A note on three dimensional good sets

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Abstract. We show that as in the case of \( n \)-fold Cartesian product for \( n \geq 4 \), even in 3-fold Cartesian product, a related component need not be full component.

Key words. Good set; full set; full component; related component; geodesic; boundary of a good set.

Introduction and Preliminaries

The purpose of this note is to answer two questions about good sets raised in [3] and [4] for the case \( n = 3 \).

Let \( X_1, X_2, \ldots, X_n \) be nonempty sets and let \( \Omega = X_1 \times X_2 \times \cdots \times X_n \) be their Cartesian product. We will write \( \overrightarrow{x} \) to denote a point \((x_1, x_2, \ldots, x_n) \in \Omega \).

For each \( 1 \leq i \leq n \), \( \Pi_i \) denotes the canonical projection of \( \Omega \) onto \( X_i \).

A subset \( S \subset \Omega \) is said to be good, if every complex valued function \( f \) on \( S \) is of the form:

\[
f(x_1, x_2, \ldots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n), \quad (x_1, x_2, \ldots, x_n) \in S,
\]

for suitable functions \( u_1, u_2, \ldots, u_n \) on \( X_1, X_2, \ldots, X_n \) respectively ([3], p. 181).

For a good set \( S \), a subset \( B \subset \bigcup_{i=1}^{n} \Pi_i S \) is said to be a boundary set of \( S \), if for any complex valued function \( U \) on \( B \) and for any \( f : S \rightarrow \mathbb{C} \) the equation (1) subject to

\[
u_i|_{B \cap \Pi_i S} = U|_{B \cap \Pi_i S}, \quad 1 \leq i \leq n,
\]

admits a unique solution. For a good set there always exists a boundary set ([3], p. 187).

A subset \( S \subset \Omega \) is said to be full, if \( S \) is maximal good set in \( \Pi_1 S \times \Pi_2 S \times \cdots \times \Pi_n S \).

A set \( S \subset \Omega \) is full if and only if it has a boundary consisting of \( n - 1 \) points ([3], Theorem 3, page 185).

If a set \( S \) is good, maximal full subsets of \( S \) form a partition of \( S \). They are called full components of \( S \) ([3], p. 183).

Two points \( \overrightarrow{x}, \overrightarrow{y} \) in a good set \( S \) are said to be related, denoted by \( \overrightarrow{x} R \overrightarrow{y} \), if there exists a finite subset of \( S \) which is full and contains both \( \overrightarrow{x} \) and \( \overrightarrow{y} \). \( R \) is an equivalence relation, whose equivalence classes are called related components of \( S \). The related components of \( S \) are full subsets of \( S \) (ref. [3]).

First we prove that when the dimension \( n = 3 \), a full component need not be a related component, by giving an example of a full set with infinitely many related components.

Consider a countable set \( T \) which consists of the following points:

\[
\begin{align*}
\overrightarrow{a_1} &= (x_1, x_2, x_3) \\
\overrightarrow{a_2} &= (y_1, y_2, x_3) \\
\overrightarrow{a_3} &= (y_1, x_2, z_3) \\
\overrightarrow{a_4} &= (\alpha_1, \alpha_2, \alpha_3)
\end{align*}
\]
\[ a_5 = (\alpha_4, \alpha_5, \alpha_3) \]
\[ a_6 = (\alpha_1, \alpha_5, \alpha_3) \]
\[ a_7 = (\alpha_4, \alpha_2, x_3) \]
\[ a_8 = (x_1, y_2, \alpha_3) \]
\[ a_9 = (\alpha_6, \alpha_7, \alpha_8) \]
\[ a_{10} = (\alpha_9, \alpha_{10}, \alpha_8) \]
\[ a_{11} = (\alpha_6, \alpha_{10}, \alpha_3) \]
\[ a_{12} = (\alpha_9, \alpha_7, x_3) \]
\[ a_{13} = (x_1, \alpha_2, \alpha_8) \]

\[ \ldots \]
\[ a_{5n-1} = (\alpha_{5n-4}, \alpha_{5n-3}, \alpha_{5n-2}) \]
\[ a_{5n} = (\alpha_{5n-1}, \alpha_{5n}, \alpha_{5n-2}) \]
\[ a_{5n+1} = (\alpha_{5n-4}, \alpha_{5n}, \alpha_{5n-7}) \]
\[ a_{5n+2} = (\alpha_{5n-1}, \alpha_{5n-3}, x_3) \]
\[ a_{5n+3} = (x_1, \alpha_{5n-8}, \alpha_{5n-2}) \]

\[ \ldots \]

Call the first three points of \( T \) as \( D_0 \) and for \( n \geq 1 \), let \( D_n \) denote the first \( 3 + n \) points of \( S \). Let \( A_0 = D_0 \) and for \( n \geq 1 \) let \( A_n = D_n \setminus D_{n-1} \). Then it is easy to see that every \( D_n \) is good and has three point boundary. All the three points of the boundary of \( D_n \) cannot come from the coordinates of points in \( D_{n-1} \): because, if all of them occur as coordinates in \( D_{n-1} \), they form a boundary for \( D_{n-1} \). Given any function \( f \) on \( D_n \), there is a solution \( u_1, u_2, u_3 \) on \( D_{n-1} \) such that

\[ f(u_1, u_2, u_3) = u_1(w_1) + u_2(w_2) + u_3(w_3), \quad (w_1, w_2, w_3) \in D_{n-1}. \]

But then \( f(a_{5n+3}) \) fixes the value of \( u_3(\alpha_{5n-2}) \) by the following equation:

\[ u_3(\alpha_{5n-2}) = f(a_{5n+3}) - u_1(x_1) - u_2(\alpha_{5n-8}) \]

When we substitute this value of \( u_3(\alpha_{5n-2}) \) in the remaining four points of \( A_n \), we get a set of linearly dependent equations. This shows that the boundary of \( D_n \) contains at least one of the five coordinates, \( \alpha_{5n-4}, \alpha_{5n-3}, \alpha_{5n-2}, \alpha_{5n-1} \) or \( \alpha_{5n} \), which are introduced in \( A_n \). One can observe the following properties of the points in the set \( A_n \): any \( k \) points of \( A_n \) has at least \( k \) coordinates introduced in \( A_n \), (i.e., they do not occur as coordinates in \( D_{n-1} \)). If we take a singleton \( \{ \tilde{a}_i \} \) in \( D_{n-1} \), any set of \( k \) points of \( A_n \) has at least \((k+1)\) coordinates which do not occur as coordinates of \( \tilde{a}_i \).

\( T \) is good as every finite subset of \( T \) is good. It cannot have a boundary \( B \) with more than two points: If \(|B| = 3\), we can choose a \( n \) sufficiently large such that all the three points of \( b \) occur as coordinates in \( D_{n-1} \). Then \( B \) is a boundary of \( D_n \) which is not possible as observed above. If \(|B| > 3\), we can choose \( n \) sufficiently large so that \( k = |B \cup \bigcup_{i=1}^{3} \Pi_i D_n| \geq 4 \). Then these \( k \) points form a boundary of \( D_n \) which is again not possible. So the boundary of \( T \) consists of only two points which shows that \( T \) is full.

We prove that no finite subset \( A \) of \( T \) other than singleton is full: Set \(|A \cap A_i| = k_i \) for \( i \geq 0 \). Let \( i_1 < i_2 < \cdots < i_l \) be such that \( k_{i_j} \neq 0 \) for \( j = 1, 2, \ldots, l \) and \( k_i = 0 \) for all other \( i \). If \( k_{i_1} > 1 \), no subset other than singleton of \( A_n \) is full, the set \( A \cap A_{i_1} \) is not full. When we add the points of \( A \cap A_{i_2} \) to \( A \cap A_{i_1} \) (as we are adding \( k_{i_2} \) points) we will be adding at least \( k_{i_2} \) new coordinates. So the set \( A \cap (A_{i_1} \cup A_{i_2}) \) is not full. Similarly when we keep adding \( A \cap A_{i_j} \) to the set \( A \cap (\bigcup_{k<j} A_{i_k}) \) the number of coordinates added is at least equal to the number of points added. So at each step \( A \cap (\bigcup_{k<j} A_{i_k}) \) is not full. In this way we get \( A = A \cap (\bigcup_{k<l} A_{i_k}) \) is also not full. If \( k_{i_2} = 1 \), in the first step when we add points of \( A \cap A_{i_2} \) to the singleton set \( A \cap A_{i_1} \) the new coordinates added is at least \( k_{i_2} + 1 \). So \( A \cap (A_{i_1} \cup A_{i_2}) \) is not full. In the remaining steps as we keep adding points from \( A \cap A_{i_j} \), the number of coordinates added is at least equal to the number of points added. So in the end we get \( A \) is not full.
For any $n$, let $\vec{b}_n = (\alpha_{5n-1}, y_2, z_3)$ and consider the set $F_n = D_n \cup \vec{b}_n$. We show that the geodesic between the points $\vec{a}_1$ and $\vec{a}_{5n+3}$ in $F_n$ is the whole set $F_n$. To show that $F_n$ is full, consider the matrix $M_n$ whose rows correspond to the points $\vec{a}_2, \vec{a}_3, \ldots, \vec{a}_{5n+3}, \vec{b}_n$ and columns correspond to the coordinates $y_1, y_2, z_3, \alpha_1, \alpha_2, \ldots, \alpha_{5n}$. This is a $5n + 3 \times 5n + 3$ matrix:

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 1 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
$$

It has an inverse given by
This shows that $F_n$ is full. To show that it is the geodesic between the points $\overline{a_1}$ and $\overline{a_{5n+3}}$ in $F_n$, we show that any proper subset $A$ of $F_n$ containing these two points is not full. If possible suppose such a set $A$ is full. Then $A$ has to contain the point $\overline{b_n}$ because no subset of $D_n$, other than singleton, is full.

Let $k = |F_n| - |A|$. As $A$ is full there exists at least $k$ coordinates of points of $F_n$ which donot occur as coordinates in the points of $A$. (Because otherwise adding these $k$ points we get $F_n$ and we will be adding less than $k$ coordinates. If $A$ is full then $F_n$ cannot be good.) Let $S$ denote these $k$ coordinates. The set $S$ cannot contain $x_1, x_2, x_3, a_{5n-1}, a_{5n-2}, a_{5n-8}, x_2$ and $z_3$ as these are used by the points of $A$. Among these $k$ coordinates let $k_i$ be the number which are introduced in $A_i$, $0 \leq i \leq n$. For $i \geq 1$, we have $0 \leq k_i \leq 5$ and $k_0 = 0$ or $1$. If $0 \leq k_1 < 5$ for some $i \geq 1$, (or if $k_0 = 1$ for $i = 0$) the $k_i$ coordinates of $S$ introduced in $A_1$ are used in at least $k_1 + 1$ points of $A_i$. So if $k_0 = 1$ or $0 < k_1 < 5$ for some $i \geq 1$, then more than $k$ points of $F_n$ cannot be in $A$ which is a contradiction.

In the case $k_1 = 0$ and $k_i = 5$ for $i \geq 1$, clearly there exists an $i \geq 1$ with $k_i = 5$. But in this case we have $k_{n-1} = k_n = 0$. If $k_i = 5$, then $A \cap A_i = \phi$ and if $k_i = 0$, then $A \cap A_i = A_i$. Let $j$ be an index such that $j_0 = 5$ and $j_{j+1} = 0$. Then $A \cap A_{j+1} = A_{j+1}$ which is a contradiction because $A_{j+1}$ uses coordinates introduced in $A_j$ which are not used by points of $A$. This shows $A$ is not full.

It can be seen that the 5 rows of $M_{5n-1}$, from $(5m-1)th$ row to $(5m+3)rd$ row, have row sums bounded by $C_1 + C_2 \sum_{j=1}^m \frac{1}{3^n}$ for some constants $C_1$ and $C_2$, independent of $n$. This shows that as in higher dimensions, in the three dimensional case also uniform boundedness of lengths of geodesics is not a necessary condition for boundedness of solutions of (1) for bounded function $f$. 

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