Investigation of chaos behaviour on damped and driven nonlinear simple pendulum motion simulated by mathematica

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Abstract. This study aimed to investigate the chaos behavior resulted from damped driven nonlinear simple pendulum motion, which was simulated by using Mathematica. The method used to solve the equation of the pendulum was the 4th order of Runge-Kutta. The solution was obtained by setting some parameters to be determined at a certain value (dimensionless), namely the gravitational acceleration ($g$) was 9.8; the length of rope ($l$) was 9.8, the damping coefficient ($\zeta$) was 0.4; the initial conditions ($\theta_0$ and $\omega_0$) were 0.8 for both; the frequency of driving force ($\Omega_D$) was 0.6; and the mass of pendulum ($m$) was 1. Due to these settings, the natural frequency of the pendulum ($\Omega_2$) was calculated to be 1. Still, the driving force ($a$) was varied in order to analyze the chaos clearly. The solution was then plotted as $\theta$ vs $t$ graphs, phase space, and Poincaré section. The results showed that the chaotic motion occurs when the driving force was in the range of 1.36-3.0, in which 3.0 was the maximum value in this simulation. In this range, the slight change in the initial condition resulted in significant differences of $\theta$ vs $t$ graph, implying sensitivity to initial conditions. The phase space depicted a chaotic attractor, while the Poincaré section resulted in many dots forming stretching and folding patterns. Based on these results, it can be said that the chaos behavior could arise from damped and driven nonlinear simple pendulum motion by varying parameters, such as driving force.

1. Introduction

After the discovery of chaotic behaviour on a 3D dynamical system for fluid flow [1], the research of chaos developed rapidly. A dynamical system is said to have chaotic behaviour if it satisfies a condition: it has sensitivity to initial condition; in other words, one cannot predict its future motion by only observing its initial condition. It can create the randomness pattern. Still, the prevailing state of a chaotic system is the results of the initial condition, the communication occurred during the process, noise, and the other external conditions that cannot be controlled [2-3]. It can be said that the chaotic system has the characteristic of randomness and sensitivity to initial conditions.

Since human lives in dynamical systems, such as electronic circuits, weather, and stock market, the study of chaos in dynamical system is expanded. Many of them are related to mechanical system, such as double pendulum [4] and triple pendulum [5]. Also, there are many research were from electromechanical systems, namely electromechanical transducer [6] and brushless DC motor (BLDCM) [2]. Besides, nowadays the study of chaos is employed to secure communications through advanced cryptography [7].
Related to that, the research on chaos in the simplest dynamical system, for example simple pendulum, is needed to understand deeply to show that the chaotic behaviour can be formed in the simple dynamical system. Basically, there were many studies related to investigation of chaos on the simple pendulum motion[8-10]. However, those research use different programming tools from the present study; therefore, the study of chaos behaviour that resulted from the solution formed by using Mathematica as the programming device is still can be discussed. In this study, the method used to solve the damped and driven simple pendulum equation was the 4th order of Runge-Kutta Method.

2. Methodology

2.1. Derivation of Pendulum’s Equation of Motion

As depicted in Figure 1, the model of simple pendulum used in this study is a system consisted of an object with mass, \( m \), which is tied to a tight light string with the length of \( l \); the string can swing freely in a vertical plane on the axis O in response to gravitation, \( g \). The forces, which is acting on the simple pendulum is tension of the string, \( T \); the pendulum’s weight, \( mg \); the damping forces, \( D \); and the external driving forces, \( F \).

\[
\begin{align*}
\theta & \quad l \\
mg & \\
F & \\
O & \\
T & \\
D & \\
\theta
\end{align*}
\]

Figure 1. The model of simple pendulum used in the study

Since the attached mass can move freely along a circle of radius \( l \) around the axis O, it can undergo rotational motion with angular acceleration, \( a \), or \( \dot{\theta} \) which is the second derivative of the angular position, \( \theta \) with respect to time. Meanwhile, the speed of the translation is given by equation (1).

\[
v = l\omega = l\dot{\theta}
\]  

(1)

The damping force, \( D \) is depicted in equation (2)

\[
D = bl\dot{\theta}
\]  

(2)

with \( b \) as the damping coefficient.

Based on the Newton’s second law to analyse all forces acting on the pendulum, one can determine that the equation of motion of the pendulum is equation (3)

\[
ds l\sin \theta + (-mgl\sin \theta) + Fl\sin \theta = I\ddot{\theta}
\]  

(3)

By setting the external driving force, \( F \), to be the function of time and \( D \) is dependent to the velocity, the equation of the pendulum motion can be written as equation (4); this equation is the second order differential equation, which describes the motion of the simple pendulum.

\[
\ddot{\theta} + \frac{b}{m} \dot{\theta} + \frac{g}{l} \sin \theta = \frac{F(t)}{ml}
\]  

(4)

The external damping forces employed in this study is the nonlinear one, given in equation (5)

\[
F(t) = A \cos \Omega_D t
\]  

(5)
By substituting equation (5) to equation (4), and by using $q = \frac{b}{m}$, $\Omega^2 = \frac{g}{l}$, and $a = \frac{A}{mg}$, we found equation (6) which gives the damped and driven nonlinear simple pendulum’s motion

$$\ddot{\theta} + q \dot{\theta} + \Omega^2 \sin \theta = a \Omega^2 \cos \Omega_D t$$

(6)

2.2. Numerical Solution Using the 4th order of Runge-Kutta

Equation (6) satisfies the necessary conditions for chaotic conditions when it is written as a set of first-order differential equations, namely equations (7)

$$\frac{d\omega}{dt} = -q\omega - \Omega^2 \sin \theta + a\Omega^2 \cos \Omega_D t$$

$$\frac{d\theta}{dt} = \omega$$

(7)

Based on equation (7), it can be said that this simple pendulum system has three variables, so the path lies in the 3-dimensional phase space, the minimum space for the formation of chaos behavior. In order for the system to display chaos behavior clearly, the parameters on the model is determined in a dimensionless state, namely $m = g = l = 1$ so that $\Omega^2 = 1$ [11]; thus, the acceleration due to gravity, $g$ is 9.8 m/s\(^2\) with the length of the string, $l$ is 9.8 m, resulted in $\Omega^2$ to be 1, and the attached mass, $m$ is considered to be equal to 1 (dimensionless); this simplification is often used in simulations.

Equation (6) also has several parameters whose values can be varied, namely, $\Omega_D$, $A$, and $b$. However, a thorough investigation of the system behavior as a function of these three parameters cannot be carried out [10]. For this reason, the parameter that was varied was amplitude of the external driving force, $A$, while others were kept constant. The value is determined at $\Omega_D \approx \Omega^2$ or $\Omega_D \approx 1$ so that the pendulum can display chaotic symptoms clearly (Baker et al, 1996). Based on that, in this study the value of $\Omega_D$ was $\frac{2}{3}$. The value of initial velocity, $\omega_0$, should be larger than $\Omega_D$, so it was set to be 0.8 rad/s, whereas the value for initial angle, $\theta_0$, was defined to be $\theta_{01}$ and $\theta_{02}$, so that the comparison of the motion when there was a slight change in the initial condition (0.01 rad) was shown clearly. At first, the initial value for $\theta_0 = \theta_{01}$ was 0.8 rad. The value for the damping coefficient was $\frac{\omega_0}{2} = 0.4$, so that the pendulum did not experience critical damped state.

In order to solve the simple pendulum’s motion equation using the 4th order of Runge-Kutta, equation (6) was defined into equation (8) and (9)

$$\frac{d\theta}{dt} = \omega$$

(8)

$$\frac{d\omega}{dt} = -q\omega - \Omega^2 \sin \theta + a\Omega^2 \cos \Omega_D t = f(t, \theta, \omega)$$

(9)

The above equations are simultaneous equations with $f_1(t, \theta, \omega) = \omega$ and $f_2(t, \theta, \omega) = f(t, \theta, \omega)$. By giving the initial condition $\theta_0$ to equation (8) and $\omega_0$ to equation (9), the angular velocity and deviation at any time were obtained. The 4th order of Runge-Kutta to solve the equations is given in a set of equation (10)

$$k_{i+} = f_1(t_n, \theta_n, \omega_n)$$

$$l_{i+} = f_2(t_n, \theta_n, \omega_n)$$

$$k_{2n} = f_1(t_n + \frac{1}{2} h, \theta_n + \frac{1}{2} k_i, \omega_n + \frac{1}{2} l_i)$$

$$k_{3n} = f_1(t_n + \frac{1}{2} h, \theta_n + \frac{1}{2} k_{2n}, \omega_n + \frac{1}{2} l_{i+})$$

$$k_{4n} = f_1(t_n + h, \theta_n + k_{3n}, \omega_n + l_{i+})$$

$$\theta_{n+1} = \theta_n + \frac{1}{6} (k_{i+} + 2k_{2n} + 2k_{3n} + k_{4n})$$

$$\omega_{n+1} = \omega_n + \frac{1}{6} (l_{i+} + 2l_{i+} + 2l_{i+} + l_{i+})$$
After yielding the value for \( k \) and \( l \) from equation (10), the values \( \theta \) and \( \omega \) were defined using equation (11) and (12)

\[
\theta_{n+1} = \theta_n + \frac{1}{6} h (k_{1n} + 2k_{2n} + 2k_{3n} + k_{4n})
\]

\[
\omega_{n+1} = \omega_n + \frac{1}{6} h (l_{1n} + 2l_{2n} + 2l_{3n} + l_{4n})
\] (11) (12)

The steps in equations (10) was repeated until the given \( t_{\text{max}} \) was obtained.

In this study, the variables used for analysis are angular velocity, \( \omega(t) \), and angular position, \( \theta(t) \), and the phase space is in the form of a plane. Physically, the paths can only move between \(-\pi\) to \(\pi\) with negligible connecting lines. Based on this, the trajectory of the system to be analyzed is truncated, so that the only trajectory remains at the \(-\pi\) to \(\pi\) limit with 150 steps.

In the phase space described above, the coordinates \( \omega(t) \), and \( \theta(t) \) are determined at \( t = 0, \Delta t, 2\Delta t, 3\Delta t \), and so on, where \( \Delta t = T / 150 \), \( T \) is the period of the driving force with value, \( T = \frac{2\pi}{D\Omega} \). For this reason, the Poincaré section is determined by plotting the intersection points in the plane for each \( t = mT \) (\( m=0,1,2,3, \ldots \)). And in this study, the dots on the first two steps were omitted to avoid transient effects of the system. From the plot of the points, it can be determined whether the system is periodic or not. If the trajectories are repeated in period \( T \), then it can be said that the system is periodic, whereas if the trajectories are not exactly repeating then the system can be said to be non-periodic.

2.3. Implementing the 4th order of Runge-Kutta to Mathematica

In order to solve the equation (6) with the 4th order of Runge-Kutta, the set of equation (10) was first defined in the program as follows:

```mathematica
RungeKutta4[___]["Step"[f_,t_,h_,y_,yp_]]:=
Block[{deltay,k1,k2,k3,k4},
k1=yp;
k2=f[t+1/2 h,y+1/2 h k1];
k3=f[t+1/2 h,y+1/2 h k2];
k4=f[t+h,y+h k3];
deltay=h (1/6 k1+1/3 k2+1/3 k3+1/6 k4);
{h,deltay}];RungeKutta4[___]["DifferenceOrder"]:=4;
```

After that definition, the equation was solved using `NDSolve` function. The graphic user interface (GUI) mode was used in this simulation, so the user can choose to analyze three different graphs, in particular, \( \theta \) vs \( t \) graphs, phase space, and Poincaré section.
3. Results and Discussion

3.1. Periodic State

For the above mentioned condition, namely $\omega_0 = 0.8$, and $\theta_0 = 0.8$, $q = 0.4$, $\Omega^2 = 1$, $\Omega_D = \frac{2}{3}$, the periodic state fulfilled when $a$ is still 0.3. This result was analysed based on the solution of the equation (6), which was depicted in Figure 2.

![Figure 2](image)

**Figure 2.** (a) Graph of $\theta$Vs $t$, (b) Graph of $\theta$Vs $t$ with a slight difference in $\theta_0$ (0.01), (c) the phase space, and (d) Poincarè section. All graphs are under the initial condition of $a = 0.3$, $q = 0.4$, $\Omega^2 = 1$, $\Omega_D = \frac{2}{3}$, $\omega_0 = 0.8$, and $\theta_0 = 0.8$.

Figure 2(a) shows the deviation of $\theta$ at any time ($t$) for the small driving force ($a = 0.3$). In this condition there are two regions of oscillation. The first oscillation occurs in the time range, 0-20 s, in this condition the effect of damping still influence the motion, so that the oscillation leads to a transient state with an exponential decrease in amplitude. The effect of this attenuation can then be anticipated by the external controlling force so that the pendulum experiences a second oscillation at a steady state, namely in the 20 s to infinity time (harmonic oscillation) with the frequency of $\Omega_D$.

If the initial state ($\theta_0$) is changed slightly 0.81, then the plot ratio of angular position vs time is shown in Figure 2(b) for $\theta_{01} = 0.8$ (Black) and $\theta_{02} = 0.81$ (Green). From this figure, one can see that the visible graph is only for $\theta_{02} = 0.81$ due to both trajectories are aligned, meaning the system is insensitive to initial conditions.
Figure 2(c) shows the state of a simple pendulum (determined by position-angular velocity coordinates) moving along a path on the phase plane while the pendulum swings. Due to the decrease in energy because of damping, the path in the transient state is twisted to the center of the plane. However, then this damping effect is anticipated by the energy absorbed from the external driving force so that the situation becomes steady with the form of a closed path. This closed path indicates that the pendulum is periodic, with its final state coming in its initial state.

In order to show the trajectory more clearly, the Poincaré section is used. In the periodic case the Poincaré section is given in Figure 2(d). Based on this figure, the path of a simple pendulum is a closed orbit, so the Poincaré section only formed one point, meaning that the trajectories of a simple pendulum only intersect the plane at a fixed point.

3.2. Quasi-Periodic State
The quasi-periodic state is a state in which a dynamic system experiences period multiplication. This situation is a way to lead to chaos, where chaos itself occurs when a system undergoes multiple period multiplication. In this study, the quasi-periodic state was reached at $a = 1.23$ for the given initial condition: $\omega_0 = 0.8$, and $\theta_0 = 0.8$, $q = 0.4$, $\Omega^2 = 1$, $\Omega_D = \frac{2}{3}$. The analysis was illustrated in Figure 3.

\[ \Omega_D = \frac{2}{3}, \omega_0 = 0.8, \text{and } \theta_0 = 0.8. \]

**Figure 3.** (a) Graph of $\theta$Vs $t$, (b) Graph of $\dot{\theta}$Vs $t$ with a slight difference in $\theta_0$ (0.01), (c) the phase space, and (d) Poincaré section. All graphs are under the initial condition of $a = 1.23$, $q = 0.4$, $\Omega^2 = 1$, $\Omega_D = \frac{2}{3}$, $\omega_0 = 0.8$, and $\theta_0 = 0.8$.

Figure 3(a) depicts that the pendulum moves in the opposite direction of the pendulum and experiences non-harmonic movements. This graph also shows that the transient state of the system occurs in the first 60 s, then the position of the angle moves in the opposite direction to the two oscillation regions, namely in the 60 s - 250 s range and in the 250 s - 400 s range, meaning that the
path continues to repeat but needs longer time to see the loop, so it appears more complex. This is said to be a doubling of the period, but this system is still periodic.

If the initial state is changed slightly, namely \( \theta_0 \) to 0.81, the two trajectories are still running in harmony, meaning that the system in this condition also has not shown sensitivity characteristics to the initial conditions. This is given in Figure 3(b).

From Figure 3(c), it can be seen that the trajectory is not in a closed loop. The trajectory in the transient state is attracted to a point in the phase space, but due to the large enough amplitude of the driving force, the energy absorbed by the pendulum becomes quite large, this energy in addition to anticipating the attenuation also causes a drastic change in state of the pendulum, this causes the orbit to rupture early, thus the trajectory moves with two distinct periods, or experiences a period doubling.

As shown in Figure 3(d), the intersection points formed are two fixed points, meaning that simple pendulum motion paths not only cross the plane at one point, but also intersect the plane at another coordinate point at some time later (two periods). For a point at the coordinates (0.5, 1), it indicates that there are trajectories that intersect the wrong plane at that point. This happens because of the transient effect of the system as is also seen in the phase space.

3.3. Chaotic State

As previously explained, chaotic conditions occur when a system experiences multiplication of periods several times. In this study, chaotic conditions have been reached at \( a = 1.36 \) for \( \omega_0 = 0.8 \), and \( \theta_0 = 0.8 \), the value of \( q = 0.4 \), \( \Omega^2 = 1 \), \( \Omega_2 = \frac{2}{3} \). This implies that the minimum driving force (based on value of \( a \) ) needed to gives chaotic behaviour is 1.36. This is shown in Figure 4.

![Figure 4](image)
From Figure 4(a), it can be seen that the motion of a simple pendulum is irregular, or in other words, the movement is never repetitive and is constantly making different movements. It can be explained that the large amplitude of the driving force causes the energy absorbed by the pendulum to be large, and because the force exerted is a force that changes harmoniously with time (nonlinear), it continues until the multiplication of periods occurs again in a denser frequency interval of a quasi periodic state. In this case, an infinite step travels only such a distance that the period becomes infinite. This is what is called chaos.

In addition to having complex movements, chaotic state also have extreme sensitivity to initial conditions. This is shown in Figure 4(b), where there are two initial states that differ slightly, namely the value of 0.01 which initially moves in harmony, but at 20 seconds later the two movements will change and spread further apart from one another. According to Walker (1991), this can be assumed as breaking a rope into two individual strands [12]. So it can be said that a chaotic system becomes unpredictable, because with the slightest disturbance, the movement of the system changes far from the initial estimate.

One other interesting point that can be written here is that the final movement of this system depends exactly on how the system starts or is deterministic. Therefore, this condition is said to be deterministic chaos. As a deterministic system, this system can be predicted for a short period of time, and as a chaotic system, this system becomes unpredictable for the long term. And this time span depends on each system. In Figure 4(b) the predictions for this simple pendulum system can be determined in about the first 20 seconds.

Figure 4(c) illustrates the simple complex pendulum motion trajectories with many periods. In contrast to phase space in a quasi-periodic state where the trajectories can still be reviewed, the trajectories in this phase space are difficult to identify due to the complex geometry of the paths. This geometry is said to be the chaotic attractor or often called the strange attractor because of its odd shape. It can be explained that the large energy from the external controlling force causes non-linearity of the system and causes the path to break and then break again into several passes, and so on. However, according to Setiawan (1991), because this attractor has a finite size, the trajectories cannot be separated exponentially, and fold towards themselves, and folds are formed in folds, this is what forms complex geometries [13].

The trajectories of a simple pendulum shown in Figure 4(c) are highlighted by the Poincarè cleavage in Figure 4(d). It can be analyzed that the trajectories of a simple pendulum intersect the plane at a very large number of points and form a pattern. Based on this, it can be said that the period of the system is very large, even infinite for an infinite time.

From Figure 4(d), it can also be explained that one of the keys to understanding chaotic behavior-sensitivity to initial conditions-is an explanation of stretching and folding operations. The stretching operation occurs between 1 and 2, and in this operation it causes increased small-scale uncertainty. Whereas the folding operation occurs between 2 and 3, this operation causes a large path separation and removes large-scale information. The next stretching operation occurs between 2 and 3, and so on until the time limit is given. Setiawan (1991) says that these operations make chaotic behavior increase microscopic fluctuations to macroscopic. After a certain time interval, the uncertainty grows and the system becomes unpredictable [14].

4. Conclusion
In summary, the damped and driven nonlinear simple pendulum-as the simplest dynamical system-can undergo the chaotic behavior for a certain value for the driving force, which is given by the value of $a$ in the simulation using Mathematica. When the chaotic state arises, the system becomes sensitive to the initial condition and forms the strange attractor in the phase space; also, it undergoes stretching and folding operation in the Poincarè section. Further study is needed to investigate the contribution of
other parameters, such as length of the strain and dumping coefficient, to the chaotic behavior of the pendulum.

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