Set-based state estimation and fault
diagnosis of linear discrete-time descriptor
systems using constrained zonotopes

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Abstract: This paper presents new methods for set-valued state estimation and active fault
diagnosis of linear descriptor systems. The algorithms are based on constrained zonotopes, a
generalization of zonotopes capable of describing strongly asymmetric convex sets, while retaining
the computational advantages of zonotopes. Additionally, unlike other set representations
like intervals, zonotopes, ellipsoids, parallelogons, among others, linear static constraints on the
state variables, typical of descriptor systems, can be directly incorporated in the mathematical
description of constrained zonotopes. Therefore, the proposed methods lead to more accurate
results in state estimation in comparison to existing methods based on the previous sets without
requiring rank assumptions on the structure of the descriptor system and with a fair trade-off
between accuracy and efficiency. These advantages are highlighted in two numerical examples.

Keywords: Set-based estimation, fault diagnosis, constrained zonotopes, descriptor systems.

1. INTRODUCTION

Many physical processes such as battery packs, robotic
systems with holonomic and nonholonomic constraints,
and socioeconomic systems (Janschek, 2011; Yang et al.,
2019), exhibit static relations between their internal vari-
able. These processes are known as descriptor systems
(or implicit systems), which have generalized dynamic and
static behaviors described through differential and
algebraic equations, respectively (Wang et al., 2018). De-
scriptor systems appear in several contexts, such as linear
control (Bender and Laub, 1987), fault-tolerant control
(Shi and Patton, 2014), and fault diagnosis (Wang et al.,
2019). However, few strategies can deal effectively with
state estimation and fault diagnosis of descriptor systems
when uncertainties with unknown probability distribution
are present (Hamdi et al., 2012).

Set-based strategies for state estimation of discrete-time
descriptor systems with unknown-but-bounded uncertain-
ties is a recent subject, often addressed using intervals,
zonotopes and ellipsoids (Efimov et al., 2015; Wang et al.,
2018a,b, Merhy et al., 2019). Interval methods are used in Efimov et al. (2015) to design a state estimator for
time-delay descriptor systems based on linear matrix in-
equalities and the Luenberger structure. However, inter-
val arithmetics can lead to conservative enclosures due
to the wrapping effect. A few methods are proposed in
Wang et al. (2018a) for set-valued state estimation of
linear descriptor systems by enclosing the intersection of
two consistent sets with a zonotope bound. Nevertheless,
since the intersection cannot be computed exactly, the
resulting bound can be conservative. These strategies are
extended in Wang et al. (2018b) for linear parameter-
varying descriptor systems, but conservative enclosures are
still present since the intersection method is maintained.
Ellipsoids are used in Merhy et al. (2019) for state estima-
tion of linear descriptor systems based on Luenberger type
observers. Despite being able to provide stable bounds,
the ellipsoidal estimation can be conservative since the
complexity of the set is fixed and static relations are not
directly incorporated. In addition, as in Wang et al. (2018a),
restrictive assumptions on the rank of the system matrices
are required to be able to design the proposed estimator.
Moreover, due to the static constraints, the reachable sets
of models of a descriptor system may be asymmetric even
if the initial set is symmetric, and therefore methods based
on the sets above can provide conservative enclosures.

Fault diagnosis aims to determine exactly which fault a
process is subject to. For the case of descriptor systems,
this problem has been considered in recent works using
zonotopes (Yang et al., 2019; Wang et al., 2019). While
Wang et al. (2019) explores the use of unknown input
observers for robust passive fault diagnosis limited to ad-

* This work has been supported by the Italian Ministry for Research
in the framework of the 2017 PRIN, Grant no. 2017YKXXXX3,
the brazilian project INCT under the grant CNPq 465755/2014-3,
FAPESP 2014/50851-0, and also by the brazilian agencies CAPES
under the grants 001 and 88887.136349/2017-00, and FAPEMIG.
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ditive faults, Yang et al. (2019) proposes a zonotope-based method for active fault diagnosis (AFD) of descriptor systems. The latter is based on the design of an input sequence for separation of the reachable sets. Unfortunately, the generator representation of zonotopes cannot incorporate exactly static relations between the state variables in general linear descriptor systems. Consequently, in practice, this may lead to a more difficult diagnosis.

This manuscript proposes strategies for set-based state estimation and AFD of linear descriptor systems based on constrained zonotopes (CZs), aiming to reduce the conservativeness described above through this asymmetric set representation. In contrast to the aforementioned sets, it is worth noting that linear static constraints, typical of descriptor systems, can be directly incorporated in the mathematical description of CZs. Thanks to this feature, set-valued methods based on CZs can provide less conservative enclosures. The proposed methods extensively explore the basic properties of CZs (Scott et al., 2016). In addition, efficient complexity reduction methods for CZs, available in the literature (Scott et al., 2016; Yang and Scott, 2018), are used in the proposed strategies allowing a direct trade-off between accuracy and computational efficiency. The superiority of the proposed approaches with respect to zonotope-based methods is highlighted in numerical examples.

1.1 Problem formulation and preliminaries

Consider a linear discrete-time descriptor system with time $k$, state $x_k \in \mathbb{R}^n$, input $u_k \in \mathbb{R}^{n_u}$, process uncertainty $w_k \in \mathbb{R}^{n_w}$, measured output $y_k \in \mathbb{R}^{n_y}$, and measurement uncertainty $v_k \in \mathbb{R}^{n_v}$. In each time interval $[k-1, k]$, $k = 1, 2, \ldots$, the system evolves according to the model

$$E x_k = A x_{k-1} + B u_{k-1} + B_w w_{k-1},$$

with $E \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $B_w \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_y \times n}$, $D \in \mathbb{R}^{n_y \times n_u}$, and $D_w \in \mathbb{R}^{n_y \times n_w}$. In the following, it is assumed that $\operatorname{rank}(E) < n$. Note that when $E$ is singular, one has $n - \operatorname{rank}(E)$ purely static constraints. It is assumed that the initial state $x_0 \in X_0$ and $(w_{k}, v_{k}) \in W \times V$ for all $k \geq 0$, where $X_0$, $W$, and $V$ are known polytopic sets. Moreover, the initial condition $(x_0, u_0, w_0, v_0)$ is assumed to be feasible, i.e., consistent with the static relations in (1), and the output $y_0$ is computed as $y_0 = C x_0 + D u_0 + D_w v_0$.

Constrained zonotopes are an extension of zonotopes, proposed in Scott et al. (2016), capable of describing also asymmetric convex polytopes, while maintaining the well-known computational benefits of zonotopes.

**Definition 1.** A set $Z \subseteq \mathbb{R}^n$ is a constrained zonotope if

$$Z = \{z \in \mathbb{R}^n : \|z\|_{\infty} \leq 1, A_z z = b_z \},$$

Equation (2) is referred as constrained generator representation (CG-rep). Each column of $G_z$ is a generator, $c_z$ is the center, and $A_z z = b_z$ are the constraints. Differently from zonotopes, the linear equality constraints in (2) permit a CZ to describe general convex polytopes (Scott et al., 2016). In the following, we use the shorthand notation $Z = \{G_z z, A_z z, b_z\}$ for CZs, $Z = \{G_z c_z\}$ for zonotopes. In addition, $B_{\infty}(A_z, b_z) = \{\xi \in \mathbb{R}^{n_y} : \|\xi\|_{\infty} \leq 1, A_z \xi = b_z\}$ and $B_{\infty}(A_z, b_z) = \{\xi \in \mathbb{R}^{n_y} : \|\xi\|_{\infty} \leq 1\}$ denote, respectively, the $n_y$-dimensional constrained unitary hypercube and the unitary hypercube

1. A few common set operations can be computed using CZs in a trivial manner. Let $Z, W \subseteq \mathbb{R}^n$, $R \subseteq \mathbb{R}^{n \times n}$, and $Y \subseteq \mathbb{R}^n$. Define the linear mapping, Minkowski sum, and generalized intersection as

$$R Z \triangleq (R Z : z \in Z),$$

$$Z \odot W \triangleq (z + w : z \in Z, w \in W),$$

$$Z \cap R Y \triangleq \{z \in Z : R z \in Y\}.$$

If $Z, W, Y$ are in CG-rep, then (3)–(5) are CZs given by

$$R Z = [RG_z, RC_z, A_z, b_z],$$

$$Z \odot W = \left\{(G_z G_w, c_z + c_w, A_z 0, 0 A_w) : [b_z, b_w]\right\},$$

$$Z \cap R Y = \left\{(G_z, 0 c_z, 0 A_z 0 A_w) \in \begin{bmatrix} b_z \ \ b_w \end{bmatrix} \right\}.$$
Consider $A$ and let $T$.

Lemma 1. For the initial set $z_k \in \mathbb{R}^{n_z \times n_z}$, the variables to time estimation the static relation (13b) can be shifted forward $\hat{\mathbf{B}}_w$.\footnote{This constraint can be incorporated in the CG-rep of the initial set $\tilde{Z}_0$ as follows. Let $T = (V^T)^{-1}$. Then, $\tilde{Z}_0 \triangleq \{(G_0, c_0, A_0, b_0)\}$, and $\tilde{Z}_0 = \mathcal{G}_0 \cap \mathcal{A}_0 \cap \mathcal{B}_0$, such that $\tilde{Z}_0$ is a set of minimal length}\footnotetext{An effective enclosure of the prediction step for the descriptor system (13) can be obtained in CG-rep as follows.}

Proof. By assumption (13a), $z_k = (\hat{\mathbf{A}} \mathbf{z}_{k-1} + \mathbf{B}_w \mathbf{w}_{k-1})$, $\hat{\mathbf{B}}_w = \mathcal{G}_0 \cap \mathcal{A}_0 \cap \mathcal{B}_0$, and $\hat{\mathbf{B}}_w = \mathcal{G}_0 \cap \mathcal{A}_0 \cap \mathcal{B}_0$. Then, $\hat{\mathbf{z}}_k = \hat{\mathbf{A}} \mathbf{z}_{k-1} + \mathbf{B}_w \mathbf{w}_{k-1} - \mathbf{B}_w \mathbf{w}_{k-1} = 0$. Consider the set $\tilde{Z}_a = T^{-1} \mathbf{X}_a = \{(T^{-1} \mathbf{G}_a, T^{-1} \mathbf{c}_a, \mathbf{A}_a, \mathbf{b}_a)\}$, and let $T^{-1} \mathbf{c}_a = \mathbf{c}_a^T$, $T^{-1} \mathbf{G}_a = \mathbf{G}_a^T \mathbf{G}_a^T$. An effective enclosure of the prediction step for the descriptor system (13) can be obtained in CG-rep as follows.

Lemma 1. For $k \geq 1$, assume $\mathbf{z}_{k-1} \in \hat{\mathbf{Z}}_{k-1} = \{(\mathbf{G}_{k-1}, \mathbf{c}_{k-1}, \mathbf{A}_{k-1}, \mathbf{b}_{k-1})\}$ and $\mathbf{w}_{k-1}, \mathbf{w}_k \in \mathcal{W} = \{(G_{w}, c_w, A_{w}, b_{w})\}$. Consider (13). If Assumption 1 holds, then $\mathbf{z}_k \in \hat{\mathbf{Z}}_{k} = \{(\mathbf{G}_{k}, \mathbf{c}_{k}, \mathbf{A}_{k}, \mathbf{b}_{k})\}$, with

$$\hat{\mathbf{G}}_k = \begin{bmatrix} \mathbf{A} \hat{\mathbf{G}}_k - \mathbf{B}_w \mathbf{G}_w & 0 & 0 \\ 0 & \mathbf{A}_0 & \mathbf{c}_0 \end{bmatrix}$$

Proof. By assumption (13a), $\mathbf{z}_k = (\hat{\mathbf{A}} \mathbf{z}_{k-1} + \hat{\mathbf{B}}_w \mathbf{w}_{k-1})$, $\hat{\mathbf{B}}_w = \mathcal{G}_0 \cap \mathcal{A}_0 \cap \mathcal{B}_0$, and $\hat{\mathbf{B}}_w = \mathcal{G}_0 \cap \mathcal{A}_0 \cap \mathcal{B}_0$. Then, $\hat{\mathbf{z}}_k = \hat{\mathbf{A}} \mathbf{z}_{k-1} + \hat{\mathbf{B}}_w \mathbf{w}_{k-1} - \hat{\mathbf{B}}_w \mathbf{w}_{k-1} = 0$. Consider the set $\hat{\mathbf{Z}}_{k} = T^{-1} \mathbf{X}_a = \{(T^{-1} \mathbf{G}_a, T^{-1} \mathbf{c}_a, \mathbf{A}_a, \mathbf{b}_a)\}$, and let $T^{-1} \mathbf{c}_a = \mathbf{c}_a^T$, $T^{-1} \mathbf{G}_a = \mathbf{G}_a^T \mathbf{G}_a^T$. An effective enclosure of the prediction step for the descriptor system (13) can be obtained in CG-rep as follows.

3. ACTIVE FAULT DIAGNOSIS

The previous section presented a method to address the problem of the set-based estimation of descriptor systems. In the following, this tool is used in the design of an AFD accounting for a finite number of possible abrupt faults. Consider a linear discrete-time descriptor system whose dynamics obeys one of possible $n_m$ known models

$$E^{[i]} \mathbf{x}_{k+1}^i = A^{[i]} \mathbf{x}_k^i + B^{[i]} \mathbf{u}_{k-1}^i + B^{[i]} \mathbf{w}_{k-1}^i$$

for $k \geq 1$, with $E^{[i]} \in \mathbb{R}^{n \times n}$, $A^{[i]} \in \mathbb{R}^{n \times n}$, $B^{[i]} \in \mathbb{R}^{n \times n}$, $B^{[i]} \in \mathbb{R}^{n \times n}$, $C^{[i]} \in \mathbb{R}^{n \times n}$, $D^{[i]} \in \mathbb{R}^{n \times n}$, and $D^{[i]} \in \mathbb{R}^{n \times n}$, $i \in I = \{1, 2, \ldots, n_m\}$. Moreover, rank($E^{[i]}$) ≤ $n$, and let $x_{k+1}^i = (x_{k+1}^i, w_{k+1}^i) \in W \times V$, and $u_k^i \in U$, with $X_0, W, V$ and $U$ being known polytopic sets. The goal of AFD is to find which model describes the process behaviour. In the following, the dynamics are assumed to not change during the diagnosis procedure, i.e. the AFD is fast enough to avoid the switching between models. In this sense, a sequence $(u_0, u_1, \ldots, u_N)$ of minimal length
$N$ is designed such that any output $y^i_N$ is consistent with only one $i \in I$. If feasible, this problem may admit multiple solutions. For this reason, we introduce a cost function and select among the feasible input sequences the optimal one.

Let $\tilde{u} = (u_0, \ldots , u_N) \in \mathbb{R}^{(N+1)n}$, $\tilde{w} = (w_0, \ldots , w_N) \in \mathbb{R}^{(N+1)n}$, and $\tilde{V} = W \times \cdots \times W$. Consider a variable transformation similar to the one used in the previous section. With a slight abuse of notation, let $z_k = (\tilde{z}_k, \tilde{z}_k)$ and $\tilde{z}_k \in \mathbb{R}^{n_s}$, $\tilde{z}_k \in \mathbb{R}^{n_u-n_s}$, with $\tilde{T}^i$ being obtained from the SVD $\tilde{E}^i = U^i \Sigma^i (V^i)^T$. Then, (9) can be rewritten as

$$z_k^i = \tilde{A}^i \tilde{z}^i_{k-1} + \tilde{B}^i u_{k-1},$$

$$(20a) \quad 0 = \tilde{A}^i \tilde{z}^i_k + \tilde{B}^i u_k,$$  

$$(20b) \quad y^i_k = \tilde{F}^i \tilde{z}^i_k + D^i u_k + D^i v_k,$$  

with $F^i = C^i T^i \mathbf{P}$, where $P = [L_n, 0_{n \times n_u}]$, $\tilde{A}^i = [\tilde{A}^i \tilde{B}^i]$, and $\tilde{A}^i = [\tilde{A}^i \tilde{B}^i]$. Note that the $(\cdot)$ and $(\cdot)$ variables are defined according to (12) for each $i$, and equation (20b) has already been shifted to time $k$.

For each model $i$, consider the CEs $Z^i_k = (T^i)^{-1} X_k \times W = \{G^i_n, c^i_n, A^i_n, b^i_n\}$, where $X_k$ satisfies Assumption 1, the set $\{G^i_n, c^i_n, A^i_n, b^i_n\}$ is $(T^i)^{-1} X_k \times W$, and define the initial feasible set $Z^i_0(u_0) = \{z \in (T^i)^{-1} X_0 \times W : (20b) \text{ holds for } k = 0\}$. This set is given by $Z^i_0(u_0) = \{G^i_n, c^i_n, A^i_n, b^i_n(u_0)\}$, where $G^i_0 = G^i_n, c^i_0 = c^i_n, A^i_0 = [\tilde{A}^i \tilde{B}^i]_0, b^i_0(u_0) = -\tilde{A}^i_{00} - \tilde{B}^i_{00}$. (21)

In addition, define the solution mappings $(\phi^i_k, \psi^i_z) : \mathbb{R}^{(k+1)n_u} \times \mathbb{R}^n \times \mathbb{R}^{(k+1)n_u} \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ such that $(\phi^i_k, \psi^i(z, w, v)) = (T^i)^{-1} X_k \times W \times \mathbf{V}$.

Using (6)–(7), and taking note that by assumption $z^i_k \in Z^i_k \subset \mathbb{R}^{n_u} \times \mathbb{R}^n \times \mathbb{R}^{n_u}$ for every $k \geq 0$, the set $Z^i_k(u)$ is given by the CEs $G^i_n, c^i_n, A^i_n, b^i_n(u)$, and $b^i_0(\tilde{u})$ are obtained by the recursive relations

$$c^i_k(\tilde{u}) = \begin{bmatrix} A^i_{00} & \tilde{B}^i_{00} \end{bmatrix} u_{k-1} + \tilde{B}^i_i u_{k-1},$$

$$(22a) \quad 0 = \tilde{A}^i \tilde{z}^i_k + \tilde{B}^i_i u_k,$$  

$$(22b) \quad y^i_k = \tilde{F}^i \tilde{z}^i_k + D^i_i u_k + D^i_i v_k,$$  

with $F^i = C^i T^i \mathbf{P}$, where $P = [L_n, 0_{n \times n_u}]$, $\tilde{A}^i = [\tilde{A}^i \tilde{B}^i]$, and $\tilde{A}^i = [\tilde{A}^i \tilde{B}^i]$. Note that the $(\cdot)$ and $(\cdot)$ variables are defined according to (12) for each $i$, and equation (20b) has already been shifted to time $k$.

Clearly, if (25) holds for all $i, j \in I, i \neq j$, then $y^i_N(u) \in Y^i_N(u)$ must hold only for one $i$. In the case that this is not valid for any $i \in I$, one concludes that the real dynamics does not belong to the set of models (20). The following theorem is based on the computation of $y^i_N(u)$ expressed by (24) and the results in Raimondo et al. (2016).
Proof. The relations below follow from (22)–(24a), (24c):
\begin{align}
c_N^{Y[i]}(\hat{u}) &= c_N^{Y[i]}(0) + F_N^{i} H_N^{i} \hat{u} + D_N^{i} u_N, \quad (27) \\
b_N^{Y[i]}(\hat{u}) &= b_N^{Y[i]}(0) + [\Omega_N^{i}] T 0^T \hat{u}. \quad (28)
\end{align}
From (8) with \( R = I \), (25) is true iff \( \hat{\xi} \in B_{\infty} \) such that
\[
\begin{bmatrix}
A_N^{Y[i]} & 0 \\
0 & A_N^{Y[i]}
\end{bmatrix}
\begin{bmatrix}
\xi \\
\hat{u}
\end{bmatrix}
= \begin{bmatrix}
b_N^{Y[i]}(\hat{u}) \\
0
\end{bmatrix},
\]
which in turn holds iff (26) is satisfied, with \( G_N^{Y[i]}(\hat{u}) \) and \( A_N^{Y[i]}(\hat{u}) \) defined as in the statement of the theorem. \( \square \)

Let \( n_q \) denote the number of all possible combinations of \( i, j \in I, i \neq j \), and define \( N_Y(i,j) = [N_T(i,j), \Omega_T(i,j)]^T \), \( Y[i,j] = Y[i,j] - b_N^{Y[i]}(i,j) + \Omega(i,j) \hat{u} \), with \( c_N^{Y[i]}(i,j), b_N^{Y[i]}(i,j), N(i,j), \) and \( \Omega(i,j) \) defined as in the statement of the theorem. Then, (25) holds iff \( \hat{\xi} \in B_{\infty} \) such that \( G_N^{Y[i,j]}(\hat{u}) \xi = -c_N^{Y[i,j]}(i,j) + N(i,j) \hat{u} \), and \( A_N^{Y[i,j]}(\hat{u}) \xi = b_N^{Y[i,j]}(i,j) + \Omega(i,j) \hat{u} \). This is equivalent to
\[
\hat{\xi} \in B_{\infty} := \left[ G_N^{Y[i,j]}(\hat{u}) \right] \xi + \left[ -c_N^{Y[i,j]}(i,j) \right] - b_N^{Y[i,j]}(i,j) = N(i,j) \hat{u},
\]

Fig. 1 shows the volumes of the CZs \( \hat{X}_k \) obtained using the estimation method in Section 2 and the zonotope method (top), as well as the projections of \( \hat{X}_k \) onto \( x_3 \) (bottom).

For the AFD of the descriptor system, we consider the design of a separate input of minimum length according to the optimization problem
\[
\min_{\hat{u} \in U} \{ J(\hat{u}) : N_Y(\hat{u}) \hat{u} \notin Y[i], \forall q = 1, 2, \ldots, n_q \}, \quad (29)
\]
with \( J(\hat{u}) \) chosen to minimize any harmful effects caused by injecting \( \hat{u} \) into (20). For simplicity we may choose \( J(\hat{u}) = \sum_{i=0}^{N} u_i^T R u_i \), where \( R \) is a weighting matrix. As in Scott et al. (2014), this is a bilevel optimization problem and can be rewritten as a mixed-integer quadratic program by defining a minimum separation threshold \( \varepsilon > 0 \) such that \( \varepsilon \leq \delta[i](\hat{u}) \), for all \( q = 1, 2, \ldots, n_q \).

4. NUMERICAL EXAMPLES

This section first evaluates the accuracy of the state estimation method proposed in Section 2 for descriptor systems using CZs. Consider system (1) with matrices \( E = \text{diag}(1, 1, 0) \), \( B_\infty = \text{diag}(0.1, 1.5, 0.6) \), \( D_\infty = \text{diag}(0.5, 1.5) \),
\[
A = \begin{bmatrix}
0.5 & 0 & 0 \\
0.8 & 0.95 & 0 \\
-1 & 0.5 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
[0] & [1] \\
[0] & [0] \\
[1] & [1]
\end{bmatrix}, \quad C = \begin{bmatrix}
[0] & [1] \\
[0] & [0] \\
[1] & [1]
\end{bmatrix}, \quad \text{and } D = 0.
\]

The initial state \( x_0 \) is bounded by the zonotope \( X_0 = \{ \text{diag}(0.1, 1.5, 0.6), [0.5 0.5 0.25]^T \} \), and the uncertainties are random uniform noises bounded by \( |w_k|_\infty \leq 1, |v_\infty|_\infty \leq 1 \). The CZ in Assumption 1 is \( \hat{X}_3 = [50 0, 0] \). The simulation is conducted for \( x_0 = [0.5, 0, 0.25]^T \), and the complexity of the CZs is limited to 15 generators and 5 constraints using the constraint elimination algorithm in Scott et al. (2016) and Method 4 in Yang and Scott (2018).

Fig. 1 shows the volumes of the CZs \( \hat{X}_k \) for \( k \in [0, 100] \) obtained using the algorithm (16)–(18). Results obtained using the zonotope method in Wang et al. (2018a) are presented for comparison. Note that the computation of the volumes was possible because this example has few states, otherwise the radii (half the length of the longest edge of the interval hull) can be used instead. The complexity of the zonotopes is limited to 15 generators and 5 constraints using the constraint elimination algorithm in Scott et al. (2016) and Method 4 in Yang and Scott (2018).

Fig. 1 shows also the projections of \( \hat{X}_k \) onto \( x_3 \). As it can be noticed, CZs provide substantially sharper bounds in comparison to zonotopes. This is possible since the enclosure in Lemma 1 takes into account the static constraints explicitly, while zonotopes provide only a conservative bound of the corresponding feasible region. However, this improved accuracy comes with an increase in computational time due to the higher set complexity (see Scott et al. (2016) for a discussion). This experiment was run 500 times consecutively on a laptop with an Intel Core i7-9750H processor, resulting in an average execution time of 3.34 ms for CZs, and of 0.33 ms for zonotopes.

We now evaluate the effectiveness of the AFD method proposed in Section 3. Consider the set of models (19) with model \( i = 1 \) described in the previous example, and

\[ \hat{\xi} \Rightarrow \delta[i](\hat{u}) \]
This paper proposes novel algorithms for set-valued state estimation and AFD of linear descriptor systems with unknown-but-bounded uncertainties. The methods use CZs, a generalization of zonotopes capable of describing strongly asymmetric convex sets. This leads to significantly tighter results than zonotope approaches. In addition, AFD was enabled without assuming rank properties on the structure of the system. The effectiveness of the new methods was corroborated by numerical examples.

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