S_1$ and $Y_1$ by $S', Y \times_S S', S'_1$ and $Y_1 \times_S S'_1$, we may assume that $S_1 \to S$ is smooth of relative dimension 1.

We consider the commutative diagram

\[
\begin{array}{c}
V_1 \quad \xrightarrow{\square} \quad Y_1 \quad \xrightarrow{\square} \quad S_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
V \quad \xrightarrow{\square} \quad Y \quad \xrightarrow{\square} \quad S
\end{array}
\]

(1.17)

where the left square is cartesian. Since $V_1 \to V \times_S S_1$ is étale, the left vertical arrow $V_1 \to V$ is also smooth of relative dimension 1. The middle vertical arrow $Y_1 \to Y$ is flat. Hence, by Lemma \textbf{1.6.1} applied to (1.17), there exists a commutative diagram

\[
\begin{array}{c}
Y'_1 \quad \xrightarrow{\square} \quad Y' \quad \xrightarrow{\square} \quad S' \\
\downarrow \quad \downarrow \quad \downarrow \\
Y_1 \quad \xrightarrow{\square} \quad Y \quad \xrightarrow{\square} \quad S
\end{array}
\]

of smooth schemes over $k$ satisfying the following conditions: The morphisms $S' \to S$, $Y' \to Y \times_S S'$ and $Y'_1 \to Y_1 \times_Y Y'$ are quasi-finite flat and dominant. The morphisms $Y' \to S'$ and $Y'_1 \to Y'$ are smooth of relative dimension 1. The inverse image $V' = V \times_Y Y'$ is the complement $Y' \subset D'$ of a divisor $D' \subset Y'$ smooth over $k$ and quasi-finite and flat over $S'$. The morphism $V' \to V \times_S S'$ is étale and the morphism $Y'_1 \times_Y Y', V' \to V_1 \times_Y V'$ is an isomorphism. The quasi-finite morphisms $D' \to D$ and $Y'_1 \times_Y D', V' \to Y_1 \times_Y D'$ are dominant. Thus the condition (1) in Theorem \textbf{1.6.2} is satisfied.

The composition $Y'_1 \to Y_1 \times_Y Y' \to A^1_{\mathbb{Q}}$ is quasi-finite and flat. We consider the cartesian diagram

\[
\begin{array}{c}
X'' \quad \xrightarrow{\square} \quad Y'_1 \\
\downarrow \quad \downarrow \\
X' \quad \xrightarrow{\square} \quad A^1_{\mathbb{Q}}, \quad \xrightarrow{\square} \quad Y'
\end{array}
\]

and the pull-back $\mathcal{F}''$ of $\mathcal{F}_1$ on $X''$. Then, since $X_1 \to Y_1$ is $SS\mathcal{F}_1$-transversal, the morphism $X'' \to Y'_1$ is $SS\mathcal{F}''$-transversal by Lemma \textbf{1.4.6}. Since $Y'_1 \to Y'$ is smooth, the
composition $X'' \to Y'$ is also $\mathcal{S}\mathcal{S}\mathcal{F}''$-transversal by Lemma \[1.4.7\]. Since $Y'_i \to A^1_{Y'}$, is quasi-finite and flat, the morphism $f': X' \to Y'$ is $\mathcal{S}\mathcal{S}\mathcal{F}'$-transversal on the image of $X''$ by the case (2) in Lemma \[1.4.9\].

Let $W' \subset A^1_{Y'}$ be the image of $Y'_i$. The image of $X'' \to X'$ equals the inverse image $X' \times_{A^1_{Y'}} W'$. Hence, $f': X' \to Y'$ is $\mathcal{S}\mathcal{S}\mathcal{F}'$-transversal on $X' \times_{A^1_{Y'}} W' \subset X'$. Since $Y'_1 \times_{Y'} D' \to Y_1 \times_Y D'$ and $D_1 \to A^1_{D'}$ are dominant, the intersection $W' \cap A^1_{D'}$ is dense in $A^1_{D'}$. Thus the condition (2') in Claim is also satisfied.

To complete the proof of the induction step, we use the following elementary lemma.

**Lemma 1.6.3.** Let $X$ be an open subset of a vector space $V$ of dimension $n$ over an infinite field $k$ regarded as a smooth scheme over $k$. Let $C \subset T^*X$ be a closed conical subset of dimension $\leq n$. Then, there exists an isomorphism $V \to A^n$ of vector spaces over $k$ such that the compositions $X \to V \to A^n \to A^1$ with the projections $p_i, i = 1, \ldots, n$ have at most isolated $C$-characteristic points.

**Proof.** Identify the cotangent bundle $T^*X$ with the product $X \times V^\vee$ with the dual and let $\mathbf{P}(C) \subset \mathbf{P}(T^*X) = X \times \mathbf{P}(V^\vee)$ be the projectivization. Then, by the assumption $\dim C \leq n$, the projection $\mathbf{P}(C) \to \mathbf{P}(V^\vee)$ is generically finite. By the assumption that $k$ is infinite, there exists a basis $p_1, \ldots, p_n$ of $V^\vee$ such that the fibers of $\mathbf{P}(C) \to \mathbf{P}(V^\vee)$ at $\bar{p}_1, \ldots, \bar{p}_n \in \mathbf{P}(V^\vee)$ are finite.

Since the projectivization $p^*_i: X \times_{A^1} T^*A^1 \to T^*X$ is the section $X \to \mathbf{P}(T^*X) = X \times \mathbf{P}(V^\vee)$ defined by $\bar{p}_i$, the closed subset of $X$ where $p_i$ is not $C$-transversal equals the image of $\bar{p}_i \times_{\mathbf{P}(V^\vee)} \mathbf{P}(C) \to X$ and is finite for each $i = 1, \ldots, n$. Hence the product of $p_1, \ldots, p_n: V \to A^1$ satisfies the condition. 

Set $C_U = \mathcal{S}\mathcal{S}\mathcal{F}_U \subset T^*U$. By the assumption that $U \subset A^n_{Y'} \to V$ is $\mathcal{S}\mathcal{S}\mathcal{F}_U$-transversal, the morphism $T^*U \to T^*U/V$ to the relative cotangent bundle is finite on $C_U$ by \[1.2\]. The image $\bar{C}_U \subset T^*U/V$ of $C_U$ and its closure $\bar{C} \subset T^*X/Y$ are closed conical subsets. Since every irreducible component of $C_U$ is of dimension $\dim X$, every irreducible component of $\bar{C}_U$ is also of dimension $\dim X$. Hence, for the generic point of each irreducible component of $Y$, the fiber of $\bar{C}_U$ is of dimension $\leq n = \dim X - \dim Y$. Consequently, for the generic point of each irreducible component of $D \subset Y$, the fiber of $\bar{C}$ is also of dimension $\leq n$.

By Lemma \[1.6.3\] applied to the fibers of the generic points of irreducible components of $D$, after replacing $S$ by a dense open subset, there exists a coordinate of $A^n_{Y'} \supset X$ such that, for each $i = 1, \ldots, n$, there exist a dense open subset $W'_i \subset A^1_{D'}$ and an open neighborhood $X_i \subset X$ of the inverse image $W'_i \times_{A^1_{D'}} X$ by the $i$-th projection $p_i$ satisfying the following condition: The inverse image of $\bar{C} \subset T^*X/Y$ by the morphism $X \times_{A^1_{Y'}} T^*A^1_{Y'}/Y \to T^*X/Y$ of the relative cotangent bundles induced by $p_i$ is a subset of the 0-section on $X_i$.

Then, the restriction $U \to A^1_{D'}$ of $p_i$ is $\mathcal{S}\mathcal{S}\mathcal{F}_U$-transversal on $X_i \cap U$. By Claim applied to the restriction $X_i \to A^1_{Y'}$ of $p_i$, there exist a commutative diagram \[1.15\] satisfying the condition (1) and for each $i = 1, \ldots, n$ a dense open subset $W'_i \subset A^1_{D'}$ satisfying the condition (2'). Hence $X' \to Y'$ is $\mathcal{S}\mathcal{S}\mathcal{F}'$-transversal on the union $W' = \bigcup_{i=1}^n p_i^{-1}W'_i \subset X' \subset A^n_{Y'}$ of the inverse images by the projections. Since $X' \to Y'$ is $\mathcal{S}\mathcal{S}\mathcal{F}'$-transversal on $U'$, it is $\mathcal{S}\mathcal{S}\mathcal{F}'$-transversal on $W' \cup U'$. By shrinking $S'$ if necessary, we may assume that $Z' = X' \setminus (W' \cup U') = \prod_{i=1}^n (A^1_{D'} \setminus (A^1_{D'} \cap W'_i)) \subset A^n_{D'}$ is quasi-finite over $S'$. 


By Corollary 1.5.6, there exists a cartesian diagram

\[
\begin{array}{ccc}
U'' & \xrightarrow{j''} & X'' \\
\downarrow & & \downarrow \\
U' & \xrightarrow{j'} & X' \\
\end{array}
\begin{array}{ccc}
\xrightarrow{f''} & \xrightarrow{f} & Y'' \\
\downarrow & & \downarrow \\
\xrightarrow{f'} & \xrightarrow{f'} & Y' \\
\end{array}
\begin{array}{ccc}
& & V'' \\
& & \downarrow \\
& & V' \\
\end{array}
\]

of smooth schemes over \( k \) satisfying the following condition: The morphism \( V'' \to V' \) is étale and \( V'' \subset Y'' \) is the complement of a divisor \( D'' \subset Y'' \) smooth over \( k \). The morphism \( D'' \to D' \) is dominant. For the pull-back \( F'' \) of \( F' \) on \( U'' \) and \( F'' = j''_* F''_{U''} \), the morphism \( f'': X'' \to Y'' \) is universally locally acyclic relatively to \( F'' \).

By Lemma 1.4.6.1 and Lemma 1.4.5.1, \( F'' \) is the pull-back of \( F' \) outside the inverse image \( Z'' \) of \( Z' \) and \( f'': X'' \to Y'' \) is \( SSF'' \)-transversal outside the inverse image \( Z'' \).

Let \( S'' \to S' \) be a quasi-finite flat dominant morphism of smooth schemes over \( k \) such that the normalization \( Y''' \) of \( Y'' \times_{S'} S'' \) is smooth over \( S'' \) of relative dimension 1 and that \( V''' = V'' \times_{Y''} Y''' \) is the complement of a divisor \( D''' \) smooth over \( k \). Let \( F''' \) be the pull-back of \( F'' \) on \( Y''' = X' \times_{Y'} Y''' \). By Proposition 1.5.2, we may assume that the morphism \( f''' = j'''_* j'''^* F''' \) is universally locally acyclic by Lemma 1.4.8.2. Further we have an isomorphism \( F''' = j'''_* j'''^* F''' \) by Lemma 1.4.5.1. Thus, the commutative diagram

\[
\begin{array}{ccc}
U''' & \xrightarrow{j'''} & X''' \\
\downarrow & & \downarrow \\
U' & \xrightarrow{j'} & X' \\
\end{array}
\begin{array}{ccc}
\xrightarrow{f'''} & \xrightarrow{f'} & Y''' \\
\downarrow & & \downarrow \\
\xrightarrow{f'} & \xrightarrow{f'} & Y' \\
\end{array}
\begin{array}{ccc}
& & S'' \\
& & \downarrow \\
& & S' \\
\end{array}
\]

where the left and middle squares are cartesian. Then the morphism \( f''' : X''' \to Y''' \) is universally locally acyclic and is \( SSF''' \)-transversal outside the inverse image \( Z''' \) of \( Z'' \) quasi-finite over \( S'' \).

Shrinking \( S'' \), we may further assume that \( X''' \to S'' \) is properly \( SSF''' \)-transversal by Lemma 1.4.8.2. Then, the morphism \( f''' : X''' \to Y''' \) is \( SSF''' \)-transversal by Lemma 1.4.8.2. Further we have an isomorphism \( F''' = j'''_* j'''^* F''' \) by Lemma 1.4.5.1. Thus, the commutative diagram

\[
\begin{array}{ccc}
V''' & \xrightarrow{f''} & Y''' \\
\downarrow & & \downarrow \\
V & \xrightarrow{f} & Y \\
\end{array}
\begin{array}{ccc}
& & S'' \\
& & \downarrow \\
& & S \\
\end{array}
\]

satisfies the required conditions.

\[ \square \]

**Corollary 1.6.4.** Let \( f : X \to Y \) be a morphism of scheme of finite type over a perfect field \( k \). Assume that \( Y \) is smooth of dimension 1. Let \( F \) be a constructible complex of \( \Lambda \)-modules on \( X \). Assume that \( f : X \to Y \) is locally acyclic relatively to \( F \) and that there exists a dense open subset \( V \subset Y \) such that \( f : V \to Y \) is \( SSF \)-transversal on \( V \). Then, there exists a cartesian diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & & \downarrow f' \\
Y & \xrightarrow{f} & Y' \\
\end{array}
\]

Therefore, the pull-back of \( F \) is a complex of \( \Lambda \)-modules on \( X' \) that is \( SSF \)-transversal. By Corollary 1.6.4, there exists a cartesian diagram
of morphisms of schemes of finite type over \( k \) satisfying the following condition: The morphism \( Y' \to Y \) is a finite generically étale morphism of smooth curves. For the pull-back \( F' \) of \( F \) on \( X' \), the morphism \( f' : X' \to Y' \) is \( SSF' \)-transversal.

**Proof.** Since the shifted vanishing cycles functor \( R\Phi[-1] \) is \( t \)-exact, we may assume that \( F \) is a perverse sheaf. Then the assertion follows from the case where \( S = \text{Spec} \ k \) in Theorem 1.6.2, Proposition 1.2.2.1 and Lemma 1.2.4. \( \square \)

## 2 Characteristic cycles and the direct image

### 2.1 Direct image of a cycle

To state the compatibility with push-forward, we fix some terminology and notations. Recall that a morphism \( f : X \to Y \) of noetherian schemes is said to be proper on a closed subset \( Z \subset X \) if its restriction \( Z \to Y \) is proper with respect to a closed subscheme structure of \( Z \subset X \).

Let \( f : X \to Y \) be a morphism of smooth schemes over a field \( k \) and we consider the diagram

\[
\begin{array}{ccc}
T^*X & \leftarrow & X \times_Y T^*Y \\
\downarrow & & \downarrow \\
T^*Y & \rightarrow & 
\end{array}
\]

as an algebraic correspondence from \( T^*X \) to \( T^*Y \). Assume that every irreducible component of \( X \) (resp. of \( Y \)) is of dimension \( n \) (resp. \( m \)). Let \( B \subset X \) be a closed subset on which \( f : X \to Y \) is proper and let \( C \subset T^*X \) be a closed subset of \( B \times_X T^*X \). Then, the closed subset \( f_0C \subset T^*Y \) is defined as the image by the right arrow in (2.1) of the inverse image of \( C \) by the left arrow. It is a closed subset by the assumption that \( f \) is proper on \( B \). The composition of the Gysin map [8, 6.6] for the first arrow and the push-forward map for the second arrow defines a morphism

\[
(2.2) \quad f_1 : CH_n(C) \longrightarrow CH_m(f_0C)
\]

since \( \dim T^*X - \dim X \times_Y T^*Y = n - m \). If every irreducible component of \( C \) (resp. \( f_0C \)) is of dimension \( \leq n \) (resp. \( \leq m \)), the morphism (2.2) defines a morphism

\[
(2.3) \quad f_1 : Z_n(C) \longrightarrow Z_m(f_0C)
\]

of free abelian groups of cycles.

**Lemma 2.1.1.** Let

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X' \\
\downarrow f & & \downarrow f' \\
Y & \rightarrow & 
\end{array}
\]

be a commutative diagram of morphisms of smooth schemes over \( k \). Assume that every irreducible component of \( X \) (resp. of \( X' \) and \( Y \)) is of dimension \( n \) (resp. \( n' \) and \( m \)). Let \( B \subset X \) be a closed subset on which \( f : X \to Y \) is proper and let \( C \subset T^*X \) be a closed subset of \( B \times_X T^*X \). Then, the diagram

\[
\begin{array}{ccc}
CH_n(C) & \xrightarrow{g_0} & CH_{n'}(g_0C) \\
\downarrow f_1 & & \downarrow f'_1 \\
& CH_m(f_0C) & 
\end{array}
\]
is commutative.

**Proof.** We consider the diagram

$$
\begin{array}{cccc}
T^*X & \leftarrow & X \times_{X'} T^*X' & \longrightarrow & T^*X' \\
\uparrow & & \uparrow & & \uparrow \\
X \times_Y T^*Y & \longrightarrow & X' \times_Y T^*Y & \longrightarrow & T^*Y
\end{array}
$$

with cartesian square. After decomposing the right vertical arrow into the composition of a smooth morphism and a regular immersion, it suffices to apply \cite[Theorem 6.2 (a)]{8].

**Lemma 2.1.2.** Let \( f: X \to Y \) be a smooth morphism of smooth irreducible schemes over a perfect field \( k \). Assume that \( X \) (resp. \( Y \)) is of dimension \( n \) (resp. \( m \)). Let \( C = \bigcup_a C_a \subset T^*X \) be a closed conical subset such that every irreducible component \( C_a \) is of dimension \( n \) and that \( f: X \to Y \) is properly \( C \)-transversal and is proper on the base \( B = C \cap T^*_X \subset X \).

Let \( A = \sum_a m_a C_a \) be a linear combination. Let \( y \in Y \) be a closed point, let \( A_y = i^!_y A \) be the pull-back \cite[Definition 7.1]{17} by the closed immersion \( i_y: X_y \to X \) of the fiber and let \( (A_y, T^*_{X_y} X_y)_{T^*Y} \) denote the intersection number. Then, we have

\[
(2.4) \quad f_*A = (-1)^m (A_y, T^*_{X_y} X_y)_{T^*Y} \cdot [T^*_Y Y]
\]

in \( Z_m(T^*_Y Y) \).

**Proof.** Since the closed immersion \( i_y: X_y \to X \) is properly \( C \)-transversal by Lemma \[1.3.6\], the pull-back \( A_y = i^!_y A \) is defined. Further by the assumption that \( f: X \to Y \) is \( C \)-transversal, we have an inclusion \( f_0 C \subset T^*_Y Y \) and \( f_*A \) is defined as an element of \( \text{CH}_m(f_0 C) = Z_m(f_0 C) \subset Z_m(T^*_Y Y) \). Hence it suffices to show that the coefficient of \( T^*_Y Y \) in \( f_*A \) equals the intersection number \((-1)^m (A_y, T^*_{X_y} X_y)_{T^*Y} \).

We consider the cartesian diagram

$$
\begin{array}{cccc}
T^*X & \leftarrow & X \times_Y T^*Y & \longrightarrow & T^*Y \\
\uparrow & & \uparrow & & \uparrow \\
X_y \times_X T^*X & \leftarrow & X_y \times_Y T^*Y & \longrightarrow & y \times_Y T^*Y \\
\downarrow & & \downarrow & & \downarrow \\
T^*X_y & \leftarrow & X_y & \longrightarrow & y.
\end{array}
$$

We regard the four sides of the exterior square of the diagram as algebraic correspondences. Since \( f \) is assumed properly \( C \)-transversal, \( f_*A \) on the upper right corner is some multiple of the 0-section \( T^*_Y Y \). Hence, the coefficient of \( f_*A \) is the image of \( A \) by the composition via the upper right corner. It equals the composition via the lower left corner by \cite[Theorem 6.2 (a)]{8} applied to the upper right and the lower left squares. The image of \( A \) on the lower left corner is \((-1)^m\)-times \( i^!_y A \) since the definition of \( i^!_y A \) in \cite[Definition 7.1]{17} involves the sign \((-1)^{\dim X - \dim X_y} \). Since the bottom line defines the intersection number \((-1)^m (A_y, T^*_{X_y} X_y)_{T^*Y} \), the assertion follows.

We study the case where \( Y \) is a smooth curve and \( \dim f_0 C = 1 \). Let \( f: X \to Y \) be a morphism of smooth schemes over \( k \). Assume that every irreducible component of \( X \)
(resp. of \(Y\)) is of dimension \(n\) (resp. 1). Let \(C \subset T^*X\) be a closed conical subset such that every irreducible component \(C_a\) of \(C = \bigcup_a C_a\) is of dimension \(n\) and that \(f: X \to Y\) is proper on the base \(B = C \cap T^*_X X \subset X\). Let \(V \subset Y\) be a dense open subscheme such that the base change \(f_V: X_V \to V\) is properly \(C_V\)-transversal for the restriction \(C_V\) of \(C\) on \(X_V\).

Let \(y \in Y \subset V\) be a closed point on the boundary and let \(t\) be a uniformizer at \(y\) and let \(df\) denote the section of \(T^*X\) defined on a neighborhood of the fiber \(X_y\) by the pull-back \(f^*dt\). Then, on a neighborhood of \(X_y\), the intersection \(C \cap df \subset T^*X\) is supported on the inverse image of the intersection \(B \cap X_y\). Hence for a linear combination \(A = \sum_a m_a C_a\), the intersection product

\[(A, df)_{T^*X, X_y}\]

supported on the fiber \(X_y\) is defined as an element of \(\text{CH}_0(B \cap X_y)\). Since \(C\) is conical, the intersection product \((A, df)_{T^*X, X_y}\) does not depend on the choice of \(t\). Thus the intersection number also denoted \((A, df)_{T^*X, X_y}\) is defined as its image by the degree mapping \(\text{CH}_0(B \cap X_y) \to \text{CH}_0(y) = \mathbb{Z}\).

**Lemma 2.1.3.** Let \(f: X \to Y\) be a morphism of smooth irreducible schemes over a perfect field \(k\). Assume that \(X\) (resp. of \(Y\)) is of dimension \(n\) (resp. 1). Let \(C = \bigcup_a C_a \subset T^*X\) be a closed conical subset as in Lemma 2.1.2.

1. The following conditions are equivalent:
   (1) \(\dim f_\circ C \leq 1\).
   (2) There exists a dense open subscheme \(V \subset Y\) such that the base change \(f_V: X_V \to V\) is \(C_V\)-transversal for the restriction \(C_V\) of \(C\) on \(X_V\).
   (3) There exists a dense open subscheme \(V \subset Y\) such that the base change \(f_V: X_V \to V\) is properly \(C_V\)-transversal for the restriction \(C_V\) of \(C\) on \(X_V\).

2. Let \(V \subset Y\) be a dense open subscheme satisfying the condition (3) above. Let \(A = \sum_a m_a C_a\) be a linear combination, let \(v \in V\) be a closed point and define the intersection number \((A_v, T^*_X X_v)_{T^*X_v}\) as in Lemma 2.1.2. Then, we have

\[f_\circ A = -(A_v, T^*_X X_v)_{T^*X_v}\cdot [T^*_v Y] + \sum_{y \in Y \subset V} (A, df)_{T^*X, X_y}\cdot [T^*_y Y]\]

in \(Z_1(f_\circ C)\).

**Proof.** 1. Since \(f_\circ C\) is a closed conical subset of the line bundle \(T^*Y\), the condition (1) is equivalent to the existence of a dense open subset \(V \subset Y\) such that \(f_\circ C \subset T^*_Y Y\) \(\cup \bigcup_{y \in Y \subset V} T^*_y Y\). This is equivalent to the condition (2). The equivalence (2) \(\Leftrightarrow\) (3) follows from Lemma 1.4.5.2.

2. It suffices to compare the coefficients of the 0-section \(T^*_v Y\) and of the fibers \(T^*_y Y\) respectively. For those of \(T^*_v Y\), it is proved in Lemma 2.1.2. For those of \(T^*_y Y\), it follows from the projection formula \([S, \text{Theorem 6.2 (a)}]\) applied to the cartesian square in the commutative diagram

\[
\begin{array}{ccc}
T^*X & \xrightarrow{df} & X \times_Y T^*Y \\
\downarrow & & \downarrow \\
X & \rightarrow & Y.
\end{array}
\]
Lemma 2.1.4. Let $X$ be a scheme of finite type of dimension $d$ over a field $k$ and let $E$ be a vector bundle on $X$ associated to a locally free $O_X$-module $E$ of rank $n$. Let $s: X \to E$ be a section, $0: X \to E$ be the zero section and $Z = Z(s) = 0(X) \cap s(X) \subset X$ be the zero locus of $s$. Let $\mathcal{K} = [O_X \to \mathcal{E}]$ be the complex of $O_X$-modules where $\mathcal{E}$ is put on degree 0 and let $c_{n-1}(\mathcal{K})$ be the localized Chern class defined in [1] Section 1. Then, we have

$$(0(X), s(X))_E = c_{n-1}(\mathcal{K}) \cap [X]$$

in $\text{CH}_{d-n}(Z)$.

Proof. We may assume that $X$ is integral and $Z \subseteq X$. By taking the blow-up at $Z$ and by [14] Proposition 2.3.1.6, we may assume that $Z$ is a Cartier divisor $D \subset X$. Then, we have an exact sequence of $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ of locally free $O_X$-modules where $\mathcal{L}$ and $\mathcal{F}$ are of rank 1 and $n-1$ respectively and $s \in \Gamma(X, \mathcal{E})$ is defined by $s \in \Gamma(X, \mathcal{L})$. Then, the right hand side equals $c_{n-1}(\mathcal{F}) \cap [D]$ by [14] Proposition (1.1) (iii)]. The left hand side also equals $c_{n-1}(\mathcal{F}) \cap [D]$ by the excess intersection formula [8] Theorem 6.3. □

We define the specialization of a cycle. Let $f: X \to Y$ be a smooth morphism of smooth schemes over a perfect field $k$ and assume that $X$ (resp. $Y$) is equidimensional of dimension $n+1$ (resp. 1). Let $y \in Y$ be a closed point, $V = Y \setminus \{y\}$ be the complement and $U = X \times_Y V$ be the inverse image. Let $C \subset T^*U$ be a closed conical subset equidimensional of dimension $n+1$ and assume that $U \to V$ is properly $C$-transversal. We define its specialization

$$\text{sp}_y C \subset T^*X_y$$

as follows. By the assumption that $U \to V$ is properly $C$-transversal and [17] Lemma 3.1], the morphism $T^*U \to T^*U/V$ to the relative cotangent bundle is finite on $C$. Hence its image $C' \subset T^*U/V$ is a closed conical subset. Let $\tilde{C}' \subset T^*X/Y$ be the closure and define $\text{sp}_y C \subset T^*X_y$ to be the fiber $\tilde{C}' \times_Y y \subset T^*X/Y \times_Y y = T^*X_y$. The specialization $\text{sp}_y C \subset T^*X_y$ is a closed conical subset equidimensional of dimension $n$.

For a linear combination $A = \sum_a m_a C_a$ of irreducible components of $C = \bigcup_a C_a$, we define its specialization

$$\text{sp}_y A \in Z_n(\text{sp}_y C)$$

as follows. First, we define $A' \in Z_{n+1}(C')$ as the push-forward of $A$ by the morphism $T^*U \to T^*U/V$ finite on $C$. Let $A' \in Z_{n+1}(\tilde{C}')$ be the unique element extending $A' \in Z_{n+1}(C')$. Then, we define $\text{sp}_y A \in Z_n(\text{sp}_y C)$ to be the minus of the pull-back of $A'$ by the Gysin map for the immersion $i_y: X_y \to X$. If $X \to Y$ is proper, for a closed point $v \in V$ and the closed immersion $i_v: X_v \to X$, we have

$$(2.7) \quad (\text{sp}_y A, T^*_{X_y}X_y)_{T^*X_y} = (i_v^t A, T^*_{X_v}X_v)_{T^*X_v}$$

since the definition of $i_v^t A$ in [17] Definition 7.1] involves the sign $(-1)^{\dim X - \dim X_v} = -1$.

Lemma 2.1.5. Let $f: X \to Y$ be a smooth morphism of smooth schemes over a perfect field $k$ and assume that $X$ (resp. $Y$) is equidimensional of dimension $n+1$ (resp. 1). Let $y \in Y$ be a closed point, $i_y: X_y \to X$ be the closed immersion of the fiber, $V = Y \setminus \{y\}$ be the complement and $U = X \times_Y V$ be the inverse image. Let $C \subset T^*X$ be a closed conical subset equidimensional of dimension $n+1$ such that $f: X \to Y$ is properly $C$-transversal.

1. For the restriction $C_U$ of $C$ on $U$, we have

$$(2.8) \quad \text{sp}_y C_U = i_y C.$$

31
2. For a linear combination \( A = \sum a_m C_a \) of irreducible components of \( C = \bigcup_a C_a \) and its restriction \( A_U \) on \( U \), we have

\[
\text{sp}_y A_U = i_U^! A.
\]

\textbf{Proof.} 1. By the assumption that \( f : X \to Y \) is properly \( C \)-transversal and [17, Lemma 3.1], the morphism \( T^*X \to T^*X/Y \) to the relative cotangent bundle is finite on \( C \) and hence its image \( C' \subset T^*X/Y \) is a closed conical subset. Further \( C' \) with reduced scheme structure is flat over \( Y \). Hence it equals the closure of the restriction \( C'_U \) and we obtain [2.8].

2. We consider the cartesian diagram

\[
\begin{array}{ccc}
T^*X & \longrightarrow & T^*X/Y \\
\uparrow & & \uparrow \\
X_y \times_X T^*X & \longrightarrow & T^*X_y.
\end{array}
\]

The right hand side is the minus of the image of \( A \) by the push-forward and the pull-back via upper right. The left hand side is the minus of the image of \( A \) by the pull-back and the push forward via lower left. Hence the assertion follows from the projection formula [8, Theorem 6.2 (a)].

\section{2.2 Characteristic cycle of the direct image}

Let \( k \) be a field and let \( \Lambda \) be a finite field of characteristic \( \ell \) invertible in \( k \). Let \( X \) be a smooth scheme over \( k \) such that every irreducible component is of dimension \( n \). Let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules on \( X \) and \( C = \text{SS} \mathcal{F} \) be the singular support. Then, every irreducible component \( C_a \) of \( C = \bigcup_a C_a \) has the same dimension as \( X \) [2, Theorem 1.3 (ii)] and the base \( B = C \cap T^*_X X \subset T^*_X X = X \) defined as the intersection with the 0-section equals the support of \( \mathcal{F} \) [2, Lemma 2.1 (i)]. Let \( f : X \to Y \) be a morphism of smooth schemes over \( k \), proper on the support of \( \mathcal{F} \). Then, we have an inclusion

\[
\text{SSR} f_* \mathcal{F} \subset f_* \text{SS} \mathcal{F}
\]

by [2] Lemma 2.2 (ii)].

We restate a conjecture from [18, Conjecture 1].

\textbf{Conjecture 2.2.1.} Let \( f : X \to Y \) be a morphism of smooth schemes over a perfect field \( k \). Assume that every irreducible component of \( X \) (resp. of \( Y \)) is of dimension \( n \) (resp. \( m \)). Let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules on \( X \) and \( C = \text{SS} \mathcal{F} \) be the singular support. Assume that \( f \) is proper on the support of \( \mathcal{F} \). Then, we have

\[
\text{CCR} f_* \mathcal{F} = f_! \text{CC} \mathcal{F}
\]

in \( \text{CH}_m(f_! \text{SS} \mathcal{F}) \).

If \( \dim f_! \text{SS} \mathcal{F} \leq m \), the equality (2.11) is an equality as cycles in \( \text{CH}_m(f_! \text{SS} \mathcal{F}) = Z_m(f_! \text{SS} \mathcal{F}) \) without rational equivalence.

A weaker version of Conjecture 2.2.1 is proved in the case \( k \) is finite and \( X \) and \( Y \) are projective in [21] using \( \varepsilon \)-factors.
If $Y = \text{Spec} \ k$, the equality (2.11) means the index formula

$$\chi(X_k, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*_X}$$

where the right hand side denotes the intersection number. Further if $X$ is projective, the equality (2.12) is proved in [17, Theorem 7.13].

**Lemma 2.2.2.** Let $f: X \to Y$ be a morphism of smooth schemes over $k$ and let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules. Assume that $f: X \to Y$ is proper on the support of $\mathcal{F}$. Assume that every irreducible component of $X$ (resp. of $Y$) is of dimension $n$ (resp. $m$).

1. Let

$$X \xrightarrow{g} X' \xrightarrow{f'} Y$$

be a commutative diagram of morphisms of smooth schemes over $k$. Then, we have

$$f_!CC\mathcal{F} = f'_!(g_!CC\mathcal{F})$$

in $\text{CH}_m(f_!SS\mathcal{F})$.

2. Assume that one of the following conditions (1) and (2) is satisfied:

   (1) $f: X \to Y$ is an immersion.
   (2) $f: X \to Y$ is quasi-projective and $SS\mathcal{F}$-transversal.

Then, we have $\dim(f_!SS\mathcal{F}) \leq m = \dim Y$ and

$$\text{CCR} f_* \mathcal{F} = f_!CC\mathcal{F}$$

in $\text{Z}_m(f_!SS\mathcal{F})$.

*Proof.* 1. It follows from Lemma 2.1.1.

2. The case (1) is proved in [17, Lemma 5.13.2]. We show the case (2). Since $f_!SS\mathcal{F}$ is a subset of the 0-section $T^*_Y Y$, we have $\dim(f_!SS\mathcal{F}) \leq m = \dim Y$. We may assume that $Y$ is connected and affine and hence $X$ is quasi-projective. Let $X \to P$ be an immersion to a projective space and factorize $X \to Y$ as the composition of an immersion $X \to Y \times P$ and the projection $Y \times P \to Y$. Then, by 1 and the case (1), we may assume that $f: X \to Y$ is projective and smooth.

By the assumption that $f: X \to Y$ is $SS\mathcal{F}$-transversal, it is locally acyclic relatively to $\mathcal{F}$ by Lemma [1.4.4]. Since $f_!SS\mathcal{F}$ is proper, the direct image $Rf_* \mathcal{F}$ is locally constant by [11, 5.2.4]. By Lemma [1.4.5.2], there exists a dense open subscheme $V \subset Y$ such that $f: X \to Y$ is properly $SS\mathcal{F}$-transversal on $V$. By [17, Lemma 5.11.1] and Lemma 2.1.2, it suffices to show the equality

$$\text{rank} \ Rf_* \mathcal{F} = (i_y^!CC\mathcal{F}, T^*_{X_y} X_y)_{T^*_{X_y}}$$

for a closed point $y \in V$. Since $\text{rank} \ Rf_* \mathcal{F} = \chi(X_y, i_y^* \mathcal{F})$, the equality (2.14) follows from the compatibility $CCi_y^* \mathcal{F} = i_y^!CC\mathcal{F}$ with the pull-back [17, Theorem 7.6] and the index formula [17, Theorem 7.13].

We consider the case where $Y$ is a smooth curve and $\dim f_!SS\mathcal{F} \leq 1$. We recall the definition of the Artin conductor and the description of the characteristic cycle of a
sheaf on a curve. Let $Y$ be a smooth irreducible curve over a perfect field $k$ and let $G$ be a constructible complex of $\Lambda$-modules on $Y$. Let $V \subset Y$ be a dense open subscheme such that the restriction $G_V$ is locally constant i.e. the cohomology sheaf $H^qG_V$ is locally constant for every integer $q$. For a closed point $y \in Y$, the Artin conductor $a_yG$ is defined by

$$a_yG = \text{rank } G_V - \text{rank } G_{\bar{y}} + \text{Sw}_yG.$$  

(2.15)

Here $\bar{y}$ denotes a geometric point above $y$ and $\text{Sw}_yG$ denotes the alternating sum of the Swan conductor. The characteristic cycle is given by

$$CCG = \left(\text{rank } G_V \cdot [T^*_Y Y] + \sum_{y \in Y \setminus V} a_yG \cdot [T^*_y Y]\right)$$  

(2.16)

by [17, Lemma 5.11.3]. Here $T^*_y Y$ is the fiber of $T^*Y$ at $y$.

Let $f: X \to Y$ be a morphism of smooth schemes over a perfect field $k$ and $y \in Y$ be a closed point. Assume that $\dim Y = 1$. Let $F$ be a constructible complex of $\Lambda$-modules on $X$. Assume that $f: X \to Y$ is proper on the support of $F$ and is properly $SSF$-transversal on a dense open subscheme $V \subset Y$. Then, we have

$$-a_yRf_*F = (CCF, df)_{T^*X, X_y},$$

(2.17)

for every closed point $y \subset Y \setminus V$ by (2.16), Lemma 2.2.2.2 (2) and Lemma 2.1.3.2, where the right hand side is defined as in (2.5).

**Theorem 2.2.3.** Let $f: X \to Y$ be a quasi-projective morphism of smooth schemes over a perfect field $k$ and $y \in Y$ be a closed point. Assume that $\dim Y = 1$. Let $F$ be a constructible complex of $\Lambda$-modules on $X$. Assume that $f: X \to Y$ is proper on the support of $F$ and is properly $SSF$-transversal on a dense open subscheme $V \subset Y$. Then, we have

$$-a_yRf_*F = (CCF, df)_{T^*X, X_y}.$$  

(2.17)

*Proof.* We may assume that $k$ is algebraically closed. By the same argument as in the proof of Lemma 2.2.2, we may assume that $f: X = Y \times P \to Y$ is the projection for a projective space $P$. By Lemma 2.1.3.1 and by replacing $Y$ by a projective smooth curve over $k$ containing $Y$ as a dense open subscheme, we may assume that $Y$ is projective and smooth.

By Lemma 2.2.2 applied to $X \to Y \to \text{Spec } k$, we obtain

$$(f_!CCF, T^*_Y Y)_{T^*Y} = (CCF, T^*_X X)_{T^*X}.$$  

By the index formula [17, Theorem 7.13], we have

$$(CCRf_*F, T^*_Y Y)_{T^*Y} = \chi(Y_k, Rf_*F) = \chi(X_k, F) = (CCF, T^*_X X)_{T^*X}.$$  

Thus, we have

$$(CCRf_*F - f_!CCF, T^*_Y Y)_{T^*Y} = 0.$$  

34
Since the coefficients of $T^*_Y Y$ in $CCR f_* F$ and $f! CCF$ are equal by (2.14), (2.16) and the index formula [17, Theorem 7.13], we obtain

\[(2.18) \quad \sum_{y \in Y - V} -a_y Rf_* F = \sum_{y \in Y - V} (CCF, df)_{T^*X, X_y}.
\]

By dévissage using Lemma [1.4, 4.3] and [17, Lemma 5.13.1], we may assume that $F$ is a perverse sheaf. Set $\hat{V} = V \cup \{y\}$ and $Z = Y - \hat{V}$. By Corollary [1.2.3] Corollary [1.6.4] and Lemma [1.2.4] there exists a faithfully flat finite morphism $Y' \to Y$ of projective smooth curves étale at $y$ satisfying the following condition: Let

\[
\begin{array}{ccc}
X & \leftarrow & X' \\
\hat{f} & \downarrow & \hat{f}' \\
Y & \leftarrow & \hat{V}' = Y' \times_Y \hat{V}
\end{array}
\]

be a cartesian diagram and set $\mathcal{F}' = \hat{f}'_* F_{\hat{V}_Y}$ for the pull-back $F_{\hat{V}_Y}$ of $F$ on $X'_{\hat{V}_Y}$. Then on $Y'_0 = Y' \to Y'$, the morphism $\hat{f}' : X' \to Y'$ is $SS\mathcal{F}'$-transversal and hence is universally locally acyclic relatively to $\mathcal{F}'$.

For each $y' \in Z' = Z \times_Y Y'$, we have $a_{y'} R\hat{f}'_* \mathcal{F}' = (CCF', df')_{T^*X', y'} = 0$. Since $Y' \to Y$ is étale at $y$, for each $y' \in Y' \times_Y y$, we have $a_{y'} R\hat{f}'_* \mathcal{F}' = a_y Rf_* F = a_y Rf_* F'$ and $(CCF, df)_{T^*X, X_y} = (CCF', df')_{T^*X', y'}$. Thus, by applying (2.18) to $\hat{f}' : X' \to Y'$ and $\mathcal{F}'$, we obtain

\[-[Y' : Y] \cdot a_y Rf_* F = [Y' : Y] \cdot (CCF, df)_{T^*X, X_y}
\]

and hence (2.17).

**Corollary 2.2.4** (cf. [4 Conjecture]). Let $f : X \to Y$ be a projective flat morphism of smooth schemes over a perfect field $k$. Assume that $\dim X = n$, $\dim Y = 1$ and that there exists a dense open subscheme $V \subset Y$ such that the base change $f_V : X \times_Y V \to V$ is smooth. Then, for a closed point $y \in V$, we have

\[(2.19) \quad -a_y Rf_* \Lambda = (-1)^n c_n X_y^*(\Omega_{X/Y}^1) \cap [X].
\]

**Proof.** Applying Theorem 2.2.3 to the constant sheaf $\mathcal{F} = \Lambda$ and $CCA = (-1)^n [T^*_X X]$, we obtain $-a_y Rf_* \Lambda = (-1)^n (T^*_X X, df)_{T^*X, X_y}$. By applying Lemma 2.1.4 to the right hand side and $[f^* \Omega_{Y/k}^1 \to \Omega_{X/k}^1]$, we obtain (2.19). 

[4 Conjecture] is stated for proper morphism. Here we need to assume $f$ to be projective since the index formula is known only for projective schemes.

**Theorem 2.2.5.** Let $f : X \to Y$ be a morphism of smooth schemes over a perfect field $k$. Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$ and $C = SS\mathcal{F}$ be the singular support. Assume that $Y$ is projective, that $f : X \to Y$ is quasi-projective and is proper on the support of $\mathcal{F}$ and that we have an inequality

\[(2.20) \quad \dim f_* C \leq \dim Y = m.
\]

Then, we have

\[(2.21) \quad CCR f_* F = f! CCF
\]
in $Z_m(f_* SS\mathcal{F})$. 

35
Proof. We may assume that \(k\) is algebraically closed. Since \(X\) is quasi-projective, there exists a locally closed immersion \(i: X \to P\) to a projective space \(P\). By decomposing \(f\) as the composition of the immersion \((i, f): X \to P \times Y\) and the second projection \(P \times Y \to Y\), we may assume that \(f\) is projective and smooth by Lemma 2.2.2. Set \(C = f_* SS\mathcal{F} \subset T^*Y\). We have \(SSR_{f_* \mathcal{F}} \subset f_* SS\mathcal{F} = C\). By the assumption, we have \(\dim C \leq m\). By the index formula [17] Theorem 7.13 and Theorem 2.2.3 the equality (2.21) is proved for \(Y\) of dimension \(\leq 1\). We show the general case by reducing to the case \(\dim Y = 1\).

We take a closed immersion of \(Y\) to a projective space \(i: Y \to P\). We use the notations \(\overrightarrow{P} \leftarrow Q \overrightarrow{P^\vee}\) in (1.7) and let \(p_X: X \times_P Q \to X\) be the projection. After replacing the immersion \(i\) by the composition with a Veronese embedding if necessary, we may assume that the restriction to \(P(i_0 C) \subset Q = P(T^*P)\) of the projection \(p^\vee: Q \to P^\vee\) is generically radial by the assumption \(\dim C \leq m = \dim Y\) and by [17] Corollary 3.21. Let \(C^\vee = p^\vee_0 p^\vee_1 C \subset T^*P\) and let \(D\) denote the image \(p^\vee(P(i_0 C)) \subset P^\vee\). By Lemma 1.3.7, Lemma 1.3.3 and the Bertini theorem, there exists a line \(L \subset P^\vee\) satisfying the following conditions: The immersion \(h: L \to P^\vee\) is properly \(C^\vee\)-transversal. The morphism \(h: L \to P^\vee\) meets \(p^\vee_0 SS\mathcal{F}\) properly. The axis \(A_L\) of \(L\) meets \(Y\) transversely and \(L\) meets \(D\) transversely.

Since the blow-up of \(P\) at \(A_L\) is the projection \(Q \times_{P^\vee} L \to P\) and since \(A_L\) meets \(Y\) transversely, the blow-up \(Y'\) of \(Y\) at \(Y \cap A_L\) fits in the cartesian diagram

\[
\begin{array}{c}
Y \leftarrow Y \times_P Q \leftarrow Y' \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
P \leftarrow Q \leftarrow Q \times_{P^\vee} L \\
\downarrow \quad \downarrow \quad \downarrow \\
P^\vee \leftarrow L
\end{array}
\]

and is smooth over \(k\). We consider the cartesian diagram

\[
\begin{array}{c}
X \leftarrow X \times_P Q \leftarrow X' \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
Y \leftarrow Y \times_P Q \leftarrow Y' \\
\downarrow \quad \downarrow \quad \downarrow \\
P^\vee \leftarrow L
\end{array}
\]

of projective smooth schemes over \(k\).

By the identification \(P(T^*P \times_P Y) \subsetneq P(T^*P) = Q = P(T^*P^\vee)\), the operation \(p^\vee_0 p^\vee_1 = p^\vee_0 p^\vee_1 i_0\) defines an injection from the set of irreducible closed conical subsets of dimension \(Y\) of \(T^*Y\) and to the set of irreducible closed conical subsets of \(T^*P^\vee\). Hence the equality (2.21) is equivalent to

\[
p^\vee_1 p^\vee_1 CCR_{f_* \mathcal{F}} = p^\vee_1 p^\vee_1 f_* CC\mathcal{F}.
\]

It suffices to compare the coefficients of \(C^\vee_a = p^\vee_0 p^\vee_0 C_a \subset T^*P^\vee\) for each irreducible component of \(C = \bigcup_a C_a\) of dimension \(m = \dim Y\).
Since the restriction $P(i_0C) \to D$ of the projection $p^\vee: Q \to \P^\vee$ is radicial on a dense open subset, irreducible components $P(i_0C_a)$ of $P(i_0C)$ correspond uniquely with irreducible components $D_a$ of $D$ defined as the images. Since $L$ meets $D$ transversely, each point of the intersection $L \cap D$ is contained in exactly one irreducible component $D_a$ of $D$. Hence, (2.23) is further equivalent to

$$
(2.24) \quad h^!p^\vee_!p^\vee_Y CCRf_*F = h^!p^\vee_Y p^\vee_Y f_iCCF.
$$

Let $\pi_X: X' \to X$ denote the composition $p_X \circ h_X$ of the top line in (2.22). We show that the equality (2.24) is further equivalent to

$$
(2.25) \quad CCR(p_L f'_*)^!\pi_X^*F = (p_L f'_*)^!CC\pi_X^*F.
$$

First, we compare the left hand sides. By [17] Corollary 7.12 applied to $i_*Rf_*F$ on $P$, the left hand side of (2.24) equals $h^!CCRP_Y^\vee p^\vee_Y Rf_*F$. Since $SSRp^\vee_Y p^\vee_Y Rf_*F \subset C^\vee$ and since $h: L \to \P^\vee$ is properly $C^\vee$-transversal, the left hand side further equals $CCR\pi_Y^*p^\vee_Y Rf_*F$ by [17] Theorem 7.6]. By proper base change theorem, this is equal to the left hand side $CCR(p_L f'_*)^!\pi_X^*F$ of (2.25).

Next, we compare the right hand sides. The right hand side of (2.24) is equal to $(p_L f'_*)^!\pi_X^*CCF$ by the projection formula [8] Theorem 6.2 (a)]. Since $h: L \to \P^\vee$ is $C^\vee$-transversal and $C^\vee = p_Y^\vee \circ p^\vee_C = (p_Y^\vee \circ f^!) \circ p^\vee_X SSF$, the immersion $h_X: X' \to X \times \P Q$ is $p^\vee_X SSF$-transversal by Lemma [1.3.6] 2. Further since $h: L \to \P^\vee$ meets $p^\vee_X SSF$ properly, the immersion $h_X: X' \to X \times \P Q$ is properly $p^\vee_X SSF$-transversal. Since $p_X$ is smooth, the composition $\pi_X = p_X \circ h_X$ is properly $SSF$-transversal. Thus by [17] Theorem 7.6], it further equals to the right hand side $(p_L f'_*)^!CC\pi_X^*F$ of (2.25).

We show the equality (2.25) by applying Theorem 2.2.3 to complete the proof. The largest open subset where the projection $p_Y^\vee: Y \times \P Q \to \P^\vee$ is $p^\vee_C$-transversal equals the largest open subset where the immersion $Y \times \P Q \to Y \times \P^\vee$ is $C \times \T^\vee$-transversal and hence is the complement of $P(i_0C) \subset Y \times \P Q = P(Y \times \P T^\vee P)$. Since $p_Y^\vee C = p_Y^\vee f_0 SSF = f_0 p^\vee_X SSF = f_0 SSF p^\vee_X F$, the composition $p^\vee_Y f: X \times \P Q \to \P^\vee$ is $SSF p^\vee_X F$-transversal on the complement $P^\vee - D$ by [17] Lemma 3.8 (1) (2)]. By Lemma [1.3.6] the morphism $p_L f'_*: X' \to L$ is $SSF\pi_X^*F$-transversal on the dense open subset $L = L \cap D$. Hence the equality (2.25) follows from Theorem 2.2.3 applied to $\pi_X^*F$ and the equality (2.21) is proved.

To prove Theorem 2.2.6 we need the assumption that $Y$ is projective, because we assume in Theorem 2.2.3 that $f$ loc. cit. is projective on the support of $F$. If we can weaken this assumption on $f$ in Theorem 2.2.3 we will be able to remove the assumption on $Y$ in Theorem 2.2.5 since the characteristic cycle is characterized by the Milnor formula (1.5).

In the case of characteristic 0, we recover the classical result as in [13] Proposition 9.4.2], under the extra assumption that $Y$ is projective and that $f$ is quasi-projective. Let $X$ be a smooth scheme equidimensional of dimension $n$ over a field $k$ and let $\omega_X \in \Omega^2(T^*X)$ denote the canonical symplectic form on the cotangent bundle $T^*X$. Let $C \subset T^*X$ be a closed conical subset. We say that $C$ is isotropic if the restriction of $\omega_X$ on $C$ is 0. We say that $C$ is Lagrangian if it is isotropic and if $C$ is equidimensional of dimension $n$.

**Lemma 2.2.6.** Let $k$ be a field of characteristic 0 and let $f: X \to Y$ be a morphism of smooth schemes over $k$. Assume that $X$ (resp. $Y$) is equidimensional of dimension $n$ (resp.
Let $C \subset T^*X$ be a closed conical subset. If $C \subset T^*X$ is isotropic, then $f_* C \subset T^*Y$ is also isotropic.

The author learned the following proof from Beilinson.

**Proof.** Let $T^*_p(X \times Y) \subset T^*(X \times Y)$ be the normal bundle of the graph $\Gamma \subset X \times Y$ of $f: X \to Y$ and let $p_2: T^*(X \times Y) = T^*X \times T^*Y \to T^*Y$ be the projection. The direct image $f_* C \subset T^*Y$ equals the image by $p_2$ of the intersection $C_1 = T^*_p(X \times Y) \cap (C \times T^*Y)$.

Since the normal bundle $T^*_p(X \times Y) \subset T^*(X \times Y)$ is isotropic and since $\omega_{X \times Y}$ equals the sum $p^*_1 \omega_X + p^*_2 \omega_Y$ of the pull-backs by projections, the assumption that $C \subset T^*X$ is isotropic implies that the restriction of $p^*_2 \omega_Y$ on $C_1$ is 0. Since $k$ is of characteristic 0, for each irreducible component $C'$ of $f_* C \subset T^*Y$, there exists a closed subset $C'_1 \subset C_1$ generically étale over $C'$. Hence the assertion follows.

**Proposition 2.2.7.** Let $k$ be a field of characteristic 0 and let $X$ be a smooth scheme over $k$. Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$.

1. The singular support $SS \mathcal{F}$ is Lagrangian.
2. Let $f: X \to Y$ be a morphism of smooth schemes over $k$. Assume that $f$ is proper on the support of $\mathcal{F}$. Then, the inequality \( (2.20) \) holds. Further if $f: X \to Y$ is quasi-projective, the equality \( (2.21) \) holds.

**Proof.** 1. We may assume that $X$ is equidimensional of dimension $n$. Since the singular support $SS \mathcal{F}$ is equidimensional of dimension $n$ [2, Theorem 1.3 (ii)], it suffices to show that $SS \mathcal{F}$ is isotropic. By devissage, we may assume that there exist a locally closed immersion $i: V \to X$ of smooth scheme, a locally constant sheaf $\mathcal{G}$ on $V$ and $\mathcal{F} = i_! \mathcal{G}$.

Since the resolution of singularity is known in characteristic 0, the immersion $i$ is decomposed by an open immersion $j: V \to W$ and a proper morphism $h: W \to X$ such that $W$ is smooth and $V$ is the complement of a divisor with simple normal crossings. Thus, by the inclusion $SS \mathcal{F} = SSRh_*j_! \mathcal{G} \subset h_! SSj_! \mathcal{G}$ and Lemma [2.2.6], it is reduced to the case where $i = j$ is an open immersion of the complement of a divisor with simple normal crossings. Since $k$ is of characteristic 0, this case is proved in [17, Proposition 4.11].

2. By 1 and Lemma [2.2.6], the direct image $f_* SS \mathcal{F}$ is isotropic. Hence the inequality $\dim f_* SS \mathcal{F} \leq \dim Y$ \( (2.20) \) holds.

We show the equality $CCRf_* \mathcal{F} = f_!CC \mathcal{F}$ \( (2.21) \). Similarly as in the proof of Theorem [2.2.5], we may assume that $Y$ is affine and $f: X = P \times Y \to Y$ is the projection for a projective smooth scheme $P$ over $k$. By resolution of singularity, we may assume that $Y$ is projective and smooth. Then since the inequality \( (2.20) \) holds, we may apply Theorem [2.2.5].

### 2.3 Index formula for vanishing cycles

We prepare some notation to formulate an index formula for vanishing cycle complex. Let $f: X \to Y$ be a smooth morphism of smooth schemes over a perfect field $k$. Assume that $X$ (resp. $Y$) is equidimensional of dimension $n + 1$ (resp. 1). Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$. Let $y \in Y$ be a closed point and $i_y: X_y \to X$ be the closed immersion of the fiber. Assume that $f: X \to Y$ is properly $SS \mathcal{F}$-transversal on the complement $X \setminus X_y$ of the fiber $X_y = f^{-1}(y)$. Then, the specialization

\[
sp_y SS \mathcal{F} \subset T^*X_y
\]

(2.26)
is defined as a closed conical subset equidimensional of dimension $n$. Further, the specialization
\[
(2.27) \quad \text{sp}_y CC\mathcal{F} \in Z_n(\text{sp}_y SS\mathcal{F})
\]
is defined as a cycle.

**Lemma 2.3.1.** Let $f: X \to Y$ be a smooth morphism of smooth schemes over a field $k$ and assume $\dim Y = 1$. Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$ and assume that $f: X \to Y$ is properly $SS\mathcal{F}$-transversal. Let $y \in Y$ be a closed point. Then, we have
\[
(2.28) \quad SSR\Psi_y \mathcal{F} = \text{sp}_y SS\mathcal{F}.
\]
Further if $k$ is perfect, we have
\[
(2.29) \quad CCR\Psi_y \mathcal{F} = \text{sp}_y CC\mathcal{F}.
\]

**Proof.** Let $i_y: X_y \to X$ denote the closed immersion of the fiber. Then, by the assumption that $f: X \to Y$ is properly $SS\mathcal{F}$-transversal, we have $\text{sp}_y SS\mathcal{F} = i_y^* SS\mathcal{F}$ and $\text{sp}_y CC\mathcal{F} = i_y^* CC\mathcal{F}$. Recall that the definitions of $\text{sp}_y$ and $i_y^*$ both involve the minus sign.

Since $f: X \to Y$ is locally acyclic relatively to $\mathcal{F}$ by Lemma 1.4.12, the canonical morphism $i_y^* \mathcal{F} \to R\Psi_y \mathcal{F}$ is an isomorphism. Hence the equalities (2.28) and (2.29) follow from Lemma 1.4.1 and [17, Theorem 7.6] respectively.

The following example shows that the inclusion $SSR\Psi \mathcal{F} \subset \text{sp}_y SS\mathcal{F}$ does not hold in general.

**Example 2.3.2.** Let $k$ be a field of characteristic $p > 2$. Let $X = \mathbb{A}^1 \times \mathbb{P}^1$ and $j: U = \mathbb{A}^1 \times \mathbb{A}^1 = \text{Spec } k[x,y] \to X$ be the open immersion. Let $\mathcal{G}$ be the locally constant sheaf of $\Lambda$-modules of rank 1 on $U$ defined by the Artin-Schreier covering $t - t = x^p y$ and by a non-trivial character $\mathcal{F}_p \to \Lambda^\times$. Then, the nearby cycles complex $R\Psi_{X_y} \mathcal{F}$ is acyclic except at the closed point $(0, \infty)$ or at degree 1 and $\dim R^1 \Psi_{X_y} \mathcal{F}(0, \infty) = 1$. Hence, the singular support $SSR\Psi_{X_y} \mathcal{F}$ equals the fiber $T^*_{(0, \infty)} X_{\infty}$ at the closed point and is not a subset of the zero-section $\text{sp}_{\infty} SS\mathcal{F} = T^*_{X_{\infty}} X_{\infty}$.

Let $Z \subset X_y$ be a closed subset. Assume that $f: X \to Y$ is properly $SS\mathcal{F}$-transversal on the complement of $Z$. Then, on the complement $X_y = Z$, we have $\text{sp}_y CC\mathcal{F} = i_y^* CC\mathcal{F} = CCI_y^* \mathcal{F}$ by Lemma 2.1.5 and the compatibility with the pull-back [17, Theorem 7.6]. Thus, the difference
\[
(2.30) \quad \delta_y CC\mathcal{F} = \text{sp}_y CC\mathcal{F} - CCI_y^* \mathcal{F}
\]
is defined as a cycle in $Z_n(\mathbb{Z} \times X (\text{sp}_y SS\mathcal{F} \cup SSi_y^* \mathcal{F}))$ supported on $Z$. If $Z$ is proper over $Y$, the intersection number $(\delta_y SS\mathcal{F}, T^*_{X_y} X_y)_{T^* X_y}$ is defined.

**Proposition 2.3.3.** Let $f: X \to Y$ be a smooth morphism of smooth schemes over a perfect field $k$. Assume that $X$ (resp. $Y$) is equidimensional of dimension $n + 1$ (resp. 1). Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$. Let $y \in Y$ be a closed point and let $Z \subset X_y$ be a closed subset. Assume that $f: X \to Y$ is properly $SS\mathcal{F}$-transversal on the complement of $Z$ and that either of the following conditions (1) and (2) is satisfied:

1. $f: X \to Y$ is projective.
2. $\dim Z = 0$.

Then, for the vanishing cycles complex $R\Phi_y \mathcal{F}$, we have
\[
(2.31) \quad \chi(Z_k, R\Phi_y \mathcal{F}) = (\delta_y CC\mathcal{F}, T^*_{X_y} X_y)_{T^* X_y}.
\]
Proof. We may assume that \( k \) is algebraically closed.

We show the case (1). Let \( v \in Y \) be a closed point different from \( y \) and let \( i_v : X_v \to X \) be the closed immersion. Then, since the projective morphism \( f : X \to Y \) is locally acyclic relative to \( \mathcal{F} \) outside \( Z \) by Lemma 1.4.4.2, the left hand side of (2.31) equals

\[
(2.32) \quad \chi(Z, R\Phi_y \mathcal{F}) = \chi(X_y, R\Psi_y \mathcal{F}) - \chi(X_y, i_y^* \mathcal{F}) = \chi(X_v, i_v^* \mathcal{F}) - \chi(X_y, i_y^* \mathcal{F})
\]

The right hand side of (2.31)

\[
(\delta_y CC \mathcal{F}, T_{X_y} X_y)_{T^* X_y} = (sp_y CC \mathcal{F}, T_{X_y} X_y)_{T^* X_y} - (CCI_y^* \mathcal{F}, T_{X_y} X_y)_{T^* X_y}
\]

equals

\[
(2.33) \quad (i_v^! CC \mathcal{F}, T_{X_v} X_v)_{T^* X_v} - (CCI_y^* \mathcal{F}, T_{X_y} X_y)_{T^* X_y}
\]

by (2.7). Since \( i_v : X_v \to X \) is properly \( SS \mathcal{F} \)-transversal by Lemma 1.3.6, the right hand side of (2.32) equals (2.33) by the compatibility with the pull-back [17, Theorem 7.6] and the index formula [17, Theorem 7.13]. Thus the equality (2.31) is proved.

We show the case (2). Since the formation of nearby cycles complex commutes with base change by [7, Proposition 3.7], we may assume that the action of the inertia group \( I_y \) on \( R\Psi_y \mathcal{F} \) is trivial. Since the vanishing cycles functor is \( t \)-exact by [12, Corollaire 4.6], we may assume that \( \mathcal{F} \) is a simple perverse sheaf.

First, we consider the case \( \mathcal{F} \) is supported on the closed fiber \( X_y \). By the assumption that \( f : X \to Y \) is properly \( SS \mathcal{F} \)-transversal on the complement of \( Z \), the morphism \( f : X \to Y \) is locally acyclic relatively to \( \mathcal{F} \) on the complement of \( Z \). Thus \( \mathcal{F} \) is supported on \( Z \) and the assertion follows in this case.

We may assume that the restriction \( \mathcal{F}|_{X_y} \) on the generic fiber is non-trivial. Then, by Proposition 1.2.2.2, the morphism \( f : X \to Y \) is locally acyclic relatively to \( \mathcal{F} \). Hence by Lemma 1.4.8.2, the morphism \( f : X \to Y \) is properly \( SS \mathcal{F} \)-transversal and the assertion follows from Lemma 2.3.1.

In the case (2) \( \dim Z = 0 \), Proposition 2.3.3 means \( CCR\Phi_y \mathcal{F} = \delta_y CC \mathcal{F} \). However, Examples 1.5.8 and 2.3.2 show that one cannot expect to have \( CCR\Psi_y \mathcal{F} = sp_y CC \mathcal{F} \) or equivalently \( CCR\Phi_y \mathcal{F} = \delta_y CC \mathcal{F} \) in general.

References

[1] M. Artin, Théorème de finitude pour un morphisme propre; dimension cohomologique des schémas algébriques affines, SGA 4 Exposé XIV, Théorie des Topos et Cohomologie Étale des Schémas, Lecture Notes in Mathematics Volume 305, 1973, pp 145-167.

[2] A. Beilinson, Constructible sheaves are holonomic, Selecta Math. New Ser. 22, Issue 4, (2016) 1797-1819.

[3] A. Beilinson, J. Bernstein, P. Deligne, O. Gabber, Faisceaux pervers, 2e éd. Astérisque 100, (2018).

[4] S. Bloch, Cycles on arithmetic schemes and Euler characteristics of curves, Algebraic geometry, Bowdoin, 1985, 421-450, Proc. Symp. Pure Math. 46, Part 2, Am. Math. Soc., Providence, RI (1987).
[5] A. J. de Jong, *Smoothness, semi-stability and alterations*, Publ. Math. IHÉS, 83 (1996), 51-93.

[6] P. Deligne, *Cohomologie à supports propres*, SGA 4 Exposé XVII, Théorie des Topos et Cohomologie Étale des Schémas, Lecture Notes in Mathematics Volume 305 (1972), 250–480.

[7] ——, *Théorèmes de finitude en cohomologie ℓ-adique*, Cohomologie étale, SGA 4¼, Springer Lecture Notes in Math. 569, (1977), 233–251.

[8] W. Fulton, *Intersection Theory*, 2nd edition (1998) Springer.

[9] A. Grothendieck, *Éléments de géométrie algébrique IV*, Étude locale des schémas et des morphismes de schémas, Publ. Math. IHES 20, 24, 28, 32 (1964-67).

[10] H. Hu, E. Yang, *Relative singular support and the semi-continuity of characteristic cycles for étale sheaves*, Selecta Math., 2018, 24-3, pp. 2235-2273.

[11] L. Illusie, *Appendice à Théorèmes de finitude en cohomologie ℓ-adique*, Cohomologie étale SGA 4¼, Springer Lecture Notes in Math. 569 (1977) 252–261.

[12] ——, *Autour du théorème de monodromie locale*, Astérisque 223 (1994), 9-57, Périodes p-adiques (Bures-sur-Yvette, 1988).

[13] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*, Springer-Verlag, Grundlehren der Math. Wissenschaften 292, (1990).

[14] K. Kato, T. Saito, *On the conductor formula of Bloch*, Publ. Math., IHES 100 (2004), 5-151.

[15] N. Katz, G. Laumon, *Transformation de Fourier et majoration de sommes exponentielles*, Publ. Math. IHÉS (1985) 62, pp 145-202.

[16] F. Orgogozo, *Modifications et cycles évanescents sur une base de dimension supérieure à un*, Int. Math. Res. Notices, (2006) No. 13, Article ID 25315, 1-38.

[17] T. Saito, *The characteristic cycle and the singular support of a constructible sheaf*, Inventiones Math. 207(2) (2017), 597-695.

[18] ——, *On the proper push-forward of the characteristic cycle of a constructible sheaf*, Proceedings of Symposia in Pure Mathematics, 97.2, 2018, pp. 485-494.

[19] M. Temkin, *Stable modification of relative curves*, J. of Algebraic Geometry, 19 (2010) 603-677.

[20] B. Toen and G. Vezzosi, *The ℓ-adic trace formula for dg-categories and Bloch’s conductor conjecture*, Boll. Unione Mat. Ital. 12 (2019), no. 1-2, 3-17.

[21] N. Umezaki, E. Yang, Y. Zhao, *Characteristic class and the ε-factor of an étale sheaf*, arXiv:1701.02841