Master and Langevin equations for electromagnetic dissipation and decoherence of density matrices

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Abstract. We set up a forward – backward path integral for a point particle in a bath of photons to derive a master equation for the density matrix which describes electromagnetic dissipation and decoherence. We also derive the associated Langevin equation. As an application we recalculate the Wigner-Weisskopf formula for the natural line width of an atomic state at zero temperature and find, in addition, the temperature broadening caused by the decoherence term. Our master equation also yields the correct Lamb shift of atomic levels. The two equations may have applications to dilute interstellar gases or to few-particle systems in cavities.

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1 Introduction

The time evolution of a quantum-mechanical density matrix $\rho(x_+;t_a;\ldots)$ of a particle coupled to an external electromagnetic vector potential $A(x,t)$ is determined by a forward – backward path integral [1]

$$(x_+,t_0|x_+,t_0) = U(x_+,x_-,t_0|x_+,x_-,t_0) = \int Dx_+ Dx_- \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} \left[ M \left( \dot{x}^2 - \dot{x}_+^2 \right) - V(x_+) + V(x_-) - \frac{e}{c} \dot{x}_+ A(x_+,t) + \frac{e}{c} \dot{x}_- A(x_-,t) \right] \right\},$$

(1.1)

where $x_+(t)$ and $x_-(t)$ are two fluctuating paths connecting the initial and final points $x_+$ and $x_-$, respectively. In terms of this expression, the density matrix $\rho(x_+,x_-;t_0)$ at a time $t_0$ is found from that at an earlier time $t_a$ by the integral

$$\rho(x_+,x_-;t_0) = \int dx_+ dx_- U(x_+,x_-,t_0|x_+,x_-,t_a)\rho(x_+,x_-;t_a).$$

(1.2)

The vector potential $A(x,t)$ appearing in the electromagnetic action $A_{\text{em}} = \int d^4x (E^2 - B^2)/2c$ in the radiation gauge via $E = A/c$, $B = \nabla \times A$, is a superposition of oscillators $X_k(t)$ of frequency $\Omega_k = c|k|$ in a volume $V$:

$$A(x,t) = \sum_k f_k(x)X_k(t), \quad f_k(x) = \frac{e^{ikx}}{\sqrt{2V\Omega_k/c}}.$$

(1.3)

These oscillators are assumed to be in equilibrium at a finite temperature $T$, where we shall write their time-ordered correlation functions as $G^{ij}_{kk}(t,t') = \langle \langle T \hat{X}_k^i(t), \hat{X}_k^j(t') \rangle \rangle = \delta^{ij}_{kk} G_{kk}(t,t') \equiv \delta_{kk}(\delta^{ij} - k^i k^j/k^2)G_{kk}(t,t')$, the transverse Kronecker symbol resulting from the sum $\sum_{\pm} e^i(k,h)\epsilon^j*(k,h)$ over the two polarization vectors of the vector potential $A(x,t)$. For a single oscillator of frequency $\Omega$, one has for $t > t'$:

$$G_{\Omega}(t,t') = \frac{1}{2} [A_{\Omega}(t,t') + C_{\Omega}(t,t')]$$

$$= \frac{\hbar}{2M\Omega} \frac{\cos \Omega \frac{t'}{2}}{\sin h(t'/2)}, \quad t > t',$$

(1.4)

which is the analytic continuation of the periodic imaginary-time Green function to $\tau = i$. The decomposition into $A_{\Omega}(t,t')$ and $C_{\Omega}(t,t')$ distinguishes real and imaginary parts, which are commutator and
anticommutator functions of the oscillator at temperature $T$: $C_G(t, t') \equiv \langle [\dot{\mathcal{X}}(t), \ddot{\mathcal{X}}(t')] \rangle_T$ and $A_{ij}(t, t') \equiv \langle [\dot{\mathcal{X}}_i(t), \dot{\mathcal{X}}_j(t')] \rangle_T$, respectively. The thermal average of the evolution kernel (1.1) is then given by the forward – backward path integral

\begin{equation}
U(x_{+b}, x_{-b}, t_b|x_{+a}, x_{-a}, t_a) = \int Dx_{+}(t) \int Dx_{-}(t) \times \exp \left\{ i \frac{\hbar}{\hbar} A_{FV}[x_+, x_-] \right\},
\end{equation}

where $\exp\{i A_{FV}[x_+, x_-]/\hbar\}$ is the Feynman-Vernon influence functional. The influence action $A_{FV}[x_+, x_-]$ is the sum of a dissipative and a fluctuating part $A_{D\text{FV}}[x_+, x_-]$ and $A_{FV}[x_+, x_-]$, respectively, whose explicit forms are

\begin{align}
A_{D\text{FV}}[x_+, x_-] &= \frac{i e^2}{2\hbar c^2} \int dt \Theta(t - t') \left( \mathcal{X}_{+b} - \mathcal{X}_{-b} \right) \\
&\quad \times \left[ \mathcal{X}_{+b} \mathcal{X}_{b}(x_+, t, x'_+, t') \mathcal{X}'_{-b} - \mathcal{X}_{-b} \mathcal{X}_{b}(x_+, t, x'_+, t') \mathcal{X}'_{-b} \right],
\end{align}

and

\begin{align}
A_{FV}[x_+, x_-] &= \frac{i e^2}{2\hbar c^2} \int dt \Theta(t - t') \left( \mathcal{X}_{+b} - \mathcal{X}_{-b} \right) \\
&\quad \times \left[ \mathcal{X}_{+b} \mathcal{X}_{b}(x_+, t, x'_+, t') \mathcal{X}'_{-b} - \mathcal{X}_{-b} \mathcal{X}_{b}(x_+, t, x'_+, t') \mathcal{X}'_{-b} \right],
\end{align}

where $x_+, x'_+$ are short for $x_{+b}(t)$, $x_{+b}(t')$, and $C_{b}(x_+, t, x'_+, t')$, $A_{b}(x_+, t, x'_+, t')$ are $3 \times 3$ commutator and anticommutator functions of the bath of photons. They are sums of correlation functions over the bath of the oscillators of frequency $\delta k$, each contributing with a weight $f_k(x)f_{-k}(x) = e^{ik(x-x')}c/2\delta k\nu$ (the normalization following from the action $\int d^4x(E^2 - B^2)/2c$). Thus we may write

\begin{align}
C_{ij}^{ij}(x, x', x') &= \sum_k f_{-k}(x)f_{k}(x') \left\langle [\dot{X}_{ij}^i(t), \dot{X}_{ij}^j(t')] \right\rangle_T \\
&= -i e^2 h \int \frac{d\omega'd^2k}{(2\pi)^4} \sigma_k(\omega') \delta_{ij} e^{ik(x-x')} \sin \omega'(t - t'),
\end{align}

(1.8)

\begin{align}
A_{ij}^{ij}(x, x', x') &= \sum_k f_{-k}(x)f_{k}(x') \left\langle \left\{ \dot{X}_{ij}^i(t), \dot{X}_{ij}^j(t') \right\} \right\rangle_T \\
&= -e^2 \int \frac{d\omega'd^2k}{(2\pi)^4} \sigma_k(\omega') \delta_{ij} \coth \left( \frac{\hbar\omega'}{2k_B T} \right) e^{ik(x-x')} \cos \omega'(t - t'),
\end{align}

(1.9)

where $\sigma_k(\omega')$ is the spectral density contributed by the oscillator of momentum $k$:

\begin{equation}
\sigma_k(\omega') \equiv \frac{2\pi e^2}{2\Omega_k} \left[ \delta(\omega' - \Omega_k) - \delta(\omega' + \Omega_k) \right].
\end{equation}

(1.10)

At zero temperature, we recognize in (1.8) and (1.9) twice the imaginary and real parts of the Feynman propagator of a massless particle for $t > t'$, which in four-vector notation with $k = (\omega/c, \mathbf{k})$ and $x = (ct, \mathbf{x})$ reads

\begin{align}
G(x, x') &= \frac{1}{2} [A(x, x') + C(x, x')] \\
&= \int \frac{d\omega d^3k}{(2\pi)^4} \frac{i e^2 h}{\omega - \Omega_k + i\eta} e^{-i\omega(t-t') - k(x-x')} \\
&= \int \frac{d^4k}{(2\pi)^4} \epsilon^{ik(x-x')} \frac{i h}{k^2 + i\eta}.
\end{align}

(1.11)

where $\eta$ is an infinitesimally small number $> 0$.

We shall now focus attention upon systems which are so small that the effects of retardation can be neglected. Then we can ignore the $\omega$-dependence in (1.8) and (1.9) and find

\begin{equation}
C_{ij}^{ij}(x, x', t') \approx C_{ij}^{ij}(x, t') = \frac{i}{2\pi c^3} \frac{\delta^{ij}}{\delta t} \delta(t - t').
\end{equation}

(1.12)

Inserting this into (1.6) and integrating by parts, we obtain two contributions. The first is a diverging term

\begin{equation}
\Delta A_{\text{loc}}[x_+, x_-] = \frac{\Delta M}{2} \int t_b dt (\delta_{ij} \dot{x}_j^i(x_+ - x_-^i)),
\end{equation}

(1.13)

where

\begin{equation}
\Delta M \equiv \frac{e^2}{c^2} \int \frac{d\omega d^3k}{(2\pi)^4} \sigma_k(\omega') \delta_{ij} \epsilon^{ik(x-x')} \dot{x}_j \left( 1 \right.
\end{equation}

(1.14)

diverges linearly. This simply renormalizes the kinetic terms in the path integral (1.5), renormalizing them to

\begin{equation}
\frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{M_{\text{ren}}}{2} (\delta_{ij} \dot{x}_j^i).
\end{equation}

(1.15)

By identifying $M$ with $M_{\text{ren}}$, this renormalization may be ignored.

The second term has the form

\begin{equation}
A_{D\text{FV}}[x_+, x_-] = -\frac{\Delta M}{2} \int t_b dt (\delta_{ij} \dot{x}_j^i (x_+ - x_-^i) R(t),
\end{equation}

(1.16)

with the friction constant

\begin{equation}
\gamma \equiv \frac{e^2}{6\pi c^2 M} = \frac{2\alpha}{3\omega M},
\end{equation}

(1.17)

where $\alpha \equiv e^2/hc \approx 1/137$ is the fine-structure constant and $\omega_M \equiv M e^2/hc$ the Compton frequency associated with the mass $M$. In contrast to the ordinary friction constant, this has the dimension 1/frequency.

Note that the retardation enforced by the Heaviside function in the exponent of (1.6) removes the left-hand half of the $\delta$-function. It expresses the causality of the dissipation forces, which is crucial for producing a probability conserving time evolution of the probability distribution [2]. The superscript $R$ in (1.16) accounts for this by