The many roles of symmetry in nuclear physics

David J Rowe
Department of Physics, University of Toronto, Toronto, ON M5S 1A7, Canada
E-mail: rowe@physics.utoronto.ca

Abstract. Much of my talk is not new. But only recently has its substance become recognized as significant due to the powerful methods, developed by J.P. Draayer and his LSU group, and the ubiquitous occurrence of deformation in essentially all nuclei.

1. Introduction
Everyone has a sense of what symmetry means in terms of patterns that are repeated under certain transformations of a system. However, the extraordinary power of group theory when allowed to include transformations that take place continuously and in time as well as in space is less widely recognised. Symmetries need not be static. For example, one finds symmetries in repeated structures in architecture and in evolving motifs in music. We are then speaking of dynamical symmetries which prove to be important in nuclear physics.

The symmetries of a system are what give it structure and form whether it be a physical theory or a work of art. Without symmetry of any kind, a system would be amorphous and chaotic. Music that is lacking in symmetry, or whose structure is not evident to the listener, is heard as noise. However, the ability to appreciate the structure of a theory or work of art depends very much on the beholder.

The focus of this presentation will be on a symmetry-based paradigm for nuclear physics, Basically, the paradigm follows the sequence: (i) Collect data. Examine it in many ways, until patterns appear. (ii) Construct models to explain the patterns. Predict consequences and identify observations that would invalidate the model or require its modification. (iii) Repeat (i) and (ii) to develop a model and correct its failings. Determine the model’s limitations and domain of validity. (iv) Show how a model, with suitable adjustment if necessary, can be embedded in a fundamental theory of the system which may also need adjustment.

This review illustrates the path by which the simplest model of nuclear collective phenomena, namely the Bohr model, evolves by use of symmetry methods into a microscopic theory.

2. Lie groups, symmetry groups, dynamical groups and spectrum generating algebras

Definition: A Lie group is a group and an analytic manifold.

The content of this definition highlights the fact that the concept of a Lie group embodies the three most important fields of mathematics that are used extensively in physics: namely algebra, analysis and geometry.
For example, the group SO(3) can be realised as the geometrical space of all orientations of a rigid body. The geometry and connectivity of this space is hard to visualize. However, once it is identified with the group SO(3), it is handled with ease. It is a smooth differentiable manifold on which rotational wave functions can be expanded by harmonic analysis (akin to Fourier expansions). As a Lie group, it has a Lie algebra of infinitesimal generators \( L_x, L_y, L_z \) with commutation relations \([L_x, L_y] = i\hbar L_z\).

**Definition:** A symmetry group \( G_{\text{sym}} \) is a group of transformations of a system that leave its Hamiltonian invariant.

Suppose, for example, that \( \hat{H} \) is a Hamiltonian and \( \{ |\alpha \lambda \nu \rangle \} \) is an orthonormal basis for its Hilbert space, where \( \lambda \) labels an irrep \( \hat{S}^\lambda \) of \( G_{\text{sym}} \), \( \nu \) labels a basis for this irrep, and \( \alpha \) indexes the multiplicity of equivalent \( \lambda \) irreps. Then

\[
\hat{S}^\lambda(g)|\alpha \lambda \nu \rangle = \sum_\mu |\alpha \lambda \mu \rangle S^\lambda_{\mu \nu}(g), \quad \forall g \in G_{\text{sym}},
\]

and

\[
\hat{H}|\alpha \lambda \nu \rangle = \sum_\beta |\beta \lambda \nu \rangle H^\lambda_{\beta \alpha}
\]

where

\[
S^\lambda_{\mu \nu}(g) = \langle \alpha \lambda \mu | \hat{S}^\lambda(g) | \alpha \lambda \nu \rangle, \quad H^\lambda_{\beta \alpha} = \langle \beta \lambda \nu | \hat{H} | \alpha \lambda \nu \rangle.
\]

It is seen that the Hamiltonian is block diagonal.

**Definition:** A dynamical group \( G_{\text{dyn}} \) for a Hamiltonian \( \hat{H} \) is a Lie group of transformations for which the Hilbert space of \( \hat{H} \) carries a sum of irreducible unitary representations with each irreducible subspace being spanned by eigenstates of \( \hat{H} \).

In both cases the Hamiltonian is partially diagonalized. The difference is that the non-zero off-diagonal matrix elements are only between states of different but equivalent irreps of a symmetry group whereas they are only between states of the same irrep of a dynamical group. Moreover the Lie algebra of the dynamical group, which is usually a spectrum generating algebra, facilitates the task of completing the diagonalization. Thus, dynamical groups and symmetry groups serve complementary purposes.

**Definition:** A spectrum generating algebra (SGA) for a class of Hamiltonians is a Lie algebra such that the Hamiltonians can be expressed (e.g., usually as polynomials) in its elements.

### 3. The Bohr model

The Bohr model was formulated [1] as a liquid-drop model with quadrupole shape coordinates \( \{ q_\nu \} \) defined by an expansion of the nuclear radius in terms of spherical harmonics

\[
R(\theta, \phi) = R_0 \left[ 1 + \sum_\nu q_\nu Y^*_{2\nu}(\theta, \phi) + \ldots \right].
\]

Corresponding quantum mechanical momenta were then defined as operators on collective model wave functions by

\[
\hat{p}^\nu = i\hbar \frac{\partial}{\partial q_\nu}, \quad \nu = 0, \pm 1, \pm 2.
\]

so that they would obey the commutation relations of a Heisenberg-Weyl \( \text{HW}(5) \) Lie algebra

\[
[\hat{q}_\mu, \hat{p}^\nu] = i\hbar \delta^\nu_\mu.
\]
An algebraic expression of the Bohr model, known as the algebraic collective model or ACM, has recently been formulated [2] with an SU(1, 1) × SO(5) ⊃ SO(3) group structure in which SU(1, 1) × SO(5) is a subgroup of a dynamical group that includes the Heisenberg-Weyl group HW(5), and SO(3) is a symmetry group. The remarkable fact is that the geometry and representations of SU(1, 1) and SO(5) enable exact calculations of SO(5) spherical harmonics and Clebsch-Gordan coefficients and give analytical expressions for all reduced matrix elements of the SO(5) spherical harmonics and other matrix elements as needed [3, 4]. Thus, the ACM provides analytical expressions for the matrix elements of any rotationally-invariant Bohr model Hamiltonian which enables it to be diagonalized quickly and easily [5]. An example, given in figure 1, illustrates the basic concepts of axially symmetric rotors with β and γ vibrational bands.

Figure 1. The Bohr model spectrum and E2 transition rates for the Hamiltonian
\[ \hat{H}(B, \alpha, \chi, \kappa) = -\frac{\hbar^2}{2M} + \frac{1}{2} B \left[ (1 - 2\alpha) \beta^2 + \alpha \beta^4 \right] - \chi \beta \cos 3\gamma \]
with \( B = 20, \alpha = 1.5 \) and \( \chi = \pm 2 \). Energies and E2 rates are given in units such that the excitation energy of the 2_1 state is 6 units and its E2 rate to the ground state is 100 units.

4. Progression to a microscopic collective model
The first step in the progression is a replacement of the surface shape coordinates \( \{ q_{\nu} \} \) by microscopic Cartesian monopole/quadrupole moments

\[ Q_{ij} = \sum_{n=1}^{A} x_{ni}x_{nj}, \quad i, j = 1, 2, 3, \quad n = 1, \ldots, A. \]  

(7)

Time derivatives and corresponding momentum observables are then given by

\[ \dot{Q}_{ij} = \frac{dQ_{ij}}{dt} = \sum_{n} (\dot{x}_{ni}x_{nj} + x_{ni}\dot{x}_{nj}), \quad \hat{P}_{ij} = \sum_{n} (p_{ni}x_{nj} + x_{ni}p_{nj}). \]  

(8)

These shape and momentum observables are now microscopic and have commutation relations, which span a known CM(3) Lie algebra [6]. The dynamical content of the CM(3) model is characterized in figure 2. This model has not been used to describe data but it represents an important step towards a more fundamental microscopic model that provides a much deeper understanding of the structure of deformed states in nuclei.
5. The symplectic model

The symplectic model \([7–9]\) is an algebraic model whose dynamical group \(\text{Sp}(3, \mathcal{R})\) is the set of all linear canonical transformations of the single-particle phase-space observables

\[
\hat{x}_{ni} \rightarrow \sum_j (a_{ij} \hat{x}_{nj} + b_{ij} \hat{p}_{nj}), \quad \hat{p}_{ni} \rightarrow \sum_j (c_{ij} \hat{x}_{nj} + d_{ij} \hat{p}_{nj}),
\]

that preserve the commutation relations \([\hat{x}_{ni}, \hat{p}_{nj}] = i\hbar\delta_{m,n}\delta_{i,j}\). Its Lie algebra comprises all elements of the CM(3) Lie algebra

\[
Q_{ij} = \sum_{n=1}^{A} x_{ni}x_{nj}, \quad P_{ij} = \sum_{n} (p_{ni}x_{nj} + x_{ni}p_{nj}), \quad \hbar\hat{L}_{ij} = \sum_{n} (\hat{x}_{ni}\hat{p}_{nj} - \hat{x}_{nj}\hat{p}_{ni}),
\]

plus the extra bilinear combinations of momentum operators

\[
\hat{K}_{ij} = \sum_{n} \hat{p}_{ni}\hat{p}_{nj}.
\]

Adding the extra \(\hat{K}_{ij}\) operators is a major step forward for the following reasons.

(i) The group \(\text{Sp}(3, \mathcal{R})\) is semi-simple (in fact, simple) whereas CM(3) is not.

(ii) The many-nucleon kinetic energy and harmonic-oscillator Hamiltonians

\[
\sum_{n=1}^{A} \sum_{i=1}^{3} \frac{1}{2m} \hat{p}_{ni}^2, \quad \hat{H}_{\text{DHO}}^{(A)} = \sum_{n=1}^{A} \left[ \frac{1}{2m} \hat{p}_{n}^2 + \frac{1}{2}m(\omega_1^2 \hat{x}_{n1}^2 + \omega_2^2 \hat{x}_{n2}^2 + \omega_3^2 \hat{x}_{n3}^2) \right],
\]

are elements of the \(\text{Sp}(3, \mathcal{R})\) Lie algebra for all values of \(\omega_1, \omega_2, \omega_3\). Thus, the symplectic model avoids the need for arbitrary inertia parameters and is compatible with both spherical and deformed harmonic-oscillator shell models

(iii) \(\text{Sp}(3, \mathcal{R})\) is a dynamical group for any Hamiltonian of the form

\[
\hat{H} = \frac{1}{2m} \sum_{i} \hat{K}_{ii} + V(Q).
\]
(iv) It contains many useful subgroup chains, e.g.

\[
\text{Sp}(3, \mathbb{R}) \supset \text{CM}(3) \supset \text{ROT}(3) \supset \text{SO}(3), \quad \text{Sp}(3, \mathbb{R}) \supset U(3) \supset SU(3) \supset SO(3),
\]

where ROT(3) is a dynamical group for a rigid-rotor model and SU(3) is the dynamical group of Elliott’s SU(3) model. The first chain reflects its content in terms of the dynamical flows of a quantum fluid while the latter is more useful for the construction of a shell-model theory of deformed nuclei.

\[
\text{Sp}(3, \mathbb{R}) \text{ irreps in nuclei are labeled by the three quantum numbers } N_0(\lambda_0 \mu_0) \text{ of its lowest weight } U(3) \text{ sub-representations, with } (\lambda_0 \mu_0) \text{ being standard SU}(3) \text{ labels. There are three classes of irreps, as illustrated in figure 3: irreps with } (\lambda_0 \mu_0) = (0 0) \text{ correspond to spherical (closed-shell) nuclei; } (\lambda_0 0) \text{ and } (0 \mu_0) \text{ irreps correspond to axially symmetric rotational nuclei; and irreps with non-zero } \lambda_0 \text{ and } \mu_0 \text{ correspond to triaxial rotational nuclei.}
\]

**Figure 3.** Basis states for the three classes of Sp(3, R) irreps

Much progress has been achieved [10, 11] at LSU by Jerry Draayer’s group in the development of multi-shell calculations for light nuclei in SU(3) and Sp(3, R) bases. However, for heavy rotational nuclei, different techniques are needed. For one thing, it is a priori less evident what shell model spaces are required for realistic calculations. For example, a comparison of independent-particle energies in the spherical shell model with those of the Nilsson model, shown in figures 4 and 5, indicate that, due to the large amount of level crossing in the neighbourhood of the Fermi surface, the Nilsson-model state for \(^{168}\text{Er}\) is already at energy \(\gtrsim 12 \hbar \omega\) above that of the spherical shell model ground state.

6. Geometrical and mean-field perspectives

It has been shown [12] that, if \(|\sigma\rangle \equiv |N_0(\lambda_0 \mu_0)\rangle\) is a lowest weight state of an Sp(3, R) irrep that is defined as a highest-weight state for its lowest-weight U(3) irrep, the Hilbert space for the Sp(3, R) irrep is spanned by the set of states

\[
\mathcal{R} = \{ \hat{R}(\Omega)\hat{U}(d)\hat{R}(\Omega')|\sigma\rangle; \quad \Omega \in \text{SO}(3), \quad d = \text{diag}(d_1, d_2, d_3), \quad \Omega' \in \text{SO}(3)\},
\]

where the product \(\Omega d\Omega'\) is the factorization of a general linear matrix \(g = \Omega d\Omega' \in \text{GL}(3, \mathbb{R})\), \(\hat{R}(\Omega)\) and \(\hat{R}(\Omega')\) are rotation operators, and \(\hat{U}(d)\) is a unitary scale transformation of the \((x, y, z)\) coordinates. The sequence of transformations \(\hat{R}(\Omega)\hat{U}(d)\hat{R}(\Omega')\) is illustrated in figure 6.

For a Hamiltonian \(\hat{H}\), the minimum-energy

\[
E(\Omega d\Omega') = \langle \sigma (\Omega d\Omega') | \hat{H} | \sigma (\Omega d\Omega') \rangle
\]

(17)
state $|\sigma(\Omega d\Omega')\rangle = \hat{R}(\Omega)\hat{U}(d)\hat{R}(\Omega')|\sigma\rangle$ on the manifold $R$, is given by mean-field theory and should satisfy a self-consistency condition \[13\]. If $\hat{H}$ is rotationally invariant, $E(\Omega d\Omega')$ is independent of $\Omega$. Consideration of figure 6(d) suggests that $\Omega'$ should be the identity matrix at the energy minimum. Thus, we seek a state $|\sigma(d)\rangle$ by self-consistency considerations.

By construction, the SU(3) highest-weight state $|\sigma\rangle$ is an eigenstate of a spherical harmonic oscillator. In contrast, the density distribution of the state $|\sigma\rangle$ is only spherical if $\lambda_0 = 0$ and $\mu_0 = 0$. Thus, a generic state $|\sigma\rangle$ does not satisfy shape-consistency with the Hamiltonian for which it is an eigenstate. However, following the scale transformation $\hat{U}(d)$, the state $|\sigma(d)\rangle$ becomes an eigenstate of a triaxial harmonic oscillator Hamiltonian with potential

$$V_d(x) = \frac{1}{2} \sum_n m(\omega_1 x_{n1}^2 + \omega_2 x_{n2}^2 + \omega_3 x_{n3}^2),$$

(18)

which can be chosen to satisfy shape consistency.

Self-consistency results were given to leading order in references \[13, 14\]. They are determined to all orders in $N_1$, $N_2$ and $N_3$ for which the state $|\sigma(d)\rangle$ becomes an eigenstate of the generally triaxial harmonic oscillator Hamiltonian with potential (18) and frequencies given by

$$N_1\omega_1 = N_2\omega_2 = N_3\omega_3 = k$$

(19)
Figure 5. Single-particle energies in the Nilsson model. The red and green boxes indicate the levels that would be occupied by protons and neutrons, respectively, in this independent-particle model.

Figure 6. The dynamical degrees of freedom of the symplectic model: row (i) for an irrep \(\langle N_0(00)\rangle\); row (ii) for a generic irrep \(\langle N_0(\lambda_0\mu_0)\rangle\). Columns (a) and (b) illustrate the basic SU(3) shape orientations; (c) and (d) illustrate the extra vibrations and rotations associated with the giant-resonance (GR) degrees of freedom.
with \( k \) determined such that, for volume conservation, \( \omega_1 \omega_2 \omega_3 = \omega^2 \). It follows that the quadrupole moments of the SU(3) approximation become doubled and the harmonic oscillator energy of the lowest-weight state, which in the SU(3) approximation is given by \( N_0 \hbar \omega = (N_1 + N_2 + N_3) \hbar \omega \) becomes

\[
E(N_1 N_2 N_3) = 3(N_1 N_2 N_3)^{1/3} \hbar \omega.
\]  

Sample results for the predicted lowest-energy \( \text{Sp}(3, \mathbb{R}) \) irreps are given in table 1. The irreps shown in blue prove to be remarkable consistent with observed low-energy states of the nuclei considered (cf. reference [15] for \( {^{168}}\text{Er} \)). The irrep indicated in red for \( {^{12}}\text{C} \) and \( {^{16}}\text{O} \) are also consistent with the rotational bands based on the first excited states of these nucleus (cf. references [14, 16]).

| \( N_0 \) | \( \lambda \) | \( \mu \) | \( E(N_1 N_2 N_3) \) |
|---|---|---|---|
| 26 | 0 | 4 | 25.3 |
| 30 | 12 | 0 | 26.0 |
| 28 | 6 | 2 | 26.3 |
| 32 | 10 | 2 | 28.6 |

| \( N_0 \) | \( \lambda \) | \( \mu \) | \( E(N_1 N_2 N_3) \) |
|---|---|---|---|
| 36 | 0 | 0 | 36 |
| 40 | 8 | 4 | 37.3 |
| 38 | 4 | 2 | 37.3 |
| 40 | 7 | 3 | 38.1 |
| 48 | 24 | 0 | 38.1 |
| 44 | 16 | 2 | 38.3 |

| \( N_0 \) | \( \lambda \) | \( \mu \) | \( E(N_1 N_2 N_3) \) |
|---|---|---|---|
| 814 | 30 | 8 | 812.6 |
| 826 | 96 | 20 | 812.9 |
| 822 | 70 | 28 | 813.0 |
| 818 | 52 | 20 | 813.1 |
| 816 | 42 | 12 | 813.1 |
| 830 | 114 | 16 | 813.2 |

### Table 1. Spherical and self-consistent harmonic oscillator energies for the lowest-weight states of various \( \text{Sp}(3, \mathbb{R}) \) irreps in units of \( \hbar \omega \).

7. A sample calculation

Figure 7 shows the results of an \( \text{Sp}(3, \mathbb{R}) \) calculation [17] for \( {^{166}}\text{Er} \). For comparison, results are also shown for SU(3) and rigid-rotor models. The remarkable fact is that they all agree with one another more precisely than they agree with the observed spectrum; in particular, none of the models exhibits the centrifugal stretching effects shown by the data. The absence of such stretching in the symplectic model is an artifact of a potential energy with an overly sharp minimum, which could be adjusted. What is important is that the microscopic symplectic model reproduces the results of the phenomenological rotor model very precisely. Even more remarkable is the fact that it returns the observed moment of inertia with a kinetic energy defined purely by the nucleon mass and with a two parameter potential energy chosen to have
a minimum at the observed deformation and a strength determined by self-consistency to get the observed E2 transition rates corresponding to such a potential. The success of the SU(3) model can be understood on the grounds that the lower angular momentum states of a large-dimensional representation of SU(3) contract to those of a rigid rotor [9, 18–20]. Nevertheless, the remarkable agreement between the three models is a strong implication that they are effectively describing the same rotational dynamics, albeit in the sense of a quasi-dynamical symmetry.

8. Quasi-dynamical symmetry

Figure 8 shows that the amplitudes in an SU(3) basis of the Sp(3,R) states of figure 7 are essentially the same for all $L \leq 10$. This is in spite of the fact that the SU(3) spectrum of figure 7 is for the lowest-weight SU(3) irrep of the Sp(3,R) calculation which, as figure 8 shows, contributes only a tiny component to any of the Sp(3,R) states shown; the amplitude of this lowest-weight SU(3) irrep appears on the far left of figure 8. Thus, the SU(3) and Sp(3,R) states of figure 7 have negligible overlaps with one another. Nevertheless, because of the near equality of the amplitudes of the many contributing SU(3) irreps, the Sp(3R) results are essentially identical to that of the average SU(3) irrep.

The equality of amplitudes from different mixed irreps of a subgroup is a characteristic of a quasi-dynamical symmetry [21]. It is also observed in a pairing + quadrupole model [22] with Hamiltonian

$$\hat{H}(\alpha) = \hat{H}_0 + (1 - \alpha) V_{su(2)} + \alpha V_{su(3)},$$

where $\hat{H}_0$ is a harmonic oscillator Hamiltonian, $V_{su(2)} = -G \hat{S}_+ \hat{S}_-$ is an SU(2) quasi-spin pairing interaction and $V_{su(3)} = -\chi \hat{Q} \cdot \hat{Q}$ is an SU(3) interaction. The model has a USp(6) dynamical group. It also has two competing dynamical subgroups, SU(3) and SU(2), and an SO(3) symmetry group. The spectrum and wave functions for this model are shown for various values of $\alpha$ in figure 9.

Quasi-dynamical symmetry underlies the collective motions of a complex system with many independent modes which, in the absence of interactions between them, vibrate or rotate independently. However, unless it overwhelms their vibrational or rotational behavior, attractive interactions between the modes, tends to make them vibrate and/or rotate in unison in their lowest frequency states. This perspective and figure 9, suggest that the existence of dominant rotational dynamics tends to suppress the mixing of states of widely different deformation. Moreover, there is expected to be a suppression of the mixing of rotational states belonging
to different Sp(3, R) irreps of large deformation when they appear among states of much lower deformation to which they are not directly connected by a two-body interaction.

9. Concluding remarks
The several avenues explored above, give hope that viable procedures for describing well-deformed nuclear states in terms of mixed symmetry shell-model calculations in an Sp(3, R) × SU(2) basis might be achievable in the near future. In progressing towards this goal many model investigations of the optimal routes to be followed are needed. The following are a few.

It is of particular importance to discover the energy distribution of lowest-weight U(3) irreps in nuclei, in terms of an empirical formula or a simple model. Equation (20) is a start on this search. However, it needs to be checked out in mean-field calculations and extended, perhaps along the lines of the Nilsson model, to include strongly-coupled spin degrees of freedom.

Studies of the distributions of nuclear shapes in nuclei, already initiated as reviewed in reference [23, 24], need to be continued so that their systematics and patterns can be understood. Methods of approach have been suggested by Jarrio et al. [15] and Carvalho [25], and extensions proposed by Hess et al. [26].

Two subjects that would naturally have been pursued in this contribution if there were time and space are: (i) a procedure for carrying out shell model calculations within the space of one or a few Sp(3, R) irreps; and (ii) a procedure for defining an effective interaction on a subspace spanned by lowest-weight U(3) irreps.

Because the states \(|\sigma(\Omega d\Omega')\rangle = \hat{R}(\Omega)\hat{U}(d)\hat{R}(\Omega')|\sigma\rangle\) of the set (16) span an Sp(3, R) irrep, a good basis for diagonalizing a given microscopic Hamiltonian in such an irrep is given by a discrete subset of states on this manifold. This should give a good approximation if the selected states are centered about the minimum-energy state. Procedures for projecting out states of good angular momentum have been much investigated, e.g., by Cusson and Lee [27],

Figure 9. Energies (left) and wave functions in an SU(3) basis (right). The SU(2) dynamical symmetry dominates for \(\alpha \lesssim 0.55\) and the SU(3) dominates for \(\alpha \gtrsim 0.6\). However, the wave functions show that this dominance is characteristic of a quasi-dynamical symmetry.
and are known to simplify for eigenstates of a deformed harmonic oscillator [28]. Moreover, if the observed low-lying states imply states of relatively strong $\beta$ rigidity, it is expected that a relative small number of states of different deformation will be sufficient. Such a method was proposed by Vassanji and Rowe [29] and pursued by Carvalho et al. [30] and others.

An effective interaction for a subspace spanned by lowest-weight U(3) irreps is, in principle, defined by the Suzuki-Lee projection method [31] and the fact that the projection of states in an $\text{Sp}(3, \mathbb{R}) \supset U(3)$ basis onto a lowest weight U(3) subspace would appear to be straightforward. However, subtle concerns will inevitably arise when mixed $\text{Sp}(3, \mathbb{R})$ states are combined with spin wave function. Thus, again model investigations with schematic interactions need to be considered.

Finally, it is worthwhile to consider how it might be possible to take advantage of quasi-dynamical symmetry to define a shell-model basis in which calculations will be most rapidly convergent. For example, quasi-dynamical symmetry would seem to imply that if a calculation can be done in an $\text{Sp}(3, \mathbb{R}) \supset U(3)$ basis for $L = 0$, or more generally, for a minimum value of the total angular momentum $J$, then at least such a calculation will have defined the $U(3) \times SU(2)$ irreps that contribute most strongly for all low-lying states.

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