RCD*(K, N) SPACES ARE SEMI-LOCALLY SIMPLY CONNECTED

JIKANG WANG

Abstract. It was shown by Mondino-Wei in [16] that any RCD*(K, N) space (X, d, m) has a universal cover. We prove that for any point x ∈ X and R > 0, there exists r < R such that any loop in B_r(x) is contractible in B_R(x); in particular, X is semi-locally simply connected and the universal cover of X is simply connected. This generalizes earlier work in [22] that any Ricci limit space is semi-locally simply connected.

1. Introduction

We are interested in local topology of non-smooth length metric spaces. Assume that (M_i, p_i) is a sequence of n-dim Riemannian manifolds with Ric ≥ -(n-1). By Gromov’s pre-compactness theorem, passing to a subsequence if necessary, (M_i, p_i) converges to (Y, p) in the Gromov-Hausdorff sense. We call all such limit spaces Y Ricci limit spaces.

The geometric structure of Ricci limit spaces were well studied by Cheeger-Colding-Naber [5, 6, 7, 8, 9, 10]. The first topological result about Ricci limit spaces was shown by Sormani-Wei that any Ricci limit space has a universal cover. Recently the author proved in [22] that any Ricci limit space is semi-locally simply connected; see also [17, 18] for the cases with volume conditions. Recall that we say a metric space Y is semi-locally simply connected if for any y ∈ Y, there exists r > 0 such that any loop in B_r(y) is contractible in Y.

In this paper we consider RCD*(K, N) spaces which generalize Ricci limit spaces, see section 2.1 for further references about RCD*(K, N) spaces. Mondino-Wei proved that any RCD*(K, N) space X has a universal cover [16] while it was unknown whether the universal cover is simply connected. We shall prove that X is semi-locally simply connected; in particular, the universal cover of X is simply connected.

Main Theorem. Assume that a measured metric space (X, d, m) is an RCD*(K, N) space for some K ∈ R and N ∈ (1, ∞). Then for any x ∈ X and R > 0, there exists r > 0 so that any loop in B_r(x) is contractible in B_R(x). In particular, X is semi-locally simply connected.

Corollary 1.1. (cf. [16]) In the setting of the main theorem, the universal cover of X is simply connected. The revised fundamental group defined in [16] (deck transformations on the universal cover) is isomorphic to π_1(X).

We should mention that Santos-Rodríguez and Zamora recently proved many properties of the fundamental group of an RCD*(K, N) space [19].

The author is supported by Fields Institute for Research in Mathematical Sciences.
Corollary 1.2. (cf. [21, 22]) Assume that a sequence of measured metric spaces $(X_i, d_i, m_i)$ are RCD$^*(K, N)$ space for some $K \in \mathbb{R}$ and $N \in (1, \infty)$ and $\text{Diam}(X_i) \leq 1$. Suppose that $(X_i, d_i, m_i)$ measured Gromov-Hausdorff converges to $(X, d, m)$. For $i$ large enough, there is an onto homomorphism $\phi_i : \pi_1(X_i) \to \pi_1(X)$.

The proof of the main theorem is similar to the proof in [22] for the Ricci limit space case: we need to show a weak homotopy control property, then use such control to construct a homotopy map. We briefly discuss the proof and point out some differences compared with [22].

Since we study local relative fundamental group, it’s natural to consider local covers (cover of a ball) and corresponding deck transformations. For a Ricci limit space case $(M_i, y_i) \xrightarrow{GH} (Y, y)$, we can consider universal cover of balls in $M_i$, saying $\tilde{B}_4(y_i)$. We have equivariant Gromov-Hausdorff convergence. Then we show that there is a slice $S$ in [17], which implies a weak homotopy control property: for any loop in a small and fixed ball of $y_i$, we can find a nearby loop in $M_i$ which is homotopic to a loop contained in a very small ball with controlled homotopy image. Here “a very small ball” means the radius converges to 0 as $i \to \infty$. If we observe $M_i$ from the limit space $Y$, a very small ball in $M_i$ has no difference between a point because of the GH-distance gap between $Y$ and $M_i$. Using the weak homotopy control, we can construct a homotopy map on $Y$ and prove that any Ricci limit space is semi-locally simply connected.

For an RCD$^*(K, N)$ space $(X, d, m)$, however, it’s unknown whether there is a sequence of manifolds converging to $(X, d, m)$. Therefore, for any $p \in X$, we shall consider relative $\delta$-cover (see section 2.2) of a ball in $X$, saying $\tilde{B}_4(y_i)$. Using Theorem 2.5, $\tilde{B}(p, 4, 40)^\delta$ are all same for all $\delta$ small enough. In particular, it implies a weak homotopy control property on $X$: for any loop $\gamma$ in a small neighborhood of $p$, $\gamma$ is homotopic to some loops, each of which is contained in a $\delta$-ball, with controlled homotopy image. The reason that we have ”some loops” instead of ”one loop” is that the fundamental group of $\tilde{B}(p, 4, 40)^\delta$ is not trivial but generated by loops in $\delta$-balls. As we said before, there is no difference between a very small ball and a point during the construction. We may choose $\delta$ arbitrarily small while $r$ is fixed due to Theorem 2.5. Therefore we can construct a homotopy map and prove that any loop in a small neighborhood of $p$ is contractible with controlled homotopy image.

The author would thank Xingyu Zhu for helpful discussions about RCD$^*$ spaces and Jiayin Pan, Jaime Santos-Rodríguez, Sergio Zamora-Barrera for some comments to simplify the proof.

2. Preliminaries

2.1. RCD$^*(K, N)$ spaces. We will consider a geodesic metric space $(X, d)$ with a $\sigma$-finite Borel positive measure $m$; the triple $(X, d, m)$ will be called a metric measure space.

RCD$^*(K, N)$ spaces are the Ricci curvature analog of the celebrated Alexandrov spaces. RCD$^*(K, N)$ spaces are the generalization of Riemannian manifolds with the volume measure, Ricci curvature bounded from below by $K$ and dimension bounded above by $N$. We do not include the definition of RCD$^*(K, N)$ spaces since it is technical and will not be used explicitly; see [1, 2, 11] for further reference. We recall some basic properties of RCD$^*(K, N)$ spaces.
Theorem 2.1. (Volume comparison, [3, 4]) Let $K \in \mathbb{R}$ and $N \geq 1$. Then there exists a function $\Lambda_{K,N}(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that if $(X, d, m)$ is an RCD$^*$$(K, N)$ space, we have
\[
\frac{m(B_r(x))}{m(B(R(x)))} \geq \Lambda_{K,N}(r, R), \forall 0 < r < R, x \in X.
\]

Theorem 2.2. (splitting, [13]) Let $(X, d, m)$ be an RCD$^*$$(0, N)$ space where $N \geq 1$. Suppose that $X$ contains a line. Then $(X, d, m)$ is isomorphic to $(X' \times \mathbb{R}, d' \times d_E, m' \times L_1)$ where $d_E$ is the Euclidean distance, $L_1$ is the Lebesgue measure and $(X', d', m')$ is an RCD$^*$$(0, N-1)$ space if $N \geq 2$ or a singleton if $N < 2$.

We call a point $x$ regular if $(r_i, x)$ converges to $\mathbb{R}^k$ for any sequence $r_i \to \infty$, where $k$ is an integer between 1 and $N$.

Theorem 2.3. (Regular points have full measure, [14, 15]) Assume that $(X, d, m)$ is an RCD$^*$$(K, N)$ space where $K \in \mathbb{R}$ and $N \geq 1$. Then $m$-a.e. $x \in X$ is a regular point.

2.2. $\delta$-cover. Let’s recall the $\delta$-cover introduced in [20]; see also [16].

Given an open covering $U$ of a length metric space $X$, there is a covering space $\tilde{X}_U$ such that $\pi_1(\tilde{X}_U, \tilde{p})$ is isomorphic to $\pi_1(X, U, p)$, where $p = \pi(\tilde{p})$ and $\pi_1(X, U, p)$ is the normal subgroup generated by all $[\alpha^{-1} \circ \beta \circ \alpha] \in \pi_1(X, p)$; $\beta$ is a loop lying in an element of $U$ and $\alpha$ is a path from $p$ to $\beta(0)$.

Definition 2.4. (relative $\delta$-cover) Let $(X, d)$ be a length metric space. The $\delta$-cover of $X$, denoted by $X^\delta$, is defined to be $\tilde{X}_U^\delta$, where $U_0$ is an open covering consisting of all $\delta$ balls in $X$.

For any $0 < r < R$ and $x \in X$, let $\tilde{B}(x, R)^\delta$ be the $\delta$ cover of $B_R(x)$. A connected component of $\pi_1^{-1}(B_r(x)) \subset \tilde{B}(x, R)^\delta$ is called a relative $\delta$-cover, denoted by $\tilde{B}(x, r, R)^\delta$.

Mondino and Wei proved the stability of relative $\delta$-cover (Theorem 4.5 in [16]). We slightly change the form for our purpose.

Theorem 2.5. ([16]) Let $(X, d, m)$ be an RCD$^*$$(K, N)$ space for some $K \in \mathbb{R}$ and $N \in (1, \infty)$ and $x \in X$. For any $R > 0$, there exists $\delta_0$ so that $\tilde{B}(x, R/10, R)^\delta$ are all same for any $\delta \leq \delta_0$.

The idea of the proof of Theorem 2.5 is to first prove the case of regular points using Halfway Lemma and Abresch-Gromoll inequality, then use the fact that regular points are dense and volume comparison theorem.

3. Proof of the Main Theorem

Recall in [22], a key lemma says that for any loop $\gamma$ in a small neighborhood of a Ricci limit space $Y$, we can find a loop $\gamma_i$ in $M_i$ so that $\gamma_i$ is point-wise close to $\gamma$ and homotopic to a very short loop by the controlled homotopy image.

A similar (and easier) idea works for an RCD$^*$$(K, N)$ space $X$. Using Theorem 2.5, we can show that any loop $\gamma$, in a small neighborhood of an RCD$^*$$(K, N)$ space, is homotopic to some loops in very small balls by a controlled homotopy image. Then we can use same construction to find a homotopy map.
Lemma 3.1. Fix $x \in \tilde{B}_{1/2}(p)$ in an RCD$^*$$(K,N)$ space $(X,p)$. For any $l < 1/2$ and small $\delta > 0$, there exists $r < l$ and $k \in \mathbb{N}$ so that any loop $\gamma \subset B_r(x)$ is homotopic to the union of some loops $\gamma_j \ (1 \leq j \leq k)$ in $\delta$-balls and the homotopy image is in $B_{40l}(x)$.

To be precise, in the unit disc $D \subset \mathbb{R}^2$, we can find $k$ disjoint discs $D_0, D_1, \ldots, D_k$ with radius less than $1/10$ and away from the boundary of $D$. Then there is a continuous map $H$ from the completion of $\bigcup_{j=1}^k D_j$ to $B_{40l}(x)$ so that $H(\partial D_j) = \gamma_j$ and $H(\partial D) = \gamma$ where each $\gamma_j$ is contained in a $\delta$-ball.

Proof. We may assume $\delta$ is small enough to apply Theorem 2.5 with $\tilde{B}(x,4l,40l)^\delta$. Let $\tilde{x}$ be a pre-image of $x$ in the stable relative $\delta$-cover. We may assume $r$ small enough so that $B_r(x)$ is isometric to $B_r(\tilde{x})$. Given a loop $\gamma$ in $B_r(x)$, we can lift $\gamma$ to a loop $\tilde{\gamma}$ in $B_r(\tilde{x})$.

Since $\tilde{\gamma}$ is a loop in the $\delta$-cover of $B_{40l}(x)$, it’s homotopic, by $\tilde{H}$, to

$$\tilde{c}_1 \tilde{\gamma}_1 \tilde{c}_1^{-1} \cdots \tilde{c}_k \gamma_k \tilde{c}_k^{-1}$$

where $\tilde{c}_j$ is a path from $\tilde{\gamma}(0)$ to a point $\tilde{x}_j \in \tilde{B}(x,40l)^\delta$ and $\tilde{\gamma}_j$ is a loop contained in $B_\delta(\tilde{x}_j), 0 \leq j \leq k$.

Let $c_j = \pi(\tilde{c}_j), \gamma_j = \pi(\tilde{\gamma}_j), 0 \leq j \leq k,$

and

$$\gamma^\delta = c_1 \gamma_1 c_1^{-1} \cdots c_k \gamma_k c_k^{-1},$$

where $\pi : \tilde{B}(x,40l)^\delta \to B_{40l}(x)$ is the projection map. So $H = \pi(\tilde{H})$ is a homotopy map between $\gamma$ and $\gamma^\delta$. Recall that here $\tilde{H}$ is a continuous map from $[0,1] \times [0,1]$ to $X$ so that $H(0,t) = H(1,t) = p, H(t,0) = \gamma(t), H(t,1) = \gamma^\delta(t).$ We shall modify the definition area of $H$ by gluing. We first glue $(0,t)$ and $(1,t)$ for each $t$, and get $H$ mapping from an annulus to $B_{40l}(x)$ so that the image of outer boundary is $\gamma$ and the image of inner boundary is $\gamma^\delta$. Then we glue the parts on the inner boundary corresponding to $c_j$ and $c_j^{-1}$ for each $j$. Now the definition area of $H$ is (up to a homeomorphism) the completion of $D - \bigcup_{j=1}^k D_j$ to $B_{40l}(x)$, where $D_j$ is a disc contained in $D$; $H(\partial D) = \gamma$ and $H(\partial D_j) = \gamma_j.$

The proof of the Main Theorem is an inductive argument using Lemma 3.1.

Proof of the main theorem. Fix $R > 0$, choose $l_i = R/10^i$ and take $l = l_i$ in Lemma 3.1, $i = 1,2,\ldots$. Although the choice of $r$ in Lemma 3.1 depends on $x$, since $X$ is locally compact, we can find $r_i$ working for all $x \in \tilde{B}_{1/2}(p)$ and $l = l_i$ in Lemma 3.1. Choose $\delta_i < r_{i+1}$. For all $x \in \tilde{B}_{1/2}(p)$, any loop in $B_{r_i}(x)$ is homotopic to loops in $\delta_i$-balls and the homotopy image is contained in $B_{\delta_i}(x)$. We shall show any loop $\gamma$ in $B_{r_i}(p)$ is contractible in $B_{R}(p)$.

Roughly speaking, we first shrink $\gamma$ to loops in $\delta_{i}$-balls, the second step is to shrink each new loop to smaller loops in $\delta_{i}$-balls, etc. Since the homotopy to shrink each loop is contained in a $\delta_{i}$-ball in the $i$-th step, this process converges to a homotopy map which contracts $\gamma$ while the image is contained in a ball with radius $\sum_{i=1}^\infty l_i < R$. We show a detailed proof in the following.

Since $\gamma$ is in $B_{r_i}(p)$, we can apply Lemma 3.1 with $\delta = \delta_i$ and $l = l_i$. There are $k_l$ loops $\gamma_j^l \subset B_{r_i}(z_j^l)$ where $z_j^l \in X, k_l$ disjoint balls $D_j^l \subset D$ with radius less than $1/10$, and a continuous map $H_1$ from the completion $D - \bigcup_{j=1}^{k_1} D_j^l$ to $B_{r_i}(p)$, $H_1(\partial D) = \gamma, H_1(\partial D_j^l) = \gamma_j^l, 1 \leq j \leq k_1$. 


If \( k_1 = 0 \), \( \gamma \) is homotopic to a point by \( H_1 \). We may assume \( k_1 > 0 \). We shall shrink \( \gamma_1^j \) in the second step. First consider \( \gamma_1^1 \) for example. Note that \( \delta_1 < \gamma_2 \), we can apply lemma 3.1 to \( \gamma_1^1 \) with \( l = l_2 \) and \( \delta = \delta_2 \). There are \( k_{2,1} \) (here the subscript (2, 1) means the first loop in the second step) loops \( \gamma_{2,1}^j \subset B_{\delta_1}(z_2^1) \), where \( z_2^1 \in X \) and \( j \leq k_{2,1} \). There are \( k_{2,1} \) disjoint balls \( D_{2,1}^j \subset D_1^1 \) with radius less than 1/100 and we can find a continuous map \( H_{2,1} \) from the completion \( D_1^1 - \sum_{j=1}^{k_{2,1}} D_{2,1}^j \) to \( B_{\delta_2}^1(z_1^1) \), so that \( H_{2,1}(\partial D_1^1) = \gamma_1^1 \) and \( H_{2,1}(\partial D_{2,1}^j) = \gamma_{2,1}^j \), \( 1 \leq j \leq k_{2,1} \). Note that \( H_1(\partial D_1^1) = H_{2,1}(\partial D_1^1) = \gamma_1^1 \), we can extend \( H_1 \) using \( H_{2,1} \).

Repeat the above process for each \( 1 \leq j \leq k_1 \) and extend \( H_1 \) to a new continuous map \( H_2 \). That is, we can find \( k_2 \) loops \( \gamma_{2}^j \subset B_{\delta_2}(z_2^j) \) where \( z_2^j \in X \), \( k_2 \) disjoint balls \( D_2^j \subset D \) with radius less than 1/100, each \( D_2^j \) is contained in one of \( D_{2,1}^j \), \( 1 \leq j' \leq k_1 \), and \( H_2 \) from the completion \( D - \sum_{j=1}^{k_{2}} D_{2,1}^j \) to \( B_{\delta_2}(p) \), \( H_2(\partial D) = \gamma_2 \), \( H_2(\partial D_{2,1}^j) = \gamma_{2,1}^j \), \( 1 \leq j \leq k_2 \). Moreover \( H_2 \) coincides with \( H_1 \) wherever \( H_1 \) is defined.

By induction, in the \( i \)-th step, we can find \( k_i \) loops \( \gamma_{i}^j \subset B_{\delta_i}(z_i^j) \) where \( z_i^j \in X \), \( k_i \) disjoint balls \( D_i^j \subset D \) with radius less than 1/100, each \( D_i^j \) is contained in one of \( D_{i-1}^{j-1} \), \( 1 \leq j' \leq k_{i-1} \), and \( H_i \) from the completion \( D - \sum_{j=1}^{k_{i}} D_{i-1}^j \) to \( X \), \( H_i(\partial D) = \gamma_i \), \( H_i(\partial D_{i-1}^j) = \gamma_{i,1}^j \). Moreover \( H_i = H_{i-1} \) where \( H_{i-1} \) is defined.

We prove that \( H_i \) converges to a continuous map \( H \) from \( D \) to \( B_R(x) \) with \( H(\partial D) = \gamma \); thus \( \gamma \) is contractible in \( B_R(x) \). For any \( q \in D \), if \( H_i(q) \) is defined for some \( i \) (thus for all large \( i \)), define \( H(q) = H_i(q) \). Then we have \( H(\partial D) = \gamma \). In this case, \( H \) is continuous at \( q \) because we may assume \( q \) is not on \( \partial D_{i-1}^j \) by taking large \( i \) (in lemma 3.1, smaller discs \( D_j \) are away from \( \partial D \) ). Otherwise we assume \( H_i \) is not defined at \( q \) for all \( i \), then for each \( i \) there is a disc \( D_i^j \) so that \( q \in D_i^j \). Moreover, \( D_i^j \subset D_{i-1}^{j-1} \) for each \( i \). Let \( S_i \subset D_i^j \) be the set where \( H_i \) is defined.

Recall that the image of \( H_i(S_i) \) is contained in a \( l_i \)-ball, thus \( H_i(S_i) \) converges to a point as \( i \to \infty \); define \( H(q) \) to be this point. The image \( H(D_i^j) \) is contained in a ball with radius \( \sum_{i=1}^{\infty} l_i \) which converges to 0 as \( i \to \infty \), thus \( H \) is continuous at \( q \). Take \( i = 1 \), \( H(D) \) is contained in a ball with radius \( \sum_{i=1}^{\infty} l_i < R \). \( \square \)

\textbf{Remark 3.2.} We actually proved that for a locally compact metric space, if any local relative \( \delta \)-cover is stable, then it is semi-locally simply connected.

We should also mention that harmonic archipelago is an example that there is a loop which can be homotopic to a loop in an arbitrarily small ball, but is not contractible [12].

\textbf{References}

[1] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala. Riemannian Ricci curvature lower bounds in metric measure spaces with \( \sigma \)-finite measure. Trans. Amer. Math. Soc., 2015.

[2] L. Ambrosio, N. Gigli, and G. Savare. Metric measure spaces with Riemannian Ricci curvature bounded from below. Duke Math. J., 2014.

[3] K. Bacher and K.-T. Sturm. Localization and tensorization properties of the curvature-dimension condition for metric measure spaces. Journal of Functional Analysis, 2010.

[4] F. Cavalletti and K.-T. Sturm. Local curvature-dimension condition implies measure-contraction property. Journal of Functional Analysis, 2012.

[5] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. i. J. Differential Geom., 46(3):406–480, 1997.

[6] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. ii. J. Differential Geom., 54(1):13–35, 2000.
[7] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. iii. J. Differential Geom., 54(1):37–74, 2000.
[8] Jeff Cheeger and Aaron Naber. Lower bounds on Ricci curvature and quantitative behavior of singular sets. Invent. Math., 191(2):321–339, 2013.
[9] Jeff Cheeger and Aaron Naber. Regularity of Einstein manifolds and the codimension 4 conjecture. Ann. of Math. (2), 182(3):1093–1165, 2015.
[10] Tobias H. Colding and Aaron Naber. Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications. Annals of Mathematics, 176(2):1173–1229, 2012.
[11] M. Erbar, K. Kuwada, and K.-T. Sturm. On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces. Invent. Math, 2015.
[12] Paul Fabel. The fundamental group of the harmonic archipelago. arXiv:math/0501426, 2005.
[13] N. Gigli. The splitting theorem in non-smooth context. Submitted paper, arXiv:1302.5555, 2013.
[14] N. Gigli, A. Mondino, and T. Rajala. Euclidean spaces as weak tangents of infinitesimally Hilbertian metric spaces with Ricci curvature bounded below. Journal fur die Reine und Angew. Math., 2015.
[15] A. Mondino and A. Naber. Structure theory of metric-measure spaces with lower Ricci curvature bounds. J. Eur. Math., 2019.
[16] Andrea Mondino and Guofang Wei. On the universal cover and the fundamental group of an RCD*(K, N)-space. J. Reine Angew. Math., 2019.
[17] Jiayin Pan and Jikang Wang. Some topological results of Ricci limit spaces. To appear in Transactions of the American Mathematical Society, 2021.
[18] Jiayin Pan and Guofang Wei. Semi-local simple connectedness of non-collapsing Ricci limit spaces. To appear in Journal of the European Mathematical Society, arXiv:1904.06877, 2019.
[19] Jaime Santos-Rodriguez and Sergio Zamora. On fundamental groups of RCD spaces. arXiv:2210.07275, 2022.
[20] Christina Sormani and Guofang Wei. Universal covers for Hausdorff limits of noncompact spaces. Transactions of the American Mathematical Society, 356(3):1233–1270, 2004.
[21] W. Tuschman. Hausdorff convergence and the fundamental group. Math. Z., 208, 1995.
[22] Jikang Wang. Ricci limit spaces are semi-locally simply connected. arXiv:2104.02160, 2021.

(Jikang Wang) Fields Institute for Research in Mathematical Sciences, Toronto, ON, Canada

Email address: jikangwang1117@gmail.com