HÖLDER ESTIMATES FOR THE $\overline{\partial}$-EQUATION ON SURFACES WITH SIMPLE SINGULARITIES

F. ACOSTA AND E. S. ZERON

Abstract. Let $\Sigma \subset \mathbb{C}^3$ be a 2-dimensional subvariety with an isolated simple (rational double point) singularity at the origin. The main objective of this paper is to solve the $\overline{\partial}$-equation on a neighbourhood of the origin in $\Sigma$, demanding a Hölder condition on the solution.

1. Introduction

Let $\Sigma \subset \mathbb{C}^3$ be a subvariety with an isolated singularity at the origin. Given a $\overline{\partial}$-closed $(0,1)$-differential form $\lambda$ defined on $\Sigma$ minus the origin, Gavosto and Fornæss proposed a general technique for solving the differential equation $\overline{\partial}g = \lambda$ on a neighbourhood of the origin in $\Sigma$. The calculations were done in the sense of distributions, and they demanded an extra Hölder condition on the solution, see [2] and [3]. Their basic idea was to analyse $\Sigma$ as a branched covering over $\mathbb{C}^2$, to solve the corresponding $\overline{\partial}$-equation on $\mathbb{C}^2$, and to lift the solution from $\mathbb{C}^2$ into $\Sigma$ again. Gavosto and Fornæss completed all the calculations in the particular case when $\Sigma \subset \mathbb{C}^3$ is defined by the polynomial $x_1x_2 = x_3^3$. That is, when $\Sigma$ is a surface with an isolated simple (rational double point) singularity of type $A_2$ at the origin, see [1, p. 60].

Let $X_N$ and $Y_N$ be two subvarieties of $\mathbb{C}^3$ defined by the respective polynomials $x_1x_2 = x_3^N$ and $y_1^2y_3 + y_2^2 = y_3^{N+1}$, for any natural number $N \geq 2$. Surface $X_N$ (respect. $Y_N$) has an isolated simple singularity of type $A_{N-1}$ (respect. $D_{N+2}$) at the origin, see [1] p. 60]. The main objective of this paper is to give an alternative and simplified solution to the equation $\overline{\partial}g = \lambda$ on both surfaces $X_N$ and $Y_N$, with an extra Hölder condition on $g$. The central idea is to consider $\mathbb{C}^2$ as a branched covering over $X_N$ and $Y_N$, instead of analysing $X_N$ as a branched covering over $\mathbb{C}^2$. In the case of $X_N$, we use the natural branched $N$-covering $\pi_N : \mathbb{C}^2 \to X_N$ defined by $\pi_N(z_1,z_2) = (z_1^N,z_2^N,z_1z_2)$, in order to obtain the following theorem. We shall explain, at the end of the third section of this paper, why we use the covering $\pi_N$ instead of a standard blow up mapping.

Theorem 1. Let $E_0(N)$ be the smallest even integer greater than or equal to $N$. Given an open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre in the origin, there exists a finite positive constant $C_1(R)$ such that: For every continuous $(0,1)$-differential form $\lambda$ defined on the compact set $\pi_N(B_R) \subset X_N$, and $\overline{\partial}$-closed on the interior
\( \pi_N(B_R) \), the equation \( \overline{\partial} h = \lambda \) has a continuous solution \( h \) on \( \pi_N(B_R) \) which also satisfies the following Hölder estimate, with \( \beta = 1/Ev(N) \),

\[
\| h \|_{\pi_N(B_R)} + \sup_{x, w \in \pi_N(B_R)} \frac{|h(x) - h(w)|}{\| x - w \|^\beta} \leq C_1(R)\| \lambda \|_{\pi_N(B_R)}.
\]

In the last section of this paper, we extend Theorem 1 to solve the \( \overline{\partial} \)-equation on the subvariety \( Y_N \) as well. The notation \( \| h \|_K \) stands for the maximum of \( |h| \) on the compact set \( K \), and \( \| x - w \| \) stands for the euclidean distance between \( x \) and \( w \). Since \( \| x - w \| \) is less than or equal to the distance between \( x \) and \( w \) measured along the surface \( X_N \), we can assert that inequality (1) is indeed a Hölder estimate on \( X_N \) itself. Finally, all differentials are defined in terms of distributions. For example, the fact that the continuous \((0, 1)\)-differential form \( \lambda \) is \( \overline{\partial} \)-closed on \( \pi_N(B_R) \) means that the integral:

\[
\int_{\pi_N(B_R)} \lambda \wedge \overline{\partial} \sigma = 0,
\]

for every smooth \((2, 0)\)-differential form \( \sigma \) defined on \( \pi_N(B_R) \setminus \{0\} \), such that both \( \sigma \) and \( \overline{\partial} \sigma \) extend continuously to the origin, and these extensions have both compact support inside \( \pi_N(B_R) \).

The proof of Theorem 1 is presented in the following two sections. The next section is devoted to introducing all the basic ideas for the particular case when \( N = 2 \). Moreover, in the third section of this paper, we shall use these ideas for solving the \( \overline{\partial} \)-equation on \( X_N \), in the extended case \( N \geq 3 \). Finally, in the last section of this paper, we extend Theorem 1 to solve the \( \overline{\partial} \)-equation on the subvariety \( Y_N \) as well.

2. Proof of Theorem 1 case \( N = 2 \).

Consider the natural branched covering \( \pi_2(z_1, z_2) = (z_1^2, z_1 z_2) \) defined from \( \mathbb{C}^2 \) onto \( X_2 := [x_1, x_2, x_2^2] \). It is easy to see that \( \pi_2 \) is a branched 2-covering, and that the origin is the only branch point of \( \pi_2 \), because the inverse image \( \pi_2^{-1}(x) \) is a set of the form \( \{ \pm z \} \), for every \( x \in X_2 \). Besides, define the antipodal automorphism \( \phi(z) = -z \) which allows us to jump between the different branches of \( \pi_2 \). In particular, we have that \( \phi^* \pi_2(z) = \pi_2(-z) = \pi_2(z) \).

We assert that the operators \( \pi_2^* \) and \( \overline{\partial} \) commute. It is easy to see that \( \pi_2^* \) and \( \overline{\partial} \) commute when \( \overline{\partial} \) is a standard differential, for \( \pi_2 \) is holomorphic. However, calculations become more complicated when \( \overline{\partial} \) is analysed in the sense of distributions. Let \( B_R \subset \mathbb{C}^2 \) be an open ball of radius \( R > 0 \). We prove the commutativity of \( \pi_2^* \) and \( \overline{\partial} \) for the particular case of a \( \overline{\partial} \)-closed \((0, 1)\)-differential form \( \lambda \) defined on \( \pi_2(B_R) \); the proof with a general differential form follows exactly the same procedure. We have that \( \overline{\partial}(\pi_2^* \lambda) = 0 \) in the sense of equation (2), and we need to prove that \( \overline{\partial}(\pi_2^* \lambda) \) is equal to \( \pi_2^*(\overline{\partial} \lambda) = 0 \) in the sense of distributions, that is:

\[
\int_{B_R} \pi_2^* \lambda \wedge \overline{\partial} v = 0,
\]

for every smooth \((2, 0)\)-differential form \( v \) with compact support in \( B_R \). The automorphism \( \phi \) preserves the orientation of \( B_R \), for it is analytic. Thus, after doing a simple change of variables, and recalling that \( \phi^* \pi_2 = \pi_2 \), we have that the integral in equation (3) is equal to \( \int_{B_R} \pi_2^* \lambda \wedge \overline{\partial} \phi^* v \). Moreover, since \( v + \phi^* v \) is constant
in the fibres of $\pi_2$ (it is invariant under the pull back $\phi^*$) there exists a second differential form $\sigma$ defined on $\pi_2(B_R)$ such that $v + \phi^*v$ is equal to $\pi_2^*\sigma$. Hence:

$$
\int_{B_R} \pi_2^* \lambda \wedge \overline{\partial} v = \int_{B_R} \pi_2^* \lambda \wedge \overline{\partial} \frac{v + \phi^*v}{2} = \int_{\pi_2(B_R)} \frac{\lambda \wedge \overline{\partial} \sigma}{2} = 0.
$$

The equality to zero follows from equation (2), and so $\overline{\partial}(\pi_2^* \lambda) = 0$ on $B_R$, as we wanted to prove. Suppose now that the differential equation $\overline{\partial}g = \pi_2^* \lambda$ has a solution $g$ on $B_R$. The sum $g + \phi^*g$ is also constant in the fibres of $\pi_2$ (it is invariant under the pull back $\phi^*$), so there exists a continuous function $f$ on $B_R$ such that $\pi_2^*f$ is equal to $g + \phi^*g$. We assert that $\overline{\partial}f = 2\lambda$ on $\pi_2(B_R)$. This result follows automatically because:

$$
\pi_2^* \overline{\partial}f = \overline{\partial}(g + \phi^*g) = \pi_2^* \lambda + \phi^* \pi_2^* \lambda = \pi_2^*(2\lambda).
$$

Previous equation demands that the operators $\phi^*$ and $\overline{\partial}$ commute as well in $B_R$, when $\overline{\partial}$ is seen as a distribution. This is an exercise based on the fact that integral $\int \phi^*\mathfrak{R} = \int \mathfrak{R}$, as we have indicated in the paragraph situated after equation (3), and because $\phi$ preserves the orientation of $B_R$. Suppose now that $\lambda$ is also continuous on the compact set $\pi_2(B_R)$. Then, we can apply Theorems 2.1.5 and 2.2.2 of [4] in order to get the following Hölder estimate.

**Theorem 2.** Given an open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre in the origin, there exist two finite positive constants $C_2(R)$ and $C_3(R)$ such that: For every continuous $(0,1)$-differential form $\lambda$ defined on $\pi_2(B_R) \subset X$, and $\overline{\partial}$-closed on the interior $\pi_2(B_R)$, the equation $\overline{\partial}g = \pi_2^* \lambda$ has a continuous solution $g$ on $B_R$ which also satisfies the following Hölder estimates,

$$
\|g\|_{B_R} + \sup_{z,\zeta \in B_R} \frac{|g(z) - g(\zeta)|}{\|z - \zeta\|^{1/2}} \leq C_2(R)\|\pi_2^* \lambda\|_{B_R}, \tag{4}
$$

and

$$
\sup_{z,\zeta \in B_R/2} \frac{|g(z) - g(\zeta)|}{\|z - \zeta\|} \leq C_3(R)\|\pi_2^* \lambda\|_{B_R}. \tag{5}
$$

**Proof.** Inequality (4) holds because of Theorem 2.2.2 in [4]. Besides, recalling the proofs of Lemma 2.2.1 and Theorem 2.2.2, in [4], we have that inequality (6) holds whenever there exists a finite positive constant $C_4(R)$ such that:

$$
\sup_{z,\zeta \in B_R/2} \frac{|E(z) - E(\zeta)|}{\|z - \zeta\|} \leq C_4(R)\|\pi_2^* \lambda\|_{B_R}, \tag{6}
$$

for every function $E(z)$ defined according to equation (2.2.7) of [4, p. 70]. Let $\Upsilon$ be the closed interval which joins $z$ and $\zeta$ inside the ball $B_{R/2}$. Then,

$$
|E(z) - E(\zeta)| \leq \int_0^1 \left| \frac{d}{dt}E(t\zeta + (1-t)z) \right| dt \leq \|z - \zeta\| \sup_{y \in \Upsilon} \sum_{k=1}^2 \left| \frac{\partial E}{\partial y_k} \right| + \left| \frac{\partial E}{\partial y_k} \right|.
$$

Finally, by equation (2.2.9) in [4], we know there exists a finite constant $C_4(R)$ such that all partial derivatives $\left| \frac{\partial E}{\partial y_k} \right|$ and $\left| \frac{\partial E}{\partial y_k} \right|$ are less than or equal to $\frac{C_4(R)}{2}\|\pi_2^* \lambda\|_{B_R}$, for every $y \in B_{R/2}$ and each index $k = 1, 2$. Notice that $D = B_R$ in
equations (2.2.7) and (2.2.9), but $y$ lies inside the smaller ball $B_{R/2}$. Thus, equation (7) automatically implies that inequalities (8) and (9) holds, as we wanted. □

The problem is now reduced to estimating the distance $\|z - \zeta\|$ with respect to the projections $\|\pi_2(z) - \pi_2(\zeta)\|$.

**Lemma 3.** Given two points $z$ and $\zeta$ in $\mathbb{C}^2$ such that $\|z - \zeta\|$ is less than or equal to $\|z + \zeta\|$, the following inequality holds:

$$2\|\pi_2(z) - \pi_2(\zeta)\| \geq \|z - \zeta\| \max\{\|z\|, \|\zeta\|, \|z - \zeta\|\}.$$

**Proof.** We know that $2\|z\|$ and $2\|\zeta\|$ are both less than or equal to $\|z + \zeta\|$. Hence, the maximum of $\|z\|$, $\|\zeta\|$ and $\|z - \zeta\|$ is also less than or equal to $\|z + \zeta\|$. The wanted result will follows after proving that $\|z - \zeta\| \cdot \|z + \zeta\|$ is less than or equal to $2\|\pi_2(z) - \pi_2(\zeta)\|$. Setting $P_1 = z_1 - \zeta_1$, $P_2 = z_2 - \zeta_2$, $Q_1 = z_1 + \zeta_1$ and $Q_2 = z_2 + \zeta_2$, allows us to write the following series of inequalities:

$$\|z - \zeta\|^2 \cdot \|z + \zeta\|^2 =$$

$$= \|P_1 Q_1\|^2 + \|P_1 Q_2\|^2 + \|P_2 Q_1\|^2 + \|P_2 Q_2\|^2$$

$$\leq 4\|P_1 Q_1\|^2 + 4\|P_2 Q_2\|^2 + \|P_1 Q_2\|^2 + \|P_2 Q_1\|^2 - 2\|P_1 Q_1 P_2 Q_2\|$$

$$\leq 4\|P_1 Q_1\|^2 + 4\|P_2 Q_2\|^2 + \|P_1 Q_2\|^2 + \|P_2 Q_1\|^2$$

$$= 4\|\pi_2(z) - \pi_2(\zeta)\|^2.$$

□

We are now in position to prove Theorem 1 for the simplest case $N = 2$.

**Proof.** (Theorem 1 case $N = 2$). Suppose that $\lambda = \sum \lambda_k d\sigma_k$. Then,

$$\pi_2^* \lambda = [2\bar{\pi} \lambda_1(\pi_2) + \bar{\pi}_2 \lambda_3(\pi_2)] d\bar{\pi}_1 + [2\pi_2 \lambda_2(\pi_2) + \pi_1 \lambda_3(\pi_2)] d\pi_2.$$

We obviously have that $|z_k| < R$ for every point $z \in B_R$. Hence,

$$\|\pi_2^* \lambda\|_{B_R} \leq 3R \|\lambda\|_{\pi_2(B_R)}.$$

Let $g$ be a continuous solution to the equation $\overline{\partial}g = \pi_2^* \lambda$ on $B_R$, and suppose that $g$ satisfies the Hölder estimates given in equations (4) and (5) of Theorem 2. Recalling the analysis done in the paragraphs situated before Theorem 2, we know there exists a continuous function $h$ defined on $\pi_2(B_R)$ such that $h \circ \pi_2$ is equal to $\frac{g + \delta^* g}{2}$. In particular, $\overline{\partial} h = \lambda$ on $\pi_2(B_R)$, and

$$\|h\|_{\pi_2(B_R)} = \frac{\|g + \delta^* g\|_{B_R}}{2} \leq \|g\|_{B_R}.$$

Notice that $\beta = 1/2$ when $N = 2$. Given two points $x, w \in \pi_2(B_R)$, choose $z, \zeta \in B_R$ such that $x = \pi_2(z)$ and $w = \pi_2(\zeta)$. Since $\pi_2(\zeta) = \pi_2(-\zeta)$, we can even choose $\zeta \in B_R$ so that $\|z - \zeta\|$ is less than or equal to $\|z + \zeta\|$. If $z$ and $\zeta$ are both inside the ball $B_{R/2}$, we may apply equation (5) of Theorem 2 and the inequality $2\|x - w\| \geq \|z - \zeta\|^2$ given in Lemma 3 in order to get:

$$\frac{|h(x) - h(w)|}{\|x - w\|^{1/2}} \leq \frac{|g(z) - g(\zeta)| + |g(-z) - g(-\zeta)|}{\|z - \zeta\|} \leq C_3(R) \|\pi_2^* \lambda\|_{B_R}.$$
On the other hand, suppose, without lost of generality, that \( z \) is not inside the ball \( B_{R/2} \); that is \( ||z|| \geq \frac{R}{2} \). Lemma 3 implies then that \( ||x - w|| \) is greater than or equal to \( \frac{R}{4} ||z - \zeta|| \). Whence, equation (4) automatically implies the following,

\[
\frac{|h(x) - h(w)|}{||x - w||^{1/2}} \leq \frac{2}{\sqrt{R}} C^2(R) \pi^*_2 \lambda ||B_R||.
\]

Finally, considering Theorem 2 and equations (9) to (12), we can deduce the existence of a bounded positive constant \( C_1(R) \) such that equation (14) holds.

We close this section with some observations about Theorem 1. Firstly, the procedure presented in this section yields a continuous solution \( h \) to the equation \( \partial h = \lambda \). Moreover, we are directly using the estimates given in (1), but we may use any integration kernel which produces estimates similar to those presented in equations (1) and (5) of Theorem 2.

On the other hand, the extension of Theorem 1 to considering a general subvariety \( \Sigma \), with an isolated singularity, does not seem to be trivial. Theorem 1 demands the existence of a branched finite covering \( \pi: W \to \Sigma \), where \( W \) is a nice non-singular manifold and the inverse image of the singular point is a singleton. It does not seem to be trivial to produce such a branched finite covering.

3. Proof of Theorem 1 case \( N \geq 3 \).

We analyse in this section the general case of the variety \( X_N \subset \mathbb{C}^2 \) defined by \( x_1, x_2 = x_3^N \), for any natural number \( N \geq 3 \). Surface \( X_N \) has an isolated simple singularity of type \( A_{N-1} \) at the origin, (11) p. 60]. Define the automorphisms \( \phi_k: \mathbb{C}^2 \to \mathbb{C}^2 \), for each natural number \( k \),

\[
\phi_k(z_1, z_2) = (\rho_k^k z_1, \rho_N^k z_2) \quad \text{where} \quad \rho_N = e^{2\pi i/N}.
\]

Consider the natural branched covering \( \pi_N(z_1, z_2) = (z_1^N, z_2^N, z_1 z_2) \) defined from \( \mathbb{C}^2 \) onto \( X_N \). It is easy to see that \( \pi_N \) is a branched \( N \)-covering, and that the origin is the only branch point of \( \pi_N \), because the inverse image \( \pi_N^{-1}(x) \) is a set of the form \( \{ \phi_k(z) \}_{1 \leq k \leq N} \), for every \( x \in X_N \). Thus, the automorphisms \( \phi_k \) allow us to jump between the different branches of \( \pi_N \). In particular, we have that \( \pi_N = \phi_k^* \pi_N \) for every \( k \). Besides, the operators \( \pi_N^* \) and \( \partial \) commute, the proof is based on the same ideas presented at the beginning of section two.

Given a \( \partial \)-closed \((0, 1)\)-differential form \( \lambda \) defined on \( X_N \), we obviously have that \( \partial(\pi_N^* \lambda) = 0 \). Suppose the differential equation \( \partial \phi = \pi_N^* \lambda \) has a solution \( \phi \) in \( \mathbb{C}^2 \). The sum \( \frac{1}{N} \sum_{k=1}^N \phi_k^* g \) is constant in the fibres of \( \pi_N \) (it is invariant under every pull back \( \phi_k^* \)), so there exists a continuous function \( h \) on \( X_N \) such that \( \pi_N^* h = \frac{1}{N} \sum \phi_k^* g \). We assert that \( \partial h = \lambda \) on \( X_N \). This result follows automatically because,

\[
\pi_N^* \partial h = \frac{1}{N} \sum \partial \phi_k^* g = \frac{1}{N} \sum \phi_k^* \pi_N^* \lambda = \pi_N^* \lambda.
\]

Let \( B_R \subset \mathbb{C}^2 \) be an open ball of radius \( R > 0 \). If \( \lambda \) is continuous on the compact set \( \pi_N(B_R) \), then, we can apply Theorem 2 in order to get a solution \( h \) which satisfies a Hölder estimate on the ball \( B_R \). Obviously, like in the second section of this paper, the central part of the proof is an estimate of the distance \( ||z - \zeta|| \) with respect to the projections \( ||\pi_N(z) - \pi_N(\zeta)|| \). This estimate is done in the next lemma. Given two points \( z \) and \( \zeta \) in \( \mathbb{C}^2 \), notation \( ||z, \zeta||_\infty \) stands for the maximum of \( ||z_1||, ||z_2||, ||z_1|| \) and \( ||z_2|| \). Moreover, \( ||z||_\infty := ||z, 0||_\infty \) as well.
Lemma 4. Let $\text{Ev}(N)$ be the smallest even integer greater than or equal to $N$. Given two points $z$ and $\zeta$ in $\mathbb{C}^2$ such that $\|z - \phi_k(\zeta)\|_\infty$ is greater than or equal to $\|z - \zeta\|_\infty$ for every automorphism $\phi_k$ defined in (13), the following inequality holds for $\delta$ equal to both $N$ and $\text{Ev}(N)/2$.

\begin{equation}
\|\pi_N(z) - \pi_N(\zeta)\| \geq \min \left\{ \frac{\|z - \zeta\|_\infty^2}{12}, \frac{\|z, \zeta\|_\infty^2 - \delta}{(8/3)N - \delta} \right\}.
\end{equation}

Proof. Set $z = (a, b)$, so that $\pi_N(z) = (a^N, b^N, ab)$. Moreover, given $\zeta = (s, t)$, we can suppose without loss of generality that $|a - s| \geq |b - t|$, and so $\|z - \zeta\|_\infty = |a - s|$. We shall prove inequality (14) by considering three cases.

Case I. Whenever $|b| \geq |s| + \frac{|a - s|}{12}$, we have the inequality,

$$|ab - st| \geq |a - s| \cdot |b| - |b - t| \cdot |s| \geq \frac{|a - s|^2}{12}.$$ 

Finally, notice that $\|\pi_N(z) - \pi_N(\zeta)\| \geq |ab - st|$, so equation (14) holds in this particular case.

Case II. If $|b| \leq |s| + \frac{|a - s|}{12}$, and there exists a natural $j$ such that $|a - \rho^j_N s|$ is less than or equal to $\frac{|a - s|}{24}$, we also have,

$$|a - s| \leq |a - \rho^j_N s| + 2|s| \leq \frac{|a - s|}{2} + 2|s|.$$ 

Consequently, $|s| \geq \frac{|a - s|}{12}$. On the other hand, we know that $\|z - \phi_j(\zeta)\|_\infty$ is equal to the maximum of $|a - \rho^j_N s|$ and $|\rho^j_N b - t|$. Recalling the hypotheses of Lemma 3 and this case (II), we have that $|a - \rho^j_N s| < |a - s|$, and that $\|z - \phi_j(\zeta)\|_\infty$ is greater than or equal to $\|z - \zeta\|_\infty = |a - s|$. Hence, both $|\rho^j_N b - t| \geq |a - s|$ and:

$$|ab - st| \geq |\rho^j_N b - t| \cdot |s| - |a - \rho^j_N s| \cdot |b| \geq \frac{|a - s|}{2} - \frac{|a - s|^2}{24} \geq \frac{|a - s|^2}{12}.$$ 

Notice that $|a - \rho^j_N s| \cdot |b|$ is less than or equal to $\frac{|a - s|}{2} + \frac{|a - s|^2}{24}$ because of the hypotheses of this case (II). We may conclude that equation (14) holds in this particular case as well, after recalling that $\|\pi_N(z) - \pi_N(\zeta)\|$ is greater than or equal to $|ab - st|$.

Case III. If $|b| \leq |s| + \frac{|a - s|}{12}$, and $|a - \rho^k_N s| \geq \frac{|a - s|}{2}$ for every natural $k$, we automatically have the following inequality,

$$|a^N - s^N| = \prod_{k=1}^{N} |a - \rho^k_N s| \geq \frac{|a - s|^N}{2^N}.$$ 

Finally, we know that $\|z - \zeta\|_\infty = |a - s|$, and that $\|\pi_N(z) - \pi_N(\zeta)\|$ is greater than or equal to $|a^N - s^N|$. The previous inequalities show that equation (14) holds for $\delta = N$. On the other hand, when $\delta = \text{Ev}(N)/2$, it is easy to deduce the existence of a subset $J$ of $\{1, 2, \ldots, N\}$ composed of at least $N - \delta$ elements and which satisfies,

\begin{equation}
|a - \rho^j_N s| \geq \max \left\{ |a|, |s|, \frac{|a - s|}{\sqrt{2}} \right\} \text{ for each } j \in J.
\end{equation}

The set $J$ can be built as follows. We may suppose, without lost of generality, that $a$ is real and $a \geq 0$, for we only need to multiply both $a$ and $s$ by an appropriate
and the hypotheses of this case (III), we can deduce the desired result.

Finally, considering all the results presented in previous paragraphs, equation (15), directly imply that:

\[ 2|a - \rho_N^j s| \geq |s| + \frac{|a - s|}{\sqrt{2}} \geq |b|, \quad \forall j \in J. \]

Moreover, since \( \frac{5}{3} > \frac{13}{12} \sqrt{2} \), and we are supposing from the beginning of this proof that \( |a - s| \geq |b - t| \), we may also deduce the following inequality,

\[ \frac{8|a - \rho_N^j s|}{3} > |s| + \frac{|a - s|}{12} + |b - t| \geq |t|, \quad \forall j \in J. \]

Finally, considering all the results presented in previous paragraphs, equation (16) and the hypotheses of this case (III), we can deduce the desired result,

\[ |a^N - s^N| = \prod_{k=1}^{N} |a - \rho_N^k s| \geq \frac{\|z, \zeta\|^{N-\delta} |a - s|^\delta}{(8/3)^N \delta 2^\delta} \]

where \( \delta = Ev(N)/2 \), the norm \( \|z - \zeta\|_\infty = |a - s| \) and \( \|z, \zeta\|_\infty \) is the maximum of \( |a|, |b|, |s| \) and \( |t| \). We can conclude that equation (14) holds when \( \delta \) is equal to \( N \) and \( Ev(N)/2 \).

We are now in position to complete the proof of Theorem 1. Notice that Lemma 4 automatically implies the following inequalities, whenever \( z \) and \( \zeta \) lie inside the compact ball \( B_R \), and \( \delta = N \),

\[ \|\pi_N(z) - \pi_N(\zeta)\| \geq \frac{\|z - \zeta\|^N}{2^N} \min \left\{ \frac{1}{3 R^{N-2}}, 1 \right\} \]

\[ \geq \frac{\|z - \zeta\|^N}{\sqrt{8^N}} \min \left\{ \frac{1}{3 R^{N-2}}, 1 \right\}. \]

**Proof.** (Theorem 1 case \( N \geq 3 \)). We shall follow step by step the proof of Theorem 1 (case \( N = 2 \)), presented in section two; so we shall only indicate the main differences. Let \( g \) be a continuous solution to the equation \( \overline{\partial}g = \pi^* \lambda \) on \( B_R \) which satisfies the Hölder estimates given in equations (4) and (5). Recalling the analysis done at the beginning of this section, we know there exists a continuous function \( h \) defined on \( \pi(B_R) \) such that \( h \circ \pi = \pi^* \lambda \) on \( B_R \), and \( \|h\|_{\pi(N(B_R))} \) is less than or equal to \( \|g\|_{B_R} \). Moreover, working like in equations (8) and (9), we may deduce the existence of a finite positive constant \( C_5(R) \) such that \( \|\pi_N \lambda\|_{B_R} \) is less than or equal to \( C_5(R) \|\lambda\|_{\pi(N(B_R))} \).

Given two points \( x, w \in \pi(B_R) \), choose \( z, \zeta \in B_R \) such that \( x = \pi_N(z) \) and \( w = \pi_N(\zeta) \). Since \( \pi_N(\zeta) = \pi_N(\phi_k(\zeta)) \) for every automorphism \( \phi_k \) defined in (13), we can even choose \( \zeta \in B_R \) so that \( \|z - \zeta\|_\infty \) is less than or equal to \( \|z - \phi_k(\zeta)\|_\infty \) for every \( \phi_k \). A direct application of Lemma 4 with \( \delta = N \), yields the existence of a finite positive constant \( C_5(R) \) such that \( \|x - w\| \) is greater than or equal to \( C_5(R) \|z - \zeta\|^{1/\beta} \). Recall that \( \beta = 1/Ev(N) \) and \( N \geq 3 \). Thus, if \( z \) and \( \zeta \) are both
inside the ball $B_{R/2}$, we may apply equation (4), in order to deduce that
\[
\frac{|h(x) - h(w)|}{|x - w|^\beta}
\]
is less than or equal to $\frac{C_3(R)}{C_3(R)} \|\pi^*\lambda\|_{B_{R}}$.

On the other hand, suppose, without lost of generality, that $z$ is not inside the ball $B_{R/2}$. A direct application of Lemma 3 with $\delta = \frac{\delta(N)}{2} = \frac{1}{2^3}$, yields the existence of a finite positive constant $C_7(R)$ such that $\|x - w\|$ is greater than or equal to $C_7(R)\|z - \zeta\|^\delta$. Whence, equation (3) automatically implies that
\[
\frac{|h(x) - h(w)|}{|x - w|^\beta}
\]
is less than or equal to $\frac{C_3(R)}{C_3(R)} \|\pi^*\lambda\|_{B_{R}}$ as well.

The analysis done in the previous paragraphs automatically implies the existence of a finite positive constant $C_1(R)$ such that equation (4) holds for every $N \geq 3$. $\square$

Finally, as we have already said at the end of section two, the proof of Theorem 1 works perfectly if we apply Theorem 2 of Henkin and Leiterer, or any other integration kernel which produces estimates similar to those posed in equations (4).

On the other hand, suppose, without lost of generality, that $z$ is not inside the ball $B_{R/2}$. A direct application of Lemma 3 with $\delta = \frac{\delta(N)}{2} = \frac{1}{2^3}$, yields the existence of a finite positive constant $C_7(R)$ such that $\|x - w\|$ is greater than or equal to $C_7(R)\|z - \zeta\|^\delta$. Whence, equation (3) automatically implies that
\[
\frac{|h(x) - h(w)|}{|x - w|^\beta}
\]
is less than or equal to $\frac{C_3(R)}{C_3(R)} \|\pi^*\lambda\|_{B_{R}}$ as well.

The analysis done in the previous paragraphs automatically implies the existence of a finite positive constant $C_1(R)$ such that equation (4) holds for every $N \geq 3$. $\square$

Finally, as we have already said at the end of section two, the proof of Theorem 1 works perfectly if we apply Theorem 2 of Henkin and Leiterer, or any other integration kernel which produces estimates similar to those posed in equations (4) and (5). For example, the hypotheses on $\lambda$ can be relaxed in Theorem 1, to consider $(0, 1)$-differential forms $\lambda$ which are bounded and continuous on $\pi(B_{R}) \setminus K$, for some compact set $K \subset B_{R}$ of zero-measure. Besides, the results presented in Theorem 1 hold as well, if we consider an arbitrary strictly pseudoconvex domain $D$, with smooth boundary and the origin in its interior, instead of the open ball $B_{R}$. In this case, the ball $B_{B_{R}/2}$ used in equation (4) of Theorem 2 would be a sufficiently small ball $B_{r}$ whose closure is contained in the interior of $D$.  

On the other hand, the work presented in this paper is strongly based on the existence of a branched finite covering $\pi_{N}$ from $C^{2}$ onto $X_{N}$, such that the inverse image of the singular point $\pi_{N}^{-1}(0) = \{0\}$ is a singleton. This property allows us to get the estimates presented in Lemmas 3 and 4, which are essential for this paper. It is obvious to consider a blow-up mapping $\eta : W \to X_{N}$ instead of the finite covering $\pi_{N}$. In any case, a blow-up is a 1-covering everywhere, except at the singular point $0$. However, since the inverse image $\eta^{-1}(0)$ is not a singleton, and it is not even finite in general, we have strong problems for calculating a Hölder solution to the equation $\overline{\partial}h = \lambda$, unless we introduce stronger hypotheses. We finish this section by analysing the case of a blow-up.

**Remark.** Let $\Sigma$ be a variety with an isolated singularity at $\sigma_{0} \in \Sigma$, and $\eta : W \to \Sigma$ be a holomorphic blow-up of $\Sigma$ at $\sigma_{0}$, such that $W$ is a smooth manifold. Given a $\overline{\partial}$-closed $(0, 1)$-form $\lambda$ defined on $\Sigma$ minus $\sigma_{0}$, we automatically have that $\eta^*\lambda$ is also $\overline{\partial}$-closed on $W$ minus $\eta^{-1}(\sigma_{0})$. Thus, suppose there exists a continuous solution $g : W \to C$ to the equation $\overline{\partial}g = \eta^*\lambda$. Since $\eta$ is a blow-up, we automatically have that $\eta^{-1}$ is well defined on $\Sigma \setminus \{\sigma_{0}\}$, and so $\lambda$ is equal to $\overline{\partial}(g \circ \eta^{-1})$ there.

Define $h := g \circ \eta^{-1}$. Unless $g$ is constant on the inverse fibre $\eta^{-1}(\sigma_{0})$, the function $h$ does not have a continuous extension to $\sigma_{0}$, and does not satisfy any Hölder condition in a neighbourhood of $\sigma_{0}$. Suppose there exists a pair of points $a$ and $b$ in $\eta^{-1}(\sigma_{0})$ such that $g(a) \neq g(b)$. Besides, take $\{a_{m}\}$ and $\{b_{m}\}$ a pair of infinite sequences in $W \setminus \eta^{-1}(\sigma_{0})$ which respectively converge to $a$ and $b$. Notice that both $\eta(a_{m})$ and $\eta(b_{m})$ converge to the same point $\sigma_{0}$. However, $g(a_{m})$ and $g(b_{m})$ converge to different points, for $g(a) \neq g(b)$. Hence, given a metric $\Delta$ on $\Sigma$, which defines the topology, we have that:
\[
\limsup_{m} \frac{|h \circ \eta(a_{m}) - h \circ \eta(b_{m})|}{\Delta[\eta(a_{m}), \eta(b_{m})]^\beta} = \infty, \quad \forall \beta > 0.
\]
That is, in order to introduce Hölder conditions on $h := g \circ \eta^{-1}$, it is essential that the solution to equation $\overline{\partial}g = \eta^*\lambda$ is constant on the inverse fibre of the singular point $\eta^{-1}(\sigma_0)$.

4. SURFACES WITH SIMPLE SINGULARITIES OF TYPE $D_{N+2}$

We finish this paper by solving the $\overline{\partial}$-equation on a neighbourhood of the origin in the subvariety $Y_N \subset \mathbb{C}^4$, defined by the polynomial $y_1^4 + y_2^2 = y_3^{N+1}$. The surface $Y_N$ has an isolated simple (rational double point) singularity of type $D_{N+2}$ at the origin, for any natural number $N \geq 2$, [1] p. 60. We extend the results presented in theorem 1 by introducing a branched 2-covering defined from $X_N := [x_1 x_2 = x_3^2 N]$ onto the surface $Y_N$. Consider the holomorphic mapping $\eta_2 : X_2 \to Y_2$, and the pair of matrices $P$ and $Q$, given by the respective equations,

$$
\eta_2(x_1, x_2, x_3) = \left( \frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2i}, x_3 \right),
$$

$$
P = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix},
Q = \frac{1}{2} \begin{bmatrix}
1 & 1 & 0 \\
-i & i & 0 \\
0 & 0 & 2
\end{bmatrix}.
$$

It is easy to see that $\eta_2(x) = \eta_2(Px)$ for every $x \in X_2$. Moreover, $\eta_2$ is a branched 2-covering, and the origin is the unique branch point, because the inverse image $\eta_2^{-1}(y)$ is a set of the form $\{x, Px\}$, for every $y \in Y_2$. For example, the inverse image of $(y_1, 0, 0)$ is composed of two points: $(2y_1, 0, 0)$ and $(0, 2y_1, 0)$. We have already defined a branched covering $\pi_2 \eta_2$ from $\mathbb{C}^2$ onto $X_2$, so the composition $\eta_2 \circ \pi_2 \eta_2$ is indeed a covering from $\mathbb{C}^2$ onto $Y_2$. The branch covering $\eta_2$ is a central part in the following result.

**Theorem 5.** Given an open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre in the origin, define the open set $E_R := \eta_2(\pi_2(B_R))$ in $Y_N$. There exists a finite positive constant $C_11(R)$ such that: for every continuous $(0,1)$-differential form $\mathfrak{K}$ defined on the compact set $\overline{E_R} \subset Y_N$, and $\overline{\partial}$-closed on $E_R$, the equation $\overline{\partial}f = \mathfrak{K}$ has a continuous solution $f$ on $E_R$ which also satisfies the following Hölder estimate, with $\beta_2 = \frac{1}{N}$,

$$
\|f\|_{E_R} + \sup_{y, \xi \in E_R} \frac{|f(y) - f(\xi)|}{\|y - \xi\|^{\beta_2}} \leq C_11(R)\|\mathfrak{K}\|_{E_R}.
$$

The proof of this theorem follows exactly the same ideas and steps presented in the proof of Theorem 1 (case $N = 2$), so we do not include it. Given a $(0,1)$-differential form $\mathfrak{K}$ continuous on $E_R \subset Y_N$, and $\overline{\partial}$-closed on $E_R$. We have that $\eta_2^* \mathfrak{K}$ is also continuous on $\pi_2(B_R)$, and $\overline{\partial}$-closed on $\pi_2(B_R)$. Therefore, we can apply Theorem 1 in order to obtain a continuous solution $h$ to the differential equation $\overline{\partial}h = \eta_2^* \mathfrak{K}$, which also satisfies the Hölder conditions given in equation 11. There exists a continuous function $f$ on $E_R$ such that $\eta_2^* f$ is equal to $\frac{\overline{\partial}f}{\beta_2}$, and so $\overline{\partial} f = \mathfrak{K}$, as we wanted. Finally, inequality 15 follows from equation 11, after noticing that there exists a pair of finite positive constants $C_8(R)$ and $C_9(R)$ such that:

$$
\|f\|_{E_R} \leq \|h\|_{\pi_2(B_R)},
\|\eta_2^* \mathfrak{K}\|_{\pi_2(B_R)} \leq C_8(R)\|\mathfrak{K}\|_{E_R},
$$

and $\frac{|f(y) - f(\xi)|}{\|y - \xi\|^{\beta_2}}$ is also less than or equal to $C_9(R)\|\eta_2^* \mathfrak{K}\|_{\pi_2(B_R)}$ for every $y$ and $\xi$ in $E_R$. We obviously need an estimate of $\|x - w\|^2$, with respect to the projections
\[ \| \eta_2(x) - \eta_2(w) \|, \] in order to show that the previous inequality above holds. This estimate is presented in the following Lemma 6. In conclusion, the proof of Theorem 5 follows the same ideas and steps of the proof of Theorem 1 (case \( N = 2 \)), we only need to apply Theorem 1 instead of Theorem 2, and the following Lemma 6 instead of Lemma 3.

**Lemma 6.** Let \( x \) and \( w \) be two points in \( X_{2N} \) whose norms \( \| x \| \) and \( \| w \| \) are both less than or equal to a finite constant \( \rho > 0 \). If the distance \( \| Q(w - Px) \| \) is greater than or equal to \( \| Q(w - x) \| \), for the matrices \( P \) and \( Q \) defined in (17), then, the following inequality holds.

\[
\begin{align*}
\| \eta_2(x) - \eta_2(w) \| & \geq C_{12}(\rho) \| x - w \|^2, \\
\text{where} \quad C_{12}(\rho) & = \frac{1}{80} \min \left\{ 4, \frac{5}{3\rho}, \frac{1}{\rho^{2N-2N}} \right\}.
\end{align*}
\]

**Proof.** Introducing the new variables \((a, b, c) := Qx \) and \((s, t, u) := Qw \), we have that \( a^2 + b^2 = c^{2N} \) and \( QPx = (a, -b, -c) \) for every \( \rho \in X_{2N} \). Moreover,

\[
\| \eta_2(x) - \eta_2(w) \| = |a - s|^2 + |b - t|^2 + |c - u|^2.
\]

A main step in this proof is to shown that the following inequality holds,

\[
\| \eta_2(x) - \eta_2(w) \| \geq 16C_{12}(\rho) \| \pi_2(b, c) - \pi_2(t, u) \|,
\]

where \( \pi_2(b, c) = (b^2, c^2, bc) \) was defined in the introduction of this paper, and \( C_{12}(\rho) \) is given in equation (19) above. We know that \( \| Q(w - Px) \| \) is greater than or equal to \( \| Q(w - x) \| \), according to the hypotheses of this lemma, so it is easy to deduce that \( \| t + b, u + c \| \) is also greater than or equal to \( \| t - b, u - c \| \), because \( QPx \) is equal to \( (a, -b, -c) \). Therefore, if equation (21) holds, a direct application of Lemma 3 yields,

\[
\| \eta_2(x) - \eta_2(w) \| \geq 8C_{12}(\rho)(|b - t|^2 + |c - u|^2).
\]

On the other hand, we can easily calculate the following upper bound for \( |a| \),

\[
|a| \leq \| Qx \| \leq \| x \| \leq \rho.
\]

A similar upper bound \( |s| \leq \rho \) holds as well. Hence, recalling equation (20), we have that \( \| \eta_2(x) - \eta_2(w) \| \) is greater than or equal to \( |a - s| \geq \frac{|a - s|^2}{2\rho} \). Adding together the inequality presented in previous statement and equation (22) yields the desired result, notice that \( \frac{1}{2\rho} > 8C_{12}(\rho) \) and \( 2\| \xi \| \geq \| Q^{-1}\xi \| \) for \( \xi \in \mathbb{C}^3 \),

\[
2\| \eta_2(x) - \eta_2(w) \| \geq 8C_{12}(\rho) \| Q(x - w) \|^2 \geq 2C_{12}(\rho) \| w - x \|^2.
\]

We may then conclude that inequality (19) holds, as we wanted. We only need to prove that equation (24) is always satisfied, in order to finish our calculations; and we are doing to prove that by considering two complementary cases.

**Case I** Whenever \( 3|c^2 - u^2| \) is greater than or equal to \( \frac{|b^2 - t^2|}{2\rho^{2N-2N}} \), the following inequality holds,

\[
|c^2 - u^2|^2 \geq \frac{16|c^2 - u^2|^2}{25} + \frac{|b^2 - t^2|^2}{(5\rho^{2N-2N})^2}.
\]

Thus, in this particular case, inequality (24) follows directly from equation (20), because \( \frac{1}{2\rho^{2N-2N}} \) and \( 4/5 \) are both greater than or equal to \( 16C_{12}(\rho) \).
Case II Whenever $|b^2 - t^2|/3$ is greater than or equal to $3|c^2 - u^2|$, we proceed as follows. The absolute values $|a|$ and $|c|$ are both bounded by $\|Qx\| \leq \rho$, according to equation (23); the same upper bound can be calculated for $|s|$ and $|u|$. Whence, the following series of inequalities hold:

$$2|b^2 - t^2|/3 \leq |b^2 - t^2| - \rho^{2N-2}N|c^2 - u^2|$$

$$\leq |b^2 - t^2| - |c^{2N} - u^{2N}|$$

$$\leq |a^2 - s^2| \leq 2\rho|a - s|.$$

Recall that $a^2 + b^2 = c^{2N}$ and that $(\xi^N - 1)$ is equal to the product of $(\xi - 1)$ times the sum $\sum_{k=0}^{N-1} \xi^k$. Inequality (21) follows then from equation (20), after noticing that $16C_{12}(\rho)|b^2 - t^2|$ is less than or equal to $|a - s|$, and obviously, $16C_{12}(\rho)$ is also less than one.

References

[1] A. Dimca. Singularities and topology of hypersurfaces, (Universitext). Springer-Verlag, New York, 1992.
[2] E. A. Gavosto. Hölder estimates for the $\overline{\partial}$-equation in some domains of finite type. J. Geom. Anal. 7 (1997), no. 4, pp. 593–609.
[3] J. E. Fornæss; E. A. Gavosto. The Cauchy Riemann equation on singular spaces. Duke Math. J. 93 (1998), no. 3, pp. 453–477.
[4] G. Henkin; J. Leiterer. Theory of functions on complex manifolds, (Monographs in Mathematics, 79). Birkhäuser-Verlag, Basel, 1984.

DEPTO. MATEMÁTICAS, CINVESTAV, APARTADO POSTAL 14-740, MÉXICO D.F., 07000, MÉXICO.

E-mail address: facosta@math.cinvestav.mx
E-mail address: eszeron@math.cinvestav.mx