Thermodynamics and weak cosmic censorship conjecture of the BTZ black holes in the extended phase space

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Abstract:
As a charged fermion drop into a BTZ black hole, the laws of thermodynamics and the weak cosmic censorship conjecture are checked in both the normal phase space and extended phase space, where the cosmological parameter and renormalization length are regarded as extensive quantities. In the normal phase space, the first law, second law, and the weak cosmic censorship are valid. While in the extended phase space, the first law as well as the weak cosmic censorship conjecture are still valid, the second law is dependent on the variation of the renormalization energy $dK$. In addition, in the extended phase space, the configurations of the extremal and near-extremal black holes will not be changed for they are stable while in the normal phase space, the extremal and near-extremal black holes will evolve into non-extremal black holes.

Keywords: weak cosmic censorship conjecture, thermodynamics, BTZ black holes

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1 Introduction

From the pioneering work of Hawking \cite{1, 2}, we now know that a black hole can be regarded as a thermodynamic system. Similar to the usual thermodynamic systems, there are four laws of thermodynamics for the black holes. The event horizons of the black holes play a key role in the thermodynamic systems for both the temperature and entropy are related to it. In addition, the event horizon will hidden the singularity of the spacetime, otherwise the weak cosmic censorship conjecture proposed by Penrose \cite{3} will be violated. The Kretschmann scalar can be used to check the weak cosmic censorship conjecture since it is independent of the choice of coordinates \cite{4, 5}. The location where the Kretschmann scalar is infinite is the singularity.

The laws of thermodynamics and weak cosmic censorship conjecture can be checked with consideration of a test particle \cite{6, 7} or a test field \cite{8, 9}. In \cite{27}, the first law, second law as well as the weak cosmic censorship conjecture of a BTZ black hole have been investigated. It was found that the first and second laws were valid and the weak cosmic censorship conjecture held for the extremal black hole. Later, the idea in \cite{27} was extended to a $D$-dimensional charged AdS black hole in the extended phase space, where the negative cosmological constant and its conjugate were regarded as the pressure and volume respectively \cite{28}. An interesting result in \cite{28} is that the second law is violated for the extremal and near-extremal black holes as the contribution of the pressure and volume are considered. In addition, the extremal black holes are found to be stable for the absorbed particles will not change the configurations of the black holes. Recently, \cite{29} and \cite{30} investigated thermodynamics and weak cosmic censorship conjecture in the Born-Infeld AdS black holes and phantom Reissner-Nordström AdS black holes. Different from the result in \cite{28}, they found that the extremal black holes will change into non-extremal black holes. The reason stems from that they did not employ any approximation while \cite{28} employed.

It should be noted that in the works mentioned above, they considered only the case that the black holes absorb scalar particles. In this paper, we will study the case of fermions with the Dirac equation. We intend to explore whether we can obtain the same result for the result is
not priori. We will take the BTZ black holes as an example.

There are two viewpoints on the thermodynamics of the BTZ black holes in the extended phase space. Identifying the cosmological constant and its conjugate as the thermodynamic pressure and thermodynamic volume, the first law was derived directly in [31]. However, in their treatment, the Reverse Isoperimetric Inequality is violated and the black holes are always superentropic. Moreover, the thermodynamic volume defined by the first law is related to the charge. Soon after, [32] introduced a new extensive quantity, namely the renormalization length, in the first law. In this framework, the Reverse Isoperimetric Inequality is satisfied and the standard definition of the thermodynamic volume is retained. In this paper, we will check whether the first law proposed by [32] can be reproduced under a charged fermion absorption. Besides the first law, we will also investigate the second law as well as the weak cosmic censorship conjecture. As a result, the first law and the weak cosmic censorship conjecture are found to be valid.

In this paper, we will set $G = c = 1$.

2 Motion of a charged fermion in the BTZ black holes

The three dimensional theory of gravity with Maxwell tensor is [33]

$$I = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R - 2\Lambda - 4\pi GF_{\mu\nu}F^{\mu\nu} \right),$$

in which $G$ is the gravitational constant, $R$ is the Ricci scalar, $g$ is determinant of the metric tensor $g_{\mu\nu}$, $\Lambda$ is the cosmological constant that relates to the AdS radius with the relation $\Lambda = -1/l^2$, and $F_{\mu\nu} = A_{\nu\mu} - A_{\mu\nu}$, where $A_{\nu}$ the electrical potential. The charged BTZ black hole solutions can be derived from Eq. (1), that is

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\phi^2,$$

where

$$f(r) = m + \frac{r^2}{l^2} - 2q^2 \log \left( \frac{r}{l} \right),$$

in which $m$ and $q$ are the parameters which relate to the mass and charge of the black hole. The non-vanishing component of the vector potential of this black hole is

$$A_t = q \log \frac{r}{l}.$$

Using the Gauss law, the electric charge of the black hole can be obtained by calculating the flux of the electric field at infinity [34], which yields

$$Q = \frac{q}{2}.$$

In addition, the total mass can be got by using the Hamiltonian approach or the counterterm method [34], which leads to

$$M = \frac{m}{8}.$$

Now we turn to investigate the dynamical of a charged fermion as it is absorbed by the BTZ black hole. We will employ the Dirac equation for electromagnetic field

$$i\gamma^\mu \left( \partial_\mu + \Omega_\mu - \frac{i}{\hbar} e A_\mu \right) \psi - \frac{m}{\hbar} \psi = 0,$$

in which $u$ is the rest mass, $e$ is the charge of the fermions, $\Omega_\mu = \frac{i}{\hbar} \Gamma_\mu^{\alpha\beta} \sum_\alpha \beta$, $\sum_\alpha \beta = \frac{1}{4} [\gamma^\alpha, \gamma^\beta]$, $\gamma^\mu$ matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$. To obtain the solution of the Dirac equation, we should choose $\gamma^\mu$ matrices firstly. In this paper, we set

$$\gamma^\mu = \left( -i f^{-\frac{1}{2}} \sigma^2, f^{-\frac{3}{2}} \sigma^1, \frac{1}{r} \sigma^3 \right),$$

in which $\sigma^\mu$ are the Pauli sigma matrix

$$\sigma^1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma^2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma^3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

For a fermion with spin 1/2, the wave function have spin up state and spin down state. In this paper, we only investigate the spin up case for the case of spin down is similar. We use the ansatz for the two-component spinor $\psi$ as

$$\psi = \begin{pmatrix} A(t, r, \phi) \\ B(t, r, \phi) \end{pmatrix} \exp \left( \frac{i}{\hbar} I(t, r, \phi) \right).$$

Inserting Eq. (10) into Eq. (7), we have the following two simplified equations

$$A \left( \mu + \frac{1}{r} \partial_\phi I \right) + B \left( \sqrt{f} \partial_t I - \frac{1}{\sqrt{f}} \partial_r I - \frac{1}{\sqrt{f}} e A_t \right) = 0,$$

$$A \left( \sqrt{f} \partial_t I + \frac{1}{\sqrt{f}} \partial_r I - \frac{1}{\sqrt{f}} e A_t \right) + B \left( \mu - \frac{1}{r} \partial_\phi I \right) = 0.$$

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These two equations have a non-trivial solution for $A$ and $B$ if and only if the determinant of coefficient matrix vanishes, which implies

$$\frac{1}{r^2} (\partial_r I)^2 - \mu^2 + \left( \sqrt{f} \partial_r I \right)^2 - \frac{1}{\sqrt{f}} (\partial_r I - \frac{1}{\sqrt{f}} e A_r )^2 = 0.$$  

(13)

There are two Killing vectors in the charged BTZ spacetime, so we can make the separation of variables for $I(t,r,\phi)$ as

$$I = -\omega t + L\phi + I(r) + K,$$  

(14)

where $\omega$ and $L$ are fermion’s energy and angular momentum respectively, and $K$ is a constant. Putting Eq.(14) into Eq.(13), we obtain

$$\partial_r I(r) = \pm \frac{1}{f} \sqrt{(\omega + e A_r)^2 + f \left( \mu^2 - \frac{L^2}{r^2} \right)}.$$  

(15)

We are interested in the radial momentum of the particle $p^r \equiv g^{rr} p_r = g^{rr} \partial_r I(r)$. In addition, we want to investigate the thermodynamics, so we will focus on the near horizon region. In this case, we get

$$\omega = \left| p^r_+ \right| - e A_r (r_+),$$  

(16)

which is obviously the same as that of the scalar particles [27]. It should be stressed that a positive sign should be endowed in front of the $\omega$ term. This choice is to assure that the signs in front of $\omega$ and $p^r_+$ are the same and positive in the positive flow of time.

3 Thermodynamics and weak cosmic censorship conjecture in the normal phase space

The electrostatic potential difference between the black hole horizon and the infinity is

$$\Phi = -2Q \log \left( \frac{r_+}{l} \right).$$  

(17)

in which $r_+$ is the event horizon of the black hole, which is determined by $f(r_+) = 0$. Based on the definition of surface gravity, the Hawking temperature can be expressed as

$$T = \frac{r_+}{2\pi l^2} - \frac{2Q^2}{\pi r_+}.$$  

(18)

For the three dimensional BTZ black hole, the black hole entropy can be expressed as

$$S = \frac{1}{2} \pi r_+^2.$$  

(19)

In addition, with Eq.(3), and Eq.(6), the mass of the BTZ black hole can be expressed as

$$M = \frac{-8l^2 Q^2 \log \left( \frac{r_+}{l} \right) + r_+^2}{8l^2}.$$  

(20)

As a charged fermion is absorbed by the black hole, the variation of the internal energy and charge of the black hole satisfy

$$\omega = dM, e = dQ,$$  

(21)

in which the energy conservation and charge conservation have been imposed. In this case, Eq.(10) can be rewritten as

$$dM = \Phi dQ + p^r_+. $$  

(22)

The absorbed fermions will change the configurations of the black holes. And there is a shift for the horizon of the black hole, labeled as $dr_+$. In the new horizon, there is also a relation, $f(r_+ + dr_+) = 0$. In other words, the change of the horizon should satisfy

$$df_+ = f(r_+ + dr_+) - f(r_+) = \frac{\partial f_+}{\partial M} dM + \frac{\partial f_+}{\partial Q} dQ + \frac{\partial f_+}{\partial r_+} dr_+ = 0.$$  

(23)

Note that here, $q$ and $m$ in Eq.3 have been substituted by $Q$ and $M$ in Eq.5 and Eq.6. Inserting Eq.(22) into Eq.(23) we can delete $dM$. Interestingly $dQ$ is eliminated meanwhile. Solving this equation, we can get $dr_+$ directly, which is

$$dr_+ = -\frac{4l^2 p^r_+ r_+}{4l^2 Q^2 - r_+^2}.$$  

(24)

Based on Eq.(24), we can get the variation of entropy by making use of Eq.(19), that is

$$dS = -\pi \frac{4l^2 p^r_+ r_+}{2 4l^2 Q^2 - r_+^2}.$$  

(25)

With Eqs.(18) and (25), we find there is a relation

$$TdS = p^r_+.$$  

(26)

In this case, the internal energy in Eq.(22) can be rewritten as

$$dM = TdS + \Phi dQ,$$  

(27)

which is the first law of black hole thermodynamics. That is to say, as a fermion drops into the black hole, the first law is valid in the normal phase space.

Next we concentrate on studying the second law of thermodynamics, which states that the entropy of the black holes never decrease in the clockwise direction. As a fermion is absorbed by the black hole, the entropy of the black hole increases according to the second law of the thermodynamics. We will employ Eq.(26) to check whether this is true.

For the extremal black holes, the temperature vanishes at the horizon for the inner horizons and outer horizons are coincident. With Eq.(18), we can get the
mass of the extremal black hole and substitute it into Eq. (25), we find
\[ dS_{\text{extreme}} = \infty. \] (28)
The divergence of \( dS \) implies the second law for the extremal black hole is meaningless since the thermodynamic system is a zero temperature system.

For the non-extremal black holes, their temperature are larger then zero, which implies
\[ r_+^2 > 4l^2Q^2, \] (29)
where we have used Eq. (13). In this case, \( dS \) in Eq. (25) is positive. The second law of thermodynamics is valid therefore.

In the normal phase space, we also can check the validity of the weak cosmic censorship conjecture, which states that the singularity of a spacetime can not be observed for an observer located at future null infinity. In other words, singularities need to be hidden by the event horizon for a black hole. So an event horizon should exist to assure the validity of the weak cosmic censorship conjecture. As a fermion is absorbed by a black hole, we intend to check whether there is an event horizon. That is, whether the equation \( f(r) = 0 \) has solutions.

For the BTZ black holes, there is a minimum value for \( f(r) \) with the radial coordinate \( r_m \). When \( f(r_m) > 0 \), there is not a horizon while when \( f(r_m) \leq 0 \), there are horizons always. At \( r_m \), the following relations \[ f(r)|_{r=r_m} = f_m = \epsilon \leq 0, \]
\[ \partial_r f(r)|_{r=r_m} = f'_m = 0, \] (30)
should satisfy. For the extremal black holes, \( \epsilon = 0, r_+ \) and \( r_m \) are coincident. For the near extreme black holes, \( \epsilon \) is a small quantity, \( r_m \) is distributed between the inner horizon and outer horizon. As a fermion drops into the black hole, the mass and charge of the black hole change into \( M + dM, Q + dQ \) respectively. Correspondingly, the locations of the minimum value and event horizon change into \( r_m + dr_m, r_+ + dr_+ \). There is also a shift for \( f(r) \), which can be written as
\[ df_m = f(r_m + dr_m) - f_m = \left( \frac{\partial f_m}{\partial M} dM + \frac{\partial f_m}{\partial Q} dQ \right), \] (31)
where we have used \( f'_m = 0 \) in Eq. (30). We first discuss the extremal black holes, for which the horizons are located at \( r_m \). In this case, Eq. (22) can be used. Inserting Eq. (22) into Eq. (31), we find \( dQ \) is deleted meanwhile. In this case, Eq. (31) can be simplified lastly as
\[ df_m = -8p'_m. \] (32)
This result shows that \( f(r_m + dr_m) \) is smaller than \( f(r_m) \) as a charged fermion is absorbed by the black hole.

For the near-extremal black holes, Eq. (22) is not valid at \( r_m \) for it holds only at the horizon. With the condition \( r_+ = r_m + \delta \), we can expand Eq. (22) at \( r_m \), which leads to
\[ dM = p'_m - 2dQQ \log \left( \frac{r}{R} \right) - \frac{2QdQ}{r} \delta + O(\delta)^2. \] (33)
Substituting Eq. (33) into Eq. (23), we get lastly
\[ df_m = -8p'_m + \frac{32dQ}{l} \delta + O(\delta)^2. \] (34)
Because \( \delta \) is a small quantity while \( l \) is a large quantity relatively, the last two terms can be neglected approximately. In this case, Eq. (34) takes the same form as Eq. (32), indicating that the weak cosmic censorship conjecture is also valid for the near-extremal black holes.

It should be stressed that the second term in Eq. (34) is small for we compare it with \( 8p'_m \). In fact, the higher order corrections are important for us to discuss the weak cosmic censorship conjecture, however in our method, we find they can be neglected after calculation strictly for the dominant term is too large.

4 Thermodynamics and weak cosmic censorship conjecture in the extended phase space

To make the charged BTZ black holes satisfy the Reverse Isoperimetric Inequality, a new thermodynamic parameter \( R \) was introduced in the first law \[ dM = TdS + VdP + PdQ + KdR, \] (35)
where
\[ M = \frac{r_+^2 - 8l^2Q^2 \log \left( \frac{r_+}{r} \right)}{8\pi}, \] (36)
\[ P = -\frac{\Lambda}{8\pi} = \frac{1}{\pi l^2}, \] (37)
\[ V = \left( \frac{\partial M}{\partial P} \right)_{S,Q,R} = \pi r_+^2, \] (38)
\[ \Phi = \left( \frac{\partial M}{\partial Q} \right)_{S,Q,R} = -2Q \log \left( \frac{r}{R} \right), \] (39)
\[ K = \left( \frac{\partial M}{\partial R} \right)_{S,Q,P} = Q^2/R, \] (40)
in which \( R \) is the renormalization length scale, and \( K \), which is the conjugate of \( R \), is the renormalized energy. Note that here the value of \( K \) is different from that in \[ . \] The reason stems from the definition of the electric charge \( Q \). In fact, with the Gauss law, we have noted that the charge parameter \( q \) is not the electric charge of the black hole, which has been shown in Eq. (5).

From Eq. (35), we know that in this framework, the volume recovers to the standard definition of the thermodynamic volume \[ , \] which is more reasonable. We are
going to explore whether the first law in Eq. (35) can be obtained by considering a charged fermion absorption. In the extended, the pressure $P$ and the renormalization length scale $R$ are also state parameters of the thermodynamic system, as a fermion is absorbed by the black hole, the pressure and the renormalization length scale will also change besides the mass, charge and entropy. In this thermodynamic system, the mass $M$ is not the internal energy but the enthalpy, which relates to the internal energy as

$$M = U + PV +KR.$$  

(41)

As a charged fermion drops into the black hole, the energy and charge are supposed to be conserved. Namely the energy and charge of the fermion equal to the varied energy and charge of the black hole, which implies

$$\omega = dU = d(M - PV - KR), \quad e = dQ.$$  

(42)

The energy in Eq. (16) changes correspondingly into

$$dU = \Phi dQ + p'_*.$$  

(43)

Considering the backreaction, the absorbed fermions will change the location of the event horizon of the black hole. However, the horizon is determined by the equation $f(r) = 0$ always as stressed in section 3. In the extended phase space, for the AdS radius $l$ and renormalization length $R$ are variables, the shift of function $f(r)$ can be expressed as

$$df_+ = \frac{\partial f_+}{\partial M} dM + \frac{\partial f_+}{\partial Q} dQ + \frac{\partial f_+}{\partial l} dl + \frac{\partial f_+}{\partial R} dR.$$  

(44)

In addition, with Eq. (11), Eq. (43) can be expressed as

$$dM - d(PV + KR) = \Phi dQ + p'_*.$$  

(45)

From Eq. (44), we can obtain $dl$. Substituting $dl$ into Eq. (44), we can delete it directly. Interestingly, $dQ$, $dR$, and $dM$ are also eliminated at the same time. In this case, there is only a relation between $p'_*$ and $dr_+$, which is

$$dr_+ = \frac{r_+(p'_* + dKR)}{Q^2}.$$  

(46)

Based on Eq. (46), the variations of entropy and volume of the black hole can be expressed as

$$dS = -\frac{\pi r_+(p'_* + dKR)}{2Q^2},$$  

(47)

$$dV = -\frac{2\pi r_+(p'_* + dKR)}{Q^2}.$$  

(48)

With Eq. (47) and Eq. (48), we find

$$TdS - PdV - RdK = p'_*.$$  

(49)

The internal energy in Eq. (13) thus would change into

$$dU = \Phi dQ + TdS - PdV - RdK.$$  

(50)

Moreover, from Eq. (11), we can get

$$dM = dU + PdV + VdP + KdR + RdK.$$  

(51)

Substituting Eq. (51) into Eq. (50), we find

$$dM = TdS + \Phi dQ + VdP + KdR,$$  

(52)

which is consistent with that in Eq. (35). That is, as a charged fermion is absorbed by the black hole, the first law of thermodynamics holds in the extended phase space.

With Eq. (17), we also can check the second law of thermodynamics in the extended phase space. It should be stressed that there is a term $dK$ in Eq. (17), which is the variation of the renormalized energy. According to Eq. (10), $dK$ is the function of $dR, dQ$. However, the existence of $dQ, dR$ would affect the definition of $\Phi$ and $K$ respectively and further violate the first law of thermodynamics. The satisfaction of the first law of thermodynamics is a necessary condition to discuss the second law of thermodynamics under a particle absorption. So, $dK$ can not be expressed as a linear relation of $dR, dQ$ though we do not know the mechanism for we know little about the renormalized energy in the extended phase space. In this paper, we will treat the variation of the renormalized energy as an independent quantity and do not care about its form.

From Eq. (17), we know the variation of the entropy depends on the variation of the renormalized energy. For the case, $dK > -p'_*/R$, $dS$ is negative and for the case $dK < -p'_*/R$, $dS$ is positive. In other words, the second law is violated for the case $dK > -p'_*/R$, and valid for the case $dK < -p'_*/R$. Moreover, for $dK = -p'_*/R$, $dS = 0$, indicating that the horizons of the black holes will not change as a charged fermion is absorbed.

We also can discuss the weak cosmic censorship conjecture in the extended phase space with the condition in Eq. (50). Because of the backreaction, the mass $M$, charge $Q$, renormalization length $R$, and AdS radius $l$ of the black hole will change into $(M + dM, Q + dQ, R + dR, l + dl)$ as a charged fermion drops into the black hole. Correspondingly, the locations of the minimum value, event horizon, AdS radius, and renormalization length will change into $r_m + dr_m, r_ + + dr_ +, l + dl, R + dR$. In this case, the shift of $f(r)$ can be written as

$$df(r_m) = \left(\frac{\partial f_m}{\partial M} dM + \frac{\partial f_m}{\partial Q} dQ + \frac{\partial f_m}{\partial l} dl + \frac{\partial f_m}{\partial R} dR\right),$$  

(53)

where we have used $f'_m = 0$ in Eq. (30). Next, we focus on finding the last result of Eq. (53). For the extremal black holes, the horizons are located at $r_m$. The energy
relation in Eq.(15) is valid. Substituting Eq.(15) into Eq.(55), we find
\[ df(r_m) = -8\rho'_\gamma - 8dK - \frac{2r_m dr_m}{l^2}. \] (54)

Substituting Eq.(19) into Eq.(54), we find
\[ df(r_m) = 0. \] (55)

That is, as fermions drop into the extremal BTZ black holes, the black holes stay at their initial states so that their configurations will not be changed. This result is quite different from that in the normal phase space where the extremal black holes will evolve into the non-extremal black holes by the absorption.

For the near-extremal black hole, Eq.(15) is not valid. But we can expand it near the lowest point with \( r_+ = r_m + \delta \). It should be stressed that \( \rho'_\gamma \) should also be expanded for it is also a function of the horizon \( r_+ \). To the first order, we get
\[
\begin{align*}
\frac{dM}{dK} &= -\frac{r^2_m}{4l^4} dl - 2Q\log\left(\frac{r_m}{R}\right) dQ + \frac{r_m dr}{4l^2} + \frac{Q^2 dr}{r_m} + \frac{Q^2 dR}{R} \\
+ \left(\frac{2r_m dl}{2l^3} + \frac{2dQ Q}{r_m} + \frac{dr}{4l^2} + \frac{Q^2 dr}{r_m^2}\right) \delta + O(\delta)^2,
\end{align*}
\] (56)

Substituting Eq.(55) into Eq.(59), we can get lastly
\[
\begin{align*}
\frac{df(r_m)}{dK} &= \left(\frac{8Q^2}{r_m} - \frac{2r_m}{l^2}\right) dr_m \\
+ \left(\frac{4r_m}{l^3} + \frac{16Q dQ}{r_m} + \frac{2dr}{l^2} - \frac{8Q^2 dr}{r_m^2}\right) \delta \\
+ O(\delta)^2,
\end{align*}
\] (57)

In addition, at the \( r_m + dr_m \), there is also a relation
\[
\partial_r f(r)|_{r=r_m+dr_m} = f'_m + df'_m = 0, \] (58)

which implies
\[
\frac{df'_m}{dQ} = \frac{\partial f'_m}{\partial l} dl + \frac{\partial f'_m}{\partial r_m} dr_m = 0. \] (59)

Solving this equation, we obtain
\[
\frac{dl}{l} = \frac{1}{2r_m^3} (-8\rho'_\gamma + 4r_m^2 Q^2 dr_m + r^2 dr_m). \] (60)

Based on the condition \( f'_m = 0 \) in Eq.(30), we can get
\[
l = \frac{r_m}{2Q}. \] (61)

Substituting Eq.(61) and Eq.(30) into Eq.(57), we find
\[
\frac{df(r_m)}{dK} = O(\delta)^2, \] (62)

which shows that the non-extremal black holes are also stable. This result is consistent with the extremal black holes in Eq.(15). So we can conclude that the weak cosmic censorship conjecture holds for both the extremal and near-extremal black holes in the extended phase space for the configurations of the black holes are not changed as fermions are adsorbed.

5 Conclusions

In the normal phase space and extended phase space, the laws of thermodynamics and weak cosmic censorship conjecture in the BTZ black holes were checked under a charged fermion absorption. We investigated firstly the motion of a fermion via the Dirac equation and obtained a relation between the energy and momentum near the horizon. With this relation, the first law was reproduced in the normal phase space. By studying the variation of the entropy, we also checked the second law of thermodynamics and found that for both the extremal black holes and near-extremal black holes, the second law was valid in the normal phase space for the variation of the entropy is positive. The weak cosmic censorship conjecture for the extremal black holes and near-extremal black holes were checked too. We found that the metric function which determine the locations of the horizons moved with the same scale, \(-8\rho'_\gamma\), implying that there are always horizon to hidden the singularity so that the weak cosmic censorship are valid for both cases.

With similar strategy, the thermodynamic laws and weak cosmic censorship conjecture were investigated in the extended phase space further. We found that the first law of thermodynamics was still valid, but the validity of the second law depended on the variation of the renormalization energy \( dK \). For the case \( dK > -\rho'_\gamma / R \), the second law is violated and for the case \( dK \leq -\rho'_\gamma / R \), the second law is valid. Though the weak cosmic censorship conjecture are valid in both the normal and extended phase space for the extremal and near-extremal black holes, their final states are different after absorption. The extremal and near-extremal black holes will evolve into non-extremal black holes in the normal phase space, while they are stable in the extended phase space.

In [27], laws of thermodynamics and the weak cosmic censorship conjecture of the BTZ black holes have been investigated. Different from [27], in this paper, the absorbed particles are fermions. In addition, laws of thermodynamics and the weak cosmic censorship conjecture were discussed not only in the normal phase space but also in the extended phase space in this paper while [27] only investigated the case of normal phase space.
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