Vortex description of the fractionalized phase in exciton bose condensate

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As a sequel to the previous work [Phys. Rev. B 72, 235104 (2005)] we present a vortex description of the fractionalized phase in exciton bose condensate. Magnetic flux line and monopole of the 3+1D emergent U(1) gauge theory are identified in the exciton picture. A bundle of vortex/anti-vortex pairs of all flavors of excitons corresponds to the magnetic flux line and a point at which the vortices and anti-vortices recombine is identified as magnetic monopole. This completes the magnetic sector of the low energy excitation in the fractionalized phase.

I. INTRODUCTION

Recently, various fractionalized phases have been studied in exciton bose condensate\cite{1}. It was shown that a single exciton model can support various fractionalized phases with either fractionalized boson or fermion along with photon. The world line representation of the fractionalized particle and the emergent photon was constructed from the world lines of the exciton. The confinement, Coulomb and Higgs phases were described in terms of the dynamics of the web made of exciton world lines in a space-time picture. The world line representation turned out to be most useful in describing the confinement/deconfinement phase transition of the fractionalized particle which is electrically charged with respect to the emergent photon. This is because the world line representation corresponds to an electric representation of the emergent gauge theory which keep tracks of the electric charge and flux line. On the other hand, it was hard to describe magnetic charge and flux in the electric representation. This is attributed to the uncertainty principle. The electric degrees of freedom need to be condensed in order for the magnetic excitations to be well defined\cite{2}. Thus it is desirable to use a magnetic description to understand magnetic excitations of the emergent gauge theory in the exciton picture. Recently, various 3+1D models have been proposed to show fractionalization\cite{3,4,5,6,7}. In Ref.\cite{3}, fractionalized phase in a 3+1D bosonic model has been studied in the magnetic description. In this picture, the emergent photon is understood as a long wavelength fluctuation of condensed excitons of multiple species. The objective of the present paper is to employ the vortex description\cite{8} to study the fractionalized phase in the exciton bose condensate\cite{1}. We identify magnetic flux and monopole of the emergent gauge theory in the exciton picture.

The fractionalized phase in the exciton system can also be understood based on conventional slave-particle approaches. The slave-particle theory inevitably involves an infinitely strong gauge coupling because the gauge field comes from the constraint field. The infinite bare gauge coupling ensures that the slave-particles are confined at high energy. Although the microscopic slave-particles are always confined within excitons, the fractionalization can still occur as the gauge coupling is renormalized to a finite value at low energy. The field theoretic argument for the gauge coupling renormalization in the slave-particle theory is consistent with the dual world line description of exciton\cite{1}. From this analysis, it is shown that the fractionalized phase occurs at least in the limit of $N$ sufficiently large where $N$ is the degeneracy of bands\cite{1}. However, the critical $N_{cr}$ above which fractionalization occurs for the exciton model, is not known. Numerical simulation is necessary to study this issue. The objective of the present paper is not to address the occurrence of fractionalization for a given finite $N$. Instead, we would like to understand magnetic excitations of the emergent gauge theory in terms of the original exciton in the fractionalized phase with large $N$. The identification of magnetic degrees of freedom also provides an alternative way of probing emergent photon in the fractionalized phase. This will be useful in numerical simulation based on the number representation of exciton. In this paper, $N$ is assumed large enough that the model has a fractionalized phase although some examples are given with a small $N$ for the sake of illustration.

Here is an overview of the present paper. We focus on 3+1D and the parameter regime where the exciton bose condensate is described by the off diagonal phase modes $\theta^{ab}$ of the $N \times N$ Hermitian matrix model,

\begin{equation}
S = -\frac{K}{4} \sum_{i,j} tr^i (\chi^i_j X_j + h.c.) + \sum_n \tilde{K}_n \sum_i tr^i \chi^n_i (1)
\end{equation}

Here $\chi^{ab} = \chi_o e^{i \theta^{ab}}$ is the Hermitian matrix element of fixed amplitude $\chi_o$ with $\theta^{ab} = -\theta^{ba}$. $\chi^{ab}$ describes the boson condensate of the exciton made of a particle in the $a$-th band and a hole in the $b$-th band where $a, b = 1, ..., N$ are band (flavor) indices. $K$ and $\tilde{K}_n$ are coupling constants determined from microscopic model. With the
with the Higgs phase where $\phi^a$ are coherent. The elementary excitations in the Higgs phase are super current modes and vortices. There are $\frac{N(N-1)}{2}$ different vortices of the off diagonal phase modes of the Hermitian matrix. However only $N$ of them remain as low energy excitations because of the dynamical constraints in Eq. (2).

Each low energy vortex can be represented as vortex of the slave boson $\phi^a$ with $a = 1, ..., N$. Note that vortex of a slave boson $\phi^a$ involves vortex (anti-vortex) of $\theta^{ab}$ ($\theta^{ba}$) for all $b$ with $a < b$ ($a > b$). Vortices of the slave bosons with different flavors attract each other. This is because two vortices of $\phi^a$ and $\phi^b$ involves a vortex (anti-vortex) of $\theta^{ab}$ at the position of the vortex of $\phi^a$ ($\phi^b$). Because of the attraction, they can form a bundle where vortices of all flavors are bound with each other within a finite distance. We will refer to this object as vortex bundle. A vortex bundle in $N = 3$ case is displayed in Fig. 1.

The vortex bundle is special among vortex excitations because it does not have net vorticity of the exciton phase. In the exciton picture the vortex bundle corresponds to pairs of vortex and anti-vortex as is shown in Fig. 2 for $N = 3$. Thus there is no long range interaction between segments of the vortex bundle. Vortex and anti-vortex of exciton can recombine at a point and the vortex bundle can end at a point in space as is shown in Fig. 3. The end point represents a point particle. In determining physical property of the particle, the orientedness of the vortex bundle is important. The vortex bundle has orientation even though a pair of vortex and anti-vortex of individual exciton does not have orientation. The orientation comes from correlation between vortex/anti-vortex pairs of different flavors. For example, there are two possibilities depending on whether the positions of vortices of $\theta^{ab}$ and $\theta^{ac}$ are coincident or the positions of anti-vortices are coincident. If the positions of vortices (anti-vortices) coincide, it is represented as a vortex (anti-vortex) of $\phi^a$ at the coincident position, which determines the orientation of the vortex bundle. Once the correlation in the vorticity of two exciton condensates is fixed, the rest are fixed by the constraint (2). Since there is an orientation in the vortex bundle, one end is identified as a particle and the other end, as an anti-particle which is distinct from the particle. In space-time picture the end points form a closed loop which represents vacuum fluctuations of the particle/anti-particle as is shown in Fig. 4. This is the world line picture of a charged particle connected by a U(1) gauge flux line. We interpret the vortex bundle as a flux line of the emergent gauge theory and the end point of the vortex bundle as a particle carrying charge.

Deep in the Higgs phase, all vortex excitations are gapped. Sizes of both the individual vortex and vortex bundle are small. The particle/anti-particle (end points of vortex bundle) are confined by short segments of the flux line (vortex bundle). As the phase stiffness of the exciton condensate decreases, the size of vortex increases. If the size of vortex bundle diverges the end point of vortex bundle emerges as deconfined particle. It can be accomplished by either condensation of individual vortices

\[ \theta^{ab} = \phi^a - \phi^b, \]
or condensation of vortex bundle without condensation of individual vortices. First, consider the former case. This corresponds to the Coulomb phase. The Goldstone modes of the exciton are gapped by the individual vortex condensation. With a finite mass gap of the deconfined particle, the fluctuations of the vortex bundle give rise to a gapless photon. The emergence of photon can be understood in the same manner as the emergence of photon in the phase transition from the confinement to Coulomb phases. The difference is that the confinement/Coulomb phase transition involves the divergences in the size of the world line web of exciton, while the Higgs/Coulomb phase transition involves the world sheet of vortex bundle. Since the world line web of exciton was identified as the world sheet of electric flux line of the emergent gauge theory, we identify the vortex bundle as magnetic flux. Accordingly the end point of the vortex bundle is identified as magnetic monopole. This identification comes from the fact that the Higgs phase is electro-magnetically dual to the confinement phase. Note that the term ‘electric’ and ‘magnetic’ has only relative meaning here. Since we choose to call the particle (flux) which is confined in the confinement as electric charge (flux), we call the charge (flux) which is confined in the Higgs phase as magnetic charge (flux). It is noted that the Coulomb phase occurs when vortices condense without the condensation of the magnetic monopoles. If magnetic monopoles are condensed, the emergent photon is gapped owing to the dual Higgs mechanism. One can understand that this Coulomb phase generically occurs in a large N limit where N becomes large while the phase stiffness of the slave bosons is fixed. (The phase stiffness of the original exciton should be scaled as 1/N in order to fix the phase stiffness of slave bosons.) In this limit, the mass of the magnetic monopole is proportional to N because superfluidity of the N slave bosons is suppressed near the position of the monopole. In this large N limit, the monopole becomes very massive and the monopole generically remains gapped when the vortices of slave bosons condense.

Second, consider the case where only vortex bundle condenses without the condensation of individual vortices. The magnetic monopole is also deconfined and the photon emerges. The difference from the first case is that the Goldstone modes of the exciton remain gapless. This is because the condensation of the vortex bundle alone does not disorder the phase of exciton. Thus we refer to this phase as Higgs phase to distinguish it from the usual Higgs phase which does not have emergent photon. This is the analogy of AF* phase where fractionalization coexists with antiferromagnetic long range order. However, it is noted that the Higgs* is less likely to occur than the Coulomb phase. This is because it needs condensation of vortex bundle while individual vortices of slave bosons are ‘gapped’. Since both the phase stiffness of the vortex bundle and the mass of the magnetic monopole are proportional to N, the Higgs* is not guaranteed to occur even in the large N limit. One may need to introduce and fine tune other interactions in the microscopic model in order to achieve the condensation of vortex bundle without the condensation of magnetic monopoles. Finally, if monopole is condensed then electric charge is confined because of the uncertainty principle. This is the confinement phase. The schematic phase diagram is shown in Fig. 1 and Fig. 2.

II. VORTEX DESCRIPTION OF THE FRACTIONALIZED PHASE IN THE HERMITIAN MATRIX MODEL OF EXCITON BOSE CONDENSATE

Since the constraint Eq. 2 satisfies the potential energy in Eq. 1, only the kinetic energy remains and the partition function of the Hermitian matrix of the exciton condensate can be written as a functional integral over the slave bosons (see. Eq. (25) of Ref. 1),

$$Z = \int D\phi^a e^{\frac{\kappa}{2N} \sum_{\nu,\sigma} \sum_{a \neq b} \cos(\nabla_\mu \phi^a(i) - \nabla_\mu \phi^b(i))}.$$  (3)

Here $\kappa = 2Kx_0^2$, $i$ is the site index in the 3+1D Euclidean lattice and $\nabla_\mu \phi^a(i) = \phi^a(i + \mu) - \phi^a(i)$ with $\mu$, the link direction. A dual transformation similar to that of the 3+1D XY-model leads to a vortex representation for the slave boson $e^{i\phi^a}$. For details of the duality transformation see Appendix A. In the vortex representation, the partition function becomes

$$Z = \sum_{F_{\rho\sigma}} \int Dg_{\rho\sigma} \exp\left(-\sum_I \frac{1}{\kappa N} \sum_{a,\mu} \left(\frac{1}{4\pi} P_{ab} \epsilon_{\mu\nu\rho\sigma} \nabla_\nu g_{\rho\sigma}^d(I) \right)^2 + \frac{i}{2} g_{\rho\sigma}^d(I) P_{ab} F_{\rho\sigma}^a(I) \right) \delta \left(\nabla_\sigma F_{\rho\sigma}^a(I) \right).$$  (4)

Here we closely followed the notations of Ref. 3. $I$ is the index of the dual lattice, $\mu, \nu, \rho, \sigma$, the link direction, and $a, b$, the flavor indices. Repeated indices are understood to be summed. $F_{\rho\sigma}^a(I)$ is an integer which represents the presence of the world sheet of a vortex of $\phi^a$ at the $\rho\sigma$ plaquette of site $I$. $g_{\rho\sigma}^d(I)$ is the two-form
lapped vortices in all space has no dynamics. This mode represents the over-
a projection operator in the flavor space. Among the N value 0 of
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dual gauge field coupled to the world sheet of vortex. It
mediates interaction between vortices. \( F_{ab} = \delta_{ab} - \frac{1}{N} \) is a
projection operator in the flavor space. Among the N
dual gauge fields, there is one zero mode which has eigen-
value 0 of \( P \). It is along the vector \( v_0^T = \frac{1}{\sqrt{N}}(1, 1, \cdots, 1) \).
Thus the flavor independent mode \( g^a_{\rho \sigma}(I) = g_{\rho \sigma}(I) \) is
projected out. The zero mode corresponds to an unphys-
ical mode with \( \phi^a = \phi \). This mode is spurious because the
flavor independent shift in every \( \phi^a \) does not make
any change in the phase of exciton \( \theta^{ab} = \phi^a - \phi^b \). The
constraint \( \nabla_\sigma F^a_{\rho \sigma}(I) = 0 \) ensures that the world sheet
of each vortex should be closed in the 3+1D space-
time. For notational simplicity, from now on we will omit the
site index \( I \) in the field variables.

Because of the projection in the dual gauge field, one
mode of \( F^a_{\rho \sigma} \) which is along the \( v_0 \) direction in the flavor
space has no dynamics. This mode represents the over-
lapped vortices in all \( \phi^a \) on a same plaquette. It is also a
spurious mode since the exciton phase is not distorted at
all by the coincident \( N \) vortices. Thus the region where
\( N \) vortices are overlapped on a same plaquette should be
regarded as vacuum. An example for the \( N = 3 \) case is
displayed in Fig. 3. With the overlapped region removed, the
world sheet of vortex is no longer closed. Lines of \( N \) vortices
can end together at a point in space as is shown in
Fig. 3 (b). In space-time picture, world sheets of the
vortex bundle made of \( N \) vortices can end on a closed
loop as is shown in Fig. 4. We interpret the closed loop
as a world line of a charged ‘particle’. The vortex bundle
which emanates from the world line of the particle is a
‘flux’. In the exciton language, the flux corresponds to a
bundle of vortex/anti-vortex pairs of all off diagonal excitons \( e^{i\theta^{ab}} \) as explained in the introduction and shown
in Fig. 2. The charged particle is the end point where
the vortex and anti-vortex recombine. Now, we discuss
the flux and the particle in the gauge theory picture. The
slave bosons \( e^{i\theta^{ab}} \) carry U(1) gauge charge and is electric-
cally coupled to a compact U(1) gauge field. Vortices of
the charged particle should involve flux of the coupled
gauge field. The energy cost is \( \rho_S \sum_\alpha (\partial_\mu \phi^\alpha - a_\mu)^2 \). The vortex bundle is accompanied by 2\( \pi \) magnetic flux of the
gauge field because it involves vortices of slave bosons of
all flavors. On the other hand, the energy of an individual vortex which involves the winding of a single \( \phi^a \) is
minimized by a 2\( \pi/N \) flux. The end point of the vor-
tex bundle is a magnetic monopole which is the source of
the 2\( \pi \) magnetic flux line. Taking into account the
open boundary of the vortex world sheets we rewrite the
partition function as

\[
Z = \sum_{F_{ab}} \sum_{I_{\rho}} \int Dg^{a}_{\rho \sigma} Dc^{a}_{\sigma} D\alpha e^{-S},
\]

where

\[
S = \sum_{\alpha} \left[ \frac{1}{12N} \sum_{\alpha,\beta} \left( \frac{1}{4\pi} P_{ab} \epsilon_{\mu \nu \rho \sigma} \nabla_\nu g^{b}_{\rho \sigma} \right)^2 + \frac{i}{2} g^{a}_{\rho \sigma} P_{ab} F^{b}_{\rho \sigma} \right]
\]

\[
+ \frac{1}{2\kappa_m} \sum_{\rho} l^2_{\rho} + \frac{1}{2\kappa_v} \sum_{\alpha} (P_{a \rho})^2
\]

\[
- ic^{a}_{\rho} (\nabla_\sigma F^{a}_{\rho \sigma} - l_{\rho}) + i\alpha (\nabla_\rho P_{\rho \rho}).
\] (6)

Here \( l_{\rho} \) represents the word line of the magnetic
monopole. \( 1/\kappa_m \) is the mass of the monopole and \( 1/\kappa_v \),
the tension of the vortex world sheet. The bare mass and
tension is zero in the lattice scale, i.e., \( 1/\kappa_m = 1/\kappa_v = 0 \).
However they are renormalized to nonzero values in long
distance scale. \( c^{a}_{\rho} \) and \( \alpha \) are Lagrangian multipliers im-
posing the flux conservation condition \( \nabla_\sigma F^{a}_{\rho \sigma} = l_{\rho} \) and
the current conservation of magnetic monopole \( \nabla_\sigma l_{\sigma} = 0 \) respectively.

Besides lattice scale \( \zeta_1 \), there are three length scales in
this theory. They are the size of the monopole world line \( \zeta_2 \), the size of individual vortex \( \zeta_3 \), and the size of
the vortex bundle \( \zeta_4 \). Note that it is possible that
\( \zeta_4 \gg \zeta_3 \). This is because \( N \) vortices can form a vortex
bundle whose effective tension is smaller than the tension of individual vortex. In the Higgs phase, all of the length scales are finite. With finite $\zeta_4$, the magnetic monopoles connected by the bundle of vortices are confined. In the long wavelength limit, the vortex and the monopole/anti-monopole excitations can be ignored and the low energy theory is described by the $N - 1$ Goldstone modes $g^b_{\rho\sigma}$. Note that there are only $N - 1$ modes because of the projection $P_{ab}$. As $\kappa$ decreases the tension (core energy) of the vortex decreases and all of the $\zeta_2$, $\zeta_3$ and $\zeta_4$ increase. There are different possibilities depending on which scale diverges.

If $\zeta_4$ diverges while $\zeta_2$ and $\zeta_3$ remain finite, the monopoles are deconfined without condensation of individual vortex. Only the bundle of vortices condenses. Since there is no long range interaction between the vortex bundle, the low energy theory is described by gauge theory. The emergence of photon is signified from the long range correlation between the loop operators,

$$\langle \delta T_{C_2} \delta T_{C_1} \rangle,$$  \hspace{1cm} (7)

where $\delta T_C = T_C - < T_C >$ and $T_C$ is a creation operator of a vortex bundle along the loop $C$ (see Fig. 1). In the gauge theory picture it creates magnetic flux line along $C$. It is the ‘t Hooft operator and is given by $T_C = e^i \oint_C \hat{a}$ where $\hat{a}$ is the vector potential associated with the dual field strength tensor. Here $(\hat{f})_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma}$ with $f_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$, the field strength tensor of the emergent gauge field. Now we represent the ‘t Hooft operator in the exciton language. In the Hamiltonian picture, we represent the ‘t Hooft loop operator as

$$T_C \sim \sum_{|C_a - C_b| < L} e^{i} \frac{\sum_{a < b} \sum_{\rho < \sigma} [\varphi(C_a, r) - \varphi(C_b, r)] \tilde{n}_{\rho\sigma}^{ab}}{|r - r'|}.$$  \hspace{1cm} (8)

Here $r$ is the 3 dimensional space vector. $|C_1 - C_2|$ is a ‘distance’ between two loops and $L$, the maximum distance between vortex and anti-vortex of exciton in a vortex bundle. $\tilde{n}_{\rho\sigma}^{ab}$ is the number operator of exciton of flavor $ab$ at site $r$. $\varphi(C, r)$ is given by

$$\nabla \varphi(C, r) = \frac{1}{2} \nabla \times \int_C \frac{dr'}{|r - r'|}.$$  \hspace{1cm} (9)

$\varphi(C, r)$ changes by $2\pi n_{C_a C'}$, when $r$ moves along a loop $C'$ which have linking number $n_{C_a C'}$ with the loop $C$. Thus $\varphi(C, r)$ can have discontinuity by $2\pi$ in space. However the ‘t Hooft loop operator is well defined because of $e^{i2\pi \tilde{n}_{\rho\sigma}^{ab}} = 1$. To see that the above operator corresponds to the ‘t Hooft operator we apply the operator to a phase eigenstate of exciton. $T_C$ create vortex (anti-vortex) along $C_a$ ($C_b$) for the phase of $\theta_{ab}$ because $e^{i \varphi(C_a, r) \tilde{n}_{\rho\sigma}^{ab}} (e^{-i \varphi(C_b, r) \tilde{n}_{\rho\sigma}^{ab}})$ rotates $\theta_{ab}$ by $\varphi(C_a, r)$ ($\varphi(C_b, r)$) which, in turn, winds by $2\pi$ ($-2\pi$) around the contour $C_a$ ($C_b$). Thus $T_C$ creates vortex of slave boson $\phi_b$ along the contour $C_a$ for each $a$. The vortex bundle is the bound state of these vortices and the position of the individual vortices are summed around a given contour $C$ within a finite length scale $L$. Since the vortex bundle corresponds to the magnetic flux line in the gauge theory picture, we identify $T_C$ as the ‘t Hooft operator of the gauge theory. It is dual to the Wilson loop which was constructed in Ref. 1. The correlation function between electric (magnetic) flux is measured by $T_C$ ($W_C$). In the Coulomb phase both of them have long range correlations. With finite $\zeta_3$ the individual vortex is not condensed. There still are long range correlation in the phases of exciton and the $(N - 1)$ Goldstone modes remain gapless. This phase is a Higgs phase but it also has the emergent photon as an additional gapless excitation apart from the Goldstone modes. Thus we call this phase as Higgs* to distinguish it from the conventional Higgs phase.

If $\zeta_3$ diverges then the individual vortex is condensed and the Goldstone modes are gapped. This corresponds to the Coulomb phase. In the Coulomb phase it is convenient to use phase representation for the vortex field rather than the world sheet of vortex. Note that $c^a_\sigma(\alpha)$ is conjugate to $f_{\rho\sigma}^a$ ($l_\rho$). Thus $c^a_\sigma(\alpha)$ represents compact phase mode of vortex (monopole) field. The summation over $f_{\rho\sigma}^a$ and $l_\rho$ leads to the effective action for the phase variables

$$S = \sum I \left[ \frac{1}{\kappa N} \sum_{a, \mu} \left( \frac{1}{4\pi} P_{ab} c_{\mu\rho\sigma} \nabla_{\nu} g^b_{\rho\sigma} \right)^2 - \kappa_v \sum_a \sum_{\rho < \sigma} \cos(\nabla_{\rho} c^a_\sigma - \nabla_{\sigma} c^a_\rho - P_{ab} g^b_{\rho\sigma}) - \kappa_m \sum_{\rho} \cos(\nabla_{\rho} \alpha - \sum_a c^a_\rho) \right].$$  \hspace{1cm} (10)

In the Coulomb phase the vortex is condensed and $c^a_\sigma$ can be regarded as non-compact variable. In a rotated basis where the projection operator becomes diagonalized as $P_{ab} = \delta_{ab} - \delta_{a1}\delta_{b1}$, the Lagrangian becomes

$$L = \frac{1}{\kappa N} \sum_{a=2}^N \sum_{\rho < \sigma} \frac{1}{4\pi} c_{\mu\rho\sigma} \nabla_{\nu} g^{a\rho}_{\nu\sigma}^a + \frac{\kappa_v}{2} \sum_{\rho < \sigma} \left( \nabla_{\rho} c^a_\sigma - \nabla_{\sigma} c^a_\rho \right)^2 + \sum_{a=2}^N \sum_{\rho \sigma} \left( \nabla_{\rho} c^a_\sigma - \nabla_{\sigma} c^a_\rho - g^{a\sigma}_{\rho\sigma} \right)^2 - \kappa_m \sum_{\rho} \cos(\nabla_{\rho} \alpha - \sqrt{N} c^a_\rho),$$  \hspace{1cm} (11)

where $c^a_\rho = A_{ab} c^b_\rho$ with $A_{ab}$, the orthogonal matrix. Especially, the first column vector $A_{1a} = \frac{1}{\sqrt{N}}$ is the eigen vector for the zero mode of $P_{ab}$. The non-zero modes $c^a_\rho$ with $a \geq 2$ are coupled to the dual gauge field $g^{a\rho}_{\nu\sigma}$. In the Coulomb phase (Higgs’ phase for the vortex field) the non-zero modes can be absorbed into the longitudinal
mode of the dual gauge field by the ‘gauge transformation’,
\[ g_{\rho\sigma}' = g_{\rho\sigma} + (\nabla_\rho e_\sigma - \nabla_\sigma e_\rho). \]
Then the dual gauge field acquire mass gap. In the low energy limit, we obtain
\[ \mathcal{L} = \frac{\kappa_v}{2N} \sum_{\rho<\sigma} (\nabla_\rho \bar{a}_\sigma - \nabla_\sigma \bar{a}_\rho)^2 - \kappa_m \sum_\rho \cos (\nabla_\rho e - \bar{a}_\rho), \]
where the photon field is given by \( \bar{a}_\rho = \sqrt{N} e_\rho' \). This is the low energy effective theory for the monopole coupled to the non-compact U(1) gauge field. Combined with the effective Lagrangian (Eq. (33) in Ref. [1]) which describes the coupling of the electrically charged particles (fractionalized bosons) to the gauge field \( a_\rho \) (not \( \bar{a}_\rho \)), they describe all of the low energy excitations in the Coulomb phase. Note that the magnetic gauge coupling increases with increasing \( N \) as \( g_m^2 \sim \frac{1}{N} \). This is due to the Dirac quantization condition \( g g_m = \frac{1}{2} \), where \( g \) is the electric gauge coupling. The electric gauge coupling decreases with \( N \) as \( g^2 \sim \frac{1}{N^2} \) in the limit where the effective hopping integral for the fractionalized boson \( t \) is fixed with increasing \( N \).

Finally, if \( \zeta \) diverges the monopoles are condensed. With the monopole condensation the gauge field \( \bar{a}_\rho \) is gapped by Higgs mechanism. Condensation of magnetic charge implies the confinement of electric charge according to the uncertainty principle. This corresponds to the confinement phase. With this we complete the phase diagram of the Hermitian matrix model from the Higgs phase side to the confinement phase side, while the other direction was studied in Ref. [1]. The schematic phase diagram is shown in Fig. 5. Fig. 6 shows an alternative phase diagram which include the Higgs* phase. The Higgs* and the Coulomb phases are exotic phases which have emergent photon. However, we emphasize that whether those phases occur or not depend on the details of dynamics. It is possible that either one or both of them are absent in the phase diagram of a specific model.

The low energy spectrum in the phase diagram of Fig. 5 is not quite symmetric despite the fact that the Higgs phase are dual to the confinement phase. This is attributed to the fact that there is only one kind of magnetically charged particle while there are \( N \) electrically charged particles. The condensation of electric charge leaves \( N-1 \) gapless modes in the Higgs phase while there is no remaining gapless mode in the confinement phase.

III. CONCLUSION

In the present paper, we studied the fractionalized phase of exciton bose condensate by using the vortex representation which is dual to the world line representation of exciton used in the previous work [1]. From this we identified the magnetic flux and monopole excitations of the emergent gauge theory in terms of the exciton picture. This completes the full identification of the low energy excitations in the fractionalized phase.
Appendix A. Vortex representation for the Hermitian matrix model

We apply the vortex transformation to Eq. (3). There are two equivalent ways of doing this. In the first way, one represents the theory as a compact U(1) gauge theory coupled with $N$ slave bosons. The resulting action is

$$ S' = -t \sum_a \sum_{i,\mu} \cos(\nabla_\mu \phi^a(i) - a_\mu(i)), \quad (A1) $$

where $t$ is the effective phase stiffness of the slave boson and $a_\mu$, the compact U(1) gauge field. Then the standard dual transformation followed by the integration of the gauge field leads to the vortex representation as is shown in Eq. (A7). In the second way, one can directly dualize the Eq. (3) without introducing gauge field in the intermediate step. The two methods give rise to the same result. Here we use the second method in order to emphasize the fact that the emergence of the U(1) gauge field is not dependent of a particular way of introducing auxiliary field, but is a consequence of the intrinsic dynamics of the model.

With the Villain approximation and the Hubbard-Stratonovich transformation, the partition function is rewritten as

$$ Z = \sum P_{\mu} \int_{-\infty}^{\infty} D\phi^a \int_{-\infty}^{\infty} D\tilde{j}^{ab}_{\mu} \exp\left(\sum_{i,\mu} \sum_{a<b} \left[\frac{1}{\kappa}(j^{ab}_{\mu}(i))^2 - i \left(\nabla_\mu \phi^a(i) - \nabla_\mu \phi^b(i) - 2\pi (p^a_{\mu}(i) - p^b_{\mu}(i))) j^{ab}_{\mu}(i)\right]\right). \quad (A2) $$

Here $i$ is the site index in the 3+1D Euclidean lattice and $\mu, \nu$ the link direction. The integer field $p^a_{\mu}(i)$ is introduced in order to restore the periodicity of the action as a function of $\phi^a$. $p^a_{\mu}$ is decomposed into the divergenceless part and the rotationless part as $p^a_{\mu} = \tilde{p}^a_{\mu} + \nabla_\mu N^a$. The summation over $N^a$ and the integration over $\tilde{\phi}^{a}$ leads to constraint

$$ \sum_b \nabla_\mu j^{(ab)}_{\mu} = 0, \quad (A3) $$

where $j^{(ab)}_{\mu} \equiv j^{ab}_{\mu}$ for $a < b, j^{(ab)}_{\mu} \equiv -j^{ba}_{\mu}$ for $a > b$ and $j^{(aa)}_{\mu} \equiv 0$. Note that the current of each exciton is not conserved. What is conserved is the flavor current $\tilde{j}^{a}_{\mu} \equiv \sum_b j^{(ab)}_{\mu}$. Introducing the flavor current we rewrite the partition function as

$$ Z = \sum P_{\mu} \int D\tilde{j}^{a}_{\mu} D\lambda^a \exp\left(\sum_{i,\mu} \sum_{a<b} \left[\frac{1}{\kappa}(j^{ab}_{\mu}(i))^2 - 2\pi i \sum_a \sum_{b} \lambda^a (\tilde{j}^{a}_{\mu} - \sum_b j^{(ab)}_{\mu}) \right] \delta(\nabla_\mu \tilde{j}^{a}_{\mu}), \quad (A4) $$

where $\lambda^a_{\mu}$ is Lagrangian multiplier field imposing the relation between the flavor current and exciton current. Here we omit the site index $i$ in the field variables. The Gaussian integration for $\tilde{j}^{ab}_{\mu}$ results in the partition function

$$ Z = \sum P_{\mu} \int D\tilde{j}^{a}_{\mu} \lambda^a \exp\left(\sum_{i,\mu} \sum_{a,b} \left[\frac{\kappa}{4N} \lambda^a P_{ab} \delta_{\mu}(i)^2 \right] \delta(\nabla_\mu \tilde{j}^{a}_{\mu}), \quad (A4) $$

Solving the constraint $\nabla_\mu \tilde{j}^{a}_{\mu} = 0$ by introducing a two-form dual gauge field $g^{a}_{\rho\sigma}, \tilde{j}^{a}_{\mu} = \frac{1}{\sqrt{N}} \epsilon_{\mu\nu\rho\sigma} \nabla_\nu g^{a}_{\rho\sigma}$, and introducing a field $F^{a}_{\rho\sigma}$ describing the world sheet of vortex, $F^{a}_{\rho\sigma} = \epsilon_{\mu\nu\rho\sigma} \nabla_\nu \tilde{j}^{a}_{\mu}$, we obtain the partition function

$$ Z = \sum F^{a}_{\rho\sigma} \int Dg^{a}_{\rho\sigma} \exp\left(\sum_{i,\mu} \sum_{a,b} \left[\frac{1}{\kappa N} \sum_{i} \left(\frac{1}{4\pi} P_{ab} \epsilon_{\mu\nu\rho\sigma} \nabla_\nu g^{a}_{\rho\sigma} \right)^2 \right] \delta(\nabla_\mu F^{a}_{\rho\sigma}), \quad (A7) $$

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