Algorithmic decomposition of filtered chain complexes

Wojciech Chachólski
KTH Stockholm, Sweden
wojtek@kth.se

Barbara Giunti
University of Modena and Reggio Emilia, Italy
bgiunti@tugraz.at

Alvin Jin
KTH Stockholm, Sweden
alvinj@kth.se

Claudia Landi
University of Modena and Reggio Emilia, Italy
claudia.landi@unimore.it

Abstract

We present an algorithm to decompose filtered chain complexes into sums of interval spheres. The algorithm's correctness is proved through principled methods from homotopy theory. Its asymptotic runtime complexity is shown to be cubic in the number of generators, e.g. the simplices of a simplicial complex, as it is based on the row reduction of the boundary matrix by Gaussian elimination. Applying homology to a filtered chain complex, one obtains a persistence module. So our method also provides a new algorithm for the barcode decomposition of persistence modules. The key differences with respect to the state-of-the-art persistent homology algorithms are that our algorithm uses row rather than column reductions, it intrinsically adopts both the clear and compress optimisation strategies, and, finally, it can process rows according to any random order.

2012 ACM Subject Classification Mathematics of computing → Algebraic topology

Keywords and phrases Filtered chain complexes, interval spheres, persistence algorithm, clear and compress

Funding Barbara Giunti: FAR2019-UniMORE
Alvin Jin: Wallenberg AI, Autonomous System and Software Program (WASP) funded by Knut and Alice Wallenberg Foundation

Acknowledgements W. Chachólski was partially supported by VR and WASP. This work was partially carried out by the second and fourth authors within the activities of ARCES (University of Bologna) and under the auspices of INdAM. The second author benefited from the hospitality by the mathematics department of NTNU Trondheim. We thank Sara Scaramuccia for useful discussions.

Introduction

The need to extract topologically meaningful information from complex data is growing. Starting from the 1990s, new methods of topological shape analysis arose [13, 14, 15, 21], and developed in what is nowadays known as persistent homology [4, 11], with applications varying from quantum physics to medicine, biology, and geology (see [19] for an extended list of references to applications). The underlying idea is that homology captures shape features

---

1 corresponding author
such as components, holes and voids, and measuring their persistence along a filtration makes it possible to infer their reliability.

A popular method to apply persistence to data sampled from an underlying shape is to build its \textit{Vietoris-Rips simplicial complex}. This construction comes with guarantees about the actual presence of the homological features detected by persistence in the underlying shape \cite{20}. However homology is an overkill and forgets about the data geometry. In \cite{7}, we have proposed a method to study not only the topology but also the geometry of the data by using \textit{filtered chain complexes} directly without applying homology to them. Among the results of \cite{7}, the most relevant here is the structure theorem stating that filtered chain complexes can be written as direct sums of indecomposables called \textit{interval spheres}. It is also important to note that filtered chain complexes do not arise only as chain complexes of filtered simplicial complexes. For example, the cover construction in \cite{7} associates a filtered chain complex with any tame parametrised chain complex. So filtered chain complexes arise also as invariants of (forward) commutative ladders \cite{12} or zigzag modules \cite{5}.

The main goal of this paper is to present an algorithm to decompose filtered chain complexes into sums of interval spheres. Our algorithm takes as input a set of generators of the filtered chain complex we want to decompose, and gives as output the list of the interval spheres decomposing it. More precisely, we identify \textit{quasi-minimal sets of generators} as a satisfactory trade-off between the complexity of the problem of pre-computing minimal sets of generators, which is equivalent to that of the decomposition itself, and more general sets of generators that may contain an arbitrarily large number of elements. In the case of a filtered chain complex built from a simplicial complex, simplices themselves form a quasi-minimal set of generators of the filtered chain complex. After presenting our algorithm, based on Gaussian elimination by rows, we prove its correctness and its runtime cost that turns out to be at most cubic in the number of generators.

From our decomposition into interval spheres, we straightforwardly obtain the barcode decomposition of the corresponding persistence module into interval modules \cite{6,20}. Therefore, our algorithm represents an alternative to usual persistence algorithms. As a second contribution of this paper, we compare our method with the standard persistence algorithm \cite{10} and some of its variants to speed computations \cite{2,3,9}. Among similarities, we detect the usage of Gaussian elimination on the boundary matrix, and therefore the asymptotic time complexity that turns out to be the same. As for the differences, we underline that using row reduction instead of column reduction we achieve the same advantages as with cohomology \cite{9} while still using homology. Besides, the splitting of interval spheres explains and intrinsically handles both the clear and the compress strategies at once \cite{3,8}. Our algorithm can process boundary matrices with rows and columns in any arbitrary order without the need for them to be sorted by entrance time and degree. Finally, proceeding by rows allows us to solve the asymmetry observed in \cite{2} between the clear and compress strategies.

\textbf{Outline.} In Section \textit{1} we introduce the necessary background on filtered chain complexes, and we review the relevant persistence algorithms. In Section \textit{2} we present our decomposition algorithm for filtered chain complexes. In Section \textit{3} we analyse such algorithm and its time complexity. In Section \textit{4} we discuss some questions that may be interesting to explore.

\section{Preliminaries}

Throughout the work, $F$ denotes a finite prime field of characteristic $p$ for a prime integer $p$. 
1. Chain complexes. Let \( \mathbb{N} = \{0, 1, \ldots \} \) be the set of natural numbers. A (non-negatively graded) chain complex of \( \mathbb{F} \)-vector spaces is a sequence of linear maps \( X = \{ \delta_{n+1}: X_{n+1} \to X_n \}_{n \in \mathbb{N}} \) of \( \mathbb{F} \)-vector spaces, called **differentials**, such that \( \delta_n \delta_{n+1} = 0 \) for all \( n \in \mathbb{N} \). A chain complex \( X \) is called **compact** if \( \bigoplus_{n \in \mathbb{N}} X_n \) is finite dimensional as a vector space \([1]\). This happens if and only if \( X_n \) is finite dimensional for all \( n \) and \( X_n \) is trivial for all \( n > N \in \mathbb{N} \). We denote by \( \text{ch} \) the category of compact chain complexes. Throughout this work, all considered chain complexes are compact.

We call an \( n \)-**sphere**, or simply a sphere, the chain complex \( S^n \) that is zero in all degrees but in degree \( n \), where it is isomorphic to \( \mathbb{F} \). We call an \( n \)-**disk**, or simply a disk, the chain complex \( D^n \), given by

\[
D^n_k = \begin{cases} 
\mathbb{F} & \text{if } k = n, n - 1 \\
0 & \text{otherwise}
\end{cases}, \quad \delta_k = \begin{cases} 
1 & \text{if } k = n \\
0 & \text{otherwise}
\end{cases}
\]

Explicitly:

\[
\begin{array}{cccccccc}
\cdots & & n+1 & n & n-1 & n-2 & \cdots \\
S^n & \cong & \cdots & \to & 0 & \to & \mathbb{F} & \to & 0 & \to & \cdots \\
D^n & \cong & \cdots & \to & 0 & \to & \mathbb{F} & \to & 1 & \to & \mathbb{F} & \to & 0 & \to & \cdots
\end{array}
\]

Since we consider only non-negatively graded chain complexes, \( D^0 \cong S^0 \).

2. Homology. The following vector spaces are called respectively the \( n \)-th **cycles** and the \( n \)-th **boundaries** of \( X \):

\[
Z_nX := \begin{cases} 
X_0 & \text{if } n = 0 \\
\ker(\delta_n: X_n \to X_{n-1}) & \text{if } n \geq 1
\end{cases}, \quad B_nX := \text{im}(\delta_{n+1}: X_{n+1} \to X_n)
\]

Since \( \delta_n \delta_{n+1} = 0 \), \( n \)-th boundaries \( B_nX \) form a vector subspace of that of \( n \)-th cycles \( Z_nX \). The quotient \( Z_nX/B_nX \) is called the \( n \)-th **homology** of \( X \) and is denoted by \( H_nX \). We write \( ZX, BX \) and \( HX \) to denote the chain complexes \( \{Z_nX\}_{n \in \mathbb{N}}, \{B_nX\}_{n \in \mathbb{N}}, \text{ and } \{H_nX\}_{n \in \mathbb{N}} \). Note that they all have trivial differentials. As an example, consider \( S^n \) and \( D^n \) defined in Paragraph \([1]\). We have \( HD^n = 0 \), \( H_nS^n = \mathbb{F} \) and \( H_kS^n = 0 \) for all \( k \neq n \).

**Filtered chain complexes.**

The symbols \( [0, \infty) \) and \( [0, \infty) \) denote the ordered set of non-negative reals and the extended non-negative reals, respectively. Functors of the form \( X: [0, \infty) \to \text{ch} \) are also referred to as **parametrised chain complexes**. The value of \( X \) at \( t \) in \( [0, \infty) \) is denoted by \( X^t \), and \( X^{s \leq t}: X^s \to X^t \) denotes the morphism in \( \text{ch} \) that \( X \) assigns to \( s \leq t \). The morphism \( X^{s \leq t} \) is also referred to as the transition morphism in \( X \) from \( s \) to \( t \).

**Definition 1.** A sequence \( \tau_0 < \cdots < \tau_k \) in \( [0, \infty) \) **discretises** \( X: [0, \infty) \to \text{ch} \) if \( X^{s \leq t}: X^s \to X^t \) may fail to be an isomorphism only when there is a \( a \in \{0, \ldots, k\} \) such that \( s < \tau_a \leq t \). A parametrised chain complex \( X: [0, \infty) \to \text{ch} \) is called **tame** if there is a sequence that discretises it.

We are especially interested in the following class of tame parametrised chain complexes.

**Definition 2.** A tame parametrised chain complex \( X \) is called a **filtered chain complex** if, for all \( s < t \in [0, \infty) \), the transition morphism \( X^{s < t} \) is a monomorphism.
Algorithmic decomposition of filtered chain complexes

The following definition provides an interesting class of filtered chain complexes, parametrised by a natural number \( n \) and an element of \( \Omega := \{ (s, e) \in [0, \infty) \times [0, \infty] \mid s \leq e \} \).

\[ \text{Definition 3.} \] Let \( n \in \mathbb{N} \) and \((s, e) \in \Omega\). An \((n\text{-dimensional})\) interval sphere is a filtered chain complex \( I^n(s, e) \)

\[ I^n(s, e)^t = \begin{cases} 0 & \text{if } t < s \\ S^n & \text{if } s \leq t < e \\ D^{n+1} & \text{if } e \leq t < \infty \end{cases} \]

equals

\[ \begin{align*} \text{Proposition 6.} & \quad \text{Relation between filtered chain complexes and persistence.} \quad \text{Applying homology to a filtered chain complex, we obtain classical persistence modules} \quad \text{[6].} \quad \text{In particular, applying } H_n \text{ to an } n\text{-dimensional interval sphere, we obtain either the trivial persistence module or a so called interval module} \quad \text{[20]: } H_n I^n[s, e] \cong 0 \text{ for } s = e, \quad \text{and } H_n I^n[s, e] \cong \mathbb{I}_{[u, e]} \text{ for } s < e. \quad \text{In general, by the additivity of the homology functor, we can state the following result linking the decomposition of filtered chain complexes into interval spheres to the decomposition of persistence modules into interval modules.} \end{align*} \]

\[ \text{Proposition 8 (Paragraph 4.5 in [7]). For each interval sphere } I^n[s, e], \text{ with } s \in [0, \infty) \text{ and } n \in \mathbb{N} \cup \{-1\}, \text{ and for each tame parametrised chain complex } X, \text{ the morphism} \quad \text{Hom}(I^n[s, e], X) \rightarrow X^s_{n+1} \text{ that assigns to a natural transformation } \varphi: I^n[s, s] \rightarrow X \text{ the element } \varphi_{n+1}(1) \in X^s_{n+1} \text{ is a linear isomorphism.} \]
As a consequence, we can identify $\varphi_i$ with $x_i := (\varphi_i)_{n+1}(1) \in X_{n+1}^s$. Therefore, $G = \{x_1, \ldots, x_m\}$ is also called a set of generators of $X$. On $G$, we can define a degree function $\deg$ and an entrance time function $\ent$ such that, given the generator $x = \varphi_i(1)$, they return $\deg(x) = n$ and $\ent(x) = t$. A set of generators $G$ is called minimal if there is no other set of generators $G'$ with $|G'| < |G|$.

Retrieving a minimal set of generators for a filtered chain complex is as hard as to decompose it into interval spheres. So, it is more convenient to consider larger sets of generators. With this goal in mind, we recall that, for each $n \in \mathbb{N}$, the radical of $X_n$ is the subfunctor $\rad X_n \subset X_n$ whose value $\rad(X_n)^t$ is the subspace of $X^t_n$ given by the sum of all the images of $X^s_{n+1}$ for all $s < t$. A set of generators $G$ of $X$ is called quasi-minimal if, for all $n \in \mathbb{N}$ and $t \in [0, \infty)$, the set $\{x \in G \mid \ent(x) = t \text{ and } \deg(x) = n\}$ is a basis of $(X_n^s/\rad X_n)^t$.

As an example, take $X$ to be an interval sphere $I^n[s, e)$, with $n \in \mathbb{N}$ and $s < e \in [0, \infty)$. A finite set of generators of $X$ is given by $G = \{x_n, x_{n+1}\}$ with $x_n^s \in X_n^s - \{0\}, x_{n+1}^c \in X_{n+1}^c - \{0\}$ arbitrarily chosen. If $s < e$, then $G$ is minimal. If $s = e$, $G$ is not minimal, since $x_n^s$ is redundant. However, it is quasi-minimal.

We use quasi-minimal sets of generators of $X$ to give a matrix encoding of $X$.

9. **Total boundary matrix.** Let $X$ be a filtered chain complex with differential $\delta$, and $G = \{x_1, \ldots, x_m\}$ be an arbitrarily ordered quasi-minimal set of generators of $X$. From the definition of $\rad$, and the assumption that $X$ is tame and all its transition morphisms are monomorphisms, it follows that, for each $n \in \mathbb{N}$ and $t \in [0, \infty)$,

$$X^t_n \cong \bigoplus_{s \leq t \in [0, \infty)} \left( X^s_n/\rad X_n \right)^s$$

(2)

By definition of a quasi-minimal set of generators $G$ of $X$, for each $x_j \in G$ with $\deg(x_j) = n + 1$ and $\ent(x_j) = t$, we can write $\delta_{n+1}^j(x_j)$ uniquely as a linear combination of elements $x_i \in G$ such that $\deg(x_i) = n$ and $\ent(x_i) \leq t$. The total boundary matrix of $X$ is the matrix $d = (d_{ij}^t)$, $1 \leq i, j \leq |G|$, such that $d_{ij}^t$ is equal to the coefficient of $x_i$ in $\delta_{n+1}^j(x_j)$. Hereafter, we adopt the notational convention by which apices denote row indices and pedicess column indices. Accordingly, $d^t_i$ denotes the $i$-th row and $d_i^t$ the $i$-th column of $d$. By construction, we can assume the rows and columns of $d$ are labelled by the degree and entrance time of the corresponding generator in $G$. The sub-matrix of $d$ corresponding to the rows and columns associated with generators with degree $n - 1$ and $n$, respectively, is the boundary matrix of the $n$-th differential $\delta_n$ of $X$. In the case when the total order of $G$ is given first by degree, $d$ is a diagonal block matrix with blocks given by the differential $\delta_n$.

In practical situations, we often obtain the filtered chain complex $X$ from its associated filtered simplicial complex $\Sigma$, in turn obtained from some construction on the data point cloud. In this case, the set of simplices of $\Sigma$ provides a quasi-minimal set of generators of $X$, and the total boundary matrix of $X$ coincides with the boundary matrix of $\Sigma$.

10. **Change of generators.** Let $X$ be a filtered chain complex and $G$ an ordered quasi-minimal set of generators of $X$. Suppose now we change one of the generators $x_i \in G$ to a new element $\tilde{x}_i = \alpha x_i + \sum_{j \neq i} \beta_j x_j$ where the real numbers $\alpha$ and $\beta_j$ are such that $\alpha \neq 0$ and $\beta_j = 0$ when $\ent(x_j) > \ent(x_i)$ or $\deg(x_j) \neq \deg(x_i)$. Taking $\tilde{G} = \{	ilde{x}_1, \ldots, \tilde{x}_m\}$ with $\tilde{x}_k = x_k$ for $k \neq i$, and $\tilde{x}_i$ as described, we claim that $\tilde{G}$ is again a quasi-minimal set of generators of $X$. Indeed, the change from $G$ to $\tilde{G}$ corresponds to a change of basis in $(X_n^s/\rad X_n)^t$, with $n = \deg(x_i)$ and $t = \ent(x_i)$. Hence, it is realised by an isomorphism $\psi: X \to X$, showing that $\tilde{G}$ is also a
Algorithmic decomposition of filtered chain complexes

set of generators of \( X \). Moreover, in \( \bar{G} \) the degree and the entrance time functions operate as follows: \( \deg(\bar{x}_j) = \deg(x_j) \) for all \( k = 1, \ldots, m, \) \( \ent(\bar{x}_j) = \ent(x_j) \) for all \( j = 1, \ldots, t, \ldots, m, \) and \( \ent(\bar{x}_i) = \max\{\ent(x_i), \max\{\ent(x_j) \mid j = 1, \ldots, i, \ldots, m \text{ such that } \beta_j \neq 0\}\} = \ent(x_i) \). Thus, for each \( n \in \mathbb{N} \) and \( t \in [0, \infty) \), the number of generators with degree \( n \) and entrance time \( t \) is the same in \( G \) and \( \bar{G} \), yielding that \( \bar{G} \) is quasi-minimal as well.

The new total boundary matrix \( \bar{d} = (\bar{d}_{ji}) \) of \( X \) relative to \( \bar{G} \) can be deduced from the original total boundary matrix \( d = (d_{ji}) \) relative to \( G \) by Gaussian elimination. In particular, the substitution of \( x_i \) by \( \bar{x}_i = \alpha x_i \), with \( \alpha \neq 0 \in \mathbb{R} \), is realized by the row operation \( d^\prime_i \rightarrow (\bar{d})^\prime_i = (\alpha)^{-1}d^\prime_i \) and by the column operation \( d_i \rightarrow (\bar{d})_i = \alpha d_i \). Similarly, the substitution of \( x_i \) by \( \bar{x}_i = x_i + \beta x_k, \) with \( \beta \neq 0 \in \mathbb{R} \) and \( k \neq i \), is realized by the row operation \( (\bar{d})^k = d^k - \beta d^i \) and by the column operation \( (\bar{d})_i = d_i + \beta d_k \). Note that, as degrees and entrance times of generators have not changed, the rows and columns of \( \bar{d} \) are still labelled with the same degrees and entrance times as in \( d \), respectively.

Persistence algorithms

In view of relating our algorithm for filtered chain complexes decomposition to persistence algorithms, we recall the standard persistence algorithm [10, Sec. VII.1] and two of its time optimisations known, respectively, as clear and compress strategies [3, 8].

For persistence algorithms, the generators need to be sorted according to first entrance times, then degrees and finally randomly to break the ties. Let \( d \) be the total boundary matrix of \( X \) associated with the differential \( \delta \) w.r.t. \( G \) ordered as described above. Let \( \text{low}(j) \) denote the index row of the lowest element of the \( j \)-th column of \( d \). If the column \( j \) is null, set \( \text{low}(j) = 0 \). The boundary matrix is called reduced if \( \text{low}(j) \neq \text{low}(j') \) for all non-zero columns \( j \neq j' \). In a reduced matrix, the lowest elements of each column are called pivots. If \( i = \text{low}(j) \) is a pivot, then the \( i \)-th column is necessarily trivial and the corresponding generator gives birth to a homology class (hence it is said to be positive), while the generator corresponding to the \( j \)-th column cause the death of that homology class (hence it is said to negative). The indices \((i, j)\) form a persistence pair. The collection of all persistence pairs of \( X \) provides the persistence diagram of \( HX \), a.k.a. its barcode decomposition [3, 10, 11].

The standard persistence algorithm [10], as well as its cohomological version [2], achieve such reduction using column operations. The worst-case runtime complexity is thus cubic in the number of generators (see [12] for an example where the worst-case is attained). To obviate this problem, time optimisation strategies have been introduced.

**11. Clear and compress.** The clear optimisation is based on the observation that if \( i \) is the lowest row of a reduced column (i.e. it is positive), then during the reduction the \( i \)-th column will eventually turn out to be trivial. Thus, instead of explicitly perform all the column operations to make it trivial, it can be directly cleared, i.e. set to zero. Symmetrically, the compress optimisation is based on the observation that if the \( j \)-th column contains a pivot (i.e. it is negative), then the row index \( j \) cannot contain any pivot. Thus, one can avoid some unnecessary computation and set the \( j \)-th row directly to zero.

Since the clear strategy requires to process columns in decreasing degree order while the compress runs in increasing degree order, they cannot be easily combined. A technique to combine them is [3], where the clear is applied first, but only limited to pairs not exceeding a threshold persistence, followed by the compress on the remaining columns. Some empirical evidence [2, 3] shows that the best performances are obtained when applying clear in cohomology, rather than using compress (either in co- or in homology).
One goal of this paper is to show that it is actually possible to clear and compress the boundary matrix simultaneously, by splitting interval spheres off of the filtered chain complex $X$ (see Paragraph [15]). In doing so, we also show that the matrix reduction can be performed by rows, rather than by columns. This way, compress in homology becomes as convenient as clear in cohomology (see Paragraph [17]). We also show that the order of rows and columns is actually irrelevant.

2 Decomposition algorithm for filtered chain complexes

Let $X$ be a filtered chain complex, and assume that we are given an arbitrarily ordered quasi-minimal set of generators $G = \{x_1, x_2, \ldots, x_m\}$ of $X$. We denote by $d = (d_i^j)$ the total boundary matrix associated with $\delta$ w.r.t. $G$, i.e. $\delta(x_j) = \sum_{i=1}^{m} d_i^j x_i$. All generators, and hence the corresponding rows and columns of $d$, are assumed to be labelled by degree and entrance time.

A pair $(x_i, x_j)$ of generators in $G$ are said to satisfy the split conditions $\text{SC1-2-3}$ if

$\text{SC1: } d_i^j \neq 0 \quad \text{SC2: } x_i = \arg \max_{h | d_i^h \neq 0} \text{ent}(x_h) \quad \text{SC3: } x_j = \arg \min_{h | d_j^h \neq 0} \text{ent}(x_h)$

In plain words, $x_i$ is the youngest element in the differential of $x_j$, and $x_j$ is the oldest element whose image under $\delta$ contains $x_i$.

If $G$ is sorted as in the standard persistence algorithm, the existence of a pair $(x_i, x_j)$ satisfying $\text{SC1-2-3}$ is straightforwardly ensured, provided $d \neq 0$. Indeed, in that case, the generator $x_j$ can be taken to correspond to any non-zero column of $d$ (which exists by the assumption $d \neq 0$), and the generator $x_i$ can be taken to correspond to the lowest non-zero element in such column (see Paragraph [18] for relations with other works).

Since we aim at using an arbitrary order of generators, we present here a more general argument to guarantee the existence of such a pair when $d \neq 0$.

$\blacktriangleright$ Proposition 12. Let $X$ be a filtered chain complex with at least one non-trivial differential, and let $G = \{x_1, x_2, \ldots, x_m\}$ be a quasi-minimal set of generators of $X$. Then there exists a pair $(x_i, x_j)$ of generators in $G$ that satisfy the split conditions $\text{SC1-2-3}$.

Proof. Since $X$ has at least one non-trivial differential, there exists an index $i_0 \in \{1, \ldots, m\}$ and a degree $n \in \mathbb{N}$ such that $x_{i_0} \in \text{im}(\delta_n)$, implying the existence of $h$ such that $d_h^{i_0} \neq 0$. Consider $x_{i_0} = \arg \max_{h | d_h^{i_0} \neq 0} \text{ent}(x_h)$. Note that $\text{ent}(x_{i_0}) \geq \text{ent}(x_h)$. Let $x_{i_1} = \arg \max_{h | d_h^{i_0} \neq 0} \text{ent}(x_{i_0})$. If $x_{i_0} = x_{i_1}$, then the claim is proved with $i = i_0$ and $j = j_0$.

Suppose then $x_{i_0} \neq x_{i_1}$. In particular, we have $\text{ent}(x_{i_1}) > \text{ent}(x_{i_0})$. Consider $x_{j_1} = \arg \min_{h | d_h^{i_1} \neq 0} \text{ent}(x_h)$. If $x_{j_1} = x_{j_0}$, the claim is proved with $i = i_1$ and $j = j_0$. Otherwise, we have $\text{ent}(x_{j_1}) < \text{ent}(x_{j_0})$ and we iterate the process, selecting at the $k$-th step generators $x_{i_k}$ and $x_{j_k}$ such that one of the following happens: (i) $x_{i_k} = x_{i_{k-1}}$, so the claim is proved with $i = i_{k-1}$ and $j = j_{k-1}$, (ii) $x_{j_k} = x_{j_{k-1}}$, so the claim is proved with $i = i_k$ and $j = j_{k-1}$, (iii) $\text{ent}(x_{i_{k-1}}) < \text{ent}(x_{i_k})$, $\text{ent}(x_{j_{k-1}}) > \text{ent}(x_{j_k})$, and the process iterates to step $k + 1$. Since $X$ is tame, this procedure eventually ends, proving the claim. $\blacktriangleright$

We are interested in a pair $(x_i, x_j)$ satisfying $\text{SC1-2-3}$ because it can be used to reduce the boundary matrix of $X$, as the proof of the following result shows.

$\blacktriangleright$ Proposition 13. Let $X$ be a filtered chain complex with at least one non-trivial differential, and $G = \{x_1, x_2, \ldots, x_m\}$ be a quasi-minimal set of generators of $X$. Let $(x_i, x_j)$ be a pair of
Algorithmic decomposition of filtered chain complexes

generators in \( \mathcal{G} \) satisfying the split conditions \( \text{SC1-2-3} \). Then there exists a quasi-minimal set of generators \( \tilde{\mathcal{G}} = \{ \tilde{x}_1, \ldots, \tilde{x}_m \} \) of \( X \), for which

1. \( \tilde{x}_j = x_j, \tilde{x}_i = \delta(x_j), \) and the pair \( (\tilde{x}_i, \tilde{x}_j) \) satisfies the split conditions \( \text{SC1-2-3} \).
2. each \( x_k \in \tilde{\mathcal{G}} \) is labelled with the same degree and entrance time as \( x_k \in \mathcal{G} \), \( k = 1, \ldots, m; \)
3. \( X \cong I^n[s,e] \oplus X', \) with \( n = \deg(\tilde{x}_i), \ s = \ent(\tilde{x}_i), \ e = \ent(\tilde{x}_j), \) and \( X' \) is the filtered chain complex generated by \( \tilde{\mathcal{G}} = \mathcal{G} \setminus \{ \tilde{x}_i, \tilde{x}_j \} \) with differential induced by \( \delta; \)
4. \( (m-2) \times (m-2) \) total boundary matrix \( \delta' \) of \( X' \) w.r.t. \( \mathcal{G}' \) is obtained from the \( m \times m \) total boundary matrix \( \delta \) of \( X \) w.r.t. \( \mathcal{G} \) by adding to each of its \( k \)-th row with \( \delta_k^j \neq 0 \) the \( i \)-th row multiplied by \( -(\delta'_j)^{-1}d_k^j \), and next removing its \( i \)-th and \( j \)-th columns, as well as its \( i \)-th and \( j \)-th rows.

**Proof.** Recall from [7] Thm. 4.2 that a sufficient condition to split an interval sphere \( I^n[s,e] \) out of a filtered chain complex \( X \) with non-trivial differential \( \delta \) is to take two non-zero elements \( x \in X_{n-1}^* \) and \( y \in X_n^* \) that satisfy the following properties:

(i) \( \delta_{n-1}(x) = 0 \)
(ii) \( \delta_n(y) = X_{n-1}^{s<e}(x) \)
(iii) \( x \notin \im(X_{n-2}^{t<e}) \) for any \( t < s \)
(iv) \( y \notin \im(X_{n-1}^{t<e}) \) for any \( t < e \)
(v) \( X_{n-2}^{s<e}(x) \notin \im(\delta_n) \) for any \( t < e \)

Under these assumptions, \( X \) is isomorphic to \( I^n[s,e] \oplus X' \) for a filtered chain complex \( X' \) uniquely determined up to isomorphism by quotienting out \( x \) and \( y \).

The existence of a pair of generators \( (x_i, x_j) \) of \( \mathcal{G} \) satisfying the split conditions \( \text{SC1-2-3} \) is ensured by Proposition 12. It is not enough to guarantee properties (i–v). Let us therefore construct a possibly different set of quasi-minimal generators \( \tilde{\mathcal{G}} \) such that the pair \( (\tilde{x}_i, \tilde{x}_j) \) satisfies properties (i–v), exploiting the properties \( \text{SC1-2-3} \) of \( (x_i, x_j) \).

The first step is to define \( \mathcal{G} = \{ \pi_1, \ldots, \pi_m \} \) with

\[
\pi_k := \begin{cases} 
\sum_r d^r_j x_r & \text{if } k = i \\
0 & \text{otherwise}
\end{cases}
\]

In other words, we take \( \pi_i \) to be the differential of \( x_j \). By the discussion in Paragraph 10, \( \mathcal{G} \) is a quasi-minimal set of generators. Moreover, \( \deg(\pi_k) = \deg(x_k) \) for all \( k, \) and \( \ent(\pi_k) = \ent(x_k) \) for all \( k \) since \( \ent(\pi_k) = \max\{\ent(x_k) | d_k^j \neq 0\} = \ent(x_i) \) by \( \text{SC2} \).

The second step is to possibly change again the set of generators \( \mathcal{G} \) to \( \tilde{\mathcal{G}} = \{ \tilde{x}_1, \ldots, \tilde{x}_m \} \) by putting \( \tilde{x}_k := \pi_k \) for \( k = j, \) and \( \tilde{x}_k := \pi_k - d_k^j \pi_j \) otherwise. \( \tilde{\mathcal{G}} \) is still a quasi-minimal set of generators, again by the discussion of Paragraph 10 and \( \deg(\tilde{x}_k) = \deg(\pi_k) \) and \( \ent(\tilde{x}_k) = \ent(\pi_k) \) for all \( h. \) Moreover, by construction, also \( (\tilde{x}_i, \tilde{x}_j) \) satisfies \( \text{SC1-2-3} \) of \( (x_i, x_j) \). So, claims 1 and 2 are verified.

To prove claim 3, it is sufficient to show that properties (i–v) are satisfied by taking \( x := \tilde{x}_i \) and \( y := \tilde{x}_j. \) Property (i) follows by \( \text{SC1-2} \) of \( (\tilde{x}_i, \tilde{x}_j) \). Property (ii) holds by construction since \( \delta_i(\tilde{x}_j) = \tilde{x}_i. \) Finally, properties (iii–v) follow by \( \text{SC2-3} \) of \( (\tilde{x}_i, \tilde{x}_j). \)

To prove claim 4, let us examine how the change of generators from \( \mathcal{G} \) to \( \tilde{\mathcal{G}} \) via \( \mathcal{G} \) changes the total boundary matrix. First, let \( \delta = (\delta_k^j) \) be the matrix of \( \delta \) w.r.t. \( \mathcal{G} \). We see that

\[
\delta_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T \quad \delta_k^j = d^k - d_k^j (d_j^k)^{-1} d^j \quad \text{for } k \neq i \quad \delta_i = 0
\]

Equalities (4) and (5) are induced by the change of generators (3), and equation (6) follows from \( \delta^2 = 0. \) Indeed, \( \delta^2 = 0 \) if and only if \( \delta^2 = 0, \) which implies \( (\delta^2)_{jk} = 0 \) for all \( k. \) On the
other hand, by (4), we have $\overline{d}_i^j = 1$ and $\overline{d}_k^j = 0$ for $k \neq i$. So, $(\overline{d}^2)_j^i = \sum_k \overline{d}_k^i \overline{d}_i^j = \overline{d}_i^j = \overline{d}^j_i$. Hence, $\overline{d}_i^k = 0$ for all $k$, and thus the column $\overline{d}_i$ is trivial.

Next, let $\overline{d}$ be the matrix associated with $\delta$ w.r.t. $G$. The change of the generators from $\overline{G}$ to $G$ affects only the $j$-th row, so that $\overline{d}^j = \overline{d}^j + \overline{d}_k^j \overline{d}^k$, and the columns $\overline{d}_k$ with $k \neq j$ and $\overline{d}_k^j = 0$. In particular, equations (4)-(6) hold also for $\overline{d}$, and in addition we have:

$$\overline{d}^j = (0, \ldots, 0, 1, 0, \ldots, 0) \quad (7)$$

$$\overline{d}^i = 0 \quad (8)$$

Since, by (4), $\overline{d}_i^j = 1$, equality (7) follows by the choice of the generators. Indeed, we have $\overline{d}_k = \overline{d}_k - \overline{d}_k \overline{d}_j$ for $k \neq j$, and thus $\overline{d}_k^i = \overline{d}_k^i - \overline{d}_k \overline{d}_j^i = \overline{d}_k^i - \overline{d}_i^j = 0$ for all $k \neq j$. Equality (8) follows by similar arguments as in the proof of (6). Claim (4) now follows by observing that the total boundary matrix w.r.t. $G' = G \setminus \{x_i, x_j\}$ corresponds to $d$ with the $i$-th row and column and the $j$-th row and column deleted. ▶

We stress that in the proof we relied on the assumption $\delta^2 = 0$. Without this assumption, the $i$-th column and the $j$-th row need not be zero, and thus cannot be directly deleted. This is why we cannot generalise such argument to the decomposition of two-parameters persistence modules. Nevertheless, this proposition applies in similar situations, such as in the case of filtered commutative ladders.

If a generator is a cycle and does not contribute to the differential, it can be directly split.

Proposition 14. Let $X$ be a filtered chain complex and $G = \{x_1, x_2, \ldots, x_m\}$ a quasi-minimal set of generators of $X$. Let $d$ be the boundary matrix associated with $\delta$ w.r.t. $G$. For each generator $x_i$ in $G$ such that

(a) $d^i = 0$ \hspace{1cm} (b) $d_i = 0$

it holds that:

1. $X$ is isomorphic to $I^n(s, \infty) \oplus X'$ for $n = \deg(x_i)$, with $s = \ent(x_i)$, and $X'$ is the filtered chain complex whose set of generators is $G' = G \setminus \{x_i\}$, with same degrees and entrance times of $G'$, and differential induced by $\delta$;

2. The $(m - 1) \times (m - 1)$ total boundary matrix $d'$ of $X'$ w.r.t. $G'$ is obtained from the $m \times m$ total boundary matrix $d$ of $X$ w.r.t. $G$ by removing its $i$-th row and column.

Proof. Consider the following exact sequence:

$$0 \rightarrow I^n[s, \infty) \xrightarrow{i} X \xrightarrow{\im(\iota)} X' \cong X/\im(\iota) \rightarrow 0 \quad (9)$$

where $\iota$ is such that $1 \in (I^n[s, \infty))_0 \mapsto x_i$, with $s = \ent(x_i)$. The hypotheses (a) and (b) are equivalent to fact that the differential of $x_i$ is trivial and $x_i \notin \im(\delta)$. Thus, they guarantee the existence of a retract $\rho: X \rightarrow I^n[s, \infty)$. We can then apply the characterisation of split exact sequences [17] Prop. 4.3 to (9) and conclude that $X \cong I^n[s, \infty) \oplus X'$, proving the first claim. The second claim follows immediately since $d' = d_i = 0$ by hypothesis. ▶

We are now ready to present Algorithm [1] that takes as input a quasi-minimal set of generators of $X$ and returns as output the list of interval spheres direct summands of $X$. By Proposition [4] it can also viewed as another algorithm for barcode decomposition.

For each instruction we display its time complexity (TC) in terms of $m$, the number of elements of $G$, which will be used in Section [3] for the analysis of the algorithm.
Algorithmic decomposition of filtered chain complexes

Algorithm 1 Interval sphere decomposition of filtered chain complexes

\begin{algorithm}
\textbf{I:} Totally ordered quasi-minimal set of generators $G$ of $X$, labelled by degree and entrance time
\textbf{O:} List of interval spheres
\begin{enumerate}
\item List $= \emptyset$
\item $d = \text{BOUNDARY}(G)$ \hfill \text{TC: b}
\item while $\exists \ i, t, \ d_i \neq 0$ do \hfill \text{TC: m}
\item \hspace{1em} $i, j = \text{PAIR}(d, i)$ \hfill \text{TC: See Method 2}
\item \hspace{1em} Append $I^{\text{deg}(i)}[\text{ent}(i), \text{ent}(j)]$ to List \hfill \text{TC: See Method 3}
\item \hspace{1em} $\text{SPLIT}(i, j, d)$ \hfill \text{TC: m}
\item \hspace{1em} for all indices $i$ of remaining rows in $d$ do \hfill \text{TC: m}
\item \hspace{2em} Append $I^{\text{deg}(i)}[\text{ent}(i), \infty]$ to List
\item Return List
\end{enumerate}
\end{algorithm}

Algorithm 1 hinges on three methods: BOUNDARY, PAIR, and SPLIT. BOUNDARY generates the boundary matrix $d$ from a minimal set of generators and is not discussed here as it depends on the data structure at hand. The boundary matrix $d$ is then a global variable and will be updated at each call of SPLIT. SPLIT performs the splitting by reducing the boundary matrix, as proved by Proposition 13. PAIR selects the pair of generators to reduce following the constructive proof of Proposition 12. We underline that PAIR does not need columns and rows of $d$ to be sorted by an a-priori-chosen order, differently to what happens for usual persistence algorithms. In general, we have made no effort to optimise PAIR as we aim at fixing a theoretical point, i.e. that it is possible to avoid sorting the generators to compute persistence. Finally, we note that Proposition 12 and consequently the PAIR method, would also work by exchanging the role of rows and columns, i.e. by first picking an arbitrary column $j$, and then looking for a row $i$ to pair it with.

Method 2 PAIR($d, i$)

\begin{algorithm}
\textbf{I:} Boundary matrix $d$, index $i$ of a non-zero row of $d$
\textbf{O:} $i, j$ satisfying SCI-2-3
\begin{enumerate}
\item repeat \hfill \text{TC: m}
\item \hspace{1em} $k = i$
\item \hspace{2em} $j = \arg \min_{h, d_k h \neq 0} \text{ent}(h)$ \hfill \text{TC: m}
\item \hspace{2em} $i = \arg \max_{h} d_{k j h \neq 0} \text{ent}(h)$ \hfill \text{TC: m}
\item until $k = i$
\item Return $i, j$
\end{enumerate}
\end{algorithm}

Method 3 SPLIT($i, j, d$)

\begin{algorithm}
\textbf{I:} Boundary matrix $d$, indices $i, j$
\textbf{O:} Updated boundary matrix $d$
\begin{enumerate}
\item for $k$ with $d_k j \neq 0$ do \hfill \text{TC: m}
\item \hspace{1em} Add to the $k$-th row the $i$-th row multiplied by $-(d_i j)^{-1} d_k j$ \hfill \text{TC: 2m}
\item Delete row $j$ from $d$
\item Delete row $i$ and column $j$ from $d$
\item Delete column $i$ from $d$
\end{enumerate}
\end{algorithm}
In conclusion, Proposition 13 and Proposition 14 guarantee the overall correctness of Algorithm 1 to obtain the decomposition of a filtered chain complex, and hence, as a by-product, the barcode decomposition of a one-parameter persistence module.

3 Algorithm’s analysis

We now analyse Algorithm 1 with regard to its relation to persistence algorithms and its time complexity.

Concerning the comparison to persistence algorithms, we confine ourselves to those originated from the standard one of [10], such us [2, 3, 9], as these algorithms rely on Gaussian elimination similarly to ours. We next show that our method naturally handles and clarifies many optimisation strategies adopted for persistence computation in the literature.

▶ 15. Geometry of clear and compress. By looking at SPLIT, we see that Algorithm 1 intrinsically adopts both the clear and compress optimisation by reducing the boundary matrix by two rows and two columns. Indeed, for an interval sphere $I_n[x_i, x_j]$, with $n = \deg(x_i)$, the only positive row (resp. column) in its boundary matrix is the one containing the generator $x_i$ of the sphere $S^n$ and the only negative row (resp. column) is the one containing the generator $x_j$ of the disk $D^{n+1}$. The fact that the generator $x_i$ of the sphere $S^n$ cannot also generate a disk $D^n$ corresponds to the fact that the index of a positive row cannot also be the index of a negative column, i.e. to the clear optimisation. Thus, Line 5 of Method 3 performs the clear. On the other hand, the fact that the generator $x_j$ of the disk $D^{n+1}$ cannot generate a sphere $S^{n+1}$ corresponds to the fact that the index of a negative column is not at the same time the index of a positive row, i.e. to the compress optimisation. We read the compress in our implementation at line 3 of Method 3.

▶ 16. Generators’ sorting. As persistence homology works with a filtered simplicial complex, the standard algorithm and its modifications work by processing generators in the order they enter the filtration, that is first by entrance time, then by degree, and then again randomly to break the ties. We refer to this ordering as the standard order.

In the case of our algorithm, fixing an order for generators is not necessary thanks to the PAIR method. However, it is still possible to work with a fixed order on the generators, if desired. It is sufficient to modify Algorithm 1 by adding the sorting instruction before line 2. By this change, the effort of PAIR is reversed on the sorting of the generators.

In details, when generators are sorted by the standard order, the PAIR method simplifies: it extracts a couple of indices $i, j$ such that $i$ is the last non-trivial row in $d$, and $j$ is the column of the leftmost non-zero element of the $i$-th row (see Paragraph 18). Alternatively, to handle smaller matrices and fully exploit Proposition 14, one can order generators first by degree, then by entrance time, and then again randomly to break the ties. In this case, the total boundary matrix $d$ is a diagonal block matrix, and we can proceed degree-wise. PAIR in this case, extracts the index $i$ of the last non-trivial row of the considered block, and the index $j$ of the leftmost non-zero column of it. However, if we process blocks in ascending (resp. descending) degree order, only the compress (resp. clear) optimisation can be adopted.

Finally, instead of reducing the boundary matrix directly, one can store the changes of generators, and compute the needed boundary columns on the fly. In this case, when generators are ordered first by degree, our algorithm obtains the ’last-step’ gain in memory as described in [2].
17. **Row versus column reductions.** While the standard algorithm computes homology by reducing the boundary matrix by column operations, the authors of [9] proposed to compute cohomology instead, and hence to reduce the coboundary matrix, though still by column operations. The advantages of cohomology over homology are achieved in combination with the clear strategy, which empirically proves to be faster than compress in combination with homology when using, in both cases, column reductions.

The coboundary matrix being the transpose of the boundary one, we underline that, to properly dualise the optimisation obtained in cohomology by the clear, it is necessary to reduce the boundary matrix via row rather than column operations. This is exactly what we do in Algorithm 1. In doing so, when generators are primarily sorted by degree (see Paragraph 16), the combination of the row reductions with the compress strategy achieves the same advantages of the cohomology with the clear. This way there is no need to actually resorting to cohomology. In [2, Sec. 3.3], an explicit calculation quantifies the computational speed-up that the column reduction with clear on the coboundary matrix provides in the barcode computation of Vietoris-Rips complexes. For the reader’s convenience, we now translate that computation into our approach, that is using row operations in homology with the compress optimisation. We underline that what is computed is the number of iterations. Each iteration has to be multiplied by the cost of a reduction, which depends on the algorithm used (see Paragraph 19).

Let $X$ be a filtered Vietoris-Rips complex, whose quasi-minimal set of generators is ordered first by degree and then by a total order of choice. Let $v$ be the number of generators in degree 0, i.e. the number of vertices, $N$ the maximal non-trivial degree of $X$, and $d_n$, for $1 \leq n \leq N + 1$, the blocks of the boundary matrix $d$ of $X$. To reduce $d$ without compress, we need to process one row per generator from degree 0 to $N$. In total, this sums up to

\begin{equation}
\sum_{n=0}^{N} \left( \frac{v}{n+1} \right) \dim(C_n X) = \sum_{n=0}^{N} \left( \frac{v-1}{n} \right) \dim(B_{n-1} X) + \sum_{n=0}^{N} \left( \frac{v-1}{n+1} \right) \dim(Z_n X) \tag{10}
\end{equation}

Note that $\dim(B_{n-1} X)$ is the number of negative generators of degree $n$. Applying the compress strategy, since the negative rows of $d_n$, for $n > 0$, are removed when processing $d_{n-1}$, (10) decreases to

\begin{equation}
\frac{v-1}{\dim(B_{n-1} X)} + \sum_{n=0}^{N} \left( \frac{v-1}{n} \right) = 1 + \sum_{n=0}^{N} \left( \frac{v-1}{n+1} \right) = 1 + \sum_{n=1}^{N+1} \left( \frac{v-1}{n} \right)
\end{equation}

that coincides with the number of generators processed in [2].

18. **Apparent pairs.** According to [2], the pairs of indices $(i,j)$ such that the lowest non-zero element of the $j$-th column is in the $i$-th row, and the leftmost non-zero element of the $i$-th row is in the $j$-th column, are called **apparent pairs**. Their identification and immediate splitting provide a significant speed-up for the barcode decomposition (see [2, Sec. 3.6] for an overview of the use of apparent pairs in the computation of persistent homology).

In the case when the total order of $G$ is taken to be the standard one, apparent pairs are precisely given by pairs of generators $(x_i, x_j)$ that satisfy the split conditions [SC1-2-3]. In the general case, the apparent pairs are the ones that exit the loop in PAIR at the first iteration. However, we underline that not all the pairs identified as splittable by PAIR are apparent pairs since some of them emerge only during the reduction of the boundary matrix.
We now study the worst-case time complexity of Algorithm 1. Recall that $m$ denotes the number quasi-minimal generators of $X$.

19. **Time complexity.** Algorithm 1 uses the methods PAIR (Method 2), SPLIT (Method 3) and BOUNDARY. We assume to be able to compute the latter with time cost $b$. For PAIR we need at most $2m^2$ operations. For SPLIT in the worst-case, we need $2m^2$ operations. Thus, the whole computation requires at most $b + m(4m^2 + 1)$ operations.

When forcing the standard order (see Paragraph 16), the time cost changes. To sort the generators, we can choose any efficient sorting algorithm as Quicksort [16], whose running time complexity is $m^2$ in the worst case. The time cost of PAIR is thus considerably reduced since, in this case, it only requires the access to the non-zero elements of the boundary matrix: its worst-case running time complexity is $2m$. Other costs remain the same, so we have at most $b + m(2m^2 + 3m + 1)$ operations. However, it is well known that, in general, Quicksort is much faster, with an average case complexity of $m \log m$, making the ordering approach more convenient. As for the cost of SPLIT, assuming now the standard order on $G$, splitting a pair of generators $(x_i, x_j)$ costs at most $cr(i)r(i, j)$, where $c$ is the number of cofaces of $x_i$, $r(i) := \#\{x_k | k \neq i, \text{ent}(x_k) \leq \text{ent}(x_i) \text{and } \text{deg}(x_k) = \text{deg}(x_i)\}$ and $r(i, j) := \#\{x_k | \text{ent}(x_k) \leq \text{ent}(x_i) \leq \text{ent}(x_j) \text{and } \text{deg}(x_k) = \text{deg}(x_i)\}$. However, as our algorithm can use any sorting, this estimate may improve.

4 **Conclusions**

At the best of our knowledge, this is the first work analysing persistence algorithms from the standpoint of filtered chain complexes. Using this approach, we have been able to geometrically explain the clear and compress optimisations and to bypass the usage of cohomology. Nevertheless, a number of questions remain open. For example, a more efficient implementation of the PAIR method, ideally tailored for different simplicial complexes, needs to be studied, to see if it can provide some speed up in computation. One could also ask if, now that we are not forced to use a specific order, there is a more efficient way to sort the generators, so to allow quick access in the memory or a faster way to calculate the (co)boundary. Yet another question is about the gain and loss of the standard reduction compared to our reduction. For example, in [18], the author describes a filtered simplicial complex $K$ whose barcode computation realises the worst-case for the standard persistence algorithm [10], but $K$ does not realise the worst-case for our Algorithm 1. However, this does not exclude the existence of another example realising the worst-case time complexity.

---

**References**

1. Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

2. Ulrich Bauer. Ripser: efficient computation of victoris-rips persistence barcodes, 2019. [arXiv:1908.02518](https://arxiv.org/abs/1908.02518).

3. Ulrich Bauer, Michael Kerber, and Jan Reininghaus. Clear and compress: Computing persistent homology in chunks. *Mathematics and Visualization*, 03 2013.

4. Gunnar Carlsson. Topology and data. *Bull. Amer. Math. Soc. (N.S.)*, 46(2):255–308, 2009.

5. Gunnar Carlsson and Vin de Silva. Zigzag persistence. *Found. Comput. Math.*, 10(4):367–405, 2010.

6. Gunnar Carlsson and Afra Zomorodian. Computing persistent homology. *Discrete Comput. Geom.*, 33(2):249–274, 2005.
Algorithmic decomposition of filtered chain complexes

7 Wojciech Chachólski, Barbara Giunti, and Claudia Landi. Invariants for tame parametrised chain complexes. Accepted for publication in *Homology, Homotopy and Applications*, 2020. arXiv:2003.03969

8 Chao Chen and Michael Kerber. Persistent homology computation with a twist. In *Proceedings 27th European Workshop on Computational Geometry*, 2011.

9 Vin de Silva, Dmitriy Morozov, and Mikael Vejdemo-Johansson. Dualities in persistent (co)homology. *Inverse Problems*, 27(12):124003, 17, 2011.

10 Herbert Edelsbrunner and John L. Harer. *Computational topology*. American Mathematical Society, Providence, RI, 2010. An introduction.

11 Herbert Edelsbrunner, David Letscher, and Afra Zomorodian. Topological persistence and simplification. *Discrete & Computational Geometry*, 28(4):511–533, Nov 2002.

12 Emerson G. Escolar and Yasuaki Hiraoka. Persistence modules on commutative ladders of finite type. *Discrete Comput. Geom.*, 55(1):100–157, 2016.

13 Massimo Ferri, Patrizio Frosini, Alessandro Verri, and Claudio Uras. On the use of size functions for shape analysis. *Biological Cybernetics*, 70(2):99–107, Dec 1993.

14 Patrizio Frosini. Measuring shapes by size functions. In David P. Casasent, editor, *Intelligent Robots and Computer Vision X: Algorithms and Techniques*, volume 1607, pages 122 – 133. International Society for Optics and Photonics, SPIE, 1992.

15 Patrizio Frosini and Claudia Landi. Size theory as a topological tool for computer vision. *Pattern Recognition and Image Analysis*, 9:596–603, 1999.

16 Charles A. R. Hoare. Algorithm 64: Quicksort. *Commun. ACM*, 4(7):321, July 1961.

17 Saunders MacLane. *Homology*. Springer-Verlag, Berlin-New York, first edition, 1967. Die Grundlehren der mathematischen Wissenschaften, Band 114.

18 Dmitriy Morozov. Persistence algorithm takes cubic time in worst case. *BioGeometry News, Dept. Comput. Sci., Duke Univ.*, 2005.

19 Database of real-world applications of TDA. https://www.zotero.org/groups/2425412/tda-applications. 2020.

20 Steve Y. Oudot. *Persistence theory: from quiver representations to data analysis*, volume 209 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

21 Vanessa Robins. Towards computing homology from finite approximations. In *Proceedings of the 14th Summer Conference on General Topology and its Applications (Brookville, NY, 1999)*, volume 24, pages 503–532 (2001), 1999.