Some Algebro-Geometric Aspects of The SL(2, R) Wess-Zumino-Witten Model of Strings on an ADS\(_3\) Background

Bogdan G. Dimitrov

Bogoliubov Laboratory for Theoretical Physics
Joint Institute for Nuclear Research, Dubna 141 980, Russia
email: bogdan@thsun1.jinr.ru

Abstract

The SL(2, R) WZW model of strings on an ADS\(_3\) background is investigated in the spirit of J.Maldacena’s and H.Ooguri’s approach (hep-th/0001053 and hep-th/0005183). Choosing a standard, but most general three-variable parametrization of the SL(2, R) group element \(g\), the system of equations for the Operator Product Expansion (OPE) relations is analysed. In the investigated SL(2, R) case, this system is consistent if each three points on the complex plane lie on a certain hypersurface in CP\(_3\). A system of three nonlinear first-order differential equations has been obtained for the parametrization functions. It was demonstrated also how the mathematical apparatus of generalized functions and integral geometry can be implemented in order to modify the integral operators, entering the Kac-Moody and Virasoro algebras.

Keywords: string theory, conformal field theory, algebraic and integral geometry, Anti-de-Sitter space

1 Introduction

Several years ago serious attempts were made to apply the originally developed in [1, 2] two-dimensional conformal WZW model to the model of strings and branes on a curved background. The basic idea was to “supply” the group element \(g\), entering the WZW action, with the properties of the chosen background Anti De-Sitter spacetime ADS\(_3\) by means of a suitable parametrization of \(g\) in terms of the ADS\(_3\) global coordinates [3, 4, 5]. However, the original WZW model, defined on an (unspecified) two-dimensional world-sheet spacetime, turned out to be not well suited to account for some specific effects, such as stretchings and windings of strings close to the boundary of ADS\(_3\). In search of a resolution, two main approaches were proposed. The first one was based on the notion of a spectral flow, which generates new solutions of the equations of motion for SL(2, R) by acting on standard representations by elements of the loop group [3, 5]. The second approach, developed in [6], had the purpose to “redefine” the integral formulation of the conserved charges

\[
J^a_0 = \oint dz J^a(z), \quad J^a_n = \oint dz J^a(z) \gamma^n(z)
\]

for the zero-mode symmetry of the algebra [7]. The peculiarity of this approach was that the Kac-Moody and Virasoro algebras should not be considered as automatically following from the OPE relations, but their consistent resolution is needed [6].
for the SL(2, R) case and written in terms of the three SL(2, R) parametrization variables. More details about the present approach are to be found in [10].

2 A Three-Parameter SL(2,R) Parametrization of the WZW Model in ADS$_3$ Background

Our starting point is the WZW effective action:

$$S = \frac{1}{4\lambda} \int \text{Tr}(\partial_\mu g^{-1}\partial_\mu g)d^2\xi + k\Gamma(g) ,$$

(2.1)

where $g$ is the group element, $\lambda$ and $k$ are dimensionless coupling constants, $\xi = (\xi_1, \xi_2)$ are the coordinates of the two-dimensional world-sheet and $\Gamma(g)$ is the boundary WZW term:

$$\Gamma(g) = \frac{1}{24} \int d^3X \, e^{\alpha\beta\gamma} \text{Tr}(g^{-1}\partial_\alpha gg^{-1}\partial_\beta gg^{-1}\partial_\gamma g) .$$

(2.2)

The integration is performed over a three-dimensional ball with coordinates $X^0$, the boundary of which is identified with the two-dimensional world-sheet. Since the third de-Rham cohomology vanishes for the SL(2, R) group, in the present case it shall be neglected.

In terms of the global coordinates $(X^0, X^1, X^2, X^3)$ on the Anti de-Sitter hyperboloid

$$-(X^0)^2 - (X^3)^2 + (X^1)^2 + (X^2)^2 = -L^2 ,$$

(2.3)

where $L$ is the de Sitter radius, one can parametrize the SL(2, R) group element $g$ as

$$g = \frac{1}{L} \left( \begin{array}{cc} X^0 + X^1 & X^2 + X^3 \\ X^2 - X^3 & X^0 - X^1 \end{array} \right)$$

(2.4)

so that $\text{det}g = 1$.

Let us now introduce three angle variables $(\Phi, \Psi, \Theta)$, which shall parametrize the SL(2,R) model. According to a well-known theorem [9], each SL(2,R) group matrix $g$ can be represented as

$$g = d_1(-e)^{s_1}s^2pd_2 ,$$

(2.6)

where $\epsilon_1, \epsilon_2 = 0$ or 1, $d_1, d_2, (-e)$ and $s$ are the diagonal matrices

$$d_1 = \left( \begin{array}{cc} e^{-\Phi} & 0 \\ 0 & e^\Phi \end{array} \right) ; \quad d_2 = \left( \begin{array}{cc} e^{-\Psi} & 0 \\ 0 & e^{\Phi} \end{array} \right) ; \quad (-e) = \left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) ; \quad s = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$$

(2.7)

The matrix $p$ in the case represents the rotations in the hyperbolic plane:

$$p = \left( \begin{array}{cc} \cosh \Theta & \sinh \Theta \\ \sinh \Theta & \cosh \Theta \end{array} \right)$$

(2.8)

and $-\infty < \Theta < +\infty$. Depending on the values of $\epsilon_1$ and $\epsilon_2$, four different cases have to be considered. Since for both the cases $(\epsilon_1 = 1, \epsilon_2 = 0$ - Case I) and for $(\epsilon_1 = \epsilon_2 = 0$ - Case II) one can write

$$(-e)^{s_1}s^2 = \varepsilon E ,$$

(2.9)

where $E$ is the unit matrix and $\varepsilon = \pm 1$, one can easily find the elements of the SL(2, R) matrix

$$g = d_1(-e)^{s_1}s^2pd_2 = \varepsilon \left( \begin{array}{cc} e^{-\Phi-\Psi}\cosh\Theta & e^{-\Phi-\Psi}\sinh\Theta \\ e^{\Phi-\Psi}\sinh\Theta & e^{\Phi+\Psi}\cosh\Theta \end{array} \right) .$$

(2.10)

Similarly, for the other two cases $(\epsilon_1 = \epsilon_2 = 1$ - Case III) and $(\epsilon_1 = 0, \epsilon_2 = 1$ - Case IV) one obtains correspondingly the matrices for $(-e)^{s_1}s^2$ and $(-e)^{s_1}s^2$, but these cases shall not be considered.

Comparing the elements of the matrices (2.5) and (2.10), the global coordinates on the de Sitter hyperboloid can be expressed as follows:

$$X^{0,1} = \varepsilon \frac{L}{2}\cosh\Theta(e^{-\Phi-\Psi} \pm e^{\Phi+\Psi}) ; \quad X^{2,3} = \varepsilon \frac{L}{2}\sinh\Theta(e^{-\Phi+\Psi} \pm e^{\Phi-\Psi})$$

(2.11)

and the + sign is for the $X^0$ and $X^2$ coordinates and the − sign for $X^1$ and $X^3$. 


3 Gauge Currents of the WZW Model in Terms of the SL(2,R) Parametrization Variables

Let us first define complex coordinates $z$ and $\tau$ on the two-dimensional world-sheet by setting up

$$ z = \xi_1 + i\xi_2 \quad \quad \tau = \xi_1 - i\xi_2 \quad \quad (3.1) $$

and thus the functional in (2.1) is defined over some two-dimensional Riemann surface.

The WZW model has a chiral $SL(2,R) \times SL(2,R)$ symmetry, characterized by an infinite number of conserved currents $J$ and $\bar{J}$, derivable from the equations

$$ \partial_z J = 0 \quad \partial_{\bar{z}} \bar{J} = 0 \quad \quad (3.2) $$

The left and right conformal currents $J_L$ and $J_R$ can be expressed as

$$ J_L = kTr(T^ag^{-1}\partial_ag) \quad J_R = kTr(T_3 g^{-1} \partial_3 g) \quad \quad (3.3) $$

and $T^a$ are the generators of the $SL(2,R)$ algebra, expressible through the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ as $T^3 = -\frac{i}{2} \sigma_2 = \left( \begin{array}{cc} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{array} \right), \quad T^+ = \frac{1}{2}(\sigma_3 + i\sigma_1) = \left( \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{array} \right) \quad \text{and} \quad T^- = \frac{1}{2}(\sigma_3 - i\sigma_1) = \left( \begin{array}{cc} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right)$. 

Now an important point should be clarified. In the initial works on CFT [1,2], there was no need to specify the two-dimensional subspace. But later on, when the WZW model of $ADS_3$ strings came out, it appeared natural, in the framework of some approximation, to relate this two-dimensional subspace to a two-dimensional subspace of the $ADS_3$ space-time, expressed in global coordinates as

$$ ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2, \quad \quad (3.4) $$

where $\phi$ is a $2\pi$-periodic coordinate. In such an approach, $z$ and $\tau$ would be defined as $z = t + i\varphi$, $\tau = t - i\varphi$ or as in [5], $t$ and $\varphi$ may be considered to be the left- and right-moving coordinates on the world-sheet, i.e. $t = u + v$ and $\varphi = u - v$. In all the above case and in many others, frequently used in the literature, the prescribed dependence between the parametrization variables and the global ones represents an approximate approach, chosen for convenience.

Since the global and the parametrization variables ($\Phi, \Psi, \Theta$) (assumed to be different from the $ADS$ global coordinates $(X^0, X^1, X^2, X^3)$ or $(t, \rho, \varphi)$) depend on the string world-sheet coordinates, one may write

$$ X^\mu \equiv X^\mu(t, \rho, \varphi) \equiv X^\mu(\Phi(z, \tau), \Psi(z, \tau), \Theta(z, \tau)) \quad \quad (3.5) $$

where $\mu = 0, 1, 2, 3$. The change of the global coordinates $X^\mu$ to $X^\mu(t, \rho, \varphi)$ from one system to another may be investigated on a purely algebraic basis, provided that the $ADS$ hyperboloid equation is given in terms of $X^\mu$. As it may be shown [10], if one has the metric in the form (3.4), there is a whole class of functions $X^\mu(t, \rho, \varphi)$, which satisfy the hyperboloid equation and which can be found as solutions of a complicated system of nonlinear differential equations in partial derivatives. This is the first basic aspect in the algebraic approach, which has not been developed until now, because usually for convenience it is preferred to choose a definite parametrization of the global coordinates $X^\mu$. There is also a second aspect of the algebraic approach, the mathematical foundation of which is developed in [11]. Suppose one works in different $ADS$ global coordinates, including also the case when the length interval is expressed through the two-dimensional world-sheet coordinates. Then, if the metric (3.4) is given, the differentials of all possible global $ADS$ coordinates have to satisfy a cubic algebraic equation, which effectively determines all possible parametrizations of the $ADS$ space-time in terms of the given coordinates $(t, \rho, \varphi)$. As a partial case [11], the elliptic Weierstrass function and its derivative can parametrize (satisfy) the parametrizable form of the cubic equation which is obtained from the original one after applying a linear-fractional transformation. However, the full “classification” of all possible parametrizations on the base of the cubic algebraic equation represents a complicated and still unresolved mathematical problem.

Concerning the dependence of the global coordinates $X^\mu$ on the parametrization variables $X^\mu \equiv X^\mu(\Phi(z, \tau), \Psi(z, \tau), \Theta(z, \tau))$, evidently some relation or equations are needed. Further this problem
shall be resolved by assuming that the gauge WZW currents are at the same time also conformal currents, therefore satisfying the Operator Product Expansion (OPE) relations, but excluding the OPE relations with the energy-momentum tensor. The above mentioned assumption is in fact frequently used in the literature, although it is not commented at all. In principle, the two types of currents are different because the gauge currents reflect the gauge symmetries of the WZW action, while the conformal currents and more especially the OPE relations follow from the conformal Ward identities \[1, 2\], which characterize the admissible transformations of the complex variable under the action of the conformal group. The identification of the gauge with the conformal currents relates the global ADS space-time symmetries with the string world-sheet symmetries, which in a sense is similar to the identifications of the global ADS coordinates with the world-sheet ones. It may be supposed that in a self-consistent WZW model of strings on an ADS3 background, the parametrization variables should not in advance be identified with the the global ADS coordinates (or with any combination of theirs), but rather should be found as a result of an combination of both the conformal theory approach and the algebraic theory approach. This, however, is very difficult, so as a first step these approaches shall be presented separately.

Let us now use expressions (2.11) and (3.3) and the formulae for the inverse matrix \(g^{-1}\) (under the condition of \(SL(2, R)\) parametrization \(detg = 1\), found from (2.4), in order to write down the corresponding WZW gauge currents \(J^a_R\) for \(a = 3, +, -\):

\[
J^3_R = a_1 \frac{\partial \Theta}{\partial z} + a_2 \frac{\partial \Psi}{\partial z} ; \quad J^+_R = b_1 \frac{\partial \Phi}{\partial z} + b_2 \frac{\partial \Psi}{\partial z} ; \quad J^-_R = b_1 \frac{\partial \Phi}{\partial z} + b_2 \frac{\partial \Psi}{\partial z} - b_3 \frac{\partial \Theta}{\partial z} \tag{3.6}
\]

where \(a_1, a_2, b_1, b_2, b_3\) are the expressions

\[
a_1 \equiv -ksinh2\Phi ; \quad a_2 \equiv ksinh2\Theta \cosh2\Phi ; \tag{3.7}
\]

\[
b_1 \equiv -L^2 ; \quad b_2 \equiv -L^2[cosh2\Theta + isinh2\Theta \sinh2\Phi] ; \quad b_3 \equiv iL^2 \cosh2\Phi . \tag{3.8}
\]

Note the appearence of an imaginary part in the currents \(J^+_R\), while the currents \(J^3_R\) contain only a real part.

### 4 An Algebraic Relation From The Three-Point Operator Product Expansion For The SL(2,R) Case

Let the currents \(J^3_R, J^+_R\) and \(J^-_R\) satisfy the OPE relations for a conformal theory with an affine \(SL(2, R) \times SL(2, R)\) Lie algebra symmetry at level \(k\). Since the OPE relations with the energy-momentum tensor contain derivatives, they shall not be used in the present investigation, but taking them into account might be an interesting problem for further research.

In their general form, the OPE relations with the conformal current (only) can be written as

\[
J^A(z)J^B(w) = \frac{1}{2} k \eta^{AB} \frac{1}{(z-w)^2} + \frac{i\epsilon^{ABC}\eta_{CD}}{z-w} J^D , \tag{4.1}
\]

where the indices \(A, B, C, D = 3, +, -\) are the structure constants of the \(SL(2, R)\) group, \(\eta^{AB}\) is the \(SL(2, R)\) group metric - in the case \(\eta^{AB} = diag.(+1, +1, -1)\) and only \(\epsilon_{012} = 1\) (the rest ones are zero). For values of \((A, B) = (3, +), (3, -)\) and \((+, -)\), the corresponding relations are

\[
J^3(z)J^+(w) = i J^-(w) \frac{1}{z-w} ; \quad J^3(z)J^-(w) = -i J^+(w) \frac{1}{z-w} ; \quad J^+(z)J^-(w) = \frac{i J^3(w)}{z-w} . \tag{4.2}
\]

For the case with equal indices, the OPE relations are

\[
J^3(z)J^3(w) = \frac{k}{2(z-w)^2} ; \quad J^+(z)J^+(w) = \frac{k}{2(z-w)^2} ; \quad J^-(z)J^-(w) = -\frac{k}{2(z-w)^2} . \tag{4.3}
\]
Multiplying both sides of the first equality in (4.2) by $J^{-}(v)$ and making use of the third equality in (4.3) (but for the points $w$ and $v$), the right hand side (R.H.S.) may be rewritten as

$$J^{3}(z)J^{+}(w)J^{-}(v) = \frac{iJ^{-}(w)J^{-}(v)}{z-w} = \frac{ik}{2(z-w)(w-v)^2} \quad (4.4)$$

Again, using (4.2) with the purpose to rewrite $J^{+}(w)J^{-}(v)$ in the left hand side (L.H.S) of (4.4) and afterwards comparing both sides of (4.4), a simple algebraic relation is obtained. In the same manner, the second equality in (4.2) can be multiplied by $J^{+}(v)$ and after transformation of the L.H.S., another algebraic relation is obtained. The two simple algebraic relations are the following

$$\frac{1}{(z-v)^2} = \frac{1}{(z-w)(w-v)} ; \quad \frac{1}{(w-v)^2} = \frac{1}{(z-w)} \quad . \quad (4.5)$$

Next, the third relation in (4.2) can be multiplied by $J^{3}(v)$, but the obtained algebraic relation will be the same as the second relation in (4.5). Therefore, making use of the above two relations (4.5), one final algebraic relation will be obtained

$$\frac{1}{(z-v)^2} + \frac{1}{(w-v)^2} = 0 \quad , \quad (4.6)$$

representing a three-dimensional algebraic surface in the complex projective plane $CP^3$.

5 Conformal OPE Relations, WZW Currents And A System of Nonlinear Differential Equations For The $SL(2,R)$ Parametrization Variables

The already found WZW currents $J^{3}(z)$, $J^{+}(z)$, $J^{-}(z)$ (3.6) depend on the symmetries of the WZW action and on the chosen $SL(2, R)$ parametrization variables. As explained in Section 3, it shall be assumed that the WZW currents are also conformal currents, satisfying the OPE relations, written for the $SL(2, R)$ group. The difficulty with the OPE relations is that they are functional relations, taken at different world-sheet points. This is the reason these relations do not give much information, they are discussed usually in regard to the Kac-Moody and Virasoro algebras, which naturally follow from them and which shall also be used further. A basic new moment in this paper will be to use the algebraic relation (4.6), which relates different points on the world sheet and thus might allow to obtain from the OPE relations some equations, taken at one and the same point. In order to achieve this, some additional assumptions have to be made.

The basic assumption for a conformal field theory is that it is invariant under Mobius transformations on the complex plane, and the Mobius transformation is

$$v(z) = \frac{az+b}{cz+d} \quad a,b,c,d \subset C \quad ad-bc = 1 \quad . \quad (5.1)$$

The assumption about Mobius invariance is consistent with the latest developments in conformal field theory [12,13], since Mobius invariance turns out to be a more fundamental concept than the conformal structure of the theory itself. Mobius invariance is enough to define the amplitudes, the vertex operators and the Virasoro algebra in a conformal field theory. In the present case, the transformation (4.6) can be written as $z = i\varepsilon w + (1 - i\varepsilon)v$, and if $v$ and $w$ can interchange their places, from the new relation and the previous it is obtained that $z = pw$ ($p$ - a complex number). Therefore, our transformation turns out to be a partial case of the Mobius one. However, note that the interchanging of places should be treated as an approximation and not as a natural operation - for example, from (4.1) it can be obtained by substracting that $[J^{A}(z), J^{B}(w)] = \frac{2\delta^{ABC}a}{z-w} J^{D} \neq 0$ in the general case.

Now it shall naturally be assumed that the global ADS coordinates $X^{\mu}$ are invariant in respect to the Mobius transformation $w(z)$, i.e.

$$X^{\mu}(z) \equiv X^{\mu}(w(z)) \quad \text{and} \quad \frac{\partial X^{\mu}(z)}{\partial z} \equiv \frac{\partial X^{\mu}(w(z))}{\partial z} \quad , \quad (5.2)$$
which of course is reminiscent of string world-sheet invariance. Indeed, if one substitutes the group element $g$ in the WZW action, the resulting action will be for a nonlinear sigma model, where the global ADS coordinates will represent the string coordinates, for which the reparametrization invariance requirement is valid. Also, the Mobius invariance concept does not contradict the existence of the algebraic relation (4.6). The important moment is that it can be generalized for $n + 1$ points, meaning that the coefficients $a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}$ in the Mobius transformation for the $(n + 1)$ point will depend on the coefficients of the remaining $n$ points. However, the number of these $n$ points (with freely varying coefficients) may go to infinity, so Mobius invariance may hold for all points. If (5.2) holds and if the derivatives $\frac{\partial X^a(z)}{\partial w}, \frac{\partial X^a(z)}{\partial \bar{\omega}}$ and $\frac{\partial X^a(z)}{\partial \eta}$ are arbitrary, then it can easily be proved [10] that it is natural to choose the parametrization variables $(\Phi, \Psi, \Theta)$ to be Mobius and reparametrization invariant, i.e.

$$(\Phi(w(z)), \Psi(w(z)), \Theta(w(z))) \equiv (\Phi(z), \Psi(z), \Theta(z)) \Rightarrow \frac{\partial \Phi(w(z), v)}{\partial w} = \frac{\partial \Phi(w(z), v)}{\partial z} \frac{\partial z}{\partial w} = -i \varepsilon \frac{\partial \Phi(z)}{\partial w}.$$  

(5.4)

In (5.4) $\frac{\partial w}{\partial z}$ can be found from the algebraic relation (4.6). The same of course holds for the derivatives of $\Psi$ and $\Theta$. Note that the assumption about arbitrariness of $\Psi$ and $\Theta$. Indeed, if one substitutes the group algebraic relation (4.6). The important moment is that it can be generalized for the third one

$J^3_R(w) = -i \varepsilon J^3_R(z) ; \quad J^+(w) = -i \varepsilon J^+(z) ; \quad J^-(w) = -i \varepsilon J^-(z).$  

(5.5)

Substituting these conformal currents into the OPE relations (4.2) for $J^3(z)J^+(w)$ and $J^3(z)J^-(w)$, it can be derived

$J^+(z) = \varepsilon_1 J^-(z) ; \quad J^3(z) = i \varepsilon_2 J^+(z),$  

(5.6)

where $\varepsilon_1$ and $\varepsilon_2$ can take values ±1 independently one from another. If the above equations are written in terms of the derivatives $\frac{\partial \Phi}{\partial z}, \frac{\partial \Psi}{\partial z}$ and $\frac{\partial \Theta}{\partial z}$, two of the derivatives $\frac{\partial \Phi}{\partial z}$ and $\frac{\partial \Psi}{\partial z}$ can be expressed through the third one $\frac{\partial \Theta}{\partial z}$

$$\frac{\partial \Phi}{\partial z} = Q \frac{\partial \Theta}{\partial z} ; \quad \frac{\partial \Psi}{\partial z} = P \frac{\partial \Theta}{\partial z},$$  

(5.7)

where $Q$ and $P$ depend on the expressions $b_1, b_2$ and $b_3$ and therefore on the parametrization variables. Next, if one uses the other set of (the first two) OPE equations (4.3) for $J^3(z)J^3(w)$ and $J^+(z)J^+(w)$, two more relations for $\frac{\partial \Phi}{\partial z}$ and $\frac{\partial \Psi}{\partial z}$ can be obtained and also, a third equation for $J^- (z) J^-(w)$. Combining the obtained system of three (quadratic) algebraic relations in respect to the derivatives $\frac{\partial \Phi}{\partial z}, \frac{\partial \Psi}{\partial z}$ and $\frac{\partial \Theta}{\partial z}$ with (5.7), finally one receives the following system of first-order nonlinear differential equations for $\Theta, \Phi$ and $\Psi$:

$$\frac{\partial \Theta}{\partial z} = T(\Theta, \Phi, \Psi) ; \quad \frac{\partial \Phi}{\partial z} = Q(\Theta, \Phi, \Psi) T(\Theta, \Phi, \Psi) ; \quad \frac{\partial \Psi}{\partial z} = P(\Theta, \Phi, \Psi) T(\Theta, \Phi, \Psi).$$  

(5.8)

where $T(\Theta, \Phi, \Psi)$ is a definite function.

6 KaC-Moody And Virasoro Algebras And Application of the Integral Geometry Approach Of Gel’fand, Graev And Vilenkin

The definition an expression of the conformal currents $J^n_R(\zeta), J^n_{\bar{R}}(\zeta)$ and $J^n_{\bar{R}}(\zeta)$ and the knowledge of the conformal stress-energy tensor $T(\zeta)$ is the first step towards constructing the relevant integral quantities, which in the notations of [2] can be written as

$$L_n A_j(z, \bar{\tau}) = \oint_C T(\zeta)(\zeta - z)^{n+1} A_j(z, \bar{\tau})d\zeta ; \quad J^n_R A_j(z, \bar{\tau}) = \oint_C J^n(\zeta)(\zeta - z)^n \frac{d\tau}{\zeta} A_j(z, \bar{\tau})d\zeta.$$  

(6.1)
where \( n \) can admit positive and negative values, \( A_j(z, \bar{\tau}) \) is some field, depending on the holomorphic and anti-holomorphic variables and \( t^a_j \) are the group generators. In this standard CFT formulation the integration is performed on an arbitrary contour \( C \) on the complex plane.

It is important to note that in standard CFT, provided that the conformal currents \( J^a \) satisfy the OPE relations, the integral operators \( L_n \) and \( J^a_n \) will satisfy the Virasoro algebra

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}c(n^3 - n)\delta_{n+m} ; \quad [L_n, J^a_m] = -mJ^a_{n+m}
\]  

(6.2)

and the Kac-Moody algebra

\[
[J^a_n, J^b_m] = f^{abc}J^c_{n+m} + \frac{1}{2}kn\delta^{ab}\delta_{n+m,0} .
\]

(6.3)

Previously we have related the CFT formulation and the WZW formulation of string theory on an ADS\(_3\) background by assuming that the WZW currents represent the conformal currents as well. Therefore it is natural to ask whether it is possible to relate the two formulations also on the level of the integral representation in terms of the Virasoro and Kac-Moody algebras. For this purpose, it shall be proposed to introduce an additional contour of integration, obtained as a result of the intersection of a set of hyperplanes with the ADS hyperboloid, and this is one of the realizations of the ADS (Lobachevsky) space. The basic problem, which shall be treated further is whether under the additionally imposed integration the Virasoro and the Kac-Moody algebras will still hold. The mathematical formulation of the additional contour integration will be given in terms of generalized functions, defined on some hypersurface [7], and the geometrical meaning might be also essentially related to the notion of integration on the orisphere (of the Lobachevsky space), developed in the framework of integral geometry [7] by Gel’fand, Graev and Vilenkin.

Let us introduce an additional contour integration \( C_1 \) as the intersection of the set of hyperplanes with the ADS hyperboloid in the expressions (6.1) for the integral operators \( L_n \) and \( J^a_n \). In the present case, since there will be an additional dependence on \( \eta = (\eta_0, \eta_1, \eta_2, \eta_3) \), these integral operators, acting on the field \( A_j(z, \bar{\tau}) \), shall be denoted by \( \tilde{L}_n(\eta) \) and \( \tilde{J}^a_n(\eta) \)

\[
\tilde{L}_n(\eta)A_j(z, \bar{\tau}) = \oint_{\tilde{C}_1} \phi \left[ \int_{\tilde{C}_1} \delta \left( \tilde{X}, \eta \right)^{n+1} \delta \left( \tilde{X}, \eta \right) + 1 \right] A_j(z, \bar{\tau}) d\tilde{X} \right] d\xi \]

(6.4)

and the integral operator \( \tilde{J}^a_n(\eta)A_j(z, \bar{\tau}) \) will be defined in a completely analogous way. In (6.4) \( \tilde{X} \) is a point on the ADS hyperboloid with an unit de-Sitter radius and \( \delta \left( \tilde{X}, \eta \right) + 1 \) is a generalized function, defined on the set of all four-dimensional hyperplanes (with coefficients \( \eta^0, \eta^3, \eta^1, \eta^2 \), determining their position) and situated on the cone, defined by the bi-linear form \( \tilde{X}, \eta \)

\[
-1 = \left[ \tilde{X}, \eta \right] = -\tilde{X}^0\eta^0 - \tilde{X}^3\eta^3 + \tilde{X}^1\eta^1 + \tilde{X}^2\eta^2 .
\]

(6.5)

The cone \( \left[ \tilde{X}, \eta \right] = 0 \) is the geometrical place of hyperplanes, separating the hyperplanes, lying inside the cone, when \( \left[ \tilde{X}, \eta \right] < 0 \) (the present case in (6.5)) and also the hyperplanes, lying outside the cone, when \( \left[ \tilde{X}, \eta \right] > 0 \).

Therefore, it can be obtained that the equality \( \tilde{L}_n, \tilde{J}^a_m \) = \( -m\tilde{J}^a_{n+m} \) in terms of the new ”hat” operators will hold only if

\[
\oint_{\left( \tilde{C}_1, X_1 \right)} F(\zeta_1, \zeta_2, z) \delta \left( \left[ X_1, \eta_1 \right] + 1 \right) d\zeta_1 dX_1 = -mJ^a(\zeta_2)(\zeta_2 - z)^{m+n}
\]

(6.6)

and \( F(\zeta_1, \zeta_2, z) \) is the function

\[
F(\zeta_1, \zeta_2, z) \equiv T(\zeta_1)J^a(\zeta_2) \left[ (\zeta_1 - z)^{n+1}(\zeta_2 - z)^{m+1} - (\zeta_1 - z)^m(\zeta_2 - z)^{n+1} \right] .
\]

(6.7)
It is important to note that this is no longer an equality, identically satisfied as a result of the OPE relations, but represents an equation. In other words, the parameters \( \eta_1^{(0)}, \eta_1^{(1)}, \eta_1^{(2)}, \eta_1^{(3)} \) (the lower subscript denotes the parameters, related to first hyperplane integration), ”fixing” the hyperplane position, have to be determined in such a way so that the two sides of (6.6) will be equal.

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References

[1] Belavin, A. A., Polyakov, A. M., Zamolodchikov, A. B. (1984) Infinite Conformal Symmetry in Two Dimensional Quantum Field Theory, Nucl. Phys. B 241 333 - 380.

[2] Knizhnik, V. G., Zamolodchikov, A. B. (1984) Current Algebra and Wess-Zumino Model in Two Dimensions, Nucl. Phys. B 247 83 - 103.

[3] Maldacena, J., Ooguri H. (2001) Strings on ADS\(3\) and the SL(2, R) WZW Model, Part 1: The Spectrum, J. Math. Phys. 42 2929 - 2960 [arXiv:hep-th/0001053].

[4] Maldacena, J., Ooguri H. (2001) Strings on ADS\(3\) and the SL(2, R) WZW Model, Part 2: Euclidean Black Hole, J. Math. Phys. 42 2961- 2977 [arXiv:hep-th/0005182].

[5] Hemming, S., Keski-Vakkuri, E. (2002) The Spectrum of Strings on BTZ Black Holes and Spectral Flow in the SL(2, R) WZW Model, Nucl. Phys. B 626 363 - 376 [arXiv:hep-th/0110252].

[6] Giveon, A., Kutasov, D., Seiberg, N. (1998) Comments on String Theory on ADS\(3\), Adv. Theor. Math. Phys. 2 733-780 [arXiv:hep-th/9806194].

[7] Gel’fand, I. M., Graev, M. I., Vilenkin, N.Y. (1968) *Generalized Functions*, vol.5, Academic Press, New York.

[8] Balasubramanian V., Boer J., Minic, D. (2002) Exploring de Sitter Space and Holography [arXiv:hep-th/0207245].

[9] Vilenkin, N. Y. (1968) *Special Functions and the Theory of Group Representations*, Transl. Math. Monogr., vol. 22, American Mathematical Society, R. I.

[10] Dimitrov, B.G. (2002) Some Algebro- and Integro- Geometric Aspects of the SL(2, R) Wess-Zumino-Witten Model of Strings on an ADS\(3\) Background, in preparation.

[11] Dimitrov, B. G. (2003) Cubic Algebraic Equations in Gravity Theory, Parametrization with the Weierstrass Function and Non-Arithmetic Theory of Algebraic Equations, submitted to Journ. Math. Phys. arXiv:hep-th/0107231.

[12] Gaberdiel, M. R., Goddard, P. (2000) Axiomatic Conformal Field Theory, Commun. Math. Phys. 209 549 - 594 [arXiv:hep-th/9810019].

[13] Gaberdiel, M. R. (2000) An introduction to Conformal Field Theory, Rept. Progr. Phys. 63 607-667 [arXiv:hep-th/9910156].