Bodies in an interacting active fluid: far-field influence of a single body and interaction between two bodies

Omer Granek¹,⁴, Yongjoo Baek², Yariv Kafri¹ and Alexandre P Solon³

¹ Department of Physics, Technion—Israel Institute of Technology, Haifa 32000, Israel
² Department of Physics and Astronomy, Seoul National University, Seoul 08826, Republic of Korea
³ Sorbonne Université, CNRS, Laboratoire de Physique Théorique de la Matière Condensée, LPTMC, F-75005 Paris, France
E-mail: omer.granek@campus.technion.ac.il

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Abstract. Because active particles break time-reversal symmetry, an active fluid can sustain currents even without an external drive. We show that when a passive body is placed in a fluid of pairwise interacting active particles, it generates long-range currents, corresponding to density and pressure gradients. By using a multipole expansion and a far-field constitutive relation, we show that the leading-order behavior of all three corresponds to a source dipole. Then, when two bodies or more are placed in the active fluid, generic long-range interactions between the bodies occur. We find these to be qualitatively different from other fluid mediated interactions, such as hydrodynamic or thermal Casimir. The interactions can be predicted by measuring a few single-body properties in separate experiments. Moreover, they are anisotropic and do not satisfy an action-reaction principle. These results extend previous results on non-interacting active particles. Our framework may point to a path towards self-assembly.

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⁴Author to whom any correspondence should be addressed.
1. Introduction

Active particles, which propel themselves by consuming stored or ambient energy, form an interesting class of far-from-equilibrium systems [1–4]. They have attracted much attention due to unusual collective phenomena which are not found in equilibrium. Examples include flocking [5–10], motility-induced phase separation (MIPS) [11–16], and the lack of an equation of state for pressure [17–19]. In particular, it is known that asymmetric obstacles immersed in a fluid of active particles (called an active fluid) create density gradients [20–22] and currents [22–25]. These phenomena are examples
of the ratchet effect: directed motion can be extracted out of fluctuations by breaking both spatial symmetry and time-reversal symmetry [26, 27]. The breaking of spatial symmetry is provided by the asymmetric obstacle, while the breaking of time-reversal symmetry stems from the dynamics of the active particles [3, 28, 29]. The active ratchet mechanism can be applied to targeted delivery [30], self-starting micromotors [31–33] and self-assembly of colloidal molecules [34–36].

Recently, the effects of arbitrary asymmetric bodies immersed in a non-interacting fluid of run-and-tumble or active Brownian particles were analyzed and quantified [37]. First, it was shown that even a single localized asymmetric body generates a long-range density disturbance which decays as a power law and whose structure is mathematically similar to the potential of an electric source dipole. The strength of the dipole is directly related to the force exerted by the body on the active fluid. In turn, a current field, whose far-field behavior is similar to the field of an electric source dipole, is generated. We note that a similar mechanism was also found to exist in diffusive systems with an asymmetric localized drive [38–42].

This led to the finding that, when multiple bodies are placed in the fluid, long-range interactions exist between the bodies. These interactions, expressed through forces and torques, are long range with a magnitude decaying with distance as a power law. They are directly related to the density and current fields produced by a single body. Hence, they differ from the previously observed confinement–induced interactions [43–47], which decay over a finite characteristic length-scale. The interactions are different from the conventional long-range interactions, such as hydrodynamic interactions [48] and similar bath-mediated interactions [49–54], which exist only among moving bodies and require interactions between the fluid particles. They also differ from thermal Casimir forces [55, 56], which require long-range correlations. The interactions generically exist even between static bodies in a fluid far from any critical point. The leading-order interactions were found to be fully determined by the single-body properties of each body involved, so that one can predict the interaction between a pair of bodies from separate observations of the individual isolated bodies. Moreover, the interactions fail to satisfy an action–reaction principle, showing that the activity of the particles compensates for the residual forces and torques. Such a non-Newtonian nature also exists in non-equilibrium depletion forces [57–61]. However, these are not truly scale-free, as they are screened on scales much larger than the body size [62, 63]. Interactions with similar scaling to this may be present in the strongly-interacting clustered phase of active matter, which exhibits almost-scale-free correlations [64–66]. On the contrary, true scale-free interactions were observed following a quenched in temperature in both passive and active matter [67]. However, in that case, the effect is only transient. Lastly, it was demonstrated [37] that the generic active fluid interactions give rise to novel dynamical phenomena involving two objects immersed in an active fluid.

In this paper, we generalize these results to the technically more demanding problem of a fluid of run-and-tumble or active Brownian particles with pairwise interactions between the active particles. Assuming that the active fluid is in a disordered phase, we show that the mathematical structure remains similar to the non-interacting problem, but with interesting differences. In particular, besides the density and current fields, one now needs to consider the pressure field. This is found to decay in a way similar to the
density field, i.e., like the potential of an electric dipole. The density field also exhibits a similar decay, but with an amplitude modified by the compressibility of the active fluid. Using the single body results, we then derive the interactions between two bodies along the lines of reference [37].

The paper is organized as follows. After defining the model of active particles in section 2, we give a brief summary of the main results in section 3. Then, we present the derivation. First, the steady-state conditions for the active particles are shown in section 4. These are used to obtain the far-field effects of a single body in section 5, which in turn allows us to derive the long-range interactions between pairs of bodies in section 6. Finally, we summarize our results and conclude in section 7.

2. Model

We consider a model of active particles which encompasses both active Brownian particles (ABPs) [68, 69] and run-and-tumble particles (RTPs) [70]. The particles propel themselves at speed $v$ and interact via pairwise central forces derived from a potential $U(|r|)$. In what follows, we consider only the two-dimensional case, and the generalization to higher dimensions is straightforward. In the overdamped limit, the position $r_i$ and the orientation $\theta_i$ of active particle $i$ are governed by the Itô–Langevin dynamics

$$\dot{r}_i = v\mathbf{e}_{\theta_i} - \mu \nabla_{r_i} [V(r_i) + \sum_{k \neq i} U(|r_i - r_k|)] + \sqrt{2D_r} \eta_i(t),$$

$$\dot{\theta}_i = \sqrt{2D_r} \xi_i(t).$$

Here $\mu$ is the mobility of particle $i$, $D_t$ and $D_r$ are translational and rotational diffusion constants, the components of $\eta_i$ and $\xi_i$ are mutually independent Gaussian white noises with unit variance, and $\mathbf{e}_{\theta_i} = (\cos \theta_i, \sin \theta_i)^T$ is a unit vector indicating the orientation of the particle. The external potential $V$, which can be written as $V = \sum_j V_j$ with the body index $j$ in the presence of multiple bodies, describes the interaction between each active particle and the bodies immersed in the active fluid. In addition to the diffusive dynamics described by the above equations, we also allow for tumbling dynamics, i.e., $\theta_i$ randomly changes its value at a rate $\alpha$. Pure ABPs correspond to $\alpha = 0$, and pure RTPs correspond to $D_r = 0$. Using this generalized model provides a unified view of active particles. In a steady state, the effect of tumbling becomes identical to the effect of active diffusion—a property used extensively in the diffusive approximation of RTP dynamics at long timescales [70–74]. This emphasizes that our results below are independent of the statistical details of the active force $v\mathbf{e}_{\theta_i}/\mu$. Rather, they rely on the existence of a typical distance traveled by the particle while keeping its orientation $l_i = v/\alpha + D_t$ (also called the run length). It is important to note that the model represents dry active matter, which is ‘dry’ in the sense that it does not conserve the momentum [1, 2, 75]. Accordingly, the model best describes particles next to a surface which can absorb the momentum, such as a layer of vibrated granular particles [6, 7, 19] and gliding bacteria [76]. Nonetheless, it has been shown that for this model, due to the reasons elucidated in [77], there is an

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equation of state for the pressure \[13\]. This will play a salient role in the derivations that follow.

All the results are valid in an adiabatic limit where it is assumed that the object or objects move on a time scale much longer than the diffusive relaxation time of the surrounding active fluid.

3. Main results

We first review our main results before presenting their derivations in detail. To do so, we first consider the case where only a single passive body is immersed in an active fluid, presenting far-field expressions for the steady-state particle density, current density and hydrostatic pressure field created by the body. Then we present results for the case where two passive bodies are placed at a large distance from each other in the same active fluid, giving expressions for the forces and torques between the bodies which are mediated by the fluid. Importantly, the interactions are expressed in terms of single-body properties.

3.1. Far-field effects of a single body

We denote by $\hat{\rho}(\mathbf{r}) \equiv \sum_i \delta(\mathbf{r} - \mathbf{r}_i)$ the empirical density and by $\hat{\mathbf{m}}(\mathbf{r}) \equiv \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \mathbf{e}_{\theta_i}$ the empirical polarization density. A hat above a symbol indicates that the symbol stands for a random variable. The hat shall be removed after taking an average over histories, so that $y = \langle \hat{y} \rangle$. We use $d$ to denote the size of the body corresponding to the potential $V(\mathbf{r})$. If the body is placed upon the origin of the coordinates, the far-field limit is defined as $\mathbf{r} \gg \max(l_r, d)$. In this limit we obtain the pressure, density, and current fields. The results are derived assuming that (i) the active fluid is homogeneous and disordered far away from the body and (ii) the dominating component of the far-field fluid stress can be expressed as a local function of the density. We justify the second assumption in the case where, in the far field, either inter-particle interactions are weak or some correlations have a mean-field structure. Importantly, we confirm this assumption using numerical simulations which verify the theoretically predicted long-range current and density profiles.

Denoting the modulated pressure field by $P(\mathbf{r})$, we find that it satisfies

$$P(\mathbf{r}) = P(\rho_b) + \frac{1}{2\pi} \frac{\mathbf{r} \cdot \mathbf{p}}{r^2} + \mathcal{O}(r^{-2}).$$

(3)

Here $\rho_b$ is the density of active particles at $r \to \infty$, and $P(\rho_b)$ is the corresponding pressure. Throughout the paper, the decay of remainders is given up to some sub-algebraic modulation. The equation of state $P(\rho_b)$ for the pressure has been derived in a few different ways \[13, 15, 77–80\] and takes the form

$$P(\rho_b) = T_{\text{eff}} \rho_b + P_0(\rho_b) + P_1(\rho_b).$$

(4)

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Here $T_{\text{eff}}\rho_b$ is the ideal-gas contribution with an effective temperature $T_{\text{eff}} \equiv D_t/\mu + v_{\text{r}}/(2\mu)$, and $P_D$ and $P_I$ are direct and indirect contributions from the interaction potential $U$, respectively. The latter two are related to the empirical density and polarization by

$$P_D (\rho_b) = -\frac{1}{4} \lim_{r \to \infty} \int d^2 r' r' U' (r') \int_0^1 d\lambda \langle \hat{\rho} (r + (1 - \lambda) r') \hat{\rho} (r - \lambda r') \rangle, \quad (5)$$

$$P_I (\rho_b) = -\frac{1}{2} \lim_{r \to \infty} \int d^2 r' \langle \hat{m} (r) \hat{\rho} (r') \rangle \cdot \nabla U (|r - r'|), \quad (6)$$

which can be written as functions of $\rho_b$ in a homogeneous and disordered fluid. We note that $P_I$ is sometimes referred to as the swim pressure of the active fluid [78, 79]. Finally, $p$ is the dipole moment given by

$$p = -\int d^2 r \rho (r) \nabla V (r). \quad (7)$$

It is equal to the net force applied on the fluid by the body, which is opposite and equal to the force applied on the body by the fluid. We note that $p = 0$ for an apolar $V$, such as one with a disk-like or rod-like shape—dipole-like long-range effects are generated only if the body has a polar asymmetric shape [18, 28].

Based on the above results, we also show that the average particle density $\rho(\mathbf{r})$ can be expanded as

$$\rho(\mathbf{r}) = \rho_b + \frac{1}{2\pi P' (\rho_b)} \frac{\mathbf{r} \cdot \mathbf{p}}{r^2} + \mathcal{O} (r^{-2}) = \rho_b \left[ 1 + \frac{\kappa (\rho_b)}{2\pi} \frac{\mathbf{r} \cdot \mathbf{p}}{r^2} \right] + \mathcal{O} (r^{-2}), \quad (8)$$

where the second equality is obtained by noting that $P' (\rho_b)$ is related to the compressibility of the active fluid by the relation $\kappa (\rho_b) = 1/[\rho_b P' (\rho_b)]$. In other words, for a given force (or dipole moment $\mathbf{p}$) exerted by the body on the surrounding active fluid, an active fluid of greater compressibility has greater density modulations.

Finally, the force generates a long-range current field $\mathbf{J}$ whose far-field expression is given by

$$\mathbf{J} (\mathbf{r}) = -\frac{\mu}{2\pi} \left[ \frac{\mathbf{p}}{r^2} \frac{2 (\mathbf{r} \cdot \mathbf{p}) \mathbf{r}}{r^4} \right] + \mathcal{O} (r^{-3}). \quad (9)$$

### 3.2. Long-range interactions between bodies

Building on the above results, we derive the interactions between two passive bodies in an active fluid. We consider the case where body 2 is separated from body 1 by a mutual far-field displacement $\mathbf{r}_{12}$. When the system is phase separated, we assume that the two bodies are immersed deep inside the same phase. We find that $\mathbf{F}_{12}$, the additional force
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exerted on body 2 due to the introduction of body 1 into the fluid, can be expressed by single-body properties. Specifically, we decompose $F_{12}$ as

$$F_{12} = F_{12}^a + F_{12}^s,$$

where $F_{12}^a$ acts only on asymmetric bodies with non-zero dipole moment (see equation (7)), while $F_{12}^s$ is present even for fully symmetric bodies with zero dipole moment. In what follows, we use subscript $j$ to denote the quantities appearing in the single-body problem of body $j$, e.g., the pressure field $P_j(r)$ and current density $J_j(r)$.

At leading order in $r_{12}$, $F_{12}^a$ and $F_{12}^s$ can be understood as the response forces induced by the pressure perturbation $\Delta P_j(r) \equiv P_j(r) - P(\rho_b)$ and the current around the body. In other words, we can write (see figure 1)

$$F_{12}^a = R^P_{2} \Delta P_1 (r_{12}) + O \left( r_{12}^{-2} \right),$$

$$F_{12}^s = R^J_2 J_1 (r_{12}) + O \left( r_{12}^{-3} \right),$$

where we used the linear response operators defined as

$$R^P_j \equiv - \partial \Delta P_j (\rho_b) [P(\rho_b), J_b] |_{J_b=0} = \int d^2r \partial \Delta P_j (\rho_b) [P(\rho_b), J_b] |_{J_b=0} \nabla V_j,$$

$$R^J_j \equiv \int d^2r (\nabla V_j) \nabla J_b \rho [P(\rho_b), J_b] |_{J_b=0}.$$

Note that equations (3), (8) and (9) imply that the forces decay with distance as $F_{12}^a \sim r_{12}^{-1}$ and $F_{12}^s \sim r_{12}^{-2}$. Importantly, $F_{12}$ can be predicted, to leading order, solely by measuring the single-body properties $p_1$, $p_2$, $R^P_2$ (or $R^s_2$) and $R_2$. In practice, for bodies with an axis of symmetry, say the $x$ axis, the measurement is reduced even further. By reflection symmetry, the dipole moment satisfies $p_2 = p_2 e_x$ (with $p_2$ not necessarily positive). Hence, $R^P_2 = R^P_2 e_x$, meaning that one has to measure only the $x$ component. In a similar manner, one of the principal axes of $R_2$ coincides with the $x$ axis, and therefore the other is the $y$ axis. This allows one to measure only two components of this tensor, instead of four. Note that $F_{12}$ is not necessarily symmetric under

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Figure 1. Schematic diagram of two interacting asymmetric passive bodies. Body 2 (blue) is placed at the origin, while body 1 (orange) is displaced by $r_{12}$ (black). Superposed dipolar currents are shown in grey streamlines, and dipole density modulations are shown in red and blue colors. In this case, the linear response operators $R^p_2$, $R^p_2$, $T_2$ and $\gamma_2$ are parallel to axis spanned by the unmodified dipole moment $p_2$ (blue), which is the $x$ axis. The $x$ and $y$ axes are the two principal axes of the operator $R_2$. The force $F_{12}$ (red) and torque $\tau_{12}$ (green) applied on body 2 by body 1 are obtained from the above quantities and the unmodified dipole moment $p_1$. See text for results and derivations.

the exchange of indices $1 \leftrightarrow 2$, indicating that an action-reaction principle for passive bodies interactions which are mediated by active particles does not hold. This property is expected, because the forces $F_{12}$ and $F_{21}$ are mediated by dry active particles, which do not conserve momentum.

The physics of equations (11) and (12) can be described as follows. To leading order, the influence of body 1 can be attributed to a local shift of the pressure field $\Delta P_1 (r_{12})$ or to a local shift of the particle density $\delta \rho_1 (r_{12})$. Since this shift is a scalar quantity, it contributes only to $F^s_{12}$ and thus can only modulate the force on an asymmetric body 2. At the next order, body 1 also generates a constant current of density $J_1 (r_{12})$. This current applies a force on body 2 even if it is fully symmetric. Thus, it provides the leading-order contribution to $F^s_{12}$, with $R_2$ being the response operator. Hence, an asymmetric body 1 can propel a fully symmetric body 2 in the direction of $J_1$.

The additional torque exerted on body 2 due to body 1, $\tau_{12}$, can be expressed in a similar manner (see figure 1). We denote the self-torque of the isolated bodies by

$$\tau_j = \int d^2r \rho_j (r) (r - X_j) \times \nabla V_j,$$

(17)
where $\mathbf{X}_j$ is the position of body $j$, satisfying $\mathbf{X}_1 - \mathbf{X}_2 = -\mathbf{r}_{12}$. Clearly, depending on the shape of the body in question, it may or may not experience a self-torque. For example, a spherically symmetric body, for which $V_j = V_j(|\mathbf{r} - \mathbf{X}_j|)$, experiences no self-torque. It has been already demonstrated, both numerically [18] and experimentally [32, 81], that asymmetric bodies generate ratchet currents that induce a self-torque.

We identify $\tau_{12}$ as the change in the self-torque of body 2 due to the introduction of body 1 into the fluid. As done for the interaction force, we decompose $\tau_{12}$ as

$$\tau_{12} = \tau_{12}^a + \tau_{12}^s, \quad (18)$$

where $\tau_{12}^a$ acts only on bodies with non-zero self-torque ($\tau_2 \neq 0$), while $\tau_{12}^s$ is present even for bodies with zero self-torque ($\tau_2 = 0$). At leading order in $r_{12}$,

$$\tau_{12}^a = T_2 \Delta P_1 (\mathbf{r}_{12}) + O (r_{12}^{-2}), \quad (19)$$
$$\tau_{12}^s = \gamma_2 \times J_1 (\mathbf{r}_{12}) + O (r_{12}^{-3}), \quad (20)$$

where we define the linear response operators

$$T_j^P \equiv \partial_{P(\rho_b)} \tau_j [P(\rho_b), J_b] |_{J_b = 0} = \int d^2r \partial_{P(\rho_b)} [P(\rho_b), J_b] |_{J_b = 0} \mathbf{r} \times \nabla V_j, \quad (21)$$

$$\gamma_j \equiv \nabla J_b \times \tau_j [P(\rho_b), J_b] |_{J_b = 0} = -\int d^2r \nabla J_b \rho [P(\rho_b), J_b] |_{J_b = 0} \cdot \mathbf{r} \nabla V_j. \quad (22)$$

In the last equality we have used the triple product vector identity. In the spirit of equations (15) and (19) can also be seen as the response to the density perturbation $\delta \rho_1$; thus

$$\tau_{12}^a = T_2^0 \delta \rho_1 (\mathbf{r}_{12}) + O (r_{12}^{-2}), \quad (23)$$

where we define the linear response operator

$$T_j^P \equiv -\partial_{\rho_b} \tau_j [\rho_b, J_b] |_{J_b = 0} = \int d^2r \partial_{\rho_b} [\rho_b, J_b] |_{J_b = 0} \mathbf{r} \times \nabla V_j. \quad (24)$$

Once more, we find that the interaction torque can be expressed, to leading order, using measurable single-body properties. For a body with the $x$ axis as an axis of symmetry, we have by reflection symmetry $T_j^P = T_j^P \mathbf{e}_x$, $T_j^0 = T_j^0 \mathbf{e}_x$ and $\gamma_j = \gamma_j \mathbf{e}_x$, which reduces the number of components required for measurement to two, instead of four. As was the case for the interaction force, because the local shift of the pressure field $\Delta P_1 (\mathbf{r}_{12})$ (or equivalently, the local shift of the particle density $\delta \rho_1 (\mathbf{r}_{12})$) is a scalar quantity, it contributes only to $\tau_{12}^a$. Thus it can modify only an already-existing self-torque on body 2 about its axis, but not generate a torque by itself. In contrast, due to the non-uniform flow in the vicinity of the body, the local current density $J_1 (\mathbf{r}_{12})$ can exert a torque on body 1 even if it has zero self-torque. Hence, an asymmetric body 1 can cause body 2 to rotate, even if it has no self-torque. As seen in equation (20), this rotation tends to align $\gamma_2$, a body dependent quantity, with $J_1 (\mathbf{r}_{12})$. 

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Finally, we note that all the results summarized in this section reproduce the non-interacting case derived in reference [37]; this can be easily checked using the equation of state for the ‘classical active gas’ \( P(\rho_b) = T_{\text{eff}} \rho_b \), for which \( R_j^\rho = p_j/(T_{\text{eff}} \rho_b) \) and \( R_j^\rho = p_j/\rho_b \). We next proceed to detailed derivations of the above results.

4. Steady-state equations

In order to obtain the above results, we use the steady-state equations for empirical distributions averaged over histories. These can be derived directly from the particle dynamics described in section 2 using standard methods [82]. This was carried out in references [15, 18] for the case of pairwise interacting ABPs (\( \alpha = 0 \)) and in reference [37] for the case of non-interacting particles (\( U = 0 \)). Here we use these references and present the equations in their general form. We also outline the explicit derivation for the case \( \alpha = 0 \) in appendix A.1.

We are interested in the marginal empirical distributions

\[
\hat{m}^{(n)}(r) \equiv \sum_i \delta(r - r_i) e_{n \theta_i} \tag{25}
\]

for integers \( n \geq 0 \), where \( e_{n \theta} = (\cos(n \theta), \sin(n \theta))^T \) is the \( n \)th harmonic unit vector. In particular, we have the identities \( \hat{m}^{(0)} = (\hat{\rho}, 0)^T \) and \( \hat{m}^{(1)} = \hat{\mathbf{m}} \). Taking an average over histories, \( \hat{m}^{(n)}(r, \theta) \equiv \langle \hat{m}^{(n)}(r, \theta) \rangle \), and considering the steady state, where \( \partial_t \hat{m}^{(n)} = 0 \), one obtains for the special case \( n = 0 \) a zero-flux condition,

\[
\nabla \cdot \mathbf{J} = 0, \tag{26}
\]

with the current density given by [15]

\[
\mathbf{J} = -\mu \rho \nabla V + \mu l \cdot \nabla \left[ (\nabla V) \mathbf{m} \right] + \mu \nabla \cdot \mathbf{\sigma}. \tag{27}
\]

Here \( \mathbf{\sigma} \) is the stress tensor given by

\[
\mathbf{\sigma} = -T_{\text{eff}} \rho \mathbb{I} + \mathbf{\sigma}^{IK} + \mathbf{\sigma}^{P}. \tag{28}
\]

In this decomposition, the ideal gas component is \(-T_{\text{eff}} \rho \mathbb{I}\), and the polarization component \( \mathbf{\sigma}^P \) is given by

\[
\mathbf{\sigma}^P(r) = l_r \int d^2 r' \left[ \nabla U(|r - r'|) \right] \langle \hat{\mathbf{m}}(r) \hat{\rho}(r') \rangle + T l_r \nabla \mathbf{m}(r) - 2(T_{\text{eff}} - T) \hat{\mathbf{Q}}(r), \tag{29}
\]

where \( T \equiv D_l/\mu \) denotes the temperature of the ambient thermal bath and

\[
\hat{\mathbf{Q}}(r) \equiv \sum_i \delta(r - r_i) \left( e_{\theta_i} e_{\theta_i} - \frac{1}{2} \mathbb{I} \right) = \frac{1}{2} \begin{pmatrix} \hat{m}_x^{(2)} & \hat{m}_y^{(2)} \\ \hat{m}_y^{(2)} & -\hat{m}_x^{(2)} \end{pmatrix}, \tag{30}
\]
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with \( 1 \) denoting the identity tensor, is the nematic order tensor. The interaction component \( \sigma^{IK} \) satisfies

\[
\nabla \cdot \sigma^{IK}(r) = - \int d^2 r' \langle \hat{\rho}(r) \hat{\rho}(r') \rangle \nabla U(|r - r'|),
\]

(31)

and is given by the standard Irving–Kirkwood formula [83, 84]

\[
\sigma^{IK}(r) = \frac{1}{2} \int d^2 r' \frac{r'r'}{r^2} U'(r') \int_0^1 d\lambda \langle \hat{\rho}(r + (1 - \lambda)r') \hat{\rho}(r - \lambda r') \rangle.
\]

(32)

We note that equations (5) and (6) are obtained from equations (29) and (32) by \( P_D = -\text{Tr}\sigma^{IK}/2 \) and \( P_I = -\text{Tr}\sigma^P/2 \).

In addition, for \( n \geq 1 \), one obtains [18, 37]

\[

m^{(n)}(r) = -\frac{1}{\alpha + n^2 D_t} \nabla \cdot \left\{ -\mu [\nabla V(r)] m^{(n)}(r) - \mu \int d^2 r' [\nabla U(|r - r'|)]
\right.
\]

\[
\langle \hat{m}^{(n)}(r) \hat{\rho}(r') \rangle - D_t \nabla m^{(n)}(r)
\left.
\right\}
\]

\[-\frac{v}{2 (\alpha + n^2 D_t)} \left[ \mathbb{D} m^{(n-1)}(r) - \mathbb{D}^\dagger m^{(n+1)}(r) \right],
\]

(33)

where \( \mathbb{D} \) and \( \mathbb{D}^\dagger \) are the antisymmetric roots of \(-\nabla^2\), defined as

\[
\mathbb{D} \equiv \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix}, \quad \mathbb{D}^\dagger \equiv \begin{pmatrix} -\partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix}.
\]

(34)

For \( n = 1 \), equation (33) implies [15]

\[
v m(r) = -l_t \nabla \cdot \left\{ -\mu [\nabla V(r)] m(r) - \mu \int d^2 r' [\nabla U(|r - r'|)]
\right.
\]

\[
\langle \hat{m}(r) \hat{\rho}(r') \rangle - D_t \nabla m(r) + vQ(r) \left. \right\} - \frac{vl_t}{2} \nabla \rho(r),
\]

(35)

with \( Q \) satisfying \( \nabla \cdot \hat{Q} = -\mathbb{D}^\dagger \hat{m}^{(2)}/2 \).
5. Far-field effects of a single body

We next consider a single passive body immersed in a homogeneous active fluid of density $\rho_b$. The body is described by the potential $V$, which is zero beyond a finite distance $\sim d$. The diameter $d$ and the run-length $l_r$ define two microscopic length scales. In the following, we derive the pressure, density, and current perturbation fields in the far-field limit $r \gg \max(l_r, d)$. We later build on these to derive the interactions between two bodies mediated by the active fluid.

5.1. Pressure field

We first derive equation (3), which describes the far-field behavior of the pressure field. Toward this goal, we examine the standard deviatoric decomposition

$$\sigma = -P \mathbf{1} + \mathbf{S},$$

(36)

where $P \equiv -\operatorname{Tr}\sigma/2$ is the pressure field, and $\mathbf{S} \equiv \sigma - I \operatorname{Tr}\sigma/2$ is the traceless deviatoric stress tensor. We can also represent $\nabla \cdot \sigma$, which is a vector, using the Helmholtz decomposition

$$\nabla \cdot \sigma = -\nabla \Phi + \nabla \times \Psi,$$

(37)

where $\Phi$ and $\Psi$ are scalar and vector potentials, respectively. Similarly, we can write the Helmholtz decomposition of $\nabla \cdot \mathbf{S}$,

$$\nabla \cdot \mathbf{S} = -\nabla \Phi_S + \nabla \times \Psi,$$

(38)

where $\Phi_S$ is the corresponding scalar potential. It is clear that the same vector potential $\Psi$ can be used in both decompositions because $\sigma$ and $\mathbf{S}$ differ only by a scalar multiple of $I$. Indeed, one can easily check that equations (36)–(38) are mutually consistent if the scalar potentials are related by

$$\Phi = P + \Phi_S.$$  

(39)

We can interpret the above relations as follows: (1) the shear stress $\mathbf{S}$ contributes to the scalar potential $\Phi_S$ and the vector potential $\Psi$; (2) the shear stress $\mathbf{S}$ also contributes to the scalar potential $\Phi$ via $\Phi_S$. With this structure in mind, we proceed by first calculating the far-field behavior of $\Phi$ and then showing that the shear-stress component $\Phi_S$ is negligible as it contributes only to higher-order corrections.

Taking the divergence of equation (27) and using the steady-state condition $\nabla \cdot \mathbf{J} = 0$, one gets

$$\partial_\alpha \partial_\beta \sigma_{\alpha\beta} = \nabla \cdot (\rho \nabla V) - l_r \partial_\alpha \partial_\beta (m_\alpha \partial_\beta V).$$

(40)

On the other hand, taking the divergence of equation (37) gives

$$\nabla^2 \Phi = -\partial_\alpha \partial_\beta \sigma_{\alpha\beta}.$$  

(41)

Combining these two equations, we obtain the Poisson equation

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To solve this equation by the method of Green’s functions, one should clarify the boundary conditions at infinity. These are fixed by assuming that the active fluid is homogeneous and disordered at $r \to \infty$. Since there is no preferred direction, $\sigma (\rho_b) = \lim_{r \to \infty} \sigma (r)$ is isotropic—there is no spontaneous shear at $r \to \infty$. Then, by the deviatoric decomposition (36), $\sigma = -P \mathds{1}$ and $S = 0$ for $r \to \infty$, which in turn implies $\Phi_S = 0$ in this limit. Thus the boundary condition at infinity is obtained as $\lim_{r \to \infty} \Phi (r) = P (\rho_b)$. It should be noted that this result relies on the spherical symmetry of the interaction potential $U$; without this symmetry, $\sigma (\rho_b)$ generally depends on the correlations among $\hat{m}^{(n)}$ with $n \geq 1$.

Based on this boundary condition, equation (42) is solved by

\[
\Phi (r) = P (\rho_b) - \frac{1}{2\pi} \int d^2 r' \ln |r - r'| \left\{ \nabla' \cdot [\rho (r') \nabla' V (r')] - l_i \partial_\alpha \partial_\beta \left[ m_\alpha (r') \partial_\beta V (r') \right] \right\},
\]

where $\nabla' = \nabla_r'$. Taking a multipole expansion, we obtain

\[
\Phi (r) = P (\rho_b) + \frac{1}{2\pi} \frac{r \cdot p}{r^2} + \mathcal{O} (r^{-2}),
\]

where the dipole moment $p$ is as defined in equation (7). We stress that the above formula relies on the assumption of a homogeneous and disordered fluid with a symmetric pairwise potential.

To obtain the far-field behavior of the pressure field from equation (44), we need information about the far-field behavior of $\Phi_S$. In general, from equations (28), (29), and (32), $\sigma$ can be expressed as a local function of $\rho$, $m^{(n)}$, $\langle \hat{\rho}^2 \rangle$, and $\langle \hat{m}^{(n)} \hat{\rho} \rangle$ with $n \geq 1$, and their spatial derivatives. However, we expect that the far-field behavior of $\sigma$ would be dominated by the local contribution from $\rho$, so that one can write

\[
\sigma (r) = \sigma (\rho (r)) + \mathcal{O} (\partial \rho),
\]

where $\partial$ stands for a spatial derivative. This can be justified mathematically based on two assumptions: (1) $U$ is short-ranged; (2) $U$ is weak or pair correlations of the empirical densities satisfy mean-field properties—see appendix $A$ for the detailed derivation. More importantly, as we show below, the results derived from equation (45) are consistent with our numerical simulation.

With equation (45), we proceed by taking a Taylor expansion

\[
\sigma (\rho) = -P (\rho_b) \mathds{1} - (\rho - \rho_b) P' (\rho_b) \mathds{1} + \mathcal{O} \left[ (\rho - \rho_b)^2 \right],
\]

which shows that the components of the deviatoric decomposition (36) satisfy

\[
P = P (\rho_b) + (\rho - \rho_b) P' (\rho_b) + \mathcal{O} \left[ (\rho - \rho_b)^2, \partial \rho \right],
\]

\[
S = \mathcal{O} \left[ (\rho - \rho_b)^2, \partial \rho \right],
\]

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in the far field. From equation (47), one observes that the far-field pressure satisfies \( P - P(\rho_b) \sim \rho - \rho_b \) and \( \partial P \sim \partial \rho \). Thus equation (48) can be rewritten as \( S = \mathcal{O}\left(\frac{[P - P(\rho_b)]^2}{\partial P}\right) \), implying

\[
\nabla \cdot S = \mathcal{O}\left(\frac{[P - P(\rho_b)]^2}{\partial P}, r^{-2}\right),
\]

(49)

\( \nabla \cdot S = \mathcal{O}\left(\frac{[P - P(\rho_b)]^2}{\partial P}, r^{-2}\right) \)

Note that \textit{a posteriori} we expect \( P - P(\rho_b) \sim r^{-1} \), meaning that \( S = \mathcal{O}(\partial P) = \mathcal{O}(r^{-2}) \) and \( \nabla \cdot S = \mathcal{O}(\partial^2 P) = \mathcal{O}(r^{-3}) \). In general, \( \Phi_S \) and \( \Psi \) satisfying the Helmholtz decomposition (38) are not local functions of \( \nabla \cdot S \), so the relation between the far-field behaviors of \( \Phi_S \) and \( S \) is not immediately obvious. However, as discussed in appendix B, we can show that the far-field behaviors of both \( \Phi_S \) and \( \Psi \) are of order \( \mathcal{O}(S, r^{-2}) \). In other words, \( \Phi_S = \mathcal{O}\left(\frac{[P - P(\rho_b)]^2}{\partial P}, r^{-2}\right) \), so equation (39) yields a far-field approximation

\[
\Phi = P + \mathcal{O}\left(\frac{[P - P(\rho_b)]^2}{\partial P}, r^{-2}\right),
\]

(50)

\( \Phi = P + \mathcal{O}\left(\frac{[P - P(\rho_b)]^2}{\partial P}, r^{-2}\right) \)

which justifies writing \( \Phi - P(\rho_b) \sim P - P(\rho_b) \) and \( \partial \Phi \sim \partial P \). Then we can invert equation (50) to obtain

\[
P = \Phi + \mathcal{O}\left(\frac{[P - P(\rho_b)]^2}{\partial P}, \frac{\Phi}{r^2}\right).
\]

(51)

\( P = \Phi + \mathcal{O}\left(\frac{[P - P(\rho_b)]^2}{\partial P}, \frac{\Phi}{r^2}\right) \)

Using equation (44) in the above relation, we finally obtain the far-field expression for the pressure field shown in equation (3).

5.2. Density and current fields

To obtain the far-field expressions for \( \rho \) and \( J \), we first note that the particle density and the pressure field are related in the far-field by \( P - P(\rho_b) \sim \rho - \rho_b \) and \( \partial P \sim \partial \rho \). Using these, equation (47) can be inverted as

\[
\rho = \rho_b + \frac{P - P(\rho_b)}{P'(\rho_b)} + \mathcal{O}\left\{\frac{[P - P(\rho_b)]^2}{\partial P}\right\}.
\]

(52)

\( \rho = \rho_b + \frac{P - P(\rho_b)}{P'(\rho_b)} + \mathcal{O}\left\{\frac{[P - P(\rho_b)]^2}{\partial P}\right\} \)

Substituting equation (3) into equation (52), we obtain the multipole expansion for \( \rho \) shown in equation (8).

We now turn to the far-field current density. Substituting equation (3) into equations (45) and (46) and noting that \((\rho - \rho_b)^2 = \mathcal{O}(r^{-2})\), we find

\[
\sigma(r) = \left[ P(\rho_b) - \frac{1}{2\pi} \frac{\mathbf{r} \cdot \mathbf{P}}{r^2} \right] 1 + \mathcal{O}(r^{-2}).
\]

(53)

\( \sigma(r) = \left[ P(\rho_b) - \frac{1}{2\pi} \frac{\mathbf{r} \cdot \mathbf{P}}{r^2} \right] 1 + \mathcal{O}(r^{-2}) \)

Substituting equation (53) into equation (27), and using the fact that outside the body equation (27) becomes \( \mathbf{J} = \mu \nabla \cdot \sigma \), we obtain equation (9). This means that, up to
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Figure 2. The simulated potential (brown colors), representing an asymmetric passive body. Theoretical prediction of a dipolar current density $\mathbf{J}(\mathbf{r})$ according to equation (9) is shown in grey streamlines. Prediction for dipolar density and pressure perturbations according to equations (3) and (8) is shown in red and blue map. The dipole moment $\mathbf{p}$ is drawn schematically in red in the negative $x$-direction.

$O(r^{-3})$, $\mathbf{J}$ is curl-free and behaves like the gradient of a scalar potential $\mu P$, with $\nabla \times \mathbf{J} = O(r^{-4})$.

The above result relies on the assumption made in equation (45) that the stress tensor at the leading order can be expressed as a function of the local density. To verify this, we numerically check the density and current fields predicted by equations (8) and (9) using a molecular dynamics simulation. For the simulation, we consider particles interacting through a short-ranged harmonic repulsion, taking $U(r) = \frac{k}{2}(1 - r)^2$ if $r < 1$ and $U(r) = 0$ otherwise as the interaction potential in equation (1). For the external potential describing the body–particle interaction, we choose an asymmetric repulsive potential, taking $V(r) = a(x)r$ if $r < 1$ and $V = 0$ otherwise. The coefficient $a(x)$ controls the asymmetry of the object (see figure 2). We take $a(x) = 0.9$ if $x > 0$ and $a(x) = 0.1$ if $x < 0$, with the other parameters set to be $v = 1$, $k = 2$, $\alpha = 5$, and $D_r = 0$. Numerical integrations of equations (1) and (2) are carried out using Euler’s method with a discrete time step $dt = 0.01$. To compute the compressibility $c(\rho_b)$ appearing in equation (8), we first independently measure the pressure as a function of density in the absence of the body. Then, after adding the body, the dipole moment $\mathbf{p}$ is measured from equation (7) for different values of the density $\rho_b$ of active particles. Given the symmetry of the problem, the dipole moment should be parallel to the $x$-axis. Finally, we compare the measured density and current fields to the theoretical prediction of equations (8) and (9). The two show an excellent agreement without any fitting parameters. Two examples are shown in figure 3: the density field along the $x$-axis (at $\psi = \arg \mathbf{r} = 0$) and the $y$-component of the current at $\psi = \pi/4$. For $\rho_b = 1$, we display data for a larger system with $L = 120$ to show that the discrepancy at large $r$ for the density field is a finite-size effect.

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It is interesting to note that we see two opposing trends in our numerical example. The dipole moment increases super linearly with the density of active particles, so that the normalized current \( |J/\rho_b| \) increases with density \( \rho_b \) in figure 3. On the contrary, the normalized disturbance in the density field \((\rho - \rho_b)/\rho_b\) decreases with \( \rho_b \) because of the decrease of the compressibility.

Finally, in appendix C, we show that by carefully taking the infinite system limit, one recovers the previously derived current-force relation [18, 37]

\[
\int d^2r \mathbf{J}(\mathbf{r}) = \mu \mathbf{p}.
\]

Moreover, we show that for periodic systems of size \( \sim L \), the correction to the particle density decays with \( L \) as \( \mathcal{O}(L^{-2}) \).

6. Long-range interactions between bodies

We now consider a pair of static bodies fixed in a fluid of density \( \rho_b \) and infinite volume. Interactions of the bodies with active particles are described by two potentials \( V_1 \) and...
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$V_2$, each localized in space. Without loss of generality, body 2 is positioned at the origin, and body 1 is located at $-r_{12}$, so that each experiences the far-field effects of the other.

6.1. Force

We are interested in the force $F_{12}$ applied by body 1, via the fluid, on body 2. More precisely, $F_{12}$ is the additional force exerted on body 2 by the active particles due to the introduction of body 1 into the fluid. To simplify notation, we use a tilde above a quantity to indicate that the value of the quantity has been modified by the presence of multiple bodies. For example, the modified force applied on body $j$ can be denoted as $\tilde{p}_j$, where $\tilde{p}_j \equiv -\int d^2r \tilde{\rho}(r) \nabla V_j(r)$. With these notations, the meaning of the interaction force can be expressed concisely by

$$F_{12} \equiv p_2 - \tilde{p}_2.$$

We obtain $F_{12}$ by taking a far-field expansion of the contribution of body 1 to the pressure field. Since the steady-state conditions derived in section 4 are valid for arbitrary $V$, we can use the two-body potential $V = V_1 + V_2$ in equation (43), which is a direct consequence of the steady-state condition $\nabla \cdot J = 0$. Thus equation (43) can be rewritten in the form of a decomposition

$$\Phi = P(\rho_b) + \Delta \tilde{\Phi}_1 + \Delta \tilde{\Phi}_2,$$

where

$$\Delta \tilde{\Phi}_j(r) \equiv -\frac{1}{2\pi} \int d^2r' \ln |r - r'| \left\{ \nabla' \cdot [\tilde{\rho}(r') \nabla' V_j(r')] - l_\alpha \partial' \partial'_\beta \left[ \tilde{m}_\alpha(r') \partial'_\beta V_j(r') \right] \right\},$$

accounts for the contribution from body $j$. Regarding body 1 as a far-field object, $\Delta \tilde{\Phi}_1$ can be expanded as

$$\Delta \tilde{\Phi}_1 = \frac{1}{2\pi} \frac{(r_{12} + r) \cdot \tilde{p}_1}{r_{12}^2} + \mathcal{O}(r_{12}^{-2})$$

$$= \frac{1}{2\pi} \left[ \frac{r_{12} \cdot \tilde{p}_1}{r_{12}^2} + \frac{r \cdot \tilde{p}_1}{r_{12}^2} - 2(r \cdot r_{12}/r_{12}^2 \cdot r_{12} \cdot \tilde{p}_1) \right] + \mathcal{O}(r_{12}^{-2}, r_{12}^{-3} r^2)$$

$$= \Delta \tilde{P}_b + \frac{\tilde{J}_b \cdot r}{\mu} + \mathcal{O}(r_{12}^{-2}, r_{12}^{-3} r^2),$$

where

$$\Delta \tilde{P}_b \equiv \frac{1}{2\pi} \frac{r_{12} \cdot \tilde{p}_1}{r_{12}^2},$$

$$\tilde{J}_b \equiv -\frac{\mu}{2\pi} \left[ \frac{\tilde{p}_1}{r_{12}^2} - 2(r_{12} \cdot \tilde{p}_1) / r_{12}^2 \right]$$

are the total pressure shift and the current induced by the presence of two bodies in the fluid. Using equation (57) in equation (55), we get

$$\Phi = P(\rho_b) + \Delta \tilde{P}_b + \frac{\tilde{J}_b \cdot r}{\mu} + \Delta \tilde{P}_2 + \mathcal{O}(r_{12}^{-2}, r_{12}^{-3} r^2).$$
\( \Delta \tilde{P}_b \) can be interpreted as a shift in the pressure around body 2, and \( \tilde{J}_b \cdot r / \mu \) as the pressure gradient across body 2 consistent with the current \( \tilde{J}_b \). Next, we note that \( \Delta \tilde{\Phi}_1 \) can be expressed in terms of single-body properties by expanding \( \Phi \left[ P (\rho_b) + \Delta \tilde{P}_b, \tilde{J}_b \right] \) with respect to \( \Delta \tilde{P}_b \) and \( \tilde{J}_b \). The expansion yields

\[
\Phi = \left[ 1 + \Delta \tilde{P}_b \partial_{\rho_b} + \tilde{J}_b \cdot \nabla \tilde{J}_b \right] \Phi \left[ P (\rho_b), J_b \right] |_{\Delta r = 0} + O \left( \Delta \tilde{P}_b^2, \tilde{J}_b^2, \tilde{J}_b \Delta \tilde{P}_b, r_{12}^{-2} \right) = \left[ 1 + \Delta \tilde{P}_b \partial_{\rho_b} + \tilde{J}_b \cdot \nabla \tilde{J}_b \right] \Phi \left[ P (\rho_b), J_b \right] |_{\Delta r = 0} + O \left( r_{12}^{-2} \right),
\]

where we have used \( \Delta \tilde{P}_b = O \left( r_{12}^{-1} \right) \) and \( \tilde{J}_b = O \left( r_{12}^{-1} \right) \). Using the multipole expansion given by equation (44) on both sides of the equation, we find

\[
\tilde{p}_2 = \tilde{p}_2 - R_2^P \Delta \tilde{P}_b - R_2 \tilde{J}_b + O \left( r_{12}^{-2} \right).
\]

This implies \( \tilde{p}_2 = \tilde{p}_2 + O \left( r_{12}^{-1} \right) \), which in turn implies \( \tilde{p}_1 = \tilde{p}_1 + O \left( r_{12}^{-1} \right) \) after exchanging the indices \( 1 \leftrightarrow 2 \). Then, substituting this back into equations (58) and (59), we obtain

\[
\Delta \tilde{P}_b = \frac{1}{2\pi} \frac{r_{12} \cdot p_1}{r_{12}^2} + O \left( r_{12}^{-2} \right) = \Delta P_1 \left( r_{12} \right) + O \left( r_{12}^{-2} \right),
\]

\[
\tilde{J}_b = -\frac{\mu}{2\pi} \left[ \frac{p_1}{r_{12}^2} - \frac{2 \left( r_{12} \cdot p_1 \right) r_{12}}{r_{12}^4} \right] + O \left( r_{12}^{-3} \right).
\]

Here \( \Delta P_1 \left( r_{12} \right) = r_{12} \cdot p_1 / \left( 4\pi r_{12}^2 \right) \), obtained from equations (44) and (51), denotes the change in the local pressure when only body 1 is present in the fluid. Using the definition \( F_{12} = p_2 - p_2 \) in equation (62), we arrive at

\[
F_{12} = R_2^P \Delta P_1 \left( r_{12} \right) - R_2 \frac{\mu}{2\pi} \left[ \frac{p_1}{r_{12}^2} - \frac{2 \left( r_{12} \cdot p_1 \right) r_{12}}{r_{12}^4} \right] + O \left( r_{12}^{-2} \right).
\]

The force \( F_{12} \) can now be decomposed according to equation (10), in which \( F_{12}^a \) acts solely on asymmetric bodies \( p_2 \neq 0 \) and \( F_{12}^s \) acts even on fully symmetric bodies \( p_2 = 0 \). We find these to be given by

\[
F_{12}^a = R_2^P \Delta P_1 \left( r_{12} \right) + O \left( r_{12}^{-2} \right)
\]

\[
F_{12}^s = -R_2 \frac{\mu}{2\pi} \left[ \frac{p_1}{r_{12}^2} - \frac{2 \left( r_{12} \cdot p_1 \right) r_{12}}{r_{12}^4} \right] + O \left( r_{12}^{-3} \right).
\]

Using the single-body result equation (9), the second equality can also be written as \( \Phi = R_2 \tilde{J}_1 \left( r_{12} \right) + O \left( r_{12}^{-3} \right) \), where \( \tilde{J}_1 \) denotes the current field induced by body 1 alone. Thus we have finally derived equations (11) and (12).

As noted before, equation (66) can be rewritten in terms of a linear response to the density modulation. Under the assumption that the fluid has only a single homogeneous phase, \( P (\rho_b) \) is bound to be a strictly monotonically increasing function of \( \rho_b \). Thus \( P (\rho_b) \) is invertible, allowing us to rewrite equation (13) as

\[
https://doi.org/10.1088/1742-5468/ab7f34
\]
\[
R_j^p = \frac{1}{P'(\rho_b)} \int d^2 r \partial_{\rho_b} \rho [\rho_b, J_b]|_{J_b=0} \nabla V_j.
\]

Combining this with equations (8) and (16) yields equation (15), which is the density version of equation (66).

### 6.2. Torque

To obtain the interaction torques mediated by the active particles, we need to derive an expression for the density shift near one body, say body 2, induced by the presence of the other body, say body 1. Substituting equation (55) in equation (51), we get

\[
P = P(\rho_b) + \Delta \tilde{\Phi}_1 + O(\Delta \tilde{\Phi}_2, \partial \Delta \tilde{\Phi}_1, \Delta \tilde{\Phi}_2).
\]

Inserting this into the inverted expansion equation (52), we obtain

\[
\tilde{\rho} = \rho_b + \frac{\Delta \tilde{\Phi}_1}{P'(\rho_b)} + O(\Delta \tilde{\Phi}_1, \partial \Delta \tilde{\Phi}_1, \Delta \tilde{\Phi}_2),
\]

where a tilde above \( \rho \) indicates that this is a solution of the two-body problem. Meanwhile, using equations (63) and (64) in equation (57) gives

\[
\Delta \tilde{\Phi}_1 = \Delta P_1(r_{12}) + \frac{J_1(r_{12}) \cdot r}{\mu} + O(r_{12}^{-2}, r_{12}^{-3}, r_{12}^{-2}).
\]

Using this relation in equation (70), we can write

\[
\tilde{\rho} = \rho_b + \frac{\Delta P_1(r_{12})}{P'(\rho_b)} + \frac{J_1(r_{12}) \cdot r}{\mu P'(\rho_b)} + O(r_{12}^{-2}, r_{12}^{-3}, r_{12}^{-2}),
\]

where the scaling of the higher-order corrections can be justified by the multipole expansion of \( \Delta \tilde{\Phi}_1 \) shown in equation (57) and the corresponding expansion of \( \Delta \tilde{\Phi}_2 \) that can be obtained by exchanging the indices 1 and 2. Using the single-body result equation (8) for body 1, we can also write

\[
\tilde{\rho} = \rho_b + \delta \rho_1(r_{12}) + \frac{J_1(r_{12}) \cdot r}{\mu P'(\rho_b)} + O(r_{12}^{-2}, r_{12}^{-3}, r_{12}^{-2}).
\]

Therefore, to leading order in \( r_{12} \), \( \tilde{\rho} \) has modified boundary conditions associated with a local density shift \( \delta \rho_1(r_{12}) \) and a local current \( J_1(r_{12}) \). An expansion with respect to these changes gives

\[
\tilde{\rho} = [1 + \Delta P_1(r_{12}) \partial_{\rho_b} \rho [P(\rho_b), J_0]|_{J_b=0} + O(r_{12}^{-2})
\]

\[
= [1 + \delta \rho_1(r_{12}) \partial_{\rho_b} \rho [\rho_b, J_0]|_{J_b=0} + O(r_{12}^{-2})],
\]

where we have used equation (72) to obtain the first equality and equation (73) to derive the second.
We can now use equation (74) or (75) to find the interaction torque $\tau_{12}$ applied by body 1 on body 2. The self-torque experienced by body $j$ in the two-body problem is

$$\tilde{\tau}_j = \int d^2r \tilde{\rho}(r) (r - X_j) \times \nabla V_j(r).$$

(76)

Using equations (74) and (75) in the above equation, we then obtain

$$\tau_{12} \equiv \tau_2 - \tilde{\tau}_2 = T_{2}^P \Delta P_1 (r_{12}) + \gamma_2 \times J_1 (r_{12}) + O(r_{12}^{-2})$$

(77)

$$\tau_{12} = T_{2}^p \delta \rho_1 (r_{12}) + \gamma_2 \times J_1 (r_{12}) + O(r_{12}^{-2}),$$

(78)

where $T_{2}^P$, $\gamma_2$, and $T_{2}^p$ are as defined in equations (21), (22), and (24), respectively. As was the case for $F_{12}$, $\tau_{12}$ can also be decomposed into two components shown in equations (18), (19), and (20), so that $\tau_{12}^a$ acts solely on bodies with a nonzero self-torque ($\tau_2 \neq 0$), whereas $\tau_{12}^s$ acts even on bodies with no self-torque ($\tau_2 = 0$).

7. Conclusions

In this paper, we have studied the long-range effects of passive bodies immersed in a fluid of mutually interacting active particles. We have shown that, to leading order in an asymptotic far-field expansion, an asymmetric body generates dipolar density and pressure gradients as well as currents, all of which decay as a power law with increasing distance. These fields mediate generic long-range interactions between the passive bodies, which also decay algebraically with distance and do not obey an action–reaction principle. Remarkably, the leading-order behaviors of these interactions can be predicted by numerically or empirically measuring a few single-body properties in separate experiments. Our results provide a natural extension of the previous results obtained for ideal active fluids [37]. While the interparticle interactions do not alter the symmetry and scaling exponents of the leading-order behaviors, they do modify the amplitudes of the long-range effects via non ideal behaviors of pressure. We recall that the interactions mediated by ideal active fluids induce interesting dynamical effects [37] with possible applications to the flocking of shaken granular media [85] and the control of self-assembly by tuning the body shapes [34–36, 78, 86, 87]. Our results clarify how such effects can be enhanced or inhibited by choosing the interparticle interactions of the active fluid. It will be very interesting to observe long-range currents and density modulations in experiment, an effort which could lead toward the useful applications described above.

Notably, our derivations of the leading-order long-range interactions rely solely on the assumption that the active fluid is deep inside the disordered phase, is far from the critical point (if any), and has a stress expansion shown in equation (45). Any overdamped system capable of demonstrating ratchet-like effects satisfying these assumptions exhibits the same phenomena, irrespective of the details of its constitutive relations. This is the case even if the interparticle interaction is dependent on the positions of arbitrarily many particles (i.e., it is not a pairwise interaction) as long as it has a short range.

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This study can still be extended in various directions. For example, it should be noted that the derivation does not work for interactions involving internal degrees of freedom, such as quorum sensing [15, 88–95], orientational alignment [96–101] and nematic alignment [3, 102–105]. It will be interesting to check if active fluids with such interparticle interactions, especially those with symmetry-breaking transitions producing orientational order, can mediate novel kinds of long-range forces and torques. Further, one can examine the consequences of introducing the bodies into a critical or super-critical fluid undergoing motility-induced phase separation (MIPS), a subject of various recent theoretical advancement [12, 14–16]. One can also consider long-range interparticle interactions, such as hydrodynamic interactions, instead of the short-range interactions considered here. This can be applicable to active particles suspended in a momentum-conserving, wet, passive bath [3, 106–113]. We also note that, unlike the leading-order components of the interactions, higher-order terms may exhibit features which are qualitatively different from the non-interacting case. In fact, previous numerical and experimental studies show that near-field interactions can be attractive, both inherently and due to depletion forces [3, 43–47, 114]. That said, considering such higher-order far-field effects and near-field effects should unveil even richer physics of bodies immersed in active fluids.

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Appendix A. Weak-interaction expansions

Here we present two explicit mathematical justifications of equation (45). The first one is obtained by taking the limit of weak pairwise interactions between active particles. This is a standard procedure, well studied in the context of equilibrium systems. We remind the reader that the important justification is that the verification of the results through the numerics, which extends outside of the weak-interactions regime. Revisiting equation (33) and using integration by parts

\[
- \int d^2 r' [\nabla U (|r - r'|)] \langle \hat{m}^{(n)}(r) \hat{\rho}(r') \rangle = - \int d^2 r' U (|r - r'|) \nabla' \langle \hat{m}^{(n)}(r) \hat{\rho}(r') \rangle.
\]

(79)

For short-ranged \( U \) we can interpret equation (33) as a recurrence relation [18, 37]

\[
m^{(n)} = m^{(n)} (\partial m^{(n-1)}(r), \partial m^{(n+1)}(r), \partial^2 m^{(n)}(r), \partial^2 \langle \hat{m}^{(n)} \hat{\rho} \rangle, \partial^3 \langle \hat{m}^{(n)} \hat{\rho} \rangle, \ldots)
\]

(80)
for \( n \geq 1 \). We note that the equation for \( n = 0 \) is set by equation (26). By unfolding equation (80), one can also write for \( n \geq 1 \)

\[
\mathbf{m}^{(n)} = \mathbf{m}^{(n)} \left( \partial^n \rho, \partial^{n+1} \rho, \ldots; \text{pair correlations and their derivatives} \right). \tag{81}
\]

Using this relation in equations (28), (29), and (32), the dependence of \( \sigma \) on the field variables can be written as

\[
\sigma = \sigma \left( \rho, \partial \rho, \partial^2 \rho, \ldots; \text{pair correlations and their derivatives} \right). \tag{82}
\]

In the far-field, where one expects the deviations from the homogeneous density \( \rho_b \) to be small, a standard dimensional analysis yields

\[
\langle \hat{m}^{(n)}(r) \hat{m}^{(n)}(r') \rangle = \mathcal{O} \left( \min \left( r^{-3}, r'^{-3} \right), U_0 \rho_b^3 \right), \tag{83}
\]

for any nonnegative integer \( n \). We achieve this by deriving the dynamics of the two-point correlations \( \langle \hat{m}^{(n)}(r) \hat{m}^{(n)}(r') \rangle \) and neglecting terms which are \( \mathcal{O} \left( \left( U_0 \rho_b / T_{\text{eff}} \right)^2 \right) \). Extension to higher orders can be done similarly by constructing a dynamical BBGKY hierarchy of correlations. Using our result equation (83), we show that two-point correlations yield only sub leading contributions to the stress tensor. Namely, we show that equation (82) reduces to

\[
\frac{1}{T_{\text{eff}} \rho_b} \sigma = \frac{1}{T_{\text{eff}} \rho_b} \sigma \left( \rho \right) + \mathcal{O} \left( \left( U_0 \rho_b / T_{\text{eff}} \right)^2 \right), \tag{84}
\]

which reproduces equation (45). Here, \( \sigma(\rho) \) denotes \( \sigma(\rho, 0, 0, \ldots) \). In A.2, we show that the first-order contribution of \( U_0 \rho_b / T_{\text{eff}} \), already contained within \( \sigma(\rho) \), changes the pressure according to

\[
\frac{P(\rho_b)}{T_{\text{eff}} \rho_b} = 1 + \frac{U_0 \rho_b}{2T_{\text{eff}}} + \mathcal{O} \left( \left( \frac{U_0 \rho_b}{T_{\text{eff}}} \right)^2 \right), \tag{85}
\]

which has the form of a standard virial expansion of a van der Waals gas at temperature \( T_{\text{eff}} \).

Instead of the weak interactions limit, we can also use a mean-field approximation to write \( \langle \hat{m}^{(n)}(r) \hat{m}^{(n)}(r') \rangle \approx m^{(n)}(r) \rho(r') \). In this case the stress tensor in equation (82) does not depend on pair correlations and can be expressed as

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\[ \sigma \approx \sigma (\rho, \partial \rho, \partial^2 \rho, \ldots) , \]  

reproducing equation (45) once more. This also yields the van der Waals equation (85), except that the mean-field approach does not require a small dimensionless parameter.

A.1. Weak-interaction expansion of the stress tensor

To derive equation (83), we need to examine the dynamics of \( \langle \hat{\rho}(r) \hat{\rho}(r') \rangle \) and impose the steady-state constraint. For simplicity, we consider the case of ABPs \( (\alpha = 0) \), so that one can make use of standard Itô calculus of continuous processes, as previously demonstrated for passive particles [82]. Our key result is the dipolar decay of \( \langle \hat{\rho}(r) \hat{m}^{(n)}(r') \rangle \), with increasing distances from the origin in the four-dimensional space \( (r, r') \), which is much faster than the corresponding decay in two dimensions. We note that similar results were also obtained in other diffusive systems [38, 40].

As a first step, we examine the time evolution of the empirical distribution of particles at position \( r \) and orientation \( \theta \), \( \hat{\psi}(r, \theta) \equiv \sum_i \delta(r - r_i) \delta(\theta - \theta_i) \). Through a standard procedure based on Itô calculus, as explicitly formulated by Dean [82] (see also [13, 18]), the time evolution of \( \hat{\psi} \) is derived from equations (1) and (2) as

\[ \partial_t \hat{\psi} = -\nabla \cdot \left[ v \psi_e - \mu \nabla V - \mu \int d^2r' \int d\theta' \hat{\psi}(r', \theta') \nabla U(|r - r'|) - D_r \nabla \right] \hat{\psi}(r, \theta) \]
\[ + \nabla \cdot \sqrt{2D_r} \hat{\eta} + \partial_{\psi} \left( D_r \partial_{\psi} \hat{\psi} + \sqrt{2D_r} \hat{\xi} \right), \]  

(87)

where \( \hat{\eta} \) and \( \hat{\xi} \) are Gaussian white noise fields with unit amplitude. It should be noted that, if one strictly carries out the derivation, \( \hat{\psi}(r, \theta) \hat{\psi}(r', \theta') \) in the above expression should be replaced with \( \hat{\psi}(r, \theta) \hat{\psi}(r', \theta') - \psi(r, \theta) \delta(r - r') / 2\pi \). The extra term reflects the fact that a particle cannot exert a force on itself. For simplicity, we eliminate this correction by assuming \( \nabla U(0) = 0 \), which is naturally true for a smooth, spherically symmetric interaction potential.

The empirical distribution \( \hat{\psi} \) can be decomposed into the Fourier components

\[ \int d\theta \hat{\psi}(r, \theta) e_{n} = \hat{m}^{(n)}(r) , \]  

(88)

with \( m^{(n)} \) being the marginal empirical distributions defined in equation (25). In particular, for \( n = 0 \) and \( 1 \), these are related to the empirical density and empirical polarization density by

\[ \hat{\rho}(r) = \int d\theta \hat{\psi}(r, \theta) , \quad \hat{m}(r) = \int d\theta \hat{\psi}(r, \theta) e_0 , \]  

(89)

which correspond to the empirical density and polarization fields, respectively. Multiplying equation (87) side by side with \( e_{n} \) and integrating over \( \theta \), one obtains the equations governing the time evolution of \( \hat{m}^{(n)} \).
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For \( n = 0 \), we obtain a noisy continuity equation

\[
\partial_t \hat{\rho} + \nabla \cdot \hat{\mathbf{J}} = 0,
\]

(90)

where the fluctuating current field is given by

\[
\hat{\mathbf{J}}(\mathbf{r}) \equiv v \hat{\mathbf{m}}(\mathbf{r}) - \mu \hat{\rho}(\mathbf{r}) \nabla \left[ V(\mathbf{r}) + \int d^2 \mathbf{r}' \hat{\rho}(\mathbf{r}') U(|\mathbf{r} - \mathbf{r}'|) \right] - D_t \nabla \hat{\rho}(\mathbf{r}) + \sqrt{2D_t} \hat{\rho}(\mathbf{r}) \chi(\mathbf{r}, t),
\]

(91)

with the Gaussian white noise field \( \sqrt{\hat{\rho}(\mathbf{r})} \chi(\mathbf{r}, t) \equiv \int d\theta \sqrt{\hat{\psi}(\mathbf{r}, \theta)} \hat{\eta}(\mathbf{r}, t) \) satisfying

\[
\left\langle \nabla \cdot \sqrt{\hat{\rho}(\mathbf{r})} \chi(\mathbf{r}, t) \nabla' \cdot \sqrt{\hat{\rho}(\mathbf{r}')} \chi(\mathbf{r}', t') \right\rangle = \frac{1}{2} I \left[ -\nabla^2_{(r,r')} \delta(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}) + \delta(\mathbf{r} - \mathbf{r}') \nabla^2 \rho(\mathbf{r}) \right] \delta(t - t').
\]

(92)

From here on, we define and use a four-dimensional differential operator \( \nabla_{(r,r')} \equiv \nabla \oplus \nabla' = (\nabla, \nabla')^T \), which also implies \( \nabla^2_{(r,r')} = \nabla^2 + \nabla'^2 \).

Similarly, for \( n = 1 \), we obtain

\[
l_t \partial_t \hat{\mathbf{m}} = -v \hat{\mathbf{m}} + \mu l_t \nabla \cdot [(\nabla V) \hat{\mathbf{m}}] + \mu \nabla \cdot \hat{\sigma}^p - \frac{v l_t}{2} \nabla \hat{\rho} + l_t \sqrt{2D_t} \hat{\xi}^{(1)},
\]

(93)

where \( \hat{\xi}^{(1)}(\mathbf{r}, t) \equiv \int d\theta \sqrt{\hat{\psi}(\mathbf{r}, \theta)} \xi(\mathbf{r}, \theta, t) e_\theta^+ \) with \( e_\theta^+ \equiv e_z \times e_\theta = (-\sin \theta, \cos \theta, 0)^T \), and we have defined the polarization component of the stress tensor (the noisy counterpart of equation (29))

\[
\hat{\sigma}^p(\mathbf{r}) \equiv l_t \int d^2 \mathbf{r}' [\nabla U(|\mathbf{r} - \mathbf{r}'|)] \hat{\mathbf{m}}(\mathbf{r}') \hat{\rho}(\mathbf{r}') + TL_t \nabla \hat{\mathbf{m}}(\mathbf{r}) - 2(T_{dd} - T) \hat{\mathbf{Q}}(\mathbf{r}) - \frac{l_t}{\mu} \sqrt{2D_t} \hat{\chi}^{(1)}(\mathbf{r}, t),
\]

(94)

with \( \hat{\chi}^{(1)}(\mathbf{r}, t) \equiv \int d\theta \sqrt{\hat{\psi}(\mathbf{r}, \theta)} \hat{\eta}(\mathbf{r}, \theta, t) e_\theta \). Finally, the nematic order tensor \( \hat{\mathbf{Q}} \) is again defined by equation (30), which can also be written in terms of \( \hat{\psi} \) as

\[
\hat{\mathbf{Q}}(\mathbf{r}) = \int d\theta \left( e_\theta e_\theta - \frac{1}{2} I \right) \hat{\psi}(\mathbf{r}, \theta).
\]

(95)

Note that by taking the average of equation (93) at steady-state, at which \( \partial_t(\hat{\mathbf{m}}) = \partial_t \hat{\mathbf{m}} = 0 \), and combining with equation (91), we recover equations (27) and (28) of section 4. Furthermore, one can reach equations (33) and (35) by using equation (88) for arbitrary \( n \) and noting that \( \nabla \cdot \hat{\mathbf{Q}} = -d\hat{\mathbf{m}}^2/(2D_t) \).

Given these results for the single-point observables, we now move on to the time evolution of two-point observables. Applying equation (92), Itô’s product rule for the time derivative of \( \langle \hat{\rho}(\mathbf{r}) \hat{\rho}(\mathbf{r}') \rangle \) reads

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\[
\partial_t \langle \dot{r}(r) \dot{r}'(r') \rangle = \langle \dot{r}(r) \partial_t \dot{r}(r) \rangle + \langle \dot{r}(r) \partial_t \dot{r}'(r') \rangle + D_t \left[ -\nabla^2_{(r,r')} \delta(r - r') \rho(r) + \delta(r - r') \nabla^2 \rho(r) \right].
\]  

(96)

Using equations (90) and (91) to calculate the first two terms on the rhs and evaluating the averages over histories, we obtain the steady-state condition

\[
D_t \delta(r - r') \nabla^2 \rho(r) = \nabla_{(r,r')} \cdot J^{(2)} (r, r'),
\]

(97)

where the four-dimensional current density \( J^{(2)} \) is given by

\[
J^{(2)} (r, r') \equiv v \left[ \langle \dot{r}(r) \dot{m}(r) \rangle \right] - \langle \hat{r}(r) \dot{r} \rangle \nabla_{(r,r')} \left[ V(r) + V(r') \right]
- \mu \int d^2r'' \langle \hat{r}(r) \dot{r}(r') \dot{r}(r'') \rangle \nabla_{(r,r')} \left[ U(|r - r''|) + U(|r' - r''|) \right]
- \delta(r - r') \rho(r),
\]

(98)

which depends on \( \langle \dot{r}(r) \dot{m}(r') \rangle \). Note that the current density \( J^{(2)} (r, r') \) associated with the two-point correlation \( \langle \dot{r}(r) \dot{r}'(r') \rangle \) has an asymmetric source, determined by \( \nabla^2 \rho \) (see figure 4). To obtain the steady-state expression for \( \langle \dot{r}(r) \dot{m}(r') \rangle \), we first need to examine its time evolution. Using the cross correlation

\[
\langle \nabla \cdot \sqrt{\rho(r)} \dot{\chi}(r, t) \nabla' \cdot \dot{\chi}^{(1)} (r', t') \rangle = \frac{1}{2} \left[ -\nabla^2_{(r,r')} \delta(r - r') \dot{m}(r) + \delta(r - r') \nabla^2 \dot{m}(r) \right] \delta(t - t'),
\]

(99)

Itô’s product rule yields

\[
\partial_t \langle \dot{r}(r) \dot{m}(r') \rangle = \langle \dot{r}(r) \partial_t \dot{m}(r') \rangle + \langle \dot{m}(r') \partial_t \dot{r}(r) \rangle
+ D_t \left[ -\nabla^2_{(r,r')} \delta(r - r') \dot{m}(r) + \delta(r - r') \nabla^2 \dot{m}(r) \right].
\]

(100)

In the steady state, using equation (90) to eliminate \( \partial_t \dot{r}(r) \) on the rhs, we obtain

\[
\langle \dot{r}(r) \partial_t \dot{m}(r') \rangle = \nabla \cdot \langle \dot{J}(r) \dot{m}(r') \rangle - D_t \left[ -\nabla^2_{(r,r')} \delta(r - r') \dot{m}(r) + \delta(r - r') \nabla^2 \dot{m}(r) \right].
\]

(101)

Meanwhile, solving equation (93) for \( v \dot{m} \) and using the result in equation (98) to rewrite the first term on its rhs, we get

\[
J^{(2)} (r, r') = -i \left[ \langle \dot{r}(r') \partial_t \dot{m}(r) \rangle \right]
+ \mu D_t \left[ \nabla \cdot \nabla V(r) \right] \langle \dot{r}(r) \dot{m}(r) \rangle
+ \mu \left[ \nabla \cdot \nabla \langle \dot{r}(r') \nabla P(r') \rangle \right] - \mu \langle \hat{r}(r) \dot{r}(r') \rangle \nabla_{(r,r')} \left[ V(r) + V(r') \right]
- \mu \int d^2r'' \langle \hat{r}(r) \dot{r}(r') \dot{r}(r'') \rangle \nabla_{(r,r')} \left[ U(|r - r''|) + U(|r' - r''|) \right]
- \nabla_{(r,r')} \left[ D_{eff} \langle \hat{r}(r) \dot{r}(r') \rangle - D_t \delta(r - r') \rho(r) \right],
\]

(102)

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where $D_{\text{eff}} \equiv D_t + \nu l / 2 = \mu T_{\text{eff}}$ is the effective diffusion constant of active particles. Using equation (101) to eliminate both $\langle \dot{\rho}(r') \partial_t \dot{m}(r) \rangle$ and $\langle \dot{\rho}(r) \partial_t \dot{m}(r') \rangle$, we find

$$
\hat{J}^{(2)} (r, r') = -\mu (\dot{\rho}(r) \dot{\rho}(r')) \nabla_{(r, r')} [V (r) + V (r')] + \mu l r \nabla_{(r, r')} \{ [\nabla V (r)] \langle \dot{\rho}(r') \dot{m}(r) \rangle + [\nabla V (r')] \langle \dot{\rho}(r) \dot{m}(r') \rangle \}
$$

$$
+ l_2 D_t \delta(r - r') \nabla_{(r, r')}^2 [m(r') \oplus m(r)] + \mu \nabla_{(r, r')} \cdot \sigma^{(2)} (r, r'),
$$

(103)

where we introduce the four-dimensional stress tensor

$$
\sigma^{(2)} (r, r') \equiv - \left[ T_{\text{eff}} \langle \dot{\rho}(r) \dot{\rho}(r') \rangle - T \delta(r - r') \rho(r) \right] \mathbb{1} + \sigma^{P(2)} (r, r') + \sigma^{IK(2)} (r, r').
$$

(104)

Here, the polarization tensor $\sigma^{P(2)}$ is given by

$$
\sigma^{P(2)} (r, r') \equiv - l_2 T \nabla_{(r, r')} \delta (r - r') [m(r') \oplus m(r)] - \frac{l}{\mu} \gamma_0 \cdot \left[ \langle \dot{J}(r) \dot{m}(r') \rangle \oplus \langle \dot{J}(r') \dot{m}(r) \rangle \right]
$$

$$
+ \langle \dot{\rho}(r') \dot{\sigma}^P (r') \rangle \oplus \langle \dot{\rho}(r) \dot{\sigma}^P (r') \rangle,
$$

(105)

where we account for the transposed ordering of the direct sum by inserting the tensor product of the exchange tensor (first Pauli matrix) $\varsigma_x$ and the two-dimensional identity tensor $\mathbb{1}_2$,

$$
\gamma_0 = \varsigma_x \mathbb{1}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
$$

(106)

which is also the zeroth Dirac matrix in the chiral basis. The interaction tensor $\sigma^{IK(2)}$ satisfies

$$
\nabla_{(r, r')} \cdot \sigma^{IK(2)} (r, r') \equiv - \int d^2 r'' \langle \dot{\rho}(r) \dot{\rho}(r') \dot{\rho}(r'') \rangle \nabla_{(r, r')} [U (|r - r''|) + U (|r' - r''|)].
$$

(107)

Up to this point, all of our results are exact. From here on, to implement the weak-interaction assumption, we neglect all terms of order $U_0$, which means that all interaction force integrals, including $\sigma^{IK(2)}$, are neglected. Noting that the steady-state condition $\nabla \cdot \hat{J} (r) = 0$ implies $\nabla_{(r, r')} \cdot [\rho(r') J (r) \oplus \rho(r) J (r')] = 0$, we can rewrite equation (97) as

$$
D_t \delta (r - r') \nabla^2 \rho(r) = \nabla_{(r, r')} \cdot \hat{J}^{(2)} (r, r'),
$$

(108)

where $\hat{J}^{(2)} (r, r') \equiv J^{(2)} (r, r') - [\rho(r') J (r) \oplus \rho(r) J (r')]$. Because equation (103) is sepa-
where we introduce the stress charge densities (see figure 4) and (108), we obtain the Poisson equation rhs of equation (105). Since at this order there is no long-range stress, we can obtain translations with second-order cumulants, with the exception of the second term on the right-hand side of equation (104). Taking the divergence of equation (109) and using equations (104) and (108), we obtain the Poisson equation

\[
\nabla^2_{(r,r')} [D_{\text{eff}} \langle \dot{\rho}(r) \dot{\rho}(r') \rangle_c - D_t \rho(r) \delta(r-r')] = \mu [\mathcal{P}(r,r') + \mathcal{I}(r,r')] + \mathcal{O}(U_0 \rho_s^2),
\]

where we introduce the stress charge densities (see figure 4)

\[
\mathcal{P}(r,r') \equiv -\nabla_{(r,r')} \cdot \{ \langle \dot{\rho}(r) \dot{\rho}(r') \rangle_c \nabla_{(r,r')} [V(r) + V(r')] \}
\]

\[
+ l_v \nabla_{(r,r')} \cdot \{ \langle \dot{\rho}(r') \dot{\mathbf{m}}(r') \rangle_c \nabla_{(r,r')} [V(r) + V(r')] \} + \mathcal{O}(U_0 \rho_s^2),
\]

\[
\mathcal{I}(r,r') \equiv -T \delta(r-r') \nabla^2 \rho(r) - l_v \nabla_{(r,r')} \cdot \delta(r-r') \left[ \nabla^2 \mathbf{m}(r') \oplus \nabla^2 \mathbf{m}(r) \right].
\]

Figure 4. Schematic representation of the leading-order charge distribution in equations (110)–(112) (red and blue colors). Left: the charge density \( \mathcal{I}(r,r') \). A pictorial description of the four-dimensional current \( J^{(2)}(r,r') \) is shown in grey arrows (see equation (97)). The charges are concentrated on the plane \( r = r' \). Within the first order in the expansion, equation (114) holds, implying that the distribution is asymmetric and of length \( \sim d \) (see text). Right: the charge density \( \mathcal{P}(r,r') \). Each dipole sheet is of thickness \( \sim d \).
Due to the separable nature equation (110), the charges are concentrated in three sheets (see figure 4). We will treat each of the two types of charge distribution separately, and show that the resulting solution decays as dipole in four-dimensions.

First, we claim that the charge density due to Itô terms, \( \mathcal{I}(\mathbf{r}, \mathbf{r}') \), is localized in space and provides leading-order dipolar contributions. At this order in the weak-interaction expansion, \( \mathbf{m} \) and \( \rho \) in equations (108) and (109) are the solutions of the corresponding non-interacting problem. In the non-interacting problem, the angular hierarchy equation (81) becomes

\[
\mathbf{m}^{(n)} = \mathbf{m}^{(n)}(\partial^n \rho, \partial^{n+1} \rho, \ldots).
\]

This allows one to represent \( \nabla^2 \rho \) and \( \nabla^2 \mathbf{m} \) as a sum of terms proportional to \( V \) and its derivatives. This procedure is the key step in writing the previously obtained solution to the non-interacting problem to arbitrary high order [37]. Specifically, for \( \nabla^2 \rho \) we have

\[
D_{\text{eff}} \nabla^2 \rho = -\mu \nabla \cdot [\rho(\mathbf{r}) \nabla V(\mathbf{r})] + \mathcal{O}(\partial^2),
\]

where \( \mathcal{O}(\partial^2) \) indicates terms which are at least of second differential order. Thus, \( \nabla^2 \rho \) and \( \nabla^2 \mathbf{m} \) are short-ranged with characteristic length \( d \), as they vanish quickly outside of the body. Moreover, the leading-order contribution from these is dipolar, as seen in equation (114). This shows that \( \mathcal{I}(\mathbf{r}, \mathbf{r}') \) is indeed a localized density of leading-order dipolar contribution.

Next, we note that the separable density \( \mathcal{P}(\mathbf{r}, \mathbf{r}') \) is concentrated within two perpendicular charge sheets of thickness \( d \). To leading-order, each sheet is a dipole sheet. By means of numerical solution and a self-consistent argument, it was previously shown that a Poisson equation with infinite sheets of multipole densities proportional to the potential, as in the above, yields a solution whose asymptotic behavior is that of a localized multipole of the same order [40]. We note that one can also verify this result using a weak-forcing expansion, where the small dimensionless parameter is \( \rho_b V_0/T_{\text{eff}} \), \( V_0 \equiv \int d^2 r V(\mathbf{r}) \). At order \( V_0^0 \), all charge densities are neglected and we obtain \( \langle \hat{\rho}(\mathbf{r}) \hat{\rho}(\mathbf{r}') \rangle_0 = \rho(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') T/T_{\text{eff}} \). At order \( V_0^1 \), \( \mathcal{P}(\mathbf{r}, \mathbf{r}') \) is obtained from the solution to the zero-order expansion, which amounts to an ideal dipole at the origin. Likewise, \( \mathcal{I}(\mathbf{r}, \mathbf{r}') \) now includes the localized dipolar contribution shown in equation (114). The resulting asymptotic decay is that of a four-dimensional dipole, namely \( \sim \left( r^2 + r'^2 \right)^{-3/2} \sim \min(\rho^{r-3}, r'^{r-3}) \). At order \( V_0^2 \), the charge density is obtained from the solution to the first order expansion, giving a charge density that decays as \( \sim r^{-3} \) and \( \sim r'^{-3} \) respectively along each sheet. Then, we invoke the argument given in reference [40], saying that a multipole density that decays faster than \( r^{-2} \) induces a potential whose asymptotic behavior is that of a localized multipole\(^6\). By induction, the dipolar decay holds up to arbitrary order in the perturbative expansion. We conclude that \( \mathcal{P}(\mathbf{r}, \mathbf{r}') \) acts as an effectively localized dipole.

In total, the far-field behavior of the solution to equation (109) is given by

\[
\langle \hat{\rho}(\mathbf{r}) \hat{\rho}(\mathbf{r}') \rangle_c = \frac{T}{T_{\text{eff}}} \rho(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') + \mathcal{O}\left( \min\left( r^{-3}, r'^{-3} \right), U_0 \rho_b^3 \right).
\]

\(^6\)In reference [40]: appendix C the argument was given for a quadrupole density. However, the proof can be generalized to any multipole density in a direct way.

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Because the correlator \( \langle \hat{\rho}(r) \hat{\rho}(r') \rangle \) appears only within the interaction force density \( \langle \hat{\rho}(r) \hat{\rho}(r') \rangle \nabla U (r - r') \), and due to our assumption that \( \nabla U (0) = 0 \), we can omit the first term in equation (115). One can skip this simplifying assumption if the above derivation is done for pair densities, e.g. \( \langle \hat{\rho}(r) \hat{\rho}(r') \rangle - \hat{\rho}(r) \delta (r - r') \), instead of correlations. We conclude that \( \langle \hat{\rho}(r) \hat{\rho}(r') \rangle = \rho(r) \rho(r') + \mathcal{O} \left( \min (r^{-3}, r'^{-3}), U_0 \rho_b^3 \right) \).

Similarly, one can derive hierarchical relations for two-point correlations, as done for single-point averages in the above, to obtain that \( \langle \hat{\rho}(r) \hat{m}^{(a)} (r') \rangle = \rho(r) m^{(a)} (r') + \mathcal{O} \left( \min (r^{-3}, r'^{-3}), U_0 \rho_b^3 \right) \). Lastly, we can utilize the fact that these correlators appear only within the interaction force densities \( \langle \hat{\rho}(r) \hat{m}^{(a)} (r') \rangle \nabla U (r - r') \) and that \( U \) is short-ranged to replace the above corrections with \( \mathcal{O} \left( r^{-3}, U_0 \rho_b^3 \right) \). This thereby confirms the stress expansion equation (84) up to \( \mathcal{O} \left( r^{-3}, \partial \rho, U_0^2 \rho_b^2 / T_{\text{df}}^2 \right) \). Since the correction is consistent with the rest of the derivation in the main text, it holds that \( \rho - \rho_b \sim r^{-1} \). It follows that the correction to the stress expansion is \( \mathcal{O} \left( \partial \rho, U_0^2 \rho_b^2 / T_{\text{df}}^2 \right) \), as written in equation (84).

The weak-interaction expansion can be extended into higher orders in the following way. Starting from equation (104), one can repeat the process depicted in section 5 to show that the pressure field \( P_c^{(2)} \equiv -\text{Tr} \sigma_c^{(2)}/4 \) is given by

\[
P_c^{(2)} (r, r') = \frac{1}{4\pi^2} \int \frac{d^2sd^2s'}{|r - s|^2 + |r' - s'|^2} \left\{ \nabla_{(s,s')} \cdot \left[ \langle \hat{\rho}(s) \hat{\rho}(s') \rangle, \nabla_{(s,s')} \left[ V(s) + V(s') \right] \right] - l \nabla_{(s,s')} \cdot \left[ \langle \hat{\rho}(s) \hat{m}(s) \rangle_c \right] + \nabla_{s} V(s) \langle \hat{\rho}(s) \hat{m}(s) \rangle_c \right\} + T \delta(s - s') \nabla_{s}^2 \rho(s) + l T \nabla_{(s,s')} \cdot \delta(s - s') \nabla_{(s,s')}^2 \left( \hat{m}(s) + \hat{m}(s) \right) + \mathcal{O} \left( U_0 \rho_b^3, r^{-2} \right),
\]

where we have utilized the homogeneous phase boundary condition, which gives \( \lim_{r,r' \to \infty} P_c^{(2)} (r, r') = 0 \). By the above considerations, \( P_c^{(2)} \) has an asymptotic behavior of a localized dipole, i.e. \( P_c^{(2)} \sim \min (r^{-3}, r'^{-3}) \). From this point, one can invert the expansion to obtain \( \langle \hat{\rho}(r) \hat{\rho}(r') \rangle_c \sim \min (r^{-3}, r'^{-3}) \), as done for \( \rho(r) \) in section 5. Following this procedure would require to assume a stress expansion of the form

\[
\sigma^{(2)} (r, r') = \sigma^{(2)} \left( \langle \hat{\rho}(r) \hat{\rho}(r') \rangle \right) + \mathcal{O} \left( \left( \partial + \partial' \right) \langle \hat{\rho}(r) \hat{\rho}(r') \rangle \right),
\]

which can be proved by computing the dynamics of three-point correlations, e.g. \( \langle \hat{\rho}(r) \hat{\rho}(r') \hat{\rho}(r'') \rangle \), and truncating the expansion at the next order by omitting four-point correlations.

### A.2. Derivation of the virial expansion

We now show that the second virial coefficient in equation (85) is 1/2. To this end, we need to calculate the first-order correction to pressure due to the leading-order behaviors of the stress components \( \sigma^{\text{IK}} \) and \( \sigma^{\text{P}} \) originating from the interactions between particles.

We first calculate \( \sigma^{\text{IK}} \) up to the leading order. In the weak-interaction regime, as previously discussed, \( \langle \hat{\rho}(r) \hat{\rho}(r') \rangle = \rho(r) \rho(r') + \mathcal{O} \left( \min (r^{-3}, r'^{-3}), U_0 \rho_b^3 \right) \) holds in the far
field. Applying this approximation, equation (32) can be expanded as

$$\sigma^{IK}(\mathbf{r}) = \frac{1}{2} \int d^2 \mathbf{r} \frac{\mathbf{r}' \cdot dU(\mathbf{r}')}{\mathbf{r}'} \int_0^1 d\lambda \rho(\mathbf{r} + (1 - \lambda) \mathbf{r}') \rho(\mathbf{r} - \lambda \mathbf{r}') + O(\mathbf{r}^{-3}, U_0^2 \rho_b^3).$$  

(118)

To simplify this expression further, we note that the integral over \(\lambda\) contains densities which can be expanded as

$$\rho(\mathbf{r} + (1 - \lambda) \mathbf{r}') = \rho(\mathbf{r}) + (1 - \lambda) \mathbf{r}' \cdot \nabla \rho(\mathbf{r}) + O((1 - \lambda)^2 \rho^2 \partial^2 \rho)$$  

(119)

and

$$\rho(\mathbf{r} - \lambda \mathbf{r}') = \rho(\mathbf{r}) - \lambda \mathbf{r}' \cdot \nabla \rho(\mathbf{r}) + O(\lambda^2 \rho^2 \partial^2 \rho).$$  

(120)

Substituting these expansions into equation (118) and carrying out the integration over \(\lambda\), we obtain

$$\sigma^{IK}(\mathbf{r}) = \frac{\rho(\mathbf{r})^2}{2} \int d^2 \mathbf{r} \frac{\mathbf{r}' \cdot dU(\mathbf{r}')}{\mathbf{r}'} + O(\mathbf{r}^{-3}, U_0^2 \rho_b^3),$$  

(121)

where we have used the far-field behavior \(\partial^2 \rho \sim \mathbf{r}^{-3}\) derived from equation (8). After evaluating the area integral over \(\mathbf{r}'\) using integration by parts, we find

$$\sigma^{IK} = -\frac{U_0}{2} \rho^2 \mathbb{1} + O(\mathbf{r}^{-3}, U_0^2 \rho_b^3) = -\frac{U_0}{2} \rho^2 \mathbb{1} - U_0 \rho_b (\rho - \rho_b) \mathbb{1} + O(\mathbf{r}^{-3}, U_0^2 \rho_b^3).$$  

(122)

Thus the contribution of \(\sigma^{IK}\) to the bulk pressure, or the direct interaction pressure \(P_D(\rho_b) = -\text{Tr} \sigma^{IK}(\rho_b)/2\), satisfies

$$\frac{P_D(\rho_b)}{T_{\text{eff}} \rho_b} = \frac{U_0 \rho_b}{2T_{\text{eff}}} + O\left(\left(\frac{U_0 \rho_b}{T_{\text{eff}}}\right)^2\right),$$  

(123)

which is an exact analog of the leading-order contribution of interparticle interactions to the bulk pressure in a passive gas, the only change being the replacement of temperature with \(T_{\text{eff}}\).

We now turn to the leading-order behavior of \(\sigma^P\), which can be obtained similarly as follows. Again assuming the weak-interaction regime, we can use the previously obtained relation \(\langle \hat{\rho}(\mathbf{r}) \hat{\mathbf{m}}^{(n)}(\mathbf{r}') \rangle = \rho(\mathbf{r}) \mathbf{m}^{(n)}(\mathbf{r}') + O(\mathbf{r}^{-3}, \mathbf{r}'^{-3}, U_0 \rho_b^3),\) so that equation (29) can be expanded as

$$\sigma^P(\mathbf{r}) = l_c \int d^2 \mathbf{r}' [\nabla U(|\mathbf{r} - \mathbf{r}'|)] \rho(\mathbf{r}') \mathbf{m}(\mathbf{r}) + Tl_c \nabla \mathbf{m} - 2(T_{\text{eff}} - T) \mathbf{Q} + O(\mathbf{r}^{-3}, U_0^2 \rho_b^3).$$  

(124)
Using integrating by parts, the area integral over $\mathbf{r}'$ can be rewritten as

$$
\int d^2 r' \rho(\mathbf{r}') \nabla U(|\mathbf{r} - \mathbf{r}'|) = - \int d^2 r' U(|\mathbf{r} - \mathbf{r}'|) \nabla' \rho(\mathbf{r}').
$$

(125)

Using this relation in equation (124) and expanding $\rho(\mathbf{r}')$ about $\mathbf{r}' = \mathbf{r}$, we can evaluate the area integral over $\mathbf{r}'$ to obtain

$$
\sigma_P = I U_0 T \rho_b (\nabla \rho) + T \rho_b \nabla m - 2 (T_{\text{eff}} - T) Q + O \left( r^{-3}, U_0^2 \rho_b^3 \right),
$$

(126)

where we again used the far-field behavior $\partial^2 \rho \sim r^{-3}$. This implies that, at order $U_0 \rho_b$, $\sigma_P$ vanishes in the bulk. As a result, the contribution of $\sigma_P$ to the bulk pressure, or the indirect interaction pressure, satisfies

$$
P_{\text{I}} (\rho_b) / (\rho_b T_{\text{eff}}) = O \left( U_0^2 \rho_b^2 / T_{\text{eff}}^2 \right).
$$

Using this result together with equations (4) and (123), we finally obtain the virial expansion (85). The derivation we have presented so far clearly shows that the virial expansions for both active and passive particles coincide up to the first order (only with the usual temperature replaced by an effective temperature) because the indirect pressure $P_I$, which captures the effects of ‘swimming’, only contributes higher-order corrections.

Appendix B. Scalar, vector and tensor shear stresses

As stated in the main text, the long-distance decay of the traceless deviatoric stress tensor $\mathbf{S}$ satisfies

$$
\mathbf{S} = O \left\{ \left[ P - P (\rho_b) \right]^2, \partial P \right\},
$$

(127)

and $\nabla \cdot \mathbf{S}$ admits the Helmholtz decomposition

$$
\nabla \cdot \mathbf{S} = - \nabla \Phi_S + \nabla \times \Psi,
$$

stated in equation (38). Here we show that both $\Phi_S$ and $\Psi$ decay with the distance as $O (S, r^{-2})$, justifying equation (50). This is not a trivial statement—due to the nonlocal nature of the Helmholtz decomposition for vectors, equation (38) does not immediately guarantee that $\mathbf{S}$, $\Phi_S$, and $\Psi$ are of the same order. In the following, we address this difficulty by applying a tensor version of the Helmholtz decomposition.

As the first step, we decompose $\mathbf{S}$ as

$$
\mathbf{S} = \mathbf{A} + \mathbf{E},
$$

(128)

where $\mathbf{A} \equiv (\mathbf{S} - \mathbf{S}^T) / 2$ and $\mathbf{E} \equiv (\mathbf{S} + \mathbf{S}^T) / 2$ are the antisymmetric and the symmetric components of $\mathbf{S}$, respectively. In analogy to linear flow, $\mathbf{A}$ can be thought of as a pure rotation, while $\mathbf{E}$ as a pure straining motion [115]. We note that, among the components of the stress tensor $\sigma$ shown in equation (28), only the polarization component $\sigma_P$ is not symmetric and can thus contribute to $\mathbf{A}$, see equation (29). Because both $\mathbf{A}$ and $\mathbf{E}$ are local functions of $\mathbf{S}$, it is evident that $\mathbf{A} = O (\mathbf{S})$ and $\mathbf{E} = O (\mathbf{S})$. It remains to show that this decay is inherited by their contributions to $\Phi_S$ and $\Psi$. We will first show this for the contributions by $\mathbf{A}$, and then for the contributions by $\mathbf{E}$. 

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Due to the constraint of antisymmetry, the rank-2 tensor $A$ has only a single free parameter, which allows the following representation:

$$A_{\alpha\beta} = -\varepsilon_{\alpha\beta\gamma} \Omega_{\gamma},$$ (129)

where $\Omega = \Omega e_z$. Then we can write $\nabla \cdot A = \nabla \times \Omega$, which means that $\nabla \cdot A$ contributes only to the solenoidal component $\nabla \times \Psi$ of $\nabla \cdot S$. Moreover, since the above representation can be inverted as $\Omega_{\alpha} = \varepsilon_{\alpha\beta\gamma} A_{\beta\gamma}/2$, $\Omega$ is clearly a local linear function of $S$. Thus, equation (127) implies

$$\Omega = \mathcal{O}(A) = \mathcal{O}(S) = \mathcal{O}\left\{ [P - P (\rho_b)]^2, \partial P \right\}$$ (130)

Hence, $A$ cannot contribute to $\Phi$ defined in equation (39), and its contributions are bound to be higher-order than the leading-order terms of equation (50).

Now, it remains to show that the contributions by $E$ also decay with distance in the same way. Applying a tensor version of the Helmholtz decomposition, also called the generalized Beltrami decomposition [116–120], the symmetric component $E$ can be decomposed as

$$E = E^S + E^I,$$ (131)

where

$$E^S = \nabla \times (\nabla \times \Pi),$$ (132)

$$E^I = \frac{1}{2} \left[ \nabla v + (\nabla v)^T \right]$$ (133)

for a symmetric rank-2 tensor $\Pi$ and a vector potential $v$. These imply $\nabla \cdot E^S = 0$ and $\nabla \times (\nabla \times E^I) = 0$. Conversely, $\nabla \cdot E = 0$ implies $E = E^S$, and $\nabla \times (\nabla \times E) = 0$ implies $E = E^I$. Thus, $E^S$ can be regarded as the solenoidal component of $E$, and $E^I$ the irrotational component. For example, linear fluids correspond to the case $E = E^I$, where $v = J/\rho$ is the fluid velocity.

Recently, it has been shown that the generalized Beltrami decomposition satisfies the following integrability rule [121]: defining $|E| \equiv \sqrt{E_{\alpha\beta} E^{\alpha\beta}}$, if $\int d^2r |E|^p < \infty$ for some fixed $p > 1$, then we also have $\int d^2r |E^S|^p < \infty$ and $\int d^2r |E^I|^p < \infty$. This stems from the fact that the space of symmetric tensors $E$ satisfying $\int d^2r |E|^p < \infty$ can be decomposed into a direct sum of two subspaces—one being the subspace of all irrotational tensors, and the other being the subspace of all solenoidal tensors. For the special case $p = 2$, this can be seen immediately, as the decomposition becomes an orthogonal one. Using integration by parts, one can verify that a tensor orthogonal to $E^S$ defined in equation (132) is of the form equation (133), with the orthogonality taken under the standard inner product $\langle E^1 | E^2 \rangle = \int d^2r E^1_{\alpha\beta}(r) E^2_{\alpha\beta}(r)$ [120]. Note that, for two-dimensional smooth fields $E$ whose derivatives vanish as $r \rightarrow \infty$, $\int d^2r |E|^p < \infty$ holds if and only if $|E|^p = o(r^{-2})$. This is equivalent to the requirement $E = o(r^{-2/p})$. Using the notation $\gamma = 2/p$, we can rewrite the integrability rule as follows:

**Rule.** If $E = o(r^{-\gamma})$ for some fixed $0 < \gamma < 2$, then $E^S = o(r^{-\gamma})$ and $E^I = o(r^{-\gamma})$ also hold.
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This result can be refined further for our purpose. To proceed, we suppose $E = O(r^{-\gamma})$ for some $\gamma > 0$. Note that we expect $P - P(\rho_b) = O(r^{-1})$, which would correspond, according to equation (127), $\gamma = 2$. Indeed, we will show using the general exponent $\gamma$ that this is the case. First, we denote $\gamma_{cf} \equiv \min(\gamma, 2)$. Then, it holds that $E = O(r^{-\gamma_{cf}})$. In particular, for any $0 < \gamma' < \gamma_{cf}$, it is true that $E = o(r^{-\gamma'})$. By our integrability rule, $E^S = o(r^{-\gamma})$ and $E^I = O(r^{-\gamma_{cf}})$. Taking the limit $\gamma' \rightarrow \gamma_{cf}$, we obtain $E^S = O(r^{-\gamma_{cf}})$ and $E^I = O(r^{-\gamma_{cf}})$, up to some sub-algebraic modulation of the decay.

Put differently, we have found that $E^I$ is of order $O(E, r^{-2})$. To apply this result to the far-field behaviors of $\Phi_S$ and $\Psi$, we go back to equation (38) and examine the far-field behavior of $\nabla \cdot S$, which is dominated by $\nabla \cdot E$, as already discussed. Taking the divergence of equation (131) side by side, the solenoidal component $E^S$ vanishes, leaving

$$\nabla \cdot E = \nabla (\nabla \cdot v) + \frac{1}{2} \left[ \nabla^2 v - \nabla (\nabla \cdot v) \right].$$  \hspace{1cm} (134)

On the rhs, one can easily find that $\nabla (\nabla \cdot v)$ is the irrotational component, while

$$\frac{1}{2} \left[ \nabla^2 v - \nabla (\nabla \cdot v) \right] = -\frac{1}{2} \nabla \times (\nabla \times v)$$  \hspace{1cm} (135)

is the solenoidal component. Combining these observations with equations (38), (128), (129), and (131), we identify

$$\Phi_S = -\nabla \cdot v, \quad \Psi = \Omega - \frac{1}{2} \nabla \times v.$$  \hspace{1cm} (136)

From equation (133) and the far-field behavior of $E^I$, we obtain $\nabla \cdot v = \text{Tr} E^I = O(E, r^{-2}) = O(S, r^{-2})$ up to a sub-algebraic modulation. Then, using the above identities, we finally conclude that $\Phi_S = O(S, r^{-2})$ and $\Psi = O(S, r^{-2})$. We have thus confirmed equation (50).

Appendix C. Finite-size effects

Here we address two different issues about how the infinite-size limit is achieved. First, we clarify the meaning of the infinite-area integral appearing in the current-force relation (54). Second, we briefly discuss how the finite-size effects modify the derivations shown in section 5, which are valid in the infinite-size limit. As an explicit example, we show that the dipole moment of a single asymmetric body in an $L \times L$ torus converges algebraically to the asymptotic value as $L \rightarrow \infty$.

C.1. Derivation of the current–force relation (54)

Integrating equation (9) side by side over the entire space, we obtain

$$\int d^2 r J(r) = \frac{\mu}{2} p,$$  \hspace{1cm} (137)
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which differs by a factor of 1/2 from the well-established current–force relation (54). As discussed below, this apparent contradiction is resolved if one properly defines the area integral over the entire system appearing in equation (54).

By integrating equation (27) side by side over an area $\mathcal{A}$ which contains all the bodies inside, the divergence theorem and $\mathbf{V} = 0$ on the boundary imply

$$
\int_{\mathcal{A}} d^2 \mathbf{r} \mathbf{J} (\mathbf{r}) = \mu \mathbf{p} + \mu \oint_{\partial \mathcal{A}} d\ell \mathbf{e}_n \cdot \mathbf{\sigma} (\mathbf{r}) ,
$$

(138)

where $d\ell$ is an infinitesimal segment on the boundary $\partial \mathcal{A}$, and $\mathbf{e}_n$ is a unit normal vector. For a finite system with periodic boundaries, if $\mathcal{A}$ covers the entire system, the boundary integral in equation (138) is carried out twice for each $d\ell$ with opposite directions of $\mathbf{e}_n$, so that its value sums to zero. As long as $\mathcal{A}$ covers the entire system, the same result still holds even in the limit $L \to \infty$. The infinite-area integral in equation (54) should be interpreted in this vein—the infinite-size limit is taken after requiring that $\mathcal{A}$ covers the entire system.

How do we then obtain equation (137) as well? Going back to equation (138), we choose $\mathcal{A}$ to be a disk $\mathcal{D}_R$ of radius $R$ centered at the origin, take the infinite-size limit, after which $R$ is sent to infinity. Using this order of limits, we can write

$$
\mu \oint_{\partial \mathcal{D}_R} d\ell \mathbf{e}_n \cdot \mathbf{\sigma} (\mathbf{r}) = \mu \int_{\mathcal{D}_R} d^2 \mathbf{r} \nabla \cdot \mathbf{\sigma} (\mathbf{r}) = -\frac{\mu}{2} \mathbf{p} + \mathcal{O} (R^{-1}) ,
$$

(139)

where the last equality is obtained by using equation (53) to evaluate $\nabla \cdot \mathbf{\sigma} (\mathbf{r})$. Using this relation in equation (138), we obtain

$$
\int_{\mathcal{D}_R} d^2 \mathbf{r} \mathbf{J} (\mathbf{r}) = \mu \frac{\mathbf{p}}{2} + \mathcal{O} (R^{-1}) ,
$$

(140)

which gives the precise meaning of equation (137). To sum up, whether one gets equations (54) or (137) is determined by whether the area integral expands with or slower than the system size.

### C.2. Finite-size corrections in a periodic system

For a finite system $\mathcal{S}$, the proper solution for equations (41) and (42) is not equation (43), but (see, for example, reference [122])

$$
\Phi (\mathbf{r}) = P (\rho_b) - \frac{1}{2\pi} \int_S d^2 \mathbf{r}' \ln |\mathbf{r} - \mathbf{r}'| \left\{ \mathbf{\nabla}' \cdot [\rho(\mathbf{r}') \mathbf{\nabla}' V (\mathbf{r}')] - l_i \partial_i ' \partial_j ' \left[ m_{\alpha} (\mathbf{r}') \partial_\alpha ' V (\mathbf{r}') \right] \right\}
\quad + \frac{1}{2\pi} \oint_{\partial \mathcal{S}} d\ell \ln |\mathbf{r} - \mathbf{r}'| \left( \mathbf{e}_n \right)_\alpha ' \partial_\alpha ' \mathbf{\sigma}_{\alpha ' \beta '} (\mathbf{r}'') ,
$$

(141)

where $\mathbf{r}''$ in the second integral is on the boundary segment $d\ell$. The boundary integral on the second line is indeed responsible for the finite-size effects observed in figure 3 near the boundary. Since it would be physically absurd if the stress diverges with the distance from the origin, it is reasonable to require that $\mathbf{\nabla} \cdot \mathbf{\sigma} (\mathbf{r}) = o (r^{-1})$. This implies that the boundary integral is $o (1)$, so that the derivations in section 5 are fully valid in the infinite-size limit.
Precisely how the boundary contributions decay with the increasing system size could be dependent on the details of the system and its boundary conditions. As an explicit example, below we show for the dipole moment that these corrections do decay with the system size $L$, namely $O(L^{-2})$.

We consider a single body described by a potential $V$ in a periodic torus of dimensions $L \times L$. Extension to mutually distant multiple bodies is straightforward. Furthermore, we assume that the boundaries are in the far field of the body, so that finite-size effects can be described using far-field effects. Given these assumptions, the system can be regarded as an infinite cubic lattice with lattice constant $L$, where an exact copy of the body is placed at the center of each cell (see figure 5). The lattice is now characterized by a periodic potential $V = \sum_i V_i$. We denote by $p_i = -\int d^2r \rho \nabla V_i$ the force applied to the fluid by body copy $i$, and $R_i$ represents the corresponding response tensor. We denote by $p$ and $R$ as the same quantities in the $L \to \infty$ limit, respectively. Following the procedure described in section 6, we obtain

$$p_j = p + R_j \sum_{i \neq j} \frac{1}{2\pi} \frac{r_{ij} \cdot p_i}{r_{ij}^2} + \sum_{i \neq j} O(r_{ij}^{-2}) . \quad (142)$$
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Since the lattice constant is \( L \), \( r_{ij} \sim n_{ij}L \) with \( n_{ij} \) designating the rescaled distance between body \( i \) and \( j \). Thus we have \( p_i = p + \mathcal{O}(L^{-1}) \) for any \( i \) and \( \sum_i \mathcal{O}(r_{ij}^2) = \mathcal{O}(L^{-2}) \). Noting that the potentials \( V_i \) are all identical to each other, equation (16) implies that the single-body response coefficient \( R_i = R \) for all \( i \). Thus, the above equation can be rewritten as

\[
p_j = p + R \sum_{i \neq j} \frac{1}{2\pi} \frac{r_{ij} \cdot p}{r_{ij}^2} + \mathcal{O}(L^{-2}). \tag{143}
\]

By reflection symmetry, the first-order terms should vanish; thus we finally obtain

\[
p_j = p + \mathcal{O}(L^{-2}). \tag{144}
\]

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