Second-order perturbations of a zero-pressure cosmological medium:
Proofs of the relativistic-Newtonian correspondence

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The dynamic world model and its linear perturbations were first studied in Einstein’s gravity. In the system without pressure the relativistic equations coincide exactly with the later known ones in Newton’s gravity. Here we prove that, except for the gravitational wave contribution, even to the second-order perturbations, equations for the relativistic irrotational zero-pressure fluid in a flat Friedmann background coincide exactly with the previously known Newtonian equations. Thus, to the second order, we correctly identify the relativistic density and velocity perturbation variables, and we expand the range of applicability of the Newtonian medium without pressure to all cosmological scales including the super-horizon scale. In the relativistic analyses, however, we do not have a relativistic variable which corresponds to the Newtonian potential to the second order. Mixed usage of different gauge conditions is useful to make such proofs and to examine the result with perspective. We also present the gravitational wave equation to the second order. Since our correspondence includes the cosmological constant, our results are relevant to currently favoured cosmology. Our result has an important practical implication that one can use the large-scale Newtonian numerical simulation more reliably even as the simulation scale approaches near horizon.

I. INTRODUCTION

Despite its algebraic and conceptual complexity in Einstein’s gravity the evolving world model and its linear structures were first studied based on Einstein’s gravity in the classic works by Friedmann in 1922 \cite{1} and Lifshitz in 1946 \cite{2}, respectively. In an interesting sequence, the much simpler and, in hindsight, more intuitive Newtonian studies followed later by Milne in 1934 \cite{3} and Bonnor in 1957 \cite{4}, respectively. In the case without pressure the Newtonian results coincide exactly with the previously derived relativistic ones for both the background world model and its first-order (linear) perturbations. The case with pressure cannot be handled in the Newtonian context despite several failed attempts in the literature to simulate it especially for the perturbation. The situation is still well described by Sachs and Wolfe in 1967 \cite{5}: “When these modified equations were perturbed to first order, their solutions did not agree with the relativistic results, even qualitatively.” In this work, we will show an additional continuation of relativistic-Newtonian correspondences in the zero-pressure medium by proving that the relativistic second-order scalar-type perturbations are described by the same equations known in Newton’s theory. That is, the Newtonian equations coincide exactly with the relativistic ones even to the second order in perturbations.

In the relativistic perturbations, due to the covariance of field equation we have freedom to fix the spacetime coordinate system by choosing some of the metric or energy-momentum variables at our disposal: this is often called the gauge choice. The original study of Lifshitz started by choosing the synchronous gauge which is still quite popular in the literature. Other gauge conditions were discovered later \cite{6,7}. It is an ironic situation that except for the widely used synchronous gauge condition, each of other gauge conditions fixes the gauge freedom completely. Thus, each has its own unique corresponding gauge invariant combination. Notice some common algebraic errors (not in Lifshitz’s work though) widespread in the literature including many textbooks due to the incomplete gauge fixing nature of the synchronous gauge, see \cite{8}.

Although infinitely many gauge conditions are available, it has been common in the literature to fix gauge conditions from the beginning. The importance of using different gauge conditions for different variables and the gauge invariance of such variables were shown by Bardeen in 1980 \cite{7}. Bardeen’s work also showed the importance of having access to many different gauge conditions which become apparent in his work in 1988 \cite{10}. In this work, the importance of having different variables evaluated in different gauges (all correspond to unique gauge-invariant combinations) will become clear as we extend Bardeen’s approach to the second-order perturbations.

Recently, we have presented a second-order perturbation formulation of the Friedmann world model considering quite general situations \cite{11}. We have resolved the gauge issue, identifying the variables to use in fixing the gauges and constructing gauge-invariant combinations, which can be easily extended even to the higher order. The basic equations are presented without fixing the temporal gauge condition thus allowing us to choose or try many available gauge conditions later depending on the situation: we call this a gauge-ready approach, see Eqs. (5)-(11) below. The Newtonian correspondence to the
linear order was made by properly arranging the equations using various gauge-invariant variables in [7,12,9]. Extending such correspondences to the second order is our task in this work. We set \( c \equiv 1 \).

II. BASIC EQUATIONS

We consider a scalar-type perturbation in the flat Friedmann background. We will consider the presence of tensor-type perturbation (gravitational waves) in §VI.

The vector-type perturbation (rotation) is not important because it always decays in the expanding phase even to the second order, see §VII.E of [11]. Our reason for considering the flat background will be explained below Eq. (4).

As the metric we take
\[
ds^2 = -a^2 (1 + 2 \alpha) d\eta^2 - 2a^2 \beta_{\alpha \beta} dx^\alpha dx^\beta + a^2 \left[ g^{\alpha \beta}_0 (1 + 2 \varphi) + 2 \gamma_{\alpha \beta} \right] dx^\alpha dx^\beta,
\]
which follows from our convention in Eqs. (49), (175), and (178) of [11]. Here, \( a(t) \) is the scale factor, and \( \alpha, \beta, \gamma \) and \( \varphi \) are spacetime dependent perturbed-order variables; we take Bardeen’s metric convention in [10] extended to the second order.

A vertical bar indicates a covariant derivative based on \( g^{\alpha \beta}_0 \) which becomes \( \delta_{\alpha \beta} \) if we take Cartesian coordinates in the flat Friedmann background. By taking \( \gamma \equiv 0 \), which we call the spatial \( C \)-gauge, the spatial gauge mode is removed completely, thus all the remaining variables we are using are spatially gauge-invariant to the second order, see §VI.B.2 of [11].

In the following we will take \( \gamma \equiv 0 \) as the spatial gauge condition and use \( \chi \equiv a \beta + a^2 \gamma \) which becomes \( \chi = a \beta \).

As the energy-momentum tensor we take
\[
\begin{align*}
\hat{T}_0^0 &= -\mu - \delta \mu + \frac{1}{a^2} \mu \chi \alpha v_\alpha, \\
\hat{T}_a^0 &= -\mu (1 - \alpha) v_\alpha, \\
\hat{T}_\beta^\alpha &= \delta \rho \delta^\beta_\beta + \frac{1}{a^2} \left( \Pi^\alpha_\beta - \frac{1}{3} \delta^\delta_\beta \Delta \Pi \right) - \frac{1}{a^2} \mu \chi \alpha v_\beta,
\end{align*}
\]
which follows from our convention in Eqs. (84), (175), and (178) of [11]; tildes indicate the covariant quantities. Here, \( \mu \) is the background energy density, and \( \delta \mu, \delta \rho, \Pi \) and \( v \) are the perturbed order energy-density, isotropic pressure, anisotropic pressure, and the flux, respectively, all based on the normal frame vector \( \hat{n}_a \) with \( \hat{n}_a \equiv 0 \).

Although we are considering a zero-pressure system (thus, \( p = 0 \) and \( \delta p = 0 = \Pi \) to the linear order), it is essential to keep the perturbed pressure terms \( \delta \rho \) and \( \Pi \) because these do not necessarily vanish to the second order in perturbation depending on the coordinate (gauge) condition we choose. This is because in [11] we have evaluated the fluid quantities based on the normal-frame \( \hat{n}_a \); we will elaborate this point in §III.

To the background order we have the Friedmann equation [1,3,13]
\[
H^2 = \frac{8\pi G}{3} \mu - \text{const.} \frac{\Lambda}{3},
\]
with the energy (mass) density \( \rho \propto a^{-3} \); \( \Lambda \) is the cosmological constant. To the linear-order perturbations we have a second-order differential equation originally derived by Lifshitz [2,4]
\[
\hat{\delta} + 2H \hat{\delta} - 4\pi G \mu \delta = 0.
\]
An overdot indicates a time derivative based on \( t (dt \equiv d\eta) \) and \( H \equiv \frac{\dot{a}}{a} \). The variable \( a(t) \) is the scale factor, and \( \delta \equiv \frac{\delta \mu}{\mu} = \frac{4\pi}{v} \) with \( \mu \) (\( \rho \)) the background and perturbed parts, respectively, of the energy (mass) density field. The “const.” part is interpreted as the spatial curvature in Einstein’s gravity and the total energy in the Newton’s gravity [13]. Equation (4) is valid even in the presence of the cosmological constant \( \Lambda \) as well as the background curvature. We will include \( \Lambda \) term in the following.

The perturbed parts of equations to the second order are presented in Eqs. (195)-(201) of [11]. In a flat background with vanishing background pressure we have
\[
\kappa - 3H \alpha + 3 \varphi + \frac{\Delta}{a^2} \chi = N_0, \quad \text{(5)}
\]
\[
4\pi G \delta \mu + H \kappa + \frac{\Delta}{a^2} \varphi = \frac{1}{4} N_1, \quad \text{(6)}
\]
\[
\kappa + \frac{\Delta}{a^2} \chi - 12\pi G \mu a v = N_2^{(s)}, \quad \text{(7)}
\]
\[
\dot{\kappa} + 2H \kappa - 4\pi G (\delta \mu + 3\delta \rho) + \left( 3 \dot{H} + \frac{\Delta}{a^2} \right) \alpha = N_3, \quad \text{(8)}
\]
\[
\hat{\chi} + H \chi - \varphi - \alpha - 8\pi G \Pi = N_4^{(s)}, \quad \text{(9)}
\]
\[
\delta \mu + 3H (\delta \mu + \delta \rho) - \mu \left( \kappa - 3H \alpha + \frac{\Delta}{v} \right) = N_5, \quad \text{(10)}
\]
\[
\left( a^4 \mu \right)' - \frac{1}{a} \frac{\rho}{a} \mu - \frac{1}{a} \frac{\Delta}{3a^2} \Pi \right) = N_6^{(s)}, \quad \text{(11)}
\]
where the pure quadratic-order terms, \( N_i \), can be read from Eqs. (99)-(105) in [11]. \( \Delta \) is a Laplacian operator. Equation (5) is a definition of \( \kappa \). Eqs. (6)-(9) follow from \( \hat{C}_0^0, \hat{G}_a^0, \hat{C}_a^0 - \hat{G}_a^0 \) and \( \hat{G}_\beta - \frac{1}{3} \delta^\delta_\beta \hat{G}_\gamma \) components of Einstein’s equation, respectively, and Eqs. (11), follow from \( \hat{T}_{0b}^b = 0 \) and \( \hat{T}_{ab}^{\alpha \beta} = 0 \), respectively. To the linear order these set of equations without fixing the temporal gauge was presented by Bardeen in [10]. All our
equations include the cosmological constant in the background. These equations are presented without fixing the temporal gauge condition and using only the spatially gauge-invariant variables even to the second order; our choice of the spatial C-gauge (\( \gamma \equiv 0 \)) guarantees such invariances of the remaining variables, see §VI.B of [11].

As the proper temporal gauge condition we can choose any of the following: \( \alpha \equiv 0 \) (the synchronous gauge), \( \chi \equiv 0 \) (the zero-shear gauge), \( \delta \equiv 0 \) (the uniform-density gauge), \( \kappa \equiv 0 \) (the uniform-expansion gauge), \( v \equiv 0 \) (the comoving gauge), \( \varphi \equiv 0 \) (the uniform-curvature gauge), etc. Except for the synchronous gauge, each of the other temporal gauge conditions completely removes the temporal gauge mode. We can also take linear combinations of the above conditions, and choose different gauge conditions to different order, see §VI.C.2 of [11]. Thus, we have infinite number of different temporal gauge choices available to each order in perturbations.

From Eqs. (5)-(11) we can derive the following set of equations expressed using gauge-invariant variables

\[
\alpha_v = -\frac{1}{2}v_X^\alpha v_{X,\alpha} - \frac{1}{\mu} \left( \delta p_v + \frac{2}{3a^2} \Pi_v \right),
\]

\[
\dot{\delta}_v - \kappa_v = \frac{1}{a} \left( \delta v_X^\alpha v_{X,\alpha} \right) - \frac{3}{\mu} H \delta p_v,
\]

\[
\kappa_v + 2H \kappa_v - 4\pi G \mu \delta_v = \frac{\Delta}{2a^2} \left( v_X^\alpha v_{X,\alpha} \right) + 12\pi G \delta p_v,
\]

\[
\frac{\Delta}{a} \left( v_X^\alpha \dot{\varphi}_X - 2\dot{\varphi}_X v_{X,\alpha} + \varphi_X^\alpha v_{X,\alpha} \right)
+ \frac{5}{2} H \left( 2v_X^\alpha \Delta v_{X,\alpha} + v_{X,\alpha}^\alpha v_{X,\alpha} \right)
+ \frac{1}{2} \nabla^\alpha \left( \delta v_X^\alpha v_{X,\alpha} \right) - \frac{3}{a} \Delta^{-1} \nabla^\alpha \left( v_{X,\alpha} \Delta \varphi_X \right),
\]

\[
\alpha_X + \varphi_X = \varphi_X^\alpha - \Delta^{-1} \left( \varphi_X \Delta \varphi_X \right)
+ 3\Delta^{-2} \nabla^\beta \left( \varphi_X \varphi_{X,\alpha}^\beta \right) - 8\pi G \Pi_X,
\]

\[
4\pi G \mu \delta_v + \frac{\Delta}{a^2} \varphi_X = \frac{1}{2} \dot{H} v_X^2 - 3aH \Delta^{-1} \nabla^\alpha \left( \delta v_{X,\alpha} v_X \right)
+ \frac{1}{a^2} \left( 4\varphi_X^\alpha \Delta \varphi_X + \frac{3}{2} \varphi_X^\alpha \varphi_{X,\alpha} \right),
\]

\[
\dot{\varphi}_X + H \varphi_X - \frac{1}{a} \alpha_X = \frac{3}{2} a H \dot{v}_X^2 + 3H \varphi_X v_X - \frac{1}{2a} \varphi_X^2
- \frac{1}{a} \Delta^{-1} \nabla^\alpha \left( \delta v_{\varphi_X} v_{\varphi_X,\alpha} \right) + \frac{1}{a \mu} \delta p_v + \frac{2}{3a^2} \Pi_v,
\]

\[
\dot{\varphi}_X - H \alpha_X + 4\pi G \mu a v_X = \varphi_X \left( \dot{\varphi}_X - \frac{3}{2} H \varphi_X \right),
\]

\[
\dot{\varphi}_v = \frac{1}{2a} \Delta^{-1} \nabla^\alpha \left( v_{X,\alpha}^\beta \varphi_v v_{\varphi_X,\alpha} + v_{X,\alpha}^\alpha \Delta \varphi_X \right).
\]

Equations (12), (13), (14), and (15) follow from Eqs. (11), (10), (8), and (7), respectively, evaluated in the comoving gauge. In Eq. (15) we used \( \chi_X + a v_X = \chi_{(s)} + a v_{(s)} \) and \( v_{(s)}^\alpha |_v = 0 \); see Sec. VI.C.2 of [11]. Equation (16) follows from Eq. (9) evaluated in the zero-shear gauge. Equation (17) follows from Eqs. (6), (7), and using \( \delta \mu_v \equiv \delta \mu - \mu a v + \delta \mu_{(s)}^v, \varphi_X \equiv \varphi - H \chi_X + \varphi_{(s)}^v \) and \( \varphi_{(s)}^v |_v = 0 \). Equation (18) follows from Eq. (11) evaluated in the zero-shear gauge. Equation (19) follows from Eqs. (5), (7), removing \( \kappa \) term and evaluating in the zero-shear gauge. Equation (20) follows from Eqs. (5), (7), removing \( \kappa \) term and evaluating in the comoving gauge. In this set of equations we located the pure quadratic terms and the possible second-order pressure terms on the RHSs.

Our notation with a perturbed-order variable as a subindex, for example, \( \delta v \) indicates a unique gauge-invariant combination of \( \delta \) and \( v \) which becomes \( \delta \) under the comoving gauge condition \( v = 0 \). Thus, \( \delta \) in the comoving gauge is equivalent to a unique gauge-invariant combination \( \delta v \). To the linear order we have \( \delta v \equiv \delta - a(\dot{\mu}/\mu)v \). An explicit form of \( \delta v \) to the second order and other gauge-invariant combinations can be found in Eqs. (280)-(284) of [11]. As we can construct many (in fact, infinitely many) gauge invariant combinations for \( \delta \), our notation apparently has the advantage of showing explicitly which gauge-invariant combination we are considering [14].

Here, we briefly discuss a conserved variable to the second order. From Eqs. (20), (18), and (16) we have

\[
\frac{1}{a^3} \left( a^3 \dot{\varphi}_v \right) = -\frac{1}{2a^2} \Delta^{-1} \nabla^\alpha \nabla^\beta \left( \varphi_{v,\alpha} \varphi_{v,\beta} \right).
\]

To the linear order we have

\[
\varphi_v = C(x).
\]

Thus, \( \varphi_v \) remains constant in time. In the large-scale limit (super-horizon scale), ignoring the quadratic-order spatial gradient terms, Eq. (22) remains valid even to the second order; for more general proof considering the pressure term see [15,11].

### III. ISSUE OF PRESSURE

Now, we discuss the role of pressure terms in a medium without pressure. From Eqs. (233), (235) of [11] we notice that the gauge (coordinate) transformation to the second order causes pressure (both isotropic and anisotropic) terms to appear even in the case without pressure originally (physically). Such a complication occurs because our fluid quantities introduced in [11] are based on the normal-frame four-vector \( \tilde{n}_a \), which differs from the fluid four-vector \( \tilde{u}_a \). In [11] we have presented the fluid quantities based on \( \tilde{n}_a \) separately as well, see Eqs. (87), (88) of [11]; by using these equations we can translate fluid quantities in the normal frame to the ones in the fluid frame, and vice versa; the gauge transformation properties of the fluid quantities in the fluid frame are presented in Eq. (238) of [11]. The isotropic and anisotropic pressures are gauge (coordinate) dependent quantities. To the linear order in the Friedmann background the anisotropic
pressure is gauge invariant and the perturbed isotropic pressure depends on the coordinate only if we have non-vanishing (and time varying) background pressure. In the normal-frame, the pure coordinate transformation to the second and higher orders will cause both pressures (i.e., isotropic and anisotropic pressure like terms in the energy-momentum tensor) generated even in the case of vanishing pressures to the background and to the linear order, see Eq. (233) of [11]; the frame dependence of fluid quantities was studied in [16]. This complication does not occur for the fluid quantities based on the fluid frame-vector ̃uα; see Eq. (238) in [11].

For vanishing pressure terms in the background and first-order perturbations we have the following gauge-invariant combinations of pressure terms (based on ̃nα) [17]

\[
\delta p_\nu = \delta p - \frac{1}{3} \mu v^\alpha v_\alpha,
\]

\[
\Pi_\nu = \Pi - \frac{3}{2} \mu a^2 \Delta^{-2} \nabla^\alpha \nabla^\beta \left( v_\alpha v_\beta - \frac{1}{3} g_{\alpha\beta} v^\gamma v_\gamma \right).
\]

(23)

From this we notice that the gauge-invariant combination \(\delta p_\nu\) is the same as \(\delta p\) in the comoving gauge. Evaluating Eq. (23) in the zero-shear gauge (\(\chi \equiv 0\)) and using \(v_\chi \equiv v - \frac{1}{a} \chi\) to the linear order, we have

\[
\delta p_\chi = \delta p + \frac{1}{3} \mu v^\alpha v_\chi^\alpha,
\]

\[
\Pi_\chi = \Pi_\nu + \frac{3}{2} a^2 \mu \Delta^{-2} \nabla^\alpha \nabla^\beta \left( v_\chi v^\beta - \frac{1}{3} g_{\alpha\beta} v_\chi v^\gamma v_\gamma \right).
\]

(24)

As the definition of fluid without pressure we set the pressure terms in the comoving gauge equal to zero, thus

\[
\delta p_\nu \equiv 0 \equiv \Pi_\nu,
\]

(25)

which are gauge-invariant (and physical) zero-pressure conditions. Thus,

\[
\delta p_\chi = \frac{1}{3} \mu v^\alpha v_\chi^\alpha,
\]

\[
\Pi_\chi = \frac{3}{2} a^2 \mu \Delta^{-2} \nabla^\alpha \nabla^\beta \left( v_\chi v^\beta - \frac{1}{3} g_{\alpha\beta} v_\chi v^\gamma v_\gamma \right).
\]

(26)

We set the pressure terms using Eqs. (25), (26). Thus, for fluid quantities based on the normal-frame, in the gauge other than the comoving gauge the physical zero-pressure condition implies presence of pressure terms in the definition of the energy-momentum tensor.

In the comoving gauge without rotation the two frames, ̃uα and ̃nα, coincide. The normal frame ̃nα has ̃nα ≡ 0. The fluid quantities are ordinarily defined in the fluid (̃uα) frame which differs in general from the normal four-vector ̃nα. In the normal-frame information about the fluid motion is present in the flux four-vector ̃qα with ̃qα ̃nα ≡ 0. In the energy frame, which takes vanishing flux ̃qα ≡ 0 as the frame condition, the comoving gauge condition takes ̃nα ≡ 0 for the fluid four-vector; here, we ignore the vector-type perturbation. Since ̃uα = 0 it coincides with the normal frame vector. Now, in the normal frame, which takes ̃nα ≡ 0 as the frame condition, the comoving gauge condition without rotation implies ̃qα ≡ 0. Thus, as long as we take the comoving gauge without rotation, in either frame we have ̃qα ≡ 0 and ̃uα = 0 = ̃nα; i.e., the fluid four-vector coincides with the normal four-vector.

IV. A PROOF

Now, we come to our main point proving the relativistic -Newtonian correspondence to the second order. Combining Eqs. (13), (14) we can derive [18]

\[
\vec{\delta}_v + 2 H \vec{\delta}_v - 4 \pi G \mu \vec{\delta}_v
\]

\[
= \frac{1}{a^2} \frac{\partial}{\partial t} \left[ a \left( \vec{a} v^\alpha \right) \right] + \frac{\Delta}{2 a^2} \left( \vec{a} v^\alpha v_\chi^\alpha \right). \tag{27}
\]

Equations (13), (14), (17), and (27) can be compared with the Newtonian perturbation equations.

The mass conservation, the momentum conservation, and the Poisson’s equation in Newtonian context give [19]

\[
\vec{\delta} \equiv \frac{1}{a} \nabla \cdot \vec{u} = - \frac{1}{a} \nabla \cdot (\delta \vec{u}), \tag{28}
\]

\[
\vec{u} + H \vec{u} + \frac{1}{a} \nabla \delta \Phi = - \frac{1}{a} \vec{u} \cdot \nabla \vec{u}, \tag{29}
\]

\[
\frac{1}{a^2} \nabla^2 \delta \Phi = 4 \pi G \rho_\delta. \tag{30}
\]

From these we have

\[
\vec{\delta} + 2 H \vec{\delta} - 4 \pi G \rho_\delta
\]

\[
= - \frac{1}{a^2} \frac{\partial}{\partial t} \left[ a \nabla \cdot (\delta \vec{u}) \right] + \frac{1}{a} \nabla \cdot (\vec{u} \cdot \nabla \vec{u}). \tag{31}
\]

In the Newtonian context Eqs. (28)-(31) are valid to fully nonlinear order; i.e., the zero-pressure Newtonian fluid equations are exact in quadratic order nonlinearity. Equation (31) has been analysed extensively in the Newtonian context, see [20,21].

To the linear order it is well known that \(\vec{\delta} v_\chi, - \nabla v_\chi\) and \(-\phi_\chi\) (or \(\alpha_\chi\)) correspond to a density perturbation \(\delta \equiv \delta \vec{v} / \vec{v}\) with \(\phi \equiv \rho + \delta \rho\) and \(\delta\rho\) (the mass density), a velocity perturbation \(\vec{u}\) and a perturbation of the gravitational potential \(\delta \Phi\), respectively. [7,12,9]. To the linear order we may identify \[9\]

\[
\delta = \delta v, \quad \delta \Phi = - \phi_\chi = \alpha_\chi,
\]

\[
\vec{u} \equiv - \nabla v_\chi, \quad - \frac{1}{a} \nabla \cdot \vec{u} = \frac{\Delta}{a} v_\chi = \kappa v_\chi. \tag{32}
\]
As we identify

$$ \delta_v = \delta, \quad \kappa_v = -\frac{1}{a} \nabla \cdot \mathbf{u}, \quad (33) $$

to the second order, Eq. (27) coincides exactly with Eq. (31). Equation (13) becomes

$$ \delta_v + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta_v \mathbf{u}), \quad (34) $$

which coincides with Eq. (28). Equation (14) gives

$$ \nabla \cdot (\mathbf{u} + H \mathbf{u}) + 4\pi G \mu \delta_v = -\frac{1}{a} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}), \quad (35) $$

which also follows from Eqs. (29), (30) in the Newtonian context. This completes our proof of the correspondence. Such identifications of density and velocity perturbations imply that we cannot identify $-\varphi_\alpha$ (or $\alpha_\chi$) with $\delta \Phi$ to the second order. This conclusion follows from a close examination of Eqs. (12)- (20). In fact, using the intrinsic three-space curvature in Eq. (265) of [11]

$$ R^{(h)} = \frac{2}{a^2} \left[ -2\Delta \varphi + 8\varphi \Delta \varphi + 3 \varphi \alpha \varphi, \alpha \right], \quad (36) $$

Eq. (17) becomes

$$ 4\pi G \mu \delta_v - \frac{1}{4} R^{(h)}_\chi = \frac{1}{2} \dot{H} \Delta \varphi^2 - 3aH \dot{H} \Delta^{-1} \nabla^\alpha (\delta_v, \alpha \varphi, \chi), \quad (37) $$

which still differs from the Newtonian Poisson’s equation. Thus, we conclude that we do not have a relativistic variable which corresponds to the Newtonian potential to the second order. Apparently, it is essentially important to employ mixed gauge conditions, i.e., take different gauge conditions for different variables, to make correspondence with the Newtonian system: in this way, correct identifications of (gauge-invariant) variables are important to show the relativistic-Newtonian correspondence.

At this point, let us clarify the meaning of the quantities involved in Eqs. (32), (33). Variables $\alpha$, $\chi$ and $\varphi$ are defined in the metric in Eq. (1). Variables $\chi$ and $\varphi$ can be further identified as the perturbed shear and perturbed three-space curvature of the normal hypersurface, respectively. From Eq. (36) we find that the intrinsic scalar curvature $R^{(h)}$ vanishes for $\varphi = 0$. From Eq. (264) of [11] we find that the tracefree part of the extrinsic curvature tensor $K_{\alpha \beta}$ (equivalently, shear tensor of the normal frame vector with a minus sign) vanishes for $\chi = 0$. The variable $\kappa$ can be interpreted as the perturbed expansion with a minus sign. From Eqs. (57), (99), and (179) of [11] we have $K = -3H + \kappa$ where $K$ is a trace of the extrinsic curvature tensor $K_{\alpha \beta}$ (equivalently, the expansion scalar, $\dot{\theta} \equiv \dot{a}_\alpha^\alpha$, with a minus sign). Variables $\delta$ and $v$ are defined in Eq. (2) and can be interpreted as the flux of the normal-frame, respectively. In the normal frame, from Eqs. (4), (76), and (175) of [11] we have the flux vector becomes $J_\alpha = -\delta_{\alpha \beta} T^\beta_\alpha = -\alpha \mu \nu, \alpha$.

Here we discuss the relation between the comoving and the synchronous gauge to the second order. Equation (12) shows that $\alpha_\alpha$, which is the same as $\alpha$ in the comoving gauge ($v \equiv 0$), does not vanish to the second order. This means that the comoving gauge does not imply our synchronous gauge to the second order in a zero-pressure medium. At this point it is important to remember that we already have fixed the spatial gauge condition using $\gamma \equiv 0$. The original synchronous gauge used by Lifshitz fixes $\delta_{00} \equiv 0 \equiv \delta_{0a}$, thus $\alpha = 0$ for the temporal gauge and $\beta = 0$ for the spatial gauge condition. We prefer to fix $\gamma \equiv 0$ (spatial C-gauge) as the spatial gauge condition instead of $\beta \equiv 0$ (spatial B-gauge) because the latter condition fails to fix the spatial gauge degree of freedom completely whereas the first one fixes it completely; this is true even to the second order, and in fact to all orders, in perturbations, see §VI.B.2 and VI.C of [11]. We can show that the comoving temporal gauge ($v \equiv 0$) together with spatial $B$-gauge ($\beta \equiv 0$) implies $\alpha = 0$ even to the second order, for a proof see [22]. By imposing the comoving ($v \equiv 0$) and the synchronous ($\alpha \equiv 0$) gauge conditions simultaneously, Kasai [23] has derived a different equation compared with ours: such a redundant choice is allowed as one takes $\beta = 0$ as the spatial gauge condition. However, in that gauge condition (the spatial $B$-gauge) the spatial gauge-mode is incompletely fixed, and the comparison with the Newtonian analyses is not available.

V. FULLY NONLINEAR EQUATIONS

By extending our comoving gauge condition to be valid to all orders we can formally derive the completely nonlinear equations for the density and velocity perturbations. We will present two methods to reach such nonlinear equations. These are based on the ADM (3 + 1) equations and the covariant (1+3) equations summarised in §II.A and II.B, respectively, of [11]. With the hindsight gained from our second-order perturbations in previous sections, it is best to take the comoving gauge condition to all orders. In the normal-frame context, only the comoving gauge allows the zero-pressure conditions to be, by definition, vanishing pressure terms to all orders. To the second order, all the equations we need to derive Eqs. (27), (34), and (35) are Eqs. (12), (13), and (14) which follow from Eqs. (8), (10), and (11); these are the Raychaudhury, the energy conservation and the momentum conservation equations, respectively. We have presented a redundant set of equations in (12)-(20) in order to show the relativistic-Newtonian correspondences with some perspective.

The complete set of ADM (3 + 1) equations is presented in Eqs. (8)-(13) of [11], see [24] for original studies. We only need Eqs. (10), (12), and (13) of [11] which are
the trace of ADM propagation equation, and the energy and momentum conservation equations, respectively. We take the comoving gauge condition to all orders which makes the flux four-vector to vanish, i.e., \( J_\alpha \equiv 0 \); here we assume vanishing vector-type perturbation, thus irrotational, which could contribute to \( J_\alpha \). Under such conditions the zero-pressure conditions (in our normal frame) imply \( S \equiv 0 \equiv S_{\alpha\beta} \) to all orders; \( S \) and \( S_{\alpha\beta} \) are the trace and tracefree parts, respectively, of the spatial part of energy-momentum tensor. Equation (13) of \([11]\) gives

\[
N_{,\alpha} = 0,
\]

where \( N \) is defined as \( \theta^{00} \equiv -N^{-2} \). Thus, we may set \( N \equiv a(t) \) to all orders. In this case we have, for example, \( \dot{E} \equiv E_\alpha N^{-1} \). Now, Eqs. (12), (10) of \([11]\) become

\[
\dot{\hat{E}} - K E = 0,
\]

\[
\dot{\hat{K}} - \frac{1}{3} K^2 - \bar{K}^{\alpha\beta} \bar{K}_{\alpha\beta} - 4\pi G E + \Lambda = 0,
\]

where \( \dot{E} \equiv \dot{E} - E_\alpha N^\alpha N^{-1} \), etc.; \( E \) is the energy density based on normal frame vector, and \( K \) and \( \bar{K}_{\alpha\beta} \) are the trace and tracefree parts, respectively, of the extrinsic curvature; \( N_{,\alpha} \) is defined as \( \theta^{0\alpha} \equiv N_{\alpha} \). The spatial indices in ADM formulation are based on the spatial metric \( h_{\alpha\beta} \) defined as \( \theta_{\alpha\beta} \equiv \theta^{0\beta} \). By combining these equations we have

\[
\left( \frac{\dot{E}}{E} \right) - \frac{1}{3} \left( \frac{\dot{E}}{E} \right)^2 - \bar{K}^{\alpha\beta} \bar{K}_{\alpha\beta} - 4\pi G E + \Lambda = 0.
\]

Notice again that Eqs. (39)-(41) are valid to all orders, i.e., these equations are fully nonlinear. From Eqs. (39)-(41), using

\[
E \equiv \mu + \delta \mu,
\]

and the quantities presented in \([11]\) we can easily derive Eqs. (34), (35), and (27), respectively; see the next section.

The complete set of covariant (1 + 3) equations is presented in Eqs. (26)-(37) of \([11]\); see \([25]\) for original studies. We only need Eqs. (26)-(28) of \([11]\) which are the energy and momentum conservations and the Raychaudhury equation, respectively. We take the energy-frame which sets the energy flux term to vanish, i.e., \( \tilde{q}_a \equiv 0 \). In this frame the frame four-vector \( \tilde{u}_a \) is the fluid four-vector. The pressure conditions imply \( \tilde{p} \equiv 0 \equiv \tilde{\pi}_{ab} \) to all orders; \( \tilde{\pi}_{ab} \) is the covariant anisotropic stress based on \( \tilde{u}_a \). Equation (27) of \([11]\) gives vanishing acceleration vector, i.e., \( \tilde{u}_a \equiv \tilde{u}_{a;b} \tilde{u}^b = 0 \) to all orders. Thus, Eqs. (26), (28) of \([11]\) become

\[
\tilde{\dot{\mu}} + \tilde{\mu} \tilde{\dot{\theta}} = 0,
\]

\[
\tilde{\dot{\theta}} + \frac{1}{3} \tilde{\theta}^2 + \tilde{\sigma}^{ab} \tilde{\sigma}_{ab} - \tilde{\omega}^{ab} \tilde{\omega}_{ab} + 4\pi G \tilde{\mu} - \Lambda = 0.
\]

where \( \tilde{\theta} \equiv \tilde{u}^a_{,a} \) is an expansion scalar, and \( \tilde{\sigma}_{ab} \) is the shear tensor. An overdot with tilde is a covariant derivative along the \( \tilde{u}_a \) vector, e.g., \( \tilde{\mu} \equiv \tilde{\mu}_{,a} \tilde{u}^a \). By combining these equations we have

\[
\left( \frac{\tilde{\mu}}{\mu} \right)^2 - \frac{1}{3} \tilde{\mu}^2 - \tilde{\sigma}^{ab} \tilde{\sigma}_{ab} + \tilde{\omega}^{ab} \tilde{\omega}_{ab} - 4\pi G \tilde{\mu} + \Lambda = 0.
\]

Notice that Eqs. (43)-(45) are valid to all orders, i.e., these equations are fully nonlinear. More general equation in a fully covariant form considering the general pressure terms can be found in Eq. (88) of \([26]\).

We take the comoving gauge condition to all orders which makes the space part of four-vector with low index to vanish, i.e., \( \tilde{u}_a \equiv 0 \); here we also assume vanishing vector-type perturbation, thus irrotational, which could contribute to \( \tilde{u}_a \). As our gauge condition (and the irrotational condition) implies \( \tilde{u}_a \equiv 0 \), the frame vector is the same as the normal frame, thus \( \tilde{u}_a = \tilde{n}_a \). In such a case we have vanishing rotation of the \( \tilde{u}_a \) flow, thus \( \tilde{\omega}_{ab} = 0 \). From Eqs. (43)-(45), using

\[
\tilde{\mu} \equiv \mu + \delta \mu,
\]

and the quantities presented in \([11]\) we can easily derive Eqs. (34), (35), and (27), respectively; see the next section.

The complete set of covariant (1 + 3) equations is presented in Eqs. (26)-(37) of \([11]\); see \([25]\) for original studies. We only need Eqs. (26)-(28) of \([11]\) which are the energy and momentum conservations and the Raychaudhury equation, respectively. We take the energy-frame which sets the energy flux term to vanish, i.e., \( \tilde{q}_a \equiv 0 \). In this frame the frame four-vector \( \tilde{u}_a \) is the fluid four-vector. The pressure conditions imply \( \tilde{p} \equiv 0 \equiv \tilde{\pi}_{ab} \) to all orders; \( \tilde{\pi}_{ab} \) is the covariant anisotropic stress based on \( \tilde{u}_a \). Equation (27) of \([11]\) gives vanishing acceleration vector, i.e., \( \tilde{u}_a \equiv \tilde{u}_{a;b} \tilde{u}^b = 0 \) to all orders. Thus, Eqs. (26), (28) of \([11]\) become

\[
\tilde{\dot{\mu}} + \tilde{\mu} \tilde{\dot{\theta}} = 0,
\]

\[
\tilde{\dot{\theta}} + \frac{1}{3} \tilde{\theta}^2 + \tilde{\sigma}^{ab} \tilde{\sigma}_{ab} - \tilde{\omega}^{ab} \tilde{\omega}_{ab} + 4\pi G \tilde{\mu} - \Lambda = 0.
\]

VI. ANOTHER DERIVATION INCLUDING THE GRAVITATIONAL WAVES

Since Eqs. (34), (35) are our main results allowing us to conclude about the relativistic-Newtonian correspondence, in the following we will derive these equations in some detail again directly from the fully nonlinear equations in §V. Now, we include the gravitational wave contribution. The metric becomes
where $C_{\alpha\beta}$ is the transverse and tracefree gravitational waves; its indices are based on $g^{(3)}_{\alpha\beta}$. We work in the temporal comoving gauge. Thus, $C_{\alpha\beta}$ is also evaluated in the comoving gauge, and equivalent to a gauge-invariant combination $C_{\alpha\beta}^{(t)}$.

We introduce

$$E \equiv \mu + \delta \mu, \quad K \equiv -3 \frac{\dot{a}}{a} + \kappa,$$  \hspace{1cm} (49)

see Eqs. (45), (72), (178), and (179) of [11]. We have

$$\dot{E} \equiv \dot{E} - E_{\alpha} N^{\alpha} N^{-1} = \dot{\mu} + \delta \mu + \frac{1}{a^{2}} \delta \mu_{,} \chi_{,\alpha}^{\alpha},$$

$$\dot{K} \equiv \dot{K} - K_{\alpha} N^{\alpha} N^{-1} = -3 \left( \frac{\dot{a}}{a} \right) + \dot{\kappa} + \frac{1}{a^{2}} \kappa_{,} \chi_{,\alpha}^{\alpha}. \hspace{1cm} (50)$$

In setting $N = a$ we already have used the comoving gauge condition. Since we take the comoving gauge we often ignore the subindex $v$ which indicates the comoving gauge choice (equivalently the unique corresponding gauge-invariant combination between the variable and $v$); for example, our $\delta$ is the same as a gauge-invariant combination $\delta_{v}$ which is the same as $\delta$ in the comoving gauge setting $v \equiv 0$. Using Eqs. (55), (57), and (175) of [11] we can show

$$K_{\alpha\beta} K_{\alpha\beta} = \frac{1}{a^{2}} \left[ \chi_{,\alpha\beta} \chi_{,\alpha\beta} - \frac{1}{3} (\Delta \chi)^{2} \right]$$

$$+ \dot{C}_{\alpha\beta}^{(t)} \left( \frac{2}{a^{2}} \chi_{,\alpha\beta} + \dot{C}_{\alpha\beta}^{(t)} \right). \hspace{1cm} (51)$$

Equations (39), (40) become

$$\left( \frac{\dot{\mu}}{\mu} + 3 \frac{\dot{a}}{a} \right) (1 + \delta) + \dot{\delta} - \kappa = \kappa \delta + \frac{1}{a^{2}} \delta_{,} \chi_{,\alpha}^{\alpha},$$

$$\frac{3}{a} \dot{a} + 4 \pi G \mu - \Lambda - \dot{\kappa} - \frac{2}{a^{2}} \kappa + 4 \pi G \mu \delta$$

$$= \frac{1}{a^{2}} \kappa_{,} \chi_{,\alpha}^{\alpha} - \frac{1}{3} \kappa^{2} - \frac{1}{a^{3}} \chi_{,\alpha}^{\alpha} \chi_{,\alpha}^{\alpha} - \frac{1}{3} (\Delta \chi)^{2}$$

$$+ \dot{C}_{\alpha\beta}^{(t)} \left( \frac{2}{a^{2}} \chi_{,\alpha\beta} + \dot{C}_{\alpha\beta}^{(t)} \right). \hspace{1cm} (53)$$

Now, we have to relate $\chi (\equiv \chi_{v})$ to our notation. Apparently, we need $\chi$ only to the linear order. To the linear order the $C_{\alpha\beta}^{(t)}$-component of Einstein equation in Eq. (15) gives $\dot{\chi}_{v} = - \kappa_{v} \equiv \nabla \cdot \chi_{v}$; we have $\chi_{v} \equiv \gamma - av \equiv - av_{\chi}$ to the linear order. As our $u_{\chi}$ is of the potential type, i.e., of the form $u_{\chi} \equiv u_{,\alpha}$, we have

$$u_{\chi} = \frac{1}{a} \nabla \chi_{v}, \hspace{1cm} (54)$$

to the linear order. Thus, we have

$$\left( \frac{\dot{\mu}}{\mu} + 3 \frac{\dot{a}}{a} \right) (1 + \delta) + \dot{\delta} + \frac{1}{a} \nabla \cdot u_{\chi} = - \frac{1}{a} \nabla \cdot (\delta_{v} u_{\chi}), \hspace{1cm} (55)$$

$$3 \frac{\dot{a}}{a} + 4 \pi G \mu - \Lambda + \frac{1}{a} \nabla \cdot \left( \dot{u}_{\chi} + \frac{\dot{a}}{a} u_{\chi} \right) + 4 \pi G \mu \delta$$

$$= - \frac{1}{a^{2}} \nabla \cdot (u_{\chi} \cdot \nabla u_{\chi}) - \dot{C}_{\alpha\beta}^{(t)} \left( \frac{2}{a} u_{,\alpha\beta} + \dot{C}_{\alpha\beta}^{(t)} \right). \hspace{1cm} (56)$$

The perturbed parts give Eqs. (34), (35) with additional contributions from the gravitational waves in Eq. (35), thus in Eq. (27) as well.

Therefore, in the presence of the tensor-type perturbation we have

$$\dot{\delta}_{v} + \frac{1}{a} \nabla \cdot u_{\chi} = - \frac{1}{a} \nabla \cdot (\delta_{v} u_{\chi}), \hspace{1cm} (57)$$

$$\frac{1}{a} \nabla \cdot (u_{\chi} + \frac{\dot{a}}{a} u_{\chi}) + 4 \pi G \mu \delta$$

$$= - \frac{1}{a^{2}} \nabla \cdot (u_{\chi} \cdot \nabla u_{\chi}) - \dot{C}_{\alpha\beta}^{(t)} \left( \frac{2}{a} u_{,\alpha\beta} + \dot{C}_{\alpha\beta}^{(t)} \right). \hspace{1cm} (58)$$

thus

$$\dot{\delta}_{v} + 2 \frac{\dot{a}}{a} \delta_{v} - 4 \pi G \mu \delta_{v} = - \frac{1}{a^{2}} \frac{\partial}{\partial \theta} \left[ a \nabla \cdot (\delta_{v} u_{\chi}) \right]$$

$$+ \frac{1}{a^{2}} \nabla \cdot (u_{\chi} \cdot \nabla u_{\chi}) + \dot{C}_{\alpha\beta}^{(t)} \left( \frac{2}{a} u_{,\alpha\beta} + \dot{C}_{\alpha\beta}^{(t)} \right). \hspace{1cm} (59)$$

The presence of linear-order gravitational waves can generate the second-order scalar-type perturbation by generating the shear terms. Here, we note the behaviour of the gravitational waves in the linear regime. To the linear order the gravitational waves are described by the well known wave equation [2]

$$\ddot{C}_{\alpha\beta}^{(t)} + \frac{3}{a^{2}} \dot{C}_{\alpha\beta}^{(t)} - \Delta \frac{1}{a^{2}} C_{\alpha\beta}^{(t)} = 0. \hspace{1cm} (60)$$

In the super-horizon scale the non-transient mode of $C_{\alpha\beta}^{(t)}$ remains constant, thus $\dot{C}_{\alpha\beta}^{(t)} = 0$, and in the sub-horizon scale, the oscillatory $C_{\alpha\beta}^{(t)}$ redshifts away, thus $C_{\alpha\beta}^{(t)} \propto a^{-1}$. Thus, we anticipate that the contribution of gravitational waves to the scalar-type perturbation is not significant to the second order.

To the second order the equation for tensor-type perturbation (gravitational waves) can be derived from Eqs. (103), (210) of [11]. Since we are ignoring the vector-type perturbation from Eqs. (211), (199) of [11] we have

$$C_{\alpha\beta}^{(t)} + \frac{3}{a^{2}} \dot{C}_{\alpha\beta}^{(t)} - \frac{\Delta}{a^{2}} C_{\alpha\beta}^{(t)} = N_{\alpha\beta},$$

$$- \frac{3}{2} \left( \nabla \cdot \nabla + \frac{1}{3} g_{(3)} \Delta \right) \Delta^{-2} \nabla \chi$$

$$\nabla \cdot \Delta^{-1} N_{\gamma\delta}, \hspace{1cm} (61)$$

where we assumed a flat background and set anisotropic stress to be zero. From Eq. (103) of [11] to the second order we have
In Eq. (62) we have ignored α and φ terms which are already quadratic order in the comoving gauge, see Eqs. (12), (20). Since we are in the comoving gauge, we have χ = χv, φ = φv, κ = κv and C(4)(αβγδ) = C(4)(αβ). Apparently, we need χv, κv and φv to the linear order. We have κv = −1/α |∇· u and u = 1/α |∇χv. For φv we have

φv ≡ φ − aHv = φχ − aHvχ,

where we have φχ = −∂Φ and u = −∇vχ in Eq. (32). Using these identifications we can express the scalar-type perturbation variables in Eq. (62) in terms of the Newtonian variables.

Equations (57), (58), and (61) provide a complete set describing the scalar- and tensor-type perturbations to the second order in the flat Friedmann background. From these equations we can see that the linear-order scalar-type (tensor-type) perturbation works as a source for the tensor-type (scalar-type) perturbation to the second order. Such effects and the presence of the gravitational waves are purely general relativistic ones.

VII. DISCUSSION

We have shown that to the second order, ignoring the gravitational wave contribution, the zero-pressure relativistic cosmological perturbation equations can be exactly identified with the known equations in Newtonian system, compare Eqs. (57)-(59) with Eqs. (28)-(31). More precisely, the relativistic equations can be identified with the continuity equation and the divergence of the Euler equation replacing the Newtonian gravitational potential using Poisson’s equation. In order to achieve such a correspondence we need correct identification of gauge-invariant density and velocity perturbation variables as in Eqs. (32), (33). It is important to notice that we have avoided using the potential-like variable in our identification. In fact, we showed that we do not have a relativistic variable which corresponds to the Newtonian potential to the second order. This is understandable because the gravitational potential introduced in Poisson’s equation reveals the action-at-a-distance nature and the static nature of Newton’s gravity theory compared with the relativistic gravity.

As a consequence, to the second order, the Newtonian hydrodynamic equations (31), (34), and (35) remain valid in all cosmological scales including the super-horizon scale. Although showing the equivalence of the zero-pressure relativistic scalar-type perturbation to the Newtonian ones to the second order, may not be entirely surprising it should not be so obvious either. It might be as well that our relativistic results give relativistic correction terms appearing to the second order which become important as we approach and go beyond the horizon scale. Our results show that there are no such correction terms appearing to the second order, and the correspondence is exact to that order. A complementary result, showing the relativistic-Newtonian correspondence in the Newtonian limit of the post-Newtonian approach, can be found in [28]. In fact, the Newtonian hydrodynamic equations appear naturally as the zeroth-order post-Newtonian limit [29].

We note that although we assumed a flat background, our equations are valid with the cosmological constant. Thus, these are compatible with current observations of the large-scale structure and the cosmic microwave background radiation which favour near flat Friedmann world model with non-vanishing Λ [30]. As we consider a flat background the ordinary Fourier analysis can be used to study the mode-couplings as in the Newtonian case in [21]. Our result also may have the following important practical cosmological implication. As we have proved that the Newtonian hydrodynamic equations are valid in all cosmological scales to the second order, our result has an important cosmological implication that large-scale Newtonian numerical simulation can be used more reliably in the general relativistic context even as the simulation scale approaches near (and goes beyond) the horizon scale.

At this point, it is important to be reminded that we showed the relativistic-Newtonian correspondence for the density and velocity perturbations, but not for the gravitational potential. Therefore, although our result assures that one can trust cold dark matter simulations at all
scales for the density and velocity fields, it does not imply that one can trust the Newtonian simulations for effects involving the gravitational potential, like the weak gravitational lensing effects. Indeed, in order to handle the lensing effects properly we often require an extra factor of two which comes from the post-Newtonian effects.*

Since the Newtonian system is exact to the second order in nonlinearity, besides the gravitational wave contribution to the second and higher order, any nonvanishing third and higher order perturbation terms in the relativistic analysis can be regarded as the pure relativistic corrections. Expanding the fully nonlinear equations in (43)-(45) or (39)-(41) to third and higher order will give the potential correction terms. Our recent investigation of this important open question shows that to the third order there occur pure relativistic correction terms which are of $\varphi_c$-order higher [31]. Thus, the corrections are independent of the horizon and are small; see the accompanying contribution in [31].

In this work we have considered an irrotational single component dust in the flat background. Extending any of these assumptions could lead to situations which deserve further attention. First, it would be interesting to see up to what point the correspondence between the two theories can be extended in the non-flat case. In this way we can identify possible relativistic effects caused by the non-flat nature of the background. Second, in this work we have ignored the vector-type perturbation because it simply decays in the expanding phase. This has to do with considering only the longitudinal part of $\mathbf{u}$ in Eqs. (35), (58). It would be interesting to include the rotational mode to see the similarity and difference between the two gravity theories. As the realistic Newtonian simulations include the whole $\mathbf{u}$ vector as the perturbed velocity it would be practically important to see the role of relativistic vector-type perturbation to the second order, and to determine whether the relativistic effect could be important. Third, the usual cosmological simulations include the cold dark matter together with the baryon, thus a system with two components. In the second and the third subjects are left for future studies.

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This equation was derived in Eq. (342) of [11] without noticing the issue of zero-pressure condition in the normal-frame choice. Since we used only Eqs. (13), (14) in [11], which are all evaluated in the comoving gauge, it does not affect the result. In [11] the Newtonian correspondence to the second order was addressed incompletely for which we need to examine the other equations in Eqs. (12)-(20). The analyses made in §VII.C of [11], although incomplete, remain correct.

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In the normal frame we have the acceleration vector becomes \( \tilde{a}_\alpha = (\alpha - \alpha^2 + \frac{1}{2} \beta \gamma^\gamma \beta_{,\gamma})_{,\alpha} \), see Eq. (69) of [11]. In that frame the temporal comoving gauge implies \( \tilde{q}_\alpha = 0 \), see Eqs. (72), (175) of [11]. The zero-pressure condition implies the momentum-conservation equation, Eq. (27) of [11], gives \( \tilde{a}_\alpha = 0 \). Thus, if we take the spatial B gauge, \( \beta \equiv 0 \), we have \( \alpha = 0 \) which is the temporal synchronous gauge.

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