Confinement potential
in dual Monopole Nambu–Jona–Lasinio model with
dual Dirac strings

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Abstract

Interquark confinement potential is calculated in the dual Monopole Nambu–
Jona–Lasinio model with dual Dirac strings suggested in Refs.[1,2] as a functional
of a dual Dirac string length. The calculation is carried out by the explicit inte-
gration over quantum fluctuations of a dual–vector field (monopole–antimonopole
collective excitation) around the Abrikosov flux line and string shape fluctuations.
The contribution of the scalar field (monopole–antimonopole collective excitation)
exchange is taken into account in the tree approximation due to the London limit
regime.

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1 Introduction

The dual Monopole Nambu–Jona–Lasinio model (MNJL) with dual Dirac strings as continuum space–time analogy of Compact Quantum Electrodynamics (CQED) [1] has been formulated in Ref.[2–4]. The MNJL–model is based on a Lagrangian, invariant under magnetic $U(1)$ symmetry, with massless magnetic monopoles self–coupled through a local four–monopole interaction [2,3]:

$$\mathcal{L}(x) = \bar{\chi}(x)i\gamma^\mu\partial_\mu\chi(x) + G[\bar{\chi}(x)\chi(x)]^2 - G_1[\bar{\chi}(x)\gamma^\mu\chi(x)][\bar{\chi}(x)\gamma^\mu\chi(x)],$$  \hspace{1cm} (1.1)

where $\chi(x)$ is a massless magnetic monopole field, $G$ and $G_1$ are positive phenomenological constants. Below we show that $G_1 = G/4$. The magnetic monopole condensation accompanies itself the creation of massive magnetic monopoles $\chi_M(x)$ with mass $M$, $\bar{\chi}\chi$–collective excitations with quantum numbers of the scalar Higgs field $\sigma$ with the mass $M_\sigma = 2M$ and the massive dual–vector field $C_\mu$ with the mass $M_C$ defined as [2,3]:

$$M_C^2 = \frac{g^2}{2G_1} - \frac{g^2}{8\pi^2}[J_1(M) + M^2J_2(M)],$$ \hspace{1cm} (1.2)

where $J_1(M)$ and $J_2(M)$ are quadratically and logarithmically divergent integrals [1,2]

$$J_1(M) = \int \frac{d^4k}{\pi^2M} = M^2\ell n\left(1 + \frac{\Lambda^2}{M^2}\right),$$

$$J_2(M) = \int \frac{d^4k}{\pi^2(M^2 - k^2)} = \ell n\left(1 + \frac{\Lambda^2}{M^2}\right) - \frac{\Lambda^2}{M^2} + \Lambda^2.$$ \hspace{1cm} (1.3)

Here $\Lambda$ is the ultra–violet cut–off. The mass of the massive magnetic monopole field $\chi_M(x)$ obeys the gap–equation [2,3]:

$$M = -2G <\bar{\chi}(0)\chi(0)> = \frac{GM}{2\pi^2}J_1(M)$$ \hspace{1cm} (1.4)

derived from the effective Lagrangian of the scalar $\sigma(x)$ and the dual–vector $C_\mu(x)$ fields by virtue of the suppression of the direct transitions $\sigma \leftrightarrow \text{vacuum}$. On the other hand, due to one–loop corrections to the mass of the monopole field derived by using the Lagrangian Eq.(1.1) the gap–equation should read

$$M = -2\left(\frac{3}{4}G + G_1\right) <\bar{\chi}(0)\chi(0)>.$$ \hspace{1cm} (1.5)

Since the level of the collective $\bar{\chi}\chi$–excitations should be completely compatible with the monopole level, the gap–equations Eq.(1.2) and Eq.(1.4) should coincide. This fixes $G_1$ in terms of $G$ as $G_1 = G/4$.

As has been shown in Refs.[2, 3] the vacuum expectation values of time ordered products of densities expressed in terms of the massless monopole field, i.e., the magnetic monopole Green functions

$$G(x_1, \ldots, x_n) = <0|\Gamma(\bar{\chi}(x_1)\Gamma_1\chi(x_1)\ldots\bar{\chi}(x_n)\Gamma_n\chi(x_n))|0>_{\text{conn.}},$$ \hspace{1cm} (1.6)
where $\Gamma_i (i = 1, \ldots, n)$ are the Dirac matrices, are given by [2,3]

$$G(x_1, \ldots, x_n) = \langle 0 | T(\bar{\chi}(x_1) \Gamma_1 \chi(x_1) \ldots \bar{\chi}(x_n) \Gamma_n \chi(x_n)) | 0 \rangle_{\text{conn.}} =$$

$$= (M) \langle 0 | T(\bar{\chi}_M(x_1) \Gamma_1 \chi_M(x_1) \ldots \bar{\chi}_M(x_n) \Gamma_n \chi_M(x_n)$$

$$\times \exp i \int d^4 x \left\{ - g \bar{\chi}_M(x) \gamma^\nu \chi_M(x) C_\nu (x) - \kappa \bar{\chi}_M(x) \chi_M(x) \sigma(x)$$

$$+ \mathcal{L}_{\text{int}}[\sigma(x)] \right\} | 0 \rangle_{\text{conn.}}. \quad (1.7)$$

Here $| 0 \rangle_{\text{conn.}}$ is the wave function of the non–perturbative vacuum of the MNJL–model in the condensed phase and $| 0 \rangle$ the wave function of the perturbative vacuum of the non–condensed phase. Then, $\mathcal{L}_{\text{int}}[\sigma(x)]$ describes self–interactions of the $\sigma$–field:

$$\mathcal{L}_{\text{int}}[\sigma(x)] = - \kappa M_\sigma \sigma^2(x) - \frac{1}{2} \sigma^4(x). \quad (1.8)$$

The self–interactions $\mathcal{L}_{\text{int}}[\sigma(x)]$ provide $\sigma$–field loop contributions and can be dropped out in the tree $\sigma$–field approximation accepted in Refs. [2–4]. The tree $\sigma$–field approximation can be justified keeping massive magnetic monopoles very heavy, i.e. $M \gg M_C$. This corresponds to the London limit $M_\sigma = 2 M \gg M_C$ in the dual Higgs model with dual Dirac strings [5–7]. The inequality $M_\sigma \gg M_C$ means also that in the MNJL model we deal with Dual Superconductivity of type II [4]. In the tree $\sigma$–field approximation the r.h.s. of Eq.(1.7) acquires the form

$$G(x_1, \ldots, x_n) = \langle 0 | T(\bar{\chi}(x_1) \Gamma_1 \chi(x_1) \ldots \bar{\chi}(x_n) \Gamma_n \chi(x_n)) | 0 \rangle_{\text{conn.}} =$$

$$= (M) \langle 0 | T(\bar{\chi}_M(x_1) \Gamma_1 \chi_M(x_1) \ldots \bar{\chi}_M(x_n) \Gamma_n \chi_M(x_n)$$

$$\exp i \int d^4 x \left\{ - g \bar{\chi}_M(x) \gamma^\nu \chi_M(x) C_\nu (x) - \kappa \bar{\chi}_M(x) \chi_M(x) \sigma(x) \right\} | 0 \rangle_{\text{conn.}}. \quad (1.9)$$

For the subsequent investigation it is convenient to represent the r.h.s. of Eq.(1.9) in terms of the generating functional of the monopole Green functions [2–4]

$$G(x_1, \ldots, x_n) = \prod_{i=1}^n \delta \frac{\delta}{\delta \bar{\eta}(x_i)} \frac{\delta}{\delta \eta(x_i)} Z[\eta, \bar{\eta}] \bigg|_{\eta = \bar{\eta} = 0}, \quad (1.10)$$

where $\bar{\eta}(\eta)$ are the external sources of the massive monopole (antimonomopole) fields, and $Z[\eta, \bar{\eta}]$ is the generating functional of the monopole Green functions defined by

$$Z[\eta, \bar{\eta}] = \frac{1}{Z} \int \mathcal{D}_M \mathcal{D} \bar{\chi}_M \mathcal{D} C_\mu \mathcal{D} \sigma \exp i \int d^4 x \left[ \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$$

$$+ \frac{1}{2} M_C^2 C_\mu(x) C^\mu(x) + \frac{1}{2} \partial_\mu \sigma(x) \partial^\mu \sigma(x) - \frac{1}{2} M_\sigma^2 \sigma^2(x)$$

$$+ \bar{\chi}_M(x) (i \gamma^\mu \partial_\mu - M - g \gamma^\mu C_\mu(x) - \kappa \sigma(x)) \chi_M(x)$$

$$+ \bar{\eta}(x) \chi_M(x) + \bar{\chi}_M(x) \eta(x) + \mathcal{L}_{\text{free quark}}(x) \right]. \quad (1.11)$$

The normalization factor $Z$ is defined by the condition $Z[0,0] = 1$. The coupling constants $g$ and $\kappa$ are related by the constraint

$$\frac{g^2}{12 \pi^2} J_2(M) = \frac{\kappa^2}{8 \pi^2} J_2(M) = 1 \quad (1.12)$$
where \( \kappa^2 = g^2/3 \) \[2,3\]. Then, \( \mathcal{L}_{\text{free quark}}(x) \) is the kinetic term for the quark and antiquark

\[
\mathcal{L}_{\text{free quark}}(x) = -\sum_{i=q,\bar{q}} m_i \int d\tau \left( \frac{dX^\mu_i(\tau)}{d\tau} \frac{dX^{\nu}_i(\tau)}{d\tau} g_{\mu\nu} \right)^{1/2} \delta^{(4)}(x - X_i(\tau)).
\] (1.13)

In our consideration quarks and antiquarks are classical point–like particles with masses \( m_q = m_{\bar{q}} = m \), electric charges \( Q_q = -Q_{\bar{q}} = Q \), and trajectories \( X^\mu_q(\tau) \) and \( X^\nu_{\bar{q}}(\tau) \), respectively. The field strength \( F^{\mu\nu}(x) \) is defined \[2–4\] as \( F^{\mu\nu}(x) = \mathcal{E}^{\mu\nu}(x) - \ast dC^{\mu\nu}(x) \), where \( dC^{\mu\nu}(x) = \partial^\sigma C^{\mu\nu}(x) - \partial^\nu C^{\mu\sigma}(x) \), and \( \ast dC^{\mu\nu}(x) \) is the dual version, i.e., \( \ast dC^{\mu\nu}(x) = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} dC_{\alpha\beta}(x) \varepsilon^{0123} = 1 \). The dual ”chromo”–electric field strength \( \mathcal{E}^{\mu\nu}(x) \), induced by a dual Dirac string, is defined following \[2–4\] as

\[
\mathcal{E}^{\mu\nu}(x) = Q \int \int d\tau d\sigma \left( \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} - \frac{\partial X^\nu}{\partial \tau} \frac{\partial X^\mu}{\partial \sigma} \right) \delta^{(4)}(x - X),
\] (1.14)

where \( X^\mu = X^\mu(\tau, \sigma) \) represents the position of a point on the world sheet swept by the string. The sheet is parameterized by internal coordinates \( -\infty < \tau < \infty \) and \( 0 \leq \sigma \leq \pi \), so that \( X^\mu(\tau,0) = X^\mu_0(\tau) \) and \( X^\mu(\tau, \pi) = X^\mu_Q(\tau) \) represent the world lines of an anti–quark and a quark \[2–7\]. Within the definition Eq.(1.14) the tensor field \( \mathcal{E}^{\mu\nu}(x) \) satisfies identically the equation of motion, \( \partial_{\mu} F^{\mu\nu}(x) = J^\nu(x) \). The electric quark current \( J^\nu(x) \) is defined as

\[
J^\nu(x) = \sum_{i=q,\bar{q}} Q_i \int d\tau \frac{dX^\nu_i(\tau)}{d\tau} \delta^{(4)}(x - X_i(\tau)).
\] (1.15)

Hence, the inclusion of a dual Dirac string in terms of \( \mathcal{E}^{\mu\nu}(x) \) defined by Eq.(1.14) satisfies completely the dual electric Gauss law of Dirac’s extension of Maxwell’s electrodynamics.

The ground state of the massive dual–vector field \( C_{\mu}(x) \) coupled to a dual Dirac string acquires the shape of the Abrikosov flux line \[2–7\]

\[
C^\nu[\mathcal{E}(x)] = -\int d^4x' \Delta(x - x') \partial_{\mu} \ast \mathcal{E}^{\mu\nu}(x'),
\] (1.16)

where \( \Delta(x - x') \) is the Green function

\[
\Delta(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x - x')}}{M^2_k - k^2 - i0}.
\] (1.17)

Integrating out the dual–vector field fluctuations \( c_{\mu}(x) \) around the shape of the Abrikosov flux line, \( C_{\mu}(x) = C_{\mu}[\mathcal{E}(x)] + c_{\mu}(x) \), and the scalar \( \sigma \)–field \[4\] we obtain the generating functional of the monopole Green functions in the following form:

\[
Z[\eta, \bar{\eta}] = \frac{1}{Z} \int \mathcal{D}x M D\bar{\chi}_M \exp i \int d^4x \left\{ \mathcal{L}_{\text{eff}} \{ \bar{\chi}_M(x), \chi_M(x), C^\nu[\mathcal{E}(x)] \} \right. \\
+ \bar{\chi}_M(x) (i \gamma^\mu \partial_{\mu} - M - g \gamma^\mu C_{\mu}[\mathcal{E}(x)]) \chi_M(x) + \bar{\eta}(x) \chi_M(x) \\
+ \bar{\chi}_M(x) \eta(x) + \mathcal{L}_{\text{free quark}}(x) \right\},
\] (1.18)

where \( \mathcal{L}_{\text{eff}} \{ \bar{\chi}_M(x), \chi_M(x), C^\nu[\mathcal{E}(x)] \} \) reads

\[
\mathcal{L}_{\text{eff}} \{ \bar{\chi}_M(x), \chi_M(x), C^\nu[\mathcal{E}(x)] \} = \mathcal{L}_{\text{string}} \{ C^\nu[\mathcal{E}(x)] \} \\
- \frac{g^2}{2M^2_\pi} \left[ \bar{\chi}_M(x) \gamma_{\mu} \chi_M(x) \right] \left[ \bar{\chi}_M(x) \gamma^\mu \chi_M(x) \right] + \frac{\kappa^2}{2M^2_\pi} \left[ \bar{\chi}_M(x) \chi_M(x) \right]^2.
\] (1.19)
The Lagrangian of the dual Dirac string \( \mathcal{L}_{\text{string}} \{ C^\nu [\mathcal{E}(x)] \} \) is defined \([3–6]\)

\[
\int d^4 x \mathcal{L}_{\text{string}} \{ C^\nu [\mathcal{E}(x)] \} = \frac{1}{4} M_C^2 \int d^4 x d^4 y \mathcal{E}_{\mu \alpha}(x) \Delta_\alpha^\nu(x-y, M_C) \mathcal{E}^\mu(x), \tag{1.20}
\]

where \( \Delta_\alpha^\nu(x-y, M_C) = (g_\alpha^\nu + 2 \partial^\alpha \partial_\nu/M_C^2) \Delta(x-y; M_C) \).

The effective Lagrangian Eq. (1.19) integrated over the massive monopole fields \( \bar{\chi}_M(x) \) and \( \chi_M(x) \) defines the string energy, i.e. the interquark potential, as a functional of the string shape.

## 2 Confinement potential

The interquark confinement potential is related to the energy of the string which is defined as follows \([2–7]\):

\[
W = - \int d^3 x \mathcal{L}_{\text{string}} \{ C^\nu [\mathcal{E}(x)] \} + \int d^3 x \ (M) < 0 \text{T} \left( \left( - \frac{g^2}{2M_C^2} [\bar{\chi}_M(x) \gamma_\mu \chi_M(x)] \right. \times [\bar{\chi}_M(x) \gamma_\mu \chi_M(x)] + \frac{\kappa^2}{2M_C^2} [\bar{\chi}_M(x) \chi_M(x)]^2 \right) \times \exp -i g \int d^4 y \bar{\chi}_M(y) \gamma_\mu C_\mu [\mathcal{E}(y)] \chi_M(y) \bigg| 0 \bigg> (M). \tag{2.1}
\]

The interaction caused by the integration over the \( \sigma \)-field fluctuations gives a trivial constant contribution to the energy of the string \([4]\) and can be dropped out. In the momentum representation of the vacuum expectation values the energy of the string is then defined by \([4]\):

\[
W = - \int d^3 x \mathcal{L}_{\text{string}} \{ C^\nu [\mathcal{E}(x)] \} - \int d^3 x \frac{g^2}{2M_C^2} \int \frac{d^4 k_1}{(2\pi)^4 i} \times \text{tr} \left\{ \frac{1}{M - k_1 + g\bar{C}[\mathcal{E}(x)]} \gamma_\mu \right\} \int \frac{d^4 k_2}{(2\pi)^4 i} \text{tr} \left\{ \frac{1}{M - k_2 + g\bar{C}[\mathcal{E}(x)]} \right\}. \tag{2.2}
\]

The momentum integrals have been calculated in Ref.\([4]\). This yields the energy of the string:

\[
W = - \int d^3 x \mathcal{L}_{\text{string}} \{ C^\nu [\mathcal{E}(x)] \}
\]

\[
- \frac{1}{2} \frac{1}{M_C^2} \left( \frac{g^2}{8\pi^2} [J_1(M) + M^2 J_2(M)] \right)^2 \int d^3 x C_\mu [\mathcal{E}(x)] C^\mu [\mathcal{E}(x)]. \tag{2.3}
\]

By using Eqs. (1.2) – (1.4), the relations \( G_1 = G/4 \) and \( M_\sigma = 2M \) we bring up the coefficient of the second term to the form

\[
- \frac{g^2}{8\pi^2} [J_1(M) + M^2 J_2(M)] = M_C^2 - \frac{g^2}{2G_1} = M_C^2 + 8 \frac{g^2 \bar{\chi} \chi}{M_\sigma}. \tag{2.4}
\]

Thus, the energy of the string containing quantum fluctuations of the scalar and dual–vector fields around the shape of the Abrikosov flux line is given by

\[
W = - \int d^3 x \mathcal{L}_{\text{string}} \{ C^\nu [\mathcal{E}(x)] \}
\]

\[
- \frac{1}{2} M_C^2 \left( 1 + \frac{8g^2 \bar{\chi} \chi}{M_C^2 M_\sigma} \right)^2 \int d^3 x C_\mu [\mathcal{E}(x)] C^\mu [\mathcal{E}(x)]. \tag{2.5}
\]
The computation of the r.h.s. of Eq. (2.3) we perform for the static straight string of the length $L$ directed along the $z$-axis. In this case the electric field strength $\mathcal{E}_{\mu\nu}(x)$ does not depend on time and is given by [6]

$$
\mathcal{E}(\vec{x}) = \vec{e}_z Q \delta(x) \delta(y) \left[ \theta \left( z - \frac{1}{2} L \right) - \theta \left( z + \frac{1}{2} L \right) \right],
$$

where a quark and an antiquark are placed at $\vec{X}_Q = (0, 0, \frac{1}{2} L)$ and $\vec{X}_{-Q} = (0, 0, -\frac{1}{2} L)$. The unit vector $\vec{e}_z$ is directed along the $z$-axis and $\theta(z)$ is the step-function. The field strength Eq. (2.7) induces the dual-vector potential

$$
\mathcal{C}(\vec{x}) = -i Q \int \frac{d^3k}{4 \pi^3} \frac{\vec{k} \times \vec{e}_z}{k_z} \frac{1}{M_C^2 + k^2} \sin \left( \frac{k_z L}{2} \right) e^{i \vec{k} \cdot \vec{x}}.
$$

For the static straight string the term $- \int d^3x \mathcal{L}_{\text{string}} \{C^\nu[\mathcal{E}(x)]\}$ reads

$$
- \int d^3x \mathcal{L}_{\text{string}} \{C^\nu[\mathcal{E}(x)]\} = -\frac{1}{4} M_C^2 \int d^3x \int d^3x' \int dx' \left[ \mathcal{E}_{0j}(\vec{x}) \left( g_j^i + \frac{2}{M_C^2} \frac{\partial}{\partial x_j} \right) \Delta(\vec{x} - \vec{x}', M_C) \mathcal{E}^{0j}(\vec{x}') + \mathcal{E}_{00}(\vec{x}) \left( g_0^0 + \frac{2}{M_C^2} \frac{\partial}{\partial x_0} \frac{\partial}{\partial x_0} \right) \Delta(\vec{x} - \vec{x}', M_C) \right] = \frac{1}{2} M_C^2 \int d^3x \int d^3x' \mathcal{E}(\vec{x}) \cdot \mathcal{E}(\vec{x}') \left( 1 - \frac{1}{M_C^2} \frac{\partial^2}{\partial z^2} \right) \Delta(\vec{x} - \vec{x}', M_C) = \frac{1}{2} Q^2 M_C^2 \int_{-L/2}^{L/2} dz \int_{-L/2}^{L/2} dz' \int dk_z \left( 1 + \frac{k_z^2}{M_C^2} \right) \frac{d^2k_\perp}{(2\pi)^3} \left( \frac{M_C^2}{M_C^2 + k_\perp^2 + k_z^2} \right) = \frac{Q^2 M_C^2}{4\pi^3} \int_{-\infty}^{\infty} \frac{dk_z}{k_z^2} \sin^2 \left( \frac{k_z L}{2} \right) \left( 1 + \frac{k_z^2}{M_C^2} \right) \int_{-\Lambda}^{\Lambda} \frac{d^2k_\perp}{M_C^2 + k_\perp^2} = \frac{Q^2 M_C^2}{4\pi^2} \int_{-\infty}^{\infty} \frac{dk_z}{k_z^2} \sin^2 \left( \frac{k_z L}{2} \right) \left( 1 + \frac{k_z^2}{M_C^2} \right) \int_{0}^{\Lambda} \frac{d^2k_\perp}{M_C^2 + k_\perp^2 + k_z^2},
$$

where $\Lambda_\perp$ is the cut-off in the plane perpendicular to the world-sheet swept by the string [2–7]. We identify $\Lambda_\perp$ with the mass of the scalar field, i.e., $\Lambda_\perp = M_\sigma = 2M$ [2–7].

For a sufficiently long string we can integrate over $k_z$ and get

$$
- \int d^3x \mathcal{L}_{\text{string}} \{C^\nu[\mathcal{E}(x)]\} = \frac{Q^2 M_C^2}{4\pi^2} \int_{-\infty}^{\infty} \frac{dk_z}{k_z^2} \sin^2 \left( \frac{k_z L}{2} \right) \left( 1 + \frac{k_z^2}{M_C^2} \right) \times \left( \ln \left( 1 + \frac{M_\sigma^2}{M_C^2} \right) - \ln \left( 1 + \frac{k_z^2}{M_C^2} \right) \right),
$$

where we have neglected $k_z$ relative to $\Lambda_\perp$. Dropping the infinite constant contributions independent of $L$ we obtain [6]:

$$
- \int d^3x \mathcal{L}_{\text{string}} \{C^\nu[\mathcal{E}(x)]\} = -\frac{L Q^2 M_C^2}{8\pi} \left[ \ln \left( 1 + \frac{M_\sigma^2}{M_C^2} \right) + 2 E_1(M_C L) \right]
$$

6
where $E_1(MCL)$ is the Exponential Integral function. For the calculation of the integral over $k_z$ we have used the auxiliary integral

$$\int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2} \ln \left( \alpha^2 + \frac{x^2}{a^2} \right) = 2\pi \ln \alpha + \frac{\pi}{a} \left( 1 - e^{-2a}\alpha \right) - 2\pi E_1(2a\alpha),$$

where in Eq.(2.10) we have set $\alpha = 1$ and $a = MCL/2$.

The first term proportional to $L$ gives the string tension $\sigma_0$ calculated in the tree–approximation [5]:

$$\sigma_0 = \frac{Q^2 M^2_C}{8\pi} \ln \left( 1 + \frac{M^2}{M^2_C} \right).$$

The last term in Eq.(2.13) induced by the quantum fluctuations of the dual–vector field $C_\mu$ around the shape of the Abrikosov flux line can be reduced to the form [6]:

$$-\frac{1}{2} M^2_C \left( 1 + \frac{8g^2 \langle \bar{\chi} \chi \rangle}{M^2_C} \right)^2 \int d^3 x C_\mu [\mathcal{E}(x)] C^\mu [\mathcal{E}(x)] = \frac{Q^2 M^2_C}{4\pi^2} \left( 1 + \frac{8g^2 \langle \bar{\chi} \chi \rangle}{M^2_C} \right)^2 \int_{-\infty}^{\infty} \frac{dk_z}{k^2} \sin^2 \left( \frac{k_z L}{2} \right) \left[ \ln \left( 1 + \frac{M^2}{M^2_C} \right) - \ln \left( 1 + \frac{k^2}{M^2_C} \right) \right] = L \frac{Q^2 M^2_C}{8\pi} \left( 1 + \frac{8g^2 \langle \bar{\chi} \chi \rangle}{M^2_C} \right)^2 \left[ \ln \left( 1 + \frac{M^2}{M^2_C} \right) + 2 E_1(MCL) - \frac{2}{MCL} \left( 1 - e^{-MCL} \right) \right].$$

Collecting the pieces together we obtain the energy of the dual Dirac string, the interquark potential, as a function of the the length of the string $L$:

$$W = L \frac{Q^2 M^2_C}{4\pi} \left( 1 + \frac{8g^2 \langle \bar{\chi} \chi \rangle}{M^2_C} + \frac{32g^4 \langle \bar{\chi} \chi \rangle^2}{M^4_C} \right) \left[ \ln \left( 1 + \frac{M^2}{M^2_C} \right) + 2 E_1(MCL) - \frac{2}{MCL} \left( 1 - e^{-MCL} \right) \right] - \frac{Q^2}{4\pi} \frac{e^{-MCL}}{L}.$$
3 String shape fluctuations

The string shape fluctuations we define as usually \[8,7\] by \(X^\mu \rightarrow X^\mu + \eta^\mu(X)\), where \(\eta^\mu(X)\) describes fluctuations around the fixed surface \(S\) swept by the shape \(\Gamma\) and obeys the constraint \(\eta^\mu(X)|_{\partial S} = 0\) \[8,7\] at the boundary \(\partial S\) of the surface \(S\). The integration over the \(\eta\)-field we perform around the shape of the static straight string with the length \(L\) tracing out the rectangular surface \(S\) with the time–side \(T\) \[8,7\]. Allowing only fluctuations in the plane perpendicular to the string world–sheet and setting \(\eta(t,z) = \eta(t,z) = 0\) \[8,7\], we arrive at the fluctuation action \(\delta S_N[\eta_x, \eta_y]\) \[7,4\]

\[
\delta S_N[\eta_x, \eta_y] = -\frac{3Q^2\Lambda^2}{32\pi} \int_{-T/2}^{T/2} dt \int_{-L/2}^{L/2} dz [\eta_x(t,z) (-\Delta) \eta_x(t,z) + (x \leftrightarrow y)], \tag{3.1}
\]

coming from the term \(\int d^4x L_{\text{string}}\{C^\nu[\mathcal{E}(x)]\}\) defined by Eq.(1.20), where \(\Delta\) is the Laplace operator in 2–dimensional space–time

\[
\Delta = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2}. \tag{3.2}
\]

The term \(\int d^4x C_\mu[\mathcal{E}(x)] C^\mu[\mathcal{E}(x)]\) in Eq.(2.3), induced by the quantum fluctuations of the dual–vector \(C_\mu\) and scalar \(\sigma\) fields around the shape of the Abrikosov flux line, does not contribute to the fluctuation action for the case of the static straight string. In order to show this we use the expression obtained in Ref.[4]:

\[
\delta C_\mu[\mathcal{E}(x)]\delta C^\mu[\mathcal{E}(x)] = \]

\[
\times Q^2 \int d^4k \int d^3q \frac{k_xq_x + k_yq_y}{k_zq_z} \sin \left( \frac{k_zL}{2} \right) \sin \left( \frac{q_zL}{2} \right) \frac{1}{M_C^2 + \vec{k}^2} \]

\[
\times 1 \frac{1}{M_C^2 + \vec{q}^2} e^{i(\vec{k} + \vec{q}) \cdot \vec{r}} \left( e^{i[(k_x + q_x)\eta_x(t,z) + (k_y + q_y)\eta_y(t,z)]} - 1 \right). \] \tag{3.3}

The contribution to the fluctuation action is given by

\[
\int d^4x \delta C_\mu[\mathcal{E}(x)]\delta C^\mu[\mathcal{E}(x)] = \]

\[
= Q^2 \int d^4x \int d^3k \int d^3q \frac{k_xq_x + k_yq_y}{k_zq_z} \sin \left( \frac{k_zL}{2} \right) \sin \left( \frac{q_zL}{2} \right) \frac{1}{M_C^2 + \vec{k}^2} \]

\[
\times 1 \frac{1}{M_C^2 + \vec{q}^2} e^{i(\vec{k} + \vec{q}) \cdot \vec{r}} \left( e^{i[(k_x + q_x)\eta_x(t,z) + (k_y + q_y)\eta_y(t,z)]} - 1 \right). \tag{3.4}
\]

Integrating over \(x\) and \(y\) we get

\[
\int d^4x \delta C_\mu[\mathcal{E}(x)]\delta C^\mu[\mathcal{E}(x)] = \]

\[
= Q^2 \int_{-T/2}^{T/2} dt \int_{-L/2}^{L/2} dz \int d^3k \frac{k_xq_x + k_yq_y}{k_zq_z} \sin \left( \frac{k_zL}{2} \right) \sin \left( \frac{q_zL}{2} \right) \frac{1}{M_C^2 + \vec{k}^2} \]

\[
\times 1 \frac{1}{M_C^2 + \vec{q}^2} e^{i(k_0 + q_0)t - i(k_x + q_x)z} \delta(k_x + q_x) \delta(k_y + q_y) \]

\[
\times (e^{i[(k_x + q_x)\eta_x(t,z) + (k_y + q_y)\eta_y(t,z)]} - 1) = 0. \tag{3.5}
\]
Thus, Eq.(3.1) defines completely the fluctuation action induced by string shape fluctuations around a static straight string with length $L$. As has been shown in Ref.[7], the fluctuation action Eq.(3.1) gives a Coulomb–like universal contribution [8] to the energy of the string:

$$W_{\text{string–shape}} = -\frac{\alpha_{\text{string}}}{L},$$

where $\alpha_{\text{string}} = \pi/12$ and $\alpha_{\text{string}} = \pi/3$ for opened and closed strings, respectively.

4 Conclusion

We have shown that in the MNJL model with dual Dirac strings the quantum fluctuations of a dual–vector field $C_\mu$ and a scalar field $\sigma$ around the shape of the Abrikosov flux line give the interquark confinement potential in the following form

$$W_{\text{tot}} = L \frac{Q^2 M^2_C}{4\pi} \left( 1 + \frac{8g^2 <\bar{\chi}\chi>}{M^2_C} + \frac{32g^4 <\bar{\chi}\chi>^2}{M^4_C M^2_\sigma} \right) \left[ \ln \left( 1 + \frac{M^2_\sigma}{M^2_C} \right) + 2 E_1(M_C L) - \frac{2}{M_C L} \ln \left( 1 - e^{-M_C L} \right) \right] - \frac{Q^2}{4\pi} \frac{e^{-M_C L}}{L} - \frac{\alpha_{\text{string}}}{L},$$

where $\alpha_{\text{string}} = \pi/12$ and $\alpha_{\text{string}} = \pi/3$ for opened and closed strings, respectively. This interquark potential resembles the result obtained in the dual Higgs model with dual Dirac strings [5–7]. Unlike the dual Higgs model with dual Dirac strings [11,12] the mass of a dual–vector field $M_C$ is not proportional to the order parameter $<\bar{\chi}\chi>$ and does not vanish in the limit $<\bar{\chi}\chi> \to 0$. This is seen from the mass formula [4]

$$M_\sigma \left( 8 M^2_C + 3 M^2_\sigma \right) = -56g^2 <\bar{\chi}\chi>,$$

which can be derived from Eq.(1.2) and the gap–equation Eq.(1.4). Thus, in the MNJL model the dual–vector field does not need a Goldston boson as a longitudinal component. This distinguishes the transition to the non–perturbative superconducting phase in the NMJL and the dual Higgs model. Indeed, in the MNJL model this transition does not accompany the appearance of Goldston bosons. The former is rather natural, since the starting $U(1)$ magnetic symmetry in the MNJL model is global and unbroken in the non–perturbative superconducting phase. Recall that in the dual Higgs model the magnetic $U(1)$ symmetry is local and becomes spontaneously broken in the superconducting phase.

Due to the independence of the mass of the dual–vector field on the monopole condensate the string tension $\sigma_0$ calculated in the tree–approximation does not depend on the monopole condensate too. The mass of the Higgs field $M_\sigma$ replaced the cut–off $\Lambda_\perp$, i.e. $\Lambda_\perp = M_\sigma$. The dependence on the magnetic monopole condensate appears by virtue of the contributions of the quantum field fluctuations of the dual–vector $C_\mu$ and the scalar $\sigma$ fields around the shape of the Abrikosov flux line. Very similar to the dual Higgs model with dual Dirac strings the quantum field fluctuations increase the value of the string tension. This implies that for the consistent investigation of the superconducting mechanism of the quark confinement within the dynamics of magnetic monopoles and dual Dirac strings one cannot deal with a classical level only and quantum contributions should be taken into account. The string shape fluctuations of dual Dirac strings induce a Coulomb–like universal contribution calculated for opened strings by Lüscher et al. [8] and for closed strings by Faber et al. [7].
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