Double phase transition of the Ising model in core–periphery networks

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Abstract. We study the phase transition of the Ising model in networks with core–periphery structures. By Monte Carlo simulations, we show that prior to the order-disorder phase transition the system organizes into an inhomogeneous intermediate phase in which core nodes are much more ordered than peripheral nodes. Interestingly, the susceptibility shows double peaks at two distinct temperatures. We find that, if the connections between core and periphery increase linearly with network size, the first peak does not exhibit any size-dependent effect, and the second one diverges in the limit of infinite network size. Otherwise, if the connections between the core and periphery scale sublinearly with the network size, both peaks of the susceptibility diverge as power laws in the thermodynamic limit. This suggests the appearance of a double transition phenomenon in the Ising model for the latter case. Moreover, we develop a mean-field theory that agrees well with the simulations.

Keywords: critical phenomena of socio-economic systems, network dynamics
1. Introduction

Phase transitions and critical phenomena in complex networks have been the subject of intense research in statistical physics and many other disciplines [1–4]. Contrary to regular lattices in the Euclidean space, complex networks are usually characterized by a highly heterogeneous connectivity among nodes, such as power-law degree distributions [5]. Owing to the heterogeneity, phase transitions and critical phenomena on complex networks are drastically different from those on regular lattices. Examples range from the anomalous behavior of an Ising model [6–10] to a vanishing percolation threshold [11, 12] and the absence of the epidemic threshold that separates healthy and endemic phases [13–15] as well as the explosive emergence of phase transitions [16, 17]. On the other hand, many real-world networks exhibit a typical mesoscopic structure such as a community structure [18, 19]. A community is a group of nodes that are densely interconnected and sparsely connected to nodes in different communities. Such a community structure is also one of essential ingredients for determining dynamics on complex networks [20, 21]. In particular, it was shown that in equilibrium [22, 23] and nonequilibrium [24] the Ising models community structure can lead to a novel metastable phase in which spin orientation aligns with those in the same community but disaligns with those in different communities.

Core–periphery structure is another mesoscale structure of networks, with which a network consists of two groups of nodes called the core and periphery. Core nodes are densely interconnected, peripheral nodes are connected to core nodes to varying extents, and peripheral nodes are sparsely interconnected [25–30]. A core–periphery structure has been found in various networks, including brain networks [31], protein interaction networks [32], social networks [25, 33], transportation networks [27, 34], and so forth. Since Borgatti and Everett [25] introduced the first quantitative formulation of core–periphery structure, many algorithms have been developed for detecting the core–periphery structure [25, 27, 33–39]. However, little attention has been paid to the dynamics on networks with core–periphery structure. Recently, Verma et al [40] proposed a simple pruning process based on the removal of underutilized links and redistribution of loads and found that such a process is responsible for the emergence of a core–periphery structure.
In the present work, we aim to study how the core–periphery structure would impact the phase transition of Ising model. By Monte Carlo (MC) simulation and a mean-field analysis, we show that an intermediate phase emerges when the temperature is lower than the critical one. Such an intermediate phase is rather inhomogeneous. That is, core nodes are much more ordered than peripheral nodes. We also find that the susceptibility exhibits a double-peak profile as the temperature varies. We show that, on the one hand, if the number of the connections between core and periphery is linear with the network size, the height of the first peak is finite and does not have a size-dependent effect, while the second one diverges in the thermodynamic limit. On the other hand, if the connections between a core and periphery increase sub-linearly with the network size, both peaks of the susceptibility diverge as power laws, which indicates the occurrence of a double phase transition in the Ising model on the disordered network systems. We should note that the double-peaked phenomenon in susceptibility was reported recently in percolation models [41, 42] and in epidemic spreading models [43, 44].

2. Model and method

We consider the Ising model on a network whose Hamiltonian is given by,

\[ H = -J \sum_{i<j} A_{ij} \sigma_i \sigma_j - h \sum_i \sigma_i, \]  

(1)

where \( \sigma_i \in \{+1, -1\} \) is the spin variable of node \( i \), \( J > 0 \) is the ferromagnetic interaction constant, and \( h \) is the external magnetic field. The network is described by an adjacency matrix whose elements \( A_{ij} \) are defined as \( A_{ij} = A_{ji} = 1 \) if nodes \( i \) and \( j \) are connected, and zero otherwise.

The network consists of \( N \) nodes and \( M = N \langle k \rangle / 2 \) undirected edges, where \( \langle k \rangle \) is the average degree of the network. We pick a fraction \( p_c \) of nodes as core nodes, and the remaining \( 1 - p_c \) fraction of nodes as peripheral nodes. We introduce the parameters \( \pi_{cc}, \pi_{cp}, \) and \( \pi_{pp} \) as the connectivity probabilities among nodes in the core–core, core–periphery, and periphery–periphery, respectively. The number of edges in the network can be computed by

\[ N \langle k \rangle / 2 = \pi_{cc} N_c (N_c - 1)/2 + \pi_{cp} N_c N_p + \pi_{pp} N_p (N_p - 1)/2, \]  

(2)

where \( N_c = p_c N \) and \( N_p = (1 - p_c) N \) are the number of core nodes and peripheral nodes, respectively. Assuming that \( N_c, N_p \gg 1 \), equation (2) can be rewritten as

\[ \tilde{\pi} = \pi_{cc} p_c^2 + 2 \pi_{cp} p_c (1 - p_c) + \pi_{pp} (1 - p_c)^2, \]  

(3)

where \( \tilde{\pi} = \langle k \rangle / (N - 1) \) is the average probability that each node is connected to the other nodes. By defining \( r_1 = \pi_{cp} / \pi_{cc} \) and \( r_2 = \pi_{pp} / \pi_{cc} \), \( \pi_{cc} \) is thus expressed as

\[ \pi_{cc} = \tilde{\pi} / (r_1^2 + 2 r_1 r_2 (1 - p_c) + r_2^2 (1 - p_c)^2), \]  

(4)

and \( \pi_{cp} = r_1 \pi_{cc}, \pi_{pp} = r_2 \pi_{cc} \). If \( r_1 = r_2 = 1 \), the resulting networks are Erdős–Rényi random graphs. If \( r_1 \sim O(1) \) and \( r_2 \sim O(0) \), the resulting networks have the characteristics.
of a core–periphery structure. The main aim of the present work is to study the phase transition behaviors of Ising model on networks with a core–periphery structure.

We perform MC simulation with the Glauber dynamics. At each elementary step, one node is randomly chosen and tries to flip its spin with the probability \( k_B T \) is the inverse temperature, \( k_B \) is the Boltzmann constant, and \( \Delta E \) is the change of the system’s energy due to the flipping trial. On each MC step (MCS), each node is tried to update its spin once on average. To characterize the phase behavior of the network, we need to define three magnetizations: the average magnetization \( m = N^{-1} \sum_{i=1}^{N} \sigma_i \) of all the nodes, the average magnetization \( m_c = N_c^{-1} \sum_{i \in C} \sigma_i \) of all the core nodes, and the average magnetization \( m_p = N_p^{-1} \sum_{i \in P} \sigma_i \) of all the peripheral nodes, where \( C \) and \( P \) denote the sets of core nodes and peripheral nodes, respectively. To make the system in equilibrium, the first \( 10^5 \) MCS are discarded and the following \( 10^5 \) MCS are used to calculate ensemble averages of the physical quantities. At the critical region, larger runs are performed with \( 2 \times 10^5 \) MCS to reach the steady state and \( 10^6 \) for computing the averages.

3. Results

Firstly, we demonstrate the results on the network with \( N = 10000 \), \( \langle k \rangle = 20 \), \( p_c = 0.2 \), \( r_1 = 0.1 \), and \( r_2 = 0.005 \). Obviously, the network has a core–periphery structure. Figure 1(a) shows \( m \), \( m_c \), and \( m_p \) as functions of the temperature \( T \) in the absence of external field, namely \( h = 0 \). As \( T \) increases from zero, \( m_p \) decreases much quicker than \( m_c \). If \( T \) is larger than a critical value \( T_c \), both \( m_p \) and \( m_c \) approach zero and a disordered paramagnetic phase emerges. For \( T \) between zero and \( T_c \), there exists an intermediate phase in which core nodes are much more ordered than peripheral nodes. Such an intermediate phase is caused by the core–periphery structure of the network where the connectivity between core nodes is much denser than that between peripheral nodes. Figure 1(b) shows the susceptibility \( \chi \) as a function of \( T \). Here \( \chi \) is calculated by the fluctuation of the magnetization \( m \) according to fluctuation-dissipation theorem, \( \chi = \beta N \left[ \langle m^2 \rangle - \langle m \rangle^2 \right] \), where \( \langle \cdot \rangle \) denotes the averages taken in the stationary regime. Interestingly, \( \chi \) exhibits double peaks at two different \( T \), \( T_{c1} \) and \( T_{c2} \) with \( T_{c1} < T_{c2} \), which seems to indicate the existence of a double phase transition.

Since phase transition actually happens in the thermodynamic limit, we consider the size effect of \( \chi \) as follows. In figure 2(a), we show \( \chi \) as a function of \( T \) for several different \( N \). One can see that the location of the first peak does not change with the network size \( N \), and its height \( \chi_{m1} \) does not change with \( N \) either. However, unlike the first peak of \( \chi \), the location of the second peak shifts to a larger temperature and its height \( \chi_{m2} \) increases as \( N \) increases. In figure 2(b), we show how \( \chi_{m1} \) and \( \chi_{m2} \) vary with \( N \). In a double logarithmic coordinate, \( \chi_{m1} \) and \( \chi_{m2} \) can be well fitted linearly, i.e. \( \chi_{m1} \sim N^{\gamma'_1/\nu} \) and \( \chi_{m2} \sim N^{\gamma'_2/\nu} \), with the exponents \( \gamma'_1/\nu \approx 0 \) and \( \gamma'_2/\nu = 0.43(5) \). This suggests that only \( \chi_{m2} \) diverges in the limit of \( N \rightarrow \infty \). We call the phenomenon a pseudo-double phase transition. The singularity in \( \chi_{m2} \) indicates an actual phase transition will occur at a certain temperature \( T_{c2} \). To determine \( T_{c2} \), we calculate the Binder’s fourth-order

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To proceed with a theoretical analysis, let us start with the mean-field equations for $m_c$ and $m_p$, given by [7, 45]

\[ m_c = \tanh \left[ \beta \left( k_{cc}m_c + k_{cp}m_p \right) + \beta h \right], \]  

\[ m_p = \tanh \left[ \beta \left( k_{pc}m_c + k_{pp}m_p \right) + \beta h \right], \]  

Figure 1. Phase transition in zero field $h = 0$. (a) The magnetizations $m$, $m_c$ and $m_p$ as functions of the temperature $T$ (in unit of $J/k_B$). (b) The susceptibility $\chi$ as a function of $T$. The network parameters are $N = 10000$, $\langle k \rangle = 20$, $p_c = 0.2$, $r_1 = 0.1$, and $r_2 = 0.005$. Symbols and lines indicate the MC simulation results and theoretical ones, respectively.

Figure 2. (a) The susceptibility $\chi$ as a function of the temperature $T$ for a different network size $N$. (b) The maximal susceptibility $\chi_m$ as functions of $N$ with a double logarithmic coordinate. The other parameters are the same as those in figure 1. The lines in (b) indicate the linear fittings. The inset in figure (b) shows the fourth-order cumulant $U$ as a function of $T$ for different $N$. $U(N)$ intercept each other at the critical temperature $T_{c2}$.

cumulant, defined as $U = 1 - \langle m^4 \rangle / \left[ 3 \langle m^2 \rangle^2 \right]$. $T_{c2}$ is determined as the point where the curves $U \sim T$ for different $N$ intercept each other. From the inset of figure 2(b), we estimate $T_{c2} \approx 55.5$.

To proceed with a theoretical analysis, let us start with the mean-field equations for $m_c$ and $m_p$, given by [7, 45]

\[ m_c = \tanh \left[ \beta \left( k_{cc}m_c + k_{cp}m_p \right) + \beta h \right], \]  

\[ m_p = \tanh \left[ \beta \left( k_{pc}m_c + k_{pp}m_p \right) + \beta h \right], \]  

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where \( k_{cc} = \pi_{cc}(N_c - 1) \) and \( k_{cp} = \pi_{cp}N_p \) are the connectivity numbers of a core node to other core nodes and peripheral nodes, respectively. Likewise, \( k_{pc} = \pi_{pc}N_c \) and \( k_{pp} = \pi_{pp}(N_p - 1) \) are the connectivity numbers of a peripheral node to core nodes and other peripheral nodes, respectively.

For \( h = 0 \), one notices that \( m_c = m_p = 0 \) is always a set of solutions of equation (5). This set of trivial solutions corresponds to the paramagnetic phase. To determine the stability of the trivial solution, we linearize equation (5) around \( m_c = m_p = 0 \), yielding

\[
\tilde{m} = J\tilde{m}.
\]

Here \( \tilde{m} = (m_c, m_p)^\top \) with \( \top \) denoting the transpose, and

\[
J = \beta \begin{pmatrix} k_{cc} & k_{cp} \\ k_{pc} & k_{pp} \end{pmatrix}
\]

is the Jacobian matrix. The nonzero solutions of \( \tilde{m} \) exist when the leading eigenvalue of \( J \) is less than one, yielding the critical temperature,

\[
T_{c2} = \frac{2 (k_{cc} k_{pp} - k_{cp} k_{pc})}{k_{cc} + k_{pp} - \sqrt{(k_{cc} - k_{pp})^2 + 4 k_{cp} k_{pc}}}.
\]

Since the susceptibility is defined as

\[
\chi(T, h) = \left( \frac{\partial m}{\partial h} \right)_T,
\]

we take the partial derivation with respect to \( h \) for equation (5), one has

\[
\chi_c = \beta (1 - m_c^2) (k_{cc} \chi_c + k_{cp} \chi_p + 1),
\]

\[
\chi_p = \beta (1 - m_p^2) (k_{pc} \chi_c + k_{pp} \chi_p + 1).
\]

Solving the above equations, one obtains

\[
\chi_c = \frac{\beta (1 - m_c^2) \left[ \beta (1 - m_p^2) (k_{cp} - k_{pp}) + 1 \right]}{(k_{cc} k_{pp} - k_{cp} k_{pc}) \beta^2 (1 - m_c^2) (1 - m_c^2) - \left[ k_{cc} (1 - m_c^2) + k_{pp} (1 - m_p^2) \right] \beta + 1},
\]

\[
\chi_p = \frac{\beta (1 - m_p^2) \left[ \beta (1 - m_c^2) (k_{pc} - k_{cc}) + 1 \right]}{(k_{cc} k_{pp} - k_{cp} k_{pc}) \beta^2 (1 - m_c^2) (1 - m_c^2) - \left[ k_{cc} (1 - m_c^2) + k_{pp} (1 - m_p^2) \right] \beta + 1}.
\]

At \( T = T_{c2} \), \( m_c = m_p = 0 \) and the denominators on the right hand side of equation (11) equal to zero, such that the susceptibilities \( \chi_c \) and \( \chi_p \) diverge at \( T = T_{c2} \). For \( T \neq T_{c2} \), one can numerically solve equation (5) to obtain \( m_c \) and \( m_p \), as well as \( m = p_c m_c + (1 - p_c) m_p \). Furthermore, substituting \( m_c \) and \( m_p \) into equation (11) one comes to \( \chi = p_c \chi_c + (1 - p_c) \chi_p \). As shown in figure 1(b) by the lines, the theory gives that the first peak of \( \chi \) occurs at \( T_{c1} = 8.2 \) and \( \chi \) diverges at \( T_{c2} = 55.3 \) that agrees well with the MC simulations.

To investigate the effect of core–periphery structure on the phase transition, we show the results for three distinct \( r_2 \), but with a fixed \( r_1 = 0.1 \), as shown in figure 3(a). The larger the value of \( r_2 \) is, the weaker core–periphery structure the network has. One can

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see that for \( r_2 = 0.05 \), the pseudo-double phase transition is still observed. For a larger \( r_2 = 0.1 \), the double-peak phenomenon is not obvious. However, for \( r_2 = 0.15 \) \( \chi \) exhibits only one peak as usual. This implies that a threshold value of \( \tilde{r}_2 \) exists, above which the pseudo-double phase transition phenomenon is destroyed. In figure 3(b), we show the two peaked temperatures, \( T_{c_1} \) and \( T_{c_2} \), as functions of \( r_2 \) with the fixed \( r_1 = 0.1 \). The first peaked-temperature \( T_{c_1} \) increases with \( r_2 \), and terminates at the threshold value of \( r_2 = \tilde{r}_2 \). Our theory predicts \( \tilde{r}_2 = 0.122 \) that is very close to the simulation value of \( \tilde{r}_2 = 0.11 \). The second peaked-temperature \( T_{c_2} \) decreases with \( r_2 \) and asymptotically approaches the average degree \( \langle k \rangle = 20 \) as \( r_2 \to 1 \). In the inset of figure 3(b), we show that as \( r_2 \) increases the threshold value \( \tilde{r}_2 \) is decreased monotonically, and vanishes for \( r_1 > 0.44 \). This implies that for \( r_1 > 0.44 \) there are no double peaks in \( \chi \) no matter what the value of \( r_2 \) is. Furthermore, we show the fraction \( p_c \) of core nodes has an impact on \( \tilde{r}_2 \), as drawn by three different \( p_c \) in the inset of figure 3(b).

As shown in [41] for the percolation model, a true double transition phenomenon is expected to occur if the number of connections among nodes in the core and periphery scale sub-linearly with the system size, i.e. as \( N^{1-\alpha} \) with \( 0 < \alpha < 1 \). In this case, \( k_{cp} \) and \( k_{pc} \) become zero in the thermodynamic limit, \( m_c \) and \( m_p \) are thus decoupled in equation (5) that allows for two distinct transition temperatures in the spirit of mean-field theory, \( T_{c_1} = k_p \) and \( T_{c_2} = k_c \), where \( k_c \) and \( k_p \) are the average degrees of a core node and a periphery node, respectively. In our notation, this is equivalent of making \( \pi_{cp} \sim N^{1-\alpha} \). Meanwhile, we leave the average degree \( \langle k \rangle \) of the network unchanged, and the ratio \( r_d = k_c / k_p \) fixed. Thus, \( k_c \) and \( k_p \) can be obtained by the equality \( \langle k \rangle = k_c p_c + k_p (1 - p_c) \). In figure 4(a), we show \( \chi \) as a function of \( T \) for five distinct \( N \) with \( \alpha = 0.5 \), \( \langle k \rangle = 20 \), \( p_c = 0.2 \), and \( r_d = 10 \). As expected, the susceptibility exhibits two peaks whose maxima, \( \chi_{m_1} \) and \( \chi_{m_2} \), both increase with \( N \) in power-law ways \( \chi_{m_{1,2}} \sim N^{\gamma_{1,2}/\nu} \) (shown in figure 4(b)). Therefore, both \( \chi_{m_1} \) and \( \chi_{m_2} \) diverge in the

Figure 3. (a) The susceptibility \( \chi \) as a function of the temperature \( T \) for a fixed \( r_1 = 0.1 \) and three different \( r_2 \). (b) The peaked temperatures, \( T_{c_1} \) and \( T_{c_2} \), as functions of \( r_2 \) for a fixed \( r_1 = 0.1 \). The other parameters are the same as those in figure 1. Symbols and lines indicate the MC simulation results and theoretical ones, respectively. The inset in (b) shows that the threshold value \( \tilde{r}_2 \) as a function of \( r_1 \) for three different \( p_c \). As \( T \) varies, \( \chi \) has two peaks in the region below the curve \( \tilde{r}_2 \sim r_1 \), and has one single peak above the curve.

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thermodynamic limit such that a true double transition phenomenon occurs in the networked Ising model with a core–periphery structure. On the other hand, the first peak is always located at \( T = 8 \) regardless of the value of \( N \). Such a position is very close to the mean-field prediction \( T_{c1} = k_p \simeq 7.14 \). The position \( T_{c2}(N) \) of the second peak shifts to larger values of \( T \) as \( N \) increases. In the limit of \( N \to \infty \), \( T_{c2}(N) \) approaches an actual critical temperature \( T_{c2} = T_{c2}(N = \infty) \). As shown in figure 4(c), the differences \( \Delta T_{c2}(N) = T_{c2} - T_{c2}(N) \) scale

**Figure 4.** (a) The susceptibility \( \chi \) as a function of the temperature \( T \) in core–periphery networks whose number of connections between core nodes and periphery nodes decays sub-linearly with the network size \( N \), i.e., as \( N^{1-\alpha} \) with \( \alpha = 0.5 \). The other networks parameters: the average degree \( \langle k \rangle = 20 \), the fraction of core nodes \( p_c = 0.2 \), and the ratio of the average degrees of a core node to a periphery node \( r_d = k_c/k_p = 10 \) are fixed. The left and right vertical dotted lines indicate two critical temperatures of mean-field prediction, \( T_{c1} = k_p \simeq 7.14 \) and \( T_{c2} = k_c \simeq 71.4 \), respectively. (b) The two peaked susceptibilities, \( \chi_{m1} \) and \( \chi_{m2} \) as functions of \( N \). The solid lines correspond to the linear fittings in a double logarithmic coordinate, as \( \chi_{m1,2} \sim N^{-\gamma_{1,2}/\nu} \). (c) The differences between \( T_{c2}(\infty) \) and \( T_{c2}(N) \) scale with \( N \) as \( \Delta T_{c2}(N) \sim N^{-1/\nu} \). The solid line shows the linear fitting.

**Figure 5.** The two peaked susceptibilities, \( \chi_{m1} \) (a) and \( \chi_{m2} \) (b), as functions of \( N \) for three values of \( \alpha = 0.2, 0.5, \) and 0.8. The solid lines correspond to the linear fittings in a double logarithmic coordinate, as \( \chi_{m1,2} \sim N^{-\gamma_{1,2}/\nu} \).
with $N$ as $\Delta T_{c_2}(N) \sim N^{-1/\nu}$, with $1/\nu = 0.319$ and $T_{c_2} = 75.4$ (approximately equals to the mean-field prediction $T_{c_2} = k_c \simeq 71.4$, as indicated by the right vertical dotted line in figure 4(a)). Finally, we consider the effect of $\alpha$ on the critical exponents $\gamma'_1/\nu$. In figure 5, we show the power-law fits of $\chi_{m_{1,2}}$ as $N$ for three distinct $\alpha$. One can see that the power-law exponent $\gamma'_1/\nu$ is increased and $\gamma'_2/\nu$ is decreased as $\alpha$ increases.

4. Conclusions

In conclusion, we have studied the phase transition of the Ising model in networks with core–periphery structure. We find that a strong core–periphery structure can lead to the occurrence of an intermediate phase prior to the order-disordered phase transition. At the intermediate phase, the spin configuration in the network is rather inhomogeneous. The core nodes are much more ordered than peripheral nodes. Interestingly, the susceptibility peaks at two distinct temperatures. We find that the susceptibility at the first peaked temperature does show any size-dependent effect if the connections between core and periphery are linear with the network size $N$. Otherwise, if the connections between core and periphery are sub-linear with $N$, the position of the first peaked susceptibility does not shift with $N$ and its height diverges as $N \rightarrow \infty$ in a power-law way. For the two cases, the height of the second peak always increases with $N$ as a power law and diverges in the limit of $N \rightarrow \infty$. The location of the second peak increases with $N$ and asymptotically approaches the critical temperature $T_{c_2}$ of order-disorder phase transition as $N \rightarrow \infty$. Therefore, the occurrence of a double phase transition in the Ising model lies on the sub-linear dependence of the connections between core and periphery on $N$, which is consistent with the conclusion of [41]. Moreover, we develop a mean-field theory for calculating the magnetization and susceptibility. The theory agrees well with the simulations. In the future, it is expected that the phase transition of other statistical physics models in networks with core–periphery structure should be considered.

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