Generators of invariant linear system on tropical curves for finite isometry group

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Abstract

For a tropical curve $\Gamma$ and a finite subgroup $K$ of the isometry group of $\Gamma$, we prove, extending the work by Haase, Musiker and Yu (\cite{3}), that the $K$-invariant part of the complete linear system associated to a $K$-invariant effective divisor on $\Gamma$ is finitely generated.

1 Introduction

Let $R(D)$ denote the set consisting of rational functions corresponding to the complete linear system $|D|$ for an effective divisor $D$ on a tropical curve $\Gamma$, where a tropical curve means a metric graph possibly with unbounded edges. $R(D)$ becomes a tropical semimodule. The projective space $R(D)/R$ is naturally identified with the complete linear system $|D|$. Haase, Musiker and Yu showed that $R(D)$ is finitely generated (\cite{3}, Theorem 6). A tropical subsemimodule $R'$ of $R(D)$ corresponds to a linear subspace $\Lambda$ of $|D|$. This linear subspace $\Lambda$ is called a linear system associated to $R'$.

In this paper, we recall some basic facts of tropical curves in Section 2. Then in Section 3, we observe the $K$-invariant set $R(D)^K$ of $R(D)$ and prove that $R(D)^K$ is actually finitely generated, where $K$ is a finite subgroup of the isometry group of $\Gamma$. Our proof is basically analogous to that of \cite{3}, but it is not perfectly compatible, i.e. the $K$-invariant set $S^K$ of the generator set $S$ of $R(D)$ defined in \cite{3}, Lemma 6] is not a generator set of $R(D)^K$. We find such a set corresponding to $S$, which we call $S(D)_K$. The condition defining $S(D)_K$ is tangibly given from geometric information. Also, the construction of a harmonic morphism with degree $|K|$ from $\Gamma$ to the quotient tropical curve $\Gamma'$ of $\Gamma$ by $K$ precedes. We follow Chan’s natural construction (\cite{2}) with a little bit of adaptation. Finally, using the harmonic morphism, we prove that
$R(D)^K$ is finitely generated as a tropical semimodule in our main theorem (Theorem 3.11). When $D$ is $K$-invariant, we can identify $R(D)^K/R$ with the $K$-invariant linear subsystem $|D|^K$ and then $|D|^K$ is finitely generated by $S(D)_K/R$.

2 Preliminaries

In this section, we briefly recall the theories of tropical curves ([5]), divisors on tropical curves ([5]), harmonic morphisms of tropical curves ([2], [3], [4]), and chip-firing moves on tropical curves ([3]), which we need later.

2.1 Tropical curves

In this paper, a graph means an unweighted, finite connected nonempty multigraph. Note that we allow the existence of loops. For a graph $G$, the sets of vertices and edges are denoted by $V(G)$ and $E(G)$, respectively. The valence $\text{val}(v)$ of a vertex $v$ of $G$ is the number of edges emanating from $v$, where we count each loop as two. A vertex $v$ of $G$ is a leaf end if $v$ has valence one. A leaf edge is an edge of $G$ having a leaf end.

An edge-weighted graph $(G, l)$ is the pair of a graph $G$ and a function $l : E(G) \to \mathbb{R}_{>0} \cup \{\infty\}$ called a length function, where $l$ can take the value $\infty$ on only leaf edges. A tropical curve is the underlying $\infty$-metric space of an edge-weighted graph $(G, l)$. For a point $x$ on a tropical curve $\Gamma$ obtained from $(G, l)$, if the distances between $x$ and all points on $\Gamma$ other than $x$ are infinity, then $x$ is called a point at infinity, else, $x$ is said to be a finite point. For the above tropical curve $\Gamma$, $(G, l)$ is said to be its model. There are many possible models for $\Gamma$. We construct a model $(G_\circ, l_\circ)$ called the canonical model of $\Gamma$ as follows: when $\Gamma$ is a circle, we determine $V(G_\circ)$ as the set consisting of one arbitrary point on $\Gamma$, else when $\Gamma$ is the $\infty$-metric space obtained from only one edge with length of $\infty$, $V(G_\circ)$ consists of the two endpoints of $\Gamma$ (those are points at infinity) and an any point on $\Gamma$ as the origin, else, generally, we determine $V(G_\circ) := \{x \in \Gamma \mid \text{val}(x) \neq 2\}$, where the valence $\text{val}(x)$ is the number of connected components of $U_x \setminus \{x\}$ with $U_x$ being any sufficiently small connected neighborhood of $x$ in $\Gamma$. Since connected components of $\Gamma \setminus V(G_\circ)$ consist of open intervals, whose lengths determine the length function $l_\circ$. If a model $(G, l)$ of $\Gamma$ has no loops, then $(G, l)$ is said to be a loopless model of $\Gamma$. For a model $(G, l)$ of $\Gamma$, the loopless model for
\((G, l)\) is obtained by regarding all midpoints of loops of \(G\) as vertices and by adding them to the set of vertices of \(G\). The loopless model for the canonical model of a tropical curve is called the \textit{canonical loopless model}.

For terminology, in a tropical curve \(\Gamma\), an edge of \(\Gamma\) means an edge of the underlying graph \(G_\circ\) of the canonical model \((G_\circ, l_\circ)\). Let \(e\) be an edge of \(\Gamma\) which is not a loop. We regard \(e\) as a closed subset of \(\Gamma\), i.e., including the endpoints \(v_1, v_2\) of \(e\). The \textit{relative interior} of \(e\) is \(e^\circ = e \setminus \{v_1, v_2\}\). For a point \(x\) on \(\Gamma\), a half-edge of \(x\) is a connected component of \(U_x \setminus \{x\}\) with any sufficiently small connected neighborhood \(U_x\) of \(x\).

For a model \((G, l)\) of a tropical curve \(\Gamma\), we frequently identify a vertex \(v\) (resp. an edge \(e\)) of \(G\) with the point corresponding to \(v\) on \(\Gamma\) (resp. the closed subset corresponding to \(e\) of \(\Gamma\)).

\section*{2.2 Divisors on tropical curves}

Let \(\Gamma\) be a tropical curve. An element of the free abelian group \(\text{Div}(\Gamma)\) generated by points on \(\Gamma\) is called a \textit{divisor} on \(\Gamma\). For a divisor \(D\) on \(\Gamma\), its \textit{degree} \(\text{deg}(D)\) is defined by the sum of the coefficients over all points on \(\Gamma\). We write the coefficient at \(x\) as \(D(x)\). A divisor \(D\) on \(\Gamma\) is said to be \textit{effective} if \(D(x) \geq 0\) for any \(x\) in \(\Gamma\). If \(D\) is effective, we write simply \(D \geq 0\). The set of points on \(\Gamma\) where the coefficient(s) of \(D\) is not zero is called the \textit{support} of \(D\) and written as \(\text{supp}(D)\).

A \textit{rational function} on \(\Gamma\) is a constant function of \(-\infty\) or a piecewise linear function with integer slopes and with a finite number of pieces, taking the value \(\pm \infty\) only at points at infinity. \(\text{Rat}(\Gamma)\) denotes the set of rational functions on \(\Gamma\). For a point \(x\) on \(\Gamma\) and \(f\) in \(\text{Rat}(\Gamma)\) which is not constant \(-\infty\), the sum of the outgoing slopes of \(f\) at \(x\) is denoted by \(\text{ord}_x(f)\). If \(x\) is a point at infinity and \(f\) is infinite there, we define \(\text{ord}_x(f)\) as the outgoing slope from any sufficiently small connected neighborhood of \(x\). Note when \(\Gamma\) is a singleton, for any \(f\) in \(\text{Rat}(\Gamma)\), we define \(\text{ord}_x(f) := 0\). This sum is 0 for all but finite number of points on \(\Gamma\), and thus

\[
\text{div}(f) := \sum_{x \in \Gamma} \text{ord}_x(f) \cdot x
\]

is a divisor on \(\Gamma\), which is called a \textit{principal divisor}. Two divisors \(D\) and \(E\) on \(\Gamma\) are said to be \textit{linearly equivalent} if \(D - E\) is a principal divisor. We handle the values \(\infty\) and \(-\infty\) as follows: let \(f, g\) in \(\text{Rat}(\Gamma)\) take the value \(\infty\) and \(-\infty\) at a point \(x\) at infinity on \(\Gamma\) respectively, and \(y\) be any
point in any sufficiently small neighborhood of \( x \). When \( \text{ord}_x(f) + \text{ord}_x(g) \) is negative, then \( (f \odot g)(x) := \infty \). When \( \text{ord}_x(f) + \text{ord}_x(g) \) is positive, then \( (f \odot g)(x) := -\infty \). Remark that the constant function of \( -\infty \) on \( \Gamma \) does not determine a principal divisor. For a divisor \( D \) on \( \Gamma \), the complete linear system \( |D| \) is defined by the set of effective divisors on \( \Gamma \) being linearly equivalent to \( D \).

The set of \( \mathbb{R} \) with two tropical operations:

\[
    a \oplus b := \max\{a, b\} \quad \text{and} \quad a \odot b := a + b
\]

becomes a semiring called the \textit{tropical semiring}, where both \( a \) and \( b \) are in \( \mathbb{R} \). For a divisor \( D \) on a tropical curve, let \( R(D) \) be the set of rational functions \( f \neq -\infty \) such that \( D + \text{div}(f) \) is effective. When \( \text{deg}(D) \) is negative, \( |D| \) is empty, so is \( R(D) \). Otherwise, from the argument in Section 3 of \cite{3}, \( D \) is not empty and consequently so is \( R(D) \). Hereafter, we treat only divisors of nonnegative degree.

**Lemma 2.2.1** \textit{(cf. \cite{3} Lemma 4)}. \( R(D) \) becomes a tropical semimodule on \( R \) by extending above tropical operations onto functions, giving pointwise sum and product.

By the definition of \( \text{ord}_x(f) \) for a point \( x \) at infinity and \( f \) in \( \text{Rat}(\Gamma) \), we can prove Lemma 2.2.1 in the same way of \cite{3} Lemma 4.

For a tropical subsemimodule \( M \) of \((\mathbb{R} \cup \{\pm \infty\})^{\Gamma} \) (or of \( \mathbb{R}^{\Gamma} \)), \( f \) in \( M \) is called an \textit{extremal} of \( M \) when it implies \( f = g_1 \) or \( f = g_2 \) that any \( g_1 \) and \( g_2 \) in \( M \) satisfies \( f = g_1 \oplus g_2 \).

**Remark 2.2.2** \textit{(\cite{3} Proposition 8)}. \textit{Any finitely generated tropical subsemimodule \( \tilde{M} \) of \( \mathbb{R}^{\Gamma} \) is generated by the extremals of \( \tilde{M} \).}

With the adaptation for \( \pm \infty \), we can prove the following lemma in same way as the above remark.

**Lemma 2.2.3.** \textit{Any finitely generated tropical subsemimodule \( M \) of \( R(D) \subset (\mathbb{R} \cup \{\pm \infty\})^{\Gamma} \) is generated by the extremals of \( M \).

2.3 Harmonic morphisms

Let \( \Gamma, \Gamma' \) be tropical curves, respectively, and \( \varphi : \Gamma \to \Gamma' \) be a continuous map. The map \( \varphi \) is called a \textit{morphism} if there exist a model \((G, l)\) of \( \Gamma \) and
a model \((G', l')\) of \(\Gamma'\) such that the image of the set of vertices of \(G\) by \(\varphi\) is a subset of the set of vertices of \(G'\), the inverse image of the relative interior of any edge of \(G'\) by \(\varphi\) is the union of the relative interiors of a finite number of edges of \(G\) and the restriction of \(\varphi\) to any edge \(e\) of \(G\) is a dilation by some non-negative integer factor \(\deg_e(\varphi)\). Note that the dilation factor on \(e\) with \(\deg_e(\varphi) \neq 0\) represents the ratio of the distance of the images of any two points \(x\) and \(y\) except points at infinity on \(e\) to that of original \(x\) and \(y\). If an edge \(e\) is mapped to a vertex of \(G'\) by \(\varphi\), then \(\deg_e(\varphi) = 0\). The morphism \(\varphi\) is said to be finite if \(\deg_e(\varphi) > 0\) for any edge \(e\) of \(G\). For any half-edge \(h\) of any point on \(\Gamma\), we define \(\deg_h(\varphi)\) as \(\deg_e(\varphi)\), where \(e\) is the edge of \(G\) containing \(h\).

Let \(\Gamma'\) be not a singleton and \(x\) a point on \(\Gamma\). The morphism \(\varphi\) is harmonic at \(x\) if the number

\[
\deg_x(\varphi) := \sum_{h \to h'} \deg_h(\varphi)
\]

is independent of the choice of half-edge \(h'\) emanating from \(\varphi(x)\), where \(h\) is a connected component of the inverse image of \(h'\) by \(\varphi\). The morphism \(\varphi\) is harmonic if it is harmonic at all points on \(\Gamma\). One can check that if \(\varphi\) is a finite harmonic morphism, then the number

\[
\deg(\varphi) := \sum_{x \to x'} \deg_x(\varphi)
\]

is independent of the choice of a point \(x'\) on \(\Gamma'\), and is said the degree of \(\varphi\), where \(x\) is an element of the inverse image of \(x'\) by \(\varphi\). If \(\Gamma'\) is a singleton and \(\Gamma\) is not a singleton, for any point \(x\) on \(\Gamma\), we define \(\deg_x(\varphi)\) as zero so that we regard \(\varphi\) as a harmonic morphism of degree zero. If both \(\Gamma\) and \(\Gamma'\) are singletons, we regard \(\varphi\) as a harmonic morphism which can have any number of degree.

Let \(\varphi : \Gamma \to \Gamma'\) be a harmonic morphism between tropical curves. For \(f\) in \(\text{Rat}(\Gamma)\), the push-forward of \(f\) is the function \(\varphi_* f : \Gamma' \to \mathbb{R} \cup \{\pm \infty\}\) defined by

\[
\varphi_* f(x') := \sum_{x \in \Gamma, \varphi(x) = x'} \deg_x(\varphi) \cdot f(x).
\]

The pull-back of \(f'\) in \(\text{Rat}(\Gamma')\) is the function \(\varphi^* f' : \Gamma \to \mathbb{R} \cup \{\pm \infty\}\) defined by \(\varphi^* f' := f' \circ \varphi\). We define the push-forward on divisors \(\varphi_* : \text{Div}(\Gamma) \to \text{Div}(\Gamma')\)
by
\[ \varphi_*(D) := \sum_{x \in \Gamma} D(x) \cdot \varphi(x). \]

One can check that \( \deg(\varphi_*(D)) = \deg(D) \) and \( \varphi_*(\text{div}(f)) = \text{div}(\varphi_*f) \) for any divisor \( D \) on \( \Gamma \) and any \( f \) in \( \text{Rat}(\Gamma) \) (cf. [1, Proposition 4.2]).

### 2.4 Chip-firing moves

In [3], Haase, Musiker and Yu used the term *subgraph* of a tropical curve as a compact subset of the tropical curve with a finite number of connected components and defined the *chip firing move* \( \text{CF}(\tilde{\Gamma}_1, l) \) by a subgraph \( \tilde{\Gamma}_1 \) of a tropical curve \( \tilde{\Gamma} \) and a positive real number \( l \) as the rational function
\[ \text{CF}(\tilde{\Gamma}_1, l)(x) := -\min(l, \text{dist}(x, \tilde{\Gamma}_1)), \]
where \( \text{dist}(x, \tilde{\Gamma}_1) \) is the infimum of the lengths of the shortest path to arbitrary points on \( \tilde{\Gamma}_1 \) from \( x \). They proved that every rational function on a tropical curve is an (ordinary) sum of chip firing moves (plus a constant) ([3, Lemma 2]) with the concept of a *weighted chip firing move*. This is a rational function on a tropical curve having two disjoint proper subgraphs \( \tilde{\Gamma}_1 \) and \( \tilde{\Gamma}_2 \) such that the complement of the union of \( \tilde{\Gamma}_1 \) and \( \tilde{\Gamma}_2 \) in \( \tilde{\Gamma} \) consists only of open line segments and such that the rational function is constant on \( \tilde{\Gamma}_1 \) and \( \tilde{\Gamma}_2 \) and linear (smooth) with integer slopes on the complement. A weighted chip firing move is an (ordinary) sum of chip firing moves (plus a constant) ([3, Lemma 1]).

With unbounded edges, their definition of chip firing moves needs a little correction. Let \( \Gamma_1 \) be a subgraph of a tropical curve \( \Gamma \) which does not have any connected components consisting only of points at infinity and \( l \) a positive real number or infinity. The *chip firing move* by \( \Gamma_1 \) and \( l \) is defined as the rational function \( \text{CF}(\Gamma_1, l)(x) := -\min(l, \text{dist}(x, \Gamma_1)) \).

**Lemma 2.4.1.** A weighted chip firing move on a tropical curve is a linear combination of chip firing moves having integer coefficients (plus a constant).

**Sketch of proof.** We use the same notations as in their proof. All we have to do is to show the construction for the case with \( l = \infty \). Especially, it is sufficient to check the case that \( \Gamma_1 \) consists only of points at infinity. Supposing that \( \Gamma_1 \) has only one point gives only two situations. Firstly, \( \Gamma_2 \) contains a finite point. Then \( f \) can be written as \( \pm s \cdot \text{CF}(\Gamma_2, \infty) \) plus a constant, where \( s \) is the slope of \( f \) on the complement. Secondly, \( \Gamma_2 \) consists only of one point at infinity. Taking a finite point \( x \), then \( f \) can be written
as $\pm s \cdot (\text{CF}(f^{-1}([f(x), \infty]), \infty) - \text{CF}([x, \infty]))$ plus a constant with same $s$ as the first situation. Suppose that $\Gamma_1$ has plural points. $\Gamma_2$ must contain at least one finite point. Let $x_i$ be the intersection of $\Gamma_1$ and the closure of $L_i$. Note that $\Gamma_1 = \{x_1, \ldots, x_k\}$, where $k$ is no less than two. With the slope $s_i$ of $f$ on $e_i := L_i \cup \{x_i\}$, $f$ is $\sum_{i=1}^{k} (\pm s_i \cdot \text{CF}(\Gamma \setminus e_i, \infty))$ plus a constant. $\square$

The next lemma is proven in the same way of [3, Lemma 2] and shows the appropriateness of this definition.

**Lemma 2.4.2.** Every rational function on a tropical curve is a linear combination of chip firing moves having integer coefficients (plus a constant).

A point on $\Gamma$ with valence two is said to be a smooth point. We sometimes refer to an effective divisor $D$ on $\Gamma$ as a chip configuration. We say that a subgraph $\Gamma_1$ of $\Gamma$ can fire on $D$ if for each boundary point of $\Gamma_1$ there are at least as many chips as the number of edges pointing out of $\Gamma_1$. A set of points on a tropical curve $\Gamma$ is said to be cut set of $\Gamma$ if the complement of that set in $\Gamma$ is disconnected.

## 3 Generators of $R(D)^K$

In this section, for an effective divisor $D$ on a tropical curve and a finite subgroup $K$ of the isometry group of the tropical curve, we find a generator set of the $K$-invariant set $R(D)^K$ of $R(D)$ and then, show that $R(D)^K$ is finitely generated as a tropical semimodule. When $D$ is $K$-invariant, $R(D)/R$ is identified with the $K$-invariant linear system $|D|^K$, so $|D|^K$ is finitely generated by the generators of $R(D)^K$ modulo tropical scaling.

**Remark 3.1 ([3, Lemma 6]).** Let $\tilde{\Gamma}$ be a tropical curve, $\tilde{D}$ be a divisor on $\tilde{\Gamma}$ and $S$ be the set of rational functions $f$ in $R(\tilde{D})$ such that the support of $\tilde{D} + \text{div}(f)$ does not contain any cut set of $\tilde{\Gamma}$ consisting only of smooth points. Then

1. $S$ contains all the extremals of $R(\tilde{D})$,
2. $S$ is finite modulo tropical scaling, and
3. $S$ generates $R(\tilde{D})$ as a tropical semimodule.
Though in the above remark they assume that $R(\tilde{D})$ is a subset of $R^\tilde{\kappa}$, the proof is applied even in the case that $R(\tilde{D})$ is a subset of $(R \cup \{\pm \infty\})^\tilde{\kappa}$ with preparations in Section 2. Also, the above remark throws the relation between $S$ and $\tilde{D}$ into relief, hence hereafter we write $S$ for $\tilde{D}$ as $S(\tilde{D})$. Consider the tropical subsemimodule of $R([0])$ corresponding to $|0| \setminus \{0\}$ on a tropical curve $[0,1]$. Let $\Gamma$ be a tropical curve, $D$ an effective divisor on $\Gamma$ and $K$ a subgroup of the isometry group of $\Gamma$. One can expect the relation between $R(D)$ and $S(D)$ to be analogous to that of their $K$-invariant counterparts $R(D)^K$ and $S(D)^K$, but in vain. Indeed, the next example objects.

Example 3.2. Let $\Gamma$ be a circle and let a map $i: \tilde{G}_0 \rightarrow \tilde{G}_0$ which transfers two edges to each other, where $\tilde{G}_0$ is the underlying graph of the canonical loopless model of $\Gamma$. For a point $x_1$ on $\Gamma$, we choose another point $x_2$ on $\Gamma$ such that $i(x_1) \neq x_2$. For the group $\tilde{K}$ generated by $i$ and the effective divisor $\tilde{D} = x_1 + x_2$, although $S(\tilde{D})^\tilde{K}$ is empty, $R(\tilde{D})^\tilde{K}$ is not empty. It means that $S(\tilde{D})^\tilde{K}$ is not a generator set of $R(\tilde{D})^\tilde{K}$.

Now, let us find a generator set for $R(D)^K$ that corresponds to $S(D)$ for $R(D)$. In the above situation, $K$ acts on $\Gamma$ naturally. We define $V_1(\Gamma)$ as the set of points $x$ on $\Gamma$ such that there exists a point $y$ in any neighborhood of $x$ whose stabilizer is not equal to that of $x$.

Lemma 3.3. $V_1(\Gamma)$ is a finite set.

Proof. We assume that $\Gamma$ is not the $\infty$-metric space obtained from only one edge with length of $\infty$. Let $\sigma : \Gamma \rightarrow \Gamma$ be an isometry. Then, for any edge $e$ of $\Gamma$, the image of $e$ by $\sigma$ agrees completely with $e$ or the intersection of $e$ and the image of $e$ by $\sigma$ is contained in the set of the endpoints of $e$. In fact, if $|e \cap \sigma(e)|$ is infinite, then $\sigma(e)$ is contained in $e$ because $e$ is an edge of $\Gamma$. It means that $\sigma(e) = e$. If $|e \cap \sigma(e)|$ is finite and $e \cap \sigma(e)$ contains a point on $\Gamma$ other than endpoints of $e$, then that point has the valence of greater than two. It contradicts to the fact that $e$ is an edge of $\Gamma$.

From the above argument, for any edge $e$ of $\Gamma$, we can roughly classify the situations into four. First, $\sigma$ is the identity map on $e$, i.e., $\sigma$ fixes all points on $e$. Second, $\sigma$ gives a mirror image of $e$. In this case, if $\Gamma$ is a circle consisting of $e$, the fixed points on $e$ by $\sigma$ are only antipodal points on the axis of symmetry of $\sigma$, otherwise, the midpoint of $e$ is fixed by $\sigma$, moreover...
when $e$ is a loop, then the vertex connected to $e$ is also fixed by $\sigma$. Third, $\sigma$ acts as a proper rotation on $e$. This is possible only when $\Gamma$ is a circle, and $\sigma$ gives no fixed points on $e$. Finally, $\sigma$ maps $e$ onto other edge of $\Gamma$, then only the endpoints of $e$ may be fixed by $\sigma$.

Consequently, under the above assumption, since $K$ is a finite set and $\Gamma$ has finite vertices and edges, $V_1(\Gamma)$ is a finite set.

Let us suppose that $\Gamma$ is the $\infty$-metric space obtained from only one edge with length of $\infty$. Since $K$ is a finite set, any $\sigma$ in $K$ is not a proper translation of $\Gamma$. Each isometry of $\Gamma$ other than translations fixes only one point on $\Gamma$. Thus, also in this case, $V_1(\Gamma)$ is a finite set. Note that there can exists only one inversion. If there were two distinct, these two can generate a translation, leading $|K|$ to infinity.

We set $(G_0, l_0)$ as the canonical loopless model of $\Gamma$. By Lemma 3.3, we obtain the model $(\tilde{G}_1, \tilde{l}_1)$ of $\Gamma$ by setting the $K$-orbit of the union of $V(G_0)$ and $V_1(\Gamma)$ as the set of vertices $V(\tilde{G}_1)$. Naturally, we can regard that $K$ acts on $V(\tilde{G}_1)$ and also on $E(\tilde{G}_1)$. Thus, the sets $V(\tilde{G}')$ and $E(\tilde{G}')$ are defined as the quotient sets of $V(\tilde{G}_1)$ and $E(\tilde{G}_1)$ by $K$, respectively. Let $\tilde{G}'$ be the graph obtained by setting $V(\tilde{G}')$ as the set of vertices and $E(\tilde{G}')$ as the set of edges. Since $\tilde{G}_1$ is connected, $\tilde{G}'$ is also connected. We obtain the loopless graph $G'$ from $\tilde{G}'$ and the loopless model $(G_1, l_1)$ of $\Gamma$ from the inverse image of $V(G')$ by the map defined by $K$. Note that $V(G_1)$ contains $V(\tilde{G}_1)$. Since $K$ is a finite subgroup of the isometry group of $\Gamma$, the length function $l' : E(G') \to \mathbb{R}_{>0} \cup \{\infty\}$, $[e] \mapsto |K_e| \cdot l_1(e)$ is well-defined, where $[e]$ and $K_e$ mean the equivalence class of $e$ and the stabilizer of $e$, respectively. Let $\Gamma'$ be the tropical curve obtained from $(G', l')$. Then, $\Gamma'$ is the quotient tropical curve of $\Gamma$ by $K$.

For any edge $e$ of $G_1$, by the Orbit-Stabilizer formula, $|K_e|$ is a positive integer. Thus, for $(G_1, l_1)$ and $(G', l')$, there exists only one morphism $\varphi : \Gamma \to \Gamma'$ that satisfies $\deg_e(\varphi) = |K_e|$ for any edge $e$ of $G_1$.

We obtain the following lemma as an extension of [2, Lemma 2.2].

**Lemma 3.4.** If both $\Gamma$ and $\Gamma'$ are not singletons, then $\varphi$ is a finite harmonic morphism of degree $|K|$.

**Proof.** Clearly, $\varphi$ is finite. Now we check that $\varphi$ is harmonic and its degree is $|K|$. Since $K$ is a finite subgroup of the isometry group of $\Gamma$, for any point $x$ on $\Gamma$ and any half-edge $h'$ of $\varphi(x)$, each connected component of $\varphi^{-1}(h')$...
has the same dilation factor \( \deg_h(\varphi) \), where \( h \) is a connected component emanating from \( x \). Therefore, for the edge \( e \) of \( G_1 \) containing \( h \) and its image \( e' \) by \( \varphi \), the following hold:

\[
\deg_x(\varphi) = \sum_{\tilde{h} \mapsto h'} \deg_{\tilde{h}}(\varphi) = \sum_{\tilde{e} \mapsto e'} \deg_{\tilde{e}}(\varphi) = |Ke| \cdot |Ke| = |K|.
\]

Where \( \tilde{h}, \tilde{e} \) and \( Ke \) denote a connected component of \( \varphi^{-1}(h') \), that of \( \varphi^{-1}(e') \) and the orbit of \( e \) by \( K \), respectively. Note that we use the Orbit-Stabilizer formula at the last equality. Accordingly, we get the conclusion.

Note that whether \( \Gamma \) is a singleton or not agrees with whether \( \Gamma' \) is a singleton.

Is \( R(D)^K \), the \( K \)-invariant set of \( R(D) \), identical to \( \varphi^*(R(\varphi_*(D))) \)? Nor is it.

**Example 3.5.** Assume the situation of Example 3.3. For a rational function \( f \) which decreases from \( \varphi(x_1) \) to \( \varphi(x_2) \) with slope one and is constant on other graph, however \( f \) is an element of \( R(\varphi_*(\tilde{D})) \), the pull-back of \( f \) by \( \varphi \) is not in \( R(\tilde{D})^K \).

Next, for \( R(D)^K \), the following holds.

**Lemma 3.6.** \( R(D)^K \) is a tropical semimodule.

**Proof.** Let \( c \) be in \( R \), \( f, g \) in \( R(D)^K \) and \( \sigma \) in \( K \). Since \( R(D) \) is a tropical semimodule by Lemma 2.2.1, \( c \odot f \) and \( f \oplus g \) are in \( R(D) \). It is obvious that \( \odot \) and \( \circ \) are associative and that \( \circ \) is distributive over \( \oplus \) from right, both \( (c \odot f) \circ \sigma \) and \( (f \oplus g) \circ \sigma \) are in \( R(D)^K \).

Note that \( R(D + \text{div}(f))^K = R(D)^K \odot (-f) \) for any \( K \)-invariant rational function \( f \).

The following lemma is an extension of [3, Lemma 5].

**Lemma 3.7.** Let \( f \) be in \( \text{Rat}(\Gamma) \). Then, \( f \) is an extremal of \( R(D)^K \) if and only if there are not two proper \( K \)-invariant subgraphs \( \Gamma_1 \) and \( \Gamma_2 \) covering \( \Gamma \) such that each can fire on \( D + \text{div}(f) \).

**Proof.** First, let us show the “if” part. Suppose that there are two such subgraphs \( \Gamma_1 \) and \( \Gamma_2 \). We can assume that each \( \Gamma_i \) does not have any connected component consisting only of points at infinity. Each \( \Gamma_i \) defines a chip firing
move \( g_i \) for a small positive number so that \( g_i \) is zero on \( \Gamma_i \) and they are non-positive. As \( \Gamma_1 \) and \( \Gamma_2 \) are \( K \)-invariant, so \( g_1 \) and \( g_2 \) are in \( R(D + \text{div}(f))^K \). Since \( g_1 \oplus g_2 = 0 \) on \( \Gamma \), we can write \( f \) as \((f + g_1) \oplus (f + g_2)\), i.e. \( f \) is not an extremal of \( R(D)^K \).

Next, let us show the “only if” part. Suppose \( f = g_1 \oplus g_2 \) for some \( g_1 \) and \( g_2 \) in \( R(D)^K \backslash \{ f \} \). For \( i = 1, 2 \), there exists \( \tilde{g}_i \) in \( R(D + \text{div}(f))^K \) such that \( g_i = \tilde{g}_i \circ f \). Let \( \Gamma_i \) be the closure of the loci where \( \tilde{g}_i = 0 \). Then, the union of \( \Gamma_1 \) and \( \Gamma_2 \) is \( \Gamma \) and each \( \Gamma_i \) is proper. Since \( \tilde{g}_i \) is \( K \)-invariant, so is \( \Gamma_i \). Then, each \( \Gamma_i \) can fire on \( D + \text{div}(f) \).

The term “a subgraph is infinite” means that the subgraph is a infinite set.

**Lemma 3.8.** Let \( A \) be a \( K \)-invariant subset of \( \text{supp}(D) \). If \( \varphi(A) \) is a cut set of \( \Gamma' \) and \( D(x) \geq \text{val}(x) - 1 \) for any \( x \) in \( A \), then there exists a \( K \)-invariant infinite subgraph \( \Gamma_1 \) of \( \Gamma \) which can fire on \( D \) and whose boundary points are in \( A \).

**Proof.** For such \( A \), let \( \Gamma_1', \ldots, \Gamma_n' \) be distinct connected components of \( \Gamma' \backslash \varphi(A) \) respectively. Note that \( n \) is no less than two since \( \varphi(A) \) is a cut set of \( \Gamma' \). Clearly, for any \( i \), the inverse image of the closure of \( \Gamma_i' \) by \( \varphi \) is a \( K \)-invariant infinite subgraph of \( \Gamma \) we want.

We call a point on \( \Gamma \) not being a vertex of \( G_1 \) a \( K \)-ordinary point. Note that if a subgraph of \( \Gamma \) has a \( K \)-ordinary point, topologically saying, it should have infinite points.

**Lemma 3.9.** Let \( \Gamma_1 \) be a \( K \)-invariant subgraph of \( \Gamma \). If \( \Gamma_1 \) is infinite and if the set of its boundary points \( \partial \Gamma_1 \) contains at least one \( K \)-ordinary point, then \( \varphi(\partial \Gamma_1) \) is a cut set of \( \Gamma' \) and contains a point on \( \Gamma' \) not being a vertex of \( G' \).

**Proof.** For such \( \Gamma_1 \), obviously \( \varphi(\partial \Gamma_1) \) contains a point on \( \Gamma' \) not being a vertex of \( G' \). It is sufficient to check that \( \varphi(\partial \Gamma_1) \) is a cut set of \( \Gamma' \). Let \( \Gamma_2 \) be the closure of the complement set of \( \Gamma_1 \) in \( \Gamma \). This \( \Gamma_2 \) is \( K \)-invariant and contains a \( K \)-ordinary point. Thus, \( \Gamma_2 \) is an infinite subgraph. Consequently, \( \Gamma' \backslash \varphi(\partial \Gamma_1) = \varphi(\Gamma_1 \cup \Gamma_2) \backslash \varphi(\partial \Gamma_1) = (\varphi(\Gamma_1) \backslash \varphi(\partial \Gamma_1)) \sqcup (\varphi(\Gamma_2) \backslash \varphi(\partial \Gamma_1)) \). Hence, \( \varphi(\partial \Gamma_1) \) is a cut set of \( \Gamma' \).

The next corollary follows from Lemma 3.8 and Lemma 3.9.
Corollary 3.10. For a subset of the support of $\varphi_*(D)$, we consider the following condition $(\ast)$:

$(\ast)$ it is a cut set of $\Gamma'$ containing no vertices of $G'$ and whose inverse image by $\varphi$ is a subset of the support of $D$.

(1) For a subset $A$ of $\text{supp}(D)$ whose image by $\varphi$ satisfies $(\ast)$, there exists a $K$-invariant infinite subgraph $\Gamma_1$ of $\Gamma$ which can fire on $D$ and whose boundary points are in $A$.

(2) Let $\Gamma_1$ be a $K$-invariant subgraph of $\Gamma$. If $\Gamma_1$ is infinite and can fire on $D$ and if the set of its boundary points consists only of $K$-ordinary points, then the image of the set of boundary points of $\Gamma_1$ by $\varphi$ satisfies $(\ast)$.

By Corollary 3.10, it is natural to define $S(D)_K$ as the set of $f$ in $R(D)_K$ such that there exist no cut sets of $\Gamma'$ contained in the support of $\varphi_*(D + \text{div}(f))$, containing no vertices of $G'$ and whose inverse image by $\varphi$ is a subset of the support of $D + \text{div}(f)$. In fact, this $S(D)_K$ is the set corresponding to $S(D)$, i.e. $S(D)_K$ is a generator set of $R(D)_K$.

Theorem 3.11. In the above situation, the following hold:

(1) $S(D)_K$ contains all the extremals of $R(D)_K$,

(2) $S(D)_K$ is finite modulo tropical scaling, and

(3) $S(D)_K$ generates $R(D)_K$ as a tropical semimodule.

Proof. (1) Suppose $f$ is in the difference set of $R(D)_K$ from $S(D)_K$, then there exists a cut set $A'$ of $\Gamma'$ contained in $\text{supp}(\varphi_*(D + \text{div}(f)))$, containing no vertices of $G'$ and such that $\varphi^{-1}(A') \subset \text{supp}(D + \text{div}(f))$. By (1) of Corollary 3.10, there exists a $K$-invariant infinite subgraph $\Gamma_1$ of $\Gamma$ which can fire on $D + \text{div}(f)$ and whose boundary points are in $\varphi^{-1}(A')$. Then, the closure of $\Gamma \setminus \Gamma_1$ can also fire on $D + \text{div}(f)$. Therefore, by Lemma 3.7, $f$ is not an extremal of $R(D)_K$.

(2) The push-forward of a rational function on $\Gamma$ induces a natural map $S(D)_K/R \to S(\varphi_*(D))/R$, $[f] \mapsto [\varphi_*(f)]$. In fact, for any $f$ in $S(D)_K$, $\varphi_*(D + \text{div}(f)) = \varphi_*(D) + \varphi_*(\text{div}(f)) = \varphi_*(D) + \text{div}(\varphi_*(f))$, thus, $\varphi_*(f)$ is in $R(\varphi_*(D))$. From $f \in S(D)_K$, there exist no cut sets of $\Gamma'$ contained
in supp(\(\varphi_*(D) + \varphi_*(\div(f))\)), containing no vertices of \(G'\) and whose inverse image by \(\varphi\) is a subset of supp\((D + \div(f))\). This means that \(\varphi_*(f)\) is in \(S(\varphi_*(D))\). Also, for any pair of \(f_1\) and \(f_2\) in \([f]\), there exists \(c\) in \(R\) satisfying \(f_2 = f_1 + c\). Since \(\varphi_*(f_2) = \varphi_*(f_1 + c) = \varphi_*(f_1) + \varphi_*(c) = \varphi_*(f_1) + c\), the map is well-defined. Now we show that the map is injective. Let \([f]\) and \([g]\) be distinct elements of \(S(D)_K/R\), thus \(\div(f)\) differs from \(\div(g)\). Since both \(f\) and \(g\) are \(K\)-invariant, so their images \(\varphi_*(\div(f))\) and \(\varphi_*(\div(g))\) are different, i.e. the map is injective. By Remark 3.1, we get the conclusion.

(3) Suppose \(f \in R(D)^K\). Let \(N(f)\) be the number of distinct \(K\)-orbits in the union of all \(K\)-invariant subsets of supp\((D + \div(f))\) which is a cut set of \(\Gamma'\) containing no vertices of \(G'\). We prove (3) by induction for \(N(f)\). If \(N(f) = 0\), then \(f \in S(D)_K\) from the definition of \(S(D)_K\). Assume that \(f \in \langle S(D)_K \rangle\) for all \(N(f) \leq k\), where \(\langle S(D)_K \rangle\) means the tropical semimodule generated by \(S(D)_K\). We consider the case where \(N(f) = k + 1\) and \(f \notin S(D)_K\). Let \(A\) be a subset of supp\((D + \div(f))\) whose image by \(\varphi\) is a cut set of \(\Gamma'\) containing no vertices of \(G'\). By (1) of Corollary 3.10 there exists a \(K\)-invariant subgraph \(\Gamma_1\) of \(\Gamma\) which can fire on \(D + \div(f)\) and whose boundary points are in \(A\). Let \(\Gamma_2\) be the closure of the complement of \(\Gamma_1\) in \(\Gamma\). For any \(x \in \partial \Gamma_i\), we write the distance between \(x\) and its closest vertex of \(G_1\) as \(l_{x_i}\). Let \(l_i := \min\{l_{x_i} | x \in \partial \Gamma_i \}\) and \(g_i := \text{CF}(\Gamma_i, l_i)\). Then, for both \(i = 1, 2\), \(f \odot g_i\) is not equal to \(f\) and is in \(R(D)^K\) since \(f, g_i \in R(D)^K\) and \(f = (f \odot g_1) \oplus (f \odot g_2)\). By the definition of \(g_i\), \(N(f) > N(f \odot g_i)\) and \(f \odot g_i \in \langle S(D)_K \rangle\), then \(f \notin \langle S(D)_K \rangle\).

By Lemma 2.2.3 and the above theorem, we obtain the following corollary, which is an extension of [3 Corollary 9].

**Corollary 3.12.** Let \(\Gamma\) be a tropical curve, \(D\) an effective divisor on \(\Gamma\) and \(K\) a finite subgroup of the isometry group of \(\Gamma\). Then, the tropical semimodule \(R(D)^K\) is generated by the extremals of \(R(D)^K\). This generating set is minimal and unique up to tropical scalar multiplication.

If \(D\) is \(K\)-invariant, \(R(D)^K/R\) is naturally identified with the \(K\)-invariant linear subsystem \(|D|^K\). In conclusion, the following statement holds from Theorem 3.11.

**Theorem 3.13.** Let \(\Gamma\) be a tropical curve, \(D\) an effective divisor on \(\Gamma\) and \(K\) a finite subgroup of the isometry group of \(\Gamma\). If \(D\) is \(K\)-invariant, then the \(K\)-invariant linear subsystem \(|D|^K\) of \(|D|\) is finitely generated by \(S(D)_K/R\).
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