Comparison of Spectral Index Determinations

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ABSTRACT

The index $n$ of a power law power spectrum of primordial density fluctuations, $P(k) \propto k^n$, has been estimated using many different techniques. The most precise compare the COBE DMR large angular scale $\Delta T$ to the amplitude of the large scale structure, but these are also the most model-dependent. The COBE DMR $\Delta T$ has also been compared to the degree-scale $\Delta T$ from several experiments. And finally, a relatively model-independent value of $n$ can derived from the COBE data alone, but the small range of angular scales covered by COBE limits the precision of these methods.

1. Introduction

In this paper I compare several different methods for determining the spectral index $n$ of the power spectrum of primordial density perturbations. All of the determinations that use COBE data are statistically compatible with the $n \approx 1$ predicted by the inflationary scenario. Because the largest scales appear as large angular scale features on the finite solid angle of sky that is available for viewing, the statistical uncertainties in the determination of $n$ cannot be conquered by the usual expedient of getting more data. A careful consideration of the statistical methods used to analyze the large-angular scale COBE data is needed.

2. Biased Statistics

As an example of the pitfalls of statistical analysis, consider the maximum likelihood method applied to determining the standard deviation of a Gaussian from a set of $N$ independent samples drawn from the distribution. The likelihood function is

$$L(\mu, \sigma) = (\sqrt{2\pi} \sigma)^{-N} \prod_{i=1}^{N} \exp[-0.5(x_i - \mu)^2/\sigma^2]$$

which is maximized at $\mu = N^{-1} \sum x_i$ and $\sigma^2 = N^{-1} \sum_i (x_i - \mu)^2$. While the sample mean is an unbiased estimate of the population mean, the variance estimate is biased by a factor of
Fig. 1.— Likelihood contours vs $Q^2$ and $n$, based on the 2 year $53 \times 90$ cross power spectrum for $3 \leq \ell \leq 30$.

$(N - 1)/N$, illustrating the fact that maximum likelihood estimators are only asymptotically unbiased. Even with an unbiased estimate of $\sigma$, the logarithm of an unbiased estimate of $\sigma$ is not an unbiased estimate of the logarithm of $\sigma$ because noise rectification by the second derivative of the logarithm leads to a bias of $-1/4N$. Since the value of $N$ when estimating the power in the $\ell$’th multipole is

$$N \approx (2\ell + 1)\Omega_{\text{obs}}/4\pi,$$

and since $\Omega_{\text{obs}} < 8\pi/3$ due to galactic contamination, these biases will be most significant for the low $\ell$’s measured by COBE. Thus any method for determining $n$ using COBE data should be tested using simulated data to calibrate these biases.

One non-recommended technique for determining $n$ is to treat the integrated likelihood $f(n) = \int L(Q, n) dQ$ as a probability density for $n$. The usual justification for this is the Bayesian rule that the probability density for $Q$ and $n$ after the experiment, $p_a(Q, n)$, is given by

$$p_a(Q, n) \propto p_p(Q, n)L(Q, n)$$

(2)
Assuming a “uniform” prior to represent prior ignorance then gives the form in Equation 2. But a uniform prior in $Q^2$ is not the same as a uniform prior in $\ln Q$, and they give different values of $n$. Different ways of expressing our prior ignorance should not affect the answer. A more dramatic example is shown in Figure 4 and Figure 2, each showing the likelihood contours for the Hauser-Peebles cross power spectrum of the 53 × 90 2 year maps. In one case they are plotted versus $Q^2 = T_2^2$, while in the other case they are plotted versus the relative mass fluctuations in $8h^{-1}$ Mpc spheres, $\sigma_8 \propto T_{650}$. The Jacobian of the transformation between $(Q^2, n)$ and $(T_{650}, n)$ depends on $n$, so a uniform prior in $(Q^2, n)$ becomes an $n$-dependent prior in $(T_{650}, n)$. Hence the $\int L(Q, n)dQ^2$ peaks at a much lower $n$ than the $\int L(Q[\sigma_8, n], n)d\sigma_8$.

There is a better way, which is to use the maximum of $L(Q, n)$ for a fixed $n$ to
generate the marginal likelihood over \( n \). This approach to “uninteresting” parameters is recommended by Avni (1976). The maximum value does not require a Jacobian when transforming to different amplitude variables, so prior ignorance of \( Q^2 \) gives the same answer as prior ignorance of \( T_{650} \).

The process of determining an amplitude parameter (usually \( \langle Q_{RMS}^2 \rangle^{0.5} \)) and the spectral index \( n \) from the COBE maps is an extreme example of data reduction. In this process one takes the \( 360 \times 10^6 \) DMR data samples per year and produces maps with \( 6 \times 6144 \) values, and from these maps one calculates a smaller number of statistics. In the final step, \( \langle Q_{RMS}^2 \rangle^{0.5} \) and \( n \) are estimated using the values of the statistics, leaving only 2 values derived from nearly \( 10^9 \) input values. This description is general enough to describe both the Górski (1994) method using linear statistics and the methods involving quadratic statistics: the correlation function used by Bennett et al. (1994) and the Hauser-Peebles power spectrum used by Wright et al. (1994). The final result of any of these analysis methods is the values \( Q_{\text{obs}} \) and \( n_{\text{obs}} \) determined from the real data, as well as an estimate \( \hat{\sigma}_1 \) for the noise standard deviation in one observation.

3. Monte Carlo Simulations

In order to test these methods for biases, it is necessary to simulate both the cosmic variance, which gives a random map with random spherical harmonic amplitudes chosen from a Gaussian distribution with a variance determined from the chosen \( Q_{in} \) and \( n_{in} \), and the experimental variance, which gives the 360 million noise values needed per year. While programs to simulate the DMR time-ordered data do exist, none of the groups mentioned above have worked at this level of detail. Instead, they have used simulations that start with the maps.

The effect of noise on the map production process can be simulated using

\[
T = \sigma_1 A^{-0.5} U
\]  

where \( \sigma_1 \) is the noise in one observation, \( U \) is an uncorrelated vector of unit variance zero mean Gaussian random variables, and \( A \) is the matrix with diagonal elements \( A_{ii} \) equal to the number of times the \( i^{th} \) pixel was observed, and off-diagonal elements \( -A_{ij} \) equal to the number of times the \( i^{th} \) pixel was referenced to the \( j^{th} \) pixel. Even though \( A \) is singular, Wright et al. (1994) give a rapidly convergent series technique for generating noise maps. Thus each noise map depends on 6144 independent Gaussian unit variance random variables and the parameter \( \sigma_1 \).

The signal map that is added to the noise maps to give the “observed” maps is generated using independent Gaussian random amplitudes. Bond & Efstathiou (1987) show that the expected variance of the coefficients \( a_{\ell m} \) in a spherical harmonic...
Fig. 3.— Each point is an input parameter set that is consistent with the real data for a given realization of the random cosmic and radiometer variance processes. The likelihood contours are at $\Delta(-2\ln L) = 1, 4$ and $9$.

The expansion of the CMBR temperature given a power law power spectrum $P(k) \propto k^n$ is $<a_{2m}^2> \propto \Gamma[\ell + (n - 1)/2]/\Gamma[\ell + (5 - n)/2]$ for $\ell < 40$, with the constant of proportionality chosen so that $5 < a_{2m}^2>/4\pi = Q^2$. The simulations done by Wright et al. (1994) included $\ell$'s up to 39, so the signal map depends on 1600 Gaussian independent unit variance random variables and the two parameters $Q$ and $n$, which I shall call $Q_{in}$ and $n_{in}$ below to distinguish them from the fitted values.

The resulting Monte Carlo map depends on a set of random variables $\{Z\}$ (1600 + 6144 elements for a one map analysis, or 1600 + 12288 for a cross-analysis needing two maps) with a known distribution, and the three parameters $Q_{in}$, $n_{in}$ and $\sigma_1$. $\sigma_1$ can be determined with great precision using the time-ordered data. Hence one needs to run many Monte Carlo simulations with different values of $Q_{in}$ and $n_{in}$ and compare the fitted values $Q_{out}$ and $n_{out}$ to the fitted values for the real data, $Q_{obs}$ and $n_{obs}$. For any given
realization of \( \{Z\} \), the fitted values \( Q_{\text{out}} \) and \( n_{\text{out}} \) are a continuous function of the input parameters \( Q_{\text{in}} \) and \( n_{\text{in}} \), and one can choose values \( Q_{\text{in}} = Q_{\text{match}} \) and \( n_{\text{in}} = n_{\text{match}} \) such that \( Q_{\text{out}} = Q_{\text{obs}} \) and \( n_{\text{out}} = n_{\text{obs}} \). By choosing many different realizations of \( \{Z\} \), one creates many different \( Q_{\text{match}}, n_{\text{match}} \) pairs. The density of the points in the \( Q_{\text{match}}, n_{\text{match}} \) plane defines a probability density function for the true parameters \( Q_{\text{true}}, n_{\text{true}} \) that does not depend on any prior knowledge but does depend on the experimental result in a reasonable way. The random element in the process comes from \( \{Z\} \), whose properties are known. Figure 3 shows this cloud of points for the 2 year 53 × 90 cross-power spectrum. The bias \( (\Delta n = 0.1) \) in the Gaussian approximation maximum likelihood method applied to the quadratic power spectrum statistics is only 0.25\( \sigma \), so the shift between the points and the contours is hard to see.

The method using linear statistics (Górski 1994) has the advantage that the Gaussian expression for the likelihood is exact. A further advantage of this method is that any non-singular linear transformation of the basis functions will give the same answer, since the covariance matrix will change to cancel the change in the values of the statistics. While this means that the original motivation for generating a set of basis functions orthonormal in the cut sky is lost, one still has the advantage that basis functions orthogonal to any number of low order multipoles are easy to find. Görski et al. (1994) find (for \( 3 \leq \ell \leq 30 \)) that the maximum of \( L(Q, n) \) occurs at \( n = 1.02 \) for the combined 2 year 53 GHz plus 90 GHz map, and Monte Carlo simulations show that the bias in this application of the maximum likelihood method is small.

Note that the \( 3 \leq \ell \leq 30 \) cross power spectrum fits in Wright et al. (1994) still include the off-diagonal effect of the quadrupole on higher \( \ell \)'s, while those in Görski et al. (1994) are completely independent of the quadrupole. The modified Hauser-Peebles method in Wright et al. (1994) uses basis function defined using

\[
G_{\ell m} = F_{\ell m} - \frac{F_{00} \langle F_{00} F_{\ell m} \rangle}{\langle F_{00} F_{00} \rangle} - \sum_{m'=-1}^{1} \frac{F_{1m'} \langle F_{1m'} F_{\ell m} \rangle}{\langle F_{1m'} F_{1m'} \rangle},
\]

where the \( F_{\ell m} \) are real spherical harmonics and the inner product \( \langle fg \rangle \) is defined over the cut sphere. These functions \( G_{\ell m} \) are orthogonal to monopole and dipole terms on the cut sphere. Call this the MD method since the basis functions are orthogonal to the monopole and dipole. Let the MDQ method use basis functions orthogonal to the monopole, dipole and quadrupole:

\[
G'_{\ell m} = F_{\ell m} - \frac{F_{00} \langle F_{00} F_{\ell m} \rangle}{\langle F_{00} F_{00} \rangle} - \sum_{m'=-1}^{1} \frac{F_{1m'} \langle F_{1m'} F_{\ell m} \rangle}{\langle F_{1m'} F_{1m'} \rangle} - \sum_{m'=-2}^{2} \frac{F_{2m'} \langle F_{2m'} F_{\ell m} \rangle}{\langle F_{2m'} F_{2m'} \rangle}.
\]

Changing from the MD method to the MDQ method causes the mean power in \( T^2_d \) for \( n = 1 \) Monte Carlo skies to go down by 31% while \( T^2_d \) for the real sky goes up by 16%. This leads to a higher \( \ell = 4 \) point and a lower value of \( n \) (\( n = 1.02 \) instead of 1.22 for the 53 × 90 cross-power spectrum). Figure 4 shows the hexadecapole power in \( \mu K^2 \) measured two different ways: on the x axis the MD method; and on the y axis \( T^2_d \) measured using...
Fig. 4.— Hexadecapole power determined using the MD method on the $x$-axis vs. the MDQ method on the $y$-axis for the real sky (open circle) and $n = 1$, $\langle Q_{RMS}^2 \rangle^{0.5} = 17 \mu K$ simulations.

the MDQ method. The real sky is shown as the open circle, while the dots are $n = 1$, $\langle Q_{RMS}^2 \rangle^{0.5} = 17 \mu K$ Monte Carlo simulations. One sees that the real sky is moderately far out on the upper edge of the cloud of simulations, and this produces the $0.5\sigma$ shift in $n$ when changing basis functions. One also sees that the distribution of $T_4^2$ is quite skewed, which explains the bias in the method that maximizes the Gaussian approximation to the likelihood.

Figure 5 shows the maximum likelihood values of $n$ from fits to $3 \leq \ell \leq 30$ for 1800 Monte Carlo runs with $n_{in} = 1$ and $Q_{in} = 17 \mu K$. The $x$-axis shows $n$ computed using the MD method, while the $y$-axis shows the results of the MDQ method. The real sky is shown as the open circle, and the mean of the 1800 Monte Carlo spectra is shown as the closed circle. This figure shows that the two methods are generally consistent, with the real sky moderately far out in the scatter. (The linear features for $n = 1.00, 1.25$ and $1.50$ are caused by the interpolation among input values spaced by $\Delta n_{in} = 0.25$ during the maximum likelihood fits.) The overall performance of the MD method is better, with a bias
Fig. 5.— Spectral index $n$ using the MD method on the $x$-axis vs. $n$ from the MDQ method for 1800 Monte Carlo skies with $n_{in} = 1$ and $Q_{in} = 17 \mu K$. The real sky is the open circle.

that is 20% smaller and a standard deviation that is 8% smaller than those given by the MDQ method. Of course fits that include the quadrupole ($2 \leq \ell \leq 30$) do even better, since they use more information. So the final result of the power spectral analysis is ambiguous: $n \approx 1.4$ if the quadrupole is included in the fit, $n \approx 1.25$ excluding the quadrupole using the MD method, or $n \approx 1.0$ when rigorously excluding the quadrupole using the MDQ method. The existence of all these options raises the specter of “optional stopping”, a time-honored method of introducing systematic errors into measurements. But fortunately this whole range of values is within the statistical uncertainty.

Even in very simple cases this level of disagreement between different estimation techniques is common. For example, the RMS difference between the median and the mean of a set of Gaussian random numbers is $3/4$ of the standard deviation of the mean.
4. **Degree-Scale**

The experiments at $\approx 1^{\circ}$ scale offer the possibility of a better determination of the primordial power spectrum index $n$, but the model-dependent effects of the wing of the Doppler peak at $\ell \approx 200$ must be allowed for. Even in the large angle region $\ell < 30$ small model-dependent corrections must be made. A Cold Dark Matter (CDM) model with a primordial spectral index $n_{pri} = 0.96$ has an apparent index $n_{app} = 1.1$ due to the “toe” of the Doppler peak that extends into the $\ell < 70$ region. Wright et al. (1994) have made this comparison with degree scale experiments, and the resulting value for $n_{pri}$ is given in Table 1.

5. **Large Scale Structure**

A comparison of the extremely large scale structure seen by COBE to the large scale structure seen in studies of the clustering of galaxies also leads to an estimate of $n_{pri}$. The uncertainty in this method is decreased because of the large range of scales covered, but also increased due uncertainties in the models of large scale structure formation. However, this comparison strongly favors $n = 1$. Prior to the COBE announcement of anisotropy, Peacock (1991) gave an implicit prediction that for $n = 1$ the amplitude of $\Delta T$ should be $\langle Q_{RMS}^2 \rangle^{0.5} = 18.8 \mu K$. Peacock & Dodds (1994) have extended this analysis of large scale structure and I get a result $n_{pri} = 0.99 \pm 0.16$ from their paper after correcting for their incorrect $\langle Q_{RMS}^2 \rangle^{0.5} = 15 \mu K$ and increasing the uncertainty to allow for the uncertainty in the IRAS bias, $b_I$. In Figure 5 I have “extended” Figure 6 from Peacock & Dodds to include the COBE datum, and show extrapolations with $n = 0.5$, 1, & 1.5 through the COBE point. This result assumes that $\Omega = 1$, but Peacock & Dodds have also found that $\Omega^{0.6}/b_I = 1.0 \pm 0.2$.

6. **Summary**

In conclusion, both the COBE $\Delta T$ data alone and the ratio of the COBE $\Delta T$ data to $1^{\circ}$ scale $\Delta T$ are consistent with the $n \approx 1$ prediction of the inflationary scenario. Furthermore, the implied level of gravitational potential perturbations is sufficient to produce the observed large scale (100 Mpc) structure if both the $n = 1$ and $\Omega = 1$ predictions of inflation are correct, and the Universe is dominated by Dark Matter.
| Method                     | COBE dataset | Q? | Result                      | Reference                  |
|----------------------------|--------------|----|-----------------------------|----------------------------|
| Correlation function      | 1 year 53×90 | N  | $n_{app} = 1.15^{+0.45}_{-0.65}$ | Smoot et al ('92)          |
| COBE: $\sigma_8$          | 1 year 53+90 | N  | $n_{pri} = 1 \pm 0.23$      | Wright et al ('92)         |
| Genus vs. smoothing       | 1 year 53   | Y  | $n_{app} = 1.7^{+1.3}_{-1.1}$ | Smoot et al ('94)          |
| RMS vs. smoothing         | 1 year 53   | Y  | $n_{app} = 1.7^{+0.3}_{-0.6}$ | Smoot et al ('94)          |
| Correlation function      | 2 year 53×90 | Y  | $n_{app} = 1.3^{+0.29}_{-0.55}$ | Bennett et al ('94)       |
| Correlation function      | 2 year 53×90 | N  | $n_{app} = 1.1^{+0.60}_{-0.55}$ | Bennett et al ('94)       |
| COBE: 1° scale            | 2 year NG    | N  | $n_{pri} = 1.15 \pm 0.2$    | Wright et al ('94)         |
| Cross power spectrum      | 2 year 53 & 90 | N  | $n_{app} = 1.25^{+0.40}_{-0.45}$ | Wright et al ('94)       |
| Cross power spectrum      | 2 year 53 & 90 | Y  | $n_{app} = 1.39^{+0.34}_{-0.39}$ | Wright et al ('94)       |
| Orthonormal functions     | 2 year 53+90 | N  | $n_{app} = 1.02 \pm 0.4$    | Górski et al ('94)         |
| COBE: $\sigma_{100}$     | 1 year 53    | Y  | $n_{pri} = 1.0 \pm 0.16$    | Peacock & Dodds           |

Table 1: Spectral index determinations

7. References

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Fig. 6.— An extended version of Figure 6 from Peacock & Dodds, showing $n = 0.5$, 1 and 1.5 extrapolations through the COBE point.