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Scalar model for frictional precursors dynamics

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Recent experiments indicate that frictional sliding occurs by nucleation of detachment fronts at the contact interface that may appear well before the onset of global sliding. This intriguing precursory activity is not accounted for by traditional friction theories but is extremely important for friction dominated geophysical phenomena as earthquakes, landslides or avalanches. Here we simulate the onset of slip of a three dimensional elastic body resting on a surface and show that experimentally observed frictional precursors depend in a complex non-universal way on the sample geometry and loading conditions. Our model satisfies Archard’s law and Amontons’ first and second laws, reproducing with remarkable precision the real contact area dynamics, the precursors envelope dynamics prior to sliding, and the normal and shear internal stress distributions close to the interfacial surface. Moreover, it allows to assess which features can be attributed to the elastic equilibrium, and which are attributed to the out-of-equilibrium dynamics, suggesting that precursory activity is an intrinsically quasi-static physical process. A direct calculation of the evolution of the Coulomb stress before and during precursors nucleation shows large variations across the sample, explaining why earthquake forecasting methods based only on accumulated slip and Coulomb stress monitoring are often ineffective.

The classical laws of friction, due to Amontons and Coulomb, postulate that a body resting on a surface can be displaced only by applying a shear force larger than a static friction force, which is proportional to the normal load and independent of the apparent area of contact. Recent research has challenged this understanding of friction, showing that macroscopic slip is due to the formation and propagation of detachment fronts through the contact interface1. The nature of these fronts and their speed depend on the way shear is applied to the sample and on its geometry2-3, a particularly compelling issue in view of the long held assumption of independence of friction on the sample shape and size. It is particularly intriguing that, in some cases, localized sliding precursors nucleate long before the applied force reaches the static friction force at which the front propagates through the entire contact interface4-6. Numerical simulations of friction models in one6-10 and two dimensions11-13 allow to study the main features of the spatio-temporal dynamics of precursors. These numerical works have mainly focused on the qualitative dynamical aspects of propagation, reproducing the different dynamical regimes observed in experiments6-9 and the nucleation of the fronts under various loading conditions11.

Based on the results of experiments8 and numerical simulations8-11 it was suggested that frictional precursors evolve according to universal laws: the sample size and normal load dependences of precursors lengths can be rescaled away and different experiments can be collapsed a single master curve. Establishing universal forms for slip precursors would be particularly important for earthquake forecasting4. Slip or stress accumulation on faults has been often observed to accelerate close to large earthquakes14-17, but detailed predictions based on this are considered to be unreliable18,19. It is therefore extremely important to better clarify the conditions leading to precursors and confirm their universality. Another puzzling aspect revealed by experiments is an apparent violation of the Amontons-Coulomb laws: direct measurement of shear \( \tau \) and normal stresses \( \sigma \) close to the frictional interface indicated regions where the Coulomb stress \( \tau_C = |\tau| - \mu \sigma \) is positive without inducing detachment2. This result suggests that the friction coefficient \( \mu \) might not be a well defined material constant as conventionally assumed.

Scalar models are commonly used to study the planar crack front propagation in disordered elastic media20,21, in quasi-two dimensional geometries22 and under antiplanar shearing conditions23. On the other hand, recent experiments have provided the evidence that classical shear cracks singular solutions, originally devised to account for brittle fractures, offer a quantitative excellent description of the static-to-dynamic friction trans-
that the precursor size. Following Ref. 4, we first study how precursors depend on sample size and load conditions. Simulations for different sample sizes and loading conditions. Results.

Simulations for different sample sizes and loading conditions. Following Ref. 4, we first study how precursors depend on $L_x$. On the normal load $F_N$ when $F_N^\text{top}$ is applied through a rod placed on the trailing edge, at height $h = 6$ mm. Experimental evidence suggests that the precursor size $\ell$ obtained for different values of $L_x$ and $F_N$ can be collapsed into a single master curve when normalized by $L_x$ and plotted versus $F_N^\text{top} / \mu F_N$. Our numerical results reproduce quantitatively the experimental findings as shown in Fig. 1(a). In our model, however, we are able to change $L_x$ over a wider range than in the experiments, revealing that data collapse is in fact only approximate (see inset of Fig. 1(a)). Similar behavior is obtained when we vary $L_y$ (Fig. S4) or $L_z$ (Fig. S5) keeping constant the other parameters: in all cases front precursors exhibit a dependence on the sample dimensions. We have also changed $L_x$ and $L_y$ holding their ratio $L_x/L_y$ unchanged (Fig. S6), $L_x$ and $L_z$ with $L_x/L_z$ constant (Fig. S7), or $L_y$ and $L_z$ with $L_y/L_z$ constant (Fig. S8). Again, data collapse is not obtained, indicating that for this loading condition the precursor length $\ell$ depends in a non-trivial way on the sample dimensions ($L_x$, $L_y$, $L_z$). The general trend however is that precursory activity tends to decrease as the varying dimension is increased: for a larger sample we typically need a larger shear force to observe a precursor of a given length.

Experimental results in Ref. 4 also suggest that the height $h$ at which the lateral force is applied to the sample trailing edge has no influence on the evolution of the front precursors. While this is true for the range of $h$ used in experiments (see Fig. 1(b)), when we increase $h$ further the curves no longer collapse. In particular, we find that the lateral force needed to nucleate the first precursor increases considerably with $h$ (see the inset of Fig. 1(b)). Remarkably, this effect persists when we increase both $h$ and $L_x$, leaving their ratio constant (Fig. S9). Yet, experiments have provided the evidence that the precursors length $\ell$ advances by periodical discrete leaps of roughly equal size, which take place at nearly constant increments of $F_N$. Moreover, this periodicity exhibits an apparent scaling with $h$, becoming larger with increasing $h^{\gamma}$. While the envelope of the curves reproduced by our model do show periodicity in the increments of the precursors sizes, the size of these discrete jumps and the corresponding increments of $F_N$ seem to remain unaffected by varying $h$, at least within the range of heights used in the experiments.

We explore further the dependence of precursors on the sample geometry by considering a different loading condition in which the lateral force is applied uniformly on the sample side surface $(2\Delta h = L_z)$. In this case, we find that the precursors are size independent when we vary $L_x$ and $L_z$ keeping their ratio $L_x/L_z$ constant (Fig. 2(a)) or $L_x$ and $L_z$ with constant $L_x/L_z$ (Fig. 2(b)). When we vary instead $L_z$ and $L_z$, keeping constant $L_x/L_z$, no universality is found and precursors again tend to disappear for large sample sizes (Fig. 2(c)). A similar effect is obtained using mixed mode loading as in Refs. 2, 3, applying simultaneously a shear force on the top surface and on the trailing edge. As the ratio between both forces $n = F_N^\text{top} / F_N^\text{lat}$ increases, the length of the precursor shrinks (Fig. 2(d)) and disappear when loading is only applied on the sample top plate.

Precursors are defined by detecting the decay of the real area of contact. This feature is perfectly reproduced by our model, and,
roughly speaking, it is ultimately due to the detachment of regions of the frictional interface satisfying the local static friction condition (19). Thus, the independence of the precursor envelope profile on the sample geometry should reflect the properties of the Coulomb stress across the entire contact interface. In the Supplementary Information (sec. IX), we discuss how some general aspects of the precursor shape can be deduced from the symmetry of the Green function.

**Normal and shear stresses at the interface and the Amontons law.**

Direct experimental measurements of shear and normal stress profiles close to the contact interface show that the Coulomb stress can exceed zero locally, without inducing any detachment front, precursor or local slip\(^2,3\). This is puzzling since it would represent a local violation of the Amontons-Coulomb law, suggesting that the friction coefficient might not be a material constant. In our model, however, the local and global friction coefficient \(\mu\) is fixed across the whole interface, and local stresses in Refs. 2, 3 are measured on a reference plane located at a height of \(z_p = 2\) mm above the frictional interface.

Thanks to the analytical solvability of our model we can compute the shear and normal stresses at any points (\(x, y, z\)) of the slider bulk: this is performed in the Suppl. Mat. sec. VIII (see Eqs. (S71), (S81) or Eqs. (S86), (S89)). Calculating the stresses on the plane \(z_p = 2\) mm yields a good quantitative agreement with experiments (Fig. 3(a), S10). In particular, the curves shown in Fig. 3 represent the shear and normal stresses averaged over the y direction (\(\tau(x, z_p) = \int_0^y dy \tau(x,y,z_p)/L_y \) and \(\sigma(x, z_p) = \int_0^y dy \sigma(x,y,z_p)/L_y\), along the entire sample \(0 < x < L_x\), just before the onset of the first precursor, i.e. when no detachment is yet present at the contact interface. This can be seen also from Fig. S10 where the full quasi-static dynamics of \(\tau(x, z_p)\) and \(\sigma(x, z_p)\) are plotted. The corresponding Coulomb stress on the same plane (\(\tau_c(x, z_p) = |\tau(x, z_p) - \mu(\sigma(x, z_p))|\) or \(\tau^{\text{CS}}_c[i] = |\tau_{zn}[i] - \mu(\sigma[z][i])|\) from Eq. (S91)) prior to the nucleation of the first precursor event is reported in Fig. 3(b) (solid green line) showing again a good agreement with the experimental data.

Our result may suggest the observed apparent violation of the Amontons first law\(^2,3\) could be due to the fluctuation undergone by the internal stresses in the material bulk, even in the vicinity (but not on) the slider frictional interface. Defining a local friction coefficient as the ratio \(\mu(x, y, z) = |\tau(x, y, z)|/|\sigma(x, y, z)|\), is not an eligible procedure if the point (\(x, y, z\)) does not lie on the frictional plane (\(x, y, 0\)). To substantiate this statement, in Fig. 3(b) we compare the y-averaged Coulomb stress on the plane \(z_p = 2\) mm above the slider-bottom surface (\(\tau_c(x, z_p)\), solid green curve) with the corresponding quantity at frictional the interface \(\tau_c(x) = \int_0^y dy \tau_c(x,y)/L_y\) (solid magenta curve). As it can be clearly seen, the Coulomb stress value may suffer large fluctuations according to the sample position where it is measured. Although the authors of the experiments in\(^4\) were careful to perform the measurements at locations \(x\) to "avoid the effects of large stress gradients", the agreement shown in Fig. 3 and the analytical

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**Figure 2** | The dependence of slip precursors on the sample aspect ratios and on the loading conditions. Rescaled precursors quasi-static evolution obtained when an uniform side shear \(F_N^{\text{lat}}\) is applied. (a) Curves exhibit the same universal behavior for different \(L_x\) and \(L_y\) but same aspect ratio \(L_x/L_y \simeq 6.1\), with \(F_N = 4\) kN and \(L_x = 10\) mm. (b) Perfect collapse of the curves is obtained when the aspect ratio \(L_x/L_y\) and \(L_y\) are kept constant. \(L_x/L_y \simeq 2\), \(L_y = 7\) mm, \(F_N = 4\) kN. (c) Precursors progressively disappear when \(L_x\) and \(L_y\) are increased by leaving unchanged the ratio \(L_x/L_y\) and \(L_y\) is constant. \(L_x/L_y \simeq 0.36\), \(L_x = 40\) mm, \(F_N = 4\) kN. This findings are consistent with the assumption that precursors evolution profiles reflect the same symmetries appearing in the shear stress at the frictional interface S97, which is function of the quantity \(R = L_x/L_y L_z\). (d) While simultaneously loading the sample from top and from the edge with a rod (\(h = 6\) mm and \(\Delta_h = 2\) mm), \(F_N^{\text{lat}} = n F_N\), precursors dynamics is suppressed for large \(n\). Sample parameters are \(F_N = 2.7\) kN, \(L_x = 201\) mm, \(L_y = 7\) mm \(L_z = 132\) mm.
largely influenced by the shear force $F_S$ by means of the FEM software. In particular we stress once again that (a), and also when a portion of the contact area is disconnected from this is seen before any detachment occurred at the interface (panel stress make any claim about the local value of $t$ considering a fully tensorial elasticity model. As a matter of fact, in calculations in Suppl. Mat. sec. VIII demonstrate that large fluctuations of Coulomb stress are also present while the droplet has formed: $D_{\text{transition from an initial stable configuration to a second on which $E_{\text{stable}}$, $F_{\text{stable}}$ disorder-induced droplet, th erefore sliding occurs either by $D_{\text{stable}}$, means which sample dimensions $L_x, L_y, L_z$ roughness $w$ and force $F_S(< \mu F_N)$ give a $\Delta E(r)$ with a positive maximum (with $r < L_x, L_y, L_z$). Unfortunately, due to the intricacy of expressions (S92) and (S94), we could obtain the answer only by numerical simulations. However, albeit one cannot completely exclude that detachment regions appear on length-scales which are well below our and experimental resolution ($\sim 1$ mm), the set of parameters used in our simulations and in the experiments does not allow for a stable disorder-induced droplet, therefore sliding occurs either by precursors nucleation from the trailing edge or as first-order phase transition for top shearing. On the other hand, it is expected that in thin films ($L_y/L_z \rightarrow \infty$, $L_x/L_z \rightarrow \infty$), interfacial disorder may induce a droplet nucleation of the kind predicted in Ref. 25. However, since the calculation of $\Delta E(r)$ involves two equilibrium configurations, a quasi-static model is the right candidate to tackle it.
Discussion

In this paper we have introduced a scalar model for the onset of frictional sliding of a three dimensional elastic object resting on a rough surface. We have devised a scalar elasticity model which allows an analytical treatment of the relevant quantities, and the straightforward implementation of the quasi-static dynamics. This model incorporates, for the first time, mesoscopic laws of contact mechanics at the frictional interface, reproducing with remarkable precision Archard’s law and Amonton’s first and second laws. Most importantly the scalar model is capable of reproducing with good accuracy the real contact area dynamics, the precursors’ envelope dynamics prior to the transition to sliding, and the normal and shear internal stress distributions close to the slider-substrate interface. The model stems from a strong Ansatz, namely that the components of displacements $u_x$ and $u_y$ are decoupled and $u_z \approx 0$. However, the solution of the model is exact: if one accepts the initial Ansatz, one has at hand

Figure 4 | Full Coulomb stress at the frictional interface. (a)–(d) Quasi-static evolution of the Coulomb stress (averaged over $y$) along the slab $x-z$ plane (on top), and on the plane $z = z_P$ (bottom panels). Color code indicates regions where $\tau_C > 0$ (yellow–red) from those for which $\tau_C < 0$ (blu), grey solid lines correspond to the set of points fulfilling $\tau_C = 0$. Panel (a) refers to the slider situation before the first precursor event nucleates, the plane $z = z_P = 2$ mm (bottom panels) is where quantities in Fig. 7 are calculated (see also Fig. S10 dashed black lines). Grey dashed lines represent the precursor envelope $\ell$ at the frictional interface, obtained from the real contact area decay (see Fig. 2(b)(inset) and Fig. 3).
that is modelled by a set of elastic asperities of height
the analytical expression for any physical observable in static equilibrium. The numerical implementation of the model is required to take into account the statistical heterogeneity inherent to the asperity disorder of the underlying rough substrate.

The first limitation of our model has been discussed previously, and consists in neglecting the Poisson expansion and the sample torque, which can have strong implications only in the case of top uniform shearing conditions, although for very large samples the scalar models conclusions should be respected. Hence no firm general conclusions nor predictions on the occurrence of frictional sliding and precursor dynamics can be drawn based on these observations. Earthquakes faults are mostly driven uniformly from an equilibrium point. A normal force $F_N$ and a shear force $F_S$ are applied uniformly on the top surface along $x$ and $y$ respectively, a lateral force $F_L$ is applied on the sample trailing edge over a rectangular region of width $2\Delta h$ at height $h$. The bottom surface of the block is discretized on a grid of size $\Delta x \times \Delta y$ (we invariably chose $\Delta y = \Delta x = 1$ mm, see also Fig. S1). Each grid element central point may form an elastic contact with the rough surface, that is modelled by a set of elastic asperities of height $h_{as}^0$ and effective transverse and normal stiffness $k_x$ and $k_{z}$ respectively.

Second, our model may assess which of the observed experimental features are due to the out-of-equilibrium dynamics and which are mostly due to equilibrium properties. For instance, our model is able to recover the precursors steps and the shape of their envelope, but fails to reproduce the increase in the precursors waiting times when the shear is applied at higher and higher $h$. Thus, we can conclude that this intriguing aspect is probably due to inertial effects present when shear is applied through an external spring (see Eq.(22)). To check this experimentally, it would be sufficient to change the spring displacement $U_s$ rate and detect any possible change in the leaps phenomenology. To the contrary, our model allows to establish that the occurrence of precursors is in fact a quasi-static physical process.

Any of the equilibrium states reached by the slider during the adiabatic evolution, is just one of the meta-stable configurations in which the system can dwell. This large number of meta-stable states is mainly due to the disorder heterogeneity of the roughness at the interface, and to a much minor degree to the rules adopted to detach the contacts when they satisfy the condition $\tau_c(x, y, 0) > 0$.

Because of its quasi-static nature, our model cannot reproduce the detachment front dynamics. According to the definition provided in Refs. 1, 2, 4, 27, 33 a detachment front indicates a drastic reduction of the real area of contact which takes place on time scales which are roughly in the millisecond range. The entire precursor experiment occurs instead over a few minutes$^{32,37}$. Experiments have revealed three different types of crack-like rupture fronts, slow, sub-Rayleigh, and interseismic (or supershear), according to their propagation velocity through the frictional interface. Precursors advance by arrested front propagation: discrete increments, indeed, occur by rupturing the contact interface at a velocity which corresponds to sub-Rayleigh fronts at the beginning, and to slow fronts close to the sliding transition$^{37}$. In particular a final slow front is responsible of the static-to-dynamics frictional sliding. Our model does not capture the crack-like propagation of fronts, since the fronts are a dynamical out-of-equilibrium processes in between two equilibrium states, namely between precursors. Nevertheless, our model might substantiate the experimental observation on the relation between precursors appearance and slow fronts triggering the frictional sliding. As a matter of fact, in Fig. 3 we were able to reproduce quantitatively the shear and normal stress profiles before any precursor nucleation occurred. In Ref. 2 these stress distributions were related to the ensuing slow rupture front (see Fig. 6A in$^{7}$). Thus it is possible to argue that whenever we observe a precursor activity, the transition to sliding is triggered by slow fronts.

To summarize the central finding of our work, three dimensional finite body scalar Green’s function makes it possible to investigate the dependence of many physical observables on any sample parameter. Our results show that the evolution of the fronts depends in a non-universal way on the loading conditions and the sample dimensions and shape. Only for some loading condition, the precursors follow a curve that allows for a simple universal rescaling in terms of the sample dimension: this prediction can be experimentally checked. Moreover we have shown that large stress gradients take place not

Figure 5 | Graphical illustration of the model. We consider a block of dimensions $L_x$, $L_y$ and $L_z$ in contact with a rough surface (sketched in the middle panel). A normal force $F_N$, and a shear force $F_S$ are applied uniformly on the top surface along $x$ and $y$ respectively, a lateral force $F_L$ is applied on the sample trailing edge over a rectangular region of width $2\Delta h$ at height $h$. The bottom surface of the block is discretized on a grid of size $\Delta x \times \Delta y$ (we invariably chose $\Delta y = \Delta x = 1$ mm, see also Fig. S1). Each grid element central point may form an elastic contact with the rough surface, that is modelled by a set of elastic asperities of height $h_{as}^0$ and effective transverse and normal stiffness $k_x$ and $k_{z}$ respectively.
only at the frictional interface but also within the material bulk. These gradients are mainly due to the way the external shear is applied to and the sample geometry, on top of frustrated Poisson expansion and elastic torque. Hence no firm general conclusions nor predictions on the occurrence of frictional sliding and precursor dynamics can be drawn based on these observations.

Earthquakes faults are mostly driven uniformly from a distance implying that, in average, precursor activity should not be present. Stress gradients and hence precursor activity could, however, arise either due to local heterogeneities or because the fault plane is tilted with respect to the earth crust 39. Measurements of local variations of the Coulomb stress around earthquake faults have been used to assess the correlation between stress accumulation and earthquake triggering 15-17. Predicting earthquakes based on slip or stress accumulation has been so far an elusive task 18-19, and the reason behind this failure can be addressed in the scenario pictured in our analysis. Indeed, as illustrated, we find that for loading condition leading to large stress gradients, the evolution of the Coulomb stress measured above the contact interface provides only a rough indicator of the ensuing detachment front dynamics, which instead appears to be very well illustrated, we find that for loading condition leading to large stress can be addressed in the scenario pictured in our analysis. Indeed, as shown in Fig. 5(a), the scalar elasticity yields that the stress tensor satisfies \( \sigma_{zz} = 0 \) (where \( k = x, y, \text{or} z \)). Thus the scalar equations for the decoupled displacement fields take the following form

\[
E \left( 1 + \frac{1}{2} \frac{\partial^2 \sigma_{zz}}{\partial z^2} + \frac{1}{2} \frac{\partial^2 \sigma_{zz}}{\partial x^2} + \frac{1}{2} \frac{\partial^2 \sigma_{zz}}{\partial y^2} \right) \geq 0
\]

(2)

where \( E \) represents the Young’s modulus and \( v \) the Poisson’s ratio. The scalar elasticity Eqs.(2) can be analytically treatable, once one specifies the proper boundary conditions. At the equilibrium, internal stresses at the surface must counterbalance the external forces acting on the sample. Since we consider a slider of dimensions \([L_x, L_y]\) the boundary conditions for Eqs.(2) are

\[
\sigma_{xx}(x,y,z) = \frac{E}{(1 + v)} \left[ \frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right] \geq 0
\]

Moreover, the scalar elasticity yields that the stress tensor satisfies \( \sigma_{uu} = 0 \) (where \( k = x, y, \text{or} z \)). Hence the scalar equations for the decoupled displacement fields take the following form

\[
\frac{E}{(1 + v)} \left[ \frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right] \geq 0
\]

(3)

As shown in Fig. 5(a), \( F_{\text{app}} \) corresponds to a shear force in the \( x \) direction, applied to the elastic slider on a portion of the plane \((0, y, z)\) of size \( L_y \times 2\Delta h \) centered around \( z = h \) and \( h(x) \) stands for the Heavyside step function; \( F_{\text{app}} \) is a shear force (also pointing to the \( x \) direction) uniformly applied on top of the slider; \( F_{\text{app}} \) is the normal force, i.e. a force applied on the entire top plane and pointing toward \(-z\); the surface stresses \( \sigma_{yy}(x,y,0) \) represent the interaction between the elastic body and the rough underlying surface at the plane \((x,y,0)\), in the \( x \) and \( z \) direction respectively (see Fig. S1). With the boundary conditions (3), we can solve the equilibrium Eqs. (2) for the displacement fields on the slider bottom plane. In technical term, we have to solve two independent Laplace equations with von Neumann boundary conditions. The solutions of Eqs.(2) are obtained by generalizing to three dimensions the corresponding solution for the von Neumann problem in two dimensions 34. The result reads

\[
u(x,y,0) = u_x + \frac{1}{E} \left[ \int_0^h \frac{\partial u_y}{\partial z} dz \frac{\partial u_x}{\partial z} dG(x_z,\zeta;y,\eta) + \int_0^h \frac{\partial u_x}{\partial z} dz \frac{\partial u_y}{\partial z} dG(x_\eta,\zeta;y,\zeta) \right]
\]

(4)
\[ u_c(x,y,0) = (u_c) + \frac{1}{E} \left\{ \int_0^{l_1} \int_0^{l_2} dy \, g(x,z,y;0;0,0) \phi_{cm}(z,\eta,0) - \int_0^{l_1} \int_0^{l_2} dy \, g(x,z,y;0;0,0)(\eta) \right\} \]

where \( G(x,z,y;\zeta,\zeta) \) is the Green function:
\[
G(x,z,y;\zeta,\zeta) = \frac{1}{8 \pi L_x L_y} \sum_{n=1}^{N} \sum_{y=1}^{N} \left( \cos \left( \frac{\pi \zeta}{L_x} \right) \cos \left( \frac{\pi \eta}{L_y} \right) \cos \left( \frac{\pi \zeta}{L_x} \right) \cos \left( \frac{\pi \eta}{L_y} \right) \right)
\]

\[ u_{cm}(x,y,0) = (u_{cm}) + \frac{1}{E} \int_0^{l_1} \int_0^{l_2} dy \, g(x,z,y;0;0,0) \phi_{cm}(z,\eta,0) \]

The last step is to provide an adequate analytical expression accounting for the effects of the interface on the slider mechanics, this is done by introducing the surface forces \( \phi_{surf} \) and \( \phi_{cm} \) in Suppl. Mat. II. We derive the expression of these forces, according to the contact mechanics theories. In first approximation they are both linear in \( P \) and \( \tau \). Indeed, we first check for the equilibrium along the x-direction, where \( \phi_{surf}(x,y) = \left\{ k_s(x,y) |u_s(x,y,0)| - u_s(x,y,0) \right\} \]

\[ u_s(x,y,0) = (u_s) + \frac{1}{E} \int_0^{l_1} \int_0^{l_2} dy \, g(x,z,y;0;0,0) k_s(x,y) u_s(x,y,0) \]

where \( k_s(x,y) = c_s \exp(-u_s(x,y,0)/u_0(x,y)) \) and \( u_s(x,y,0) < 0 \). The constant \( c_s \) appearing in the expression of \( k_s \) and \( k_c \) are the only two adjustable parameters that our model encompasses (see the next subsection and Suppl. Mat. VI). The expressions for \( k_s \) and \( k_c \) respect the laws of contact mechanics\(^{20,21} \) and are entirely motivated by: the transverse (or tangential) stiffness \( k_s \) of PMMA is indeed proportional to the normal load\(^{22} \), which in general decreases exponentially with the vertical elastic displacement\(^{23} \); the normal stiffness \( k_c \sim -dP/du_0 \), where \( P \sim -u_0/u_0^2 \) is the squeezing pressure\(^{24} \). Therefore, internal stresses are not decoupled at the interface, as they are connected via the local normal pressure entering the definition of the tangential stiffness \( k_s \).

Finally, introducing the linear expressions (8) and (9) into the Eq.s (5) and (4) respectively, we obtain closed equations for the displacements at the contact plane:
\[ u_c(x,y,0) = (u_c) + \frac{1}{E} \left\{ \int_0^{l_1} \int_0^{l_2} dy \, g(x,z,y;0;0,0) k_s(x,y) u_s(x,y,0) + \right\}
\]

where \( \phi_{surf}(x,y) = -k_c(x,y) u_s(x,y,0) \)

and
\[ u_s(x,y,0) = (u_s) + \frac{1}{E} \int_0^{l_1} \int_0^{l_2} dy \, g(x,z,y;0;0,0) k_c(x,y) u_s(x,y,0) \]

In the former expressions for the interfacial displacements, terms involving the contributions arising from the external shear or normal forces \( F_x \) and \( F_y \) can be calculated analytically. This is, indeed, one of the novelties that our model introduces and the complete expression of the Green function (see Eq.(6)) allows the determination of any of the force-induced components in the interfacial displacements equations. It will be clear in the next sections that this property entails the critical interpretation of the experimental and numerical results and, in particular, it furnishes precise predictions on the precursors’ appearance and dynamics and their dependence on the slider dimensions. In the Suppl. Mat. III we give the full analytical derivation of the terms proportional to \( F_x \) and \( F_y \) appearing in Eq.(10) and (11). By this, to simplify the displacements expressions, we introduce the following shorthand notations:
\[ u_{cm}(x,y,0) = \left( - \frac{1}{E} \int_0^{l_1} \int_0^{l_2} dy \, g(x,z,y;0;0,0) P \right) \]

\[ u_{cm}(x,y,0) = \left( - \frac{1}{E} \int_0^{l_1} \int_0^{l_2} dy \, g(x,z,y;0;0,0) F_y \right) \]

\[ u_s(x,y,0) = \left( - \frac{1}{E} \int_0^{l_1} \int_0^{l_2} dy \, g(x,z,y;0;0,0) F_x \right) \]

thanks to which the Eq.s (10) and (11) take the form
\[ u_s(x,y,0) = (u_s) \]

Discretization, numerical solution and quasi-static dynamics. The slider interfacial normal, i.e. the solutions of the Eq.s (15) and (16), are achieved by discretization of the slider bottom plane. As a matter of fact, we place a square grid on the contact plane, composed by elements having an area \( \Delta x \times \Delta y \), so that \( L_x = N_x \Delta x \) and \( L_y = N_y \Delta y \) with \( \Delta x = 1 \) mm (see Fig. S1). Albeit the terms in Eq.s (15) and (16) are defined on the entire contact plane \( (x, y) \), we calculate them only on each single point \( (x, y) \), which is the center of the grid element. This is the case, by instance, of the surface forces \( \phi_s \) and \( \phi_c \) in Eqs.(9) and (8) respectively, which are formally distributed: we interpret them as acting effectively only on the grid center point, representative of the enclosed area \( \Delta x \times \Delta y \), as shown in Fig. S1.

In the Suppl. Mat. IV we report the formal derivation of the discretization technique, which leads to the following expressions for the linear inversion Eq.s (15) and (16):
\[ u_s(x,y,0) = \sum_{j=1}^{N_x} A_{1}^s \left[ u_{cm}(x,y,0) + u_{cm}(x,y,0) + (u_s) \right] \]

\[ u_s(x,y,0) = \sum_{j=1}^{N_x} A_{1}^s \left[ u_{cm}(x,y,0) + u_{cm}(x,y,0) + (u_s) \right], \]

where the matrices \( A_{1}^s \) and \( A_{1}^s \) are defined in Eqs.(S44) and (S45) respectively and the vector \( u_{cm}^S \) in Eq.(S46). With the former expressions at hand, we can calculate the equilibrium displacements along \( \hat{x} \) and \( \hat{z} \), compatible with given values of the external normal and shear forces: we hereby recall that the two constants \( (u_s) \) and \( (u_z) \) are set to ensure that the surface forces counterbalance the external loads (see Eqs.(S48) and (S50)).

In a typical simulation, external shear forces are increased quasi-statically and the actual values of the local interfacial displacements are calculated numerically at the discretized bottom interface thanks to Eqs.(18) and (17) respectively (see Fig. S1). Indeed, we first check for the equilibrium along \( \hat{x} \) and secondly along \( \hat{z} \). Contact springs are disconnected, i.e. irreversibly broken, whenever the local Coulomb stress satisfies
\[ \tau_c(x,y,0) = |\tau_{cm}(x,y,0) - \mu | \]

where \( \mu \) represents the static local friction coefficient set to \( \mu = 0.5 \). When the condition (19) is fulfilled, the corresponding interface portion is detached from the underlying surface resulting in a local slip event. Every time a spring is broken, the equilibrium Eq.s (18) and (17) are recalculated with the new boundary condition, i.e. setting to 0 the interfacial forces corresponding to the broken spring.

The overall sliding occurs when one of the interfacial contacts has survived, i.e. when \( \tau_c(x,0) > 0 \) across the entire bottom plane (see the flowchart in Fig. S1). This happens when the static friction force is equal to \( F_{\text{SN}} \), satisfying the Amonton’s first
Figure 7 | Real area of contact and determination of precursors size. (a) Successive snapshots of the contact area, normalized to its initial value, show the advancement of the slip precursors as $F_{st}^{\text{lat}}$ is increased. Dark (pale) blue indicates a decrease (increase) in the contact area. (a)–(c) Quasi-static evolution of the average real area of contact $A_R(F_S) = \int_0^{L_y} dy A_R(x,y)/L_y$ (normalized to the zero-shear value $A_R(0)$) for three type of loading conditions: with a rod ($h = 6 \, \text{mm}$, $\Delta_n = 2 \, \text{mm}$) (b), uniformly from a side (c), and uniformly from top (d). $F_N = 2.7 \, \text{kN}$, $L_x = 200 \, \text{mm}$, $L_y = 7 \, \text{mm}$, $L_z = 75 \, \text{mm}$.

Color map goes from blue ($A_R(F_S)/A_R(0)<1$) to red ($A_R(F_S)/A_R(0)>1$): for any value of $F_S$, blue region corresponds to the precursor size, and the boundary between blue and red/yellow regions represents the precursor size $\ell$. Regions close to the trailing edge experience a decay of the real contact area as $F_S$ is adiabatically increased, whereas the real area of contact considerably grows on the opposite side ((a) and (b)). When the sample is loaded uniformly from top, the sliding takes place without precursors appearance (c).

The normal stiffness is obtained by measuring the real contact area as a function of the normal load when no shear is applied, and tuning $c_z$ until the resulting area matches that reported in Ref. 31 (see Fig. 6(a)). Indeed, as detailed in the Suppl. Mat. sec. VI, we define the total real area of contact (20) in the discrete form as

$$A_R = \sum_{t=1}^{N_T} \Delta x \Delta y \int_{-1}^{1} \int_{-1}^{1} dA_R(x,y).$$

Changing the value of the constant $c_z$ corresponds to change the equilibrium set of $u_i[\ell]$ given by Eq.(18): higher is $c_z$, stiffer are the interfacial springs, smaller will get the corresponding real contact area. The best value for $c_z = 1.65 \times 10^5 \, \text{N/m}^3$ yields the curve reported in Fig. 6(a), showing a remarkable agreement with the experimental observation.

To determine the transverse stiffness, we compare the quasi-static evolution of $A_R$ detected in experiments with the corresponding one obtained from simulations (Fig. 7(a)). In particular, we consider a block of dimensions $L_y = 140 \, \text{mm}, L_z = 6 \, \text{mm}$, and $L_x = 75 \, \text{mm}$ under a normal load $F_N = 3.3 \, \text{kN}$ and increase adiabatically the lateral force $F_{st}^{\text{lat}}$ applied at height $h = 6 \, \text{mm}$ as in Ref. 4. As shown in Fig. 6(b), as the lateral shear force is increased, the portion of contact area close to the trailing edge decreases drastically. According to the definition furnished in Ref. 4, precursors correspond to the regions of the frictional interface which undergo a reduction of the area of real contact, for values of the applied shear well before the static frictional force. A pictorial view of the adiabatic precursor evolution is reported in the inset of Fig. 6(b), where the color code represents the variation of the average local contact area with respect to its value at $F_S = 0$. The boundary between the portion of contact surface which has decreased and that which has increased during the shearing process, determines the precursor size $\ell$. This yields a curve that we compare with experiments to estimate the best value of $c_z = 1.65 \times 10^5 \, \text{N/m}^3$ (see the caption of Fig. 6(b) and Suppl. Mat. sec. VI for more details).

Loading mode and stick-slip events. Throughout the paper, we will consider the conceptually simple case in which the sample is loaded by imposing a constant shear force on the appropriate boundaries. This means that we will adopt $F_S$ as the adiabatic variable (quasi-static parameter), and calculate the equilibrium interfacial
displacement from Eqs.(17) and (18) each time that $F_2$ is slowly increased. This leads to discrete “leapfrog” precursors for which we study the continuum envelope as reported for instance in Fig. 6(b). However, the discrete nature of the precursors dynamics is more apparent if we drive the system as in the experiments reported in Ref. 4, where the lateral force is applied through a spring with elastic constant $K_S = 4 \times 10^9 \text{N/m}$. To model this case, we replace the external force appearing in Eq.(4) with the expression

$$F^\text{ext} = K_S(U_s - (u_s))$$

(22)

where $U_s$ is the externally applied displacement, which now corresponds to the adiabatic adjustable parameter, $(u_s)$, on the other hand, has the same meaning as the force-controlled protocol. In Fig. 8, we report the evolution of the spring force (Eq.(22)) as a function of the applied displacement for a typical simulation. Small stick-slip events, corresponding to discrete precursors leaps, are shown in the inset of Fig. 8, closely resembling the experimental observations. Increasing the displacement further leads to larger stick-slip events that in the constant-force case correspond to the last system size spanning event, that leads to the slip of the entire block. More details on the solution of the elastic equations for this particular case can be found in Suppl. Mat. sec. VII.

Our model does not encompass the rejuvenation of the real area of contact once the precursor has passed, because once a spring is broken no rehealing is allowed. However, in previous models such as those in Refs. 10, 11, 13, 40, once a precursor has detached a portion of interface, the corresponding interfacial contacts always reform, and the whole previous precursor path is broken again by each new precursor. This clearly contradicts the experimental evidence, where no rehealing of the real area of contact can be appreciated between subsequent precursor jumps, and the discrete jumps in the precursors dynamics can be observed only by displaying the derivative $dA_s/dt$ (see by instance Fig. 6(a) of Ref. 4 or Fig.14 of Ref. 27).

1. Rubinstein, S. M., Cohen, G. & Fineberg, J. Detachment fronts and the onset of dynamic friction. Nature 430, 1005 (2004).
2. Ben-David, O., Cohen, G. & Fineberg, J. The Dynamics of the Onset of Frictional Slip. Science 330, 211 (2010).
3. Ben-David, O. & Fineberg, J. Static Friction Coefficient Is Not a Material Constant. Phys. Rev. Lett. 106, 254301 (2011).
4. Rubinstein, S. M., Cohen, G. & Fineberg, J. Dynamics of Precursors to Frictional Sliding. Phys. Rev. Lett. 98, 226103 (2007).
5. Maegawa, S., Suzuki, A. & Nakano, K. Precursors of Global Slip in a Longitudinal Line Contact Under Non-Uniform Normal Loading. Tribology Letters 38, 313–323 (2010). URL http://dx.doi.org/10.1007/s11249-010-9611-7.
6. Braun, O. M., Barai, I. & Urbakh, M. Dynamics of transition from static to kinetic friction. Phys Rev Lett 103, 194301 (2009).
7. Bouchbinder, E., Brener, E. A., Barai, I. & Urbakh, M. Slow Cracklike Dynamics at the Onset of Frictional Sliding. Phys. Rev. Lett. 107, 235501 (2011). URL http://link.aps.org/doi/10.1103/PhysRevLett.107.235501.
8. Amundsen, D., Scheibert, J., Thegersen, K., Tromborg, J. & Malthe-Sørenssen, A. 1D Model of Precursors to Frictional Stick-Slip Motion Allowing for Robust Comparison with Experiments. Tribology Letters 45, 357–369 (2012). URL http://dx.doi.org/10.1007/s11249-011-9894-3.
9. Bar Sinai, Y., Brener, E. A. & Bouchbinder, E. Slow rupture of frictional interfaces. Geophysical Research Letters 39, L03308–1–6 (2012).
10. Caporozza, R. & Urbakh, M. Static friction and the dynamics of interfacial rupture. Phys. Rev. B 86, 085430 (2012). URL http://link.aps.org/doi/10.1103/PhysRevB.86.085430.
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Author contributions
A.T., A.B. and S.Z. designed the model, A.T. performed analytical calculations, A.B. wrote the code, A.B. and A.T. performed numerical simulations, A.B. and A.T. analyzed the data and prepared the figures, A.B. prepared the movies, A.T. wrote the supplementary text, S.Z. and A.T. wrote the manuscript. S.S. elaborated and performed the simulations on FEM model. All authors discussed the results and implications and commented on the manuscript at all stages.

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