Probing Hawking and Unruh effects and quantum field theory in curved space by geometric invariants.

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II. AHARONOV–ANANDAN INVARIANT AND VACUUM CONDENSATES

In order to generate the Aharonov–Anandan phase in the course of the time evolution of an isolated system it is necessary and sufficient that its state \( |\phi(t)\rangle \) is not a stationary state, i.e. it has a nonzero value of the uncertainty \( \Delta E(t) \) in energy, \( \Delta E^2(t) = \langle \phi(t)|H^2|\phi(t)\rangle - (\langle \phi(t)|H|\phi(t)\rangle)^2 \). The AAI is then defined as \( S = \langle 2(2\pi)^{-1} \rangle \int_0^t \Delta E(t') dt' \).

In the phenomena mentioned in the Introduction, the physically relevant states \( |\Psi(\theta)\rangle \), with \( \theta \equiv \theta(\xi, t) \) and \( \xi \) some physically relevant parameter, are related to the original ones \( |\psi(t)\rangle \) by the Bogoliubov transformation \( |\Psi(\theta)\rangle = J^{-1}(\theta)|\psi(t)\rangle \) (see Appendix A). For them the...
variance of the energy is always different from zero. The operator \( J^{-1}(\theta) \) is the generator of the Bogoliubov transformation,

\[
\alpha_k^*(\theta) = J^{-1}(\theta) \alpha_k(t) J(\theta) = U_k(\theta) \alpha_k(t) + V_k(\theta) a_{-k}^{-1}(t).
\]

Our discussion, however, is not limited to the single mode \( \alpha_k^* \). It also covers distinct mode cases \( \{a_k^* \text{ and } b_k^* \} \). In the following we consider Bogoliubov transformations at the quantum level (not at the classical one) and we do not specify further the form of the Bogoliubov coefficients \( U_k \) and \( V_k \), except for the fact that they satisfy the relation \(|U_k|^2 \pm |V_k|^2 = 1\), with + for fermions and − for bosons, which guarantees that the transformation is a canonical one. For bosons these coefficients enter the energy variance of \( |\Psi(\theta)\rangle \) as the energy variance is \( \Delta E(t) = \sqrt{2\hbar \omega_k}|U_k(\theta)|^2|V_k(\theta)|^2 \) and the geometric invariant is

\[
S(t) = 2\sqrt{2} \int_0^t \omega_k(U_k(\theta'))|V_k(\theta')|d\theta',
\]

with \( \theta' = \theta(\xi, t') \). \( S \) is related to the number of particles condensed in the vacuum \( |0(\theta)\rangle = J^{-1}(\theta)|0\rangle \), \( |0\rangle \) is the vacuum annihilated by \( \alpha_k^* \) and \( a_{-k}^* \) given by

\[
N_{ak}(\theta) = \langle 0(\theta)|a_{k}^* a_{k}^*|0(\theta)\rangle = |V_k(\theta)|^2.
\]

Indeed \( \Delta E(\theta) \) can be written as

\[
\Delta E(\theta) = \sqrt{2\hbar \omega_k \sqrt{1 + N_{ak}(\theta)} N_{ak}(\theta)},
\]

then the corresponding AAI is

\[
S(t) = 2\sqrt{2} \int_0^t \omega_k \sqrt{1 + N_{ak}(\theta')} N_{ak}(\theta') dt'.
\]

For fermions one obtains relations similar to Eqs.\([1]-[4]\). In the cases of Hawking and Urruch effects, the AAI are related to the number of particle produced in the vacuum. For these phenomena, we get a non-zero \( S \) even though the final amount of particle created are extremely small. This represents another important characteristic of the AAI.

Finally, we note that the noise does not affect AAIs for any system which presents a condensate structure. For such systems, the background noise is given by the non-zero energy of the vacuum condensate \( |0(\theta)\rangle \). Denoting with \( H \) the Hamiltonian of the system, the noise is

\[
\Lambda = \langle 0(\theta)| : H : |0(\theta)\rangle
\]

where the symbol \( : \ldots : \) denotes the normal ordering with respect to \( |0\rangle \). In general the noise \( \Lambda \) is a c-number, (we will compute it explicitly for thermal state) an then it does not modify the uncertainty \( \Delta E(t) \) in the energy of the system and consequently it does not modify the AAI. Indeed, it is immediate to show that the shift \( H \rightarrow H + \Lambda \) does not affect the value of \( \Delta E(\theta) \) computed on the Bogoliubov transformed state \( |\Psi(\theta)\rangle \):

\[
\Delta E^2(\theta) = \langle \Psi(\theta)|(H + \Lambda)^2|\Psi(\theta)\rangle - \langle \Psi(\theta)|(H + \Lambda)|\Psi(\theta)\rangle^2 = \Delta E^2(\theta).
\]

This fact leaves unchanged AAI, as can be seen by the definition of \( S \). On the contrary, the noise affects Berry-like phases since \( \Lambda \) contributes to the value of the Hamiltonian responsible of the system evolution.

We now analyze specific cases.

### III. AHARONOV–ANANDAN INARIANT AND HAWKING EFFECT

**Thermal state** – In the formalism of Thermo Field Dynamics (TFD) \([3]\), (for the convenience of the reader we comment on the TFD formalism in Appendix B) the Bogoliubov parameter \( \theta \) is related to temperature and for bosons (for fermions one can proceed in a similar way) \( U_k = \sqrt{e^{\beta \hbar \omega_k} / (e^{\beta \hbar \omega_k} - 1)} \), \( V_k = \sqrt{1 / (e^{\beta \hbar \omega_k} - 1)} \), with \( \beta = 1/k_B T \). We remark that here and in the following cases \( U_k \) and \( V_k \) are real quantities.

The uncertainty in the energy of the temperature dependent state \( |\Psi(\theta)\rangle = \alpha^\dagger(\theta)|0(\theta)\rangle \), is given by

\[
\Delta E(\theta) = \sqrt{2\hbar \omega_k U_k(\theta)V_k(\theta)} = \sqrt{2\hbar \omega_k \left( e^{\hbar \omega_k / 2k_B T} / (e^{\hbar \omega_k / k_B T} - 1) \right)},
\]

and the AAI is

\[
S_T(t) = 2\sqrt{2} \omega_k t \left( e^{\hbar \omega_k / 2k_B T} / (e^{\hbar \omega_k / k_B T} - 1) \right).
\]

Then, the difference of geometric invariants existing between two thermal states at different temperature \( T_1 \) and \( T_2 \) is

\[
\Delta S_{T}(t) = 2\sqrt{2} \omega_k t \left[ e^{\hbar \omega_k / 2k_B T_1} / (e^{\hbar \omega_k / k_B T_1} - 1) - e^{\hbar \omega_k / 2k_B T_2} / (e^{\hbar \omega_k / k_B T_2} - 1) \right].
\]

**Hawking effect** – The geometric invariant difference \([4]\) could help to detect the Hawking radiation in acoustic black hole created in Bose-Einstein condensate. According to the no-hair conjecture, Hawking radiation emitted by black holes depends only on the mass, angular momentum and charge of the black hole. The thermal bath observed outside the event horizon of a black hole has temperature \( T_H = \hbar c^3 / (8\pi G M k_B) \), where \( G \) is the gravitational constant and \( M \) the black hole mass.

Recently, an acoustic black hole has been created in a Bose-Einstein condensate of \( 10^5 \) atoms of \(^{87}\text{Rb} \)[23]. In this system, the condensate is accelerated by a step-like potential to velocities which cross and exceed the speed of sound. The sonic event horizon is represented by the point where the flow velocity equals the speed of sound and the sound waves cannot escape the event.
The relevant temperatures are now \( T_H = \mu/k_B \pi \lambda \), where \( \mu = mc^2 \) is the chemical potential of the condensate with \( m \) atomic mass, \( c \) speed of sound in the condensate (typically, a few \( 10^{-3} \text{m/s} \) in Bose-Einstein condensate) and \( \lambda \) is related to the number of correlation lengths needed to have a thermal spectrum for the Hawking radiation. \( \lambda = 7 \) and \( T_H \in (2 - 10) \text{nK} \) in \cite{23}. Since the temperature of the condensate \( T_{cond} \) is \( (20 - 170) \text{nK} \), the Hawking radiation is very difficult to identify since it is indistinguishable from thermal noise. However, we remark that a geometric invariant is associated to Hawking radiation. This invariant is analogous to the phase studied in the thermal state case. Now the temperature is \( T \equiv T_H \). The geometric invariant could be detected via interferometry measuring the difference between geometric invariants associated to two flows of one-dimensional Bose-Einstein condensate, one in which is realized an acoustic horizon and the other in which the stream is subsonic. The realization of such configuration can be obtained by using two devices like the one presented in \cite{23}. In this way the Hawking radiation can be revealed by means of the presence of a difference of geometric invariants \( \Delta S_H \) given by Eq. \( 7 \).

The relevant temperatures are now \( T_1 \equiv T_H + T_{cond} \) and \( T_2 \equiv T_{cond} \) since the second flow is subsonic and there is no Hawking radiation. Fig. 1 is thus derived by plotting \( \Delta S_H \) vs the excitation energy for different black hole temperatures. In our computation we have taken into account that Hawking radiation should have a wavelength shorter than the dimension of the black hole \cite{24} and that the minimum trapped wavelength in the experiment of ref. \cite{23} is \( \sim 1.6 \mu \text{m} \). By using Bogoliubov relation of dispersion \( \omega^2 = c^2 k^2 (1 + k^2/k_c^2) \), where \( k_c = (mc)/\hbar \) is the acoustic Compton wavenumber, we have derived the minimum value of the excitation energy in acoustic black hole. In Fig. 1 we considered the fact that the black hole horizon is maintained for about 20 ms in the experiments reported in \cite{23}.

Fig. 1 shows that in a wide frequency interval, AAI are in principle detectable and can represent a new method to reveal Hawking effect in Bose-Einstein condensate.

We note that the finite temperature effects and deviations from Bogoliubov treatment, e.g. at Bogoliubov-Hartree-Fock-Popov or Zaremba-Nikuni-Griffin levels, do not modify considerably the results above presented. Indeed, for \( T \ll T_{cond} \), finite temperature effects are going to be negligible for Bose-Einstein condensate since the depletion of the Bose-Einstein condensate is negligible \cite{23}.

Then, the number of condensed particle can be assumed approximatively constant in the time interval of about 20 ms \cite{23}. Thus, even in the non-equilibrium situations and in particular, in the case of Hawking radiation emission, the Bose occupation number can be assumed almost equal to the one obtained in the thermal equilibrium case at temperature \( T_{cond} + T_H \),

\[
N_{\text{cond}}(\theta) \sim \frac{1}{(e^{\beta \hbar \omega} - 1)}.
\]  

In this case, the noise \( \Lambda \) accounting for the background (original) noise and for the additional noise amplified by the horizon is obtained by computing the energy of the quanta condensed in the thermal vacuum \( |0(\theta)\rangle \) at \( T = T_{cond} + T_H \equiv 1/k_B \beta \), i.e.

\[
\Lambda = \int \omega \hbar \omega \rho_{\omega} \sim \int \omega \hbar \omega \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1}.
\]

This is a c-number leaving unchanged AAI, thus AAI provides in principle a tool able to distinguish Hawking radiation from thermal noise.

**IV. AHARONOV–ANANDAN INVARIANT AND GRAPHENE PHYSICS**

Recently as been shown that a sheet of graphene shaped as the Beltrami pseudosphere, \( d_2 = du^2 + r^2 e^{2u/r} dv^2 \), with \( v \in [0, 2\pi] \), \( u \in [-\infty, 0] \) and \( r \) radius of curvature, displays a finite temperature electronic local density of states \cite{27} given by

\[
\rho = \frac{4}{\pi (\hbar v_F)^2} \frac{E e^{-2u/r}}{e^{E/(k_B T_0 e^{-u/r})} - 1},
\]

with \( E = \hbar \omega \). It has been shown that the electric properties can be expressed in terms of massless, neutral \((2+1)\)-dim Dirac pseudoparticles, which in turn leads us to consider a general relativistic-like space-time. In particular, the electronic thermal spectrum appears to be of Hawking-Unruh type and depends on the curvature of the surface. The temperature in this case is \( T = T_0 e^{u/r} \), with, \( T_0 = \hbar v_F/(2\pi k_B r) \), and \( v_F \) Fermi velocity. In this sense, graphene provides a realization of QFT in curved space-time.

By resorting to the above results presented in detail in \cite{27}, we now proceed as in the thermal state case (with
a temperature, \( T = T_0 e^{a/\tau} \) and derive the AAI for electrons in graphene. In [27] an experiment with a Scanning Tunneling Microscope (STM) has been also proposed to detect the effect there provided. Here, by limiting ourselves to the Beltrami pseudosphere metrics, we show that such an effect could be in principle revealed also by means of AAI. Indeed, from the analysis of AAI, one can obtain the values of the temperature \( T_0 \).

The AAI in graphene shaped as the Beltrami pseudosphere is, as in the thermal state case,

\[
S_g(t) = 2\sqrt{2}\omega_k t^2 e^{\omega_k/2k_B T} e^{-2u/\tau} e^{\omega_k/k_B T - 1},
\]

where the factor \( e^{-2u/\tau} \) is due to the Weyl symmetry transformation of the \( U_k \) and \( V_k \) Bogoliubov coefficients.

Notice that in the \( r \rightarrow \infty \) limit the finite temperature electronic local density of states does not match the flat one \([27]\). Consequently, in such a limit Eq. (10) does not reduce to the geometric invariant of the flat space case.

At given \( E \) and given \( r \), the difference of invariants \( \Delta S_g \) between the tunneling currents along two different meridians \( u \) and \( u' \) can be measured. One derives the value of \( T_0 \) from such a measured \( \Delta S_g \) value. In Fig. 2 we plot \( \Delta S_g \) vs \( T_0 \) for the values of the electron energy shown in the inset, and for \( u = -r/2 \) and \( u' = -r/3 \), with variable \( r \). We see that a non-zero difference of invariants \( \Delta S_g \) in graphene appears for a wide range of values of \( T_0 \) and then it could help to verify quantum field theory in curved space in table top experiments.

V. AHARONOV–ANANDAN INVARIANT AND UNRUH EFFECT

In the Unruh effect, the ground state for an inertial observer is seen by the uniformly accelerated observer as in thermodynamic equilibrium with a non-zero temperature. The Bogoliubov coefficients allow to express the Minkowski vacuum in terms of Rindler states and the temperature of thermal bath depends on the acceleration \( a \) of the observer, \( T_r = h a/(2\pi c k_B) \). The detection of such phenomenon is very hard: acceleration of the order of \( 2.5 \times 10^{20} m/s^2 \) corresponds to a temperature of 1K. It has been shown that the Berry phase variation due to the acceleration of a two level atom, which can be observed through interference with an inertial atom, may produce evidence of the Unruh effect [28, 29]. We observe, however, that the Berry phase is defined only for systems which have an adiabatic and cyclic evolution and, in contrast to AAI, it is affected by thermal noise. Here we suggest that the AAI, which is independent of the particular time evolution of the system and generalize the Berry phase to the noncyclic case, may provide a useful tool in the laboratory detection of the Unruh effect.

By resorting to the results of [28], we consider a nonunitary evolution of an accelerated two level atom and we study the interaction of the atom with all vacuum modes of the electromagnetic field in the multipolar scheme. The accelerated atom is treated as an open system in a reservoir of electromagnetic fields. The environment induces quantum mechanical decoherence and dissipation thus implying a nonunitary evolution of the atom.

In the following, we compute the AAI of the two level system in the presence of an acceleration and in the inertial case by using the discussion presented in [28]. The difference between the two invariant gives the geometric invariant due only to the atom acceleration.

The Hamiltonian of the system (atom plus reservoir) is

\[
H = \frac{\hbar}{2} \omega_0 \sigma_3 + H_\phi - e \sum_{mn} r_{mn} \cdot E(x(t)) \sigma_{mn},
\]

where \( \sigma_3 \) is the Pauli matrix, \( \omega_0 \) is the energy level spacing of the atom, \( H_\phi \) is the Hamiltonian of the electromagnetic field, and \( E \) is the electric field strength. We assume a weak interaction between atom and field, and study the evolution of the total density matrix \( \rho_{tot} = \rho(0) \otimes |0\rangle \langle 0| \), in the frame of the atom. \( \rho(0) \) is the initial reduced density matrix of the atom and \( |0\rangle \) is the vacuum. The evolution, in regime of weak interaction, can be expressed in the Kossakowski-Lindblad form as,

\[
\frac{\partial \rho(\tau)}{\partial \tau} = -i \frac{\hbar}{2} [H_{eff}, \rho(\tau)] + L[\rho(\tau)]
\]

where \( \tau \) is the proper time. Here, \( H_{eff} \) is the effective hamiltonian,

\[
H_{eff} = \frac{\hbar}{2} \Omega \sigma_3 = \frac{\hbar}{2} \left[ \omega_0 + i \left( K(-\omega_0) - K(\omega_0) \right) \right]
\]

where \( \Omega \) is the renormalized energy level spacing which contains the Lamb shift term [28], \( K(\omega_0) \) is the Hilbert transform of the correlation functions

\[
G^+(x-y) = \frac{e^2}{\hbar^2} \sum_{i,j=1}^3 \langle +|r_i|+\rangle (-|r_j|+) \langle 0| E_i(x) E_j(x) |0\rangle,
\]
and the AAI in Eq.(27) becomes
\[ S_a = \pm \int_0^t \frac{e^{2A_a\tau} \sin \theta}{\sqrt{e^{4A_a\tau} \sin^2 \theta + (R - R e^{4A_a\tau} + \cos^2 \theta)^2}} \, \Omega_a \, d\tau , \]
where $R_a = B_a/A_a$, with $\gamma_0 = e^2 \langle \langle \sigma^+ \rangle \rangle^2 / 3\pi \varepsilon_0 \hbar c^3$ the spontaneous emission rate, and the effective level spacing of the atoms given by

$$\Omega_a = \omega_0 + \frac{\gamma_0}{2\pi \omega_0^3} P \int_0^\infty d\omega \omega^3 \left( \frac{1}{\omega + \omega_0} - \frac{1}{\omega - \omega_0} \right) \left( 1 + \frac{a^2}{c^2 \omega^2} \right) \left( 1 + \frac{2}{e^{2\pi\omega_0/a} - 1} \right). \tag{33}$$

In free field case, there is no Aharonov-Anandan phase for the inertial system, then the geometric invariant difference between accelerated and inertial systems, which describes the AAI in the Unruh effect, reduces to (cf. Eq. 7)) $\Delta S_U(t) = 2\sqrt{2} \omega \kappa e^{\pi \omega \kappa / a} / (e^{2\pi \omega \kappa / a} - 1)$.

VI. AHARONOV–ANANDAN INVARIANT AND QUANTUM THERMOMETER

We now remark that the AAI may be used in order to obtain very precise temperature measurement. We observe that the invariant (32) is acquired by the atom also if it interacts with a thermal state. In this case $A_a$ and $B_a$ are replaced by

$$A_T = \frac{\gamma_0}{4} \left( 1 + 4\pi^2 \frac{k_B^2 T^2}{\hbar^2 \omega_0^2} \right) \frac{e^{E_0 / k_B T} + 1}{e^{E_0 / k_B T} - 1}, \tag{34}$$

and

$$B_T = \frac{\gamma_0}{4} \left( 1 + 4\pi^2 \frac{k_B^2 T^2}{\hbar^2 \omega_0^2} \right). \tag{35}$$

with $E_0 = \hbar \omega_0$. Then, a thermometer can be built by means of an atomic interferometer in which a single atom follows two different paths and interacts with two thermal states (samples) at different temperatures. The difference between the geometric invariants obtained in the two paths allows to determine the temperature of one sample once known the temperature of the other one (see also [30]).

Assuming that the temperature $T_h$ of the hotter source is known, for fixed values of $\omega_0$ and of time, one can obtain precise measurements of the temperature $T_c$ of the colder cavity. For the atomic transition frequencies $\omega_0$ and the temperatures of the hot source reported in Fig. 4, and for time intervals of the order of $t = 4 \times 2\pi / \omega_0$ s, measurement of cold source temperatures can be obtained of about 2 orders of magnitude below the reference temperature of the hot source, as shown in Fig. 4.

VII. CONCLUSIONS

We have shown that all the phenomena where vacuum condensates appear exhibit non-cyclic geometric Aharonov-Anandan invariants. We have discussed their use as novel tools in laboratory detection of phenomena particularly hard to be detected, such as Hawking and Unruh effects, thermal field theories and graphene physics.
in the Beltrami pseudosphere metrics, which makes AAI also interesting in the study of QFT in curved space. Finally, we have suggested that a very precise quantum thermometer can be built by exploiting geometric invariants properties. The AAI is a geometric invariant that can be detected in interferometric experiments when the dynamical phase is much smaller than the geometric one. Paths of slightly different lengths can be chosen in order to let the AAI dominate over the relative dynamical phase. Present technologies allow for dynamical phase measurements within a precision of $\Delta \phi \sim 10^{-8}$, which is far smaller than the AAI values obtainable in the cases considered in this paper.

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Appendix A: Bogoliubov transformations

Consider a set of bosonic ladder operators $a_k$ and $b_k$. The canonical commutation relations (CCRs) are:

\[ [a_k, a_p^\dagger] = [b_k, b_p^\dagger] = \delta^{kp} (k - p), \]

with all other commutators vanishing. The vacuum $|0\rangle$ is defined by $a_k|0\rangle = b_k|0\rangle = 0$, and the Fock space is built out of it in the well known way. Such a space provides an irreducible representation of the algebra of the CCRs.

The Bogoliubov transformations on the quantum operators $a_k$ and $b_k$ with transformation parameter $\theta_k$ have the form:

\[
\begin{align*}
\alpha_k(\theta) &= U_k(\theta) a_k - V_k(\theta) b_k^\dagger, \\
\beta_k(\theta) &= U_k(\theta) b_k - V_k(\theta) a_k^\dagger.
\end{align*}
\]

The requirement that they leave the CCRs invariant, i.e. are canonical transformations, implies that $|U_k|^2 - |V_k|^2 = 1$, and then $U_k = e^{i\phi_k} \cosh \theta_k$ and $V_k = e^{i\phi_k} \sinh \theta_k$. For simplicity we put $\phi_k = 0$.

The transformations (A1) can be rewritten as $\alpha_k(\theta) = J^{-1}(\theta) a_k J(\theta)$, and $\beta_k(\theta) = J^{-1}(\theta) b_k J(\theta)$, with the generator $J(\theta) = \exp \left[-\sum_k \theta_k (a_k b_k^\dagger - a_k^\dagger b_k)\right]$.

The vacuum $|0(\theta)\rangle$ relative to the transformed operators $\alpha_k(\theta)$ and $\beta_k(\theta)$ is defined by $\alpha_k(\theta)|0(\theta)\rangle = \beta_k(\theta)|0(\theta)\rangle = 0$ and is related to the vacuum $|0\rangle$ by $|0(\theta)\rangle = J^{-1}(\theta)|0\rangle$. The Stone–von Neumann theorem in Quantum Mechanics guaranties that for a finite number of degrees of freedom, i.e. for discrete $k$, all the representations of the CCRs are unitarily equivalent representations, i.e. in such a case $J^{-1}(\theta)$ is a unitary operator. For discrete $k$ the vacua $|0\rangle$ and $|0(\theta)\rangle$ and the corresponding Fock spaces on them constructed are unitary equivalent vacua and spaces. Thus any vector in one space can be expressed in terms of vectors in the other space. On the other hand, in the $k$ continuum limit, the Stone–von Neumann theorem does not hold and the transformation $|0(\theta)\rangle = J^{-1}(\theta)|0\rangle$ is only a formal relation since $|0\rangle$ and $|0(\theta)\rangle$ (and the corresponding Fock spaces) are unitarily inequivalent states (and spaces). An explicit example is provided by the thermo field dynamics (TFD) formalism briefly summarized below. Thus, in the $k$ continuum limit, Bogoliubov transformations relate sets of operators whose corresponding vacuum states belong to different (inequivalent) representations of the CCRs and provide an “exact” formalism, different from and not to be confused with the Bogoliubov approximation formalism (discussed in full detail in ref. [31]), of which the conclusions presented in the main text of the paper are independent.

Appendix B: Thermo Field Dynamics

In TFD the statistical average $\langle A \rangle$ of an observable $A$ is expressed as the expectation value in the temperature dependent state $|0(\theta)\rangle$:

\[
\langle A \rangle \equiv \frac{\text{Tr}[A e^{-\beta H}]}{\text{Tr}[e^{-\beta H}]} = \langle 0|A|0(\theta)\rangle,
\]

with $H = H - \mu N$, $\mu$ is the chemical potential, $\beta = 1/k_B T$ and $\theta \equiv \theta(\beta)$ is the transformation parameter related to temperature. For simplicity, $\mu$ will be neglected in the following. The possibility to construct such a state $|0(\theta)\rangle$ satisfying Eq. (B1) is guaranteed by the very same structure of QFT allowing multi-vacua states, i.e. infinitely many unitarily inequivalent representations of the CCRs. In TFD different representations are labeled by different temperature values. One can show that Eq. (B1) is obtained by doubling the operator algebra $A \rightarrow A \otimes A$ and, correspondingly, doubling the state space. Here, as customary in the TFD formalism, we denote the doubled states by using the tilde notation $|n\rangle \rightarrow |n\rangle \otimes \tilde{|n\rangle}$, where $|n\rangle$ and $\tilde{|n\rangle}$ are eigenstates of the Hamiltonian $\hat{H}$ and $\hat{H}$, respectively: $\hat{H}|n\rangle = E_n|n\rangle$, $\langle n|m\rangle = \delta_{nm}$, and $\hat{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle$, $\langle \tilde{n}|\tilde{m}\rangle = \delta_{nm}$. $\hat{H}$ has the same operatorial
form of $\mathcal{H}$. Non-tilde and tilde operators are assumed to be commuting boson operators. The state $|0(\theta)\rangle$ is found to be

$$|0(\theta)\rangle = \frac{1}{\sqrt{\text{Tr}[e^{-\beta\mathcal{H}}]}} \sum_n e^{-\beta \epsilon_n} |n, \tilde{n}\rangle . \quad \text{(B2)}$$

Let us consider the example of the number operator $N = a^\dagger a$ for the boson case. For notational simplicity we neglect the momentum index. We introduce the tilde operators and the commutation relations are $[\tilde{a}, a^\dagger] = 1$, $[\tilde{a}, \tilde{a}^\dagger] = 1$. All other commutators are zero and $a$ and $\tilde{a}$ commute among themselves. The state $|0(\theta)\rangle$ is formally given (at finite volume) by

$$|0(\theta)\rangle = J^{-1}(\theta)|0\rangle = \frac{1}{U(\theta)} \exp \left( \frac{V(\theta)}{U(\theta)} \right) a^\dagger \tilde{a}^\dagger |0\rangle , \quad \text{(B3)}$$

where $|0\rangle \equiv |0, \tilde{0}\rangle$ is the vacuum for $a$ and $\tilde{a}$, $J^{-1}(\theta) = e^{i\theta \mathcal{G}}$, with $\mathcal{G} \equiv -i (a^\dagger \tilde{a}^\dagger - a \tilde{a})$, and $U(\theta) \equiv 1 + f_B(\omega)$, $V(\theta) \equiv \sqrt{f_B(\omega)}$, i.e. $U^2(\theta) - V^2(\theta) = 1$, so that

$$U(\theta) = \cosh \theta , \quad V(\theta) = \sinh \theta . \quad \text{(B4)}$$

In these relations $f_B(\omega)$, with $\omega$ the energy of the quantum $a$, denotes the Bose-Einstein distribution. The expectation value of $N$ in $|0(\theta)\rangle$ gives then its statistical average (cf. Eq. (11) with $\mathcal{H} = H = \omega a^\dagger a$):

$$\langle N \rangle = \langle 0(\theta)|N|0(\theta)\rangle = \frac{1}{\cosh \omega - 1} = f_B(\omega) . \quad \text{(B5)}$$

$\mathcal{G}$ is the generator of the Bogoliubov transformations

$$\alpha(\theta) = e^{-i\theta \mathcal{G}} a e^{i\theta \mathcal{G}} = a \cosh \theta - \tilde{a}^\dagger \sinh \theta , \quad \text{(B6a)}$$

$$\tilde{\alpha}(\theta) = e^{-i\theta \mathcal{G}} \tilde{a} e^{i\theta \mathcal{G}} = a \cosh \theta - \tilde{a}^\dagger \sinh \theta . \quad \text{(B6b)}$$

The commutation relations $[\alpha(\theta), \tilde{\alpha}^\dagger(\theta)] = \tilde{\alpha}(\theta), \tilde{\alpha}^\dagger(\theta)] = 1$. All other commutators are vanishing. $\alpha(\theta)$ and $\tilde{\alpha}(\theta)$ commute among themselves and annihilate the state $|0(\theta)\rangle$:

$$\alpha(\theta)|0(\theta)\rangle = 0 , \quad \tilde{\alpha}(\theta)|0(\theta)\rangle = 0 . \quad \text{(B7)}$$

$|0(\theta)\rangle$ is called the thermal vacuum and is the zero energy state for the Hamiltonian $\bar{H} \equiv H - \bar{H} = \omega(a^\dagger a - \tilde{a}^\dagger \tilde{a})$, i.e. $\bar{H}|0(\theta)\rangle = 0$.

When the momentum $k$ index is restored the operator $\mathcal{G}$ and the thermal vacuum are formally (at finite volume) given by

$$\mathcal{G} = -i \sum_k (a_k^\dagger \tilde{a}_k^\dagger - a_k \tilde{a}_k) , \quad \text{(B8)}$$

$$|0(\theta)\rangle = \prod_k \frac{1}{\cosh \theta_k} \exp \left( \tanh \theta_k \ a_k^\dagger \tilde{a}_k^\dagger \right) |0\rangle , \quad \text{(B9)}$$

with $\langle 0(\theta)|0(\theta)\rangle = 1$, $\forall \theta$. It can be shown that $|0(\theta)\rangle$ is an $SU(1,1)$ generalized coherent state $^{[32]}$. In the infinite volume limit, the continuous limit relation $\sum_k \to V(2\pi) \int d^3k$ gives

$$\langle 0(\theta)|0\rangle \to 0 \quad \text{as } V \to \infty \quad \forall \theta = \{\theta_k\} \neq 0 , \quad \text{(B10)}$$

for $d^3k \ln \cosh \theta_k$ finite and positive. In general, $\langle 0(\theta)|0(\theta')\rangle \to 0$ as $V \to \infty \quad \forall \theta$ and $\theta'$, with $\theta' \neq \theta$. The meaning of these relations is that for each $\theta \equiv \{\theta_k\}$ the representation $\{0(\theta)\}$ of the CCRs is unitarily equivalent to the representations $\{0(\theta')\}$, $\forall \theta' \neq \theta$ in the infinite volume limit. The state $|0(\theta)\rangle$ is a condensate of pairs of $a$ and $\tilde{a}$ quanta and

$$\mathcal{N}_{0,\theta} = \langle 0(\theta)|a_k^\dagger \tilde{a}_k|0(\theta)\rangle = |V_k(\theta)|^2 = \sinh^2 \theta_k. \quad \text{(B11)}$$

Similar expression is obtained for $\mathcal{N}_{0,\theta}$. A copy $\{\alpha_k(\theta), \alpha_k^\dagger(\theta), \tilde{\alpha}_k(\theta), \tilde{\alpha}_k^\dagger(\theta) : |0(\theta)\rangle \mid k\}$ thus exists for each $\theta$ of the original algebra induced by the Bogoliubov generator. It generates the group of automorphisms of $\mathfrak{su}(1,1)_k$ parameterized by $\theta_k$, namely a realization of the operator algebra at each $\theta$, which can be implemented by Gelfand-Naimark-Segal construction in the $C^*$-algebra formalism $^{[23], [33]}$.

In non-equilibrium systems or dissipative systems, for time dependent Bogoliubov parameter $\theta(t)$, since the $\tilde{a}$ particle can be thought of as the “hole” (the anti-particle) of the $a$ particle, the energy may flow out of the $a$-system into the $\tilde{a}$-system, or vice-versa $^{[9]}$. The tilde system thus represents the thermal bath or the environment into which the $a$-system is embedded.

The TFD formalism above presented may be extended to the fermion case $^{[2]}$, which here we do not present for brevity.

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