Approximation by Lexicographically Maximal Solutions in Matching and Matroid Intersection Problems

Kristóf Bérczi, Tamás Király, Yutaro Yamaguchi, and Yu Yokoi

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Abstract

We study how good a lexicographically maximal solution is in the weighted matching and matroid intersection problems. A solution is lexicographically maximal if it takes as many heaviest elements as possible, and subject to this, it takes as many second heaviest elements as possible, and so on. If the distinct weight values are sufficiently dispersed, e.g., the minimum ratio of two distinct weight values is at least the ground set size, then the lexicographical maximality and the usual weighted optimality are equivalent. We show that the threshold of the ratio for this equivalence to hold is exactly 2. Furthermore, we prove that if the ratio is less than 2, say $\alpha$, then a lexicographically maximal solution achieves $(\alpha/2)$-approximation, and this bound is tight.

1 Introduction

Matching in bipartite graphs is one of the fundamental topics in combinatorics and optimization. Due to its diverse applications, various optimality criteria of matchings have been proposed based on the number of edges, the total weight of edges, etc. The concept of rank maximality is one of them, which is especially studied in the context of matching problems under preferences \cite{2,7}. In this setting, we are given a partition
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\{E_1, E_2, \ldots, E_k\} of the edge set \(E\) that represents a priority order, and a matching \(M \subseteq E\) is rank-maximal if \(|M \cap E_1|\) is maximized, and subject to this, \(|M \cap E_2|\) is maximized, and so on. As pointed out in several papers [1, 2, 4, 7] (on generalized problems), the problem of finding a rank-maximal solution can be reduced to the usual weight-maximization setting by using sufficiently dispersed weights, e.g., by assigning \(|E_i|^{k-i}\) for each element in \(E_i\). (Clearly, it is not enough to assign arbitrary weights that are consistent with the priority order.)

In this paper, we explore the relation between rank-maximality and optimality in the weighted setting for two generalizations of bipartite matching: matching in general graphs and matroid intersection. To avoid confusion, we hereafter replace the term “rank” with “lex(igraphical)” because we also deal with ranks in matroids. The main results of the paper (Theorems 2.1 and 2.2) state that, in both problems, the equivalence between lex-maximality and weighted optimality holds if the minimum ratio of two distinct weight values is larger than 2. This implies that we can choose the base of exponential weights as any constant larger than 2 (instead of \(|E|\)) in the aforementioned reduction. Furthermore, we show that if the minimum ratio is at most 2, say \(\alpha\), then a lex-maximal solution achieves \((\alpha/2)\)-approximation of the maximum weight, and this bound is tight.

2 Results

We assume the basic notation and terminology on graphs and matroids (see, e.g., [8]). Throughout the paper, let \(E\) be a finite ground set and \(w:E \to \mathbb{R}_{>0}\) be a positive weight function on \(E\). For a subset \(X \subseteq E\), the weight \(w(X)\) is defined as the sum of its elements. Let \(w_1, w_2, \ldots, w_k\) be the distinct values of \(w\) in descending order. We assume \(k \geq 2\) (otherwise, the results are trivial), and define

\[ \alpha_w := \min_{1 \leq i \leq k-1} \frac{w_i}{w_{i+1}}. \]

For a subset \(X \subseteq E\) and \(1 \leq i \leq k\), we denote by \(X_i = \{ e \in X \mid w(e) = w_i \}\) the restriction of \(X\) to the elements of weight \(w_i\), and define \(X_{\leq i} := \bigcup_{j \leq i} X_j\) and accordingly \(X_{\geq i}, X_{<i}, \text{ and } X_{>i}\). For two subsets \(X, Y \subseteq E\), we say that \(X\) is lex-larger than \(Y\) (or \(Y\) is lex-smaller than \(X\)) if there exists an index \(i\) such that \(|X_j| = |Y_j|\) for any \(j < i\) and \(|X_i| > |Y_i|\). For a family \(\mathcal{F} \subseteq 2^E\) of subsets of \(E\), we say that \(X \in \mathcal{F}\) is lex-maximal if no \(Y \in \mathcal{F}\) is lex-larger than \(X\). Note that a lex-maximal subset \(X \in \mathcal{F}\) may not be unique but the sequence \(|X_1|, |X_2|, \ldots, |X_k|\) is unique.

A weighted matching instance consists of an undirected graph \(G = (V, E)\) and a weight function \(w\) on the edge set. We denote by \(\text{opt}(G, w)\) the optimal value, i.e., the maximum weight of a matching in \(G\). In addition, we denote by \(\text{lexopt}(G, w)\) the weight of a lex-maximal matching in \(G\) (where \(\mathcal{F}\) is the family of matchings in \(G\)), which takes as many edges of weight \(w_1\) as possible, and subject to this, takes as many edges of weight \(w_2\) as possible, and so on.

A weighted matroid intersection instance consists of two matroids \(M_1 = (E, \mathcal{I}_1)\) and \(M_2 = (E, \mathcal{I}_2)\) and a weight function \(w\) on the common ground set. We denote by
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opt(M_1, M_2, w) the optimal value, i.e., the maximum weight of a common independent set in \(I_1 \cap I_2\). In addition, we denote by lexopt(M_1, M_2, w) the weight of a lex-maximal common independent set (where \(F = I_1 \cap I_2\)), which takes as many elements of weight \(w_1\) as possible, and subject to this, takes as many elements of weight \(w_2\) as possible, and so on.

The main theorems are stated as follows.

**Theorem 2.1.** Let \((G, w)\) be a weighted matching instance. If \(\alpha_w \leq 2\), then

\[
\text{lexopt}(G, w) \geq \frac{\alpha_w}{2} \cdot \text{opt}(G, w).
\]

Otherwise, a lex-maximal matching is a maximum-weight matching, and vice versa.

**Theorem 2.2.** Let \((M_1, M_2, w)\) be a weighted matroid intersection instance. If \(\alpha_w \leq 2\), then

\[
\text{lexopt}(M_1, M_2, w) \geq \frac{\alpha_w}{2} \cdot \text{opt}(M_1, M_2, w).
\]

Otherwise, a lex-maximal common independent set is a maximum-weight common independent set, and vice versa.

We remark that the approximation ratio is tight as the weighted bipartite matching problem is included as a common special case. Let \(G = (V, E)\) be a bipartite graph with \(V = \{1, 2, 3, 4\}\) and \(E = \{e_1 = \{1, 3\}, e_2 = \{2, 3\}, e_3 = \{2, 4\}\}\). Define \(w(e_1) = 1, w(e_2) = x \in (1, 2],\) and \(w(e_3) = 1\). We then have \(\alpha_w = x\) (as \(w_1 = x\) and \(w_2 = 1\)), \(\text{opt}(G, w) = 2\), and \(\text{lexopt}(G, w) = x\).

## 3 Proofs

We prove both theorems using the same strategy. The key definition and lemmas are as follows.

**Definition 3.1.** Let \(X \subseteq E\) be a feasible solution that is not lex-maximal in a weighted matching or matroid intersection instance, and \(i\) be the smallest index such that \(X_{\leq i}\) is not lex-maximal in the restricted instance whose ground set is \(E_{\leq i}\). We say that a feasible solution \(Y \subseteq E\) in the original instance is *eligible* if the following three conditions are satisfied:

- \(|Y_j| = |X_j|\) for any \(j < i\),
- \(|Y_i| = |X_i| + 1\), and
- \(|X_{>i} \setminus Y_{>i}| \leq 2\).

Intuitively, an eligible solution lexicographically improves the original solution at the most significant improvable class by sacrificing at most two lighter elements.

**Lemma 3.2.** For any weighted matching instance \((G, w)\) and any matching \(X\) that is not lex-maximal, there exists an eligible matching \(Y\).
3.1 Matching: Proof of Lemma 3.2

Lemma 3.3. For any weighted matroid intersection instance \((M_1, M_2, w)\) and any common independent set \(X\) that is not lex-maximal, there exists an eligible common independent set \(Y\).

We here prove Theorems 2.1 and 2.2 using Lemmas 3.2 and 3.3, whose proofs are given later.

Proof of Theorems 2.1 and 2.2. Let \(X^*\) be an optimal solution. Starting with \(X = X^*\), we repeatedly update \(X\) to an eligible \(Y\) until it becomes lex-maximal. Let \(Y^*\) be the lex-maximal solution that is finally obtained.

In any update from \(X\) to \(Y\), we have

\[
w(Y) = w(Y_{\leq i}) + w(Y_{> i}) \geq w(X_{\leq i}) + w_i + w(X_{> i}) - 2w_{i+1} \geq w(X) - \frac{2 - \alpha_w}{\alpha_w} \cdot w_i.
\]

If \(\alpha_w > 2\), then we have \(w(Y) > w(X)\), which cannot happen at the beginning when \(X = X^*\). Thus, \(X^*\) is lex-maximal. As any lex-maximal solution has the same weight, we also see that any lex-maximal solution is optimal, and we are done for the second statements.

Suppose that \(\alpha_w \leq 2\). Since we always have \(|Y_j| = |X_j| = |Y_j^*| (j < i)\) and \(|Y_i| = |X_i| + 1 \leq |Y_i^*|\), we see that \(i\) is nondecreasing during the process and each \(w_i\) appears in the right-hand side at most \(|Y_i^*|\) times in total. Thus, by repeating the above inequalities, we obtain

\[
w(Y^*) \geq w(X^*) - \frac{2 - \alpha_w}{\alpha_w} \sum_{i=1}^{k} (w_i \cdot |Y_i^*|) = w(X^*) - \frac{2 - \alpha_w}{\alpha_w} \cdot w(Y^*),
\]

which implies the first statements.

3.1 Matching: Proof of Lemma 3.2

We prove Lemma 3.2 by contradiction.\(^1\) Suppose to the contrary that there exists a counterexample, and take a minimal one in the following sense. First, the ground set \(E\) is minimized as the first priority, and subject to this, a counterexample matching \(X\) (that is not lex-maximal but admits no eligible matching \(Y\)) and a lex-maximal matching \(Z\) are taken so that the symmetric difference \(X \triangle Z = (X \setminus Z) \cup (Z \setminus X)\) is minimized.

By the minimality, we have \(E = X \cup Z = X \triangle Z\). Indeed, if there exists an edge \(e \in E \setminus (X \cup Z)\), then we obtain a smaller counterexample by removing \(e\) (no eligible matching can newly appear), a contradiction. Similarly, if there exists \(e \in X \cap Z\), then we obtain a smaller counterexample again by removing \(e\) (note that no edge is adjacent to \(e\) as \(X\) and \(Z\) are matchings), a contradiction.

Claim 3.4. There are no adjacent edges of the heaviest weight.

\(^1\)We could give an algorithmic proof similar to the matroid intersection case in the next section, but we here describe an alternative proof because it is slightly simpler and may be extended to other problems.
3.2 Matroid Intersection: Proof of Lemma 3.3

Proof. Suppose to the contrary that $e_1 = \{v,u_1\} \in X$ and $e_2 = \{v,u_2\} \in Z$ are adjacent (at $v$) and $w(e_1) = w(e_2) = w_1$. If $u_1 = u_2$ (i.e., $e_1$ and $e_2$ are parallel), then $X' = (X \setminus \{e_1\}) \cup \{e_2\}$ is a smaller counterexample, a contradiction. Otherwise, consider the instance obtained by contracting the two edges $e_1$ and $e_2$, i.e., by merging $u_1$ and $u_2$ into a single vertex, removing the vertex $v$ together with the incident edges, and restricting the weight function to the remaining set of edges.

As $X$ is a matching that is not lex-maximal, this is true for $X' = X \setminus \{e_1\}$ after the contraction. By the minimality of the counterexample, there exists an eligible matching $Y' \subseteq E \setminus \{e_1, e_2\}$. Since at most one of $u_1$ and $u_2$ is matched by $Y'$ in the original graph, we can add $e_1$ or $e_2$ to $Y'$ to obtain a matching $Y$ in the original instance. Moreover, the $Y$ thus obtained is eligible by definition even if $e_2$ is added since $e_1 \in X_{\leq 1}$ for any $i \geq 1$, contradicting our indirect assumption.

Claim 3.5. $X_1 = \emptyset$.

Proof. Suppose to the contrary that there exists an edge $e \in X_1$. Then, by Claim 3.4, we have $w(f) < w(e) = w_1$ for each adjacent edge $f \in Z$. Thus, we can obtain a lex-larger matching $Z'$ from $Z$ by adding $e$ and by removing all the (at most two) adjacent edges, a contradiction.

By Claim 3.5, we have $|X_1| = 0 < |Z_1|$. Let $f \in Z$ be an edge with $w(f) = w_1$. Then, we can obtain an eligible matching $Y$ from $X$ by adding $f$ and by removing all the (at most two) adjacent edges, a contradiction. This concludes the proof of the lemma.

3.2 Matroid Intersection: Proof of Lemma 3.3

The proof relies on (the correctness of) an augmenting path algorithm for the weighted matroid intersection problem, which was first described in [6]. We first review the basic facts based on [8, Sections 41.2 and 41.3].

Let $(M_1, M_2, w)$ be a weighted matroid intersection instance. For each matroid $M_i$ ($i \in \{1, 2\}$), we denote the independent set family, the rank function, and the span function$^2$ by $\mathcal{I}_i \subseteq 2^E$, $r_i : 2^E \rightarrow \mathbb{Z}_{\geq 0}$, and $\text{span}_i : 2^E \rightarrow 2^E$, respectively. For a set $X$ and elements $x \in X$ and $y \notin X$, we write $X \setminus \{x\}$ and $X \cup \{y\}$ as $X - x$ and $X + y$, respectively.

Let $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ be a common independent set. The exchangeability graph with respect to $I$ is a directed bipartite graph $D = (E \setminus I, I; A)$ defined as follows. Let $A := A_1 \cup A_2$, where

$$A_1 := \{(y,x) \mid x \in E \setminus I, \ y \in I, \ I + x - y \in \mathcal{I}_1\},$$

$$A_2 := \{(x,y) \mid x \in E \setminus I, \ y \in I, \ I + x - y \in \mathcal{I}_2\}.$$ 

We also define

$$S := \{s \in E \setminus I \mid I + s \in \mathcal{I}_1\},$$

$$T := \{t \in E \setminus I \mid I + t \in \mathcal{I}_2\},$$

$^2$This is also called the closure function; we follow the terminology of [8] for simplicity.
3.2 Matroid Intersection: Proof of Lemma 3.3

where elements in \( S \) and in \( T \) are called sources and sinks, respectively. Note that \( A_1 \) and \( S \) depend only on \( M_1 \), and \( A_2 \) and \( T \) depend only on \( M_2 \). Let \( c: E \to \mathbb{R} \) be a cost function on the vertex set defined as follows:

\[
c(e) := \begin{cases} w(e) & (e \in I), \\ -w(e) & (e \in E \setminus I). \end{cases}
\] (3.1)

An \( S-T \) path \( P \) in \( D \) is cheapest if the total cost of its vertices is minimized. Subject to this, \( P \) is shortest if the number of its vertices is minimized.

For a nonnegative integer \( \ell \), a common independent set \( I \in I_1 \cap I_2 \) is said to be \( \ell \)-extreme if \( w(I) \) is maximized subject to \( I \in I_1 \cap I_2 \) and \( |I| = \ell \). The following lemma leads to a simple augmenting path algorithm for the weighted matroid intersection problem. The key is that one can augment any \( \ell \)-extreme solution to some \((\ell + 1)\)-extreme solution (if exists) by simply exchanging elements along a path.

**Lemma 3.6** (cf. [8, Theorems 41.3, 41.5, and 41.6]). Let \( I \in I_1 \cap I_2 \) be an \( \ell \)-extreme common independent set, and suppose that there exists a common independent set \( J \in I_1 \cap I_2 \) with \(|J| > |I|\). Then, \( D \) contains an \( S-T \) path, which may consist of a single vertex in \( S \cap T \). Let \( P \) be a shortest cheapest \( S-T \) path in \( D \) with respect to the cost function \( c \) defined as (3.1). Then, no inner vertex of \( P \) is a source or a sink, and \( I \Delta P \) is an \((\ell + 1)\)-extreme common independent set.

Now we start the proof of Lemma 3.3. Let \( X \in I_1 \cap I_2 \) be a common independent set that is not lex-maximal, and let \( i \) be the smallest index such that \( X_{\leq i} \) is not lex-maximal in the restricted instance whose ground set is \( E_{\leq i} \).

**Claim 3.7.** There exists a common independent set \( Y' \subseteq E_{\leq i} \) such that \( |Y'_j| = |X_j| \) for any \( j < i \), \( |Y'_i| = |X_i| + 1 \), \( \text{span}_1(X_{\leq i}) \subseteq \text{span}_1(Y') \), and \( \text{span}_2(X_{\leq i}) \subseteq \text{span}_2(Y') \).

**Proof.** Define an auxiliary weight function \( w': E_{\leq i} \to \mathbb{R}_{\geq 0} \) by \( w'(e) := n^{i-j} \) for each \( e \in E_j \) (\( j = 1, 2, \ldots, i \)), where \( n = |E_{\leq i}| \). As remarked in the introduction, the lexicographical order coincides with the weighted order, i.e., for any two subsets \( Z, Z' \subseteq E_{\leq i} \), \( Z \) is lex-larger than \( Z' \) if and only if \( w'(Z) > w'(Z') \).

Let \( \ell := |X_{\leq i}| \). Then, \( X_{\leq i} \) is \( \ell \)-extreme in the restricted instance \( (M_1|E_{\leq i}, M_2|E_{\leq i}, w') \), which has a larger common independent set. By Lemma 3.6, one can obtain an \((\ell + 1)\)-extreme common independent set \( Y' \subseteq E_{\leq i} \) by flipping \( X_{\leq i} \) along a source-sink path \( P \) in the exchangeability graph. Note that \( |Y'_j| = |X_j| \) for any \( j < i \) and \( |Y'_i| = |X_i| + 1 \) by the choice of \( i \) and the definition of \( w' \).

If \( P \) consists of a single vertex (which is a source and a sink), then the claim immediately follows since \( Y' = X_{\leq i} + y \) for some element \( y \in E_i \). Otherwise, let \( s \) and \( t \) be the first and last vertices of \( P \), respectively. By the definitions of the exchangeability graph and the sources and sinks, \( y \in \text{span}_1(X_{\leq i}) \subseteq \text{span}_1(X_{\leq i} + s) \) for every \( y \in P - s \) and \( y \in \text{span}_2(X_{\leq i}) \subseteq \text{span}_2(X_{\leq i} + t) \) for every \( y \in P - t \) (recall that any \( y \in P - s \) is not a source and any \( y \in P - t \) is not a sink). Hence, we have
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span₁(X ≤₁ ∪ P) = span₁(X ≤₁ + s) and span₂(X ≤₁ ∪ P) = span₂(X ≤₁ + t), and then

\[ \ell + 1 = |Y'| = r₁(Y') \]
\[ \leq r₁(X ≤₁ ∪ P) = r₁(X ≤₁ + s) \]
\[ \leq r₁(X ≤₁) + r₁(s) = |X ≤₁| + 1 = \ell + 1, \]
\[ \ell + 1 = |Y'| = r₂(Y') \]
\[ \leq r₂(X ≤₁ ∪ P) = r₂(X ≤₁ + t) \]
\[ \leq r₂(X ≤₁) + r₂(t) = |X ≤₁| + 1 = \ell + 1, \]

which implies that the equality holds everywhere. Thus we have span₁(Y') = span₁(X ≤₁ + s) ⊇ span₁(X ≤₁) and span₂(Y') = span₂(X ≤₁ + t) ⊇ span₂(X ≤₁), which completes the proof.

Take a subset Y' ⊆ E ≤₁ satisfying the conditions in Claim 3.7. We then obtain an eligible common independent set Y ⊆ E from Y' ∪ X >₁ by removing at most two elements in X >₁ as follows. If r₁(Y' ∪ X >₁) = r₂(Y' ∪ X >₁) = |X| + 1, then Y' ∪ X >₁ ∈ I₁ ∩ I₂ and we do not need to remove any element. Suppose that r₁(Y' ∪ X >₁) < |X| + 1. As span₁(X ≤₁) ⊆ span₁(Y'), we have r₁(Y' ∪ X >₁) ≥ r₁(X) = |X|, and hence r₁(Y' ∪ X >₁) = |X|. This implies that Y' ∪ X >₁ contains exactly one circuit of M₁, which must intersect X >₁ since Y' ∈ I₁. Hence, there exists an element x ∈ X >₁ such that Y' ∪ (X >₁ \ x) ∈ I₁. The same holds for M₂. Thus we obtain a common independent set Y ⊆ Y' ∪ X >₁ with |X >₁ \ Y >₁| ≤ 2, which is eligible.

4 Concluding Remarks

In this paper, we have analyzed how good a lex-maximal solution is in the weighted matching and matroid intersection problems based on how dispersed the distinct weight values are. It is well-known that, subject to a single matroid, a lex-maximal solution is always optimal, which can be found by a greedy algorithm. For more general independence systems, Jenkyns [3] and Korte and Hausmann [5] independently analyzed the worst-case approximation ratio of a greedy algorithm. In particular, the worst ratio is 2 in the weighted matching and matroid intersection problems, while a lex-maximal solution (which is a possible output of a greedy algorithm) always achieves the maximum weight if the distinct weight values are sufficiently dispersed. From this perspective, we have filled the gap between these two situations.

A natural question is as follows: how about a further (common) generalization, e.g., the weighted matroid parity problem? We just remark that it seems difficult to extend our algorithmic proof straightforwardly because no counterpart of augmentation from any ℓ-extreme solution to some (ℓ + 1)-extreme solution along a path (Lemma 3.6) is known. A minimal counterexample proof (like the matching case) also seems nontrivial; in particular, we have found no counterpart of Claim 3.4.
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