I first met Professor Lu Qikeng in the Mittag-Leffler Institute in Stockholm in 1988 during the Year of Several Complex Variables organized by John-Eric Fornaess and Christer Kiselman. When I entered the beautiful building of this institute placed in Stockholm suburbs the first men whom I met were Professors Lu and Kiselman sitting under the portrait of Sofya Kovalevskaya. After I was introduced to Professor Lu we started to discuss the topics of our mutual mathematical interest. At that time it was complex analysis in matrix spaces — a traditional theme for Chinese mathematicians starting from Professor Hua Lookeng. I was looking for a generalization of the well known criterion of holomorphic convexity of \( n \)-circled (or Reinhardt) domains to the matrix case. When I told Professor Lu about this problem he decided to propose it to his student Zhou Xiangyu. Shortly after that Zhou Xiangyu has solved the problem (it was solved independently also by Eric Bedford and Jiri Dadok).

It was the beginning of our collaboration and friendship with Professor Lu and Zhou Xiangyu. In 90’s (I do not remember the year precisely) Professor Lu has visited Steklov Institute where he met my teacher Professor V.S.Vladimirov who told him about a long-standing conjecture in axiomatic quantum field theory. It was the so called "extended future tube conjecture" posed by A.S. Wightman. V.S.Vladimirov was a great enthusiast of this problem and proposed it to all of his students working in quantum field theory and several complex variables. Professor Lu became also interested in this problem and invited me to come to Beijing and give a mini-course on it in the Institute of Mathematics of Academia Sinica. I came to Beijing on the eve of 1999 in the middle of a cold and windy winter and gave three lectures on the extended future tube conjecture presenting several reformulations of this problem. From this time my relations with Professor Lu were mostly via Zhou Xiangyu. He visited Steklov Institute several times almost for three years in total and became the first Chinese mathematician who got the Doctor of Science (second doctor) degree from this Institute for the solution of the extended future tube conjecture and other results on invariant domains of holomorphy.

*While preparing it the author was partially supported by the RFBR grants 16-01-00117, 13-02-91330 and Scientific Program of Presidium of RAS "Nonlinear Dynamics"
This story shows how Professor Lu cared about his students and how far-seeing he was as a mathematician. I should add that being a very nice person we was easy to deal with. I always felt relaxed while meeting and talking to him despite certain language problems. His untimely death is a big loss for me personally and, I believe, for all mathematics.

**Introduction**

In their papers [7],[8] Seiberg and Witten have proposed what is called now the Seiberg–Witten theory. Motivated by this theory, Witten [11] introduced the Seiberg–Witten equations which were used to define a new kind of invariant of smooth 4-dimensional manifolds. The Seiberg–Witten equations, opposite to the Yang–Mills duality equations which are conformally invariant, are not invariant under scale transformation so in order to produce invariants from these equations one should plug the scale parameter \( \lambda \) into them and consider the scale limit \( \lambda \to \infty \).

If we consider such a limit in the case of 4-dimensional symplectic manifolds a solution of Seiberg–Witten equations will concentrate in a neighborhood of some pseudoholomorphic curve (more precisely, a pseudoholomorphic divisor) while the equations reduce to families of static Abelian Higgs equations defined in the normal planes to the limiting pseudoholomorphic curve. Such a limit is called adiabatic as well as the reduced Seiberg–Witten equations. Solutions of these equations may be considered as families of static solutions of the Abelian Higgs model in the complex plane with a complex parameter \( z \) running along the pseudoholomorphic curve. This parameter plays the role of complex time while the reduced Seiberg–Witten equations have the form of a nonlinear \( \bar{\partial} \)-equation with respect to \( z \).

It turns out that this construction has a non-trivial \((2+1)\)-dimensional analogue. Namely, if we consider in the \((2 + 1)\)-dimensional Abelian Higgs model the "slow-time" limit then Abelian Higgs equations will reduce to the adiabatic equations with solutions given by the geodesics on the moduli space of static Abelian Higgs solutions (called otherwise vortices) in the metric generated by the kinetic energy functional.

Thus we may consider the reduced Seiberg–Witten equations as a \((2 + 2)\)-dimensional analogue of the adiabatic equations in Abelian Higgs case. Solutions of these equations can be treated as complex analogues of adiabatic geodesics while the nonlinear \( \bar{\partial} \)-equation may be considered as a complex analogue of the Euler equation for these geodesics.

**1 Seiberg–Witten theory**

In this section we recall briefly some basic facts from Seiberg–Witten theory(cf. [1],[6],[9] for a more detailed presentation).

**1.1 Spinor algebra**

**Clifford algebras.** Let \( V = \mathbb{R}^n \) be an \( n \)-dimensional Euclidean vector space provided with the Euclidean metric and an orthonormal basis \( \{e_i\}, i = 1, \ldots, n \). The *Clifford algebra* \( \text{Cl}(n) \) is generated by the elements \( 1, e_1, e_2, \ldots, e_n \), satisfying the following relations:

\[
e_i^2 = -1, \quad e_i e_j + e_j e_i = 0 \quad \text{for} \ i \neq j.
\]
As a real vector space, $\text{Cl}(n)$ has dimension $2^n$ with the basis given by the elements of the form $1, e_I := e_{i_1}e_{i_2}\ldots e_{i_k}$ where $I = \{i_1, i_2, \ldots, i_k\}$ is a subset of $\{1, 2, \ldots, n\}$ consisting of indices $i_1 < i_2 < \ldots < i_k$ with $k = 1, 2, \ldots, n$. We denote by $\text{Cl}^c(n) := \text{Cl}(n) \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of this algebra.

**Spin group.** Denote by $\text{Pin}(n)$ the subgroup of the multiplicative group $\text{Cl}^*(n)$ of the Clifford algebra, generated by the unit vectors $v \in V$, i.e. by vectors $v$ with $|v| = 1$. The *spinor group* $\text{Spin}(n)$ is the identity connected component of $\text{Pin}(n)$. There is an exact sequence:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}(n) \xrightarrow{\pi} \text{SO}(n) \longrightarrow 0.$$  

**Spin representation.** In the case of even $n = 2m$ there is a spinor representation of the group $\text{Spin}(2m)$ in the $(2^m)$-dimensional Hermitian complex vector space $W$:

$$\Gamma : \text{Spin}(2m) \longrightarrow \text{End} W,$$

which extends to the representation of the whole complexified Clifford algebra

$$\Gamma : \text{Cl}^c(V) \rightarrow \text{End} W.$$  

The action of $\text{Cl}^c(V)$ on $W$ is called the *Clifford multiplication* while elements of $W$ are called the *spinors*.

**Semi-spinor spaces.** Define the *Clifford volume element* $\omega$ by setting

$$\omega := e_1e_2\ldots e_{2m} \in \text{Cl}_{2m}(V).$$

Then $\omega^2 = (-1)^m$ so we can introduce the *semispinor spaces*

$$W^\pm := \{w \in W : \Gamma(\omega)w = \pm i^m w\}.$$  

Thus we obtain a decomposition $W = W^+ \oplus W^-$ into the direct sum of semi-spinor spaces which are interchanged under the Clifford multiplication by vectors $v \in V$.

**Relation with exterior algebra.** Consider the exterior algebra $\Lambda^*V$ of the space $V$. There is a linear isomorphism

$$\text{Alt} : \Lambda^*V \rightarrow \text{Cl}(V)$$

defined by associating with a form $v_1 \wedge \ldots \wedge v_k$ the element of $\text{Cl}(V)$ given by the alternating sum

$$v_1 \wedge \ldots \wedge v_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)v_{\sigma(1)}\cdot \ldots \cdot v_{\sigma(k)}.$$  

By duality, we have also the isomorphism

$$\text{Alt}' : \Lambda^*(V') \rightarrow \text{Cl}(V') \cong \text{Cl}(V)$$

where $V'$ is the (real) dual vector space of $V$.

Using the spin representation $\Gamma : \text{Cl}(V) \rightarrow \text{End} W$, we can define

$$\rho := \Gamma \circ \text{Alt}' : \Lambda^*(V') \rightarrow \text{End} W.$$  

The introduced map $\rho$ determines the Clifford multiplication by forms from $\Lambda^*(V')$ in the space $W$. In particular, the Clifford multiplication by 2-forms leaves the subspaces $W^\pm$
invariant. More precisely, the map $\rho$ associates with the real-valued 2-forms the skew-symmetric traceless endomorphisms of subspaces $W^\pm$, and with imaginary-valued 2-forms the Hermitian traceless endomorphisms of these subspaces.

**Dimension 4.** If $\dim V = 4$ then the subspace $\Lambda^2(V')$ decomposes into the direct sum

$$\Lambda^2(V') = \Lambda^2_+ \oplus \Lambda^2_-$$

of subspaces of selfdual and anti-selfdual forms with respect to Hodge $*$-operator. In this case the map $\rho$ induces the isomorphisms: $\Lambda^2_+ \to \text{su}(W^\pm)$ and

$$\rho^\pm : \Lambda^2_\pm \otimes i\mathbb{R} \to \text{Herm}_0(W^\pm).$$

The isomorphisms, inverse to $\rho^\pm$, are denoted by

$$\sigma^\pm = (\rho^\pm)^{-1} : \text{Herm}_0(W^\pm) \to \Lambda^2_\pm \otimes i\mathbb{R}.$$  

**Complex case.** In the case when $V = \mathbb{C}^n$ provided with an Hermitian metric there exists a canonical spin representation. The corresponding spinor space is given by

$$W_{\text{can}} = \Lambda^0,(V') := \bigoplus_{q=0}^n \Lambda^{0,q}(V').$$

The spin representation $\Gamma_{\text{can}}$ on vectors $v \in V$ is given by the formula

$$\Gamma_{\text{can}}(v)w^{0,q} = v^{0,1} \wedge w^{0,q} - v^{1,0} \cdot w^{0,q}$$

where $v' = v^{1,0} + v^{0,1}$ is the dual covector of $v$ and $w^{0,q} \in \Lambda^{0,q}(V')$. The semi-spinor spaces coincide with

$$W^\pm_{\text{can}} = \Lambda^{0,\text{ev}}(V'), \quad W^-_{\text{can}} = \Lambda^{0,\text{od}}(V').$$

**Spin$^c$ group.** The group Spin$^c(n)$ is a circle extension of the group Spin$(n)$ defined as

$$\text{Spin}^c(n) = \{ z = e^{i\theta} x : x \in \text{Spin}(n), \theta \in \mathbb{R} \}.$$  

There is an exact sequence

$$0 \longrightarrow \text{Spin}(V) \longrightarrow \text{Spin}^c(V) \overset{\delta}{\longrightarrow} U(1) \longrightarrow 0$$

where $\delta : xe^{i\theta} \mapsto e^{2i\theta}$. So Spin$^c(V) = \text{Spin}(V) \times_{\mathbb{Z}_2} U(1)$.

### 1.2 Spin$^c$-structures

**Definition.** Let $X$ be an oriented $n$-dimensional Riemannian manifold and $P_{\text{SO}(n)} \to X$ is a principal SO$(n)$-bundle of orthonormal bases on $X$. The Spin$^c$-structure on $P_{\text{SO}(n)}$ is an extension of this bundle to a principal Spin$^c(n)$-bundle $P_{\text{Spin}^c(n)} \to X$ together with a Spin$^c$-invariant bundle epimorphism:

$$P_{\text{Spin}^c(n)} \longrightarrow P_{\text{SO}(n)}$$

$$\downarrow \quad \downarrow$$

$$X \quad X.$$
where Spin\(^c\)(n) acts on \(P_{SO(n)}\) by the homomorphism \(\pi : \text{Spin}^c(n) \to \text{SO}(n)\).

**Characteristic bundle.** We can associate with the bundle \(P_{\text{Spin}^c(n)}\) the principal U(1)-bundle \(P_{U(1)} \to X\) so that the following diagram is commutative:

\[
\begin{array}{ccc}
P_{\text{Spin}^c(n)} & \xrightarrow{\delta} & P_{U(1)} \\
\downarrow & & \downarrow \\
X & \equiv & X
\end{array}
\]

where Spin\(^c\)(n) acts on \(P_{U(1)}\) by the homomorphism \(\delta : \text{Spin}^c(n) \to \text{U}(1)\). The complex line bundle \(L \to X\), associated with \(P_{U(1)} \to X\), is called the *characteristic bundle* of the given Spin\(^c\)-structure, and its first Chern class \(c_1(L)\) is called the *characteristic class* of the Spin\(^c\)-structure.

**Spin\(^c\)-structures on vector bundles.** In a similar way one can define a Spin\(^c\)-structure on an oriented Riemannian vector bundle \(V \to X\) of rank \(n\), associated with a principal bundle \(P_{SO(n)} \to X\). The *Spin\(^c\)-structure* on \(V \to X\) is the extension of its structure group from \(SO(n)\) to \(\text{Spin}^c(n)\). In other words, the bundle \(V \to X\) admits a Spin\(^c\)-structure if it is associated with a principal Spin\(^c\)-(n)-bundle \(P_{\text{Spin}^c(n)} \to X\), i.e. there exists a bundle isomorphism

\[
P_{\text{Spin}^c(n)} \times_{\text{Spin}^c(n)} \mathbb{R}^n \to V
\]

where Spin\(^c\)(n) acts on \(\mathbb{R}^n\) by the homomorphism \(\pi : \text{Spin}^c(n) \to \text{SO}(n)\).

In particular, one can take for such \(V\) the tangent bundle \(TX\). In this case the Spin\(^c\)-structure on \(TX\) is called the *Spin\(^c\)-structure on the manifold* \(X\). Such a structure exists on any 4-dimensional oriented Riemannian manifold \(X\).

**Definition in terms of spin representation.** In the case when the rank of \(V\) is even, i.e. \(n = 2m\), we can give an equivalent definition of the Spin\(^c\)-structure on \(V\) in terms of the spin representation. Namely, a Spin\(^c\)-structure on the bundle \(V\) of rank \(2m\) is a pair \((W, \Gamma)\), consisting of a complex Hermitian vector bundle \(W \to X\) of rank \(2m\) (spinor bundle) and a bundle homomorphism \(\Gamma : V \to \text{End}W\) subject to relations

\[
\Gamma^*(v) + \Gamma(v) = 0, \quad \Gamma^*(v)\Gamma(v) = |v|^2 \text{Id}.
\]

Such a homomorphism extends to a bundle homomorphism \(\Gamma : \text{Cl}^c(V) \to \text{End}W\) where \(\text{Cl}^c(V)\) is the bundle of complexified Clifford algebras, associated with the oriented Riemannian vector bundle \(V\). In particular, \(W\) can be decomposed into the direct sum \(W = W^+ \oplus W^-\) of semi-spinor bundles. The characteristic line bundle of the Spin\(^c\)-structure \((W, \Gamma)\) coincides with the bundle

\[
L_\Gamma := P_{\text{Spin}^c(2m)} \times_{\text{Spin}^c(2m)} \mathbb{C} \to X
\]

where the action of the group Spin\(^c\)(2m) on \(\mathbb{C}\) is given by the homomorphism \(\delta : \text{Spin}^c(2m) \to \text{U}(1)\).

**Associated Spin\(^c\)-structures.** Suppose that an oriented Riemannian vector bundle \(V \to X\) of rank \(2m\) has a Spin\(^c\)-structure \((W, \Gamma)\). Then with any complex line bundle \(E \to X\) we can associate a new Spin\(^c\)-structure \((W_E, \Gamma_E)\) by setting

\[
W_E := W \otimes E, \quad \Gamma_E := \Gamma \otimes \text{Id}.
\]
This new Spin$^c$-structure $(W_E, \Gamma_E)$ corresponds to the principal Spin$^c(2m)$-bundle

$$P_{\Gamma_E} = P_{\Gamma} \otimes U(1) \, P_E$$

where $P_\Gamma$ is the principal Spin$^c(2m)$-bundle, associated with $(W, \Gamma)$ and $P_E$ is the principal $U(1)$-bundle, associated with $E$. The characteristic bundle of Spin$^c$-structure $(W_E, \Gamma_E)$ coincides with

$$L_{\Gamma_E} := L_{\Gamma} \otimes E \otimes 2$$

where $L_{\Gamma}$ is the characteristic bundle of $(W, \Gamma)$.

**Complex case.** In the case when $V \to X$ is a complex (or almost complex) vector bundle of (complex) rank $n$, provided with a complex (resp. almost complex) structure $J$, compatible with Riemannian metric and orientation of $V$ we can construct a canonical Spin$^c$-structure $(W_{\text{can}}, \Gamma_{\text{can}})$ on $V$. For this structure

$$W_{\text{can}} := \Lambda^{0,*}(V')$$

where $V'$ is provided with the dual almost complex structure $J'$. The Clifford multiplication map $\Gamma_{\text{can}}$ is given by the same formula, as in the case of complex vector spaces. The characteristic bundle $L_{\text{can}}$ coincides with the anticanonical bundle $K'$ of $V$:

$$K' = \Lambda^{0,n}(V').$$

### 1.3 Spin$^c$-connections and Dirac operator

**Spin$^c$-connection.** Suppose that $X$ is an oriented Riemannian manifold of dimension $2m$ for which the tangent bundle $TX$ can be provided with a Spin$^c$-structure $(W, \Gamma)$. Denote by $\nabla_0$ the Levi-Civita connection on $TX$. Then the Spin$^c$-connection $\nabla$ on $X$ is an extension of the Levi-Civita connection $\nabla_0$ to $W$. In other words, it is a linear first order differential operator on the space $C^\infty(X, W)$ of smooth sections of $W$ satisfying the following Leibniz rule

$$\nabla_u (\Gamma(v)s) = \Gamma(v)\nabla_us + \Gamma(\nabla_{0,u})s$$

for any vector fields $u, v$ on $X$ and any smooth section $s$ of $W$.

**Spin$^c$-connection form.** Denote by $A$ the connection form of the introduced Spin$^c$-connection $\nabla$. It is a 1-form on the principal Spin$^c(2m)$-bundle $P_\Gamma$ with values in the Lie algebra $\text{spin}^c(2m)$ of the group Spin$^c(2m)$. This Lie algebra is equal to

$$\text{spin}^c(2m) = \text{so}(2m) \oplus i\mathbb{R}$$

where $\text{so}(2m)$ is the Lie algebra of the orthogonal group $\text{SO}(2m)$. It follows from this representation that

$$A = A_0 + A$$

where $A_0$ is the connection form of the Levi-Civita connection on $TX$ and $A$ is the trace part of the form $A$. The form $2A$ generates a connection on the characteristic bundle $L_{\Gamma}$. In the case when $L_{\Gamma}$ has a square root, i.e. a line bundle $L_{\Gamma}^{1/2} \to X$ such that $L_{\Gamma}^{1/2} \otimes L_{\Gamma}^{1/2} = L_{\Gamma}$ (this condition is fulfilled, e.g., for spin manifolds $X$) the form $A$ generates a connection on $L_{\Gamma}^{1/2}$.

**Dirac operator.** Denote by $\nabla_A$ (resp. $d_A$) the covariant derivative (resp. exterior covariant derivative) on sections from $C^\infty(X, W)$ generated by the connection $A$. The Dirac
operator $D_A : C^\infty(X, W^+) \to C^\infty(X, W^-)$, associated with the connection $A$, is defined by the formula

$$D_A s = \sum_{j=1}^{2m} \Gamma(e_j) \nabla_{A,e_j} s$$

where $s \in C^\infty(X, W)$ and $\{e_j\}$ is a local orthonormal basis of $TX$. (This definition does not depend on the choice of $\{e_j\}$.)

**Complex case.** If $X$ is a complex (or almost complex) manifold of dimension $n$ then it can be provided with a canonical Spin$^c$-structure $(W_{\text{can}}, \Gamma_{\text{can}})$ and associated canonical Spin$^c$-connection $A_{\text{can}}$. If $E \to X$ is a complex line bundle over $X$, provided with a Hermitian connection $B$, then we can construct a Spin$^c$-connection $A_E$ on the associated principal bundle $P_{\Gamma_E}$ by setting

$$A_E := A_{\text{can}} \otimes \text{Id} + \text{Id} \otimes B$$

where $2A_{\text{can}}$ is the connection form of the canonical connection on $L_{\text{can}} = \Lambda^{0,n}(T^*X)$. The corresponding spinor space $W_E = \Lambda^{0,*}(X, E)$ is decomposed into the direct sum

$$W_E = W_E^+ \oplus W_E^-$$

with

$$W_E^+ = \Lambda^{0,\text{ev}}(X, E), \quad W_E^- = \Lambda^{0,\text{od}}(X, E).$$

The associated Dirac operator $D_{A_E} : C^\infty(X, W_E^+) \to C^\infty(X, W_E^-)$ coincides in this case with the operator $D_{A_E} = \bar{\partial}_B + \bar{\partial}_B^*$ where $\partial_B$ is the covariant $\partial$-operator and $\bar{\partial}_B^*$ is the adjoint of $\bar{\partial}_B$.

### 1.4 Seiberg–Witten equations on 4-dimensional Riemannian manifolds

**Seiberg–Witten equations.** Let $X$ be an oriented compact Riemannian 4-manifold provided with a Spin$^c$-structure $(W, \Gamma)$ and Spin$^c$-connection $\nabla_A$. Then the associated *Seiberg–Witten equations* (briefly: SW-equations) have the form

$$\begin{cases}
D_A \Phi = 0 \\
F_A^+ = (\Phi \otimes \Phi^*)_0
\end{cases}$$

where $\Phi \in C^\infty(X, W^+)$ and $(\Phi \otimes \Phi^*)_0 := \Phi \otimes \Phi^* - \frac{1}{2} |\Phi|^2 \text{Id}$ is the traceless Hermitian endomorphism associated with $\Phi$. The term $F_A^+ \in \Omega^2_+ (X, i\mathbb{R})$ is the selfdual part of the curvature $F_A$.

The first SW-equation is the covariant Dirac equation. To explain the meaning of the second SW-equation recall that for 4-dimensional manifolds $X$ we have the decomposition

$$\Lambda^2(T^*X) = \Lambda^2_+ \oplus \Lambda^2_-$$

of the bundle $\Lambda^2(T^*X)$ of 2-forms on $X$ into the direct sum of subbundles $\Lambda^2_{\pm} \equiv \Lambda^2_{\pm}(T^*X)$ of selfdual (resp. anti-selfdual) 2-forms with respect to Hodge $*$-operator. Then the Clifford multiplication determines an isomorphism

$$\sigma_+ : \text{Herm}_0(W^+) \to \Omega^2_+(X, i\mathbb{R})$$
where $\Omega^2_+(X, i\mathbb{R})$ is the space of sections of the bundle $\Lambda^2_+(T^*X) \otimes i\mathbb{R}$ over $X$.

**Seiberg–Witten action functional.** The SW-equations are the Euler–Lagrange equations for the following Seiberg–Witten action functional

$$S(A, \Phi) = \frac{1}{2} \int_X \left\{ |F_A|^2 + |\nabla_A \Phi|^2 + \frac{|\Phi|^2}{4} (s(g) + |\Phi|^2) \right\} \text{vol}(g)$$

where $s(g)$ is the scalar curvature of $(X, g)$ and $\text{vol}(g)$ is its volume form.

The Seiberg–Witten equations, as well as Seiberg–Witten action, are invariant under gauge transforms given by the formula:

$$A \mapsto A + g^{-1} dg, \quad \Phi \mapsto g^{-1} \Phi$$

where $g = e^{i\chi} \in C^\infty(X, U(1))$.

**Perturbed Seiberg–Witten equations.** In order to guarantee the solvability of these equations we consider the perturbed equations by plugging an appropriate self-dual 2-form $\eta \in \Omega^2(X, i\mathbb{R})$ into the second equation. As a result we shall obtain the following equations

$$\begin{cases} D_A \Phi = 0 \vspace{0.5cm} \\ F^+_A + \eta = (\Phi \otimes \Phi^*)_0. \end{cases}$$

### 1.5 Seiberg–Witten equations on 4-dimensional symplectic manifolds

**Dirac operator in symplectic case.** Let $X$ be a compact symplectic 4-manifold provided with the symplectic form $\omega$ and compatible almost complex structure $J$. Let $E \rightarrow X$ be a complex Hermitian line bundle with a Hermitian connection $B$. We suppose that $E$ is provided with the Spin$^c$-structure $(W_E, \Gamma_E)$ and Spin$^c$-connection $\nabla_A$ where $A \equiv A_E$ is the tensor product of the canonical Spin$^c$-connection $A_{\text{can}}$ and $B$, determined by the connection form

$$A_E := A_{\text{can}} \otimes \text{Id} + \text{Id} \otimes B.$$ 

In this case the corresponding Dirac operator $D_A$ coincides with $\bar{\partial}_B + \bar{\partial}^*_B$ and a section $\Phi \in C^\infty(X, W^+_E)$ is given by the pair $\Phi = (\varphi_0, \varphi_2) \in \Omega^0(X, E) \oplus \Omega^{0,2}(X, E)$.

**Seiberg–Witten equations in symplectic case.** The complexified bundle $\Lambda^2_+ \otimes \mathbb{C}$ of selfdual 2-forms in the considered case is decomposed into the direct sum of subbundles

$$\Lambda^2_+ \otimes \mathbb{C} = \Lambda_{2,0} \oplus \mathbb{C}[\omega] \oplus \Lambda^{0,2}.$$ 

Respectively, the second SW-equation for the curvature decomposes into the sum of three equations — the one for the component, parallel to $\omega$, the $(0, 2)$-component and $(2, 0)$-component. The latter one is conjugate to the $(0, 2)$-component and by this reason is omitted below.

So the Seiberg–Witten equations take on the form

$$\begin{cases} \bar{\partial}_B \varphi_0 + \bar{\partial}^*_B \varphi_2 = 0 \vspace{0.5cm} \\ F^2_{A_{\text{can}}} + F_\omega^\omega = \frac{i}{4} (|\varphi_0|^2 - |\varphi_2|^2) - \eta^\omega \vspace{0.5cm} \\ F^0_{B} + \eta^{0,2} \varphi_0 \varphi_2 = \frac{\varphi_0 \varphi_2}{2}. \end{cases}$$
For the solvability of these equations we should impose some topological condition on the first Chern class of the line bundle $E$. Namely, we shall require that the following inequality is satisfied (cf. [10]):

$$0 \leq c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega]$$  \hspace{1cm} (1)

where $K(X) = \Lambda^* T^* X$ is the canonical bundle of $X$ and $[\omega]$ is the cohomology class of the form $\omega$.

## 2 Abelian Higgs model

### 2.1 Ginzburg–Landau Lagrangian

Static $(2 + 1)$-dimensional Abelian Higgs model is governed by the *Ginzburg–Landau Lagrangian*, defined on the plane $\mathbb{R}^2_{(x_1, x_2)}$ with coordinates $(x_1, x_2)$, having the form

$$\mathcal{L}(A, \Phi) = |F_A|^2 + |d_A\Phi|^2 + \frac{1}{4}(1 - |\Phi|^2)^2$$

where $A$ is a $U(1)$-connection on $\mathbb{R}^2_{(x_1, x_2)}$, represented by the 1-form

$$A = A_1 dx_1 + A_2 dx_2$$

with smooth pure imaginary coefficients. The curvature $F_A$ of this connection is given by the 2-form

$$F_A = dA = \sum_{i,j=1}^2 F_{ij} dx_i \wedge dx_j = 2F_{12} dx_1 \wedge dx_2$$

with coefficients $F_{ij} = \partial_i A_j - \partial_j A_i$, $\partial_j := \partial/\partial x_j$.

The variable $\Phi$ is the scalar field, given by a smooth complex-valued function $\Phi = \Phi_1 + i\Phi_2$ on $\mathbb{R}^2_{(x_1, x_2)}$. The covariant exterior derivative $d_A \Phi$ in the second term of Ginzburg–Landau Lagrangian is given by the formula

$$d_A \Phi = d\Phi + A \Phi = \sum_{i=1}^2 (\partial_i + A_i) \Phi dx_i.$$  

The term $\frac{1}{4}(1 - |\Phi|^2)^2$ is the most important ingredient in Ginzburg–Landau Lagrangian. It is responsible for the nonlinear character of the “self-interaction” of the field $\Phi$. We require that $|\Phi| \to 1$ for $|x| \to \infty$. In a neighborhood of a zero of $\Phi$ the vector field $\vec{v} = \nabla \theta$ behaves like the hydrodynamical vortex, by this reason solutions of the considered model are also called *vortices*.

### 2.2 Vortices

**Definition.** The potential energy of our model is given by the integral of Ginzburg–Landau Lagrangian

$$U(A, \Phi) = \frac{1}{2} \int \mathcal{L}(A, \Phi) d^2 x.$$
The condition $|\Phi| \to 1$ implies that the considered problem has an integer-valued topological invariant $d$, given by the rotation number of the map $\Phi$, sending the circles of sufficiently large radius to the topological circles.

Mathematically, vortices are the pairs $(A, \Phi)$, minimizing the potential energy $U(A, \Phi) < \infty$ in a given topological class, fixed by the value of $d$. If $d > 0$ (resp. $d < 0$) such pairs are called $d$-vortices (resp. $|d|$-antivortices).

**Gauge transforms.** The potential energy $U(A, \Phi)$ is invariant under the gauge transforms, given by the formula:

$$A \mapsto A + i d \chi, \quad \Phi \mapsto e^{-i \chi} \Phi$$

where $\chi$ is an arbitrary smooth real-valued function on $\mathbb{R}^2(x_1, x_2)$.

The moduli space of $d$-vortices is defined as the quotient

$$\mathcal{M}_d = \frac{\{d\text{-vortices } (A, \Phi)\}}{\{\text{gauge transforms}\}}$$

described by the following theorem of Taubes.

**Taubes theorem.** Introduce the complex coordinate $z = x_1 + i x_2$ in the plane $\mathbb{R}^2(x_1, x_2)$, identifying $\mathbb{R}^2(x_1, x_2)$ with the complex plane $\mathbb{C}$.

**Theorem 1** (Taubes cf. [2]). For any unordered collection $z_1, \ldots, z_d$ of $d$ points on the complex plane $\mathbb{C}$, some of which may coincide, there exists a unique (up to gauge transforms) $d$-vortex $(A, \Phi)$ such that the map $\Phi$ vanishes precisely at the points $z_1, \ldots, z_d$ with the same multiplicities as for the collection $z_1, \ldots, z_d$.

Moreover, any critical point $(A, \Phi)$ of the functional $U(A, \Phi) < \infty$ with vortex number $d > 0$ is gauge equivalent to some $d$-vortex. In other words, all solutions of the Euler–Lagrange equations for the functional $U(A, \Phi)$ with finite energy are stable and have minimal energy in their topological class.

**Description of moduli space of vortices.** The Taubes theorem implies that the moduli space $\mathcal{M}_d$ of $d$-vortices may be identified with the vector space $\mathbb{C}^d$ by associating with the collection $z_1, \ldots, z_d$ the monic polynomial, having its zeros precisely at given points $z_1, \ldots, z_d$ with given multiplicities. The antivortices with $d < 0$ admit an analogous description.

This result has the following physical interpretation. Solutions of the Euler–Lagrange equations for the functional $U(A, \Phi)$ consist either of vortices, or antivortices. Our model cannot contain simultaneously both vortices and antivortices — such bound states should "annihilate" before the system is transformed to the static state.

### 2.3 Dynamical Ginzburg–Landau equations

**Ginzburg–Landau action functional.** Now we switch on the time in our model by adding the variable $x_0 = t$. In this case the Higgs field $\Phi = \Phi(t, x_1, x_2)$ is given by a smooth complex-valued function on the space $\mathbb{R}^3(t, x_1, x_2)$, and the 1-form $A$ is replaced by the form $A = A_0 dt + A_1 dx_1 + A_2 dx_2$ with coefficients $A_\mu = A_\mu(t, x_1, x_2), \mu = 0, 1, 2$, being smooth functions with pure imaginary values on the space $\mathbb{R}^3(t, x_1, x_2)$. Denote by $A^0 = A_0 dt$ the time component of $A$ and by $A = A_1 dx_1 + A_2 dx_2$ its space component.
The potential energy of the system is given by the same formula, as before, i.e. \( U(A, \Phi) = U(A, \Phi) \), while the kinetic energy has the form

\[
T(A, \Phi) = \frac{1}{2} \int \left\{ |F_{01}|^2 + |F_{02}|^2 + |d_{A0}\Phi|^2 \right\} \, dx_1 dx_2
\]

where \( F_{0j}, j = 1, 2, \) are defined in the same way, as before, i.e. \( F_{0j} = \partial_0 A_j - \partial_j A_0 \), and \( d_{A0}\Phi = d\Phi + A_0 \, dt \).

The described dynamical model is governed by the Ginzburg–Landau action functional:

\[
S(A, \Phi) = \int_0^{T_0} \left( T(A, \Phi) - U(A, \Phi) \right) \, dt,
\]

and the Euler–Lagrange equations for this functional, called otherwise the Ginzburg–Landau equations (briefly: GL-equations) have the form

\[
\begin{cases}
\partial_t F_{01} + \partial_2 F_{02} = -i \text{Im}(\bar{\Phi} \nabla_{A,0}\Phi) \\
\partial_0 F_{0j} + \sum_{k=1}^2 \varepsilon_{jk} \partial_k F_{12} = -i \text{Im}(\bar{\Phi} \nabla_{A,j}\Phi), \quad j = 1, 2 \\
(\nabla_{A,0}^2 - \nabla_{A,1}^2 - \nabla_{A,2}^2)\Phi = \frac{1}{2}\Phi(1 - |\Phi|^2),
\end{cases}
\]

where

\[
\nabla_{A,\mu} = \partial_\mu + A_\mu, \quad \mu = 0, 1, 2,
\]

and \( \varepsilon_{12} = -\varepsilon_{21} = 1, \varepsilon_{11} = \varepsilon_{22} = 0 \). The first of these equations is of constraint type while the last one is a nonlinear wave equation.

**Dynamical gauge transforms.** The GL-equations, as well as the action \( S(A, \Phi) \), are invariant under the dynamical gauge transforms, given by the same formula, as in the static case:

\[
A_\mu \mapsto A_\mu + i\partial_\mu \chi, \quad \Phi \mapsto e^{-i\chi}\Phi, \quad \mu = 0, 1, 2,
\]

only now \( \chi \) is a smooth real-valued function on \( \mathbb{R}^3 \times [t, x_1, x_2] \).

Our main goal is to describe solutions of the above GL-equations up to dynamical gauge transforms. The quotient of the space of dynamical solutions modulo gauge transforms is called the moduli space of dynamical solutions.

For the analysis of dynamical solutions it is convenient to choose the gauge function \( \chi \) so that the time component of the potential will vanish, i.e. \( A_0 = 0 \) (temporal gauge). Note that after imposing this condition we still have gauge freedom with respect to static gauge transforms.

**Configuration space.** In the temporal gauge the dynamical solution of the GL-equations may be considered as a trajectory of the form \( \gamma : t \mapsto [A(t), \Phi(t)] \) where \( [A, \Phi] \) denotes the gauge class of the pair \( (A, \Phi) \) with respect to static gauge transforms. This trajectory lies in the configuration space

\[
\mathcal{N}_d = \frac{\{ (A, \Phi) \text{ with } U(A, \Phi) < \infty \text{ and vortex number } d \}}{\{ \text{static gauge transforms} \}}
\]

which contains, in particular, the moduli space of \( d \)-vortices \( \mathcal{M}_d \).
The configuration space $N_d$ may be thought of as a canyon the bottom of which is occupied by the moduli space $M_d$ of $d$-vortex solutions. Respectively, one may consider a dynamical solution as the trajectory $\gamma(t)$ of a small ball rolling along the walls of the canyon. The lower is the kinetic energy of the ball, the closer lies its trajectory to the bottom. Our ball may even hit the bottom of the canyon but cannot stop there since, having a non-zero kinetic energy, it should ascend the canyon wall again.

3 Adiabatic limit construction

Here we present only a brief formulation of the adiabatic limit construction and its properties for the Ginzburg–Landau and Seiberg–Witten equations (for a more detailed exposition cf. [9] and [10]).

3.1 Adiabatic limit in Ginzburg–Landau equations

Adiabatic equation. Consider a family of dynamical solutions $\gamma_\epsilon$ of GL-equations, depending on a parameter $\epsilon > 0$, with trajectories $\gamma_\epsilon : t \mapsto [A_\epsilon(t), \Phi_\epsilon(t)]$. Suppose that the kinetic energy of these trajectories

$$T(\gamma_\epsilon) := \int_0^{T_0} T(\gamma_\epsilon(t))dt$$

tends to zero for $\epsilon \to 0$, proportional to $\epsilon$. Then in the limit $\epsilon \to 0$ the trajectory $\gamma_\epsilon$ converts into a static solution, i.e. a point of $M_d$. However, if we introduce on $\gamma_\epsilon$ the ”slow time” $\tau = \epsilon t$ and consider the limit of the ”rescaled” trajectories $\gamma_\epsilon(\tau)$ for $\epsilon \to 0$ then in this limit we shall obtain a trajectory $\gamma_0$, lying in $M_d$. Of course, such trajectories cannot be solutions of the original dynamical equations since any of their points is a static solution. However, they describe approximately dynamic solutions with small kinetic energy.

This procedure is called the adiabatic limit. In this limit the original dynamical equations reduce to the adiabatic equation whose solutions are called the adiabatic trajectories.

Adiabatic trajectories. The following theorem gives an intrinsic description of adiabatic trajectories in terms of the space $M_d$.

Theorem 2. The kinetic energy functional generates a Riemannian metric on the space $M_d$, called the kinetic or $T$-metric. The adiabatic trajectories $\gamma_0$ are the geodesics of this metric.

Adiabatic principle. The idea of the approximate description of ”slow” dynamical solutions in terms of the moduli space of static solutions was proposed on an heuristic level by Manton [3] who postulated the following adiabatic principle: for any geodesic trajectory $\gamma_0$ on the moduli space of $d$-vortices $M_d$ it should exist a sequence $\{\gamma_\epsilon\}$ of dynamical solutions, converging to $\gamma_0$ in the adiabatic limit.

A rigorous mathematical formulation and the proof of this principle were given recently by Palvelev in [4] (cf. also [5]).

Adiabatic correspondence. The adiabatic principle reduces the description of scattering of vortices in our model to the description of geodesics on the moduli space of $d$-vortices $M_d$ in the kinetic metric, i.e. to the solution of Euler geodesic equation on the space $M_d$ provided with $T$-metric.
In other words we have the following correspondence, established by the adiabatic limit:

\[
\{ \text{solutions of GL-equations} \} \leftrightarrow \{ \text{geodesics of the moduli space of vortices in } T\text{-metric} \}
\]

### 3.2 Adiabatic limit in SW-equations on 4-dimensional symplectic manifolds

**SW-equations with scale parameter.** In order to study the adiabatic limit in SW-equations we plug the scale parameter into them. For that we set the perturbation \( \eta \) equal to \( \eta = -F^+_{\text{can}} + \pi i \lambda \omega \) where \( \lambda \) is the scale parameter and introduce the normalized sections \( \alpha := \frac{\phi_0}{\sqrt{\lambda}} \), \( \beta := \frac{\phi_2}{\sqrt{\lambda}} \). The perturbed Seiberg–Witten equations will take the form

\[
\begin{align*}
\bar{\partial}_B \alpha + \bar{\partial}_B^* \beta &= 0 \\
\frac{4i}{\lambda} F^\omega_B &= 4\pi + |\beta|^2 - |\alpha|^2 \\
\frac{2}{\lambda} F^{0,2}_B &= \bar{\alpha} \beta.
\end{align*}
\]

and will be called briefly the SW\(\lambda\)-equations.

**Apriori estimates.** Suppose that the necessary solvability condition (1) is satisfied and SW-invariant of \( X \) does not vanish. Then these equations have a solution \((\alpha_\lambda, \beta_\lambda)\) for sufficiently large \( \lambda \). This solution has the following behavior for \( \lambda \to \infty \).

**Theorem 3** (Taubes \[10\]).

1. \( |\alpha_\lambda| \to 1 \) everywhere outside the set of zeros \( \alpha^{-1}_\lambda(0) \);
2. \( |\beta_\lambda| \to 0 \) everywhere together with its derivatives of the 1st order.

**Taubes construction.** Denote by \( C_\lambda := \alpha^{-1}_\lambda(0) \) the zero set of the section \( \alpha_\lambda \). Then the curves \( C_\lambda \) converge in the sense of currents to some pseudoholomorphic divisor Poincaré dual to the Chern class \( c_1(E) \), i.e. a chain \( \sum d_k C_k \), consisting of connected pseudoholomorphic curves \( C_k \) taken with multiplicities \( d_k \). Simultaneously, the original SW-equations reduce to a family of static GL-equations defined in the complex planes normal to the curves \( C_k \). The chain \( \sum m_k C_k \) may be considered as a complex analogue of adiabatic trajectory in the \((2+1)\)-dimensional case.

Conversely, in order to reconstruct a solution of Seiberg–Witten equations from the chain \( \sum m_k C_k \), the family of vortex solutions in normal planes should satisfy a nonlinear \( \bar{\partial} \)-equation which may be considered as a complex analogue of the Euler equation for adiabatic geodesics with ”complex time”.

**Adiabatic correspondence.** Thus, in this case we have the following correspondence, established by the adiabatic limit:

\[
\{ \text{solutions of SW-equations} \} \leftrightarrow \{ \text{families of vortex solutions in normal planes} \}
\]

\[1\text{For the precise definition of SW-invariant cf. } \[1\], \[6\], \[11\] \]
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