Screening in anisotropic superfluids and the superfluid density in underdoped cuprates

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We examine the nature of the collective excitations in a strongly anisotropic system of bosons interacting via Coulomb interaction. Such a system has often been used in the past to model the effects of quantum and classical phase fluctuations on the superfluid density of underdoped cuprates. Depending on the anisotropy and the effective strength of the interaction we find four different regimes for the temperature dependence of the superfluid density. Coulomb interaction in underdoped cuprates is argued to be effectively short-ranged, and less then unity in appropriately defined units.

Temperature dependence of the superfluid density $\rho(T)$ is one of the key properties of a superfluid. This is particularly true in high-temperature superconductors, where the linear low-temperature behaviour of $\rho(T)$ at optimal doping served as early evidence for the d-wave symmetry of the superconducting order parameter. Another intriguing aspect of $\rho(T)$ is its evolution with underdoping, the effect of which may be expected to increase the importance of fluctuations. Indeed, it has been argued that for a low value of $\rho(0)$ as is observed in the underdoped regime the further reduction of $\rho(T)$ with temperature should be primarily due to classical phase fluctuations. This argument has been criticized for its neglect of Coulomb interaction, which is expected to strongly suppress classical phase fluctuations below the plasmon energy gap and thus reinstate quasiparticles as the source of the linear temperature dependence of $\rho(T)$ at low $T$.

The competition between the quasiparticles and phase fluctuations for the form of $\rho(T)$ becomes quite explicit in the so-called Ioffe-Larkin rule

$$\rho^{-1}(T) = \rho_{qp}^{-1}(T) + \rho_{fluc}^{-1}(T),$$

where $\rho_{qp}(T)$ is the standard BCS quasiparticle contribution in a d-wave state, and $\rho_{fluc}(T)$ is the (model-dependent) fluctuation component. This transparent result was first derived within the context of effective gauge theories of the t-J model, but it may be expected to apply more generally to strongly fluctuating quasi-two-dimensional superconductors with $\rho(0) < \rho_{qp}(0)$.

Cuprates, however, are strongly anisotropic materials, with the anisotropy in the ab-plane and the c-axis increasing with underdoping. It is well known that under these conditions the plasmon dispersion becomes very anisotropic as well, and the large plasmon energy gap gets replaced by a much lower one proportional to the interlayer coupling. One may therefore expect Coulomb interaction in such highly anisotropic superconductors to become less efficient in gapping the phase mode. In this paper we study in greater detail the combined effect of large anisotropy and Coulomb interactions on $\rho(T)$. We model the fluctuation component in Eq. (1) by a layered system of bosons interacting via Coulomb interactions and with the density proportional to doping. Such an effective theory arises naturally in several theories of underdoped cuprates, and provides a rather general representation of a charged layered superfluid. We begin with the simple case of a two-layer system and demonstrate that, when the layers are Josephson-decoupled, there exists a linearly dispersing phase mode at low wavevectors. This is essentially a consequence of the perfect screening of interactions in one layer by fluctuations in the other. Generalizing to a system with an infinite number of layers, for a weak Coulomb interaction we find that there are four discernible regimes for the temperature dependence of the superfluid density, controlled by the ratio between the boson density and the Josephson coupling (Fig. 1). An estimate of the relevant parameters places the underdoped YBa$_2$Cu$_3$O$_{6+x}$ (YBCO) with $T_c > 5K$ firmly in the regime I, where the Coulomb interaction acts as an effective short-range interaction; the long-range nature of the Coulomb interaction in this regime is irrelevant except at extremely low temperatures. We also find that the value of the dimensionless interaction strength $\lambda$ in YBCO is $\lambda < 1$, and argue that it is possible that $\lambda < 1$.

Let us define the quantum mechanical action for a layered system of interacting bosons as $S = \int_0^\beta d\tau \int d^2\vec{x} \mathcal{L}$ with

$$\mathcal{L} = \sum_{i=1}^{N} \Phi_i^\dagger(\vec{x},\tau)(\partial_\tau - \frac{\nabla^2}{2m} - \mu)\Phi_i(\vec{x},\tau) + \frac{1}{2} \sum_{i,j} \int d^2\vec{x} |\Phi_i(\vec{x},\tau)|^2 V_{ij}(\vec{x} - \vec{x}') |\Phi_j(\vec{x}',\tau)|^2$$

FIG. 1: Four regimes for the temperature dependence of the superfluid density $\rho(T)$ in a layered bosonic system with weak Coulomb interactions: I) quasi-two-dimensional (2D) regime with weak and effectively short-range (screened) interaction for $K(0)/t \gg 1$, II) quasi-2D regime with weak long-range interaction for $1 \ll K(0)/t \ll 1/\lambda$, III) three-dimensional (3D) regime with weak long-range interaction for $\lambda^3 \ll K(0)/t \ll 1$, and IV) 3D regime with strong long-range interaction for $K(0)/t \gg \lambda^3$. $K(0) = \hbar^2 \rho(0)/m$, and $t$ is the interlayer Josephson coupling. $\lambda = (2\pi c^2/(ed))/(\hbar^2/(md^2)) \ll 1$ is the dimensionless strength of the Coulomb interaction, and $d$ is the inter-layer separation.
\[-i \sum \sum_{i,j=1}^{2} \Phi_i^*(x, \tau) \Phi_j(x, \tau),\]

where \( N \) is the number of two-dimensional (2D) layers, \( t \) is a weak Josephson coupling between the layers, \( \beta = 1/T \), and we set \( k_B = h = 1 \) throughout, unless otherwise noted. The Coulomb interaction is

\[ V_{ij}(x - x') = e^2/\epsilon \sqrt{|x - x'|^2 + |i - j|^2 d^2}, \]

\( d \) being the separation between the layers and \( \epsilon \) the static background dielectric constant. We assume the presence of a neutralizing background of density \( \rho_0 \) equal to the average areal density of bosons at the chemical potential \( \mu \).

The nature of the excitations in the above superfluid system is most explicit in a system of only two layers, which differs already from a single layer in an important way. Let us introduce first the usual density-phase variables as \( \Phi_i(x, \tau) = \sqrt{\rho_0 + \Pi_i(x, \tau)} e^{i \theta_i(x, \tau)} \) and expand the Lagrangian to the quadratic order in \( \Pi_i \) and \( \theta_i \).

\[
\mathcal{L} = \sum_{i=1}^{2} \left( \frac{\rho_0}{2m} (\nabla \theta_i)^2 + \frac{(\nabla \Pi_i)^2}{8m \rho_0} + i(\rho_0 + \Pi_i) \dot{\theta}_i \right) + \frac{1}{2} \sum_{i,j} \int d^2 x \Pi_i(x, \tau) V_{ij}(x - x') \Pi_j(x', \tau) \\
+ t \rho_0 (\theta_1 - \theta_2)^2 + \frac{t}{4 \rho_0} (\Pi_1 - \Pi_2)^2.
\]

We assumed here the usual periodic boundary conditions in imaginary time: \( \Pi_i(x, \beta) = \Pi_i(x, 0) \) and \( \theta_i(x, \beta) = \theta_i(x, 0) + 2 \pi n_i(x) \), with \( n_i(x) \) integer. By rotating the fields as \( \theta_{\pm} = (\theta_1 \pm \theta_2)/\sqrt{2} \), \( \Pi_{\pm} = (\Pi_1 \pm \Pi_2)/\sqrt{2} \), we can decouple the above Lagrangian as \( \mathcal{L} = \mathcal{L}_{\pm} + \mathcal{L}_{-} \), with

\[
\mathcal{L}_{\pm} = \frac{\rho_0}{2m} (\nabla \theta_{\pm})^2 + i \sqrt{2} \rho_0 + \Pi_{\pm} \dot{\theta}_{\pm} + \frac{(\nabla \Pi_{\pm})^2}{8m \rho_0} \\
+ \frac{1}{2} \int d^2 x \Pi_{\pm}(x, \tau) V_{\pm}(x, \tau - x') \Pi_{\pm}(x', \tau),
\]

\[
\mathcal{L}_{-} = \frac{\rho_0}{2m} (\nabla \theta_{-})^2 + i \Pi_{-} \dot{\theta}_{-} + \frac{(\nabla \Pi_{-})^2}{8m \rho_0} \\
+ \frac{1}{2} \int d^2 x \Pi_{-}(x, \tau) V_{-}(x, \tau - x') \Pi_{-}(x'),
\]

where \( V_{\pm}(x) = V_{11}(x) \pm V_{12}(x) \). The Gaussian integration over \( \Pi_{\pm} \) yields two branches of excitations with the energies

\[
\omega_{\pm}^2 = \frac{k^2}{2m} (2\rho_0 V_+(k) + \frac{k^2}{2m}),
\]

\[
\omega_{-}^2 = \frac{k^2}{2m} + 2t)(2\rho_0 V_-(k) + \frac{k^2}{2m} + 2t).
\]

\[ V_{\pm}(k) = \frac{2\pi e^2}{\epsilon k} (1 \pm e^{-k d}). \]

The branch \( \omega_{\pm} \) describes the usual two-dimensional plasmon, \( \omega_{\pm} \approx \sqrt{4\pi \epsilon^2 \rho_0 k/\epsilon m} \) at low momenta. The two layers oscillate in phase and, as a consequence, \( \omega_{\pm} \) is independent of the Josephson coupling. The canonically conjugate variable to \( \theta_{\pm} \) is the sum of two densities, and therefore the Coulomb interaction affects the energy of this mode, as is usual in systems with long-range interactions. In contrast, the conjugate variable to \( \theta_{-} \) is the difference between the two densities which can oscillate without any cost in Coulomb energy. As a result, for \( t = 0 \), \( \omega_{-} \approx (2\pi \rho_0 e^2 d/\epsilon m)^{1/2} k \) at low momenta, and the dispersion of the lower branch is the same as if the system had only a short-range interaction of strength \( \sim 2\pi e^2 d/\epsilon \). With \( t = 0 \) the layers cannot exchange particles and the density in each layer therefore may oscillate so as to perfectly screen the Coulomb interaction in the other. The oscillations, however, are then out of phase, and consequently when \( t \neq 0 \) this mode becomes gapped, with \( \omega_- \approx \sqrt{2\pi t^2 e^2 d/\epsilon} \) at low momenta. We will refer hereafter to this energy as the Josephson gap.

The remarkable feature of the above result is that in a system with negligible Josephson interaction between the layers, the Coulomb interaction becomes effectively short-ranged as far as the low-energy excitation spectrum is concerned. More precisely, when \( t = 0 \), \( \omega_- \) is linear and deviates from \( \omega_+ \) significantly for \( k \ll 1/d \). For a large separation between the layers and for \( 1/d \ll k \ll (8\pi m \rho_0 e^2/\epsilon)^{1/3} \), \( \omega_- \approx \omega_+ \approx \sqrt{k} \). Finally, for \( (8\pi m \rho_0 e^2/\epsilon)^{1/3} \ll k \), \( \omega_- \approx \omega_+ \approx k^2/2m \) if the layers are brought close together so that \( 1/d \gg (8\pi m \rho_0 e^2/\epsilon)^{1/3} \), \( \omega_- \approx k \ll \sqrt{8\pi m \rho_0 e^2 d/\epsilon} \), and \( \omega_- \approx k^2/2m \) otherwise, without the intermediate region \( \omega_- \approx \sqrt{k} \). In this

\[ \text{FIG. 2: The two branches } \omega_{\pm} \text{ of the excitation spectrum of the two-layer system, with } t = 0 \text{ and weak Coulomb interaction. The lowest mode crosses over from linear behaviour at low } k \text{ to } k^2 \text{ behaviour. The plasmon starts out as } \sqrt{k} \text{ before crossing to } k^2. \]
regime $\omega_-$ becomes identical to the phonon spectrum of the weakly interacting Bose gas. This is illustrated in Fig. 2. With a finite Josephson coupling $\omega_-$ approaches the Josephson gap for $k \ll \sqrt{4m t}$. If the Josephson coupling were strong, of course, $\omega_- \gg \omega_+$, and the plasmon would reside its place as the low-energy mode of the system.

In a system with $N$ layers, for $t = 0$ there are $N - 1$ modes with linear dispersion and only a single plasmon. This is easily established by considering the interaction matrix $V_{ij}(k)$. When $k \rightarrow 0$, $V_{ij}(k) = \left(2\pi e^2 / ek \right) \left(1 + O(kd)\right)$. So in the limit $k \rightarrow 0$,

$$V(k) = \frac{2\pi e^2}{ek} (1,1,...1)^T \otimes (1,1,...1),$$

(10)

and the interaction matrix has one eigenvector with the eigenvalue $2\pi Ne^2 / ek$, and $N - 1$ degenerate eigenvectors with zero eigenvalue. The former eigenvector is the total density which is canonically conjugate to the sum of the phases and describes the plasmon. The latter $N - 1$ modes, being orthogonal to the plasmon, are electrically neutral and consequently cross from linear dispersion at low momenta to the Josephson gap at $k = 0$.

The existence of linearly dispersing modes below the usual plasmon modifies the behavior of the superfluid density at low temperatures. To be specific, we focus on the system with infinitely many layers which is relevant in the context of high-temperature superconductivity. Imposing periodic boundary conditions in the direction orthogonal to the layers the excitation spectrum becomes

$$\omega^2(k, z) = e(k, z)(2\rho_0 V(k, z) + e(k, z)),$$

(11)

with $e(k, z) = (k^2 / 2m) + t \sin^2(kz/d)$, and

$$V(k, z) = \frac{2\pi e^2}{ek} \frac{\sinh(kd)}{\cosh(kd) - \cos(kz/d)}.$$ 

(12)

For $k_z = 0$ one finds the usual three-dimensional plasmon at $\omega^2(0, 0) = \omega_p^2 = 4\pi e^2 \rho_0 / dm e$, while when $k_z \neq 0$ and $t = 0$, for $k \ll 1/d$, $\omega(k, z) = \omega_p k / k_z$. The latter modes become gapped when $t \neq 0$ and $\omega(k \rightarrow 0, k_z \neq 0) = \omega_p / \sqrt{md^2 / 2}$.

The temperature dependence of the areal in-plane superfluid density in Landau’s two-fluid model is given by

$$\rho(T) = \rho(0) + \frac{d}{2m} \int \frac{d^2 k}{(2\pi)^2} \int_{-\pi/d}^{\pi/d} dk_z \frac{\partial n_b(\omega(k, z))}{\partial \omega(k, z)},$$

(13)

where $n_b(\omega)$ is the usual boson occupation number. We will find it convenient to express the superfluid density in units of energy by defining $K(T) = h^2 \rho(T) / m$; we have also restored the dimensionful Planck constant. The rescaled superfluid density $\tilde{K}(T)$ may then be expressed entirely in terms of dimensionless quantities as

$$\tilde{K}(T) = K(0) - \frac{T}{8\pi},$$

(14)

$$\int_0^\infty y dy \left\{ \frac{1}{2} \left[ f(y, z) \left( f(y, z) + 2\lambda K(0) \sinh \left( \frac{2\pi y}{T} \right) \right) \right] \right\}^{1/2},$$

where $f(y, z) = y + (p/\tilde{T}) \sin^2(\pi z / 2); \tilde{X} = X / T_d$. Dimensionless, with $T_d = h^2 / (md^2)$ as the characteristic energy scale in the problem. The parameter $\lambda = 2\pi e^2 / (kdT_d)$ is the dimensionless measure of the Coulomb interaction’s strength.

Equation 13 or 14 expected to be valid for $\lambda \ll 1$ and not too close to the critical temperature, leading to four distinct regimes of temperature dependence of the superfluid density. Take $t/K(0) \ll 1$, as is relevant to the cuprates, and consider the function $K(T)$ as $K(0)$ is decreased at fixed $t$. We will approximate this to crudely correspond to underdoping a high-temperature superconductor, as we discuss shortly.

I) For $(0) / t > 1$, the system is quasi-2D. Assuming $\lambda \ll 1$, $\tilde{T}_c$ to the zeroth order in $\lambda$ the superfluid density in Eq. 13 is easily seen to equal the Bose condensate in the layered non-interacting system [7],

$$K(T) \approx K(0) - \frac{T}{2\pi} \left[ \ln \frac{T_1}{t} + 1.386 + O(t/T_1) \right],$$

(15)

over most of the temperature range. The deviations from Eq. 15 are most significant below the crossover temperature $T_1 = \lambda K(0)$, where $\Delta K(T) = K(0) - K(T) \sim T^3$, and within the critical region of width $\sim \lambda T_c$ around $T_c$ with $T_c \approx 2\pi K(0) / \ln(2\pi K(0) / t)$. Besides $T_1$, there exists also a lower crossover temperature $T'_1$ of the order of the Josephson gap, where $\Delta K(T)$ becomes exponentially suppressed. The latter temperature is

$$\frac{T'_1}{T_c} = \frac{1}{2\pi} \sqrt{\frac{\lambda}{K(0)} \ln \frac{2\pi K(0)}{t}},$$

(16)

whereas

$$\frac{T_1}{T_c} = \frac{\lambda}{2\pi} \ln \frac{2\pi K(0)}{t}.$$ 

(17)

The three characteristic temperature scales for variations of $K(T)$ will therefore satisfy the inequalities

$$T'_1 \ll T_1 \ll T_c,$$

(18)

for

$$\frac{T}{K(0)} \ll \lambda \ll \frac{2\pi}{\ln(2\pi K(0) / t)}.$$ 

(19)

For such an interval for $\lambda$ to exist we obviously need $t/K(0) \ll 2\pi / \ln(2\pi K(0) / t)$, which is comfortably satisfied for $K(0) / t > 1$. When the inequality 15 is satisfied the long-range nature of the Coulomb interaction is relevant only at very low temperatures and the interaction
appears in \( \rho(T) \) as being effectively short-ranged, and weak. Furthermore, as the ratio \( K(0)/t \) is reduced the relative temperature range over which \( K(T) \) behaves as a power law, \( T_1/T_c \), decreases as well, albeit logarithmically slowly. One may interpret this decrease as that the interaction is being slowly renormalized towards zero with the reduction of the boson density \( \lambda \). The regime in which the inequalities hold is marked as I in Fig. 4.

II) If \( K(0) \) is decreased further so that \( K(0)/t \sim 1/\lambda \), one finds \( T_1' \approx T_1 \ll T_c \). The low-temperature behavior of the superfluid density corresponds to the long-range interaction. This regime could be named the 2D, weakly interacting, long-range regime, and is labeled II in Fig. 4.

III) For \( K(0)/t \sim 1 \), \( K(T) \) to the zeroth order in \( \lambda \), has the form of the fully three-dimensional Bose condensate over most of the temperature range, and

\[
K(T) \approx K(0) - 1.306 \frac{T^{3/2}}{\pi^{3/2} t^{1/2}} .
\]

In this regime the exponential behavior of \( \Delta K(T) \) sets in below

\[
T_2' \approx \frac{\lambda^{1/2}}{\pi} \frac{t}{K(0)}^{1/6} T_c.
\]

The effective short-range behaviour \( \Delta K(T) \sim T^4 \), on the other hand, would appear below

\[
T_2 = \lambda K(0) \approx \frac{\lambda}{\pi} \left( \frac{K(0)}{t} \right)^{1/3} T_c.
\]

Note that \( T_2/T_c \) now decreases as a power of \( K(0) \), which is again equivalent to the infrared renormalization group flow of the short-range coupling constant in a weakly interacting system of 3D bosons \( \lambda \). However, since \( T_2' > T_2 \) for \( K(0)/t < 1/\lambda \), and \( 1/\lambda \gg 1 \), by the time the system enters the 3D regime where \( K(0)/t \sim 1 \) the temperature dependence of \( K(T) \) crosses over directly from exponential at low temperatures to \( \sim T^{3/2} \) at higher temperatures. This is then the 3D, still weakly-interacting, long-range regime, labeled III in Fig. 4.

IV) Eventually, by reducing the density further one enters the regime where the Josephson gap becomes comparable to \( T_c \). From Eq. 21 this occurs when

\[
\frac{K(0)}{t} \approx \frac{\lambda^3}{\pi^4}.
\]

This is the strongly-interacting regime (labeled IV in Fig. 4) in which the superfluid density varies exponentially over the scale of \( \sim T_c \). In this regime the system may be expected to eventually suffer the phase transition into a Wigner crystal.

To determine the relevant regime for cuprates one needs an estimate of the dimensionless coupling constant \( \lambda \). We find that \( 2\pi e^2/ed \approx 2500K \) assuming \( \varepsilon \approx 30 \) and \( d = 12A \) in YBCO. The estimate of the temperature scale \( T_d \) requires some assumptions on the relation between the superfluid density of the bosons and the measured superfluid density. For a finite interaction

\[
K(0) \leq \frac{\varepsilon^2}{m_0},
\]

where the right-hand side is the undepleted superfluid density of the non-interacting system. In the effective gauge theories of the \( t-J \) model \( \lambda \), or of the fluctuating d-wave superconductor \( \lambda \), density of bosons at low doping equals the density of holes, and thus \( \rho_0 = x/\alpha^2 \), with \( \alpha \) the lattice constant in the ab-plane. So

\[
T_d > \frac{(a/d)^2}{x} K(0).
\]

Since \( (a/d)^2 \approx 0.1 \) in YBCO it is convenient to take the doping \( x = 0.1 \) as the reference point, so that \( T_d > K(0) \) at this particular doping. Furthermore, the total superfluid density is related to the superfluid density of the bosons via the Ioffe-Larkin rule in Eq. 1, written more precisely as

\[
K_{\text{tot}}^{-1}(T) = K_{qp}^{-1}(T) + K_{\text{th}}^{-1}(T)
\]

where \( K_{qp}(T) = K_{qp}(0) - \alpha^2(2\ln(2)/\pi)(v_F/v_A)T + O(T^2) \) is the contribution from nodal quasiparticles, with \( \alpha \) as the corresponding Fermi liquid parameter \( \lambda \). Expanding to the first order in \( T \) one finds

\[
K_{\text{tot}}(T) = K_{\text{tot}}(0) - (Z\alpha)^2 \frac{2(\ln 2)}{\pi^2} \frac{v_F}{v_{\Delta}} T,
\]

where the \( Z = K(0)/K_{qp}(0) \). Measurements of the superfluid density and thermal conductivity lead to an estimate \( Z\alpha \approx 0.8 \) at \( x \approx 0.1 \), and therefore

\[
K(0) = \frac{K_{\text{tot}}(0)}{1 - Z} \approx 5K_{\text{tot}}(0),
\]
assuming conservatively that $\alpha = 1$. Estimating $T_c \approx 65K$ at $x = 0.1$ in YBCO, the interpolation of known results on the penetration depth yields $1/\lambda^2(0) \approx 50/\mu m^2$ [19], which expressed in Kelvins [20] leads to $K_{tot}(0) \approx 400K$. This leads to the estimate

$$T_d > 2000K.$$ \hspace{1cm} (29)

A similar value of the lower bound is obtained by repeating the exercise at optimal doping. The value of the dimensionless coupling $\lambda$ in YBCO is then

$$\lambda < 1.25.$$ \hspace{1cm} (30)

There are at least two reasons to suspect that the value of the interaction parameter $\lambda$ may lie significantly below our estimated upper bound in the last equation. First, for $\lambda \sim 1$ $K(0)$ would be well below the non-interacting value of $\hbar^2\rho_0/m$, which would in turn yield a lower value of $\lambda$. Second, our estimate is evidently very sensitive to the value of $Z$ in Eq. 28; assuming $\alpha < 1$, for example, would bring $Z$ closer to unity and significantly increase the energy scale $T_d$, and thus decrease the value of $\lambda$. So the Eq. 30 should be understood as a very comfortable upper bound, with $\lambda$ most likely lying well below it.

The Josephson coupling $t$ may be related to the superfluid density along the $c$-axis $K^c(0)$ as

$$\frac{t}{2T_d} = \frac{K^c(0)}{K(0)}. \hspace{1cm} (31)$$

Assuming that the measured superfluid density in very underdoped cuprates is dominated by the bosonic component, the above ratio is $10^{-4}$, and appears to become doping independent at low dopings [21]. So, we estimate

$$\frac{K(0)}{t} \approx 10^4 \hspace{1cm} (32)$$

in the underdoped regime.

As the critical temperature changes from $T_c = 92K$ at optimal doping to $T_c = 9K$ in the extremely underdoped regime in YBCO, the $ab$-plane superfluid density changes by roughly two orders of magnitude [22]. Assuming a constant Josephson coupling in this range leads to the left-hand inequality in Eq. 19, $\lambda > t/K(0)$, being comfortably satisfied by our estimates, and the Coulomb interaction may be safely considered to be effectively short-ranged. Allowing the Josephson coupling to also decrease with underdoping, which may be closer to reality, only strengthens the above conclusion. The right-hand inequality, however, would not be quite satisfied for $\lambda \approx 1$.

As we argued, however, this is only an upper bound, and $\lambda$ may in fact be significantly smaller. To see the effect of the coupling strength on the form of the superfluid density, we plot in Fig. 3 $K(T)/K(0)$ from Eq. 14 for various values of $\lambda$ [23], together with the experimental points [24] on YBCO with $9K \leq T_c \leq 22K$. The best fit is achieved for $\lambda = 10^{-3}$, although it is clear that any value $\lambda < 0.1$ would be almost equally good. This supports the recent proposal [5] by one of us that the superfluid density in very underdoped cuprates is essentially the Bose condensate of the non-interacting layered bosonic system. The reader should also remember that whereas Eq. 14 yields the correct temperature dependence at low temperatures, it does not include the critical fluctuations within the critical region of width $\sim \Delta T_c$ near $T_c$. These are known to modify the superfluid density into $K(T) \sim (T_c - T)^{\nu_{xy}}$, with $\nu_{xy} \approx 0.67$, [24], and thus contribute to an additional rounding of the curve $K(T)$ from its non-interacting form. The absence of any such discernible critical region in the data in Fig. 3 additionally supports our suggestion that $\lambda$ in YBCO may be rather small.

If $\lambda \sim 1$, the system crosses from a 2D, short-range regime, for $K(0)/t \ll 1$, to a 3D long-range regime, for $K(0)/t \sim 1$, and the regimes II, III, and IV from Fig. 1, well separated for weak coupling, now overlap significantly. The regime I, however, even in this case remains wide and distinct. In fact, it is well known that in $^4$He, which is a strongly interacting Bose liquid, the variation of the superfluid density with temperature is well described by Landau’s two-fluid model, except in the critical region. It thus seems likely that Eq. 14 would remain qualitatively correct over most of the temperature region even if the bosonic system is not quite weakly interacting, as long as it is reasonably far from solidification.

In conclusion, we discussed the nature of the collective modes in a strongly anisotropic bosonic superfluid with Coulomb interaction between bosons. In particular, the influence of the anisotropic dispersion of these modes on the temperature dependence of the superfluid density was analyzed. Depending on relative values of the anisotropy and the interaction four different regimes can be discerned for a weak interaction. A crude estimate of relevant parameters for cuprates shows that the Coulomb interaction in underdoped YBCO may be considered to effectively be short-ranged and at least marginally weak, and the system to be quasi-two-dimensional.

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