MULTIPLICATION OPERATOR ON THE BERGMAN SPACE BY A PROPER HOLOMORPHIC MAP

GARGI GHOSH

Abstract. Suppose that $f := (f_1, \ldots, f_d) : \Omega_1 \rightarrow \Omega_2$ is a proper holomorphic map between two bounded domains in $\mathbb{C}^d$. In this paper, we find a non-trivial minimal joint reducing subspace for the multiplication operator (tuple) $M_f = (M_{f_1}, \ldots, M_{f_d})$ on the Bergman space $\mathbb{A}^2(\Omega_1)$, say $\mathcal{M}$. We further show that the restriction of $(M_{f_1}, \ldots, M_{f_d})$ to $\mathcal{M}$ is unitarily equivalent to Bergman operator on $\mathbb{A}^2(\Omega_2)$.

1. Introduction

For $d \geq 1$, let $\Omega$ be a bounded domain in $\mathbb{C}^d$. The Bergman space on $\Omega$, denoted by $\mathbb{A}^2(\Omega)$ is the vector space of square integrable holomorphic functions on $\Omega$ with respect to Lebesgue measure on $\Omega$. The Bergman space $\mathbb{A}^2(\Omega)$ is a Hilbert space with reproducing kernel, called the Bergman kernel. We study multiplication operators induced by holomorphic functions. Thus for $g = (g_1, \ldots, g_d) \in H^\infty(\Omega, \mathbb{C}^d)$, the Banach space of bounded holomorphic functions on $\Omega$, we define $M_{g_i} : \mathbb{A}^2(\Omega) \rightarrow \mathbb{A}^2(\Omega)$ by

$$M_{g_i} \varphi = g_i \varphi, \quad \varphi \in \mathbb{A}^2(\Omega) \text{ for } i = 1, \ldots, d.$$ 

Clearly, $M_{g_i}$ is a bounded linear operator

$$\|M_{g_i}\| = \|g_i\|_\infty = \sup \{|g_i(z)| : z \in \Omega\} \text{ for } i = 1, \ldots, d.$$ 

Let $\mathbf{M}_g$ denote the tuple of operators $(M_{g_1}, \ldots, M_{g_d})$. We put $\mathbf{M} := (M_1, \ldots, M_d)$, where $M_i$ denotes the multiplication by the $i$-th coordinate function of $\Omega$ on $\mathbb{A}^2(\Omega)$ for $i = 1, \ldots, d$, $\mathbf{M}$ is called the Bergman operator on $\mathbb{A}^2(\Omega)$ or the Bergman operator associated to the domain $\Omega$. A joint reducing subspace for a tuple $\mathbf{T} = (T_1, \ldots, T_d)$ of operators on a Hilbert space $\mathcal{H}$ is a closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that

$$T_i \mathcal{M} \subseteq \mathcal{M} \text{ and } T_i^* \mathcal{M} \subseteq \mathcal{M} \text{ for } i = 1, \ldots, d.$$ 

It is easy to see that $\mathbf{T}$ has a joint reducing subspace of if and only if there exists an orthogonal projection $P$ ($P : \mathcal{H} \rightarrow \mathcal{H}, P^2 = P = P^*$) such that

$$PT_i = T_i P \text{ for } i = 1, \ldots, d.$$ 

Here $\mathcal{M} = P \mathcal{H}$ is the corresponding joint reducing subspace of $\mathbf{T}$. A joint reducing subspace $\mathcal{M}$ of $\mathbf{T}$ is called minimal or irreducible if only joint reducing subspaces contained in $\mathcal{M}$ are $\mathcal{M}$ and $\{0\}$.

It is natural to start with a biholomorphic multiplier $f \in H^\infty(\Omega, \mathbb{C}^d)$ on $\mathbb{A}^2(\Omega)$. It is easy to see that $\mathbf{M}_f$ is irreducible as follows. Let $\Omega_1, \Omega_2$ be bounded domains in $\mathbb{C}^d$
and \( f : \Omega_1 \to \Omega_2 \) be a biholomorphism. If \( J_f \) denotes the jacobian of \( f \), the linear map \( U_f : \mathbb{A}^2(\Omega_2) \to \mathbb{A}^2(\Omega_1) \) defined by
\[
U_f \varphi = (\varphi \circ f) J_f \quad \text{for} \quad \varphi \in \mathbb{A}^2(\Omega_2)
\]
is a surjective isometry, that is, a unitary which satisfies \( U_f M_i = M_i U_f \) for \( i = 1, \ldots, d \), where \( M = (M_1, \ldots, M_d) \) is the Bergman operator on \( \mathbb{A}^2(\Omega_2) \). Therefore, \( M_f \) is unitarily equivalent to the Bergman operator on \( \mathbb{A}^2(\Omega_2) \). Since the Bergman operator associated to any domain is irreducible, it follows that \( M_f \) is irreducible.

This motivates us to consider next nicely behaved class of holomorphic multipliers in \( H^\infty(\Omega_1, \mathbb{C}^d) \), namely, the proper holomorphic multipliers in \( H^\infty(\Omega_1, \mathbb{C}^d) \). A holomorphic map \( f : \Omega_1 \to \Omega_2 \) is said to be \textit{proper} if \( f^{-1}(K) \) is compact in \( \Omega_1 \) whenever \( K \subseteq \Omega_2 \) is compact. Clearly, a biholomorphic map proper, basic properties of proper holomorphic maps are discussed in [18, Chapter 15]. The problem of understanding the reducing subspaces of multiplication operators by finite Blaschke products (same as the class of proper holomorphic self maps of the unit disc \( \mathbb{D} \) in \( \mathbb{C} \)) on the Bergman space \( \mathbb{A}^2(\mathbb{D}) \) in and parametrizing the number of minimal reducing subspaces has been studied profusely, see [10] and references therein. Analogous problems for several variables are pursued in [12, 13]. In this paper, we give a bare hand derivation of the two results:

1. If \( \Omega_1, \Omega_2 \subseteq \mathbb{C}^d \) are two bounded domains and \( f = (f_1, \ldots, f_d) : \Omega_1 \to \Omega_2 \) is a proper holomorphic mapping, then \( M_f = (M_{f_1}, \ldots, M_{f_d}) \) has a non-trivial minimal joint reducing subspace \( \mathcal{M} \).
2. The restriction of \( M_f \) to \( \mathcal{M} \) is unitarily equivalent to the Bergman operator on \( \mathbb{A}^2(\Omega_2) \).

We draw attention to the simplicity of our methods and to the fact that we do not require any regularity on the domains \( \Omega_1, \Omega_2 \) to obtain the aforementioned results. Therefore, it allows us to specialize the domains \( \Omega_1, \Omega_2 \) to be different standard domains and obtain explicit description of the non-trivial minimal reducing subspace and the restriction operator \( M_f|_{\mathcal{M}} \). In particular, we specialize to

1. \( \Omega_2 \) to be a complete Reinhardt domain.
2. \( \Omega_1 = \Omega_2 = \mathbb{D} \), the result obtained here is one of the main results in [11, Theorem 25], [20, Theorem 15, p. 393] and the main result in [14].
3. \( \Omega_1 = \Omega_2 = \mathbb{G}_d \), the symmetrized polydisc in \( \mathbb{C}^d \). This domain has been studied extensively during last two decades from the viewpoint of function theory and operator theory, for example, see [5, 6, 9] and references therein.

For two domains \( D_1 \) and \( D_2 \) in \( \mathbb{C}^d \), and an analytic covering \( p : D_1 \to D_2 \), a deck transformation or an automorphism of \( p \) is a biholomorphism \( h : D_1 \to D_1 \) such that \( p \circ h = p \). The set of all deck transformations of \( p \) forms a group. We call it the group of deck transformations of \( p \) or group of automorphisms of \( p \) and denote it by \( \text{Deck}(p) \) or \( \text{Aut}(p) \). It is well known that a proper holomorphic mapping \( f : \Omega_1 \to \Omega_2 \) is a (unbranched) analytic covering
\[
f : \Omega_1 \setminus f^{-1}(f(V_f)) \to \Omega_2 \setminus f(V_f),
\]
where \( V_f := \{ z \in \Omega_1 : J_f(z) = 0 \} \) is the branch locus of \( f \) [18, Subsection 15.1.10]. The group \( \text{Deck}(f) \) of this (unbranched) analytic covering is called the group of deck transformations of the proper map \( f : \Omega_1 \to \Omega_2 \). An analytic covering \( p \) is called Galois if \( \text{Deck}(p) \) acts transitively on the fibre set \( p^{-1}(\{w\}) \) of \( w \in D_2 \) for some (and thus for all) \( w \in D_2 \). Equivalently, cardinality of the set \( p^{-1}(\{w\}) \) equals to the order of \( \text{Deck}(p) \). In other words, for every \( w \in D_2 \), the set \( p^{-1}(\{w\}) \) is a \( \text{Deck}(p) \)-orbit, and
vice versa. Briefly, $D_2 = D_1/\text{Deck}(p)$. Hence an analytic cover $p$ is Galois if and only if $D_1/\text{Deck}(p) \cong D_2$.

We recall a notion from [2]. We say that $f : D_1 \to D_2$ is factored by automorphisms if there exists a finite subgroup $G \subseteq \text{Aut}(D_1)$ such that

$$f^{-1}f(z) = \bigcup_{\rho \in G} \rho(z) \text{ for all } z \in D_1. \quad (1.1)$$

Form this it follows that $f$ is $G$-invariant, that is, $f \circ \rho = f$ for $\rho \in G$. If such a group $G$ exists then $f$ factors a $f \circ \eta$, where $\eta : D_1 \to D_1/G$ is a quotient map and $\hat{f} : D_1/G \to D_2$ is a biholomorphism. That is, $G \cong \text{Deck}(f)$.

Now we describe a procedure developed in [4, 3] to obtain joint reducing subspaces of $\mathbf{M}_f$ acting on a Hilbert space $\mathcal{H} \subseteq \text{Hol}(\Omega_1, \mathbb{C}^d)$ (in particular, $\mathbb{A}^2(\Omega_1)$) with $G$-invariant reproducing kernel $K$, that is.

$$K(\rho \cdot z, \rho \cdot w) = K(z, w) \text{ for } z, w \in \Omega_1, \rho \in G.$$  

for a proper holomorphic map $f = (f_1, \ldots, f_d) : \Omega_1 \to \Omega_2$ which factors through automorphisms $G \subseteq \text{Aut}(\Omega_1)$, here $G \subseteq \text{Aut}(\Omega_1)$ is the group which appears in (1.1). If $\hat{G}$ denotes the set of equivalence class of irreducible representations $G$, then it is shown that

$$\mathcal{H} = \bigoplus_{\rho \in \hat{G}} \mathbb{P}_\rho \mathcal{H},$$

where $\mathbb{P}_\rho$ is an orthogonal projection satisfying $\mathbb{P}_\rho M_{f_i} = M_{f_i} \mathbb{P}_\rho$ for $\rho \in \hat{G}$ and $i = 1, \ldots, d$. Thus we obtain a family $\{\mathbb{P}_\rho \mathcal{H} : \rho \in \hat{G}\}$ of joint reducing subspaces for the multiplication tuple $\mathbf{M}_f$.

A factorization of a proper holomorphic map does not always exist (see [8, p. 223]). We have justified by explicit examples that proper holomorphic self-maps of $\mathbb{D}, \mathbb{D}^d$ and $\mathbb{G}_d$ do not factor through automorphisms, in general, by showing their groups of deck transformations are trivial (since $G \cong \text{Deck}(f)$, if $f : \Omega_1 \to \Omega_2$ factors through automorphisms $G \subseteq \text{Aut}(\Omega_1)$). We are able to produce a non-trivial joint minimal reducing subspace $\mathcal{M}$ of $\mathbf{M}_f$ acting on $\mathbb{A}^2(\Omega_1)$ and describe the restriction operator $\mathbf{M}_f|\mathcal{M}$ even if $f$ does not factor through automorphisms.

2. PROPER HOLONMPHRIC MAPS AND REDUCING SUBSPACE

Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic map of multiplicity $m$ between two bounded domains in $\mathbb{C}^d$. We denote the Bergman space on $\Omega_i$ by $\mathbb{A}^2(\Omega_i)$ and the reproducing kernels by $K_i$ for $i = 1, 2$, respectively.

Let $\Gamma_f : \mathbb{A}^2(\Omega_2) \to \mathbb{A}^2(\Omega_1)$ be the linear map defined by the rule:

$$\Gamma_f \psi = \frac{1}{\sqrt{m}} (\psi \circ f) J_f, \; \psi \in \mathbb{A}^2(\Omega_2), \quad (2.1)$$

where $J_f$ is the jacobian of the proper map $f$. Next we note that $f(N)$ has measure zero with respect to the Lebesgue measure on $\mathbb{C}^n$, where $N = Z(J_f)$ is the zero set of $J_f$. By the change of variables formula, we obtain,

$$\int_{\Omega_1} |J_f|^2 |\psi \circ f|^2 dV = \int_{\Omega_1 - f^{-1}(f(N))} |J_f|^2 |\psi \circ f|^2 dV = m \int_{\Omega_2 - f(N)} |\psi|^2 dV = m \int_{\Omega_2} |\psi|^2 dV, \quad (2.2)$$

where $\psi \in \mathbb{A}^2(\Omega_2)$.
where, $\varphi \in \mathcal{A}^2(\Omega_2)$ and $dV$ is the Lebesgue measure. This shows that the map $\Gamma_f$ is an isometry.

In the following proposition, we provide a criterion for a proper holomorphic map $f$ to be biholomorphic in terms of the isometry $\Gamma_f$.

**Proposition 2.1.** The linear map $\Gamma_f : \mathcal{A}^2(\Omega_2) \to \mathcal{A}^2(\Omega_1)$ is surjective if and only if $f$ is biholomorphic.

**Proof.** Clearly, $\text{ran } \Gamma_f = \{(\psi \circ f)J_f : \psi \in \mathcal{A}^2(\Omega_2)\}$. Since $\Gamma_f$ is an isometry $\text{ran } \Gamma_f$ is a closed subspace of $\mathcal{A}^2(\Omega_1)$. Put $V_f := \{z \in \Omega_1 : J_f(z) = 0\}$, so $V_f$ is an analytic hypersurface in $\Omega_1$ and the subspace $\mathcal{H} := \{g \in \mathcal{A}^2(\Omega_1) : g|_{V_f} = 0\}$ of $\mathcal{A}^2(\Omega_1)$ is closed. Obviously, $\text{ran } \Gamma_f \subseteq \mathcal{H}$. If $f$ is not biholomorphic, then $f$ is not injective. Consequently, $V_f \neq \emptyset$. Since $\Omega_1$ is bounded, $\mathcal{A}^2(\Omega_1)$ contains constant functions but non-zero constant functions do not belong to $\mathcal{H}$, hence do not belong to $\text{ran } \Gamma_f$.

Conversely, if $f$ is biholomorphic, then $f^{-1} : \Omega_2 \to \Omega_1$ is also biholomorphic. It follows from the chain rule that $\Gamma_{f^{-1}} : \mathcal{A}^2(\Omega_1) \to \mathcal{A}^2(\Omega_2)$ is the inverse of $\Gamma_f$. Hence $\Gamma_f$ is surjective. □

Now onward by a proper map we mean a proper holomorphic map which is not biholomorphic. For a function $f = (f_1, \ldots, f_d) : \Omega_1 \to \Omega_2$, the symbol $M_f$ is defined to be the commuting tuple $(M_{f_1}, \ldots, M_{f_d})$. Let $M := (M_1, \ldots, M_d)$ denote the commuting tuple of multiplication operators by the coordinate functions, that is, $M_i$ denotes multiplication by the $i$-th coordinate function, $i = 1, \ldots, d$.

The following theorem exhibits explicitly a non-trivial joint reducing subspace (consequently, at least two) of the tuple of multiplication operators $M_f$ by a proper map $f : \Omega_1 \to \Omega_2$ on the Bergman space $\mathcal{A}^2(\Omega_1)$.

**Theorem 2.2.** Suppose that $\Omega_1, \Omega_2$ are bounded domains in $\mathbb{C}^d$ and $f : \Omega_1 \to \Omega_2$ is a proper holomorphic map of multiplicity $m$. If $\Gamma_f : \mathcal{A}^2(\Omega_2) \to \mathcal{A}^2(\Omega_1)$ is defined by

$$
\Gamma_f \psi = \frac{1}{\sqrt{m}}(\psi \circ f)J_f, \quad \psi \in \mathcal{A}^2(\Omega_2),
$$

where $J_f$ is the jacobian of the proper map $f$. Then $\Gamma_f(\mathcal{A}^2(\Omega_2))$ is a joint reducing subspace for the commuting tuple $M_f = (M_{f_1}, \ldots, M_{f_d})$ on $\mathcal{A}^2(\Omega_1)$.

**Proof.** Being the image of a Hilbert space under the isometry $\Gamma_f$, $\Gamma_f(\mathcal{A}^2(\Omega_2))$ is a closed subspace of $\mathcal{A}^2(\Omega_1)$. The orthogonal projection $P$ from $\mathcal{A}^2(\Omega_1)$ onto $\Gamma_f(\mathcal{A}^2(\Omega_2))$ is given by the following formula [21, p. 551]

$$
P \varphi = \frac{1}{m} \sum_{k=1}^m (\varphi \circ f^k \circ f)J_{f^k \circ f}, \quad \varphi \in \mathcal{A}^2(\Omega_1),
$$

where $\{f^j\}_{j=1}^m$ are the local inverses of $f$. Let $e_i : \Omega_2 \to \mathbb{C}$ denote the coordinate projections defined by $e_i(w) = w_i$ for $i = 1, \ldots, d$. So that

$$
e_i(f(z)) = (e_i \circ f)(z) = f_i(z).
$$

(2.3)

$$(PM_{f_i})\varphi = P(f_i \varphi) = \frac{1}{m} \sum_{k=1}^m (f_i \circ f^k \circ f) (\varphi \circ f^k \circ f) J_{f^k \circ f}.
$$

(2.4)

Repetitive use of the relation (2.3) and the fact $(f \circ f^k)(z) = z$, lead us to

$$
f_i \circ f^k \circ f = (e_i \circ f) \circ (f^k \circ f) = e_i \circ (f \circ f^k) \circ f = e_i \circ f = f_i.
$$
Hence from (2.4), it follows immediately that
\[(PM_f)\varphi = f_1 P\varphi = (M_{f_1} P)\varphi \text{ for } \varphi \in \mathbb{A}^2(\Omega_1).\]
This shows that $P$ commutes with each component of $(M_{f_1}, \ldots, M_{f_n})$. Hence $\Gamma_f(\mathbb{A}^2(\Omega_2))$ is a joint reducing subspace of the tuple $M_f$.

In the next theorem, we offer a canonical description of the restriction of $M_f$ to the joint reducing subspace $\text{ran} \Gamma_f$.

**Theorem 2.3.** The restriction of $M_f$ to $\Gamma_f(\mathbb{A}^2(\Omega_2))$ is unitarily equivalent to $M$ on $\mathbb{A}^2(\Omega_2)$. Consequently, the restriction of $M_f$ to $\Gamma_f(\mathbb{A}^2(\Omega_2))$ is irreducible.

**Proof.** The map $\Gamma_f : \mathbb{A}^2(\Omega_2) \rightarrow \Gamma_f(\mathbb{A}^2(\Omega_2))$ is surjective and it intertwines $M_i$ on $\Gamma_f(\mathbb{A}^2(\Omega_2))$ and $M_i$ on $\mathbb{A}^2(\Omega_2)$ for all $i = 1, \ldots, n$, because
\[M_f \Gamma_f \psi = f_i (\psi \circ f) J_f = (e_i \circ f)(\psi \circ f) J_f = \Gamma_f(e_i \psi) = \Gamma_f M_i \psi \text{ for } \psi \in \mathbb{A}^2(\Omega_2).\]
By Equation (2.2), $\Gamma_f$ is an isometry. This completes the proof.

The following corollary follows immediately from the Theorem above.

**Corollary 2.4.** If $\Omega \subseteq \mathbb{C}^d$ is a domain such that $f$ is a proper holomorphic self-map of $\Omega$ of multiplicity $m$. Then the restriction of $M_f$ to the joint reducing subspace $\Gamma_f(\mathbb{A}^2(\Omega))$ is unitarily equivalent to $M$ on $\mathbb{A}^2(\Omega)$.

In the next section, we provide examples where the Corollary above is applicable.

Before proceeding further, we recall a relevant definition.

**Definition 2.5.** A domain $D$ in $\mathbb{C}^d$ is called a complete Reinhardt domain if whenever $z \in D$, the closed polydisc $\{|w| \leq |z| : j = 1, \ldots, d\}$ is also contained in $D$.

In such a domain $D$, the monomials form a complete orthogonal system for $\mathbb{A}^2(D)$. Let $\mathbb{Z}_+$ denote the set of nonnegative integers and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$ be a multi-index. For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, $z^\alpha := \prod_{j=1}^d z_j^{\alpha_j}$. We make a note of an observation in the following proposition.

**Proposition 2.6.** Suppose that $\Omega_1, \Omega_2$ are bounded domains in $\mathbb{C}^d$ such that $\Omega_2$ is a complete Reinhardt domain and $f = (f_1, \ldots, f_d) : \Omega_1 \rightarrow \Omega_2$ is a proper holomorphic map of multiplicity $m$. Then \[\left\{\frac{1}{\sqrt{m}} z^\alpha f^\alpha \right\}_{\alpha \in \mathbb{Z}_+^d} \text{ is an orthonormal basis for } \Gamma_f(\mathbb{A}^2(\Omega_2)).\]

**Proof.** We observe that $\Gamma_f$ is an isometry onto its range, hence it maps an orthonormal basis of $\mathbb{A}^2(\Omega_2)$ to an orthonormal basis in its range. Since $\Omega_2$ is a complete Reinhardt domain, \[\left\{\frac{z^\alpha}{\|z^\alpha\|} \right\}_{\alpha \in \mathbb{Z}_+^d} \text{ is an orthonormal basis of } \mathbb{A}^2(\Omega_2).\] The result follows noting that the desired orthonormal basis is the image of \[\left\{\frac{z^\alpha}{\|z^\alpha\|} \right\}_{\alpha \in \mathbb{Z}_+^d} \text{ under } \Gamma_f.\]

The following corollary is immediate from Corollary 2.4 and Proposition 2.6.

**Corollary 2.7.** If $\Omega \subseteq \mathbb{C}^d$ is a complete Reinhardt domain such that $f$ is a proper holomorphic self-map of $\Omega$ of multiplicity $m$. Then the restriction of $M_f$ to the joint reducing subspace $\Gamma_f(\mathbb{A}^2(\Omega))$ is unitarily equivalent to $M$ on $\mathbb{A}^2(\Omega)$. Moreover,
\[\Gamma_f(\mathbb{A}^2(\Omega)) = \overline{\text{span}} \left\{J_f f^\alpha : \alpha \in \mathbb{Z}_+^d \right\} \text{ and } \left\{\frac{1}{\sqrt{m}} z^\alpha f^\alpha \right\}_{\alpha \in \mathbb{Z}_+^d}\]
is an orthonormal basis of $\Gamma_f(\mathbb{A}^2(\Omega))$. 
Before proceeding to applications, it is interesting to compute reproducing kernel for the subspace \( \Gamma_f(\mathbb{R}^2(\Omega_2)) \) in terms of the reproducing kernel \( K_2 \) of \( \mathbb{R}^2(\Omega_2) \).

**Proposition 2.8.** The reproducing kernel of \( K_f \) of \( \Gamma_f(\mathbb{R}^2(\Omega_2)) \) is given by

\[
K_f(z, w) = \frac{1}{m} J_f(z) K_2(f(z), f(w)) J_f(w) \text{ for } z, w \in \Omega_1.
\]

**Proof.** Since \( K_2 \) is the reproducing kernel of \( \mathbb{R}^2(\Omega_2) \), it follows that

\[
(K_2)_{f(w)} := K_2(\cdot, f(w)) \in \mathbb{R}^2(\Omega_2) \text{ for } w \in \Omega_1.
\]

For every \( w \in \Omega_1 \), \( \frac{1}{m} J_f(w) J_f((K_2)_{f(w)} \circ f) \in \Gamma_f(\mathbb{R}^2(\Omega_2)) \). For \( \psi \in \mathbb{R}^2(\Omega_2) \), we note that

\[
\langle J_f(\psi \circ f), \frac{1}{m} J_f(w) J_f((K_2)_{f(w)} \circ f) \rangle = J_f(w) \langle (K_2)_{f(w)}(f), \psi \rangle = J_f(w) \langle \psi, (K_2)_{f(w)} \rangle = J_f(w) \langle \psi \circ f, w \rangle.
\]

Therefore, for every \( w \in \Omega_1 \), \( \frac{1}{m} J_f(w) J_f((K_2)_{f(w)} \circ f) \) has reproducing property. By the uniqueness of the reproducing kernel of a Hilbert space with a reproducing kernel, we conclude that \( \frac{1}{m} J_f(w) J_f((K_2)_{f(w)} \circ f), w \in \Omega_1 \) is the reproducing kernel function of \( \Gamma_f(\mathbb{R}^2(\Omega_2)) \). If we denote the reproducing kernel of \( \Gamma_f(\mathbb{R}^2(\Omega_2)) \) by \( K_f \), then

\[
K_f(z, w) = \langle (K_f)_w, (K_f)_z \rangle = \langle \frac{1}{m} J_f(w) J_f((K_2)_{f(w)} \circ f), \frac{1}{m} J_f(z) J_f((K_2)_{f(z)} \circ f) \rangle = \frac{1}{m} J_f(z) J_f(w) \langle (K_2)_{f(w)}, (K_2)_{f(z)} \rangle = \frac{1}{m} J_f(z) K_2(f(z), f(w)) J_f(w) \text{ for } z, w \in \Omega_1.
\]

\[\Box\]

**Remark 2.9.** Here is an alternative way of arriving at the formula for \( K_f \). Let \( \{e_\alpha\}_{\alpha \in I} \) be an orthonormal basis for \( \mathbb{R}^2(\Omega_2) \). Since \( \Gamma_f \) is an isometry, \( \{\Gamma_f e_\alpha\}_{\alpha \in I} \) is an orthonormal basis for \( \Gamma_f(\mathbb{R}^2(\Omega_2)) \). Therefore the reproducing kernel \( K_f \) of \( \Gamma_f(\mathbb{R}^2(\Omega_2)) \) is given by the following formula

\[
K_f(z, w) = \sum_{\alpha \in I} (\Gamma_f e_\alpha)(z)(\Gamma_f e_\alpha)(w) = \frac{1}{m} \sum_{\alpha \in I} J_f(z) (e_\alpha(f(z))(J_f(w)(e_\alpha(f(w)))) = \frac{1}{m} J_f(z) \left( \sum_{\alpha \in I} e_\alpha(f(z)) e_\alpha(f(w)) \right) J_f(w) = \frac{1}{m} J_f(z) K_2(f(z), f(w)) J_f(w) \text{ for } z, w \in \Omega_1.
\]
Computation of the reproducing kernel of $\Gamma_f(\mathbb{A}^2(\Omega_2))$ for a proper holomorphic map $f : \Omega_1 \to \Omega_2$ plays a crucial role in computing the Bergman kernel of $\mathbb{A}^2(\Omega_2)$ in the technique developed in [15] and generalized in [21].

3. Applications

3.1. Example (Unit disc). In order to describe all possible proper holomorphic self-maps of the unit disc $D$, we recall a definition.

**Definition 3.1.** A finite Blaschke product is a rational function of the form

$$B(z) = e^{i\theta} \prod_{j=1}^{n} \left( \frac{z - a_j}{1 - \bar{a}_j z} \right)^{k_j},$$

where $a_1, \ldots, a_n \in \mathbb{D}$ are the distinct zeros of $B$ with multiplicities $k_1, \ldots, k_n$, respectively and $\theta \in \mathbb{R}$.

Finite Blaschke products are examples of proper holomorphic self-maps of the unit disc $D$. In fact, the following proposition shows that they constitute the set of all possible proper holomorphic self-maps of the unit disc $D$, whose proof can be found in [17, Remarks 3, p. 142].

**Proposition 3.2.** Let $f : D \to D$ be a proper holomorphic map. Then $f$ is a finite Blaschke product.

In [7, Example 2.1, p. 334], authors produce one example of finite Blaschke product given by

$$\tilde{B}(z) = z^4 \left( \frac{z - 1/2}{1 - z/2} \right)^2.$$

such that Deck($\tilde{B}$) is trivial. In [19, 4.7, p. 711], Rudin exhibits another example of Blaschke product

$$\hat{B}(z) = \prod_{j=1}^{3} \frac{z - a_j}{1 - \bar{a}_j z}, \text{ where } a_1 = -\frac{1}{2}, a_2 = 0, a_3 = \frac{3}{4},$$

with trivial Deck($\hat{B}$). It is clear that any finite Blaschke product $B$ which has either $\tilde{B}$ or $\hat{B}$ as a factor will have trivial Deck($B$).

Theorem 2.2 and Corollary 2.7 yield the following result which is one of the main results in [11]. The same result was also obtained differently in [20, Theorem 15, p. 393] and it is the main result in [14]. Put $e_n(z) = \sqrt{n + 1} z^n$ for $n \geq 0$ and $z \in D$, recall that $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal basis for $\mathbb{A}^2(D)$. The multiplication operator $M$ by $z$ on $\mathbb{A}^2(D)$ is a weighted shift operator, known as the Bergman shift $Me_n = \sqrt{\frac{n+1}{n+2}} e_{n+1}$.

**Theorem 3.3.** Suppose that $B$ is a finite Blaschke product on $D$ of order $m$. Then $\Gamma_B(\mathbb{A}^2(D))$ is a non-trivial minimal reducing subspace of $M_B$ on $\mathbb{A}^2(D)$. Moreover, the restriction of $M_B$ to $\Gamma_B(\mathbb{A}^2(D))$ is unitarily equivalent to the Bergman shift. In fact,

$$\Gamma_B(\mathbb{A}^2(D)) = \overline{\text{span}}\{B^n B' : n \geq 0\} \text{ and } \left\{ \sqrt{\frac{n+1}{m}} B^n B' \right\}_{n=0}^{\infty}$$

is an orthonormal basis of $\Gamma_B(\mathbb{A}^2(D))$. 

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3.2. Example (polydisc). Proper holomorphic self maps of polydisc is described by the following theorem (see [1], [16]).

**Theorem 3.4.** If \( \Omega_1, \ldots, \Omega_d, \Delta_1, \ldots, \Delta_d \subset \mathbb{C} \) are bounded domains and if \( f : \Omega_1 \times \cdots \times \Omega_d \to \Delta_1 \times \cdots \times \Delta_d \) is a proper mapping, then there exist a permutation \( \sigma \) of \( \{1, \ldots, d\} \) and proper maps \( f_i : \Omega_{\sigma(i)} \to \Delta_j \) such that
\[
 f(z_1, \ldots, z_d) = (f_1(z_{\sigma(1)}), \ldots, f_n(z_{\sigma(d)})).
\]

Since Proposition 3.2 reads that any proper holomorphic self-map of the open unit disc \( \mathbb{D} \subseteq \mathbb{C} \) is given by some finite Blaschke product, hence by Theorem 3.4 we get that any proper holomorphic self-map \( B \) of the polydisc \( \mathbb{D}^d \) is given by
\[
 B(z) = (B_1(z_1), \ldots, B_d(z_d)),
\]
(3.3)
where each \( B_i \) is a finite Blaschke product.

Before proceeding further, we produce examples of proper holomorphic self-maps \( B \) of \( \mathbb{D}^d \) with trivial \( \text{Deck}(B) \).

**Proposition 3.5.** There is a proper holomorphic self-map \( B \) of \( \mathbb{D}^d \) with \( \text{Deck}(B) = \{\text{identity}\} \).

**Proof.** Choose the finite Blaschke product \( \hat{B} \) given in (3.2) and define \( B : \mathbb{D}^d \to \mathbb{D}^d \) by
\[
 B(z) = (\hat{B}(z_1), \ldots, \hat{B}(z_d))
\]
(3.4)
If \( \varphi \in \text{Deck}B \) for some \( \varphi \in \text{Aut}(\mathbb{D}^d) \), we claim that \( \varphi = \text{identity} \). It is well known that \( \text{Aut}(\mathbb{D}^d) \cong \text{Aut}(\mathbb{D})^d \times \mathfrak{S}_d \), where \( \mathfrak{S}_d \) is the permutation group on \( d \) symbols. So \( \varphi = (\varphi_1, \ldots, \varphi_d) \), where \( \varphi_i \in \text{Aut}(\mathbb{D}) \) for \( i = 1, \ldots, d \). If \( B \circ \varphi = B \), then
\[
 B(\varphi_1(z_1), \ldots, \varphi_d(z_d)) = B(z_1, \ldots, z_d)
\]
for all \( z = (z_1, \ldots, z_d) \in \mathbb{D}^d \).

By our choice of \( B \) as in (3.4), we get
\[
 (\hat{B}(\varphi_1(z_1)), \ldots, \hat{B}(\varphi_d(z_d))) = (\hat{B}(z_1), \ldots, \hat{B}(z_1)) \text{ for all } z = (z_1, \ldots, z_d) \in \mathbb{D}^d.
\]
Hence \( \hat{B} \circ \varphi_i = \hat{B} \) for \( i = 1, \ldots, d \). Therefore, by the choice of \( \hat{B} \) in (3.2), we have \( \varphi_i = \text{identity} \) for \( i = 1, \ldots, d \). Thus \( \varphi = \text{identity} \). \( \blacksquare \)

**Remark 3.6.** From the proof of the Proposition above it is clear that if \( B \) is a proper holomorphic self-map each of whose components contains \( \hat{B} \) as a factor then \( \text{Deck}(B) \) is trivial.

Put \( e_m(z) = \sqrt{\prod_{i=1}^d (m_i + 1)} z^m \) for \( z \in \mathbb{D}^d \). We recall that \( \{e_m\}_{m \in \mathbb{Z}^d} \) is an orthonormal basis for the Bergman space \( A^2(\mathbb{D}^d) \). If \( M_i \) denotes the multiplication operator by \( z_i \) on \( A^2(\mathbb{D}^d) \), then \( M = (M_1, \ldots, M_d) \) is a several variable weighted shift on \( A^2(\mathbb{D}^d) \), also called Bergman multishift whose multi-weight sequence is \( (\sqrt{m_1 + 1}, \ldots, \sqrt{m_d + 1}) \).

Let \( M_B \) be the operator tuple \( (M_{B_1}, \ldots, M_{B_d}) \) acting on \( A^2(\mathbb{D}^d) \). Now Theorem 2.2 and Corollary 2.7 immediately generalize Theorem 3.3 to the case of the polydisc \( \mathbb{D}^d \).

**Theorem 3.7.** Suppose that \( B : \mathbb{D}^d \to \mathbb{D}^d \) is given by \( B(z) = (B_1(z_1), \ldots, B_d(z_d)) \), where each \( B_i \) is a finite Blaschke product of order \( m_i \). Then \( \Gamma_B(A^2(\mathbb{D}^d)) \) is a non-trivial minimal reducing subspace of \( M_B \) on \( A^2(\mathbb{D}^d) \). Moreover, the restriction of \( M_B \)
to $\Gamma_B(A^2(\mathbb{D}^d))$ is unitarily equivalent to the Bergman multishift. Moreover,

$$
\Gamma_B(A^2(\mathbb{D}^d)) = \text{span}\{J_BB^\alpha : \alpha \in \mathbb{Z}_+^d\}
$$

and

$$
\left\{ \prod_{j=1}^d \frac{\alpha_j + 1}{m_j} J_BB^\alpha \right\}_{\alpha \in \mathbb{Z}_+^d}
$$

is an orthonormal basis of $\Gamma_B(A^2(\mathbb{D}^d))$.

### 3.3. Example (Symmetrized Polydisc).

Let $s : \mathbb{C}^d \to \mathbb{C}^d$ be the symmetrization map defined by $s(z) = (s_1(z), \ldots, s_d(z))$, where $s_i$ is the elementary symmetric polynomial in $d$ variables of degree $i$, that is, $s_i$ is the sum of all products of $i$ distinct variables $z_i$ so that

$$
s_i(z) = \sum_{1 \leq k_1 < k_2 < \ldots < k_i \leq d} z_{k_1} \cdots z_{k_i}.
$$

The symmetrization map $s$ is a proper holomorphic map of multiplicity $d!$ (see [19, Theorem 5.1]). The domain $G_d := s(\mathbb{D}^d)$ is known as the symmetrized polydisc. It is pointed out in [4, p. 771] that $G_d$ is not a Reinhardt domain. In [9, Theorem 1], authors characterize proper holomorphic self-maps and automorphisms of the symmetrized polydisc $G_d$. In particular, they proved that $G_d$ admits proper holomorphic self-maps which are not automorphisms of $G_d$. The theorem states as following.

**Theorem 3.8** (Edigarian-Zwonek). Let $f : G_d \to G_d$ be a holomorphic mapping. Then $f$ is proper if and only if there exists a finite Blaschke product $B$ such that

$$
f(s(z)) = s(B(z_1), \ldots, B(z_d)) \text{ for } z = (z_1, \ldots, z_d) \in \mathbb{D}^d,
$$

where $s$ is symmetrization map. In particular, $f$ is an automorphism if and only if

$$
f(s(z)) = s(\varphi(z_1), \ldots, \varphi(z_d)) \text{ for } z = (z_1, \ldots, z_d) \in \mathbb{D}^d,
$$

where $\varphi$ is an automorphism of $\mathbb{D}$.

Before describing reducing subspaces of $M_f$, we exhibit examples of proper holomorphic self-maps $f$ of $G_d$ with trivial $\text{Deck}(f)$.

**Proposition 3.9.** There is a proper holomorphic self-map $f$ of $G_d$ with $\text{Deck}(f) = \{\text{identity}\}$.

**Proof.** Choose a proper holomorphic self-map $f$ of $G_d$ given by

$$
f(s(z)) = s(\hat{B}(z_1), \ldots, \hat{B}(z_d)) \text{ for } z = (z_1, \ldots, z_d) \in \mathbb{D}^d,
$$

(3.5)

where $\hat{B}$ is the finite Blaschke product given in (3.2). If $\varphi \in \text{Deck}(f)$ for some $\varphi \in \text{Aut}(G_d)$, we claim that $\varphi = \text{identity}$. From Theorem 3.8, we know that $\text{Aut}(G_d) \cong \text{Aut}(\mathbb{D})$. For $z = (z_1, \ldots, z_d) \in \mathbb{D}^d$, we have

$$
(f \circ \varphi)(s(z)) = f\left(s(\varphi(z_1), \ldots, \varphi(z_d))\right) = s\left(\hat{B}(\varphi(z_1)), \ldots, \hat{B}(\varphi(z_d))\right).
$$

Since $f \circ \varphi = f$, we must have

$$
(f \circ \varphi)(s(z)) = s\left(\hat{B}(\varphi(z_1)), \ldots, \hat{B}(\varphi(z_d))\right) = s(\hat{B}(z_1), \ldots, \hat{B}(z_d)).
$$

This forces $\hat{B} \circ \varphi = \hat{B}$, hence by (3.2) $\varphi = \text{identity}$. ~\qed
Remark 3.10. From the proof of the Proposition above it is clear that if \( f \) is a proper holomorphic self-map whose associated finite Blaschke product \( B \) contains \( B \) as a factor then \( \text{Deck}(f) \) is trivial.

Theorem 2.2 and Corollary 2.4 enables us to describe a non-trivial minimal reducing subspace of \( M_f \) acting on \( \mathbb{A}^2(\mathbb{G}_d) \) for a proper holomorphic self-map \( f \) of \( \mathbb{G}_d \).

**Theorem 3.11.** Suppose that \( f \) is a proper holomorphic self-map of \( \mathbb{G}_d \). Then \( \Gamma_f(\mathbb{A}^2(\mathbb{G}_d)) \) is a non-trivial minimal reducing subspace of \( M_f \) on \( \mathbb{A}^2(\mathbb{G}_d) \). Moreover, the restriction of \( M_f \) to \( \Gamma_f(\mathbb{A}^2(\mathbb{G}_d)) \) is unitarily equivalent to the Bergman operator \( M \) on \( \mathbb{A}^2(\mathbb{G}_d) \).

It is pointed out in [4, Corollary 3.19] that the Bergman operator \( M \) on \( \mathbb{A}^2(\mathbb{G}_d) \) is not unitarily equivalent to a joint weighted shift. Let \( \mathbb{A}^2_{\text{anti}}(\mathbb{D}^d) \) be the subspace of \( \mathbb{A}^2(\mathbb{D}^n) \) consisting of anti-symmetric functions, that is
\[
\mathbb{A}^2_{\text{anti}}(\mathbb{D}^d) = \{ f \in \mathbb{A}^2(\mathbb{D}^d) : f \circ \sigma^{-1} = sgn(\sigma)f \text{ for } \sigma \in \mathbb{S}_d \},
\]
where \( \mathbb{S}_d \) is the permutation group on \( d \) symbols and \( sgn(\sigma) \) is 1 or \(-1\) according as \( \sigma \) is an even or an odd permutation, respectively. It follows from [15, p. 2363] that the Bergman operator \( M \) on \( \mathbb{A}^2(\mathbb{G}_n) \) is unitarily equivalent to \( M_s = (M_{s_1}, \ldots, M_{s_d}) \) on \( \mathbb{A}^2_{\text{anti}}(\mathbb{D}^d) \). Hence from Theorem 3.11 we conclude the following result.

**Theorem 3.12.** If \( f \) is a proper holomorphic self-map of \( \mathbb{G}_d \) Then the restriction of \( M_f \) to \( \Gamma_f(\mathbb{A}^2(\mathbb{G}_d)) \) is unitarily equivalent to \( M_s \) on \( \mathbb{A}^2_{\text{anti}}(\mathbb{D}^d) \). Consequently, \( M_s \) on \( \mathbb{A}^2_{\text{anti}}(\mathbb{D}^d) \) is irreducible.

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(Ghosh) Department of Mathematics and Statistics, Indian Institute of Science Education and Research Kolkata, Mohanpur 741246, Nadia, West Bengal, India

E-mail address, Ghosh: gg13ip034@iiserkol.ac.in