Input-output theory with time-reversal

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We incorporate unitary transformations to the photon wave packet propagating between cascaded quantum systems in input-output theory. This is a basis for quantum state transfer between different quantum systems (quantum transduction), from system 1 to system 2. We present a transformation, \( U \), that time reverses, frequency translates, and stretches the photon wave packet. \( U \) can be tuned such that a wave packet emitted by system 1 is absorbed by system 2. We find concomitant modifications to the standard input-output theory that can best be understood in terms of a change to the state’s time argument, \( \rho(t) = \rho_1(t) \otimes \rho_2(t) \), where \( t \) is a fictitious time for system 1, which runs backwards, as determined by \( U \).

I. INTRODUCTION

We consider two quantum systems, labeled ‘1’ and ‘2,’ that are coupled indirectly by the ability to exchange photons or phonons. We assume this coupling is unidirectional, such that system 1 can affect system 2, but not vice versa. That is, system 1 can produce a photon—for simplicity we will henceforth talk about photons only—that travels to system 2, but no photon travels from system 2 back to system 1. This is the standard situation analyzed in the theory of cascaded quantum systems [1–3]. In standard theory the single-photon wave packet freely propagates from system 1 to 2. That photonic degree of freedom can then easily be eliminated to obtain an effective description solely in terms of quantum operators acting on the Hilbert spaces of systems 1 and 2.

The unidirectional character of the theory is clear in the quantum trajectory description, in which there are two types of time evolution. One is by discrete jumps, the other is continuous time evolution governed by a Schrödinger-like equation, but with a non-Hermitian effective Hamiltonian [4–6]. This effective Hamiltonian will contain a term proportional to \( \sigma_j^+ \otimes \sigma_j^+ \) without the Hermitian conjugate term. Here \( \sigma_j^\pm \) are the creation and annihilation operators for system \( j = 1, 2 \). That is, there is a term that annihilates an excitation in system 1 and creates an excitation in system 2, but not vice versa.

Here we consider what happens when we manipulate the single-photon wave packet by applying a (unitary) transformation to it. One question we answer is, for the class of unitary transformations we consider, do we still get a simple effective description of our two systems? The second question we answer is more subtle: the fact that the effective Hamiltonian contains just a term proportional to \( \sigma_1^+ \otimes \sigma_2^+ \) (naively) seems to indicate that the photon emitted by system 1 will be absorbed by system 2. However, we know that this alone does not suffice. In fact, the optimal sort of wave packet that would be absorbed by system 2 would have to be the time-inverse of the wave packet system 2 would emit itself. So, suppose we apply a unitary transformation that time-reverses the wave packet emitted by system 1. Then, how can we see from the evolution equation for the state of the systems that system 2 will absorb the photon more efficiently?

We obtain the answers by slightly changing the standard interpretation of the equations obtained in the quantum trajectory picture. Doing that will lead us to define a mathematical object \( \rho(t) \) consisting of two parts. One part is simply the state of system 2 at time \( t \), the other is the state of a fictitious system 1 at a time \( t = f(t) \) that decreases as a function of \( t \) if we apply a time-reversal transformation. The time evolution of that mathematical object thus corresponds to system 2 evolving forward in time in a standard way, but the fictitious system 1 evolves backwards in time. This is similar in spirit to the theory of the “past quantum state” [7], which likewise introduces a mathematical object consisting of one forward-evolving standard quantum state and a backwards evolving part. In the latter case the backwards evolving part describes retrodiction, whereas in our case it results only if we time-reverse the single-photon wave packet.

The motivation for our work arises from the recent interest in and progress towards hybrid quantum systems [8–15] and quantum transduction [16–18]. Interfacing different quantum systems would allow for their individual strengths to be leveraged so as to create more robust and scalable quantum information architectures. For instance, one could use fast gate solid state qubits as processors and long coherence time trapped ion qubits for memory [9]. Connecting different quantum systems with high fidelity is challenging because a wave packet emitted by one system generally differs, in terms of its spectral properties (shape, resonance frequency, and decay rate), from that which another system would likely absorb. This challenge can be dealt with using photon manipulation to transform the wave packet emitted by one system, tuning its spectral properties, so that it will be absorbed by another system. The relevant photon manipulations are optically reversing the temporal envelope of a photon wave packet, quantum frequency conversion, and pulse shaping [19–21].

In Sec. II we rehearse the input-output methods of Gardiner [2] to model the two systems interacting with

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the same heat bath. We determine the field resulting from the systems interaction, which is the physical entity we transform in Sec. III. We introduce the composite state of systems 1 and 2, \( \rho(t) \), adding a new interpretation of the state’s time argument. In Sec. III we describe and analyze a scheme for quantum state transfer using photon manipulation that could be used for quantum transduction. Modeling the photon manipulation as a unitary transformation, we show how to include field transformations in input-output theory. We analyze how the field is transformed, showing that we can produce a wave packet that will be absorbed by system 2, and how the equation for system 2 is affected. In Sec. IV we separate the transformation into different phases to analyze and illustrate it in more detail. In Sec. V we determine the effect of the transformation in the Schrödinger picture. We discuss our results and give some conclusions in Sec. VI.

II. SETUP

Consider a one-dimensional (1D) setup, where systems 1 and 2 are coupled to the same heat bath, which has boson annihilation operators \( b(\omega) \). Let the positions of systems 1 and 2 be \( x = 0 \) and \( x = \epsilon \tau \), respectively, with \( \tau > 0 \). Based on the formalism developed and used in Refs. 5 and 2 we have the Hamiltonian

\[
H = H_{\text{sys}} + H_B + H_{\text{int}},
\]

where \( H_{\text{sys}} \),

\[
H_B = \int d\omega |\omega| b^\dagger(\omega)b(\omega),
\]

and

\[
H_{\text{int}} = i \int d\omega \left\{ \kappa_1(\omega) \left( \sigma^+_1 b^\dagger(\omega) - \sigma^-_1 b(\omega) \right) \\
+ \kappa_2(\omega) \left( \sigma^+_2 b^\dagger(\omega)e^{-i\omega\tau} - \sigma^-_2 b(\omega)e^{i\omega\tau} \right) \right\}
\]

are the Hamiltonians for the systems, bath, and their interaction, respectively (\( \hbar = 1 \)). We have omitted tensor product signs here for brevity, as is standard. We work in the Heisenberg picture so the operators \( \sigma^\pm_1 \) and \( b \) depend on time while the couplings \( \kappa_j \) do not. Integrals without explicit bounds are taken to be from \(-\infty\) to \( \infty \). Note this is a pseudo 1D setup in that the only propagating degree of freedom is the longitudinal mode characterized by \( b(\omega) \) and the other three quantum numbers (polarization and two transverse spatial modes) are fixed.

Making the Markov approximation of a flat coupling \( \kappa_j = \sqrt{\gamma_j}/2\pi \) [2], the physical 1D electric field [22] component

\[
A^+(x,t) = \frac{1}{\sqrt{2\pi}} \int d\omega b(\omega,t)e^{i\omega x/c}
\]

can be expressed in terms of the right and left ‘in fields,’

\[
b^{r}_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} d\omega e^{-i\omega(t-t_0)} b(\omega,t_0)
\]

and

\[
b^{l}_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_0^{-\infty} d\omega e^{-i\omega(t-t_0)} b(\omega,t_0),
\]

respectively, for \( t_0 < t \) or in terms of the ‘out fields’ \( b^{r,l}_{\text{out}} \) defined similarly with \( t_0 \rightarrow t_1 > t \) [1]. After solving the Heisenberg equation of motion (EOM) for \( b(\omega,t) \) with initial condition at \( t_0 \) one finds

\[
A^+(x,t) = b^{r}_{\text{in}}(t_-) + b^{l}_{\text{in}}(t_+) + \sqrt{\gamma_1}u(x)\sigma^-_1(t_-) + \sqrt{\gamma_2}u(x-c\tau)\sigma^-_2(t_- + \tau),
\]

where \( t_{\pm} = t \pm x/c \) and \( u(x) \) is the Heaviside step function with \( u(0) = 1/2 \). Using a final condition at \( t_1 \) one finds a similar expression for \( A^+(x,t) \) in terms of the out fields. Equating these expressions and using the independence of \( x \) and \( t \) then implies

\[
b^{l}_{\text{out}}(t) - b^{r}_{\text{in}}(t) = 0
\]

and

\[
b^{r}_{\text{out}}(t) - b^{l}_{\text{in}}(t) = \sqrt{\gamma_1}\sigma^-_1(t) + \sqrt{\gamma_2}\sigma^-_2(t + \tau).
\]

Physically, Eq. (8) says that the leftward propagating free fields are not changed and Eq. (9) says that the rightward propagating free fields are changed by the radiation from the two systems. This qualitatively matches the modeled unidirectional coupling between our two systems.

As the leftward propagating free field is unimpeded and trivial, we focus on the rightward propagating field and we abbreviate \( b_{\text{in,out}} \equiv b^{r,l}_{\text{in,out}} \). Then with

\[
s_1(t) := \frac{\gamma_1}{2} \sigma^-_1(t) + \sqrt{\gamma_1}b_{\text{in}}(t)
\]

and

\[
s_2(t) := \frac{\gamma_2}{2} \sigma^-_2(t) + \sqrt{\gamma_1}\gamma_2\sigma^-_1(t - \tau) + \sqrt{\gamma_2}b_{\text{in}}(t - \tau)
\]

the general Heisenberg EOM for a system operator \( a(t) \) is [2]

\[
\dot{a}(t) = \sum_{j=1}^{2} \left( s_j^\dagger [a, \sigma^-_j] - [a, \sigma^+_j] s_j \right) - i [a, H_{\text{sys}}]
\]

in which each operator has the same time argument \( t \), which we omit for brevity. Let \( a_j \) denote a system \( j \) operator. Then \( [a_1, \sigma^-_2] = 0 \) so

\[
\dot{a}_1(t) = s_1^\dagger [a_1, \sigma^-_1] - [a_1, \sigma^+_1] s_1 - i [a_1, H_{\text{sys}}],
\]

which shows that \( \dot{a}_1(t) \) depends on system 1 and field operators at the same time \( t \). Similarly, \( [a_2, \sigma^-_1] = 0 \) so

\[
\dot{a}_2(t) = s_2^\dagger [a_2, \sigma^-_2] - [a_2, \sigma^+_2] s_2 - i [a_2, H_{\text{sys}}],
\]
which shows that $\dot{a}_2(t)$ depends on system 2 operators at the same time $t$ and on the delayed output of system 1 at time $t - \tau$ through $\varrho_2(t)$. That is, the evolution of $a_1$ depends on a single time $t$ while the evolution of $a_2$ depends on two times $t$ and $t - \tau$.

As $a_1$’s EOM depends on a single time we can shift this time to $t - \tau$ to match the time arguments of the system 1 and field operators in $\varrho_2$’s EOM. Using this structure, we can write the composite state of both systems in the Schrödinger picture as

$$\rho(t) := \rho_1(f(t)) \otimes \rho_2(t)$$  

where $f(t) = t - \tau$ an invertible function of $t$. The point of writing $f(t)$ is that both the equations for system 1 and system 2 can refer to the same time $f(t)$ for the state of system 1. Importantly, this is true both for Gardiner’s original equations and for our transformed equations (as we will find in Sec. V B), where $f(t)$, the time argument of system 1, takes a more complicated form.

### III. FIELD TRANSFORMATION

Suppose system 1 starts in a superposition of an excited state and a stable ground state and that system 2 starts in its ground state. Then, laser pulses induce a coherent superposition of a spontaneously emitted photon (if system 1 was excited) and the vacuum (if system 1 was in the ground state). This transfers the state to the field mode, where the coherent superposition then propagates along a 1D channel to system 2, so that with some probability we can transfer the state of the field to that system, thus transferring quantum information from system 1 to 2. We want to maximize that probability.

The probability of absorption of a photon depends on the spectral properties of the wave packet. System 2 will absorb a wave packet emitted by system 1 with high fidelity if it is similar to the time-reversed wave packet a lone system 2 would emit [23]. Thus, we apply a unitary transformation at position $0 < x < ct$ along the channel that time reverses, frequency translates, and stretches the photon wave packet, transforming it into the time-reversed output of system 2. Finally, laser pulses induce absorption in system 2, thus completing quantum state transfer, as illustrated in Fig. 1.

Ref. 23 developed a scheme to achieve quantum state transfer between distant identical atoms by producing time-symmetric photon wave packets. Incorporating photon manipulation into our scheme makes it versatile so that we can transfer information between systems with distinct spectral properties [24] and the wave packets need not be time-symmetric [25]. (For completeness we note that Ref. 26 demonstrates a method for producing a photon wave packet with a rising exponential shape, thus mimicking a time-reversed wave packet, by producing two photons and measuring one of them. It is not clear whether that method could be adapted to our context in which we need an appropriate superposition of the vacuum state and a single-photon state in order to transmit the quantum state of one material system to another.)

![FIG. 1. Scheme to transfer quantum information between distinct quantum systems along a 1D channel. The black arrows indicate laser pulses that induce emission in system 1 and absorption in system 2 (see the text for an explanation). Here the systems are taken to be three-level A systems, as an example, which we discuss at the end of Sec. III.](image)

We compound the photon manipulations into a single unitary transformation

$$U(\nu, \nu') = \sqrt{\alpha} \delta(\nu' + \alpha(\nu - \omega_0)) e^{i\nu T},$$

which acts on a function in frequency space by time-reversing it, scaling it by $\alpha$, and shifting it by $\omega_0$:

$$\tilde{f}(\nu) = \int d\nu' U(\nu, \nu') f(\nu') = \sqrt{\alpha} f(-\alpha(\nu - \omega_0)) e^{i\nu T}.$$  

It satisfies the unitarity condition

$$\int d\nu'' U(\nu'', \nu'') U^*(\nu', \nu'') = \alpha \delta(\nu - \nu') e^{i(\nu - \nu') T} = \delta(\nu - \nu'),$$

where we used the property $\delta(ax) = \delta(x)/|a|$. The time parameter $T$ in the phase sets the time the transformation begins, as justified later in this section. Thus, the parameters $\alpha$, $\omega_0$, and $T$ can be tuned such that the transformation time-reverses the slowly varying envelope of the wave packet and shifts the resonance frequency and decay rate of system 1 to those of system 2.

The frequency scaling transformation is centered at $\omega = 0$ so a frequency $\omega$ is mapped to $-\omega$ under time reversal, then to $-\omega/\alpha$ by the scaling. Thus, to time reverse a wave packet emitted from system 1 with resonance frequency $\omega_1$, scale it by $\alpha$, and shift its resonance frequency to that of system 2, $\omega_2$, we need

$$\omega_0 = \omega_2 + \omega_1/\alpha$$

in Eq. (16). Transformations of this form are physically justified in Ref. 19, which uses the parameters $\alpha = -m > 0$, $\omega_1 = \omega_\chi$, and $\omega_2 = \omega_\gamma$.

Consider the part of the field in Eq. (7) that describes the radiation emitted by system 1,

$$A_{1i}^+(x, t) := \sqrt{\gamma_1} u(x) \sigma_1^-(t-),$$

which will be transformed before it is incident on system 2. The subscript $i$ denotes that this is the initial field.
This part of the field transforms at \( x = X \) in frequency space according to Eq. (17) as

\[
\tilde{A}^+_1(X,\nu) := \int d\nu' U(\nu,\nu')A^+_1(X,\nu')
\]

so taking the inverse temporal Fourier transform yields

\[
\tilde{A}^+_1(X,t) = \frac{1}{\sqrt{\alpha}} e^{-i\omega_0(t-T)} A^+_1(X,\tilde{t}_0) \tag{22}
\]

with

\[
\tilde{t}_0 := (T - t)/\alpha. \tag{23}
\]

We can now determine the interval of time for which this transformed outgoing field is valid. For the transformation to be causal the field must depend on past times, \( t \geq \tilde{t}_0 \), such that \( t \geq t_s = T/(1 + \alpha) \), which defines the physical meaning of \( T \). Physically, the transformation will be driven by a pump laser pulse of length \( L \) over a total integration time \( \Delta = L/c \) [19]. As the wave packet is broadened by \( \alpha \) in the time domain the upper bound for the transformation is \( t \leq t_f \equiv t_s + \alpha \Delta \).

The field will freely propagate after the transformation as

\[
\tilde{A}^+_1(x,t) = \tilde{A}^+_1(X) \left( 1 - \frac{x}{X} \right) \tag{24}
\]

such that letting \( T = t + ((1 + \alpha) X - x)/c \)

\[
\tilde{A}^+_1(x,t) = \sqrt{\frac{\gamma_1}{\alpha}} e^{i\omega_0 X/c} e^{-i\omega_0 T} \tilde{\sigma}^-_1 \left( \frac{-T}{\alpha} \right). \tag{25}
\]

Let us define

\[
\begin{align*}
\tilde{\sigma}^+_1(t) &:= e^{-i\omega_0 t} \sigma^-_1(-t/\alpha), \tag{26a} \\
\tilde{\sigma}^+_1(t) &:= e^{i\omega_0 t} \sigma^+_1(-t/\alpha), \tag{26b} \\
\tilde{\sigma}^+_1(t) &:= \sigma^+_1(-t/\alpha), \tag{26c}
\end{align*}
\]

which preserve the Pauli spin algebra as \( \{\tilde{\sigma}^+_1(t), \tilde{\sigma}^-_1(t)\} = \tilde{\sigma}^+_1(t) \) and \( \{\tilde{\sigma}^+_1(t), \tilde{\sigma}^+_1(t)\} = \pm 2\tilde{\sigma}^+_1(t) \). Then we find that

\[
\tilde{A}^+_1(x,t) = \sqrt{\frac{\gamma_1}{\alpha}} \tilde{\sigma}^-_1(T) e^{i\omega_0 X/c} \tag{27}
\]

is the transformed field being produced at \( x = X \) during the interval \( t_s < t < t_f \), which is specified by \( \alpha, T, \) and \( \Delta \).

To produce a field with a decay rate matching systems 2’s we select \( \alpha = \gamma_1/\gamma_2 \) as then \( \sqrt{\gamma_1/\alpha} \rightarrow \sqrt{\gamma_2} \) in Eq. (27). With this choice for \( \alpha \) the field at the transformation device is

\[
\tilde{A}^+_1(X,t) = \sqrt{\gamma_2} e^{i\omega_0 (T-t)} \tilde{\sigma}^-_1(\tilde{t}_0 - X/c), \tag{28}
\]

which we can compare to the field just before the transformation

\[
A^+_1(X,t) = \sqrt{\gamma_1} \tilde{\sigma}^-_1(t - X/c). \tag{29}
\]

Thus, under the transformation the initial field for system 1 is scaled, acquires a phase, and \( t \rightarrow \tilde{t}_0 \) in \( \sigma^-_1 \). This tells us that during the transformation system 2 operators satisfy Eq. (14) with the field transformation

\[
\sqrt{\gamma_1} \sigma^-_1(t - \tau) \rightarrow \sqrt{\gamma_2} \sigma^-_1(\tilde{t}_0 - \tau) \tag{30}
\]

in the rotating frame where the fast-rotating terms disappear (and similarly for \( b_{in}(t - \tau) \) in \( s_2(t) \)).

The transformed operator \( \tilde{\sigma}^-_1 \) can be thought of as corresponding to a fictitious system 1 that emits the time-reversed output of system 2 [27]. It satisfies the EOM

\[
\dot{\tilde{\sigma}}^-_1(t) = \frac{1}{\alpha} \left\{ i \left[ \tilde{\sigma}^-_1(t), H_{\text{sys}} \left( -\frac{t}{\alpha} \right) \right] + \frac{\gamma_1}{2} \tilde{\sigma}^-_1(t) \right\}
- \sqrt{\gamma_1} e^{-i\omega_0 t} \tilde{\sigma}^-_1(t) b_{in} \left( -\frac{t}{\alpha} \right) - i\omega_0 \tilde{\sigma}^-_1(t), \tag{31}
\]

where we used Eq. (13) to find \( \tilde{\sigma}^-_1 \).

### A. Two-level atoms

As an illustrative example, we consider two-level atom systems with

\[
H_{\text{sys}} = \frac{1}{2} (\omega_1 \sigma^+_1 + \omega_2 \sigma^+_2) \tag{32}
\]

so Eqs. (13) and (14) imply

\[
\dot{\tilde{\sigma}}^-_1 = -\left( \frac{\gamma_1}{2} + i\omega_1 \right) \tilde{\sigma}^-_1 + \sqrt{\gamma_1} \tilde{\sigma}^-_1 b_{in}, \tag{33}
\]

\[
\dot{\tilde{\sigma}}^-_2 = -\left( \frac{\gamma_2}{2} + i\omega_2 \right) \tilde{\sigma}^-_2 + \sqrt{\gamma_2} \tilde{\sigma}^-_2 (b_{in}(t - \tau) \tag{34}
+ \sqrt{\gamma_1} \tilde{\sigma}^-_1 (t - \tau)) \right)
\]

Note atom 2 is driven by atom 1 via the \( \sqrt{\gamma_1} \tilde{\sigma}^-_1 \) term in Eq. (34) yet there is not an analogous term for atom 1 being driven by atom 2. From Eq. (31) the EOM for \( \tilde{\sigma}^-_1 \) is

\[
\dot{\tilde{\sigma}}^-_1(t) = \left\{ \frac{\gamma_1}{2\alpha} + i \left( \frac{\omega_1}{\alpha} - \omega_0 \right) \right\} \tilde{\sigma}^-_1(t)
- \sqrt{\gamma_1} e^{-i\omega_0 t} \tilde{\sigma}^-_1(t) b_{in} \left( -\frac{t}{\alpha} \right). \tag{35}
\]

Shifting the frequency of the wave packet emitted by atom 1 according to Eq. (19) and scaling the decay rate by \( \alpha = \gamma_1/\gamma_2 \) yields

\[
\dot{\tilde{\sigma}}^-_1(t) = \left( \frac{\gamma_2}{2} - i\omega_2 \right) \tilde{\sigma}^-_1(t)
- \frac{\gamma_2}{\sqrt{\gamma_1}} e^{-i(\omega_2 + \omega_1\gamma_2/\gamma_1) t} \tilde{\sigma}^-_1(t) b_{in} \left( -\frac{-\gamma_2 t}{\gamma_1} \right). \tag{36}
\]

Comparing the first term in Eq. (36), \( (\gamma_2/2 - i\omega_2) \tilde{\sigma}^-_1 \), with that of Eq. (34), \( (\gamma_2/2 + i\omega_2) \tilde{\sigma}^-_2 \), we see that the \( \gamma_2 \) terms have opposite signs while the \( \omega_2 \) terms have
the same sign. Thus, as desired, \( \tilde{\sigma}^-_1(t) \) satisfies a time-reversed equation in terms of its slowly varying envelope function as compared to \( \sigma^-_2(t) \) (we cannot time-reverse the fast oscillations \[19\]).

This procedure would extend to other setups in which systems 1 and 2 have the same level structure such as the three-level \( \Lambda \) systems depicted in Fig. (1) above. In this case, the laser pulse that induces absorption in system 2 must be time reversed relative to the pulse inducing the emission of system 1 \[23\]. This procedure can then be generalized to systems with more levels, where more laser pulses and their time reversed counterparts need to be used to induce emission and absorption in the systems, respectively.

**IV. ILLUSTRATION**

We can write the state of the field emitted by system 1 at any point and time by thinking about different phases of the transformation:

- Phase 1 before transformation,
- Phase 2 input field processing,
- Phase 3 transformed field production,
- Phase 4 transformation complete.

In this section we will work in a coordinate system where \( t = 0 \) corresponds to the time at which the leading edge of a photon wave packet is emitted from system 1. The initial field freely propagates during Phase 1 until \( t = t_i = t_s - \Delta \) at which point the first part of the field to be transformed enters the transformation device at \( x = X \).

The portion of the field to be transformed will have passed through \( X \) by \( t = t_s \) at which point the transformed field starts to be produced. That is, Phase 2 occurs over the interval \( t_i < t < t_s \) during which the input field is “processed” by the transformation device as the field intercepts a leftward propagating laser pulse of duration \( \Delta \) (that drives the transformation) at position \( X \). This results in a gap \[28\] in the field of duration \( \Delta \) during which a vacuum field \( V(x,t) \), \( \langle V \rangle = 0 \), is produced because our system is “buffering” until \( t = t_s \) when the time-reversed stretched/compressed field is produced.

The transformed pulse is produced during Phase 3 for \( t_s < t < t_f \). During this field production we block any incident beam from passing through \( X \). Doing so ensures that there is always a single \( f(t) \) describing the time argument of the field (making it a well-defined function) as later discussed in Sec. V B. This will result in some loss yet the times \( T \) and \( \Delta \) can be tuned to capture almost the entirety of the initial wave packet making the loss arbitrarily small. After the transformation is complete at \( t = t_f \), i.e. during Phase 4, any of the original field (including noise) passing through \( X \) will be unchanged and the transformed portion of the field will freely propagate.

The field due to atom 1 at any point and time is

\[
A^-_1(x,t) = \begin{cases} 
V(x,t), & x \geq X, t_i < t < \frac{X}{c} < t_s \\
\tilde{A}^+_1(x,t), & x \geq X, t_s < t < \frac{X}{c} < t_f \\
A^+_1(x,t), & \text{elsewise}
\end{cases}
\]

(37)

which accounts for all of the Phases. The free field \( b_n(t) \) will be similarly transformed as it passes through the transformation device. The first case in Eq. (37) captures the processing of the portion of the wave packet to be transformed in Phase 2, during which a vacuum field \( V(x,t) \) is produced \[28\]. The second and third cases capture the free propagation of the transformed and initial fields, respectively.

We illustrate these phases for an exponentially decaying wave packet, which is common to spontaneous emission processes, in Fig. 2 below. Specifically, we consider the two-level atom systems of Eq. (32) in vacuum in a rotating frame with frequency \( \omega_1 \). Then taking expectation values in Eq. (33) yields

\[
\langle \tilde{\sigma}^-_1 \rangle = -\frac{\gamma_t}{2} \langle \sigma^-_1 \rangle \implies \langle \sigma^-_1(t) \rangle = \langle \sigma^-_1(0) \rangle e^{-\gamma_1 t/2}.
\]

(38)

Note that \( t_a \equiv X/c + \Delta \) is the time at which a length \( L \) of the wave packet emitted from system 1 will have passed through \( X \) arriving at the transformation device. Thus, for the exponentially decaying wave packet \( t_a \) is the ideal starting time of transformation as if \( t_s = t_a \) then the first portion of the wave packet to pass through \( X \) is transformed. This can be accomplished by letting \( T = (1 + \alpha)t_a \).

**V. SCHRODINGER PICTURE EVOLUTION**

**A. Before transformation**

The identity

\[
\text{Tr}(\rho_H \dot{a}(t)) = \text{Tr}(\dot{\rho}_S(t) a)
\]

(39)

connects the Heisenberg picture operator and state on the left to the Schrödinger picture operator and state on the right. Using this identity, the cyclic nature of the trace, and Eq. (12) one finds the evolution equation for the systems and field to be

\[
\dot{\rho}_S = \sum_{j=1}^{2} \left[ \left[ \sigma^-_j, \rho_S s^+_j \right] - \left[ \sigma^+_j, s_j \rho_S \right] \right] - i[H_{\text{sys}}, \rho_S].
\]

(40)

In the Schrödinger picture the operators no longer depend on time with all the time dependence shifted onto \( \rho_S(t) \). Consider coherent input states of the field \( |\beta\rangle \langle \beta| \), where \( b_{\text{in}} |\beta\rangle = \beta |\beta\rangle \) with \( \beta \in \mathbb{C} \). Tracing over the field we obtain the state of the systems

\[
\rho(t) = \text{Tr}_{\text{field}}(\rho_S),
\]

(41)
which is the same as the state in Eq. (15), and then
\[
\text{Tr}_{\text{field}}(b_n \rho_S) = \beta \rho
\]
such that \(\rho\) satisfies Eq. (40) with \(b_n \to \beta\). After some algebra, we can write a master equation for \(\hat{\rho}\) in Lindblad form
\[
\dot{\rho} = i[\rho, H_0] + J \rho J^\dagger - \frac{1}{2} \{\rho, J^\dagger J\}, \tag{43}
\]
where \(\{\cdot, \cdot\}\) is the anticommutator,
\[
J = \sqrt{\gamma_1}\sigma_1^- + \sqrt{\gamma_2}\sigma_2^- + \beta \tag{44}
\]
is the jump operator, and \(H_0 = H_{\text{sys}} + H_{\text{ex}}\) with
\[
H_{\text{ex}} = -\frac{i}{2} \left[\sqrt{\gamma_1}\gamma_2\sigma_2^+ \sigma_1^- + \left(\sqrt{\gamma_1}\sigma_1^+ + \sqrt{\gamma_2}\sigma_2^+\right) \beta\right] + \text{He}c, \tag{45}
\]
where He denotes the Hermitian conjugate of the previous term.

Now the connection to the quantum trajectory method can be made. Letting \(H' = -iJ^\dagger J/2\), there is smooth evolution governed by a Schrödinger-like equation with a non-Hermitian effective Hamiltonian
\[
H_{\text{eff}} = H_0 + H'
\]
\[
= H_{\text{sys}} - \frac{i}{2} \left[\gamma_1\sigma_1^+ \sigma_1^- + \gamma_2\sigma_2^+ \sigma_2^- + 2\sqrt{\gamma_1\gamma_2}\sigma_1^+ \sigma_2^- + 2\beta \left(\sqrt{\gamma_1}\sigma_1^+ + \sqrt{\gamma_2}\sigma_2^+\right) + |\beta|^2\right], \tag{46}
\]
where the \(\sigma_1^+ \sigma_2^-\) term survives, not its Hermitian conjugate. This describes an excitation in the first system leading to absorption in the second, but not the other way around. In addition to this smooth evolution there are also random quantum jumps, where the operator \(J\) is applied to the state of our two systems (and then the state has to be normalized) \([5, 6]\).

\section{B. After transformation}

Once we introduce a transformation between the systems, the output of system 1 will transform according to Eq. (37) in the Heisenberg picture. Thus, as in Eq. (30), the transformed EOM for \(a_2\) depends on system 1 and field operators at time \(t_0 - \tau\) and on system 2 operators at the same time \(t\). The transformed EOM for \(a_1\) only depends on operators at a single time \(t_0\), which can be shifted to \(t_0 - \tau\). Hence we can determine the reduced density operator describing the composite state of the systems corresponding to the transformed wave packet, \(\rho(t)\), in the same manner as Sec. V A. Finding the evolution equation for the total transformed state in the Schrödinger picture, \(\tilde{\rho}_S(t)\), then tracing over the field we obtain
\[
\rho(t) = \text{Tr}_{\text{field}}(\tilde{\rho}_S(t)) = \rho_1(t_0 - \tau) \otimes \rho_2(t) \tag{47}
\]
at \(x = X\). That is, during Phase 3 \(\rho_1\) describes the fictitious system 1 at time \(t_0 - \tau\) and system 2 is still described by \(\rho_2(t)\) as in Eq. (48). A key point is that \(\rho_1\)’s time argument has slope \(-1/\alpha\) corresponding to a fictitious system that is evolving backwards in time (as the slope is negative) at a new decay rate due to the \(\alpha\) scaling. This indicates that system 2 will absorb the photon more efficiently. This state satisfies the same EOM as before (Eq. (43)) with the change in the time argument of system 1 accounting for transformation. Thus, we can write the state of the systems at any time as \(\rho(t) = \rho_1(t) \otimes \rho_2(t)\) using Eqs. (15) and (47) as
\[
\hat{t} = f(t) = \begin{cases} 
\text{undefined,} & t_i < t < t_s \\
\hat{t}_0 - \tau, & t_s < t < t_f \\
\hat{t} - \tau, & \text{elsewise}
\end{cases} \tag{48}
\]
which describes the time argument of system 1 (or \(\hat{t}\) for the portion of the field that has been transformed) at
This gives an invertible but discontinuous transformation. The inverse function is
\[ f^{-1}(t) = \begin{cases} 
T - \alpha(t + \tau), & t_i < t + \tau < t_s \\
\text{undefined,} & t_s < t + \tau < t_f \\
t + \tau, & \text{elsewise}
\end{cases} \] (49)
which is a fictitious time for system 2 as a function of the time for system 1. (See Fig. 3 below.)

Naively, these jumps seem to indicate that system 2 operators would have discontinuities in their evolution as they are being driven by system 1, where Eq. (30) holds for the transformed field. Yet at \( t_i \) system 1 is just leaving one of its ground states and at \( t_s \) the decay of the first system (corresponding to the \( \sigma_1^- \) operator) is nearly complete (assuming \( \Delta \) and \( T \) are chosen appropriately) so again system 1 will be in a ground state. Hence \( \sigma_1^- \)'s coupling is effectively zero in both cases as system 1 is (at least nearly) in a ground state, which yields zero if acted on by an annihilation operator. This likewise applies to the vertical jump of duration \( \alpha \Delta \) in \( f(t) \). Thus, system 2 operators evolve smoothly even though \( f(t) \) has discontinuities. Moreover, \( f \)'s evolution will be smoothed out by the gradual switching to and from the vacuum field \( V(x,t) \) during Phase 2 [28].

This Heisenberg ↔ Schrödinger conversion works in a similar way as before because the unitary transformation to the field has the effect of transforming system 1 and field operator arguments from \( t - \tau \) to \( t_0 - \tau \). In the Schrödinger picture the same unitary is applied to the state, with the same effect on its argument \( t \). Note that the value of \( \beta \), the eigenvalue of \( b_{in} \) for a coherent input state, does not depend on time here as the usual phase factor is transformed away in the rotating frame.

VI. DISCUSSION

In this paper we incorporate photon manipulation into the standard input-output theory. We consider the transformation of a photon wave packet emitted by system 1 into the time-reversed and rescaled output of system 2. This can be used to ensure (ideally) that the emission of system 1 results in system 2 becoming excited. Hence this leads to a potentially more versatile scheme for quantum state transfer as it allows information to be transferred between non-identical systems.

A main result is a new interpretation of system 1’s time argument. In particular, we find that after the transformation we still have the standard evolution equation
\[ \dot{\rho} = i[\rho, H_0] + \mathcal{L}[J] \rho \] (50)
with
\[ \mathcal{L}[J] \rho = J \rho J^\dagger - \frac{1}{2} \{ \rho, J^\dagger J \} \] (51)
a Lindblad superoperator and \( J \) a jump operator (see Eq. (44)). Yet crucially, the scaling by a factor \( \alpha \) and time reversal are accounted for by the change in the time argument of the state of system 1. That is, the state of the system is \( \rho(t) = \rho_1((T - t)/\alpha - \tau) \otimes \rho_2(t) \) during the transformation. Thus, the effective description in the quantum trajectory method is unchanged yet the interpretation of the state does change.

Clearly, implementing the unitary transformation device in our scheme would be challenging. Accordingly, modeling a more realistic, imperfect transformation device is an important extension of this work. Thus, in future work we plan to analyze limitations in the transformation, losses, as well as non-unitary transformations. This will involve using numerical simulations to justify the robustness of using photon manipulation to increase the fidelity of quantum state transfer.

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We refer to the field operator $A^+$ as the electric field to be consistent with Ref. 2 and to emphasize that it is what the systems interact with. We can do this as the vector potential and electric field are proportional to $A^+$ due to the Markov approximation [5].

The buffering during Phase 2 of the transformation can be modeled as the activation of a $c$ mode in the vacuum state. The $c$ mode has creation and annihilation operators $c_{in}$ and $c_{in}$, respectively, that satisfy the same commutation relations as $b_{in}$. Then the field being transmitted through the unitary transformation device at $x = X$ is $\cos \phi(t) b_{in}(t) - i \sin \phi(t) c_{in}(t)$, where $\phi(t)$ is a switching function that determines whether the $c$ mode is active. Thus, $\phi$ should be a smoothed out (for continuity) square wave that is $\pi/2$ during Phase 2 of the transformation and zero elsewise.