HODGE COHOMOLOGY CRITERIA FOR AFFINE VARIETIES

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Abstract. We give several new criteria for a quasi-projective variety to be affine. In particular, we prove that an algebraic manifold $Y$ with dimension $n$ is affine if and only if $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$, $i > 0$ and $\kappa(D, X) = n$, i.e., there are $n$ algebraically independent nonconstant regular functions on $Y$, where $X$ is the smooth completion of $Y$, $D$ is the effective boundary divisor with support $X - Y$ and $\Omega^j_Y$ is the sheaf of regular $j$-forms on $Y$. This proves Mohan Kumar’s affineness conjecture for algebraic manifolds and gives a partial answer to J.-P. Serre’s Steinness question [36] in algebraic case since the associated analytic space of an affine variety is Stein [15, Chapter VI, Proposition 3.1].

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1. Introduction

J.-P. Serre [36] raised the following question in 1953: If $Y$ is a complex manifold with $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$ and $i > 0$, then what is $Y$? Is $Y$ Stein? Here $\Omega^j_Y$ is the sheaf of holomorphic $j$-forms and the cohomology is Čech cohomology. A complex space $Y$ is Stein if and only if $H^i(Y, G) = 0$ for every analytic coherent sheaf $G$ on $Y$ and all positive integers $i$. For a holomorphic variety $Y$, it is Stein if and only if it is both holomorphically convex and holomorphically separable [13, Page 14]). We say that $Y$ is holomorphically convex if for any discrete sequence $\{y_n\} \subset Y$, there is a holomorphic function $f$ on $Y$ such that the supremum of the set $\{|f(y_n)|\}$ is $\infty$. $Y$ is holomorphically separable if for every pair $x, y \in Y$, $x \neq y$, there is a holomorphic function $f$ on $Y$ such that $f(x) \neq f(y)$.

To investigate Serre’s question, we are interested in the classification of algebraic manifolds (i.e., irreducible smooth algebraic varieties defined over $\mathbb{C}$) with vanishing Hodge cohomology. Since the associated analytic variety of an affine variety is Stein [15, Chapter VI, Proposition 3.1], it is natural to ask the algebraic analogue of Serre’s question: if $Y$ is a smooth quasi-projective variety with $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$ and $i > 0$ (from now on $\Omega^j_Y$ is the sheaf of regular $j$-forms and the cohomology is still Čech cohomology), then is $Y$ affine? The answer is no even in surface case [24]. There are three types of surfaces and two of them are not affine since they have no nonconstant regular functions [24, Lemma 1.8]. Curve case is trivial since $Y$ is not complete by Serre duality and any noncomplete curve...
is affine [15, Chapter II, Proposition 4.1]. We have classified threefolds with vanishing Hodge cohomology in our previous papers [42, 43, 44, 45]. It again shows that $Y$ may not be affine.

It is obvious that $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$ and $i > 0$ is a necessary condition for $Y$ being affine since $\Omega^j_Y$ is coherent [14, Chapter III, Section 5]. Conversely, since the vanishing Hodge cohomology is not a sufficient condition, a further question is: What condition should we add such that $Y$ is affine? This question is very interesting on its own because among algebraic varieties, affine varieties are basic, natural and important. To study a nonaffine variety, we can cover it by affine varieties and examine the local data then glue them together to get global information by cohomology.

Now we translate our question to a purely algebraic geometry question: When is a quasi-projective variety (or a scheme) affine? There are many criteria for affineness. Here we only mention what we know. There may be other results we are not aware of.

Serre’s well-known criterion says that a variety $Y$ is affine if and only if $H^i(Y, F) = 0$ for every coherent sheaf $F$ on $Y$ and all positive integers $i$ [15, Chapter II, Section 1, Theorem 1.1]. Goodman and Hartshorne proved that $Y$ is affine if and only if $Y$ contains no complete curves and the dimension $h^1(Y, F)$ of the linear space $H^1(Y, F)$ is bounded for all coherent sheaf $F$ on $Y$ [11]. Let $X$ be the completion of $Y$. Goodman also proved that $Y$ is affine if and only if after suitable blowing up the closed subvariety on the boundary $X - Y$, the new boundary $X' - Y$ is the support of an ample divisor, where $X' \to X$ is the blowing up with center in $X - Y$ [10; 15, Chapter 2, Theorem 6.1]. Let $D$ be the effective boundary divisor with support $X - Y$, then $Y$ is affine if $D$ is ample. So if we can show the ampleness of $D$, $Y$ is affine. There are two important criteria for ampleness due to Nakai-Moishezon and Kleiman [22; 23, Chapter 1, Section 1.5]. Another sufficient condition is that if $Y$ contains no complete curves and the linear system $|nD|$ is base point free, then $Y$ is affine [15, Chapter 2, Page 64]. Therefore we can apply base point free theorem if we know the numerical condition of $D$ [32; 23, Chapter 3, Page 75, Theorem 3.3]. Neeman proved that if $Y$ (the associated scheme of $Y$) can be embedded in an affine scheme, then $Y$ is affine if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$ [22].

In higher dimension (at least, in our problem), it is very hard to check the ampleness of a big (even big and nef) divisor $D$ and the base point freeness of the linear system $|nD|$. In [15], only for a threefold and a prime divisor $D$, we show that $|nD|$ is base point free for all sufficiently large $n$.

The theory of $D$-dimension and Iitaka fibration is widely used in the classification of algebraic varieties [17-20, 26, 39]. The notion of $D$-dimension is due to Iitaka [17]: If for all $m > 0$, $H^0(X, \mathcal{O}_X(mD)) = 0$, then the $D$-dimension $\kappa(D, X) = -\infty$. Otherwise,

$$\kappa(D, X) = \text{tr.deg}_C \oplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)) - 1.$$ 

Based on the classification of surfaces [24], for threefolds, Mohan Kumar conjectured that $Y$ is affine if and only if $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$, $i > 0$ and
\( \kappa(D, X) = 3 \), i.e., there are 3 algebraically independent nonconstant regular functions on \( Y \), where \( X \) is the smooth completion of \( Y \) and \( D \) is the boundary divisor with support \( X - Y \). If \( Y \) is a smooth threefold and \( X - Y \) is a smooth projective surface, we have proved this conjecture in [45]. In fact, we can drop the smoothness assumption when \( Y \) is a threefold.

We work over the complex number field \( \mathbb{C} \).

**Theorem 1.1.** If \( Y \) is a quasi-projective threefold, then \( Y \) is affine if and only if \( H^i(Y, \Omega^j_Y) = 0 \) for all \( j \geq 0 \), \( i > 0 \) and \( \kappa(D, X) = 3 \), where \( X \) is a completion of \( Y \) and \( D \) is an effective boundary divisor with support \( X - Y \).

In higher dimension, we have the following conjecture.

**Conjecture 1.2.** An algebraic manifold \( Y \) of dimension \( n \) is affine if and only if \( H^i(Y, \Omega^j_Y) = 0 \) for all \( j \geq 0 \), \( i > 0 \) and \( \kappa(D, X) = n \).

In fact, we can prove a more general theorem since we do not require the smoothness of the variety.

**Definition 1.3.** We say that \( Y \) satisfies the condition A if the following three conditions hold

1. the boundary \( X - Y \) is connected;
2. \( X - Y \) is of pure codimension 1 in \( X \);
3. every closed subvariety \( Z \) of codimension 1 in \( Y \) is affine.

**Theorem 1.4.** If \( Y \) is an irreducible quasi-projective variety with dimension \( n \geq 1 \), then \( Y \) is affine if and only if the following three conditions are satisfied:

1. \( Y \) satisfied condition A;
2. \( \kappa(D, X) \geq 1 \);
3. \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \).

**Theorem 1.5.** If \( Y \) is a smooth quasi-projective variety with dimension \( n \), then \( Y \) is affine if and only if \( H^i(Y, \Omega^j_Y) = 0 \) for all \( j \geq 0 \), \( i > 0 \) and \( \kappa(D, X) = n \).

By the result in [42], if \( Y \) is an algebraic manifold with \( H^i(Y, \Omega^j_Y) = 0 \) for all \( j \geq 0 \), \( i > 0 \) and \( \kappa(D, X) \geq 1 \), then there is a surjective morphism from \( Y \) to a smooth affine curve \( C \) such that a general fibre is smooth and irreducible. In [42], we asked whether \( Y \) is affine if the general fibre is affine. By the above theorem, it is easy to see that the answer is yes.

**Theorem 1.6.** If \( Y \) is a smooth quasi-projective variety with dimension \( n \), then \( Y \) is affine if and only if \( H^i(Y, \Omega^j_Y) = 0 \) for all \( j \geq 0 \), \( i > 0 \), \( \kappa(D, X) \geq 1 \) and the general fibre of \( Y \to C \) is affine.

**Theorem 1.7.** If \( Y \) is a smooth quasi-projective variety with dimension \( n \), then \( Y \) is affine if and only if \( H^i(Y, \Omega^j_Y) = 0 \) for all \( j \geq 0 \), \( i > 0 \), \( \kappa(D, X) \geq 1 \) and every closed subvariety of codimension 1 is affine.

The following corollary is a partial answer to J.-P. Serre’s Steinness question.
Corollary 1.8. If an n-dimensional algebraic manifold Y with $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$ and $i > 0$ has $D$-dimension $n$, then $Y$ is Stein.

Definition 1.9. An algebraic variety $Y$ is regularly separable if for any two distinct points $y_1$ and $y_2$ on $Y$, there is a regular function $f$ on $Y$ such that $f(y_1) \neq f(y_2)$.

Corollary 1.10. If an algebraic manifold $Y$ of dimension $n$ satisfies $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$ and $i > 0$, then the following conditions are equivalent
1. $Y$ is affine;
2. $Y$ is regularly separable;
3. any closed subvariety of codimension 1 in $Y$ is affine;
4. $\kappa(D, X) = n$.

This paper is organized as follows. We will give two preliminary lemmas and review some known results in Section 2 and prove the above theorems in Section 3. The main idea of our proof is to show that $Y$ is regularly separable, i.e., for any two distinct points $y_1$ and $y_2$, there is a regular function $H$ on $Y$ such that $H(y_1) \neq H(y_2)$. Then we have an injective birational morphism from the associated scheme of $Y$ to $\text{Spec}^f(Y, \mathcal{O}_Y)$. Therefore we can apply Zariski’s Main Theorem [27, Chapter III, Section 9] and Neeman’s result [29].

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2. Preliminary Lemmas

The proof of Lemma 2.1 is essentially due to Mohan Kumar [24]. We include a proof here for completeness.

Lemma 2.1. Let $Y$ be a smooth quasi-projective variety with $H^i(Y, \Omega^j_Y) = 0$ for every $j \geq 0$ and $i > 0$. Let $X$ be the projective variety containing $Y$, then $X - Y$ is connected.

Proof. If $X - Y = Z$ is not connected, write $Z = Z_1 + Z_2$, where both $Z_1$ and $Z_2$ are closed in $Z$, nonempty and $Z_1 \cap Z_2 = \emptyset$. Let $\dim Y = n$. We have a long exact sequence of local cohomology [14, Page 212]

$$0 = H^{n-1}(Y, \Omega^n_Y) \longrightarrow H^n(Z, \Omega^n_X) \longrightarrow H^n(X, \Omega^n_X) \longrightarrow H^n(Y, \Omega^n_Y) = 0.$$ 

By Serre duality

$$H^0_Z(X, \Omega^n_X) = H^n(X, \Omega^n_X) = H^n(X, \mathcal{O}_X) = \mathbb{C}.$$ 

But by Mayer-Vietoris sequence [14, Page 212],

$$H^n_Z(X, \Omega^n_X) \cong H^n_{Z_1}(X, \Omega^n_X) \oplus H^n_{Z_2}(X, \Omega^n_X).$$ 

Both summands are at least one dimensional since

$$H^n_{Z_1}(X, \Omega^n_X) \longrightarrow H^n(X, \Omega^n_X) = \mathbb{C} \longrightarrow H^n(X - Z_i, \Omega^n_X) = 0,$$ 

Therefore we can apply Zariski’s Main Theorem [27, Chapter III, Section 9] and Neeman’s result [29].
where the last cohomology vanishes because $X - Z_i$ is not complete [21, 37]. This is a contradiction.

Q.E.D.

**Remark 2.2.** The above proof also works for singular varieties because for singular complete varieties, we have Grothendieck duality [1, Chapter I, 1.3]: $H^n(X, \omega_X) \cong \mathbb{C}$.

Recall an equivalent definition of $D$-dimension. If $h^0(X, \mathcal{O}_X(mD)) > 0$ for some $m \in \mathbb{Z}$ and $X$ is normal, choose a basis $\{f_0, f_1, \cdots, f_n\}$ of the linear space $H^0(X, \mathcal{O}_X(mD))$, it defines a rational map $\Phi_{|mD|}$ from $X$ to the projective space $\mathbb{P}^n$ by sending a point $x$ on $X$ to $(f_0(x), f_1(x), \cdots, f_n(x))$ in $\mathbb{P}^n$. By definition of $D$-dimension [39, Definition 5.1],

$$\kappa(D, X) = \max_m \{\dim(\Phi_{|mD|}(X))\}.$$

**Lemma 2.3.** Let $X, Y$ be as above and $n$ be the dimension of $Y$. Then $Y$ contains no complete curves if $\kappa(D, X) = n$, where $D$ is an effective divisor with support $X - Y$.

**Proof.** We will prove the claim by induction on the dimension of $Y$. The claim is trivial if $Y$ is a curve since $Y$ is not complete by Serre duality. The surface case was proved in [24]. Assume that the claim holds for every $(n - 1)$-dimensional variety with the same property. Suppose now $\dim Y = n$. If $Y$ has a connected complete curve $C$, pick a point $p$ on $C$. Let $Z$ be a smooth prime principal divisor passing through $p$ (for the existence of $Z$, see Step 2 of proof of Theorem 1.5 in next section). Let $f \in H^0(Y, \mathcal{O}_Y)$ be the defining function of $Z$, then we have the following short exact sequence

$$0 \longrightarrow \Omega^j_Y \longrightarrow \Omega^j_Y|_Z \longrightarrow 0,$$

where the first map is defined by $f$. Thus for all $i > 0$ and $j \geq 0$, we have

$$H^i(Z, \Omega^j_Y|_Z) = 0.$$

In particular, $H^i(Z, \mathcal{O}_Z) = 0$. From the exact sequence [14, Chapter II, Theorem 8.17; 12, Page 157],

$$0 \longrightarrow \Omega^{j-1}_Z \longrightarrow \Omega^j_Y|_Z \longrightarrow \Omega^j_Z \longrightarrow 0,$$

we have

$$H^i(Z, \Omega^j_Z) = 0$$

for all $i > 0$ and $j \geq 0$. By the inductive assumption, $Z$ contains no complete curves.

Consider the morphism from $Y$ to $\mathbb{C}$

$$f : Y \longrightarrow \mathbb{C}.$$

We know $p \in Z = f^{-1}(0)$, a prime principal divisor defined by $f$. Since $C$ is connected, $C \subset Z$. This is in contradiction with the fact that $Z$ has no complete curves.

The proof is completed.
Theorem 2.4. [Fujita] Let $D$ be an effective $\mathbb{Q}$-divisor on a normal projective surface $X$. Then there exists a unique decomposition
\[ D = P + N \]

satisfying the following conditions:
1. $N$ is an effective $\mathbb{Q}$-divisor and either $N = 0$ or the intersection matrix of the irreducible components of $N$ is negative definite;
2. $P$ is a nef $\mathbb{Q}$-divisor and the intersection of $P$ with each irreducible component of $N$ is zero.

Lemma 2.5. [Sakai] Let $D = P + N$ be the Zariski decomposition of an effective divisor $D$ on a normal complete surface $X$, then $\kappa(D, X) = 2$ if and only if $P^2 > 0$.

Theorem 2.6. [Iitaka] Let $X$ be a normal projective variety and let $D$ be an effective divisor on $X$. Then there exist two positive numbers $\alpha$ and $\beta$ such that for all sufficiently large $n$ we have
\[ \alpha n^{\kappa(D, X)} \leq h^0(X, \mathcal{O}_X(nD)) \leq \beta n^{\kappa(D, X)}. \]

We say that $D$ is a big divisor if $h^0(X, \mathcal{O}_X(nD)) \geq \alpha n^d$, where $d$ is the dimension of $X$.

A fibre space is a proper surjective morphism $f: V \to W$ between two varieties $V$ and $W$ such that the general fibre is connected. We know the following two ways to calculate the $D$-dimension of a variety.

Theorem 2.7. [Iitaka] Let $f: V \to W$ be a surjective morphism of two varieties and let $D$ be a Cartier divisor on $W$, then we have
\[ \kappa(f^*D, V) = \kappa(D, W). \]

Theorem 2.8. [Iitaka] Let $f: V \to W$ be a fibre space from a complete non-singular variety $V$ to a variety $W$. If $D$ is a divisor on $V$, then there is an open dense subset (in complex topology) $U$ in $W$ such that for every point $u \in U$
\[ \kappa(D, V) \leq \kappa(D_u, V_u) + \dim W, \]

where $V_u = f^{-1}(u)$ and $D_u = D|_{V_u}$.

For the proof of Iitaka’s theorems, see [18, Lecture 3] or [39, Chapter II, Section 5].

We will frequently change the coefficients of the effective divisor $D$ or blow up a closed subvariety in $D$ without changing the $D$-dimension because of the following two properties [17; 39, Chapter II, Section 5].

Let $f: X' \to X$ be a surjective morphism between two complete varieties $X'$ and $X$, let $D$ be a divisor on $X$ and $E$ an effective divisor on $X'$ such that $\text{codim} f(E) \geq 2$, then
\[ \kappa(f^{-1}(D) + E, X') = \kappa(D, X), \]

where $f^{-1}(D)$ is the reduced transform of $D$, defined to be $f^{-1}(D) = \sum D_i$, $D_i$’s are the irreducible components of $D$. The second property of $D$-dimension is
that it does not depend on the coefficients of $D$ under a mild condition which is true in our case since we always choose effective boundary divisor $D$ with normal crossings. Let $D_1, D_2, \cdots, D_n$ be any divisor on $X$ such that for every $i$, $0 \leq i \leq n$, $\kappa(D_i, X) \geq 0$, then for integers $p_1 > 0, \cdots, p_n > 0$, we have [20, Section 5]
\[
\kappa(D_1 + \cdots + D_n, X) = \kappa(p_1 D_1 + \cdots + p_n D_n, X).
\]
In particular, if $D_i$’s are irreducible components of $D$ and $D$ is effective, then we can change its coefficients and do not change the $D$-dimension.

For a projective manifold $M$, let $L$ be a line bundle on $M$, then there is a Cartier divisor $D$ determined by $L$. We define $\kappa(L, M) = \kappa(D, M)$.

**Lemma 2.9 (Fujita).** Let $M$ and $S$ be two projective manifolds. Let $\pi : M \to S$ be a fibre space and let $L$ and $H$ be line bundles on $M$ and $S$ respectively. Suppose that $\kappa(H, S) = \dim S$ and that $\kappa(aL - b\pi^*(H)) \geq 0$ for certain positive integers $a, b$. Then $\kappa(L, M) = \kappa(L|_F, F) + \kappa(H, S)$ for a general fibre $F$ of $\pi$.

**Lemma 2.10.** Let $X$ be normal proper over an algebraically closed field $k$. If there is an $m_0 > 0$ such that for all $m > m_0$, $h^0(X, \mathcal{O}_X(mD)) > 0$, then
\[
\mathbb{C}(\Phi|_{mD})(X) = Q((X, D)).
\]
In particular, if $\kappa(D, X) = \dim X$, then $\Phi|_{mD}$ is birational for all $m \gg 0$.

There are many versions of Zariski’s Main Theorems. The proof of the following Zariski’s Main Theorem can be found in [27, Chapter III, Section 9].

**Theorem 2.11. [Zariski’s Main Theorem]** Let $X$ be a normal variety over a field $k$ and let $f : X' \to X$ be a birational morphism with finite fibre from a variety $X'$ to $X$. Then $f$ is an isomorphism of $X'$ with an open subset $U \subset X$.

**Theorem 2.12. [Neeman]** Let $X = \text{Spec} A$ be a scheme, $U \subset X$ a quasi-compact Zariski open subset. Then $U$ is affine if and only if $H^i(U, \mathcal{O}_U) = 0$ for $i \geq 1$.

3. Proof of the Theorems

Recall our notation: $Y$ is an open subset of a projective variety $X$ and $D$ is the effective boundary divisor with support $X - Y$. We may assume that the boundary divisor $D$ has simple normal crossings by further blowing up suitable closed subvariety of $X - Y$. 
Proof of Theorem 1.1. One direction is trivial. If $Y$ is affine, then $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$, $i > 0$ since $\Omega^j_Y$ is a coherent sheaf [14, Chapter 3, Theorem 3.7]; the affineness of $Y$ also implies that $D$ is big since after further blowing up the boundary $X - Y$, it is the support of an ample divisor [15, Chapter 2, Section 6, Theorem 6.1]. Therefore $\kappa(D, X) = 3$.

Conversely, we need to prove that $Y$ is affine if $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$, $i > 0$ and $\kappa(D, X) = 3$. It is sufficient to prove the theorem for normal varieties. The reason is the following.

By Chevally’s theorem [14, Page 222; 15, Chapter 2, Page 63], $Y$ is affine if and only if its normalization $Y'$ is affine since we have a finite morphism from $Y'$ to $Y$. Because a finite morphism is an affine morphism [14, Page 128], $Y'$ also satisfies the vanishing cohomology $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$ and $i > 0$ [14, page 222]. By the definition and property of $D$-dimension, the new triple $(Y', X', D')$ after the normalization also satisfies the hypothesis of Theorem 1.1 [17]. So we may assume that $Y$ is normal.

We divide our proof into several steps.

Step 1. Every principal divisor $Z = \{ f = 0 \}$ defined by a nonconstant regular function $f \in H^0(Y, \mathcal{O}_Y)$ satisfies $H^i(Z, \Omega^j_Z) = 0$ for all $j \geq 0$, $i > 0$.

Proof. For $j = 0, 1, 2$, the function $f$ gives a short exact sequence

$$0 \longrightarrow \Omega^j_Y \longrightarrow \Omega^j_Y \longrightarrow \Omega^j_Y|_Z \longrightarrow 0,$$

where the first map is defined by $f$. Since $H^i(Y, \Omega^j_Y) = 0$ for every $i > 0$ and $j \geq 0$, $H^i(Z, \Omega^j_Z) = 0$. In particular, $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$. We have an exact sequence [14, Chapter 2, Section 8],

$$0 \longrightarrow A \longrightarrow \Omega^j_Y|_Z \longrightarrow \Omega^j_Z \longrightarrow 0,$$

where $A$ is a coherent sheaf on $Z$. Since $Z$ is not complete, $H^2(Z, A) = 0$ [21]. Thus we have $H^i(Z, \Omega^j_Z) = 0$ for all $j \geq 0$, $i > 0$.

Remark 3.1. If the dimension of $Y$ is higher than 3, in the last exact sequence, in general $H^2(Z, A)$ does not vanish. Only when $Z$ is smooth, for all $i > 0$, $H^i(Z, A) = 0$ if $\dim Y > 3$.

Step 2. Let $\bar{Z} = X_0 \subset X$ be a projective surface in $X$ such that $Z$ is an open subset of $\bar{Z}$. Let $D_0$ be the boundary divisor with support $\bar{Z} - Z$. If $Z$ is a surface with $H^i(Z, \Omega^j_Z) = 0$ for all $j \geq 0$, $i > 0$ and $\kappa(D_0, X_0) = 2$, then $Z$ is affine.

Proof. We proved this result in [44]. We include a proof here for completeness.

By the same proof of Lemma 1.1 and Lemma 1.4 in [24], we know that $Z$ contains no complete curves and the boundary $D_0$ is connected. We will prove that $Z$ is affine if it satisfies the following three conditions

1. $Z$ contains no complete curves;
2. $\bar{Z} - Z$ is connected;
3. $\kappa(D_0, X_0) = 2$. 


The idea of proof is to show that the boundary \( \bar{Z} - Z \) is the support of an ample divisor \( P \) on \( \bar{Z} \).

Again we may assume that \( Z \) is normal. For a normal surface \( X \), the intersection theory is due to Mumford [28]. For any effective divisor on a complete normal surface, we have Zariski decomposition [34].

Write the Zariski decomposition \( D = P + N \), where \( N \) is negative definite, \( P \) is effective and nef and any prime component of \( N \) does not intersect \( P \) \[11, 34\]. We may assume that both \( P \) and \( N \) are integral by multiplying a positive integer to \( D \). Let \( \text{Supp}D = \{D_1, D_2, \ldots, D_n\} = X - Y \). Since \( \kappa(D, X) = 2 \), \( P^2 > 0 \) [3 Corollary 14.18]. First we claim that \( \text{Supp}P = \text{Supp}D = X - Y \).

If \( \text{Supp}P \neq X - Y \), then there is a prime component \( D_1 \) in \( X - Y \) such that \( P \cdot D_1 > 0 \) and \( D_1 \) is not a component of \( P \) since \( X - Y \) is connected. Let

\[
Q = mP + D_1,
\]

where \( m \) is a big positive integer. Then \( Q \) is an effective divisor and \( \text{Supp}Q = \text{Supp}P \cup D_1 \). We may choose \( m \) such that

\[
Q^2 = m^2P^2 + 2mP \cdot D_1 + D_1^2 > 0.
\]

For every prime component \( E \) in \( P \), since \( P \) is nef and \( D_1 \) is not contained in \( \text{Supp}P \), we can choose sufficiently large \( m \) such that

\[
Q \cdot E = mP \cdot E + D_1 \cdot E \geq 0, \quad D_1 \cdot Q = mD_1 \cdot P + D_1^2 > 0.
\]

Thus we get a new effective divisor \( Q \) such that \( Q \) is nef and \( Q^2 > 0 \). We may replace \( P \) by \( Q \) and still call it \( P \). By finitely many such replacements, we can get an effective nef divisor \( P \) such that \( P^2 > 0 \) and \( \text{Supp}P = \text{Supp}D = X - Y \).

We claim that the boundary \( X - Y \) is the support of an ample divisor. In fact, the following three conditions imply the ampleness: (1) \( X - Y \) is connected; (2) \( Y \) contains no complete curves; (3) There is an effective nef divisor \( P \) with \( \text{Supp}P = X - Y \) and \( P^2 > 0 \).

If \( P \) is not ample, then there is an irreducible curve \( C \) in \( X \) such that \( P \cdot C = 0 \) by Nakai-Moisshon’s ampleness criterion [14, Chapter V, Theorem 1.10]. Since \( Y \) has no complete curves, \( C \) must be one of the \( D_i 's \). Rearrange the order, we may assume \( D_i \cdot P = 0 \) for \( i = 1, 2, \ldots, r \) and \( D_j \cdot P > 0 \) for \( j = r + 1, \ldots, n \). Write

\[
P = \sum_{i=1}^{r} a_i D_i + \sum_{j=r+1}^{n} b_j D_j = A + B,
\]

where \( A = \sum_{i=1}^{r} a_i D_i \), \( B = \sum_{j=r+1}^{n} b_j D_j = A + B \). Then for \( i = 1, \ldots, r \),

\[
0 = P \cdot D_i = A \cdot D_i + B \cdot D_i.
\]

Since \( D_i \) is not a component of \( B \) for \( i = 1, \ldots, r \), \( B \cdot D_i \geq 0 \). So \( A \cdot D_i \leq 0 \) for every \( i = 1, \ldots, r \). Thus the intersection matrix \( [D_s \cdot D_t]_{1 \leq s, t \leq r} \) is negative semi-definite \[2\]. Since \( A \cup B = X - Y \) is connected, there is at least one component of \( A \), say, \( D_{i_0} \), such that \( D_{i_0} \cdot B > 0 \). Hence \( D_{i_0} \cdot A < 0 \). This implies that the intersection matrix \( [D_{i_0} \cdot D_i]_{1 \leq s, t \leq r} \) is negative definite \[2\]. Therefore there is an effective divisor \( E = \sum_{i=1}^{r} \alpha_i D_i \) such that \( E \cdot D_i < 0 \) for all \( i = 1, \ldots, r \) \[2\].
So there are positive numbers $\alpha_i$, $i = 1, \ldots, r$, such that for every $i$, $E \cdot D_i < 0$, where $E = \sum_{i=1}^{r} \alpha_i D_i$. Let $P_1 = mP - E$, $m \gg 0$, then $P_1^2 > 0$, $P_1$ is nef and if $1 \leq i \leq r$, 

$$P_1 \cdot D_i = -E \cdot D_i > 0.$$ 

If $r + 1 \leq j \leq n$, then choose sufficiently large $m$ such that 

$$P_1 \cdot D_j = mP \cdot D_j - E \cdot D_j > 0.$$ 

Thus $P_1$ is an effective ample divisor with support $X - Y$. Replace $P$ by $P_1$, we have shown that $X - Y$ is the support of an ample divisor $P$. Therefore $Y$ is an affine surface.

**Step 3.** An irreducible principal divisor $Z = \{f = 0, f \in H^0(Y, \mathcal{O}_Y)\}$ is affine.

*Proof.* In Step 1, we showed that $H^1(Z, \mathcal{O}_Z^1) = 0$ for all $j \geq 0$, $i > 0$. We will prove $\kappa(D_0, X_0) = 2$ then apply Step 2, where $\bar{Z}$ is the closure of $Z$ is $X$, $X_0 = \bar{Z} \subset X$, supp$D = \bar{Z} - Z$.

By [32], the regular function $f$ determines the following commutative diagram

$$
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow{f|_Y} & & \downarrow{f} \\
\bar{C} & \rightarrow & \bar{C}
\end{array}
$$

where $C$ is a smooth affine curve embedded in a smooth projective curve $\bar{C}$, and $f$ is proper and surjective, every fibre of $f$ over $\bar{C}$ is connected, general fibre is smooth. Also general fibre of $f|_Y$ is connected and smooth. In particular, $Z = \{f = 0\}$ is one fibre in $Y$. Let $Z = f^{-1}(0)$ be the inverse image of $0$ in $X$, then $Z = Z \cap Y$.

Let $\pi : X' \rightarrow X$ be the proper surjective morphism from a smooth projective variety $X'$ to $X$ with connected fibres ($\pi$ is birational and generically finite), then we have a new fibre space $f' : X' \rightarrow \bar{C}$ with the same property as the original fibre space $f : X \rightarrow \bar{C}$. By [39, Theorem 5.1],

$$\kappa(\pi^*D, X') = \kappa(D, X) = 3.$$ 

By [39, Theorem 5.11], for a general fibre $X'_t = f'^{-1}(t)$ in $X'$, $t \in C$, we have 

$$3 \leq \kappa(f'^*D, X') \leq \kappa(f'^*D|_{X'_t}, X'_t) + 1,$$

where $X'_t = f'^{-1}(t)$ is the the general fibre in $X'$. So $\kappa(f'^*D|_{X'_t}, X'_t) = 2$. Apply [39, Theorem 5.13] to the fibre $X'_t$, we have

$$\kappa(D_1, X_t) = \kappa(f'^*D|_{X'_t}, X'_t) = 2.$$ 

By upper semi-continuity theorem [14, Chapter 3, Section 12] or [39, Chapter 1], $\kappa(D_0, X_0) = 2$. By Step 2, $Z$ is affine.

**Step 4.** For every point $y \in Y$, there is a principal divisor $Z$ passing through $y$ and the closure $\bar{Z}$ of $Z$ in $X$ is connected.

*Proof.* The following construction of fibre space is due to Ueno [40, Page 46]. Since $\kappa(D, X) = 3$, let $\Phi_{nD}$ be the rational map from $X$ to $\mathbb{P}^N$ defined by $\{\phi_0, \phi_1, \ldots, \phi_N\}$, a basis of $H^0(X, \mathcal{O}_X(nD))$, where $N = \dim|nD|$ and $\dim(\text{im}(\Phi_{nD})) \geq$
2. Choose three hyperplane divisors $H_a$, $H_b$, and $H_c$ such that the rational functions $\eta_1$ and $\eta_2$ on $X$ induced by rational functions

$$\frac{a_0X_0 + a_1X_1 + \cdots + a_N X_N}{b_0X_0 + b_1X_1 + \cdots + b_N X_N}$$

and

$$\frac{c_0X_0 + c_1X_1 + \cdots + c_N X_N}{b_0X_0 + b_1X_1 + \cdots + b_N X_N}$$
on $\mathbb{P}^N$ are algebraically independent, where $a = (a_0, a_1, \ldots, a_N)$, $b = (b_0, b_1, \ldots, b_N)$, $c = (c_0, c_1, \ldots, c_N)$, $a, b, c \in \mathbb{C}^{N+1}$. By the rational map $\Phi_{\eta D}$, we can consider that $\eta_1$ and $\eta_2$ are elements of $\mathbb{C}(X)$. By Zariski’s lemma [16, Chapter X, Section 13, Theorem 1, Page 78], there exists a constant $d$ such that the field $\mathbb{C}(\eta_1 + d\eta_2)$ is algebraically closed in $\mathbb{C}(X)$. Define a rational map $f$ from $X$ to $\mathbb{P}^1$ by sending points $x$ in $X$ to $(1, \eta_1(x) + d\eta_2(x))$ in $\mathbb{P}^1$. We can choose $\eta_1$ and $\eta_2$, such that $\eta_1 + d\eta_2$ only has poles in $D$, that is, when restricted to $Y$, $f$ is a morphism. By [39, Corollary 1.10], we have diagram

$$
\begin{array}{ccc}
Y & \hookrightarrow & X \\
\downarrow f_Y & & \downarrow f \\
C & \hookrightarrow & \mathbb{P}^1,
\end{array}
$$

where $f$ is proper and surjective, every fibre of $f$ over $C$ is connected, general fibre is smooth. Also general fibre of $f|_Y$ is connected and smooth.

For a point $y \in Y$, let $f(y) = a$, then $y \in S_a = f^{-1}(a) \cap Y$. By [42], $S_a$ satisfies $H^i(S_a, \Omega^j_{S_a}) = 0$ for all $i > 0$ and $j \geq 0$. So $S_a$ contains no complete curves. Let $X_a = f^{-1}(a)$, then the boundary $X_a - S_a$ is connected. By the same argument as in the Step 3, we claim that $\kappa(D_a, X_a) = 2$, where $D_a$ is the effective boundary divisor supported in $X_a - S_a$. By Step 2, $S_a$ is affine. Let $Z = S_a$, we are done.

**Step 5.** There is an injective morphism from the associate scheme of $Y$ to $\text{Spec} \Gamma(Y, \mathcal{O}_Y)$.

**Proof.** First let $P$ be a point on the quasi-projective variety $Y$, then it is a closed point on the associated scheme (we still call it $Y$) of $Y$. Let $\mathcal{M}_P$ be the maximal ideal of the local ring $\mathcal{O}_P$. Let $n = \dim_{\mathbb{C}} \mathcal{M}_P/\mathcal{M}_P^2$, then there are $n$ regular functions $f_1, \ldots, f_n$ on an open subset containing $P$ such that

$$\mathcal{M}_P = (f_1, \ldots, f_n).$$

Since $D$ is big, we claim that each $f_i$ can be extended to a regular function on $Y$ (See Step 1, proof of Theorem 1.5). Then these $n$ regular functions give a maximal ideal of $\Gamma(Y, \mathcal{O}_Y)$.

For an irreducible closed subset $A$ of $Y$, let $f_1, \ldots, f_r$ be the local defining functions of $A$, then we may assume that all of them are regular on $Y$. So the ideal $I = (f_1, \ldots, f_n)$ is a prime ideal of $\Gamma(Y, \mathcal{O}_Y)$. Naturally we can define a map from $Y$ to $\text{Spec} \Gamma(Y, \mathcal{O}_Y)$ by sending a closed point to the corresponding maximal ideal in $\Gamma(Y, \mathcal{O}_Y)$ and a nonclosed point on the scheme $Y'$ to the associated prime ideal in $\Gamma(Y, \mathcal{O}_Y)$. Let $g$ be the morphism defined by this map. We will prove its injectivity.
g is injective if and only if for any two distinct points $y_1$ and $y_2$ in $Y$, there is a regular function $f$ on $Y$ separating these two points, i.e., $f(y_1) \neq f(y_2)$. To see this, let $A$ and $B$ be two distinct irreducible closed subsets of $Y$ (as a quasi-projective variety). There is a point $p$ in one of them, say $A$, such that $p$ is not a point of $B$. Then there is a regular function $f$ on $Y$ such that $f(y_1) = 0$ but $f(y_2) = 1$. Therefore $f$ is one of the defining functions of $A$ but not $B$. Write the prime ideals defining $A$ and $B$ as follows

$$g(A) = (f, f_1, \ldots, f_m), \quad g(B) = (g_1, \ldots, g_n).$$

Then $f \notin g(B)$. So the images $g(A) \neq g(B)$. This shows the injectivity of $g$.

Let $y_1$ and $y_2$ be two distinct points in $Y$ and consider their images in the fibre space constructed in Step 4

$$Y \hookrightarrow X \xrightarrow{f} \mathbb{P}^1.$$

If $f(y_1) \neq f(y_2)$, then $g(y_1) \neq g(y_2)$. Assume $f(y_1) = f(y_2) = a \in C = f(Y) \subset \mathbb{C}$. Since $S_a = f^{-1}(a) \cap Y$ is affine, there is a regular function $r$ on $S_a$ such that $r(y_1) \neq r(y_2)$.

From the short exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{S_a} \longrightarrow 0,$$

where the first map is defined by $f - a$, we have surjective map from $H^0(Y, \mathcal{O}_Y)$ to $H^0(Z, \mathcal{O}_Z)$ since $H^1(Y, \mathcal{O}_Y) = 0$. Lift $r$ to a regular function $R$ on $Y$, we find a function separating $y_1$ and $y_2$. So $g$ is injective.

**Remark 3.2.** We may use the following alternating construction of the morphism $g$ from $Y$ to $\text{Spec} \Gamma(Y, \mathcal{O}_Y)$ as follows.

Since $\kappa(D, X) = 3$, we have a dominant morphism $h$ from $Y$ to $\mathbb{A}^d_C$ defined by three algebraically independent nonconstant functions on $Y$. Let $Z$ be the normalization of $\mathbb{A}^3$ in $Y$, then we have a morphism $g$ from $Y$ to $Z = \text{Spec} \Gamma(Y, \mathcal{O}_Y)$ and $g$ is birational since $Y$ and $Z$ have the same function field.

**Step 6.** $Y$ is affine.

*Proof.* The above morphism $g$ is birational since $Y$ and $\text{Spec} \Gamma(Y, \mathcal{O}_Y)$ have the same function field. By Zariski’s Main Theorem [27, Chapter 3, Section 9], $g$ is an open immersion from $Y$ to $\text{Spec} \Gamma(Y, \mathcal{O}_Y)$. By Neeman’s theorem [29], $Y$ is affine since $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. In fact, as a scheme, $Y = \text{Spec} \Gamma(Y, \mathcal{O}_Y)$ since they have the same coordinate ring.

Q.E.D.

**Proof of Theorem 1.4.** If $Y$ is an affine variety with dimension $d$, then it satisfies the following four conditions [15, Chapter 2, Section 1, 3, 6]:

1. $Y$ satisfied condition $A$;
2. $\kappa(D, X) \geq 1$;
3. $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.
Suppose now that the quasi-projective variety $Y$ satisfies the above three conditions, we will prove that it is affine. We will use the induction on the dimension of $Y$. If $Y$ is a curve, it is trivial [15, Chapter 2, Section 4, Proposition 4.1]. If $Y$ is a surface, the proof is the same as in the proof of Theorem 1.1, Step 2. Assume that the claim holds for every $d - 1$ dimensional variety. Let $\dim Y = d$. We may assume that $Y$ is normal [14, Page 128, Page 222].

**Step 1.** $\kappa(D, X) = d.$

*Proof.* Since $\kappa(D, X) \geq 1$, we can construct a fibre space $f : X \to \bar{C}$, where $\bar{C}$ is a smooth projective curve. We also have the following commutative diagram

\[
\begin{array}{ccc}
Y & \hookrightarrow & X \\
\downarrow f|_Y & & \downarrow f \\
C & \hookrightarrow & \bar{C}, \\
\end{array}
\]

where $C = f(Y)$ is a smooth affine curve, $f$ is proper and surjective, every fibre of $f$ over $C$ is connected.

We will compute the $D$-dimension of $X$ by Fujita’s formula. For the condition in Fujita’s formula, see the proof of Theorem 1.6.

Let $\pi : X' \to X$ be a proper surjective morphism such that $X'$ is a smooth projective variety and every fibre is connected. In fact, $\pi$ is birational and generically finite. Then we have [39, Chapter II, Theorem 5.13]

\[\kappa(\pi^*D, X') = \kappa(D, X).\]

Consider a general fibre $Y_t = f^{-1}(t) \cap Y$, $t \in C \subset \bar{C}$. By condition A, $Y_t$ is affine. Let $X_t = f^{-1}(t)$ and $D_t$ be an effective boundary divisor with support $X_t - Y_t$, then $\kappa(D_t, X_t) = d - 1$. Pull this divisor $D_t$ back to $X'_t = \pi^{-1}(X_t)$, we have [39, Chapter II, Theorem 5.13]

\[\kappa(\pi^*D_t, X'_t) = \kappa(D_t, X_t) = d - 1.\]

Consider the new fibre space $X' \to \bar{C}$, by Fujita’s formula we have

\[\kappa(\pi^*D, X') = \kappa(\pi^*D_t, X'_t) + 1 = d.\]

Therefore $\kappa(D, X) = d$.

**Step 2.** For any point $y$ in $Y$, there is a principal divisor $Z$ passing through $y$.

*Proof.* We may assume $d \geq 2$. Consider Ueno’s fibre space [39, Page 46] in Step 4, proof of Theorem 1.1

\[
\begin{array}{ccc}
Y & \hookrightarrow & X \\
\downarrow f|_Y & & \downarrow f \\
C & \hookrightarrow & \mathbb{P}^1, \\
\end{array}
\]

We know that every fibre of $f$ is connected. Let $f(y) = a$, then $a \in f^{-1}(a) = X_t$. Let $Z = Y_t = X_t \cap Y$, then $Z = \{f - a = 0, f \in H^0(Y, \mathcal{O}_Y)\}$ is a principal divisor passing through $y$.

By condition A, $Z$ is affine.

**Step 3.** $Y$ contains no complete curves.
Proof. If $Y$ has a complete curve $F$, we may assume that it is connected. Let $p \in F$ be a point on $F$. Let $f(p) = a$, where $f$ is the morphism in Step 1. Then $a$ is a point on $C = f(Y)$. Let $X_a = f^{-1}(a), Y_a = X_a \cap Y$, then $p$ is a point on the open fibre $Y_a$. Since $F$ is connected and $X_a$ is also connected, $F \subset Y_a$. But $Y_a$ is a closed subvariety of $Y$, so $Y_a$ is affine by condition A. This is a contradiction since an affine variety does not have any complete curve. Thus $Y$ contains no complete curves.

Step 4. There is an injective morphism $g$ from $Y$ to $\text{Spec}\Gamma(Y, \mathcal{O}_Y)$.

Proof. We again define the morphism from $Y$ to $\text{Spec}\Gamma(Y, \mathcal{O}_Y)$ as in the Step 5, proof of Theorem 1.1. We send a closed irreducible subvariety on $Y$ (as a quasi-projective variety) to its defining prime ideal. We know this morphism $g$ is injective if and only if for any two distinct points $y_1$ and $y_2$ on $Y$, there is a regular function $H$ on $Y$ such that $H(y_1) \neq H(y_2)$. Let $Z$ be the principal divisor in Step 2 passing though $y_1$ and $f \in \Gamma(Y, \mathcal{O}_Y)$ be its defining function. If $f(y_2) \neq f(y_1)$, then we are done. Suppose $f(y_2) = f(y_1) = 0$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

we can lift any regular function from $Z$ to $Y$. In particular, we can choose $h \in \Gamma(Z, \mathcal{O}_Z)$ such that $h(y_1) = 0$ but $h(y_2) = 1$. Thus there is a regular function $H$ on $Y$ such that $H(y_1) \neq H(y_2)$.

Step 5. $Y$ is affine.

Proof. By Zariski’s Main Theorem, $Y$ is isomorphic to an open subset of $\text{Spec}\Gamma(Y, \mathcal{O}_Y)$. By Neeman’s theorem, $Y$ is affine.

Q.E.D.

Proof of Theorem 1.5. If $Y$ is affine, then $H^i(Y, \mathcal{O}_Y^1) = 0$ for all $j \geq 0$, $i > 0$ and $\kappa(D, X) = n$ [15, Chapter 2, Theorem 1.1, Theorem 6.1]. Let $Y$ be a smooth quasi-projective variety with $H^i(Y, \mathcal{O}_Y^1) = 0$ for all $j \geq 0$, $i > 0$ and $\kappa(D, X) = d$, we need to prove that it is affine. We will use the induction on the dimension of $Y$.

The claim holds for curves [15, Chapter 2, proposition 4.1] and surfaces by the proof of Theorem 1.1, Step 2. Assume that the claim holds for $d-1 > 0$ dimensional smooth varieties. Let $\dim Y = d$.

Step 1. We first claim that for any point $y$ in $Y$, there is a smooth prime principal divisor $Z = \{ f = 0, f \in H^0(Y, \mathcal{O}_Y) \}$ passing through $y$.

Proof. Since $\kappa(D, X) = \dim X = d > 0$, there is $a > 0$ such that for all $m \gg 0$, $h^0(X, \mathcal{O}_X(mD)) \geq am^d$ [39, Theorem 8.1]. Recall our notation in Section 2. Choose a basis $\{ f_0, f_1, \cdots, f_n \}$ of the linear space $H^0(X, \mathcal{O}_X(mD))$, it defines a rational map $\Phi_{mD}$ from $X$ to the projective space $\mathbb{P}^n$ by sending a point $x$ on $X$ to $(f_0(x), f_1(x), \cdots, f_n(x))$ in $\mathbb{P}^n$. By definition of $D$-dimension,

$$\kappa(D, X) = \max_m \{ \dim(\Phi_{mD}(X)) \} = d.$$
Let \( C(X) \) be the function field of \( X \). Let
\[
R(X, D) = \bigoplus_{\gamma \geq 0} H^0(X, \mathcal{O}_X(\gamma D))
\]
be the graded \( C \)-domain and \( R^* \subset R \) the multiplicative subset of all nonzero homogeneous elements. Then the quotient ring \( R^{* - 1}R \) is a graded \( C \)-domain and its degree 0 part \( (R^{* - 1}R)_0 \) is a field denoted by \( Q((R)) \), or \( Q((X, D)) \), i.e.,
\[
Q((R)) = Q((X, D)) = (R^{* - 1}R)_0.
\]
It is easy to see that \( Q((X, D)) = Q((X, aD)) \) and \( \kappa(D, X) = \kappa(aD, X) \) for all positive integer \( a \). Notice that the quotient field \( Q(R) \) of \( R \) is different from \( Q((R)) \).

By [26, Proposition 1.4], the function field of the image \( \Phi_{[mD]}(X) \) is \( Q((X, D)) \), i.e., \( \mathbb{C}(\Phi_{[mD]}(X)) = Q((X, D)) \). By [26, Proposition 1.9], \( \Phi_{[mD]} \) is birational for all \( m \gg 0 \) since \( D \) is effective and \( \kappa(X, D) = \text{dim} X \). So the two function fields \( \mathbb{C}(\Phi_{[mD]}(X)) \) and \( \mathbb{C}(X) \) are isomorphic
\[
\mathbb{C}(\Phi_{[mD]}(X)) = \mathbb{C}(X).
\]
Therefore
\[
\mathbb{C}(X) = Q((X, D)).
\]
This implies that any rational function \( r \) on \( X \) can be written as a quotient \( f/g \) of two regular functions \( f \) and \( g \) on \( Y \) with \( g \neq 0 \), where there is some \( m > 0 \) such that \( f, g \in H^0(X, \mathcal{O}_X(mD)) \). If \( r \) is regular on an open subset \( U \) in \( Y \), then \( r = 0 \) on a closed subset \( A \) of \( U \) gives \( f = 0 \) on \( A \).

For any point \( y \) on \( Y \), there is a smooth prime divisor \( Z \) passing through \( y \). Since \( Y \) is smooth, it is locally factorial. Let \( r \) be the local defining function of \( Z \) near \( y \), then there is a regular function \( f \) on \( Y \) such that \( f \) also defines \( Z \) near \( y \). We define the smooth prime divisor \( Z = \{ f = 0 \} \).

**Step 2.** The above smooth principal divisor \( Z \) satisfies the same vanishing condition \( H^i(Z, \Omega^j_Z) = 0 \) for all \( i > 0 \) and \( j \geq 0 \). Moreover, \( Z \) is affine.

**Proof.** Let \( f \) be the defining function of \( Z \), then
\[
0 \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_Y|_Z \longrightarrow 0,
\]
where the first map is defined by \( f \). So \( H^i(Z, \Omega^j_Y|_Z) = 0 \) since \( H^i(Y, \Omega^j_Y) = 0 \) for all \( j \geq 0 \), \( i > 0 \). In particular, \( H^i(Z, \mathcal{O}_Z) = 0 \) for all \( i > 0 \). Since both \( Y \) and \( Z \) are smooth, We have an exact sequence [14, Chapter 2, Section 8; 12, Page 157],
\[
0 \longrightarrow \Omega^i_Z \longrightarrow \Omega^i_Y|_Z \longrightarrow \Omega^i_Z \longrightarrow 0.
\]
So \( H^i(Z, \Omega^j_Z) = 0 \) for all \( j \geq 0 \), \( i > 0 \).

By Theorem 5.11 and Theorem 5.13, [39]
\[
d = \kappa(D, X) \leq \kappa(D_t, X_t) + 1,
\]
where \( X_t = f^{-1}(t) \) is a general fibre and \( D_t \) is the boundary divisor on \( X_t \) supported in \( X_t - Y \) for \( t \in C \subset \mathbb{C} \). So \( \kappa(D_t, X_t) = d - 1 \). Let \( Z = X_0 \) be the completion of \( Z \) in \( X \), let \( D_0 \) be the effective boundary divisor supported in \( X_0 - Y \), then by Grauert’s upper semi-continuity theorem [39, Chapter 1], \( \kappa(D_0, \bar{Z}) = d - 1 \).
By the inductive assumption, $Z$ is affine.

**Step 3.** $Y$ is affine.

*Proof.* We construct the morphism $g$ from $Y$ to $\text{Spec}(Y, \mathcal{O}_Y)$ as in Step 5, proof of Theorem 1.1. $g$ is birational since $Y$ and $\text{Spec}(Y, \mathcal{O}_Y)$ have the same function field. It is also injective. In fact, for any two distinct points $y_1$ and $y_2$ on $Y$ there is a regular function $R$ on $Y$ such that $R(y_1) \neq R(y_2)$. This is the consequence of the affineness of the smooth principal divisor $Z = \{ f = 0 \}$ passing through $y_1$. More precisely, if $y_2 \notin Z$, then $f(y_1) \neq f(y_2)$, we are done. Assume $y_2 \in Z$, i.e., $f(y_1) = f(y_2) = 0$. By the affineness of $Z$ and the short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0,$$

we can lift a regular function $r$ from $Z$ to $Y$, where $r(y_1) \neq r(y_2)$. So if $R$ is the lifting, then $R(y_1) \neq R(y_2)$. This proves the injectivity of $g$. By Zariski’s Main Theorem and Neeman’s theorem, $Y$ is affine.

Q.E.D.

**Proof of Theorem 1.6.** We will prove that $\kappa(D, X) = \dim X = n$ then apply Theorem 1.5.

Let $f : Y \to C$ be the surjective morphism from $Y$ to the smooth affine curve $C$. Then $f$ gives a rational map from $X$ to the completion $\bar{C}$ of $C$. Resolve the indeterminacy of $f$ on the boundary $X - Y$. We may replace $X$ by its suitable blowing up and assume that $f : X \to \bar{C}$ is surjective and proper morphism. Notice that this procedure does not change $Y$. $Y$ is still an open subset of $X$.

By Stein factorization, we may assume that every fibre is connected and general fibre is smooth. Pick a point $t_1 \in \bar{C} - C$, then $\kappa(t_1, \bar{C}) = 1$. For a general point $t \in C$, by the Riemann-Roch formula, there is a positive integer $m$, such that $h^0(\bar{C}, \mathcal{O}(mt_1 - t)) > 1$. Let $s$ be a nonconstant section of $H^0(\bar{C}, \mathcal{O}(mt_1 - t))$, then

$$\text{divs} + mt_1 - t \geq 0.$$ 

Pull it back to $X$, we have

$$f^*(\text{divs} + mt_1 - t) = \text{div} f^*(s) + mf^*(t_1) - f^*(t) \geq 0.$$ 

Let $D_1 = f^*(t_1)$ and $F = f^*(t)$, then $h^0(X, \mathcal{O}_X(mD_1 - F)) > 0$. Choose an effective divisor $D$ with support $X - Y$ such that $D_1 \leq D$, then we have

$$h^0(X, \mathcal{O}_X(mD - F)) \geq h^0(X, \mathcal{O}_X(mD_1 - F)) > 0.$$ 

Since $F|_Y$ is a smooth affine subvariety of codimension 1 [15, Chapter 2, Proposition 4.1], $\kappa(D|_F, F) = n - 1$. By Fujita’s equality,

$$\kappa(D, X) = \kappa(mD, X) = \kappa(mD|_F, F) + \kappa(t_1, \bar{C}) = n.$$ 

By Theorem 1.5, $Y$ is affine.

Q.E.D.

It is easy to see that Theorem 1.7 follows from Theorem 1.6.
Remark 3.3. In Theorem 1.5, if we drop one of the two conditions in the theorem, the theorem is no longer true.

Example 1. A threefold \( Y \) with \( H^i(Y, \Omega^j_Y) = 0 \) for all \( i > 0 \) and \( j \geq 0 \) but \( \kappa(D, X) = 1 \).

Let \( C \) be an elliptic curve and \( E \) the unique nonsplit extension of \( \mathcal{O}_C \) by itself. Let \( Z = \mathbb{P}_C(E) \) and \( D \) be the canonical section, then \( H^i(S, \Omega^j_S) = 0 \) for all \( i > 0 \) and \( j \geq 0 \), where \( S = Z - D \) \[24\]. Let \( F \) be a smooth affine curve and \( Y = S \times F \), then \( H^i(Y, \Omega^j_Y) = 0 \) by Künneth formula \[35\]. Let \( X \) be the closure of \( Y \) and \( D \) be the effective boundary divisor, then \( \kappa(D, X) = 1 \) \[42\]. So \( Y \) is not affine.

Example 2. A threefold \( Y \) with \( \kappa(D, X) = 3 \) but \( Y \) is not affine.

For instance, remove a hyperplane section \( H \) and a line \( L \) from \( \mathbb{P}^3 \), where \( L \) is not contained in \( H \). Let \( Y = \mathbb{P}^3 - H - L \). Then \( Y \) contains no complete curves. Let \( f : X \to \mathbb{P}^3 \) be the blowing up of \( \mathbb{P}^3 \) along \( L \). Then \( X \) is a smooth projective threefold and \( Y \) is an open subset of \( X \). Let \( D = f^{-1}(H) + E \), where \( E \) is the exceptional divisor. Then by Iitaka’s formula in Section 2, \( \kappa(D, X) = \kappa(H, \mathbb{P}^3) = 3 \). But \( Y \) is not affine since the boundary \( \mathbb{P}^3 - Y \) is not of pure codimension 1 \[15, Chapter 2, Proposition 3.1\].

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