Topological Zak Phase in Strongly-Coupled LC Circuits

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(Dated: November 27, 2017)

We show the emergence of topological Bogoliubov bosonic excitations in the relatively strong coupling limit of an LC (inductance-capacitance) one-dimensional quantum circuit. This dimerized chain model reveals a $\mathbb{Z}_2$ local symmetry as a result of the counter-rotating wave (pairing) terms. The topology is protected by the sub-lattice symmetry, represented by an anti-unitary transformation. We present a method to measure the winding of the topological Zak phase across the Brillouin zone by a reflection measurement of (microwave) light. Our method probes bulk quantities and can be implemented even in small systems. We study the robustness of edge modes towards disorder.

Topological Bloch bands are characterized by a topological number which is manifested in the appearance of protected edge states. For non-interacting fermions this results in the celebrated integer quantum Hall effect \cite{1}, conducting surface states of topological insulators \cite{2,3} and semimetals \cite{2,4}. Topological properties can also be accessed with bosonic systems such as cold atoms \cite{5}, photons \cite{6-8} and polaritons \cite{9}.

The Su-Schrieffer-Heeger (SSH) model defined on the dimerized one-dimensional lattice with two sites per unit cell is one of the simplest models demonstrating topological properties \cite{10,11}. The edge states are protected by the bipartite nature of the system (particles can hop only between the two sublattices). Arbitrary long range interaction which respect the bipartite nature of the SSH model may change the topological number but not the robustness of the edge states to disorder \cite{12}. The topology of the SSH model is described by the Zak phase \cite{13}, which was measured in cold atoms by introducing an artificial gauge field and mimicking Bloch oscillations \cite{5}, in photonic quantum walk \cite{14} and in photonic crystals \cite{15}. The midgap edge state were observed in dielectric resonators \cite{16}, polariton systems \cite{17} and classical LC chains \cite{18}, and their wave function was explored in Ref. \cite{18,19}. A two-leg ladder of SSH chains supports fractional excitations and shows a rich topology with two types of corner edge states \cite{20,21}. The ladder is a one-dimensional version of a two-dimensional quadrupole insulator \cite{22} realized in Refs. \cite{23-25}. In the non-linear regime a dimered chain shows topologically enforced bifurcations \cite{26}.

Here, we study topology in the strong coupling regime of quantum circuits in which the rotating wave approximation (RWA) is not applicable, leading to the appearance of counter rotating (pairing) terms. Such strong coupling limit also leads to the evolution of the Jaynes-Cummings model towards the Rabi model when describing the qubit-cavity interaction \cite{7,27}, and to the super radiant phase in Dicke model with a macroscopic number of photons in the ground state \cite{28,29} but has not been studied in the framework of topological systems. We start by showing that the counter rotating terms do not change the topology of the system although they modify the nature of the excitations from pure particles to Bogoliubov quasi-particles. We also find the wave-function of the midgap states demonstrating their edge localization. Then we propose a method to observe the winding of the topological Zak phase and probe the strong coupling regime via a light scattering experiment. Topological Bogoliubov excitations in other interacting bosonic systems have been studied in Refs. \cite{26,30-32}.

In the following, we study a chain of LC resonators with alternating capacitive coupling (see Fig. 1). The unit cell of the chain is composed of two sites, denoted by indices $a$ and $b$ that belong respectively to sublattices $A$ and $B$. Similar systems have been achieved experimentally \cite{33} and some topological aspects can already be measured in the classical limit (without quantification of the phase and charge in the circuit) \cite{23-25,34}.

The charge on the capacitor of a quantum LC resonator can be quantized introducing bosonic creation and annihilation operators $Q \propto (a + a^\dagger)$ \cite{35}. The capacitive coupling between resonators $i$ and $i + 1$ is $\propto Q_i Q_{i+1}$. In this description the dimer chain of $LC$ resonators leads...
to the following Hamiltonian
\[ H = \sum_n \epsilon_0 (a_n^\dagger a_n + b_n^\dagger b_n) + \sum_n v (a_n + a_n^\dagger) (b_n + b_n^\dagger) \\
+ \sum_n w (a_{n+1} + a_{n+1}^\dagger) (b_{n+1} + b_{n+1}^\dagger), \] (1)

where \((a_n^\dagger, b_n^\dagger)\) are the creation operators in the \(n\)th unit cell, \(\epsilon_0 = \sqrt{1/2} \sqrt{C + C_w + C_v}\) is the frequency of the resonators and \(v, w = \epsilon_0 \sqrt{C + C_w + C_v}\) are the hopping matrix elements. The inductance \(L\) and capacitances \(C, C_v\) and \(C_w\) are defined in Fig. 1. In the interaction picture, the pairing terms reduce the Dicke model. It is important to note that in the weak coupling limit \(h(k) \ll \epsilon_0\), the spectrum of the SSH chain is reproduced:
\[ \epsilon_{\pm}(k) \rightarrow \epsilon_0 \pm h(k) \text{ and the excitations are } \text{particle-like } \nu_k^\pm \rightarrow (1, 0). \]

The Bogoliubov quasi-particles operators are given by
\[ \gamma_k^\pm = \psi_k^\pm (a_k, a_k^\dagger, b_k, b_k^\dagger) \]
\[ \psi_k^\pm = \frac{1}{\sqrt{2}} \left( \pm e^{i\varphi(k)} \nu_k^+ + \nu_k^- \right) \]
\[ \nu_k^\pm = (\cosh \eta_k^\pm, \sinh \eta_k^\pm), \]

where the Bogoliubov rotation \(\eta_k^\pm\) is defined by \(\tanh 2\eta_k^\pm = \pm h(k)/\epsilon_0 \pm h(k)\). Indeed in the weak coupling regime the spectrum is not symmetric with respect to \(\epsilon_0\). However, in the weak coupling limit the spectrum is symmetric since the transformation that diagonalizes the Hamiltonian is unitary. In contrast, in the strong coupling regime, the symmetry is broken due to the symplecticity of the Bogoliubov transformation (see Fig. 1).

Detecting whether the system is in the strong or the weak coupling regime can therefore be achieved by studying the asymmetry of the two bands of the energy spectrum. The difference between the two bandwidths (see Fig. (2d)), to lowest order in \(v\) and \(w\), is
\[ W^- - W^+ = \frac{4vw}{\epsilon_0}, \] (6)

which vanishes in the weak coupling limit.

In the strong coupling regime, the ground state of the system is not the vacuum state of the \(a\) and \(b\) oscillators but rather a two mode squeezed vacuum given by
\[ |GS\rangle = \frac{1}{N} \prod_k \exp \left( -\tanh \eta_k^+ \alpha_k^\dagger \alpha_{-k} - \tanh \eta_k^- \beta_k^\dagger \beta_{-k} \right) |0\rangle \] (7)

where \(\alpha_k/\beta_k = \frac{1}{\sqrt{2}} \left( \pm e^{i\varphi(k)} a_k + b_k \right)\) are the eigen-operators in the absence of the pairing terms and \(|0\rangle\) is the vacuum state of \(a_n\) and \(b_n\). In the weak coupling limit \(\eta_k^\pm \rightarrow 0\) and the ground state identifies with the vacuum of the oscillators \(|GS\rangle \rightarrow |0\rangle\). The zero point motion of the quantum oscillators is manifested in an average mean photon number in the ground state. To lowest order in \(v\) and \(w\) it is given by \(\langle a_n^\dagger a_n\rangle = \langle b_n^\dagger b_n\rangle = (v^2 + w^2)/(2\epsilon_0^2)\).

The topology of one-dimensional systems with inversion symmetry is characterized by the Zak phase [13]
\[ \varphi_{Zak}^{\pm} = i \int_{-\pi/\ell}^{\pi/\ell} \psi_k^\pm \partial_k \psi_k^\pm dk. \] (8)
The correlation function \( D \) takes the form
\[ D = \text{diag}(1, 1, 1, 1) \] in the trivial (c) and topological (d) phases for a ring composed of 20 unit cells. The envelope of the resonances reveals the winding of the Zak phase as explained in the text.

Since the Bogoliubov transformation for boson is symplectic rather than unitary as for fermions, the scalar product is modified \( \psi^\dagger \psi = \psi^\dagger D \psi \). For our choice of basis (Eq. 5), the diagonal matrix \( D \) take the form
\[ D = \text{diag}(1, 1, 1, 1) \] and the Zak phases of the two bands are identical and equal to
\[ \varphi_{Zak}^\pm = \frac{1}{2} \int_{-\pi/l}^{\pi/l} \partial_k \varphi(k) dk \] independently of the Bogoliubov rotation \( \eta_k^\pm \). Therefore the Zak phase does not depend on the coupling strength. For \( v > w \) the system is in a trivial phase with \( \varphi_{Zak}^z = 0 \) while for \( v < w \) the system is in a topological phase with \( \varphi_{Zak}^z = \pi \text{ (mod 2\pi)} \). At the transition point \( v = w \) the gap between the band closes.

An equivalent and perhaps more transparent way to characterize the topology of the system is by looking at the winding number of the phase \( \varphi(k) \), i.e. the number of times \( e^{i\varphi(k)} \) winds around the origin for \( k \in [-\pi/l, \pi/l] \). The function \( h(k)e^{i\varphi(k)} = v + we^{-ikl} \) defines a circle in the complex plane of radius \( w \) around \((v, 0)\) (see Fig. 2). In the topological phase \( \psi < w \), the circle encloses the origin once and \( \varphi(k) \) is a surjective function onto \([-\pi, \pi)\). In the trivial phase \( \psi > w \), the circle does not enclose the origin and \( |\varphi(k)| < \pi/2 \). Hereafter, we propose a method to observe the winding of the phase \( \varphi(k) \) based on a reflection measurement.

The fact that the topology does not depend on the Bogoliubov rotation \( \eta_k^\pm \) can be understood using adiabatic switching of the pairing terms. A transition between two gapped topological phases implies closing of the gap at the transition point. We consider a Hamiltonian which interpolates between the SSH chain and our chain
\[ H_{ad} = \sum_n [a_n^d a_n + b_n^d b_n] + \sum_n (\delta v a_n b_n^\dagger + \omega a_n b_n + h.c.) \] for \( \delta v = \delta w = 0 \) the Hamiltonian identifies with the SSH chain Hamiltonian while for \( \delta v = v, \delta w = w \) it yields Eq. (1). The spectrum of the Bogoliubov quasi-particles for any \( \delta v, \delta w \) is given by
\[ \epsilon_{ad}^\pm(k) = \sqrt{(\epsilon_0 \pm h(k))^2 - \delta h^2(k)} \] where \( \delta h = |\delta v + \delta w e^{-ik}| \). For any real \( \delta v, \delta w \) the pairing terms do not close the energy gap, therefore the topology remains the same as in the weak coupling regime.

Next we propose an experimental protocol to measure the Zak phase by probing the winding of \( \varphi(k) \) which is based on the measurement of the correlation function \( \chi_n(t) = -i\Theta(t) \langle [q(t), q(0)] \rangle \) of the single cell (charge) operator \( q_n = a_n^d a_n + b_n^d b_n \). This correlation function can be probed by the reflection coefficient of a transmission line (capacitively) coupled to a single cell in the chain which is given by \( r(\omega) = 1 + \lambda^2 \chi_n(\omega) \), here \( \lambda \) is the (weak) coupling constant between the transmission line and the chain. In the periodic boundary condition \( \chi_n(0) = \epsilon_0 \) does not depend on \( n \). The Fourier transform of the correlation function yields
\[ \chi_n(\omega) = \frac{1}{N} \sum_k (1 + \cos \varphi(k)) \frac{\epsilon_0}{\epsilon_+} \frac{1}{\epsilon_+(\omega - \epsilon_+(k) + i0^+)} + \frac{1}{N} \sum_k (1 - \cos \varphi(k)) \frac{\epsilon_0}{\epsilon_-} \frac{1}{\epsilon_-\omega - \epsilon_-(k) + i0^+} \] where \( N \) means the number of unit cells.

The imaginary part \( \chi_n(\omega) \) (See Fig. 2) exhibits resonances at the eigen-energies \( \epsilon_{\pm}(k) \), thus measuring the band structure of the chain. The strong-coupling limit can be zeroed through the asymmetry of the bandwidths (Eq. 6). The envelope of the resonances reflects the winding of the phase \( \varphi(k) \) through the terms \( 1 \mp \cos \varphi(k) \). Let us consider the envelope function of the first band \( 1 - \cos \varphi(k) \). The bottom of the first band is at \( k = 0 \) where \( \varphi(0) = 0 \) and its top is at \( kl = \pi \) where \( \varphi(\pi) = 0 \) in the trivial phase and \( \varphi(\pi) = \pi \) in the topological phase. Therefore, in the topological phase the envelope function of the first band is a monotonically increasing function whereas in the trivial phase it starts and ends at zero. The behaviour of the envelope of the
second band resonances can be understood in the same way, noticing that the bottom of the second band is at \( kl = \pi \) and its top is at \( k = 0 \). The general behaviour of the envelope function is similar in the weak and strong coupling regimes.

A single \( k \) mode can be observed by coupling the transmission line to all the cells of the chain. The reflection coefficient is then related to the correlation function \( \chi_{k=0}(\omega) \) of \( q_{k=0} = \frac{1}{N^2} \sum_n q_n \), given by the \( k = 0 \) mode of Eq. (12). The \( k \) mode can be scanned across the Brillouin zone in a way similar to Bloch oscillations by introducing an effective gauge field [39].

The bulk-edge correspondence asserts the existence of midgap localized states at the boundaries of a finite topological system. In contrast to the bulk Bogoliubov excitations, these edge states are particle like. We show that the SSH ansatz for the edge operators introduces an effective gauge field [39].

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break the sublattice symmetry, the topological properties would remain the same.

Acknowledgements: We acknowledge discussions with Camille Aron, Jérôme Esteve, Julien Gabelli, Shyam Shankar, Loïc Henriet, Loïc Herviou, Guillaume Roux and Ronny Thomale. This work has also benefitted from Shankar, Loïc Henriet, Loïc Herviou, Guillaume Roux, Camille Aron, Jérôme Esteve, Julien Gabelli, Shyam Shankar, Loïc Henriet, Loïc Herviou, Guillaume Roux, Camille Aron, Jérôme Esteve, Julien Gabelli, Shyam Shankar, Loïc Henriet, Loïc Herviou, Guillaume Roux, Camille Aron, Jérôme Esteve, Julien Gabelli, Shyam Shankar, Loïc Henriet, Loïc Herviou, Guillaume Roux, and M. Schirò, C. R. Physique 17, 808 (2016).

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