Multibump nodal solutions for an indefinite superlinear elliptic problem

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Abstract
We define some Nehari-type constraints using an orthogonal decomposition of the Sobolev space $H^1_0$ and prove the existence of multibump nodal solutions for an indefinite superlinear elliptic problem.

1 Introduction
Consider a Lipschitz bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and a function $a \in C(\bar{\Omega})$, with $a = a^+ - a^-$, where $a^+ = \max\{a, 0\}$ as usual. Assume the set $a^+ > 0$ is the union of a finite number, $L \geq 1$, of open connected and disjoint Lipschitz components. We separate the components arbitrarily into three families
\[
\Omega^+ = \{x \in \Omega : a^+(x) > 0\} = (\bigcup_{i=1}^I \tilde{\omega}_i) \cup (\bigcup_{j=1}^J \tilde{\omega}_j) \cup (\bigcup_{k=1}^K \tilde{\omega}_k) = \tilde{\Omega} \cup \hat{\Omega} \cup \bar{\Omega},
\]
so that $L = I + J + K$; we also assume
\[
\Omega^- = \{x \in \Omega : a^-(x) > 0\} = \Omega \setminus \Omega^+.
\]
Let $\mu > 0$ and $p$ be a superquadratic and subcritical exponent, $2 < p < 2^*$, with $2^* = 2N/(N-2)$ for $N \geq 3$, and $2^* = +\infty$ for $N = 1$ or 2. Our main result is

Theorem 1.1. For every large $\mu$, there exists an $H^1_0(\Omega)$ weak solution $u_\mu$ of
\[
-\Delta u = (a^+ - \mu a^-)|u|^{p-2}u \quad \text{in } \Omega.
\]
Furthermore, the family \( \{u_\mu\} \) has the property that (modulo a subsequence)

\[
  u_\mu \rightharpoonup u \quad \text{in } H_0^1(\Omega) \quad \text{as } \mu \to +\infty,
\]

where

\[
\begin{cases}
  -\Delta u = a^+|u|^{p-2}u & \text{in } \tilde{\omega}_i, \\
  u^\pm \not\equiv 0 & \text{in } \tilde{\omega}_i, \quad i = 1, \ldots, I,
\end{cases}
\]

\[
\begin{cases}
  -\Delta u = a^+|u|^{p-2}u & \text{in } \tilde{\omega}_j, \\
  u^+ \not\equiv 0, \ u^- \equiv 0 & \text{in } \tilde{\omega}_j, \quad j = 1, \ldots, J,
\end{cases}
\]

and

\[
  u \equiv 0 \quad \text{in } \bar{\omega}_k, \quad k = 1, \ldots, K,
\]

and

\[
  u \equiv 0 \quad \text{in } \Omega^-.
\]

The one-dimensional version of \( \text{(1)} \) was studied in [15] with topological shooting arguments and phase-plane analysis. Theorem [16] extends the main result in [7] where the case \( \tilde{\Omega} = \emptyset \) was considered, so that the function \( u \) in [2] was positive. The authors used a volume constrain regarding the \( L^p \) norm, rescaling and a min-max argument based on the Mountain Pass Lemma. A careful analysis allowed them to distinguish between the solutions that arise from the \( 2^L \) different possible partitionings of \( \Omega^+ = \hat{\Omega} \cup \bar{\Omega} \). However, the argument in [7] does not seem either to extend easily to the present situation or to be suited to non-homogeneous nonlinearities.

Our approach is adapted from the work [18] regarding a system of equations related to

\[
\begin{cases}
  -\epsilon^2 \Delta u + V(x)u = f(u) & \text{in } \Omega, \\
  u > 0 & \text{in } \Omega,
\end{cases}
\]

when \( \epsilon \) is small and the functions \( V \) and \( f \) satisfy appropriate conditions. The positive function \( V \) was assumed to have a finite number of minima. In particular, the authors proved the existence of multipeak positive solutions by defining a Nehari-type manifold which, roughly speaking, imposes that the derivative of the associated Euler-Lagrange functional at a function \( u \) should vanish when applied to a truncation of \( u \) around a minimum of the potential function \( V \).

The perspective of [18] is related to the one of [16] which, using Nehari conditions and a cut-off operator, simplifies the original techniques for gluing together mountain-pass type solutions of [12], [13] and [20].

Our method consists in defining a Nehari-type set, \( \mathcal{N}_\mu \), by imposing that the derivative of the associated Euler-Lagrange functional at a function \( u \) should vanish when applied to the positive and negative parts of some projections of \( u \). The idea to use these projections is borrowed from [7], where they are also used, but in a different way.
We prove that the Euler-Lagrange functional associated to (1) has a minimum over the set $\mathcal{N}_\mu$ using an argument similar to the one found in [8]. Since our set $\mathcal{N}_\mu$ is not a manifold (see [5, Lemma 3.1]), one has to demonstrate, as in [9], that the minima are indeed critical points. As mentioned above, in the case that $\hat{\Omega} = \emptyset$ we recover the main result of [7], but with a simpler proof.

Our results are somewhat parallel to the ones of singular perturbation problems like in [14]. The large parameter $\mu$ in (1) plays the role of the small parameter $\epsilon$. The solutions concentrate in the set $\hat{\Omega} \cup \hat{\Omega}$ and vanish in the set $\Omega \cup \Omega^-$ as $\mu \to +\infty$.

In [1] flow invariance properties together with a weak splitting condition proved the existence of infinitely many geometrically distinct two bump solutions of a periodic superlinear Schrödinger equation. The paper [1] is concerned with the singular perturbed equation above. As a special case, the authors observed the existence of multiple pairs of concentrating nodal solutions at an isolated minimum of the potential.

There has been much interest in elliptic problems with a sign changing weight. We refer to [2], [3], [6], [11], [17], [19], [21] and the references therein.

For simplicity we restrict the proof to the case where $I = J = K = 1$, but it extends to the other ones as well. The work is organized as follows. In Section 2 we provide estimates for minimizing sequences on the set $\mathcal{N}_\mu$. In Section 3 we prove the existence of a minimizer in the set $\mathcal{N}_\mu$. Finally, in Section 4 we prove that a minimizer in the set $\mathcal{N}_\mu$ is a critical point using a local deformation and a degree argument similar to the one in [10].

## 2 Estimates for minimizing sequences on a Nehari-type set $\mathcal{N}_\mu$

As mentioned in the Introduction, we consider a Lipschitz bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and a function $a \in C(\Omega)$. We assume the set $a^+ > 0$ is the union of three Lipschitz components,

\[
\{ x \in \Omega : a^+(x) > 0 \} = \hat{\omega} \cup \hat{\omega} \cup \hat{\omega},
\]

and

\[
\{ x \in \Omega : a^-(x) > 0 \} = \Omega \setminus (\hat{\omega} \cup \hat{\omega} \cup \hat{\omega}). \tag{3}
\]

We introduce a positive parameter $\mu$ and consider $2 < p < 2^*$. We denote by $\langle \cdot, \cdot \rangle$ the usual inner product on the Sobolev space $H^1_0(\Omega)$, i.e., $\langle u, v \rangle = \int \nabla u \cdot \nabla v$ for $u, v \in H^1_0(\Omega)$. When the region of integration is
not specified it is understood that the integrals are over $\Omega$. We denote by $\| \|$ the induced norm. We define the spaces

$$
\begin{align*}
H(\tilde{\omega}) &= \{ u \in H^1_0(\Omega) : u = 0 \text{ in } \Omega \setminus \tilde{\omega} \}, \\
H(\hat{\omega}) &= \{ u \in H^1_0(\Omega) : u = 0 \text{ in } \Omega \setminus \hat{\omega} \}, \\
H(\bar{\omega}) &= \{ u \in H^1_0(\Omega) : u = 0 \text{ in } \Omega \setminus \bar{\omega} \},
\end{align*}
$$

which can be obtained from the spaces $H^1_0(\tilde{\omega})$, $H^1_0(\hat{\omega})$, $H^1_0(\bar{\omega})$ by extending functions as zero on $\Omega \setminus \tilde{\omega}$, $\Omega \setminus \hat{\omega}$, $\Omega \setminus \bar{\omega}$, respectively.

Each $u \in H^1_0(\Omega)$ can be decomposed as

$$
u = \tilde{u} + \hat{u} + \bar{u} + u,$$

with $\tilde{u}$, $\hat{u}$ and $\bar{u}$ the projections of $u$ on $H(\tilde{\omega})$, $H(\hat{\omega})$ and $H(\bar{\omega})$, respectively.

We recall the projections are defined by

$$
\begin{align*}
\tilde{u} \in H(\tilde{\omega}) : \forall \varphi \in H(\tilde{\omega}), \quad \langle u, \varphi \rangle &= \langle \tilde{u}, \varphi \rangle, \\
\hat{u} \in H(\hat{\omega}) : \forall \varphi \in H(\hat{\omega}), \quad \langle u, \varphi \rangle &= \langle \hat{u}, \varphi \rangle, \\
\bar{u} \in H(\bar{\omega}) : \forall \varphi \in H(\bar{\omega}), \quad \langle u, \varphi \rangle &= \langle \bar{u}, \varphi \rangle.
\end{align*}
$$

Clearly, these projections are orthogonal and continuous with respect to the weak topology. The function $u$ is harmonic in $\tilde{\omega} \cup \hat{\omega} \cup \bar{\omega}$.

The following is Theorem 1.1 in the case when $I = J = K = 1$.

**Proposition 2.1.** For every large $\mu$, there exists an $H^1_0(\Omega)$ weak solution $u_\mu$ of

$$
-\Delta u = (a^+ - \mu a^-)|u|^{p-2}u \quad \text{in } \Omega. \quad (4)
$$

Furthermore, the family $\{u_\mu\}$ has the property that, modulo a subsequence,

$$
u_\mu \rightharpoonup u \quad \text{in } H^1_0(\Omega) \text{ as } \mu \to +\infty, \quad (5)
$$

where

$$
u = \tilde{u} + \hat{u}, \quad (6)
$$

and

$$
\begin{align*}
-\Delta \hat{u} &= a^+ |\hat{u}|^{p-2}\hat{u} \quad \text{in } \hat{\omega}, \\
\hat{u}^\pm &\equiv 0, \quad (7)
\end{align*}
$$

and

$$
\begin{align*}
-\Delta \tilde{u} &= a^+ |\tilde{u}|^{p-2}\tilde{u} \quad \text{in } \tilde{\omega}, \\
\tilde{u}^+ &\not\equiv 0, \quad \tilde{u}^- \equiv 0. \quad (8)
\end{align*}
$$
The solutions of (4) are the critical points of the $C^2$ functional $I_{\mu} : H^1_0(\Omega) \to \mathbb{R}$, defined by

$$I_{\mu}(u) = \frac{1}{2} \|u\|^2 - \frac{1}{p} \int (a^+ - \mu a^-) |u|^p.$$  

We fix a function $v$ such that $v = \hat{v} + \hat{v}^+$, with $\hat{v}^+, \hat{v}^- \neq 0$ and

$$I_{\mu}'(v)(\hat{v}^+) = I_{\mu}'(v)(\hat{v}^-) = I_{\mu}'(v)(\hat{v}) = 0$$  

for some (and hence all) $\mu > 0$.

The restriction of $I_{\mu}$ to $H(\hat{\omega}) \oplus H(\bar{\omega})$ is independent of $\mu$ and has a strict local minimum at zero. We fix a small $\rho_0 > 0$ such that zero is the unique minimizer of $I_{\mu}$ in $\{u \in H(\hat{\omega}) \oplus H(\bar{\omega}) : \max\{\|\hat{u}\|, \|\bar{u}\|\} \leq \rho_0\}$. For $0 < \rho \leq \rho_0$, we denote by $c_{\rho}$ the positive constant

$$c_{\rho} := \inf_{u \in H(\hat{\omega}) \oplus H(\bar{\omega})} \frac{I_{\mu}(u)}{\rho \leq \max\{\|\hat{u}\|, \|\bar{u}\|\} \leq \rho_0}.$$  

(9)

The solutions of (4) will be obtained by minimizing the functional $I_{\mu}$ on the following Nehari-type set, $N_{\mu}$. Let $\rho_0$ be as above and $R > \|v\|$.

Definition 2.2. $N_{\mu}$ is the set of functions $u = \hat{u} + \hat{u} + \bar{u} + \bar{u} \in H^1_0(\Omega)$ such that

1. $\hat{u}^+, \hat{u}^-, \hat{u}^+ \neq 0$,
2. $I_{\mu}'(u)(\hat{u}^+) = I_{\mu}'(u)(\hat{u}^-) = I_{\mu}'(u)(\hat{u}^+) = 0$,
3. $I_{\mu}(u) \leq I_{\mu}(v) + 1$,
4. $\|u\| \leq \min\{\|\hat{u}^+\|, \|\hat{u}^-\|, \|\hat{u}^+\|\} < \|\hat{u} + \hat{u}^+\| \leq R$,
5. $\max\{\|\hat{u}^-\|, \|\bar{u}\|\} \leq \rho_0$.

We remark that $v \in N_{\mu}$ for all $\mu > 0$.

The square of the $H^1_0(\Omega)$ norm of $u$ is equal to the sum of the squares of the $H^1_0(\Omega)$ norms of the components of $u$, but the $p$-th power of the $L^p(\Omega)$ norm of $u$ does not have such a nice property. However, the next lemma says that this is almost the case when $\mu$ is large.

Lemma 2.3. Let $\delta > 0$ be given. There exists $\mu_\delta$ such that, if $\mu > \mu_\delta$,

$$\forall u \in N_{\mu}, \quad \int |u|^p < \delta.$$  

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Proof. Suppose, by contradiction, that for some \( \delta > 0 \) there exists \( \mu_n \to +\infty \) and \( u_n \in \mathcal{N}_{\mu_n} \) with

\[
\int |u_n|^p \geq \delta. \tag{10}
\]

As \( \|u_n\| \) is bounded, we may suppose \( u_n \rightharpoonup u \). We have \( u_n \rightharpoonup u \) and \( u \equiv 0 \) in \( \Omega \setminus (\hat{\omega} \cup \hat{\hat{\omega}} \cup \hat{\tilde{\omega}}) \). Otherwise, by (13) and modulo a subsequence,

\[
\int a^-|u_n|^p \geq c > 0.
\]

This would contradict \((N_{iii})\) for sufficiently large \( n \):

\[
\frac{1}{2} \|u_n\|^2 - \frac{1}{p} \int a^+|u_n|^p + \frac{\mu_n}{p} \int a^-|u_n|^p \leq I_{\mu}(v) + 1.
\]

So the function \( u \) belongs to \( H(\hat{\omega}) \oplus H(\hat{\hat{\omega}}) \oplus H(\hat{\tilde{\omega}}) \) and is harmonic in \( \hat{\omega} \cup \hat{\hat{\omega}} \cup \hat{\tilde{\omega}} \).

It follows that \( u \) must be identically equal to zero in \( \Omega \). This contradicts (10). \( \square \)

Usually one may obtain a lower bound for the \( H_0^1(\Omega) \) norm of \( \hat{\omega} \), \( \hat{\hat{\omega}} \) and \( \hat{\tilde{\omega}} \) from \((N_{i})\) and a condition like \((N_{ii})\). Here, in addition, we require the first inequality in \((N_{iv})\) to prove Lemma 2.4.

There exists a constant \( \kappa \), independent of \( \mu \), such that

\[
\forall u \in \mathcal{N}_{\mu}, \quad \min \{ \|\hat{u}^+\|, \|\hat{u}^-\|, \|\hat{u}^+\| \} \geq \kappa > 0. \tag{11}
\]

Proof. Let \( w \) be one of the three functions \( \hat{u}^+ \), \( -\hat{u}^- \) or \( \hat{u}^+ \). Denote by \( \chi \) the characteristic function of the set \( \{ x \in \Omega : w(x) \neq 0 \} \) and let \( c \) be the Sobolev constant

\[
\left( \int |v|^p \right)^{1/p} \leq c \|v\|, \quad \forall v \in H_0^1(\Omega).
\]

From \( I'_{\mu}(u)w = 0 \),

\[
\|w\|^2 = \int a^+|u|^{p-2}uw \leq \|a\|_{\infty} \left( \int |u|^p \right)^{\frac{p-1}{2}} \left( \int |w|^p \right)^{\frac{1}{2}} \\
\leq \|a\|_{\infty} c^p (\|u\| + \|w\|) \left( \int |w|^p \right)^{\frac{p-1}{p}} \|w\| \leq 2^{p-1} \|a\|_{\infty} c^p \|w\|^p,
\]

because of the first inequality in \((N_{iv})\). Since \( w \neq 0 \), due to \((N_{i})\), we may take

\[
\kappa = \left( 2^{p-1} \|a\|_{\infty} c^p \right)^{-1/(p-2)}.
\]

Now we fix a \( \mu \) and turn to minimizing sequences \( (u_n) \) for \( I_{\mu} \) restricted to \( \mathcal{N}_{\mu} \). Later it will be important that the limit of such a sequence has a neighborhood whose points satisfy \((N_{i})\), \((N_{iii})\), \((N_{iv})\) and \((N_{v})\). This follows from

\[\]
Lemma 2.5. Let $\overline{R}$ be fixed, $\|v\| < \overline{R} < R$, and $\delta$ be given, $0 < \delta < \rho_0$. There exists $\mu_\delta > 0$ such that for every $\mu > \mu_\delta$ and every minimizing sequence $(u_n)$ for $I_\mu$ restricted to $\mathcal{N}_\mu$, we have, for large $n$,

(a) $I_\mu(u_n) \leq I_\mu(v) + \frac{1}{2}$,
(b) $\|\hat{u}_n + \check{u}_n^+\| < \overline{R}$,
(c) $\max\{\|\hat{u}_n^+\|, \|\bar{u}_n\|\} < \delta$,
(d) $\|u_n\| < \delta$;
also
(e) $\frac{\mu}{p} \int a^- |u_n|^p < \delta$.

Proof. (a) Immediate since $(u_n)$ is minimizing and $v \in \mathcal{N}_\mu$ for all $\mu$.
(b) Suppose $\|\hat{u}_n + \check{u}_n^+\| \geq \overline{R}$ (12) for large $n$.

\[
I_\mu(u_n) = \frac{1}{2} \|\check{u}_n + \hat{u}_n^+\|^2 + \frac{1}{2} \|\hat{u}_n^+\|^2 + \frac{1}{2} \|\bar{u}_n\|^2 + \frac{1}{2} \|u_n\|^2
\]

\[-\frac{1}{p} \int a^+ |u_n|^{p-2} u_n (\check{u}_n + \hat{u}_n^+) + \frac{1}{p} \int a^+ |u_n|^{p-2} u_n \check{u}_n
\]

\[-\frac{1}{p} \int a^+ |u_n|^{p-2} u_n \bar{u}_n - \frac{1}{p} \int a^+ |u_n|^{p-2} u_n + \frac{\mu}{p} \int a^- |u_n|^p \geq \left(1 - \frac{1}{p}\right) \|\check{u}_n + \hat{u}_n^+\|^2 + o(1).\]

Here and henceforth $o(1)$ denotes a value, independent of $u \in \mathcal{N}_\mu$, that can be made arbitrarily small by choosing $\mu$ sufficiently large. For the proof of the last inequality we used $\mathcal{N}_\mu$,

\[
\frac{1}{2} \|\hat{u}_n^+\|^2 + \frac{1}{p} \int a^+ |u_n|^{p-2} u_n \hat{u}_n^+ \geq o(1)
\]

and

\[
\frac{1}{2} \|\bar{u}_n\|^2 - \frac{1}{p} \int a^+ |u_n|^{p-2} u_n \bar{u}_n \geq o(1)
\]

(consequences of $\mathcal{N}_\mu$) and Lemma 2.3,

\[-\frac{1}{p} \int a^+ |u_n|^{p-2} u_n \bar{u}_n = o(1)\]
\( \frac{1}{2} \| \nu_n \|^2 + \frac{\mu}{p} \int a^- |\nu_n|^p \geq 0. \)

We now use (12) and the definition of \( \mathcal{R} \). For sufficiently large \( \mu \),

\[ I_\mu(u_n) \geq \left( \frac{1}{2} - \frac{1}{p} \right) \mathcal{R}^2 + o(1) > \left( \frac{1}{2} - \frac{1}{p} \right) \| v \|^2 + c = I_\mu(v) + c, \]

for some \( c > 0 \). This contradicts the fact that \( (u_n) \) is minimizing.

(c) Suppose \( \| \tilde{u}_n \| \geq \delta \) for large \( n \). As in (b), we have

\[ I_\mu(u_n) = I_\mu(u_n + \tilde{u}_n) + \frac{1}{2} \| \tilde{u}_n \|^2 - \frac{1}{p} \int a^- |\tilde{u}_n|^p + o(1) \geq I_\mu(u_n + \tilde{u}_n) + c\delta + o(1), \]

due to Lemma 2.3 and then (9). This implies that

\[ \lim I_\mu(u_n) > \lim \inf I_\mu(u_n + \tilde{u}_n), \]

for sufficiently large \( \mu \), and contradicts the assumption that \( (u_n) \) is minimizing, because \( u_n + \tilde{u}_n \in \mathcal{N}_\mu \). Similarly, one proves that \( \| \bar{u}_n \| \geq \delta \) for large \( n \) leads to a contradiction, for sufficiently large \( \mu \), because \( u_n - \bar{u}_n \in \mathcal{N}_\mu \).

(d) Suppose \( \| \nu_n \| \geq \delta \) for large \( n \). From \( (N_{ii}) \) and Lemma 2.3 we know

\[ \| \tilde{u}_n^+ \|^2 = \int a^+ |\tilde{u}_n^+|^p + o(1), \]
\[ \| \tilde{u}_n^- \|^2 = \int a^+ |\tilde{u}_n^-|^p + o(1), \]
\[ \| \hat{u}_n^+ \|^2 = \int a^+ |\hat{u}_n^+|^p + o(1). \]

We define \( \tilde{r}_n, \tilde{s}_n \) and \( \hat{t}_n \) by

\[ \tilde{r}_n = \left( \frac{\| \tilde{u}_n^+ \|^2}{\int a^+ |\tilde{u}_n^+|^p} \right)^{\frac{1}{p-2}}, \]
\[ \tilde{s}_n = \left( \frac{\| \tilde{u}_n^- \|^2}{\int a^+ |\tilde{u}_n^-|^p} \right)^{\frac{1}{p-2}}, \]
\[ \hat{t}_n = \left( \frac{\| \hat{u}_n^+ \|^2}{\int a^+ |\hat{u}_n^+|^p} \right)^{\frac{1}{p-2}}, \]

so that \( \tilde{r}_n, \tilde{s}_n, \hat{t}_n = 1 + o(1) \) by Lemma 2.4 and

\[ v_n := \tilde{r}_n \tilde{u}_n^+ - \tilde{s}_n \tilde{u}_n^- + \hat{t}_n \hat{u}_n^+ - \hat{u}_n^- + \bar{u}_n. \]
Provided $\mu$ is large, we can guarantee $v_n \in \mathcal{N}_\mu$ for large $n$ due to (a), (b), (c) and Lemma 2.4. We now obtain an upper bound for $I_\mu(v_n)$:

$$I_\mu(v_n) = I_\mu(\tilde{u}_n + \tilde{u}_n + \tilde{u}_n) + o(1) \leq I_\mu(u_n) + o(1) - \frac{1}{2} \left( \frac{1}{2} \|u_n\|_2^2 - \frac{1}{p} \int a^+ |u_n|^p - \frac{1}{p} \int a^- |u_n|^p \right) \quad (13)$$

$$\leq I_\mu(u_n) + o(1) - \frac{1}{2} \|u_n\|_2^2 \leq I_\mu(u_n) + o(1) - \frac{1}{2} \delta^2.$$

This implies that $\lim \inf I_\mu(v_n) < \lim I_\mu(u_n)$ for sufficiently large $\mu$, which is impossible.

(e) Follows from inequality (13).

3 Existence of a minimizer in $\mathcal{N}_\mu$

For each $u \in \mathcal{N}_\mu$, we consider the 3-dimensional manifold with boundary in $H^1_0(\Omega)$ parametrized by

$$\varsigma(\tilde{r}, \tilde{s}, \hat{t}) = \tilde{r}\tilde{u}^+ - \tilde{s}\tilde{u}^- + \hat{t}\hat{u}^+ - \hat{u}^- + \bar{u} + u \quad (14)$$

We call $f$ the function $I_\mu \circ \varsigma$, so that

$$f(\tilde{r}, \tilde{s}, \hat{t}) = \frac{\tilde{r}^2}{2} \|\tilde{u}^+\|^2 + \frac{s^2}{2} \|\tilde{u}^-\|^2 + \frac{\hat{t}^2}{2} \|\hat{u}^+\|^2 + K$$

$$\leq \frac{1}{p} \int a^+ |\tilde{r}\tilde{u}^+ + \tilde{s}\tilde{u}^-| - \frac{1}{p} \int a^+ |\tilde{u}^-| - \frac{1}{p} \int a^+ |\hat{t}\hat{u}^+ + \bar{u}|,$$

with

$$K = \frac{1}{2} \|\tilde{u}^-\|^2 + \frac{1}{2} \|\tilde{u}^+\|^2 + \frac{1}{2} \|\bar{u}\|^2 - \frac{1}{p} \int a^+ |u - \tilde{u}|^p - \frac{1}{p} \int a^+ |\tilde{u}^+ + \tilde{u}^-|^p + \frac{\mu}{p} \int a^- |\tilde{u}|^p.$$

Two properties of $f$ are immediate, namely $f(1, 1, 1) = I_\mu(u)$ and $\nabla f(1, 1, 1) = 0$ by $(\mathcal{N}_\mu)$. The critical point $(1, 1, 1)$ is characterized in

**Lemma 3.1.** For $\mu$ sufficiently large, independent of $u \in \mathcal{N}_\mu$, the point $(1, 1, 1)$ is an absolute maximum of $f$. Furthermore, if

$$|(\tilde{r}, \tilde{s}, \hat{t}) - (1, 1, 1)| \geq \theta > 0,$$
then
\[ f(\tilde{r}, \tilde{s}, \hat{t}) \leq f(1,1,1) - d_\theta. \] (15)

The constant \( d_\theta > 0 \) may be chosen independent of \( u \) and \( \mu \).

Proof. We define an auxiliary function \( g : [0,2]^3 \to \mathbb{R} \) by
\[
g(\tilde{r}, \tilde{s}, \hat{t}) := \left( \frac{\tilde{r}^2}{2} - \frac{\tilde{r}^p}{p} \right) \| \tilde{u}^+ \|^2 + \left( \frac{\tilde{s}^2}{2} - \frac{\tilde{s}^p}{p} \right) \| \tilde{u}^- \|^2 + \left( \frac{\hat{t}^2}{2} - \frac{\hat{t}^p}{p} \right) \| \hat{u}^+ \|^2 + K,
\]

which satisfies \( \nabla g(1,1,1) = 0 \) and
\[
D^2g(1,1,1) = -(p-2) \text{diag} \left\{ \| \tilde{u}^+ \|^2, \| \tilde{u}^- \|^2, \| \hat{u}^+ \|^2 \right\} \leq -(p-2)\kappa I,
\]
where \( \kappa \) was defined in Lemma 2.4. One easily checks that in a small neighborhood of \((1,1,1)\) the second derivative \( D^2g \) is below a negative definite matrix which is independent of \( u \in N_\mu \). We also have that, for any derivative \( D^\alpha \) with \( |\alpha| \leq 2 \),
\[
|D^\alpha f - D^\alpha g| = o(1), \tag{16}
\]

by Lemma 2.3. Notice that the right-hand-side is uniform in \( u \) and \( \mu \). Thus, by (16) with \( |\alpha| = 2 \), \( f \) has a strict local maximum at \((1,1,1)\). We take \( \alpha = 0 \) to conclude this maximum is absolute. Of course, the previous two statements hold provided \( \mu \) is sufficiently large.

Let \( \mu \) be fixed and \( (u_n) \) be a minimizing sequence for \( I_\mu \) restricted to \( N_\mu \). Since \( N_\mu \) is bounded in \( H^1_0(\Omega) \), we may assume
\[ u_n \rightharpoonup u \quad \text{in} \quad H^1_0(\Omega). \]

Lemma 3.2. If \( \mu \) is sufficiently large, the function \( u \) belongs to \( N_\mu \). Therefore (by the lower semi-continuity of the norm) the function \( u \) is a minimizer of \( I_\mu \) restricted to \( N_\mu \).

Proof. We may assume \( \tilde{u}_n^+ \rightharpoonup \tilde{u}^+ \), \( \tilde{u}_n^- \rightharpoonup \tilde{u}^- \), \( \hat{u}_n^+ \rightharpoonup \hat{u}^+ \) in \( H^1_0(\Omega) \), since \( w_n \rightharpoonup w \) in \( H^1_0(\Omega) \) implies a subsequence of \( w_n \) converges pointwise a.e. to \( w \). From (\( N_\mu \)) and Lemma 2.3,
\[
\min \left\{ \int a^+ |u|^{p-2} u \tilde{u}^+, - \int a^+ |u|^{p-2} u \tilde{u}^-, \int a^+ |u|^{p-2} u \hat{u}^+ \right\} \geq \kappa.
\]

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These three integrals are also bounded above by a constant independent of \( \mu \) because \( \mathcal{N}_\mu \) is bounded. It follows from Lemma 2.3 that the integrals
\[
\int a^+|\tilde{u}^+|^p, \quad \int a^+|\tilde{u}^-|^p, \quad \int a^+|\hat{u}^+|^p
\]
are bounded below by a positive constant independent of \( \mu \). The Sobolev inequality now implies that the norms
\[
\|\tilde{u}^+\|, \quad \|\tilde{u}^-\|, \quad \|\hat{u}^+\|
\]
amare bounded below by a positive constant independent of \( \mu \). From the lower-semicontinuity of the norm,
\[
\|\tilde{u}^+\| \leq \lim \inf \|\tilde{u}^+_n\|, \quad \|\tilde{u}^-\| \leq \lim \inf \|\tilde{u}^-_n\|, \quad \|\hat{u}^+\| \leq \lim \inf \|\hat{u}^+_n\|.
\]
(17)
We wish to prove that equalities hold. Otherwise, choose \((\tilde{r}, \tilde{s}, \hat{t})\), defined by
\[
\begin{align*}
\tilde{r} &= \left(\frac{\|\tilde{u}^+\|^2}{\int a^+|u|^{p-2}u\tilde{u}^+}\right)^{\frac{1}{p-2}}, & \tilde{s} &= \left(\frac{\|\tilde{u}^-\|^2}{\int a^+|u|^{p-2}u\tilde{u}^-}\right)^{\frac{1}{p-2}}, \\
\hat{t} &= \left(\frac{\|\hat{u}^+\|^2}{\int a^+|u|^{p-2}u\hat{u}^+}\right)^{\frac{1}{p-2}},
\end{align*}
\]
so that the function
\[
w := \tilde{r}\tilde{u}^+ - \tilde{s}\tilde{u}^- + \hat{t}\hat{u}^+ - \hat{u}^- + \tilde{u} + u
\]
satisfies \( \mathcal{N}_u \). By (17), the strong convergence in \( L^p(\Omega) \), and what we have just seen,
\[
(\tilde{r}, \tilde{s}, \hat{t}) \in [c,1]^3 \setminus \{(1,1,1)\},
\]
for some \( c > 0 \) independent of \( \mu \). The function \( w \) clearly satisfies \( \mathcal{N}_i \) and \( \mathcal{N}_r \). Lemma 2.3 guarantees that \( \mathcal{N}_{iv} \) is satisfied for sufficiently large \( \mu \).

Consider the estimate
\[
I_\mu(\tilde{r}\tilde{u}^+ - \tilde{s}\tilde{u}^- + \hat{t}\hat{u}^+ - \hat{u}^- + \tilde{u} + u) < \lim \inf I_\mu(\tilde{r}\tilde{u}^+_n - \tilde{s}\tilde{u}^-_n + \hat{t}\hat{u}^+_n - \hat{u}^-_n + \tilde{u}_n + u_n)
\]
\[
\leq \lim I_\mu(u_n),
\]
where the last inequality is due to Lemma 3.1. It shows that \( w \) satisfies \( \mathcal{N}_{iii} \). Therefore \( w \in \mathcal{N}_\mu \) and \( I_\mu(w) < \lim I_\mu(u_n) \). This is a contradiction. We have established that equality holds in all three of (17). Therefore \( u \in \mathcal{N}_\mu \) for large \( \mu \).
4 A minimizer in $\mathcal{N}_\mu$ is a critical point

In the previous section we obtained a minimizer $u$ of $I_\mu$ on $\mathcal{N}_\mu$. We will now prove that this minimizer is indeed a critical point of $I_\mu$. This will be done by using a deformation argument on the manifold introduced above. Let $\sigma$ be the restriction to the interval $[1/2, 2]^3$ of the $\varsigma$ corresponding to the minimizer $u$. Recall $\varsigma$ was defined in (14). We define a negative gradient flow in a neighborhood of $u$ in the following way. Let $B_\rho(u) := \{w \in H^1_0(\Omega) : \|w - u\| < \rho\}$, where $\rho$ is chosen small enough so that

$$\sigma(\tilde{r}, \tilde{s}, \hat{t}) \in B_\rho(u) \Rightarrow \frac{1}{2} < \tilde{r}, \tilde{s}, \hat{t} < 2$$

and $w \in B_\rho(u)$ implies that $w$ satisfies (N$_i$), (N$_{iii}$), (N$_w$) and (N$_r$), for sufficiently large $\mu$. Such a $\rho$ exists because the function $u$ satisfies (11) and (a), (b), (c) and (d) of Lemma 2.5. Let $\varphi$ be a Lipschitz function, $\varphi: H^1_0(\Omega) \rightarrow [0, 1]$, such that $\varphi = 1$ on $B_{\rho/2}(u)$ and $\varphi = 0$ on the complement of $B_\rho(u)$. Consider the Cauchy problem

$$\begin{cases}
\frac{d\eta}{d\tau} = -\varphi(\eta) \nabla I_\mu(\eta), \\
\eta(0) = w,
\end{cases}$$

whose solution we denote by $\eta(\tau; w)$. For $\tau \geq 0$, let

$$\sigma(\tau; w) = \eta(\tau; \sigma(\tilde{r}, \tilde{s}, \hat{t})).$$

Lemma 4.1. The set $\sigma(\tau; w)$ intersects $\mathcal{N}_\mu$ in an nonempty set.

Proof. Consider the maps $\tilde{\phi}^+, \tilde{\phi}^-, \tilde{\psi}^+, \tilde{\psi}^-$ from $\{w \in H^1_0(\Omega) : \tilde{w}^\pm \neq 0, \hat{w}^+ \neq 0\}$ to $\mathbb{R}$, defined by

$$\tilde{\phi}^+(w) = \frac{\pm \int a^+ |w|^{p-2} w \tilde{w}^+}{\|\tilde{w}^+\|^2}, \quad \hat{\phi}(w) = \frac{\int a^+ |w|^{p-2} \hat{w}^+}{\|\hat{w}^+\|^2},$$

$$\tilde{\psi}^+(w) = \frac{\int a^+ |\tilde{w}^+|^p}{\|\tilde{w}^+\|^2}, \quad \hat{\psi}(w) = \frac{\int a^+ |\hat{w}^+|^p}{\|\hat{w}^+\|^2}.$$ 

These maps are well defined on $\sigma(\tau; w)$, because if $w \in B_\rho(u)$, then $w$ satisfies (N$_r$). We finally define

$$\Phi_r := \left(\tilde{\phi}^+, \tilde{\phi}^-, \hat{\phi}\right) \circ \sigma$$

and

$$\Psi := \left(\tilde{\psi}^+, \tilde{\psi}^-, \hat{\psi}\right) \circ \sigma.$$
from \( ([1/2, 2]^3) \) to \( \mathbb{R}^3 \). Since \( \int u|p = o(1) \) uniformly in \( u \) and \( \mu \) and the value of \( \kappa \) in Lemma 2.4 is independent of \( \mu \),

\[
\Psi(\tilde{r}, \tilde{s}, \tilde{t}) = \left( \tilde{r}^{p-2} \tilde{\psi}^+(u), \tilde{s}^{p-2} \tilde{\psi}^-(u), \tilde{t}^{p-2} \tilde{\psi}(u) \right) = \\
(1 + o(1))\tilde{r}^{p-2}, (1 + o(1))\tilde{s}^{p-2}, (1 + o(1))\tilde{t}^{p-2})
\]

with the last three \( o(1) \) independent of \( u \) and \( \mu \). As a consequence,

\[
\text{dist} \left( \Psi \left( \partial [1/2, 2]^3 \right), (1, 1, 1) \right) \geq c > 0,
\]

the constant \( c \) being independent of \( u \) and \( \mu \). We deduce from (20) that for large \( \mu \),

\[
\deg \left( \Psi, [1/2, 2]^3, (1, 1, 1) \right) = 1.
\]

Notice that condition (18) and the definition of the flux (19) guarantee

\[
\Phi_{\tau|_{[1/2, 2]^3}} = \Phi_{0|_{[1/2, 2]^3}} = \Psi|_{[1/2, 2]^3} + o(1)
\]

and therefore

\[
\deg \left( \Phi_{\tau}, [1/2, 2]^3, (1, 1, 1) \right) = 1.
\]

for \( \mu \) large enough. This proves that

\[
\sigma_{\tau} \left( [1/2, 2]^3 \right) \cap N_{\mu} \neq \emptyset.
\]

\( \square \)

We are ready to give the Proof of Proposition 2.1. Let \( \mu \) be large and \( u_{\mu} \) be a minimizer of \( I_{\mu} \) restricted to \( N_{\mu} \). The existence of such a \( u_{\mu} \) was proven in Lemma 3.2. Suppose that \( I'_{\mu}(u_{\mu}) \neq 0 \). By Lemma 3.1 with \( u = u_{\mu} \), max \( I_{\mu} \circ \sigma \left( [1/2, 2]^3 \right) = I_{\mu}(u_{\mu}) \), and so for any small \( \tau > 0 \),

\[
\max I_{\mu} \circ \sigma_{\tau} \left( [1/2, 2]^3 \right) < I_{\mu}(u_{\mu}).
\]

This contradicts Lemma 4.1. So \( I'_{\mu}(u_{\mu}) = 0 \), and the minimizer of \( I_{\mu} \) on \( N_{\mu} \) is a weak solution of (4).

Consider now \( u \) as in (5). Properties (6), (7) and (8) follow from Lemma 2.4 and Lemma 2.5(c), (d), as

\[
\min \left\{ \int a^+ |u_{\mu}|^{p-2} u_{\mu} \hat{u}_{\mu}^+, - \int a^+ |u_{\mu}|^{p-2} u_{\mu} \hat{u}_{\mu}^-, \int a^+ |u_{\mu}|^{p-2} u_{\mu} \hat{u}_{\mu}^+ \right\} \geq \kappa.
\]

\( \square \)

Theorem 1.1 can be proved as Proposition 2.1 with obvious adaptations.
References

[1] Ackermann, N.; Weth, T.. Multibump solutions of nonlinear periodic Schrödinger equations in a degenerate setting. Commun. Contemp. Math. 7 (2005), no. 3, 269–298.

[2] Alama, S.; Del Pino, M.. Solutions of elliptic equations with indefinite nonlinearities via Morse theory and linking. Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996), no. 1, 95–115.

[3] Alama, S.; Tarantello, G.. On semilinear elliptic equations with indefinite nonlinearities. Calc. Var. Partial Differential Equations 1 (1993), no. 4, 439–475.

[4] Bartsch, T.; Clapp, M.; Weth, T.. Configuration spaces, transfer, and 2-nodal solutions of a semiclassical nonlinear Schrödinger equation. Math. Ann. to appear.

[5] Bartsch, T.; Weth, T.. A note on additional properties of sign changing solutions to superlinear elliptic equations. Topol. Methods Nonlinear Anal. 22 (2003), no. 1, 1–14.

[6] Berestycki, H.; Capuzzo-Dolcetta, I.; Nirenberg, L.. Variational methods for indefinite superlinear homogeneous elliptic problems. NoDEA Nonlinear Differential Equations Appl. 2 (1995), no. 4, 553–572.

[7] Bonheure, D.; Gomes, J.M.; Habets, P.. Multiple positive solutions of superlinear elliptic problems with sign-changing weight. J. Differential Equations 214 (2005), no. 1, 36–64.

[8] Castro, A.; Cossio, J.; Neuberger, J.M.. A sign-changing solution for a superlinear Dirichlet problem. Rocky Mountain J. Math. 27 (1997), no. 4, 1041–1053.

[9] Cerami, G.; Solimini, S.; Struwe, M.. Some existence results for superlinear elliptic boundary value problems involving critical exponents. J. Funct. Anal. 69 (1986), no. 3, 289–306.

[10] Clapp, M.; Weth, T.. Minimal nodal solutions of the pure critical exponent problem on a symmetric domain. Calc. Var. Partial Differential Equations 21 (2004), no. 1, 1–14.

[11] Costa, D.G.; Ramos, M.; Tehrani, H.. Non-zero solutions for a Schrödinger equation with indefinite linear and nonlinear terms. Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), no. 2, 249–258.
[12] Coti Zelati, V.; Rabinowitz, P.H.. Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials. J. Amer. Math. Soc. 4 (1991), no. 4, 693–727.

[13] Coti Zelati, V.; Rabinowitz, P.H.. Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^n$. Comm. Pure Appl. Math. 45 (1992), no. 10, 1217–1269.

[14] Del Pino, M.; Felmer, P.L.. Multi-peak bound states for nonlinear Schrödinger equations. Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998), no. 2, 127–149.

[15] Gaudenzi, M.; Habets, P.; Zanolin, F.. A seven-positive-solutions theorem for a superlinear problem. Adv. Nonlinear Stud. 4 (2004), no. 2, 149–164.

[16] Li, Y.; Wang, Z.Q.. Gluing approximate solutions of minimum type on the Nehari manifold. Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Viña del Mar-Valparaiso, 2000), 215–223, Electron. J. Differ. Equ. Conf., 6, Southwest Texas State Univ., San Marcos, TX, 2001.

[17] Ramos, M.. Remarks on a priori estimates for superlinear elliptic problems. Topological methods, variational methods and their applications (Taiyuan, 2002), 193–200, World Sci. Publ., River Edge, NJ, 2003.

[18] Ramos, M.; Tavares, H.. Solutions with multiple spike patterns for an elliptic system. Preprint.

[19] Ramos, M.; Terracini, S.; Troestler, C.. Superlinear indefinite elliptic problems and Pohožaev type identities. J. Funct. Anal. 159 (1998), no. 2, 596–628.

[20] Séré, É.. Existence of infinitely many homoclinic orbits in Hamiltonian systems. Math. Z. 209 (1992), no. 1, 27–42.

[21] Tehrani, H.. On indefinite superlinear elliptic equations. Calc. Var. Partial Differential Equations 4 (1996), no. 2, 139–153.