Goldstone Bosons in Josephson Junctions

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Abstract: For a microscopic model of a Josephson junction the normal coordinates of the two junction Goldstone bosons are constructed and their dynamical spectrum is computed. The explicit dependence on the phase difference of the two superconductors is calculated.

KEY WORDS: superconductors, Josephson junctions, Goldstone bosons

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1 Introduction

In 1962, Josephson [1] predicted a novel phenomenon in superconductivity, namely when two different superconductors were brought into close contact. Based on elementary quantum mechanics, he predicted the existence of a supercurrent with a peculiar current-voltage dependence. He argued that there would emerge a current of Cooper pairs which is proportional to the sine of the phase difference of the order parameters of both superconductors. The success of this prediction was immediate when indeed this phenomenon was experimentally observed one year later [2]. It counts as one of the greatest successes of quantum mechanics in physics and you will find a chapter on

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The Josephson effects in almost every textbook on superconductivity. The increase of knowledge on this subject in theoretical solid state physics in the following decades has been tremendous and applications of Josephson junctions in electronic devices have been developed. Progress in conceiving a microscopic theory for the Josephson effect in rigorous quantum statistical mechanics was made when Sewell obtained the Josephson and Meissner effects in an model independent approach from the assumption of off-diagonal long range order and local gauge covariance.

In we considered a concrete microscopic quantum model yielding an ab initio and rigorous understanding of the emerging of a Josephson current, which is computed and numerically calculated. The model consists of two two-dimensional superconducting plates having a common one-dimensional contact surface through which Cooper pairs can tunnel and as such induce a current. In section 2 we repeat the essentials of the model for which we construct a non-equilibrium steady state (NESS). One of the attractive aspects of the construction is that our NESS has the nice property of having a finite interaction area. We derive an analytical expression for the current in the case that the phases of the two superconductors are not too large. We find back the perfect sine-behavior.

Section 3 is devoted to the study of the appearance of Goldstone bosons in the junction, due to the interaction of the two bulk superconductors and a direct consequence of the gauge symmetry breaking. We apply the general result of where one finds the general explicit construction of the normal
coordinates of the Goldstone boson as a consequence of the spontaneous symmetry breaking which appears in our model. The Goldstone bosons of the bulk superconductors can also be found in that paper. Here we find two supplementary Goldstone bosons. We construct their normal coordinates and their dynamics induced by the micro-dynamics of the model. We consider their dynamics in diagonal form and analyze its spectrum, again as in section 2 as a function of the phase difference of the two bulk superconductors. Here one finds a cosine-behavior.

2 Micro model for Josephson Junctions

As said above we consider the model [5] for two two-dimensional superconducting plates I and II with a common one-dimensional contact surface (line) through which the Cooper pairs can travel in order to induce a current.

The two superconductors are modeled by the strong coupling BCS-model on a square lattice using the Anderson quasi-spin formalism and described
by the Hamiltonians $H_{i,N}$ with $i = I, II$

$$H_{i,N} = \sum_{k,l=1}^{N} \epsilon_i \sigma^z(k, l) - \frac{1}{N} \sum_{k,l,m,n=1}^{N} \sigma^+_i(k, l) \sigma^-_i(m, n), \quad \epsilon_i > 0 \quad (2.1)$$

acting on the Hilbert space $\bigotimes_{j=1}^{N^2} \mathbb{C}_j^2$, $\sigma^\pm_i$ and $\sigma^z = \sigma^+_i \sigma^-_i$ are copies of the Pauli matrices. The $\sigma^+$ and $\sigma^-$ represent the creation and annihilation operators of the Cooper pairs of the superconductors; the $\epsilon$ do represent the kinetic energies of the Cooper pairs.

The junction between the superconductors $I$ and $II$ is modeled by the interaction

$$V_N = -\frac{\gamma}{N} \sum_{k_1, k_2=1}^{N} (\sigma^+_I(k_1, 1) \sigma^-_I(k_2, 1) + h.c.), \quad \gamma > 0 \quad (2.2)$$

which is responsible for the Cooper pair tunneling through the barrier. A pair at the site $(k_I, 1)$ of the first superconductor can tunnel through the junction and create a pair at the site $(k_2, 1)$ of the second superconductor and vice-versa. The coupling constant $\gamma$ governs the rate of this process. Note that only Cooper pairs on the contact surfaces of $I$ and $II$ participate in this process. Remark that only $N$ sites of each superconductor are interacting with $N$ sites of the other one. The lattice permutation invariance of the Hamiltonians $(2.1)$ and the interaction $(2.2)$ make the model given by the
total Hamiltonian of the system

\[ H_N = H_{I,N} + H_{II,N} + V_N \]  \hspace{1cm} (2.3)

exactly soluble in the thermodynamic limit \( N \) tending to infinity \([6]\).

2.1 Equilibrium states of the non-interacting superconductors

For completeness we discuss here the equilibrium states of the Hamiltonians (2.1). We treat the first (I) one explicitly, the second is analogous and obtained by replacing the index \( I \) by the index \( II \). Following \([6]\), the extremal equilibrium states at inverse temperature \( \beta_I \) in the thermodynamic limit are the product states \( \omega_{\varphi_I} \) with the expectation values of all tensor product observables \( X = X_{x_1} \otimes X_{x_2} \otimes ... \); \( x_1, x_2, ... \in \mathbb{N}^2 \) and all \( X_{x_j} \in M_2(2 \text{ by } 2 \text{ complex matrices}) \), given by

\[
\omega_{\varphi_I}(X) = \prod_{x \in \mathbb{N}^2} \text{Tr} \rho_{\varphi_{I,x}} X_x
\]  \hspace{1cm} (2.4)

Here \( \rho_{\varphi_{I,x}} \) is the x-copy of the 2 by 2 density matrix \( \rho_{\varphi_I} \in M_2 \), solution of the selfconsistency equation

\[
\rho_{\varphi_I} = \frac{\exp -\beta_I h_{\varphi_I}}{\text{Tr} \exp -\beta_I h_{\varphi_I}}
\]  \hspace{1cm} (2.5)
with the one-site effective Hamiltonian $h_{\varphi I}$ given by

$$h_{\varphi I} = \epsilon_I \sigma^z_I - \lambda_I (e^{i\varphi_I} \sigma^-_I + \text{h.c.}), \quad \lambda_I \geq 0$$  \hspace{1cm} (2.6)

Clearly the density matrix $\rho$ is also determined by the equivalent self-consistency equation for the order parameter $\lambda_I = |\omega_{\varphi_I}(\sigma^-)|$ which by explicit computation becomes

$$\lambda_I (1 - \frac{1}{\mu_I} \tanh \beta_I k_I) = 0, \quad \mu_I = \sqrt{\epsilon^2_I + \lambda^2_I}$$  \hspace{1cm} (2.7)

Remark that $\{\pm \mu_I\}$ constitutes the spectrum of the effective Hamiltonian $h_{\varphi I}$ which is independent of the phase angle $\varphi_I$.

It can readily be seen that (2.1) admits always a solution $\lambda_I = 0$. It yields the normal phase state of the superconductor. For $\epsilon_I < \frac{1}{2}$ and $\beta_I$ large enough or temperature low enough, there exists also a solution $\lambda_I \neq 0$. These solutions yield the superconducting phase states. The phase $\varphi_I$ can be fixed freely, yielding an infinite degeneration of the equilibrium states under the conditions mentioned above.

The second superconductor (II) has analogous phase states with phases which can be chosen independently from the first one.

In the following we fix for each superconductor such a superconducting phase state denoted by $\omega_{I,\varphi_I}$ and $\omega_{II,\varphi_{II}}$ with $\varphi_I \neq \varphi_{II}$.
2.2 Non-equilibrium steady state (NESS)

Now we construct a non-equilibrium but steady state (NESS) for the total interacting system 2.3. We start from the product state $\omega = \omega_{I,\varphi_I} \otimes \omega_{II,\varphi_{II}}$ on the system, i.e. the state of the system of the two superconductors in their respective superconducting phase states characterized by their phases, inverse temperatures and kinetic energies. In this state $\omega$ one can compute the global dynamics yielding a time evolution

$$\alpha_t(.) = \omega - \lim_{N} \exp itH_N \exp itH_N \quad (2.8)$$

where $\omega - \lim$ is the weak limit under the state $\omega$. Now we are ready to look for a state $\tilde{\omega}$ which is invariant under the dynamics 3.13.

Due to the specific lattice permutation symmetry of the model, it is natural to choose this state $\tilde{\omega}$ among the product states [6], i.e.

$$\tilde{\omega}_{\varphi_I}(X) = \text{Tr} \prod_{x \in \mathbb{N}^2} \tilde{\rho}_x X_x \quad (2.9)$$

As the state should be time invariant, it should have the same lattice invariance as the Hamiltonian 2.3. The symmetry of the Hamiltonian divides the total system into four parts.

There are the bulk parts of the superconductors I and II which we denote by $I_a$ and $II_a$, and there are the contact or surface parts denoted by $I_b$ and $II_b$. Therefore one can write the state $\tilde{\omega}$ as a tensor product of four symmetric
Figure 2: Division of the system in four subsystems

product states on their different regions:

\[ \tilde{\omega} = \tilde{\omega}_{I_a} \otimes \tilde{\omega}_{I_b} \otimes \tilde{\omega}_{II_b} \otimes \tilde{\omega}_{II_a} \] (2.10)

Finally we require that the state \( \tilde{\omega} \) is a steady state

\[ \lim_{N} \tilde{\omega}([H_N, X]) = 0 \] (2.11)

Due to the product structure of the state, the Hamiltonian can be identified

with an effective Hamiltonian \[ 6 \] of the type \( \tilde{H}_N = \sum_{x \in I_N \cup II_N} \tilde{h}_x \), where \( \tilde{h}_x \in M_{2,x} \). Imposing this time invariance yields

\[
\tilde{h}_x = \begin{cases} 
\epsilon_I \sigma^z(i) - \tilde{\omega}(\sigma_{I_a}^+)\sigma_{I_a}^-(i) - \tilde{\omega}(\sigma_{I_a}^-)\sigma_{I_a}^+(i), & i \in I_a; \\
\epsilon_I \sigma_{I_b}^z(i) - \tilde{\omega}(\sigma_{I_b}^+)\sigma_{I_b}^-(i) - \tilde{\omega}(\sigma_{I_b}^-)\sigma_{I_b}^+(i) \\
-\gamma \left( \tilde{\omega}(\sigma_{II_b}^+)\sigma_{II_b}^-(i) + \tilde{\omega}(\sigma_{II_b}^-)\sigma_{II_b}^+(i) \right), & i \in I_b; \\
\epsilon_{II} \sigma_{II_b}^z(i) - \tilde{\omega}(\sigma_{II_a}^+)\sigma_{II_a}^-(i) - \tilde{\omega}(\sigma_{II_a}^-)\sigma_{II_a}^+(i) \\
-\gamma \left( \tilde{\omega}(\sigma_{II_b}^+)\sigma_{II_b}^-(i) + \tilde{\omega}(\sigma_{II_b}^-)\sigma_{II_b}^+(i) \right), & i \in II_b; \\
\epsilon_{II} \sigma_{II_b}^z(i) - \tilde{\omega}(\sigma_{II_a}^+)\sigma_{II_a}^-(i) - \tilde{\omega}(\sigma_{II_a}^-)\sigma_{II_a}^+(i), & i \in II_a,
\end{cases} \] (2.12)
We use also the notation

\[ \tilde{\Lambda}_I = \tilde{\lambda}_I \exp i \tilde{\phi}_I = \tilde{\omega}(\sigma^+_I), \quad \tilde{\phi}_I = \arg \tilde{\omega}(\sigma^+_I) \quad (2.13) \]

and analogously for the second superconductor with \( I \) replaced by \( II \).

The local density matrices \( \tilde{\rho}_x \) for \( x \in I \cup II \) of the state \( \tilde{\omega} \) are the \( \tilde{h}_x \) invariant projections of the density matrices \( \rho_x \) of \( \omega \) (2.4). For more details of the construction, see [5]. In any case, for \( x \in I_a \cup II_a \) : \( \rho_x = \tilde{\rho}_x \) as follows from 2.12. For the lattice point \( x \in I_b \cup II_b \) one readily computes the selfconsistency non-equilibrium equations

\[ \begin{align*}
\tilde{\lambda}_I e^{i \tilde{\phi}_I} &= \left( \lambda_I e^{i \phi_I} + \gamma \tilde{\lambda}_{II} e^{i \tilde{\phi}_{II}} \right) \frac{\epsilon^2_I + |\lambda_I e^{i \phi_I} + \gamma \tilde{\lambda}_{II} e^{i \tilde{\phi}_{II}}| \cos(\phi_I - \tilde{\phi}_I)}{\epsilon^2_I + |\lambda_I e^{i \phi_I} + \gamma \tilde{\lambda}_{II} e^{i \tilde{\phi}_{II}}|^2} \\
\tilde{\lambda}_{II} e^{i \tilde{\phi}_{II}} &= \left( \lambda_{II} e^{i \phi_{II}} + \gamma \tilde{\lambda}_I e^{i \tilde{\phi}_I} \right) \frac{\epsilon^2_{II} + |\lambda_{II} e^{i \phi_{II}} + \gamma \tilde{\lambda}_I e^{i \tilde{\phi}_I}| \cos(\phi_{II} - \tilde{\phi}_I)}{\epsilon^2_{II} + |\lambda_{II} e^{i \phi_{II}} + \gamma \tilde{\lambda}_I e^{i \tilde{\phi}_I}|^2}
\end{align*} \quad (2.14) \]

Together with the selfconsistency equations 2.1 for \( \lambda_I \) and \( \lambda_{II} \), the equations 2.14 form a set of six coupled equations whose solutions determine the non-equilibrium steady state \( \tilde{\omega} \) of the total system.

This state divides the system into four parts. The bulk parts of both superconductors away from the contact surface do not feel each other nor do they feel the surface. They behave as stable reservoirs. On the contact surfaces \( I_b \) and \( II_b \) the system is effectively perturbed and influenced by the properties of the states of both superconductors.

Remark that we limited the interaction to take place only on a contact surface of one layer thickness. It is clear that the whole construction can be
generalized to the case of any fixed finite number of layers.

In [5] we considered the currents of Cooper pairs emerging in the system traveling from the superconductor I to II and vice versa. One considers the relative particle number operator of the Cooper pairs

\[ Q_N = \sum_{x \in \mathbb{N}^2} (\sigma_i^+ \sigma^-_i - \sigma'^+_{II} \sigma^-_{II}) \]  

(2.15)

The local relative current is then

\[ J(Q_N) = i[H_N, Q_N] = -\frac{2i\gamma}{N} \sum_{i,j}^N (\sigma^-_i (i, 1) \sigma'^+_{II} (j, 1) - h.c.) \]  

(2.16)

Remark that in the observable current there is no direct contribution from the bulk of the two superconductors, only the two contact layers are contributing. Moreover one remarks that this current is of the same order of magnitude as the contact surface, namely \( N \).

The Josephson current measured in the thermodynamic limit (\( N \to \infty \)) state \( \tilde{\omega} \), called NESS, is readily calculated and given by

\[ j(Q) = \lim_{N} \frac{\tilde{\omega}(J(Q_N))}{N} = -4\gamma \tilde{\lambda}_{I} \tilde{\lambda}_{II} \sin (\tilde{\varphi}_I - \tilde{\varphi}_{II}) \]  

(2.17)

This result was obtained in [5], here it is written in a more concise form. However one has to realize that the quantities \( \tilde{\lambda} \) and \( \tilde{\varphi} \) are functions of the originally given parameters \( \lambda_I, \lambda_{II}, \varphi_I \) and \( \varphi_{II} \), given by (2.14). First of all it is easy to see that these equations are shift invariant for an arbitrary shift of the two originally given angles. This means that, without loss of generality, one can take one of the angels, say \( \varphi_I \), equal to zero. Furthermore, it is natural
to assume the coupling constant $\gamma$ to be very small, i.e. $\gamma \ll \min(\epsilon_I, \epsilon_{II})$. Therefore it is reasonable to compute the quantities $\tilde{\lambda}$ and $\tilde{\varphi}$ only up to first order in this parameter $\gamma$.

By multiplying the two selfconsistency equations with each other, using the fact that $\varphi_I = 0$ and taking $\varphi_{II} < \frac{\pi}{2}$ one gets that the phase difference $\tilde{\varphi}_I - \tilde{\varphi}_{II}$ is proportional to the phase difference $\varphi_I - \varphi_{II}$. Suppose now that also the second phase $\varphi_{II}$ is small, corresponding to the usual experimental regime. Then one remarks that also $\tilde{\varphi}_I = 0$. It follows that

$$\tilde{\varphi}_{II} \approx \varphi_{II} \quad (2.18)$$

After substitution of (2.18) in (2.14) one gets

$$\tilde{\lambda}_I = \lambda_I - \gamma \frac{\lambda_I^2 \lambda_{II}}{\mu_I^2} \quad (2.19)$$
$$\tilde{\lambda}_{II} = \lambda_{II} + \gamma \frac{\lambda_I^2 \epsilon_{II}^2}{\mu_{II}^2}$$

After substitution of all these equations in the formula (2.17) one gets the expected formula for the Josephson current

$$j(Q) = -4\gamma \lambda_I \lambda_{II} \sin(\varphi_I - \varphi_{II}) \quad (2.20)$$

yielding an analytical expression for the current for small phase differences between the two bulk superconductors. A numerical computation of the current for arbitrary phase differences is found in [5].
3 Symmetry breaking and Goldstone bosons

As is well known, spontaneous symmetry breakdown (SSB) is one of the basic features accompanying collective phenomena. It became a representative tool for the analysis of many phenomena in modern physics. For long range interactions, it is typical that SSB is accompanied also by the breaking of the symmetry of the dynamics. The latter phenomenon is known to be accompanied by the occurrence of oscillations of a Goldstone boson with a non-vanishing energy spectrum. These oscillations together with the Goldstone boson disappear if the SSB disappears. In [7] one was able to construct explicitly the normal coordinates of these new particles called Goldstone particles. In particular for mean field systems such as the BCS-model [8], the Overhauser model [9], a spin density wave model [10], the anharmonic crystal model [13], and the jellium model [11], one has constructed these Goldstone boson normal coordinates.

Our two-dimensional model consisting of two interacting superconductors also shows the phenomenon of SSB. As the main contribution of this paper we consider the construction and the calculation of the spectrum of the corresponding Goldstone bosons.

As far as the Josephson current, computed in the previous section, is concerned we remark that the bulk parts $I_a$ and $II_a$ of the superconductors are not contributing to it. Therefore it is reasonable to look for the Goldstone particles within the contact areas $I_b$ and $II_b$. In particular we compute the
normal coordinates and the dynamics of the Goldstone bosons of this junction area. From [7] we know that the canonical coordinates of these bosons are given by the fluctuation operators of the generator of the broken symmetry and of the order parameter operator.

As the following gauge transformation holds

$$e^{i\alpha \sigma^z} \sigma^+ e^{-i\alpha \sigma^z} = \sigma^+ e^{2i\alpha}, \quad \alpha \in \mathbb{R} \quad (3.1)$$

the $\sigma^z$ are the local generators of the broken gauge symmetry of the effective Hamiltonian 2.12. Indeed for all $\alpha$

$$\bar{\omega}(e^{i\alpha \sigma^z} \sigma^+ e^{-i\alpha \sigma^z}) = \bar{\omega}(\sigma^+) e^{2i\alpha} \quad (3.2)$$

proving that the state is not invariant under the $U(1)$ gauge group.

Therefore we consider the local operators, for $i \in I_b, II_b$

$$\tilde{Q}_i = \frac{|\Lambda_i^2|}{\bar{\mu}_i^2} \tilde{\sigma}_i^z + \frac{\epsilon_i}{\bar{\mu}_i^2} (\bar{\Lambda}_i \tilde{\sigma}_i^+ + h.c.) \quad (3.3)$$

where for all $i \in I_b$:

$$\bar{\Lambda}_i = \bar{\Lambda}_{I_b} = \bar{\omega}(\sigma^+_{I_b}) + \gamma \bar{\omega}(\sigma_{II_b}) \quad (3.4)$$

$$\bar{\mu}_i = \bar{\mu}_{I_b} = \sqrt{\epsilon_i^2 + |\Lambda_{I_b}|^2} \quad (3.5)$$

$$\epsilon_I = \epsilon_i \quad (3.6)$$

and equivalently with $i \in II_b$ i.e. by substitution of $I$ by $II$ and vice versa.
Remark that the operator $\tilde{Q}_i$ is indeed the generator of the gauge transformations, namely up to a constant equal to $\sigma^z$, but normalized to zero expectation value

$$\tilde{\omega}(\tilde{Q}_i) = \frac{|\tilde{\Lambda}_i|^2}{\mu_i^2} \tilde{\omega}(\sigma^z) + \frac{\epsilon_i}{\mu_i^2} 2|\tilde{\Lambda}|^2 = 0$$ (3.7)

We consider also essentially the order parameter operator $\sigma^\pm$ fluctuation

$$\tilde{P}_j = \frac{i}{\mu_j} (\overline{\tilde{\Lambda}_j} \sigma^+_j - h.c.)$$ (3.8)

Again remark that $\tilde{\omega}(\tilde{P}_j) = 0$ i.e. also this operator is duly normalized to zero.

Using the general quantum fluctuation theory for product states [12], one computes the following quantum central limits in the given state $\tilde{\omega}$ and obtain the normal coordinates of two Goldstone bosons. For the region $I_b$ one gets the normal coordinates

$$b_{I_b}(Q) = \lim_{N} \frac{1}{\sqrt{N}} \sum_{j \in I_b, j=1}^{N} \tilde{Q}_j$$ (3.9)

$$b_{I_b}(P) = \lim_{N} \frac{1}{\sqrt{N}} \sum_{j \in I_b, j=1}^{N} \tilde{P}_j$$

and for the region $II_b$ one gets the normal coordinates

$$b_{II_b}(Q) = \lim_{N} \frac{1}{\sqrt{N}} \sum_{j \in II_b, j=1}^{N} \tilde{Q}_j$$ (3.10)

$$b_{II_b}(P) = \lim_{N} \frac{1}{\sqrt{N}} \sum_{j \in II_b, j=1}^{N} \tilde{P}_j$$
In 3.9 one gets the normal coordinates of a first Goldstone boson, and in 3.10 the normal coordinates of a second independent Goldstone boson. Indeed, by a straightforward computation one checks readily the following canonical commutation relations

\[
[b_{I_b}(Q), b_{I_{II_b}}(Q)] = [b_{I_b}(Q), b_{II_b}(P)] = 0,
\]

\[
[b_{I_b}(P), b_{II_b}(Q)] = [b_{I_b}(P), b_{II_b}(P)] = 0,
\]

\[
[b_{I_b}(Q), b_{I_b}(P)] = 4i \frac{\tilde{\lambda}_{I_b}^2}{\mu_{I_b}},
\]

\[
[b_{II_b}(Q), b_{II_b}(P)] = 4i \frac{\tilde{\lambda}_{II_b}^2}{\mu_{II_b}}.
\]

Remark that in the case of temperatures above the critical ones of the bulk superconductors the order parameters vanish: \(\tilde{\lambda}_{I_b} = \tilde{\lambda}_{II_b} = 0\) such that all commutators in (3.11) vanish. Also one has

\[
\tilde{\omega}(b_{I_b}(Q)^2) = \tilde{\omega}(b_{I_{II_b}}(Q)^2) = \tilde{\omega}(b_{I_b}(P)^2) = \tilde{\omega}(b_{II_b}(P)^2) = 0
\]

(3.12)

and hence all the operators themselves vanish: \(b_{I_b}(Q) = b_{I_b}(P) = b_{II_b}(Q) = b_{II_b}(P) = 0\), i.e. the Goldstone bosons disappear in the normal phases.

Next we consider the dynamics of the Goldstone bosons in the case of superconducting phases for the bulk superconductors. We consider the time evolution of the normal modes \(3.3\) and \(3.8\), which is induced by the initial micro-dynamics given by the effective Hamiltonian \(2.12\). In general, let \(A\) be a local observable situated at the lattice point \(x \in I_b\) or \(II_b\), then denote \(\tilde{\alpha}_t\)
the time evolution of the fluctuation of $A$ after time $t$. It is given by

\begin{align}
\tilde{\alpha}_t b_{I_b}(A) &= b_{I_b}(\exp(it\tilde{h}_x)A \exp(-it\tilde{h}_x)) \tag{3.13} \\
\tilde{\alpha}_t b_{II_b}(A) &= b_{II_b}(\exp(it\tilde{h}_x)A \exp(-it\tilde{h}_x))
\end{align}

Of course the operator $A$ stands for the operators $\tilde{Q}_x \tag{3.3}$ and $\tilde{P}_x \tag{3.8}$. A straightforward computation of the dynamics using $\tag{3.3}$ yields the simple solutions

\begin{align}
\tilde{\alpha}_t b_{I_b}(Q) &= b_{I_b}(Q) \cos(2\tilde{\mu}_{I_b} t) + b_{I_b}(P) \sin(2\tilde{\mu}_{I_b} t) \tag{3.14} \\
\tilde{\alpha}_t b_{I_b}(P) &= -b_{I_b}(Q) \sin(2\tilde{\mu}_{I_b} t) + b_{I_b}(P) \cos(2\tilde{\mu}_{I_b} t)
\end{align}

and analogously for the surface $II_b$ one gets the same dynamics for the second Goldstone boson by replacing the index $I_b$ by the index $II_b$.

The two bosons behave dynamically as two independent quantum harmonic oscillators with frequencies

\begin{align}
\tilde{\nu}_I &= 2\tilde{\mu}_{I_b} \tag{3.15} \\
\tilde{\nu}_{II} &= 2\tilde{\mu}_{II_b}
\end{align}

For the bulk superconductors $I_a$ and $II_a$ the Goldstone bosons dynamics were computed before \[8\]. The frequencies $\nu_I = 2\mu_I$ and $\nu_{II} = 2\mu_{II}$ are clearly phase independent.

However for the frequencies of the Goldstone bosons considered in this paper, the situation is completely different. The frequencies $\tilde{\nu}_I$, $\tilde{\nu}_{II}$ computed above
do depend on the phase difference $\varphi_I - \varphi_{II}$ of the two phases of the bulk superconductors.

We consider the phase dependence explicitly for the region $I_b$, the computation for the second region is analogous:

$$\tilde{\nu}_I = 2\sqrt{\varepsilon_I^2 + |\tilde{\Lambda}_I|^2} = 2\sqrt{\varepsilon_I^2 + |\lambda_I \exp i\varphi_I + \gamma \tilde{\lambda}_{II} \exp i\tilde{\varphi}_{II}|^2} \quad (3.16)$$

Remark that the $\tilde{\lambda}_I$ and $\tilde{\varphi}_{II}$ are determined by the parameters $\varepsilon_I$, $\varepsilon_{II}$, $\lambda_I$, $\lambda_{II}$ and the phases $\varphi_I$ and $\varphi_{II}$ through the selfconsistency equations.

It is instructive again to get an explicit form of the frequencies in terms of the given parameters of the system. We derive again the formula for the frequency $\tilde{\nu}_I$ up to first order in the coupling constant $\gamma$, and in the case that the phase difference $\varphi_I - \varphi_{II}$ is small, a situation explored before for the current.

We get from (2.18), (2.19) and (3.15)

$$\tilde{\mu}_I^2 = \varepsilon_I^2 + \lambda_I^2 + 2\gamma \lambda_I \lambda_{II} \cos(\varphi_I - \varphi_{II}) \quad (3.17)$$

Hence one gets the following expressions for the dynamical frequencies of the Goldstone modes

$$\tilde{\nu}_I = \nu_I + 4\gamma \frac{\lambda_I \lambda_{II}}{\nu_I} \cos(\varphi_I - \varphi_{II}) \quad (3.18)$$

$$\tilde{\nu}_{II} = \nu_{II} + 4\gamma \frac{\lambda_I \lambda_{II}}{\nu_{II}} \cos(\varphi_I - \varphi_{II}) \quad (3.19)$$
From these expressions the dependence of the Goldstone frequencies on the phase differences of the bulk superconductors is explicitly given. We learn that these frequencies decrease when the phase difference increase in contradistinction with the current. The current has a sine-behavior, the frequencies a cosine-behavior. This point may be interesting from the experimental point of view.

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