AN ASYMPTOTIC SHAPE THEOREM FOR RANDOM LINEAR GROWTH MODELS

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Abstract. In this paper, we define a class of random growth models whose growth is at least and at most linear and prove an asymptotic shape theorem for these models. This proof generalizes already known proofs for the classical contact process [Har74], [DG82] or some of its variants [GM12a], [Des14] and allows us to obtain conjectured asymptotic shape theorems for several models: the contact process in a randomly evolving environment [SW08], the oriented percolation with hostile immigration [GM12b], the bounded modified contact process [DS00], Richardson’s model with stirring and the contact process with stirring [MMS24+].

1. Introduction

In 1974, Harris [Har74] introduced the classical contact process as an interacting particle system modeling the spread of an epidemic through the grid $\mathbb{Z}^d$ with the following dynamics: an infected site recovers at rate 1, and infects its neighbors at rate $\lambda$. Harris showed that, for each dimension $d$, this process exhibits a phase transition. This means that there exists a non trivial value $\lambda_c(d)$, called critical value, such that if $\lambda > \lambda_c(d)$, then the probability that, starting from a single infected point, the infection persists over time is positive. In dimension 1, thanks to a coupling with an oriented percolation, Durrett and Griffeath [DG83] proved that the rightmost infected particle has linear speed. In higher dimensions, the analogous result is called asymptotic shape theorem: denoting by $H_t$ the set of particles infected at least once before time $t$, is there a deterministic set $B$ such that $H_t$ converges to $B$ (in a sense to be made precise)? Durrett and Griffeath [DG83] proved this result provided they had linear some estimates on the growth of the contact process on the whole supercritical zone, which were only known for large $\lambda$ at the time. Then, Bezuidenhout-Grimmett [BG90] made a smart block construction to prove these estimates for the contact process, thus completing the proof of the asymptotic shape theorem for the classical contact process in all. An overview can be found in [Dur91] or [Lig99].

Since then, a lot of extensions appeared in the literature: the two stage contact process by Krone [Kro99], the boundary modified contact process by Durrett and Schinazi [DS00], the contact process in randomly evolving environment by Broman [Bro07], Remenik [Rem08], Steif and Warfirheimer [SW08], the contact process with aging [Des14], the contact process with stirring [MMS24+], etc... For all these processes, we can exhibit

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a property of interest \( P \) about the state of the sites of \( \mathbb{Z}^d \), then define \( H_t \) as the subset of \( \mathbb{Z}^d \) consisting of all points satisfying the property at least once before \( t \). All these processes are linear random growth models in following sense: there exists deterministic compact sets \( H_- \) and \( H_+ \) such that, for \( t \) large enough, with high probability:

\[
    tH_- \subset H_t \subset tH_+.
\]

The first inclusion expresses the at least linear growth and the second one the at most linear growth. These models also share characteristics like attractiveness, phase transition phenomenon etc. In this paper, we want to highlight their common properties which lead to prove that a general class of linear random growth models satisfy an asymptotic shape theorem. We call hitting time of \( x \) the first time \( t(x) \) that \( x \) satisfies the property \( (P) \). Models for which once a site \( x \) satisfies the property \( (P) \), it is forever, are called permanent models. It includes for example first passage percolation (introduced by Hammersley and Welsh [HW65]) for the property "being wet". For permanent models, Kingman’s theorem is generally used on the sequence of hitting times \((t(nx))_{n \in \mathbb{N}}\) for all \( x \in \mathbb{Z}^d \), thanks to the hypotheses of sub-additivity, stationarity and integrability. In our case of random linear growth models, we consider non permanent models: the fact that the property can disappear means that the standard integrability conditions are not satisfied, since \( t(nx) \) can be infinite with positive probability. If we condition on the property’s survival, then stationarity and subadditivity properties can be lost. To overcome such lacks, we introduce a suitable quantity \( \sigma(x) \) called essential hitting time of \( x \), inspired by [GM12a]. It can been seen as a regeneration time: it is a time when the site \( x \) satisfies the property, and spreads it forever. It turns out that this function \( \sigma \) satisfies adequate stationarity properties as well as the almost-subadditivity conditions involved in Kesten and Hammersley’s theorem (see [Kes73] and [Ham74]), a well-known extension of Kingman’s seminal result. Therefore, we can use the same techniques as the ones involved in [Des14], in order to prove an asymptotic shape theorem, that is the existence of a norm \( \mu \) such that, for all \( \varepsilon > 0 \), almost surely on the event "the property persists over time", for all \( t \) large enough, we have that

\[
    (1 - \varepsilon)B_{\mu} \subset \frac{H_t}{t} \subset (1 + \varepsilon)B_{\mu},
\]

with \( H_t = H_t + [0,1]^d \), and \( B_{\mu} \) the unit ball associated to \( \mu \).

In Section 2, we define our class of random linear growth models, and we give several examples of models belonging to our class. After that, we state the general asymptotic shape theorem. We deduce from it, as particular cases, asymptotic shape theorems that were conjectured for several models: the oriented percolation with hostile immigration [GM12a], Richardson’s model with stirring and contact process with stirring [MMS24+], the contact process in a randomly evolving environment [SW08] and the bounded modified contact process [DS00]. The linear estimates needed to apply our result are not proved for the last two models, but they can be obtained with classical restart techniques. The proof of the general asymptotic shape theorem is the subject of the following sections, and is cut in two parts. In Section 3, we define the essential hitting time, and we give its properties. In Section 4, we prove the general asymptotic shape theorem for the essential hitting time, and deduce from it the general asymptotic shape theorem for the hitting time.
2. A class of random linear growth models

2.1. Notations and definitions.

2.1.1. State space and property of interest. We will work on interacting particle systems on \( \mathbb{Z}^d \). For \( x \) and \( y \) in \( \mathbb{Z}^d \), we say that \( x \) and \( y \) are neighbors, and we denote by \( x \sim y \), if \( ||x-y||_1 = \sum_{i=1}^{d} |x_i - y_i| = 1 \). Let \( S \) be a countable totally ordered set, which will represent the possible states (or types) of a particle \( x \in \mathbb{Z}^d \). In most cases, \( S \) will be finite. We denote by \( S^{zd} \) the set of mappings \( \xi : \mathbb{Z}^d \rightarrow S \). Let \( C(S^{zd}) \) be the set of continuous functions from \( S^{zd} \) to \( \mathbb{R} \), and \( C_0(S^{zd}) \) be the subset of continuous functions depending on only finitely many coordinates of \( \xi \). The set \( S^{zd} \) will be considered as the space of configurations:

\[
\xi \in S^{zd} \iff \left\{ \begin{array}{l}
\xi : \mathbb{Z}^d \rightarrow S \\
y \rightarrow \xi(y)
\end{array} \right.
\]

Let \( (P) \) be a property of interest about elements of \( S \), that is, a non-empty subset of \( S \). We can think about the property \( (P) \) as 'to be in a specific state' (for example 'to be infected' for the classical contact process). Since \( S \) is totally ordered, then for all \( x \in \mathbb{Z}^d \), then we can define a minimal configuration \( \delta_x \) among configurations \( \xi \) satisfying \( \{y \in \mathbb{Z}^d, \xi(y) \in P\} = \{x\} \).

2.1.2. Markov process. We work with a time space \( T \) which can be discrete \( (T = \mathbb{N}) \) or continuous \((T = \mathbb{R}_+)\). Let \( (\xi_t)_{t \in T} \) be a stationary Feller Markov process taking values in \( S^{zd} \). This process gives the state of each site at each time. We denote by \( (\mathcal{F}_t)_{t \in T} \) its associated filtration. For \( f \in S^{zd} \), \( (\xi^f_t) \) is the process with initial configuration \( \xi_0 = f \). For \( y \in \mathbb{Z}^d \), \( (\xi^y_t) \) is the process with initial configuration \( \xi_0 = \delta_y \) the minimal configuration previously defined where only \( y \) satisfies \( (P) \). If the initial configuration is not precised, it means that we start with the configuration \( \xi_0 = \delta_{y_{zd}} \).

We define the processes \((A_t)_{t \in T}\) and \((H_t)_{t \in T}\), taking values in \( \mathcal{P}(\mathbb{Z}^d) \), by

\[
A_t = \{y \in \mathbb{Z}^d, \xi_t(y) \in P\} \quad \text{and} \quad H_t = \bigcup_{s \leq t} A_s;
\]

\(A_t\) is the set of points which satisfies the property \( (P) \) at time \( t \), which has no reason to be non decreasing, and \( H_t \) is the set of points which have satisfied \( (P) \) before time \( t \).

The process \((H_t)\) is non decreasing: in the sense that once \( x \) belongs to \( H_{t_0} \), it belongs to \( H_t \) fort all \( t > t_0 \). The processes \((A^f_t), (H^f_t), (A^y_t), (H^y_t)\) are defined as before according to the initial configuration.

In general, the process \((A_t)\) only gives a partial information: if we want to know the set \( A_{t+dt} \) of sites satisfying the property at time \( t + dt \), we need the full knowledge of the states of each site at time \( t \). Therefore, we should be careful when using the Markov property with the process \((A_t)\).

2.1.3. Interacting particle system. We are interested in a subclass of nearest neighbor interacting (NNI) particle systems; a general definition and the proof of their existence can be found in the book [Lig05]. In order to specify the dynamics of the process, let us introduce some notations. For \( x, y \in \mathbb{Z}^d \), \( s \in S \), \( \bar{s} = (s_1, s_2) \in S^2 \) and \( \xi \in S^{zd} \), we set:
form of Harris’ (see [Des14] for an example of such a construction). For a lot of models, such a construction is done to obtain a graphical construction in the graph

\begin{equation}
A_f(\xi) = \sum_{x \in \mathbb{Z}^d} \sum_{s \in S} c(x, \xi, s) \left( f(\xi(x,s)) - f(\xi) \right) + \sum_{x \neq y} \sum_{s \in S^2} c(x, y, \xi, s) \left( f(\xi(x,y,s)) - f(\xi) \right),
\end{equation}

for \( f \in C_0(\mathbb{Z}^d) \) and \( \xi \in \mathbb{Z}^d \). The quantity \( c(x, \xi, s) \geq 0 \) (resp. \( c(x, y, \xi, s) \geq 0 \)) is the intensity for a jump \( \xi \to \xi(x,s) \) (resp. \( \xi \to \xi(x,y,s) \)), and depends on \( \xi \) only through \( \{ \xi(z), z \sim x \} \) and \( \xi(x) \) (resp. through \( \{ \xi(z), z \sim x \} \) or \( z \sim y \)). For simplicity’s sake, we take \( c(x, \xi, s) = 0 \) if \( \xi(x) = s \) and \( c(x, y, \xi, s) = 0 \) if \( \xi(x) = s_1 \) or \( \xi(y) = s_2 \).

Note that, in most of the models we will mention after, only one site changes state at a time, so the second term of the generator is equal to zero. This term appears for models which incorporate stirring [MMS24+]. We believe that our proofs can be generalized for finite range interactions instead of nearest neighbors interactions, but all our examples have a generator of the previous form.

We say that the process \( (\xi_t) \) can be constructed from (marked) Poisson point processes if there exists families \( (\mathcal{N}_e)_{e \in \mathbb{E}^d} \) (where \( \mathbb{E}^d \) is the set of oriented edges between nearest neighbors), and \( (\mathcal{N}_x)_{x \in \mathbb{Z}^d} \) of mutually independent Poisson point processes on \( \mathbb{R}^+ \times \mathcal{M} \), where \( \mathcal{M} \) is a finite set of marks, such that:

- the set of jump times of the process \( (\xi_t) \) is a subset of the times given by these families,
- if \( (T, M) \in \mathcal{N}_e \) (resp. \( (T, M) \in \mathcal{N}_x \)) is a jump time of \( (\xi_t) \) of mark \( M \in \mathcal{M} \), then we have \( \xi_T = f_e(\xi_{T^-}, M) \) (resp. \( x_T = f_x(\xi_{T^-}, M) \)), where \( f_e \) and \( f_x \) are deterministic functions.

For a lot of models, such a construction is done to obtain a graphical construction in the form of Harris’ (see [Des14] for an example of such a construction).

For \( f \in \mathbb{Z}^d \), we denote by \( (\xi^f_t) \) the process \( (\xi_t) \) starting from \( \xi_0 = f \). The classical contact process and a lot of its variations are additive processes, that is: for all \( f, g \in \mathbb{Z}^d \), we can couple the processes \( (\xi^f_t) \), \( (\xi^g_t) \) and \( (\xi^{f \vee g}_t) \) in such a way that \( \xi^f_t = \xi^g_t = \xi^{f \vee g}_t \) for all \( t \in \mathcal{T} \). Harris [Harris] proved that additive processes can me constructed from Poisson processes and graphically represented.

An important property we ask for our models is attractiveness for the set of sites which verifies the property: the process \( (A_t) \) is attractive if for all \( E_-, E_+ \subset \mathcal{P}(\mathbb{Z}^d) \) with \( E_- \subset E_+ \), we can couple the processes \( A_t^{E_-} \) and \( A_t^{E_+} \) in such a way that we have \( A_t^{E_-} \subset A_t^{E_+} \) for all \( t \in \mathcal{T} \), where for all \( E \in \mathcal{P}(\mathbb{Z}^d) \), \( (A_t^E) \) is the process \( (A_t) \) conditioned to start from \( A_0 = E \). Note that additivity implies attractiveness.

Let us summarize the class \( \mathcal{C} \) of models we are considering:
**Definition 1.** Let $(\xi_t)$ be a stationary Feller Markov process taking values in $S^d$, and $(P)$ a property of interest. We say that $((\xi_t), (P))$ belongs to the class $\mathcal{C}$ if it satisfies the following conditions:

- The generator of the process $(\xi_t)$ is of the form $\mathbf{1}$.
- The process $(\xi_t)$ can be constructed with Poisson point processes.
- The process $(A_t)$ is attractive.
- $\{A_t = \emptyset\}$ is an absorbing state.

**Remark.** An other way to construct a Feller Markov process on $S^d$ from a Poisson point process is given by Swart ([Swa22], Section 4.3). When possible, he constructs the process from a Poisson point process $\omega$ on $\mathcal{G} \times \mathbb{R}_+$, where $\mathcal{G}$ is a set of local maps from $S^d$ to itself such that the generator $A$ of $(\xi_t)$ can be written as:

$$\forall \xi \in S^d, \quad Af(\xi) = \sum_{m \in \mathcal{G}} r_m [f(m(\xi)) - f(\xi)],$$

where for all $m \in \mathcal{G}$, $r_m$ is a positive constant. Note that such a construction can replace our hypothesis of the existence of a construction with Poisson point processes.

**2.1.4. Probability space and translations.** We denote by $\mathbb{E}^d$ the set of oriented edges between nearest neighbors of $\mathbb{Z}^d$. Let $((\xi_t), (P)) \in \mathcal{C}$, $(N_e)_{e \in \mathbb{E}^d}$ and $(N_x)_{x \in \mathbb{Z}^d}$ be mutually independent Poisson point processes on $\mathbb{R}_+ \times \mathcal{M}$ such that the process $(\xi_t)$ can be constructed from these families. We endow $\mathbb{R}^+ \times \mathcal{M}$ with the $\sigma$-algebra $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{P}(\mathcal{M})$, and we denote by $M$ the set of locally finite counting measures $m = \sum_{i=0}^{+\infty} \delta_{(t_i, m_i)}$ on $\mathbb{R}^+ \times \mathcal{M}$. We endow $M$ with the $\sigma$-algebra $\mathcal{F}_M$ generated by the maps $m \mapsto m(B)$, with $B \in \mathcal{B}(\mathbb{R}^+)$. We define the measurable spaces $(\Omega, \mathcal{F})$ by

$$(\Omega, \mathcal{F}) = (M^\mathbb{E}^d \times M^\mathbb{Z}^d, \mathcal{F}^\mathbb{E}^d \otimes \mathcal{F}^\mathbb{Z}^d).$$

We define the probability measure $\mathbb{P} = \mathbb{E}\mathbb{E}^d \otimes \mathbb{E}\mathbb{Z}^d \mathcal{M}$ on this measurable space. For any $t \geq 0$, we define the time translation operator $\theta_t$ on a locally finite counting measure $m = \sum_{i=1}^{+\infty} \delta_{t_i}$ on $\mathbb{R}_+$ by setting

$$\theta_t m = \sum_{i=1}^{+\infty} \mathbf{1}_{\{t_i \geq t\}} \delta_{t_i - t}.$$

It induces an operator on $\Omega$, still denoted $\theta_t$: we call it time translation. It is defined as follows:

$$\theta_t((\omega_e)_{e \in \mathbb{E}^d}, (\omega_x)_{x \in \mathbb{Z}^d}) = ((\theta_t \omega_e)_{e \in \mathbb{E}^d}, (\theta_t \omega_x)_{x \in \mathbb{Z}^d}).$$

The Poisson point processes being translation invariant, the probability measure $\mathbb{P}$ is stationary under the action of $\theta_t$. For all $z \in \mathbb{Z}^d$, we also define a spatial translation $T_z$ on $\Omega$ by:

$$T_z((\omega_e)_{e \in \mathbb{E}^d}, (\omega_x)_{x \in \mathbb{Z}^d}) = ((\omega_{(x+z,y+z)})_{e \in \mathbb{E}^d}, (\omega_{x+z})_{x \in \mathbb{Z}^d}).$$

**2.1.5. Extinction time and hitting time.** We define, for all $x, y \in \mathbb{Z}^d$:

$$\tau^x = \inf\{t > 0, A_t^x = \emptyset\} \quad \text{and} \quad t^x(y) = \inf\{t > 0, y \in A_t^x\}.$$

Starting from $\xi_0 = \delta_x$, $\tau^x$ is the extinction time of the property, that is, the first time when no one satisfies $(P)$. The quantity $t^x(y)$ is the hitting time of $y$, that is,
the first time when $y$ satisfies $(P)$. Finally, we denote by $\mathbb{P}_x$ the probability measure $\mathbb{P}(\cdot | \tau^x = +\infty)$, the probability conditioned to the survival of the property, starting from $\xi_0 = \delta_x$. When $x = 0_{\mathbb{Z}^d}$, we will omit the $x$ in the notations: we will simply write $\tau$, $t(y)$ and $\mathbb{P}$.

2.1.6. Linear growth models. We work on a subclass $C_L$ of $C$ which contains processes for which the set of sites satisfying the property grows linearly with time, in a sense we precise in the following definition.

**Definition 2.** We say that $((\xi_t), (P))$ belongs to the class $C_L$ if it satisfies the following conditions:

1. we have $\inf_{x \in \mathbb{Z}^d} \mathbb{P}(\tau^x = +\infty) > 0$,
2. there exist $C_1, C_2, M_1, M_2 > 0$ such that for all $t \in T$ and $x, y \in \mathbb{Z}^d$,

   (AML) $\mathbb{P} \left( \exists y \in \mathbb{Z}^d : t^x(y) \leq t \text{ and } \|y - x\|_1 \geq M_1 t \right) \leq C_1 \exp(-C_2 t)$,

   (SC) $\mathbb{P}(t < \tau^x < \infty) \leq C_1 \exp(-C_2 t)$,

   (ALL) $\mathbb{P}(t^x(y) \geq M_2 \|y - x\|_1 + t, \tau = \infty) \leq C_1 \exp(-C_2 t)$.

The condition (2) means that the growth of the set of points satisfying $(P)$ is at most linear (AML), at least linear (ALL) and, if the extinction time $\tau^x$ is finite, then it is small (SC for small clusters, in analogy with percolation vocabulary). A model in $C \cap C_L$ is called a random linear growth model.

2.2. Results and examples. We will prove the following general asymptotic shape theorem.

**Theorem 1.** Let $((\xi_t), (P))$ be a Markov process taking values in $S^{\mathbb{Z}^d}$ and $(P)$ an associated property of interest. If $((\xi_t), (P))$ belongs to classes $C$ and $C_L$, then there exists a norm $\mu$ on $\mathbb{R}^d$ such that for all $\varepsilon > 0$, $\mathbb{P}(\cdot | \tau = +\infty)$ almost surely, for all $t$ large enough,

$$(1 - \varepsilon)B_\mu \subset \frac{\tilde{H}_t}{t} \subset (1 + \varepsilon)B_\mu,$$

with $\tilde{H}_t = H_t + [0, 1)^d$, and $B_\mu$ the unit ball associated to $\mu$.

The proof of this theorem is given in Section 4. In the following, we give 6 examples of models to which this theorem applies. This asymptotic shape theorem is already proved for examples 2.2.1 and 2.2.2 and conjectured for example 2.2.4. In each statement mentioned below, we denote by $B_\mu$ the unit ball associated to a norm $\mu$ and $\mathbb{P}$ the probability conditioned to survival of the property $(P)$ starting from the configuration $\delta_0$.

2.2.1. Classical contact process. The generator of the contact process with infection rate $\lambda > 0$ and recovery rate 1 can be written in the form \(\Pi\), with $S = \{0, 1\}$, $T = \mathbb{R}_+$ and

$$c(x, \xi, 0) = \mathbbm{1}_{\xi(x) = 1}$$
$$c(x, \xi, 1) = \mathbbm{1}_{\xi(x) = 0} \times \lambda \sum_{y \sim x} \mathbbm{1}_{\xi(y) = 1}.$$
As detailed in the introduction, the asymptotic shape theorem has already been proved for \((P) = \{1\}\), that is the property ‘to be infected’, by Bezuidenhout, Durrett, Griffeath and Grimmett, see \([DG82]\) and \([BG90]\).

The next examples are extensions of the classical contact process from the literature; we give here definitions with the notations of our context. They belong to class \(C\) by construction. Several authors have worked hard to show that they belong to the class \(C_L\).

### 2.2.2. Contact process with aging (CPA)

Here, we interpret the contact process dynamics as birth/death instead of infection/recovery. \([Des14]\) introduced the contact process with aging where the particles have an integer ‘age’ impacting their birth rate so \(S = \{0, 1, \ldots\}\) and \(T = \mathbb{R}_+\). The parameters of the model are the aging rate \(\gamma > 0\) and the infection rates \((\lambda_i)_{i \geq 1}\) which are non-decreasing and limited when \(i\) goes to infinity.

The jump intensities in the generator are

\[
\begin{align*}
c(x, \xi, 0) &= \mathbb{1}_{\xi(x) > 0} \\
c(x, \xi, 1) &= \mathbb{1}_{\xi(x) = 0} \times \sum_{i \geq 1} \lambda_i N_i(x, \xi) \\
c(x, \xi, k) &= \gamma \mathbb{1}_{\xi(x) = k - 1}
\end{align*}
\]

where \(k \geq 2\) and \(N_i(x, \xi)\) is the number of neighbors of \(x\) with age \(i\) in \(\xi\). Let \(P = \{1, \ldots\}\) be the property ‘to be alive’. The process \((A_t)\) represents the set of alive points regardless of their age. Here, \(\delta_0\) is the configuration where all particles are dead except the origin which is alive with age 1. From the results obtained in \([Des14]\), \(((\xi_t), (P))\) belongs to \(C \cap C_L\) and verify the asymptotic shape theorem in the supercritical phase. This work has inspired our proof of an asymptotic shape theorem for a general class of random linear growth models. We deduce from this result an asymptotic shape theorem for Krone’s model \([Kro99]\), which is a particular case of the contact process with aging.

### 2.2.3. Contact process in a randomly evolving environment (CPREE)

The contact process in a randomly evolving environment was introduced by Braman \([Bro07]\), and studied in particular by Steif and Warfheimer \([SW08]\). For this model, \(T = \mathbb{R}_+\) and \(S = \{(0, 0), (0, 1), (1, 0), (1, 1)\}\). An element of \(S\) represents a pair (type, stage) where the type can be favorable (1) or unfavorable (0) and the stage can be alive (1) or dead (0). We can write the jump intensities:

| \(\xi(x) \rightarrow s\) | \(c(x, \xi, s)\) |
|-----------------|-----------------|
| birth           |                 |
| (type independent) | \(0, 0 \rightarrow (0, 1)\) | \(\sum_{y \sim x} \xi(y)\) |
|                 | \(1, 0 \rightarrow (1, 1)\) | \(\sum_{y \sim x} \xi(y)\) |
| death           |                 |
| (type dependent) | \(0, 1 \rightarrow (0, 0)\) | \(\delta_0\) |
|                 | \(1, 1 \rightarrow (1, 0)\) | \(\delta_1\) |
| evolution       |                 |
| of type         | \(0, 0 \rightarrow (1, 0)\) | \(\gamma p\) |
|                 | \(0, 1 \rightarrow (1, 1)\) | \(\gamma p\) |
|                 | \(1, 0 \rightarrow (0, 0)\) | \(\gamma (1 - p)\) |
|                 | \(1, 1 \rightarrow (0, 1)\) | \(\gamma (1 - p)\) |

with \(0 \leq \delta_1 < \delta_0\), \(p \in [0, 1]\) and \(\gamma \geq 0\) being the four parameters of the system.
Let \((P) = \{(0,1), (1,1)\}\) be the property 'to be alive'. We consider the lexicographic order on \(S\) so here, \(\delta_0\) is the configuration where 0 is in state (0,1) and all the other sites are in state (0,0). The process \((A_t)\) represents the set of alive points regardless of their type (favorable or unfavorable) and corresponds to the process \((C_L)\) in Steif and Warfheimer’s article \[SW08\]; our \(\tau\) corresponds to their \(\tau^{0,0}\). Using the Steif-Warfheimer construction and a restart argument like the one in section 5.3 of \[Des14\], it is easy to see that the conditions (AML), (SC) and (ALL) are satisfied, so \(((\xi_t), (P))\) belongs to \(C \cap C_L\).

**Corollary 2** (Asymptotic shape theorem for CPREE). Let \((\xi_t)\) be a contact process in a randomly evolving environment. If \(p > p_c(d)\) (critical value for bond percolation on \(\mathbb{Z}^d\)), then there exists a norm \(\mu\) on \(\mathbb{R}^d\) such that for all \(\varepsilon > 0\), \(\mathbb{P}(|\xi| = +\infty)\) almost surely, for all \(t\) large enough,

\[
(1 - \varepsilon)B_\mu \subset \frac{H_t}{t} \subset (1 + \varepsilon)B_\mu,
\]

where \(\tilde{H}_t = H_t + [0,1)^d\), with \(H_t\) the set of points born before time \(t\), regardless of their type.

For \(p \leq p_c(d)\), the process can survive without satisfying an asymptotic shape theorem.

2.2.4. **Dependent oriented percolation (DOP).** In \[GM12\], Garet and Marchand introduced a dependent oriented percolation model where fertile bacterium (represented by a type 1 particle) is submerged in a population of immune cells (type 2 particle) that are going to impede its development. The immune cells are not very fertile but benefit from a constant immigration process. Take \(T = \mathbb{N}\) and \(S = \{0,1,2\}\). The system is described by a discrete time Markov chain depending on 3 parameters \(p,q, \alpha \in (0,1)\).

Firstly, between time \(n\) and time \(n + 1/2\), each particle attempts to infect its neighbors with its type: it succeeds with probability \(p\) if the particle is of type 1 particle, and with probability \(q\) if it is of type 2 particle (taking the place of the type 1 particle if there is one). Secondly, between time \(n + 1/2\) and time \(n + 1\), there is an immigration of type 2 particles on each site with probability \(\alpha > 0\) (again eventually taking the place of a type 1 particle). Let \((P) = \{1\}\) be the property 'to be a fertile particle'. The process \((A_t)\), denoted by \((\eta_{1,t})\) in \[GM12\], represents alive fertile particles. Here, \(\delta_0\) is the configuration where all particles are dead except the origin which is alive and fertile (of type 1) and our \(\tau\) is their \(\tau^{0,0}\). Theorem 1.2 of \[GM12\] ensures that the conditions (AML), (SC) and (ALL) are satisfied if the process survives (that is if \(\alpha\) is smaller than a critical value \(\alpha_c(p,q)\)). So, \(((\xi_t), (P))\) belongs to \(C \cap C_L\). Therefore, using Theorem 4 proves Conjecture 1.3 of \[GM12\].

**Corollary 3** (Asymptotic shape theorem for DOP). Let \((\xi_t)\) be a dependent oriented percolation. If \(p > \hat{p}_c(d)\) (critical value for oriented bond percolation on \(\mathbb{Z}^d \times \mathbb{N}\)), \(q < \hat{p}_c(d)\) and \(\alpha \in (0, \alpha_c(p,q))\), then there exists a norm \(\mu\) on \(\mathbb{R}^d\) such that for all \(\varepsilon > 0\), \(\mathbb{P}(\ . \ | \tau = +\infty)\) almost surely, for all \(t\) large enough,

\[
(1 - \varepsilon)B_\mu \subset \frac{H_t}{t} \subset (1 + \varepsilon)B_\mu,
\]

where \(\tilde{H}_t = H_t + [0,1)^d\), with \(H_t\) the set of sites where particles of type 1 are before \(t\).
2.2.5. **Boundary modified contact process (BMCP).** Durrett and Schinazi [DS00] introduced a Boundary Modified Contact Process where particles can be infected (state 1), susceptible that have never been infected (state $-1$) and susceptible that have previously infected (state 0). Take $T = \mathbb{R}^+$, $S = \{-1, 0, 1\}$ and the jump intensities as follows:

$$c(x, \xi, 1) = \mathbb{1}_{\xi(x) = -1} \times \lambda_e \sum_{y \sim x} \mathbb{1}_{\xi(y) = 1} + \mathbb{1}_{\xi(x) = 0} \lambda_i \sum_{y \sim x} \mathbb{1}_{\xi(y) = 1},$$

$$c(x, \xi, 0) = \mathbb{1}_{\xi(x) = 1}.$$

where the quantities $\lambda_e$ and $\lambda_i$ are the two non-negative parameters of the system. We consider $P = \{1\}$ that is the property ‘to be infected’, and the process $(A_t)$ represents infected particles at time $t$. Here, $\delta_0$ is the configuration where all site are susceptible that have never been infected (i.e. in state $-1$) except the origin which is infected (state 1). We think that the renormalization work done by Durrett and Schinazi in [DS00] leads to conditions (AML), (SC) and (ALL) thanks to restart techniques (like in section 5.3 of [Des14]) for $\lambda_i > \max(\lambda_c(d), \lambda_e)$, where $\lambda_c(d)$ is the critical value of the classical contact process on $\mathbb{Z}^d$. So, we deduce from Theorem 1 the following result:

**Corollary 4** (Asymptotic shape theorem for BMCP). Let $(\xi_t)$ be a boundary modified contact process, and $P = \{1\}$ the property ‘to be infected’. If $((\xi_t), (P))$ belongs to the class $C_L$, then there exists a norm $\mu$ on $\mathbb{R}^d$ such that for all $\varepsilon > 0$, $\mathbb{P}(\tau = +\infty)$ almost surely, for all $t$ large enough,

$$(1 - \varepsilon)B_{\mu} \subset \frac{\bar{H}_t}{t} \subset (1 + \varepsilon)B_{\mu},$$

where $\bar{H}_t = H_t + [0, 1)^d$, with $H_t$ the set of particles infected before time $t$.

2.2.6. **Richardson’s model with stirring (RMS) and contact process with stirring (CPS).** Richardson’s model is a contact process without healing. The stirring dynamic corresponds to an exchange of the states of two sites. For both RMS and CPS, we have $S = \{0, 1\}$, $T = \mathbb{R}^+$ and $P = \{1\}$ the property ‘to be infected’. Here, the initial configuration $\delta_0$ is the classical one where all site are healthy except the infected origin. For these two models, stirring represents the movements of the infected particles. The generators of RMS and CPS can be written in the form $\mathbb{1}$. For RMS with infection rate $\lambda > 0$ and stirring rate 1, the jump intensities are:

$$c(x, \xi, 1) = \mathbb{1}_{\xi(x) = 0} \times \lambda \sum_{y \sim x} \mathbb{1}_{\xi(y) = 1},$$

$$c(x, y, \xi, 1, 0) = \mathbb{1}_{\{\xi(x) = 1, \xi(y) = 0\}}.$$  

For CPS with infection rate $\lambda > 0$, healing rate 1 and stirring rate $N > 0$, the jump intensities are:

$$c(x, \xi, 1) = \mathbb{1}_{\xi(x) = 0} \times \lambda \sum_{y \sim x} \mathbb{1}_{\xi(y) = 1},$$

$$c(x, y, \xi, 1, 0) = N \times \mathbb{1}_{\{\xi(x) = 1, \xi(y) = 0\}},$$

$$c(x, \xi, 0) = \mathbb{1}_{\xi(x) = 1}.$$
Durrett and Neuhauser [DN94] studied the asymptotic behavior of some interacting particle systems in which the particles are stirred at fast rate. After that, Katori [Kat94] and Konno [Kon94], and more recently [BM14], [LV17] and [MS21] studied the particular case of the contact process with stirring, and obtained results on the speed of convergence of the critical parameter $\lambda_c(N)$, seen as a function of the stirring rate $N$.

Marchand, Marcovici and Siest [MMS24+] prove that both RMS and CPS are in the class $C \cap C_L$ for large enough infection rates, therefore Theorem 1 can be applied to obtain an asymptotic shape theorem in these cases. We denote by $\lambda_c(d)$ the critical infection parameter for the classical contact process with healing rate 1 in dimension $d$.

**Corollary 5** (Asymptotic shape theorem for RMS and CPS). (1) Let $(\xi_t)$ be RMS with infection rate $\lambda > 2d\lambda_c(d)$ and stirring rate 1. There exists a norm $\mu$ on $\mathbb{R}^d$ such that for all $\varepsilon > 0$, $\mathbb{P}$-almost surely, for all $t$ large enough,

$$ (1 - \varepsilon)B_\mu \subset \tilde{H}_t \subset (1 + \varepsilon)B_\mu, $$

where $\tilde{H}_t = H_t + [0,1]^d$, with $H_t$ the set of particles infected before time $t$.

(2) Let $(\xi_t)$ be a CPS of stirring rate $\nu$, infection rate $\lambda > (2d\nu + 1)\lambda_c(d)$ and healing rate 1. There exists a norm $\mu$ on $\mathbb{R}^d$ such that for all $\varepsilon > 0$, $\mathbb{P}(\cdot ; \tau = +\infty)$-almost surely, for all $t$ large enough,

$$ (1 - \varepsilon)B_\mu \subset \frac{\tilde{H}_t}{t} \subset (1 + \varepsilon)B_\mu, $$

where $\tilde{H}_t = H_t + [0,1]^d$, with $H_t$ the set of particles infected before time $t$.

For contact processes (with aging, boundary modified, in a randomly evolving environment, with stirring), $t(x)$ is the first time when the particle $x$ is alive (or infected) regardless its age, type or memory. For dependent oriented percolation, $t(x)$ is the first time when the site $x$ is a type 1 particle.

### 2.3. Discussion on non-attractive models.

Attractivity property is a key ingredient in our proofs, in particular for the construction of the essential hitting time. There are non-attractive models for which an asymptotic shape theorem is conjectured but for which, to our knowledge, there are only proofs in dimension 1. The kinetically constrained model FA1-f is one of them: it is an interacting particle system on $\{0,1\}^Z$, where each site can update its state (at rate $q$ in state $1$, at rate $1 - q$ in state 0) if it has at least one 1 in its neighbors. This model is not attractive. Blondel, Deshayes and Toninelli [BDT19] have shown linear edge growth in dimension 1 (for $q$ large enough), but the question remains open in higher dimensions. More recently, Velasco [Vel24+] introduced two new extensions of the contact process. In both models, $S = \{-1,0,1\}$, where $-1$ represents sterile particle (unable to give birth):

- the inherited sterility process (IS), where 1’s give birth to 1 at rate $\lambda$ with probability $p$, and to $-1$ with probability $1 - p$.
- the spontaneous sterility process (Spont) where 1’s give birth to 1 at rate $\lambda$ (as in the classical CP), but where $-1$’s can appear spontaneously in the place of 0’s (without neighborhood condition).
Spont is attractive, and Velasco’s renormalization work could make it possible to obtain the estimates (ALL), (AML) and (SC), implying that Spont is in the class \( \mathcal{C} \cap \mathcal{C}_L \), and by Theorem 1 we verify an asymptotic shape theorem. The IS, on the other hand, is not attractive and our techniques cannot be applied.

From now, \( (\xi_t, (P)) \) belongs to \( \mathcal{C} \cap \mathcal{C}_L \). In Section 3 we introduce the essential hitting time \( \tau \), a quantity that has good properties regarding to the dynamical system, and show that the difference between \( \tau \) and the hitting time \( t \) is not too large. In Section 4 we prove Theorem 1 thanks to a subadditive ergodic theorem applied to the essential hitting time.

### 3. Essential hitting time

#### 3.1. Definition and properties

With non permanent models like contact process extensions, the hitting times can be infinite (because extinction is possible), and if we condition on the survival, we can lose independence, stationarity and even subadditivity properties required by Kingman’s theory. We are inspired by the construction of Garet and Marchand [GM12a], for the contact process in random environment, to build the essential hitting time for a random linear growth model \((\xi_t, (P)) \in \mathcal{C} \cap \mathcal{C}_L \). In the following, we will use the different properties of \((\xi_t)\) detailed in Definitions 1 and 2.

We set \( u_0(x) = v_0(x) = 0 \) and we define by induction two sequences of stopping times \((u_n(x))_n\) and \((v_n(x))_n\) as follows.

- Suppose that \( v_k(x) \) is defined. We set
  \[
  u_{k+1}(x) = \inf \{ t \geq v_k(x) : x \in A_t \},
  \]
  If \( v_k(x) \) is finite, then \( u_{k+1}(x) \) is the first time after \( v_k(x) \) where the site \( x \) satisfies \((P)\); otherwise, \( u_{k+1}(x) = +\infty \).

- Suppose that \( u_k(x) \) is defined, with \( k \geq 1 \). We set \( v_k(x) = u_k(x) + \tau^x \circ \theta_{u_k(x)} \). If \( u_k(x) \) is finite, then the time \( \tau^x \circ \theta_{u_k(x)} \) is the (possibly infinite) extinction time of \((P)\) starting at time \( u_k(x) \) from the configuration \( \delta_x \); otherwise, \( v_k(x) = +\infty \).

We have \( u_0(x) = v_0(x) = 0 \leq u_1(x) \leq v_1(x) \leq \ldots \leq u_i(x) \leq v_i(x) \ldots \) We then define \( K(x) \) to be the first step when \( v_k(x) \) or \( u_{k+1}(x) \) becomes infinite:

\[
K(x) = \min \{ k \geq 0 : v_k(x) = +\infty \text{ or } u_{k+1}(x) = +\infty \}.
\]

The following lemma says that when the property survives in the entire process \((\xi_t)\), then the procedure stops after a finite number of steps almost surely, and at a time \( u_K(x) \) when the site \( x \) verifies the property \((P)\), and spreads it forever.

**Lemma 6.** Let \( \rho := \inf_{x \in \mathbb{Z}^d} \mathbb{P}(\tau^x = +\infty) \). We have:

1. \( \forall x \in \mathbb{Z}^d, \forall k \in \mathbb{N}, \mathbb{P}(K(x) > k) \leq (1 - \rho)^k \).
2. Almost surely, for all \( x \in \mathbb{Z}^d \),
   \[
   (K(x) = k \text{ and } \tau = +\infty) \iff (u_k(x) < +\infty \text{ and } v_k(x) = +\infty).
   \]
Proof. First item. Since \( u_{k+1}(x) \) is a stopping time, we can apply the strong Markov property at time \( u_{k+1}(x) \). Therefore we have:

\[
\mathbb{P}(K(x) \geq k + 1) = \mathbb{P}(u_{k+2}(x) < +\infty) \\
\leq \mathbb{P}(u_{k+1}(x) < +\infty, v_{k+1} < +\infty) \\
\leq \mathbb{P}(u_{k+1}(x) < +\infty, \tau^x \circ \theta_{u_{k+1}} < +\infty) \\
\leq \mathbb{P}(u_{k+1}(x) < +\infty) \mathbb{P}(\tau^x < +\infty) \\
\leq \mathbb{P}(K(x) > k)(1 - \rho).
\]

Second item. Suppose that \( K(x) = k \). Since \( v_k(x) \) is a stopping time, we can apply the strong Markov property at time \( v_k(x) \). Therefore, denoting by \( \eta \) the configuration \( \xi_{v_k(x)} \), we obtain:

\[
\mathbb{P}(\tau = +\infty, v_k(x) < +\infty, u_{k+1} = +\infty | F_{v_k(x)}) \\
= \mathbb{1}_{\{v_k(x) < +\infty\}} \mathbb{P}(\tau^\eta = +\infty, \tilde{\theta}^\eta(x) = +\infty) \\
= 0,
\]

since the growth is at least linear (ALL) then the first implication follows.

Now suppose that \( u_k(x) < +\infty \) and \( v_k(x) = +\infty \). By definition, we have \( K(x) = k \) and \( \tau^x \circ \theta_{u_k(x)} = +\infty \). Since the model is attractive, and since \( x \) satisfies the property \( (P) \) at time \( u_k(x) \), then we have \( \tau = +\infty \), and we have the second implication. \( \square \)

Now we can define our essential hitting time \( \sigma(x) \), and a translation related to it. See Figure 1 for an illustration of the construction.

Definition 3. For \( x \in \mathbb{Z}^d \), we call essential hitting time of \( x \) by the property \( (P) \) the quantity \( \sigma(x) = u_{K(x)} \). We define the operator \( \tilde{\theta}_x \) on \( \Omega \) by setting:

\[
\tilde{\theta}_x = \begin{cases} 
T_x \circ \theta_{\sigma(x)} & \text{if } \sigma(x) < +\infty, \\
T_x & \text{otherwise.}
\end{cases}
\]

In the following lemma, we state that the essential hitting time \( \sigma(x) \) and its associated translation \( \tilde{\theta}_x \) verify the independence and invariance properties we need to prove our asymptotic shape theorem. We recall that we denote by \( \mathbb{P}_x \) the probability measure \( \mathbb{P}(\cdot | \tau^x = +\infty) \), the probability conditioned to the survival of the property, starting from \( \xi_0 = \delta_x \).

Lemma 7. Let \( x, y \in \mathbb{Z}^d \), with \( x \neq 0_{\mathbb{Z}^d} \).

1. For all \( A \) in the \( \sigma \)-algebra generated by \( \sigma(x) \), and \( B \in F \), we have:

\[
\mathbb{P}(A \cap \tilde{\theta}_x^{-1}(B)) = \mathbb{P}(A)\mathbb{P}_x(B).
\]

2. For all \( A \in F \), we have \( \mathbb{P}(\tilde{\theta}_x^{-1}(A)) = \mathbb{P}(A) \).

3. Under \( \mathbb{P}_x \), \( \sigma(y) \circ \tilde{\theta}_x \) is independent from \( \sigma(x) \), and it has the same law as \( \sigma(y) \).

4. Under \( \mathbb{P}_x \), the random variables \( (\sigma(x) \circ (\tilde{\theta}_x)^j)_{j \geq 0} \) are independent and identically distributed.
Figure 1. Construction of the essential hitting time. At time $u_1(x)$, the site $x$ verifies the property for the first time. At time $v_1(x)$, the process starting from $\delta_x$ at time $u_1(x)$ does not contain any sites checking the property; since it is an absorbing state, the property disappears forever in this process. However, the initial process spreads the property to $x$ at time $u_2(x)$, so the procedure restart at this time. At time $v_2(x)$, the process starting from $\delta_x$ at time $u_2(x)$ does not contain any sites checking the property. Note that instead of pursuing the procedure at the red circle, we start at the first time after the extinction time $v_2(x)$ when $x$ checks the property. If we do not do that, then the process obtained does not have the independence properties we want. Finally, the process starting from $\delta_x$ at time $u_3(x)$ spread the property forever, therefore the procedure stops.

Proof. For the first point, the proof is very similar to the one of Lemma 8 of [GMT12a]. We just have to show that, for all $k \in \mathbb{N}^*$, we have

$$P(A \cap \tilde{\theta}_x^{-1}(B) \cap \{K(x) = k\}) = P(A \cap \{K(x) = k\}) P_x(B).$$

Let $A' \in \mathbb{R}$ be a Borel set such that $A = \{\sigma(x) \in A'\}$. We have:

1. $P(\{\tau = +\infty\} \cap A \cap \tilde{\theta}_x^{-1}(B) \cap \{K(x) = k\})$
2. $P(\tau = +\infty, \sigma(x) \in A', T_x \circ \theta_{\sigma(x)} < +\infty, u_k(x) < +\infty, v_k(x) = +\infty)$
3. $P(\tau = +\infty, u_k(x) \in A', T_x \circ \theta_{u_k(x)} < +\infty, \tau \circ \theta_{u_k(x)} = +\infty)$
4. $P(u_k(x) \in A', u_k(x) < +\infty)P(T_x \in B, \tau = +\infty)$
5. $P(u_k(x) \in A', u_k(x) < +\infty)P_x(\{\tau = +\infty\} \cap B)$.

Line (3) comes from the second point of Lemma 6. The attractivity property implies that, for any stopping time $T$, we have

$$\{T < +\infty, x \in \xi_T, T_x \circ \theta_T = +\infty\} \subset \{\tau = +\infty\},$$
from which we deduce line (11). We obtain line (12) with the strong Markov property at time \( u_k(x) \). Dividing by \( \mathbb{P}(\tau = +\infty) \), we obtain:

\[
\mathbb{P}(A \cap \tilde{\theta}_x^{-1}(B) \cap \{K(x) = k\}) = \beta \mathbb{P}_x(B),
\]

where \( \beta \) is a constant that does not depend on \( B \). Taking \( B = \Omega \), we obtain:

\[
\beta = \mathbb{P}(A \cap \{K(x) = k\}),
\]

and item (1) follows. Items (2), (3) and (4) follow from item (1), see the proof of Corollary 9 of [GM12a] for details.

Item (2) of Lemma 7 implies that, for all \( x \in \mathbb{Z}^d \), \((\Omega, \mathcal{F}, \mathbb{P}, \tilde{\theta}_x)\) is a dynamical system.

Now, our goal is to prove that the essential hitting time has the good properties we wanted. Since on \( \{\tau = +\infty\} \), we have almost surely

\[
\sigma(x) = u_1(x) + \sum_{i=1}^{K(x)-1} (v_i(x) - u_i(x)) + \sum_{i=1}^{K(x)-1} (u_{i+1}(x) - v_i(x)),
\]

by definition of \( \sigma(x) \) and by Lemma 6, then it amounts to study these two sums. The following lemma addresses the first one.

**Lemma 8.** There exist \( C_3, C_4 > 0 \) such that, for all \( x \in \mathbb{Z}^d \) and \( t > 0 \), we have:

\[
\mathbb{P}(\exists i < K(x), v_i(x) - u_i(x) > t) \leq C_3 \exp(-C_4 t).
\]

**Proof.** Using the strong Markov property at time \( u_i(x) \), Lemma 6 and property (SC), we have:

\[
\begin{align*}
\mathbb{P}(\exists i < K(x), v_i(x) - u_i(x) > t) &= \mathbb{P}(\bigcup_{i=1}^{+\infty} \{v_i(x) - u_i(x) > t\} \cap \{K(x) > i\} \cap \{\tau = +\infty\}) \\
&\leq \frac{1}{\rho} \sum_{i=1}^{+\infty} \mathbb{P}(\{t < v_i(x) - u_i(x) < +\infty\} \cap \{u_i(x) < +\infty\} \cap \{\tau = +\infty\}) \\
&\leq \frac{1}{\rho} \sum_{i=1}^{+\infty} \mathbb{P}(\theta_{u_i(x)}^{-1}(\{t < \tau^x < +\infty\}) \cap \{u_i(x) < +\infty\}) \\
&\leq \frac{1}{\rho} \mathbb{P}(t < \tau^x < +\infty) \sum_{i=1}^{+\infty} \mathbb{P}(u_i(x) < +\infty) \\
&\leq \frac{1}{\rho^2} \mathbb{P}(t < \tau^x < +\infty) \sum_{i=1}^{+\infty} \mathbb{P}(K(x) > i - 1) \\
&\leq \frac{1}{\rho^2} \mathbb{P}(t < \tau^x < +\infty) \leq C_3 \exp(-C_4 t),
\end{align*}
\]

with \( C_3 = \frac{1}{\rho^2} C_1 \) and \( C_4 = C_2 \).  \( \Box \)
3.2. Bad growth points. The second sum of Equation (7) is harder to bound: the "reinfection time" $u_{i+1}(x) - v_i(x)$ depends on the configuration at time $v_i(x)$, therefore the strong Markov property will not be sufficient. We will introduce the notion of bad growth points in order to treat this case: if there is none of them at time $v_i(x)$ in a certain box centered on $x$ with high probability, then the site $x$ will be reinfected quickly enough, which bounds the reinfection time $u_{i+1}(x) - v_i(x)$.

Let $\nu_y$ be the counting measure giving the times of all possible changes of state of $y$. Thanks to the construction with Poisson point processes, the measure $\nu_y$ can be expressed as the law of a Poisson point process on $\mathbb{R}^+$. For every $x \in \mathbb{Z}^d$ and $t > 0$, we say that $y$ has bad growth (with respect to $x$ parameterized by $t$) if at least one of the following events occurs:

1. the spread of the property starting from $y$ is faster than expected by (AML),
2. $y$ has a finite but too long offspring,
3. $y$ has an infinite offspring but takes too long to reinfect $x$,
4. there is not a single event occurring at $y$ in the time interval $[0, t/2]$.

We denote by $E^y(x, t)$ the corresponding event:

$$E^y(x, t) = \{ H^y_t \not\subset y + B_{M_1 t} \} \cup \{ t/2 < \tau^y < +\infty \}$$
$$\cup \{ \tau^y = +\infty \text{ and } \inf\{ s \geq 2t : x \in A^y_s \} > \kappa t \} \cup \{ \nu_y[0, t/2] = 0 \}.$$

where $\kappa = 3M_1(1 + M_2)$, $M_1$ and $M_2$ being the constants respectively given in (AML) and (ALL). For a laps time $L > 0$, the measure $\nu_y$ allows us to count the number of bad growth points $(y, s) \in (x + B_{M_1 t + 2}) \times [0, L]$:

$$N_L(x, t) = \sum_{y \in x + B_{M_1 t + 2}} \int_0^L \mathbb{1}_{E^y(x, t)} \circ \theta_s d(\nu_y + \delta_0)(s).$$

Going back to our examples:

- For DOP (which is a discrete model), $\nu$ is the counting measure on $\mathbb{N}$.
- For CPA we consider:

$$\nu_y = \omega^0_y + \omega^\gamma_y + \sum_{e \in \mathcal{E} : y \in e} \omega^\infty_e,$$

where $\omega^0_y$, $\omega^\gamma_y$ and $\omega^\infty_e$ are the Poisson processes respectively giving the possible death times at $y$, the possible maturation times and the possible birth times through $e$.
- For CPREE, we consider:

$$\nu_y = \omega^0_y + \omega^1_y + \sum_{e \in \mathcal{E} : y \in e} \omega_e,$$

where $\omega^0_y$ is the Poisson process associated to type 0 deaths, $\omega^1_y$ the one associated to type 1 deaths and $\omega_e$ the one associated to births.

We control the probability that a space-time box contains no bad growth point.

**Lemma 9.** There exist $A, B > 0$ such that for all $L > 0$, $x \in \mathbb{Z}^d$ and $t > 0$ one has

$$\mathbb{P}(N_L(x, t) \geq 1) \leq A \exp(-Bt).$$
Proof. Let \( x \in \mathbb{Z}^d, t > 0 \) and \( y \in x + B_{Mt+2} \). First, we control the probability of \( E^y(x, t) \). There exist \( A_1, B_1, A_2, B_2 > 0 \) such that, for all \( t > 0 \), one has

- \( \mathbb{P}(H^y_t \not\subset y + B_{Mt}) \leq A_1 \exp(-B_1t) \) by property (ALL),
- \( \mathbb{P}(t < \tau^y < \infty) \leq A_2 \exp(-B_2t) \) by property (SC),
- If \( \tau^y = +\infty \), there exists \( z \in A^y_{2t} \) with \( \tau^z \circ \theta_{2t} = +\infty \). Thus, the choice of \( \kappa \) implies that

\[
\{\tau^y = +\infty, \inf\{s \geq 2t : x \in A^y_3\} > \kappa t\} \\
\subset \{A^y_{2t} \not\subset y + B_{2Mt}\} \cup \bigcup_{z \in y + B_{2Mt}} \{t^z(x) \circ \theta_{2t} > (\kappa - 2Mt) t\} \\
\subset \{A^y_{2t} \not\subset y + B_{2Mt}\} \cup \bigcup_{z \in y + B_{2Mt}} \{t^z(x) \circ \theta_{2t} > C\|x - z\| + Mt - 3C\}.
\]

Hence, with properties (ALL) and (AML),

\[
\mathbb{P}(\tau^y = +\infty, \inf\{s \geq 2t : x \in A^y_3\} > \kappa t) \\
\leq A \exp(-2BMt) + (1 + 4Mt)^d A \exp(-B(Mt - 3C))\).
\]

In the case of continuous processes, we also have to bound \( \mathbb{P}(\nu_y([0, t/2]) = 0) \). Since \( \nu_y \) has the law of a Poisson point process on \( \mathbb{R}_+ \), then there exists a constant \( A_3 > 0 \) such that \( \mathbb{P}(\nu_y([0, t/2]) = 0) = \exp(-A_3t/2) \).

We obtain the existence of \( A_4, B_4 > 0 \) such that for all \( x \in \mathbb{Z}^d, t > 0 \) and for all \( y \in x + B_{Mt+2} \) we have

\[
(8) \quad \mathbb{P}(E^y(x, t)) \leq A_4 \exp(-B_4t)
\]

For \( y \in x + B_{Mt+2} \), we set \( T^y_0 = 0 \) and denote by \((T^y_n)_{n \geq 1}\) the increasing sequence of the times given by \( \nu_y \). We have

\[
\mathbb{P}(N_L(x, t) \geq 1) \leq \mathbb{E}[N_L(x, t)] = \sum_{y \in x + B_{Mt+2}} \mathbb{E}\left[ \int_0^L \mathbb{1}_{E^y(x, t)} \circ \theta_s d(\nu_y + \delta_0)(s) \right]
\]

\[
= \sum_{y \in x + B_{Mt+2}} \mathbb{E}\left[ \sum_{n=0}^{+\infty} \mathbb{1}_{\{T^y_n \leq L\}} \mathbb{1}_{E^y(x, t)} \circ \theta_{T^y_n} \right]
\]

\[
= \sum_{y \in x + B_{Mt+2}} \sum_{n=0}^{+\infty} \mathbb{E}\left[ \mathbb{1}_{\{T^y_n \leq L\}} \mathbb{P}(E^y(x, t)) \right]
\]

\[
\leq \sum_{y \in x + B_{Mt+2}} \left( 1 + \mathbb{E}[\nu_y([0, L])] \right) \mathbb{P}(E^y(x, t))
\]

\[
\leq (2Mt + 5)^d (1 + LI) \mathbb{P}(E^y(x, t)),
\]

where \( I \) is the total intensity of the Poisson point process \( \nu_y \). Using Equation (8) we obtain the announced upper bound. \( \square \)

If there are no such bad growth points, we can bound the reinfection time:

**Lemma 10.** Let \( x \in \mathbb{Z}^d \) and \( t \geq 2 \). If \( L, s \) are positive integers such that \( N_L(x, t) \circ \theta_s = 0 \) and \( u_i(x) \in [s + t, s + L] \), then \( v_i(x) = +\infty \) or \( u_{i+1}(x) - u_i(x) \leq \kappa t \).
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\[ s \leq s' \leq \frac{t}{2} \leq \frac{t}{2} \leq t' \leq \kappa t \]

\[ \lfloor u_i(x) - t \rfloor = t_0 \]

\[ t_2 \geq u_{i+1}(x) \]

\[ v_i(x) - u_i(x) \leq \frac{t}{2} \text{ thanks to (3).} \]

\[ t_0 \leq u_i(x) - \frac{t}{2} \text{ thanks to (4).} \]

\[ u_i(x) - t_0 \geq \frac{t}{2} \]

\[ \sigma(x + y) - \left( \sigma(x) + \sigma(y) \circ \theta_x \right) \geq t \]

\[ \mathbb{P} \left( \sigma(x + y) - \left( \sigma(x) + \sigma(y) \circ \theta_x \right) \geq t \right) \leq A \exp \left( -B \sqrt{t} \right). \]

**Figure 2.** The gray block does not contain any bad growth point, so for any point \( y \) in this area, for all \( u \in [0, L] \), the event \( E^u(x, t) \circ \theta_{s+u} \) is not realised. We have \( v_i(x) - u_i(x) \leq \frac{t}{2} \) thanks to (3). The site \( x_0 \) is in the gray block because of (1). We have \( t_0 \leq u_i(x) - \frac{t}{2} \) thanks to (4). Therefore, \( u_i(x) - t_0 \geq \frac{t}{2} \), and (3) implies that the process starting from \( (x_0, t_0) \) survives. Finally, the site \( x_0 \) reinfects \( x \) (after \( v_i(x) \), but before \( u_i(x) + \kappa t \)), because of (3).

This lemma is a direct consequence of the definition. A detailed proof can be found in [GM12a]. See Figure 10 to have an idea of the proof. Iterating Lemma 10, we will be able to dominate \( \sigma(x) \), which will be useful to prove almost sub-additivity in Proposition 11.

### 3.3. Subadditivity and difference between \( \sigma \) and \( t \).  

**Proposition 11.** There exists \( A, B > 0 \) such that for all \( x, y \in \mathbb{Z}^d \),

\[ \forall t > 0, \quad \mathbb{P} \left( \sigma(x + y) - \left( \sigma(x) + \sigma(y) \circ \theta_x \right) \geq t \right) \leq A \exp \left( -B \sqrt{t} \right). \]
Moreover, for \( p \geq 1 \), there exists \( M_p > 0 \) such that for all \( x, y \in \mathbb{Z}^d \),
\[
\mathbb{E}[(\sigma(x + y) - (\sigma(x) + \sigma(y) \circ \tilde{\theta}_z))^p] \leq M_p.
\]

Proof. Let \( x, y \in \mathbb{Z}^d \), \( t > 0 \) and let \( s = \sigma(x) + \sigma(y) \circ \tilde{\theta}_z \). We have:
\[
\mathbb{P}\left(\sigma(x + y) > \sigma(x) + \sigma(y) \circ \tilde{\theta}_z + t\right) \leq \mathbb{P}\left(K(x + y) > \frac{\sqrt{t}}{\kappa}\right)
+ \mathbb{P}\left(K(x + y) \leq \frac{\sqrt{t}}{\kappa}, \sigma(x + y) \geq s + t\right).
\]

Thanks to (11) of Lemma \([11]\) the first right hand term is bounded by \( \frac{1}{\rho} \exp\left(\frac{\sqrt{t}}{\kappa}\ln(1 - \rho)\right) \).

For the second term we will iterate Lemma [11] to prove that \( \sigma \) is unlikely that big. Note that if \( K(x + y) \leq \frac{\sqrt{t}}{\kappa} \), then \( t \geq K(x + y)\kappa \sqrt{t} \) and
\[
\{N_{K(x+y)\kappa \sqrt{t}}(x+y, \sqrt{t} \geq 1) \subset \{N_t(x+y, \sqrt{t} \geq 1)\}.
\]

So,
\[
\mathbb{P}\left(K(x + y) \leq \frac{\sqrt{t}}{\kappa}, \sigma(x + y) \geq s + t\right)
\]
\[
\leq \mathbb{P}\left(K(x + y) \leq \frac{\sqrt{t}}{\kappa}, \sigma(x + y) \geq s + K(x + y)\kappa \sqrt{t}\right)
\]
\[
\leq \mathbb{P}\left(N_t\left(x + y, \sqrt{t}\right) \circ \theta_s \geq 1\right)
\]
\[
+ \mathbb{P}\left(\exists i < K(x + y) : v_i(x + y) - u_i(x + y) > \sqrt{t}\right).
\]

Indeed, if \( N_{K(x+y)\kappa \sqrt{t}}(x+y, \sqrt{t}) = 0 \) and \( \forall i \leq K(x + y) \), then there are two possibilities:
- either \( \forall i \leq K(x + y) \), \( u_i(x + y) \leq s + \sqrt{t} \) and then \( \sigma(x + y) \leq s + \sqrt{t} \leq s + K(x + y)\kappa \sqrt{t} \),
- or we define \( i_0 = \max\{i : u_i(x + y) \leq s + \sqrt{t}\} \) then \( v_{i_0}(x + y) \leq s + 2\sqrt{t} \) and \( u_{i_0+1}(x + y) \leq s + \kappa \sqrt{t} \) and \( u_{i_0+j}(x + y) \geq s + \sqrt{t} \) for all \( j \). We can iterate Lemma [11] to conclude that for all \( j \leq K(x + y) - i_0, u_{i_0+j}(x + y) \leq s + j\kappa \sqrt{t} \) and then \( \sigma(x + y) \leq s + K(x + y)\kappa \sqrt{t} \).

Since \( N_t(x + y, \sqrt{t}) \circ \theta_s = N_t(0, \sqrt{t}) \circ T_{x} \circ T_{y} \circ \theta_{\sigma(z)} \circ \theta_{\sigma(y) \circ \tilde{\theta}_z} \), we have
\[
N_t\left(x + y, \sqrt{t}\right) \circ \theta_s = N_t\left(0, \sqrt{t}\right) \circ \theta_y \circ \tilde{\theta}_x.
\]

Thus, \( \mathbb{P}(N_t(x + y, \sqrt{t}) \circ \theta_s \geq 1) = \mathbb{P}(N_t(0, \sqrt{t}) \geq 1) \), which is controlled by Lemma [9] and the second term of the sum (9) is bound by Lemma [6] \( \Box \)

Proposition 12. There exist \( A, B, \alpha > 0 \) such that for every \( z > 0 \) and every \( x \in \mathbb{Z}^d \),
\[
\mathbb{P}(\sigma(x) \geq t(x) + K(x)(\alpha \log(1 + ||x||) + z)) \leq A \exp(-Bz).
\]

Proof. We proceed in the same spirit as previously. First, we introduce a new box of bad growth points. Then, we show that the probability of these points is small and we
conclude by the fact that the quantity to control is dominated by this probability. For further details, see Proposition 17 of [GM12a]. □

**Proposition 13.** $\mathbb{P}$-almost surely, it holds that 
\[
\lim_{\|x\| \to +\infty} \frac{|\sigma(x) - t(x)|}{\|x\|} = 0.
\]

**Proof.** We want to bound $\mathbb{E}(|\sigma(x) - t(x)|^p)$. Using Minkowski inequality, Proposition 12 and item 1 of Lemma 6, we obtain
\[
\mathbb{E}(|\sigma(x) - t(x)|^p) \leq C(p)(\log(1 + \|x\|))^p.
\]
So, the sequence \(\left(\frac{|\sigma(x) - t(x)|}{1 + \|x\|}\right)_{x \in \mathbb{Z}^d}\) is in $\ell^p(\mathbb{Z}^d)$ and tends to 0 almost surely. □

**Corollary 14.** (I) There exist $A, B, C > 0$ such that 
\[
\forall x \in \mathbb{Z}^d, \forall t > 0, \mathbb{P}(\sigma(x) \geq C\|x\| + t) \leq A \exp(B\sqrt{t}).
\]
(II) For $p \geq 1$, there exists a constant $C(p) > 0$ such that 
\[
\forall x \in \mathbb{Z}^d, \mathbb{P}(\sigma(x)^p) \leq C(p)(1 + \|x\|)^p.
\]
(III) For every $\epsilon > 0$, $\mathbb{P}$-a.s., there exists $R > 0$ such that 
\[
\forall x, y \in \mathbb{Z}^d, (\|x\| \geq R \text{ and } \|x - y\| \leq \epsilon \|x\|) \implies (|\sigma(x) - \sigma(y)| \leq C\epsilon \|x\|).
\]

**Proof.** The first item (I) is proven writing
\[
\mathbb{P}(\sigma(x) > (C + 1)\|x\| + t) \leq \mathbb{P}(t(x) \geq C\|x\| + t/2) + \mathbb{P}(K(x) > \frac{1}{2\alpha}\sqrt{\|x\| + t/2})
\]
\[
+ \mathbb{P}(\sigma(x) > (C + 1)\|x\| + t, t(x) < C\|x\| + t/2, K(x) \leq \frac{1}{2\alpha}\sqrt{\|x\| + t/2})
\]
and then using (ALL), Lemma 6 and Proposition 12. The second one (II) comes from Minkowski inequality and (I). Then, the last item (III) follows from (I), Proposition 11 and Borel-Cantelli lemma. □

4. **Proof of the asymptotic shape theorem**

In this section we will use the following almost subadditive ergodic theorem, which is a reformulation of the theorem of Kesten and Hammersley (see [Ham74] and [Kes73]), adapted for the essential hitting time and the properties it verifies.

**Proposition 15** (Theorem 39 of [Des14]). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(\sigma(x))_{x \in \mathbb{Z}^d}$ be random variables with finite second moments and such that, for every $x \in \mathbb{Z}^d$, $\sigma(x)$ and $\sigma(-x)$ have the same distribution. Let $(s(y))_{y \in \mathbb{Z}^d}$ and $(r(x,y))_{x,y \in \mathbb{Z}^d}$ be collections of random variables such that:

\begin{itemize}
  \item **Hyp 1:** $\forall x, y \in \mathbb{Z}^d$, $\sigma(x + y) \leq \sigma(x) + s(y) + r(x,y)$ with $s(y)$ having the same law as $\sigma(y)$, and being independent from $\sigma(x)$, \n  \item **Hyp 2:** $\forall x, y \in \mathbb{Z}^d$, $\exists C_{x,y}$ and $\alpha_{x,y} < 2$ such that 
  \[\forall n, p, \mathbb{E}[r(nx, py)^2] \leq C_{x,y}(n + p)^{\alpha_{x,y}},\]
  \item **Hyp 3:** $\exists C > 0$ such that $\forall x \in \mathbb{Z}^d$, $\mathbb{P}(\sigma(nx) > C\|x\|) \xrightarrow{n \to \infty} 0$,
  \item **Hyp 4:** $\exists K > 0$ such that $\forall x \geq 0$, $\mathbb{P}$-p.s. $\exists M$ such that $(\|x\| \geq M \text{ and } \|x - y\| \leq K\|x\|) \implies \|\sigma(x) - \sigma(y)\| \leq \epsilon\|x\|$, \n\end{itemize}
Hyp 5: \( \exists c > 0 \) such that \( \forall x \in \mathbb{Z}^d, \ P(\sigma(nx) < cn\|x\|) \to 0 \).

Then there exists \( \mu : \mathbb{Z}^d \to \mathbb{R}^+ \) such that
\[
\lim_{\|x\| \to \infty} \frac{\sigma(x) - \mu(x)}{\|x\|} = 0 \ a.s.
\]
Moreover, \( \mu \) can be extended to a norm on \( \mathbb{R}^d \) and we have the following asymptotic shape theorem: for all \( \epsilon > 0 \), \( P \) almost surely, for all \( t \) large enough,
\[
(1 - \epsilon)B_\mu \subset \tilde{G} \subset (1 + \epsilon)B_\mu,
\]
where \( \tilde{G} = \{ x \in \mathbb{Z}^d : \sigma(x) \leq t \} + [0,1]^d \) and \( B_\mu \) is the unit ball for \( \mu \).

We now deduce the expected asymptotic shape theorem for the hitting time \( t \):

**Proposition 16.** There exists a norm \( \mu \) on \( \mathbb{R}^d \) such that almost surely under \( P \),
\[
\lim_{\|x\| \to +\infty} \frac{t(x) - \mu(x)}{\|x\|} = 0,
\]
and for every \( \epsilon > 0 \), \( P \)-a.s., for every large \( t \),
\[
(1 - \epsilon)B_\mu \subset \tilde{H} \subset (1 + \epsilon)B_\mu
\]
where \( \tilde{H} = \{ x \in \mathbb{Z}^d / t(x) \leq t \} + [0,1]^d \) and \( B_\mu \) is the unit ball for \( \mu \).

Theorem 1 is contained in the previous result.

**Proof.** First, we use Proposition 15 to show that \( \sigma \) satisfies an asymptotic shape theorem. We check the hypotheses of Proposition 15 using the controls of Corollary 14. Thanks to (I), \( \sigma \) has finite second moment required. We take \( s(y) = \sigma(y) \circ \tilde{\theta}_x \). The hypotheses 1 and 2 are satisfied thanks to properties of \( \tilde{\theta}_x \) (Lemma 7) and Proposition 11. The hypothesis 3 is the at least linear growth (I). The hypothesis 4 is the control (III). We deduce the result for \( t \) from the result for \( \sigma \) thanks to Proposition 13. Finally, hypothesis 5 is immediately checked thanks to the at most linear growth (AML):
\[
P(\sigma(nx) < M_2 \|x\|) \leq P(t(nx) < M_2n\|x\|) \leq \frac{A}{\rho} \exp(-BM_2n\|x\|).
\]
\( \Box \)

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