A Note on Near-factor-critical Graphs

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Abstract

A near-factor of a finite simple graph $G$ is a matching that saturates all vertices except one. A graph $G$ is said to be near-factor-critical if the deletion of any vertex from $G$ results in a subgraph that has a near-factor. We prove that a connected graph $G$ is near-factor-critical if and only if it has a perfect matching. We also characterize disconnected near-factor-critical graphs.

Keywords: perfect matching, factor, near-factor, near-factor-critical, Tutte’s theorem

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1 Introduction

All graphs $G$ considered in this note are finite and simple with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of $G$ is called the order of $G$. A matching $M$ of $G$ is a subset of $E(G)$ such that no two edges in $M$ share a common endpoint. A matching $M$ saturates a vertex $v$ of $G$, or $v$ is said to be $M$-saturated, if $v$ is an endpoint of some edge in $M$. Otherwise, $v$ is said to be $M$-unsaturated. A matching $M$ is called perfect if

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every vertex of $G$ is $M$-saturated. A 1-factor is synonymous with a perfect matching. A near-factor is a matching that saturates all vertices except one.

Let $S \subseteq V(G)$. The subgraph of $G$ obtained from $G$ by deleting all vertices of $S$ is denoted by $G \setminus S$. In particular, if $S$ is a singleton $\{v\}$, we denote $G \setminus \{v\}$ by $G - v$. A graph $G$ is said to be factor-critical if $G - v$ has a 1-factor for every $v \in V(G)$. A factor-critical graph is necessarily of odd order. This notion was first introduced by Gallai [1] and has been intensively studied, e.g. [3, 4]. We call a graph $G$ near-factor-critical if $G - v$ has a near-factor for every $v \in V(G)$. A near-factor-critical graph $G$ is necessarily of even order.

Li et al. [2] showed that, for a graph $G$ with a vertex of degree one, $G$ has a 1-factor if and only if $G$ is near-factor-critical. They asked whether this result could be generalized to any connected graph $G$. In this note, we are going to give a positive solution to this question. Since the union of two disjoint odd cycles has no 1-factor and is near-factor-critical, the connectedness of $G$ is essential for a generalization.

2 Main results

A component of a graph $G$ is a maximal connected subgraph of $G$. An odd, or even, component is a component having odd, or even, number of vertices. The number of odd components is denoted by $o(G)$. If a graph $G$ has a near-factor $M$, then there exists a unique $M$-unsaturated vertex in $G$ and is denoted by $u(M)$.

**Lemma 1** Let $G$ be a connected graph. Suppose that, for $S \subseteq V(G)$, there is an odd component $H$ of $G \setminus S$. For any vertex $v \notin V(H)$, if $M$ is a near-factor of $G - v$ and $u(M) \notin V(H)$, then there is an edge of $M$ joining a vertex of $S$ with a vertex of $H$.

**Proof.** Since $u(M) \notin V(H)$, every vertex of $H$ is $M$-saturated. Since $v \notin V(H)$, if each edge of $M$ has zero or two endpoints in $H$, then $H$ is of even order, a contradiction. By the connectedness of $G$, there must exist an edge of $M$ having one endpoint in $H$ and one endpoint in $S$.

**Theorem 2** The following conditions are equivalent for a connected graph $G$.

1. $G$ has a 1-factor.
2. $o(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.
3. $G$ is near-factor-critical.

**Proof.** The equivalence between conditions 1 and 2 is the well-known Tutte’s Theorem [6]. It is clear that condition 1 implies condition 3. It remains to prove that condition 3 implies condition 2.
Suppose that condition 2 fails for $G$. Then there is a subset $S \subseteq V(G)$ such that $o(G \setminus S) > |S|$. Choose an arbitrary vertex $v \in S$. Then $G - v$ has a near-factor $M$. By Lemma 4, $G \setminus S$ has at least $o(G \setminus S) - 1$ odd components $H$ such that there is an edge of $M$ having one endpoint in $H$ and one endpoint in $S$. Thus, $|S| \geq 1 + o(G \setminus S) - 1 = o(G \setminus S) > |S|$, a contradiction.

**Theorem 3** Let $G$ be a disconnected graph. Then $G$ is near-factor-critical if and only if one of the following holds.

1. All components of $G$ are even and each of them has a 1-factor.
2. There are only two components $H_1$ and $H_2$ of $G$ and each of them is factor-critical.

**Proof.** The sufficiency is straightforward. We now prove the necessity. Suppose that there exist an even component $F$ and an odd component $H$ of $G$. Choose an arbitrary vertex $v$ from $F$. Then $G - v$ has a near-factor $M$. Since both $F - v$ and $H$ have odd number of vertices, each of them should contain an $M$-unsaturated vertex. This contradicts the uniqueness of $u(M)$. Therefore, $G$ consists of either all even or all odd components. In the latter case, the number of odd components is even since the order of $G$ is even.

Suppose that $G$ consists of even components $F_1, F_2, \ldots, F_p$, where $p \geq 2$. Choosing an arbitrary vertex $x$ of $F_1$, $G - x$ has a near-factor $M_1$. Since $F_1 - x$ is of odd order, $u(M_1)$ belongs to $F_1 - x$. Thus, $M_1$ restricted to $F_i$ is a 1-factor of $F_i$ for each $i \geq 2$. Similarly, $F_1$ can be argued to have a 1-factor if the vertex $x$ was chosen from $F_2$.

Next, suppose that $G$ consists of odd components $H_1, H_2, \ldots, H_{2q}$ for some $q \geq 1$. Choosing an arbitrary vertex $z$ of $H_1$, $G - z$ has a near-factor $M_2$ such that $u(M_2) \not\in V(H_1)$ since $H_1 - z$ is of even order. Each odd component $H_j$, $j \geq 2$, should contain an $M_2$-unsaturated vertex. It follows that $q = 1$ and $H_1 - z$ has a 1-factor. Similarly, $H_2 - z$ can be argued to have a 1-factor if the vertex $z$ was chosen from $H_2$.

**Remark.** It follows from Theorems 2 and 3 that the time complexity for recognizing near-factor-critical graphs is dominated by the complexity for recognizing factor-critical graphs. The latter was determined in [3], using the well-known maximum matching algorithm of [5], to run in $O(n^{1/2}m)$ time, where $n = |V(G)|$ and $m = |E(G)|$.

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