On a Mathematical Model of the Rotating Atmosphere of the Earth

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Abstract

In meteorology the analysis of motions of the atmosphere on the Earth has been done using various mathematical models and using various approximations. In this article as the simplest model the compressible Euler equations with barotropic equation of state of the ideal gas is analyzed under the co-ordinate system which rotates with constant angular velocity. Mathematically rigorous inquiry is tried. Although problems remain to be open, some fundamental results are exhibited.

Key Words and Phrases. Compressible Euler equations, Atmospheric motion, Rotations, Hille-Yosida theory, Variational principle,

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1 Introduction

In meteorology the analysis of motions of the atmosphere on the Earth has been done using various mathematical models and using various approximations. Actually models used for numerical weather prediction or climate simulation must take into account various factors besides the fundamental state variables of the gas and must meet practical computational efficiency with moderate accuracy in the process of the numerical simulations. Discussions of choice of models have been accumulated as an enormous collection in the meteorological literatures. However fundamental mathematically rigorous inquiries of the structure of solutions of the equations adopted as models in atmospheric dynamics have not yet been thoroughly done even for the simplest model. In this article we try to investigate the fundamental mathematical properties of the simplest model with the compressible Euler equations and the barotropic, or, isentropic motion of the ideal gas described in the co-ordinate system which rotates with a constant angular velocity. Although there remains open problems to be solved in mathematically rigorous way, some elementary aspects of the inquiry are exhibited in

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In the following in this section the equations and the boundary condition to be considered are going to be described, and the definition of some concepts of solutions are going to be given. In Section 2 stationary solutions with compactly supported density and zero relative velocity will be discussed. The limit of allowable magnitude of the angular velocity of the rotation should be noted. In Section 3 Lagrangian co-ordinate description of the equations for the perturbations near the stationary solutions will be discussed, and in Section 4 we shall show that the linearized wave equation for the perturbations allows an application of the Hille-Yosida theory of existence of solutions. Higher order regularities of the solutions remains in an open problem. In Section 5 the so called ‘variational principle’ will be formulated and its efficiency will be discussed. Although the concept of eigenfrequency and eigenvectors to the wave equations for the perturbations is remarkable, the existence and completeness of eigenvectors is in an open problem when the rotation is present.

Let us describe the situation to be considred precisely.

We consider motions of an atmosphere governed by the compressible Euler equations described by the uniformly rotating coordinate system \((t, x)\):

\[
\frac{D\rho}{Dt} + \rho(\nabla|v|) = 0, \tag{1.1a}
\]

\[
\rho \left[ \frac{Dv}{Dt} + 2\Omega \times v \right] + \nabla P + \rho \nabla \Phi^{(\Omega)} = 0 \tag{1.1b}
\]

on \(t \in \mathbb{R}, x = (x^1, x^2, x^3) \in \mathbb{R}^3 \setminus \mathcal{B}_0 = \{ x \in \mathbb{R}^3 | r := \|x\| > R_0 \}\), where \(\mathcal{B}_0 = \{ x \in \mathbb{R}^3 | r := \|x\| \leq R_0 \}\), \(R_0\) being a positive number. The variables \(\rho, P\) are the density, the pressure and \(v = (v^1, v^2, v^3)^T\) is the velocity field. We are denoting

\[
\frac{D}{Dt} := \frac{\partial}{\partial t} + (v|\nabla) = \frac{\partial}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k} \tag{1.3}
\]

and \((\nabla|v|) = \sum_k \frac{\partial v^k}{\partial x^k}\). On the other hand

\[
\Omega = \Omega \frac{\partial}{\partial x^3}, \tag{1.4}
\]

\(\Omega\) being a constant, and \(\Phi^{(\Omega)}\) is the geopotential given as

\[
\Phi^{(\Omega)}(x) = -\frac{GM_0}{r} - \frac{\Omega^2}{2} \varpi^2 \tag{1.5}
\]

where

\[
r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \quad \varpi = \sqrt{(x^1)^2 + (x^2)^2},
\]
$G M_0$ being a positive constant. Since we are concerned with the value of the potential $\Phi\langle \Omega \rangle$ only on the domain $\mathbb{R}^3 \setminus B_0$, we assume that $\Phi^{(1)} \in C(\mathbb{R}^3)$ and holds for $r = \|x\| > R_0$ by removing the singularity $\frac{1}{r}$ at $r = 0$ by a smooth function near $r = 0$, namely, we consider, instead of (1.5),

$$
\Phi^{(1)}(x) = \chi\left(\frac{r}{R_0}\right) \left[-\frac{G M_0}{r} - \frac{\Omega^2}{2}\right],
$$

(1.6)

where $\chi \in C(\mathbb{R})$ such that $0 < \chi < 1$.

The atmosphere is surrounding the Earth of radius $R_0$, mass $M_0$ and $G$ is the gravitational constant. $\Omega$ is the angular velocity of the rotation.

We assume

(A) : $P$ is the function of $\rho$ defined by

$$
P = A \rho^\gamma \quad \text{for} \quad \rho > 0,
$$

(1.7)

$A, \gamma$ being positive constants such that $1 < \gamma < 2$.

Under this assumption we introduce the variable $\Upsilon$ by

$$
\Upsilon := \int_0^\rho \frac{dP}{\rho} = \frac{A \gamma}{\gamma - 1} \rho^{\gamma - 1} \quad \text{for} \quad \rho > 0.
$$

(1.8)

Since we are interested in solutions with $\rho$ which has a vacuum region on which $\rho = 0$, the concept of classical solutions needs a precise definition. The following are the definitions adopted in this article.

Definition 1
An open connected subset $\mathcal{D}$ of $\mathbb{R}^3$ is said to be an admissible domain with cover $\mathcal{D}$ if $\mathcal{D} = \mathcal{D} \cup B_0$ is an open connected subset of $\mathbb{R}^3$ such that $\mathcal{D} = \mathcal{D} \setminus B_0$ and $\partial \mathcal{D}$ is smooth.

Here and hereafter we use the following

Notation 1
Let $A, B$ be subsets of $\mathbb{R}^3$. $\overline{A}$ denotes the complement of $A$ and $A \setminus B$ stands for $A \cap B$. $\text{Cl} A$ stands for the closure of $A$, $\text{Int} A$ stands for the interior of $A$, and $\partial A$ stands for the boundary of $A$, namely, $\partial A = \text{Cl} A \setminus \text{Int} A$.

Note that, if $\mathcal{D}$ is an admissible domain, then, for a sufficiently small positive $\delta$, we have

$$
\{R_0 < r \leq (1 + \delta)R_0\} \subset \mathcal{D}.
$$

Definition 2
Let $\mathcal{D}$ be an admissible domain with cover $\mathcal{D}$. A vector field $\mathbf{v}$ defined on $[0, T] \times \mathcal{D}$ is said to be a classical admissible velocity field on $[0, T] \times \mathcal{D}$
if 1) \(v\) has an extension onto \([0, T] \times \tilde{\mathcal{D}}\) of class \(C^1([0, T] \times \tilde{\mathcal{D}})\) and 2) the boundary condition

\[
(n | v) = 0 \quad \text{on} \quad [0, T] \times \partial B_0
\]  

(1.9)

holds, where \(n = n(x) = x/R_0\) is the inward normal vector at the boundary point \(x \in \partial B_0\).

**Definition 3** Let \(\mathcal{D}\) be an admissible domain with cover \(\tilde{\mathcal{D}}\). A function \((\rho, v)\) defined on \([0, T] \times \mathcal{D}\) is said to be a classical \((\rho, v)\)-solution on \([0, T] \times \mathcal{D}\) if 1) \((\rho, v)\) has an extension onto \([0, T] \times \tilde{\mathcal{D}}\) of class \(C^1([0, T] \times \tilde{\mathcal{D}})\), 2) \(\rho \geq 0\) on \([0, T] \times \mathcal{D}\), 3) \((\rho, v)\) satisfies (1.1a), (1.1b) on \([0, T] \times \mathcal{D}\), and 4) \(\rho > 0\) on \([0, T] \times \partial B_0\) and the boundary condition

\[
(n | v) = 0 \quad \text{on} \quad [0, T] \times \partial B_0,
\]  

(1.10)

holds, where \(n = n(x) = x/R_0\) is the inward normal vector at the boundary point \(x \in \partial B_0\).

We want to permit the variable \(\Upsilon\) to take negative values somewhere. In order to do this we extend the equations divided by \(\rho\) where \(\rho > 0\) to the equations:

\[
\frac{D\Upsilon}{Dt} + (\gamma - 1) \Upsilon (\nabla | v) = 0,
\]

(1.11a)

\[
\frac{Dv}{Dt} + 2\Omega \times v + \nabla (\Upsilon + \Phi^{(\Omega)}) = 0
\]

(1.11b)

**Definition 4** Let \(\mathcal{D}\) be an admissible domain with cover \(\tilde{\mathcal{D}}\). A function \((\Upsilon, v)\) defined on \([0, T] \times \mathcal{D}\) is said to be a classical \((\Upsilon, v)\)-solution on \([0, T] \times \mathcal{D}\) if 1) \((\Upsilon, v)\) has an extension onto \([0, T] \times \tilde{\mathcal{D}}\) of class \(C^1([0, T] \times \tilde{\mathcal{D}})\), 2) \((\Upsilon, v)\) satisfies (1.1a), (1.1b) on \([0, T] \times \mathcal{D}\), and 4) \(\Upsilon > 0\) on \([0, T] \times \partial B_0\) and the boundary condition

\[
(n | v) = 0 \quad \text{on} \quad [0, T] \times \partial B_0,
\]  

(1.12)

holds, where \(n = n(x) = x/R_0\) is the inward normal vector at the boundary point \(x \in \partial B_0\).

As for the variable \(\Upsilon\), \(\Upsilon \geq 0\) is not assumed. When \((\Upsilon, v)\) is a classical \((\Upsilon, v)\)-solution on the admissible domain \([0, T] \times \mathcal{D}\), then \((\rho, v)\) defined by

\[
\rho = \left(\frac{\gamma - 1}{A_\gamma} \Upsilon \lor 0\right)^{\frac{1}{\gamma - 1}}
\]

(1.13)

turns out to be a classical \((\rho, v)\)-solution on \([0, T] \times \mathcal{D}\). Here and hereafter we use the

**Notation 2** We denote

\[
Q \lor Q' = \max\{Q, Q'\}, \quad Q \land Q' = \min\{Q, Q'\}.
\]

(1.14)
But the inverse may be impossible, namely, we cannot expect that any classical \((\rho, v)\)-solution comes from classical \((\Upsilon, v)\)-solution by the above procedure, since \(1 < \frac{1}{\gamma - 1} < +\infty\) but \(0 < \gamma - 1 < 1\) so that \(\rho \in C^1\) does not imply \(\rho^{\gamma - 1} \in C^1\) at the vacuum boundary. In this sense the concept of classical \((\rho, v)\)-solutions is weaker than that of classical \((\Upsilon, v)\)-solutions.

Moreover we note that the uniqueness of solution cannot be expected under the concept of classical \((\rho, v)\)-solutions in the following sense:

Let \((\rho, v)\) be a classical \((\rho, v)\)-solution on \([0, T] \times \Omega\), \(\Omega\) being an admissible domain with cover \(\tilde{\Omega}\). Suppose that \((0, T] \times \Omega \setminus \{\rho > 0\} \neq \emptyset\), and consider a velocity field \(v' \in C^1([0, T] \times \tilde{\Omega})\) such that \(v'(t, x) = v(t, x)\) on \((0, T] \times \Omega \setminus \{x | \rho(t, x) > 0\}\), where \(\Omega\) is a non-empty open subset of \((0, T] \times \tilde{\Omega} \setminus \{\rho > 0\}\). Clearly \((\rho, v')\) is another classical \((\rho, v)\)-solution such that it coincides with the original \((\rho, v)\) on \([0, t_1]\) but it is different from the original \((\rho, v)\) after \(t = t_1\) with \(0 < t_1 < T\).

We are interested in motions with compactly supported \(\rho\). Namely we use

**Definition 5** Let \(\Omega\) be an admissible domain. A classical \((\rho, v)\)-solution \((\rho, v)\) on \([0, T] \times \Omega\) is said to be compactly supported if, for any fixed \(t \in [0, T]\), \(\text{Cl}\{x | \rho(t, x) > 0\}\) is a compact subset of \(\partial \Omega_0 \cup \Omega\).

## 2 Stationary solutions

We are looking for stationary solutions \(\Upsilon = \Upsilon(x), v = v(x)\). The equation to be satisfied are

\[
(v|\nabla)\Upsilon + (\gamma - 1)\Upsilon |(\nabla |v) = 0, \tag{2.1a}
\]

\[
(v|\nabla)v + 2\Omega \times v + \nabla(\Upsilon + \Phi^{(\Omega)}) = 0. \tag{2.1b}
\]

Let us use the cylindrical co-ordinate system \((\varpi, \phi, z)\) defined by

\[
x_1 = \varpi \cos \phi, \quad x_2 = \varpi \sin \phi, \quad x_3 = z.
\]

The basis of the co-ordinates consists of the unit vectors

\[
e_{\varpi} = \frac{1}{\varpi} \frac{\partial}{\partial \varpi} = \frac{1}{\varpi} \left( \cos \phi \frac{\partial}{\partial x_1} + \sin \phi \frac{\partial}{\partial x_2} \right),
\]

\[
e_{\phi} = \frac{1}{\varpi} \frac{\partial}{\partial \phi} = -\sin \phi \frac{\partial}{\partial x_1} + \cos \phi \frac{\partial}{\partial x_2},
\]

\[
e_z = \frac{\partial}{\partial x_3}.
\]

Suppose that the velocity field \(v\) is of the form

\[
v = V^\phi e_{\phi} = \frac{V^\phi}{\varpi} \frac{\partial}{\partial \phi}, \tag{2.2}
\]
Since

\[(v|\nabla) = \frac{V^\phi}{\omega} \frac{\partial}{\partial \phi}, \quad (\nabla|v) = \frac{1}{\omega} \frac{\partial V^\phi}{\partial \phi},\]

the equation of continuity (2.1a) reduces to

\[\frac{V^\phi}{\omega} \frac{\partial \Upsilon}{\partial \phi} + (\gamma - 1) \frac{\Upsilon}{\omega} \frac{\partial V^\phi}{\partial \phi} = 0.\]

This equation holds if

\[\frac{\partial V^\phi}{\partial \phi} = 0, \quad \frac{\partial \Upsilon}{\partial \phi} = 0. \tag{2.3}\]

So, supposing (2.3), we solve the equation of motion (2.1b). Since

\[(v|\nabla)v = -(V^\phi)^2 e_\omega + \frac{V^\phi}{\omega} \frac{\partial V^\phi}{\partial \phi} e_\phi = -(V^\phi)^2 e_\omega,\]

\[2\Omega \times v = -2\Omega \omega V^\phi e_\omega,\]

\[\nabla (T + \Phi^{(\Omega)}) = \frac{\partial}{\partial \omega} (T + \Phi^{(\Omega)}) e_\omega + \frac{1}{\omega} \frac{\partial}{\partial \phi} (T + \Phi^{(\Omega)}) e_\phi + \frac{\partial}{\partial z} (T + \Phi^{(\Omega)}) e_z,\]

the equation (2.1b) reduces to

\[-(V^\phi)^2 - 2\Omega \omega V^\phi + \frac{\partial}{\partial \omega} (T + \Phi^{(\Omega)}) = 0, \quad (2.4a)\]

\[\frac{\partial}{\partial z} (T + \Phi^{(\Omega)}) = 0. \tag{2.4b}\]

Here recall that \(\frac{\partial}{\partial \phi} \Phi^{(\Omega)} = 0\) and that we are supposing \(\frac{\partial}{\partial \phi} T = 0\).

Taking

\[\frac{\partial}{\partial z} (2.4a) - \frac{\partial}{\partial \omega} (2.4b) = 0,\]

we see \(\partial V^\phi/\partial z = 0\). Therefore there should exist a function \(\omega\) such that \(V^\phi = \omega \omega(\omega)\), namely, we consider

\[v = \omega(\omega) \frac{\partial}{\partial \phi} = \omega \omega(\omega) e_\phi. \tag{2.5}\]

Integration of (2.4a), (2.4b) gives

\[T + \Phi^{(\Omega)} + \frac{\Omega^2}{2} \omega^2 - B(\omega) = \text{Const.}, \tag{2.6}\]

where

\[B(\omega) := \int_0^\omega (\omega(\omega') + \Omega)^2 \omega' d\omega'. \tag{2.7}\]
Recall
\[ \Phi^{(\Omega)} + \frac{\Omega^2}{2} \omega^2 = -\frac{GM_0}{r}. \]

Therefore (2.6) reads
\[ \Upsilon + \Phi^{(\Omega, \omega)} = \text{Const.}, \quad (2.8) \]
where
\[ \Phi^{(\Omega, \omega)} := -\frac{GM_0}{r} - B(\varpi) \]
\[ \quad = -\frac{GM_0}{r} - \int_0^\infty (\omega(\varphi) + \Omega)^2 \varphi d\varphi. \quad (2.9) \]

As in Section 2, let us specify the constant so that
\[ \Upsilon = GM_0 \left( \frac{1}{r} - \frac{1}{R} \right) + B(\varpi). \quad (2.10) \]

Thus we have stationary solution
\[ \Upsilon = GM_0 \left( \frac{1}{r} - \frac{1}{R} \right) + B(\varpi), \quad v = \varpi \omega(\varpi) e^\varphi. \quad (2.11) \]

Here, in order to fix the idea, let us suppose the following assumption:

(B): The function \( \omega \) belongs to the class \( C^1([0, +\infty]) \cap L^\infty(0, +\infty) \).

Now let us consider the static stationary solution \( (\rho, v) = (\rho(x), 0) \): that is, \( \omega(\varpi) = 0 \), and
\[ \Upsilon := -\Phi^{(\Omega)} - \frac{GM_0}{R} \quad (2.12) \]
so that
\[ \Upsilon = GM_0 \left( \frac{1}{r} - \frac{1}{R} \right) + \frac{\Omega^2}{2} \omega^2 \quad \text{for} \quad r > R_0, \]
where \( R \) is a positive constant at the present. If \( R \geq R_0 \), then \( R - R_0 \) is the height of the stratosphere at the North and South Poles, and
\[ \Upsilon_P := GM_0 \left( \frac{1}{R_0} - \frac{1}{R} \right) \geq 0 \quad (2.13) \]
gives the density of the atmosphere at the Poles
\[ \rho_P := \left( \frac{\gamma - 1}{A\gamma} \Upsilon_P \right)^{-\frac{1}{\gamma - 1}} = \left( \frac{\gamma - 1}{A\gamma} GM_0 \left( \frac{1}{R_0} - \frac{1}{R} \right) \right)^{-\frac{1}{\gamma - 1}}. \quad (2.14) \]
Now we are going to observe the shape of the set

\[ \mathcal{U} := \{(\varpi, z) \mid 0 \leq \varpi, |z| < \infty, R_0 < r, -\Phi^{(\Omega)} - \frac{GM_0}{R} > 0\} \tag{2.15} \]

Let us introduce the non-dimensional variables \( X = \varpi/R_0, Z = z/R_0 \) and put

\[ F(X^2, Z^2; \kappa) := \frac{1}{\sqrt{X^2 + Z^2}} + \kappa X^2, \tag{2.16} \]

\[ \kappa = \frac{\Omega^2 R_0^3}{2GM_0}. \tag{2.17} \]

Then

\[ \Phi^{(\Omega)} = -\frac{GM_0}{R_0} F(X^2, Z^2; \kappa) \tag{2.18} \]

and \( \{ -\Phi^{(\Omega)} - \frac{GM_0}{R} > 0 \} = \{ F > \frac{R_0}{R} \} \). Let us observe the shape of the level set \( \{ F(X^2, Z^2; \kappa) = \lambda \} \) of \( F \), \( \lambda \) being a positive number. To do so, solving

\[ F(X^2, Z^2; \kappa) = \lambda, \tag{2.19} \]

we consider

\[ Z^2 = \frac{g(X^2)}{(\lambda - \kappa X^2)^2}, \tag{2.20} \]

for \( X^2 < \frac{\lambda}{\kappa} \), where

\[ g(Q) = g(Q; \kappa, \lambda) := 1 - \lambda^2 Q + 2\lambda \kappa Q^2 - \kappa^2 Q^3. \tag{2.21} \]

We see

\[ g(0) = 1, \quad g\left(\frac{\lambda}{\kappa}\right) = 1, \]

and

\[ Dg(Q) = (\lambda - \kappa Q)(3\kappa Q - \lambda). \]

Note that

\[ g\left(\frac{\lambda}{3\kappa}\right) > 0 \iff \lambda^3 < \frac{27}{4}. \]

We see the shape of the graph of the function \( g \) according to the following three cases:

**Case(L):** \( \lambda^3 < \frac{27}{4} \kappa. \)

Then \( g(Q) > 0 \) for \( 0 \leq Q < Q_\infty \), \( g(Q_\infty) = 0 \), \( g(Q) < 0 \) for \( Q_\infty < Q < +\infty \). Here \( Q_\infty = Q_\infty\left(\frac{\lambda}{3\kappa}\right) \) is a number such that \( Q_\infty > \frac{\lambda}{\kappa}. \)
Case (M): $\lambda^3 = \frac{27}{4}\kappa$.

Then $g(Q) > 0$ for $0 \leq Q < \frac{\lambda}{3\kappa} (= \frac{9}{\kappa^2})$, $g(\frac{\lambda}{3\kappa}) = 0$, $g(Q) > 0$ for $\frac{\lambda}{\kappa} < Q < Q_\infty$, $g(Q_\infty) = 0$, $g(Q) < 0$ for $Q_\infty < Q < +\infty$.

Case (H): $\lambda^3 > \frac{27}{4}\kappa$.

Then $g(Q) > 0$ for $0 \leq Q < Q_-$, $g(Q_-) = 0$, $g(Q) < 0$ for $Q_- < Q < Q_+$, $g(Q_+) = 0$, $g(Q) > 0$ for $Q_+ < Q < Q_\infty$, $g(Q_\infty) = 0$, $g(Q) < 0$ for $Q_\infty < Q < +\infty$. Here $Q_\pm = Q_\pm (\frac{\lambda}{3\kappa})$ are numbers such that $0 < Q_- < \frac{\lambda}{\kappa} < Q_+ < \frac{\lambda}{\kappa}$ and $Q_\pm \to \frac{\lambda^2}{\kappa} \sim \frac{27}{4}$.

Recall that we need the function $g(Q)$ only for $0 \leq Q \leq \frac{1}{\kappa}$, and we do not take care of the behavior of $g(Q)$ beyond $Q = \frac{1}{\kappa}$, a fortiori, neither near nor beyond $Q = Q_\infty$. Anyway, correspondingly we see the shape of the set $\{ F > \lambda \}$ as follows:

In Case (L), the set $\{ F > \lambda \}$ is unbounded.

In Case (M), the set $\{ F > \lambda \}$ consists of two connected components, say, $\Omega_0$ and $\Omega_\infty$, where $\Omega_0$ is bounded and included in $\{ 0 \leq X < \sqrt{\frac{\lambda}{2\kappa}} (= \frac{1}{\kappa}) \}$ but $\Omega_\infty$ is an unbounded subset of $\{ \sqrt{\frac{\lambda}{2\kappa}} < X < \sqrt{\frac{\lambda}{\kappa}} \}$. Note that $Dg(\frac{\lambda}{3\kappa}) = 0$ so that $\partial\Omega_0 = \{ F = \lambda, X \leq \frac{3\lambda}{2\kappa} \}$ has a corner at the point $(X,Z) = (\sqrt{\frac{\lambda}{\kappa}},0)$. Namely $\partial\Omega_0$ near this point can be described as

$$
Z = \pm C \left( \frac{3}{2\lambda} - X \right) \left( 1 + \left[ X - \frac{3}{2\lambda} \right] \right) \quad \text{as} \quad X \to \frac{3}{2\lambda} - 0,
$$

where $C = C(\kappa)$ is a positive constant and $[Y]_1$ stands for a convergent power series of the form $\sum_{k \geq 1} a_k Y^k$.

In Case (H), the set $\{ F > \lambda \}$ consists of two connected components, say $\Omega_0 \subset \{ 0 \leq X < \sqrt{Q_-} \}$ and $\Omega_\infty \subset \{ \sqrt{Q_+} < X < \sqrt{\frac{\lambda}{\kappa}} \}$. $\Omega_0$ is bounded but $\Omega_\infty$ is unbounded. Note that $X$ along $\partial\Omega_0$ can be solved as a smooth (real analytic) function of $Z,|Z| \ll 1$, near the point $(X,Z) = (\sqrt{Q_-},0)$. Namely the shape of $\partial\Omega_0 = \{ F = \lambda, X < \sqrt{\frac{\lambda}{2\kappa}} \}$ near this point is

$$
Z = \pm C \sqrt{Q_- - X} \left( 1 + |X - Q_-| \right) \quad \text{as} \quad X \to Q_- - 0,
$$

$C = C(\kappa,\lambda)$ being a positive constant. The point is not a corner, and $\partial\Omega_0$ is smooth. However it should be noticed that $\Omega_0$ is not an ellipse.

Therefore, applying the above observations to $\lambda = \frac{R_0}{R}$, we claim

**Theorem 1** Compact supported axially and equatorially symmetric static stationary solutions with the height of the stratosphere at the North Pole $R > R_0$
and constant angular velocity $\Omega$ exist if and only if 
$\kappa = \Omega^2 \text{ is constant }$ 

$$\frac{\Omega^2 R_0^3}{2GM_0} \leq \frac{4}{27} \left( \frac{R_0}{R} \right)^3.$$ 

In this sense we should restrict ourselves to the case $0 \leq \kappa < \frac{4}{27} \lambda^3$, that is, we require the following assumption 

(K): $R > R_0$ and it holds 

$$0 \leq \left( \frac{R}{R_0} \right)^3 \kappa = \frac{\Omega^2 R_0^3}{2GM_0} < \frac{4}{27}. \quad (2.22)$$ 

We put 

$$\mathcal{R}_1(\Omega^2) := \begin{cases} +\infty & \text{for } \Omega^2 = 0 \\ \frac{2}{3} \left( \frac{GM_0}{\Omega^2} \right)^{3/4} & \text{for } \Omega^2 > 0 \end{cases}. \quad (2.23)$$ 

Then the condition (K) reads 

$$R_0 < R < \mathcal{R}_1(\Omega^2). \quad (2.24)$$ 

In order that $R_0 < \mathcal{R}_1(\Omega^2)$ it is necessary that 

$$\Omega^2 < \left( \frac{4GM_0}{9(R_0)^2} \right)^2. \quad (2.25)$$ 

Under the assumption (K), or, in Case (H) with $\lambda = \frac{R_0}{R}$, we consider the number $\frac{3R}{2R_0} = \frac{3}{2\lambda}$. Since $\frac{3}{2\lambda} < \sqrt{\frac{1}{\lambda^2}}$ and since $g\left( \frac{9}{4\lambda^2} \right) < 0$, which is not obvious but can be shown, we see that $\sqrt{\mathcal{Q}} - \frac{3}{2\lambda}$. On the other hand, we see obviously that $\frac{3}{2\lambda} < \sqrt{Q} < \sqrt{\mathcal{Q}^+}$. Therefore we have the estimate 

$$\sqrt{\mathcal{Q}} < \frac{3}{2\lambda} = \frac{3R}{2R_0} < \sqrt{\mathcal{Q}^+}.$$ 

Thus the closure of $\mathcal{U}_0$ is included in $\{ R_0 \leq r, \varpi < 3R/2 \}$, and $\mathcal{Y}$ is a classical solution on $[0, +\infty] \times \mathfrak{D}$, where $\mathfrak{D} = \{ R_0 < r, \varpi < 3R/2 \}$. So 

$$\tilde{\rho} = \left( \frac{\gamma - 1}{A_\gamma} (\mathcal{Y} \mid \mathfrak{D}) \vee 0 \right)^{\frac{1}{\gamma - 1}}$$ 

turns out to be a classical solution on $[0, +\infty] \times \mathfrak{D}$. In other words, if we put 

$$\mathcal{Y} = \begin{cases} \mathcal{Y} & \text{on } \mathcal{U}_0 \\ 0 & \text{on } \{ R_0 < r \} \setminus \mathcal{U}_0, \end{cases} \quad (2.26)$$
that is,
\[
\mathcal{Y}_* = (\mathcal{Y} \vee 0) \cdot 1_{\mathcal{W}_{<3R/2}} = (\mathcal{Y} \cdot 1_{\mathcal{W}_{<3R/2}}) \vee 0
\]
\[
= \begin{cases} 
\mathcal{Y} & \text{if } \mathcal{Y} > 0 \text{ and } \mathcal{W} < \frac{3R}{2} \\
0 & \text{otherwise}
\end{cases},
\tag{2.27}
\]
the density distribution \(\bar{\rho}\) given by
\[
\bar{\rho} = \left(\frac{\gamma - 1}{\Lambda_0^2} \mathcal{Y}_*\right)^{\frac{1}{\gamma - 1}}
\tag{2.28}
\]
gives a classical solution \((\bar{\rho}, 0)\) on \([0, \infty[ \times (\mathbb{R}^3 \setminus \mathcal{B}_0)\). Note that \((\mathcal{Y} \cdot 1_{\mathcal{W}_{<3R/2}}, 0)\)
is not a classical \((\mathcal{Y}, \mathbf{v})\)-solution on \([0, +\infty[ \times (\mathbb{R}^3 \setminus \mathcal{B}_0)\) in the sense of Definition 4.

We are considering the uniformly rotating atmosphere which occupies
\[
\mathcal{R} := \{ x \in \mathbb{R}^3 \mid R_0 < r, \rho^0(x) = \rho(\mathcal{W}, z) > 0\},
\tag{2.29}
\]
provided \((\mathbf{K})\). Then the boundary \(\partial \mathcal{R}\) of the atmosphere consists of the two connected components \(\Sigma_0 = \partial \mathcal{B}_0 = \{ r = R_0 \}\), the surface of the Earth, and
\[
\Sigma_1 = \{ \frac{1}{\sqrt{X^2 + Z^2}} + \kappa X^2 = \frac{R_0}{R}, \text{ with } \mathcal{W} = R_0 X, z = R_0 Z\},
\]
the stratosphere of the atmosphere. We are going to show that the boundary \(\Sigma_1\) is a physical vacuum boundary, that is, at each boundary point \(P \in \Sigma_1\) we have
\[
(\nabla \mathcal{Y}^0|\mathbf{n}) < 0,
\]
where \(\mathbf{n}\) is the outer normal vector at \(P\) and \(\mathcal{Y}^0(x) = \mathcal{Y}(\mathcal{W}, z)\).
Recall that \(\mathcal{Y} = \frac{1}{\gamma - 1} \frac{dP}{d\rho}\), while \(dP/d\rho\) is the square of the sound speed.

In fact if we consider the situation in the \((X, Z)\)-plane at \(P : X = X_1, Z = Z_1\) with \(0 < X_1 < \sqrt{Q_-}, Z_1 = f(X_1)\) where
\[
f(X) := \frac{\sqrt{g(X^2)}}{\lambda - \kappa X^2} \text{ with } \lambda = \frac{R_0}{R}.
\]
By a straight calculation, it can be shown that \(Df(X) < 0\) for \(0 < X < \sqrt{Q_-}\). We have
\[
\mathbf{n} = \frac{1}{1 + Df(X_1)^2} \left( - Df(X_1) \frac{\partial}{\partial X} + \frac{\partial}{\partial Z} \right)
\]
so that
\[
(\nabla \mathcal{Y}^0|\mathbf{n}) = \frac{1}{1 + Df(X_1)^2} \left( - \frac{\partial \mathcal{Y}}{\partial X} Df(X_1) + \frac{\partial \mathcal{Y}}{\partial Z} \right)
\]
\[
= \frac{1}{1 + Df(X_1)^2} \frac{GM_0}{R_0} \left( X Df(X) - f(X) \right) \frac{\left( \lambda^2 + f(X)^2 \right)^{3/2}}{\lambda^2 + f(X)^2} - 2\kappa X Df(X) \bigg|_{X = X_1}.
\]
Since \(0 < X_1, 0 < f(X_1), Df(X_1) < 0, 0 < \kappa\), we see \((\nabla \mathcal{Y}^0|\mathbf{n}) < 0\) at \(P\). The exceptional cases, the North Pole \((X = 0)\) and the Equator \((X = \sqrt{Q_-})\), can
be checked easily.

Now we consider a not static stationary solution

$$\Upsilon = GM_0 \left( \frac{1}{r} - \frac{1}{R} \right) + B(\varpi), \quad v = \varpi \omega(\varpi) e_\phi$$  

(2.30)

under the assumption:

(B): The function $\omega$ belongs to the class $C^1([0, +\infty]) \cap L^\infty(0, +\infty)$.

Under this assumption, we put

$$\tilde{\kappa}(X^2) = \frac{R_0}{GM_0} B(R_0 X) \frac{1}{X^2} \int_0^{R_0 X} (\omega(\varpi) + \Omega)^2 \varpi d\varpi.$$  

(2.31)

Note $X^2 \mapsto \tilde{\kappa}(X^2) X^2$ is continuous and monotone nondecreasing.

Put

$$\kappa := \sup_{X^2 > 0} \tilde{\kappa} = \frac{R_0^3}{2GM_0} \| \omega + \Omega \|^2,$$  

(2.32)

where

$$\| \omega + \Omega \| = \sup_{\varpi > 0} | \omega(\varpi) + \Omega |.$$  

(2.33)

Let us consider the stationary solution with compactly supported $\rho$ under the assumption

(\bar{K}): It holds that

$$R_0 < R, \quad \left( \frac{R}{R_0} \right)^3 \kappa = \frac{R_0^3}{2GM_0} \| \omega + \Omega \|^2 < \frac{4}{27}.$$  

(2.34)

The density is given by

$$\rho = \left( \frac{\gamma - 1}{A\gamma} \right)^{\frac{1}{\gamma - 1}} (Y^{\bullet})^{-\frac{1}{\gamma - 1}} \quad \text{with} \quad Y^{\bullet} = (Y \lor 0) \cdot 1_{\varpi < 3R/2}.$$  

(2.35)

In fact, $\{ Y > 0 \}$ is $\{ F(X^2, Z^2; \tilde{\kappa}) > \lambda = \frac{R_0}{R} \}$, where

$$F(X^2, Z^2; \tilde{\kappa}) = \frac{1}{\sqrt{X^2 + Z^2}} + \tilde{\kappa}(X^2) X^2.$$  

(2.36)

Here we are using the change of variable $\varpi = R_0 X, z = R_0 Z$.

Then $F(X^2, Z^2; \tilde{\kappa}) > \lambda$ if and only if either 1) $\tilde{\kappa}(X^2) X^2 \geq \lambda$, or 2)

$$\tilde{\kappa}(X^2) X^2 < \lambda \quad \text{and} \quad Z^2 < f(X^2; \tilde{\kappa})$$

where

$$f(X^2; \tilde{\kappa}) = \frac{1}{(\lambda - \tilde{\kappa}(X^2) X^2)^2} - X^2.$$  

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Note that \( f(X^2; \tilde{\kappa}) \leq f(X^2; \kappa) \). Now we are supposing
\[
\tilde{\kappa}(X^2) \leq \frac{4}{27} \left( \frac{R_0}{R} \right)^3.
\]

So, \( \{ T > 0 \} \) has a bounded connected component of the form \( \{ 0 < X^2 < Q_-, Z^2 < f(X^2, \tilde{\kappa}) \} \). Here \( Q_- \) is a positive number \( \leq Q_-(\frac{\lambda^3}{\kappa}) < \frac{1}{2} \lambda \) such that \( f(X^2; \tilde{\kappa}) > 0 \) for \( 0 < X^2 < Q_- \), and \( f(Q_-; \tilde{\kappa}) = 0 \).

**Remark 1** In this situation with \( R(> R_0) \) being fixed, it is sufficient that the function \( \omega \) is given on the finite interval \( [0, 3R/2] \) as a function of \( C^1 \)-class on this interval, under a weaker assumption than (B), and we can put
\[
\| \omega + \Omega \|_\infty = \sup_{0 \leq \tau \leq 3R/2} | \omega(\tau) + \Omega |
\]
instead of (2.33).

### 3 Description of the flow by the Lagrangian coordinate

Let \( D_0 \) be an admissible domain with cover \( \tilde{D}_0 \), and let \( v \) be a classical admissible velocity field on \( [0, T_0] \times D_0 \). Let \( D \) with a cover \( \tilde{D} \) be an admissible subdomain of \( D \), which we shall call a proper admissible subdomain of \( D \), such that \( \tilde{D} \subset D_0 \), namely, there is a compact set \( K \) such that \( \text{Cl} \tilde{D} \subset \text{Int} K \subset K \subset \tilde{D}_0 \).

Then there is a positive number \( T(< T_0) \) such that for any \( (\tau, \bar{x}) \in [0, T] \times \tilde{D} \) the initial value problem of the ordinary differential equation
\[
\begin{align*}
\frac{dx}{dt} &= v(t, x), \\
\bar{x}|_{t=\tau} &= \bar{x}
\end{align*}
\]
(3.1a)
(3.1b)

admits the unique solution \( x = \varphi(t, \tau, \bar{x}) \) which exists on \( t \in [0, T] \) while \( x \in \tilde{D}_0 \). We shall call \( \varphi \) the flow associated with the velocity field \( v \). The flow \( \varphi \), as a function, belongs to the class \( C^1([0, T] \times [0, T] \times \text{Cl} \tilde{D} : \tilde{D}_0) \).

In fact the fundamental theorems of ordinary differential equations (see e.g., [6]) applied to the vector field \( v \) considered as a \( C^1 \)-field on \( [0, T_0] \times D_0 \) guarantee the existence and uniqueness of the flow on \( [0, T] \) valued in \( D_0 \) for \( x \in \text{Int} K \). On the other hand the boundary condition (1.9) of \( v \) on \( \partial D_0 \) guarantees that \( \partial D_0 \) is an invariant set of the equation so that the flow starting with \( \bar{x} \in D \) cannot touch \( \partial D_0 \) and must remain inside of \( \tilde{D}_0 \), namely, \( x \in D \setminus \partial D_0 = D \).

Of course, abstractly speaking, we can consider the existence domain for \( \varphi \) of the form \( \mathcal{O} = \bigcup_{(\tau, \bar{x}) \in [0, T_0] \times \tilde{D}_0} I_{\tau, \bar{x}} \times \{ \tau \} \times \{ \bar{x} \} \), where \( I_{\tau, \bar{x}} \subset [0, T_0] \) is the
maximal interval of existence of the solution of (3.1a) (3.1b). The above observation says that $[0, T] \subset I_{\tau, \bar{x}}$ uniformly for $\forall (\tau, \bar{x}) \in [0, T] \times \bar{D}$.

Hereafter, we consider $\varphi(t, 0, \bar{x})$ and denote, for the sake of brevity,

$$\varphi(t, \bar{x}) := \varphi(t, 0, \bar{x}).$$

Thus

$$\frac{\partial}{\partial t} \varphi(t, \bar{x}) = v(t, \varphi(t, \bar{x})), \quad (3.2a)$$

$$\varphi(0, \bar{x}) = \bar{x}, \quad (3.2b)$$

for $(t, \bar{x}) \in [0, T] \times \bar{D}$.

Now let us suppose that the velocity field $v$ is that of a classical $(T, v)$-solution $(\bar{T}, \bar{v})$ on $[0, T_0] \times \bar{D}_0$. We consider the associated flow $\varphi \in C^1([0, T] \times \bar{D})$ which satisfies (3.2a) for $\forall (t, \bar{x}) \in [0, T] \times \bar{D}$.

Hereafter we denote

$$(D_x\varphi)(t, \bar{x}) = D\varphi(t, \bar{x}) = \left(\frac{\partial}{\partial t} \varphi(t, \bar{x})\right)_{j, \alpha} \quad (3.3)$$

We shall use the inverse matrix of $D_x\varphi(t, \bar{x})$. Actually there is a positive number $T_1(\leq T)$ such that $D_x\varphi(t, \bar{x})$ is invertible when $(t, \bar{x}) \in [0, T_1] \times \bar{D}$. (Proof. Let

$$\Delta = D\varphi(t, \bar{x}) - I = D\varphi(t, \bar{x}) - D\varphi(t, \bar{x}).$$

Since $\varphi \in C^1([0, T] \times \bar{D})$, for any $\Theta, 0 < \Theta < 1$, there is $T_1 \leq T$ such that $\|\Delta\| < \Theta$ for $(t, \bar{x}) \in [0, T_1] \times \bar{D}$. Then $(D\varphi(t, \bar{x}))^{-1}$ is given by the Neumann series $\sum_{k=0}^{\infty} (-\Delta)^k$. □

If we denote by $\psi_t$ the inverse mapping of $\varphi(t, \cdot) \mid_{\bar{D}}$, $\bar{D}$ being a neighborhood of $\bar{x}_0 \in \bar{D}, t \in [0, T_1]$, then for $\bar{x} \in \bar{D}, (D_x\varphi(t, \bar{x}))^{-1}$ is nothing but the matrix

$$\left(\frac{\partial}{\partial x_j} (\psi_t(x))^\alpha\right)_{\alpha, j}$$

at $x = \varphi(t, \bar{x})$.

We note that it holds

$$D_x\varphi(t, \bar{x}) = D\varphi(t, \bar{x}) = \exp \left[ \int_0^t (D_x v)(s, \varphi(s, \bar{x})) ds \right] \quad (3.4)$$

for $(t, \bar{x}) \in [0, T] \times \bar{D}$. Here, of course, $D_x v$ stands for the matrix $\left(\frac{\partial v^k}{\partial x^j}\right)_{k, j}$.

In fact, we see

$$\frac{\partial}{\partial t} D_x \varphi(t, \bar{x}) = \frac{\partial}{\partial t} D\varphi(t, \bar{x}) =$$

$$= D_x \frac{\partial}{\partial t} \varphi(t, \bar{x}) = D_x v(t, \varphi(t, \bar{x})) =$$

$$= D_x v(t, \bar{x}) \bigg|_{x = \varphi(t, \bar{x})} . D_x \varphi(t, \bar{x}).$$

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and 

\[ D_{\bar{x}}(0, \bar{x}) = I. \]

Let us put

\[ \overline{\Upsilon}(x) = \Upsilon(0, x). \tag{3.5} \]

and

\[ \Upsilon^L(t, \bar{x}) = \Upsilon(t, \varphi(t, \bar{x})) \tag{3.6} \]

for \((t, \bar{x}) \in [0, T] \times \bar{D} \).

Integrating the equation of continuity, we claim

**Proposition 1** It holds that

\[ \Upsilon^L(t, \bar{x}) = \Upsilon(t, \varphi(t, \bar{x})) = \overline{\Upsilon}(\bar{x}) \det D\varphi(t, \bar{x}) - (\gamma - 1) \int_0^t (\nabla_{\bar{x}}|v)(t', \varphi(t', \bar{x})) dt'. \tag{3.7} \]

for \(t \in [0, T], \bar{x} \in \bar{D} \).

Proof. The equation (1.11a) reads

\[ \frac{D}{Dt} \log \Upsilon = - (\gamma - 1)(\nabla_{\bar{x}}|v)(t, x) \]

where \(\Upsilon \neq 0\). Therefore

\[ \Upsilon(t, x) = \overline{\Upsilon}(\bar{x}) \exp \left[ - (\gamma - 1) \int_0^t (\nabla_{\bar{x}}|v)(t', \varphi(t', \bar{x})) dt' \right] \]

We note

\[ (\nabla_{\bar{x}}|v)(t, x) = \text{tr} D_{\bar{x}} v(t, x). \]

But, since

\[ \left( \frac{\partial}{\partial t} \right)_{\bar{x}} D\varphi(t, \bar{x}) = D_{\bar{x}} \left( \frac{\partial}{\partial t} \right)_{\bar{x}} x(t, 0, \bar{x}) = D_{\bar{x}} v(t, \bar{x}) \]

\[ = D_{\bar{x}} v(t, \bar{x}) \cdot D_{\bar{x}} x = D_{\bar{x}} v(t, \bar{x}) \cdot D\varphi(t, \bar{x}), \]

we have

\[ D_{\bar{x}} v(t, \bar{x}) = \left( \frac{\partial}{\partial t} \right)_{\bar{x}} D\varphi(t, \bar{x}) \cdot D\varphi(t, \bar{x})^{-1}. \]

Thus

\[ (\nabla_{\bar{x}}|v)(t, x) = \text{tr} \left( \left( \frac{\partial}{\partial t} \right)_{\bar{x}} D\varphi(t, \bar{x}) \cdot D\varphi(t, \bar{x})^{-1} \right) = \left( \frac{\partial}{\partial t} \right)_{\bar{x}} \log \det D\varphi(t, \bar{x}). \]

Since \(D\varphi(0, \bar{x}) = I\), it follows that

\[ \int_0^t (\nabla_{\bar{x}}|v)(t', \varphi(t', \bar{x})) dt' = \log \det D\varphi(t, \bar{x}). \]
Let us consider the equation of motion (3.11b), namely

$$\frac{Dv}{Dt} + Bv + \text{grad}_x(\Upsilon + \Phi^{(\Omega)}) = 0,$$  

where we denote

$$Bv = 2\Omega \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \times v = 2\Omega \begin{bmatrix} -v^2 \\ v^1 \\ 1 \\ 0 \end{bmatrix} = 2\Omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} v.$$  

We put

$$v^L := \left( D\varphi(t, \bar{x}) \right)^{-1} v(t, \varphi(t, \bar{x})) = \left( D\varphi(t, \bar{x}) \right)^{-1} \frac{\partial}{\partial t} \varphi(t, \bar{x}).$$  

Then (3.11) reads

$$\frac{\partial v^L}{\partial t} + (D_x x)^{-1} \frac{\partial}{\partial t}(D_x x)v^L + (D_x x)^{-1} B(D_x x)v^L + (D_x x)^{-1}((D_x x)^{-1})^\top \text{grad}_x(\Upsilon^L + \Phi^{(\Omega)L}) = 0$$

on $(t, \bar{x}) \in [0, T] \times \bar{D}$. Here

$$\Phi^{(\Omega)L}(t, \bar{x}) := \Phi^{(\Omega)}(\varphi(t, \bar{x})).$$

Let us consider the boundary condition

$$(n(x)|v(t, x)) = 0 \quad \forall (t, x) \in [0, T] \times \partial \mathcal{B}_0,$$  

where

$$n(x) = \frac{1}{R_0} x \quad \text{for} \quad x \in \partial \mathcal{B}_0 = \{\|x\| = R_0\}.$$  

Put

$$n^L(t, \bar{x}) = \frac{1}{R_0} (D_x x(t, \bar{x}))^\top \varphi(t, \bar{x})$$

for $(t, \bar{x}) \in [0, T] \times \partial \mathcal{B}_0$. Then (3.13) reads

$$(n^L(t, \bar{x})|v^L(t, \bar{x})) = 0 \quad \text{for} \quad (t, \bar{x}) \in [0, T] \times \partial \mathcal{B}_0.$$
Note that (3.16) means
\[
\frac{1}{2} \frac{\partial}{\partial t} \|\varphi(t, \bar{x})\|^2 = \left( \varphi \frac{\partial}{\partial t} \varphi \right) = 0 \quad \text{if } \bar{x} \in \partial \mathcal{B}_0,
\]
so that, if \( \bar{x} \in \partial \mathcal{B}_0 \), then \( \varphi(t, \bar{x}) \in \partial \mathcal{B}_0 \) for \( \forall t \).

Now let us suppose that there is a vector field \( \varphi(t, \bar{x}) \) for \( (t, \bar{x}) \in [0, T] \times \hat{\mathcal{D}} \), such that \( \varphi \in \bigcap_{\ell=1,2} C^{2-\ell}([0, T]; C^{\ell}(\text{Cl}{\hat{\mathcal{D}}})) \) and
\[
\varphi(0, \bar{x}) = \bar{x}
\]
which satisfies the equation (3.11), where we read
\[
D_{\bar{x}} \mathbf{x} = D_{\varphi}(t, \bar{x}), \quad (3.18a)
\]
\[
v^L = \left( D_{\varphi}(t, \bar{x}) \right)^{-1} \frac{\partial}{\partial t} \varphi(t, \bar{x}), \quad (3.18b)
\]
\[
\Upsilon^L = \Upsilon(\bar{x}) \det \left( D_{\varphi}(t, \bar{x}) \right)^{-(\gamma-1)}, \quad (3.18c)
\]
\[
\Phi^{(\ell)}(\Upsilon) = \Phi^{(\ell)}(\varphi(t, \bar{x})), \quad (3.18d)
\]
and the boundary condition (3.16), where we read
\[
n^L(t, \bar{x}) = \frac{1}{R_0} (D_{\varphi}(t, \bar{x}))^\top \varphi(t, \bar{x}), \quad (3.19)
\]
forgetting that the vector field \( \varphi \) comes from the flow generated by the velocity field \( v \) of a classical solution \( (T, v) \). Namely, we use the following definitions:

**Definition 6** Let \( \mathcal{D} \) be an admissible domain with cover \( \hat{\mathcal{D}} \). A scalar field \( \Upsilon \) defined on \( \hat{\mathcal{D}} \) is called an admissible \( \Upsilon \)-data, if \( \Upsilon \in C^1(\text{Cl}{\hat{\mathcal{D}}}) \) and \( \Upsilon > 0 \) on \( \partial \mathcal{B}_0 \).

**Definition 7** Let \( \mathcal{D} \) be an admissible domain with cover \( \hat{\mathcal{D}} \), and let \( \Upsilon \) be an admissible \( \Upsilon \)-data on \( \mathcal{D} \). A vector field \( \varphi \) is said to be an admissible flow on \( [0, T] \times \mathcal{D} \) associated with \( \Upsilon \), if 1) \( \varphi \in \bigcap_{\ell=1,2} C^{2-\ell}([0, T]; C^{\ell}(\text{Cl}{\hat{\mathcal{D}}})) \), 2) the equation (3.11) and the boundary condition (3.16) are satisfied on \( [0, T] \times \mathcal{D} \) and on \( [0, T] \times \partial \mathcal{B}_0 \), and 3) the initial condition
\[
\varphi(0, \bar{x}) = \bar{x} \quad \text{for } \forall \bar{x} \in \hat{\mathcal{D}}
\]
holds.

As for the existence of the inverse mapping of \( \varphi(t, \cdot) \), we have

**Proposition 2** Let \( \varphi \in \bigcap_{\ell=1,2} C^{2-\ell}([0, T]; C^{\ell}(\text{Cl}{\hat{\mathcal{D}}})) \) be a vector field such that
\[
\varphi(0, \bar{x}) = \bar{x} \quad \text{for } \forall \bar{x} \in \hat{\mathcal{D}},
\]
and
Let $\Omega, \Omega_0, \Omega_1$ be connected open subsets of $\mathbb{R}^3$ such that

$$\mathcal{B}_0 \subset \Omega_0 \subset \Omega \subset \Omega_1 \subset \breve{\Omega}$$

(3.20)

and $\Omega_1$ is convex. Then for a sufficiently small $T_1(\ll T)$ there is a mapping $\psi(t, \cdot) : \Omega \to \Omega_1$, $t$ being $\in [0, T_1]$, such that

$$x = \varphi(t, \psi(t, x))$$

(3.21)

for $\forall (t, x) \in [0, T_1] \times \Omega$. Moreover

$$\Omega_0 \subset \psi(t, \Omega) = \{ \psi(t, x) | x \in \Omega \}$$

(3.22)

for $\forall t \in [0, T_1]$.

Proof. Fixing $t$, we are going to solve the equation for unknown $\bar{x}$

$$x = \varphi(t, \bar{x})$$

for given $x$. Writing

$$F(\bar{x}) = x + \bar{x} - \varphi(t, \bar{x}),$$

we convert the equation to the fixed point problem

$$\bar{x} = F(\bar{x}).$$

Note

$$\| F(\bar{x}) - x \| = \| \bar{x} - \varphi(t, \bar{x}) \| = o(1),$$

$$\| DF(\bar{x}) \| = \| I - D\varphi(t, \bar{x}) \| = o(1)$$

uniformly for $\bar{x} \in \Omega_1$ as $t \to 0$, since $\varphi(0, \bar{x}) = \bar{x}$. Therefore, if $x \in \Omega$, we see $F(\bar{x}) \in \Omega_1$ for $\bar{x} \in \Omega_1$ and $\| F(\bar{x}') - F(\bar{x}) \| \leq \Theta \| \bar{x}' - \bar{x} \|$ for $\bar{x}', \bar{x} \in \Omega_1$, $\Theta$ being $\in [0, 1]$, provided that $0 \leq t \leq T_1 \ll 1$. Here recall that $\Omega_1$ is supposed to be convex. Therefore the fixed point problem admits the unique solution $\bar{x} = \psi(t, x) \in \Omega_1$ for $(t, x) \in [0, T_1] \times \Omega$. Then

$$x = \varphi(t, \psi(t, x))$$

for $(t, x) \in [0, T_1] \times \Omega$.

Since $\mathcal{C} \mathcal{O}_0 \subset \breve{\Omega}$, taking $T_1$ smaller if necessary, we can assume that $\varphi(t, \bar{x}) \in \breve{\Omega}$ if $(t, \bar{x}) \in [0, T_1] \times \mathcal{O}_0$. Thus, if $x \in \mathcal{O}_0$, there is $x \in \breve{\Omega}$ such that $\bar{x} = \psi(t, x)$ provided that $t \in [0, T_1]$. □

In this situation, we can claim

**Theorem 2** Let $\Omega$ be an admissible domain with cover $\breve{\Omega}$, $\breve{\varrho}$ be an admissible $\breve{\varrho}$-data on $\breve{\Omega}$, and $\varphi$ be an admissible flow associated with $\breve{\varrho}$. Let $T_1, \Omega$ be those...
of Proposition 2. Put

$$\Upsilon(t, x) = \left. \psi(t, x) \left( \frac{\det D\varphi(t, \bar{x})}{\det D\varphi(t, \bar{x})} \right)^{-(\gamma - 1)} \right|_{\bar{x} = \psi(t, x)}$$  \hspace{1cm} (3.23)

$$v(t, x) = \left. \frac{\partial}{\partial t} \varphi(t, \bar{x}) \right|_{\bar{x} = \psi(t, x)}$$  \hspace{1cm} (3.24)

for \((t, x) \in [0, T_1] \times \mathcal{O}\). Then \((\Upsilon, v)\) is a classical \((\Upsilon, v)\)-solution on \([0, T_1] \times \mathcal{O}\), where \(\mathcal{O} = \mathcal{O} \setminus \mathcal{B}_0\).

• Now let us derive the linearized approximation of (3.7), (3.11), (3.16). Namely, we fix a stationary solution \((\bar{\Upsilon}, 0)\), say, we take

$$\bar{\Upsilon}(\bar{x}) = \Phi(\bar{x}) - \frac{G M_0}{R}$$  \hspace{1cm} (3.25)

under the assumption (K). Let us denote

$$\mathcal{R} := \{ \bar{T} > 0, \bar{\varpi} < 3R/2, r > R_0 \} = \{ \bar{\rho} > 0, r > R_0 \}.$$  \hspace{1cm} (3.26)

Considering small \(\varepsilon\), a quantity \(Q\) will denoted by \(O(\varepsilon)\) if \(Q\) and its derivatives are of order \(O(\varepsilon)\) uniformly on each bounded interval of \(t\). We assume that \(\Upsilon - \bar{\Upsilon}, v\) are of \(O(\varepsilon)\).

Then (3.4) shows

$$D_{\bar{x}} x(t, \bar{x}) = I + \int_0^t (D_{\bar{x}} x(s, \varphi(s, \bar{x}))) ds + O(\varepsilon^2)$$  \hspace{1cm} (3.27)

$$= I + O(\varepsilon),$$  \hspace{1cm} (3.28)

$$D_{\bar{x}} \bar{x}(t, x) = I + O(\varepsilon),$$  \hspace{1cm} (3.29)

and so on.

The equation (3.7) gives

$$\Upsilon^L(t, \bar{x}) = \left. \left( 1 - (\gamma - 1) \text{div}_{\bar{x}} (\varphi(t, \bar{x}) - \bar{x}) \right) \right|_{\bar{x} = \psi(t, x)} + O(\varepsilon^2).$$  \hspace{1cm} (3.30)

Recalling (3.25), we have

$$\Upsilon^L(t, \bar{x}) + \Phi^{(1)}(\varphi(t, \bar{x})) = \Upsilon^L(t, \bar{x}) - \bar{\Upsilon}_L(\bar{x}) - \frac{GM_0}{R}. \hspace{1cm} (3.31)$$

Here we define \(\bar{\Upsilon}_L\) by

$$\bar{\Upsilon}_L(\bar{x}) = \bar{\Upsilon}(\varphi(t, \bar{x})).$$  \hspace{1cm} (3.32)

Then

$$\bar{\Upsilon}_L(\bar{x}) = \bar{\Upsilon}(\bar{x}) + (\text{grad}_{\bar{x}} \bar{\Upsilon}_L(\bar{x})) |_{\varphi(t, \bar{x})} + O(\varepsilon^2).$$  \hspace{1cm} (3.33)
Summing up, putting
\[ G(t, \bar{x}) := \Upsilon(t, \varphi(t, \bar{x}) + \Phi^{(\bar{\nu})} (\varphi(t, \bar{x})) + \frac{GM_0}{R}, \]
\[ = \Upsilon^L(t, \bar{x}) - \bar{\Upsilon}_L(\bar{x}), \] (3.34)
we see
\[ G(t, \bar{x}) = \bar{\Upsilon}(\bar{x}) - (\gamma - 1)\bar{\Upsilon}(\bar{x}) \text{div}_{\bar{x}} (\varphi(t, \bar{x}) - \bar{x}) + \]
\[ - (\text{grad}_{\bar{x}} \bar{\Upsilon}_L(\bar{x})|\varphi(t, \bar{x}) - \bar{x}) + O(\varepsilon^2). \] (3.35)

Let a constant vector field \( \hat{\xi}(\bar{x}) = O(\varepsilon) \) on \( \mathcal{R} \) be given arbitrarily. We take \( \bar{\Upsilon} \) given by
\[ \bar{\Upsilon}(\bar{x}) = \varphi(t, \bar{x}) - \bar{x} - (\gamma - 1)\bar{\Upsilon}(\bar{x}) \text{div}_{\bar{x}} \hat{\xi} - \text{grad}_{\bar{x}} \bar{\Upsilon}_L(\bar{x})|\hat{\xi}. \] (3.36)

Then it turns out to be that
\[ G(t, \bar{x}) = -(\gamma - 1)\bar{\Upsilon}(\bar{x}) \text{div}_{\bar{x}} \bar{\Upsilon} - (\text{grad}_{\bar{x}} \bar{\Upsilon}_L(\bar{x})|\bar{\Upsilon}) + O(\varepsilon^2), \] (3.37)
\[ = -(\gamma - 1)\bar{\Upsilon}(\bar{x}) \text{div}_{\bar{x}} \hat{\xi} - (\text{grad}_{\bar{x}} \bar{\Upsilon}(\bar{x})|\hat{\xi}) + O(\varepsilon^2), \] (3.38)
where
\[ \hat{\xi}(t, \bar{x}) = \varphi(t, \bar{x}) - \bar{x} + \hat{\xi}(\bar{x}) \] (3.39)
and \( \text{grad}_{\bar{x}} \bar{\Upsilon}(\bar{x}) \) means \( \left( \text{grad}_{\bar{x}} \bar{\Upsilon}(\bar{x}) \right)|_{\bar{x} = \bar{x}} \) so that
\[ \text{grad}_{\bar{x}} \bar{\Upsilon}_L(\bar{x}) = (D_{\bar{x}} \varphi(t, \bar{x})) \text{grad}_{\bar{x}} \bar{\Upsilon}(\bar{x}) \]
\[ = \text{grad}_{\bar{x}} \bar{\Upsilon}(\bar{x}) + O(\varepsilon). \]

Then we see
\[ G(t, \bar{x}) = -\bar{\sigma}(\bar{x}) \text{div}_{\bar{x}} (\bar{\rho}(\bar{x})\hat{\xi}(t, \bar{x})) + O(\varepsilon^2) \text{ on } \mathcal{R}, \] (3.40)
where
\[ \sigma = \frac{d\Upsilon}{d\rho} = A\gamma \rho^{-2}. \] (3.41)

Moreover we introduce
\[ \xi(t, \bar{x}) := (D\varphi(t, \bar{x}))^{-1}\hat{\xi}(t, \bar{x}) - \int_0^t \frac{\partial}{\partial s} \left[ (D\varphi(s, \bar{x}))^{-1} \right] \hat{\xi}(s, \bar{x}) ds. \] (3.42)

Then we have
\[ \xi = \hat{\xi} + O(\varepsilon^2), \] (3.43)
\[
\frac{\partial \xi}{\partial t} = v^L, \quad (3.44)
\]
and
\[
\xi(0, \bar{x}) = \xi(\bar{x}) \quad (3.45)
\]

and
\[
G(t, \bar{x}) = -\bar{\sigma}(\bar{x}) \text{div}_\mathbb{R}(\bar{\rho}(\bar{x})\xi(t, \bar{x})) + O(\varepsilon^2) \quad (3.46)
\]

The approximation of the equation of motion (3.11) is clearly given by
\[
\frac{\partial v^L}{\partial t} + Bv^L + \text{grad}_\mathbb{R}G = O(\varepsilon^2). \quad (3.47)
\]

Inserting (3.44), (3.46) into (3.47), we have
\[
\frac{\partial^2 \xi}{\partial t^2} + B\frac{\partial \xi}{\partial t} + \text{grad}_\mathbb{R}\left(-\bar{\sigma}(\bar{x}) \text{div}_\mathbb{R}(\bar{\rho}(\bar{x})\xi)\right) = O(\varepsilon^2). \quad (3.48)
\]

The approximation of the boundary condition (3.16) is given by
\[
\left(n(\bar{x}) \bigg| \frac{\partial \xi}{\partial t}(t, \bar{x}) \right) = 0 \quad \text{for} \quad (t, \bar{x}) \in [0, T] \times \partial \mathcal{B}_0. \quad (3.49)
\]

In fact we have
\[
n^L = \frac{1}{R_0} (I + O(\varepsilon))(\bar{x} + O(\varepsilon)) = n(\bar{x}) + O(\varepsilon)
\]
and
\[
v^L = \frac{\partial \xi}{\partial t} = O(\varepsilon).
\]

Of course (3.49) is equivalent to
\[
\left(n(\bar{x}) | \xi(t, \bar{x}) \right) = 0 \quad \text{for} \quad (t, \bar{x}) \in [0, T] \times \partial \mathcal{B}_0, \quad (3.50)
\]
provided that
\[
\left(n(\bar{x}) | \xi(\bar{x}) \right) = 0 \quad \text{on} \quad \partial \mathcal{B}_0. \quad (3.51)
\]

Now the domain \(\mathcal{R} = \{\bar{\rho} > 0\} = \{\bar{T} > 0\} \cap \{\bar{x} < 3R/2\}\) has the form
\[
\mathcal{R} = \{R_0 < r < R \cdot H(\zeta^2; \kappa, \lambda), \quad -1 \leq \zeta := \frac{z}{r} \leq 1\},
\]
where \(\zeta^2 \mapsto H(\zeta^2; \kappa, \lambda)\) is a smooth monotone function on \([0, 1]\) such that
\(1 = H(1; \kappa, \lambda) \leq H(0; \kappa, \lambda)\). (Note \(H(1) = H(0) \iff \kappa = 0\), that is, \(\Omega = 0\).)
Given a small $\xi$, we consider $\widetilde{\mathcal{Y}}$ determined by (3.36). Of course $\widetilde{\mathcal{Y}} - \mathcal{Y}$ is small, but the topology of $\{\mathcal{Y} > 0\} \cap \{\varpi < 3R/2\}$ is not clear, generally speaking. In fact, as for $\{\mathcal{Y} > 0\} = \left\{ \left( 1 - (\gamma - 1)(\nabla|^\mathcal{Y}|_\xi) \right) \mathcal{Y} > (\nabla|^\varpi|_\xi) \right\}$, we have that $\nabla|^\mathcal{Y}|/\mathcal{Y}$ may diverge along the vacuum boundary $\Sigma_1 = \{ \mathcal{Y} = 0, \varpi < 3R/2 \}$ of $\varrho$. At least, if, for example, $\tilde{\xi}_1 \in C^\infty_0(\Sigma_0 \cup R)$, then for $\tilde{\xi} = \varepsilon \tilde{\xi}_1$ with $\varepsilon \ll 1$, it is guaranteed that $\{\mathcal{Y} > 0\} \cap \{\varpi < 3R/2\} = R(= \{\mathcal{Y} > 0\} \cap \{\varpi < 3R/2\})$. More generally, if $\frac{1}{\varrho} (\nabla|\varrho \tilde{\xi}_1|) = (\nabla \log \varrho \tilde{\xi}_1) + (\nabla|^\mathcal{Y}|_\xi) \subset \frac{1}{\gamma - 1} (\nabla \log \mathcal{Y} \tilde{\xi}_1) + (\nabla|^\varpi|_\xi)$ is bounded on a neighborhood of the vacuum boundary $\Sigma_1$ of $\varrho$, then $\varrho > 0 \Leftrightarrow \mathcal{Y} > 0$ there for $\tilde{\xi} = \varepsilon \tilde{\xi}_1$ with $\varepsilon \ll 1$.

**Historical Remark:** The derivation of the linearized approximation of the equations in Lagrangian co-ordinate system can be found [14, Sect. 56], [1, pp. 139-140,], [2, p.11, (A)], [16], [13, p.500, (1)] and so on. But there was considered only the case of $\varrho = \varrho$ and $\xi = 0$.

### 4 Linearized equations for perturbations from a static stationary solution

Let us fix a static stationary solution

$$\mathcal{Y} = -\Phi^{(n)} - \frac{GM_0}{R}, \quad \varrho = 0$$

under the assumption (K). We are concerned with the domain

$$\mathcal{R} = \{ \varrho > 0 \} = \{ \mathcal{Y} > 0, \varpi < \frac{3R}{2}, R_0 < r \}.$$

We consider the initial boundary problem

$$\begin{align*}
\frac{\partial^2 \xi}{\partial t^2} + B \frac{\partial \xi}{\partial t} + L \xi &= 0 \quad \text{on } [0, +\infty) \times \mathcal{R}, \\
(n |\xi|) &= 0 \quad \text{on } [0, +\infty) \times \Sigma_0, \\
\xi \big|_{t=0} &= \tilde{\xi}(x), \quad \frac{\partial \xi}{\partial t} \big|_{t=0} = \tilde{\nu}(x) \quad \text{on } \mathcal{R}.
\end{align*}$$

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Here

\[ Bv = 2\Omega Jv, \quad J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]  \hspace{1cm} (4.3)

\[ L\xi = \nabla G, \quad G = -\frac{dT}{d\rho}g, \quad g = (\nabla |\bar{\rho}\xi), \]  \hspace{1cm} (4.4)

\[ n = \frac{x}{R_0} \]  \hspace{1cm} (4.5)

and \( \xi, \bar{v} \) are given vector fields on \( \mathcal{R} \) such that \((n|\xi) = (n|\bar{v}) = 0\) on \( \Sigma_0 \), while \( \Sigma_0 = \partial \mathcal{B}_0 = \{ r = R_0 \} \).

Here and hereafter the Lagrangian coordinate, denoted by \( \bar{x} \) in the previous section, is denoted by \( x \), since we do not refer the Eulerian coordinate so far.

We consider the differential operator \( L \) in the Hilbert space \( \mathcal{H} \) of all measurable functions \( \xi \) defined on \( \mathcal{R} \) such that \( \| \xi \|_\mathcal{H} < \infty \), where

\[ \| \xi \|_\mathcal{H}^2 = \int_\mathcal{R} \| \xi(x) \|^2 \bar{\rho}(x) dx, \]  \hspace{1cm} (4.6)

that is, the inner product of \( \mathcal{H} \) is

\[ (\xi_1|\xi_2)_\mathcal{H} = \int_\mathcal{R} (\xi_1(x)|\xi_2(x))\bar{\rho}(x) dx. \]  \hspace{1cm} (4.7)

Of course

\[ (\xi_1(x)|\xi_2(x)) := \sum_k \xi_1^k(x)(\xi_2^k(x))^* \quad \text{for} \quad \xi_\mu(x) = \begin{bmatrix} \xi_1^\mu(x) \\ \xi_2^\mu(x) \\ \xi_3^\mu(x) \end{bmatrix}, \mu = 1, 2. \]

Here and hereafter \( Z^* \) denotes the complex conjugate \( X - iY \) of \( Z = X + iY \), while \( i \) stands for the imaginary unit, \( \sqrt{-1} \).

Briefly speaking, we consider \( \mathcal{H} = L^2(\mathcal{R}, \bar{\rho} dx; \mathbb{C}^3) \).

For \( \xi_\mu \in C_0^\infty(\mathcal{R}), \mu = 1, 2 \), we have

\[ (L\xi_1|\xi_2)_\mathcal{H} = \int_{\mathcal{R}} \frac{dT}{d\rho}g_1g_2^* dx, \quad \text{with} \quad g_\mu = (\nabla |\bar{\rho}\xi_\mu). \]  \hspace{1cm} (4.8)

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So we put
\[ Q(\xi_1, \xi_2) = \int_{\mathbb{R}} \frac{d \Omega}{d \rho} g^*_{\rho} d\mathbf{x} \quad \text{with} \quad g_\mu = (\nabla |\bar{\rho} \xi_\mu), \] (4.9)
and
\[ Q[\xi] = Q(\xi, \xi) = \int_{\mathbb{R}} \frac{d \Omega}{d \rho} |g|^2 d\mathbf{x} \quad \text{with} \quad g = (\nabla |\bar{\rho} \xi). \] (4.10)

We start from the operator \( T_{1c} \) in \( H \) defined by
\[ T_{1c}: \xi \mapsto \mathbf{L}_\xi \] on the domain \( \mathcal{D}(T_{1c}) = C_0^\infty(\mathbb{R}) \). Then \( T_{1c} \) is densely defined, symmetric and bounded from below as
\[ (T_{1c} \xi | \xi) = Q[\xi] \geq 0. \]

Therefore, seeing, e.g., [12, Chapter VI, Section 2.3], we have that \( T_{1c} \) admits the Friedrichs extension \( T_1 \) which is a self-adjoint operator in \( H \). The domain of \( T_1 \) is
\[ \mathcal{D}(T_1) = \{ \xi \in \mathcal{H}_0^{\text{div}} \mid \mathbf{L}_\xi \in \mathcal{H} \} \] in the sense of distribution. \( (4.11) \)

Here \( \mathcal{H}_0^{\text{div}} \) is the Hilbert space of all \( \xi \in \mathcal{H} \) such that there is a sequence \( \varphi_n \in C_0^\infty(\mathbb{R}) \) such that
\[ \varphi_n \rightarrow \xi \quad \text{in} \quad \mathcal{H} \quad \text{as} \quad n \rightarrow \infty, \]
\[ Q[\varphi_n - \varphi_m] \rightarrow 0 \quad \text{as} \quad n, m \rightarrow \infty. \]

In order to fix the idea we use

**Definition 8** We put
\[ \mathcal{H}_1^{\text{div}} = \{ \xi \in \mathcal{H} \mid g = (\nabla |\bar{\rho} \xi) \in L^2(\mathbb{R}; \frac{d \Omega}{d \rho} d\mathbf{x}) \}, \] (4.12)
and regard it as a Hilbert space endowed with the inner product
\[ (\xi_1 | \xi_2)_{\mathcal{H}_1^{\text{div}}} = Q(\xi_1, \xi_2) + (\xi_1 | \xi_2)_{\mathcal{H}}. \] (4.13)

Thus we are saying

**Definition 9** \( \mathcal{H}_0^{\text{div}} \) is the closure of \( C_0^\infty(\mathbb{R}) \) in \( \mathcal{H}_1^{\text{div}} \).

Hereafter we shall denote by \( \mathbf{L} \) the Friedrichs extension \( T_1 \), diverting the letter. So we have

**Theorem 3** The operator \( \mathbf{L} \) is a self-adjoint operator bounded from below by 0 in the Hilbert space \( \mathcal{H} \), whose domain is
\[ \mathcal{D}(\mathbf{L}) = \{ \xi \in \mathcal{H}_0^{\text{div}} \mid \mathbf{L}_\xi \in \mathcal{H} \}. \] (4.14)
Note that $\xi \in D(L) \subset H_0^\text{div}$ enjoys the boundary condition
\[(n|\xi) = 0 \quad \text{on} \quad \Sigma_0 = \{r = R_0\} \tag{4.15}\]
in the following sense: There is known to uniquely exist the ‘normal trace operator’ $\gamma_n$ which maps $\hat{\mathcal{S}}^\text{div}$ into $H^{-1/2}(\Sigma_0)$ continuously such that $\gamma_n \xi = - (n|\xi) \in C^1(\mathcal{R} \cup \partial \mathcal{R})$, and $\gamma_n(\xi) = 0$ when $\xi \in H_0^\text{div}$. (See e.g., [2] Chapter I.)

Here we have used the fact that $\bar{\rho}, \beta$ in the following sense: There is known to uniquely exist the ‘normal trace operator’ on $H$ being a real number. When $\mathcal{B} \{\xi \in H\}$ on the functional spaces $\gamma$ normal trace operator since the above estimates guarantee that the space $\hat{\mathcal{S}}$ bounded linear inverse operator $(\mathcal{B} + \lambda)$ is defined by an arbitrary smooth matrix-valued function $\hat{\mathcal{B}} := \bar{\rho} \xi$, $C$ is a sufficiently large finite positive number, and the theory on the functional spaces $H(\text{div}; \mathcal{R}), H_0(\text{div}; \mathcal{R}), H^{1/2}(\partial \mathcal{R}), H^{-1/2}(\partial \mathcal{R})$ and the normal trace operator $\gamma_n$, seen in [2] can be applied to the vector field $\xi = \bar{\rho} \xi$, since the above estimates guarantee that the space $\hat{\mathcal{S}}^\text{div} := \{\xi \in \hat{\mathcal{S}}^\text{div}\}, \hat{\mathcal{S}}^\text{div} := \{\xi \xi \in \hat{\mathcal{S}}^\text{div}\}$ are continuously imbedded into $H(\text{div}; \mathcal{R}), H_0(\text{div}; \mathcal{R})$.

Therefore, if $\xi(\cdot, \cdot) \in D(L) \quad \forall \tau$, then the boundary condition $\{4.2b\}$ is satisfied in this sense.

As for the operator $J_\cdot: v \mapsto Jv$, it is clear that it is a bounded linear operator on $\bar{\mathcal{S}}$. But, for the sake of generality, we replace $2\Omega$ by the operator $B: v \mapsto Bv$ defined by an arbitrary smooth matrix-valued function $B: x \mapsto (B_j(x))_{i,j} \in C^\infty(\mathcal{R} \cup \partial \mathcal{R}; \mathbb{R}^{3 \times 3})$. Of course $B$ is a bounded operator on $\bar{\mathcal{S}}$, and
\[\|B(v)\|_B \leq \beta \|v\|_B, \tag{4.16}\]
where $\beta = \|B(\cdot)\|_\infty$. The adjoint operator $B^*$ is defined by the transposed matrix $B(x)^\top$. Later we shall use the following densely defined closed operator
\[L + cB, \quad D(L + cB) = D(L), \tag{4.17}\]
c being a real number. When $c \neq 0$, we cannot say that it is self-adjoint, in general, but we can claim the following

Proposition 3 For any $c \in \mathbb{R}$ and $\lambda > |c\beta|$ the operator $L + cB + \lambda$ has the bounded linear inverse operator $(L + cB + \lambda)^{-1}$ defined on the whole space $\bar{\mathcal{S}}$ such that
\[\|L + cB + \lambda\|^{-1}\|_{\mathcal{B}(\mathcal{S})} \leq \frac{1}{\lambda - |c\beta|}, \tag{4.18}\]
Proof. First we see \( L + cB + \lambda \) is invertible. In fact, if
\[
(L + cB + \lambda)\xi = f, \quad \xi \in D(L), \quad f \in \mathcal{H},
\]
then we see
\[
(\lambda - |c\beta|)\|\xi\|_{\mathcal{H}}^2 \leq Q(\xi) + (cB\xi, \xi)_{\mathcal{H}} + \lambda\|\xi\|_{\mathcal{H}}^2 = (\xi, f)_{\mathcal{H}} \leq \|\xi\|_{\mathcal{H}}\|f\|_{\mathcal{H}},
\]
therefore we have
\[
\|\xi\|_{\mathcal{H}} \leq \frac{1}{\lambda - |c\beta|}\|f\|_{\mathcal{H}}.
\]
We claim that the range \( R(L + cB + \lambda) \) is dense in \( \mathcal{H} \). In fact, suppose
\[
((L + cB + \lambda)\xi, f) = 0 \quad \forall \xi \in D(L),
\]
Then
\[
(L\xi, f) = -((cB + \lambda)\xi, f)
\]
\[
= (\xi, -cB^* - \lambda f)
\]
for \( \forall \xi \in D(L) \). Hence \( f \in D(L^*) \) and
\[
L^* f = -cB^* - \lambda f.
\]
Since \( L = L^* \), this means that \( f \in D(L) \) and
\[
(L + cB^* + \lambda) f = 0.
\]
Since \( L + cB^* + \lambda \) is invertible, for \( \|B^T\| = \|B\| = \beta \), it follows that \( f = 0 \). Summing up, we have the assertion. □

We are going to apply the Hille-Yosida theory to the initial-boundary value problem
\[
\frac{\partial^2 \xi}{\partial t^2} + B \frac{\partial \xi}{\partial t} + L \xi = 0,
\]
\[
\xi = \xi_0, \quad \frac{\partial \xi}{\partial t} = \bar{v} = \begin{bmatrix} v^1(0, x) \\ v^2(0, x) \\ v^3(0, x) \end{bmatrix} \quad \text{at} \quad t = 0,
\]
\[
\xi(t, \cdot) \in D(L) \quad \text{for} \quad \forall t \geq 0.
\]
(4.19)
We put

\[ U = \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix}, \quad \dot{\xi} = \frac{\partial \xi}{\partial t}, \tag{4.20} \]

\[ A U = \begin{bmatrix} O & -I \\ L & B \end{bmatrix} U = \begin{bmatrix} -\dot{\xi} \\ B\dot{\xi} + L\xi \end{bmatrix}, \tag{4.21} \]

\[ \mathcal{E} = \mathcal{H}_0^{\text{div}} \times \mathcal{H} \]

with

\[ (U_1|U_2)\mathcal{E} = (\xi_1|\xi_2)_{\mathcal{H}^{\text{div}}} + (\dot{\xi}_1|\dot{\xi}_2)_{\mathcal{H}} = \]

\[ = Q(\xi_1, \xi_2) + (\xi_1|\xi_2)_{\mathcal{H}} + (\dot{\xi}_1|\dot{\xi}_2)_{\mathcal{H}}, \tag{4.22} \]

\[ D(A) = D(L) \times \mathcal{H}_0^{\text{div}}. \tag{4.23} \]

Then the initial-boundary value problem \[4.19\) can be written as

\[ \frac{dU}{dt} + A U = 0, \quad U|_{t=0} = U_0, \tag{4.24} \]

where

\[ U_0 = \begin{bmatrix} \xi \\ 0 \end{bmatrix}. \tag{4.25} \]

Applying \[3\] Theorem 7.4, we can claim

**Proposition 4** If \( U_0 \in D(A) \), say, if \( \xi^{0} \in D(L) \) and \( \dot{v} \in \mathcal{H}_0^{\text{div}} \), then there exists a unique solution \( U \in C^1([0, +\infty[; \mathcal{E}) \cap C([0, +\infty[, D(A)) \) to the problem \[4.24\].

Moreover \( E(t) = \|U(t)\|_{\mathcal{E}}^2 \) enjoys

\[ E(t) \leq e^{2\Lambda t} E(0), \tag{4.26} \]

where \( \Lambda = 1 \lor \beta \).

Here we consider that \( D(A) \) is equipped with the operator norm \( (\|U\|_{\mathcal{E}}^2 + \|A U\|_{\mathcal{E}}^2)^{1/2} \).

Proof of Proposition. Firstly \( A + 1 \lor \beta \) is monotone, that is, for \( \forall U \in D(A) \) we have

\[ \Re[(AU|U)_{\mathcal{E}}] + (1 \lor \beta)\|U\|_{\mathcal{E}}^2 = \Re[-Q(\dot{\xi}, \xi) - (\ddot{\xi}|\xi)_{\mathcal{H}} + (L\xi|\dot{\xi})_{\mathcal{H}} + (B\dot{\xi}|\xi)_{\mathcal{H}}] + 
\]

\[ + (1 \lor \beta)(Q|\xi) + \|\xi\|_{\mathcal{H}}^2 + \|\dot{\xi}\|_{\mathcal{H}}^2) = 
\]

\[ \geq -\Re[(\ddot{\xi}|\xi)_{\mathcal{H}}] - \beta\|\dot{\xi}\|_{\mathcal{H}}^2 + (1 \lor \beta)\|\xi\|_{\mathcal{H}}^2 + (1 \lor \beta)\|\dot{\xi}\|_{\mathcal{H}}^2 
\]

\[ \geq \|\xi\|_{\mathcal{H}}^2 - \Re[(\ddot{\xi}|\xi)_{\mathcal{H}}] + \|\xi\|_{\mathcal{H}}^2 
\]

\[ \geq 0, \]

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since \((L\xi, \dot{\xi}) = Q(\xi, \dot{\xi})^*\) and \(|\Re[(B\dot{\xi}|\xi)]| \leq \beta \|\dot{\xi}\|_{L^2}^2\).

If \(\Lambda > \beta\), then the operator \(A + \Lambda\) has the bounded inverse defined on \(E\).

Actually the equation

\[
\Lambda U + \Lambda U = F = \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{E}
\]

means

\[
\begin{cases}
\dot{\xi} + \Lambda \xi = f \in \mathcal{S}_0^{\text{div}} \\
B\dot{\xi} + L\xi + \Lambda \dot{\xi} = g \in \mathcal{S},
\end{cases}
\]

which can be solved as

\[
\begin{cases}
\xi = (L + \Lambda B + \Lambda^2)^{-1}(Bf + \Lambda f + g) \in \mathcal{D}(L), \\
\dot{\xi} = (L + \Lambda B + \Lambda^2)^{-1}(Bf + \Lambda f + g) - f \in \mathcal{S}_0^{\text{div}},
\end{cases}
\]

thanks to Proposition 3, since \(\Lambda^2 > |\Lambda\beta|\) for \(\Lambda > \beta\). □

Therefore, considering the problem

\[
\frac{\partial^2 \xi}{\partial t^2} + B \frac{\partial \xi}{\partial t} + L\xi = 0,
\]

\(\xi(t, \cdot) \in \mathcal{D}(L)\) for \(\forall t \geq 0\),

\(\xi = \hat{\xi}, \quad \frac{\partial \xi}{\partial t} = \hat{v}\) at \(t = 0\),

we can claim

**Theorem 4** Suppose \(\hat{\xi} \in \mathcal{D}(L)\) and \(\hat{v} \in \mathcal{S}_0^{\text{div}}\). Then the initial-boundary value problem (4.27) admits a unique solution

\[
\xi \in C^2([0, +\infty[, \mathcal{S}) \cap C^1([0, +\infty[, \mathcal{S}_0^{\text{div}}) \cap C([0, +\infty[, \mathcal{D}(L))
\]

and the energy

\[
E(t) = \|\xi\|^2_{\mathcal{S}_0^{\text{div}}} + \|\dot{\xi}\|^2_{\mathcal{S}}
\]

\[
= \|\xi\|^2_{\mathcal{S}} + Q|\xi| + \|\frac{\partial \xi}{\partial t}\|^2_{\mathcal{S}}
\]

enjoys the estimate

\[
\sqrt{E(t)} \leq e^{\Lambda t} \cdot \sqrt{E(0)},
\]

where \(\Lambda = 1 \lor \beta\).
Here \( D(L) \) is equipped with the norm \( (\|\xi\|_{\mathcal{H}^{2\text{div}}}^2 + \|L\xi\|_{\mathcal{H}}^2)^{1/2} \).

Correspondingly we may consider the inhomogeneous initial-boundary value problem
\[
\frac{dU}{dt} + AU = F(t), \quad U|_{t=0} = U_0. \tag{4.30}
\]
We can claim

**Proposition 5** If \( U_0 \in D(A) \) and \( F \in C([0, +\infty[; \mathcal{E}) \), then there exists a unique solution

\[
U \in C^1([0, +\infty[; \mathcal{E}) \cap C([0, +\infty[; D(A))
\]

to the problem (4.30), and it enjoys the estimate
\[
\|U(t)\|_\mathcal{E} \leq e^{\Lambda t} \left( \|U_0\|_\mathcal{E} + \int_0^t e^{-\Lambda s} \|F(s)\|_\mathcal{E} ds \right), \tag{4.31}
\]
where \( \Lambda = 1 \vee 2|\Omega| \).

Therefore, considering the problem
\[
\frac{\partial^2 \xi}{\partial t^2} + B\frac{\partial \xi}{\partial t} + L\xi = f(t, \mathbf{x}),
\]
\[
\xi = \xi, \quad \frac{\partial \xi}{\partial t} = \mathbf{v} \quad \text{at} \quad t = 0,
\]
\[
\xi(t, \cdot) \in D(L) \quad \text{for} \quad \forall t \geq 0., \tag{4.32}
\]
we can claim

**Theorem 5** Suppose \( \mathbf{\hat{v}} \in D(L), \mathbf{\hat{v}} \in \mathcal{H}^{0\text{div}}_0 \) and \( f \in C([0, +\infty[; \mathcal{H}) \). Then the initial-boundary value problem (4.32) admits a unique solution

\[
\xi \in C^2([0, +\infty[, \mathcal{H}) \cap C^1([0, +\infty[, \mathcal{H}^{0\text{div}}_0) \cap C([0, +\infty[, D(L))
\]

and the energy
\[
E(t) = E(t, \xi) := \|\xi\|_{\mathcal{H}^{2\text{div}}}^2 + \|\xi\|_{\mathcal{H}}^2
\]
\[
= \|\xi\|_{\mathcal{H}}^2 + Q|\xi| + \left\| \frac{\partial \xi}{\partial t} \right\|_{\mathcal{H}}^2 \tag{4.33}
\]
enjoys the estimate
\[
\sqrt{E(t)} \leq e^{\Lambda t} \left( \sqrt{E(0)} + \int_0^t e^{-\Lambda s} \|f(s)\|_{\mathcal{H}} ds \right) \tag{4.34}
\]
for \( \Lambda = 1 \vee \beta \).
5 Eigenfrequency, eigenvector, the variational principle

Astrophysicists used to discuss on the so called ‘variational principle’. See [4], [5], [10], and so on. Although they discuss about self-gravitating gaseous masses, we would like to follow their discussions by applying them to the present case of the model of rotating atmosphere on the Earth, namely, we consider the linearized wave equation (4.2a).

Since the operator \( J : v \mapsto \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v \) is skew symmetric, we introduce the operator \( J_* \) defined by

\[
J_* = iJ.
\] (5.1)

Then the operator \( J_* \) is a bounded self-adjoint operator on \( H \). Recall

\[
(J_* \xi | \xi)_H = -2 \int_\Omega \text{Im}[\xi^1 (\xi^2)^*] \bar{\rho}(x) dx.
\] (5.2)

The equation (4.2a) reads

\[
\frac{\partial^2 \xi}{\partial t^2} - 2\Omega i J_* \frac{\partial \xi}{\partial t} + L \xi = 0
\] (5.3)

Let us suppose that there exists a solution \( \xi \) to (5.3) of the form

\[
\xi(t, x) = e^{i\sigma t} \Xi(x)
\] (5.4)

where \( \sigma \in \mathbb{C} \) and \( \Xi \in \text{D}(L) \). Then the equation (5.3) reduces to

\[
-\sigma^2 \Xi + \sigma^2 \Omega J_* \Xi + L \Xi = 0,
\] (5.5)

or the equation (4.2a) reduces to

\[
-\sigma^2 \Xi + \text{i} \sigma^2 \Omega J \Xi + L \Xi = 0.
\] (5.6)

So, we use

**Definition 10** When the equation (5.5) and (5.6) is satisfied for \( \sigma \in \mathbb{C} \) and \( \Xi \in \text{D}(L) \), \( \neq 0 \), then \( \sigma \) is called an eigenfrequency of the wave equation (4.2a) and \( \Xi \) is called an eigenvector associated with the eigenfrequency \( \sigma \).

Note that 0 is an eigenfrequency. In fact the vector

\[
\Xi = \frac{1}{\rho} \nabla \times a
\]
belongs to $\text{Ker} \mathbf{L}$ for any $a \in C_0^\infty (\mathbb{R}; \mathbb{C}^3)$, and turns out to be an eigenvector associated with the eigenfrequency 0 provided that $\Xi \neq 0$.

Here let us recall the operator $\mathbf{A}$ defined as

$$\mathbf{A} \mathbf{U} = \begin{bmatrix} O & -I \\ L & 2\Omega J \end{bmatrix} \mathbf{U} = \begin{bmatrix} -\dot{\Xi} \\ 2\Omega J \dot{\Xi} + L\Xi \end{bmatrix}$$

for

$$\mathbf{U} = \begin{bmatrix} \Xi \\ \Xi \end{bmatrix} \in \mathcal{D}(\mathbf{A}) = \mathcal{D}(L) \times \mathcal{S}_0^{\text{div}},$$

which was introduced in order to apply the Hille-Yosida theory to the linear evolution equation in the preceding section. Obviously we can claim

**Proposition 6**  
If and only if $\sigma \in \mathbb{C}$ is an eigenfrequency of the equation (5.3), $\Lambda = i\sigma$ is an eigenvalue of the operator $-\mathbf{A}$.

Let us introduce the following

**Definition 11**  
We denote by $\mathfrak{L}$ the one parameter family of operators $(-\sigma^2 + \sigma^2 \Omega J + L)_{\sigma \in \mathbb{C}}$, which is called 'quadratic pencil'. We denote

$$\mathfrak{L}(\sigma) := -\sigma^2 + \sigma^2 \Omega J + L.$$  \hfill (5.7)

If the operator $\mathfrak{L}(\sigma) = -\sigma^2 + \sigma^2 \Omega J + L$ admits the bounded inverse defined on $\mathcal{S}$, $\sigma$ is said to belong to the resolvent set $\varrho(\mathfrak{L})$. We denote $\sigma(\mathfrak{L}) = \mathbb{C} \setminus \varrho(\mathfrak{L})$, and call it the spectrum of the quadratic pencil $\mathfrak{L}$.

If $\sigma$ is an eigenfrequency, then it belongs to the spectrum $\sigma(\mathfrak{L})$, so, $0 \in \sigma(\mathfrak{L})$, but belonging to $\sigma(\mathfrak{L})$ does not mean being an eigenfrequency a priori, of course.

**Proposition 7**  
$\varrho(\mathfrak{L})$ is an open subset of $\mathbb{C}$, and $\sigma(\mathfrak{L})$ is closed.

Proof. Let us consider $\sigma \in \varrho(\mathfrak{L})$. Then

$$\mathfrak{L}(\sigma + \Delta \sigma) = \mathfrak{L}(\sigma) \left[ I + \mathfrak{L}(\sigma)^{-1}(\Delta \sigma(-2\sigma + 2\Omega J, -\Delta \sigma) \right]$$

admits the bounded inverse and $\sigma + \Delta \sigma \in \varrho(\mathfrak{L})$, if

$$\|\mathfrak{L}(\sigma)^{-1}\Delta \sigma(-2\sigma + 2\Omega J, -\Delta \sigma)\|_{\mathcal{B}(\ell)} < 1.$$  

For this inequality, it is sufficient that

$$\|\mathfrak{L}(\sigma)^{-1}\|_{\mathcal{B}(\ell)} \cdot |\Delta \sigma| \cdot \|(-\sigma + 2\Omega J, -\sigma)\| < 1,$$

or

$$|\Delta \sigma| < \frac{1}{\mathfrak{L}(\sigma)^{-1}} \cdot ((|\sigma| + |\Omega|) + |\Delta \sigma|)^2.$$  

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This means $\varrho(\mathcal{L})$ is open. □

We claim

**Proposition 8** It holds that

$$i\varrho(\mathcal{L}) = \varrho(-A), \quad i\sigma(\mathcal{L}) = \sigma(-A).$$

(5.8)

Here $\varrho(-A), \sigma(-A)$ stand for the usual resolvent set, the spectrum of the operator $-A$ in the Hilbert space $\mathcal{E} = \mathcal{S}^{\text{div}}_0 \times \mathcal{H}.$

Proof. Let $\sigma \in \varrho(\mathcal{L})$ and $\Lambda = i\sigma$. Consider the equation

$$AU + \Lambda U = F = \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{E},$$

or,

$$\begin{cases}
\bar{\Xi} + i\sigma \Xi = f \in \mathcal{S}^{\text{div}}_0,
\\
2\Omega J\bar{\Xi} + L\Xi + i\sigma \Xi = g \in \mathcal{H}.
\end{cases}$$

This system of equations can be solved as

$$\begin{cases}
\Xi = (-\sigma^2 - i\sigma 2\Omega J + L)^{-1}(2\Omega J f - i\sigma f + g) \in \mathcal{D}(L),
\\
\bar{\Xi} = (-\sigma^2 - i\sigma 2\Omega J + L)^{-1}(2\Omega J f - i\sigma f + g) - f \in \mathcal{S}^{\text{div}}_0,
\end{cases}$$

since $\sigma \in \varrho(\mathcal{L})$, while $F \mapsto U$ is continuous. Therefore $\Lambda = i\sigma \in \varrho(-A)$, or, $i\varrho(\mathcal{L}) \subset \varrho(-A)$.

Inversely let $\Lambda \in \varrho(-A)$ and $\sigma = -i\Lambda$. Consider the equation

$$(\sigma^2 + i\sigma 2\Omega J + L)\Xi = f \in \mathcal{H},$$

or

$$(\Lambda^2 + \Lambda 2\Omega J + L)\Xi = f,$$

which is equivalent to the system of equations

$$\begin{cases}
\Lambda \Xi + 2\Omega J\bar{\Xi} + L\Xi = f,
\\
\hat{\Xi} = \Lambda \Xi.
\end{cases}$$

But this is nothing but

$$AU + \Lambda U = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$
Since $\Lambda \in \sigma(-A)$ is supposed, this admits the solution

$$U = \begin{bmatrix} \Xi \\ \Xi \end{bmatrix} = (A + \Lambda)^{-1} \begin{bmatrix} 0 \\ f \end{bmatrix},$$

and $f \mapsto \Xi$ is continuous, that is, $\sigma = -i\Lambda \in \sigma(\mathcal{L})$. $\square$

When $\Omega = 0$, we have

$$\sigma(\mathcal{L}) = \{ \sigma \in \mathbb{C} \mid \lambda = \sigma^2 \in \sigma(L) \},$$

where $\sigma(L)$ is the spectrum of the self-adjoint operator $L$ in the Hilbert space $\mathcal{H}$. Since $L$ is self-adjoint and $L \geq 0$, we have $\sigma(L) \subset \{ \lambda \in \mathbb{R} \mid \lambda \geq 0 \}$. Thus it holds that

$$\sigma(\mathcal{L}) \subset \mathbb{R} \quad \text{and} \quad \{ \Lambda \in \mathbb{C} \mid \Re[\Lambda] \neq 0 \} \subset \rho(-A), \quad (5.9)$$

when $\Omega = 0$. But, when $\Omega \neq 0$, the situation is not so evident. At least, we can claim

$$\sigma(\mathcal{L}) \subset \mathbb{C} \setminus [-\infty, -2|\Omega]|i,$$

since

$$\{ \Lambda \in \mathbb{R} \mid \lambda > 2|\Omega| \} \subset \rho(-A),$$

as shown in the proof of Proposition 4. Moreover, it is clear that $\mathcal{L}(\sigma^*) = (\mathcal{L}(\sigma))^*$, since $2\Omega J_*$ and $L$ are self-adjoint. Therefore we have

$$\sigma \in \rho(\mathcal{L}) \iff \sigma^* \in \rho(\mathcal{L}). \quad (5.10)$$

This means that $\rho(\mathcal{L})$ and $\sigma(\mathcal{L})$ are symmetric re the real axis in the complex number plane. Correspondingly, $\sigma(-A)$ and $\rho(-A)$ are symmetric re the imaginary axis. Thus we can claim

$$\sigma(\mathcal{L}) \subset \mathbb{C} \setminus \{|-\infty, -2|\Omega|, 2|\Omega|, +\infty\}|i \quad \text{and} \quad \{ \Lambda \in \mathbb{R} \mid |\Lambda| > 2|\Omega| \} \subset \rho(-A). \quad (5.11)$$

However the gap between the information (5.9) for $\Omega = 0$ and that (5.11) for $\Omega \neq 0$ is too much. So, we are want to strengthen (5.11) when $\Omega \neq 0$. In order to do it, we use the following

**Proposition 9** If $\sigma \in \rho(\mathcal{L})$, then it holds that

$$\|\mathcal{L}(\sigma)^{-1}\|_{\mathcal{B}(\mathcal{H})} \geq \frac{1}{d(2(|\sigma| + |\Omega|) + d)}, \quad (5.12)$$

where $d := \text{dist}(\sigma, \sigma(\mathcal{L}))$.

**Proof.** Let $\sigma \in \rho(\mathcal{L})$. Then, for $\Delta \sigma \in \mathbb{C}$, the operator

$$\mathcal{L}(\sigma + \Delta \sigma) = \mathcal{L}(\sigma) \left[ I + \mathcal{L}(\sigma)^{-1} \Delta \sigma (-2\sigma + 2\Omega J_* - \Delta \sigma) \right]$$
admits the bounded inverse in $B(\mathcal{H})$ and $\sigma + \Delta \sigma \in \partial \sigma(\mathcal{L})$, if

$$||\mathcal{L}(\sigma)^{-1} \Delta \sigma(-2\sigma + 2\Omega J_* - \Delta \sigma)||_{B(\mathcal{H})} < 1.$$ 

For this inequality, it is sufficient that

$$||\mathcal{L}(\sigma)^{-1}||_{B(\mathcal{H})} \cdot |\Delta \sigma| \cdot |(-2\sigma + 2\Omega J_* - \Delta \sigma)| < 1.$$ 

In other words, if $\sigma + \Delta \sigma \in \sigma(\mathcal{L})$, then it should hold

$$||\mathcal{L}(\sigma)^{-1}||_{B(\mathcal{H})} \cdot |\Delta \sigma| \cdot |(-2\sigma + 2\Omega J_* - \Delta \sigma)| \geq 1,$$

and necessarily

$$||\mathcal{L}(\sigma)^{-1}||_{B(\mathcal{H})} \cdot |\Delta \sigma| \cdot (|\sigma| + |\Omega|) + |\Delta \sigma| \geq 1.$$ 

If $d < +\infty$, then there is a sequence $\sigma + (\Delta \sigma)_n \in \sigma(\mathcal{L})$ such that $(\Delta \sigma)_n \to d$, and the assertion follows. \square

Let us fix $\sigma_\infty \in \partial \sigma(\mathcal{L})$. We are going to show $\sigma_\infty \in \mathbb{R}$.

Let us consider a sequence $(\sigma_n)_n$ such that $\sigma_n \in \partial \sigma(\mathcal{L})$ and $\sigma_n \to \sigma_\infty$ as $n \to \infty$. By Proposition 3 we have $||\mathcal{L}(\sigma_n)^{-1}||_{B(\mathcal{H})} \to +\infty$, therefore there are $f_n \in \mathcal{H}$ such that $||f_n||_B = 1$ and $||\mathcal{L}(\sigma_n)^{-1} f_n||_B \to +\infty$ as $n \to \infty$. Put $\xi_n = \mathcal{L}(\sigma_n)^{-1} f_n (\in D(L))$ and $\eta_n = \xi_n / ||\xi_n||_B$. Then $||\eta_n||_B = 1$ and

$$||\mathcal{L}(\sigma_n)\eta_n|\eta_n||_B| = \left|\frac{1}{||\xi_n||_B^2} (f_n|\xi_n)_B\right| \leq \frac{1}{||\xi_n||_B^2} \to 0.$$ 

But we see

$$\mathcal{L}(\sigma_n)\eta_n|\eta_n)_B = -(\sigma_n)^2 + \sigma_n b_n + c_n,$$

where

$$b_n := 2\Omega(J_\ast \eta_n|\eta_n)_B, \quad c_n = (L\eta_n|\eta_n)_B = Q[\eta_n].$$

Therefore $b_n, c_n$ are real and

$$|b_n| \leq 2|\Omega|, \quad c_n \geq 0.$$ 

Hence, by taking a subsequence if necessary, we can suppose that $b_n$ tends to a limit $b_\infty$ such that $|b_\infty| \leq 2|\Omega|$. Put $c_\infty := (\sigma_\infty)^2 - \sigma_\infty b_\infty$. Then we see $c_n \to c_\infty$. Hence $c_\infty$ is real and $\geq 0$, and $\sigma_\infty$ tends to enjoy the quadratic equation

$$-(\sigma_\infty)^2 + b_\infty \sigma_\infty + c_\infty = 0.$$ 

Consequently,

$$\sigma_\infty = \frac{b_\infty}{2} + \sqrt{\frac{b_\infty^2}{4} + c_\infty} \quad \text{or} \quad \sigma_\infty = \frac{b_\infty}{2} - \sqrt{\frac{b_\infty^2}{4} + c_\infty},$$

so, anyway, $\sigma_\infty \in \mathbb{R}$. This was to be proved.

Summing up, we can claim
Proposition 10 It holds that

$$\partial\sigma(\mathcal{L}) \subset \mathbb{R}.$$  

(5.13)

This conclusion owes to [8, Theorem 1]. But their original proof is little bit logically weak, and we have needed to edit it as above.

Let us consider $\sigma_0 = \alpha_0 + \beta_0 i \in \sigma(\mathcal{L})$, where $\alpha_0, \beta_0 \in \mathbb{R}$. We are going to show that $|\beta_0| \neq 0$ implies a contradiction. By the symmetricity of $\sigma(\mathcal{L})$ we can suppose $\beta_0 > 0$ without loss of generality. Choosing $K > 2|\Omega| \vee \beta_0$, we consider the segment

$$I = [\sigma, K i]$$

$$= \{ \sigma(t) = (1 - t)\alpha_0 + (\beta_0 + (K - \beta_0)t)i \ | \ 0 \leq t \leq 1 \}.$$

Note that $\sigma(0) = \sigma_0 \in \sigma(\mathcal{L})$ and $\sigma(1) = K i \in \sigma(\mathcal{L})$ by (5.11), since $K > 2|\Omega|$. Put

$$\bar{t} := \sup \{ t \in [0, 1] \ | \ \sigma(t) \in \sigma(\mathcal{L}) \}.$$

Then $0 \leq \bar{t} < 1, \sigma(\bar{t}) \in \sigma(\mathcal{L})$ and $\sigma(t) \in \sigma(\mathcal{L})$ for $t > \bar{t}$. Hence $\sigma(\bar{t}) \in \partial\sigma(\mathcal{L})$.

But $\Im[\sigma(\bar{t})] \geq \beta_0 > 0$, a contradiction to $\partial\sigma(\mathcal{L}) \in \mathbb{R}$. □

Therefore we can claim

Theorem 6 It holds that

$$\sigma(\mathcal{L}) \subset \mathbb{R}$$

(5.14)

even when $\Omega \neq 0$.

Now we note that it is known that there is a sequence of eigenfrequencies $\sigma_n, \pm \sqrt{\lambda_n}n \in \mathbb{N}$, when $\Omega = 0$. (See the discussion given later.) However, up to now, we have no knowledge on the existence of eigenfrequencies when $\Omega \neq 0$. When $\Omega = 0$, then (5.9) reads

$$-\sigma^2 \Xi + L \Xi = 0,$$

so that the eigenvector $\Xi$ associated with the eigenfrequency $\sigma \neq 0$ can be supposed to be real, since $\sigma \in \mathbb{R}$. However, when $\Omega \neq 0$, the situation is different. Namely, we claim

Proposition 11 Suppose $\Omega \neq 0$. Let $\sigma \neq 0$ be an eigenfrequency of the equation

$$\| \Xi \| \neq 0, \| \Xi \| \text{ and } \Xi \text{ be an associated eigenvector. Then the eigenvector } \Xi \text{ is impossible to be real, that is, } \Im[\Xi(x)] \text{ cannot vanish everywhere.}$$

Proof. By Theorem 1 we see $\sigma \in \mathbb{R}$. Let us look at (5.6):

$$-\sigma^2 \Xi + i\sigma 2\Omega \times \Xi + L \Xi = 0.$$  

(5.15)
Let us denote \( X(x) = \Re[\Xi(x)], Y(x) = \Im[\Xi(x)] \) so that \( X(x), Y(x) \in \mathbb{R}, \Xi = X + iY \). Suppose that \( Y = 0 \) and deduce a contradiction. Now (5.15) means

\[
-\sigma^2 X - \sigma^2 \Omega \times Y + LX = 0, \quad (5.16a)
\]
\[
-\sigma Y + \sigma^2 \Omega \times X + LY = 0. \quad (5.16b)
\]

Since \( Y = 0 \) is supposed, this reads

\[
-\sigma^2 X + LX = 0, \quad (5.17a)
\]
\[
\sigma^2 \Omega \times X = 0. \quad (5.17b)
\]

Since \( \sigma \neq 0, \Omega \neq 0 \), (5.17a) implies

\[
X^1 = X^2 = 0, \quad (5.18)
\]

where \( X = (X^1, X^2, X^3)\). Then (5.17b) reads

\[
\frac{\partial}{\partial x^1} \left( -\frac{\partial T}{\partial \rho} \frac{\partial}{\partial x^3} (\bar{\rho}X^3) \right) = 0, \quad (5.19a)
\]
\[
\frac{\partial}{\partial x^2} \left( -\frac{\partial T}{\partial \rho} \frac{\partial}{\partial x^3} (\bar{\rho}X^3) \right) = 0, \quad (5.19b)
\]
\[
-\sigma^2 X^3 + \frac{\partial}{\partial x^3} \left( -\frac{\partial T}{\partial \rho} \frac{\partial}{\partial x^3} (\bar{\rho}X^3) \right) = 0. \quad (5.19c)
\]

Consequently, (5.19a) and (5.19b) imply that \( -\frac{\partial T}{\partial \rho} \frac{\partial}{\partial x^3} (\bar{\rho}X^3) \) is a function of \( x^3 \) independent of \( x^1, x^2 \), and, since \( \sigma \neq 0 \), (5.19a) implies that \( X^3 \) is so, too. However \( \Xi = X \in D(L) \) suppose the boundary condition

\[
\left( \Xi \frac{\partial}{\partial r} \right) = 0 \quad \text{on} \quad r = R_0.
\]

Namely,

\[
X^3(x^3)x^3 = 0 \quad \text{on} \quad \Sigma_0 = \{ \ r = R_0 \},
\]

therefore \( X^3 = 0 \), and \( \Xi = (X^1, X^2, X^3)^\top = 0 \), a contradiction. □

Let \( \sigma \) be an eigenfrequency of the equation (4.2a) and \( \Xi \) be an associated eigenvector.

Then \( \xi(t, x) = e^{i\sigma t} \Xi(x) \) is a solution of the equation (3.22). Since the coefficients of the equation (4.2a) are real and the equation is linear, we can claim that \( \xi(t, x)^*, \Re[\xi(t, x)], \Im[\xi(t, x)] \) are solutions of (3.22), too. But, since \( \sigma \) is
real by Theorem 6, we see
\[ \xi_R(t, x) := \Re[e^{i\sigma t} \Xi(x)] = \]
\[ = \cos(\sigma t) \Xi_R(x) - \sin(\sigma t) \Xi_I(x) = \]
\[ = \left[ \begin{array}{c}
|\Xi(x)|^2 \cos(\sigma t + \alpha_1(x)) \\
|\Xi(x)|^2 \cos(\sigma t + \alpha_2(x)) \\
|\Xi(x)|^2 \cos(\sigma t + \alpha_3(x))
\end{array} \right], \]
where
\[ \Xi_R(x) := \Re[\Xi(x)], \quad \Xi_I(x) := \Im[\Xi(x)] \]
\[ \tan \alpha_j(x) = \frac{\Xi_I(x)}{\Xi_R(x)}, \quad \text{so that} \quad \Xi(x)^j = |\Xi(x)^j|e^{i\alpha_j(x)}. \]

Note that the field \( \xi_R(t, x) \) is a real-valued solution of (4.2a) such that \( \xi_R(0, x) = \Xi_R(x) \).

Let \( \sigma, \Xi(\neq 0) \) be an eigenfrequency and an associated eigenvector. Multiplying (5.5) by \( \Xi^* \bar{\rho}(x) \) and integrating it, we have
\[ -\sigma^2 \|\Xi\|^2_H + \sigma^2 \Omega(J, \Xi)_{\beta_H} + Q[\Xi] = 0. \] (5.20)
Recall the quadratic form \( Q \) is defined by (4.10). If we write
\[ a = \|\Xi\|^2_H, \quad b = 2\Omega(J, \Xi)_{\beta_H}, \quad c = Q[\Xi], \] (5.21)
then \( a, b, c \) are real numbers, and \( \sigma \) satisfies the quadratic equation
\[ -a\sigma^2 + b\sigma + c = 0, \] (5.22)
whose roots are
\[ \sigma = \frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \] (5.23)

Here D. Lynden-Bell and J. P. Ostriker, [16, p.301, line 18], say:

Equation (36) [read (5.23)] shows that the system is stable if \( c \) is positive for each eigen \( \xi \) [read \( \Xi \)]. This assured if \( C \) [read \( L \)] is positive definite. Thus:

A sufficient condition for stability is that \( C \) [read \( L \)] is positive definite. This is the condition for secular stability.

This saying sounds strange. In fact, we may suppose the meaning of the words ‘stability’ and ‘secular stability’ as C. Hunter [10] defines:
A general system is said to be ordinarily or dynamically unstable if the amplitude of some mode grows exponentially in time, but ordinarily stable if every mode is oscillatory in time. An ordinarily stable system can be said to be secularly unstable if small additional dissipative forces can cause some perturbation to grow. Otherwise, the system is said to be secularly stable.

As C. Hunter says in the same article, the definition of secular instability does not always confirm to that given above, so, we consider the (ordinary) stability. If there is an eigenfrequency $\sigma$ which is not real, then the system described by (5.3) is unstable. It is true. But this means that there is an eigenfrequency $\sigma$ and an associated eigenvector $\Xi$ such that

$$\frac{b^2}{4a^2} + \frac{c}{a} < 0.$$ 

Normalizing $a = \|\Xi\|_H^2 = 1$, this means

$$\frac{b^2}{4} + c < 0.$$ 

Therefore we can claim that the system is unstable if there is an eigenvector $\Xi$ such that

$$\frac{b^2}{4} + c = \Omega^2((J, \Xi(\Xi))_5) + Q(\Xi) < 0.$$ 

Of course, in the situation under consideration, this cannot happen, since $Q(\Xi) \geq 0$ for $\forall \Xi \in D(L)$ and $b \in \mathbb{R}$.

However, logically speaking, the condition $c = Q(\Xi) > 0$ for each eigenvector $\Xi$ is far from the condition of the stability, contrary to the saying of D. Lynden-Bell and J. P. Ostriker.

Moreover let us note the following fact: $Q(\Xi) \geq 0$ for any $\Xi \in \mathcal{H}^{\text{div}}$, but $Q(\Xi) = 0$ does not imply $\Xi = 0$; In fact

$$\Xi(x) = \frac{1}{\rho(x)} \nabla \times a,$$

$a$ being an arbitrary vector field belonging to $C^0_0(\mathbb{R})$, belongs to the kernel of $L$, that is, $Q(\Xi) = 0$.

Anyway, we are going to describe the ‘variational principle’.

Let us suppose that there exist $\sigma_0 \in \mathbb{R}$ and $\Xi_0 \in \mathcal{H}^{\text{div}} (\neq 0)$ such that

$$-\sigma_0^2(\Xi_0)_5^2 + \sigma_0^2(\Omega(J, \Xi_0(\Xi_0))_5 + Q(\Xi_0) = 0.$$  

(5.24)

Of course if $\sigma_0, \Xi_0$ are real eigenfrequency and an associated eigenvector then (5.24) is satisfied. Now we assume that

$$a_0 = ||\Xi_0||_5^2, \quad b_0 = 2\Omega(J, \Xi_0(\Xi_0))_5, \quad c_0 = Q(\Xi_0)$$  

(5.25)
satisfies
\[
\frac{b_0}{4a_0^2} + \frac{c_0}{a_0} > 0.
\] (5.26)

Then
\[
\sigma_0 = \frac{b_0}{2a_0} + \sqrt{\frac{b_0^2}{4a_0^2} + \frac{c_0}{a_0}}
\] (5.27a)
or
\[
\sigma_0 = \frac{b_0}{2a_0} - \sqrt{\frac{b_0^2}{4a_0^2} + \frac{c_0}{a_0}}
\] (5.27b)

In order fix the idea, suppose that (5.27a) is the case. Then we can consider \(\sigma\) as a functional of \(\Xi \in \mathcal{H}_{\text{div}}\) near \(\Xi_0\), say, \(\|\Xi - \Xi_0\|_{\mathcal{H}_{\text{div}}} \leq \delta (\ll 1)\), defined by
\[
\sigma(\Xi) = \frac{b(\Xi)}{2a(\Xi)} + \sqrt{\frac{b(\Xi)^2}{4a(\Xi)^2} + \frac{c(\Xi)}{a(\Xi)}}
\] (5.28)
with
\[
a(\Xi) = \|\Xi\|_{\mathcal{H}_{\text{div}}}^2, \quad b(\Xi) = 2\Omega(J, \Xi|\Xi)_{\mathcal{H}_{\text{div}}} + c(\Xi) = Q[\Xi].
\] (5.29)

Here we take \(\delta\) so small that
\[
\frac{b(\Xi)^2}{4a(\Xi)^2} + \frac{c(\Xi)}{a(\Xi)} > 0 \quad \text{for} \quad \|\Xi - \Xi_0\|_{\mathcal{H}_{\text{div}}} \leq \delta.
\] (5.30)

The variation \(\delta \sigma = \delta \sigma(\Xi)\) of \(\sigma\) at \(\Xi\), \(\|\Xi - \Xi_0\|_{\mathcal{H}_{\text{div}}} \leq \delta\), is the linear functional on \(\mathcal{H}_{\text{div}}\) defined by
\[
\langle \delta \sigma(\Xi) | h \rangle = \lim_{\tau \to 0} \frac{1}{\tau} (\sigma(\Xi + \tau h) - \sigma(\Xi)).
\] (5.31)

It follows from (5.28) the equation
\[
-\sigma^2\|\Xi\|_{\mathcal{H}_{\text{div}}}^2 + 2\Omega(J, \Xi|\Xi)_{\mathcal{H}_{\text{div}}} + Q[\Xi] = 0.
\] (5.32)
holds for \(\sigma = \sigma(\Xi), \Xi \in \mathcal{H}_{\text{div}}, \|\Xi - \Xi_0\|_{\mathcal{H}_{\text{div}}} \leq \delta\). Therefore we have
\[
(-2\sigma\|\Xi\|_{\mathcal{H}_{\text{div}}}^2 + 2\Omega(J, \Xi|\Xi)_{\mathcal{H}_{\text{div}}}) \delta \sigma +
-\sigma^2 \delta\|\Xi\|_{\mathcal{H}_{\text{div}}}^2 + \sigma \delta(2\Omega(J, \Xi|\Xi)_{\mathcal{H}_{\text{div}}} + \delta Q[\Xi]) = 0,
\]
or, precisely writing,
\[
(-2\sigma\|\Xi\|_{\mathcal{H}_{\text{div}}}^2 + 2\Omega(J, \Xi|\Xi)_{\mathcal{H}_{\text{div}}}) \langle \delta \sigma(\Xi) | \delta \Xi \rangle +
+ 2\Re \left[ -\sigma^2(\Xi|\delta \Xi)_{\mathcal{H}_{\text{div}}} + \sigma 2\Omega(J, \Xi|\delta \Xi)_{\mathcal{H}_{\text{div}}} + Q(\Xi, \delta \Xi) \right] = 0
\]
for \( \forall \delta \Xi \in \mathcal{F}_{\text{div}} \). Here we note that

\[
-2\sigma \| \Xi \|^2 + 2\Omega(J, \Xi|\Xi)_{\delta} = -2\sigma a(\Xi) + b(\Xi) = -\sqrt{\frac{b(\Xi)^2}{4a(\Xi)^2} + \frac{c(\Xi)}{a(\Xi)}} \neq 0
\]

for \( \| \Xi - \Xi_0 \|_{\text{div}} \leq \delta \). Therefore \( \delta \sigma(\Xi) = 0 \) if and only if

\[
-\sigma^2(\Xi|\Xi)_{\delta} + \sigma 2\Omega(J, \Xi|\Xi)_{\delta} + Q(\Xi, h) = 0
\]

for \( \forall h \in \mathcal{F}_{\text{div}} \). Thus we can claim the following ‘variational principle’:

**Theorem 7** Let \( \Xi_0 \in \mathcal{F}_{\text{div}} \) satisfy

\[
\| \Xi_0 \|^2_{\delta} > 0, \quad \frac{(2\Omega(J, \Xi_0|\Xi_0)_{\delta})^2}{4\| \Xi_0 \|^2_{\delta}} + \frac{Q(\Xi_0)}{\| \Xi_0 \|^2_{\delta}} > 0.
\]

The variation \( \delta \sigma \) of \( \sigma \) (specified by (5.28)) vanishes at \( \Xi_0 \) if and only if \( \Xi_0 \in \mathcal{D}(L) \) and enjoys the equation (5.3):

\[
-\sigma^2_0 \Xi_0 + \sigma_0 2\Omega J, \Xi_0 + L \Xi_0 = 0.
\]

Then \( e^{i\sigma_0 t} \Xi_0(x) \) is a solution of the equation (5.3). Here \( \sigma_0 = \sigma(\Xi_0) \).

This principle tells us that, if we want to find an eigenfrequency, we may try to find a stationary point of the functional

\[
\sigma(\Xi) = \frac{b(\Xi)}{2} \pm \sqrt{\frac{b(\Xi)^2}{4a(\Xi)^2} + \frac{c(\Xi)}{a(\Xi)}}
\]

under the constraint \( \| \Xi \|_{\delta} = 1 \). But it seems that this principle is far from the solution of the problem to establish the existence and completeness of the system of eigenvectors.

For example, as a ‘practical use of the variational principle’, D. Lynden-Bell and J. P. Ostriker, [16, Section 2.6], proposed the following scheme:

Take sufficiently many functions \( \Xi_{(i)}, i = 1, \ldots, N \) as those who consist a base, and consider the trial function

\[
\Xi = \sum_{i=1}^{N} a^i \Xi_{(i)}, \quad a = \begin{bmatrix} a^1 \\ \vdots \\ a^N \end{bmatrix}.
\]

Put

\[
A = (A_{ij})_{i,j}, \quad A_{ij} = (\Xi_{(i)}|\Xi_{(j)})_{\delta},
\]

\[
B = (B_{ij})_{i,j}, \quad B_{ij} = 2\Omega(J, \Xi_{(i)}|\Xi_{(j)})_{\delta},
\]

\[
C = (C_{ij})_{i,j}, \quad C_{ij} = Q(\Xi_{(i)}, \Xi_{(j)}).
\]
Then
\[ \sigma(\Xi) = \left( (-\sigma^2A + \sigma B + C) | a \right).\]

Thus, D. Lynden-Bell and J. P. Ostriker say, the variational principle reads
\[ \delta \left( (-\sigma^2A + \sigma B + C) | a \right) = 0\]

and, varying \( a \), we obtain the ‘secular determinant’
\[ \det(-\sigma^2A + \sigma B + C) = 0 \quad (5.33)\]

for the determination of the variationally best eigenfrequencies \( \sigma \).

But it is doubtful that this scheme is so practical, since we have no confidence that the equation (5.33), which is an algebraic equation for the unknown \( \sigma \) of degree is \( 2N \), can be numerically solved to determine an approximating eigenfrequency \( \sigma_N \) so that they converge to a true eigenfrequency as \( N \to \infty \).

On the other hand, since
\[ \sigma(\Xi) = \frac{b(\Xi)}{2} + \sqrt{\frac{b(\Xi)^2}{4} + c(\Xi)} \geq 0 \]

for \( \forall \Xi \in \mathcal{S}_{\text{div}} \), we can consider
\[ \sigma_* := \inf \{ \sigma(\Xi) | \Xi \in \mathcal{S}_{\text{div}}, \| \Xi \|_{\beta} = 1 \} \]

and we may expect that the minimum might give an eigenfrequency. But, when \( \Omega \neq 0 \), we are not sure about the existence of a \( \Xi^* \) which attains the infimum, namely \( \sigma_* = \sigma(\Xi^*) \), in general, since the imbedding \( \mathcal{S}_{\text{div}} \hookrightarrow \mathcal{H} \) is not compact and we may be unable to extract convergent subsequences from a minimizing sequence, say, a sequence \( (\Xi_n)_{n \in \mathbb{N}} \) such that \( \Xi_n \in \mathcal{S}_{\text{div}}, \| \Xi_n \|_{\beta} = 1, \sigma(\Xi_n) \to \sigma_* \) as \( n \to \infty \).

When \( \Omega = 0 \), then \( b(\Xi) = 0 \), and the variational problem
\[ \lambda_* (= \sigma_*^2) = \inf \{ c(\Xi)(= Q(\Xi)) | \Xi \in \mathcal{S}_{\text{div}}, \| \Xi \|_{\beta} = 1 \} \]

actually admits the trivial solution \( \lambda_* = 0 \) with any eigenvector \( \in \text{Ker}(L) \). But the Mini-Max principle does not work, for, since \( \dim \text{Ker}(L) = \infty \), we cannot go ahead across the 0 eigenvalue eternally, although there actually remain infinitely many positive eigenvalues. Thus also in this case the variational principle seems to be not so useful.

However, when \( \Omega = 0 \), the equation (5.33) reduces to
\[ -\lambda \Xi + LE = 0, \quad (5.34)\]

where \( \lambda = \sigma^2 \), and this eigenvalue problem is completely solved as follows:
Suppose $\Omega = 0$. We note that the background stationary solution $\bar{\rho}$ is a spherically symmetric equilibrium, say,

$$\bar{\rho}(x) = \left(\frac{(\gamma - 1)GM_0}{A\gamma} \left(\frac{1}{r} - \frac{1}{R}\right)\right)^{\frac{1}{\gamma - 1}}$$

(5.35)

and $\mathcal{R} = \{\bar{\rho} > 0\} = \{R_0 < r < R\}$ is an annulus. Taking the divergence of $\bar{\rho}$ times (5.34), the problem reduces to

$$- \lambda g - \text{div}(\frac{\partial T}{\partial \rho} g) = 0,$$

(5.36)

where

$$g = \text{div}(\bar{\rho} \Xi).$$

(5.37)

Note that we can treat the operator $\mathcal{N} : g \mapsto \text{div}(\frac{\partial T}{\partial \rho} g)$ as that with the similar property to the Laplacian operator $\Delta = \text{div grad}$, taking into account the singular behaviors of $\bar{\rho}$ and $\frac{\partial T}{\partial \rho}$ at the physical vacuum boundary, say,

$$\bar{\rho} \sim \left(\frac{(\gamma - 1)GM_0}{A\gamma}\right)^{\frac{1}{\gamma - 1}}(R - r)^{\frac{1}{\gamma - 1}},$$

$$\frac{\partial T}{\partial \rho} \sim A\gamma \left(\frac{(\gamma - 1)GM_0}{A\gamma}\right)^{\frac{2}{\gamma - 2}}(R - r)^{-\frac{2}{\gamma - 2}}$$

near $\Sigma_1 = \{r = R\}$. Therefore we can claim:

The operator $\mathcal{N}$ can be considered as a self-adjoint operator in the Hilbert space $\mathcal{G}$ and its spectrum $\sigma(\mathcal{N})$ is of the Sturm-Liouville type, that is, $\sigma(\mathcal{N}) = \{\lambda^N_n | n \in \mathbb{N}\}$, where $\lambda^N_n$ is an eigenvalue with finite multiplicity, $0 < \lambda^N_0 < \cdots < \lambda^N_n < \lambda^N_{n+1}$, and $\lambda^N_n \to +\infty$ as $n \to \infty$. Here

$$\mathcal{G} = L^2(\mathcal{R}; \frac{\partial T}{\partial \rho} dx) \cap \{g \mid \int_{\mathcal{R}} g dx = 0\};$$

Note that the imbedding of $L^2(\mathcal{R}; \frac{\partial T}{\partial \rho} dx)$ into $L^2(\mathcal{R}; dx) (\mapsto L^1(\mathcal{R}; dx))$ is continuous. For a proof see the proof of [11, Theorem 2]. Hence the argument of [11] leads us to the following conclusion:

When $\Omega = 0$, $\mathcal{L}$ can be considered as a self-adjoint operator in the Hilbert space $\mathfrak{F}$ and its spectrum $\sigma(\mathcal{L})$ coincides with $\sigma(\mathcal{N}) \cup \{0\}$, while dim Ker($\mathcal{L}$) = $\infty$ and $\lambda = \lambda^N_n \neq 0$ is an eigenvalue with finite multiplicity. Here

$$\mathfrak{F} = \{\Xi \in \mathfrak{F} \mid \text{div}(\bar{\rho} \Xi) \in \mathcal{G}\} = \{\Xi \in \mathfrak{F}^{\text{div}} \mid \int_{\mathcal{R}} \text{div}(\bar{\rho} \Xi) dx = 0\}.$$
Note that $\mathfrak{H}$ is a closed subspace of $\mathcal{H}^{\text{div}}$, since $G \ni g \mapsto \int_{\Omega} gd\mathbf{x}$ is continuous. In fact, we have
\[
\left| \int g \right| \leq \left[ \int \frac{d\rho}{d\mathbf{T}} |g| \right]^{1/2} \left[ \int \frac{d\rho}{d\mathbf{T}} \right]^{1/2} \lesssim \|g\|_{L^2(\Omega d\mathbf{x})}.
\]
We have $\mathcal{H}_0^{\text{div}} \subset \mathfrak{H}$, since
\[
\int \text{div}(\bar{\rho}\varphi)d\mathbf{x} = 0 \quad \text{for } \forall \varphi \in C_c^\infty(\Omega).
\]
Therefore by the well-known theorem (\cite{7} p.905, X.3.4. Theorem) complemented by \cite{12} p. 177, Chapter III, Theorem 6.15), we can say that the eigenvectors of a CONS of $\text{Ker}(\mathbf{L})$ and all
\[
\psi_n = \frac{1}{\lambda_n} \bar{\rho} \text{grad} \left( - \frac{d\mathbf{T}}{d\rho} \varphi_n \right) \left\| \frac{1}{\lambda_n} \bar{\rho} \text{grad} \left( - \frac{d\mathbf{T}}{d\rho} \varphi_n \right) \right\|^{-1},
\]
$\varphi_n$ being an eigenvalue of $\mathcal{N}$ associated with the eigenvalue $\lambda_n \neq 0$ form a complete orthogonal system of the Hilbert space $\mathfrak{H}$.

In this sense, when $\Omega = 0$, the eigenfrequency problem is completely solved.

**Remark 2** Let $\Omega = 0$. If we consider the operator $\mathbf{L}$ in the space $\mathfrak{H}$, we can claim that $\{0\} \cap \sigma(\mathcal{N}) = \{0, \lambda_0, \lambda_1, \cdots\} \subset \sigma(\mathbf{L})$, but we do not know whether they coincide or not, say, we do not know whether there are real continuous spectrum between the eigenvalues or not.

However, when $\Omega \neq 0$, the above discussion seems not to work. In fact the term $2\Omega(\nabla|\bar{\rho}\mathbf{J}, \Xi)$ may cause trouble, since it cannot be reduced to a quantity determined by $g = (\nabla|\bar{\rho}\mathbf{J})$.

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No new data were created or analyzed in this study.

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