On the fourth-order Leray-Lions problem with indefinite weight and nonstandard growth conditions

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Abstract

We prove the existence of at least three weak solutions for the fourth-order problem with indefinite weight involving the Leray-Lions operator with nonstandard growth conditions. The proof of our main result uses variational methods and the critical theorem of Bonanno and Marano (Appl. Anal. 89 (2010), 1-10).

Keywords: Leray-Lions type operator, critical theorem, generalized Sobolev space, variable exponent.

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1 Introduction

In this paper, we shall show the existence of three weak solutions for the following interesting problem

\[
\begin{cases}
\Delta (a(x, \Delta u)) = \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega, \\
u = \Delta u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \((N \geq 2)\) with a smooth boundary \( \partial \Omega \), \( \lambda > 0 \) is a parameter, \( V \) is a function in a generalized Lebesgue space \( L^{s(x)}(\Omega) \), functions \( p, q, s \in C(\overline{\Omega}) \) satisfy the inequalities

\[1 < \min_{x \in \Omega} q(x) < \max_{x \in \Omega} q(x) < \min_{x \in \Omega} p(x) \leq \max_{x \in \Omega} p(x) \leq \frac{N}{2} < s(x) \text{ for all } x \in \Omega,\]

and \( \Delta (a(x, \Delta u)) \) is the Leray-Lions operator of the fourth-order, where \( a \) is a Carathéodory function satisfying some suitable supplementary conditions. For more details about this kind of operators the reader is referred to Boureanu \cite{4} and Leray-Lions \cite{17} (and the references therein).
Note that the study of this type of operators is very active in several fields, e.g. in electrorheological fluids (Růžička [22]), elasticity (Zhikov [21]), stationary thermorheological viscous flows of non-Newtonian fluids (Rajagopal-Růžička [21]), image processing (Chen-Levine-Rao [5]), and mathematical description of the processes filtration of barotropic gas through a porous medium (Antontsev-Shmarev [2]).

Similar problems have been studied before by various authors, see e.g. recent papers of Afrouzi-Chung-Mirzapour [1], Kefi-Rădulescu [14], Kong [15, 16], and Chung-Ho [6]. In particular, Kefi [13] studied the following problem

$$\begin{cases}
-\text{div}(|\nabla u|^{p(x)-2} u) = \lambda V(x)|u|^{q(x)-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases} \quad (1.2)$$

Under the condition in problem (1.1), he has shown that problem (1.2) has a continuous spectrum and his main argument was the Ekeland variational principal.

Before introducing our main result, we define

$$C_+(\Omega) := \{ h \mid h \in C(\Omega), h(x) > 1, \text{ for all } x \in \Omega \},$$

and for $\eta > 0$, $h \in C_+(\Omega)$, we set

$$h^- := \inf_{x \in \Omega} h(x), \quad h^+ := \sup_{x \in \Omega} h(x)$$

and

$$[\eta]^h := \sup\{ \eta^{h^-}, \eta^{h^+} \}, \quad [\eta]_h := \inf\{ \eta^{h^-}, \eta^{h^+} \}.$$  

**Remark 1.1.** It is easy to verify that the following holds

$$[\eta]^\frac{1}{h} = \sup\{ \eta^{\frac{1}{h^-}}, \eta^{\frac{1}{h^+}} \}, \quad [\eta]_h^{\frac{1}{h}} = \inf\{ \eta^{\frac{1}{h^-}}, \eta^{\frac{1}{h^+}} \}.$$  

We denote

$$\delta(x) := \sup\{ \delta > 0 \mid B(x, \delta) \subseteq \Omega, \text{ for all } x \in \Omega \},$$

where $B$ is the ball of radius $\delta$ centered at $x$. One can prove that there exists $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$, where $D := \sup_{x \in \Omega} \delta(x)$.

Throughout this paper, we shall need the following hypotheses:

- $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $a(x, 0) = 0$, for a.e. $x \in \Omega$.
(H₂) There exist $c_1 > 0$ and a nonnegative function $\alpha \in L^{\frac{p(x)}{p(x)-1}}(\Omega)$ such that
\[ |a(x,t)| \leq c_1(\alpha(x) + |t|^{p(x)-1}), \] for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

(H₃) The following inequality holds
\[ (a(x,t) - a(x,s))(t-s) \geq 0, \] for a.e. $x \in \Omega$, and all $s, t \in \mathbb{R}$
with equality if and only if $s = t$.

(H₄) The following inequality holds
\[ |t|^{p(x)} \leq \min \{a(x,t)p(x)A(x,t)\}, \] for a.e. $x \in \Omega$ and all $s, t \in \mathbb{R}$,
where $A : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ represents the antiderivative of $a$, that is,
\[ A(x,t) := \int_0^t a(x,s)ds. \]

(H₅) Assume that $V \in L^{s(x)}(\Omega)$ satisfies the following
\[ V(x) := \begin{cases} 
\leq 0, & \text{for } x \in \Omega \setminus B(x_0,D), \\
\geq v_0, & \text{for } x \in B(x_0,\frac{D}{2}), \\
> 0, & \text{for } x \in B(x_0,D) \setminus B(x_0,\frac{D}{2}),
\end{cases} \]
where $B(x_0,D)$ is the ball of radius $D$ centered at $x_0$ and $v_0$ is a positive constant.

Remark 1.2. We note the following facts:

(1) $A(x,t)$ is a $C^1$-Carathéodory function, i.e., for every $t \in \mathbb{R}$, $A(.,t) : \Omega \to \mathbb{R}$ is measurable and $A(x,.) \in C^1(\mathbb{R})$, for a.e. $x \in \Omega$.

(2) By hypothesis (H₂), there exists a constant $c_3$ such that
\[ |A(x,t)| \leq c_3 \left( \alpha(x)|t| + |t|^{p(x)} \right), \] for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

In the sequel, let
\[ L := w(D^N - \left(\frac{D}{2}\right)^N), \quad w := \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \]
where $\Gamma$ denotes the Euler function. Furthermore, let $k > 0$ be the best constant for which the inequality (2.2) below holds. The main result of this paper now reads as follows.

\[3\]
Theorem 1.1. Assume that hypotheses \((H_1) - (H_5)\) are fulfilled and that there exist \(r > 0\) and \(d > 0\) such that
\[
r < \frac{1}{p^+} \left[ \frac{2d(N - 1)}{D^2} \right] L,
\]
and
\[
\frac{v_0[d]}{c_3(2N - 1)} \left( |\alpha| \frac{4d(N-1)}{D^2} L^{\frac{1}{p^+}-1} + \left[ \frac{4d(N-1)}{D^2} \right]^{\frac{1}{p^+}} \right)
\]
Then for every \(\lambda \in \Lambda_r := \left( \frac{1}{\gamma_d}, \frac{1}{w_r} \right)\), problem (1.1) admits at least three weak solutions.

Remark 1.3. If we set \(r = 1\), then conditions of Theorem 1.1 read as follows: There exists \(d > 0\) such that
\[
p^+ < \frac{2d(N - 1)}{D^2} L
\]
and
\[
\frac{v_0[d]}{c_3(2N - 1)} \left( |\alpha| \frac{4d(N-1)}{D^2} L^{\frac{1}{p^+}-1} + \left[ \frac{4d(N-1)}{D^2} \right]^{\frac{1}{p^+}} \right)
\]
Remark 1.4. We are interested in the Leray-Lions type operators because they are quite general. Indeed, consider
\[
a(x, t) := \theta(x)|t|^{p(x)-2}t,
\]
where \(p \in C_+(\Omega)\), \(p^+ < +\infty\), and choose \(\theta \in L^\infty(\Omega)\) such that there exists \(\theta_0 > 0\) with \(\theta(x) \geq \theta_0 > 0\), for a.e. \(x \in \Omega\). One can then see that (1.6) satisfies hypotheses \((H_1) - (H_4)\) and we arrive at the following operator
\[
\Delta(\theta(.)|\Delta|^{p(.)-2}\Delta u).
\]
Note that when \(\theta \equiv 1\), we get the well-known \(p(x)\)-biharmonic operator \(\Delta_{p(.)}^2(u)\), see Kefi-Rădulescu [14]. Moreover, we can make the choice
\[
a(x, t) := \theta(x)(1 + |t|^2)^{\frac{p(x)}{p(x)+2}}t,
\]
and obtain the following operator

\[ \Delta \left( \theta(.,) (1 + |\Delta u|^{\frac{p(.)}{p(.)-2}} \Delta u) \right), \]

where \( p \) and \( \theta \) are as in (1.6).

In the sequel, define \( a(x,t) \) as in (1.6) with \( \theta \equiv 1 \). Then problem (1.1) becomes

\[
\begin{aligned}
\Delta (|\Delta u|^{p(x)-2} \Delta u) &= \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega, \\
u = \Delta u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

(1.7)

and we obtain the following result.

**Corollary 1.1.** Assume that there exist \( r, d > 0 \) such that

\[ r < \frac{1}{p^+} \left[ \frac{2d(N-1)}{D^2} \right] p, \]

(1.8)

and

\[ \overline{w}_r < \gamma_d := \frac{\nu_0[d]_q}{c_3(2N-1)} \left[ \frac{4d(N-1)}{D^2} \right]^p. \]

(1.9)

Then for every

\[ \lambda \in \overline{\lambda}_r := \left( \frac{1}{\gamma_d}, \frac{1}{\overline{w}_r} \right), \]

problem (1.7) admits at least three weak solutions.

This paper is organized as follows: in Section 2 we give some preliminaries and necessary background results on the Sobolev spaces with variable exponents, whereas Section 3 is devoted to the proof of our main result.

## 2 Preliminaries and Background

In this section, we recall some definitions and basic properties of variable exponent Sobolev spaces. For a deeper treatment of these spaces, we refer the reader to Fan-Zhao [10], R˘adulescu [19], and R˘adulescu-Repovˇs [20], and for the other background material to Papageorgiou-R˘adulescu-Repovˇs [18].
Let \( p \in C_+(\Omega) \) be such that

\[
1 < p^- := \min_{x \in \Omega} p(x) \leq p^+ := \max_{x \in \Omega} p(x) < +\infty.
\]

We define the Lebesgue space with variable exponent as follows

\[
L^{p(x)}(\Omega) := \{ u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \},
\]

which is equipped with the so-called Luxemburg norm

\[
|u|_{p(x)} := \inf \left\{ \mu > 0 \mid \int_{\Omega} \frac{|u(x)|^{p(x)}}{\mu} \, dx \leq 1 \right\}.
\]

Variable exponent Lebesgue spaces are like classical Lebesgue spaces in many respects: they are Banach spaces and are reflexive if and only if \( 1 < p^- \leq p^+ < \infty \). Moreover, the inclusion between Lebesgue spaces is generalized naturally: if \( q_1, q_2 \) are such that \( p_1(x) \leq p_2(x) \), a.e. \( x \in \Omega \), then there exists a continuous embedding

\[
L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega).
\]

For \( u \in L^{p(x)}(\Omega) \) and \( v \in L^{p'(x)}(\Omega) \), the Hölder inequality holds

\[
\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)},
\]

where \( \frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \).

The modular on the space \( L^{p(x)}(\Omega) \) is the map \( \rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R} \) defined by

\[
\rho_{p(x)}(u) := \int_{\Omega} |u|^{p(x)} \, dx.
\]

For any positive integer \( m \), we define the Sobolev space with variable exponents as follows:

\[
W^{m,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) \mid D^\alpha u \in L^{p(x)}(\Omega) , |\alpha| \leq m \right\},
\]

where \( \alpha := (\alpha_1, \alpha_2, ..., \alpha_N) \) is a multi-index and

\[
|\alpha| := \sum_{i=1}^N \alpha_i, \quad D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_N} x_N}.
\]
Then $W^{m,p(x)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm
\[ \|u\|_{m,p(x)} := \sum_{|\alpha| \leq m} |D^\alpha u|_{p(x)}. \]

The space $W^{m,p(x)}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{m,p(x)}(\Omega)$. It’s well-known that both $W^{2,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$ are separable and reflexive Banach spaces. It follows that
\[ X := W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega), \]
is also a separable and reflexive Banach space, when equipped with the norm
\[ \|u\|_X := \|u\|_{W^{2,p(x)}(\Omega)} + \|u\|_{W^{1,p(x)}_0(\Omega)}. \]

Let
\[ \|u\| := \inf \left\{ \mu > 0 \mid \int_\Omega \frac{|\Delta u|^{p(x)}}{\mu} dx \leq 1 \right\}, \]
represent a norm which is equivalent to $\|\cdot\|_X$ on $X$ (see El Amrouss-Ourraoui [9, Remark 2.1]). Therefore in what follows, we shall consider the normed space $(X, \|\cdot\|)$.

The modular on the space $X$ is the map $\rho_{p(x)} : X \to \mathbb{R}$ defined by
\[ \rho_{p(x)}(u) := \int_\Omega |\Delta u|^{p(x)} dx. \]

This mapping satisfies some useful properties and we cite some below.

**Lemma 2.1.** (El Amrous-Moradi-Moussaoui [3]) For every $u, u_n \in W^{2,p(x)}(\Omega)$, the following statements hold:

1. $\|u\| < 1$ (resp. $= 1$, $> 1$) $\iff \rho_{p(x)}(u) < 1$ (resp. $= 1$, $> 1$);

2. $\|[u]\| := \min\{\|u\|^{p_-}, \|u\|^{p_+}\} \leq \rho_{p(x)}(u) \leq \max\{\|u\|^{p_-}, \|u\|^{p_+}\} := \|[u]\|^p$;

3. $\|u_n\| \to 0$ (resp. $\to \infty$) $\iff \rho_{p(x)}(u_n) \to 0$ (resp. $\to \infty$).

**Proposition 2.1.** (Edmunds-Rakosnik [7]) Let $p$ and $q$ be measurable functions such that $p \in L^\infty(\Omega)$, and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^q(x)(\Omega)$, $u \neq 0$. Then
\[ \|[u]_{p(x)q(x)}\|_p := \min\{|u|_{p(x)q(x)}^{p_-}, |u|_{p(x)q(x)}^{p_+}\} \leq |u|_{p(x)}^{p(x)} \leq \max\{|u|_{p(x)q(x)}^{p_-}, |u|_{p(x)q(x)}^{p_+}\}. \]
We recall that the critical Sobolev exponent is defined as follows:

\[ p^*(x) := \begin{cases} 
\frac{Np(x)}{N-2p(x)}, & \text{if } p(x) < \frac{N}{2}, \\
+\infty, & \text{if } p(x) \geq \frac{N}{2}.
\end{cases} \]

**Remark 2.1.** (Kefi [13]) Denote the conjugate exponent of the function \( s(x) \) by \( s'(x) \) and set \( \beta(x) \) := \( \frac{s(x)q(x)}{s(x) - q(x)} \). Then there exist compact and continuous embeddings \( X \hookrightarrow L^{s'(x)q(x)}(\Omega) \) and \( X \hookrightarrow L^{\beta(x)}(\Omega) \) and the best constant \( k > 0 \) such that

\[ |u|_{s'(x)q(x)} \leq k\|u\|. \quad (2.2) \]

In order to formulate the variational approach to problem (1.1), let us recall the definition of a weak solution for our problem.

**Definition 2.1.** We say that \( u \in X \setminus \{0\} \) is a weak solution of problem (1.1) if \( \Delta u = 0 \) on \( \partial \Omega \) and

\[ \int_{\Omega} a(x, \Delta u) \Delta v dx - \lambda \int_{\Omega} V(x)|u|^{q(x)-2}uv dx = 0, \quad \text{for all } v \in X. \]

We state the following proposition which will be needed in Section 3.

**Proposition 2.2.** (Gasiński-Papageorgiou [17]) If \( X \) is a reflexive Banach space, \( Y \) is a Banach space, \( Z \subset X \) is nonempty, closed and convex subset, and \( J : Z \rightarrow Y \) is completely continuous, then \( J \) is compact.

Our main tool will be the following critical theorem Bonanno-Marano [3], which we restate in a more convenient form.

**Theorem 2.1.** (Bonanno-Marano [3, Theorem 3.6]) Let \( X \) be a reflexive real Banach space and \( \Phi : X \rightarrow \mathbb{R} \) a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on \( X \). Let \( \Psi : X \rightarrow \mathbb{R} \) be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that

\[ (a_0) \quad \inf_{x \in X} \Phi(x) = \Phi(0) = \Psi(0) = 0. \]

Assume that there exist \( r > 0 \) and \( \overline{x} \in X \), with \( r < \Phi(\overline{x}) \), such that:

\[ (a_1) \quad \frac{\sup_{\Phi(x) \leq r} \Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}. \]
for each $\lambda \in \Lambda_r := \left( \frac{\Phi(x)}{\Psi(x)}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)} \right)$, the functional $\Phi - \lambda \Psi$ is coercive.

Then for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda \Psi$ has at least three distinct critical points in $X$.

3 Proof of the Main Result

In this section, we present the proof of Theorem 1.1. To begin, let us denote

$$\Psi(u) := \int_{\Omega} \frac{1}{q(x)} V(x)|u|^q(x)dx.$$ 

The Euler-Lagrange functional corresponding to problem (1.1) is then defined by $I_\lambda : X \to \mathbb{R}$,

$$I_\lambda(u) := \phi(u) - \lambda \Psi(u), \text{ for all } u \in X,$$

where

$$\Phi(u) := \int_{\Omega} A(x, \Delta u)dx.$$ 

It is clear that condition (a0) in Theorem 2.1 is fulfilled, and by virtue of Proposition 2.1, $\Psi$ is well-defined since we have for all $u \in X$,

$$|\Psi(u)| \leq \frac{1}{q} \int_{\Omega} V(x)|u|^q(x)dx \leq \frac{1}{q} |V(x)|_{s(x)}||u|^q(x)|_{s'(x)} \leq \frac{1}{q} |V(x)|_{s(x)}|[u]|_{s'(x)q(x)}^q.$$ 

Moreover, by inequality (2.2) in Remark 2.1, one has

$$|\Psi(u)| \leq \frac{1}{q} |V(x)|_{s(x)}|[u]|^q,$$

therefore $\Psi$ is indeed well-defined. We shall also need the following lemma.

Lemma 3.1. (i) The functional $\Phi$ is a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional (BOUREANU [4]) whose Gâteaux derivative admits a continuous inverse on $X$. 

(ii) The functional $\Psi$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.

Proof. The proof splits into two parts:
(i) It is clear from Lemma 2.1 and hypothesis (H₄) that for every \( u \in X \) such that \( \| u \| > 1 \), one has
\[
\Phi(u) \geq \int_\Omega \frac{1}{p(x)} |\Delta u|^p(x) dx \geq \frac{1}{p^+} p_p(x) (u) \geq \frac{1}{p^+} \| u \|^{p^-},
\]
and thus \( \Phi \) is coercive.

For the rest of the proof, we will use the same argument as in the proof of HO-Sim [12, Lemma 3.2]. First, we shall show that \( \Phi' \) is strictly monotone. Using (H₃) and integrating over \( \Omega \), we obtain for all \( u, v \in X \) with \( u \neq v \),
\[
0 < \int_\Omega (a(x, \Delta u) - a(x, \Delta v)) (\Delta u - \Delta v) dx = <\Phi'(u) - \Phi'(v), u - v>,
\]
which means that \( \Phi' \) is strictly monotone.

Note that the strict monotonicity of \( \Phi' \) implies that \( \Phi' \) is an injection. From the assertion (H₄) it is clear that for any \( u \in X \) with \( \| u \| > 1 \), one has
\[
<\Phi'(u), u > \geq \frac{\| u \|^{p^-}}{\| u \|} = \| u \|^{p^- - 1},
\]
and thus \( \Phi' \) is coercive. Therefore it is a surjection in view of Minty-Browder Theorem for reflexive Banach space (cf. Zeidler [23]), so \( \Phi' \) has a bounded inverse mapping \( (\Phi')^{-1} : X^* \rightarrow X \).

Let \( f_n \rightarrow f \) as \( n \rightarrow +\infty \) in \( X^* \) and set \( u_n = (\Phi')^{-1}(f_n), u = (\Phi')^{-1}(f) \). Then the boundedness of \( (\Phi')^{-1} \) and \( \{ f_n \} \) imply that \( \{ u_n \} \) is bounded. Without loss of generality, we can assume that there exists a subsequence, again denoted by \( u_n \), and \( \tilde{u} \) such that \( u_n \rightharpoonup \tilde{u} \) (weakly) in \( X \), which implies
\[
| < f_n - f, u_n - \tilde{u} > | \leq \| f_n - f \|_{X^*} \| u_n - \tilde{u} \|.
\]

We can now infer that
\[
\lim_{n \rightarrow +\infty} <\Phi'(u_n), u_n - \tilde{u} > = \lim_{n \rightarrow +\infty} < f_n, u_n - \tilde{u} > = \lim_{n \rightarrow +\infty} < f_n - f, u_n - \tilde{u} > = 0,
\]
which implies that
\[
\lim_{n \rightarrow +\infty} \int_\Omega a(x, \Delta u_n)(\Delta u_n - \Delta \tilde{u}) dx = 0.
\]

By invoking Boureanu [4, Theorem 3.2], one can conclude that \( u_n \rightharpoonup \tilde{u} \) (strongly) as \( n \rightarrow +\infty \) in \( X \). This yields \( f_n = \Phi'(u_n) \rightarrow \Phi'(\tilde{u}) \) and thus \( f = \Phi'(\tilde{u}) \), by the injectivity of \( \Phi' \), we obtain \( u = \tilde{u} \) and hence \( (\Phi')^{-1}(f_n) \rightarrow (\Phi')^{-1}(f) \) and the proof of Lemma 3.1 is thus completed.
(ii) Next, we show that $\Psi'(u)$ is compact. Let $v_n \rightharpoonup v$ in $X$. Then

$$
|\langle \Psi'(u), v_n \rangle| - |\langle \Psi'(u), v \rangle| \leq \int_{\Omega} |V(x)||u|^q(x-1) |v_n - v| dx
$$

$$
\leq |V(x)||u|^q(x-1) \frac{q(x)}{q(x)-1} |v_n - v|_{\beta(x)}.
$$

As a consequence of Remark 2.1 and due to the compact embedding $X \hookrightarrow L^{\beta(x)}(\Omega)$, we have $|\langle \Psi'(u), v_n \rangle| \to |\langle \Psi'(u), v \rangle|$, as $n \to +\infty$. This means that $\Psi'(u)$ is completely continuous. So, by Proposition 2.2, $\Psi'$ is indeed compact.

\[\square\]

**Proof of Theorem 1.1.** As we have observed above, the functionals $\Phi$ and $\Psi$ satisfy the regularity assumptions of Theorem 2.1. Now, let $v_d \in X$ be the function defined by

$$v_d := \begin{cases} 0, & \text{if } x \in \Omega \setminus B(x_0, D), \\ \frac{2d}{D} (D - |x - x_0|), & \text{if } x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}), \\ d, & \text{if } x \in B(x_0, \frac{D}{2}), \end{cases}$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^N$. It is then easy to see that

$$\Delta v_d = \begin{cases} 0, & \text{if } x \in \Omega \setminus B(x_0, D) \cup B(x_0, \frac{D}{2}), \\ - \frac{2d(N-1)}{D(x-x_0)}, & \text{if } x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}). \end{cases}$$

Using Lemma 2.1 and the continuity of the embedding $L^{p^+} (\Omega) \hookrightarrow L^{p(x)}(\Omega)$, we can conclude that

$$\frac{1}{p^+} \left[ \frac{2d(N-1)}{D^2} \right]_p \leq \Phi(v_d) \leq c_3 L^{\frac{1}{p^+}} |\alpha(x)| \frac{4d(N-1)}{D^2} + c_3 \left[ \frac{4d(N-1)}{D^2} \right]^p L,$$

$$\Psi(v_d) \geq \int_{B(x_0, \frac{D}{2})} \frac{V(x)}{q(x)} |v_d|^q(x) dx \geq \frac{1}{q^+} v_0 [d_q] w(D) \frac{1}{2} N,$$

and hence

$$\frac{\Psi(v_d)}{\Phi(v_d)} \geq \frac{\frac{1}{q^+} v_0 [d_q] w(D) \frac{1}{2} N}{c_3 L^{\frac{1}{p^+}} |\alpha(x)| \frac{4d(N-1)}{D^2} + c_3 \left[ \frac{4d(N-1)}{D^2} \right]^p L} = \gamma_d.$$
Next, from \( r < \frac{1}{p^+} \left[ \frac{2d(N-1)}{D^2} \right] r^p \) \( L \), we get \( r < \Phi(v_d) \). Now, for each \( u \in \Phi^{-1}((-\infty, r]) \), due to condition \((H_4)\), one has that

\[
\frac{1}{p^+} \|u\|_p \leq r.
\]  

(3.2)

Proposition 2.1 and inequalities (3.2) and (2.2) now yield

\[
\Psi(u) \leq \frac{1}{q} \int_{\Omega} |V_{s(x)}||u|^{q(x)} \, dx \leq \frac{1}{q} \int_{\Omega} |V_{s(x)}[k\|u\|]^{q} \, dx \leq \frac{(p^+)^2}{q^+} \|u\|^{q-1} \int_{\Omega} |V_{s(x)}[r]^q \, dx.
\]  

(3.3)

Therefore

\[
\frac{1}{r} \sup_{\Phi(u) \leq r} \Psi(u) \leq \bar{w}.
\]

In the next step, we shall prove that for each \( \lambda > 0 \), the energy functional \( \Phi - \lambda \Psi \) is coercive. By Remark 2.1 we have

\[
\Psi(u) \leq \frac{1}{q} \int_{\Omega} |V(x)|u|^{q(x)} \, dx \leq \frac{1}{q} \int_{\Omega} |V_{s(x)}[k\|u\|]^{q} \, dx.
\]  

(3.4)

For \( \|u\| > 1 \), relations (3.1) and (3.4) give the following

\[
\Phi(u) - \lambda \Psi(u) \geq \frac{1}{p^+} \|u\|^{p^-} - \lambda \frac{1}{q} \int_{\Omega} |V_{s(x)}[k\|u\|]^{q}.
\]

Since \( 1 \leq q^- \leq q^+ < p^- \), it follows that \( \Phi(u) - \lambda \Psi(u) \) is coercive. Finally, due to the fact that

\[
\Lambda := \left( \frac{1}{\gamma_d}, \frac{1}{\bar{w}} \right) \subseteq \left( \frac{\Phi(v_d)}{\Psi(v_d)}, \frac{r}{\sup_{\Phi(u) \leq r} \Psi(u)} \right),
\]

Theorem 2.1 implies that for each \( \lambda \in \Lambda \), the functional \( \Phi - \lambda \Psi \) admits at least three critical points in \( X \) which are weak solutions for problem (1.1). This completes the proof of Theorem 1.1. \( \square \)

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