The Brauer group is not a derived invariant

Nicolas Addington

Abstract

In this short note we observe that the recent examples of derived-equivalent Calabi–Yau 3-folds with different fundamental groups also have different Brauer groups, using a little topological K-theory.

Some years ago Gross and Popescu [12] studied a simply-connected Calabi–Yau 3-fold $X$ fibered in non-principally polarized abelian surfaces. They expected that its derived category would be equivalent to that of the dual abelian fibration $Y$, which is again a Calabi–Yau 3-fold but with $\pi_1(Y) = (\mathbb{Z}_8)^2$, the largest known fundamental group of any Calabi–Yau 3-fold. This derived equivalence was later proved by Bak [2] and Schnell [23]. Ignoring the singular fibers it is just a family version of Mukai’s classic derived equivalence between an abelian variety and its dual [19], but of course the singular fibers require much more work. As Schnell pointed out, it is a bit surprising to have derived-equivalent Calabi–Yau 3-folds with different fundamental groups, since for example the Hodge numbers of a 3-fold are derived invariants [22, Cor. C].

Gross and Pavanelli [11] showed that $\text{Br}(X) = (\mathbb{Z}_8)^2$, the largest known Brauer group of any Calabi–Yau 3-fold. In this note we will show that the finite abelian group $H_1(X, \mathbb{Z}) \oplus \text{Br}(X)$ is a derived invariant of Calabi–Yau 3-folds; thus in this example we must have $\text{Br}(Y) = 0$, and in particular the Brauer group alone is not a derived invariant. This too is a bit surprising, since the Brauer group is a derived invariant of K3 surfaces: if $X$ is a K3 surface then $\text{Br}(X) \cong \text{Hom}(T(X), \mathbb{Q}/\mathbb{Z})$ [7, Lem. 5.4.1], where $T(X) = \text{NS}(X)^\perp \subset H^2(X, \mathbb{Z})$ is the transcendental lattice, which is a derived invariant by work of Orlov [20].

Since an earlier version of this note first circulated, Hosono and Takagi [14] have found a second example of derived-equivalent Calabi–Yau 3-folds with different fundamental groups. Their $X$ and $Y$ are constructed from spaces of $5 \times 5$ symmetric matrices in what is likely an instance of homological projective duality [15], and one has $\pi_1(X) = \mathbb{Z}_2$ and $\pi_1(Y) = 0$. While $\text{Br}(X)$ and $\text{Br}(Y)$ are not known, from our result we see that $\text{Br}(Y) \cong$
$\mathbb{Z}_2 \oplus \text{Br}(X)$, so they are different. An explicit order-2 element of $\text{Br}(Y)$ arises naturally in Hosono and Takagi’s construction [14, Prop. 3.2.1].

It is worth mentioning that both $\pi_1$ and Br are birational invariants, so while birational Calabi–Yau 3-folds are derived equivalent [5], the converse is not true. In addition to the two examples just mentioned, there is the Pfaffian–Grassmannian derived equivalence of Borisov and Căldăraru [4]. In that example $X$ is a complete intersection in a Grassmannian, so $H_1(X, \mathbb{Z}) = \text{Br}(X) = 0$, so from our result we see that $H_1(Y, \mathbb{Z}) = \text{Br}(Y) = 0$ as well; to show that $X$ and $Y$ are not birational Borisov and Căldăraru use a more sophisticated minimal model program argument.

Before proving our result we fix terminology.

**Definition.** A *Calabi–Yau 3-fold* is a smooth complex projective 3-fold $X$ with $\omega_X \cong O_X$ and $b_1(X) = 0$. In particular $H_1(X, \mathbb{Z})$ may be torsion.

This is in contrast to the case of surfaces, where $\omega_X \cong O_X$ and $b_1(X) = 0$ force $\pi_1(X) = 0$ [17, Thm. 13]. There are several reasons not to require $\pi_1(X) = 0$ for Calabi–Yau 3-folds. As we have just seen, a simply-connected Calabi–Yau 3-fold may be derived equivalent to a non-simply-connected one; it may also be mirror to a non-simply-connected one. Perhaps the best reason is that families of simply-connected and non-simply-connected Calabi–Yau 3-folds can be connected by “extremal transitions,” that is, by performing a birational contraction and then smoothing; most known families of Calabi–Yau 3-folds can be connected by extremal transitions [10, 18].

**Definition.** The *Brauer group* of a smooth complex projective variety $X$ is

$$\text{Br}(X) = \text{tors}(H^2_{\text{an}}(X, O_X^\times)),$$

where tors denotes the torsion subgroup.

This used to be called the *cohomological Brauer group* until it was shown to coincide with the honest Brauer group [8]. From the exact sequence

$$H^2(X, O_X) \to H^2(X, O_X^\times) \to H^3(X, \mathbb{Z}) \to H^3(X, O_X)$$

we see that if $X$ is a Calabi–Yau 3-fold then

$$\text{Br}(X) = \text{tors}(H^3(X, \mathbb{Z})).$$

That is, the Brauer group of a Calabi–Yau 3-fold is entirely topological, in contrast to that of a K3 surface which is entirely analytic.
**Proposition.** Let $X$ and $Y$ be Calabi–Yau 3-folds with $D^b(X) \cong D^b(Y)$. Then $H_1(X, \mathbb{Z}) \oplus \text{Br}(X) \cong H_1(Y, \mathbb{Z}) \oplus \text{Br}(Y)$.

**Proof.** Brunner and Distler [6, §2.5] analyzed the boundary maps in the Atiyah–Hirzebruch spectral sequence and saw that for a Calabi–Yau 3-fold $X$, or indeed any closed oriented 6-manifold with $b_1(X) = 0$, it degenerates at the $E_2$ page. Thus there is a short exact sequence

$$0 \to H^5(X, \mathbb{Z}) \to K^1_{\text{top}}(X) \to H^3(X, \mathbb{Z}) \to 0,$$

where $K^*_{\text{top}}(X)$ is topological K-theory. Since $H^5(X, \mathbb{Z}) = H_1(X, \mathbb{Z})$ is torsion, this gives an exact sequence

$$0 \to H_1(X, \mathbb{Z}) \to \text{tors}(K^1_{\text{top}}(X)) \to \text{Br}(X) \to 0. \quad (1)$$

While it is not strictly necessary for our purposes, they also got an exact sequence

$$0 \to \text{Br}(X)^* \to \text{tors}(K^0_{\text{top}}(X)) \to H_1(X, \mathbb{Z})^* \to 0; \quad (2)$$

here if $A$ is a finite abelian group then the dual group $A^* := \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$, which is non-canonically isomorphic to $A$.

Doran and Morgan [9, §4] analyzed $K^*_{\text{top}}(X)$ more carefully using the fact that $c_1(X) = 0$ and showed that the sequences (1) and (2) are in fact split.

Now the proposition follows from the fact that $K^0_{\text{top}}$ and $K^1_{\text{top}}$ are derived invariants [1, §2.1]. In a bit more detail, if $\Phi: D^b(X) \to D^b(Y)$ and $\Psi: D^b(Y) \to D^b(X)$ are inverse equivalences, then by [20, Thm. 2.2] there are objects $E, F \in D^b(X \times Y)$ such that

$$\Phi(-) = \pi_{Y*}(E \otimes \pi^*_X(-)) \quad \Psi(-) = \pi_{X*}(F \otimes \pi^*_Y(-)),$$

and arguing as in [16, Lem. 5.32] we find that the same formulas define inverse isomorphisms $K^*(X) \to K^*(Y)$ and $K^*(Y) \to K^*(X)$: use the fact that the pushforward on $K^*$ satisfies a projection formula and is compatible with the pushforward on $D^b$.

We conclude with a remark on $H_1$ and Br in mirror symmetry. Batyrev and Kreuzer [3] predicted that mirror symmetry exchanges $H_1$ and Br, having calculated both groups for all Calabi–Yau hypersurfaces in 4-dimensional toric varieties. In all their examples the groups are quite small: either $H_1 = 0$ and Br $= \mathbb{Z}_2$, $\mathbb{Z}_3$, or $\mathbb{Z}_5$, or vice versa. This prediction does not seem
to be right in general. On the one hand it is contradicted by a prediction of Gross and Pavanelli [11, Rem. 1.5], based on calculations in Pavanelli’s thesis [21], that if $X$ is the abelian fibration above, with $H_1(X) = 0$ and $Br(X) = (\mathbb{Z}_8)^2$, then its mirror $\hat{X}$ has $\pi_1(\hat{X}) = Br(\hat{X}) = \mathbb{Z}_8$. Even more seriously, Hosono and Takagi’s $X$ and $Y$ have the same mirror according to [13], but different $H_1$ and $Br$ as we have discussed. Mirror symmetry is expected to exchange $K^0_{\text{top}}$ and $K^1_{\text{top}}$, however, so mirror Calabi–Yau 3-folds should have the same $H_1 \oplus Br$.

I thank P. Aspinwall and A. Căldăraru for helpful discussions, and S. Hosono and H. Takagi for encouraging me to publish this note.

References

[1] N. Addington and R. P. Thomas. Hodge theory and derived categories of cubic fourfolds. Preprint, arXiv:1211.3758.

[2] A. Bak. The spectral construction for a (1,8)-polarized family of abelian varieties. Preprint, arXiv:0903.5488.

[3] V. Batyrev and M. Kreuzer. Integral cohomology and mirror symmetry for Calabi-Yau 3-folds. In Mirror symmetry. V, volume 38 of AMS/IP Stud. Adv. Math., pages 255–270. Amer. Math. Soc., Providence, RI, 2006. Also math/0505432.

[4] L. Borisov and A. Căldăraru. The Pfaffian-Grassmannian derived equivalence. J. Algebraic Geom., 18(2):201–222, 2009. Also math/0608404.

[5] T. Bridgeland. Flops and derived categories. Invent. Math., 147(3):613–632, 2002. Also math/0009053.

[6] I. Brunner and J. Distler. Torsion D-branes in nongeometrical phases. Adv. Theor. Math. Phys., 5(2):265–309, 2001. Also hep-th/0102018.

[7] A. Căldăraru. Derived categories of twisted sheaves on Calabi–Yau manifolds. PhD thesis, Cornell University, 2000. Available at math.wisc.edu/~andreic/publications/ThesisSingleSpaced.pdf.

[8] A. J. de Jong. A result of Gabber. Available at math.columbia.edu/~dejong/papers/2-gabber.pdf.
[9] C. Doran and J. Morgan. Algebraic topology of Calabi-Yau threefolds in toric varieties. *Geom. Topol.*, 11:597–642, 2007. Also math.AG/0605074.

[10] P. Green and T. Hübsch. Connecting moduli spaces of Calabi–Yau threefolds. *Comm. Math. Phys.*, 119(3):431–441, 1988.

[11] M. Gross and S. Pavanelli. A Calabi-Yau threefold with Brauer group $(\mathbb{Z}/8\mathbb{Z})^2$. *Proc. Amer. Math. Soc.*, 136(1):1–9, 2008. Also math/0512182.

[12] M. Gross and S. Popescu. Calabi-Yau threefolds and moduli of abelian surfaces. I. *Compositio Math.*, 127(2):169–228, 2001. Also math/0001089.

[13] S. Hosono and H. Takagi. Determinantal quintics and mirror symmetry of Reye congruences. Preprint, arXiv:1208.1813.

[14] S. Hosono and H. Takagi. Double quintic symmetroids, Reye congruences, and their derived equivalence. Preprint, arXiv:1302.5883.

[15] S. Hosono and H. Takagi. Duality between $S^2\mathbb{P}^4$ and the double quintic symmetroid. Preprint, arXiv:1302.5881.

[16] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2006.

[17] K. Kodaira. On the structure of compact complex analytic surfaces. I. *Amer. J. Math.*, 86:751–798, 1964.

[18] M. Kreuzer and H. Skarke. Complete classification of reflexive polyhedra in four dimensions. *Adv. Theor. Math. Phys.*, 4(6):1209–1230, 2000. Also hep-th/0002240.

[19] S. Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Math. J.*, 81:153–175, 1981.

[20] D. Orlov. Equivalences of derived categories and K3 surfaces. *J. Math. Sci. (New York)*, 84(5):1361–1381, 1997. Also alg-geom/9606006.

[21] S. Pavanelli. *Mirror symmetry for a two parameter family of Calabi-Yau threefolds with Euler characteristic 0*. PhD thesis, University of Warwick, 2003.
[22] M. Popa and C. Schnell. Derived invariance of the number of holomorphic 1-forms and vector fields. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(3):527–536, 2011. Also arXiv:0912.4040.

[23] C. Schnell. The fundamental group is not a derived invariant. In *Derived categories in algebraic geometry*, EMS Ser. Congr. Rep., pages 279–285. Eur. Math. Soc., Zürich, 2012. Also arXiv:1112.3586.

Department of Mathematics  
Duke University, Box 90320  
Durham, NC 27708-0320

adding@math.duke.edu