ODE/IM correspondence and modified affine Toda field equations

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July 31, 2015@National Taiwan University

KI and C. Locke, arXiv:1312.6759, 1502.00906
Introduction

- The ODE/IM correspondence is a relation between spectral analysis of ordinary differential equation (ODEs), and the “functional relations” approach to 2d quantum integrable model (IM). [Dorey-Tateo]
- This is an example of the correspondence between classical and quantum integrable models
- has many applications
  - gluon scattering amplitudes/null polygonal Wilson loops in $\mathcal{N} = 4$ SYM at strong coupling and minimal surface in AdS spacetime [Alday-Maldacena]
  - BPS spectrum in $\mathcal{N} = 2$ SUSY gauge theories [Gaiotto-Moore-Neitzke]
  - PT-symmetric quantum mechanics $\mathcal{H} = p^2 + ix^3$
  - $t - t^*$ equations in SUSY field theories
Dorey-Tateo (1998) studied the spectral determinant of the quartic potential and its relation to the $A_3$-related Y-system.

ODE/IM correspondence for classical Lie algebras
Dorey-Dunning-Masoero-Suzuki-Tateo, 2006

Lukyanov-Zamolodchikov (2010) studied the linear problem associated with the modified sinh-Gordon equation in the context of ODE/IM correspondence ($A^{(1)}_1$: $\varphi_{tt} - \varphi_{xx} + \sinh \varphi = 0$)

The results were generalized to the case of Tzitzéica-Bullough-Dodd equation by Dorey et al. (2012). ($A^{(2)}_2$: $\varphi_{tt} - \varphi_{xx} + e^{2\varphi} - e^{-\varphi} = 0$)

We will

- Introduce the affine Toda field equation and its linear problem
- Discuss the conformal limit and its relation to the ODE/IM correspondence
- Study the Bethe ansatz equations for affine Lie algebras
The general scheme of the ODE/IM correspondence for affine Toda equation is [Dorey-Faldella-Negro-Tateo]

\[
\text{ODE} \leftrightarrow \text{ODE/IM} \leftrightarrow \text{BAE} \leftrightarrow \text{CFT}
\]

↑ Conformal limit

↑ UV limit

↑ Linear problem

↑ affine Toda equation

↑ Conformal limit

↑ UV limit

↑ Linear problem

↑ affine Toda equation
1 Introduction

2 ODE/IM correspondence and modified sinh-Gordon equation

3 affine Toda field equations

4 Conformal Limit and ODE/IM correspondence

5 ψ-system and Bethe ansatz equations

6 Outlook
OED/IM correspondence

[Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov]

- **ODE**

\[
- \frac{d^2}{dx^2} + \frac{\ell(\ell + 1)}{x^2} + x^{2M} - E \right] y(x, E, \ell) = 0
\]

- **large, real positive** \(x\) **asymptotics**: 
  \[ y \sim \frac{x^{-\frac{M}{2}}}{\sqrt{2i}} \exp \left( - \frac{x^{M+1}}{M+1} \right) \]

- **subdominant (small) solution** in the sector \(|\arg x| < \frac{\pi}{2M+2}\)

\[ y_k(x, E, \ell) = \omega^k y(\omega^{-k} x, \omega^{2k} E, \ell) \quad (\omega = \exp(\frac{2\pi i}{2M+2})) \]

\( \{y_k, y_{k+1}\} \) **forms a basis** of solutions for the ODE:

- **\(y_k\)** obeys the **Stokes relation**

\[ C(E, \ell)y_0(x, E, \ell) = y_{-1}(x, E, \ell) + y_1(x, E, \ell) \]

The coefficient \(C(E, \ell)\) is called the **Stokes multiplier**.
small $x$ asymptotics: $\psi(x, E, \ell) \sim x^{\ell+1}$ (other solution is $x^{-\ell}$)

Take the Wronskian ($W[f, g] := fg' - f'g$) of both sides of the Stokes relation with $\psi$

$$C(E, \ell)W[y_0, \psi](E, \ell) = W[y_{-1}, \psi](E, \ell) + W[y_1, \psi](E, \ell)$$

Setting $D(E, \ell) = W[y_0, \psi]$, the above relation is

$$C(E, \ell)D(E, \ell) = \omega^{-(\ell+\frac{1}{2})}D(\omega^{-2}E, \ell) + \omega^{\ell+\frac{1}{2}}D(\omega^{2}E, \ell)$$

**T-Q relation:** ($D$: Q-function (spectral determinant), $C$: T-function)

$\psi_+ = \psi(x, E, \ell)$, $\psi_- = \psi(x, E, -\ell - 1)$ are linearly independent solutions. The Wronskian $W[\psi_+, \psi_-]$ yields the quantum Wronskian relations for $D$.

$$(2\ell + 1) = \omega^{-(\ell+\frac{1}{2})}D_-(\omega^{-1}E)D_+(\omega E) - \omega^{\ell+\frac{1}{2}}D_-(\omega E)D_+(\omega^{-1}E)$$
One can then derive the Bethe ansatz equation from the T-Q or quantum Wronskian relation.

$E_n^\pm$: zeros of $D^\pm(E)$

$$\frac{D_\pm(\omega^{-2} E_n^\pm)}{D_\pm(\omega^2 E_n^\pm)} = -\omega^{\pm(\ell+\frac{1}{2})}$$

Expanding $D_\pm(E) \sim \prod_{m=1}^{\infty} (1 - \frac{E}{E_m^{\pm}})$, we get the BA eq.

$$\prod_{m=1}^{\infty} \frac{E_{m}^{\pm} - \omega^{-2} E_n^{\pm}}{E_{m}^{\pm} - \omega^2 E_n^{\pm}} = -\omega^{\pm(2\ell+1)}$$
### Dictionary

| ODE | I(ntegrable) M(odel) |
|-----|----------------------|
| \[
- \frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^2M - E \]
\[ y = 0 \]
energy \( E \)
degree of potential \( M \)
angular momentum \( \ell \)
Stokes multiplier \( C(E, \ell) \)
spectral determinant \( D(E, \ell) \)
the Stokes relation | 6-vertex model
spectral parameter
anisotropy
twist parameter
Transfer matrix (T-function)
Q-operator
T-Q relation |
modified sinh-Gordon equation: $A_1^{(1)}$

We discuss the relation between the ODE and the modified Sinh-Gordon equation [Lukyanov-Zamolodchikov 1003.5333]

\[
\partial_z \partial_{\bar{z}} \eta - e^{2\eta} + p(z) \bar{p}(\bar{z}) e^{-2\eta} = 0, \quad p(z) = z^2M - s^2M
\]

zero curvature condition $[\partial + A, \bar{\partial} + \bar{A}] = 0$

\[
A = \frac{1}{2} \partial_z \eta \sigma^3 - e^\theta (\sigma^+ e^n + \sigma^- pe^{-n})
\]

\[
\bar{A} = -\frac{1}{2} \partial_{\bar{z}} \eta \sigma^3 - e^{-\theta} (\sigma^+ e^n + \sigma^- \bar{p}e^{-n})
\]

asymptotic behavior of $\eta(z, \bar{z})$ at $\rho \to 0, \infty$ ($z = \rho e^{i\phi}$)

- $\eta \to M \log \rho$ ($\rho \to \infty$)
- $\eta \to \ell \log \rho$ ($\rho \to 0$)

We introduce a new parameter $\ell$ for the boundary condition at $\rho = 0$. 
linear system and its solutions

- linear problem \((\partial + A)\Psi = (\bar{\partial} + \bar{A})\Psi = 0\)
- linear problem is invariant under
  \[\Omega: \phi \rightarrow \phi + \frac{\pi}{M}, \theta \rightarrow \theta - \frac{i\pi}{M}\]
  \[\Pi: \theta \rightarrow \theta + i\pi, \hat{\Pi}[A] = \sigma^3 A \sigma^3\]
- \(\rho \rightarrow \infty\), from the WKB analysis, subdominant solution is
  \[\Xi \sim \begin{pmatrix} e^{iM\phi} \\ e^{-iM\phi} \end{pmatrix} \exp \left( -\frac{2\rho^{M+1}}{M+1} \cosh(\theta + i(M + 1)\phi) \right)\]
- \(\rho \rightarrow 0\) basis \(\Psi_+(\rho, \phi|\theta) \rightarrow \begin{pmatrix} 0 \\ e^{(i\phi+\theta)\ell} \end{pmatrix}\), \(\Psi_-(\rho, \phi|\theta) \rightarrow \begin{pmatrix} e^{(i\phi+\theta)\ell} \\ 0 \end{pmatrix}\)

\[\Xi = Q_-(\theta)\Psi_+ + Q_+(\theta)\Psi_-\]

\(Q_\pm(\theta)\) are the Q-function of the quantum Sinh-Gordon model
From MShG to ODE

- take the light-cone limit $\bar{z} \to 0$. Then linear system reduced to a differential equation.

$$
\Psi = \begin{pmatrix}
  e^{\frac{\theta}{2}} e^{\frac{n}{2}} \psi \\
  e^{-\frac{n}{2}} e^{\frac{\theta}{2}} (\partial_z + \partial_z \eta) \psi
\end{pmatrix}
$$

$$
[\partial_{\bar{z}}^2 - u - e^{\theta} p] \psi = 0, \quad u = (\partial_z \eta)^2 - \partial_z^2 \eta
$$

- conformal limit: $z \to 0$, $\theta \to \infty$

$$
x = z e^{\frac{\theta}{M+1}}, \quad E = s^{2M} e^{\frac{2\theta M}{1+M}}, \quad \text{fixed}
$$

$$
\left[-\partial_x^2 + \frac{\ell(\ell + 1)}{x^2} + x^{2M}\right] \psi = E\psi
$$

Schrödinger type ODE [Dorey-Tateo, BLZ]
affine Toda field equations (1)

\( \mathfrak{g} \): a simple Lie algebra of rank \( r \)

\[
\begin{align*}
[E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta}, & \text{for } \alpha + \beta \neq 0, \\
[E_\alpha, E_{-\alpha}] &= \frac{2\alpha \cdot H}{\alpha^2}, \\
[H^i, E_\alpha] &= \alpha^i E_\alpha.
\end{align*}
\]

\( \alpha_1, \cdots, \alpha_r \): the simple roots of \( \mathfrak{g} \)
\( \alpha^\vee_1, \cdots, \alpha^\vee_r \): simple coroots
\( \alpha_0 = -\theta \) (\( \theta \): the highest root)
(dual) Coxeter labels: \( \sum_{i=0}^r n_i \alpha_i = \sum_{i=0}^r n^\vee_i \alpha^\vee_i = 0 \).
(dual) Coxeter number \( h, h^\vee \):

\[
h = \sum_{i=0}^r n_i, \quad h^\vee = \sum_{i=0}^r n^\vee_i.
\]
affine Toda field equations (2)

\[ \mathcal{L} = \frac{1}{2} \partial^\mu \phi \cdot \partial_\mu \phi - \left( \frac{m}{\beta} \right)^2 \sum_{i=0}^{r} n_i \left[ \exp(\beta \alpha_i \cdot \phi) - 1 \right], \]

\[ \partial^\mu \partial_\mu \phi + \left( \frac{m^2}{\beta} \right) \sum_{i=0}^{r} n_i \alpha_i \exp(\beta \alpha_i \phi) = 0. \]

complex coordinates: \( z = \frac{1}{2}(x^0 + ix^1), \quad \bar{z} = \frac{1}{2}(x^0 - ix^1) \quad (z = \rho e^{i\theta}) \)

conformal transformation (\( \rho^\vee \): co-Weyl vector)

\[ z \rightarrow \bar{z} = f(z), \quad \phi \rightarrow \bar{\phi} = \phi - \frac{1}{\beta} \rho^\vee \log(\partial f \bar{\partial} f), \]

modified affine Toda equations:

\[ \partial \bar{\partial} \phi + \left( \frac{m^2}{\beta} \right) \left[ \sum_{i=1}^{r} n_i \alpha_i \exp(\beta \alpha_i \phi) + p(z)\bar{p}(\bar{z})n_0 \alpha_0 \exp(\beta \alpha_0 \phi) \right] = 0, \]

\[ p(z) = (\partial f)^h, \quad \bar{p}(\bar{z}) = (\bar{\partial} f)^h. \]
Lax formalism

- The modified affine Toda equation can be expressed as a linear problem: \((\partial + A)\Psi = 0\) and \((\bar{\partial} + \bar{A})\Psi = 0\).

\[
A = \frac{\beta}{2} \partial \phi \cdot H + me^\lambda \left\{ \sum_{i=1}^{r} \sqrt{n_i^\vee} E_{\alpha_i} e^{\frac{\beta}{2} \alpha_i \phi} + p(z) \sqrt{n_0^\vee} E_{\alpha_0} e^{\frac{\beta}{2} \alpha_0 \phi} \right\},
\]

\[
\bar{A} = -\frac{\beta}{2} \bar{\partial} \phi \cdot H - me^{-\lambda} \left\{ \sum_{i=1}^{r} \sqrt{n_i^\vee} E_{-\alpha_i} e^{\frac{\beta}{2} \alpha_i \phi} + \bar{p}(\bar{z}) \sqrt{n_0^\vee} E_{-\alpha_0} e^{\frac{\beta}{2} \alpha_0 \phi} \right\}
\]

- zero-curvature condition: \([\partial + A, \bar{\partial} + \bar{A}] = 0 \implies \text{affine Toda field equations}\)
symmetries and $p(z)$

- Motivated by the ODE/IM correspondence, we put

$$p(z) = z^{hM} - s^{hM}, \quad \bar{p}(\bar{z}) = \bar{z}^{hM} - \bar{s}^{hM}$$

$h$: the Coxeter number, and $M$ is some positive real parameter

- We define the transformation $\hat{\Omega}_k$ and $\hat{\Pi}$

$$\hat{\Omega}_k : \left\{ \begin{array}{l}
z \rightarrow ze^{\frac{2\pi ki}{hM}} \\
s \rightarrow se^{\frac{2\pi ki}{hM}} \\
\lambda \rightarrow \lambda - \frac{2\pi ki}{hM} \end{array} \right.$$

$$\hat{\Pi} : \left\{ \begin{array}{l}
\lambda \rightarrow \lambda - \frac{2\pi i}{h} \\
A \rightarrow SAS^{-1}, \quad S = \exp\left(\frac{2\pi i}{h} \rho^\vee \cdot H\right) \end{array} \right.$$

- The equation of motion and linear problem are invariant under $\hat{\Omega}_k$ for integer $k$.

- $SE_{\alpha_i}S^{-1} = e^{2\pi i/h} E_{\alpha_i}$ ($i = 1, \cdots, r$)
asymptotic behavior of the Toda field

- In the large $|z|$ region, asymptotic solution is

$$\phi(z, \bar{z}) = \frac{M}{\beta} \rho^\nu \log(z\bar{z}) + O(1)$$

- For small $|z|$, we assume logarithmic behavior, with expansion

$$\phi(z, \bar{z}) = g \log(z\bar{z}) + \phi^{(0)}(g) + \gamma(z, \bar{z}, g) + \sum_{i=0}^{r} \frac{C_i(g)}{(c_i(g) + 1)^2} (z\bar{z})^{c_i(g) + 1} + \ldots.$$  

$g$ is an $r$-component vector

- Substituting this expansion into the Toda equation, we can determine the constants $C_i$

- The exponents are found to be $c_i + 1 = 1 + \beta \alpha_i \cdot g > 0$.  

$A_r^{(1)}$ modified affine Toda [KI-Locke, Adamopuolou-Dunning]

- $A_r^{(1)}$ is the simplest algebra to start with, and includes the sinh-Gordon model as a specific example.
- The fundamental representation with highest weight $\omega_1$ weights are $h_1 = \omega_1$, $h_i = \omega_i - \omega_{i+1}$, $h_{r+1} = -\omega_r$, where $\omega_i$ are the fundamental weights defined by $\omega_i \cdot \alpha_j^\vee = \delta_{ij}$.
- The linear problem $(\partial_z + A)\Psi = 0$, $\Psi = t(\psi_1, \cdots, \psi_{r+1})$

holomorphic connection:

$$A = \begin{pmatrix}
\frac{\beta}{2} h_1 \cdot \partial \phi & me^\lambda e^{\frac{\beta}{2} \alpha_1 \cdot \phi} & 0 & \cdots & 0 \\
0 & \frac{\beta}{2} h_2 \cdot \partial \phi & me^\lambda e^{\frac{\beta}{2} \alpha_2 \cdot \phi} & \ddots & \\
\vdots & & \ddots & \ddots & \frac{\beta}{2} h_r \cdot \partial \phi & me^\lambda e^{\frac{\beta}{2} \alpha_r \cdot \phi} \\
me^\lambda p(z) e^{\frac{\beta}{2} \alpha_0 \cdot \phi} & \cdots & 0 & \frac{\beta}{2} h_{r+1} \cdot \partial \phi
\end{pmatrix}.$$
• gauge transformation: \( U = \text{diag}(e^{-\frac{\beta}{2} h_1 \cdot \phi}, \ldots, e^{-\frac{\beta}{2} h_{r+1} \cdot \phi}) \)

\[
\tilde{A} = UAU^{-1} + U \partial U^{-1}, \quad \tilde{\Psi} = U \Psi,
\]

\[
\tilde{A} = \begin{pmatrix}
\beta h_1 \partial \phi & me^\lambda & 0 & \cdots & 0 \\
0 & \beta h_2 \partial \phi & me^\lambda & \ddots & \\
0 & \ddots & \ddots & \ddots & \\
me^\lambda p(z) & \ddots & \beta h_r \partial \phi & me^\lambda & 0 \\
& & & \beta h_{r+1} \partial \phi & \end{pmatrix}.
\]

• the linear problem becomes a single \((r + 1)\)-th order differential equation

\[
D(h_{r+1}) \cdots D(h_1) \tilde{\psi}_1 = (-me^\lambda)^h p(z) \tilde{\psi}_1.
\]

\[
D(h) \equiv \partial + \beta h \cdot \partial \phi
\]

• scalar Lax operator (Gelfand-Dickii, Drinfeld-Sokolov reduction)
For the barred linear equation, a different gauge transformation is used to simplify the equations

\[ U = \text{diag}(e^{\frac{\beta}{2} h_1 \cdot \phi}, \ldots, e^{\frac{\beta}{2} h_{r+1} \cdot \phi}), \quad \sim = U \Psi \]

The full linear problem gives the differential equations

\[ D(h_{r+1}) \cdots D(h_1) \psi = (-me^\lambda)^h p(z) \psi \]
\[ \bar{D}(-h_1) \cdots \bar{D}(-h_{r+1}) \bar{\psi} = (me^{-\lambda})^h \bar{p}(\bar{z}) \bar{\psi} \]

where \( \psi = \tilde{\psi}_1 \) and \( \bar{\psi} = \tilde{\psi}_{r+1} \)
Massive ODE/IM correspondence

- For small $|z|$ solution $\psi^{(i)} \sim z^{\mu_i}$ define the vector $\Psi^{(i)}$ with
  $$(\Psi^{(i)})_j \sim \delta_{ij}(\bar{z}/z)^{\frac{\beta}{2}} h_i$$

- For large $|z|$ the small solution is
  $$\Xi(\rho, \theta|\lambda) \sim C \begin{pmatrix} e^{-\frac{irM\theta}{4}} \\ e^{-\frac{i(r-2)M\theta}{4}} \\ \vdots \\ e^{-\frac{i\rho M\theta}{4}} \end{pmatrix} \exp \left(-\frac{2\rho^{M+1}}{M+1} m \cosh(\lambda + i\theta(M+1))\right)$$

- we can expand $\Xi$ as
  $$\Xi = \sum_{i=0}^{r} Q_i(\lambda) \Psi^{(i)}.$$

For $A_1^{(1)}$ (sinh-Gordon) $A_2^{(2)}$ (Tzitzéica-Bullough-Dodd), the Q-coefficients correspond to the Q-function of a 2D massive QFT.
$A_r^{(1)}$: KI-Locke, Adamopoulou-Dunning, 1401.1187
Conformal Limit and ODE/IM correspondence

- First we take the light-cone limit \( \bar{z} \to 0 \) and we define the conformal limit \( z \to 0, \lambda \to \infty \) with fixed

\[
x = (me^\lambda)^{(M+1)}z, \quad E = s^{hM}(me^\lambda)^{hM/(M+1)}
\]

- The differential equation becomes

\[
\left[ D_x(h_{r+1}) \cdots D_x(h_1) - (-1)^hp(x, E) \right] \psi(x, E, g) = 0
\]

where \( D_x(a) = \partial_x + \beta \frac{a \cdot g}{x} \) and \( p(x, E) \equiv x^{hM} - E \).

- This is the ODE for \( A_r \)-type Lie algebra Suzuki, Dorey-Dunning-Tateo

- By writing out the unique asymptotically decaying solution \( \xi(x, E, g) \) to this equation in terms of the small \( x \) basis \( \chi^{(i)} \sim x^{\mu_i} + O(x^{\mu_i+h}) \), we have

\[
\xi(x, E, g) = \sum_{i=0}^{r} Q^{(i)}(E) \chi^{(i)}(x, E, g)
\]
• Symanzik rotation \( \psi_k(x, E, g) = \psi(\omega^k x, \Omega^k E, g) \) with \( \Omega = \exp(i \frac{2\pi M}{M+1}) \) and \( \omega = \exp(i \frac{2\pi}{n}) \)

• auxiliary functions: \( \psi^{(a)} = W^{(a)}[\psi^{\frac{1-a}{2}}, \ldots, \psi^{\frac{a-1}{2}}] \) (\( a = 2, \ldots, r \))

• \( A_n \) \( \psi \)-system (Plücker relations)

\[ \psi^{(a-1)} \psi^{(a+1)} = W[\psi^{\frac{a-1}{2}}, \psi^{\frac{a}{2}}], \quad \psi^{(0)} = \psi^{(n)} = 1 \]

• quantum Wronskian relation

\[
Q^{(a+1)}(a-1) = \omega^{\frac{1}{2}}(\mu_a - \mu_{a-1}) Q^{(a)}(a) \bar{Q}^{(a)}(\frac{a}{2}) - \omega^{\frac{1}{2}}(\mu_{a-1} - \mu_a) Q^{(a)}(\frac{a}{2}) \bar{Q}^{(a)}(\frac{a-1}{2})
\]

• Bethe ansatz equation

\[
\omega^{\mu_{i-1} - \mu_i} \frac{Q^{(i-1)}(E^{(i)}_n) Q^{(i)}(E^{(i)}_n) Q^{(i+1)}(E^{(i)}_n)}{Q^{(i-1)}(E^{(i)}_n) Q^{(i)}(E^{(i)}_n) Q^{(i+1)}(E^{(i)}_n)} = -1.
\]

where \( E^{(i)}_n \) are zeros of \( Q^{(i)}(E) \).
We will consider the other affine Lie algebras and find the (pseudo-)differential equations associated to the linear problem for the fundamental representation.

| Lie Algebra | Equation |
|-------------|----------|
| $A_{r}^{(1)}$ | $D(h)\psi = (-me^\lambda)^h p(z)\psi$ |
| $D^{(1)}_r$ | $D(h^{\dagger})\partial^{-1} D(h)\psi = 2^{r-1}(me^\lambda)^h \sqrt{p(z)}\partial \sqrt{p(z)}\psi$ |
| $B_r^{(1)}$ | $D(h^{\dagger})D(h)\psi = 2^r(me^\lambda)^h \sqrt{p(z)}\partial \sqrt{p(z)}\psi$ |
| $A_{2r-1}^{(2)}$ | $D(h^{\dagger})D(h)\psi = -2^{r-1}(me^\lambda)^h \sqrt{p(z)}\partial \sqrt{p(z)}\psi$ |
| $C_r^{(1)}$ | $D(h^{\dagger})D(h)\psi = (me^\lambda)^h p(z)\psi$ |
| $D^{(2)}_{r+1}$ | $D(h^{\dagger})\partial D(h)\psi = 2^{r+1}(me^\lambda)^{2h} p(z)\partial^{-1} p(z)\psi$ |
| $A_{2r}^{(2)}$ | $D(h^{\dagger})\partial D(h)\psi = -2^r \sqrt{2}(me^\lambda)^h p(z)\psi$ |
| $G_2^{(1)}$ | $D(h^{\dagger})\partial D(h)\psi = 8(me^\lambda)^h \sqrt{p(z)}\partial \sqrt{p(z)}\psi$ |
| $D^{(3)}_4$ | $D(h^{\dagger})\partial D(h)\psi + (\omega + 1)2\sqrt{3}(me^\lambda)^4 D(h^{\dagger})p(z)\psi$ 
$- (\omega + 1)2\sqrt{3}(me^\lambda)^4 pD(h) - 8\sqrt{3}\omega(me^\lambda)^3 D(-h_1)\sqrt{p}\partial \sqrt{p}D(h_1)$ 
$+ (\omega - 1)^3 12(me^\lambda)^8 p\partial^{-1} p \} \psi = 0$ |

$D(h) = D(h_r) \cdots D(h_1)$, $D(h^{\dagger}) = D(-h_1) \cdots D(-h_r)$ for $h = (h_r, \cdots, h_1)$
Langlands duality

Langlands (GNO) dual: $\hat{\mathfrak{g}}$: simple roots $\alpha_i \iff \hat{\mathfrak{g}}^\vee$: simple coroots

- $\hat{\mathfrak{g}}^\vee = X_r^{(1)}$ for $X = ADE$
- $\hat{\mathfrak{g}}^\vee = X_r^{(s)}$ for non-simply laced $\hat{\mathfrak{g}}$ (twisted affine Lie algebra)
  
  \[ (B_r^{(1)})^\vee = A_{2r-1}^{(2)}, \quad (C_r^{(1)})^\vee = D_{r+1}^{(2)}, \quad (F_4^{(1)})^\vee = E_6^{(2)}, \quad (G_2^{(1)})^\vee = D_4^{(3)}, \]
  
  \[ (A_{2r}^{(2)})^\vee = A_{2r}^{(2)} \]
In Dorey-Dunning-Masoero-Suzuki-Tateo (2007), they found a set of pseudo-differential equations associated to classical Lie algebras.

| affine Toda equation | ODE(Dorey et al.) |
|----------------------|-------------------|
| $A^{(1)}_r$          | $A_r$             |
| $(B^{(1)}_r) \vee = A^{(2)}_{2r-1}$ | $B_r$             |
| $(C^{(1)}_r) \vee = D^{(2)}_{r+1}$ | $C_r$             |
| $D^{(1)}_r$          | $D_r$             |
We want to prove that the modified affine Toda equation for the Langlands dual $\hat{\mathfrak{g}}^\vee$ corresponds to the $\mathfrak{g}$-type Bethe ansatz equation.

- ODE is complicated for higher-rank affine Lie algebras ($E$-type ODE?)
- One can derive the $\psi$-system based on the linear system (classical Lie algebra in the conformal limit [Sun,1201.1614])

We consider the affine Toda field equations for $\hat{\mathfrak{g}}$

- Applying the gauge transformation

$$ U_A = z^M \rho^\vee \cdot H e^{-\beta \phi \cdot H/2} $$

gives a simple form of the linear problem in the large $\rho$ limit,

$$ \tilde{A} = me^\lambda z^M \Lambda_+ , \quad \tilde{\bar{A}} = me^{-\lambda} \bar{z}^M \Lambda_- $$

$$ \Lambda_{\pm} = \sqrt{n_0^\vee} E_{\pm \alpha_0} + \sum_{i=1}^r \sqrt{n_i^\vee} E_{\pm \alpha_i} $$

\[ \]
We will consider the fundamental representations $V^{(a)}$ with the highest weight $\omega_a \ (a = 1, \cdots, r)$ of $\hat{g}$.

$e_i^{(a)}$: a basis of $V^{(a)}$ and $e_1^{(a)}$ is the highest weight vector.

asymptotic form for a subdominant solutions along the positive real axis

$$
\Psi^{(a)}(z, \bar{z}|\lambda) = \exp \left( -2\mu^{(a)} \frac{\rho^{M+1}}{M+1} m \cosh (\lambda + i\theta(M + 1)) \right) e^{-i\theta M \rho^{V} \cdot H} \mu^{(a)}.
$$

$\mu^{(a)}$ and $\mu^{(a)}$ are the eigenvalues of $\Lambda_+^{(a)} = (\Lambda_-^{(a)})^T$ with the largest real part and its eigenvector in module $V^{(a)}$.

For small $z$,

$$
\Psi^{(a)}(z, \bar{z}|\lambda, g) = \sum_{i=1}^{\dim(V^{(a)})} Q_i^{(a)}(\lambda, g) \chi_i^{(a)}(z, \bar{z}|\lambda, g)
$$

$$
\chi_i^{(a)} = e^{-(\lambda+i\theta)\beta g \cdot h_i^{(a)}} e_i^{(a)} + O(\rho) \text{ as } \rho \to 0
$$
**ψ-system**

- embedding map:

\[ \iota : \bigwedge^2 V^{(a)} \rightarrow \bigotimes_{b=1}^{r} (V^{(b)})^{B_{ab}}. \]

- highest weight: \( 2\omega_a - \alpha_a = \sum_{b=1}^{r} B_{ab} \omega_b \)
  \( A_{ab} \): Cartan matrix of \( g \) and \( B_{ab} \) the incidence matrix

\[ B_{ab} = 2\delta_{ab} - A_{ab}. \]

- take the anti-symmetric product of for \( \Psi_{\pm 1/2}^{(a)} \) and decompose it by the embedding map such that large \( \rho \) asymptotics matches

- the largest eigenvalues \( \mu^{(a)} \) of \( \Lambda^{(a)}_{\pm} \) constrained such that the two asymptotics coincide with each other
ψ-system for $\hat{g}^\vee$

$\hat{g} = A, D, E$ [Sun, Masoero-Raimondo-Valeri, KI-Locke]

- ψ-system

$$\iota \left( \Psi_{-1/2}^{(a)} \wedge \Psi_{1/2}^{(a)} \right) = \bigotimes_{b=1}^{r} \left( \Psi^{(b)} \right)^{B_{ab}}.$$

- $\mu^{(a)}$ satisfy the equations

$$2 \cos \left( \pi / h \right) \mu^{(a)} = \sum_{b=1}^{r} B_{ab} \mu^{(b)}.$$

- These eigenvalues coincide with the mass spectrum of affine Toda field theories for $\hat{g}$ [Braden-Corrigan-Dorey-Sasaki]

| Eigenvalues $\mu^{(a)}$ of $\Lambda_+^{(a)}$ | $\Longleftrightarrow$ | Eigenvalues of $(m^2)^{ab} = \sum_{i=0}^{r} \alpha_i^a \alpha_i^b$ |
|-------------------------------------------|-----------------|--------------------------------------------------|
| For $A_r^{(1)}$, $\mu^{(a)} = \sin \frac{\pi a}{r+1} / \sin \frac{\pi}{r+1}$ |
\( \psi\)-system for \( \hat{g} \) \((\hat{g} = B_r^{(1)}, C_r^{(1)}, F_4^{(1)}, G_2^{(1)})\)

\[
(B_r^{(1)})^\vee = A_{2r-1}^{(2)} : \quad \iota \left( \Psi_{-1/2}^{(a)} \wedge \Psi_{1/2}^{(a)} \right) = \Psi^{(a-1)} \otimes \Psi^{(a+1)} \quad \text{for } a = 1, \ldots, r - 1,
\]

\[
\iota \left( \Psi_{-1/4}^{(r)} \wedge \Psi_{1/4}^{(r)} \right) = \Psi_{-1/4}^{(r-1)} \otimes \Psi_{1/4}^{(r-1)}.
\]

\[
(C_r^{(1)})^\vee = D_{r+1}^{(2)} : \quad \iota \left( \Psi_{-1/4}^{(a)} \wedge \Psi_{1/4}^{(a)} \right) = \Psi^{(a-1)} \otimes \Psi^{(a+1)} \quad \text{for } a = 1, \ldots, r - 2,
\]

\[
\iota \left( \Psi_{-1/4}^{(r-1)} \wedge \Psi_{1/4}^{(r-1)} \right) = \Psi^{(r-2)} \otimes \Psi_{-1/4}^{(r)} \otimes \Psi_{1/4}^{(r)},
\]

\[
\iota \left( \Psi_{-1/2}^{(r)} \wedge \Psi_{1/2}^{(r)} \right) = \Psi^{(r-1)}.
\]

\[
(F_4^{(1)})^\vee = E_6^{(2)} : \quad \iota \left( \Psi_{-1/2}^{(1)} \wedge \Psi_{1/2}^{(1)} \right) = \Psi^{(2)}, \quad \iota \left( \Psi_{-1/2}^{(2)} \wedge \Psi_{1/2}^{(2)} \right) = \Psi^{(1)} \otimes \Psi^{(3)},
\]

\[
\iota \left( \Psi_{-1/4}^{(3)} \wedge \Psi_{1/4}^{(3)} \right) = \Psi^{(2)} \otimes \Psi_{-1/4}^{(4)} \otimes \Psi_{1/4}^{(4)}, \quad \iota \left( \Psi_{-1/4}^{(4)} \wedge \Psi_{1/4}^{(4)} \right) = \Psi^{(3)}.
\]

\[
(G_2^{(1)})^\vee = D_4^{(3)} : \quad \iota \left( \Psi_{1/2}^{(1)} \wedge \Psi_{1/2}^{(1)} \right) = \Psi^{(2)},
\]

\[
\iota \left( \Psi_{1/6}^{(2)} \wedge \Psi_{1/6}^{(2)} \right) = \Psi_{-2/6}^{(1)} \otimes \Psi_0^{(1)} \otimes \Psi_{2/6}^{(1)}.
\]
Comments

- $\hat{g} = B_r^{(1)}, C_r^{(1)}$ [Sun, 1201.1614] (massless limit)

- For $\hat{g} = F_4^{(1)}, G_2^{(1)}$, the $\psi$-system coincides with that conjectured by [Dorey-Dunning-Masoero-Suzuki-Tateo, 0612298]

- The eigenvalues $\mu^{(a)}$ do not coincide with those of mass matrix of affine Toda field theories.
  - $A_{2r-1}^{(2)}$: $\mu^{(a)} = \frac{\sqrt{2}}{\sin \frac{\pi}{2r-1}} \sin \frac{\pi a}{2r-1}$
    
    Affine Toda field theory:
    $m_a = 2\sqrt{2}m \sin \frac{a\pi}{2r-1} (a = 1, \cdots, r - 1), \quad m_r = \sqrt{2}m$
  - $D_4^{(3)}$: $\mu^{(2)}/\mu^{(1)} = \sqrt{2}$
    
    AFT: $\mu^{(2)}/\mu^{(1)} = \sqrt{\frac{3+\sqrt{3}}{3-\sqrt{3}}}$
Bethe-ansatz equations for affine Toda field equations

- **conformal limit** We first take the light-cone limit $\bar{z} \to 0$. Then consider the limit $\lambda \to \infty$ and $z, s \to 0$ with fixed $x = (me^\lambda)^{1/(M+1)} z$, $E = s^h M (me^\lambda)^h M/(M+1)$,

- the solution $\Psi^{(a)}$ becomes

$$
\psi^{(a)}(x, E) = Q^{(a)}(E) \chi_1^{(a)}(x, E) + \tilde{Q}^{(a)}(E) \chi_2^{(a)}(x, E) + \cdots,
$$

with $\chi_i^{(a)} \sim x^i \lambda_i^{(a)}$

- Substituting the $\psi$-system we obtain the quantum Wronskian relations. For $A_r^{(1)}$ case, for example, we find that

$$
\omega - \frac{1}{2} (\lambda_1^{(a)} - \lambda_2^{(a)}) Q_{-1/2}^{(a)} \tilde{Q}_{1/2}^{(a)} - \omega \frac{1}{2} (\lambda_1^{(a)} - \lambda_2^{(a)}) Q_{1/2}^{(a)} \tilde{Q}_{-1/2}^{(a)} = Q^{(a-1)} Q^{(a+1)}.
$$

where $Q_k^{(a)}(E) = Q^{(a)}(\omega^h M^k E)$ ($\omega = \exp(2\pi i / h(M + 1))$)

- Let us denote the zeros of $Q^{(a)}(E)$ as $E_k^{(a)}$. Then substituting the $\pm 1/2$ Symanzik rotation of the quantum Wronskian relations yields the Bethe ansatz equations.
\[ A_r^{(1)}, D_r^{(1)}, E_r^{(1)}: \]
\[
\prod_{b=1}^{r} \frac{Q_{A_{a_b}/2}}{Q_{-A_{a_b}/2}} \bigg|_{E_k^{(a)}} = -\omega^{1+\beta \alpha_a \cdot g}.
\]

\[ (B_r^{(1)})^\vee = A_{2r-1}^{(2)}: \]
\[
\frac{Q_{-1/2}^{(a-1)} Q_1^{(a)} Q_{-1/2}^{(a+1)}}{Q_{1/2}^{(a-1)} Q_1^{(a)} Q_{1/2}^{(a+1)}} \bigg|_{E_i^{(a)}} = -\omega^{1+\beta \alpha_a \cdot g} \quad \text{for } a = 1, \ldots, r - 1,
\]
\[
\frac{Q_{-1/2}^{(r-1)} Q_1^{(r)}}{Q_{1/2}^{(r-1)} Q_1^{(r)}} \bigg|_{E_i^{(r)}} = -\omega^{1/2(1+\beta \alpha_r \cdot g)}.
\]

\[ (C_r^{(1)})^\vee = D_{r+1}^{(2)}: \]
\[
\frac{Q_{-1/4}^{(a-1)} Q_{1/2}^{(a)} Q_{-1/4}^{(a+1)}}{Q_{1/4}^{(a-1)} Q_{1/2}^{(a)} Q_{1/4}^{(a+1)}} \bigg|_{E_i^{(a)}} = -\omega^{1/2(1+\beta \alpha_a \cdot g)} \quad \text{for } a = 1, \ldots, r - 2,
\]
\[
\frac{Q_{-1/4}^{(r-2)} Q_{1/2}^{(r-1)} Q_{-1/4}^{(r)}}{Q_{1/4}^{(r-2)} Q_{1/2}^{(r-1)} Q_{1/4}^{(r)}} \bigg|_{E_i^{(r-1)}} = -\omega^{1/2(1+\beta \alpha_{r-1} \cdot g)} \quad \text{for } a = 1, \ldots, r - 2,
\]
\[
\frac{Q_{-1/2}^{(r-1)} Q_1^{(r)}}{Q_{1/2}^{(r-1)} Q_1^{(r)}} \bigg|_{E_i^{(r)}} = -\omega^{1+\beta \alpha_r \cdot g}.
\]
These are the Bethe ansatz equations for $\hat{g}$ classified by Reshetikhin and Wiegmann (1987) and Kuniba-Suzuki (1995) based on $U_q(\hat{g})$. 
Twisted affine Lie algebra $A_{2r}^{(2)}$

- $\psi$-system

$$t \left( \Psi^{(a)}_{-1/2} \wedge \Psi^{(a)}_{1/2} \right) = \Psi^{(a-1)} \otimes \Psi^{(a+1)},$$
$$t \left( \Psi^{(r)}_{-1/2} \wedge \Psi^{(r)}_{1/2} \right) = \Psi^{(r-1)} \otimes \Psi^{(r)}.$$

- Bethe ansatz equation:

$$Q^{(a-1)}_{a} Q^{(a)}_{1} Q^{(a+1)}_{-1/2} \left| E^{(a)}_{i} \right| = -\omega^{1+\beta_{a}} \cdot g,$$
$$Q^{(r-1)}_{1} Q^{(r)}_{-1/2} Q^{(r)}_{1} \left| E^{(r)}_{i} \right| = -\omega^{1+\beta_{r}} \cdot g.$$

for $a = 1, \ldots, r - 1$

- For $r = 1$, Tzitzéica-Bullough-Dodd model $\leftrightarrow$ BA eq. for Izergin-Korepin model [Dorey-Tateo, Dorey-Faldella-Negro-Tateo]

- $r \geq 1$ BA eqs. for $U_{q}(A_{2r}^{(2)})$ [Reshetikhin-Wiegmann, Kuniba-Suzuki]
ODE or PDE

$\iff$

ODE/IM

$\iff$

BAE

massless TBA

CFT

$\uparrow$ Conformal limit

$\psi$-system

$\uparrow$ UV limit

Linear problem

$\iff$

massive ODE/IM

$\iff$

BAE

massive TBA

massive QFT

$\uparrow$

affine Toda equation
Outlook

- Our dictionary of ODE/IM correspondence is not yet complete
  - general $p(z)$ Bazhanov-Lukyanov
  - ODE/IM for $B^{(1)}_r, C^{(1)}_r, G^{(1)}_2, F^{(1)}_4$
    Bethe-ansatz(-like) equations (IM?)
  - affine Lie superalgebra
  - generalized Drinfeld-Sokolov reduction
    [Balog-Feher-O’Raifeartaigh-Forgacs-Wipf]
  - Gaudin-type Bethe equations [Feigin-Frenkel]
  - T-system, Y-system, nonlinear integral equations

- Why this correspondence holds? Why Langlands dual?
  We need to understand this correspondence in a stringy setup.
  - AdS$_4$ and AdS$_5$ minimal surface (null-poly Wilson loop, form factor)
    affine $B^{(1)}_2$ Toda field equation KI-Locke-Satoh-Shu, work in progress
  - gauge/Bethe correspondence [Nekrasov-Shatashvili, ..., Chen-Hsin-Koroteev]
  - Langlands dual and Hitchin system [Kapustin-Witten]