An Explicit Nash Equilibrium to a Multi-Leader-Follower Game

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The multi-leader-follower game (MLFG) is a generalization of the Stackelberg game which considers bilevel games with a single leader. Here, the individuals (players) are divided into two groups, namely leaders and followers, according to their position (role) in the game. Mathematically, this yields a hierarchical Nash game, where further minimization problems appear in the leaders’ optimization problems as constraints. A Nash equilibrium is then given by a multistrategy vector of all players, where no player has the incentive to change his chosen strategy unilaterally.

We derive a Nash equilibrium for a quadratic multi-leader-follower game using the nonsmooth best response function of the follower. The existence and uniqueness of solutions are proven for an example and its Nash equilibrium is explicitly computed.

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1 Introduction and the Problem Formulation

We consider an MLFG, where the follower’s problem is given by the convex optimization problem

$$\min_{y \in \mathbb{R}^n} \frac{1}{2} y^T Q_y y - x^T B y \quad \text{s.t.} \quad y \geq L^T x,$$

(1)

where $Q_y \in \mathbb{R}^{m \times m}$ is a positive definite diagonal matrix and $B, L \in \mathbb{R}^{n \times m}$ are the matrices that describe the coupling of the player variables: $y$ denotes the follower’s strategy and $x$ the leaders’, respectively. In particular, the unique solution of (1) can be derived by its KKT conditions and is given by the best response function:

$$y^*(x) = \max \{Q_y^{-1} B^T x, L^T x\}.$$

(2)

Furthermore, the leaders are modeled for $\nu = 1, \ldots, N$ by the minimization problems

$$\min_{x_{\nu} \in \mathbb{R}^n} \frac{1}{2} x_{\nu}^T Q_{\nu} x_{\nu} + c_{\nu}^T x_{\nu} + a^T y \quad \text{s.t.} \quad x_{\nu} \in X_{\nu},$$

(3)

with nonempty, convex, and closed strategy sets $X_{\nu}$, symmetric positive definite Hessians $Q_{\nu} \in \mathbb{R}^{n_{\nu} \times n_{\nu}}$, and the vectors $c_{\nu} \in \mathbb{R}^{n_{\nu}}, a \in \mathbb{R}^{n_{\nu}}$. We denote the multistrategy vector of all leaders by $x = (x_{\nu})_{\nu=1}^N \in \mathbb{R}^n$ and the joint leader strategy set by $X = X_1 \times \cdots \times X_N$, respectively. In particular, $B_{\nu,i}, L_{\nu,i} \in \mathbb{R}^n$ denote the $i$-th column, and $B_{\nu,:}, L_{\nu,:}$ their submatrices of the rows referring to the variable of leader $\nu$.

2 The Nash Equilibrium Problem Reformulation and Theoretical Results

The MLFG (1-3) is formulated as a Nash equilibrium problem (NEP) with nonsmooth objective functions for $\nu = 1, \ldots, N$:

$$\min_{x_{\nu} \in \mathbb{R}^n} \frac{1}{2} x_{\nu}^T Q_{\nu} x_{\nu} + c_{\nu}^T x_{\nu} + \sum_{i=1}^{m} a_i \max \{Q_y^{-1} B^T x, L^T x\} \quad \text{s.t.} \quad x_{\nu} \in X_{\nu}.$$

(4)

However, existence (and uniqueness) of Nash equilibria is not immediately obvious for this game, because standard results typically rely—besides convexity like assumptions—either on differentiability of the objectives or on compactness of the strategy sets. If compactness of $X$ is additionally assumed, the existence of Nash equilibria to (4) is guaranteed by the existence theorem of Nikaido and Isoda [3].

The remainder of this article aims to illustrate the challenges of noncompact strategy sets in combination with nonsmooth objectives. We consider the NEP formulation in (4) of the MLFG in (1-3) and assume the strategy set to be the nonnegative orthant, i.e. $X = \mathbb{R}_+^n$. Furthermore, we restrict ourselves to settings, where the kinks, i.e. nonsmoothness, of the objectives appear on the boundary of $X$ and introduce some additional technical assumptions on the data $B$ and $L$. We derive these by reformulating the nonsmooth part of the objective as follows:

$$\sum_{i=1}^{m} a_i \max \{Q_y^{-1} B x, L^T x\} = \sum_{i=1}^{m} a_i \left[ \max \left\{ (L^T - Q_y^{-1} B^T)_{:,i} x, 0 \right\} + (Q_y^{-1} B^T)_{:,i} x \right].$$

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Since we assumed $x$ to be nonnegative, the differentiability of the objectives clearly depends columnwise on the entries of the matrix $(L^T - Q_y^{-1}B^T)$. Therefore, we split the sum in the objective accordingly, let

$$I_\geq = \left\{ i \in \{1, \ldots, m\} \left| (L^T - Q_y^{-1}B^T)_{:,i} \geq 0 \right. \right\}, \quad I_\leq = \left\{ i \in \{1, \ldots, m\} \left| 0 \neq (L^T - Q_y^{-1}B^T)_{:,i} \leq 0 \right. \right\}, \quad (5)$$

and $I_{ns} = \{1, \ldots, m\} \setminus (I_\geq \cup I_\leq)$ be disjoint sets which are independent of the decision variables and depend on the data only. If all entries of one column of $(L^T - Q_y^{-1}B^T)$ are of the same sign, the nonsmoothness can be easily handled.

We separate the sum and drop the $\max$ operator for the index sets $I_\geq$ and $I_\leq$. This yields an equivalent formulation to (4):

$$\min_{x_\nu \in \mathbb{R}^m} \frac{1}{2} x_\nu^T Q_\nu x_\nu + c_\nu^T x_\nu + \sum_{i \in I_\geq} a_i L^T_{:,i} x + \sum_{i \in I_\leq} a_i (Q_y^{-1}B^T)_{:,i} x + \sum_{i \in I_{ns}} a_i \max \{ (Q_y^{-1}B^T)_{:,i} x, L^T_{:,i} x \}. \quad (6)$$

With this reformulation, we are able to state an existence and uniqueness result for noncompact strategy set in case $I_{ns} = \emptyset$, which is a condition on $Q_y$, $B$, and $L$.

**Lemma 2.1 (Existence and Uniqueness)** Let the multi-leader-follower game (1-3) be reformulated as in (6) with the index sets $I_\geq$, $I_\leq$, $I_{ns}$ as in (5). If $I_{ns} = \emptyset$, then there exists a unique Nash equilibrium of (1-3).

**Proof.** Formulate (6) as variational inequality (VI), then show that the concatenated gradients of the strictly convex objective are uniformly monotone. Due to [1, Theorem 2.3.3], the VI has a unique solution, which is the unique Nash equilibrium of (1-3) [1, Proposition 1.4.2].

That lemma illustrates that compactness is not a necessary condition for the existence of a Nash equilibrium to (1-3). We conclude by showing that the Nash equilibrium for diagonal $Q_\nu$ can be computed explicitly.

**Lemma 2.2** Let the multi-leader-follower game (1-3) be reformulated as in (6) with the index sets $I_\geq$, $I_\leq$, $I_{ns}$ defined as in (5). Furthermore, let $I_{ns} = \emptyset$ and assume $Q_\nu$ to be diagonal. Then the Nash equilibrium for $\nu = 1, \ldots, N$ is given by:

$$x_\nu = -\min \left\{ Q_\nu^{-1} c_\nu + \sum_{i \in I_\geq} a_i Q_\nu^{-1} L^T_{:,i} + \sum_{i \in I_\leq} a_i \max \{ (Q_y^{-1}B^T)_{:,i} x, L^T_{:,i} x \}, 0 \right\}.$$

**Proof.** We consider the nonnegative orthant as strategy set; thus Guignard constraint qualification holds in every feasible point and we formulate the KKT system of the leaders’ problems for $\nu = 1, \ldots, N$

$$0 = Q_\nu x_\nu + c_\nu + \sum_{i \in I_\geq} a_i L^T_{:,i} + \sum_{i \in I_\leq} a_i (Q_y^{-1}B^T)_{:,i} - \lambda_\nu \quad \text{and} \quad 0 \leq \lambda_\nu \perp x_\nu \geq 0,$$

$$\Leftrightarrow 0 \leq Q_\nu^{-1} \lambda_\nu = x_\nu + Q_\nu^{-1} c_\nu + \sum_{i \in I_\geq} a_i Q_\nu^{-1} L^T_{:,i} + \sum_{i \in I_\leq} a_i Q_\nu^{-1} (Q_y^{-1}B^T)_{:,i} \perp x_\nu \geq 0,$$

The inverse of $Q_\nu$ exists, because it is assumed to be positive definite. This complementary condition can be rewritten by the minimum function. The follower’s solution is obtained by plugging the multistrategy vector $x$ in the best response (2).

## 3 Further Work on MLFG

For a more detailed study of quadratic MLFG of this type, we refer to [2]. A more general model is presented there which also incorporates nonlinear data and more general strategy sets. Besides existence theory, numerical methods are discussed. Also, an extensive literature review may be found in there.

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**References**

[1] F. Facchinei and J. S. Pang. Finite-Dimensional Variational Inequalities and Complementarity Problems (Springer, New York, 2007).

[2] H. Herty, S. Steffensen, and A. Thünen, Solving Quadratic Multi-Leader-Follower Games by Smoothing the Follower’s Best Response, arXiv:1808.07941 (2018).

[3] H. Nikaidô and K. Isoda, Note on non-cooperative convex game, Pacific Journal of Mathematics 5(5), pp. 807–815 (1955).