Holomorphic functions in generalized Cayley-Dickson algebras

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Abstract. In this paper we investigated some properties of holomorphic functions (belonging to the kernel of the Dirac operator) defined on domains of the real Cayley-Dickson algebras. For this purpose, we study first some properties of these algebras, especially multiplication tables for certain elements of the basis. Using these properties, we provided an algorithm for constructing examples of the class of functions under consideration.

Keywords: Cayley-Dickson and generalized Cayley-Dickson algebras; Dirac operator; holomorphic functions.

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0. Introduction

The theory of quaternionic differentiable functions has many applications in different areas of mathematics, physics and in other applied sciences (see, for example, [3], [4]). This theory has its origins in the paper [7] in which the authors proposed, for the first time, an analogue of the Cauchy-Riemann conditions in three-dimensional case. For the four-dimensional case, an analogue of these conditions was considered in the paper [2] and, as a next step of this generalization, the differentiable functions in the octonionic algebra was considered in the papers [11], [12].

Generalization of the Cauchy-Riemann conditions in all algebras obtained by the Cayley-Dickson process (called Cayley-Dickson algebras) was done in the paper [6], where differentiable functions of variables belonging to Cayley-Dickson algebras were defined. For such functions, was established analogues results with the main results of complex analysis, results which can be successfully used in the further studies of special functions of variables with values in Cayley–Dickson algebras.

Comparing with [6], in the present paper, we investigate another class of differentiable functions (using the Dirac operator) in Cayley-Dickson algebras and, more important, we provide an example of this kind of functions and an
algorithm to find such as examples. Since these functions are rather complicated objects, it is quite important to have a way to generate examples.

The paper is organized in two sections. In the first section, we briefly presented some properties of algebras obtained by the Cayley-Dickson process and the algorithm described by J. W. Bales regarding an easy way to multiply the elements from a basis in such algebras (by using *exclusive or* operation and a *twist map*). In the second section, by description the multiplication tables for certain elements of the basis (Propositions 2.2 and 2.3), we obtained the main result of this work: an example of a left hyperholomorphic function in generalized Cayley-Dickson algebras (Theorem 2.12). Moreover, in Theorem 2.10 we proved that for studying left $A_t$-holomorphic functions in generalized Cayley-Dickson algebras $A_t = \left( \frac{\text{sign}(\gamma_1), \ldots, \text{sign}(\gamma_t)}{R} \right)$, that means this study depends only by the sign of the real numbers $\gamma_1, \gamma_2, \ldots, \gamma_t$.

1. Preliminaries

Let $K$ be a commutative field with $\text{char}K \neq 2$ and $A$ be an algebra over the field $K$. A unitary algebra $A \neq K$ such that we have $x^2 + \alpha_x x + \beta_x = 0$, for each $x \in A$, with $\alpha_x, \beta_x \in K$, is called a *quadratic algebra*.

In the following, we briefly present the *Cayley-Dickson process* and the properties of the algebras obtained. For details about the Cayley-Dickson process, the reader is referred to [9] and [10].

Let $A$ be a finite dimensional unitary algebra over a field $K$ with a scalar involution

$$\overline{a} : A \to A, \quad a \to \overline{a},$$

i.e. a linear map satisfying the following relations:

$$\overline{ab} = \overline{b} \overline{a}, \quad \overline{a} = a,$$

and

$$a + \overline{a}, a\overline{a} \in K \cdot 1 \text{ for all } a, b \in A.$$

The element $\overline{a}$ is called the *conjugate* of the element $a$, the linear form

$$t : A \to K, \quad t(a) = a + \overline{a}$$

and the quadratic form

$$n : A \to K, \quad n(a) = a\overline{a}$$

are called the *trace* and the *norm* of the element $a$. Hence an algebra $A$ with a scalar involution is quadratic.

Let $\gamma \in K$ be a fixed non-zero element. We define the following algebra multiplication on the vector space

$$A \oplus A : (a_1, a_2) (b_1, b_2) := (a_1 b_1 + \gamma b_2 \overline{a_2}, \overline{a_1} b_2 + b_1 a_2). \quad (1)$$
We obtain an algebra structure over $A \oplus A$, denoted by $(A, \gamma)$ and called the algebra obtained from $A$ by the Cayley-Dickson process or simply generalized Cayley-Dickson algebra. We have $\dim (A, \gamma) = 2 \dim A$.

Let $x \in (A, \gamma)$, $x = (a_1, a_2)$. The map

$$\overline{\cdot} : (A, \gamma) \to (A, \gamma), \ x \to \overline{x} = (\overline{a_1}, -a_2),$$

is a scalar involution of the algebra $(A, \gamma)$, extending the involution of the algebra $A$.

If we take $A = K$ and apply this process $t$ times, $t \geq 1$, we obtain an algebra over $K$, $A_t = (\gamma_1, ..., \gamma_t_K)$. By induction in this algebra, the set $\{ e_0 = 1, e_1, ..., e_{n-1} \}$, $n = 2^t$, generates a basis with the properties:

$$e_i^2 = \gamma_i 1, \ \gamma_i \in K, \gamma_i \neq 0, \ i = 1, ..., n - 1 \quad (2)$$

and

$$e_ie_j = -e_j e_i = \beta_{ij} e_k, \ \beta_{ij} \in K, \ \beta_{ij} \neq 0, \ i \neq j, \ i, j = 1,...n - 1, \quad (3)$$

$\beta_{ij}$ and $e_k$ being uniquely determined by $e_i$ and $e_j$.

From [10], Lemma 4, it results that in an algebra $A_t$ with the basis $B = \{ e_0 = 1, e_1, ..., e_{n-1} \}$ satisfying relations (2) and (3) we have:

$$e_i (e_i x) = \gamma_i^2 x = (xe_i)e_i, \quad (4)$$

for all $i \in \{ 1, 2, ..., n - 1 \}$ and for every $x \in A$.

The algebras $A_t$, in general, are neither commutative and nor associative algebras, but are flexible (i.e. $x(yx) = (xy)x = xyx$, for all $x, y \in A_t$) quadratic and power associative (i.e. the subalgebra $<x>$ of $A$, generated by any element $x \in A$, is associative).

Remark 1.1. For $\gamma_1 = ... = \gamma_t = -1$ and $K = \mathbb{R}$, in [1], the author described how we can multiply the basis vectors in the algebra $A_t, \dim A_t = 2^t = n$. He used the binary decomposition for the subscript indices.

Let $e_p, e_q$ be two vectors in the basis $B$ with $p, q$ representing the binary decomposition for the indices of the vectors, that means $p, q$ are in $\mathbb{Z}_2^n$. We have that $e_p e_q = \gamma_n(p, q) e_{p \oplus q}$, where:

i) $p \oplus q$ are the sum of $p$ and $q$ in the group $\mathbb{Z}_2^n$ or, more precisely, the "exclusive or" for the binary numbers $p$ and $q$;

ii) $\gamma_n$ is a function $\gamma_n : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \{-1, 1\}$.

The map $\gamma_n$ is called the twist map.

The elements of the group $\mathbb{Z}_2^n$ can be considered as integers from 0 to $2^n - 1$ with multiplication "exclusive or" of the binary representations. Obviously, this operation is equivalent with the addition in $\mathbb{Z}_2^n$. 

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From now on, in whole the paper, we will consider $K = \mathbb{R}$. Using the same notations as in the Bales’s paper, we consider the following matrices:

$$A_0 = A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$ \hspace{1cm} (5)

In the same paper [1], the author find the properties of the twist map $\gamma_n$ and put the signs of this map in a table. He partitioned the twist table for $\mathbb{Z}_2^n$ into $2 \times 2$ matrices and obtained the following result:

**Theorem 1.2.** ([1], Theorem 2.2., p. 88-91) For $n > 0$, the Cayley-Dickson twist table $\gamma_n$ can be partitioned in quadratic matrices of dimension 2 of the form $A, B, C, -B, -C$, defined in the relation (5). Relations between them can be found in the below twist trees:

![Twist trees](image)

**Fig. 1: Twist trees([1], Table 9)**

**Definition 1.3.** Let $x = x_0, x_1, x_2, \ldots$ and $y = y_0, y_1, y_2, \ldots$ be two sequences of real numbers. The ordered pair

$$(x, y) = x_0, y_0, x_1, y_1, x_2, y_2, \ldots$$

is a sequence obtained by **shuffling** the sequences $x$ and $y$.

Using Theorem 1.2, in [1], the author gave the below algorithm for find $\gamma_n(s, r)$, where $s, r \in \mathbb{Z}_2^n$:

i) We find the shuffling sequence $(s, r)$.

ii) Starting with the root $A_0$, we can find $\gamma_n(s, r)$ using the twist tree. We remark that "$00" = unchanged, "$01" =left $\rightarrow$right, "$10" =$right$ $\rightarrow$left, "$11" =$right$ $\rightarrow$ right.

Let $\mathbb{H}(\gamma_1, \gamma_2)$ be the generalized quaternion algebra and $\mathbb{H}(-1, -1)$ be the quaternion division algebra. Below, you can see the multiplication tables:
Example 1.4. Let $A_4$ be the real sedenion algebra. That means $\dim A_4 = 16$ with $\{1, e_1, ..., e_{15}\}$ a basis in this algebra. Let compute $e_7e_{13} = \gamma_4(7_2, 13_2)e_7 \otimes e_{13}$. We have the following binary decompositions:

$$
\begin{align*}
7_2 &= 0111, \text{ since } 7 = 2^2 + 2 + 1 \text{ and } \\
13_2 &= 1101, \text{ since } 13 = 2^3 + 2^2 + 1.
\end{align*}
$$

Since $0111 \otimes 1101 = 1010(= 2^3 + 2 = 10)$, it results that $7 \otimes 13 = 10$.

Now, we compute $\gamma_4(e_7, e_{13})$. First, we shuffle the sequences 0111 and 1101.

We obtain 01 11 10 11. Starting with $A_0$, it results: $A_0 \overset{01}{\rightarrow} A \overset{11}{\rightarrow} -C \overset{10}{\rightarrow} C \overset{11}{\rightarrow} -C$, then $\gamma_4(e_7, e_{13}) = -1$ and $e_7e_{13} = -e_{10}$.
2. Main results

In this section, for a generalized Cayley-Dickson algebra $A_t$, writing the basis’s elements in a convenient way, we can obtain multiplication tables for certain elements of the basis. Using these results, in Theorem 2.12 we provide an example of a left hyperholomorphic function in generalized Cayley-Dickson algebras.

RemarK 2.1. i) In the generalized quaternion algebra, $\mathbb{H}(\gamma_1, \gamma_2)$, the basis can be written as

$$\{1 = e_0, e_1, e_2, e_1 e_2\}.$$ 

For the generalized octonion algebra, $\mathbb{O}(\gamma_1, \gamma_2, \gamma_3)$, the basis can be written

$$\{1 = e_0, e_1, e_2, e_4, e_1 e_4, e_2 e_4, (e_1 e_2) e_4\}.$$ 

Therefore $e_3 = e_1 e_2, e_7 = e_3 e_4 = (e_1 e_2) e_4, e_2 e_4 = e_6$ and, when compute them, in these products do not appear any of the elements $\gamma_1, \gamma_2, \gamma_3$, or products of some of them at the end.

We remark that in the algebra $A_t = (\mathbb{R}^{2^t-2})$ in the products of the form

$$e_1 e_2, (e_1 e_2) e_4, \ldots, ((e_2 e_2^{r+1}) \ldots e_2^{k}) e_2^i,$$

when compute them, do not appear any of the elements $\gamma_1, \gamma_2, \ldots, \gamma_t$ or products of some of them at the end.

ii) Using above remarks, the basis in the algebra $A_t = (\mathbb{R}^{2^t-2})$ can be written under the form

$$\{1 = e_0, e_1, e_2, \ldots, e_2^{t-1}, e_3 e_2^{t-1}, e_4 e_2^{t-1}, \ldots, e_2^{t-1-}, e_2^{t-1} e_2^{t-1}\}$$

with

$$e_i e_2^{t-1} = -e_2^{t-1} e_i = e_2^{t-1} T_i, \quad i \in \{1, 2, \ldots, 2^{t-1} - 1\}.$$  

(7)

Proposition 2.2. Let $A_t = (\mathbb{R}^{2^t-2})$ be an algebra obtained by the Cayley-Dickson process and $\{e_0 = 1, e_1, \ldots, e_{2^t-1}\}, n = 2^t$ be a basis. Let $r \geq 1, r < k \leq i < t$. Therefore

$$(e_2 e_2^{r+1}) \ldots e_2^i) e_2^i = (-1)^{k-r+2} e_T,$$ 

(8)

$$(e_1 e_2^r) e_2^{r+1}) \ldots e_2^k) e_2^i = (-1)^{k-r+3} e_{T+1},$$ 

(9)

where $T = 2^r + 2^{r+1} + \ldots + 2^k + 2^i$ and

$$e_1 e_2^i = e_2^{i+1}.$$ 

(10)
Proof. From Remark 2.1, it results that we can use Theorem 1.2 for \( \gamma_1, \gamma_2, \ldots, \gamma_t \) arbitrary. From Remark 1.1, it results \( T = 2^r + 2^{r+1} + \ldots + 2^k + 2^i \). For \( T \), we have the binary decomposition

\[
T_2 = 100 \ldots 0111 \ldots 10 \ldots 0.
\]

Using the same remark, we obtain \( e_2^r e_2^{r+1} = \gamma_n \begin{pmatrix} 01 \ldots 0 & 10 \ldots 0 \\ r+2 & r+2 \end{pmatrix} e_2^{r+2}. \) We "shuffling" \( 01 \ldots 0 \) and \( 10 \ldots 0 \) and we obtain \( 01 \ \underbrace{10 \ldots 0 \ldots 0 \ldots 0}_{r \text{ pairs}} 00 \ldots 0. \) Starting with \( A_0 \), it results:

\[
A_0 \xrightarrow{01} A \xrightarrow{10} C,
\]

then \( \gamma_n \begin{pmatrix} 01 \ldots 0 & 10 \ldots 0 \\ r+2 & r+2 \end{pmatrix} = 1 \) and \( e_2^r e_2^{r+1} = e_2^{r+2}. \)

We compute \( (e_2^r e_2^{r+1}) e_2^{r+2}. \) We obtain

\[
(e_2^r e_2^{r+1}) e_2^{r+2} = e_2^{r+2} + e_2^{r+1} e_2^{r+2} = \gamma_n \begin{pmatrix} 01 \ldots 0 & 10 \ldots 0 \\ r+3 & r+3 \end{pmatrix} e_2^{r+2} + e_2^{r+3}.
\]

Shuffling \( 011 \ldots 0 \) and \( 10 \ldots 0 \), we get \( 01 \ \underbrace{10 \ldots 0 \ldots 0 \ldots 0}_{r \text{ pairs}} 00 \ldots 0 \). Starting with \( A_0 \), it results: \( A_0 \xrightarrow{01} A \xrightarrow{10} C \xrightarrow{01} -C, \) then

\[
\gamma_n \begin{pmatrix} 01 \ldots 0 & 10 \ldots 0 \\ r+3 & r+3 \end{pmatrix} = -1,
\]

therefore \( e_2^{r+2} + e_2^{r+1} e_2^{r+2} = -e_2^{r+2} - e_2^{r+3} - e_2^{r+4} \). Continuing this procedure, we remark that the number of "1" in the "shuffling" obtained influences the sign. Since \( T = 2^r + 2^{r+1} + \ldots + 2^k + 2^i \) has binary decomposition

\[
T_2 = 100 \ldots 0111 \ldots 10 \ldots 0.
\]

in which we have \( k - r + 2 \) elements equal with 1, we obtain relation (8). In the same way it results relations (9) and (10). □

Proposition 2.3. With the same notations as in Proposition 2.2, for the algebra \( A_t = \left( \begin{smallmatrix} 1 \ldots 1 \\ -1 \ldots -1 \end{smallmatrix} \right) \), we have:

\[
\begin{array}{c|cc}
& e_T & e_{T+1} \\
e_{T_1} & (-1)^{t-r+1} e_2^i & -( -1)^{t-r+1} e_2^i \\
e_{T_1+1} & -( -1)^{t-r+1} e_2^i + e_2^{i+1} & -( -1)^{t-r+1} e_2^{i+1}
\end{array}
\]

(11)
for $r < k$, where $T = 2^r + 2^{r+1} + \ldots + 2^k$ and $T_1 = 2^r + 2^{r+1} + \ldots + 2^k$ and 

\[ \begin{array}{c|cc}
 \epsilon_T & \epsilon_{T+1} \\
 \hline
 \epsilon_{2^k} & \epsilon_M & -\epsilon_{M+1} \\
 \epsilon_{2^k+1} & -\epsilon_{M+1} & -\epsilon_M \\
 \end{array} \]  

(12)

where $M = 2^k + 2^l$.

**Proof.** Case 1: $r < k$. We compute $e_T e_T$. We have $e_T e_T = \gamma(s, q) e_M$, where $s, q$ are the binary decomposition of $T_1$ and $T$. The binary decomposition of $M$ is $M_2 = T_1 \otimes T$. It results $M = 2^l$,

\[ s = \underbrace{00\ldots0111}_i \underbrace{\ldots10}_r, \quad q = \underbrace{100\ldots0111}_i \underbrace{\ldots10}_r \].

By “shuffling” $s \otimes q$, we obtain

\[ \underbrace{01}_i \underbrace{00}_i \ldots \underbrace{00}_i, \quad \underbrace{11}_i \underbrace{11}_i \ldots \underbrace{11}_i, \quad \underbrace{00}_i \ldots \underbrace{00}_i, \quad \underbrace{00}_i \ldots \underbrace{00}_i. \]

Starting with $A_0$, we get:

\[ A_0 \xrightarrow{01} A \xrightarrow{00} \ldots \xrightarrow{00} A \xrightarrow{11} C \xrightarrow{11} C \xrightarrow{11} C \xrightarrow{11} \ldots \xrightarrow{11} (-1)^{k-r+1} C \xrightarrow{00} \ldots \xrightarrow{00} (-1)^{k-r+1} C. \]

Therefore $\gamma(s, q) = (-1)^{k-r+1}$.

Now, we compute $e_T e_{T+1}$. For this, we will "shuffling" $00\ldots0111\ldots10\ldots0$ with

\[ \underbrace{100\ldots0111\ldots10\ldots1}_i \underbrace{\ldots10\ldots0}_r. \]

It results

\[ \underbrace{01}_i \underbrace{00}_i \ldots \underbrace{00}_i, \quad \underbrace{11}_i \underbrace{11}_i \ldots \underbrace{11}_i, \quad \underbrace{00}_i \ldots \underbrace{00}_i, \quad \underbrace{00}_i \ldots \underbrace{00}_i. \]

Starting with $A_0$, we get:

\[ A_0 \xrightarrow{01} A \xrightarrow{00} \ldots \xrightarrow{00} A \xrightarrow{11} C \xrightarrow{11} C \xrightarrow{11} C \xrightarrow{11} \ldots \xrightarrow{11} (-1)^{k-r+1} C \xrightarrow{00} \ldots \xrightarrow{01} \xrightarrow{-(-1)^{k-r+1} C. \]

For $e_{T+1} e_T$, "shuffling" $00\ldots0111\ldots10\ldots1$ with $100\ldots0111\ldots10\ldots0$, it results

\[ \underbrace{01}_i \underbrace{00}_i \ldots \underbrace{00}_i, \quad \underbrace{11}_i \underbrace{01}_i \ldots \underbrace{01}_i, \quad \underbrace{00}_i \ldots \underbrace{00}_i, \quad \underbrace{00}_i \ldots \underbrace{10}_r. \]

Starting with $A_0$, we get:

\[ A_0 \xrightarrow{01} A \xrightarrow{00} \ldots \xrightarrow{00} A \xrightarrow{11} C \xrightarrow{11} C \xrightarrow{11} C \xrightarrow{11} \ldots \xrightarrow{11} (-1)^{k-r+1} C \xrightarrow{00} \ldots \xrightarrow{10} \xrightarrow{-(-1)^{k-r+1} C. \]

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For \( e_{T_1 + 1} e_{T + 1} \), we compute first \((T_1 + 1) \otimes (T + 1)\). We obtain:

\[
(2^r + 2^{r+1} + \ldots + 2^k + 1) \otimes (2^r + 2^{r+1} + \ldots + 2^k + 2^k + 1) = \\
\left( \begin{array}{c}
00\ldots0111\ldots10\ldots1
\end{array} \right)_{i-k \quad k-r+1 \quad r} \otimes \left( \begin{array}{c}
100\ldots0111\ldots10\ldots1
\end{array} \right)_{i-k \quad k-r+1 \quad r} = \\
\left( \begin{array}{c}
10\ldots0000\ldots000
\end{array} \right)_{i-k \quad k-r+1 \quad r} = 2^i.
\]

Now, "shuffling" \(00\ldots0111\ldots10\ldots1\) with \(100\ldots0111\ldots10\ldots1\), it results
\(01\hspace{1pt}00\hspace{1pt}00\ldots00\hspace{1pt}11\hspace{1pt}01\hspace{1pt}01\ldots01\hspace{1pt}00\hspace{1pt}00\ldots00\hspace{1pt}11\).

Starting with \(A_0\), we get:

\[
A_0 \xrightarrow{01} A \xrightarrow{00} \ldots \xrightarrow{00} A \xrightarrow{11} -C \xrightarrow{11} C \xrightarrow{11} -C \xrightarrow{11} C \xrightarrow{11} \ldots \xrightarrow{11} (-1)^{k-r+1} C \xrightarrow{00} \ldots \xrightarrow{11} (-1)^{k-r+1} C.
\]

Case 2: \(r = k\). We have \(M = 2^k \otimes T = 2^i + 2^k\). For \(e_{2^k} e_T\), "shuffling" \(00\ldots010\ldots0\)
with \(100\ldots00\ldots0\), it results
\(01\hspace{1pt}00\hspace{1pt}00\ldots00\hspace{1pt}01\hspace{1pt}00\hspace{1pt}00\ldots00\hspace{1pt}10\hspace{1pt}00\hspace{1pt}00\ldots00\hspace{1pt}10\hspace{1pt}00\hspace{1pt}00\ldots00\).

Starting with \(A_0\), we get:

\[
A_0 \xrightarrow{01} A \xrightarrow{00} \ldots \xrightarrow{00} A \xrightarrow{10} C \xrightarrow{00} C \xrightarrow{00} \ldots \xrightarrow{00} C.
\]

For \(e_{2^k} e_{T+1}\), "shuffling" \(00\ldots010\ldots0\) with \(100\ldots00\ldots0\), it results
\(01\hspace{1pt}00\hspace{1pt}00\ldots00\hspace{1pt}01\hspace{1pt}00\hspace{1pt}00\ldots00\hspace{1pt}10\hspace{1pt}00\hspace{1pt}00\ldots00\hspace{1pt}10\hspace{1pt}00\hspace{1pt}00\ldots00\).

Starting with \(A_0\), we get:

\[
A_0 \xrightarrow{01} A \xrightarrow{00} \ldots \xrightarrow{00} A \xrightarrow{10} C \xrightarrow{00} C \xrightarrow{00} \ldots \xrightarrow{01} -C.
\]
Proposition 2.4. Let $A_t = \left(\frac{\gamma_1 \ldots \gamma_t}{R}\right)$ be an algebra obtained by the Cayley-Dickson process. For any $x_1, x_2, \ldots, x_t \in R - \{0\}$, we have that

\[
\left(\frac{\gamma_1 \ldots \gamma_t}{R}\right) \sim \left(\frac{\gamma_1 x_1^2 \ldots \gamma_t x_t^2}{R}\right).
\]

Proof. Let $A_t = \left(\frac{\gamma_1 \ldots \gamma_t}{R}\right)$ with the basis $\{e_0 = 1, e_1, \ldots, e_{n-1}\}$, $n = 2^t$ and let $A'_t = \left(\frac{\gamma_1 x_1^2 \ldots \gamma_t x_t^2}{R}\right)$ with the basis $\{e'_0 = 1, e'_1, \ldots, e'_{n-1}\}$ such that $(e'_i)^2 = \gamma_i x_i^2$, $i \in \{1, 2, \ldots, n-1\}$. We remark that $(x_i e_i)^2 = x_i^2 \gamma_i$ and from here, it results that the map $\tau : A'_t \to A_t$, $\tau (e'_i) = e_i x_i$ is an algebra isomorphism. □

Remark 2.5. From Proposition 2.4, it results that for each $n = 2^t$ there are only $n$ non-isomorphic algebras $A_t$. These algebras are of the form $A_t = \left(\frac{\gamma_1 \ldots \gamma_t}{R}\right)$, with $\gamma_1, \ldots, \gamma_t \in \{-1, 1\}$.

Definition 2.6. Let $\{e_0 = 1, e_1, \ldots, e_{n-1}\}$ be a basis in $A_t = \left(\frac{\gamma_1 \ldots \gamma_t}{R}\right)$, $n = 2^t$. To domain $\Omega \subset R^{2^t-1}$ we will associate the domain $\Omega_\xi := \{\xi = x_1 e_1 + \ldots + x_{n-1} e_{n-1} : (x_1, x_2, \ldots, x_{n-1}) \in \Omega\}$ included in $A_t$.

Consider a function $\Phi : \Omega_\xi \to A_t$ of the form

\[
\Phi(\xi) = \sum_{k=1}^{n-1} \Phi_k(x_1, x_2, \ldots, x_{n-1}) e_k,
\]

where $(x_1, x_2, \ldots, x_{n-1}) \in \Omega$ and $\Phi_k : \Omega \to R$.

We say that a function of the form (13) is left $A_t$-holomorphic in a domain $\Omega_\xi$ if the first partial derivatives $\partial \Phi_k / \partial x_k$ exist in $\Omega$ and the following equality is fulfilled in every point of $\Omega_\xi$:

\[
D[\Phi](\xi) = \sum_{k=1}^{2^t-1} e_k \frac{\partial \Phi}{\partial x_k} = 0.
\]

The operator $D$ is called Dirac operator. Note that if $A_t$ is the generalized quaternion algebra, then the left $A_t$-holomorphic functions is also called hyperholomorphic. We also note that every hyperholomorphic function $\Phi$ in a domain $\Omega_\xi$ is a solution of the equation

\[
\gamma_1 \frac{\partial^2 \Phi}{\partial x_1^2} + \gamma_2 \frac{\partial^2 \Phi}{\partial x_2^2} + \gamma_1 \gamma_2 \frac{\partial^2 \Phi}{\partial x_3^2} = 0.
\]

Remark 2.7. Let $\mathbb{H}(\gamma_1, \gamma_2)$ be the generalized quaternion algebra with the basis $\{1, e_1, e_2, e_3\}$, $\gamma_1 > 0$, $\gamma_2 > 0$ and $\mathbb{H}(-1, -1)$ be the usual quaternion division algebra with the basis $\{1, i, j, k\}$. Let $\Omega$ be a domain in $R^3$, and let
Ω := \{ζ = xi + yj + zk : (x, y, z) ∈ Ω \} be a corresponding domain in \(\mathbb{H}(-1, -1)\). The function \(\Phi : Ω_ζ \rightarrow \mathbb{H}(-1, -1)\) of the form

\[
\Phi(ζ) = u_1(x, y, z) + u_2(x, y, z) i + u_3(x, y, z) j + u_4(x, y, z) k.
\]
is hyperholomorphic in the domain \(Ω\) if

\[
D[Φ](ζ) = i \frac{∂Φ}{∂x} + j \frac{∂Φ}{∂y} + k \frac{∂Φ}{∂z} = 0.
\]

For another domain \(Δ \subset \mathbb{R}^3\), we associate the domain \(Δ_ζ := \{\tilde{ζ} = \tilde{x}e_1 + \tilde{y}e_2 + \tilde{z}e_3 : (\tilde{x}, \tilde{y}, \tilde{z}) \in Δ \} \) in the algebra \(\mathbb{H}(γ_1, γ_2)\). The Dirac operator in \(\mathbb{H}(γ_1, γ_2)\), denoted by \(\tilde{D}\), is

\[
\tilde{D} := e_1 \frac{∂}{∂\tilde{x}} + e_2 \frac{∂}{∂\tilde{y}} + e_3 \frac{∂}{∂\tilde{z}}.
\]

The elements of bases in \(\mathbb{H}(-1, -1)\) and \(\mathbb{H}(γ_1, γ_2)\) satisfy the following equalities:

\[
e_1 = i\sqrt{γ_1}, \quad e_2 = j\sqrt{γ_2}, \quad e_3 = k\sqrt{γ_1γ_2}.
\]  

(14)

Now we establish a connection between hyperholomorphic functions in the algebras \(\mathbb{H}(-1, -1)\) and \(\mathbb{H}(γ_1, γ_2)\), where \(γ_1 > 0, γ_2 > 0\). For this, we denote

\[
x = \frac{1}{\sqrt{γ_1}} \tilde{x}, \quad y = \frac{1}{\sqrt{γ_2}} \tilde{y}, \quad z = \frac{1}{\sqrt{γ_1γ_2}} \tilde{z}.
\]

These relations give us the operator equalities:

\[
\frac{∂}{∂x} = \frac{1}{\sqrt{γ_1}} \frac{∂}{∂\tilde{x}}, \quad \frac{∂}{∂y} = \frac{1}{\sqrt{γ_2}} \frac{∂}{∂\tilde{y}}, \quad \frac{∂}{∂z} = \frac{1}{\sqrt{γ_1γ_2}} \frac{∂}{∂\tilde{z}}.
\]  

(15)

Now, using relations (14) and (15), we obtain

\[
\tilde{D}[Φ](ζ) = e_1 \frac{∂Φ}{∂\tilde{x}} + e_2 \frac{∂Φ}{∂\tilde{y}} + e_3 \frac{∂Φ}{∂\tilde{z}} =
\]

\[
= i \frac{∂Φ}{∂x} \frac{1}{\sqrt{γ_1}} \sqrt{γ_1} + j \frac{∂Φ}{∂y} \frac{1}{\sqrt{γ_2}} \sqrt{γ_2} + k \frac{∂Φ}{∂z} \frac{1}{\sqrt{γ_1γ_2}} \sqrt{γ_1γ_2} =
\]

\[
= i \frac{∂Φ}{∂x} + j \frac{∂Φ}{∂y} + k \frac{∂Φ}{∂z} = D[Φ](ζ) = 0.
\]

Using the above notations, we obtain the following theorem:

**Theorem 2.8.** Let \(Ω\) be an arbitrary domain in \(\mathbb{R}^3\) and \(Δ\) be a domain in \(\mathbb{R}^3\) such that the coordinates of the corresponding points \(ζ = xi + yj + zk \in Ω_ζ\) and \(\tilde{ζ} = \tilde{x}e_1 + \tilde{y}e_2 + \tilde{z}e_3 \in Δ_ζ\) satisfy the following relations:

\[
x = \frac{1}{\sqrt{\text{sign}(γ_1)γ_1}} \tilde{x}, \quad y = \frac{1}{\sqrt{\text{sign}(γ_2)γ_2}} \tilde{y}, \quad z = \frac{1}{\sqrt{\text{sign}(γ_1)\text{sign}(γ_2)γ_1γ_2}} \tilde{z}.
\]
where \( \text{sign}(a) \) is the sign of the non-zero real number \( a \). Then if the function \( \Phi : \Omega_\zeta \to \mathbb{H}(\text{sign}(\gamma_1), \text{sign}(\gamma_2)) \) is hyperholomorphic in the domain \( \Omega_\zeta \), then the same function \( \Phi \), of \( \zeta \), is hyperholomorphic in the domain \( \Delta_\zeta \in \mathbb{H}(\gamma_1, \gamma_2) \). The converse is also true.

**Proof.** Since \( e_1 = i\sqrt{\text{sign}(\gamma_1)\gamma_1}, \ e_2 = j\sqrt{\text{sign}(\gamma_2)\gamma_2}, \ e_3 = k\sqrt{\text{sign}(\gamma_1)\text{sign}(\gamma_2)\gamma_1\gamma_2} \), the result directly follows from Remark 2.7. \( \square \)

**Remark 2.9.** (i) The above Theorem tell us that for studying hyperholomorphic functions in generalized quaternion algebras \( \mathbb{H}(\gamma_1, \gamma_2) \) it is suffices to consider hyperholomorphic functions only in the algebras \( \mathbb{H}(\text{sign}(\gamma_1), \text{sign}(\gamma_2)) \).

(ii) The result similar to the previous remark was established in the paper [8] (Theorem 5) in a three-dimensional commutative associative algebra.

**Theorem 2.10.** Let \( A_t = \left( \frac{2^{1,...,2t}}{\mathbb{R}} \right) \) be a generalized Cayley-Dickson algebra. Let \( \Omega \) be an arbitrary domain in \( \mathbb{R}^{d-1} \) and \( \Delta \) be a domain in \( \mathbb{R}^{d-1} \) such that the coordinates of the corresponding points \( \zeta = x_1e_1 + \ldots + x_{2n-1}e_{2n-1} \in \Omega_\zeta \) and \( \tilde{\zeta} = \tilde{x}_1\tilde{e}_1 + \tilde{x}_2\tilde{e}_2 + \ldots + \tilde{x}_{2n-1}\tilde{e}_{2n-1} \in \Delta_\zeta \) satisfy the following relations

\[
x_1 = \frac{1}{\sqrt{\text{sign}(\gamma_1)\gamma_1}} \tilde{x}_1, \quad x_2 = \frac{1}{\sqrt{\text{sign}(\gamma_2)\gamma_2}} \tilde{x}_2, \ldots
\]

\[
\ldots, x_n = \frac{1}{\sqrt{\text{sign}(\gamma_1)\ldots\text{sign}(\gamma_t)\gamma_1\ldots\gamma_t}} \tilde{x}_n.
\]

If the function \( \Phi : \Omega_\zeta \to \left( \frac{\text{sign}(\gamma_1)\ldots\text{sign}(\gamma_t)}{\mathbb{R}} \right) \) is left \( A_t \)-holomorphic in the domain \( \Omega_\zeta \), then the same function \( \Phi \), but depending of \( \tilde{\zeta} \) is left \( A_t \)-holomorphic in the domain \( \Delta_\zeta \in A_t \). The converse is also true.

**Proof.** Let \( \{1, e_1, \ldots, e_{n-1}\} \) be a basis in \( \left( \frac{\text{sign}(\gamma_1)\ldots\text{sign}(\gamma_t)}{\mathbb{R}} \right) \) and \( \{1, \tilde{e}_1, \ldots, \tilde{e}_{n-1}\} \) be a basis in \( A_t = \left( \frac{2^{1,...,2t}}{\mathbb{R}} \right) \).

Since

\[
e_1 = e_1\sqrt{\text{sign}(\gamma_1)\gamma_1}, \quad \tilde{e}_2 = e_2\sqrt{\text{sign}(\gamma_2)\gamma_2}, \ldots,
\]

\[
\ldots, \tilde{e}_{n-1} = e_{n-1}\sqrt{\text{sign}(\gamma_1)\ldots\text{sign}(\gamma_t)\gamma_1\ldots\gamma_t},
\]

the result is obtained from a simple computation as in Remark 2.7. \( \square \)

**Remark 2.11.** Using above Theorem, it is obvious that, for studying left \( A_t \)-holomorphic functions in generalized Cayley-Dickson algebras \( A_t = \left( \frac{2^{1,...,2t}}{\mathbb{R}} \right) \) it is suffices to consider left \( A_t \)-holomorphic functions only in the algebras \( \left( \frac{\text{sign}(\gamma_1)\ldots\text{sign}(\gamma_t)}{\mathbb{R}} \right) \).

Now we consider another class of differentiable functions. Let \( A_t = \left( \frac{2^{1,...,\gamma_t}}{\mathbb{R}} \right) \), with \( \gamma_1 = \ldots = \gamma_t = -1 \), and the domain \( \Omega \subset \mathbb{R}^d \). We denote with \( \Omega_\zeta := \{ \zeta = x_0 + x_1e_1 + \ldots + x_{n-1}e_{n-1} : (x_0, x_1, \ldots, x_{n-1}) \in \Omega \} \) a domain in \( A_t \). This domain is called congruent with the domain \( \Omega \).
We consider a function $\Phi : \Omega_\zeta \to A_4$ of the form
\[
\Phi(\zeta) = \sum_{k=0}^{n-1} \Phi_k(x_0, x_1, \ldots, x_{n-1})e_k,
\tag{16}
\]
where $(x_0, x_1, \ldots, x_{n-1}) \in \Omega$ and $\Phi_k : \Omega \to \mathbb{R}$.

We say that a function of the form $\Phi_k$ is left $A_t$–hyperholomorphic in a domain $\Omega_\zeta$ if the first partial derivatives $\partial \Phi_k / \partial x_k$ exist in $\Omega$ and the following equality is fulfilled in every point of $\Omega_\zeta$:
\[
\sum_{k=0}^{2^t-1} e_k \frac{\partial \Phi}{\partial x_k} = 0.
\]

In the following, we will provide an algorithm to constructing a left $A_t$–hyperholomorphic functions. Using the above notations, let $v(x, y)$ be a rational function defined in a domain $G \subset \mathbb{R}^2$. In the following, using some ideas given in Theorem 3 from [12], we will give an example of left $A_t$–hyperholomorphic function, for all $t \geq 1$, $t \in \mathbb{N}$. For this, we consider the following functions:

$$
\begin{align*}
\phi_1 &= x_0 + e_1 x_1, \\
\phi_2 &= \frac{1}{e_1} (x_0 + e_1 x_1),
\end{align*}
$$

$$
\begin{align*}
\rho_{2s-1} &= x_{2s} - e_1 x_{2s+1}, \\
\rho_{2s} &= -\frac{1}{e_1} (x_{2s} - e_1 x_{2s+1}), \quad s \in \{1, 2, \ldots, 2^{t-1} - 1\},
\end{align*}
$$

$$
\begin{align*}
F_t(\zeta) &= v(\phi_1, \phi_2) + v(\rho_1, \rho_2) e_2 + v(\rho_3, \rho_4) e_4 + [v(\rho_5, \rho_6) e_2] e_4 + \\
&\quad + v(\rho_7, \rho_8) e_8 + (v(\rho_9, \rho_{10}) e_2) e_8 + (v(\rho_{11}, \rho_{12}) e_4) e_8 + [v(\rho_{13}, \rho_{14}) e_2] e_8 + \\
&\quad \cdots + \sum_{i=1}^{t-1} \sum_{k=1}^{i-1} v(\rho_{M_{isk}}, \rho_{M_{is}k}) e_2^i e_2^{i+1} \cdots e_2^k e_2^k + \sum_{i=1}^{t-1} (v(\rho_{2^i-1, \rho_{2^i}}) e_2^i),
\end{align*}
$$

where $M_{isk} = 2^r + 2^{r+1} + \cdots + 2^k + 2^i$.

It results
\[
F_t(\zeta) = v(\phi_1, \phi_2) + \\
+ \sum_{i=1}^{t-1} \sum_{k=1}^{i-1} v(\rho_{M_{isk}}, \rho_{M_{is}k}) e_2^i e_2^{i+1} \cdots e_2^k e_2^k + \sum_{i=1}^{t-1} (v(\rho_{2^i-1, \rho_{2^i}}) e_2^i),
\]
or
\[
F_t(\zeta) = F_{t-1}(\zeta) + \\
+ \sum_{k=1}^{t-2} \sum_{r=1}^{k-1} v(\rho_{M_{k(r-1)}}, \rho_{M_{k(r-1)}}) e_2^r e_2^{r+1} \cdots e_2^k e_2^k + v(\rho_{2^t-1}, \rho_{2^t-1}) e_2^{2t-1}.
\]
We denote with \( \mathbb{C}_2 \), the "complex" planes \( \{x_{2s} + e_1x_{2s+1} : x_{2s}, x_{2s+1} \in \mathbb{R}\} \) and with \( D_{2s} := \{ (x_{2s}, x_{2s+1}) : x_{2s} + e_1x_{2s+1} \in \mathbb{C}_2 \}, s \in \{0, 1, 2, ..., 2^{t-1} - 1\} \) the Euclidian planes. Let \( G_{2s} \) be a domains in \( \mathbb{C}_2 \) and let \( \bar{G}_{2s} \) be the corresponded domains in \( D_{2s} \). We have the following theorem:

**Theorem 2.12.** With the above notations, we consider the functions \( v(\phi_1, \phi_2) \) and \( v(\rho_{2s-1}, \rho_{2s}) \) defined in the corresponding domains \( G_0 \subset \mathbb{C}_0 \) and \( G_{2s} \subset \mathbb{C}_2, s \in \{1, 2, ..., 2^{t-1} - 1\} \). Then the map \( F_t(\zeta) \) is a left \( A_t \)–hyperholomorphic function in the domain \( \Theta \subset A_t \) which is congruent with the domain \( \bar{G}_0 \times \bar{G}_2 \times G_4 \times \ldots \times \bar{G}_{2^{t-1} - 1} \subset \mathbb{R}^{2t} \), for \( t \geq 1 \).

**Proof.** For \( t = 1 \), we have \( F_1(\zeta) = v(\phi_1, \phi_2) \), which is an holomorphic function in \( D_0 \subset \mathbb{C}_0 \), as we can see in [12], Theorem 3.

For \( t = 2 \), we obtain \( F_2(\zeta) = v(\phi_1, \phi_2) + v(\rho_1, \rho_2) e_2 \) and for \( t = 3 \), we get \( F_3(\zeta) = v(\phi_1, \phi_2) + v(\rho_1, \rho_2) e_2 + v(\rho_3, \rho_4) e_4 \). \( F_2(\zeta) \) and \( F_3(\zeta) \) are hyperholomorphic, respectively octonionic hyperholomorphic function, from Remark 2.1 and Theorem 3 from [12].

For \( t \geq 4 \), using induction steps, supposing that \( F_{t-1}(\zeta) \) is a left \( A_{t-1} \)–hyperholomorphic function, we will prove that \( F_t(\zeta) \) is \( A_t \)-hyperholomorphic. That means \( D[F_t] = 0 \). From induction steps, we obtain \( D[F_{t-1}] = 0 \). We will prove that

\[
\sum_{k=0}^{2^{t-1}-1} e_k \frac{\partial F_t}{\partial x_k} = 0.
\]

This sum has \( 2^{t-1} \) terms. First two terms are:

\[
\left( \frac{\partial F_t}{\partial x_{2^{t-1}}}, e_1 \frac{\partial F_t}{\partial x_{2^{t-1}+1}} \right) =
\]

\[
\frac{\partial v}{\partial \rho_{2^{t-1}-1}} \frac{\partial \rho_{2^{t-1}-1}}{\partial x_{2^{t-1}}} + \frac{\partial v}{\partial \rho_{2^{t-1}}} \frac{\partial \rho_{2^{t-1}}}{\partial x_{2^{t-1}}} - e_1 \left( \frac{\partial v}{\partial \rho_{2^{t-1}-1}} \frac{\partial \rho_{2^{t-1}-1}}{\partial x_{2^{t-1}+1}} + \frac{\partial v}{\partial \rho_{2^{t-1}}} \frac{\partial \rho_{2^{t-1}}}{\partial x_{2^{t-1}+1}} \right) =
\]

\[
\frac{\partial v}{\partial \rho_{2^{t-1}-1}} - \frac{\partial v}{\partial \rho_{2^{t-1}}} e_1 - e_1 \frac{\partial v}{\partial \rho_{2^{t-1}}} = 0.
\]
Since \( e_1^2 = \gamma_1, \gamma_1^2 = 1 \), \( \frac{\partial v}{\partial \rho_{T-1}} \) and \( \frac{\partial v}{\partial \rho_T} \) can be written as \( a_{2r-1-1}(\zeta) + b_{2r-1-1}(\zeta) e_1 \), respectively \( a_{2r-1}(\zeta) + b_{2r-1}(\zeta) e_1 \) where \( a_{2r-1-1}(\zeta), b_{2r-1-1}(\zeta), a_{2r-1}(\zeta), b_{2r-1}(\zeta) \) are real valued functions.

Case 1: \( r < k \). In the general case, we denote \( T = 2r + 2^{r+1} + \ldots + 2^k + 2^{l-1} \) and \( T_1 = 2^r + 2^{r+1} + \ldots + 2^k \), for \( r < k \). We will compute the terms

\[
-\epsilon_{T_1} \frac{\partial F_1}{\partial x_T} - \epsilon_{T_1+1} \frac{\partial F_1}{\partial x_{T+1}}.
\]

We compute first \( \frac{\partial F_1}{\partial x_T} \). It results

\[
\frac{\partial F_1}{\partial x_T} = \left( \ldots \left( \frac{\partial v}{\partial \rho_{T-1}} \frac{\partial T-1}{\partial x_T} + \frac{\partial v}{\partial \rho_T} \frac{\partial T}{\partial x_T} \right) + \right. e_{2r})e_{2r+1})e_{2r-1} =
\]

\[
= \left( \ldots \left( \frac{\partial v}{\partial \rho_{T-1}} + \frac{\partial v}{\partial \rho_T} e_1 \right) + \right. e_{2r})e_{2r+1})e_{2r-1} =
\]

Since we can write \( \frac{\partial v}{\partial \rho_{T-1}} \) under the form \( a_{T-1}(\zeta) + b_{T-1}(\zeta) e_1 \) and \( \frac{\partial v}{\partial \rho_T} \) under the form \( a_T(\zeta) + b_T(\zeta) e_1 \), where \( a_{T-1}, b_{T-1}, a_T, b_T \) are real valued functions, using Proposition 2.2, we obtain:

\[
\frac{\partial F_1}{\partial x_T} = \left( \ldots \left( \frac{\partial v}{\partial \rho_{T-1}} + \frac{\partial v}{\partial \rho_T} e_1 \right) + \right. e_{2r})e_{2r+1})e_{2r-1} =
\]

\[
= \left( \ldots (a_{T-1}(\zeta)e_{2r})e_{2r+1})e_{2r-1} + \right. (b_{T-1}(\zeta)e_{2r})e_{2r+1})e_{2r-1} +
\]

\[
+ (a_T(\zeta)e_1)e_{2r})e_{2r+1})e_{2r-1} + (b_T(\zeta)e_1)e_{2r})e_{2r+1})e_{2r-1} =
\]

\[
= a_{T-1}(\zeta)(-1)^{k-r+2}e_T + b_{T-1}(\zeta)(-1)^{k-r+3}e_{T+1} +
\]

\[
+ a_T(\zeta)(-1)^{k-r+3}e_{T+1} - b_T(\zeta)(-1)^{k-r+2}e_T.
\]

Using Proposition 2.3, relation (11), we compute \(-\epsilon_{T_1} \frac{\partial F_1}{\partial x_T} \).
\[-(a_{T-1}(\zeta)(-1)^{k-r+2}(-1)^{k-r+1}e_{2i} - b_{T-1}(\zeta)(-1)^{k-r+3}(-1)^{k-r+1}e_{2i+1}) - \\
-a_T(\zeta)(-1)^{k-r+3}(-1)^{k-r+1}e_{2i+1} - b_T(\zeta)(-1)^{k-r+2}(-1)^{k-r+1}e_{2i} = \\
= -(a_{T-1}(\zeta)(-1)^{2k-2r+3}e_{2i} - b_{T-1}(\zeta)(-1)^{2k-2r+4}e_{2i+1}) - \\
-a_T(\zeta)(-1)^{2k-2r+4}e_{2i+1} - b_T(\zeta)(-1)^{2k-2r+3}e_{2i}.
\]

Now, we compute $\frac{\partial F_i}{\partial x_{T+1}}$. We obtain

\[
\frac{\partial F_i}{\partial x_{T+1}} = \left(\ldots \left( \frac{\partial v}{\partial \rho_T} e_1 + \frac{\partial v}{\partial \rho_T} e_2 \right) e_2 \ldots e_{2i} e_{2i+1} = \\
= \left(\ldots \left( - \frac{\partial v}{\partial \rho_{T-1}} e_1 + \frac{\partial v}{\partial \rho_{T-1}} e_2 \right) e_2 \ldots e_{2i} e_{2i+1} = \\
\right)\left(\ldots (a_{T-1}(\zeta)e_{2i}e_{2i+1})\ldots e_{2k} e_{2i-1} - (b_{T-1}(\zeta)e_{2i}e_{2i+1})\ldots e_{2k} e_{2i-1} + \\
+ (a_T(\zeta)e_{2i}e_{2i+1})\ldots e_{2k} e_{2i-1} + (b_T(\zeta)e_{2i}e_{2i+1})\ldots e_{2k} e_{2i-1} = \\
= -a_{T-1}(\zeta)(-1)^{k-r+3}e_{T+1} + b_{T-1}(\zeta)(-1)^{k-r+2}e_T + \\
+ a_T(\zeta)(-1)^{k-r+2}e_T + b_T(\zeta)(-1)^{k-r+3}e_{T+1}.
\]

Using Proposition 2.3, we compute $-e_{T+1} \frac{\partial F_i}{\partial x_{T+1}}$.

\[-e_{T+1} \frac{\partial F_i}{\partial x_{T+1}} = -e_{T+1} \left( -a_{T-1}(\zeta)(-1)^{k-r+3}e_{T+1} + b_{T-1}(\zeta)(-1)^{k-r+2}e_T + \\
\right)\]
\[ +a_T(\zeta) (-1)^{k-r+2} e_T + b_T(\zeta) (-1)^{k-r+3} e_{T+1} \]  
\[ = - \left( a_{T-1}(\zeta) (-1)^{k-r+3} (-1)^{k-r+1} e_{2i} - b_{T-1}(\zeta) (-1)^{k-r+2} (-1)^{k-r+3} e_{2i+1} \right) - \]  
\[ - \left( - a_T(\zeta) (-1)^{k-r+2} (-1)^{k-r+3} e_{2i+1} - b_T(\zeta) (-1)^{k-r+3} (-1)^{k-r+1} e_{2i} \right) = \]  
\[ = - \left( a_{T-1}(\zeta) (-1)^{2k-2r+4} e_{2i} - b_{T-1}(\zeta) (-1)^{2k-2r+3} e_{2i+1} \right) - \]  
\[ - \left( - a_T(\zeta) (-1)^{2k-2r+3} e_{2i+1} - b_T(\zeta) (-1)^{2k-2r+4} e_{2i} \right). \]

Now, we can compute \(-e_T \frac{\partial F_i}{\partial x_T} - e_{T+1} \frac{\partial F_i}{\partial x_{T+1}}\). It results \(-e_T \frac{\partial F_i}{\partial x_T} - e_{T+1} \frac{\partial F_i}{\partial x_{T+1}} = \)  
\[ = - \left( a_{T-1}(\zeta) (-1)^{2k-2r+3} e_{2i} - b_{T-1}(\zeta) (-1)^{2k-2r+4} e_{2i+1} \right) - \]  
\[ - \left( - a_T(\zeta) (-1)^{2k-2r+4} e_{2i+1} - b_T(\zeta) (-1)^{2k-2r+3} e_{2i} \right) - \]  
\[ - \left( a_{T-1}(\zeta) (-1)^{2k-2r+4} e_{2i} - b_{T-1}(\zeta) (-1)^{2k-2r+3} e_{2i+1} \right) - \]  
\[ - \left( - a_T(\zeta) (-1)^{2k-2r+3} e_{2i+1} - b_T(\zeta) (-1)^{2k-2r+4} e_{2i} \right) = 0. \]

**Case 2:** \(r = k\), we use Proposition 2.2 and Proposition 2.3, relation \(\{12\}\) and it easy to show that \(-e_T \frac{\partial F_i}{\partial x_T} - e_{T+1} \frac{\partial F_i}{\partial x_{T+1}} = 0. \)

\[ \square \]

**Remark 2.13.** The above proposition generalizes Theorem 3 from [12].

**The Algorithm**

1) Input \(t\).
2) Input functions \(v, \phi_1, \phi_2\).
3) For \(i \in \{1, ..., t - 1\}, \ k \in \{1, ..., i\}, \ r \in \{1, ..., k - 1\}\), compute \(M_{rki} = 2^r + ... + 2^k + 2^i\), 
\(v(\rho_{M_{rki}-1}, \rho_{M_{rki}}) = \alpha_{M_{rki}} + \beta_{M_{rki}}e_1\).

4) For \(i \in \{1, ..., t - 1\}, \ k \in \{1, ..., i\}, \ r \in \{1, ..., k - 1\}\),

- if \(r < k\), we compute

\[
(... (\alpha_{M_{rki}} + \beta_{M_{rki}} e_1) e_{2^r} e_{2^{r+1}} ... e_{2^k}) e_{2^i}) = (-1)^{k-r+2} (\alpha_{M_{rki}} e_{M_{rki}} - \beta_{M_{rki}} e_{M_{rki},-1})
\]

- if \(r = k\), we compute

\[
v(\rho_{2^i-1}, \rho_{2^i}) e_{2^i} = (\alpha_{2^i-1} + \beta_{2^i-1} e_1) e_{2^i} = \alpha_{2^i-1} e_{2^i} + \beta_{2^i-1} e_{2^i+1}.
\]

5) Output function

\[
F_t(\zeta) = v(\phi_1, \phi_2) + \sum_{i=4}^{t-1} \left( \sum_{k=1}^{i-1} \left( \sum_{r=1}^{k-1} (-1)^{k-r+2} (\alpha_{M_{rki}} (\zeta) e_{M_{rki}} - \beta_{M_{rki}} (\zeta) e_{M_{rki},-1}) ) \right) \right)
\]

\[
+ \sum_{i=1}^{t-1} (\alpha_{2^i-1} (\zeta) e_{2^i} + \beta_{2^i-1} (\zeta) e_{2^i+1}) .
\]

**Conclusion.** In this paper, we generalized the notion of left \(A_t\)-holomorphic functions from quaternions to all algebras obtained by the Cayley-Dickson process and we provided an algorithm to find examples of left \(A_t\)-hyperholomorphic functions, using the *shuffling* procedure given by Bales in [1].

The theory of the right \(A_t\)-holomorphic functions and the theory of the right \(A_t\)-hyperholomorphic functions are similarly to the corresponding theories for the left functions and can be easy treated, using the above ideas and procedures.

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