Nonlinear quantum gravity on the constant mean curvature foliation

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Abstract
A new approach to quantum gravity is presented based on a nonlinear quantization scheme for canonical field theories with an implicitly defined Hamiltonian. The constant mean curvature foliation is employed to eliminate the momentum constraints in canonical general relativity. It is, however, argued that the Hamiltonian constraint may be advantageously retained in the reduced classical system to be quantized. This permits the Hamiltonian constraint equation to be consistently turned into an expectation value equation on quantization that describes the scale factor on each spatial hypersurface characterized by a constant mean exterior curvature. This expectation value equation augments the dynamical quantum evolution of the unconstrained conformal three-geometry with a transverse traceless momentum tensor density. The resulting quantum theory is inherently nonlinear. Nonetheless, it is unitary and free from a nonlocal and implicit description of the Hamiltonian operator. Finally, by imposing additional homogeneity symmetries, a broad class of Bianchi cosmological models are analysed as nonlinear quantum minisuperspaces in the context of the proposed theory.

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1. Introduction

The term ‘quantization’ is ill defined. Ideally, one starts with a quantum theory and may wish to classcize it. However, until the ultimate quantum theory of everything is found, one continues to ‘prepare’ a classical system for quantization under a consistent scheme. Further justifications of the quantized system may then be sought. In the canonical approach, this requires a choice of phase variables. Gauge or other auxiliary degrees of freedom, if any, must be factored out from the redundant phase space so that only the physically significant degrees of freedom are quantized. The corresponding type of variables are often referred to as...
dynamical variables whose identification from a given theory is not always straightforward and can be ambiguous [1]. Thus, a central question to ask in quantizing a classical system is what is the right point of departure for quantization? Once this is decided, the actual quantization may then be addressed in terms of its physical consequence and mathematical expediency. The problem of quantization is particularly perplexing for gravity. Starting with the Dirac–Arnowitt–Deser–Misner (Dirac–ADM) formalism of general relativity [2], the implementation of canonical quantization has been impeded by the presence of the Hamiltonian and momentum constraints that signal the redundancy of the conventional geometrodynamical variables. While the momentum constraints are clearly related to the invariance of the formalism under spatial diffeomorphism, a gauge interpretation of the Hamiltonian constraint is more evasive. In particular, the Dirac algebra satisfied by these gravitational constraints fails to be a Lie algebra of any gauge group.

Two divergent strategies have been adopted in dealing with the gravitational constraints. The first strategy aims to exploit them as first class constraints to generate the quantum dynamics of the spatial geometry through the Dirac constraint quantization [3–7]. In essence, this scheme turns both Hamiltonian and momentum constraints into quantum annihilation equations for the physical state functional. The second strategy is based on the reduced phase space method, whose basic theory and history may be found in [8–15]. When applied to canonical general relativity this strategy advocates the elimination of all gravitational constraints in order to obtain a nonvanishing effective Hamiltonian that generates the evolution of certain unconstrained geometric variables [16–19]. The reduced system would then be amenable to the conventional Schrödinger quantization in principle.

The Dirac constraint quantization strategy gains support from the confirmation that the Dirac algebra guarantees the hypersurface independence of the dynamic evolution and hence the general covariance of any parametrized field theories [20, 21]. Together with the interconnection between the Hamiltonian and momentum constraints [22] this property suggests that, under Dirac constraint quantization, a hypersurface-independent quantum evolution of gravity could be constructed without the need for gauge fixing. As the Hamiltonian is quadratic in the canonical momenta of the spatial metric, this strategy results in the Wheeler–DeWitt equation [23, 24], which is a functional Klein–Gordon equation for three-geometries that have no obvious probabilistic interpretation. While this nonunitary quantum evolution may lead to interesting predictions such as the tunnelling phenomena in certain models of the early universe [25], considerable efforts, notably by Kuchař [26–28], have been made to find alternative set of gravitational variables in which Dirac quantization yields the Tomonaga–Schwinger-type equations [29, 30] that generate unitary quantum evolution of geometry. For a broad class of midisuperspace models with two commuting spatial isometries, the symmetry-reduced general relativity can be shown to be equivalent to a parametrized scalar doublet field theory in two-dimensional Minkowski space [31]. However, it is not clear whether Kuchař’s variables can be found in full general relativity [32]. Further discussion on Kuchař’s method may be found in [53 and references therein].

The strategy of applying the Schrödinger quantization to a reduced gravitational phase space by means of gauge fixing was initially explored by Dirac [34] and later by ADM [2]. The ADM procedure amounts to choosing four out of the six spatial metric components as special spacetime coordinates. The gravitational constraints are solved (and are thereby eliminated) for the canonical momenta of these coordinates, leading to a nonvanishing ‘true’ Hamiltonian. The remaining two spatial metric components with the corresponding momenta are then regarded as dynamical variables. Note that this procedure does not by itself single out any preferred dynamical variables. The analogy that ADM drew between the gravitational and electromagnetic fields led them to introduce a background Minkowski metric, with respect to
which the transverse traceless part of the general spatial metric components and their momenta are treated as dynamical variables to be quantized. However, this scheme relies on an \textit{ad hoc} reference metric at the expense of losing covariance.

In this regard, a much more satisfactory gauge fixing scheme that singles out a ‘spin-2’ part of gravitational field was developed by York [35–37]. Built upon previous works by Lichnerowicz, Brill and Choquet-Bruhat [38–40], York defined a decomposition of general second rank symmetric tensors of any weight into transverse traceless, longitudinal and trace parts, in a conformably covariant fashion. The application of this powerful scheme to canonical general relativity is most revealing if the time slicing endows each spatial hypersurface with a constant mean exterior curvature. Under this constant mean curvature (CMC) foliation, the spatial metric is written in terms of the scale factor (of the volume element) and a conformal metric. The latter is used to decompose the metric momentum tensor. The resulting longitudinal part of the metric momentum tensor must vanish to satisfy the decoupled momentum constraints. The trace part of this tensor is proportional to the mean exterior curvature which labels the spatial hypersurfaces like time. This ‘exterior time’ may also be considered as a manifestation of the geometric carrier of information about time originally suggested by Baierlein et al [41]. Therefore, the transverse traceless part of the metric momentum tensor, along with the conformal metric, naturally carries dynamical degrees of freedom and can be interpreted as gravitational waves [37]. The scale factor is to be determined by solving the reduced Hamiltonian constraint as an elliptic equation [42, 43]. Furthermore, it has been shown by Isenberg, O’Murchadha, York and Nester that York’s conformal treatment is capable of accommodating scalar, electromagnetic, spinor fields [44] and indeed ‘virtually all’ known physically relevant forms of matter field [45].

The clear geometrical meaning and physical interpretation of York’s identification of the dynamical variables of general relativity raised the hope for quantum gravity by quantizing the conformal three-geometry with transverse traceless momentum. The diffeomorphisms of the three-geometry and the longitudinal part of the metric momentum (nullified by solving the momentum constraints using the CMC condition) may be regarded as gauge variables. In contrast, no simplicity seems to exist in dealing with the remaining Hamiltonian constraint. If this constraint is also solved for the scale factor before quantization as per the ADM procedure, then a nonvanishing Hamiltonian for canonical general relativity may be constructed, which turns out to be the volume integral of the universe. As a nontrivial elliptic equation must be solved, this Hamiltonian is a nonlocal functional of the dynamical variables and extrinsic time and is only defined implicitly. This somewhat disturbing feature has been noted by Choquet-Bruhat, York, Fischer and Moncrief [43, 46–48]. It presents at least a technical obstacle to quantization as issues such as factor ordering ambiguities are hard to tackle with an implicit and nonlocal Hamiltonian [49].

Under this state of the affair, alternative quantization schemes should not be ruled out where York’s identification of the dynamic part of gravity may be utilized without having to solve the Hamiltonian constraint prior to quantization. The purpose of this paper is to explore one such possibility. Specifically it is proposed that the scale factor may be treated on an equal footing with the lapse function as a Lagrangian multiplier as befits their roles in formulating the kinematics of the spatial hypersurfaces. These nondynamical variables are conceptually akin to Dirac’s surface variables in his analysis of the generic surface kinematics of the parametrized classical field theory [51].

The rest of this paper is devoted to developing the above new strategy for canonical quantum gravity, with emphasis on its physical interpretation and geometrical basis. In section 2, the previously proposed nonlinear generalization of a finite-dimensional physical system based on a quantum action principle [52, 53] is extended to a field-theoretical
description. This generalization is motivated by the ‘implicit nature’ of the gravitational Hamiltonian on the CMC foliation. Before proceeding, it is necessary to lay out the essential prerequisites for classical canonical gravity. This is given in the first half of section 3, with a focus on the elimination of the momentum constraints leading to a reduced system of the pattern discussed in section 2. The second half of section 3 addresses the quantization of the ‘CMC-reduced’ canonical general relativity in section 3, using the nonlinear quantization method developed in section 2. The corresponding quantum action generates the quantum evolution of the conformal three-geometry augmented with two elliptic equations that determine the scale factor and lapse function. In order to establish the essential framework of the proposed theory with a view to more complete treatment, some of the discussion given in this paper remains formal. As in the development of other quantum gravity ideas, a ‘toy model’ in an even simpler mathematical setting will help to grasp the salient features of the new approach. Of course, one should always bear in mind that the usefulness of this way of extracting guidance is limited [54, 55]. Despite this caveat, it is demonstrated in section 4 that the class A Bianchi cosmologies and the Kantowski–Sachs universe can be treated as nonlinear quantum minisuperspaces of the proposed theory. The conclusion with discussion on future work is given in section 5. Below units of $c=\bar{\hbar}=16\pi G=1$ are adopted.

2. Nonlinear quantization of a field theory with an ‘implicit Hamiltonian’

Let $M$ be a three-dimensional space with coordinates $x^i \ (i = 1, 2, 3)$. With respect to a preferred time $\tau$ consider the canonical evolution of a field $\phi$ and its conjugate momentum $\pi$, each with $n$ components, namely $\phi^a = \phi^a(x, \tau)$ and $\pi_a = \pi_a(x, \tau)$ ($a = 1, 2, \ldots, n$). The time-dependent Hamiltonian density $h = h(\phi, \dot{\phi}, \pi, \dot{\pi}, \tau)$ is assumed to be explicitly and locally defined. Here and henceforth, the subscript comma and over dot denote differentiation with respect to a spatial coordinate and time $\tau$, respectively. The canonical field equations are generated by varying the action integral

$$S[\phi, \pi] = \int \int_M \{\pi \cdot \dot{\phi} - h\} \, d^3 x \, d\tau \quad (1)$$

where $\pi \cdot \dot{\phi} := \pi_a \dot{\phi}^a$, with respect to $\pi$ and $\phi$ to be

$$\dot{\phi} = \frac{\delta H}{\delta \pi} \quad \dot{\pi} = -\frac{\delta H}{\delta \phi} \quad (2)$$

in terms of the Hamiltonian

$$H = \int_M h \, d^3 x. \quad (3)$$

The treatment of a canonical field theory using an implicit Hamiltonian can be developed as follows. In terms of an arbitrary time $t$ an action equivalent to (1) is

$$S[\phi, \pi, \sigma, N] = \int \int_M \{\pi \cdot \partial_t \phi - \sigma \partial_t \tau - N H\} \, d^3 x \, dt \quad (4)$$

where

$$H = -\sigma + h \quad (5)$$

and $\tau = \tau(t)$ together with two Lagrangian multipliers $\sigma = \sigma(x, t)$ and $N = N(x, t)$. Note that $H$ has no explicit dependence on the parameter time $t$. The integral

$$\Sigma = \int_M \sigma \, d^3 x \quad (6)$$

is introduced for later use.
Now put \( t = \tau \) but retain \( \tau \) and \( N \). The action (4) becomes
\[
S[\phi, \pi, \sigma, N] = \int_M \{ \pi \cdot \dot{\phi} - \sigma - N\mathcal{H} \} d^3x \, d\tau.
\] (7)

This form suggests a generalization from the restricted form of \( \mathcal{H} \) in (5) a generic \( \mathcal{H} = \mathcal{H}(\phi, \phi_i, \ldots, \pi, \pi_j, \ldots, \sigma, \sigma_j, \ldots, \tau) \). For the purpose of the subsequent discussion, it suffices to assume that this expression is also explicitly and locally defined. With this generic \( \mathcal{H} \), the variations of the action (7) in \( \pi, \phi, N, \sigma \) generate, respectively, the equations
\[
\dot{\phi} = \frac{\delta H}{\delta \pi}, \quad \dot{\pi} = -\frac{\delta H}{\delta \phi} \quad (8)
\]
\[
\mathcal{H} = 0 \quad (9)
\]
\[
1 + \frac{\delta H}{\delta \sigma} = 0 \quad (10)
\]
where
\[
H = \int_M N\mathcal{H} \, d^3x. \quad (11)
\]

Equations (8) govern the evolution of \( \phi \) and \( \pi \). Through \( H \) these equations are coupled with (9) and (10) that can be regarded as algebraic or elliptic equations determining \( \sigma \) and \( N \), respectively. For \( \mathcal{H} \) to be interrelated as the energy density’ of the system, it should be non-negative. Clearly, the above equations reduce to (2) for the ‘simple case’ where (5) holds, with (9) and (10) yielding \( \sigma = h \) and \( N = 1 \). Given a generic \( \mathcal{H} \), although in principle (9) may be solved for the ‘true’ nonvanishing Hamiltonian density \( \sigma \), this will bring about a high degree of nonlocality into the reduced canonical theory. Such an implicit and nonlocal Hamiltonian may in turn cause severe conceptual and technical problems for quantization.

The standard canonical quantization of the system with the action (1) yields the Schrödinger equation
\[
i\dot{\Psi} = \hat{H}\Psi \quad (12)
\]
for the state functionals \( \Psi = \Psi[\phi] \) in terms of the Hamiltonian operator
\[
\hat{H} = \int_M \hat{h} \, d^3x \quad (13)
\]
where \( \hat{h} \) is obtained by substituting \( \pi \rightarrow \hat{\pi} = -i\hbar \frac{\delta}{\delta \phi} \) into \( h \) with a suitable choice of factor ordering. The Schrödinger equation (12) can be shown to follow from the quantum action integral [53, 56]
\[
S_Q[\Psi, \Psi^*] = \int M \left\{ \text{Re} \langle \Psi, i\dot{\Psi} \rangle - \langle \Psi, \hat{H}\Psi \rangle \right\} \, d\tau \quad (14)
\]
using the inner product of any state functionals \( \Psi_1 \) and \( \Psi_2 \) in terms of the functional integration
\[
\langle \Psi_1, \Psi_2 \rangle := \int \Psi_1^* \Psi_2 \, D\phi \quad (15)
\]
in an appropriate sense. The operator \( \hat{h} \) and other physical observables are assumed to be Hermitian with respect to this inner product. The expectation value of any Hermitian operator \( \hat{O} \) with respect to any state functional \( \Psi \) is defined, in a standard manner, as
\[
\langle \hat{O} \rangle := \frac{\langle \Psi, \hat{O}\Psi \rangle}{\langle \Psi, \Psi \rangle}. \quad (16)
\]
By the same token as the steps leading from (1) to (4), an action equivalent to (14) can be written, using the Lagrangian multipliers $\sigma(x,t)$ and $N(x,t)$, as

$$S_Q[\Psi, \Psi^*, \sigma, N] = \int \{\text{Re} \langle \Psi, i\dot{\Psi} \rangle - \Sigma \langle \Psi, \Psi \rangle - \langle \Psi, \hat{H} \Psi \rangle\} \, d\tau$$

where

$$\hat{H} = \int_M N \hat{\mathcal{R}} \, d^3x$$

with

$$\hat{\mathcal{R}} = -\sigma + \hat{h}.$$  \hfill (19)

Here $\hat{\mathcal{R}}$ is the operator of $\mathcal{R}$ given in (5). For a generic $\mathcal{H} = \mathcal{H}(\phi, \phi_j, \pi_j, \sigma, \pi, \sigma, \tau)$ as discussed above that admits a corresponding Hermitian operator $\hat{\mathcal{H}}$, the use of this operator in the quantum action (17) will, under variations with respect to $\Psi$, its conjugate, $N$ and $\sigma$, generate the following generalized form of quantum evolution equations:

$$i\dot{\Psi} = \hat{H}\Psi$$

$$\langle \hat{\mathcal{H}} \rangle = 0$$

$$1 + \left\{ \frac{\delta \hat{\mathcal{H}}}{\delta \sigma} \right\} = 0$$

respectively, up to a (physically irrelevant) overall time-dependent phase of $\Psi$.

The Schrödinger-type equation (20) thus describes the unitary quantum evolution of $\Psi[\phi]$. The ‘Hamiltonian operator’ $\hat{H}$ depends on $\sigma$ and $N$, for which (21) and (22), must be solved. As these two equations are nonlinear in the state functional, the overall quantum evolution is nonlinear. Although the introduction of nonlinearity into quantum gravity is relatively new, the nonlinear modification of quantum mechanics has long been receiving considerable attention in addressing the measurement problem [57–59]. Interestingly, the energy incurred by the nonlinear term in certain ‘dynamical reduction models’ can be shown to be comparable to the gravitational self-energy of the measurement apparatus [57]. This leads one to suspect that gravity might be related to quantum evolution and measurement in a nonlinear manner. For further discussion on nonlinear quantum theories, see e.g. [60–62]. In view of this, the nonlinear quantum formalism presented in this section has been developed with a view to quantizing general relativity. It will be demonstrated in the following section that the action integral for classical canonical gravity does indeed reduce to the form of (7) under the CMC foliation. The nonlinear quantization scheme may therefore be employed for quantum gravity as discussed below.

### 3. Canonical general relativity on the CMC foliation and its nonlinear quantization

In this section the essentials of the Dirac–ADM formulation of canonical general relativity are first recapitulated. The theory is then gauge fixed by choosing the spatial hypersurfaces to possess a constant mean exterior curvature. The resulting ‘CMC-reduced’ formulation is readily organized into the form of the generalized canonical field theory discussed in the foregoing section that may be quantized in a nonlinear fashion.

Under the ADM 3+1 split of spacetime, each spatial hypersurface arbitrarily labelled by time $t$ has a spatial metric $g$ with components $g_{ij}$ in coordinates $x^i$ ($i = 1, 2, 3$). The inverse
(i.e. contravariant) metric $g^{-1}$ has components $g^{ij}$. In terms of the lapse function $N$ and shift vector $X$ with components $X^i$, the spacetime line element takes the form

$$ds^2 = -N^2 dt^2 + g^{ij}(dx^i + X^i dt)(dx^j + X^j dt).$$  \hfill (23)

Here indices are lowered or raised using $g_{ij}$ and $g^{ij}$, respectively.

The extrinsic curvature tensor $K$ has components given by

$$K_{ij} = \frac{1}{2N}(-\dot{g}_{ij} + X_{i;j} + X_{j;i})$$  \hfill (24)

where the semicolon denotes a covariant derivative using the Levi-Civita connection of $g$.

In terms of $K$ and $\mathcal{K}$ the ‘metric momentum’ tensor density $p$ is given by

$$p^{ij} = -\mu(K^{ij} - g^{ij}\mathcal{K})$$  \hfill (26)

where $\mu = (\det g)^{1/2}$ is the scale factor for the spatial volume element.

The well-known Dirac–ADM action integral for canonical gravity takes the form

$$S = \int \int_M \{ p \cdot \dot{g} - N\mathcal{H} - X \cdot \mathcal{J} \} \, d^3x \, dt$$  \hfill (27)

where $p \cdot \dot{g} = p^{ij}\dot{g}_{ij}$, $X \cdot \mathcal{J} = X^i\mathcal{J}_i$ in terms of the Hamiltonian and momentum constraints $\mathcal{H}$ and $\mathcal{J}_i$ given by

$$\mathcal{H} = \frac{1}{4\mu} \{ g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}\} p^{ij}p^{kl} - \mathcal{R}\mu$$  \hfill (28)

$$\mathcal{J}_i = -2g_{ij}p^{jk,k} = -2g_{ij}p^{jk,k} - 2g_{ij,k}p^{jk} + g_{jk,i}p^{jk}$$  \hfill (29)

respectively.

The roles of $N$ and $X^i$ as Lagrangian multipliers result in the Hamiltonian constraint equation

$$\mathcal{H} = 0$$  \hfill (30)

and the momentum constraint equations

$$\mathcal{J}_i = 0.$$  \hfill (31)

York’s treatment of canonical gravity begins by expressing the metric $g$ as

$$g = \varphi^4 \gamma$$  \hfill (32)

in terms of a conformal factor $\varphi$ and a conformal metric $\gamma$ satisfying a ‘normalization’ condition. This may be done by fixing its volume scale factor $\mu_\gamma = (\det \gamma)^{1/2}$ [43, 50] or its scalar curvature $\mathcal{R}_\gamma$ whereby regarding $\gamma$ as a Yamabe metric [46, 63, 64]. For example

$$\mu_\gamma = 1$$  \hfill (33)

or

$$\mathcal{R}_\gamma = \pm 1, 0$$  \hfill (34)

depending on the topology of the spatial hypersurface $M$. The present discussion does not depend explicitly on the choice of such normalization conditions. However, boundary terms in various integrations over $M$ are dropped at will for the sake of mathematical convenience. That is to say, either a compact $M$ or an adequate fall-off of the relevant fields over $M$ at infinity is assumed.
An interesting result of York’s is that the momentum constraints decouples from the Hamiltonian constraint under the CMC foliation. Furthermore, the momentum constraint equations (31) are satisfied if and only if the longitudinal part of the metric momentum \( p \) vanishes, leading to

\[
p = \varphi^{-4} \sigma + \frac{2}{3} \varphi^3 K \gamma^{-1} \mu \gamma
\]  

(35)

where \( \sigma \) is a transverse traceless tensor density with respect to \( \gamma \) [35, 37].

The Hamiltonian constraint equation gives rise to the Lichnerowicz equation [36, 38, 46, 50]

\[
\Delta \varphi - \frac{3}{64} \varphi^5 - \frac{1}{8} R \varphi + \frac{1}{8} \gamma_{ij} \gamma^{ij} \sigma^{kl} \sigma^{kl} \mu^{-2} \varphi^{-7} = 0
\]  

(36)

regarded as the elliptic equation for the conformal factor \( \varphi \) if (30) is also to be solved. The present analysis will, however, defer this process by retaining \( \mathcal{H} \) in the action. Since

\[
\mu = \varphi^{-1} \mu \gamma
\]  

(37)

in the subsequent discussion the conformal factor \( \varphi \) will be eliminated using

\[
\varphi = \mu^{-1} \mu \gamma.
\]  

(38)

Following [46, 47] one writes

\[
p \cdot \dot{g} = \sigma \cdot \dot{\gamma} + \frac{4}{3} \gamma K \mu
\]  

(39)

where \( \sigma \cdot \dot{\gamma} = \sigma^{ij} \dot{\gamma}_{ij} \). This, together with the fact that (31) has been solved by (35) and an integration by parts, reduces the Dirac–ADM action (27) to the form

\[
S[\gamma, \sigma, \mu, K, N] = \int \int_M \left\{ \sigma \cdot \dot{\gamma} - \frac{4}{3} \mu K - N \mathcal{H} \right\} d^3 x \ d t.
\]  

(40)

Here the CMC-reduced Hamiltonian constraint \( \mathcal{H} = \mathcal{H}(\gamma, \sigma, \mu, \tau) \) is obtained by substituting (32) and (35) into (28) and has the form

\[
\mathcal{H} = -\frac{3}{64} \mu^{-1} \gamma^{ij} \sigma^{kl} \sigma^{kl} - \mathcal{R} \mu
\]  

(41)

\[
= -\frac{3}{64} \mu^{-1} \gamma^{ij} \sigma^{kl} \sigma^{kl} - \mathcal{R} \mu \gamma \mu^{-2} \mathcal{L} \left( \mu^{-1} \mu \gamma \right)
\]  

(42)

where \( \Delta \gamma \) is the Laplace–Beltrami operator on scalar functions with respect to \( \gamma \) (following the sign convention of [43, 50] as opposed to that of [37, 46]). Note that the action (40) has a structure analogous to that of (4). To bring (40) to the form of (7), where \( \sigma, \phi \) and \( \pi \) may be identified as \( \mu, \gamma \) and \( \sigma \), respectively, the time parametrization \( t = \tau := \frac{2}{3} K \) is adopted, yielding the action

\[
S[\gamma, \sigma, \mu, N] = \int \int_M \{ \sigma \cdot \dot{\gamma} - \mu - N \mathcal{H} \} d^3 x \ d \tau
\]  

(43)

\[
= \int \left\{ \int_M \sigma \cdot \dot{\gamma} d^3 x - \text{Vol} - H \right\} d \tau
\]  

(44)

where

\[
H = \int_M N \mathcal{H} d^3 x
\]  

(45)

and

\[
\text{Vol} = \int_M \mu d^3 x
\]  

(46)
representing the ‘volume of the universe’. It follows that
\[
\frac{\delta H}{\delta \mu} = \frac{4}{3} \Delta N - N \left( \frac{3}{8} \tau^2 + \mu^{-2} \gamma_{ik} \gamma_{jl} \varpi_{ij} \varpi_{kl} + \frac{1}{3} \mathcal{R} \right)
\]
(47)

\[
= \frac{4}{3} \mu \gamma \frac{5}{6} \mu - \frac{5}{6} \Delta_1 \gamma \left( N \mu \gamma - \frac{1}{6} \mu \right)
\]
(48)

where \(\Delta\) is the Laplace–Beltrami operator on scalar functions with respect to \(g\).

In complete analogy with (8), (9) and (10), the variations of the CMC-reduced action (43) with respect to \(\varpi, \gamma, N\) and \(\mu\) generate the canonical-type field equations

\[
\dot{\gamma} = \frac{\delta H}{\delta \varpi}, \quad \dot{\varpi} = -\frac{\delta H}{\delta \gamma}
\]
(49)

for the conformal metric \(\gamma\) and its transverse traceless momentum density \(\varpi\), supplemented with

\[
\mathcal{H} = 0
\]
(50)

and

\[
1 + \frac{\delta H}{\delta \mu} = 0
\]
(51)

as the elliptic equations for the scale factor \(\mu\) and lapse function \(N\), respectively.

Proceeding with the nonlinear quantization scheme formulated in section 2, one considers state functionals \(\Psi = \Psi_1[\gamma]\) depending on the conformal metric \(\gamma\) modulo diffeomorphisms of \(M\) [49]. The inner product of any two such state functionals \(\Psi_1\) and \(\Psi_2\) are given in terms of an appropriate functional integration of the form

\[
\langle \Psi_1, \Psi_2 \rangle := \int \Psi_1^* \Psi_2 D\gamma
\]
(52)

over the equivalent classes of the conformal metric \(\gamma\) related also by diffeomorphisms of \(M\). Corresponding to the classical action (44), one further writes down the quantum action for the above \(\Psi_1[\gamma], \mu(x, \tau)\) and \(N(x, \tau)\) as

\[
S_Q[\Psi_1, \Psi^*, \mu, N] = \int \left\{ \text{Re} \langle \Psi, i \dot{\Psi} \rangle - \text{Vol} \langle \Psi, \mathcal{H} \rangle - \langle \Psi, \hat{H} \rangle \right\} d\tau.
\]
(53)

This action in turn generates the following Schrödinger-type equation:

\[
i \dot{\Psi} = \hat{H} \Psi
\]
(54)

for \(\Psi\) (up to an overall time-dependent phase), coupled with the elliptic-type equations

\[
\langle \hat{\mathcal{H}} \rangle = 0
\]
(55)

\[
1 + \frac{\delta \hat{H}}{\delta \mu} = 0
\]
(56)

which serve to determine \(\mu\) and \(N\), respectively. In the above equations, the operator \(\hat{\mathcal{H}}\) is obtained by substituting \(\sigma \rightarrow \hat{\sigma} := -i \frac{\delta}{\delta \mu}\) into (42) where the term quadratic in \(\sigma\) naturally becomes the Laplace–Beltrami operator in the space of conformal three-geometries.

The Schrödinger-type equation (54) provides a unitary quantum description of the conformal three-geometry. Through the transverse and traceless nature of the corresponding momenta, it is gravitational waves that are quantized. The quantum evolution depends on \(\mu\) and
which are determined by the elliptic-type equations (55) and (56). These two equations make the overall quantum evolution nonlinear and nonlocal in the state functional. Nevertheless, the Hamiltonian constraint operator is locally defined. Its kinetic part is quadratic in the momentum operator and should be represented by the Laplacian in the space of conformal three-geometries. It is worth noting that the above procedure still carries through if the conformal factor $\varphi$ is chosen in place of $\mu$, with (56) slightly modified as $\mu$ becomes an operator in terms of $\gamma$.

4. Nonlinear quantum minisuperspaces

The nonlinear framework for quantum gravity presented in the previous section is complicated by the presence of the Laplacian of $\mu$ and $N$ as well as the field-theoretical nature of the problem. This complexity can be avoided in the cosmological approach with finite degrees of freedom. At this oversimplified but concrete level, one expects to gain intuition about the full theory for being able to carry out explicit analysis.

For this purpose, the class A Bianchi-type models and Kantowski–Sachs universe [65, 66] will be treated as nonlinear quantum minisuperspaces of the full theory under discussion. These cosmological models are described by the lapse function $N(t)$, length scale factor $R(t)$, as well as two additional functions $\beta^+ (t)$ and $\beta^- (t)$ to allow for the dynamical anisotropy of the spatial hypersurface at any parameter time $t$. The spacetime line element takes the following common form:

$$\text{d} s^2 = -N^2 \text{d} t^2 + R^2 (e^{2\beta})_{ij} e^i e^j$$

for $i, j = 1, 2, 3$, where $e^i$ denote some basis 1-forms of the spatial hypersurface and $\beta$ is a traceless matrix with elements given by

$${\beta}_{ij} = \text{diag}(\beta_+ + \sqrt{3} \beta_-, \beta_+ - \sqrt{3} \beta_-, -2 \beta_+).$$

The basis 1-forms $e^i$ satisfy the structure equation

$$\text{d} e^i = \frac{1}{2} C^i_{jk} e^j \wedge e^k$$

with certain model-specific structure constants $C^i_{jk}$ [65]. Evidently, the spatial metric $g$ may be written as

$$g = R^2 \gamma$$

where

$${\gamma}_{ij} = (e^{2\beta})_{ij}$$

satisfying the normalization condition (33). Hence $\mu = R^3$.

It follows from (24), (25), (26) and (35) that

$$\tau = \frac{4}{3} \mathcal{K} = -\frac{4 \dot{R}}{NR}$$

with the over dot indicating differentiation with respect to $t$, and

$$\omega^{ij} = \frac{R^3}{N} \text{diag}[(\dot{\beta}_+ + \sqrt{3} \dot{\beta}_-) e^{-2\beta_- - 2\sqrt{3} \beta_-}, (\dot{\beta}_+ - \sqrt{3} \dot{\beta}_-) e^{-2\beta_- + 2\sqrt{3} \beta_-}, -2 \dot{\beta}_+ e^{4\beta_-}].$$

Therefore,

$$\omega \cdot \dot{\gamma} = \frac{12 \mu}{N} (\dot{\beta}_+^2 + \dot{\beta}_-^2) = \omega_+ \dot{\beta}_+ + \omega_- \dot{\beta}_-$$

where

$$\omega_{\pm} = \frac{12 \mu}{N} \dot{\beta}_{\pm}.$$
With the choice of time $t = \tau$ and the substitution (as simple volume normalization) $\int_M d^3x \to 1$ the action (43) reduces to

$S[\beta, \sigma, \mu, N] = \int_0^\infty [\sigma_+ \dot{\beta}_+ + \sigma_- \dot{\beta}_- - \mu - N \mathcal{H}] d\tau$ (66)

where

$\mathcal{H} = -\frac{3}{8} \mu \tau^2 + \frac{\sigma^2}{24\mu} - \mu \hat{\mathcal{R}}_\gamma$ (67)

with $\sigma^2 := \sigma_+^2 + \sigma_-^2$.

As no factor ordering ambiguity arises, the cosmological model may now be quantized using the standard substitutions: $\sigma_+ \to \hat{\sigma}_+ = -i\partial_{\beta_+}$ and $\sigma_- \to \hat{\sigma}_- = -(\partial_{\beta_-}^2 + \partial_{\rho_-}^2)$. These operators are to act on a wavefunction $\Psi = \Psi(\beta, \tau)$. The inner product of two such wavefunctions $\Psi_1, \Psi_2$ is simply

$\langle \Psi_1, \Psi_2 \rangle := \int_{-\infty}^\infty \int_{-\infty}^\infty \Psi_1^* \Psi_2 d\beta_+ d\beta_-$. (68)

As per preceding discussions, the quantum action of (66) can be written as

$S_Q[\Psi, \Psi^*, \mu, N] = \int [\text{Re} \langle \Psi, i\dot{\Psi} \rangle - \mu \langle \Psi, \Psi \rangle - N \langle \Psi, \hat{\mathcal{R}}_\gamma \Psi \rangle] d\tau$. (69)

This generates the Schrödinger-type equation

$i\dot{\Psi} = N \left( \frac{\hat{\sigma}^2}{24\mu} - \mu \hat{\mathcal{R}}_\gamma \right) \Psi$ (70)

for $\Psi(\beta, \tau)$ (up to an overall time-dependent phase), and following two equations:

$-\frac{3}{8} \mu \tau^2 + \frac{\langle \sigma_+^2 \rangle}{24\mu} - \mu \hat{\mathcal{R}}_\gamma = 0$ (71)

$1 - N \left( -\frac{\tau^2}{8} + \frac{\langle \sigma_-^2 \rangle}{24\mu} + \frac{1}{3} \mu \hat{\mathcal{R}}_\gamma \right) = 0$ (72)

algebraic in $\mu(\tau)$ and $N(\tau)$. Equation (70) describes the unitary quantum evolution of the anisotropy (i.e. ‘truncated gravitational waves’) of the cosmological model in a mechanical fashion. In particular, it involves a ‘Hamiltonian operator’ consisting of the sum of a kinetic term and a potential term. However, the presence of the supplementary conditions (71) and (72) introduces nonlinearity as well as nonlocality into the description. In this manner, equations (70)–(72) constitute a consistent system of nonlinear integro-partial differential equations. Such equations can be solved at least numerically in a fashion similar to [52] where a computational approach to the nonlinearily quantized Friedmann universe is detailed. In the case of the Bianchi I model, the properties $\mathcal{R}_\gamma = 0$ and $\langle \sigma_+^2 \rangle$ being a constant enable one to find exact solutions to (70)–(72) using the analytical method developed in [53].

5. Concluding remarks

A discussion has been given of a nonlinear quantization scheme for canonical field theories with an implicitly defined Hamiltonian. Two Lagrangian multipliers are involved in the formulation: one representing the value of the energy density and the other affecting this value as an implicit function of the canonical variables allowing for explicit time dependence. This structure has been shown to naturally arise from canonical general relativity on the
constant mean curvature foliation, providing a basis for a nonlinear quantum theory of gravity. As the scale factor (or conformal factor) and lapse function remain unquantized, they act as Lagrangian multipliers in a quantum action principle. In particular, the scale factor represents the effective positive energy density. The true dynamical degrees of freedom for general relativity, as identified by York, are carried by the conformal three-geometries with transverse traceless momenta and are quantized nonlinearly in the proposed theory. York’s exterior time, namely the constant mean exterior curvature, or physically the (local) rate of expansion of the universe, is also adopted here as the preferred time. This work has set the stage for more explicit and detailed investigations. Indeed, work is underway to transcribe the present geometrodynamical description to a ‘connectodynamical’ description so as to assimilate powerful functional techniques offered by the loop quantum gravity approach. As the presented framework features inherent nonlinearity as well as nonlocality in the state functional, it is of interest to explore the resulting physical consequences. This will form a subject for future research, together with other issues such as the inclusion of matter interaction. On this last point, it is reasonable to anticipate much of the essential methodology of this work for pure gravity to be carried over, owing to the versatility of the constant mean curvature analysis in accommodating matter fields.

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