WEAK SHADOWING PROPERTY FOR IFS

MEHDI FATEHI NIA

Abstract. In this paper we extend the notion of weak shadowing property to parameterized iterated function systems IFS and prove some related theorems on these notion. It is proved that every uniformly contracting and every uniformly expanding IFS has the weak shadowing property. Then, as an example, we give an IFS which has the shadowing property, but fails to have the weak shadowing property. As a main result, we show that the weak shadowing property is a generic property in the set of all iterated function systems.

1. Introduction

The notion of shadowing is an important tool for studying properties of discrete dynamical systems. From numerical point of view, if a dynamical system has the shadowing property, then numerically obtained orbits reflect the real behavior of trajectories of the systems (see [4, 8, 10]). The so-called weak shadowing property is introduced and studied in [3, 6] and this is proved that shadowing and weak shadowing are $C^0$- generic properties in $H(X)$, where $H(X)$ is the set of all homeomorphisms of a compact topological space $X$. Specially, the space $X$ is one of the following:

(i) a topological manifold with boundary ($\text{dim}(X) \geq 2 \text{ if } \partial X \neq \emptyset$),
(ii) a Cartesian product of a countably infinite number of manifolds with nonempty boundary,

2010 Mathematics Subject Classification. 37C50, 37C15.

Key words and phrases. Iterated function systems, generic property, weak shadowing, uniformly contracting.
(iii) a Cantor set, then weak shadowing is a generic property in $H(X)$ [6]. Iterated function systems (IFS), are used for the construction of deterministic fractals and have found numerous applications, in particular to image compression and image processing [1]. Important notions in dynamics like attractors, minimality, transitivity, and shadowing can be extended to IFS (see [2, 5]). Gutu and Glavan defined the shadowing property for a parameterized iterated function system and prove that if a parameterized IFS is uniformly contracting, then it has the shadowing property [5].

The present paper concerns the weak shadowing property for parameterized IFS and some important result about weak shadowing property are extended to iterated function systems. Firstly, we introduce the weak shadowing property and conjugacy on IFS. In Theorem 3.8 we show that if an IFS has the shadowing property it has the weak shadowing property. So every uniformly contracting (expanding) IFS has the weak shadowing property. In Example 3.5 we give an IFS which has the weak shadowing property but does not have the shadowing property. Then we prove that the weak shadowing is a generic property in $H_\Lambda(X)$.

2. PRELIMINARIES

Let $(X, d)$ be a complete metric space. Let us recall that a parameterized Iterated Function System (IFS) $\mathcal{F} = \{X; f_\lambda|\lambda \in \Lambda\}$ is any family of continuous mappings $f_\lambda : X \to X$, $\lambda \in \Lambda$, where $\Lambda$ is a finite nonempty set (see [5]).

Let $T = \mathbb{Z}$ or $T = \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$ and $\Lambda^T$ denote the set of all infinite sequences $\{\lambda_i\}_{i \in T}$ of symbols belonging to $\Lambda$. A typical element of $\Lambda^\mathbb{Z}_+$ can be denoted as $\sigma = \{\lambda_0, \lambda_1, \ldots\}$ and we use the shorted notation

$$\mathcal{F}_{\sigma_n} = f_{\lambda_n} \circ \ldots \circ f_{\lambda_1} \circ f_{\lambda_0}.$$
Definition 2.1. [5] A sequence \( \{x_n\}_{n \in T} \) in \( X \) is called an orbit of the IFS \( F \) if there exist \( \sigma \in \Lambda^T \) such that \( x_{n+1} = f_{\lambda_n}(x_n) \), for each \( \lambda_n \in \sigma \).

Given \( \delta > 0 \), a sequence \( \{x_n\}_{n \in T} \) in \( X \) is called a \( \delta \)-pseudo orbit of \( F \) if there exist \( \sigma \in \Lambda^T \) such that for every \( \lambda_n \in \sigma \), we have \( d(x_{n+1}, f_{\lambda_n}(x_n)) < \delta \).

One says that the IFS \( F \) has the shadowing property (on \( T \)) if, given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( \delta \)-pseudo orbit \( x = \{x_n\}_{n \in T} \) there exist an orbit \( y = \{y_n\}_{n \in T} \), satisfying the inequality \( d(x_n, y_n) \leq \epsilon \) for all \( n \in T \). In this case one says that the \( \{y_n\}_{n \in T} \) or the point \( y_0, \epsilon \)- shadows the \( \delta \)-pseudo orbit \( \{x_n\}_{n \in T} \).

Definition 2.2. One says that the IFS \( F \) has the weak shadowing property (on \( T \)) if, given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( \delta \)-pseudo orbit \( x = \{x_n\}_{n \in T} \) there exist an orbit \( y = \{y_n\}_{n \in T} \subset B_\epsilon(x) \), satisfying the inequality \( d(x_n, y_n) \leq \epsilon \) for all \( n \in T \).

Where \( B_\epsilon(S) \) denote the set of all \( x \in X \) such that \( d(x, S) < \epsilon \).

Please note that if \( \Lambda \) is a set with one member then the parameterized IFS \( F \) is an ordinary discrete dynamical system. In this case the shadowing property for \( F \) is ordinary shadowing property for a discrete dynamical system.

The parameterized IFS \( F = \{X; f_\lambda|\lambda \in \Lambda\} \) is uniformly contracting if there exists

\[ \beta = \sup_{\lambda \in \Lambda} \sup_{x \neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x, y)} \]

and this number called also the contracting ratio, is less than one.

Respectively, we shall say that \( F \) is uniformly expanding if

\[ \alpha = \inf_{\lambda \in \Lambda} \inf_{x \neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x, y)} > 1. \]

We call \( \alpha \) the expanding ratio [5].

Suppose \( f, g \) are two homeomorphism on \( X \) we define \( d_0(f, g) = \max\{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x))\} \) for \( x \in X \).
Let $\mathcal{H}_\Lambda(X)$ denote the set of all IFS, $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ such that each $f_\lambda$ is a homeomorphism and for $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}, \mathcal{G} = \{X; g_\lambda | \lambda \in \Lambda\} \in \mathcal{H}_\Lambda(X)$ Let

$$\rho(\mathcal{F}, \mathcal{G}) = \max_{\lambda, \mu \in \Lambda} \{d(f_\lambda(x), g_\mu(x)), d(f^{-1}_\lambda(x), g^{-1}_\mu(x)) : x \in X\}.$$ 

Clearly $\rho$ is a complete metric on $\mathcal{H}_\Lambda(X)$.

We recall that the space $X$ is homogeneous if for $\epsilon > 0$ we can find $\delta > 0$ which if $\{x_1, ..., x_n\}, \{y_1, ..., y_n\} \subset X$ are two sets of disjoint points satisfying $d(x_i, y_i) \leq \delta$, for all $1 \leq i \leq n$, then there exist a homeomorphism $h : X \to X$ with $d_0(h, id_X) \leq \epsilon$ and $h(x_i) = y_i$, $1 \leq i \leq n$.

**Definition 2.3.** Suppose $(X, d)$ and $(Y, d')$ are compact metric spaces and $\Lambda$ is a finite set. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ are two IFS which $f_\lambda : X \to X$ and $g_\lambda : Y \to Y$ are continuous maps for all $\lambda \in \Lambda$. We say that $\mathcal{F}$ is said to be topologically conjugate to $\mathcal{G}$ if there is a homeomorphism $h : X \to Y$ such that $g_\lambda = hof_\lambda oh^{-1}$, for all $\lambda \in \Lambda$. In this case, $h$ is called a topological conjugacy.

**3. Weak shadowing property for iterated function systems**

In this section we investigate the structure of parameterized IFS with the weak shadowing property. It is well known that if $f : X \to X$ and $g : Y \to Y$ are conjugated then $f$ has the weak shadowing property if and only if so does $g$. In the next theorem we extend this property for *iterated function systems*.

**Theorem 3.1.** Suppose $(X, d)$ and $(Y, d')$ are compact metric spaces and $\Lambda$ is a finite set. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | i \in \Lambda\}$ are two IFS which $f_\lambda : X \to X$ and $g_\lambda : X \to X$ are continuous maps for all $\lambda \in \Lambda$. Suppose that $\mathcal{F}$ is topologically conjugate to $\mathcal{G}$, then $\mathcal{F}$ has the weak shadowing property if and only if so does $\mathcal{G}$.
Proof. Suppose that \( \mathcal{F} \) has the weak shadowing property, we prove that \( \mathcal{G} \) also have this property. Fix \( \epsilon > 0 \) and consider \( h : X \rightarrow Y \) as the conjugacy map between \( \mathcal{F} \) and \( \mathcal{G} \). Since \( h \) is a homeomorphism then there exists \( \epsilon_1 > 0 \) such that \( d(a, b) < \epsilon_1 \), implies \( d'(h(a), h(b)) < \epsilon \). Let \( \delta_1 > 0 \) be an \( \epsilon_1 \) modulus of weak shadowing for \( \mathcal{F} \), there is \( \delta > 0 \) such that \( d'(x, y) < \delta \) implies that \( d(h^{-1}(x), h^{-1}(y)) < \delta_1 \).

Now, Suppose that \( x = \{x_i\}_{i \geq 0} \) is a \( \delta \)-pseudo orbit for \( \mathcal{G} \). Then \( x' = \{h^{-1}(x_i)\}_{i \geq 0} \) is a \( \delta_1 \)-pseudo orbit for \( \mathcal{F} \). Since \( \mathcal{F} \) has the weak shadowing property then there exist an orbit \( y' = \{y_i\}_{i \geq 0} \) in \( \mathcal{F} \) such that \( y' \subset B_{\epsilon_1}(x') \).
So, \( y \subset B_\epsilon(x) \), where \( y = \{h(y_i)\}_{i \geq 0} \) is an orbit of \( \mathcal{G} \). \( \square \)

By shadowing and weak shadowing definitions for IFS we have the following theorem.

**Theorem 3.2.** Let \( X \) be a complete metric space, if \( \mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\} \) has the shadowing property then it has the weak shadowing property.

So, Theorem 3.8, Theorems 2.1 and 2.2 in \([5]\) we have the following results.

**Corollary 3.3.** If a parameterized IFS \( \mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\} \) is uniformly contracting, then it has the weak shadowing property on \( \mathbb{Z}_+ \).

**Corollary 3.4.** If a parameterized IFS \( \mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\} \) is uniformly expanding and if each function \( f_\lambda(\lambda \in \Lambda) \) is surjective, then the IFS has the weak shadowing property on \( \mathbb{Z}_+ \).

The following example shows that the inverse of Theorem is not true.

**Example 3.5.** Consider the unit circle \( S^1 \) with coordinate \( x \in [0, 1) \). Suppose that \( 0 < \beta_1, \beta_2 < 1 \) are two distinct irrational numbers and \( f_i \) is homeomorphisms on \( S^1 \) defined by \( f_i(x) = x + \beta_i \), for \( i \in \{0, 1\} \). Let \( \mathcal{F} = \{S^1; f_1, f_2\} \). Since every orbit of \( f_1 \) is an orbit of \( \mathcal{F} \) that is dense in \( S^1 \), then \( \mathcal{F} \) has the
weak shadowing property \[9\].

Now, suppose that \(\beta_2 - \beta_1 = \frac{1}{2}\). We show that \(F\) does not have the shadowing property.

To obtain a contradiction, we assume that \(F\) has the shadowing property. Take \(\epsilon = \frac{1}{5}\) and \(\delta > 0\) be the corresponding number for shadowing property. Let \(\alpha\) be a rational number which \(|\alpha - \beta_1| < \delta\) and \(g : S^1 \to S^1\) be a homeomorphism defined by \(g(x) = x + \alpha\). This is clear that every orbits of \(g\) is a \(\delta\)-pseudo orbit of \(f_1(x) = x + \beta\). Since \(\alpha\) is a rational number, then there is \(m \in \mathbb{N}\) such that \(g^m\) is identity map. Let \(\sigma = \{\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots\}\) be an arbitrary sequence in \(\{1, 2\}^\mathbb{Z}\).

**Claim:** For every \(p \in S^1\), the sets \(\{F_{\sigma_{mk}}(p)\}_{k \geq 0}\) is dense in \(S^1\).

So, for any \(x, p \in S^1\), \(\{x, g(x), \ldots, g^{m-1}(x), \ldots\}\) is a \(\delta\)-pseudo orbit of \(F\); but there exists \(k > 0\) that \(d(g^{km}(x), F_{\sigma_{mk}}(p)) > \frac{1}{5}\).

Hence \(F\) does not have the shadowing property.

**Proof of Claim.** For any \(n > 0\), let \(A_n = \{\lambda_i; \lambda_i = 2, 1 \leq i \leq n\}\) and \(n_2\) be the cardinality of the set \(A_n\). Then \(F_{\sigma_{mk}}(p) = p + mk\beta_1 + \frac{mk\beta_2}{2} (mod 1)\).

By a similar argument to that given in (\[7\] Example 2), we can show that \(\{F_{\sigma_{mk}}(p)\}_{k \geq 0}\) is dense in \(S^1\).

**Lemma 3.6.** Suppose \(x = \{x_i\}_{i=0}^n\) is a \(\delta\)-pseudo orbit. there exist a \(2\delta\)-pseudo trajectory \(y = \{y_i\}_{i=0}^n\) such that \(x \subset B_\epsilon(y)\) and \(y_i \neq y_j\) for all \(i \neq j\).

**Proof.** Consider a finite sequence \(\{\lambda_i\}_{i=0}^{n-1} \subset \Lambda\) such that \(d(f_{\lambda_i}(x_i, x_{i+1}) < \delta\), for all \(0 \leq i \leq n - 1\). Since \(X\) is a compact space and has no isolated points then we can find a sequence \(y = \{y_i\}_{i=0}^n\) of distinct points such that \(d(f_{\lambda_i}(x_i), y_i) < \frac{\delta}{2}\) and \(d(x_{i+1}, y_{i+1}) < \frac{\delta}{2}\), for all \(\lambda \in \Lambda\) and all \(0 \leq i \leq n - 1\). So \(d(f_{\lambda_i}(y_i), y_{i+1}) \leq d(f_{\lambda_i}(y_i), f_{\lambda_i}(x_i)) + d(f_{\lambda_i}(x_i), x_{i+1}) + d(x_{i+1}, y_{i+1}) < \frac{\delta}{2} + \frac{\delta}{2} + \delta = 2\delta\), for \(0 \leq i \leq n - 1\). \(\square\)
Lemma 3.7. Let $x = \{x_i\}_{i \in \mathbb{Z}}$ be a sequence in $X$ and $x_n = \{x_i\}_{i = -n}^n$, for every $n \geq 1$. For each $\epsilon > 0$ there exist $k \in \mathbb{N}$ such that $x \subset B_\epsilon(x_k)$.

Proof. This is clear that $\{X - x, B_\epsilon(x_n) : n \geq 1\}$ where $x$ is closure of $x$, is an open cover for $X$ and has a finite subcover. Suppose $X \subset (X - x) \cup B_\epsilon(x_{n_1}) \cup \ldots \cup B_\epsilon(x_{n_l})$, for some $n_1 < n_2 < \ldots < n_l$. Since $x \cap (X - x) = \emptyset$ and $B_\epsilon(x_{n_1}) \subset B_\epsilon(x_{n_2}) \subset \ldots \subset B_\epsilon(x_{n_l})$, then $x \subset B_\epsilon(x_{n_l})$. □

Next theorem is the main result of this paper and the main idea of proof is the same as that of [6].

Theorem 3.8. If the space $X$ is a generalized homogeneous then the $F$–weak shadowing property is generic in $\mathcal{H}_\Lambda(X)$.

Proof. Given $\epsilon > 0$ and $V = \{V_1, \ldots, V_k\}$ be a cover of $X$ consisting of open sets with diameters less than $\epsilon$. Suppose $F \in \mathcal{H}_\Lambda(X)$ and take $J_F$ is the family of sets $L \subset \{1, 2, \ldots, k\}$ such that there exist an obit $x = \{x_i\}_{i \in \mathbb{Z}}$ of $F$ satisfying $x \cap V_j \neq \emptyset$, for all $j \in L$.

Claim. For any $F \in \mathcal{H}_\Lambda(X)$ there is a neighborhood $U$ of $F$ such that $J_F \subset J_G$ for $F \in U$.

Take

$$C_V = \{F \in \mathcal{H}_\Lambda(X) : J_F = J_G \text{ for } G \text{ sufficiently close to } F\}.$$

By definition $C_V$ is an open subset of $\mathcal{H}_\Lambda(X)$. Now we show that $C_V$ is dense in $\mathcal{H}_\Lambda(X)$. Consider an arbitrary open set $W \subset \mathcal{H}_\Lambda(X)$ and $J_F$ is a maximal element of $J_W = \{J_F : F \in W\}$, i.e., for every $G \in W$, $J_F \subset J_G$ implies that $F = G$. Thus, by claim $F \in C_V \cap W$. So $C_V$ is an open dense subset of $\mathcal{H}_\Lambda(X)$.

Take $F \in C_V$ we prove that $F$ has the weak shadowing property.

Since $C_V$ is an open set there is $\gamma > 0$ such that for $G \in \mathcal{H}_\Lambda(X)$, $\rho(F, G) < \gamma$ implies that $J_F = J_G$. Suppose $\beta > 0$ is a $\gamma$–modulus of homogeneity of $X$. 
Let \( x = \{ x_i \}_{i \in \mathbb{Z}} \) be a \( \frac{\beta}{2} \)-pseudo orbit of \( \mathcal{F} \). Because of the Lemma 3.7 there exist \( k \in \mathbb{N} \) such that \( x \subset B_\epsilon(x_k) \). By Lemma 3.6 there exist a \( \beta \)-pseudo trajectory \( y = \{ y_i \}_{i=-n}^n \) (belong to \( \mathcal{F} \)) such that \( x_k \subset B_\epsilon(y) \) and \( y_i \neq y_j \) for all \( i \neq j \). Suppose \( 0 < \tau < \gamma \) is a number that \( d(a, b) < \tau \) implies that \( d(f^{-1}_\lambda(a), f^{-1}_\lambda(b)) < \gamma \), for all \( a, b \in X \) and all \( \lambda \in \Lambda \). Also, suppose \( h \in \mathcal{H}(X) \), \( d_0(h, id_X) < \tau \) is a homeomorphism connecting \( f_\lambda(y_i) \) with \( y_{i+1} \) for all \( -n \leq i \leq n - 1 \). Consider IFS \( \mathcal{G} = \{ X; g_\lambda = hof_\lambda | \lambda \in \Lambda \} \) and let \( \sigma = \{ \mu_0, \mu_1, ... \} \) be an arbitrary element of \( \Lambda^{\mathbb{Z}}^+ \). So the sequence

\[
z = \{ ..., g_{\mu_1}^{-1}(g_{\mu_0}^{-1}(y_{-n})), g_{\mu_0}^{-1}(y_{-n}), y_{-n}, y_{-(n-1)}, ..., y_n, g_{\mu_0}(y_n), g_{\mu_1}(g_{\mu_0}(y_n)), ... \}
\]

is an orbit of \( \mathcal{G} \).

This is clear that \( \rho(\mathcal{F}, \mathcal{G}) < \gamma \) and hence \( J_\mathcal{F} = J_\mathcal{G} \). So there is an orbit \( z' \) of \( \mathcal{F} \) such that for any \( 1 \leq i \leq k \) if \( z' \cap V_i \neq \emptyset \) then \( z \cap V_i \neq \emptyset \). Thus \( z \subset B_\epsilon(z') \) and consequently \( z' \subset B_\epsilon(x) \).

**Proof of Claim:** Suppose that \( J_\mathcal{F} = \{ L_1, L_2, ..., L_m \} \). For any \( 1 \leq j \leq m \), there is an orbit \( x^j \) such that \( x^j \cap U_{j_i} \neq \emptyset \) for all \( j_i \in L_j \). So there exist \( \epsilon_j > 0 \) such that \( \rho(\mathcal{F}, \mathcal{G}) < \epsilon_j \) implies that, for an orbit \( y^j \) of \( \mathcal{G} \) such that \( y^j \cap U_{j_i} \neq \emptyset \) for all \( j_i \in L_j \). Thus \( L_j \in \). Take \( \epsilon = \min \{ \epsilon_1, \epsilon_2, ..., \epsilon_m \} \), similar argument shows that if \( \rho(\mathcal{F}, \mathcal{G}) < \epsilon \) then \( L_j \in J_\mathcal{G} \) for all \( 1 \leq j \leq m \).

By Theorem 3.8 and proof of Theorem 2 in [6] we have the following theorem.

**Theorem 3.9.** If the space \( X \) is one of the following:

(i) a topological manifold with boundary (\( \dim(X) \geq 2 \) if \( \partial X \neq \emptyset \)),

(ii) a Cartesian product of a countably infinite number of manifolds with nonempty boundary,

(iii) a Cantor set,

then \( \mathcal{F} \)-weak shadowing is a generic property in \( H_\Lambda(X) \).
WEAK SHADOWING PROPERTY FOR IFS

REFERENCES

[1] M.F. Barnsley, Fractals everywhere, Academic Press, Boston, (1988).

[2] M. F. Barnsley and A. Vince, The Conley attractor of an iterated function system, 
    \texttt{arXiv:1206.6319v1}

[3] R. Corless and S.Yu. Pilyugin, Approximate and real trajectories for generic dynamical 
    systems. J. Math. Anal. Appl. 189 (1995) 409-423.

[4] E. M. Coven, I. Kan, and J. A. Yorke, Pseudo-orbit shadowing in the family of tent 
    maps, Trans. Amer. Math. Soc. 308 (1988), 227-241.

[5] V. Glavan and V. Gutu, Shadowing in parameterized IFS, Fixed Point Theory, 7 (2006), 
    no. 2, 263-274.

[6] M. Mazure, Weak shadowing for discrete dynamical systems on nonsmooth manifolds, 
    J. Math. Anal. Appl. 281 (2003), 657-662.

[7] J. Palis Jr. and W. de Melo, Geometric theory of dynamical systems: an introduction, 
    Springer-Verlage, New York, 1982.

[8] K. Palmer, Shadowing in Dynamical Systems. Theory and Applications, Kluwer Acad. 
    Publ., Dordrecht, (2000).

[9] S. Yu. Pilyugin, Shadowing in dynamical systems, Speringer, Berlin, (1999).

[10] K. Sakai, Various shadowing properties for positively expansive maps, Topology. Appl. 
    131(2003), no. 10, 15-31.

DEPARTMENT OF MATHEMATICS, YAZD UNIVERSITY, P. O. BOX 89195-741 YAZD, 
IRAN, E-MAIL:FATEHNIAM@YAZD.AC.IR