EXISTENCE OF WEAK SOLUTIONS FOR PARTICLE-LADEN FLOW WITH SURFACE TENSION

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Abstract. We prove the existence of solutions for a coupled system modeling the flow of a suspension of fluid and negatively buoyant non-colloidal particles in the thin film limit. The equations take the form of a fourth-order non-linear degenerate parabolic equation for the film height $h$ coupled to a second-order degenerate parabolic equation for the particle density $\psi$. We prove the existence of physically relevant solutions, which satisfy the uniform bounds $0 \leq \psi/h \leq 1$ and $h \geq 0$.

1. Introduction. In lubrication theory, the free surface height of a thin liquid film is governed by a degenerate fourth-order parabolic equation which in one dimension typically has the form

$$h_t + (f_0(h))_x = -(f_1(h)h_{xxx})_x + (f_2(h)h_x)_x,$$

where the coefficients $f_0, f_1, f_2$ depend on the relevant physics (e.g. $f_0 = f_1 = f_2 = h^3$ for the flow of a fluid driven by gravity down an incline) [17]. Equations of this type have been the subject of considerable theoretical study; the tools for analysis can provide insight into important phenomena such as instabilities in spreading films [7, 9, 18, 4], and can be utilized to design efficient numerical schemes [20]. Bernis and Friedman [5] first demonstrated existence and positivity of solutions to the equation $h_t = -(h^n h_{xxx})_x$ through the use of energy and entropy estimates.

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In later work, Bertozzi and Pugh explained the theory for the equation (1) with \(f_0 = 0\), using different choices of regularization and entropy functions to study regularity, long-time behavior [3] and the growth of singularities [4].

There are a wide variety of problems in multiphase thin-film flows that lead to more complicated systems. Lubrication models of such flows reduce to coupled systems for the film height and a quantity tracking the second phase, whose complex dynamics have been the subject of considerable interest in recent research [11]. Here we consider one such model for gravity-driven suspension flow in one dimension that accounts for the non-uniform distribution of particles within the bulk of the fluid, proposed in [16] and recently extended to include surface tension in [15]. The model equations [15] for the film height \(h(x, t)\) and depth-integrated particle density \(\psi(x, t)\) have the form

\[
\begin{align*}
    h_t + (h^3 f_1(\phi))_x &= -\beta(h^3 f_1(\phi) h_{xxx})_x + (h^3 (f_2(\phi) \psi_x + f_3(\phi) h_x))_x, \\
    \psi_t + (h^3 g_1(\phi))_x &= -\beta(h^3 g_1(\phi) h_{xxx})_x + (h^3 (g_2(\phi) \psi_x + g_3(\phi) h_x))_x,
\end{align*}
\]

where \(\phi = \psi/h\) is the depth-averaged concentration of particles that cannot exceed a maximum packing fraction \(\phi_m\), normalized here so that \(\phi_m = 1\). The fluxes vanish at the maximum packing fraction (i.e. \(f_i(1) = g_i(1) = g_i(0) = 0\)), where flow of the suspension is completely inhibited by the particles. This adds an additional degeneracy into the equations (along with the standard degeneracy for thin films as \(h \to 0\)), which has been studied in the related problem of non-linear diffusion equations for sedimenting particles [2].

The flux functions in the model equations (2) have a particular behavior in the dilute limit (\(\phi \to 0\)) and the high-concentration limit (\(\phi \to 1\)). In particular, for negatively buoyant particles,

\[
\begin{align*}
    f_i(\phi) &\sim \frac{1}{3}, & g_i(\phi) &\sim b_i \phi^{3/2} \text{ as } \phi \to 0, & i &= 0, 1, 3 \\
    f_2(\phi) &\sim a_2 \phi, & g_2 &\sim b_2 \phi^2 \text{ as } \phi \to 0, \\
    f_i(\phi) &\sim c_i (1 - \phi)^2, & g_i(\phi) &\sim d_i (1 - \phi)^2 \text{ as } \phi \to 1, & i &= 0, 1, 2, 3
\end{align*}
\]

for constants \(a_i, b_i, c_i, d_i > 0\) [15, 19]. These fluxes arise from depth-integrating the suspension \((f)\) and particle \((g)\) volume fluxes, which depend on the distribution of particles in the fluid depth. The exponents of \(\phi\) in the dilute limit are a consequence of the particle accumulation towards the substrate of the fluid [15]. The suspension evolves as a settled layer of particles with a clear fluid layer above; the height of this layer can be shown to scale with \(\phi^{3/2}\) as \(\phi \to 0\), so the fluxes for the particle transport in the lubrication model, which scale with the cube of the height, gain a factor of \(\phi^{3/2}\). The quadratic decay in the high concentration limit is due to the singularity in the suspension viscosity \(\mu \sim (1 - \phi)^{-2}\), a law that captures the inhibiting of the flow near the maximum packing fraction [6].

The system (2) is closely related to the equations governing transport of insoluble surfactant on the fluid surface [11], for which the concentration satisfies an equation with a non-degenerate diffusion term. Existence and positivity of weak solutions was established in [13, 1] using a finite element approach and studied for more general systems in later work by [8, 14, 10]. The techniques employed there are almost applicable to (2), but must be modified to account for a few key differences in the structure of the equations. First, we do not include the non-degenerate Brownian diffusion term for \(\psi\), leaving only the degenerate diffusion term for the \(\psi\) equation which vanishes when \(\phi = 0, \phi = 1\) or \(h = 0\). Second, the fluxes depend on the ratio
\( \psi/h \) of the conserved variables \( h \) and \( \psi \) and vanish when \( \psi/h \geq 1 \), so it is critical to establish this bound.

Here, we are concerned with the existence of physically relevant solutions in the sense that \( h \geq 0 \) and \( 0 \leq \psi/h \leq 1 \) when the initial data satisfies the same, with periodic boundary conditions. Under assumptions on the behavior of the flux coefficients \( f_i \) and \( g_i \) compatible with the properties (3) of the physical model, we prove existence of such solutions and the bound \( \phi \leq 1 \) when \( f_i = g_i = 0 \) for \( \phi \geq 1 \).

In Section 2, the governing system and assumptions used in the existence result are introduced. In Section 3 the relevant notion of a weak solution is defined and the main result is stated, which is proven in Section 4.

2. System and assumptions. Next, we assume that \( f_3 = g_3 = 0 \) for simplicity (due to the fourth order diffusion, this second order term is not important to the existence result). Let us consider the following system of equations:

\[
\begin{align*}
ht + ([h^3|Bf_1(h\psi)|_{x} x + f_0(h\psi)])_x = (D_1(h, \psi)\psi)_x, \\
(\psi)_{t} + ([h^3|Bg_1(h\psi)|_{x} x + g_0(h\psi)])_x = (D_2(h, \psi)\psi)_x
\end{align*}
\]

in \( QT = (0, T) \times \Omega \) with periodic boundary conditions

\[
\frac{\partial x}{\partial\tau} (-a, t) = \frac{\partial x}{\partial\tau} (a, t), \quad \frac{\partial x}{\partial\tau} (a, t) \forall t > 0,
\]

where \( \Omega := (-a, a) \subset \mathbb{R}^1 \) is bounded domain,

\[
h_0(x) \in H^1(\Omega), \quad \psi_0(x) \in L^2(\Omega), \quad 0 \leq \psi_0(x) \leq h_0(x),
\]

and \( D_1, f_i, g_i \) are continuous functions such that

\[
0 \leq f_1(z) \leq a_0(1 + |z|)^{-m}, \quad |f_0(z)| \leq a_1f_1(\zeta), \quad \text{where} \ m \geq 0,
\]

\[
|g_1(z)| \leq b_0f_1(\zeta)|z|^{\beta}, \quad |g(z)| \leq b_1|z|^{\beta} \forall |z| \leq 1,
\]

\[
|f_2(z)| \leq a_2f_1(\zeta)|z|^{\beta}, \quad b_2|z|^{\beta} \leq g_2(z) \forall |z| \leq 1,
\]

\[
D_1(a, b) := |a|^3f_2(\zeta), \quad D_2(a, b) := |a|g_2(\zeta),
\]

where \( a_2, b_0, b_2 \) satisfy the following restrictions:

\[
\frac{a_2^2}{\sqrt{\beta b_2}} < \sqrt{1 - \frac{\beta b_2}{4\beta}} \quad \text{and} \quad \frac{\beta b_2^2}{4\beta b_2} < 1, \quad \text{or} \quad \frac{a_2^2}{\sqrt{\beta b_2}} > \max\left\{ \frac{\beta b_2^2}{4\beta b_2} - 1, \sqrt{1 + \frac{\beta b_2}{4\beta}} \right\}.
\]

The assumptions (9)-(13) are motivated by the behavior of the fluxes in the physical model given by (3) (which also satisfy the assumptions here), but are modified slightly for the sake of the proof. Assumption (9) is a (not particularly restrictive) technical assumption on the growth rate of the fluxes that is easily satisfied by the physical model. Of particular note is (10), which is a bound on the size of the first-order flux of particles in the dilute limit \( \phi \to 0 \); this matches the limiting behavior \( g_1(\phi) \sim b_1\phi^{3/2} \) for the physical model. The lower bound required on the diffusion
coefficient \( g_2 \) given by (11) is slightly more general than the limit \( g_2 \sim b_2 \phi^2 \) in the physical model.

Integrating (4) on \( \Omega \), due to periodic boundary conditions (6), we obtain the mass conservation
\[
\int_{\Omega} h \, dx = \int_{\Omega} h_0 \, dx. \tag{14}
\]

3. Main result.

**Definition 3.1.** [weak solution] A generalized weak solution of the problem (4)–(7) with initial data \((h_0, \psi_0)\) satisfying (8) is a pair \((h, \psi)\) has the following regularity properties
\[
h \geq 0 \text{ in } Q_T, \quad 0 \leq \psi \leq h \text{ a.e. in } Q_T, \\
h \in C^{\frac{1}{4}, 1}_{x,T}(\bar{Q}_T) \cap L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))^*), \\
\psi \in L^\infty(0, T; L^2(\Omega)) \cap W^1_2(0, T; (W^1_2(\Omega))^*), \\
I := \beta f_1(\phi)h^3h_{xxx} + f_0(\phi)h^3 - D_1(h, \psi)\psi_x \in L^2(\{h > 0\}), \\
\beta g_1(\phi)h^3h_{xxx} - D_2(h, \psi)\psi_x \in L^2(\{\psi > 0\}), \\
g_0(\phi)h^3 \in L^6(\{h > 0\}),
\]
where \( \phi := \frac{\psi}{h} \). Furthermore, \((h, \psi)\) satisfies (4)–(5) in the following sense:
\[
\int_{0}^{T} \int_{\{h > 0\}} (h_t(t), \xi(t)) \, dt - \int_{\{h > 0\}} I_{\xi_x} \, dxdt = 0, \\
\int_{0}^{T} \int_{\{\psi > 0\}} (\psi_t(t), \zeta(t)) \, dt - \int_{\{\psi > 0\}} g_0(\phi)h^3\zeta_x \, dxdt \\
- \int_{\{\psi > 0\}} (\beta g_1(\phi)h^3h_{xxx} - D_2(h, \psi)\psi_x)\zeta_x \, dxdt = 0
\]
for all \( \xi \in L^2(0, T; H^1(\Omega)) \) and \( \zeta \in L^3(0, T; W^3_2(\Omega)): \xi(-a, t) = \xi(a, t), \zeta(-a, t) = \zeta(a, t) \) for all \( t \in (0, T) \). Moreover, the initial conditions for \( h \) and \( \psi \) are attained in the sense of traces in the spaces \( H^1(0, T; (H^1(\Omega))^*) \) and \( W^1_2(0, T; (W^1_2(\Omega))^*) \), respectively.

**Theorem 3.2.** [existence] Let (9)–(13) hold. Assume that the initial data \((h_0, \psi_0)\) satisfy (8) and (14). Then, for any time \( T > 0 \), there exists a weak solution \((h, \psi)\) of the problem (4)–(7) in the sense of Definition 3.1.

4. Proof of Theorem 3.2.

4.1. Auxiliary problems. We regularize the degeneracy which is apparent for \( h = 0, \psi = 0 \) and \( \psi = h \). For this purpose we approximate the system by a family of non-degenerate equations:
\[
h_t + (\beta F_\delta(h)f_1(\frac{\psi}{h}))h_{xxx} + F_\delta(h)f_0(\frac{\psi}{h}))_x = (D_1, \delta(h, \psi)\psi_x)_x, \tag{15}
\]
\[
\psi_t + (F_\delta(h)(\beta g_1(\frac{\psi}{h}))h_{xxx} + g_0(\frac{\psi}{h}))_x = (D_2, \delta(h, \psi)\psi_x)_x \tag{16}
\]
in $Q_T = (0, T) \times \Omega$ with periodic boundary conditions
\[
\frac{\partial h}{\partial x_i}(-a, t) = \frac{\partial h}{\partial x_i}(a, t), \quad \frac{\partial h}{\partial x_i}(-a, t) = \frac{\partial h}{\partial x_i}(a, t) \quad \forall \ t > 0,
\]
\[i = 0, 3, \ k = 0, 1, \text{ and initial conditions}
\]
\[
h(x, 0) = h_0(x) \geq h_0 + \delta, \quad \psi(x, 0) = \psi_0(x) \geq \psi_0 + \varepsilon
\]
for all $\theta \in (0, \frac{2}{s+1})$ and $\mu \in (0, 1)$, where $\varepsilon > 0$, $\delta > 0$, and
\[
F_{\delta}(z) := F_{\delta}(z) + \varepsilon = \frac{|z|^{s+3}}{|z|^s + |z|^s} + \varepsilon, \ s \geq 8, \ f_{1, \delta}(z) := f_1(z) + \delta;
\]
\[
|g_{1, \delta}(z)| \leq b_0 f_{1, \delta}(z)|z|^3 \text{ if } |z| \leq 1, \ g_{1, \delta}(z) = \delta \text{ if } |z| > 1;
\]
\[
D_{1, \delta}(h, \psi) = F_{\delta}(h) f_2(\frac{\psi}{h}), \quad D_{2, \delta}(h, \psi) = D_2(h, \psi) + \varepsilon.
\]
Here $h_{0, \delta}$ and $\psi_{0, \epsilon}$ are smooth enough approximation functions, i.e. $h_{0, \delta} \in H^1(\Omega)$ and $\psi_{0, \epsilon} \in L^2(\Omega)$. Integrating (15) and (16) in $Q_T$ by (17), we get the mass conservation
\[
\int_{\Omega} h(x, t) \, dx = \int_{\Omega} h_{0, \delta}(x) \, dx, \quad \int_{\Omega} \psi(x, t) \, dx = \int_{\Omega} \psi_{0, \epsilon}(x) \, dx.
\]
Let us denote by $\phi := \frac{\psi}{h}$, and
\[
\chi_{\phi} = 1 \text{ if } |\phi| \leq 1, \quad \chi_{\phi} = 0 \text{ if } |\phi| > 1.
\]
Note that
\[
D_{1, \delta}(h, \psi) = 0 \quad \forall |\phi| > 1,
\]
\[
D_{2, \epsilon}(h, \psi) \geq D_{2, \epsilon}(\psi) := b_2 |\psi|^3 + \varepsilon \quad \forall |\phi| \leq 1, \text{ and } D_{2, \epsilon}(h, \psi) = \varepsilon \quad \forall |\phi| > 1.
\]

4.2. **Galerkin approximation.** Now we use a Galerkin approximation which transforms the system of partial differential equations into a system of ordinary differential equations. As basis functions for the finite dimensional space we select an $L^2$-orthonormal basis of eigenfunctions which are solutions of the periodic boundary value problem:
\[
-v_i'' = \lambda_i v_i \text{ in } \Omega, \quad v_i(-a) = v_i(a).
\]
We make a Galerkin ansatz for $h_{\epsilon, \delta}^N(x, t)$ and $\psi_{\epsilon, \delta}^N(x, t)$ of the form
\[
h_{\epsilon, \delta}^N = \sum_{i=0}^{N} a_i(t) v_i(x), \quad \psi_{\epsilon, \delta}^N = \sum_{i=0}^{N} b_i(t) v_i(x).
\]
According to (15) and (16) the functions $a_i(t)$ and $b_i(t)$ are subject to the following Galerkin equations which have to hold for $j = 0, N$:
\[
\dot{a}_j(t) = -\beta \delta \varepsilon \lambda_j ||v_j'||_2^2 a_j(t) - \beta \sum_{i=0}^{N} \lambda_i a_i(t) \int_{\Omega} (F_{\delta}(h_{\epsilon, \delta}^N) f_{1, \delta}(\frac{\psi_{\epsilon, \delta}^N}{h_{\epsilon, \delta}^N}) + \varepsilon f_{1}(\frac{\psi_{\epsilon, \delta}^N}{h_{\epsilon, \delta}^N})) v'_i v'_i \, dx + \int_{\Omega} F_{\delta}(h_{\epsilon, \delta}^N) f_{0}(\frac{\psi_{\epsilon, \delta}^N}{h_{\epsilon, \delta}^N}) v'_i v'_i \, dx - \sum_{i=0}^{N} b_i(t) \int_{\Omega} D_{1, \delta}(h_{\epsilon, \delta}^N, \psi_{\epsilon, \delta}^N) v'_i v'_i \, dx,
\]
\[
\dot{h}(t) = -\varepsilon \lambda_i b_j(t) - \sum_{i=0}^{\frac{\varepsilon}{\delta_0}} b_i(t) \int_{\Omega} D_2(h^N_{\varepsilon \delta}, \psi_i^N) v_i v_j \, dx - \\
\beta \sum_{i=0}^{\frac{\varepsilon}{\delta_0}} \lambda_i a_i(t) \int_{\Omega} F_\delta(h^N_{\varepsilon \delta}) g_1(\psi_i^N) v_i v_j \, dx + \int_{\Omega} F_\delta(h^N_{\varepsilon \delta}) g_0(\psi_i^N) v_i v_j \, dx
\]

with
\[
a_j(0) = (h_0, \delta, v_j)_{L^2(\Omega)}, \quad b_j(0) = (\psi_0, \varepsilon, v_j)_{L^2(\Omega)}.
\]

Due to (9)–(12), the right-hand side of this system is Lipschitz continuous on $a_j$ and $b_j$. Thus, by the Picard-Lindelöf theorem a unique local solution of the system exists. Solvability for some $T > 0$ can be proved using a priori estimates (uniformly in $N$, $\varepsilon$ and $\delta$).

For brevity, we denote by $h := h^N_{\varepsilon \delta}$, $\psi := \psi^N_{\varepsilon \delta}$ and $\phi = \psi / \varepsilon$. Multiplying (15) by $h - h_{xx}$ and integrating on $\Omega$, we deduce that

\[
\frac{1}{2} \frac{d}{dt} \|h\|^2_{H^1(\Omega)} + \beta \int_{\Omega} f_1, \delta(h) h^2_{xx} \, dx = \\
\beta \int_{\Omega} f_1, \delta(h) h_{xx} h_{xxx} \, dx - \int_{\Omega} f_0(h) F_\delta(h) h_{xx} \, dx + \int_{\Omega} f_0(h) F_\delta(h) h_x \, dx + \\
\int_{\Omega} D_1, \delta(h, \psi) \psi_x h_{xx} \, dx - \int_{\Omega} D_1, \delta(h, \psi) \psi_x h_x \, dx \leq \\
\beta \left( \int_{\Omega} f_1, \delta(h) F_\delta(h) h^2_{xx} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} f_1, \delta(h) F_\delta(h) h^2_x \, dx \right)^{\frac{1}{2}} + \\
\left( \int_{\Omega} f_1, \delta(h) F_\delta(h) h^2_{xx} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} f_0(h) F_\delta(h) h^2_x \, dx \right)^{\frac{1}{2}} + \\
\left( \int_{\Omega} f_0(h) F_\delta(h) h^2_x \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} h^2_x \, dx \right)^{\frac{1}{2}} + \\
\left( \int_{\Omega} f_0(h) F_\delta(h) h^2_x \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} h^2_x \, dx \right)^{\frac{1}{2}} + \\
C(\beta(a_0 + \delta)^{\frac{1}{2}} (\|h\|^2_{H^1(\Omega)} + \|h\|^2_{H^1(\Omega)}) + a_1 \chi \|h\|^2_{H^1(\Omega)}) \left( \int_{\Omega} f_1, \delta(h) F_\delta(h) h^2_{xx} \, dx \right)^{\frac{1}{2}} + \\
Ca_1 a_0 \chi \|h\|^2_{H^1(\Omega)} + \left( \int_{\Omega} f_1, \delta(h) F_\delta(h) h^2_{xx} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} f_0(h) F_\delta(h) h^2_x \, dx \right)^{\frac{1}{2}} + \\
C \frac{1}{\|h\|^2_{H^1(\Omega)}} \left( \int_{\Omega} D^2_1, \delta(h, \psi) \psi_x^2 \, dx \right)^{\frac{1}{2}}.
\]

whence we find that
\[ \frac{1}{2} \frac{d}{dt} \|h\|_{H^1(\Omega)}^2 + (\beta - \varepsilon_1 - \varepsilon_2) \int_{\Omega} f_{1,\delta}(\phi) F_{\delta \varepsilon}(h) h_{xx}^2 \, dx \]

\[ \leq \frac{C_2}{\varepsilon_5} (\beta^2(a_0 + \delta)(\|h\|_{H^1(\Omega)}^2 + \|h\|_{H^1(\Omega)}^2) + a_1^2 \varepsilon_5 \|h\|_{H^1(\Omega)}^2 + Ca_1 a_0 \varepsilon_5 \|h\|_{H^1(\Omega)}^2 + \frac{1}{\varepsilon_3} \int_{\Omega} D_{1,\delta}(h, \psi) \psi_x^2 \, dx + \varepsilon_3 \int_{\Omega} D_{2,\delta}(h, \psi) \psi_x^2 \, dx + \frac{C_2}{\varepsilon_5} \|h\|_{H^1(\Omega)}^2 \]

\[ \leq C_1 \max\{1, \|h\|_{H^1(\Omega)}^2\} + a_2^2 \varepsilon_5 \|h\|_{H^1(\Omega)}^2 + \varepsilon_3 \int_{\Omega} |\psi|^3 \psi_x^2 \, dx, \quad (21) \]

where \( C_1 > 0 \) is independent of \( N, \varepsilon \) and \( \delta < \delta_0 \).

Next, multiplying (16) by \( \Phi''_\varepsilon(\psi) \), we deduce that

\[ \frac{d}{dt} \int_{\Omega} \Phi_\varepsilon(\psi) \, dx + \int_{\Omega} D_{2,\varepsilon}(h, \psi) \Phi''_\varepsilon(\psi) \psi_x^2 \, dx = \]

\[ \beta \int_{\Omega} F_\delta(h) h_{xx} g_{1,\delta}(\phi) \Phi''_\varepsilon(\psi) \psi_x \, dx + \int_{\Omega} F_\delta(h) g_{0,\delta}(\phi) \Phi''_\varepsilon(\psi) \psi_x \, dx \leq \]

\[ \beta \left( \int_{\Omega} F_\delta(h) \frac{g_{2,\delta}(\phi) \Phi''_\varepsilon(\psi)}{f_{1,\delta}(\phi) D_{2,\varepsilon}(h, \psi)} D_{2,\varepsilon}(h, \psi) \Phi''_\varepsilon(\psi) \psi_x^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} f_{1,\delta}(\phi) F_{\delta \varepsilon}(h) h_{xx}^2 \, dx \right)^{\frac{1}{2}} + \]

\[ \left( \int_{\Omega} D_{2,\varepsilon}(h, \psi) \Phi''_\varepsilon(\psi) \psi_x^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} F_\delta(h) g^2(\phi) \Phi''_\varepsilon(\psi) \psi_x^2 \, dx \right)^{\frac{1}{2}}, \]

where \( \Phi''_\varepsilon(z) = \frac{|z|^{-3} D_{2,\varepsilon}(z) = b_2 \chi_\varepsilon + \varepsilon |z|^{-3} > 0, \) i.e. \( \Phi_\varepsilon(z) = \frac{b_2}{2} \chi_\varepsilon z^2 + \frac{\varepsilon}{2} |z|^{-1} \).

Note that

\[ F_\delta(h) \frac{g_{2,\delta}(\phi) \Phi''_\varepsilon(\psi)}{f_{1,\delta}(\phi) D_{2,\varepsilon}(h, \psi)} \leq |h|^{3} \frac{g_{2,\delta}(\phi)}{f_{1,\delta}(\phi)} \frac{\Phi''_\varepsilon(\psi)}{\psi_x} \leq \delta |\phi|^{1} \]

Then, due to (9)–(13), we obtain that

\[ \frac{d}{dt} \int_{\Omega} \Phi_\varepsilon(\psi) \, dx + (1 - \varepsilon_4 - \varepsilon_5) \int_{\Omega} D_{2,\varepsilon}(h, \psi) \Phi''_\varepsilon(\psi) \psi_x^2 \, dx \leq \]

\[ \frac{\beta^2 (b_5^2 + \delta)}{4 \varepsilon_4} \int_{\Omega} f_{1,\delta}(\phi) F_{\delta \varepsilon}(h) h_{xx}^2 \, dx + \frac{\beta^2 \chi_\varepsilon}{4 \varepsilon_5} \int_{\Omega} |h|^3 \, dx \leq \]

\[ \frac{\beta^2 (b_5^2 + \delta)}{4 \varepsilon_4} \int_{\Omega} f_{1,\delta}(\phi) F_{\delta \varepsilon}(h) h_{xx}^2 \, dx + C_2 \|h\|_{H^1(\Omega)}^2, \quad (22) \]

where \( C_2 > 0 \) is independent of \( N, \varepsilon \) and \( \delta < \delta_0 \).

Summing (21) and (22), we have

\[ \frac{1}{2} \frac{d}{dt} \|h\|_{H^1(\Omega)}^2 + \frac{d}{dt} \int_{\Omega} \Phi_\varepsilon(\psi) \, dx + (\beta - \varepsilon_1 - \varepsilon_2 - \frac{\beta^2 (b_5^2 + \delta)}{4 \varepsilon_4}) \int_{\Omega} f_{1,\delta}(\phi) F_{\delta \varepsilon}(h) h_{xx}^2 \, dx + \]

\[ \frac{\beta^2 (b_5^2 + \delta)}{4 \varepsilon_4} \int_{\Omega} f_{1,\delta}(\phi) F_{\delta \varepsilon}(h) h_{xx}^2 \, dx + C_2 \|h\|_{H^1(\Omega)}^2. \]
where $C_3 = \max\{1, C_1, C_2\} > 0$. Choosing $\varepsilon_i$ such that

$$
\beta - \varepsilon_1 - \varepsilon_2 - \frac{\beta^2(b_0^2 + \delta)}{4\varepsilon_4} > 0, \quad 1 - \varepsilon_4 - \varepsilon_5 - \chi_0 \frac{a_2^2}{4\varepsilon_5} (\frac{1}{\varepsilon_5} + \varepsilon_3) > 0,
$$

namely,

$$
0 < \varepsilon_1, \varepsilon_2, \varepsilon_3 \ll 1, \quad \chi_0 \frac{a_2^2}{4\varepsilon_5} < \varepsilon_2 < \beta - \frac{\beta^2(b_0^2 + \delta)}{4\varepsilon_4},
$$

$$
\max\{\frac{\beta(b_0^2 + \delta)}{4}, \frac{1}{\varepsilon_5}(4\beta + \beta^2(b_0^2 + \delta) - \chi_0 \frac{a_2^2}{4\varepsilon_5} - \sqrt{(4\beta + \beta^2(b_0^2 + \delta) - \chi_0 \frac{a_2^2}{4\varepsilon_5})^2 - 16\beta^3(b_0^2 + \delta)}\} < \varepsilon_4 < \min\{1, \frac{1}{\varepsilon_5}(4\beta + \beta^2(b_0^2 + \delta) - \chi_0 \frac{a_2^2}{4\varepsilon_5} + \sqrt{(4\beta + \beta^2(b_0^2 + \delta) - \chi_0 \frac{a_2^2}{4\varepsilon_5})^2 - 16\beta^3(b_0^2 + \delta)}\}
$$

provided

$$
|4\beta + \beta^2(b_0^2 + \delta) - \chi_0 \frac{a_2^2}{4\varepsilon_5}| > 4(b_0^2 + \delta)^\frac{3}{2}, \beta^ \frac{3}{2},
$$

we get

$$
\frac{1}{2} \frac{d}{dt} \|h\|_{H^1(\Omega)}^2 + \frac{d}{dt} \int_\Omega \Phi_\varepsilon(\psi) dx + C \int_\Omega \int f_{1, \beta}(\phi) F_{\delta\varepsilon}(h) h_{xx^2}^2 dx dt + C \int_\Omega D_{2, \varepsilon}(h, \psi) \Phi_\varepsilon''(\psi) \psi_2^2 dx \leq C_3 \max\{1, \|h\|_{H^1(\Omega)}^2\},
$$

(24)

Applying Grönwall’s lemma to (24) with $y(t) = \max\{1, \|h\|_{H^1(\Omega)}^2\} + 2\|\Phi_\varepsilon(\psi)\|_1$, we obtain that

$$
\|h\|_{H^1(\Omega)}^2 \leq \frac{\max\{1, \|h_N\|_{H^1(\Omega)}^2\} + 2\|\Phi_\varepsilon(\psi_N)\|_1}{(1 - 3C_3 \max\{1, \|h_N\|_{H^1(\Omega)}^2\} + 2\|\Phi_\varepsilon(\psi_N)\|_1)^\frac{3}{2}}.
$$

(25)

for all $t < T_N := [3C_3 \max\{1, \|h_N\|_{H^1(\Omega)}^2\} + 2\|\Phi_\varepsilon(\psi_N)\|_1]^\frac{3}{2} - 1$. Because $h_{0c}^N \to h_0$ strongly in $H^1(\Omega)$ and $\Phi_\varepsilon(\psi_{0c}^N) \to \Phi_0(\psi_0)$ strongly in $L^1(\Omega)$ as $N \to +\infty$ and $\varepsilon \to 0$ then we can select a time $T_0 := [6C_3 \max\{1, \|h_0\|_{H^1(\Omega)}^2\} + 2\|\Phi_0(\psi_0)\|_1]^\frac{3}{2} - 1 < T_N$ which is independent of $N, \varepsilon$ and $\delta$. As a result, we have the following a priori estimate

$$
\|h\|_{H^1(\Omega)}^2 + \int_\Omega \Phi_\varepsilon(\psi) dx + C \int_\Omega \int f_{1, \beta}(\phi) F_{\delta\varepsilon}(h) h_{xx^2}^2 dx dt + C \int_\Omega D_{2, \varepsilon}(h, \psi) \Phi_\varepsilon''(\psi) \psi_2^2 dx dt \leq C_4
$$

(26)

for all $T \leq T_0$, where $C_4$ is independent of $N, \varepsilon$ and $\delta$.

Hence, from (26) it follows that the solution $(a_i(t), b_i(t))$ can be extended up to $T_0$. As a conclusion, we have shown that the Galerkin equations have solutions

$$
h_{0c}^N, \psi_{0c}^N \in C^1(0, T; C^\infty(\Omega))$$

for all $T \leq T_0$. 

4.3. Limit processes.

4.3.1. Limits of $N \to +\infty$ and $\varepsilon \to 0$. Let $\phi_{\varepsilon \delta}^N = \frac{\psi_{\varepsilon \delta}^N}{h_{\varepsilon \delta}^N}$. Next, we have to show that in the following weak formulation we can pass to the limit for $N \to +\infty$:

$$
\int_0^T \langle h_{\varepsilon \delta \delta}^N(t), \xi^N(t) \rangle\,dt - \beta \int_0^T \left[ \int f_1(h_{\varepsilon \delta}^N) f_{\delta}(h_{\varepsilon \delta}^N) \epsilon_{x,x}^N \,dx\,dt \right. \\
\left. - \int D_1(h_{\varepsilon \delta}^N, \psi_{\varepsilon \delta}^N) \psi_{\varepsilon \delta}^N \epsilon_{x}^N \,dx\,dt \right]
$$

(27)

$$
\int_0^T \langle \psi_{\varepsilon \delta \delta}^N(t), \zeta^N(t) \rangle\,dt - \beta \int_0^T \left[ \int g_1(h_{\varepsilon \delta}^N) g_{\delta}(h_{\varepsilon \delta}^N) \epsilon_{x}^N \,dx\,dt \right. \\
\left. - \int D_2(h_{\varepsilon \delta}^N, \psi_{\varepsilon \delta}^N) \psi_{\varepsilon \delta}^N \epsilon_{x}^N \,dx\,dt \right]
$$

(28)

for all $\xi^N \in L^2(0,T;H^1(\Omega))$ and $\zeta^N \in L^3(0,T;W_3^1(\Omega))$ such that $\xi^N \to \xi$ in $L^2(0,T;H^1(\Omega))$ and $\zeta^N \to \zeta$ in $L^3(0,T;W_3^1(\Omega))$ with $\xi(-a,t) = \xi(a,t)$, $\zeta(-a,t) = \zeta(a,t)$ for all $t \in (0,T)$.

Next, we have to establish appropriate convergence properties. By (26) we have the following (uniformly in $N$, $\varepsilon$ and $\delta$) boundedness for all $T \leq T_0$

$$
\{h_{\varepsilon \delta}^N\} \text{ in } L^\infty(0,T;H^1(\Omega)),
$$

(29)

$$
\{\Phi_{\varepsilon}(\psi_{\varepsilon \delta}^N)\} \text{ in } L^\infty(0,T;L^1(\Omega)),
$$

(30)

$$
\{(f_1(h_{\varepsilon \delta}^N) f_{\delta}(h_{\varepsilon \delta}^N))^{1/2} h_{\varepsilon \delta,xxxx}^N\} \text{ in } L^2(Q_T),
$$

(31)

$$
\{D_1(h_{\varepsilon \delta}^N, \psi_{\varepsilon \delta}^N) \psi_{\varepsilon \delta,xx}^N\} \text{ in } L^2(Q_T),
$$

(32)

$$
\{(D_2(h_{\varepsilon \delta}^N, \psi_{\varepsilon \delta}^N) \Phi_{\varepsilon}(\psi_{\varepsilon \delta}^N))^{1/2} \psi_{\varepsilon \delta,xx}^N\} \text{ in } L^2(Q_T),
$$

(33)

$$
\{(\delta)^{1/2} h_{\varepsilon \delta,xxxx}^N\} \text{ in } L^2(Q_T).
$$

(34)

By (29) and the embedding theorem, we have

$$
\{h_{\varepsilon \delta}^N\} \text{ is uniformly bounded in } L^\infty(Q_T).
$$

(35)

Note that from (30) it follows

$$
\{\chi_{\varepsilon} \psi_{\varepsilon \delta}^N\} \text{ in } L^\infty(0,T;L^2(\Omega)),
$$

(36)

and from (33), (36) we have

$$
\{\chi_{\varepsilon} \psi_{\varepsilon \delta}^N\} \text{ in } L^2(0,T;H^1(\Omega)),
$$

(37)

By (37) and (36), due to the embedding theorem for parabolic function spaces from [12, Proposition 3.2, p. 8] applied to $w = \chi_{\varepsilon} \psi_{\varepsilon \delta}^N \frac{1}{2} \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, we can derive the following estimate for $\psi_{\varepsilon \delta}^N$:

$$
\{\chi_{\varepsilon} \psi_{\varepsilon \delta}^N\} \text{ in } L^9(Q_T).
$$

(38)
The previous statements allow us to prove that
\[
I_{\varepsilon \delta}^N \coloneqq \beta (F_{\varepsilon \delta}^N(h_{\varepsilon \delta}^N) + \varepsilon) f_{1,\delta}(\phi_{\varepsilon \delta}^N) h_{\varepsilon \delta,xxx}^N + F_{\delta}(h_{\varepsilon \delta}^N) f_0(\phi_{\varepsilon \delta}^N) - D_{1,\delta}(h_{\varepsilon \delta}^N, \psi_{\varepsilon \delta}^N) \psi_{\varepsilon \delta,x}^N \text{ is u. b. in } L^2(Q_T),
\]
(39)
and therefore by (29), (31) and (34), we find that
\[
\{h_{\varepsilon \delta,t}^N\} \text{ is uniformly bounded in } L^2(0,T;(H^1(\Omega))^*),
\]
(41)
and by (36) and (33), we deduce that
\[
\{\psi_{\varepsilon \delta,t}^N\} \text{ is uniformly bounded in } L^2(0,T;(W^{1,3}_3(\Omega))^*).
\]
(42)
By (29) and (41) we find (see [5]) that
\[
\{h_{\varepsilon \delta}^N\} \text{ is uniformly bounded in } C^1_\infty(\bar{Q}_T).
\]
(43)
Therefore, we conclude that there exists a subsequence \(N = N_k, \varepsilon = \varepsilon_l\) such that
\[
h_{\varepsilon \delta}^N \to h_{\delta} \text{ uniformly as } N \to +\infty, \varepsilon \to 0.
\]
(44)
Moreover, by (36) we have
\[
h_{\varepsilon \delta}^N \to h_{\varepsilon \delta} \text{ weakly in } L^2(0,T;W^3_2(\Omega)) \text{ as } N \to +\infty.
\]
(45)
From (36), (19) and (37) it follows that there exists a subsequence such that
\[
\psi_{\varepsilon \delta}^N \to \psi_{\delta} \text{ *-weakly in } L^\infty(0,T;L^1(\Omega)) \text{ and a. e. in } Q_T,
\]
(46)
\[
\chi_{\phi} \psi_{\varepsilon \delta}^N \to \chi_{\phi} \psi_{\delta} \text{ *-weakly in } L^\infty(0,T;L^2(\Omega)) \text{ and a. e. in } Q_T,
\]
(47)
\[
\chi_{\phi}(\psi_{\varepsilon \delta}^N)^{\frac{1}{2}} \to \chi_{\phi}(\psi_{\delta})^{\frac{1}{2}} \text{ weakly in } L^2(0,T;H^1(\Omega))
\]
(48)
as \(N \to +\infty, \varepsilon \to 0\). Thus, by (41) and (42), we have for correspondent subsequences
\[
h_{\varepsilon \delta,t}^N \to h_{\delta,t} \text{ *-weakly in } L^2(0,T;(H^1(\Omega))^*),
\]
(49)
\[
\psi_{\varepsilon \delta,t}^N \to \psi_{\delta,t} \text{ *-weakly in } L^\frac{2}{3}(0,T;(W^{1,3}_3(\Omega))^*).
\]
(50)
In particular, by (44) and (46) we get
\[
\phi_{\varepsilon \delta}^N := \frac{\psi_{\varepsilon \delta}^N}{h_{\varepsilon \delta}^N} \to \phi_{\delta} := \frac{\psi_{\delta}}{h_{\delta}} \text{ a. e. on } \{|h_{\delta}| > \mu\}
\]
(51)
for all \(\mu > 0\) as \(N \to +\infty\) and \(\varepsilon \to 0\). Due to (51), we can take limit in all terms of (27)–(28), connected with \(\phi_{\varepsilon \delta}^N\), as \(N \to +\infty\) and \(\varepsilon \to 0\) on the set \(|h_{\delta}| > \mu\).

On the next subsections, we will prove that \(h_{\delta} > 0\) and \(\psi_{\delta} > 0\). For this reason, instead of convergence (51) on the set \(|h_{\delta}| > \mu\), we obtain this convergence a. e. in \(Q_T\).
Applying these convergence results to (27)–(28) we get that the Galerkin solutions \( h_{\epsilon_0}^N, \psi_{\epsilon_0}^N \) converge for any fixed \( \delta > 0 \) to a weak solution \((h_\delta, \psi_\delta)\) of the degenerate problem

\[
\int_0^T \langle h_{\delta,t}(t), \xi(t) \rangle dt - \beta \iint_{Q_T} f_1(\phi_\delta) F_\delta(h_\delta) h_{\delta,xxx} \xi_x dx dt -
\iint_{Q_T} f_0(\phi_\delta) F_\delta(h_\delta) \xi_x dx dt = - \iint_{Q_T} D_1(\delta, h_\delta, \psi_\delta) \psi_{\delta,x} \xi_x dx dt,
\]

\[
\iint_{Q_T} g_1(\phi_\delta) F_\delta(h_\delta) h_{\delta,xxx} \zeta_x dx dt -
\iint_{Q_T} g_0(\phi_\delta) F_\delta(h_\delta) \zeta_x dx dt = - \iint_{Q_T} D_2(h_\delta, \psi_\delta) \psi_{\delta,x} \zeta_x dx dt
\]

for all \( T \leq T_0 \) and \( \xi \in L^2(0,T; H^1(\Omega)) \), and \( \zeta \in L^3(0,T; W^1_3(\Omega)) \) with \( \xi(-a,t) = \xi(a,t) \), \( \zeta(-a,t) = \zeta(a,t) \) for all \( t \in (0,T_0) \).

4.3.2. Positivity of \( h_\delta \). Next, we show \( h_\delta > 0 \) for all \( \delta < \delta_0 \). This allows us to extend the corresponding integrals in (52), (53) on all \( Q_T \). Multiplying (15) by \( G''_{\delta x}(h) \), we get

\[
\frac{d}{dt} \int_{\Omega} G_{\delta x}(h) dx = \beta \int_{\Omega} f_{1,\delta}(\phi) F_{\delta x}(h) G''_{\delta x}(h) h_x h_{xxx} dx +
\int_{\Omega} f_0(\phi) F_\delta(h) G''_{\delta x}(h) h_x dx - \int_{\Omega} D_{1,\delta}(h, \psi) \psi_x G''_{\delta x}(h) h_x dx =
\beta \int_{\Omega} f_{1,\delta}(\phi) |h|^\alpha h_x h_{xxx} dx + \int_{\Omega} \frac{f_0(\phi) F_\delta(h)}{F_{\delta x}(h)} |h|^\alpha h_x dx - \int_{\Omega} \frac{f_2(\phi) F_\delta(h)}{F_{\delta x}(h)} |h|^\alpha h_x \psi_x dx,
\]

where \( G''_{\delta x}(z) = \frac{|z|^\alpha}{F_{\delta x}(z)} \). Using (9), we have

\[
\frac{d}{dt} \int_{\Omega} G_{\delta x}(h) dx \leq a_1 \left( \int_{\Omega} h_x^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} f_{1,\delta}(\phi) |h|^{2\alpha} dx \right)^{\frac{1}{2}} +
\beta \left( \int_{\Omega} f_{1,\delta}(\phi) F_{\delta x}(h) h_{xxx}^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} f_{1,\delta}(\phi) \frac{|h|^{2\alpha}}{F_{\delta x}(h)} h_x^2 dx \right)^{\frac{1}{2}} +
a_2 \chi_\phi \left( \int_{\Omega} |h|^{2\alpha-3} h_x^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\psi|^3 \psi_x^2 dx \right)^{\frac{1}{2}}.
\]

Choose \( \alpha \geq \frac{3}{2} \) and \( s \geq 3 \). Then

\[
\frac{d}{dt} \int_{\Omega} G_{\delta x}(h) dx \leq C a_1 a_0^{\frac{3}{2}} \|h\|_{H^1(\Omega)}^{\alpha + 1} +
\]
where

\[
\int f_{1,\delta}(\phi)F_{\delta}(h)h_{xx}^2 \, dx \leq C_2 N_0 h_{xx}^{2a-1} + C_2 (\int |\phi|^b \psi_x^2 \, dx)^{\frac{1}{2}}.
\]

Integrating this inequality in time, taking into account (26), we deduce that

\[
\int_\Omega G_{\delta}(h) \, dx \leq \int_\Omega G_{\delta}(h_{0,\delta}) \, dx + C_4(T)
\]  

for all \( T \leq T_0 \), where \( C_4(T) \) is independent of \( N, \epsilon, \delta \) and \( \delta < \delta_0 \). By Fatou’s lemma, (44) and from the uniform (in \( N, \epsilon, \delta \)) bound of \( \int G_{\delta}(h_{0,\delta}) \, dx \) we deduce that \( \int G_{\delta}(h) \, dx \) is uniformly bounded in \( N, \epsilon, \delta \).

First of all, we show that \( h_{\delta} \geq 0 \) in \( Q_{T_0} \) when \( s \geq 4 \). If this is not true, then there is a point \((x_0, t_0) \in Q_{T_0}\) such that \( h_{\delta}(x_0, t_0) < 0 \). Since convergence \( h_{\delta} \to h_{\delta} \) is uniform as \( \epsilon \to 0 \) then there exist \( \gamma > 0 \) and \( \epsilon_0 > 0 \) such that \( h_{\delta}(x, t_0) < -\gamma \) if \( |x - x_0| < \gamma \) and \( \epsilon < \epsilon_0 \). But for such \( x \), by the monotone convergence theorem

\[
G_{\delta}(h_{\delta}(x, t_0)) = \int_A \int_A \frac{|z|^a}{F_{\delta}(z)} \, dz \, dv \geq \int_{-\gamma}^{0} \int_A \frac{|z|^a}{F_{\delta}(z)} \, dz \, dv \to \infty \quad \text{as} \quad \epsilon \to 0
\]

where \( A = \max |h_{\delta} \cdot \epsilon| \) for all small \( \delta, \epsilon \). Hence, \( \lim_{\epsilon \to 0} \int G_{\delta}(h_{\delta}) \, dx = \infty \) and this is in contradiction with (54).

Next, we show that \( h_{\delta} > 0 \) on \( \Omega \) when \( s \geq 4 \). Indeed, if \( h_{\delta} \) is not positive everywhere in \( Q_{T_0} \), then there exists a point \((x_0, t_0) \in Q_{T_0}\) such that \( h_{\delta}(x_0, t_0) = 0 \). Then by the Hölder continuity of \( h_{\delta} \in C^{1/2} \), we have \( |h_{\delta}(x, t)| = |h_{\delta}(x, t) - h_{\delta}(x_0, t)| \leq C|x - x_0|^{1/2} \). Hence, taking into account \( G_{\delta}(z) \sim \frac{|h_{\delta}|^a}{(|\alpha - s + 1|)(|\alpha - s + 2|)} \) for \( |z| \ll 1 \), we come to a contradiction

\[
\infty > \int \infty \geq C \int |x - x_0|^{\frac{a - s + 2}{2}} \, dx = \infty \quad \text{if} \quad s \geq 8, \quad \alpha \in \left[ \frac{s}{2}, s - 4 \right].
\]

As \( G_{\delta}(z) - G_0(z) = \frac{\delta^{a - s + 2}}{(|\alpha - s + 1|)(|\alpha - s + 2|)} \) for all \( z \geq 0 \) then, due to (18), \( G_{\delta}(h_{\delta}) - G_0(h_{0,\delta}) \leq \frac{\delta^{a - s + 2}}{(|\alpha - s + 1|)(|\alpha - s + 2|)} \) as \( \delta \to 0 \). Hence, \( \int G_0(h) \, dx \) is bounded provided

\[
\int h_0^{a - 1} \, dx < \infty,
\]

hence it follows that \( h \geq 0 \) if \( h_0 \geq 0 \) in \( \Omega \).

4.3.3. Nonnegativity of \( \psi_\delta \). Now, we can use the bound for \( \int \Phi_{\epsilon}(\psi_{\epsilon,\delta}) \, dx \) (see (26)) to derive the lower bound \( \psi_\delta \geq 0 \). If \( z < 0 \) and \( 0 < \epsilon < \epsilon_0 \), then, due to \( \Phi_{\epsilon}'(z) = \chi_{\delta} |b_2 + \epsilon||z|^{-3} \) and \( \Phi_{\epsilon}''(z) = \frac{b_2 \chi_{\delta}}{2z^2} z^2 + \frac{\epsilon}{2} |z|^{-1} \), we have

\[
\Phi_{\epsilon}(z) \geq \Phi_{\epsilon}(\epsilon) + \Phi_{\epsilon}'(\epsilon)(z - \epsilon) + \frac{1}{2} \Phi_{\epsilon}''(\epsilon)(z - \epsilon)^2 \geq \Phi_{\epsilon}'(\epsilon)(z - \epsilon) + \frac{\chi_{\delta} b_2 z^2 + 1}{2z^2} z^2.
\]
It follows that
\[ z^2 \leq \frac{2\varepsilon^2}{\chi_0 b_2 \varepsilon^2 + 1} (\Phi_\varepsilon(z) - \Phi_\varepsilon'(\varepsilon)(z - \varepsilon)). \]
This implies
\[ \int_\Omega (-\psi_\varepsilon) dx \leq \frac{2\varepsilon^2}{\chi_0 b_2 \varepsilon^2 + 1} \int_\Omega \Phi_\varepsilon(\psi_\varepsilon) dx - \frac{2\varepsilon^2 \Phi_\varepsilon'(\varepsilon)}{\chi_0 b_2 \varepsilon^2 + 1} \int_\Omega (\psi_\varepsilon - \varepsilon) dx \leq 2\varepsilon^2 \int_\Omega \Phi_\varepsilon(\psi_\varepsilon) dx + \varepsilon \int_\Omega \psi_{0,\varepsilon} dx + 2|\Omega|\varepsilon^2. \]

Then, taking into account (30) and (19), passing to the limit in this inequality as \( \varepsilon \to 0 \) yields \( \psi_\delta \geq 0 \) a.e. in \( Q_{T_0} \).

4.3.4. Estimate \( \psi_\delta \leq h_\delta \). Let us denote by
\[ v := h_\delta - \psi_\delta. \]
We want to show that \( v \geq 0 \). Subtracting (16) from (15) with \( \varepsilon = 0 \), we arrive at
\[ v_t + (L_\delta)_x = (\tilde{D}_\delta(h, \psi)v_x)_x \text{ in } Q_{T_0}, \] \( \tag{55} \)
\[ v(x, 0) = v_{0,\delta}(x) := h_{0,\delta} - \psi_0 \geq 0, \] \( \tag{56} \)
\[ v(-a, t) = v(a, t), \quad v_x(-a, t) = v_x(a, t) \quad \forall \, t \in (0, T_0), \] \( \tag{57} \)
where
\[ L_\delta := \beta F_\delta(h)(f_1,\delta(\phi) - g_1,\delta(\phi))h_{xx} + F_\delta(h)(f_0(\phi) - g_0(\phi)) + \tilde{D}_\delta(h, \psi)h_x, \]
\[ \tilde{D}_\delta(h, \psi) := D_2(h, \psi) - D_1,\delta(h, \psi) = h^3 g_2(\phi) - F_\delta(h) f_2(\phi). \]
By (13) we find that
\[ L_\delta = \tilde{D}_\delta = 0 \text{ if } v < 0, \text{ i.e. } \phi_\delta = \frac{\psi_\delta}{h_\delta} > 1. \] \( \tag{58} \)
Choose
\[ r \in C^1(\mathbb{R}) : r(z) < 0, \quad r'(z) \geq 0 \text{ if } z < 0, \quad r(z) = 0 \text{ if } z \geq 0. \]
Then
\[ R(z) := \int_0^z r(s) ds = 0 \text{ if } z \geq 0, \quad R(z) > 0 \text{ if } z < 0, \]
and in particular,
\[ \int_\Omega R(v_{0,\delta}(x)) dx = 0. \]
Multiplying (55) by \( R'(v) = r'(v) \), we deduce that
\[ \frac{d}{dt} \int_\Omega R(v) dx + \int_\Omega \tilde{D}_\delta(h, \psi)r'(v)v_x^2 dx = - \int_\Omega r(v)(L_\delta)_x dx, \]
whence by (58) we have
\[ 0 \leq \int_\Omega R(v) dx \leq \int_\Omega R(v_{0,\delta}) dx = 0. \]
This implies that \( R(v) = 0 \), thus
\[ v \geq 0, \text{ i.e. } \psi_\delta \leq h_\delta \iff \phi_\delta \leq 1, \text{ a.e. in } \Omega \text{ for any } t \in [0, T_0]. \] \( \tag{59} \)
Passing to the limit in (59) as \( \delta \to 0 \) yields
\[
0 \leq \psi \leq h \text{ a.e. in } Q_{T_0}.
\] (60)

4.3.5. Global existence. Using the mass conservation (19), we can extend our local solution for all times. We consider the approximation solutions \((h_\delta, \psi_\delta)\), where \(h_\delta > 0\). Next, instead of (26), taking into account (59), we obtain more exact a priori estimates for \((h_\delta, \psi_\delta)\). For brevity, we denote by \(h := h_\delta, \psi := \psi_\delta\) and \(\phi := \phi_\delta\).

Multiplying (15) with \(\varepsilon = 0\) by \(-h_{xx}\), we deduce
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} h_{xx}^2 dx + \beta \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx = \]
\[
- \int_{\Omega} f_0(\phi) F_\delta(h) h_{xxx} dx + \int_{\Omega} D_{1,\delta}(h, \psi) \psi_x h_{xxx} dx \leq \]
\[
\left( \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} F_\delta(h) \frac{f_0^2(\phi)}{F_\delta(h)} dx \right)^{\frac{1}{2}} + \]
\[
\left( \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} D_{2,\delta}(h, \psi) h_{xx}^2 dx \right)^{\frac{1}{2}} \leq \]
\[
\left( \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx \right)^{\frac{1}{2}} \left( a_1^2 \int_{\Omega} h^3 dx \right)^{\frac{1}{2}} + \]
\[
\left( \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx \right)^{\frac{1}{2}} \left( a_2^2 \int_{\Omega} \psi^2 \psi_x^2 dx \right)^{\frac{1}{2}},
\]
whence we find that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} h_{xx}^2 dx + (\beta - \varepsilon_1 - \varepsilon_2) \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx \leq a_1^2 \int_{\Omega} h^3 dx + a_2^2 \int_{\Omega} \psi^2 \psi_x^2 dx.
\] (61)

Using the Nirenberg-Gagliardo interpolation inequality
\[
\|v\|_p \leq c_0 \|v_x\|_{L^p(\Omega)}^{\frac{2(p-1)}{3p}} + c_1 \|v\|_1
\] (62)
for all \(v \in H^1(\Omega)\) with \(v = h \geq 0, p = 3\) and the mass conservation \(\int h dx = \|h_{0,\delta}\|_1\), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} h_{xx}^2 dx + (\beta - \varepsilon_1 - \varepsilon_2) \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx \leq \]
\[
\frac{Ca_1^2}{4\varepsilon_1} \left( \int_{\Omega} h_{xx}^2 dx \right)^{\frac{2}{3}} + \frac{a_2^2}{4\varepsilon_2 b_2} \int_{\Omega} D_2(h, \psi) \psi_x^2 dx,
\] (63)
where the \(C\)'s is independent of \(\delta < \delta_0\).
Next, multiplying (16) with $\varepsilon = 0$ by $\psi$, we deduce that

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \psi^2 dx + \int_{\Omega} D_2(h, \psi) \psi_x^2 dx = \beta \int_{\Omega} F_\delta(h) h_{xxx} g_{1,\delta}(\phi) \psi_x dx + \int_{\Omega} F_\delta(h) g_0(\phi) \psi x dx \leq \beta \left( \int_{\Omega} F_\delta(h) \frac{\psi_x^2}{f_{1,\delta}(\phi)} dx \right)^{\frac{1}{2}} + \frac{\beta^2}{b_2} \left( \int_{\Omega} D_2(h, \psi) \psi_x^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \frac{\psi_x^2}{D_2(h, \psi)} dx \right)^{\frac{1}{2}} + \beta \left( \frac{\beta^2}{b_2} \int_{\Omega} D_2(h, \psi) \psi_x^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx \right)^{\frac{1}{2}} + \left( \int_{\Omega} D_2(h, \psi) \psi_x^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} h^3 dx \right)^{\frac{1}{2}}.
$$

Then we obtain that

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \psi^2 dx + (1 - \varepsilon_3 - \varepsilon_4) \int_{\Omega} D_2(h, \psi) \psi_x^2 dx \leq \frac{\beta^2 h_0^3}{4 \varepsilon_3 b_2} \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx + \frac{b_2^2}{4 \varepsilon_4 b_2} \int_{\Omega} h^3 dx \leq \frac{\beta^2 h_0^3}{4 \varepsilon_3 b_2} \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx + \frac{C h_0^2}{4 \varepsilon_4 b_2} \left( \int_{\Omega} h^2 dx \right)^{\frac{3}{2}} + \frac{C h_0^2}{4 \varepsilon_4 b_2}. \quad (64)
$$

Summing (63) and (64), we have

$$
\frac{1}{2} \frac{d}{dt} (\|h_x\|_{L^2(\Omega)}^2 + ||\psi||_{L^2(\Omega)}^2) + (\beta - \varepsilon_1 - \varepsilon_2 - \frac{\beta^2 h_0^3}{4 \varepsilon_3 b_2}) \int_{\Omega} f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx + (1 - \varepsilon_3 - \varepsilon_4 - \frac{a^2}{4 \varepsilon_2 b_2}) \int_{\Omega} D_2(h, \psi) \psi_x^2 dx \leq C \|h_x\|_{L^2(\Omega)}^2 + C_5. \quad (65)
$$

Choosing $\varepsilon_i$ such that

$$
\beta - \varepsilon_1 - \varepsilon_2 - \frac{\beta^2 h_0^3}{4 \varepsilon_3 b_2} > 0, \quad 1 - \varepsilon_3 - \varepsilon_4 - \frac{a^2}{4 \varepsilon_2 b_2} > 0,
$$

namely,

$$
0 < \varepsilon_1, \varepsilon_4 \ll 1, \quad \frac{\beta^2 h_0^3}{4 b_2 (\beta - \varepsilon_2)} < \varepsilon_3 < 1 - \frac{a^2}{4 \varepsilon_2 b_2},
$$

$$
\frac{\beta}{2} \left[ \frac{a^2}{4 \varepsilon_2 b_2} - \frac{\beta h_0^2}{4 b_2} + 1 - \sqrt{\left( \frac{a^2}{4 \varepsilon_2 b_2} - \frac{\beta h_0^2}{4 b_2} \right)^2 - \frac{a^2}{\beta b_2}} \right] < \varepsilon_2 < \frac{\beta}{2} \left[ \frac{a^2}{4 \varepsilon_2 b_2} - \frac{\beta h_0^2}{4 b_2} + 1 + \sqrt{\left( \frac{a^2}{4 \varepsilon_2 b_2} - \frac{\beta h_0^2}{4 b_2} + 1 \right)^2 - \frac{a^2}{\beta b_2}} \right].
$$
provided
\[ \frac{a_1^2}{t_0^2} < \sqrt{1 - \frac{\beta_0}{t_0^2}} \quad \text{and} \quad \frac{\beta_0}{t_0^2} < 1 \quad \text{or} \quad \frac{a_1^2}{t_0^2} > \max\left\{ \frac{\beta_0}{t_0^2} - 1, \sqrt{1 + \frac{\beta_0}{t_0^2}} \right\}, \]
we have
\[ \frac{4}{h^2} (\|h_x\|_{L^2(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2) + C \int \Omega f_{1,\delta}(\phi)F_5(h)h_{x,x,x}^2dx + C \int \Omega D_2(h,\psi)\psi_{x}^2 dx \leq C_6 \max\{1, \|h_x\|_{L^2(\Omega)}^4\}, \] (66)
where \( C_6 > 0 \) is independent of \( \delta > 0 \).

By Grönwall’s lemma applied to \( y(t) := \max\{1, \|h_x\|_{L^2(\Omega)}^2\} + \|\psi\|_{L^2(\Omega)}^2 \), we obtain that
\[ \|h_x\|_{L^2(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 \leq \left[ (\max\{1, \|h_{0,x}\|_{L^2(\Omega)}^2\} + \|\psi_0\|_{L^2(\Omega)}^2)^{\frac{1}{2}} + \frac{1}{2} C_6 t \right]^3 \] (67)
for all \( t \geq 0 \). As a result, we have
\[ \|h_x\|_{L^2(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + C \iint_{Q_T} f_{1,\delta}(\phi)F_5(h)h_{x,x,x}^2 dx dt + C \iint_{Q_T} D_2(h,\psi)\psi_{x}^2 dx dt \leq C_7(T) \forall T > 0, \] (68)
where \( C_7(T) \) is independent of \( \delta < \delta_0 \). The a priori estimate (68) allows us to construct the limit solution \((h,\psi)\) as \( \delta \to 0 \) for all \( T > 0 \).

4.3.6. Limit process for \( \delta \to 0 \). Similar to (51), in view of (59) and (60), we can take limit in all terms of (52)–(53), connected with \( \phi_\delta = \frac{\psi_\delta}{\mu_\delta} \), as \( \delta \to 0 \) on the set \( \{h > \mu\} \). On the sets \( \{\psi \leq h \leq \mu\} \) and \( \{\psi \leq \mu < h\} \), we can find an upper bound of the corresponding integrals on this set. Really, if \( \delta \) is sufficiently small, depending on \( \mu \), then
\[ \left| \iint_{\{h \leq \mu\}} f_{1,\delta}(\phi_\delta)F_5(h_\delta)h_{\delta,x,x,x}^2 \xi_x dx dt \right| \leq \left( \iint_{Q_T} f_{1,\delta}(\phi_\delta)F_5(h_\delta)h_{\delta,x,x,x}^2 dx dt \right)^{\frac{1}{2}} \left( \iint_{\{h \leq \mu\}} f_{1,\delta}(\phi_\delta)F_5(h_\delta)C_\mu^2 dx dt \right)^{\frac{1}{2}} \leq C \mu^2, \]
\[ \left| \iint_{\{h \leq \mu\}} f_0(\phi_\delta)F_5(h_\delta)\xi_x dx dt \right| \leq \left( \iint_{Q_T} \xi_x^2 dx dt \right)^{\frac{1}{2}} \left( \iint_{\{h \leq \mu\}} (f_0(\phi_\delta)F_5(h_\delta))^2 dx dt \right)^{\frac{1}{2}} \leq C \mu^3, \]
\[ \left| \iint_{\{h \leq \mu\}} D_1(\delta,\psi_\delta)\psi_{x,x}^2 \xi_x dx dt \right| \leq \left( \iint_{Q_T} D_2(h_\delta,\psi_\delta)\psi_{x,x}^2 dx dt \right)^{\frac{1}{2}} \times \left( \frac{\int_{\{h \leq \mu\}} D_3^2(\phi_\delta,\psi_\delta) dx dt}{\int_{\{h \leq \mu\}} \xi_x^2 dx dt} \right)^{\frac{1}{2}} \leq C \mu^2, \]
Applying these convergence results to (52)–(53), we get that the solutions $(h, \psi)$ converge to a weak nonnegative solution of the degenerate problem

\[
\begin{aligned}
\int_{\{h > 0\}} \beta f_1(\phi) h^3 h_{xxx} \xi_x dxdt &= 0,
\int_{\{h > 0\}} g_0(\phi) h^3 \zeta_x dxdt &= 0,
\int_{\{\psi > 0\}} (\beta g_1(\phi) h^3 h_{xxx} - D_2(h, \psi) \psi_x) \zeta_x dxdt &= 0
\end{aligned}
\]
for all $T > 0$.

5. **Conclusions.** We have obtained an existence result for a coupled system of degenerate parabolic equations governing the height $h$ and particle concentration $\psi$ of a viscous suspension under the effect of surface tension. The solution satisfies the physical bounds $h \geq 0$ and $0 \leq \psi/h \leq 1$ corresponding to the boundedness of the particle concentration (which is bounded above by the maximum packing fraction for the suspension). The existence result depends on certain bounds on the flux coefficients, particularly on the degeneracy in the $\psi$-diffusion term as $\psi \to 0$, that are consistent with the asymptotic results obtained for the physical system.

The result established here may be useful in future study of this system, for example in developing numerical methods that preserve the bounds on the solution as done for other equations from lubrication theory [20] or in studying the growth of singularities and long-time behavior of advancing fronts.
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