\[|Z_{\text{Kup}}| = |Z_{\text{Henn}}|^2 \text{ FOR LENSES SPACES} \]

LIANG CHANG AND ZHENGHAN WANG

Abstract. M. Hennings and G. Kuperberg defined quantum invariants \(Z_{\text{Henn}}\) and \(Z_{\text{Kup}}\) of oriented 3-manifolds based on certain Hopf algebras, respectively. We prove that \(|Z_{\text{Kup}}| = |Z_{\text{Henn}}|^2\) for lens spaces when both invariants are based on factorizable finite dimensional ribbon Hopf algebras. Recently a fermionic generalization of the Turaev-Viro state sum TQFTs is proposed using Grassmann variables. We conjecture that the Kuperberg invariants for non-semisimple super-Hopf algebras are the partition functions of such “TQFTs for systems with fermions”.

1. Introduction

A Turaev-Viro-type topological quantum field theory (TQFT) based on a spherical fusion category \(\mathcal{C}\) is equivalent to the Reshetikhin-Turaev-TQFT based on the Drinfeld center \(Z(\mathcal{C})\) of \(\mathcal{C}\) \cite{TV2,BK}. Consequently, \(Z_{\text{TV}}(M) = |Z_{\text{RT}}(M)|^2\) for any closed oriented 3-manifold \(M\). As Hennings invariants are non-semisimple generalizations of Reshetikhin-Turaev-invariants \cite{Ker}, and Kuperberg invariants are non-semisimple generalizations of Turaev-Viro-invariants \cite{BW} (In this paper, by Kuperberg invariant, we mean the one from noninvolutory Hopf algebras in \cite{Ku}), a similar relation might exist for the Kuperberg and Hennings invariants. Such a connection is first suggested in \cite{Ker}.

Problem: Prove a non-semisimple generalization of the relation between Turaev-Viro and Reshetikhin-Turaev invariants to Kuperberg and Hennings invariants.

One issue with the above problem is that the Kuperberg invariant \(Z_{\text{Kup}}\) depends on a combing or framing of the 3-manifold \(M\), while there is no such explicit dependence of comings or framings for the Hennings invariant \(Z_{\text{Henn}}\). Ideally, the conjectured relation would follow from a similar relation between two kinds of nonsemisimple (2+1)-TQFTs. As a first step, we prove from the following relation between Kuperberg \(Z_{\text{Kup}}\) and Hennings \(Z_{\text{Henn}}\) invariants for the lens spaces \(L(p,q)\). Given an oriented manifold \(M\), \(\overline{M}\) denotes \(M\) with the opposite orientation.

Theorem: Let \(H\) be a factorizable finite dimensional ribbon Hopf algebra and \(L(p,q)\) be an oriented lens space. Then \(Z_{\text{Kup}}(L(p,q), f, H) = Z_{\text{Henn}}(L(p,q)\#\overline{L(p,q)}, H)\) for some suitably chosen framing \(f\) of \(L(p,q))\).

Different choices of framings change the Kuperberg invariant by a multiplication of roots of unity. Using \(Z_{\text{Henn}}(\overline{M}, H) = \overline{Z_{\text{Henn}}(M, H)}, Z_{\text{Henn}}(M_1\#M_2, H) = Z_{\text{Henn}}(M_1, H)Z_{\text{Henn}}(M_2, H)\), we have

Corollary 1. \(|Z_{\text{Kup}}| = |Z_{\text{Henn}}|^2\)

The contents of the paper are as follows. In Section 2, we recall the definitions of the Hennings and Kuperberg invariants and set up our notations. In Section 3, we prove our
2. Hennings and Kuperberg invariants

2.1. Some facts about Hopf algebras. Let \( H = (\mathbb{C}, \Delta, S, \lambda, \epsilon) \) be a finite dimensional Hopf algebra with multiplication \( \lambda \), comultiplication \( \Delta \), antipode \( S \), unit \( \lambda \), and counit \( \epsilon \). We also use \( \lambda \) to denote the identity map \( \text{id} \) on \( H \) sometimes.

Recall that a Hopf algebra \( H \) is quasitriangular if there exists an \( R \)-matrix \( R \in \mathbb{C} \otimes H \otimes H \) and \( R_{ij} \in \mathbb{C} \otimes H \otimes H \) is obtained from \( R = \sum_s s_i \otimes t_i \) by inserting the unit \( \lambda \) into the tensor factor labeled by the index \( i \). In a quasitriangular Hopf algebra with \( R \)-matrix \( R = \sum_i s_i \otimes t_i \), there exists a special element \( u = \sum_i S(t_i) s_i \) satisfying \( S^2(x) = uxu^{-1} \) for \( x \in H \). Suppose \( P \) is the permutation of the two tensor factors of \( H \otimes H \), we use \( R^r \) to denote \( P \cdot R = \sum t_i \otimes s_i \).

A quasitriangular Hopf algebra is ribbon if there exists a central element \( \theta \) satisfying the relations

\[
\Delta(\theta) = (R^rR)^{-1}(\theta \otimes \theta), \quad \epsilon(\theta) = 1, \quad \text{and} \quad S(\theta) = \theta.
\]

It can be shown that \( G = u\theta^{-1} \) is a grouplike element and \( S^2(x) = GxG^{-1} \) for \( x \in H \).

2.1.1. Integrals, cointegrals and unimodular Hopf algebras. A left integral \( \lambda^L \) (respectively, right integral \( \lambda^R \)) for \( H \) is an element in \( H^* \) which satisfies \((\text{id} \otimes \lambda^L) \Delta(h) = \lambda^L(h) \cdot 1\) (respectively, \((\lambda^R \otimes \text{id}) \Delta(h) = \lambda^R(h) \cdot 1\)) for all \( h \in H \). Dually, a left cointegral \( \lambda^L \) (respectively, right cointegral \( \lambda^R \)) for \( H \) is an element in \( H \) which satisfies \( h\lambda^L = \epsilon(h)\lambda^L \) (respectively, \( \lambda^R h = \epsilon(h)\lambda^R \)) for all \( h \in H \). A Hopf algebra \( H \) is called unimodular if the space of left cointegrals for \( H \) is the same as the space of right cointegrals for \( H \).

For finite dimensional Hopf algebras, the left and right integrals (respectively, left and right cointegrals) are unique up to scalar multiplication, and we may choose a normalization that \( \lambda^R(\lambda^L) = \lambda^R(s(\lambda^L)) = 1 \). From this, there is an algebra homomorphism \( \alpha \in H^* \), called modulus of \( H \), independent of the choice of \( \lambda^L \), such that \( \lambda^L h = \alpha(h)\lambda^L \) for all \( h \in H \). Likewise, there is a grouplike element \( g \in H \), called comodulus of \( H \), independent of the choice of \( \lambda^R \), such that \( (id \otimes \lambda^R) \Delta(h) = \lambda^R(h)g \) for all \( h \in H \). They are of finite order and \( \omega = \alpha(g) \) is a root of unity.

2.1.2. Drinfeld map and factorizable Hopf algebras. Given \( Q = \sum_i Q_i^{(1)} \otimes Q_i^{(2)} \in H \otimes H \), we define a map \( f_Q : H^* \to H \) by \( f_Q(p) = \sum_i p(Q_i^{(1)})Q_i^{(2)} \) for \( p \in H^* \). The conditions for \( R \)-matrix implies that \( f_R \) is an algebra homomorphism and \( f_{R^r} \) is an algebra antimorphism. The map \( f_{R^r} : H^* \to H \) is called the Drinfeld map. If the Drinfeld map for a quasitriangular Hopf algebra \( H \) is nondegenerate, then \( H \) is called factorizable.

Proposition 1. If a quasitriangular Hopf algebra \( H \) is factorizable, then it is unimodular.

For a proof, see Prop. 3 on page 224 of [Ra2].

For a factorizable Hopf algebra \( H \), \( f_{R^r}(\lambda^R) = \lambda^L \) and \( \langle \lambda^R, \lambda^L \rangle = 1 \) under some normalization (see [CW2]). This relates the left cointegral \( \lambda^L \) with the right integral \( \lambda^R \). We will use such a pair of related integral and cointegral throughout this paper.

In this paper, we work with factorizable finite dimensional ribbon Hopf algebras. For such Hopf algebras, we use \( \Lambda \) to denote the left and right cointegrals for \( H \). The comodulus \( \alpha \) is the counit \( \epsilon \). The right integral for \( H^* \), denoted by \( \lambda \), satisfies the following properties for all \( x \) and \( y \) in \( H \) [Ra1]:

\[
\lambda(\lambda^L) = \lambda^L(s(\lambda^L)) = 1.
\]
Given such a $\lambda$, we can define a trace-like functional $tr : H \to \mathbb{C}$ by $tr(x) = \lambda(xG) = \lambda(Gx)$ such that $tr(xy) = tr(yx)$ and $tr(S(x)) = tr(x)$ for all $x$ and $y$ in $H$.

The following lemma is important for the proof of the main theorem. Let $\Delta^{(n-1)}(x) = \sum(x) x_{(1)} \otimes \ldots \otimes x_{(n)}$ be in the Sweedler notation. In this paper, we omit the summation symbol, i.e. $\Delta^{(n-1)}(x) = x_{(1)} \otimes \ldots \otimes x_{(n)}$.

**Lemma 1.** Let $H$ be a factorizable finite dimensional ribbon Hopf algebra. Then for $p \in H^*$ and $n \in \mathbb{N}$,

$$\Delta^{(n-1)}(f_{R^R}(p)) = f_{R^R}(p(1)) f_{R^R}(p(2n-1)) \otimes f_{R^R}(p(2)) \otimes \ldots \otimes f_{R^R}(p(n-1)) f_{R^R}(p(n+1)) \otimes f_{R^R}(p(n))$$

In particular, since $f_{R^R}(\lambda^R) = \Lambda^L$, we have

$$\Lambda^L_{(k)} = f_{R^R}(\lambda^R_{(k)}) f_{R^R}(\lambda^R_{(2n-k)}) \text{ for } k = 1, \ldots, n-1$$

and $\Lambda^L_{(n)} = f_{R^R}(\lambda^R_{(n)})$.

**Proof.** Recall that $f_R$ is an anti-coalgebra map and $f_{R^R}$ is a coalgebra map, i.e. for $p \in H^*$,

$$\Delta(f_R(p)) = (f_R \otimes f_R)(\Delta(p)) = f_R(p(2)) \otimes f_R(p(1)),$$

$$\Delta(f_{R^R}(p)) = (f_{R^R} \otimes f_{R^R})(\Delta^R(p)) = f_{R^R}(p(1)) \otimes f_{R^R}(p(2)).$$

These two properties follow from the definition of the $R$-matrix: $(\Delta \otimes id)(R) = R_{13}R_{23}$ and $(id \otimes \Delta)(R) = R_{13}R_{12}$.

The proof of the lemma can be done by induction. When $n = 2$,

$$\Delta(f_{R^R}(p)) = \Delta(f_{R^R}(p(1)) f_{R^R}(p(2)))$$

$$= \Delta(f_{R^R}(p(1))) \Delta(f_{R^R}(p(2)))$$

$$= f_{R^R}(p(1)) f_{R^R}(p(4)) \otimes f_{R^R}(p(2)) f_{R^R}(p(3))$$

$$= f_{R^R}(p(1)) f_{R^R}(p(3)) \otimes f_{R^R}(p(2))$$

Suppose the lemma is valid for $n = k$, then when $n = k + 1$,

$$\Delta^{(k)}(f_{R^R}(p)) = (id \otimes \ldots \otimes id \otimes \Delta)(\Delta^{(k-1)}(f_{R^R}(p)))$$

$$= (id \otimes \ldots \otimes id \otimes \Delta)(f_{R^R}(p(1)) f_{R^R}(p(2k-1)) \otimes f_{R^R}(p(2k-2)) \otimes \ldots \otimes f_{R^R}(p(k-1)) f_{R^R}(p(k+1)) \otimes f_{R^R}(p(k)))$$

$$= f_{R^R}(p(1)) f_{R^R}(p(2k-1)) \otimes f_{R^R}(p(2k-2)) \otimes \ldots \otimes f_{R^R}(p(k-1)) f_{R^R}(p(k+1)) \otimes \Delta(f_{R^R}(p(k)))$$

$$= f_{R^R}(p(1)) f_{R^R}(p(2k+1)) \otimes f_{R^R}(p(2k)) f_{R^R}(p(2k+2)) \otimes \ldots \otimes f_{R^R}(p(k-1)) f_{R^R}(p(k+3)) \otimes f_{R^R}(p(k)) f_{R^R}(p(k+2)) \otimes f_{R^R}(p(k+1))$$

Hence, the lemma holds for all $n \in \mathbb{N}$. $\square$

2.1.3. **Examples.** Factorizable finite dimensional Hopf algebras include:

1. $U_q sl(2, \mathbb{C})$ at an odd root of unity. Let $q$ be an $l$-th primitive root of unity with $l$ odd. $U_q sl(2, \mathbb{C})$ is generated by $E$, $F$ and $K$ with the following relations:

$$E^l = F^l = 0, \quad K^l = 1$$
and the Hopf algebra structure given by
\[ KE = q^2EK, \quad KF = q^{-2}FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \]
\[ \Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K, \]
\[ \varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = 1, \]
\[ S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}. \]

(2) The Drinfeld double \( D(H) \) of a finite dimensional Hopf algebra \( H \).

2.2. **Hennings invariant.** Let \( H \) be a unimodular finite dimensional ribbon Hopf algebra with \( \lambda(\theta)\lambda(\theta^{-1}) \neq 0 \).

2.2.1. **Kauffman-Radford version of the Hennings invariant.** Given a unimodular finite dimensional Hopf algebra \((H, R, \theta)\), the Kauffman-Radford version of the Hennings invariant \([KR]\) is defined as follows. First \((H, R, \theta)\) leads to a regular isotopy invariant \( TR(L, H) \) for links \( L \): decorate each crossing with the tensor factors from the \( R \)-matrix \( R = \sum s_i \otimes t_i \) as below.

Once all the crossings have been decorated, the resulting diagram is a labeled diagram immersed in the plane. The Hopf algebra elements may be moved across maxima or minima at the expense of the application of the antipode or its inverse. Passing through the extremum in a clockwise direction introduces \( S^{-1} \) and passing through the extremum in a counterclockwise direction introduces \( S \) as below.

To define \( TR(L) \), slide all the Hopf algebra elements on the same component into one vertical portion of the diagram. Along a vertical line, all the Hopf algebra elements are multiplied together.

The final juxtaposition of labeled elements at the chosen points gives rise to a product \( w_i \in H \) for the \( i \)-th component. Let \( d_i \) be the Whitney degree of this component obtained by traversing it upward from the chosen vertical portion. The Whitney degree is the total number of turns of the tangent vector as one traverses the curve in the given direction. For
Define $TR(L, H) = \text{tr}(w_1G^d_1) \cdots \text{tr}(w_nG^d_n)$. This is a regular isotopy invariant of links. Moreover, if $\lambda(\theta)\lambda(\theta^{-1}) \neq 0$, which is always true when $H$ is factorizable, then

$$Z_{\text{Henn}}(M(L), H) = \frac{\lambda(\theta)\lambda(\theta^{-1})}{\lambda(\theta)/\lambda(\theta^{-1})} - c(L)^2 TR(L, H)$$

is an invariant of the oriented 3-manifold $M(L)$ obtained from surgery on the framed link $L$ with the blackboard framing, where $c(L)$ denotes the number of components of $L$, and $\sigma(L)$ denotes the signature of the framing matrix of $L$.

2.2.2. Properties of Hennings invariant. We have the following from [H]:

1. $Z_{\text{Henn}}(M_1 \# M_2, H) = Z_{\text{Henn}}(M_1, H)Z_{\text{Henn}}(M_2, H)$,
2. $Z_{\text{Henn}}(M, H) = Z_{\text{Henn}}(\overline{M}, H)$

2.3. Kuperberg invariant. Let $H$ be any finite dimensional Hopf algebra.

2.3.1. Kuperberg combings. Kuperberg uses minimal Heegaard diagrams to construct his invariants for framed oriented 3-manifolds [Ku]. Given a Heegaard diagram of a closed connected oriented 3-manifold $M$, Kuperberg named the attaching curves $c_l$'s of the 1-handles as lower circles and the attaching curves $c_u$'s of the 2-handles as upper circles. A Heegaard diagram on a surface $F$ of genus $g$ is called minimal if has $g$ lower circles and $g$ upper circles. The orientation of $M$ induces an orientation on its Heegaard surface $F$, by the convention that a positive tangent basis at a point on $F$ extends to a positive basis for $M$ by appending a normal vector that points from the lower side to the upper side. Heegaard circles are also oriented. Indeed, the orientation of the lower circles are induced by the right-hand rule. Define a combing on a minimal Heegaard diagram on surface $F$ to be a vector field on $F$ with $2g$ singularities of index $-1$, one on each circle, and one singularity of index $+2$ disjoint from all circles. The singularity of index $-1$ on a given circle, which is called the base point of the circle, should not lie on a crossing and the two outward-pointing vectors should be tangent to the circle. Kuperberg uses the combings of Heegaard diagrams to represent combings of 3-manifolds by the following fact:

**Proposition 2.** Any combing $b$ of a minimal Heegaard diagram of $M$ can be extended to a combing $\overline{b}$ of $M$. Conversely, any combing of $M$ is homotopic to the Kuperberg extension of some combing of the minimal Heegaard diagram.

2.3.2. Twist front and rotation numbers. Given a combing $b_1$ of $M$, by Prop. 2 we may assume it is extended from some combing of the minimal Heegaard diagram. To describe a framing $(b_1, b_2)$ of $M$, where $b_2$ is an orthogonal combing to $b_1$, it suffices to describe $b_1$ as a diagram combing and then to describe $b_2$ on the Heegaard surface $F$ and on all upper and lower disks. Kuperberg introduces twist fronts to indicate the position of $b_2$. A twist front is an arc along which $b_2$ is normal to $F$ and points from the lower to the upper handlebody. A twist front is transversely oriented in the direction that $b_2$ rotates by the right-hand rule relative to $b_1$ and transverse orientation is presented by the zigzag symbol as in Fig. [H]
To define the Kuperberg invariant, orient all circles according to the orientation of \( M \). Let \( f = (b_1, b_2) \) be a framing from the minimal Heegaard diagram \( D \). For each point \( p \) on some circle \( c \) of \( D \) with base point \( o \), we define \( \psi(p) \) to be the counterclockwise rotation of the tangent to \( c \) relative to \( b_1 \) from \( o \) to \( p \) in units of \( 1 = 360^\circ \). If \( p \) is a crossing, then two rotation angles \( \psi_l(p) \) and \( \psi_u(p) \) are defined. \( \psi_l \) and \( \psi_u \) are defined to be the total counterclockwise rotation.

Let \( \phi(p) \) be the total right-handed twist of \( b_2 \) around \( b_1 \) from \( o \) to \( p \), and similarly define \( \phi_l(p) \) and \( \phi_u(p) \). Using twist fronts, \( \phi(p) \) can be computed as the total signs of all fronts crossed from \( o \) to \( p \), not counting the front that terminates at \( o \) itself.

**2.3.3. The Kuperberg invariant.** First we need some special elements constructed from the integral and cointegral. For any integer \( n \), define \( \lambda_{n-\frac{1}{2}} \in H^* \) such that \( \lambda_{n-\frac{1}{2}}(x) = \lambda^R(xg^n) \) and \( \Lambda_{n-\frac{1}{2}} = (id \otimes \alpha^n)\Delta(\Lambda^R) \in H \). Also define the tilt map \( T \) to be \( T(x) = (\alpha \otimes id \otimes \alpha^{-1})\Delta^2(S^{-2}(x)) \) for \( x \in H \).

In Kuperberg’s tensor notation, the algorithm for his invariant \( Z_{Kup}(M, f, H) \) is as follows: replace each upper circle \( c_u \) with the multiplication tensor \( \mu \) with one inward arrow for each crossing and the outward arrow with \( \lambda_m \) at the base point, with the arrows ordered as indicated. Here \( m = -\psi(c_u) \). Replace each lower circle \( c_l \) with a comultiplication tensor \( \Delta \) with an outward arrow for each crossing and the inward arrow with \( \Lambda_n \) at the base point, with the arrow ordered as indicated. Here \( n = \psi(c_l) \). Replace each crossing by the tensor \( -1 \rightarrow S^aT^b \rightarrow \) where \( a = 2(\psi_l(p) - \psi_u(p)) - \frac{1}{2} \), \( b = \phi_l(p) - \phi_u(p) \), and \( p \) is the crossing point.

Finally, contract all tensor corresponding to circles and crossings according to incidence. The Kuperberg invariant is then a big summation:

\[
Z_{Kup}(M, f, H) = \sum \langle \lambda, \cdots S^{m_1}T^{k_1}(\Lambda_{(m_1)}) \cdots g^{i_1} \rangle \cdots \langle \lambda, \cdots S^{m_i}T^{k_i}(\Lambda_{(m_i)}) \cdots g^{i_N} \rangle
\]

Here the order for multiplication and comultiplication follows the orientations of the upper and lower circles.
For a factorizable finite dimensional ribbon Hopf algebra $H$, we have $\alpha = \varepsilon$. So $\Lambda_{n+\frac{1}{2}} = \Lambda$ for all integer $n$ and $T = S^{-2}$. Thus, the invariant is of the following form:

\[ Z_{Kup}(M, f, H) = \sum \langle \lambda, \cdots S^{n_1}(1)(\Lambda_{m_1}(1)), \cdots g_1 \rangle \cdots \langle \lambda, \cdots S^{n_2}(N)(\Lambda_{m_2}(N)), \cdots g_N \rangle \]

\subsection*{3.3.4. Basic properties of Kuperberg invariant}

With suitable choices of framings, we have [Kül]:

1. $Z_{Kup}(M_1 \# M_2, H) = Z_{Kup}(M_1, H)Z_{Kup}(M_2, H)$,
2. $Z_{Kup}(M, H) = Z_{Kup}(\overline{M}, H^{op}) = Z_{Kup}(\overline{M}, H^{cop}) = Z_{Kup}(M, H)$.

\section*{3. A relation between Kuperberg and Hennings invariants}

In this section, we prove our main theorem:

**Theorem 1.** Let $H$ be a factorizable finite dimensional ribbon Hopf algebra and $L(p, q)$ be an oriented lens space, then

\[ Z_{Kup}(L(p, q), f, H) = Z_{Henn}(L(p, q) \# \overline{L(p, q)}, H) \]

for some suitably chosen framing $f$ of $L(p, q)$.

To prove Theorem 1, we will calculate $Z_{Kup}(L(p, q), f, H)$ and $Z_{Henn}(L(p, q) \# \overline{L(p, q)}, H)$ through the framed Heegaard diagram and the chain-mail link, respectively. Since $L(p, q)$ is homeomorphic to $L(p, q + kp)$ for each integer $k$, it suffices to show the theorem for the case $p > q > 0$.

\subsection*{3.1. Chain-mail links}

Let $M$ be a closed connected oriented 3-manifold. A bridge from a Heegaard diagram of $M$ to a surgery diagram of $M \# \overline{M}$ is the chain-mail link introduced in [Ro]. Let $(F, H_1, H_2)$ be a minimal Heegaard decomposition of $M$. Push the upper circles $c_\alpha$'s into $H_1$ slightly, then the upper circles and the lower circles form a link in $H_1$. All these curves are framed by thickening them into bands parallel to the Heegaard surface $F$. This results in a chain-mail link $C(M) \subseteq H_1$, which is in fact a surgery presentation for $M \# \overline{M}$ ([Ro]). Fig. 2 and Fig. 3 are the Heegaard diagram and the corresponding chain-mail link for the Lens space $L(5, 2)$.

In Fig. 2, a 1-handle (not drawn) is attached to the two round circles in the 2-sphere $S^2$ regarded as the plane together with the point at infinity. In general, the picture Fig. 4 gives us a minimal Heegaard diagram for $L(p, q)$ with $p > q > 0$, where $r$ is the remainder, i.e.
\[ r = p - \left\lfloor \frac{p}{q} \right\rfloor q \text{ and } 0 < r < q. \] In the picture, the horizontal line presents the lower circle \( c_l \) (Our lower circles are above the plane). The upper circle \( c_u \) has \( q \) strands coming out from the right circle and then going clockwise around the circle for \( \left\lfloor \frac{q}{q} \right\rfloor - 1 \) times until meeting the \( q \) strands from the left circle. To make \( p \) intersections with \( c_l \), we let the first \( r \) strands of the left \( q \) strands go around counterclockwise to match the right \( q \) strands from above. Fig. 5 is the corresponding chain-mail link.

3.2. \( Z_{K_{up}}(L(p,q), f, H) \). In this section, we calculate the Kuperberg invariant for \( L(p,q) \) with some framing. Since the Kuperberg invariants depend on the framings of 3-manifolds, in order to match them with the Hennings invariants, we need to choose a particular framing for \( L(p,q) \) which depends on the value of \( p \) and \( q \). First, define \( N_1 = \frac{q + 1}{2} \) if \( q \) is odd; \( N_1 = \frac{p + q + 1}{2} \) if \( q \) is even. Since \( p \) and \( q \) are relatively prime, \( N_1 \) is always a natural number. Let \( N_j \equiv N_1 + (j - 1)q(\text{mod} p) \) such that \( N_j \in \{1, \ldots, p\} \) for \( j = 1, \ldots, p \). And set \( N_{k_1}, \ldots, N_{k_q} \) to be the sequential indices starting from \( N_1 \) satisfying that they are all between 1 and \( q \). In the following, we write \( \Lambda_j = \Lambda(N_j) \) for \( j = 1, \ldots, n \) for short.
We set up a framed Heegaard diagram for $L(p,q)$ shown in Fig. 6 and Fig. 13. The framing $f$ is represented by the dashed flows and the twist front. The twist fronts vary depending on whether $q$ is odd or even. Two index $-1$ singularities are located at the two ends of twist front on the horizontal line. The right $-1$ singularity is at the $N_1$-th intersection of the horizontal line and the upper circle $c_u$. The lower circle $c_l$ is presented by the horizontal line with base point the left $-1$ singularity and oriented towards right from the base point. To avoid the right singularity, let $c_l$ turn around slightly when it meet this singularity. Likewise, The upper circle $c_u$ with base point the right singularity is oriented towards down from its base point. The index $+2$ singularity is located at the infinity. The orientation of circles are shown in Fig. 4.

3.2.1. $Z_{K_{up}}(L(p,q), f, H)$ when $q$ is odd. We choose a framed Heegaard diagram for $L(p,q)$ as shown in Fig. 6. In this case, $N_{k_1} = N_1$. Let us first analyze the pattern of powers of the
$S_{mj} - 2(\Lambda_{j+1})$ and $S_{mj}(\Lambda_j)$

**Figure 7.** Power of $S$ changing when $q$ is odd

$S_{mj}(\Lambda_j)$ and $S_{mj}(\Lambda_{j+1})$

**Figure 8.** Power of $S$ changing when $q$ is odd

$S_{mj}(\Lambda_j)$ and $S_{mj}(\Lambda_{j+1})$

**Figure 9.** Power of $S$ changing when $q$ is odd: case 1 and 2

$S_{mj}(\Lambda_j)$ and $S_{mj}(\Lambda_{j+1})$

**Figure 10.** Power of $S$ changing when $q$ is odd: case 3 and 4

$S_{mj}(\Lambda_j)$ and $S_{mj}(\Lambda_{j+1})$

**Figure 11.** Power of $S$ changing when $q$ is odd: case 5 and 6

$S_{mj}(\Lambda_j)$ and $S_{mj}(\Lambda_{j+1})$

**Figure 12.** Power of $S$ changing when $q$ is odd: case 7
antipode $S$ in the product in (1). For $\Lambda_k$, which is the starting point to do the multiplication along $c_u$, the power of $S$ is $2(\psi_l(k_1) - \psi_u(k_1)) - \frac{1}{2} = 2(-\frac{1}{4} - 0) - \frac{1}{2} = -1$.

**Lemma 2.** The powers of $S$ change as shown in Fig. 9 to Fig. 12. Namely the powers of $S$ from $\Lambda_j$ to the next factor $\Lambda_{j+1}$ remain the same or decrease by 2, depending on which way that the upper circle goes from $\Lambda_j$ to the next factor.

**Proof.** Suppose the $j$-th term in the summation is $S^{m_j}(\Lambda_j)$, we calculate the difference $m_{j+1} - m_j$ case by case, which is

$$m_{j+1} - m_j = 2(\psi_l(\Lambda_{j+1}) - \psi_l(\Lambda_j)) - 2(\psi_u(\Lambda_{j+1}) - \psi_u(\Lambda_j)) + 2(\phi_u(\Lambda_{j+1}) - \phi_u(\Lambda_j))$$

Here $2(\phi_l(\Lambda_{j+1}) - \phi_l(\Lambda_j))$ makes no contribution since the lower circle does not intersect with the twist fronts. Note that $T = S^{-2}$ for factorizable Hopf algebras.

The patterns shown in Fig. 7 include five cases:

1. This case is shown in the first picture in Fig. 9

$$m_{j+1} - m_j = 2(0 - (-\frac{1}{2})) - 2(\frac{1}{2}) + 2(-1) = -2$$

2. This case is shown in the second picture in Fig. 9

$$m_{j+1} - m_j = 2((-\frac{1}{2}) - (-\frac{1}{2})) - 2(\frac{1}{2} + \frac{1}{2}) + 2(0) = -2$$

3. and (4) These two cases are shown in the first picture in Fig. 10 and they share the same calculation.

$$m_{j+1} - m_j = 2(0 - 0) - 2(-\frac{1}{2} + \frac{1}{2}) + 2(-1) = -2$$

5. This case is shown in the first picture in Fig. 11

$$m_{j+1} - m_j = 2(-\frac{1}{2} - 0) - 2(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) + 2(0) = -2$$

The patterns shown in Fig. 8 include the following two cases:

6. This case is shown in the second picture in Fig. 11

$$m_{j+1} - m_j = 2((-\frac{1}{2}) - (-\frac{1}{2})) - 2(0) + 2(0) = 0$$

7. This case is shown in the second picture in Fig. 12

$$m_{j+1} - m_j = 2((-\frac{1}{2} - 0) - 2(-\frac{1}{2})) + 2(0) = 0$$

Two more data to write down the Kuperberg invariant are $\psi(c_l)$ and $\psi(c_u)$. It is easy to see that $\psi(c_l) = -\frac{1}{2}$.

$$\psi(c_u) = -\frac{1}{4} + (N_1 - 1)(\frac{1}{2} - \frac{1}{2}) + (q - N_1)(\frac{1}{2} + \frac{1}{2}) + \frac{1}{2} + \frac{1}{4} = \frac{q}{2}$$

and then the power of $g$ is $-\psi(c_u) + \frac{1}{2} = -\frac{q+1}{2}$. 

Now we can write down the Kuperberg invariant $Z_{Kup}(L(p,q), f, H)$

$$Z_{Kup} = \langle \lambda, S^{-1}(\Lambda_{k_1}) \cdots S^{-1}(\Lambda_{k_2-1}) S^{-3}(\Lambda_{k_2}) \cdots S^{-3}(\Lambda_{k_3-1}) \cdots S^{-2q+1}(\Lambda_{k_p}) \rangle$$

$$= \langle \lambda, S^{2q-2}(\Lambda_{p-1}) \cdots S^{2q-2}(\Lambda_{p+2-k_3}) S^{2q-4}(\Lambda_{p+1-k_3}) \cdots S^{2q-4}(\Lambda_{p+2-k_3}) \rangle$$

$$= \langle \lambda, S^{2q-2}(\Lambda_{p-1}) \cdots S^{2q-2}(\Lambda_{p+2-k_3}) S^{2q-4}(\Lambda_{p+1-k_3}) \cdots S^{2q-4}(\Lambda_{p+2-k_3}) \rangle$$

Here we have used the unimodular property that $S(\Lambda) = \Lambda$ and so $\sum S(\Lambda_{(p)}) \otimes \cdots \otimes S(\Lambda_{(p+1-j)}) \otimes \cdots \otimes S(\Lambda_{(1)}) = \sum \Lambda_{(1)} \otimes \cdots \otimes \Lambda_{(j)} \otimes \cdots \otimes \Lambda_{(p)}$ and by symmetry $k_j + k_{q+2-j} = p+2$ for $j = 2, \ldots, q$.

3.2.2. $Z_{Kup}(L(p,q), f, H)$ when $q$ is even. Fig. 13 is a framed Heegaard diagram for $L(p,q)$ when $q$ is even. In this case, the twist front is different from the case when $q$ is odd, so it makes the pattern of the power changes of $S$ different.

**Lemma 3.** As shown in Fig. 14 and Fig. 15, the powers of $S$ from $\Lambda_{(N_j)}$ to the next factor $\Lambda_{(N_{j+1})}$ remain the same or increase by 2 depending on which way that the upper circle goes from $\Lambda_{(N_j)}$ to the next factor.
Proof. Similar to the odd case, we calculate the change \( m_{j+1} - m_j \) of the powers of \( S \) case by case shown from Fig. 16 to Fig. 18.

(1) as shown in the first picture in Fig. 16

\[
m_{j+1} - m_j = 2(0 - (-\frac{1}{2})) - 2(\frac{1}{2}) + 2(0) = 0.
\]

(2) as shown in the second picture in Fig. 16

\[
m_{j+1} - m_j = 2(0 - 0) - 2(-\frac{1}{2} + \frac{1}{2}) + 2(0) = 0.
\]

(3) as shown in the first picture in Fig. 17

\[
m_{j+1} - m_j = 2((-\frac{1}{2}) - 0) - 2(-\frac{1}{2}) + 2(1) = 2.
\]
(4) as shown in the second picture in Fig. 17
\[ m_{j+1} - m_j = 2(0 - 0) - 2\left(-\frac{1}{2} - \frac{1}{2}\right) + 2(0) = 2. \]
(5) as shown in the second picture in Fig. 18
\[ m_{j+1} - m_j = 2\left(-\frac{1}{2}\right) - 2(0 - 0) + 2(0) = 2. \]

\[ \square \]

For \( \psi(c_u) \), we have
\[ \psi(c_u) = -\frac{1}{4} + q\left(-\frac{1}{2} + \frac{1}{2}\right) + (N_1 - 1 - q)(1 + \frac{1}{2}) - \frac{1}{4} = \frac{p - q - 1}{2} - \frac{1}{2}, \]
and then the power of \( g \) is \(-\psi(c_u) + \frac{1}{2} = \frac{p - q + 1}{2}\).
Thus the Kuperberg invariant is
\[ Z_{K_{up}} = \langle \lambda, S^{-1}(\Lambda_1)S(\Lambda_2) \cdots S^{2k_1-5}(\Lambda_{k_1-1})S^{2k_1-5}(\Lambda_{k_1})S^{2k_1-3}(\Lambda_{k_1+1}) \cdots S^{2k_2-7}(\Lambda_{k_2-1}) \cdots \rangle \]
\[ \cdots S^{2k_2-2q-3}(\Lambda_{k_2})S^{2k_2-2q-1}(\Lambda_{k_2+1}) \cdots S^{2p-2q-3}(\Lambda_p)g^{\frac{p-q+1}{2}} \]
\[ = \langle \lambda, \Lambda_pS^2(\Lambda_{p-1}) \cdots S^{2k_1-4}(\Lambda_{p+2-k_1})S^{2k_1-4}(\Lambda_{p+1-k_1})S^{2k_1-2}(\Lambda_{p-k_1}) \cdots S^{2k_2-6}(\Lambda_{p-k_2+1}) \cdots \rangle \]
\[ \cdots S^{2k_2-2q-2}(\Lambda_{p+1-k_2})S^{2k_2-2q}(\Lambda_{p-k_2}) \cdots S^{2p-2q-2}(\Lambda_1)g^{\frac{p-q+1}{2}} \]
\[ = \langle \lambda, S^{-2p+2q+2}(\Lambda_p)S^{-2p+2q+4}(\Lambda_{p-1}) \cdots S^{-2p+2q+2k_1-2}(\Lambda_{p+2-k_1}) \cdots S^{-2p+2q+2k_2-4}(\Lambda_{p-k_2+1}) \cdots S^{-2p+2k_2}(\Lambda_{p+2-k_2})S^{-2p+2k_2}(\Lambda_{p+1-k_2}) \cdots S^{-2}(\Lambda_2)A_1g^{\frac{p-q+1}{2}} \rangle \]
\[ = \langle \lambda, S^{-2p+2q+2}(\Lambda_p)S^{-2p+2q+4}(\Lambda_{p-1}) \cdots S^{-2p+2q+2k_1-2}(\Lambda_{p+2-k_1}) \cdots S^{-2p+2q+2k_2-4}(\Lambda_{p-k_2+1}) \cdots S^{-2p+2k_2}(\Lambda_{p+2-k_2})S^{-2p+2k_2}(\Lambda_{p+1-k_2}) \cdots S^{-2}(\Lambda_2)A_1g^{\frac{p-q+1}{2}} \rangle \]
\[ = \langle \lambda, S^{-2p+2q+2}(\Lambda_p)S^{-2p+2q+4}(\Lambda_{p-1}) \cdots S^{-2p+2q+2k_1-2}(\Lambda_{p+2-k_1}) \cdots S^{-2p+2q+2k_2-4}(\Lambda_{p-k_2+1}) \cdots S^{-2p+2k_2}(\Lambda_{p+2-k_2})S^{-2p+2k_2}(\Lambda_{p+1-k_2}) \cdots S^{-2}(\Lambda_2)A_1g^{\frac{p-q+1}{2}} \rangle \]

Here we have used that \( \sum S^{-1}(\Lambda(p)) \otimes \cdots \otimes S^{-1}(\Lambda(p+1-j)) \otimes \cdots \otimes S^{-1}(\Lambda(1)) = \sum \Lambda(1) \otimes \cdots \otimes \Lambda(j) \otimes \cdots \otimes \Lambda(p) \) and \( k_j + k_{q+1-j} = p + 2 \) for \( j = 1, \ldots, q. \)

3.3. \( Z_{Henn}(L(p,q)\#L(p,q), H) \). We use the chain-mail link \( L \) in Fig. 5 to calculate the Hennings invariant for \( L(p,q)\#L(p,q) \). First note that, the signature \( \sigma(L) \) of the framing matrix of the chain-mail link is zero, so the normalization factor
\[ [\lambda(\theta)\lambda(\theta^{-1})]\frac{c(L)}{\sigma(L)}[\lambda(\theta)/\lambda(\theta^{-1})]\frac{c(L)}{\sigma(L)} = 1 \]
because of the fact that \( \lambda(\theta)\lambda(\theta^{-1}) = 1 \) for a factorizable ribbon Hopf algebra (see [CW2]). It is sufficient to find the link invariant \( TR(L, H) \). First, the contribution of the lower circle \( c_l \) to the Hennings invariant is equivalent to decorate the upper circle \( c_u \) with cointegrals. That is
Lemma 4.

\[
\sum \langle \lambda_1, t_{2p} t_{2p-1} \cdots t_{p+2} t_{p+1} s_p s_{p-1} \cdots s_2 s_1 \rangle s_{2p} t_1 \otimes s_{2p-1} t_2 \otimes \cdots \otimes s_{p+2} t_{p-1} \otimes s_{p+1} t_p 
\]

\[
= \sum \langle \lambda_1, t_{2p} \rangle \langle \lambda_2, s_1 \rangle s_{2p} t_1 \otimes \langle \lambda_2, t_{2p-1} \rangle \langle \lambda_2, s_2 \rangle s_{2p-1} t_2 \otimes \cdots \otimes \langle \lambda_{p-1}, t_{p+2} \rangle \langle \lambda_{p+1}, s_{p-1} \rangle s_{p+2} t_{p-1} \otimes \langle \lambda_{p-1}, t_{p+1} s_p \rangle s_{p+1} t_p 
\]

\[
= f_{R'}(\lambda_1) f_R(\lambda_2) \otimes f_{R'}(\lambda_2) f_R(\lambda_2-1) \otimes \cdots \otimes f_{R'}(\lambda_{p-1}) f_R(\lambda_{p-1}) \otimes f_{R'} f_R(\lambda_p) 
\]

\[
= \sum \Lambda_1 \otimes \cdots \otimes \Lambda_p 
\]

The next step is to resolve the crossings where the upper circle \(c_u\) crosses itself. A typical crossing in the chain-mail is shown in Fig. 21.
Lemma 5.

\[
\begin{array}{c|c|c|c|c|c}
\Lambda(1) & \Lambda(2) & \Lambda(p-1) & \Lambda(p) \\
\hline
s_1t_1 & \cdots & s_{p-1}t_{p-1} & s_p + 1t_p \\
\end{array}
\] = \[
\begin{array}{c|c|c|c|c|c}
\Lambda(1) & \Lambda(2) & \Lambda(p-1) & \Lambda(p) \\
\hline
s_1 \cdot \cdots \cdot s_{2} & t_{1} \\
\end{array}
\]

Proof. The corresponding immersed diagram is

\[
\begin{array}{c|c|c|c|c|c}
\Lambda(1) & \Lambda(2) & \Lambda(p-1) & \Lambda(p) \\
\hline
s_1t_1 & \cdots & s_{p-1}t_{p-1} & s_p + 1t_p \\
\end{array}
\] = \[
\begin{array}{c|c|c|c|c|c}
\Lambda(1) & \Lambda(2) & \Lambda(p-1) & \Lambda(p) \\
\hline
s_1 \cdot \cdots \cdot s_{2} & t_{1} \\
\end{array}
\]

From this immersed diagram, we obtain the following tensor element

\[
\sum (s_p s_{p-1} \cdots s_2 s_1) \otimes \Lambda(1) t_1 \otimes \Lambda(2) t_2 \otimes \cdots \otimes \Lambda(p-1) t_{p-1} \otimes \Lambda(p) t_p
\]

\[
= \sum s \otimes \Lambda(1) t_1 \otimes \Lambda(2) t_2 \otimes \cdots \otimes \Lambda(p-1) t_{p-1} \otimes \Lambda(p) t_p
\]

\[
= \sum \varepsilon(t) s \otimes \Lambda(1) \otimes \Lambda(2) \otimes \cdots \otimes \Lambda(p-1) \otimes \Lambda(p)
\]

\[
= \sum 1 \otimes \Lambda(1) \otimes \Lambda(2) \otimes \cdots \otimes \Lambda(p-1) \otimes \Lambda(p)
\]

\[
= \sum \Lambda(1) \otimes \Lambda(2) \otimes \cdots \otimes \Lambda(p-1) \otimes \Lambda(p)
\]

Here we have used the property of the $R$-matrix that $(\varepsilon \otimes id)(R) = 1$ and $(id \otimes \Delta^{(p-1)})(R) = R_{12} \cdots R_{12}$. This comes from $(id \otimes \Delta)(R) = R_{12}$. 

\[\square\]
3.3.1. $Z_{Henn}(L(p,q) \# L(p,q), H)$ when $q$ is odd. In this case, we push all the labels $\Lambda_{(i)}$’s to $\Lambda_{(N_0)}$ along the upper circle and write down the following equality for $Z_{Henn}(L(p,q) \# L(p,q), H)$.

$$Z_{Henn} = \langle \lambda, S^{-2p+2q}(\Lambda_p) \cdots S^{-2k_q+2q}(\Lambda_{k_q}) \cdots \cdots S^{-2k_3+6}(\Lambda_{k_3-1}) \cdots S^{-2k_2+4}(\Lambda_{k_2}) \cdots \cdots S^{-2k_2+4}(\Lambda_{k_2-1}) \cdots S^{-2}(\Lambda_{k_1+1})\Lambda_{k_1}G^{p-q+1} \rangle$$

$$= \langle \lambda, S^{-2p+2q}(\Lambda_p)G^{-1} \cdots S^{-2k_q+2q}(\Lambda_{k_q})G^{-1} \cdots \cdots S^{-2k_3+6}(\Lambda_{k_3-1})G^{-1} \cdots S^{-2k_2+4}(\Lambda_{k_2})G^{-1} \cdots S^{-2k_2+4}(\Lambda_{k_2-1})G^{-1} \cdots S^{-2}(\Lambda_{k_1+1})G^{-1} \Lambda_{k_1}G^{p-q+1} \rangle$$

$$= \langle \lambda, S^{2q-2}(\Lambda_p) \cdots S^{2q-2}(\Lambda_{k_q}) \cdots \cdots S^2(\Lambda_{k_3-1}) \cdots S^2(\Lambda_{k_2}) \cdots \cdots S^2(\Lambda_{k_2-1}) \cdots S^2(\Lambda_{k_1+1})\Lambda_{k_1}G^{-q+1} \rangle$$

Here we have used the fact that $G$ is grouplike and $G^{-1}S^2(x) = xG^{-1}$ and $\Lambda G^{-1} = \Lambda$. Note that $G^2 = g$. Hence we obtain that when $q$ is odd, $Z_{Kup}(L(p,q), f, H) = Z_{Henn}(L(p,q) \# L(p,q), H)$. 

The last step is to push all the labels $\Lambda_{(i)}$’s in Fig. 22 to where $\Lambda_{(N_1)}$ is located and then do the evaluation by $\lambda$ to get the Hennings invariants. It will be done case by case according to whether $q$ is odd or even.
3.3.2. $Z_{\text{Henn}}(L(p,q)\#\overline{L(p,q)}, H)$ when $q$ is even. Now, we push all the labels $\Lambda_{(i)}$’s to $\Lambda_1$ along the upper circle and obtain

$$Z_{\text{Henn}} = \langle \lambda, S^{-2p+2q+2}(\Lambda_p)S^{-2p+2q+4}(\Lambda_{p-1}) \cdots S^{-2k_q+2q+2}(\Lambda_{k_q}) \rangle$$

$$S^{-2k_q+2q+2}(\Lambda_{k_q-1})S^{-2k_q+2q+4}(\Lambda_{k_q-2}) \cdots S^{-2k_{q-1}+2q}(\Lambda_{k_{q-1}-1})$$

$$\ldots \ldots S^{-2k_1+4}(\Lambda_{k_1})S^{-2k_1+4}(\Lambda_{k_1-1}) \cdots S^{-2}(\Lambda_2)\Lambda_1 G^{p-q+1}$$

Thus $Z_{\text{Kup}}(L(p,q), f, H) = Z_{\text{Henn}}(L(p,q)\#\overline{L(p,q)}, H)$ when $q$ is even.

3.4. Remarks on the general case. For a general oriented 3-manifold $M$, it is natural to divide the conjecture into two cases:

1. If $H_1(M, \mathbb{Q}) \neq 0$, prove that both invariants of $M$ are 0,
2. If $H_1(M, \mathbb{Q}) = 0$, then carry out a similar computation.

Unfortunately, the choice of a suitable framing for the Kuperberg invariant so that a comparison with the Hennings invariant is straightforward is extremely hard to come by. Even for the lens spaces, we are lucky to find the suitable framings. Other choices lead to expressions that are hard to compare the two invariants.

### 4. Hamiltonian Model for Kuperberg Invariant

It is said that R. Bott once asked why there were differential forms. While we cannot answer this deep question, we point out one possibly related question: why are there electrons? The reason that the two questions might be related is that the description of electrons in the second quantized language uses the so-called Grassmann numbers (in physics jargon), which are just differential 1-forms. By the De Rham theory, closed differential forms represent cohomology classes, so fermions are aware of the topology of the space that they occupy. Therefore, TQFTs for fermion systems are different from purely bosonic ones such as double Witten-Chern-Simons theories.

Recently in condensed matter physics, there are great interests to study topological phases of matter which are modeled by TQFTs [NSSFD][Wan]. Conventional phases of matter are described by their symmetries (e.g. 3D crystals are classified by pointed space-groups) and characterized by local order parameters (e.g. magnetization)[Land]. Topological phases of matter lack conventional group symmetries and local order parameters, and posses an elusive order: topological order, which remains invariant under smooth small deformations of Hamiltonians [Wen]. A working definition for a topological phase of matter is a state of matter whose low energy physics is captured by a TQFT. The algebraic data of a bosonic $(2 + 1)$-TQFT is a modular category $[T]$. Therefore, in dimension 2 modular category is an adequate framework for bosonic properties in a topological phase of matter. Topological properties in fractional quantum Hall liquids that originate from the fermionic nature of the underlying electrons are more subtle. This seemingly minor point has implication for an algebraic framework for topological phases of matter of fermion systems due to locality and other considerations.

The fermi statistics between electrons is non-local. Locality requires quantum states on a space manifold $X$ to be determined by local properties. In bosonic $(2 + 1)$-TQFTs, locality is encoded by the gluing axiom and by that disjoint union of spaces corresponding to tensor product of vector spaces. But fermions are always related to each other by exhibiting a minus sign when they are exchanged. Moreover path integral for fermions contains Grassmann numbers, not just complex numbers. Therefore, an extension of the modular category
framework is necessary to axiomatize TQFTs for fermions, which are TQFTs with special kinds of non-local properties. Spin TQFTs capture many properties of fermion systems, but it is not clear that such a framework is adequate.

There are two kinds of bosonic (2+1)-TQFTs: Turaev-Viro type and Reshetikhin-Turaev type \cite{TV1,T}. Turaev-Viro type TQFTs have a lattice Hamiltonian formulation which is understood mathematically \cite{LW}.

A local quantum model for bosons on a graph \(\Gamma\) with vertices (or sites) \(\{i\}\) is defined through its Hilbert space \(V\) and a local bosonic Hamiltonian \(H\). The Hilbert space \(V\) has a tensor structure \(V = \bigotimes V_i\), where \(V_i\) is the local Hilbert space on the vertex \(i\). A \(k\)-local bosonic operator is an operator that acts nontrivially only on \(\leq k\) nearby sites of \(\Gamma\) for some fixed \(k\), e.g., \(k = 2\) or \(3\) for real systems. A \(k\)-local bosonic Hamiltonian \(H\) is a sum of \(k\)-local bosonic Hermitian operators.

A local quantum model for fermions is also defined through its Hilbert space \(V\) and a local fermionic Hamiltonian \(H_f\). Let \(c_i^\alpha\) be a fermionic operator at site \(i\), i.e., operators that satisfy the anticommutation relation for different sites. The Hilbert space \(V\) for fermions is the space spanned by states generated from the action of the fermionic operators and their hermitian conjugates on the groundstate \(|0\rangle\). The anticommutation relation implies that \(V\) also has the tensor form \(V = \bigotimes V_i\), where \(V_i\) is the local Hilbert space on the site \(i\). It follows that the total Hilbert space for a fermion system has the same structure as the total Hilbert space for a boson system. But for a fermion system, each local Hilbert space \(V_i\) is \(\mathbb{Z}_2\)-graded and an ordering of the site labels \(\{i\}\) is required. A \(k\)-local fermionic Hamiltonian \(H_f\) is a sum of terms: \(H = \sum_P O_P\), where \(\sum_P\) sums over \(\leq k\) nearby sites. Each term \(O_P\) is a product of an even number of local fermionic operators and any number of local bosonic Hermitian operators on the few nearby sites \(P\).

Beyond 1D, a local fermionic Hamiltonian \(H_f\) in general is not a local bosonic Hamiltonian. Thus classifying quantum phases of local fermion systems corresponds to classifying quantum phases of some particular kind of non-local boson systems.

In \cite{GWW}, a generalization of the Levin-Wen model to fermion systems is introduced using Grassmann variables.

**Conjecture:** A state sum formulation for the Kuperberg invariant based on non-semisimple super-Hopf algebras exists using the generalization of the Levin-Wen model to fermions introduced in \cite{GWW}.

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E-mail address: liangchang@math.ucsb.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106, U.S.A.

E-mail address: zhenghwa@microsoft.com

MICROSOFT STATION Q, CNSI BLDG RM 2237, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106-6105, U.S.A.