Dynamics of an Ensemble of Noisy Bistable Elements with Global Time-Delayed Coupling

D. Huber and L. S. Tsimring

Institute for Nonlinear Science, University of California, San Diego, La Jolla, CA 92093-0402

(Dated: March 22, 2022)

The dynamics of an ensemble of bistable elements with global time-delayed coupling under the influence of noise is studied analytically and numerically. Depending on the noise level the system undergoes ordering transitions and demonstrates multi-stability. That is, for a strong enough positive feedback it exhibits a non-zero stationary mean field and a variety of stable oscillatory mean field states are accessible for positive and negative feedback. The regularity of the oscillatory states is maximal for a certain noise level, i.e., the system demonstrates coherence resonance. While away from the transition points the system dynamics is well described by a Gaussian approximation, near the bifurcation points a description in terms of a dichotomous theory is more adequate.

PACS numbers: 05.40.-a, 02.30.Ks, 02.50.Ey, 05.65.+b

Noise induced hopping events in bi- or multi-stable systems form the basis of many interesting phenomena observed in physics, biology, chemistry, as well as social science. The study of such rate processes has thus been a subject of great interest and various techniques such as Langevin, Fokker-Planck and master equations have been used to describe the dynamics of stochastic systems [1, 2, 3].

In this article we generalize a well-studied stochastic model consisting of an ensemble of interconnected noise driven bistable oscillators by introducing time-delayed couplings. Such an extension is important since it has been realized that time-delays are ubiquitous in nature and often change fundamentally the dynamics of the system [4, 5, 6].

The dynamics of the network is studied numerically by using Langevin equations and analytically by two complementary mean field descriptions which are derived from the corresponding Fokker-Planck and master-equations, respectively.

We assume that the bistable elements are highly interconnected, so that the connectivity can be approximated by a global all-to-all coupling. In such a system a bistable element may for instance model the basic properties of a neuron that can either be in a firing or a non-firing state, a person that can opt between two choices by means of individual voting, or a gene that is either expressed or non-expressed. The globally coupled system may then represent a highly interconnected neural network, a social group in which the individual voting behavior is influenced by opinion polls, or a genetic regulatory network, respectively. Other examples are catalytic surface reactions and allosteric enzymic reactions.

The properties of globally coupled elements have been a subject of many studies [2, 5, 12, 13]. In particular, Desai and Zwanzig studied the synchronization of thermally activated bistable elements with instantaneous coupling and found an exact mean field solution in the thermodynamic limit $N \to \infty$, where $N$ is the number of elements in the network. This system exhibits a second order phase transition to an ordered state with non-zero stationary mean field. The effect of a time-delayed coupling has been studied by Yeung and Strogatz for a globally coupled network of periodic phase oscillators and Tsimring and Pikovsky investigated the dynamics of a single bistable element driven by noise and time-delayed feedback.

Here, as in [13], it is assumed that the time delays between the bistable elements are uniform. Such an approximation is justified in certain neural networks, whose time-delays are remarkably constant, as suggested by recent findings. Similarly, for certain regulatory genetic networks the response time lag is determined by a single time constant.

Our prototype system for the study of rate processes in extended systems consists of $N$ Langevin equations, each describing the overdamped noise driven motion of a particle in a bistable potential $V = -x^3/2 + x^4/4$, whose symmetry is distorted by a global coupling to the time-delayed mean field $X(t-\tau) = N^{-1} \sum_{i=1}^{N} x_i(t-\tau)$,

$$\dot{x}_i(t) = x_i(t) - x_i(t)^3 + \varepsilon X(t-\tau) + \sqrt{2D}\xi(t),$$

(1)

where $\tau$ is the time delay, $\varepsilon$ is the coupling strength of the feedback and $D$ denotes the variance of the Gaussian fluctuations $\xi(t)$, which are mutually independent and uncorrelated $\langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij}\delta(t-t')$.

We first study system (1) numerically. The simulations are carried out using a fixed-step fifth-order Runge-Kutta method with linear interpolation for the evaluations at intermediate steps required for the delayed variables. If not otherwise stated, the time-step and the number of elements is $\Delta t = 0.01 - 0.05$ and $N = 2500$, respectively.

For $\varepsilon = 0$, the elements are decoupled from each other. They jump from one potential well to the other randomly and independently of each other. Therefore, in this case the mean field $X = 0$. For small $|\varepsilon|$, the mean field remains zero. At a certain $\varepsilon = \varepsilon_{st} > 0$ which depends
on the noise intensity $D$, but is independent of the time delay, the system undergoes a second order (continuous) phase transition and adopts a non-zero stationary mean field.

For a negative feedback, a transition to a periodically oscillating mean field solution is observed at a certain $\varepsilon = \varepsilon_{osc-} < 0$. Here and for the rest of this paper a ($-/+$) index means that the corresponding value is associated with a negative/positive feedback.

Above a certain noise level $D_H$ the transition at $\varepsilon_{osc-}$ is second order as well. However, for $D < D_H$ the system exhibits a first order (discontinuous) transition associated with hysteretic behavior. The critical noise strength $D_H$ depends on the time delay and is $D_H = 0.07$ for $\tau = 100$.

For large time delays $\tau \gg \tau_K$ ($\tau_K$ is the inverse Kramers escape rate from one well into the other), depending on the initial state the system adopts one of the many accessible oscillatory states featuring different periods. Even for a positive feedback, besides the stationary solution several oscillatory states with periods $T \lesssim \tau$ are observed for $\varepsilon > \varepsilon_{osc+} \gtrsim \varepsilon_{st}$. If the feedback is negative, the system only has oscillatory non-trivial solutions. The observed periods are $T \lesssim 2\tau$ for $\varepsilon < \varepsilon_{osc-}$.

The simulations show that for a negative feedback all oscillating states have a vanishing time-averaged mean field $\langle X \rangle_t$. However, for a positive feedback besides symmetric periodic solutions with $\langle X \rangle_t = 0$, states with significant non-zero temporal mean are possible.

In order to theoretically study the dynamical properties of a globally coupled set of noisy bistable elements (with no time delay), Desai and Zwanzig derived a hierarchy of equations for the cumulant moments of the distribution function from the multi-dimensional Fokker-Planck equation for the joint probability distribution for all elements. For large noise intensities, when the statistics of individual elements are approximately Gaussian, this hierarchy can be truncated. Applying this approach to our system yields the following set of equations for the mean field $X$ and the variance $M = N^{-1} \sum (x_i - X)^2$:

$$\dot{X}(t) = X(t) - X^3(t) - 3X(t)M(t) + \varepsilon X(t - \tau),$$

$$\frac{1}{2} \dot{M}(t) = \dot{M}(t) - 3X^2(t)M(t) - 3M^2(t) + D. \quad (2)$$

To compare the predictions of the Gaussian approximation with the original Langevin model, we calculate the power of the main peak $P_{peak}$ in the power spectrum of $X(t)$. It is proportional to the amplitude of the mean field oscillations and can thus be used to analyze the Hopf bifurcation, which describes the transition to the oscillating mean-field states. The pitchfork bifurcation to the non-zero stationary mean field for a large enough positive coupling is characterized by the dependence of the temporal mean of the mean field $\langle X \rangle_t$ on the system parameters.

For $\tau = 100$ the peak power $P_{peak}$ and the temporal mean $\langle X \rangle_t$ resulting from the Gaussian approximation and the Langevin model are shown in Fig. 1 as a function of the coupling strength $\varepsilon$. The phase diagrams of these models are shown in Fig. 2 in the $(D, \varepsilon)$-parameter plane.

Fig. 1 shows that away from the bifurcation points the Gaussian approximation describes the Langevin dynamics correctly. However, near the transition points the Langevin dynamics is strongly non-Gaussian even for large noise temperatures. For instance, the Gaussian approximation predicts that both bifurcations are subcritical for the entire noise range $D = 0.03 - 1.0$ considered in this study (see Fig. 2), while in the original Langevin model the bifurcations are subcritical only for $D < 0.07$.

Including higher-order cumulant equations only leads to a very slow convergence towards the true solution of...
the Langevin model. Thus, in order to describe the behavior of the system near the bifurcation points, we apply a complementary dichotomous approximation, which is valid in the limit of small noise, where the characteristic Kramers transition time \( \tau_K \gg 1 \). In the dichotomous approximation intra-well fluctuations of \( x \) are neglected. Thus each bistable element can be replaced by a discrete two-state system which can only take the values \( x_{1,2} = \pm 1 \). Then the collective dynamics of the entire system is described by the master equations for the occupation probabilities of these states \( n_{1,2} \). This approach has been successfully used in studies of stochastic resonance and coherence resonance [e.g., 2, 14, 17, 18]. For instance, using this approach Jung et al. [3] found stationary mean field solutions in a globally coupled, time delay free network of bistable elements.

The dynamics of a single element is determined by the hopping rates \( p_{12} \) and \( p_{21} \), i.e., by the probabilities to hop over the potential barrier from \( x_1 \) to \( x_2 \) and from \( x_2 \) to \( x_1 \), respectively. In a globally coupled system, \( n_{1,2} \) and \( p_{12,21} \) are identical for all elements. Then the master equations for the occupation probabilities read

\[
\dot{n}_1 = -p_{12} n_1 + p_{21} n_2, \quad \dot{n}_2 = p_{12} n_1 - p_{21} n_2. \tag{3}
\]

In the dichotomous approximation the mean field \( \dot{X} = x_1 n_1 + x_2 n_2 = n_2 - n_1 \), and making use of the probability conservation \( n_1 + n_2 = 1 \) we obtain the equation for the mean field

\[
\dot{X}(t) = p_{12} - p_{21} - (p_{21} + p_{12}) X(t). \tag{4}
\]

The hopping probabilities \( p_{12,21} \) are given by the Kramers transition rate [10] for the instantaneous potential well, which in the limit of small noise \( D \) and coupling strength \( \varepsilon \) reads [cf. 14],

\[
p_{12,21} = \sqrt{\frac{1}{2} \mp 3\varepsilon X(t-\tau)} \exp \left( \frac{-1 \mp 4\varepsilon X(t-\tau)}{4D} \right). \tag{5}
\]

A linear stability analysis of Eq. (4) near the trivial state \( X = 0 \) yields the transcendent equation for the complex eigenvalue \( \lambda \),

\[
\lambda = \frac{\sqrt{7}}{\pi} e^{-1/4D} \left( \frac{(4-3D)}{4D} e^{-\lambda \tau} - 1 \right). \tag{6}
\]

For a positive coupling this equation always has a real solution. At a certain critical coupling \( \varepsilon_{\text{crit}} = 4D/(4-3D) \) the eigenvalue becomes positive indicating the pitchfork bifurcation observed in the Langevin system [14]. Besides this real solution, Eq. (6) possesses an infinite number of complex solutions corresponding to oscillating mean fields. However, only a finite number of them corresponds to unstable modes at finite \( \tau \) and \( \varepsilon \) as we will see below. The critical values \( \varepsilon \) of the corresponding instabilities are found by substituting \( \lambda = \mu + i\omega \) into Eq. (6), separating real and imaginary part and setting \( \mu = 0 \):

\[
\omega \tau = \frac{\sqrt{7}}{\pi} \exp(-1/4D) \tau \tan \omega \tau \tag{7}
\]

\[
\varepsilon_{\text{osc}} = \frac{\varepsilon_{\text{st}}}{\cos \omega \tau}. \tag{8}
\]

This set of equations has a multiplicity of solutions, indicating that multi-stability occurs in the globally coupled system beyond a certain coupling strength. For finite time delays and positive coupling, besides the stationary solution, several oscillatory states with periods \( T_k \) close to but slightly larger than \( \tau / k \) are observed for \( \varepsilon > \varepsilon_{\text{osc}+} \) \( \{k = 1, 2, \ldots\} \), where the transition points are ordered as follows, \( 0 < \varepsilon_{\text{st}} < \varepsilon_{\text{osc}+} < \varepsilon_{\text{osc}+}^2 \ldots \) If the feedback is negative, the system has oscillatory solutions with periods \( T_l \) close to but slightly larger than \( 2\tau / (2l+1) \) for \( \varepsilon < \varepsilon_{\text{osc}−} \) \( \{l = 0, 1, \ldots\} \), where \( 0 > \varepsilon_{\text{osc}−}^0 > \varepsilon_{\text{osc}−}^1 \ldots \)

Let us now discuss the bifurcation properties in the limit of large and small time delays as well as vanishing noise and compare them with those of a single oscillator system. The critical coupling \( \varepsilon_{\text{st}} \) of the pitchfork bifurcation is time delay independent and goes to zero for vanishing noise. However, the critical coupling of the Hopf bifurcation behaves like time delay dependent and goes to zero for vanishing noise. This should be contrasted to the dynamics of a single noise-free oscillator with time-delayed feedback that only exhibits oscillations at strong negative feedback (\( \varepsilon < -1 \)). For very small time delays \( \tau \to 0 \), the critical coupling strength \( \varepsilon_{\text{osc}−}^0 \to +\infty \).

In order to compare the predictions of the dichotomous model with the Langevin dynamics, the peak power \( P_{\text{peak}} \) and the temporal mean \( \langle X \rangle t \), resulting from the dichotomous theory, are also plotted in Fig. 1. The phase diagram for the dichotomous theory is shown in Fig. 2.

Fig. 1 and 2 show that the dichotomous theory agrees with the Langevin dynamics quite well for small noise in the range \( D \approx 0.07 \) in the neighborhood of the bifurcation points. The theory also correctly describes the bifurcation type. Indeed, the dichotomous theory predicts accurately the noise strength \( D_{\text{hf}} (=0.07) \) at which the Hopf bifurcation changes from supercritical to subcritical. However, for very small \( D \) the Kramers time becomes very large, and the accuracy of numerics becomes insufficient for a comparison with the theory.

Let us point out that the system studied in this paper exhibits the phenomena of coherence resonance [e.g., 20] and array-enhanced resonance. Since both Kramers random switching rate \( p \) (see Eq. 5) and the frequency of the oscillatory states \( f = \omega / (2\tau) \) (see Eq. 6) depend on the noise strength, i.e., \( p = p(D) \) and \( f = f(D) \), the noise
can be tuned so that the random hopping between the potential wells of the bistable oscillators synchronizes with the periodic modulation of the mean field. This statistical synchronization takes place when $f = p/2$ [18], where the regularity of the oscillatory state becomes maximal. Here, this regularity is quantified by $\beta = H f_p / \Delta f$, where $H$ is the height of the spectral peak at $f_p$ and $\Delta f$ is the half width of the peak. The coherence measure $\beta$ as a function of the noise strength is shown in Fig. 3(a). We observe that the regularity of the oscillatory states increases with increasing $N$, a property which was reported for other systems and is sometimes referred to as array-enhanced resonance [21]. Interestingly, the enhancement of the temporal regularity with increasing system size is only observed for macroscopic mean field oscillations, while the inverse holds for “subcritical coherence”. That is, the coherence observed in the power spectra of subcritical mean field fluctuations (i.e., for $|\varepsilon| < |\varepsilon_{osc}|$) decays inversely proportional to the number of elements in the network, and becomes negligible for $N > 10$. This is shown in Fig. 3(b). Qualitatively, the same dependency on the system size is found if the delayed average does not include the delayed element itself, i.e., the element $x_i$ is coupled to $X_i(t - \tau) = \sum_{j=1,j\neq i}^{N-1} x_j$.

In summary, we have shown that a network of noisy bistable elements with global time-delayed coupling possesses a multiplicity of stable oscillatory states for both positive and negative feedback in addition to a non-zero stationary mean field for a strong enough positive feedback which also occurs in a non-delayed system. These novel oscillatory states have a maximum regularity for a certain noise strength. The bifurcations of the trivial equilibrium are well described by the dichotomous theory in the limit of small noise and coupling strength. Far away from the bifurcation points the mean-field properties of the system are well described by the Gaussian approximation. However, the quantitative theory for the large noise strength near the bifurcation points is still lacking. In this paper the effect of uniform time delays on the dynamics of a globally coupled network of bistable elements has been studied. However, many real networks have sparse coupling and non-uniform time delays. These properties should thus be included in a more general description. Preliminary results of our simulations with nonuniform time-delays suggest, that the bifurcation properties do qualitatively not change for a wide range of Gaussian distributed time delays, which substantiates the generic nature of the here considered model. This issue will be discussed in detail in an upcoming article.

We are grateful to A. Pikovsky for many useful discussions. This work was supported by the Swiss National Science Foundation (D.H.) and by the U.S. Department of Energy, Office of Basic Energy Sciences under grants DE-FG03-95ER14516 and DE-FG-03-96ER14592 (L.T.).

[1] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. 62, 251 (1990).
[2] R. C. Desai and R. Zwanzig, J. Stat. Phys. 19, 1 (1978).
[3] P. Jung, U. Behn, E. Pantazelou, and F. Moss, Phys. Rev. A 46, R1709 (1992).
[4] P. C. Bressloff and S. Coombes, Phys. Rev. Lett. 80, 4815 (1998).
[5] Y. Nakamura, F. Tominaga, and T. Munakata, Phys. Rev. E 49, 4849 (1994).
[6] M. Y. Choi and B. A. Huberman, Phys. Rev. B 31, 2862 (1985).
[7] H. Sompolinsky, D. Golomb, and D. Kleinfeld, Phys. Rev. A 43, 6990 (1991).
[8] D. H. Zanette, Phys. Rev. E 55, 5315 (1997).
[9] T. Gardner, C. Cantor, and J. Collins, Nature 403, 339 (2000).
[10] K. C. Rose, D. Battogtokh, A. Mikhailov, R. Inhibil, W. Engel, and A. M. Bradshaw, Phys. Rev. Lett. 76, 3582 (1996).
[11] B. Hess and A. Mikhailov, Biophys. Chem. 58, 365 (1996).
[12] Y. Kuramoto, Chemical Oscillations, Waves and Turbulence (Springer, Berlin, 1991).
[13] M. K. S. Yeung and S. H. Strogatz, Phys. Rev. Lett. 82, 648 (1999).
[14] L. S. Tsimring and A. Pikovsky, Phys. Rev. Lett. 87, 250602 (2001).
[15] M. Salami, C. Itami, T. Tsumoto, and F. Kimura, PNAS 100, 6174 (2003).
[16] J. Paulsson and M. Ehrenberg, Q. Rev. Biophys. 34, 1 (2001).
[17] B. McNamara and K. Wiesenfeld, Phys. Rev. A 39, 4854 (1989).
[18] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. 70, 223 (1998).
[19] H. Kramers, Physica (Utrecht) 7, 284 (1940).
[20] A. S. Pikovsky and J. Kurths, Phys. Rev. Lett. 78, 775 (1997).
[21] C. Zhou, J. Kurths, and B. Hu, Phys. Rev. Lett. 87, 98101 (2001).