STOCHASTIC CONVECTIVE BRINKMAN-FORCHHEIMER EQUATIONS

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Abstract. The stochastic convective Brinkman-Forchheimer (SCBF) equations or the tamed Navier-Stokes equations in bounded or periodic domains are considered in this work. We show the existence of a pathwise unique strong solution (in the probabilistic sense) satisfying the energy equality (Itô formula) to the SCBF equations perturbed by multiplicative Gaussian noise. We exploited a monotonicity property of the linear and nonlinear operators as well as a stochastic generalization of the Minty-Browder technique in the proofs. The energy equality is obtained by approximating the solution using approximate functions constituting the elements of eigenspaces of the Stokes operator in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously. We further discuss about the global in time regularity results of such strong solutions in periodic domains. The exponential stability results (in mean square and pathwise sense) for the stationary solutions is also established in this work for large effective viscosity. Moreover, a stabilization result of the stochastic convective Brinkman-Forchheimer equations by using a multiplicative noise is obtained. Finally, we prove the existence of a unique ergodic and strongly mixing invariant measure for the SCBF equations subject to multiplicative Gaussian noise, by making use of the exponential stability of strong solutions.

1. Introduction

Let \( \mathcal{O} \subset \mathbb{R}^n \) \((n = 2, 3)\) be a bounded domain with a smooth boundary \( \partial \mathcal{O} \). The convective Brinkman-Forchheimer (CBF) equations or the tamed Navier-Stokes equations are given by (see [26] for Brinkman-Forchheimer equations with fast growing nonlinearities)

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \alpha \mathbf{u} + \beta |\mathbf{u}|^{r-1} \mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \mathcal{O} \times (0, T), \\
\nabla \cdot \mathbf{u} &= 0, \quad \text{in } \mathcal{O} \times (0, T), \\
\mathbf{u} &= 0 \quad \text{on } \partial \mathcal{O} \times (0, T), \\
\mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in } \mathcal{O}, \\
\int_{\mathcal{O}} p(x, t) dx &= 0, \quad \text{in } (0, T).
\end{align*}
\]
The convective Brinkman-Forchheimer equations (1.1) describe the motion of incompressible fluid flows in a saturated porous medium. Here $u(t, x) \in \mathbb{R}^n$ represents the velocity field at time $t$ and position $x$, $p(t, x) \in \mathbb{R}$ denotes the pressure field, $f(t, x) \in \mathbb{R}^n$ is an external forcing. The final condition in (1.1) is imposed for the uniqueness of the pressure $p$. The constant $\mu$ represents the positive Brinkman coefficient (effective viscosity), the positive constants $\alpha$ and $\beta$ represent the Darcy (permeability of porous medium) and Forchheimer (proportional to the porosity of the material) coefficients, respectively. The absorption exponent $r \in [1, \infty)$ and $r = 3$ is known as the critical exponent. For $\alpha = \beta = 0$, we obtain the classical 3D Navier-Stokes equations (see [27, 48, 49, 13, 20, 21, 41], etc). The nonlinearity of the form $|u|^{r-1}u$ can be found in tidal dynamics as well as non-Newtonian fluid flows (see [2, 40, 39], etc and references therein).

Let us now discuss some of the solvability results available in the literature for the 3D CBF and related equations in the whole space as well as periodic domains. The authors in [7], showed that the Cauchy problem for the Navier-Stokes equations with damping $r|u|^{r-1}u$ in the whole space has global weak solutions, for any $r \geq 1$, global strong solutions, for any $r \geq 7/2$ and that the strong solution is unique, for any $7/2 \leq r \leq 5$. An improvement to this result was made in [51] and the authors showed that the above mentioned problem has global strong solutions, for any $r > 3$ and the strong solution is unique, when $3 < r \leq 5$. Later, the authors in [52] proved that the strong solution exists globally for $r \geq 3$, and they established two regularity criteria, for $1 \leq r < 3$. For any $r \geq 1$, they proved that the strong solution is unique even among weak solutions. The existence and uniqueness of a smooth solution to a tamed 3D Navier-Stokes equation in the whole space is proved in [43]. A simple proof of the existence of global-in-time smooth solutions for the CBF equations (1.1) with $r > 3$ on a 3D periodic domain is obtained in [24]. The authors also proved that global, regular solutions exist also for the critical value $r = 3$, provided that the coefficients satisfy the relation $4\beta\mu \geq 1$. Furthermore, they showed that in the critical case every weak solution verifies the energy equality and hence is continuous into the phase space $L^2$. The authors in [25] showed that the strong solutions of three dimensional CBF equations in periodic domains with the absorption exponent $r \in [1, 3]$ remain strong under small enough changes of initial condition and forcing function.

The authors in [1] considered the Navier-Stokes problem in bounded domains with compact boundary, modified by the absorption term $|u|^{r-2}u$, for $r > 2$. For this problem, they proved the existence of weak solutions in the Leray-Hopf sense, for any dimension $n \geq 2$ and its uniqueness for $n = 2$. But in three dimensions, they were not able to establish the energy equality satisfied by the weak solutions. The existence of regular dissipative solutions and global attractors for the system (1.1) with $r > 3$ is established in [26]. As a global smooth solution exists for $r > 3$, the energy equality is satisfied in bounded domains. Recently, the authors in [19] were able to construct functions that can approximate functions defined on smooth bounded domains by elements of eigenspaces of linear operators (e.g., the Laplacian or the Stokes operator) in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously. As a simple application of this result, they proved that all weak solutions of the critical CBF equations ($r = 3$) posed on a bounded domain in $\mathbb{R}^3$ satisfy the energy equality.

Let us now discuss some results available in the literature for the stochastic counterpart of the (1.1) and related models in the whole space or on a torus. The existence of a unique
strong solution

\[ u \in L^2(\Omega; L^\infty(0, T; H^1(\mathcal{O}))) \cap L^2(0, T; H^2(\mathcal{O})), \quad (1.2) \]

with \( \mathbb{P} \)-a.s. continuous paths in \( C([0, T]; H^1(\mathcal{O})) \), for \( u_0 \in L^2(\Omega; H^1(\mathcal{O})) \), to the stochastic tamed 3D Navier-Stokes equation in the whole space as well as in the periodic boundary case is obtained in [44]. They also prove the existence of a unique invariant measure for the corresponding transition semigroup. Recently, [5] improved their results for a slightly simplified system. The authors in [31] established the local and global existence and uniqueness of solutions for general deterministic and stochastic nonlinear evolution equations with coefficients satisfying some local monotonicity and generalized coercivity conditions. In [32], the author showed the existence and uniqueness of strong solutions for a large class of SPDE, where the coefficients satisfy the local monotonicity and Lyapunov condition, and he provided the stochastic tamed 3D Navier-Stokes equations as an example. A large deviation principle of Freidlin-Wentzell type for the stochastic tamed 3D Navier-Stokes equations driven by multiplicative noise in the whole space or on a torus is established in [45]. All these works established the existence and uniqueness of strong solutions in the regularity class given in (1.2). The global solvability of 3D Navier-Stokes equations in the whole space with a Brinkman-Forchheimer type term subject to an anisotropic viscosity and a random perturbation of multiplicative type is described in the work [4].

Unlike whole space or periodic domains, in bounded domains, there is a technical difficulty in getting the regularity given in (1.2) of the velocity field \( u(\cdot) \) appearing in (1.1). It is mentioned in the paper [26] that the major difficulty in working with bounded domains is that \( P_H(|u|^{-1}u) \) (\( P_H : L^2(\mathcal{O}) \to H \) is the Helmholtz-Hodge orthogonal projection) need not be zero on the boundary, and \( P_H \) and \( -\Delta \) are not necessarily commuting (see Example 2.19, [42]). Moreover, \( \Delta u \big|_{\partial \mathcal{O}} \neq 0 \) in general and the term with pressure will not disappear (see [26]), while taking inner product with \( \Delta u \) to the first equation in (1.1). Therefore, the equality

\[
\int_{\mathcal{O}} (-\Delta u(x)) \cdot |u(x)|^{-1}u(x)dx \\
= \int_{\mathcal{O}} |\nabla u(x)|^2|u(x)|^{-1}dx + 4 \left[ \frac{r - 1}{(r + 1)^2} \right] \int_{\mathcal{O}} |\nabla|u(x)|^{-\frac{r}{2}}|^2dx \\
= \int_{\mathcal{O}} |\nabla u(x)|^2|u(x)|^{-1}dx + \frac{r - 1}{4} \int_{\mathcal{O}} |u(x)|^{-3}|\nabla|u(x)|^2|^2dx,
\]

(1.3)

may not be useful in the context of bounded domains. The authors in [46] showed the existence and uniqueness of strong solutions to stochastic 3D tamed Navier-Stokes equations on bounded domains with Dirichlet boundary conditions. They also proved a small time large deviation principle for the solution. The author in [30] proved the existence of a random attractor for the three-dimensional damped Navier-Stokes equations in bounded domains with additive noise by verifying the pullback flattening property. The existence of a random attractor \( (r > 3, \text{ for any } \beta > 0) \) as well as the existence of a unique invariant measure \( (3 < r \leq 5, \text{ for any } \beta > 0 \text{ and } \beta \geq \frac{1}{2} \text{ for } r = 3) \) for the stochastic 3D Navier-Stokes equations with damping driven by a multiplicative noise is established in the paper [28]. By using classical Faedo-Galerkin approximation and compactness method, the existence of martingale solutions for the stochastic 3D Navier-Stokes equations with nonlinear damping is obtained in [29]. The works [30], [18], etc considered various stochastic problems related
to the equations similar to stochastic CBF equations in bounded domains with Dirichlet boundary conditions. As far as strong solutions are concerned, some of these works proved regularity results in the space given in (1.2), by using the estimate given in (1.3), which may not hold true always.

In this work, we consider the stochastic convective Brinkman-Forchheimer (SCBF) equations and show the existence and uniqueness of strong solutions in the probabilistic sense in a larger space than (1.2) and discuss about some asymptotic behavior. The main motivation of this work comes from the papers [24] and [19]. The work [24] helped us to construct functions that can approximate functions defined on smooth bounded domains by elements of eigenspaces of Stokes operator in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously. On the n-dimensional torus, one can approximate functions in \( L^p \)-spaces using truncated Fourier expansions (see [19]). We also got inspiration from the work [23], where the Itô formula for processes taking values in intersection of finitely many Banach spaces has been established. Due to the difficulty explained above, one may not expect regularity of \( u(\cdot) \) given in (1.2) in bounded domains. The novelties of this work are:

- The existence and uniqueness of strong solutions to SCBF equations \((r > 3, \text{ for any } \mu \text{ and } \beta, r = 3 \text{ for } 2\beta\mu \geq 1)\) in bounded domains with \( u_0 \in L^2(\Omega; L^2(\mathcal{O})) \) is obtained in the space \( L^2(\Omega; L^\infty(0, T; L^2(\mathcal{O}))) \cap L^2(0, T; H^1_0(\mathcal{O}))) \cap L^{r+1}(\Omega; L^{r+1}(0, T; L^{r+1}(\mathcal{O}))) \), with \( \mathbb{P} \)-a.s. paths in \( C([0, T]; L^2(\mathcal{O})) \).
- The energy equality (Itô’s formula) satisfied by the SCBF equations is established by approximating the strong solution using the finite-dimensional space spanned by the first \( n \) eigenfunctions of the Stokes operator.
- The exponential stability (in the mean square and almost sure sense) of the stationary solutions is obtained for large \( \mu \) and the lower bound on \( \mu \) does not depend on the stationary solutions. A stabilization result of the stochastic convective Brinkman-Forchheimer equations by using a multiplicative noise is also obtained.
- The existence of a unique ergodic and strongly mixing invariant measure for the SCBF equations perturbed by multiplicative Gaussian noise is established by using the exponential stability of strong solutions.

The organization of the paper is as follows. In the next section, we define the linear and nonlinear operators, and provide the necessary function spaces needed to obtain the global solvability results of the system (1.1). For \( r > 3 \), we show that the sum of linear and nonlinear operators is monotone (Theorem 2.2), and for \( r = 3 \) and \( 2\beta\mu \geq 1 \), we show that the sum is globally monotone (Theorem 2.3). The demicontinuity and hence the hemicontinuity property of these operators is also obtained in the same section (Lemma 2.5). The SCBF equations perturbed by Gaussian noise is formulated in section 3. After providing an abstract formulation of the SCBF equations in bounded or periodic domains, we establish the existence and uniqueness of global strong solution by making use of the monotonicity property of the linear and nonlinear operators as well as a stochastic generalization of the Minty-Browder technique (see Proposition 3.5 for a-priori energy estimates and Theorem 3.7 for global solvability results). We overcame the major difficulty of establishing the energy equality for the SCBF equations by approximating the solution using the finite-dimensional space spanned by the first \( n \) eigenfunctions of the Stokes operator. Due to the technical difficulties explained earlier, we prove the regularity results of the global strong solutions under
smoothness assumptions on the initial data and further assumptions on noise co-efficient, in periodic domains only (Theorem 3.11). The section 4 is devoted for establishing the exponential stability (in the mean square and almost sure sense) of the stationary solutions (Theorems 4.7 and 4.8) for large $\mu$. In both Theorems, the lower bound of $\mu$ does not depend on the stationary solutions. A stabilization result of the stochastic convective Brinkman-Forchheimer equations by using a multiplicative noise is also obtained in the same section (Theorem 4.9). In the final section, we prove the existence of a unique ergodic and strongly mixing invariant measure for the SCBF equations by using the exponential stability of strong solutions (Theorem 5.5).

2. Mathematical Formulation

The necessary function spaces needed to obtain the global solvability results of the system (1.1) is provided in this section. We prove monotonicity as well as hemicontinuity properties of the linear and nonlinear operators in the same section. In our analysis, the parameter $\alpha$ does not play a major role and we set $\alpha$ to be zero in (1.1) in the entire paper.

2.1. Function spaces. Let $C_0^\infty(\mathcal{O}; \mathbb{R}^n)$ be the space of all infinitely differentiable functions ($\mathbb{R}^n$-valued) with compact support in $\mathcal{O} \subset \mathbb{R}^n$. Let us define

$$
\mathcal{V} := \{ u \in C_0^\infty(\mathcal{O}, \mathbb{R}^n) : \nabla \cdot u = 0 \},
$$

$$
\mathbb{H} := \text{the closure of } \mathcal{V} \text{ in the Lebesgue space } L^2(\mathcal{O}) = L^2(\mathcal{O}; \mathbb{R}^n),
$$

$$
\mathcal{V} := \text{the closure of } \mathcal{V} \text{ in the Sobolev space } H_0^1(\mathcal{O}) = H_0^1(\mathcal{O}; \mathbb{R}^n),
$$

$$
\tilde{L}^p := \text{the closure of } \mathcal{V} \text{ in the Lebesgue space } L^p(\mathcal{O}) = L^p(\mathcal{O}; \mathbb{R}^n),
$$

for $p \in (2, \infty)$. Then under some smoothness assumptions on the boundary, we characterize the spaces $\mathbb{H}$, $\mathcal{V}$ and $\tilde{L}^p$ as $\mathbb{H} = \{ u \in L^2(\mathcal{O}) : \nabla \cdot u = 0, u \cdot n|_{\partial \mathcal{O}} = 0 \}$, with norm $\|u\|_{\mathbb{H}} := \int_{\mathcal{O}} |u(x)|^2 \, dx$, where $n$ is the outward normal to $\partial \mathcal{O}$, $\mathcal{V} = \{ u \in H_0^1(\mathcal{O}) : \nabla \cdot u = 0 \}$, with norm $\|u\|_{\mathcal{V}} := \int_{\mathcal{O}} |\nabla u(x)|^2 \, dx$, and $\tilde{L}^p = \{ u \in L^p(\mathcal{O}) : \nabla \cdot u = 0, u \cdot n|_{\partial \mathcal{O}} \}$, with norm $\|u\|_{\tilde{L}^p} := \int_{\mathcal{O}} |u(x)|^p \, dx$, respectively. Let $(\cdot, \cdot)$ denotes the inner product in the Hilbert space $\mathbb{H}$ and $(\cdot, \cdot)$ denotes the induced duality between the spaces $\mathcal{V}$ and its dual $\mathcal{V}'$ as well as $\tilde{L}^p$ and its dual $\tilde{L}'$. Note that $\mathbb{H}$ can be identified with its dual $\mathbb{H}'$. We endow the space $\mathcal{V} \cap \tilde{L}^p$ with the norm $\|u\|_{\mathcal{V}} + \|u\|_{\tilde{L}^p}$, for $u \in \mathcal{V} \cap \tilde{L}^p$ and its dual $\mathcal{V}' + \tilde{L}'$ with the norm

$$
\inf \left\{ \max \left( \|v_1\|_{\mathcal{V}}, \|v_1\|_{\tilde{L}'^p} \right) : v = v_1 + v_2, \ v_1 \in \mathcal{V}', \ v_2 \in \tilde{L}' \right\}.
$$

Moreover, we have the continuous embedding $\mathcal{V} \cap \tilde{L}^p \hookrightarrow \mathbb{H} \hookrightarrow \mathcal{V}' + \tilde{L}'$. For the functional set up in periodic domains, interested readers are referred to see [49, 36, 24], etc.

2.2. Linear operator. Let $P_0 : L^2(\mathcal{O}) \to \mathbb{H}$ denotes the Helmholtz-Hodge orthogonal projection (see [27, 9]). Let us define

$$
\begin{align*}
Au &= -P_0 \Delta u, \quad u \in D(A), \\
D(A) &= \mathcal{V} \cap H^2(\mathcal{O}).
\end{align*}
$$
It can be easily seen that the operator $A$ is a non-negative self-adjoint operator in $H$ with $V = D(A^{1/2})$ and

$$
\langle Au, u \rangle = \|u\|_V^2, \quad \text{for all } u \in V, \quad \text{so that } \|Au\|_{V'} \leq \|u\|_V. \quad (2.1)
$$

For a bounded domain $O$, the operator $A$ is invertible and its inverse $A^{-1}$ is bounded, self-adjoint and compact in $H$. Thus, using spectral theorem, the spectrum of $A$ consists of an infinite sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$, with $\lambda_k \to \infty$ as $k \to \infty$ of eigenvalues. Moreover, there exists an orthogonal basis $\{e_k\}_{k=1}^\infty$ of $H$ consisting of eigenvectors of $A$ such that $Ae_k = \lambda_k e_k$, for all $k \in \mathbb{N}$. We know that $u$ can be expressed as $u = \sum_{k=1}^\infty (u, e_k)e_k$ and $Au = \sum_{k=1}^\infty \lambda_k (u, e_k)e_k$. Thus, it is immediate that

$$
\|\nabla u\|^2_H = \langle Au, u \rangle = \sum_{k=1}^\infty \lambda_k |(u, e_k)|^2 \geq \lambda_1 \sum_{k=1}^\infty |(u, e_k)|^2 = \lambda_1 \|u\|_H^2. \quad (2.2)
$$

It should be noted that, in this work, we are not using the Gagliardo-Nirenberg, Ladyzhenskaya or Agmon’s inequalities. Thus, the results obtained in this work are true for $2 \leq n \leq 4$ in bounded domains (see the discussions above (3.38)) and $n \geq 2$ in periodic domains. The following interpolation inequality is also frequently in the paper. Assume $1 \leq s \leq r \leq t \leq \infty$, $\theta \in (0, 1)$ such that $\frac{1}{r} = \frac{s}{t} + \frac{1-\theta}{t}$ and $u \in L^s(O) \cap L^t(O)$, then we have

$$
\|u\|_{L^t} \leq \|u\|_{L^s}^{\theta}|u|_{L^r}^{1-\theta}. \quad (2.3)
$$

2.3. Bilinear operator. Let us define the trilinear form $b(\cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R}$ by

$$
b(u, v, w) = \int_O (u(x) \cdot \nabla)v(x) \cdot w(x)dx = \sum_{i,j=1}^n \int_O u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x)dx.
$$

If $u, v$ are such that the linear map $b(u, v, \cdot)$ is continuous on $V$, the corresponding element of $V'$ is denoted by $B(u, v)$. We also denote (with an abuse of notation) $B(u) = B(u, u) = P_H(u \cdot \nabla)u$. An integration by parts gives

$$
\begin{cases}
    b(u, v, v) = 0, \quad \text{for all } u, v \in V, \\
    b(u, v, w) = -b(u, w, v), \quad \text{for all } u, v, w \in V. 
\end{cases} \quad (2.4)
$$

In the trilinear form, an application of Hölder’s inequality yields

$$
|b(u, v, w)| = |b(u, w, v)| \leq \|u\|_{L^{r+1}} \|v\|_{L^{-2(r+1)}} \|w\|_V, 
$$

for all $u \in V \cap \widehat{L}^{r+1}$, $v \in V \cap \widehat{L}^{-2(r+1)}$ and $w \in V$, so that we get

$$
\|B(u, v)\|_{V'} \leq \|u\|_{L^{r+1}} \|v\|_{L^{-2(r+1)}}. \quad (2.5)
$$

Hence, the trilinear map $b : V \times V \times V \to \mathbb{R}$ has a unique extension to a bounded trilinear map from $(V \cap \widehat{L}^{r+1}) \times (V \cap \widehat{L}^{-2(r+1)}) \times V$ to $\mathbb{R}$. It can also be seen that $B$ maps $V \cap \widehat{L}^{r+1}$ into $V' + \widehat{L}^{-2(r+1)}$ and using interpolation inequality (see (2.3)), we get

$$
|\langle B(u, u), v \rangle| = |b(u, v, u)| \leq \|u\|_{L^{r+1}} \|u\|_{L^{-2(r+1)}} \|v\|_V \leq \|u\|_{L^{r+1}} \|u\|^{\frac{r+1}{r+2}}_{L^{r+1}} \|v\|_{L^{-2(r+1)}}, \quad (2.6)
$$

for all $v \in V \cap \widehat{L}^{r+1}$. Thus, we have

$$
\|B(u)\|_{V' + \widehat{L}^{-2(r+1)}} \leq \|u\|_{L^{r+1}} \|u\|^{\frac{r+1}{r+2}}_{L^{r+1}}. \quad (2.7)
$$
Using (2.5), for \( u, v \in \mathbb{V} \cap \tilde{\mathbb{L}}^{r+1} \), we also have

\[
\|B(u) - B(v)\|_{\mathbb{V}' + \tilde{\mathbb{L}}^{r+1}} \leq \|B(u - v, u)\|_{\mathbb{V}'} + \|B(v, u - v)\|_{\mathbb{V}'} \\
\leq \left( \|u\|_{L^2_{r-r+1}^\infty} + \|v\|_{L^2_{r-r+1}^\infty} \right) \|u - v\|_{\mathbb{L}^{r+1}} \\
\leq \left( \|u\|_{H^r}^{r-3} \|u\|_{\mathbb{L}^{r+1}}^{2} + \|v\|_{H^r}^{r-3} \|v\|_{\mathbb{L}^{r+1}}^{2} \right) \|u - v\|_{\mathbb{L}^{r+1}}, \tag{2.8}
\]

for \( r > 3 \), by using the interpolation inequality. For \( r = 3 \), a calculation similar to (2.8) yields

\[
\|B(u) - B(v)\|_{\mathbb{V}' + \tilde{\mathbb{L}}^{r+1}} \leq \left( \|u\|_{L^3} + \|v\|_{L^3} \right) \|u - v\|_{L^4}, \tag{2.9}
\]

hence \( B(\cdot) : \mathbb{V} \cap \tilde{\mathbb{L}}^{4} \rightarrow \mathbb{V}' + \tilde{\mathbb{L}}^{3} \) is a locally Lipschitz operator.

2.4. Nonlinear operator. Let us now consider the operator \( C(u) := P_{\mathbb{H}}(|u|^{-1}u) \). It is immediate that \( \langle C(u), u \rangle = \|u\|_{L^2_{r-r+1}}^{r+1} \) and the map \( C(\cdot) : \tilde{\mathbb{L}}^{r+1} \rightarrow \tilde{\mathbb{L}}^{r+1} \) is Gateaux differentiable with Gateaux derivative \( C'(u)v = r|u|^{-1}v \), for \( v \in \mathbb{L}^{r+1} \). For \( u, v \in \tilde{\mathbb{L}}^{r+1} \), using Taylor’s formula, we have

\[
\langle P_{\mathbb{H}}(|u|^{-1}u) - P_{\mathbb{H}}(|v|^{-1}v), w \rangle \leq \|(|u|^{-1}u) - (|v|^{-1}v)\|_{\mathbb{L}^{r+1}} \|w\|_{\mathbb{L}^{r+1}} \\
\leq \sup_{0<\theta<1} r\|u - v\| \theta |u| + (1 - \theta) |v|^{-1} \|w\|_{\mathbb{L}^{r+1}} \\
\leq \sup_{0<\theta<1} r\|\theta u + (1 - \theta)v|^{-1}\|u - v\|_{\mathbb{L}^{r+1}} \|w\|_{\mathbb{L}^{r+1}} \\
\leq r\left( \|u\|_{\mathbb{L}^{r+1}} + \|v\|_{\mathbb{L}^{r+1}} \right)^{-1} \|u - v\|_{\mathbb{L}^{r+1}} \|w\|_{\mathbb{L}^{r+1}}, \tag{2.10}
\]

for all \( u, v, w \in \tilde{\mathbb{L}}^{r+1} \). Thus the operator \( C(\cdot) : \tilde{\mathbb{L}}^{r+1} \rightarrow \tilde{\mathbb{L}}^{r+1} \) is locally Lipschitz. Moreover, for any \( r \in [1, \infty) \), we have

\[
\langle P_{\mathbb{H}}(u|u|^{-1}) - P_{\mathbb{H}}(v|v|^{-1}), u - v \rangle \\
= \int_\Omega (|u(x)| |u(x)|^{-1} - |v(x)| |v(x)|^{-1}) \cdot (u(x) - v(x)) dx \\
= \int_\Omega (|u(x)|^{-1} - |u(x)|^{-1} u(x) \cdot v(x) - |v(x)|^{-1} u(x) \cdot v(x) + |v(x)|^{-1} u(x) \cdot v(x) + |v(x)|^{-1}) dx \\
\geq \int_\Omega (|u(x)|^{-1} - |u(x)|^{-1} |v(x)| - |v(x)|^{-1} |u(x)| + |v(x)|^{-1}) dx \\
= \int_\Omega (|u(x)|^{-1} - |v(x)|^{-1}) (|u(x)| - |v(x)|) dx \geq 0. \tag{2.11}
\]

Furthermore, we find

\[
\langle P_{\mathbb{H}}(u|u|^{-1}) - P_{\mathbb{H}}(v|v|^{-1}), u - v \rangle \\
= \langle |u|^{-1}, |u - v|^2 \rangle + \langle |v|^{-1}, |u - v|^2 \rangle + \langle |u|^{-1} - u|v|^{-1}, u - v \rangle \\
= \|u|^{-1} \|_{\mathbb{H}}^2 + \|v|^{-1} \|_{\mathbb{H}}^2 + \langle u \cdot v, |u|^{-1} + |v|^{-1} - |u|^2, |v|^{-1} - |v|^2 \rangle. \tag{2.12}
\]
But, we know that

\[ \langle u \cdot v, |u|^{r-1} + |v|^{r-1} \rangle - \langle |u|^2, |v|^{r-1} \rangle - \langle |v|^2, |u|^{r-1} \rangle \]

\[ = -\frac{1}{2}\|u|^{\frac{r}{2}}(u - v)\|_H^2 - \frac{1}{2}\|v|^{\frac{r}{2}}(u - v)\|_H^2 + \frac{1}{2}\left(\|u|^{r-1} - |v|^{r-1}\|, (|u|^2 - |v|^2)\right) \]

\[ \geq -\frac{1}{2}\|u|^{\frac{r}{2}}(u - v)\|_H^2 - \frac{1}{2}\|v|^{\frac{r}{2}}(u - v)\|_H^2. \]

From (2.12), we finally have

\[ \langle P_{\overline{H}}(u|u|^{r-1}) - P_{\overline{H}}(v|v|^{r-1}), u - v \rangle \geq \frac{1}{2}\|u|^{\frac{r}{2}}(u - v)\|_H^2 + \frac{1}{2}\|v|^{\frac{r}{2}}(u - v)\|_H^2 \geq 0, \] for \( r \geq 1. \)

2.5. **Monotonicity.** Let us now show the monotonicity as well as the hemicontinuity properties of the linear and nonlinear operators, which plays a crucial role in this paper.

**Definition 2.1** ([3]). Let \( \mathbb{X} \) be a Banach space and let \( \mathbb{X}' \) be its topological dual. An operator \( G : D \to \mathbb{X} \), \( D = D(G) \subset \mathbb{X} \) is said to be monotone if

\[ \langle G(x) - G(y), x - y \rangle \geq 0, \text{ for all } x, y \in D. \]

The operator \( G(\cdot) \) is said to be hemicontinuous, if for all \( x, y \in \mathbb{X} \) and \( w \in \mathbb{X}' \),

\[ \lim_{\lambda \to 0} \langle G(x + \lambda y), w \rangle = \langle G(x), w \rangle. \]

The operator \( G(\cdot) \) is called demicontinuous, if for all \( x \in D \) and \( y \in \mathbb{X} \), the functional \( x \mapsto \langle G(x), y \rangle \) is continuous, or in other words, \( x_k \to x \) in \( \mathbb{X} \) implies \( G(x_k) \rightharpoonup G(x) \) in \( \mathbb{X}' \).

Clearly demicontinuity implies hemicontinuity.

**Theorem 2.2.** Let \( u, v \in \mathbb{V} \cap \mathbb{L}^{r+1} \), for \( r > 3 \). Then, for the operator \( G(u) = \mu A u + B(u) + \beta C(u) \), we have

\[ \langle (G(u) - G(v), u - v) + \eta\|u - v\|_V^2 \rangle \geq 0, \]

where

\[ \eta = \frac{r - 3}{2\mu(r - 1)} \left( \frac{2}{\beta \mu(r - 1)} \right)^{\frac{2}{r - 3}}. \]

That is, the operator \( G + \eta I \) is a monotone operator from \( \mathbb{V} \cap \mathbb{L}^{r+1} \) to \( \mathbb{V}' + \mathbb{L}^{r+1}_{\mathbb{L}} \).

**Proof.** We estimate \( \langle A u - A v, u - v \rangle \) by using an integration by parts as

\[ \langle A u - A v, u - v \rangle = \|u - v\|_V^2. \]

From (2.13), we easily have

\[ \beta \langle C(u) - C(v), u - v \rangle \geq \frac{\beta}{2}\|v|^{\frac{r}{2}}(u - v)\|_H^2. \]

Note that \( \langle B(u, u - v), u - v \rangle = 0 \) and it implies that

\[ \langle B(u) - B(v), u - v \rangle = \langle B(u, u - v), u - v \rangle + \langle B(v, u - v), u - v \rangle \]

\[ = \langle B(u - v, v), u - v \rangle = -\langle B(u - v, v), u - v \rangle. \]

Using Hölder’s and Young’s inequalities, we estimate \( |\langle B(u - v, u - v), v \rangle| \) as

\[ |\langle B(u - v, u - v), v \rangle| \leq \|u - v\|_V \|v(u - v)\|_H. \]
We estimate \( \|v(u - v)\|_V^2 \) from (2.18) and use Hölder’s and Young’s inequalities to estimate it as (see [24] also)

\[
\int_\Omega |v(x)|^2 |u(x) - v(x)|^2 \, dx = \int_\Omega |v(x)|^2 |u(x) - v(x)|^{r-1} |u(x) - v(x)|^{2(r-3)/(r-1)} \, dx
\leq \left( \int_\Omega |v(x)|^{r-1} |u(x) - v(x)|^2 \, dx \right)^{\frac{2}{r}} \left( \int_\Omega |u(x) - v(x)|^2 \, dx \right)^{\frac{r-3}{r-1}}
\leq \beta \mu \left( \int_\Omega |v(x)|^{r-1} |u(x) - v(x)|^2 \, dx \right) + \frac{r-3}{r-1} \left( \frac{2}{\beta \mu (r-1)} \right)^{\frac{2}{r-3}} \left( \int_\Omega |u(x) - v(x)|^2 \, dx \right),
\tag{2.19}
\]

for \( r > 3 \). Using (2.19) in (2.18), we find

\[
|\langle B(u - v, u - v), v \rangle| \leq \frac{\mu}{2} \|u - v\|_V^2 + \frac{\beta}{2} \|v\|_V^{r+1} (u - v)\frac{2}{2\mu (r-1)} (\frac{2}{\beta \mu (r-1)}) \leq \frac{\mu}{2} \|u - v\|_V^2.
\tag{2.20}
\]

Combining (2.16), (2.17) and (2.20), we get

\[
\langle (G(u) - G(v), u - v) + \frac{r-3}{2\mu (r-1)} \left( \frac{2}{\beta \mu (r-1)} \right)^{\frac{2}{r-3}} \|u - v\|_V^2 \|u - v\|_H^2 \geq \frac{\mu}{2} \|u - v\|_V^2 \geq 0,
\tag{2.21}
\]

for \( r > 3 \) and the estimate (2.14) follows. \( \square \)

**Theorem 2.3.** For the critical case \( r = 3 \) with \( 2 \beta \mu \geq 1 \), the operator \( G(\cdot) : \mathbb{V} \cap \mathbb{L}^{r+1} \rightarrow \mathbb{V}' + \mathbb{L}^{r+1} \) is globally monotone, that is, for all \( u, v \in \mathbb{V} \), we have

\[
\langle (G(u) - G(v), u - v) \geq 0.
\tag{2.22}
\]

**Proof.** From (2.13), we have

\[
\beta \langle C(u) - C(v), u - v \rangle \geq \frac{\beta}{2} \|v(u - v)\|_H^2.
\tag{2.23}
\]

We estimate \( |\langle B(u - v, u - v), v \rangle| \) using Hölder’s and Young’s inequalities as

\[
|\langle B(u - v, u - v), v \rangle| \leq \|v(u - v)\|_H\|u - v\|_V \leq \mu\|u - v\|_V^2 + \frac{1}{4\mu} \|v(u - v)\|_H^2.
\tag{2.24}
\]

Combining (2.16), (2.23) and (2.24), we obtain

\[
\langle (G(u) - G(v), u - v) \geq \frac{1}{2} \left( \beta - \frac{1}{2\mu} \right) \|v(u - v)\|_H^2 \geq 0,
\tag{2.25}
\]

provided \( 2 \beta \mu \geq 1 \). \( \square \)

**Remark 2.4.** 1. As in [52], for \( r \geq 3 \), one can estimate \( \|B(u - v, u - v)\| \) as

\[
|\langle B(u - v, u - v), v \rangle|
\[
\leq \mu \|u - v\|^2_V + \frac{1}{4\mu} \int_\Omega |v(x)|^2 |u(x) - v(x)|^2 \, dx \\
= \mu \|u - v\|^2_V + \frac{1}{4\mu} \int_\Omega |u(x) - v(x)|^2 \left( |v(x)|^{r-1} + 1 \right) \frac{|v(x)|^2}{|v(x)|^{r-1} + 1} \, dx \\
\leq \mu \|u - v\|^2_V + \frac{1}{4\mu} \int_\Omega |v(x)|^{r-1} |u(x) - v(x)|^2 \, dx + \frac{1}{4\mu} \int_\Omega |u(x) - v(x)|^2 \, dx,
\]

where we used the fact that \( \left\| \frac{|v|^2}{|v|^{r-1} + 1} \right\|_{L^\infty} < 1 \), for \( r \geq 3 \). The above estimate also yields the global monotonicity result given in (2.22), provided \( 2\beta \mu \geq 1 \).

2. For \( n = 2 \) and \( r = 3 \), one can estimate \( |\langle B(u - v, u - v), v \rangle| \) using Hölder’s, Ladyzhenskaya and Young’s inequalities as
\[
|\langle B(u - v, u - v), v \rangle| \leq \|u - v\|_{L^4} \|u - v\|_V \langle v \rangle |v|_{L^4} \\
\leq 2^{1/4} \|u - v\|_{L^2}^{1/2} \|u - v\|_V^{3/2} |v|_{L^4} \\
\leq \frac{\mu}{2} \|u - v\|^2_V + \frac{27}{32\mu} \|v\|^4_{L^4} \|u - v\|^2_H.
\]

Combining (2.16), (2.17) and (2.22), we obtain
\[
\langle (G(u) - G(v), u - v) + \frac{27}{32\mu^3} N^4 \|u - v\|^2_H \geq 0,
\]

for all \( v \in \hat{B}_N \), where \( \hat{B}_N \) is an \( \hat{L}^4 \)-ball of radius \( N \), that is, \( \hat{B}_N := \{ z \in \hat{L}^4 : \|z\|_{L^4} \leq N \} \). Thus, the operator \( G(\cdot) \) is locally monotone in this case (see [35, 37], etc).

Lemma 2.5. The operator \( G : V \cap \hat{L}^{r+1} \to V' + \hat{L}^{r+1} \) is demicontinuous.

Proof. Let us take a sequence \( u^n \to u \) in \( V \cap \hat{L}^{r+1} \). For any \( v \in V \cap \hat{L}^{r+1} \), we consider
\[
\langle G(u^n) - G(u), v \rangle = \mu \langle A(u^n) - A(u), v \rangle + \langle B(u^n) - B(u), v \rangle - \beta \langle C(u^n) - C(u), v \rangle.
\]

Next, we take \( \langle A(u^n) - A(u), v \rangle \) from (2.29) and estimate it as
\[
|\langle A(u^n) - A(u), v \rangle| = |\langle (v(u^n) - u, \nabla v) \rangle| \leq \|u^n - u\|_V \|v\|_V \to 0, \quad \text{as} \quad n \to \infty,
\]

since \( u^n \to u \) in \( V \). We estimate the term \( \langle B(u^n) - B(u), v \rangle \) from (2.29) using Hölder’s inequality as Hölder’s inequality as
\[
|\langle B(u^n) - B(u), v \rangle| = |\langle B(u^n, u^n - u), v \rangle + \langle B(u^n - u, u), v \rangle| \\
\leq |\langle B(u^n, v), u^n - u \rangle| + |\langle B(u^n - u, v), u \rangle| \\
\leq \left( \|u^n\|_{L^{2(r+1)}} + \|u\|_{L^{2(r+1)}} \right) \|u^n - u\|_{L^{r+1}} \|v\|_V \\
\leq \left( \|u^n\|_{H^1}^{r-1} + \|u\|_{H^1}^{r-1} \right) \|u^n - u\|_{H^{r+1}} \|v\|_V \\
\to 0, \quad \text{as} \quad n \to \infty,
\]

since \( u^n \to u \) in \( \hat{L}^{r+1} \) and \( u^n, u \in V \cap \hat{L}^{r+1} \). Finally, we estimate the term \( \langle C(u^n) - C(u), v \rangle \) from (2.29) using Taylor’s formula and Hölder’s inequality as
\[
|\langle C(u^n) - C(u), v \rangle| \leq \sup_{0 < \theta < 1} r \|u^n - u\| \theta |u^n + (1 - \theta) u|^{r-1} \|v\|_{L^{r+1}}
\]
\[ \leq r \|u^n - u\|_{\tilde{L}^{r+1}} \left( \|u^n\|_{\tilde{L}^{r+1}} + \|u\|_{\tilde{L}^{r+1}} \right)^{r-1} \|v\|_{\tilde{L}^{r+1}} \to 0, \text{ as } n \to \infty, \tag{2.32} \]

since \( u^n \to u \) in \( \tilde{L}^{r+1} \) and \( u^n, u \in \tilde{L}^{r+1} \), for \( r \geq 3 \). From the above convergences, it is immediate that \( \langle G(u^n) - G(u), v \rangle \to 0 \), for all \( v \in \mathcal{V} \cap \tilde{L}^{r+1} \). Hence the operator \( G : \mathcal{V} \cap \tilde{L}^{r+1} \to \mathcal{V}' + \tilde{L}^{r+1} \) is demicontinuous, which implies that the operator \( G(\cdot) \) is also hemicontinuous. \( \square \)

3. Stochastic convective Brinkman-Forchheimer equations

In this section, we consider the following stochastic convective Brinkman-Forchheimer equations perturbed by multiplicative Gaussian noise:

\[
\begin{cases}
\d u(t) - \mu \Delta u(t) + (u(t) \cdot \nabla)u(t) + \beta |u(t)|^{r-1}u(t) + \nabla p(t) = \Phi(t, u(t))dW(t), \text{ in } \mathcal{O} \times (0, T), \\
\nabla \cdot u(t) = 0, \text{ in } \mathcal{O} \times (0, T), \\
u(t) = 0 \text{ on } \partial \mathcal{O} \times (0, T), \\
u(0) = u_0 \text{ in } \mathcal{O},
\end{cases}
\tag{3.1}
\]

where \( W(\cdot) \) is an \( \mathbb{H} \)-valued Wiener process.

3.1. Noise coefficient. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space equipped with an increasing family of sub-sigma fields \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) of \( \mathcal{F} \) satisfying:

(i) \( \mathcal{F}_0 \) contains all elements \( F \in \mathcal{F} \) with \( \mathbb{P}(F) = 0 \),

(ii) \( \mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s \geq t} \mathcal{F}_s \), for \( 0 \leq t \leq T \).

**Definition 3.1.** A stochastic process \( \{W(t)\}_{0 \leq t \leq T} \) is said to be an \( \mathbb{H} \)-valued \( \mathcal{F}_t \)-adapted Wiener process with covariance operator \( Q \) if

(i) for each non-zero \( h \in \mathbb{H} \), \( \langle Q^{1/2}h \rangle^{-1}(W(t), h) \) is a standard one dimensional Wiener process,

(ii) for any \( h \in \mathbb{H} \), \( \langle W(t), h \rangle \) is a martingale adapted to \( \mathcal{F}_t \).

The stochastic process \( \{W(t) : 0 \leq t \leq T\} \) is a \( \mathbb{H} \)-valued Wiener process with covariance \( Q \) if and only if for arbitrary \( t \), the process \( W(t) \) can be expressed as \( W(t, x) = \sum_{k=1}^{\infty} \sqrt{\mu_k} e_k(x) \beta_k(t) \), where \( \beta_k(t), k \in \mathbb{N} \) are independent one dimensional Brownian motions on \((\Omega, \mathcal{F}, \mathbb{P})\) and \( \{e_k\}_{k=1}^{\infty} \) are the orthonormal basis functions of \( \mathbb{H} \) such that \( \mathbb{E}[e_k] = Q e_k \). If \( W(\cdot) \) is an \( \mathbb{H} \)-valued Wiener process with covariance operator \( Q \) with \( \text{Tr} \ Q = \sum_{k=1}^{\infty} \mu_k < +\infty \), then \( W(\cdot) \) is a Gaussian process on \( \mathbb{H} \) and \( \mathbb{E}[W(t)] = 0, \text{ Cov}[W(t)] = t Q, \ t \geq 0 \). The space \( \mathbb{H}_0 = Q^{1/2} \mathbb{H} \) is a Hilbert space equipped with the inner product \( \langle \cdot, \cdot \rangle_0 \),

\[ (u, v)_0 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} (u, e_k)(v, e_k) = (Q^{-1/2}u, Q^{-1/2}v), \text{ for all } u, v \in \mathbb{H}_0, \]

where \( Q^{-1/2} \) is the pseudo-inverse of \( Q^{1/2} \).

Let \( \mathcal{L}(\mathbb{H}) \) denotes the space of all bounded linear operators on \( \mathbb{H} \) and \( \mathcal{L}_Q := \mathcal{L}_Q(\mathbb{H}) \) denotes the space of all Hilbert-Schmidt operators from \( \mathbb{H}_0 := Q^{1/2} \mathbb{H} \) to \( \mathbb{H} \). Since \( Q \) is a trace class operator, the embedding of \( \mathbb{H}_0 \) in \( \mathbb{H} \) is Hilbert-Schmidt and the space \( \mathcal{L}_Q \) is a Hilbert space equipped with the norm \( \|\Phi\|_{\mathcal{L}_Q}^2 = \text{Tr}(\Phi Q \Phi^*) = \sum_{k=1}^{\infty} \|Q^{1/2} \Phi^* e_k\|^2_\mathbb{H} \) and
Hypothesis 3.2. The noise coefficient $\Phi(\cdot, \cdot)$ satisfies:

(H.1) The function $\Phi \in C([0, T] \times \mathcal{V}; L^2(\mathbb{H}))$.

(H.2) (Growth condition) There exists a positive constant $K$ such that for all $t \in [0, T]$ and $u \in \mathbb{H}$,

$$\|\Phi(t, u)\|^2_{L^2} \leq K(1 + \|u\|^2_{\mathbb{H}}),$$

(H.3) (Lipschitz condition) There exists a positive constant $L$ such that for any $t \in [0, T]$ and all $u_1, u_2 \in \mathbb{H}$,

$$\|\Phi(t, u_1) - \Phi(t, u_2)\|^2_{L^2} \leq L\|u_1 - u_2\|^2_{\mathbb{H}}.$$

3.2. Abstract formulation of the stochastic system. On taking orthogonal projection $P_H$ onto the first equation in (3.1), we get

$$
\begin{cases}
    du(t) + [\mu Au(t) + B(u(t))]dt = \Phi(t, u(t))dW(t), & t \in (0, T), \\
    u(0) = u_0,
\end{cases}
$$

(3.2)

where $u_0 \in L^2(\Omega; \mathbb{H})$. Strictly speaking one should write $P_H \Phi$ instead of $\Phi$.

Let us now provide the definition of a unique global strong solution in the probabilistic sense to the system (3.2).

Definition 3.3 (Global strong solution). Let $u_0 \in L^2(\Omega; \mathbb{H})$ be given. An $\mathbb{H}$-valued $(\mathcal{F}_t)_{t \geq 0}$-adapted stochastic process $u(\cdot)$ is called a strong solution to the system (3.2) if the following conditions are satisfied:

(i) the process $u \in L^2(\Omega; L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathcal{V})) \cap L^{r+1}(0, T; \tilde{\mathbb{L}}^{r+1})$ and $u(\cdot)$ has a $\mathcal{V} \cap \tilde{\mathbb{L}}^{r+1}$-valued modification, which is progressively measurable with continuous paths in $\mathbb{H}$ and $u \in C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \cap L^{r+1}(0, T; \tilde{\mathbb{L}}^{r+1}), \mathbb{P}$-a.s.,

(ii) the following equality holds for every $t \in [0, T]$, as an element of $\mathcal{V} + \tilde{\mathbb{L}}^{r+1}$, $\mathbb{P}$-a.s.

$$u(t) = u_0 - \int_0^t [\mu Au(s) + B(u(s)) + \beta C(u(s))]ds + \int_0^t \Phi(s, u(s))dW(s).$$

(3.3)

(iii) the following Itô formula holds true:

$$
\begin{align*}
    &\|u(t)\|^2_{\mathbb{H}} + 2\mu \int_0^t \|u(s)\|^2_{\mathcal{V}}ds + 2\beta \int_0^t \|u(s)\|^{r+1}_{\tilde{\mathbb{L}}^{r+1}}ds \\
    &= \|u_0\|^2_{\mathbb{H}} + \int_0^t \|\Phi(s, u(s))\|^2_{L^2}ds + 2\int_0^t \Phi(s, u(s))dW(s), u(s)),
\end{align*}
$$

(3.4)

for all $t \in (0, T), \mathbb{P}$-a.s.

An alternative version of condition (3.3) is to require that for any $v \in \mathcal{V} \cap \tilde{\mathbb{L}}^{r+1}$:

$$
(u(t), v) = (u_0, v) - \int_0^t \langle \mu Au(s) + B(u(s)) + \beta C(u(s)), v \rangle ds \\
+ \int_0^t \langle \Phi(s, u(s))dW(s), v \rangle, \mathbb{P}$$-

a.s.

(3.5)
Definition 3.4. A strong solution $u(\cdot)$ to (3.2) is called a pathwise unique strong solution if $\tilde{u}(\cdot)$ is an another strong solution, then

$$\mathbb{P}\left\{ \omega \in \Omega : u(t) = \tilde{u}(t), \text{ for all } t \in [0, T] \right\} = 1.$$  

3.3. Energy estimates. In this subsection, we formulate a finite dimensional system and establish some a-priori energy estimates. Let $\{e_1, \ldots, e_n, \ldots\}$ be a complete orthonormal system in $\mathbb{H}$ belonging to $V$ and let $\mathbb{H}_n$ be the span$\{e_1, \ldots, e_n\}$. Let $P_n$ denote the orthogonal projection of $V'$ to $\mathbb{H}_n$, that is, $P_n x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$. Since every element $x \in \mathbb{H}$ induces a functional $x^* \in \mathbb{H}$ by the formula $\langle x^*, y \rangle = \langle x, y \rangle$, $y \in V$, then $P_n \|x\|_V$ the orthogonal projection of $\mathbb{H}$ onto $\mathbb{H}_n$ is given by $P_n x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$. Hence in particular, $P_n$ is the orthogonal projection from $\mathbb{H}$ onto span$\{e_1, \ldots, e_n\}$. We define $B^n(u^n) = P_n B(u^n), C^n(u^n) = P_n C(u^n)$, and $\Phi^n(\cdot, u^n) = P_n \Phi(\cdot, u^n)P_n$. We consider the following system of ODEs:

$$\begin{cases}
\text{d}(u^n(t), v) = -\langle \mu A u^n(t) + B^n(u^n(t)) + \beta C^n(u^n(t)), v \rangle \text{d}t + \langle \Phi^n(t, u^n(t))dW^n(t), v \rangle, \\
u^n(0) = u^n_0,
\end{cases}$$

with $u^n_0 = P_n u_0$, for all $v \in \mathbb{H}_n$. Since $B^n(\cdot)$ and $C^n(\cdot)$ are locally Lipschitz (see (2.8) and (2.10)), and $\Phi^n(\cdot, \cdot)$ is globally Lipschitz, the system (3.6) has a unique $\mathbb{H}_n$-valued local strong solution $u^n(\cdot)$ and $u^n \in L^2(\Omega; L^\infty(0, T; \mathbb{H}_n))$ with $F_t$-adapted continuous sample paths. Now we discuss about the a-priori energy estimates satisfied by the system (3.6). Note that the energy estimates established in the next proposition is true for $r \geq 1$.

Proposition 3.5 (Energy estimates). Let $u^n(\cdot)$ be the unique solution of the system of stochastic ODE’s (3.6) with $u_0 \in L^2(\Omega, \mathbb{H})$. Then, we have

$$\mathbb{E}\left[ \sup_{t \in [0, T]} \|u^n(t)\|_V^2 + 4\mu \int_0^T \|u^n(t)\|_V^2 \text{d}t + 4\beta \int_0^T \|u^n(t)\|_{L^{r+1}} \text{d}t \right] \leq (2\mathbb{E}[\|u_0\|_V^2] + 14KT)e^{2SKT}. \quad (3.7)$$

Proof. Step (1): Let us first define a sequence of stopping times $\tau_N$ by

$$\tau_N := \inf\{t : \|u^n(t)\|_\mathbb{H} \geq N\}, \quad (3.8)$$

for $N \in \mathbb{N}$. Applying the finite dimensional Itô formula to the process $\|u^n(\cdot)\|_\mathbb{H},$ we obtain

$$\|u^n(t \wedge \tau_N)\|_\mathbb{H}^2 = \|u^n(0)\|_\mathbb{H}^2 - 2 \int_0^{t \wedge \tau_N} \langle \mu A u^n(s) + B^n(u^n(s)) + \beta C^n(u^n(s)), u^n(s) \rangle \text{d}s$$

$$+ \int_0^{t \wedge \tau_N} \|\Phi^n(s, u^n(s))\|_{L^2_Q}^2 \text{d}s + 2 \int_0^{t \wedge \tau_N} \langle \Phi^n(s, u^n(s))dW^n(s), u^n(s) \rangle. \quad (3.9)$$

Note that $\langle B^n(u^n), u^n \rangle = \langle B(u^n), u^n \rangle = 0$. Taking expectation in (3.9), and using the fact that final term in the right hand side of the equality (3.9) is a martingale with zero expectation, we find

$$\mathbb{E}\left[ \|u^n(t \wedge \tau_N)\|_\mathbb{H}^2 \right] \leq 2\mu \int_0^{t \wedge \tau_N} \|u^n(s)\|_V^2 \text{d}s + 2\beta \int_0^{t \wedge \tau_N} \|u^n(s)\|_{L^{r+1}} \text{d}s$$

$$\leq \mathbb{E}[\|u^n(0)\|_\mathbb{H}^2] + \mathbb{E}\left[ \int_0^{t \wedge \tau_N} \|\Phi^n(s, u^n(s))\|_{L^2_Q}^2 \text{d}s \right].$$
where we used the Hypothesis 3.2 (H.2). Applying Gronwall’s inequality in (3.10), we get
\[
\mathbb{E}[\|u^n(t \wedge \tau_N)\|_{H}^2] \leq (\mathbb{E}[\|u_0\|_{H}^2] + KT)e^{KT},
\]
for all \( t \in [0, T] \). Note that for the indicator function \( \chi \),
\[
\mathbb{E}[\chi_{\{\tau_N < t\}}] = \mathbb{P}\{ \omega \in \Omega : \tau_N^\prime(\omega) < t \},
\]
and using (3.8), we obtain
\[
\mathbb{E}[\|u^n(t \wedge \tau_N)\|_{H}^2] = \mathbb{E}[\|u^n(\tau_N^\prime)\|_{H}^2 \chi_{\{\tau_N^\prime < t\}}] + \mathbb{E}[\|u^n(t)\|_{H}^2 \chi_{\{\tau_N^\prime \geq t\}}] \\
\geq \mathbb{E}[\|u^n(\tau_N^\prime)\|_{H}^2 \chi_{\{\tau_N^\prime < t\}}] \geq N^2 \mathbb{P}\{ \omega \in \Omega : \tau_N^\prime < t \}.
\]
Using the energy estimate (3.11), we find
\[
\mathbb{P}\{ \omega \in \Omega : \tau_N < t \} \leq \frac{1}{N^2} \mathbb{E}[\|u^n(t \wedge \tau_N)\|_{H}^2] \leq \frac{1}{N^2} (\mathbb{E}[\|u_0\|_{H}^2] + KT)e^{KT}.
\]
Hence, we have
\[
\lim_{N \to \infty} \mathbb{P}\{ \omega \in \Omega : \tau_N < t \} = 0, \quad \text{for all } t \in [0, T],
\]
and \( t \wedge \tau_N \to t \) as \( N \to \infty \). Taking limit \( N \to \infty \) in (3.11) and using the monotone convergence theorem, we get
\[
\mathbb{E}[\|u^n(t)\|_{H}^2] \leq (\mathbb{E}[\|u_0\|_{H}^2] + KT)e^{KT},
\]
for \( 0 \leq t \leq T \). Substituting (3.15) in (3.10), we arrive at
\[
\mathbb{E}\left[ \|u^n(t)\|_{H}^2 + 2\mu \int_0^t \|u^n(s)\|_{H}^2 ds + 2\beta \int_0^t \|u^n(s)\|_{L_r+1}^{r+1} ds \right] \leq (\mathbb{E}[\|u_0\|_{H}^2] + KT)e^{2KT},
\]
for all \( t \in [0, T] \).

**Step (2):** Let us now prove (3.7). Taking supremum from 0 to \( T \wedge \tau_N \) before taking expectation in (3.9), we obtain
\[
\mathbb{E}\left[ \sup_{t \in [0, T \wedge \tau_N]} \|u^n(t)\|_{H}^2 + 2\mu \int_0^{T \wedge \tau_N} \|u^n(t)\|_{H}^2 dt + 2\beta \int_0^{T \wedge \tau_N} \|u^n(t)\|_{L_r+1}^{r+1} dt \right] \\
\leq \mathbb{E}[\|u_0\|_{H}^2] + \mathbb{E}\left[ \int_0^{T \wedge \tau_N} \|\Phi^n(t, u^n(t))\|_{L_2}^2 dt \right] \\
+ 2\mathbb{E}\left[ \sup_{t \in [0, T \wedge \tau_N]} \left| \int_0^t (\Phi^n(s, u^n(s)) dW^n(s), u^n(s)) \right| \right].
\]
Now we take the final term from the right hand side of the inequality (3.17) and use Burkholder-Davis-Gundy (see Theorem 1, [16] for the Burkholder-Davis-Gundy inequality and Theorem 1.1, [6] for the best constant), H"older and Young’s inequalities to deduce that
\[
2\mathbb{E}\left[ \sup_{t \in [0, T \wedge \tau_N]} \left| \int_0^t (\Phi^n(s, u^n(s)) dW^n(s), u^n(s)) \right| \right]
\]
Substituting (3.18) in (3.17), we find
\[
\leq 2\sqrt{3}\mathbb{E}\left[ \int_0^{T^\wedge \tau_N} \|\Phi^n(t, u^n(t))\|_{L^q}^2 \|u^n(t)\|_{H^2}^2 dt \right]^{1/2}
\]
\[
\leq 2\sqrt{3}\mathbb{E}\left[ \sup_{t \in [0, T^\wedge \tau_N]} \|u^n(t)\|_{H^2} \left( \int_0^{T^\wedge \tau_N} \|\Phi^n(t, u^n(t))\|_{L^q}^2 dt \right)^{1/2} \right]
\]
\[
\leq \frac{1}{2}\mathbb{E}\left[ \sup_{t \in [0, T^\wedge \tau_N]} \|u^n(t)\|_{H^2}^2 \right] + 6\mathbb{E}\left[ \int_0^{T^\wedge \tau_N} \|\Phi^n(t, u^n(t))\|_{L^q}^2 dt \right].
\]
(3.18)

Substituting (3.18) in (3.17), we find
\[
\mathbb{E}\left[ \sup_{t \in [0, T^\wedge \tau_N]} \|u^n(t)\|_{H^2}^2 \right] + 4\mu \int_0^{T^\wedge \tau_N} \|u^n(t)\|_r^2 dt + 4\beta \int_0^T \|u^n(t)\|_{E^r+1}^2 dt 
\]
\[
\leq 2\mathbb{E}[\|u_0\|_{H^2}^2] + 14K \int_0^{T^\wedge \tau_N} \mathbb{E}[1 + \|u^n(t)\|_{H^2}^2] dt,
\]
(3.19)

where we used the Hypothesis 3.2 (H.2). Applying Gronwall’s inequality in (3.19), we obtain
\[
\mathbb{E}\left[ \sup_{t \in [0, T^\wedge \tau_N]} \|u^n(t)\|_{H^2}^2 \right] \leq (2\|u_0\|_{H^2}^2 + 14KT) e^{14KT}.
\]
(3.20)

Passing \(N \to \infty\), using the monotone convergence theorem and then substituting (3.20) in (3.19), we finally obtain (3.17).

\[\square\]

**Lemma 3.6.** For \(r > 3\) and any functions
\[
\mathbf{u}, \mathbf{v} \in L^2(\Omega; L^\infty(0, T; H) \cap L^2(0, T; V)) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{L}^{r+1})),
\]
we have
\[
\int_0^T e^{-\eta t} \left[ \mu \langle A(\mathbf{u}(t) - \mathbf{v}(t)), \mathbf{u}(t) - \mathbf{v}(t) \rangle + \langle B(\mathbf{u}(t)) - B(\mathbf{v}(t)), \mathbf{u}(t) - \mathbf{v}(t) \rangle 
\right.
\]
\[
+ \beta \langle \mathcal{C}(\mathbf{u}(t)) - \mathcal{C}(\mathbf{v}(t)), \mathbf{u}(t) - \mathbf{v}(t) \rangle \bigg] dt + \left( \eta + \frac{L}{2} \right) \int_0^T e^{-\eta t} \|\mathbf{u}(t) - \mathbf{v}(t)\|_{H^2}^2 dt
\]
\[
\geq \frac{1}{2} \int_0^T e^{-\eta \eta t} \|\Phi(t, \mathbf{u}(t)) - \Phi(t, \mathbf{v}(t))\|_{L^q}^2 dt,
\]
(3.21)

where \(\eta\) is defined in (2.15).

**Proof.** Multiplying (2.14) with \(e^{-\eta t}\), integrating over time \(t \in (0, T)\), and using the Hypothesis 3.2 (H.3), we obtain (3.21).

\[\square\]

3.4. **Existence and uniqueness of strong solution.** Our next aim is to show that the system (3.2) has a unique global strong solution by exploiting the monotonicity property (see (3.21)) and a stochastic generalization of the Minty-Browder technique. The local monotonicity property of the linear and nonlinear operators and a stochastic generalization of the Minty-Browder technique has been used to obtain solvability results in the works [35, 47, 12, 37, 38], etc.
Theorem 3.7. Let \( \mathbf{u}_0 \in L^2(\Omega; \mathbb{H}) \), for \( r > 3 \) be given. Then there exists a pathwise unique strong solution \( \mathbf{u}(\cdot) \) to the system (3.2) such that

\[
\mathbf{u} \in L^2(\Omega; \mathbb{L}^{r+1}(0, T; \mathbb{L}^{r+1})),
\]

with \( \mathbb{P} \)-a.s., continuous trajectories in \( \mathbb{H} \).

Proof. The global solvability results of the system (3.2) is divided into the following steps.

**Step (1):** Finite-dimensional (Galerkin) approximation of the system (3.2): Let us first consider the following Galerkin approximated Itô stochastic differential equation satisfied by \( \{\mathbf{u}^n(\cdot)\} \):

\[
\left\{
\begin{array}{l}
d\mathbf{u}^n(t) = -G(\mathbf{u}^n(t))dt + \Phi^n(t, u^n(t))dW^n(t), \\
\mathbf{u}^n(0) = \mathbf{u}_0^n,
\end{array}
\right.
\]

where \( G(\mathbf{u}^n(\cdot)) = \mu \mathbf{u}^n(\cdot) + B^n(\mathbf{u}^n(\cdot)) + \beta C^n(\mathbf{u}^n(\cdot)) \). Applying Itô’s formula to the finite dimensional process \( e^{-2nt}||\mathbf{u}^n(\cdot)||_2^2 \), we obtain

\[
e^{-2nt}||\mathbf{u}^n(t)||_2^2 = ||\mathbf{u}^n(0)||_2^2 - \int_0^t e^{-2ns}(2G(\mathbf{u}^n(s)) + 2\eta \mathbf{u}^n(s), \mathbf{u}^n(s))ds
\]

\[
+ \int_0^t e^{-2ns}\|\Phi^n(s, \mathbf{u}^n(s))\|_{L^2_Q}^2 ds + 2 \int_0^t e^{-2ns}(\Phi^n(s, \mathbf{u}^n(s))dW^n(s), \mathbf{u}^n(s)),
\]

for all \( t \in [0, T] \). Note that the final term from the right hand side of the equality (3.23) is a martingale and on taking expectation, we get

\[
\mathbb{E}[e^{-2nt}||\mathbf{u}^n(t)||_2^2] = \mathbb{E}[||\mathbf{u}^n(0)||_2^2] - \mathbb{E}\left[\int_0^t e^{-2ns}(2G(\mathbf{u}^n(s)) + 2\eta \mathbf{u}^n(s), \mathbf{u}^n(s))ds\right]
\]

\[
+ \mathbb{E}\left[\int_0^t e^{-2ns}\|\Phi^n(s, \mathbf{u}^n(s))\|_{L^2_Q}^2 ds\right],
\]

for all \( t \in [0, T] \).

**Step (2):** Weak convergence of the sequences \( \mathbf{u}^n(\cdot) \), \( G(\mathbf{u}^n(\cdot)) \) and \( \Phi^n(\cdot, \cdot) \). We know that \( L^2(\Omega; L^\infty(0, T; \mathbb{H})) \cong (L^2(\Omega; L^1(0, T; \mathbb{H}))') \) and the space \( L^2(\Omega; L^1(0, T; \mathbb{H})) \) is separable. Moreover, the spaces \( L^2(\Omega; L^2(0, T; \mathbb{V})) \) and \( L^{r+1}(\Omega; L^{r+1}(0, T; \mathbb{L}^{r+1})) \) are reflexive \( (X'' = X) \). From the energy estimate (3.7) given in Proposition 3.5, we know that the sequence \( \{\mathbf{u}^n(\cdot)\} \) is bounded independent of \( n \) in the spaces \( L^2(\Omega; L^\infty(0, T; \mathbb{H})) \), \( L^2(\Omega; L^2(0, T; \mathbb{V})) \) and \( L^{r+1}(\Omega; L^{r+1}(0, T; \mathbb{L}^{r+1})) \). Then applying the Banach-Alaoglu theorem, we can extract a subsequence \( \{\mathbf{u}^{n_k}\} \) of \( \{\mathbf{u}^n\} \) such that (for simplicity, we denote the index \( n_k \) by \( n \)):

\[
\left\{
\begin{array}{l}
\mathbf{u}^n \xrightarrow{w^*} \mathbf{u} \text{ in } L^2(\Omega; L^\infty(0, T; \mathbb{H})), \\
\mathbf{u}^n \xrightarrow{w} \mathbf{u} \text{ in } L^2(\Omega; L^2(0, T; \mathbb{V})), \\
\mathbf{u}^n \xrightarrow{w} \mathbf{u} \text{ in } L^{r+1}(\Omega; L^{r+1}(0, T; \mathbb{L}^{r+1})), \\
G(\mathbf{u}^n) \xrightarrow{w} G_0 \text{ in } L^2(\Omega; L^2(0, T; \mathbb{V}')) + L^\infty(\Omega; L^\infty(0, T; \mathbb{L}^{r+1})),
\end{array}
\right.
\]

(3.25)
Using Hölder’s inequality, interpolation inequality (see (2.3)) and Proposition 3.5, we justify the final convergence in (3.5) in the following way:

\[
\mathbb{E} \left[ \left| \int_0^T \langle G(u^n(t)), v(t) \rangle dt \right| \right] \\
\leq \mu \mathbb{E} \left[ \left| \int_0^T (\nabla u^n(t), \nabla v(t)) dt \right| \right] + \mathbb{E} \left[ \left| \int_0^T \langle B(u^n(t), v(t)), u^n(t) \rangle dt \right| \right] \\
+ \beta \mathbb{E} \left[ \left| \int_0^T |u^n(t)|^{-1} u^n(t), v(t) \rangle dt \right| \right] \\
\leq \mu \mathbb{E} \left[ \left( \int_0^T \|\nabla u^n(t)\|_\mathbb{H} \right) \left( \int_0^T \|\nabla v(t)\|_\mathbb{H} dt \right) \right] + \mathbb{E} \left[ \left( \int_0^T \|u^n(t)\|_{L^{r+1}} \|v^n(t)\|_{L^{2(r+1)}} \|v(t)\|_{V} dt \right) \right] \\
+ \beta \mathbb{E} \left[ \left( \int_0^T \|u^n(t)\|_{L^{r+1}} \|v^n(t)\|_{L^{2(r+1)}} \|v(t)\|_{V} dt \right) \right] \\
\leq \mu \left\{ \mathbb{E} \left( \int_0^T \|u^n(t)\|_{V}^2 dt \right) \right\}^{1/2} \left\{ \mathbb{E} \left( \int_0^T \|v(t)\|_V^2 dt \right) \right\}^{1/2} \\
+ T^{r-3} \left\{ \mathbb{E} \left( \int_0^T \|u^n(t)\|_{L^{r+1}}^2 dt \right) \right\}^{1/2} \left\{ \mathbb{E} \left( \sup_{t \in [0,T]} \|u^n(t)\|_{\mathbb{H}}^2 \right) \right\}^{1/2} \left\{ \mathbb{E} \left( \int_0^T \|v(t)\|_V^2 dt \right) \right\}^{1/2} \\
+ \beta \left\{ \mathbb{E} \left( \int_0^T \|u^n(t)\|_{L^{r+1}}^2 dt \right) \right\}^{1/2} \left\{ \mathbb{E} \left( \int_0^T \|v(t)\|_{L^{r+1}}^2 dt \right) \right\}^{1/2} \\
\leq C(\mathbb{E}[\|u_0\|_{\mathbb{H}}^2], \mu, T, \beta, K) \left\{ \mathbb{E} \left( \int_0^T \|v(t)\|_V^2 dt \right) \right\}^{1/2} + \mathbb{E} \left( \int_0^T \|v(t)\|_{L^{r+1}}^2 dt \right) \right\}^{1/2},
\]

(3.26)

for all \( v \in L^2(\Omega; L^2(0, T; V)) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{L}^{r+1})) \). Using the Hypothesis 3.2 (H.2) and energy estimates in Proposition 3.5, we also have

\[
\mathbb{E} \left[ \int_0^T \|\Phi^n(t, u^n(t))\|_{L^2_Q} dt \right] \leq K \mathbb{E} \left[ \int_0^T (1 + \|u^n(t)\|_{\mathbb{H}}^2) dt \right] \\
\leq KT (1 + (2\mathbb{E}[\|u_0\|_{\mathbb{H}}^2] + 14KT) e^{26KT}) < +\infty.
\]

(3.27)

The right hand side of the estimate (3.27) is independent of \( n \) and thus we can extract a subsequence \( \{\Phi_{n_k}(\cdot, u_{n_k})\} \) of \( \{\Phi^n(\cdot, u^n)\} \) such that (reabeled as \( \{\Phi^n(\cdot, u^n)\}) \)

\[
\Phi^n(\cdot, u^n) \overset{w}{\rightharpoonup} \Phi(\cdot) \quad \text{in} \quad L^2(\Omega; L^2(0, T; \mathcal{L}_Q(\mathbb{H}))).
\]

(3.28)
Step (3): Itô stochastic differential satisfied by $u(\cdot)$. We make use of the discussions in Theorem 7.5 \[11\], to obtain the Itô stochastic differential satisfied by $u(\cdot)$. Note that we cannot apply the results available in Theorem 7.5 \[11\] here directly as we have the weak convergence of the nonlinear term in $L^2(\Omega; L^2(0, T; U')) + L^{2+1}(\Omega; L^{2+1}(0, T; \tilde{L}^{2+1}))$. Due to technical reasons, we extend the time interval from $[0, T]$ to an open interval $(-\varepsilon, T + \varepsilon)$ with $\varepsilon > 0$, and set the terms in the equation (3.22) equal to zero outside the interval $[0, T]$. Let $\phi \in H^1(-\mu, T + \mu)$ be such that $\phi(0) = 1$. For $v_m \in \mathcal{V} \cap \tilde{L}^{r+1}$, we define $v_m(t) = \phi(t)v_m$. Applying Itô’s formula to the process $(u^n(t), v_m(t))$, one obtains

$$
(u^n(T), v_m(T)) = (u^n(0), v_m) + \int_0^T (u^n(t), \dot{v}_m(t))dt - \int_0^T (G(u^n(t)), v_m(t))dt
$$

$$
+ \int_0^T (\Phi^n(t, u^n(t))dW^n(t), v_m(t)),
$$

(3.29)

where $\dot{v}_m(t) = \frac{d\phi(t)}{dt}v_m$. Note that, we can take the term by term limit $n \to \infty$ in (3.29) by making use of the weak convergences given in (3.25) and (3.28). For instance, we consider the stochastic integral present in the equality (3.29), with $d\phi(t)v_m$. Passing to limits term wise in the equation (3.29), we get

$$
(\Phi^n(\cdot, u^n)P_n)_{\mathcal{P}_T} \to (\Phi(\cdot)Q\Psi(t))_{\mathcal{P}_T}, \text{ for all } \Phi, \Psi \in \mathcal{P}_T.
$$

Let us define a map $v : \mathcal{P}_T \to L^2(\Omega; L^2(0, T))$ by

$$
v(\Phi) = \int_0^T (\Phi(s)dW(s), v_m(s)),
$$

for all $t \in [0, T]$. Note that the map $v$ is linear and continuous. The weak convergence of $\Phi^n(\cdot, u^n)P_n \Rightarrow \Phi(\cdot)$ in $L^2(\Omega; L^2(0, T; \mathcal{L}_Q(\mathbb{H})))$ (see (3.28)) implies that $(\Phi^n(t, u^n(t))P_n, \zeta)_{\mathcal{P}_T} \to (\Phi(t)dW(t), \zeta)_{\mathcal{P}_T}$, for all $\zeta \in \mathcal{P}_T$ as $n \to \infty$. From this, as $n \to \infty$, we conclude that

$$
v(\Phi^n(t, u^n(t))) = \int_0^t (\Phi^n(t, u^n(t))dW(s), v_m(s)) \to \int_0^t (\Phi(t)dW(s), v_m(s)),
$$

for all $t \in [0, T]$ and for each $m$. Passing to limits term wise in the equation (3.29), we get

$$
(\xi, v_m)\phi(T) = (u_0, v_m) + \int_0^T (u(t), \dot{v}_k)dt - \int_0^T \phi(t)(G_0(t), v_m)dt + \int_0^T \phi(t)(\Phi(t)dW(t), v_m).
$$

(3.30)

Let us now choose a subsequence $\{\phi_k\} \subset H^1(-\mu, T + \mu)$ with $\phi_k(0) = 1$, for $k \in \mathbb{N}$, such that $\phi_k \to \chi_t$ and the time derivative of $\phi_k$ converges to $\delta_t$, where $\chi_t(s) = 1$, for $s \leq t$ and 0 otherwise, and $\delta_t(s) = \delta(t - s)$ is the Dirac $\delta$-distribution. Using $\phi_k$ in place of $\phi$ in (3.30) and then letting $k \to \infty$, we obtain

$$
(u(t), v_m) = (u_0, v_m) - \int_0^t \langle G_0(s), v_m \rangle ds + \int_0^t (\Phi(s)dW(s), v_m),
$$

(3.31)

for all $0 < t < T$ with $(u(T), v_m) = (\xi, v_m)$ and for any $v_m \in \mathcal{V} \cap \tilde{L}^{r+1}$. Remember that $\mathcal{V} \subset \mathcal{V} \cap \tilde{L}^{r+1} \subset \mathbb{H}$ and $\mathcal{V}$ is dense in $\mathbb{H}$. Therefore, $\mathcal{V} \cap \tilde{L}^{r+1}$ is dense in $\mathbb{H}$ and the above
equation holds for any $v \in V \cap \mathbb{L}^{r+1}$. Thus, we have

$$
(u(t), v) = (u_0, v) - \int_0^t \langle G_0(s), v \rangle ds + \int_0^t \langle \Phi(s) dW(s), v \rangle, \ \mathbb{P}\text{-a.s.},
$$

(3.32)

for all $0 < t < T$ with $(u(T), v) = (\xi, v)$, for all $v \in V \cap \mathbb{L}^{r+1}$. Hence, $u(\cdot)$ satisfies the following stochastic differential:

$$
\begin{cases}
    du(t) = -G_0(t) dt + \Phi(t) dW(t), \\
    u(0) = u_0,
\end{cases}
$$

(3.33)

for $u_0 \in L^2(\Omega; \mathbb{H})$.

**Step (4): Energy equality satisfied by $u(\cdot)$.** Let us now establish the energy equality satisfied by $u(\cdot)$. Note that such an energy equality is not immediate due to the final convergence in (3.25) and we cannot apply the infinite dimensional Itô formula available in the literature for semimartingales (see Theorem 1, [22], Theorem 6.1, [34]). We follow [19] to obtain an approximation of $u(\cdot)$ in bounded domains such that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously (one can see [24] for such an approximation in periodic domains). We approximate $u(t)$, for each $t \in (0, T)$ and $\mathbb{P}$-a.s. by using the finite-dimensional space spanned by the first $n$ eigenfunctions of the Stokes operator as (Theorem 4.3, [19])

$$
u_n(t) := P_{1/n} u(t) = \sum_{\lambda_j < n^2} e^{-\lambda_j/n} \langle u(t), e_j \rangle e_j.
$$

(3.34)

For notational convenience, we use $u_n$ for approximations of the type (3.34) and $u^n$ for Galerkin approximations. Note first that

$$
\|u_n\|^2_H = \|P_{1/n} u\|^2_H = \sum_{\lambda_j < n^2} e^{-2\lambda_j/n} \langle u, e_j \rangle^2 \leq \sum_{j=1}^{\infty} \langle u, e_j \rangle^2 = \|u\|^2_H < +\infty,
$$

(3.35)

for all $u \in H$. It can also be seen that

$$
\|(I - P_{1/n}) u\|^2_H = \|u\|^2_H - 2\langle u, P_{1/n} u \rangle + \|P_{1/n} u\|^2_H
$$

$$
= \sum_{j=1}^{\infty} \langle u, e_j \rangle^2 - 2 \sum_{\lambda_j < n^2} e^{-\lambda_j/n} \langle u, e_j \rangle^2 + \sum_{\lambda_j < n^2} e^{-2\lambda_j/n} \langle u, e_j \rangle^2
$$

$$
= \sum_{\lambda_j < n^2} (1 - e^{-\lambda_j/n})^2 \langle u, e_j \rangle^2 + \sum_{\lambda_j \geq n^2} \langle u, e_j \rangle^2,
$$

(3.36)

for all $u \in H$. Note that the final term in the right hand side of the equality (3.36) tends zero as $n \to \infty$, since the series $\sum_{j=1}^{\infty} \langle u, e_j \rangle^2$ is convergent. The first term on the right hand side of the equality can be bounded by

$$
\sum_{j=1}^{\infty} (1 - e^{-\lambda_j/n})^2 \langle u, e_j \rangle^2 \leq 4 \sum_{j=1}^{\infty} \langle u, e_j \rangle^2 = 4\|u\|^2_H < +\infty.
$$

By using dominated convergence theorem, we can interchange the limit and sum and hence we get

$$
\lim_{n \to \infty} \sum_{j=1}^{\infty} (1 - e^{-\lambda_j/n})^2 \langle u, e_j \rangle^2 = \sum_{j=1}^{\infty} \lim_{n \to \infty} (1 - e^{-\lambda_j/n})^2 \langle u, e_j \rangle^2 = 0.
$$
Hence \( \|(I - P_{1/n})u\|_{\mathbb{H}} \to 0 \) as \( n \to \infty \) and \( \|I - P_{1/n}\|_{L(\mathbb{H})} \to 0 \) as \( n \to \infty \). Moreover, for \( u \in \mathbb{V}' \), we have
\[
\|(I - P_{1/n})u\|_{\mathbb{V}'}^2 = \|(I - P_{1/n})A^{-1/2}u\|_{\mathbb{H}}^2 \leq \|I - P_{1/n}\|_{L(\mathbb{H})}^2 \|u\|_{\mathbb{V}'}^2 \to 0 \quad \text{as} \quad n \to \infty.
\]

The authors in [19] showed that such an approximation satisfies:

1. \( u_n(t) \to u(t) \) in \( \mathbb{H}^1 \) with \( \|u_n(t)\|_{\mathbb{H}^1} \leq C\|u(t)\|_{\mathbb{H}^1}, \) \( \mathbb{P}\)-a.s. and a.e. \( t \in [0, T] \),
2. \( u_n(t) \to u(t) \) in \( L^p(O) \) with \( \|u_n(t)\|_{L^p} \leq C\|u(t)\|_{L^p}, \) for any \( p \in (1, \infty), \) \( \mathbb{P}\)-a.s. and a.e. \( t \in [0, T] \),
3. \( u_n(t) \) is divergence free and zero on \( \partial O, \) \( \mathbb{P}\)-a.s. and a.e. \( t \in [0, T] \).

In (1) and (2), \( C \) is an absolute constant. Note that for \( n \leq 4, \) \( D(A) \subset \mathbb{H}^2 \subset L^p, \) for all \( p \in (1, \infty) \) (cf. [19]). Since \( e_j \)'s are the eigenfunctions of the Stokes’ operator \( A, \) we get \( e_j \in D(A) \subset \mathbb{H}^{r+1}. \) Taking \( v = e_j \) in (3.32), multiplying by \( e^{-\lambda_j/n}e_j \) and then summing over all \( j \) such that \( \lambda_j < n^2, \) we see that \( u_n(\cdot) \) satisfies the following Itô stochastic differential:
\[
u_n(t) = u_{0n} - \int_0^t G_{0n}(s)ds + \int_0^t \Phi_n(s)dW(s), \tag{3.38}
\]
where \( u_{0n} = P_{1/n}u_0, \) \( G_{0n} = P_{1/n}G_0 \) and \( \Phi_n = P_{1/n}\Phi. \) It is clear that the equation (3.38) has a unique solution \( u_n(\cdot) \) (see [14]). One can apply Itô’s formula to the process \( \|u_n(\cdot)\|_{\mathbb{H}}^2 \)

\[
\|u_n(t)\|_{\mathbb{H}}^2 = \|u_{0n}\|_{\mathbb{H}}^2 - 2\int_0^t (G_{0n}(s), u_n(s))ds + 2\int_0^t (\Phi_n(s)dW(s), u_n(s)) \tag{3.39}
\]

for all \( t \in (0, T). \) With the convergence given in (3.36), one can show that
\[
\|\Phi_n - \Phi\|_{L_\mathbb{Q}}^2 = \sum_{j=1}^{\infty} \|\Phi_n - \Phi\|_{\mathbb{H}}^2 = \sum_{j=1}^{\infty} (P_{1/n} - I)\Phi e_j\|_{\mathbb{H}}^2 \leq \|I - P_{1/n}\|_{L(\mathbb{H})}^2 \|\Phi\|_{L_\mathbb{Q}}^2 \to 0, \tag{3.40}
\]
as \( n \to \infty. \) Thus \( \|\Phi_n(t) - \Phi(t)\|_{L_\mathbb{Q}}^2 \to 0, \) for a.e. \( t \in (0, T) \) and \( \mathbb{P}\)-a.s. Moreover, we have
\[
\|\Phi_n\|_{L_\mathbb{Q}}^2 = \sum_{j=1}^{\infty} \|\Phi_n e_j\|_{\mathbb{H}}^2 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} e^{-2\lambda_j/k} |(\Phi e_j, e_k)|^2 \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |(\Phi e_j, e_k)|^2
\]
\[
= \sum_{j=1}^{\infty} \|\Phi e_j\|_{\mathbb{H}}^2 = \|\Phi\|_{L_\mathbb{Q}}^2. \tag{3.41}
\]

Once again, using the convergence given in (3.36), we get
\[
E\left[ \sup_{t \in [0, T]} \|u(t) - u_n(t)\|_{\mathbb{H}}^2 \right] \leq \|I - P_{1/n}\|_{L(\mathbb{H})}^2 E\left[ \sup_{t \in [0, T]} \|u(t)\|_{\mathbb{H}}^2 \right] \to 0, \tag{3.42}
\]
as \( n \to \infty, \) since \( u \in L^2(\Omega; L^\infty(0, T; \mathbb{H})). \) Thus, it is immediate that
\[
E\left[ \|u_n(t)\|_{\mathbb{H}}^2 \right] \to E\left[ \|u(t)\|_{\mathbb{H}}^2 \right], \quad \text{as} \quad n \to \infty, \tag{3.43}
\]
which follows from (2). Since \( u \in L^{r+1}(\Omega; L^{r+1}(0, T; L^{r+1})) \) and the fact that \( \|u^n(t, \omega) - u(t, \omega)\|_{L^{r+1}} \to 0 \), for a.e. \( t \in [0, T] \) and \( \mathbb{P} \)-a.s., one can obtain the above convergence by an application of the dominated convergence theorem (with the dominating function \((1 + C)\|u(t, \omega)\|_{L^{r+1}}\)).

Since \( G_0 \in L^2(\Omega; L^2(0, T; \mathcal{V}')) + L^{r+1}(\Omega; L^{r+1}(0, T; L^{r+1})) \), we can write down \( G_0 = G^1_0 + G^2_0 \), where \( G^1_0 \in L^2(\Omega; L^2(0, T; \mathcal{V}')) \) and \( G^2_0 \in L^{r+1}(\Omega; L^{r+1}(0, T; L^{r+1})) \). Note that \( 1 < \frac{r+1}{r} < \frac{4}{3} \). We use the approximation

\[
G^1_{0,n}(t) := P_{1/n} G^1_0(t) = \sum_{\lambda_j < n^2} e^{-\lambda_j/n} \langle G^1_0(t), e_j \rangle e_j,
\]

and by using (3.37), we get

\[
\mathbb{E} \left[ \int_0^T \|G^1_{0,n}(t) - G^1_0(t)\|_{\mathcal{V}'}^2 dt \right] \to 0, \quad \text{as} \quad n \to \infty.
\] (3.46)

Similarly, for \( G^2_0 \), we use the approximation

\[
G^2_{0,n}(t) := P_{1/n} G^2_0(t) = \sum_{\lambda_j < n^2} e^{-\lambda_j/n} \langle G^2_0(t), e_j \rangle e_j,
\]

and by using (2), we get

\[
\mathbb{E} \left[ \int_0^T \|G^2_{0,n}(t) - G^2_0(t)\|_{L^{r+1} \to L^{r+1}} dt \right] \to 0, \quad \text{as} \quad n \to \infty.
\] (3.47)

By defining \( G_{0,n} = G^1_{0,n} + G^2_{0,n} \), one can easily see that

\[
\|G_{0,n} - G_0\|_{L^2(\Omega; L^2(0, T; \mathcal{V}')) + L^{r+1}(\Omega; L^{r+1}(0, T; L^{r+1}))} \to 0, \quad \text{as} \quad n \to \infty.
\] (3.48)

Let us now consider

\[
\mathbb{E} \left[ \int_0^t (G_{0,n}(s), u_n(s))ds - \int_0^t (G_0(s), u(s))ds \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^t (G_{0,n}(s) - G_0(s), u_n(s))ds \right] + \mathbb{E} \left[ \int_0^t (G_0(s), u_n(s) - u(s))ds \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^t \langle G^1_{0,n}(s) - G^1_0(s), u_n(s) \rangle ds \right] + \mathbb{E} \left[ \int_0^t \langle G^2_{0,n}(s) - G^2_0(s), u_n(s) \rangle ds \right]
\]

\[
+ \mathbb{E} \left[ \int_0^t \langle G^1_0(s), u_n(s) - u(s) \rangle ds \right] + \mathbb{E} \left[ \int_0^t \langle G^2_0(s), u_n(s) - u(s) \rangle ds \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^t \|G^1_{0,n}(s) - G^1_0(s)\|_{\mathcal{V}'} \|u_n(s)\|_{\mathcal{V}'} ds \right] + \mathbb{E} \left[ \int_0^t \|G^2_{0,n}(s) - G^2_0(s)\|_{L^{r+1}} \|u_n(s)\|_{L^{r+1}} ds \right]
\]

\[
+ \mathbb{E} \left[ \int_0^t \|G^1_0(s)\|_{\mathcal{V}'} \|u_n(s) - u(s)\|_{\mathcal{V}'} ds \right] + \mathbb{E} \left[ \int_0^t \|G^2_0(s)\|_{L^{r+1}} \|u_n(s) - u(s)\|_{L^{r+1}} ds \right]
\]
\[
\leq C \left[ E \left( \int_0^t \| G_{0n}(s) - G_0(s) \|_p^2 \, ds \right) \right]^{\frac{p}{p+1}} \left[ E \left( \int_0^t \| u(s) \|_q^2 \, ds \right) \right]^{\frac{1}{p+1}} \\
+ C \left[ E \left( \int_0^t \| G_{0n}(s) - G_0(s) \|_p^2 \, ds \right) \right]^{\frac{p}{p+1}} \left[ E \left( \int_0^t \| u_n(s) - u(s) \|_q^2 \, ds \right) \right]^{\frac{1}{p+1}} \\
+ \left[ E \left( \int_0^t \| G_0(s) \|_p^2 \, ds \right) \right]^{\frac{p}{p+1}} \left[ E \left( \int_0^t \| u_n(s) - u(s) \|_q^2 \, ds \right) \right]^{\frac{1}{p+1}} \rightarrow 0, \tag{3.49}
\]
as \( n \to \infty \), for all \( t \in (0, T) \), where we used (3.45) and (3.48) (as one can show the above convergence by taking supremum over time \( 0 \leq t \leq T \)). Next, we establish the convergence of the stochastic integral. In order to do this, we consider
\[
E \left[ \int_0^t (\Phi_n(s)dW(s), u_n(s)) - \int_0^t (\Phi_n(s)dW(s), u_n(s)) \right] \\
\leq E \left[ \left| \int_0^t ((\Phi_n(s) - \Phi(s))dW(s), u_n(s)) \right| \right] + E \left[ \left| \int_0^t (\Phi(s)dW(s), u_n(s) - u(s)) \right| \right]. \tag{3.50}
\]
Applying the Burkholder-Davis-Gundy inequality, we get
\[
E \left[ \sup_{t \in [0,T]} \left| \int_0^t ((\Phi_n(s) - \Phi(s))dW(s), u_n(s)) \right| \right] \\
\leq \sqrt{3} \left[ \int_0^T \| \Phi_n(s) - \Phi(s) \|_H^2 \| u_n(s) \|_H^2 \, ds \right]^{1/2} \\
\leq \sqrt{3} \left[ \sup_{t \in [0,T]} \| u(s) \|_H \left( \int_0^T \| \Phi_n(s) - \Phi(s) \|_H^2 \, ds \right)^{1/2} \right] \\
\leq \sqrt{3} \left\{ \left[ \sup_{t \in [0,T]} \| u(s) \|_H^2 \right] \right\}^{1/2} \left\{ \left[ \int_0^T \| \Phi_n(s) - \Phi(s) \|_H^2 \, ds \right] \right\}^{1/2} \\
\rightarrow 0 \quad \text{as} \quad n \to \infty, \tag{3.51}
\]
using (3.40) and (3.35) and Lebesgue’s dominated convergence theorem. Once again an application of the Burkholder-Davis-Gundy inequality yields
\[
E \left[ \sup_{t \in [0,T]} \left| \int_0^t (\Phi(s)dW(s), u_n(s) - u(s)) \right| \right] \\
\leq \sqrt{3} \left\{ \left[ \sup_{t \in [0,T]} \| u_n(s) - u(s) \|_H^2 \right] \right\}^{1/2} \left\{ \left[ \int_0^T \| \Phi(s) \|_H^2 \, ds \right] \right\}^{1/2} \rightarrow 0, \tag{3.52}
\]
as \( n \to \infty \), for all \( t \in (0, T) \), where we used (3.32). Combining (3.51) and (3.52), and then substituting it in (3.50), we obtain
\[
E \left[ \int_0^t (\Phi_n(s)dW(s), u_n(s)) - \int_0^t (\Phi_n(s)dW(s), u_n(s)) \right] \rightarrow 0. \tag{3.53}
\]
as $n \to \infty$, for all $t \in (0, T)$. Finally, we consider
\[
\mathbb{E} \left[ \int_0^t \|\Phi_n(s)\|_{L^2_Q}^2 \, ds - \int_0^t \|\Phi(s)\|_{L^2_Q}^2 \, ds \right] \\
\leq \mathbb{E} \left[ \int_0^t \|\Phi_n(s) - \Phi(s)\|_{L^2_Q} \left( \|\Phi_n(s)\|_{L^2_Q} + \|\Phi(s)\|_{L^2_Q} \right) \, ds \right] \\
\leq 2 \left\{ \mathbb{E} \left[ \int_0^t \|\Phi_n(s) - \Phi(s)\|_{L^2_Q}^2 \, ds \right] \right\}^{1/2} \left\{ \mathbb{E} \left[ \int_0^t \|\Phi(s)\|_{L^2_Q}^2 \, ds \right] \right\}^{1/2} \to 0, \tag{3.54}
\]
as $n \to \infty$, for all $t \in (0, T)$, where we used (3.40) and (3.41). Using the convergences given in (3.43), (3.44), (3.49), (3.53) and (3.54), along a subsequence one can pass to limit in (3.39) to get the energy equality:
\[
\|u(t)\|_{H^1}^2 = \|u_0\|_{H^1}^2 - 2 \int_0^t \langle G_0(s), u(s) \rangle \, ds + \int_0^t \|\Phi(s)\|_{L^2_Q}^2 \, ds + 2 \int_0^t (\Phi(s)dW(s), u(s)), \tag{3.55}
\]
for a.e. $t \in (0, T)$, $\mathbb{P}$-a.s. Note that the above Itô’s formula holds true only for a.e. $t \in [0, T]$, due to the convergence (3.43). Now, we show that the Itô formula (3.55) holds true for all $t \in [0, T]$. Let $\eta(t)$ be an even, positive, smooth function with compact support contained in the interval $(-1, 1)$, such that \( \int_{-\infty}^{\infty} \eta(s) \, ds = 1 \). Let us denote by $\eta^h$, a family of mollifiers related to the function $\eta$ as
\[
\eta^h(s) := h^{-1} \eta(s/h), \quad \text{for } h > 0.
\]
In particular, we get \( \int_0^h \eta^h(s) \, ds = \frac{1}{h} \). For any function $v \in L^p(0, T; \mathbb{X})$, $\mathbb{P}$-a.s., where $\mathbb{X}$ is a Banach space, for $p \in [1, \infty)$, we define its mollification in time $v^h(\cdot)$ as
\[
v^h(s) := (v * \eta^h)(s) = \int_0^T v(\tau) \eta^h(s - \tau) \, d\tau, \quad \text{for } h \in (0, T).
\]
From Lemma 2.5, [21], we know that this mollification has the following properties. For any $v \in L^p(0, T; \mathbb{X}), v^h \in C^k([0, T]; \mathbb{X})$, $\mathbb{P}$-a.s. for all $k \geq 0$ and
\[
\lim_{h \to 0} \|v^h - v\|_{L^p(0, T; \mathbb{X})} = 0, \quad \mathbb{P}$-a.s. \tag{3.56}
\]
Moreover, $\|v^h(t) - v(t)\|_{\mathbb{X}} \to 0$, as $h \to 0$, for a.e. $t \in [0, T]$, $\mathbb{P}$-a.s. For some time $t_1 > 0$, we set
\[
u^h(t) = \int_0^{t_1} \eta^h(t - s) u(s) \, ds =: (\eta^h * u)(t),
\]
with the parameter $h$ satisfying $0 < h < T - t_1$ and $h < t_1$, where $\eta_h$ is the even mollifier given above. Note that $\nu^h(\cdot)$ satisfies the following Itô stochastic differential:
\[
u^h(t) = \nu^h(0) + \int_0^t (\eta^h * u)(s) \, ds \tag{3.57}
\]
Applying Itô’s product formula to the process $(\nu^h(\cdot), u(\cdot))$, we obtain
\[
(u^h(t), u(t)) = (u(0), u^h(0)) - \int_0^t (u^h(s), G_0(s)) \, ds + \int_0^t (u^h(s), \Phi(s)dW(s))
\]
where \([u^h, u]_t\) is the quadratic variation between the processes \(u^h(\cdot)\) and \(u(\cdot)\). Using stochastic Fubini’s theorem ([14, Theorem 4.33]), we find

\[
[u^h, u]_t = \left[ \int_0^{t_1} \eta^h(t - s) \left( \int_0^s \Phi(\tau) dW(\tau) \right) ds, \int_0^t \Phi(\tau) dW(\tau) \right]_t
\]

and hence

\[
[u^h, u]_t = \int_0^{t_1} \left( \int_\tau^{t_1} \eta^h(t - s) ds \right) \Phi(\tau) dW(\tau), \quad \int_0^t \Phi(\tau) dW(\tau)
\]

Since the function \(\eta^h\) is even in \((-h, h)\), we obtain \(\dot{\eta}^h(r) = -\dot{\eta}^h(-r)\). Changing the order of integration, we get (see [24])

\[
\int_0^{t_1} (u(s), (\dot{\eta}^h * u)(s)) ds = \int_0^{t_1} \int_0^{t_1} \dot{\eta}^h(s - \tau)(u(s), u(\tau)) ds d\tau
\]

\[
= - \int_0^{t_1} \int_0^{t_1} \dot{\eta}^h(\tau - s)(u(s), u(\tau)) ds d\tau
\]

\[
= - \int_0^{t_1} \int_0^{t_1} \dot{\eta}^h(\tau - s)(u(s), u(\tau)) d\tau ds
\]

\[
= - \int_0^{t_1} \int_0^{t_1} \dot{\eta}^h(s - \tau)(u(\tau), u(s)) ds d\tau = 0.
\]

Thus, from (3.58), it is immediate that

\[
(u(t_1), u^h(t_1)) = (u(0), u^h(0)) - \int_0^{t_1} \langle u^h(s), G_0(s) \rangle ds + \int_0^{t_1} (u^h(s), \Phi(s) dW(s))
\]

\[
+ \int_0^{t_1} \left( \int_\tau^{t_1} \eta^h(t_1 - s) ds \right) \|\Phi(\tau)\|_{\mathcal{L}_Q}^2 d\tau.
\]

Now, we let \(h \to 0\) in (3.42). For \(G_0 = G_0^1 + G_0^2\), where \(G_0^1 \in L^2(\Omega; L^2(0, T; \mathbb{V}'))\) and \(G_0^2 \in L^{r+1}_r(\Omega; L^{t+1}_{t+1}(0, T; L^{r+1}_r))\), we consider

\[
\mathbb{E} \left[ \left| \int_0^{t_1} \langle G_0(s), u^h(s) \rangle ds - \int_0^{t_1} \langle G_0(s), u(s) \rangle ds \right| \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^{t_1} |\langle G_0^1(s), u^h(s) - u(s) \rangle| ds \right] + \mathbb{E} \left[ \int_0^{t_1} |\langle G_0^2(s), u^h(s) - u(s) \rangle| ds \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^{t_1} \|G_0^1(s)\|_{\mathbb{V}} \|u^h(s) - u(s)\|_{\mathbb{V}} ds \right] + \mathbb{E} \left[ \int_0^{t_1} \|G_0^2(s)\|_{L^{r+1}_r} \|u^h(s) - u(s)\|_{L^{r+1}_r} ds \right]
\]

\[
\leq \left\{ \mathbb{E} \left[ \int_0^{t_1} \|G_0^1(s)\|^2_{\mathbb{V}} ds \right] \right\}^{1/2} \left\{ \mathbb{E} \left[ \int_0^{t_1} \|u^h(s) - u(s)\|^2_{\mathbb{V}} ds \right] \right\}^{1/2}
\]
Using Burkholder-Davis-Gundy and Hölder's inequalities, we have
\[
\frac{d}{dt} \left( \int_0^t \|G_0^h(s)\|_{L^2_{\mu+1}}^2 \, ds \right) \leq \frac{\varepsilon}{\varepsilon + 1} \left( \int_0^t \|u^h(s) - u(s)\|_{L^2_{\mu+1}}^2 \, ds \right)^{\frac{\varepsilon}{\varepsilon + 1}} \to 0 \quad \text{as} \quad h \to 0.
\]
(3.62)

Thus, along a subsequence, we obtain
\[
\lim_{h \to 0} \int_0^{t_1} (G_0(s), u^h(s)) \, ds = \int_0^{t_1} (G_0(s), u(s)) \, ds, \ P\text{-a.s.}
\]
(3.63)

Using Burkholder-Davis-Gundy and Hölder’s inequalities, we have
\[
E \left[ \sup_{t_1 \in [0, T]} \left| \int_0^{t_1} (u^h(s), \Phi(s) \, dW(s)) - \int_0^{t_1} (u(s), \Phi(s) \, dW(s)) \right|^2 \right] \leq \sqrt{3} E \left[ \int_0^T \|u^h(s) - u(s)\|_H^2 \|\Phi(s)\|_{L^q}^2 \, ds \right]^{1/2}
\]
\[
\leq \sqrt{3} E \left[ \sup_{s \in [0, T]} \|u^h(s) - u(s)\|_H^2 \left( \int_0^T \|\Phi(s)\|_{L^q}^2 \, ds \right)^{1/2} \right]
\]
\[
\leq \sqrt{3} \left\{ E \left[ \sup_{s \in [0, T]} \|u^h(s) - u(s)\|_H^2 \right] \right\}^{1/2} \left\{ E \left[ \int_0^T \|\Phi(s)\|_{L^q}^2 \, ds \right] \right\}^{1/2} \to 0,
\]
(3.64)
as \( h \to 0 \). Thus, along subsequence, we get
\[
\lim_{h \to 0} \int_0^{t_1} (u^h(s), \Phi(s) \, dW(s)) = \int_0^{t_1} (u(s), \Phi(s) \, dW(s)), \ P\text{-a.s.}
\]
(3.65)

Finally, using the fact that \( \int_0^h \eta^h(s) \, ds = \frac{1}{2} \), we estimate
\[
E \left[ \int_0^{t_1} \left( \int_0^{t_1} \eta^h(t_1 - s) \, ds \right) \|\Phi(\tau)\|_{L^q}^2 \, d\tau \right]
\]
\[
= E \left[ \int_0^{t_1} \eta^h(s) \int_0^{t_1 - s} \|\Phi(\tau)\|_{L^q}^2 \, d\tau \, ds \right]
\]
\[
= E \left[ \int_0^h \eta^h(s) \left( \int_0^{t_1} \|\Phi(\tau)\|_{L^q}^2 \, d\tau - \int_0^{t_1 - s} \|\Phi(\tau)\|_{L^q}^2 \, d\tau \right) \, ds \right]
\]
\[
= \frac{1}{2} E \left[ \int_0^{t_1} \|\Phi(\tau)\|_{L^q}^2 \, d\tau \right] - E \left[ \int_0^h \eta^h(s) \int_0^{t_1 - s} \|\Phi(\tau)\|_{L^q}^2 \, d\tau \, ds \right]
\]
\[
\to \frac{1}{2} E \left[ \int_0^{t_1} \|\Phi(\tau)\|_{L^q}^2 \, d\tau \right], \ \text{as} \ \ h \to 0,
\]
(3.66)
using the continuity of the integral in the final term. Thus, along a subsequence, we further have
\[
\lim_{h \to 0} \int_0^{t_1} \left( \int_0^{t_1} \eta^h(t_1 - s) \, ds \right) \|\Phi(\tau)\|_{L^q}^2 \, d\tau = \frac{1}{2} \int_0^{t_1} \|\Phi(\tau)\|_{L^q}^2 \, d\tau, \ P\text{-a.s.}
\]
(3.67)

Using the convergences (3.63)-(3.67) in (3.61), along a subsequence, we get
\[
\int_0^{t_1} (u(s), G_0(s)) \, ds - \int_0^{t_1} (u(s), \Phi(s) \, dW(s)) - \frac{1}{2} \int_0^{t_1} \|\Phi(s)\|_{L^q}^2 \, ds
\]
Combining the above convergences, we finally obtain the energy equality as

\[
= - \lim_{h \to 0} \langle \mathbf{u}(t_1), \mathbf{u}^h(t_1) \rangle + \lim_{h \to 0} \langle \mathbf{u}(0), \mathbf{u}^h(0) \rangle, \quad \mathbb{P}\text{-a.s.}
\]  

(3.68)

It should be noted that continuous in time and

\[
\lim_{s \to 0} (\mathbf{u}(t) - \mathbf{u}(t-s), \mathbf{v}) = \int_{t-s}^{t} \langle G_0(r), \mathbf{v} \rangle dr + \int_{t-s}^{t} \langle \Phi(r) \rangle dW(r), \quad \mathbb{P}\text{-a.s.,}
\]

(3.69)

and \( \lim_{s \to 0} (\mathbf{u}(t) - \mathbf{u}(t-s), \mathbf{v}) = 0, \quad \mathbb{P}\text{-a.s.}, \) for all \( \mathbf{v} \in \tilde{H}^{r+1} \cap \mathbb{V}. \) Using the fact that \( \mathbf{u} \) is \( L^2\)-weakly continuous in time and \( \int_0^h \eta^h(s) ds = \frac{1}{2}, \) we get

\[
(\mathbf{u}(t_1), \mathbf{u}^h(t_1)) = \int_0^{t_1} \eta^h(s)(\mathbf{u}(t_1), \mathbf{u}(t_1-s)) ds
\]

\[
= \frac{1}{2} ||\mathbf{u}(t_1)||_H^2 + \int_0^{h} \eta^h(s)(\mathbf{u}(t_1), \mathbf{u}(t_1-s) - \mathbf{u}(t_1)) ds \to \frac{1}{2} ||\mathbf{u}(t_1)||_H^2,
\]

(3.70)

as \( h \to 0, \quad \mathbb{P}\text{-a.s.} \) Similarly, we obtain

\[
(\mathbf{u}(0), \mathbf{u}^h(0)) \to \frac{1}{2} ||\mathbf{u}(0)||_H^2 \quad \text{as} \quad h \to 0, \quad \mathbb{P}\text{-a.s.}
\]

(3.71)

Combining the above convergences, we finally obtain the energy equality

\[
||\mathbf{u}(t_1)||_H^2 = ||\mathbf{u}(0)||_H^2 - 2 \int_0^{t_1} \langle G_0(s), \mathbf{u}(s) \rangle ds + 2 \int_0^{t_1} \langle \mathbf{u}(s), \Phi(s) dW(s) \rangle
\]

\[
+ \int_0^{t_1} ||\Phi(s)||_{\mathbb{L}_Q}^2 ds,
\]

(3.72)

for all \( t_1 \in (0, T). \)

Taking expectation and noting the fact that the final term in the right hand side of the equality (3.55) is a martingale, we find

\[
\mathbb{E}[||\mathbf{u}(t)||_H^2] = \mathbb{E}[||\mathbf{u}_0||_H^2] - 2 \mathbb{E} \left[ \int_0^t \langle G_0(s), \mathbf{u}(s) \rangle ds \right] + \mathbb{E} \left[ \int_0^t ||\Phi(s)||_{\mathbb{L}_Q}^2 ds \right].
\]

(3.73)

Thus an application of Itô’s formula to the process \( e^{-2nt} ||\mathbf{u}(\cdot)||_H^2 \) yields

\[
\mathbb{E}[e^{-2nt} ||\mathbf{u}(t)||_H^2] = \mathbb{E}[||\mathbf{u}_0||_H^2] - \mathbb{E} \left[ \int_0^t e^{-2ns}(2G_0(s) + 2\eta(s), \mathbf{u}(s)) ds \right]
\]

\[
+ \mathbb{E} \left[ \int_0^t e^{-2ns} ||\Phi(s)||_{\mathbb{L}_Q}^2 ds \right],
\]

(3.74)

for all \( t \in [0, T]. \) Note that the initial value \( \mathbf{u}^n(0) \) converges to \( \mathbf{u}_0 \) strongly in \( L^2(\Omega; H) \), that is,

\[
\lim_{n \to \infty} \mathbb{E}[||\mathbf{u}^n(0) - \mathbf{u}_0||_H^2] = 0.
\]

(3.75)

**Step (5): Minty-Browder technique and global strong solution.** It is now left to show that

\[ G(\mathbf{u}(\cdot)) = G_0(\cdot) \quad \text{and} \quad \Phi(\cdot, \mathbf{u}(\cdot)) = \Phi(\cdot). \]

In order to do this, we make use of the Lemma 3.6. For \( \mathbf{v} \in L^2(\Omega; L^\infty(0, T; H_m)) \) with \( m < n, \) using the local monotonicity result (see (3.21)), we get

\[
\mathbb{E} \left[ \int_0^T e^{-2nt}(2 \langle G(\mathbf{v}(t)) - G(\mathbf{u}^n(t)), \mathbf{v}(t) - \mathbf{u}^n(t) \rangle + 2\eta(t) \langle \mathbf{v}(t) - \mathbf{u}^n(t), \mathbf{v}(t) - \mathbf{u}^n(t) \rangle dt \right]
\]
Rearranging the terms in (3.76) and then using energy equality (3.24), we obtain

\[
\begin{align*}
\mathbb{E} \left[ \int_0^T e^{-2\eta t} \| \Phi^n(t, v(t)) - \Phi^n(t, u^n(t)) \|^2_{L_Q} dt \right].
\end{align*}
\] (3.76)

Let us now discuss the convergence of the terms involving noise coefficient. Note that

\[
\begin{align*}
\mathbb{E} \left[ \int_0^T e^{-2\eta t} \langle 2G(t), v(t) - u^n(t) \rangle dt \right] \\
- \mathbb{E} \left[ \int_0^T e^{-2\eta t} \| \Phi^n(t, v(t)) \|^2_{L_Q} dt \right] + 2 \mathbb{E} \left[ \int_0^T e^{-2\eta t} (\Phi^n(t, v(t)), \Phi^n(t, u^n(t)))_{L_Q} dt \right] \\
\geq \mathbb{E} \left[ \int_0^T e^{-2\eta t} \langle 2G(t), v(t) - u^n(t) \rangle dt \right] \\
- \mathbb{E} \left[ \int_0^T e^{-2\eta t} \| \Phi^n(t, v(t)) \|^2_{L_Q} dt \right] + \mathbb{E} \left[ \int_0^T e^{-2\eta t} \| \Phi^n(t, u^n(t)) \|^2_{L_Q} dt \right] \\
= \mathbb{E} \left[ \int_0^T e^{-2\eta t} \langle 2G(u^n(t)), v(t) - u^n(t) \rangle dt \right] + \mathbb{E} \left[ e^{-2\eta T} \| u^n(T) \|^2_H - \| u^n(0) \|^2_H \right].
\end{align*}
\] (3.77)

Let us now discuss the convergence of the terms involving noise co-efficient. Note that

\[
\begin{align*}
\mathbb{E} \left[ \int_0^T e^{-2\eta t} \langle 2(\Phi^n(t, v(t)), \Phi^n(t, u^n(t)))_{L_Q} - \| \Phi^n(t, v(t)) \|^2_{L_Q} \rangle dt \right] \\
= \mathbb{E} \left[ \int_0^T e^{-2\eta t} 2(\Phi(t, v(t)), \Phi^n(t, u^n(t)))_{L_Q} dt \right] \\
+ \mathbb{E} \left[ \int_0^T e^{-2\eta t} 2(\Phi^n(t, v(t)) - \Phi(t, v(t)), \Phi^n(t, u^n(t)))_{L_Q} dt \right] \\
- \mathbb{E} \left[ \int_0^T e^{-2\eta t} \| \Phi^n(t, v(t)) \|^2_{L_Q} dt \right] \\
\leq \mathbb{E} \left[ \int_0^T e^{-2\eta t} 2(\Phi(t, v(t)), \Phi^n(t, u^n(t)))_{L_Q} dt \right] \\
+ 2C \left( \mathbb{E} \left[ \int_0^T e^{-4\eta t} \| \Phi^n(t, v(t)) - \Phi(t, v(t)) \|^2_{L_Q} dt \right] \right)^{1/2} \\
- \mathbb{E} \left[ \int_0^T e^{-2\eta t} \| \Phi^n(t, v(t)) \|^2_{L_Q} dt \right],
\end{align*}
\] (3.78)

where \( C = \left( \mathbb{E} \left[ \int_0^T e^{-4\eta t} \| \Phi^n(t, u^n(t)) \|^2_{L_Q} dt \right] \right)^{1/2} \). Then, applying the weak convergence of \( \{ \Phi^n(\cdot, u^n(\cdot)) : n \in \mathbb{N} \} \) given in (3.28) to the first term and using the Lebesgue dominated convergence theorem to the second and final terms on the right hand side of the inequality (3.78), we deduce that

\[
\begin{align*}
\mathbb{E} \left[ \int_0^T e^{-2\eta t} \langle 2(\Phi^n(t, v(t)), \Phi^n(t, u^n(t)))_{L_Q} - \| \Phi^n(t, v(t)) \|^2_{L_Q} \rangle dt \right] \\
- \mathbb{E} \left[ \int_0^T e^{-2\eta t} \langle 2(\Phi(t, v(t)), \Phi(t))_{L_Q} - \| \Phi(t, v(t)) \|^2_{L_Q} \rangle dt \right],
\end{align*}
\] (3.79)
as \( n \to \infty \). Taking \( \liminf \) on both sides of (3.77), and using (3.79), we obtain
\[
\mathbb{E} \left[ \int_0^T e^{-2n\eta} \langle 2G(v(t)) + 2n v(t), v(t) - u(t) \rangle dt \right] \\
- \mathbb{E} \left[ \int_0^T e^{-2n\eta} \| \Phi(t, v(t)) \|_{L_q}^2 dt \right] + 2 \mathbb{E} \left[ \int_0^T e^{-2n\eta} \langle \Phi(t, v(t)), \Phi(t) \rangle_{L_q} dt \right] \\
\geq \mathbb{E} \left[ \int_0^T e^{-2n\eta} \langle 2G_0(t) + 2n u(t), v(t) \rangle dt \right] + \liminf_{n \to \infty} \mathbb{E} \left[ e^{-2n\eta} \| u^n(T) \|_{H}^2 - \| u^n(0) \|_{H}^2 \right].
\] (3.80)

Making use of the lower semicontinuity property of the \( H \)-norm and the strong convergence given in (3.75), the second term on the right hand side of the inequality (3.80) satisfies:
\[
\liminf_{n \to \infty} \mathbb{E} \left[ e^{-2n\eta} \| u^n(T) \|_{H}^2 - \| u^n(0) \|_{H}^2 \right] \geq \mathbb{E} \left[ e^{-2\eta\lambda} \| u(T) \|_{H}^2 - \| u_0 \|_{H}^2 \right].
\] (3.81)

We use the energy equality (3.74) and (3.81) in (3.80) to have
\[
\mathbb{E} \left[ \int_0^T e^{-2n\eta} \langle 2G(v(t)) + 2n v(t), v(t) - u(t) \rangle dt \right] \\
\geq \mathbb{E} \left[ \int_0^T e^{-2n\eta} \| \Phi(t, v(t)) \|_{L_q}^2 dt \right] - 2 \mathbb{E} \left[ \int_0^T e^{-2n\eta} \langle \Phi(t, v(t)), \Phi(t) \rangle_{L_q} dt \right] \\
+ \mathbb{E} \left[ \int_0^T e^{-2n\eta} \| \Phi(t, v(t)) \|_{L_q}^2 dt \right] + \mathbb{E} \left[ \int_0^T e^{-2n\eta} \langle 2G_0(t) + \eta u(t), v(t) - u(t) \rangle dt \right].
\] (3.82)

Rearranging the terms in (3.82), we obtain
\[
\mathbb{E} \left[ \int_0^T e^{-2n\eta} \langle 2G(v(t)) - 2G_0(t) + 2n(v(t) - u(t)), v(t) - u(t) \rangle dt \right] \\
\geq \mathbb{E} \left[ \int_0^T e^{-2n\eta} \| \Phi(t, v(t)) - \Phi(t) \|_{L_q}^2 dt \right] \geq 0.
\] (3.83)

Note that the estimate (3.83) holds true for any \( v \in L^2(\Omega; L^\infty(0, T; H_m)) \) and for any \( m \in \mathbb{N} \), since the estimate is independent of \( m \) and \( n \). Using a density argument, the inequality (3.83) remains true for any
\[
v \in L^2(\Omega; L^\infty(0, T; H) \cap L^2(0, T; V)) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{V}^{r+1})) =: \mathcal{G}.
\]

Indeed, for any \( v \in \mathcal{G} \), there exists a strongly convergent subsequence \( v_m \in \mathcal{G} \), which satisfies the inequality (3.83). Taking \( v(\cdot) = u(\cdot) \) in (3.83) immediately gives \( \Phi(\cdot, v(\cdot)) = \Phi(\cdot) \). Next, we take \( v(\cdot) = u(\cdot) + \lambda w(\cdot) \), \( \lambda > 0 \), where \( w \in \mathcal{G} \), and substitute for \( v \) in (3.83) to find
\[
\mathbb{E} \left[ \int_0^T e^{-2\eta\lambda} \langle G(u(t) + \lambda w(t)) - G_0(t) + \eta \lambda w(t), \lambda w(t) \rangle dt \right] \geq 0.
\] (3.84)

Dividing the above inequality by \( \lambda \), using the hemicontinuity property of \( G(\cdot) \) (see Lemma 2.5), and passing \( \lambda \to 0 \), we obtain
\[
\mathbb{E} \left[ \int_0^T e^{-2\eta \lambda} \langle G(u(t)) - G_0(t), w(t) \rangle dt \right] \geq 0,
\] (3.85)
since the final term in (3.84) tends to 0 as \( \lambda \to 0 \). Thus from (3.85), we get \( G(u(t)) = G_0(t) \) and hence \( u(\cdot) \) is a strong solution of the system (3.2) and \( u \in \mathcal{G} \). From (3.55), it is immediate that \( u(\cdot) \) satisfy the following energy equality (Itô’s formula):

\[
\begin{align*}
\|u(t)\|^2_{H^2} + 2\mu \int_0^t \|u(s)\|^2_Y ds + 2\beta \int_0^t \|u(s)\|^r_{L^{r+1}} ds \\
= \|u_0\|^2_{H^2} + \int_0^t \|\Phi(s, u(s))\|^2_{L^q} ds + 2 \int_0^t (\Phi(s, u(s))dW(s), u(s)),
\end{align*}
\]

for all \( t \in (0, T) \), \( \mathbb{P} \)-a.s. Moreover, the following energy estimate holds true:

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u(t)\|^2_{H^2} + 4\mu \int_0^T \|u(t)\|^2_Y dt + 4\beta \int_0^T \|u(t)\|^r_{L^{r+1}} dt \right] \\
\leq (2\mathbb{E}[\|u_0\|^2_{H^2}] + 14KT )e^{28KT}.
\end{align*}
\]

Furthermore, since \( u(\cdot) \) satisfies the energy equality (3.86), one can show that the \( \mathcal{F}_t \)-adapted paths of \( u(\cdot) \) are continuous with trajectories in \( C([0, T]; \mathbb{H}) \), \( \mathbb{P} \)-a.s. (see [22, 34, 23], etc).

**Step (6): Uniqueness.** Finally, we show that the strong solution established in step (5) is unique. Let \( u_1(\cdot) \) and \( u_2(\cdot) \) be two strong solutions of the system (3.2). For \( N > 0 \), let us define

\[
\tau^1_N = \inf_{0 \leq t \leq T} \left\{ t : \|u_1(t)\|_{H^1} \geq N \right\}, \quad \tau^2_N = \inf_{0 \leq t \leq T} \left\{ t : \|u_2(t)\|_{H^1} \geq N \right\} \quad \text{and} \quad \tau_N := \tau^1_N \wedge \tau^2_N.
\]

Using the energy estimate (3.87), it can be shown in a similar way as in step (1), Proposition 3.5 that \( \tau_N \to T \) as \( N \to \infty \), \( \mathbb{P} \)-a.s. Let us define \( w(\cdot) := u_1(\cdot) - u_2(\cdot) \) and \( \tilde{\Phi}(\cdot) := \Phi(\cdot, u_1(\cdot)) - \Phi(\cdot, u_2(\cdot)) \). Then, \( w(\cdot) \) satisfies the following system:

\[
\begin{cases}
\text{dw}(t) = -[\mu Aw(t) + B(u_1(t)) - B(u_2(t))] + \beta(C(u_1(t)) - C(u_2(t)))dt + \tilde{\Phi}(t)dW(t), \\
w(0) = w_0.
\end{cases}
\]

Then, \( w(\cdot) \) satisfies the following energy equality:

\[
\begin{align*}
\|w(t \wedge \tau_N)\|^2_{H^2} + 2\mu \int_0^{t \wedge \tau_N} \|w(s)\|^2_Y ds \\
= \|w(0)\|^2_{H^2} - 2 \int_0^{t \wedge \tau_N} \langle B(u_1(s)) - B(u_2(s)), w(s) \rangle ds \\
- 2 \int_0^{t \wedge \tau_N} \langle C(u_1(s)) - C(u_2(s)), u_1(s) - u_2(s) \rangle ds + \int_0^{t \wedge \tau_N} \|\tilde{\Phi}(s)\|^2_{L^q} ds \\
+ 2 \int_0^{t \wedge \tau_N} \langle \tilde{\Phi}(s)dW(s), w(s) \rangle.
\end{align*}
\]

From (2.20), we obtain

\[
|\langle B(u_1) - B(u_2), w \rangle| \leq \frac{\mu}{2}\|w\|^2_Y + \frac{\beta}{2}\|u_1\|^r_{H^2} w + \eta\|w\|^2_{H^2}.
\]

and from (2.17), we get

\[
\beta|C(u_1) - C(u_2), w \rangle \geq \frac{\beta}{2}\|u_2\|^r_{H^2} w.
\]
Thus, using the above two estimates in (3.89), we infer that
\[
\left\| \mathbf{w}(t \wedge \tau_N) \right\|^2_{E_T} + \mu \int_0^{t \wedge \tau_N} \left\| \mathbf{w}(s) \right\|^2_{\mathbb{H}} ds
\]
\[
\leq \left\| \mathbf{w}(0) \right\|^2_{E_T} + 2\eta \int_0^t \left\| \mathbf{w}(s) \right\|^2_{\mathbb{H}} ds + \int_0^{t \wedge \tau_N} \left\| \Phi(s) \right\|^2_{L^Q} ds + 2 \int_0^{t \wedge \tau_N} \left( \Phi(s) dW(s), \mathbf{w}(s) \right).
\]
(3.90)
It should be noted that the final term in the right hand side of the inequality (3.90) is a local martingale. Taking expectation in (3.90), and then using the Hypothesis 3.2 (H.2), we obtain
\[
\mathbb{E} \left[ \left\| \mathbf{w}(t \wedge \tau_N) \right\|^2_{E_T} + \mu \int_0^{t \wedge \tau_N} \left\| \mathbf{w}(s) \right\|^2_{\mathbb{H}} ds \right] \leq \mathbb{E} \left[ \left\| \mathbf{w}(0) \right\|^2_{E_T} + (L + 2\eta) \int_0^t \left\| \mathbf{w}(s) \right\|^2_{\mathbb{H}} ds \right].
\]
(3.91)
Applying Gronwall’s inequality in (3.91), we arrive at
\[
\mathbb{E} \left[ \left\| \mathbf{w}(t \wedge \tau_N) \right\|^2_{E_T} \right] \leq \mathbb{E} \left[ \left\| \mathbf{w}_0 \right\|^2_{E_T} e^{(L+2\eta)T} \right].
\]
(3.92)
Thus the initial data \( \mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0 \) leads to \( \mathbf{w}(t \wedge \tau_N) = 0 \), \( \mathbb{P} \)-a.s. But using the fact that \( \tau_N \to T \), \( \mathbb{P} \)-a.s., implies \( \mathbf{w}(t) = 0 \) and hence \( \mathbf{u}_1(t) = \mathbf{u}_2(t) \), \( \mathbb{P} \)-a.s., for all \( t \in [0, T] \), and hence the uniqueness follows. \( \square \)

**Remark 3.8.** Recently authors in [23] (Theorem 1) obtained Itô’s formula for processes taking values in intersection of finitely many Banach spaces. But it seems to the author that this result may not applicable in our context for establishing the energy equality (3.86), as our operator \( B(\cdot) : \mathbb{V} \cap \mathbb{L}^{r+1} \to \mathbb{V} + \mathbb{L}^{r+1} \) (see (2.6)) and one can show the local integrability in the sum of Banach spaces only.

**Theorem 3.9.** For \( r = 3 \) and \( 2\beta \mu \geq 1 \), let \( \mathbf{u}_0 \in L^2(\Omega; \mathbb{H}) \) be given. Then there exists a pathwise unique strong solution \( \mathbf{u}(\cdot) \) to the system (3.2) such that
\[
\mathbf{u} \in L^2(\Omega; L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V})) \cap L^4(\Omega; L^4(0, T; \mathbb{V})),
\]
with \( \mathbb{P} \)-a.s., continuous trajectories in \( \mathbb{H} \).

**Proof.** A proof of the Theorem 3.9 follows similarly as in the Theorem 3.7 by using the global monotonicity result (2.22) and the fact that
\[
\int_0^T \langle G(\mathbf{u}(t)) - G(\mathbf{v}(t)), \mathbf{u}(t) - \mathbf{v}(t) \rangle + \frac{L}{2} \int_0^T \left\| \mathbf{u}(t) - \mathbf{v}(t) \right\|^2_{\mathbb{H}} dt 
\]
\[
\geq \frac{1}{2} \int_0^T \left\| \Phi(t, \mathbf{u}(t)) - \Phi(t, \mathbf{v}(t)) \right\|^2_{L^Q} dt,
\]
(3.93)
for \( 2\beta \mu \geq 1 \). Uniqueness also follows easily by using the estimate (2.24). \( \square \)

3.5. **Regularity of strong solution.** In order to get the regularity results of the strong solution to (3.2), we restrict ourselves to periodic domains with \( \int_O \mathbf{u}(x) dx = 0 \). The main difficulty in working with bounded domains is that \( P_\mathbb{H}(|\mathbf{u}|^{r-1}|\mathbf{u}) \) need not be zero on the boundary, and \( P_\mathbb{H} \) and \( -\Delta \) are not necessarily commuting (see [12]). Thus applying Itô’s formula for the process \( \| A^{1/2} \mathbf{u}(\cdot) \|^2_{\mathbb{H}} \) may not work. Moreover, \( \Delta \mathbf{u}^0 \neq 0 \) in general and the term with pressure will not disappear (see [26]) and hence applying Itô’s formula for the
stochastic process \( \| \nabla \mathbf{u}(\cdot) \|^2_{L^2} \) (for \( \mathbf{u}(\cdot) \) appearing in (3.1)) also may not work. In a periodic
domain or in whole space \( \mathbb{R}^n \), the operators \( P_1 \) and \( -\Delta \) commute and we have the following
result (see Lemma 2.1, [24]):

\[
0 \leq \int_{\Omega} |\nabla \mathbf{u}(x)|^2 |\mathbf{u}(x)|^{-1} dx \leq \int_{\Omega} |\mathbf{u}(x)|^{-1} \mathbf{u}(x) \cdot A \mathbf{u}(x) dx \leq r \int_{\Omega} |\nabla \mathbf{u}(x)|^2 |\mathbf{u}(x)|^{-1} dx.
\]

(3.94)

Using Lemma 2.2, [24], we further have

\[
\| \mathbf{u} \|_{L^{r+1}(\mathbb{R}^n)} \leq C \int_{\Omega} |\nabla \mathbf{u}(x)|^2 |\mathbf{u}(x)|^{-1} dx, \quad \text{for } r \geq 1.
\]

(3.95)

Note that the estimate (3.94) is true even in bounded domains (with Dirichlet boundary
conditions) if one replaces \( A \mathbf{u} \) with \( -\Delta \mathbf{u} \) and (3.95) holds true in bounded domains as well
as whole space \( \mathbb{R}^n \). We also assume that the noise coefficient satisfies the following:

**Hypothesis 3.10.** There exist a positive constant \( \tilde{K} \) such that for all \( t \in [0, T] \) and \( \mathbf{u} \in \mathbb{V} \),

\[
\| A^{1/2} \Phi(t, \mathbf{u}) \|_{L^2}^2 \leq \tilde{K}(1 + \| \mathbf{u} \|_{\mathbb{V}}^2).
\]

**Theorem 3.11.** Let \( \mathbf{u}_0 \in L^2(\Omega; \mathbb{V}) \) be given. Then, for \( r > 3 \), the pathwise unique strong
solution \( \mathbf{u}(\cdot) \) to the system (3.2) satisfies the following regularity:

\[
\mathbf{u} \in L^2(\Omega; L^\infty(0, T; \mathbb{V}) \cap L^2(0, T; D(A))) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \mathbb{L}^{3(r+1)})),
\]

with \( \mathbb{P} \)-a.s., continuous trajectories in \( \mathbb{V} \).

**Proof.** Let us take \( \mathbf{u}_0 \in L^2(\Omega; \mathbb{V}) \), to obtain the further regularity results of the strong
solution to (3.2) with \( r \geq 3 \). Applying the infinite dimensional Itô formula to the process
\( \| A^{1/2} \mathbf{u}(\cdot) \|_{H^1}^2 \), we get

\[
\| \mathbf{u}(t) \|_{\mathbb{V}}^2 + 2\mu \int_0^t \| A \mathbf{u}(s) \|_{ H^1}^2 ds
\]

\[
= \| \mathbf{u}_0 \|_{\mathbb{V}}^2 - 2 \int_0^t (B(\mathbf{u}(s)), A \mathbf{u}(s)) ds - 2\beta \int_0^t (C(\mathbf{u}(s)), A \mathbf{u}(s)) ds
\]

\[
+ \int_0^t \| A^{1/2} \Phi(s, \mathbf{u}(s)) \|_{L^2}^2 ds + 2 \int_0^t (A^{1/2} \Phi(s, \mathbf{u}(s)) dW(s), A^{1/2} \mathbf{u}(s)).
\]

(3.96)

We estimate \( |(B(\mathbf{u}), A \mathbf{u})| \) using Hölder’s, and Young’s inequalities as

\[
|B(\mathbf{u}, A \mathbf{u})| \leq \| \mathbf{u} \| \| \nabla \mathbf{u} \|_{H^1} \| A \mathbf{u} \|_{H^1} \leq \frac{\mu}{2} \| A \mathbf{u} \|_{H^1}^2 + \frac{1}{2\mu} \| \mathbf{u} \| \| \nabla \mathbf{u} \|_{H^1}^2.
\]

(3.97)

For \( r > 3 \), we estimate the final term from (3.97) using Hölder’s and Young’s inequalities as

\[
\int_{\Omega} |\mathbf{u}(x)|^2 |\nabla \mathbf{u}(x)|^2 dx
\]

\[
= \int_{\Omega} |\mathbf{u}(x)|^2 |\nabla \mathbf{u}(x)|^{\frac{r}{r-1}} |\nabla \mathbf{u}(x)|^{\frac{2(r-3)}{r-1}} dx
\]

\[
\leq \left( \int_{\Omega} |\mathbf{u}(x)|^{r-1} |\nabla \mathbf{u}(x)|^2 dx \right)^{\frac{2}{r-1}} \left( \int_{\Omega} |\nabla \mathbf{u}(x)|^2 dx \right)^{\frac{r-3}{r-1}}
\]
\[
\leq \beta \mu \left( \int_\mathcal{O} |u(x)|^{r-1} |\nabla u(x)|^2 \, dx \right) + \frac{r-3}{r-1} \left( \frac{2}{\beta \mu (r-1)} \right)^{\frac{2}{r-3}} \left( \int_\mathcal{O} |\nabla u(x)|^2 \, dx \right).
\]

Making use of the estimate (3.94) in (3.97), taking supremum over time from 0 to \( T \), we have
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u(t)\|_V^2 + \mu \int_0^T \|Au(t)\|_H^2 \, dt + \beta \int_0^T \|u(t)\|^{-\frac{1}{2}}_2 |\nabla u(t)|^2_2 \, ds \right]
\leq \mathbb{E} \left[ \|u_0\|_V^2 + 2\eta \mathbb{E} \left[ \int_0^T \|u(t)\|_2^2 \, dt \right] + \mathbb{E} \left[ \int_0^T \|A^{1/2} \Phi(t, u(t))\|^2_{L^2} \, dt \right] + 2\mathbb{E} \left[ \sup_{t \in [0,T]} \int_0^t \left( A^{1/2} \Phi(s, u(s)) \, dW(s), A^{1/2} u(s) \right) \right],
\]
(3.98)

where \( \eta \) is defined in (2.15). Applying Burkholder-Davis-Gundy inequality to the final term appearing in the inequality (3.98), we obtain
\[
2\mathbb{E} \left[ \sup_{t \in [0,T]} \int_0^t \left( A^{1/2} \Phi(s, u(s)) \, dW(s), A^{1/2} u(s) \right) \right] \leq 2\sqrt{3} \mathbb{E} \left[ \int_0^T \|A^{1/2} \Phi(t, u(t))\|_{L^2}^2 \|A^{1/2} u(t)\|_{H^2}^2 \, dt \right]^{1/2}
\leq 2\sqrt{3} \mathbb{E} \left[ \sup_{t \in [0,T]} \|A^{1/2} u(t)\|_{H} \left( \int_0^T \|A^{1/2} \Phi(t, u(t))\|_{L^2}^2 \, dt \right)^{1/2} \right]
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0,T]} \|A^{1/2} u(t)\|_{H}^2 \right] + 6\mathbb{E} \left[ \int_0^T \|A^{1/2} \Phi(t, u(t))\|_{L^2}^2 \, dt \right].
\]
(3.99)

Substituting (3.99) in (3.98), we find
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u(t)\|_V^2 + 2\mu \int_0^T \|Au(t)\|_H^2 \, dt + 2\beta \int_0^T \|u(t)\|^{-\frac{1}{2}}_2 |\nabla u(t)|^2_2 \, dt \right]
\leq 2\mathbb{E} \left[ \|u_0\|_V^2 + 4\eta \mathbb{E} \left[ \int_0^T \|u(t)\|_2^2 \, dt \right] + 14\mathbb{E} \left[ \int_0^T \|A^{1/2} \Phi(t, u(t))\|_{L^2}^2 \, dt \right] \right]
\leq 2\mathbb{E} \left[ \|u_0\|_V^2 + 4\eta \mathbb{E} \left[ \int_0^T \|u(t)\|_2^2 \, dt \right] + 14\widetilde{K} \mathbb{E} \left[ \int_0^T (1 + \|u(t)\|_2^2) \, dt \right] \right],
\]
(3.100)

where we used Hypothesis 3.2. Applying Gronwall’s inequality in (3.100), we obtain
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u(t)\|_V^2 \right] \leq \left\{ 2\mathbb{E} \left[ \|u_0\|_V^2 \right] + 14\widetilde{K} T \right\} e^{(4\eta + 14\widetilde{K})T}.
\]
(3.101)

Using (3.101) in (3.100), we finally get
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u(t)\|_V^2 + 2\mu \int_0^T \|Au(t)\|_H^2 \, dt + 2\beta \int_0^T \|u(t)\|^{-\frac{1}{2}}_2 |\nabla u(t)|^2_2 \, dt \right]
\leq \left\{ 2\mathbb{E} \left[ \|u_0\|_V^2 \right] + 14\widetilde{K} T \right\} e^{(2\eta + 7\widetilde{K})T},
\]
(3.102)
which completes the proof for \( r > 3 \).

For \( r = 3 \), we estimate \(|(B(\mathbf{u}), A\mathbf{u})|\) as

\[
|(B(\mathbf{u}), A\mathbf{u})| \leq \|(\mathbf{u} \cdot \nabla)\mathbf{u}\|_{\mathcal{H}} A\mathbf{u} \leq \frac{1}{4\theta} \|A\mathbf{u}\|_{\mathcal{H}}^2 + \theta \|\nabla\mathbf{u}\|_{\mathcal{H}}^2. 
\] (3.103)

A calculation similar to (3.100) gives

\[
E\left[ \sup_{t \in [0,T]} \|\mathbf{u}(t)\|_{\mathcal{V}}^2 + 2\left(\mu - \frac{1}{2\theta}\right) \int_0^T \|A\mathbf{u}(t)\|_{\mathcal{H}}^2 dt + 4(\beta - \theta) \int_0^T \|\mathbf{u}(t)\|_{\mathcal{H}}^2 dt \right] 
\leq 2E\left[\|\mathbf{u}_0\|_{\mathcal{V}}^2\right] + 14\tilde{K}E\left[\int_0^T (1 + \|\mathbf{u}(t)\|_{\mathcal{V}}^2) dt\right], 
\] (3.104)

For \( 2\beta\mu \geq 1 \), it is immediate that

\[
E\left[ \sup_{t \in [0,T]} \|\mathbf{u}(t)\|_{\mathcal{V}}^2 + 2\left(\mu - \frac{1}{2\theta}\right) \int_0^T \|A\mathbf{u}(t)\|_{\mathcal{H}}^2 dt + 4(\beta - \theta) \int_0^T \|\mathbf{u}(t)\|_{\mathcal{H}}^2 dt \right] 
\leq \left\{ 2E\left[\|\mathbf{u}_0\|_{\mathcal{V}}^2\right] + 14\tilde{K}T \right\} e^{28\tilde{K}T}. 
\] (3.105)

Hence \( \mathbf{u} \in L^2(\Omega; L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; D(A))) \) and using the estimate (3.95), we also get \( \mathbf{u} \in L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{L}^3(r+1))) \). Moreover, the \( \mathcal{F}_t \)-adapted paths of \( \mathbf{u}(\cdot) \) are continuous with trajectories in \( C([0, T]; \mathcal{V}) \), \( \mathbb{P} \)-a.s.

\section{Stationary solutions and stability}

In this section, we consider the stationary system (in the deterministic sense) corresponding to the convective Brinkman-Forchheimer equations. We show the existence and uniqueness of weak solutions to the steady state equations and discuss about exponential stability as well as stabilization by noise results.

\subsection{Existence and uniqueness of weak solutions to the stationary system}

Let us consider the following stationary system:

\[
\begin{align*}
-\mu \Delta \mathbf{u}_\infty + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u}_\infty + \beta |\mathbf{u}_\infty|^{r-1}\mathbf{u}_\infty + \nabla p_\infty &= \mathbf{f}, \quad \text{in } \mathcal{O}, \\
\nabla \cdot \mathbf{u}_\infty &= 0, \quad \text{in } \mathcal{O}, \\
\mathbf{u}_\infty &= 0 \quad \text{on } \partial \mathcal{O}.
\end{align*}
\] (4.1)

Taking the Helmholtz-Hodge orthogonal projection onto the system (4.1), we can write down the abstract formulation of the system (4.1) as

\[
\mu A\mathbf{u}_\infty + B(\mathbf{u}_\infty) + \beta C(\mathbf{u}_\infty) = \mathbf{f} \quad \text{in } \mathcal{V}'. 
\] (4.2)

We show that there exists a unique weak solution of the system (4.2) in \( \mathcal{V} \cap \tilde{L}^{r+1} \), for \( r \geq 3 \). Given any \( \mathbf{f} \in \mathcal{V}' \), our problem is to find \( \mathbf{u}_\infty \in \mathcal{V} \cap \tilde{L}^{r+1} \) such that

\[
\mu(\nabla \mathbf{u}_\infty, \nabla \mathbf{v}) + (B(\mathbf{u}_\infty), \mathbf{v}) + \beta(C(\mathbf{u}_\infty), \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle, \quad \text{for all } \mathbf{v} \in \mathcal{V} \cap \tilde{L}^{r+1},
\] (4.3)

is satisfied. Our next aim is to discuss about the existence and uniqueness of weak solutions of the system (4.1).
Theorem 4.1. For every $f \in V'$ and $r \geq 1$, there exists at least one weak solution of the system (4.1). For $r > 3$, if

$$\mu > \frac{2\eta}{\lambda_1},$$

(4.4)

where $\eta$ is defined in (2.15), then the solution of (4.2) is unique. For $r = 3$, the condition given in (4.4) becomes $\mu \geq \frac{1}{\eta^3}$.

Proof. (i) We show the existence of weak solution to (4.1) (equivalently the existence of (4.2)) by applying a Faedo-Galerkin approximation technique. Let $\{e_k\}$ be eigenfunctions of the Stokes operator defined in the previous section. For a fixed positive integer $m$, we look for a function $u_m \in V$ of the form

$$u_m^\infty = \sum_{k=1}^m \xi_k e_k, \quad \xi_k \in \mathbb{R},$$

(4.5)

and

$$\mu(\nabla u_m^\infty, \nabla e_k) + (B(u_m^\infty), e_k) + \beta(C(u_m^\infty), e_k) = (f, e_k),$$

(4.6)

for $k = 1, \ldots, m$. Equivalently, the equation (4.6) can also be written as

$$\mu A u_m^\infty + P_m B(u_m^\infty) + \beta P_m C(u_m^\infty) = P_m f.$$  

(4.7)

Note that the equations (4.5)-(4.6) are system of nonlinear equations for $\xi_1, \ldots, \xi_m$ and the existence of solutions is proved in the following way. We use Lemma 1.4, Chapter 2, [48] to obtain the existence of solution to the system of equations (4.5)-(4.6). We take $W = \text{Span}\{e_1, \ldots, e_m\}$ and the scalar product on $W$ is the scalar product $[\cdot, \cdot] = (\nabla \cdot, \nabla \cdot)$ induced by $V$ and $P = P_m$ is defined by

$$[P_m(u), v] = (\nabla P_m(u), \nabla v) = \mu(\nabla u, \nabla v) + b(u, u, v) + \beta(C(u), v) - (f, v),$$

(4.8)

for all $u, v \in W$. The continuity of the mapping $P_m : W \to W$ is easy to verify, as in finite dimensional space all norms are equivalent. In order to apply Lemma 1.4, Chapter 2, [48], we need to establish that

$$[P_m(u), u] > 0, \quad \text{for} \quad [u] = k > 0,$$

where $[\cdot]$ denotes the norm on $W$. Note that it is the norm induced by $V$. Next, we consider

$$[P_m(u, u)] = \mu \|\nabla u\|^2_H + \beta \|u\|_{L^{r+1}}^{r+1} - (f, u) \geq \frac{\mu}{2} \|\nabla u\|^2_H - \frac{1}{2\mu} \|f\|^2_{V'},$$

(4.9)

where we used Cauchy-Schwarz and Young's inequalities. It follows that $[P_m(u, u)] > 0$ for $[u] = k$ and $k$ is sufficiently large, more precisely $k > \frac{1}{\mu} \|f\|_{V'}$. Hence, the hypotheses of Lemma 1.4, Chapter 2, [48] are satisfied and a solution $u_m$ of (4.6) exists.

Multiplying (4.6) by $\xi_k$ and then adding from $k = 1, \ldots, m$, we find

$$\mu \|\nabla u_m\|^2_H + \beta \|u_m\|_{L^{r+1}}^{r+1} = (P_m f, u_m) \leq \|f\|_{V'} \|\nabla u_m\|_H \leq \frac{\mu}{2} \|\nabla u_m\|^2_H + \frac{1}{2\mu} \|f\|^2_{V'},$$

(4.10)

where we used Hölder's and Young's inequalities. From (4.10), we deduce that

$$\mu \|\nabla u_m\|^2_H + 2\beta \|u_m\|_{L^{r+1}}^{r+1} \leq \frac{1}{\mu} \|f\|^2_{V'}.$$  

(4.11)
Thus, we get \( \|u_m\|_V^2 \) is bounded uniformly and independent of \( m \). Since \( V \) and \( L^{r+1} \) are reflexive, using the Banach-Alaoglu theorem, we can extract a subsequence \( \{u_{m_k}\} \) of \( \{u_m\} \) such that

\[
\begin{align*}
    u_{m_k} & \to u_\infty, \quad \text{in} \ V, \\
    u_{m_k} & \to u_\infty \quad \text{in} \ L^{r+1},
\end{align*}
\]

as \( k \to \infty \). Since the embedding of \( V \subset H \) is compact, one can extract a subsequence \( \{u_{m_{kj}}\} \) of \( \{u_{m_k}\} \) such that

\[
u_{m_{kj}} \to u_\infty, \quad \text{in} \ H, \ \text{a.e.} \ x \in \mathcal{O}, \tag{4.14}
\]
as \( j \to \infty \). Passing to limit in (4.16) along the subsequence \( \{m_{kj}\} \), we find that \( u_\infty \) is a solution to (4.13) and \( u_\infty \) satisfies

\[
\mu \|u_\infty\|_V^2 + 2\beta \|u_\infty\|_{L^{r+1}}^{r+1} \leq \frac{1}{\mu} \|f\|_{V^r},
\]

for \( r \geq 1 \).

(iii) Let us now establish the uniqueness for \( r > 3 \). We take \( u_\infty \) and \( v_\infty \) as two weak solutions of the system (4.3). We define \( w_\infty := u_\infty - v_\infty \). Then \( w_\infty \) satisfies:

\[
\mu \langle \nabla w_\infty, \nabla v \rangle + \langle B(u_\infty) - B(v_\infty), v \rangle + \beta \langle C(u_\infty) - C(v_\infty), v \rangle = 0,
\]

for all \( v \in V \). Taking \( v = w_\infty \) in (4.16), we obtain

\[
\mu \| \nabla w_\infty \|_H^2 = - \langle B(u_\infty) - B(v_\infty), w_\infty \rangle - \beta \langle C(u_\infty) - C(v_\infty), w_\infty \rangle \\
\leq \frac{\mu}{2} \| w_\infty \|_V^2 + \eta \| w_\infty \|_H^2,
\]

where we used (2.17) and (2.20). Using (4.17) in (4.17), we get

\[
\left( \frac{\mu}{2} - \frac{\eta}{\lambda_1} \right) \| \nabla w_\infty \|_V^2 \leq 0.
\]

and for \( \mu > \frac{2\eta}{\lambda_1} \), we have \( u_\infty = v_\infty \), for a.e. \( x \in \mathcal{O} \). For \( r = 3 \) and \( 2\beta\mu \geq 1 \), one can use the estimate (2.24) to obtain the uniqueness. \( \square \)

**Remark 4.2.** We can get uniqueness for \( r \geq 1 \) also. For \( n \leq 4 \), we know that \( \|u\|_{L^4} \leq C\|u\|_V \), by using Sobolev inequality. One can estimate \( -\langle B(w_\infty, v_\infty), w_\infty \rangle \) using Hölder’s, and Sobolev’s inequalities as

\[
-\langle B(w_\infty, v_\infty), w_\infty \rangle \leq \|w_\infty\|_{L^4}^2 \|v_\infty\|_V \leq C\|v_\infty\|_V \|w_\infty\|_V.
\]

Using (4.19) and (2.11) in (4.17), we get

\[
\left( \mu - C\|v_\infty\|_V \right) \| \nabla w_\infty \|_V^2 \leq 0.
\]

Since \( v_\infty \) satisfies (4.15), from (4.20), we obtain

\[
\left( \mu - C\|f\|_{V^r} \right) \| \nabla w_\infty \|_V^2 \leq 0.
\]

If the condition \( \mu > C\sqrt{\|f\|_{V^r}} \) is satisfied, then we have \( u_\infty = v_\infty \).
4.2. **Exponential stability.** Let us now discuss about the exponential stability of the stationary solution obtained in Theorem 4.1. Let us first discuss about the deterministic case.

**Definition 4.3.** A *weak solution* \( u(t) \) of the system of the deterministic system

\[
\begin{cases}
\frac{du(t)}{dt} + \mu Au(t) + B(u(t)) + \beta C(u(t)) = f, \\
u(0) = u_0,
\end{cases}
\] (4.22)

converges to \( u_\infty \) is exponentially stable in \( \mathbb{H} \) if there exist a positive number \( \kappa > 0 \), such that

\[ ||u(t) - u_\infty||_\mathbb{H} \leq ||u_0 - u_\infty||_\mathbb{H} e^{-\kappa t}, \quad t \geq 0.\]

In particular, if \( u_\infty \) is a stationary solution of system (4.2), then \( u_\infty \) is called exponentially stable in \( \mathbb{H} \) provided that any weak solution to (4.22) converges to \( u_\infty \) at the same exponential rate \( \kappa > 0 \).

**Theorem 4.4.** Let \( u_\infty \) be the unique solution of the system (4.2). If \( u(\cdot) \) is any weak solution to the system (4.22) with \( u_0 \in \mathbb{H} \) and \( f \in V' \) arbitrary, then we have \( u_\infty \) is exponentially stable in \( \mathbb{H} \) and \( u(t) \to u_\infty \) in \( \mathbb{H} \) as \( t \to \infty \), for \( \mu > \frac{2\nu}{\lambda_1} \), for \( r > 3 \) and \( \mu \geq \frac{1}{2\beta} \), for \( r = 3 \).

**Proof.** Let us define \( w = u - u_\infty \), so that \( w \) satisfies the following system:

\[
\begin{cases}
\frac{dw(t)}{dt} + \mu Aw(t) + \beta(B(u(t)) - B(u_\infty)) + \beta(C(u(t)) - C(u_\infty)) = 0, \quad t \in (0, T), \\
w(0) = u_0 - u_\infty.
\end{cases}
\] (4.23)

Taking inner product with \( w(\cdot) \) to the first equation in (4.23), we find

\[
\frac{1}{2} \frac{d}{dt} ||w(t)||_V^2 + \mu ||w(t)||_V^2 \\
= -\beta \langle (B(u(t)) - B(u_\infty)), w(t) \rangle - \beta \langle (C(u(t)) - C(u_\infty)), w(t) \rangle \\
\leq \frac{\mu}{2} ||w(t)||_V^2 + \eta ||w(t)||_H^2,
\] (4.24)

for \( r > 3 \), where we used (2.17) and (2.20). Thus, it is immediate that

\[
\frac{d}{dt} ||w(t)||_H^2 + (\lambda_1 \mu - 2\eta) ||w(t)||_H^2 \leq 0.
\] (4.25)

Thus, an application of variation of constants formula yields

\[
||u(t) - u_\infty||_H^2 \leq e^{-\kappa t} ||u_0 - u_\infty||_H^2,
\] (4.26)

where \( \kappa = (\lambda_1 \mu - 2\eta) > 0 \), for \( \mu > \frac{2\nu}{\lambda_1} \) and the exponential stability of \( u_\infty \) follows. For \( r = 3 \) and \( \mu \geq \frac{1}{2\beta} \) one can use the estimates (2.28) and (2.24) to get the required result. \( \square \)

Now we discuss about the exponential stability results in the stochastic case.

**Definition 4.5.** A *strong solution* \( u(t) \) of the system (3.3) converges to \( u_\infty \in \mathbb{H} \) is exponentially stable in mean square if there exist two positive numbers \( a > 0 \), such that

\[
\mathbb{E}[||u(t) - u_\infty||_H^2] \leq \mathbb{E}[||u_0 - u_\infty||_H^2] e^{-at}, \quad t \geq 0.
\]

\[1\] For \( r \geq 3 \), the existence and uniqueness of weak solutions can be obtained from [1, 26, 19], etc.
In particular, if $\mathbf{u}_\infty$ is a stationary solution of system (4.1), then $\mathbf{u}_\infty$ is called exponentially stable in the mean square provided that any strong solution to (3.2) converges in $L^2$ to $\mathbf{u}_\infty$ at the same exponential rate $\alpha > 0$.

**Definition 4.6.** A strong solution $\mathbf{u}(t)$ of the system (3.2) converges to $\mathbf{u}_\infty \in \mathbb{H}$ almost surely exponentially stable if there exists $\alpha > 0$ such that

$$
\lim_{t \to +\infty} \frac{1}{t} \log \| \mathbf{u}(t) - \mathbf{u}_\infty \|_{\mathbb{H}} \leq -\alpha, \text{ P-a.s.}
$$

In particular, if $\mathbf{u}_\infty$ is a stationary solution of system (4.1), then $\mathbf{u}_\infty$ is called almost surely exponentially stable provided that any strong solution to (3.2) converges in $\mathbb{H}$ to $\mathbf{u}_\infty$ with the same constant $\alpha > 0$.

Let us now show the exponential stability of the stationary solutions to the system (4.1) in the mean square as well as almost sure sense. The authors in [30] obtained similar results, but with more regularity on the stationary solutions as well as the lower bound of $\mu$ depends on the stationary solutions. The following results are true for all $r \geq 3$, and one has to take $2\beta \mu \geq 1$, for $r = 3$. The system under our consideration is

$$
\begin{cases}
\text{d}\mathbf{u}(t) + \mu A\mathbf{u}(t) + B(\mathbf{u}(t)) + \beta C(\mathbf{u}(t)) = \mathbf{f} + \Phi(t, \mathbf{u}(t))dW(t), & t \in (0, T), \\
\mathbf{u}(0) = \mathbf{u}_0
\end{cases}
$$

(4.27)

where $\mathbf{f} \in \mathbb{V}'$ and $\mathbf{u}_0 \in L^2(\Omega; \mathbb{H})$.

**Theorem 4.7.** Let $\mathbf{u}_\infty$ be the unique stationary solution of (4.1) and $\Phi(t, \mathbf{u}_\infty) = 0$, for all $t \geq 0$. Suppose that the conditions in Hypothesis 3.2 are satisfied, then for $\theta = \mu \lambda_1 - (2\eta + L) > 0$, we have

$$
\mathbb{E}\left[ \| \mathbf{u}(t) - \mathbf{u}_\infty \|^2_{\mathbb{H}} \right] \leq e^{-\theta t} \mathbb{E}\left[ \| \mathbf{u}_0 - \mathbf{u}_\infty \|^2_{\mathbb{H}} \right],
$$

(4.28)

provided

$$
\mu > \frac{2\eta + L}{\lambda_1},
$$

(4.29)

where $L$ is the constant appearing in Hypothesis 3.2 (H.3), $\eta$ is defined in (2.13) and $\lambda_1$ is the Poincaré constant.

**Proof.** Let us define $\mathbf{w} := \mathbf{u} - \mathbf{u}_\infty$ and $\theta = \mu \lambda_1 - (2\eta + L) > 0$. Then $\mathbf{w}$ satisfies the following Itô stochastic differential:

$$
\begin{cases}
\text{d}\mathbf{w}(t) + \mu A\mathbf{w}(t) + (B(\mathbf{u}(t)) - B(\mathbf{u}_\infty)) + \beta (C(\mathbf{u}(t)) - C(\mathbf{u}_\infty)) \\
= (\Phi(t, \mathbf{u}(t)) - \Phi(t, \mathbf{u}_\infty))dW(t), & t \in (0, T), \\
\mathbf{w}(0) = \mathbf{u}_0 - \mathbf{u}_\infty
\end{cases}
$$

(4.30)

since $\Phi(t, \mathbf{u}_\infty) = 0$, for all $t \in (0, T)$. Then $\mathbf{w}(\cdot)$ satisfies the following energy equality:

$$
e^{\theta t}\| \mathbf{w}(t) \|^2_{\mathbb{H}} = \| \mathbf{w}_0 \|^2_{\mathbb{H}} - 2 \int_0^t e^{\theta s}(B(\mathbf{u}(s)) - B(\mathbf{u}_\infty(s)), \mathbf{w}(s))ds + \theta \int_0^t e^{\theta s}\| \mathbf{w}(s) \|^2_{\mathbb{H}}ds
$$

$$- 2 \int_0^t e^{\theta s}(C(\mathbf{u}(s)) - C(\mathbf{u}_\infty(s)), \mathbf{w}(s))ds + \int_0^t e^{\theta s}\| \Phi(s) \|^2_{\mathbb{L}_Q}ds
$$

$$+ 2 \int_0^t e^{\theta s}(\tilde{\Phi}(s)dW(s), \mathbf{w}(s)),
$$

(4.31)
where $\Phi(\cdot) = \Phi(\cdot, u(\cdot)) - \Phi(\cdot, u_\infty)$. A calculation similar to (3.90) yields
\[
e^{\theta t}\|w(t)\|^2_H \leq \|w_0\|^2_H + \left(\theta + 2\eta - \mu\lambda_1\right) \int_0^t e^{\theta s}\|w(s)\|^2_H ds + \int_0^t e^{\theta s}\|\Phi(s)\|^2_{L_Q} ds
+ 2 \int_0^t e^{\theta s}(\tilde{\Phi}(s)dW(s), w(s)).
\]
(4.32)

Taking expectation, and using Hypothesis (H.3) and the fact that the final term is a martingale, we find
\[
e^{\theta t}\mathbb{E}[\|w(t)\|^2_H] \leq \mathbb{E}[\|w_0\|^2_H] + \left(\theta + \eta + L - \mu\lambda_1\right) \int_0^t e^{\theta s}\mathbb{E}[\|w(s)\|^2_H] ds.
\]
(4.33)

Since $\mu$ satisfies (4.29) implies $\theta = \mu\lambda_1 - (2\eta + L) > 0$ and an application of Gronwall’s inequality yields (4.28) is satisfied and hence $u(t)$ converges to $u_\infty$ exponentially in the mean square sense. \hfill \Box

**Theorem 4.8.** Let all conditions given in Theorem 4.7 are satisfied and $\mu > \frac{2\eta + 3L}{\lambda_1}$. (4.34)

Then the strong solution $u(\cdot)$ of the system (4.22) converges to the stationary solution $u_\infty$ of the system (4.2) almost surely exponentially stable.

**Proof.** Let us take $n = 1, 2, \ldots, h > 0$. Then the process $\|u(t) - u_\infty\|^2_H$, for $t \geq nh$ satisfies:
\[
\|u(t) - u_\infty\|^2_H + 2\mu \int_{nh}^t \|u(s) - u_\infty\|^2_Y ds \leq \|u(nh) - u_\infty\|^2_H + 2 \int_{nh}^t \langle B(u(s)) - B(u_\infty), u(s) - u_\infty\rangle ds
- 2 \int_{nh}^t \langle C(u(s)) - C(u_\infty), u(s) - u_\infty\rangle ds
+ 2 \int_{nh}^t ((\Phi(s, u(s)) - \Phi(s, u_\infty))dW(s), u(s) - u_\infty)
+ \int_{nh}^t \|\Phi(s, u(s)) - \Phi(s, u_\infty)\|^2_{L_Q} ds.
\]
(4.35)

Taking supremum from $nh$ to $(n+1)h$ and then taking expectation in (4.35), we find
\[
\mathbb{E}\left[\sup_{nh \leq t \leq (n+1)h} \|u(t) - u_\infty\|^2_H + \mu \int_{nh}^{(n+1)h} \|u(s) - u_\infty\|^2_Y ds\right]
\leq \mathbb{E}[\|u(nh) - u_\infty\|^2_H] + 2\eta\mathbb{E}\left[\int_{nh}^{(n+1)h} \|u(s) - u_\infty\|^2_Y ds\right]
+ \mathbb{E}\left[\int_{nh}^{(n+1)h} \|\Phi(s, u(s)) - \Phi(s, u_\infty)\|^2_{L_Q} ds\right]
+ 2\mathbb{E}\left[\sup_{nh \leq t \leq (n+1)h} \int_{nh}^t ((\Phi(s, u(s)) - \Phi(s, u_\infty))dW(s), u(s) - u_\infty)\right],
\]
(4.36)
Thus by the Borel–Cantelli lemma, there is a finite integer $n_0(\omega)$ such that
\[
\sup_{nh \leq t \leq (n+1)h} \|u(t) - u_\infty\|_H \leq e^{-\frac{1}{2}(\theta-\epsilon)nh}, \quad \mathbb{P}\text{-a.s.,}
\]
for all $n \geq n_0$, which completes the proof. \qed
4.3. Stabilization by a multiplicative noise. It is an interesting question to ask about the exponential stability of the stationary solution for small values of $\mu$. For the 2D stochastic Navier-Stokes equations, the authors in [8] obtained such a stabilization result with the system perturbed by a one dimensional Wiener process $W(t)$ and for $\Phi(t, u(t)) = \sigma(u(t) - u_\infty)$, where $\sigma$ is a real number. The same method we apply to obtain the stabilization of the stochastic convective Brinkman-Forchheimer equations by using the same multiplicative noise. We have the following stabilization result of the stochastic convective Brinkman-Forchheimer equations by noise:

**Theorem 4.9.** Let the equations (3.2) be perturbed by an one dimensional Wiener process $W(t)$ with $\Phi(t, u(t)) = \sigma(u(t) - u_\infty)$, where $\sigma$ is a real number. Then, there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 0$, such that for all $\omega \notin \Omega_0$, there exists $T(\omega) > 0$ such that any strong solution $u(t)$ to (3.2) satisfies

$$\|u(t) - u_\infty\|_{L^2}^2 \leq \|u_0 - u_\infty\|_{L^2}^2 e^{-\zeta t}, \text{ for any } t \geq T(\omega),$$

where $\zeta = \frac{1}{2} (\sigma^2 + 2\mu \lambda_1 - 2\eta) > 0$. In particular, the exponential stability of sample paths with probability one holds if $\zeta > 0$.

**Proof.** We know that the process $u(\cdot) - u_\infty$ satisfies the following energy equality:

$$\|u(t) - u_\infty\|_{L^2}^2 + 2\mu \int_0^t \|u(s) - u_\infty\|_{H^1}^2 \, ds$$

$$= \|u_0 - u_\infty\|_{L^2}^2 - 2 \int_0^t (\mathcal{B}(u(s)) - B(u_\infty), u(s) - u_\infty) \, ds$$

$$- 2 \int_0^t (\mathcal{C}(u(s)) - \mathcal{C}(u_\infty), u(s) - u_\infty) \, ds$$

$$+ 2 \int_0^t ((\Phi(s, u(s))) \, dW(s), u(s) - u_\infty) + \int_0^t \|\Phi(s, u(s))\|_{L^2}^2 \, ds.$$ (4.44)

Applying Itô’s formula to the process $\log \|u(\cdot) - u_\infty\|_{L^2}^2$, we find

$$\log \|u(t) - u_\infty\|_{H^1}^2$$

$$= \log \|u_0 - u_\infty\|_{H^1}^2 - 2\mu \int_0^t \frac{\|u(s) - u_\infty\|_{H^1}^2}{\|u(s) - u_\infty\|_{H^1}^2} \, ds$$

$$- 2 \int_0^t \frac{(\mathcal{B}(u(s)) - B(u_\infty), u(s) - u_\infty)}{\|u(s) - u_\infty\|_{H^1}^2} \, ds - 2 \int_0^t \frac{(\mathcal{C}(u(s)) - \mathcal{C}(u_\infty), u(s) - u_\infty)}{\|u(s) - u_\infty\|_{H^1}^2} \, ds$$

$$+ 2 \int_0^t \frac{\sigma \|u(s) - u_\infty\|_{H^1}^2}{\|u(s) - u_\infty\|_{H^1}^2} \, dW(s) + \int_0^t \frac{\sigma^2 \|u(s) - u_\infty\|_{H^1}^2}{\|u(s) - u_\infty\|_{H^1}^2} \, ds - \frac{1}{2} \int_0^t \frac{4\sigma^2 \|u(s) - u_\infty\|_{H^1}^4}{\|u(s) - u_\infty\|_{H^1}^4} \, ds$$

$$\leq \log \|u_0 - u_\infty\|_{H^1}^2 + (-2\mu \lambda_1 + 2\eta - \sigma^2) t + 2\sigma W(t),$$ (4.45)

where we used (2.17) and (2.20). We know that $\lim_{t \to \infty} \frac{W(t)}{t} = 0$, $\mathbb{P}$-a.s. Thus, one can assure the existence of a set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 0$, such that for every $\omega \notin \Omega_0$, there exists $T(\omega) > 0$ such that for all $t \geq T(\omega)$, we have

$$\frac{2\sigma W(t)}{t} \leq \frac{1}{2} (2\mu \lambda_1 - 2\eta + \sigma^2).$$
Hence from (4.45), we finally have
\[
\log \| u(t) - u_\infty \|_{H}^2 \leq \log \| u_0 - u_\infty \|_{H}^2 - \frac{1}{2}(\sigma^2 + 2\mu\lambda_1 - 2\eta)t,
\]
for any \( t \geq T(\omega) \), which completes the proof. \( \square \)

5. INVARIANT MEASURES AND ERGODICITY

In this section, we discuss the existence and uniqueness of invariant measures and ergodicity results for the stochastic convective Brinkman-Forchheimer equations (3.2). Let us first provide the definitions of invariant measures, ergodic, strongly mixing and exponentially mixing invariant measures. Let \( X \) be a Polish space (complete separable metric space).

Definition 5.1. A probability measure \( \eta \) on \( (X, \mathcal{B}(X)) \) is called an invariant measure or a stationary measure for a given transition probability function \( P(t, x, dy) \) if it satisfies
\[
\eta(A) = \int_X P(t, x, A) d\eta(x),
\]
for all \( A \in \mathcal{B}(X) \) and \( t > 0 \). Equivalently, if for all \( \varphi \in C_b(X) \) (the space of bounded continuous functions on \( X \)), and all \( t \geq 0 \),
\[
\int_X \varphi(x) d\eta(x) = \int_X (P_t \varphi)(x) d\eta(x),
\]
where the Markov semigroup \( (P_t)_{t \geq 0} \) is defined by
\[
P_t \varphi(x) = \int_X \varphi(y) P(t, x, dy).
\]

Definition 5.2 (Theorem 3.2.4, Theorem 3.4.2, [15], [36]). Let \( \eta \) be an invariant measure for \( (P_t)_{t \geq 0} \). We say that the measure \( \eta \) is an ergodic measure, if for all \( \varphi \in \tilde{L}^2(X; \eta) \), we have
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T (P_t \varphi)(x) dt = \int_X \varphi(x) d\eta(x) \quad \text{in} \quad \tilde{L}^2(X; \eta).
\]
The invariant measure \( \eta \) for \( (P_t)_{t \geq 0} \) is called strongly mixing if for all \( \varphi \in \tilde{L}^2(X; \eta) \), we have
\[
\lim_{t \to +\infty} P_t \varphi(x) = \int_X \varphi(x) d\eta(x) \quad \text{in} \quad \tilde{L}^2(X; \eta).
\]
The invariant measure \( \eta \) for \( (P_t)_{t \geq 0} \) is called exponentially mixing, if there exists a constant \( k > 0 \) and a positive function \( \Psi(\cdot) \) such that for any bounded Lipschitz function \( \varphi \), all \( t > 0 \) and all \( x \in X \),
\[
\left| P_t \varphi(x) - \int_X \varphi(x) d\eta(x) \right| \leq \Psi(x) e^{-kt} \| \varphi \|_{Lip},
\]
where \( \| \cdot \|_{Lip} \) is the Lipschitz constant.

Clearly exponentially mixing implies strongly mixing. Theorem 3.2.6, [15] states that if \( \eta \) is the unique invariant measure for \( (P_t)_{t \geq 0} \), then it is ergodic. The interested readers are referred to see [15] for more details on the ergodicity for infinite dimensional systems.

Let us now show that there exists a unique invariant measure for the Markovian transition probability associated to the system (3.2). Moreover, we establish that the invariant measure is ergodic and strongly mixing (in fact exponentially mixing). Let \( u(t, u_0) \) denotes the unique
strong solution of the system (3.2) with the initial condition \( u_0 \in \mathbb{H} \). Let \((P_t)_{t \geq 0}\) be the Markovian transition semigroup in the space \( C_b(\mathbb{H}) \) associated to the system (3.2) defined by

\[
P_t\varphi(u_0) = \mathbb{E}[\varphi(u(t, u_0))] = \int_{\mathbb{H}} \varphi(y) P(t, u_0, dy) = \int_{\mathbb{H}} \varphi(y) \eta_t(u_0)(dy), \quad \varphi \in C_b(\mathbb{H}),
\]

where \( P(t, u_0, dy) \) is the transition probability of \( u(t, u_0) \) and \( \eta_t(u_0) \) is the law of \( u(t, u_0) \). The semigroup \((P_t)_{t \geq 0}\) is Feller, since the solution to (3.2) depends continuously on the initial data. From (5.1), we have

\[
P_t\varphi(u_0) = \langle \varphi, \eta_t(u_0) \rangle = \langle P_t \varphi, \eta \rangle,
\]

where \( \eta \) is the law of the initial data \( u_0 \in \mathbb{H} \). Thus from (5.2), we have \( \eta_{t,u_0} = P_t^* \eta \). We say that a probability measure \( \eta \) on \( \mathbb{H} \) is an invariant measure if

\[
P_t^* \eta = \eta, \quad \text{for all } t \geq 0.
\]

That is, if a solution has law \( \eta \) at some time, then it has the same law for all later times. For such a solution, it can be shown by Markov property that for all \((t_1, \ldots, t_n)\) and \( \tau > 0, (u(t_1 + \tau, u_0), \ldots, u(t_n + \tau, u_0)) \) and \((u(t_1, u_0), \ldots, u(t_n, u_0))\) have the same law. Then, we say that the process \( u \) is stationary. For more details, the interested readers are referred to see [15, 17], etc.

**Theorem 5.3.** Let \( u_0 \in \mathbb{H} \) be given. Then, for \( \mu > \frac{K}{\lambda_1} \), there exists an invariant measure for the system (3.2) with support in \( \mathbb{V} \).

**Proof.** Let us use the energy equality obtained in (3.86) to find

\[
\|[u(t)]^2_{\mathbb{H}} + 2\mu \int_0^t \|[u(s)]^2_{\mathbb{V}} ds + 2\beta \int_0^t \|[u(s)]^{r+1}_{\mathbb{L}_{r+1}} ds
\]

\[
= \|[u_0]^2_{\mathbb{H}} + \int_0^t \|[\Phi(s, u(s))]^2_{\mathbb{L}_2} ds + 2\beta \int_0^t \|[\Phi(s, u(s))]^{r+1}_{\mathbb{L}_{r+1}} ds.
\]

Taking expectation in (5.4), using Hypothesis (3.2) (H.3), Poincaré inequality and the fact that the final term is a martingale having zero expectation, we obtain

\[
\mathbb{E}\left\{ \|[u(t)]^2_{\mathbb{H}} + \left(2\mu - \frac{K}{\lambda_1}\right) \int_0^t \|[u(s)]^2_{\mathbb{V}} ds + 2\beta \int_0^t \|[u(s)]^{r+1}_{\mathbb{L}_{r+1}} ds \right\} \leq \mathbb{E}[\|[u_0]^2_{\mathbb{H}}].
\]

Thus, for \( \mu > \frac{K}{\lambda_1} \), we have

\[
\frac{\left(2\mu - \frac{K}{\lambda_1}\right)}{t} \mathbb{E}\left[ \int_0^t \|[u(s)]^2_{\mathbb{V}} ds \right] \leq \frac{1}{T_0} \|[u_0]^2_{\mathbb{H}}, \quad \text{for all } t > T_0.
\]

Using Markov’s inequality, we get

\[
\lim_{R \to \infty} \sup_{T > T_0} \left[ \frac{1}{T} \int_0^T \mathbb{P}\left\{ \|[u(t)]_{\mathbb{V}} > R \right\} dt \right] \leq \lim_{R \to \infty} \sup_{T > T_0} \frac{1}{R^2} \mathbb{E}\left[ \frac{1}{T} \int_0^T \|[u(t)]^2_{\mathbb{V}} dt \right] = 0.
\]

Hence along with the estimate in (5.7), using the compactness of \( \mathbb{V} \) in \( \mathbb{H} \), it is clear by a standard argument that the sequence of probability measures

\[
\eta_{t,u_0}(\cdot) = \frac{1}{t} \int_0^t \Pi_{s,u_0}(\cdot) ds, \quad \text{where } \Pi_{t,u_0}(\Lambda) = \mathbb{P}(u(t, u_0) \in \Lambda), \ \Lambda \in \mathcal{B}(\mathbb{H}),
\]
is tight, that is, for each \( \delta > 0 \), there is a compact subset \( K \subset \mathbb{H} \) such that \( \eta_n(K^c) \leq \delta \), for all \( t > 0 \), and so by the Krylov-Bogoliubov theorem (or by a result of Chow and Khasminskii see [10]) \( \eta_{t_n, u_0} \to \eta \), weakly for \( n \to \infty \), and \( \eta \) results to be an invariant measure for the transition semigroup \( (P_t)_{t \geq 0} \), defined by

\[
P_t \varphi(u_0) = \mathbb{E}[\varphi(u(t, u_0))],
\]

for all \( \varphi \in C_b(\mathbb{H}) \), where \( u(\cdot) \) is the unique strong solution of (3.2) with the initial condition \( u_0 \in \mathbb{H} \).

Now we establish the uniqueness of invariant measure for the system (3.2). Similar results for 2D stochastic Navier-Stokes equations is established in [17] and for the stochastic 2D Oldroyd models is obtained in [38]. The following result provide the exponential stability for all \( \varphi \).

**Theorem 5.4.** Let \( u(\cdot) \) and \( v(\cdot) \) be two solutions of the system (3.2) with \( r > 3 \) and the initial data \( u_0, v_0 \in \mathbb{H} \), respectively. Then, for the condition given in (4.29), we have

\[
\mathbb{E} \left[ \|u(t) - v(t)\|_{\mathbb{H}}^2 \right] \leq \|u_0 - v_0\|_{\mathbb{H}}^2 e^{-r(\mu - (2\eta + L))t},
\]

where \( \eta \) is defined in (2.15).

**Proof.** Let us define \( w(t) = u(t) - v(t) \). Then, \( w(\cdot) \) satisfies the following energy equality:

\[
\begin{align*}
\|w(t)\|_{\mathbb{H}}^2 &= \|w_0\|_{\mathbb{H}}^2 - 2\mu \int_0^t \|w(s)\|_{\mathbb{H}}^2 ds - 2\beta \int_0^t \langle C(u(s)) - C(v(s)), w(s) \rangle ds \\
&\quad - 2 \int_0^t \langle B(u(s)) - B(v(s)), w(s) \rangle ds + \int_0^t \|\tilde{\Phi}(s)\|_{L_2^Q}^2 ds \\
&\quad + 2 \int_0^t \langle \tilde{\Phi}(s) dW(s), w(s) \rangle,
\end{align*}
\]

where \( \Phi(\cdot, u(\cdot)) - \Phi(\cdot, v(\cdot)) \). Taking expectation in (5.9) and then using the Poincaré inequality, Hypothesis (H.3), (2.17) and (2.20), one can easily see that

\[
\mathbb{E} \left[ \|w(t)\|_{\mathbb{H}}^2 \right] \leq \|w_0\|_{\mathbb{H}}^2 - \mu \lambda_1 \int_0^t \mathbb{E} \left[ \|w(s)\|_{\mathbb{H}}^2 \right] ds + (2\eta + L) \int_0^t \mathbb{E} \left[ \|w(s)\|_{\mathbb{H}}^2 \right] ds,
\]

where \( \eta \) is defined in (2.15). Thus, an application of the Gronwall’s inequality yields

\[
\mathbb{E} \left[ \|w(t)\|_{\mathbb{H}}^2 \right] \leq \|w_0\|_{\mathbb{H}}^2 e^{-r(\mu - (2\eta + L))t},
\]

and for \( \mu > \frac{2\eta + L}{\lambda_1} \), we obtain the required result (5.8). \( \square \)

For \( 2\beta \mu \geq 1 \), the results obtained in the Theorem 5.4 can be established for \( \mu > \frac{1}{\lambda_1} \), using the estimate (2.26). Let us now establish the uniqueness of invariant measures for the system (3.2) obtained in Theorem 5.3. We prove the case of \( r > 3 \) only and the case of \( r = 3 \) follows similarly.

**Theorem 5.5.** Let the conditions given in Theorem 5.4 hold true and \( u_0 \in \mathbb{H} \) be given. Then, for the condition given in (4.29), there is a unique invariant measure \( \eta \) to system (3.2). The measure \( \eta \) is ergodic and strongly mixing, i.e.,

\[
\lim_{t \to \infty} P_t \varphi(u_0) = \int_\mathbb{H} \varphi(v_0) d\eta(v_0), \ \text{\( \eta \)-a.s., for all} \ u_0 \in \mathbb{H} \ \text{and} \ \varphi \in C_b(\mathbb{H}).
\]
Proof. For \( \varphi \in \text{Lip}(\mathbb{R}) \) (Lipschitz \( \varphi \)), since \( \eta \) is an invariant measure, we have

\[
\left| P_t \varphi(u_0) - \int_{\mathbb{R}} \varphi(v_0) \eta(dv_0) \right| = \left| \mathbb{E}[\varphi(u(t, u_0))] - \int_{\mathbb{R}} P_t \varphi(v_0) \eta(dv_0) \right| = \left| \int_{\mathbb{R}} \mathbb{E}[\varphi(u(t, u_0)) - \varphi(u(t, v_0))] \eta(dv_0) \right| \\
\leq L \varphi \int_{\mathbb{R}} \mathbb{E}[\|u(t, u_0) - u(t, v_0)\|_{\mathbb{R}}] \eta(dv_0) \\
\leq L \varphi e^{-\frac{(\mu \lambda - (2n+L))t}{2}} \int_{\mathbb{R}} \|u_0 - v_0\|_{\mathbb{R}} \eta(dv_0) \\
\leq L \varphi e^{-\frac{(\mu \lambda - (2n+L))t}{2}} \left( \|u_0\|_{\mathbb{R}} + \int_{\mathbb{R}} \|v_0\|_{\mathbb{R}} \eta(dv_0) \right) \rightarrow 0 \text{ as } t \to \infty, \quad (5.13)
\]

since \( \int_{\mathbb{R}} \|v_0\|_{\mathbb{R}} \eta(dv_0) < +\infty \). Hence, we deduce (5.12), for every \( \varphi \in C_b(\mathbb{R}) \), by the density of \( \text{Lip}(\mathbb{R}) \) in \( C_b(\mathbb{R}) \). Note that, we have a stronger result that \( P_t \varphi(u_0) \) converges exponentially fast to equilibrium, which is the exponential mixing property. This easily gives uniqueness of the invariant measure also. Indeed, if \( \tilde{\eta} \) is another invariant measure, then

\[
\left| \int_{\mathbb{R}} \varphi(u_0) \eta(du_0) - \int_{\mathbb{R}} \varphi(v_0) \tilde{\eta}(dv_0) \right| = \left| \int_{\mathbb{R}} P_t \varphi(u_0) \eta(du_0) - \int_{\mathbb{R}} P_t \varphi(v_0) \tilde{\eta}(dv_0) \right| = \left| \int_{\mathbb{R}} \int_{\mathbb{R}} [P_t \varphi(u_0) - P_t \varphi(v_0)] \eta(du_0) \tilde{\eta}(dv_0) \right| \\
\leq L \varphi e^{-\frac{(\mu \lambda - (2n+L))t}{2}} \int_{\mathbb{R}} \|u_0 - v_0\|_{\mathbb{R}} \eta(du_0) \tilde{\eta}(dv_0) \rightarrow 0 \text{ as } t \to \infty. \quad (5.14)
\]

By Theorem 3.2.6, [15], since \( \eta \) is the unique invariant measure for \( (P_t)_{t \geq 0} \), we know that it is ergodic.

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