The background photon temperature \( \bar{T} \) is one of the fundamental cosmological parameters. Despite its significance, \( \bar{T} \) has never been allowed to vary in the data analysis, owing to the precise measurement of the cosmic microwave background (CMB) temperature by COBE FIRAS. However, even in future CMB experiments, \( \bar{T} \) will remain unknown due to the unknown monopole contribution \( \Theta_0 \) at our position to the observed (angle-averaged) temperature \( \langle T \rangle_{\text{obs}} \). By fixing \( \bar{T} \equiv \langle T \rangle_{\text{obs}} \), the standard analysis underestimates the error bars on cosmological parameters, and the best-fit parameters obtained in the analysis are biased in proportion to the unknown amplitude of \( \Theta_0 \). Using the Fisher formalism, we find that these systematic errors are smaller than the error bars from the Planck satellite. However, with \( \bar{T} \equiv \langle T \rangle_{\text{obs}} \), these systematic errors will always be present and irreducible, and future cosmological surveys might misinterpret the measurements.

**Introduction.** Cosmology has seen enormous development in recent decades (see, e.g., [1] for a review). In particular, the cosmic microwave background (CMB) experiments have greatly improved in recent years with the Wilkinson Microwave Anisotropy Probe (WMAP) and the Planck satellites [2, 3]. The primary cosmological parameters are now constrained at the sub-percent level [4, 5], and the angular scale of the acoustic peak is even better constrained by an order-of-magnitude. This level of precision in cosmological parameter estimation demands a matching accuracy in our theoretical predictions.

The background CMB temperature \( \bar{T} \) is one of the fundamental cosmological parameters that characterize the evolution of the Universe. In particular, it is tantamount to the photon energy density \( \omega_\gamma \), and it sets the total radiation density \( \omega_r \) (hence the epoch \( z_{eq} \) of the matter-radiation equality), once the other cosmological parameters such as the matter density \( \omega_m \) and the neutrino masses \( m_\nu \) are provided. Despite its significant role in cosmology, the background CMB temperature \( \bar{T} \) has never been treated as a free cosmological parameter in literature, because of the pioneering work [2, 3] by the COBE Far Infrared Absolute Spectrometer (FIRAS) in 1990, which provided the precise measurements of the observed CMB temperature \( \langle T \rangle_{\text{obs}} \) by averaging the CMB temperature measurements over the sky.

The final release [7] of the COBE FIRAS measurements is \( \langle T \rangle_{\text{obs}} = 2.728 \pm 0.004 \) K, and the measurements were later further calibrated in Ref. [8] by using the WMAP differential temperature measurements [9]: \( \langle T \rangle_{\text{obs}} = 2.7255 \pm 5.7 \cdot 10^{-4} \) K. This measurement of the CMB temperature with exquisite precision underpins the standard practice, in which the background CMB temperature \( \bar{T} \) is set equal to the observed CMB temperature \( \langle T \rangle_{\text{obs}} \). In this Letter, we show that this practice is fundamentally incorrect even in the era of future CMB experiments with virtually no measurement errors in \( \langle T \rangle_{\text{obs}} \), and it leads to underestimation of the error bars on the cosmological parameters and systematic biases in the cosmological parameter estimation.

The cosmological parameter \( \bar{T} \).— The background CMB temperature \( \bar{T} \) is really one of the other cosmological parameters such as the background matter density \( \omega_m \) or the (background) Hubble parameter \( H_0 \) that are defined in a homogeneous and isotropic universe and control the evolution of the perturbations in an inhomogeneous universe. These cosmological parameters, however, differ from the, so called, observed cosmological parameters. For instance, it is well known that the local Hubble parameter obtained by using the background Hubble law with the local velocity and distance measurements differs from the (background) Hubble parameter. The background cosmological parameters can be inferred from the data analysis, but these are not referred to as the observed cosmological parameters.

The discrepancy between the cosmological parameters and the observed cosmological parameters exists due to the inhomogeneities in our Universe, affecting our measurements or the observations (hence justifying the superscript “obs”). The observed CMB temperature \( \langle T \rangle_{\text{obs}} \) from the COBE FIRAS is obtained by averaging the CMB temperature measurements on the sky, and it differs from the background CMB temperature \( \bar{T} \) due to the monopole perturbation \( \Theta_0 \). As any other physical quantities, the CMB temperature at a given position \( x \) and direction \( \hat{n} \) in general includes not only the background \( \bar{T} \), but also the perturbation \( \Theta(x, \hat{n}) \), and the separation of the background and perturbation is made for our theoretical convenience. Therefore, when averaged over the sky at our position \( x_o \), the observed CMB temperature can be expressed as \( \langle T \rangle_{\text{obs}} = \bar{T}(1 + \Theta_0) \), where the monopole perturbation is

\[
\Theta_0 := \int \frac{d^3 \hat{n}}{4\pi} \Theta(x_o, \hat{n}) ,
\]

and we suppressed the dependence of \( \Theta_0 \) on the observer position \( x_o \).

Compared to the other multipole moments \( \Theta_l \) \((l \geq 1)\) in CMB, the monopole is not an observable, as it is absorbed into
the observed CMB temperature $\langle T \rangle_{\text{obs}}$ together with the background temperature $\bar{T}$. Despite this peculiarity, the monopole perturbation $\Theta_0$ at our position is very unlikely to be zero. The Ergodic theorem states that once the fluctuations are averaged over a sufficiently large volume, the resulting average is equivalent to the ensemble average or the average over many realizations of our Universe. While the ensemble average of the monopole is zero, it is shown in Ref. [10] that the angle average is not quite the ensemble average, as it is obtained only at our own position. This implies that if we were to perform the angle average of the CMB temperature at the Andromeda galaxy, we would obtain $\langle T \rangle_{\text{obs}}$ different from the COBE FIRAS result, due to the fluctuation of the monopole from place to place. Only if we could average the CMB temperature $\langle T \rangle_{\text{obs}}(x)$ over all the possible observer positions, we would be able to replace the average with the ensemble average and obtain the background CMB temperature $\bar{T}$. As this procedure is impossible, the background CMB temperature $\bar{T}$ can never be measured and needs to be treated as a free cosmological parameter as the other cosmological parameters.

**CMB observations and theoretical predictions.**—In observations, the CMB temperature map as well as the polarization map obtained in the CMB experiments is decomposed with spherical harmonics $Y_{lm}$ as $T_{\text{obs}}(\hat{n}) := \sum_{lm} T_{lm} Y_{lm}(\hat{n})$, and the angular multipoles $T_{lm}$ are used to construct the observed CMB power spectra $D_l := \sum_m T_{lm}^2/(2l+1)$ for $l \geq 1$. The angle average of the CMB temperature is equivalent to the monopole $\langle T \rangle_{\text{obs}} := T_{00}/\sqrt{4\pi}$. The theoretical predictions are, however, based on the separation of the background and the perturbation around it, so that the CMB temperature is modeled as $T_{\text{obs}}(\hat{n}) := \bar{T}(1+\Theta)$ and the angular decomposition of the temperature anisotropies $\Theta(\hat{n}) := \sum_{lm} a_{lm} Y_{lm}(\hat{n})$ yields the angular multipole $a_{lm}$ and their power spectra $C_l := \langle |a_{lm}|^2 \rangle$, where the angular multipoles and the power spectra are both dimensionless, as opposed to the dimensionful quantities $T_{lm}$ and $D_l$ in observation.

The conversion between these quantities is trivial in theory: $T_{lm} = \bar{T} a_{lm}$ and $D_l = \bar{T}^2 C_l$ for $l \geq 1$, but it is impossible in observation, as the background CMB temperature $\bar{T}$ is unknown. However, this poses no problem, as we can include an additional cosmological parameter $\bar{T}$ in our data analysis and obtain the best-fit value for $\bar{T}$ as the other (unknown) cosmological parameters in a given model. The problems arise because the data analysis is performed by fixing $T := \langle T \rangle_{\text{obs}}$ by hand. This incorrect procedure results in two problems: 1) the background evolution in our theoretical predictions never matches the correct background in our Universe, unless the monopole at our position happens to be zero, and 2) by using $\langle T \rangle_{\text{obs}}$ instead of $T$, the observed temperature and the CMB power spectra are in practice compared to $T_{\text{obs}}/\langle T \rangle_{\text{obs}} = a_{lm}/(1+\Theta_0)$ and $C^\text{biased}_l := \frac{\langle |a_{lm}|^2 \rangle}{(1+\Theta_0)^2} = C_l \left( 1 + \frac{3}{4\pi} C_0 + \cdots \right), \quad (2)$

where the monopole of the power spectrum is $C_0 \simeq 1.7 \cdot 10^{-9}$ in our fiducial $\Lambda$CDM model. Though the latter is negligible, the former causes the dominant systematic errors in the standard data analysis.

**Underestimation of the error bars.**—One immediate consequence of the standard practice with $\bar{T} = \langle T \rangle_{\text{obs}}$ is the underestimation of the error bars on the cosmological parameters in a given model, as there exists one less degree of freedom in the parameter estimation than in reality. The correct error bars on the cosmological parameters can be estimated by considering the full model with extra cosmological parameter $p_0 := \ln \bar{T}$, in addition to the standard model parameters $p_i (i = 1, \cdots, N)$ and by marginalizing over the nuisance parameter $p_0$. To estimate the inflation of the error bars, we adopt the Fisher information matrix formalism. For the Gaussian fluctuations on the sky, the Fisher matrix takes the standard form with one critical difference: the observables contain both the background and the perturbation. For CMB, the observables are $T^\text{obs}_l$ and $D^\text{obs}_l$, and the Fisher matrix is then obtained in Ref. [11] as

$$F_{00} = \frac{4\pi}{C_0} + \sum_{l=2}^{\infty} \frac{2l+1}{2} C_l^2 \left( 2C_l + \frac{\partial C_l}{\partial \ln \bar{T}} \right)^2, \quad (3)$$

$$F_{i0} = \sum_{l=2}^{\infty} \frac{2l+1}{2} C_l^2 \left( \frac{\partial}{\partial p_i} C_l \right) \left( 2C_l + \frac{\partial C_l}{\partial \ln \bar{T}} \right), \quad (4)$$

$$F_{ij} = \sum_{l=2}^{\infty} \frac{2l+1}{2} C_l^2 \left( \frac{\partial}{\partial p_i} C_l \right) \left( \frac{\partial}{\partial p_j} C_l \right), \quad (5)$$

where the standard Fisher analysis corresponds to the submatrix of the full Fisher matrix ($F^\text{estd}_{ij} \equiv F_{ij}$). The correct error bars on the cosmological parameters after marginalizing over $p_0$ can be obtained as the diagonal elements of the $N\times N$ sub-matrix

$$\sigma^2_p = \text{diag} \left( F_{ij} - F_{i0} F_{0j} F_{00}^{-1} \right), \quad (6)$$

of the inverse of the full Fisher information matrix. This estimate, however, relies on the critical assumption that the correct $\bar{T}$ is $\langle T \rangle_{\text{obs}}$, despite our complete ignorance.

For the proof of concept, we apply the Fisher formalism to a CMB experiment like the Planck satellite, where we used the temperature $C_{TT}^l$ at $l = 2 \sim 2500$, the polarization $C_{TE}^l$ at $l = 2 \sim 2000$, and the cross $C_{EE}^l$ power spectra at $l = 30 \sim 2000$ as our CMB observables. The Fisher matrix is computed by accounting for the covariance among the temperature and the polarization observables [12,13]. We adopt that the sky coverage is $f_{\text{sky}} = 0.86$, the detector pixel noise is $\Delta^2_\nu = (0.55 \mu K \text{ deg})^2$, and the beam size is $\sigma_\theta = 7.22 \text{ arcmin}$ in FWHM for 143 GHz channel. These specifications are taken into consideration in the Fisher matrix by modifying the factor $(2l+1)/2C_l^2$. Finally, for our fiducial cosmological parameters, we adopted the best-fit $\Lambda$CDM model parameters reported in Table 7 of the Planck 2018 results [5] (Planck alone). The CMB power spectra are computed by using the Class Boltzmann code [14].
FIG. 1. Inflation of the error bars on the $\Lambda$CDM cosmological parameters, after the unknown background temperature $\bar{T}$ is accounted for. The errors are relative, e.g., 1% in the plot means that the correct error bar $\sigma$ is larger than $\sigma_{\mathrm{std}}$ in the standard practice by 1%: $\sigma = 1.01 \sigma_{\mathrm{std}}$. By incorrectly fixing $\bar{T} \equiv \langle T \rangle_{\mathrm{obs}}$, the error bars on the cosmological parameters are underestimated in the standard practice. The amplitude inflation of error bars in Figure 1 is relative to the error bar in the initial COBE FIRAS measurement is used. Note that the inflation of the error bars (dotted) is less than a few percents for the three cases. What is important is to note that the error bars are always underestimated (solid lines) in the standard data analysis, even with no measurement uncertainty in $\langle T \rangle_{\mathrm{obs}}$ from future CMB experiments.

**Cosmological parameter bias.**—By fixing $\bar{T} \equiv \langle T \rangle_{\mathrm{obs}}$, the standard data analysis contains systematic errors in terms of biases in the cosmological parameter estimation. Assuming that the systematic errors are small, the best-fit cosmological parameters $p^\mu_{\mathrm{sf}}$ are characterized by the parameter biases $\delta p^\mu$ from the true parameter set $p^\mu_{\mathrm{tr}}$ as $p^\mu_{\mathrm{sf}} := p^\mu_{\mathrm{tr}} + \delta p^\mu \equiv \ln \bar{T} + \Theta_0$, and $\Theta_0$, so that the parameter bias for $p_0 = \ln \bar{T}$ is the unknown monopole at our position: $\delta p_0 \equiv \Theta_0$.

The relation between two parameter sets can be obtained by considering that the likelihood $\mathcal{L}(p^\mu_{\mathrm{tr}})$ of the CMB observables is maximized at the best-fit parameters $p^\mu_{\mathrm{sf}}$:

$$0 = \frac{\partial}{\partial p_i} \mathcal{L} \bigg|_{p^\mu_{\mathrm{sf}}} = \text{Tr} \left[ \tilde{C}_i^{-1} \tilde{C} \right] - \text{Tr} \left[ \tilde{C}_i^{-1} \tilde{C} \left( \delta_{\text{obs}} - \tilde{\mu} \right) \left( \delta_{\text{obs}} - \tilde{\mu} \right)^T \right],$$

where the commas represent derivative of the covariance matrix $\mathcal{C}$ with respect to the parameter $p_i$ and the observed data set $\delta_{\text{obs}}$ includes the observed temperature and polarization anisotropies. The covariance matrix $\mathcal{C}(p^\mu_{\mathrm{tr}})$ and the mean $\tilde{\mu}(p^\mu_{\mathrm{tr}})$ are the theoretical predictions in a given model, where $\mu = \bar{T} + \Theta_0$ for temperature anisotropies and $\mu = 0$ for polarization anisotropies. However, due to the incorrect assumption $(\bar{T} \equiv \langle T \rangle_{\mathrm{obs}})$ in the standard practice, the theoretical predictions for $\mathcal{C}$ and $\mu$ depend only on the model parameters $p^\mu_{\mathrm{tr}}$, but not on $\bar{T}$, and we used tilde to represent that the theoretical predictions are incorrect in this regard and evaluated at $p^\mu_{\mathrm{sf}}$, not at $p^\mu_{\mathrm{tr}}$.

Using the spherical harmonics decomposition, the condition for the best-fit parameter set is expressed as

$$0 = \sum_{l=2}^{\infty} (2l + 1) [\tilde{C}_l^{-1} \tilde{\mu} - \tilde{C}_l] \left[ \frac{\bar{T}^2}{(\langle T \rangle_{\text{obs}})^2} \right],$$

where the power spectra $\tilde{C}_l$ account for the covariance among the temperature, the polarization, and their cross power spectra together with the detector noise and beam smoothing [12,13]. To make further progress, we take the ensemble average to replace the ratio of $a^\mu_{\text{obs}}/(\langle T \rangle_{\text{obs}})^2$ with $C^\mu_{\text{obs}}$ and

FIG. 2. Fractional deviation of the cosmological parameter bias $\delta p_i$ in the standard data analysis, if multiplied by the amplitude of the monopole $\Theta_0$ at our position. While the monopole at our position is unknown, its 1-$\sigma$ rms fluctuation is $\sim 1.2 \times 10^{-5}$. The cosmological parameter bias is independent of the measurement uncertainty in $\langle T \rangle_{\text{obs}}$.

\[ C_i^{\text{biased}}(p_i^b) \simeq \tilde{C}_i \left(1 + \frac{3}{4\pi} \hat{C}_0 - \frac{\partial \ln \tilde{C}_i}{\partial \ln T} \Theta_0 - \frac{\partial \ln \tilde{C}_i}{\partial p_i} \delta p_i\right), \]

where the first correction arises from $C_i^{\text{biased}}$ and the remaining corrections arise due to the difference between $p_i^b$ and $p_i^c$. Ignoring the small correction due to the first term, the cosmological parameter bias can be neatly expressed as

\[ \delta p_i = - \left(F_{\text{std}}^{-1}\right)_{ij} F_{j0} \Theta_0, \]

and it is in proportion to the amplitude of the unknown monopole at our position, while it is independent of the measurement uncertainty in $\langle T \rangle_{\text{obs}}$, given our assumption $p_i^b \simeq p_i^c$.

Figure 2 shows the fractional deviation of the cosmological parameter bias $\delta p_i$, but scaled with the unknown amplitude of the monopole $\Theta_0$. If the monopole happens to vanish at our position, there would be no bias in the cosmological parameters by using the standard practice. However, if the monopole at our position is non-zero, the standard analysis yields the biases in the best-fit cosmological parameters in proportion to the unknown amplitude of the monopole. For instance, if the monopole is at 10-$\sigma$ fluctuation at our position, i.e., $\Theta_0 \simeq 1.2 \cdot 10^{-4}$, the baryon density parameter $\omega_b$ is off by 0.06% level (\(\delta \omega_b/\omega_b = 6 \cdot 10^{-4}\)). Given that $\omega_b$ is constrained at 0.67% level (\(\sigma_{\text{std}}^{\text{obs}}/\omega_b = 7 \cdot 10^{-4}\)), this level of bias is still tolerable today. While the bias in $\omega_c$ is of similar magnitude, the error bar on $\omega_c$ is larger, hence the impact is smaller. Similar to the case for the error inflation, three cosmological parameters $\omega_b$, $\omega_c$, $A_s$ are more susceptible to the systematic errors due to the monopole $\Theta_0$.

Conclusions.— We showed that in principle the background CMB temperature $\bar{T}$ has to be considered as an unknown cosmological parameter, because the observed (angle-average) CMB temperature $\langle T \rangle_{\text{obs}}$ includes the unknown monopole contribution at our position. We investigated the impact of this “new” cosmological parameter $\bar{T}$ on the CMB data analysis. With the current uncertainty in $\langle T \rangle_{\text{obs}}$, the standard data analysis underestimates the error bars on the cosmological parameters by a relative amount of up to 2%. Furthermore, if the monopole is non-vanishing at our position, the best-fit cosmological parameters in the standard analysis are biased in proportion to the unknown amplitude of the monopole.

We conclude that these systematic errors, albeit negligible today, are always present and irreducible in the standard data analysis. Therefore, cosmological measurements might be misinterpreted in future experiments with better precision than the Planck satellite. Of course, these systematic errors can be readily avoided by including one extra cosmological parameter $\bar{T}$. A proper analysis of the real Planck data is in preparation [15].

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