Robust Performance Analysis of Cooperative Control Dynamics via Integral Quadratic Constraints

Adwait Datar ©, Christian Hespe ©, and Herbert Werner ©, Member, IEEE

Abstract—In this article, we study cooperative control dynamics with gradient-based forcing terms. As a specific example, we focus on source-seeking dynamics with vehicles embedded in an unknown scalar field with a subset of agents having gradient information. We consider time-invariant and uncertain interaction potentials common in formation control and flocking. We leverage the framework of α-integral quadratic constraints (IQCs) to obtain convergence rate estimates whenever exponential stability can be achieved. Sufficient conditions take the form of linear matrix inequalities independent of the size of network. A derivation (purely in time domain) of the so-called hard Zames–Falb α-IQCs involving general noncausal higher order multipliers is given along with a suitably adapted parameterization of the multipliers to the α-IQC setting. Numerical examples illustrate the application of theoretical results.

Index Terms—Cooperative control, linear matrix inequalities, robust control.

I. INTRODUCTION AND RELATED WORK

A variety of cooperative control algorithms, such as formation control, flocking, consensus, coverage control, etc., extremum-seeking control problems, or real-time optimization problems lead to closed-loop dynamics, which could be represented as gradient flows or dynamics involving gradient-based forcing terms (see [1], [2], and [3]). The central theme of this article is to analyze the performance of such dynamics using tools from robust control. We focus on the source-seeking problem, which involves one or more vehicles located at arbitrary locations in an underlying scalar field with the goal of moving toward the minimum (source) of the field. Our main tools for the analysis are integral quadratic constraints (IQCs) [4], [5]. The exponential versions of IQCs for systems in discrete time (ρ-IQCs) are introduced in [6] and in continuous time (α-IQCs) in [7], which also introduces the soft Zames–Falb (ZF) α-IQCs corresponding to causal multipliers. While Zhang et al. [8] extended [6] to less conservative noncausal ZF multipliers, Freeman [9] presented the extension in the continuous-time setting. The theory developed in [9] is in a very general setting of Bochner spaces and covers Lemma 12 in this article. However, the proof of [9, Lemma 3] (Lemma 12 here) is not available and we therefore present a self-contained proof making all arguments in the time domain building on ideas presented in a separate paradigm [10]. This article goes further by considering multiplier parameterization proposed in [5, Sec. 5.8.3] adapted to the α-IQCs setting. The analysis of distributed optimization algorithms [11], [12] leads to similar closed-loop dynamics. In contrast to these works, we allow for a subset of agents to have the gradient information, relax the requirement on connectivity of the graph, and instead require a connectivity to at least one agent with gradient information. As a result, the decomposition approach used in [12] does not go through as it is. More importantly, since we do not look at time-varying graphs, we can use dynamic multipliers instead of the static multipliers used in [12]. The analysis of flocking dynamics considered in this article is closely related to the analysis of optimization algorithms over nonconvex functions with nonexponential convergence rates. Functions that are not strongly convex are considered in [13] whereas Fazlyab et al. [14] considered nonconvex functions to obtain nonexponential convergence rates. In contrast to the work in [14], we use a time-invariant storage function, make use of LaSalle’s invariance principle to conclude asymptotic stability, and give a local result specifically geared toward vehicles with on-board tracking controllers. Another closely related work is of Arcak et al. [15] where sparse linear matrix inequalities (LMIs) are derived, which are then solved via distributed optimization algorithms. In contrast, we derive LMIs that have a complexity corresponding to the analysis of a single subsystem. The main contributions of this article are as follows.

1) Building on the work of Hu and Seiler [7], a time-domain derivation of general noncausal exponential ZF IQCs is given in Theorem 5 along with a suitable adaptation of multiplier parameterization from [5, Sec. 5.8.3] in Theorem 6. This article extends the work in [16], which focuses on a single agent and does not contain proof of Lemma 12, which we give in Appendix B of this article.

2) Source-seeking dynamics involving vehicle agents under formation control are analyzed. The equilibria of the dynamics are characterized in Lemmata 3 and 4. LMIs (independent of network size) are derived for estimating the exponential convergence rates in Lemma 8 and Theorem 9, and supporting results are given in Lemmata 1 and 2. Numerical examples suggest that estimates obtained here are tight and are given in Section VI.

3) Theorem 10 and Corollary 11 analyze source-seeking dynamics with flocking agents to give a local sufficient condition for asymptotic stability in the form of an LMI independent of network size generalizing [17], [18].

Except for Lemma 12, we give key ideas behind the proofs so that the interested reader can work out the details if necessary. Nevertheless, detailed proofs are given in [19].

II. NOTATION

Let $\mathbb{S}_n$ denote the set of symmetric matrices of size $n$. For $X \in \mathbb{S}_n$, $X \succeq (\succeq) 0$ means that $X$ is symmetric positive definite (semidefinite) and $X \prec (\preceq) 0$ means $-X \succeq (\succeq) 0$. Let $0$ and $I$ denote the vectors or matrices of all zeroes and ones of appropriate sizes, respectively. $\text{diag}(x)$ is the diagonal matrix with elements of $x$ placed on the diagonal and $\text{blkdiag}(A_1, \ldots, A_n)$ is a block-diagonal matrix with blocks $A_i$ on the diagonal. Let $I_d$ be the identity matrix of dimension $d$. The Kronecker product is denoted by $\otimes$. For positive constants $0 < \alpha \leq$
Let $S(m, L)$ denote the set of continuously differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ for some positive integer $d$, which satisfy
\[ m \|y_1 - y_2\|^2 \leq \langle \nabla f(y_1) - \nabla f(y_2) \rangle^T (y_1 - y_2) \leq L \|y_1 - y_2\|^2 \]
for all $y_1, y_2 \in \mathbb{R}^d$. The set of vector valued functions, which are square-integrable over $[0, T]$ for any finite $T$, is denoted by $L^2_\mathbb{R}(0, \infty)$. The set of functions $h : \mathbb{R} \to \mathbb{R}$, such that $\int_0^\infty |h(t)| dt < \infty$, we use $\mathbb{K}_\infty$ to represent an linear time invariant (LTI) system with state-space matrices $A, B, C, D$. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected unweighted graph of order $N$ with the set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The set of informed agents denoted by $\mathcal{V}_I \subseteq \mathcal{V}$ is a subset of nodes with the gradient information and is introduced later in detail. We use $\mathcal{G}^N_{\mathcal{V}_I} = (\{1, \ldots, N\}, \{(1, 1), (1, 2), (1, N)\})$ to denote a star, and $\mathcal{G}^N_{\mathcal{V}_I} = (\{1, \ldots, N\}, \{(1, 2), (2, 3), \ldots, (N - 1, N), (N, 1)\})$ denotes a cycle. The notation $\mathcal{X}$ denotes the matrix $I_N \otimes X$ and let $\mathcal{X}_{(i)}$ denote the matrix $X \otimes I_2$ for any matrix $X$. For an ordered set of vectors $(x_1, x_2, \ldots, x_N)$, the vector formed by stacking these vectors be denoted by $x = [x_1^T x_2^T \ldots x_N^T]^T$.

### III. Problem Setup

Consider a source-seeking scenario where $N$ vehicle agents moving in $\mathbb{R}^d$ space (typically $d \in \{1, 2, 3\}$) are embedded in an external differentiable scalar field $\psi : \mathbb{R} \to \mathbb{R}$ with the goal of cooperatively moving toward a minimizer (source) of $\psi$. To this end, assume that a tracking controller has been designed and the closed-loop dynamics of the $i$th vehicle agent with reference position $q_i(t)$ and reference velocity $p_i(t)$ can be described for a given initial condition $x_i(0)$ by
\[
\begin{align*}
\dot{x}_i(t) &= Ax_i(t) + B_q \begin{bmatrix} \dot{q}_i(t) \\ \dot{p}_i(t) \end{bmatrix} \\
y_i(t) &= Cx_i(t)
\end{align*}
\]
(1)

where $y_i(t)$ is the position output of agent $i$. Since these dynamics need a desired reference position $q_i$ and a desired reference velocity $p_i$ as inputs, we need a mechanism to generate these desired trajectories. This is achieved by augmenting the above closed-loop tracking dynamics by a second-order prefilter of the form
\[
\begin{align*}
\dot{q}_i(t) &= p_i(t) \\
\dot{p}_i(t) &= -k_d \cdot p_i(t) - k_v \cdot u_i(t)
\end{align*}
\]
(2)

where $u_i(t) \in \mathbb{R}^d$ is the external force, $q_i(0) = Cx_i(0)$, and $p_i(0) = 0$. This augmentation allows the interpretation of the input $u$ exciting the overall system as a physical force. Moreover, the prefilter acts as a crude approximation of a vehicle explicitly bringing in integral action and the notions of mass and friction with the environment. This effectively produces smooth reference trajectories that are fed to the local tracking controllers. If the vehicle is able to track the generated trajectories well enough, i.e., $y_i \approx q_i$, we end up with a system of masses with friction interconnected with springs, which is well studied in the literature. Thus, with good tracking controllers, stability properties of interconnected systems of masses (second-order vehicle models) can be expected to extend to higher order vehicle models. While this prefilter structure plays a central role in the analysis involving flocking dynamics (see Section V), the analysis involving formation control dynamics (see Section IV) does not rely on this structure. However, in both situations, this structure provides a pragmatic approach to synthesize controllers and analyze them a posteriori via the developed LMIIs. Furthermore, one can design tracking controllers by any method of choice and plug in the closed-loop dynamics in this framework for a systematic performance analysis. Finally, these prefilter dynamics also provide us with simple interpretable tuning knobs, e.g., the damping coefficient $k_d$, which can be used to slow down or speed up the overall dynamics. The overall dynamics with state $y_i = [q_i^T \dot{q}_i^T \dot{p}_i^T]^T$ and suitable initial condition $\eta_i(0)$ are represented by the LTI system $G$ described by
\[
\begin{align*}
\dot{\eta}_i(t) &= A_G \eta_i(t) + B_G u_i(t) \\
y_i(t) &= C_G \eta_i(t)
\end{align*}
\]
(3)

The interaction between different agents is captured by an interaction potential $V : \mathbb{R}^{Nd} \to \mathbb{R}$ (see [20] or Section VI-B for an example), which captures the cooperation/disagreement between agents and is minimized when the relative positions of the agents are as desired. Thus, $V$ can be interpreted as a disagreement cost. Let a nonempty subset $\mathcal{V}_I \subseteq \mathcal{V}$ of informed agents have access to the gradient $\nabla \psi$ evaluated at their respective positions. These informed agents have an additional forcing term in the direction of the negative gradient that drives them toward the source. For a compact representation of the overall dynamics, define a function $f : \mathbb{R}^{Nd} \to \mathbb{R}$ as follows.

**Definition 1**: For a given interaction potential $V$, the set of informed agents $\mathcal{V}_I$, and a scalar field $\psi$, define a function $f : \mathbb{R}^{Nd} \to \mathbb{R}$ by
\[
f(y) = V(y) + \sum_{i \in \mathcal{V}_I} \psi(y_i).
\]

Note that the term $\sum_{i \in \mathcal{V}_I} \psi(y_i)$ can be thought of as the source-seeking cost such that it is minimized when all informed agents are at the source. The function $f$ can thus be seen as the sum of the disagreement cost and the source-seeking cost bringing out a tradeoff that might be present if there are multiple informed agents with a nonzero desired relative displacement (see the discussion after Lemma 3).

The overall dynamics can be described using the Kronecker notation from Section II as
\[
\begin{align*}
\dot{\eta}_i(t) &= A_G \eta_i(t) + B_G u_i(t), \quad \eta_i(0) = \eta_0 \\
y_i(t) &= C_G \eta_i(t) \\
u_i(t) &= \nabla \psi(y_i(t)) + u_\psi(t)
\end{align*}
\]
(4)

where $u_\psi(t)$ is given by stacking up
\[
u_\psi_i(t) = \begin{cases} 
\nabla \psi(y_i(t)), & \text{if } i \in \mathcal{V}_I \\
0, & \text{otherwise.}
\end{cases}
\]
(5)

### IV. Analysis of Formation Control Dynamics

The special case of formation control dynamics [21] is obtained if $V(y) = \frac{1}{2} (y - r)^T(L \otimes I_N)(y - r)$, where $L$ is the graph Laplacian matrix and $r$ is the formation reference encoding the desired formation shape (see [21] for details). The objective is to make the agents form a desired shape and move toward the source of the external field at the same time. The assumptions and the first problem statement are given next.

**Assumption 1**: The scalar field $\psi$ belongs to $S(m, L)$ and $y_{opt}$ minimizes $\psi$, i.e., $\psi(y) \geq \psi(y_{opt})$ for all $y \in \mathbb{R}^d$.

**Assumption 2**: For every node $i \in \mathcal{V}$, there is a node $j \in \mathcal{V}_I$ such that $G$ contains a path from $i$ to $j$.

**Problem 1**: Assume that $\psi$, $G$, and $\mathcal{V}_I$ satisfy Assumptions 1 and 2 and $V(y) = \frac{1}{2} (y - r)^T(L \otimes I_N)(y - r)$. Characterize the equilibria of dynamics (4) and derive sufficient conditions independent of the network size $N$ under which the state trajectories generated by (4) remain bounded for all $t \geq 0$ and $y$ converges exponentially with a rate $\alpha$ to the minimizer $y_\ast$ of $f$, i.e., $\exists \alpha \geq 0 : \|y(t) - y_\ast\| \leq \kappa e^{-\alpha t}, \forall t \geq 0$.

This section addresses Problem 1. Conditions ensuring $f \in S(m, L)$ and equilibria of the dynamics (minimizers of $f$) are studied in Sections IV-A and IV-B, respectively, where the decomposition of $f$ (see
Definition 1) plays a central role. This is followed by the main analysis in Section IV-C, where we only need \( f \in S(m, L) \) without any further restrictions.

A. Smoothness and Convexity Properties of \( f \)

We now define two grounded Laplacians to characterize the convexity and smoothness properties of \( f \). Consider an undirected graph \( G_z \) by adding an \((N + 1)\)th node to the vertex set of \( G \) such that this added node is grounded \([22]\) and has an edge with all informed agents in \( V_i \) with an edge weight \( m_{G_z} \) (or \( L_{G_z} \)). The grounded Laplacian for \( G_z \) as defined in \([22]\) equals \( L_z \) (or \( L_z \)) defined below.

Definition 2: For a given graph \( G \) of order \( N \) (with its corresponding Laplacian \( L \)), a set of informed agents \( V_i \), and constants \( 0 < m_{G_z} \leq L_{G_z} \), the grounded Laplacians \( L_z \) and \( L_{G_z} \) are defined as \( L_z = \lambda + m_{G_z}E \) and \( L_{G_z} = \lambda + L_{G_z}E \), where \( E \) is a diagonal matrix with the \( i \)th diagonal entry equal to \( 1 \) if \( i \in V_i \) and equal to 0 otherwise.

Lemma 1: For a given graph \( G \) of order \( N \) (with its corresponding Laplacian \( L \)), a set of informed agents \( V_i \), a scalar field \( \psi \), a formation reference vector \( r \), and constants \( m \) and \( L \) such that \( 0 < m \leq L \), let \( f \) be as defined in Definition 1 with \( V(y) = \frac{1}{2}(y - r)^T(L \otimes I_{d})(y - r) \) and \( L_z \) as defined in Definition 2. Then, for constants \( m \) and \( L \) such that \( 0 < m \leq L \), the following statements are equivalent:

1) \( f \in S(m, L) \) for all \( \psi \in S(m_{G_z}, L_{G_z}) \);
2) \( f \in L_z \) and \( L_{G_z} \leq L \).

The above lemma gives a condition to verify \( f \in S(m, L) \). The direction \( 1) \Rightarrow 2) \) can be seen by picking a quadratic \( \psi \), whereas the reverse direction is seen as a direct consequence of the definition of \( S(m, L) \).

We emphasize that Lemma 1 provides a way to estimate the constants \( m \) and \( L \) from structural properties of the underlying graph (see Section VI-A). The next lemma shows that Assumption 2 is necessary and sufficient for \( f \in S(m, L) \). The main idea behind the proof is to see that \( L_z \succ 0 \) if Assumption 2 is satisfied while it has a nonempty kernel if the assumption is not satisfied.

Lemma 2: Assume that the constants \( m_{G_z} \) and \( L_{G_z} \) satisfy \( 0 < m_{G_z} \leq L_{G_z} \). There exist constants \( m \) and \( L \) with \( 0 < m \leq L \) such that \( f \) as in Definition 1 with \( V(y) = \frac{1}{2}(y - r)^T(L \otimes I_{d})(y - r) \) belongs to \( S(m, L) \) for every \( \psi \in S(m_{G_z}, L_{G_z}) \) if and only if the graph \( G \) and the set of informed agents \( V_i \) satisfy Assumption 2.

B. Minimizers of \( f \)

Recall that Problem 1 also requires us to characterize the equilibria of dynamics \((4)\). Since we have integral action in the prefilter augmented in series with a stable tracking plant, it can be shown that equilibria of the dynamics are characterized by the minima of \( f \). Furthermore, since \( f \) is composed of the source-seeking cost and the disagreement cost (see Definition 1 and the paragraph succeeding it), a geometric characterization of the minima of \( f \) is essential to evaluate how well the source-seeking goals and the formation forming goals are achieved by the equilbrium. This is analyzed next.

Lemma 3 characterizes the minimizers for the case of consensus \((r = 0)\) and formation control with a single informed agent \((\lvert V_i \rvert = 1)\) and only necessary conditions are obtained for the general case in Lemma 4. The main tools in deriving these lemmas are definitions of convexity, \( \nabla \psi(z) = 0 \) and the fact that \( f \) has a unique minimizer \( z \) satisfying \( \nabla V(z) = 0 \).

Theorem 5: Assume that the scalar field \( \psi \) satisfies Assumption 1, graph \( G \) and set \( V_i \) satisfy Assumption 2, and \( f \) be as defined in Definition 1 with \( V(y) = \frac{1}{2}(y - r)^T(L \otimes I_d)(y - r) \). Then, the following statements hold.

1) If \( r = 0 \), then \( y = [y_1^T \ldots y_N^T]^T \) is the minimizer of \( f \) if only if \( y = 1_N \times y_{opt} \).
2) If \( V_i = \{i\} \) for some \( i \in V_i \), then \( y = [y_1^T \ldots y_N^T]^T \) is the minimizer of \( f \) if \( y_i = y_{opt} + (r_j - r_i) \) for all \( j \).

Note that for item 2), if \( r \) is chosen such that \( r_i = 0 \) for the informed agent \( i \in V_i \), then \( r \) just encodes the desired positions of the agents with the coordinate system such that the source and the informed agent are located at the origin. Scenarios involving multiple informed agents and a nonzero formation reference \( r \) are difficult to characterize because the terms \( \frac{1}{2}(y - r)^T(L \otimes I_d)(y - r) \) and \( \sum_{i \in V_i} \psi(y_i) \) have competing objectives. The equilibrium can thus result in a situation where none of the informed agents are at the source, and agents are not at optimal relative distances.

Lemma 4: Assume that \( \psi \), \( G \), and \( V_i \) satisfy Assumptions 1 and 2, \( f \) be as per Definition 1 with \( V(y) = \frac{1}{2}(y - r)^T(L \otimes I_d)(y - r) \), and let \( z = [z_1^T \ldots z_N^T]^T \) be a minimizer of \( f \). Then, \( [y_1^T \psi(z_1)^T z_1 - y_1^T] \geq 0 \) for all \( i \in V_i \) contains \( y_{opt} \). Furthermore, we obtain the following stronger conclusions if \( \psi \) is quadratic or radially symmetric.

1) If \( \psi \) is radially symmetric around the source, i.e., it has the form \( \psi(y) = \psi(|r - y_{opt}|) \) for some function \( \psi : \mathbb{R} \to \mathbb{R} \), then the minimizer \( z \) of \( f \) such that \( y_{opt} \) lies in the convex hull of \( \{z_i \mid i \in V_i\} \).
2) If \( \psi \) is quadratic, i.e., it has the form \( \psi(y) = y^TQy + b^Ty + c \), then the minimizer \( z \) of \( f \) satisfies \( y_{opt} = \frac{1}{\|y_{opt}\|^2} \sum_{i \in V_i} z_i \), i.e., the center of mass of informed agents is at the minimizer \( y_{opt} \) of \( \psi \).

C. Robust Analysis

We first derive properties between the signals \( u(t) \) and \( y(t) \) related by the map \( u(t) = \nabla f(y(t)) \), where \( f \) is an arbitrary function in \( S(m, L) \) and \( y_i \), minimizes \( f \) such that \( \nabla f(y_i) = 0 \). The standard (nonexponential) ZF IQCs have been well studied. We give a purely time-domain derivation of the general noncausal higher order ZF alpha-IQCs.

For convenience, let the deviation signals be defined by

\[
\hat{y}(t) = y(t) - \mu, \quad \hat{u}(t) = u(t) - u_s = \nabla f(y(t)) - \nabla f(y_s) = \nabla f(\hat{y}(t) + y_s),
\]

and for constants \( m \) and \( L \), define

\[
\hat{p}(t) = \hat{u}(t) - m\hat{y}(t) \quad \text{and} \quad q(t) = L\hat{y}(t) - \hat{u}(t).
\]

Let \( h \) belonging to \( L_1(-\infty, \infty) \) be such that for some \( H \in \mathbb{R} \)

\[
h(s) \geq 0 \quad \forall s \in \mathbb{R} \quad \text{and} \quad \int_{-\infty}^{\infty} h(s)ds \leq H
\]

define signals \( w_1(t) \) and \( w_2(t) \) by

\[
w_1(t) = \int_{0}^{t} e^{-2\alpha(t-\tau)}h(t-\tau)q(\tau)d\tau
\]

\[
w_2(t) = \int_{0}^{t} e^{-2\alpha(t-\tau)}h(-t-\tau)p(\tau)d\tau.
\]

Theorem 5: Assume that the function \( h \) satisfying \((7)\) is fixed and assume that \( \alpha \geq 0 \). Let \( \hat{u}, \hat{y} \in L_{2}[0, \infty) \), related by \( \hat{u}(t) = \nabla f(\hat{y}(t) + y_s) \), where \( f \in S(m, L) \) and \( y_i \) minimizes \( f \). Then, the signals defined in \((6)\) and \((8)\) satisfy the following inequality \( \forall t \geq 0 \):

\[
\int_{0}^{T} e^{2\alpha t} (Hp(t)^Tq(t) - p(t)^Tw_1(t) - q(t)^Tw_2(t)) dt \geq 0.
\]

Note that \( h(t) \equiv 0 \) and \( H = 1 \) satisfies \((7)\) giving us \( \int_{0}^{T} e^{2\alpha t} p(t)^Tq(t)dt \geq 0 \), the sector IQC involved in the circle criterion (CC). Theorem 5 thus produces a larger set of IQCs depending on the choice of \( h \).
The function $h$ can be parameterized following the ideas in [5] and making suitable adaptations to accommodate the $\alpha$-IQC setting. This is discussed in detail in [16, Sec. 3] (where the notation $\Psi$ is used instead of $\Pi$). We give here the final result. Let $L = \frac{A_{\varepsilon}}{\varepsilon}, B_{\varepsilon} = \varepsilon B_{0}, P = \varepsilon P_{0}, \pi = \pi_{m,L} \otimes I_{N_{d}},$ where $\pi_{m,L} \otimes P$ are defined in Appendix A. Let us define for any $\tilde{u}, \tilde{y} \in L_{2}\varepsilon[0, \infty)$

$$\tilde{z}(t) = \int_{0}^{t} C_{\varepsilon} e^{A_{\varepsilon} (t - r)} B_{\varepsilon} \left[ \tilde{y}(r) \tilde{u}(r) \right] dr + D_{\varepsilon} \left[ \tilde{y}(t) \tilde{u}(t) \right]. \quad (9)$$

The function $h$ is parameterized as the impulse response of an LTI system, and constraints on $h$ are captured via LMIs such that corresponding to every $P \in \mathbb{P}$ is a function $h$ and constant $H$ satisfying (7). Finally, the integral from Theorem 5 without the factor $e^{\alpha t}$ can be rearranged as $\tilde{z}^{T} (P \otimes \varepsilon) \tilde{z} = H p^{T} q - p^{T} w_{1} - q^{T} w_{2}$ to give us the next result as an immediate consequence of Theorem 5.

**Theorem 6:** Let $\tilde{u}, \tilde{y} \in L_{2}\varepsilon[0, \infty)$ be related by $\tilde{u}(t) = \nabla f(\tilde{y}(t) + y_{r}, \ldots), f \in S(m, L)$ and $y_{r}$ minimizes $f$. Then, for any $\alpha \geq 0$, the signal $\tilde{z}(t)$ as defined in (9) satisfies

$$\int_{0}^{T} e^{\alpha t} \tilde{z}(t)^{T} (P \otimes \varepsilon) \tilde{z}(t) dt \geq 0 \quad \forall P \in \mathbb{P} \quad \forall T \geq 0. \quad (10)$$

Once we have the IQC result in the form of Theorem 6 at our disposal, we can apply standard arguments as in [7] to obtain the analysis result.

Let $A_{\varepsilon} \otimes B_{\varepsilon} = \Pi$ $I_{\varepsilon} \otimes I_{\varepsilon} \otimes B_{\varepsilon} \otimes I_{\varepsilon}$, which is used in the next result from [7].

**Remark 1:** Note that although (11) is not linear in $\alpha$ and $C$ due to the product $A\varepsilon \otimes B_{\varepsilon} \otimes I_{\varepsilon}$ it can be solved efficiently using a backtracking over $\alpha$ as commonly suggested in the literature [6].

Note that when $N$ is large, LMI (11) becomes computationally intractable. However, a smaller LMI independent of $N$ can be derived without any additional conservatism. This is possible due to the specific diagonal and repeated structure of the multiplier and the plant (see [6, Sec. 4.2]). The key idea is that once the uncertainty consisting of the interconnections is characterized by an IQC with a diagonal repeated multiplier, the nominal plant and the multiplier form repeated decoupled systems leading to repeated decoupled verification LMIs.

**Lemma 8:** Let $\left[ \begin{array}{c} \tilde{A}_{0} / \varepsilon \\ \tilde{B}_{0} / \varepsilon \\ \end{array} \right] = \left( \pi_{m,L} \otimes I_{\varepsilon} \right) \left[ \begin{array}{c} G \\ I_{\varepsilon} \end{array} \right] P_{0} \left( \begin{array}{c} \tilde{A}_{0} / \varepsilon \\ \tilde{B}_{0} / \varepsilon \\ \end{array} \right) \left( \begin{array}{c} \tilde{A}_{0} / \varepsilon \\ \tilde{B}_{0} / \varepsilon \\ \end{array} \right)$, then the following statements are equivalent.

1) $\exists \lambda \in (0, P \in \mathbb{P}$ such that (11) is satisfied.
2) $\exists \lambda > 0, P \in \mathbb{P}$ such that

$$\left[ \begin{array}{c} \tilde{A}_{0} / \varepsilon \tilde{X}_{0} + \tilde{X}_{0} \tilde{A}_{0} + 2 \alpha \tilde{X}_{0} \\ \tilde{B}_{0} / \varepsilon \tilde{X}_{0} + 0 \\ \end{array} \right] + \left[ \begin{array}{c} C_{\varepsilon} / \varepsilon \tilde{X}_{0} \\ \end{array} \right] P_{0}(\varepsilon) \left( \begin{array}{c} \tilde{A}_{0} / \varepsilon \\ \tilde{B}_{0} / \varepsilon \\ \end{array} \right) \leq 0. \quad (12)$$

Finally, let us define an appropriate uncertainty set for dynamics (4).

**Lemma 9:** Let the graph $\mathcal{G}$, the set of informed agents $V_{l}$, and the scalar field $\psi$ be such that $(\mathcal{G}, V_{l}, \psi) \in \Delta_{m,L}$ for some $0 < m \leq L$. Let $y_{l}$ be the minimizer of $f$. If $\exists y_{l} > 0, P \in \mathbb{P}$ such that (12) is satisfied, then, under dynamics (4), $y$ converges exponentially to $y_{l}$ with rate $\alpha$, i.e., $\exists \lambda \geq 0$ such that $\|y(t) - y_{l}\| \leq ke^{-\alpha t}$ holds for all $t \geq 0$.

**V. ANALYSIS OF FLOCKING DYNAMICS**

Flocking dynamics, such as from [20], involve distance-based non-convex interaction potentials but satisfy a Lipschitz condition on the gradients. The objective here is to make the agents flock toward the source of the external field. Nonconvexity of $f$ prevents us from proving exponential stability with the help of ZF-IQCs. We therefore make less strict assumptions on $f$ and only ask for asymptotic stability.

**Assumption 3:** The function $f : \mathbb{R}^{N_{d}} \rightarrow \mathbb{R}$ and an open set $S$ containing a minimizer $y_{l}$ of $f$ satisfy the following.

1) $f$ is differentiable on $S$.
2) $y_{l} \in S$ is a minimizer of $f$ on $S$, i.e., $\nabla f(y_{l}) = 0$ and $f(y_{l}) = f(y) \forall y \in S$.
3) There exists a symmetric matrix $M_{l}$ such that

$$\left( M_{10} \otimes I_{\varepsilon} \right) \left( y - y_{l} \right) - \nabla f(y) \geq 0 \quad \forall y \in S.$$

4) There exists a symmetric matrix $M_{20}$ such that

$$\left( M_{20} \otimes I_{\varepsilon} \right) \left( y - y_{l} \right) - \nabla f(y) \geq 0 \quad \forall y \in S.$$

5) There exist positive constants $c_{1}$ and $c_{2}$ such that

$$S_{0} = \{ y + e | f(y) - f_{min} | \leq c_{1}, \|e\|^{2} \leq c_{2} \}$$

is bounded and contained in $S$.

Properties of $f$ need to be captured via a suitable selection of matrices $M_{10}$ and $M_{20}$ in 3) and 4). For example, $M_{20} = \text{blkdiag}(L^{2}I_{1}, -I)$ captures the property that $\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|$. Note that 4) is stricter than 3), i.e., if 4) holds with some $M_{10}$, then 3) also holds with $M_{10} = M_{20}$. See [14, Sec. 3.2.2 and 6.2] for more examples. If global properties of $f$ are known, one can set $S = \mathbb{R}^{N_{d}}$. The constants $c_{1}$ and $c_{2}$ from Assumption 3 item 5) are used to find an allowable set of initial states so that trajectories do not leave $S$ [see (15) from Theorem 10]. An additional assumption on the vehicle dynamics is made next, which states that the tracking controller is stabilizing and has zero steady-state error for step position references $q(t)$.

**Assumption 4:** The vehicle dynamics (1) are such that every eigenvalue of the matrix $A$ has strictly negative real part and $-C A^{-1} B_{q} = I$.

**Problem 2:** Assume that the function $f$ and the vehicle dynamics (1) satisfy Assumptions 3 and 4. Derive sufficient conditions independent of the network size $N$ under which the state trajectories of the dynamics (4) remain bounded for all $t \geq 0$ and $y$ converges asymptotically to a minimizer of $f$.

The analysis for Problem 2 is based on standard dissipativity-based arguments, which do not rely on the decomposition of $f$ (see Definition 1). To exploit the IQC descriptions from Assumption 3, items 3) and 4), dynamics (4) are transformed into a suitable form to facilitate the analysis. Specifically, we write the input as $\nabla f(y(t)) = \nabla f(q(t)) - (\nabla f(q(t)) - \nabla f(y(t)))$ and consider as output channels $q(t)$ and $q(t) - y(t)$. In addition, the state $\eta = [\eta_{1}^{T} \cdots \eta_{N_{d}}^{T}]^{T}$ is permuted to a new state vector $\tilde{\eta} = [y^{T} q^{T} p^{T}]^{T}$ to obtain equivalent dynamics

$$\begin{bmatrix} \dot{\tilde{\eta}}(t) \\ \tilde{q}(t) \\ \tilde{q}(t) - y(t) \end{bmatrix} = \begin{bmatrix} A_{C} & B_{C} & -B_{C} \\ C_{G_{1}} & 0 & 0 \\ C_{G_{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\eta}(t) \\ \tilde{q}(t) \\ \tilde{q}(t) - y(t) \end{bmatrix} + \begin{bmatrix} d_{1}(t) \\ d_{2}(t) \end{bmatrix}$$
\[ d_1(t) = \nabla f(q(t)) \]
\[ d_2(t) = \nabla f(q(t)) - \nabla f(y(t)). \] (13)

Drawing motivation from [14, Sec. 3], a nonnegative function \( V_\eta \) of the state \( \dot{\eta} \) is constructed such that it is 0 only when \( q = y \), \( x_0 = A_0^{-1}B_0 y_0 \), and \( p_0 = 0 \), where \( y_0 \in S \) is the minimizer as per Assumption 3. In this regard, for given matrices \( A \) and \( B_0 \), and matrix variables \( R \geq 0 \) and \( Q > 0 \), construct a block \( 3 \times 3 \) matrix \( X_0 \) as

\[
\begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{bmatrix} = R + \begin{bmatrix}
Q & QA^{-1}B_q & 0 \\
0 & B_q^TQA^{-1}B_q & 0 \\
0 & 0 & I_d
\end{bmatrix}.
\]

Define the storage function \( V(x, q, p) \) as

\[
(\dot{\eta} - \ddot{\eta})^T \begin{bmatrix}
\dot{X}_{11} & \dot{X}_{12} & \dot{X}_{13} \\
\dot{X}_{21} & \dot{X}_{22} & \dot{X}_{23} \\
\dot{X}_{31} & \dot{X}_{32} & \dot{X}_{33}
\end{bmatrix} (\dot{\eta} - \ddot{\eta}) + 2\mu(f(q) - f_{\text{max}})
\]

where \( \dot{\eta} = [x_q^T \ q^T \ p_0^T]^T \) and the notation \( : \) is given in Section II. With an energy function \( E(t) := V(x(t), q(t), p(t)) \), it can be shown that if \( E(t) \) is below some prescribed value, \( \dot{\eta} \) is bounded and \( y(t) \) belongs to \( S_0 \subset S \). Feasibility of \( Z \leq 0 \) from (14) shown at the bottom of this page, which corresponds to the dissipation inequality

\[
\dot{E} + \epsilon\|p\|^2 + \begin{bmatrix}
q - q_0 \\
\nabla f(q)
\end{bmatrix} \begin{bmatrix}
\lambda_1 M_{10} \otimes I_{N_d} \\
\lambda_2 M_{20} \otimes I_{N_d}
\end{bmatrix} \begin{bmatrix}
q - q_0 \\
\nabla f(q) - \nabla f(y)
\end{bmatrix} \leq 0
\]

which implies that if \( y(t) \in S, E(t) \) is nonincreasing. If \( E(0) \) is small enough and initial output is in \( S_0 \), a forward invariance of set \( S_0 \) and thus boundedness of state and output trajectories is established. LaSalle’s invariance principle then implies convergence. In comparison to [14], our analysis is motivated by the proposed modular architecture of augmenting an off-the-shelf closed-loop tracking system in series with the prefilter block (see Section III). The constant \( c_2 \) from Assumption 3 item 5) plays the role of an upper bound on the tracking error \( q - y \) and ensures that the set \( S \), where properties of \( f \) are known, is not exited even under imperfect tracking. The storage function is thus composed of an additional “energy” term capturing tracking performance of the tracking system. The main result is summarized next.

Theorem 10 (Analysis for Problem 2): Assume that the function \( f : \mathbb{R}^{N_d} \to \mathbb{R} \) and an open set \( S \subset \mathbb{R}^{N_d} \) satisfy Assumption 3 and assume that the vehicle dynamics (1) satisfy Assumption 4. If there exist \( R \geq 0, Q > 0, \mu > 0, \lambda_1 \geq 0, \lambda_2 \geq 0, \) and \( \epsilon > 0 \) such that \( Z \) defined in (14) is negative semidefinite and if the initial conditions \( x_0, q_0, \) and \( p_0 \) satisfy

\[
V_s(x_0, q_0, p_0) \leq \min \left\{ \frac{2c_1 \mu}{\epsilon^2}, \frac{c_2 \lambda_{\text{max}}(Q)}{\|C\|^2} \right\} \]

where \( c_1 \) and \( c_2 \) are constants involved in Assumption 3 item 5), then the state trajectory \( y(t) \) generated by the dynamics (4) remains bounded for all \( t \geq 0 \) and \( y(t) \) converges to the set \( \{ y_0 \in S | \nabla f(y_0) = 0 \} \).

Remark 3: Note that the sufficient condition \( Z \leq 0 \) [see (14)] contains model matrices appearing in the dynamics of a single agent and is independent of the network-size \( N \).

Corollary 1: Assume that \( f \) and \( S = \mathbb{R}^{N_d} \) satisfy Assumption 3 and assume that the vehicle dynamics (1) satisfy Assumption 4. In addition, assume that \( f \) is proper, i.e., \( \{ q | f(q) \leq c \} \) is compact for all \( c \in \mathbb{R} \). If there exist \( R \geq 0, Q > 0, \mu > 0, \lambda_1 \geq 0, \) and \( \lambda_2 \geq 0 \) such that \( Z \) defined in (14) is negative semidefinite, then, for any initial condition, the state trajectory \( \eta \) under dynamics (4) remains bounded for all \( t \geq 0 \) and \( y(t) \) converges to the set \( \{ y_0 | \nabla f(y_0) = 0 \} \).

VI. NUMERICAL EXAMPLES

A. Numerical Results With Formation Control

Consider a linearized quadrotor model with an linear-quadratic-regulator (LQR)-based state-feedback controller tuned for zero steady-state error for step position references. The closed-loop system in the form of (1) can be augmented with dynamics (2) to obtain \( G \). Assume that \( \psi \in \mathcal{S}(m_{\psi}, L_{\psi}) \) and \( V(y) = \lambda_1 (y - r)^T(L \otimes I_d)(y - r) \), where \( L \) is the graph Laplacian corresponding to \( G \) with the set of informed agents \( V_i \). Consider the setup of Problem 1 such that \( f \) as per Definition 1 belongs to \( \mathcal{S}(m, L) \). We apply Theorem 9 to estimate the rate of convergence for dynamics (4).

1) Conservatism analysis with known \( m \) and \( L \): For a free parameter \( L \), consider the uncertainty set \( \Delta_{0, 3, L} \) of tuples \( \{G, V_i, \psi\} \) such that \( f \) as per Definition 1 with \( V(y) = \lambda_1 (y - r)^T(L \otimes I_d)(y - r) \) belongs to \( S(0, 3, L) \). In order to estimate the conservatism, we find by trial and error, nontrivial examples with performance as close as possible to the performance guaranteed by the theory. For this purpose, define

\[
\Delta_1 = \{ (G, V_i, \psi) | G = G^2 \text{ and } V_i = \{ \psi \in \mathcal{S}(m_{\psi}, L_{\psi}) \}
\]

\[
\Delta_2 = \{ (G^2, V_i, \psi) | \psi = 1.85\|y - y_0\|^2 \}
\]

Fig. 1 (left-hand side) shows the convergence rate estimates obtained from Theorem 9 guaranteed by the ZF multipliers and the CC for different values of \( L \). While CC can certify stability for \( 0.3 \leq L \leq 7 \), the ZF multipliers can certify stability for \( 0.3 \leq L \leq 17.64 \). Since convergence rates for examples in Delta 1 and Delta 2 coincide with the estimates, these estimates are tight.

2) Robust analysis with unknown \( m \) and \( L \): We now show how constants \( m \) and \( L \) can be estimated if they are unknown. Assume

\[
Z = \begin{bmatrix}
A_0 X_0 + X_0 A_0 \\
B_0 X_0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \varepsilon I_d & \mu I_d & 0 & 0 \\
0 & 0 & \mu I_d & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} + \sum_{i=1}^{2} \begin{bmatrix}
\lambda_1 M_{i0} \otimes I_d
\end{bmatrix} =: C_{10} \cdot D_{1d}, \quad \text{where}
\]

\[
A_0 = \begin{bmatrix}
A & B_q & B_p \\
0 & 0 & I_d \\
0 & 0 & -k_p I_d
\end{bmatrix}, \quad B_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad C_{10} = \begin{bmatrix}
0 & I_d & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad D_{10} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad C_{20} = \begin{bmatrix}
-C & I_d & 0 & 0 \\
I_d & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \text{and } D_{20} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 & I_d
\end{bmatrix}.
\]
that \( \psi \in S(m_{\psi}, L_{\psi}) \) and a minimal structure in the graph is known in the form of essential edges that are present in all allowable graphs. This means that all allowable graphs could be obtained by adding edges to the minimal graph. Let the graph Laplacian associated with this minimal graph be denoted by \( L_0 \). The grounded graph Laplacian (see Definition 2) associated with this minimal graph is \( L_m := L_0 + m_{\psi} E \). Using the fact that adding edges can only increase the eigenvalues of the graph Laplacian (see [24, Lemma 6.9]), we have that \( L_m = L_0 + m_{\psi} E \leq L + m_{\psi} E = L \). Furthermore, assuming that the maximum degree, denoted by \( d_{\text{max}} \), is known and using the fact that \( 2d_{\text{max}} \) is an upper bound on the largest eigenvalue of the graph Laplacian (Gershgorin discs theorem) together with \( \psi \in S(m_{\psi}, L_{\psi}) \), we get \( L_0 \geq (2d_{\text{max}} + L_{\psi}) I_{\nu} \). The uncertainty set can then be defined to be \( \Delta \), with \( \nu = \lambda_{\min}(L_0) \) and \( L = (2d_{\text{max}} + L_{\psi}) I_{\nu} \). This is now illustrated on a concrete example. Let \( \Delta \) be the set of all \((G, V, \psi)\) such that \( \psi \in S(3, L_{\psi}) \) and \( G, V \) satisfy the following.

1. At least one third of total number of agents are informed.
2. Every agent that is not an informed agent has an edge with at least one informed agent.
3. Maximum degree of all agents is 2.

Since any informed agent \( i \in V_i \) is either connected to 0, 1, or 2 other agents, a suitable ordering of the agents will lead to a minimal grounded Laplacian of the form \( L_m = \text{blkdiag}(m_{\psi}, L_1, \ldots, L_1, L_2, \ldots, L_2) \), where

\[
L_1 = \begin{bmatrix} 1 + m_{\psi} & -1 \\ -1 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2 + m_{\psi} & -1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.
\]

Therefore, with \( m = \min(m_{\psi}, \lambda_{\min}(L_1), \lambda_{\min}(L_2)) = 0.4116 \) and \( L = L_{\psi} + 2d_{\text{max}} = L_{\psi} + 4 \), we get that \( \Delta \subseteq L_{0.4116, L_{\psi} + 4} \). Fig. 1 (right-hand side) shows the convergence rate estimates for different values of \( L_{\psi} \). We can again find worst-case examples by trial and error that coincide with the estimates showing that the analysis is without any conservatism here as well. With \( \psi_0(x) = (x - y_{\text{opt}})^T \text{blkdiag}(3, L_{\psi})(x - y_{\text{opt}}) \), the worst-case examples are in

\[
\Delta_3 = \{(G, V, \psi) \mid G = G_{c_{\text{cycle}}}, V_i = V, \quad \psi = \psi_0\} \subseteq \Delta
\]

\[
\Delta_4 = \{(G, V, \psi) \mid G = G_{\text{cycle}}, V_i = \{1\}, \quad \psi = \psi_0\} \subseteq \Delta.
\]

### B. Numerical Results With Flocking

Now consider three quadrotors from Section VI embedded in \( \psi \in S(0.5, 1) \) with the minimizer \( y_{\text{opt}} = [60 \ 30]^T \). Let \( V(y) = \sum_{(i,j)} k_{\psi, L}(\|y_i - y_j\|_\infty - d)^2 \), where the sigma-norm \( \|\cdot\|_\infty \) is defined in [20] (with \( \varepsilon = 1 \) and \( 0 \leq k \leq 1 \). We assume that at least one agent has access to the gradient. One can then show that \( f \) (see Definition 1) is proper since \( \psi \) is strongly convex, \( V \) is nonnegative, and agents that are not informed agents, if any, are connected to the informed agent at all times. Furthermore, \( \nabla^2 f(y) = \nabla^2 V(y) + \nabla^2(\sum_{i\in V} \psi(y_i)) \leq 2I + I \leq 3I \), which implies that \( f \) satisfies Assumption 3 with \( S = \mathbb{R}^N \) and \( M_0 = M_{20} = \text{blkdiag}(9I, 1) \). Therefore, the local LQR controller is designed to satisfy Assumption 4. Applying Corollary 11, a sufficient condition for stability of the overall dynamics is the feasibility of the LMI (14). Numerical studies verify that the LMI (14) is feasible for \( k_d \geq 4.8 \). Fig. 2 shows trajectories with different values of \( k_d \) for \( \psi(z) = \frac{1}{2}\|z - y_{\text{opt}}\|_2^2 \) and \( k = 1 \). The trajectories for \( k_d \leq 2 \) do not converge, whereas with \( k_d = 5 \), the trajectories converge. The trajectories with \( k_d = 4 \) converge showing the conservativeness in this example.

### VII. Conclusion

We analyze cooperative control dynamics involving gradient-based forcing terms using the framework of IQCs and demonstrate its application on the problem of source seeking with formation control and flocking. All LMIs are independent of the network size, and the convergence rate estimates for formation control are found to be tight in the studied examples. These results can be extended to nonlinear systems with a quasi-linear parameter varying (LPV) representation as in [25].

### Appendix

#### A. Definitions of \( \pi_{m,L} \) and \( \mathbb{P} \)

With \( \beta = -1 \), a positive integer \( \nu \), and some \( \alpha > 0 \), define

\[
A^\nu_{\psi} = \begin{bmatrix} \beta - 2\alpha & 0 & \cdots & 0 \\ 1 & \beta - 2\alpha & \cdots & 0 \\ 0 & 1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \beta - 2\alpha \end{bmatrix}, \quad B_{\nu} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]
Finally, define the set $\mathcal{F}$ as

$$
\mathcal{F} = \begin{cases} 
0 & 0 & H & -P_3 \\
0 & 0 & -P_3^T & 0 \\
* & * & 0 & 0 \\
* & * & 0 & 0
\end{cases}
H, P_1, P_3 \text{ satisfy (16), (17)}.
$$

}\]

\section{B. Supporting Lemma}

This section proves a lemma that is central in the derivation of the ZF IQCAs and is used in the proof of Theorem 5.

\textbf{Lemma 12:} Assume that $\alpha \geq 0$ and let $\beta(\tau) = \min\{1, e^{-2\alpha \tau}\}$ for $\tau \in \mathbb{R}$. Let $\tilde{u}, \tilde{y} \in L_2(0, \infty)$ be related by $\tilde{u} = \nabla f(\tilde{y} + y_*)$, where $f : S(m, L)$ and $y_*$ minimizes $f$. Then, the signals $p$ and $q$ defined in (6) satisfy

$$
\int_0^T e^{2\alpha \tau} p(t)^T (q(t) - \beta(\tau) q_T(t - \tau)) dt \geq 0
$$

(18)

$\forall \tau \in \mathbb{R} \forall T \geq 0$, where $q_T$ denotes the extension defined by

$$
q_T(t) = \begin{cases} 
q(t), & t \in [0, T] \\
0, & t \in \mathbb{R} \setminus [0, T].
\end{cases}
$$

Note that the asymmetry of the function $\beta(\tau)$ (also present in [9]) arises from the factor $e^{2\alpha \tau}$ in alpha-IQC formulation.

\textbf{Proof of Lemma 12:} The proof goes along the lines of [6] and borrows some ideas from [10]. The central idea behind the proof is to exhibit a nonnegative function $F : \mathbb{R} \rightarrow \mathbb{R}$ with support $[0, T]$ such that the integrand of (18) can be lower bounded for all $\tau \in \mathbb{R}$ and all $t \in [0, T]$ as

$$
e^{2\alpha \tau} p(t)^T (q(t) - \beta(\tau) q_T(t - \tau)) \geq F(t) - F(t - \tau).
$$

(20)

\textbf{If such a function $F$ exists, integrating both sides of (20) from 0 to $T$ and using the nonnegativity of $F$ along with the fact that $F$ is zero outside $[0, T]$, we get the desired result (18). The remainder of this proof constructs function $F$.}

For convenience, let the dimension of $\tilde{y}$ be denoted by $n_y$, i.e., $\tilde{y} \in \mathbb{R}^{n_y}$. Let $g : \mathbb{R}^{n_y} \rightarrow \mathbb{R}$ be defined as

$$
g(\tilde{y}) = f(\tilde{y} + y_*) - f(y_*) - \frac{m}{2} \|\tilde{y}\|^2
$$

where $f \in S(m, L)$ and $y_*$ minimizes $f$. This can be used to show that $g \in S(0, L - m)$, $g(0) = 0$, and $\nabla g(0) = 0$. It can be further shown (see [6]) that for all $\tilde{y}, \tilde{y}_1, \tilde{y}_2 \in \mathbb{R}^{n_y}$

$$
(L - m) g(\tilde{y}) - \frac{1}{2} \|\nabla g(\tilde{y})\|^2 \geq 0
$$

(21)

$$
(L - m) \nabla g(\tilde{y})^T \tilde{y} \geq (L - m) g(\tilde{y}) + \frac{1}{2} \|\nabla g(\tilde{y})\|^2
$$

(22)

Using (21), define a nonnegative function $r : \mathbb{R}^{n_y} \rightarrow \mathbb{R}$ as

$$
r(\tilde{y}) = (L - m) g(\tilde{y}) - \frac{1}{2} \|\nabla g(\tilde{y})\|^2.
$$

Using definitions (6) and $u = \nabla f(\tilde{y} + y_*)$, we have

$$
p(t) = \nabla g(\tilde{y}(t))
$$

$$
q(t) = (L - m) \tilde{y}(t) - \nabla g(\tilde{y}(t)).
$$

Since $\nabla g(0) = 0$ and $f$ is a static map, we can consider signal extensions $\tilde{u} \tilde{y}, \tilde{y} T, \tilde{p} T, \tilde{q} T$ using (19) to have $t \in \mathbb{R}$

$$
\tilde{p}(t) = \tilde{u}(t) - m \tilde{y}(t) = \nabla g(\tilde{y}(t))
$$

$$
\tilde{q}(t) = L \tilde{y}(t) - \tilde{u}(t) = (L - m) \tilde{y}(t) - \nabla g(\tilde{y}(t)).
$$

This, together with (22), shows that for any $t_1 \in \mathbb{R}$

$$
\tilde{p}(t_1)^T \tilde{q}(t_1) \geq r(\tilde{y}(t_1)).
$$

(24)

\textbf{Along the same lines, using (23), we get for $t_1, t_2 \in \mathbb{R}$}

$$
\tilde{p}(t_1)^T \tilde{q}(t_1) - \tilde{q}(t_2) \geq r(\tilde{y}(t_1)) - r(\tilde{y}(t_2)).
$$

(25)

For any $\beta \in [0, 1]$, multiplying (24) by $(1 - \beta)$, multiplying (25) by $\beta$, and then adding them together, we get

$$
\beta(\tau) \tilde{p}(t_1)^T \tilde{q}(t_1) - \beta \tilde{q}(t_2) \geq \beta(\tau) r(\tilde{y}(t_1)) - \beta r(\tilde{y}(t_2)).
$$

Since $\beta(\tau)$ as defined in the statement of this lemma lies in $[0, 1]$, we can use the above inequality along with nonnegativity of $r$ and $\beta(\tau) \leq e^{-2\alpha \tau} \forall \tau \in \mathbb{R}$ to obtain

$$
\beta(\tau) \tilde{p}(t_1)^T \tilde{q}(t_1) - \beta(\tau) \tilde{q}(t_2) \geq r(\tilde{y}(t_1)) - \beta(\tau) r(\tilde{y}(t_2)).
$$

(26)

Multiplying both sides by $e^{2\alpha \tau}$ and using (19), we get that for all $\tau \in \mathbb{R}$ and $t \in [0, T]$

$$
e^{2\alpha \tau} p(t)^T (q(t) - \beta(\tau) q_T(t - \tau)) \geq F(t) - F(t - \tau)
$$

where $F(t) := e^{2\alpha \tau} r(\tilde{y}(t))$ is the sought function.

\section{References}

[1] J. Cortés and M. Egerstedt, “Coordinated control of multi-robot systems: A survey,” SICE J. Control, Meas., Syst. Integration, vol. 10, no. 6, pp. 495–503, 2017.

[2] S. Michalski and C. Ebenbauer, “Extremum control of linear systems based on output feedback,” in 2016 IEEE 55th Conf. Decis. Control, 2016, pp. 2963–2968.

[3] Z. E. Nelson and E. Mallada, “An integral quadratic constraint framework for real-time steady-state optimization of linear time-invariant systems,” in 2018 Annu. Amer. Control Conf., 2018, pp. 597–603.

[4] A. Megretski and A. Rantzer, “System analysis via integral quadratic constraints,” IEEE Trans. Autom. Control, vol. 42, no. 6, pp. 819–830, Jun. 1997.
[5] J. Veenman, C. W. Scherer, and H. Köroğlu, “Robust stability and performance analysis based on integral quadratic constraints,” *Eur. J. Control*, vol. 31, pp. 1–32, 2016.

[6] L. Lessard, B. Recht, and A. Packard, “Analysis and design of optimization algorithms via integral quadratic constraints,” *SIAM J. Optim.*, vol. 26, no. 1, pp. 57–95, 2016.

[7] B. Hu and P. Seiler, “Exponential decay rate conditions for uncertain linear systems using integral quadratic constraints,” *IEEE Trans. Autom. Control*, vol. 61, no. 11, pp. 3631–3637, Nov. 2016.

[8] J. Zhang, P. Seiler, and J. Carrasco, “Noncausal FIR Zames-Falb multiplier search for exponential convergence rate.” Accessed: Feb. 25, 2019. [Online]. Available: https://arxiv.org/pdf/1902.09473

[9] R. A. Freeman, “Noncausal Zames-Falb multipliers for tighter estimates of exponential convergence rates,” in *Annu. Amer. Control Conf.*, 2018, pp. 2984–2989.

[10] C. W. Scherer, “Dissipativity and integral quadratic constraints: Tailored computational robustness tests for complex interconnections,” *IEEE Control Syst. Mag.*, vol. 42, no. 3, pp. 115–139, Jun. 2022.

[11] A. Sundararajan, B. Hu, and L. Lessard, “Robust convergence analysis of distributed optimization algorithms,” in *2017 55th IEEE Annu. Allerton Conf. Commun., Control, Comput.*, 2017, pp. 1206–1212.

[12] A. Sundararajan, B. Van Scoy, and L. Lessard, “Analysis and design of first-order distributed optimization algorithms over time-varying graphs,” *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 4, pp. 1597–1608, Dec. 2020.

[13] B. Hu and L. Lessard, “Dissipativity theory for Nesterov’s accelerated method,” in *Proc. Int. Conf. Mach. Learn.*, 2017, pp. 1549–1557.

[14] M. Fazlyab, A. Ribeiro, M. Morari, and V. M. Preciado, “Analysis of optimization algorithms via integral quadratic constraints: Nonstrongly convex problems,” *SIAM J. Optim.*, vol. 28, no. 3, pp. 2654–2689, 2018.

[15] M. Arcak, C. Meissen, and A. Packard, *Networks of Dissipative Systems: Compositional Certification of Stability, Performance, and Safety*. Berlin, Germany: Springer, 2016.

[16] A. Datar and H. Werner, “Robust performance analysis of source-seeking dynamics with integral quadratic constraints,” in *2022 Amer. Control Conf.*, 2022, pp. 5229–5234.

[17] A. Datar, P. Paulsen, and H. Werner, “Flocking towards the source: Indoor experiments with quadrotors,” in *2020 IEEE Eur. Control Conf.*, pp. 1638–1643.

[18] A. Attallah, A. Datar, and H. Werner, “Flocking of linear parameter varying agents: Source seeking application with underwater vehicles,” *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 7305–7311, 2020.

[19] A. Datar, C. Hespe, and H. Werner, “Robust performance analysis of cooperative control dynamics via integral quadratic constraints,” 2022, arXiv:2206.04650.

[20] R. Olfati-Saber, “Flocking for multi-agent dynamic systems: Algorithms and theory,” *IEEE Trans. Autom. Control*, vol. 51, no. 3, pp. 401–420, Mar. 2006.

[21] J. A. Fax and R. M. Murray, “Information flow and cooperative control of vehicle formations,” *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.

[22] W. Xia and M. Cao, “Analysis and applications of spectral properties of grounded Laplacian matrices for directed networks,” *Automatica*, vol. 80, pp. 10–16, 2017.

[23] A. Datar, C. Hespe, and H. Werner, “Code for the paper on robust performance analysis of source-seeking dynamics with integral quadratic constraints,” Zenodo Repository, 2022, doi: 10.5281/zenodo.6840534.

[24] F. Bullo, *Lectures on Network Systems*, 1.7 ed., Kindle Direct Publishing, 2024. [Online]. Available: https://fbullo.github.io/lns

[25] A. Datar, A. M. Gonzalez, and H. Werner, “Gradient-based cooperative control of quasi-linear parameter varying vehicles with noisy gradients,” *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 8030–8035, 2023.