ON SIBUYA-LIKE DISTRIBUTIONS IN BRANCHING AND
BIRTH-AND-DEATH PROCESSES

LEV B. KLEBANOV AND MICHAL ŠUMBERA

ABSTRACT. We report some properties of heavy-tailed Sibuya-like distributions related
to thinning, self-decomposability and branching processes. Extension of the thinning
operation of non-negative integer-valued random variables to scaling by arbitrary posi-
tive number leads to a new class of probability distributions with generating function
\( Q(w) \) expressible as a Laplace transform \( \varphi(1 - w) \) and probability mass function \( p_n \)
satisfying simple one step recurrence relation between \( p_{n+1} \) and \( p_n \). We show that the
compound Poisson-Sibuya and the shifted Sibuya distributions belong to this class.
Using the fact that the same Markov property is present in stationary solutions of
the birth and death equations we identify the Sibuya distribution and some of its
variants as particular solutions of these equations. We also establish condition when
integer-valued non-negative heavy-tailed random variable has finite \( r \)-th absolute mo-
moment (\( 0 < r < a < 1 \)).

1. SELF-DECOMPOSABILITY AND SCALING OF NON-NEGATIVE INTEGER-VALUED
RANDOM VARIABLES

Infinitely divisible distributions \( \text{Feller (1957); Norman L. Johnson (2005); F.W. Steutel}
(2004) \) play nowadays an important role in several parts of the probability theory. They
are also frequently used by the models based on the sum of several independent quanti-
ties with the same distribution because its infinite divisibility appears to be a convenient
assumption \( \text{F.W. Steutel (2004)} \). In the simplest case of the non-negative integer ran-
dom variable (rv) \( Y \in \mathbb{Z}_+ \) the condition of infinite divisibility means that \( \forall n \in \mathbb{N} \) it can
be written as a random sum \( Y_n = X_1 + X_2 + \ldots + X_n \) of independent identically distributed
(iid) rvs \( X_i \in \mathbb{Z}_+, i = 1, n \). Consequently, if \( \mathbb{P}(N = n) = p_n \) is the probability mass
function (pmf) and \( Q(w) = \sum_n p_n w^n \) the probability generating function (pgf) of the rv
\( N \) and \( G(w) = \mathbb{E}[w^{X_i}] \) is the pgf of the rvs \( X_i \) then the \( n \)-th root of the compound pgf
\begin{equation}
H(w) = G(Q(w)) := G \circ Q
\end{equation}
is \( \forall n \) also the pgf. This is equivalent to saying that \( H(w) \) can be expressed as the pgf
of compound Poisson distribution \( \text{Feller (1957)} \):
\begin{equation}
H(w) = G \circ Q = P \circ R, \quad P(\lambda, w) = e^{-\lambda(1-w)}, \quad R(w) = 1 + \frac{\log H(w)}{\lambda}
\end{equation}
where \( P(w) \) is the pgf of the Poisson distribution and \( R(w) \) is the pgf of positive discrete
rv with \( R(0) = 0 \).

Important class of the infinitely divisible distributions on \( \mathbb{Z}_+ \) are those which are self-
decomposable \( \text{F.W. Steutel (2004)} \). Let us recall that discrete random variable \( X \in \mathbb{Z}_+ \)

\textbf{Date:} May 3, 2022.
with the pgf $G(w)$ is called self-decomposable if it can be written as a sum of two independent variables.\cite{Steutel2004}

\begin{equation}
X = a \odot X + Y_a; \quad \forall a \in (0, 1).
\end{equation}

The thinning operator $\odot$ \cite{Renyi1956, Steutel1979} is defined in the following way

\begin{equation}
a \odot X := \sum_{i=1}^{X} Z_i, \quad Z_i \overset{d}{=} B_a, \quad G_{a \odot X}(w) := G(B_a(w)) = G(1 - a(1 - w))
\end{equation}

where $\overset{d}{=} \text{ means equality in distribution}, B_a$ is two-valued Bernoulli-distributed random variable with $p_0 = 1 - a$ and $p_1 = a$ and $B_a(w)$ is its pgf. In terms of pgfs Eq. (3) reads:

\begin{equation}
G(w) = G(B_a(w)) H_a(w),
\end{equation}

where $H_a(w)$ is the pgf of the rv $Y_a$. It can be shown that the function $H_a(w)$ is absolutely monotone and infinitely divisible.\cite{Steutel2004}. Obviously if $G_1(w)$ and $G_2(w)$ are self-decomposable pgfs then their product $G_1(w)G_2(w)$ is also self-decomposable pgf. More general definitions of thinning operators were given in \cite{KlebanovSlamova2014, Klebanov2021}.

The importance of the thinning operation (4) for the rvs with finite mean is explained by the following theorem.

**Theorem 1.** Let $Q(w)$ be a pgf of the rv $X$ such that $EX = Q^{(1)}(1) < \infty$ and with scaled factorial moments $F_j$ of order $j$

\begin{equation}
F_j := \frac{EX^{(j)}}{(EX)^j} = \frac{EX(X-1)\ldots(X-j+1)}{(EX)^j} = \frac{Q^{(j)}(1)}{(Q^{(1)}(1))^j}, \quad j = 1, 2, 3, \ldots
\end{equation}

Then the rv $Y = a \odot X$ with the pgf $G(w) = Q_{a \odot X}(w)$ has identical scaled factorial moments as $X$.

**Proof.**

\begin{equation}
G^{(j)}(w) = a^j Q^{(j)}(1 - a + aw); \quad \frac{G^{(j)}(1)}{(G^{(1)}(1))^j} = \frac{a^j Q^{(j)}(1)}{(aQ^{(1)}(1))^j} = F_j
\end{equation}

**Example 1.** Consider rv $X$ with $\langle n \rangle := EX < \infty$ which has the negative binomial distribution (NBD) with the pgf

\begin{equation}
Q_{\text{NBD}}(w) = \left(1 + \frac{\langle n \rangle}{k} (1 - w)\right)^{-k}; \quad k > 0.
\end{equation}

Its factorial moments $F_j = (k + j - 1)/k!$, where $(x)_n = \prod_{\ell=0}^{n-1} (x - \ell)$ is the falling factorial, do not depend on $\langle n \rangle$ and therefore its thinned version $Y = \odot X$ with the pgf

\begin{equation}
Q_{\text{NBD}}(B_a(w)) = \left(1 + \frac{\langle n \rangle}{k} \left(1 - B_a(w)\right)\right)^{-k} = \left(1 + \frac{a\langle n \rangle}{k} (1 - w)\right)^{-k}
\end{equation}

has not only the same scaled factorial moments as the rv $X$ but is also form-invariant w.r.t. scaling $\langle n \rangle \to a\langle n \rangle$.
Returning back to the self-decomposable distributions we note, that there is only one distribution, discrete stable distribution (DSD), for which the infinitely divisible component $Y_a$ is also self-decomposable

\begin{equation}
X = a \circ X + Y_a = a \circ X + b \circ X, \quad a^\delta + b^\delta = 1, \quad \forall \delta \in (0, 1).
\end{equation}

The canonical form (2) of its pgf reads

\begin{equation}
Q_{DSD}(\lambda, \gamma, w) = P \circ S = e^{-\lambda (1-w)^\gamma}, \quad S(\gamma, w) = 1 - (1 - w)^\gamma, \quad \gamma \in (0, 1),
\end{equation}

where $P(\lambda, w)$ and $S(\gamma, w)$ are the pgfs of Poisson and Sibuya distributions Sibuya (1979); Devroye (1993); Huillet & Martinez (2018), respectively.

Let us note that there is a wide class of self-decomposable pgfs which similarly to Poisson (2), Bernouilly (5), negative binomial (8) and discrete stable distributions (11) depend explicitly on the argument $1 - w$, i.e. allow expansion of the type

\begin{equation}
Q(w) = \sum_{n \geq 0} p_n w^n = 1 + \sum_{n > 0} b_n (1 - w)^n, \quad \sum_{n > 0} b_n > -1.
\end{equation}

This category comprises also some less frequently used ones like the Mittag-Leffler Huillet & Martinez (2018) or Discrete Linnik Devroye (1993); Huillet & Martinez (2018) distributions. Rather general construction of such pgfs is given in Klebanov (2021). More precisely, let $\varphi(s)$ be the Laplace transform of a positive random variable $X$. Then

\begin{equation}
Q(w) = \varphi(1 - w)
\end{equation}

is a probability generating function.

The change of variable $w = (1 - a) + az$ in (13) gives us

\begin{equation}
Q(1 - a + az) = \varphi(a(1 - z)).
\end{equation}

In other words, application of the thinning operator $w \to 1 - a + az$ is equivalent to the scale change from 1 to $a$ in the case of $a \in (0, 1)$. We can apply the thinning operator to arbitrary positive integer random variable but for the case of $0 < a < 1$. However, $\varphi(as)$ remains to be Laplace transform for any positive $a$. Naturally, we came to the problem: Describe all probability generating functions $Q(w)$ such that $Q(1 - a + aw)$ is again the pgf for all $a > 0$. The solution is given by the following result.

**Theorem 2.** Let $Q(w)$ be a probability generating function. $Q(1 - a + aw)$ is the pgf for all $a > 0$ if and only if there exists a Laplace transform of a positive random variable $\varphi(s)$ such that the representation (13) holds.

**Proof.** Without loss of generality we may consider the case $a > 1$.

1. Suppose that $\varphi(s)$ is Laplace transform of a distribution function and $Q(w) = \varphi(1 - w)$ is probability generating function. However, $\varphi(as)$ with arbitrary $a > 0$ is Laplace transform as well. According to the result from Klebanov (2021) mentioned above, $\varphi(a(1 - w)) = Q(1 - a + aw)$ is probability generating function for any $a > 0$.

2. Suppose that $Q(1 - a + aw)$ is probability generating function for any $a > 0$. Define $\varphi(s) = Q(1 - s)$. It is necessary to proof that $\varphi(s)$ is Laplace transform of a distribution function or, equivalently, that it is absolutely monotone function. In other words we have to prove that

\begin{equation}
\varphi^{(k)}(s) = (-1)^k A_k(s), \quad k = 0, 1, \ldots,
\end{equation}
where the functions $A_k(s)$ are non-negative for $s > 0$. For any $w \in (0, 1)$ and any $a > 0$ we have $\varphi(a(1 - w)) = Q(1 - a + aw)$. Therefore,

\begin{equation}
\frac{d^k}{d w^k} \varphi(a(1 - w)) = (-1)^k a^k \varphi^{(k)}(a(1 - w)) = (-1)^k a^k Q^{(k)}(1 - a + aw).
\end{equation}

Because $Q(1 - a + aw)$ is probability generating function for any $a > 0$ the terms $a^k Q^{(k)}(1 - a + aw)$ are non-negative for all $0 < w < 1$ and all positive $a$. It proves absolutely monotones of $\varphi(s)$. □

**Corollary 1.** Let $\mathbb{P}(N = n) = p_n = Q^{(n)}(0)/n!$, $n \in \mathbb{Z}_+$ be a pmf of random variable $N$ with the pgf $Q(w)$ such that

\begin{equation}
Q(w) = \varphi(1 - w) = \int_0^\infty e^{-(1-w)x} f(x) dx
\end{equation}

where $\varphi(s)$ is the Laplace transform of a positive random variable $X$. Then $p_n$ satisfies the one-step recurrence relation

\begin{equation}
p_{n+1} = \frac{p_n}{n+1} g(n); \quad g(n) = \frac{Q^{(n+1)}(0)}{Q^{(n)}(0)} = \frac{(n+1)p_{n+1}}{p_n} = \int_0^\infty e^{-x} x^{n+1} f(x) dx = \int_0^\infty e^{-x} x^n f(x) dx.
\end{equation}

Let us note that if pgf $Q(w)$ has the representation \[17\] then for any $b \in (0, 1]$ the function $Q(bw)/Q(b)$ is pgf and has the same representation with corresponding function $f_b(x)$. Namely,

\begin{equation}
f_b(x) = e^{-(1-b)x/b} f(x/b) / Z(b), \quad Z(b) = b \int_0^\infty e^{-(1-b)u} f(u) du.
\end{equation}

It is also worth mentioning that the rv $Y \in \mathbb{Z}_+$ with the pgf $Q(w)$ satisfying Eq. \[17\] can be also interpreted as Poisson–distributed random variable whose mean $\mathbb{E}Y = X$ fluctuates according to the probability density $f(x)$ or, equivalently, as a mixture of the Poisson pgfs $P(x, w) = e^{-x(1-w)}$ with the mixture weight $f(x)$. Let us give few examples.

**Example 2.** Consider pgf of the Hermite distribution\[Norman L. Johnson (2005)\] $Q(w) = e^{-\lambda_1(1-w)} e^{-\lambda_2(1-w)^2}$ describing sum of two independent random variables $X_1, X_2$ having pgf $e^{-\lambda_1(1-w)}$ and $e^{-\lambda_2(1-w)^2}$ correspondingly. Values of $X_1$ are concentrated on $n = 0, 1, 2, 3, \ldots$ of $X_2$ on even values only $n = 0, 2, 4, 6, \ldots$. Hermite distribution is infinitely divisible. Its pmf satisfies recursion relation $(n+1)p_{n+1} = \lambda_1 p_n + 2\lambda_2 p_{n-1}$. For $\lambda_2 = 0$ the recursion simplifies to $(n+1)p_{n+1} = \lambda_1 p_n$ and pgf $Q(w)$ can be obtained from Eq. \[13\] with $f(x) = \delta(x - \lambda_1)$. However, for $\lambda_2 \neq 0$ the density $f(x)$ does not exist. On the opposite side is the notoriously known example of the negative binomial distribution with the pgf \[8\] which can be obtained from Eq. \[13\] taking for $f(x)$ the gamma distribution density $f(x) = (\beta^k / \Gamma(k)) x^{k-1} e^{-\beta x}$ with $\beta = (1 - q)/q$.

**Example 3.** Let’s compare the pgf \[11\] of discrete stable distribution with the Laplace integral \[Pollard (1946)\]

\begin{equation}
e^{-\alpha x} = \int_0^\infty e^{-\tau x} g_\alpha(x) dx, \quad g_\alpha(x) = \frac{1}{\pi} \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j! x^{1+\alpha j}} \Gamma(1+\alpha j) \sin(\pi \alpha j),
\end{equation}

where $g_\alpha(x)$ is the probability density function of the one-sided continuous stable distribution. Exact and explicit expressions for $g_\alpha(x)$, for all $\alpha = l/k < 1$, with $k$ and $l$
positive integers can be found in Penson & Görski (2010). The substitutions \( \gamma = \alpha \) and \( r = \lambda - \gamma (1 - w) \) in Eq. (20) enable us to express the pgf (11) as the Laplace transform (17) of the function

\[
f(x) = \lambda^\gamma g_\gamma(\lambda^\gamma x).
\]

Hence the pmf of the discrete stable distribution satisfies the one step recurrence relation

\[
(n + 1)p_{n+1} = g(n)p_n \quad \text{with} \quad g(n) \quad \text{given by Eq. (18)}.
\]

It is worth mentioning that since a closed-form expression for \( p_n \) using elementary functions is unknown the proof of this relation by the other methods is almost impossible.

**Example 4.** Consider random variable \( Y \overset{d}{=} X + 1 \) where \( Y \in \mathbb{N} \) has Sibuya distribution. Then the rv \( X \in \mathbb{Z}_+ \) has shifted Sibuya distribution which is self-decomposable Christoph & Schreiber (2000) and therefore also infinitely divisible F.W. Steutel (2004). Its pgf can be expressed via Eq. (17)

\[
S_0(\gamma, w) = S(\gamma, w) = \frac{1 - (1 - w)^\gamma}{w} = \int_0^\infty e^{-(1-w)x} f(x)dx;
\]

\[
f(x) = -\frac{e^x \Gamma(-\gamma, x)}{\Gamma(-\gamma)} \approx x^{1-\gamma} \left( \frac{\gamma}{\Gamma(1 - \gamma)} + O \left( \frac{1}{x} \right) \right),
\]

where \( \Gamma(a, x) \) is the incomplete gamma function with \( \Gamma(a, 0) = \Gamma(a) \). From Eq. (18) we obtain

\[
g(n) = \frac{\int_0^\infty x^{n+1} \Gamma(-\gamma, x)dx}{\int_0^\infty x^n \Gamma(-\gamma, x)dx} = \frac{(n + 1)(n + 1 - \gamma)}{n + 2}.
\]

It is worth mentioning that the \( r \)-th absolute moment of the probability density function \( f(x) \) (23)

\[
\int_0^\infty x^r f(x)dx = \frac{\sin(\pi\gamma)\Gamma(r + 1)}{\sin(\pi(\gamma - r))}
\]

diverges for \( r > \gamma \). This well-known property of the continuous heavy-tailed distributions is also shared by the family of discrete distribution represented by the Sibuya distribution. In the latter case the mean and higher moments \( \langle n^r \rangle = Q^{(r)}(w = 1), \ r \geq 1 \) do not exist. One may naively expect that in this case also the Shannon entropy \( S = - \sum_n p_n \log p_n \) of these distributions diverges. However, this is not the case. The entropy \( S \) is finite either if \( \langle \log n \rangle = \sum_n p_n \log n < \infty \) or if \( \exists \ r > 0 \) such that \( \langle n^r \rangle = \sum_{n=1}^\infty n^r p_n < \infty \) Baccetti & Visser (2013). The following theorem establishes the condition when \( \langle n^r \rangle < \infty, \ 0 < r < 1 \).

**Theorem 3.** Integer-valued non-negative random variable \( Z \) with probability generating function \( Q(w) \) has finite \( r \)-th absolute moment \( (0 < r < a < 1) \) iff

\[
\int_0^1 \frac{1 - Q(w)}{(\log w)^{1+r}w}dw < \infty
\]
Proof. For continuous non-negative random variable $X$ with Laplace transform $\varphi(u) = E e^{-uX}$ its $r$-th absolute moment $E X^r$, $0 < r < 1$ can be calculated using identity
\[
(27) \int_0^{\infty} \frac{1 - e^{-Xu}}{u^{1+r}} du = X^r \int_0^{\infty} \frac{1 - e^{-z}}{z^{1+r}} dz = X^r (-\Gamma(-r)); \quad z = Xu, \quad 0 < r < 1
\]
Similarly, for the integer-valued non-negative random variable $Z$ with Laplace transform $\varphi(u) = Q(e^{-u})$ we have
\[
(29) \quad E Z^r = \frac{1}{-\Gamma(-r)} \int_0^{\infty} \frac{1 - Q(e^{-u})}{u^{1+r}} du = \frac{1}{-\Gamma(-r)} \int_0^{1} \frac{1 - Q(w)}{(-\log w)^{1+r}} dw.
\]

Example 5. Consider $N$ extended objects–particles each of the same unit volume $v_0 = 1$. Particles are incompressible and densely packed occupying the total volume $V = N$ in three dimensional space. If $N$ is the Sibuya-distributed rv with the pgf $S(\gamma, w)$ then the integral (26) and hence the corresponding moments $E N^r$ are finite only for $0 < r < \gamma < 1$. Consequently the mean $E V$ of the fluctuating volume $V \overset{d}{=} N$ is non-existent. However, for $\gamma > 1/2$ the mean value of the enclosing surface $S \overset{d}{=} N^{1/3}$ as well as of the linear extension $R \overset{d}{=} N^{1/3}$, where $\overset{d}{=} \text{means proportionality in distribution,}$ of the 3d-space occupied by particles are finite. Moreover for $1/2 \geq \gamma > 1/3$ $E S$ does not exist but $E R < \infty$.

Similarly to the pgfs of Bernoulli and geometric distributions Sibuya pgfs form a commutative semi-group under the operation of compound $\circ$
\[
(30) \quad S(\gamma_1) \circ S(\gamma_2) = S(\gamma_1\gamma_2)
\]
The following theorem establishes the additional similarity with Bernoulli distribution.

**Theorem 4.** Sapatinas (1995) Let $Q(w)$ be the self-decomposable pgf of the random variable $X$ on $\mathbb{Z}_+$ and $S(\gamma, w)$ the pgf of Sibuya distribution. Then $\forall \gamma \in (0, 1)$ the function $G(w) = Q \circ S(\gamma)$ is also a self-decomposable pgf on $\mathbb{Z}_+$.

Proof. We need to prove that $\forall a$ the function
\[
(31) \quad \frac{G(w)}{G(B_a(w))} = \frac{Q(S(\gamma, w))}{Q(1-a^{\gamma}(1-S(\gamma, w)))} = \frac{Q(S(\gamma, w))}{Q(B_a(\gamma, S(\gamma, w)))} = Q_a^\gamma(S(\gamma, w))
\]
is also the pgf. This is true since $Q_a^\gamma(S(w))$ is a compound pgf $\forall a \in (0, 1)$.

Remark 1. Obviously, sum of $n$ independent self-decomposable rvs $X_1 + \ldots + X_n$ is also self-decomposable. The non-trivial character of the above theorem consists in the statement that this remains true also for the sum iid self-decomposable rvs when $n$ fluctuates according to the Sibuya distribution.

Remark 2. Note that the thinned version of the Sibuya pgf appearing in Eq. (31)
\[
(32) \quad 1 - a^{\gamma}(1 - S(\gamma, w)) = 1 - \lambda(1 - w)^{\gamma} := S(\lambda, \gamma, w); \quad \lambda > 0
\]
is the pgf of the scaled Sibuya distribution Christoph & Schreiber (2000). The latter is known to be infinitely divisible if and only if $\lambda \leq 1 - \gamma$ and self-decomposable if and
only if \( \lambda \leq (1 - \gamma)/(1 + \gamma) \) \cite{Christoph & Schreiber (2000)}. In this case, as follows from the Theorem \ref{th:1}, the pgf \( S(\lambda, \gamma) \circ S(\alpha) \) is also self-decomposable.

**Corollary 2.** Every Sibuya-distributed random variable \( X_1 \) with parameter \( \gamma_1 \) can be decomposed into the sum of two rvs \( X_1 = X_2 + Y \), where \( X_2 \) has the Sibuya distribution with parameter \( \gamma_2 < \gamma_1 \) and \( Y \) is self-decomposable.

**Proof.** Let \( Q(w) \) be the pgf of the rv \( Y \) and \( \gamma_1 = \alpha \gamma_2, 0 < \alpha < 1 \). Then

\[
S(\alpha \gamma_2, w) = S(\alpha) \circ S(\gamma_2) = S(\gamma_2, w) Q(w).
\]

and hence

\[
Q(w) = \frac{S(\alpha) \circ S(\gamma_2)}{S(\gamma_2, w)} = \left( \frac{S(\alpha, w)}{w} \right) \circ S(\gamma_2) = S_0(\alpha) \circ S(\gamma_2),
\]

where \( S_0(\alpha, w) \) is the pgf of the shifted Sibuya distribution \cite{Christoph & Schreiber (2000)}. The latter is self-decomposable \cite{Christoph & Schreiber (2000)} and so we can apply the Theorem \ref{th:1} to obtain the needed result. \( \square \)

**Example 6.** Skipping the trivial case of the discrete stable distribution \cite{christoph & schreiber (2000)} which is self-decomposable by construction let us consider the negative binomial distribution with parameter \( k = 1 \), so called geometric distribution, with the pgf \( Q(w) = (1 + \lambda(1 - w))^{-1} \), cf. Eq. \ref{eq:35}. Then

\[
G(w) = Q \circ S(\gamma) = \frac{1}{1 + \lambda(1 - w)^\gamma}
\]

is the pgf of Mittag-Leffler distribution. Since \( Q(w) \) is self-decomposable it is also \( G(w) \). The random variable \( Y = X + 1 \) with the pgf \( wG(w) \) where \( X \) has the pgf given by Eq. \ref{eq:35} with \( \gamma = \frac{1}{2} \) and \( \lambda = 1 \) describes the number of trials until the first return to the origin in one-dimensional symmetric random walk on the lattice \cite{Feller (1957)}.

**2. Birth-and-death process**

Let us consider continuous-time birth-and-death (B-D) process \cite{Feller (1957); Anderson (2012)} consisting of random events of the same kind – e.g. creations and disintegrations of particles, populations of interacting species with indistinguishable members, queueing networks (interconnected queues) etc. The system represents a continuous-time Markov chain with a discrete state space \( n \in \mathbb{N} \) which changes only through transitions from states to their nearest neighbors \( n \to n \pm 1 \). It is described by stochastic process \( \{X(t) : t \geq 0\} \)

\[
\mathbb{P}(X(s + t) = j | X(s) = i) = \mathbb{P}(X(t) = j | X(0) = i) = P_{i,j}(t); \quad \forall i, j, t > 0, s > 0
\]

with transition probabilities \( P_{i,j}(t) \) satisfying Chapman-Kolmogorov equation

\[
P_{i,j}(s + t) = \sum_k P_{i,k}(s) P_{k,j}(t).
\]

Evolution of the probability distribution \( p_j(t) \equiv \mathbb{P}(X(t) = j) = \sum_i \mathbb{P}(X(0) = i) P_{i,j}(t) \) to find the system in state \( j \) at time \( t \) if at time \( t = 0 \) it was in state \( i \) is controlled by the time-independent transition probabilities per unit time

\[
\lambda_j = \left. \frac{dP_{j,j+1}(t)}{dt} \right|_{t=0+}, \quad \mu_j = \left. \frac{dP_{j,j-1}(t)}{dt} \right|_{t=0+}.
\]
and satisfies differential-difference equation

\[
\frac{dp_j(t)}{dt} = -(\lambda_j + \mu_j)p_j(t) + \lambda_{j-1}p_{j-1}(t) + \mu_{j+1}p_{j+1}(t), \quad j > i,
\]

\[
\frac{dp_i(t)}{dt} = -\lambda_ip_i(t) + \mu_{i+1}p_{i+1}(t), \quad i \in \mathbb{N}.
\]

The last equation in (39) reflects the initial condition \( p_j(0) = \delta_{ij} \).

One of the commonly used models specifying functional dependence of the transitional probabilities \( \lambda_j \) and \( \mu_j \) on \( j \) considers \( j \to j \pm 1 \) transition as resulting from several underlying microscopic (particle-level) processes \( k \to k \pm 1, \quad k = 0, \ldots, j \) with the elementary transition probabilities \( \alpha_k \) and \( \beta_k \), respectively. Thus one particle is born or dies with the elementary probability per unit of time \( \alpha_0 \) or \( \beta_0j \), one particle can split into two particles with the rate \( \alpha_1j \), two particles can fuse into a single particle with the rate \( \beta_1j(j - 1) \), etc. The relations between the transitional probabilities \( \lambda_j \) and \( \mu_j \) and elementary ones are given by the following formulas

\[
\lambda_j = \sum_{k=0}^{\ell} \alpha_k(j)_k, \quad \mu_j = \sum_{k=0}^{m} \beta_k((j - 1)k); \quad \ell \leq j; \quad m \leq j; \quad (j)_k = \prod_{i=0}^{k-1}(j-i).
\]

For application of this approach to multiparticle production in high energy physics see e.g. [Biyajima et al. (1987); Zborovský (1996)].

2.1. **Stationary B-D equations.** It is well known that for large \( t \) solution of the B-D Eq. (39) can be well approximated by the limiting probabilities \( p_j = \lim_{t \to \infty} p_j(t) \) [Feller (1957); Anderson (2012)] which solve the Eq. (39) with \( dp_j(t)/dt = 0 \). The latter is in this case transformed into two functional equations for one unknown function \( F(j) \)

\[
F(j+1) \equiv \mu_{j+1}p_{j+1} - \lambda_jp_j = \mu_jp_j - \lambda_{j-1}p_{j-1} \equiv F(j), \quad j > i
\]

\[
F(i + 1) = \mu_{i+1}p_{i+1} - \lambda_ip_i = 0, \quad i \in \mathbb{N}.
\]

Solution of the first equation \( F(j+1) = F(j) \) is fixed to zero by the second equation \( F(i + 1) = 0 \). Consequently

\[
\frac{p_{j+1}}{p_j} = \frac{\lambda_j}{\mu_{j+1}} = \frac{\sum_{k=0}^{\ell} \alpha_k(j)_k}{(j + 1) \sum_{k=0}^{m} \beta_k((j - 1)k)}.
\]

We have thus proven the following theorem.

**Theorem 5.** The unique solution of Eqs. (42) with \( \lambda_j \) and \( \mu_j \) given by Eq. (40) reads:

\[
p_j = p_i \prod_{n=i}^{j-1} \frac{g(n)}{n+1}; \quad g(n) = \frac{(n+1)p_{n+1}}{p_n} = \frac{\sum_{k=0}^{\ell} \alpha_k(n)_k}{\sum_{k=0}^{m} \beta_k((n - 1)k)}; \quad \ell \leq j, \quad m \leq j
\]

where \( p_i \) is the first non-zero value of the positive sequence defining the pmf \( p_j \), i.e. \( \min_{i \geq 0} p_i > 0 \) and hence satisfying \( g(i) > 0 \) and \( g(i - 1) \leq 0 \).

**Remark 3.** Class of discrete distributions on \( \mathbb{Z}_+ \) satisfying the recursion formula (43) is substantially wider then usually discussed Kotz or Ord family of distributions or their generalizations [Norman L. Johnson (2005); Chakraborty (2015)]. It is also worth nothing the similarity between Eq. (43) and pgfs satisfying Eq. (13). However, not every pgf satisfying Eq. (13) has its \( g(n) \)-function so simple that it can be written as in Eq. (43).
To illustrate this point let us give two examples which are related to Sibuya-like distributions. The first is the thinned version of the shifted Sibuya distribution (22) with the pgf $S_0(B_a(w))$. It is self-decomposable and so it can be expressed as the Laplace integral (17). Using Eq. (18) we can calculate its $g$-function to obtain

$$g(n) = \frac{(n+1)(n+1-\gamma)}{(n+2)} 2F_1(1, \gamma + 1; n + 3; 1 - a)$$

(44)

where $2F_1(a_1, a_2; b_1; z)$ is the Gauss hypergeometrical function. Obviously only for $a = 1$ the function $g(n)$ can be expressed as a ratio of two polynomials in $n$.

Another example is the pgf of the discrete stable distribution (11). Although its integral (17) using Eq. (18) we can calculate its $g(n)$

$$g(n) = \frac{\sum_{i=0}^{n+1} (\lambda \gamma)^{i+1} F_i(\gamma)}{\sum_{i=0}^{n} (\lambda \gamma)^{i+1} F_i(\gamma)}$$

(45)

where the functions $F_i(\gamma)$ are some polynomials in $\gamma$ satisfying $F_i(1) = 0, i > 0$ with $F_0 = 1$, it does not allow to be written in the form of Eq. (13).

**Corollary 3.** The pmf (43) is the generalized hypergeometric probability distributions [2005] with the pgf:

$$Q(w) = \frac{H(zw)}{H(z)} = \sum_{n=0}^{\infty} \frac{z^n n!}{n!} \prod_{i=j}^{n-1} g(i) = \ell F_m(a, b, z)$$

(46)

where $\ell F_m(a, b, z)$ represents the generalized hypergeometric function with series expansion

$$\ell F_m(a_1, \ldots, a_l; b_1, \ldots, b_m; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_l)_n z^n}{(b_1)_n \cdots (b_m)_n n!}.$$  

The constants $a_1, \ldots, a_l$ and $b_1, \ldots, b_m$ in Eq. (17) are some rational functions of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ and $z = \alpha_\ell / \beta_m$ where $\alpha_\ell$ and $\beta_m$ are the last non-zero transition amplitudes.

**Proof.** Using Eq. (13) together with the identity $n! = j! \prod_{i=j}^{n-1} (i + 1)$ and $z = \alpha_\ell / \beta_m$ we obtain

$$Q(w) = \sum_{n=0}^{\infty} w^n \prod_{i=j}^{n-1} g(i) = w_j z^{-j} j! \sum_{n=0}^{\infty} \left( zw \right)^n \prod_{i=j}^{n-1} \frac{(a_1)_n \cdots (a_l)_n z^n}{(b_1)_n \cdots (b_m)_n n!} = H(zw) / H(z).$$

(48)

The last equality with $1/H(z) = p_j z^{-j} j!$ follows from the condition $Q(1) = 1$. Thus the pgf (18) corresponds to the power series distribution with the function $H(z)$ which is of the type of Eq. (17).  

**Remark 4.** Note the certain arbitrariness when associating the coefficients $a_i$ and $b_i$ with $\alpha_i$ and $\beta_i$, respectively. This is due to the permutation symmetry of the generalized hypergeometric function (17) w.r.t. its arguments $a_i$ and $b_i$ separately.

**Example 7.** Consider Eq. (13) with $l = 1, j = 0$, i.e. the case with the all coefficients $\alpha_0, \alpha_1, \beta_0, \beta_1$ positive. Then

$$g(n) = \frac{\alpha_0 + \alpha_1 n}{\beta_0 + \beta_1 n}; \quad H \left( \frac{\alpha_1}{\beta_1} \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1 / \beta_1)^n}{n!} \prod_{i=0}^{n-1} \frac{\alpha_1 + i}{\beta_1 + i} = 1F_1 \left( \frac{\alpha_0}{\alpha_1}; \frac{\beta_0}{\beta_1}; \frac{\alpha_1}{\beta_1} \right)$$

(49)
where $\text{PF}_1(a; b; z)$ is Kummer’s (confluent hypergeometric) function.

Neglecting in $g(n)$ of Eq. ($49$) $\alpha_1$ and setting $\beta_0 = \beta_1$ with $z = \alpha_0/\beta_1 = \theta$ yields the pmf of the Conway-Maxwell-Poisson (CMP) distribution with parameter $r = 2$ et al. (2005)

$$g(n) = \frac{\theta^{n+1}}{n+1}, \quad p_n = p_0 \prod_{i=0}^{n-1} \frac{g(i)}{i+1} = p_0 \frac{\theta^n}{(n!)^2}, \quad H(z) = \frac{1}{p_0} = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^2}. \tag{50}$$

Expression for the scaled factorial moments (6) of this pmf obtained in Daly & Gaunt (2016) reads

$$F_j = \frac{I_j(2\sqrt{\theta})}{(I_1(2\sqrt{\theta}))^j}; \quad I_j(x) = \sum_{k=0}^{\infty} \frac{1}{k!(j+k+1)} \left( \frac{x}{2} \right)^{j+k} \tag{51}$$

It is worth mentioning that the pmf (50) has appeared in Ko et al. (2001) as a stationary solution of the kinetic master equation describing the production of charged particles which are created or destroyed only in pairs due to the conservation of their charge.

On the other hand neglecting in $g(n)$ of Eq. ($49$) the constant $\beta_1$ we obtain the NBD

$$g(n) = \frac{\alpha_0 + \alpha_1 n}{\beta_0}; \quad p_n = p_0 \frac{q^n n!}{n!}; \quad q = \frac{\alpha_1}{\alpha_0}, \quad k = \frac{\alpha_0}{\alpha_1} \tag{52}$$

$$\sum_{n=0}^{\infty} p_n = 1 = p_0 \left( 1 + \sum_{n=1}^{\infty} \frac{q^n n!}{n!} \right) = (1-q)^{-k}; \quad p_n = \left( \frac{n+k-1}{k} \right) (1-q)^k q^n. \tag{53}$$

**Example 8.** Consider now more general case of Eq. (43) with $\ell = 2, j = 0$

$$g(n) = \frac{\alpha_0 + \alpha_1 n + \alpha_2 n(n-1)}{\beta_0 + \beta_1 n} \tag{54}$$

With $z = \alpha_2/\beta_1$, $\alpha_0 = \beta_1 = \beta_0/2$ and $\alpha_1 = 3\alpha_2 = \beta_0\beta_2/2$ we obtain the zero-inflated logarithmic distribution

$$g(n) = \frac{(n+1)^2}{n+2}; \quad p_n = p_0 \prod_{i=1}^{n-1} \frac{g(i)}{i+1} = -\ln(1-\theta) \frac{\theta^{n+1}}{n+1}. \tag{55}$$

On the other hand the logarithmic distribution can be reconstructed from

$$g(n) = n\theta; \quad p_n = p_1 \prod_{i=1}^{n-1} \frac{g(i)}{i+1}; \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_1} = \theta, \alpha_0 = \beta_0 = 0. \tag{56}$$

And similarly to the above case for the geometric distribution with $g(n) = (n+1)\theta$ the only non-zero coefficients are $\alpha_1 = 2\beta_1\theta, \alpha_2 = \beta_1\theta$.

**Example 9.** While the NBD (53) is frequently used in different areas of science e.g. in medicine Carter & Potts (2014), biology Lloyd-Smith et al. (2003) or in particle physics Biyajima et al. (1987); Giovannini & Van Hove (1986); Chliapnikov & Chikhilev (1989), the pmf obtained from Eq. (43) with non-zero $\alpha_0, \alpha_2, \beta_0, \beta_2$ and thus with the pgf Eq. (46) where

$$H(z) = 2F_2(a_1, a_2; b_1, b_2; z); \quad a_1+a_2 = b_1+b_2 = -1, \quad a_1a_2 = \frac{\alpha_0}{\alpha_2}, \quad b_1b_2 = \frac{\beta_0}{\beta_2}, \quad z = \frac{\alpha_2}{\beta_2} \tag{57}$$

is seldom. Nevertheless, as shown in Zborovsky (1990) it is important when describing the data with a secondary maximum (bump).
3. Sibuya–like solutions of the stationary birth-death equations

3.1. Generalized Sibuya distribution. The Sibuya distribution with the probability mass function

\[ P(N = n) = \frac{\gamma(\gamma - 1)\ldots(\gamma - n + 1)}{n!}(-1)^{n+1} = \left(\frac{\gamma}{n}\right)(-1)^{n+1}, \quad n \in \mathbb{N}, \]

first appeared in Sibuya (1979). The generalized Sibuya distribution with parameters \( \nu \geq 0 \) and \( 0 < \gamma < \nu + 1 \) has been introduced in Kozubowski & Podgorski (2017) as the distribution of the number \( N \) of trials until series of the Bernoulli trials with the varying success probability \( \gamma/(\nu + n - 1) \) and thus having the probability mass function

\[ p_n = \left(1 - \frac{\gamma}{\nu + 1}\right)\ldots\left(1 - \frac{\gamma}{\nu + n - 1}\right)\frac{\gamma}{\nu + n}. \]

One connection of the generalized Sibuya distribution with the classical Sibuya variable \( N \) is through the probability \( P(N - m = n | N > m) \), which has the generalized Sibuya distribution with \( \nu = m \). This extends to the generalized Sibuya variable \( N \) for which \( P(N - m = n | N > m) \) has also the generalized Sibuya distribution with \( \nu + m \) as its parameter.

Writing the pmf \( p_n \) as

\[ p_n = p_1 \prod_{i=1}^{n-1} \frac{g(i)}{i+1}; \quad p_1 = \frac{\gamma}{\nu + 1}; \quad g(i) = \frac{(i + 1)(i + \nu - \gamma)}{i + \nu + 1}. \]

we can see that function \( g(i) \) of the generalized Sibuya distribution is a special case of Eq. (54) with \( \beta_2 = 0 \) and thus solves the stationary B-D equations (42). The corresponding non-zero coefficients are

\[ \frac{\alpha_2}{\beta_1} = 1, \quad \frac{\alpha_1}{\beta_1} = 2 + \nu - \gamma, \quad \frac{\alpha_0}{\beta_1} = \nu - \gamma, \quad \frac{\beta_0}{\beta_1} = \nu + 1. \]

Moreover for \( \nu = 0 \) we have \( p_1 = \gamma \) and Eq. (60) reduces to the pmf of Sibuya distribution (11).

The pgf of the generalized Sibuya distribution calculated from Eq. (60) reads

\[ Q(w) = \frac{w\gamma}{\nu + 1} \cdot \frac{w}{\nu + 2} \cdot 2F_1(1, \nu - \gamma + 1; \nu + 2; w), \]

where \( 2F_1(a_1, a_2; b_1; z) \) is the Gauss hypergeometric function.

3.2. Extended Sibuya distribution. Consider now Eq. (13) with \( \ell = 2, j = 1 \) and hence with the function \( g(n) \) given by Eq. (54) with \( \alpha_0 = \beta_0 = 0 \)

\[ g(n) = \frac{\alpha_2 n(n-1) + \alpha_1 n}{\beta_1 n} = b(n - \gamma); \quad \frac{\alpha_2}{\beta_1} = b, \quad \frac{\alpha_1}{\beta_1} = b(1 - \gamma), \quad \gamma = 1 - \frac{\alpha_1}{\alpha_2}. \]

Using Corollary 3 it is easy to verify that the corresponding pgf

\[ R(b, \gamma, w) = \frac{H(bw)}{H(b)} = \frac{1 - (1 - bw)^\gamma}{1 - (1 - b)^\gamma} = \frac{S(\gamma, bw)}{S(\gamma, b)}. \]
represents yet another generalization of the Sibuya pgf $S(\gamma, w)$. For $\gamma = -k < 0$ and fixed $b = q$ the pgf (63) coincides with the NBD pgf (8) truncated at zero

$$R(w) = \frac{1 - (1 - qw)^{-k}}{1 - (1 - q)^{-k}} = \frac{Q_{\text{NBD}}(w) - Q_{\text{NBD}}(0)}{1 - Q_{\text{NBD}}(0)}; \quad Q_{\text{NBD}}(w) = \left(\frac{1 - q}{1 - qw}\right)^k.$$  

For $\gamma \to 0$, i.e. when $\alpha_2 \to \alpha_1$, the pgf (63) approaches the logarithmic distribution

$$\lim_{\gamma \to 0} \frac{1 - (1 - bw)^{\gamma}}{1 - (1 - b)^{\gamma}} = \frac{\log(1 - bw)}{\log(1 - b)}$$

and for $\gamma \to 1$ it converges to distribution with the pmf $p_n = \delta_{1n}$.

Let us note that the pgf (63) was already mentioned in Letac (2019) (see Eq. (6) therein) as the natural exponential family extension of the Sibuya distribution. Since no name was given to it nor its relation with the other known distributions was discussed we take a liberty and call it the pgf of extended Sibuya distribution stressing its smooth transition in $\gamma$ to the pgfs of several fundamental discrete distributions.

Let us now study the factorial moments $\mathbb{E} X^{(j)} := \langle n(j) \rangle := \langle n(n-1) \ldots (n-j+1) \rangle$, $j \geq 1$, see Eq. (6), of the pgf $R(w)$. Ignoring the region with $\gamma < 0$ which corresponds to the zero-truncated NBD let us analyze the behavior of

$$F_2 = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2} = \frac{(1 - \gamma) (1 - (1 - b)^\gamma)}{\gamma (1 - b)^\gamma} = \frac{1 - \gamma}{\gamma} \cdot (e^{\delta \gamma} - 1); \quad \delta := -\log(1 - b).$$

Ignoring the region with $\gamma < 0$ which corresponds to the zero-truncated NBD let us analyze the behavior of

$$\frac{dF_2}{d\gamma} = \frac{e^{\delta \gamma} (\gamma (1 - \gamma) - 1) + 1}{\gamma^2}.$$  

For $\gamma \to 0$, i.e. for the logarithmic distribution, $F_2$ has a minimum at $\delta = 2$. Further up, in the region $(\gamma > 0, \delta > 2)$, the extreme of $F_2(\gamma)$ moves along the trajectory $dF_2/d\gamma|_{\gamma_0} = 0$ or equivalently $\psi'(\gamma_0, \delta) = 0$, where $\psi(\gamma, \delta) = (e^{\delta \gamma} (\gamma (1 - \gamma) - 1) + 1)$. While $F_2(\gamma)$ increases for $0 < \gamma < \gamma_0$ it decreases for $\gamma > \gamma < 1$. When approaching the singularity at $b = 1$ the second scaled factorial moment $F_2$ and hence also its maximum goes to infinity. Let us add that the same maxima appear also in the higher scaled factorial moments $F_j, j > 2$.

3.3. The shifted variants of extended and generalized Sibuya distributions. Consider random variable $Y \overset{d}{=} X + 1$ where $Y \in \mathbb{N}$ has either extended or generalized Sibuya distribution. Then the rv $X \in \mathbb{Z}_+$ has shifted extended or generalized Sibuya distribution, respectively. The case when $Y$ has ordinary Sibuya distribution was already discussed in Example 4. Note that contrary to the original extended or generalized Sibuya distributions their shifted variants are now defined on $\mathbb{Z}_+$ and so they may be self-decomposable.
3.3.1. The shifted extended Sibuya distribution. If \( p_n \) and \( \mathcal{R}(b, \gamma, w) \) are the pmf and the pgf of the extended Sibuya distribution \( [63] \), respectively, then \( r_n = p_{n+1} \) and \( \mathcal{R}_0(b, \gamma, w) = \mathcal{R}(b, \gamma, w)/w \), cf. Eq. (22), are the pmf and the pgf of the shifted extended Sibuya distribution. Corresponding function \( g(n) \) thus reads

\[
g_r(n) = \frac{(n+1)r_{n+1}}{r_n} = \frac{n+1}{n+2} \cdot \frac{(n+2)p_{n+2}}{p_{n+1}} = \frac{n+1}{n+2}b(n+1-\gamma); \quad b \leq 1
\]

cf. Eq. (24). The elementary transition amplitudes \( [13] \) derived from Eq. (71) are

\[
\frac{\alpha_2}{\beta_1} = b, \quad \frac{\alpha_1}{\beta_1} = b(3-\gamma), \quad \frac{\alpha_0}{\beta_1} = b(1-\gamma), \quad \frac{\beta_0}{\beta_1} = 2.
\]

Theorem 6. The shifted extended Sibuya distribution is self-decomposable.

Proof. Following Christoph & Schreiber [2000] we use

Lemma 1. Bondesson [1992] A strictly decreasing pmf \( r_n, n = 0, 1, 2, \ldots \) such that

\[
\max_{0 \leq n \leq j} \frac{r_{n+1}}{r_n} \leq \frac{j+2}{j+1} \cdot \frac{r_{j+1} - r_{j+2}}{r_j - r_{j+1}}, \quad \forall j = 0, 1, 2, \ldots
\]

is discrete self-decomposable.

In our case the sequence

\[
\frac{r_{n+1}}{r_n} = \frac{p_{n+2}}{p_{n+1}} = b \left( 1 - \frac{1}{n+2} \right)
\]

is strictly decreasing and therefore the left hand side of Eq. (73) gives:

\[
\max_{0 \leq n \leq j} \frac{r_{n+1}}{r_n} = b \left( 1 - \frac{1}{j+2} \right).
\]

For the right hand side of the inequality (73) we have

\[
\frac{j+2}{j+1} \cdot \frac{r_{j+1} - r_{j+2}}{r_j - r_{j+1}} = \frac{(j+2)}{(j+1)} \cdot \frac{b(j+1-\gamma)(j+3-b(j+2-\gamma))}{(j+3)(j+2-b(j+1-\gamma))}.
\]

Dividing now the r.h.s. of the Eq. (73) by its l.h.s. we obtain

\[
R(b, j) = \frac{j+2}{j+1} \cdot \frac{r_{j+1} - r_{j+2}}{r_j - r_{j+1}} / \max_{0 \leq n \leq j} \frac{r_{n+1}}{r_n} = \frac{(j+2)^2(j+3-b(j+2-\gamma))}{(j+1)(j+3)(j+2-b(j+1-\gamma))}.
\]

Now we need to prove that \( R(b, j) \geq 1 \). It is easy to verify that \( R(b, j) \) is decreasing function with respect to \( b \) for each fixed \( j \). Therefore, its minimum is attained at point \( b = 1 \) where

\[
\frac{j^2 + 4j + 4}{j^2 + 4j + 2} > 1.
\]

Hence Eq. (73) is satisfied and the shifted extended Sibuya distribution is indeed self-decomposable and thus also infinitely divisible F.W. Steutel [2004]. \( \square \)

For \( b < 1 \) the Theorem 1 tells us that the scaled factorial moments of the shifted extended Sibuya distribution with the pgf \( \mathcal{R}_0(b, \gamma, w) \) and of its thinned version \( \mathcal{R}_0(b, \gamma, 1-a+aw) \) are the same. Moreover, since \( \mathcal{R}_0(b, \gamma, w) = S_0(\gamma, bw)/S_0(\gamma, b) \), see Eq. (64), it is easy to check using Eq. (19) with \( f(x) \) given by Eq. (23) that the pgf of the shifted extended Sibuya distribution satisfies the Theorem 2 and hence the scaling variable \( a \) can be any positive number.
3.3.2. The shifted generalized Sibuya distribution. If \( p_n \) is the pmf of the generalized Sibuya distribution \([59]\) then \( r_n = p_{n+1} \) is the pmf of the shifted generalized Sibuya distribution with the \( g \)-function

\[
g_r(n) = \frac{(n+1)r_{n+1}}{r_n} = \frac{n+1}{n+2} \cdot \frac{(n+2)p_{n+2}}{p_{n+1}} = \frac{(n+1)(n+1+\nu-\gamma)}{n+2+\nu}
\]

and hence

\[
g_r(n) = \frac{\alpha_2}{\beta_1} = 1, \quad \frac{\alpha_1}{\beta_1} = (3+\nu-\gamma), \quad \frac{\alpha_0}{\beta_1} = (1+\nu-\gamma), \quad \frac{\beta_0}{\beta_1} = \nu + 2.
\]

Interestingly, the only difference between the coefficients of the original \([61]\) and the shifted distribution \([75]\) consists in replacement \( \nu \to \nu + 1 \).

To check the self-decomposability we once again use the Lemma \([\text{I}]\) to find that

\[
M = \max_{0 \leq n \leq j} \frac{r_{n+1}}{r_n} = \left(1 - \frac{1+\gamma}{j+2+\nu}\right).
\]

Using Eq. \([74]\) we rewrite the r.h.s. of the Eq. \([73]\) as

\[
N = \frac{j+2}{j+1} \cdot \frac{r_{j+1} - r_{j+2}}{r_j - r_{j+1}} = g_r(j) \cdot \frac{j+2-g_r(j+1)}{j+1-g_r(j)} = -\frac{(j+2)(\gamma+2\nu-1)^2}{(j+1)(j-\nu+2)(j-\nu+3)}
\]

Dividing now the r.h.s. of the Eq. \([73]\) by its l.h.s. and neglecting terms \( O(j^{-3}) \) we observe that there always \( \exists j_0 > 0 \) such that \( \forall j \geq j_0 \) the ratio

\[
\frac{N}{M} \approx \frac{(\gamma+2\nu-1)^2}{j^2} < \frac{(3\nu)^2}{j^2} < 1
\]

where in that last equation we have used the inequality \( 0 < \gamma < \nu + 1 \). Thus contrary to the previous case the shifted generalized Sibuya distribution is not self-decomposable.

3.4. Beyond the Sibuya distribution. Let us consider the following four-parameter generalization of the Sibuya pgf

\[
Q(w) = \frac{1 - (1 - (1 - (1 - bw)^\gamma)^\ell)^k}{1 - (1 - (1 - (1 - b)^\gamma)^\ell)^k} = \frac{1 - (1 - (H(b, \gamma, w)^\ell)^k}{1 - (1 - (H(b, \gamma, 1)^\ell)^k}.
\]

In \([\text{Klebanov 2021}]\) it was shown that Eq. \([76]\) represents a probability generating function in the case of \( \gamma = 1/m, \ k = m, \ b = 1 \) where \( m, \ \ell \in \mathbb{N} \). It is not difficult to prove this statement remains true for the case of positive integer \( k \leq m \) and \( b \in (0, 1] \).

For the case of \( k = \ell = 1 \) the function \( Q(w) \) coincides with probability generating function \( R(w) \) of the extended Sibuya distribution \([64]\). For the case of \( k = 1 \) the pgf Eq. \([76]\) represents the sum of \( \ell \) extended Sibuya– distributed random variables. In particular, for \( \ell = 2 \), the function \( g(n) \) is easily calculable

\[
g(n) = \frac{b(n+1) (\binom{\gamma}{n+1} - 2 \binom{\gamma}{n+1})}{2 \binom{\gamma}{n} - \binom{2\gamma}{n}}.
\]
Let us note that for $\gamma \geq 1/2$ the function (77) is positive and for $n \geq 2$ it is close to linear function of $n g(n) \approx a_1(\gamma)(n + a_2(\gamma))$. In particular for $\gamma = 1/2$ it coincides with the extended Sibuya distribution (63).

On the other hand for $k = \ell = 2$, $m \geq 2$ the ratio $g(n)$ (13) takes the form

$$g(n) = -b \frac{(1 + n) \left(4^{2/m + 1} - 4(3/m + 1) + \frac{4}{1 + n}\right)}{2 \left(4^{2/m} - 4(3/m) + \frac{4}{m}\right)}.$$  

Most simple case is $m = 2$ with $g(n) = \frac{b}{4}(2n - 3)$ which is positive for $n \geq 2$. The pmf then reads

$$p_n = p_2 \frac{b 2i - 3}{4i + 1} = \frac{\alpha_3 n(n - 1)(n - 2) + \alpha_2 n(n - 1)}{\beta_2 n(n - 1)}; \quad \frac{\alpha_3}{\beta_2} = \frac{b}{2}, \quad \frac{\alpha_2}{\beta_2} = \frac{7b}{4}.$$  

3.5. The extended Sibuya as progeny. Consider branching process $(Z_n)_{n \geq 0}$ governed by the pmf $p_k, k \in \mathbb{Z}$. Feller (1957); Harris (2012). Markov property of the process determines how the direct descendants of the $n$th generation form the $(n + 1)$st generation. If $Z_n$ is the size of the $n$th generation and $H_n(w)$ is its pgf then rv $Y_n = 1 + Z_1 + \ldots + Z_n$ with the pgf $Q_n(w)$ represents the total number of descendants up to the $n$th generation including the ancestor (zero generation). By assumption $Z_0 = 1$ and $H_1(w) = H(w) = \sum_{k=0}^{\infty} p_k w^k$ and therefore

$$Q_1(w) = w H(w); \quad Q_n(w) = w H(Q_{n-1}(w)).$$

If $H^{(1)}(1) = \sum_{k=0}^{\infty} k p_k \leq 1$ then the rv $Y = \sum_{n=0}^{\infty} Z_n$ is finite and is called progeny (Feller 1957). The pgf of the progeny then corresponds to a fixed point of Eq. (80)

$$\lim_{n \to \infty} Q_n(w) = Q(w) = w H(Q(w)).$$

By inverting the last equation we can express $H(w)$ in terms of $Q(w)$

$$Q(w) = w H(Q(w)); \quad w = Q^{-1}(u) = g(u); \quad H(u) = \frac{u}{g(u)}.$$  

Note that although such inversion is in principle always possible the rv $Y$ is the progeny only when all the expansion coefficients of the function $H(u)$ in powers of $u$ are positive. In particular, for the case of Sibuya pgf (11) Eq. (82) with $Q(w) = S(\gamma, w)$ the progeny was shown to exists only for $\frac{1}{2} \leq \gamma < 1$ (Letac 2019).

For the extended Sibuya distribution Eq. (82) with $Q(w) = R(w)$ yields

$$H(u, b, \gamma) = \frac{ub}{1 - (1 - uc)^{1/\gamma}}; \quad c = 1 - (1 - b)^{\gamma}; \quad \gamma \neq 0.$$  

Expanding $H(u, b, \gamma)$ in powers of $u$

$$H(u, b, \gamma) = \frac{b\gamma}{c} - \frac{1}{2} b(\gamma - 1)u - \frac{bc(\gamma^2 - 1)}{12\gamma} u^2 - \frac{(bc^2(\gamma^2 - 1))}{24\gamma} u^3 + O(u^4).$$

we observe that for $\gamma < -1$ the coefficient $\sim -(\gamma^2 - 1)u^2$ becomes negative (taking into account that $bc/\gamma > 0$) and for $-1 < \gamma < 0$ the same is true for the coefficient $\sim -(bc^2(\gamma^2 - 1))u^3$. On the other hand for $\gamma = -1$ with $H(u) = -\frac{b}{c} + bu$ the corresponding Geometric distribution pgf $R(w, b, -1)$ is progeny.
Further on at $\gamma = 0$

$$H(u, b, 0) = \frac{bu}{1 - (1 - b)^u} = -\frac{b}{\log(1 - b)} + \frac{bu}{2} - \frac{1}{12} u^2 (b \log(1 - b)) + \frac{1}{720} bu^4 \log^3 (1 - b) + O(u^5)$$

it is now coefficient $\sim \log^3 (1 - b) u^4$ which is negative. Finally for $\gamma > 0$ substitution $t = bw \to w$ in Eq. (64) yields the pgf of Sibuya distribution (14) $S(t)$ multiplied by positive factor $1/(1 - (1 - b)^\gamma)$ and hence with the identical signs of all expansion coefficients. Therefore the results obtained in Letac (2019) for $\gamma > 0$ are unchanged and extended Sibuya pgf is progeny for $\gamma \in (\frac{1}{2}, 1)$ while for $0 < \gamma < \frac{1}{2}$ it is not. In terms of first and second (factorial) moments of the pgf (83)

$$\langle k \rangle = H^{(1)}(1, b, \gamma) = 1 + \frac{(1 - b)c}{b\gamma} \leq 1$$

$$\langle k(k - 1) \rangle = H^{(2)}(1, b, \gamma) = \frac{c(1 - b)^{1 - 2\gamma}[2(1 - b)(c - b(\gamma + 1)(c - \gamma)]}{b^2 \gamma^2}.$$  

While in the regions $-1\gamma < 0$ and $0 < \gamma < \frac{1}{2}$ the function $H^{(2)}(1, b, \gamma)$ has non-monotonic dependence on the parameter $b$ with a maximum at $\frac{3}{4} < b_{max} < 1$ rapidly moving towards 1, for $\gamma \geq \frac{1}{2}$ it becomes a monotonously increasing function of $b$. The change in the vicinity of $\gamma = \frac{1}{2}$ is quite dramatic:

$$\lim_{b \to 1^-} H^{(2)}(1, b, \gamma < \frac{1}{2}) = 0; \lim_{b \to 1^-} H^{(2)}(1, b, \gamma \geq \frac{1}{2}) = \infty.$$  

4. Conclusion

The goal of this article was to discuss the properties of Sibuya-like random variables related to their scaling and birth-death processes. We have shown that their pmf satisfy a one-step recurrence relation $g(n) = (n + 1)p_{n+1}/p_n$ with some of them representing the stationary solutions of the birth-death equations and most of them being self-decomposable. In addition to already known Sibuya-like distribution (shifted Sibuya, generalized Sibuya, discrete stable) we have found a new one - extended Sibuya distribution. The latter is very similar to ordinary Sibuya distribution having the same bounds of its validity as a progeny in branching process.

Acknowledgements

The work of Lev B. Klebanov was partially supported by Grant 19-28231X GA ČR. Michal Šumbera is partially supported by the grants LTT17018 and LTT18002 of the Ministry of Education of the Czech Republic.

References

Anderson, William J. 2012. Continuous-time Markov chains. 1991 edn. Springer series in statistics. New York, NY: Springer.

Baccetti, Valentina, & Visser, Matt. 2013. Infinite Shannon entropy. Journal of Statistical Mechanics: Theory and Experiment, 2013(04), P04010.

Biyajima, Minoru, Kawabe, Tetsuji, & Suzuki, Naomichi. 1987. Two Probability Distributions in the Pure Birth Process and Their Applications to Hadron Hadron Collisions. Phys. Lett. B, 189, 466.
Bondesson, Lennart. 1992. *Generalized gamma convolutions and related classes of distributions and densities*. Lecture Notes in Statistics. New York, NY: Springer.

Carter, Evelene M, & Potts, Henry W W. 2014. Predicting length of stay from an electronic patient record system: a primary total knee replacement example. *BMC Med. Inform. Decis. Mak.*, 14(1), 26.

Chakraborty, Subrata. 2015. Generating discrete analogues of continuous probability distributions-A survey of methods and constructions. *J. Stat. Distrib. Appl.*, 2(1).

Chilappnikov, P. V., & Chikilev, O. G. 1989. Negative Binomial Distribution and Stationary Branching Processes. *Phys. Lett. B*, 222, 152–154.

Christoph, G., & Schreiber, K. 2000. Scaled Sibuya distribution and discrete self-decomposability. *Statistics and Probability Letters*, 48, 181.

Carter, Evelene M, & Potts, Henry W W. 2014. Predicting length of stay from an electronic patient record system: a primary total knee replacement example. *BMC Med. Inform. Decis. Mak.*, 14(1), 26.

Chakraborty, Subrata. 2015. Generating discrete analogues of continuous probability distributions-A survey of methods and constructions. *J. Stat. Distrib. Appl.*, 2(1).

Chilappnikov, P. V., & Chikilev, O. G. 1989. Negative Binomial Distribution and Stationary Branching Processes. *Phys. Lett. B*, 222, 152–154.

Christoph, G., & Schreiber, K. 2000. Scaled Sibuya distribution and discrete self-decomposability. *Statistics and Probability Letters*, 48, 181.

Daly, Fraser, & Gaunt, Robert E. 2016. The Conway-Maxwell-Poisson distribution: Distributional theory and approximation. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13(1), 635.

Devroye, L. 1993. A triptych of discrete distributions related to the stable law. *Statistics & Probability Letters*, 18, 349–351.

Huillet, T., & Martinez, S. 2018. Regenerative mutation processes related to the self-decomposability of Sibuya distributions. *Probability in the Engineering and Informational Sciences*, 1.

Klebanov, L. 2021. On normalizations of integer-valued random variables (in Russian). *Zapiski Nauchnih Seminarov POMI*, 505, 138–146.

Klebanov, L., & Slamova, L. 2014. Discrete Stable and Casual Stable Random Variables. *arXiv*, 1406.3748v1, 1–8.

Ko, C. M., Koch, V., Lin, Zi-wei, Redlich, K., Stephanov, Misha A., & Wang, Xin-Nian. 2001. Kinetic equation with exact charge conservation. *Phys. Rev. Lett.*, 86, 5438–5441.

Kozubowski, Tomasz, & Podgorski, Krzysztof. 2017. A generalized Sibuya distribution. *Annals of the Institute of Statistical Mathematics*, 06, 855–887.

Letac, Gérard. 2019. Is the Sibuya distribution a progeny? *J. Appl. Probab.*, 56(01), 52–56.

Lloyd-Smith, J O, Schreiber, S J, Kopp, P E, & Getz, W M. 2005. Superspreading and the effect of individual variation on disease emergence. *Nature*, 438(7066), 355–359.

Norman L. Johnson, Adrienne W. Kemp, Samuel Kotz. 2005. *Univariate Discrete Distributions*. John Wiley and Sons.
Penson, K. A., & Górska, K. 2010. Exact and Explicit Probability Densities for One-Sided Lévy Stable Distributions. *Phys. Rev. Lett.*, **105**(Nov), 210604.

Pollard, Harry. 1946. The representation of $e^{-x^3}$ as a Laplace integral. *Bull. New Ser. Am. Math. Soc.*, **52**(10), 908–910.

Rényi, A. 1956. A characterization of Poisson processes. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, **1**, 519.

Sapatinas, T. 1995. Characterization of probability distributions based on discrete p-monotonicity. *Statistics & Probability Letters*, **24**, 339.

Sibuya, M. 1979. Generalized hypergeometric, digamma, and trigamma distributions. *Annals of the Institute of Statistical Mathematics*, **31**, 373–390.

Zborovsky, I. 1996. Structure of multiplicity distributions in high-energy p p/p anti-p collisions and cascade processes. Pages 342–349 of: *7th International Workshop on Correlation and Fluctuations in Multiparticle Production*.

**Department of Probability and Mathematical Statistics, Faculty of Mathematics and Physics of the Charles University, Sokolovská 49/83, 186 75 Prague, Czech Republic**

*Email address: Lev.Klebanov@mff.cuni.cz*

**Nuclear Physics Institute, Czech Academy of Sciences, 25068 Řež, Czech Republic**

*Email address: sumbera@ujf.cas.cz*