Integrability and Holographic Aspects of
Six-Dimensional $\mathcal{N} = (1, 0)$ Superconformal Field
Theories

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Abstract: In the framework of six-dimensional conformal field theories with $\mathcal{N} = (1, 0)$
supersymmetry we develop the map between the holographic description, the field theoretical
description and the associated Hanany-Witten set-ups. General expressions that calculate
various observables are presented. The study of string solitons singles out a special back-
ground of Massive IIA on which we show (by explicitly finding a Lax pair) that the Neveu-
Schwarz part of the string sigma model is classically integrable. We study the particular dual
conformal field theory and compute some of its observables.

Keywords: Six-dimensional SCFTs, Holography, Integrability, $\lambda$-deformation.
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1 Introduction and General Idea

Quantum field theories are usually thought of in the context of renormalisation group flows from short distances in the UV to long distances in the IR. The endpoints of these flows called fixed points are scale invariant. In \(d\)-dimensional relativistic field theories, it is common to assume that the fixed-point theory is a conformal field theory (CFT), whose spacetime symmetry \(SO(1,d-1)\) is enhanced to the conformal algebra \(SO(2,d)\). Aside from free CFTs, there is compelling evidence for a vast landscape of interacting CFTs in diverse dimensions, with many of these theories being non-Lagrangian (CFTs that do not have a known representation in terms of fields and a Lagrangian). In such cases one typical route to learn about the dynamics of these CFTs is to use algebraic methods. These are 'kinematic' in nature and hence limited for the goal of calculating detailed dynamical information. In this sense, the best way to learn about these field theories is to use Maldacena’s Conjecture [1] and find the holographic dual background to calculate field theory correlators with it.

Geometrical Engineering, that is the interplay between the geometry of the string theory’s extra dimensions, and the resulting low energy field theories on branes probing this geometry, is by now a well-established research tool. In the case of geometries with singularities, such methods have led to the construction and study of conformal field theories in diverse dimensions. Of particular significance for this paper, are the conformal field theories in six dimensions, which resist a Lagrangian description. The key ingredients of these theories are tensionless strings coupled to dynamical tensor modes. Examples of such theories include the ADE \(\mathcal{N} = (2,0)\) theories [2], realised on a stack of M5-branes. Less is known about the \(\mathcal{N} = (1,0)\) theories. These CFTs do not have a weakly coupled UV Lagrangian. However, one can always go to the tensor branch of these theories, which corresponds to giving vacuum expectation values (VEVs) to scalars in tensor multiplets. In such cases one can find an effective Lagrangian description for \(\mathcal{N} = (1,0)\) theories in terms of a weakly coupled quiver gauge theory, where the scalars in the tensor multiplets controlling the coupling constants of the corresponding gauge groups, are promoted to a set of dynamical fields. Moving to the origin of the tensor branch typically leads to a strongly coupled 6d SCFT with \(\mathcal{N} = (1,0)\) supersymmetry. Some of these systems have a realisation in string theory [3],[4]. The description of these CFTs is well advanced, see for example [5]-[24], for a sample of papers that deal with the system from the field theoretical, brane picture, or holographic perspectives.

The general idea of this paper is to develop important new formal tools in the mapping between six-dimensional CFTs with \(\mathcal{N} = (1,0)\) SUSY and their holographic description. In particular, we will discuss aspects of the CFTs that are not suitable to be understood with either Geometrical Engineering or algebraic methods, hence rely heavily on the holographic description.

In more detail, the contents of this work are the following: in Section 2, we summarise the known field theoretical and holographic description, we describe the mapping between these
descriptions and provide new holographic expressions for some of the characteristic quantities that follow from a Hanany-Witten description [25] of the CFTs. For example we give new expressions that calculate the number of branes, linking numbers and entanglement entropy of a given quiver in terms of the holographic description. For the benefit of the readers new to the topic, we present detailed examples of the mapping between the quiver CFT and the holographically dual background. In Section 3, we fully rely on the holographic description and discuss the possibility of finding a particularly special background in Massive IIA on which the string theory is classically integrable. The treatment uses semiclassical string solitons and studies its dynamical evolution. A particular background is then singled out as special, on which the non-integrable characteristics of the soliton are absent. In Section 4, we present the Lax pair from which the equations of motion of a bosonic string on such particular background are derived. We connect this Lax pair with the so called $\lambda$-deformation of a Wess-Zumino-Witten (WZW) model. In Section 5, we study some of the field theoretical observables associated with this special background. These observables can only be accessed through the holographic description and provide a definition of the particular integrable $\mathcal{N} = (1,0)$ six-dimensional CFT. In fact, an alternative to giving the precise colour and flavour groups at the origin of the tensor branch is to display the observables of the conformal field theory. We summarise our findings and conclude in Section 6, where we also lay-out some ideas to work in the future. Our work is complemented by very extensive and dense appendixes for the benefit of the readers willing to work on these topics.

2 Six dimensional SCFTs and their holographic description

Let us start with a summary of six-dimensional $\mathcal{N} = (1,0)$ conformal field theories and their holographic description. It is useful to remind the reader the main issue afflicting higher dimensional ($d > 4$) field theories. Consider a simple interacting field theory in six dimensions with action,

$$ S = \int d^6x \left[ -\frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) \right]. $$

Here $\phi$ represents a real scalar field with classical dimension $[\phi] = m^2$. The potential can be a mass term $V = \frac{m^2}{2} \phi^2$ or more interestingly a classically marginal interaction term, like $V = g\phi^3$, but this would lead to a system without ground state (for $\phi < 0$). On the other hand a potential like $V = \lambda \phi^4$ has a well defined vacuum, but the interaction is irrelevant, hence the theory is not well defined without a UV completion. The Wilsonian logic, according to which we start from a conformal (not necessarily weakly coupled) field theory and deform it by inserting relevant operators into the Lagrangian, flowing to interesting field theories at low energies, does not seem to apply here.

Nevertheless, different string theoretic constructions have suggested that supersymmetric field theories of scalars coupled to gauge fields have an interacting UV fixed point. In fact,
for a Lagrangian like
\[ \mathcal{L} \sim -\frac{1}{2}(\partial_{\mu}\phi)^2 - c \phi F_{\mu\nu}^2 + \text{fermions}, \] (2.1)
when \( \langle \phi \rangle \to 0 \) we are dealing with the strong coupling limit of a gauge field theory (since the scalar \( \phi \) takes the role of the inverse coupling of the gauge theory). The presence of fermions in the supersymmetric theory imply the possible existence of gauge anomalies that need to be cancelled. This cancellation is possible if the scalar \( \phi \) belongs to a tensor multiplet [5], [6] (see below for a description of the relevant multiplets) and a certain tuning between the amount of adjoint and fundamental matter must be imposed.

This picture was realised in brane constructions. The relevant Hanany-Witten set-ups [25] were presented in [7]. The associated field theories preserve eight Poincare supercharges, have \( SO(1,5) \) Lorentz and \( SU(2) \) R-symmetries. In more detail, the field theories with \( \mathcal{N} = (1,0) \) SUSY are constructed in terms of the following multiplets:

- Tensor multiplets with field content \( (B_{\mu\nu}, \lambda_1, \lambda_2, \phi) \). A two form with self-dual curvature \( H_3 = dB_2 \), two fermions and a real scalar.
- Vector multiplets with field content \( (A_\mu, \hat{\lambda}_1, \hat{\lambda}_2) \), a six dimensional vector and two fermions.
- Hypermultiplets with field content \( (\varphi_1, \varphi_2, \psi_1, \psi_2) \), two scalars and two fermions.
- Linear multiplets with field content \( (\vec{\pi}, c, \bar{\xi}) \) an \( SU(2) \) triplet and a singlet, together with a fermion.

The field theories have a 'tensor branch' when the scalar \( \phi \) gets a non-zero VEV. In this case the \( SU(2)_R \) symmetry is preserved. On the other hand when the scalars inside the hyper or the linear multiplet get VEVs, we explore the Higgs branch breaking the R-symmetry. In what follows we will be concerned with the tensor branch only.

To reproduce the Lorentz and R-symmetry mentioned above, the authors of [7] distributed D6, NS5 and D8 branes according to Table 1

|   | \( t \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) | \( x_7 \) | \( x_8 \) | \( x_9 \) |
|---|---|---|---|---|---|---|---|---|---|
| NS5 | • | • | • | • | • | • | • | • | • |
| D6 | • | • | • | • | • | • | • | • | • |
| D8 | • | • | • | • | • | • | • | • | • |

**Table 1:** The generic brane set-ups. All the branes are extended on the Minkowski \( R^{1,5} \) directions. The D6-branes also extend over \( x_6 \) where they have finite size extension between NS5-five branes. The D8-branes also extend along the \( x_7, x_8 \) and \( x_9 \) directions, preserving the \( SO(3)_R \) symmetry.

There are some key-differences with Hanany-Witten set-ups in lower dimensions,
The dimension of the field theory on the NS5-branes is the same as that on the bounded D6-branes. The non-decoupling of the five-branes dynamics adds the dynamical tensor multiplets to the field theories. These are absent in lower dimensional set-ups.

The bending of the NS5-branes due to other \( p \)-branes ending on them leads to a Laplace equation in \( 6-p \) dimensions. In this case as \( p = 6 \), there is no-bending. The field content is always such that anomalies are cancelled, namely

\[
N_{D6,R} + N_{D6,L} + N_{D8} = 2N_{D6,c},
\]

being \( N_{D6,R/L} \) the number of six branes to the right/left of a given stack with \( N_{D6,c} \) branes.

We can consider D2 branes on \((t,x_1, x_6)\) that end on the NS5-branes. These branes represent one dimensional magnetically charged defects identified with the instantonic strings charged under the self-dual \( H_3 \).

When the system is in the tensor branch (the difference between the scalars in different tensor multiplets \( \langle \phi_i - \phi_{i-1} \rangle \) is non-zero) the instantonic strings are massive and the field theory can be described by an anomaly-free quiver. When \( \langle \phi_i - \phi_{i-1} \rangle \to 0 \), the theory is proposed [5] to flow to a strongly coupled six dimensional CFT with \((1,0)\) SUSY. These are the theories that we study in this paper.

### 2.1 Holographic description

Let us now discuss the holographic description of the CFTs that appear when we move to the origin of the tensor branch. This description was developed in a set of papers most notably [18]-[24]. We adopt the notation of [24].

The six dimensional SCFTs have \( SO(2,6) \times SU(2)_R \) bosonic symmetries, see for example [26]. They are realised as the isometries of a Massive Type IIA background of the form,

\[
ds^2 = f_1(z)ds^2_{AdS_7} + f_2(z)dz^2 + f_3(z)\Omega_2^2(\chi, \xi), \\
B_2 = f_4(z)\text{Vol}_{\Omega_2}, \quad F_2 = f_5(z)\text{Vol}_{\Omega_2}, \quad e^\phi = f_6(z), \quad F_0 = F_0(z).
\]

We have defined \( d\Omega_2^2 = d\chi^2 + \sin^2\chi d\xi^2 \) and \( \text{Vol}_{\Omega_2} = \sin\chi d\chi \wedge d\xi \).

If we impose that \( \mathcal{N} = (1,0) \) SUSY is preserved by the background, we need the functions \( f_i(z) \) to satisfy some first order and nonlinear differential equations. These BPS equations are solved if the functions \( f_i(z) \) in eq.(2.3) are all defined in terms of a function \( \alpha(z) \) and its derivatives—see [19]-[24] for the details,

\[
f_1(z) = 8\sqrt{2\pi} \sqrt{-\frac{\alpha}{\alpha'}} , \quad f_2(z) = \sqrt{2\pi} \sqrt{-\frac{\alpha''}{\alpha}} , \quad f_3(z) = \sqrt{2\pi} \sqrt{-\frac{\alpha''}{\alpha}} \left( \frac{\alpha^2}{\alpha'^2 - 2\alpha\alpha''} \right), \\
f_4(z) = \pi \left( -z + \frac{\alpha\alpha'}{\alpha'^2 - 2\alpha\alpha''} \right) , \quad f_5(z) = \left( \frac{\alpha''}{162\pi^2} + \frac{\pi F_0\alpha\alpha'}{\alpha'^2 - 2\alpha\alpha''} \right), \\
f_6(z) = 2\frac{5}{4}\pi \frac{\pi^2 3^4}{\sqrt{\alpha'^2 - 2\alpha\alpha''}} \left( -\frac{\alpha}{\alpha'} \right)^{\frac{3}{2}}.
\]
Where $\alpha(z)$ has to satisfy the differential equation

$$\alpha''' = -162\pi^3 F_0.$$  \hspace{1cm} (2.5)

The function $\alpha(z)$ must be piece-wise continuous, this implies that $F_0$ can be piece-wise constant and discontinuous. The internal space $\mathcal{M}_3 = (z, \Omega_2)$ is a two-sphere 'fibered' over the z-interval. The warp factor $f_3(z)$ must vanish at the beginning and at the end of the z-interval ($z = 0$ and $z = z_f$ by convention), in such a way that the two-sphere shrinks smoothly at those points.

For a piece-wise constant and possibly discontinuous $F_0(z)$, the general solution to eq.(2.5) in each interval of constant $F_0$ is,

$$\alpha(z) = a_0 + a_1 z + \frac{a_2}{2} z^2 - \frac{162\pi^3 F_0}{6} z^3.$$  

As we observed above, the function $\alpha(z)$ is in general piece-wise continuous and generically a polynomial solution like the one above should be proposed for each interval $[z_i, z_{i+1}]$. Imposing that the two-sphere shrinks smoothly at $z = 0$ and $z = z_f$ implies that $\alpha(0) = \alpha(z_f) = 0$. We shall discuss a generic solution below. Before that, let us find general expressions for the brane-charges associated with the backgrounds in eq.(2.3).

### 2.1.1 Page charges

We define the conserved Page charges,

$$Q_{Dp} = \left(\frac{1}{2\pi}\right)^{7-p} g_s(\alpha')^{(7-p)/2} \int F_{8-p} - B_2 \wedge F_{6-p}, \hspace{0.5cm} Q_{NS5} = \frac{1}{4\pi^2 g_s^2 \alpha'} \int H_3.$$  \hspace{1cm} (2.6)

In what follows we set $g_s = \alpha' = 1$. Calculating explicitly for the NS5-brane charge. Using that $\alpha(0) = \alpha(z_f) = 0$ we find,

$$Q_{NS5} = \frac{1}{4\pi^2} \int_{z, \Omega_2} \partial_z f_4 = \frac{1}{\pi} \int_{z=0}^{z=z_f} \partial_z f_4 = \frac{f_4(z_f) - f_4(0)}{\pi} = -z_f.$$  \hspace{1cm} (2.7)

Up to an orientation-related sign, the size of the z-interval equals the number of five branes. Hence we need to choose $z_f$ to be a positive integer. We shall take $Q_{NS5} = z_f = N_5$ in what follows.

Calculating the charge of D6-branes, we find

$$Q_{D6} = \frac{1}{2\pi} \int_{(\lambda, \xi)} F_2 - F_0 B_2 = \left[ \frac{\alpha'' + 162\pi^3 F_0 z}{81\pi^2} \right]_{81\pi^2} = \frac{\alpha'' - z\alpha'''}{81\pi^2}.$$  \hspace{1cm} (2.8)

The charge in eq.(2.8) computes the charge of D6 branes but also includes the charge of D6-brane induced on the D8-branes. To avoid this 'overcounting', we propose a formula

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that calculates only the number of D6 branes in the associated Hanany-Witten set up. Our proposal reads,

\[ N_{D6} = -\frac{1}{81\pi^2} \int_0^{z_f} \alpha''(z) dz. \]  

(2.9)

This can be derived by explicitly performing the integral in any generic \( \alpha''(z) \), observing that it counts the sum of the ranks of the gauge groups.

We also present an expression that calculates the number of D8 branes in any given Hanany-Witten set-up. Our proposed new expression reads,

\[ N_{D8} = \frac{1}{81\pi^2} \left[ \alpha'''(0) - \alpha'''(z_f) \right]. \]  

(2.10)

In other words, the jumps in \( \alpha'''(z) \) across any interval counts D8-branes according to eq.(2.5). Adding these jumps leads to eq.(2.10).

These expressions are analogous to those derived in [27], for the case of Hanany-Witten set-ups associated with four dimensional \( \mathcal{N} = 2 \) SCFTs. In Section 2.2 we test the new expressions in eqs.(2.9),(2.10) on some examples.

### 2.1.2 Linking numbers

One interesting quantity characterising the Hanany-Witten set-ups are the linking numbers. For the case at hand, with D6, D8 and NS5 branes these topological invariants (unchanged under Hanany-Witten moves) are defined for the j-th D8 brane (\( L_j \) is the linking number) and the i-th Neveu-Schwarz five brane (\( K_i \) being the corresponding linking number) by counting the number of the other branes to the left and right. More precisely, we have

\[
L_j = N_{D6}^{\text{right}} - N_{D6}^{\text{left}} + N_{NS}^{\text{left}},
\]

\[
K_i = N_{D6}^{\text{right}} - N_{D6}^{\text{left}} - N_{D8}^{\text{right}}.
\]

(2.11)

They must satisfy a consistency relation

\[
\sum_{j=1}^{N_8} L_j + \sum_{i=1}^{N_5} K_i = 0.
\]

(2.12)

In the Hanany-Witten set-ups that are relevant for the CFTs we study in this paper, all the linking numbers for the different Neveu-Schwarz five-branes are equal. We have found that they can be holographically calculated by very simple expressions. Our proposal is that for these CFTs we calculate the linking numbers as,

\[
K_1 = K_2 = \ldots = K_{N_5} = \frac{1}{81\pi^2} \alpha'''(z_f) \rightarrow \sum_{i=1}^{N_5} K_i = \frac{1}{81\pi^2} \alpha'''(z_f) z_f.
\]

(2.13)

\[
L_i = z_i \rightarrow \sum_{j=1}^{N_8} L_j = -\frac{1}{81\pi^2} \alpha'''(z_f) z_f.
\]

(2.14)
These expressions satisfy eq.(2.12) and are analogous to those presented in the case of four dimensional CFTs with eight supercharges \[27\]. In Section 2.2 we test these expressions in a couple of examples. The reader is invited to apply the expressions of eqs.(2.9),(2.10),(2.13),(2.14) to the examples of the paper \[28\].

2.1.3 Entanglement Entropy

We briefly discuss this interesting observable characterising CFTs. Our treatment is an extension of that presented in \[29\] for backgrounds of the form in eq.(2.3). The entanglement entropy $S_{EE}$ for a rectangular region is calculated by solving a minimisation problem for an eight manifold hanging from infinity in the AdS-radial direction. The two regions of the space are separated by a distance $L_{EE}$ (calculated in terms of the background functions as shown below). A regularisation is needed analogously to what happens when computing Wilson loops, see \[29\] for the details. In particular for the backgrounds of the form in eq.(2.3). The eight surface is parametrised by the coordinates

$$\Sigma_8 = [x_1, x_2, x_3, x_4, x_5, z, \chi, \xi], \quad R = R(x_1).$$

Using Poincaré coordinates for the AdS$_7$ space, the induced metric of the eight-surface is

$$ds^2_{8,ind} = f_1 \left[ R^2 dx_4^2 + dx_1^2 \left( R^2 + \frac{R'^2}{R^2} \right) \right] + f_2 dz^2 + f_3 (d\chi^2 + \sin^2 \chi d\xi^2).$$

The entanglement entropy is

$$S_{EE} = \frac{1}{4G_N} \int d^8 \sigma \ e^{-2\phi} \sqrt{\det g_{8,ind}}, \quad G_N = 8\pi^6 g_s^2 \alpha'^4 = 8\pi^6$$

$$S_{EE} = \frac{128V_4}{6561G_N} \int_0^{z_f} \alpha''(z)\alpha(z) dz \int dx_1 R^5 \sqrt{1 + \frac{R'^2}{R^4}}, \quad V_4 = \int dx_2 dx_3 dx_4 dx_5.$$

Following the formalism of the works \[29\], we find the regularised version of the entanglement entropy, $S_{EE}^{reg}$ and the separation between the regions $L_{EE}$ to be,

$$S_{EE}^{reg} = \frac{V_4}{4G_N} \int_0^{\infty} dy \ \left( \frac{y^8}{\sqrt{y^4 - 1}} - y^3 \right) N R_0^4 = \mu_1 N R_0^4,$$

$$L_{EE} = \left[ 2 \int_1^{\infty} \frac{dy}{\sqrt{y^4 - 1}} \right] \frac{1}{R_0} = \mu_2 R_0,$$

$$S_{EE}^{reg} = N \left( \frac{\mu_1 \mu_2^4}{L^4} \right), \quad N = -\frac{512}{6561} \int_0^{z_f} \alpha(z)\alpha''(z) dz.$$

(2.16)

The factors $\mu_1 \mu_2^4$ are common to all six dimensional conformal field theories. The power $L^{-4}$ is the only possible one given conformality and the dimension of the CFT. All the information about the particular CFT in consideration is in the factor $N \sim \int \alpha \alpha''$. Notice that this factor also appears when computing the central charge of the CFT, see \[28\]. This is not a surprise as both quantities measure the number of degrees of freedom.
2.2 Connecting the holographic background with the CFT

Let us discuss the connection between a quiver field theory, and the geometry in eq.(2.3).

The problem can be organised as follows: first, we consider a non-anomalous quiver with bifundamental matter, gauge and flavour groups satisfying the relation in eq.(2.2). Then, we define the function \( R(z) \), a piecewise continuous linear function such that at \( z = j \) (with \( j \) being a positive integer number) the value \( R(j) = N_j \) is the rank of the \( j \)-th gauge group. It was shown in [24] that this rank-function must be convex to satisfy the anomaly cancellation condition in eq.(2.2).

The link with the holographic description is given by the identification,

\[
R(z) = -\frac{1}{81\pi^2} \alpha''(z). \tag{2.17}
\]

Finally, after this identification, we need to determine the function \( \alpha(z) \) by imposing boundary conditions and continuity of \( \alpha \) and \( \alpha' \).

Working out examples is possibly the clearest way to explain the procedure to the reader not acquainted with this formalism. We first present full details for a simple example and then we consider a more generic situation. The interested reader can consult the examples in Section 2.1.1 of the paper [28] \(^1\).

A simple example

Consider the Hanany-Witten set-up, quiver and Rank function \( R(z) \) in Figures 1-3.

![Figure 1](image)

**Figure 1:** The Hanany-Witten set-up for the field theory. The vertical lines denote individual Neveu-Schwarz branes extended on the \((x_4, x_5)\) space. The horizontal ones D6 branes, that extend on \(x_6\), in between five branes. The crossed-circles represent D8 branes, that extend on the \((x_7, x_8, x_9)\) directions. All the branes share the Minkowski directions. This realises the isometries \( SO(1,5) \times SO(3) \).

\(^1\) In order for the background to capture faithfully the CFT dynamics one should work with long linear quivers and with large ranks. In this sense the examples of eqs.(2.6) and (2.8) of [28] are rigorously trustable. Our examples in this section should be taken as illustrative of the procedure.
Figure 2: The quiver corresponding to the Hanany-Witten set-up above.

Figure 3: The rank function $R(z)$ corresponding to the field theory.

In this example, the rank function and the function $\alpha''(z)$ are given by,

$$R(z) = -\frac{1}{81\pi^2} \alpha''(z) = N \begin{cases} 
  z & 0 \leq z \leq 1 \\
  (z - 1) + 1 & 1 \leq z \leq 2 \\
  (z - 2) + 2 & 2 \leq z \leq 3 \\
  3 & 3 \leq z \leq 4 \\
  3 - 3(z - 4) & 4 \leq z \leq 5.
\end{cases}$$

This implies that the generic function $\alpha(z)$ for this example is,

$$\alpha(z) = -81\pi^2N \begin{cases} 
  a_0 + a_1z + \frac{z^3}{6} & 0 \leq z \leq 1 \\
  b_0 + b_1(z - 1) + \frac{1}{2}(z - 1)^2 + \frac{1}{6}(z - 1)^3 & 1 \leq z \leq 2 \\
  c_0 + c_1(z - 2) + 2\frac{(z-2)^2}{2} + \frac{1}{6}(z - 2)^3 & 2 \leq z \leq 3 \\
  d_0 + d_1(z - 3) + 3\frac{(z-3)^2}{2} & 3 \leq z \leq 4 \\
  p_0 + p_1(z - 4) + 3\frac{(z-4)^2}{2} - 3\frac{(z-4)^3}{6} & 4 \leq z \leq 5.
\end{cases}$$
To determine the ten integration constants, we need to impose:

- That $\alpha(0) = \alpha(5) = 0$. This is to have an internal space that shrinks smoothly at the beginning and end of the $z$-interval. These conditions imply

$$a_0 = 0, \quad p_0 + p_1 + \frac{3}{2} - \frac{3}{6} = 0$$

- That the function $\alpha(z)$ is continuous, this implies the equations,

$$a_1 + \frac{1}{6} = b_0, \quad b_0 + b_1 + \frac{1}{2} + \frac{1}{6} = c_0, \quad c_0 + c_1 + 1 + \frac{1}{6} = d_0, \quad d_0 + d_1 + \frac{3}{2} = p_0.$$  

- That the function $\alpha'(z)$ is continuous. This implies,

$$a_1 + \frac{1}{2} = b_1, \quad b_1 + 1 + \frac{1}{2} = c_1, \quad c_1 + 2 + \frac{1}{2} = d_1, \quad d_1 + 3 = p_1.$$  

Solving these equations we find,

$$a_0 = 0, \quad -5a_1 = 19, \quad -30b_0 = 109, \quad -10b_1 = 33, \quad -15c_0 = 94, \quad -5c_1 = 9,$$

$$-10d_0 = 69, \quad 10d_1 = 7, \quad -10p_0 = 47, \quad 10p_1 = 37.$$  

In this way, we have a well defined function $\alpha(z)$.

We can apply our expressions for the number of NS, D6 and D8 branes and linking numbers. Using eqs.(2.7), (2.9) and (2.10) we find,

$$N_{NS5} = z_f = 5, \quad N_{D8} = \frac{1}{81\pi^2} \left[ \alpha'''(0) - \alpha'''(z_f) \right] = 4N,$$

$$N_{D6} = -\frac{1}{81\pi^2} \int_0^{z_f} \alpha''(z)dz = 9N. \quad \text{(2.18)}$$

Notice that this coincides with the numbers we count from the Hanany-Witten set-up in Figure 1.

We can also calculate the linking numbers using eqs.(2.13),(2.14). We find,

$$K_1 = K_2 = \ldots = K_5 = \frac{1}{81\pi^2} \alpha'''(z_f) = -3N \rightarrow \sum_{i=1}^{N_5} K_i = -15N.$$

$$L_1 = L_2 = \ldots = L_N = 3, \quad \hat{L}_1 = \ldots = \hat{L}_{3N} = 4 \rightarrow \sum_{i=1}^{N_8} L_i + \hat{L}_i = 3N + 4 \times 3N = 15N.$$  

We have denoted by $L_i, \hat{L}_i$ the linking numbers of the D8 branes in the first and second stacks. These numbers are also obtained by simple inspection of the Hanany-Witten diagram.

The relation in eq.(2.12) is satisfied. The entanglement entropy can be calculated straightforwardly using eq.(2.16).

Let us now study a more generic example.
A more generic example
Following the same logic, we can work out a slightly more generic situation. Consider the
field theory represented by the Hanany-Witten set-up in Figure 4 or equivalently, the quiver
in Figure 5.

![Figure 4](image-url)

**Figure 4:** The Hanany-Witten set-up corresponding to the generic field theory studied here.

![Figure 5](image-url)

**Figure 5:** The quiver corresponding to the generic field theory studied here.

For the gauge anomalies to cancel, we need

\[2N_1 - N_2 = F_1, \quad 2N_2 - N_1 - N_3 = F_2,\]
\[2N_3 - N_2 - N_4 = F_3, \quad 2N_4 - N_3 = F_4.\]  \hfill (2.19)

We construct the rank-function

\[R(z) = \begin{cases} 
N_1 z & 0 \leq z \leq 1 \\
(N_2 - N_1)(z - 1) + N_1 & 1 \leq z \leq 2 \\
(N_3 - N_2)(z - 2) + N_2 & 2 \leq z \leq 3 \\
(N_4 - N_3)(z - 3) + N_3 & 3 \leq z \leq 4 \\
-N_4(z - 4) + N_4 & 4 \leq z \leq 5.
\end{cases}\]
The function \( \alpha(z) \) reads,
\[
\alpha(z) = -81 \pi^2 \begin{cases} 
a_0 + a_1 z + N_1 \frac{z^3}{6} & 0 \leq z \leq 1 
\quad b_0 + b_1 (z-1) + N_1 \frac{(z-1)^3}{6} + (N_2 - N_1) \frac{(z-1)^3}{6} & 1 \leq z \leq 2 
\quad c_0 + c_1 (z-2) + N_2 \frac{(z-2)^3}{6} + (N_3 - N_2) \frac{(z-2)^3}{6} & 2 \leq z \leq 3 
\quad d_0 + d_1 (z-3) + N_3 \frac{(z-3)^3}{6} + (N_4 - N_3) \frac{(z-3)^3}{6} & 3 \leq z \leq 4 
\quad p_0 + p_1 (z-4) + N_4 \frac{(z-4)^3}{6} - N_4 \frac{(z-4)^3}{6} & 4 \leq z \leq 5.
\end{cases}
\]

We determine the ten coefficients by imposing that \( \alpha(0) = \alpha(5) = 0 \), the continuity of \( \alpha(z) \) and \( \alpha'(z) \). The resolution of the system is straightforward, we do not quote the result here.

We calculate the number of branes in the Hanany-Witten set-up. We find,
\[
N_{NS5} = z_f = 5, \quad N_{D8} = \frac{1}{81 \pi^2} \left[ \alpha'''(0) - \alpha'''(5) \right] = N_1 + N_4 = F_1 + F_2 + F_3 + F_4,
\]
\[
N_{D6} = -\frac{1}{81 \pi^2} \int_0^{z_f} \alpha'' dz = N_1 + N_2 + N_3 + N_4,
\]
this coincides with the counting from the Hanany-Witten set-up. Similarly, for the linking numbers we obtain
\[
K_1 = K_2 = \ldots = K_5 = -N_4 \rightarrow \sum_{i=1}^{N_5} K_i = -\frac{1}{81 \pi^2} \alpha'''(z_f) z_f = -5N_4.
\]

For the i-th stack of D8 branes we find the linking number \( L_{(i)} \),
\[
L_{(1)} = 1 \rightarrow \sum_{i=1}^{F_1} L_{(i)} = F_1, \quad L_{(2)} = 2 \rightarrow \sum_{i=1}^{F_2} L_{(i)} = 2F_2,
\]
\[
L_{(3)} = 3 \rightarrow \sum_{i=1}^{F_3} L_{(i)} = 3F_3, \quad L_{(4)} = 4 \rightarrow \sum_{i=1}^{F_4} L_{(i)} = 4F_4,
\]
\[
\sum_{i=1}^{N_5} L_i = F_1 + 2F_2 + 3F_3 + 4F_4 = 5N_4.
\]

Again, we find agreement between the holographic calculation and the direct inspection of the Hanany-Witten set-up, validating our proposed expressions (2.9),(2.10),(2.13),(2.14).

In what follows, we analytically study the integrability of strings in generic backgrounds dual to six dimensional \( \mathcal{N} = (1, 0) \) CFTs.

### 3 Study of the integrability of strings on generic string backgrounds

In this section we discuss the classical integrability of the string sigma model on the backgrounds given by eqs.(2.3),(2.4). The strategy we adopt is the following: suppose the sigma model for the String Theory on the background is integrable. Then, any subsector of the
sigma model (in particular, any string soliton) must also be integrable. The idea is then to encounter a contradiction by finding a subsector that is not (Liouville) integrable. If such a subsector is found, the whole sigma model is not integrable. This strategy was applied to a large variety of examples, see the papers [30], [31] and citations to them. Of course, not finding such a subsector does not guarantee the integrability of the sigma model.

In the papers [28], [32] the authors presented particular string solitons for which the classical equations of motion were solvable and the non-integrability was shown under generic circumstances. We would like to revisit the treatment in those works. Indeed, we present a generalisation of [28], [32] that leads to a condition on the background for which the string soliton fails to detect non-integrability. In other words, the string soliton proposed in the papers [28], [32] and its generalisation presented below are very 'efficient' at detecting non-integrable subsectors of the sigma model. Imposing that the soliton fails to detect non-integrability. In other words, the string soliton proposed in the papers [28], [32] and its generalisation presented below are very 'efficient' at detecting non-integrable subsectors of the sigma model. To proceed, we write the relevant part of the Neveu-Schwarz sector of the background in eq.(2.3) choosing global coordinates for AdS7. We have

$$ds^2 = f_1(z) \left[ -dt^2 \cosh \rho + d\rho^2 + \sinh^2 \rho (d\varphi^2 + \cos^2 \varphi d\theta^2 + \sin^2 \varphi d\Omega_3) \right] + f_2(z) dz^2 + f_3(z) (d\chi^2 + \sin^2 \chi d\xi^2), \quad B_2 = f_4(z) \sin \chi d\chi \wedge d\xi. \quad (3.1)$$

We propose a string solution of the form,

$$t = t(\tau), \quad \rho = \rho(\tau), \quad \varphi = \varphi(\tau), \quad \theta = \mu \sigma, \quad z = z(\tau), \quad \chi = \chi(\tau), \quad \xi = \kappa \sigma. \quad (3.2)$$

Here the integers $\kappa$ and $\mu$ indicate how many times the string wraps around the $\xi$ and $\theta$-directions respectively.

We study the equations of motion of this soliton derived from the Polyakov action, supplemented by the Virasoro constraint,

$$S_P = -\frac{1}{4\pi \alpha'} \int d^2 \sigma \left( \eta^{ab} G_{\mu\nu} + \epsilon^{ab} B_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu, \quad (3.3)$$

$$T_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{1}{2} \eta_{ab} \eta^{cd} \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} = 0.$$}

The equations of motion are,

$$f_1(z) \dot{t} = \frac{E}{\cosh^2 \rho},$$

$$f_1(z) \dot{\rho} = -\frac{E^2}{f_1(z) \cosh^3 \rho} + f_1(z) \sinh \rho \cosh \rho \left( \varphi^2 - \mu^2 \sin^2 \varphi \right) - f_1'(z) \rho \dot{\rho},$$

$$f_1(z) \dot{\varphi} = -f_1(z) \left( \frac{2 \cosh \rho}{\sinh \rho} \dot{\rho} + \mu^2 \cos \varphi \sin \varphi \right) - f_1'(z) \dot{\rho} \dot{\varphi}, \quad (3.4)$$

$$f_3(z) \ddot{\rho} = \kappa f_4'(z) \dot{\rho} \sin \chi - f_3'(z) \dot{\rho} \dot{\chi} - \kappa^2 f_3(z) \sin \chi \cos \chi,$$

$$2 f_2(z) \ddot{z} = f_1'(z) \left( -\frac{E^2}{f_1(z)^2 \cosh^2 \rho} + \dot{\rho}^2 + \sinh^2 \rho \left( \dot{\varphi}^2 - \mu^2 \sin^2 \varphi \right) \right) - f_2'(z) \dot{z}^2 + f_3'(z) \left( \dot{\chi}^2 - \kappa^2 \sin^2 \chi \right) - 2 \kappa \dot{\chi} \sin \chi f_4'(z).$$
Here the dots indicate derivatives with respect to \( \tau \) and the primes indicate derivatives with respect to \( z \). We have used the first equation above, to replace for \( \dot{t} \) in the other four equations. Notice that when we set \( \rho = \varphi = 0 \) this system of equations reduces to the system that was studied in eq.(3.5) of the paper [28].

The Virasoro constraints for the string soliton are,

\[
T_{\tau\tau} = 0, \quad T_{\sigma\sigma} = 0, \quad T_{\tau\sigma} = 0 \rightarrow \quad (3.5)
\]

\[
f_1(z) \left( -\cosh^2 \rho \dot{t}^2 + \rho^2 + \sinh^2 \rho \left( \varphi^2 + \mu^2 \sin^2 \varphi \right) \right) + f_2(z) \dddot{z}^2 + f_3(z) \left( \dddot{\chi}^2 + \kappa^2 \sin^2 \chi \right) = 0.
\]

The reader can check that the equations of motion imply that the string soliton satisfies the Virasoro constraints by making an appropriate choice for the integration constant \( E \).

We proceed with the strategy outlined above. The reader unfamiliar with the technical details should consult the papers [28], [32] for a clear explanation of the procedure. First, we find a simple solution by solving the equation for \( \dddot{z} \), choosing configurations with,

\[
\dddot{\varphi}(\tau) = \dddot{\rho}(\tau) = \dddot{\chi}(\tau) = \dddot{\rho}(\tau) = \dddot{\chi}(\tau) = \dddot{\rho}(\tau) = \dddot{\chi}(\tau) = \dddot{\rho}(\tau) = \dddot{\chi}(\tau) = 0.
\]

In fact, the equations of motion in (3.4) are automatically solved, aside from the equation for \( \dddot{z} \) that reads,

\[
2f_2(z) \dddot{z} = -\frac{f_1'(z)}{f_1(z)^2} E^2 - \dddot{z}^2 f_2'(z) \quad (3.6)
\]

After inserting the expressions for the functions \( f_1(z) \) and \( f_2(z) \) in terms of \( \alpha(z) \), the above equation for \( \dddot{z} \) reads,

\[
\dddot{z} - \left( \frac{\alpha''' - \alpha' \alpha''}{4 \alpha^2} \right) \left( \frac{\alpha}{\alpha''} \right) \left( \frac{E^2}{16 \pi^2} - \dddot{z}^2 \right) = 0, \quad (3.7)
\]

which has a simple solution

\[
z_{sol}(\tau) = \frac{E}{4 \pi} \tau. \quad (3.8)
\]

This also solves the constraint in eq.(3.5) using the first of eqs.(3.4) for \( \dot{t} \) and the expressions for \( f_1(z), f_2(z) \).

We then have a ‘base solution’ around which we perturb the other variables. This leads to writing the Normal Variational Equation (NVE) for the different coordinates \( \varphi, \rho, \chi \). These NVEs are second order linear equations with variable coefficients, for which there are well developed criteria for the existence of Liouville integrable solutions—see [28], [32] for a summary of these criteria. Below, we study the NVE for the \( \rho \)-variable. The detailed study of this NVE and those for the other coordinates together with the analysis of the Liouville integrability are given in Appendix A.
3.1 NVE for the ρ-coordinate

We allow for small fluctuations in \( \rho(\tau) = 0 + \epsilon r(\tau) \) and insert the \( z_{sol}(\tau) \) in (3.8) into the equation of motion for \( \ddot{r}(\tau) \), we find for the NVE, at leading order in the small parameter \( \epsilon \),

\[
\ddot{r}(\tau) + B_r(\tau) \dot{r}(\tau) + A_r(\tau) r(\tau) = 0
\]

A detailed analysis of the Liouville integrability of this equation is relegated to Appendix A. Here, we make a simple observation: if the warp factor \( f_1(z) \) is equal to a constant, then \( B_r = 0 \), and the above differential equation is that of a harmonic oscillator, which admits a Liouvillian solution of the form \( r(\tau) = \exp(iE\tau) \). On the other hand, if \( f_1(z) \) is not equal to a constant, we show in Appendix A that the NVE in eq.(3.9) does not admit Liouvillian solutions.

In summary, the analysis above strongly suggests that the situation of constant AdS\(_7\) warp factor is quite special. This implies that the background-defining function \( \alpha(z) = A\sin(\omega z) \), for which the functions \( f_1(z), f_2(z) \) are constant, is special. This solution does not fall within the class of solutions studied in Section 2.1. In spite of this we study below the background and the integrability of the string sigma model for strings moving on this special solution. We postpone to Section 5 a more detailed analysis of the CFT dual to this special solution.

4 Integrability of strings on the special background

As pointed out above, there is only one particular function \( \alpha(z) = A\sin(\omega z) \), that makes constant the warp factor of AdS\(_7\). We gave reasons to suspect that strings might be integrable on the generated background. We analyse this in what follows.

Let us first write the complete Massive IIA solution and then show that the equations of motion of the sigma model can be derived from a Lax pair.

When \( \alpha(z) = A\sin(\omega z) \), the \( z \)-coordinate varies in the interval \( 0 \leq z \leq \frac{\pi}{\omega} \). We choose \( \omega = \frac{\pi}{N_5} \), being \( N_5 \) a large integer number. The full background in eqs.(2.3), (2.4) reads,

\[
ds^2 = \frac{\sqrt{2}\pi}{\omega} \left( 8\text{AdS}_7 + \omega^2 dz^2 + \frac{\sin^2 \omega z}{1 + \sin^2 \omega z} \right) d\Omega_2, \\
e^{-2\phi} = e^{-2\alpha}(1 + \sin^2 \omega z), \\
F_0 = \frac{A\omega^3 \cos \omega z}{162\pi^3}, \\
F_2 = -\frac{A\omega^2 \sin^3 \omega z}{81\pi^2} \left( 1 + \sin^2 \omega z \right) d\Omega_2.
\]

The expression for \( F_0 \) suggests that we have a continuous distribution of D8 branes. Indeed, in contrast with the results of the examples discussed in Section 2.1, the \( F_0 \) in eq.(4.3) is a
continuous function, instead of a piece-wise constant and discontinuous one, characteristic of
localised D8 branes. Postponing to Section 5 the discussion of the dual field theoretical
understanding of the Massive IIA background in eqs.(4.1)-(4.3), we focus here on the integrability
of the string sigma model in this solution.

As it is well-known, the way to show integrability of the string sigma model is to find a
Lax pair that encodes the classical equations of motion of this sigma model. We present a
Lax connection for the Polyakov action of strings moving on a background with the Neveu-
Schwarz sector of eq.(4.1)-(4.2).

As discussed above, the warp-factor $f_1(z)$ as defined in eq.(2.3) is constant for this background,
making the metric in eq.(4.1) a direct product of the AdS$_7$ and $\mathcal{M}^3$ spaces. This will simplify
proving integrability of the string on this background considerably, as the oscillations of the
string on these different spaces decouple. We can write the Polyakov action in eq.(3.3), for
the string on this background as the sum of the action for a string on a seven-dimensional
AdS$_7$ geometry, and the action for the string on the three-dimensional internal space $\mathcal{M}^3$, with a $B_2$-field. This reads,

$$S_P = S_P^{\text{AdS}_7} + S_P^{\mathcal{M}^3}$$

$$= \frac{1}{4\pi\alpha'} \int d^2\sigma \, \eta^{ab} G_{\alpha\beta}^{\text{AdS}_7} \partial_a X^\alpha \partial_b X^\beta - \frac{1}{4\pi\alpha'} \int d^2\sigma \left( \eta^{ab} G_{\mu\nu}^{\mathcal{M}^3} + \epsilon^{ab} B_\mu^\mathcal{M}^3 \right) \partial_a X^\mu \partial_b X^\nu$$

where the Latin indices range over the worldsheet coordinates, and the Greek indices range
over the target space. In particular, $\alpha, \beta$ range over the AdS$_7$ directions, and $\mu, \nu$ range over
the coordinates $z, \chi$ and $\xi$ of the internal space $\mathcal{M}^3$. We reinstated the constant $\alpha' = 1$. Let
us study in turn the Lax connection for each part of the action.

**Lax pair for AdS$_7$**

The Polyakov action on an AdS$_n$ target space without a $B_2$-field, is known to be integrable (as
is the action of the string on the other symmetric space dS$_n$ and the Euclidean counterparts
H$^n$ and S$^n$, see Appendix B for a more detailed explanation and references). We can think
of these symmetric spaces as cosets $F = G/H$ of a Lie group $G$ by a Lie subgroup $H$.

Therefore, we first introduce the Principal Chiral Model (PCM) on a semisimple Lie
group $G$,

$$S_{\text{PCM}} = -\frac{\kappa^2}{2\pi} \int d^2\sigma \, \text{Tr} \left[ \partial_a g \partial^a g^{-1} \right], \quad g \in G.$$  \hspace{1cm} (4.5)

which exhibits a $G_L \times G_R$ global symmetry and can be written in terms of the Maurer-Cartan
form $j_a$, a Lie algebra valued connection on the group manifold,

$$S_{\text{PCM}} = \frac{\kappa^2}{2\pi} \int d^2\sigma \, \text{Tr} \left[ j_a j^a \right], \quad j_a = g^{-1} \partial_a g \in \mathfrak{g}.$$  \hspace{1cm} (4.6)

This Maurer-Cartan form is by construction flat. The flatness condition together with the
equations of motion for the action in eq.(4.6) read

$$\partial_+ j_- + \partial_- j_+ = 0,$$

$$\partial_+ j_- - \partial_- j_+ - [j_+, j_-] = 0$$  \hspace{1cm} (4.7)
Here we used lightcone coordinates on the string worldsheet. The above eqs. (4.7) combine to construct a Lax connection
\[ L_\pm = \frac{1}{1 \mp Z} j_\pm, \]  
(4.8)
where \( Z \in \mathbb{C} \) is the spectral parameter, such that the flatness of the Lax connection
\[ dL + L \wedge L = 0, \]  
(4.9)
is equivalent to the equations of motion obtained from the action in eq. (4.6).

However, our case at hand is not a group manifold \( G \), but a symmetric coset \( F = G/H \). That means that there is a \( \mathbb{Z}_2 \) automorphism of the algebra of \( G \), under which the latter decomposes as \( \mathfrak{g} = \mathfrak{f} \oplus \mathfrak{h} \). Thus, the right action of \( H \) is realized as a gauge symmetry and, by introducing a \( \mathfrak{h} \)-valued gauge field \( B_a \), the new gauge invariant PCM action reads
\[ S_{PCM} = \frac{\kappa^2}{2\pi} \int d^2\sigma \text{Tr} [J_a J^a], \quad J_a = j_a - B_a \]  
(4.10)
where we have defined the projection \( J_a = P_f(j_a) \). The resulting equations of motion are
\[ D_a J^a = 0, \quad D_a = \partial_a + [B_a, \cdot] \]  
(4.11)
while the new flatness condition
\[ \partial_a B_b - \partial_b B_a + [B_a, B_b] + D_a J_b - D_b J_a + [J_a, J_b] = 0 \]  
(4.12)
uses the commutation relations \([\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, [\mathfrak{h}, \mathfrak{f}] \subset \mathfrak{f}\) and \([\mathfrak{f}, \mathfrak{f}] \subset \mathfrak{h}\) to decompose into two separate projections on the algebras \( \mathfrak{h}, \mathfrak{f} \) as
\[ \partial_a B_b - \partial_b B_a + [B_a, B_b] + [J_a, J_b] = 0 \]
\[ D_a J_b - D_b J_a = 0 \]  
(4.13)
As before, the flatness eq. (4.13) together with eq. (4.11) combine in a Lax connection for the coset space,
\[ L_\pm = B_\pm + Z^{\pm 1} J_\pm \]  
(4.14)
whose flatness condition is equivalent the equations of motion of the symmetric PCM. This demonstrates that the string on a symmetric space is classically integrable in the absence of a \( B_2 \)-field.

In Appendix B we introduce a more natural environment to realise the symmetric PCM. The reader unfamiliar with these technical details should consult this Appendix. We now construct the Lax pair on the \( \mathcal{M}^3 \) part of the space.
Lax pair for $M^3$

Here, we state the Lax connection whose flatness condition gives the equations of motion for the three coordinates in $M^3$ derived from the action $S_{Pol}^M$ in eq.(4.4). We will elaborate more on the derivation of this Lax connection in Section 4.1. The Lax connection is of the form

$$\mathcal{L}_\pm = 2 \left(1 + \sqrt{2} \right) \frac{A_\pm}{1 \mp Z},$$

(4.15)

where

$$A_\pm = \begin{pmatrix}
\pm \sin \chi \sin \xi z_\pm + \frac{\sin 2z}{2(1+\sin^2 z)} (\cos \chi \sin \chi z_\pm + \sin \chi \cos \chi \xi z_\pm) - \frac{\sin^2 z}{\sqrt{2}(1+\sin^2 z)} (2 \cos \chi \chi z_\pm - \sin 2\chi \sin \chi \xi z_\pm) \\
\mp \sin \chi \cos \xi z_\pm + \frac{\sin 2z}{2(1+\sin^2 z)} (\cos \chi \cos \chi z_\pm - \sin \chi \sin \chi \xi z_\pm) - \frac{\sin^2 z}{\sqrt{2}(1+\sin^2 z)} (2 \sin \chi \chi z_\pm + \sin 2\chi \cos \chi \xi z_\pm) \\
\mp \cos \chi \chi z_\pm + \frac{\sin \chi}{\sqrt{2}(1+\sin^2 z)} (\pm \sin 2\chi \chi z_\pm + 2\sqrt{2}\sin^2 z \sin \chi \xi z_\pm)
\end{pmatrix}.
$$

We denoted by $Z$ the spectral parameter, and use the notation $f_+ = \partial_+ f$, $f_- = \partial_- f$ and $f_{+++} = f_{++} = \partial_+ \partial_- f$. We wrote $A_\pm$ in the adjoint representation of $SU(2)$. One can check that the flatness condition for this connection is equivalent to the equations of motion for the Polyakov action on the internal space $M^3$ with $B_2$-field. These equations of motion take the form,

$$z_{+-} = \frac{1}{2\sqrt{2}} \left(z_+ z_- + \sin^2 \chi \xi_+ \xi_- + (5 + \sin^2 z) \sin^2 z \sin \chi (\chi_+ \chi_- - \chi_+ \chi_-)\right),$$

$$\chi_{+-} = \frac{1}{2} \sin 2\chi \xi_+ \xi_- + \frac{1}{2\sqrt{2}} \left(1 + \frac{4}{1 + \sin^2 z}\right) \sin \chi (z_+ \xi_- - z_- \xi_+) - \frac{\cot z}{1 + \sin^2 z} (z_+ \chi_- + z_+ \chi_+),$$

$$\xi_{+-} = -\frac{1}{2\sqrt{2}} \left(1 + \frac{4}{1 + \sin^2 z}\right) \sin^{-1} \chi (z_+ \chi_- - z_- \chi_+) - \frac{\cot z}{1 + \sin^2 z} (z_+ \xi_- + z_- \xi_+) - \frac{\cos \chi}{\sin \chi} (\chi_+ \xi_- + \chi_- \xi_+).$$

It is illuminating to discuss the reason why this background is integrable or more precisely, where the form of the Lax pair on $M^3$ in eq.(4.15) comes from.

### 4.1 Relation with $\lambda$-deformation

We explain more intuitively the reason why the bosonic sector of the string worldsheet on the background (4.1)-(4.3) is integrable.

It turns out that the Neveu-Schwarz sector part of the internal space $M^3$ for this background—e.g.s.(4.1)-(4.2) is exactly equal to that of the $\lambda$-deformed WZW-model on SU(2). The $\lambda$-deformation is an integrable deformation of the WZW-model proposed by Sfetsos in [33]. The WZW model is given by an action of the form,

$$S_{WZW,k} = \frac{k}{2\pi} \int_{\partial \mathcal{B}} \text{Tr}[j_a j^a] + \frac{k}{6\pi} \int_{\mathcal{B}} \epsilon_{abc} \text{Tr}[j^a j^b j^c]$$

(4.16)

Here the first term is the action of the PCM and describes the fluctuations of the string on a group manifold. Just like the PCM, the WZW-model on any Lie group $G$ is integrable.
The λ-deformation is an integrable deformation of the WZW-model. Namely, a deformation term added to the action in eq. (4.16) that preserves the integrability. The action of the λ-deformed WZW-model is given by

\[ S_\lambda = S_{WZW,k} + \frac{k}{\pi} \int_{\partial B} j^A_A \left( \lambda^{-1} - D^T \right)^{-1} j^B_B \]

(4.17)

where \( j_a \) as before in eq. (4.16) and (4.6) is the left invariant current, and \( \tilde{j}^a_a = \partial_a g g^{-1} \) is the right invariant current. Notice these currents are algebra valued and the indices \( A \) and \( B \) range over the components of the algebra of the group on which we study this action. The matrix \( D_{AB} = \text{Tr} [T_A^g T_B^g]^{-1} \) relates the left and right invariant currents as \( j^A_A = D_{AB} \tilde{j}^B_B \). Here the \( T_A \) are the generators of the group.

If we analyse the λ-deformed WZW-model on the Lie group SU(2), the action (4.17) is equivalent to the Polyakov action of the string on a target space of the form [34],

\[ ds^2 = 2k \left( \frac{1 + \lambda}{1 - \lambda} dz^2 + \frac{1 - \lambda^2}{\Delta} \sin^2 z d\Omega_2^2 \right), \]

\[ B_2^\lambda = -2k \left( z - \frac{(1 - \lambda)^2}{\Delta} \cos z \sin z \right) \text{vol}_2, \]

\[ e^{-2\Phi^\lambda} = e^{-2\Phi_0} \Delta, \]

where \( \Delta = 1 + \lambda^2 - 2\lambda \cos 2z \) and \( \lambda \in [0, 1] \). For \( \lambda = 0 \) we obtain the original WZW model. The action we obtain for \( \lambda \to 1 \) is related to the non-Abelian T-dual of the WZW-model in eq. (4.16). See [34] for a detailed explanation.

The λ-deformation for \( \lambda = 3 - 2\sqrt{2} \)

The connection between the λ-deformation of the WZW-model on SU(2) in eq.(4.18) and the background solution in eqs. (4.1)-(4.3) is made by noting that for \( \lambda = 3 - 2\sqrt{2} \), which implies \( \Delta = 4\lambda \left( 1 + \sin^2 z \right) \), the geometry of eq.(4.18) reads,

\[ ds^2 = 2\sqrt{2} k \left( dz^2 + \frac{\sin^2 z}{1 + \sin^2 z} d\Omega_2^2 \right), \]

\[ B_2^\lambda = -2k \left( z - \frac{\sin z \cos z}{1 + \sin^2 z} \right) \text{vol}_2, \]

\[ e^{-2\Phi^\lambda} = e^{-2\Phi_0^\lambda} (12 - 8\sqrt{2}) \left( 1 + \sin^2 z \right). \]

This is identical to the Neveu-Schwarz sector of the internal space \( \mathcal{M}^3 \) in eqs. (4.1), (4.2) if we identify \( \omega = \frac{\pi}{2k} \) and choose conveniently \( e^{-2\Phi_0^\lambda} \). The holographic limit \( \omega \to 0 \) associated to long quivers, corresponds to \( k \to \infty \), the semi-classical limit of the WZW model. We write the metric, dilaton and \( B_2 \)-field for this solution as,

\[ ds^2_{10} = 8\sqrt{2} \pi ds^2_{AdS_7} + \frac{\pi}{2} ds^2_\lambda, \quad e^{-2\Phi} = e^{-2\Phi^\lambda}, \quad B_2 = \pi B_2^\lambda. \]

(4.20)
In summary, for the function $\alpha(z) = A \sin(\omega z)$, the geometry becomes a direct product of $\text{AdS}_7 \times \mathcal{M}^3$. The sigma model for the string factorises into a sigma model on $\text{AdS}_7$ times a sigma model on $\mathcal{M}^3$ coupled to a $B_2$-field. The first is integrable, and a Lax pair can be written as explained in detail in Appendix B. The sigma model on $\mathcal{M}^3$ is the $\lambda$-deformation of the WZW model on $S^3$—see [34]—for a particular value of the parameter $\lambda = 3 - 2\sqrt{2}$. This implies the existence of a Lax pair, given in eq. (4.15), for this part of the space. As a consequence, the Neveu-Schwarz sector of the string sigma model on the whole solution of eqs. (4.1), (4.3) is integrable.

There are other examples in the literature of integrable supergravity backgrounds where it is observed that those geometries are a direct product of integrable sub-spaces, with constant warp factors. Examples of this are the Sfetsos-Thompson solution [35] in the Gaiotto-Maldacena class of supergravity backgrounds, the Lunin-Maldacena real $\beta$-deformations [36], etc. It would be interesting to derive a no-go theorem for the integrability of a string background with non-trivial warp factors. Some similar ideas have been presented in [37].

To complement this analytical proof of integrability, in Appendix C, we perform a careful numerical treatment of the string soliton in eq. (3.2), analysing its dynamics and finding results in agreement with the integrability of the solitons, like trajectories in phase space, Poincaré sections, power spectrum, Lyapunov exponents.

We will now present a short analysis of the background in eqs. (4.1)-(4.3) from the point of view of the dual $\mathcal{N} = (1,0)$ six-dimensional CFT.

5 Field theory interpretation of the special background

In this section we present a first approach to the conformal field theory dual to the background in eqs. (4.1)-(4.3). Since the function $R'(z) = -\frac{\alpha'''(z)}{81\pi^2}$ is not piece-wise discontinuous and constant, a description in terms of well defined six-dimensional gauge and flavour groups as that given in Section 2 is not the most suitable. Instead, we will define the CFT by calculating some of its observables. The use of the background to calculate these observables defining the CFT, is the main message of this section.

It is illustrative to first present a different way to arrive to the function $\alpha(z) = A \sin(\omega z)$, than that of our presentation of Section 3 was purely based on integrability of the sigma model. In the paper [28], the authors gave a way to write solutions to the equation of motion (2.5)—see Section 2.3 in [28]. The idea was to choose a quiver, write the rank function $R(z)$ and the function $F_0$ (typically piece-wise constant and discontinuous). An even periodic extension of $F_0$ was proposed and a Fourier series expansion of $F_0$ found. By integration, the function $\alpha(z)$ was written as,

$$\alpha(z) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi}{N_5} z \right).$$

While the infinite sum of harmonics reproduces the piecewise continuous function $\alpha(z)$ made out of cubic polynomials in each interval, it is natural to wonder what is the physical content
of each harmonic in the sum (since the dynamical equation (2.5) is linear). As we discussed around eqs. (4.1)-(4.3), this leads to a background that can be interpreted as if the D8 sources are smeared all along the z-coordinate, instead of sharply localised as the expression for $\alpha''$ suggests in the examples of Section 2.1. A situation of this sort was also suggested (though not analysed in detail) in the work of Cremonesi and Tomasiello [24]. These authors observed that a possible scaling under which the backgrounds of the form in eq. (2.3) are trustable representations of $\mathcal{N} = (1, 0)$ SCFTs, involved taking the number of D8 branes to infinity and creating a continuous distribution. The authors of [24] emphasise that their treatment of anomalies still holds true in this case. Below, we analyse the different observables discussed in Section 2.1 for the particular solution of eqs. (4.1)-(4.3).

We consider the solution derived from $\alpha(z) = A\sin(\omega z)$. We choose $\omega = \frac{n\pi}{N_5}$ which makes the coordinate range in $0 \leq z \leq \frac{N_5}{n}$. We work with $n = 1$ only (the first harmonic) in what follows. The expression of eq. (2.7) indicates that $N_{NS5} = N_5$. We can calculate the number of D6 and D8 branes in this background. Using eqs. (2.9), (2.10) we find,

$$N_{D6} = -\frac{1}{81\pi^2} \int_0^{N_5} \alpha''(z) dz = \frac{2A}{81\pi N_5},$$

$$N_{D8} = \frac{1}{81\pi^2} [\alpha'''(0) - \alpha'''(N_5)] = -\frac{2A\pi}{81N_5^2}.$$  

In absolute value, these expressions imply relations among the quantities,

$$A = \frac{81\pi}{2} N_5 N_{D6}, \quad A = \frac{81}{2\pi} N_5^3 N_{D8}, \quad \pi^2 N_{D6} = N_{D8} N_5^2.$$  

We can use the expressions for the linking numbers—eqs. (2.13), (2.14) and the expression for $A$ in terms of the number of colour D6 branes,

$$\sum_{i=1}^{N_5} K_i = \frac{1}{81\pi^2} \alpha'''(N_5) N_5 = \frac{A\pi}{81N_5^2} = -\frac{N_5 N_{D8}}{2}.$$  

In the paper [28], the authors found an expression for the central charge of the conformal quiver—see eq. (2.14) in [28]. This coincides with the holographic central charge $a$ found in [24], derived by field theoretical means. Let us apply this expression for the case at hand. For $\alpha(z) = A\sin(\omega z)$, we find

$$c = -\frac{2^8}{3^8 \times 16 \times G_N} \int_0^{\frac{\pi}{2}} \alpha(z) \alpha''(z) dz = \frac{8}{3^8 \times G_N} A^2 \omega^2 N_5 = \frac{N_{D6}^2 N_5}{4\pi^2}.$$  

We have used the expression for $A$ in terms of the number of colour D6 branes and that in our conventions $G_N = 8\pi^6$. We can compute the entanglement entropy. Using the expression in eq. (2.16), we find that for this particular CFT

$$S_{EE}^{\text{reg}} = \left(\frac{\mu_1 \mu_2}{L^4}\right) \times 64\pi^4 N_{D6}^2 N_5,$$  

(5.6)
that, as anticipated, has the same scaling with \( N_5 \) and \( N_{D6} \) as the central charge.

An interesting observation is that these expressions for the linking numbers, central charge and entanglement entropy in eqs.(5.4),(5.5),(5.6) have the same scaling with \( N_{D6} \) and \( N_5 \) as a four-dimensional \( \mathcal{N} = 2 \) quiver that starts with a flavour group of rank \( N_{D6} \), continues with \( N_5 - 1 \) colour groups of rank \( N_{D6} \) and closes with a flavour group of rank \( N_{D6} \). See around eq.(3.16) of the paper [27].

Another interesting observable in all CFTs is the Wilson loop, in particular those for which the non-dynamical quarks transform under the internal symmetries. We find it interesting to study a fundamental string on a generic background of the form in eq.(2.3), parametrised by

\[
t = \tau, \quad x = \sigma, \quad R = R(\sigma), \quad z = z(\sigma).
\]  

(5.7)

We use Poincaré coordinates for the AdS\(_7\) space, parametrised by \((t, \vec{x}, R)\). The Nambu-Goto action of the fundamental string on a generic background is,

\[
S_{NG} = \frac{1}{2\pi} \int d\tau d\sigma \sqrt{f_1^2 R^4 + f_1^2 R^2 + f_1 f_2 R^2 z'^2} = \frac{T}{2\pi} \int d\sigma \sqrt{f_1^2 R^4 + f_1^2 R^2 + f_1 f_2 R^2 z'^2}.
\]  

(5.8)

This action does not depend explicitly on the 'time variable' \( \sigma \) and this implies the conserved 'Hamiltonian',

\[
\frac{f_1^2 R^4}{\sqrt{f_1^2 R^4 + f_1^2 R^2 + f_1 f_2 R^2 z'^2}} = C.
\]  

(5.9)

At this point, it is interesting to analyse three situations:

- The situation for which the coordinate \( z(\sigma) \) is constant. In this case, we are back to the usual Wilson loop calculation in strongly coupled CFTs [38], that gives \( E_{QQ} \sim \frac{\sqrt{\lambda}}{L_{QQ}} \).

- The situation in which \( R(\sigma) = R_0 \) is constant. In this case we find the Nambu-Goto action,

\[
S_{NG} = \frac{T}{2\pi} \int d\sigma \sqrt{f_1^2(z) R_0^4 + f_1(z) f_2(z) R_0^2 z'^2}.
\]  

(5.10)

That leads to more a conventional minimisation problem, equivalent to the calculation of the 'usual' rectangular Wilson loop in a background of the form \( ds^2 \sim R_0^2 f_1(z) [dx_1^2 + f_2(z) dz^2] \). Using eq.(2.4) we find that \( f_1(z) f_2(z) = 16\pi^2 \). The main difference with the situations calculated previously in the bibliography is that the \( z \)-coordinate is bounded.

- More interesting than the general study presented above is to consider the action in eq.(5.8) for the case of our special background in eq.(4.1), for which \( f_1(z) = \frac{8\sqrt{2} \pi}{\omega} \) and \( f_2(z) = \sqrt{2} \pi \omega \). Using these values, the action in eq.(5.8) reads,

\[
S_{NG} = \frac{\sqrt{32} T}{\omega} \int d\sigma \sqrt{R^4(\sigma) + R^2(\sigma) + \frac{\omega^2}{8} R^2(\sigma) z'(\sigma)^2}.
\]  

(5.11)
We can redefine the variable \( \tilde{z} = \omega z \sqrt{8} \) and from the action in eq.(5.11) we find two conserved quantities,
\[
\frac{R^4}{\sqrt{R^4 + R^2 + R^2 \tilde{z}^2}} = E, \quad \frac{R^2 \tilde{z}'}{\sqrt{R^4 + R^2 + R^2 \tilde{z}^2}} = J, \quad R^4 \frac{J^2}{E^2} = \tilde{z}^2. \tag{5.12}
\]
Following the usual procedure to write the separation of the external quarks in the \( x \)-direction \( L_{QQ,x} \) and in the \( \tilde{z} \)-direction \( L_{QQ,\tilde{z}} \),
\[
L_{QQ,x} = \frac{E}{R^0} \int_1^\infty dy \frac{1}{y^2 \sqrt{y^4 - J^2 y^2 - \frac{E^2}{R^0}}} , \quad L_{QQ,\tilde{z}} = \frac{J}{R^0} \int_1^\infty dy \frac{1}{\sqrt{y^4 - J^2 y^2 - \frac{E^2}{R^0}}} . \tag{5.13}
\]
The Energy of the quark-antiquark pair \( E_{QQ} \) is after regularisation,
\[
E_{QQ} = \frac{\sqrt{32} R^0}{\omega} \left[ \int_1^\infty dy \left( \frac{y^2}{\sqrt{y^4 - J^2 y^2 - \frac{E^2}{R^0}}} - 1 \right) - 1 \right] . \tag{5.14}
\]
After redefining \( \tilde{J} R^0 = J \) and \( \tilde{E} R^2_0 = E \), we observe that these expressions in eqs.(5.13)-(5.14) are the same as those obtained by Maldacena in [38] when considering quarks that are charged under the R-symmetry. In our background the \( z \)-coordinate is not isometric, but the fundamental string in the configuration of eq.(5.7) does see it as part of the R-symmetry.

We close this section hoping to have given the reader a flavour of the many things that can be holographically computed with the background of eqs.(4.1)-(4.3). These observables serve as a definition of the six dimensional \( \mathcal{N} = (1,0) \) CFT, even when the precise ranks of the colour and flavour groups are not easy to determine. Let us present some summary and conclusions.

### 6 Conclusions and Future Directions

In this paper we have studied six-dimensional superconformal field theories with \( \mathcal{N} = (1,0) \) SUSY. The main goal was to learn about these non-Lagrangian, strongly coupled, field theories using holography.

In particular, we have found new expressions calculating (in holographic language) the number of NS5, D6 and D8-branes, and their linking numbers, that characterise the Hanany-Witten set-ups associated with the CFTs. We also found a closed expression calculating the entanglement entropy of a rectangular region, explicitly dependent on the matter content of the CFT. Interestingly, we have found a particular background in Massive Type IIA on which the NS sector of the string sigma model is classically integrable. We have written the Lax pair from which the sigma model equations of motion are derived. We related this special background to a \( \lambda \)-deformation of a WZW model. Our study was complemented with an intensive numerical...
analysis and a careful discussion of the Liouville integrability of string solitons. Various explicitly worked out examples and detailed appendixes complement our study. Let us comment on the natural lines of investigation suggested by this work. It is interesting to understand in detail the character of our special background in eqs.\((4.1)-(4.3)\). In fact, as we commented, D8-branes are smeared in this solution. Finding the precise smearing form and fitting the solution in the framework developed in the past, see for example the works [39], may be illuminating and useful for further progress.

It would be good to exploit the integrable background of eqs.\((4.1)-(4.3)\), by repeating various of the studies that in the paradigmatic case of AdS\(_5 \times S^5\) gave insightful results. It would also be interesting to learn about the applicability of the formalism that we presented for the situation in which a flow from the six dimensional CFT to a lower dimensional field theory is realised by a background solution.

This work and previous experience suggest that when the pre-factor in front of the AdS-space is independent of the coordinates of the internal space, the study of the string sigma model decouples between an AdS\(_p\) part and a \(\Sigma_{10-p}\) one. We suggest that searching for backgrounds with these characteristics is a good guide to find integrable solutions. Along these lines, it should be interesting to understand the conditions that allow to formulate a no-go theorem for integrability.

The study of these issues is fascinating and we hope to report on them in the near future.

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A Non-integrability of Strings on General Backgrounds

In this Appendix, we analyse in more detail the eqs. \((3.4)\) for a string soliton on the AdS\(_7\)-backgrounds defined in eqs. \((2.3)-(2.5)\), where we take for \(\alpha(z)\) a general third order polynomial of the form \(a_3 z^3 + a_2 z^2 + a_1 z + a_0\). A similar analysis was originally performed in [28], but we will generalise the results found there in two ways:

- First we will derive two relations between the coefficients \(a_0\), \(a_1\), \(a_2\) and \(a_3\). When either of these two relations - \((A.8)\) and \((A.9)\) - are met, the string soliton will be non-integrable for these coefficients.
Second, from these relations we then derive that the string soliton will always be non-integrable at the beginning and end of the $z$-interval for any generic quiver.

We start from the equations of motion for the string soliton that are listed in eq. (3.4). As is explained in Section 3, we can solve the equations of motion for $\ddot{z}(\tau)$ by first choosing the solutions $\ddot{\varphi}(\tau) = \dot{\varphi}(\tau) = \varphi(\tau) = 0$, $\ddot{\chi}(\tau) = \dot{\chi}(\tau) = \chi(\tau) = 0$, and $\ddot{\rho}(\tau) = \dot{\rho}(\tau) = \rho(\tau) = 0$. These solutions simplify the equation for $\ddot{z}$ to a new expression that has the solution $z_{sol}(\tau) = \frac{E}{4\pi} \tau$.

**NVE for $\rho$**

If we now allow for small fluctuations in $\rho(\tau) = 0 + \epsilon r(\tau)$ and insert the solution for $z_{sol}(\tau)$, we find for the NVE

$$\ddot{r}(\tau) + B_r(\tau) \dot{r}(\tau) + A_r(\tau)r(\tau) = 0$$

$$B_r(\tau) = \frac{f'_1(z)}{f_1(z)} \frac{E}{4\pi} \bigg|_{z_{sol}} = \frac{E}{8\pi} \left( \frac{\alpha'}{\alpha} - \frac{\alpha''}{\alpha^2} \right) \bigg|_{z_{sol}}$$

$$A_r(\tau) = \frac{E^2}{f_1(z)^2} \bigg|_{z_{sol}} = \frac{-E}{128\pi^2} \frac{\alpha''}{\alpha} \bigg|_{z_{sol}}$$

When only considering a string that moves along the $z$ and $\rho$-directions, it is now easy to see that if the warp factor $f_1(z)$ is equal to a constant, $B_r = 0$, and the above differential equation admits a Liouvillian solution of the form $r(\tau) = \exp(iE\tau)$.

When we allow for a warp factor between the AdS$_7$ and $\mathcal{M}^3$ spaces, such that $f_1(z)$ is no longer equal to a constant, we can use Kovacic’s algorithm [40] to try to determine if the resulting differential equation (A.1) will still admit any Liouvillian solutions. This can be done by combining the coefficients $A(\tau)$ and $B(\tau)$ of a second order differential equation into a new function $V(\tau)$, defined below. By applying Galois theory to second order differential equations, one can find if the solutions to the differential equation will be Liouvillian by studying the pole structure and the behaviour at infinity of this function $V(\tau)$. For the differential equation (A.1) to admit Liouvillian solutions, the function $V(\tau)$ has to satisfy at least one of following necessary but not sufficient conditions:

- The poles of $V(\tau)$ are all either of order 1 or of even order. At infinity the function $V(\tau)$ is of even order, or of order greater than two.
- The function $V(\tau)$ has at least one single pole that is either of odd order greater than 2, or of order 2.
- The order of the poles does not exceed 2, and the order of $V(x)$ at infinity is at least 2.

Here the order of $V(\tau)$ at infinity is equal to the degree of the denominator minus the degree of the numerator. For a more detailed summary of Kovacic’s procedure, see Appendix B of
[28]. In this case the NVE for \( \rho \), in eq. (A.1), has a corresponding function \( V_r(\tau) \) of the form

\[
V_r = \frac{1}{4} \left( 2B_r' + B_r^2 - 4A_r \right)
\]

\[
= \frac{E^2}{256\pi^2a^2(\alpha')^2} \left( -3(\alpha')^2(\alpha'')^2 - 2\alpha\alpha'\alpha''\alpha''' + \alpha \left( 6(\alpha''')^3 + 5\alpha(\alpha'')^2 - 4\alpha\alpha'\alpha''\alpha''' \right) \right)
\]  

(A.2)

If we now consider a function \( \alpha(z) \) corresponding to a general background, \( \alpha(z) \) will be a piece wise continuous polynomial along the \( z \)-direction of at most order 3 at every point.

\[
\alpha(z) = a_3z^3 + a_2z^2 + a_1z + a_0
\]  

(A.3)

Inserting this solution for \( \alpha(z) \) into the potential in eq. (A.2) will allow us to analyse the pole structure of \( V_r(\tau) \) corresponding to a general Massive IIA background,

\[
V_r(\tau) = \frac{3E^2}{2} \frac{3a_3^3E^6\tau^6 + 24a_2a_3^2E^5\pi^5\tau^5 + \ldots}{(4a_2\pi + 3a_3E\tau)^2(64a_0\pi^3 + 16a_1E\pi^2\tau + 4a_2E^2\pi^2\tau^2 + a_3E^3\tau^3)^2}.
\]  

(A.4)

The dots in the numerator are quartic terms in \( \tau \), the order of \( V(\tau) \) at infinity is thus equal to two. Note that \( V_r(\tau) \) has one pole of order two at \( \tau = r_0 = -4a_2\pi/3a_3E \). The other poles come from the cubic polynomial that is the second term in the denominator, one of the real roots of this cubic polynomial can coincide with the earlier pole \( \tau = r_0 \).

Let us first examine the possible roots coming from this cubic polynomial: A cubic polynomial of the form \( a\tau^3 + b\tau^2 + c\tau + d = 0 \) has three complex roots, the multiplicity of which can be obtained from the determinant

\[
\Delta = a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abcd
\]

\[
= -4096 \left( -a_1^2a_2^2 + 4a_1^3a_3 - 18a_0a_1a_2a_3 + a_0 \left( 4a_2^3 + 27a_0a_3^2 \right) \right) E^6\pi^6.
\]  

(A.5)

If \( \Delta > 0 \) the polynomial has three distinct real roots, for \( \Delta < 0 \) there is one real root and two complex conjugate roots. When \( \Delta = 0 \) there are roots with multiplicity larger than one, this can be either a single real root of multiplicity 3, or one real root of multiplicity 2 and another additional root. If we list all possible options we arrive at the following table where the order of the poles of \( V_r(\tau) \) are listed in both the case when first pole \( r_0 \) does coincide with one of the real poles from the cubic term \( (r_0 = r_1) \), and when this does not happen. We see that in all cases all poles will be of even order, and that the resulting \( V_r(\tau) \) thus might pass the first of Kovacic’s criteria. These conditions are however necessary but not sufficient to guarantee the existence of Liouvillian solutions. Let us next turn to the string soliton moving along the \( \varphi \) direction.

**NVE for \( \varphi \)**

We now examine the equation of motion for \( \tilde{\varphi} \). If we allow for small fluctuations in \( \varphi(\tau) = 0 + \epsilon f(\tau) \), and we insert the solution for \( z(\tau) \) from eq. (3.8) while we now freeze the string
### Table 2: The orders of the poles of $V(\tau)$, depending on both the determinant $\Delta$ of the cubic polynomial in the denominator, and on whether the additional root $r_0$ coincides with one of the roots of the cubic polynomial or not.

| $\Delta > 0$ | $r_0 \neq r_1$ | $r_0 = r_1$ |
|--------------|----------------|-------------|
| $(\tau - r_1)(\tau - r_2)(\tau - r_3)$ | 2, 2, 2, 2 | 4, 2, 2 |
| $(z - r_1)^3$ | 2, 6 | 8 |
| $(z - r_1)^2(z - r_2)$ | 2, 4 | 6, 2 |
| $(z - r_1)(z - r_2)^2$ | 2, 2, 4 | 4, 4 |
| $\Delta < 0$ | $(\tau - r_1)(\tau - c_2)(\tau - c_3)$ | 2, 2, 2 | 4, 2, 2 |

along the other directions such that $\rho = \dot{\rho} = \ddot{\rho}$ we find for the NVE

$$
\ddot{f}(\tau) + B(\tau)\dot{f}(\tau) + A(\tau)f(\tau) = 0
$$

$$
B_f(\tau) = \frac{f_1'(z) E}{f_1(z) 4\pi |z|_{z_{sol}}} = \frac{E}{8\pi} \left( \frac{\alpha'}{\alpha} - \frac{\alpha''}{\alpha'} \right) \bigg|_{z_{sol}}
$$

$$
A_f(\tau) = \mu^2
$$

Again, we see that when $f_1(z)$ is equal to a constant, the above NVE will reduce to the harmonic oscillator with solution $a(\tau) = \exp(i\mu \tau)$.

When we allow for a more general warp factor where $f_1(z)$ is no longer constant, we can again use Kovacic’s algorithm to determine if the above differential equation will still admit Liouvillian solutions. Inserting again for $\alpha(z)$ a general third order polynomial as given in eq. (A.3) gives us for the potential

$$
V_f(\tau) = \frac{-36a_3^4E^8\mu^2\tau^8 - 384a_2a_3^3E^7\pi\mu^2\tau^7 + \ldots}{4(4a_2\pi + 3a_3E\tau)^2 (64a_0\pi^3 + 16a_1E^2\pi^2\tau + 4a_2E^2\pi^2\tau^2 + a_3E^3\tau^3)^2}
$$

Notice that though the order of the numerator is different from $V_r(\tau)$, the pole structure in the denominator is identical to that in eq. (A.4). Since the numerator does now contain terms $\tau^8$, the order of $V_f(\tau)$ at infinity is zero, as $V_f(\tau) \sim \mu^2$ when $\tau \to \infty$. We thus see immediately that $V_f(\tau)$ fails to meet the first and third of Kovacic’s criteria. Only when both $\Delta \neq 0$ and $r_0 \neq r_1$ all poles are of order 2, we can pass the second of Kovacic’s criteria.

Requiring that $r_0$ is not a root of the cubic polynomial in the denominator implies, after inserting $\tau = r_0$, that

$$
c = 2a_2^3 - 9a_1a_2a_3 + 27a_0a_3^2
$$

Here $c = 0$, when $r_0$ coincides with one of the roots of the cubic polynomial. We can use the expression for $c$ to simplify the expression for $\Delta$ in eq. (A.5).

$$
d = \frac{\Delta}{-4096E^6\pi^6} = a_1 (c - 27a_0a_3^2) - a_1^2 (a_2^2 - 4a_1a_3)
$$
We have thus found two constraints relating the constants $a_0$, $a_1$, $a_2$ and $a_3$ in a general solution for $\alpha(z)$ of the form in eq. (A.3), such that when either $c = 0$ or $d = 0$, Kovacic’s criteria guarantees the non-integrability of the string soliton.

From this we can immediately conclude that for every function $\alpha(z)$ corresponding to a quiver diagram, on the first part of the $z$-interval the string-soliton is guaranteed to be non-integrable. This is because any quiver, starting with an SU($N$) flavour group will have $\alpha(z) = -81\pi^2 N \left( \frac{1}{N} z^3 - a_1 z \right)$ for $z \in [0, 1]$, for which $c = 0$. The pole at $r_0$ will thus coincide with one of the roots of the cubic polynomial, giving us 2 poles of order two, and one pole at $r_0$ of order 4. The function $V_f(\tau)$ corresponding to this solution will fail to meet any of Kovacic’s criteria and the NVE (A.6) will have non-Liouvillian solutions.

**NVE for $\chi$**

If we allow for small fluctuations in $\chi(\tau) = 0 + \epsilon x(\tau)$ and insert the solution in (3.8) in the equation of motion for $\ddot{x}(\tau)$, we find for the NVE for $\ddot{x}(\tau)$

$$\ddot{x}(\tau) + B_x(\tau) \dot{x}(\tau) + A_x(\tau) x(\tau) = 0$$

$$B_x(\tau) = \left. \frac{E f_3'(z)}{4\pi f_3(z)} \right|_{z=\text{sol}} = \left. \frac{E}{8\pi} \left( 3 \frac{\alpha'}{\alpha} + \frac{(\alpha'^2 + 2\alpha\alpha'')\alpha'''}{\alpha'^2 - 2\alpha\alpha''} \right) \right|_{z=\text{sol}} \quad \text{(A.10)}$$

$$A_x(\tau) = \left. \left( \kappa^2 - \kappa \frac{E f_4'(z)}{4\pi f_3(z)} \right) \right|_{z=\text{sol}} = \left. \left( \kappa^2 - \kappa \frac{1}{4\pi} \frac{6\alpha''^2 - 2\alpha'\alpha'' - 3\alpha'^2 \alpha''}{2\alpha\alpha'' - \alpha'^2} \right) \right|_{z=\text{sol}} \quad \text{(A.11)}$$

We now only consider fluctuations of the string along the $z$ and $\chi$-direction, which is the same situation that was considered in [28]. Because the function $V_x(\tau)$ now involves $f_3(z)$ and $f_4(z)$ and their derivatives, it is far less obvious to see from this result that the string soliton will directly fail to be integrable when $f_1(z)$ is not equal to a constant (as we obtained from the NVE’s for $\rho$ and $\phi$). If we would again insert a general function $\alpha(z)$ of the form given in eq. (A.3), the resulting $V_x(\tau)$ will be a complicated sum of large fractions. We will omit the result here, but is it difficult to see from this result what functions $\alpha(z)$ would give rise to an integrable string soliton. For this reason the authors in [28] did not make a general argument, but instead studied various examples. It is quite difficult to extract general expressions for these cases, as could be nicely done for the NVE’s for $\rho$ and $\phi$, where we saw that the only backgrounds for which the string soliton could have a Liouvillian solution is when $f_1(z)$ is constant.

**B Integrability on the symmetric $\sigma$-model**

Classical Liouvillian integrability for a Hamiltonian dynamical system or for a field theory, emerges over the existence of a flat Lax connection $\mathcal{L}$, i.e.

$$d\mathcal{L} + \mathcal{L} \wedge \mathcal{L} = 0 \quad \text{(B.1)}$$
on the cotangent bundle $T^*M$ (phase space), together with the involution of all the analogous independent conserved quantities. Generally, there is no particular prescription for finding such a connection and one has to rely on their inspiration to address the problem. However, given a two-dimensional scalar field theory in a homogeneous space for a connected semisimple Lie group $G$, the action can be reformulated in terms of its underlying group structure as

$$S_{PCM} \equiv -\frac{\kappa^2}{\pi} \int \text{Tr} j_a j^a$$  \hspace{1cm} (B.2)

where the Lie-algebra-valued current $j \in \mathfrak{g}(G)$,

$$j_\pm \equiv g^{-1} \partial_\pm g = j_\pm^i t_i \quad g \in G, \quad t_i \in \mathfrak{g}$$  \hspace{1cm} (B.3)

is defined over the group element $g = e^{X^i t_i}$, that is all the point transformations on the scalar field worldsheet, on the group manifold. This one-form current is by construction flat and its flatness condition, together with the equations of motion,

$$\partial_\pm j_- + \partial_- j_+ = 0,$$

$$\partial_+ j_- - \partial_- j_+ - [j_+, j_-] = 0,$$  \hspace{1cm} (B.4)

can be combined in a parametrised Lax connection

$$\mathcal{L}_\pm = \frac{j_\pm}{1 + Z},$$  \hspace{1cm} (B.5)

where $Z \in \mathbb{C}$ is the spectral parameter. The flatness condition in eq.(B.1) is equivalent to the equations of motion in eq.(B.4). One also defines the holonomy of $\mathcal{L}$ for constant time, i.e. the monodromy

$$M(Z) = \mathcal{P} \exp \int \mathcal{L},$$  \hspace{1cm} (B.6)

which defines a parallel transport on the group manifold $\Sigma(G)$ and is constant in time, i.e. $\partial t M(Z) = 0$. This means that by expanding in $Z$ at infinity we can obtain an infinite set of conserved charges. This system is known in the literature as the Principal Chiral Model (PCM), it exhibits a global $G_L \times G_R$ symmetry and it is obviously integrable.

The string $\sigma$-model (in the presence of a $B_2$ field) on an homogeneous space for a group $G$ can be represented by the Wess-Zumino-Witten (WZW) model as,

$$S_{WZW,k} = \frac{k}{2\pi} \int_{\partial B} \text{Tr} j_a j^a + \frac{k}{6\pi} \int_B \epsilon_{abc} \text{Tr} j^a j^b j^c \quad j \in \mathfrak{g}(G)$$  \hspace{1cm} (B.7)

which exhibits an $G_{L,\text{cur}} \times G_{R,\text{cur}}$ current algebra symmetry, it is an exact CFT and thus integrable.
The situation becomes more elegant in the case of the non linear $\sigma$-model on a symmetric homogeneous space. Symmetric spaces are backgrounds with rich underlying group structure, which can be exploited in a natural way to make the integrability of the $\sigma$-model manifest. From the group theoretical point of view, a symmetric space is a coset space $G/H$, where the isometry $G$ is a connected Lie group and the subgroup $H \subset G$ is its isotropy group. Then the $\sigma$-model can be recast as a gauged WZW model, for which we introduce non-dynamical gauge fields belonging to the algebra $\mathfrak{h}(H)$ before we integrate them out of the action.

In what follows we will illustrate the classical integrability of the string worldsheet on a symmetric space. To study this in more detail see [41], [42]. For a more general review of integrability in the context of string theory see [43] and in the context of the AdS/CFT correspondence, see [44].

**Integrability of AdS space**

The $\sigma$-model on AdS space is integrable. We know this as a fact, since, as we illustrated above, the $\sigma$-model is integrable on every symmetric homogeneous space. One might ask for an explicit Lax formulation given a specific background. This proves to be quite challenging as we go higher in the dimension of the AdS-space. The difficulty rests on the fact that finding the gauged group element (matrix) of the coset space becomes an involved task in higher dimensions.

Indeed, one can realise the element of a group $G$ abstractly as

$$g \equiv \exp X^i t_i$$  \hspace{1cm} (B.8)

where $t_i \in \mathfrak{g}(G)$ and $X^i$ parametrise the adjoint space, which produces another formulation of the PCM action as

$$S_{PCM} = -\frac{\kappa^2}{\pi} \int d^2 \sigma \, \eta_{ij} e^i_\mu(X) e^j_\nu(X) \partial_+ X^\mu \partial_- X^\nu,$$  \hspace{1cm} (B.9)

where $\eta_{ij} = \langle t_i, t_j \rangle$ is the metric on the Lie algebra $\mathfrak{g}$, defined by $[t_i, t_j] = f_{ijk} t_k$, while the vielbein

$$e^i_\mu = \frac{\partial X^i}{\partial X^\mu}$$  \hspace{1cm} (B.10)

represents the relationship between the adjoint and the target space$^2$.

In this context, the vielbeins $e^i_\mu$ represent the components of the symmetry transformations of $G$ or, equivalently, the Killing vectors of the manifold at hand. Subsequently, the vielbein is realised as the Maurer-Cartan connection

$$j^i_\pm \equiv e^i_\pm = e^i_\mu \partial_\pm X^\mu$$  \hspace{1cm} (B.11)

$^2$The vielbeins represent a relationship between different bases, i.e. they express an object in different frames. As such, this relationship can exist between any kind of spaces.
where \( j_\pm = j_{\pm}^i t_i \), and satisfies the structural flatness condition
\[
\partial_{\mu} e_i^{\nu} - \partial_{\nu} e_i^{\mu} + f^{i}_{jk} e_j^{\mu} e_k^{\nu} = 0. \tag{B.12}
\]
This flatness identity together with the equations of motion of the PCM
\[
\partial_+ \left( e_\mu^i \partial_- X^\mu \right) + \partial_- \left( e_\mu^i \partial_+ X^\mu \right) = 0, \tag{B.13}
\]
allows us to construct the Lax connection,
\[
\mathcal{L}_\pm = \frac{j_\pm}{1 \mp Z} \tag{B.14}
\]
where \( Z \in \mathbb{C} \) is the spectral parameter. The flatness condition
\[
[\partial_+ + \mathcal{L}_+, \partial_- + \mathcal{L}_-] = 0, \tag{B.15}
\]
is equivalent to equations of motion.

Thus, we conclude that in order to specify a particular Lax connection for the \( \sigma \)-model on a symmetric space, one only needs the Killing vectors of the background manifold.\(^3\)

A symmetric space can always be realised as an embedding in a higher dimensional space, the former inheriting most of the isometries of the latter. A standard example is \( S^2 \) which inherits the SO(3) isometries from \( \mathbb{R}^3 \) (but not the translations).

AdS\(_n\) space is a hypersurface in \( \mathbb{R}^{2,n-1} \) onto which only the Lorentz group is tangent. Therefore, the boosts and the rotations of \( \mathbb{R}^{2,n-1} \)
\[
V_i \equiv V_i^A \partial_{Y^A} \tag{B.16}
\]
where \( Y^A, A = 0, \ldots, n \) are the embedding coordinates which build the hypersurface
\[
\eta_{AB} Y^A Y^B = -l^2 \tag{B.17}
\]
with \( \eta_{AB} = \text{diag}(-1,1,\ldots,-1) \), are inherited into AdS\(_n\) as the Killing vectors
\[
\xi_i \equiv \xi_i^\mu \partial_\mu = g^{\mu\nu} \left( \frac{\partial Y^A}{\partial x^\nu} V_A \right)_i \partial_\mu \tag{B.18}
\]
where \( x^\mu, \mu = 0, \ldots, n-1 \) are the AdS\(_n\) coordinates and \( g_{\mu\nu} \) its metric, while \( i \) runs in the vector space. By choosing one of the solutions to eq.(B.17), like the global embedding
\[
Y^0 = l \cosh \rho \cos t \\
Y^j = l \sinh \rho \Omega^j, \quad j = 1, \ldots, n-1 \\
Y^n = l \cosh \rho \sin t \tag{B.19}
\]
\(^3\)One could be naively troubled about the fact that a symmetric space has less degrees of freedom that the number of its Killing vectors, e.g. \( S^2 \) has two d.o.f. and three Killing vectors. In reality, the independent Killing vectors are as many as the d.o.f. of a certain space, but we often talk about the generalized Killing vectors, which are a linear combination of the independent ones. The reason we use the generalized form is to associate these vectors with the space isometries, e.g. in \( S^2 \) with the SO(3) transformations.
where $\Omega^j$ are the Euclidean coordinates for the unit sphere ($\Omega^j \Omega_j = 1$), one can find each one of the $n(n + 1)/2$ Killing vectors of $\text{AdS}_n$.

It’s worth emphasising that the Killing vectors that are inherited into a symmetric space, through an embedding, are the generalized ones. This means that they match the isometry group of the symmetric space, while their independent formulation reveals the actual degrees of freedom of the space. Thus, while in a matrix realization of the PCM we would, traditionally, have to gauge the isotropy group $H$ out of the isometry group $G$ to obtain the element of the coset $G/H$, the Killing vectors constitute a natural environment to describe a symmetric space.

Since we have identified the Killing vectors $\xi_i$ of the background space of the PCM with the vielbeins $e_i$ in eq.(B.9), then one can explicitly check that the equations of motion of this action, eq.(B.13), are equivalent to the standard equations of motion of the $\sigma$-model in the same background, as they should. Therefore, the Killing vectors can be used to build up an explicit Lax connection through equations (B.11) and (B.14), as anticipated.

**The AdS$_3$ example**

While $\text{AdS}_n$ can give frustratingly long results as we increase the value of $n$, AdS$_3$ constitutes a relatively compact example for the above methodology. Choosing a global AdS$_3$ embedding in $\mathbb{R}^{2,2}$ as

$$
Y^0 = \cosh \rho \cos t \\
Y^1 = \sinh \rho \cos t \sin \phi \\
Y^2 = \sinh \rho \cos t \cos \phi \\
Y^3 = \cosh \rho \sin t,
$$

(B.20)

then the six corresponding Killing vectors are

$$
\begin{align*}
\xi_1 &= \partial_t \\
\xi_2 &= \partial_\phi \\
\xi_3 &= \tanh \rho \sin t \sin \phi \partial_t + \coth \rho \cos t \cos \phi \partial_\phi + \cosh 2\rho \cos t \sin \phi \partial_\rho \\
\xi_4 &= \tanh \rho \sin t \cos \phi \partial_t - \coth \rho \cos t \sin \phi \partial_\phi + \cosh 2\rho \cos t \cos \phi \partial_\rho \\
\xi_5 &= \tanh \rho \cos t \sin \phi \partial_t + \coth \rho \sin t \cos \phi \partial_\phi + \sin t \sin \phi \partial_\rho \\
\xi_6 &= \tanh \rho \cos t \cos \phi \partial_t - \coth \rho \sin t \sin \phi \partial_\phi + \sin t \cos \phi \partial_\rho
\end{align*}
$$

where the curved indices of the components $\xi^i_{\mu}$ can be lowered, as usual, with the global AdS$_3$ metric $g_{\mu\nu}$. These Killing vectors $\xi_i$, as discussed before, are the vielbeins $e_i$ of the PCM action eq.(B.9) that construct the flat current eq.(B.11), namely

$$
j^i_{\pm} = \xi^i_{\mu} \partial_{\pm} X^\mu
$$

(B.22)
from which the Lax connection in eq. (B.14) is built as

$$\mathcal{L}_i^j = \frac{j^i_\pm}{1 \mp Z}. \quad \text{(B.23)}$$

The flatness eq. (B.15) of the PCM Lax connection results in two sets of equations. The first being the flatness eq. (B.12) of the Maurer-Cartan current, which is a structural fact as it can be easily checked by the reader. This is an identity to be expected, since this flatness equation can be realised as just the Cartan’s first structure equation applied on Killing vectors.

For the second set of equations we obtain the equations of motion eq. (B.13) of the PCM, the necessary condition for an integrable model.

To further validate all of the above, we have to secure that the equations of motion of the PCM coincide with the equations of motion of the bosonic string on AdS$_3$. For that purpose, we use the AdS$_3$ Killing vectors, eq. (B.21), on the PCM equations of motion eq. (B.13). That is

$$\partial_\tau (\xi^i_\mu \partial_- X^\mu) + \partial_- (\xi^i_\mu \partial_\tau X^\mu) = 0 \quad \text{(B.24)}$$

In particular, $\xi_1 = e_1$ (which lifts to a boost in the $Y^0 - Y^3$ plane of $\mathbb{R}^{2,2}$) gives

$$\cosh \rho \partial_\tau \partial_- t = - \sinh \rho (\partial_\tau \rho \partial_- t + \partial_\tau t \partial_- \rho) \quad \text{(B.25)}$$

which is the correct equation of motion for $t$, while $\xi_2 = e_2$ (which lifts to a rotation in the $Y^1 - Y^2$ plane of $\mathbb{R}^{2,2}$) gives,

$$\sinh \rho \partial_\tau \partial_- \phi = - \cosh \rho (\partial_\tau \rho \partial_- \phi + \partial_\tau \phi \partial_- \rho), \quad \text{(B.26)}$$

which is the correct equation of motion for $\phi$. Last but not least, $\xi_6 = e_6$ (which lifts to a rotation in the $Y^2 - Y^3$ plane of $\mathbb{R}^{2,2}$), supplemented with the above equations for $t$ and $\phi$, gives

$$\partial_\tau \partial_- \rho = \cosh \rho \sinh \rho (\partial_\tau \rho \partial_- \phi - \partial_\tau \phi \partial_- t) \quad \text{(B.27)}$$

which, of course, is the correct equation of motion for $\rho$.

In accordance with what we have discussed so far, the fact that it took just three of the six Killing vectors to deduce the equations of motion is just another manifestation of the actual degrees freedom of AdS$_3$, the same number as its independent Killing vectors.

### C Numerical analysis of the string soliton on the background of eqs. (4.1)-(4.3)

In this Appendix we will complement our analytical study of the integrability of the string worldsheet on the background (4.1-4.3) with a numerical analysis, following [30]. Our numerical analysis indeed confirms the integrability of the string worldsheet on this background.
This underlines the reliability of the numerical methods used in [28], where these same numerical methods were used to show that the dynamics of string solitons on the more general quiver solutions that we discussed in Appendix A, were non-integrable.

We will analyse the dynamics of a string soliton wrapping around the $\xi$-direction, and moving along the $\chi$ and $z$-directions of the internal space $\mathcal{M}_3$. This amounts to studying the numerical evolution of the last two equations of motion in eqs. (3.4), setting $\rho = \varphi = \theta = 0$. We study the solution where the function $\alpha(z)$ is given by

$$\alpha(z) = -81\pi^2 \sin\left(\frac{\pi z}{4}\right) \quad (C.1)$$

We will first study how the string moves through the $(z, \chi)$-plane. Note that the ‘energy’ of the classical string soliton - the integration constant $E$, that has to be tuned to satisfy the Virasoro constraint (3.5) - is given by

$$E^2 = f_1(z)^2 (\rho^2 + \sinh^2 \rho (\varphi^2 + \mu^2 \sin^2 \varphi)) + f_2(z) \dot{z}^2 + f_3(z) (\chi^2 + \kappa^2 \sin^2 \chi) \quad (C.2)$$

Note the energy minimises for the solutions $\rho = \varphi = \chi = 0$. We have seen in our analysis that if we move the string away from the points $\rho = 0$ and $\varphi = 0$ it will oscillate around these points. Moving the string away from the point $\chi = 0$ results in more complicated dynamics. Here we will first study the dynamics of the string when we increase $\chi$ away from the stable point $\chi = 0$ and study the fluctuations in the $z$ and $\chi$ directions. Below, we show plots of various observables. These plots are calculated using the special background in eqs (4.1)-(4.3). The reader should compare them with the figures displayed in Section 4 of [28], where an analysis of strings on generic (non-integrable) backgrounds was performed.

In figure 6a we see that if we start with an initial value that is very close to the poles of the 2-sphere ($\chi = 0$ and $\chi = \pi$), like $\chi(0) = 0.01$, the string tends to move back and forth linearly along the $z$-direction. Every time the string hits the end points on the $z$-domain it will flip to the other pole on the 2-sphere and start moving back along the $z$-direction.

As we increase the energy and allow the string to move further away from the poles, in Figure 6b, it will tend to oscillate around the poles on the 2-sphere and still flips to the other pole every time the string reaches the end of the $z$-domain. As we increase the energy of the string in Figure 6c we see that the string starts to oscillate stronger around the poles on the 2-sphere until the string gets enough energy to start moving freely around the entire 2-sphere, in Figure 6d.

When we look at the motion of the string in the $(z, \chi)$-plane it does appear that the motion of the string solition is quasi-periodic. As a comparison, compare the plots for the numerical evolution in Figure 6 with those of Figure 8 in [28]. There we see similar figures for the dynamics of a string on the $(z, \chi)$-plane on a background for which the analytical approach - based on Kovacic’s criteria - shows the solutions are non-Liouvilleian. At a first glance the dynamics for the non-Liouvilleian solution presented there appear much more chaotic.

**Power Spectra** We now discuss the power spectra of the trajectories $\chi(t)$ and $z(t)$ that are shown in Figure 7. Analysing these power spectra will allow us to see more clearly
if the dynamics is (quasi-)periodic, which will be characterised by having peaks at various frequencies - or chaotic, which will typically be characterised in the power spectra by a broad band of 'noise'.

In the first figure we see the lowest peak is located at a frequency of 0.02, corresponding to the motion of the string along the $z$-axis which takes roughly 55$t$. In addition to this, we see the higher harmonics that give rise to the exact shape of the trajectories.

As we increase the energy of the string soliton and allow it to probe a larger region of the 2-sphere we see that more higher harmonics show up, indicating that the dynamics might stay quasi-periodic, but that the string now follows a much more complicated trajectory.

As we go to even higher energies we see that over the course of one oscillation along the $z$-direction, the string starts to circle around in small loops. This is reflected in the power spectrum where we see clusters of higher peaks appear at several frequencies; For the fundamental frequency of 0.02 there is still a large peak. Second at 0.3 there is a group of smaller peaks, corresponding to the small fluctuations of the string that take roughly 3$t$ and can be seen in Figure 6c.

As we increase the energy even more, these smaller oscillations of roughly 3$t$ start to
Figure 7: Power spectra for the trajectories in Figures 6a-6d. Here the spectra for both $z(t)$ and $\chi(t)$ are shown in yellow and blue respectively. To calculate these spectra we ran a numerical evolution up to $t = 5000$ (roughly 100 oscillations along the $z$-direction), with a resolution of 10 data-points per time-unit.

dominate as the string now constantly oscillates through the entire 2-sphere and no longer seems to moves back and forth along the $z$-directions. This can again be seen in the Fourier spectrum where the largest peak at the fundamental frequency of 0.02 has now almost vanished, while there is a large and relatively broad peak at a frequency of 0.3, indicating the quasi-periodic behaviour that we see in Figure 6d. Compare this again with Figure 9 in [28], where we see similar figures for the dynamics of a string-soliton whose trajectories are non-Liouvillian. The power spectra presented for that string-soliton are characterised by wide bands of noise, corresponding to the more chaotic trajectories.

**Poincaré plots** - To study if we are indeed dealing with an integrable system where the dynamics is quasi-periodic we make a Poincaré plots for these four different energies. To make a Poincaré plot we choose different initial conditions in the $(\chi, p_z)$-plane, that all correspond...
to the same energy (C.2). We then run the numerical evolution for all these initial points and monitor the \((z, p_z)\)-plane every time the trajectories pass through the point \(\chi(t) = 0\).

If the dynamics of the string solitons we are studying was integrable, this classical mechanical system with \(2 \times 2\) degrees of freedom would have 2 independent integrals of motion that are in involution (meaning their Poisson bracket vanishes). The trajectories of this system would then be confined to the surfaces of a series of embedded 2-dimensional KAM tori in the \((z, p_z, \chi, p_\chi)\) phase-space.

We see this is exactly the case for the plots in Figure 8.

**LCE spectrum** - One other typical characteristic of an integrable classical mechanical system is a vanishing Lyapunov exponent. The Lyapunov exponent is a measure of the sensitivity of the system to its initial conditions. Typically for a chaotic system, two nearby initial points will diverge during the dynamical evolution as

\[
\Delta F(x_i, p_i, t) \sim \Delta F(x_i, p_i, 0) e^{\lambda_i x_i + \tilde{\lambda}_i p_i}
\]  

Figure 8: Poincaré sections for the \((z, p_z)\)-plane at \(\chi(t) = 0\), for different energies.
where the $\lambda$ are the Lyapunov exponents associated with the position directions in phase-space, and the $\tilde{\lambda}$ those associated with the momentum directions. Since our string soliton is a Hamiltonian system the initial volume in phase space will be conserved (as a consequence of Liouville’s theorem). The Lyapunov exponents will therefore satisfy the additional constraint that

$$\sum_{i=1}^{N} (\lambda^i + \tilde{\lambda}^i) = 0$$

where $N$ indicated the dimensionality of the Hamiltonian system. We see that the Lyapunov exponents in Figure 9 do converge to zero. This together with the results for the Poincaré plots, and the fact that the geometry with $\alpha(z)$ equal to (C.1) passes Kovacic’s criteria for all NVE seems to suggest that the string soliton might be integrable on this background.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{LCE_spectra.png}
\caption{LCE spectrum, for $t = 0$, $p_t = 6$, $z = 0.1$, $p_z = 1.58$, $\chi = 0.05$, $p_\chi = 0.05$. (b) LCE spectrum, for $t = 0$, $p_t = 6$, $z = 0.2$, $p_z = 0.73$, $\chi = 0.1$, $p_\chi = 0.25$.}
\end{figure}

\textbf{Figure 9}: Lyapunov exponents for different initial conditions.

\section*{References}

[1] J. M. Maldacena, Int. J. Theor. Phys. \textbf{38}, 1113 (1999) [Adv. Theor. Math. Phys. \textbf{2}, 231 (1998)] [hep-th/9711200].

[2] E. Witten, Nucl. Phys. B \textbf{443}, 85 (1995) [hep-th/9503124].

[3] N. Seiberg and E. Witten, Nucl. Phys. B \textbf{471}, 121 (1996) [hep-th/9603003].

[4] J. D. Blum and K. A. Intriligator, Nucl. Phys. B \textbf{506}, 199 (1997) [hep-th/9705044].

[5] N. Seiberg, Phys. Lett. B \textbf{390}, 169 (1997) [hep-th/9609161].

[6] U. H. Danielsson, G. Ferretti, J. Kalkkinen and P. Stjernberg, Phys. Lett. B \textbf{405}, 265 (1997) [hep-th/9703098].

[7] A. Hanany and A. Zaffaroni, Nucl. Phys. B \textbf{529}, 180 (1998) [hep-th/9712145].

[8] I. Brunner and A. Karch, JHEP \textbf{9803}, 003 (1998) [hep-th/9712143].
[9] K. Ohmori, H. Shimizu and Y. Tachikawa, JHEP 1408, 002 (2014) [arXiv:1404.3887 [hep-th]].
[10] M. Del Zotto, J. J. Heckman, A. Tomasiello and C. Vafa, JHEP 1502, 054 (2015) [arXiv:1407.6359 [hep-th]].
[11] C. Cordova, T. T. Dumitrescu and K. Intriligator, JHEP 1610, 080 (2016) [arXiv:1506.03807 [hep-th]].
[12] K. Intriligator, JHEP 1410, 162 (2014) [arXiv:1408.6745 [hep-th]].
[13] K. Ohmori, H. Shimizu, Y. Tachikawa and K. Yonekura, PTEP 2014, no. 10, 103B07 (2014) [arXiv:1408.5572 [hep-th]].
[14] M. Beccaria and A. A. Tseytlin, JHEP 1601, 001 (2016) [arXiv:1510.02685 [hep-th]].
[15] L. Bhardwaj, JHEP 1511, 002 (2015) [arXiv:1502.06594 [hep-th]].
[16] I. Bandos, H. Samtleben and D. Sorokin, Phys. Rev. D 88, no. 2, 025024 (2013) [arXiv:1305.1304 [hep-th]].
[17] C. M. Chang, arXiv:1810.04169 [hep-th].
[18] D. Gaiotto and A. Tomasiello, JHEP 1412, 003 (2014) [arXiv:1404.0711 [hep-th]].
[19] F. Apruzzi, M. Fazzi, A. Passias, D. Rosa and A. Tomasiello, JHEP 1411, 099 (2014) Erratum: [JHEP 1505, 012 (2015)] [arXiv:1406.0852 [hep-th]].
[20] F. Apruzzi, M. Fazzi, A. Passias, A. Rota and A. Tomasiello, Phys. Rev. Lett. 115, no. 6, 061601 (2015) [arXiv:1502.06616 [hep-th]].
[21] A. Passias, A. Rota and A. Tomasiello, JHEP 1510, 187 (2015) [arXiv:1506.05462 [hep-th]].
[22] F. Apruzzi and M. Fazzi, JHEP 1801, 124 (2018) [arXiv:1712.03235 [hep-th]].
[23] N. Bobev, G. Dibitetto, F. F. Gautason and B. Truijen, JHEP 1702, 116 (2017) doi:10.1007/JHEP02(2017)116 [arXiv:1612.06324 [hep-th]].
[24] S. Cremonesi and A. Tomasiello, JHEP 1605, 031 (2016) [arXiv:1512.02225 [hep-th]].
[25] A. Hanany and E. Witten, Nucl. Phys. B 492, 152 (1997) [hep-th/9611230].
[26] W. Nahm, Nucl. Phys. B 135, 149 (1978). S. Minwalla, Adv. Theor. Math. Phys. 2, 783 (1998) [hep-th/9712074].
[27] C. Nunez, D. Roychowdhury, S. Speziali and S. Zacarias, arXiv:1901.02888 [hep-th].
[28] C. Nunez, J. M. Penin, D. Roychowdhury and J. van Gorsel, JHEP 1806, 078 (2018) [arXiv:1802.04269 [hep-th]].
[29] U. Kol, C. Nunez, D. Schofield, J. Sonnenschein and M. Warschawski, JHEP 1406, 005 (2014) [arXiv:1403.2721 [hep-th]]. N. T. Macpherson, C. Nunez, L. A. Pando Zayas, V. G. J. Rodgers and C. A. Whiting, JHEP 1502, 040 (2015) [arXiv:1410.2650 [hep-th]].
[30] P. Basu and L. A. Pando Zayas, Phys. Lett. B 700, 243 (2011) [arXiv:1103.4107 [hep-th]].
[31] P. Basu and L. A. Pando Zayas, Phys. Rev. D 84, 046006 (2011) [arXiv:1105.2540 [hep-th]].
[32] C. Nunez, D. Roychowdhury and D. C. Thompson, JHEP 1807, 044 (2018) [arXiv:1804.08621 [hep-th]].
[33] K. Sfetsos, Nucl. Phys. B 880, 225 (2014) [arXiv:1312.4560 [hep-th]].
[34] K. Sfetsos and D. C. Thompson, JHEP 1412, 164 (2014) [arXiv:1410.1886 [hep-th]]. S. Driezen, A. Sevrin and D. C. Thompson, JHEP 1809, 015 (2018) [arXiv:1806.10712 [hep-th]].

[35] K. Sfetsos and D. C. Thompson, Nucl. Phys. B 846, 21 (2011) [arXiv:1012.1320 [hep-th]]. Y. Lozano and C. Nunez, JHEP 1605, 107 (2016) [arXiv:1603.04440 [hep-th]].

[36] O. Lunin and J. M. Maldacena, JHEP 0505, 033 (2005) [hep-th/0502086].

[37] L. Wulff, Phys. Rev. D 96, no. 10, 101901 (2017) [arXiv:1708.09673 [hep-th]]. L. Wulff, JHEP 1802, 106 (2018) [arXiv:1711.00296 [hep-th]].

[38] J. M. Maldacena, Phys. Rev. Lett. 80, 4859 (1998) [hep-th/9803002].

[39] C. Nunez, A. Paredes and A. V. Ramallo, Adv. High Energy Phys. 2010, 196714 (2010) [arXiv:1002.1088 [hep-th]]. R. Casero, C. Nunez and A. Paredes, Phys. Rev. D 73, 086005 (2006) [hep-th/0602027]. P. Koerber and D. Tsimpis, JHEP 0708, 082 (2007) [arXiv:0706.1244 [hep-th]]. A. Barranco, J. Gaillard, N. T. Macpherson, C. Nunez and D. C. Thompson, JHEP 1308, 018 (2013) [arXiv:1305.7229 [hep-th]].

[40] J. J. Kovacic, J. Symbolic Computation (1986) 2, 3-43.

[41] A. Torrielli, J. Phys. A 49, no. 32, 323001 (2016) [arXiv:1606.02946 [hep-th]].

[42] K. Zarembo, arXiv:1712.07725 [hep-th].

[43] G. Arutyunov and S. Frolov, JHEP 0502, 059 (2005) [hep-th/0411089].

[44] N. Beisert et al., Lett. Math. Phys. 99, 3 (2012) [arXiv:1012.3982 [hep-th]].
1 A general quiver and its associated holographic dual

Let us consider the most general quiver field theory. The gauge group is

$$SU(N_1) \times SU(N_2) \times \ldots \times SU(N_P).$$

The Flavour group, with each element associated with a colour node, is

$$SU(F_1) \times SU(F_2) \times \ldots \times SU(F_P).$$

The condition for gauge anomalies cancellation is,

$$2N_k - N_{k+1} - N_{k-1} = F_k. \quad (1.1)$$

We are using that $N_{-1} = N_{P+1} = 0$. The Rank function $R(z) = -\frac{\alpha''(z)}{81\pi^2}$ is,

$$R(z) = \begin{cases} 
N_1 z & 0 \leq z \leq 1 \\
N_1 + (N_2 - N_1)(z - 1) & 1 \leq z \leq 2 \\
N_2 + (N_3 - N_2)(z - 2) & 2 \leq z \leq 3 \\
N_k + (N_{k+1} - N_k)(z - k) & k \leq z \leq k + 1, \quad k := 3, \ldots, P \\
N_P - N_P(z - P) & P \leq z \leq P + 1. 
\end{cases} \quad (1.2)$$

From here we write the third derivative of $\alpha(z)$, satisfying $\alpha'''(z) = -162\pi^3 F_0(z)$ is,

$$\alpha'''(z) = -81\pi^2 \begin{cases} 
N_1 & 0 \leq z \leq 1 \\
(N_2 - N_1) & 1 \leq z \leq 2 \\
(N_3 - N_2) & 2 \leq z \leq 3 \\
(N_{k+1} - N_k) & k \leq z \leq k + 1, \quad k := 3, \ldots, P \\
-N_P & P \leq z \leq P + 1. 
\end{cases} \quad (1.3)$$

This function, piece-wise constant and discontinuous indicates the presence of D8 branes associated with the different flavour groups.

We would like to write the associated function $\alpha(z)$. This would need the triple integration of eq.(1.3) and the determination of the integration constants to ensure continuity of $\alpha(z)$ and $\alpha'(z)$. As was suggested in [1] we will instead periodically even-extend the function $\alpha'''(z)$ and integrate it three times (with vanishing integration constants). The result of this is,

$$\alpha(z) = \frac{162(P + 1)^3}{\pi^2} \sum_{n=1}^{P} \sum_{k=1}^{P} c(k, n) \sin \left( \frac{n\pi z}{(P + 1)} \right), \quad (1.4)$$

$$c(k, n) = \frac{N_k}{n^2} \left[ 2\sin \left( \frac{nk\pi}{(P + 1)} \right) - \sin \left( \frac{n(k + 1)\pi}{(P + 1)} \right) - \sin \left( \frac{n(k - 1)\pi}{(P + 1)} \right) \right].$$

The sum over the discrete variable $n$ can be performed and the function $\alpha(z)$ is written as a sum of various Polylogarithmic functions $Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$.

We can derive nice expressions for $N_{D6}$, $N_{D8}$ and the central charge or entanglement entropy,
using the formulas proposed or derived in the main body of the paper. See eqs. (??), (??), (??).

In fact, after some algebra we can write,

\[
N_{D6} = \frac{2(P + 1)^2}{\pi^3} \sum_{k=1}^{P} \sum_{n=1}^{\infty} c_{k,n} n (1 - (-1)^n) = \frac{4(P + 1)^2}{\pi^3} \sum_{k=1}^{P} \sum_{l=0}^{\infty} c_{k,(2l+1)} (2l + 1),
\]

\[
N_{D8} = -\frac{2}{\pi} \sum_{k=1}^{P} \sum_{n=1}^{\infty} c_{k,n} n^3 (1 - (-1)^n) = -\frac{4}{\pi} \sum_{k=1}^{P} \sum_{l=0}^{\infty} c_{k,(2l+1)} (2l + 1)^3, \tag{1.5}
\]

\[
c_{k,(2l+1)} = \frac{N_k}{(2l + 1)^4} \left[ 2 \sin \left( \frac{(2l + 1)k\pi}{(P + 1)} \right) - \sin \left( \frac{(2l + 1)(k + 1)\pi}{(P + 1)} \right) - \sin \left( \frac{(2l + 1)(k - 1)\pi}{(P + 1)} \right) \right].
\]

Instead of calculating the entanglement entropy, we find an expression for the central charge. In the paper [1], see eq.(2.14), it was derived that the central charge can be holographically calculated as,

\[
c = -\frac{g_s}{3^8} \times 16 \times G_N \int_0^{P+1} \alpha(z) \alpha''(z) dz.
\]

After some algebra we find,

\[
c = \sum_{k=1}^{P} \sum_{j=1}^{P} N_k N_j C_{kj}. \tag{1.6}
\]

\[
C_{kj} = \frac{4}{\pi^3} (P + 1)^3 \sum_{n=1}^{\infty} \frac{1}{n^8} \left[ 2 \sin \left( \frac{n k \pi}{(P + 1)} \right) - \sin \left( \frac{n (k + 1) \pi}{(P + 1)} \right) - \sin \left( \frac{n (k - 1) \pi}{(P + 1)} \right) \right] \times
\]

\[
\left[ 2 \sin \left( \frac{n j \pi}{(P + 1)} \right) - \sin \left( \frac{n (j + 1) \pi}{(P + 1)} \right) - \sin \left( \frac{n (j - 1) \pi}{(P + 1)} \right) \right].
\]

As was explained in [2], the supergravity approximation for the holographic dual to these SCFTs is trustable if the quivers are long. This implies that \( P \to \infty \). We can evaluate the expression in eq.(1.6) in the limit of large \( P \).

**References**

[1] C. Nunez, J. M. Penin, D. Roychowdhury and J. Van Gorsel, JHEP 1806, 078 (2018) [arXiv:1802.04269 [hep-th]].

[2] S. Cremonesi and A. Tomasiello, JHEP 1605, 031 (2016) [arXiv:1512.02225 [hep-th]].