1 Introduction

Some 2300 years ago the great Archimedes proved that the volume of the orthogonal intersection of two circular cylinders of equal radius, today called a bicylinder, is two thirds of the volume of its circumscribed cube \cite{1}. (It seems have remained unnoticed that if the two cylinders intersect at an angle, then the volume of the solid is two thirds that of the circumscribed box. This follows at once by making an affine transformation of $\mathbb{R}^3$, which preserves ratios of volumes.) Later, modern mathematicians found the volume of the intersection of three such cylinders, a so-called tricylinder or “Steinmetz solid”, and its computation has caused countless desperation headaches in generations of calculus students.

In 2017, Oliver Knill proposed, under the title “Archimedes’ Revenge,” that one prove that the volume of the intersection $R$ of the three (solid) hyperboloids

\begin{equation}
    x^2 + y^2 - z^2 \leq 1; \quad y^2 + z^2 - x^2 \leq 1; \quad z^2 + x^2 - y^2 \leq 1
\end{equation}

is equal to $\log 256$.

He proposed it as a challenge for the Harvard Maths 21A summer school in 2017. One of Knill’s students (the third author) offered an integral \cite{2} which computes the volume (see below); this motivated the solution we give here.

Figure 1: The solid contained by the three hyperboloids
2 The volume in the first octant

By symmetry, the intersection $R$ of the three hyperboloids is the union of eight congruent solids, one in each octant. We will compute the volume of the component solid $R_1$ in the first octant. The intricate internal complexity of the solid $R_1$ is shown by a brute-force triple integral for its volume $V_1$, namely:

\[
V_1 = \int_{0}^{1/\sqrt{2}} \int_{0}^{1/\sqrt{2}} \int_{0}^{1/\sqrt{2}} \left\{ \int_{0}^{\sqrt{x^2+y^2}} \int_{\sqrt{y^2-x^2}}^{\sqrt{x^2+y^2}} \int_{\sqrt{x^2-y^2}}^{\sqrt{1-x^2+y^2}} \right\} 1 \, dy \, dx \, dz
\]

This is by no means the end of the story. The first two integrals in the second and third lines cannot be computed directly, but one must change the order of integration to calculate them. Thus a complete computation of the volume $V_1$ by this direct approach is quite daunting and tedious.

3 Symmetry

We will show how symmetry considerations reduce the computation of $V_1$ to a single integral!

The solid $R$, whose total volume must be determined, is shown in Figure 1 – that appears on the webpage [2] of Oliver Knill, who kindly contributed the image. This Mathematica graphic shows its main features: a highly symmetrical solid whose boundary is a framework with several line segments supporting negatively curved surface patches taken from the three hyperboloids. (These line segments also form the edges of Kepler’s *stella octangula*.)

The origin of $\mathbb{R}^3$ is located at the (hidden) center of the solid. It evidently has the reflection symmetries $x \leftrightarrow -x$, $y \leftrightarrow -y$, $z \leftrightarrow -z$; and the cyclic symmetry $(x, y, z) \mapsto (y, z, x)$ of the $120^\circ$ rotation around the diagonal line $x = y = z$.

The key to understanding the solid in Figure 1 is that a one-sheeted hyperboloid is a ruled surface (indeed, a doubly ruled surface: it can be generated by either of two families of nonintersecting straight lines). Consider the intersection of any two of the hyperboloids of Eq. (1):

\[
\begin{align*}
\{ y^2 + z^2 - x^2 = 1 \} & \iff \{ z^2 = 1 \} \iff \{ x = \pm 1 \} \iff \{ z = \pm 1 \} \\
\{ z^2 + x^2 - y^2 = 1 \} & \iff \{ x^2 = y^2 \} \iff \{ x \pm y = 0 \}.
\end{align*}
\]

The two lines $z = 1$, $x \pm y = 0$ are the horizontal lines at the upper boundary in Figure 1; the other two form the lower boundary at the bottom. The solid is constrained to lie inside the third hyperboloid $x^2 + y^2 - z^2 \leq 1$, which cuts off these four lines at $x^2 + y^2 \leq 2$, yielding four line segments with eight endpoints $(\pm 1, \pm 1, \pm 1)$, all signs being allowed.
It should now be clear that the portion of the solid in the first octant, $R_1$, contains one vertex $(1, 1, 1)$ of Kepler’s star, and three cross-points of the boundary segments, namely, the standard basis vectors of $\mathbb{R}^3$.

The solid $R_1$ is composed of five pieces:

- two back-to-back tetrahedra $\Pi_1$ and $\Pi_2$ with a common base, namely the equilateral triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; and opposite vertices $(0, 0, 0)$ and $(1, 1, 1)$, respectively. Their union $\Pi_1 \cup \Pi_2$ is a triangular dipyramid (Figure 2).

- three congruent curved pieces $S_1$, $S_2$, $S_3$, each of which is bounded by an outer face of the larger tetrahedron $\Pi_2$ and one of the hyperboloids.

The volumes of the tetrahedra are: $\text{Vol}(\Pi_1) = \frac{1}{6}$ and $\text{Vol}(\Pi_2) = \frac{1}{3}$, giving a total of $\frac{1}{2}$.

4 Volume of one curved piece

We shall compute the volume of the solid piece $S_2$ bounded by: the face of $\Pi_2$ with vertices $(1, 0, 0)$, $(0, 0, 1)$, $(1, 1, 1)$; a segment of the unit circle in the $xz$-plane; and the hyperboloid.
\[ z^2 + x^2 - y^2 = 1. \] The face of \( \Pi_2 \) lies in the plane \( x - y + z = 1 \). The projection of this tetrahedral face onto the \( xy \)-plane is the triangle with vertices \((0, 0), (1, 0), (1, 1)\). See Figure 3.

The portion of the hyperboloid that forms the roof of \( S_2 \) is shown in Figure 3 with both rulings by line segments. Its curved boundary (a quarter circle) can be parametrized by either \((\sin \theta, 0, \cos \theta)\) or \((\cos \phi, 0, \sin \phi)\) with the angles in the interval \([0, \pi/2]\). The rulings proceed from there to either of the straight sides, as follows:

\[
t \mapsto (\sin \theta + t \cos \theta, 0, \cos \theta - t \sin \theta), \quad 0 \leq t \leq \sec \theta - \tan \theta; \\
s \mapsto (\cos \phi - s \sin \phi, 0, \sin \phi + s \cos \phi), \quad 0 \leq s \leq \sec \phi - \tan \phi.
\]

The triple integral for the volume of the curved piece \( S_2 \) is thus:

\[
I := \int_{0}^{1} \int_{y}^{1} \int_{1-x+y}^{\sqrt{1+y^2-x^2}} 1 \, dz \, dx \, dy.
\]

The inner integration is immediate:

\[
I = \int_{0}^{1} \int_{y}^{1} \sqrt{1+y^2-x^2} - (1-x+y) \, dx \, dy.
\]

We write

\[
I_1 := \int_{0}^{1} \int_{y}^{1} \sqrt{1+y^2-x^2} \, dx \, dy \quad \text{and} \quad I_2 := \int_{0}^{1} \int_{y}^{1} (1-x+y) \, dx \, dy.
\]

A routine calculation gives \( I_2 = \frac{1}{3} \).

Using the indefinite integral of \( \sqrt{a^2 - x^2} \) with \( a = \sqrt{1+y^2} \) and then integrating by parts twice, we obtain

\[
I_1 = \int_{0}^{1} \left[ \frac{x}{2} \sqrt{1+y^2-x^2} + \frac{1+y^2}{2} \arcsin \left( \frac{x}{\sqrt{1+y^2}} \right) \right]_{x=y}^{x=1} \, dy
\]

\[
= \int_{0}^{1} \left\{ \frac{1+y^2}{2} \arcsin \left( \frac{1}{\sqrt{1+y^2}} \right) - \frac{1+y^2}{2} \arcsin \left( \frac{y}{\sqrt{1+y^2}} \right) \right\} \, dy
\]

\[
= \frac{\pi}{6} + \frac{1}{2} \int_{0}^{1} \left( \frac{y + \frac{1}{3}y^3}{1+y^2} \right) \, dy - \frac{\pi}{6} - \frac{1}{2} \int_{0}^{1} \left( \frac{y + \frac{1}{3}y^3}{1+y^2} \right) \, dy
\]

\[
= \int_{0}^{1} \left( \frac{y + \frac{1}{3}y^3}{1+y^2} \right) \, dy = \frac{1}{3} \int_{0}^{1} \left( \frac{2y}{1+y^2} + y \right) \, dy = \frac{\log 2}{3} + \frac{1}{6}.
\]

Thus the volume of the curved piece \( S_2 \) is

\[
I = I_1 - I_2 = \frac{\log 2}{3} + \frac{1}{6} - \frac{1}{3} = \frac{\log 2}{3} - \frac{1}{6}.
\]
5  Total Volume

Therefore the volume of the solid $R_1$ in the first quadrant is

$$V_1 = 3I + \frac{1}{2} = 3\left(\frac{\log 2}{3} - \frac{1}{6}\right) + \frac{1}{2} = \log 2;$$

and the total volume of the solid $R$ in Archimedes’ revenge is:

$$\text{Vol}(R) = 8V_1 = 8\log 2 = \log 256. \quad \text{qed!}$$

6  Comment

The idea to exploit the symmetry of a component in one octant was already suggested by the third author [2]. His solution stated that the solid $R_1$ is composed of a tetrahedron (actually a dipyramid) of volume $\frac{1}{2}$ and the three congruent curved pieces. He computed the volume of a curved piece using the following integral:

$$I := \frac{1}{2} \int_0^1 \left( (z^2 + 1)\left(\frac{\pi}{2} - 2\arctan z\right) + z^2 - 1 \right) dz \quad (\ast)$$

which the first two authors found somewhat mysterious, and indeed their attempt to decipher it led to the solution offered here.

Acknowledgements

We are grateful to Oliver Knill for sharing the image in Figure 1 that appears on his webpage [2] (which also includes the student’s suggested solution), and for helpful comments. We also acknowledge support from the Vicerrectoría de Investigación of the Universidad de Costa Rica.

References

[1] Archimedes, The Works of Archimedes: The Method, T. L. Heath, ed., Dover Publications, New York, 1960.

[2] From the webpages of Oliver Knill, 2017: http://people.math.harvard.edu/~knill/teaching/summer2017/exhibits/revenge/

Postscript

A shortened version of the above solution was published as a Classroom Note by the first two authors in the College Mathematical Journal, vol. 56 (2024), 257–259. Shortly afterward, in an email exchange, the third author sent them his original solution, employing the aforementioned integral $(\ast)$. We append this elegant solution below.
\section{The original calculation}

Firstly, we can divide the solid into 8 parts, one part is shown in Figure 4:

![Figure 4: The curved piece $S_1$ in the first octant](image)

Next, we need to calculate the volume of the dark part in the picture. (This is $S_1$, one of the curved pieces mentioned above.)

If we slice it with a horizontal plane (at height $z$ between 0 and 1), we can get a shape like the one below (Figure 5). Call $S(z)$ the area of the shaded shape.

![Figure 5: A horizontal slice of the curved piece $S_2$](image)

Here $A = (0, 0, z)$, $B = (1, z, z)$, $C = (z, 1, z)$. The parameters in Figure 5 are related by

$$z = \tan \theta, \quad \alpha = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2 \arctan z,$$

and for convenience we put $r := \sqrt{1 + z^2}$, the radius of the circular arc $BC$ (which is a slice of the hyperboloid $x^2 + y^2 - z^2 = 1$). The area $S(z)$ is that of the circle sector $ABC$ minus the area of the triangle $\triangle ABC$. Since $\overrightarrow{AB} = (1, z, 0)$ and $\overrightarrow{AC} = (z, 1, 0)$, the triangle has area

$$\text{Area}(\triangle ABC) = \frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \| = \frac{1}{2} (1 - z^2).$$

The area of the circle sector $ABC$ is $\frac{1}{2}r^2\alpha$, and so

$$S(z) = \frac{1}{2} \left( (z^2 + 1) \left( \frac{\pi}{2} - 2 \arctan z \right) + (z^2 - 1) \right).$$

The volume of the curved piece $S_2$ is obtained by integrating $S(z)$ from $z = 0$ to $z = 1$:

$$\text{Vol}(S_2) = \int_0^1 S(z) \, dz = \frac{1}{2} \int_0^1 \left( \frac{\pi}{2} (z^2 + 1) + (z^2 - 1) - 2(z^2 + 1) \arctan z \right) \, dz$$

$$= \frac{1}{2} - \frac{1}{3} - \left[ \frac{z^3}{3} \arctan z + z \arctan z \right]_{z=0}^{z=1} + \int_0^1 \left( \frac{z^3}{3} + z \right) \, d(\arctan z)$$

$$= \frac{\pi - 1}{3} - \frac{\pi}{12} - \frac{\pi}{4} + \int_0^1 \frac{\frac{1}{2}z^2 + z}{z^2 + 1} \, dz = \frac{1}{3} + \frac{1}{3} \int_0^1 \left( z + \frac{2z}{z^2 + 1} \right) \, dz$$

$$= -\frac{1}{3} + \frac{1}{3} \left[ \frac{z^2}{2} + \log(z^2 + 1) \right]_{z=0}^{z=1} = -\frac{1}{6} + \frac{\log 2}{3}.$$

Figure 6: The unit cube with three equal pyramids removed

Then we can calculate the volume $V$ of the remaining part in the first octant, which is the cube of side 1 with three equal pyramids removed (Figure 6):

$$V = 1 \times 1 \times 1 - 3 \left( \frac{1}{3} \left( \frac{1}{2} \times 1 \times 1 \right) \right) = \frac{1}{2}.$$

So the total volume of the part in the first octant (see Figure 4) is:

$$V + 3 \left( -\frac{1}{6} + \frac{\log 2}{3} \right) = \frac{1}{2} - \frac{1}{2} + \log 2 = \log 2.$$

Finally, the total volume of the whole object is

$$8 \log 2 = \log(2^8) = \log 256.$$