ON THE WELL-POSEDNESS AND DECAY RATES OF STRONG SOLUTIONS TO A MULTI-DIMENSIONAL NON-CONSERVATIVE VISCOS COMPRESSIBLE TWO-FLUID SYSTEM

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ABSTRACT. The present paper deals with the Cauchy problem of a multi-dimensional non-conservative viscous compressible two-fluid system. We first study the well-posedness of the model in spaces with critical regularity indices with respect to the scaling of the associated equations. In the functional setting as close as possible to the physical energy spaces, we prove the unique global solvability of strong solutions close to a stable equilibrium state. Furthermore, under a mild additional decay assumption involving only the low frequencies of the data, we establish the time decay rates for the constructed global solutions. The proof relies on an application of Fourier analysis to a complicated parabolic-hyperbolic system, and on a refined time-weighted inequality.

1. Introduction. It is well known that models of two-phase or multiphase flows are widely applied to study the hydrodynamics in industry, for example, in manufacturing, engineering, and biomedicine, where the fluids under investigation contain more than one component. In fact, it has been estimated that over half of everything produced in a modern industrial society depends, to some degree, on a multiphase flow process for its optimum design and safe operation. In nature, there is a variety of different multiphase flow phenomena, such as sediment transport, geysers, volcanic eruptions, clouds, and rain [2, 4]. In addition, models of multiphase flows also naturally appear in many contexts within biology, ranging from tumor biology and anticancer therapies to developmental biology and plant physiology[17]. The principles of single-phase flow fluid dynamics and heat transfer are relatively well

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understood; however, the thermoﬂuid dynamics of two-phase ﬂows is an order of magnitude more complicated than that of the single-phase ﬂow due to the existence of a moving and deformable interface and its interactions with two phases [27, 20, 21].

In this paper, we are concerned with the following mathematical model of multiphase ﬂow, namely a multi-dimensional non-conservative viscous compressible two-ﬂuid system \( \mathbb{R}^N(N \geq 2) \):

\[
\begin{cases}
\alpha^+ + \alpha^- = 1, \\
\partial_t(\alpha^+ \rho^+) + \text{div}(\alpha^+ \rho^+ u^+) = 0, \\
\partial_t(\alpha^- \rho^- u^-) + \text{div}(\alpha^- \rho^- u^- \otimes u^-) + \alpha^\pm \nabla P^\pm(\rho^\pm) = \text{div}(\alpha^\pm \tau^\pm), \\
P^+/(\rho^+) = A^+(\rho^+)^{\gamma^+} = P^-/(\rho^-) = A^-(\rho^-)^{\gamma^-},
\end{cases}
\]

(1.1)

where the variable \( 0 \leq \alpha^+(x,t) \leq 1 \) is the volume fraction of ﬂuid + in one of the two gases, and \( 0 \leq \alpha^-(x,t) \leq 1 \) is the volume fraction of the other ﬂuid −. Moreover, \( \rho^\pm(x,t) \geq 0, u^\pm(x,t) \), and \( P^\pm(\rho^\pm) = A^\pm(\rho^\pm)^{\gamma^\pm} \) are, respectively, the densities, the velocities, and the two pressure functions of the ﬂuids. It is assumed that \( \gamma^\pm \geq 1, A^\pm > 0 \) are constants. In what follows, we set \( A^+ = A^- = 1 \) without loss of any generality. Also, \( \tau^\pm \) are the viscous stress tensors

\[
\tau^\pm := 2\mu^\pm D(u^\pm) + \lambda^\pm \text{div}u^\pm \text{Id},
\]

(1.2)

where \( D(u^\pm) \) stand for the deformation tensor, the constants \( \mu^\pm \) and \( \lambda^\pm \) are the (given) shear and bulk viscosity coefﬁcients satisfying \( \mu^\pm > 0 \) and \( \lambda^\pm + 2\mu^\pm > 0 \). This model is known as a two-ﬂuid ﬂow system with algebraic closure, and we refer readers to Refs [2, 3, 14] for more discussions on the system.

From the viewpoint of partial differential equations, system (1.1) is a highly nonlinear system coupling between hyperbolic equations and parabolic equations. As a matter of fact, there is no diffusion on the mass conservation system satisfying hyperbolic equations, whereas velocity evolves according to the parabolic equations due to the viscosity phenomena. We should point out that the system (1.1) includes important single phase ﬂow models such as the compressible Navier-Stokes equations when one of the two phases volume fraction tends to zero (i.e., \( \alpha^+ = 0 \) or \( \alpha^- = 0 \) ). As an extremely important system to describe compressible ﬂuids (e.g., gas dynamics), the compressible Navier-Stokes equations have attracted a lot of attention among many analysts and many important results have been developed. Here we brieﬂy review some of the most relevant papers about global well-posedness and large time behaviors of the solutions to the system. Lions [24] proved the global existence of weak solutions for large initial data. However, the question of uniqueness of weak solutions remains open, even in the two dimensional case. Matsumura and Nishida [25, 26] ﬁrst studied the global existence of classical solutions to the compressible Navier-Stokes equations for data \((\rho_0, u_0)\) with high regularity order and close to a stable equilibrium in the 3D whole space and obtained the time decay rates based on the \( L^2 \)-framework. Later, Ponce [28] established the optimal \( L^p \) \( (2 \leq p \leq \infty) \) decay rates. Applying Fourier analysis to the linearized homogeneous system and capturing the dissipation of the hyperbolic component in the solution, Kawashima [18, 19] and Shizuta and Kawashima [29] developed a general approach to obtain the time-decay of solutions. It is worth mentioning that Li and Zhang [23] obtained the optimal \( L^p \) time-decay rates for the compressible Navier-Stokes equations in three dimensions when initial data belong to some space \( H^s \cap \dot{B}^{-s}_{1,\infty} \).
(see Definition 2.2 for details) and \( s \in [0, 1] \). Guo and Wang [16] obtained the optimal decay rates for the compressible Navier-Stokes equations when the initial data are close to a stable equilibrium state in negative Sobolev spaces by using a pure energy method. In [9], Danchin first proved the existence and uniqueness of the global strong solution for initial data close to a stable equilibrium state in critical Besov spaces. Later, Danchin [11] further established the time decay rates of the global solutions constructed in [9].

Since a single phase flow model may be considered as a special case of a two-phase flow model in the limit when one of the two phases volume fractions tends to zero, the mathematical structure of the two-phase system is much more complex than that in the case of single phase flow model. So, extending the currently available results for single phase flow models to two-phase models is not an easy task. Nowadays, more and more researchers pay more attention to the mathematical problems of the generic two-phase model. In [2], Bresch et al. first established the existence of global weak solutions to the 3D generic two-fluid flow model with capillary pressure effects in terms of a third order derivative of \( \alpha^\pm \rho^\pm \). Based on detailed analysis of the Green function to the linearized system and on elaborate energy estimates to the nonlinear system, the authors in [8] obtained global existence of smooth solutions and the time decay rates to the 3D model where the initial data are close to an equilibrium state in \( H^s(\mathbb{R}^3_+) \) with high Sobolev regularity and belong to \( L^1(\mathbb{R}^3_+) \). More recently, Lai-Wen-Yao [22] studied the vanishing capillarity limit of the smooth solutions to the 3D model with unequal pressure functions. When a generic two-fluid flow model does not include capillary pressure effects, the model reduces to the system (1.1). Bresch-Huang-Li [3] extended the result in [2] and proved the existence of global weak solutions to (1.1) in one space dimension. In 2016, Evje-Wang-Wen [14] proved the global existence of strong solutions to the model (1.1) with constant viscosity coefficients and unequal pressure functions by the standard energy method under the condition that the initial data are close to the constant equilibrium state in \( H^1(\mathbb{R}^3_+) \) and obtained the optimal decay rates for the constructed global strong solutions in \( L^2 \)-norm if the initial data belong to \( L^1 \) additionally. However, to the best of our knowledge, very few results have been established on the global well-posedness and the decay rates of strong solutions to a multi-dimensional non-conservative viscous compressible two-fluid system in critical regularity framework. The purpose of this work is to investigate the mathematical properties of system (1.1) in critical regularity framework. More specifically, we address the question of whether available mathematical results such as the global well-posedness and time decay rate in critical Besov spaces to a single fluid governed by the compressible barotropic Navier-Stokes equations may be extended to multi-dimensional non-conservative viscous compressible two-fluid systems in critical regularity framework.

First, we will derive another expression of the pressure gradient in terms of the gradients of \( \alpha^\pm \rho^\pm \) and \( \alpha^- \rho^- \) by using the pressure equilibrium assumption. The method comes from [2]. For the convenience of the reader, we also show some derivations in this part. The relation between the pressures of (1.1) implies the following differential identities

\[
\begin{align*}
dP^+ &= s_+^2 \, d\rho^+, & dP^- &= s_-^2 \, d\rho^-,
\end{align*}
\]

where \( s_\pm \) denote the sound speed of each phase respectively.

\[
(1.3)
\]
Let
\[ R^\pm = \alpha^\pm \rho^\pm. \]  
(1.4)

Resorting to (1.1)_1, we have
\[ d\rho^+ = \frac{1}{\alpha^+} (dR^+ - \rho^+ d\alpha^+), \quad d\rho^- = \frac{1}{\alpha^-} (dR^- + \rho^- d\alpha^+). \]  
(1.5)

Combining with (1.4) and (1.5), we conclude that
\[ d\alpha^+ = \frac{\alpha^- s_+^2}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2} dR^+ - \frac{\alpha^+ s_-^2}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2} dR^-. \]

Substituting the above equality into (1.5), we obtain
\[ dp^+ = \frac{\rho^+ \rho^- s_-^2}{R^- (\rho^-)^2 s_+^2 + R^+(\rho^-)^2 s_-^2} \left( \rho^- dR^+ + \rho^+ dR^- \right), \]
and
\[ dp^- = \frac{\rho^+ \rho^- s_-^2}{R^- (\rho^-)^2 s_+^2 + R^+(\rho^-)^2 s_-^2} \left( \rho^- dR^+ + \rho^+ dR^- \right), \]
which give, for the pressure differential \( dP^\pm \),
\[ dP^+ = C^2 (\rho^- dR^+ + \rho^+ dR^-), \]
and
\[ dP^- = C^2 (\rho^- dR^+ + \rho^+ dR^-), \]
where
\[ C^2 \overset{\text{def}}{=} \frac{s_-^2 s_+^2}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2}. \]

Recalling \( \alpha^+ + \alpha^- = 1 \), we get the following identity:
\[ \frac{R^+}{\rho^+} + \frac{R^-}{\rho^-} = 1, \quad \text{and therefore} \quad \rho^- = \frac{R^- \rho^+}{\rho^+ - R^+}. \]  
(1.6)

Then it follows from the pressure relation (1.1)_4 that
\[ \varphi(\rho^+) := P^+(\rho^+) - P^- (\frac{R^- \rho^+}{\rho^+ - R^+}) = 0. \]  
(1.7)

Differentiating \( \varphi \) with respect to \( \rho^+ \), we have
\[ \varphi'(\rho^+) = s_+^2 + s_-^2 \cdot \frac{R^- R^+}{(\rho^+ - R^+)^2}. \]

By the definition of \( R^\pm \), it is natural to look for \( \rho^+ \) which belongs to \( (R^+,-\infty) \).
Since \( \varphi' > 0 \) in \( (R^+,-\infty) \) for any given \( R^\pm > 0 \), and \( \varphi : (R^+,+\infty) \rightarrow (-\infty,+\infty) \),
this determines that \( \rho^+ = \rho^+(R^+, R^-) \in (R^+,+\infty) \) is the unique solution of the equation (1.7).
Due to (1.5), (1.6) and (1.1)_1, \( \rho^- \) and \( \alpha^\pm \) are defined as follows:
\[ \rho^- (R^+, R^-) = \frac{R^- \rho^+(R^+, R^-)}{\rho^+(R^+, R^-) - R^+}, \]
\[ \alpha^+ (R^+, R^-) = \frac{R^+}{\rho^+(R^+, R^-)}, \]
\[ \alpha^- (R^+, R^-) = 1 - \frac{R^+}{\rho^+(R^+, R^-)} = \frac{R^-}{\rho^- (R^+, R^-)}. \]
Based on the above analysis, the system (1.1) is equivalent to the following form
\[
\begin{aligned}
\partial_t R^\pm + \text{div}(R^\pm u^\pm) &= 0, \\
\partial_t (R^+ u^+) + \text{div}(R^+ u^+ \otimes u^+) + \alpha^+ C^2 [\rho^- \nabla R^+ + \rho^+ \nabla R^-] &= \text{div}(\alpha^+ [\mu^+(\nabla u^+ + \nabla^t u^+) + \lambda^+ \text{div} u^+ \text{Id}]), \\
\partial_t (R^- u^-) + \text{div}(R^- u^- \otimes u^-) + \alpha^- C^2 [\rho^- \nabla R^+ + \rho^+ \nabla R^-] &= \text{div}(\alpha^- [\mu^- (\nabla u^- + \nabla^t u^-) + \lambda^- \text{div} u^- \text{Id}]).
\end{aligned}
\]
(1.8)

In this paper, we are concerned with the Cauchy problem of the system (1.8) in \(\mathbb{R}^+ \times \mathbb{R}^N\) subject to the initial data
\[
(R^+, u^+, R^-, u^-)(x,t)\big|_{t=0} = (R_0^+, u_0^+, R_0^-, u_0^-)(x), \quad x \in \mathbb{R}^N,
\]
and
\[
u^+(x,t) \to 0, \quad \nu^-(x,t) \to 0, \quad R^+ \to R_\infty^+, \quad R^- \to R_\infty^- > 0 , \quad \text{as} \ |x| \to \infty,
\]
where \(R_\infty^\pm\) denote the background doping profile, and in the present paper \(R_\infty^\pm\) are taken as 1 for simplicity.

At this stage, we are going to use scaling considerations for (1.1) to guess which spaces may be critical. One can check that if \((\alpha^+ \rho^+, u^+, \alpha^- \rho^-, u^-)\) solves (1.1), so does \(((\alpha^+ \rho^+), u^+_\lambda, (\alpha^- \rho^-)_\lambda, u^-_\lambda)\) where:
\[
\begin{aligned}
(\alpha^+ \rho^+)_\lambda(t,x) &= (\alpha^+ \rho^+)(\lambda^2 t, \lambda x), \quad u^+_\lambda(t,x) = \lambda u^+(\lambda^2 t, \lambda x), \\
(\alpha^- \rho^-)_\lambda(t,x) &= (\alpha^- \rho^-)(\lambda^2 t, \lambda x), \quad u^-_\lambda(t,x) = \lambda u^-(\lambda^2 t, \lambda x).
\end{aligned}
\]
(1.10)

provided that the pressure laws \(P\) have been changed into \(\lambda^2 P\). This suggests us to choose initial data \(((\alpha^+ \rho^+)_0, u^+_0, (\alpha^- \rho^-)_0, u^-_0)\) in critical spaces whose norm is invariant for all \(\lambda > 0\) by the transformation \(((\alpha^+ \rho^+)_0, u^+_0, (\alpha^- \rho^-)_0, u^-_0)\) \(\to\) \(((\alpha^+ \rho^+)_0(\lambda \cdot), \lambda u^+_0(\lambda \cdot), (\alpha^- \rho^-)_0(\lambda \cdot), \lambda u^-_0(\lambda \cdot))\). Due to the mixed hyperbolic-parabolic property of the partial differential system (1.1), motivated by Danchin’s excellent work in [9], the different dissipative mechanisms of low frequencies and high frequencies inspire us to deal with \((\alpha^+ \rho^+ - 1, u^+, \alpha^- \rho^- - 1, u^-)\) in \(B_{2,1}^{\frac{N}{2} - 1, \frac{N}{2}} \times B_{2,1}^{\frac{N}{2} - 1, \frac{N}{2}} \times B_{2,1}^{\frac{N}{2} - 1, \frac{N}{2}} \times B_{2,1}^{\frac{N}{2} - 1, \frac{N}{2}}\) (see Definitions 2.2 and 2.5 for details). However, we cannot obtain the desired bounds directly in critical regularity framework if the convection terms are treated as perturbations. More precisely, there exists a difficulty coming from the convection terms \(u^\pm \cdot \nabla R^\pm\) in the transport equations without any diffusion in high frequencies, as one derivative loss about the function \(R^\pm\) will appear no matter how smooth is \(u^\pm\) if they are viewed as perturbation terms. To overcome the difficulty, employing the Littlewood-Paley theory and some commutator estimates, we shall, as in [9] for the standard barotropic Navier-Stokes equations, study a complicated hyperbolic-parabolic linear system including convection terms and then deduce the smooth effect for \((R^+ - 1, u^+, R^- - 1, u^-)\) in the low frequencies regime and the \(L^1\) decay on the density \(R^\pm\) in the high frequencies regime. In particular, based on the damping effect of \(R^\pm\), we further exploit the smooth effect for \((u^+, u^-)\) in the high frequencies regime with \(\nabla R^\pm\) being viewed as perturbation terms and finally establish a uniform priori estimate for the complicated system (see the following Lemma 3.1 for details). Here, it should be pointed out that, different from the standard barotropic compressible Navier-Stokes equations, we need to make more careful analyses to cancel some mixed terms from the two-phase flows. Next, one may wonder how global strong solutions constructed above look like for
large time. Under a suitable additional condition involving only the low frequencies of the data and in the $L^2$-critical regularity framework, we exhibit the time decay rates for the constructed global strong solutions. In this part, our main ideas are based on the low-high frequency decomposition and a refined time-weighted energy functional. In low frequencies, making good use of Fourier localization analysis to a linearized parabolic-hyperbolic system in order to obtain smoothing effect of the Green function in the low-frequency part and avoid some complicate analysis of the Green function (see Lemma 6.1), which is a $8 \times 8$ matrix. In high frequencies, we deal with the estimates of the nonlinear terms in the system employing the Fourier localization method and commutator estimates. Finally, in order to close the energy estimates, we exploit some decay estimates with gain of regularity for the high frequencies of $\nabla u^\pm$.

Now we state our main results as follows:

**Theorem 1.1.** Assume that $(R_0^+ - 1, u_0^+, R_0^- - 1, u_0^-) \in \dot{B}_{2,1}^{\frac{\nu}{2} - 1, \frac{\nu}{2}} \times \dot{B}_{2,1}^{\frac{\nu}{2} - 1}$. Then there exists a constant $\eta > 0$ such that if

$$
\|(R_0^+ - 1, R_0^- - 1)\|_{\dot{B}_{2,1}^{\frac{\nu}{2} - 1}} + \|(u_0^+, u_0^-)\|_{\dot{B}_{2,1}^{\frac{\nu}{2} - 1}} \leq \eta, \quad (1.11)
$$

then the Cauchy problem (1.8)-(1.9) admits a unique global solution $(R^+ - 1, u^+, R^- - 1, u^-)$ satisfying that for all $t \geq 0$,

$$
X(t) \lesssim \|(R_0^+ - 1, R_0^- - 1)\|_{\dot{B}_{2,1}^{\frac{\nu}{2} - 1}} + \|(u_0^+, u_0^-)\|_{\dot{B}_{2,1}^{\frac{\nu}{2} - 1}}, \quad (1.12)
$$

where

$$
\begin{align*}
X(t) & \overset{\text{def}}{=} \|(R^+ - 1, u^+, R^- - 1, u^-)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{\nu}{2} - 1})}^\ell \\
&+ \|(R^+ - 1, u^+, R^- - 1, u^-)\|_{L_t^1(\dot{B}_{2,1}^{\frac{\nu}{2} - 1})}^h \\
&+ \|(u^+, u^-)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{\nu}{2} - 1})}^h + \|(R^+ - 1, R^- - 1)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{\nu}{2} - 1})}^h \\
&+ \|(u^+, u^-)\|_{L_t^1(\dot{B}_{2,1}^{\frac{\nu}{2} - 1})}^h + \|(R^+ - 1, R^- - 1)\|_{L_t^1(\dot{B}_{2,1}^{\frac{\nu}{2} - 1})}^h. 
\end{align*} \quad (1.13)
$$

**Theorem 1.2.** Let the data $(R_0^+ - 1, u_0^+, R_0^- - 1, u_0^-)$ satisfy the assumptions of Theorem 1.1. Denote $\langle \tau \rangle \overset{\text{def}}{=} \sqrt{1 + \tau^2}$ and $\alpha \overset{\text{def}}{=} \min\{2 + \frac{\nu}{4}, \frac{\nu}{2} + \frac{1}{2} - \varepsilon\}$ with $\varepsilon > 0$ arbitrarily small. There exists a positive constant $c$ such that if in addition

$$
D_0 \overset{\text{def}}{=} \|(R_0^+ - 1, u_0^+, R_0^- - 1, u_0^-)\|_{\dot{B}_{2,\infty}^{\frac{\nu}{2} - 1}} \leq c, \quad (1.14)
$$

then the global solution $(R^+ - 1, u^+, R^- - 1, u^-)$ given by Theorem 1.1 satisfies for all $t \geq 0$,

$$
D(t) \leq C(D_0 + \|(\nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-)\|_{\dot{B}_{2,\infty}^{\frac{\nu}{2} - 1}}^h) \quad (1.15)
$$

with

$$
\begin{align*}
D(t) & \overset{\text{def}}{=} \sup_{\tau \in (\varepsilon - \frac{\nu}{4}, \varepsilon)} \|(\tau^{\frac{\nu}{2} + \frac{\varepsilon}{2}}(R^+ - 1, u^+, R^- - 1, u^-))\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{\nu}{2} - 1})}^\ell \\
&+ \|(\tau^\alpha(\nabla R^+, u^+, \nabla R^-, u^-))\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{\nu}{2} - 1})}^h \\
&+ \|\tau(\nabla u^+, \nabla u^-)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{\nu}{2} - 1})}^h. \quad (1.16)
\end{align*}
$$
Remark 1. In Theorem 1.2, we obtain the time decay rates for multi-dimensional non-conservative viscous compressible two-fluid system (1.1) in critical regularity framework. Additionally, the regularity index $s$ can take both negative and non-negative values, rather than only nonnegative integers, which improves the classical decay results in high Sobolev regularity, such as [14] when $f = P^+(\rho^+) = P^-(\rho^-)$. In fact, for the solution $(R^+ - 1, u^+, R^- - 1, u^-)$ constructed in Theorem 1.1, applying to homogeneous Littlewood-Paley decomposition for $R^+ - 1$, we have
\[
\|\Lambda^s(R^+ - 1)\|_{L^2} \lesssim \sum_{q \in \mathbb{Z}} \|\Delta_q\Lambda^s(R^+ - 1)\|_{L^2} = \|\Lambda^s(R^+ - 1)\|_{\dot{B}_{2,1}^s}.
\]

Based on Bernstein’s inequalities and the low-high frequencies decomposition, we may write
\[
\sup_{t \in [0,T]} \langle t \rangle^{\frac{N}{4} + \frac{\alpha}{2}} \|\Lambda^s(R^+ - 1)\|_{\dot{B}_{2,1}^s} \lesssim \|\langle t \rangle^{\frac{N}{4} + \frac{\alpha}{2}} (R^+ - 1)\|_{L_t^p(\dot{B}_{2,1}^s)} + \|\langle t \rangle^{\frac{N}{4} + \frac{\alpha}{2}} (R^+ - 1)\|_{L_t^p(\dot{B}_{2,1}^s)},
\]

If follows from Inequality (1.15) and definitions of $D(t)$ and $\alpha$ that
\[
\|\langle t \rangle^{\frac{N}{4} + \frac{\alpha}{2}} (R^+ - 1)\|_{L_t^p(\dot{B}_{2,1}^s)} \lesssim D_0 + \|\langle \nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\rangle\|_{\dot{B}_{2,1}^{s-1}},
\]
and that, because we have $\alpha \geq \frac{N}{4} + \frac{s}{2}$ for all $s \leq \min\{2, N/2\}$,
\[
\|\langle t \rangle^{\frac{N}{4} + \frac{\alpha}{2}} (R^+ - 1)\|_{L_t^p(\dot{B}_{2,1}^s)} \lesssim D_0 + \|\langle \nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\rangle\|_{\dot{B}_{2,1}^{s-1}}.
\]

This yields the following desired result for $R^+ - 1$
\[
\|\Lambda^s(R^+ - 1)\|_{L^2} \leq C(D_0 + \|\langle \nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\rangle\|_{\dot{B}_{2,1}^{s-1}}) \langle t \rangle^{-\frac{N}{4} - \frac{\alpha}{2}}
\]
if $-N/2 < s \leq \min\{2, N/2\}$,

where the fractional derivative operator $\Lambda^\ell$ is defined by $\Lambda^\ell f \equiv \mathcal{F}^{-1}(|\cdot|^\ell \mathcal{F} f)$.

Similarly, we have
\[
\|\Lambda^s u^+\|_{L^2} \leq C(D_0 + \|\langle \nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\rangle\|_{\dot{B}_{2,1}^{s-1}}) \langle t \rangle^{-\frac{N}{4} - \frac{\alpha}{2}}
\]
if $-N/2 < s \leq \min\{2, N/2 - 1\}$,
\[
\|\Lambda^s (R^- - 1)\|_{L^2} \leq C(D_0 + \|\langle \nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\rangle\|_{\dot{B}_{2,1}^{s-1}}) \langle t \rangle^{-\frac{N}{4} - \frac{\alpha}{2}}
\]
if $-N/2 < s \leq \min\{2, N/2\}$,
\[
\|\Lambda^s u^-\|_{L^2} \leq C(D_0 + \|\langle \nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\rangle\|_{\dot{B}_{2,1}^{s-1}}) \langle t \rangle^{-\frac{N}{4} - \frac{\alpha}{2}}
\]
if $-N/2 < s \leq \min\{2, N/2 - 1\}$.

In particular, taking $s = 0$ leads back to the standard optimal $L^1-L^2$ decay rate of $(R^+ - 1, u^+, R^- - 1, u^-)$ as in [14] when $N = 3$.

Remark 2. Due to the embedding $L^1(\mathbb{R}^3) \hookrightarrow \dot{B}_{2,\infty}^{-\frac{3}{2}}(\mathbb{R}^3)$, our results in Theorem 1.2 extend the known conclusions in [14]. In particular, our condition involves only the low frequencies of the data and is based on the $L^2(\mathbb{R}^3)$-norm framework. In particular, the decay rates of strong solutions is in the so-called critical Besov spaces in any dimension $N \geq 2$ and the dimension of space is more extensive and is not limited to $N = 3$. 


Remark 3. In this paper, we can not deal with the case of the model (1.1) with unequal pressure functions as in [14] since we take advantages of the symmetrizers methods in our process of the proof.

Notations. We assume $C$ be a positive generic constant throughout this paper that may vary at different places and denote $A \leq CB$ by $A \lesssim B$. We shall also use the following notations

$$z^\ell \overset{\text{def}}{=} \sum_{j \leq k_0} \hat{\Delta}_j z \quad \text{and} \quad z^h \overset{\text{def}}{=} z - z^\ell, \quad \text{for some } j_0.$$  

$$\|z\|_{B^{\ell}_{2,1}} \overset{\text{def}}{=} \sum_{j \leq k_0} 2^{js} \|\hat{\Delta}_j z\|_{L^2} \quad \text{and} \quad \|z\|_{B^{h}_{2,1}} \overset{\text{def}}{=} \sum_{j \geq k_0} 2^{js} \|\hat{\Delta}_j z\|_{L^2}, \quad \text{for some } j_0.$$  

Noting the small overlap between low and high frequencies, we have

$$\|z^\ell\|_{B^{h}_{2,1}} \lesssim \|z\|_{B^{\ell}_{2,1}} \quad \text{and} \quad \|z^h\|_{B^{\ell}_{2,1}} \lesssim \|z\|_{B^{h}_{2,2}}.$$  

2. Littlewood-Paley theory and some useful lemmas. Let us introduce the Littlewood-Paley decomposition. Choose a radial function $\varphi \in \mathcal{S}(\mathbb{R}^N)$ supported in $\mathcal{C} = \{ \xi \in \mathbb{R}^N, \frac{3}{4} \leq |\xi| \leq \frac{5}{3} \}$ such that

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \text{for all } \xi \neq 0.$$  

The homogeneous frequency localization operators $\hat{\Delta}_q$ and $\hat{S}_q$ are defined by

$$\hat{\Delta}_q f = \varphi(2^{-q}D)f, \quad \hat{S}_q f = \sum_{k \leq q-1} \hat{\Delta}_k f \quad \text{for } q \in \mathbb{Z}.$$  

With our choice of $\varphi$, one can easily verify that

$$\hat{\Delta}_q \hat{\Delta}_k f = 0 \quad \text{if } |q - k| \geq 2 \quad \text{and} \quad \hat{\Delta}_q (\hat{S}_{k-1} f \hat{\Delta}_k f) = 0 \quad \text{if } |q - k| \geq 5.$$  

We denote the space $\mathcal{Z}'(\mathbb{R}^N)$ by the dual space of $\mathcal{Z}(\mathbb{R}^N) = \{ f \in \mathcal{S}(\mathbb{R}^N); D^\alpha f(0) = 0; \forall \alpha \in \mathbb{N}^N \text{ multi-index} \}$. It also can be identified by the quotient space of $\mathcal{S}'(\mathbb{R}^N)/\mathcal{P}$ with the polynomials space $\mathcal{P}$. The formal equality

$$f = \sum_{q \in \mathbb{Z}} \hat{\Delta}_q f$$

holds true for $f \in \mathcal{Z}'(\mathbb{R}^N)$ and is called the homogeneous Littlewood-Paley decomposition.

The following Bernstein’s inequalities will be frequently used.

Lemma 2.1. [5] Let $1 \leq p_1 \leq p_2 \leq +\infty$. Assume that $f \in L^{p_1}(\mathbb{R}^N)$, then for any $\gamma \in (\mathbb{N} \cup \{0\})^N$, there exist constants $C_1$, $C_2$ independent of $f$, $q$ such that

$$\supp \hat{f} \subseteq \{ |\xi| \leq A_0 2^q \} \Rightarrow \|D^\gamma f\|_{p_2} \leq C_1 2^{q|\gamma|} q^{N(\frac{1}{p_1} - \frac{1}{p_2})} \|f\|_{p_1},$$

$$\supp \hat{f} \subseteq \{ A_1 2^q \leq |\xi| \leq A_2 2^q \} \Rightarrow \|f\|_{p_1} \leq C_2 2^{-q|\gamma|} \sup_{|\beta| = |\gamma|} \|\partial^\beta f\|_{p_1}.$$  

Definition 2.2. Let $s \in \mathbb{R}$, $1 \leq p, r \leq +\infty$. The homogeneous Besov space $\dot{B}^{s}_{p,r}$ is defined by

$$\dot{B}^{s}_{p,r} = \left\{ f \in \mathcal{Z}'(\mathbb{R}^N) : \|f\|_{\dot{B}^{s}_{p,r}} < +\infty \right\}.$$
Remark 4. Some properties about the Besov spaces are as follows

- Derivation:
  \[ \|f\|_{\dot{B}_{p,r}^s} \overset{\text{def}}{=} \left\| 2^{qs} \|\Delta_q f(t)\|_{L^p} \right\|_{t_r}. \]

- Algebraic properties: for \( s > 0, \dot{B}_{p,1}^s \cap L^\infty \) is an algebra;
- Interpolation: for \( s_1, s_2 \in \mathbb{R} \) and \( \theta \in [0,1] \), we have
  \[ \|f\|_{\dot{B}_{p,1}^{\theta s_1 + (1-\theta)s_2}} \leq \|f\|_{\dot{B}_{p,1}^{s_1}}^{\theta} \|f\|_{\dot{B}_{p,1}^{s_2}}^{(1-\theta)}. \]

Definition 2.3. Let \( s \in \mathbb{R}, 1 \leq p, \rho, r \leq +\infty \). The homogeneous space-time Besov space \( \dot{L}_T^p(B_{p,r}^s) \) is defined by

\[ \dot{L}_T^p(B_{p,r}^s) = \left\{ f \in \mathbb{R}^+ \times \mathbb{R}^+ : \|f\|_{L_T^p(B_{p,r}^s)} < +\infty \right\}, \]

where

\[ \|f\|_{L_T^p(B_{p,r}^s)} \overset{\text{def}}{=} \left\| \|2^{qs} \|\Delta_q f\|_{L^p} \right\|_{L_T^p}. \]

We next introduce the Besov-Chemin-Lerner space \( \tilde{L}_T^q(B_{p,r}^s) \) which is initiated in [6].

Definition 2.4. Let \( s \in \mathbb{R}, 1 \leq p, q, r \leq +\infty, 0 < T \leq +\infty \). The space \( \tilde{L}_T^q(B_{p,r}^s) \) is defined by

\[ \tilde{L}_T^q(B_{p,r}^s) = \left\{ f \in \mathbb{R}^+ \times \mathbb{R}^+ : \|f\|_{\tilde{L}_T^q(B_{p,r}^s)} < +\infty \right\}, \]

where

\[ \|f\|_{\tilde{L}_T^q(B_{p,r}^s)} \overset{\text{def}}{=} \left\| 2^{qs} \|\Delta_q f(t)\|_{L^q(0,T;L^r)} \right\|_{t_r}. \]

Obviously, \( \tilde{L}_T^1(B_{p,1}^s) = \dot{L}_T^1(B_{p,1}^s) \). By a direct application of Minkowski’s inequality, we have the following relations between these spaces

\[ L_T^p(B_{p,r}^s) \hookrightarrow \tilde{L}_T^p(B_{p,r}^s), \quad \text{if} \quad r \geq p, \]

\[ \tilde{L}_T^p(B_{p,r}^s) \hookrightarrow L_T^p(B_{p,r}^s), \quad \text{if} \quad \rho \geq r. \]

To deal with functions with different regularities for high frequencies and low frequencies, motivated by [11, 9], it is more effective to work in hybrid Besov spaces. We remark that using hybrid Besov spaces has been crucial for proving global well-posedness for compressible systems in critical spaces (see [7, 9, 11]).

Definition 2.5. Let \( s, t \in \mathbb{R} \). We set

\[ \|f\|_{\dot{B}_{2,1}^{s,t}} = \sum_{q \geq 0} 2^{qs} \|\Delta_q f\|_{L^2} + \sum_{q > 0} 2^{qt} \|\Delta_q f\|_{L^2}. \]

For \( m = \left[ \frac{s}{2} + 1 - s \right] \), we define

\[ \dot{B}_{2,1}^{s,t} = \left\{ f \in \mathcal{S}'(\mathbb{R}^N) : \|f\|_{\dot{B}_{2,1}^{s,t}} < \infty \right\}, \quad \text{if} \quad m < 0, \]

\[ \dot{B}_{2,1}^{s,t} = \left\{ f \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m : \|f\|_{\dot{B}_{2,1}^{s,t}} < \infty \right\}, \quad \text{if} \quad m \geq 0. \]

Remark 5. Some properties about the hybrid Besov spaces are as follows...
• \( \tilde{B}^{s,t}_{2,1} = \tilde{B}^{s}_{2,1} \).
• If \( s \leq t \), then \( \tilde{B}^{s,t}_{2,1} = \tilde{B}^{s}_{2,1} \cap \tilde{B}^{s}_{2,1} \). Otherwise, \( \tilde{B}^{s,t}_{2,1} = \tilde{B}^{s}_{2,1} + \tilde{B}^{s}_{2,1} \). In particular, \( \tilde{B}^{s}_{2,1} \rightarrow L^\infty \) as \( s \leq \frac{N}{2} \).
• Interpolation: for \( s_1, s_2, t_1, t_2 \in \mathbb{R} \) and \( \theta \in [0, 1] \), we have
\[
\|f\|_{\tilde{B}^{s_1s_2+(1-\theta)t_1,t_2}_{2,1}} \leq \|f\|_{\tilde{B}^{s_1}_{2,1}}^{\theta} \|f\|_{\tilde{B}^{s_2}_{2,1}}^{1-\theta};
\]
• If \( s_1 \leq s_2 \) and \( t_1 \geq t_2 \), then \( \tilde{B}^{s_1,t_1}_{2,1} \rightarrow \tilde{B}^{s_2,t_2}_{2,1} \).

We have the following properties for the product in Besov spaces and hybrid Besov spaces.

**Proposition 1.** [13] For all \( 1 \leq r, p, p_1, p_2 \leq +\infty \), there exists a positive universal constant such that
\[
\|fg\|_{\tilde{B}^{s}_{p,r}} \lesssim \|f\|_{L^\infty} \|g\|_{\tilde{B}^{s}_{p,r}} + \|g\|_{L^\infty} \|f\|_{\tilde{B}^{s}_{p,r}}, \quad \text{if } s > 0;
\]
\[
\|fg\|_{\tilde{B}^{s_1s_2-p}_{p,r}} \lesssim \|f\|_{\tilde{B}^{s_1}_{p,r}} \|g\|_{\tilde{B}^{s_2}_{p,r}}, \quad \text{if } s_1, s_2 < \frac{N}{p}, \text{ and } s_1 + s_2 > 0;
\]
\[
\|fg\|_{\tilde{B}^{s}_{p,r}} \lesssim \|f\|_{\tilde{B}^{s}_{p,r}} \|g\|_{\tilde{B}^{s}_{p,r} \cap L^\infty}, \quad \text{if } |s| < \frac{N}{p};
\]
\[
\|fg\|_{\tilde{B}^{s}_{2,1}} \lesssim \|f\|_{\tilde{B}^{s}_{2,1}} \|g\|_{\tilde{B}^{s}_{2,1}}, \quad \text{if } s \in (-N/2, N/2).
\]

**Proposition 2.** [10] For all \( s_1, s_2 > 0 \), there exists a positive universal constant such that
\[
\|fg\|_{\tilde{B}^{s_1s_2}_{2,1}} \lesssim \|f\|_{L^\infty} \|g\|_{\tilde{B}^{s_1s_2}_{2,1}} + \|g\|_{L^\infty} \|f\|_{\tilde{B}^{s_1s_2}_{2,1}}.
\]

For all \( s_1, s_2 \leq \frac{N}{2} \) such that \( \min\{s_1+t_1, s_2+t_2\} > 0 \), there exists a positive universal constant such that
\[
\|fg\|_{\tilde{B}^{s_1+t_1-s_2}_{2,1}} \lesssim \|f\|_{\tilde{B}^{s_1}_{2,1}} \|g\|_{\tilde{B}^{s_2}_{2,1}}.
\]

For the composition of functions, we have the following estimates.

**Proposition 3.** [9] Let \( s > 0, 1 \leq p, r \leq \infty \) and \( u \in \tilde{B}^{s}_{p,r} \cap L^\infty \).

(i) If \( F \in W^{1,\infty}_{loc}(\mathbb{R}^N) \) with \( F(0) = 0 \), then \( F(u) \in \tilde{B}^{s}_{p,r} \). Moreover, there exists a function of one variable \( C_0 \) depending only on \( s \) and \( F \), such that
\[
\|F(u)\|_{\tilde{B}^{s}_{p,r}} \leq C_0(\|u\|_{L^\infty}) \|u\|_{\tilde{B}^{s}_{p,r}}.
\]

(ii) If \( u, v \in \tilde{B}^{N}_{2,1}, (v-u) \in \tilde{B}^{N}_{2,1} \) for \( s \in (-\frac{N}{2}, \frac{N}{2}] \) and \( G \in W^{[\frac{N}{2}]+3,\infty}_{loc}(\mathbb{R}^N) \) satisfies \( G'(0) = 0 \), then \( G(v) - G(u) \in \tilde{B}^{N}_{2,1} \) and there exists a function of two variables \( C \) depending only on \( s, N \) and \( G \), such that
\[
\|G(v) - G(u)\|_{\tilde{B}^{N}_{2,1}} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty})(\|u\|_{\tilde{B}^{N}_{2,1}} + \|v\|_{\tilde{B}^{N}_{2,1}})\|v-u\|_{\tilde{B}^{N}_{2,1}}.
\]

(iii) If \( u, v \in \tilde{B}^{N}_{p,\infty} \cap L^\infty, (v-u) \in \tilde{B}^{N}_{p,\infty} \) with \( s \in (-\frac{N}{p}, \frac{N}{p}] \) and \( G \in W^{[\frac{N}{2}]+3,\infty}_{loc}(\mathbb{R}^N) \) satisfies \( G'(0) = 0 \), then \( G(v) - G(u) \in \tilde{B}^{N}_{p,\infty} \) and there exists a function of two variables \( C \) depending only on \( s, N \) and \( G \), such that
\[
\|G(v) - G(u)\|_{\tilde{B}^{N}_{p,\infty}} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty})(\|u\|_{\tilde{B}^{N}_{p,\infty} \cap L^\infty} + \|v\|_{\tilde{B}^{N}_{p,\infty} \cap L^\infty})\|v-u\|_{\tilde{B}^{N}_{p,\infty}}.
\]
Throughout this paper, the following estimates for the convection terms arising in the linearized systems will be used frequently.

**Proposition 4.** [9] Let $F$ be an homogeneous smooth function of degree $m$. Suppose that $-N/2 < s_1, t_1, s_2, t_2 \leq 1 + N/2$. Then, the following two estimates hold

$$
|(F(D)\partial_q (v \cdot \nabla u), F(D)\partial_q a)| \\
\leq C\gamma q 2^{-q(\phi^{\alpha,\beta}(q)-m)}\|v\|_{B_{2,1}^s} \|a\|_{B_{2,1}^{s_2}} \|F(D)\partial_q a\|_2,
$$

$$
|(F(D)\partial_q (v \cdot \nabla a), \partial_q b) + (\partial_q (v \cdot \nabla b), F(D)\partial_q a)| \\
\leq C\gamma q \|v\|_{B_{2,1}^s} \times (2^{-q(\phi^{\alpha,\beta}(q)-m)}\|F(D)\partial_q a\|_2 \|b\|_{B_{2,1}^{s_2}} \\
+ 2^{-q(\phi^{\alpha,\beta}(q)-m)}\|a\|_{B_{2,1}^{s_2}} \|\partial_q b\|_2),
$$

where $(\cdot, \cdot)$ denotes the 2-inner product, $\sum_{q \in \mathbb{Z}} \gamma q \leq 1$ and the operator $F(D)$ is defined by $F(D)f := F^{-1}F(\xi)Ff$, $\phi^{\alpha,\beta}(q)$ is the following characteristic function on $\mathbb{Z}$

$$
\phi^{\alpha,\beta}(q) = \begin{cases} 
\alpha, & \text{if } q \leq 0, \\
\beta, & \text{if } q \geq 1.
\end{cases}
$$

**Proposition 5.** [11] Let $1 \leq p, p_1 \leq \infty$, $1 \leq r \leq \infty$ and $\sigma \in \mathbb{R}$. There exists a constant $C > 0$ depending only on $\sigma$ such that for all $q \in \mathbb{Z}$, we have

$$
\|\|v \cdot \nabla, \partial_t \partial_q^s a\|\|_L^p \leq C\gamma q 2^{-q(\sigma-1)}\|\nabla v\|_{B_{2,1}^s} \|\nabla a\|_{B_{p,1}^{\sigma-1}},
$$

for $\min(\frac{N}{p_1}, \frac{N}{p}) < \sigma \leq 1 + \min(\frac{N}{p_1}, \frac{N}{p})$,

$$
\|\|v \cdot \nabla, \partial_q^s a\|\|_L^p \leq C\gamma q 2^{-q\sigma}\|\nabla v\|_{B_{p_1,\infty}^s \cap L^\infty} \|a\|_{B_{p_1}^{\sigma}},
$$

for $\min(\frac{N}{p_1}, \frac{N}{p}) < \sigma < 1 + \frac{N}{p_1}$,

where the commutator $[\cdot, \cdot]$ is defined by $[f, g] = fg - gf$ and $(c_j)_{j \in \mathbb{Z}}$ denotes a sequence such that $\sum_{q \in \mathbb{Z}} c_q \leq 1$.

**Proposition 6.** [1] Assume $\mu > 0$, $\sigma \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$ and $1 \leq p_2 \leq p_1 \leq \infty$. Let $u$ satisfy

$$
\begin{cases} 
\partial_t u - \mu \Delta u = f, \\
u |_{t=0} = u_0.
\end{cases}
$$

Then for all $T > 0$ the following a priori estimate is fulfilled

$$
\mu^{\frac{2}{\gamma}} \|u\|_{L^\infty_T L^{s_1, \frac{m}{s_1}}(B_{p_1, r})} \lesssim \|u_0\|_{B_{p_1, r}^{s_1}} + \mu^{\frac{2}{\gamma}-1} \|f\|_{L^\infty_T L^{s_2, \frac{m}{s_2}}(B_{p_1, r})},
$$

**Remark 6.** The solutions to the following Lamé system

$$
\begin{cases} 
\partial_t u - Au = f, \\
u |_{t=0} = u_0,
\end{cases}
$$

also satisfy (2.18). Here, $Au = \mu \Delta u + (\lambda + \mu)\nabla \text{div} u$. 

Proposition 7. [10] Let \( s \in \left( -N \min\left( \frac{1}{p}, \frac{1}{p'} \right), 1 + \frac{N}{p} \right) \), \( 1 \leq p, r \leq +\infty \), and \( s = 1 + \frac{N}{p} \) if \( r = 1 \). Let \( v \) be a vector field such that \( \nabla v \in L_T^1(\mathcal{B}_{p,r}^s \cap L^\infty) \). Assume that \( u_0 \in \mathcal{B}_{p,r}^s \), \( g \in L_T^1(\mathcal{B}_{p,r}^s) \) and \( u \) is the solution of the following transport equation
\[
\begin{aligned}
\frac{\partial}{\partial t} u + v \cdot \nabla u &= g, \\
|u|_{t=0} &= u_0.
\end{aligned}
\] (2.19)

Then there holds for \( t \in [0, T] \),
\[
\|u\|_{L_T^p(\mathcal{B}_{p,r}^s)} \leq e^{Cv(t)} \left( \|u_0\|_{\mathcal{B}_{p,r}^s} + \int_0^t e^{-Cv(\tau)} \|g(\tau)\|_{\mathcal{B}_{p,r}^s} \, d\tau \right),
\] (2.20)
where \( V(t) \) is defined as
\[
V(t) = \int_0^t \|\nabla v(\tau)\|_{\mathcal{B}_{p,r}^s \cap L^\infty} \, d\tau.
\] If \( r < +\infty \), then \( f \) belongs to \( C([0, T]; \mathcal{B}_{p,r}^s) \).

We finish this subsection by listing an elementary but useful inequality.

Lemma 2.6. [26] Let \( r_1, r_2 > 0 \) satisfy \( \max\{r_1, r_2\} > 1 \). Then
\[
\int_0^t \frac{1}{(1 + t - \tau)^{-r_1}(1 + \tau)^{-r_2}} \, d\tau \leq C(r_1, r_2)(1 + t)^{-\min\{r_1, r_2\}}.
\]

3. Reformulation of the system (1.8) and A priori estimates for the linearized system with convection terms.

3.1. Reformulation of the system (1.8). To make it more convenient to study, we need some reformulations of (1.8). More precisely, taking a change of variables by
\[
c^\pm = R^\pm - 1.
\]
Then, the system (1.8) can be rewritten as
\[
\begin{aligned}
\partial_t c^+ + \text{div} u^+ &= H_1, \\
\partial_t u^+ + \beta_1 \nabla c^+ + \beta_2 \nabla c^- - \nu_1^+ \Delta u^+ - \nu_2^+ \nabla \text{div} u^+ &= H_2, \\
\partial_t c^- + \text{div} u^- &= H_3, \\
\partial_t u^- + \beta_3 \nabla c^+ + \beta_4 \nabla c^- - \nu_1^- \Delta u^- - \nu_2^- \nabla \text{div} u^- &= H_4,
\end{aligned}
\] (3.21)
with initial data
\[
(c^+, u^+, c^-, u^-)|_{t=0} = (c^+_0, u^+_0, c^-_0, u^-_0),
\] (3.22)
where
\[
\beta_1 = \frac{c^2(1,1)\rho^-(1,1)}{\rho^+(1,1)}, \quad \beta_2 = \beta_3 = C^2(1,1), \quad \beta_4 = \frac{c^2(1,1)\rho^+(1,1)}{\rho^-(1,1)}, \quad \nu_1^+ = \frac{\mu^+}{\rho^-(1,1)}, \quad \nu_2^+ = \frac{\mu^+ + \lambda^+}{\rho^+(1,1)}
\]
and the source terms are
\[
H_1 = -\text{div}(c^+ u^+), \quad H_2^+ = -g_+(c^+, c^-) \partial_t c^+ - \tilde{g}_+(c^+, c^-) \partial_t c^- - (u^+ \cdot \nabla) u_i^+ + \mu^+ h_+(c^+, c^-) \partial_j c^+ \partial_j u_i^+ + \mu^+ k_+(c^+, c^-) \partial_j c^- \partial_j u_i^+ + \mu^+ l_+(c^+, c^-) \partial_j c^- \partial_j u_i^+, \quad i, j \in \{1, 2, \cdots, N\},
\]
\[
H_3 = -\text{div}(c^- u^-), \quad H_4^+ = -g_-(c^+, c^-) \partial_t c^- - \tilde{g}_-(c^+, c^-) \partial_t c^+ - (u^- \cdot \nabla) u_i^-.
\] (3.23)
with convection terms. Next, we investigate some a priori estimates for the following linearized system with convection terms

\[
\begin{align*}
\partial_t c^+ + v^+ \cdot \nabla c^+ + \text{div} u^+ &= H_{31}(c^+, u^+), \\
\partial_t u^+ + v^+ \cdot \nabla u^+ + \beta_1 \nabla c^+ + \beta_2 \nabla c^- - \nu_1^+ \Delta u^+ - \nu_2^+ \nabla \text{div} u^+ &= H_{21}(c^+, u^+, c^-), \\
\partial_t c^- + v^- \cdot \nabla c^- + \text{div} u^- &= H_{31}(c^-, u^-), \\
\partial_t u^- + v^- \cdot \nabla u^- + \beta_3 \nabla c^- + \beta_4 \nabla c^+ - \nu_1^- \Delta u^- - \nu_2^- \nabla \text{div} u^- &= H_{41}(c^-, u^-, c^+), \\
(c^+, u^+, c^-, u^-)|_{t=0} &= (c_0^+, u_0^+, c_0^-, u_0^-).
\end{align*}
\]

We will establish a uniform estimate for a mixed hyperbolic-parabolic linear system (3.32) with convection terms. What is crucial in this work is to exploit the smoothing effects on the velocity $u^+, u^-$ and the $L^1$ decay on $c^+, c^-$, which play a key role to control the pressure term in the proof of the Theorem 1.1.

**Lemma 3.1.** Denote

\[
V(t) := \int_0^t \|(v^+, v^-)(\tau)\|_{B_{2,1}^{s_2}}^2 d\tau.
\]
Let $T > 0$ and $(c^+, u^+, c^-, u^-)$ be a solution of the system (3.32). Then the following estimates hold for $t \in [0, T)$

$$
\|(c^+, c^-)\|_{L^\infty([0, t] ; B_{2,1}^{2} ; \mathbb{R}^2)} + \|(u^+, u^-)\|_{L^\infty([0, t] ; B_{2,1}^{2} ; \mathbb{R}^2)}
\leq \int_0^t \|(c^+, c^-)\|_{B_{2,1}^{2}} \, dt + \int_0^t \|(u^+, u^-)\|_{B_{2,1}^{2}} \, dt + \int_0^t \|(H_{11}, H_{31})\|_{B_{2,1}^{2}} \, dt.
$$

(3.33)

Proof. Applying the operator $\Delta_q$ to the system (3.32), we deduce that $(\Delta_q c^+, \Delta_q u^+, \Delta_q c^-, \Delta_q u^-)$ satisfies

$$
\begin{cases}
\partial_t \Delta_q c^+ + \Delta_q (v^+ \cdot \nabla c^+) + \text{div} \Delta_q u^+ = \Delta_q H_{11}, \\
\partial_t \Delta_q u^+ + \Delta_q (v^+ \cdot \nabla u^+) - \nu_1^+ \Delta \Delta_q u^+ - \nu_2^+ \nabla \text{div} \Delta_q u^+ + \beta_1 \nabla \Delta_q c^+ + \beta_2 \nabla \Delta_q c^- = \Delta_q H_{21}, \\
\partial_t \Delta_q c^- + \Delta_q (v^- \cdot \nabla c^-) + \text{div} \Delta_q u^- = \Delta_q H_{31}, \\
\partial_t \Delta_q u^- + \Delta_q (v^- \cdot \nabla u^-) - \nu_1^- \Delta \Delta_q u^- - \nu_2^- \nabla \text{div} \Delta_q u^- + \beta_1 \nabla \Delta_q c^+ + \beta_2 \nabla \Delta_q c^- = \Delta_q H_{41}.
\end{cases}
$$

(3.34)

Taking the $L^2$-scalar product of the first equation of (3.34) with $\Delta_q c^+$ and $-\Delta \Delta_q c^+$, the second equation with $\Delta_q u^+$, the third equation with $\Delta_q c^-$ and $-\Delta \Delta_q c^-$ and the fourth equation with $\Delta_q u^-$ respectively, we obtain the following six identities:

$$
\frac{1}{2} \frac{d}{dt} \|\Delta_q c^+\|_{L^2}^2 + (\Delta_q (v^+ \cdot \nabla c^+) | \Delta_q c^+) + (\text{div} \Delta_q u^+ | \Delta_q c^+) = (\Delta_q H_{11} | \Delta_q c^+),
$$

(3.35)

$$
\frac{1}{2} \frac{d}{dt} \|\nabla \Delta_q c^+\|_{L^2}^2 + (\Delta_q (v^+ \cdot \nabla c^+) | \Delta_q c^+) + (\text{div} \Delta_q u^+ | \Delta_q c^+) = (\Delta_q H_{11} | \Delta_q c^+) - \Delta \Delta_q c^+),
$$

(3.36)

$$
\frac{1}{2} \frac{d}{dt} \|\Delta_q u^+\|_{L^2}^2 + (\Delta_q (v^+ \cdot \nabla u^+) | \Delta_q u^+) + \nu_1^+ \|\nabla \Delta_q u^+\|_{L^2}^2 + \nu_2^+ \|\text{div} \Delta_q u^+\|_{L^2}^2 + \beta_1 (\nabla \Delta_q c^+ | \Delta_q u^+) + \beta_2 (\nabla \Delta_q c^- | \Delta_q u^+) = (\Delta_q H_{21} | \Delta_q u^+),
$$

(3.37)

$$
\frac{1}{2} \frac{d}{dt} \|\Delta_q c^-\|_{L^2}^2 + (\Delta_q (v^- \cdot \nabla c^-) | \Delta_q c^-) + (\text{div} \Delta_q u^- | \Delta_q c^-) = (\Delta_q H_{31} | \Delta_q c^-),
$$

(3.38)

$$
\frac{1}{2} \frac{d}{dt} \|\nabla \Delta_q c^-\|_{L^2}^2 + (\Delta_q (v^- \cdot \nabla c^-) | \Delta_q c^-) + (\text{div} \Delta_q u^- | \Delta_q c^-) = (\Delta_q H_{31} | \Delta_q c^-),
$$

(3.39)
where $(\cdot, \cdot)$ stands for the $L^2$ inner product.

In order to obtain a second energy estimate, we need to derive some identities involving $(\Delta_q c^+ | \Delta_q c^-)$, $(\nabla \Delta_q c^+ | \Delta_q u^+)$, $(\nabla \Delta_q c^- | \Delta_q u^-)$. Taking the $L^2$-scalar product of the first equation of (3.34) with $\Delta_q c^-$ and the third equation with $\Delta_q c^+$ and then summing the results, which yields

\[
\frac{d}{dt} (\Delta_q c^+ | \Delta_q c^-) + (\Delta_q (v^+ \cdot \nabla c^+) | \Delta_q c^-) + (\Delta_q (v^- \cdot \nabla c^-) | \Delta_q c^+ ) + (\text{div} \Delta_q u^+ | \Delta_q c^-) + (\text{div} \Delta_q u^- | \Delta_q c^+ ) = (\Delta_q H_{11} | \Delta_q c^-) + (\Delta_q H_{21} | \Delta_q c^+ ).
\]

On the other hand, applying the operator $\nabla$ to the first equation in (3.2) and taking the $L^2$ scalar product with $\Delta_q u^+$, then calculating the scalar product of the second equation in (3.2) with $\nabla \Delta_q c^+$, and then summing up the results, we get

\[
\frac{d}{dt} (\nabla \Delta_q c^+ | \Delta_q u^+) - \| \text{div} \Delta_q u^+ \|_{L^2}^2 + |\nu_1^+ + \nu_2^+| (\text{div} \Delta_q u^+ | \nabla \Delta_q c^+) + \beta_1 \| \nabla \Delta_q c^+ \|_{L^2}^2
\]

\[+ \beta_2 (\nabla \Delta_q c^- | \nabla \Delta_q c^+) + (\nabla \Delta_q (v^+ \cdot \nabla c^+) | \Delta_q u^+) + (\nabla \Delta_q (v^- \cdot \nabla c^-) | \Delta_q u^- ) = (\nabla \Delta_q H_{11} | \Delta_q u^+) + (\Delta_q H_{21} | \nabla \Delta_q c^+).
\]

Similarly,

\[
\frac{d}{dt} (\nabla \Delta_q c^- | \Delta_q u^-) - \| \text{div} \Delta_q u^- \|_{L^2}^2 + |\nu_1^- + \nu_2^-| (\text{div} \Delta_q u^- | \nabla \Delta_q c^-) + \beta_3 (\nabla \Delta_q c^+ | \nabla \Delta_q c^-) + \beta_4 \| \nabla \Delta_q (v^- \cdot \nabla c^-) | \Delta_q u^- )
\]

\[+ (\nabla \Delta_q (v^+ \cdot \nabla u^+) | \nabla \Delta_q c^-) = (\nabla \Delta_q H_{31} | \Delta_q u^-) + (\Delta_q H_{41} | \nabla \Delta_q c^-).
\]

Define

\[
\alpha_q^2 = \beta_1 \| \Delta_q c^+ \|_{L^2}^2 + \beta_2 \| \Delta_q c^- \|_{L^2}^2 + A (\nu_1^+ + \nu_2^+) \| \nabla \Delta_q c^+ \|_{L^2}^2
\]

\[+ A (\nu_1^- + \nu_2^-) \| \nabla \Delta_q c^- \|_{L^2}^2 + \| \Delta_q u^+ \|_{L^2}^2 + \| \Delta_q u^- \|_{L^2}^2 + 2 \beta_2 (\Delta_q c^+ | \Delta_q c^-) + 2 A (\nabla \Delta_q c^+ | \Delta_q u^+) + 2 A (\nabla \Delta_q c^- | \Delta_q u^-),
\]

where $A = \frac{1}{4} \min \{\nu_1^+, \nu_2^-\} > 0$. Taking $\mu_1 \in (\frac{1}{11}, \frac{1}{3})$ and employing Young’s inequality, we obtain

\[
|2 A (\nabla \Delta_q c^+ | \Delta_q u^+)| \leq M_1 A \| \Delta_q u^+ \|_{L^2}^2 + \frac{A}{M_1} \| \nabla \Delta_q c^+ \|_{L^2}^2
\]

\[\leq \| \Delta_q u^+ \|_{L^2}^2 + A (\nu_1^+ + \nu_2^+) \| \nabla \Delta_q c^+ \|_{L^2}^2,
\]

\[
|2 A (\nabla \Delta_q c^- | \Delta_q u^-)| \leq M_1 A \| \Delta_q u^- \|_{L^2}^2 + \frac{A}{M_1} \| \nabla \Delta_q c^- \|_{L^2}^2
\]

\[\leq \| \Delta_q u^- \|_{L^2}^2 + A (\nu_1^- + \nu_2^-) \| \nabla \Delta_q c^- \|_{L^2}^2.
\]
Using further $\beta_2^2 = \beta_3^2 = \beta_1 \beta_4$ and Young’s inequality, and choosing $M_2 = \frac{\beta_1}{\beta_2}$, we get

$$2\beta_2 \|\Delta_qc^+|\Delta_qc^-\| \leq M_2\beta_2 \|\Delta_qc^+\|_{L^2}^2 + \frac{\beta_2}{M_2} \|\Delta_qc^-\|_{L^2}^2,$$

$$\leq \beta_1 \|\Delta_qc^+\|_{L^2}^2 + \beta_4 \|\Delta_qc^-\|_{L^2}^2,$$

$$2\beta_2 \|\nabla \Delta_qc^+|\nabla \Delta_qc^-\| \leq M_2\beta_2 \|\nabla \Delta_qc^+\|_{L^2}^2 + \frac{\beta_2}{M_2} \|\nabla \Delta_qc^-\|_{L^2}^2,$$

$$\leq \beta_1 \|\nabla \Delta_qc^+\|_{L^2}^2 + \beta_4 \|\nabla \Delta_qc^-\|_{L^2}^2.$$

Then, there exist two positive constants $c_1$ and $c_2$ such that

$$c_1 \alpha_q^2 \leq \|\Delta_qc^+\|_{L^2}^2 + \|\Delta_qc^-\|_{L^2}^2 + \|\nabla \Delta_qc^+\|_{L^2}^2 + \|\nabla \Delta_qc^-\|_{L^2}^2$$

$$+ \|\Delta_qu^+\|_{L^2}^2 + \|\Delta_qu^-\|_{L^2}^2 \leq c_2 \alpha_q^2.$$ 

Thus, for some fix $q_0$,

$$\alpha_q \approx \begin{cases} \|\Delta_qc^+, \Delta_qc^-, \Delta_qu^-\|_{L^2}, & \text{for } q \leq q_0, \\ \|\nabla \Delta_qc^+, \nabla \Delta_qc^-, \nabla \Delta_qc^-\|_{L^2}, & \text{for } q > q_0. \end{cases}$$

Combining with (3.35)-(3.43), it yields, with the help of Proposition 4, that

$$\frac{1}{2} d q^2 + (\nu_2 - A)\|\nabla \Delta_qu^+\|_{L^2}^2 + (\nu_2 - A)\|\nabla \Delta_qu^-\|_{L^2}^2 + \nu_1 \|\nabla \Delta_qu^+\|_{L^2}^2$$

$$+ 2\beta_2 \|\nabla \Delta_qc^+|\nabla \Delta_qc^-\|$$

$$= -\beta_1 (\Delta_q(v^+ \cdot \nabla c^+)|\Delta_qc^+) + 2A(\nu_1^+ + \nu_2^-)(\Delta_q(v^+ \cdot \nabla c^+)|\Delta_qc^+)$$

$$- (\Delta_q(v^+ \cdot \nabla c^+)|\Delta_qu^+) - \beta_4 (\Delta_q(v^- \cdot \nabla c^-)|\Delta_qc^+)$$

$$+ 2A(\nu_1^- + \nu_2^-)(\Delta_q(v^- \cdot \nabla c^-)|\Delta_qc^+) - (\Delta_q(v^- \cdot \nabla u^-)|\Delta_qu^-)$$

$$- \beta_2 (\Delta_q(v^+ \cdot \nabla c^+)\Delta_qc^-) - \beta_3 (\Delta_q(v^- \cdot \nabla c^-)|\Delta_qc^+)$$

$$- A(\nabla \Delta_q(v^+ \cdot \nabla c^+)|\Delta_qu^+) - A(\Delta_q(v^+ \cdot \nabla u^+)|\nabla \Delta_qc^+)$$

$$- A(\nabla \Delta_q(v^- \cdot \nabla c^-)|\Delta_qu^-) - A(\Delta_q(v^- \cdot \nabla u^-)|\nabla \Delta_qc^-)$$

$$+ (\Delta_qH_{11}|\Delta_qc^+) = 2A(\nu_1^+ + \nu_2^-)(\Delta_qH_{11}) - \Delta \Delta_qc^+$$

$$+ (\Delta_qH_{21}|\Delta_qu^+) + (\Delta_qH_{31}|\Delta_qc^-) - 2A(\nu_1^- + \nu_2^-)(\Delta_qH_{21}) - \Delta \Delta_qc^-$$

$$+ (\Delta_qH_{31}|\Delta_qu^-) + \beta_2 (\Delta_qH_{11}|\Delta_qc^-) + \beta_3 (\Delta_qH_{31}|\Delta_qc^+)$$

$$+ A(\nabla \Delta_qH_{11}|\Delta_qu^+)$$

$$+ A(\Delta_qH_{21}|\nabla \Delta_qc^+) + A(\nabla \Delta_qH_{31}|\Delta_qu^-) + A(\Delta_qS_{31}|\nabla \Delta_qc^-)$$

$$\lesssim \gamma_q \left( \|\Delta_qH_{11}\|_{L^2} + \|\nabla \Delta_qH_{11}\|_{L^2} + \|\Delta_qH_{21}\|_{L^2} + \|\Delta_qH_{31}\|_{L^2} + \|\nabla \Delta_qH_{31}\|_{L^2} + \|\Delta_qH_{41}\|_{L^2} 

+ 2^{-\frac{3}{2}}(\gamma_q)^2 \|\tilde{\gamma_q}V\|_{(\Delta_qu^+, \Delta_qu^+, \Delta_qu^-)} \right).$$

(3.44)

with $(\gamma_q)_{q \in \mathbb{Z}}$ in the unit sphere of $\ell^1(\mathbb{Z})$. 
Thus, by Gronwall’s inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \alpha_q^2 + c_0 \min(2^{2q}, 1) \alpha_q^2 \\lesssim \gamma_q 2^{\left(\frac{q}{2} - 1\right)q} \left[ \|H_{11}\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} + \|H_{21}\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} + \|H_{31}\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} + \|H_{41}\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} + \|V'||\|c^+, u^+, c^-, u^-\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} \times \dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} \times \dot{B}^{\frac{q}{2} - 1, q}_{2, 1})\right] \alpha_q,
\]
which implies that
\[
2^{\left(\frac{q}{2} - 1\right)q} \alpha_q + c_0 \int_0^t \min(2^{2q}, 1)2^{\left(\frac{q}{2} - 1\right)q} \alpha_q(\tau) d\tau \lesssim 2^{\left(\frac{q}{2} - 1\right)q} \alpha_q(0) + C \gamma_q \int_0^t \left[ \|H_{11}\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} + \|H_{21}\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} + \|H_{31}\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} + \|H_{41}\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} + \|V'||\sum_q 2^{\left(\frac{q}{2} - 1\right)q} \alpha_q(\tau) \right] d\tau.
\]
Thus, by Gronwall’s inequality, we have
\[
\left\| (c^+, c^-) \right\|_{L^\infty([0,t]; \dot{B}^{\frac{q}{2} - 1, q}_{2, 1})} + \left\| (u^+, u^-) \right\|_{L^\infty([0,t]; \dot{B}^{\frac{q}{2} - 1, q}_{2, 1})} \\
+ \int_0^t \left\| (c^+, c^-)(\tau) \right\|_{\dot{B}^{\frac{q}{2} + 1, q}_{2, 1}} d\tau + \int_0^t \left\| (u^+, u^-)(\tau) \right\|_{\dot{B}^{\frac{q}{2} + 1, q}_{2, 1}} d\tau \lesssim e^{C\nu(t)} \left( \left\| (c_0^+, c_0^-) \right\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} + \left\| (u_0^+, u_0^-) \right\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} \right) \]
\[
+ \int_0^t \|H_{11}, H_{31}\)(\tau)\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} d\tau + \int_0^t \|H_{21}, H_{41}\)(\tau)\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}} d\tau).
\]
Based on the damping effects for $c^+$ and $c^-$, we further exploit the smoothing effects of $u^+$ and $u^-$ in high frequencies regime by considering (3.32) with $\nabla c^+$ and $\nabla c^-$ being viewed as source terms. From (3.37) and (3.40), we have
\[
\frac{1}{2} \frac{d}{dt} \|\tilde{\Delta} q u^+\|_{L^2}^2 + C 2^{2q} \|\tilde{\Delta} q u^+\|_{L^2}^2 \\
\lesssim -(\tilde{\Delta} q (v^+ \cdot \nabla u^+)) \|\tilde{\Delta} q u^+\| - \beta_1 (\nabla \tilde{\Delta} q e^+ \|\tilde{\Delta} q u^+\|) - \beta_2 (\nabla \tilde{\Delta} q e^- \|\tilde{\Delta} q u^+\|) \\
+ (\tilde{\Delta} q H_{21}\|\tilde{\Delta} q u^+\|),
\]
\[
\frac{1}{2} \frac{d}{dt} \|\tilde{\Delta} q u^-\|_{L^2}^2 + C 2^{2q} \|\tilde{\Delta} q u^-\|_{L^2}^2 \\
\lesssim -(\tilde{\Delta} q (v^- \cdot \nabla u^-)) \|\tilde{\Delta} q u^-\| - \beta_3 (\nabla \tilde{\Delta} q e^+ \|\tilde{\Delta} q u^-\|) - \beta_4 (\nabla \tilde{\Delta} q e^- \|\tilde{\Delta} q u^-\|) \\
+ (\tilde{\Delta} q H_{41}\|\tilde{\Delta} q u^-\|).
\]
It follows that from Proposition 4
\[
\frac{d}{dt} \sum_{q \geq q_0} 2^{\left(\frac{q}{2} - 1\right)q} \|\tilde{\Delta} q u^+\|_{L^2} + C \sum_{q \geq q_0} 2^{\left(\frac{q}{2} - 1\right)q} 2^{2q} \|\tilde{\Delta} q u^+\|_{L^2} \\
\lesssim \sum_{q \geq q_0} 2^{\left(\frac{q}{2} - 1\right)q} \left(2^q \|\tilde{\Delta} q e^+\|_{L^2} + 2^q \|\tilde{\Delta} q e^-\|_{L^2} + \|\tilde{\Delta} q H_{21}\|_{L^2} \\
+ V'(t) \gamma_q 2^{-\left(\frac{q}{2} - 1\right)q} \|u^+\|_{\dot{B}^{\frac{q}{2} - 1, q}_{2, 1}}\right)
Combining with (3.46) and (3.47), we finally conclude that (3.33). Thus, we complete the proof of Lemma 3.1.

\[ \int_0^t C \sum_{q \geq q_0} 2^q (\| \Delta_q u^+ \|_{L^2} + \| \Delta_q u^- \|_{L^2}) \, dt \]

\[ \lesssim e^{C_{\text{CV}(t)}} \left( \| (c_0^+ + c_0^-) \|_{\dot{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \| (u_0^+ + u_0^-) \|_{\dot{B}^{\frac{N}{2}-1}} \right) \]

\[ \quad + \int_0^\tau \| (H_{11}, H_{31})(\tau) \|_{\dot{B}^{\frac{N}{2}-1, \frac{N}{2}}} \, d\tau + \int_0^\tau \| (H_{21}, H_{41})(\tau) \|_{\dot{B}^{\frac{N}{2}-1}} \, d\tau. \] \hspace{1cm} (4.47)

Combining with (3.46) and (4.47), we finally conclude that (3.33). Thus, we complete the proof of Lemma 3.1. \[ \square \]

4. Global existence for initial data near equilibrium. In this section, we show that if the initial data satisfy

\[ \| (R_0^+ - 1, R_0^- - 1) \|_{\dot{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \| (u_0^+ + u_0^-) \|_{\dot{B}^{\frac{N}{2}-1}} \leq \eta, \]

for some sufficiently small \( \eta \), then there exists a positive constant \( M \) such that

\[ X(t) \leq M\eta. \]

This uniform estimate will enable us to extend the local solution \( (R^+ - 1, u^+, R^- - 1, u^-) \) obtained within an iterative scheme as in [9] to a global one. To this end, we use a contradiction argument. Define

\[ T_0 = \sup \{ T \in [0, \infty) : X(T) \leq M\eta \}, \]

with \( M \) to be determined later. Suppose that \( T_0 < \infty \). We apply the linear estimates in Lemma 3.1 to the solutions of the reformulated system (3.21) such that for all \( t \in [0, T_0] \), the following estimates hold

\[ X(T_0) \lesssim e^{C_{\text{CV}(t)}} \left( \| (c_0^+, c_0^-) \|_{\dot{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \| (u_0^+, u_0^-) \|_{\dot{B}^{\frac{N}{2}-1}} \right) \]

\[ \quad + \int_0^{T_0} \| (H_{11}, H_{31})(\tau) \|_{\dot{B}^{\frac{N}{2}-1, \frac{N}{2}}} \, d\tau + \int_0^{T_0} \| (H_{21}, H_{41})(\tau) \|_{\dot{B}^{\frac{N}{2}-1}} \, d\tau. \] \hspace{1cm} (4.48)

where

\[ V(T_0) = \int_0^{T_0} \| (u^+, u^-)(\tau) \|_{\dot{B}^{\frac{N}{2}+1}} \, d\tau. \]
In what follows, we derive some estimates for the nonlinear terms $H_{11} - H_{41}$. First, by Proposition 2, we have

$$
\| (H_{11}, H_{31}) \|_{L^0_t(B^{s_{-1}, \frac{N}{2}}_{2,1})} \lesssim \| c^+ \|_{L^0_t(B^{s_{-1}, \frac{N}{2}}_{2,1})} \| \text{div} u^+ \|_{L^1_t(B^{s, \frac{N}{2}}_{2,1})} + \| c^- \|_{L^0_t(B^{s_{-1}, \frac{N}{2}}_{2,1})} \| \text{div} u^- \|_{L^1_t(B^{s, \frac{N}{2}}_{2,1})} \tag{4.49}
$$

Next, we bound the terms $H_{21}$ and $H_{41}$. By the embedding $B^{s_{-1}, \frac{N}{2}}_{2,1} \hookrightarrow B^{s, \frac{N}{2}}_{2,1} \hookrightarrow L^\infty$ and Proposition 3 (ii), we get

$$
\| l_+(c^+, c^-) \|_{B^{s, \frac{N}{2}}_{2,1}} \lesssim C_0 \left( \| c^+ \|_{L^\infty}, \| c^- \|_{L^\infty} \right) \left( \| c^+ \|_{B^{s, \frac{N}{2}}_{2,1}} + \| c^- \|_{B^{s, \frac{N}{2}}_{2,1}} \right)
$$

thus, thanks to Proposition 1, we easily infer

$$
\| l_+(c^+, c^-) \|_{L^1_t(B^{s, \frac{N}{2}}_{2,1})} \lesssim \left( \| c^+ \|_{L^\infty_t(B^{s_{-1}, \frac{N}{2}}_{2,1})} + \| c^- \|_{L^\infty_t(B^{s_{-1}, \frac{N}{2}}_{2,1})} \right) \| \partial_j^2 u^+ \|_{L^1_t(B^{s, \frac{N}{2}}_{2,1})} \lesssim M^2 \eta^2.
$$

Similarly,

$$
\| k_+(c^+, c^-) \|_{L^1_t(B^{s, \frac{N}{2}}_{2,1})} \lesssim M^2 \eta^2 + M^3 \eta^3.
$$
Hence, we gather that

\[ \| u^+ \cdot \nabla u^+ \|_{L^1_t(B_{2,1}^\infty)} \lesssim \| u^+ \|_{L^\infty_t(B_{2,1}^\infty)} \| \nabla u^+ \|_{L^1_t(B_{2,1}^\infty)} \]

\[ \lesssim \| u^+ \|_{L^\infty_t(B_{2,1}^\infty)} \| u^+ \|_{L^1_t(B_{2,1}^\infty)} \lesssim M^2 \eta^2. \]

According to Proposition 3(ii) and interpolation inequality, we have

\[ \| g_+(c^+, c^-) \partial_i c^+ - \tilde{g}_+(c^+, c^-) \partial_i c^- \|_{L^2_t(B_{2,1}^\infty)} \lesssim \| g_+(c^+, c^-) \|_{L^2_t(B_{2,1}^\infty)} \| \partial_i c^+ \|_{L^\infty_t(B_{2,1}^\infty)} + \| \tilde{g}_+(c^+, c^-) \|_{L^2_t(B_{2,1}^\infty)} \| \partial_i c^- \|_{L^\infty_t(B_{2,1}^\infty)} \]

\[ \lesssim \| g_+(c^+, c^-) \|_{L^2_t(B_{2,1}^\infty)} \| \partial_i c^+ \|_{L^\infty_t(B_{2,1}^\infty)} + \| \tilde{g}_+(c^+, c^-) \|_{L^2_t(B_{2,1}^\infty)} \| c^- \|_{L^\infty_t(B_{2,1}^\infty)} \]

\[ \lesssim C_0 \left( \| c^+ \|_{L^\infty_t(B_{2,1}^\infty)} \| c^- \|_{L^\infty_t(B_{2,1}^\infty)} \right)^2 \lesssim M^2 \eta^2. \]

Hence, we gather that

\[ \| H_{21} \|_{L^2_t(B_{2,1}^\infty)} \leq C(M^2 \eta^2 + M^3 \eta^3). \] (4.50)

Similarly, we also have

\[ \| H_{41} \|_{L^2_t(B_{2,1}^\infty)} \leq C(M^2 \eta^2 + M^3 \eta^3). \] (4.51)

Substituting (4.49)-(4.51) into (4.48), we obtain that

\[ X(T_0) \leq C_1 e^{C_1 M \eta} \left( \eta + M^2 \eta^2 + M^3 \eta^3 \right). \]

Choose \( M = 8 C_1 \), for sufficiently small \( \eta \) such that

\[ e^{C_1 M \eta} \leq 2, \quad (1 + M \eta) M^2 \eta \leq 1, \]

which implies that

\[ X(T_0) \leq \frac{M \eta}{2}. \]

This is a contradiction with the definition of \( T_0 \). As a consequence, we conclude that \( T_0 = \infty \). Based on the above global uniform estimates, employing a classical Friedrich’s approximation and compactness method (cf. [9, 10, 11]), we can establish the global existence of strong solutions of the system (1.8)-(1.9). Here, we omit it. This completes the proof of the existence of a global solution to the system (1.8)-(1.9) in Theorem 1.1.

5. Uniqueness. In this section, we prove the uniqueness of the solution for the system (1.8)-(1.9). First, let us recall the Osgood Lemma (see [15]), which allows us to deduce uniqueness of the solution in the critical case.

Lemma 5.1. [15] Let \( f \geq 0 \) be a measurable function, \( \gamma \) be a locally integrable function and \( \mu \) be a positive, continuous and non decreasing function which verifies the following condition

\[ \int_0^1 \frac{dr}{\mu(r)} = + \infty. \]
Let also $a$ be a positive real number and let $f$ satisfy the inequality
\[ f(t) \leq a + \int_0^t \gamma(s) \mu(f(s)) \, ds. \]

Then,

(i) if $a$ is equal to zero, the function $f$ vanishes;

(ii) if $a$ is not zero, then we have
\[ -\mathcal{M}(f(t)) + \mathcal{M}(a) \leq \int_0^t \gamma(s) \, ds, \quad \text{with} \quad \mathcal{M}(x) = \int_0^x \frac{1}{t} \, d\tau. \]

Next, we need the following result of logarithmic interpolation.

**Lemma 5.2.** [13] Let $s \in \mathbb{R}$. Then for any $1 \leq p, \rho \leq +\infty$ and $0 < \epsilon \leq 1$, we have
\[ \|g\|_{L^p_{\epsilon}(\tilde{B}_{p,\infty}^N)} \lesssim \frac{\|g\|_{L^p_{\epsilon}(\tilde{B}_{p,\infty}^N)}}{\epsilon} \log(\epsilon + \frac{\|g\|_{L^p_{\epsilon}(\tilde{B}_{p,\infty}^N)} + \|g\|_{L^p_{\epsilon}(\tilde{B}_{p,\infty}^N)}}{\|g\|_{L^p_{\epsilon}(\tilde{B}_{p,\infty}^N)}}). \]

We assume that $(c_i^+, u_i^+, c_i^-, u_i^-)$, $(c_i^+, u_i^+, c_i^-, u_i^-)$ are two solutions of the system (3.21) with the same initial data satisfying (1.12). Observe that $\partial_t c_i^+ \in L_{loc}^1(\tilde{B}_{2,1}^N)$, hence $c_i^+ \in C(\tilde{B}_{2,1}^N) \cap L^\infty(\tilde{B}_{2,1}^N)(i = 1, 2)$. This entails $c_i^+ \in C([0, \infty) \times \mathbb{R}^N)$. On the other hand, if $\eta$ is sufficiently small, we have
\[ |c_i^+(t, x)| \leq \frac{1}{4} \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad x \in \mathbb{R}^N. \]

The continuity in time for $c_2^+$ thus yields the existence of a time $T > 0$ such that
\[ \|c_i^+(t)\|_{L^\infty} \leq \frac{1}{2} \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad t \in [0, T]. \]

From the embedding theorem and (1.12), we have
\[ \|c_i^+\|_{L^\infty(B_{N,1}^1)} \leq \eta. \quad (5.52) \]

Set $\delta c^+ = c_i^+ - c_i^-$, $\delta u^+ = u_i^+ - u_i^-$, $\delta c^- = c_i^- - c_i^-$ and $\delta u^- = u_i^- - u_i^-$. Then \((\delta c^+, \delta u^+, \delta c^-, \delta u^-)\) satisfies the following system
\[
\begin{align*}
\partial_t \delta c^+ + u_i^+ \cdot \nabla \delta c^+ &= -\text{div} \delta u^- + \delta H_{11}, \\
\partial_t \delta u^+ - \nu_i^+ \Delta \delta u^+ - \nu_i^+ \text{div} \delta u^+ &= -\delta u^- \cdot \nabla u_i^+ - u_i^+ \cdot \text{div} \delta u^- - \beta_1 \nabla \delta c^+ - \beta_2 \nabla \delta c^- + \delta H_{21}, \\
\partial_t \delta c^- + u_i^- \cdot \nabla \delta c^- &= -\text{div} \delta u^+ + \delta H_{31}, \\
\partial_t \delta u^- - \nu_i^- \Delta \delta u^- - \nu_i^- \text{div} \delta u^-= & -\delta u^- \cdot \nabla u_i^- - u_i^- \cdot \text{div} \delta u^+ - \beta_3 \nabla \delta c^+ - \beta_4 \nabla \delta c^- + \delta H_{41}, \\
(\delta c^+, \delta u^+, \delta c^-, \delta u^-)|_{t=0} &= 0,
\end{align*}
\]
where
\[
\begin{align*}
\delta H_{11} &= H_{11}(c_i^+, u_i^+) - H_{11}(c_i^+, u_i^-), \quad \delta H_{21} = H_{11}(c_i^+, u_i^+, c_i^-) - H_{11}(c_i^+, u_i^+, c_i^-), \\
\delta H_{31} &= H_{31}(c_i^-, u_i^+) - H_{31}(c_i^-, u_i^-), \quad \delta H_{41} = H_{41}(c_i^+, u_i^+, c_i^-) - H_{41}(c_i^+, u_i^-, c_i^-).
\end{align*}
\]

In what follows, we set $U_i^+(t) = \int_0^t \|u_i^+(\tau)\|_{B_{2,1}^N} \, d\tau$ for $i = 1, 2$, and denote by $A_T$ a constant depending on $\|c_i^+\|_{L^\infty(B_{N,1}^1)}$ and $\|c_i^+\|_{L^\infty(B_{N,1}^1)}$. To begin with, we shall
prove uniqueness on the time interval $[0, T]$ by estimating $(\delta c^+, \delta u^+, \delta c^-, \delta u^-)$ in the following functional space:

$$F_T = L^\infty([0, T]; \dot{B}^{1}_{N, \infty}) \times (L^\infty([0, T]; \dot{B}^{-1}_{N, \infty}) \cap \dot{L}^1([0, T]; \dot{B}^1_{N, \infty}))^N \times L^\infty([0, T]; \dot{B}^{0}_{N, \infty}) \times (L^\infty([0, T]; \dot{B}^{-1}_{N, \infty}) \cap \dot{L}^1([0, T]; \dot{B}^1_{N, \infty}))^N.$$

We apply Proposition 7 to get for any $t \in [0, T],$

$$\|\delta c^+(t)\|_{\dot{B}^1_{N, \infty}} \lesssim e^{CU_2(t)} \int_0^t (\|\delta H_{11}(\tau)\|_{\dot{B}^1_{N, \infty}} + \|\text{div}\delta u^+(\tau)\|_{\dot{B}^1_{N, \infty}}) d\tau,
\|\delta c^-(t)\|_{\dot{B}^0_{N, \infty}} \lesssim e^{CU_2(t)} \int_0^t (\|\delta H_{11}(\tau)\|_{\dot{B}^0_{N, \infty}} + \|\text{div}\delta u^-(\tau)\|_{\dot{B}^0_{N, \infty}}) d\tau. \tag{5.54}$$

From Proposition 1, we have

$$\|\delta H_{11}(\tau)\|_{\dot{B}^0_{N, \infty}} + \|\text{div}\delta u^+(\tau)\|_{\dot{B}^1_{N, \infty}} \lesssim \|u_2^+(\tau)\|_{\dot{B}^1_{0, 1}}\|\delta c^+(\tau)\|_{\dot{B}^1_{N, \infty}} + (1 + \|c_1^+(\tau)\|_{\dot{B}^1_{0, 1}})\|\delta u^+(\tau)\|_{\dot{B}^1_{N, \infty}},$$

$$\|\delta H_{11}(\tau)\|_{\dot{B}^0_{N, \infty}} + \|\text{div}\delta u^-(\tau)\|_{\dot{B}^1_{N, \infty}} \lesssim \|u_2^-(\tau)\|_{\dot{B}^1_{0, 1}}\|\delta c^-(\tau)\|_{\dot{B}^1_{N, \infty}} + (1 + \|c_1^-(\tau)\|_{\dot{B}^1_{0, 1}})\|\delta u^-(\tau)\|_{\dot{B}^1_{N, \infty}}.$$ Plugging the above two inequalities into (5.54), we get by Gronwall’s inequality that

$$\|(\delta c^+, \delta c^-)(t)\|_{\dot{B}^1_{0, \infty}} \lesssim e^{CU_2(t)} \int_0^t \left(1 + \|(c_1^+, c_1^-)(\tau)\|_{\dot{B}^1_{0, 1}}\right)\|(\delta u^+, \delta u^-)(\tau)\|_{\dot{B}^1_{0, 1}} d\tau, \tag{5.55}$$

where $U_2(t) = \max \{U_{21}(t), U_{22}(t)\}.$

Applying Remark 6 to the second equation and the fourth equation of (5.53), we have

$$\|\delta u^+\|_{\dot{L}^1(\dot{B}^1_{0, \infty})} + \|\delta u^-\|_{\dot{L}^1(\dot{B}^1_{0, \infty})} \lesssim \|\delta H_{21}\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} \|\nabla u_1^+\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} + \|\delta u^+\cdot \nabla u_1^+\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} + \|\delta u^-\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} + \|\nabla \delta c^+\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} \tag{5.56}$$

$$\|\delta u^-\|_{\dot{L}^1(\dot{B}^1_{0, \infty})} + \|\delta u^-\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} \lesssim \|\delta H_{41}\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} \|\nabla u_1^+\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} + \|\delta u^-\cdot \nabla u_1^-\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} + \|\nabla \delta c^+\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} + \|\nabla \delta c^-\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} \tag{5.57}$$

Employing Proposition 1 and Proposition 3, we have

$$\|\delta H_{21}\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} + \|\delta u^+\cdot \nabla u_1^+\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} + \|\delta u^-\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} \lesssim A_T \int_0^T \left(1 + \|u_2^+(\tau)\|_{\dot{B}^1_{0, 1}}\right) \|(\delta c^+, \delta c^-)(\tau)\|_{\dot{B}^1_{0, \infty}} d\tau$$

$$+ \|u_2^+(\tau)\|_{\dot{L}^1(\dot{B}^{-1}_{N, \infty})} \|\delta u^+\|_{\dot{L}^1(\dot{B}^1_{0, \infty})} + A_T \|(c_1^+, c_1^-)\|_{\dot{L}^\infty(\dot{B}^1_{0, 1})} \|\delta u^+\|_{\dot{L}^1(\dot{B}^1_{0, \infty})}, \tag{5.58}$$
\[ \| \delta H_{41} \|_{L_t^1(B_{N,\infty}^{-1})} + \| \delta u^- \cdot \nabla u_1^+ \|_{L_t^1(B_{N,\infty}^{-1})} + \| u_2^- \cdot \nabla \delta u^- \|_{L_t^1(B_{N,\infty}^{-1})} \]
\[ + \| \nabla \delta c^+ \|_{L_t^1(B_{N,\infty}^{-1})} + \| \nabla \delta c^- \|_{L_t^1(B_{N,\infty}^{-1})} \]
\[ \lesssim A_T \int_0^T \left( 1 + \| u_2^+ (\tau) \|_{B_{N,1}^2} \right) \| \delta c^+, \delta c^- (\tau) \|_{B_{N,\infty}^\infty} d\tau \]
\[ + \| (u_1^-, u_2^-) \|_{L_t^1(B_{N,1}^1)} \| \delta u^- \|_{L_t^1(B_{N,\infty}^\infty)} + A_T \| (c_1^+, c_1^-) \|_{L_t^1(B_{N,1}^1)} \| \delta u^- \|_{L_t^1(B_{N,\infty}^\infty)}. \]
(5.59)

Employing (5.52) and taking T small enough such that
\[ A_T \| (c_1^+, c_1^-) \|_{L_t^1(B_{N,1}^1)} + \| (u_1^-, u_2^-) \|_{L_t^1(B_{N,1}^1)} \ll 1. \]
(5.60)

Combining with (5.56)-(5.60), we have
\[ \| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)} \lesssim \int_0^T \left( 1 + \| (u_2^+, u_2^-) (\tau) \|_{B_{N,1}^2} \right) \| (\delta c^+, \delta c^- (\tau) \|_{B_{N,\infty}^\infty} d\tau. \]
(5.61)

From Lemma 5.2, it follows that
\[ \| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,1}^1)} \lesssim \| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)} \]
\[ \times \log \left( e + \frac{\| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)} + \| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)}}{\| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)}} \right), \]
which together with (5.55) yields that for any \( t \in [0, T] \),
\[ \| (\delta c^+, \delta c^-) (t) \|_{B_{N,\infty}^\infty} \lesssim \| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)} \]
\[ \times \log \left( e + \frac{\| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)} + \| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)}}{\| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)}} \right). \]
(5.62)

Combining the above inequality with (5.61), we have
\[ \| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)} \lesssim \int_0^t \left( 1 + \| (u_1^-, u_2^-) (\tau) \|_{B_{N,1}^2} \right) V(\tau) \log \left( e + \frac{C_T}{V(\tau)} \right) d\tau, \]
(5.63)

where \( V(t) = \| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)} \), \( C_T = \| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)} + \| (\delta u^+, \delta u^-) \|_{L_t^1(B_{N,\infty}^1)} \).

Notice that \( 1 + \| (u_2^+, u_2^-) (t) \|_{B_{N,1}^2} \) is integrable on \([0, T]\), and
\[ \int_0^1 \frac{dr}{r \log (e + C_T r^{-1})} dr = +\infty. \]

From the Osgood lemma 5.1, we conclude that \( (\delta u^+, \delta u^-) = 0 \) on \([0, T]\). This gives by inequality (5.55), that \( (\delta c^+, \delta c^-) = 0 \). A standard continuity argument ensures that \((c_1^+, u_1^+, c_1^-, u_1^-) = (c_2^+, u_2^+, c_2^-, u_2^-)\) on \([0, +\infty)\).

6. **Time decay estimates.** In this section, we will establish the time decay rates of the global strong solutions constructed in Theorem 1.1. We divide the proof into several steps.
Step 1: Low frequencies. We first exhibit the smoothing properties of the system (3.21) in the low frequencies regime. The key to these remarkable properties is given by the following lemma.

Lemma 6.1. Let \((c^+, u^+, c^-, u^-)\) be a solution of the system (3.21). Then, there exist two positive constants \(c_0\) and \(C\) depending only on \(\beta_i (i = 1, 2, 3, 4)\) and \(\nu_i^\pm (i = 1, 2)\) respectively, such that the following inequality holds for all \(t \geq 0\),

\[
\left\| (\hat{\Delta}_q c^+, \hat{\Delta}_q u^+, \hat{\Delta}_q c^-, \hat{\Delta}_q u^-) \right\|_{L^2}^\ell \leq C \left(e^{-c_0 2^{2q+1}t}\left\| (\hat{\Delta}_q c_0^+, \hat{\Delta}_q u_0^+, \hat{\Delta}_q c_0^-, \hat{\Delta}_q u_0^-) \right\|_{L^2} + \int_0^t e^{-c_0 2^{2q}(t-\tau)} \left(\Delta_q H_1, \Delta_q H_2, \Delta_q H_3, \Delta_q H_4\right)(\tau) \right\|_{L^2}^\ell d\tau \right) .
\]

(6.64)

Proof. By the same derivation process of (3.45), in the case \(H_1 \equiv H_2 \equiv H_3 \equiv H_4 \equiv 0\), we have

\[
\frac{1}{2} \frac{d}{dt} \alpha_q + c_0 2^{2q} \alpha_q \leq 0,
\]

where

\[
\alpha_q \approx \left\| \hat{\Delta}_q c^+ \right\|_{L^2} + \left\| \hat{\Delta}_q c^- \right\|_{L^2} + \left\| \nabla \hat{\Delta}_q c^+ \right\|_{L^2} + \left\| \nabla \hat{\Delta}_q c^- \right\|_{L^2} + \left\| \hat{\Delta}_q u^+ \right\|_{L^2} + \left\| \hat{\Delta}_q u^- \right\|_{L^2}.
\]

Thus,

\[
\left\| (\hat{\Delta}_q c^+, \hat{\Delta}_q u^+, \hat{\Delta}_q c^-, \hat{\Delta}_q u^-) \right\|_{L^2}^\ell \leq C e^{-c_0 2^{2q+1}t} \left(\left\| (\hat{\Delta}_q c_0^+, \hat{\Delta}_q u_0^+, \hat{\Delta}_q c_0^-, \hat{\Delta}_q u_0^-) \right\|_{L^2} + \int_0^t e^{-c_0 2^{2q}(t-\tau)} \left(\Delta_q H_1, \Delta_q H_2, \Delta_q H_3, \Delta_q H_4\right)(\tau) \right\|_{L^2}^\ell \right) .
\]

(6.66)

Furthermore, taking advantage of the Duhamel formula, we can readily deduce (6.64).

Denoting by \(A(D)\) the semi-group associated to the system (3.21), we have for all \(q \in \mathbb{Z}\),

\[
\begin{pmatrix}
\hat{\Delta}_q c^+(t) \\
\hat{\Delta}_q u^+(t) \\
\hat{\Delta}_q c^-(t) \\
\hat{\Delta}_q u^-(t)
\end{pmatrix} = e^{tA(D)} \begin{pmatrix}
\hat{\Delta}_q c_0^+ \\
\hat{\Delta}_q u_0^+ \\
\hat{\Delta}_q c_0^- \\
\hat{\Delta}_q u_0^-
\end{pmatrix} + \int_0^t e^{(t-\tau)A(D)} \begin{pmatrix}
\hat{\Delta}_q H_1(\tau) \\
\hat{\Delta}_q H_2(\tau) \\
\hat{\Delta}_q H_3(\tau) \\
\hat{\Delta}_q H_4(\tau)
\end{pmatrix} d\tau .
\]

(6.67)

Based on (6.66) and (6.67), we get for all \(q \leq q_0\),

\[
\left\| e^{tA(D)} \hat{\Delta}_q U \right\|_{L^2} \lesssim e^{-c_0 2^{2q}t} \left\| \hat{\Delta}_q U \right\|_{L^2} .
\]

Hence, multiplying by \(t^{\frac{\alpha}{2} + \frac{s}{2} + \frac{1}{2}}\) and summing up on \(q \leq q_0\), we get

\[
t^{\frac{\alpha}{2} + \frac{s}{2} + \frac{1}{2}} \sum_{q \leq q_0} 2^{qs} \left\| e^{tA(D)} \hat{\Delta}_q U \right\|_{L^2} \lesssim \sum_{q \leq q_0} 2^{qs} e^{-c_0 2^{2q}t} \left\| \hat{\Delta}_q U \right\|_{L^2} t^{\frac{\alpha}{2} + \frac{s}{2} + \frac{1}{2}} \lesssim \sum_{q \leq q_0} 2^{qs(\frac{s}{2} + \frac{1}{2})} e^{-c_0 2^{2q}t} \left\| \hat{\Delta}_q U \right\|_{L^2} t^{\frac{\alpha}{2} + \frac{s}{2} + \frac{1}{2}} \lesssim \left\| U \right\|_{L^2}^{\ell - \frac{\alpha}{2} - \frac{s}{2}} \sum_{q \leq q_0} 2^{qs(\frac{s}{2} + \frac{1}{2})} e^{-c_0 2^{2q}t} t^{\frac{\alpha}{2} + \frac{s}{2} + \frac{1}{2}} .
\]

(6.68)
As for any \( \sigma > 0 \) there exists a constant \( C_\sigma \) so that
\[
\sup_{t \geq 0} \sum_{q \in \mathbb{Z}} t^{\frac{2q}{2}} 2^{q \sigma} e^{-c_0 2^q t} \leq C_\sigma. \tag{6.69}
\]
We get from (6.68) and (6.69) that for \( s > -N/2 \),
\[
\sup_{t \geq 0} t^{\frac{s}{q} + \frac{\lambda}{2}} \| e^{tA(D)} U \|_{\dot{B}^{2,1}_{2,1}}^\ell \lesssim \| U \|_{\dot{B}^\frac{N}{2}, \infty}^\ell.
\]
Furthermore, it is obvious that for \( s > -N/2 \),
\[
\| e^{tA(D)} U \|_{\dot{B}^{2,1}_{2,1}} \lesssim \| U \|_{\dot{B}^\frac{N}{2}, \infty} \sum_{q \leq q_0} 2^{q(s + \frac{\lambda}{2})} \lesssim \| U \|_{\dot{B}^\frac{N}{2}, \infty}^\ell.
\]
Hence, setting \( \langle t \rangle \overset{\text{def}}{=} \sqrt{1 + t^2} \), we get
\[
\sup_{t \geq 0} \langle t \rangle^{\frac{s}{q} + \frac{\lambda}{2}} \| e^{tA(D)} U \|_{\dot{B}^{2,1}_{2,1}} \lesssim \| U \|_{\dot{B}^\frac{N}{2}, \infty}^\ell. \tag{6.70}
\]
Thus, from (6.64) and (6.70), we have
\[
\| (c^+, u^+, c^-, u^-) \|_{\dot{B}^{2,1}_{2,1}}^\ell \lesssim \sup_{t \geq 0} \langle t \rangle^{-(\frac{s}{q} + \frac{\lambda}{2})} \| (c^+_0, u^+_0, c^-_0, u^-_0) \|_{\dot{B}^\frac{N}{2}, \infty}^\ell + \int_0^t \langle t - \tau \rangle^{-(\frac{s}{q} + \frac{\lambda}{2})} \| (H_1, H_2, H_3, H_4)(\tau) \|_{\dot{B}^\frac{N}{2}, \infty}^\ell d\tau. \tag{6.71}
\]
We claim that for all \( s \in (-N/2, 2] \) and \( t \geq 0 \), then
\[
\int_0^t \langle t - \tau \rangle^{-(\frac{s}{q} + \frac{\lambda}{2})} \| (H_1, H_2, H_3, H_4)(\tau) \|_{\dot{B}^\frac{N}{2}, \infty}^\ell d\tau \lesssim \langle t \rangle^{-(\frac{s}{q} + \frac{\lambda}{2})} \left( X^2(t) + D^2(t) + D^3(t) + D^4(t) \right), \tag{6.72}
\]
where \( X(t) \) and \( D(t) \) have been defined in (1.13) and (1.16), respectively.

Owing to the embedding \( L^1 \hookrightarrow \dot{B}^\frac{N}{2}_{2, \infty} \), it suffices to prove (6.72) with \( \| (H_1, H_2, H_3, H_4)(\tau) \|_{L^1}^\ell \) instead of \( \| (H_1, H_2, H_3, H_4)(\tau) \|_{\dot{B}^\frac{N}{2}, \infty}^\ell \).

To bound the term with \( H_1 \), we use the following decomposition:
\[
H_1 = u^+ \cdot \nabla c^+ + c^+ \text{ div } (u^+)^\ell + c^+ \text{ div } (u^+)^h.
\]
Now, from Hölder’s inequality, the embedding \( \dot{B}^{2,1}_{2,1} \hookrightarrow L^2 \), the definitions of \( D(t) \), \( \alpha \) and Lemma 2.6, one may write for all \( s \in (\varepsilon - \frac{N}{2}, 2] \),
\[
\int_0^t \langle t - \tau \rangle^{-(\frac{s}{q} + \frac{\lambda}{2})} \| (u^+ \cdot \nabla c^+)(\tau) \|_{L^1} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-(\frac{s}{q} + \frac{\lambda}{2})} \| u^+ \|_{L^2} \| \nabla c^+ \|_{L^2} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-(\frac{s}{q} + \frac{\lambda}{2})} \| u^+ \|_{\dot{B}^{2,1}_{2,1}} \| \nabla c^+ \|_{\dot{B}^{2,1}_{2,1}} d\tau.
\]
The term $c^+ \Div (u^+)^t$ may be treated along the same lines, and we have

$$\int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \|c^+ \Div (u^+)^t\|_{L^1} \, d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \|c^+\|_{B^2_{\text{r}, 1}}^\ell \|\nabla u^+\|_{B^2_{\text{r}, 1}}^\ell \, d\tau
\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{N}{2}} \|c^+(\tau)\|_{B^2_{\text{r}, 1}}^\ell \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{N}{2} + \frac{3}{2}} \|u^+(\tau)\|_{B^2_{\text{r}, 1}}^\ell \right)
\int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \, d\tau + \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\alpha} \|c^+(\tau)\|_{B^2_{\text{r}, 1}}^\ell \right)
\int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \, d\tau
\lesssim D^2(t) \int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\min(\frac{N}{2} + \frac{3}{2}, \frac{\alpha}{2} + \frac{3}{2})} \, d\tau
\lesssim \langle t \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} D^2(t).$$

Regarding the term with $c^+ \Div (u^+)^h$, we get for all $t \geq 2$ that

$$\int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \|c^+ \Div (u^+)^h(\tau)\|_{L^1} \, d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \|c^+(\tau)\|_{B^2_{\text{r}, 1}}^h \|\Div u^+(\tau)\|_{B^2_{\text{r}, 1}}^h \, d\tau
\lesssim \int_0^1 \langle t - \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \|c^+(\tau)\|_{B^2_{\text{r}, 1}}^h \|\Div u^+(\tau)\|_{B^2_{\text{r}, 1}}^h \, d\tau + \int_1^t \langle t - \tau \rangle^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \|c^+(\tau)\|_{B^2_{\text{r}, 1}}^h \|\Div u^+(\tau)\|_{B^2_{\text{r}, 1}}^h \, d\tau \overset{\text{def}}{=} I_1 + I_2.$$
From the definitions of $X(t)$ and $D(t)$, we obtain

$$I_1 \lesssim \langle t \rangle^{-\frac{N}{2} + \frac{d}{2}} \sup_{0 \leq \tau \leq 1} \|c^+(\tau)\|_{\dot{B}^0_{2,1}} \int_0^1 \|\text{div} u^+(\tau)\|^h_{B^0_{2,1}} d\tau$$

$$\lesssim \langle t \rangle^{-\frac{N}{2} + \frac{d}{2}} \sup_{0 \leq \tau \leq 1} \|c^+(\tau)\|_{\dot{B}^0_{2,1}} \int_0^1 \|u^+(\tau)\|^h_{B^0_{2,1}} d\tau$$

$$\lesssim \langle t \rangle^{-\frac{N}{2} + \frac{d}{2}} D(1) X(1),$$

and, using the fact that $\langle \tau \rangle \approx \tau$ when $\tau \geq 1$, we get

$$I_2 \lesssim \int_1^t \langle t - \tau \rangle^{-\frac{N}{2} + \frac{d}{2}} \langle \|c^+(\tau)\|^h_{B^0_{2,1}} + \|c^+(\tau)\|^h_{\dot{B}^0_{2,1}} \rangle \|\text{div} u^+(\tau)\|^h_{B^0_{2,1}} d\tau$$

$$\lesssim \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^{-\frac{N}{2}} \|c^+(\tau)\|^h_{B^0_{2,1}} \right) \left( \sup_{1 \leq \tau \leq t} \|\tau \text{div} u^+(\tau)\|^h_{\dot{B}^0_{2,1}} \right) \int_1^t \langle t - \tau \rangle^{-\frac{N}{2} + \frac{d}{2}} \langle \tau \rangle^{-\frac{N}{2} + 1} d\tau + \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^\alpha \|c^+(\tau)\|^h_{B^0_{2,1}} \right)$$

$$\lesssim D^2(t) \int_1^t \langle t - \tau \rangle^{-\frac{N}{2} + \frac{d}{2}} \langle \tau \rangle^{-\alpha \min(\alpha + 1, \frac{N}{2} + 1)} d\tau$$

$$\lesssim \langle t \rangle^{-\frac{N}{2} + \frac{d}{2}} D^2(t).$$

Thus, for $t \geq 2$, we conclude that

$$\int_0^t \langle t - \tau \rangle^{-\frac{N}{2} + \frac{d}{2}} \|c^+ \text{div} (u^+)^h\|_{L^1} d\tau$$

$$\lesssim \langle t \rangle^{-\frac{N}{2} + \frac{d}{2}} \left( D^2(t) + X^2(t) \right). \quad (6.75)$$

The case $t \leq 2$ is obvious as $\langle t \rangle \approx 1$ and $\langle t - \tau \rangle \approx 1$ for $0 \leq \tau \leq t \leq 2$, and

$$\int_0^t \|c^+ \text{div} (u^+)^h\|_{L^1} d\tau$$

$$\lesssim \|c^+\|_{L^\infty(L^2)} \|\text{div} (u^+)^h\|_{L^1(L^2)}$$

$$\lesssim \|c^+\|_{L^\infty(B^0_{2,1})} \|\text{div} (u^+)^h\|_{L^1(B^0_{2,1})}$$

$$\lesssim \|c^+\|_{L^\infty(B^0_{2,1})} \|u^+\|^h_{\dot{B}^0_{2,1}}$$

$$\lesssim \|c^+\|_{L^\infty(B^0_{2,1})} \|u^+\|^h_{\dot{B}^0_{2,1}}$$

$$\lesssim X(t) D(t). \quad (6.76)$$

From (6.73)-(6.76), we get

$$\int_0^t \langle t - \tau \rangle^{-\frac{N}{2} + \frac{d}{2}} \|H_1(\tau)\|^h_{\dot{B}^0_{2,\infty}} d\tau \lesssim \langle t \rangle^{-\frac{N}{2} + \frac{d}{2}} \left( X^2(t) + D^2(t) \right).$$

The term $H_3$ may be treated along the same lines, and we obtain

$$\int_0^t \langle t - \tau \rangle^{-\frac{N}{2} + \frac{d}{2}} \|H_3(\tau)\|^h_{\dot{B}^0_{2,\infty}} d\tau \lesssim \langle t \rangle^{-\frac{N}{2} + \frac{d}{2}} \left( X^2(t) + D^2(t) \right).$$
Next, to bound the first term of $H_2$, we write that

$$
\int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)} \|g_+ (c^+, c^-) \partial_t c^-(\tau)\|_{L^1} \ d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)} \|g_+ (c^+, c^-)\|_{L^2} \|\nabla c^+\|_{L^2} \ d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)} \|g_+ (c^+, c^-)\|_{\dot{B}^{2,1}_2} \|\nabla c^+\|_{\dot{B}^{2,1}_2} \ d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)} \|\nabla c^+\|_{\dot{B}^{2,1}_2} \ d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)} \left(\|\nabla c^+\|_{\dot{B}^{2,1}_2}^h + \|\nabla c^+\|_{\dot{B}^{2,1}_2}^h\right) \ d\tau
$$

(6.77)

where $g_+$ stands for some smooth function vanishing at 0.

Similar to (6.77), we have

$$
\int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)} \|g_+ (c^+, c^-) \partial_t c^- (\tau)\|_{L^1} \ d\tau \\
\lesssim D^2(t) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)} \langle \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)} \|\nabla c^+\|_{\dot{B}^{2,1}_2} \ d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)} D^2(t).
$$

(6.78)

To bound the term with $(u^+ \cdot \nabla) u^+_i$, we employ the following decomposition:

$$(u^+ \cdot \nabla) u^+_i = (u^+ \cdot \nabla) (u^+_i) + (u^+ \cdot \nabla) (u^+_i)^h.$$ 

For the term $(u^+ \cdot \nabla) (u^+_i)^e$, we have

$$
\int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)} \|(u^+ \cdot \nabla) (u^+_i)^e\|_{L^1} \ d\tau
$$
\[
\begin{align*}
\lesssim & \int_0^t (t - \tau)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \left( \|u^+\|_{B_{\alpha, 1}^0}^\ell + \|u^+\|_{B_{\alpha, 1}^0}^h \right) \left\|\nabla u^+\right\|_{B_{\alpha, 1}^0}^\ell \, d\tau \\
\lesssim & \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^\frac{N}{4} \|u^+(\tau)\|_{B_{\alpha, 1}^0}^\ell \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^\frac{N}{4} + \frac{5}{4} \|u^+(\tau)\|_{B_{\alpha, 1}^0}^\ell \right) \\
& \times \int_0^t (t - \tau)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \langle \tau \rangle^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \, d\tau \\
& + \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^\alpha \|u^+(\tau)\|_{B_{\alpha, 1}^0}^h \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^\frac{N}{4} + \frac{5}{4} \|u^+(\tau)\|_{B_{\alpha, 1}^0}^\ell \right) \\
& \times \int_0^t (t - \tau)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \langle \tau \rangle^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \, d\tau \\
\lesssim & D^2(t) \int_0^t (t - \tau)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \langle \tau \rangle^{-\min\left(\frac{N}{4} + \frac{5}{4}, \alpha + \frac{N}{4} + \frac{5}{4}\right)} \, d\tau \\
\lesssim & (t)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} D^2(t).
\end{align*}
\]

Regarding the term with \((u^+ \cdot \nabla)(u_i^+)\)^h, we have for all \(t \geq 2\) that

\[
\begin{align*}
\int_0^t (t - \tau)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \left\|\left(u^+ \cdot \nabla\right)(u_i^+)\right\|^h_{L^1} \, d\tau \\
\lesssim & \int_0^t (t - \tau)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \|u^+(\tau)\|_{L^2} \left\|\nabla u^+(\tau)\right\|^h_{L^2} \, d\tau \\
\lesssim & \int_0^t (t - \tau)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \|u^+(\tau)\|_{B_{\alpha, 1}^0} \left\|\nabla u^+(\tau)\right\|^h_{B_{\alpha, 1}^0} \, d\tau \\
\lesssim & \int_0^t (t - \tau)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \|u^+(\tau)\|_{B_{\alpha, 1}^0} \left\|\nabla u^+(\tau)\right\|^h_{B_{\alpha, 1}^0} \, d\tau \\
& + \int_1^t (t - \tau)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \|u^+(\tau)\|_{B_{\alpha, 1}^0} \left\|\nabla u^+(\tau)\right\|^h_{B_{\alpha, 1}^0} \, d\tau \\
\overset{\text{def}}{=} & K_1 + K_2.
\end{align*}
\]

From the definitions of \(X(t)\) and \(D(t)\), we get

\[
K_1 \lesssim \langle t \rangle^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \sup_{0 \leq \tau \leq t} \|u^+(\tau)\|_{B_{\alpha, 1}^0} \int_0^1 \left\|\nabla u^+(\tau)\right\|^h_{B_{\alpha, 1}^0} \, d\tau \lesssim \langle t \rangle^{-\left(\frac{N}{4} + \frac{5}{4}\right)} D(1)X(1),
\]

and, using the fact that \(\langle \tau \rangle \approx \tau\) when \(\tau \geq 1\),

\[
K_2 \lesssim \int_1^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \left( \|u^+(\tau)\|_{B_{\alpha, 1}^0} \|u^+(\tau)\|_{B_{\alpha, 1}^0}^h \right) \left\|\nabla u^+(\tau)\right\|^h_{B_{\alpha, 1}^0} \, d\tau \\
\lesssim \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^\frac{N}{4} \|u^+(\tau)\|_{B_{\alpha, 1}^0}^\ell \right) \left( \sup_{1 \leq \tau \leq t} \|\nabla u^+(\tau)\|_{B_{\alpha, 1}^0}^\ell \right) \int_1^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \langle \tau \rangle^{-\left(\frac{N}{4} + 1\right)} \, d\tau \\
& + \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^\alpha \|u^+(\tau)\|_{B_{\alpha, 1}^0}^h \right) \left( \sup_{1 \leq \tau \leq t} \|\nabla u^+(\tau)\|_{B_{\alpha, 1}^0}^h \right) \int_1^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \langle \tau \rangle^{-\left(\alpha + 1\right)} \, d\tau \\
\lesssim & D^2(t) \int_1^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{5}{4}\right)} \langle \tau \rangle^{-\min\left(\alpha + 1, \frac{N}{4} + 1\right)} \, d\tau \\
\lesssim & (t)^{-\left(\frac{N}{4} + \frac{5}{4}\right)} D^2(t).
\]
Therefore, for $t \geq 2$, we deduce that
\[ \int_0^t (t - \tau)^{-\frac{m}{2} + \frac{3}{4}} \tau \| (u^+ \cdot \nabla)(u_i^+)^h(\tau) \|_{L^1} d\tau \lesssim (t)^{-\frac{m}{2} + \frac{3}{4}} (D^2(t) + X^2(t)). \] (6.79)

The case $t \leq 2$ is obvious as $(t) \approx 1$ and $(t - \tau) \approx 1$ for $0 \leq \tau \leq t \leq 2$, and
\[ \int_0^t \|(u^+ \cdot \nabla)(u_i^+)^h\|_{L^1} d\tau \lesssim \|u^+\|_{L^\infty_t(L^2)} \|\nabla u^+\|_{L^1_t(L^2)} \lesssim \|u^+\|_{L^\infty_t(B_{2,1}^0)} \|\nabla u^+\|_{L^1_t(B_{2,1}^0)} \lesssim \|u^+\|_{L^\infty_t(B_{2,1}^0)} \|u^+\|^h_{L^1_t(B_{2,1}^{\frac{m}{2} + 1})} \lesssim X(t) D(t). \] (6.80)

To deal with the term $\mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+$, we take the following decomposition:
\[ \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+ = \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j (u_i^+)^\ell + \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j (u_i^+) h. \]

For the term $\mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j (u_i^+)^\ell$, we have
\[ \int_0^t (t - \tau)^{-\frac{m}{2} + \frac{3}{4}} \|\mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j (u_i^+)^\ell\|_{L^1} d\tau \lesssim \int_0^t (t - \tau)^{-\frac{m}{2} + \frac{3}{4}} \|h_+ (c^+, c^-) \partial_j c^+ \partial_j (u_i^+)^\ell\|_{L^2} d\tau \]
\[ \lesssim \int_0^t (t - \tau)^{-\frac{m}{2} + \frac{3}{4}} \|h_+ (c^+, c^-)\|_{B_{2,1}^0} \|\nabla c^+ \nabla (u_i^+)^\ell\|_{B_{2,1}^0} d\tau \]
\[ \lesssim \int_0^t (t - \tau)^{-\frac{m}{2} + \frac{3}{4}} (1 + \|c^+, c^-\|_{B_{2,1}^0}) \|\nabla c^+ \|_{B_{2,1}^0} \|\nabla (u_i^+)^\ell\|_{B_{2,1}^{\frac{m}{2} + 1}} d\tau \]
\[ \lesssim \int_0^t (t - \tau)^{-\frac{m}{2} + \frac{3}{4}} \|\nabla c^+\|_{B_{2,1}^{\frac{m}{2} + 1}} \|\nabla (u_i^+)^\ell\|_{B_{2,1}^{\frac{m}{2} + 1}} + \int_0^t (t - \tau)^{-\frac{m}{2} + \frac{3}{4}} \|c^+, c^-\|_{B_{2,1}^0} \|\nabla c^+\|_{B_{2,1}^0} \|\nabla (u_i^+)^\ell\|_{B_{2,1}^{\frac{m}{2} + 1}} \overset{\text{def}}{=} L_1 + L_2. \]

We bound the two terms $L_1$ and $L_2$ as follows respectively,
\[ L_1 \lesssim \int_0^t (t - \tau)^{-\frac{m}{2} + \frac{3}{4}} \|\nabla c^+\|_{B_{2,1}^{\frac{m}{2} + 1}} \|\nabla c^+\|^h_{B_{2,1}^0} + \|\nabla c^+\|_{B_{2,1}^{\frac{m}{2} + 1}} \|\nabla u^+\|^\ell_{B_{2,1}^0} d\tau \]
\[ \lesssim \left( \sup_{0 \leq \tau \leq T} \|c^+\|_{B_{2,1}^{\frac{m}{2} + 1}} \right) \left( \sup_{0 \leq \tau \leq T} \|c^+\|_{B_{2,1}^{\frac{m}{2} + 1}} \right) \left( \sup_{0 \leq \tau \leq T} \|\nabla u^+\|^\ell_{B_{2,1}^{\frac{m}{2} + 1}} \right) \]
\[ \times \int_0^t (t - \tau)^{-\frac{m}{2} + \frac{3}{4}} \langle \tau \rangle^{-\frac{m}{2} + \frac{3}{4}} \|\nabla u^+\|_{B_{2,1}^{\frac{m}{2} + 1}} d\tau \]
\[ + \left( \sup_{0 \leq \tau \leq T} \|c^+\|_{B_{2,1}^{\frac{m}{2} + 1}} \right) \left( \sup_{0 \leq \tau \leq T} \|\nabla u^+\|^h_{B_{2,1}^{\frac{m}{2} + 1}} \right) \left( \sup_{0 \leq \tau \leq T} \|\nabla u^+\|^\ell_{B_{2,1}^{\frac{m}{2} + 1}} \right) \]
\[ \times \int_0^t (t - \tau)^{-\frac{m}{2} + \frac{3}{4}} \langle \tau \rangle^{-\frac{m}{2} + \frac{3}{4}} \|\nabla u^+\|_{B_{2,1}^{\frac{m}{2} + 1}} d\tau \]
\[ \lesssim D^2(t) \int_0^t (t - \tau)^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right) \langle \tau \rangle - \min\left(\frac{\Delta}{2} + 1, \frac{\Delta}{2} + \frac{1}{2} + \alpha\right)} d\tau \]
\[ \lesssim \langle t \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} D^2(t), \]
and
\[ L_2 \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \left( \| (c^+, c^-) \|_{\dot{H}^2_{2,1}} + \| (c^+, c^-) \|_{\dot{B}^h_{2,1}} \| \nabla c^+ \|_{\dot{H}^2_{2,1}} \| \nabla u^+ \|_{\dot{H}^2_{2,1}} \right) d\tau \]
\[ \lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{\Delta}{2}} \| (c^+, c^-) \|_{\dot{B}^h_{2,1}} \right) \]
\[ \int_0^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\frac{1}{2}} \left( \| \nabla c^+ \|_{\dot{H}^2_{2,1}} \| \nabla (u^+) \|_{\dot{H}^2_{2,1}} \right) d\tau \]
\[ + \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\alpha} \| (c^+, c^-) \|_{\dot{B}^h_{2,1}} \right) \]
\[ \int_0^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\alpha} \| \nabla c^+ \|_{\dot{H}^2_{2,1}} \| \nabla (u^+) \|_{\dot{H}^2_{2,1}} d\tau \]
\[ \lesssim D^3(t) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\min(N + 1, \frac{\Delta}{2} + \frac{1}{2} + \alpha, \frac{\Delta}{2} + \frac{1}{2})} d\tau \]
\[ \lesssim \langle t \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} D^3(t). \]

Thus,
\[ \int_0^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \| \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j (u_i^+) \|_{L^1} d\tau \lesssim \langle t \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} (D^2(t) + D^3(t)). \]

(6.81)

Regarding the term \( \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j (u_i^+) \), we also get,
\[ \int_0^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \| \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j (u_i^+) \|_{L^1} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \| h_+ (c^+, c^-) \|_{L^2} \| \partial_j c^+ \partial_j (u_i^+) \|_{L^2} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} (1 + \| (c^+, c^-) \|_{\dot{B}^h_{2,1}}) \| \nabla c^+ (\tau) \|_{\dot{B}^0_{2,1}} \| \nabla u^+ (\tau) \|_{\dot{H}^2_{2,1}} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \| \nabla c^+ (\tau) \|_{\dot{B}^0_{2,1}} \| \nabla u^+ (\tau) \|_{\dot{H}^2_{2,1}} d\tau \]
\[ + \int_0^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \| (c^+, c^-) \|_{\dot{B}^h_{2,1}} \| \nabla c^+ (\tau) \|_{\dot{B}^0_{2,1}} \| \nabla u^+ (\tau) \|_{\dot{H}^2_{2,1}} d\tau \]
\[ \overset{\text{def}}{=} M_1 + M_2. \]

We deal with the two terms \( M_1 \) and \( M_2 \) in the following, if \( t \geq 2 \),
\[ M_1 \lesssim \int_0^1 \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \| \nabla c^+ (\tau) \|_{\dot{B}^0_{2,1}} \| \nabla u^+ (\tau) \|_{\dot{H}^2_{2,1}} d\tau \]
\[ + \int_1^t \langle t - \tau \rangle^{-\left(\frac{\Delta}{2} + \frac{1}{2}\right)} \| \nabla c^+ (\tau) \|_{\dot{B}^0_{2,1}} \| \nabla u^+ (\tau) \|_{\dot{H}^2_{2,1}} d\tau \]
\[ \overset{\text{def}}{=} M_{11} + M_{12}. \]
Using the definitions of $X(t)$ and $D(t)$, we obtain

\[
M_{11} \lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \sup_{0 \leq \tau \leq 1} \|\nabla c^+(\tau)\|_{B_{2,1}^0} \int_0^1 \|\nabla u^+(\tau)\|^h_{B_{2,1}^0} d\tau \lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} D(1) X(1),
\]

and, employing the fact that $\langle \tau \rangle \approx \tau$ when $\tau \geq 1$, we have

\[
M_{12} \lesssim \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^{\frac{\alpha}{2} + \frac{1}{2}} \|\nabla c^+(\tau)\|_{B_{2,1}^0} \right) \left( \sup_{1 \leq \tau \leq t} \|\tau \nabla u^+(\tau)\|^h_{B_{2,1}^0} \right) 
\times \int_1^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} d\tau 
+ \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^\alpha \|\nabla c^+(\tau)\|^h_{B_{2,1}^0} \right) \left( \sup_{1 \leq \tau \leq t} \|\tau \nabla u^+(\tau)\|^h_{B_{2,1}^0} \right) 
\times \int_1^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-(\alpha + 1)} d\tau 
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} D^2(t).
\]

For the term $M_2$, we have

\[
M_2 \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \|\nabla c^+(\tau)\|_{B_{2,1}^0} \|\nabla u^+(\tau)\|^h_{B_{2,1}^0} d\tau 
+ \int_1^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \|\nabla c^+(\tau)\|^h_{B_{2,1}^0} \|\nabla u^+(\tau)\|_{B_{2,1}^0} d\tau 
= M_{21} + M_{22}.
\]

Remembering the definitions of $X(t)$ and $D(t)$, we obtain

\[
M_{21} \lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \sup_{0 \leq \tau \leq 1} \|\nabla c^+(\tau)\|_{B_{2,1}^0} \sup_{0 \leq \tau \leq 1} \|\nabla c^+(\tau)\|_{B_{2,1}^0} \int_0^1 \|\nabla u^+(\tau)\|^h_{B_{2,1}^0} d\tau 
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} D^2(1) X(1),
\]

and

\[
M_{22} \lesssim \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^{\frac{\alpha}{2}} \|\nabla c^+(\tau)\|_{B_{2,1}^0} \right) 
\int_1^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\frac{\alpha}{2}} \|\nabla c^+(\tau)\|^h_{B_{2,1}^0} d\tau 
+ \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^\alpha \|\nabla c^+(\tau)\|^h_{B_{2,1}^0} \right) 
\int_1^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-(\alpha + 1)} \|\nabla u^+(\tau)\|^h_{B_{2,1}^0} d\tau 
\lesssim D^3(t) \int_1^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \|\nabla c^+(\tau)\|^h_{B_{2,1}^0} d\tau 
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} D^3(t).
\]

Therefore, for $t \geq 2$, we obtain

\[
\int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \|\nabla c^+(\tau)\|^h_{B_{2,1}^0} \|\nabla c^+(\tau)\|_{B_{2,1}^0} d\tau 
\approx \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \left( \int_0^t \|\nabla c^+(\tau)\|^h_{B_{2,1}^0} d\tau \right) 
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} (X^2(t) + D^2(t) + D^3(t) + D^4(t)). \tag{6.82}
\]
The case $t \leq 2$ is obvious as $(t) \approx 1$ and $(t - \tau) \approx 1$ for $0 \leq \tau \leq t \leq 2$, and

$$\int_0^t \|\mu^+ h_+(c^+, c^-) \partial_j c^+ \partial_j u^+_i(\tau)\|_{L^2} \, d\tau \lesssim \|h_+(c^+, c^-)\|_{L^\infty_t(\mathbb{R}^2)} \|\nabla c^+ \nabla (u^+)^h\|_{L^1_t(\mathbb{R}^2)} \lesssim \left( \|h_+(c^+, c^-)\|_{L^\infty_t(\mathbb{R}^2)} - \frac{(C^2\alpha^2 - (1, 1))}{s_+^2(1, 1)} \right) \|\nabla c^+\|_{L^\infty_t(\mathbb{R}^2)} \|\nabla u^+\|_{L^1_t(\mathbb{R}^2)} \|H_1(B_1)\| (6.83)$$

From (6.81)-(6.83), we finally conclude that

$$\int_0^t \|\mu^+ h_+(c^+, c^-) \partial_j c^+ \partial_j u^+_i(\tau)\|_{L^2} \, d\tau \lesssim (t)^{-\left(\frac{N}{2} + \frac{3}{2}\right)} (X^2(t) + D^2(t) + D^3(t) + D^4(t)).$$

Similarly, we also obtain the corresponding estimates of other terms $\mu^+ k_+(c^+, c^-) \partial_j c^+ \partial_j u^+_j$, $\mu^+ k_+(c^+, c^-) \partial_j c^+ \partial_j u^+_j$, $\mu^+ k_+(c^+, c^-) \partial_j c^+ \partial_j u^+_j$, $\mu^+ k_+(c^+, c^-) \partial_j c^+ \partial_j u^+_j$, $\mu^+ k_+(c^+, c^-) \partial_j c^+ \partial_j u^+_j$, $\mu^+ k_+(c^+, c^-) \partial_j c^+ \partial_j u^+_j$. Here, we omit the details.

From the low-high frequency decomposition for $\mu^+ l_+(c^+, c^-) \partial^2 u^+_i$, we have

$$\mu^+ l_+(c^+, c^-) \partial^2 u^+_i = \mu^+ l_+(c^+, c^-) \partial^2 (u^+_i)^\ell + \mu^+ l_+(c^+, c^-) \partial^2 (u^+_i)^h,$$

where $l_+$ stands for some smooth function vanishing at $0$. Thus,

$$\int_0^t (t - \tau)^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \|\mu^+ l_+(c^+, c^-) \partial^2 (u^+_i)^\ell\|_{L^2} \, d\tau \lesssim \int_0^t (t - \tau)^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \|l_+(c^+, c^-)\|_{L^2} \|\nabla^2 u^+\|_{L^2} \, d\tau \lesssim \int_0^t (t - \tau)^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \|\nabla^2 u^+\|_{L^2} \, d\tau \lesssim \left( \sup_{\tau \in [0, t]} \langle \tau \rangle^\frac{N}{2} \|\|c^+, c^-\|\|_{L^2} \right) \left( \sup_{\tau \in [0, t]} \langle \tau \rangle^\frac{N}{2} \|\nabla^2 u^+\|_{L^2} \right)

\times \int_0^t (t - \tau)^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\left(\frac{N}{2} + 1\right)} \, d\tau \left(6.85\right)$$

$$+ \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^\alpha \|\|c^+, c^-\|\|_{L^\infty_{\tau}} \right) \left( \sup_{\tau \in [0, t]} \langle \tau \rangle^\frac{N}{2} \|\nabla^2 u^+\|_{L^2} \right)

\times \int_0^t (t - \tau)^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\left(\frac{N}{2} + \alpha + 1\right)} \, d\tau \lesssim D^2(t) \int_0^t (t - \tau)^{-\left(\frac{N}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\min\left(\frac{N}{2} + 1, \frac{N}{2} + \alpha + 1\right)} \, d\tau \lesssim (t)^{-\left(\frac{N}{2} + \frac{3}{2}\right)} D^2(t).$$

To handle the term $\mu^+ l_+(c^+, c^-) \partial^2 (u^+_i)^h$, we consider the cases $t \geq 2$ and $t \leq 2$ respectively. When $t \geq 2$, then we have
\[
\int_0^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \mu^+ l_+ (c^+, c^-) \partial_2^2 (u_i^+)^h \|_{L^1} \, d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \mu^+ l_+ (c^+, c^-) \|_{\bar{B}^2_{2,1}} \| \nabla^2 u^+ \|^h_{\bar{B}^2_{2,1}} \, d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \| (c^+, c^-) \|_{\bar{B}^0_{2,1}} \| \nabla^2 u^+ \|^h_{\bar{B}^2_{2,1}} \, d\tau
\]
\[
+ \int_1^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \| (c^+, c^-) \|_{\bar{B}^2_{2,1}} \| \nabla^2 u^+ \|^h_{\bar{B}^2_{2,1}} \, d\tau
\]
\[
def \equiv N_1 + N_2.
\]

From the definitions of \( X(t) \) and \( D(t) \), we obtain
\[
N_1 = \int_0^1 \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \| (c^+, c^-) \|_{\bar{B}^2_{2,1}} \| \nabla^2 u^+ \|^h_{\bar{B}^2_{2,1}} \, d\tau \, d\tau
\]
\[
\lesssim \langle t \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \left( \sup_{\tau \in [0,1]} \| (c^+, c^-) \|_{\bar{B}^2_{2,1}} \right) \int_0^1 \| u^+ \|^h_{\bar{B}^2_{2,1}} \, d\tau
\]
\[
\lesssim \langle t \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} D(1) X(1),
\]
and, using the fact that \( \langle \tau \rangle \approx \tau \) when \( \tau \geq 1 \), we have
\[
N_2 = \int_1^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \| (c^+, c^-) \|_{\bar{B}^2_{2,1}} \| \nabla^2 u^+ \|^h_{\bar{B}^2_{2,1}} \, d\tau
\]
\[
\lesssim \int_1^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \| (c^+, c^-) \|_{\bar{B}^2_{2,1}} \| \nabla^2 u^+ \|^h_{\bar{B}^2_{2,1}} \, d\tau
\]
\[
\lesssim \int_1^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \| (c^+, c^-) \|_{\bar{B}^2_{2,1}} \| \nabla^2 u^+ \|^h_{\bar{B}^2_{2,1}} \, d\tau
\]
\[
+ \int_1^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \| (c^+, c^-) \|_{\bar{B}^2_{2,1}} \| \nabla^2 u^+ \|^h_{\bar{B}^2_{2,1}} \, d\tau
\]
\[
\lesssim \left( \sup_{0 \leq \tau \leq t} \langle t \rangle^{-\frac{1}{2}} \right) \| (c^+, c^-) \|_{\bar{B}^2_{2,1}} \left( \sup_{0 \leq \tau \leq t} \| \tau \nabla u(\tau) \|^h_{\bar{B}^2_{2,1}} \right)
\]
\[
\int_1^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right) + 1} \, d\tau
\]
\[
+ \left( \sup_{0 \leq \tau \leq t} \| \nabla (c^+, c^-) \|_{\bar{B}^2_{2,1}}^{-\alpha} \right) \left( \sup_{0 \leq \tau \leq t} \| \tau \nabla u(\tau) \|^h_{\bar{B}^2_{2,1}} \right)
\]
\[
\int_1^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\alpha + \frac{1}{4} + \frac{1}{2}\right)} \, d\tau
\]
\[
\lesssim D^2(t) \int_1^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\alpha + \frac{1}{4} + \frac{1}{2}\right)} \, d\tau
\]
\[
\lesssim \langle t \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} D^2(t).
\]

Thus, for \( t \geq 2 \), we arrive at
\[
\int_0^t \langle t - \tau \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} \mu^+ l_+ (c^+, c^-) \partial_2^2 (u_i^+)^h \|_{L^1} \, d\tau \lesssim \langle t \rangle^{-\left(\frac{1}{4} + \frac{1}{2}\right)} (X^2(t) + D^2(t)).
\]
The case \( t \leq 2 \) is obvious as \( (t) \approx 1 \) and \( (t - \tau) \approx 1 \) for \( 0 \leq \tau \leq t \leq 2 \),
\[
\int_0^t \| \mu^+ t^+ (c^+, c^-) \partial_j \partial_j (u^+_1) \|^h_{L^1} \ d\tau \\
\lesssim \int_0^t \| t^+ (c^+, c^-) \|^h_{ \dot{B}_{2,1}^{\alpha}} \| \nabla^2 u^+_1 \|^h_{ \dot{B}^{\alpha}_{2,1}} \ d\tau \\
\lesssim (\sup_{\tau \in [0,1]} \| (c^+, c^-)(\tau) \|^h_{ \dot{B}^{\alpha}_{2,1}} ) \int_0^1 \| u \|^h_{ \dot{B}^{\alpha}_{2,1}} \ d\tau \\
\lesssim D(t) X(t).
\]

From (6.85)-(6.87), we get
\[
\int_0^t \langle t - \tau \rangle^{-(\frac{\alpha}{2} + \frac{\alpha}{2})} \| \mu^+ t^+ (c^+, c^-) \partial_j \partial_j (u^+_1) \|^h_{L^1} \ d\tau \lesssim \langle t \rangle^{-(\frac{\alpha}{2} + \frac{\alpha}{2})} (D^2(t) + X^2(t)). \tag{6.88}
\]

Similarly,
\[
\int_0^t \langle t - \tau \rangle^{-(\frac{\alpha}{2} + \frac{\alpha}{2})} \| (\mu^+ + \lambda^+) t^+ (c^+, c^-) \partial_1 \partial_j u^+_1 \|^h_{L^1} \ d\tau \lesssim \langle t \rangle^{-(\frac{\alpha}{2} + \frac{\alpha}{2})} (D^2(t) + X^2(t)). \tag{6.89}
\]

Thus,
\[
\int_0^t \langle t - \tau \rangle^{-(\frac{\alpha}{2} + \frac{\alpha}{2})} \| H_2(\tau) \|^h_{ \dot{B}_{2,\infty}^{\alpha}} \ d\tau \lesssim \langle t \rangle^{-(\frac{\alpha}{2} + \frac{\alpha}{2})} (X^2(t) + D^2(t) + D^3(t) + D^4(t)).
\]

The term \( H_4 \) may be treated along the same lines, and we have
\[
\int_0^t \langle t - \tau \rangle^{-(\frac{\alpha}{2} + \frac{\alpha}{2})} \| H_4(\tau) \|^h_{ \dot{B}_{2,\infty}^{\alpha}} \ d\tau \lesssim \langle t \rangle^{-(\frac{\alpha}{2} + \frac{\alpha}{2})} (X^2(t) + D^2(t) + D^3(t) + D^4(t)).
\]

Thus, we complete the proof of (6.72). Combining with (6.70) and (6.72), we conclude that for all \( t \geq 0 \) and \( s \in (\theta, 2) \),
\[
\langle t \rangle^{\frac{\alpha}{2} + \frac{\alpha}{2}} \| (c^+, u^+, c^-, u^-) \|^h_{ \dot{B}_{2,1}^{\alpha}} \lesssim D_0 + X^2(t) + D^2(t) + D^3(t) + D^4(t). \tag{6.90}
\]

**Step 2: High frequencies.** Now, the starting point is Inequality (3.44) which implies that for \( q \geq q_0 \) and for some \( c_0 = c(q_0) > 0 \), we have
\[
\frac{1}{2} \frac{d}{dt} \alpha_q^2 + c_0 \alpha_q^2 \\
\leq \left( \| (\Delta_q H_{11}, \Delta_q H_{21}, \Delta_q H_{31}, \Delta_q H_{41}, \nabla \Delta_q H_{11}, \nabla \Delta_q H_{31}) \|_{L^2} + \| R_q(u^+, c^+) \|_{L^2} \\
+ \| R_q(u^+, u^+) \|_{L^2} + \| R_q(u^-, c^-) \|_{L^2} + \| R_q(u^-, u^-) \|_{L^2} \\
+ \| \tilde{R}_q(u^+, c^+) \|_{L^2} + \| \tilde{R}_q(u^-, c^-) \|_{L^2} + \| \nabla (u^+, u^-) \|_{L^\infty} \alpha_q \right) \alpha_q,
\]
in which
\[
R_q(u, b) \overset{\text{def}}{=} [u^+ \cdot \nabla, \Delta_q] b = u^+ \cdot \nabla \Delta_q b - \Delta_q (u^+ \cdot \nabla b) \quad \text{for} \quad b \in \{c^\pm, u^\pm\},
\]
\[
\tilde{R}_q(u, b) \overset{\text{def}}{=} [u^+ \cdot \nabla, \partial_1 \Delta_q] c^\pm = u^+ \cdot \nabla \partial_1 \Delta_q c^\pm - \partial_1 \Delta_q (u^+ \cdot \nabla c^\pm).
\]

After time integration, we have
\[
e^{c_0 t} \alpha_q(t) \\
\leq \alpha_q(0) + \int_0^t e^{c_0 \tau} \left( \| (\Delta_q H_{11}, \Delta_q H_{21}, \Delta_q H_{31}, \Delta_q H_{41}, \nabla \Delta_q H_{11}, \nabla \Delta_q H_{31}) \|_{L^2} \\
+ \| R_q(u^+, c^+) \|_{L^2} + \| R_q(u^+, u^+) \|_{L^2} + \| R_q(u^-, c^-) \|_{L^2} + \| R_q(u^-, u^-) \|_{L^2} \\
+ \| \tilde{R}_q(u^+, c^+) \|_{L^2} + \| \tilde{R}_q(u^-, c^-) \|_{L^2} + \| \nabla (u^+, u^-) \|_{L^\infty} \alpha_q \right) \alpha_q dt.
\]
+ \|R_q(u^+, c^+)\|_{L^2} + \|R_q(u^+, u^+)\|_{L^2} + \|R_q(u^-, c^-)\|_{L^2} + \|R_q(u^-, u^-)\|_{L^2} \\
+ \|\tilde{R}_q(u^+, c^+)\|_{L^2} + \|\tilde{R}_q(u^-, c^-)\|_{L^2} + \|\nabla(u^+, u^-)\|_{L^2} \simeq \alpha_q \, dt.

For \( q \geq q_0 \), we have \( \alpha_q \approx \|(\nabla \Delta_q c^+, \hat{\Delta}_q u^+, \nabla \Delta_q c^-, \hat{\Delta}_q u^-)\|_{L^2} \). Then,

\[
\langle t \rangle^{\alpha} \|(\nabla \Delta_q c^+, \hat{\Delta}_q u^+, \nabla \Delta_q c^-, \hat{\Delta}_q u^-)(t)\|_{L^2} \\
\lesssim \langle t \rangle^{\alpha} e^{-\alpha t} \|(\nabla \Delta_q c^+, \hat{\Delta}_q u^+, \nabla \Delta_q c^-, \hat{\Delta}_q u^-)(0)\|_{L^2} \\
+ \langle t \rangle^{\alpha} \int_0^t e^{\alpha(t-s)} \left( \|\Delta_q H_{11}, \Delta_q H_{21}, \Delta_q H_{31}, \Delta_q H_{41}, \nabla \Delta_q H_{11}, \nabla \Delta_q H_{31}, \nabla \Delta_q H_{41}\|_{L^2} \\
+ \|R_q(u^+, c^+)\|_{L^2} + \|R_q(u^+, u^+)\|_{L^2} + \|R_q(u^-, c^-)\|_{L^2} + \|R_q(u^-, u^-)\|_{L^2} \right) \\
+ \|\tilde{R}_q(u^+, c^+)\|_{L^2} + \|\tilde{R}_q(u^-, c^-)\|_{L^2} + \|\nabla(u^+, u^-)\|_{L^2} \lesssim \alpha_q \, dt,
\]

and thus, by multiplying both sides by \( 2^{q(\frac{N}{2} - 1)} \), taking the supremum on \([0, T]\), and then summing up over \( q \geq q_0 \),

\[
\langle t \rangle^{\alpha} \|(\nabla c^+, u^+, \nabla c^-, u^-)\|_{H^\infty(B_{\frac{N}{2} - 1}^{2,1})} \\
\lesssim \|\nabla_0^+, u_0^+, \nabla_0^-, u_0^-\|_{H^\infty(B_{\frac{N}{2} - 1}^{2,1})} \\
+ \sum_{q \geq q_0} \sup_{0 \leq t \leq T} \langle t \rangle^{\alpha} \int_0^t e^{\alpha(t-s)} 2^{q(\frac{N}{2} - 1)} S_q \, d\tau
\]

with \( S_q \equiv \sum_{i=1}^8 S_q^i \) and

\[
S_q^1 \equiv \|(\Delta_q H_{11}, \Delta_q H_{21}, \Delta_q H_{31}, \Delta_q H_{41}, \nabla \Delta_q H_{11}, \nabla \Delta_q H_{31}, \nabla \Delta_q H_{41})\|_{L^2}, \\
S_q^2 \equiv \|R_q(u^+, c^+)\|_{L^2}, \quad S_q^3 \equiv \|R_q(u^+, u^+)\|_{L^2}, \quad S_q^4 \equiv \|R_q(u^-, c^-)\|_{L^2}, \\
S_q^5 \equiv \|R_q(u^-, u^-)\|_{L^2}, \quad S_q^6 \equiv \|\tilde{R}_q(u^+, c^+)\|_{L^2}, \quad S_q^7 \equiv \|\tilde{R}_q(u^+, u^+)\|_{L^2}, \\
S_q^8 \equiv \|\nabla(u^+, u^-)\|_{L^\infty} \|(\nabla \Delta_q c^+, \hat{\Delta}_q u^+, \nabla \Delta_q c^-, \hat{\Delta}_q u^-)\|_{L^2}.
\]

Bounding the sum, for \( 0 \leq t \leq 2 \), and taking advantage of Proposition 5, we end up with

\[
\sum_{q \geq q_0} \sup_{0 \leq t \leq 2} \langle t \rangle^{\alpha} \int_0^t e^{\alpha(t-s)} 2^{q(\frac{N}{2} - 1)} S_q(\tau) \, d\tau \lesssim \int_0^2 \sum_{q \geq q_0} 2^{q(\frac{N}{2} - 1)} S_q(\tau) \, d\tau \\
\lesssim \int_0^2 \left( \|H_{11}, H_{21}, H_{31}, H_{41}, \nabla H_{11}, \nabla H_{31}\|_{H^{\infty}(B_{\frac{N}{2} - 1}^{2,1})} \\
+ \|\nabla(u^+, u^-)\|_{L^\infty(B_{\frac{N}{2} - 1}^{2,1})} \|(c^+, u^+, c^-, u^-, \nabla c^+, \nabla c^-)\|_{L^{\infty}(B_{\frac{N}{2} - 1}^{2,1})} \right) \, d\tau
\]

\[
\lesssim \int_0^2 \left( \|(\nabla H_{11}, H_{21}, \nabla H_{31}, H_{41})\|_{L^{\infty}(B_{\frac{N}{2} - 1}^{2,1})} \\
+ \|\nabla(u^+, u^-)\|_{L^\infty(B_{\frac{N}{2} - 1}^{2,1})} \|(c^+, u^+, c^-, u^-, \nabla c^+, \nabla c^-)\|_{L^{\infty}(B_{\frac{N}{2} - 1}^{2,1})} \right) \, d\tau \equiv Q_1 + Q_2.
\]
From Propositions 1-3, we bound the terms $Q_1$ and $Q_2$ as follows

\[
\int_0^t \| \nabla H_{11} \|_{B_{2,1}^{\frac{N}{2},-1}}^h \, d\tau \lesssim \int_0^t \| H_{11} \|_{B_{2,1}^{\frac{N}{2}}}^h \, d\tau \\
\lesssim \int_0^t \| c^+ \text{div} u^+ \|_{B_{2,1}^{\frac{N}{2}}}^h \, d\tau \\
\lesssim \int_0^t \| c^+ \|_{B_{2,1}^{\frac{N}{2}}} \| \text{div} u^+ \|_{B_{2,1}^{\frac{N}{2}}} \, d\tau \\
\lesssim \| c^+ \|_{L^\infty(B_{2,1}^{\frac{N}{2}+1,1/2})} \| u^+ \|_{L^1(B_{2,1}^{\frac{N}{2}+1})} \\
\lesssim X^2(2).
\]

For the term $\nabla H_{31}$, along the same lines, we have

\[
\int_0^t \| \nabla H_{31} \|_{B_{2,1}^{\frac{N}{2},-1}}^h \, d\tau \lesssim X^2(2).
\]

Combining interpolation inequality and Hölder’s inequality, we deduce that

\[
\int_0^t \| g_+(c^+, c^-) \partial_t c^+ (\tau) \|_{B_{2,1}^{\frac{N}{2},-1}} \, d\tau \\
\lesssim \| (c^+, c^-) \|_{L^2_t(B_{2,1}^{\frac{N}{2}})} \| \nabla c^+ \|_{L^2_t(B_{2,1}^{\frac{N}{2}+1})} \\
\lesssim \left( \| (c^+, c^-) \|_{L^1_t(B_{2,1}^{\frac{N}{2}+1,1/2})} \right)^{1/2} \left( \| (c^+, c^-) \|_{L^\infty_t(B_{2,1}^{\frac{N}{2}+1,1/2})} \right)^{1/2} \\
\times \left( \| c^+ \|_{L^1_t(B_{2,1}^{\frac{N}{2}+1,1/2})} \right)^{1/2} \left( \| c^+ \|_{L^\infty_t(B_{2,1}^{\frac{N}{2}+1,1/2})} \right)^{1/2} \\
\lesssim X^2(2),
\]

where $g_+$ stands for some smooth function vanishing at 0.

Similarly,

\[
\int_0^t \| \tilde{g}_+(c^+, c^-) \partial_t c^- \|_{B_{2,1}^{\frac{N}{2},-1}} \, d\tau \lesssim X^2(2).
\]

For the term $\mu^+ h_+(c^+, c^-) \partial_j c^+ \partial_j u_1^+$, we obtain

\[
\int_0^t \| \mu^+ h_+(c^+, c^-) \partial_j c^+ \partial_j u_1^+ (\tau) \|_{B_{2,1}^{\frac{N}{2},-1}} \, d\tau \\
\lesssim \int_0^t \| h_+(c^+, c^-) \|_{B_{2,1}^{\frac{N}{2}}} \| \nabla c^+ \nabla u^+ (\tau) \|_{B_{2,1}^{\frac{N}{2}}} \, d\tau \\
\lesssim \int_0^t \| h_+(c^+, c^-) \|_{B_{2,1}^{\frac{N}{2}}} \| \nabla c^+ \|_{B_{2,1}^{\frac{N}{2}}} \| \nabla u^+ (\tau) \|_{B_{2,1}^{\frac{N}{2}}} \, d\tau \\
\lesssim (1 + \| (c^+, c^-) \|_{L^\infty_t(B_{2,1}^{\frac{N}{2}+1,1/2})} \| c^+ \|_{L^\infty_t(B_{2,1}^{\frac{N}{2}+1,1/2})} \| u^+ \|_{L^1_t(B_{2,1}^{\frac{N}{2}+1})} \, d\tau \\
\lesssim X^2(2) + X^3(2).
\]
The estimates of other terms such as $\mu^+ k_+ (c^+, c^-) \partial_x c^- \partial_x u_j^+$, $\mu^+ h_+ (c^+, c^-) \partial_x c^+ \partial_x u_j^+$, $\mu^+ k_+ (c^+, c^-) \partial_x c^- \partial_x u_j^+$, $\lambda^+ h_+ (c^+, c^-) \partial_x c^+ \partial_x u_j^+$ and $\lambda^+ k_+ (c^+, c^-) \partial_x c^- \partial_x u_j^+$ are similar to (6.95). Here, we omit the details.

For $\mu^+ l_+ (c^+, c^-) \partial_x^2 u_j^+$, thus

$$
\int_0^t \| \mu^+ l_+ (c^+, c^-) \partial_x^2 u_j^+ (\tau) \|_{B_{x,1}^{\frac{N}{2} - 1}} d\tau \\
\lesssim \|(c^+, c^-)\|_{L_t^\infty (B_{x,1}^{\frac{N}{2}})} \| \nabla^2 u_j^+ \|_{L_t^1 (B_{x,1}^{\frac{N}{2} - 1})} \\
\lesssim \|(c^+, c^-)\|_{L_t^\infty (B_{x,1}^{\frac{N}{2}})} \| u_j^+ \|_{L_t^1 (B_{x,1}^{\frac{N}{2} + 1})}
$$

(6.96)

where $l_+$ stands for some smooth function vanishing at 0.

Similarly,

$$
\int_0^2 \| H_{21} \|_{B_{x,1}^{\frac{N}{2} - 1}} d\tau \lesssim X^2(2) + X^3(2),
$$

$$
\int_0^2 \| H_{41} \|_{B_{x,1}^{\frac{N}{2} - 1}} d\tau \lesssim X^2(2) + X^3(2),
$$

and

$$
\int_0^2 \| \nabla (u^+, u^-) \|_{B_{x,1}^{\frac{N}{2}}} \| (c^+, u^+, c^-, u^-, \nabla c^+, \nabla c^-) \|_{B_{x,1}^{\frac{N}{2} - 1}} d\tau \\
\lesssim \| \nabla (u^+, u^-) \|_{L_t^1 (B_{x,1}^{\frac{N}{2}})} \| (c^+, u^+, c^-, u^-, \nabla c^+, \nabla c^-) \|_{L_t^\infty (B_{x,1}^{\frac{N}{2} - 1})} \\
\lesssim \| (u^+, u^-) \|_{L_t^1 (B_{x,1}^{\frac{N}{2} + 1})} \| (c^+, c^-) \|_{L_t^\infty (B_{x,1}^{\frac{N}{2} - 1})} + \| (u^+, u^-) \|_{L_t^\infty (B_{x,1}^{\frac{N}{2} - 1})}
$$

$\lesssim X^2(2)$.

Therefore, for the case $t \leq 2$,

$$
\sum_{q \geq q_0} \sup_{0 \leq t \leq 2} \left( \langle t \rangle^\alpha \int_0^t e^{c_0 (\tau - t)} 2^q (\frac{N}{2} - 1) S_q d\tau \right) \lesssim X^2(2) + X^3(2).
$$

(6.97)

To bound the supremum on $[2, T]$, we split the integral on $[0, t]$ into integrals on $[0, 1]$ and $[1, t]$, respectively. The $[0, 1]$ part of the integral is easy to handle, and we have

$$
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \langle t \rangle^\alpha \int_0^1 e^{c_0 (\tau - t)} 2^q (\frac{N}{2} - 1) S_q d\tau \right) \\
\lesssim \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \langle t \rangle^\alpha e^{-\frac{\alpha t}{2}} \int_0^1 2^q (\frac{N}{2} - 1) S_q d\tau \right) \\
\lesssim \int_0^1 \sum_{q \geq q_0} 2^q (\frac{N}{2} - 1) S_q d\tau.
$$

Hence

$$
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \langle t \rangle^\alpha \int_0^t e^{c_0 (\tau - t)} 2^q (\frac{N}{2} - 1) S_q d\tau \right) \lesssim X^2(1) + X^3(1).
$$

(6.98)
Let us finally consider the \([1,t]\) part of the integral for \(2 \leq t \leq T\). We shall use repeatedly the following inequalities

\[
\| \tau \nabla u^\pm \|_{L^p(B_{R_1}^{\infty})} \lesssim D(t),
\]

which is straightforward as regards to the high frequencies of \(u^\pm\) and stem from

\[
\| \tau \nabla u^\pm \|_{L^p(B_{R_1}^{\infty})} \lesssim \|(\tau)^{\frac{N}{2} + \frac{1}{2}} \nabla u^\pm \|_{L^p(B_{R_1}^{\infty})} \lesssim \|(\tau)^{\frac{N}{2} + \frac{1}{2}} u^\pm \|_{L^p(B_{R_1}^{\infty})} \lesssim D(t),
\]

for the low frequencies of \(u^\pm\).

Regarding the contribution of \(\tau\), by Lemma 2.6 we first notice that

\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \langle t \rangle^\alpha \int_1^t e^{c_0(\tau-t)2^q(\frac{N}{2}-1)} S_q^1(\tau) d\tau \right)
\]

\[
= \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \langle t \rangle^\alpha \int_1^t e^{c_0(\tau-t)2^q(\frac{N}{2}-1)} \right)
\]

\[
\lesssim \| \tau^\alpha (H_{11}, H_{21}, H_{31}, H_{41}, \nabla H_{11}, \nabla H_{31}) \|_{L^p(B_{R_1}^{\infty})} \lesssim \| \tau^\alpha (H_{11}, H_{21}, H_{31}, H_{41}, \nabla H_{11}, \nabla H_{31}) \|_{L^p(B_{R_1}^{\infty})} \lesssim \| \tau^\alpha (\nabla H_{11}, H_{21}, \nabla H_{31}, H_{41}) \|_{L^p(B_{R_1}^{\infty})} \lesssim \| \tau^\alpha (\nabla H_{11}, H_{21}, \nabla H_{31}, H_{41}) \|_{L^p(B_{R_1}^{\infty})} \lesssim \| \tau^\alpha (\nabla H_{11}, H_{21}, \nabla H_{31}, H_{41}) \|_{L^p(B_{R_1}^{\infty})}.
\]

Now, the product laws in tile spaces ensures that

\[
\| \tau^\alpha \nabla H_{11} \|_{L^p(B_{R_1}^{\infty})} \lesssim \| \tau^\alpha - 1 \|_{L^p(B_{R_1}^{\infty})} \| \tau \|_{L^p(B_{R_1}^{\infty})} \lesssim D(T).
\]

The high frequencies of the first term is obviously bounded by \(D(T)\). That is,

\[
\| \tau^\alpha - 1 \|_{L^p(B_{R_1}^{\infty})} \lesssim \| \tau \|_{L^p(B_{R_1}^{\infty})} \lesssim D(T),
\]

as for the low frequencies, we notice that if \(N \leq 4\) for all small enough \(\varepsilon > 0\),

\[
\| \tau^\alpha - 1 \|_{L^p(B_{R_1}^{\infty})} \lesssim \| \tau^\alpha - 1 \|_{L^p(B_{R_1}^{\infty})} \lesssim D(T),
\]

and if \(N \geq 5\),

\[
\| \tau^\alpha - 1 \|_{L^p(B_{R_1}^{\infty})} \lesssim \| \tau^\alpha - 1 \|_{L^p(B_{R_1}^{\infty})} \lesssim D(T).
\]

Combining with (6.100), (6.101) and (6.102), we obtain

\[
\| \tau^\alpha - 1 \|_{L^p(B_{R_1}^{\infty})} \lesssim D(T).
\]
Therefore, using (6.99) and (6.103) we get
\[
\|\tau^\alpha \nabla H_1\|_{L_T^\infty(B_{2,1}^N)} \lesssim D^2(T).
\]
Similarly,
\[
\|\tau^\alpha \nabla H_3\|_{L_T^\infty(B_{2,1}^N)} \lesssim D^2(T).
\]
Next, we shall use repeatedly the following inequality
\[
\|c^+\|_{L_T^\infty(B_{2,1}^N/2)} \lesssim \|c^+\|_{L_T^\infty(B_{2,1}^N)},
\]
\[
\|c^-\|_{L_T^\infty(B_{2,1}^N/2)} \lesssim \|c^-\|_{L_T^\infty(B_{2,1}^N)}.
\]
For the first term of \(H_{21}\), employing (6.104) and the definition of \(D(t)\), we have
\[
\|\tau^\alpha g_+(c^+, c^-)\|_{L_T^\infty(B_{2,1}^N)} \lesssim \|\tau^\alpha \nabla c^+\|_{L_T^\infty(B_{2,1}^N)} \lesssim X(T)D(T).
\]
According to (6.101), (6.102) and the definition of \(D(t)\), we have
\[
\|\tau^\alpha g_+(c^+, c^-)\|_{L_T^\infty(B_{2,1}^N)} \lesssim \|\tau^\alpha \nabla c^+\|_{L_T^\infty(B_{2,1}^N)} \lesssim X(T)D(T).
\]
Thus,
\[
\|\tau^\alpha g_+(c^+, c^-)\|_{L_T^\infty(B_{2,1}^N)} \lesssim X^2(T) + D^2(T).
\]
Thus, the terms $\mu^+ k_+(c^+, c^-) \partial_j c^- \partial_j u^+_i$, $\mu^+ h_+(c^+, c^-) \partial_j c^+ \partial_j u^+_i$, $\mu^+ k_+(c^+, c^-) \partial_j c^- \partial_j u^+_j$, $\lambda^+ h_+(c^+, c^-) \partial_j c^+ \partial_j u^+_j$ and $\lambda^+ k_+(c^+, c^-) \partial_j c^- \partial_j u^+_j$ may be treated along the same lines.

From (6.99) and (6.103), we also see that

$$\|\tau^{\alpha} \mu^+ l_+(c^+, c^-) \partial^2_j u^+_i \|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})} \lesssim \|\nabla^2 u^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})} \|\tau^{\alpha-1}(c^+, c^-)\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}})} \lesssim D^2(T),$$

$$\|\tau^{\alpha}(\mu^+ + \lambda^+) l_+(c^+, c^-) \partial_i \partial_j u^+_j \|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})} \lesssim \|\nabla^2 u^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})} \|\tau^{\alpha-1}(c^+, c^-)\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}})} \lesssim D^2(T).$$

Thus,

$$\|\tau^{\alpha} H_{21}\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})} \lesssim X^2(T) + D^2(T) + D^4(T),$$

$$\|\tau^{\alpha} H_{11}\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})} \lesssim X^2(T) + D^2(T) + D^4(T).$$

To bound the term with $S_q^{2\delta}$, we use the following fact that

$$\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \tau^{\alpha} \int_1^t e^{c_0(\tau-t)} 2q(\frac{N}{2}-1) S_q^2(\tau) d\tau \right) \lesssim \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \tau^{\alpha} \int_1^t e^{c_0(\tau-t)} 2q(\frac{N}{2}-1) \|R_q(u^+, c^+)^\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})} d\tau \right) \lesssim \|\tau^\alpha u^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}})} \|\tau^{\alpha-1} c^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})}.$$

The first term on the right-side of the above inequality may be bounded from (6.99), and the high frequencies of the last one on the right-side are obviously bounded by $D(T)$. To bound the term $\|\tau^{\alpha-1} c^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})}$, if $N \leq 6$ we have the following inequality

$$\|\tau^{\alpha-1} c^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})} \lesssim \|\tau^{\alpha-1} c^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1-2\epsilon})} \lesssim \|\tau^{\alpha-\frac{N}{2}-\frac{\epsilon}{2}+2\epsilon} c^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1-2\epsilon})} \lesssim D(T).$$

If $N \geq 7$,

$$\|\tau^{\alpha-1} c^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}-1})} \lesssim \|\tau^{\alpha-1} c^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}})} \lesssim \|\tau^{\alpha-\frac{N}{2}-2\epsilon} c^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}})} \lesssim \|\tau^{\frac{N}{2}} c^+\|_{L_t^\infty(B_{2,1}^{\frac{N}{2}})} \lesssim D(T).$$
We eventually get
\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \tau^\alpha \int_1^t e^{\gamma_0 (\tau-t)} 2^{q (\frac{2}{d} - 1)} S_q^2(\tau) \, d\tau \right) \lesssim D^2(T). \tag{6.107}
\]

Similarly,
\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \tau^\alpha \int_1^t e^{\gamma_0 (\tau-t)} 2^{q (\frac{2}{d} - 1)} \left( S_q^3(\tau) + S_q^4 + S_q^5(\tau) \right) \, d\tau \right) \lesssim D^2(T). \tag{6.108}
\]

Finally, using product laws, (6.99), (6.103) and Lemma 2.6, we obtain
\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{\gamma_0 (\tau-t)} 2^{q (\frac{2}{d} - 1)} S_q^6(\tau) \, d\tau \right)
\lesssim \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{\gamma_0 (\tau-t)} 2^{q (\frac{2}{d} - 1)} \| \tilde{R}_q(u^+, c^+) \|_{L^2} \, d\tau \right) \tag{6.109}
\]
\[
\lesssim \| \tau \nabla u^+ \|_{\tilde{L}^\infty(B_{\frac{N}{2} T}^2)} \| \tau^{\alpha - 1} \nabla c^+ \|_{\tilde{L}^\infty(B_{\frac{N}{2} T}^2)} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{\gamma_0 (\tau-t)} \tau^{-\alpha} \, d\tau \right)
\lesssim D^2(T),
\tag{6.110}
\]
and
\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{\gamma_0 (\tau-t)} 2^{q (\frac{2}{d} - 1)} S_q^8(\tau) \, d\tau \right)
\lesssim \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{\gamma_0 (\tau-t)} 2^{q (\frac{2}{d} - 1)} \| \nabla (u^+, u^-) \|_{L^\infty} \right.
\left. \| \nabla \Delta_q c^+, \Delta_q u^+, \nabla \Delta_q c^-, \Delta_q u^- \|_{L^2(\tau)} \, d\tau \right) \tag{6.111}
\]
\[
\lesssim \| \tau \nabla (u^+, u^-) \|_{\tilde{L}^\infty(B_{\frac{N}{2} T}^2)} \| \tau^{\alpha - 1} (\nabla c^+, u^+, \nabla c^-, u^-) \|_{\tilde{L}^\infty(B_{\frac{N}{2} T}^2)} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{\gamma_0 (\tau-t)} \tau^{-\alpha} \, d\tau \right)
\lesssim D^2(T).
\]

Putting all the above inequalities together, we conclude that
\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{\gamma_0 (\tau-t)} 2^{q (\frac{2}{d} - 1)} S_q(\tau) \, d\tau \right) \lesssim X^2(T) + X^3(T) + D^2(T) + D^4(T). \tag{6.112}
\]

Then plugging (6.97), (6.98) and (6.112) into (6.91) yields
\[
\| \langle \tau \rangle^{\alpha} (\nabla c^+, u^+, \nabla c^-, u^-) \|_{L^\infty(B_{\frac{N}{2} T}^2)} \lesssim \| (\nabla c_0^+, u_0^+, \nabla c_0^-, u_0^-) \|_{B_{\frac{N}{2} T}^2} + X^2(T) + X^3(T) + D^2(T) + D^4(T). \tag{6.113}
\]
Step 3: Decay estimates with gain of regularity for the high frequencies of $\nabla u^+, \nabla u^-$. This step is devoted to bounding the last two terms of $D(t)$. We first deal with the term $\|\tau \nabla u^+\|_{L^\infty_t(B^s_{2,1})}^h$ and shall use the fact that the velocity $u^+$ satisfies

$$\partial_t u^+ - Au^+ = F := -\beta_1 \nabla c^+ - \beta_2 \nabla c^- + H_2;$$

where $A = \nu_1^+ \Delta - \nu_2^+ \nabla \text{div}$.

So,

$$\partial_t(tAu^+) - A(tAu^+) = Au^+ + tAF.$$

We deduce from Remark 6 that

$$\|\tau A u^+\|_{L^\infty_t(B^s_{2,1})}^h \lesssim \|A u^+\|_{L^1_t(B^s_{2,1})}^h + \|\tau AF\|_{L^\infty_t(B^s_{2,1})}^h,$$

whence, using the bounds given by Theorem 1.1, we have

$$\|\tau \nabla u^+\|_{L^\infty_t(B^s_{2,1})}^h \lesssim X(0) + \|\tau F\|_{L^\infty_t(B^s_{2,1})}^h.$$ 

In order to bound the first two terms of $F$, we notice that, from $\alpha \geq 1$ and (6.113), we have

$$\|\tau (\nabla c^+, \nabla c^-)\|_{L^\infty_t(B^s_{2,1})}^h \lesssim \|\tau^\alpha (\nabla c^+, \nabla c^-)\|_{L^\infty_t(B^s_{2,1})}^h \lesssim X(0) + X^2(t) + X^3(t) + D^2(t) + D^3(t).$$

Next, the product and composition estimates adapted to tilde spaces give

$$\|\tau g_+(c^+, c^-)\partial_t c^+\|_{L^\infty_t(B^s_{2,1})}^h \lesssim \|\tau^\alpha_2 (c^+, c^-)\|_{L^\infty_t(B^s_{2,1})}^h \|\tau^\alpha_2 \nabla c^+\|_{L^\infty_t(B^s_{2,1})}^h \lesssim D^2(t).$$

Similarily, we have

$$\|\tau \tilde{g}_+(c^+, c^-)\partial_t c^-\|_{L^\infty_t(B^s_{2,1})}^h \lesssim D^2(t).$$

Furthermore, from (6.99) and the definition of $X(t)$, we have

$$\|\tau (u^+ \cdot \nabla) u^+\|_{L^\infty_t(B^s_{2,1})}^h \lesssim \|u^+\|_{L^\infty_t(B^s_{2,1})}^h \|\tau \nabla u^+\|_{L^\infty_t(B^s_{2,1})}^h \lesssim X(t) D(t).$$

Employing (6.99) and (6.104), we get

$$\|\tau \mu^+ h_+(c^+, c^-)\partial_j c^+ \partial_j u^+_i\|_{L^\infty_t(B^s_{2,1})}^h \lesssim \left(1 + \|(c^+, c^-)\|_{L^\infty_t(B^s_{2,1})}^h\right) \|\nabla c^+\|_{L^\infty_t(B^s_{2,1})}^h \|\tau \nabla u^+\|_{L^\infty_t(B^s_{2,1})}^h \lesssim X^2(t) + D^2(t) + X^4(t).$$

The terms $\mu^+ h_+(c^+, c^-)\partial_j c^- \partial_j u^+_i$, $\mu^+ h_+(c^+, c^-)\partial_j c^+ \partial_j u^-_i$, $\mu^+ k_+(c^+, c^-)\partial_j c^- \partial_j u^-_i$, $\mu^+ k_+(c^+, c^-)\partial_j c^+ \partial_j u^+_i$ and $\lambda^+ k_+(c^+, c^-)\partial_j c^- \partial_j u^+_i$ and $\lambda^+ k_+(c^+, c^-)\partial_j c^+ \partial_j u^-_i$ may be treated along the same lines.

From (6.99) and (6.104), we have

$$\|\tau \mu^+ l_+(c^+, c^-)\partial_j^2 u^+_i\|_{L^\infty_t(B^s_{2,1})}^h \lesssim \|(c^+, c^-)\|_{L^\infty_t(B^s_{2,1})}^h \|\tau \nabla u^+\|_{L^\infty_t(B^s_{2,1})}^h \lesssim X(t) D(t),$$
\[ \| \tau (\mu^+ + \lambda^+) l_+ (c^+, c^-) \partial_i \partial_j u_j^+ \|_{L^\infty_t(B_{2,1}^{\infty,-1})} \lesssim X(t) D(t). \]

Therefore,
\[ \| \tau \nabla u^+ \|_{L^\infty_t(B_{2,1}^{\infty,-1})} \lesssim X(0) + X^2(t) + X^4(t) + D^2(t). \quad (6.117) \]

Similarly,
\[ \| \tau \nabla u^- \|_{L^\infty_t(B_{2,1}^{\infty,-1})} \lesssim X(0) + X^2(t) + X^4(t) + D^2(t). \quad (6.118) \]

Finally, adding up these obtained inequalities (6.117) and (6.118) to (6.90) and (6.113) yields for all \( t \geq 0 \),
\[ D(t) \lesssim X(0) + D_0 + \| \nabla c_0^+ + u_0^+ , \nabla c_0^- , u_0^- \|_{B_{2,1}^{\infty,-1}}^h \]
\[ \quad + X^2(t) + X^3(t) + X^4(t) + D^2(t) + D^3(t) + D^4(t) \]
\[ \lesssim D_0 + \| \nabla c_0^+ + u_0^+ , \nabla c_0^- , u_0^- \|_{B_{2,1}^{\infty,-1}}^h \]
\[ \quad + X^2(t) + X^3(t) + X^4(t) + D^2(t) + D^3(t) + D^4(t) , \]

where we have used \( X(0)^{\frac{1}{2}} = \| (c_0^+ + u_0^+ , c_0^- , u_0^-) \|_{B_{2,1}^{\infty,-1}}^h \lesssim \| (c_0^+ , u_0^+ , c_0^- , u_0^-) \|_{B_{2,1}^{\infty,-1}}^h \). As Theorem 1.1 ensures that \( X(t) \) is small, one can conclude that (1.15) is fulfilled for all time if \( D_0 \) and \( \| \nabla R_0^+ , u_0^+ , \nabla R_0^- , u_0^- \|_{B_{2,1}^{\infty,-1}}^h \) are small enough. This completes the proof of Theorem 1.2. \( \square \)

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