Conjecture: 100% of elliptic surfaces over $\mathbb{Q}$ have rank zero

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Abstract
Based on an equation for the rank of an elliptic surface over $\mathbb{Q}$ which appears in the work of Nagao, Rosen, and Silverman, we conjecture that 100% of elliptic surfaces have rank 0 when ordered by the size of the coefficients of their Weierstrass equations, and present a probabilistic heuristic to justify this conjecture. We then discuss how it would follow from either understanding of certain $L$-functions, or from understanding of the local behaviour of the surfaces. Finally, we make a conjecture about ranks of elliptic surfaces over finite fields, and highlight some experimental evidence supporting it.

1 Introduction
Let $E$ be an elliptic surface over $\mathbb{Q}$. Fix a Weierstrass equation

$$E : y^2 = x^3 + A(T)x + B(T)$$

with $A(T), B(T) \in \mathbb{Z}[T]$ and $4A(T)^3 + 27B(T)^2 \neq 0$. Given an integer $t$ such that $4A(t)^3 + 27B(t)^2 \neq 0$, let $E_t$ denote the elliptic curve over $\mathbb{Q}$ with Weierstrass equation

$$E_t : y^2 = x^3 + A(t)x + B(t).$$

Given an elliptic curve $E/\mathbb{Q}$ and a prime $p$, define $a_p(E)$ to be the $p^{th}$ coefficient of the $L$-function attached to $E$.

It was conjectured by Nagao [12] that

$$\text{rank } E(\mathbb{Q}(T)) = -\lim_{X \to \infty} \frac{1}{X} \sum_{p < X} \sum_{t=1}^{p} a_p(E_t) \frac{\log p}{p}. \quad (1)$$

Rosen and Silverman [13] proved that (1) held if one assumed Tate’s conjecture and that a certain $L$-function didn’t vanish on the right edge of its critical strip. In particular, (1) holds unconditionally for rational elliptic surfaces.

In light of (1), we think it’s likely that the average rank of elliptic surfaces over $\mathbb{Q}$ is 0, essentially because the average value of $a_p(E_t)$ should be 0. We give a more thorough justification for this belief in section 3.

There are several ways the belief that the average rank of elliptic surfaces is 0 can be formulated as a precise conjecture. For convenience, we introduce the following framework for discussing possible formulations of such a conjecture.

Let $\mathcal{S}(M)$ with $m \in \mathbb{Z}_{>0}$ be a sequence of subsets of the set of all elliptic surfaces over $\mathbb{Q}$, with the properties that $\mathcal{S}(M)$ is finite for every $M$, and $\mathcal{S}(M) \subseteq \mathcal{S}(M')$ whenever $M \leq M'$. We’ll say that $\mathcal{S} = \bigcup_{M=1}^{\infty} \mathcal{S}(M)$ is a family of elliptic surfaces, and the filtration given by $M$ is an ordering of that family. This parallels the language used when discussing statistics of ranks of elliptic curves, where, for example, people might discuss “the family of quadratic twists of a fixed elliptic curve, ordered by conductor”.

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Define the average rank of the family $S$ to be the quantity
\[
\lim_{M \to \infty} \frac{1}{\# S(M)} \sum_{E \in S(M)} \text{rank} E(\mathbb{Q}(T)).
\]
We believe that the average rank of many families $S$ will be $0$. In formulating conjecture 1.1 below, we pick a specific family for the sake of concreteness, but there are many other reasonable choices.

Conversely, there are several examples in the literature of families of elliptic surfaces over $\mathbb{Q}$ which were constructed to have strictly positive rank [5] [6]. Average ranks of the elliptic curves coming from specializations of elliptic surfaces have been studied as well [1] [6] [15], and in some situations the average ranks of the specializations can exhibit rich and possibly unexpected behaviour.

To formulate conjecture 1.1 below, we first define, for positive integers $d$ and $M$, the set
\[
P_d(M) = \{ p \in \mathbb{Z}[T] : \deg(p) = d, \mu(p) < M \},
\]
where $\mu(p)$ is the Mahler measure of $p$ (using the height of $p$ would also be reasonable here). Set
\[
S_{m,n}(M) := \{ E : y^2 = x^3 + A(T)x + B(T) : A \in P_m(M^2), B \in P_n(M^3), 4A(T)^3 + 27B(T)^2 \neq 0 \}.
\]
Then we believe the average rank of the family $S = S_{m,n}$ will be $0$ whenever $m$ and $n$ are both positive, i.e.

**Conjecture 1.1.** For any fixed positive integers $m$ and $n$, we have
\[
\lim_{M \to \infty} \frac{1}{\# S_{m,n}(M)} \sum_{E \in S_{m,n}(M)} \text{rank} E(\mathbb{Q}(T)) = 0.
\]

In section 4 we discuss the main obstacles for proving conjecture 1.1 using the analytic framework in [13]. In section 5 we outline an approach to conjecture 1.1 based on investigating statistics of ranks of elliptic surfaces over finite fields.

## 2 Acknowledgments

We thank Noam Elkies, Bjorn Poonen, and Michael Snarski for helpful discussions. This work was supported by grant 550031 from the Simons Foundation.

## 3 Probabilistic heuristics

For a fixed prime $p$, Birch [2] gave all moments of the distribution of $a_p(E)$ when $E$ is chosen by selecting a (possibly singular) Weierstrass equation with coefficients in $\mathbb{F}_p$ uniformly at random. Let $(A_{p,t})$ be a sequence of independent random variables indexed by a prime $p$ and a positive integer $t$, with the property that, for every $t$, the random variable $A_{p,t}$ has the same distribution as the values $a_p(E)$ for $E$ a Weierstrass equation in $\mathbb{F}_p$ chosen uniformly at random, as in Birch’s work. We highlight the following property of the sequence $(A_{p,t})$:

**Proposition 3.1.** For any $\varepsilon > 0$, the series
\[
\frac{1}{X^{2+\varepsilon}} \sum_{p<X} \sum_{t=1}^p A_{p,t} \log \frac{p}{p}
\]
converges to $0$ with probability $1$ as $X$ goes to infinity.
Proof. By construction we have
\[ \text{Var} \left( \frac{1}{X^{1+\epsilon}} \sum_{p<X} \sum_{t=1}^p A_{p,t} \log \frac{p}{p} \right) \ll \frac{(\log p)^2}{X^{1+\epsilon}}. \]

Thus
\[ \text{Var} \left( \frac{1}{X^{1+\epsilon}} \sum_{p<X} \sum_{t=1}^p A_{p,t} \log \frac{p}{p} \right) \ll \frac{1}{X^{1+\epsilon}} \sum_{p<X} (\log p)^2 \ll 1. \]

By Kolmogorov’s three-series theorem, as stated in [4, theorem 2.5.8] for example, it follows that the series
\[ \frac{1}{X^{1+\epsilon}} \sum_{p<X} \sum_{t=1}^p A_{p,t} \log \frac{p}{p} \]
converges almost surely. Hence the series
\[ \frac{1}{X^{1+\epsilon}} \sum_{p<X} \sum_{t=1}^p A_{p,t} \log \frac{p}{p} = \frac{1}{X^{1+\epsilon}} \left( \frac{1}{X^{1+\epsilon}} \sum_{p<X} \sum_{t=1}^p A_{p,t} \log \frac{p}{p} \right) \]
converges to 0 almost surely. \(\square\)

Proposition [5.1] is relevant because it suggests that, if we believe that the values \(a_{p}(E_t)\) are distributed in the same way \(a_{p}(E)\) is for elliptic curves \(E/\mathbb{F}_p\) chosen uniformly at random, then we should expect that 100% of the time the quantity in (1) is 0.

There is evidence to support the belief that the values \(a_{p}(E_t)\) will be distributed in this way. If, instead of fixing \(p\) and letting \(t\) vary initially, we fix \(t\) and let \(p\) vary, then, for all \(\epsilon > 0\) the series
\[ \frac{1}{X^{1+\epsilon}} \sum_{p<X} A_{p,t} \log \frac{p}{p} \]
converges to 0 as \(X\) goes to infinity with probability 1. This suggests that, for a fixed elliptic curve \(E_t/\mathbb{Q}\), we should expect that
\[ \sum_{p<X} a_{p}(E_t) \log \frac{p}{p} \ll X^{\epsilon} \]
for all \(\epsilon > 0\), because we should expect that the reductions \(E_t/\mathbb{F}_p\) will be distributed uniformly at random. The bound (3) was proven by Heath-Brown [8], so the reductions \(E_t/\mathbb{F}_p\) do indeed behave uniformly randomly in this case.

The popular belief that the average rank of elliptic curves over \(\mathbb{Q}\) is \(\frac{1}{2}\) can also support the idea that these sums of \(a_{p}\) values behave randomly in the way described. The Birch and Swinnerton-Dyer conjecture (BSD) connects the rank of an elliptic curve \(E/\mathbb{Q}\) to the coefficients \(a_{p}(E)\) of its \(L\)-function. The original formulation of BSD was
\[ \prod_{p<X} \frac{p+1-a_{p}(E)}{p} \sim C_{E}(\log X)^{\text{rank } E(\mathbb{Q})} \]
for some constant \(C_E\) which depends on \(E\). See work of Goldfeld [7], K. Conrad [3], and Kuo-R. Murty [10] for some treatment of this specific formulation of BSD. One consequence of this form of BSD is [14]
\[ \text{rank } E(\mathbb{Q}) = \frac{1}{2} - \frac{1}{\log X} \int_1^X \frac{1}{x} \sum_{p<x} a_{p}(E) \log \frac{dx}{x} + O \left( \frac{1}{\log X} \right). \]

It is widely believed that many families of elliptic curves over \(\mathbb{Q}\) have average rank \(\frac{1}{2}\). This belief suggests that the quantity
\[ \frac{1}{X} \sum_{p<x} a_{p}(E) \log p \]
(5)
appearing in equation (4) should average to 0 in those families. However, because the error term in (4) depends on $E$, even knowing that the average rank of elliptic curves was $\frac{1}{2}$ wouldn’t be enough to conclude that the quantity $\text{5}$ averages to 0. While it would be surprising if the error terms in (4) did not average to 0 as well, controlling these error terms is difficult, and is the main obstacle in proving anything about average ranks from this perspective.

4 Obstacles for analytic proofs

In this section we discuss what obstacles exist which make proving conjecture 1.1 difficult within the framework established in [13]. In proving Nagao’s conjecture, Rosen and Silverman first prove an analytic version of his conjecture:

$$\text{Res}_{s=2} \sum_p \sum_{t=1}^p a_p(E_t) \frac{\log p}{p^s} = -\text{rank } E(\mathbb{Q}(T)).$$

(6)

To do this, they introduce the following notation:

- $L_2(E/Q, s)$ is the Hasse-Weil $L$-function of $E/Q$ attached to $H^2_{et}(E/\overline{\mathbb{Q}}, \mathbb{Q}_l)$.
- $\text{NS}(E/\overline{\mathbb{Q}})$ is the Néron-Severi group of $E/\overline{\mathbb{Q}}$.
- $\mathcal{S}$ is the trivial part of $\text{NS}(E/\overline{\mathbb{Q}}) \otimes \mathbb{Q}$, generated by the image of the zero section and by all components of all fibers.
- $\mathcal{S}_l(1)$ is the Tate twist $\mathcal{S} \otimes T_l(\overline{\mathbb{Q}}^*)$ of the Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-module $\mathcal{S}$.
- $L(\mathcal{S}_l(1), s)$ is the Artin $L$-function attached to the representation $\mathcal{S}_l(1)$.
- $N(E/Q, s) = L_2(E/Q, s) / L(\mathcal{S}_l(1), s)$.

Then, assuming the Tate conjecture, Silverman and Rosen prove that both sides of (6) are equal to

$$\text{Res}_{s=2} \frac{d}{ds} \log N(E/Q, s).$$

The conjecture (11) then follows from standard analytic techniques applied to the series $\frac{d}{ds} \log N(E/Q, s)$.

We now discuss what might be involved in a proof of conjecture 1.1 using these analytic techniques. Suppose that there exist constants $\delta_E > 0$ and $C_E$ such that

$$\left| \frac{1}{X} \sum_{p<X} \sum_{t=1}^p a_p(E_t) \frac{\log p}{p} + \text{rank } E(\mathbb{Q}(T)) \right| < C_E X^{-\delta_E},$$

as is typical in this kind of setting. To prove conjecture (11) it would be sufficient to find, for every elliptic surface $E \in S$, a real number $X_E > 0$ depending only on $E$, such that both

$$\lim_{M \to \infty} \frac{1}{\#S(M)} \sum_{E \in S(M)} \frac{1}{X_E} \sum_{p<X_E} \sum_{t=1}^p a_p(E_t) \frac{\log p}{p} = 0$$

(7)

and

$$\lim_{M \to \infty} \frac{1}{\#S(M)} \sum_{E \in S(M)} C_E X_E^{-\delta_E} = 0$$

(8)

simultaneously. Condition (7) requires that $M$ be “large” relative to the values $X_E$ so that the averages

$$\frac{1}{\#S(M)} \sum_{E \in S(M)} a_p(E_t)$$
for fixed $p$ and $t$ are small. Condition (8), on the other hand, would like for $M$ to be “small” relative to the values $X_\ell$. The difficulty, then, is to find some choice of $X_\ell$’s such that both of these conditions hold at once. We expect that showing that there is such a choice will be difficult. Condition (7) might require a Pólya-Vinogradov style result, where one shows that there are no “conspiracies” among the values $a_\ell(E)$ that might cause them to behave differently than random variables. Condition (8) might require subconvexity bounds, and the number $\delta_\ell$ will depend on the locations of the zeroes of $N(E/\mathbb{Q}, s)$.

5 An approach via finite fields

Let $\mathcal{S}_{t,m,n}$ denote the set of elliptic surfaces $E/\mathbb{F}_t : y^2 = x^3 + A(T)x + B(T)$ with $\deg(A) = m$ and $\deg(B) = n$. This set is finite. Let $\rho_\ell(m,n)$ denote the proportion of elliptic surfaces in $\mathcal{S}_{t,m,n}$ which have positive rank.

**Conjecture 5.1.** For every prime $\ell_0$, and every pair of integers $m_0, n_0 > 0$,

$$\lim_{\ell \to \infty} \rho_\ell(m_0, n_0) = \lim_{m \to \infty} \rho_\ell(m, n_0) = \lim_{n \to \infty} \rho_\ell(m_0, n) = \frac{1}{2}.$$

This conjecture, beyond being interesting in its own right, provides an approach for proving conjecture [11] See [11] for experimental evidence towards conjecture [5.1] where $\rho_\ell(m,n)$ is estimated computationally for $\ell = 7$, $n = 6, 12, 18, 24, 30$, and $m \leq n/2$.

Let $N$ be a squarefree positive integer. Let $\mathcal{S}^{(N)}(M)$ denote the subset of $\mathcal{S}(M)$ for which $\gcd(N,4A(T)^3 + 27B(T)^2) = 1$. Then, for any $m, n, M$,

$$\frac{\#\{E \in \mathcal{S}^{(N)}(M) : E/\mathbb{F}_\ell \text{ has positive rank for all } \ell | N\}}{\#\mathcal{S}^{(N)}(M)} = \prod_{\ell | M} \rho_\ell(m,n) + O(M^{-1})$$

by the Chinese remainder theorem, where $E/\mathbb{F}_\ell$ denotes the reduction mod $\ell$ of $E/\mathbb{Q}$.

If $E/\mathbb{Q}$ has positive rank, then either the reduction $E/\mathbb{F}_\ell$ has positive rank, or the kernel of the reduction $E(\mathbb{Q}(T)) \to E(\mathbb{F}_\ell(T))$ is of finite index in $E(\mathbb{Q}(T))$. If this kernel was never of finite index then conjecture [11] would follow from conjecture [5.1] (as well as much weaker versions of this conjecture), via observation [9]. The kernel of the reduction $E(\mathbb{Q}(T)) \to E(\mathbb{F}_\ell(T))$ is of finite index in $E(\mathbb{Q}(T))$ occasionally, but presumably not nearly enough for this approach to fail. However, proving as much seems difficult. The generators of $E(\mathbb{Q}(T))$ will map to the identity of $E(\mathbb{F}_\ell(T))$ if their denominators are divisible by $\ell$, so one is naturally lead to investigate the dependence of the height of the generators of $E(\mathbb{Q}(T))$ on the size of the coefficients of the Weierstrass model of $E/\mathbb{Q}$.

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