Pluripotential theory and convex bodies: large deviation principle

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Abstract. We continue the study in [2] in the setting of weighted pluripotential theory arising from polynomials associated to a convex body $P$ in $(\mathbb{R}^+)^d$. Our goal is to establish a large deviation principle in this setting specifying the rate function in terms of $P$—pluripotential-theoretic notions. As an important preliminary step, we first give an existence proof for the solution of a Monge-Ampère equation in an appropriate finite energy class. This is achieved using a variational approach.

1. Introduction

As in [2], we fix a convex body $P \subseteq (\mathbb{R}^+)^d$ and we define the logarithmic indicator function

$$H_P(z) := \sup_{J \in P} \log |z^J| := \sup_{(j_1, \ldots, j_d) \in P} \log [|z_1|^{j_1} \ldots |z_d|^{j_d}].$$

We assume throughout that

$$\Sigma \subseteq kP \text{ for some } k \in \mathbb{Z}^+$$

where

$$\Sigma := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, \sum_{j=1}^d x_i \leq 1\}.$$ 

Then

$$H_P(z) \geq \frac{1}{k} \max_{j=1, \ldots, d} \log^+ |z_j|$$

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where \( \log^+ |z_j| = \max[0, \log |z_j|] \). We define

\[
L_P = L_P(C^d) := \{ u \in PSH(C^d) : u(z) - H_P(z) = O(1), \ |z| \to \infty \},
\]

and

\[
L_{P,+} = L_{P,+}(C^d) = \{ u \in L_P(C^d) : u(z) \geq H_P(z) + C_u \}.
\]

These are generalizations of the classical Lelong classes when \( P = \Sigma \). We define the finite-dimensional polynomial spaces

\[
Poly(nP) := \{ p(z) = \sum_{J \in nP \cap (\mathbb{Z}^d)} c_J z^J : c_J \in \mathbb{C} \}
\]

for \( n=1, 2, \ldots \) where \( z^J = z_1^{j_1} \cdots z_d^{j_d} \) for \( J = (j_1, \ldots, j_d) \). For \( p \in Poly(nP) \), \( n \geq 1 \) we have \( \frac{1}{n} \log |p| \in L_P \); also each \( u \in L_{P,+}(C^d) \) is locally bounded in \( \mathbb{C}^d \). For \( P = \Sigma \), we write \( Poly(nP) = P_n \).

Given a compact set \( K \subset \mathbb{C}^d \), one can define various pluripotential-theoretic notions associated to \( K \) related to \( L_P \) and the polynomial spaces \( Poly(nP) \). Our goal in this paper is to prove some probabilistic properties of random point processes on \( K \) utilizing these notions and their weighted counterparts. We require an existence proof for the solution of a Monge-Ampère equation in an appropriate finite energy class; this is done in Theorem 2.8 using a variational approach and is of interest on its own. The third section recalls appropriate definitions and properties in \( P \)-pluripotential theory, mostly following [2]. As in [2], our spaces \( Poly(nP) \) do not necessarily arise as holomorphic sections of tensor powers of a line bundle. Subsection 3.3 includes a standard elementary probabilistic result on almost sure convergence of probability measures associated to random arrays on \( K \) to a \( P \)-pluripotential-theoretic equilibrium measure. Section 4 sets up the machinery for the more subtle large deviation principle (LDP), Theorem 5.1, for which we provide two proofs (analogous to those in [9]). As in [9], the first proof was inspired by [6] and the second proof was utilized by Berman in [5]. The reader will find far-reaching applications and interpretations of LDP’s in the appropriate settings of holomorphic line bundles over a compact, complex manifold in [5]. In particular, the case where \( P \) is a convex integral polytope (vertices in \( \mathbb{Z}^d \)) which is the moment polytope for a toric manifold (\( P \) is Delzant) is covered in [5].

2. Monge-Ampère and \( P \)-pluripotential theory

2.1. Monge-Ampère equations with prescribed singularity

In this section, \( (X, \omega) \) is a compact Kähler manifold of dimension \( d \).
2.1.1. Quasi-plurisubharmonic functions

A function \( u : X \to \mathbb{R} \cup \{-\infty\} \) is called quasi-plurisubharmonic (quasi-psh) if locally \( u = \rho + \varphi \), where \( \varphi \) is plurisubharmonic and \( \rho \) is smooth.

We let \( PSH(X, \omega) \) denote the set of \( \omega \)-psh functions, i.e. quasi-psh functions \( u \) such that \( \omega u := \omega + dd^c u \geq 0 \) in the sense of currents on \( X \).

Given \( u, v \in PSH(X, \omega) \) we say that \( u \) is more singular than \( v \) (and we write \( u \prec v \)) if \( u \leq v + C \) on \( X \), for some constant \( C \). We say that \( u \) has the same singularity as \( v \) (and we write \( u \asymp v \)) if \( u \prec v \) and \( v \prec u \).

Given \( \phi \in PSH(X, \omega) \), we let \( PSH(X, \omega, \phi) \) denote the set of \( \omega \)-psh functions \( u \) which are more singular than \( \phi \).

2.1.2. Nonpluripolar Monge-Ampère measure

For bounded \( \omega \)-psh functions \( u_1, \ldots, u_d \), the Monge-Ampère product \((\omega + dd^c u_1) \wedge \ldots \wedge (\omega + dd^c u_d)\) is well-defined as a positive Radon measure on \( X \) (see [14], [3]). For general \( \omega \)-psh functions \( u_1, \ldots, u_d \), the sequence of positive measures

\[
1 \cap \{ u_j > -k \} \left( \omega + dd^c \max(u_1, -k) \right) \wedge \ldots \wedge \left( \omega + dd^c \max(u_d, -k) \right)
\]

is non-decreasing in \( k \) and the limiting measure, which is called the nonpluripolar product of \( \omega_{u_1}, \ldots, \omega_{u_d} \), is denoted by

\[
\omega_{u_1} \wedge \ldots \wedge \omega_{u_d}.
\]

When \( u_1 = \ldots = u_d = u \) we write \( \omega_u^d := \omega_u \wedge \ldots \wedge \omega_u \). Note that by definition \( \int_X \omega_u \wedge \ldots \wedge \omega_u \leq \int_X \omega^d \).

It was proved in [20, Theorem 1.2] and [11, Theorem 1.1] that the total mass of nonpluripolar Monge-Ampère products is decreasing with respect to singularity type. More precisely,

**Theorem 2.1.** Let \( \omega_1, \ldots, \omega_d \) be Kähler forms on \( X \). If \( u_j \prec v_j \), \( j = 1, \ldots, d \), are \( \omega_j \)-psh functions then

\[
\int_X (\omega_1 + dd^c u_1) \wedge \ldots \wedge (\omega_d + dd^c u_d) \leq \int_X (\omega_1 + dd^c v_1) \wedge \ldots \wedge (\omega_d + dd^c v_d).
\]

As noted above, for a general \( \omega \)-psh function \( u \) we have the estimate \( \int_X \omega_u^d \leq \int_X \omega^d \). Following [15] we let \( \mathcal{E}(X, \omega) \) denote the set of all \( \omega \)-psh functions with maximal total mass, i.e.

\[
\mathcal{E}(X, \omega) := \left\{ u \in PSH(X, \omega) : \int_X \omega_u^d = \int_X \omega^d \right\}.
\]
Given $\phi \in PSH(X, \omega)$, we define

$$E(X, \omega, \phi) := \left\{ u \in PSH(X, \omega, \phi) : \int_X \omega^d = \int_X \omega_\phi^d \right\}.$$  

**Proposition 2.2.** Let $\phi \in PSH(X, \omega)$. The following are equivalent:

1. $E(X, \omega, \phi) \cap E(X, \omega) \neq \emptyset$;
2. $\phi \in E(X, \omega)$;
3. $E(X, \omega, \phi) \subseteq E(X, \omega)$.

**Proof.** We first prove (1) $\implies$ (2). If $u \in E(X, \omega, \phi) \cap E(X, \omega)$ then $\int_X \omega^d = \int_X \omega_\phi^d$. On the other hand, since $u$ is more singular than $\phi$, Theorem 2.1 ensures that

$$\int_X \omega^d = \int_X \omega_\phi^d \leq \int_X \omega_u^d \leq \int_X \omega^d,$$

hence equality holds, proving that $\phi \in E(X, \omega)$.

Now we prove (2) $\implies$ (3). If $\phi \in E(X, \omega)$ and $u \in E(X, \omega, \phi)$ then

$$\int_X \omega_u^d = \int_X \omega_\phi^d = \int_X \omega^d,$$

hence $u \in E(X, \omega)$.

Finally (3) $\implies$ (1) is obvious. ∎

**Proposition 2.3.** Assume that $\phi_j \in PSH(X, \omega_j), \ j = 1, \ldots, d$ with $\int_X (\omega_j + dd^c \phi_j)^d > 0$. If $u_j \in E(X, \omega_j, \phi_j), \ j = 1, \ldots, d$, then

$$\int_X (\omega_1 + dd^c u_1) \wedge \cdots \wedge (\omega_d + dd^c u_d) = \int_X (\omega_1 + dd^c \phi_1) \wedge \cdots \wedge (\omega_d + dd^c \phi_d).$$

**Proof.** Theorem 2.1 gives one inequality. The other one follows from [11, Proposition 3.1 and Theorem 3.14]. ∎

### 2.1.3. Model potentials

For a function $f : X \to \mathbb{R} \cup \{-\infty\}$, we let $f^*$ denote its uppersemicontinuous (usc) regularization, i.e.

$$f^*(x) := \limsup_{X \not
exists y \to x} f(y).$$

Given $\phi \in PSH(X, \omega)$, following J. Ross and D. Witt Nyström [18], we define

$$P_\omega[\phi] := \left( \lim_{t \to +\infty} P_\omega(\min(\phi + t, 0)) \right)^*.$$
Here, for a function $f$, $P_\omega(f)$ is defined as

$$P_\omega(f) := (x \mapsto \sup\{u(x) : u \in PSH(X, \omega), u \leq f\})^*.$$  

It was shown in [11, Theorem 3.8] that the nonpluripolar Monge-Ampère measure of $P_\omega[\phi]$ is dominated by Lebesgue measure:

$$\omega + dd^c P_\omega[\phi] \leq 1_{\{P_\omega[\phi]=0\}} \omega^d \leq \omega^d. \quad (2.1)$$

This fact plays a crucial role in solving the complex Monge-Ampère equation. For the reader’s convenience, we note that in the notation of [11] (on the left)

$$P_{\omega,\phi}(0) = P_\omega[\phi].$$

**Definition 2.4.** A function $\phi \in PSH(X, \omega)$ is called a model potential if $\int_X \omega^d > 0$ and $P_\omega[\phi]=\phi$. A function $u \in PSH(X, \omega)$ has model type singularity if $u$ has the same singularity as $P_\omega[u]$; i.e., $u - P_\omega[u]$ is bounded on $X$.

There are plenty of model potentials. If $\varphi \in PSH(X, \omega)$ with $\int_X \omega^d > 0$ then, by [11, Theorem 3.12], $P_\omega[\varphi]$ is a model potential. In particular, if $\int_X \omega^d = \int_X \omega^d$ (i.e. $\varphi \in E(X, \omega)$) then $P_\omega[\varphi]=0$.

We will use the following property of model potentials proved in [11, Theorem 3.12]: if $\phi$ is a model potential then

$$u \in PSH(X, \omega, \phi) \implies u - \sup_{X} u \leq \phi. \quad (2.2)$$

In the sequel we always assume that $\phi$ has model type singularity and small unbounded locus; i.e., $\phi$ is locally bounded outside a closed complete pluripolar set, allowing us to use the variational approach of [7] as explained in [11].

### 2.1.4. The variational approach

We call a measure which puts no mass on pluripolar sets a nonpluripolar measure. For a positive nonpluripolar measure $\mu$ on $X$ we let $L_\mu$ denote the following linear functional on $PSH(X, \omega, \phi)$:

$$L_\mu(u) := \int_X (u - \phi) \, d\mu.$$

For $u \in PSH(X, \omega)$ with $u \simeq \phi$, we define the Monge–Ampère energy

$$E_\phi(u) := \frac{1}{(d+1)} \sum_{k=0}^{d} \int_X (u - \phi) \omega_u^k \wedge \omega_{\phi}^{d-k}. \quad (2.3)$$
It was shown in [11, Theorem 4.10] (by adapting the arguments of [7]) that $E_\phi$ is non-decreasing and concave along affine curves, giving rise to its trivial extension to $PSH(X, \omega, \phi)$.

We define

$$(2.4) \quad E^1(X, \omega, \phi) := \{ u \in PSH(X, \omega, \phi) : E_\phi(u) > -\infty \}. $$

The following criterion was proved in [11, Theorem 4.13]:

**Proposition 2.5.** Let $u \in PSH(X, \omega, \phi)$. Then $u \in E^1(X, \omega, \phi)$ iff $u \in E(X, \omega, \phi)$ and $\int_X (u - \phi) \omega^d > -\infty$.

**Lemma 2.6.** If $E$ is pluripolar then there exists $u \in E^1(X, \omega, \phi)$ such that $E \subset \{ u = -\infty \}$.

**Proof.** Without loss of generality we can assume that $\phi$ is a model potential. Then (2.1) gives $\int_X |v| \omega^d = 0$. It follows from [7, Corollary 2.11] that there exists $v \in E^1(X, \omega, 0)$, $v \leq 0$, such that $E \subset \{ v = -\infty \}$. Set $u := P_\omega(\min(v, \phi))$. Then $E \subset \{ u = -\infty \}$ and we claim that $u \in E^1(X, \omega, \phi)$. For each $j \in \mathbb{N}$ we set $v_j := \max(v, -j)$ and $u_j := P_\omega(\min(v_j, \phi))$. Then $u_j$ decreases to $u$ and $u_j \simeq \phi$. Using [11, Theorem 4.10 and Lemma 4.15] it suffices to check that $\{ \int_X |u_j - \phi| \omega^d_{u_j} \}$ is uniformly bounded. It follows from [11, Lemma 3.7] that

$$\int_X |u_j - \phi| \omega^d_{u_j} \leq \int_X |u_j| \omega^d_{u_j} \leq \int_X |v_j| \omega^d_{v_j} + \int_X |\phi| \omega^d_{\phi}$$

$$= \int_X |v_j| \omega^d_{v_j}.$$

The fact that $\int_X |v_j| \omega^d_{v_j}$ is uniformly bounded follows from [15, Corollary 2.4] since $v \in E^1(X, \omega, 0)$. This concludes the proof. □

**Lemma 2.7.** Assume that $E^1(X, \omega, \phi) \subset L^1(X, \mu)$. Then, for each $C > 0$, $L_\mu$ is bounded on

$$E_C := \{ u \in PSH(X, \omega, \phi) : \sup_X u \leq 0 \text{ and } E_\phi(u) \geq -C \}.$$

**Proof.** By concavity of $E_\phi$ the set $E_C$ is convex. We now show that $E_C$ is compact in the $L^1(X, \omega^d)$ topology. Let $\{ u_j \}$ be a sequence in $E_C$. We claim that $\{ \sup_X u_j \}$ is bounded. Indeed, by [11, Theorem 4.10]

$$E_\phi(u_j) \leq \int_X (u_j - \phi) \omega^d_{\phi}$$

$$\leq (\sup_X u_j) \int_X \omega^d_{\phi} + \int_X (u_j - \sup_X u_j - \phi) \omega^d_{\phi}.$$
It follows from (2.2) that $u_j - \sup_X u_j \leq P_\omega[\phi] \leq \phi + C_0$, where $C_0$ is a constant. The boundedness of $\{\sup_X u_j\}$ then follows from that of $\{E_\phi(u_j)\}$ and the above estimate. This proves the claim.

A subsequence of $\{u_j\}$, still denoted by $\{u_j\}$, converges in $L^1(X, \omega^d)$ to $u \in PSH(X, \omega)$ with $\sup_X u \leq 0$. Since $u_j - \sup_X u_j \leq \phi + C_0$, we have $u - \sup_X u \leq \phi + C_0$. This proves that $u \in PSH(X, \omega, \phi)$. The upper semicontinuity of $E_\phi$ (see [11, Proposition 4.19]) ensures that $E_\phi(u) \geq -C$, hence $u \in EC$. This proves that $EC$ is compact in the $L^1(X, \omega^d)$ topology.

The result then follows from [7, Proposition 3.4]. □

The goal of this section is to prove the following result:

**Theorem 2.8.** Assume that $\mu$ is a nonpluripolar positive measure on $X$ such that $\mu(X) = \int_X \omega^d_\phi$. The following are equivalent

1. $\mu$ has finite energy, i.e., $L_\mu$ is finite on $E^1(X, \omega, \phi)$;
2. there exists $u \in E^1(X, \omega, \phi)$ such that $\omega^d_u = \mu$;
3. there exists a unique $u \in E^1(X, \omega, \phi)$ such that

$$F_\mu(u) = \max_{v \in E^1(X, \omega, \phi)} F_\mu(v) < +\infty$$

where $F_\mu = E_\phi - L_\mu$.

**Remark 2.9.** It was shown in [11, Theorem 4.28] that a unique (normalized) solution $u$ in $E(X, \omega, \phi)$ always exists (without the finite energy assumption on $\mu$). But that proof does not give a solution in $E^1(X, \omega, \phi)$. Below, we will follow the proof of [11, Theorem 4.28] and use the finite energy condition, $E^1(X, \omega, \phi) \subset L^1(X, \mu)$, to prove that $u$ belongs to $E^1(X, \omega, \phi)$.

**Lemma 2.10.** Assume that $E^1(X, \omega, \phi) \subset L^1(X, \mu)$. Then there exists a positive constant $C$ such that, for all $u \in E^1(X, \omega, \phi)$ with $\sup_X u = 0$,

$$L_\mu(u) \geq -C(1 + |E_\phi(u)|^{1/2}).$$

(2.5)

The proof below uses ideas in [7], [15].

**Proof.** Since $\phi$ has model type singularity, it follows from [11, Theorem 4.10] that $E_\phi - E_{P_\omega[\phi]}$ is bounded. Without loss of generality we can assume in this proof that $\phi = P_\omega[\phi]$. Fix $u \in E^1(X, \omega, \phi)$ such that $\sup_X u = 0$ and $|E_\phi(u)| > 1$. Then, by [11, Theorem 3.12], $u \leq \phi$. Set $a = |E_\phi(u)|^{-1/2} \in (0, 1)$, and $v := au + (1 - a)\phi \in$
$\mathcal{E}^1(X, \omega, \phi)$. We estimate $E_\phi(v)$ as follows

$$(d+1)E_\phi(v) = a \sum_{k=0}^{d} \int_X (u-\phi)\omega^k_v \wedge \omega^d_\phi$$

$$= a \sum_{k=0}^{d} \int_X (u-\phi)(a\omega_u + (1-a)\omega_\phi)^k \wedge \omega^d_\phi$$

$$\geq C(d)a \int_X (u-\phi)\omega^d_\phi + C(d)a^2 \sum_{k=0}^{d} \int_X (u-\phi)\omega^k_u \wedge \omega^d_\phi,$$

where $C(d)$ is a positive constant which only depends on $d$. It follows from $\phi = P_\omega[\phi]$ and [11, Theorem 3.8] that $\omega^d \leq \omega^d$ (recall (2.1)). This together with [14, Proposition 2.7] give

$$\int_X (u-\phi)\omega^d_\phi \geq -C_1,$$

for a uniform constant $C_1$. Therefore,

$$(d+1)E_\phi(v) \geq -C_1 C(d)a + C_2 a^2 E_\phi(u) \geq -C_3.$$

It thus follows from Lemma 2.7 that $L_\mu(v) \geq -C_4$ for a uniform constant $C_4>0$. Thus

$$\int_X (u-\phi) d\mu \geq -C_4/a,$$

which gives (2.5). □

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. Without loss of generality we can assume that $\phi$ is a model potential. We first prove $(1) \Rightarrow (2)$. We write $\mu = f\nu$, where $\nu$ is a non-pluripolar positive measure satisfying, for all Borel subsets $B \subset X$,

$$\nu(B) \leq A \text{Cap}_\phi(B),$$

for some positive constant $A$, and $0 \leq f \in L^1(X, \nu)$ (cf., [11, Lemma 4.26]). Here $\text{Cap}_\phi$ is defined as

$$\text{Cap}_\phi(B) := \sup \left\{ \int_B \omega^d_u : u \in \text{PSH}(X, \omega), \phi - 1 \leq u \leq \phi \right\}.$$
For $k$ large enough, $1 \leq c_k \leq 2$ and $c_k \to 1$ as $k \to +\infty$. It follows from [11, Theorem 4.25] that there exists $u_j \in \mathcal{E}^1(X, \omega, \phi)$, sup$_X u_j = 0$, such that $\omega^d_{u_j} = \mu_j$; by [11, Theorem 3.12], $u_j \leq \phi$. A subsequence of $\{u_j\}$ which, by abuse of notation, will be denoted by $\{u_j\}$, converges in $L^1(X, \omega^d)$ to $u \in PSH(X, \omega)$ with $u \leq \phi$. Define $v_k := (\sup_{j \geq k} u_j)^*$. Then $v_k \leq u$ and sup$_X v_k = 0$. It follows from (2.5) and [11, Theorem 4.10] that

$$|E_\phi(u_j)| \leq \int_X |u_j - \phi| \omega^d_{u_j} \leq 2 \int_X |u_j - \phi| d\mu$$

$$\leq 2C(1 + |E_\phi(u_j)|^{1/2}).$$

Therefore $\{|E_\phi(u_j)|\}$ is bounded, hence so is $\{|E_\phi(v_j)|\}$ since $E_\phi$ is non-decreasing. It then follows from [11, Lemma 4.15] that $u \in \mathcal{E}^1(X, \omega, \phi)$.

Now, repeating the arguments of [11, Theorem 4.28] we can show that

$$L_\mu(v) = \int_X (v - \phi) \omega^d_u$$

$$= \int_X (v - u) \omega^d_u + \int_X (u - \phi) \omega^d_u$$

$$\geq E_\phi(v) - E_\phi(u) + \int_X (u - \phi) \omega^d_u > -\infty.$$ 

Hence $L_\mu$ is finite on $\mathcal{E}^1(X, \omega, \phi)$. Now, for all $v \in \mathcal{E}^1(X, \omega, \phi)$, by [11, Theorem 4.10] and Proposition 2.5 we have

$$F_\mu(v) - F_\mu(u) = E_\phi(v) - E_\phi(u) - \int_X (v - u) \omega^d_u \leq 0.$$

This gives (3). Finally, (3) $\implies$ (1) is obvious. □

2.2. Monge-Ampère equations on $\mathbb{C}^d$ with prescribed growth

As in the introduction we let $P$ be a convex body contained in $(\mathbb{R}^+)^d$ and fix $r > 0$ such that $P \subset r \Sigma$. We assume (1.2); i.e., $\Sigma \subset kP$ for some $k \in \mathbb{Z}^+$. This ensures that $H_P$ in (1.1) is locally bounded on $\mathbb{C}^d$ (and of course $H_P \in L_P^+(\mathbb{C}^d)$). Let $u \in L_P(\mathbb{C}^d)$ and define

$$\hat{u}(z) := u(z) - \frac{r}{2} \log(1 + |z|^2), z \in \mathbb{C}^d.$$
Consider the projective space \( \mathbb{P}^d \) equipped with the Kähler metric \( \omega := r \omega_{FS} \), where
\[
\omega_{FS} = dd^c \frac{1}{2} \log(1 + |z|^2)
\]
on \( \mathbb{C}^d \). Then \( \tilde{u} \) is bounded from above on \( \mathbb{C}^d \). It thus can be extended to \( \mathbb{P}^d \) as a function in \( PSH(\mathbb{P}^d, \omega) \).

For a plurisubharmonic function \( u \) on \( \mathbb{C}^d \), we let \( (dd^c u)^d \) denote its nonpluripolar Monge-Ampère measure; i.e., \( (dd^c u)^d \) is the increasing limit of the sequence of measures \( 1_{\{u > -k\}} (dd^c \max(u, -k))^d \). Then
\[
\omega^d = (\omega + dd^c \tilde{u})^d = (dd^c u)^d \text{ on } \mathbb{C}^d.
\]

If \( u \in L_P(\mathbb{C}^d) \) then
\[
\int_{\mathbb{C}^d} (dd^c u)^d \leq \int_{\mathbb{C}^d} (dd^c H_P)^d = d! Vol(P) =: \gamma_d = \gamma_d(P)
\]
(cf., equation (2.4) in [2]). We define
\[
\mathcal{E}_P(\mathbb{C}^d) := \left\{ u \in L_P(\mathbb{C}^d) : \int_{\mathbb{C}^d} (dd^c u)^d = \gamma_d \right\}.
\]

By the construction in (2.6) we have that \( \tilde{H}_P \in PSH(\mathbb{P}^d, \omega) \). We define
\[
\tilde{\Phi}_P := P_{\omega}[\tilde{H}_P].
\]
The key point here, which follows from [12, Theorem 7.2], is that \( \tilde{H}_P \) has model type singularity (recall Definition 2.4) and hence the same singularity as \( \tilde{\Phi}_P \). Defining \( \Phi_P \) on \( \mathbb{C}^d \) using (2.6); i.e., for \( z \in \mathbb{C}^d \),
\[
\Phi_P(z) = \tilde{\Phi}_P(z) + \frac{r}{2} \log(1 + |z|^2),
\]
we thus have \( \Phi_P \in L_{P,+}(\mathbb{C}^d) \). The advantage of using \( \Phi_P \) is that, by (2.1), \( (dd^c \Phi_P)^d \leq \omega^d \) on \( \mathbb{C}^d \). Note that \( L_{P,+}(\mathbb{C}^d) \subset \mathcal{E}_P(\mathbb{C}^d) \). For \( u, v \in L_{P,+}(\mathbb{C}^d) \) we define
\[
E_v(u) := \frac{1}{(d+1)} \sum_{j=0}^{d} \int_{\mathbb{C}^d} (u - v)(dd^c u)^j \wedge (dd^c v)^{d-j}. \tag{2.7}
\]
The corresponding global energy (see (2.3)) is defined as
\[
E_v(\tilde{u}) := \frac{1}{(d+1)} \sum_{j=0}^{d} \int_{\mathbb{P}^d} (\tilde{u} - \tilde{v})(\omega + dd^c \tilde{u})^j \wedge (\omega + dd^c \tilde{v})^{d-j}. 
\]
Then $E_v$ is non-decreasing and concave along affine curves in $L_{P,+}(\mathbb{C}^d)$. We extend $E_v$ to $L_P(\mathbb{C}^d)$ in an obvious way. Note that $E_v$ may take the value $-\infty$. We define

$$\mathcal{E}_P^1(\mathbb{C}^d) := \{ u \in L_P(\mathbb{C}^d) : E_{H_P}(u) > -\infty \}.$$ 

We observe that in the above definition we can replace $E_{H_P}$ by $E_{\Phi_P}$, since for $u \in L_{P,+}(\mathbb{C}^d)$, by the cocycle property (cf. Proposition 3.3 [2]),

$$E_{H_P}(u) - E_{H_P}(\Phi_P) = E_{\Phi_P}(u).$$

We thus have the following important identification (see (2.4)):

(2.8)  \hspace{1cm} u \in \mathcal{E}_P^1(\mathbb{C}^d) \iff \tilde{u} \in \mathcal{E}_P^1(\mathbb{P}^d, \omega, \tilde{\Phi}_P).$$

We then have the following local version of Proposition 2.5:

**Proposition 2.11.** Let $u \in L_P(\mathbb{C}^d)$. Then $u \in \mathcal{E}_P^1(\mathbb{C}^d)$ iff $u \in \mathcal{E}_P(\mathbb{C}^d)$ and $\int_{\mathbb{C}^d} (u - H_P)(dd^c u)^d > -\infty$. In particular, if $supp(dd^c u)^d$ is compact, $u \in \mathcal{E}_P^1(\mathbb{C}^d)$ iff $\int_{\mathbb{C}^d} (dd^c u)^d = \gamma_d$ and $\int_{\mathbb{C}^d} u (dd^c u)^d > -\infty$.

**Proof.** Since $\tilde{H}_P \simeq \tilde{\Phi}_P$,

$$\int_{\mathbb{P}^d} (\tilde{u} - \tilde{H}_P) \omega_{\tilde{u}}^d > -\infty \iff \int_{\mathbb{P}^d} (\tilde{u} - \tilde{\Phi}_P) \omega_{\tilde{u}}^d > -\infty$$

where $\tilde{u} \in PSH(\mathbb{P}^d, \omega)$ and $u$ are related by (2.6). Moreover, $\Phi_P \in L_{P,+}(\mathbb{C}^d)$ implies $u \leq \Phi_P + C$ so that $\tilde{u} \in PSH(\mathbb{P}^d, \omega, \tilde{\Phi}_P)$. But

$$\int_{\mathbb{P}^d} (\tilde{u} - \tilde{H}_P) \omega_{\tilde{u}}^d = \int_{\mathbb{C}^d} (u - H_P)(dd^c u)^d$$

and the result follows from (2.8) by applying Proposition 2.5 to $\tilde{u}$. For the last statement, note that for general $u \in L_P(\mathbb{C}^d)$ we may have $\int_{\mathbb{C}^d} H_P(dd^c u)^d = +\infty$, but if $(dd^c u)^d$ has compact support then $\int_{\mathbb{C}^d} H_P(dd^c u)^d$ is finite. \(\square\)

Note that Theorem 2.1 and Proposition 2.3 give the following result:

**Theorem 2.12.** Let $u_1, \ldots, u_d$ be functions in $\mathcal{E}_P(\mathbb{C}^d)$. Then

$$\int_{\mathbb{C}^d} dd^c u_1 \wedge \ldots \wedge dd^c u_d = \gamma_d.$$ 

For $u_1, \ldots, u_n \in L_{P,+}(\mathbb{C}^d)$ Theorem 2.12 was proved in [1, Proposition 2.7].

Having the correspondence (2.8) we can state a local version of Theorem 2.8; this will be used in the sequel. Let $\mathcal{M}_P(\mathbb{C}^d)$ denote the set of all positive Borel measures $\mu$ on $\mathbb{C}^d$ with $\mu(\mathbb{C}^d) = d!Vol(P) = \gamma_d$. 


Theorem 2.13. Assume that $\mu \in \mathcal{M}_P(\mathbb{C}^d)$ is a positive nonpluripolar Borel measure. The following are equivalent

(1) $\mathcal{E}^1_P(\mathbb{C}^d) \subset L^1(\mathbb{C}^d, \mu)$;
(2) there exists $u \in \mathcal{E}^1_P(\mathbb{C}^d)$ such that $(dd^c u)^d = \mu$;
(3) there exists $u \in \mathcal{E}^1_P(\mathbb{C}^d)$ such that

$$\mathcal{F}_{\mu}(u) = \max_{v \in \mathcal{E}^1_P(\mathbb{C}^d)} \mathcal{F}_{\mu}(v) < +\infty.$$ 

A priori the functional $\mathcal{F}_{\mu}$ is defined for $u \in \mathcal{E}^1_P(\mathbb{C}^d)$ by

$$\mathcal{F}_{\mu, \Phi_P}(u) := E_{\Phi_P}(u) - \int_{\mathbb{C}^d} (u - \Phi_P) d\mu.$$ 

However, using this notation, since

$$\mathcal{F}_{\mu, \Phi_P}(u) - \mathcal{F}_{\mu, H_P}(u) = \mathcal{F}_{\mu, \Phi_P}(H_P),$$ 

in statement (3) of Theorem 2.13 we can take either of the two definitions $\mathcal{F}_{\mu, \Phi_P}$ or $\mathcal{F}_{\mu, H_P}$ for $\mathcal{F}_{\mu}$.

Remark 2.14. If $\mu$ has compact support in $\mathbb{C}^d$ then $\int_{\mathbb{C}^d} \Phi_P d\mu$ and $\int_{\mathbb{C}^d} H_P d\mu$ are finite. Therefore, the functional $\mathcal{F}_{\mu}$ can be replaced by

$$u \mapsto E_{H_P}(u) - \int_{\mathbb{C}^d} u d\mu.$$ 

Using the remark, for $\mu \in \mathcal{M}_P(\mathbb{C}^d)$ with compact support, it is natural to define the Legendre-type transform of $E_{H_P}$:

$$E^*(\mu) := \sup_{u \in \mathcal{E}^1_P(\mathbb{C}^d)} [E_{H_P}(u) - \int_{\mathbb{C}^d} u d\mu].$$

This functional, which will appear in the rate function for our LDP, will be given a more concrete interpretation using $P-$pluripotential theory in section 4; cf., equation (4.18).

Finally, for future use, we record the following consequence of Lemma 2.6 and the correspondence (2.8).

Lemma 2.15. If $E \subset \mathbb{C}^d$ is pluripolar then there exists $u \in \mathcal{E}^1_P(\mathbb{C}^d)$ such that $E \subset \{u = -\infty\}$. 

3. $P$–pluripotential theory notions

Given $E \subset \mathbb{C}^d$, the $P$–extremal function of $E$ is

$$V_{P,E}^*(z) := \limsup_{\zeta \to z} V_{P,E}(\zeta)$$

where

$$V_{P,E}(z) := \sup\{u(z) : u \in L_P(\mathbb{C}^d), \ u \leq 0 \text{ on } E\}.$$ 

For $K \subset \mathbb{C}^d$ compact, $w:K \to \mathbb{R}^+$ is an admissible weight function on $K$ if $w \geq 0$ is an uppersemicontinuous function with $\{z \in K : w(z) > 0\}$ nonpluripolar. Setting $Q := -\log w$, we write $Q \in \mathcal{A}(K)$ and define the weighted $P$–extremal function

$$V_{P,K,Q}^*(z) := \limsup_{\zeta \to z} V_{P,K,Q}(\zeta)$$

where

$$V_{P,K,Q}(z) := \sup\{u(z) : u \in L_P(\mathbb{C}^d), \ u \leq Q \text{ on } K\}.$$ 

If $Q=0$ we write $V_{P,K,Q}=V_{P,K}$, consistent with the previous notation. For $P=\Sigma$,

$$V_{\Sigma,K,Q}(z) = V_{K,Q}(z) := \sup\{u(z) : u \in L(\mathbb{C}^d), \ u \leq Q \text{ on } K\}$$

is the usual weighed extremal function as in Appendix B of [19].

We write (omitting the dependence on $P$)

$$\mu_{K,Q} := (dd^c V_{P,K,Q}^*)^d \text{ and } \mu_K := (dd^c V_{P,K}^*)^d$$

for the Monge-Ampère measures of $V_{P,K,Q}^*$ and $V_{P,K}^*$ (the latter if $K$ is not pluripolar). Proposition 2.5 of [2] states that

$$\text{supp}(\mu_{K,Q}) \subset \{z \in K : V_{P,K,Q}^*(z) \geq Q(z)\}$$

and $V_{P,K,Q}^* = Q$ q.e. on $\text{supp}(\mu_{K,Q})$, i.e., off of a pluripolar set.

3.1. Energy

We recall some results and definitions from [2]. For $u,v \in L_{P,+}(\mathbb{C}^d)$, we define the mutual energy

$$\mathcal{E}(u,v) := \int_{\mathbb{C}^d} (u-v) \sum_{j=0}^d (dd^c u)^j \wedge (dd^c v)^{d-j}.$$
For simplicity, when \( v = H_P \), we denote the associated (normalized) energy functional by \( E \):

\[
E(u) := E_H(u) = \frac{1}{d+1} \sum_{j=0}^{d} \int_{\mathbb{C}^d} (u - H_P) dd^c v^j \wedge (dd^c H_P)^{d-j}
\]

(recall (2.7)).

For \( u, u', v \in L_P(C^d) \), and for \( 0 \leq t \leq 1 \), we define

\[
f(t) := E(u + t(u' - u), v),
\]

From Proposition 3.1 in [2], \( f'(t) \) exists for \( 0 \leq t \leq 1 \) and

\[
f'(t) = (d+1) \int_{\mathbb{C}^d} (u' - u)(dd^c(u + t(u' - u)))^d
\]

Hence, taking \( v = H_P \), we have, for \( F(t) := E(u + t(u' - u)) \), that

\[
F'(t) = \int_{\mathbb{C}^d} (u' - u)(dd^c(u + t(u' - u)))^d.
\]

Thus \( F'(0) = \int_{\mathbb{C}^d} (u' - u)(dd^c u)^d \) and we write

\[
(3.1) \quad <E'(u), u' - u> := \int (u' - u)(dd^c u)^d.
\]

We need some applications of a global domination principle. The following version, sufficient for our purposes, follows from [11], Corollary 3.10 (see also Corollary A.2 of [8]).

**Proposition 3.1.** Let \( u \in L_P(C^d) \) and \( v \in E_P(C^d) \) with \( u \leq v \) a.e. \( (dd^c v)^d \). Then \( u \leq v \) in \( C^d \).

This will be used to prove an approximation result, Proposition 3.3, which itself will be essential in the sequel. First we need a lemma.

**Lemma 3.2.** Assume that \( \varphi \leq u, v \leq H_P \) are functions in \( E^1_P(C^d) \). Then for all \( t > 0 \),

\[
\int_{\{u \leq H_P - 2t\}} (H_P - u)(dd^c v)^d \leq 2^{d+1} \int_{\{\varphi \leq H_P - t\}} (H_P - \varphi)(dd^c \varphi)^d.
\]

In particular, the left hand side converges to 0 as \( t \to +\infty \) uniformly in \( u, v \).
Proof. For $s > 0$, we have the following inclusions of sets:

$$(u \leq H_{P} - 2s) \subset \left( \varphi \leq \frac{v + H_{P}}{2} - s \right) \subset (\varphi \leq H_{P} - s).$$

We first note that the left hand side in the lemma is equal to

$$\int_{\{u \leq H_{P} - 2t\}} (H_{P} - u)(dd^{c}v)^{d}$$

(3.2)

$$= 2t \int_{\{u \leq H_{P} - 2t\}} (dd^{c}v)^{d} + \int_{2t}^{\infty} \left( \int_{\{u \leq H_{P} - s\}} (dd^{c}v)^{d} \right) ds.$$

We claim that, for all $s > 0$,

$$\int_{\{u \leq H_{P} - 2s\}} (dd^{c}v)^{d} \leq 2^{d} \int_{\{\varphi \leq H_{P} - s\}} (dd^{c}\varphi)^{d}. \quad (3.3)$$

Indeed, the comparison principle ([11, Corollary 3.6]) and the inclusions of sets above give

$$\int_{\{u \leq H_{P} - 2s\}} (dd^{c}v)^{d} \leq \int_{\{\varphi \leq \frac{v + H_{P}}{2} - s\}} (dd^{c}v)^{d} \leq 2^{d} \int_{\{\varphi \leq \frac{v + H_{P}}{2} - s\}} \left( dd^{c}\left( \frac{v + H_{P}}{2} \right) \right)^{d}$$

$$\leq 2^{d} \int_{\{\varphi \leq \frac{v + H_{P}}{2} - s\}} (dd^{c}\varphi)^{d} \leq 2^{d} \int_{\{\varphi \leq H_{P} - s\}} (dd^{c}\varphi)^{d}.$$ 

The claim is proved. Using (3.3) and (3.2) we obtain

$$\int_{\{u \leq H_{P} - 2t\}} (H_{P} - u)(dd^{c}v)^{d}$$

$$\leq 2^{d+1} t \int_{\{\varphi \leq H_{P} - t\}} (dd^{c}\varphi)^{d} + 2^{d+1} \int_{t}^{\infty} \left( \int_{\{\varphi \leq H_{P} - s\}} (dd^{c}\varphi)^{d} \right) ds$$

$$= 2^{d+1} \int_{\{\varphi \leq H_{P} - t\}} (H_{P} - \varphi)(dd^{c}\varphi)^{d}. \quad \Box$$

**Proposition 3.3.** Let $u \in E_{P}^{1}(\mathbb{C}^{d})$ with $(dd^{c}u)^{d} = \mu$ having support in a non-pluripolar compact set $K$ so that $\int_{K} u d\mu > -\infty$ from Proposition 2.11. Let $\{Q_{j}\}$ be a sequence of continuous functions on $K$ decreasing to $u$ on $K$. Then $u_{j} := V_{P,K,Q_{j}}^{*} u$ on $\mathbb{C}^{d}$ and $\mu_{j} := (dd^{c}u_{j})^{d}$ is supported in $K$. In particular, $\mu_{j} \rightarrow \mu = (dd^{c}u)^{d}$ weak-*.

Moreover,

$$\lim_{j \rightarrow \infty} \int_{K} Q_{j} d\mu_{j} = \lim_{j \rightarrow \infty} \int_{K} Q_{j} d\mu = \int_{K} u d\mu > -\infty. \quad (3.4)$$
Proof. We can assume \( \{Q_j\} \) are defined and decreasing to \( u \) on the closure of a bounded open neighborhood \( \Omega \) of \( K \). By adding a negative constant we can assume that \( Q_1 \leq 0 \) on \( \Omega \). Since \( \{Q_j\} \) is decreasing, so is the sequence \( \{u_j\} \). Moreover, by \cite[Proposition 5.1]{4} \( u_j \leq Q_j \) on \( K \setminus E_j \) where \( E_j \) is pluripolar. But \( u \) is a competitor in the definition of \( V_{P,K,Q} \), so that \( u \leq u_j \) on \( K \setminus E_j \). Thus \( \bar{u} := \lim_{j \to \infty} u_j \geq u \) everywhere and \( \bar{u} \leq u \) on \( K \setminus E \), where \( E := \bigcup_j E_j \) is a pluripolar set. Since \( (dd^c u)^d \) puts no mass on pluripolar sets,

\[
\int_{\{u < \bar{u}\}} (dd^c u)^d \leq \int_{E \cup (\mathbb{C}^d \setminus K)} (dd^c u)^d = 0.
\]

It thus follows from Proposition 3.1 that \( \bar{u} \leq u \), hence \( \bar{u} = u \) on \( \mathbb{C}^d \). The second equality in (3.4) follows from the monotone convergence theorem. It remains to prove that

\[
\lim_{j \to \infty} \int_K (-Q_j) \, d\mu_j = \int_K (-u) \, d\mu.
\]

For each \( k \) fixed and \( j \geq k \) we have

\[
\int_K (-Q_j) \, d\mu_j \geq \int_K (-Q_k) \, d\mu_j = \int_{\Omega} (-Q_k) \, d\mu_j,
\]

hence \( \liminf_{j \to \infty} \int_K (-Q_j) \, d\mu_j \geq \int_K (-Q_k) \, d\mu \) since \( \Omega \) is open and \( \mu_j, \mu \) are supported on \( K \). Letting \( k \to +\infty \) we arrive at

\[
\liminf_{j \to \infty} \int_K (-Q_j) \, d\mu_j \geq \int_K (-u) \, d\mu.
\]

It remains to prove that

\[
\limsup_{j \to \infty} \int_K (-Q_j) \, d\mu_j \leq \int_K (-u) \, d\mu.
\]

The sequence \( \{u_j\} \) is not necessarily uniformly bounded below on \( K \). However, using the facts that \( Q_j \geq u \) and \( H_P \) is continuous in \( \mathbb{C}^d \), it suffices to prove that

\[
(3.5) \quad \limsup_{j \to \infty} \int_K (H_P - u) (dd^c u_j)^d \leq \int_K (H_P - u) (dd^c u)^d.
\]

To verify (3.5), we use Lemma 3.2.

By adding a negative constant we can assume that \( u_j \leq H_P \). For a function \( v \) and for \( t > 0 \) we define \( v^t := \max(v, H_P - t) \). Note that for each \( t \) the sequence \( \{u_j^t\} \) is locally uniformly bounded below. Define

\[
a(t) := 2^{d+1} \int_{\{u \leq H_P - t/2\}} (H_P - u) (dd^c u)^d.
\]
Since \( u \in \mathcal{E}_P^1(\mathbb{C}^d) \), from Proposition 2.11 we have \( a(t) \to 0 \) as \( t \to +\infty \). By Lemma 3.2 we have

\[
(3.6) \quad \sup_{j \geq 1} \int_{\{u \leq H_P - t\}} (H_P - u)(dd^c u_j)^d \leq a(t).
\]

By the plurifine property of non-pluripolar Monge-Ampère measures [10, Proposition 1.4] and (3.6) we have

\[
\int_{K} (H_P - u)(dd^c u_j)^d \leq \int_{K \cap \{u > H_P - t\}} (H_P - u)(dd^c u_j)^d + a(t) \\
= \int_{K \cap \{u > H_P - t\}} (H_P - u^t)(dd^c u^t_j)^d + a(t) \\
\leq \int_{K} (H_P - u^t)(dd^c u^t_j)^d + a(t).
\]

Since \( H_P \) is bounded in \( \Omega \), it follows from [16, Theorem 4.26] that the sequence of positive Radon measures \((H_P - u^t)(dd^c u^t_j)^d\) converges weakly on \( \Omega \) to \((H_P - u^t)(dd^c u^t)^d\). Since \( K \) is compact it then follows that

\[
\limsup_j \int_{K} (H_P - u)(dd^c u_j)^d \leq \int_{K} (H_P - u^t)(dd^c u^t)^d + a(t).
\]

We finally let \( t \to +\infty \) to conclude the proof in the following manner:

\[
\int_{K} (H_P - u^t)(dd^c u^t)^d \leq \int_{K \cap \{u > H_P - t\}} (H_P - u^t)(dd^c u^t)^d + a(t) \\
\leq \int_{K} (H_P - u)(dd^c u)^d + a(t),
\]

where in the first estimate we have used \( \{u \leq H_P - t\} = \{u^t \leq H_P - t\} \) and Lemma 3.2 and in the last estimate we use again the plurifine property.  

We now give an alternate description of the Legendre-type transform \( E^* \) from (2.9) which will be related to the rate function in a large deviation principle. Given \( K \subset \mathbb{C}^d \) compact, we let \( \mathcal{M}_P(K) \) denote the space of positive measures on \( K \) of total mass \( \gamma_d \) and we let \( C(K) \) denote the set of continuous, real-valued functions on \( K \).

**Proposition 3.4.** Let \( K \) be a nonpluripolar compact set and \( \mu \in \mathcal{M}_P(K) \). Then

\[
E^*(\mu) = \sup_{v \in C(K)} \left[ E(V^*_P, K, v) - \int_K v \, d\mu \right].
\]
Proof. We first treat the case when $E^*(\mu)=+\infty$. By Theorem 2.13 there exists $u \in \mathcal{E}_P^1(\mathbb{C}^d)$ such that $\int_K u \, d\mu = -\infty$. We take a decreasing sequence $Q_j \in C(K)$ such that $Q_j \downarrow u$ on $K$ and set $u_j := V_{P,K,Q_j}^*$. Then $\{u_j\}$ are decreasing; since $u \in \mathcal{E}_P^1(\mathbb{C}^d)$ and $E$ is non-decreasing, $\{E(u_j)\}$ is uniformly bounded and we obtain

$$E(V_{P,K,Q_j}^*) - \int_K Q_j \, d\mu \to +\infty,$$

proving the proposition in this case.

Assume now that $E^*(\mu)<+\infty$. Theorem 2.13 ensures that $\int_{\mathbb{C}^d} u \, d\mu > -\infty$ for all $u \in \mathcal{E}_P^1(\mathbb{C}^d)$. By Lemma 2.15, $\mu$ puts no mass on pluripolar sets. From monotonicity of $E$ and the definition of $E^*$ in (2.9) we have

$$E^*(\mu) \geq \sup_{v \in C(K)} [E(V_{P,K,v}^*) - \int_K v \, d\mu].$$

Here we have used that

$$V_{P,K,v}^* \leq v \text{ q.e. on } K \text{ for } v \in C(K).$$

For the reverse inequality, fix $u \in \mathcal{E}_P^1(\mathbb{C}^d)$. Let $\{Q_j\}$ be a sequence of continuous functions on $K$ decreasing to $u$ on $K$ and set $u_j := V_{P,K,Q_j}^*$. Given $\varepsilon > 0$, we can choose $j$ sufficiently large so that, by monotone convergence,

$$\int_K Q_j \, d\mu \leq \int_K u \, d\mu + \varepsilon;$$

and, by monotonicity of $E$,

$$E(V_{P,K,Q_j}^*) \geq E(u).$$

Hence

$$E(V_{P,K,Q_j}^*) - \int_K Q_j \, d\mu \geq E(u) - \int_K u \, d\mu - \varepsilon$$

so that

$$\sup_{v \in C(K)} [E(V_{P,K,v}^*) - \int_K v \, d\mu] \geq E^*(\mu)$$

and equality holds. □
3.2. Transfinite diameter

Let $d_n=d_n(P)$ denote the dimension of the vector space $\text{Poly}(nP)$. We write

$$\text{Poly}(nP) = \text{span}\{e_1, \ldots, e_{d_n}\}$$

where $\{e_j(z)=z^{\alpha(j)}\}_{j=1,\ldots,d_n}$ are the standard basis monomials. Given $\zeta_1, \ldots, \zeta_{d_n} \in \mathbb{C}^d$, let

$$V DM(\zeta_1, \ldots, \zeta_{d_n}) := \det[e_i(\zeta_j)]_{i,j=1,\ldots,d_n}$$

(3.7)

$$= \det \begin{bmatrix}
  e_1(\zeta_1) & e_1(\zeta_2) & \cdots & e_1(\zeta_{d_n}) \\
  \vdots & \vdots & \ddots & \vdots \\
  e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \cdots & e_{d_n}(\zeta_{d_n})
\end{bmatrix}$$

and for $K \subset \mathbb{C}^d$ compact let

$$V_n = V_n(K) := \max_{\zeta_1, \ldots, \zeta_{d_n} \in K} |V DM(\zeta_1, \ldots, \zeta_{d_n})|.$$  

It was shown in [2] that

(3.8) $$\delta(K) := \delta(K, P) := \lim_{n \to \infty} V_n^{1/l_n}$$

exists where

$$l_n := \sum_{j=1}^{d_n} \deg(e_j) = \sum_{j=1}^{d_n} |\alpha(j)|$$

is the sum of the degrees of the basis monomials for $\text{Poly}(nP)$. We call $\delta(K)$ the $P$–transfinite diameter of $K$. More generally, for $w$ an admissible weight function on $K$ and $\zeta_1, \ldots, \zeta_{d_n} \in K$, let

$$V DM_n^Q(\zeta_1, \ldots, \zeta_{d_n}) := V DM(\zeta_1, \ldots, \zeta_{d_n}) w(\zeta_1) \cdots w(\zeta_{d_n})$$

(3.9)

$$= \det \begin{bmatrix}
  e_1(\zeta_1) & e_1(\zeta_2) & \cdots & e_1(\zeta_{d_n}) \\
  \vdots & \vdots & \ddots & \vdots \\
  e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \cdots & e_{d_n}(\zeta_{d_n})
\end{bmatrix} \cdot w(\zeta_1) \cdots w(\zeta_{d_n})$$

be a weighted Vandermonde determinant. Let

$$W_n(K) := \max_{\zeta_1, \ldots, \zeta_{d_n} \in K} |V DM_n^Q(\zeta_1, \ldots, \zeta_{d_n})|.$$  

An $n$–th weighted $P$–Fekete set for $K$ and $w$ is a set of $d_n$ points $\zeta_1, \ldots, \zeta_{d_n} \in K$ with the property that

$$|V DM_n^Q(\zeta_1, \ldots, \zeta_{d_n})| = W_n(K).$$
The limit
\[ \delta^Q(K) := \delta^Q(K, P) := \lim_{n \to \infty} W_n(K)^{1/l_n} \]
exists and is called the \textit{weighted} $P-$\textit{transfinite diameter}. The following was proved in \cite{2}.

\textbf{Theorem 3.5. (Asymptotic Weighted $P-$Fekete Measures)} Let $K \subset \mathbb{C}^d$ be compact with admissible weight $w$. For each $n$, take points $z_1^{(n)}, z_2^{(n)}, \ldots, z_{d_n}^{(n)} \in K$ for which

\[ \lim_{n \to \infty} \left[ |VDM_n^Q(z_1^{(n)}, \ldots, z_{d_n}^{(n)})| \right]^\frac{1}{l_n} = \delta^Q(K) \]

(\textit{asymptotically weighted} $P-$\textit{Fekete arrays}) and let $\mu_n := \frac{1}{d_n} \sum_{j=1}^{d_n} \delta_{z_j^{(n)}}$. Then

\[ \mu_n \rightharpoonup \frac{1}{\gamma_d} \mu_{K,Q} \text{ weak-}. \]

Another ingredient we will use is a Rumely-type relation between transfinite diameter and energy of $V^*_P,K,Q$ from \cite{2}.

\textbf{Theorem 3.6.} Let $K \subset \mathbb{C}^d$ be compact and $w = e^{-Q}$ with $Q \in C(K)$. Then

\[ \log \delta^Q(K) = \frac{-1}{\gamma_d dA} E(V^*_P,K,Q, H_P) = \frac{-(d+1)}{\gamma_d dA} E(V^*_P,K,Q). \]

Here $A = A(P,d)$ was defined in \cite{2}; we recall the definition. For $P = \Sigma$ so that $Poly(n \Sigma) = \mathcal{P}_n$, we have

\[ d_n(\Sigma) = \binom{d+n}{d} = 0(n^d/d!) \text{ and } l_n(\Sigma) = \frac{d}{d+1} n d_n(\Sigma). \]

For a convex body $P \subset (\mathbb{R}^+)^d$, define $f_n(d)$ by writing

\[ l_n = f_n(d) \frac{nd}{d+1} d_n = f_n(d) \frac{l_n(\Sigma)}{d_n(\Sigma)} d_n. \]

Then the ratio $l_n/d_n$ divided by $l_n(\Sigma)/d_n(\Sigma)$ has a limit; i.e.,

\[ \lim_{n \to \infty} f_n(d) =: A = A(P,d). \]
3.3. Bernstein-Markov

For $K \subset \mathbb{C}^d$ compact, $w = e^{-Q}$ an admissible weight function on $K$, and $\nu$ a finite measure on $K$, we say that the triple $(K, \nu, Q)$ satisfies a weighted Bernstein-Markov property if for all $p_n \in \mathcal{P}_n,$

$$\|w^n p_n\|_K \leq M_n \|w^n p_n\|_{L^2(\nu)} \text{ with } \limsup_{n \to \infty} M_n^{1/n} = 1.$$  

(3.13)

Here, $\|w^n p_n\|_K := \sup_{z \in K} |w(z)^n p_n(z)|$ and

$$\|w^n p_n\|_{L^2(\nu)}^2 := \int_K |p_n(z)|^2 w(z)^{2n} d\nu(z).$$

Following [1], given $P \subset (\mathbb{R}^+)^d$ a convex body, we say that a finite measure $\nu$ with support in a compact set $K$ is a Bernstein-Markov measure for the triple $(P, K, Q)$ if (3.13) holds for all $p_n \in \text{Poly}(nP)$.

For any $P$ there exists $A = A(P) > 0$ with $\text{Poly}(nP) \subset \mathcal{P}_A n$ for all $n$. Thus if $(K, \nu, Q)$ satisfies a weighted Bernstein-Markov property, then $\nu$ is a Bernstein-Markov measure for $(P, K, \tilde{Q})$ where $\tilde{Q} = AQ$. In particular, if $\nu$ is a strong Bernstein-Markov measure for $K$; i.e., if $\nu$ is a weighted Bernstein-Markov measure for any $Q \in C(K)$, then for any such $Q$, $\nu$ is a Bernstein-Markov measure for the triple $(P, K, Q)$. Strong Bernstein-Markov measures exist for any nonpluripolar compact set; cf., Corollary 3.8 of [9]. The paragraph following this corollary gives a sufficient mass-density type condition for a measure to be a strong Bernstein-Markov measure.

Given $P$, for $\nu$ a finite measure on $K$ and $Q \in A(K)$, define

$$Z_n := Z_n(P, K, Q, \nu) := \int_K \cdots \int_K |VDM^n_Q(z_1, \ldots, z_{d_n})|^2 d\nu(z_1) \cdots d\nu(z_{d_n}).$$

(3.14)

The main consequence of using a Bernstein-Markov measure for $(P, K, Q)$ is the following:

**Proposition 3.7.** Let $K \subset \mathbb{C}^d$ be a compact set and let $Q \in A(K)$. If $\nu$ is a Bernstein-Markov measure for $(P, K, Q)$ then

$$\lim_{n \to \infty} \frac{1}{n} \log Z_n^{1/n} = \delta^Q(K).$$

(3.15)

**Proof.** That $\limsup_{n \to \infty} \frac{1}{n} \log Z_n^{1/n} \leq \delta^Q(K)$ is clear. Observing from (3.7) and (3.9) that, fixing all variables but $z_j$,

$$z_j \mapsto VDM^n_Q(z_1, \ldots, z_j, \ldots, z_{d_n}) = w(z_j)^n p_n(z_j)$$

for some $p_n \in \text{Poly}(nP)$, to show $\liminf_{n \to \infty} \frac{1}{n} \log Z_n^{1/n} \geq \delta^Q(K)$ one starts with an $n-$th weighted $P-$Fekete set for $K$ and $w$ and repeatedly applies the weighted Bernstein-Markov property. $\Box$
Recall $\mathcal{M}_P(K)$ is the space of positive measures on $K$ with total mass $\gamma_d$. With the weak-* topology, this is a separable, complete metrizable space. A neighborhood basis of $\mu \in \mathcal{M}_P(K)$ can be given by sets

$$G(\mu, k, \varepsilon) := \{\sigma \in \mathcal{M}_P(K) : |\int_K \left(\text{Re}z\right)^\alpha \left(\text{Im}z\right)^\beta (d\mu - d\sigma)| < \varepsilon \}$$

for $0 \leq |\alpha| + |\beta| \leq k$

(3.16)

where $\text{Re} = (\text{Re}z_1, ..., \text{Re}z_n)$ and $\text{Im} = (\text{Im}z_1, ..., \text{Im}z_n)$.

Given $\nu$ as in Proposition 3.7, we define a probability measure $\text{Prob}_n$ on $K^{d_n}$ via, for a Borel set $A \subset K^{d_n}$,

$$\text{Prob}_n(A) := \frac{1}{Z_n} \cdot \int_A |VD M^Q_n(z_1, ..., z_{d_n})|^2 \cdot d\nu(z_1) ... d\nu(z_{d_n}).$$

(3.17)

We immediately obtain the following:

**Corollary 3.8.** Let $\nu$ be a Bernstein-Markov measure for $(P, K, Q)$. Given $\eta > 0$, define

$$A_{n, \eta} := \{(z_1, ..., z_{d_n}) \in K^{d_n} : |VD M^Q_n(z_1, ..., z_{d_n})|^2 \geq (\delta Q(K) - \eta)^{2l_n}\}.$$

Then there exists $n^* = n^*(\eta)$ such that for all $n > n^*$,

$$\text{Prob}_n(K^{d_n} \setminus A_{n, \eta}) \leq \left(1 - \frac{\eta}{2\delta Q(K)}\right)^{2l_n}.$$

**Remark 3.9.** Corollary 3.8 was proved in [9], Corollary 3.2, for $\nu$ a probability measure but an obvious modification works for $\nu(K) < \infty$.

Using (3.17), we get an induced probability measure $P$ on the infinite product space of arrays $\chi := \{X = \{x_j^{(n)}\}_{n=1,2,...; j=1,...,d_n} : x_j^{(n)} \in K\}$:

$$(\chi, P) := \prod_{n=1}^{\infty} (K^{d_n}, \text{Prob}_n).$$

**Corollary 3.10.** Let $\nu$ be a Bernstein-Markov measure for $(P, K, Q)$. For $P$-a.e. array $X = \{x_j^{(n)}\} \in \chi$,

$$\nu_n := \frac{1}{d_n} \sum_{j=1}^{d_n} \delta_{x_j^{(n)}} \rightharpoonup \frac{1}{\gamma_d} \mu_{K,Q} \text{ weak-*}.$$
Proof. From Theorem 3.5 it suffices to verify for P-a.e. array \( X = \{ x_j^{(n)} \} \)

\[
\liminf_{n \to \infty} \left( \frac{|VD_{M_n} Q (x_1^{(n)}, \ldots, x_{d_n}^{(n)})|}{n} \right)^{\frac{1}{n}} = \delta^Q (K).
\]

Given \( \eta > 0 \), the condition that for a given array \( X = \{ x_j^{(n)} \} \) we have

\[
\liminf_{n \to \infty} \left( \frac{|VD_{M_n} Q (x_1^{(n)}, \ldots, x_{d_n}^{(n)})|}{n} \right)^{\frac{1}{n}} \leq \delta^Q (K) - \eta
\]

means that \( (x_1^{(n)}, \ldots, x_{d_n}^{(n)}) \in K_{d_n} \setminus A_{n, \eta} \) for infinitely many \( n \). Setting

\[
E_n := \{ X \in \mathcal{X} : (x_1^{(n)}, \ldots, x_{d_n}^{(n)}) \in K_{d_n} \setminus A_{n, \eta} \},
\]

we have

\[
P(E_n) \leq \text{Prob}_n (K_{d_n} \setminus A_{n, \eta}) \leq \left( 1 - \frac{\eta}{2 \delta^Q (K)} \right)^{2t_n}
\]

and \( \sum_{n=1}^{\infty} P(E_n) < +\infty \). By the Borel-Cantelli lemma,

\[
P(\limsup_{n \to \infty} E_n) = P(\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k) = 0.
\]

Thus, with probability one, only finitely many \( E_n \) occur, and (3.19) follows. \( \square \)

The main goal in the rest of the paper is to verify a stronger probabilistic result – a large deviation principle – and to explain this result in \( P \)-pluripotential-theoretic terms.

4. Relation between \( E^* \) and \( J, J^Q \) functionals

We define some functionals on \( \mathcal{M}_P (K) \) using \( L^2 \)-type notions which act as a replacement for an energy functional on measures. Then we show these functionals \( \overline{J} (\mu) \) and \( J (\mu) \) defined using a “lim sup” and a “lim inf” coincide (see Definitions 4.1 and 4.2); this is the essence of our first proof of the large deviation principle, Theorem 5.1. Using Proposition 3.4, we relate this functional with \( E^* \) from (2.9).

Fix a nonpluripolar compact set \( K \) and a strong Bernstein-Markov measure \( \nu \) on \( K \). For simplicity, we normalize so that \( \nu \) is a probability measure. Recall then for any \( Q \in C (K) \), \( \nu \) is a Bernstein-Markov measure for the triple \( (P, K, Q) \). Given \( G \subset \mathcal{M}_P (K) \) open, for each \( s = 1, 2, \ldots \) we set

\[
\tilde{G}_s := \{ \mathbf{a} = (a_1, \ldots, a_s) \in K^s : \frac{\gamma d}{s} \sum_{j=1}^{s} \delta_{a_j} \in G \}.
\]
Define, for $n=1,2,...$,

$$J_n(G) := \left[ \int_{\tilde{G}_{dn}} |VD M_n(a)|^2 d\nu(a) \right]^{1/2n}.$$ 

**Definition 4.1.** For $\mu \in \mathcal{M}_P(K)$ we define

$$\overline{J}(\mu) := \inf_{G \ni \mu} \overline{J}(G) \text{ where } \overline{J}(G) := \limsup_{n \to \infty} J_n(G);$$

$$\underline{J}(\mu) := \inf_{G \ni \mu} \underline{J}(G) \text{ where } \underline{J}(G) := \liminf_{n \to \infty} J_n(G).$$

The infima are taken over all neighborhoods $G$ of the measure $\mu$ in $\mathcal{M}_P(K)$. A priori, $\overline{J}, \underline{J}$ depend on $\nu$. These functionals are nonnegative but can take the value zero. Intuitively, we are taking a “limit” of $L^2(\nu)$ averages of discrete, equally weighted approximants $1 \sum_{j=1}^s \delta_{a_j}$ of $\mu$. An “$L^\infty$” version of $\overline{J}, \underline{J}$ was introduced in [8] where $J_n(G)$ is replaced by

$$W_n(G) := \sup_{a \in \tilde{G}_{dn}} |VD M_n(a)|^{1/ln} \geq J_n(G).$$

The weighted versions of these functionals are defined for $Q \in \mathcal{A}(K)$ using

$$J^Q_n(G) := \left[ \int_{\tilde{G}_{dn}} |VD M_n^Q(a)|^2 d\nu(a) \right]^{1/2n}.$$ 

**Definition 4.2.** For $\mu \in \mathcal{M}_P(K)$ we define

$$\overline{J}^Q(\mu) := \inf_{G \ni \mu} \overline{J}^Q(G) \text{ where } \overline{J}^Q(G) := \limsup_{n \to \infty} J^Q_n(G);$$

$$\underline{J}^Q(\mu) := \inf_{G \ni \mu} \underline{J}^Q(G) \text{ where } \underline{J}^Q(G) := \liminf_{n \to \infty} J^Q_n(G).$$

The uppersemicontinuity of $\overline{J}, \overline{J}^Q, \underline{J}$ and $\underline{J}^Q$ on $\mathcal{M}_P(K)$ (with the weak-* topology) follows as in Lemma 3.1 of [8]. Set

$$b_d = b_d(P) := \frac{d+1}{Ad\gamma_d}.$$ 

**Proposition 4.3.** Fix $Q \in C(K)$. Then

1. $\overline{J}^Q(\mu) \leq \delta^Q(K);$ 
2. $\overline{J}(\mu) = \overline{J}^Q(\mu) \cdot (e^{J K Q d\mu})^{b_d};$
3. $\log \overline{J}(\mu) \leq \inf_{v \in C(K)} [\log \delta^K(1) + b_d \int_K \nu d\mu];$
4. $\log \overline{J}^Q(\mu) \leq \inf_{v \in C(K)} [\log \delta^K(1) + b_d \int_K \nu d\mu] - b_d \int_K Q d\mu.$

Properties (1)-(4) also hold for the functionals $\underline{J}, \underline{J}^Q$. 
Proof. Property (1) follows from
\[ J_n^Q(G) \leq \sup_{a \in \tilde{G}_{d_n}} |VDM_n^Q(a)|^{1/ln} \leq \sup_{a \in K^{d_n}} |VDM_n^Q(a)|^{1/ln}. \]

The proofs of Corollary 3.4, Proposition 3.5 and Proposition 3.6 of [8] work mutatis mutandis to verify (2), (3) and (4). The relevant estimation, replacing the corresponding one which is two lines above equation (3.2) in [8], is, given \( \varepsilon > 0 \), for \( a \in \tilde{G}_{d_n} \),
\[
|VDM_n^Q(a)| e^{\frac{nd_n}{\gamma_d} (\varepsilon - \int_K Q d\mu)} \leq |VDM_n^Q(a)| \leq |VDM_n^Q(a)| e^{\frac{nd_n}{\gamma_d} (\varepsilon + \int_K Q d\mu)}.
\]

To see this, we first recall that
\[ |VDM_n^Q(a)| = |VDM_n^Q(a)| e^{\sum_{j=1}^{d_n} Q(a_j)}. \]

For \( \mu \in \mathcal{M}_P(K) \), \( Q \in C(K) \), \( \varepsilon > 0 \), there exists a neighborhood \( G \) of \( \mu \) in \( \mathcal{M}_P(K) \) with
\[-\varepsilon < \int_K Q d\mu - \frac{\gamma_d}{d_n} \sum_{j=1}^{d_n} Q(a_j) < \varepsilon \]
for \( a \in \tilde{G}_{d_n} \). Plugging this double inequality into the previous equality we get (4.4).

Moreover, from (3.12),
\[
\lim_{n \to \infty} \frac{nd_n}{\gamma_d} l_n = \frac{d+1}{Ad} = b_d \gamma_d
\]
so that \( \frac{nd_n}{\gamma_d} \sim l_n b_d \) as \( n \to \infty \). Taking \( l_n \)-th roots in (4.4) accounts for the factor of \( b_d \) in (2), (3) and (4).

Remark 4.4. The corresponding \( W, W^Q, \hat{W}, \hat{W}^Q \) functionals, defined using (4.2), clearly dominate their “\( J \)” counterparts; e.g., \( W^Q \geq \hat{J}^Q \).

Note that formula (3.11) can be rewritten:
\[
\log \delta^Q(K) = -b_d E(V^*_{P,K,Q}).
\]

Thus the upper bound in Proposition 4.3 (3) becomes
\[
\log \hat{J}^Q(\mu) \leq -b_d \sup_{v \in C(K)} \left[ E(V^*_{P,K,v}) - \int_K v d\mu \right] = -b_d E^*(\mu).
\]

For the rest of section 4 and section 5, we will always assume \( Q \in C(K) \). Theorem 4.5 shows that the inequalities in (3) and (4) are equalities, and that the \( \hat{J}, \hat{J}^Q \) functionals coincide with their \( J, J^Q \) counterparts. The key step in the proof of Theorem 4.5 is to verify this for \( \hat{J}^Q(\mu_{K,v}) \) and \( J^Q_{P}(\mu_{K,v}) \).
Theorem 4.5. Let \( K \subset \mathbb{C}^d \) be a nonpluripolar compact set and let \( \nu \) satisfy a strong Bernstein-Markov property. Fix \( Q \in C(K) \). Then for any \( \mu \in \mathcal{M}_P(K) \),

\[
\log J(\mu) = \log J(\mu) = \inf_{v \in C(K)} \{ \log \delta^v(K) + b_d \int_K v \, d\mu \}
\]

and

\[
\log J^Q(\mu) = \log J^Q(\mu) = \inf_{v \in C(K)} \{ \log \delta^v(K) + b_d \int_K v \, d\mu \} - b_d \int_K Q \, d\mu.
\]

Proof. It suffices to prove (4.8) since (4.9) follows from (2) of Proposition 4.3. We have the upper bound

\[
\log J(\mu) \leq \inf_{v \in C(K)} \{ \log \delta^v(K) + b_d \int_K v \, d\mu \}
\]

from (3); for the lower bound, we consider different cases.

Case I: \( \mu = \mu_{K,v} \) for some \( v \in C(K) \).

We verify that

\[
\log J(\mu_{K,v}) = \log J(\mu_{K,v}) = \log \delta^v(K) + b_d \int_K v \, d\mu_{K,v}
\]

which proves (4.8) in this case.

To prove (4.10), we use the definition of \( J(\mu_{K,v}) \) and Corollary 3.8. Fix a neighborhood \( G \) of \( \mu_{K,v} \). For \( \eta > 0 \), define \( A_{n,\eta} \) as in (3.18) with \( Q = v \). Set

\[
\eta_n := \max \left( \frac{\delta^v(K)}{n Z_n^{1/2} n^{1/2}}, \frac{Z_n^{1/2} n^{1/2}}{n+1} \right).
\]

By Proposition 3.7, \( \eta_n \to 0 \). We claim that we have the inclusion

\[
A_{n,\eta_n} \subset \tilde{G}_{d_n} \quad \text{for all } n \text{ large enough}.
\]

We prove (4.12) by contradiction: if false, there is a sequence \( \{n_j\} \) with \( n_j \to \infty \) and

\[
x^j=(x^j_1, \ldots, x^j_{d_{n_j}}) \in A_{n_j,\eta_{n_j}} \setminus \tilde{G}_{d_{n_j}}.
\]

However \( \mu_j := \frac{\gamma_n}{d_{n_j}} \sum_{i=1}^{d_{n_j}} \delta_{x^j_i} \notin G \) for \( j \) sufficiently large contradicts Theorem 3.5 since \( x^j \in A_{n_j,\eta_{n_j}} \) and \( \eta_{n_j} \to 0 \) imply \( \mu_j \to \mu_{K,v} \) weak-*.

Next, a direct computation using (4.11) shows that, for all \( n \) large enough,

\[
\text{Prob}_n(K^{d_n} \setminus A_{n,\eta_n}) \leq \frac{(\delta^v(K) - \eta_n)^{2/n}}{Z_n} \leq \left( \frac{n}{n+1} \right)^{2/n} \leq \frac{n}{n+1}
\]
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(recall $\nu$ is a probability measure). Hence

$$\frac{1}{Z_n} \int_{\tilde{G}_{d_n}} |VDM_n^\nu(z_1, \ldots, z_{d_n})|^2 \cdot \nu(z_1) \ldots \nu(z_{d_n})$$

$$\geq \frac{1}{Z_n} \int_{A_n, \eta_n} |VDM_n^\nu(z_1, \ldots, z_{d_n})|^2 \cdot \nu(z_1) \ldots \nu(z_{d_n})$$

$$\geq \frac{1}{n+1}.$$ 

Since $P \subset r \Sigma$ and $\Sigma \subset kP$ for some $k \in \mathbb{Z}^+$, $t_n = 0(n^{d+1})$ and we have $\frac{1}{Z_n} \log(n+1) \to 0$. Since $\nu$ satisfies a strong Bernstein-Markov property and $v \in C(K)$, using Proposition 3.7 and the above estimate we conclude that

$$\liminf_{n \to \infty} \frac{1}{2n} \log \int_{\tilde{G}_{d_n}} |VDM_n^\nu(z_1, \ldots, z_{d_n})|^2 \nu(z_1) \ldots \nu(z_{d_n})$$

$$\geq \log \delta^v(K).$$ 

Taking the infimum over all neighborhoods $G$ of $\mu_K, v$ we obtain

$$\log J^v(\mu_K, v) \geq \log \delta^v(K).$$ 

From (1) Proposition 4.3, $\log \overline{J}^v(\mu_K, v) \leq \log \delta^v(K)$; thus we have

$$\log J^v(\mu_K, v) = \log \overline{J}^v(\mu_K, v) = \log \delta^v(K).$$ 

Using (2) of Proposition 4.3 with $\mu = \mu_K, v$ we obtain (4.10).

**Case II:** $\mu \in \mathcal{M}_P(K)$ with the property that $E^*(\mu) < \infty$.

From Theorem 2.13 and Proposition 2.11 there exists $u \in L_P(\mathbb{C}^d)$ – indeed, $u \in \mathcal{E}_{P}^{\mu}(\mathbb{C}^d)$ – with $\mu = (dd^c u)^d$ and $\int_K u \, d\mu > -\infty$. However, since $u$ is only use on $K$, $\mu$ is not necessarily of the form $\mu_{K,v}$ for some $v \in C(K)$. Taking a sequence of continuous functions $\{Q_j\} \subset C(K)$ with $Q_j \downarrow u$ on $K$, by Proposition 3.3 the weighted extremal functions $V^*_P, K, Q_j$ decrease to $u$ on $\mathbb{C}^d$;

$$\mu_j := (dd^c V^*_P, K, Q_j)^d \longrightarrow \mu = (dd^c u)^d \text{ weak-*;}$$

and

$$\lim_{j \to \infty} \int_K Q_j \, d\mu_j = \lim_{j \to \infty} \int_K Q_j \, d\mu = \int_K u \, d\mu.$$ 

From the previous case we have

$$\log \overline{J}(\mu_j) = \log J(\mu_j) = \log \delta^{Q_j}(K) + b_d \int_K Q_j \, d\mu_j.$$
Using uppersemicontinuity of the functional $\mu \to J(\mu)$,
\[
\limsup_{j \to \infty} J(\mu_j) = \limsup_{j \to \infty} J(\mu_j) \leq J(\mu).
\]
Since $Q_j \downarrow u$ on $K$,
\[
(4.16) \quad \limsup_{j \to \infty} \log \delta^{Q_j}(K) = \limsup_{j \to \infty} \log \delta^{Q_j}(K).
\]
Therefore
\[
M := \lim_{j \to \infty} \log J(\mu_j) = \lim_{j \to \infty} \left( \log \delta^{Q_j}(K) + b_d \int_K Q_j \, d\mu_j \right)
\]
exists and is less than or equal to $\log J(\mu)$. We want to show that
\[
(4.17) \quad \inf_v [\log \delta^v(K) + b_d \int_K v \, d\mu] \leq M.
\]
Given $\varepsilon > 0$, by (4.15) for $j \geq j_0(\varepsilon)$,
\[
\int_K Q_j \, d\mu_j \geq \int_K Q_j \, d\mu - \varepsilon \quad \text{and} \quad \log J(\mu_j) < M + \varepsilon.
\]
Hence for such $j$,
\[
\inf_v [\log \delta^v(K) + b_d \int_K v \, d\mu] \leq \log \delta^{Q_j}(K) + b_d \int_K Q_j \, d\mu
\]
\[
\leq \log \delta^{Q_j}(K) + b_d \int_K Q_j \, d\mu_j + b_d \varepsilon
\]
\[
= \log J(\mu_j) + b_d \varepsilon < M + (b_d + 1)\varepsilon,
\]
yielding (4.17). This finishes the proof in Case II.

**Case III:** $\mu \in \mathcal{M}(K)$ with the property that $E^*(\mu) = +\infty$.

It follows from Proposition 3.4 and Theorem 3.6 that the right-hand side of (4.8) is $-\infty$, finishing the proof. □

**Remark 4.6.** From now on, we simply use the notation $J, J^Q$ without the overline or underline. Using Proposition 3.4 and Theorem 3.6, we have
\[
\log J(\mu) = \inf_{Q \in C(K)} [\log \delta^Q(K) + b_d \int_K Q \, d\mu]
\]
\[
= - \sup_{Q \in C(K)} [- \log \delta^Q(K) - b_d \int_K Q \, d\mu]
\]
\[
= - \sup_{Q \in C(K)} [b_d E(V^*_{P,K,Q}) - b_d \int_K Q \, d\mu] = -b_d \sup_{Q \in C(K)} [E(V^*_{P,K,Q}) - \int_K Q \, d\mu]
\]
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(recall (4.6)) which one can compare with

\[ E^*(\mu) = \sup_{Q \in C(K)} \left[ E(V_{P,K,Q}^*) - \int_K Q \, d\mu \right] \]

from Proposition 3.4 to conclude

\[ \log J(\mu) = -b_d E^*(\mu). \]

In particular, \( J, J^Q \) are independent of the choice of strong Bernstein-Markov measure for \( K \).

Following the idea in Proposition 4.3 of [9], we observe the following:

**Proposition 4.7.** Let \( K \subset \mathbb{C}^d \) be a nonpluripolar compact set and let \( \nu \) satisfy a strong Bernstein-Markov property. Fix \( Q \in C(K) \). The measure \( \mu_{K,Q} \) is the unique maximizer of the functional \( \mu \rightarrow J^Q(\mu) \) over \( \mu \in \mathcal{M}_P(K) \); i.e.,

\[ J^Q(\mu_{K,Q}) = \delta^Q(K) \quad \text{(and } J(\mu_K) = \delta(K)). \]

**Proof.** The fact that \( \mu_{K,Q} \) maximizes \( J^Q \) (and \( \mu_K \) maximizes \( J \)) follows from (4.10), (4.14) and Proposition 4.3.

Assume now that \( \mu \in \mathcal{M}_P(K) \) maximizes \( J^Q \). From Remark 4.4 and the definitions of the functionals, for any neighborhood \( G \subset \mathcal{M}_P(K) \) of \( \mu \),

\[ \overline{J^Q}(\mu) \leq \overline{W^Q}(\mu) \leq \sup_{n \to \infty} \left\{ \limsup_n |VD_M^Q(a^{(n)})|^{1/l_n} \right\} \leq \delta^Q(K) \]

where the supremum is taken over all arrays \( \{a^{(n)}\}_{n=1,2,...} \) of \( d_n \)-tuples \( a^{(n)} \) in \( K \) whose normalized counting measures \( \mu_n := \frac{1}{d_n} \sum_{j=1}^{d_n} \delta_{a_j^{(n)}} \) lie in \( G \). Since \( J^Q(\mu) = \delta^Q(K) \) there is an asymptotic weighted Fekete array \( \{a^{(n)}\} \) as in (3.10). Theorem 3.5 yields that \( \mu_n := \frac{1}{d_n} \sum_{j=1}^{d_n} \delta_{a_j^{(n)}} \) converges weak-* to \( \mu_{K,Q} \), hence \( \mu_{K,Q} \in G \). Since this is true for each neighborhood \( G \subset \mathcal{M}_P(K) \) of \( \mu \), we must have \( \mu = \mu_{K,Q} \).

5. Large deviation

As in the previous section, we fix \( K \subset \mathbb{C}^d \) a nonpluripolar compact set; \( Q \in C(K) \); and a measure \( \nu \) on \( K \) satisfying a strong Bernstein-Markov property. For \( x_1, ..., x_{d_n} \in K \), we get a discrete measure \( \frac{\gamma_d}{d_n} \sum_{j=1}^{d_n} \delta_{x_j} \in \mathcal{M}_P(K) \). Define \( j_n : K^{d_n} \to \mathcal{M}_P(K) \) via

\[ j_n(x_1, ..., x_{d_n}) := \frac{\gamma_d}{d_n} \sum_{j=1}^{d_n} \delta_{x_j}. \]
From (3.17), \( \sigma_n := (j_n)_* (\text{Prob}_n) \) is a probability measure on \( \mathcal{M}_P(K) \): for a Borel set \( B \subset \mathcal{M}_P(K) \),

\[
(5.1) \quad \sigma_n(B) = \frac{1}{Z_n} \int_{\tilde{B}_{d_n}} |VDM_n^Q(x_1, \ldots, x_{d_n})|^2 \nu(x_1) \ldots \nu(x_{d_n})
\]

where \( \tilde{B}_{d_n} := \{ a = (a_1, \ldots, a_{d_n}) \in K^{d_n}; \sum_{j=1}^{d_n} a_j \in B \} \) (recall (4.1)). Here, \( Z_n := Z_n(P, K, Q, \nu) \). Note that

\[
(5.2) \quad \sigma_n(B)^{1/2l_n} = \frac{1}{Z_n^{1/2l_n}} J_n^Q(B).
\]

For future use, suppose we have a function \( F: \mathbb{R} \to \mathbb{R} \) and a function \( v \in C(K) \). We write, for \( \mu \in \mathcal{M}_P(K) \),

\[
<v, \mu> := \int_K v \, d\mu
\]

and then

\[
(5.3) \quad \int_{\mathcal{M}_P(K)} F(<v, \mu>) d\sigma_n(\mu)
\]

\[
:= \frac{1}{Z_n} \int_K \ldots \int_K |VDM_n^Q(x_1, \ldots, x_{d_n})|^2 F \left( \frac{\gamma_d}{d_n} \sum_{j=1}^{d_n} v(x_j) \right) \nu(x_1) \ldots \nu(x_{d_n}).
\]

With this notation, we offer two proofs of our LDP, Theorem 5.1. We state the result; define LDP in Definition 5.2; and then proceed with the proofs. This closely follows the exposition in section 5 of [9].

**Theorem 5.1.** The sequence \( \{ \sigma_n = (j_n)_* (\text{Prob}_n) \} \) of probability measures on \( \mathcal{M}_P(K) \) satisfies a large deviation principle with speed \( 2l_n \) and good rate function

\( \mathcal{I} := \mathcal{I}_{K,Q} \) where, for \( \mu \in \mathcal{M}_P(K) \),

\[
\mathcal{I}(\mu) := \log J_n^Q(\mu_{K,Q}) - \log J_n^Q(\mu).
\]

This means that \( \mathcal{I}: \mathcal{M}_P(K) \to [0, \infty] \) is a lowersemicontinuous mapping such that the sublevel sets \( \{ \mu \in \mathcal{M}_P(K); \mathcal{I}(\mu) \leq \alpha \} \) are compact in the weak-* topology on \( \mathcal{M}_P(K) \) for all \( \alpha \geq 0 \) (\( \mathcal{I} \) is “good”) satisfying (5.4) and (5.5):

**Definition 5.2.** The sequence \( \{ \mu_n \} \) of probability measures on \( \mathcal{M}_P(K) \) satisfies a large deviation principle (LDP) with good rate function \( \mathcal{I} \) and speed \( 2l_n \) if for all measurable sets \( \Gamma \subset \mathcal{M}_P(K) \),

\[
(5.4) \quad \inf_{\mu \in \Gamma} \mathcal{I}(\mu) \leq \liminf_{n \to \infty} \frac{1}{2l_n} \log \mu_n(\Gamma) \quad \text{and}
\]
\begin{equation}
\limsup_{n \to \infty} \frac{1}{2l_n} \log \mu_n(\Gamma) \leq - \inf_{\mu \in \Gamma} I(\mu).
\end{equation}

In the setting of $\mathcal{M}_P(K)$, to prove a LDP it suffices to work with a base for the weak-* topology. The following is a special case of a basic general existence result for a LDP given in Theorem 4.1.11 in [13].

**Proposition 5.3.** Let $\{\sigma_\varepsilon\}$ be a family of probability measures on $\mathcal{M}_P(K)$. Let $\mathcal{B}$ be a base for the topology of $\mathcal{M}_P(K)$. For $\mu \in \mathcal{M}_P(K)$ let

\[ I(\mu) := - \inf_{\{G \in \mathcal{B}, \mu \in G\}} \left( \liminf_{\varepsilon \to 0} \varepsilon \log \sigma_\varepsilon(G) \right). \]

Suppose for all $\mu \in \mathcal{M}_P(K)$,

\[ I(\mu) = - \inf_{\{G \in \mathcal{B}, \mu \in G\}} \left( \limsup_{\varepsilon \to 0} \varepsilon \log \sigma_\varepsilon(G) \right). \]

Then $\{\sigma_\varepsilon\}$ satisfies a LDP with rate function $I(\mu)$ and speed $1/\varepsilon$.

There is a converse to Proposition 5.3, Theorem 4.1.18 in [13]. For $\mathcal{M}_P(K)$, it reads as follows:

**Proposition 5.4.** Let $\{\sigma_\varepsilon\}$ be a family of probability measures on $\mathcal{M}_P(K)$. Suppose that $\{\sigma_\varepsilon\}$ satisfies a LDP with rate function $I(\mu)$ and speed $1/\varepsilon$. Then for any base $\mathcal{B}$ for the topology of $\mathcal{M}_P(K)$ and any $\mu \in \mathcal{M}_P(K)$

\[ I(\mu) := - \inf_{\{G \in \mathcal{B}, \mu \in G\}} \left( \liminf_{\varepsilon \to 0} \varepsilon \log \sigma_\varepsilon(G) \right) \\
= - \inf_{\{G \in \mathcal{B}, \mu \in G\}} \left( \limsup_{\varepsilon \to 0} \varepsilon \log \sigma_\varepsilon(G) \right). \]

**Remark 5.5.** Assuming Theorem 5.1, this shows that, starting with a strong Bernstein-Markov measure $\nu$ and the corresponding sequence of probability measures $\{\sigma_n\}$ on $\mathcal{M}_P(K)$ in (5.1), the existence of an LDP with rate function $I(\mu)$ and speed $2l_n$ implies that necessarily

\begin{equation}
I(\mu) = \log J^Q(\mu_{K,Q}) - \log J^Q(\mu).
\end{equation}

Uniqueness of the rate function is basic (cf., Lemma 4.1.4 of [13]).

We turn to the first proof of Theorem 5.1, using Theorem 4.5, which gives a pluripotential theoretic description of the rate functional.
Proof. As a base for the topology of \( \mathcal{M}_P(K) \), we can take the sets from (3.16) or simply all open sets. For \( \{\sigma_\varepsilon\} \), we take the sequence of probability measures \( \{\sigma_n\} \) on \( \mathcal{M}_P(K) \) and we take \( \varepsilon = \frac{1}{2\ln n} \). For \( G \in \mathcal{B} \), from (5.2),

\[
\frac{1}{2\ln n} \log \sigma_n(G) = \log J^Q_n(G) - \frac{1}{2\ln n} \log Z_n.
\]

From Proposition 3.7, and (4.14) with \( v = Q \),

\[
\lim_{n \to \infty} \frac{1}{2\ln n} \log Z_n = \log \delta^Q(K) = \log J^Q_\mu(K,Q);
\]

and by Theorem 4.5,

\[
\inf_{G \ni \mu} \limsup_{n \to \infty} \log J^Q_n(G) = \inf_{G \ni \mu} \liminf_{n \to \infty} \log J^Q_n(G) = \log J^Q_\mu(K,Q).
\]

Thus by Proposition 5.3 \( \{\sigma_n\} \) satisfies an LDP with rate function

\[
\mathcal{I}(\mu) := \log J^Q_\mu(K,Q) - \log J^Q_\mu(K)
\]

and speed \( 2\ln n \). This rate function is good since \( \mathcal{M}_P(K) \) is compact. \( \square \)

Remark 5.6. From Proposition 4.7, \( \mu_{K,Q} \) is the unique maximizer of the functional

\[
\mu \mapsto \log J^Q_\mu(K)
\]

over all \( \mu \in \mathcal{M}_P(K) \). Thus

\[
\mathcal{I}_{K,Q}(\mu) \geq 0 \quad \text{with} \quad \mathcal{I}_{K,Q}(\mu) = 0 \iff \mu = \mu_{K,Q}.
\]

To summarize, \( \mathcal{I}_{K,Q} \) is a good rate function with unique minimizer \( \mu_{K,Q} \). Using the relations

\[
\log J(\mu) = -b_d \sup_{Q \in C(K)} \left[ E(V^*_P,K,Q) - \int_K Q d\mu \right]
\]

\[
J(\mu) = J^Q(\mu) \cdot (e^{\int_K Q d\mu})^{b_d}, \quad \text{and} \quad J^Q(\mu_{K,Q}) = \delta^Q(K)
\]

(the latter from (4.19)), we have

\[
\mathcal{I}(\mu) := \log \delta^Q(K) - \log J^Q(\mu)
\]

\[
= \log \delta^Q(K) - \log J(\mu) + b_d \int_K Q d\mu
\]

\[
= b_d \sup_{Q \in C(K)} \left[ E(V^*_P,K,Q) - \int_K Q d\mu \right] + \log \delta^Q(K) + b_d \int_K Q d\mu
\]

\[
= b_d \sup_{v \in C(K)} \left[ E(V^*_P,K,v) - \int_K v d\mu \right] - b_d [E(V^*_P,K,Q) - \int_K Q d\mu]
\]

from (4.6).
The second proof of our LDP follows from Corollary 4.6.14 in [13], which is a general version of the Gärtner-Ellis theorem. This approach was originally brought to our attention by S. Boucksom and was also utilized by R. Berman in [5]. We state the version of the [13] result for an appropriate family of probability measures.

**Proposition 5.7.** Let $C(K)^*$ be the topological dual of $C(K)$, and let $\{\sigma_\varepsilon\}$ be a family of probability measures on $\mathcal{M}_P(K) \subset C(K)^*$ (equipped with the weak-* topology). Suppose for each $\lambda \in C(K)$, the limit

$$
\Lambda(\lambda) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log \int_{C(K)^*} e^{\lambda(x)/\varepsilon} d\sigma_\varepsilon(x)
$$

exists as a finite real number and assume $\Lambda$ is Gâteaux differentiable; i.e., for each $\lambda, \theta \in C(K)$, the function $f(t) := \Lambda(\lambda + t\theta)$ is differentiable at $t=0$. Then $\{\sigma_\varepsilon\}$ satisfies an LDP in $C(K)^*$ with the convex, good rate function $\Lambda^*$.

Here

$$
\Lambda^*(x) := \sup_{\lambda \in C(K)} (\langle \lambda, x \rangle - \Lambda(\lambda))
$$

is the Legendre transform of $\Lambda$. The upper bound (5.5) in the LDP holds with rate function $\Lambda^*$ under the assumption that the limit $\Lambda(\lambda)$ exists and is finite; the Gâteaux differentiability of $\Lambda$ is needed for the lower bound (5.4). To verify this property in our setting, we must recall a result from [2].

**Proposition 5.8.** For $Q \in \mathcal{A}(K)$ and $u \in C(K)$, let

$$
F(t) := E(V_P^*_t, Q + tu)
$$

for $t \in \mathbb{R}$. Then $F$ is differentiable and

$$
F'(t) = \int_{C^d} u(\text{dd} V_P^*_t, Q + tu)^d.
$$

In [2] it was assumed that $u \in C^2(K)$ but the result is true with the weaker assumption $u \in C(K)$ (cf., Theorem 11.11 in [16] due to Lu and Nguyen [17], see also [11, Proposition 4.20]).

We proceed with the second proof of Theorem 5.1. For simplicity, we normalize so that $\gamma_d=1$ to fit the setting of Proposition 5.7 (so members of $\mathcal{M}_P(K)$ are probability measures).

**Proof.** We show that for each $v \in C(K)$,

$$
\Lambda(v) := \lim_{n \to \infty} \frac{1}{2l_n} \log \int_{C(K)^*} e^{2l_n \langle v, \mu \rangle} d\sigma_n(\mu)
$$
exists as a finite real number. First, since \( \sigma_n \) is a measure on \( \mathcal{M}_P(K) \), the integral can be taken over \( \mathcal{M}_P(K) \). Consider
\[
\frac{1}{2l_n} \log \int_{\mathcal{M}_P(K)} e^{2l_n \langle v, \mu \rangle} d\sigma_n(\mu).
\]
By (5.3), this is equal to
\[
\frac{1}{2l_n} \log \frac{1}{Z_n} \cdot \int_{K^{d_n}} |VD\mathcal{M}_n^{Q-\frac{l_n}{nd_n}v}(x_1, \ldots, x_{dn})|^2 d\nu(x_1) \ldots d\nu(x_{dn}).
\]
From (4.5), with \( \gamma_d = 1, \frac{l_n}{nd_n} \to \frac{1}{bd} \); hence for any \( \epsilon > 0 \),
\[
\frac{1}{bd} + \epsilon \leq \frac{l_n}{nd_n} v \leq \frac{1}{bd} - \epsilon \quad \text{on} \ K
\]
for \( n \) sufficiently large. Recall that
\[
Z_n = \int_{K^{d_n}} |VD\mathcal{M}_n^{Q}(x_1, \ldots, x_{dn})|^2 d\nu(x_1) \ldots d\nu(x_{dn}).
\]
Define
\[
\tilde{Z}_n := \int_{K^{d_n}} |VD\mathcal{M}_n^{Q-v/bd}(x_1, \ldots, x_{dn})|^2 d\nu(x_1) \ldots d\nu(x_{dn}).
\]
Then we have
\[
\lim_{n \to \infty} \frac{\tilde{Z}_n}{Z_n} = \delta^{Q-v/bd}(K) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{Z_n} = \delta^Q(K).
\]
from (3.15) in Proposition 3.7 and the assumption that \( (K, \nu, \tilde{Q}) \) satisfies the weighted Bernstein-Markov property for all \( \tilde{Q} \in C(K) \). Thus
\[
\Lambda(v) = \lim_{n \to \infty} \frac{1}{2l_n} \log \frac{\tilde{Z}_n}{Z_n} = \log \frac{\delta^{Q-v/bd}(K)}{\delta^Q(K)}.
\]
Define now, for \( v, v' \in C(K) \),
\[
f(t) := E(V^*_P,K,Q-(v+tv')).
\]
Proposition 5.8 shows that \( \Lambda \) is Gâteaux differentiable and Proposition 5.7 gives that \( \Lambda^* \) is a rate function on \( C(K)^* \).

Since each \( \sigma_n \) has support in \( \mathcal{M}_P(K) \), it follows from (5.4) and (5.5) in Definition 5.2 of an LDP with \( \Gamma \subset C(K)^* \) that for \( \mu \in C(K)^* \setminus \mathcal{M}_P(K) \), \( \Lambda^*(\mu) = +\infty \). By Lemma 4.1.5 (b) of [13], the restriction of \( \Lambda^* \) to \( \mathcal{M}_P(K) \) is a rate function. Since \( \mathcal{M}_P(K) \) is compact, it is a good rate function. Being a Legendre transform, \( \Lambda^* \) is convex.
To compute $\Lambda^*$, we have, using (5.7) and (3.11),

$$\Lambda^*(\mu) = \sup_{v \in C(K)} \left( \int_K v \, d\mu - \log \frac{\delta^Q - v/b_d}{\delta^Q(K)} \right)$$

$$= \sup_{v \in C(K)} \left( \int_K v \, d\mu - b_d[E(V_{P,K,Q}^* - E(V_{P,K,Q,v}/b_d))] \right).$$

Thus

$$\Lambda^*(\mu) + b_d E(V_{P,K,Q}^*) = \sup_{u \in C(K)} \left( \int_K u \, d\mu + b_d E(V_{P,K,Q,v}^*) \right) - b_d\int_K u \, d\mu \quad \text{(taking } u = -v/b_d).$$

Rearranging and replacing $u$ in the supremum by $v=u+Q$,

$$\Lambda^*(\mu) = \sup_{u \in C(K)} \left( b_d E(V_{P,K,Q,v}^*) - b_d\int_K u \, d\mu \right) - b_dE(V_{P,K,Q}^*)$$

$$= b_d \left[ \sup_{v \in C(K)} E(V_{P,K,v}^*) - \int_K v \, d\mu \right] - b_d \left[ E(V_{P,K,Q}^*) - \int_K Q \, d\mu \right]$$

which agrees with the formula in Remark 5.6 (since $\mu$ is a probability measure). □

**Remark 5.9.** Thus the rate function can be expressed in several equivalent ways:

$$\mathcal{I}(\mu) = \Lambda^*(\mu) = \log J^Q(\mu_{K,Q}) - \log J^Q(\mu)$$

$$= b_d \left[ \sup_{v \in C(K)} E(V_{P,K,v}^*) - \int_K v \, d\mu \right] - b_d \left[ E(V_{P,K,Q}^*) - \int_K Q \, d\mu \right]$$

which generalizes the result equating (5.3), (5.10) and (5.11) in [9] for the case $P=\Sigma$ and $bd=1$. Note in the last equality we are using the slightly different notion of $E^*$ in (2.9) and Proposition 3.4 than that used in [9].

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