Quantum effects in gravitational wave signals from cuspy superstrings

Diego Chialva\textsuperscript{1,2} and Thibault Damour\textsuperscript{3}

\textsuperscript{1} International School for Advanced Studies (SISSA), Via Beirut 2-4, I-34013 Trieste, Italy
\textsuperscript{2} INFN Sezione di Trieste, Italy
\textsuperscript{3} Institut des Hautes Études Scientifiques, 35 route de Chartres, F-91440 Bures-sur-Yvette, France
E-mail: chialva@sissa.it and damour@ihes.fr

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Abstract. We study the gravitational emission, in superstring theory, from fundamental strings exhibiting cusps. The classical computation of the gravitational radiation signal from cuspy strings features strong bursts in the special null directions associated with the cusps. We perform a quantum computation of the gravitational radiation signal from a cuspy string, as measured in a gravitational wave detector using matched filtering and located in the special null direction associated with the cusp. We study the quantum statistics (expectation value and variance) of the measured filtered signal and find that it is very sharply peaked around the classical prediction. Ultimately, this result follows from the fact that the detector is a low-pass filter which is blind to the violent high-frequency quantum fluctuations of both the string worldsheet $X^\mu(\tau, \sigma)$ and the incoming gravitational field $h^\mu_{\nu}(x)$.

Keywords: gravity waves/theory, string theory and cosmology, gravity waves/experiments

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1. Introduction

Recently, there has been a renewed interest in the possibility [1] that fundamental strings of superstring theory may have astronomical sizes and play the role of cosmic strings [2]. There are indeed viable models of brane inflation [3, 4] where stable strings, of horizon size, are produced at the end of inflation [5]–[11]. The tension of these strings is expected [5, 4], [8]–[10] to lie in the range $10^{-11} \lesssim G\mu \lesssim 10^{-6}$. The upper bound of this range is however already very constrained both by pulsar timing observations [12]–[14] and by measurements of the anisotropy of the cosmic microwave background [15].

Until recently, it was thought that the gravitational effects of strings with tension $G\mu \ll 10^{-6}$ were too weak to be observable. However, it has been shown in [16]–[18] that strings with tensions in the large range $10^{-13} \lesssim G\mu \lesssim 10^{-6}$ (which includes the range expected from brane inflation models) could be detected by the gravitational wave interferometers LIGO and LISA through the observation of the gravitational wave bursts associated with cusp formation. It has long been known that cusps periodically form during the oscillatory evolution of a generic smooth string loop [19]. Geometrically, a cusp corresponds to an (isolated) point on the string worldsheet where the tangent plane to the worldsheet is null (i.e. tangent to the light-cone) instead of being time-like, as it generically is. In other words, a future-directed local light-cone (with vertex on the worldsheet) generically intersects the worldsheet along two distinct null vectors. A cusp is a special point on the worldsheet where these two null vectors coincide (or are parallel). The common null direction, say $\ell^{\mu}$, of these coinciding null vectors defines the null direction of strongest emission of the gravitational wave bursts studied in [16]–[18].

The crucial feature that makes the gravitational wave bursts associated with cusps sensitive probes of tensions as small as $G\mu \gtrsim 10^{-13}$ is the fact that, in the Fourier domain, their gravitational wave amplitude $h$ is proportional to the inverse cubic root of the frequency $f$ of observation: $h(f) \propto |f|^{-1/3}$ [16, 17].

This corresponds to a time-domain waveform proportional to $h(t) \propto |t - t_c|^{1/3}$. Note that $h(f)$ denotes here the logarithmic Fourier transform of $h(t)$: $h(f) \equiv |f| h(2\pi f)$ where $h(\omega) \equiv \int dt e^{-i\omega t} h(t)$ is the usual Fourier transform.
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This (weak) power-law dependence on the frequency, $\propto |f|^{-1/3}$, is directly related to the (weak) geometrical singularity which exists at the isolated points of the worldsheet where cusps form. As the geometrical definition of these cusp singularities depends on a classical description of the worldsheet, one might worry that they be blurred by quantum fluctuation effects (associated with the effective string length $\ell_s = (\hbar/(2\pi \mu))^{1/2}$). The main purpose of the present paper is to perform a quantum computation of the observable gravitational radiation signal from a cuspy string to investigate to what extent quantum effects might blur the special feature that makes the corresponding classical signal such a sensitive probe of small tensions. Our conclusion will be that quantum effects (both in the string dynamics which sources the signal, and in the emitted gravitational field) are utterly negligible and jeopardize in no way the measurability estimates made in [16]–[18].

The computations of the present work are quite different from the ones of previous studies of quantum effects in gravitational radiation from superstrings [20, 21]. Indeed previous studies considered the quantum spectrum of massless emission, i.e. the probability for certain massive string energy eigenstates to decay into another massive string energy eigenstate and a (unique) massless (graviton) state. Such computations would be mostly relevant if one had initially prepared the string into a particular energy eigenstate and had a detector that could observe individual outgoing gravitons. However, we are concerned here with quite a different physical situation. The detectors we are interested in (LIGO, LISA, ...) do not detect individual gravitons but measure instead a certain (quasi-classical) filtered wave amplitude. In addition, we shall argue that the massive string states we are interested in are not typical energy eigenstates but, instead, some quasi-classical coherent states. As a consequence of this special physical situation we shall not be able to express our quantum computation within the usual string perturbation formalism, but will resort to a mixture of first quantized strings and second quantized gravitational field. We leave to future work a derivation of our approximate results from a fully consistent string theory framework.

2. The two types of quantum effects in filtered gravitational wave signals from strings

Let us motivate our discussion by considering, as model problem, a second quantized field theory where a massless field $h(x)$ is coupled to a quantum source $J(x)$,

$$S = \int d^Dx \left[ \frac{1}{2} h(x) \Box h(x) + h(x) J(x) + \cdots \right],$$

where the ellipsis concern the dynamics of the quantum variables entering the definition of the source $J(x)$. (In our application the latter variables will be the string worldsheet coordinates $X^\mu(\tau, \sigma)$.) For simplicity, we suppress all Lorentz indices and write equations

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5 It will sometimes be convenient to consider units where $c = 1$ but where $\hbar$ is not set to one, and to correspondingly define the string tension $\mu$ with (classical) units [mass]/[length]. In these units the combination $G\mu$ is dimensionless, independently of $\hbar$, and naturally enters classical gravitational wave calculations.

6 Let us note that [22] has shown that the waveform of gravitational wave bursts from cusps is robust against the presence of (classical) small-scale wiggles on the string.

7 The fact that superstring theory is, essentially, only defined as a first quantized theory of string states makes it technically difficult to start from string transition amplitudes to discuss all the quantum effects we want to discuss. This is why we find convenient to use such a second quantized field theory model.
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as if the field \( h(x) \) were a scalar. We have, however, in mind a massless spin-2 field \( h_{\mu\nu}(x) \) (so that the kinetic operator \( \Box \) in equation (2.1) should be replaced by a suitably gauge-fixed version of the Einstein–Pauli–Fierz kinetic operator; we shall define our normalization of \( h_{\mu\nu} \) below). The Heisenberg quantum equation of motion for the field \( h(x) \) reads

\[
\Box h(x) = J(x). \tag{2.2}
\]

If \( G_{\text{ret}} \) denotes the retarded Green’s function \( (-\Box G_{\text{ret}}(x) = \delta^D(x)) \) the second quantized field operator \( h(x) \) can be written in the form (see, e.g., [23]\(^9\))

\[
h(x) = h_{\text{in}}(x) + \int d^D y G_{\text{ret}}(x - y) J(y). \tag{2.3}
\]

In this equation the \( \text{in} \) field operator \( h_{\text{in}}(x) \) is a free field \( (\Box h_{\text{in}}(x) = 0) \) which describes the incoming vacuum fluctuations of \( h(x) \). We work here in the Heisenberg picture and assume that the quantum state of the field \( h(x) \) is the \( \text{in vacuum} \ |0\rangle_{\text{in}} \).

In the following we shall consider a gravitational wave detector such as LIGO or LISA, and express its measurement in terms of the field \( h(x) \), equation (2.3), considered in the far radiation zone of the source \( J \).

As always when discussing quantum effects, it is crucial to make clear what experimental situation, and precise observable, one is considering. Here, we are considering, as basic quantum observable, the \textit{filtered output} of a gravitational wave detector looking for bursts of the form predicted by the (classical) computation of [16, 17]. The instantaneous output of a gravitational wave detector is (after suitable normalization) of the form

\[
o(t) = \zeta^{\mu\nu} \hat{h}_{\mu\nu}(t, \mathbf{x}_0) + n(t), \tag{2.4}
\]

where \( n(t) \) is the noise in the detector, \( \hat{h}_{\mu\nu}(x) \) a gauge-invariant projection\(^{10}\) of the gravitational field of equation (2.3), considered in the radiation zone, \( \mathbf{x}_0 \) the spatial location of the (centre of mass of the) detector, and \( \zeta^{\mu\nu} \) the suitably normalized\(^{11}\) polarization tensor to which the detector is sensitive. When looking, in the noisy time series of the output \( o(t) \), for a signal having a certain classically predicted behaviour \( h^{\text{predict}}(t) \propto |t - t_c|^{1/3} \), or, in the Fourier domain\(^{12}\) \( \hat{h}^{\text{predict}}(\omega) \propto |\omega|^{-4/3} \), the optimal linear filter consists in considering as basic observable the \textit{filtered output}

\[
o_t \equiv \int dt f(t) o(t) = \int dt f(t) \zeta^{\mu\nu} \hat{h}_{\mu\nu}(t, \mathbf{x}_0) + \int dt f(t) n(t) \tag{2.5}
\]

\(^8\) We use the signature ‘mostly plus’.

\(^9\) Chapter IV of the textbook [23] considers a quantum field \( h(x) \) interacting with a \textit{classical} source \( J(x) \). We simply extend here the use of the general (Heisenberg picture) result (2.3) to the case of a \textit{quantum} source \( J(x) \), associated with the dynamics of a quantized string. Rigorously speaking, one would need to start from a second quantized approach to string theory to make full sense of the formulae we write. Their physical meaning (discussed below) is, however, so transparent that we are confident of the correctness (to leading order in \( G\mu \)) of our final results.

\(^{10}\) A gravitational wave detector is sensitive only to a gauge-invariant measure of the gravitational field. For instance, one can define \( h_{\mu\nu} \), in the rest frame of the detector, by \( h_{00} = 0 \) and \( R_{0\alpha j} = -\frac{1}{2} \partial_{0j} h_{ij} \).

\(^{11}\) Say \( \zeta^{\mu\nu} \zeta^\mu_\nu = 1 \).

\(^{12}\) We use here the usual Fourier transform \( \hat{h}(\omega) \equiv \int dt e^{i\omega t} h(t) \).
where $f(t)$ is a time-domain filter function\(^{13}\) defined, in the Fourier domain, as

\[
\hat{f}(\omega) = \frac{\hat{h}^{\text{predict}}(\omega)}{S_n(\omega)},
\]

(2.6)

where $\hat{h}^{\text{predict}}(\omega)$ is the Fourier transform of the predicted signal, and $S_n(\omega)$ the noise spectral density, i.e. the Fourier transform of the correlation function of the detector’s noise:

\[
\langle n(t_1) n(t_2) \rangle = C_n(t_1 - t_2) = \int \frac{d\omega}{2\pi} S_n(\omega) e^{-i\omega(t_1-t_2)}.
\]

(2.7)

When discussing the detection of gravitational wave bursts of the form predicted in [16, 17], i.e. $\hat{h}^{\text{predict}}(t) \propto |t - t_c|^{1/3}$, one should define a filter function $f(t)$ by taking $\hat{h}^{\text{predict}}(\omega) = |\omega|^{-4/3}$ in equation (2.6), and then taking the inverse Fourier transform:

\[
f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{h}^{\text{predict}}(\omega)/S_n(\omega).
\]

Note that the optimal filter $f(t)$ differs from the expected time-domain signal $\hat{h}^{\text{predict}}(t)$ by the convolution action of a ‘whitening kernel’ with Fourier transform $1/S_n(\omega)$.

When inserting the decomposition (2.3) of the radiation field $h_{\mu\nu}(x)$ entering the filtered output of equation (2.5), we see that the basic quantum observable $\alpha_t$ can be decomposed into three contributions:

\[
\alpha_t = A_t^{\text{in}} + A_t^J + n_t,
\]

(2.8)

where, after introducing the $J$-generated field entering equation (2.3) namely,

\[
h_{\mu\nu}^J(x) \equiv \int d^3y G_{\text{ret}}(x - y) J_{\mu\nu}(y),
\]

(2.9)

we have defined

\[
A_t^{\text{in}} \equiv \int dt f(t) \zeta^{\mu\nu} \hat{h}_{\mu\nu}^{\text{in}}(t, x_0),
\]

(2.10)

\[
A_t^J \equiv \int dt f(t) \zeta^{\mu\nu} \hat{h}_{\mu\nu}^J(t, x_0),
\]

(2.11)

\[
n_t \equiv \int dt f(t) n(t).
\]

(2.12)

Viewed in the vacuum $|0\rangle_{\text{in}}$ appropriate to the Heisenberg picture we are using, the in field $h_{\mu\nu}^{\text{in}}$ has vanishing expectation value. Therefore, the first contribution $A_t^{\text{in}}$, equation (2.10), which we might call the ‘filtered in amplitude’, describes the effect of the quantum vacuum fluctuations in $h(x)$, as seen in the filtered output. In other words, the fluctuating quantum observable $A_t^{\text{in}}$ describes the shot noise due to the quantized nature of the gravitational field $h_{\mu\nu}(x)$.\(^{14}\) The second contribution $A_t^J$, equation (2.11), which is the ‘filtered gravitational wave amplitude generated by the source $J$', will be the main focus

\(^{13}\) More precisely one considers a bank of time-translated filter functions $f(t - t_0)$, with varying ‘arrival times’ $t_0$.

\(^{14}\) Our use of a second quantized description of the gravitational field $h_{\mu\nu}(x)$ allowed us to derive with ease the expression of the contribution $A_t^{\text{in}}$ to the observed filtered signal $\alpha_t$. A derivation of $A_t^{\text{in}}$ within first quantized string perturbation theory would be more cumbersome because one would need to study multi-graviton amplitudes.
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of this work, in the case where the source \( J_{\mu \nu} \) is the stress–energy tensor \( T_{\mu \nu} \) of a cosmic size superstring. Finally, the third contribution \( n_f \), equation (2.12), is simply the filtered effect of the detector’s noise (which is usually treated as a classical random variable).

Note that, contrary to the two contributions \( A^{in}_I \) and \( n_f \) which are purely fluctuating, i.e. have zero expectation values, the contribution \( A^f \) generated by the quantum stress–energy tensor \( J_{\mu \nu} \propto T_{\mu \nu}(X) \) of a superstring (considered in a highly excited state) will have a non-zero expectation value (hopefully corresponding to the classical computation of [16,17]) around which quantum effects linked to the quantized dynamics of the string will introduce fluctuations. The aim of our computation is to quantitatively estimate the magnitude of the latter fluctuations, i.e. the probability distribution function of the quantum observable \( A^f \), to see to what extent quantum fluctuations of the string dynamics might blur the classically expected signal.

3. Light-cone gauge description of cuspy string states in superstring theory

In order to compute the probability distribution function of the filtered, source-generated gravitational wave amplitude \( A^f \), equation (2.11) with (2.9), we need two ingredients: (i) a description of the quantum operator \( J_{\mu \nu} \) in terms of the string dynamical variables, and (ii) a description of the quantum state of a cuspy string. In this section, we shall discuss this second ingredient.

Let us first remark that the experimental situation we are interested in is a special one. We consider a state where, as seen from the small subsystem we are interested in (made of a massive string and of some gravitational excitations\(^{15}\)), a macroscopic external agency (the cosmological expansion) has stretched an initially microscopic size string state into a quasi-classical, macroscopic size state. If the system we were interested in was (to give a simple example) an harmonic oscillator \( \hat{X}(t) \), we would describe this blowing up, by a time-dependent external agency, to the macroscopic level, of an initially microscopic quantum state (say the ground state), by coupling the harmonic oscillator to a large, classical, time-dependent external force \( F(t) \). It is then well known that the final state of the oscillator is described (in the Schrödinger picture) by a coherent state, namely: \( |F \rangle = \exp \left[ \int dt \ F(t) \ \hat{X}(t) \right] |0 \rangle \), where \( F(t) \) is the classical force and \( \hat{X}(t) = (2\omega_0)^{-1/2} (\hat{a} e^{-i\omega_0 t} + \hat{a}^\dagger(t) e^{i\omega_0 t}) \) the position operator of the (unit mass) oscillator. Thinking of the string as a collection of oscillators, this simple example motivates us to considering that an appropriate description of the quantum state of a superstring stretched to macroscopic sizes by the cosmological expansion is a certain coherent state\(^{16}\) of the infinite set of oscillators describing \( X^\mu(\tau, \sigma) \). However, we need to take care of the constraints which gauge away the ‘time-like’ and ‘longitudinal’ oscillators to leave only the \( D - 2 \) transverse oscillators. The simplest way to do so is to work in a light-cone gauge

\[
\begin{align*}
n_{\mu} X^\mu(\tau, \sigma) &= \alpha'(n_{\mu} p^\mu) \tau,
\end{align*}
\]

\(^{15}\) That is, some extra massless string states.

\(^{16}\) We shall work henceforth within the approximation where, when discussing the distribution function of the observable \( A^f \), equation (2.11), the coupling of the string state to gravity (and, in particular, its decay under back reaction) is neglected, so that we consider that the state of the string is some given coherent state which determines the statistics of \( A^f \). This is similar to saying that, in ordinary Schrödinger quantum mechanics, the statistics of some observable \( f(\hat{X}) \) is determined by the wavefunction \( \psi(\hat{X}; t) \), supposedly known at the time \( t \) where the observable \( f(\hat{X}) \) is observed.
where $n_\mu$ is a certain fixed null vector. In this gauge, if we choose, say, $n_\mu = (\frac{1}{\sqrt{2}}, 0, \ldots, 0, \frac{1}{\sqrt{2}})$ so that $n_\mu X^\mu \equiv (X^0 + X^{D-1})/\sqrt{2} \equiv X^+$, the $X^+$ oscillators are set to zero ($X^+ = \alpha^+ p^+$), the $X^i$ ‘transverse’ oscillators (with $i = 1, 2, \ldots, D - 2$) are unconstrained, and the $X^-$ oscillators are expressed as quadratic combinations of the infinite set of transverse oscillators.

We shall therefore consider string states of the $|\alpha\rangle_R |\tilde{\alpha}\rangle_L$ where

$$
|\alpha\rangle_R = \prod_{\{n\}, \{i\}} \exp \left( -\frac{\alpha_n^i \alpha_n^{i \dagger}}{2} + \alpha_n^i \tilde{\alpha}_n^{i \dagger} \right) |0\rangle
$$

$$
|\alpha\rangle_L = \prod_{\{n\}, \{i\}} \exp \left( -\frac{\tilde{\alpha}_n^i \alpha_n^{i \dagger}}{2} + \tilde{\alpha}_n^i \tilde{\alpha}_n^{i \dagger} \right) |0\rangle.
$$

(3.2)

Beware of the somewhat unconventional notation used in this equation. The quantities $\alpha_n^i$ (with $i = 1, 2, \ldots, D - 2$ and $n = 1, 2, \ldots$) denote some given c-numbers parametrizing the right-moving part of the considered coherent state, while the $\tilde{\alpha}_n^i, \tilde{\alpha}_n^{i \dagger}$ denote the annihilation and creation operators of the $n$th right-moving mode. They are such that $[\tilde{\alpha}_n^i, \alpha_m^j] = [\alpha_n^i, \tilde{\alpha}_m^j] = \delta^{ij} \delta_{n-m}^{0}$. The quantities $\tilde{\alpha}_n^i$, $\tilde{\alpha}_n^{i \dagger}$ denote the corresponding quantities for the left-moving part of the string. In other words, the derivatives of the transverse string coordinates read

$$
\partial_{\pm} X^i (\tau \pm \sigma) = \sqrt{\frac{\alpha_n^i}{2}} \sum_n \sqrt{n} \left( \tilde{\alpha}_n^i e^{-i(n \pm \sigma)} + \tilde{\alpha}_n^{i \dagger} e^{i(n \pm \sigma)} \right)
$$

(3.3)

where $\sigma^\pm \equiv \tau \pm \sigma$, $\partial_\pm \equiv \partial/\partial \sigma^\pm = \frac{1}{2} (\partial_{\tau} \pm \partial_{\sigma})$, and where the ‘tilde’ quantities correspond to $\partial_-$ and $\tau + \sigma$ (left-movers).

Because of the Lorentz covariance of light-cone quantization (in the critical dimension\(^{17}\)) we have full freedom in selecting the special null direction $n^\mu$ entering equation (3.1). We shall follow here the following logic. We start from some given coherent state, i.e. two sequences of complex numbers $\alpha_n^i, \tilde{\alpha}_n^i$. These numbers define both a quantum state and a classical solution of the string equations of motion (in a generic light-cone gauge (3.1)). We know from [19] that, generically, this classical solution will exhibit cusps, with certain associated null directions $\ell^\mu(\alpha, \tilde{\alpha})$ of intense classical gravitational radiation emission. To simplify our computation of the quantum observable $A^\mu_I$ associated with some cusp null direction $\ell^\mu(\alpha, \tilde{\alpha})$, we wish to ensure that $\ell^\mu(\alpha, \tilde{\alpha})$ is parallel to $n^\mu$.\(^{18}\) Such a parallelism can always be realized (simply by rotating appropriately $n^\mu$). However, after such a rotation, the values of the sequences of c-numbers $\alpha_n^i, \tilde{\alpha}_n^i$ will change in a complicated manner. Actually, the new ‘specially aligned’ sequences $(\alpha^{new}_n, \tilde{\alpha}^{new}_n)$ must now behave in a non-generic way, that we need to specify.

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\(^{17}\) In the formal developments we try to keep a general $D$ and assume $D = 10$ (for the superstring). However, we shall later assume that the considered coherent state has a macroscopic extension only in $D = 4$ uncompactified dimensions. The compactified dimensions will correspond to small values of the $\alpha$s and $\tilde{\alpha}$s and will have a negligible contribution to our final results.

\(^{18}\) Actually, three different special null vectors will enter our study: the cusp null direction $\ell^\mu$, the light-cone gauge null direction $n^\mu$ and the ‘graviton momentum’ $k^\mu$ (see below). We recall that, in the light-cone gauge, the operator $X^- (\tau, \sigma) = (X^0 - X^{D-1})/\sqrt{2}$ is a complicated (quadratic) functional of the transverse oscillators. To avoid having a term $exp(ik^\mu X^-)$ in the source-generated $h_{\mu}^J$ (see below) we need $0 = k^+ = k^0 n^\mu$, which implies the parallelism of the two real null vectors $k^\mu$ and $n^\mu$. Finally, as we wish to study gravitational emission in the ‘cusp’ direction $\ell^\mu$, we shall have to require that the three null vectors $\ell^\mu, n^\mu$ and $k^\mu$ are all parallel.
We recall (see, e.g., [17]) that, in a generic conformal gauge \( \sigma^{old}_+, \sigma^{old}_- \), the left and right string coordinates\(^{19} \) behave, near a cusp (located at \( X^\mu = 0 \) and occurring at \( \sigma_+ = 0 = \sigma_- \), as

\[
X^\mu_\pm(\sigma^{old}_\pm) = \ell^\mu_\pm \sigma^{old}_\pm + \frac{1}{2} \dot{X}^\mu_\pm(\sigma^{old}_\pm)^2 + \frac{1}{6} X^{(3)\mu}_\pm(\sigma^{old}_\pm)^3 + \cdots
\]

so that (remembering that \( \ell^\mu_\pm \ddot{X}^\mu_\pm = 0 \) at the cusp)

\[
\ell^\mu_\pm X^\mu_\pm(\sigma^{old}_\pm) = \frac{1}{6} \ell^\mu_\pm X^{(3)\mu}_\pm(\sigma^{old}_\pm)^3 + \cdots.
\]

(3.4)

On the other hand, in the specially aligned 'new' light-cone gauge \( n^\mu \propto \ell^\mu \), we have, from equation (3.1), the property that

\[
\ell^\mu_\pm X^\mu_\pm = \alpha'(\ell^\mu_\mu p^\mu) \sigma^{new}_\pm.
\]

(3.5)

The conclusion is therefore that \( \sigma^{new}_\pm \sim (\sigma^{old}_\pm)^3 \). Considering the transverse components of \( X^\mu_\pm \), equation (3.4), in the new gauge, we then have

\[
X^i_\pm(\sigma^{new}_\pm) = \frac{1}{2} \dot{X}^i_\pm(\sigma^{old}_\pm)^2 + \cdots \sim (\sigma^{new}_\pm)^{2/3},
\]

so that

\[
\partial_\pm X^i_\pm(\sigma^{new}_\pm) \sim (\sigma^{new}_\pm)^{-1/3}.
\]

(3.6)

Note that, geometrically, it means that the transverse projection of the string \( X^i(\tau^{new}, \sigma^{new}) \) draws, at the light-cone moment where the cusp forms (\( \tau^{new} = 0 \)), a cuspy curve in transverse space. This is a simple geometrical consequence of having aligned our light-cone gauge precisely with the null direction associated with the cusp.

The result (3.6) applies to the classical expectation value of \( \partial_\pm X^i_\pm \). For a coherent state, this is simply given by replacing the operators \( \alpha_n^i, \tilde{\alpha}_n^i \) in equation (3.3) by \( \alpha_n^i, \tilde{\alpha}_n^i \) respectively. This means that the sequences of c-numbers \( \alpha_n^i, \tilde{\alpha}_n^i \) must have a certain power-law behaviour

\[
\alpha_n^i \sim \tilde{\alpha}_n^i \sim n^{-\gamma}
\]

as \( n \to \infty \) with (considering the right-moving modes)

\[
\partial_- X^i(\sigma_-) \propto \sum_n \sqrt{n} \alpha_n^i e^{-i n \sigma_-} \sim \sum_{n=1}^{1/\sigma_-} n^{(1/2)-\gamma} \sim \left( \frac{1}{\sigma_-} \right)^{(3/2)-\gamma}.
\]

(3.7)

Comparing to equation (3.8) determines the power index \( \gamma \) of (3.9) to be

\[
\gamma = \frac{7}{6}.
\]

(3.8)

In conclusion, we shall consider coherent string states of the form (3.2) with sequences of c-numbers \( \alpha_n^i, \tilde{\alpha}_n^i \) satisfying (3.9) with (3.11). Such states will describe a quantum version of a cusp aligned with our chosen direction of observation \( n^\mu \propto \ell^\mu \).

\(^{19} \) We follow here [17] in writing \( X^\mu = \frac{1}{2}(X^\mu_+ + X^\mu_-) \).
4. Gravitational wave signals from cuspy string states

We turn now to a description of the filtered string-generated signal $A^I_t$, equation (2.11), with $h^I_{\mu\nu}$ given by equation (2.9). Before discussing the exact form of the source $J_{\mu\nu}$ corresponding to the gravitational coupling of the string, let us consider the effect of the retarded Green’s function $G_{\text{ret}}$ in the integral (2.9). When considering a string state which is localized around some centre of mass worldline $y^\mu \simeq x^\mu_0(\tau)$ (at a cosmological distance from the Earth), and a detector located in the solar system, we can approximate the (Fourier transformed) retarded Green’s function in equation (2.9) (considered for concreteness in the physically relevant case of $D = 4$ uncompactified dimensions)

$$G_{\text{ret}}(t, x; t, y) = \int \frac{d\omega}{2\pi} C^\omega_{\text{ret}}(x; y) e^{-i\omega(t-t_\nu)}, \quad (4.1)$$

in the following well-known way

$$C^\omega_{\text{ret}}(x; y) = \frac{e^{i\omega|x-y|}}{4\pi|x-y|} \simeq \frac{e^{i\omega|x|}}{4\pi|x|} e^{-i\omega n \cdot y}, \quad (4.2)$$

where $n \equiv x/|x|$ is the unit vector directed from the origin (located, say, at the centre of mass of the string) towards the detector.

Inserting equations (4.1), (4.2) into equation (2.9) yields (see [17])

$$h^I_{\mu\nu}(t, x) \simeq \frac{1}{4\pi r} \int \frac{d\omega}{2\pi} e^{-i\omega(t-r)} \tilde{J}^\omega_{\mu\nu}(\omega, \omega n) \quad (4.3)$$

where $r \equiv |x|$ and where $\tilde{J}^\omega_{\mu\nu}(k^\lambda)$, with $k^\lambda = (\omega, k)$, is the spacetime Fourier transform of the source $J_{\mu\nu}$:

$$\tilde{J}^\omega_{\mu\nu}(k^\lambda) = \int d^D x e^{-ikx} J_{\mu\nu}(x). \quad (4.4)$$

Note the important fact that the integral in (4.3) contains only an integration over the frequency $\omega$. The integral over $d^D k$ has been effectively already performed and has led to the fact, apparent on the right-hand side (rhs) of equation (4.3), that the only $k$s entering the final result are related to the frequency by $k = \omega n$. In other words $k^\mu = (\omega, k) = (\omega, \omega n)$ is a null vector directed from the source towards the detector.

If we normalize the gravitational field $h^I_{\mu\nu}(x)$ in the geometrical (Einsteinian) way, i.e. $g_{\mu\nu}(x) = \eta_{\mu\nu} + h^I_{\mu\nu}(x)$, the source $J_{\mu\nu}$ of $h^I_{\mu\nu}$ (with $-\Box h^I_{\mu\nu} = J_{\mu\nu}$) will be

$$J_{\mu\nu}(x) = +16 \pi G \left( T_{\mu\nu}(x) - \frac{1}{D-2} \eta_{\mu\nu} T(x) \right) \quad (4.5)$$

where $G$ is Newton’s constant, and where $T_{\mu\nu}(x)$ denotes the stress–energy tensor of the source. Then, we can write $h^I_{\mu\nu}$ as (in $D = 4$)

$$h^I_{\mu\nu}(t, x) \simeq 4G \left( \frac{1}{r} \int \frac{d\omega}{2\pi} e^{-i\omega(t-r)} \left( \tilde{T}^\omega_{\mu\nu}(k^\lambda) - \frac{1}{2} \eta_{\mu\nu} \tilde{T}(k^\omega) \right) \right), \quad (4.6)$$

where $\tilde{T}^\omega_{\mu\nu}(k^\lambda)$ is the spacetime Fourier transform of $T_{\mu\nu}(x^\lambda)$, and where $k^\lambda = (\omega, \omega n)$. 

The coupling \( \int \frac{1}{2} h_{\mu \nu} T^{\mu \nu} \) between a fundamental\(^{20}\) string, of tension\(^{21}\) \( \mu \) and the gravitational field is proportional to

\[
\mu \int d^2\sigma [h_{\mu \nu}(X)(\partial_+ X^\mu \partial_- X^\nu) + h_{\mu \nu}(X)(\psi_+^\mu \partial_- \psi_+^\nu) + h_{\mu \nu}(X)(\psi_-^\mu \partial_+ \psi_-^\nu) + R_{\mu \nu \rho \sigma}(X)\psi_+^\mu \psi_-^\nu \psi_+^\rho \psi_-^\sigma].
\]

(4.7)

Because of the on-shell constraint \( \psi_\pm = \psi_\pm(\tau \pm \sigma) \) the terms \( \psi_+^\mu \partial_- \psi_+^\nu \) and \( \psi_-^\mu \partial_+ \psi_-^\nu \) do not contribute to \( T^{\mu \nu} \). As for the term quartic in the fermions, it does, \( \text{a priori} \), contribute a term \( \alpha(\delta R_{\alpha \beta \gamma \delta}/\delta h_{\mu \nu}) \psi_+^\alpha \psi_+^\beta \psi_-^\gamma \psi_-^\delta \). This term contributes to the Fourier transformed \( \tilde{T}_{\mu \nu}(k^\lambda) \) a term proportional to \( (k \cdot \psi_+)(k \cdot \psi_-) \psi_+^\mu \psi_-^\nu \).

We shall further simplify our computation by considering a detector which is, as seen from the source, precisely at the centre of the gravitational burst emitted by the cusp. In other words, we shall require that the basic null direction \( n^\mu \) defining the light-cone gauge (equation (3.1)) is not only parallel to the null direction \( \ell^\mu \) defined by the cusp, but also to the null direction (1, \( n \)) connecting the source to the detector. This implies that all the graviton momenta \( k^\mu = (\omega, \omega n) \) entering the radiation field (4.6) are also parallel to \( n^\mu \). As one sets \( \psi_\pm = 0 \) in the light-cone gauge, \( k \cdot \psi_\pm = -k^0 \psi_\pm \) vanishes in the light-cone gauge.

Finally, it is enough to consider the bosonic contribution to the stress–energy tensor \( (d^2\sigma \equiv d\tau \, d\sigma) \)

\[
T^{\mu \nu}(x) = \mu \int d^2\sigma \delta^D(x - X(\tau, \sigma))(\partial_\tau X^\mu \partial_\sigma X^\nu - \partial_\sigma X^\mu \partial_\tau X^\nu)
\]

\[
= 4\mu \int d^2\sigma \delta^D(x - X(\sigma_+, \sigma_-)) \partial_+ X^{(\mu} \partial_- X^{\nu)},
\]

(4.8)

and to its Fourier transform

\[
\tilde{T}^{\mu \nu}(k) = 4\mu \int d^2\sigma \, e^{-ik \cdot X} \partial_+ X^{(\mu} \partial_- X^{\nu)},
\]

(4.9)

which is the vertex operator for the emission or absorption of a graviton of momentum \( k \) by a string.

If we simplify formulae by rewriting the filter function \( f(t) \) in equation (2.11) as \( f^{\text{new}}(t - r_0) \) (where \( r_0 = |x_0| \) is the radial distance from the source to the detector), the filtered source-generated amplitude (2.11) reads (taking into account the tracelessness of \( \zeta^{\mu \nu} \))

\[
A^\mu_f = \frac{4G}{r_0} \int \frac{d\omega}{2\pi} \hat{f}(-\omega) \zeta^{\mu \nu} \tilde{T}^{\mu \nu}(k^\lambda),
\]

(4.10)

where \( \hat{f}(\omega) = \int dt \, e^{+i\omega t} f^{\text{new}}(t) \) is the Fourier transform of \( f^{\text{new}}(t) = f^{\text{old}}(t + r_0) \). As explained above the ‘hat’ over \( \tilde{T}^{\mu \nu} \) denotes the projection over the gauge-invariant gravitational wave amplitude affecting the detector (which is only sensitive to tidal forces). We can use the remaining freedom in our light-cone frame to ensure that the detector is ‘at

\(^{20}\) The effects linked to the quantum fluctuations around the Nambu–Goto dynamics investigated here are clearly present (possibly together with other effects) in all types of strings: fundamental, Dirichlet or even gauge theory ones.

\(^{21}\) The tension \( \mu \) denotes the effective four-dimensional tension, including eventual warp factors.
rest’ with respect to the usual Lorentz frame \((x^0, x^1, x^2, x^3)\) behind the light-cone frame (i.e. with \(x^+ = (x^0 + x^3)/\sqrt{2}\)). Then the detector’s polarization tensor will have only spatial components \(\zeta_{ij}\), \(I, J = 1, 2, 3\). In addition, the ‘hat’ projection simply consists in projecting \(h_{\mu\nu}\) on its physically active transverse traceless (TT) components \(h_{ij}, i, j = 1, 2\) only. Let us define, as usual, the TT projection operator of a symmetric spatial tensor projecting \(\equiv I_{IJ}\).

Equation (2.8) exhibited a decomposition of the filtered output, \(o_t\), of a gravitational wave detector into three terms: (i) the quantum noise \(A_{tt}^p\), equation (2.10), which describes the

\[
A_{tj}^f = \frac{4G}{r_0} \int \frac{d\omega}{2\pi} \tilde{c}_{TT}^{ij}(\frac{\omega}{2}) \frac{\omega}{2\pi} \frac{d\tau}{\omega} \zeta_{ij}^{TT} e^{i\omega/2\pi} \partial_+ X^i \partial_- X^j,
\]

in which one should insert the oscillator expansions (3.3).

Equations (4.12) and (3.3) define the quantum observable \(A_{tj}^f\) as a certain operator in the string dynamics Hilbert space. It is easy to perform explicitly the triple integration in equation (4.12): (i) the integral over \(\sigma\) yields a Kronecker \(\delta_{\text{sym}}\) between the left and right mode numbers \(e^{i\sigma(t_\mu - t_\mu')}\), \(e^{-i\sigma(t_\mu - t_\mu')}\); (ii) the integral over \(\tau\) then yields some delta functions of the frequency \(\delta((\ell_\omega/2\pi - 2n)\) or \(\delta((\ell_\omega/2\pi + 2n)\); and (iii) the integral over \(\omega\) then yields a series over \(n\). Finally, defining (as in [17]) the fundamental frequency \(\omega_1 \equiv 4\pi/\ell\), we get

\[
A_{tj}^f = 16\pi \frac{G}{r_0} \zeta_{ij}^{TT} \sum_{n=1}^{\infty} n \left[ \tilde{f}(n - \omega_1) a_n^i a_n^j + \tilde{f}(n + \omega_1) a_n^i \tilde{a}_n^j \right].
\]

5. Quantum noise contributions to gravitational wave signals

Equation (2.8) exhibited a decomposition of the filtered output, \(o_t\), of a gravitational wave detector into three terms: (i) the quantum noise \(A_{tt}^p\), equation (2.10), which describes the

\footnote{To avoid any confusion: beware that the transverse components of \(\zeta_{ij}^{TT}\) differ from the restriction \(\zeta_{ij}\) of \(\zeta_{tj}\) to its transverse components \(i, j = 1, 2\).}

\footnote{Note that, contrary to what happened in the time gauge calculation of [16,17], there is not any more a ‘cubic’ saddle point in the phase factor \(e^{-ik\cdot x}\) of equation (4.12). The fact that a cusp is a strong emitter of high-frequency gravitational waves shows up, when using a ‘cusp-aligned’ light-cone gauge, in the singular behaviour \(\partial_x X \sim (\sigma_x)^{-1/3}\) of the string coordinate gradients entering the vertex operator \(\partial_+ X^i \partial_- X^j\).}
filtered vacuum fluctuations of the gravitational field, (ii) the filtered, string-generated signal $A^I_f$, explicitly expressed as equation (4.13), and (iii) the filtered detector noise $n_f$, equation (2.12). Let us consider, for concreteness, the case of the LIGO detector. As discussed in [17], and recalled above, the optimal filter for detecting gravitational wave bursts from cuspy strings is equation (2.6) with $\tilde{h}^{\text{predict}}(\omega) \propto |\omega|^{-4/3} \cdot e^{i\omega t_0}$. The division by $S_n(\omega)$ provides a Fourier domain filter $\tilde{f}(\omega)$, equation (2.6), which is peaked around the characteristic (circular) frequency $\omega_\ast \approx (2\pi) \times 150$ Hz [17]. It is then convenient to normalize the filter function $f(t)$ entering equations (2.10)–(2.12) so that the modulus of $\tilde{f}(\omega_\ast) = \int dt f(t) e^{i\omega_\ast t}$ is equal to one. With this normalization, equation (6.5) of [17] says that the filtered detector noise $n_f$ of initial LIGO can be roughly modelled as a Gaussian variable with standard deviation $\sigma_n^i \approx 1.7 \times 10^{-22}$. Advanced LIGO might reach a level smaller by a factor 13.5, i.e. $\sigma_n^i \approx 1.3 \times 10^{-24}$. Figure 1 of [17] (and their multi-parameter generalizations in [18]) shows that the classical estimate of the string-generated signal $A^I_f$ might be comparable (and hopefully larger) than these noise levels in a wide range of string tensions $10^{-12} \lesssim G\mu \lesssim 10^{-6}$. Let us now estimate the quantum statistics of both $A^I_n$ and $A^I_f$, to see whether quantum noise can play any significant role.

Let us start by considering the vacuum fluctuation term $A^I_n$, equation (2.10), which reads, when using a TT gauge ($I, J = 1, 2, 3$),

$$A^I_n = \int dt f(t) \zeta^{IJ} h^{\text{TT}}_{IJ}(t, x_0). \quad (5.1)$$

Here, $h^{\text{TT}}_{IJ}(x)$ is a free gravitational radiation field, considered in its vacuum state. Therefore, $A^I_n$, equation (5.1), is a quantum Gaussian noise. To compute its standard deviation, let us use the two-point (Wightman) correlation function of $h^{\text{in}}(x)$:

$$\langle h^{\text{in}}_{IJ} (t, x) h^{\text{in}}_{IJ'} (t', x') \rangle = 32 \pi G \int \frac{d^3k}{(2\pi)^3} \frac{P^{\text{TT}}_{IJJ'}(k) e^{-i\omega_k(t-t') + ik(x-x')}}{2 \omega_k}, \quad (5.2)$$

where the factor $32 \pi G$ comes from the ‘geometric’ (instead of ‘canonical’) normalization of $h_{\mu\nu}$ and where $P^{\text{TT}}$ denotes the TT projector used above (here considered for the direction $k = k/\omega_k$ with $\omega_k = |k|$). We then find that the variance of $A^I_n$ (using $\zeta^{IJ} \zeta_{IJ} = 1$) is of order

$$(\sigma(A^I_n))^2 \sim G \int d\omega \omega |\tilde{f}(\omega)|^2 \sim G \omega_\ast^2 |\tilde{f}(\omega_\ast)|^2; \quad (5.3)$$

where we use the fact that the Fourier filter $\tilde{f}(\omega)$ is peaked around $\omega_\ast$. As said above, we normalize the filter $f(t)$ so that $|\tilde{f}(\omega_\ast)| = 1$. Hence, the standard deviation of the filtered gravitational vacuum noise is of order

$$\sigma(A^I_n) \sim G^{1/2} \omega_\ast \sim 2\pi f_s t_{\text{Planck}} \sim 5 \times 10^{-41} \left( \frac{f_s}{150 \text{ Hz}} \right). \quad (5.4)$$

This is clearly too small to worry about.

Let us now consider the statistical properties of the string-generated signal (4.13). As explained above, we model the state of the string by a coherent state $|\alpha\rangle_R |\bar{\alpha}\rangle_L$. In such a state the expectation values of the annihilation and destruction operators $a^+_n$, $\bar{a}^+_n$, $a^-_n$, $\bar{a}^-_n$ are, by definition, the c-numbers $\alpha_n^\dagger, \bar{\alpha}_n^\dagger$, and their complex conjugates (c.c.) $\bar{\alpha}_n^\dagger, \bar{\alpha}_n^\dagger$. 


so that the expectation value of $A_f^J$ reads

$$\langle A_f^J \rangle = 16\pi \frac{G}{\ell r_0} \zeta_{TT}^{ij} \sum_{n=1}^{\infty} (n \tilde{f}(n \omega_1) \alpha_n^J \tilde{\alpha}_n^J + \text{c.c.}).$$

(5.5)

This result is evidently equivalent to replacing the operator $X^i$ in equation (4.12) by its classical value (obtained by replacing $\alpha$ by $\alpha$ in equations (3.3)), and therefore equivalent to the results of [16, 17].

The principal novel result of this study is now obtained by considering the variance of the operator $A_f^J$ in the coherent state $|\alpha\rangle |\tilde{\alpha}\rangle$. Using $[a_n^i, a_n^J] = [\tilde{a}_n^i, \tilde{a}_n^J] = \delta^{ij} \delta_{nm}$ (and $[a, \tilde{a}] = 0$ etc), it is easily computed from equation (4.13) and found to be

$$\langle \sigma(A_f^J)^2 \rangle = \left(16\pi \frac{G}{\ell r_0} \right)^2 \sum_{n=1}^{\infty} n^2 |\tilde{f}(n \omega_1)|^2 \left| \zeta_{TT}^{ij} \zeta_{TT}^{\alpha_n^J \tilde{\alpha}_n^J} + \zeta_{TT}^{ij} \zeta_{TT}^{\alpha_n^J \tilde{\alpha}_n^J} \right|^2.$$  

(5.6)

The first important thing to notice in the variance (5.6) is its crucial dependence on the detector’s filter function $\tilde{f}(t)$. If one had considered a time-sharp filter, $\tilde{f}(t) = \delta(t - t_0)$, i.e. a frequency-flat $|\tilde{f}(\omega)| = 1$, the variance (5.6) would be infinite. Both the last, state-independent ‘vacuum’ contribution ($\alpha \Sigma n^2$), and the first, state-dependent one ($\alpha \Sigma n^2(|\alpha_n|^2 + |\tilde{\alpha}_n|^2)$) would diverge for a cuspy string state (i.e. $|\alpha_n|^2 \sim |\tilde{\alpha}_n|^2 \sim n^{-3/2}$ according to equations (3.9), (3.11)). Note, by contrast, that the expectation value (5.5) would remain convergent (like $\Sigma n^{-1/3}$) even for a time-sharp filter function. This shows that quantum fluctuations in a gravitational wave ‘cusp’ burst are pretty violent on short timescales. On the other hand, if one takes into account the fact that the detecting filter is well peaked around some optimal frequency $\omega$, we see from equation (5.6) that the variance of the filtered cusp signal will be finite and of order (we use $|\tilde{f}(\omega_*)| = 1$, $(\zeta_{TT})^2 \sim 1$ and consider $\Delta n \sim n_*$ terms around the peak value $n_*$ such that $n_* \omega_1 = \omega_*$)

$$\langle \sigma(A_f^J)^2 \rangle \sim \left(16\pi \frac{G}{\ell r_0} \right)^2 n_*^2(|\alpha_n|^2 + |\tilde{\alpha}_n|^2 + 1).$$

(5.7)

Comparing this result with the expectation value (5.5), i.e. $\langle A_f^J \rangle \sim (16\pi G/(\ell r_0)) n_*^2 |\alpha_n| |\tilde{\alpha}_n|$, we can approximately write the standard deviation of $A_f^J$ in the form

$$\sigma(A_f^J) \sim \frac{|\langle A_f^J \rangle|}{n_*^{1/2}} \left( \frac{1}{|\alpha_n|^2} + \frac{1}{|\tilde{\alpha}_n|^2} + \frac{1}{|\alpha_n|^2 |\tilde{\alpha}_n|^2} \right)^{1/2} \sim \frac{|\langle A_f^J \rangle|}{n_*^{1/2} |\alpha_n|}.$$

(5.8)

In the last expression we have assumed that $|\tilde{\alpha}_n| \sim |\alpha_n| \gg 1$, so that one can neglect the ‘vacuum’ contribution.

Let us now estimate how large $n_*$ and $|\alpha_n|$ are for the type of cosmic superstring that one might expect to detect via their gravitational wave bursts. Correspondingly to the assumption $|X_\pm^L|$ time gauge $\sim 2n_*/\ell$ of [17] (i.e. that the string is not too wiggly), we can assume that, in the special cusp-related light-cone gauge we are using, the asymptotic

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24 The detector’s noise spectrum $S_n(\omega)$ increases like $\omega^2$ for large frequencies. Therefore the filter function (2.6) decreases like $\omega^{-10/3}$. 

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behaviour $|\alpha_n| \sim |\tilde{\alpha}_n| \sim A/n^\gamma$ with $\gamma = 7/6$ is roughly valid from $n = 1$ to infinity. Then we can use the closed string mass formula

$$\frac{1}{4} \alpha' M^2 = \sum n \langle a_n^i a_n^i \rangle = \sum \langle n \tilde{a}_n^i \tilde{a}_n^i \rangle \sim \sum_{n=1}^{\infty} |A|^2 n^{-4/3},$$

(5.9)

together with $\alpha' = 1/(2\pi \mu)$ and $M = \mu \ell$ to estimate the coefficient $A$: $|A|^2 \sim \mu \ell^2$. Using also the link $\omega_* \sim 4\pi n_* / \ell$, we have the estimates $n_* \sim f_* \ell$ and $|\alpha_n| \sim \ell \sqrt{\mu} (f_* \ell)^{-7/6}$ where $f_* \equiv \omega_*/2\pi$ denotes the detector’s optimal frequency. In other words the ratio between the standard deviation and the expectation value of $A_f^i$ can be estimated as

$$R = \frac{\sigma(A_f^i)}{\langle A_f^i \rangle} \sim \frac{1}{n_*^{1/2} |\alpha_n|} \sim f_* \ell_* (f_* \ell)^{-1/3} \sim f_* t_{\text{Planck}} (f_* \ell)^{-1/3} (G\mu)^{-1/2},$$

(5.10)

where $\ell_* \sim \mu^{-1/2}$ is the quantum string length, and $\ell \equiv M/\mu$ the invariant length of the considered coherent-state macroscopic string.

In terms of cosmologically relevant dimensionless ratios, this yields

$$R \sim 10^{-43} \left( \frac{\ell}{10^{10} \text{yr}} \right)^{-1/3} \left( \frac{f_*}{150 \text{ Hz}} \right)^{2/3} \left( \frac{G\mu}{10^{-9}} \right)^{-1/2}.$$  

(5.11)

As this is a ratio, the corresponding absolute value of $\sigma(A_f^i)$ (for a detectable amplitude $\langle A_f^i \rangle \sim 10^{-25}$) will be down to the $\sim 10^{-65}$ level! This is clearly negligibly small, even with respect to the already negligibly small graviton shot noise term (5.4).

In summary, the main conclusions of this study are: the gravitational radiation ‘cusp’ signal $h^{\text{cusp}}(f_*)$ emitted by a string (in a coherent state) and detected by a low-frequency detector (with characteristic frequency $f_*$) such as LIGO or LISA is affected by two types of quantum noise. On the one hand, the graviton shot noise $\delta h^\text{shot}(f_*) \sim f_* t_{\text{Planck}}$ (where $t_{\text{Planck}} = (hG)^{1/2}$), and, on the other hand, the signal coming from quantum fluctuations of the string stress–energy tensor near a cusp $\delta h^\text{string}(f_*) \sim G\mu (f_* \ell)^{1/3} (\ell_*/r_0)$ (where $\ell_* \sim (h/\mu)^{1/2}$). Both types of noise (and especially the second one) are totally negligible compared to the classical signal $h^{\text{cusp}}(f_*) \sim G\mu (f_* \ell)^{-1/3} (\ell/r_0)$. The negligible character of the quantum noises crucially depends on considering the ‘frequency windowing’ due to the detector, around a characteristic frequency $f_*$. Indeed, if the detector were allowed to make time-sharp measurements, i.e. if we allow the bandwidth $\Delta f \sim f_*$ to go to infinity, both types of quantum noises would diverge like a positive power of $f_*$. In other words, our computation highlights the fact that the observable cusp signal comes not from short distance scales (UV) near the cusp singularity (which do undergo violent fluctuations), but from length scales of order $f_*^{-1}$ which are intermediate between the UV scales $\sim \ell_*$ and the IR cut-off $\sim \ell$ associated with the overall size of the string.

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25 We recall that the stretching effect of the cosmological expansion, together with the smoothing effects of loop production and, possibly, radiation damping, are expected to lead to a network made of loops whose overall shape is dominated (in the cosmic time gauge) by the first few modes $\alpha_n, \tilde{\alpha}_n$. The fact that such string states might look rather special within the ensemble of possible quantum string states (see, e.g., [21]) is not necessarily relevant within the physical context that we consider.

26 Note that recent work [24] suggests that $\ell$ is only ten times smaller than the cosmological horizon.

27 We consider only the quantum effects carried by the gravitational wave signal $h_{\mu\nu}$. There are evidently many (much more relevant) quantum noise effects coming from the measuring process in the detector.
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