Supersymmetric D2 anti-D2 Strings

Dongsu Bak\textsuperscript{a} and Nobuyoshi Ohta\textsuperscript{b}

\textsuperscript{a}Physics Department, University of Seoul, Seoul 130-743, Korea
\textsuperscript{b}Department of Physics, Osaka University, Toyonaka, Osaka 560-0043, Japan

(dsbak@mach.uos.ac.kr, ohta@phys.sci.osaka-u.ac.jp)

We consider the flat supersymmetric D2 and anti-D2 system, which follows from ordinary non-commutative D2 anti-D2 branes by turning on an appropriate worldvolume electric field describing dissolved fundamental strings. We study the strings stretched between D2 and anti-D2 branes and show explicitly that the would-be tachyonic states become massless. We compute the string spectrum and clarify the induced noncommutativity on the worldvolume. The results are compared with the matrix theory description of the worldvolume gauge theories.
1 Introduction

Recently an interesting class of 1/4 BPS brane configurations has been constructed including the supertubes and the supersymmetric brane-antibrane system [1, 2, 3, 4, 5]. The supertubes are tubular configurations embedded in flat 10 dimensional space and self supported from collapse by the contribution of angular momentum produced by their own worldvolume gauge fields [1]. Many tubes separated parallelly are again 1/4 BPS and there is no static force between them, which is shown in the matrix theory description [2, 4] or the supergravity analysis [3]. The tubes involve a critical electric field as well as a magnetic field, which respectively correspond to dissolved fundamental strings and D0 branes. The tubes have no net D2 brane charge. Instead they carry nonvanishing dipole components of D2 charges [3, 4].

The elliptic deformation of tubes discussed in Ref. [5] still preserves 1/4 supersymmetries and, in the limit where the ellipse becomes two parallel lines, the system becomes a flat D2 and anti-D2 system which is supersymmetric. (One may actually show that there are many other supersymmetric solutions with various shapes like hyperbola for example.) As long as the E-fields are critical and B-fields come with opposite sign, the supersymmetries satisfied by the brane turn out to be the same as those of the antibrane. By the matrix theory description of the worldvolume theory, one can indeed prove that the would-be tachyons in the ordinary D2 and anti-D2 system with B-fields disappear in the supersymmetric case [5]. Thus the system is stable against decay. In the corresponding supergravity solutions, the D2 and anti-D2 can easily be identified separately when they are separated. One might ask what happens when D2 and anti-D2 are brought together and become coincident. Clearly in this limit the dipole moment of the D2 charges becomes zero and massless modes are expected to appear between D2 and anti-D2. When D2 and anti-D2 are on top of each other, should we view that they disappear to nothing? In the matrix model description, however, nothing particular happens other than the massless degrees of freedom and the corresponding nonabelian structure due to brane and antibrane remains.

In this note, we would like to probe the coincident limit of the supersymmetric D2 and anti-D2 system by studying the worldsheet CFT of strings connecting the D2 and anti-D2. Without E-field the tachyonic modes of D2 anti-D2 strings are computed in Ref. [6] and shown to agree with those of the matrix theory description [3, 4, 5] in the zero slope limit. We begin with a generic value of E-field and show how would-be tachyonic modes disappear in the string spectrum when the E-field approach to the critical value. The limit we are taking is not that straightforward; the coordinate fields X become singular in their overall factors. However the modes and their correlation functions do not share these apparent singularity, so the worldsheet CFT is well defined in the limit in spite of the apparent coordinate-like singularity. We shall clarify the noncommutativity arising in the low-energy worldvolume theories by the analysis of the commutators of the coordinate fields. We also compute the fluctuation spectra from the matrix theory and show that a consistent picture emerges as a consequence.

In Section 2, we study the mode expansion and spectrum of the D2 anti-D2 strings as well as D0-D2 string spectrum. In Section 3, we investigate the noncommutativity induced by the background gauge field on D2 and anti-D2 and discuss the Seiberg-Witten limit. We compare the above results to those of the matrix theory analysis focused on the fluctuation spectra. Last section comprises

*The critical value of the electric field here does not mean the tensionless limit due to the nonvanishing magnetic components.
conclusions and remarks.

2 D2 anti-D2 Strings

We are interested in the D2-anti-D2 system extended to 1, 2 directions, which is supersymmetric due to the presence of the specific background $E$ and $B$-field $B$. For the supersymmetric configurations, the electric component, e.g. $E = B_{20}$ should be critical and the magnetic part $B = B_{12}$ should come with opposite signs on D2 and anti-D2. We begin with generic values of electric components and take the critical limit, i.e. $E = 1/(2\alpha')$. Also we take $B$ to have the same magnitude on the D2 and anti-D2. Since $B$ comes with opposite signature on the D2 and anti-D2, only D0-branes (not D0) are induced on them with the same densities. The same magnitude condition of $B$-field may be relaxed as far as they are nonvanishing on the D2 and anti-D2.

Let us examine the open string spectrum between the D2 and anti-D2. The background field at $\sigma = 0$ is specifically taken as

$$B^{(0)} = \frac{1}{2\pi\alpha'}\begin{pmatrix} 0 & 0 & -e \\ 0 & 0 & -b \\ e & b & 0 \end{pmatrix}; \quad g_{ij} = \eta_{ij},$$

and $B^{(\pi)}$ at $\sigma = \pi$ is obtained by reversing the sign of $b$. The boundary condition is given as

$$g_{ij}\partial_{\sigma}X^j + 2\pi\alpha' B_{ij}\partial_t X^j = 0. \quad (2)$$

In our case of D2 at $\sigma = 0$ and anti-D2 at $\sigma = \pi$, this is written as

$$\partial_{\sigma}X^0 + e\partial_t X^2 = 0, \quad \partial_{\sigma}X^1 \mp b\partial_t X^2 = 0, \quad \partial_{\sigma}X^2 + e\partial_t X^0 \pm b\partial_t X^1 = 0, \quad \text{at} \quad \sigma = \begin{cases} 0, \\ \pi. \end{cases} \quad (3)$$

The boundary conditions for fermions are similar and the mode oscillators can be easily obtained once the bosonic part is found.

With the specified boundary conditions, we solve the equation of motion and find

$$X^0 = i \sqrt{\frac{2\alpha'}{1 - e^2}} \sum_{n \neq 0} \frac{a_n e^{-int} \cos n\sigma}{n} + \frac{1}{\sqrt{1 - e^2}} (2\alpha' p^0 t + x^0)
- e \sqrt{\frac{\alpha'}{1 - e^2}} \sum_{n \neq 0} \left( \frac{c_{n+\nu} e^{-i(n+\nu)t - i\frac{n\pi}{2}} + d_{n-\nu} e^{i(n+\nu)t + i\frac{n\pi}{2}}}{n + \nu} \right) \sin \left[ (n + \nu)\sigma - \frac{\pi\nu}{2} \right],$$

$$X^1 = \sqrt{\alpha'} \sum \left( \frac{c_{n+\nu} e^{-i(n+\nu)t - i\frac{n\pi}{2}} + d_{n-\nu} e^{i(n+\nu)t + i\frac{n\pi}{2}}}{n + \nu} \right) \cos \left[ (n + \nu)\sigma - \frac{\pi\nu}{2} \right] + x^1, \quad (4)$$

$$X^2 = -e \sqrt{\frac{2\alpha'}{1 - e^2}} \sum_{n \neq 0} \frac{a_n e^{-int} \sin n\sigma}{n} - \frac{e}{\sqrt{1 - e^2}} 2\alpha' p^0 \left( \sigma - \frac{\pi}{2} \right) + x^2
+ i \sqrt{\frac{\alpha'}{1 - e^2}} \sum \left( \frac{c_{n+\nu} e^{-i(n+\nu)t - i\frac{n\pi}{2}} - d_{n-\nu} e^{i(n+\nu)t + i\frac{n\pi}{2}}}{n + \nu} \right) \cos \left[ (n + \nu)\sigma - \frac{\pi\nu}{2} \right],$$

2
where $\nu$ is defined by

$$\tan \frac{\pi \nu}{2} = \frac{b}{\sqrt{1 - e^2}}, \quad (0 \leq \nu < 1).$$

Here $c_{n+\nu}$'s are complex oscillator and the reality condition of the mode expansion says that $a_n = a_{-n}^\dagger$ and $c_{n+\nu} = d_{-n-\nu}^\dagger$.

This result shows that $\nu = 1$ for $e = 1$. This implies that the level becomes integer, which is in accordance with the supersymmetry restoration in the limit $e \rightarrow 1$. Furthermore if one takes the limit $e \rightarrow 0$, Eq. (4) reduces to those in Ref. [6] for the ordinary noncommutative D2 and anti-D2 system which is of course unstable. Though the $e \rightarrow 1$ limit for $X^0$ and $X^2$ looks singular in Eq. (4), there is no singularity in the worldsheet CFT. In fact, the mode expansions perfectly make sense in the combinations

$$\left(\partial_\sigma X^0 + e \partial_t X^2\right)/\sqrt{1 - e^2}, \quad \left(e \partial_\sigma X^0 + \partial_t X^2\right)/\sqrt{1 - e^2},$$

$$\left(e \partial_\sigma X^0 + \partial_t X^2\right)/\sqrt{1 - e^2}, \quad \left(\partial_t X^0 + e \partial_\sigma X^2\right)/\sqrt{1 - e^2},$$

even in the limit. This is just like considering, in CFT, $\partial \sigma X^i$, which has good conformal property. Alternatively, one may explicitly compute the correlation functions $\langle X^i(z) X^j(z') \rangle$ and verify that they are nonsingular in the limit $e \rightarrow 1$.

Upon quantization, we posit the nonvanishing commutation relations between oscillators to be

$$[a_m, a_n] = -m \delta_{m+n}, (m \neq 0); \quad [x^0, p^0] = -i; \quad [c_{m+\nu}, d_{n-\nu}] = (m + \nu) \delta_{m+n}.$$  (7)

The justification of the quantization here is relegated to the appendix. One thing to note is that we have not yet specified the commutation relation between $x^i$'s and these will be fixed in the next section by requiring the equal-time commutators of $X^i(\sigma)$ to commute with each other for $0 < \sigma < \pi$. The corresponding Virasoro generators can be found as

$$L_m = \frac{1}{2} \sum_n \left(-a_{m-n}a_n + c_{m-n+\nu}d_{n-\nu} + d_{n-\nu}c_{m-n+\nu}\right).$$  (8)

with $L_0$ being the Hamiltonian of the system.

We define vacuum by $c_{n+\nu}|0\rangle = 0$ for $n \geq 0$ and $d_{n-\nu}|0\rangle = 0$ for $n > 0$. Hence the corresponding vacuum energy from $L_0$ becomes

$$E_\nu = -\frac{1}{2} \sum_{n=1}^{\infty} n + \frac{1}{2} \sum_{n=1}^{\infty} (n - \nu) + \frac{1}{2} \sum_{n=1}^{\infty} (n - (1 - \nu)) = +\frac{1}{24} + \left(\frac{1}{24} - \frac{1}{8}(2\nu - 1)^2\right).$$  (9)

Note that this is the contribution to the vacuum energy from $X^0, X^1, X^2$.

The vacuum energy for the Ramond sector is trivial; the bosonic one is canceled by the fermionic contribution and the total is zero. In the Neveu-Schwarz sector, the vacuum energy from the NS-fermions associated with 0, 1, 2 components is given by the substitution $\nu \rightarrow |\nu - 1/2|$. In the “light-cone” gauge, the contributions from $a_n$ and one transverse oscillator cancel with each other.
Summing the contributions of the rest of oscillators, we find the ground state energy

\[
E^{\text{total}}_\nu = \left( \frac{1}{24} - \frac{1}{8} (2\nu - 1)^2 \right) - \left( \frac{1}{24} - \frac{1}{8} (2|\nu - 1/2| - 1)^2 \right) - \frac{6}{24} - \frac{6}{48}
\]

\[
= -\frac{1}{4} - \frac{|\nu - 1/2|}{2}.
\] (10)

For \( \nu = 0 \), this is consistent with the vacuum energy of NS sector without background field.

The ground state energy (10) gives two lower states with energies

\[
E_1 = -\frac{1}{2} \nu, \quad \text{for} \quad 1 \geq \nu \geq \frac{1}{2},
\]

\[
E_0 = \frac{1}{2} (\nu - 1), \quad \text{for} \quad \frac{1}{2} \geq \nu \geq 0.
\] (11)

When \( \nu = 0 \), \( |E_0\rangle \) gives the true ground state and \( |E_1\rangle \) is the first excited state, but the energy changes when \( \nu \) is increased. For \( \nu \geq \frac{1}{2} \), \( |E_1\rangle \) becomes the true ground state and \( |E_0\rangle \) is the first excited state. For \( \nu = 0 \) and D2-D2, the ground state \( |E_0\rangle \) is projected out by GSO projection and \( |E_1\rangle \) is kept. However, our D2-D2 system has opposite GSO projection, and the state with \( E_1 \) is projected out and \( |E_0\rangle \) is kept, giving tachyonic state. This state becomes massless for \( \nu = 1 \), giving a stable system.

Let us consider what would be the spectrum at lower levels. The ground state is denoted by \( |\nu/2\rangle \). Corresponding to the mode oscillators \( c_{n+\nu} \) and \( d_{-n-\nu} \), we have fermionic oscillators \( \psi_{n+\nu} \) and \( \bar{\psi}_{-n-\nu} \). For \( \nu \geq \frac{1}{2} \), \( \psi_{\nu} \) and \( \bar{\psi}_{\nu} \) and transverse oscillators \( \psi_\nu(s = 3, \ldots, 8) \) give lower states (which remain after GSO projection)

\[
|\nu/2\rangle \equiv \psi_{-\nu+\frac{1}{2}} |\nu/2\rangle, \quad |3(1-\nu)/2\rangle \equiv \psi_{\nu-\frac{1}{2}} |\nu/2\rangle, \quad |(1-\nu)/2\rangle \equiv \bar{\psi}_{\nu-\frac{1}{2}} |\nu/2\rangle.
\] (12)

For \( \nu < 1 \), the first state is the ground state discussed above and gives tachyonic one. All these states give 8 massless for \( \nu = 1 \), and the level is degenerate with Ramond sector, in accordance with the restoration of supersymmetry. In Ref. 5, it was argued that 1/4 supersymmetry is restored. Our result of 8 massless bosonic states is consistent with this claim because the representation of 8 supercharges contains 2^4 states in total, half of which are bosonic.

It is interesting that the limit \( \nu = 1 \) can be achieved just by sending \( e \to 1 \) but \( b \) finite. In the absence of electric background, \( b \) must be sent to infinity in order to achieve supersymmetry. However, that is incompatible with the Seiberg-Witten limit to be discussed in the next section.

The physical picture of the system can be most easily understood by combining T-duality [3, 4, 9] and Lorentz boost. Let us make T-duality in the \( X^2 \)-direction, and then make the Lorentz boost in the same direction by

\[
\begin{pmatrix} X^{0'} \\ X^{2'} \end{pmatrix} = \begin{pmatrix} \cosh \beta & \sinh \beta \\ \sinh \beta & \cosh \beta \end{pmatrix} \begin{pmatrix} X^0 \\ X^2 \end{pmatrix}; \quad \cosh \beta = \frac{1}{\sqrt{1 - e^2}}, \quad \sinh \beta = \frac{e}{\sqrt{1 - e^2}}.
\] (13)

The boundary conditions (3) are then cast into

\[
\partial_\sigma X^{0'} = 0, \quad \partial_\sigma (\sqrt{1 - e^2} X^1 \mp b X^{2'}) = 0, \quad \text{at} \quad \sigma = \left\{ 0, \pi \right\}.
\] (14)
This means that the system consists of two D1-branes, one tilted in the \((X^1, X^{2'})\) plane by the angle \(\mp \nu\) at \(\sigma = 0\) and another by \(-\mp \nu\) at \(\sigma = \pi\). There is no supersymmetry in the tilted two D1-branes. In the limit \(\nu \to 1\), however, this system reduces to boosted parallel branes, and supersymmetry is restored. The number of restored supersymmetry cannot be determined by simply looking at these boundary conditions, as is usual for rotated brane configuration. It was determined \(1/4\) by Killing spinor analysis \([5]\).

We would like to comment upon the 0-2 string spectra at this point. We consider D0 at \(\sigma = 0\) and D2 at \(\sigma = \pi\), for which the boundary conditions become

\[
\begin{cases}
  \partial_\sigma X^0 = 0, \\
  \partial_\tau X^2 = 0, \\
  \partial_\tau X^1 = 0,
\end{cases}
\quad \text{at } \sigma = 0;
\begin{cases}
  \partial_\sigma X^0 + e\partial_\tau X^2 = 0, \\
  \partial_\sigma X^1 - b\partial_\tau X^2 = 0, \\
  \partial_\sigma X^2 + e\partial_\tau X^0 + b\partial_\tau X^1 = 0,
\end{cases}
\quad \text{at } \sigma = \pi. \tag{15}
\]

For this case, we find

\[
X^0 = i \sqrt{\frac{2\alpha'}{1 - e^2}} \sum_{n \neq 0} \frac{a_n}{n} e^{-in\nu} \cos n\sigma + \frac{1}{\sqrt{1 - e^2}} (2\alpha' p^0 t + x^0) \\
+ e \sqrt{\frac{\alpha'}{1 - e^2}} \sum_{n \neq 0} \left( \frac{c_{n+\nu'}}{n + \nu'} e^{-i(n+\nu')\sigma} + \frac{d_{n-\nu'}}{n + \nu'} e^{i(n+\nu')\sigma} \right) \cos(n + \nu')\sigma,
\]

\[
X^1 = \sqrt{\alpha'} \sum_{n \neq 0} \left( \frac{c_{n+\nu'}}{n + \nu'} e^{-i(n+\nu')\sigma} + \frac{d_{n-\nu'}}{n + \nu'} e^{i(n+\nu')\sigma} \right) \sin(n + \nu')\sigma + x^1, \tag{16}
\]

\[
X^2 = -e \sqrt{\frac{2\alpha'}{1 - e^2}} \sum_{n \neq 0} \frac{a_n}{n} e^{-in\nu} \sin n\sigma - e \sqrt{\frac{\alpha'}{1 - e^2}} 2\alpha' p^0 \left( \sigma - \frac{\pi}{2} \right) + x^2 \\
+ i \sqrt{\frac{\alpha'}{1 - e^2}} \sum_{n \neq 0} \left( \frac{c_{n+\nu'}}{n + \nu'} e^{-i(n+\nu')\sigma} - \frac{d_{n-\nu'}}{n + \nu'} e^{i(n+\nu')\sigma} \right) \sin(n + \nu')\sigma,
\]

where \(\nu'\) is defined by

\[
e^{2i\pi\nu'} = \frac{-\sqrt{1 - e^2} + ib}{\sqrt{1 - e^2} - ib} = e^{i\pi(\nu + 1)}, \quad (1/2 \leq \nu' < 1). \tag{17}
\]

Then the expressions for the commutation relation between oscillators, the Hamiltonian, the ground state, the excited spectrum take the same forms as before. So the ground states after GSO projection give 8 massless bosonic degrees again but with the newly defined \(\nu'\).

### 3 Noncommutatitivity and Seiberg-Witten Limit

We shall compute here the noncommutatitivity induced due to the background gauge fields of the D2 anti-D2 system and investigate the Seiberg-Witten decoupling limit involved with the system.

For this let us first compute the equal-time commutators of \(X^i(\sigma)\). Using the commutation relations in \([6]\), we have

\[
[X^i(\sigma), X^j(\sigma')] = [x^i, x^j] - \frac{2i\alpha'}{\sqrt{1 - e^2}} \sum_{n \neq 0} \frac{1}{n + \nu} \cos \left( (n+\nu)\sigma - \frac{\pi\nu}{2} \right) \cos \left( (n+\nu)\sigma' - \frac{\pi\nu}{2} \right) \tag{18}
\]
To evaluate this, we use the identities

\[
\sum_{n=-\infty}^{\infty} \frac{\cos n\theta}{n+a} = \frac{\pi}{\sin a\pi} \cos (a(2m+1)\pi - a\theta) \quad \text{for } 2m\pi \leq \theta \leq 2(m+1)\pi \\
\sum_{n=-\infty}^{\infty} \frac{\sin n\theta}{n+a} = \begin{cases} 
\frac{\pi}{\sin a\pi} \sin (a(2m+1)\pi - a\theta) & \text{for } 2m\pi < \theta < 2(m+1)\pi \\
0 & \text{for } \theta = m\pi 
\end{cases}
\]  

(19)

with an integer \(m\) and \(a \neq 0\). One then finds that

\[
\sum_{n=1}^{\infty} \frac{1}{n+\nu} \cos \left( (n+\nu)\sigma - \frac{\pi\nu}{2} \right) \cos \left( (n+\nu)\sigma' - \frac{\pi\nu}{2} \right) = \pi \cot \frac{\nu\pi}{2} \begin{cases} 
1 & \text{if } 0 < \sigma + \sigma' < 2\pi \\
\cos \nu\pi & \text{if } \sigma = \sigma' = 0, \pi 
\end{cases}
\]

(20)

By requiring the bulk contribution to be zero, we fix the commutator \([x^1, x^2]\) as

\[
x^1, x^2 = \frac{i\alpha'\pi}{b}.
\]

(21)

The commutation relation becomes

\[
[X^1(\sigma), X^2(\sigma')]|_{\sigma=\sigma'=0,\pi} = \frac{i2\alpha' b}{1+b^2-e^2},
\]

(22)

while vanishing in the bulk.

For the 0–2 commutator, we get

\[
[X^0(\sigma), X^2(\sigma')] = \frac{1}{\sqrt{e^2-1}} [x^0, x^2] + i \frac{2\alpha' e}{1-e^2} (\sigma' - \pi/2) + i \frac{2\alpha' e}{1-e^2} \sum_{n \neq 0} \frac{1}{n} \cos n\sigma \sin n\sigma' \\
+ i \frac{2\alpha' e}{1-e^2} \sum_{n} \frac{1}{n+\nu} \sin \left( (n+\nu)\sigma - \frac{\pi\nu}{2} \right) \cos \left( (n+\nu)\sigma' - \frac{\pi\nu}{2} \right).
\]

(23)

In addition to (19), we shall use also the identity

\[
\sum_{n \neq 0} \frac{\sin n\theta}{n} = \begin{cases} 
\pi - \theta & \text{for } 0 < \theta < 2\pi \\
-\pi - \theta & \text{for } -2\pi < \theta < 0 \\
0 & \text{for } \theta = 0 
\end{cases}
\]

(24)

The straightforward evaluation leads to

\[
[X^0(\sigma), X^2(\sigma')]|_{\sigma=\sigma'=0} = -[X^0(\sigma), X^2(\sigma')]|_{\sigma=\sigma'=\pi} = -i \frac{2\alpha' e}{1+b^2-e^2},
\]

(25)

where \([x^0, x^2] = 0\) is chosen such that there is again no bulk contribution to the commutator. Finally, with \([x^0, x^1] = 0\), \([X^0(\sigma), X^1(\sigma')] = 0\) including boundaries. Hence we see that the end points of the string become noncommutative. In particular, the contribution of \([X^1(\pi), X^2(\pi)]\) has the opposite signature to the one given in Ref. [11]. This is because they consider strings ending on D-branes with the same \(b\) while we are considering here strings from D2 with \(b\) ending on anti-D2 with \(-b\).

The result can be neatly written as

\[
[X^i(\sigma), X^j(\sigma')] = i\Theta^{ij, (\sigma)} \epsilon_{\sigma, \sigma'},
\]

(26)
where $\Theta^{ij, (\sigma)}$ is the Seiberg-Witten expression of noncommutativity
\[ \Theta^{ij, (\sigma)} = 2\pi\alpha' \left( \frac{1}{g + 2\pi\alpha' B(\sigma)} \right)^{ij}_A = -(2\pi\alpha')^2 \left( \frac{1}{g + 2\pi\alpha' B(\sigma)} \right) B^{(\sigma)} \left( \frac{1}{g - 2\pi\alpha' B(\sigma)} \right)^{ij}, \] (27)

where the background $B^{(\sigma)}$ is specified in Eq. (1) and is nonvanishing only at $\sigma = 0, \pi$, and $\epsilon_{\sigma, \sigma'}$ is defined to be $\pm 1$ for $\sigma = 0, \pi$.

To discuss the Seiberg-Witten decoupling limit, we restore $g_{ij} = \text{diag}(-|g_{00}|, g_{11}, g_{22})$ and take the limit where $g_{11}, g_{22} \sim \epsilon$ and $\alpha' \sim \sqrt{\epsilon}$ while fixing $B_{02}, B_{12}$ and $g_{00}$. The critical condition becomes $\epsilon^2 = |g_{00}|g_{22}$, for which the system is supersymmetric. The noncommutativity at the end points of the open string becomes

\[ [X^1(0), X^2(0)] = [X^1(\pi), X^2(\pi)] = i \frac{2\pi\alpha' b}{g_{11} g_{22} \left( 1 - \frac{\epsilon^2}{|g_{00}|g_{22}} \right) + b^2}, \]

\[ [X^0(0), X^2(0)] = [X^0(\pi), X^2(\pi)] = -i \frac{2\pi\alpha' \epsilon}{|g_{00}|g_{22} \left( 1 - \frac{\epsilon^2}{|g_{00}|g_{22}} + \frac{b^2}{g_{11} g_{22}} \right)}. \] (28)

Now using the critical condition and taking the limit in which $g_{11}, g_{22} \sim \epsilon$ and $\alpha' \sim \sqrt{\epsilon}$ while fixing $B_{02}, B_{12}$ and $g_{00}$, we obtain

\[ [X^1(0), X^2(0)] = [X^1(\pi), X^2(\pi)] = \frac{i}{B}, \] (29)

with $B = b/(2\pi\alpha')$ and all other coordinate commutators vanish especially at the end points.

Thus we end up only with the spatial noncommutativity in the Seiberg-Witten limit. This implies that the worldvolume theory in the zero slope limit may be described by the noncommutative gauge theory with spatial noncommutativity $[x, y] = i\theta$. In this case, it looks like that the noncommutativities on both branes have the same signature. But, taking into account of the fact that the two ends of open strings are oppositely charged, the noncommutativity geometry on the D2 is with $\theta = 1/B$ while on the anti-D2 with $\theta = -1/B$. This suggests the the effective low-energy theory on this system is a noncommutative super Yang-Mills theory from D2-branes and that with opposite noncommutativity from anti-D2-branes, together with bifundamental scalars from the open strings between them. For the field theory description of the worldvolume theory, the use of different noncommutativities on D2 and anti-D2 is inconvenient and not conventional. Instead, one may use the description where $\theta = 1/B$ for both D2 and anti-D2 and the effects of opposite noncommutativity on the anti-D2 is described by the background magnetic field. Here the background magnetic field is not decoupled in general, but the $U(2)$ noncommutative gauge symmetries are still in effect. As shown in Refs. [1, 3, 6], the background magnetic field then makes the spectrum tachyonic for the ordinary noncommutative D2 and anti-D2. In our case, the would-be tachyonic degrees disappear, in spite of the presence of the same background magnetic field, and the massless continuum spectra appear as the above analysis of the string spectrum indicates. To study the details, one has to look at the interaction amplitudes obtained from the worldsheet, which goes beyond the scope of this short note. The worldvolume gauge theory has been obtained in the matrix theory description [5] and we shall use this for further check of the continuum spectra of the fluctuation.
4 Comparison to the Matrix Theory Description

In the matrix theory description [5], it was shown that the supersymmetric D2 anti-D2 system is described, in the gauge $A_0 = Y$, by the background

$$X = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad Y = \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix}$$

(30)

with

$$x + iy = \sqrt{2\theta} \sum_{n=0}^{\infty} \sqrt{n+1} |n\rangle \langle n+1|.$$  

(31)

This satisfies $[x, y] = i\theta$ with the noncommutativity scale $\theta$ being $2\pi\alpha'/b$ in the previous sections. The dynamics of strings connecting D2 and anti-D2 is described by the off-diagonal fluctuations of the above matrices. Hence we shall turn on these off diagonal components by

$$X = \begin{pmatrix} x & H_1 \\ H_1^\dagger & x \end{pmatrix}, \quad Y = \begin{pmatrix} y & H_2 \\ H_2^\dagger & -y \end{pmatrix}, \quad X_s = \begin{pmatrix} 0 & H_s \\ H_s^\dagger & 0 \end{pmatrix},$$

(32)

where the index $s = 3, 4, \cdots, 9$ refers to the transverse matrix coordinates including $Z = X_3$. The linear fluctuations are governed by the equations of motion

$$D_0D_0X_J + [X_I, [X_I, X_J]] = 0$$

(33)

together with the Gauss law $[X_I, D_0X_I] = 0$. In the gauge $A_0 = Y$ again, we find that the Gauss law leads to

$$\theta \partial_y \dot{H}_1 - 2iy\dot{H}_2 - \theta^2 \partial_y^2 H_2 - 4i\theta H_1 - 2i\theta y \partial_y H_1 = 0,$$

(34)

where we have used $[x, ] = i\theta \partial_y$ and transformed all the matrix variables to ordinary functions using Weyl-Moyal mapping. Consequently, the variables $H$’s appearing in the above equation are now ordinary functions with all the products here ordinary. Similarly from the other components, one obtains

$$\ddot{H}_1 - 4iy\dot{H}_1 - \theta \partial_y \ddot{H}_2 = 0,$$

$$\ddot{H}_2 - 2iy\dot{H}_2 - \theta^2 \partial_y^2 H_2 - 4i\theta H_1 - 2i\theta y \partial_y H_1 = 0,$$

$$\ddot{H}_s - 4iy\dot{H}_s - \theta^2 \partial_y^2 H_s = 0.$$  

(35)

Combining (34) and (35), one finds that $H_2$ is related to $H_1$ by

$$\ddot{H}_2 = \theta \partial_y H_1,$$

(36)

while $H_1$ and $H_s$ satisfies

$$\ddot{H} - 4iy\dot{H} - \theta^2 \partial_y^2 H = 0.$$

(37)

\footnote{Unlike the notation in [5], we use here $Y$ as the second worldvolume matrix coordinate and $Z$ as a transverse direction.}
Hence there are eight independent degrees are present in the fluctuations. For the case of the harmonic time dependence $H = h(x, y)e^{-i\omega t}$, the equation becomes

$$-\partial_y^2 h + \frac{4\omega}{\theta^2} \left( y - \frac{\omega}{4} \right) h = 0. \quad (38)$$

which corresponds to the Airy equation. The matrix Hamiltonian is proportional to $\omega^2 H^2$ and clearly nonnegative definite for the fluctuation.

Since the spectrum is continuous, we conclude that the degrees involved are massless. However the equation is not a free wave equation but involves a peculiar background proportional to $y$ that is not enough to make the spectrum discrete. Hence the results here are consistent with the string theory analysis especially in the number of the massless degrees. Of course, there are no tachyons in the spectrum as found from the string spectrum. Finally, we note that a multiplication of an arbitrary function of $x$ only to $h$ still solves the equation without affecting $\omega$. This property will disappear if one considers nonlinear corrections to the equations of motion.

5 Conclusions

In this note, we have considered the strings connecting D2 and anti-D2 that are supersymmetric due to the background $E$ and $B$ fields. Beginning with generic values of $E$-field, we have shown that the tachyons disappear in the critical limit corresponding to the supersymmetric configurations. Although there appear apparent singularities in the coordinate fields, the worldsheet CFT is well defined. In the limit, the ground states after GSO projection become massless, which is consistent with the restoration of the supersymmetries. For the low-energy description implied by the worldsheet CFT, one has to compute the three and four point amplitudes to extract the interaction terms in the zero slope limit. Instead of going this direction, we investigate the matrix theory fluctuation spectra corresponding to the D2 and anti-D2 strings and found that the massless degrees indeed appear in spite of the presence of the background fields. Consequently a coherent picture from both descriptions emerges.

The original supertube has a circular geometry and the elliptic deformation preserves again a quarter of the supersymmetries. Probing these tubes by strings are of interest though the analysis are expected to be more involved than those presented here. One may ask how the radial size and the background $E$ and $B$ fields are related to the disappearance of tachyonic modes or how the 0-tube strings behave. These kinds of information ought to be helpful in resolving the dynamical issues involved with the appearance of tubes out of F1 and D0’s or what governs the shapes \cite{4,12}.

The dynamics we are ultimately interested in are the dynamical processes in which an initial collection of D0’s and F1 flows into the tubular branes or the supersymmetric D2 and anti-D2 systems or vice versa. Such deformations are large in the sense that we call only local bounded-energy fluctuations small. Along these large deformation including the change of the radius, $E$ and $B$-field, it is particularly of interest to know how the stability of the systems are affected.

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A Quantization of the System

We justify the quantization appearing in (7) by the following reverse procedure. Note first that we do not yet specify our system by a specified action. We define our system by the following first order form

\[
S = \int dt \left( -\frac{i}{2} \sum_{n \neq 0}^{\infty} \frac{1}{n} \dot{a}_n a_{-n} - \dot{x}^0 p^0 + i \sum_n \frac{1}{n + \nu} \dot{c}_{n+\nu} d_{-n-\nu} - L_0 \right),
\]  

(A.1)

where arbitrary time dependence of the oscillator variables are to be understood. Upon quantization, the equal time commutation relations in (7) follows. In connection with the original system, we define \(X^i\) such that the oscillator variables in (A.1) replaces the corresponding oscillator variables absorbing the time dependent phase factors of (4). Then the Heisenberg equation of motion of \(X^i\) implies that \(X^i\) satisfies the desired free wave equation. Furthermore the time development of \(X^i\) is consistent with the boundary conditions in (3) on shell. If one turns the action in (A.1) into second order form and rewrite oscillator variables in terms of \(X^i\), one then can show in fact that the second order action is proportional to \(T_2 \int dtd\sigma (\dot{X} \cdot \dot{X} - \partial_\sigma X \cdot \partial_\sigma X)\) up to boundary terms that may be relevant in specifying the boundary conditions. This completes the definition of our system of the D2 anti-D2 strings.

Another way to derive the commutation relations of the oscillators is to use the “canonical” momenta defined by

\[
P_i = \frac{1}{2\pi \alpha'} (g_{ij} \partial_t X^j + 2\pi \alpha' B_{ij}^{(\sigma)} \partial_{\sigma} X^j),
\]

(A.2)

and impose the equal-time commutation relations

\[
[X^i(\sigma, t), P_j(\sigma', t)] = i \delta_{ij} \delta(\sigma - \sigma'),
\]

(A.3)

with the delta function specified by the Neumann boundary conditions. Or again reversing the procedure, the momenta defined above satisfy the equal time commutation relations (A.3) provided the commutation relations of the oscillator variables in (3) holds.

In this procedure, care must be taken of the fact that the background changes depending on \(\sigma\), as specified in Eq.(3). In fact as far as the boundary value of \(B_{ij}(\sigma)\) is specified as in Eq. (3), how to extend it inside bulk does not matter. One may prove this fact by the explicit computation using the commutation relation.

For the sake of illustration, we here present a proof only for the case \(i = j = 1\). First note that

\[
[X^1(\sigma), \dot{X}^1(\sigma')] = 2i\alpha' \sum_n \cos \left( (n + \nu)\sigma - \frac{\pi \nu}{2} \right) \cos \left( (n + \nu)\sigma' - \frac{\pi \nu}{2} \right),
\]

(A.4)
where we have used the commutation relations. Using $\sum_n \sin n \theta = 0$, one may rearrange the above as

$$[X^1(\sigma), \dot{X}^1(\sigma')] = i\alpha' \sum_n \left[ \cos n(\sigma + \sigma') \cos(\nu + \sigma' - \pi) + \cos n(\sigma - \sigma') \cos \nu(\sigma - \sigma') \right]. \quad (A.5)$$

Now we use the fact that

$$\sum_n \cos nx = 2\pi \sum_k \tilde{\delta}(x - 2k\pi) \quad (A.6)$$

where $\tilde{\delta}(x)$ is the usual delta function defined as $\int_{-\infty}^\infty \tilde{\delta}(x - x_0)f(x) = f(x_0)$ for any continuous function $f$. One then obtains

$$[X^1(\sigma), \dot{X}^1(\sigma')] = 2\pi\alpha' i \left[ \tilde{\delta}(\sigma - \sigma') + (\tilde{\delta}(\sigma + \sigma') + \tilde{\delta}(\sigma + \sigma' - 2\pi)) \right] \cos \nu\pi \quad (A.7)$$

for $0 \leq \sigma, \sigma' \leq \pi$. By a similar computation, one has

$$[X^1(\sigma), \partial_{\sigma'} X^2(\sigma')] = \frac{4\pi\alpha'ib}{1 + b^2 - c^2} \left( -\tilde{\delta}(\sigma + \sigma') + \tilde{\delta}(\sigma + \sigma' - 2\pi) \right) \quad (A.8)$$

again for $0 \leq \sigma, \sigma' \leq \pi$. Combining these two results, one finally gets

$$[X^1(\sigma), P_1(\sigma')] = i \left[ \tilde{\delta}(\sigma - \sigma') + \tilde{\delta}(\sigma + \sigma') + \tilde{\delta}(\sigma + \sigma' - 2\pi) \right] = i\delta(\sigma - \sigma') \quad (A.9)$$

for $0 \leq \sigma, \sigma' \leq \pi$. This completes the proof for $i = j = 1$ case and the remaining can be proved in a similar way.

Finally using again (7) and regulating appropriately, one can show that the momenta commutes with each other at equal time, i.e. $[P_1(\sigma, t), P_2(\sigma', t)] = 0$.

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