Computational Thresholds for the Fixed-Magnetization Ising Model

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ABSTRACT
The ferromagnetic Ising model is a model of a magnetic material and a central topic in statistical physics. It also plays a starring role in the algorithmic study of approximate counting: approximating the partition function of the ferromagnetic Ising model with uniform external field is tractable at all temperatures and on all graphs, due to the randomized algorithm of Jerrum and Sinclair.

Here we show that hidden inside the model are hard computational problems. For the class of bounded-degree graphs we find computational thresholds for the approximate counting and sampling problems for the ferromagnetic Ising model at fixed magnetization (that is, fixing the number of +1 and −1 spins).

In particular, letting $β_\Lambda(\Delta)$ denote the critical inverse temperature of the zero-field Ising model on the infinite $\Delta$-regular tree, and $\eta^{\Delta,1}_{\Lambda,1}$ denote the mean magnetization of the zero-field + measure on the infinite $\Lambda$-regular tree at inverse temperature $β$, we prove, for the class of graphs of maximum degree $\Delta$: (i) for $β < β_\Lambda(\Delta)$ there is an FPRAS and efficient sampling scheme for the fixed-magnetization Ising model for all magnetizations $\eta$. (ii) For $β > β_\Lambda(\Delta)$, there is an FPRAS and efficient sampling scheme for the fixed-magnetization Ising model for magnetizations $\eta$ such that $|\eta| > η^{\Delta,1}_{\Lambda,1}$. (iii) For $β > β_\Lambda(\Delta)$, there is no FPRAS for the fixed-magnetization Ising model for magnetizations $\eta$ such that $|\eta| < η^{\Delta,1}_{\Lambda,1}$ unless NP=RP.

1 INTRODUCTION

The Ising model is a mathematical model of a magnetic material, fundamental in the study of phase transitions in statistical physics. The Ising model is a probability distribution over spins in a graph, and its partition function is the weighted sum over all spin configurations. In the field of approximate counting in computer science, the ferromagnetic Ising model plays a special role along with the monomer-dimer model as models for which approximating the partition function is tractable on all graphs and at all temperatures [30, 31].

Conditioning on the magnetization of the model corresponds to fixing the balance of the random cut generated. In particular, at zero magnetization (an equal number of plus and minus spins), the Ising model is a probability distribution on bisections of the graph. In the study of spin models on sparse random graphs in physics, it has long been known that conditioning on zero magnetization can turn a ferromagnetic system into a glassy system [42] (i.e. fixing the magnetization can drastically change the model and induce slow dynamics). This suggests that lurking inside the tractable computational problems associated to the Ising model there may be hard problems accessible by fixing the magnetization.

We make this idea concrete in a complexity-theoretic sense by reducing NP-hard balanced cut problems to approximating the...
1.1 The Ising Model on Graphs and Trees

The Ising model on a finite graph $G$ at inverse temperature $\beta$ and activity $\lambda$ is the probability distribution $\mu_{G,\beta,\lambda}$ on $\Sigma_G := \{\pm 1\}^V(G)$ defined by

$$\mu_{G,\beta,\lambda}(\sigma) = \frac{e^{\beta \sum_{\langle u,v\rangle \in E(G)} \sigma_u \sigma_v \lambda M(\sigma)}}{Z_G(\beta,\lambda)}$$

where $M(\sigma) = \sum_{\sigma \in V(G)} \sigma \sigma_0$ and

$$Z_G(\beta,\lambda) = \sum_{\sigma \in \Sigma_G} e^{\beta \sum_{\langle u,v\rangle \in E(G)} \sigma_u \sigma_v \lambda M(\sigma)}.$$

The distribution $\mu_{G,\beta,\lambda}$ is the Gibbs measure and $Z_G(\beta,\lambda)$ is the partition function. When $\beta \geq 0$ the model is ferromagnetic, and we will always assume this in what follows. In statistical physics the activity is often written as $\lambda = e^h$ where $h$ is the external field, and so we will call the unbiased case $\lambda = 1$ the zero-field Ising model.

The quantity $M(\sigma)$ is the magnetization of the configuration $\sigma$. The normalized mean magnetization of the Ising model is

$$\eta_G(\beta,\lambda) = \frac{\langle M(\sigma) \rangle_{G,\beta,\lambda}}{|V(G)|},$$

where $\langle \cdot \rangle_{G,\beta,\lambda}$ denotes expectation with respect to the Ising model.

We can also define the Ising model at fixed magnetization. For $k \equiv |V(G)| \mod 2$, $|k| \leq |V(G)|$, let $\Sigma_G(k) = \{\sigma \in \Sigma_G : M(\sigma) = k\}$ be the subset of Ising configurations with magnetization $k$. Then the Ising model on $G$ at inverse temperature $\beta$ and fixed magnetization $k$ is the distribution $\nu_{G,\beta,k}$ on $\Sigma_G(k)$ defined by

$$\nu_{G,\beta,k}(\sigma) = \frac{e^{\beta \sum_{\langle u,v\rangle \in E(G)} \sigma_u \sigma_v \lambda M(\sigma)}}{Z^\text{fix}_G(\beta,\lambda)}$$

where

$$Z^\text{fix}_G(\beta,\lambda) = \sum_{\sigma \in \Sigma_G(k)} e^{\beta \sum_{\langle u,v\rangle \in E(G)} \sigma_u \sigma_v \lambda M(\sigma)}.$$

The distribution $\nu_{G,\beta,k}$ is simply the Ising model at inverse temperature $\beta$ (and arbitrary activity $\lambda > 0$) conditioned on the event $\sigma \in \Sigma_G(k)$. The fixed-magnetization partition function $Z^\text{fix}_G(\beta,\lambda)$ is the coefficient of $\lambda^k$ when interpreting $Z_G(\beta,\lambda)$ as a Laurent polynomial in $\lambda$.

The Ising model can be defined on the infinite $\Lambda$-regular tree $\mathbb{T}_\Lambda$ via the DLR equations \cite{[16,37]} or as a weak limit of Ising models on finite-depth trees with given boundary conditions. Infinite regular trees are important in computer science as “optimal” expanders, and here we use a known relationship between the Ising model on random regular graphs and on the infinite tree. Depending on the parameters $\beta, \lambda$ there may be a unique infinite-volume Gibbs measure on $\mathbb{T}_\Lambda$ or there may be multiple measures. The critical inverse temperature is $\beta_\Lambda(\Lambda) = \log \frac{\lambda}{\lambda - 1}$; for $\beta < \beta_\Lambda$ there is a unique Gibbs measure for all $\lambda$ and for $\beta > \beta_\Lambda$ there can be multiple measures if $\lambda$ is close enough to 1 \cite{[40]}. We will be interested in one particular Gibbs measure on $\mathbb{T}_\Lambda$, the “+” measure induced by the weak limit of finite-depth trees with all + boundary conditions. We denote this measure $\mu^+_{\Lambda,\beta,\lambda}$. By the FKG inequality $\mu^+_{\Lambda,\beta,\lambda}$ stochastically dominates all other Gibbs measures on $\mathbb{T}_\Lambda$ with the same parameters.

We let $\eta^+_{\Lambda,\beta,\lambda}$ denote the expected value of the spin at the root of $\mathbb{T}_\Lambda$ under $\mu^+_{\Lambda,\beta,\lambda}$ (equivalently, the expected value of the spin at any fixed vertex since $\mu^+_{\Lambda,\beta,\lambda}$ is translation invariant). Then the magnetization of the measure $\mu^+_{\Lambda,\beta,\lambda}$ is

$$\eta^+_{\Lambda,\beta,\lambda} = \text{tanh} \left( L^* + \text{artanh}(\text{tanh} L^* \tanh \frac{\beta}{2}) \right),$$

where $L^*$ is the largest solution to

$$L^* = \log \lambda + (\Lambda - 1) \text{artanh}(\text{tanh} L^* \tanh \frac{\beta}{2}).$$

See Section 4 for more details and a derivation.

The phase transition on $\mathbb{T}_\Lambda$ manifests itself via the following “spontaneous magnetization” phenomenon \cite{[40]}:

1. For $\beta < \beta_\Lambda(\Lambda)$, $\eta^+_{\Lambda,\beta,1} = 0$.
2. For $\beta > \beta_\Lambda(\Lambda)$, $\eta^+_{\Lambda,\beta,1} > 0$.

1.2 Computational Problems and Thresholds

There are two main computational problems associated to a spin model like the Ising model. The approximate counting problem asks for an $\epsilon$-relative approximation to the partition function $Z_G$; that is, a number $\hat{Z}$ so that $(1-\epsilon)Z_G \leq \hat{Z} \leq (1+\epsilon)Z_G$. An FPTAS is an algorithm that provides such an approximation and runs in time polynomial in $|V(G)|$ and $1/\epsilon$. An FPRAS is a randomized algorithm that provides such an approximation with probability at least 2/3 and runs in time polynomial in $|V(G)|$ and $1/\epsilon$. The approximate sampling problem is to output a sample $\sigma$ with distribution $\hat{\mu}$ so that $\|\mu_G - \hat{\mu}\|_V < \epsilon$. An efficient sampling scheme is a randomized algorithm that satisfies this guarantee and runs in time polynomial in $|V(G)|$ and $\log(1/\epsilon)^4$.

Jerrum and Sinclair gave an FPRAS for the ferromagnetic Ising model for all graphs, all inverse temperatures $\beta$, and all choices of the activity $\lambda$ \cite{[31],2}. Via self-reducibility of the random cluster representation of the Ising model, this gives an efficient sampling scheme as well \cite{[47]}.

On the other hand, for the anti-ferromagnetic Ising model (and the hard-core model of weighted independent sets), the approximate counting and sampling problems are NP-hard in general, and for the class of bounded degree graphs precise computational thresholds

\footnote{Sometimes the required dependence of the running time for an efficient approximate sampler is taken to be polynomial in $1/\epsilon$ instead of $\log(1/\epsilon)$; we use the stronger definition here.}

\footnote{In fact the algorithm works in the case of non-uniform activities, as long as they are consistent: all at least 1 or all at most 1. The general case of approximating the partition function with non-uniform activities is #BIS-hard \cite{[22]}.}
are known. The results of Weitz [55], Sly [52], Sly–Sun [53], Galanis–Stefankovič–Vigoda [19], and Sinclair–Srivastava–Thurley [51] show that for these models (and for $\beta < -\beta_c(\Delta)$ in the case of the anti-ferromagnetic Ising model) there is a computational threshold at some critical activity $\lambda_c = \lambda_c(\Delta, \beta)$. In the case of the hard-core model, there is an FPTAS for $Z_\Delta(\lambda)$ for $\lambda < \lambda_c$ and graphs $G$ of maximum degree $\Delta$ while for $\lambda > \lambda_c(\Delta)$ there is no FPTAS unless NP=RP.

For the ferromagnetic Ising model there are no such computational thresholds. But one can ask instead for approximation algorithms for coefficients of the partition function or approximate sampling algorithms for the Ising model at fixed magnetization.

For $\beta \geq 0$ and $\eta \in [-1, 1]$, let $\text{Fixed-Ising}(G, \beta, \eta)$ be the problem of computing the partition function $Z^\text{fix}_G(\beta, k)$ of the $n$-vertex graph $G$, where $k$ is the largest integer such that $k \equiv n \mod 2$ and $k \leq \eta n$. In other words, $k = 2\lceil (\eta + 1)n/2 \rceil - n$. The associated sampling problem is to sample spin assignments from the measure $v_{G, \beta, k}$. The restriction on the parity of $k$ is simply to ensure that configurations of magnetization $k$ exist. Abusing notation slightly we will refer to both $\eta$ and $k$ as the magnetization, but it will be clear from context what is meant.

Fixed magnetization is the setting of the Kawasaki dynamics for the Ising model [33, 34]: a conservative dynamics with stationary distribution $v_{G, \beta, k}$ that at each step proposes a swap of nearest-neighbor spins. Understanding the convergence properties of the Kawasaki dynamics on subsets of $\mathbb{Z}^d$ is a deep mathematical problem [5, 6, 39, 56]. In this paper we address the problem on general graphs from the perspective of computational complexity.

### 1.3 Our Results

In what follows we always assume $\beta \geq 0$ and $\Delta \geq 3$. When $\beta < \beta_c(\Delta)$ we give efficient approximate counting and sampling algorithms for all magnetizations.

**Theorem 1.** Let $\Delta \geq 3$ and $\beta < \beta_c(\Delta)$. For all $\eta \in [-1, 1]$ there is an FPRAS and efficient sampling scheme for $\text{Fixed-Ising}(G, \beta, \eta)$ for graphs of maximum degree $\Delta$.

**Theorem 2.** Let $\Delta \geq 3$, $\beta > \beta_c(\Delta)$, and $\eta_c = \eta^*_c(\Delta, \beta)$ the mean magnetization of the zero-field + measure on $T_\Delta$.

**Theorem 2.** Let $\Delta \geq 3$, $\beta > \beta_c(\Delta)$, and $\eta_c = \eta^*_c(\Delta, \beta)$ the mean magnetization of the zero-field + measure on $T_\Delta$.

(a) For all $\eta$ with $|\eta| > \eta_c$ there is an FPRAS and efficient sampling scheme for $\text{Fixed-Ising}(G, \beta, \eta)$ for graphs of maximum degree $\Delta$.

(b) Unless NP=RP, for all $\eta$ with $|\eta| \leq \eta_c$ there is no FPRAS for $\text{Fixed-Ising}(G, \beta, \eta)$ for graphs of maximum degree $\Delta$.

In (b) our proof in fact shows that given $\Delta$, $\beta$ there is some $\zeta > 0$ such that unless NP=RP, there is no polynomial-time algorithm for $\text{Fixed-Ising}(G, \beta, \eta)$ which achieves a multiplicative approximation of $e^{\zeta n}$ on $n$-vertex graphs $G$ of maximum degree $\Delta$.

The infinite regular tree plays several roles in the proof of Theorem 2. For the hardness results, non-uniqueness for the zero-field Ising model on the tree at $\beta > \beta_c$ corresponds to “phase coexistence” of the model on the random $\Delta$-regular graph [12]. Phase coexistence allows us to use random graphs as gadgets, as Sly does in establishing a computational threshold for the hard-core model [52] (and as is done in subsequent hardness proofs, e.g. [4, 18, 53]).

For the algorithmic results, the $+$ measure on the infinite regular tree appears as the solution to a problem from extremal graph theory that is essential for the proof of Theorem 2. For the ferromagnetic Ising model with activity $\lambda > 1$, the extremal problem we must solve is to find the maximum mean magnetization over all graphs of maximum degree $\Delta$? We prove that the magnetization of the $+$ measure on the infinite $\Delta$-regular tree is an upper bound, and this value is approached by that of the random $\Delta$-regular graph in the $n \to \infty$ limit. The following result is the main combinatorial result of our paper.

**Theorem 3.** For all graphs $G$ of maximum degree $\Delta$, all $\lambda \geq 1$, and all $\beta \geq 0$,

$$\eta_G(\beta, \lambda) \leq \eta^*_c(\Delta, \beta, \lambda).$$

By integrating the mean magnetization from $\lambda = 1$ to $\infty$, this theorem implies the $\Delta$-regular case of a result of Ruozzi which states that the “Bethe approximation” is a lower bound on the normalized partition function of the ferromagnetic Ising model [49]. In combinatorics, results of this type belong to the field of extremal problems for bounded-degree graphs: maximizing or minimizing observables of statistical physics models over given classes of graphs, like the occupancy fraction of the hard-core or monomer-dimer models [10].

The area is surveyed by Zhao in [58] and Csikvári describes several cases in which the optimal bound on a partition function is given by an analogous quantity on an infinite regular tree [8]. Bounds on observables such as the mean magnetization or occupancy fraction are stronger than bounds on the partition function, and to the best of our knowledge Theorem 3 is the first case in which the infinite tree is proved to be extremal for an observable.

Theorem 3 implies the following extremal spontaneous magnetization result, which is what we use to guarantee the effectiveness of our algorithm. Define

$$\eta^*_c(\Delta, \beta) = \lim_{\lambda \to 1^+} \sup_{G \in \tilde{G}_\Delta} \eta_G(\beta, \lambda).$$
where $G_\Delta$ is the class of graphs of maximum degree $\Delta$. Then $\eta^*(\Delta, \beta) = \eta^*_\lambda.\beta.1$. The lower bound comes from taking a sequence of random $\Delta$-regular graphs, while the upper bound follows from Theorem 3. We describe in the next section the content of our algorithmic results for $\beta > \beta_c$: that $\eta_e(\Delta, \beta) = \eta^*(\Delta, \beta) = \eta^*_\lambda.\beta.1$.

1.4 Overview of the Techniques

1.4.1 Algorithms. For the algorithmic results of Theorem 2, we aim to apply the same type of algorithm as in Theorem 1: find an activity $\lambda$ so the mean magnetization is close to the target magnetization, and prove that the probability of hitting the mean is not too small. Again by continuity, there is an activity with the correct mean magnetization, but the distribution may not be concentrated around its mean. For instance, taking $\lambda = 1$ gives 0 mean magnetization by symmetry, but if $\beta > \beta_c$, then hitting 0 magnetization on the random regular graph is exponentially unlikely. So our question becomes: given an arbitrary graph of maximum degree $\Delta$ and a desired magnetization $\eta$, is the magnetization under $\mu_G(\beta, \lambda)$ guaranteed to be concentrated around its mean when $\lambda$ is chosen so that the mean magnetization is (close to) $\eta$? The answer to this question is given by the Lee–Yang theorem [38] in combination with Theorem 3, which guarantees that to achieve a mean magnetization $\eta > \eta^*(\Delta, \beta)$ we can pick an activity $\lambda$ bounded away from 1 independent of $n$. The Lee–Yang theorem then gives the zero-freeness result that provides us with the required central limit theorem.

Our proof of Theorem 3 is an extension of an approach used by Krinsky [36] to prove the result for infinite lattices like $\mathbb{Z}^d$ (or more generally graphs satisfying vertex and edge transitivity). The theorem (and the paper [36] that inspired it) may be of independent interest in combinatorics and algorithms. The proof of Theorem 3 relies heavily on correlation inequalities, namely the GKS inequalities [23, 35], and identities due to Thompson [54]. The techniques are distinct from previous approaches in this area of extremal graph theory such as the entropy method [32], occupancy method [10], and inductive approaches [9, 50].

1.4.2 Hardness. To prove a matching hardness result, we must overcome the barrier of the tractability of approximating the Ising partition function. This rules out the approach used in [11] for proving hardness of approximating the number of independent sets of a given size, namely reducing approximating the partition function to approximating a fixed coefficient of the partition function. Instead, we use the fact that imposing the fixed-magnetization constraint fundamentally alters the behavior of the model. When highly connected components of a graph are connected with a relatively sparse set of edges, the fixed-magnetization, zero-field ferromagnetic Ising model exhibits a kind of global anti-ferromagnetic behavior due to the constraint on the magnetization: the spins on each highly connected component will align, but the number of components that pick each spin will be essentially determined by the constraint. This behavior is what allows us to prove hardness. We use a probabilistic analysis of the fixed-magnetization Ising model to show that a gadget construction based on that of [52] can be used to reduce an NP-hard cut problem to approximating the fixed-magnetization Ising partition function. To illustrate our methods we sketch a simplified version of the proof for zero magnetization.

Similar to previous approaches, our gadget $G$ is essentially a random $\Delta$-regular bipartite graph with some edges removed and trees attached to create “terminal vertices” of degree $\Delta - 1$. Given an instance $H$ of Min-Bisection, we replace each vertex of $H$ by a copy of the gadget $G$ and then join a number of terminal vertices of the appropriate copies of the gadget graph for each edge of $H$. When $\beta > \beta_c$, the Ising model on a single gadget $G$ exhibits phase coexistence, with a bimodal distribution of either many more $+$ spins than $-$ spins or vice versa. The phase coexistence property of each gadget is so strong that when we take the collection of gadgets joined by the crossing edges and condition the Ising model on zero magnetization, the phase coexistence property on each gadget persists, and zero-magnetization is achieved (with high probability) by having an equal number of gadgets in each phase. Showing this involves proving a local central limit theorem and large deviation results for the magnetization of a collection of gadgets conditioned on an arbitrary spin assignment to the set of terminal vertices. This shows that the dominant contribution to the zero-magnetization partition function is given by configurations whose gadget phase assignments encode minimum bisections of $H$, and this in turn implies that a good approximation algorithm for the partition function can recover a minimum bisection.

The proof of the local central limit theorem conditioned on the phases of the gadgets is a new technical ingredient in our proof. It involves bounding the moments of the magnetization on a single gadget, conditioned on a phase, and employing a Fourier analytic proof of a local central limit theorem.

The full proof and the general case of $\eta \neq 0$ are only slightly more complex. Broadly, the same approach works except we reduce from a generalization of Min-Bisection, $\gamma$-Min-Exact-Balanced-Cut ($\gamma$-MEBC), that requires the partition of a vertex set of size $N$ to have part sizes $\gamma N$ and $(1 - \gamma)N$. It is convenient to add to the collection of gadget graphs some isolated vertices which smooth out certain parts of the analysis. In particular, it helps in proving the local central limit theorem. We choose $\gamma$ as a function of $\Delta, \beta$, and $\eta$, and we prove that when the Ising model on the collection of gadget graphs is conditioned to have magnetization $\eta$, with high probability the phases of the gadgets are split in fractions $\gamma$ and $1 - \gamma$. Then a good approximation algorithm for the $\eta$-magnetization partition function can recover a minimum $\gamma$-balanced cut.

1.5 Related Work

The algorithmic problem of sampling configurations of a fixed magnetization (or fixed size, in the case of independent sets) is the problem of sampling from the “canonical ensemble” in the language of statistical physics (in contrast to the “grand canonical ensemble” of the usual Ising or hard-core model). Work on this problem goes back to the very first Markov Chain Monte Carlo algorithm designed to sample from the canonical ensemble of hard spheres [41]. Conservative dynamics such as these are still among the most used in current scientific applications (e.g., [2]). Grand canonical ensembles are generally more amenable to mathematical analysis due to their conditional independence properties, and much is known about both specific algorithms for sampling from these distributions (e.g. Glauber dynamics [44], random-cluster dynamics [24]) and
about the computational complexity of the approximate counting and sampling problems for these models.

The computational complexity of approximately counting and sampling independent sets of a given size in bounded-degree graphs was recently addressed by Davies and Perkins who proved a computation threshold for these problems [11]. As in Theorem 2, the threshold is given in terms of an extremal graph theory problem: that of minimizing the occupancy fraction over \( G \in \mathcal{G}_\Delta \). Faster algorithms and an FPTAS up to the threshold for this problem were recently given in [26].

The use of random graphs as gadgets in hardness reductions was pioneered by Dyer, Frieze, and Jerrum [17] and used by Sly in identifying the computational threshold for the hard-core model [52], with further applications in [4, 18, 20, 53] among others. In particular, a detailed understanding of the moments of the partition function \( Z_G \) for random regular graphs is now known, and, via the small subgraph conditioning method, concentration results for \( Z_G \). We use this understanding extensively in Section 3.

Finally, the Ising model at fixed magnetization has been studied extensively in both mathematics and physics, on \( \mathbb{Z}^d \) and on random graphs [42]. Conditioning the ferromagnetic Ising model on zero magnetization has the effect of introducing “frustration”: the impossibility of satisfying all edge constraints simultaneously.

At zero temperature (\( \beta = \infty \)), the zero-magnetization Ising model is simply the uniform distribution on min-bisections of a graph; finding the size of the min bisection has long been known to be \( \mathsf{NP} \)-hard [21]. The min-bisection problem is also studied on random graphs from the perspective of statistical physics [13, 14, 45, 57]. Our work is an exploration of the worst-case computational complexity of the positive temperature regime of this problem.

1.6 Questions and Future Directions

Though we do not pursue it in this extended abstract, it is likely that the techniques of Jain, Perkins, Sah, and Sawhney [26] can be used to improve the algorithmic results of Theorems 1 and 2 in two ways:

1. Obtain an FPTAS (efficient deterministic approximation algorithm) for Fixed-Ising(\( G, \beta, \eta \)) for the same range of parameters for which we obtain an FPRAS.
2. Improve the running time of our approximate sampling algorithm to \( O(n \log n) \).

We have shown here a computational threshold for the fixed-magnetization Ising model. One can also ask what is achievable with a specific algorithm widely used in scientific applications, namely the Kawasaki dynamics. We conjecture that the Kawasaki dynamics mix rapidly on all graphs of maximum degree \( \Delta \) for the same set of parameters for which we provide an FPRAS. In fact there are two versions of the Kawasaki dynamics: the local flip dynamics in which at each step a swap of spins across an edge is proposed; and the global flip dynamics in which at each step a swap of arbitrary spins in the graph is proposed. We conjecture that both versions mix in polynomial time for the parameters above; we further conjecture that the global flip dynamics mix in time \( O(n \log n) \).

**Conjecture 1.** For \( \beta < \beta_c(\Delta) \), the Kawasaki dynamics mix in time polynomial in \( n \) for any fixed magnetization and any graph \( G \) of maximum degree \( \Delta \) on \( n \) vertices.

For \( \beta > \beta_c(\Delta) \) and \( |\eta| > \eta_c(\Delta, \beta) \) the Kawasaki dynamics mix in time polynomial in \( n \) for any fixed magnetization \( k \geq \eta n \) and any graph \( G \) of maximum degree \( \Delta \) on \( n \) vertices.

For the global flip dynamics, the mixing time in both cases is \( O(n \log n) \).

In the previous uses of random (bipartite) graphs as gadgets in hardness reductions for approximate counting problems, the gadgets themselves are not in general hard instances for the given problems. In particular, recent results [7, 25, 28, 29] show that for parameters sufficiently deep in the given non-uniqueness regimes, random regular graphs are tractable instances for approximate counting and sampling. We ask whether for random graphs there are efficient algorithms anywhere inside the NP-hardness regime.

**Question 1.** For \( \Delta \geq 3 \), \( \beta > \beta_c(\Delta) \), is there some \( |\eta| < \eta_c(\Delta, \beta) \) so that there exist efficient approximate counting and sampling algorithms for Fixed-Ising(\( G, \beta, \eta \)) for random \( \Delta \)-regular graphs?

1.7 Organization

In this extended abstract we give the details of the hardness result, Theorem 2(b). See the full version of the paper for the algorithmic results, Theorems 1 and 2(a). In Section 2 we provide some of the results we will use in our algorithms and hardness reductions. In Section 3 we give the hardness reduction. In Section 4 we prove Theorem 3, solving the extremal problem that identifies the limit of our algorithmic approach. In Section 5 we sketch our algorithmic approach.

2 PRELIMINARIES

Recall that \( \mathcal{G}_\Delta \) denotes the class of graphs of maximum degree \( \Delta \). We use \( \mu_{\beta, \lambda} \) to denote the Ising model on \( G \) at inverse temperature \( \beta \) and activity \( \lambda \). We will often drop \( \beta \) from the notation when it remains fixed and we will drop \( \lambda \) from the notation in the case \( \lambda = 1 \) (so \( \mu_{\beta} = \mu_{\beta, \lambda = 1} \) when \( \beta \) is understood from the context). We use the bracket notation \( \langle \cdots \rangle_{G, \beta, \lambda} \) to denote expectations with respect to the Ising model, in part to distinguish these expectations from expectations over random graphs in Section 3. For a graph \( G \), let \( \Sigma_G = \{ \pm 1 \}^{V(G)} \). Slightly abusing notation, for \( U \subset V(G) \), let \( \Sigma_U = \{ \pm 1 \}^U \). We let \( M(\sigma) \) denote the magnetization of a configuration \( \sigma \) and \( X(\sigma) \) denote the number of + spins (so \( M(\sigma) = 2X(\sigma) - |V(G)| \)). We let \( X \) denote the random variable \( X(\sigma) \) when \( \sigma \) is drawn from \( \mu_{\beta, \lambda} \).

We now collect a number of results that we use in the proofs that follow. The first ones give zero-free regions for the Ising model partition function, viewed as a (Laurent) polynomial in \( \lambda \).

**Theorem 4 (Lee–Yang [38]).** For \( \beta \geq 0 \), \( \lambda \in \mathbb{C} \), and any graph \( G \), \( Z_G(\beta, \lambda) = 0 \) only if \( |\lambda| = 1 \).

**Theorem 5 (Peters–Rechts [46]).** Let \( \Delta \geq 3 \) and \( \beta \in (0, \beta_c(\Delta)) \). Then there exists \( \theta = \theta(\beta) \in (0, \pi) \) such that for any \( \lambda \in \mathbb{C} \) with \( |\arg(\lambda)| < \theta \) and any graph \( G \in \mathcal{G}_\Delta \), we have \( Z_G(\beta, \lambda) \neq 0 \).

By the following general result of Michelen and Sahasrabudhe, these zero-freeness results imply central limit theorems for the
magnetization of the ferromagnetic Ising model on graphs in $G_\Lambda$ when $\beta < \beta_c(\Lambda)$ or when $\lambda > 1$. We apply this result to a random variable counting the number of $+1$ spins in a sample from the Ising model; its generating function is a scaling of $Z_G(\beta, \lambda)$.

**Theorem 6** (Michele-Sahasrabudhe [43]). For $n \geq 1$ let $X_n$ be a random variable taking values in $\{0, \ldots, n\}$ with mean $\mu_n$, standard deviation $\sigma_n$, and probability generating function $f_n$. If the roots of $f_n$ satisfy $\arg(\zeta) \geq \sigma_n \delta n \to \infty$, then $(X_n - \mu_n) / \sigma_n$ converges in distribution to a standard normal random variable.

A central limit theorem for the magnetization in fact implies a local central limit theorem, following the approach of Dobrushin and Tirozzi [15] (for spin models on $\mathbb{Z}^d$) and the results of [26] for the hard-core model. We give the proof in the full version of the paper. Let $X$ denote the number of $+1$ spins in a sample from the Ising model.

**Proposition 7.** Fix $\lambda > 1$ and $\beta \geq 0$. Then for any graph $G \in G_\Lambda$ on $n$ vertices and any non-negative integer $\ell$,

$$
\mu_{G,\beta,\lambda}(X = \ell) = \frac{1}{\sqrt{2\pi \text{var}(X)}} \exp\left(-\frac{(\ell - (X_{G,\beta,\lambda})^2)^2}{2 \text{var}(X)}\right) + o(n^{-1/2}),
$$

where $\text{var}(X) = (X_{G,\beta,\lambda}^2 - (X_{G,\beta,\lambda})^2)/G,\beta,\lambda$ and where the constant in the error term depend only on $\Lambda$, $\beta$, $\lambda$. The same holds for $\beta < \beta_c$, $\lambda \geq 1$, and any $G \in G_\Lambda$.

Moreover, under the conditions above $\text{var}(X) = \Theta(n)$ where again the implied constants depend only on $\Lambda, \beta, \lambda$.

For the hardness results, we reduce an NP-hard cut problem to the problem of approximating the Ising model at fixed magnetization. The $\gamma$-MIN-EXACT-BALANCED-CUT ($\gamma$-MEBC) problem is the problem of finding the minimum of $|E(S, S')|$ over all $S \subset V(G)$, $|S| = [\gamma n]$, where $n = |V(G)|$. For stronger inapproximability in Theorem 2(b), we apply an inapproximability result due to Bui and Jones [3], though this is not essential to our method: we can reduce from exactly solving $\gamma$-MEBC instead.

**Theorem 8** (Bui–Jones [3]). Let $\gamma$ be a rational number in $(0, 1)$ and let $\varepsilon > 0$. Then $\gamma$-MEBC is NP-hard to approximate within an additive error $n^{2 - \varepsilon}$ on $n$-vertex graphs.

Key ingredients in the algorithmic results are efficient approximate counting and sampling algorithms for the ferromagnetic Ising model provided by Jerrum and Sinclair and Randall and Wilson.

**Theorem 9** (Jerrum–Sinclair [31], Randall–Wilson [47]). For all inverse temperatures $\beta$ and all activities $\lambda$, there is an FPRAS and efficient sampling scheme for the Ising model for all graphs $G$.

## 3 HARDNESS

### 3.1 The Reduction and Its Properties

Given $\Delta \geq 3$, $\beta > \beta_c(\Lambda)$ and $\eta \in (0, \eta_c)$, our goal is to reduce the problem $\gamma$-MIN-EXACT-BALANCED-CUT to approximating a fixed-magnetization Ising partition function, for some rational number $\gamma \in ((1 + \eta/\eta_c)/2, 1)$. (By symmetry we need only consider $\eta \geq 0$.

For the reduction we require a gadget $G = G(\Lambda, n, \theta, \psi)$ where $\theta, \psi \in (0, 1/8)$ are constants that can be determined later in terms of $\Delta, \beta$. The gadget is identical to the constructions in [20, 52], which is a balanced bipartite graph on $n_G = (2 + o(1))n$ vertices. The majority of the vertices have degree $\Delta$, and $m = O(n^4)$ vertices on each side of $G$ are designated terminal vertices of degree $\Delta - 1$. We detail the construction of the gadget and state its properties after showing how it is used in the reduction.

Let $H$ be a graph on $h = \lceil \eta^{\theta(\Delta - 1)} \rceil$ vertices, which is the input for $\gamma$-MEBC. Given $G$ as above and an integer $s$, we construct a graph $H^G_s$ of maximum degree $\Delta$ on $N := hn_G + s$ vertices as follows:

- We include a copy $G^x$ of $G$ for each vertex $x \in V(H)$.
- We include $s$ isolated vertices.
- For each edge $xy \in E(H)$, we include a matching of size $k = \lceil n^{\theta(\Delta - 1)} \rceil$ between the left terminals of $G^x$ and left terminals of $G^y$ and a matching of size $k$ between the right terminals of $G^x$ and right terminals of $G^y$. We do this in such a way that each terminal is used at most once (which is possible since $kh \leq m$).

For reference, our parameter choices are listed here. The parameters $\Delta, \beta$ are fixed and we can compute $\eta_c$ from them (to arbitrary precision). The parameter $\eta$ is fixed and satisfies the conditions of the theorem. From these parameters we compute an arbitrary rational number $\gamma$ such that

$$
\frac{1 + \eta/\eta_c}{2} < \gamma < 1,
$$

which is possible because $\eta \in (0, \eta_c)$. Suitable choices of $\theta, \psi \in (0, 1/8)$ can be made in terms of $\Delta, \beta$ (see Lemma 11). We are then given an instance $H$ of $\gamma$-MIN-EXACT-BALANCED-CUT on $h$ vertices with $h$ sufficiently large, and we choose an $n$ large enough that $h \leq n^{\theta/4}(\Delta - 1)$. Let $h_+ = \lceil \eta h \rceil$ and $h_- = \lfloor (1 -\eta) h \rfloor$ so that the $\gamma$-MIN-EXACT-BALANCED-CUT problem is to find the minimum of $|E(S, S')|$ over $S \subset V(H)$ with $|S| = h_+$. We insist that $h$ is large enough that $\min\{h_+, h_-\} \geq 1$. Now let

- $m = (\Delta - 1)\lceil \theta \log_2 1/\eta-c \rceil = o(n^{1/4})$,
- $n' = (\Delta - 1)\lceil \theta \log_2 1/\eta-c \rceil + |\psi \log_2 1/\eta-c \rceil = o(n^{1/4})$,
- $n_G = 2(n + m + \frac{1}{(\Delta - 1)\lceil \theta \log_2 1/\eta-c \rceil + n_G}) = (2 + o(1))n$,
- $k = n^{\theta/4}$,
- $s$ be a non-negative integer such that

$$
2n(h_+ - h_-) \eta_c - \eta [hn_G + s] \leq \eta n h,
$$

and $s = \Theta(n h)$,
- $N = hn_G + s$,
- $M' = h_+ - h_-,$
- $\delta > 0$ be small enough as a function of $\gamma$ and $\eta_c$,
- $\varepsilon = \lceil N(\eta + 1/2) \rceil$.

The parameters $m, n, n_G, k$ relate to the gadget construction that we detail below. There exists an $s$ satisfying (1) with $s = \Theta(n h)$ which can be found in time polynomial in $h$ because our parameter choices mean that (as $h \to \infty$)

$$
2n(h_+ - h_-) \eta_c = (2 + o(1)) \cdot (2\gamma - 1) \eta_c \cdot nh,
$$

and

$$
\eta [hn_G + s] = (2 + o(1)) \cdot \eta \cdot nh + O(s).
$$

Since $(2\gamma - 1) \eta_c > \eta$ for some non-negative integer $s = \Theta(n h)$ the latter can be made within an additive term $O(1)$ of the former (and hence within $\eta n h$). Finally, $N$ is the number of vertices in the graph.
which we construct in the reduction, $M'$ is the magnetization of the cuts considered for the $γ$-MEBC problem on $H$, and $t$ is such that on $H^G_{\Psi}$ FIXED-ISING$(G, \beta, \eta)$ asks for configurations $σ$ with $X(σ) = t$ (and the desired fixed magnetization is thus $2t - N$). Throughout this section there are many absolute constants (depending only on $A, \beta, \eta$) used and defined. For ease of reading we will make ample use of $O(\cdot)$ and $\Omega(\cdot)$ notation as well as reusing constants $C, c$ etc. The main result of this section is the following.

Theorem 10. Given $ε \in (0, 1)$ there exists $ξ > 0$ such that there is a randomized, polynomial-time algorithm to construct a graph $G$ as above so that with probability at least $2/3$ the following holds: given an $e^{N^ε}$-relative approximation to $Z^N_{G, α}$ one can compute, in time polynomial in $h$, an additive $h^{2 - ε}$ approximation to the $γ$-MIN-EXACT-BALANCED-CUT of $H$.

Theorem 10 together with Theorem 8 immediately gives Theorem 2(b).

3.2 The Gadget

We use the same gadget construction as in [20, 52]. The construction is defined by the maximum degree $Δ$, an integer $n$ and constants $θ, ψ \in (0, 1/8)$ which then determine the parameters $m, m', n_G, k$ listed above. To construct $G = G(Δ, n, θ, ψ)$, let $G' = G'(Δ, n, θ, ψ)$ be a random bipartite graph with $n + m'$ vertices on each side obtained by choosing $Δ$ perfect matchings between the sides uniformly at random, and from the final matching removing $m'$ of the edges. With high probability the matchings will be pairwise disjoint sets of edges, so $G'$ is a simple graph. Let $U_0$ be the set of vertices of degree $Δ$ in $G'$, and $W_0$ be the set of vertices of degree $Δ - 1$.

To form $G$ from $G'$, on each side partition the $m'$ vertices of degree $Δ - 1$ into $m$ equal-sized sets, and attach the leaves of a copy of a $(Δ - 1)$-ary tree of depth $[ψ \log Δ - 1 - n]$ to each set. Then each side of $G'$ has had $m$ trees each of which contains $O(n^θ)$ vertices added. The roots of these trees are now the only vertices of degree $Δ - 1$ in $G$, and there are $m$ roots that were added to each side. These are the terminal vertices which allow us to connect the gadgets together. Let $V_0$ be the vertex set of $G$, and $R_0$ be the terminal vertices.

Constructed in this way, we want to show that various properties of $G'$ and $G$ hold with sufficiently high probability. Many of these properties were verified in [20, 52], but we require additional control of statistics of the number of $+1$ spins.

Throughout this entire section we fix the inverse temperature $β = β_0$ and take $λ = 1$. Recall that $X$ denotes the number of $+1$ spins in a sample from the Ising model. We will condition on various phases of the Ising model on $G$ and on $H^G_{\Psi}$. For $G$, given an Ising configuration $σ \in Σ_G$, we say the phase is $+$ if $\sum_{u \in U} σ_u > 0$ and $-$ if $\sum_{u \in U} σ_u < 0$. If the sum is 0 then we take the phase to be the spin of some distinguished vertex $u_1 \in U_0$ fixed in advance (arbitrarily). Note that neither the spins of $W_0$ nor the spins of the trees added to $G'$ in the construction of $G$ appear in the definition of the phase. By symmetry, the probability under $μ_G$ of each phase is exactly $1/2$. We denote the Ising model on $G$ conditioned on the $+$ phase and $-$ phase respectively by $μ_{G,+}$ and $μ_{G,-}$.

operators. For a spin assignment $τ \in Σ_R$ to the terminals of $G$, we include an additional subscript $τ$ to denote conditioning on the event $\{σ_{R_0} = τ\}$ (that is, $σ$ restricted to $R_0$ is equal to $τ$).

Let $Z_{G,α}$ be the contribution to the partition function $Z_G$ of the Ising model on the random gadget $G$ from spin assignments in which there are precisely $2an$ vertices in $U_0$ of spin $+$. Let $Z_{G,+}$ and $Z_{G,-}$ be the contributions from the $+$ and $-$ phase respectively to $Z_G$. We have

$$Z_{G,+} = \sum_{1/2 < α < 1} Z_{G,α} + \frac{1}{Z} Z_{G,1/2};$$

because all contributions with $α > 1/2$ belong to the $+$ phase, but for $α = 1/2$ we break the tie symmetrically so that half the contribution $Z_{G,1/2}$ goes to the $+$ phase. Usually, it suffices to use the upper bound on $Z_{G,α}$ obtained by taking the entire contribution from $α = 1/2$ to the phase at hand.

We now state some results from [18, 20, 52] which are obtained by sophisticated versions of the first and second moment methods and an application of the small subgraph conditioning method [27, 48]. We will state the results for the $+$ phase, the $-$ phase is completely analogous. Let $α^+ = (1 + n_1)/2$, $α^- = (1 - n_1)/2$, and

$$q = \frac{(α^- + β_0 + 1 - α^+)α^-1}{(α^- e^β_0 + 1 - α^+)α^-1 + (α^- + (1 - α^+)e^β_0)α^-1},$$

which means $q$ is the probability that the root of a $(Δ - 1)$-ary tree gets spin $+1$ in the zero-field "$+$ measure" on the infinite $(Δ - 1)$-ary tree (defined analogously to the $+$ measure on the infinite $Δ$-regular tree). Note that $1/2 < q < α^+$. We use $Q^G(\cdot)$ to denote the product measure on $Σ_G$ that assigns probability $q$ to $+1$ spins and probability $1 - q$ to $-1$ spins, and vice versa for $Q^G(\cdot)$.

The following lemma collects previous results on the gadget, most notably the near independence of the terminal spins conditioned on a phase.

Lemma 11 ([52, Proof of Thm. 2.1], [18, Proof of Lem. B.3], [20, Lem. 22]). Let $G$ be the random graph described above with parameters $θ, ψ \in (0, 1/8)$. Then there exist constants $C, c > 0$ so that for all $α \in [1/2, 1]$,

$$\frac{\mathbb{E} Z_{G,α}}{\mathbb{E} Z_{G,+}} \leq \frac{C}{\sqrt{n}} e^{-cn(α-α^-)^2},$$

and for all $α \in [0, 1/2]$

$$\frac{\mathbb{E} Z_{G,α}}{\mathbb{E} Z_{G,-}} \leq \frac{C}{\sqrt{n}} e^{-cn(α-α^-)^2}.$$
• For every \( \tau_{W_0} \in \Sigma_{W_0} \setminus B \),
\[
\max_{\tau_0 \in \Sigma_{R_0}} \left| \frac{\mu_{G,s}(\sigma_0 = \tau_0 | \sigma_{W_0} = \tau_{W_0})}{Q_{R_0}(\tau_{R_0})} - 1 \right| \leq n^{-30}
\]
\((iii)\)
\[Z_{G,s} > \frac{1}{C} \mathbb{E}Z_{G,s}^+ .\]
The same also hold with \( + \) replaced by \( - \).

In previous works \([20, 52]\) a version of \((iii)\) with \(1/C\) replaced by a function \(o(1)\) as \( n \to \infty \) is used (along with the fact that this weaker bound holds with high probability), but the stated version follows from the small subgraph conditioning method used therein \([27, 48]\). In order to handle the fixed-magnetization constraint in our reduction, we show that certain additional properties hold with good probability.

**Lemma 12.** For sufficiently large \( n \), with probability at least \( 8/10 \) over the choice of the gadget described above, the following hold simultaneously, for all choices of \( \tau \in \Sigma_{R_0} \) (and for \( + \) replaced by \( - \) as well):

\( a) \langle X - 2na^+ \rangle_{G,s,\tau} = O(\sqrt{n}) \)
\( b) \langle X - 2na^+ \rangle_{G,s,\tau} = O(n) \)
\( c) \langle X - 2na^+ \rangle_{G,s,\tau} = O(\sqrt{n^2}) \)
\( d) \) For \( \delta > 0 \) as specified above and \( t_0 = \delta/(2\omega h) \) for some constant \( c_0 > 1/4 \),
\[\langle e^{it(X - 2a^+ n)} \rangle_{G,s,\tau} \leq e^{c t_0^2 n} \]
and
\[\langle e^{i(t(X - 2a^+ n - X))} \rangle_{G,s,\tau} \leq e^{c t_0^2 n} \]

We prove Lemma 12 in Section 3.4.

### 3.3 Proof of Theorem 10

Here we prove Theorem 10 given Lemmas 11 and 12. Let \( G \) be a gadget which satisfies these lemmas, and let \( H^G \) denote the graph constructed from \( H \) and isolated vertices as above. Let \( \hat{H}^G \) denote the same graph but without the edges between gadgets (so \( \hat{H}^G \) consists of \( h \) disjoint copies of \( G \) and isolated vertices). We write \( E \) for the set of edges \( E(H^G) \setminus E(\hat{H}^G) \) that lie between gadgets.

Given \( \sigma \in \Sigma_{H^G} \), let \( Y(\sigma) \in \Sigma_{H} \) be a vector denoting the phases of the gadgets \( G^x, \sigma \in V(H) \). We will call such a \( Y \) a phase vector. For a given \( Y \in \Sigma_H \) let \( \mu_{G,Y} \) be the Ising model on \( H^G \) conditioned on \( \{ Y(\sigma) = Y \} \). Let \( \langle \cdot \rangle_{H^G,Y} \) be the corresponding expectation operator. Define \( \mu_{G,Y}^\tau \) and \( \langle \cdot \rangle_{H^G,Y} \rangle^\tau \) analogously. For \( x \in V(H) \) let \( R^x \) be the set of terminals in the gadget \( G^x \), and let \( B \) be the union of the terminal vertices in all the copies of the gadget. For a spin assignment \( \tau \in \Sigma_B \) to the terminals, we include an additional subscript \( \tau \) to indicate conditioning on the event that \( \{ \sigma_B = \tau \} \).

We need two probabilistic results before proving Theorem 10. The first is a large deviation bound for \( X \) conditioned on any phase vector \( Y \) and any assignment of terminal spins \( \tau \). The second is a local central limit theorem for \( \mu_{H^G,Y,\tau}(X = t) \) when \( M(Y) = M^* \), and for an arbitrary terminal spin assignment \( \tau \).

**Lemma 13.** Assume the gadget \( G \) satisfies Lemma 12. Then for any phase vector \( Y \), any \( \tau \in \Sigma_W \), with
\[v = \frac{s}{2} + 2n \sum_{x \in V(H)} \alpha^x Y_x,\]
we have
\[\mu_{H^G,Y,\tau}(X \in [-\delta n, \delta n]) \leq \exp(-\Omega(n/h)),\]
where \( \delta > 0 \) is a constant defined above.

**Proof.** Note that to leading order \( v \) is the mean \( \langle X \rangle_{H^G,Y,\tau} \).
We prove the bound on the upper tail; the proof for the lower tail is identical. Let \( X_0 \sim 
Bin(s, 1/2) \) be the number of \( + \) spins among the \( s \) isolated vertices. Then \( \langle e^{it(X_0 - s/2)} \rangle \leq e^{t^2/4} \leq e^{t^2} \) since \( c_0 > 1/4 \). Using this along with Lemma 12 we can bound the moment generating function,
\[\langle e^{it(X_0 - s)} \rangle_{H^G,Y,\tau} \leq e^{c t_0^2 n h},\]
for some constant \( c > 0 \). Then we have
\[\mu_{H^G,Y,\tau}(X \geq \delta n) \leq e^{-t_0 \delta n + c t_0^2 n h} = e^{-\frac{2 t_0}{h}} \]
since \( t_0 = \delta/(2h) \).

The next lemma is a local central limit theorem for \( X \) with respect to \( \mu_{H^G,Y,\tau} \).

**Lemma 14.** Assume the gadget \( G \) satisfies Lemma 12. Then for any phase vector \( Y \in \Sigma_W \) with \( M(Y) = M^* \), any \( \tau \in \Sigma_W \), and any integer \( t \),
\[\mu_{H^G,Y,\tau}(X = t) = \frac{1}{\sqrt{2\pi \kappa^2}} \exp \left[ - \frac{(t - \langle X \rangle_{H^G,Y,\tau})^2}{2\kappa^2} \right] + o\left( \frac{1}{\sqrt{n}} \right),\]
where \( \kappa^2 = \operatorname{var}_{H^G,Y,\tau}(X) \). In particular,
\[\mu_{H^G,Y,\tau}(X = t) = \Omega\left( \frac{1}{\sqrt{n}} \right).\]

**Proof.** We have \( s = \Theta(nh) \), and by the independence of disjoint gadgets and the second moment bound in Lemma 12, we have \( \kappa^2 = \Theta(nh) \). Moreover, by our choice of \( s \) and the fact that \( M(Y) = M^* \), we have \( \langle X \rangle_{H^G,Y,\tau} - t = O(\sqrt{n}) \), and so the second statement follows from the first.

We start by proving a central limit theorem with the standard method of characteristic functions. Let \( \overline{X} = \frac{1}{2} (X - \langle X \rangle_{H^G,Y,\tau}) \) and \( \phi_X(t) = \langle e^{it \overline{X}} \rangle_{H^G,Y,\tau} \). Then
\[\phi_X(t) = \frac{1 + e^{it/\kappa}}{2} e^{-t/(2\kappa)} \prod_{x \in V(H)} \left( \mu_{H^G,Y,\tau}(X \in [-\kappa, \kappa]) \right)_{G,Y,x} \]
\[= \prod_{x \in V(H)} \left( 1 + \frac{t^2}{8\kappa^2} + O(\kappa^{-3}) \right)^{\frac{s}{2}} \]
\[= \prod_{x \in V(H)} \left( 1 + \frac{t^2 \operatorname{var}_{G,Y,x} (X)}{2\kappa^2} + O(\kappa^{-3/2}) \right) \]
\[= e^{-t^2/2 + o(1)},\]
since \( h \gamma^{3/2} k^{-3} = O(k^{-1/2}) \to 0 \). Here we used the bound on the third moment of \( X \) in a gadget given by Lemma 12. This proves that \( \Xi \Rightarrow N(0, 1). \)

Let \( L \) denote the lattice \( \langle X \rangle \mu_G, Y, r + Z/\kappa \). Let \( N(x) = \frac{1}{2 \sqrt{\pi}} e^{-x^2/2} \).

We want to show that
\[
\sup_{x \in L} |\kappa \mu_G, Y, r + Z/\kappa (X = x) - N(x)| = o(1),
\]
since \( \kappa = \Theta(\sqrt{n}h) \). Using Fourier inversion, we have for any \( K > 0 \),
\[
2\pi \sup_{x \in L} |\kappa \mu_G, \beta, \lambda (X = x) - N(x)|
\leq \sup_{x \in L} \int_{-\pi K}^{\pi K} \phi_\beta(t) e^{-i t x} dt - \int_{-\pi K}^{\pi K} e^{-t^2/2} e^{-i t x} dt
\leq \int_{-\pi K}^{\pi K} \phi_\beta(t) e^{-t^2/2} dt + \int_{|t| > \pi K} e^{-t^2/2} dt
\leq \int_{|t| \geq K} e^{-t^2/2} dt + \int_{|t| \geq K} |\phi_\beta(t)| dt
\leq A_1 + A_2 + A_3.
\]

Because \( \phi_\beta(t) = e^{-t^2/2} + o(1) \), applying the bounded convergence theorem gives that \( A_1 \to 0 \) as \( n \to \infty \) for any fixed \( K \), so we can choose \( n \) large enough to guarantee \( A_1 < \varepsilon/3 \). We can pick \( K \) large enough to ensure \( A_2 < \varepsilon/3 \). For \( A_3 \), we use the fact that the portion of the characteristic function coming from the isolated vertices has nice behavior. In particular,
\[
|\phi_\beta(t)| \leq \left( 1 + \frac{e^{it/\kappa}}{2} e^{-t/(2\kappa)} \right)^3
\leq e^{-\frac{t^2}{4\kappa}} = e^{-\Omega(t^2)}
\]
since \( s = \Theta(k^2) \). Then by choosing \( K \) large enough again we can make \( A_3 < \varepsilon/3 \) as well.

With these ingredients we can prove Theorem 10.

Proof of Theorem 10. Recall that we assume \( \Delta \geq 3, \beta > \beta_c(\Delta) \), and that \( \eta \in (0, \eta_\beta) \). Let \( G \) be the gadget graph that satisfies Lemmas 11 and 12, and recall the notation of Section 3.1 which includes the graph \( H_{G_0}^0 \) on \( N \) vertices formed from copies of \( G \) and \( s \) isolated vertices.

Let \( b \) be the value of \( \gamma \)-MEBC on the graph \( H \), and let \( \varepsilon > 0 \). We will obtain upper and lower bounds on \( b \) in terms of \( Z_{H_{G_0}^0}^\text{fix}(\beta, 2\ell - N) \) such that for suitably small \( \zeta \), an \( e^{-N^{\zeta^2}} \)-relative approximation to \( Z_{H_{G_0}^0}^\text{fix}(\beta, 2\ell - N) \) constrains \( b \) to an interval of length at most \( h^{\ell + \varepsilon} \).

Since \( \beta \) and the magnetization \( 2\ell - N \) are fixed, we will write \( Z_{H_{G_0}^0}^\text{fix}(\beta, 2\ell - N) \) for \( Z_{H_{G_0}^0}^\text{fix}(\beta, 2\ell - N) \). Moreover, for a phase vector \( Y \in \Sigma_H \), we write \( Z_{H_{G_0}^0}^\text{fix}(Y) \) for the contribution to \( Z_{H_{G_0}^0}^\text{fix}(\beta, 2\ell - N) \) from spin assignments with phase vector \( Y \). For \( \tau \in \Sigma_R \) we write \( Z_{H_{G_0}^0}^\text{fix}(Y, \tau) \) for the contribution to \( Z_{H_{G_0}^0}^\text{fix}(Y) \) from spin assignments \( \sigma \) which agree with \( \tau \) on the terminals \( R \). Similarly, since we only consider the usual Ising model with no external field we write \( Z_{H_{G_0}^0}^\text{fix}(1) \) for \( Z_{H_{G_0}^0}^\text{fix}(\beta, 1) \).

We start by bounding the partition function \( Z_{H_{G_0}^0}^\text{fix}(Y) \) from above. The first step is to split the partition function into sums over \( Y \) according to whether \( M(Y) = M^* \). We have
\[
Z_{H_{G_0}^0}^\text{fix}(Y) = \sum_{Y: M(Y) = M^*} Z_{H_{G_0}^0}^\text{fix}(Y) + \sum_{Y: M(Y) \neq M^*} Z_{H_{G_0}^0}^\text{fix}(Y).
\]

For an arbitrary phase vector \( Y \) we split \( Z_{H_{G_0}^0}^\text{fix}(Y) \) into a sum over spin assignments \( \tau \in \Sigma_R \) to the terminals and pull out the factor of the summand contributed by edges in \( E \), giving
\[
Z_{H_{G_0}^0}^\text{fix}(Y) = \sum_{\tau \in \Sigma_R} Z_{H_{G_0}^0}^\text{fix}(Y, \tau) \prod_{e \in E} e^{\beta t_e \tau_e}.
\]

To handle the fixed-magnetization constraint, observe that when \( \sigma \) is drawn from the Ising model \( \mu_{H_{G_0}^0, Y, \tau} \) we have
\[
Z_{H_{G_0}^0}^\text{fix}(Y, \tau) = Z_{H_{G_0}^0}^\text{fix}(Y, \tau) \cdot \mu_{H_{G_0}^0, Y, \tau}(X = t),
\]
which we can control with Lemmas 13 and 14. In the case \( M(Y) = M^* \) we have
\[
Z_{H_{G_0}^0}^\text{fix}(Y, \tau) = Z_{H_{G_0}^0}^\text{fix}(Y, \tau) \cdot \Omega(1/\sqrt{n}h),
\]
and in the case \( M(Y) \neq M^* \) we use
\[
Z_{H_{G_0}^0}^\text{fix}(Y, \tau) = Z_{H_{G_0}^0}^\text{fix}(Y, \tau) \cdot \exp(-\Omega(n/h)).
\]

For the sum over \( Y \) with \( M(Y) = M^* \) this means for some constant \( C > 0 \),
\[
\sum_{Y: M(Y) = M^*} Z_{H_{G_0}^0}^\text{fix}(Y) \leq C \sqrt{nh} \sum_{\tau \in \Sigma_R} Z_{H_{G_0}^0}^\text{fix}(Y, \tau) \prod_{e \in E} e^{\beta t_e \tau_e}.
\]

Now we can apply the phase-conditioned, nearly-independent terminal spins property of the gadget. Using Lemma 16 for the inequality (and the fact that \( (1 + O(n^{-2\delta}))^h = 1 + o(1) \)), we have
\[
Z_{H_{G_0}^0}^\text{fix}(Y, \tau) = Z_{H_{G_0}^0}^\text{fix}(Y) \cdot \mu_{H_{G_0}^0, Y}(\sigma_R = \tau) \leq (1 + o(1)) Z_{H_{G_0}^0}^\text{fix}(Y) Q_R^Y(\tau),
\]
where \( Q_R^Y(\tau) \) is the probability measure on \( \Sigma_R \) such that
\[
Q_R^Y(\tau) = \prod_{x \in V(H)} Q_{H_{G_0}^0}^Y(\tau_x).
\]

Continuing from (3) and absorbing factors into the constant, we have
\[
\sum_{Y: M(Y) = M^*} Z_{H_{G_0}^0}^\text{fix}(Y) \leq C \sqrt{nh} \sum_{\tau \in \Sigma_R} Z_{H_{G_0}^0}^\text{fix}(Y) \prod_{x \in V(H)} e^{\beta t_e \tau_e},
\]
and the final sum over \( \tau \) can be expressed in terms of the number cut \( Y \) of edges of \( H \) which are cut by the phase vector \( Y \). This observation appears in [52] and is precisely why nearly-independent phase-correlated spins are important in reductions such as these.

Recall \( q \) defined in (2). For every edge \( xy \in E(H) \) cut by \( Y \), there are precisely \( 2k \) edges in \( \mathcal{E} \) such that the measure \( Q_{H_{G_0}^0}^Y(\tau) \) gives one endpoint spin +1 with probability \( q \) and the other endpoint spin +1 with probability \( 1 - q \). Such edges are monochromatic with probability \( 2q(1-q) \). Similarly, for every edge \( xy \in E(H) \) not cut by \( Y \) there are precisely \( 2k \) edges in \( \mathcal{E} \) which are monochromatic with probability \( q^2 + (1-q)^2 \).

\[ \text{For constants } \Theta = 2q(1-q)e^{\beta t_e/2} + \text{for } b = \beta, \ell - N, \text{ and the final sum over } \tau \text{ can be expressed in terms of the number cut } \]
\[
(q^2 + (1 - q)^2)e^{-\beta/2} \quad \text{and} \quad \Gamma = 2q(1 - q)e^{-\beta/2} + (q^2 + (1 - q)^2)e^{\beta/2}
\]
we have
\[
\sum_{\tau \in \mathbb{E}} Z_{\mathcal{H}C}^\text{fix}(\tau) \prod_{u \in \mathcal{E}} e^{\beta \tau_{ru}} \geq \Gamma r^{2k|E(H)|}(|\Theta/\Gamma|^{2k} \text{cut}(Y)).
\]

Note that $\Theta < 1$ so that smaller cuts give larger quantities above. Finishing the upper bound started in (3), we have
\[
\sum_{Y : M(Y) = M^*} Z_{\mathcal{H}C}^\text{fix}(Y) \leq C_{\text{h}} \sum_{Y : M(Y) = M^*} Z_{\mathcal{H}C}^\text{fix}(Y) \leq C_{\text{h}} \sum_{Y : M(Y) = M^*} Z_{\mathcal{H}C}^\text{fix}(Y) \leq C_{\text{h}} \sum_{Y : M(Y) = M^*} Z_{\mathcal{H}C}^\text{fix}(Y) \leq C_{\text{h}} \sum_{Y : M(Y) = M^*} Z_{\mathcal{H}C}^\text{fix}(Y),
\]

because the $\gamma$-MEBC $b$ of $H$ gives the largest contribution of the form $(\Theta/\Gamma)^{2k|E(H)|}$, and the partition function $Z_{\mathcal{H}C}^\text{fix}$ is an upper bound on $\sum_{Y : M(Y) = M^*} Z_{\mathcal{H}C}^\text{fix}(Y)$.

For phase vectors $Y$ with $M(Y) \neq M^*$ it suffices to consider the worst-case contribution from edges between gadgets. For such $Y$, $|E| = 2k|E(H)|$. Our construction ensures that $k|E(H)| \leq k\ell^2 = o(n/h)$ so that this is a negligible fraction of $Z_{\mathcal{H}C}^\text{fix}$. Combining these bounds, for all large enough $n$ we have
\[
Z_{\mathcal{H}C}^\text{fix} \leq \left(\frac{C_{\text{h}}}{\sqrt{\mathbb{E}}} \right)^{2k|E(H)|} (\Theta/\Gamma)^{2k|E(H)|} \leq Z_{\mathcal{H}C}^\text{fix} \leq C_{\text{h}} \sum_{Y : M(Y) = M^*} Z_{\mathcal{H}C}^\text{fix}(Y),
\]

because $|E| = 2k|E(H)|$. The construction ensures that $k|E(H)| \leq k\ell^2 = o(n/h)$ so that this is a negligible fraction of $Z_{\mathcal{H}C}^\text{fix}$. Combining these bounds, for all large enough $n$ we have
\[
Z_{\mathcal{H}C}^\text{fix} \leq \left(\frac{C_{\text{h}}}{\sqrt{\mathbb{E}}} \right)^{2k|E(H)|} (\Theta/\Gamma)^{2k|E(H)|} \leq Z_{\mathcal{H}C}^\text{fix} \leq C_{\text{h}} \sum_{Y : M(Y) = M^*} Z_{\mathcal{H}C}^\text{fix}(Y).
\]

where we apply Lemma 14 to obtain the second line and the lower bound in Lemma 12(i) to obtain the third. Finally, since we have perfect symmetry between the phases we have $Z_{\mathcal{H}C}^\text{fix}(Y^*) = 2^{-\frac{h}{2}} Z_{\mathcal{H}C}^\text{fix}$ and
\[
Z_{\mathcal{H}C}^\text{fix} \geq \frac{C_{\text{h}}}{\sqrt{\mathbb{E}}} \Gamma r^{2k|E(H)|} (\Theta/\Gamma)^{2k|E(H)|} Z_{\mathcal{H}C}^\text{fix}.
\]

The upper bound from (4) and the lower bound above combine to give
\[
T - \frac{\log(Z_{\mathcal{H}C}^\text{fix}(Y^*)/Z_{\mathcal{H}C}^\text{fix})}{2\log(\Theta/\Gamma)} \leq b \leq T + \frac{\log C_{\text{h}}}{2\log(\Theta/\Gamma)}
\]
on the min-bisection of $H$, where
\[
T = \frac{\log(Z_{\mathcal{H}C}^\text{fix}(Y^*)/Z_{\mathcal{H}C}^\text{fix}) + 2k|E(H)| \log \Gamma - \log \sqrt{n}}{2\log(\Theta/\Gamma)}.
\]

We can approximate $Z_{\mathcal{H}C}^\text{fix}$ to within an absolute constant factor in (randomized) time polynomial in $N$ (which is polynomial in $h$) by Theorem 9. For the theorem we suppose that we have a relative $e^{\Omega(h^3)}$-approximation of $Z_{\mathcal{H}C}^\text{fix}$, and hence if $T$ is given by the definition of $T$ above with $Z_{\mathcal{H}C}^\text{fix}$ replaced by these approximate values, we have $|T - T| \leq O(N^5/k)$ and hence we can find $b$ exactly.

\[
\begin{align*}
(3.4) \quad \text{Proof of Lemma 12} \\
\quad \text{We prove the lemma in the case of the $+$ phase as the $-$ phase is the same. Note that we can ignore the contribution of vertices in $R_0$ and the attached trees to $X$ in the bounds since there are $o(n^{1/2})$ of these vertices. For this section $X$ and $X(\sigma)$ will refer to the number of $+$ spins in the vertices of $G$.} \\
\quad \text{We first prove the three bounds of the lemma without conditioning on $[\sigma_{R_0} = \tau]$. We will show that for large enough $n$, with probability at least $8/10$ over the choice of gadget, we have the following bounds:}
\end{align*}
\]

\[
\begin{align*}
\langle |X - 2na^+| \rangle_{G^+} &= O(\sqrt{n}) \quad (5) \\
\langle |X - 2na^{+2} \rangle_{G^+} &= O(n) \quad (6) \\
\langle |X - 2na^{+3} \rangle_{G^+} &= O(n^{3/2}) \quad (7) \\
\langle e^{a(X-2na^+)} \rangle_{G^+} &\leq O(c(n^{1/3})^\epsilon) \quad (8)
\end{align*}
\]
Let \( \xi : \mathbb{R} \to \mathbb{R} \) be a non-negative function that satisfies \( \xi(x) \leq e^{\|x\|} \) for some constant \( c > 0 \). We aim to prove bounds on \( \langle \xi(X - 2a^+ n) \rangle_{G^+} \) for all four choices of functions \( \xi : \mathbb{R} \to \mathbb{R} \) for \( k \in \{1, 2, 3\} \), and \( \xi(x) = e^{\|x\|} \).

For such a function \( \xi \), we can write

\[
\langle \xi(X - 2a^+ n) \rangle_{G^+} \leq \frac{\sum_{a \geq 1/2} \xi(2a - 2a^+ n) \mathbb{E} Z_{G^+}}{\mathbb{E} Z_{G^+}},
\]

(this is an inequality instead of an equality simply because we include all configurations with \( \sigma = 1/2 \)).

By (iii) of Lemma 11 we have \( Z_{G^+} \geq \frac{1}{C} \mathbb{E} Z_{G^+} \), with probability at least \( 1 - 1/10 \). By Markov’s inequality we have

\[
\sum_{a \geq 1/2} \xi(2a - 2a^+ n) \mathbb{E} Z_{G^+} \leq 100 \sum_{a \geq 1/2} \xi(2a - 2a^+ n) \mathbb{E} Z_{G^+},
\]

for all four choices of \( \xi \) with probability at least \( 1 - 4/100 \). Thus with probability at least \( 1 - 1/10 - 4/100 \geq 8/10 \) we have

\[
\langle \xi(X - 2a^+ n) \rangle_{G^+} \leq 100 \frac{\sum_{a \geq 1/2} \xi(2a - 2a^+ n) \mathbb{E} Z_{G^+}}{\mathbb{E} Z_{G^+}},
\]

so to prove (5), (6), (7), (8) it is enough to show that

\[
\sum_{a \geq 1/2} \xi(2a - 2a^+ n) \mathbb{E} Z_{G^+}
\]

satisfies the desired bounds.

Now using the bound

\[
\mathbb{E} Z_{G^+} = O(n^{-1/2}) e^{-\Omega(n(a - a^+)^2)}
\]

from Lemma 11, we can bound

\[
\sum_{a \geq 1/2} \xi(2a - 2a^+ n) \mathbb{E} Z_{G^+}
\]

\[
\leq O(n^{-1/2}) \sum_{n(1 - 2a^+)} \xi(t)e^{-\Omega(t^2/n)} + o(1)
\]

\[
= O(n^{-1/2}) \int_{n(1 - 2a^+)}^{2n(1 - a^+)} \xi(u)e^{-\Omega(u^2/n)} du
\]

\[
= O(n^{1/2}) \int_{1 - 2a^+}^{2(1-a^+)} \xi(u)e^{-\Omega(u)} du
\]

\[
= O(n^{1/2}) \int_{-\infty}^{\infty} \xi(x)e^{-\Omega(x^2)} dx
\]

Since

\[
\int_{-\infty}^{\infty} |x|^k e^{-\Omega(x^2)} dx = O(n^{k - 1/2})
\]

and

\[
\int_{-\infty}^{\infty} e^{\|x\|} e^{-\Omega(x^2)} dx = e^{O(\|x\|)}
\]

we obtain (5), (6), (7), (8).

Now we transfer these bounds to the measure conditioned on \( \{\sigma_{R_0} = \tau\} \). For the moment generating function we have

\[
\langle e^{\delta_1(X - 2a^+ n)} \rangle_{G^+, \tau} = \frac{\langle e^{\delta_1(X - 2a^+ n)} \mathbb{I}_{(\sigma_{R_0} = \tau)} \rangle_{G^+}}{\mathbb{P}_{G^+}(\tau)},
\]

\[
\leq \frac{\langle e^{\delta_1(X - 2a^+ n)} \mathbb{I}_{(\sigma_{R_0} = \tau)} \rangle_{G^+}}{Q_{R_0}(\tau)(1 - O(n^{-2\theta}))}
\]

\[
\leq e^{O(\|\gamma_{R_0}\|)} e^{O(\|\gamma_{R_0}\|)} = e^{O(\|\gamma_{R_0}\|)}
\]

where we used that \( t_{\gamma_{R_0}}^2 = O(n^{-\theta} + O(n^{1/2}) \) and \( |R_0| = O(n^{\theta/2}) \).

For the \( k \)th moment,

\[
\langle (X - 2a^+ n)^k \rangle_{G^+, \tau} = \frac{\langle (X - 2a^+ n)^k \mathbb{I}_{(\sigma_{R_0} = \tau)} \rangle_{G^+}}{\mu_{G^+}(\{\sigma_{R_0} = \tau\})}
\]

\[
\leq \frac{\langle (X - 2a^+ n)^k \mathbb{I}_{(\sigma_{R_0} = \tau)} \rangle_{G^+}}{Q_{R_0}(\tau)(1 - O(n^{-2\theta}))}
\]

\[
= \frac{\langle (X - 2a^+ n)^k \mathbb{I}_{(\sigma_{R_0} = \tau)} \rangle_{G^+}}{Q_{R_0}(\tau)(1 - O(n^{-2\theta}))}
\]

\[
= \frac{\sum_{x \in B^c} |X(\sigma) - 2a^+ n|^k \mathbb{I}_{(\sigma_{R_0} = \tau)} \cdot \mu_{G^+}(\sigma)}{Q_{R_0}(\tau)(1 - O(n^{-2\theta}))}
\]

\[
= \frac{Q_{R_0}(\tau)(1 - O(n^{-2\theta})) \langle (X - 2a^+ n)^k \rangle_{G^+, \tau}}{Q_{R_0}(\tau)(1 - O(n^{-2\theta}))}
\]

\[
= O(n^{(k-1)/2})
\]

where we have used from Lemma 11 that for all \( \gamma_{R_0} \in B^c \),

\[
\mu_{G^+}(\{\sigma_{R_0} = \tau\} | \gamma_{R_0} \in B^c) = Q_{R_0}(\tau)(1 + O(n^{-2\theta}))
\]

and we have used the fact that conditioned on \( \gamma_{R_0} \), \( X(\sigma) \) and \( \sigma_{R_0} \) are independent (using here that in this section \( X(\sigma) \) only counts + spins to the vertices of \( G^\tau \)).

4 Extremal bounds on the mean magnetization

The proof of Theorem 3 is based on that of Knizhnik in [36] which applies to lattices such as \( \mathbb{Z}^d \). We require some simple calculus facts recorded in the following lemma, see the full version of the paper for the proof.

Lemma 15. Let \( \beta > 0, h_1(x) = \text{artanh} \left( \tanh x \frac{\tan \beta}{2} \right) \), and let \( h_2(x, y) = \tanh \left( x + \text{artanh} \left( \tanh y \frac{\tan \beta}{2} \right) \right) \). Then \( h_1 \) is strictly concave on \( (0, \infty) \), \( h_2 \) is strictly concave when \( x, y \in (0, \infty) \), and \( h_2(x, x) \) is an increasing function of \( x \).
Proof of Theorem 3. Let $G = (V, E)$ be a graph and consider the ferromagnetic Ising model with partition function

$$Z_G(\beta, \lambda) = \sum_{\sigma \in \Xi_G} e^{\frac{1}{2} \sum_{u \in E(G)} \beta_{uv} \sigma_u \sigma_v M(\sigma)},$$

where we allow each edge $uv \in E$ to have its own inverse temperature parameter $\beta_{uv}$. Specializing to $\beta_{uv} = \beta$ for all edges $uv$, we recover the definition used elsewhere in this work.

When $\beta$ and $\lambda$ are understood from context, given a function $f$ with domain $\Sigma_G$, let $(f)_{G}$ be the expected value of $f$ with respect to the Ising model on $G$. We also write $(f)_{G-uv}$ for the expected value of $f$ with respect to the Ising model on the graph formed from $G$ by removing the edge $uv$, which is equivalent to setting the parameter $\beta_{uv}$ to zero. We extend this notation to $(f)_{G-F}$ when we want to remove some set $F$ of edges.

A key feature of the ferromagnetic Ising model with non-negative external field is the following list of Griffiths’ inequalities [23], also known as the GKS inequalities after Kelly and Sherman who generalized Griffiths’ work [35]. For $A \subset V$, let $\sigma_A = \prod_{v \in A} \sigma_v$. Then we have for any graph $G = (V, E), A \subset V, \text{and } uv \in E,$

$$\langle \sigma_A \rangle \geq 0 \quad (9)$$

$$\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \geq 0 \quad (10)$$

$$\langle \sigma_A \rangle - \langle \sigma_A \rangle_{G-uv} \geq 0. \quad (11)$$

In fact, (10) implies (11) by considering the derivative of $\langle \sigma_A \rangle$ with respect to $\beta_{uv}$. With $B = \{u, v\}$ we have

$$\frac{\partial}{\partial \beta_{uv}} \langle \sigma_A \rangle = \frac{1}{2} \langle \sigma_A \sigma_B \rangle - \frac{1}{2} \langle \sigma_A \rangle \langle \sigma_B \rangle \geq 0,$$

where the inequality is by (10).

We will apply Griffiths’ inequalities during some careful manipulation of expectations using the following identities. For $s = \pm 1$ and any $\beta$ we have

$$e^{\beta s} = \left(1 + s \tanh \frac{\beta}{2}\right) \cosh \frac{\beta}{2}, \quad (12)$$

which follows from the definitions of the hyperbolic functions in terms of exponential functions. We also use the addition formula

$$\text{artanh} \left( \frac{x + y}{1 + xy} \right) = \text{artanh} x + \text{artanh} y.$$

From now on, we work with $\beta_{uv} = \beta$ for all $uv \in E$. Let $u \in V$ and $v, w \in N(u)$ with $v \neq w$. Applying (12) to the term $e^{\beta \sigma_u \sigma_v}$ which occurs in both the numerator and the denominator of $\langle \sigma_u \rangle$ gives

$$\langle \sigma_u \rangle = \frac{\langle \sigma_u \rangle_{G-uv} + \langle \sigma_u \rangle_{G-uw} \tanh \frac{\beta}{2}}{1 + \langle \sigma_u \rangle_{G-uvw} \tanh \frac{\beta}{2}}, \quad (13)$$

and the same identity applied to the edge $uw$ in $G-uv$ gives

$$\langle \sigma_u \rangle_{G-uv} = \frac{\langle \sigma_u \rangle_{G-uvw} + \langle \sigma_u \rangle_{G-uw} \tanh \frac{\beta}{2}}{1 + \langle \sigma_u \rangle_{G-uw} \tanh \frac{\beta}{2}}. \quad (14)$$

To each of these we apply (10) to the expectation in the denominator, giving

$$\langle \sigma_u \rangle \leq \frac{\langle \sigma_u \rangle_{G-uv} + \langle \sigma_u \rangle_{G-uw} \tanh \frac{\beta}{2}}{1 + \langle \sigma_u \rangle_{G-uvw} \tanh \frac{\beta}{2}}, \quad (15)$$

$$\langle \sigma_u \rangle_{G-uw} \leq \frac{\langle \sigma_u \rangle_{G-uvw} + \langle \sigma_u \rangle_{G-uw} \tanh \frac{\beta}{2}}{1 + \langle \sigma_u \rangle_{G-uw} \tanh \frac{\beta}{2}}. \quad (16)$$

Applying $g = \text{artanh}$ to both sides of (15) and using the addition formula, we obtain

$$g(\langle \sigma_u \rangle) \leq g(\langle \sigma_u \rangle_{G-uv}) + g(\langle \sigma_u \rangle_{G-uw} \tanh \frac{\beta}{2}). \quad (17)$$

Doing this again with (16), we also use (11) with the edge $uw$ in the graph $G-uv$, giving $\langle \sigma_u \rangle_{G-uw} \leq \langle \sigma_u \rangle_{G-uw} \tanh \beta$ and hence

$$g(\langle \sigma_u \rangle) \leq g(\langle \sigma_u \rangle_{G-uw}) + g(\langle \sigma_u \rangle_{G-uw} \tanh \beta). \quad (18)$$

Observe that we can iterate the process used to obtain (18) over each $w \in N(u) \setminus \{v\}$ in turn to obtain

$$g(\langle \sigma_u \rangle) \leq \log \lambda + \sum_{w \in N(u) \setminus \{v\}} g(\langle \sigma_u \rangle_{G-uw} \tanh \frac{\beta}{2}). \quad (19)$$

where we have used the fact that removing the set $F_u := \{uv : v \in N(u)\}$ of edges incident to $u$ we have

$$\langle \sigma_u \rangle_{G-F_u} = \frac{\lambda - 1}{\lambda + 1} = \text{tanh}(\log \lambda)$$

because $u$ is an isolated vertex in $G-F_u$.

At this point, Krinsky assumes that $G$ is both edge and vertex transitive so that for some $L > 0$ and for all $uv \in E$ we have

$$\langle \sigma_u \rangle_{G-uv} = \langle \sigma_u \rangle_{G-uw} = \tanh L.$$ 

In this special case, (19) becomes

$$L \leq \log \lambda + (\lambda - 1) g(\tanh L \tanh \frac{\beta}{2}),$$

where $\lambda$ is the degree of a vertex in $G$. This implies that $L$ is bounded above by the largest solution $L^*$ to

$$L^* = \log \lambda + (\lambda - 1) g(\tanh L^* \tanh \frac{\beta}{2}). \quad (20)$$

Plugging this into (17) and observing that for any vertex-transitive graph $\langle \sigma_u \rangle$ is the mean magnetization $\mu_G$, we have

$$\eta_G \leq \tanh (L^* + g(\tanh L^* \tanh \frac{\beta}{2})).$$

The right-hand side is precisely $\eta^*_\lambda \beta$, the mean magnetization of the + measure on the infinite $\Delta$-regular tree, which one can derive from first principles (as in, e.g., [1]). In fact, it suffices to observe that every inequality we applied to obtain this bound holds with equality in the tree. That is, in the infinite $\Delta$-regular tree $\langle \sigma_u \sigma_v \rangle_{G-uv} = \langle \sigma_u \rangle_{G-uv} \langle \sigma_v \rangle_{G-uv}$ since removing $uv$ leaves $u$ and $v$ in different connected components so $\sigma_u$ and $\sigma_v$ are independent. Similarly, we have $\langle \sigma_u \sigma_v \rangle_{G-uw} = \langle \sigma_u \rangle_{G-uvw} \langle \sigma_v \rangle_{G-uw}$ and since $w$ is in a different component from the edge $uw$ after $uw$ is removed, $\langle \sigma_w \rangle_{G-uvw} = \langle \sigma_w \rangle_{G-uvw}$. The tree is edge and vertex transitive, proving that the derived upper bound is given by the mean magnetization of some measure on the tree. As the + measure stochastically dominates all other translation-invariant measures on the tree, it corresponds to the largest solution $L^*$ to (20).

We now apply this argument to a finite graph $G$ that is not necessarily vertex or edge transitive. First, we observe that we can
reduce to the case that $G$ is regular by a well-known construction. Suppose that $G$ has maximum degree $\Delta$ but minimum degree $\delta \leq \Delta - 1$. We construct a graph $H$ with minimum degree $\delta + 1$ such that $\eta_G \leq \eta_H$. Let $H_b$ be formed from the disjoint union of two copies of $G$, so that the mean magnetization of $H_b$ is equal to that of $G$. For $i \geq 0$, if there is a vertex $u$ of degree $\delta$ in $H_i$ let $H_{i+1}$ be formed from $H_i$ by connecting $u$ to its copy in the other copy of $G$. The inequality (11) shows that $\eta_{H_b}$ is non-decreasing as $i$ increases because adding the edge can only create any term $\langle \sigma_u \rangle$, and the mean magnetization is the average of these terms over all vertices $u$. When the process terminates at some $H$, the minimum degree of $H$ is $\delta + 1$, and we cannot have decreased the mean magnetization. Iterating this construction, we can obtain a $\Delta$-regular graph $H$ whose mean magnetization is an upper bound on the mean magnetization of $G$, hence it suffices to prove the theorem in the case of a $\Delta$-regular graph.

In a $\Delta$-regular graph, careful averaging and some applications of Jensen’s inequality allow us to recover the result obtained for edge and vertex transitive graphs. We can interpret (19) as a property of an oriented edge $uv$, and average over all edges incident to $u$ oriented away from $u$. Let $L_{uv}$ be given by $\langle \sigma_u \rangle_{G-uv}$, so that averaging (19) over $u \in V(G)$ gives

$$
\frac{1}{\Delta} \sum_{u \in V(G)} L_{uv} \leq \log \lambda + \frac{1}{\Delta} \sum_{u \in V(G)} g(\tanh L_{uv} \tan \beta). \quad (21)
$$

By Lemma 15, the function $x \mapsto g(\tanh x \tan \beta)$ is concave on $(0, \infty)$. This means that (21) and Jensen’s inequality give

$$
\frac{1}{\Delta} \sum_{u \in V(G)} L_{uv} \leq \log \lambda + (\Delta - 1) g \left( \tanh \left( \frac{1}{\Delta} \sum_{u \in V(G)} L_{uv} \right) \tan \frac{\beta}{2} \right). \quad (22)
$$

To clean this up, we define

$$
A_u := \frac{1}{\Delta} \sum_{u \in V(G)} L_{uv}, \quad B_u := \frac{1}{\Delta} \sum_{u \in V(G)} L_{uv},
$$

so that (22) gives

$$
A_u \leq \log \lambda + (\Delta - 1) g \left( \tanh B_u \tan \frac{\beta}{2} \right). \quad (23)
$$

This we average over a uniform random $u \in V$ and again appeal to concavity. Here we finally obtain the desired equation because the averages satisfy

$$
L := \frac{1}{\Delta n} \sum_{u \in V} (L_{\sigma u} + L_{\sigma v}) = \frac{1}{\Delta n} \sum_{u \in V} \sum_{v \in V} L_{uv}
$$

so an application of Jensen’s inequality gives for $L$ what we had for $l$ in the case of a transitive graph,

$$
L \leq \log \lambda + (\Delta - 1) g(\tanh L \tan \frac{\beta}{2}).
$$

As before, this means that $L \leq L^*$. To conclude the argument in the $\Delta$-regular case, we apply the same averaging trick to (17). For any edge $uv \in E$,

$$
g(\langle \sigma_u \rangle) \leq g(\langle \sigma_u \rangle_{G-uv}) + g(\langle \sigma_v \rangle_{G-uv} \tan \frac{\beta}{2}).
$$

so fixing $u$ and averaging over $v \in V(G)$ gives

$$
g(\langle \sigma_u \rangle) \leq \frac{1}{\Delta} \sum_{v \in V(G)} g(\langle \sigma_u \rangle_{G-uv}) + \frac{1}{\Delta} \sum_{v \in V(G)} g(\langle \sigma_v \rangle_{G-uv} \tan \frac{\beta}{2}).
$$

Applying Jensen’s inequality again, we have

$$
\langle \sigma_u \rangle \leq A_u + g(\tanh B_u \tan \frac{\beta}{2}),
$$

and so

$$
\langle \sigma_u \rangle \leq \tanh(A_u + g(\tanh B_u \tan \frac{\beta}{2})). \quad (24)
$$

By Lemma 15, the right-hand side is concave as a function of $A_u$ and $B_u$. Averaging (24) over $u \in V$ and applying Jensen’s inequality, we conclude

$$
\eta_G = \frac{1}{n} \sum_{u \in V} \langle \sigma_u \rangle \leq \tanh(L^* + g(\tanh L^* \tan \frac{\beta}{2}))) = \eta^*_G
$$

using the fact that the function $x \mapsto \tanh(x + g(\tanh x \tan \frac{\beta}{2}))$ is non-decreasing proved in Lemma 15.

\section{5 ALGORITHMS}

In this section we sketch the proofs of Theorems 1 and 2(a). By symmetry, it suffices to consider the case when $\eta \geq 0$. We can exclude the trivial case $\eta = 1$ since there is just a single spin configuration in that case.

We will use several ingredients from Section 2. Fix $\beta, \eta, \Delta$ satisfying the conditions of either theorem, and let $G$ be a graph of maximum degree $\Delta$ on $n$ vertices. Let $\ell = [\frac{24}{\eta - 2}]$ so that our goal is to sample an Ising configuration $\sigma$ with $X(\sigma) = \ell$.

Since we can efficiently sample from $\mu_{G,\beta,\lambda}$ for any $\lambda$ via Theorem 9, we can perform a binary search on values of $\lambda$, estimating $X(\sigma)$, to find a $\lambda$ so that

$$
|X(\sigma) - \ell| = o(\sqrt{n}). \quad (25)
$$

Given such an activity $\lambda$, we will approximately sample from $\mu_{G,\beta,\lambda}$ until we sample a configuration $\sigma$ with $X(\sigma) = \ell$ and then output $\sigma$. For this algorithm to be efficient we must ensure that the probability of hitting this value is not too small; in fact, we will show that it is $\Theta(n^{-1/2})$.

For Theorem 1 this follows immediately from Proposition 7 which provides a local central limit theorem and $\Theta(n)$ variance for $X$ for $\beta < \beta_c(\Delta)$ and any activity $\lambda$.

For Theorem 2, we need to ensure that we can find $\lambda$ satisfying (25) that is bounded away from 1 independently of $n$ so we can apply Proposition 7. This is guaranteed by the extremal result, Theorem 3, and the conditions of the Theorem 2. In particular, because $\eta > \eta_{\mu}(\beta, \Delta)$ and by continuity of the magnetization of the + measure on the tree) there is some $\lambda_{\text{min}} > 1$ so that $\eta = \eta_{\mu}(\beta, \lambda_{\text{min}})$. Theorem 3 then says that to achieve mean magnetization $\eta$ on any $G \in \cal{G}_\Delta$ we must take $\lambda \geq \lambda_{\text{min}}$, thus giving the required uniform bound away from 1.

See the full version of the paper for details of the approach. We note that the running time of our algorithm could certainly be improved by using a faster Ising sampler (e.g. [44]) or by using the techniques of [26], but here we will not try to optimize the running time beyond finding polynomial-time algorithms.
