Solving an inverse obstacle problem for the wave equation by using the boundary control method

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Abstract
We introduced in Oksanen (2011 arXiv:1106.3204) a method to locate discontinuities of a wave speed in dimension 2 from acoustic boundary measurements modeled by the hyperbolic Neumann-to-Dirichlet operator. Here we extend the method for sound hard obstacles in arbitrary dimension. We present numerical experiments with simulated noisy data suggesting that the method is robust against measurement noise.

(Some figures may appear in colour only in the online journal)

1. Introduction
Nondestructive obstacle reconstruction through wave propagation motivates a number of mathematical problems with several applications such as medical and seismic imaging. There is a large body of literature concerning obstacle detection using time harmonic waves, and we refer the reader to the review articles [9, 21] and to the monographs [10, 15]. Recently there has also been interest in reconstruction methods from acoustic measurements in the time domain [7, 8, 17, 18]. In this paper we present a numerical method of the latter type.

To illustrate our results, let us consider acoustic boundary measurements on a compact domain $M \subset \mathbb{R}^n$ that models a medium with a constant wave speed. Physically, $M$ could correspond, for example, to a water tank. Let $\Sigma \subset M^{\text{int}}$ be another compact domain that models a sound hard obstacle in $M$. Then the acoustic waves in $M$ are solutions of the wave equation,

$$\partial_t^2 u(t, x) - \Delta u(t, x) = 0, \quad (t, x) \in (0, \infty) \times (M \setminus \Sigma),$$

subject to the boundary condition $\partial_n u |_{(0, \infty) \times \Sigma} = 0$. We consider boundary measurements of acoustic waves that are encoded by the operator

$$\Lambda : \partial_n u |_{(0, \infty) \times \partial M} \mapsto \tilde{u} |_{(0, \infty) \times \partial M},$$

where $u$ is a solution to the above wave equation with vanishing initial conditions at $t = 0$. Physically, this data can be obtained in finite precision by using, for example, pulse-echo
measurements $/Lambda_1$ have smooth boundaries. In this case, the proposed method recovers the surface $\partial \Sigma$ given the measurements $\Lambda$.

The method is very versatile with respect to the acoustic medium considered, and instead of the constant wave speed as above, we can allow the wave speed to be anisotropic and non-homogeneous. In fact, we prove our results in the case where the medium is modeled by a weighted Laplace operator on a Riemannian manifold. This allows us to cover the most important non-homogeneous cases, such as the isotropic wave equation

$$\partial_t^2 u - c(x)^2 \Delta u = 0,$$

while retaining the ease of computations due to the geometric interpretation. Moreover, the method provides a partial reconstruction of $\partial \Sigma$ given measurement data only on a part of the boundary $\partial M$.

The proposed method is based on the boundary control (BC) method. By using the BC method, a smooth wave speed can be fully reconstructed from the Neumann-to-Dirichlet operator. In the isotropic case, this uniqueness result is by Belishev who introduced the BC method in [2]. In the anisotropic case, the uniqueness follows from [5] when the measurement data are given on an infinite time interval, and is proved in [1] when the measurement data are given on a finite time interval with optimal length. Moreover, it is known that by using singular harmonic functions, the BC method is able to recover obstacles; see the discussion in [3, page R42]. Although the results in this paper are not new as uniqueness results, our formulation of the method is novel from the point of view of practical computations. In particular, no singular harmonic functions are employed, and the formulation leads to a straightforward numerical implementation. To our knowledge, no previous formulation of the BC method for obstacle detection has been tested in numerical experiments. We refer to the monograph [14] and to the review article [4] for further details on the BC method.

2. Statement of the results

Let $M$ be a compact smooth manifold with smooth boundary $\partial M$ and let $g$ be a smooth Riemannian metric tensor on $M$. Let $\Sigma \subset M^m$ be a compact set with nonempty interior and smooth boundary, and let $\mu \in C^\infty(M)$ be strictly positive. We consider the following wave equation on $M$:

$$\begin{align*}
\partial_t^2 u(t, x) - \Delta_{g, \mu} u(t, x) &= 0, & (t, x) &\in (0, \infty) \times (M \setminus \Sigma), \\
\partial_{\nu, \mu} u(t, x) &= f(t, x), & (t, x) &\in (0, \infty) \times \partial M, \\
\partial_{\nu, \mu} u(t, x) &= 0, & (t, x) &\in (0, \infty) \times \partial \Sigma, \\
u|_{t=0} (x) &= 0, & \partial_t u|_{t=0} (x) &= 0, & x &\in M \setminus \Sigma,
\end{align*}$$

where $\Delta_{g, \mu}$ is the weighted Laplace–Beltrami operator and $\partial_{\nu, \mu}$ is the normal derivative corresponding to $\Delta_{g, \mu}$. That is, if we let $(g^k(\chi))_{j=1}^n$ and $|g(\chi)|$ denote the inverse and determinant of $g(\chi)$ in local coordinates, respectively, then we have

$$\begin{align*}
\Delta_{g, \mu} u &= \mu^{-1} \text{div}(\mu \text{ grad } u) \\
&= \sum_{j,k=1}^n \mu(\chi)^{-1} |g(\chi)|^{-\frac{1}{2}} \frac{\partial}{\partial \chi^j} \left( \mu(\chi)|g(\chi)|^{\frac{1}{2}} g^{jk}(\chi) \frac{\partial u}{\partial \chi^k} \right), \\
\partial_{\nu, \mu} u &= \mu(\text{grad } u, v)_{TM \times T^*M} = \sum_{j,k=1}^n \mu(\chi) v_k(\chi) g^{jk}(\chi) \frac{\partial u}{\partial \chi^j}.
\end{align*}$$
where \( \nu = (v_1, \ldots, v_n) \) is the exterior co-normal vector of \( \partial M \) normalized with respect to \( g \), that is, \( \sum_{j,k=1}^n g^{kj} v_j v_k = 1 \).

In what follows, we consider \( L^2(S) \) as the subspace of functions \( f \) in \( L^2((0, T) \times \partial M) \) satisfying \( \text{supp}(f) \subset S \) whenever \( S \subset (0, T) \times \partial M \) is closed. Let us denote the solution of (1) by \( u^f(t, x) = u(t, x) \). For \( T > 0 \) and an open \( \Gamma \subset \partial M \) we define the Neumann-to-Dirichlet operator

\[
\Lambda_{T, \Gamma} : f \mapsto u^f|_{(0, T) \times \Gamma}, \quad f \in L^2((0, T) \times \Gamma).
\]

Let us assume that the metric tensor \( g \) and the weight function \( \mu \) are known but \( \Sigma \) is unknown. We will describe a method to locate \( \Sigma \) from the measurements \( \Lambda_{T, \Gamma} \).

Let us point out that if \( M \subset \mathbb{R}^n, g = c(x)^{-2} dx^2 \) and \( \mu(x) = c(x)^{n-2} \) where \( c \in C^\infty(M) \) is strictly positive, then \( \Delta_{g, \mu} = c(x)^2 \Delta \), where \( \Delta \) is the Euclidean–Laplacian. Thus the isotropic wave equation is covered by the theory. The more general equation (1) allows for an anisotropic wave speed to be modeled.

Note that the operator \( \Delta_{g, \mu} \) with the domain \( H^2(M) \cap H^1_0(M) \) is self-adjoint on the space \( L^2(M; \mu dV_g) \), where \( dV_g \) is the Riemannian volume measure of \( (M, g) \), that is, \( \mu dV_g = |g|^{1/2} dx \) in local coordinates. We call \( \mu dV_g \) the measure corresponding to \( \Delta_{g, \mu} \) and denote it also by \( V \).

We define for a function \( \tau : \partial M \to \mathbb{R} \) the domain of influence with and without the obstacle,

\[
M_{\Sigma}(\tau) := \{ x \in M \setminus \Sigma \mid \text{there is } y \in \partial M \text{ such that } d_{\Sigma}(x, y) \leq \tau(y) \},
\]

\[
M(\tau) := \{ x \in M \mid \text{there is } y \in \partial M \text{ such that } d(x, y) \leq \tau(y) \},
\]

where \( d_{\Sigma} \) is the Riemannian distance function of \( (M \setminus \Sigma, g) \) and \( d \) is that of \( (M, g) \). As \( (M, g) \) is known, we can compute the shape of the domain of influence \( M(\tau) \) for any \( \tau : \partial M \to \mathbb{R} \). Our main theorem is the following.

**Theorem 1.** Let \( T > 0 \) and let \( \Gamma \subset \partial M \) be open. For a function \( \tau \) in

\[
C_T(\Gamma) := \{ \tau : \partial M \to \mathbb{R} \mid \tau|_{\Gamma} \in C(\Gamma), \ 0 \leq \tau \leq T, \ \tau|_{\partial M \setminus \Gamma} = 0 \},
\]

the volume \( V(M_{\Sigma}(\tau)) \) can be computed from \( \Lambda_{2T, \Gamma} \) by solving a sequence of linear equations on \( L^2((0, T) \times \Gamma) \). Moreover,

\[
M(\tau) \setminus \Sigma = \emptyset \quad \text{if and only if} \quad V(M_{\Sigma}(\tau)) < V(M(\tau)).
\]

Theorem 1 allows us to probe the obstacle with the known domains of influence \( M(\tau) \), \( \tau \in C_T(\Gamma) \). We will illustrate this probing method in section 4 via numerical experiments in the two-dimensional case. In section 3 we give a proof of theorem 1 that is based on ideas from the BC method. The BC method depends on Tataru’s hyperbolic unique continuation result [22], whence it is expected to have only logarithmic type stability. Also, our result depends on [22] and this is reflected in the fact that the linear equations solved as a part of the method need to be regularized; see formula (10) below. From the computational point of view, our method can be seen as a modification of that in [6], and the iterative time-reversal control strategy introduced there can be adapted to give an efficient implementation of our method.

### 3. Proof of the main theorem

We begin by showing that the volumes \( V(M_{\Sigma}(\tau)) \), \( \tau \in C_T(\Gamma) \), can be computed from \( \Lambda_{2T, \Gamma} \) by solving a sequence of linear equations on \( L^2((0, T) \times \partial M) \). Our proof relies on general
results from regularization theory and it can also be adapted to simplify the arguments in [20].

We define the operator

\[ K := J\Lambda_{2T,\Gamma} \Theta_{2T} - RA_{T,\Gamma} RJ\Theta_{2T}, \]

where \( \Theta_{2T} \) is the extension by zero from \((0, T)\) to \((0, 2T)\), \( R \) is the time reversal on \((0, T)\),

that is \( RF(t) := f(T - t) \), and

\[ Jf(t) := \frac{1}{2} \int_{t}^{2T-t} f(s) \, ds, \quad f \in L^2(0, 2T), \quad t \in (0, T). \]

The operator \( K \) is the principal object of the BC method and it is called the connecting operator [4]. We recall that \( K \) is a compact operator on \( L^2((0, T) \times \Gamma) \) since, see [23],

\[ \Lambda_{T,\Gamma} : L^2((0, T) \times \Gamma) \to H^{2/3}((0, T) \times \Gamma). \]

Let \( f \in C_0^\infty((0, T) \times \Gamma) \) and let \( \phi \in C^\infty(M \setminus \Sigma) \). Moreover, let \( t \in (0, \infty) \) and integrate by parts

\[ \partial_t^2 (u^i(t), \phi)_{L^2(M \setminus \Sigma; d\Sigma')} = (\Lambda_{x,\mu} u^i(t), \phi)_{L^2(M \setminus \Sigma; d\Sigma')} - (\partial_{x,\mu} u^i(t), \phi)_{L^2(M \setminus \Sigma; d\Sigma')}, \]

(3)

where \( d\Sigma' \) denotes the Riemannian surface measure on \((\partial M, g)\). Note that the boundary term on \( \partial \Sigma \) vanish as \( u^i \) satisfies the homogeneous Neumann boundary condition there.

In particular, for \( f, h \in C_0^\infty((0, T) \times \Gamma), t \in (0, T) \) and \( s \in (0, 2T), \)

\[ (\partial_t^2 - \partial_s^2)(u^i(t), u^h(s))_{L^2(M \setminus \Sigma; d\Sigma')} = (f(t), \Lambda_{2T,\Gamma} h(s))_{L^2(0; d\partial M; d\Sigma')} - (\Lambda_{T,\Gamma} f(t), h(s))_{L^2(0; d\partial M; d\Sigma')} - (\Lambda_{T,\Gamma} u^i(t), \phi)_{L^2(M \setminus \Sigma; d\Sigma')}. \]

By solving this wave equation with vanishing initial conditions at \( t = 0 \) and noting that \( \Lambda_{T,\Gamma}^* = RA_{T,\Gamma} R, \) we obtain Blagoveščenski's identity

\[ (u^i(T), u^h(T))_{L^2(M \setminus \Sigma; d\Sigma')} = (f, Kh)_{L^2(0; d\partial M; d\Sigma')} - (\Lambda_{T,\Gamma} f(t), h(s))_{L^2(0; d\partial M; d\Sigma')}. \]

(4)

which holds for all \( f, h \in L^2((0, T) \times \Gamma) \) by continuity of \( K \) and density of smooth functions in \( L^2 \). The identity (4) has its roots in the work by Blagoveščenski who observed that for a wide class of hyperbolic equations, the inner products of waves satisfy the string equation with the right-hand side determined by the measurement data on the boundary.

Moreover, by letting \( \phi = 1 \) identically in (3), we obtain

\[ \partial_t^2 (u^i(t), 1)_{L^2(M \setminus \Sigma; d\Sigma')} = (\partial_{x,\mu} u^i(t), 1)_{L^2(M \setminus \Sigma; d\Sigma')}. \]

(5)

Note that this identity does not hold if \( u^i \) satisfies the homogeneous Dirichlet boundary condition on \( \partial \Sigma \), instead of the Neumann one. This is why our method does not extend to the detection of sound soft obstacles in a straightforward manner. We obtain the identity

\[ (u^i(T), 1)_{L^2(M \setminus \Sigma; d\Sigma')} = (f, b)_{L^2(0; d\partial M; d\Sigma')}, \]

(6)

where \( b(t, x) = T - t, \) by solving the ordinary differential equation (5) with vanishing initial conditions at \( t = 0 \).

Let \( \tau \in C_1(\Gamma) \) and let us define the set

\[ S_\tau := \{(t, x) \in [0, T] \times \overline{\Gamma}; \ t \in [T - \tau(x), T]\}. \]

We define the operator

\[ W_\tau f := u^i(T), \quad W_\tau : L^2(S_\tau) \to L^2(M \setminus \Sigma). \]

It follows from [16] that \( W_\tau \) is compact. Moreover, we may consider a restriction of \( K \),

\[ K_\tau f = Kf|_{S_\tau}, \quad K_\tau : L^2(S_\tau) \to L^2(S_\tau). \]
Then the equations (4) and (6) yield that on $L^2(S_t)$
\[ W_t^* W_t = K_t, \quad W_t^* 1 = b. \tag{7} \]

Let us now consider the control equation
\[ W_t f = 1, \quad \text{for } f \in L^2(S_t). \tag{8} \]

We have $\text{supp}(W_t f) \subset M_{2\Sigma}(\tau)$ since the wave equation (1) has finite speed of propagation. Moreover, it can be shown using Tataru’s unique continuation [22] that the inclusion
\[ \{W_t f : f \in L^2(S_t)\} \subset L^2(M_{2\Sigma}(\tau)) \tag{9} \]
is dense; see the appendix below. In particular, if there is a least-squares solution $f_0$ to (8) then we would have $W_t f_0 = 1_{M_{2\Sigma}(\tau)}$. However, as $W_t$ is compact, the range of $W_t$ is a proper dense subset of $L^2(M_{2\Sigma}(\tau))$ and (8) typically fails to have a least-squares solution. In fact, in the one-dimensional case it is easy to show that (8) never has a least-squares solution.

The standard technique to remedy the nonexistence of a least-squares solution to a linear equation is to use a regularization method. As $W_t$ is compact and we have the information (7) at our disposal, there are several ways to regularize that are available to us. For example, we could use a regularization by projection [11, section 3.3] or a regularization based on a spectral approximation of the inverse [11, theorem 4.1]. Here we will consider only the classical Tikhonov regularization,
\[ f_\alpha := (W_t^* W_t + \alpha)^{-1} W_t^* 1 = (K_t + \alpha)^{-1} b, \quad \alpha > 0. \tag{10} \]

We have the following abstract lemma.

**Lemma 1.** Suppose that $X$ and $Y$ are Hilbert spaces. Let $y \in Y$ and let $A : X \to Y$ be a bounded linear operator with the range $R(A)$. Then $Ax_\alpha \to Py$ as $\alpha \to 0$, where $x_\alpha = (A^* A + \alpha)^{-1} A^* y$, $\alpha > 0$, and $P : Y \to R(A)$ is the orthogonal projection.

**Proof.** Note that for all $x \in X$
\[ \|Ax - y^2\| = \|Ax - Py^2 + (1 - P)y^2\|. \]

By [11, theorem 5.1] we know that $x_\alpha$ is the unique minimizer of
\[ \|Ax - y^2 + \alpha\| x. \]

Let $\epsilon > 0$ and let $x^\epsilon \in X$ satisfy $\|Ax^\epsilon - Py^2\| < \epsilon$. Then
\[ \|Ax_\alpha - Py^2\| \leq \|Ax_\alpha - y^2 - (1 - P)y^2\| \\ \leq \|Ax_\alpha - y^2 + \alpha\| x_\alpha - \| (1 - P)y^2\| \\ \leq \|Ax^\epsilon - y^2 + \alpha\| x^\epsilon - \| (1 - P)y^2\| \\ = \|Ax^\epsilon - Py^2 + \alpha\| x^\epsilon < \epsilon + \alpha\| x^\epsilon \leq 2\epsilon, \]
for $\alpha \leq \epsilon / \|x^\epsilon\|$.

By the density (9) we have that $R(W_t) = L^2(M_{2\Sigma}(\tau))$. Thus, the above lemma implies that $W_t f_\alpha \to 1_{M_{2\Sigma}(\tau)}$ in $L^2(M \setminus \Sigma)$ as $\alpha$ tends to zero. In particular, we may compute the volume $V(M_{2\Sigma}(\tau))$ from the boundary data $A_{2\Sigma;\Lambda}$ by the formula
\[ V(M_{2\Sigma}(\tau)) = \lim_{\alpha \to 0^+} ((K_t + \alpha)^{-1} b, b)_{L^2(S_t; d\sigma dS_t)}. \tag{11} \]

**Lemma 2.** Let $T > 0$, $\Gamma \subset \partial M$ be open and let $\tau \in C_T(\Gamma)$. Then
\[ M(\tau)^{\text{int}} \cap \Sigma^{\text{int}} \neq \emptyset \quad \text{if and only if} \quad V(M_{2\Sigma}(\tau)) < V(M(\tau)). \]
Proof. Note that $d_U(x,y) \geq d(x,y)$ for any $x,y \in M \setminus \Sigma$. Hence $M_\Sigma(\tau) \subset M(\tau)$. Moreover, $M_\Sigma(\tau) \cap \Sigma = \emptyset$ by definition. In particular, if the open set $M(\tau)^{\text{int}} \cap \Sigma^{\text{int}}$ is nonempty, then
\begin{equation*}
V(M_\Sigma(\tau)) \leq V(M(\tau) \setminus \Sigma) < V(M(\tau) \setminus \Sigma) + V(M(\tau)^{\text{int}} \cap \Sigma^{\text{int}}) \leq V(M(\tau)).
\end{equation*}
Thus, we have shown the implication from left to right in (2).

Let us now suppose that $V(M_\Sigma(\tau)) < V(M(\tau))$. Then $M(\tau) \setminus M_\Sigma(\tau)$ is not a null set (that is, a set of measure 0). But $\partial M(\tau)$ is a null set [20], whence there is $x \in M(\tau)^{\text{int}} \setminus M_\Sigma(\tau)$. Thus there is $y \in \partial M$ and a path $\gamma : [0, \ell] \to M$ from y to x such that the length of $\gamma$ satisfies $l(\gamma) \leq \tau(y)$. The path $\gamma$ intersects $\Sigma$ since otherwise we would have $x \in M_\Sigma(\tau)$. Let $z \in \Sigma \cap \gamma([0, \ell])$. Then $z \in M(\tau)^{\text{int}}$ since $x \in M(\tau)^{\text{int}}$, and there is a neighborhood $U \subset M(\tau)^{\text{int}}$ of $z$ such that $U \cap \Sigma^{\text{int}} \neq \emptyset$. Hence also $M(\tau)^{\text{int}} \cap \Sigma^{\text{int}} \neq \emptyset$. \qed

Theorem 1 follows from formula (11) and lemma 2.

4. Numerical results

4.1. Simulation of the data

In all our numerical examples $(M, g)$ is the two-dimensional unit square with the Euclidean metric, that is,
$$M = [0, 1]^2, \quad g = (dx^1)^2 + (dx^2)^2.$$ Moreover, $T = 1$ and the accessible part of the boundary $\Gamma$ is the bottom edge of $M$,
$$\Gamma = \{(x^1, 0) \in M; x^1 \in (0, 1)\}.$$

For the computation of the Dirichlet-to-Neumann map we discretize in space by using finite elements, and solve the resulting system of ordinary differential equations by a backward differentiation formula (BDF). To be very specific, we use the commercial Consol solver with quadratic Lagrange elements and BDF time-stepping with maximum order of 2. Both the maximum element size and time step size are set to the constant value $h = 0.0025$.

We discretize the measurement $\Lambda_{\Gamma, T} f, f \in L^2((0, T) \times \Gamma)$, by taking the point values on the uniform grid of temporal points $t_j \in [0, 2T], j = 1, 2, \ldots, N_t$, and spatial points $x_k \in \Gamma$, $k = 1, 2, \ldots, N_s$, where $N_t = 20$ and $N_s = 800$. The higher precision in time reflects the fact that a measurement of this type can be realized by using $N_t$ receivers (e.g. microphones) with the sampling rate $h$.

We model noisy measurements by adding white Gaussian noise to the signal
$$\lambda_f(j, k) := \Lambda_{\Gamma, T} f(t_j, x_k), \quad j = 1, 2, \ldots, N_t, \quad k = 1, 2, \ldots, N_s.$$ To be very specific, we use the Matlab function awgn both to measure the power of the signal $\lambda_f$ and to add noise with a specified signal-to-noise ratio (SNR). We have used SNRs 14 dB and 7 dB corresponding to 4% and 20% noise power levels, respectively.

4.2. Solving the control equation

The operator $K_\tau$ is self-adjoint and positive-semidefinite by (7), whence $K_\tau + \alpha$ positive-definite for $\alpha > 0$. We solve the Tikhonov regularized control equation
\begin{equation}
(K_\tau + \alpha) f = b
\end{equation}
by using the conjugate gradient (CG) method on a finite-dimensional subspace $C_\tau \subset L^2(S_\tau)$ that we will define below. We have used the initial value $f = 0$ in all our CG iterations.
We denote by \( \Gamma_k \subset \Gamma \) the Voronoi cell corresponding to the measurement point \( x_k \), \( k = 1, 2, \ldots, N_x \), that is,
\[
\Gamma_k := \{ x \in \Gamma; |x - x_k| \leq |x - x_l|, \ l = 1, 2, \ldots, N_x \}.
\]
Moreover, we denote by \( C \) the space of piecewise constant sources \( f \) that can be represented as a linear combination of the functions
\[
f_k(t, x) := 1_{[0,h]}(t)1_{\Gamma_k}(x), \quad k = 1, 2, \ldots, N_t,
\]
and their time translations by an integer multiple of \( h \). Finally, we define
\[
C_\tau := \{ f \in C; \text{supp}(f) \subset S_\tau \}, \quad \tau \in C_T(\Gamma).
\]
As the wave equation (1) is invariant with respect to translations in time, we can compute \( \lambda_f \) for arbitrary \( f \in C_\tau \) and \( \tau \in C_T(\Gamma) \) if we are given the measurements
\[
\lambda_{f_k}, \quad k = 1, 2, \ldots, N_t.
\]
To summarize, we employ \( N_t = 20 \) measurements that can be realized by using \( N_t \) receivers with the sampling rate \( h = 0.0025 \).

### 4.3. Regularization and calibration

As the control equation (12) may be ill-posed for \( \alpha = 0 \), we terminate the CG iteration early after \( N_{cg} \) steps. This amounts to regularization of the problem [12]. To calibrate the method we probed the empty space case, \( \Sigma = \emptyset \), with half-spaces. That is, we chose the profile function
\[
\tau_r(x) := r, \quad x \in \Gamma, \quad r \in [1/10, 1/2].
\]
In this case, the CG iteration essentially converges after ten steps even when not using the Tikhonov regularization, that is, \( \alpha = 0 \), see figure 1. For this reason, we have chosen \( N_{cg} = 10 \) in all our further simulations.

In addition to the empty space case, we have experimented with the disk and the square-shaped obstacles defined as follows: \( \Sigma_d \) is the disk with radius \( 3/10 \) and center \( p := (1/2, 1/2) \) and \( \Sigma_s \) is the square with side length 0.424, center \( p \) and axes rotated by \( \pi/4 \) with respect to the axes of \( M \), see figure 4.

It is not clear to us, why the method underestimates the volume \( V(M_p(\tau_r)) \), see figure 2 (the leftmost plot). One possibility is that we use too few spatial basis functions; however, the smallness of \( N_t \) is motivated by applications. Moreover, the underestimation is systematic and is canceled when considering the volume differences,
\[
V(M_{\Sigma}(\tau)) - V(M_p(\tau)), \quad (14)
\]
0.1 0.3 0.5

Figure 2. Reconstructed volumes $V(M(\tau_r))$ as a function of $r$ compared to the real volume (dotted red). The two reconstructions (solid blue and dashed blue) correspond to two different realizations of noise. From left, first: noiseless, $\alpha = 0$; second: SNR = 14 dB, $\alpha = 0$; third: SNR = 7 dB, $\alpha = 0$; fourth: SNR = 7 dB, $\alpha = 10^{-3}$.

0.1 0.3 0.5

Figure 3. Reconstructed volume differences (14) with $\tau = \tau_r$ as a function of $r$ compared to the real difference (dotted red) in the case of the disk-shaped obstacle $\Sigma = \Sigma_r$. The two reconstructions (solid blue and dashed blue) correspond to two different realizations of noise. From left, first: noiseless, $\alpha = 0$; second: SNR = 14 dB, $\alpha = 0$; third: SNR = 7 dB, $\alpha = 10^{-3}$.

see figure 3. In terms of applications, this means that we should calibrate the method in a known background before probing a region that possibly contains an obstacle.

According to our experiments the method reconstructs volumes reliably when SNR = 14 dB and we regularize only through the early termination of the CG iteration. When SNR = 7 dB and $\alpha = 0$, a reconstruction can be seriously disrupted even in the empty space case. After introducing Tikhonov regularization with $\alpha = 10^{-3}$, the effect of noise vanishes but a large systematic error appears, see figure 2 (the two rightmost plots). We see that considering the volume differences (14) becomes even more essential when $\alpha > 0$.

4.4. Probing with disk-shaped domains of influence

We will now describe our experiments concerning reconstruction of the shape of an obstacle. To this purpose, we chose the profile function $\tau \in C_2(\Gamma)$ to be of the form,

$$\tau_r^y(x) := r - |x - y|, \quad x \in \Gamma, \quad y \in \Gamma, \quad r \in [1/10, 1/2].$$

Then $M(\tau_r^y) = B(y, r)$, that is, the intersection of $M$ and the closed disk of radius $r$ centered at $y$. Probing with disks has been considered in the context of electrical impedance tomography in [13] and our numerical results are comparable to the results therein.

Analogously to [13] and [19], let us define the largest region $H_{\Sigma}(\Gamma)$ on which we can conclude the absence of obstacles by probing with the sets $B(y, r) \cap M, y \in \Gamma, r \in (0, T)$. We denote

$$R_T(y) := \sup\{r \in (0, T); B(y, r) \cap \Sigma^\text{int} = \emptyset\} = \sup\{r \in (0, T); V(M(\tau_r^y)) = V(M(\tau_r^y'))\},$$
and define

$$H_\Sigma (\Gamma) := \bigcup_{y \in \Gamma} \{ B(y, R_T(y)) \cap \mathcal{M} \}.$$ 

Let us describe next how we approximate $R_T(y)$ when computing with finite precision. Let $\epsilon > 0$, $N_r \in \mathbb{N}$ and let $r_l \in [0, T]$, $l = 1, 2, \ldots, N_r$, be a uniform grid of points. We denote

$$L(\epsilon, N_r) := \max\{ l = 1, 2, \ldots, N_r ; V(M_\Sigma (\tau_l)) - V(M_\mathcal{M} (\tau_l)) \geq -\epsilon \},$$
and define the approximation \( r_T(y; \epsilon, N_r) = r_{L(\epsilon, N_r)}(y) \). We have used the threshold \( \epsilon = 5 \times 10^4 \) in noiseless cases and \( \epsilon = 4 \times 10^3 \) when SNR = 14 dB. According to our numerical experiments the method reconstructs \( H_2/(\Gamma_1) \) reliably when using these values of \( \epsilon \) and \( N_r = 500 \), see figure 5, where a white pixel means that the center point of the pixel is erroneously identified to be in \( H_2/(\Gamma_1) \) (false positive) and a black pixel means erroneous identification of not being in \( H_2/(\Gamma_1) \) (false negative).

Computationally, the shape reconstruction amounts to solving a large number of independent systems of linear equations by running a few number of CG steps for each of them. Our implementation with parameters as above and \( r_l(s) \) restricted in \([1/10, 1/2]\) led to 4020 systems with the number of unknowns varying between 30 and 1000. The run time for the full reconstruction on a single processor was about 10 min; however, as the systems are independent, the method allows for an efficient parallel implementation.

**Appendix. Approximate controllability**

In this section we show that the inclusion (9) is dense, that is we prove the following lemma.

**Lemma 3.** Let \( T > 0 \), let \( \Gamma \subset \partial M \) be open and let \( \tau \in CT(\Gamma) \). Then
\[ \{ u^\tau(T); f \in C^\infty_0(S_\tau) \} \]

is dense in \( L^2(M_{\Sigma}(\tau)) \).

A density result of this type is called approximate controllability in the control theoretic literature. To our knowledge, lemma 3 is proved previously only in the case of a constant function \( \tau \), see e.g. [14, theorem 3.10]. We will give a proof in the general case \( \tau \in CT(\Gamma) \) by reducing it to the constant function case. To simplify the notation we consider only the case \( \Sigma = \emptyset \), since the general case follows by replacing \( M \) by \( M \setminus \Sigma \) in the proofs below.

**Lemma 4.** Let \( T > 0 \), \( J \in \mathbb{N} \), let \( \Gamma_j \subset \partial M \) be open and let \( h_j \in CT(\Gamma_j) \) for \( j = 1, 2, \ldots, J \). We define
\[ h^J(y) := \begin{cases} \max\{h_j(y); j \, \text{satisfies} \, \Gamma_j \ni y\}, & y \in \bigcup_{j=1}^J \Gamma_j, \\ 0, & \text{otherwise}. \end{cases} \]  

If for all \( j = 1, \ldots, J \) the functions
\[ \{ u^J(T); f \in C^\infty_0(S_{h_j}) \} \]
are dense in \( L^2(M(h_j)) \), then the functions
\[ \{ u^J(T); f \in C^\infty_0(S_{h^J}) \} \]
are dense in \( L^2(M(h^J)) \).

**Proof.** Note that \( \partial M \subset M(\tau) \) if \( \tau(y) \geq 0 \) for all \( y \in \partial M \). Abusing the notation slightly, we will consider \( M(\tau) \) as a subset of \( M^\text{int} \). This does not affect the density since \( \partial M \) is a null set. We denote \( \Gamma^J := \bigcup_{j=1}^J \Gamma_j \) and have
\[ M(h^J) = \{ x \in M^\text{int}; \, \text{there is} \, y \in \bigcap_{j=1}^J \Gamma_j \, \text{s.t.} \, d(x, y) \leq h(y) \} \]
\[ = \bigcup_{j=1}^J \{ x \in M^\text{int}; \, \text{there is} \, y \in \bigcap_{j=1}^J \Gamma_j \, \text{s.t.} \, d(x, y) \leq h_j(y) \} \]
\[ = \bigcup_{j=1}^J M(\Gamma_j, h_j). \]
We will now prove the density by induction with respect to $J$. The case $J = 1$ is trivial. Let us denote $M_0 := M(h^J)$ and $M_1 := M(h_{J+1})$. Let $\psi \in L^2(M_0 \cup M_1)$. By induction hypothesis there is a sequence of smooth functions $(f^0_k)_{k=1}^\infty$ supported in $S_{h^J}$ such that
$$u^0(T) \to 1_{M_0} \psi.$$ Moreover, there is a sequence of smooth functions $(f^1_k)_{k=1}^\infty$ supported in $S_{h_{J+1}}$ such that
$$u^1(T) \to 1_{M_1} (\psi - 1_{M_0} \psi).$$ Thus,
$$u^{0+1}(T) \to 1_{M_0} (1 - 1_{M_1}) \psi + 1_{M_1} \psi = \left(1_{M_0 \setminus M_1} + 1_{M_1}\right) \psi = 1_{M_0 \cup M_1} \psi = \psi.$$ Moreover, $f^0_k + f^1_k$ is supported in $S_{h^J} \cup S_{h_{J+1}} \subset S_{h^{J+1}}$.

**Proof of lemma 3.** Let $\psi \in L^2(M(h^J))$ and $\epsilon > 0$. There is a simple function
$$h_\epsilon(y) = \sum_{j=1}^J T_j 1_{\Gamma_j}(y),$$ where $J \in \mathbb{N}$, $T_j \in (0, T)$ and $\Gamma_j \subset \Gamma$ are open and disjoint, such that $\tau < h_\epsilon + \epsilon$ almost everywhere on $\Gamma$ and $h_\epsilon < \tau$ on $\overline{\Gamma}$, see e.g. [20, proof of lemma 4.2]. We denote $h_j := T_j 1_{\Gamma_j}$ and define $\tau_\epsilon = h^J$ as the maximum (A.2). By the construction $\tau_\epsilon < \tau$ and $\tau_\epsilon > h_\epsilon$.

The functions (A.1) for $\tau = h_j$ are dense in $L^2(M(h^J))$ by [14, proof of theorem 3.10]. Lemma 4 implies that there is a smooth function $f$ supported in $S_{\tau_\epsilon} \subset S_{\tau}$ such that
$$\|1_{M(\tau_\epsilon)} \psi - u^\epsilon(T)\|_{L^2(M)}^2 < \epsilon.$$ Thus,
$$\|\psi - u^\epsilon(T)\|_{L^2(M)}^2 < \epsilon + \int_{M(\tau_\epsilon) \setminus M(\tau_\epsilon)} \psi^2 \, dV.$$ We have $V(M(\tau_\epsilon)) \to V(M(\tau))$ as $\epsilon \to 0$, see [20]. Thus the second term in (A.3) tends to zero as $\epsilon \to 0$.

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