Simple decompositions of simple special Jordan superalgebras

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Abstract

We classify decompositions of simple special finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero into the sum of two proper simple subsuperalgebras.

1 Introduction

In our previous work with T.Tvalavadze [17] we considered special simple finite-dimensional Jordan algebras decomposable as the sum of two proper simple subalgebras. The main result in [17] is the following.

Theorem. Let \( J \) be a finite-dimensional special simple Jordan algebra over an algebraically closed field \( F \) of characteristic not two. The only possible decompositions of \( J \) as the sum of two simple subalgebras \( J_1 \) and \( J_2 \) are the following:

1. \( J \cong F \oplus V \) and \( J_1 \cong F \oplus V_1, \ J_2 \cong F \oplus V_2 \), where \( V, V_1, V_2 \) are vector spaces.
2. Either \( J \cong H(\mathcal{R}_3) \) and \( J_1 \cong H(F_3), \ J_2 \cong F \oplus V \), or \( J \cong H(\mathcal{R}_n) \), \( n \geq 3 \), \( J_1 \cong H(F_n) \) and \( J_2 \) is isomorphic to one of the following algebras: \( H(F_{n-1}), H(F_n) \) or \( H(\mathcal{R}_{n-1}) \).
3. \( J \cong H(\mathcal{Q}_n) \) and \( J_1, J_2 \cong H(\mathcal{R}_n) \).

Actually the problem of simple decompositions of simple algebras first arises in the paper of Onishchik (see [12]) in which he classified all possible types of simple decompositions of simple complex and real Lie algebras. Later for associative algebras over an arbitrary field \( F \) the same problem was formulated and then solved by Bahturin and Kegel in [2]. According to [2], no full-matrix...
algebra can be written as the sum of two full-matrix subalgebras. Note that if $F$ is algebraically closed with zero characteristic, then this follows from [12].

To begin with we briefly remind the classification of simple Jordan superalgebras obtained by Kac (see [4]) over an algebraically closed field $F$ with zero characteristic. If $\mathcal{J}$ is a simple special finite-dimensional Jordan superalgebra over algebraically closed field $F$ with zero characteristic, then $\mathcal{J}$ is isomorphic to one of the following superalgebras:

1. $M_{n,m}(F)^{(+)},$ the set of all matrices of order $n + m$ with respect to the natural $\mathbb{Z}_2$-gradation under the Jordan supermultiplication;
2. $osp(n,m),$ the set of all matrices of order $n + 2m$ symmetric with respect to the orthosymplectic superinvolution. The superalgebra consists of matrices $\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}$ where $A^t = A,$ $D$ is symplectic, $B,$ $C$ are skew-symmetric;
3. $P(n) = \left\{ \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \right\}, B^t = B, C^t = -C \in M_n(F);$ 
4. $Q(n) = \left\{ \begin{pmatrix} A & B \\ A & D \end{pmatrix} \right\}$ where $A$ and $B$ are any square matrices of order $n.$
5. $\mathcal{J}$ be a proper subalgebra $\mathcal{J}_0 = F + V_0,$ $\mathcal{J}_1 = V_1.$

Let $V = V_0 + V_1$ be a $\mathbb{Z}_2$-graded vector space with a non-singular symmetric bilinear superform $f(,): V \times V \to F.$ Consider the direct sum of $F$ and $V,$ $\mathcal{J} = F \oplus V,$ and determine multiplication according to $(\alpha + v)(\beta + w) = (\alpha \beta + f(v,w)) + (\alpha w + \beta v).$

Then $\mathcal{J}$ becomes a Jordan superalgebra of the type $J(V, f)$ with respect to the following $\mathbb{Z}_2$-gradation: $\mathcal{J}_0 = F + V_0,$ $\mathcal{J}_1 = V_1.$

6. The 3-dimensional Kaplansky superalgebra $K_3,$ $(K_3)_0 = Fe,$ $(K_3)_1 = Fx + Fy,$ with the multiplication $e^2 = e,$ $ex = \frac{1}{2}x,$ $ey = \frac{1}{2}y,$ $[x,y] = e.$

7. The 1-parametric family of 4-dimensional superalgebras $D_t,$ $D_t = (D_t)_0 + (D_t)_1,$ where $(D_t)_0 = Fe_1 + Fe_2,$ $(D_t)_1 = Fx + Fy,$ where $e_1^2 = e_1,$ $e_1 e_2 = 0,$ $e_1x = \frac{1}{2}x,$ $e_1y = \frac{1}{2}y,$ $[x,y] = e_1 + te_2,$ $i = 1, 2.$ A superalgebra $D_t$ is simple only if $t \neq 0.$ If $t = -1,$ then $D_{-1}$ isomorphic $M_{1,1}(F).$

Next we cite some important Lemmas and Theorems from [17] which will be repeatedly used later.

**Lemma 1.1** Let a Jordan algebra $\mathcal{J}$ of the type $H(D'_m)$ be a proper subalgebra of $H(D_n)$ such that the identity of $H(D_n)$ is an element of this subalgebra. If either
1. $D' = F$ and $D = F,$ or
2. $D' = R,$ $Q,$ and $D = R,$
then $m \leq \frac{n}{2}$

**Lemma 1.2** Let $V$ be a vector space with a non-singular symmetric bilinear form $f,$ and $v_0$ a fixed non-trivial vector in $V.$ Let $S$ be the set of all linear operators which are symmetric with respect to $f.$ Then, $S v_0 = V.$
Indeed, the universal associative enveloping superalgebra of the above Jordan
an associative superalgebra can be considered as an associative alg ebra. The

Theorem 2.2

later discussion.

following Theorem by C.Martinez and E.Zelmanov \[8\] plays a key role in the

J

for a Jordan superalgebra

Theorem 2.1

A

2. Decompositions of superalgebras of the type

M_{n,m}(F)^{(+)}

Our main goal is to prove the following.

Theorem 2.1 Let \( A \) be a superalgebra of the type \( M_{n,m}(F)^{(+) \} \) where \( n,m > 0 \).
If both \( n,m \) are odd, then \( A \) has no decompositions into the sum of two proper nontrivial simple subsuperalgebras. If one of the indices, for example \( m \), is an even number and the other is odd, then the only possible simple decom position is the following: \( A = B + C \) where \( B \) and \( C \) have types \( osp(n, \frac{m}{2}) \) and \( M_{n-1,m}(F)^{(+) \}, \) respectively. If both indices are even, then \( A \) admits two types of decompositions of the following forms:

1. \( A = B_1 + C_1 \) where \( B_1 \) and \( C_1 \) have types \( osp(n, \frac{m}{2}) \) and \( M_{n-1,m}(F)^{(+) \}, \)
2. \( A = B_2 + C_2 \) where \( B_2 \) and \( C_2 \) have types \( osp(m, \frac{n}{2}) \) and \( M_{m-1,n}(F)^{(+) \).

Before the discussion of various properties of \( M_{n,m}(F)^{(+) \} \) we recall a defini-
tion of the universal associative enveloping superalgebra of a Jordan superalgebra which will be frequently used later.

An associative specialization \( u : J \rightarrow U(J) \) where \( U(J) \) is an associative superalgebra is said to be universal if \( U(J) \) is generated by \( u(J) \), and for any other specialization \( \varphi : J \rightarrow A \) where \( A \) is an associative superalgebra there exists a homomorphism \( \psi : U(J) \rightarrow A \) such that \( \varphi = \psi u \). Then \( U(J) \) is called a universal associative enveloping superalgebra of \( J \). It is worth noting that an associative superalgebra can be considered as an associative algebra. The following Theorem by C.Martinez and E.Zelmanov \[8\] plays a key role in the later discussion.

Theorem 2.2 Let \( U(J) \) denote a universal associative enveloping superalgebra
for a Jordan superalgebra \( J \). Then \( U(M_{k,l}^{(+) \} \cong M_{k,l}(F) \oplus M_{k,l}(F) \) where \( (k,l) \neq (1,1) \); 
\( U(Q(k)) \cong Q(k) \oplus Q(k), \) \( k \geq 2 \); \( U(\text{osp}(m,n)) \cong M_{m,2n}(F), \) \( (m,n) \neq (1,1) \); 
\( U(P(n)) \cong M_{n,n}(F), \) \( n \geq 3 \).

Remark 1 (see \[8\]) In the case where \( J \cong M_{1,1}(F)^{(+) \), \( P(2), \text{osp}(1,1), K_3 \)
or \( D_4 \) the universal enveloping superalgebras have more complicated structure.
Indeed, the universal associative enveloping superalgebras of the above Jordan
superalgebras are no more finite-dimensional. Also we note that if the characteristic of the basic field $F$ equals zero, then $K_3$ has no non-zero finite-dimensional associative specializations.

The following Theorem by Martínez and Zelmanov (see $\mathbb{S}$) describes all irreducible one-sided bimodules of $D(t)$ where $t \neq -1, 0, 1$.

**Theorem 2.3** Let $F$ be an algebraically closed field with zero characteristic. If $t = -\frac{m}{m+1}$, $m \geq 1$, then $D(t)$ has two irreducible finite-dimensional one sided bimodules $V_1(t)$ and $V_1(t)^{op}$.

If $t = -\frac{m}{m+1}$, $m \geq 1$, then $D(t)$ has two irreducible finite-dimensional one sided bimodules $V_2(t)$ and $V_2(t)^{op}$.

If $t$ cannot be represented as $-\frac{m}{m+1}$ and $-\frac{m+1}{m}$ where $m$ is a positive integer, then $D(t)$ does not have non-zero finite-dimensional specializations.

**Remark 2** If $\text{char} F = p > 2$, then for an arbitrary $t$ the superalgebra $D(t)$ can be embedded in a finite-dimensional associative superalgebra.

**Remark 3** If $t = -\frac{m}{m+1}$, then $\dim V_1(t)_0 = m$, $\dim V_1(t)_1 = m + 1$. If $t = -\frac{m+1}{m}$, then $\dim V_2(t)_0 = m + 1$, $\dim V_2(t)_1 = m$.

Now we look at the case when $J \cong J(V, f)$. Let $V = V_0 + V_1$ be a $Z_2$-graded vector space, $\dim V_0 = m$, $\dim V_1 = 2n$. Let $f(\ , \ ) : V \times V \to F$ be a supersymmetric bilinear form on $V$. The universal associative enveloping algebra of the Jordan algebra $F + V_0$ is the Clifford algebra $C(V_0, f) = \langle 1, e_1, \ldots, e_m | e_i e_j + e_j e_i = 0, i \neq j, e_i^2 = 1 \rangle$. In $V_1$ we can find a basis $v_1, w_1, \ldots, v_n, w_n$ such that $f(v_i, w_j) = \delta_{ij}$, $f(v_i, v_j) = f(w_i, w_j) = 0$. Consider the Weyl algebra $W_n = \langle 1, v_i, w_i, 1 \leq i \leq n, [v_i, v_j] = [v_i, w_j] = 0, [v_i, w_j] = 0 \rangle.$ According to $\mathbb{S}$, the universal associative enveloping algebra of $F + V$ is isomorphic to the (super)tensor product $C(V, f) \otimes_F W_n$. We will utilize this fact in the following Lemma.

**Lemma 2.4** There are no subsuperalgebras $B$ isomorphic to $J(V, f)$, where $V = V_0 + V_1$, $V_1 \neq \{0\}$ in a finite-dimensional Jordan superalgebra $A^{(+)}$, where $A$ is an associative superalgebra.

**Proof.** We assume the contrary, that is, there exists a subsuperalgebra $B$ of the type $J(V, f)$ in $A^{(+)}$. For $B$, we consider the universal associative enveloping superalgebra $U(B)$. According to the above fact, $U(B) = C(V_0, f) \otimes_F W_n$, where $C(V_0, f)$ is a Clifford algebra for $V_0$, $f$ is a bilinear form on $V_0$, $W_n$ is a Weyl algebra, $n = \frac{1}{2} \dim V_1$. Let $\varphi$ denote the identity embedding of $B$ in $A$. As a direct consequence of the definition of universal enveloping algebra, $\varphi$ can be uniquely extended to a homomorphism $\tilde{\varphi} : U(B) \to A$. Note that $\varphi(x) = \varphi(x) = x$ where $x \in V_1$. In other words, $\varphi(V_1) \neq 0$. However, since $V_1$ generates $W_n$, $\varphi(W_n) \neq 0$. It follows that $\varphi(W_n) \cong W_n$. Therefore, $A$ has an infinite-dimensional subsuperalgebra. This contradicts our assumptions.

In the next Lemma we will prove that no simple decompositions in which one of the components has either the type $K_3$ or $D_t$ are possible.
Lemma 2.5 Any superalgebra $\mathcal{J}$ of the type $M_{n,m}(F)^{(+)}$, $n, m > 0$ cannot be represented as the sum of two proper simple subsuperalgebras one of which has the type $K_3$ or $D_t$.

Proof. First of all, we note that if char $F = 0$, then $K_3$ has no non-trivial finite-dimensional associative specializations (see Remark 1). Therefore, we can directly pass to the second case when one of the subsuperalgebras is isomorphic to $D_t$. Next we suppose that $\mathcal{J}$ is given in the canonical form which is the set of all matrices of order $(n + m)$ with respect to the natural $Z_2$-gradation. Let $V = V_0 + V_1$ denote the $Z_2$-graded module corresponding to the natural representation of $M_{n,m}(F)^{(+)}$, dim $V_0 = n$, dim $V_1 = m$. Any decomposition of $M_{n,m}(F)^{(+)}$ induces that of $(M_{n,m}(F)^{(+}))(\mathcal{J}) = H(\mathcal{R}_n) \oplus H(\mathcal{R}_m)$ given by

$$H(\mathcal{R}_n) \oplus H(\mathcal{R}_m) = (Fe_1 \oplus Fe_2) + \mathcal{B}_0,$$

where $e_1, e_2$ are pairwise orthogonal idempotents in $\mathcal{A} \cong D_t$. Next we define a pair of homomorphisms denoted as $\pi_1, \pi_2$ which are the projections on the ideals $H(\mathcal{R}_n)$ and $H(\mathcal{R}_m)$, respectively. Then the above decomposition can be rewritten in the following way:

$$H(\mathcal{R}_n) = \pi_1(Fe_1 \oplus Fe_2) + \pi_1(\mathcal{B}_0),$$

$$H(\mathcal{R}_m) = \pi_2(Fe_1 \oplus Fe_2) + \pi_2(\mathcal{B}_0).$$

Next we estimate the dimension of $\pi_1(\mathcal{B}_0)$. We know that $\pi_1(\mathcal{B}_0)$ is either a simple or a non-simple semisimple subalgebra. Therefore, in the first case we have $\dim \pi_1(\mathcal{B}_0) \leq n^2 - 2n + 1$, and in the second case $\dim \pi_1(\mathcal{B}_0) \leq n^2 - 2n + 2$. It follows that $\dim H(\mathcal{R}_n) \leq 2 + \dim \pi_1(\mathcal{B}_0), n^2 \leq n^2 - 2n + 4, n \leq 2$. Using the same arguments as above we can prove that $m \leq 2$. As a result, we have four possible cases $\mathcal{J} = M_{1,1}(F)^{(+)}$, $M_{1,2}(F)^{(+)}$, $M_{2,1}(F)^{(+)}$ or $M_{2,2}(F)^{(+)}$. Notice that the first case can be immediately excluded because $\dim M_{1,1}(F)^{(+)} = 4$. Since $M_{1,2}(F)^{(+)} \cong M_{2,1}(F)^{(+)}$ the second and forth are the only cases of interest to us. Let $\mathcal{J} = M_{1,2}(F)^{(+)}$. By the dimension argument, $\dim \mathcal{B} \geq 5$, and, moreover, $\rk \mathcal{B} \leq 3$. Clearly, there are only three appropriate choices for $\mathcal{B} : osp(2,1), P(2)$ or $Q(2)$. However, for all cases, $U(\mathcal{B})$ is isomorphic to $M_{2,2}(F)$, that is, cannot be a subsuperalgebra of $M_{1,2}(F)$.

Finally, let $\mathcal{J} = M_{2,2}(F)^{(+)}$. Again by the dimension argument, $\dim \mathcal{B} \geq 12$ and $\rk \mathcal{B} \leq 4$. Considering all possible cases we come to the conclusion that there are no appropriate subsuperalgebras in $\mathcal{J}$. This proves our Lemma.

Lemma 2.6 Any superalgebra of the type $M_{n,m}(F)^{(+)}$, $n, m > 0$, cannot be represented as the sum of two proper non-trivial simple subsuperalgebras one of which has either the types $M_{1,1}(F)^{(+)}$, $osp(1,1)$ or $P(2)$.

Proof. Assume that $M_{n,m}(F)^{(+)} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ satisfy all the above conditions. Let $\mathcal{A}$ have one of types $M_{1,1}(F)^{(+)}$, $osp(1,1)$ or $P(2)$. The even part of the above decomposition can be rewritten as follows:

$$H(\mathcal{R}_n) = \pi_1(\mathcal{A}_0) + \pi_1(\mathcal{B}_0),$$
The Lemma is proved.

Let $\mathcal{A}$ be isomorphic to either $M_{1,1}(F)(^+)$ or $osp(1,1)$. Then $\dim \pi_1(\mathcal{A}_0) = 2$.

Acting in the same manner as in Lemma 2.5, we obtain the following possibilities: $n = m = 1, n = 1, m = 2 \ (m = 2, n = 1), n = m = 2$. Obviously, there are no possible simple decompositions in the first case due to the low dimension of $M_{1,1}(F)(^+)$. In the second and third cases we have the following restrictions on the dimension and the rank of $\mathcal{B}_0$: $\dim \mathcal{B}_0 \geq 5, \ rk \mathcal{B}_0 \leq 3; \ \dim \mathcal{B}_0 \geq 12, \ rk \mathcal{B}_0 \leq 4$. Considering all cases one after another we conclude that there is no suitable choice for $\mathcal{B}_0$. Therefore, $M_{n,m}(F)(^+) \neq \mathcal{A} + \mathcal{B}$.

In the last case when $\mathcal{A} \cong P(2)$ there are the following restrictions on indices: $n \leq 3, m \leq 3$. In other words, $n = 1, m = 2; n = m = 2; n = 1, m = 3; n = 2, m = 3; n = m = 3$. By the dimension and rank arguments there is no such $\mathcal{B}_0$. The Lemma is proved.

Next taking into account all previous Lemmas we list simple decompositions that might exist in $M_{n,m}(F)(^+)$. Let $\mathcal{A}$ and $\mathcal{B}$ stand for the simple non-trivial Jordan subsuperalgebras of $M_{n,m}(F)(^+)$.\[
\begin{array}{c|c|c}
\mathcal{A} & \mathcal{B} \\
\hline
1 & M_{k,l}(F)(^+) & M_{p,q}(F)(^+) \\
2 & M_{k,l}(F)(^+) & P(q) \\
3 & M_{k,l}(F)(^+) & Q(p) \\
4 & P(k) & Q(l) \\
5 & P(k) & P(l) \\
6 & Q(k) & Q(l) \\
7 & osp(k,l) & M_{p,q}(F)(^+) \\
8 & osp(k,l) & Q(p) \\
9 & osp(k,l) & P(q) \\
10 & osp(k,l) & osp(p,q) \\
\end{array}
\]

Considering associative subalgebras $S(\mathcal{A})$ and $S(\mathcal{B})$ generated by $\mathcal{A}$ and $\mathcal{B}$, respectively, we obtain a new decomposition of the form $M_{n+m}(F) = S(\mathcal{A}) + S(\mathcal{B})$ where $S(\mathcal{A})$ and $S(\mathcal{B})$ are associative subalgebras of $M_{n+m}(F)$. Note that $S(\mathcal{A})$ is a homomorphic image of $U(\mathcal{A})$. As a direct consequence of Theorem 2.2, $U(\mathcal{A})$ is either an associative simple algebra or a direct sum of two or more simple pairwise isomorphic associative algebras.

**Lemma 2.7** Let $\mathcal{A}$ be a proper non-trivial simple subsuperalgebra in $M_{n,m}(F)(^+)$ where $n, m > 0$. Then $S(\mathcal{A})$ coincides with $M_{n+m}(F)$ if and only if one of the following conditions hold

1. Either $\mathcal{A} \cong osp(p,q), p + 2q = n + m$, or
2. $\mathcal{A} \cong P(n)$ for the case when $n = m$.

**Proof.** First, we note that the converse of this Lemma is obvious (see Theorem 2.2). To prove that one of the above conditions holds in the case when $S(\mathcal{A}) = M_{n,m}(F)$, then we first show that $\mathcal{A}$ cannot be of type $M_{k,l}(F)(^+)$ or $Q(p)$. If $\mathcal{A}$ has the type $M_{k,l}(F)(^+)$, then $k + l < n + m$. By Theorem 2.2, $S(\mathcal{A})$ is either
a proper simple subalgebra of the type $M_{k+l}(F)$ or a non-simple semisimple subalgebra of the type $M_{k+l}(F) \oplus M_{k+l}(F)$. In both cases, $S(A) \neq M_{n+m}(F)$.

If $A \cong Q(k)$, then its associative enveloping algebra is a non-simple semisimple subalgebra which is the direct sum of two or more simple ideals of the type $M_k(F)$. Therefore, $S(A) \neq M_{n+m}(F)$.

For the rest cases, $A$ can either have the type $osp(p,q)$ or $P(k)$. If $A \cong osp(p,q)$, then $S(A) \cong M_{p+2q}(F)$. Hence $S(A) = M_{n+m}(F)$ if and only if $p + 2q = n + m$. This yields (1).

Next we continue our proof by assuming that $n \neq m$, say, $n < m$. We let $A$ have the type $P(k)$. Then its even component $A_0$, which is isomorphic to $H(R_k)$, is a proper subalgebra in $M_{n,m}(F)$ of the form $I_1 \oplus I_2$, where $I_1 \cong H(R_n)$, $I_2 \cong H(R_m)$. As previously, let $\pi_1$ and $\pi_2$ denote the projections on $I_1$ and $I_2$, respectively.

Suppose that $S(A) = M_{n+m}(F)$. Since $A \cong P(k)$, then $S(A) \cong M_{2k}(F)$. This implies $2k = n + m$, $k = \frac{n+m}{2}$. In particular, $k > n$. Hence $\pi_2(A_0) = \{0\}$ and $\pi_2(A_0) \cong H(R_k)$. It follows that $A_0 \subseteq I_2$. Thus the identity $e$ of $A$ is an element of $I_2$. For any $x \in A_1$, $xe + ex = 2xe$ where the multiplication is associative. Multiplying both sides of this equation by $e$, we obtain the following $exe + ex = 2ex$. Since $exe = 0$, we have $ex = 2ex$. Similarly, $xe = 0$, that is, $x = 0$, for any $x \in A_1$, a contradiction.

In conclusion, it remains to consider the case when $n = m$ and $A \cong P(n)$. However, it is obvious that $S(A) \cong M_{2k}(F)$ and $S(A) = M_{2n}(F)$ if and only if $k = n$. This completes our proof.

**Lemma 2.8** Let $M_{n,m}(F)^{(+)} = A + B$, $n, m > 0$. Then one of the subsuperalgebras in the given decomposition has either the type $osp(p,q)$ where $p + 2q = n + m$ or $P(n)$ (only if $n = m$).

**Proof.** Let us assume the contrary, that is, neither $A$ nor $B$ is a subsuperalgebra of any of the above types. Then, by Lemma 2.7, $S(A)$ and $S(B)$ are proper associative subalgebras in $M_{n+m}(F)$. Theorem 2.2 states that both $S(A)$ and $S(B)$ are either simple associative algebras or non-simple semisimple associative algebras decomposable into the sum of two or more pairwise isomorphic simple algebras. Therefore, $\dim S(A) \leq k^2 \left(\frac{n+m}{k}\right) = (n+m)k$ where $k^2$ is a dimension of a simple ideal, $k > 1$. If one of the subsuperalgebras in the decomposition of $M_{n+m}(F)$ has a non-zero annihilator then by Proposition in [2] no such decomposition exists. Hence the identity of $M_{n+m}(F)$ is contained in the intersection of $S(A)$ and $S(B)$. On the other hand, the dimension of $S(A)$ as well as $S(B)$ is strictly greater than $\frac{(n+m)^2}{2}$.

Thus, by the dimension argument, the sum of $S(A)$ and $S(B)$ is a proper vector subspace of $M_{n+m}(F)$. Therefore, $M_{n+m}(F) \neq S(A) + S(B)$. This implies that our hypothesis was wrong.

**Lemma 2.9** Let $M_{n,m}(F)^{(+)} = A + B$, $n, m > 0$. Then, in the case when $m$ is even, and $n$ is odd, $A \cong osp(n, \frac{m}{2})$ and $B \cong M_{k,l}(F)^{(+)}$ where either $k = n - 1, n$ or $l = m$. On the contrary, if $m$ is odd, and $n$ is even, then $A \cong osp(m, \frac{n}{2})$ and $B \cong M_{k,l}(F)^{(+)}$ where either $k = m - 1, m$ or $l = n$. 

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Proof.
Since the proof remains the same for both cases, we consider only the first case. First, let $n \neq m$. In view of Lemma 2.8, one of the subsuperalgebras in $M_{n,m}(F)^{(+) = A + B,}$ for example $A$, is isomorphic to $osp(p,q)$ where

$$p + 2q = n + m.$$  

The decomposition of $M_{n,m}(F)^{(+) given above induces the following representation of the even component $M_{n,m}(F)^{(+) = A_0 + B_0$ where $M_{n,m}(F)^{(+) = H(R_n) \oplus H(R_m), A_0 \cong H(F_p) \oplus H(Q_q),}$ If for some $i, \pi_1(A_0) \cong H(F_p) \oplus H(Q_q),}$ then either $p+2q \leq n$ or $p+2q \leq m$. However these inequalities conflict with condition (1). Hence either $\pi_1(A_0) \cong H(F_p), \pi_2(A_0) \cong H(Q_q),$ or $\pi_1(A_0) \cong H(Q_q), \pi_2(A_0) \cong H(F_p).$ If the first possibility holds true, then

1. $H(R_n) = \pi_1(A_0) + \pi_1(B_0), \pi_1(A_0) \cong H(F_p), p \leq n, \pi_1(B_0) \neq 0.$
2. $H(R_m) = \pi_2(A_0) + \pi_2(B_0), \pi_2(A_0) \cong H(Q_q), q \leq \frac{m}{2}, \pi_2(B_0) \neq 0.$

Since $p + 2q = n + m$, it follows that $p = n$ and $q = \frac{m}{2}$, $A$ has the type $osp(n, \frac{m}{2}).$ If the second possibility holds true, then acting in the same manner, we can show that $p = m, q = \frac{n}{2}$. However, we assumed that $n$ is odd. Hence it remains to prove that $B \cong M_{k,l}(F)^{(+) where a pair of indices $k,l$ satisfies the conditions given in the Lemma. To prove this, we consider all possible types for $B$ in a step-by-step manner.

If $A \cong osp(n, \frac{m}{2}), B \cong P(k)$, then the decomposition induces the following representation of the odd part: $M_{n,m}(F)^{(+) = A_1 + B_1$ where dim $A_1 = nm$, dim $B_1 = k^2$, that is, $2nm \leq nm + k^2$, $nm \leq k^2.$ Conversely, $k \leq n, k \leq m$ since both projections $\pi_1(B_0), \pi_2(B_0)$ are non-zero. Moreover, one of the inequalities should be strict since $n \neq m$. Therefore, $k^2 < nm$, which is a contradiction.

If $A \cong osp(n, \frac{m}{2}), B \cong Q(k)$, then, acting in the same manner as in the previous case, we can prove that $M_{n,m}(F)^{(+) \neq A + B.$

If $A \cong osp(n, \frac{m}{2}), B \cong osp(p,q)$, then, by the dimension argument, we have the following inequality $n + m < 2\left(\frac{n(n+1)}{2} + \frac{m(m-1)}{2} + nm\right).$ Simplifying the last inequality, we obtain that $m \leq n.$ Clearly, the opposite inequality $m \geq n$ also holds true. Therefore, $m = n$ which contradicts our hypothesis. Overall, it remains to consider the case when $A \cong osp(n, \frac{m}{2}), B \cong M_{k,l}(F)^{(+)$. Again the decomposition of $M_{n,m}(F)^{(+) induces that of $M_{n,m}(F)^{(+) as follows: $M_{n,m}(F)^{(+) = A_0 + B_0.$ Moreover, $M_{n,m}(F)^{(+) = H(R_n) \oplus H(R_m), A_0 \cong H(F_n) \oplus H(Q_m), B_0 \cong H(R_k) \oplus H(R_l).$ If both $\pi_1(B_0)$ and $\pi_2(B_0)$ are non-simple semisimple, that is, $\pi_1(B_0) \cong B_0$ and $\pi_2(B_0) \cong B_0$, then we have the following restrictions: $k + l \leq n$ and $k + l \leq m$. Since $n \neq m$, we can assume without any loss of generality that $n < m$. Hence the dimension of $\pi_i(B_0), i = 1, 2,$ is less than $n^2 - 2n + 2$. It follows from $\pi_1(B_0) \cong B_0 \cong B_0$ that dim $B_0 \leq n^2 - 2n + 2.$

As a result, dim $M_{n,m}(F)^{(+) = n^2 + m^2 \leq \frac{n(n+1)}{2} + \frac{m(m-1)}{2} + n^2 - 2n + 2, m(m+1) \leq \frac{n(n+1)}{2} + 2 - 2n,$ which is wrong. Therefore, we have only two possibilities: either $\pi_1(B_0)$ or $\pi_2(B_0)$ is a simple algebra. According to 17.
Therefore, $k \leq n - 1$ and, for the second, $l = m$. Thus the Lemma is proved for the case when $n \neq m$.

To complete our proof we consider the case when $n = m$. First, we assume that neither $A$ nor $B$ has the type $P(n)$. By the previous Lemma one of the subsuperalgebras, for example $A$, is isomorphic to $osp(p,q)$, $p + 2q = 2n$, that is, $p = n$, $q = \frac{n}{2}$.

Then
\[
H(\mathcal{R}_n) = \pi_1(A_0) + \pi_1(B_0), \quad \pi_1(A_0) \cong H(F_n), \pi_1(B_0) \neq 0
\]
\[
H(\mathcal{R}_n) = \pi_2(A_0) + \pi_2(B_0), \quad \pi_2(A_0) \cong H(Q_{\frac{n}{2}}), \pi_2(B_0) \neq 0.
\]

For some $i$, let $\pi_i(B_0)$ be a non-simple semisimple subalgebra, then

\[
\pi_i(B_0) \cong \begin{cases} 
H(\mathcal{R}_k) \oplus H(\mathcal{R}_l), & k + l \leq n \\
H(F_k) \oplus H(Q_l), & k + 2l \leq n 
\end{cases}
\]

Therefore, $\dim \pi_i(B_0) \leq n^2 - 2n + 2$. However $\dim M_{n,n}(F)(\+)
\leq \dim A_0 + \dim B_0$, $2n^2 \leq n^2 - 2n + 2 + \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = 2n^2 - 2n + 2$, that is, $n \leq 1$.

As mentioned above, there are no simple decompositions in $M_{1,1}(F)(\+)$. Hence both $\pi_1(B_0)$ and $\pi_2(B_0)$ are simple. It follows that $\pi_1(B_0) \cong H(\mathcal{R}_{n-1}), \pi_2(B_0) \cong H(\mathcal{R}_n)$, that is, $B_0 \cong M_{n-1,n}(F)$.

Next we let $A$ be of the type $P(n)$. Then $B \cong \begin{cases} 
P(k) \\
Q(k) \\
osp(k,l) \\
M_{k,l}(F)
\end{cases}$, for some integers $k$ and $l$.

1. $B \cong P(k)$, hence $\dim B = 2k^2$, $k \leq n$. For $\dim M_{n,n}(F)(\+)
\leq \dim A + \dim B$, it is clear that $k = n$, and the sum in the decomposition is direct. However since both subsuperalgebras have the type $P(n)$, they contain the identity of $M_{n,n}(F)(\+)$, a contradiction.

2. $B \cong Q(k)$. In this case the proof is the same as in Case 1.

3. $B \cong osp(k,l)$. Clearly, for some $i$, $\pi_i(B_0)$ is non-simple semisimple. Therefore, $k + 2l \leq n$. In particular, $\pi_i(B_0) \cong B_0$. Hence, $\dim B_0 \leq n^2 - 2n + 2$. Thus $\dim M_{n,n}(F)(\+) = 2n^2 \leq 2n^2 - 2n + 2$, a contradiction. As a result, $\pi_i(B_0)$, $i = 1, 2$, is a simple subalgebra, that is, $k \leq n$, $l \leq \frac{n}{2}$. Then $\dim B \leq 2n^2$. By the dimension argument, $k = n, l = \frac{n}{2}$ and the sum in the given decomposition is direct. However, this contradicts the fact that both subsuperalgebras in the given decomposition contain the identity of $M_{n,n}(F)(\+)$. 

4. $B \cong M_{k,l}(F)(\+), k + l < 2n$. The even part of $M_{n,n}(F)(\+)$, that is, $M_{n,n}(F)(\+)$ equals to the sum of two orthogonal ideals denoted as $I_1$ and $I_2$, both ideals isomorphic to $H(\mathcal{R}_n)$. By the dimension argument, $\dim M_{n,n}(F)(\+) \leq 2n^2 + (k+l)^2$, $4n^2 \leq 2n^2 + (k+l)^2$, $k+l \geq \sqrt{2n}$. In particular, $\pi_1(B_0), \pi_2(B_0)$ are simple. Therefore, acting by a appropriate automorphism of $M_{n,n}(F)(\+)$, $B_0$ can be reduced to the block-diagonal form. Moreover, $I_1$ and $I_2$ contain all simple ideals isomorphic to $H(\mathcal{R}_k)$ and $H(\mathcal{R}_l)$, respectively.

Suppose that the identity of $M_{n,n}(F)(\+)$ is an element of $B$. This implies that $kk_1 = ll_1 = n$ where $k_1$ and $l_1$ are the numbers of blocks which have types
$H(R_k)$ and $H(R_l)$, respectively. In view of the inequality $k + l \geq \sqrt{2}n$ this result implies that either $k_1 = 2$, $l_1 = 1$ or $k_1 = 1$, $l_1 = 2$, that is, $\mathcal{B} \cong M_{n,n}(F)$ up to the order of indices. By Theorem 2.2, $S(\mathcal{B})$ is isomorphic to $M_{n,n}(F)$ or $M_{3n}(F) \oplus M_{3n}(F)$. Obviously, $S(\mathcal{B})$ cannot be non-simple semisimple of the indicated type because its rank is greater than $2n$. However, by Lemma 1.1, the first case is also impossible because $\frac{3n}{2} > n$. Hence the identity of $M_{n,n}(F)$ is not an element of $\mathcal{B}$. In other words, $\mathcal{B}$ as well as $\mathcal{B}_0$ has a non-zero annihilator.

Acting by appropriate automorphism of $M_{n,n}(F)$ we can reduce $\mathcal{B}$ to the following form:

$$
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \vdots & T_1 & T_2 & \\
0 & 0 & \vdots & T_3 & T_4 \\
0 & 0 & \vdots & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
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& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & &
\end{pmatrix}
$$

where $T_1$, $T_2$, $T_3$ and $T_4$ are matrices of orders $(n-1) \times (n-1)$, $(n-1) \times m$, $m \times (n-1)$ and $m \times m$, respectively.

This implies that $\mathcal{A}_0$ takes the form:

$$
\left\{ \begin{pmatrix} X & C^{-1}X^t \end{pmatrix} \right\},
$$

for some $C$, $\det C \neq 0$. Then using the automorphism $\varphi(Y) = C'^{-1}YC'$ where

$$
C' = \begin{pmatrix} I & 0 \\ 0 & C^{-1} \end{pmatrix},
$$

$\mathcal{A}$ can be reduced to the form where

$$
\mathcal{A}_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix} \right\}
$$

while $\mathcal{B}$ remains the same. Obviously, this decomposition is not possible. The Lemma is proved.

**Example 1** A Jordan superalgebra of the type $M_{n,m}(F)^{(+)}$ where $m$ is even can be represented as the sum of two proper simple subsuperalgebras $\mathcal{A}$ and $\mathcal{B}$ which have types $osp(n, \frac{m}{2})$ and $M_{n-1,m}(F)^{(+)}$, respectively.

**Proof.** To prove, we consider the first subsuperalgebra in the standard realization:

$$
\left\{ \begin{pmatrix} A & C \\ S^{-1}C^t & B \end{pmatrix} \right\}
$$

where $A$ is a symmetric matrix of order $n$, $B$ is a symplectic matrix of order $m$, $C$ is any matrix of order $n \times m$, $S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where $I$ is identity matrix of
As usual, we assume the contrary, that is, there exists some other simple order \( \frac{m}{n} \). The second subalgebra can be viewed in the following form:

\[
\begin{pmatrix}
A & B \\
0 & 0 & 0 \\
C & D
\end{pmatrix}
\]

where \( A \) and \( C \) of orders \( (n-1) \times n \) and \( m \times n \), respectively, have the last two columns equal, \( B \) and \( D \) are any matrices of orders \( (n-1) \times n \), \( m \times m \), respectively. By straightforward calculations \( \dim(A_1 + B_1) = \dim(A_1 + \dim B_1 - \dim(A_1 \cap B_1) = mn + 2m(n-1) - m(n-2) = 2mn \) This proves our Lemma.

**Example 2** In the case when \( n \) is even, a Jordan algebra of the type \( M_{n,m}(F)^{(+)} \) can also be decomposed into the sum of \( \mathcal{A} \) and \( \mathcal{B} \) where \( \mathcal{A} \cong osp(\frac{m}{n}) \) and \( \mathcal{B} \cong M_{n-1,n}(F)^{(+)} \). This decomposition can be constructed in the same manner as in Example 1.

**Proposition 1** Example 1 and 2 are the only possible decompositions of \( M_{n,m}(F)^{(+)} \), \( n, m > 0 \) into the sum of two proper simple non-trivial subsuperalgebras for appropriate values of \( n, m \).

**Proof.** As usual, we assume the contrary, that is, there exists some other simple decomposition of \( M_{n,m}(F)^{(+)} \) different from one in Example 1. By Lemma 2.9, this decomposition takes the following form:

1. If \( m \) is even, then \( M_{n,m}(F)^{(+)} = \mathcal{A} + \mathcal{B} \), \( \mathcal{A} \cong osp(n, \frac{m}{n}) \), \( \mathcal{B} \cong M_{l,k}(F)^{(+)} \) where either \( l = n - 1, m \) or \( k = m \).

2. If \( n \) is even, then \( M_{n,m}(F)^{(+)} = \mathcal{A} + \mathcal{B} \), \( \mathcal{A} \cong osp(m, \frac{n}{m}) \), \( \mathcal{B} \cong M_{k,l}(F)^{(+)} \) where either \( k = m - 1, m \) or \( l = n \).

Then \( M_{n,m}(F)_1 = \mathcal{A}_1 + \mathcal{B}_1 \). It follows that \( \dim M_{n,m}(F)_1 \leq \dim \mathcal{A}_1 + \dim \mathcal{B}_1 \), that is, \( 2mn \leq mn + 2lk, \) \( mn \leq 2lk \). Hence, for even \( m, l \geq \frac{n}{2} \), in the case \( k = m \), and \( k \geq \frac{m}{2} \), in the case \( l = n - 1 \) or \( n \). Likewise, if \( n \) is even, then \( k \geq \frac{n}{2} \), in the case \( l = n \), and \( l \geq \frac{n}{2} \), in the case \( k = m - 1 \) or \( m \). For definiteness, we consider the case when \( m \) is even, and \( l = n - 1 \) because the proof remains the same for all other cases.

Let \( V = V_0 + V_1 \) denote a \( \mathbb{Z}_2 \)-graded vector space where \( \dim V_0 = n \) and \( \dim V_1 = m \). By its definition, \( M_{n,m}(F)^{(+)} \) coincides with the set of all linear transformations acting in \( V \). Then let \( \rho \) stand for the natural representation of \( \mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 \) in \( V \). It follows from the definition of this action that \( \rho(\mathcal{B}_0)(V_0) \subseteq V_0, \rho(\mathcal{B}_0)(V_1) \subseteq V_1, \rho(\mathcal{B}_1)(V_0) \subseteq V_1, \rho(\mathcal{B}_1)(V_1) \subseteq V_0 \). Since \( \mathcal{B} \) is a non-simple semisimple Jordan algebra it acts completely reducibly in \( V \). Next we describe this action in more details. For this, we identify \( V \) with a \( \mathbb{Z}_2 \)-graded vector space of the form \( W = \langle v_0 \rangle \oplus (V_0' \otimes F^r) \oplus V_1' \), \( r \geq 1 \) where \( v_0 \) is a vector in \( V_0 \) annihilated by \( \mathcal{B}_0 \), \( V_0' \) is an invariant complementary subspace of \( \langle v_0 \rangle \), \( \rho(\mathcal{B}_0)|_{V_0'} \cong H(R_n, V_1') \) an invariant subspace of \( V_1 \) such that \( \mathcal{B}_0 \), \( \rho(\mathcal{B}_0)|_{V_1'} \cong H(R_n, V_1') \)
\(H(\mathcal{R}_k)\). Moreover, \(W_0 = \langle v_0 \rangle \oplus V_0' \otimes e_0, W_1 = V_0' \otimes \langle e_1, \ldots, e_r \rangle \oplus V'_1\) where \(\langle e_0, e_1, \ldots, e_r \rangle\) is a basis for \(F^{r+1}\). Then, \(\rho(\mathcal{B}_0) = \rho(\mathcal{B}_0)|_{\langle v_0 \rangle} \oplus \rho(\mathcal{B}_0)|_{V_0'} \otimes \text{Id}_{r+1} \oplus \rho(\mathcal{B}_0)|_{V'_1}\). Note that \(\rho(\mathcal{B}_0)|_{\langle v_0 \rangle} = 0\). In other words, by choosing an appropriate basis in \(V_0\) and \(V_1\), \(\rho(\mathcal{B}_0)\) can be written in a block-diagonal form in which the first block of order 1 is zero, the last block has order \(k\), and the other blocks have order \(r + 1\). Next we consider the representation of the odd part \(\mathcal{B}_1\). For this, we choose any \(a \in \mathcal{B}_0\) such that
\[
\rho(a)(V_0' \otimes F^{r+1}) = 0, \quad \rho(a)(V'_1) \neq 0. \tag{2}
\]
All such elements form an ideal of \(\mathcal{B}_0\) isomorphic to \(H(\mathcal{R}_k)\). Then we choose any non-zero \(x \in \mathcal{B}_1\). Let \(e\) denote the identity of \(\mathcal{B}\), \(e \in \mathcal{B}_0\). Then \(\rho(x)v_0 = \rho(x \circ e)v_0 = \rho(x \circ e)v_0 = \rho \left( \frac{xy + yx}{2} \right) v_0 = \frac{1}{2} (\rho(x) \rho(e) v_0 + \rho(e) \rho(x) v_0) = \frac{1}{2} \rho(x) v_0\), that is, \(\rho(x)v_0 = 0\), for any \(x \in \mathcal{B}_1\). Next we find the representation of \(a \circ x \in \mathcal{B}_1\). As mentioned above, \(\rho(a \circ x)(v_0) = 0\). Besides, \(2 \rho(a \circ x)(V_0' \otimes e_0) = \rho(a) \rho(x)(V_0' \otimes e_0) + \rho(a \circ x)(V_0' \otimes e_0) \subseteq V'_1\), \(\rho(a \circ x)(V_0' \otimes \langle e_1, \ldots, e_r \rangle) = 0\), \(\rho(a \circ x)(V'_1) \subseteq V'_0 \otimes e_0\). Clearly, we can find \(c \in \mathcal{B}_0\) whose action is given by the following formulae:
\[
\rho(c)(V_0' \otimes F^{r+1}) \neq 0, \quad \rho(c)(V'_1) = 0. \tag{3}
\]
Now we need to determine
\[
c \circ (x \circ a). \tag{4}
\]
Since \(2 \rho(c \circ (a \circ x)) = \rho(c) \rho(a \circ x) + \rho(a \circ x) \rho(c)\), we have the following:
\[
\rho(c) \circ (a \circ x)(v_0) = 0, \quad \rho(c \circ (a \circ x))(V_0' \otimes \langle e_1, \ldots, e_r \rangle) = 0. \tag{5}
\]
Besides,
\[
\rho(c \circ (x \circ a))(V_0' \otimes e_0) = \rho(c) \rho(x) \rho(a)(V_0' \otimes e_0) + \rho(x) \rho(c) \rho(a)(V_0' \otimes e_0) + \rho(a) \rho(c) \rho(x)(V_0' \otimes e_0) + \rho(a) \rho(x) \rho(c)(V_0' \otimes e_0) = \rho(a) \rho(x) \rho(c)(V_0' \otimes e_0) \subseteq V'_1. \tag{6}
\]
Similarly,\[
\rho(c \circ (x \circ a))(V'_1) = \rho(c) \rho(x) \rho(a)(V'_1) + \rho(x) \rho(c) \rho(a)(V'_1) + \rho(a) \rho(c) \rho(x)(V'_1) + \rho(a) \rho(x) \rho(c)(V'_1) = \rho(c) \rho(x) \rho(a)(V'_1) \subseteq V'_0 \otimes e_0.
\]
Assume that \(\rho(x)(V'_1) \neq 0\), \(\rho(x)(V_0' \otimes e_0) \neq 0\) (mod \(V_0' \otimes \langle e_1, \ldots, e_r \rangle\)). Then \(\rho(c \circ (x \circ a))\) has the following matrix form:
\[
\begin{pmatrix}
0 & \cdots & 0 & \cdots & 0 \\
\vdots & 0 & 0 & \cdots & XY_1Z \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & ZY_2X & 0 & \cdots & 0
\end{pmatrix},
\]
where \(X\) is an arbitrary square matrix of order \(k\), \(Y_1\) and \(Y_2\) are some fixed non-zero matrices of order \(k \times (n - 1)\) and \((n - 1) \times k\), respectively, \(Z\) is any
square matrix of order $n - 1$. Next we choose any $y \in B_1$. We have seen that there exists an element $y$ of form (4) such that $\rho(y - a \odot (x \odot c))(V'_1) = 0$ or $\rho(y - a \odot (x \odot c))(V'_0 \otimes e_0) = 0$ (mod $V'_0 \otimes \langle e_1, \ldots, e_r \rangle$). Suppose that one of the above equations does not hold. Without any loss of generality we let $\rho(y')(V'_0 \otimes e_0) \neq 0$ (mod $V'_0 \otimes \langle e_1, \ldots, e_r \rangle$), where $y' = y - a \odot (x \odot c)$. Multiplying $y'$ by the elements of the form (2) and then (3) we obtain $a' \odot (y' \odot c') \in B_1$, where $a'$ and $c'$ ran relevant sets, and $\rho(a' \odot (y' \odot c'))(V'_0 \otimes F^{r+1}) = 0$, $\rho(a' \odot (y' \odot c'))V'_1 = \rho(a')\rho(y')\rho(c')V'_1 \subseteq V'_0 \otimes e_0$. Moreover, $\rho(a' \odot (y' \odot c')) : V'_1 \to V'_0 \otimes e_0$ represents all linear transformations from $k$-dimensional vector space into $(n - 1)$-dimensional vector space. Besides, all such elements are linearly independent from all the elements (4). Therefore, we found $2(n - 1)k$ linearly independent elements of $B_1$, (dim $B_1 = 2(n - 1)k$). If there is at least one element $\tilde{y} \in B_1$ such that $\rho(\tilde{y})(V'_0 \otimes e_0) \neq 0$ (mod $V'_1$) or $\rho(\tilde{y})(V'_0 \otimes \langle e_1, \ldots, e_r \rangle) \neq 0$, then it will be also linearly independent with all above elements. Hence, by dimension arguments, there is no $\tilde{y}$ satisfying the above conditions. Consequently, for all elements in $A_1$, the following $\rho(\tilde{y})(V'_0) = 0$ (mod $V'_1$), $\rho(\tilde{y})(V'_0 \otimes \langle e_1, \ldots, e_r \rangle) = 0$, $\rho(\tilde{y})(V'_1) \subseteq V'_0 \otimes e_0$ hold true. If we fix a basis in $V$ such that in this basis the even part has the diagonal form:

$$
\begin{pmatrix}
X & \text{0} \\
\text{0} & \ddots & \text{0} \\
\text{0} & \cdots & \text{0} & \text{0} \\
\end{pmatrix}
$$

then the odd part becomes the following:

$$
\begin{pmatrix}
\text{0} & \text{0} & \cdots & Z \\
\text{0} & \cdots & \text{0} & \text{0} \\
\text{0} & \text{0} & \cdots & \text{0} \\
\end{pmatrix}
$$

(7)

where $Z, Z'$ are any matrices of order $(n - 1) \times k$ and $k \times (n - 1)$, respectively. Then it follows from $B_1 \cap B_1 \subseteq B_0$ that $B_1 = 0$, a contradiction.

We henceforth assume that the equations $\rho(y - a \odot (x \odot c))(V'_1) = 0$ and $\rho(y - a \odot (x \odot c))(V'_0 \otimes e_0) = 0$ (mod $V'_0 \otimes \langle e_1, \ldots, e_r \rangle$) hold true simultaneously. Then multiplying $y - a \odot (x \odot c)$ by the elements (4), we obtain some elements of $B_0$ which act on $V'_1$ and $V'_0 \otimes \langle e_1, \ldots, e_r \rangle$ non-invariantly. Hence, $y - a \odot (x \odot c) = 0$. Therefore, the odd component of $A_1$ has form (7). As proved before, this is not possible.

Next we assume that $\rho(x)(V'_0 \otimes e_0) = 0$ (mod $V'_0 \otimes \langle e_1, \ldots, e_r \rangle$) for all $x \in B_1$, and for at least one element $x' \in B_1$, $\rho(x')(V'_1) \neq 0$.

Acting in the same manner as before, we obtain $a' \odot (x' \odot c') \in B_1$ which acts trivially on all subspaces except for $V'_1$, which it carries into $V'_0 \otimes e_0$. Considering the difference between an arbitrary element $y \in B_1$ and a corresponding element $a'' \odot (x'' \odot c'')$, we can show that $\rho(y - a'' \odot (x'' \odot c''))(V'_0 \otimes e_0) = 0$ (mod $V'_0 \otimes F^{r+1}$).
\[ \langle e_1, \ldots, e_r \rangle \oplus V'_1, \rho(y - a'' \circ (x'' \circ c'))(V'_1) = 0. \]
Again multiplying \( a' \circ (x' \circ c') \) and \( y - a'' \circ (x'' \circ c'') \), we obtain some elements from \( B_0 \) acting on \( V'_1 \) non-trivially. Then we conclude that \( B_1 \) consists of all elements which act on \( V'_0 \otimes e_0 \) trivially and carry the other subspaces into \( V'_0 \otimes e_0 \). Hence \( B_1 \otimes B_1 = 0 \), a contradiction.

Finally, if \( \rho(x)(V'_1) = 0, \rho(x)(V'_0 \otimes e_0) = 0(\mod (V'_0 \otimes F^{r+1})) \), then it follows that \( B_1 \otimes B_0 = 0 \), which is clearly a wrong statement. The Proposition is proved.

Based on all above Lemmas and Proposition 1, we conclude that Theorem 1 is true. In other words, \( M_{n,m}(F)^{(+)} \) where \( n, m > 0 \), \( m \) is even, and \( n \) is odd admits only one decomposition into the sum of two proper simple subsuperalgebras. If both \( n, m \) are odd, then \( M_{n,m}(F)^{(+)} \) cannot be represented as the sum of two proper simple subsuperalgebras. If both indices are even, then \( A \) admits two different types of decompositions of the following forms:

1. \( A = B_1 + C_1 \) where \( B_1 \) and \( C_1 \) have types \( osp(n, m) \) and \( M_{n-1,m}(F)^{(+)} \),
2. \( A = B_2 + C_2 \) where \( B_2 \) and \( C_2 \) have types \( osp(n, m) \) and \( M_{m-1,n}(F)^{(+)} \).

3 Decompositions of superalgebras of the type \( osp(n, m) \)

This section is dedicated to the study of simple decompositions of \( osp(n, m) \). Actually, we will show that there are no such decompositions over algebraically closed field \( F \) of zero characteristic. Our main purpose is to prove the following.

**Theorem 3.1** Let \( J \) be a superalgebra of the type \( osp(n, m) \) where \( n, m > 0 \). Then \( J \) cannot be written as the sum of two proper nontrivial simple subsuperalgebras \( A \) and \( B \).

The proof of this Theorem is based on the following Lemmas.

**Lemma 3.2** Let \( J \) be a superalgebra of type \( osp(n, m) \) where \( n, m > 0 \), and \( A, B \) are two proper simple subsuperalgebras none of which has any of the types \( K_3 \) or \( D_t \). Then \( J \) cannot be represented as the sum of \( A \) and \( B \).

**Proof.** First we identify \( J \) with \( osp(n, m) \) which can be considered in the canonical form. Next we assume the contrary, that is,

\[ osp(n, m) = A + B, \quad (8) \]

The decomposition (8) generates the following decomposition of the associative enveloping algebra into the sum of three non-zero subspaces.

\[ M_{n+2m}(F) = S(osp(n, m)) = S(A) + S(B) + S(A)S(B), \quad (9) \]

where \( S(A), S(B) \) denote the associative enveloping algebras of \( A, B \), respectively. Let 1 denote the identity of \( osp(n, m) \). Then we consider the following cases.
Case 1. Let $1 \not\in A$, $1 \not\in B$. This implies that there exist non-zero $a_0$ and $b_0$ in Ann($A$) and Ann($B$), respectively. Then multiplying every term of (9) by $a_0$ on the left and $b_0$ on the right, the following equation $a_0M_{n+2m}(F)b_0 = 0$ takes place, which is clearly wrong.

Case 2. $1 \in A$, $1 \in B$. Six cases arise:

(a) $A \cong M_{k,l}(F)^{(+)}$, $B \cong M_{p,q}(F)^{(+)}$. The given decomposition induces the decomposition of the even part $osp(n,m)_0 = A_0 + B_0$ which in turn can be projected on the ideals of the even component. In particular, $H(F_n) = \pi_1(A_0) + \pi_1(B_0)$. By Theorem 1.3, both projections cannot be simultaneously simple. Therefore, at least one of the components is non-simple semisimple. For definiteness, let $\pi_1(A_0) \cong H(R_k) \oplus H(R_l)$. By Lemma 1.1, $k + l \leq \frac{n}{2}$. Then, $\dim \pi_1(A_0) = k^2 + l^2 \leq (\frac{n}{2} - 1)^2 + 1 = \frac{n^2}{4} - n + 2$. If $\pi_1(B_0) \cong H(R_p) \oplus H(R_q)$, then, by Lemma 3.3, $p \leq \frac{n}{2}$ or $q \leq \frac{n}{2}$. If $\pi_1(B_0) \cong H(R_p) \oplus H(R_q)$, then $\dim \pi_1(B_0) = p^2 + q^2 \leq (\frac{n}{2} - 1)^2 + 1 = \frac{n^2}{4} - n + 2$. As a result, $\dim H(F_n) = \frac{n^2 + n}{2} - 2(\frac{n}{2} - n + 2), \frac{5n^2}{2} \leq 4, n \leq 1$ or $\dim H(F_n) = \frac{n^2 + n}{2} - \frac{n^2}{4} - n + 2 + \frac{n^2}{4}, \frac{5n^2}{2} \leq 2, n \leq 1$.

There remains one case where $n = 1$. The decomposition takes the following form: $osp(1,m) = A + B$ where $A \cong M_{1,1}(F)^{(+)}$, $B \cong M_{p,q}(F)^{(+)}$. Then $\pi_1(A_0) = \pi_2(A_0) + \pi_2(B_0)$. If $\pi_2(A_0) \cong F \oplus H(R_l)$, $\pi_2(B_0) \cong H(R_p) \oplus H(R_q)$, then $1 + l \leq m$, $p + q \leq m$, $\dim \pi_2(A_0) \leq m^2 - 2m + 2d$, $\dim \pi_2(B_0) \leq m^2 - 2m + 2$. As a result, $2m^2 - 4m + 4 \geq 4(2m^2 - 2m + 4) = 3m, m \leq 1$, that is, $m = 1$. However, it is clear that both subsuperalgebras in $osp(1,1) = A + B$ are isomorphic to $M_{1,1}(F)^{(+)}$, and, by dimension argument, $A, B$ coincide with $M_{1,1}(F)^{(+)}$.

If one of the projections is non-simple semisimple and the other is simple, then $m(2m - 1) \leq m^2 + 2m^2 - 2m + 2 = 2m^2 + 2m - 2 + 2$, $m \leq 2$. Therefore, $osp(1,2) = A + B$, $A \cong M_{1,2}(F)^{(+)}$, $B \cong M_{p,q}(F)^{(+)}$, $H(Q_2) = \pi_2(A_0) + \pi_2(B_0)$, where $\pi_2(A_0)$ is simple, and $\pi_2(B_0)$ is non-simple. By the dimension argument, the sum in the above decomposition is direct, that is, one of the subalgebras does not contain the identity, which is obviously wrong. If both projections are simple, then $osp(1,m) = A + B$, $A \cong M_{1,1}(F)^{(+)}$. However, by the dimension argument, the latter does not hold. Otherwise, $A_1 = B_1 = osp(1,m)$ because their dimensions are equal. It follows that $A_0 = (A_1 \cap A_1) = (B_1 \cap B_1) = B_0$, that is, $A = B = osp(1,m)$.

(2) $A \cong M_{k,l}(F)^{(+)}$, $B \cong P(q)$ where $Q(q) \cong P(q) > 1)$. Therefore, $H(F_n) = \pi_1(A_0) + \pi_1(B_0)$ where $\pi_1(A_0) \cong H(R_k) \oplus H(R_l)$, $\pi_1(B_0) \cong H(R_q)$. Again, by the same arguments as in the previous case, $n \leq 1$. Let $n = 1$ then $\pi_1(A_0) \cong F$. In this case, $k = 1$ (or $l = 1$) and the following decomposition holds true: $H(Q_1) = \pi_2(A_0) + \pi_2(B_0)$ where $\pi_2(A_0) \cong F \oplus H(R_l)$, $\pi_2(B_0) \cong H(R_q)$ or $\pi_2(A_0) \cong H(R_l)$, $\pi_2(B_0) \cong H(R_q)$. In the first case, we have proved that $m = 2$, $l = 1$. Hence $osp(1,2) = A + B$ where $A \cong M_{1,1}(F)^{(+)}$, $B \cong P(2)$, which induces the following: $H(Q_1) = \pi_2(A_0) + \pi_2(B_0)$ where $A_0 \cong F \oplus B_0 \cong H(R_2)$. The sum in the last decomposition is direct, and both subalgebras contain 1, which is a contradiction. In the second case, $1 = q = m$, that is, $osp(1,m) = A + B$, $A \cong M_{1,1}(F)^{(+)}$.
Since \( \dim \text{osp}(1, m) \geq \dim B_1 \), then \( 2m \geq m^2 \), that is, \( m = 2 \). If \( m = 2 \), then \( \text{osp}(1, 2) = A + B \) where \( A \cong M_{1,2}(F)^{(+)} \), \( B \cong P(2) \) which induces the equality \( H(Q_2) = \pi_2(A_0) + \pi_2(B_0) \) where \( A_0 \cong H(R_2), B_0 \cong H(R_2) \).

Notice that the identity of \( \text{osp}(1, 2) \) is an element of \( A \), that is, \( A \) has trivial two-sided annihilator. Consider an associative enveloping algebras of \( \text{osp}(1, 2) \) and \( A \) denoted as \( S(\text{osp}(1, 2)) \) and \( S(A) \), respectively. It can be shown that \( S(A) \cong M_2(F) \) is a subalgebra of \( S(\text{osp}(1, 2)) = M_5(F) \) (see Theorem 2.2). By Lemma 1.1, \( S(A) \) contains no identity of \( M_5(F) \), therefore, has a non-zero two-sided annihilator, and so does \( A \), a contradiction.

(c) \( A, B \) have types \( P(q) \) or \( Q(p) \). Then the decomposition leads to the sum of two proper subalgebras, which does not exist.

(d) \( A \cong \text{osp}(k, l), B \cong M_{p,q}(F^{(+)} \). Since \( S(A) \cong M_{k+2l}(F) \) contains the identity of the entire superalgebra, \( k + 2l \leq \frac{n+2m}{2} \). Similarly, \( p + q \leq \frac{n+2m}{2} \).

Thus \( \dim \text{osp}(n, m) \leq \dim A + \dim B \), that is, \( \frac{n^2+m^2}{2} + m(2m-1) + 2nm \leq k^2+l + l(2l-1) + 2kl + \frac{(n+2m)^2}{2} \). By straightforward calculations we obtain \( \frac{n^2}{4} + \frac{m^2}{2} + m \leq 3m \), which is true if and only if \( m = n = 1 \). Obviously, \( \text{osp}(1, 1) \) has no simple decompositions.

(e) \( A \cong \text{osp}(k, l), B \cong P(q) \). Then, we have \( k + 2l \leq \frac{n+2m}{2}, 2q \leq \frac{n+2m}{2} \). Therefore, \( \dim B = 2q^2 \leq 2(\frac{n+2m}{2})^2 \). Again, by the dimension argument, this decomposition is not possible.

(f) \( A \cong \text{osp}(k, l), B \cong \text{osp}(p, q) \). Then \( k + 2l \leq \frac{n+2m}{2}, p + 2q \leq \frac{n+2m}{2} \). Comparing \( \dim \text{osp}(n, m) \) with \( \dim A + \dim B \) we have \( \frac{n^2}{4} + 2nm + m^2 \leq 4m \), a contradiction.

Case 3 Let \( 1 \in A, 1 \notin B \). As mentioned above, the given decomposition induces the following decompositions of the ideals of the even component:

\[
H(F_n) = \pi_1(A_0) + \pi_1(B_0),
\]

\[
H(Q_m) = \pi_2(A_0) + \pi_2(B_0).
\]

If either \( \pi_1(A_0), \pi_2(B_0) \) or \( \pi_1(B_0), \pi_2(A_0) \) are non-simple semisimple, then \( \dim A_0 = \dim \pi_1(A_0) < \dim H(F_n), \dim B_0 = \dim \pi_2(B_0) < \dim H(Q_m) \). This implies that \( \dim A_0 + \dim B_0 < \dim (H(F_n) \oplus H(Q_m)) \), which is wrong. Likewise we have a contradiction in the second case. Therefore, there is a simple algebra in each pair: \( (\pi_1(A_0), \pi_2(B_0)), (\pi_1(B_0), \pi_2(A_0)) \). Since \( 1 \) is not an element of \( B \), \( A \) has a non-zero two-sided annihilator, and so does \( B_0 \). It follows that one of \( \pi_1(B_0), \pi_2(B_0) \) has a non-zero two-sided annihilator. Let us assume the first possibility, that is, \( \pi_1(B_0) \) can be embedded in the simple subalgebra which also has a non-zero annihilator. Since \( H(F_n) \) cannot be written as the sum of two simple subalgebras, \( \pi_1(A_0) \) should be non-simple semisimple. This implies that

\[
\pi_1(A_0) \cong \begin{cases} H(F_k) \oplus H(Q_l), \\ H(R_k) \oplus H(R_l) \end{cases}
\]
In other words, we represent $H(F_n)$ as the sum of a non-simple semisimple subalgebra of form (12) and a subalgebra which has a non-zero two-sided annihilator. 

Let $V$ denote the $n$-column vector space. Then, there exists a non-zero vector $v \in V$ annihilated by the second subalgebra. By Lemma 1.2, $\dim H(F_n)v = n$. It follows from (10) that $\dim \pi_1(A_0)v = n$.

If $\pi_1(A_0) \cong H(F_k) \oplus H(Q_l)$, then by some automorphism of $F_n$ it can be reduced to the following form:

$$
\begin{pmatrix}
  X & \ldots & 0 & \ldots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & \ldots & X & \ldots & 0 \\
  0 & \ldots & 0 & Y & \ldots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
  0 & \ldots & 0 & 0 & \ldots & Y
\end{pmatrix},
$$

where $X$ is a symmetric matrix of order $k$, $Y$ is a symplectic matrix of order $2l$. Let $v$ be equal to $(v_{11}, \ldots, v_{1k_1}, v_{21}, \ldots, v_{2l})^t$ where $v_{11}$ is a vector of dimension $k$, $i = 1, \ldots, k_1$, $v_{2j}$ is a vector of dimension $2l$, $j = 1, \ldots, l_1$. Since $\pi_1(A_0)$ contains $1$, $kk_1 + 2ll_1 = n$. Then, $\dim\{Xv_{1i} | X \in H(F_k)\} = k$, $\dim\{Yv_{2j} | Y \in H(Q_l)\} = 2l - 1$ (see Lemma 1.4). Therefore, $\dim \pi_1(A_0)v = kk_1 + (2l - 1)l_1 < n$, a contradiction. If $\pi_1(A_0) \cong H(R_k) \oplus H(R_l)$, then by some automorphism of $F_n$ it can be reduced to

$$
\begin{pmatrix}
  X & 0 & \ldots & 0 & 0 \\
  0 & X^t & \ldots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \ldots & Y & 0 \\
  0 & 0 & \ldots & 0 & Y^t
\end{pmatrix}.
$$

Hence, $\pi_2(B_0)$ has a non-trivial two-sided annihilator, that is, can be embedded in the simple algebra with a non-zero annihilator. Therefore, $\pi_2(A_0)$ is non-simple semisimple because $H(Q_m)$ cannot be written as the sum of two simple subalgebras one of which has a non-zero two-sided annihilator (see [17]). As a result, we have the decomposition of the form: $H(Q_m) = \pi_2(A_0) + \pi_2(B_0)$ which in turn induces the following

$$F_{2m} = F_{2m-1} + \langle \pi_2(A_0) \rangle,$$

in which the first subalgebra clearly has a non-zero two-sided annihilator, and the second is non-simple semisimple. According to [2], such decomposition cannot exist. The Lemma is proved.

**Lemma 3.3** A superalgebra $\mathcal{J}$ of the type $osp(n, m)$ where $n, m > 0$ cannot be decomposed into the sum of two proper simple subsuperalgebras one of which has either the type $K_3$ or $D_4$. 

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Proof. First we identify $J$ with $osp(n, m)$. Since Kaplansky’s superalgebra $K_3$ has no finite-dimensional associative specializations, $K_3$ cannot be a subsuperalgebra of a superalgebra of the type $osp(n, m)$. Assume that $osp(n, m) = A + B$ where, for example, $A \cong D_t$. Then, the above decomposition induces the following:

$$H(F_n) = \pi_1(Fe_1 \oplus Fe_2) + \pi_1(B_0),$$
$$H(Q_m) = \pi_2(Fe_1 \oplus Fe_2) + \pi_2(B_0).$$

Let us note that $dim\pi_1(B_0) \leq \frac{n(n-1)}{2}$ if it is a simple subalgebra and $dim\pi_1(B_0) \leq \frac{n^2-3n+2}{2} + 2$ if it is a non-simple semisimple subalgebra. This implies that $dim H(F_n) = \frac{n(n+1)}{2} \leq 4 + \frac{n^2}{2} - \frac{3n}{2} + 1$, $2n \leq 5$, $n \leq 2$. Similarly, $dim \pi_2(B_0) \leq 2m^2 - 5m + 3$ if $\pi_2(B_0)$ is a simple subalgebra, and $dim \pi_2(B_0) \leq 2m^2 - 5m + 4$ if $\pi_2(B_0)$ is a non-simple semisimple subalgebra. Thus $dim H(Q_m) = 2m^2 - m \leq 2 + 2m^2 - 5m + 4$, $2m \leq 6$, $m \leq \frac{3}{2}$. Therefore, either $J \cong osp(1, 1)$ or $J \cong osp(2, 1)$. Since $dim osp(1, 1) = 4$, the first case is not possible. Let $A$ be isomorphic to $D_t$. By the dimension argument, either $B \cong M_{1,1}(F)$ or $osp(1, 1)$.

In turn, $A$ acts completely reducibly in the 4-dimensional column vector space $V$ because $dim V_0 = dim V_1 = 2$ (see Theorem 2.3). Moreover, $V = W_1 \oplus W_2$, $dim W_1 = 1$, $dim W_2 = 3$, $W_1$, $W_2$ are invariant subspaces with respect to the action of $A$. Besides, $A$ acts in $W_1$ trivially and in $W_2$ irreducibly. It follows that, by some graded automorphism of $osp(n, m)$, $A$ can be reduced to the following form:

$$A = \left\{ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & & A' \end{pmatrix} \right\}.$$

Since $dim A = 4$, $A \cong osp(1, 1)$. Finally, we obtain a decomposition of a superalgebra of the type $osp$ into the sum of two subsuperalgebras of the same type. As proved before, such decomposition does not exist. The Lemma is proved.

4 Decompositions of superalgebras of types $Q(n)$ and $P(n)$

First of all we recall the canonical realizations of Jordan superalgebras of types $P(n)$ and $Q(n)$. A Jordan superalgebra of the type $Q(n)$ can be represented as the set of all matrices of order $2n$ which have the following form:

$$\left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}$$
where $A$ and $B$ are any square matrices of order $n$. A canonical realization of a Jordan algebra of the type $P(n)$ consists of all matrices of the form:

$$\left\{ \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \right\}$$

where $A$ is any square matrix of order $n$, $B$ is a symmetric matrix of order $n$, $C$ is a skew-symmetric matrix of order $n$. Notice here that the even part of $Q(n)$ as well as $P(n)$ is isomorphic to a simple Jordan algebra of the type $H(R_n)$. Besides, $\dim Q(n) = \dim P(n) = 2n^2$. Later on only these two properties will be primarily used, hence, all Lemmas proved in this section are true for Jordan superalgebras of both types. For definiteness, we consider only Jordan subalgebras of type $Q(n)$. In several steps, we prove that no Jordan superalgebras of types $P(n)$ or $Q(n)$ can be represented as the sum of two proper simple subsuperalgebras.

**Lemma 4.1** Let $A$ of the type $osp(p,q)$ be a proper subsuperalgebra of $Q(n)$. Then $\dim A \leq \frac{n^2 + n}{2}$.

**Proof.** It follows from the Lemma conditions that $A_0 \cong H(F_p) \oplus H(Q_q)$ is a proper subalgebra of $Q(n)_0$ which is isomorphic to $H(R_n)$. Therefore, $p + 2q \leq n$, $p, q > 0$. It is easy to see that the subalgebra takes on its maximum value when $p + 2q = n$. Then $\dim A = \frac{p^2 + p}{2} + q(2q - 1) + 2pq$. This implies that $\dim A \leq \frac{n^2 + n}{2}$. The Lemma is proved.

**Lemma 4.2** Let $A$ of the type $M_{k,l}(F)^{(+)1}$ where $k, l > 0$ be a proper subsuperalgebra of $Q(n)$. Then $\dim A \leq n^2$.

**Proof.** Since $A$ is proper, $k + l \leq n$, hence $(k + l)^2 \leq n^2$.

**Lemma 4.3** A superalgebra $\mathcal{J}$ of either the type $P(n)$ or $Q(n)$, $n > 1$, cannot be represented as the sum of two proper nontrivial subsuperalgebras one of which has either the type $K_3$ or $D_4$.

**Proof.** For definiteness, we assume that $\mathcal{J}$ has the type $P(n)$. Next, in order to simplify our notation we identify $\mathcal{J}$ with its canonical realization denoted as $P(n)$. By Remark 1 in Section 2 no superalgebra of the type $K_3$ can be a subsuperalgebra of $P(n)$. Therefore, let $A$ be of type $D_4$. The given decomposition of $P(n)$ induces that of the form: $P(n)_0 = A_0 + B_0$ where $A_0 = Fe_1 \oplus Fe_2$, $e_1$ and $e_2$ are pairwise orthogonal idempotents. Next we estimate the dimension of $B_0$. If $B_0$ is simple, then $\dim B_0 \leq n^2 - 2n + 1$. If $B_0$ is non-simple semisimple, then $\dim B_0 \leq n^2 - 2n + 2$. As a result, $\dim P(n)_0 = n^2 \leq 2 + n^2 - 2n + 2$, $n \leq 2$. The only case which remains to be proved is when $n = 2$. By the dimension and rank arguments, either $B \cong M_{1,1}(F)$ or $B \cong osp(1,1)$. In both cases, $B_0$ is isomorphic to $Fe_1' \oplus Fe_2'$ where $e_1'$ and $e_2'$ are pairwise orthogonal idempotents. Hence $P(2)_0 = H(R_2) = Fe_1 \oplus Fe_2 + Fe_1' \oplus Fe_2'$. Both subalgebras contain the identity. Thus, $\dim H(R_2) = 4 > \dim A + \dim B_0$. The Lemma is proved.
Lemma 4.4 Let $J$ of the type $P(n)$ or $Q(n)$ be represented as the sum of two proper non-trivial subsuperalgebras $A$ and $B$ whose even components are semisimple Jordan algebras and one of them has a non-zero two-sided annihilator. Then $J \neq A + B$.

Proof. Let $J = A + B$, and $A$ have a two-sided annihilator. Then $J_0 = A_0 + B_0$ where $A_0, B_0$ are semisimple Jordan subalgebras, Ann $A_0 \neq \{0\}$. Since $J_0 \cong H(R_n)$, $J_0$ can be represented as the set of all matrices of order $n$, denoted as $F_n^{(+)}$, under the Jordan multiplication. Obviously, $F_n = \langle A_0 \rangle + \langle B_0 \rangle$ where $\langle A_0 \rangle$ and $\langle B_0 \rangle$ denote associative enveloping algebras for $A_0$ and $B_0$, respectively. This implies that $F_n$ can be written as the sum of two semisimple subalgebras $\langle A_0 \rangle$ and $\langle B_0 \rangle$ one of which has a non-zero two-sided annihilator. This contradicts Proposition 1 in [2]. The Lemma is proved.

The following table summarizes all the information obtained above.

| $\mathcal{A}$   | Max dim   |
|-----------------|-----------|
| 1 $M_{k,l}(F)^{(+)}$ | $n^2$     |
| 2 $osp(p,q)$    | $n^2 + n$|
| 3 $Q(k)$        | $2(n-1)^2$|
| 4 $P(k)$        | $2(n-1)^2$|

In the second column we list all possible types which subsuperalgebras of $P(n)$ and $Q(n)$ can have. In the third column we point out the maximal dimension corresponding to each subsuperalgebra.

Theorem 4.5 Let $J$ have type either $Q(n)$ or $P(n)$, where $n > 1$. Then $J$ cannot be represented as the sum of two proper simple non-trivial subsuperalgebras $A$ and $B$.

Proof. Since the case $P(n)$ is completely similar to the case $Q(n)$, we give proof only for $Q(n)$. We have the following cases.

Case 1. $Q(n) = A + B$, $A \cong M_{k,1}(F)^{(+)}$, $B \cong M_{k,1}(F)^{(+)}$. Besides, the dimensions of both subsuperalgebras are not greater than $n^2$. This implies that the sum in the above decomposition is direct. As a consequence of this fact, the given decomposition induces the decomposition of $Q(n)_0$ into the direct sum of two semisimple subalgebras $A_0$ and $B_0$ one of which does not contain the identity of the whole superalgebra or, equivalently, has a non-trivial two-sided annihilator. By Lemma 4.4, no such decomposition is possible.

Case 2. $Q(n) = A + B$, $A \cong osp(p,q)$, $B \cong \begin{cases} \text{osp}(p',q') \\ M_{k,1}(F)^{(+)} \\ Q(k) \\ P(k) \end{cases}$.

Taking into account Lemma 4.1, we can conclude that the decomposition into the sum of two subsuperalgebras of the type $osp$ is not possible. According to the above estimates, the
with Lemma 4.4. If \( 1 \neq \frac{n}{2} \) and \( \dim(A + B) \leq \frac{n^2}{2} + \frac{n^2}{2} < 2n^2 \). If \( 1 \notin H(R_k) \), then we have a contradiction with Lemma 4.4.

**Case 3.** \( Q(n) = A + B, A \cong M_{k,1}(F)(+), B \cong \{ \frac{Q(m)}{P(m)} \}. \)

This decomposition induces that of the even part: \( H(R_n) = A_0 + B_0, A_0 \cong H(R_k) \oplus H(R_1), B_0 \cong H(R_m) \). If \( 1 \notin B_0 \), then again we have a contradiction with Lemma 4.4. If \( 1 \in B_0, k \leq \frac{n}{2}, \) that is, \( \dim B \leq \frac{n^2}{2} \). However \( \dim(A + B) \leq n^2 + \frac{n^2}{2} < 2n^2 \), which is wrong.

**Case 4.** Let \( Q(n) = A + B, A \cong P(k), B \cong Q(l), k, l < n \). As above, this decomposition induces the decomposition of the even part \( H(R_n) \) into the sum of two subalgebras of types \( H(R_k) \) and \( H(R_l) \). However it follows from the classification of simple decompositions of simple Jordan algebras that no such decomposition exists. The Theorem is proved.

## 5 Decompositions of superalgebras of types \( J(V, f), K_3, D_t \)

**Theorem 5.1** Let a superalgebra \( J \) have type \( J(V, f) \), and \( A, B \) be simple non-trivial subsuperalgebras of \( J \). Then \( J = A + B \) implies that \( A, B \) are isomorphic to some superalgebras of non-singular symmetric bilinear superforms. Moreover, for any decomposition of \( V \) into the sum of nontrivial graded subspaces \( V = W_1 + W_2 \) with nondegenerate restrictions \( f_1, f_2 \) of \( f \) one has \( J(V, f) = J(W_1, f_1) + J(W_2, f_2) \).

**Proof.** As usual, we identify \( J \) with \( J(V, f) = (F + V_0) + V_1 \) where \( J(V, f)_0 = F + V_0, J(V, f)_1 = V_1 \). It follows from the rule of multiplication, that \( J_1, J_1 = F_1 \), where \( 1 \) denotes the identity in \( J(V, f) \). In particular, \( A_1, A_1 \subseteq F_1 \) and \( B_1B_1 \subseteq F_1 \). Note that the idempotents in \( J_0 \) have the form: 1 or \( \frac{1}{2} + v \) where \( f(v, v) = \frac{1}{2}, v \in V_0 \). In particular, if \( v_1 \) and \( v_2 \) are pairwise orthogonal idempotents in \( J_0, v_1 = \frac{1}{2} + v, v_2 = \frac{1}{2} - v \) where \( v \in V_0 \).

Next we consider the following cases.

(a) \( A \cong K_3 \) where \( K_3 = \langle e, x, y \rangle, [x, y] = e, ex = \frac{e}{2}, ey = \frac{e}{2}, e^2 = e, e \in A_0 \).

As mentioned above, \( e = 1 \) or \( e = \frac{1}{2} + v \), \( v \in V_0 \). If \( e = 1 \), then, obviously, \( ex = x \neq \frac{e}{2} \). Hence, \( e = \frac{1}{2} + v \). Then, on the one hand, \( [x, y] = f(x, y)1 \neq 0 \). Conversely, \( [x, y] = \frac{1}{2} + v \) where \( v \neq 0 \). Hence \( \frac{1}{2} + v = f(x, y)1 \). However \( f(x, y)1 \in F, v \in F \), which is wrong.

(b) \( A \cong D_t, t \neq -1, 0, 1 \) where \( D_t = \langle e_1, e_2, x, y \rangle, [x, y] = e_1 + te_2, e_1, e_2 \) are pairwise orthogonal idempotents in \( A_0 \). Hence there exists \( v \in V_0 \), such that \( e_1 = \frac{1}{2} + v, e_2 = \frac{1}{2} - v \). Then \( [x, y] = \frac{1}{2} + 1 + (1 - t)v \). On the other hand, \( [x, y] = f(x, y)1 \neq 0 \). However \( (1 - t)v = 0 \), that is \( t = 1 \), but \( D_1 \cong J(V', f') \).

(c) \( A \cong osp(n, m) \). Since \( A_0 \leq 2 \), then \( n = 1, m = 1 \). It is easy to check that \( osp(1, 1)_1 \oplus osp(1, 1)_1 \) cannot be generated by one idempotent.
(d) \( \mathcal{A} \cong M_{n,m}(F) \). As in the previous case, \( \text{rk} \mathcal{A}_0 \leq 2 \), that is, \( n, m = 1 \). By simple calculation we can show that \( M_{1,1}(F) \odot M_{1,1}(F) \) cannot be a linear span of one idempotent.

(e) \( \mathcal{A} \cong P(n) \) or \( Q(n) \). In this case, the proof is similar to one in cases (c) and (d).

It remains to prove the second part of Theorem 5.1, that is, the existence of the decomposition. A \( Z_2 \)-graded vector space \( V = V_0 + V_1 \) can be represented as the sum of two \( Z_2 \)-graded vector subspaces \( W_1 \) and \( W_2 \) in such a way that \( V_0 = (W_1)_0 + (W_2)_0 \) and the restriction of \( f \) to \((W_1)_0 \), \((W_2)_0 \), \((W_1)_1 \), \((W_2)_1 \) are non-singular. Thus, \( \mathcal{J} = (F + V_0) + V_1 = (F + (W_1)_0 + (W_2)_0) + ((W_1)_1 + (W_2)_1) \) is the sum of two proper simple subsuperalgebras of types \( J(W_1, f_1) \) and \( J(W_2, f_2) \), respectively.

**Decompositions of \( K_3 \)**

Let \( \mathcal{A} \) be a subsuperalgebra of \( K_3 \). Then we have the following restrictions: \( \dim \mathcal{A} \leq 3 \) and \( \text{rk} \mathcal{A}_0 = 1 \). Considering all cases one after another, we obtain that \( \mathcal{A} \cong J(V, f) \) is the only possible case, and \( \dim \mathcal{A} = 2 \). Hence \( \dim \mathcal{A}_0 = 1 \), \( \mathcal{A}_0 = (K_3)_0 = \langle e \rangle \), where \( e \) is an idempotent. In other words, \( e \) is the identity of \( \mathcal{A} \). However, if we consider some element of the form \( \alpha x + \beta y \) belonging to \( \mathcal{A}_1 \), then \( e(\alpha x + \beta y) = \frac{\alpha \beta + \beta \alpha}{2} \), that is, \( e \) cannot be the identity. This implies that \( K_3 \) has no subsuperalgebras of the type \( J(V, f) \). As a direct consequence, we have

**Theorem 5.2** A Jordan superalgebra of the type \( K_3 \) has no decompositions into the sum of two proper simple non-trivial subsuperalgebras.

**Decompositions of \( D_t \)**

Acting in the same manner as in the previous case, we come to the conclusion that the only possible subsuperalgebras in \( D_t \) can be of types \( K_3 \) and \( J(V, f) \). Let \( \mathcal{A} \cong K_3 \) be a subsuperalgebra of \( D_t \). Then we can choose a basis in \( \mathcal{A} \) in such a way that \( \mathcal{A} = \langle e', x', y' \rangle \), \( e'x' = \frac{x'}{2}, e'y' = \frac{y'}{2}, [x', y'] = e' \), \( e'^2 = e' \). Moreover, \( \mathcal{A}'_0 = \langle e' \rangle, \mathcal{A}'_1 = \langle x', y' \rangle \). Therefore, \( e' \) is an idempotent in \( \langle D_t \rangle_0 = \langle e_1, e_2 \rangle \).

Hence either \( e' = e_i \), \( i = 1, 2 \), or \( e' = e_1 + e_2 \). In the last case, we have \( e'(\alpha x + \beta y) = (e_1 + e_2)(\alpha x + \beta y) = (\alpha x + \beta y) \neq \frac{(\alpha \beta + \beta \alpha)}{2} \). Hence \( e' = e_i \). On the other hand, \( [(\alpha x + \beta y), (\alpha' x + \beta' y)] = (\alpha \beta' - \beta \alpha')(e_1 + e_2) \neq e' = e_i \). This implies that \( \mathcal{A} \) of the type \( K_3 \) cannot be a subsuperalgebra \( D_t \).

Let \( \mathcal{A} \cong J(V, f) \) be a subsuperalgebra of \( D_t \). Then \( \mathcal{A}_0 \subseteq (D_t)_0 = \langle e_1, e_2 \rangle \), \( \mathcal{A}_1 \subseteq (D_t)_1 = \langle x, y \rangle \). It is well-known that a subsuperalgebra of type \( J(V, f) \) has the identity \( e, e \in \langle e_1, e_2 \rangle \), that is, \( e = e_1 + e_2 \). If \( \dim \mathcal{A}_0 > 1 \), then we can always choose some element of form \( (\alpha e_1 + \beta e_2) \) which is linearly independent with \( e_1 + e_2 \), and \( (\alpha e_1 + \beta e_2)^2 \) is proportionate to \( e_1 + e_2 \). This implies that \( \alpha = \beta \). If \( \dim \mathcal{A}_0 = 1 \). However if \( D_t = \mathcal{A} + \mathcal{B} \) where \( \mathcal{A} \cong J(V_1, f_1), \mathcal{B} \cong J(V_2, f_2) \), then \( \mathcal{A}_0 = \mathcal{B}_0 = \langle e_1 + e_2 \rangle \), that is, \( (D_t)_0 \neq \mathcal{A}_0 + \mathcal{B}_0 \).

**Theorem 5.3** A Jordan superalgebra of the type \( D_t \) has no simple decomposition into the sum of two non-trivial subsuperalgebras.
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