On type IIB supergravity action on $M^5 \times X^5$ solutions

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Abstract

While the 10d type IIB supergravity action evaluated on $\text{AdS}_5 \times S^5$ solution vanishes, the 5d effective action reconstructed from equations of motion using $M^5 \times S^5$ compactification ansatz is proportional to the $\text{AdS}_5$ volume. The latter is consistent with the conformal anomaly interpretation in AdS/CFT context. We show that this paradox can be resolved if, in the case of $M^5 \times X^5$ topology, the 10d action contains an additional 5-form dependent “topological” term $\int F_{5M} \wedge F_{5X}$. The presence of this term is suggested also by gauge-invariance considerations in the PST formulation of type IIB supergravity action. We show that this term contributes to the 10d action evaluated on the D3-brane solution.

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1 Introduction

Many discussions of applications of the maximally supersymmetric case of AdS/CFT duality [1] start with a classical action of 5d gauged supergravity or simply 5d gravity with a cosmological term

\[ S_5 = -\frac{1}{2\kappa_5^2} \int d^5x \sqrt{g} \left( R_5 + 12L^{-2} + \ldots \right). \]  

(1.1)

Evaluating this action on the AdS\(_5\) vacuum solution with radius \(L\) gives a factor of volume of AdS\(_5\) space. Assuming \(S_4\) as a boundary of AdS\(_5\), the regularized value of the volume reproduces the planar part of the UV divergent (conformal \(a\)-anomaly) term in the free energy of \(\mathcal{N} = 4\) \(SU(N)\) super Yang-Mills theory on \(S^4\) (see, e.g.,[2, 3, 4])

\[ S_5 = \frac{8L^4}{2\kappa_5^2} \text{vol(AdS}_5) = N^2 \log(\Lambda r). \]  

(1.2)

The action like (1.1) is also a starting point of investigations of AdS black hole thermodynamics [5, 6, 7, 8].

The 5d gauged supergravity action is assumed to follow from the 10d type IIB supergravity action compactified on \(S^5\) [9, 10]. However, the actual compactification procedure involves starting with the 10d field equations [11, 12], substituting there an \(S^5\) compactification ansatz and then reconstructing the corresponding action for the 5d fields (cf. [13]). The bosonic part of the 10d type IIB action may be written as

\[ S_{10} = -\frac{1}{2\kappa_{10}^2} \{ \int d^{10}x \sqrt{G} \left( e^{-2\phi} \left[ R + 4(\partial_\mu \phi)^2 - \frac{1}{4}|H_3|^2 \right] ight. \\
\left. - \frac{1}{2}|F_1|^2 - \frac{1}{2}|F_3|^2 - \frac{1}{4}|F_5|^2 \right) - \frac{1}{2} \int B_2 \wedge F_3 \wedge F_5 \} + \ldots, \]  

(1.3)

\[ F_1 = dC_0, \quad F_3 = dC_2 - C_0 H_3, \quad F_5 = dC_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3. \]  

(1.4)

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\(^1\) Here we use that \(\frac{1}{2\kappa_5^2} = \frac{L^4 \text{vol}(S^5)}{2\kappa_{10}^2}, 2\kappa_{10}^2 = (2\pi)^7 g_5^2 \alpha'^4, L^4 = 4\pi g_s \alpha'^2 N.\) To recall, \(\text{vol}(S^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \to \gamma_n = \pi^n, \text{vol(AdS}_{2n+1}) = \frac{2(\pi^n)^{n+1}}{(n+1)!} \log(\Lambda r) \to \gamma_{2n+1} = \pi^n \log(\Lambda r) \) and \(R_5 = -20L^{-2}, r\) is the radius of boundary 4-sphere and \(\Lambda\) is an IR cutoff on the AdS side (corresponding to UV cutoff on the SYM side).

\(^2\) Here \(|F_p|^2 = \frac{1}{2}F_{\mu_1...\mu_p}F^{\mu_1...\mu_p}\). Extra \(\frac{1}{2}\) in the normalization of the \(F_5\) kinetic term has to do with the requirement that the corresponding analog of the Einstein equation should contain the contribution of the stress tensor of only the self-dual half of \(F_5\).
Here, as usual, the self-duality condition $F_5 = \ast F_5$ is relaxed [14] and is imposed by hand at the level of equations of motion (alternative approaches that involve auxiliary fields where the self-duality condition follows from the equations of motion are discussed in [15, 16, 17, 18, 19, 20]).

Comparing (1.1) and (1.3) we arrive at the following apparent paradox: the 10d action (1.3) evaluated on the vacuum $\text{AdS}_5 \times S^5$ solution

$$ds_{10}^2 = L^2(ds_{\text{AdS}_5}^2 + d\Omega_5^2) , \quad F_5 = 4L^{-1}(\epsilon_5 + \ast \epsilon_5) , \quad L^4 = 4\pi \alpha'^2 g_s N ,$$

is clearly vanishing ($R = -20L^{-2} + 20L^{-2} = 0, |F_5|^2 = 0)^3$ while the value of the 5d action (1.1) on the $\text{AdS}_5$ solution is non-zero (1.2) and consistent with the AdS/CFT duality.

It is of course well known that substituting some special-symmetry ansatz for a subset of fields into the action is not the same as doing this in the equations of motion and then reconstructing the corresponding dimensionally reduced action for the remaining field variables. However, the values of the actions on the full solutions are expected to match. Furthermore, the problem is that the 10d action and, in particular, its on-shell value should be more fundamental: it should follow from (a properly defined) quantum string theory path integral. Thus using the 10d approach is important if one is to go beyond the leading order in $\alpha'$, in particular, in the context of AdS/CFT.

One may wonder if this issue has to do with the subtlety of implementing self-duality of $F_5$. However, this is not the case: similar disagreement between the on-shell values of the reduced 3d action and the 10d action is found in the case of $\text{AdS}_3 \times S^3 \times T^4$ background supported by a 3-form flux. Here the 10d action is well defined off-shell for a generic 3-form field and the effective 6d self-duality of the latter (implying the vanishing of the 10d action) is just a feature of a particular solution.

A natural way to resolve this problem is to assume that the 10d action (1.3) is missing some “boundary term” that restores the equivalence of its on-shell value with that of the 5d action (1.1). However, such term cannot be one of the familiar choices like the Gibbons-Hawking-York (GHY) one [21, 22]\footnote{This term does not contribute in the case of AdS asymptotics.} or boundary terms that may be added to the 5d action (1.1) to make it IR finite when evaluated on a classical solution with $\text{AdS}_5$ asymptotics (see, e.g., [2, 23, 7, 24]).

An important general point is that boundary or topological terms may not be universal: they may depend on a choice of vacuum (near which one expands in order to find an effective action for fluctuations) or asymptotic boundary conditions. For example, in the type IIB string theory there are two maximally supersymmetric vacua – the flat space $R^{1,9}$ and $\text{AdS}_5 \times S^5$ [11] – that have different asymptotic symmetries. The corresponding effective actions may, in principle, contain different boundary terms.

In what follows we will be interested in the case when the topology of 10d space-time is that of a product $M^5 \times X^5$ where $M^5$ is non-compact and $X^5$ is a compact space. We will suggest a novel 5-form dependent “topological” term that should be added to the 10d action (1.3) to restore its on-shell equivalence with the reduced 5d action (1.2).\footnote{Note that a self-dual 5-form is real in the case of Minkowski 10d signature but is imaginary in the Euclidean signature case.}

Let us stress again that the reason why one would like to understand the 10d origin of the on-shell value of the reduced action like (1.2) is that it should have a string theory origin (being related to string partition function on a 2-sphere). For example, the tree-level bosonic string
effective action may be written as\(^6\)

\[
S_D = S_{\text{bulk}} + S_{\text{bndry}}, \quad S_{\text{bulk}} = \hat{S}_D = \kappa \int d^D x \sqrt{G} \ e^{-2\phi} \tilde{\beta}^\phi, \quad \kappa = \frac{2}{\kappa_5^2 \alpha'}, \quad (1.6)
\]

\[
\tilde{\beta}^\phi = c_0 - \frac{1}{4}\alpha' \left[ R + 4\nabla^2 \phi - 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} |H_3|^2 \right] + \mathcal{O}(\alpha'^2), \quad c_0 = \frac{1}{6}(D - 26), \quad (1.7)
\]

\[
S_{\text{bndry}} = -\frac{1}{2} \alpha' \int d^{D-1} x \sqrt{\gamma} e^{-2\phi} (K - 2\partial_n \phi) = -\frac{1}{2} \alpha' \int d^{D-1} x \sqrt{\gamma} \nabla_a (e^{-2\phi} n^a). \quad (1.8)
\]

Here the integrand (1.7) of the bulk part is proportional to the generalized conformal anomaly coefficient \(\tilde{\beta}^\phi\) and thus must vanish on-shell\(^7\) not only to first two leading orders [25] but also to all orders in \(\alpha'\) [26].\(^8\) The boundary term (1.8) which is a dilatonic generalization [28] of the standard GHY term may, in general, produce a non-zero on-shell value for the total action.

Similar remarks apply to the NS-NS part of the type IIB superstring effective action. Note that the boundary term that should be added in general to the bulk type IIB action (1.3) (with the second-derivative dilaton term in (1.7) integrated by parts and thus not automatically vanishing on solutions with non-constant dilaton) is given by (1.8) without the \(\partial_n \phi\) term, i.e.

\[
S_{\text{bndry}} = -\frac{1}{\kappa_{10}^2} \int d^{D-1} x \sqrt{\gamma} e^{-2\phi} K. \quad (1.9)
\]

As for the R-R terms in the second line of (1.3), they may lead to additional non-trivial boundary contributions when evaluated on a classical solution.\(^9\) Given that the bulk \(|F_3|^2\) term vanishes identically upon use of the on-shell self-duality condition, an extra \(F_5\)-dependent contribution to 10d action would be required to get a non-zero contribution for solutions with only \(F_5\)-flux being non-zero. This new term should not change the equations of motion, i.e. it should be a “topological” or “boundary” term.\(^10\)

We shall suggest such a topological term in section 2. In section 3 we shall compute the value of the full 10d action (containing the bulk term (1.3), the boundary term (1.9) as well as the topological term) on the extremal D3-brane solution and its non-extremal generalization. We shall also note the non-zero value of the topological term on solutions describing BPS intersections of two and four D3-branes that in the near-core limit reduce to AdS\(_3 \times S^3 \times T^4\) and AdS\(_2 \times S^2 \times T^6\) backgrounds respectively. Section 4 will contain some concluding remarks.

In Appendix A we shall argue that the presence of the same topological term is suggested also by gauge invariance requirement in the PST formulation [16, 17] of type IIB supergravity action. In Appendix B we shall discuss the computation of the value of the 10d action on fundamental string, NS5-brane and D5-brane solutions.

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\(^6\)This action may be reconstructed also from scattering amplitudes near asymptotically flat vacuum, with the boundary term required for a consistent definition of the graviton/dilaton S-matrix.

\(^7\)Strictly speaking, this is true for backgrounds for which there is no source in the dilaton equation, cf. discussion of brane solutions in Appendix B.

\(^8\)The same conclusion was reached for the on-shell value of the closed bosonic string field theory action [27].

\(^9\)For example, \(|F_3|^2\) term reduces to a boundary term upon use of the field equation \(\nabla_\mu F^{\mu\nu\lambda} + \ldots = 0\), cf. also [29].

\(^{10}\)Note that the fact that particular topological or boundary terms may or may not be relevant depending on boundary asymptotics of the fields is not unfamiliar. For example, the GHY boundary term complementing the Einstein action is relevant in the asymptotically flat space but may not be contributing in the AdS case (e.g. it vanishes for the AdS Schwarzschild black hole because the black hole correction to the AdS metric vanishes too rapidly at infinity [5]).
2 Topological term

While the obvious guess for the 10d topological invariant $\int F_5 \wedge F_5$ is identically zero, a non-trivial candidate is possible if we assume that the 10d space has a particular topological structure. Namely, let us specify to the backgrounds for which the 10d space-time is a product $M^5 \times X^5$ where $M^5$ is non-compact (e.g., asymptotically $\text{AdS}_5$) while $X^5$ is compact and a similar factorization applies to the 5-form field strength (for simplicity, we shall ignore all other fields)

$$M^{10} = M^5 \times X^5,$$  \hspace{1cm}  

$$F_5 = F_{5M} \oplus F_{5X},$$  \hspace{1cm}  

and also its potential $C_4 = C_{4M} \oplus C_{4X}$. Then consider the following “topological” term

$$S_{\text{top}} = \gamma \int F_{5M} \wedge F_{5X}.$$  \hspace{1cm}  

As $M^5$ is non-compact and $F_{5M} = dC_{4M}$ while $dF_{5X} = 0$ this term reduces to a boundary contribution and thus does not affect the bulk equations of motion.

Integrating over the compact $X^5$ then gives

$$S_{\text{top}} = \gamma q \int_M F_{5M}, \ \ \ \ q = \int_X F_{5X}.$$  \hspace{1cm}  

The integral of a 5-form $F_{5M}$ is effectively equivalent to an extra $M^5$ volume term. Equivalently, using the on-shell condition of selfduality of $F_5$ giving $F_{5X} = *F_{5M}$ we conclude that $S_{\text{top}} = \gamma \int F_{5M} \wedge *F_{5M} \sim \text{vol}(X^5) \int_M |F_{5M}|^2$, which again produces, as is well known [30], a contribution to 5d cosmological term.

More generally, the assumption of simple “5+5” factorization of $F_5$ may be relaxed: provided $F_5$ can be split into an “electric” part (involving time differential) and its dual magnetic part the topological term may be written as

$$S_{\text{top}} = \gamma \int F^{(el)}_5 \wedge F^{(mag)}_5, \ \ \ \ F^{(mag)}_5 = *F^{(el)}_5.$$  \hspace{1cm}  

The value of the coefficient $\gamma$ in (2.2),(2.4) required to match the coefficient of the cosmological term in (1.1) is\textsuperscript{11}

$$\gamma = -\frac{1}{4 (5!)^2 \kappa_{10}^2},$$  \hspace{1cm}  

so that the topological term in (2.4) takes the form

$$S_{\text{top}} = -\frac{1}{4 (5!)^2 \kappa_{10}^2} \int F^{(el)}_5 \wedge *F^{(el)}_5 = \frac{1}{4 \kappa_{10}^2} \int d^{10}x \sqrt{G} |F^{(el)}_5|^2.$$  \hspace{1cm}  

The total 10d action is then given by the sum of the bulk term (1.3), the new topological term (2.2) and the boundary term (1.9)

$$S_{10} = \tilde{S}_{10} + S_{\text{top}} + S_{\text{bndry}}.$$  \hspace{1cm}  

Note that the $|F_5|^2$ term in the bulk action (1.3) may be written (before imposing self-duality) as $\frac{1}{8 \kappa_{10}^2} \int d^{10}x \sqrt{G} (|F^{(el)}_5|^2 + |F^{(mag)}_5|^2)$. Adding the topological term (2.6) corresponds effectively to the 10d action $\frac{1}{8 \kappa_{10}^2} \int d^{10}x \sqrt{G} |F_5|^2$.

\textsuperscript{11}Note that in our notation (with Minkowski signature 10d metric) for a general 5-form one has:

$$\int F_5 \wedge *F_5 = -(5!)^2 \int d^{10}x \sqrt{G} |F_5|^2.$$
to reversing the sign of the magnetic part in $|F_5|^2$, thus doubling the contribution of the electric part once going on-shell (the self-duality condition implies $|F_5^{(el)}|^2 = -|F_5^{(max)}|^2$).

Let us note that a similar procedure of inverting the sign of the square of the “electric components” of field strength in the action was used also in the discussion of flux compactifications (cf. [31]). This may be interpreted as implied by the “democratic” formulation of supergravity [32] with doubled number of RR fields.\(^{12}\)

As a result, the value (1.2) of the tree-level type IIB action on the AdS\(_5 \times S^5\) vacuum solution comes entirely from the topological term (2.2),(2.5): using (1.5) we get

$$S_{10}|_{\text{AdS}_5 \times S^5} = S_{\text{top}}|_{\text{AdS}_5 \times S^5} = -\frac{1}{4 (5!)^2 \kappa_{10}^2} \int F_{\text{AdS}_5} \wedge F_{S^5} = -\frac{4L^8}{\kappa_{10}^2} \text{vol(AdS}_5) \ , \quad (2.8)$$

which is the same result that follows from the 5d action (1.2).

This has straightforward generalization to the case of AdS\(_5 \times X^5\) solutions where \(X^5\) is an Einstein manifold as in [3]; instead of (1.2) one gets \(S_5 = k N^2 \log(Ar)\), with \(k \equiv \frac{\text{vol}(S^5)}{\text{vol}(X^5)}\) and \(L^4 = 4\pi\alpha'^2g_s k N\).

As we will show in Appendix A, the same term (2.2) with precisely the same coefficient (2.5) is also required for gauge invariance in the PST formulation [16, 17] of the 10d supergravity action where the 5-form self-duality condition follows from the equations of motion.

To provide further evidence that adding the term (2.2) to the type IIB action (1.3) restores its on-shell equivalence with the 5d reduced action like (1.1) let us consider the following \(M^5 \times S^5\) ansatz for the metric and \(F_5\) (with its self-duality condition relaxed and all other fields set to zero)

$$ds_{10}^2 = L^2 \left[ e^{-\frac{40}{3} \nu(x)} g_{nn}(x) dx^m dx^n + e^{2\nu(x)} d\Omega_5^2 \right] , \quad F_5 = 4L^{-1} [a(x) w_5 + b w_5] \ . \quad (2.9)$$

Here \(x = \{ x^m \} (m = 0, 1, \ldots 4)\), \(w_5\) and \(w_5\) are the volume forms on \(M^5\) (with metric \(g_{mn}\)) and \(S^5\) and we extracted the factors of the overall scale \(L\). Following [33] we introduced the warp factors depending on a “fixed scalar” \(\nu(x)\).\(^{13}\) The condition \(d F_5 = 0\) implies that \(a = a(x)\) and \(b = \text{const}\). Then the \(R - \frac{1}{4}|F_5|^2\) part of the 10d action (1.3) compactified on \(S^5\) becomes

$$\hat{S}_5 = -\frac{1}{2 \kappa_5^2} \int d^5x \sqrt{g} \left[ R_5 - \frac{40}{3} (\partial_5 \nu)^2 - V(\nu) + \ldots \right] , \quad (2.10)$$

$$V(\nu) = L^{-2} \left( -20 e^{-\frac{40}{3} \nu} - 4a^2 e^{\frac{40}{3} \nu} + 4b^2 e^{-\frac{40}{3} \nu} \right) . \quad (2.11)$$

The 3 terms in the potential \(V\) originate from the scalar curvature of \(S^5\) and the \(|F_5|^2\) term in (1.3) (cf. [33, 1]). Using the on-shell self-duality of \(F_5\) that gives \(a = e^{-\frac{40}{3} \nu}b\) we find that the last two terms in the potential (2.11) mutually cancel and thus, as was already mentioned above, we do not reproduce the value of the cosmological constant in (1.1).

If instead one plugs the ansatz (2.9) into the 10d equations of motion for (1.3) (that imply that \(b^2 = 1\), \(a = e^{-\frac{40}{3} \nu}\)) and then reconstructs the corresponding effective action for the remaining 5d fields \(g_{mn}(x)\) and \(\nu(x)\) one finds instead the action (2.10) with the following potential [33]

$$V(\nu) = L^{-2} \left( -20 e^{-\frac{16}{3} \nu} + 8 e^{-\frac{40}{3} \nu} \right) . \quad (2.12)$$

\(^{12}\)For example, one may start with an action containing two unconstrained 5-form field strengths \(F_5^a\) and \(F_5^b\) and consider configurations in which \(F_5^a\) has electric part only, and \(F_5^b\) the magnetic part only. Dualizing electric \(F_5^a\) will convert it to magnetic one and thus effectively double the total magnetic contribution.

\(^{13}\)The specific dependence on \(\nu\) in the metric is required to decouple \(\nu\) from the 5-d graviton; this generalizes the graviton mode decomposition in [34, 10] where \(\nu\) was identified with the zero mode of the trace of the perturbation of the metric of \(S^5\).
This potential has the minimum at $\nu = 0$ and where it reproduces the cosmological term $12L^{-2}$ in (1.1). Comparing to (2.11), the potential (2.12) has the sign of the middle $a^2$ term in (2.11) effectively reversed so that it doubles the coefficient of the last $\beta^2$ term upon use of the on-shell condition $a = e^{-\frac{2\alpha}{\pi}}$.

This is precisely what happens if we add to (2.10) the contribution of the topological term (2.2),(2.5) and then use the self-duality of $F_5$. We conclude that adding this term to the type IIB action ensures the equivalence between the 10d and 5d actions not only for AdS$_5 \times S^5$ but also for more general solutions of $M^5 \times S^5$ topology.

### 3 10d action on D3-brane solutions

Let us now generalize the above discussion of the on-shell value of the type IIB action (1.3) with the topological term (2.2) added to the case of the extremal and non-extremal D3-brane solutions that also have the product topology as in (2.1).

The extremal D3-brane solution is given by [35, 36]

$$ds^2_{10} = h^{-1/2}(r) dy^{\mu} dy_{\mu} + h^{1/2}(r) (dt^2 + r^2 d\Omega_5^2) , \quad h(r) = 1 + \frac{L^4}{r^4} , \quad L^4 = 4\pi \alpha'^2 g_s N , \quad (3.1)$$

$$C_4^{(ed)} = \left[h^{-1}(r) - 1\right] dt \wedge dy^1 \wedge dy^2 \wedge dy^3 , \quad F_5 = F_5^{(el)} + F_5^{(mag)} , \quad F_5^{(mag)} = * F_5^{(el)} , \quad (3.2)$$

$$F_5^{(el)} = \frac{4r^3 L^4}{(r^4 + L^4)^2} dt \wedge dy^1 \wedge dy^2 \wedge dy^3 \wedge dr , \quad F_5^{(mag)} = 4L^{-1}w_5 . \quad (3.3)$$

Here $y^{\mu} = (y^0 \equiv t, y^1, y^2, y^3)$ are coordinates along the D3-brane and $w_5 = \sqrt{g_{ss^5}} dz^5 \wedge \ldots \wedge dz^9$ is the volume form of $S^5$. The near-core limit $h \to \frac{L^4}{r^4}$ corresponds to the AdS$_5 \times S^5$ case.

As discussed in the Introduction, the bulk part of the type IIB action (1.3) has zero on-shell value (once again, the self-duality of $F_5$ implies $|F_5|^2 = 0$ and thus also $R = 0$). A non-trivial contribution may come from the topological term (2.2) and also from the GHY boundary term (1.9) that may be non-vanishing in this asymptotically flat case. From (2.2),(2.3) we find (cf. (2.8))

$$S_{\text{top}} \bigg|_{\text{D3}} = -\frac{1}{4(5!)} \frac{\kappa_{10}^2}{r_0^2} \int F_5^M \wedge F_5^X = -\frac{1}{4(5!)} \frac{\kappa_{10}^2}{r_0^2} \int F_5^{(el)} \wedge F_5^{(mag)}$$

$$= -\frac{\text{vol}(S^5)}{2 \kappa_{10}^2} \int_0^\infty dr \frac{8L^8 r^3}{(r^4 + L^4)^2} \int d^4y = -\frac{\text{vol}(S^5)}{\kappa_{10}^2} L^4 \int d^4y = -\frac{1}{2} N \mu_3 \int d^4y . \quad (3.4)$$

Here

$$\mu_3 = \frac{2\text{vol}(S^5) L^4}{N \kappa_{10}^2} = \frac{1}{(2\pi)^3 g_s \alpha'^2} \quad (3.5)$$

is tension of a unit-charge D3-brane (cf. footnote 1) and $\int d^4y$ is the integral over the D3 world volume directions. Compactifying $(y^1, y^2, y^3)$ on a torus with volume $V_3$ we get

$$S_{\text{top}} \bigg|_{\text{D3}} = -\frac{1}{2} N M_3 \int dt , \quad M_3 = \mu_3 V_3 , \quad V_3 = \int d^3y , \quad (3.6)$$

where $M_3$ is the mass of a single D3-brane.

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14If we focus on the near-core limit ($r \ll L$) of (3.1),(3.2) we get the same expression as in (2.8) with the volume of AdS$_5$ written in Poincare coordinates.
The GHY boundary term (1.9) (that did not contribute in the AdS$_5 \times S^5$ case) happens to give the same result as in (3.4) (here the asymptotic boundary is at $r = \infty$)\(^{15}\)

\[
S_{\text{bdry}}\big|_{D3} = -\frac{\text{vol}(S^5)}{\kappa_{10}^2} \left. \frac{L^4}{1 + \frac{L^4}{r^4}} \right|_{r \to \infty} \int d^4y = -\frac{1}{2} N \mu_3 \int d^4y .
\]  

(3.7)

Then the on-shell value of the 10d action (2.7) on the D3-brane solution is given by

\[
S_{10}\big|_{D3} = (S_{\text{top}} + S_{\text{bdry}})\big|_{D3} = -N \mu_3 \int d^4y .
\]  

(3.8)

In addition, one may consider the value of the D3-brane source action that provides the delta-function in the equation for the harmonic function $h(r)$

\[
S_{\text{source}} = -N \mu_3 \int d^4y \sqrt{G_4} + N \mu_3 \int C_4 .
\]  

(3.9)

More generally, considering this as an action of a static probe D3-branes placed at distance $r$ parallel to the source branes at $r = 0$ one finds from (3.1),(3.2) that the $h^{-1}$ factors from the two terms in (3.9) cancel each other\(^{16}\) leaving simply

\[
S_{\text{source}}\big|_{D3} = -N \mu_3 \int d^4y
\]  

(3.10)

coming from the $-1$ in $C_4$ in (3.2). This is equal to the free brane action at $r = \infty$ and the same expression is thus also at $r \to 0$.

As a result, the total action on D3-brane solution is given by

\[
S_{\text{tot}} = S_{\text{bulk}} + S_{\text{top}} + S_{\text{bdry}} + S_{\text{source}} , \quad S_{\text{tot}}\big|_{D3} = -2N \mu_3 \int d^4y .
\]  

(3.11)

Similar computations of the value of 10d action on some other p-brane solutions are presented in Appendix B.

Next, let us consider the non-extremal (black) D3-brane solution \([35]\) generalizing (3.1)--(3.3)\(^{17}\)

\[
ds_{10}^2 = h^{-1/2}(r) \left[ -f(r)dt^2 + (dy^i dy^j) + h^{1/2}(r) \left[ f^{-1}(r)dr^2 + r^2 d\Omega_5^2 \right] \right] ,
\]  

(3.12)

\[
h(r) = 1 + \frac{\tilde{L}^4}{r^4} , \quad f(r) = 1 - \frac{r_0^4}{r^4} , \quad \tilde{L}^4 = \sqrt{L^8 + \frac{1}{4} r_0^8 - \frac{1}{2} r^4} ,
\]  

(3.13)

\[
C_4^{(el)} = \sigma \left[ h^{-1}(r) - 1 \right] dy^0 \wedge \cdots \wedge dy^3 , \quad \sigma = \frac{L^4}{\tilde{L}^4} = \sqrt{1 + \frac{r_0^4}{L^4}} ,
\]  

(3.14)

\[
F_5^{(el)} = \frac{4\sigma \tilde{L}^4 r^3}{(r^4 + \tilde{L}^4)^2} dy^0 \wedge \cdots \wedge dy^3 \wedge dr , \quad F_5^{(mag)} = 4\sigma \tilde{L}^{-1} w_5 , \quad F_5 = F_5^{(el)} + F_5^{(mag)} ,
\]

where $L$ is the same as in (3.1). We shall consider this solution for $r_0 \leq r < \infty$ and should not introduce an explicit brane source.

The value of the topological term (2.3) is found as in (3.4)

\[
S_{\text{top}}\big|_{\text{black D3}} = -\frac{1}{4(5!)^2 \kappa_{10}^2} \int F_5^{(el)} \wedge F_5^{(mag)} = -\frac{\text{vol}(S^5)}{2\kappa_{10}^2} \sigma^2 \int_{r_0}^{\infty} dr \frac{8 \tilde{L}^8 r^3}{(r^4 + \tilde{L}^4)^2} \int d^4y
\]  

\[
= -\frac{\text{vol}(S^5)}{\kappa_{10}^2 L^4} \int d^4y .
\]  

(3.15)

\(^{15}\)Here and below when evaluating the boundary term (1.9) we neglect contributions that are independent of the parameters of the solution.

\(^{16}\)This is, of course, a manifestation of the BPS condition of the vanishing force, see, e.g., \([37]\).

\(^{17}\)We use the same parametrization as in \([38]\).
Once again we see that the topological term gives a non-trivial contribution to the action.

The expression (3.15) may be written also as
\[ S_{\text{top}}|_{\text{black D3}} = \frac{1}{2} N \mu_3 C_4^{(el)}(r_{0}) \int d^4 y , \quad \text{(3.16)} \]
\[ C_4^{(el)}(r_{0}) = - \frac{\sigma \tilde{L}^4}{r_{0}^4 + \tilde{L}^4} , \quad N \mu_3 = \frac{2 \text{vol}(S^5)L^4}{\kappa_{10}^2} = \frac{2 \text{vol}(S^5)\sigma \tilde{L}^4}{\kappa_{10}^2} , \quad \text{(3.17)} \]
i.e. is proportional to a product of the electric potential \( C_4^{(el)} \) at the horizon and the black D3-brane charge. This is analogous to what one finds in the case of the Reissner–Nordstrom black hole [22].

The calculation of the asymptotic \( r \to \infty \) boundary GHY term (1.9) here gives (cf. (3.7))
\[ S_{\text{bndry}}|_{\text{black D3}} = - \frac{\text{vol}(S^5)}{\kappa_{10}^2} \left[ \tilde{L}^4 \left( \frac{1 - \frac{r_{0}^4}{L^4}}{1 + \frac{r_{0}^4}{L^4}} + 3r_{0}^4 \right) \right] \int d^4 y = - \frac{\text{vol}(S^5)}{\kappa_{10}^2} \left( \tilde{L}^4 + 3r_{0}^4 \right) \int d^4 y . \quad \text{(3.18)} \]
As the bulk 10d action (1.3) is again vanishing, the total action (2.7) computed on the non-extremal D3-brane solution then follows by combining (3.15) and (3.18)
\[ S_{10}|_{\text{black D3}} = (S_{\text{top}} + S_{\text{bndry}})|_{\text{black D3}} = - \frac{2 \text{vol}(S^5)}{\kappa_{10}^2} \left( \tilde{L}^4 + \frac{3}{2} r_{0}^4 \right) \int d^4 y \]
\[ = - \frac{2 \text{vol}(S^5)}{\kappa_{10}^2} \left( \sqrt{L^8 + \frac{1}{4} r_{0}^8} + r_{0}^4 \right) \int d^4 y . \quad \text{(3.19)} \]
The same result should be found by first compactifying on \( S^5 \), finding the reduced 5d action generalizing (1.1) and then evaluating it on the corresponding 5d black brane solution.\(^{19}\)

Similar discussion can be repeated for the type IIB solutions describing BPS intersections of D3-branes – D3\( \perp \)D3 [39] and D3\( \perp \)D3\( \perp \)D3\( \perp \)D3 [40]. In the near core limit they reduce (in the extremal case) to AdS\(_3 \times S^3 \times T^4\) and AdS\(_2 \times S^2 \times T^6\) backgrounds respectively. Here the bulk part of type IIB action is again vanishing, with possible non-zero contribution coming from the topological term (2.4) defined in terms of
\[ F_5^{(el)} = dC_4^{(el)} , \quad F_5^{(mag)} = * F_5^{(el)} , \quad F_5 = F_5^{(el)} + F_5^{(mag)} , \quad \text{(3.20)} \]
and also the GHY term (in the case of the full asymptotically flat solution).

The D3\( \perp \)D3 solution is the following generalization of the D3 background (3.1)–(3.3):
\[ ds_{10}^2 = (h_1 h_2)^{1/2} \left[ (h_1 h_2)^{-1} \left( -dt^2 + dy_1^2 \right) + h_1^{-1} (dy_2^2 + dy_3^2) + h_2^{-1} (dy_4^2 + dy_5^2) \right] + dr^2 + r^2 d\Omega_3^2 , \quad h_i = 1 + \frac{L_i^2}{r^2} , \quad \text{(3.21)} \]
Here \((y^1, y^2, y^3)\) and \((y^1, y^3, y^4)\) are spatial coordinates along the two D3-branes intersecting over \( y^i \) direction. In the near-core limit \( h_i \to \frac{L_i^2}{r^2} \) this background reduces to AdS\(_3 \times S^3 \times T^4\)
\(^{18}\)In the context of black brane thermodynamics the topological term will thus contribute to the part of the “thermodynamic potential” related to the product of the chemical potential and the corresponding conserved charge.

\(^{19}\)For comparison with the extremal case (3.6) let us note that the ADM mass of black D3-brane is given by
\[ M_3 = \mu_3 V_3 \frac{L_1^4 + \frac{r_0^4}{L_4^4}}{L_4^4} = M_3 \left[ \sqrt{1 + \frac{r_0^4}{4L_4^4}} + \frac{3r_0^4}{4L_4^4} \right] . \]
with $ds^2_{AdS_3} = \frac{r^2}{L^2} (-dt^2 + dy_1^2) + \frac{L^2}{r^2} dr^2$, $ds^2_{S^3} = L^2 d\Omega^2_3$ (where $L^2 = L_1 L_2$) and $ds^2_{T^4} = \frac{L^2}{L_1} (dy_2^2 + dy_3^2) + \frac{L^2}{L_2} (dy_4^2 + dy_5^2)$.

Note that here $F_5$ does not have a simple 5+5 decomposition so the topological term term is defined by (2.4) or, equivalently, (2.6). Computing it gives a non-zero value consistent with the Maxwell theory. Here the AdS term in the effective 2d action. This example is related to the near-core limit of the four D3-brane background discussed in the previous section.

Similarly, the four D3-brane solution is given by

$$ds^2_{10} = (h_1 h_2 h_3 h_4)^{1/2} \left[ - (h_1 h_2 h_3 h_4)^{-1} dt^2 + (h_1 h_2) -1 dy_1^2 + (h_1 h_3) -1 dy_2^2 + (h_1 h_4) -1 dy_3^2 ight. 
+ (h_2 h_3) -1 dy_4^2 + (h_2 h_4) -1 dy_5^2 + (h_3 h_4) -1 dy_6^2 + dr^2 + r^2 d\Omega^2_2 \big], \quad h_i = 1 + \frac{L_i}{r},$$

$$C_4^{(el)} = [h_4^{-1} - 1] dt \wedge dy_1 \wedge dy_2 \wedge dy_3 + [h_2^{-1} - 1] dt \wedge dy_1 \wedge dy_4 \wedge dy_5 
+ [h_3^{-1} - 1] dt \wedge dy_2 \wedge dy_4 \wedge dy_6 + [h_4^{-1} - 1] dt \wedge dy_3 \wedge dy_5 \wedge dy_6. \quad (3.22)$$

This reduces to AdS$_2 \times S^2 \times T^6$ with the 6-torus formed by $(y_1, ..., y_6)$. Here the topological term (2.4), (2.6) produces again a non-zero contribution to 10d action.

## 4 Concluding remarks

Depending on topology of space-time or asymptotic boundary conditions, the 10d supergravity action (or, more generally, string effective action) may need to be supplemented by particular boundary or “topological” terms specific to a type of backgrounds considered.

Here we considered the case of $M^{10} = M^5 \times X^5$ with 5-form flux and showed that adding the “topological” term (2.2) or (2.4) to the bulk type IIB action (1.3) restores its equivalence with the 5d reduced action (obtained via equations of motion by compactifying on $X^5$). This leads to consistent on-shell values of the full 10d action (e.g., for AdS$_5 \times X^5$ or D3-brane solution). Similar terms are to be added in cases of other topologies, e.g., $\int_6 F_{3M} \wedge F_{3X}$ for $M^{10} = M^3 \times X^3 \times T^4$.\footnote{Note also that an analogous example is found in the case of the extremal dyonic black hole in 4d Einstein-Maxwell theory. Here the AdS$_3 \times S^2$ vacuum is supported by $F_2 = F_2^{(el)} + F_2^{(mag)}$ and the on-shell value of the action is zero ($R_4 = 0, |F_2|^2 = 0$). Adding the standard topological term $\int F_2 \wedge F_2$ then produces a cosmological term in the effective 2d action. This example is related to the near-core limit of the four D3-brane background discussed in the previous section.}

String theory origin of the term (2.2) and whether it may receive $\alpha'$ corrections remains to be understood. One particular case when the contribution of this term may be important is the computation of $\alpha'$ corrections to near-extremal D3-brane entropy as in [41].

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A Topological term in $F_5$ action on $M^5 \times X^5$ from PST formulation

In the PST formulation [16, 17] of the 5-form action the condition of self-duality is derived from an action. This is achieved by introducing an extra scalar field $a(x)$ along with extra gauge invariance so that the number of dynamical degrees of freedom is unchanged. For a closed 5-form $F_5$ let us consider the following action:

$$S_{\text{PST}} = \int (F_5 \wedge *F_5 + i_v \mathcal{F} \wedge * i_v \mathcal{F}) = - \int 2v \wedge F_5 \wedge i_v (F_5 - *F_5) , \quad \mathcal{F} \equiv F_5 - *F_5 . \quad (A.1)$$

We assume that $F_5$ can be expressed locally as $F_5 = dC_4$ (we ignore all the other fields that may contribute to $F_5$ in (1.4)). $v = v_\mu dx^\mu$ is defined in terms of a scalar $a(x)$ as

$$v_\mu = \frac{1}{\sqrt{-|\partial a|^2}} \partial_\mu a , \quad v^\mu v_\mu = -1 . \quad (A.2)$$

The variation of (A.1) over $C_4$ and $a$ then leads to equations that imply the self-duality condition $F_5 = *F_5$. The dependence on the scalar $a$ drops out of the equations of motion.

The reason for this is that apart from the standard gauge symmetry of a 4-form potential $C_4 \rightarrow C_4 + d\xi_3$, the action (A.1) is invariant (up to boundary terms, see below) under the following gauge transformations

$$\delta_\eta a = \eta , \quad \delta_\eta C_4 = - \frac{1}{\sqrt{-|\partial a|^2}} i_v (F_5 - *F_5) \eta , \quad (A.3)$$
$$\delta_\xi a = 0 , \quad \delta_\xi C_4 = \xi_3 \wedge da , \quad \delta_\xi F_5 = d\xi_3 \wedge da . \quad (A.4)$$

Here the scalar $\eta(x)$ and the 3-form $\xi_3(x)$ are the gauge parameters. The first symmetry (A.3) implies that $a$ is a pure gauge field. The second is effectively reflecting the fact that the number of degrees of freedom of $C_4$ is halved on-shell (where $F_5$ becomes self-dual).

Let us consider the variation of the action under arbitrary $\delta C_4$ and $\delta a$:

$$\delta S_{\text{PST}} = - \int \frac{2v}{\sqrt{-|\partial a|^2}} \wedge d\delta a \wedge i_v \mathcal{F} \wedge i_v \mathcal{F} - \int 4v \wedge \delta F_5 \wedge i_v \mathcal{F} - \int 2F_5 \wedge \delta F_5$$
$$= - \int 2\delta a \left[ \frac{v}{\sqrt{-|\partial a|^2}} \wedge i_v \mathcal{F} \wedge i_v \mathcal{F} \right] - \int 4\delta C_4 \wedge d[v \wedge i_v \mathcal{F}]$$
$$+ \int_\partial \left[ \frac{2v}{\sqrt{-|\partial a|^2}} \delta a \wedge i_v \mathcal{F} \wedge i_v \mathcal{F} + 4\delta C_4 \wedge v \wedge i_v \mathcal{F} \right] - \int 2F_5 \wedge \delta F_5 . \quad (A.5)$$

Assuming that $\delta C_4 = 0$ at the boundary, the resulting equations of motion may be written as:

$$\delta a : \quad d \left[ \frac{v}{\sqrt{-|\partial a|^2}} \wedge i_v (F_5 - *F_5) \wedge i_v (F_5 - *F_5) \right] = 0 , \quad (A.6)$$
$$\delta C_4 : \quad d \left[ v \wedge i_v (F_5 - *F_5) \right] = 0 . \quad (A.7)$$

Under the transformation (A.4) the expression in brackets in (A.7) changes as:

$$\delta_\xi \left[ v \wedge i_v (F_5 - *F_5) \right] = - \delta_\xi F_5 = - d\xi_3 \wedge da , \quad (A.8)$$

so that (A.4) is a symmetry of (A.7). Furthermore, using (A.2) we may choose such $\xi_3$ that $i_v (F_5 - *F_5) = 0$. Then

$$F_5 - *F_5 = - v \wedge i_v (F_5 - *F_5) + * (v \wedge i_v (F_5 - *F_5)) = 0 . \quad (A.9)$$

\[21\]As usual, $i_v$ denotes the contraction of a differential form with a vector field, obtained from the coefficient of 1-form $v$ by raising the index with the help of the metric. In (A.1) we ignore an overall normalization factor.
Therefore, the symmetry (A.4) makes all solutions of (A.7) equivalent to the self-dual solution $F_5 = \ast F_5$ (and all of them lead to the vanishing on-shell value of $S_{\text{PST}}$).

Under (A.4) the integrand of (A.1) changes as:\footnote{Here $v \wedge i_v \delta \xi F_5 \wedge F_5 = - \delta \xi F_5 \wedge F_5 + v \wedge \delta \xi F_5 \wedge i_v F_5$ and $v \wedge i_v \delta \xi F_5 \wedge F_5 = v \wedge (\delta \xi F_5 \wedge v) \wedge F_5 = - \delta \xi F_5 \wedge v \wedge i_v \ast F_5$.}

$$
\delta \xi \mathcal{L} = -2 \left[ v \wedge \iota_v (\delta \xi F_5 - \ast \delta \xi F_5) \wedge F_5 + v \wedge \iota_v (F_5 - \ast F_5) \wedge \delta \xi F_5 + v \wedge \iota_v (\delta \xi F_5 - \ast \delta \xi F_5) \wedge \delta \xi F_5 \right].
$$
(A.10)

Using that $\delta \xi F_5 \wedge da = 0$, the variation of the action (A.1) may be written as

$$
\delta \xi S_{\text{PST}} = -2 \int F_5 \wedge \delta \xi F_5.
$$
(A.11)

This vanishes if 10d space has no boundary (as $dF_5 = 0$ we have $F_5 \wedge d\xi_3 \wedge da = -d(F_5 \wedge \xi_3 \wedge da)$) but otherwise produces a boundary term.

Let us now assume as in (2.1) that the 10d space has a product structure, i.e. $M^{10} = M^5 \times X^5$ where $X^5$ is a compact Euclidean space with no boundary while $M^5$ (with Minkowski signature metric) may be non-compact, and also that a similar factorization applies to the 4-form potential and the parameters of the transformations in (A.4), i.e.

$$
C_4 = C_{4M} \oplus C_{4X}, \quad F_5 = F_{5M} \oplus F_{5X}, \quad \delta \xi C_4 = \delta \xi C_{4M} \oplus \delta \xi C_{4X}.
$$
(A.12)

In this case, (A.11) takes the form

$$
\delta \xi S_{\text{PST}} = -2 \int F_{5X} \wedge \delta \xi F_{5M} = 2 \int \delta \xi F_M \wedge F_{5X} = 2 \int X F_{5X} \int_M \delta \xi F_{5M},
$$
(A.13)

where we used that $\delta \xi F_{5X}$ is exact so its integral over $X^5$ vanishes. The integral $\int_M \delta \xi F_{5M} = \int_{\partial M} \xi_3 \wedge da$ depends on the boundary values of the gauge parameter $\xi_3$ and the scalar field $a$. If these are non-trivial and if $F_5$ has a non-trivial value of the “magnetic” charge $\int_X F_{5X} \neq 0$, then the variation (A.13) may be non-zero.

A way to maintain the invariance of the action (A.1) under (A.4) is to add to (A.1) the topological term defined in (2.2)

$$
S_{\text{top}} = -2 \int_M F_{5M} \wedge F_{5X} = -2 \int_X F_{5X} \int_M F_{5M}.
$$
(A.14)

The variation of this term under the gauge transformation (A.4) will then cancel the change (A.13) of the PST action. Assuming $F_{5M} = dC_{5M}$ is valid globally on $M^5$, the term (A.14) may be expressed as an integral over the boundary $\partial M^5 \times X^5$ and thus does not affect the equations of motion for $F_5$. Let us note that a similar argument suggesting to add the term (A.14) to maintain gauge invariance can be given [20] also in the formulation of self-dual $F_5$ field suggested in [19].

Using that the equations of motion for (A.1) imply the self-duality of $F_5$, i.e. $F_{5M} = \ast F_{5X}$, the on-shell value of (A.1) plus (A.14) may be written also as

$$
(S_{\text{PST}} + S_{\text{top}}) \bigg|_{F_5 = \ast F_5} = S_{\text{top}} \bigg|_{F_5 = \ast F_5} = 2 \int F_{5X} \wedge \ast F_{5X} = -2 \int F_{5M} \wedge \ast F_{5M}.
$$
(A.15)

Replacing the $|F_5|^2$ term in the 10d action (1.3) by (A.1) one gets the corresponding PST analog of the type IIB action to which now we should add also (A.14) with the corresponding coefficient being as in (2.5). It is interesting to note that the condition of the symmetry under (A.4) fixes also the relative coefficient between the kinetic 5-form term (A.1) and the Chern-Simons type term ($\int B_2 \wedge F_3 \wedge F_5$ in (1.3)) in the resulting version of type IIB action [17].
B 10d action on F1, NS5 and D5 brane solutions

For comparison with the case of the D3-brane solution discussed in section 3 here we will discuss the values of the 10d action (3.11) on the fundamental string, NS5-brane and D5-brane extremal solutions. In these cases $F_5 = 0$ so the topological term (2.2) will not play a role. These $p$-brane solutions are supported by sources given by the corresponding brane actions that have the structure (see, e.g., [42])

$$S_{\text{source}} = -NT_p \int d^{p+1}y e^{-q_\phi} \sqrt{G_{p+1}} + NT_p \int A_{p+1},$$

where $T_p$ is a tension of a single brane and $A_{p+1}$ is the corresponding NS-NS or R-R potential. The dilaton coupling constant is $q = 0, 2,$ and $1$ for F1, NS5 and D5 cases respectively. The total action will be

$$S_{\text{tot}} = S_{\text{bulk}} + S_{\text{bndry}} + S_{\text{source}}, \quad S_{\text{bulk}} = \hat{S}_{10},$$

where the bulk part is given by (1.3) and the boundary one by (1.9).

The F1 solution [43] is electrically charged with the respect to the $B_2$ field ($T_1 = \frac{1}{2\pi \alpha^\prime}$)

$$ds^2 = H^{-1}(r)(-dy_0^2 + dy_1^2) + dx^a dx^a, \quad H(r) = 1 + \frac{Q}{r^6}, \quad Q = \frac{NT_1 \kappa_0^3}{3 \text{vol}(S^5)} = 32N\pi^2 \alpha^3 g_s^2,$$

$$B_2 = \left[H^{-1}(r) - 1\right] dy^0 \wedge dy^1, \quad e^{2\phi} = H^{-1}(r).$$

Substituting this solution into (B.2) we find

$$S_{\text{source}} = -NT_5 \int d^6y e^{-2\phi} \sqrt{G_6} + NT_5 \int \tilde{B}_5, \quad T_5 = \frac{1}{(2\pi)^5 \alpha^3 g_s}.$$

The corresponding background is ($\mu = 0, ..., 5; \ a = 6, 7, 8, 9; \ r^2 = x^a x^a$)

$$ds^2 = \eta_{\mu\nu} dy^\mu dy^\nu + H(r) dx^a dx^a, \quad H(r) = 1 + \frac{Q}{r^2}, \quad Q = \frac{NT_5 \kappa_0^3}{\text{vol}(S^5)} = \alpha' N,$$

$$\tilde{B}_5 = \left[H^{-1}(r) - 1\right] dy^0 \wedge ... \wedge dy^6, \quad e^{2\phi} = H(r).$$

Here we find

$$S_{\text{source}} = -NT_5 \int d^6y, \quad S_{\text{source}}|_{\text{NS5}} = -NT_5 \int d^6y,$$

$$S_{\text{bndry}}|_{\text{NS5}} = -NT_5 \int d^6y.$$

Evaluating the bulk term here and in (B.4) we used the explicit form of the solution: note that the NS-NS part of the bulk action (1.3) or (1.6) automatically vanishes only for solutions without a source term in the dilaton equation.

In the case of D5-brane solution that has magnetic charge with respect to the RR 3-form $F_3$ we may again introduce the dual electric potential $\tilde{C}_6$ ($dC_6 = * F_3$) and consider

$$S_{\text{source}} = -N \mu_5 \int d^6y e^{-\phi} \sqrt{G_6} + N \mu_5 \int \tilde{C}_6, \quad \mu_5 = \frac{1}{(2\pi)^5 \alpha^3 g_s}.$$
The D5-solution supported by the corresponding source at \( x^a = 0 \) is \([35]\)

\[
    ds^2 = H^{-\frac{1}{2}}(r) \eta_{\mu\nu} dy^\mu dy^\nu + H^{\frac{1}{2}}(r) dx^a dx^a, \quad H(r) = 1 + \frac{Q}{r^2}, \quad Q = \frac{N\mu_5\kappa_{10}^2}{\text{vol}(S^3)} = \alpha' N g_s,
\]

\[
    \tilde{C}_6 = [H^{-1}(r) - 1] dy^0 \wedge ... \wedge dy^6, \quad e^{-2\phi} = H(r).
\]  

(B.10)

The resulting contributions to the total action (B.2) here are

\[
    S_{\text{bulk}}\big|_{D5} = -\frac{1}{2} N \mu_5 \int d^6 y, \quad S_{\text{bdry}}\big|_{D5} = -\frac{1}{2} N \mu_5 \int d^6 y, \quad S_{\text{source}}\big|_{D5} = -N \mu_5 \int d^6 y,
\]

\[
    S_{\text{tot}}\big|_{D5} = -2 N \mu_5 \int d^6 y.
\]  

(B.11)

The values of the total actions for NS5 (B.8) and D5 (B.11) cases have the same structure as for the D3-brane solution in (3.8) and also are consistent with the S-duality relation between the two 5-branes.

Note that the bulk and boundary contributions match only in sum: one can show that the S-duality transformation in the formulation using the string frame metric leaves invariant only the sum of the bulk (1.3) and boundary (1.9) terms in the type IIB action.

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