1. Definitions and statements

Throughout this paper $K$ is a field, $\bar{K}$ its algebraic closure and $\text{Gal}(K) = \text{Aut}(\bar{K}/K)$ the absolute Galois group of $K$.

If $X$ is an abelian variety over $\bar{K}$ then we write $\text{End}(X)$ for the ring of all its $\bar{K}$-endomorphisms; the notation $1_X$ stands for the identity automorphism of $X$. If $Y$ is an abelian variety over $\bar{K}$ then we write $\text{Hom}(X, Y)$ for the corresponding group of all $\bar{K}$-homomorphisms.

Let $f(x) \in K[x]$ be a polynomial of degree $n \geq 2$ with coefficients in $K$ and without multiple roots, $\mathfrak{R}_f \subset \mathcal{K}$ the $(n$-element) set of roots of $f$ and $K(\mathfrak{R}_f) \subset \mathcal{K}$ the splitting field of $f$. We write $\text{Gal}(f) = \text{Gal}(f/K)$ for the Galois group $\text{Gal}(K(\mathfrak{R}_f)/K)$ and call it the Galois group of $f(x)$ over $K$; it permutes roots of $f$ and may be viewed as a certain permutation group of $\mathfrak{R}_f$, i.e., as a subgroup of the group $\text{Perm}(\mathfrak{R}_f) \cong S_n$ of permutation of $\mathfrak{R}_f$. (It is well known that $\text{Gal}(f)$ is transitive if and only if $f$ is irreducible.) Let us put

$$g = \left\lfloor \frac{n-1}{2} \right\rfloor.$$  

Clearly, $g$ is a nonnegative integer and either $n = 2g + 1$ or $n = 2g + 2$.

Let us assume that $\text{char}(K) \neq 2$. We write $C_f$ for the genus $g$ hyperelliptic $K$-curve $y^2 = f(x)$ and $J(C_f)$ for its jacobian. Clearly, $J(C_f)$ is a $g$-dimensional abelian variety that is defined over $K$. In particular, $J(C_f) = \{0\}$ if and only if $n = 2$. The abelian variety $J(C_f)$ is an elliptic curve if and only if $n = 4$.

Let us assume that $K$ is a subfield of the field $\mathbb{C}$ of complex numbers (and $\bar{K}$ is the algebraic closure of $K$ in $\mathbb{C}$). Then one may view $J(C_f)$ as a complex abelian variety and consider its first rational homology group $H_1(J(C_f), \mathbb{Q})$ and the Hodge group $\text{Hdg}(J(C_f))$ of $J(C_f)$, which is a certain connected reductive algebraic $\mathbb{Q}$-subgroup of the general linear group $\text{GL}(H_1(J(C_f), \mathbb{Q}))$. The canonical principal polarization on $J(C_f)$ gives rise to the nondegenerate alternating bilinear form $H_1(J(C_f), \mathbb{Q}) \times H_1(J(C_f), \mathbb{Q}) \to \mathbb{Q}$ and the corresponding symplectic group $\text{Sp}(H_1(J(C_f), \mathbb{Q}))$ contains $\text{Hdg}(J(C_f))$ as a (closed) algebraic $\mathbb{Q}$-subgroup. In addition, $\text{End}(J(C_f))$ coincides with the endomorphism ring of the complex abelian variety $J(C_f)$ and $\text{End}^0(J(C_f))$ coincides with the centralizer of $\text{Hdg}(J(C_f))$ in $\text{End}_{\mathbb{Q}}(H_1(J(C_f), \mathbb{Q}))$.

The following result was obtained by the author in [19, Th. 2.1], [20, Sect. 10]. (See also [24], [22], [23].)

**Theorem 1.1.** Suppose that $K \subset \mathbb{C}$, $n \geq 5$ (i.e., $g \geq 2$) and $\text{Gal}(f) = S_n$ or the alternating group $A_n$. Then $\text{End}(J(C_f)) = \mathbb{Z}$ and $\text{Hdg}(J(C_f)) = \text{Sp}(H_1(J(C_f), \mathbb{Q}))$. 

\[ \text{Proof.} \]
Every Hodge class on each self-product of $J(C_f)$ can be presented as a linear combination of products of divisor classes. In particular, the Hodge conjecture is valid for each self-product of $J(C_f)$.

**Remark 1.2.** The assertion that $\text{Hdg}(J(C_f)) = \text{Sp}(H_1(J(C_f), \mathbb{Q}))$ was not stated explicitly in [20]. However, it follows immediately from the description of the Lie algebra $\mathfrak{m}_t$ of the corresponding Mumford-Tate group [20, p. 429] as the direct sum of the scalars $\mathbb{Q}\text{Id}$ and the Lie algebra of the symplectic group, because the Lie algebra of the Hodge group coincides with the intersection of Lie algebras of the Mumford-Tate group and the symplectic group. (The same arguments prove the equality $\text{Hdg}(J(C_f)) = \text{Sp}(H_1(J(C_f), \mathbb{Q}))$ for all $f(x)$ that satisfy the conditions of Theorem 10.1 of [20].)

Our next result that was obtained in [21, Th. 1.2 and Theorem 2.5] deals with homomorphisms of hyperelliptic jacobians.

**Theorem 1.3.** Suppose that $\text{char}(K) \neq 2$, $n \geq 3$ and $m \geq 3$ are integers and let $f(x)$ and $h(x)$ be irreducible polynomials over $K$ of degree $n$ and $m$ respectively. Suppose that

$$\text{Gal}(f) = S_n, \quad \text{Gal}(h) = S_m$$

and the corresponding splitting fields $K(\mathfrak{R}_f)$ and $K(\mathfrak{R}_h)$ are linearly disjoint over $K$. Then either

$$\text{Hom}(J(C_f), J(C_h)) = \{0\}, \quad \text{Hom}(J(C_h), J(C_f)) = \{0\}$$

or $\text{char}(K) > 0$ and both $J(C_f)$ and $J(C_h)$ are supersingular abelian varieties.

**Remark 1.4.** If $K \subset \mathbb{C}$ then Theorem 1.3 implies that (under its assumptions) there are no nonzero homomorphisms between complex abelian varieties $J(C_f)$ and $J(C_h)$.

The main results of the present paper are the following statements.

**Theorem 1.5.** Suppose that $n = 2g + 2 = \deg(f) \geq 8$. Let $\tilde{C}_f \to C_f$ be an unramified double cover of complex smooth projective irreducible curves and let $P$ be the corresponding Prym variety, which is a $(g-1)$-dimensional (principally polarized) complex abelian variety.

If $\text{Gal}(f) = S_n$ then:

- $\text{End}(P) = \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$.
- Every Hodge class on each self-product of $P$ can be presented as a linear combination of products of divisor classes. In particular, the Hodge conjecture holds true for each self-product of $P$.

**Theorem 1.6.** Suppose that $n = 2g + 2 \geq 10$, $K \subset \mathbb{C}$ and $f(x) = (x - a)h(x)$ where $a \in K$ and $h(x) \in K[x]$ is an irreducible degree $n-1$ polynomial with $\text{Gal}(h) = S_{n-1}$. Let $\tilde{C}_f \to C_f$ be an unramified double cover of complex smooth projective irreducible curves and let $P$ be the corresponding Prym variety, which is a $(g-1)$-dimensional (principally polarized) complex abelian variety.

Then:

- $\text{End}(P) = \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$.
- every Hodge class on each self-product of $P$ can be presented as a linear combination of products of divisor classes. In particular, the Hodge conjecture holds true for each self-product of $P$. 
Our proof is based on the explicit description of Prym varieties of hyperelliptic curves \cite{11, 2} and our results about Hodge groups of hyperelliptic jacobians mentioned above.

Remark 1.7. If \( n = 2g + 2 \leq 10 \) then \( \dim(P) = g - 1 \leq 3 \). Notice that if \( A \) is a complex abelian variety of dimension \( \leq 3 \) then it is well known that every Hodge class on each self-product of \( A \) can be presented as a linear combination of products of divisor classes \cite[Th. 0.1(iv)]{9}. In particular, the Hodge conjecture holds true for each self-product of \( A \).

The paper is organized as follows. In Section 2 we discuss an elementary construction from Galois theory and apply it in Section 3 to homomorphisms of hyperelliptic jacobians. Section 4 deals with Hodge groups of hyperelliptic jacobians. In Section 5 we discuss hyperelliptic prymians and prove the main results.

2. Galois theory

Throughout this Section, \( K \) is an arbitrary field and \( n \geq 3 \) is an integer, \( f(x) \in K[x] \) is a degree \( n \) irreducible polynomial, whose Galois group

\[ \text{Gal}(f) = \text{Gal}(K(\mathcal{R}_f)/K) \]

is the full symmetric group \( \text{Perm}(\mathcal{R}_f) = S_n \). If \( T \subset \mathcal{R}_f \) is a non-empty subset then we put

\[ f_T(x) = \prod_{\alpha \in T} (x - \alpha) \in K(\mathcal{R}_f)[x]. \]

By definition,

\[ \deg(f_T) = \#(T), \quad \mathcal{R}_f T = T. \]

We view \( \text{Perm}(T) \) as a subgroup of \( \text{Perm}(\mathcal{R}_f) = \text{Gal}(f) \) that consists of all permutations that leave invariant every element outside \( T \).

Remark 2.1. Let us consider the subfield \( E_0 = K(\mathcal{R}_f)_{\text{Perm}(T)} \) of \( \text{Perm}(T) \)-invariants. Since \( \text{Perm}(T) \) leaves invariant \( T = \mathcal{R}_f T \),

\[ f_T(x) \in E_0[x]. \]

Clearly,

\[ \text{Gal}(K(\mathcal{R}_f)/E_0) = \text{Perm}(T) \subset \text{Perm}(\mathcal{R}_f) = \text{Gal}(\mathcal{R}_f/K). \]

Let us prove that the splitting field \( E_0(\mathcal{R}_f T) = E_0(T) \) of \( f_T(x) \) over \( E_0 \) coincides with \( K(\mathcal{R}_f) \). Indeed, \( \text{Gal}(K(\mathcal{R}_f)/E_0(T)) \) consists of all elements of \( \text{Perm}(T) = \text{Gal}(K(\mathcal{R}_f)/E_0) \) that leave invariant every element of \( T \). Since every element of \( \text{Perm}(T) \) leaves invariant every element of \( \mathcal{R}_f \setminus T \), \( \text{Gal}(K(\mathcal{R}_f))/E_0(T)) = \{1\} \), i.e.,

\[ K(\mathcal{R}_f) = E_0(T) = E_0(\mathcal{R}_f T). \]

This implies that the Galois group

\[ \text{Gal}(E_0(\mathcal{R}_f T)/E_0) = \text{Gal}(K(\mathcal{R}_f)/E_0) = \text{Perm}(T). \]

Lemma 2.2. Let \( T \) and \( S \) be two nonempty disjoint subsets of \( \mathcal{R}_f \). Then there exists a field subextension \( E/K \subset K(\mathcal{R}_f)/K \) that enjoys the following properties.

(i) The Galois group \( \text{Gal}(K(\mathcal{R}_f)/E) \) coincides with the subgroup

\[ \text{Gal}(T) \times \text{Gal}(S) \subset \text{Perm}(\mathcal{R}_f) = \text{Gal}(K(\mathcal{R}_f)/K), \]

which consists of all permutations that leave invariant \( T \), \( S \) and every element outside \( T \cup S \).

(ii) Both \( f_T(x) \) and \( f_S(x) \) lie in \( E[x] \), i.e., all their coefficients belong to \( E \).
(iii) Let \( E(\mathcal{R}_T) = E(T) \) and \( E(\mathcal{R}_S) = E(S) \) be the splitting fields over \( E \) of \( f_T(x) \) and \( f_S(x) \) respectively. Then the natural injective homomorphisms
\[
\text{Gal}(E(T)/E) \hookrightarrow \text{Perm}(T), \quad \text{Gal}(E(S)/E) \hookrightarrow \text{Perm}(S)
\]
are group isomorphisms, i.e.,
\[
\text{Gal}(E(T)/E) = \text{Perm}(T), \quad \text{Gal}(E(S)/E) = \text{Perm}(S).
\]

(iv) \( E(T) \) and \( E(S) \) are linearly disjoint over \( E \).

(v) The compositum \( E(T)E(S) \) coincides with \( K(\mathcal{R}_f) \).

Proof. Recall that
\[
\mathcal{R}_{f_T} = T, \quad \mathcal{R}_{f_S} = S.
\]
We define \( E \) as the subfield \( K(\mathcal{R}_f)^{\text{Perm}(T) \times \text{Perm}(S)} \) of \( \text{Perm}(T) \times \text{Perm}(S) \)-invariants. Now Galois theory gives us (i). The subgroup \( \text{Perm}(T) \times \text{Perm}(S) \) leaves invariant both sets \( T = \mathcal{R}_T \) and \( S = \mathcal{R}_S \). This implies that all the coefficients of \( f_T(x) \) and \( f_S(x) \) are \( \text{Perm}(T) \times \text{Perm}(S) \)-invariant, i.e., lie in \( E \). This proves (ii). Clearly,
\[
[K(\mathcal{R}_f) : E] = \#(\text{Perm}(T) \times \text{Perm}(S)).
\]

Clearly, the subgroup of \( \text{Perm}(T) \times \text{Perm}(S) \) that consists of all permutations that act identically on \( S \) coincides with \( \text{Perm}(T) \). Similarly, the subgroup of \( \text{Perm}(T) \times \text{Perm}(S) \) that consists of all permutations that act identically on \( T \) coincides with \( \text{Perm}(S) \). This implies that
\[
\text{Gal}(E(T)/E) = [\text{Perm}(T) \times \text{Perm}(S)]/\text{Perm}(S) = \text{Perm}(T),
\]
\[
\text{Gal}(E(S)/E) = [\text{Perm}(T) \times \text{Perm}(S)]/\text{Perm}(T) = \text{Perm}(S),
\]
which proves (iii). This implies that
\[
[E(T) : E] = \#(\text{Perm}(T)), \quad [E(S) : E] = \#(\text{Perm}(S)).
\]

Let \( L \) be the compositum \( E(T)E(S) \). Clearly, \( L \) contains \( T, S \) and \( E \). Therefore \( \text{Gal}(K(\mathcal{R}_f)/L) \) consists of elements of \( \text{Perm}(T) \times \text{Perm}(S) = \text{Gal}(K(\mathcal{R}_f)/E) \) that act identically on \( T \) and \( T \). Since all elements of \( \text{Perm}(T) \times \text{Perm}(S) \) act identically on the complement to \( T \cup S \), we conclude that \( \text{Gal}(K(\mathcal{R}_f)/L) = \{1\} \), i.e.,
\[
K(\mathcal{R}_f) = L = E(T)E(S).
\]

This proves (v). We also obtain that
\[
[E(T)E(S) : E] = [K(\mathcal{R}_f) : E] = \#(\text{Perm}(T) \times \text{Perm}(S)) = [E(T) : E][E(S) : E],
\]
i.e.
\[
[E(T)E(S) : E] = [E(T) : E][E(S) : E],
\]
which means that \( E(T)/E \) and \( E(S)/E \) are linearly disjoint. This proves (iv). \( \Box \)

3. Homomorphisms of hyperelliptic jacobians

We keep the notation and assumptions of Section 2. Also we assume that \( \text{char}(K) \neq 2 \).

**Theorem 3.1.** Let \( T \) and \( S \) be disjoint nonempty subsets of \( \mathcal{R}_f \) and consider the hyperelliptic curves
\[
C_{f_T} : y^2 = f_T(x), \quad C_{f_S} : y^2 = f_S(x)
\]
and their jacobians \( J(C_{f_T}) \) and \( J(C_{f_S}) \). Then either
\[
\text{Hom}(J(C_{f_T}), J(C_{f_S})) = \{0\}, \quad \text{Hom}(J(C_{f_S}), J(C_{f_T})) = \{0\}
\]
or char($K$) > 0 and both $J(C_{f_T})$ and $J(C_{f_S})$ are supersingular abelian varieties.

Proof. If $\#(T) < 3$ (resp. $\#(S) < 3$) then $C_{f_T}$ (resp. $C_{f_S}$) has genus zero and
$J(C_{f_T}) = 0$ (resp. $J(C_{f_S}) = 0$), which implies that there are no non-zero homomorphisms between $J(C_{f_T})$ and $J(C_{f_S})$. So, further we assume that

\[ n_1 := \#(T) \geq 3, \quad n_2 := \#(S) \geq 3. \]

By Lemma 2.2 there exists a field $E$ such that both $f_T(x)$ and $f_S(x)$ lie in $E[x]$, their Galois groups are $\text{Perm}(T) \cong S_{n_1}$ and $\text{Perm}(S) \cong S_{n_2}$ respectively. In addition, their splitting fields are linearly disjoint over $E$. Now the result follows from Theorem 1.3 applied to $E, f_T(x), f_S(x)$ instead of $K, f(x), h(x)$. □

Remark 3.2. Suppose that $m := \#(S) = 2r + 1$ is odd and let $b$ be an arbitrary element of $K$. Let us consider the hyperelliptic curve $C^b_{f_S} : y^2 = (x - b)f_S(x)$.

By Remark 2.1 there exists a field $E_0 \subset K(\mathcal{R}_T)$ such that $f_S(x)$ lies in $E_0[x]$ and $\text{Gal}(E_0(\mathcal{R}_{f_S})/E_0) = \text{Perm}(S) = S_m$. Then the standard substitution [24 p. 25]

\[ x_1 = \frac{1}{x - b}, \quad y_1 = \frac{y}{(x - b)^{r+1}} \]

gives us a degree $m$ irreducible polynomial $h(x_1) \in E[x_1]$ such that

\[ E_0(\mathcal{R}_{h}) = E_0(\mathcal{R}_{f_S}), \quad \text{Gal}(E_0(\mathcal{R}_{h})/E_0) = \text{Gal}(E_0(\mathcal{R}_{f_S})/E_0) = S_m \]

and $C^b_{f_S}$ is $E_0$-birationally isomorphic to the hyperelliptic curve $C_h : y^2 = h(x_1)$. (It is assumed in [24 p. 25] that $m \geq 5$ but the substitution works for any positive odd $m$.)

Corollary 3.3. Let $T$ and $S$ be disjoint nonempty subsets of $\mathcal{R}_T$ and assume that $\#(S)$ is odd. Let $b$ be an arbitrary element of $K$. Let us consider the hyperelliptic curves

\[ C_{f_T} : y^2 = f_T(x), \quad C^b_{f_S} : y^2 = (x - b)f_S(x) \]

and their jacobians $J(C_{f_T})$ and $J(C^b_{f_S})$. Then either

\[ \text{Hom}(J(C_{f_T}), J(C^b_{f_S})) = \{0\}, \quad \text{Hom}(J(C^b_{f_S}), J(C_{f_T})) = \{0\} \]

or char($K$) > 0 and both $J(C_{f_T})$ and $J(C^b_{f_S})$ are supersingular abelian varieties.

Proof. We may assume that both $m_1 = \#(T)$ and $m_2 = \#(S)$ are, at least, 3. Let $E$ be as in Lemma 2.2. In particular, the splitting fields $E(T)$ and $E(S)$ are linearly disjoint over $E$ and

\[ \text{Gal}(E(T)/E) \cong S_{m_1}, \quad \text{Gal}(E(S)/E) \cong S_{m_2}. \]

Using Remark 3.3 over $E$ (instead of $E_0$), we obtain that there is a degree $m$ irreducible polynomial $b(x) \in E[x]$ such that

\[ E(\mathcal{R}_{h}) = E(\mathcal{R}_{f_S}), \quad \text{Gal}(E(\mathcal{R}_{h})/E) = \text{Gal}(E(\mathcal{R}_{f_S})/E) = S_{m_2} \]

and $C^b_{f_S}$ is birationally $E$-isomorphic to the hyperelliptic curve $C_h : y^2 = h(x)$. Clearly, the jacobians $J(C^b_{f_S})$ and $J(C_h)$ are isomorphic. Applying Theorem 1.3 to $E, f_T(x), h(x)$, we conclude that either

\[ \text{Hom}(J(C_{f_T}), J(C_h)) = \{0\}, \quad \text{Hom}(J(C_h), J(C_{f_T})) = \{0\} \]

or char($K$) > 0 and both $J(C_{f_T})$ and $J(C^b_{f_S})$ are supersingular abelian varieties. Since $J(C^b_{f_S})$ and $J(C_h)$ are isomorphic, we are done. □
Theorem 3.4. Let $T$ and $S$ be disjoint nonempty subsets of $\mathcal{R}_f$ and assume that both $\#(T)$ and $\#(S)$ are odd. Let $a$ and $b$ be arbitrary (not necessarily distinct) elements of $K$. Let us consider the hyperelliptic curves
\[ C^a_{T^r} : y^2 = (x - a)f_T(x), \quad C^b_{S^r} : y^2 = (x - b)f_S(x) \]
and their jacobians $J(C^a_{T^r})$ and $J(C^b_{S^r})$. Then either
\[ \text{Hom}(J(C^a_{T^r}), J(C^b_{S^r})) = \{0\}, \quad \text{Hom}(J(C^b_{S^r}), J(C^a_{T^r})) = \{0\} \]
or $\text{char}(K) > 0$ and both $J(C^a_{T^r})$ and $J(C^b_{S^r})$ are supersingular abelian varieties.

Proof. We may assume that both $m_1 := \#(T)$ and $m_2 := \#(S)$ are, at least, 3. Again, let $E$ be as in Lemma 2.2. Applying Remark 3.3 two times over $E$ (instead of $E_0$) to the polynomials $(x - a)f_T(x)$ and $(x - b)f_S(x)$, we conclude that there are degree $m$ irreducible polynomials $h_1(x) \in E[x]$ and $h_2(x) \in E[x]$ such that
\[ E(\mathcal{R}_{h_1}) = E(T), \quad E(\mathcal{R}_{h_2}) = E(S), \]
\[ \text{Gal}(E(\mathcal{R}_{h_1})/E) = S_{m_1}, \quad \text{Gal}(E(\mathcal{R}_{h_2})/E) = S_{m_2}. \]
$C^a_{T^r}$ is $E$-birationally isomorphic to $C_{h_1}$ and $C^b_{S^r}$ is $E$-birationally isomorphic to $C_{h_2}$. Clearly, $J(C^a_{T^r}) \cong J(C_{h_1})$ and $J(C^b_{S^r}) \cong J(C_{h_2})$. Applying Theorem 3.4 to $E, h_1(x), h_2(x)$, we conclude that either
\[ \text{Hom}(J(C_{h_1}), J(C_{h_2})) = \{0\}, \quad \text{Hom}(J(C_{h_2}), J(C_{h_1})) = \{0\} \]
or $\text{char}(K) > 0$ and both $J(C_{h_1})$ and $J(C_{h_2})$ are supersingular abelian varieties. The rest is clear.

Remark 3.5. Let $K_2/K$ be the only quadratic subextension of $K(\mathcal{R}_f)/K$. Clearly, $K_2(\mathcal{R}_f) = K(\mathcal{R}_f)$ and the Galois group $\text{Gal}(K_2(\mathcal{R}_f)/K_2)$ coincides with the alternating group $A_n$.

Theorem 3.6. Suppose that $\text{char}(K) \neq 2$. Let $T \subset \mathcal{R}_f$ be a 4-element subset. Let us consider the corresponding elliptic curve
\[ C_{f_T} : y^2 = f_T(x) \]
and its jacobian $J(C_{f_T})$. If $n \geq 8$ then one of the following conditions holds:
\begin{itemize}
  \item $\text{End}(J(C_{f_T})) = \mathbb{Z}$ for all $T$.
  \item $\text{char}(K) > 0$ and all $J(C_{f_T})$’s are supersingular elliptic curves mutually isomorphic over $\overline{K}$.
\end{itemize}

Proof. Let $j_T$ be the $j$-invariant of the elliptic curve $J(C_{f_T})$ ([17], [7] Ch. III, Sect. 2). Clearly,
\[ j_T \in K(T) \subset K(\mathcal{R}_f) \]
and
\[ j_{\sigma T} = \sigma j_T \forall \sigma \in \text{Gal}(K(\mathcal{R}_f)/K) = \text{Gal}(f). \]
Suppose that $J(C_{f_T})$ admits complex multiplication. Then one of the following two conditions holds.
\begin{itemize}
  \item[(i)] $p = \text{char}(K) > 0$. Then a classical result of M. Deuring asserts that $j_T$ is algebraic, i.e., lies in a finite field $F_q$ where $q$ is a power of the prime $p$. (See [4], [13] Sect. 3.2), [8] Ch. 13, Sect. 5.) In particular, $K(j_T)/K$ is an abelian field extension.
(ii) \( \text{char}(K) = 0 \). Then there exists an imaginary quadratic field \( k \) such that \( \text{End}^0(J(C_{f_T})) = k \). In addition, a classical result of the theory of complex multiplication asserts that \( j_T \) is an algebraic number such that the field extension \( k(j_T)/k \) is abelian. (See [16 Sect. 5.4], [8 Ch. 10, Sect. 3].)

Let us consider the overfield \( K' \) of \( K \) that is defined as follows. If \( \text{char}(K) > 0 \) then \( K' = K_2 \). If \( \text{char}(K) = 0 \) then \( K' \) is the compositum \( K_2k \) of \( K_2 \) and the imaginary quadratic field \( k \); in particular, \( K' \) contains \( k \).

Since \( A_n = \text{Gal}(K'(\mathcal{R}_f)/K') \) is simple nonabelian, the field extension \( K'(\mathcal{R}_f)/K' \) does not contain nontrivial abelian subextensions. However, \( j_T \in K'(\mathcal{R}_f) \) and the field (sub)extension \( K'(j_T)/k \) is abelian. This implies that this subextension is trivial, i.e., \( j_T \in K' \). This means that for all \( \sigma \in \text{Gal}(K'(\mathcal{R}_f)/K') = A_n \)

\[
j_T = \sigma j_T = j_{\sigma T}.
\]

Since \( n \geq 8 \), the permutation group \( A_n \) is 4-transitive and therefore the jacobians \( J(C_{f_T}) \)'s are mutually isomorphic over \( \bar{K} \) for all 4-element subsets \( T \subset \mathcal{R}_f \).

Let \( T_1 \) and \( T_2 \) be two disjoint 4-element subsets of \( \mathcal{R}_f \). (Since \( n \geq 8 \), such \( T_1 \) and \( T_2 \) do exist.) Applying Theorem 3.4 to \( T_1 \) and \( T_2 \) (instead of \( T \) and \( S \)) and taking into account that \( J(C_{f_{T_1}}) \) and \( J(C_{f_{T_2}}) \) are mutually isomorphic over \( \bar{K} \), we conclude that \( \text{char}(K) > 0 \) and both \( J(C_{f_{T_1}}) \) and \( J(C_{f_{T_2}}) \) are supersingular elliptic curves.

**Theorem 3.7.** Suppose that \( \text{char}(K) \neq 2 \). Let \( a \) be an arbitrary element of \( K \). Let \( T \subset \mathcal{R}_f \) be a 3-element subset. Let us consider the corresponding elliptic curve

\[
C^n_{f_T} : y^2 = (x - a)f_T(x)
\]

and its jacobian \( J(C^n_{f_T}) \). If \( n \geq 6 \) then one of the following conditions holds:

1. \( \text{End}(J(C^n_{f_T})) = \mathbb{Z} \) for all \( T \).
2. \( \text{char}(K) > 0 \) and all \( J(C^n_{f_T}) \)'s are supersingular elliptic curves mutually isomorphic over \( \bar{K} \).

**Proof.** Let \( j_{T,a} \) be the \( j \)-invariant of the elliptic curve \( J(C^n_{f_T}) \). Clearly,

\[
j_{T,a} \in K(T) \subset K(\mathcal{R}_f)
\]

and

\[
j_{\sigma T,a} = \sigma j_{T,a} \forall \sigma \in \text{Gal}(K(\mathcal{R}_f)/K) = \text{Gal}(f).
\]

Suppose that \( J(C^n_{f_T}) \) admits complex multiplication. Then, as in the proof of Theorem 3.6 there exists an overfield \( K' \supset K_2 \) such that either \( K' = K_2 \) or \( K' \) is a quadratic extension of \( K_2 \) and in both cases \( K'(j_{T,a}) \subset K'(\mathcal{R}_f) \) and the field (sub)extension \( K'(j_{T,a})/K' \) is abelian. Again, \( A_n = \text{Gal}(K'(\mathcal{R}_f)/K') \) is simple nonabelian and therefore there are no nontrivial abelian subextensions of \( K'(\mathcal{R}_f)/K' \). This implies that \( j_{T,a} \in K' \), i.e., for all \( \sigma \in \text{Gal}(K'(\mathcal{R}_f)/K') = A_n \)

\[
j_{T,a} = j_{\sigma T,a} = j_{\sigma T,a}.
\]

Since \( n \geq 6 \), the permutation group \( A_n \) is 3-transitive and therefore the jacobians \( J(C^n_{f_{T_1}}) \)'s are mutually isomorphic over \( \bar{K} \) for all 3-element subsets \( T \subset \mathcal{R}_f \).

Let \( T_1 \) and \( T_2 \) be two disjoint 3-element subsets of \( \mathcal{R}_f \). (Since \( n \geq 6 \), such \( T_1 \) and \( T_2 \) do exist.) Applying Theorem 3.4 to \( T_1 \), \( a \) and \( T_2 \), \( a \) (instead of \( T \), \( a \) and \( S \), \( b \)) and taking into account that \( J(C^n_{f_{T_1}}) \) and \( J(C^n_{f_{T_2}}) \) are isomorphic over \( \bar{K} \) (i.e.,
Hdg\((J(C_{f_T})), J(C_{f_{T'}})) \neq \{0\}\), we conclude that char\((K) > 0\) and both \(J(C_{f_T})\) and \(J(C_{f_{T'}})\) are supersingular elliptic curves. The rest is clear.  

\section{Hodge Groups of Hyperelliptic Jacobians}

We keep the notation and assumptions of Sections 3 and 4. Also we assume that \(K \subset \mathbb{C}\).

**Definition 4.1.** We say (as in \cite{9} Sect. 1.8) that a complex abelian variety \(X\) satisfies property \((D)\) if every Hodge class on each self-product \(X^r\) of \(X\) can be presented as a linear combination of products of divisor classes. If this condition is satisfied then the Hodge conjecture is true for all \(X^r\).

**Remark 4.2.** Abelian varieties that satisfy \((D)\) are also called stably nondegenerate \cite{5}; see also \cite{12}.

**Example 4.3.** If \(Y\) is an elliptic curve over \(\mathbb{C}\) with \(\text{End}(Y) = \mathbb{Z}\) then it is well known \cite{9} Th. 0.1(iv) that \(Y\) satisfies \((D)\) and \(\text{Hdg}(Y) = \text{Sp}(H_1(Y, \mathbb{Q}))\).

**Theorem 4.4.** Let \(X_1\) and \(X_2\) be complex abelian varieties of positive dimension and \(X = X_1 \times X_2\). Suppose that

\[\text{End}(X_1) = \mathbb{Z}, \text{End}(X_2) = \mathbb{Z}, \text{Hom}(X_1, X_2) = \{0\}\]

Then:

1. \(\text{End}(X_1 \times X_2) = \mathbb{Z} \oplus \mathbb{Z}\).
2. If both \(X_1\) and \(X_2\) satisfy \((D)\) then \(\text{Hdg}(X) = \text{Hdg}(X_1) \times \text{Hdg}(X_2)\) and \(X\) satisfies \((D)\).

**Proof.** (i) is obvious. (ii) follows from \cite{6} Th. 0.1 and Prop. 1.8 (see also Theorem 3.2(i) of \cite{9}). \(\square\)

**Remark 4.5.** Since \(\text{End}(X_i) = \mathbb{Z}\) and \(X_i\) satisfies \((D)\),

\[\text{Hdg}(X_i) = \text{Sp}(H_1(X_i, \mathbb{Q}))\]

\cite{5} \cite{12}. (See also \cite{9} Sect. 1.8.)

**Lemma 4.6.** Suppose that \(T\) is a subset of \(R_f\) with \(#(T) \geq 5\). Let us consider the hyperelliptic curve \(C_{f_T} : y^2 = f_T(x)\) and its jacobian \(J(C_{f_T})\). Then \(\text{End}(J(C_{f_T})) = \mathbb{Z}\) and \(\text{Hdg}(J(C_{f_T})) = \text{Sp}(H_1(J(C_{f_T}), \mathbb{Q}))\). In addition, \(J(C_{f_T})\) satisfies \((D)\).

**Proof.** Let us put \(m = \#(T)\). We have \(m \geq 5\) and \(\deg(f_T) = m \geq 5\).

By Remark 2.1, there exists a (sub)field

\[E_0 \subset K(R_f) \subset \bar{K} \subset \mathbb{C}\]

such that \(f_T(x) \in E_0(T)\) and the Galois group of \(f_T(x)\) over \(E_0\) is \(\text{Perm}(T) \cong S_m\). Now the result follows from Theorem 4.4 applied to \(m, E_0, f_T(x)\) instead of \(n, k, f(x)\). \(\square\)

**Lemma 4.7.** Suppose that \(T\) is a subset of \(R_f\) with \(#(T) \geq 5\). Suppose that \(m\) is odd and let \(a\) be an arbitrary element of \(K\). Let us consider the hyperelliptic curve \(C_{f_T}^a : y^2 = (x - a)f_T(x)\) and its jacobian \(J(C_{f_T}^a)\). Then \(\text{End}(J(C_{f_T}^a)) = \mathbb{Z}\) and \(\text{Hdg}(J(C_{f_T}^a)) = \text{Sp}(H_1(J(C_{f_T}^a), \mathbb{Q}))\). In addition, \(J(C_{f_T}^a)\) satisfies \((D)\).
Proof. By Remark 3.2 there exists a field
\[ E_0 \subset K(\mathcal{R}_f) \subset \bar{K} \subset C \]
and a degree \( m \) irreducible polynomial \( h(x) \in E_0[x] \) such that
\[ E(\mathcal{R}_h) = E(\mathcal{R}_{f_1}), \quad \text{Gal}(E(\mathcal{R}_h)/E) = \text{Gal}(E(\mathcal{R}_{f_1})/E) = S_m \]
and \( C_{f_1} \) is birationally \( E \)-isomorphic to the hyperelliptic curve \( C_h : y^2 = h(x) \).
Clearly, the jacobians \( J(C_{f_1}) \) and \( J(C_h) \) are isomorphic. It follows from Lemma 4.6 applied to \( m, E_0, h(x) \) (instead of \( n, K, f(x) \)) that \( \text{End}(J(C_h)) = \mathbb{Z}, \text{Hdg}(J(C_h)) = \text{Sp}(H_1(J(C_h), \mathbb{Q})) \) and \( J(C_h) \) satisfies (D). Since \( J(C_{f_1}) \) and \( J(C_h) \) are isomorphic, we are done. \( \square \)

5. Prym varieties

Following [11, 2], let us give an explicit description of hyperelliptic prymians \( P \), assuming that \( \text{char}(K) \neq 2 \). Suppose that \( n = 2g + 2 \geq 6 \) and
\[ f(x) \in K[x] \subset \bar{K}[x] \]
is a degree \( n \) polynomial without multiple roots. Let us split the \( n \)-element set \( \mathcal{R}_f \) of roots of \( f(x) \) into a disjoint union
\[ \mathcal{R}_f = \mathcal{R}_1 \sqcup \mathcal{R}_2 \]
of non-empty sets \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) of even cardinalities \( n_1 \) and \( n_2 \) respectively. Further we assume that
\[ n_1 \geq n_2 \geq 2. \]
and put
\[ f_1(x) = \prod_{\alpha \in \mathcal{R}_1} (x - \alpha), \quad f_2(x) = \prod_{\alpha \in \mathcal{R}_2} (x - \alpha). \]
We have \( n_1 + n_2 = n \) and define nonnegative integers \( g_1 \) and \( g_2 \) by
\[ n_1 = 2g_1 + 2, \quad n_2 = 2g_2 + 2. \]
Clearly,
\[ g_1 + g_2 = g - 1. \]
Let us consider the hyperelliptic curves \( C_{f_1} : y^2 = f_1(x) \) and \( C_{f_2} : y^2 = f_2(x) \) of genus \( g_1 \) and \( g_2 \) respectively and corresponding hyperelliptic jacobians \( J(C_{f_1}) \) and \( J(C_{f_2}) \) of dimension \( g_1 \) and \( g_2 \) respectively. Then the prymians \( P \) of \( C_f : y^2 = f(x) \) are just the products \( J(C_{f_1}) \times J(C_{f_2}) \) for all the partitions \( \mathcal{R}_f = \mathcal{R}_1 \sqcup \mathcal{R}_2 \).

Now Theorem 1.5 becomes an immediate corollary of the following statement.

**Theorem 5.1.** Suppose that \( n = 2g + 2 \geq 8 \), \( K \subset C \) and \( \text{Gal}(f) = S_n \). Let us put \( P = J(C_{f_1}) \times J(C_{f_2}) \). Then:
(i) \( \text{Hom}(J(C_{f_1}), J(C_{f_2})) = \{0\} \), \( \text{Hom}(J(C_{f_2}), J(C_{f_1})) = \{0\} \).
(ii) Suppose that \( g_1 \geq 1 \), i.e., \( n_1 \geq 4 \). Then
\[ \text{End}(J(C_{f_1})) = \mathbb{Z}, \quad \text{Hdg}(J(C_{f_1})) = \text{Sp}(H_1(J(C_{f_1}), \mathbb{Q})) \]
and \( J(C_{f_1}) \) satisfies (D).
(iii) If \( n_2 = 2 \) then \( P = J(C_{f_1}) \). In particular, \( \text{End}(P) = \mathbb{Z}, \quad \text{Hdg}(P) = \text{Sp}(H_1(P, \mathbb{Q})) \) and \( P \) satisfies (D).
Now the assertion (i) follows from Corollary 3.3 applied to \( \text{Hdg}(J(C_{f_1})) \times \text{Hdg}(J(C_{f_2})) = \text{Sp}(H_1(J(C_{f_1}), Q)) \times \text{Sp}(H_1(J(C_{f_2}), Q)) \) and \( P \) satisfies (D).

**Proof of Theorem 5.1.** The assertion (i) follows from Theorem 5.1.

If \( n_i \geq 6 \) then (ii) follows from Lemma 4.6. Suppose that \( n_i = 4 \). Then \( J(C_{f_i}) \) is an elliptic curve. It follows from Theorem 5.6 that \( \text{End}(J(C_{f_i})) = \mathbb{Z} \). Now the assertion about its Hodge group and property (D) follows from Example 4.3. This completes the proof of (ii).

Let us prove (iii). If \( n_2 = 2 \) then \( J(C_{f_2}) = 0 \) and therefore \( P = J(C_{f_1}) \). Now the assertion follows from (ii).

Let us prove (iv). We assume that \( n_1 \geq n_2 \geq 4 \).

By already proven (i) and (ii),

\[
\text{End}(J(C_{f_1})) = \mathbb{Z}, \quad \text{End}(J(C_{f_2})) = \mathbb{Z}, \quad \text{Hom}(J(C_{f_1}), J(C_{f_2})) = \{0\}.
\]

Now (iv) follows from Theorem 4.4 applied to \( X_1 = J(C_{f_1}), X_2 = J(C_{f_2}) \) and \( X = P \).

Theorem 1.6 is an immediate corollary of the following statement.

**Theorem 5.2.** Suppose that \( n = 2g + 2 \geq 10, K \subset \mathbb{C} \) and \( f(x) = (x-a)h(x) \) where \( a \in K \) and \( h(x) \in K[x] \) is an irreducible degree \( n-1 \) polynomial with \( \text{Gal}(h) = S_{n-1} \). Let us put \( P = J(C_{f_1}) \times J(C_{f_2}) \). Then:

(i) \( \text{Hom}(J(C_{f_1}), J(C_{f_2})) = \{0\}, \quad \text{Hom}(J(C_{f_2}), J(C_{f_1})) = \{0\} \).

(ii) Suppose that \( n_i \geq 1, i.e., n_i \geq 4 \). Then

\[
\text{End}(J(C_{f_i})) = \mathbb{Z}, \quad \text{Hdg}(J(C_{f_i})) = \text{Sp}(H_1(J(C_{f_i}), Q))
\]

and \( J(C_{f_i}) \) satisfies (D).

(iii) If \( n_2 = 2 \) then \( P = J(C_{f_1}) \). In particular, \( \text{End}(P) = \mathbb{Z}, \quad \text{Hdg}(P) = \text{Sp}(H_1(P, Q)) \) and \( P \) satisfies (D).

(iv) If \( n_2 \geq 4 \) then \( \text{End}(P) = \mathbb{Z} \oplus \mathbb{Z}, \quad \text{Hdg}(P) = \text{Hdg}(J(C_{f_2})) \times \text{Hdg}(J(C_{f_2})) = \text{Sp}(H_1(J(C_{f_1}), Q)) \times \text{Sp}(H_1(J(C_{f_2}), Q)) \)

and \( P \) satisfies (D).

**Proof.** Clearly, \( \mathfrak{R}_f = \mathfrak{R}_h \cup \{a\} \); in particular, \( a \) belongs to precisely one of \( \mathfrak{R}_1 \) and \( \mathfrak{R}_2 \). Suppose that \( a \) lies in \( \mathfrak{R}_j \) and does not belong to \( \mathfrak{R}_k \) and put

\[
T = \mathfrak{R}_k \subset \mathfrak{R}_h, \quad S = \mathfrak{R}_j \setminus \{a\} \subset \mathfrak{R}_h.
\]

Now the assertion (i) follows from Corollary 3.3 applied to \( h(x) \) (instead of \( f(x) \)).

If \( n_k \geq 6 \) then the assertion (ii) for \( J(C_{f_k}) \) follows from Lemma 4.6. Suppose that \( n_k = 4 \), i.e., \( J(C_{f_k}) \) is an elliptic curve. Then it follows from Theorem 3.6 applied to \( m = n-1 \geq 9 \) and \( h(x) \) (instead of \( f(x) \)) that \( \text{End}(J(C_{f_k})) = \mathbb{Z} \). Now the assertion about its Hodge group and property (D) follows from Example 4.3.

If \( n_j \geq 6 \) then the assertion (ii) for \( J(C_{f_j}) \) follows from Lemma 4.7. Suppose that \( n_j = 4 \), i.e., \( J(C_{f_j}) \) is an elliptic curve. Then it follows from Theorem 3.7 applied to \( m = n-1 \) and \( h(x) \) (instead of \( f(x) \)) that \( \text{End}(J(C_{f_j})) = \mathbb{Z} \). Now the assertion about its Hodge group and property (D) follows from Example 4.3. This ends the proof of (ii).
The proof of the remaining assertions (iii) and (iv) goes literally as the proof of the corresponding assertions of Theorem \ref{main-theorem}. \hfill \Box

**Examples 5.3.** Let us take $K = \mathbb{Q}$ and $f_n(x) = x^n - x - 1$. It is known \cite{15} p. 42 that $\text{Gal}(f_n) = S_n$. Let $a$ be a rational number. Suppose that $n = 2g + 2$ and let us consider the hyperelliptic genus $g$ curves $C_{f_n} : y^2 = f_n(x)$ and $C_{f_n+1} : y^2 = (x-a)f_{n-1}(x)$. Then:

(i) If $n = 2g + 2 \geq 8$ then all $(2^{2g} - 1)$ Prym varieties $P$ of $C_{f_n}$ satisfy (D). Among them there are exactly $n(n-1)/2$ complex abelian varieties with $\text{End}(P) = \mathbb{Z}$; for all others $\text{End}(P) = \mathbb{Z} \oplus \mathbb{Z}$.

(ii) If $n = 2g + 2 \geq 10$ then all $(2^{2g} - 1)$ Prym varieties $P$ of $C_{f_n+1}$ satisfy (D). Among them there are exactly $n(n-1)/2$ complex abelian varieties with $\text{End}(P) = \mathbb{Z}$; for all others $\text{End}(P) = \mathbb{Z} \oplus \mathbb{Z}$.

**Example 5.4.** Let $z_1, \ldots, z_n$ be algebraically independent (transcendental) complex numbers and $L = \mathbb{Q}(z_1, \ldots, z_n) \subset \mathbb{C}$ the corresponding subfield of $\mathbb{C}$, which is isomorphic to the field of rational functions in $n$ variables over $\mathbb{Q}$. Let $K \subset L$ be the (sub)field of symmetric rational functions. Then

$$f(x) = \prod_{i=1}^n (x - z_i) \in K[x], \quad \mathfrak{H}_f = \{z_1, \ldots, z_n\}, \quad \text{Gal}(f) = S_n.$$ 

Suppose that $n = 2g + 2$ and let us consider the hyperelliptic genus $g$ curve $C_f : y^2 = f(x)$.

If $g \geq 3$ (i.e., $n \geq 8$) then all $(2^{2g} - 1)$ Prym varieties $P$ of $C_f$ satisfy (D). Among them there are exactly $n(n-1)/2$ complex abelian varieties with $\text{End}(P) = \mathbb{Z}$; for all others $\text{End}(P) = \mathbb{Z} \oplus \mathbb{Z}$.

If $n = 6$ (i.e., $g = 2$) then all fifteen Prym varieties $P$ are elliptic curves $y^2 = \prod_{z \in T}(x - z)$ where $T$ is a 4-element subset of $\{z_1, \ldots, z_6\}$. The algebraic independence of $z_1, \ldots, z_6$ implies that the $j$-invariants of these elliptic curves are transcendental numbers and therefore all $P$ have no complex multiplication, i.e., $\text{End}(P) = \mathbb{Z}$.

**Remark 5.5.** The property (D) and equality $\text{End}(P) = \mathbb{Z}$ for general (not necessarily unramified) Prym varieties $P$ of arbitrary smooth projective curves were proven in \cite{1}.

**References**

[1] I. Biswas, K. H. Paranjape, *The Hodge conjecture for general Prym varieties*. J. Algebraic Geometry 11 (2002), 33–39.

[2] S. G. Dalaljan, *The Prym variety of an unramified double covering of a hyperelliptic curve*. (Russian) Uspehi Mat. Nauk 29 (1974), no. 6(180), 165–166. MR0404270 (53 #8073).

[3] P. Deligne, *Hodge cycles on abelian varieties* (notes by J.S. Milne). Lecture Notes in Math., vol. 900 (Springer-Verlag, 1982), pp. 9–100.

[4] M. Deuring, *Die Typen der Multiplikatorenringe elliptischer Funktionenkörper*. Abh. Math. Sem. Hansischen Univ. 14 (1941), 197-272. MR0005125 (3,104f).

[5] F. Hazama, *Algebraic cycles on certain abelian varieties and powers of special surfaces*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 31 (1985), no. 3, 487–520.

[6] F. Hazama, *Algebraic cycles on nonsimple abelian varieties*. Duke Math. J. 58 (1989), 31–37.

[7] A. Knapp, *Elliptic curves*. Princeton University Press, Princeton, 1992.

[8] S. Lang, *Elliptic functions*, Second edition. Springer-Verlag, New York, 1987.

[9] B. Moonen, Yu. G. Zarhin, *Hodge classes on abelian varieties of low dimension*. Math. Ann. 315 (1999), 711–733.
[10] D. Mumford, A note of Shimura’s paper “Discontinuous groups and abelian varieties”. Math. Ann. 181 (1969), 345-351.
[11] D. Mumford, Prym varieties I. In: Contributions to Analysis, pp. 325–350, Academic Press, 1974; Selected Papers, vol. I, pp. 545–570, Springer Verlag, New York, 2004.
[12] V. Kumar Murty, Exceptional Hodge classes on certain abelian varieties. Math. Ann. 268 (1984), no. 2, 197–206.
[13] F. Oort, The isogeny class of a CM-type abelian variety is defined over a finite extension of the prime field. J. Pure Appl. Algebra 3 (1973), 399–408.
[14] K. Ribet, Hodge classes on certain abelian varieties. Amer. J. Math. 105 (1983), 523–538.
[15] J.-P. Serre, Topics in Galois Theory. Jones and Bartlett Publishers, Boston-London, 1992.
[16] G. Shimura, Introduction to the arithmetic theory of automorphic functions. Publ. Math. Soc. Japan 11, Iwanami Shoten and Princeton University Press, Princeton, 1971.
[17] J. Tate, Algebraic formulas in arbitrary characteristic. Appendix 1 to [8], pp. 299–306.
[18] Yu.G. Zarhin, Weights of simple Lie algebras in the cohomology of algebraic varieties. Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984), 264–304; Math. USSR Izv. 24 (1985), 245 - 281.
[19] Yu.G. Zarhin, Hyperelliptic Jacobians without complex multiplication. Math. Res. Letters 7 (2000), 123–132.
[20] Yu.G. Zarhin, Very simple 2-adic representations and hyperelliptic Jacobians. Moscow Math. J. 2 (2002), issue 2, 403-431.
[21] Yu.G. Zarhin, Homomorphisms of hyperelliptic Jacobians. In: Number Theory, Algebra and Algebraic Geometry (Shafarevich Festschrift). Trudy Mat. Inst. Steklov 241 (2003), 90–104; Proc. Steklov Inst. Math. 241 (2003), 79–92.
[22] Yu.G. Zarhin, Non-supersingular hyperelliptic Jacobians. Bull. Soc. Math. France 132 (2004), no. 4, 617–634.
[23] Yu.G. Zarhin, Homomorphisms of abelian varieties. In: Y. Aubry, G. Lachaud (ed.), Arithmetic, Geometry and Coding Theory (AGCT 2003), Séminaires et Congrès 11 (2005), 189–215.
[24] Yu.G. Zarhin, Families of absolutely simple hyperelliptic Jacobians. Proc. Lond. Math. Soc. 100 (2010), no. 1, 24–54.

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA
E-mail address: zarhin@math.psu.edu