Exact Quantum Correlations of Conjugate Variables From Joint Quadrature Measurements

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We demonstrate that for two canonically conjugate operators $\hat{q}, \hat{p}$, the global correlation $\langle \hat{q} \hat{p} + \hat{p} \hat{q} \rangle - 2\langle \hat{q} \rangle \langle \hat{p} \rangle$, and the local correlations $\langle \hat{q} \rangle \langle \hat{p} \rangle - \langle \hat{q} \rangle$ and $\langle \hat{p} \rangle \langle \hat{q} \rangle - \langle \hat{p} \rangle$ can be measured exactly by Von Neumann-Arthurs-Kelly joint quadrature measurements. These correlations provide sensitive experimental test of quantum phase space probabilities quite distinct from the probability densities of $q, p$. E.g. for EPR states, and entangled generalized coherent states, phase space probabilities which reproduce the correct position and momentum probability densities have to be modified to reproduce these correlations as well.

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We use the Von Neumann-Arthurs-Kelly Joint Measurements, realizable as heterodyne measurements in quantum optics. As a first application these correlations are used to experimentally test proposed phase space probabilities and to constrain construction of such correlations are used to experimentally test proposed phase space probabilities and to constrain construction of such correlations.

Arthurs-Kelly Results. Their idea is that the system (with position and momentum operators $\hat{q}, \hat{p}$) interacts with an apparatus which has two commuting observables $x_1, x_2$ and approximate values of system position and momentum are extracted from accurate joint observation of $x_1, x_2$. The Von Neumann-Arthurs-Kelly interaction during the time interval $(t_0, t_0 + T)$ is,

$$H = K(\hat{q}\hat{p}_1 + \hat{p}\hat{p}_2),$$  (1)

where $\hat{p}_1, \hat{p}_2$ are canonical conjugates of $x_1, x_2$ respectively, the coupling $K$ is large, and $T$ is small, with $KT = 1$. During interaction time, $H$ is so strong that the free Hamiltonians of the system and apparatus are neglected. Arthurs and Kelly start with the system-apparatus initial state,

$$\psi(q, x_1, x_2, t_0) = \phi(q)\chi_1(x_1)\chi_2(x_2)$$  (2)

where, $\phi(q)$ is the system state and the apparatus state is given by,

$$\chi_1(x_1) = \pi^{-1/4}b^{-1/2} \exp(-x_1^2/(2b^2)),$$  (3)

$$\chi_2(x_2) = \pi^{-1/4}(2b)^{1/2} \exp(-2b^2 x_2^2),$$  (4)

and $b/\sqrt{2}$ is the uncertainty of $x_1$ in the initial apparatus state. They solve the Schrödinger equation exactly and obtain the final joint probability density of the apparatus variables to be just the Husimi function $\mathcal{H}$,

$$P(x_1, x_2) = |\langle \phi_0(x_1, x_2) | \phi \rangle|^2 / (2\pi),$$  (5)

where

$$\phi_0(x_1, x_2) = (2\pi b^2)^{-1/4} \exp(iq x_2 - (x_1 - q)^2/(4b^2))$$  (6)

is a minimum uncertainty system state centred at $q = x_1, p = x_2$. Note that for any value of $b$, $\langle x_1 \rangle = \langle \hat{q} \rangle$, and $\langle x_2 \rangle = \langle \hat{p} \rangle$, but the dispersions in $x_1, x_2$ are larger than those for the corresponding system variables $q, p$.

$$\Delta x_1^2 = (\Delta q)^2 + b^2, \Delta x_2^2 = (\Delta p)^2 + 1/4b^2.$$  (7)
they obey the “measurement or noise” uncertainty relation, (units $\hbar = 1$),
\[
\Delta x_1 \Delta x_2 \geq 1
\]
(8)
obtained by varying $b$. Here the minimum uncertainty is twice the usual “preparation uncertainty”. Arthurs and Goodman [4] gave a beautiful proof of this fundamental uncertainty relation, independent of any particular choice of the Hamiltonian. Further, for the $x_1$ distribution to approximate $q$ distribution closely, we need $b \ll \Delta q$; for the $x_2$ distribution to approximate $p$ distribution closely, we need $b \gg (\Delta p)^{-1}$.

\[
P_1(x_1) \equiv \int P(x_1, x_2)dx_2 = (2\pi)^{-1/2}b^{-1} \\
\int dq|\phi(q)|^2 \exp \left(-\left(x_1 - q\right)^2/(2b^2)\right) \to b \to 0 |\phi(x_1)|^2, \tag{9}
\]
\[
P_2(x_2) \equiv \int P(x_1, x_2)dx_1 = (2\pi)^{-1/2}b \\
\int dp|\tilde{\phi}(p)|^2 \exp \left(-\left(x_2 - p\right)^2/(2b^2)\right) \to b \to \infty |\phi(x_2)|^2, \tag{10}
\]
where $\tilde{\phi}(p)$ denotes the Fourier transform of $\phi(q)$.

**Exact Measurement of Quantum Correlations Between Conjugate Variables.** The above equations show that the exact position and momentum probability densities of the system are recovered by the Arthurs-Kelly measurement in the limits $b \to 0$ and $b \to \infty$ respectively, i.e. in two experiments with very different initial apparatus states. It is a pleasant surprise that the joint measurement can nevertheless give local and global correlations between $\hat{q}$ and $\hat{p}$ exactly. We define,

\[
\langle \hat{p}\rangle(q) \equiv \frac{\langle \Lambda(q)\hat{p} + \hat{p}\Lambda(q)\rangle}{2\langle \Lambda(q)\rangle}; \langle \hat{q}\rangle(p) \equiv \frac{\langle \Lambda(p)\hat{q} + \hat{q}\Lambda(p)\rangle}{2\langle \Lambda(p)\rangle},
\]
(11)
where $\langle A \rangle$ denotes the quantum expectation value of a self-adjoint operator $A$, and the projection operators $\Lambda(q), \Lambda(p)$ are defined by,

\[
\Lambda(q) = |\psi\rangle\langle q|, \Lambda(p) = |p\rangle\langle p|.
\]
(12)
For a pure state $|\psi\rangle$ we have the explicit expressions,

\[
\langle \hat{p}\rangle(q) = \frac{\text{Re}(\hat{p}^*\langle q\rangle q\hat{p} \partial \phi(q)/\partial q)}{|\phi(q)|^2}, \tag{13}
\]
\[
\langle \hat{q}\rangle(p) = \frac{\text{Re}(\hat{q}^*\langle p\rangle p\hat{q} \partial \phi(p)/\partial p)}{|\phi(p)|^2}. \tag{14}
\]
We shall see that the local correlations $\langle \hat{p}\rangle(q) - \langle \hat{p}\rangle$ and $\langle \hat{q}\rangle(p) - \langle \hat{q}\rangle$ can be measured exactly for arbitrary $q$ and $p$ respectively for appropriate values of $b$. The global correlation $(\hat{q}\hat{p} + \hat{p}\hat{q}) - 2\langle \hat{q}\rangle\langle \hat{p}\rangle$ is in fact exactly measurable for any value of $b$.

For the Arthurs-Kelly measurement we define as for a classical distribution,

\[
\langle x_2\rangle_{A-K}(x_1) \equiv \int x_2P(x_1, x_2)dx_2/P(x_1), \tag{15}
\]
\[
\langle x_1\rangle_{A-K}(x_2) \equiv \int x_1P(x_1, x_2)dx_1/P(x_2), \tag{16}
\]
\[
\langle x_1x_2\rangle_{A-K} \equiv \int x_1x_2P(x_1, x_2)dx_1dx_2. \tag{17}
\]
Substituting the value of $P(x_1, x_2)$, and doing the integral over $x_2$ we obtain,

\[
\langle x_2\rangle_{A-K}(x_1) = \frac{(b\sqrt{2\pi})^{-1}}{4b^2} \int dq \phi(q)\phi^*(q') \exp \left(-\frac{(x_1 - q)^2 + (x_1 - q')^2}{2b^2}\right) \frac{\partial \delta(q - q')}{\partial q} \\
= \text{Re} \int \frac{dq}{b\sqrt{2\pi}} \phi(q)\phi^*(q)(-i) \frac{\partial \delta(q)}{\partial q} \partial p. \tag{18}
\]
where $\delta(q - q')$ is the Dirac delta function. Similarly, we obtain,

\[
\langle x_1\rangle_{A-K}(x_2) \to b \to 0 \langle \hat{p}\rangle(q = x_1), \tag{20}
\]
\[
\langle x_1x_2\rangle_{A-K} \to b \to \infty \langle \hat{q}\rangle(p = x_2). \tag{21}
\]
Thus we have proved that the quantum position probability density and the local correlation $\langle \hat{p}\rangle(q) - \langle \hat{p}\rangle$ can be measured exactly with the initial condition $b \to 0$; the quantum momentum probability density and the local correlation $\langle \hat{q}\rangle(p) - \langle \hat{q}\rangle$ can be measured exactly with the very different initial condition $b \to \infty$. A similar calculation shows that for any value of $b$,

\[
\langle 2x_1x_2\rangle_{A-K} = \langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle, \tag{22}
\]
the global correlation is exactly measured in the Arthurs-Kelly (A-K) measurement. Thus, the A-K measurements with $b \to 0$ and $b \to \infty$ equip us with exact probability densities of position and momentum as well as their exact local and global correlations.

**Experimental test of phase space probabilities by correlation measurements.** We demonstrate that exact measurement of the correlations is a valuable tool to discriminate between various phase space probability densities which may give exactly the same position and momentum probability densities. The tremendous progress initiated by research on Bell inequalities,

\[
\int d\rho P(x_1, x_2)dx_1dx_2 = (2\pi)^{-1} (2\pi)^{-1}b^{-1} \\
\int \frac{dp\phi(p)}{b\sqrt{2\pi}} \phi^*(p)\frac{\partial \phi(p)}{\partial p}.
\]
and quantum contextuality [11], and their extension to phase space [12] teaches us that in $2N$ dimensional phase space, a positive density can have a maximum of $N + 1$ marginals reproducing quantum probability densities for arbitrary states. (E.g. for $N = 2$, probability densities of $(q_1, q_2), (p_1, q_2), (p_1, p_2)$ can be reproduced.) Of course we know that all marginals of Wigner’s quasi-probability distribution [13] agree with the corresponding quantum probability densities for the state $|\psi\rangle$. But we shall only consider positive densities as candidates for a probability interpretation. De Broglie and Bohm [10] proposed a positive phase space density which reproduces the quantum position probability density but fails to agree with the quantum momentum probability density [11]. The most general positive densities with two marginals reproducing quantum position and momentum probabilities [11], and with $N + 1$ marginals reproducing the corresponding quantum probabilities are also known [12]. Roy and Singh [15] built a new causal quantum mechanics symmetric in $q, p$ in which the phase space density obeys positivity and the marginal conditions on momentum and position probabilities. For example for $N = 1$, the two densities

$$\rho_\epsilon(q, p) = |\hat{\varphi}(q)|^2 |\hat{\phi}(p)|^2$$

$$\delta \left( \int_{-\infty}^{\infty} dp' |\hat{\varphi}(p')|^2 - \int_{-\infty}^{\infty} dq' |\hat{\phi}(\epsilon q', t)|^2 \right),$$

where $\epsilon = \pm 1$ clearly reproduce the quantum position and momentum probabilities as marginals.

$$\int \rho_\epsilon(q, p) dp = |\hat{\varphi}(q)|^2; \int \rho_\epsilon(q, p) dq = |\hat{\phi}(p)|^2.$$  

(24)

To demonstrate the discriminatory power of the quantum correlation measurements we shall use them in several concrete examples to test these two phase space densities (for $\epsilon = \pm 1$), as well as the correlationless phase space density $|\hat{\varphi}(q)|^2 |\hat{\phi}(p)|^2$, all of which reproduce quantum $q, p$ probability densities.

(i) Free particle spreading wave packets for a non-relativistic particle of mass $m$. At the time $t_0$ of the A-K measurement, let

$$\tilde{\varphi}(p) = (\alpha)^{-1/4} \exp[-(p - \beta)^2 / 2\alpha] - it_0 p^2 / 2m]$$

$$\Delta p^2 = \frac{\alpha}{2}, \Delta q^2 = \frac{1}{2\alpha} + \frac{\alpha t_0 / m}{2\alpha}.$$  

(25)

The Roy-Singh $q, p$ symmetric causal quantum mechanics gives, for $\epsilon = \pm 1$,

$$\langle p \rangle(q)_\pm - \langle \tilde{q} \rangle = \pm \frac{\Delta q}{\Delta p} (p - \beta t_0 / m),$$

(26)

whereas the Arthurs-Kelly correlation is,

$$\langle x_2 \rangle(x_1)_{A-K} - \langle \tilde{p} \rangle = \frac{\sqrt{(\Delta q \Delta p)^2 - 1}}{(\Delta q)^2 + b^2} (x_1 - \beta t_0 / m),$$

(27)

and it’s limit $b \to 0$ is the true quantum correlation. The correlationless phase space density gives 0 for the above correlation and the $\epsilon = -1$ case gives a negative correlation, both disagreeing with the quantum correlation, whereas the ratio of the correlation in the A-K measurement to that in the $\epsilon = 1$ Roy-Singh causal density (see figure) approaches unity for $b/\Delta q \ll 1$ and $\Delta q \Delta p \gg 1/2$; i.e. there is agreement with quantum mechanics only when the uncertainty product is large. Similarly, for correlation relations at given $p$, the Roy-Singh causal quantum mechanics gives, for $\epsilon = \pm 1$,

$$\langle q \rangle(p)_\pm - \langle \tilde{q} \rangle = \pm \frac{\Delta q}{\Delta p} (p - \beta),$$

(28)

which agrees only for $\epsilon = 1$ and only for large $\Delta q \Delta p$ with the quantum correlation which is the $b \to \infty$ limit of the Arthurs-Kelly correlation,

$$\langle x_1 \rangle(x_2)_{A-K} - \langle \tilde{p} \rangle = \frac{\sqrt{(\Delta q \Delta p)^2 - 1/4}}{(\Delta p)^2 + (4b^2)^{-1}} (x_2 - \beta).$$

(29)

For the global correlation, the Roy-Singh causal quantum mechanics with $\epsilon = \pm$ gives,

$$2\langle qp \rangle_\pm - 2\langle q \rangle(p) = \pm 2\Delta q \Delta p,$$  

(30)

of which only the $\epsilon = 1$ correlation agrees with the quantum correlation,

$$\langle q \dot{p} + \dot{q} \rangle - 2\langle q \rangle(\dot{p}) = \sqrt{(2\Delta q \Delta p)^2 - 1},$$

(31)

provided that $2\Delta q \Delta p \gg 1$. 

FIG. 1: For the free particle expanding Gaussian wave packet, the ratio of the correlation $\langle x_2 \rangle(x_1) - \langle x_2 \rangle$ in the Arthurs-Kelly measurement to $\langle p \rangle(q) - \langle \tilde{p} \rangle$ in the $\epsilon = 1$ Roy-Singh causal phase space density is plotted for various values of $b/\Delta q$ and of $\Delta q \Delta p$. The causal correlation agrees with the quantum correlation (i.e. the $b \to 0$ limit of the A-K correlation) only for large values of the uncertainty product. A convex combination of the $\epsilon = 1$ and $\epsilon = -1$ causal phase space densities can reproduce quantum correlations exactly. (Figure computed by Arunabha S. Roy.)
(ii) Generalized coherent states of light \[ |n\rangle \text{ are displaced } n\text{-th excited states of the oscillator of frequency } \omega, \]
\[
\phi_{n,\alpha}(q, t_0) = (q - \bar{q}|n\rangle \exp (-i\omega t_0 (n + \frac{1}{2}) + i\bar{q}(q - \bar{q}), \]
where \( a = A \exp (-i(\omega t_0 + \theta)), q = \bar{Re}, \bar{p} = \bar{Im}, \) and \( A, \theta \) are real constants. Here the Roy-Singh causal quantum mechanics with \( \epsilon = \pm 1 \) gives,
\[
\langle p|q\rangle \pm (\bar{p}) = \pm (q - \bar{q}), \langle q|p\rangle \pm (\bar{q}) = \pm (p - \bar{p}), \]
\[
(2qp)\pm - 2\langle q|p\rangle = \pm (2n + 1). \]
In contrast quantum mechanics gives zero for the above three correlations and thus agrees with the correlationless phase space density.

**Phase space probabilities reproducing quantum position and momentum probabilities and correlations exactly.** Surprisingly, in both the examples considered above, convex combinations of the Roy-Singh phase space densities with \( \epsilon = \pm 1 \),
\[
\rho(q, p) = \lambda_+ \rho_+(q, p) + \lambda_- \rho_-(q, p), \]
where the state dependent constants \( \lambda_\pm \) are chosen to reproduce the quantum global correlation \( \langle \hat{q}\hat{p} + \hat{p}\hat{q} - 2\langle \hat{q}\rangle\langle \hat{p}\rangle \) yield local correlations also equal to the corresponding quantum local correlations. Explicitly, in cases (i) and (ii) of Gaussian packets and generalized coherent states,
\[
(i) \lambda_\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - (2\Delta q\Delta p)^2}, \quad (ii) \lambda_\pm = 1/2. \]

**EPR states.** A normalizable version of the original EPR state \[ |q_1 - q_2 = q_0\rangle |p_1 + p_2 = p_0\rangle \] of two particles is,
\[
\phi(q_1 - q_2, p_1 + p_2) = \phi_1(q_1 - q_2) \phi_2(p_1 + p_2), \]
where the individual Gaussian wave functions,
\[
\phi_1(q_1 - q_2) = (\pi \alpha_1)^{-1/4} \exp (-\frac{(q_1 - q_2 - q_0)^2}{2\alpha_1}), \]
\[
\phi_2(p_1 + p_2) = (\pi \alpha_2)^{-1/4} \exp (-\frac{(p_1 + p_2 - p_0)^2}{2\alpha_2}) \]
are sharply peaked at \( q_1 - q_2 - q_0 = 0 \) and \( p_1 + p_2 - p_0 = 0 \) respectively in the limits \( \alpha_1 \rightarrow 0, \alpha_2 \rightarrow 0 \). We now construct the phase space density,
\[
\rho_{1c}\langle q_1 - q_2, (p_1 + p_2)/2 \rangle \rho_{2c}(q_1 + q_2, (p_1 + p_2)/2), \]
with the two factors \( \rho_{1c} \) and \( \rho_{2c} \) made to fit respectively the \( \langle q_1 - q_2, (p_1 + p_2)/2 \rangle \) and \( \langle q_1 + q_2, (p_1 + p_2)/2 \rangle \) correlations in the Gaussian states \( \phi_1, \phi_2 \) using convex combinations of the Roy-Singh phase space densities described above. This phase space density reproduces exactly, the above quantum correlations as well as quantum joint probability densities of the four commuting pairs of variables \( q_1 - q_2, (q_1 + q_2)/2, q_1 - q_2, p_1 + p_2, (q_1 + q_2)/2, p_1 - p_2, p_1 + p_2, p_1 - p_2 )/2 \).

**Entangled generalized coherent states.** For two modes of light with the same frequency an exact solution of the Schrödinger equation at time \( t_0 \) is the entangled generalized coherent state
\[
\phi_{m,\alpha}((q_1 + q_2)/\sqrt{2}, t_0) \phi_{n,\beta}((q_1 - q_2)/\sqrt{2}, t_0), \]
where \( m, n \) are integers, \( \alpha, \beta \) complex constants and the factors \( \phi_{m,\alpha}, \phi_{n,\beta} \) are generalized coherent states defined before. A phase space probability reproducing the relevant quantum correlations and probabilities exactly is
\[
\rho_{mC}((q_1 + q_2)/\sqrt{2}, (p_1 + p_2)/\sqrt{2}), \rho_{nC}((q_1 - q_2)/\sqrt{2}, (p_1 - p_2)/\sqrt{2}), \]
where \( \rho_{mC}, \rho_{nC} \) are arithmetic means of the \( \epsilon = \pm \) Roy-Singh phase space densities for the states \( \phi_{m,\alpha}, \phi_{n,\beta} \) respectively.

**Future directions.** The central point is the exact measurability of local and global correlations between conjugate observables. Actual joint quadrature measurements to test their correlations will be very interesting. The Arthurs-Kelly joint measurements and hence the possibilities of exact measurements of quantum correlations between conjugate variables can be generalized to \( 2N \)-dimensional phase space. An interesting question is triggered by the success in exact reproduction of chosen quantum correlations and probabilities in the special states (including entangled states) considered. Can we construct phase space probabilities reproducing quantum position and momentum probabilities and their correlations exactly for every quantum state?

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