A four-wing chaotic attractor and its circuit implementation

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Abstract. By introducing a state feedback control to a proposed four dimensional chaotic system, an extremely complex four-wing chaotic attractor is derived having larger positive Lyapunov Exponent (LE) than other chaotic systems. Spectral analysis shows that the system in the four-wing chaotic mode has very broad frequency bandwidth, verifying its random nature, and indicating the prospect for engineering applications such as secure communications. Finally, an analog circuit, implementing the new four-wing chaotic system, is presented.

1. Introduction
In the investigation of chaos theory and applications, it is very important to generate new chaotic systems or to enhance complex dynamics and topological structure based on the existing chaotic attractors. Generalizing Chua’s circuit with multi-scroll attractors and generalizing the Lorenz system with double-wing in a butterfly shape are the results of two important research efforts [1]. Lü et al. [2] surveyed the generation of the multi-scroll chaotic attractors, including some fundamental theories, design methodologies, circuit implementations, and practical applications. Therefore, it is hence no longer a very difficult task to generalize Chua’s circuit with multi-scroll attractors. Much investigation also focused on the generalization of the Lorenz system [6]. The Chen system, the generalized Lorenz system family, and the hyperbolic-type of generalized Lorenz canonical form [2-8] has been generated. All these Lorenz-like systems are smooth with two quadratic terms, all have three equilibria, and produce a double-wing attractor. Therefore, it is very significant to generate multi-wing chaotic attractors from an autonomous chaotic system with complicated topological structures including multi-wings, wider frequency bandwidths, and more complex dynamics such as rich bifurcations.

Recently, we have found a new four-dimensional autonomous chaotic system, in which each equation contains a cubic term [9]. This system has very rich nonlinear dynamics, including chaos, period-doubling bifurcations, sinks, sources, etc. It can display: two coexisting symmetric single-wing chaotic attractor, two coexisting double-wing chaotic attractors [10].

In this paper, by adding a simple linear term to the system, we are able to generate a real four-wing attractor. It is very significant that the new four-wing chaotic attractor has a more complicated topological structure and dynamics characterized by a larger positive Lyapunov exponent. Spectral analysis shows that the system in the four-wing chaotic mode has a very broad frequency bandwidth, verifying its random nature, and indicating the prospect for engineering applications such as secure communications.

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communications. Finally, an electronic circuit, implementing the new four-wing chaotic system is also presented.

2. The new system with a four-wing attractor
Qi et al. [9] proposed a 4-D autonomous system with the cubic nonlinearities which is described by

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_2 x_3 x_4, \\
\dot{x}_2 &= b(x_1 + x_2) - x_1 x_3 x_4, \\
\dot{x}_3 &= -c x_1 + x_1 x_2 x_4, \\
\dot{x}_4 &= -d x_4 + x_1 x_2 x_3,
\end{align*}
\]

(1)

Here, \( x_i \) (\( i = 1, 2, 3, 4 \)) are the state variables and \( a, b, c, d \) are positive real constants. It displays simultaneously the upper attractor and the lower attractor dependent of initial subspace [10], but cannot produce a four-wing chaotic attractor. By further study, we found that it is the key of generation of four-wing chaotic attractor to destroy the exchangeable and symmetric properties of system (1) between the third or fourth equations.

Now, introducing a simple linear state-feedback in the third equation of system (1), we get the following system,

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_2 x_3 x_4, \\
\dot{x}_2 &= b(x_1 + x_2) - x_1 x_3 x_4, \\
\dot{x}_3 &= -c x_1 + e x_2 + x_1 x_2 x_4, \\
\dot{x}_4 &= -d x_4 + x_1 x_2 x_3,
\end{align*}
\]

(2)

where \( e \) is a constant parameter. The new linear term will produce some new interesting complex dynamics.

When \( a = 50, \ b = 4.3, \ c = 13, \ d = 20, \ e = 6 \) , system (2) is in a chaotic mode with the largest LE \( l_1 = 2.0158 \), as shown in Figs. 1(a, b). One sees that the orbit is in chaos independent of the initial region, and some orbits move among equilibria \( S_{1,3} \) and \( S_{1,4} \) and gradually form a four-wing chaotic attractor, in which the symbols ‘○’, ‘□’, ‘◇’, ‘*’, ‘+’ denote \( S_1, S_3, S_2, S_3, S_4 \), respectively. With further changes of parameters and an increase of the first positive LE, till \( a = 50, \ b = 10, \ c = 13, \ d = 20, \ e = 6 \) with largest LE \( l_1 = 7.9812 \), a very complicated four-wing chaotic attractor are clearly generated. Fig. 2 shows the observation in different projective planes. It is notable that there exist many flows freely running around \( S_{1,4}, S_{1,2} \) and \( S_{1,4} \). The diagonal and anti-diagonal orbits play an important role in forming the real four-wing attractor, since they connect the upper and the lower sub-attractors together. Every variable of all orbits can freely move across the boundary to the opposite side. The system equilibria \( S_i, i = 1, \cdots, 4 \), are located at the centers of the four wings of the attractor, and the origin is the center of the whole chaotic attractor, as shown in Fig. 2.

The orbits between any two equilibria resemble the butterfly shape of the Lorenz chaotic attractor, which as a whole forms a singular tornado-like shape with four inner holes, as shown in Fig. 2(d).

According to the distribution of the dots ‘·’ marked in Figs. 2(c), one can see that the density at the center of each wing is greater than that at their peripheries, which implies that the running speed of near-center orbits of each wing is lower than that of the outer orbits. Moreover, the folding and stretching of near-center orbits are more severe than those of the outer orbits.

It is well known that the largest positive LE of an attractor is compatible with exponential stretching and folding of its orbits in a bounded space [11], i.e the value of the largest positive LE measures the degree of the disorder of the attractor. The largest LE of the four-wing chaotic system is \( l_1 = 10.5 \) by calculation at parameters values \( a = 50, b = 16, \ c = 13, d = 20, e = 6 \) , but that of the Lorenz system is about 0.9 . Therefore, the four-wing chaotic system has extremely rich frequency properties which will
be further proved by the frequency analysis in section 3.

Fig. 1. The four-wing chaotic attractor: $a = 50, b = 4.3, c = 13, d = 20, e = 6$, and $l_1 = 2.0158$.

Fig. 2. The four-wing chaotic attractor: $a = 50, b = 10, c = 13, d = 20, e = 6$, and $l_1 = 7.9812$.

3. Frequency spectral analysis

Many proposed chaos-based encryption schemes have been totally or partially broken by different attacks. One of the reasons is that the degree of randomness of simple chaotic signals is not high enough, as reflected by the narrow bandwidths of those signals [12].

Fig. 4(a) shows the frequency spectra of the Lorenz system with largest LE $l_1 = 0.9$. For comparison, the Runge-Kutta method was used to solve all systems, with sampling time 0.001(s), running time 0-50 (s), the number of spectral averages 50, and all spectra are normalized, where only the frequency ranges with spectral values greater than $10^{-1}$ are considered.

The bandwidth of the Lorenz system is approximately 4Hz. Therefore the chaotic signals cannot be used to sufficiently mask the messages in real communication applications. Once intercepted, there is a high possibility that the messages can be extracted.

System (2) generates a simple four-wing chaotic attractor with a positive LE $l_1 = 2.0158$ for the parameters $a = 50, b = 4.2, c = 13, d = 20, e = 6$, as shown in Fig. 1. Its bandwidth is about 15Hz, as...
shown in Fig. 4(b). It is obvious that the value of the largest positive LE and the complex structure of a dynamical system play an important role in determining bandwidth and degree of disorder of the orbit of the chaotic system. When \( a = 50, b = 10, c = 13, d = 20, e = 6 \), system (2) generates complicated four-chaotic attractor with largest LE \( I_1 = 7.9812 \), as shown in Fig. 2. Its bandwidth increases at 38 Hz. When using the parameters \( a = 50, b = 16, c = 13, d = 20, e = 6 \), for system (2) has an extremely complicated four-wing chaotic attractor with largest LE \( I_1 = 10.5 \), and its frequency spectrum bandwidth is at 67 Hz which is 16 times than that of Lorenz system.

![Image of frequency spectra](image)

(a) The Lorenz system: \( I_1 = 0.9 \)  
(b) The simple four-wing chaotic attractor: \( I_1 = 2.0158 \)

![Image of frequency spectra](image)

(c) The complicated four-wing chaotic attractor: \( I_1 = 7.9812 \)  
(d) The extremely complicated four-wing chaotic attractor: \( I_1 = 10.5 \)

Fig. 4. Frequency spectra about the Lorenz system and three groups of four-wing chaotic attractor of system (2) at different parameter points.

4. Circuit realization of the four-wing chaotic system

The electronic circuit in Fig. 5 has been designed and built to realize system (1) with four channels to perform the integration of the three state variables, \( x_1, x_2, x_3, x_4 \), respectively. The circuit employs four analog multipliers used to realize the three quadratic terms in the system, and has 14 operational amplifiers, along with some linear resistors and capacitors to perform additions, subtractions, multiplications, and integrations. The circuit computes the original equations directly.
Fig. 6 shows the experimental results observed on an oscilloscope. These results are qualitatively identical to the numerical four-wing chaotic attractor shown in Fig. 2, confirming the existence of the four-wing chaotic attractor.

Fig. 5. The diagram of implementation of the four-wing

(a) Projection on $x_1 - x_2$

Fig. 6. The phase portraits of the hyperchaotic chaotic system observed on the oscilloscope.

5. Conclusions
A new four-wing chaotic attractor discovered numerically and was confirmed experimentally by introducing state feedback control to a known chaotic system. The largest positive LE has demonstrated that the new chaotic system has extremely complicated dynamics and structure. Spectral analysis has shown that the system in the four-wing chaotic mode has a very broad frequency bandwidth, implying that
it is promising in engineering applications. The generation of an eight-wing chaotic system attractor is under investigation.

Fig. 6. (Continued)

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