Hawking radiation from rotating AdS black holes in conformal gravity

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Abstract. Research on Hawking radiation before was almost limited in Einstein gravity, Horava-Lifshitz gravity and the like, exploring that in conformal gravity is a glaring blank. We first extend research on Hawking radiation to conformal gravity. We adopt two methods including the Parikh-Wilczek’s semi-classical tunneling method and the method of complex-path integral to investigate Hawking radiation from the new rotating AdS black holes in conformal gravity. As an improvement, we employ the latter method to explore Hawking radiation from rotating black holes in a universal coordinate system, thus avoiding dragging coordinate transformations even in rotating case. The result shows that the tunneling probability remains different due to different prerequisites. Moreover, as for deriving geodesic equations, almost all of related works before existed one shortcoming, different from the massless case, they used a different and very lame approach which violated the variation principle of action to derive the geodesic equation of massive particles. To remedy the shortcoming, we improve treatment to deduce the geodesic equations of massive and massless particles in a unified and self-consistent way, while the geodesic equation of massless particles can be derived only by taking the proper limit of that of massive particles. In addition, we also obtain the Hawking temperature resorting to the complex-path integral method.

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1 Introduction

Conformal gravity, which is an invariant gravity theory under the Weyl conformal transformation, had been advanced [1] for a long time. As an intriguing higher-derivative theory [2,3], it was initially hoped for an alternative candidate for a renormalizable quantum gravity beyond Einstein’s theory of general relativity, and had attracted a lot of attention [4-6,8-10] during the past decades. An attractive property of conformal gravity is that it is a gauge covariant theory [11] which shares many similar characters to that of Yang-Mills gauge theory. In particular, with exact static black hole solutions to conformal gravity found in [11], Birkhoff’s theorem was investigated [12] and an explanation of galactic rotating curves was made [11] in the context of the theory. In recent years, renewed interest in conformal gravity theory has stirred people’s nerves once again [13-15,16,17,18,19,20,21]. Among these investigations, there is an attempt to give an interpretation of the accelerating expansion of the Universe without introducing dark matter and dark energy [16]. Meanwhile, with the help of exact vacuum static solution to conformal gravity, the rotating curves of galaxies [17,18] and the perihelion precession [19] as well as light deflection [21] have been researched once more.

It is established that any (A)dS black hole solution in pure Einstein gravity can be embedded into that of conformal gravity. Although static black hole solution to conformal gravity was known [10] long ago, the construction of a rotating AdS black hole solution remains obscure until recently. Therefore, it is of particular interest to obtain new rotating black holes in conformal gravity that is beyond Einstein’s gravity theory. Only quite recently, an exact solution that describes a rotating and charged AdS black hole has been constructed in [22] where its global structure and some thermodynamical properties with an unusual Bekenstein-Smarr formula are analyzed.

As an intriguing viewpoint, regarding the Hawking radiation [23] as a tunneling process initiated by Kraus and Wilczek [24,25] has received so popular attentions that it can be applicable in various black holes. However, seen from large numbers of previous works concentrating on Hawking radiation as tunneling, they were almost limited in Einstein gravity, Horava-Lifshitz gravity and the like, unfortunately, research on the tunneling radiation in conformal gravity was a glaring blank so far. Instead, understanding about Hawking radiation would be more per-
fect if only extending to study that in conformal gravity. What’s more, as for the new rotating AdS black hole \[22\], some other properties, for example quantum thermal effect etc., are so attractive that deserving further study. Motivated by these facts, in this paper one of focuses is on extending to explore the Hawking radiation as tunneling from the new rotating AdS black hole in conformal gravity, thus filling the blank.

As far as the relatively recent research on the tunneling picture of Hawking radiation is concerned, two different approaches are currently very popular, one is the Parikh-Wilczek’s semi-classical tunneling method \[26\] (based upon the early developments \[24,25,27\]), the other is the Hamilton-Jacobi tunneling method \[28\] which is developed from the method of complex-path integral proposed in \[29,30\]. Soon after the appearance of Parikh-Wilczek’s work \[26\] on the semi-classical treatment of Hawking radiation as tunneling, a large amount of work has been focused on directly applying it to various cases of black holes such as those in de Sitter \[31,32,33,34\], anti-de Sitter \[35,36,37\] space-times, static \[38,39,40,41\], rotating \[42,43,44,45\] and spherical symmetric \[46\], rotating charged \[47,48,49,50,51\], accelerating and rotating \[52,53\] black holes in Einstein-Maxwell theory, black holes in Einstein-Maxwell-dilaton gravity \[54\], Black Holes in Horava-Lifshitz-tz gravity \[55,56\], and those in Einstein-Gauss-Bonnet Gravity \[57\] as well as dynamic black holes in noncommutative gravity \[58,59\]. Although both methods are used to calculate the tunneling probability of Hawking radiation, they are essentially different in one important aspect: the Parikh-Wilczek’s semi-classical tunneling method considers the fluctuation of the background spacetime with the total conserved quantities (mass, charge, and angular momentum) fixed, while the method of complex-path integral neglection the back reaction on the black hole geometry.

In the Parikh-Wilczek’s semi-classical tunneling method, one often needs to derive the geodesic equation of test particles in a Painlevé-like coordinate system which is well-behaved at the event horizon. However, in almost all of previous works, the derivation of geodesic of massive particles adopted the relation \(v_p = v_{sp}/2\) between the group velocity and the phase velocity. In General Relativity, both the null and timelike geodesics can be derived from the Lagrangian with the help of the variation principle. This is a very lame approach which violates the variation principle of action and is quite different from the derivation of the massless particles which follow the null geodesic. Moreover, in the process of deriving the massless and massive geodesic equations in the rotating black holes, one has to be limited to a dragging coordinate system. But it is well-known that according to the principle of covariance, all physics laws should not depend on a concrete reference system. To study Hawking radiation as tunneling from rotating black holes, a generic non-dragging coordinate system should also be permissible. Therefore, the points mentioned above become two apparent defects in the Parikh-Wilczek’s semi-classical tunneling approach. These two shortcomings had not been overcome until our recent paper \[60\] appeared. In \[60\], we have presented a new improved treatment to reinvestigate the tunneling radiation of Kerr black holes in Einstein gravity by deriving the massive particles’ geodesic equation from the classical Lagrangian action both in the dragging and non-dragging coordinate systems, while the massless geodesic can be obtained by taking an appropriate limit of the massive one.

In this paper, we shall develop our previous work and extend that to doing research on Hawking radiation as tunneling from a neutral rotating AdS black hole in conformal gravity theory. Starting with the classical Lagrangian action of a massive particle in the rotating black hole, we will make use of three conserved integral constants of motion to deduce the geodesic equations of tunneling particles both in the non-dragging and dragging coordinate systems. Moreover, the geodesic equation of massless particles can be coherently derived by taking a proper limit of that of massive particles. That is to say, both the geodesic equation of massive and that of massless particles can be obviously deduced in a unified and self-consistent way rather than using the aforementioned lame approach, thus remedying one of the above defects of previous work, even if in conformal gravity. By the way, in order to facilitate investigation of the tunneling radiation from rotating AdS black holes in conformal gravity, we will introduce a superior coordinate transformation from various coordinate transformations and recast the new neutral black hole solution into a beautiful form which can apparently satisfy Landau’s condition of the coordinate clock synchronization \[53\] in the dragging coordinate system, so that the simultaneity of coordinate clocks could be transmitted from one place to another and has nothing to do with the integration path. In quantum mechanics, particle tunneling across a barrier is an instantaneous process, therefore this property is necessary for us to investigate the tunneling process. In contrast to this, in the Appendix we will show another kind of the coordinate transformation which can not fulfill Landau’s condition of the clock synchronization.

With regard to the method of complex-path integral, it is only focused on the leading term to calculate the tunneling probability of Hawking radiation while the background spacetime is considered to be fixed to simplify the integration of the Hamilton’s principal function. However, in the method of complex-path integral \[28,51,62\], one often employs a dragging coordinate system rather than a non-dragging one to study Hawking radiation from rotating black holes. However it is reasonable that investigations on the Hawking radiation could have still been completed in a generic non-dragging coordinate system without using a dragging one. Motivated by this, as an improvement, our one attempt is focused on using it without a coordinate transformation, even for rotating black holes. Actually, it’s available. We will apply this method in a non-dragging coordinate system to calculating the tunneling probability of rotating AdS black holes in conformal gravity for the sake of contrast, in the present paper we will investigate the tunneling radiation by means of the Parikh-Wilczek’s
semi-classical tunneling method in a dragging coordinate system and the method of complex-path integral without a dragging coordinate transformation in turn. What’s more, it is worth noting that the trick of applying the first law of black hole thermodynamics will greatly simplify the tunneling integration calculation.

Our paper is organized as follows. To begin with, we shall review the rotating neutral solution and its thermodynamics of AdS black holes recently constructed [22] in the context of conformal gravity in Section 2. After that, geodesic equations of massive and massless particles which tunnel across the event horizon of rotating AdS black holes in conformal gravity will be derived in a unified and self-consistent way in Section 3. As an example, in Section 4 we will calculate the tunneling probability of the rotating AdS black holes by adopting the Parikh-Wilczek’s semi-classical tunneling method in a dragging coordinate system and the method of complex-path integral in a non-dragging one respectively. In Section 5 we shall work out the Hawking temperature by means of complex-path integral and make a comparison with the temperature given in [22] via the standard method. Section 6 is devoted to our conclusions. The Appendix will give another kind of coordinate transformation.

2 Rotating AdS black holes in conformal gravity and its thermodynamics

Recently, a new rotating charged AdS black hole solution in conformal gravity that is beyond Einstein gravity was constructed in [22]. In this section, we will review the thermodynamical aspect of the neutral rotating charged AdS black hole that is relevant to our research. Introducing the coordinate transformations \( t \rightarrow \tilde{t}, \phi \rightarrow \tilde{\phi} = ag^2\tilde{t} \), the neutral rotating AdS black hole solution can be written as

\[
ds^2 = -\frac{\Delta g}{\Delta_\phi} (\Delta_\phi \tilde{t} - a \sin^2 \theta d\tilde{\phi})^2 + \Sigma (\frac{dr^2}{\Delta_\phi} + \frac{\sin^2 \theta}{\Delta_\phi} a^2 d\tilde{\phi})^2, \tag{1}
\]

where

\[
\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta_\phi = 1 - g^2 a^2 \cos^2 \theta, \quad \Delta = (r^2 + a^2)(1 + g^2 r^2) - 2\mu^3, \quad \Xi = 1 - g^2 a^2,
\]

in which \( a, \mu \) and \( g^2 \) are parameters related to the mass, angular momentum and the cosmological constant \( \Lambda = -3g^2 \) respectively.

Accompanying with this new neutral rotating black hole solution, thermodynamical quantities were also given in [22]. The entropy and temperature are given by

\[
T = \frac{g^2 r_+^2 - 1 - a^2 (3 + g^2 r_+^2)}{4\pi r_+ (r_+^2 + a^2)}, \quad \tag{2}
\]

\[
S = -\frac{2\pi a a^2 (1 + g^2 r_+^2)}{\Xi r_+^2}. \tag{3}
\]

Note that the parameter \( \alpha \) should be negative for a positive entropy in this neutral black hole. The energy and the angular momentum are computed as

\[
E = \frac{2\alpha a^2 g^2 \mu}{\Xi}, \quad J = -\frac{2\alpha a \mu}{\Xi}, \tag{4}
\]

and the angular velocity at the horizon is

\[
\Omega = \frac{a(1 + g^2 r_+^2)}{(r_+^2 + a^2)}. \tag{5}
\]

Conjugate to the cosmological constant \( \Lambda \), the corresponding thermodynamical potential \( \Theta \) is given by

\[
\Theta = \frac{a a^2 (r_+^2 + a^2)(1 + g^2 r_+^2)}{6\Xi r_+^2}. \tag{6}
\]

These quantities obey the differential form of the first law of black hole thermodynamics

\[
dE = T dS + \Omega dJ + \Theta d\Lambda, \tag{7}
\]

while the so-called “Smarr formula” in Ref. [22] takes an unusual form

\[
E = 2\Theta \Lambda. \tag{8}
\]

It can be verified that this formula is indeed valid, whilst the usual integral Smarr formula can not hold true anymore in conformal gravity theory. This feature might hint an important difference [20] of conformal gravity from that of the ordinary General Relativity.

In order to explore Hawking radiation as tunneling from black holes in conformal gravity in a convenient way, it needs to make a proper coordinate transformation. Undoubtedly, there are various possible coordinate transformations, such as a different one given in the Appendix. Here we choose a superior one as follows, its superiority will be shown in the dragging coordinate system later on.

\[
d\tilde{t} = dt - \frac{\sqrt{r^2 + a^2(1 + g^2 r^2) - \Delta}}{\Delta} dr, \quad \tag{9}
\]

\[
d\tilde{\phi} = d\phi - \frac{a \sqrt{r^2 + a^2(1 + g^2 r^2) - \Delta}}{\Delta} dr. \tag{10}
\]

After performing the above coordinate transformations, the line element (11) is changed to the following form,

\[
ds^2 = -\frac{(1 + g^2 r^2)\Delta_\phi}{\Xi} dt^2 + \frac{\Delta_\phi}{\Xi} d\theta^2 + \frac{(r^2 + a^2)^2}{\Xi} \sin^2 \theta d\phi^2 + \sqrt{\frac{(r^2 + a^2)^2(1 + g^2 r^2) - \Delta_\phi}{\Xi}} (\Delta_\phi dt - a \sin^2 \theta d\phi) \nonumber \\
+ \sqrt{\frac{(r^2 + a^2)^2(1 + g^2 r^2)}{\Xi}} dr^2. \nonumber \tag{11}
\]

It is obvious that in the new form (11), the metric is well behaved at the horizon. In addition, if we set \( g^2 = 0 \) it will degenerate into a form which is analogous to the Kerr solution [HS86] if expressed in the Painlevé-Gullstrand coordinate system.
3 Geodesic equations of tunneling particles

In this section, we work out geodesic equations of massive particles in the new neutral rotating AdS black holes in conformal gravity in a non-dragging coordinate system. Then, we do a similar analysis in the dragging coordinate system.

3.1 Geodesic equations of tunneling particles in a generic non-dragging coordinate system

Associated with the stationary black hole solutions, there, in general, exist three conserved integration constants that can be deduced via the variational principle. And these will give rise to the null and timelike geodesic equation as well. Here we restrict the Hamiltonian \( \mathcal{H} = mg_{\mu\nu}\dot{x}^\mu\dot{x}^\nu /2 \) to be a constant, namely \( \mathcal{H} = -mk /2 \), \( (k = 0, 1) \) in which the constant \( k \) can take two different values corresponding to the 4-velocity normalization condition of null and timelike geodesic, respectively. Adding two more constants \( E \) (energy) and \( L \) (angular momentum), one can completely determine the geodesic equations of massive particles. In addition, of special interest is that we can derive the geodesic equations of massless particles if we set \( k = 0 \), or the rest mass \( m = 0 \) from the geodesic equations of massive particles. As a consequence, we can write out geodesic equations of massive and massless particles in a unified way, thus extending the original work [60] to the case in conformal gravity.

At first, we plan to derive the geodesic equation of massive particles from the above metric (11) which is expressed in a non-dragging coordinate system.

According to Chandrasekhar’s book [64], the corresponding Lagrangian that we consider for a massive particle is,

\[
\mathcal{L} = \frac{m}{2} \left\{ \left[ 1 + \frac{1 + g^2 r^2}{\Xi} \right] \frac{\Delta \theta}{\Xi} \dot{t} + \frac{\Delta \varphi}{\Xi} \dot{\varphi} + \frac{(r^2 + a^2) \sin^2 \theta}{\Xi} \dot{\varphi} \right. \\
\left. + \frac{\sqrt{\left( r^2 + a^2 \right) \left( 1 + g^2 r^2 \right) - \Delta}}{\sqrt{\Xi}} \left( \Delta \varphi \dot{t} - a \sin^2 \theta \dot{\varphi} \right) \right. \\
\left. + \dot{r} \sqrt{\frac{\Sigma \Xi}{(r^2 + a^2) \left( 1 + g^2 r^2 \right)}} \right\}^2 ,
\]

in which \( m \) is the rest mass of particle. We derive the Lagrangian (12) from universal theory which is not dependent on specific gravity, it’s also available for conformal gravity. From it the generalized momenta \( P_\alpha = \partial \mathcal{L} / \partial \dot{\alpha} \) can be deduced and given by

\[
P_t = \frac{m \Delta \theta}{\Xi} \left\{ - (1 + g^2 r^2) \dot{t} \\
+ \dot{r} \sqrt{1 - \frac{(r^2 + a^2) \Delta }{(r^2 + a^2) \left( 1 + g^2 r^2 \right)}} \right. \\
+ \left[ \frac{(r^2 + a^2) \left( 1 + g^2 r^2 \right) - \Delta}{\sqrt{\Xi}} \right] \dot{\varphi} \\
\left. + \frac{\dot{r}}{(r^2 + a^2) \left( 1 + g^2 r^2 \right)} \right\} ,
\]

and

\[
P_r = \frac{m \Sigma \Xi}{\Delta \varphi} \left\{ (r^2 + a^2) \left( 1 + g^2 r^2 \right) \dot{r} \\
+ \frac{\Delta}{\Xi} \right\} ,
\]

\[
P_\theta = \frac{m \Delta \varphi}{\Xi} \left\{ (r^2 + a^2) \left( 1 + g^2 r^2 \right) \dot{\varphi} \\
- a \sin^2 \theta \left[ \dot{r} \sqrt{1 - \frac{(r^2 + a^2) \Delta}{(r^2 + a^2) \left( 1 + g^2 r^2 \right)}} \\
+ \left( r^2 + a^2 \right) \left( 1 + g^2 r^2 \right) \Delta \right] \left( \Delta \varphi \dot{t} - a \sin^2 \theta \dot{\varphi} \right) \right\} .
\]

The corresponding Hamiltonian can be obtained via the Legendre transformation, \( \mathcal{H} = i P_t + \dot{r} P_r + \dot{\varphi} P_\varphi + \dot{\theta} P_\theta - \mathcal{L} = mg_{\mu\nu}\dot{x}^\mu\dot{x}^\nu /2 \).

In consequence, we have

\[
\mathcal{H} = \frac{m}{2} \left\{ - (1 + g^2 r^2) \frac{\Delta \theta}{\Xi} \dot{t} + \frac{\Delta \varphi}{\Xi} \dot{\varphi} + \frac{(r^2 + a^2) \sin^2 \theta}{\Xi} \dot{\varphi} \right. \\
+ \left[ \sqrt{\left( r^2 + a^2 \right) \left( 1 + g^2 r^2 \right) - \Delta} \right] \left( \Delta \varphi \dot{t} - a \sin^2 \theta \dot{\varphi} \right) \\
+ \dot{r} \sqrt{\frac{\Sigma \Xi}{(r^2 + a^2) \left( 1 + g^2 r^2 \right)}} \right\}^2 
\]

Here \( t \) and \( \theta \) act as ignorable coordinates, and they correspond to the conserved generalized momenta \( P_t \) and \( P_\theta \). Let them be two integral constants \( E \) and \( L \) respectively,

\[
P_t = E , \quad P_\theta = L ,
\]

we also have the 4-velocity normalization condition,

\[
\mathcal{H} = -mk /2 .
\]
Solving these three conditions for $\dot{r}$, $\dot{t}$ and $\dot{\phi}$, we can obtain
\[
\dot{r} = \pm \sqrt{\frac{W}{m\Sigma}},
\] (19)
\[
\dot{t} = \pm \sqrt{\frac{(r^2 + a^2)Y + a\Delta_r \left( L + \frac{aE\sin^2 \theta}{\Delta_\theta} \right)}{m\Sigma \Delta_r}},
\] (20)
\[
\dot{\phi} = \pm \sqrt{\frac{(1 + g^2r^2)}{m\Sigma \Delta_r} \left( \frac{\Delta_r \Delta_\theta \left( L + \frac{aE\sin^2 \theta}{\Delta_\theta} \right)}{(1 + g^2r^2)\sin^2 \theta} \right) + aY + \frac{\Delta_r \Delta_\theta \left( L + \frac{aE\sin^2 \theta}{\Delta_\theta} \right)}{(1 + g^2r^2)\sin^2 \theta}}.
\] (21)
where we denote
\[
W = Y^2 - \frac{\Delta_r (L\Delta_\theta + aE\sin^2 \theta)^2}{\Delta_\theta \sin^2 \theta} - \Delta_r (m^2 k\Sigma + \Delta_\theta P_\Sigma^2),
\] 
\[
Y = (r^2 + a^2)E + aL(1 + g^2r^2).
\]
With these equations in hand, we proceed to derive the geodesic equations of massive particles in a pretty form in a non-dragging coordinate system. For the radial part, we get
\[
\dot{\bar{r}} = \frac{d\bar{r}}{dt} = \frac{\dot{r}}{l}
\] 
\[
= \Delta_r \left[ (r^2 + a^2)\sqrt{1 - \frac{\Delta_r}{(r^2 + a^2)(1 + g^2r^2)}} \right] - (r^2 + a^2)Y + a\Delta_r \left( L + \frac{aE\sin^2 \theta}{\Delta_\theta} \right)^{-1},
\] (22)
and for the angular part,
\[
\dot{\bar{\phi}} = \frac{d\bar{\phi}}{dt} = \frac{\dot{\phi}}{l}
\] 
\[
= (1 + g^2r^2) \left\{ - aY + \frac{\Delta_r \Delta_\theta \left( L + \frac{aE\sin^2 \theta}{\Delta_\theta} \right)}{(1 + g^2r^2)\sin^2 \theta} \left[ 1 - \frac{\Delta_r}{(r^2 + a^2)(1 + g^2r^2)} \right] \right\}
\] 
\[
\times \left\{ - (r^2 + a^2)Y + a\Delta_r \left( L + \frac{aE\sin^2 \theta}{\Delta_\theta} \right) \right\}
\] 
\[
\pm a\sqrt{\left[ 1 - \frac{\Delta_r}{(r^2 + a^2)(1 + g^2r^2)} \right] W}
\] (23)
where the signs “±” correspond to the geodesics of ingoing and outgoing particles from the event horizon.

Next, we will also check their asymptotic behavior of the outgoing particles near the event horizon. When $r$ goes to $r_+$, the radial equation (22) and angular equation (23) behave like
\[
\bar{r} = \frac{dr}{dt} \xrightarrow{r \to r_+} \frac{\Delta'(r_+)(r - r_+)}{2(r_+^2 + a^2)} = \kappa(r - r_+),
\] (24) and
\[
\bar{\phi} = \frac{d\phi}{dt} \xrightarrow{r \to r_+} \frac{a(1 + g^2r_+^2)}{r_+^2 + a^2},
\] (25)
where $\Delta'(r_+)$ denotes the first derivative of $\Delta_r$ at horizon radius $r_+$ and $\kappa$ is the surface gravity on the horizon.

### 3.2 Geodesics equations of tunneling particles in a dragging coordinate system

Now we shall calculate geodesic equation of the new neutral rotating AdS black holes in conformal gravity in a dragging coordinate system.

With regard to the metric (11), in order to make the event horizon coincide with the infinite red-shift surface in the dragging frame, we first perform a coordinate transformation $d\phi = \Omega dt$ in which the dragging angular velocity $\Omega$ reads
\[
\Omega = \frac{a\Delta_\theta}{\Delta_r (r^2 + a^2) - \Delta_r a^2 \sin^2 \theta},
\] (26)
then the metric (11) is sent to the following form
\[
d^2 s = \frac{\Delta_r \Delta_\theta \Sigma}{\Delta_\theta (r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta} dt^2 + 2\Delta_\theta \Sigma \left[ (r^2 + a^2)(1 + g^2r^2) \right] dt dr + \frac{\Sigma}{(r^2 + a^2)(1 + g^2r^2) dr^2 + \frac{\Sigma}{\Delta_\theta} \Sigma^2 d\theta^2}.
\] (27)
Perfectly, the new form (27) satisfies Landau’s condition of the coordinate clock synchronisation [63]. In other words, if the simultaneity of coordinate clocks can be transmitted from one place to another and has nothing to do with the integration path, the metric components in a dragging coordinate system should satisfy [10]
\[
\frac{\partial}{\partial x^i} \left( - \frac{g_{00}}{g_{0i}} \right) = \frac{\partial}{\partial x^j} \left( - \frac{g_{00}}{g_{0j}} \right) (i, j = 1, 2, 3).
\] (28)
In fact, because the covariant component $g_{\theta}$ is zero, it only needs to demand the ratio $g_{0i}/g_{0t}$ be a function which is independent of the variable $\theta$. Indeed, its consistency turns out to be true.

In addition, in the new form (27), there is no any coordinate singularity at the event horizon in the dragging coordinate system, and what is more, the event horizon and the infinite red-shift surface coincide with each other so that the geometrical optical limit can be applied. These properties are very advantageous for us to discuss Hawking radiation via tunneling from rotating black holes and work out the tunneling probability at the event horizon.
Likewise, the corresponding Lagrangian in this dragging system is given by
\[
\mathcal{L} = \frac{m\Sigma}{2} \left\{- \frac{\Delta r \Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_t^2 - \frac{\Delta_\alpha}{2} \frac{\Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_r^2 \right\} + \frac{\Delta_\alpha}{2} \frac{\Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_r^2 \right\} + \frac{\Delta_\alpha}{2} \frac{\Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_r^2 \right\} ,
\]
and we can obtain the generalized momenta \( P_\alpha = \partial \mathcal{L} / \partial \dot{x}_\alpha \) as follows
\[
P_t = m \Sigma \left\{ \frac{1}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_t^2 + \frac{\Delta_\alpha}{2} \frac{\Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_r^2 \right\} ,
\]
\[
P_r = m \Sigma \left\{ \frac{1}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_r^2 + \frac{\Delta_\alpha}{2} \frac{\Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_r^2 \right\} ,
\]
\[
P_\theta = m \Sigma \left\{ \frac{\Delta_\alpha}{2} \frac{\Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_r^2 \right\} .
\]
Hence the Hamiltonian is
\[
\mathcal{H} = i P_t + i P_r + \dot{\theta} P_\theta + \dot{\phi} P_\phi - \mathcal{L} = m g_{\alpha \nu} \dot{x}^\nu \dot{x}^\nu / 2
\]
\[
= m \Sigma \left\{ - \frac{\Delta r \Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_t^2 + \frac{\Delta_\alpha}{2} \frac{\Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_r^2 \right\} + \frac{\Delta_\alpha}{2} \frac{\Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_r^2 \right\} + \frac{\Delta_\alpha}{2} \frac{\Delta \theta}{\Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \partial_r^2 \right\} ,
\]
Similarly, corresponding to the ignorable coordinate \( t \) one can introduce a conservation constant \( P_t = E \) (energy) and impose the 4-velocity normalization condition \( \mathcal{H} = - m k / 2 \), \((k = 0, 1)\). Solving these equations which corresponds to these two integral constants, we get
\[
\dot{r} = \pm \frac{E \sqrt{\mathcal{W}}}{m \Sigma} ,
\]
\[
i = - \frac{E}{m \Sigma} \cfrac{\Delta_\alpha}{\Delta_\alpha \Delta \theta \Sigma z} \left\{ \Delta \theta (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta \right\} + \Delta \theta \left( \frac{(r^2 + a^2)(1 + g_2 r^2) - \Delta_\alpha a^2 \sin^2 \theta}{1 + g_2 r^2} \right) W ,
\]
in which
\[
\mathcal{W} = \frac{(r^2 + a^2)(1 + g_2 r^2)}{\Delta_\alpha a^2 \sin^2 \theta} \left[ \Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta \right] \frac{1}{E} \frac{\Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta}{(r^2 + a^2)^2} \left[ \Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta \right] \right\} .
\]
Thus the geodesic equations of massive particles in the dragging coordinate system can be derived from Eqs. \((34)\) and \((35)\).
\[
\frac{1}{r} = \frac{dt}{dr} = \frac{i}{r} = \frac{r^2 + a^2}{\Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \left[ \Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta \right] \frac{1}{E} \frac{\Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta}{(r^2 + a^2)^2} \left[ \Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta \right] \right\} .
\]
where the signs \( \pm \) correspond to the geodesic equations of ingoing and outgoing particles from event horizon.

Now we consider the geodesic equation of massless particles in the dragging coordinate system. Let both \( m \) and \( P_\theta \) approach zero, we find
\[
\frac{1}{r} = \frac{dt}{dr} = \frac{i}{r} = \frac{r^2 + a^2}{\Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta} \left[ \Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta \right] \frac{1}{E} \frac{\Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta}{(r^2 + a^2)^2} \left[ \Delta_\alpha (r^2 + a^2)^2 - \Delta_\alpha a^2 \sin^2 \theta \right] \right\} .
\]
It is worth noting that the result by setting \( g_2 = 0 \) in \((35)\) completely coincides with the previous result \((15)\) which is actually the case of massless particles. As a consequence, we can get geodesic equations of massive and massless particles in a unified way in the conformal gravity.

There is no doubt that the asymptotic behavior of outgoing particles near the event horizon in the dragging coordinate system needs to be studied. Therefore, on the basis of the radial equation \((37)\), when \( r \) goes to \( r_+ \), we have
\[
\dot{r} = \frac{dr}{dt} = r \rightarrow r_+ \overset{!}{=} \frac{\Delta_\alpha (r_+)(r - r_+)}{2(r_+^2 + a^2)} = \kappa (r - r_+) ,
\]
and the angular velocity at the event horizon is given by
\[
\dot{\phi} = \frac{d\phi}{dt} = r \rightarrow r_+ \overset{!}{=} a \frac{(1 + g_2 r_+^2)}{r_+^2 + a^2} = \Omega_+ .
\]
So far, we have in turn worked out the geodesic equations of particles which tunnel across the event horizon of
rotating AdS black holes in conformal gravity in a non-dragging coordinate system and a dragging one. It is worth mentioning that the geodesic equations of particles have been successfully derived from the Lagrangian \( \mathcal{L} \), rather than the relation \( v_p = v_p/2 \) between the group velocity and the phase velocity, especially for massive particles, we have effectively avoided aforementioned shortcoming in the process of deriving the massive and massless particles’ geodesic equations and deduced geodesic equations of massive and massless particles in a unified and self-consistent way in conformal gravity.

Compare Eqs. (30) and (40) with (24) and (25), it is indicated that the asymptotic behaviors of tunneling particles across the event horizon in the dragging coordinate system are the same as that in a non-dragging one, that’s just what we would look forward to.

4 Tunneling probability of particles via adopting different methods and coordinate systems

Parikh-Wilczek’s semi-classical tunneling method and the complex-path integral method have been frequently to explore Hawking radiation ever before. However, most of work following these two methods did research on the Hawking radiation with the help of coordinate transformations such as a dragging coordinate transformation. In this section, we will extend both methods to the case of the rotating AdS black holes in conformal gravity, not being limited to those of Einstein theory any longer. We will first use the Parikh-Wilczek’s semi-classical tunneling method to compute the tunneling rate in a dragging coordinate system. Often times in rotating case, almost all of works before calculated tunneling probability in a dragging coordinate system. However, with improving the treatment, we shall make use of the complex-path integral method to calculate the tunneling probability without performing a dragging coordinate transformation.

4.1 Parikh-Wilczek’s semi-classical tunneling method in a dragging coordinate system

At first, we start with the Parikh-Wilczek’s semi-classical tunneling method in a dragging coordinate system to derive the tunneling probability. In the process of Hawking radiation, the energy of black holes would reduce with particles tunneling out to the infinity, meanwhile, the event horizon surface would shrink. When taking the self-gravitation between particles into consideration, with the permission for mass fluctuation of black holes under the conditions of energy conservation as well as angular momentum conservation, and assuming that the energy of particles tunneling out black holes is \( \omega \), then the corresponding line element in the dragging coordinate system should be written as

\[
\begin{align*}
ds^2 &= - \frac{\Delta_r \Delta_\theta \Sigma}{\Delta_\theta (r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta} dt^2 \\
&+ 2 \Delta_\theta \sqrt{(r^2 + a^2) [(r^2 + a^2)(1 + g^2 r^2) - \Delta_r]} dt dr \\
&+ \frac{\Sigma}{(r^2 + a^2)(1 + g^2 r^2)^2} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2,
\end{align*}
\]

where \( \Delta_r = (r^2 + a^2)(1 + g^2 r^2) + (M - \omega) \Sigma^2 r^3 / (\alpha a^2 g^2) \), in which \( M \) is the mass of the black hole. In fact, if a particle carrying an energy \( \omega \) tunnel out black hole, the mass and the angular momentum of the rotating AdS black hole in conformal gravity will change to \( (M - \omega) \) and \( (M - \omega) / (\alpha a^2 g^2) \), respectively.

What’s more, the radial geodesic equation of the outgoing particle is given by

\[
\begin{align*}
\frac{1}{r} = \frac{r^2 + a^2}{\Delta_r} \left\{ \frac{\Delta_r}{(r^2 + a^2)^2 (1 + g^2 r^2)} - \frac{\Delta_r a^2 \sin^2 \theta}{\Delta_\theta (r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta} \right\},
\end{align*}
\]

According to the WKB approximation, the emission rate of the tunneling particle is related to the action of the tunneling process by

\[
\Gamma \sim e^{-2 \text{Im} S}.
\]

It can be easily seen from the line element (41) that \( \phi \) is an ignorable coordinate in the Lagrangian function \( \mathcal{L} \). To eliminate this degree of freedom, the imaginary part of the action should be rewritten as

\[
\text{Im} S = \text{Im} \int_{r_1}^{r_f} (\mathcal{L} - P_\phi \dot{\phi}) dt = \text{Im} \left[ \int_{r_1}^{r_f} P_\phi dr - \int_{\phi_1}^{\phi_f} P_\phi d\phi \right] = \text{Im} \left[ \int_{r_1}^{r_f} P_\phi dr - \int_{\phi_1}^{\phi_f} P_\phi d\phi \right],
\]

where, \( r_1 \) and \( r_f \) correspond to the horizon radius before and after the shrinkage, \( P_r \) and \( P_\phi \) are the canonical momenta conjugate to \( r \) and \( \phi \), respectively. To proceed with an explicit calculation, we can utilize the Hamiltonian equations of motion as follows

\[
\begin{align*}
\dot{r} &= \frac{dH}{dP_r} |_{(r_\phi, P_\phi)} = \frac{d(M - \omega)}{dP_r}, \\
\dot{\phi} &= \frac{dH}{dP_\phi} |_{(\phi, r, P_r)} = \frac{\Omega}{a} \frac{d(M - \omega)}{dP_\phi},
\end{align*}
\]

where we have made use of equations \( dH |_{(\phi, r, P_r)} = \Omega dJ = \frac{\partial J}{\partial \phi} d(M - \omega) \) and \( P_\phi = J = \frac{M - \omega}{\alpha g^2} \). On the basis of previous
analysis, we have
\[ \text{Im} S = \text{Im} \int_{\tilde{r}_i}^{\tilde{r}_f} \int_{M}^{M-\omega} (d\tilde{r} - \tilde{\Omega} dJ) \frac{dr}{r} \]
\[ = \text{Im} \int_{\tilde{r}_i}^{\tilde{r}_f} \int_{M}^{M-\omega} \left[ d(M - \omega) - \frac{\tilde{\Omega}}{ag^2} d(M - \omega) \right] \frac{dr}{r}, \tag{47} \]

In order to complete the above integral, we need the help of the asymptotic behavior of the outgoing particles near the event horizon
\[ \tilde{r} \approx \tilde{k}(r - \tilde{r}_+), \quad \tilde{k} = \frac{\Delta'(\tilde{r}_+)}{2(\tilde{r}_+^2 + a^2)}, \tag{48} \]
where \( \tilde{k} \) denotes the surface gravity on the horizon after the particles emission. The angular velocity is given by
\[ \tilde{\Omega} = \frac{a(1 + g^2 \tilde{r}_+^2)}{\tilde{r}_+^2 + a^2}, \tag{49} \]
in addition, the entropy is
\[ S = -\frac{2\pi a^2 (1 + g^2 \tilde{r}_+^2)}{2\tilde{r}_+}. \tag{50} \]

One can verify that these thermodynamical quantities comply with the differential form of the first law of black hole thermodynamics \[22\] when \( \Lambda = -3g^2 \) is treated as a constant, namely,
\[ d(M - \omega) = \frac{\tilde{k}}{2\pi} dS + \frac{\tilde{\Omega}}{ag^2} d(M - \omega). \tag{51} \]

Substituting Eqs. (48) and (51) into (47), the imaginary part of action can be written as
\[ \text{Im} S \approx \text{Im} \int_{\tilde{r}_i}^{\tilde{r}_f} \int_{M}^{M-\omega} d(M - \omega) - \frac{\tilde{\Omega}}{ag^2} d(M - \omega) \frac{dr}{r} \]
\[ = -\frac{1}{2} \int_{S_{BH}(M-\omega)} S_{BH} dS = -\frac{1}{2} \Delta S_{BH}. \tag{52} \]

Therefore, the tunneling probability of particles is
\[ \Gamma \sim e^{-2\text{Im} S} = e^{-\Delta S_{BH}}, \tag{53} \]
which indicates that the Parikh-Wilczek’s result is also true in the conformal gravity.

So far, we have got the tunneling probability of particles via adopting Parikh-Wilczek’s semi-classical tunneling method. In contrast to the conventional treatment frequently used in tunneling integration, the superiority of using the first law of black hole thermodynamics is highlighted. There is no doubt that just as in Einstein gravity, the tunneling probability of a rotating AdS black hole in conformal gravity is also related to the change of Bekenstein-Hawking entropy and the real radiation spectrum from a black hole is not pure thermal any longer. Here our study is done in a dragging coordinate system, however, it is, in fact, not limited to do the same analysis in such a system, one can proceed the same procedure in a generic non-dragging system. To be not repeated, we shall not present the details here. Instead, in the following we shall proceed with the complex-path integral method in a non-dragging coordinate system. It is worth mentioning that our derivation process is different from most of previous treatments by means of the same method in a dragging coordinate system.

### 4.2 Complex-path integral method in a non-dragging coordinate system

To develop the complex-path integral method, unlike almost all of related works, here much effort is focused on using it to work out the tunneling probability in a non-dragging coordinate system, thus to avoid a dragging coordinate transformation, even in rotating black hole in conformal gravity. We now consider a scalar particle moving in the classical black hole with the background spacetime fixed. Therefore, we neglect the particle’s self-gravitation when considering the semi-classical approximation and only the leading contribution. As a consequence, the corresponding action \( I \) satisfies the classical Hamilton-Jacobi equation
\[ g^{\mu\nu} \partial_\mu I = m^2 = 0. \tag{54} \]

Substituting the metric components \[11\] into Eq. \( 54 \), the Hamilton-Jacobi equation becomes
\[ \frac{1}{\Delta_r} \left[ (r^2 + a^2) \partial_t I + a(1 + g^2 r^2) \partial_\phi I \right]^2 = \frac{\Delta_r}{\Sigma} \left( \partial_r I \right)^2 \]
\[ + \frac{\Delta_\theta}{\Sigma} \left( \partial_\theta I \right)^2 + \left( \frac{a \sin \theta \partial_t I + \frac{\partial_r}{\sin \theta} \partial_\phi I}{\Delta_\theta} \right)^2 = m^2 = 0. \tag{55} \]

Use the following ansatz for variable separation
\[ I = -E \tilde{t} + R(r) + X(\theta) + J \tilde{\phi}, \tag{56} \]
accordingly, we denote
\[ \partial_\tilde{r} I = -E, \quad \partial_\tilde{\phi} I = J, \quad \partial_r I = R'(r) = dR(r)/dr, \]
\[ \partial_\theta I = X'(\theta) = dX(\theta)/d\theta. \tag{57} \]

Substituting Eq. \( 57 \) into \( 55 \) and solving the radial part for the outgoing particles, we have
\[ R'_+(r) = \frac{1}{\Delta_r} \left[ a(1 + g^2 r^2) J - (r^2 + a^2) E \right]^2 + m^2 \Sigma \Delta_r \]
\[ - \Delta_r \left[ \frac{J}{\sin \theta} - E \frac{a \sin \theta}{\Delta_\theta} \right]^2 + X'^2 \right]^{1/2}. \tag{58} \]
Since the leading contribution near the horizon is dominant, we may use the following near-horizon approximation,
\[
\Delta_t = (r - r_+) \Delta'(r_+) + \cdots ,
\] (59)
thus, Eq. (58) turns into
\[
R'_+(r) = - \frac{(r_+^2 + a^2)E - (1 + g^2 r_+^2)aJ}{(r - r_+) \Delta'(r_+)} .
\] (60)
Using the asymptotic behavior of the outgoing particles near the event horizon, it is not difficult to get
\[
R_+(r_+) = \frac{i \pi E - \Omega J}{2 \kappa} .
\] (61)
Taking into account the contribution of both the ingoing and outgoing particles and noting that \(Im I = Im R(r_+)\) [28], the tunneling probability of particles is
\[
\Gamma \sim e^{-\frac{\pi \Sigma}{\kappa}(E - \Omega J)} .
\] (62)

It is obvious that Eq. (62) is different from Eq. (63) the reason for this is that here the particles’ self-gravitation has been neglected when considering the semi-classical approximation and only the leading contribution. This is different from our previous treatment.

As a conclusion, to calculate the tunneling probability, both the Parikh-Wilczek’s semi-classical tunneling method and the complex-path integral method are available even in conformal gravity. It should be pointed out that attempt to calculate tunneling probability of Hawking radiation without dragging coordinate transformations turns out to be a success finally. In other words, calculating the tunneling probability in a non-dragging coordinate system is indeed feasible, not limited in a dragging system any longer.

5 Hawking temperature by using the method of complex-path integral

As a supplementary verification of the feasibility of the complex-path integration method in conformal gravity, in this section we shall use it to calculate Hawking temperature of rotating AdS black holes in conformal gravity.

For our aim, the neutral rotating AdS black hole solutions in conformal gravity [11] is rewritten in a different form
\[
\begin{align*}
\text{ds}^2 &= - \frac{\Delta_t \Delta_t}{(r^2 + a^2)^2 \Delta_\theta - \Delta_\theta a^2 \sin^2 \theta} \, dt^2 + \frac{\Sigma}{\Delta_t} \, dr^2 \\
&\quad + \frac{\Delta_t}{\Delta_\theta} \, d\theta^2 + \frac{\Sigma}{\Delta_\theta} \sin^2 \theta \, d\phi^2 \\
&\quad \times \left\{ \Delta_\theta \left[ \frac{r_+^2}{(r_+^2 + a^2)^2} (1 + g^2 r_+^2) - \Delta_t \right] \frac{d\phi}{(r^2 + a^2)^2 \Delta_\theta - \Delta_\theta a^2 \sin^2 \theta} \right\}^2 .
\end{align*}
\] (63)

where \(\Sigma, \Xi, \Delta_t, \Delta_\theta\) are the same as those given in [11]. Considering the near-horizon approximation of the metric, from (63) we have
\[
\begin{align*}
ds^2 &= - \frac{\Delta_t \Delta_t}{(r^2 + a^2)^2 \Delta_\theta - \Delta_\theta a^2 \sin^2 \theta} \, dt^2 + \frac{\Sigma}{\Delta_t} \, dr^2 \\
&\quad + \frac{\Delta_t}{\Delta_\theta} \, d\theta^2 + \frac{\Sigma}{\Delta_\theta} \sin^2 \theta \, d\phi^2 \\
&\quad \times \left\{ \Delta_\theta \left[ \frac{r_+^2}{(r_+^2 + a^2)^2} (1 + g^2 r_+^2) - \Delta_t \right] \frac{d\phi}{(r^2 + a^2)^2 \Delta_\theta - \Delta_\theta a^2 \sin^2 \theta} \right\}^2 ,
\end{align*}
\] (64)
where \(\Delta_t = r_+^2 + a^2 \cos^2 \theta\) and \(\Omega_\theta = a(1 + g^2 r_+^2)/(r_+^2 + a^2)\) is the angular velocity of the horizon. According to the work [28], we get
\[
\beta = \frac{4 \pi (r_+^2 + a^2)}{\Delta_\theta (r_+)} = \frac{2 \pi}{\kappa} ,
\] (65)
therefore the Hawking temperature is
\[
T = \frac{r_+^2 (g^2 r_+^2 - 1) - a^2 (3 + g^2 r_+^2)}{4 \pi r_+ (r_+^2 + a^2)} .
\] (66)

It agrees with the expression given in [2] which is derived by means of standard method. To some extent, thus this conclusion provides further evidence for feasibility of the complex-path integration method in conformal gravity.

6 Conclusion

In this paper, we have filled the blank and, as an extension, chiefly studied the Hawking radiation as tunneling in conformal gravity, especially from the new neutral rotating AdS black holes [22], by using the Parikh-Wilczek’s semi-classical tunneling method and the complex-path integral method, thus providing a more perfect understanding for Hawking radiation as tunneling.

Starting with reviewing the new rotating AdS black hole in conformal gravity, in order to make the simultaneity of coordinate clocks be able to be transmitted from one place to another and have nothing to do with the integration path, we have recast the metric into a superior form which satisfies Landau’s condition of the coordinate clock synchronization [63] in the dragging coordinate system. It’s also necessary after all.

For the purpose of deriving geodesic equations of massive and massless particles in a unified and self-consistent way in conformal gravity, we have rectified shortcoming of almost all of related works before in deducing the corresponding geodesic equations by using a different and very lame approach which violated the variation principle of action. As an improvement, We have derived them both from the Lagrangian, and the geodesic of massless particles can be derived by taking the proper limit of that of massive particles. In fact, the improvement is very universal and can also be used to investigate Hawking radiation as tunneling from black holes in any other gravity theory,
because process of deriving the geodesic equations from the Lagrangian is not dependent on specific gravity theory.

Afterwards, we have adopted two different methods, namely, the Parikh-Wilczek’s semi-classical tunneling method and the method of complex-path integral to work out the tunneling probability, respectively. It should be pointed out that, in this paper, the method of complex-path integral have been used in a universal coordinate system, not limited to a dragging coordinate system as before any longer even in the rotating case. Our results show that the tunneling probability when taking into account self-gravitation between particles is related to the change of Bekenstein-Hawking entropy and the real radiation spectrum from black hole is not pure thermal spectrum any longer, however, the tunneling probability is not in connection with the entropy change directly as one considers the semi-classical approximation and the leading contribution while neglecting particle self-gravitation. Of course, the trick of applying the first law of black hole thermodynamics has provided a great convenience for tunneling integration calculation. Moreover, the Hawking temperature derived by the method of complex-path integral is in accordance with that obtained by the standard method.

It is of interest to extend the present work to probe further into the Hawking radiation from the charged rotating AdS black holes in conformal gravity. We leave this for the future plan.

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Appendix

In this Appendix, we will show that there is another kind of coordinate system that can do the tunneling calculation.

If we introduce the following coordinate transformations

\[ d\tilde{t} = dt - \frac{\sqrt{r^2 + a^2} \sqrt{(r^2 + a^2) \Delta \omega - \Delta r}}{\Delta r} dr, \]

\[ d\tilde{\phi} = d\phi - \frac{a \sqrt{1 + g^2 r^2} \sqrt{(r^2 + a^2) \Delta \omega - \Delta r}}{\Delta r \sqrt{(r^2 + a^2) \Delta \omega}} dr, \]

the metric \( \Pi \) turns into a new form

\[
ds^2 = -\Delta_\omega dt^2 + \frac{\Sigma d\theta^2}{\Delta_\omega} + \frac{\Delta \phi (r^2 + a^2) \sin^2 \theta}{\Sigma} (d\phi - ag^2 dt)^2 
+ \left[ \sqrt{\Delta_\omega (r^2 + a^2) - \Delta r} \right] \left( \Delta_\omega dt - a \sin^2 \theta d\phi \right) 
+ \left[ \frac{\Sigma}{\Delta_\omega (r^2 + a^2)} dr \right]^2.
\] (A3)

One can repeat the same procedure in the main text to derive the geodesic equations and the tunneling rate of massive particles, which is omitted here. Now the dragging angular velocity is

\[
\Omega = -\frac{g \phi}{g \phi} = a \frac{\Delta \phi}{\Delta \omega} \frac{(r^2 + a^2)(1 + g^2 r^2) - \Delta r}{\Delta \omega (r^2 + a^2)^2 - \Delta r a^2 \sin^2 \theta}.
\] (A4)

After performing a dragging coordinate transformation \( d\phi = \Omega dt \), the metric (A3) is then changed into the following form

\[
ds^2 = -\Delta_\omega \Delta \omega \frac{\Sigma}{\Delta \omega (r^2 + a^2)^2 - \Delta r a^2 \sin^2 \theta} dt^2 
+ \frac{2 \Sigma \sqrt{\Delta \omega (r^2 + a^2) \left( (r^2 + a^2)(1 + g^2 r^2) - \Delta r \right)}}{\Delta \omega (r^2 + a^2)^2 - \Delta r a^2 \sin^2 \theta} dr dt 
+ \frac{\Sigma}{(r^2 + a^2) \Delta \omega} dr^2.
\] (A5)

The metric (A3) components in the dragging coordinate system can’t satisfy Eq. (28) due to

\[
\frac{\partial}{\partial \phi} \left( -\frac{g \phi}{g \phi} \right) \neq \frac{\partial}{\partial \phi} \left( -\frac{g \phi}{g \phi} \right) = 0.
\] (A6)

Different from the metric (27), the metric (A3) can’t satisfy Landau’s condition of the coordinate clock synchronization [35], however, one can still finish the tunneling analysis without any difficulty in this coordinate system too.

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