EXPRESSING MATRICES INTO PRODUCTS OF COMMUTATORS OF INVOLUTIONS, SKEW-INVOLUTIONS, FINITE ORDER AND SKEW FINITE ORDER MATRICES

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Abstract. Let $R$ be an associative ring with unity 1 and consider that $2, k$ and $2k \in \mathbb{N}$ are invertible in $R$. For $m \geq 1$ denote by $UT_n(m, R)$ and $UT_\infty(m, R)$, the subgroups of $UT_n(R)$ and $UT_\infty(R)$ respectively, which have zero entries on the first $m - 1$ super diagonals. We show that every element on the groups $UT_n(m, R)$ and $UT_\infty(m, R)$ can be expressed as a product of two commutators of involutions and also, can be expressed as a product of two commutators of skew-involutions and involutions in $UT_\infty(m, R)$. Similarly, denote by $UT_\infty^{(s)}(R)$ the group of upper triangular infinite matrices whose diagonal entries are $s$th roots of 1. We show that every element of the groups $UT_n(\infty, R)$ and $UT_\infty(m, R)$ can be expressed as a product of $4k - 6$ commutators all depending of powers of elements in $UT_\infty^{(k)}(m, R)$ of order $k$ and, also, can be expressed as a product of $8k - 6$ commutators of skew finite matrices of order $2k$ and matrices of order $2k$ in $UT_\infty^{(2k)}(m, R)$. If $R$ is the complex field or the real number field we prove that, in $SL_n(R)$ and in the subgroup $SL_{\omega}(R)$ of the Vershik-Kerov group over $R$, each element in these groups can be decomposed into a product of commutators of elements as described above.

1. Introduction

It is a classical question whether the elements of a ring or a group can be expressed as sums or products of elements of some particular set. For example, expressing matrices as a product of involutions was studied by several authors \cite{6,7,11,12}. In case of product of commutators we can see \cite{5,8,9}. Also in \cite{5} the author shows the necessary and sufficient condition for a matrix over a field to be the product of an involution and a skew-involution.

Bier and Waldemar in \cite{1} studied the commutators of elements of the group $UT_\infty(m, R)$ of infinite unitriangular matrices over an associative ring $R$ with unity 1 containing exactly all these matrices, which have zero entries on the first $m - 1$ superdiagonals. They prove that every unitriangular matrix of a specified form is a commutator of two other unitriangular matrices. Considering $\omega$ a $k$th root of unity in $R$, recently Gargate in \cite{2} prove that every element of the group $UT_\infty(R)$ can be expressed as a product of $4k - 6$ commutators all depending of powers of elements in $UT_\infty^{(k)}(R)$ of order $k$.

In the section 3 following the same direction, we study the subgroup $UT_n(m, R)$ and $UT_\infty(m, R)$ of $UT_n(R)$ and $UT_\infty(m, R)$ respectively which have zero entries on the first $m - 1$ superdiagonals.

The main result of this paper is stated as follows:

\textit{Key words and phrases.} Upper triangular matrices; finite order; commutators.
Theorem 1.1. Let $R$ be an associative ring with unity $1$ and suppose that $2, k$ and $2k$ are invertible in $R$. Then every matrix in $UT_{\infty}(m, R)$ and $UT_n(m, R)$ can be expressed as a product of at most:

1. Two commutators of involutions in $\pm UT_{\infty}^{(2)}(m, R)$.
2. Two commutators of skew-involutions and involutions both in $\pm UT_{\infty}^{(2)}(m, R)$.
3. $4k - 6$ commutators of matrices of order $k$ in $UT_{\infty}^{(k)}(m, R)$.
4. $8k - 6$ commutators of matrices of skew order $2k$ and matrices of order finite $2k$ in $\pm UT_{\infty}^{(2k)}(m, R)$.

Paras and Salinasan in [5] shows the necessary and sufficient condition for a matrix $A$ over a field to be the product of an involution and a skew-involution.

In the section 4, the authors give a necessary and sufficient condition for every matrix in $SL_n(C)$ to be written as product of commutators of one involution and one skew-involution. Also, we give conditions for what every matrix in $SL_n(C)$ to be written as products of commutators into matrices of order finite and skew-order finite. Here we have the following results:

Theorem 1.2. Let $A \in SL_n(C)$. Then $A$ is product of commutators of one involution and one skew-involution if and only if there is a matrix $B \in SL_n(C)$ such that $A = -B^2$ and $B$ is similar to $-B^{-1}$.

And in the general case:

Theorem 1.3. Let $A \in SL_n(C)$. If there is a matrix $B \in SL_n(C)$ such that $A = B^k$ and $B$ is the product of two matrices of order $k$ then $A$ is product of $2k - 3$ commutators of elements of order $k$.

Also,

Theorem 1.4. Let $A \in SL_n(C)$. If there is a matrix $B \in SL_n(C)$ such that $A = -B^{2k}$ and $B$ is the product of one skew order $2k$ matrix and one matrix of order $2k$ then $A$ is product of $4k - 3$ commutators of these elements.

Considering the results obtained in Gargate [2], we have the following

Theorem 1.5. All element in $SL_n(C)$ can be written as a product of at most:

1. Two commutators of involutions in $\pm UT_{n}^{(2)}(C)$.
2. Two commutators of skew-involutions and involutions both in $\pm UT_{n}^{(2)}(C)$.
3. $4k - 6$ commutators of matrices of order $k$ in $UT_{n}^{(k)}(C)$.
4. $8k - 6$ commutators of matrices of skew order $2k$ and matrices of order finite $2k$ in $\pm UT_{n}^{(2k)}(C)$.

Also we consider $GL_{V,K}(\infty, C)$ the Vershik-Kerov group and we have the following result:

Theorem 1.6. Assume that $R = C$ is a complex field or the real number field. Then every element of the group $SL_{V,K}(\infty, m, C)$ can be expressed as a product of at most:

1. Two commutators of involutions.
2. Two commutators of skew-involutions and involutions.
3. $4k - 6$ commutators of matrices of order $k$.  

4.) 8k − 6 commutators of matrices of skew order 2k and matrices of order finite 2k.
in $GL_{V,K}(\infty, m, \mathbb{C})$.

2. Preliminaries

Let $R$ be an associative ring with identity 1. Denote by $T_n(R)$ and $T_\infty(R)$ the groups of $n \times n$ and infinite upper triangular matrices over a ring $R$ and denote by $UT_n(R)$ and $UT_\infty(R)$ the subgroups of $T_n(R)$ and $T_\infty(R)$, respectively, whose entries on the main diagonal are equal to unity 1.

For $m \geq 1$ we denote the subgroups $T_n(m, R)$ and $T_\infty(m, R)$ of $T_n(R)$ and $T_\infty(R)$, respectively, containing exactly all those matrices which have zero entries on the first $m − 1$ super diagonals. Analogously we denote by $UT_n(m, R)$ and $UT_\infty(m, R)$ the subgroups of $T_n(m, R)$ and $T_\infty(m, R)$ respectively whose entries on the $m$th super diagonal are equal to unity 1.

Also, define the subgroups, for any $s \in \mathbb{N}$ and $m \geq 1$:

$±UT_\infty^{(s)}(R) = \{ g \in T_\infty(R), \ g_{ii}^s = ±1 \}$,

$±UT_\infty^{(s)}(m, R) = \{ g \in T_\infty(m, R), \ g_{ii}^s = ±1 \}$

$±D_\infty^{(s)}(R) = \{ g \in UT_\infty^{(s)}(R), \ g_{ij} = 0, \ if \ i \neq j \}$,

$±D_\infty^{(s)}(m, R) = \{ g \in UT_\infty^{(s)}(m, R), \ g_{ij} = 0, \ if \ i \neq j \}$.

Consider $k \geq 3$, a matrix $A \in M_n(R)$ (or in $T_\infty(R)$) is called an involution if $A^2 = I$, a skew-involution if $A^2 = −I$, a finite order $k$ if $A^k = I$ and a skew finite order $k$ if $A^k = −I$, where $I = I_n$ is the identity matrix in $M_n(R)$ (or $I = I_\infty$ in $T_\infty(R)$, respectively).

Denote by $E_{ij}$ the finite or infinite matrix with a unique nonzero entry equal to 1 in the position $(i, j)$, so $A = \sum_{1 \leq i \leq j \leq n} a_{ij} E_{ij}$ ($A = \sum_{i,j \in \mathbb{N}} a_{ij} E_{i,j}$) is the $n \times n$ (infinite $\mathbb{N} \times \mathbb{N}$) matrix with $a_{ij}$ in the position $(i, j)$. Denote by $[\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}$ the commutator of two elements $\alpha$ and $\beta$ of a group $G$.

Denote by $J_m(\infty, R)$ the set of all infinite matrices in $T_\infty(R)$ in which all the entries outside of the $m$th super diagonal equal 0. Let $A \in UT_\infty(m, R)$ and denote by $J_m(A)$ the matrix of $J_m(\infty, R)$ that has the same entries in the $m$th super diagonal as $A$. Denote by $Z$ the center of the ring $R$, and by $D_\infty(Z)$ the subring of all diagonal infinite matrices with entries in $Z$.

We say that $A \in UT_\infty(m, R)$ is $m$-coherent if there is a sequence $\{D_i\}_{i \geq 0}$ of elements of $D_\infty(Z)$ such that

$A = \sum_{i=0}^{\infty} D_i J_m(A)^i$.

If $D_0 = D_1 = I_\infty$ the the sequence is called normalized. If $m = 1$ the matrix is called coherent.

We will denote by $(*, *, *, \cdots)_m$ the $m$th super diagonal of a matrix $A \in T_\infty(m, R)$, for $m \geq 1$. 


3. Expressing Matrix into Products of Commutators

In this section we assume that 2, $k$ or $2k$ are invertible elements in $R$, this the according to each case we will study. Also we will can assume that $i \in R$.

**Remark 3.1.** Let $G = M_n(R)$ or $T_\infty(R)$, then

1.) If $\alpha \in G$ is a product of $r$ involution (or skew-involutions and involutions, or elements of order $k$, or skew-finite order $2k$ and of order $2k$), then for every $\beta \in G$ the conjugate $\beta \alpha \beta^{-1}$ is a product of $r$ involutions (or skew-involutions and involutions, or elements of order $k$, or skew-finite order $2k$ and of order $2k$, respectively).

2.) If $\alpha$ is a product of $r$ commutators of involutions (or skew-involutions and involutions, or elements of order $k$, or skew-finite order $2k$) then for every $\beta \in G$ the conjugate $\beta \alpha \beta^{-1}$ is a product of $r$ commutators of involutions (or skew-involutions and involutions, or elements of order $k$, or skew-finite order $2k$ and of order $2k$, respectively) as well.

The following results we adapted from Hou [9].

**Remark 3.2.** If $A \in UT_\infty(m, R)$ is $m$-coherent then, for $r \geq 2$, $A^r$ is $m$-coherent as well.

**Proof.** If $D = \text{diag}(a_1, a_2, a_3, \cdots) \in D(\infty, Z)$ define $S(D) = \text{diag}(a_2, a_3, a_4, \cdots) \in D(\infty, Z)$ and in this case we have that

$$\forall J \in J_m(\infty, R), JD = S^m(D)J,$$

then

$$A^2 = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} D_k J_m(A)^k D_i J_m(A)^i$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} D_k S^{km}(D_i) J^{k+i}$$

$$= \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} D_i S^{im}(D_{k-i}) \right) J(A)^k,$$

so, $A^2$ is $m$-coherent. For the case $k \geq 3$ follows similarly from Gargate in [2]. □

**Lemma 3.3.** Let $J \in J_1(\infty, R)$, then there exists a coherent matrix $A \in UT_\infty(R)$ such that $J(A) = J$ and $A$ is the commutator of one skew-involution and one involution in $\pm UT_\infty(R)$.

**Proof.** Suppose that

$$J = \sum_{i=1}^{\infty} a_{i,i+1} E_{i,i+1},$$

and consider

$$B = \begin{bmatrix} -i & -\frac{1}{2}ia_{23} & \cdots \\ i & -\frac{1}{2}ia_{45} & \cdots \\ -i & -\frac{1}{2}ia_{45} & \cdots \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & \frac{1}{2}a_{12} & -1 \\ 1 & \frac{1}{2}a_{34} & -1 \end{bmatrix},$$

where $i$ is the imaginary number.
We can observe that $B \in \pm UT_\infty^{(2)}(R)$ is a skew-involution and $C \in \pm UT_\infty^{(2)}(R)$ is an involution. Also, define $A = [B, C] = BCB^{-1}C^{-1} = -(BC)^2$, is not difficult to verify that $J(A) = J$. Now, observe that $BC$ is coherent, because

$$BC = -iI_\infty - \frac{i}{2} \sum_{i=1}^{\infty} a_{i,i+1}E_{i,i+1} - \frac{1}{4} \sum_{i=1}^{\infty} a_{2i,2i+1}a_{2i+1,2i+2}E_{2i,2i+2} = \sum_{k=0}^{2} D_k J(BC)^k$$

where $D_0 = -iI_\infty, D_1 = I_\infty$ and $D_2 = \sum_{i=1}^{\infty} E_{2i,2i}$ and by Remark 3.2 we conclude that $A$ is coherent. 

And, for the group $UT_\infty(m, R)$, we have the following Lemma:

**Lemma 3.4.** Let $J \in J_m(\infty, R)$, then there exists a $m$-coherent matrix $A \in UT_\infty(m, R)$ such that $J_m(A) = J$ and $A$ is the commutator of

1.) Two involutions in $\pm UT_\infty^{(2)}(m, R)$.
2.) One skew-involution and one involution both in $\pm UT_\infty^{(2)}(m, R)$.
3.) Matrices of order $k$ in $UT_\infty^{(k)}(m, R)$.
4.) One skew finite order $2k$ and one matrix of order finite $2k$ in $\pm UT_\infty^{(2k)}(m, R)$.

**Proof.**
1.) Consider

$$J = \sum_{i=1}^{\infty} a_{i,m+i}E_{i,m+i}$$

and define $B, C \in \pm UT_\infty(m, R)$ as

$$B = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & \cdots & 0 & \frac{1}{2}a_{2,2} & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{2}a_{4,4} & \cdots \\
\vdots & \vdots & & & & & & \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & & & & & & \\
\end{bmatrix}$$

where

$$\text{diag}(B) = (1, 1, 1, \cdots, 1, -1, -1, \cdots, -1, 1, 1, \cdots)_m$$

and $J_m(B)$ the corresponding matrix with entries in the $m$th super diagonal equal to $(0, \frac{1}{2}a_{2,2}, 0, 0, \frac{1}{2}a_{4,4}, 0, \cdots)_m$ and
It is not difficult to prove that $BC$ and $C$ are involutions, and $J_m(BCB^{-1}C^{-1}) = J$.

In order to prove that $BC$ is $m$-coherent we can observe two cases:

- If $m$ is an odd number then
  
  $$BC = I_\infty + \frac{1}{2} \sum_{i=1}^{\infty} a_{i,i+m} E_{i,i+m},$$

  and, in this case we consider $D_0 = D_1 = I_\infty$.

- If $m$ an even number we have
  
  $$BC = I_\infty + \frac{1}{2} \sum_{i=1}^{\infty} a_{i,i+m} E_{i,i+m} + \frac{1}{4} \sum_{i=1}^{\infty} a_{2i,2i+m} a_{2i+m,2i+2m} E_{2i,2i+2m},$$

  and we consider $D_0 = D_1 = I_\infty$ and $D_2 = \sum_{i=1}^{\infty} E_{2i,2i}$. So we prove that $BC$ is $m$-coherent.

2.) We can consider the same form of matrix $B$ and $C$ but with some different entries, for instance, in $B$

$$\text{diag}(B) = \left( -i, -i, \ldots, -i, i, i, \ldots, i, -i, -i, \ldots \right),$$

where $i$ is the imaginary number and the corresponding matrix $J_m(B)$ with entries in the $m$th super diagonal equals to $(\frac{1}{2}a_{1,m+1}, 0, \frac{1}{2}a_{3,m+3}, 0, \frac{1}{2}a_{5,m+5}, 0, \cdots)_m$. Also

$$\text{diag}(C) = \left( 1, 1, \cdots, 1, -1, -1, \cdots, -1, 1, 1, \cdots \right),$$

and the corresponding matrix $J_m(C)$ with entries in the $m$th super diagonal equals to $(\frac{1}{2}a_{12}, 0, \frac{1}{2}a_{34}, 0, \frac{1}{2}a_{56}, \cdots)_m$.

Observe that $B$ is skew-involution and $C$ is an involution and the proof that $BC$ is $m$-coherent follows similarly from item (1) above.

3.) For $k > 2$, consider $\omega$ an $k$th root of unity in $R$. Then we consider the matrix $B$ and $C$ of the item (1) but with diagonal

$$\text{diag}(B) = \left( 1, 1, \cdots, 1, \omega, \omega, \cdots, \omega, 1, 1, \cdots \right),$$
and the corresponding $J_m(B)$ with entries in the $m$th super diagonal equals to $(0, \frac{1}{k}a_{2,m+2}, 0, \frac{1}{k}a_{4,m+4}, 0, \frac{1}{k}a_{6,m+6}, \cdots)_m$, also

$$\text{diag}(C) = \left(1, 1, \cdots, 1, \omega^{-1}, \omega^{-1}, \cdots, \omega^{-1}, 1, 1, \cdots\right)$$

and the entries of $J_m(C)$ in the $m$th super diagonal equals to

$$\left(\frac{1}{k}a_{1,m+1}, 0, \frac{1}{k}a_{3,m+3}, 0, \frac{1}{k}a_{5,m+5}, \cdots\right)_m.$$  

We can observe that $B^k = C^k = I$ and similarly to Lemma 3.5 in Gargate we define $A = (BC)^k$ that is product of $2k - 3$ commutators.

4.) For $k \geq 2$ all solutions of the equation $X^{2k} = 1$ are of the form $(i\omega)^j$ where $i$ is the imaginary number, $\omega$ be an $k$th root of unity such that $i, \omega \in R$ and $j = 0, 1, 2, \cdots, 2k - 1$. In this case, we consider the above matrices $B$ and $C$ but with diagonal entries

$$\text{diag}(B) = \left(-i, -i, \cdots, -i, i\omega, i\omega, \cdots, i\omega, -i, -i, \cdots\right)$$

and the corresponding $J_m(B)$ with entries in the $m$th super diagonal equals to

$$(0, -\frac{1}{k}i\omega a_{2,m+2}, 0, -\frac{1}{k}i\omega a_{4,m+4}, 0, \frac{1}{k}i\omega a_{6,m+6}, \cdots)_m$$

and the corresponding $J_m(C)$ with entries in the $m$th super diagonal equals to

$$(\frac{1}{k}a_{1,m+1}, 0, \frac{1}{k}a_{3,m+3}, 0, \frac{1}{k}a_{5,m+5}, \cdots)_m.$$  

Here we can observe that $B^{2k} = -I$ and $C^{2k} = I$. From the Lemma 3.6 in Gargate we can observe that, if $B^{2k} = -I$ in the proof of this Lemma, we obtain that $(BC)^{2k} = -F_{2k}(B,C)$ where $F_{2k}(B,C)$ is the product of $4k - 3$ commutators whose entries are powers of $B$ and $C$. In this case we define $A = -(BC)^k$ and obtain the result.

**Lemma 3.5.** Let $A, B$ be $m$-coherent matrices of $UT_\infty(m, R)$ such that $J_m(A) = J_m(B)$. Then $A$ and $B$ are conjugated in the group $UT_\infty(m, R)$.

**Proof.** Consider $J = J_m(A) = J_m(B)$ and let $(X_n)_{n \geq 0}$ be a sequence of elements of $D_\infty(R)$ such that $X_0 = I_\infty$. Suposse that $X = \sum_{k=0}^{\infty} X_k J^k \in UT_\infty(m, R)$. Now, choose two normalized sequences $(D_k)_{k \geq 0}$ and $(D'_k)_{k \geq 0}$ in $D_\infty(R)$ such that $A = \sum_{k=0}^{\infty} D_k J^k$ and $B = \sum_{k=0}^{\infty} D'_k J^k$, and suppose that $AX = XB$, then

$$AX = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} D_i S^{im}(X_{k-i})\right) J^k \text{ and } XB = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} X_i S^{im}(D'_{k-i})\right) J^k.$$  

so to have the result it is enough to prove that $\forall k \geq 2$

$$\sum_{i=0}^{k} D_i S^{im}(X_{k-i}) = \sum_{i=0}^{k} X_i S^{im}(D'_{k-i}).$$

Check this condition for $k = 0, 1$. If $k \geq 2$ then we have
Proof. For the case Lemma 3.8. Let 

\[ X_k + S^m(X_{k-1}) + \sum_{i=2}^{k} D_i S^m(X_{k-i}) = X_k + X_{k-1} S^{(k-1)m}(D'_1) + \sum_{i=0}^{k-2} X_i S^m(D'_{k-i}) \]

but \( D'_1 = I_\infty \), then rewrite the last equality, for all \( k \geq 2 \)

\[ S^m(X_k) - X_k = \sum_{i=0}^{k-1} X_i S^m(D'_{k+1-i}) - \sum_{i=2}^{k+1} D_i S^m(X_{k+1-i}), \]

and consider the first \( m \)th super diagonal entries in the each \( X_k \) be zero for all \( k \geq 1 \), then we can define a sequence \((X_k)_{k \geq 0}\) inductively and with this we proof that \( AX = XB \).

From Lemmas 3.3 and 3.5 we obtain the follow result.

**Corollary 3.6.** Let \( A \in UT_\infty(R) \) whose entries except in the main diagonal and the first super diagonal are all equal to zero. Then \( A \) is a commutator of one skew-involution and one involution.

**Proof.** We know that \( A \) is coherent. Consider \( J = J_1(A) \) and by Lemma 3.3 there is \( T \in UT_\infty(R) \) coherent such that \( J = J_1(T) \) and \( T = [S, P] \) with \( S \) an skew-involution and \( T \) and involution respectively. Finally, by the Lemma 3.6 we concluded that \( A \) and \( T \) are conjugated.

And, for the subspace \( UT_\infty(m, R) \) we have similar results

**Corollary 3.7.** Assume that \( R \) is an associative ring with identity 1 and 2 (also \( k \) or 2\( k \)) is an invertible element of \( R \). Then for every \( A \in UT_\infty(m, R) \) (or \( UT_\infty(m, R) \)), whose entries except in the main diagonal and the \( m \)th super diagonal are all to zero, is a commutator of

1.) Two involutions in \( \pm UT_\infty^{(2)}(m, R) \).
2.) One skew-involution and one involution both in \( \pm UT_\infty^{(2)}(m, R) \).
3.) Matrices of order \( k \) in \( UT_\infty^{(k)}(m, R) \).
4.) One skew finite order 2\( k \) and one matrix of order finite 2\( k \) in \( \pm UT_\infty^{(2k)}(m, R) \).

**Proof.** Similar to the Corollary 3.6 and Corollary 3.7 in [2].

For the group \( UT_\infty(m, R) \) we have the following Lemma.

**Lemma 3.8.** Let \( R \) be an associative ring with identity 1 and let \( n \in \mathbb{N} \).

1.) If \( A, B \in UT_n(m, R) \) such that \( a_{i,m+i} = b_{i,m+i} = 1 \) for all \( 1 \leq i \leq n - 1 \), then \( A \) and \( B \) are conjugated in \( UT_n(m, R) \).
2.) If \( A, B \in UT_\infty(m, R) \) such that \( a_{i,m+i} = b_{i,m+i} = 1 \) for all \( 1 \leq i \), then \( A \) and \( B \) are conjugated in \( UT_\infty(m, R) \).

**Proof.** For the case \( m = 1 \) see the Lemma 2.6 in Hou [4]. Now consider \( m > 1 \).

1.) Consider \( A = (a_{ij}) \in UT_n(m, R) \) and

\[
J = \begin{pmatrix}
1 & 0 & \cdots & 0 & 1 \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & \cdots & 0 & 1 \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & \cdots & 0 & 1
\end{pmatrix},
\]

and conclude that \( AX = XB \).
where all blank entries are equal to 0. We need only to prove that $A$ is conjugated to the matrix $J$, for this we can constructed a matrix $X = (x_{ij}) \in UT_n(R)$ such that $X^{-1}AX = J$ or $AX = XJ$.

Let $X$ be the matrix

$$X = \begin{pmatrix}
1 & x_{12} & \cdots & x_{1,n-m} \\
& \ddots & \ddots & \vdots \\
& & 1 & \cdots & x_{m,n-m} \\
& & & \ddots & \vdots \\
& & & & \ddots & \ddots \\
& & & & & 1 & x_{n-1,n} \\
& & & & & & 1
\end{pmatrix}$$

where $X'$ is a matrix of order $m \times m$ with arbitrary entries and the other entries of the matrix $X$ are related as follows

(a) For the first super diagonal entries of $X$ where $j = i + 1$ we choose

$$x_{ij} = x_{i+m,j+m} + a_{i,j+m}$$

(b) For $j = i + k$ with $k \geq 2$, the following super diagonal entries are obtained from

$$x_{i,j} = x_{i+m,j+m} + a_{i,j+m} + \sum_{s=1}^{k} a_{i,j+m-(k-s)} \cdot x_{i+m+s,j+m}.$$

2. If $A = (a_{ij}) \in UT_\infty(m, R)$ then $A$ is conjugated to the matrix $J \in UT_\infty(m, R)$ as in (1) and here constructed a matrix $X = (x_{ij}) \in UT_\infty(m, R)$ such that $AX = XJ$. The relations of $(x_{ij})$ in the super diagonal entries are the same as those given in (1) with the difference that the matrix $X'$ disappears.

Thus the Lemma 3.8 is proved. \(\square\)

From Corollary 3.7 and Lemma 3.8 we have

**Corollary 3.9.** Let $R$ be an associative ring with unity 1 and that 2 (k or 2k respectively) is an invertible element in $R$. Every matrix in $UT_\infty(m, R)$ whose entries in the diagonal and the mth super diagonal are all equal to the identity 1, is a commutator of

1. Two involutions in $\pm UT_\infty^{(2)}(m, R)$.
2. One skew-involution and one involution both in $\pm UT_\infty^{(2)}(m, R)$.
3. Two matrices of order $k$ in $UT_\infty^{(k)}(m, R)$.
4. One skew finite order $2k$ and one matrix of order finite $2k$ in $\pm UT_\infty^{(2k)}(m, R)$.

**Proof.** Follows similar from Hou in [9] and Gargate in [2]. \(\square\)

Finally, we proved the Main Theorem:
Proof of Theorem 1.1. Consider $A \in UT_\infty(m, R)$ then we can write
\[ A = I_\infty + \sum_{i=1}^\infty \sum_{j=m+i}^\infty a_{i,j} E_{i,j} \]
and consider
\[ A_1 = I_\infty + \sum_{i=1}^\infty (a_{i,m+i} - 1) E_{i,m+i} \in UT_\infty(m, R), \]
then, by corollary 3.6, $A_1$ is a commutator as desired. Observe that $A_2 = A_1^{-1} A \in UT_\infty(m, R)$ is a matrix whose entries in the diagonal and the $m$th super diagonal are all equal to 1, then by Corollary 3.7, $A_2$ is also a commutator as desired. So $A$ is product of commutators according to each case.

4. Case $R = \mathbb{C}$ the complex field and the group Vershik-Kerov

In this section, we consider the case $R = \mathbb{C}$ a complex field and the group $SL_n(\mathbb{C})$. We proved the Theorem 1.2:

Proof of Theorem 1.2. ($\Rightarrow$) If $A = [S, T]$ with $S$ one involution and $T$ one skew-involution then $A = -(ST)^2$. Consider $B = ST$ then $B^2 = -A$ and $-B^{-1} = TS$. Hence $B = ST = S(TS)S^{-1} = S(-B^{-1})S^{-1}$,

then, $B$ is similar to $-B^{-1}$.

($\Leftarrow$) By hypothesis, if $B \in SL_n(\mathbb{C})$ is such that $A = -B^2$ and $B$ is similar to $-B^{-1}$ then, by the Theorem 5 in [3], $B$ is product of one involution $S$ and one skew-involution $T$. Then we have
\[ A = -B^2 = -(ST)^2 = -STST = -STS^{-1}(-T^{-1}) = STS^{-1}T^{-1} = [S, T]. \]

An immediate consequence is the following corollary:

Corollary 4.1. Let $A \in SL_n(\mathbb{C})$. Then

1.) $A$ is a product of commutators of involutions if there is a matrix $B \in SL_n(\mathbb{C})$ such that $A = B^2$ and $B$ is product of two involutions.

2.) $A$ is a product of commutators of one skew-involution and one involution if there is a matrix $B \in SL_n(\mathbb{C})$ such that $A = -B^2$ and $B$ is a product of one skew-involutions and one involution.

Proof. Follows immediately from proof of Theorem 1.2

Next, we proved the Theorem 1.3:

Proof of Theorem 1.3. In this case, we can rewrite $A = B^k = (CD)^k$ with $C^k = D^k = I$ and the result follows from Lemma 3.6 in Gargate [2].

And, in the general case we have as a result the Theorem 1.4:

Proof of Theorem 1.4. Follows immediately from Lemma 3.6 in Gargate [2].

Then, in $SL_n(\mathbb{C})$ we proved the Theorem 1.5.
Proof of Theorem 1.3. Consider $A \in SL_n(\mathbb{C})$ not a scalar matrix, then by Theorem 1 in [13], we can find a lower-triangular matrix $L$ and an upper-triangular matrix $U$ such that $A$ is similar to $LU$, and both $L$ and $U$ are unipotent. By the Theorem 1.1 it follows that each one of the matrices $L$ and $U$ is a product of commutators as desired. For the scalar case $A = \alpha I$ with $\det(A) = 1$ it suffices to consider the case when $n$ is exactly the order of $\alpha$ (see [9]). We using the techniques of [10] for the proof in each case. Observe that, if $n$ is even we have

$$
\alpha I = \begin{bmatrix}
\alpha & 0 & 0 & \cdots & 0 \\
0 & \alpha^{-1} & 0 & \cdots & 0 \\
0 & 0 & \alpha & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha^{n-2} & \alpha^{-1} \\
0 & 0 & \cdots & 0 & \alpha
\end{bmatrix} \begin{bmatrix}
1 \\
\alpha^2 \\
\alpha^{-2} \\
\vdots \\
\alpha^{-n+2} \\
1
\end{bmatrix},
$$

and if $n$ is odd

$$
\alpha I = \begin{bmatrix}
\alpha & 0 & 0 & \cdots & 0 \\
0 & \alpha^{-1} & 0 & \cdots & 0 \\
0 & 0 & \alpha & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha^{n-2} & \alpha^{-1} \\
0 & 0 & \cdots & 0 & \alpha
\end{bmatrix} \begin{bmatrix}
1 \\
\alpha^2 \\
\alpha^{-2} \\
\vdots \\
\alpha^{-n+2} \\
1
\end{bmatrix}.
$$

Thus, if $n$ is even then denote by

$$
S = \text{diag}(\alpha, \alpha^{-1}, \alpha^3, \cdots, \alpha^{n-1}, \alpha^{-n+1}) \quad \text{and} \quad T = \text{diag}(1, \alpha^2, \alpha^{-2}, \cdots, \alpha^{-n+2}, 1),
$$

so $\alpha I = ST$. The objective is analyse, in each case, the decomposition in products of the matrix

$$
\begin{bmatrix}
a & 0 \\
0 & a^{-1}
\end{bmatrix}
$$

with $a \in \mathbb{C}$, $a \notin \{0, 1, -1\}$.

1.) For involutions, it’s enough to find two matrices $B$ and $C$ such that $\alpha I = B^2C^2$ and both are products of involutions. For this case consider

$$
B = \text{diag}(\alpha^{1/2}, \alpha^{-1/2}, \alpha^{3/2}, \cdots, \alpha^{(-n+2)/2}, 1),
$$

and observe that $B^2 = S$. Here

$$
\begin{bmatrix}
a & 0 \\
0 & a^{-1}
\end{bmatrix} = J_1(a) \cdot J_2(a),
$$

with $J_1(a) = \begin{bmatrix}
0 & ay \\
1 & ay
\end{bmatrix}$ and $J_2(a) = \begin{bmatrix}
0 & y \\
1 & y
\end{bmatrix}$, $y \in \mathbb{C} \setminus \{0\}$ and both are involutions. Then it follows that $B$ is product of two involutions, so $S$ is a commutator of these two involutions. Similarly we can proof the same result for $T$. Observe that for the case $n$ odd number the proof is similar, so we can conclude that $\alpha I$ is a product of two commutators of involutions. For other proof of this case see Hou in [9].
2.) For this case we can see the following decomposition, if $n$ is even
\[ \alpha I = \text{diag}(-\alpha, -\alpha^{-1}, -\alpha^3, \cdots, -\alpha^{n-1}, -\alpha^{-n+1}) \times \text{diag}(-1, -\alpha^2, -\alpha^{-2}, \cdots, -\alpha^{-n+2}, -1) \]

and if $n$ is odd
\[ \alpha I = -\text{diag}(-\alpha, -\alpha^{-1}, -\alpha^3, \cdots, -\alpha^{n+2}, -1) \times \text{diag}(-1, -\alpha^2, -\alpha^{-2}, \cdots, -\alpha^{-n-1}, -\alpha^{-n+1}). \]

For $n$ even, consider $S = \text{diag}(-\alpha, -\alpha^{-1}, -\alpha^3, \cdots, -\alpha^{n-1}, -\alpha^{-n+2}, -1) = -B^2$ with $B = \text{diag}(ia^{1/2}, ia^{-1/2}, ia^{3/2}, \cdots, ia^{(n+2)/2}, i)$. And, observe that, for all $a \in \mathbb{C}$ $a \neq 0$,
\[
\begin{bmatrix}
a & 0 \\
0 & ia^{-1}
\end{bmatrix} = (iJ_1(a)) \cdot J_2(a)
\]

with $J_1(a)$ and $J_2(a)$ the same matrices in the item (1). Observe that, in this case $iJ_1(a)$ is an skew-involution and $J_2(a)$ is an involution. So, $B$ is the product of one skew-involution and one involution, then by the Corollary we have that $S$ is a commutator of one skew-involution and one involution. Follows similarly for $T$ and so we can conclude that $\alpha I$ is a product of two commutators of skew-involutions and involutions. For the case $n$ odd number follows immediately.

3.) Observe that for $a \in \mathbb{C}$, $a \notin \{0, 1, -1\}$ then
\[
\begin{bmatrix}
a & 0 \\
0 & a^{-1}
\end{bmatrix} = J_1(a) \cdot J_2(a)
\]

with
\[
J_1(a) = \begin{bmatrix}
at/a+1 & a - \left(\frac{at}{a+1}\right)^2 \\
-1/a & t/a+1
\end{bmatrix} \quad \text{and} \quad J_2(a) = \begin{bmatrix}
at/a+1 & a \left(\frac{at}{a+1}\right)^2 - 1 \\
1 & t/a+1
\end{bmatrix},
\]

where $t = \theta + \theta^{-1}$, $\theta^k = 1$, $\theta \neq 1$. Observe that $J_1(a)^k = J_2(a)^k = I$ (see [10]).

Then, if $n$ is even we have
\[
\alpha I = \begin{bmatrix}
\alpha & \alpha^{-1} & \alpha^3 & \cdots & \alpha^{n-1} \\
\alpha^{-n+1} & \alpha^{-n+1} & \alpha^{-n+1} & \cdots & \alpha^{-n+2}
\end{bmatrix} \begin{bmatrix}
1 & \alpha^2 & \alpha^{-2} & \cdots & \alpha^{-(n+2)} \\
\alpha & \alpha^{-1} & \alpha^3 & \cdots & \alpha^{n-1}
\end{bmatrix},
\]

and if $n$ is odd
In the Vershik-Kerov group, any matrix of the form $GL_n$ is a complex field or the real number field. Let $S$ be an infinite unitriangular matrix. By Theorem 1.4 we conclude that $S^n$ is a product of $2k - 3$ commutators of $J_1$ and $J_2$. Similarly we can obtain the same results for the others matrices. Therefore, by the decomposition, we conclude that $\alpha I_n$ is a product of $4k - 6$ commutators. The case of $n$ odd number follows immediately.

4.) For $n$ even, consider

$$\alpha I = -\operatorname{diag}(-\alpha, -\alpha^{-1}, -\alpha^3, \ldots, -\alpha^{n-1}, -\alpha^{-n+1}) \times$$

$$\operatorname{diag}(-1, -\alpha^2, -\alpha^{-2}, \ldots, -\alpha^{-n+2}, -1),$$

and $S = \operatorname{diag}(-\alpha, -\alpha^{-1}, -\alpha^3, \ldots, -\alpha^{n-1}, -\alpha^{-n+1}) = -B^{2k}$ with

$$B = \operatorname{diag}(i^{1/k} \alpha^{1/(2k)}, i^{1/k} \alpha^{-1/(2k)}, \ldots, i^{1/k} \alpha^{-1/(2k+1)}),$$

where $i$ is the imaginary number.

Observe that

$$\begin{pmatrix} i^{1/k} a & 0 \\ 0 & i^{1/k} a^{-1} \end{pmatrix} = (i^{1/k} J_1(a)) \cdot J_2(a),$$

and $i^{1/k} J_1(a)$ is a skew order $2k$ matrix. By Theorem 1.4 we conclude that $S$ is a product of $4k - 3$ commutators. By the same observations in the above items we conclude the proof.

Let $n$ a positive integer and consider $GL_n(C)$ the general linear group over $C$. The Vershik-Kerov group $GL_{VK}(\infty, C)$ is the group consisting of all infinite matrices of the form

$$\begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix}$$

(1)

where $M_1 \in GL_n(C)$ and $M_3 \in T_{\infty}(C)$. Also, denote by $GL_{VK}(\infty, m, R)$ the subgroup of $GL_{VK}(\infty, C)$ such that $M_3 \in T_{\infty}(m, R)$ and denote by $SL_{VK}(\infty, m, R)$ the subgroup of $GL_{VK}(\infty, m, R)$ such that $M_1 \in SL_n(C)$ and $M_3 \in UT_{\infty}(m, R)$.

We use the following lemma proof in [2].

**Lemma 4.2.** Assume that $C$ is a complex field or the real number field. Let $A \in GL_n(C)$ of which $1$ is no eigenvalue, and let $T$ be an infinite unitriangular matrix. In the Vershik-Kerov group, any matrix of the form

$$\begin{pmatrix} A & B \\ 0 & T \end{pmatrix}$$

(2)
is conjugated to
\[
\begin{pmatrix}
A & 0 \\
0 & T
\end{pmatrix}
\]

Then, in this case we shown the Theorem 1.6

Proof of Theorem 1.6. Consider \( M \in SL_{V,K}(m, \mathbb{C}) \) in the form \( M = \begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix} \), with \( M_1 \in SL_n(\mathbb{C}) \) and \( M_3 \in UT_\infty(m, \mathbb{C}) \). From the proof of Theorem 1.3 in [9], \( M \) is conjugated to an infinite matrix of the form \( \begin{pmatrix} A & 0 \\ 0 & T \end{pmatrix} \), with \( A \in SL_n(\mathbb{C}) \) for which 1 is no eigenvalue and \( T \in UT_\infty(m, \mathbb{C}) \). By the Theorem 1.1 and the Theorem 1.5, both are products of commutators and we know that the direct sum of \( A \) and \( T \) is also a product of commutators of elements as desired. \( \square \)

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