New texture-zero models for lepton mixing

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Abstract
I systematically consider, in the context of the type-I seesaw mechanism, all the predictive cases in which both the Dirac mass matrix connecting the left-handed neutrinos to the right-handed neutrinos, and the Majorana mass matrix of the latter neutrinos, feature texture zeros, while the mass matrix of the charged leptons is diagonal. I have found a few cases which had not been discussed in the literature previously.

1 Introduction
There has recently been renewed interest in the possibility of ‘texture’ zeros in the fermion mass matrices [1, 2, 3]. Texture zeros are well grounded in renormalizable field theories, since they can always be implemented through models with suitable Abelian symmetries and (possibly many) scalar fields with vacuum expectation values [4]. The revival of interest was partially motivated by the realization of the fact that a popular alternative approach, where lepton mixing is completely determined by a non-Abelian symmetry, seems to have been fully explored [5].

With three Majorana neutrinos, the mass Lagrangian is

\[ \mathcal{L}_{\text{mass}} = -\bar{\ell}_L M_{\ell} \ell_R - \bar{\ell}_R M^\dagger_{\ell} \ell_L + \frac{1}{2} \left( \nu^T C^{-1} M \nu - \bar{\nu} M^* C \bar{\nu}^T \right), \]  

(1)

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where $C$ is the charge-conjugation matrix in Dirac space. The column-vector $\nu$ contains the three left-handed light-neutrino fields. The neutrino Majorana mass matrix $M$ acts in flavour space and is symmetric.

In ref. [6], $M_\ell$ was assumed to be diagonal while $M$ had two zero matrix elements. This was later generalized to the situation wherein $M_\ell$ is diagonal and $M^{-1}$ has two zero matrix elements [7]; mixed situations in which $M$ and $M^{-1}$ have one zero matrix element each, while $M_\ell$ remains diagonal, were studied in ref. [8]. Recently, all the cases in which both $M_\ell$ and $M$ sport texture zeros were mapped [1, 2].

It is known that the three light neutrinos are exceedingly light; one of them may actually be massless. Possibly the most popular theory for explaining that extreme lightness is the (type I) seesaw mechanism. In that theory, $M$ is not really a fundamental mass matrix, rather

$$M = -M_D M_R^{-1} M_D^T$$

is just the effective (approximate) mass matrix for the light neutrinos arising out of the Dirac mass matrix $M_D$ connecting the standard neutrinos to some gauge-singlet (“right-handed”) neutrino fields and of the Majorana mass matrix $M_R$ of the latter. In this context, assuming the presence of texture zeros in $M$ seems unwarranted; one should rather consider texture zeros in $M_D$ and $M_R$. Indeed, that was the rationale for ref. [7], where $M_D$ was assumed to be diagonal (which means that it has six texture zeros) and two texture zeros were enforced in $M_R$.

In this paper I want to map all the cases in which there are texture zeros in $M_D$ and $M_R$ while $M_\ell$ remains diagonal (which in itself means that $M_\ell$ has six texture zeros). I will look for predictive cases, *i.e.* for cases which lead to non-trivial fits for the lepton mixing (PMNS) matrix and/or for the neutrino mass ratios. The case in which $M_D$ has six texture zeros and $M_R$ has two texture zeros was considered in ref. [7]; here I consider additional cases in which $M_D$ has either five or four texture zeros and, correspondingly, $M_R$ has either three or four texture zeros, respectively. In my search I have recovered the cases studied in refs. [8] and [9]; additionally, I have uncovered some extra cases which had not been, to my knowledge, studied before.

Since in my search $M_\ell$ is kept diagonal, the light neutrinos in the column vector $\nu = (\nu_e, \nu_\mu, \nu_\tau)^T$ may be labelled through their flavour. Then, $M = [M_{\alpha\beta}]$, where $\alpha$ and $\beta$ may be either $e$, $\mu$, or $\tau$. In some of the new cases

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1The matrix $M$ is $3 \times 3$ symmetric and therefore it has, in general, six independent matrix elements. We say that “$M$ has $n$ zero matrix elements” if $n$ out of those six independent matrix elements vanish. The actual total number of zero entries in $M$ will be larger than $n$ if some of the vanishing entries are off-diagonal.

2We shall assume in this paper that the number of right-handed neutrinos is three.
that I present in this paper the constraints on $M$ may most conveniently be written in terms of the matrix $A = [A_{\alpha\beta}]$ defined by

$$A_{\alpha\beta} \equiv M_{\alpha\beta} (M^{-1})_{\beta\alpha}.$$  

(I do not use the summation convention in this paper.) The matrix $A$ was first used in the context of lepton mixing in refs. [10, 11]. It has the properties that the sum of its matrix elements over any of its rows or columns is equal to one and that it is invariant either under a rephasing of $M$,

$$M_{\alpha\beta} \rightarrow e^{i(\xi_{\alpha}+\chi_{\beta})}M_{\alpha\beta},$$

or under the multiplication of $M$ by any number. In our case, since $M$ is symmetric, $A$ is symmetric too. I have found that some texture-zero models predict one diagonal matrix element of $A$ to be one and another diagonal matrix element of $A$ to be zero; thus, but for a permutation of its rows and columns,

$$A = \begin{pmatrix} 0 & t & 1-t \\ t & 1 & -t \\ 1-t & -t & 2t \end{pmatrix},$$

where $t$ is some complex number. In this case, $A$ has only two degrees of freedom (the real and the imaginary parts of $t$), instead of the six degrees of freedom of the general case.

## 2 Possibilities for $M_D$

I shall consider the possibility of a permutation of the rows and columns of $M$ only at the end. Such a permutation corresponds to a transformation

$$M \rightarrow ZMZ^T,$$

where

$$Z \in S_3 \equiv \{ A^2, A, B, ABA, AB, BA \},$$

where

$$A \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$
The transformation (6) is equivalent to

\[ M_D \rightarrow ZM_D, \]  

which is a permutation of the rows of \( M_D \). Since I am going to consider the possibility of a transformation (6) at the end, I do not need to consider the possibility of a transformation (9) now at the beginning. So, I shall take two matrices \( M_D \) which only differ through a permutation of their rows to be equivalent.

Under this proviso, a matrix \( M_D \) with four texture zeros must be of one of the following forms:

\[
\begin{align*}
(0 & 0 0), \quad (0 & 0 0), \quad (0 & 0 0), \\
(0 & x x), \quad (x & 0 x), \quad (x & x x), \\
(x & x x), \quad (x & x x), \quad (x & x x)
\end{align*}
\]

\[ (10) \]

\[
\begin{align*}
(0 & 0 x), \quad (0 & 0 x), \quad (x & 0 0), \\
(x & x x), \quad (x & x x), \quad (x & x x)
\end{align*}
\]

\[ (11) \]

\[
\begin{align*}
(0 & x 0), \quad (0 & 0 x), \quad (0 & x 0), \\
(0 & x x), \quad (x & 0 x), \quad (0 & x x)
\end{align*}
\]

\[ (12) \]

\[
\begin{align*}
(0 & x 0), \quad (0 & x 0), \quad (0 & x 0), \\
(x & x 0), \quad (x & x 0), \quad (x & x 0)
\end{align*}
\]

\[ (13) \]

\[
\begin{align*}
(0 & 0 x), \quad (x & 0 0), \quad (x & 0 0), \\
(x & x 0), \quad (x & x 0), \quad (x & x 0)
\end{align*}
\]

\[ (14) \]

\[
\begin{align*}
(0 & x x), \quad (0 & x x), \quad (x & 0 0), \\
(0 & x x), \quad (x & 0 x), \quad (x & x x)
\end{align*}
\]

\[ (15) \]

\[
\begin{align*}
(0 & 0 x), \quad (0 & x x), \quad (0 & 0 x), \\
(0 & x x), \quad (0 & x x), \quad (x & 0 0)
\end{align*}
\]

\[ (16) \]
In matrices (10–16) the symbol $\times$ represents a non-zero entry.

If we consider the possibility of transformations (9), then there are three matrices of each of the types (11), (13), and (14), and six matrices of each of the types (10), (12), (15), and (16). So, altogether there are $3 \times 12 + 6 \times 15 = 126$ possible matrices $M_D$ with four texture zeros; this is as it should be, since $M_D$ has nine independent matrix elements and $(9 \times 8 \times 7 \times 6) / 4! = 126$.

Matrices $M_D$ with a row of zeros are uninteresting since they yield one massless, decoupled neutrino. Therefore, the matrices (10) may be neglected. The matrices (11) may also be neglected because they lead to ‘scaling’, i.e. the matrix $M$ has a right-eigenvector with one zero entry corresponding to the eigenvalue zero [12]; this implies that the PMNS matrix has one zero matrix element, which contradicts experiment.

In order to write down all the possible forms of matrices $M_D$ with five textures zeros, one simply has to interchange the zeros with the $\times$ symbols in the matrices (10–16). One obtains

$$
\begin{pmatrix}
\times & \times & \times \\
\times & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & \times \\
0 & \times & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & \times \\
0 & 0 & \times \\
0 & 0 & 0
\end{pmatrix},
$$ (17)

$$
\begin{pmatrix}
\times & \times & 0 \\
\times & \times & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & 0 \\
\times & 0 & \times \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & 0 \\
0 & \times & \times \\
0 & 0 & 0
\end{pmatrix},
$$ (18)

$$
\begin{pmatrix}
\times & \times & 0 \\
\times & 0 & \times \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & 0 \\
\times & 0 & \times \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & \times \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
$$ (19)

$$
\begin{pmatrix}
\times & \times & 0 \\
\times & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & 0 \\
0 & \times & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & \times \\
0 & \times & 0 \\
0 & 0 & 0
\end{pmatrix},
$$ (20)

$$
\begin{pmatrix}
\times & \times & 0 \\
0 & \times & \times \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & 0 \\
0 & \times & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & \times \\
0 & \times & 0 \\
0 & 0 & 0
\end{pmatrix},
$$ (21)

$$
\begin{pmatrix}
\times & \times & 0 \\
\times & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & 0 \\
0 & \times & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\times & \times & \times \\
0 & \times & 0 \\
0 & 0 & 0
\end{pmatrix},
$$ (22)
Matrices $M_D$ with a row of zeros are uninteresting. Therefore, forms (17), (18), and (19) may be neglected. Forms (20) and (21) may also be neglected because they lead to scaling. Therefore, only the nine forms (22) and (23) should be considered.

3 Possibilities for $M_R$

The matrix $M$ is invariant under

$$M_D \rightarrow M_D Z, \quad M_R \rightarrow Z^T M_R Z$$

(24)

because $Z^T = Z^{-1}$, $\forall Z \in S_3$. Therefore, a permutation of the rows and columns of $M_R$ is equivalent to a permutation of the columns of $M_D$. Since in the preceding section I have not chosen any particular order for the columns of $M_D$, I am free in this section to restrict the order of the rows and columns of $M_R$. Moreover, if one particular form of $M_R$ is invariant under some $S_2$ subgroup of $S_3$, then one may disconsider the action of that $S_2$ on the columns of $M_D$.

If one wants to obtain a model as predictive as those in the literature\(^5\) and if $M_D$ has four texture zeros, then $M_R$ should also have four texture zeros. Since $\det M_R$ must be nonzero\(^6\), there is only one possible form for $M_R$,

$$\begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix},$$

(25)

but for permutations of the rows and columns—which are equivalent to permutations of the columns of $M_D$. Equation (25) leads to

$$M_R^{-1} = \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & y \\ 0 & y & 0 \end{pmatrix}.$$ 

(26)

\(^5\)The models in ref. [7] have six zeros in $M_D$ and two zeros in $M_R$. Models as predictive should have either five zeros in $M_D$ and three zeros in $M_R$ or four zeros in $M_D$ and four zeros in $M_R$.

\(^6\)If $M_R$ is a singular matrix then one right-handed neutrino is massless and the see-saw mechanism is not fully operative; see ref. [13].
The form (25) of $M_R$ is invariant under the interchange of the second and third rows and columns. Therefore, when $M_R$ is of that form, one may disconsider the possibility of a permutation of the second and third columns of $M_D$. Thus, out of the 21 forms (12–16) of $M_D$ only the following 13 must be considered:

$$M_D = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & d & e \end{pmatrix}, \quad (27a)$$

$$M_D = \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ c & d & e \end{pmatrix}; \quad (27b)$$

$$M_D = \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ 0 & d & e \end{pmatrix}, \quad (28a)$$

$$M_D = \begin{pmatrix} 0 & 0 & a \\ b & 0 & c \\ d & 0 & e \end{pmatrix}, \quad (28b)$$

$$M_D = \begin{pmatrix} a & 0 & 0 \\ b & 0 & c \\ d & 0 & e \end{pmatrix}; \quad (28c)$$

$$M_D = \begin{pmatrix} 0 & 0 & a \\ b & c & 0 \\ d & e & 0 \end{pmatrix}, \quad (29a)$$

$$M_D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}; \quad (29b)$$
\( M_D = \begin{pmatrix}
0 & 0 & a \\
0 & b & c \\
d & 0 & e
\end{pmatrix} \), \hspace{1cm} (30a)

\( M_D = \begin{pmatrix}
0 & 0 & a \\
0 & b & c \\
d & e & 0
\end{pmatrix} \), \hspace{1cm} (30b)

\( M_D = \begin{pmatrix}
0 & 0 & a \\
b & 0 & c \\
d & e & 0
\end{pmatrix} \), \hspace{1cm} (30c)

\( M_D = \begin{pmatrix}
0 & a & 0 \\
b & 0 & c \\
d & e & 0
\end{pmatrix} \), \hspace{1cm} (30d)

\( M_D = \begin{pmatrix}
a & 0 & 0 \\
0 & b & c \\
d & 0 & e
\end{pmatrix} \), \hspace{1cm} (30e)

\( M_D = \begin{pmatrix}
a & 0 & 0 \\
b & 0 & c \\
d & e & 0
\end{pmatrix} \). \hspace{1cm} (30f)

If \( M_D \) has five texture zeros then \( M_R \) should have three texture zeros.
Since \( \det M_R \) must be nonzero, the following are the only possible forms for \( M_R \):

\[
\begin{pmatrix}
\times & 0 & 0 \\
0 & \times & 0 \\
0 & 0 & \times
\end{pmatrix}, \hspace{1cm} \begin{pmatrix}
0 & \times & \times \\
\times & 0 & \times \\
\times & \times & 0
\end{pmatrix}, \hspace{1cm} (31a)
\]

\[
\begin{pmatrix}
\times & \times & 0 \\
\times & 0 & 0 \\
0 & 0 & \times
\end{pmatrix}, \hspace{1cm} \begin{pmatrix}
\times & \times & 0 \\
\times & 0 & 0 \\
0 & 0 & \times
\end{pmatrix}. \hspace{1cm} (31b)
\]
They correspond to

\[ M_{R}^{-1} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}, \quad (32a) \]

\[ M_{R}^{-1} = \begin{pmatrix} -x & \sqrt{xy} & \sqrt{xz} \\ \sqrt{xy} & -y & \sqrt{yz} \\ \sqrt{xz} & \sqrt{yz} & -z \end{pmatrix}, \quad (32b) \]

\[ M_{R}^{-1} = \begin{pmatrix} 0 & x & 0 \\ x & y & 0 \\ 0 & 0 & z \end{pmatrix}, \quad (32c) \]

\[ M_{R}^{-1} = \begin{pmatrix} x & 0 & \sqrt{xz} \\ 0 & 0 & \sqrt{xz} \\ \sqrt{xz} & y & z \end{pmatrix}, \quad (32d) \]

respectively.

The forms of the matrices \((31a)\) are invariant under \(S_3\) while the forms of the matrices \((31b)\) are not invariant under any non-trivial permutation of their rows and columns. Therefore, with either eq. \((32a)\) or eq. \((32b)\) one may take, instead of the nine possibilities \((22, 23)\) for \(M_D\), just the following two possibilities:

\[ M_D = \begin{pmatrix} a & b & 0 \\ c & 0 & 0 \\ 0 & d & 0 \end{pmatrix}, \quad (33a) \]

\[ M_D = \begin{pmatrix} a & b & 0 \\ c & 0 & 0 \\ 0 & 0 & d \end{pmatrix}, \quad (33b) \]

With eqs. \((32c)\) and \((32d)\), on the other hand, one must use the full set of nine possibilities for \(M_D\):

\[ M_D = \begin{pmatrix} a & b & 0 \\ c & 0 & 0 \\ 0 & d & 0 \end{pmatrix}, \quad (34a) \]

\[ M_D = \begin{pmatrix} a & 0 & b \\ c & 0 & 0 \\ 0 & 0 & d \end{pmatrix}, \quad (34b) \]

\[ M_D = \begin{pmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix}, \quad (34c) \]
\[ M_D = \begin{pmatrix} a & b & 0 \\ c & 0 & 0 \\ 0 & 0 & d \end{pmatrix}, \quad (35a) \]
\[ M_D = \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix}, \quad (35b) \]
\[ M_D = \begin{pmatrix} a & 0 & b \\ c & 0 & 0 \\ 0 & d & 0 \end{pmatrix}, \quad (35c) \]
\[ M_D = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix}, \quad (35d) \]
\[ M_D = \begin{pmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & d & 0 \end{pmatrix}, \quad (35e) \]
\[ M_D = \begin{pmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & 0 & d \end{pmatrix}. \quad (35f) \]

4 Constraints on \( M \)

4.1 Possibilities with eq. (32a)

With this form of \( M_R^{-1} \) one should use the two options (33) for \( M_D \). With eq. (33b) one neutrino decouples; this is incompatible with experiment. With eq. (33a) one obtains\(^7\)

\[ \det M = 0, \quad M_{\alpha\beta} = 0 \ (\alpha \neq \beta), \quad (36) \]

which are constraints on \( M \) which have not yet been considered in the literature.

\(^7\)To be sure, from eqs. (32a) and (33a) one obtains \( \det M = 0 \) and \( M_{23} = 0 \). But now I generalize and consider other options for \( M_D \) that differ from eq. (33a) through a permutation of the rows; then I obtain, in general, the conditions (36). In the same fashion, throughout this section I shall consider, for each particular case, the results that follow after considering all possible permutations of the rows of \( M_D \).
4.2 Possibilities with eq. (32b)

With this form of $M_R^{-1}$ one should once again use the two options (33) for $M_D$. With eq. (33b) one obtains

$$(M^{-1})_{\alpha\alpha} = (M^{-1})_{\beta\beta} = 0 \quad (\alpha \neq \beta),$$

which has already been considered in ref. [7]. With eq. (33a) one obtains

$$\det M = 0, \quad M_{\alpha\alpha} M_{\beta\beta} - (M_{\alpha\beta})^2 = 0 \quad (\alpha \neq \beta),$$

which is new and potentially interesting.

4.3 Possibilities with eq. (32c)

With this form of $M_R^{-1}$ one should use either one of the nine options (34–35) for $M_D$.

With eq. (34b) one of the neutrinos decouples. With eq. (34c) one recovers the conditions (36). With eq. (34a) one obtains

$$\det M = 0, \quad M_{\alpha\alpha} = 0,$$

which must be studied.

Both eq. (35a) and eq. (35b) lead to the decoupling of one neutrino. With either eq. (35c) or eq. (35f) one gets

$$M_{\alpha\alpha} = M_{\alpha\beta} = 0 \quad (\alpha \neq \beta),$$

which has been studied in ref. [6]. With eq. (35d) one has

$$(M^{-1})_{\alpha\alpha} = 0, \quad M_{\alpha\beta} = 0 \quad (\alpha \neq \beta),$$

while with eq. (35e) one has

$$M_{\alpha\alpha} = 0, \quad (M^{-1})_{\alpha\beta} = 0 \quad (\alpha \neq \beta).$$

Both constraints (41) and (42) have already been studied in ref. [8].

4.4 Possibilities with eq. (32d)

With this form of $M_R^{-1}$ one should use either one of the nine options (34–35) for $M_D$.

Equation (34a) leads to one neutrino decoupling and eq. (34b) leads to scaling. Equation (34c) reproduces the conditions (39).
Equation (35a) leads once again to the conditions (37). Both eq. (35b) and eq. (35e) reproduce the conditions (40). Equation (35d) leads to
\[ M_{\alpha\alpha} = 0, \quad (M^{-1})_{\alpha\alpha} = 0. \] (43)
These conditions have also been studied in ref [8]. Equation (35c) leads to
\[ M_{\alpha\alpha} = M_{\alpha\beta} = 0, \quad (M^{-1})_{\alpha\alpha} = 0 \quad (\alpha \neq \beta). \] (44)
This is one constraint too many, but I shall consider it later. Equation (35f) gives
\[ (M^{-1})_{\alpha\alpha} = 0, \quad A_{\beta\beta} = 1, \quad M_{\gamma\gamma} \neq 0 \quad (\alpha \neq \beta \neq \gamma \neq \alpha), \] (45)
which is new. I have explicitly written down the condition \( M_{\gamma\gamma} \neq 0 \) in conditions (45) in order to distinguish this model from cases \( A_1,2 \) and \( B_3,4 \) of ref. [6]. For instance, \( M_{11} = M_{12} = 0 \) in case \( A_1 \); this leads to \( (M^{-1})_{33} = 0 \) and \( A_{22} = 1 \), corresponding to \( t = 0 \) in eq. (5).

4.5 Possibilities with eq. (26)

With eq. (26) one must use for \( M_D \) the 13 options in eqs. (27–30).

Equation (27a) yields
\[ M_{\alpha\alpha} = M_{\beta\beta} = 0 \quad (\alpha \neq \beta). \] (46)
These conditions have been studied in ref. [6] (see also ref. [14]). Equation (27b) reproduces the conditions (40).

Equation (28a) yields once again the conditions (39). Equations (28b) and (28c) lead to two massless neutrinos.

Equation (29a) reproduces the conditions (43). Equation (29b) makes one neutrino decouple.

Equations (30a), (30b), and (30d) reproduce the conditions (40). Equation (30e) reproduces the conditions (41). Finally, eq. (30f) yields
\[ M_{\alpha\alpha} = 0, \quad A_{\beta\beta} = 1, \quad (M^{-1})_{\gamma\gamma} \neq 0 \quad (\alpha \neq \beta \neq \gamma \neq \alpha), \] (47)
which is new.

4.6 Summary

Most matrices \( M \) that I have found embody conditions that have already been treated in the literature. A few matrices \( M \), though, present features
which, to my knowledge, have not yet been studied. These are

\[ \begin{align*}
\text{det} \, M & = 0 \quad \text{and} \quad M_{\alpha\alpha} = 0; \\
\text{det} \, M & = 0 \quad \text{and} \quad M_{\alpha\beta} = 0; \\
\text{det} \, M & = 0 \quad \text{and} \quad M_{\alpha\alpha}M_{\beta\beta} - (M_{\alpha\beta})^2 = 0; \\
M_{\alpha\alpha} & = M_{\alpha\beta} = 0 \quad \text{and} \quad (M^{-1})_{\alpha\alpha} = 0, \\
M_{\alpha\alpha} & = 0, \quad A_{\beta\beta} = 1, \quad \text{and} \quad (M^{-1})_{\gamma\gamma} \neq 0; \\
(M^{-1})_{\alpha\alpha} & = 0, \quad A_{\beta\beta} = 1, \quad \text{and} \quad M_{\gamma\gamma} \neq 0.
\end{align*} \]

In eqs. (49–53) it should be understood that \( \alpha \neq \beta \neq \gamma \neq \alpha \).

According to ref. [8], the possibility (51) should be excluded because \( M_{\alpha\alpha} = 0 \) together with \( (M^{-1})_{\alpha\alpha} = 0 \) is experimentally excluded for any value of \( \alpha = e, \mu, \tau \). So in the next section I shall only consider conditions (48–50), (52), and (53).

5 Comparison with the data

5.1 Introduction

PMNS matrix: Since \( M_\ell \) is diagonal, the unitary matrix that diagonalizes \( M \) is the lepton mixing (PMNS) matrix \( U \):

\[ M = U^* \text{diag}(\mu_1, \mu_2, \mu_3) U^\dagger, \]

where the \( \mu_j \) are complex; the neutrino masses are \( m_j = |\mu_j| \) \( (j = 1, 2, 3) \). The matrix \( U \) is written

\[ U = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & \epsilon^* \\
-s_{12}c_{23} - \epsilon c_{12}s_{23} & c_{12}c_{23} - \epsilon s_{12}s_{23} & s_{23}c_{13} \\
s_{12}s_{23} - \epsilon c_{12}c_{23} & -c_{12}s_{23} - \epsilon s_{12}c_{23} & c_{23}c_{13}
\end{pmatrix}, \]

where \( \epsilon \equiv s_{13} \exp(i\delta) \). In eq. (55), \( s_{jj'} \equiv \sin \theta_{jj'} \) and \( c_{jj'} \equiv \cos \theta_{jj'} \).

The data: I define

\[ r_{\text{solar}} = \sqrt{\frac{m_2^2 - m_1^2}{m_3^2 - m_1^2}}, \]

(56)
I use the $3\sigma$ ranges \[ (57) \]
\[
0.278 \leq s_{12}^2 \leq 0.375, \\
0.0177 \leq s_{13}^2 \leq 0.0294, \\
0.392 \leq s_{23}^2 \leq 0.643, \\
0.0268 \leq r_{\text{solar}}^2 \leq 0.0356
\]
for ‘normal’ ordering of the neutrino masses ($m_3 > m_2 > m_1$), and
\[
0.278 \leq s_{12}^2 \leq 0.375, \\
0.0183 \leq s_{13}^2 \leq 0.0297, \\
0.403 \leq s_{23}^2 \leq 0.640, \\
0.0280 \leq r_{\text{solar}}^2 \leq 0.0372
\]
for ‘inverted’ ordering ($m_2 > m_1 > m_3$).

**Neutrino mass observables:** The models in this paper cannot predict the absolute value of the neutrino masses, since all the predictions in eqs. \( (48) \)–\( (53) \) are invariant under $M \to c M$, where $c$ is an arbitrary complex number. They may, though, predict the relative value of any two neutrino mass observables. Most conveniently, one of those observables should be chosen to be the square root of the atmospheric squared-mass difference,

\[
m_{\text{atmospheric}} \equiv \sqrt{|m_3^2 - m_1^2|} \approx 0.05 \text{ eV}. \quad (59)
\]

The other relevant mass observables—besides $m_{\text{solar}} \equiv \sqrt{m_2^2 - m_1^2}$—are
\[
m_{\text{cosmological}} \equiv m_1 + m_2 + m_3, \\
m_{\beta\beta} \equiv |M_{ee}| = \left| \sum_{j=1}^{3} \mu_j^2 (U_{ej})^2 \right|, \\
m_{\nu_e} \equiv \sum_{j=1}^{3} m_j |U_{ej}|^2.
\]

Indeed, $m_{\text{cosmological}}$ may be derived from various cosmological observations; $m_{\beta\beta}$ may be derived from the rates of neutrinoless double-$\beta$ decay of various nuclides; and $\langle m_{\nu_e} \rangle$ is the average mass of the electron neutrino to be measured in experiments on the electron energy end-point of tritium $\beta$ decay. I define
\[
r_{\text{cosmological}} \equiv \frac{m_{\text{cosmological}}}{m_{\text{atmospheric}}}, \quad r_{\beta\beta} \equiv \frac{m_{\beta\beta}}{m_{\text{atmospheric}}}, \quad r_{\nu_e} \equiv \frac{m_{\nu_e}}{m_{\text{atmospheric}}}. \quad (61)
\]

There are other phenomenological fits to the data—see refs. [16] [17].
5.2 The conditions (50)

When \( \det M = 0 \) either \( \mu_1 = 0 \) (normal ordering) or \( \mu_3 = 0 \) (inverted ordering). With normal ordering one has

\[
0 = M_{aa}^* M_{bb}^* - (M_{ab}^*)^2 = [\mu_2^* (U_{a2})^2 + \mu_3^* (U_{a3})^2] [\mu_2^* (U_{b2})^2 + \mu_3^* (U_{b3})^2]
\]

\[ - (\mu_2^* U_{a2} U_{b3} + \mu_3^* U_{a3} U_{b3})^2. \]

(62)

This gives

\[
0 = \mu_2^* \mu_3^* (U_{a2} U_{b3} - U_{a3} U_{b2})^2, \tag{63}
\]

hence \( U_{a2} U_{b3} - U_{a3} U_{b2} = 0 \). But \( U \) is a unitary matrix, therefore

\[
|U_{a2} U_{b3} - U_{a3} U_{b2}| = |U_{\gamma 1}|, \tag{64}
\]

where \( \gamma \neq \alpha, \beta \). One concludes that the conditions (50) predict, in the case of normal ordering, one matrix element of the first column of \( U \) to vanish. This contradicts experiment. In the case of inverse ordering, conditions (50) predict a matrix element of the third column of \( U \) to vanish. This also contradicts experiment.

Thus, conditions (50) are experimentally excluded.

5.3 The conditions (48)

With normal ordering one has \( \mu_1 = 0 \) and

\[
0 = M_{aa}^* = \mu_2^* (U_{a2})^2 + \mu_3^* (U_{a3})^2. \tag{65}
\]

Therefore,

\[
\frac{|U_{\alpha 3}|^2}{|U_{\alpha 2}|} = \frac{m_2}{m_3} = r_{\text{solar}}. \tag{66}
\]

Equation (66) is incompatible with experiment for any \( \alpha = e, \mu, \tau \).

With inverted ordering one has instead \( \mu_3 = 0 \) and

\[
0 = M_{aa}^* = \mu_1^* (U_{a1})^2 + \mu_2^* (U_{a2})^2. \tag{67}
\]

Therefore,

\[
\frac{|U_{\alpha 1}|^2}{|U_{\alpha 2}|} = \frac{m_2}{m_1} = \sqrt{1 + r_{\text{solar}}^2}. \tag{68}
\]

Equation (68) can fit the phenomenology in the cases \( \alpha = \mu \) and \( \alpha = \tau \). If \( \alpha = \mu \), then \( s_{12} \) should not be too low; \( s_{23} \) and (to a lesser extent) \( s_{13} \) are
also preferably above their central values; moreover, \( \cos \delta \gtrsim 0.5 \) is predicted. If \( \alpha = \tau \), then \( s_{12} \) and \( s_{13} \) should be at or above their best-fit values while \( \theta_{23} \) lies preferably in the first octant; \( \cos \delta \lesssim -0.5 \) in this case. In both cases, the predictions for the neutrino mass ratios are

\[
2.014 \leq r_{\text{cosmological}} \leq 2.018, \quad 0.24 \leq r_{\beta \beta} \leq 0.42, \quad 0.974 \leq r_{\nu_e} \leq 0.988.
\]

(69)

To summarize, conditions (48) can only hold with an inverted neutrino mass spectrum, with either \( \alpha = \mu \) or \( \alpha = \tau \), and with large \( |\cos \delta| \).

### 5.4 The conditions (49)

With normal ordering the conditions (49) produce

\[
\left| \frac{U_{\alpha 3} U_{\beta 3}}{U_{\alpha 2} U_{\beta 2}} \right| = r_{\text{solar}}
\]

(70)

while with inverted ordering one obtains instead

\[
\left| \frac{U_{\alpha 1} U_{\beta 1}}{U_{\alpha 2} U_{\beta 2}} \right| = \sqrt{1 + r_{\text{solar}}^2}.
\]

(71)

Equation (70) is incompatible with experiment. Condition (71) may agree with the phenomenology in the cases \( \alpha = e \) and either \( \beta = \mu \) or \( \beta = \tau \); the mixing angles are free but there is a stringent prediction \( |\cos \delta| < 0.1 \). The predictions for \( r_{\text{cosmological}} \) and \( r_{\nu_e} \) are the same as in inequalities (69) while \( 0.948 \leq r_{\beta \beta} \leq 0.983 \) is higher in this case.

To summarize, conditions (49) can only hold if the neutrino mass spectrum is inverted and if \( (\alpha, \beta) \) is either \( (e, \mu) \) or \( (e, \tau) \). A tiny \( |\cos \delta| \) is predicted.

### 5.5 The conditions (52)

I firstly define

\[
V_{\beta j} = \left( U_{\beta j}^* \right)^2, \quad x = \frac{\mu_1}{\mu_3}, \quad y = \frac{\mu_2}{\mu_3}.
\]

(72)
I then define

\[ \begin{align*}
  c_1 & \equiv V_{\alpha 1}, \\
  c_2 & \equiv V_{\alpha 2}, \\
  c_3 & \equiv V_{\alpha 3}, \\
  c_4 & \equiv V_{\beta 1}V_{\beta 2}^{*}, \\
  c_5 & \equiv c_4^{*}, \\
  c_6 & \equiv V_{\beta 1}V_{\beta 3}^{*}, \\
  c_7 & \equiv c_6^{*}, \\
  c_8 & \equiv V_{\beta 2}V_{\beta 3}^{*}, \\
  c_9 & \equiv c_8^{*}, \\
  c_{10} & \equiv \sum_{j=1}^{3} |V_{\beta j}|^2 - 1. 
\end{align*} \] (73)

Then, from the first condition (52),

\[ 0 = M_{\alpha\alpha} = \mu_1 V_{\alpha 1} + \mu_2 V_{\alpha 2} + \mu_3 V_{\alpha 3}. \] (74)

Therefore,

\[ 0 = c_1 x + c_2 y + c_3. \] (75)

The second condition (52) is

\[ \begin{align*}
  1 & = A_{\beta\beta} \\
  & = M_{\beta\beta} \left( M^{-1} \right)_{\beta\beta} \\
  & = (\mu_1 V_{\beta 1} + \mu_2 V_{\beta 2} + \mu_3 V_{\beta 3}) \left( \frac{V_{\beta 1}^{*}}{\mu_1} + \frac{V_{\beta 2}^{*}}{\mu_2} + \frac{V_{\beta 3}^{*}}{\mu_3} \right) \\
  & = |V_{\beta 1}|^2 + |V_{\beta 2}|^2 + |V_{\beta 3}|^2 + \frac{y}{x} V_{\beta 1}^{*} V_{\beta 2} + \frac{x}{y} V_{\beta 1} V_{\beta 2}^{*} \\
  & \quad + \frac{1}{x} V_{\beta 1}^{*} V_{\beta 3} + x V_{\beta 1} V_{\beta 3}^{*} + \frac{1}{y} V_{\beta 2}^{*} V_{\beta 3} + y V_{\beta 2} V_{\beta 3}^{*}. 
\end{align*} \] (76)

Therefore,

\[ 0 = c_4 x^2 + c_5 y^2 + c_6 x^2 y + c_7 y + c_8 x y^2 + c_9 x + c_{10} x y. \] (77)

Equations (75) and (77) determine \( x \) and \( y \) through

\[ \begin{align*}
  0 & = c_2 (c_2 c_6 - c_1 c_8) y^3 \\
  & \quad + (2 c_2 c_3 c_6 - c_1 c_2 c_{10} - c_1 c_3 c_8 + c_1^2 c_5 + c_2^2 c_4) y^2 \\
  & \quad + (2 c_2 c_3 c_4 - c_1 c_3 c_{10} - c_1 c_2 c_9 + c_1^2 c_7 + c_2^2 c_6) y \\
  & \quad + c_3 (c_3 c_4 - c_1 c_9), \quad (78a) \\
  x & = -\frac{c_2 y - c_3}{c_1}. \quad (78b)
\end{align*} \]
In this way, the first two conditions (52) allow one to, by using as input the PMNS matrix, exactly determine both the Majorana phases and the ratios among the neutrino masses. One must still impose the third condition (52), viz.
\[ V_{\gamma_1}^* y + V_{\gamma_2}^* x + V_{\gamma_3}^* xy \neq 0, \]  
(79)
on the values of $x$ and $y$ that have been determined.

One must choose the input, viz. the PMNS matrix, in such a way that the resulting $x$ and $y$ satisfy $|x| < |y|$, i.e. $m_{\text{solar}} > 0$, and that
\[ r_{\text{solar}} = \sqrt{\frac{|y|^2 - |x|^2}{1 - |x|^2}} \]  
(80)
is in its experimentally allowed range. If $1 > |x|$ the neutrino mass spectrum is normal; it is inverted if $|x| > 1$. For the neutrino mass ratios one has
\[ r_{\text{cosmological}} = \frac{|x| + |y| + 1}{\sqrt{1 - |x|^2}}, \]  
(81a)\[ r_{\beta\beta} = \frac{|xV_{e1} + yV_{e2} + V_{e3}|}{\sqrt{1 - |x|^2}}, \]  
(81b)\[ r_{\nu e} = \frac{|xV_{e1} + |yV_{e2}| + |V_{e3}|}{\sqrt{1 - |x|^2}}. \]  
(81c)

Numerically, I have found that there are two types of cases in which the conditions (52) are able to fit the experimental values. In the first type of cases, $|\cos \delta| \gtrsim 0.5$ must be close to unity\footnote{In this paper, the expression \textquotedblleft a \gtrsim b\textquotedblright means the following: the quantity a has a lower bound that is approximately equal to b, but a may as well be much larger than b.} and the neutrino masses are of order $m_{\text{atmospheric}}$. This type of cases occurs for an inverted neutrino mass spectrum when $\beta = e$ and either $\alpha = \mu$ or $\alpha = \tau$; in the first case $\cos \delta \gtrsim 0.5$ and in the second one $\cos \delta \lesssim -0.5$. One obtains for these cases
\[ 2.019 < r_{\text{cosmological}} < 2.035, \]  
\[ 0.24 < r_{\beta\beta} < 0.44, \]  
\[ 0.974 < r_{\nu e} < 0.989. \]  
(82)

In the second type of cases neutrino masses are quasi-degenerate, $\theta_{23}$ is in a well-defined octant, and $\cos \delta$ is extremely close to zero. Moreover,
when $\theta_{23} \to \pi/4$, the neutrino masses grow towards infinity and $\cos \delta \to 0$\footnote{In ref. \cite{18} it had already been noted that models $B_{3,A}$ of ref. \cite{6} display the property that $\theta_{2,3} \to \pi/4$ and $\cos \delta \to 0$ when the neutrino masses become quasi-degenerate. Our models, though, do not coincide with models $B_{3,A}$. In those models $A_{ee} = 1$ and $A_{\mu\mu} = A_{\tau\tau} = 0$; in our models $A_{ee} = 1$ and either $A_{\mu\mu}$ vanishes or $A_{\tau\tau}$ vanishes, but they do not both vanish.}. These cases occur when $\beta = e$ and either $\alpha = \mu$ for a normal neutrino mass spectrum or $\alpha = \tau$ for an inverted neutrino mass spectrum; in both cases $s_{23}^2 < 0.5$. If one wants to have $s_{23}^2 > 0.5$ instead, then we must interchange $\alpha = \mu$ with $\alpha = \tau$. In all these cases $|\cos \delta| < 0.1$ (approaching zero when $s_{23}^2$ approaches 0.5) and $r_{\text{cosmological}} \gtrsim 2.5$, $r_{\beta\beta} \approx r_{\nu e} \gtrsim 0.5$ (approaching infinity when $s_{23}^2$ approaches 0.5).

5.6 The conditions (53)

Equations (53) are the same as eqs. (52) with $M \leftrightarrow M^{-1}$, i.e. with $\mu_j \to \mu_j^{-1}$ and $V \to V^\ast$. In practice, this means that, for each input PMNS matrix, one may use eqs. (78) in the previous subsection, but now they will yield $1/x^\ast$ and $1/y^\ast$ instead of $x$ and $y$, respectively. Thus, one should use

$$V_{\beta j} = (U_{\beta j}^\ast)^2, \quad x \equiv \frac{\mu_3^\ast}{\mu_1^\ast}, \quad y \equiv \frac{\mu_3^\ast}{\mu_2^\ast}$$ (83)

instead of eqs. (72). The condition $m_{\text{solar}} > 0$ now requires $|x| > |y|$ and

$$r_{\text{solar}} = \sqrt{\frac{|x|^2 - |y|^2}{|y|^2 ||x|^2 - 1|}}$$ (84)

must be in its experimentally allowed range. If $1 > |x|$ then the neutrino mass spectrum is inverted; it is normal if $|x| > 1$. For the neutrino mass ratios one has in this case

$$r_{\text{cosmological}} = \frac{|x| + |y|}{|y|} \sqrt{1 - |x|^2},$$

$$r_{\beta\beta} = \frac{|y^*V_{e1} + x^*V_{e2} + x^*y^*V_{e3}|}{|y| \sqrt{1 - |x|^2}},$$

$$r_{\nu e} = \frac{|y^*V_{e1}| + |x^*V_{e2}| + |x^*y^*V_{e3}|}{|y| \sqrt{1 - |x|^2}}.$$ (85a, 85b, 85c)

Numerically, I have found that the conditions (53) fit experiment in the same two types of cases as the conditions (52). Thus, for an inverted neutrino
mass spectrum and $\alpha = e$, one has either $\cos \delta \gtrsim 0.4$ for $\beta = \mu$ or $\cos \delta \lesssim -0.4$ for $\beta = \tau$. In both cases
\begin{align}
2.058 &< r_{\text{cosmological}} < 2.087, \\
0.35 &< r_{\beta\beta} < 0.54, \\
0.977 &< r_{\nu_e} < 0.990. \tag{86}
\end{align}

On the other hand, for $\beta = e$ there is another set of cases, with $|\cos \delta| \lesssim 0.15$, $r_{\text{cosmological}} \gtrsim 2$, and $r_{\beta\beta} \approx r_{\nu_e} \gtrsim 0.5$. These cases may have $\theta_{23}$ either in the first octant—for a normal neutrino mass spectrum when $\alpha = \tau$ and for an inverted neutrino mass spectrum when $\alpha = \mu$—or in the second octant—interchanging $\alpha = \mu$ with $\alpha = \tau$. The remarkable feature of this second set of cases is that the neutrino masses become almost degenerate when $\theta_{23}$ approaches $\pi/4$.\footnote{In ref. [19] a model was discussed in which the neutrino masses approach degeneracy when $s_{12}^2 \to 1/3$, with $s_{12}^2 < 1/3$ for a normal neutrino mass spectrum and $s_{12}^2 > 1/3$ for an inverted spectrum. The present cases display analogous features, with $s_{12}$ replaced by $s_{23}$ and $1/3$ replaced by $1/2$ just as in ref. [18].}

6 Conclusions

In this paper I have exhaustively classified all the predictive cases where a type-I see-saw mechanism based on three right-handed neutrinos has a diagonal charged-lepton mass matrix $M_\ell$ and both the neutrino Dirac mass matrix $M_D$ and the right-handed-neutrino Majorana mass matrix $M_R$ have texture zeros. Most of the cases with predictive power had already been studied in the literature, but I have discovered a few new ones. The new cases predict either $|\cos \delta| \gtrsim 0.5$ or $|\cos \delta| \lesssim 0.1$; some of the latter cases feature quasi-degenerate neutrinos when $\theta_{23}$ is very close to $\pi/4$.

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