Diamagnetic field states in cosmological plasmas

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Using a generally covariant Electro-Vortic (magnetofluid) formalism for relativistic plasmas, the dynamical evolution of a generalized vorticity (a combination of the magnetic and kinematic parts) is studied in a cosmological context. We derive macroscopic vorticity and magnetic field structures that can emerge in spatial equilibrium configurations of the relativistic plasma. These fields, however, evolve in time. These magnetic and velocity fields fields are self-consistently sustained in a diamagnetic state in the expanding Universe, and do not require an external seed for their existence. In particular, we explore a special class of magnetic/velocity field structures supported by a plasma in which the generalized vorticity vanishes. We derive a highly interesting characteristic of such “superconductor-like” fields in a cosmological plasmas in the radiation-era in early Universe. In that case, the fields grow proportional to the scale factor, establishing a deep connection between the expanding universe and the primordial magnetic fields.

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I. INTRODUCTION

Exploring the interaction of gravitational fields and inhomogeneous plasma thermodynamics as a possible source of primordial magnetic fields has, recently, received considerable attention [1, 2]. Much of this work has been carried out within the framework of what has been called a unified magnoeto-fluid, recently generalized to, Electro-Vortic (EV) formalism [3, 9]. The primary new construct of this formalism is the EV ten- sor $F_{\mu\nu} = F^{\mu\nu} + (m/q)S_{\mu\nu}$, a weighted sum of the Electromagnetic $F^{\mu\nu}$ and the Vortical $S^{\mu\nu}$ field tensors; the latter representing both the kinematic and thermal content of the relativistic hot fluid. In the EV formalism, the fluid dynamics reduces to a simple Helmholtz vortical form in terms of new composite variables, the most familiar being the so called generalized vorticity $\Omega$ (such that $\Omega^i = e^{\lambda j}M_{jk}$) that has both magnetic and thermal-kinetic parts. The most important message is that in the considerably complicated dynamics of a hot relativistic fluid, $\Omega$ plays the same role as the magnetic field $B$ does in the much simpler magnetohydrodynamics (MHD). Since many of our familiar concepts emerged from an MHD description, it is important to note the connections between EV dynamics and MHD.

The earlier special relativistic theory [3] was extended to explore general relativistic effects in Refs. [4, 5, 6]. It was shown that a combination of gravity modified Lorentz factor of the fluid element and the spatial variation of plasma thermodynamics, leads to an additional (to the special relativistic mechanism [11, 12]) source that creates a vorticity seed out of a state with no initial vortic-
mordial magnetic field problem in an expanding universe described by the Friedmann-Robertson-Walker (FRW) metric.

Our aim is to seek self-consistent magnetic-velocity field solutions (in the spirit of \[1\]) that evolve with the universe through an explicit dependence on the scale factor. Interestingly enough we find that the magnetic field strength increases with the scale factor, at least partially compensating the automatic dilution caused by the expanding universe.

The paper is organized as follows. In Sec. II, we review some of the essentials of the the ElectroVortic formalism. In Sec. III, we develop the 3+1 formalism used to describe the cosmological fluid, and derive the dynamical equation for the generalized vorticity. In Sec. IV and V, we study time varying spatial “equilibrium” solutions with their spatial structure resembling that of a perfect diamagnetic plasma state. The results are discussed in Sec. VI.

II. UNIFIED PLASMA DYNAMICS IN CURVED SPACETIME

The dynamics of an ideal plasma (a charged fluid), immersed in an electromagnetic field \(E_{\mu\nu}\), is contained in the conservation law

\[
\nabla_{\nu} T^\mu_{\nu} = qnF^{\mu\nu}U_{\nu},
\]

where \(U_{\mu}\) (such that \(U_{\mu}U^{\mu} = -1\)) is the four velocity, \(\nabla_{\nu}\) is the covariant derivative for the metric \(g_{\mu\nu}\) describing a curved spacetimes (\(c = 1\)), and

\[
T^{\mu\nu} = mnfU^{\mu}U^{\nu} + pg^{\mu\nu},
\]

is the energy-momentum tensor for an ideal plasma \[2\]: the charge \(q\) and the mass \(m\) of the fluid element are scalar invariants. The definition of the energy-momentum \[2\] involves three thermodynamic scalars: the scalar number density \(n\) (the rest frame density), the scalar pressure \(p\), and enthalpy density \(h = mnf\), where \(f\) is a function of temperature \(T\) that in the special case of a relativistic Maxwell distribution becomes \(f = K_{3}(m/k_{B}T)/K_{2}(m/k_{B}T)\) where \(K_{j}\) is the modified Bessel functions of order \(j\), and \(k_{B}\) is the Boltzmann constant. The system is completed with the continuity

\[
\nabla_{\mu} (nU^{\mu}) = 0,
\]

and Maxwell equations

\[
\nabla_{\nu} F^{\mu\nu} = 4\pi q nU^{\mu}.
\]

The global dynamics given by Eqs. \[1\-4\] is more conveniently studied in terms of a unified field tensor \[1\ 3\ 12\],

\[
M^{\mu\nu} = F^{\mu\nu} + \frac{m}{q} S^{\mu\nu},
\]

in which all kinematic and thermal (through \(f\)) aspects of the fluid are now represented by the antisymmetric tensor \[1\ 3\ 12\]

\[
S^{\mu\nu} = \nabla^{\mu} (fU^{\nu}) - \nabla^{\nu} (fU^{\mu}).
\]

The resulting equation of motion takes the form \((\partial^{\mu} p)\) is the four derivative)

\[
q U_{\nu} M^{\mu\nu} = -T\partial^{\mu}\sigma.
\]

where \(\sigma\) is the scalar entropy density of the fluid, and it is related to pressure through

\[
\partial^{\mu}\sigma = \frac{mn\partial^{\mu} f - \partial^{\mu} p}{nT},
\]

We can see that due to the antisymmetry of \(M_{\mu\nu}\), the fluid description presented here is always isentropic

\[
U_{\mu} \partial^{\mu}\sigma = 0.
\]

This general formulation of plasma dynamics in curved spacetime allows even a more compact expression for a special class of velocity fields. If the four velocity were derivable from a Clebsch potential \(Q\), \[10\],

\[
U^{\mu} = \frac{1}{T} \partial^{\mu} Q,
\]

then \[4\] reduces to

\[
U_{\nu} M^{\mu\nu} = 0,
\]

where the new field tensor

\[
M^{\mu\nu} = M^{\mu\nu} - \frac{1}{q} \partial^{\mu} (\sigma \partial^{\nu} Q) + \frac{1}{q} \partial^{\nu} (\sigma \partial^{\mu} Q)
\]

\[
= M^{\mu\nu} - \partial^{\mu} (\frac{\sigma T}{q} U^{\nu}) + \partial^{\nu} (\frac{\sigma T}{q} U^{\mu}).
\]

represents complete Electro-Vortic unification of the ideal relativistic dynamics.

III. EXPLICIT PLASMA DYNAMICS IN COSMOLOGY

For an explicit formulation of plasma dynamics in the cosmological background, we introduce the FRW metric. Since the focus of this calculation is to figure out how the plasma dynamics is affected by the expansion of the Universe, we will restrict ourselves to a spatially flat universe; the corresponding FRW metric will be \[10\ 17\]

\[
ds^{2} = -dt^{2} + a^{2} \gamma_{ij} dx^{i} dx^{j}, \quad (i, j = 1, 2, 3)
\]

\(a = a(t)\) is the time-dependent scale factor of the Universe, and \(\gamma_{ij} = (1, 1, 1)\) is the 3-metric of the spacelike hypersurfaces of the flat spacetime.

Following the procedure described in Refs. \[1\ 2\ 18\], we perform a 3 + 1 decomposition of the covariant fluid
plasma equations (Eq. (7) with the FRW metric) in order to put them in a more intuitive vectorial form. We define a normalized timelike vector field \( n^\mu \), obeying \( n^\mu n_\mu = -1 \) and \( n^\mu \gamma_{\mu\nu} = 0 \), such as \( n_\mu = (1, 0, 0, 0) \) and \( n^\mu = (-1, 0, 0, 0) \). The projection into time-like and space-like hypersurfaces, then, is readily achieved by contracting every tensor with \( n^\mu \) and \( a^2 \gamma_{\mu\nu} \).

We may write the four-velocity \( U^\mu = (\Gamma, \Gamma v^i) \)

\[
U^\mu = -\Gamma n^\mu + a^2 \Gamma \gamma_{\mu\nu} v^\nu , \tag{14}
\]

where \( v^i = dx^i/dt \) corresponds to the spatial component of the fluid velocity \( \mathbf{v} \). It follows that \( n_\mu U^\mu = \Gamma \), where the Lorentz factor is given by

\[
\Gamma = (1 - a^2 v^2)^{-1/2} , \tag{15}
\]

where \( v^2 = \gamma_{ij} v^i v^j = \mathbf{v} \cdot \mathbf{v} \). Note that the scale factor modifies \( \Gamma \) from its special relativistic value. Using the 3+1 decomposition \([14]\) of the FRW metric, the continuity equation \([33]\) becomes

\[
\frac{1}{a^3} \frac{\partial}{\partial t} (a^3 \Gamma) + \nabla \cdot (n \Gamma \mathbf{v}) = 0 , \tag{16}
\]

where \( \nabla \) is the flat spatial gradient operator.

Now, neglecting the plasma back-reaction on spacetime, one may write down the decomposition of the field equations. The 3+1 decomposed electric and magnetic fields are obtained as

\[
E^\mu = n_\nu F^{\nu\mu} , \quad B^\mu = \frac{1}{2} n_\rho \epsilon^{\rho\mu\sigma\tau} F_{\sigma\tau} , \tag{17}
\]

where \( \epsilon^{\alpha\beta\gamma\delta} \) is the totally antisymmetric tensor. Notice that the electric \( E^\mu \) and the magnetic \( B^\mu \) fields are spacelike tensors \((n_\nu E^\nu = 0 \) and \( n_\nu B^\nu = 0 \)) implying that \( E^0 = 0 = B^0 \) for a cosmological background. With the previous definitions we decompose the electromagnetic field tensor as

\[
F^{\mu\nu} = E^\nu n^\mu - E^\mu n^\nu - \epsilon^{\mu\nu\sigma\tau} B_{\sigma\tau} . \tag{18}
\]

This allows us to explicitly write the Maxwell equations \([4]\) in terms of electric and magnetic fields for a cosmological background by projecting them into time-like and space-like hypersurfaces. Substituting Eq. \([18]\) into \([4]\), and projecting it onto \( n_\mu \), we find \([18, 20]\)

\[
\nabla \cdot \mathbf{E} = 4 \pi q n \Gamma . \tag{19}
\]

Also, projecting Eq. \([4]\) onto space-like hypersurfaces yields

\[
\nabla \times \mathbf{B} = 4 \pi q n \Gamma \mathbf{v} + \frac{1}{a^3} \frac{\partial}{\partial t} (a^3 \mathbf{E}) . \tag{20}
\]

Similarly, one can define the dual electromagnetic tensor

\[
F^{*\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = B^\nu n^\mu - B^\mu n^\nu - \epsilon^{\mu\nu\rho\sigma} E_{\rho} n_{\sigma} , \tag{21}
\]

that satisfies \( F^{*\mu\nu} = 0 \) by its antisymmetry. When projected onto \( n_\mu \), we find the time-like component \([18, 20]\)

\[
\nabla \cdot \mathbf{B} = 0 , \tag{22}
\]

whereas the spacelike projection has the vectorial equivalent \([18, 20]\)

\[
\frac{1}{a^3} \frac{\partial}{\partial t} (a^3 \mathbf{B}) = - \nabla \times \mathbf{E} . \tag{23}
\]

Eqs. \([14, 20, 22]\) and \([24]\) correspond to the Maxwell equations for a cosmological plasma in an expanding universe.

For plasma dynamics, a similar decomposition can be performed on the antisymmetric unified tensor \([\mathbf{E}]\) yielding the generalized electric \( \xi^\mu \) and magnetic \( \Omega^\mu \) fields,

\[
\xi^\mu = n_\nu M^{\mu\nu} , \quad \Omega^\mu = \frac{1}{2} n_\rho \epsilon^{\rho\mu\sigma\tau} M_{\sigma\tau} , \tag{24}
\]

that are both spacelike \((n_\nu \xi^\nu = 0 \) and \( n_\nu \Omega^\nu = 0 \)). Equivalently, \( M^{\mu\nu} \) may be written as

\[
M^{\mu\nu} = \xi^\mu n^\nu - \xi^\nu n^\mu - \epsilon^{\mu\nu\rho\sigma} \Omega_{\rho\sigma} . \tag{25}
\]

The \( n_\mu \) projection will give the (three-vector) generalized electric \( \xi^\mu \) and magnetic fields \( \Omega^\mu \)

\[
\xi^\mu = \mathbf{E} - \frac{m}{qa^2} \left[ \nabla (f \Gamma) + \frac{1}{a^2} (f a^2 \Gamma \mathbf{v}) \right] , \tag{26}
\]

\[
\Omega^\mu = \mathbf{B} + \frac{ma^2}{q} \nabla \times (f \Gamma \mathbf{v}) , \tag{27}
\]

the curved spacetime generalization of the corresponding vector fields defined in Refs. \([11, 12]\). We emphasize that the generalized magnetic field \( \Omega \) allows an interpretation as a generalized vorticity because it is, indeed, the curl of a potential \((\Omega = \nabla \times \mathbf{A})\), where

\[
\mathbf{A} = \mathbf{A} + \frac{a^2 m f \Gamma}{q} \mathbf{v} , \tag{28}
\]

and \( \mathbf{A} \) is the vector potential of the electromagnetic field. From now, the names “generalized magnetic fields” and “generalized vorticity” are used interchangeably.

The generalized electric field \([20]\) and the generalized magnetic field \([27]\) are the key to writing the equation of motion \([7]\) in an insightful form. With previous definitions, Eq. \([7]\) becomes

\[
\xi^\mu - a^2 \gamma_{ij} \xi^i \gamma^j n^\mu + n_\chi \epsilon^{\lambda\mu\sigma\nu} v_\nu \Omega^\sigma = - \frac{T}{q \Gamma} \partial^\mu \sigma . \tag{29}
\]

This is the covariant form of the equation of motion from where the 3+1 equations can be obtained by appropriated projections on the time-like and space-like hypersurfaces. The \( n^\mu \) projection gives rise to the equation for energy conservation

\[
a^2 \mathbf{v} \cdot \xi = \frac{T}{q \Gamma} \frac{\partial}{\partial t} \sigma , \tag{30}
\]
while the spacelike $\gamma^\beta\mu$ projection yields the vectorial momentum evolution equation

$$\xi + v \times \Omega = -\frac{T}{q\Gamma} \nabla \sigma.$$  (31)

Eqs. (30) and (31) are equivalent to the usual 3+1 plasma equations [10, 20] invoked in plasma literature. Notice that there exist effects of the interaction of the fluid with the spacetime expansion hidden in the definition of the unified fields.

This unified magnetofluid approach leads us directly to the general vortical form of depicting the plasma dynamics. In this formalism the sources of general vorticity (where the magnetic field is just a component) are explicitly revealed. The vortical plasma dynamics can be completely described by using the antisymmetric properties of the unified tensor $M^{\mu\nu}$. Its dual tensor follows the conservation equation

$$\nabla_\nu M^{\mu\nu} = 0,$$  (32)

where $M^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta}M_{\alpha\beta}$ (in analogy with the electromagnetic tensor). A 3+1 decomposition of this equation provides physical insights on vortical dynamics. The dual tensor,

$$M^{\mu\nu} = \Omega^{\mu} n^\nu - \Omega^{\nu} n^\mu + \epsilon^{\mu\nu\alpha\beta}\xi_\alpha n_\beta,$$  (33)

on 3 + 1 decomposition, leads to the equation for the timelike hypersurface

$$\nabla \cdot \Omega = 0.$$  (34)

This equation represents the generalization of the divergence-free nature of the magnetic field. On the other hand, the spacelike projection of Eq. (32),

$$\frac{\partial}{\partial t} (a^3\Omega) + a^3 \nabla \times \xi = 0,$$  (35)

represents the constraint linking the generalized electric and magnetic fields (Generalized Faraday law).

**IV. PLASMAS IN CLASSICAL PERFECT DIAMAGNETISM STATE- MAGNETIC FIELD STRUCTURES**

In formulating plasma dynamics beyond MHD, we noticed that if generalized vorticity were to replace the magnetic field, the MHD like vortical structure of the dynamics is fully retained. Since the velocity and magnetic fields are the measurable of interest, we will seek self-consistent solutions for $v$ and $B$ for a specified thermodynamics (the current model does not evolve thermodynamics). A variety of such solutions for the special relativistic dynamics were worked out in Ref. [10]. In this section, we will explore the appropriate translations of some of these solutions in the context of cosmological plasmas in an expanding universe.

An interesting (and exact) class of solutions, accessible to cosmological plasmas, belong to the general category of states that display Classical Perfect Diamagnetism (CPD) [12]. In CPD states, the generalized vorticity is fully expelled from the plasma interior. It is worthwhile to remark here that it is the vanishing of the canonical vorticity that leads to the London equation (implying Meissner Ochsenfeld effect) describing a standard superconductor.

Let us now derive the equations that define the CPD state pertinent to the cosmological plasma. In its simplest manifestation in a homentropic plasma ($\partial^\mu \sigma = 0$), Eq. (7) allows the solution

$$M^{\mu\nu} = 0, \quad F^{\mu\nu} = -\frac{m}{q} S^{\mu\nu},$$  (36)

that has a vanishing generalized vorticity, $\Omega = 0$. More explicitly, this condition relates the magnetic and velocity fields [see Eq. (21)],

$$B = -\frac{ma^2}{q} \nabla \times (f\Gamma v).$$  (37)

Note that [37] and the Maxwell equations [20]–[23] form a self-consistent system for the magnetic and velocity fields. In fact, the combination yields

$$\nabla^2 B = \frac{4\pi q^2}{mf^2} \hat{a} + \frac{1}{a^2} \frac{\partial^2}{\partial t^2} (a^3 B),$$  (38)

that can be solved for the magnetic field as long as the thermodynamic functions and the scale parameter $a$ are specified. There are no explicit external drives, relativistic or otherwise, needed to catapult the system from a zero to finite magnetic field state.

Unlike the standard equations associated with superconducting states, Eq. (38) is time dependent. An exactly solvable set emerges if we assume that the thermodynamic quantities ($n$ and $f$) are functions of time only (through the scale factor $a$). In such a case, the ansatz, $B(x, t) = b(x)T(t)/a(t)^3$ splits [38] into two ordinary differential equations

$$\nabla^2 b = \lambda_\rho^2 b,$$  (39)

$$\frac{\partial^2 T}{\partial t^2} + \omega_p^2 \left( \frac{\hat{n}}{a^2 f} - 1 \right) T = 0,$$  (40)

for the spatial and the temporal parts. The spatial part [39] is, precisely, the London equation predicting the spatial decay of $b$ on a collisionless thermally corrected skin depth $\lambda_\rho = c/\omega_p$, where $\omega_p = \sqrt{4\pi q^2 n_0/(f_0 m)}$ is the thermally corrected plasma frequency. For the temporal part of the magnetic field, we have assumed that $n = \hat{n}n_0$, $f = \hat{f} f_0$, with $\hat{n}$ and $\hat{f}$ denoting the temporal variation of the profiles.

To explicitly evaluate the temporal behavior of the magnetic field, let us study the solution in the radiation dominated era when the scale factor increases as
\(a = a_0 t^{1/2}\) \[16\]. Similarly, a hot plasma has a temperature that increases as \(T \propto a^{-1}\), where its density is \(n \propto T^3\). For high temperatures, \(f \approx 4k_B T/m\), and therefore \(\hat{n} = a^{-3}\), and \(\hat{f} = a^{-1}\). Thereby, Eq. (40) becomes

\[
\frac{\partial^2 T}{\partial t^2} + \omega_p^2 \left(\frac{1}{a_0^2 t^2} - 1\right) T = 0 , \tag{41}
\]

which can be solved exactly in terms of Bessel functions. Since the era of interest for the current enquiry belongs to relatively smaller times \((a_0^2 t \ll 1)\), the approximate but explicit solution

\[
T(t) \approx t^{\lambda/2} \tag{42}
\]

where

\[
\lambda = 1 - \sqrt{1 - \frac{4\omega_p^2}{a_0^2}} , \quad 0 < \lambda < 1 , \tag{43}
\]

is more instructive. Notice that for real \(\lambda (\omega_p < a_0^2/2)\) such a solution persists only for a fraction of the plasma time (inverse of the plasma frequency).

The magnetic field, described by Eqs. (39) and (41) (and their solutions), though characterizing a CPD state, does not describe an equilibrium state; it evolves with the expansion of the universe. However, the spatial configuration remains as usual, described by the London equation. Physically, the solution \(B(x,t) \approx b(x)t^{(\lambda-3)/2}\), for the radiation-dominated era, describes a perfect diamagnetic state of the primordial plasma that expels the magnetic field from its inner region. As the Universe expand, and the time grows, this CPD state is more feasible to achieve, as the magnetic field decays faster in the plasma region.

VI. DISCUSSION

We have worked out, in this paper, the cosmological version (in an expanding universe) of the self-consistent magnetic and velocity fields that can exist even in plasmas with vanishing generalized vorticies. Such “superconducting” macroscopic states do not require any external seed mechanisms, such as Biermann battery \[1, 7, 30, 31\] or its generalizations.

Both classes of previously known diamagnetic states (magnetic fields nonzero only over a skin depth), when translated into the cosmic context, yield fields that grow with the expansion as \(T \propto a\) (while the scale factor goes up as \(a \propto t^{1/2}\)). The temporal growth of \(B\) tends to compete with the dilution (as \(a^{-3}\)) caused by the cosmological expansion. These new states with initially growing fields (whose detailed nature is yet to be explored) add a brand new element towards advancing our understanding of the cosmological magnetic fields and flows.

We have demonstrated, using the ElectroVortic formulation of the dynamics of relativistic plasmas in a time dependent metric, that the growing magnetic and vorticity fields are automatic solutions in an expanding universe. This newly established characteristic can and does bring additional insights into the comprehension of the “origin” of primordial magnetic fields in the Universe. This new set of solutions enriches the study of cosmological plasmas in general \[32, 33\], and can provide highly relevant initial conditions for dynamo action \[34, 38\].

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