Quantum Interference Effects in the Detection Probability of Charged Leptons Produced in Charged Current Weak Interactions

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Abstract

The phases of quantum interference effects in charged lepton production by neutrinos (neutrino oscillations) following pion decay at rest and in flight, muon decay and nuclear $\beta$-decay at rest as well as for ‘muon oscillations’ following pion decay at rest and in flight are calculated. The same phase is found for neutrino and muon oscillations following pion decay at rest. The results found disagree with the conventionally used value: $\phi_{12} = \Delta m^2 L/(2P)$. Differences to previous treatments are briefly commented on.

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This letter sketches briefly a derivation of spatially dependent interference effects (conventionally called ‘neutrino oscillations’) in the detection probability of charged leptons produced in neutrino interactions following the weak decays of unstable ‘source’ particles: pions, muons and $\beta$–radioactive nuclei. Similar effects for the detection probability of the decay products of muons produced in pion decay (‘muon oscillations’) are also considered. The calculations are based on Feynman’s path amplitude formulation of quantum mechanics [1] in which the probability of transition from a set of initial states $I = \sum_{i} i_{t}$ to a set of final states $F = \sum_{m} f_{m}$ is given by the relation:

$$ P_{FI} = \sum_{m} \sum_{l} \left| \sum_{k_{1}} \sum_{k_{2}} \cdots \sum_{k_{n}} (f_{m} | k_{1} \rangle \langle k_{1} | k_{2} \rangle \cdots \langle k_{n} | i_{t} \rangle \right|^{2} $$

where $k_{j}, j = 1, n$ are (unobserved) intermediate quantum states. Only the essential elements of the calculation and the results are presented here. Full details are given elsewhere [2].

The application of Eqn(1) to pion decay at rest is illustrated in the ideal experiment shown in Fig.1. A $\pi^{+}$ comes to rest in a stopping target T at time $t_{0}$ as recorded by the counter $C_{A}$ (Fig.1a)). The pion at rest constitutes the initial state of the path amplitudes. In Fig.1b) and Fig.1c) are shown two alternative histories of the stopped pion. In Fig.1b) the pion decays at time $t_{1}$ into the neutrino mass eigenstate $|\nu_{1} \rangle$, of mass $m_{1}$, and in Fig.1c) into the neutrino mass eigenstate $|\nu_{2} \rangle$, of mass $m_{2}$, at the later time $t_{2}$. If
$m_1 > m_2$, then, for a suitable choice of the times $t_1$ and $t_2$, interference between the path amplitudes corresponding to the different physical processes shown in Fig.1b) and Fig.1c) will occur when a neutrino interaction ($\nu_1, \nu_2)n \rightarrow e^- p$ takes place at time $t_D$ and distance $L$ from the pion decay point (Fig.1d)). The final state of the neutrino interaction event is also that of the path amplitudes. For this experiment, the path amplitudes corresponding to the two alternative histories of the decaying pion are:

$$A_i = <e^-p|T_R|n\nu_i> U_{ei}D(x_f - x_i, t_D - t_i, m_i) U_{i\mu}$$

$$<\nu_i\mu^+|T_R|\pi^+ > e^{-\frac{ip}{m}(t_i-t_0)}D(0, t_i - t_0, m_\pi) \quad i = 1, 2$$

(2)

Here $U_{ai}$ is the Maki-Nakagawa-Sakata (MNS) [3] matrix element giving the charged current coupling strength of the charged lepton of flavour $\alpha$ ($\alpha = e, \mu, \tau$) to the neutrino mass eigenstate $\nu_i$. The ‘reduced’ matrix elements $<e^-p|T_R|n\nu_i>$ and $<\nu_i\mu^+|T_R|\pi^+ >$ are given by setting $U_{ai} = 1$ in the charged current, so that, for example, $<e^-p|T|n\nu_i > = U_{ei} <e^-p|T_R|n\nu_i >$. Since the purely kinematical effects of non-vanishing neutrino masses are expected to be negligible, then, to a very good approximation, $|\nu_1 >$ and $|\nu_2 >$ may be replaced, in the reduced matrix elements, by the massless neutrino wavefunction $|\nu_0 >$. The former are then independent of neutrino flavour. In the case of a $2 \times 2$ MNS matrix for the first two generations of leptons, the unitarity of the matrix implies that it is completely defined by a single real parameter, conventionally chosen to be an angle, $\theta$, such that:

$$U_{e1} = U_{1e} = U_{\mu 2} = U_{2\mu} = \cos \theta$$

$$U_{e2} = U_{2e} = -U_{\mu 1} = -U_{1\mu} = \sin \theta$$

(3)

(4)

In Eqn(2), $m_\pi$ and $\Gamma_\pi$ are the pion pole mass and decay width and $D$ is the Lorentz invariant configuration space propagator [4, 5] of a neutrino or the pion. In the limit of large time-like separations, or of on-shell particles, appropriate to the experiment shown in Fig.1, $D \simeq \exp[-im\Delta \tau]$ where $m$ is the pole mass of the particle and $\Delta \tau$ the increment of proper time corresponding to the path. In the following, the additional functional dependence $\simeq (m/\Delta \tau)^{3/2}$ of $D$ in the asymptotic region (leading to solid angle correction factors) is neglected. With these approximations:

$$D(\Delta x, \Delta t, m) \simeq \exp[-im\sqrt{(\Delta t)^2 - (\Delta x)^2}]$$

$$= \exp[-im\Delta \tau]$$

$$\equiv \exp[-i\Delta \phi]$$

(5)

The phase increments, $\Delta \phi$, corresponding to the paths of the neutrinos and the pion in the amplitudes $A_i$ are:

$$\Delta \phi_i^\nu = m_i \Delta \tau_i = \frac{m_i^2}{E_i} \Delta t_i = \frac{m_i^2}{p_i} L$$

$$\Delta \phi_i^\pi = m_\pi (t_i - t_0) = m_\pi (t_D - t_0) - \frac{m_\pi L}{v_i}$$

$$\simeq m_\pi (t_D - t_0) - m_\pi L \left(1 + \frac{m_i^2}{2P_0^2} \right)$$

(6)

(7)
where \( m_i, E_i, P_i \) and \( v_i \) are the mass, energy, momentum and velocity of the mass eigenstate \( \nu_i \) and

\[
P_0 = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = 29.8 \text{MeV}
\]  

(8)

The neutrinos are assumed to follow classical rectilinear trajectories such that \( \Delta t_i = L/v_i = E_i L/P_i \) and the time dilatation formula \( \Delta t = \gamma \Delta \tau = E \Delta \tau / m \) has been used in Eqn(6). In Eqn(7) the neutrino velocity\(^a\) \( v_i \) is expressed in terms of the neutrino mass to order \( m_i^2 \). Using Eqns(5-7), neglecting terms of \( O(m^4) \) and higher, and replacing the massive neutrino wavefunctions in the reduced matrix elements by those of massless neutrinos, the path amplitudes of Eqn(2) may be written as:

\[
A_i = \langle e^- p | T_R | m_\nu > U_{e\nu} U_{\nu \mu} < \nu_\mu \mu^+ | T_R | \pi^+ > \exp [i \phi_0 - \frac{\Gamma_\pi}{2} (t_D - t_0 - t^f_i)] \exp \left[ i \frac{m_i^2 - m_\mu^2}{P_0} \left( \frac{m_\pi}{2P_0} - 1 \right) L \right] i = 1, 2
\]  

(9)

where the neutrino times-of-flight \( t^f_i = t_D - t_i \) have been introduced and

\[
\phi_0 \equiv m_\pi (L - t_D + t_0)
\]  

(10)

Using now Eqn(1) to calculate the transition probability, and integrating over the detection time \( t_D \) [2], gives, for the probability to observe the reaction \( (\nu_1, \nu_2)m \to e^- p \) at distance \( L \) from the decay point:

\[
P(e^- p | L) = C_N(\nu; \pi) \sin^2 \theta \cos^2 \theta (1 - F^\nu(\Gamma_\pi) \cos \phi_{12}^{\nu,\pi})
\]  

(11)

where

\[
\phi_{12}^{\nu,\pi} = \frac{\Delta m^2}{P_0} \frac{m_\pi}{2P_0} \left( \frac{m_\pi}{2P_0} - 1 \right) L = \frac{2m_\pi m_\mu^2 \Delta m^2 L}{(m_\pi^2 - m_\mu^2)^2}
\]  

(12)

\[
F^\nu(\Gamma_\pi) = \exp \left( -i \frac{\Gamma_\pi m_\mu^2}{2m_\mu^2} \phi_{12}^{\nu,\pi} \right)
\]  

(13)

and Eqns(3) and (4) have been used. Here \( C_N(\nu; \pi) \) is an \( L \) independent normalisation factor and \( \Delta m^2 \equiv m_1^2 - m_2^2 \). The first and second terms in the second member of Eqn(12) are the contributions of \( \Delta \phi^\pi_i \) and \( \Delta \phi^\nu_i \) to the interference phase. The latter is a factor of two larger than in the conventional value [6, 7] \( \phi_{12} = \Delta m^2 L/(2P) \). \( \Delta \phi^\pi_i \) gives a numerically large \( (m_\pi/2P_0 = 2.34) \) contribution to the oscillation phase.

The above calculation is readily repeated for the case of detection of muon decay \( \mu^+ \to e^+ (\nu_1, \nu_2) (\pi_1, \pi_2) \) at distance \( L \) from the \( \pi^+ \) decay point. The phase increments analogous to Eqns(6) and (7) are:

\[
\Delta \phi^\mu_i = \frac{m_\pi^2 L}{P_0} \left[ 1 + \frac{m_\mu^2 E^\mu_0}{2m_\pi^2 P_0^2} \right]
\]  

(14)

\[
\Delta \phi^{\pi(\mu)}_i = m_\pi (t_D - t_0) - \frac{m_\pi L}{v^\mu_0} \left[ 1 + \frac{4m_\pi^2 m_\mu^2}{(m_\pi^2 - m_\mu^2)^2(m_\pi^2 + m_\mu^2)} \right]
\]  

(15)

where

\[
E^\mu_0 = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} \quad \text{and} \quad v^\mu_0 = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}
\]  

\(^a\)Units with \( \hbar = c = 1 \) are used.
The result found for the time-integrated decay probability is:

\[ P(e^+\nu\bar{\nu}|L) = C_N(\mu; \pi)(1 - F^\nu(\Gamma_{\pi})\sin 2\theta \cos \phi_{12}^{\nu,\pi}) \]  

(16)

where

\[ \phi_{12}^{\nu,\pi} = \frac{m_\mu^2 \Delta m^2}{2P_{\pi}^2} \left( 1 - \frac{E_0}{m_\pi} \right) L = \frac{2m_\pi m_\mu^2 \Delta m^2 L}{(m_\pi^2 - m_\mu^2)^2} = \phi_{12}^{\nu,\pi} \]

(17)

\[ F^\nu(\Gamma_{\pi}) = \exp \left( -\frac{\Gamma_{\pi} m_\pi}{(m_\pi^2 - m_\mu^2) \phi_{12}^{\nu,\pi}} \right) \]

(18)

The first and second terms in the second member of Eqn(17) are the contributions of \( \Delta \phi_i^{\pi(\mu)} \) and \( \Delta \phi_i^\nu \) to the interference phase. It is interesting to note that neutrino and muon oscillation phases are the same for given values of \( \Delta m^2 \) and \( L \). For oscillation phases \( \phi_{12}^{\nu,\pi} = \phi_{12}^{\mu,\pi} = 1 \), the damping factors of the oscillation term, due to the non-vanishing pion lifetime take the values \( F^\nu(\Gamma_{\pi}) = 1 - 1.58 \times 10^{-16} \) and \( F^\mu(\Gamma_{\pi}) = 1 - 4.4 \times 10^{-16} \). This damping effect is thus completely negligible in typical neutrino oscillation experiments with oscillation phases of order unity.

Formulae like Eqn(11) have been derived in a similar manner \cite{2} for ‘\( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \)’ oscillations \footnote{If the neutrinos are massive, the ‘lepton flavour eigenstates’ \( \bar{\nu}_e \) and \( \bar{\nu}_\mu \) are unphysical. However the phrase ‘\( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) oscillations’ is still a useful and compact way of describing experiments where anti-neutrinos, produced in association with a muon, give rise to a detection event containing an electron. The amplitudes for all physical processes contain, however, only the physical states \( \nu_i \).} following muon decay at rest, detected via the process \((\bar{\nu}_1, \bar{\nu}_2)p \rightarrow e^+n \) and ‘\( \bar{\nu}_e \rightarrow \bar{\nu}_e \)’ oscillations following nuclear \( \beta-e \)-decay, detected via the same process. The results found, for the time-integrated decay probabilities, are:

\[ P(e^+n, \mu|L) = C_N(\bar{\nu}; \mu) \sin^2 \theta \cos^2 \theta \left[ 1 - \cos \frac{\Delta m^2}{P_{\bar{\nu}}} \left( \frac{m_\mu}{2P_{\bar{\nu}}} - 1 \right) L \right] \]

(19)

\[ P(e^+n, \beta|L) = C_N(\bar{\nu}; \beta) \left[ \sin^4 \theta + \cos^4 \theta + 2 \sin^2 \theta \cos^2 \theta \cos \frac{\Delta m^2}{P_{\bar{\nu}}} \left( \frac{E_\beta}{2P_{\bar{\nu}}} - 1 \right) L \right] \]

(20)

In these formulae, \( P_{\bar{\nu}} \) is the momentum of the detected \( \bar{\nu}_1, \bar{\nu}_2 \), \( E_\beta \) is the total energy release in the \( \beta-e \)-decay process, and damping corrections due to the finite lifetimes of the decaying particles are neglected.

Finally, in Ref. \cite{2}, the cases of neutrino and muon oscillations following the decay in flight of ultrarelativistic pions were considered. The oscillation probability formulae are the same as Eqns(11) and (16) respectively, with the phases:

\[ \phi_{12}^{\nu,\pi} \text{ (in flight)} = \frac{m_\mu^2 \Delta m^2 L}{(m_\pi^2 - m_\mu^2)E_\nu \cos \theta_\nu} \]

(21)

\[ \phi_{12}^{\mu,\pi} \text{ (in flight)} = \frac{2m_\mu^2 \Delta m^2(m_\mu^2E_{\pi} - m_\pi^2E_\mu)L}{(m_\pi^2 - m_\mu^2)^2 E_\mu \cos \theta_\mu} \]

(22)

The neutrinos or muons are detected at distance \( L \) from the decay point, along the direction of flight of the parent pion, with decay angles \( \theta_\nu \) and \( \theta_\mu \) respectively.

Because the parent pion and the daughter muon in the decay process \( \pi \rightarrow \mu \nu \) are unstable particles, their physical masses \( W_\pi \) and \( W_\mu \) have distributions depending, through Breit-Wigner amplitudes, on their pole masses \( m_\pi \) and \( m_\mu \) and decay widths \( \Gamma_{\pi} \) and \( \Gamma_{\mu} \).
Energy-momentum conservation in the pion decay process then leads to a distribution of path amplitudes with different neutrino momenta, an effect neglected in the above discussion. In Ref.[2] corrections resulting from the (coherent) momentum smearing due to $W_\mu$ and (incoherent) smearing due to $W_\pi$ are calculated using a Gaussian approximation for the Breit-Wigner amplitudes. The resulting damping corrections due to $W_\pi$ to the interference terms in Eqns(11) and (16) are found to be vanishingly small. For $\Delta m^2 = (1\text{eV})^2$ and $L = 30\text{m}$ (typical of the LNSD [8] or KARMEN [9] experiments) the corresponding damping factors are found to be $1 - 1.3 \times 10^{-29}$ (for neutrino oscillations) and $1 - 5.2 \times 10^{-30}$ (for muon oscillations) [2]. The damping corrections from the variation of $W_\mu$ are even smaller.

Corrections to Eqns(11) and (16) due to thermal motion of the decaying pion and finite target and detector sizes have also been evaluated in Ref.[2]. For neutrino oscillations with $\Delta m^2 = (1\text{eV})^2$ and $L = 30\text{m}$ and a room-temperature target, the damping factor of the interference term is found to be $1 - 6.7 \times 10^{-10}$ and the shift in the oscillation phase to be $1.2 \times 10^{-9}\text{rad}$. In summary, all known sources of damping of the interference terms in Eqn(10) and (15) are expected to be completely negligible in any foreseeable neutrino or muon oscillation experiment.

The first published calculation of the neutrino oscillation phase [10] gave a result (only the contribution of the neutrino propagators was taken into account) in agreement with Eqn(6) above. A later calculation [11] assumed instead that the phase of the neutrino propagator evolves as $Et$, i.e. according to the non-relativistic Schrödinger equation. The assumptions were also made that the different neutrino mass states are produced at the same time and have equal momenta. The first assumption leads to the ‘standard’ formula for the oscillation phase: $\phi_{12} = \Delta m^2 L/(2P)$ [6, 7] which has subsequently been used for the analysis of all neutrino oscillation experiments. The Lorentz-invariant phase of Eqn(6) $\simeq m^2 t/E$ evidently agrees with the result of Ref.[11] in the non-relativistic limit where $E \simeq m$, but such a limit is clearly inappropriate to describe experiments with ultra-relativistic neutrinos.

The most important difference in the treatment given in the present paper to previous ones that have appeared in the literature is allowing the possibility for the different neutrino mass eigenstates to be produced at different times. Only in this way can the constraints, of both space-time geometry (the detection event is at a unique space-time point), and exact energy-momentum conservation in the decay process [12], be satisfied.

The other new feature is the inclusion of the important contribution to the oscillation phase from the propagator of the decaying particle, a necessary consequence of the different production times of the different mass eigenstates.

In the standard treatment, the common production time of the mass eigenstates follows from the assumption that a ‘neutrino flavour eigenstate’, that is a coherent superposition of mass eigenstates, is produced in the decay of the source particle. It has been shown that the production of such states in pion decay is incompatible with the measured branching ratio $\Gamma(\pi \to e\nu)/\Gamma(\pi \to \mu\nu)$ and the values of the MNS matrix elements deduced from atmospheric and solar neutrino oscillations [13]. Actually, as first pointed out by Shrock [14, 15], as a consequence of the Standard Model structure of the charged lepton current:

$$J_\mu(CC)^{\text{ lept}} = \sum_{\alpha,i} \bar{\psi}_\alpha \gamma_\mu (1 - \gamma_5) U_{\alpha i} \psi_{\nu_i}$$  \hspace{1cm} (23)
the different neutrino mass eigenstates are produced incoherently in different physical processes. This is the basic assumption of the calculations presented above. Clearly, in this case, the different mass eigenstates may be produced at different times in the (classically) alternative histories corresponding to the different path amplitudes. It has also been recently demonstrated [16] that the factor of two difference in the contribution of neutrino propagation to the oscillation phase between the calculation presented above and the standard formula is a necessary consequence of the assumption of equal production times for all mass eigenstates and hence equal space-time (as contrasted to kinematical \(^{c}\)) velocities. In the same paper [16] it is also shown that different kinematical hypotheses that lead to different, but unequal, kinematical velocities (equal momenta, equal energies, or exact energy-momentum conservation in the production process) differ in their predictions for the oscillation phase only by terms of \(O(m_4^4)\), and so are all equivalent at \(O(m_2^2)\).

It may be remarked that the physical interpretation of ‘neutrino oscillations’ provided by the path amplitude description is different from the conventional one in terms of ‘lepton flavour eigenstates’. In the latter, the amplitudes of different lepton flavours in the neutrino are supposed to vary harmonically as a function of time. In the amplitudes for the different physical processes in the path amplitude treatment there is, instead, no variation of lepton flavour in the propagating neutrinos. Only in the detection process itself, the associated MNS matrix elements give different coupling strengths between the different neutrino mass eigenstates and the final state charged lepton and the interference effect occurs between the different path amplitudes that is described as ‘neutrino oscillations’. In the case of the observation of the recoil muons no such ‘lepton flavour projection’ occurs in the detection event, but exactly similar interference effects are predicted to occur. As previously emphasised [17], the ‘flavour oscillations’ of neutrinos, neutral kaons and b-mesons are just special examples of the universal phenomenon of quantum mechanical superposition that is the physical basis of Eqn (1).

The most important practical conclusion of the work presented here is that identical information on neutrino masses is given by the observation of either neutrinos or muons from pion decay at rest. In view of the possibility of detecting muons simply and with essentially 100% efficiency, in contrast to the tiny observable event rates of neutrino interactions, the recent indications for ‘\(\bar{\nu}_\mu \rightarrow \bar{\nu}_e\)’ oscillations with \(\Delta m^2 \simeq (1\text{eV})^2\) following \(\mu^+\) decay at rest [8] could be easily checked by a search for muon oscillations following \(\pi^+\) decay at rest. For \(\Delta m^2 \simeq (1\text{eV})^2\), the first absolute maximum of the interference term in Eqn (16) occurs at \(L \simeq 8m\). Note that Eqn(16) is valid for any muon detection process that does not distinguish between events where \(\nu_1\) or \(\nu_2\) are produced.

\(^{c}\)The ‘kinematical’ velocity, \(v_{\text{kin}}\), is defined as \(v_{\text{kin}} = p/E\).
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Figure 1: The space-time description of an experiment in which neutrinos produced in the processes $\pi^+ \rightarrow \mu^+(\nu_1, \nu_2)$ are detected at distance, $L$, via the processes $(\nu_1, \nu_2)n \rightarrow e^- p$ (see text).