BPS States and Vacuum Manifold of $SU_q(n)$ Georgi-Glashow Model

F.P. Zen, B.E. Gunara, R. Muhamad and D. P. Hutasoit

Theoretical High Energy Physics Group,
Theoretical Physics Laboratory,
Department of Physics, Bandung Institute of Technology,
Jl. Ganesha 10 Bandung, 40132, INDONESIA.

Abstract

We construct the Georgi-Glashow Lagrangian for gauge group $SU_q(n)$. Breaking this symmetry spontaneously gives $q$-dependent masses of gauge field and vacuum manifold. It turned out that the vacuum manifold is parameterized by the non-commutative quantities. We showed that the monopole solutions exist in this model, which is indicated by the presence of the BPS states.
I. Introduction

The notion of the Lie group has been generalized by Drinfel’d[1], Jimbo[2], and Woronowicz[3]. Their generalized Lie group, i.e., noncommutative and non-cocommutative Hopf algebra, is now known as the quantum group under an enthusiastic study by lot of mathematicians and physicists. Several authors have attempted to quantize or $q$-deform the Lorentz group[4].

On the other hand, the Georgi-Glashow model, which is a simple theory in 3+1 dimensional, has been studied by many authors such as ’t Hooft-Polyakov[5] and Julia-Zee[6]. This model has a solitonic solution, called monopole, in the Higgs vacuum[5, 6].

The purpose of this paper is to generalize the Georgi-Glashow model for the case of quantum group, and to show that there exists a solitonic solution in general case. We shall be concerned only quantum group $SU_q(n)$ with the simplest example of quantum group, $SU_q(2)$, since it reduces to $SU(2)$ for $q = 1$. We construct the Georgi-Glashow Lagrangian (also with $\theta$-term), and then we define the variation of Lagrangian. We find that the equation of motion, besides fields, depends on the quantities which are independent of the representation of the gauge group and a noncommutative factor. Breaking the gauge symmetry spontaneously gives $q$-dependent masses of gauge and vacuum manifold. Vacuum manifold is parameterized by the gauge invariant quantity, which is similar to Seiberg-Witten theory[10] (an excellent review on this subject is given by Alvarez-Gaume-Hassan[7]), and for this model, in which the gauge group is quantum group, the parameter of vacuum manifold is noncommutative. We also derive the field strength corresponding to the unbroken subgroup and the $q$-dependent BPS bound mass.

The basis of theory presented here are the notion of the differential calculus on $SU_q(2)$ which was developed by Woronowicz[3] and the $SU_q(2)$ Yang-Mills theory which was constructed by Hirayama[8].

This paper is organized as follows. First we review the $SU_q(2)$ theory in section II, this review section is taken almost verbatim from Hirayama[8]. In section III we present the $SU_q(n)$ Georgi-Glashow theory. The discussion of BPS states and the vacuum manifold of the model is presented in section IV. Section V is devoted for conclusion and outlook.
II. The $SU_q(2)$ and Yang-Mills Theory

A. $SU_q(2)$ Transformation

We briefly review the $SU_q(2)$ theory which was developed by Woronowicz\[3\]. The fundamental representation of $SU_q(2)$ is given by

$$w = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$ (2.1)

where $\alpha, \gamma, \alpha^*, \gamma^*$ are operators satisfying certain algebras\[8\].

We denote $R$ as polynomial rings which are generated by $I, \alpha, \gamma, \alpha^*$, and $\gamma^*$ and $M_N (B)$ as a set of $N \times N$ matrices whose entries belong to the set $B$. Let $R'$ be the set of the representations of $R$ whose operators act on a Hilbert space $H$[3]. We define the product of $w_1$ and $w_2$ as\[3\]

$$w_1 \oplus w_2 = \begin{pmatrix} \alpha_2 \otimes \alpha_1 - q\gamma_2^* \otimes \gamma_1 & -q (\gamma_2^* \otimes \alpha_1^* + \alpha_2 \otimes \gamma_1^*) \\ \gamma_2 \otimes \alpha_1 + \alpha_2^* \otimes \gamma_1 & \alpha_2^* \otimes \alpha_1 - q\gamma_2 \otimes \gamma_1^* \end{pmatrix} \in M_2 (R' \otimes R')$$ (2.2)

which acts on $H \otimes H$ and it is closely related to the coproduct defined on $R$. The $*$-operation is the complex conjugate for complex numbers. The $\otimes$-product of operators $R$ or $R'$ is defined by

$$(a_1 \otimes a_2 \otimes \ldots \otimes a_n) (b_1 \otimes b_2 \otimes \ldots \otimes b_n) = a_1 b_1 \otimes a_2 b_2 \otimes \ldots \otimes a_n b_n.$$ (2.3)

We denote the set of $w_m \oplus w_{m-1} \oplus \ldots \oplus w_1$ as $C_m$ and it has inverse $C_m^{-1}$.

B. Group Theoretic Representation of $w$

In this subsection we review the group-theoretic representation of $SU_q(2)$ which was studied by Woronowicz\[3\]. It turned out that the representation theory of $SU_q(2)$ is quite similar to that of $SU(2)$. The matrix $W \in M_N (R)$ is said to be the representation of $w$ if it satisfies\[3\]

$$\Delta (W_{ij}) = (W \oplus W)_{ij} = \sum_{k=1}^{N} W_{ik} \otimes W_{kj} \quad ; i, j = 1, 2, \ldots, N$$ (2.4)
where $\Delta$ is the coproduct which is defined as

$$
\Delta (w) \equiv \begin{pmatrix}
\Delta (\alpha) & \Delta (-q\gamma^*) \\
\Delta (\gamma) & \Delta (\alpha^*)
\end{pmatrix} = w \oplus w,
$$

(2.5)

We define the set $C_N^m$ by

$$
C_N^m = \{ W_m \oplus W_{m-1} \oplus ... \oplus W_1 \},
$$

(2.6)

where $W_i \in M_{N^i} (R')$, $i = 1, 2, ..., m$ are the canonical representations of $w_i \in M_2 (R')$, $i = 1, 2, ..., m$, respectively.

C. 3D Calculus of $SU_q (2)$

The differential calculus of $SU_q (2)$ is discussed in Woronowicz[3], in which coordinates are non-commutative operators. The 3D calculus which was studied by Woronowicz[3], is not only left-covariant but also has simple structure and mysteriously works well even for the higher order differential calculi. Here we briefly recapitulate the 3D calculus of Woronowicz.

The linear functionals $\chi_0, \chi_1, \chi_2, f_0, f_1, f_2, e$ on $R$ are defined by

$$
\chi_0 (w) = \begin{pmatrix}
\chi_0 (\alpha) & \chi_0 (-q\gamma^*) \\
\chi_0 (\gamma) & \chi_0 (\alpha^*)
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix},
$$

(2.7)

$$
\chi_1 (w) = \begin{pmatrix}
1 & 0 \\
0 & -q^2
\end{pmatrix}, \chi_2 (w) = \begin{pmatrix}
0 & 0 \\
-q & 0
\end{pmatrix}, \chi_k (I) = 0,
$$

$$
f_0 (w) = f_2 (w) = \begin{pmatrix}
q^{-1} & 0 \\
0 & q
\end{pmatrix}, f_1 (w) = \begin{pmatrix}
q^{-2} & 0 \\
0 & q^2
\end{pmatrix}, f_k (I) = 1,
$$

(2.8)

$$
e (w) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, e (I) = 1, \quad k = 0, 1, 2.
$$

(2.9)

The convolution product of a linear functional $X$ on $R$, and $a \in R$, is defined by

$$
X \ast a = \sum_i X (a_i') a_i'' \in R
$$

(2.10)
where $a'_i$ and $a''_i$ are given by $\Delta (a) = \sum_i a'_i \otimes a''_i$. The differential operator $d$ is defined by

$$da = \sum_{k=0}^{2} (\chi_k \ast a) \omega_k, \quad a \in R,$$

where $\omega_k$, $k = 0, 1, 2$, are the bases of the space of differential 1-forms. The higher order differential calculus can be defined to maintain the property

$$d^2 = 0.$$

The Hermitian $\chi_k (W)^\dagger$ of $\chi_k (W) \in M_N (C)$ is given by

$$\chi_k (W)^\dagger = \sum_{j=0}^{2} t_{kj} \chi_j (W),$$

where $t_{11} = 1$, $t_{02} = -\frac{1}{q}$, and $t_{20} = -q$.

### D. Local $SU_q (2)$

Let $x = (x^0, x^1, x^2, x^3)$ be coordinates of the four dimensional Minkowski spacetime and $\alpha (x), \gamma (x), \alpha^* (x), \gamma^* (x) \in R$ be the $x$-dependent representations of $\alpha, \gamma, \alpha^*, \gamma^* \in R$ respectively, as operators acting on the Hilbert space $H$ introduced in 2.1. To discuss the field theory of $SU_q (2)$, it is inevitable to consider the functions of $x, \alpha (x), \gamma (x), \alpha^* (x), \gamma^* (x)$ and their derivatives with respect to $x^\mu$. We denote the set of functions of the form $g [x] \equiv g (x, \alpha (x), \gamma (x), \alpha^* (x), \gamma^* (x))$ by $R^x$. The functional $X^x$ on $R^x$ should be introduced so that $X^x (w (x)) = X (w)$, e.g.,

$$\chi_k^x (w (x)) = \chi_k (w), \quad f_k^x (w (x)) = f_k (w), \quad e^x (w (x)) = e (w), \quad k = 0, 1, 2,$$

where $w (x)$ is defined by

$$w (x) = \begin{pmatrix} \alpha (x) & -q \gamma^* (x) \\ \gamma (x) & \alpha^* (x) \end{pmatrix}.$$
and \( w, \chi_k, f_k, \) and \( e \) are those defined hitherto. Recalling (2.11), the differential operator \( d^x \) should be defined to act on \( g[x] \) as

\[
d^x g[x] = \sum_{k=0}^{2} (\chi_k^x * g[x]) \omega_k^x + (\partial_\mu g[x]) dx^\mu, \quad (2.16)
\]

where \( \omega_k^x, k = 0, 1, 2 \) are the analogue of the previous \( \omega_k \) and \( \partial_\mu g[x] \) is the conventional partial derivative of \( g[x] \) with respect to the explicit \( x \)-dependence of \( g[x] \). A consistent set of rules is derived from the result of Woronowicz\[3\] by supposing that \( \omega_k^x \) and \( d^x g[x] \) decompose as the \( \omega_k^x, \mu \) and \( (D_\mu g[x]) dx^\mu \) respectively, and assuming that \( \{dx^\mu, dx^\nu\} = [dx^\mu, \omega_k^x,\nu] = [dx^\mu, a] = 0, \ a \in R^x, \mu, \nu = 0, 1, 2, 3. \) We call the above procedure as the Z-procedure\[8\]. The Z-procedure leads us to the following definition of the partial derivative \( D_\mu g[x] \) of \( g[x] \in R^x, \)

\[
D_\mu g[x] = \sum_{k=0}^{2} (\chi_k^x * g[x]) \omega_k^x,\mu + \partial_\mu g[x]. \quad (2.17)
\]

E. The \( SU_q(2) \) Yang-Mills Theory

In this subsection, we briefly review the \( SU_q(2) \) Yang-Mills theory constructed by Hirayama\[8\]. We suppose that the components of the gauge field, \( A_{k,\mu}(x), i = 0, 1, 2, \mu = 0, 1, 2, 3 \) respectively. We postulate that

\[
A_{\mu,i}(x)W(x) = W(x)f_i(\omega)A_{\mu,i}(x), \quad (2.18)
\]

\[
A_{\mu,i}(x)A_{\nu,j}(x) = c_{ji}A_{\nu,j}(x)A_{\mu,i}(x), \quad (2.19)
\]

\[
A_{i,\mu}^+(x) = \sum_{k=0}^{2} t_{ji}A_{j,\mu}(x), \quad (2.20)
\]

where \( t_{02} = -q, t_{20} = -q^{-1}, \) and \( t_{11} = 1. \)

Throughout this subsection we denote

\[
W(x) \equiv W_m(x) \oplus W_{m-1}(x) \oplus ... \oplus W_1(x),
\]
The vector field \( A_{k,\mu}(x) \) transform as

\[
A^W_{\mu}(x) = W(x)(I_{m-1} \otimes \chi_k(W_1)) - \frac{1}{ig} (D_{\mu}W(x))W^{-1}(x),
\]

where

\[
A_{\mu}(x) = \sum_{k=0}^{2} A_{\mu,k}(x) \chi_k(W_1),
\]

the field in the gauge \( W(x) \). In (2.21), \( g \) is the gauge coupling constant, \( D_{\mu}W(x) \) is defined by

\[
D_{\mu}W(x) = \sum_{l=1}^{m} W_m(x) \oplus ... \oplus D_{\mu}W_l(x) \oplus ... \oplus W_1(x),
\]

and \( \chi_k(W_1) \) is equal to \( \chi_k(W_1) \). The gauge transform \( (A^W_{\mu}(x))^W' \) of \( A^W_{\mu}(x) \) by \( W'(x) \) is defined by

\[
(A^W_{\mu}(x))^W' = (W'(x) \otimes I_m)(I_n \otimes A^W_{\mu}(x))(W'^{-1}(x) \otimes I_m)
\]

\[
- \frac{1}{ig} (D_{\mu}W'(x))W'^{-1}(x) \otimes I_m.
\]

Then we have

\[
(A^W_{\mu}(x))^W' = A^{W'\oplus W}_{\mu}(x).
\]

We define the field strength \( F^{W}_{\mu\nu}(x) \) in the gauge \( W(x) \) by

\[
F^{W}_{\mu\nu}(x) = [\nabla^W_{\mu}, \nabla^W_{\nu}], \quad \nabla^W_{\mu} = D_{\mu} + igA^W_{\mu}(x),
\]

then we find that

\[
F^{W}_{\mu\nu}(x) = W(x)(I_{m-1} \otimes F_{\mu\nu}(x))W^{-1}(x),
\]
\[ F_{\mu\nu}(x) = \sum_{k=0}^{2} F_{k,\mu\nu}(x) \chi_k(W_1). \] (2.28)

The transformation law of \( F_{\mu\nu}(x) \) is given by

\[
(F_{\mu\nu}(x))^{W'} = (W'(x) \otimes I_m)(I_n \otimes F_{\mu\nu}(x))(W'^{-1}(x) \otimes I_m)
= F_{\mu\nu}^{W' \otimes W}(x). \tag{2.29}
\]

The Lagrangian density of the local SU\(_q\)(2) invariant field theory should be independent of the choice of \( W(x) \), the dimensionality \( N \) and the integer \( m \).

We begin with defining \( S_{kl}^W \) by

\[
S_{kl}^W = \text{tr}(\rho^N \chi_k(W) (\rho^N)^2 \chi_l(W)), W \in C_1^N, k, l = 0, 1, 2, \tag{2.30}
\]
where \( \rho^N \) is given by \( \rho^N = (\sigma^N)^{-1} \). If we define \( K_N \) by

\[
K_N = -\frac{q}{8} (S_{20}^W)^{-1}, W \in C_1^N, \tag{2.31}
\]

then the product \( K_N S_{kl}^W \) is independent of \( W \). Then the gauge invariant Lagrangian is

\[
L_{GG}^W(x) = K_N \text{tr}(\sigma^N \tau(W) F_{\mu\nu}^{W,\mu\nu}(W) \tau(W) F_{\mu\nu}^{W} \tau(W)) , \tag{2.32}
\]

where

\[
\tau(W) = (W_m(x) \oplus \ldots \oplus W_2(x) \oplus \rho^N I) W^{-1}(x). \tag{2.33}
\]

III. The SU\(_q\)(\(n\)) Georgi-Glashow Theory

A. Gauge Field and Scalar Field

We introduce the gauge field and the scalar field which are fields that present in the Georgi-Glashow model. We consider the components of gauge
fields and scalar fields are $A_{k,\mu}(x)$ and $\phi_i(x)$, where $i = 0, 1, ..., n^2 - 2$, and $\mu = 0, 1, 2, 3$ respectively.

If $\Psi_i, i = 0, 1, ..., n^2 - 2$ are fields, then we generalize the Hirayama’s postulate to

$$
\Psi_i(x)W(x) = W(x)f_i(\omega)\Psi_i(x), \quad (3.1)
$$

$$
\Psi_i(x)\Psi_j(x) = c_{ji}\Psi_j(x)\Psi_i(x). \quad (3.2)
$$

From the equation (2.22), we generalize the vector field $A_{k,\mu}(x)$ to

$$
A_{\mu}(x) = \sum_{k=0}^{n^2-2} A_{\mu,k}(x) \chi_k(W_1), \quad (3.3)
$$

and its transformation are given by

$$
A^{W}_{\mu}(x) = W(x)(I_{m-1} \otimes A_{\mu}(x))W^{-1}(x) - \frac{1}{ig}(D_{\mu}W(x))W^{-1}(x), \quad (3.4)
$$

and for the scalar field $\phi(x)$

$$
\phi(x) = \sum_{k=0}^{n^2-2} \phi_k(x) \chi_k(W_1), \quad (3.5)
$$

$$
\phi^{W}(x) = W(x)(I_{m-1} \otimes \phi(x))W^{-1}(x). \quad (3.6)
$$

The additional property of the gauge field is

$$
A_{i,\mu}^{\dagger}(x) = \sum_{k=0}^{n^2-2} t_{ji} A_{j,\mu}(x), \quad (3.7)
$$

and for the generator is

$$
\chi_k(W_1)^{\dagger} = \sum_{j=0}^{n^2-2} t_{kj} \chi_j(W_1). \quad (3.8)
$$
If we transform $A^W_\mu(x)$ to $(A^W_\mu(x))^{W'}$ and $\phi^W(x)$ to $(\phi^W(x))^{W'}$, then we get

\[
(A^W_\mu(x))^{W'} = (W'(x) \otimes I_m)(I_n \otimes A^W_\mu(x))(W'^{-1}(x) \otimes I_m) \tag{3.9}
\]

\[-\frac{1}{ig} (D_\mu W'(x)) W'^{-1}(x) \otimes I_m,
\]

\[
(\phi^W(x))^{W'} = (W'(x) \otimes I_m)(I_n \otimes \phi^W(x))(W'^{-1}(x) \otimes I_m). \tag{3.10}
\]

Thus we have

\[
(A^W_\mu(x))^{W'} = A^{W'\oplus W}_\mu(x), \tag{3.11}
\]

\[
(\phi^W(x))^{W'} = \phi^{W'\oplus W}(x), \tag{3.12}
\]

which have the same form as the equation (2.25).

For the scalar part, the transformation law of $\nabla_\mu^W \phi^W$ is also given by

\[
(\nabla_\mu^W \phi^W(x))^{W'} = (W'(x) \otimes I_m)(I_n \otimes \nabla_\mu^W \phi^W(x))(W'^{-1}(x) \otimes I_m) \tag{3.13}
\]

\[
= (\nabla_\mu^W \phi(x))^{W'\oplus W}.
\]

\section{B. The Construction of Georgi-Glashow Lagrangian}

A similar reason from the previous section can be applied that the Lagrangian density of the local $SU_q(n)$ invariant field theory should be independent of the choice of $W(x)$, the dimensionality $N$ and the integer $m$.

We begin with defining $S^W_{kl}$ by

\[
S^W_{kl} = tr(\rho^N \chi_k(W)) (\rho^N)^2 \chi_l(W), W \in C_1^N, \quad k, l = 0, 1, \ldots, n^2 - 2 \tag{3.14}
\]

where $\rho^N$ is given by $\rho^N = (\sigma^N)^{-1}$. If we define $K_N$ by
\[ K_N = -\frac{q}{8} (S_W^W)^{-1}, W \in \mathbb{C}^N_1, \quad i, j = 0, 1, ..., n^2 - 2 \quad (3.15) \]

then the product \( K_N S^W_{kl} \) is independent of \( W \).

We now define the \( \tau \)-quantities of any function \( F^W(x) \) by

\[
(F^W(x))^\tau = \tau(W) F^W(x) \tau^\dagger(W),
\]

where

\[
\tau(W) = (W_m(x) \oplus ... \oplus W_2(x) \oplus \rho N I) W^{-1}(x),
\]

and \( F^W(x) \) is function \( F(x) \) and transform with respect to the gauge transformation \( W(x) \in \mathbb{C}^N_m \). If we transform the \( \tau \)-quantities of any function \( F^W(x) \) by the gauge transformation \( W'(x) \in \mathbb{C}^N_n \), then its transform to

\[
(F^W(x))^\tau \rightarrow (F^{W' \oplus W}(x))^\tau = \tau(W' \oplus W) F^{W' \oplus W}(x) \tau^\dagger(W' \oplus W). \quad (3.18)
\]

Then we construct the Georgi-Glashow Lagrangian by

\[
L^W_{GG}(x) = K_N tr (\sigma^N ( - \frac{1}{4} (F^W_{\mu\nu})^\tau (F^W_{,\mu\nu})^\tau \\
+ (\nabla^W_{\mu} \phi^W(x))^{\tau,\dagger} ((\nabla^W_{\mu} \phi^W)^{\tau} + \lambda (\{\phi^W, \phi^W\}^\tau)^2))
\]

where

\[
(\nabla^W_{\mu} \phi^W(x))^{\tau,\dagger} = (\tau(W) \nabla^W_{\mu} \phi^W(x) \tau^\dagger(W))^{\dagger}
\]

\[
= \tau(W) (\nabla^W_{\mu} \phi^W(x))^{\dagger} \tau^\dagger(W)
\]

(3.20)

As we expect, the pseudoscalar quantity is also gauge invariant, i.e.,

\[
K_N tr \left( \sigma^N (F^W_{\mu\nu})^\tau (\tilde{F}^W_{\mu\nu})^\tau \right),
\]

where \( \tilde{F}^W_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F^W_{\rho\sigma} \) and \( \varepsilon^{\mu\nu\rho\sigma} \) is the Levi-Civita symbol for 3+1 dimension.

11
Then according to Witten[9], we can construct Georgi-Glashow model with an additional $\theta$-term

$$L^{W}_{GG}(x) = K_{N} \text{tr} \left( \sigma^{N} \left( - \frac{1}{4} (F^{W}_{\mu \nu})^{\tau} (F^{W,\mu \nu})^{\tau} \right) + (\nabla^{W}_{\mu} \phi^{W})^{\tau} \left( \nabla^{W,\mu} \phi^{W} \right)^{\tau} + \lambda \left( \left[ \phi^{W,\dagger}, \phi^{W} \right] \right)^{\tau} \right)^{2} + \frac{\theta e^{2}}{32 \pi^{2}} (F^{W}_{\mu \nu})^{\tau} \left( \tilde{F}^{W,\mu \nu} \right)^{\tau} \right)$$

where $\theta$ is a real parameter and $e$ is the charge unit.

### C. Equation of Motion

Before we derive the equation of motion from Lagrangian (3.19) and (3.22), first we must define the variation of the Lagrangian. Given any function of $\Psi_{i}$ and $D_{\mu} \Psi_{i}$, where $\Psi_{i}$ is a field with $i = 0, 1, ..., n^{2} - 2$, then variation of $L(\Psi_{i}, D_{\mu} \Psi_{i})$ is defined by

$$\delta L(\Psi_{i}, D_{\mu} \Psi_{i}) = \sum_{i} \left( \delta \Psi_{i} \frac{\partial L}{\partial \Psi_{i}} + \delta (D_{\mu} \Psi_{i}) \frac{\partial L}{\partial (D_{\mu} \Psi_{i})} \right) = 0, \quad (3.23)$$

where $\frac{\partial}{\partial \Psi_{i}}$ and $\frac{\partial}{\partial (D_{\mu} \Psi_{i})}$ are the usual partial derivative of $\Psi_{i}$ and $D_{\mu} \Psi_{i}$, respectively. From the above definition we get the equation of motion

$$\frac{\partial L}{\partial \Psi_{i}} - D_{\mu} \left( \frac{\partial L}{\partial (D_{\mu} \Psi_{i})} \right) = 0. \quad (3.24)$$

We can write the equation (3.19) in component fields

$$L^{W}_{GG} = \sum_{k, l=0}^{2} K_{N} S_{kl} \left( - \frac{1}{4} F_{k,\mu \nu} F_{l}^{\mu \nu} + \sum_{j} t_{jk} (\nabla_{\mu} \phi)_{j}^{\dagger} (\nabla^{\mu} \phi)_{l} + \lambda \left[ \phi^{\dagger}, [\phi, \phi] \right] \right). \quad (3.25)$$

From equation (3.23), we can define the energy-momentum tensor as
\[ T_{\mu\nu} \equiv \sum_{l} \sum_{(i)=1}^{n} \left( D_{\mu} \Psi_{l}^{(i)} \right) \frac{\partial L}{\partial \left( D_{\nu} \Psi_{l}^{(i)} \right)} - \eta_{\mu\nu} L, \quad (i) = 1, 2, \ldots n, \]  
\[ (3.26) \]

where index \((i)\) are numbers of fields and \(\eta_{\mu\nu} = diag(-1, 1, 1, 1)\) is the metric tensor. Then by equation (3.24), we get the equation of motion for the Lagrangian (3.25)

\[- \frac{1}{2} \sum_{k,l} K_{N}(S_{kl}^{W} + c_{kl}S_{lk}^{W}) \left[ \delta_{ik} D_{\nu} F_{l}^{\nu\rho} - i g \sum_{p} d_{kip} A_{p,\nu} F_{l}^{\nu\rho} \right] = j_{\rho}^{i}, \]  
\[ (3.27) \]

where

\[ j_{\rho}^{i} = - ig \sum_{k,l} K_{N} S_{kl}^{W} \left[ \sum_{j} t_{jk} \left( \sum_{m,n} d_{jmn} t_{im} \phi_{l}^{\dagger} \nabla_{\rho} \phi \right) + \delta_{ik} D_{\nu} F_{l}^{\nu\rho} \right. \]  
\[ \left. - \sum_{n} d_{lin} \left( (D_{\rho} \phi_{j}^{\dagger}) c_{ji} - i g \sum_{p,q} d_{jpq} c_{qi} \phi_{l}^{\dagger} A_{p,\nu}^{\dagger} \phi_{q} \right) \right] \]  
\[ (3.28) \]

If we compare with the equation of motion for the Lagrangian (3.22), that is

\[ \sum_{k,l} K_{N}(S_{kl}^{W} + c_{kl}S_{lk}^{W}) \left[ \frac{1}{2} \delta_{ik} D_{\nu} F_{l}^{\nu\rho} - i g \sum_{p} d_{kip} A_{p,\nu} F_{l}^{\nu\rho} \right] \left. \right] + \frac{\theta e^{2}}{16\pi^{2}} \left( \delta_{ik} D_{\nu} \tilde{F}_{l}^{\nu\rho} - i g \sum_{p} d_{kip} A_{p,\nu} \tilde{F}_{l}^{\nu\rho} \right) \]  
\[ = j_{\rho}^{i}, \]  
\[ (3.29) \]

where \(j_{\rho}^{i}\) is the same as (3.28), except there is an additional \(\theta\)-term. If we want to preserve Witten’s theory\[9\], i.e., the \(\theta\)-term does not affect the equation of motion, then we must impose the constraint

\[ \sum_{k,l} K_{N}(S_{kl}^{W} + c_{kl}S_{lk}^{W}) \left( \delta_{ik} D_{\nu} \tilde{F}_{l}^{\nu\rho} - i g \sum_{p} d_{kip} A_{p,\nu} \tilde{F}_{l}^{\nu\rho} \right) = 0. \]  
\[ (3.30) \]

13
We called the equation (3.30) as the Bianchi constraint.

In the case of classical Lie group, the Bianchi identity, $\nabla_\mu \tilde{F}^{\mu\nu} = 0$, where $\nabla_\mu = \partial_\mu + igA_\mu$, and the Bianchi constraint, equation (3.30), coincides. But in the case of quantum group they are different, because the Bianchi identity comes from the geometry while the Bianchi constraint comes from the variation of Lagrangian. The relation between them is not clear until now.

We see from equations (3.25) until (3.30), there are always appear quantities $K_N S^W_{kl}$ which are always independent of the choice of the representation of the gauge group and a noncommutative factor $c_{lk}$. We will see later that both quantities are also appear in the parameter of the vacuum manifold, i.e. $u_2$, in the field strength corresponding with unbroken subgroups, and in the BPS bound mass.

### IV. BPS States and Vacuum Manifold of the Model

#### A. Vacuum Manifold of the Model

In this section we begin to find the vacuum configuration in this theory. We start with the energy-momentum tensor of the model which can be derived from Lagrangian (3.25), that is (without the $\theta$-term)

$$ T_{\mu\nu} = \sum_{p,q,l} K_N S^W_{pq} \left[ -\frac{1}{2} (\delta_{lp} + c_{lp}\delta_{lq}) (D_\mu A_\rho^l) F_{q,\nu\rho} \right. $$

$$ + \sum_j t_{jp} ((D_\mu \phi)_l (\nabla_\nu \phi)_j^\dagger \delta_{lq} + (D_\mu \phi)_l^\dagger (\nabla_\nu \phi)_q \delta_{lj}) - \eta_{\mu\nu} L_{GG}. \right. \tag{4.1} $$

Then we define a norm denoted by $\| \|$, i.e., $\| \| : F_{nc} \to \mathbb{R}$, where $F_{nc}$ is a noncommutative field and $\mathbb{R}$ is a real field, such that

$$ \| T_{\mu\nu} \| \geq 0, \tag{4.2} $$

and it vanishes only if

$$ F^\mu_{\alpha\nu} = 0, \quad \nabla_\mu \phi = 0, \quad V(\phi) = 0. \tag{4.3} $$
The first equation in (4.3) implies that in the vacuum, $F^\mu\nu_a$ is pure gauge and the last two equations define the Higgs vacuum. The structure of the space of vacua is determined by

$$V(\phi) = \sum_{k,l} K_N S_{kl}^W [\phi^\dagger, \phi]_k [\phi^\dagger, \phi]_l = 0. \quad (4.4)$$

Therefore, the Higgs vacuum is defined by $[\phi^\dagger, \phi] = 0$, which implies that $\phi$ takes values in the Cartan subalgebra of the gauge group $SU_q(n)$. We denote by $U_q(1)^{n-1}$ a subgroup of $SU_q(n)$, which is generated by elements of the Cartan subalgebra of the gauge group $SU_q(n)$. It is clear that $U_q(1)^{n-1}$ is the unbroken subgroup of $SU_q(n)$ which keeps the Higgs vacuum invariant.

There the Georgi-Glashow model has a family of vacuum states. Vacuum manifold, which is formed by the potential (4.4), parameterized by gauge invariant quantities. For this model, we have the gauge invariant quantity parameterizing the space of vacua, that is

$$u_n = K_N \text{tr}(\sigma^N \left( [\phi^W]^T \right)^n), \quad (4.5)$$

which is similar to Seiberg-Witten theory [10]. For $SU_q(2)$ gauge group, the parameter $u_n$ in (4.5) is

$$u_2 = K_N \text{tr}(\sigma^N \left( [\phi^W]^T \right)^2). \quad (4.6)$$

As we mention above, if we write the above equations in their components, then quantities $K_N S_{kl}$ appears in the parameter of the vacuum manifold. Then, up to a gauge transformation, we can take $\phi = a\chi_1$, so the parameter $u_2$ in (4.6) becomes

$$u_2 = \frac{1}{8} \left(1 + q^2\right) a^2 \equiv u, \quad (4.7)$$

where $a \in \mathbb{C}_{nc}$ and $\mathbb{C}_{nc}$ is the noncommutative complex field.

If we take values of $\phi$ in Cartan subalgebra, i.e., $\phi = a\chi_1$, then the Lagrangian (3.25) becomes

$$L_{GG} = -2q K_N S_{02}^W \left[ A_{2,\mu}^* (D_\nu D^\nu) A_{2,\mu}^\dagger + \frac{1}{2} A_{2,\mu}^* (q^{-6} (1 + q^2))^2 a^* a A_{2,\mu}^\dagger \right] \quad (4.8)$$

$$-2q^{-1} K_N S_{20}^W \left[ A_{0,\mu}^* (D_\nu D^\nu) A_{0,\mu}^\dagger + \frac{1}{2} A_{0,\mu}^* (q^6 (1 + q^2))^2 a^* a A_{0,\mu}^\dagger \right]$$
+2K_NS_{11}^{W}A_{1,\mu}^{*}(D_{\nu}D^{\nu})A_{i}^{\mu} + ...$

where dots denote higher order terms. From the above Lagrangian we can read off the masses of the gauge fields as follow

\[
m_0 = q^6 (1 + q^2) \|a^* a\|^{1/2} \quad (4.9)
m_1 = 0
m_2 = q^{-6} (1 + q^2) \|a^* a\|^{1/2}.
\]

B. \(U_q(1)^{n-1}\)-Field Strength and BPS States

1. \(U_q(1)^{n-1}\)-Field Strength

In this subsection, we derive the solution of the second equation in (4.3), then find the field strength corresponding to the unbroken part of the gauge group, i.e., \(U_q(1)^{n-1}\).

Let \(\phi_i^{(v)}\) denote the field \(\phi_i\) in a Higgs vacuum. It then satisfies the equations

\[
\left[ \phi_i^{(v)} \right]^{\dagger} = 0,
D_\mu \phi_i^{(v)} + ig \left[ A_\mu, \phi_i^{(v)} \right] = 0. \quad (4.10)
\]

We find that the solution of the second equation in (4.9) is

\[
A_{j,\mu} = \frac{1}{ig} \left( -\sum_{p,q,i} (M^{-1})_{jp} d_{pq} \phi_q^{(v)} \right) (D_\mu \phi_i^{(v)} + \sum_{p,k} (M^{-1})_{jp} M'_{pk} \phi_k^{(v)}) A_\mu \right),
\quad (4.11)
\]

where

\[
M_{pq} = \sum_{q,i,k} d_{pq} d_{ij} c_{kj} \phi_q^{(v)} \phi_k^{(v)}, \quad (4.12)
\]

\[
M'_{pq} = \sum_{q,i,j,l} t_{kj} d_{pq} d_{ij} c_{kl} \phi_q^{(v)} \phi_l^{(v)}, \quad (4.13)
\]
and $M^{-1}$ is the inverse of $M$ with $\det fM = \sum_{\sigma} (-f(c_{i(\sigma)}))^{l(\sigma)}M^\dagger_{\sigma(1)}...M^\dagger_{\sigma(3)}$, where $l(\sigma)$ is the minimal number of inversions in permutation $\sigma$ [1].

If we define

$$h_{j,\mu}(\phi^{(v)}, \phi^{(v)^\dagger}, D_\mu \phi^{(v)}) \equiv \sum_{p,q,l} (M^{-1})_{jp} d_{pqi} \phi^{(v)^\dagger}_q D_\mu \phi^{(v)}_l;$$

$$\tilde{g}_j(\phi^{(v)}, \phi^{(v)^\dagger}) \equiv \sum_{p,k} (M^{-1})_{jp} M'_{pk} \phi^{(v)^\dagger}_k,$$

then we get

$$f_{j,\mu\nu} = -\frac{1}{ig}(D_\mu h_{j,\nu} - D_\nu h_{j,\mu}) + \frac{1}{ig} \sum_{k,l} d_{jkl} h_{k,\mu} A_\nu + \frac{1}{ig} \tilde{g}_j (D_\mu A_\nu - D_\nu A_\mu),$$

with constraints

$$\sum_{k,l} d_{jkl} \tilde{g}_k A_\mu \tilde{g}_l A_\nu = 0,$$

$$D_\mu \tilde{g}_j + \frac{1}{ig} \sum_{k,l} d_{jkl} h_{k,\mu} \tilde{g}_l = 0.$$

The field strength $F_{\mu\nu}$ corresponding to the unbroken part of $SU_q(n)$ can be identified as

$$F_{\mu\nu} = K_N tr \left( \sigma^N \left( \phi^{(v)W} \right)^\dagger \left( F^W_{\mu\nu} \right)^\dagger \right),$$

$$= \sum_{k,l} K_N S^{W\phi^{(v)}}_{kl} F_{l,\mu\nu}.$$

2. BPS States

In this subsection, we derive the Bogomol’nyi bound[14] on the mass of dyon in terms of its electric and magnetic charge, which are sources for the equation (4.18). We define the electric and magnetic charge as

$$q \equiv \oint \phi E_a dS^a = \int_X \nabla_a E^a \cdot d^3x,$$
\[
g \equiv \oint_{\partial X} B_a dS^a = \int_X \nabla_a B^a d^3x, \quad a = 1, 2, 3, \quad (4.20)
\]
respectively, where \( X \) is a manifold and \( \partial X \) is the boundary of \( X \). From equation (4.18) and the Bianchi identity, \( \nabla_{\mu} \tilde{F}^{\mu\nu} = 0 \), the electric and magnetic charge can be written as
\[
q = \sum_{k,l} K_N S_{kl}^W \int [ (\nabla_a \phi_k) E_l^a + \phi_k (\nabla_a E_l^a) ] d^3x, \quad (4.21)
\]
\[
g = \sum_{k,l} K_N S_{kl}^W \int (\nabla_a \phi_k) B_l^a d^3x, \quad (4.22)
\]
where \( E_l^a = F_l^{0a} \) and \( B_l^a = -\frac{1}{2} \epsilon^{abc} F_{bc,l} \), \( a, b, c = 1, 2, 3 \).

Now the dyon mass is given by
\[
M \equiv \left\| \int T_{00} d^3x \right\| \geq \left\| \int L_{GGG} d^3x \right\| \quad (4.23)
\]
\[
\geq \left\| \sum_{k,l} K_N S_{kl}^W \left[ \sqrt{2} \int \left( \sum_j t_{jk} (\nabla_a \phi_j) E_l^a + c_{lk} (\nabla_a \phi_l) E_k^a \right) d^3x \sin \theta \right.ight.
\]
\[
\left. + \sqrt{2} \int \left( \sum_j t_{jk} (\nabla_a \phi_j) B_l^a + c_{lk} (\nabla_a \phi_l) B_k^a \right) d^3x \cos \theta \right] \right\|
\]
\[
\geq \left\| \sum_{k,l} K_N S_{lk}^W \left( \sqrt{2} c_{lk} \int \left[ (\nabla_a \phi_k) E_l^a d^3x \sin \theta + (\nabla_a \phi_k) B_l^a d^3x \cos \theta \right] \right) \right\|. \quad (4.24)
\]
We see that there exist \( U \) transformation such that \( (S^W)^T = S^W U \), where \( T \) denotes transpose of a matrix. Using this, the equation (4.23) can be written as
\[
M \geq \left\| \sum_{k,l} K_N U_{ji} S_{kj}^W \sqrt{2} c_{lk} \left[ \int (\nabla_a \phi_k) E_l^a d^3x \sin \theta + \int (\nabla_a \phi_k) B_l^a d^3x \cos \theta \right] \right\|. \quad (4.24)
\]
We propose that there exists \( \epsilon > 0 \) such that the above equation becomes
\[
M \geq \frac{1}{\epsilon} \left\| U \right\| \left\| c \right\| \left\| \sqrt{2} (q \sin \theta + g \cos \theta) \right\|, \quad (4.25)
\]
and it turns out that \( \epsilon = \left\| U \right\|_{q=1} \left\| c \right\|_{q=1} \) because the model will reduce to the \( SU(2) \) case when \( n = 2 \) and \( q = 1[5,6,7,11,12] \).
V. Conclusion and Outlook

In this paper, we have constructed the $SU_q(n)$ Georgi-Glashow model (also with $\theta$-term). The equation of motion, besides the fields, depends on quantities $K_N S_{kl}^W$ which are independent of the representation of the gauge group and a noncommutative factor $c_{ik}$. In the case of classical Lie group, the Bianchi identity and the Bianchi constraint, equation (3.30), coincides. But in the case of quantum group they are different, because the Bianchi identity comes from the geometry while the Bianchi constraint comes from the variation of Lagrangian. The relation between them is not clear until now. We break the gauge symmetry spontaneously and this gives rise to the masses of gauge field which depend on $q$ and $a$, where $a$ is the vacuum parameter. The vacuum manifold is parameterized by the gauge invariant quantity which depends on a scalar field $\phi$. We get the field strength corresponding to the unbroken subgroup, $U_q(1)^{n-1}$, and the $q$-dependent BPS bound mass.

For further work, we extend the problem to the supersymmetric case, especially the Seiberg-Witten theory.[13]

VI. Acknowledgement

One of us (F.Z) would like to thank YANBINBANG SDM-IPTEK (Habibie Foundation) for financial support. Work of F.Z and B.G is supported by Hibah Bersaing VII/2, 1999-2000 project of DIKTI, Minister of Education and Culture of the Republic of Indonesia. F.Z, B.G and R.M thank the Abdus Salam International Centre for Theoretical Physics for their warm hospitality during our visit, where we initiated this project. We also thank P.Silaban for his encouragement.

VII. References

1. V. G. Drinfel’d, in Proceedings of the International Congress of Math (Berkeley, CA, USA, 1986), p.286.
2. M. Jimbo, Lett. Math. Phys. 10 (1985), p.63.
3. S. L. Woronowicz, Publ. RIMS, Kyoto Univ. 23 (1987), p.117.
4. P. Podles and S. L. Woronowicz, Commun. Math. Phys. 130 (1990),
5. G. ’t Hooft, Nuc. Phys. B79 (1974) p.276, A. M. Polyakov, JETP Lett. 20 (1974), p.194.
6. B. Julia and A. Zee, Phys. Rev. D11 (1975) p.2227.
7. L. Alvarez-Gaume and S. F. Hassan, CERN-TH/96-371.
8. M. Hirayama, Prog. Theor. Phys. 88 (1992) p.111.
9. E. Witten, Phys. Lett. 86B (1979) p.283.
10. N. Seiberg and E. Witten, Nuc. Phys. B426 (1994) p.19, N. Seiberg and E. Witten, Nuc. Phys. B431 (1994) p.484.
11. E. B. Bogomol’nyi, Sov. J. Nuc. Phys. 24 (1976) p.449.
12. M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35 (1975) p.760.
13. F. P. Zen, B. E. Gunara, D. P. Hutasoit, R. Muhamad, in progress.