PREFACE: DIFFUSION ON FRACTALS AND NON-LINEAR DYNAMICS

This special issue is dedicated to the memory of B. O. Stratmann (1957–2015), who played a major role in developing the theory of dynamical systems, fractal and hyperbolic geometry, and thermodynamic formalism; who continued the long standing traditions of L. Arnold from the 1970s of dynamics in Bremen; and who, together with K. Falk and M. Kesseböhmer, built the current and active dynamics research group at the University of Bremen.

The overarching theme of this special issue is diffusion on fractals and non-linear dynamics, and is aimed at both newcomers to the area, as well as specialists. The contributions reflect different aspects of these topics and form new connections. The articles featured here have been carefully selected and peer reviewed.

This special issue, in part, complements and records the outcomes of the 2015 Winter School and Symposium on Diffusion on Fractals and Non-linear Dynamics hosted by the groups Dynamical Systems and Geometry and Applied Analysis within the Department of Mathematics and Computer Science at the University of Bremen. The event successfully brought together experts from 11 countries and was attended by over 60 academics. During the meeting new collaborative projects were formed – results of which are hereby featured. We hope to have captured the exciting atmosphere of the event in this special issue through the original research, survey and expository articles.

We are indebted and grateful to a number of anonymous referees for their invaluable help and suggestions in preparing this special issue. We express our gratitude to the University of Bremen and to the German Research Council who sponsored the winter school and symposium through the grants M8 Post-Doc-Initiative PLUS, Gastdozentenprogramm and Scientific Network – Skew Product Dynamics and Multifractal Analysis; without their support this special issue and the event could not have come to fruition.

This special issue centres around four themes: fractal geometry and applications of thermodynamic formalism, quasicrystals, skew-product dynamical systems, and non-linear dynamics. Below we give a short synopsis of these topics and the articles hereby featured.

1. **Fractal geometry and applications of thermodynamic formalism.** Structures in nature, such as galaxies and landscapes, aggregates and colloids, and polymers and proteins, often possess complexity irregularity and randomness on large and small scales. Fractal geometry is a mathematical theory used to describe and analyse properties of such structures, which are often called fractal. Geometric characteristics which allow for their detailed analysis include dimensions, such as Assouad, Fourier, Hausdorff or Minkowski; measures, like Hausdorff, Gibbs or Patterson-Sullivan; or volume forms such as the Minkowski content.

A prominent approach to geometrically describing a fractal object is to consider the decay of volume of particular $\epsilon$-covers as $\epsilon$ tends to zero. In this way the Hausdorff and Minkowski dimensions are defined as the corresponding exponents, and
the Hausdorff measure and Minkowski content as the respective pre-factors of the leading asymptotic term. These quantities, as well as the exponents and pre-factors of the lower-order asymptotic terms (related to curvature), are important geometric characteristics. A useful method to determine such characteristics of particular fractals relies on considering the Mellin transform of the \( \epsilon \)-dependent volume function of the \( \epsilon \)-cover. This leads to a description of the problem in terms of \( \zeta \)-functions and is related to a statement that is equivalent to the Riemann hypothesis. It is in the analysis of these \( \zeta \)-functions that renewal theory and thermodynamic formalism have been applied successfully, yielding new and meaningful results. In the article by M. Kesseböhmer and S. Kombrink a complex Ruelle-Perron-Frobenius theorem for Markov shifts over an infinite alphabet is proven, whence extending results by M. Pollicott from the finite to the infinite alphabet setting. As an application they obtain an extension of renewal theory in symbolic dynamics, as developed by S. P. Lalley. This work is complemented by the paper of M. Rauch, where new notions of topological pressure for measurable potentials are introduced and studied. By proving a corresponding variational principle, the author also establishes a Bowen formula for the Hausdorff dimension of cookie-cutters with discontinuous geometric potentials.

The thermodynamic formalism also makes apparent the link between geometry and the theory of Diophantine approximation. It has played a crucial role in the understanding of regularity of singular maps. A beautiful survey of this is presented in the article by J. J. Miao and S. Munday. The exposition exposes a number of open questions some of which are explicitly stated, while others are only suggested.

The theory of complex dimensions, as introduced by M. L. Lapidus and M. van Frankenhuysen, is a \( \mathbb{C} \)-valued extension of non-integer notions of dimension such as the Hausdorff dimension and the Minkowski dimension for sets \( U \subseteq \mathbb{R}^d \). These non-integer notions of dimension assign a single number to \( U \). By contrast, the set of complex dimensions provides a richly structured geometric invariant of \( U \), typically an infinite but discrete subset of \( \mathbb{C} \) of which the Minkowski dimension is a distinguished member. The complex dimensions describe not just the order of magnitude of the scaling properties of \( U \), but also the oscillatory aspects. K. Dettmers et al. provide an articulate account of this topic and present an extension of the theory to subsets of \( n \)-dimensional Euclidean space, with an emphasis on self-similar sets.

We conclude this topic with the article by K. Hattori et al. concerning non-Markovian random walks on the Sierpiński gasket. As a main result the authors show that the exponent \( \nu \) governing the short-time behaviour of the scaling limit varies continuously in the parameter. The limit process is almost surely self-avoiding, while it has path Hausdorff dimension equal to \( 1/\nu \), which is strictly greater than one.

2. Quasicrystals. Quasicrystals, being arrangements with long range aperiodic order, were discovered by D. Shechtman in 1982. This discovery (honoured with the Nobel prize in Chemistry in 2011) came as a complete surprise to physicists, chemists and material scientists. It also stimulated a tremendous amount of research in mathematics, in particular, questions concerning characterisation of order versus disorder, diffraction theory, and quantum mechanics of transport in solids. The involved methods range from number theory and geometry, over ergodic theory to harmonic analysis and operator theory. Dynamical systems and self-similarity feature very prominently. Indeed, all manifestations of ‘the same’ quasicrystals can
be conveniently gathered in the form of a dynamical system (called the hull of the quasicrystal) and many concrete models display self-similarity. Here we present an elegant survey article by M. Baake and D. Lenz. The authors give a careful exposition of definitions and results, with a strong emphasis on the underlying physical motivation. Much of the focus of their paper is on comparing two well-studied methods for understanding diffraction, the classical definitions involving the Fourier transform of the autocorrelation measure, and the dynamical spectrum coming from the natural translation action on the hull of the point set. Throughout the article the authors highlight open problems in the field. Notably, inspired by the winter school and symposium, the authors include a section on the structural differences of quasi-crystals generated by abstract point sets and those emerging in non-linear evolution equations for soft matter.

3. Skew-product dynamical systems. Skew products form a class of dynamical systems that have proved relevant both from a theoretical and a practical point of view. In the description of real-world processes, they are used to model dynamical systems which are subject to the influence of external time-varying factors, a situation which naturally appears in many applications. On the theoretical side, the study of skew products is an important step towards understanding higher dimensional discrete dynamical systems and also gives an impetus to many other areas of mathematics, including, but not restricted to, hyperbolic dynamics, rotation theory of toral homeomorphisms, Schrödinger operators with quasiperiodic potential, and quasicrystals. M. G. Gharaei and A. J. Homburg use a relatively simple but quintessential class of skew product maps to illustrate and prove several important phenomena in skew product dynamical systems: intermingled basins of attraction, synchronization, on-off intermittency and random walks with drift. These concepts are further studied in the article by G. Keller, who provides an insightful and detailed analysis of the stability index and uncertainty exponent of two-dimensional skew-product systems highlighting the intriguing behaviour of intermingled basins. These skew-product systems consist of one-dimensional fibre maps with negative Schwarzian derivative which are driven by a one-dimensional mixing Markov map. Through an application of the thermodynamic formalism the author obtains the stability index and a natural upper estimate of the uncertainty exponent. In addition, the author’s calculation of the corresponding exponent, which is the upper bound for the uncertainty exponent, yields a nice application of the theory and provides a possible explanation for the discrepancy of theoretical and numerical calculations of the uncertainty exponent observed in previous literature on this topic.

4. Non-linear dynamics. The alluring expository article by M. Beck provides an accessible introduction to the use of pointwise estimates in establishing non-linear stability for certain non-linear waves. Non-linear waves are an ubiquitous phenomenon in infinite dimensional dynamical systems generated by (non-linear) evolutionary equations on spatially extended domains. The viewpoint taken here is that of spatial dynamics, where the dynamics acts in the unbounded space direction and the actual time direction is constrained to a compact domain. For the stability problem one may then regard the linearisation in the non-linear wave together with the non-linear equations as a skew-product system with continuous (spatial) time. Parabolic systems, as considered here, appear frequently in models of soft matter or more generally as macroscopic models for effectively diffusing particles
on smooth domains. Such systems naturally generate spatial patterns, and non-linear waves are a particularly ordered or coherent type of these. The article also summarises the application of these techniques in two more complex cases: time-periodic shocks and defects. In two space dimensions, the linear problem readily allows for more disordered ‘quasi’-patterns with almost or quasi-periodic quasi-crystalline structures. However, the rigorous non-linear existence is an essentially open problem, and much more so the stability of these. Even in low dimensions bifurcations and the emergence of complex attractors occur and provide difficult problems.

The expository article by M. Beck is complemented by the paper of S. Gonchenkov and I. Ovsiannikov on discrete Lorenz attractors. These arise in unfoldings of certain homoclinic tangencies, and the authors provide an introduction to the various types and notions of chaotic attractors in this context. Indeed, the bifurcation theoretical perspective on complex dynamics taken here complements the viewpoint of topological and measure theoretic dynamics of this special issue’s first three themes. The authors follow the tradition of the Shilnikov school for classifying dynamical systems based on the recurrence induced by a homoclinic orbit with given properties. By carefully studying the organisation and intersection of invariant manifolds the authors unravel the structures that generate the attractor upon unfolding the organising centre as well as geometric details of the loci in parameter space.

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