SADDLE-POINT THEOREMS
FOR THE GENERALIZED CONE-CONVEX OPTIMIZATIONS
OF SET-VALUED FUNCTIONS

Renying Zeng

School of Mathematical Sciences, Chongqing Normal University, Chongqing, China
Email: renying.zeng@saskpolytech.ca

Abstract: This paper works with preconvexlike set-valued vector optimization problems in topological linear spaces. A Farkas-Minkowski alternative theorem, a scalarization theorem, some vector saddle-point theorems and some scalar saddle point theorem are proved.

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1. Introduction and Preliminary Results

Convexity and generalized convexities have played very important roles in optimization (including mathematical programming, and multi-objective mathematical programming). Convexity and generalized convexities are also very important in other areas such as multi-criteria decision, non-smooth analysis, and control theory, etc. In this paper, we work with preconvexlike vector optimization problems of set-valued mappings in topological linear spaces.

Let $X$ be a real topological linear space, $X^*$ the topology dual of $X$. A subset $X_+$ of $X$ is said to be a convex cone if

$$\alpha x^1 + \beta x^2 \in X_+, \forall x^1, x^2 \in X_+, \forall \alpha, \beta \geq 0.$$ 

A real topological linear space $Y$ with a convex cone is said to be an ordered topological linear space. We denote $int X_+$ the topological interior of $X_+$. The partial order on $X$ is defined by

$$x^1 \leq x^2, \text{ if } x^1 - x^2 \in X_+,$$
$$x^1 < x^2, \text{ if } x^1 - x^2 \in int X_+.$$ 

The subset of $X^*$

$$X^*_+ = \{ \xi \in X^* : \langle x, \xi \rangle \geq 0, \forall x \in X_+ \}$$

is said to be the dual cone of the cone $X_+$, where $\langle x, \xi \rangle = \xi(x)$.
Suppose that $X$ and $Y$ are two real topological linear spaces. Let $f: X \rightarrow 2^Y$ be a set-valued function, where $2^Y$ denotes the power set of $Y$.

Let $D$ be a nonempty subset of $X$. Setting

$$f(D) = \bigcup_{x \in D} f(x),$$
$$\langle f(x), \eta \rangle = \{\{y, \eta\} : y \in f(x)\},$$
$$\langle f(D), \eta \rangle = \bigcup_{x \in D} \langle f(x), \eta \rangle.$$

For $x \in D, \eta \in Y^*$, write

$$\langle f(x), \eta \rangle \geq 0, \text{ if } \langle y, \eta \rangle \geq 0, \forall y \in f(x),$$
$$\langle f(D), \eta \rangle \geq 0, \text{ if } \langle f(x), \eta \rangle \geq 0, \forall x \in D.$$

A set-valued function $f: X \rightarrow 2^Y$ is said to be $Y_+$-convex on $D$ if $\forall x^1, x^2 \in D, \forall \alpha \in [0, 1]$, one has

$$\alpha f(x^1) + (1-\alpha) f(x^2) \subseteq f(\alpha x^1 + (1-\alpha) x^2) + Y_+.$$

A set-valued function $f: X \rightarrow 2^Y$ is said to be $Y_+$-convexlike on $D$ if $\forall x^1, x^2 \in D, \forall \alpha \in [0, 1]$, $\exists x^3 \in D$ such that

$$\alpha f(x^1) + (1-\alpha) f(x^2) \subseteq f(x^3) + Y_+.$$

We introduce below the concept of preconvexlike about vector-valued functions to set-valued functions.

**Definition 1.1** A set-valued function $f: X \rightarrow 2^Y$ is said to be $Y_+$-preconvexlike on $D$ if $\forall x^1, x^2 \in D, \forall \alpha \in (0, 1)$, $\exists x^3 \in D$, and $\exists \tau > 0$, such that

$$\alpha f(x^1) + (1-\alpha) f(x^2) \subseteq \tau f(x^3) + Y_+.$$

Any set- and real-valued function is convexlike, it is known that a set-valued convexlike function is not necessary to be a convex function.

Let $X = Y = R^2$ (the two dimensional Euclidean space), $Y_+$ be the first quadrant of $R^2$. Example 1.1 illustrates that a set-valued preconvexlike function is not necessary to be a convexlike function.

**Example 1.1** Take $D = [Y_+ \setminus \{(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}] \cup \{(0,1),(1,0)\}$, and define a set-valued function $f: X \rightarrow 2^Y$ by

$$f(x_1, x_2) = D.$$

Then $f$ is preconvexlike not convexlike.
We also introduce the concept of preaffine set-valued functions here, which extends the definition of affine functions.

**Definition 1.2** A set-valued function $f: X \rightarrow 2^Y$ is said to be preaffine on $D$ if $\forall x^1, x^2 \in D$, $\forall \alpha \in (0, 1)$, $\exists x^3 \in D$, and $\exists \tau > 0$, such that

$$\alpha f(x^1) + (1 - \alpha) f(x^2) \subseteq \tau f(x^3).$$

Assume that $X$, $Y$, $Z$ and $W$ are topological linear spaces with convex cones $X_+, Y_+, Z_+$, and $W_+$ respectively; $X_+, Y_+, Z_+$, and $W_+$ have nonempty interiors $\text{int} \, X_+$, $\text{int} \, Y_+$, $\text{int} \, Z_+$, and $\text{int} \, W_+$.

**Lemma 1.1** [12] Suppose that $Y$ is a topological linear space. If $\xi \in X^* \setminus \{O\}$, and $x^0 \in \text{int} \, X_+$, then $\langle x^0, \xi \rangle > 0$.

**Theorem 1.2** (Fakas-Minkowski Theorem of Alternate) Suppose

(a) $f: X \rightarrow 2^Y$ is $Y_+$-preconvexlike on $D$,
(b) $h: X \rightarrow 2^W$ is preaffine on $D$, and $\text{int} \, [h(D)] \neq \emptyset$,

and, (i) and (ii) denote the systems

(i) $\exists x^0 \in D$ such that $f(x^0) \cap (-\text{int} \, Y_+) \neq \emptyset$, and $g(x^0) \cap (-Z_+) \neq \emptyset$, and $O \in h(x^0)$, where $O$ is the zero element of a topological linear space;

(ii) $\exists (\xi, \eta, \zeta) \in Y^*_+ \times Z^*_+ \times W^*_+$, $(\xi, \eta, \zeta) \neq O$, such that

$$\xi(f(x)) + \eta(g(x)) + \zeta(h(x)) \geq 0, \forall x \in D.$$

If (i) has no solution then (ii) has a solution.

If (ii) has a solution with $\xi \neq O$, then (i) has no solution.

**Proof.** At first, aim to show that the Cartesian product

$$A = [\bigcup_{i=1}^{3} tf(D) + \text{int} \, Y_i] \times [\bigcup_{i=0}^{2} fg(D) + Y_i] \times [\bigcup_{i=0}^{2} th(D)]$$

is convex.

Take $c^i = (t^i, f(x^i) + y^i, t^i g(x^i) + z^i, t^i h(x^i)) \in A$ where $(y^i, z^i) \in (\text{int} \, Y_i) \times Z_i$, $t^i, t^i g(x^i) + z^i, t^i h(x^i)$ are positive numbers $(i = 1, 2, 3)$. For $\alpha \in (0, 1)$, set $y^0 = \alpha y^1 + (1 - \alpha) y^2$, $z^0 = \alpha z^1 + (1 - \alpha) z^2$. Noting that $\text{int} \, Y_i$ and $Z_i$ are convex cones, we have $y^0 \in \text{int} \, Y_i$, $z^0 \in Z_i$. Since $f$ and $g$ are preconvexlike and $g$ is preaffine, $\exists x_3', x_3'', x_3''' \in D$, and $\exists \tau', \tau'', \tau''' > 0$, such that
Similarly
\[
\alpha_1'\, g(x^1) + (1-\alpha)\xi_2\, g(x^2) + \alpha z^1 + (1-\alpha)z^2
\]
\[
\subseteq t'' g(x_3^2) + Z_+ + Z_+ \subseteq t'' g(x_3^2) + Z_+.
\] (1.2)

And
\[
\alpha_1''\, h(x^1) + (1-\alpha)\xi_2\, h(x^2) \subseteq t''' h(x_3^3) \subseteq t''' h(x_3^3).
\] (1.3)

From (1.1), (1.2) and (1.3), if \( c^1, c^2 \in A \), we have
\[
\alpha c^1 + (1-\alpha)c^2 \in [t' f(x_3') + \text{int} Y_+] \times [t'' g(x_3'') + Z_+] \times [t''' h(x_3''')] \subseteq A
\]
which means \( A \) is convex.

Secondly, suppose that (i) has no solution, and want to show (ii) has a solution.
Because (i) has no solution, \( O \not\subseteq A \).
And because we assume that \( \text{int} Y_+ \neq \emptyset \), \( \text{int} Z_+ \neq \emptyset \), and \( \text{int}[ h(D)] \neq \emptyset \), we have
\( \text{int} A \neq \emptyset \).

By the separation theorem of convex sets of a topological linear space
\[
\exists \gamma = (\xi, \eta, \zeta) \in Y^* \times Z^* \times W^*, \gamma = (\xi, \eta, \zeta) \neq O, \quad (1.4)
\]
such that
\[
\gamma(A) \geq 0,
\]
i.e., \( \forall t', t'', t''' > 0, \forall y^0 \in \text{int} Y_+, \forall z^0 \in Z_+ \), we have
\[
\left\langle t' f(x) + y^0, \xi \right\rangle + \left\langle t'' g(x) + z^0, \eta \right\rangle + \left\langle t''' h(x), \zeta \right\rangle \geq 0, \forall x \in D.
\]

Noting that \( \forall y \in Y_+, \exists y_n \in \text{int} Y_+, (n = 1, 2,...) \) such that \( y_n \to y(n \to \infty) \), and \( \forall s' > 0, s'' > 0, s'y_n \in \text{int} Y_+, s''z^0 \in Z_+ \), we have
\[
\left\langle t' f(x) + s'y_n, \xi \right\rangle + \left\langle t'' g(x) + s''z^0, \eta \right\rangle + \left\langle t''' h(x), \zeta \right\rangle \geq 0, \forall x \in D.
\] (1.5)

Let \( t' = t'' = s' = s'' = 1, t''' = 0, n \to \infty \), we get \( \left\langle y, \xi \right\rangle + \left\langle z^0, \eta \right\rangle \geq 0, \forall y \in Y_+, \forall z^0 \in Z_+ \).

Therefore
\[
(\xi, \eta, \zeta) \in Y^*_+ \times Z^*_+ \times W^*_+.
\]

Let \( t' = t'' = t''' = 1, s', s'' \to 0 \) in (1.5) we will get
\[
\left\langle f(x), \xi \right\rangle + \left\langle g(x), \eta \right\rangle + \left\langle h(x), \zeta \right\rangle \geq 0, \forall x \in D.
\]

Which shows us that (ii) has a solution.

Thirdly, assume that (ii) has a solution \((\xi, \eta, \zeta)\) with \( \xi \neq O \), i.e.
\[
\left\langle f(x), \xi \right\rangle + \left\langle g(x), \eta \right\rangle + \left\langle h(x), \zeta \right\rangle \geq 0, \forall x \in D.
\]

We are going to prove (i) has no solution.

Otherwise, if (i) has a solution \( x \in D \), there would exist \( y \in f(x), z \in g(x) \), and \( w \in h(x) \) such that \( y \in \text{int} Y_+, z \in -Z_+, w = O \). By Lemma 1.1 we would have
\[
\left\langle y, \xi \right\rangle + \left\langle z, \eta \right\rangle + \left\langle w, \zeta \right\rangle < 0, \forall x \in D.
\]

Which is a contradiction. \( \square \)

2. Scalarization

Consider the following vector optimization problem with set-valued functions:

\[
\begin{align*}
\text{(VP)} & \quad Y_+ \text{ - min } f(x), \\
& \quad s.t., g(x) \cap (-Z_+) \neq \emptyset, 0 \notin h(x), \\
& \quad x \in X.
\end{align*}
\]
Let $D$ be the feasible set of (VP), i.e., $D = \{x \in X : g(x) \cap (-Y_+) \neq \emptyset, 0 \in h(x)\}$.

**Definition 2.1** $\bar{x} \in D$ is said to be a weakly efficient solution of (VP) if $\exists \bar{y} \in f(\bar{x})$ such that $(\bar{y} - f(D)) \cap \text{int}Y_+ = \emptyset$.

**Definition 2.2** The problem (VP) is said to satisfy the Slater constraint qualification (SC) if $\forall (\eta, \zeta) \in (Z_+^* \times W^*) \setminus \{0\}, \exists x \in D$ such that

$(-R_+) \cap [\eta(g(x)) \cap \zeta(h(x))] \neq \emptyset$,

where $R_+$ is the set of all positive real numbers.

**Definition 2.3** $\bar{x} \in D$ is said to be an optimal solution of the scalar optimization problem (VPS), if $\exists \bar{y} \in f(\bar{x})$, and $\exists y_+ \in Y_+$ such that

$\xi(f(x)) \subseteq \xi(\bar{y}) + Y_+, \forall x \in D$.

By use of the following Theorem of Scalarization, we may convert a vector optimization problem into a scalar optimization problem.

**Theorem 2.1** (Scalarization) Suppose $\bar{x} \in D$, and

(a) $f(x) - f(\bar{x}) : X \rightarrow 2^\mathbb{Y}$ is $\mathbb{Y}_+$-preconvexlike on $D$, and $g(x) : Y \rightarrow 2^Z$ is $\mathbb{Z}_+$-preconvexlike on $D$,

(b) $h(x) : X \rightarrow 2^W$ is preaffine on $D$, and

(c) (VP) satisfies the Slater constraint qualification (SC),

then $\bar{x}$ is a weakly efficient solution of (VP) if and only if $\exists \xi \in Y_+^* \setminus \{0\}$ such that $\bar{x}$ is an optimal solution of the following scalar optimization problem (VPS):

$$(\text{VPS}) \quad \min_{x \in D} \xi(f(x)).$$

**Proof.** If $\exists \xi \in Y_+^* \setminus \{0\}$ and if $\bar{x} \in D$ is an optimal solution of the scalar optimization problem (VPS), then $\exists \bar{y} \in f(\bar{x})$, such that $\xi(f(x)) \subseteq \xi(\bar{y}) + Y_+, \forall x \in D$. So

$$(\bar{y} - f(D)) \cap \text{int}Y_+ = \emptyset.$$ 

Therefore $\bar{x}$ is a weakly efficient solution of (VP).

On the other hand, suppose $\bar{x}$ is a weakly efficient solution of (VP), we want to show that $\bar{x}$ is an optimal solution of the scalar optimization problem (VPS).

From Definition 2.1 $\exists \bar{y} \in f(\bar{x})$ such that the following system

$$(f(x) - \bar{y}) \cap (-\text{int}Y_+) \neq \emptyset, g(x) \cap (-Z_+) \neq \emptyset, 0 \in h(x)$$


has no solution for \( x \in D \). Hence, Theorem 1.2 implies

\[
\exists \xi \in Y^*, \eta \in Z^*, \zeta \in W^* \text{ with } (\xi, \eta, \zeta) \neq O
\]  

(2.1)

such that

\[
\xi(f(x) - \bar{y}) + \eta(g(x)) + \zeta(h(x)) \geq 0, \forall x \in D.
\]

i.e.,

\[
\xi(f(x)) + \eta(g(x)) + \zeta(h(x)) \geq \zeta(\bar{y}), \forall x \in D. \tag{2.2}
\]

If \( \xi = 0 \), then by (2.1) we get \((\xi, \eta) \neq O\). And (2.2) yields

\[
\eta(g(x)) + \zeta(h(x)) \geq 0, \forall x \in D.
\]

This is contradicting to the Slater constraint qualification (SC). Therefore \( \xi \neq O \).

Hence, from \( \bar{x} \in D \), i.e., \( g(x) \cap (-Z^+) \neq \emptyset, O \in h(x) \), (2.1) gives

\[
\xi(f(x)) \geq \zeta(\bar{y}), \forall x \in D.
\]

Which means \( \bar{x} \) is an optimal solution of (VPS).

3. Vector Saddle-Point Theorems

Write

\[
P\min[A, Y_+] = \{ y \in A : (v - A) \cap \text{int} Y_+ = \emptyset \},
\]

\[
P\max[A, Y_+] = \{ y \in A : (A - y) \cap \text{int} Y_+ = \emptyset \}.
\]

We familiar with that

**Lemma 3.1** \( \bar{x} \in D \) is a weakly efficient solution of (VP), if and only if

\[
f(\bar{x}) \cap P\min[f(D), Y_+] \neq \emptyset.
\]

**Definition 3.1** A triple \((\bar{x}, \bar{S}, \bar{T}) \in X \times B^+ (Z, Y) \times B(W, Y)\) is said to be a vector saddle-point of \( L \) if

\[
L(\bar{x}, \bar{S}, \bar{T}) \cap P\min[L(X, \bar{S}, \bar{T}), Y_+] \cap P\max[L(\bar{x}, B^+ (Z, Y), B(W, Y)), Y_+] \neq \emptyset.
\]

Where

\[
L(\bar{x}, \bar{S}, \bar{T}) = f(\bar{x}) + \bar{S}(g(\bar{x})) + \bar{T}(h(\bar{x})).
\]

**Definition 3.1** A convex cone \( Y_+ \) is said to be pointed if \( Y_+ \cap (-Y_+) = \{O\} \).
From now on, suppose all convex cones are pointed and closed.

**Theorem 3.1** A triple \((\bar{x}, \bar{y}, \bar{z}) \in X \times B^+(Z, Y) \times B(W, Y)\) is a vector saddle-point of \(L\), if and only if \(\exists \bar{y} \in f(\bar{x}), \bar{z} \in g(\bar{x})\), such that

(i) \(\bar{y} \in P \min[L(X, \bar{y}, \bar{z}), Y_+]\),

(ii) \(g(\bar{x}) \subset -Z_+, h(\bar{x}) = \{O\}\),

(iii) \((f(\bar{x}) - \bar{y} - S(\bar{z})) \cap \text{int} Y_+ = \emptyset\).

**Proof.** The necessity. Assume that \((\bar{x}, \bar{y}, \bar{z}) \in X \times B^+(Z, Y) \times B(W, Y)\) is a vector saddle-point of \(L\). From Definition 3.1

\[L(\bar{x}, \bar{y}, \bar{z}) \cap P \min[L(X, \bar{y}, \bar{z}), Y_+] \cap P \max[L(\bar{x}, B^+(Z, Y) \times B(W, Y)), Y_+] \neq \emptyset.\]

So, \(\exists \bar{y} \in f(\bar{x}), \bar{z} \in g(\bar{x}), \bar{w} \in h(\bar{x})\), i.e.,

\[
\bar{y} + S(\bar{z}) + T(\bar{w}) \in L(\bar{x}, \bar{y}, \bar{z}) = f(\bar{x}) + S(g(\bar{x})) + T(h(\bar{x})), \tag{3.1}
\]

such that

\[
\{f(\bar{x}) + S(g(\bar{x})) + T(h(\bar{x})) - [\bar{y} + S(\bar{z}) + T(\bar{w})]\} \cap \text{int} Y_+ = \emptyset, \tag{3.2}
\]

and

\[(\bar{y} + S(\bar{z}) + T(\bar{w}) - [f(X) + S(g(X)) + T(h(X))]) \cap \text{int} Y_+ = \emptyset. \tag{3.3}\]

Taking \(T = \bar{T}\) in (3.2) we get

\[S(z) - S(\bar{z}) \notin \text{int} Y_+, \forall z \in g(\bar{x}), \forall S \in B^+(Z, Y). \tag{3.4}\]

Aim to show that \(-\bar{z} \in Z_+\).

Otherwise, since \(O \in -Z_+\), if \(-\bar{z} \notin Z_+\), we would have \(-\bar{z} \neq O\).

Because \(Z_+\) is a closed convex set, by the separate theorem \(\exists \eta \in Z^* \setminus \{O\}\)

\[\eta(z_+) > \eta(-\bar{z}), \forall z \in Z_+, \forall t > 0. \tag{3.5}\]

i.e.,

\[\eta(z_+) > \frac{1}{t} \eta(-\bar{z}), \forall z \in Z_+, \forall t > 0. \tag{3.6}\]

Let \(t \to \infty\) we obtain \(\eta(z_+) \geq 0, \forall z \in Z_+\). Which means that \(\eta \in Z_+^* \setminus \{O\}\). Meanwhile, \(O \in Z_+\) and (3.5) yield that \(\eta(\bar{z}) > 0\). Given \(\bar{z} \in \text{int} Z_+\) and let
\[ S(z) = \frac{\eta(z)}{\eta(\bar{z})} \bar{z} + \bar{S}(z). \]

Then \( \bar{S} \in B^+(Z,Y) \) and
\[ S(\bar{z}) - \bar{S}(\bar{z}) = \bar{z} \in \text{int } Y. \]

Contradicting to (3.4). Therefore
\[ -\bar{z} \in Z. \]

Now, aim to prove that \(-g(\bar{x}) \subseteq Z_+\).
Otherwise, if \(-g(\bar{x}) \not\subseteq Z_+\), then \( \exists z_0 \in g(\bar{x}) \) such that \( O \neq z_0 \not\in Z_+ \). Similar to the above \( \exists \eta_0 \in Z_+ \setminus \{O\} \) such that \( \eta_0 \in Z_+ \setminus \{O\}, \eta_0(z_0) > 0 \). Given \( \bar{z} \in \text{int } Z_+ \) and let
\[ S_0(z) = \frac{\eta_0(z)}{\eta_0(z_0)} \bar{z}. \]

Then \( S_0 \in B^+(Z,Y) \) and \( S_0(z_0) = \bar{z} \in \text{int } Y_+ \). And we have proved that \(-\bar{z} \in Z_+\), so
\[ -\bar{S}(\bar{z}) \in Y_+. \]
Therefore
\[ S_0(z_0) - \bar{S}(\bar{z}) \in \text{int } Y_+, Y_+ \subseteq \text{int } Y_+. \]

Again, contradicting to (3.4).

Therefore \(-g(\bar{x}) \subseteq Z_+\). Similarly, one has \(-h(\bar{x}) \subseteq W_+\). From (3.2) we get
\[ [T(h(\bar{x})) - T(\bar{w})] \cap \text{int } Y_+ = \emptyset. \]
Hence
\[ T(\bar{w}) - T(\bar{w}) \notin \text{int } Y_+, \forall T \in B(W,Y). \tag{3.6} \]

Similarly, from (3.2) again we have
\[ T(w) - T(\bar{w}) \notin \text{int } Y_+, \forall w \in h(\bar{x}), \forall T \in B(W,Y). \tag{3.7} \]

If \( \bar{w} \neq O \), since \(-h(\bar{x}) \subseteq W_+\) and \( W_+ \) is a pointed cone, we have \( \bar{w} \notin W_+\). Because \( Y_+ \) is a closed convex set, by the separation theorem \( \exists \zeta \in W^* \), such that
\[ \zeta(w) < \zeta(\bar{w}), \forall w \in W_+. \tag{3.8} \]
So \( \zeta(\bar{w}) \neq 0 \) since \( O \in W_+ \). Taking \( y^0 \in \text{int } Y_+ \) and define \( T^0 \in B^+(W,Y) \) by
\[
T^0(w) = \frac{\zeta(w)}{\bar{\zeta}(w)} y^0 + \bar{T}(w).
\]

Then
\[
T^0(\bar{w}) = y^0 \in \text{int} \ Y_+,
\]
Contradicting to (3.6). Therefore \( \bar{w} = O \). Thus
\[
O \in h(\bar{x}).
\]

Now, we’d like to prove \( h(\bar{x}) = \{O\} \).

Otherwise, if \( w^0 \in h(\bar{x}): w^0 \neq O \), similar to (3.8) \( \exists \zeta^0 \in W^* \), such that \( \zeta^0(w) < \zeta^0(w^0), \forall w \in W_+ \). So \( \zeta^0(w^0) \neq 0 \). Given \( y_0 \in \text{int} \ Y_+ \) and define \( T_0 \in B(W,Y) \), by
\[
T_0(w) = \frac{\zeta^0(w)}{\zeta^0(w^0)} y_0.
\]
Then \( T_0(w^0) = y_0 \in \text{int} \ Y_+ \), i.e., \( \bar{w} = O \) \( T^0(w^0) - \bar{T}(\bar{w}) \in \text{int} \ Y_\). Contradicting to (3.7). Therefore we must have
\[
h(\bar{x}) = \{O\}. \tag{3.9}
\]
Combining (3.2), (3.3), (3.9), and we conclude that
\[
\bar{y} \in P \min \{L(X,\bar{S},\bar{T}),Y_+\}, \tag{3.10}
\]
and
\[
(f(\bar{x}) - \bar{y} - \bar{S}(\bar{x})) \cap \text{int} \ Y_+ = \emptyset.
\]

We have proved that, if \( (\bar{x},\bar{S},\bar{T}) \in X \times B^+(Z,Y) \times B(W,Y) \) is a vector saddle-point of \( L \), then the conditions (i)-(iii) hold.

The sufficiency. Suppose that the conditions (i)-(iii) are satisfied. Note that \( -g(\bar{x}) \subseteq Z_+, h(\bar{x}) = \{O\} \) means
\[
-\bar{S}(g(\bar{x})) \subseteq Y_+, \quad T(h(\bar{x})) = \{O\}, \quad \forall (S,T) \in B^+(Z,Y) \times B(W,Y), \tag{3.11}
\]
and the condition (i) states that
\[
\{\bar{y} - [f(X) + \bar{S}(g(X)) + T(h(X))]\} \cap \text{int} \ Y_+ = \emptyset,
\]
So $Y_+ + \text{int} \ Y_+ \subseteq Y_+$ and $-S(\zeta) \in Y_+$ together imply

$$\{\gamma + \overline{S}(\zeta) + \overline{T}(\omega) - [f(X) + \overline{S}(g(X)) + \overline{T}(h(X))]\} \cap \text{int} \ Y_+ = \emptyset.$$ 

Hence

$$\gamma + \overline{S}(\zeta) + \overline{T}(\omega) \in P \min[L(X, \overline{S}, \overline{T}), Y_+] .$$

On the other hand, since $(f(\overline{x}) - [\gamma + S(\zeta)]) \cap \text{int} \ Y_+ = \emptyset$, from (3.11), and from $\text{int} \ Y_+ + \text{int} \ Y_+ \subseteq \text{int} \ Y_+$ we conclude that

$$\bigcup_{(x, T) \in B^+(Z, Y) \times B(W, Y)} \{f(\overline{x}) + S(g(\overline{x})) + T(h(\overline{x})) - [\gamma + \overline{S}(\zeta) + \overline{T}(\omega)]\} \cap \text{int} \ Y_+ = \emptyset .$$

Hence

$$\gamma + \overline{S}(\zeta) + \overline{T}(\omega) \in P \max[L(\overline{x}, B^+(Z, Y), B(W, Y)), Y_+]$$

Consequently,

$$L(\overline{x}, \overline{S}, \overline{T}) \cap P \min[L(X, \overline{S}, \overline{T}), Y_+] \cap P \max[L(\overline{x}, B^+(Z, Y), B(W, Y)), Y_+] \neq \emptyset.$$ 

Therefore $(\overline{x}, \overline{S}, \overline{T}) \in X \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L$. □

**Theorem 3.2** If $(\overline{x}, \overline{S}, \overline{T}) \in X \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L$, and if $O \in \overline{S}(g(\overline{x}))$, then $\overline{x}$ is a weak efficient solution of (VP).

**Proof.** Assume that $(\overline{x}, \overline{S}, \overline{T}) \in D \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L$, from Theorem 3.1 we have

$$-S(g(\overline{x})) \subseteq Y_+, h(\overline{x}) = \{O\} . \quad (3.12)$$

So $\overline{x} \in D$ (the feasible solution of (VP)). And $\exists \gamma \in f(\overline{x})$ such that $\gamma \in P \min[L(X, \overline{S}, \overline{T}), Y_+]$, i.e.

$$(\gamma - [f(X) + \overline{S}(g(X)) + \overline{T}(h(X))] \cap \text{int} \ Y_+ = \emptyset .$$

Thus

$$(\gamma - [f(D) + \overline{S}(g(\overline{x})) + \overline{T}(h(\overline{x}))] \cap \text{int} \ Y_+ = \emptyset .$$

By (3.12) and note that $O \in \overline{S}(g(\overline{x}))$ we get
Therefore by Definition 2.1 $\bar{x}$ is a weakly efficient solution of (VP). □

**Theorem 3.3** Suppose $\bar{x} \in D$ is a weakly efficient solution of (VP), and

(a) $f: X \to 2^Y$ is $Y_+$-preconvexlike on $D$, and $g: Y \to 2^Z$ is $Z_+$-preconvexlike on $D$,

(b) $h: X \to 2^W$ is preaffine on $D$, and $\text{int}[h(D)] \neq \emptyset$,

(c) (VP) satisfies the Slater constrained qualification (SC).

If $-g(\bar{x}) \subseteq Z_+$, $h(\bar{x}) = \{O\}$, and if $\exists \bar{y} \in f(\bar{x})$ for which $(f(\bar{x}) - \bar{y}) \cap \text{int} Y_+ = \emptyset$, then $(\bar{s}, \bar{t}) \in B^+(Z, Y) \times B(W, Y)$ such that $(\bar{x}, \bar{s}, \bar{t}) \in X \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L$ and $O \in \bar{s}(g(\bar{x}))$.

**Proof.** Assume that $\bar{x} \in D$ is a weakly efficient solution of (VP), then $\exists \bar{y} \in f(\bar{x})$ for which there is not any $x \in D$ such that $f(x) - \bar{y} \in -\text{int} Y_+$. So, there is not any $x \in X$ such that $f(x) - \bar{y} \in -\text{int} Y_+, g(x) \in -Z_+, O \in h(x)$.

By Theorem 1.2 $\exists (\xi, \eta, \xi) \in Y_+^* \times Z_+^* \times W^* \setminus \{O\}$ such that

$$
\xi(f(x) - \bar{y}) + \eta(g(x)) + \xi(h(x)) \geq 0, \forall x \in D. \quad (3.13)
$$

Since $\bar{y} \in f(\bar{x})$ and $O \in h(\bar{x})$, take $x = \bar{x}$ in the above we obtain $\eta(g(\bar{x})) \geq 0$. And $\bar{x} \in D$ and $\eta \in Z_+^*$ imply that $\exists \bar{y} \in g(\bar{x}) \cap (-Z_+)$ for which $\eta(\bar{y}) \leq 0$. Hence $\eta(\bar{x}) = 0$, which means that

$$
0 \in \eta(g(\bar{x})). \quad (3.14)
$$

Know that $x \in D$ means $O \in h(x)$; and $\exists \bar{z} \in g(x) \cap (-Z_+)$, which yields $\eta(z) \leq 0$. These and (3.13) deduce that $\xi(f(x) - \bar{y}) \geq 0, \forall x \in D$. According to the Slater constraint qualification, we have $\xi \neq O$. So we may take $y_0 \in \text{int} Y_+$ such that $\xi(y_0) = 1$. Define the operator $S: Z \to Y$ and $T: W \to Y$ by

$$
S(z) = \eta(z)y_0, T(w) = \xi(w)y_0. \quad (3.15)
$$

It is easy to see that

$$
S \in B^+(Z, Y), S(Z_+) = \eta(Z_+)y_0 \subseteq Y_+, T \in B(W, Y).
$$
And (3.14) implies that
\[ S(g(\bar{x})) = \eta(g(\bar{x}))y_0 \in 0 \cdot Y_+ = O. \] (3.16)

Since \( \bar{x} \in D \), we have \( O \in h(\bar{x}) \). Hence
\[ O \in T(h(\bar{x})). \] (3.17)

Therefore by (3.16) and (3.17)
\[ \bar{y} \in f(\bar{x}) \subseteq f(\bar{x}) + S(g(\bar{x})) + T(h(\bar{x})). \]

From (3.13) and (3.14)
\[ \xi[f(x) + S(g(x)) + T(h(x))] = \xi(f(x)) + \eta(g(x))\xi(y_0) + \zeta(h(x))\xi(y_0) = \xi(f(x)) + \eta(g(x)) + \zeta(h(x)) \geq \xi(\bar{y}), \forall x \in D, \]
i.e.,
\[ \xi[f(x) - \bar{y} + S(g(x)) + T(h(x))] \geq 0, \forall x \in D. \] (3.18)

Take \( F(x) = f(x) + S(g(x)) + T(h(x)), \ G(x) = \{O\} \) and \( H(x) = \{O\} \), applying Theorem 1.2 to the functions \( F(x) - \bar{y}, G(x), H(x) \), then (3.18) deduces that
\[ (\bar{y} - [f(D) + S(g(D)) + T(h(D))] \cap \text{int} \ Y_+ = \emptyset. \] (3.19)

Hence
\[ \bar{y} \in \text{P min}\{L(X, \overline{S}, \overline{T}), Y_+\}. \] (3.20)

On the other hand, since \( O \in \overline{S}(g(\bar{x})) \) (3.16), \( \exists \overline{z} \in g(\bar{x}) \) for which \( \overline{S}(\overline{z}) = O \). This and \( (f(\bar{x}) - \bar{y}) \cap \text{int} \ Y_+ = \emptyset \) together deduce that
\[ (f(\bar{x}) - \bar{y} - S(\overline{z})) \cap \text{int} \ Y_+ = \emptyset. \] (3.21)

Combining the assumption \( -g(\bar{x}) \subseteq Z_+, \ h(\bar{x}) = \{O\} \) and (3.20), (3.21), by Theorem 3.1 we conclude that \((\bar{x}, \overline{S}, \overline{T}) \in X \times B^*(Z,Y) \times B(W,Y)\) is a vector saddle-point of \( L \).

**Remark 3.1** For vector-valued functions, the condition \( g(\bar{x}) \subseteq -Z_+, \ h(\bar{x}) = \{O\} \) in Theorem 3.2, 3.3 are always satisfied if \( \bar{x} \in D \).
4. Scalar Saddle-Point Theorems

**Definition 4.1** Given \( \overline{\xi} \in Y^*_+ \setminus \{O\} \). The real-valued Lagrangian function of (VP) \( l_{\overline{\xi}} : X \times Z^*_+ \times W^* \rightarrow R \) is defined by
\[
l_{\overline{\xi}}(x, \eta, \zeta) = \overline{\xi}(f(x)) + \eta(g(x)) + \zeta(h(x)).
\]

**Definition 4.2** Given \( \overline{\xi} \in Y^*_+ \setminus \{O\} \). A triple \((\overline{x}, \overline{\eta}, \overline{\zeta})\) is said to be a scalar saddle-point of the Lagrangian function \( l_{\overline{\xi}} \), if
\[
l_{\overline{\xi}}(x, \eta, \zeta) \leq l_{\overline{\xi}}(\overline{x}, \overline{\eta}, \overline{\zeta}) \leq l_{\overline{\xi}}(x, \overline{\eta}, \overline{\zeta}),
\]
\(\forall x \in D, \forall (\eta, \zeta) \in Z^*_+ \times W^*\).

The definition of a scalar saddle-point is the common definition of a saddle-point. To compare with the concept of a vector saddle-point in the previous section of this paper, we call it here a scalar saddle-point.

**Theorem 4.1** Suppose \( \overline{x} \in D \), and

(a) \( f : X \rightarrow 2^Y \) is \( Y_+ \)-preconvexlike on \( D \), and \( g : Y \rightarrow 2^Z \) is \( Z_+ \)-preconvexlike on \( D \),
(b) \( h : X \rightarrow 2^W \) is preaffine on \( D \), and \( \text{int}[h(D)] \neq \emptyset \),
(c) (VP) satisfies the Slater constrained qualification (SC).

If \( \overline{x} \in D \) is a weakly efficient solution of (VP) for which \( g(\overline{x}) \subseteq -Z_+ \), \( h(\overline{x}) = \{O\} \), then \( \exists (\overline{\xi}, \overline{\eta}, \overline{\zeta}) \in (Y^*_+ \setminus \{O\}) \times Z^*_+ \times W^* \) such that \((\overline{x}, \overline{\eta}, \overline{\zeta})\) is a scalar saddle-point of the Lagrangian function \( l_{\overline{\xi}} \) and \( \overline{\eta}(g(\overline{x})) = \{0\} \).

**Proof.** Suppose that \( \overline{x} \in D \) is a weakly efficient solution of (VP). Similar to the proof of (3.18) in Theorem 3.3 (in fact, \( f(\overline{x}) \) takes place of \( \overline{y} \) from (3.12) to (3.18)) \( \exists (\overline{S}, \overline{T}) \in B^+(Z,Y) \times B(W,Y) \) such that
\[
O \in \overline{S}(g(\overline{x})), \quad (4.1)
\]
and (note that \( g(\overline{x}) \subseteq -Z_+ \), \( h(\overline{x}) = \{O\} \))
\[
\bar{\xi}(f(\bar{x})) + \bar{S}(g(\bar{x})) + \bar{T}(h(\bar{x})) \\
\leq \bar{\xi}(f(\bar{x})) \\
\leq \bar{\xi}[f(x)] + \bar{S}(g(x)) + \bar{T}(h(x)) \\
= \bar{\xi}(f(x)) + \bar{\xi} \circ \bar{S}(g(x)) + \bar{\xi} \circ \bar{T}(h(x)), \forall x \in D. \tag{4.2}
\]

Take \( \bar{\eta} = \bar{\xi} \circ \bar{S}, \bar{\zeta} = \bar{\xi} \circ \bar{T} \), then \((\bar{\eta}, \bar{\zeta}) \in Z_{+}^{*} \times W^{*} \). And from (4.2) we attain

\[
l_{\bar{\xi}}(\bar{x}, \bar{\eta}, \bar{\zeta}) \\
\leq \bar{\xi}(f(\bar{x})) + \bar{\eta}(g(\bar{x})) + \bar{\zeta}(h(\bar{x})) \\
= \bar{\xi}[f(\bar{x}) + \bar{S}(g(\bar{x})) + \bar{T}(h(\bar{x})] \\
\leq \bar{\xi}[f(x) + \bar{S}(g(x)) + \bar{T}(h(x)] \\
= \bar{\xi}(f(x)) + \bar{\eta}(g(x)) + \bar{\zeta}(h(x)) \\
= l_{\bar{\xi}}(x, \bar{\eta}, \bar{\zeta}), \forall x \in D. \tag{4.3}
\]

And

\[
\bar{\xi}(f(\bar{x})) + \bar{\eta}(g(\bar{x})) \leq \bar{\xi}(f(x)) + \bar{\eta}(g(x)) + \bar{\zeta}(h(x)), \forall x \in D. \tag{4.4}
\]

Since \( x \in D \) we have \( 0 \in \bar{\zeta}(h(x)) \), and (4.1) states that \( 0 \in \bar{\eta}(g(\bar{x})) \), therefore from (4.4)

\[
\bar{\xi}(f(\bar{x})) \leq \bar{\xi}(f(x)) + \bar{\eta}(g(x)), \forall x \in D.
\]

Taking \( x = \bar{x} \) in above we attain \( 0 \leq \bar{\eta}(g(\bar{x})) \). However, the assumption \( g(\bar{x}) \subseteq -Z_{+} \) implies that \( \bar{\eta}(g(\bar{x})) \leq 0 \). Thus

\[
\bar{\eta}(g(\bar{x})) = \{0\}. \tag{4.5}
\]

Hence from (4.5), noting that \( \eta(g(\bar{x})) \leq 0 \), and \( h(\bar{x}) = \{O\} \), we obtain

\[
l_{\bar{\xi}}(\bar{x}, \eta, \zeta) \\
= \bar{\xi}(f(\bar{x})) + \eta(g(\bar{x})) + \zeta(h(\bar{x})) \\
\leq \bar{\xi}(f(\bar{x})) + \bar{\eta}(g(\bar{x})) + \bar{\zeta}(h(\bar{x})) \\
= l_{\bar{\xi}}(\bar{x}, \bar{\eta}, \bar{\zeta}), \forall (\eta, \zeta) \in Y_{+}^{*} \times W^{*}. \tag{4.6}
\]

Combining (4.3) and (4.6), \((\bar{x}, \bar{\eta}, \bar{\zeta}) \) is a saddle-point of the Lagrangian function \( l_{\bar{\xi}} \). \( \square \)

**Remark 4.1** For vector-valued functions, the condition \( g(\bar{x}) \subseteq -Z_{+}, h(\bar{x}) = \{O\} \) in Theorem 4.1 are always satisfied if \( \bar{x} \in D \).
Theorem 4.2 Let $\bar{x} \in D$. If $\exists (\vec{\xi}, \vec{\eta}, \vec{\zeta}) \in (Y_+ \setminus \{O\}) \times Z_+^* \times W^*$ for which $(\bar{x}, \vec{\eta}, \vec{\zeta})$ is a saddle-point of the Lagrangian function $l_{\vec{\zeta}}$, then $\bar{x} \in D$ is a weakly efficient solution of (VP) and $\vec{\eta}(g(\bar{x})) = \{0\}$, $\vec{\zeta}(h(\bar{x})) = \{0\}$.

Proof. Suppose $\exists (\vec{\xi}, \vec{\eta}, \vec{\zeta}) \in (Y_+ \setminus \{O\}) \times Z_+^* \times W^*$ such that $(\bar{x}, \vec{\eta}, \vec{\zeta})$ is a scalar saddle-point of the Lagrangian function $l_{\vec{\zeta}}$, i.e.,

$$l_{\vec{\zeta}}(\bar{x}, \eta, \zeta) \leq l_{\vec{\zeta}}(\bar{x}, \vec{\eta}, \vec{\zeta}) \leq l_{\vec{\zeta}}(x, \vec{\eta}, \vec{\zeta}), \forall (\eta, \zeta) \in Z_+^* \times W^*, \forall x \in D.$$  

That is to say

$$\vec{\zeta}(f(\bar{x})) + \eta(g(\bar{x})) + \zeta(h(\bar{x})) \leq \vec{\zeta}(f(x)) + \vec{\eta}(g(x)) + \vec{\zeta}(h(x)), \forall x \in D. \forall (\eta, \zeta) \in Z_+^* \times W^*.$$  

Then

$$\eta(g(\bar{x})) + \zeta(h(\bar{x}))) \leq \vec{\eta}(g(\bar{x})) + \vec{\zeta}(h(\bar{x})), \forall (\eta, \zeta) \in Z_+^* \times W^*.$$  

On the other hand

$$\vec{\zeta}(f(\bar{x})) + \vec{\eta}(g(\bar{x})) + \vec{\zeta}(h(\bar{x})) \leq \vec{\zeta}(f(x)) + \vec{\eta}(g(x)) + \vec{\zeta}(h(x)), \forall x \in D. \quad (4.7)$$

Take $\eta = \vec{\eta}$, or $\zeta = \vec{\zeta}$ in (4.7) we have

$$\eta(g(\bar{x})) \leq \vec{\eta}(g(\bar{x})), \forall \eta \in Z_+^*,$$

$$\zeta(h(\bar{x})) \leq \vec{\zeta}(h(\bar{x})), \forall \zeta \in W^*.$$  

Therefore, taking $\eta = O$ in we get $\vec{\eta}(g(\bar{x})) \geq 0$, but taking $\eta = 2\vec{\eta}$ in we get $\vec{\eta}(g(\bar{x})) \leq 0$. Hence

$$\vec{\eta}(g(\bar{x})) = \{0\}. \quad (4.8)$$

Similarly,

$$\vec{\zeta}(h(\bar{x})) = \{0\}. \quad (4.9)$$

Noting that $g(x) \cap (-Z_+) \neq \emptyset$ and $0 \in \vec{\zeta}(h(x))$ (since $x \in D$), according to (4.7), (4.8), and (4.9) we obtain

$$\vec{\zeta}(f(\bar{x})) \leq \vec{\zeta}(f(x)), \forall x \in D.$$  

Therefore, by Theorem 2.1, $\bar{x}$ is a weakly efficient solution of (VP). $\square$
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