$\eta'$ Mass and Chiral Symmetry Breaking at Large $N_c$ and $N_f$

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Abstract

We propose a method for implementing the large-$N_c$, large-$N_f$ limit of QCD at the effective Lagrangian level. Depending on the value of the ratio $N_f/N_c$, different patterns of chiral symmetry breaking can arise, leading in particular to different behaviors of the $\eta'$-mass in the combined large-$N$ limit.
1. Large-$N_c$ considerations successfully explain many non-perturbative aspects of confining gauge theories [1,2]. However, there are at least two exceptions—both related to a strong OZI rule violation—in which the $1/N_c$ expansion apparently fails: (i) In the scalar channel the spectrum is not dominated by a nonet of ideally mixed states and chiral symmetry breaking exhibits an important dependence on the number $N_f$ of light quark flavors [3]; (ii) At large $N_c$, the $\eta'$-field becomes massless due to its relation to the U(1) anomaly while Nature realizes it like a heavy state. In this note we reconsider these problems in the limit in which both $N_f$ and $N_c$ tend to infinity with fixed ratio [4]. Since, at least at lowest orders of perturbation theory, the (rescaled) QCD $\beta$-function (with $g^2 N_c \equiv \text{const.}$) only depends on the ratio $N_f/N_c$, we might expect that the hadronic spectrum resembles the physical one, in particular with chiral symmetry breakdown and $\Lambda_H \sim 1 \text{ GeV}$. On the other hand, several hints (e.g. from the study of the conformal window, in QCD and its supersymmetric version), suggest a non trivial phase structure of the theory as a function of $N_f$ and $N_c$. One can in principle distinguish three different phases, characterized by different symmetries of the vacuum, depending on the ratio $N_f/N_c$: (a) for low $N_f/N_c$ only the $\text{SU}_V(N_f)$ remains unbroken; (b) for higher $N_f/N_c$ the vacuum is invariant under a larger group, $\text{SU}_V(N_f) \times Z_{\text{chiral}}(N_f)$, where $Z_{\text{chiral}}(N_f)$ is the center of the chiral symmetry group $\text{SU}_L(N_f) \times \text{SU}_R(N_f)$ [5,6]; (c) for high $N_f/N_c$ no spontaneous symmetry breaking takes place (and hence no confinement) and the symmetry of the vacuum is the whole $\text{SU}_L(N_f) \times \text{SU}_R(N_f)$. Notice that case (b) corresponds to the maximal possible symmetry of the vacuum in a confining vector-like theory. The existence of this phase is an assumption related to the issue of the non-perturbative renormalization of the bare Weingarten’s inequalities comparing axial-axial and pseudoscalar-pseudoscalar two point functions [6,7]. We model the combined large-$N_f$, large-$N_c$ limit by adding to the usual light flavors $q = (u, d, s)$, a set of $N$ auxiliary flavors $Q = (Q_1, \ldots, Q_N)$ of common mass $M \gg m_q$, but still $M \ll \Lambda_H$. This mass $M$ should be considered sufficiently small so that a power series expansion makes sense, but simultaneously much larger than any of the light quark masses $m_q$, thus the auxiliary fields can be integrated out at sufficiently low-energy. We then formally deal with $N_f = N + 3 \equiv n \to \infty$.
flavors, but only the three lightest ones are physical. The rôle of these auxiliary flavors should be analogous to the one of the strange quark, when one considers the SU(2) × SU(2) chiral dynamics of u and d quarks.

2. Let $N_f/N_c$ be subcritical, so that we are in the Z-chiral($n$)-asymmetric phase. The $n^2 - 1$ (pseudo) Goldstone bosons (GB) can be collected in a matrix $\hat{U}(x) \in SU(n)$, (hereafter $n \times n$ matrices will be denoted by a hat) and their low-energy dynamics can be described by the effective Lagrangian

$$L_{\text{sub}} = \frac{F^2}{4} \left\{ \langle D_\mu \hat{U} D^\mu \hat{U}^\dagger \rangle + 2B_0 \langle \hat{U}^\dagger \hat{\chi} + \hat{\chi}^\dagger \hat{U} \rangle \right\}. \tag{1}$$

where $\hat{\chi}$ is the scalar-pseudoscalar source,

$$L_{\chi}^{\text{QCD}} = -\bar{\Psi}_L \hat{\chi} \Psi_R - \bar{\Psi}_R \hat{\chi}^\dagger \Psi_L, \quad \Psi = \begin{pmatrix} q \\ Q \end{pmatrix}, \tag{2}$$

and $\langle \ldots \rangle$ denotes flavor trace. The $n \times n$ source matrix will be chosen as

$$\hat{\chi} = \begin{pmatrix} \chi & 0 \\ 0 & M e^{i\theta/N} \mathbf{1}_{N \times N} \end{pmatrix}, \tag{3}$$

where $\chi$ is the $3 \times 3$ light quark source (mass term), $\theta$ is the vacuum angle and $M$ is real and positive. Notice that there are no sources attached to the $N$ auxiliary flavors and the corresponding GB degrees of freedom are frozen: in the tree approximation the $n \times n$ GB field matrix becomes

$$\hat{U} = \begin{pmatrix} U e^{i\varphi/3} & 0 \\ 0 & e^{-i\varphi/N} \mathbf{1}_{N \times N} \end{pmatrix} \in SU(n), \tag{4}$$

where $U \in SU(3)$ collects the eight physical GB fields. The remaining U(1) field $\varphi$ will be interpreted as the $\eta'$-field: for $\chi = m \mathbf{1}_{3 \times 3}$ (no mixing),

$$\eta' = F \sqrt{\frac{n}{6N}} \varphi \tag{5}$$

and the corresponding mass, for $m = 0$ is
\[ M^2_{\eta'} = \frac{6B_0M}{n} \to 0 , \]  

which vanishes in the (combined) large-\( N \) limit. Hence, in this phase, \( \eta' \)-mass behaves as in the standard \( N_c \to \infty, N_f \)-fixed limit.

3. For higher \( N_f/N_c \) we expect the \( Z_{\text{chiral}}(n) \)-symmetric phase to occur: the vacuum is symmetric under

\[ \Psi_{L,R} \to e^{2\pi i k_{L,R}/n} \Psi_{L,R}, \quad k_{L,R} = 1, \ldots, n - 1 , \]  
in addition to the usual \( SU_V(n) \). Notice that this \( Z_{\text{chiral}}(n) \) is also a subgroup of the (anomalous) \( U_L(1) \times U_R(1) \). This additional symmetry of the vacuum finds its natural interpretation within the effective theory described by the Lagrangian \( \mathcal{L}(\hat{U}, \hat{\chi}, \theta) \). The GB field \( \hat{U}(x) \in SU(n) \) is usually understood as simply connected to \( 1 \): \( \hat{U}(x) = 1 + i\varphi_a(x)T^a + \ldots \) and, in the corresponding effective theory, the integration measure \( D\hat{U} \) is treated accordingly. However \( SU(n) \) is not simply connected. We are free to choose an integration measure treating all sectors of \( SU(n) \) alike:

\[ \int D\hat{U} e^{i\int dx \mathcal{L}(\hat{U}, \hat{\chi}, \theta)} \to \int D\hat{U} \sum_{k=0}^{n-1} e^{i\int dx \mathcal{L}(\hat{U}e^{-2\pi i k/n}, \hat{\chi}, \theta)} , \]  

where \( \hat{U} \) and \( D\hat{U} \) again concern the connected vicinity of \( 1 \). This freedom derives from the fact that an effective theory is merely constrained by Ward identities (WI), which fix its local but not its global aspects. In particular the usual solution of the anomalous \( U(1) \) WI only guarantees \( \mathcal{L}(\hat{U}, \hat{\chi}, \theta) = \mathcal{L}(\hat{U}, \hat{\chi}e^{i\theta/n}) \) for \( \theta \sim 0 \). However, under the \( Z_{\text{chiral}}(n) \) transformation one has

\[ \mathcal{L}(\hat{U}e^{-\frac{2\pi ik}{n}}, \hat{\chi}e^{i\theta/n}) = \mathcal{L}(\hat{U}, \hat{\chi}e^{i\theta+2\pi k/n}) , \]  
i.e. the above prescription \((8)\) restores the \( 2\pi \)-periodicity in the vacuum angle \([8]\). The \( Z_{\text{chiral}}(n) \)-symmetry of the vacuum expressed in terms of a local Lagrangian amounts to the constraint
\[ \mathcal{L}(\hat{U}e^{-\frac{2\pi ik}{n}}, \hat{\chi}, \theta) = \mathcal{L}(\hat{U}, \hat{\chi}, \theta) = \mathcal{L}(\hat{U}, \hat{\chi}e^{i\theta/n}) , \]  

which is not necessarily true in general and it will be taken as a definition of the Z\textsubscript{chiral}(-symmetric phase. The effective Lagrangian exhibiting Z\textsubscript{chiral}(n)-symmetry consists of two parts, \( \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \). \( \mathcal{L}_1 \) has the whole continuous U\textsubscript{A}(1) symmetry, \( \hat{\chi} \rightarrow e^{i\alpha} \hat{\chi} \), i.e. it is independent of the \( \theta \) angle,

\[ \mathcal{L}_1 = \frac{F^2}{4} \left\{ \langle D_\mu \hat{U} D^\mu \hat{U}^\dagger \rangle + Z \langle \hat{U}^\dagger \hat{\chi} \rangle \langle \hat{\chi} \hat{U} \rangle + \ldots \right\} , \tag{11} \]

where dots stand for pure source and higher orders terms. \( \mathcal{L}_2 \) consists of terms which break U\textsubscript{A}(1) down to Z\textsubscript{chiral}(n). The lowest order term with this property reads

\[ \mathcal{L}_2 = \sum_{k} A^{(k)}_{\{j_1 \ldots j_k\}} \langle (\hat{U}^\dagger \hat{\chi})^{j_1} \rangle \ldots \langle (\hat{U}^\dagger \hat{\chi})^{j_k} \rangle + \text{h.c.} , \tag{12} \]

where \( j_1, \ldots, j_k = 1, \ldots, n, \quad j_1 + \ldots + j_k = n \).

Despite the fact that the Lagrangians (11) and (12) are of different orders in \( \hat{\chi} \) they can coexist, since they describe two independent sectors of the effective theory: Eq. (12) is holomorphic and is not renormalized by loops arising from the U(1)-invariant sector as represented by Eq. (11). Eq. (12) becomes more transparent using Eqs. (3)-(4) and expanding in powers of the light quark masses \( \chi \). Denoting by \( W \) the 3 × 3 matrix and by \( \zeta \) the phase factor,

\[ W = U^\dagger \chi e^{-i\frac{\varphi}{2}}, \quad \zeta = e^{i\frac{\theta + \varphi}{N}}, \tag{13} \]

Eq. (12) can be rewritten as

\[ \mathcal{L}_2 = a_n (M\zeta)^n + b_n \langle W \rangle (M\zeta)^{n-1} + c_n \langle W^2 \rangle (M\zeta)^{n-2} \]

\[ + d_n \langle W \rangle^2 (M\zeta)^{n-2} + \text{h.c.} + \mathcal{O}(W^3) . \tag{14} \]

Similarly, the reduction SU(n)→SU(3)×U(1) of the component \( \mathcal{L}_1 \) can be expressed in terms of the variables (13) as

\[ \mathcal{L}_1 = \frac{F^2}{4} \left\{ \langle D_\mu \hat{U}^\dagger D^\mu \hat{U} \rangle + \frac{n}{3N} \partial_\mu \varphi \partial^\mu \varphi \right. \]

\[ + MNZ \langle W \zeta^\dagger + W^\dagger \zeta \rangle + Z \langle W^\dagger \rangle \langle W \rangle \right\} + \ldots . \tag{15} \]
FIG. 1. Representation of the contribution to the induced condensate Eq. (17). The cross refers to $\overline{Q}Q$ insertion.

In the tree approximation the $\eta'$-mass merely arises from the holomorphic part $\mathcal{L}_2$, $(m_u = m_d = m_s = 0)$,

$$M_{\eta'}^2 = \frac{12N}{n} \frac{1}{F^2} a_n M^n. \quad (16)$$

In contrast to the $Z_{\text{chiral}}(n)$-asymmetric phase, [cf. Eq. (1)], which contains a genuine condensate term $B_0$, such a term is absent in the $Z_{\text{chiral}}(n)$-symmetric phase. However, in the reduction (3)-(4), an induced condensate appears through the OZI violation terms [see Fig. (1)] (the $Z_{\text{chiral}}(n)$-symmetry is explicitly broken by the $Q$-mass term),

$$F^2 B_{\text{induced}} = \frac{1}{2} F^2 M N Z + 2b_n M^{n-1}, \quad (17)$$

where the first term on the r.h.s. arises from $\mathcal{L}_1$, whereas the second term comes from the holomorphic Lagrangian. The quadratic terms in $W$ of Eq. (14) contribute to the subleading low-energy constants $L_6, L_7, L_8$ [9], denoted by hat

$$B_{\text{induced}}^2 (\hat{L}_6 - \hat{L}_7) = \frac{1}{32} F^2 Z,$$

$$B_{\text{induced}}^2 \hat{L}_8 = \frac{1}{4} c_n M^{n-2}, \quad B_{\text{induced}}^2 (\hat{L}_6 + \hat{L}_7) = \frac{1}{4} d_n M^{n-2}. \quad (18)$$

4. We now turn to the leading behavior of all these induced low-energy constants in the combined large-$N$ limit. We consider (connected) correlators of quark bilinears $\bar{\Psi} \Gamma \Psi$. 6
Usual large-$N_c$ counting rules are maintained. In addition, every quark loop gives rise to a flavor trace involving all flavor matrices contained in that loop. Consequently, each internal ("sea") quark loop will be enhanced by a factor $N_f = n$ and suppressed (as usual) by a factor $1/N_c$. (However, if a quark loop is valence, i.e. attached to an external flavor source, it will not lead to a flavor-enhancement factor.) In particular, the constants $F^2$ and $Z$ in Eq. (11) behave as $F^2 \sim N$ and $Z \sim 1/N_c$. The contribution to the fermionic determinant can be formally written (in a large Euclidean box) like

$$
\Delta = \exp \sum_k \log \left(1 + \frac{\lambda_k^2 - \omega_k^2}{\omega_k^2 + M^2}\right)^n,
$$

(19)

where $\lambda_k$ are Dirac operator eigenvalues and $\omega_k$ the corresponding eigenvalues in the absence of interactions. Hence at large-$N_c$ one expects $\lambda_k^2 - \omega_k^2 \sim \mathcal{O}(g^2) \sim \mathcal{O}(1/N_c)$. This illustrates the mechanism by which the fermionic determinant stays non trivial and finite in the combined large-$N$ limit, merely depending on the ratio $N_f/N_c$. The large-$N$ counting of the holomorphic part is more subtle: we deal with a large-$N_f$, large-$N_c$ behavior of a large-$N$-point function. In the tree approximation, the holomorphic part of the Lagrangian (at $\hat{U} = 1$) is connected with a QCD correlation function in a Euclidean finite volume $V$

$$
-\mathcal{L}_2(1, \hat{\chi}) = -\sum_{k}^{n-1} \sum_{\{j_1 \ldots j_k\}}^{} A_{\{j_1 \ldots j_k\}}^{(k)} \langle \hat{\chi}^{j_1} \rangle \ldots \langle \hat{\chi}^{j_k} \rangle = \frac{1}{n!} \frac{1}{V} \langle \int dx \bar{\Psi} L \hat{\chi} \Psi R(x) \rangle^n_{\text{con}}.
$$

(20)

The average on r.h.s. of Eq. (20) can be evaluated at non zero masses $m$ and $M$. The other chirality part $\bar{\Psi}_R \Psi_L$ will contribute but not to the lowest order $\hat{\chi}^n$ of the holomorphic part of the Lagrangian. Let us introduce the notation

$$
K = \int dx \sum_{i=1}^{N} \bar{Q}_L^i(x) Q_R^i(x),
$$

$$
k_i = \int dx \bar{q}_L^i(x) q_R(x), \quad i = 1, 2, 3.
$$

(21)

Choosing in Eq. (20) $\hat{\chi} = \text{diag}(m_1, m_2, m_3, M, \ldots, M)$, and combining it with Eq. (14) one gets

$$
- \left\{ a_n M^n + b_n M^{n-1}(m_1 + m_2 + m_3) \right\}
$$
Comparing the coefficients of $M^n$

$$ a_n M^n = -\frac{1}{n!} M^n \frac{1}{V} \langle K^n \rangle_{\text{con}}. \quad (23) $$

The single quark loop contribution to Eq. (23) reads

$$ \frac{1}{V} \langle K^n \rangle = -(n - 1)! \frac{1}{V} \langle \left\langle \int dx_1 \ldots dx_n \text{Tr} \left\{ S^{RL}(x_1, x_2) \right\} \right. $$
$$ S^{RL}(x_2, x_3) \ldots \left. S^{RL}(x_n, x_1) \right\} \rangle \text{Tr}(1_{N \times N}), \quad (24) $$

where $S^{RL}(x, y)$ denotes the chiral part of the fermion propagator

$$ S^{RL}(x, y) = \left( \frac{1 + \gamma_5}{\sqrt{2}} \right) \sum_{\lambda_k \geq 0} \frac{M}{M^2 + \lambda_k^2} \varphi_k(x) \varphi_k^\dagger(y) \left( \frac{1 + \gamma_5}{\sqrt{2}} \right) $$

in terms of the orthonormal Fujikawa chiral basis [10] ($\varphi_k$ is the Dirac eigenvector belonging to the eigenvalue $\lambda_k$) and $\langle \langle \ldots \rangle \rangle$ stands for the average over gluon configurations with insertion of fermionic determinant. The factor $(n - 1)!$ in Eq. (24) counts the different ways of connecting $n$ points by a single one quark loop. In fact a closer examination of the combinatorics of multiloop diagrams’ contributions to $\langle K^n \rangle$ in Eq. (24) reveals that none of them is more important that the one with the least number of quark loops. The integrals in Eq. (24) can be performed with the result

$$ a_n M^n \sim \lim_{V \to \infty} \frac{1}{V} \langle \left\langle \sum_{\lambda_k \geq 0} \left( 1 + \frac{\lambda_k^2}{M^2} \right)^{-n} \right\rangle \rangle. \quad (25) $$

Even if we do not consider here the chiral limit $M \to 0$, the behavior of Eq. (25) is merely controlled by the average density of small Dirac eigenvalues. Indeed, for any fixed $M$ and (arbitrary small) $\epsilon$ the eigenvalues $\lambda_k^2 \geq \epsilon$ do not contribute to the large-$N$ limit of Eq. (25).

The latter should be of the order of the average number of states $N_{\epsilon}$ with $\lambda_k^2 \leq \epsilon$. On general grounds one expects $N_{\epsilon} \sim VN_{\epsilon}$ [11]. (The density of states should grow proportionally with $N_{\epsilon}$.) Hence, in the combined large-$N$ limit $a_n M^n \sim O(N_{\epsilon})$ and according to Eq. (16)
\[ M_{\eta'}^2 \sim \text{const}, \quad (26) \]

thus not suppressed anymore. Similar conclusions have been reached also in different contexts \[12,13\]. The remaining coefficients in Eq. (22) can be found similarly:

\[ b_n M_n^{n-1} \sim -\frac{M^{n-1}}{(n-1)!} \frac{1}{V} \langle K^{n-1}k_1 \rangle_{\text{con}}, \quad (27) \]

\[ (c_n + d_n) M_n^{n-2} \sim -\frac{M^{n-2}}{(n-2)!} \frac{1}{V} \langle K^{n-2}k_1^2 \rangle_{\text{con}}, \quad (28) \]

receiving leading contribution from at least two quark loops and consequently suppressed by \(1/N_c\) relative to Eq. (25). Finally

\[ d_n M_n^{n-2} \sim -\frac{M^{n-2}}{(n-2)!} \frac{1}{V} \langle K^{n-2}k_1^2 k_2 \rangle_{\text{con}}, \quad (29) \]

which involves at least three quark loops. This leads to the final estimate

\[ b_n M_n^{n-1} \sim c_n M_n^{n-2} \sim O(1), \quad d_n M_n^{n-2} \sim O(1/N_c). \quad (30) \]

As a consequence, the holomorphic contribution to the induced condensate, Eq. (17), is suppressed relative to the non-holomorphic one. The latter is given by the OZI rule violating constant \(Z\) which is suppressed by \(1/N_c\) but this suppression is compensated by a flavor enhancement factor \(N = n - 3\). As a result

\[ F^2 B_{\text{induced}} \sim O(N) + O(1), \quad (31) \]

where the first term is the non-holomorphic and the second one the holomorphic contribution. For the tree contribution quoted in Eq. (18) the large-\(N\) counting reads

\[ B_{\text{induced}}^2 (\hat{L}_6 - \hat{L}_7) \sim B_{\text{induced}}^2 \hat{L}_8 \sim O(1), \]

\[ B_{\text{induced}}^2 (\hat{L}_6 + \hat{L}_7) \sim O(1/N). \quad (32) \]

5. More comments on the large-\(N\) behavior of the \(\eta'\)-mass are in order. The usual argument for finding the behavior of the \(\eta'\)-mass \[2\] derives from the necessity to cancel
the $\theta$-dependence of the pure gluodynamics when massless quarks are added. This is only possible when the $1/N_c$ suppression of the internal quark loops is compensated by the $\eta'$-pole contribution $M_{\eta'}^{-2}$. This leads to the Veneziano-Witten's formula and the vanishing of $M_{\eta'}^2$ as $1/N_c$. On the other hand, in the combined large-$N$ limit internal quark loops are not suppressed, and it is not possible to isolate pure glue contributions by large-$N$ arguments. As a consequence the $\eta'$-mass does not vanish anymore and its relation to the topological susceptibility is lost. We may as well consider the $\eta'$-field as heavy and integrate it out. At tree level this amounts to the shift in the constant $L_7$

$$L_7 = \hat{L}_7 - \frac{F^2}{48 M_{\eta'}^2}, \quad L_i = \hat{L}_i, \quad (i \neq 7),$$

(33)
as it is seen by evaluating the singlet minus octet pseudoscalar two-point function [14].

6. Let us summarize the results of this work. We have asked whether the combined large-$N$ limit (as defined in this paper) helps understanding the peculiar properties of QCD in the vacuum and $\eta'$-channels. (i) In the scalar channel, this limit suggests a flavor enhancement of the OZI rule violation, leading in particular to the emergence of an induced quark condensate [Eq. (17)] on top of the genuine condensate $B_0$ [c.f. Eq. (1)]. In the large $N_f/N_c$ phase in which the genuine condensate is forbidden due to the $Z_{\text{chiral}}(n)$-symmetry, the induced condensate plays the rôle of $B_0$ in describing chiral symmetry breaking. An induced condensate would manifest itself by an important flavor dependence [as in Eq. (17)] and it could be, in principle, disentangled from $B_0$ in this way [3]. (ii) A distinctive feature of the $Z_{\text{chiral}}(n)$-symmetric phase is the non-vanishing $\eta'$-mass in the combined large-$N$ limit. (Since this phase is expected for large $N_f/N_c$, the usual large-$N_c$, fixed-$N_f$ arguments do not apply.) In particular, the relation of the $M_{\eta'}$ to the axial anomaly and to the topological susceptibility of the pure YM theory are now modified. (iii) The place of the scale $M$ in building $\eta'$-mass, induced condensate and the low-energy constants $L_6, L_7$ and $L_8$ is important and not entirely understood. Special attention should be payed to the rôle of $M$ in the low-energy expansion and in the renormalization of the whole effective theory. We have
calculated the one-loop contribution to all the above mentioned quantities. Qualitatively, they do not modify any of our conclusions.

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REFERENCES

[1] G. t’Hooft, Nucl. Phys. B**72**, 461 (1974); ibid. B**75**, 461 (1974).

[2] G. Veneziano, Nucl. Phys. B**159** (1979) 213; E. Witten, Nucl. Phys. B**156**, 269 (1979); ibid. B**160**, 57 (1979).

[3] B. Moussallam, Eur. Phys. J. C **14** (2000) 111; S. Descotes, L. Girlanda and J. Stern, JHEP**0001** (2000) 041; S. Descotes, JHEP**0103** (2001) 002.

[4] G. Veneziano, Nucl. Phys. B**117**, 519 (1976).

[5] R. Dashen, Phys. Rev. **183** (1969) 1245.

[6] I. I. Kogan, A. Kovner and M. Shifman, Phys. Rev. D **59**, 016001 (1999).

[7] D. Weingarten, Phys. Rev. Lett. **51** (1983) 1830.

[8] I. Halperin and A. Zhitnitsky, Phys. Rev. Lett. **81**, 4071 (1998); Phys. Rev. D **58**, 054016 (1998).

[9] J. Gasser and H. Leutwyler, Nucl. Phys. B**250**, 465 (1985).

[10] K. Fujikawa, Phys. Rev. D **29**, 285 (1984).

[11] H. Leutwyler and A. Smilga, Phys. Rev. D **46**, 5607 (1992).

[12] S. D. Hsu, F. Sannino and J. Schechter, Phys. Lett. B **427** (1998) 300.

[13] P. Minkowski, Phys. Lett. B **423** (1998) 157.

[14] H. Leutwyler, Nucl. Phys. B **337** (1990) 108.