COMPLEX $b$-MANIFOLDS

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Abstract. A complex $b$-structure on a manifold $M$ with boundary is an involutive subbundle $bT^{0,1}M$ of the complexification of $\overline{b}T_M$ with the property that $\mathbb{C}bT_M = bT^{0,1}M + \overline{b}T^{0,1}M$ as a direct sum; the interior of $M$ is a complex manifold. The complex $b$-structure determines an elliptic complex of $b$-operators and induces a rich structure on the boundary of $M$. We study the cohomology of the indicial complex of the $b$-Dolbeault complex.

1. Introduction

A complex $b$-manifold is a smooth manifold with boundary together with a complex $b$-structure. The latter is a smooth involutive subbundle $bT^{0,1}M$ of the complexification $\mathbb{C}bT_M$ of Melrose’s $b$-tangent bundle [5, 6] with the property that $\mathbb{C}bT_M = bT^{0,1}M + \overline{b}T^{0,1}M$ as a direct sum. Manifolds with complex $b$-structures generalize the situation that arises as a result of spherical and certain anisotropic (not complex) blowups of complex manifolds at a discrete set of points or along a complex submanifold, cf. [2], Section 2, [9], as well as (real) blow-up of complex analytic varieties with only point singularities.

The interior of $M$ is a complex manifold. Its $\overline{\partial}$-complex determines a $b$-elliptic complex, the $b\overline{\partial}$-complex, on sections of the exterior powers of the dual of $bT^{0,1}M$, see Section 2. The indicial families $D(\sigma)$ of the $\overline{\partial}$-operators at a connected component $N$ of $\partial M$ give, for each $\sigma$, an elliptic complex, see Section 6. Their cohomology at the various values of $\sigma$ determine the asymptotics at $N$ of tempered representatives of cohomology classes of the $\overline{\partial}$-complex, in particular of tempered holomorphic functions.

Each boundary component $N$ of $M$ inherits from $bT^{0,1}M$ the following objects in the $C^\infty$ category:

1. an involutive vector subbundle $\mathcal{V} \subset \mathbb{C}TN$ such that $\mathcal{V} + \overline{\mathcal{V}} = \mathbb{C}TN$;
2. a real nowhere vanishing vector field $\mathcal{T}$ such that $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$;
3. a class $\mathcal{B}$ of sections of $\mathcal{V}^*$, where the elements of $\mathcal{B}$ have additional properties, described in (4) below. The vector bundle $\mathcal{V}$, being involutive, determines a complex of first order differential operators $\mathcal{D}$ on sections of the exterior powers of $\mathcal{V}^*$, elliptic because of the second property in (1) above. To that list add

4. If $\beta \in \mathcal{B}$ then $\overline{\partial}\beta = 0$ and $\Im(\beta, \mathcal{T}) = -1$, and if $\beta, \beta' \in \mathcal{B}$, then $\beta' - \beta = \mathcal{D}u$ with $u$ real-valued.

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These properties, together with the existence of a Hermitian metric on $\mathcal{V}$ invariant under $\mathcal{T}$ make $\mathcal{N}$ behave in many ways as the circle bundle of a holomorphic line bundle over a compact complex manifold. These analogies are investigated in [10, 11, 12, 13]. The last of these papers contains a detailed account of circle bundles from the perspective of these boundary structures. The paper [8], a predecessor of the present one, contains some facts studied here in more detail.

The paper is organized as follows. Section 2 deals with the definition of complex $b$-structure and Section 3 with holomorphic vector bundles over complex $b$-manifolds (the latter term just means that the $b$-tangent bundle takes on a primary role over that of the usual tangent bundle). The associated Dolbeault complexes are defined in these sections accordingly.

Section 4 is a careful account of the structure inherited by the boundary. In Section 5 we show that complex $b$-structures have no formal local invariants at boundary points. The issue here is that we do not have a Newlander-Nirenberg theorem that is valid in a neighborhood of a point of the boundary, so no explicit local model for $b$-manifolds.

Section 6 is devoted to general aspects of $b$-elliptic first order complexes $A$. We introduce here the set $\text{spec}^q_{b,N}(A)$, the boundary spectrum of the complex in degree $q$ at the component $N$ of $M$, and prove basic properties of the boundary spectrum (assuming that the boundary component $N$ is compact), including some aspects concerning Mellin transforms of $A$-closed forms. Some of these ideas are illustrated using the $b$-de Rham complex.

Section 7 is a systematic study of the $\overline{\partial}_b$-complex of CR structures on $N$ associated with elements of the class $\beta$. Each $\beta \in \beta$ defines a CR structure, $\mathcal{K}_\beta = \ker \beta$. Assuming that $\mathcal{V}$ admits a $\mathcal{T}$-invariant Hermitian metric, we show that there is $\beta \in \beta$ such that the CR structure $\mathcal{K}_\beta$ is $\mathcal{T}$-invariant.

In Section 8 we assume that $\mathcal{V}$ is $\mathcal{T}$-invariant and show that for $\mathcal{T}$-invariant CR structures, a theorem proved in [13] gives that the cohomology spaces of the associated $\overline{\partial}_b$-complex, viewed as the kernel of the Kohn Laplacian at the various degrees, split into eigenspaces of $-i\mathcal{L}_T$. The eigenvalues of the latter operator are related to the indicial spectrum of the $\overline{\partial}_b$-complex.

In Section 9 we prove a precise theorem on the indicial cohomology and spectrum for the $\overline{\partial}_b$-complex under the assumption that $\mathcal{V}$ admits a $\mathcal{T}$-invariant Hermitian metric.

Finally, we have included a very short appendix listing a number of basic definitions in connection with $b$-operators.

2. Complex $b$-structures

Let $M$ be a smooth manifold with smooth boundary. An almost CR $b$-structure on $M$ is a subbundle $\mathcal{W}$ of the complexification, $\mathcal{C}^0T\mathcal{M} \rightarrow \mathcal{M}$ of the $b$-tangent bundle of $M$ (Melrose [5, 6]) such that

\begin{equation}
\mathcal{W} \cap \overline{\mathcal{W}} = 0
\end{equation}

with $\mathcal{W} = \overline{\mathcal{W}}$. If in addition

\begin{equation}
\mathcal{W} + \overline{\mathcal{W}} = \mathcal{C}^0T\mathcal{M}
\end{equation}

then we say that $\mathcal{W}$ is an almost complex $b$-structure and write $b^0,1\mathcal{M}$ instead of $\overline{\mathcal{W}}$ and $b^1,0\mathcal{M}$ for its conjugate. As is customary, the adverb “almost” is dropped
if \( \mathcal{W} \) is involutive. Note that since \( C^\infty(\mathcal{M}; T\mathcal{M}) \) is a Lie algebra, it makes sense to speak of involutive subbundles of \( T\mathcal{M} \) (or its complexification).

**Definition 2.3.** A complex \( b \)-manifold is a manifold together with a complex \( b \)-structure.

By the Newlander-Nirenberg Theorem [14], the interior of complex \( b \)-manifold is a complex manifold. However, its boundary is not a CR manifold; rather, as we shall see, it naturally carries a family of CR structures parametrized by the defining functions of \( \partial \mathcal{M} \) in \( \mathcal{M} \) which are positive in \( \mathcal{M} \).

That \( C^\infty(\mathcal{M}; bT\mathcal{M}) \) is a Lie algebra is an immediate consequence of the definition of the \( b \)-tangent bundle, which indeed can be characterized as being a vector bundle \( bT\mathcal{M} \to \mathcal{M} \) together with a vector bundle homomorphism
\[
ev : bT\mathcal{M} \to T\mathcal{M}
\]
covering the identity such that the induced map
\[
ev_* : C^\infty(\mathcal{M}; bT\mathcal{M}) \to C^\infty(\mathcal{M}; T\mathcal{M})
\]
is a \( C^\infty(\mathcal{M}; \mathbb{R}) \)-module isomorphism onto the submodule \( C^\infty_{\text{tan}}(\mathcal{M}, T\mathcal{M}) \) of smooth vector fields on \( \mathcal{M} \) which are tangential to the boundary of \( \mathcal{M} \). Since \( C^\infty_{\text{tan}}(\mathcal{M}, T\mathcal{M}) \) is closed under Lie brackets, there is an induced Lie bracket on \( C^\infty(\mathcal{M}; bT\mathcal{M}) \) The homomorphism \( \ev \) is an isomorphism over the interior of \( \mathcal{M} \), and its restriction to the boundary,
\[
(2.4) \quad \ev_{\partial \mathcal{M}} : b\partial \mathcal{M} \to \partial \mathcal{M}
\]
is surjective. Its kernel, a fortiori a rank-one bundle, is spanned by a canonical section denoted \( r_{\partial} \). Here and elsewhere, \( r \) refers to any smooth defining function for \( \partial \mathcal{M} \) in \( \mathcal{M} \), by convention positive in the interior of \( \mathcal{M} \).

Associated with a complex \( b \)-structure on \( \mathcal{M} \) there is a Dolbeault complex. Let \( b\Lambda^q \mathcal{M} \) denote the \( q \)-th exterior power of the dual of \( bT^{0,1} \mathcal{M} \). Then the operator
\[
\cdots \to C^\infty(\mathcal{M}; b\Lambda^q \mathcal{M}) \xrightarrow{\overline{\partial}} C^\infty(\mathcal{M}; b\Lambda^{q+1} \mathcal{M}) \to \cdots
\]
is defined by
\[
(2.5) \quad (q + 1) \overline{\partial}\phi(V_0, \ldots, V_q) = \sum_{j=0}^q V_j \phi(V_0, \ldots, \hat{V}_j, \ldots, V_q)
\]
\[
+ \sum_{j<k} (-1)^{j+k} \phi([V_j, V_k], V_0, \ldots, \hat{V}_j, \ldots, \hat{V}_k, \ldots, V_q)
\]
as with the standard de Rham differential (see Helgason [3, p. 21]) whenever \( \phi \) is a smooth section of \( b\Lambda^q \mathcal{M} \) and \( V_0, \ldots, V_q \in C^\infty(\mathcal{M}; bT^{0,1} \mathcal{M}) \). In this formula \( V_j \) acts on functions via the vector field \( \ev_* V_j \). The involutivity of \( bT^{0,1} \mathcal{M} \) is used in the the terms involving brackets, of course. The same proof that \( d \circ d = 0 \) works here to give that \( \overline{\partial}^2 = 0 \). The formula
\[
(2.6) \quad \overline{\partial}(f\phi) = f \overline{\partial}\phi + \overline{\partial} f \wedge \phi \quad \text{for } \phi \in C^\infty(\mathcal{M}; b\Lambda^q \mathcal{M}) \text{ and } f \in C^\infty(\mathcal{M}),
\]
implies that \( \overline{\partial} \) is a first order operator.

Since we do not have at our disposal holomorphic frames (near the boundary) for the bundles of forms of type \((p, q)\) for \( p > 0 \), we define \( \overline{\partial} \) on forms of type \((p, q)\) with \( p > 0 \) with the aid of the \( b \)-de Rham complex, exactly as in Folland and Kohn [2] for
standard complex structures and de Rham complex. The \( b \)-de Rham complex, we recall from Melrose [7], is the complex associated with the dual, \( \mathcal{C}^\blacksquare T^* \mathcal{M} \), of \( \mathcal{C}^\blacksquare T \mathcal{M} \),
\[
\cdots \to C^\infty(\mathcal{M}; \mathcal{A}^1 \mathcal{M}) \xrightarrow{\mathcal{b} d} C^\infty(\mathcal{M}; \mathcal{A}^{1+1} \mathcal{M}) \to \cdots
\]
where \( \mathcal{A}^p \mathcal{M} \) denotes the \( r \)-th exterior power of \( \mathcal{C}^\blacksquare T^* \mathcal{M} \). The operators \( \mathcal{b} d \) are defined by the same formula as \( (2.5) \), now however with the \( V_j \in C^\infty(\mathcal{M}; \mathcal{C}^\blacksquare T \mathcal{M}) \). On functions \( f \) we have
\[
\mathcal{b} d f = \text{ev}^* df.
\]
More generally,
\[
\text{ev}^* \circ d = \mathcal{b} d \circ \text{ev}^*
\]
in any degree. Also,
\[
(2.7) \quad \mathcal{b} d(f \phi) = f \mathcal{b} d \phi + \mathcal{b} d f \wedge \phi \quad \text{for } \phi \in C^\infty(\mathcal{M}; \mathcal{A}^r \mathcal{M}) \text{ and } f \in C^\infty(\mathcal{M}).
\]
It is convenient to note here that for \( f \in C^\infty(\mathcal{M}) \),
\[
(2.8) \quad \mathcal{b} d f \text{ vanishes on } \partial \mathcal{M} \text{ if } f \text{ does.}
\]
Now, with the obvious definition,
\[
(2.9) \quad \mathcal{A}^r \mathcal{M} = \bigoplus_{p+q=r} \mathcal{A}^{p,q} \mathcal{M}.
\]
Using the special cases
\[
\mathcal{b} d : C^\infty(\mathcal{M}; \mathcal{A}^{0,1} \mathcal{M}) \to C^\infty(\mathcal{M}; \mathcal{A}^{1,1} \mathcal{M}) + C^\infty(\mathcal{M}; \mathcal{A}^{0,2} \mathcal{M}),
\]
\[
\mathcal{b} d : C^\infty(\mathcal{M}; \mathcal{A}^{1,0} \mathcal{M}) \to C^\infty(\mathcal{M}; \mathcal{A}^{2,0} \mathcal{M}) + C^\infty(\mathcal{M}; \mathcal{A}^{1,1} \mathcal{M}),
\]
consequences of the involutivity of \( \mathcal{b} T^{0,1} \mathcal{M} \) and its conjugate, one gets
\[
\mathcal{b} d \phi \in C^\infty(\mathcal{M}; \mathcal{A}^{p+1,q} \mathcal{M}) \oplus C^\infty(\mathcal{M}; \mathcal{A}^{p,q+1} \mathcal{M}) \quad \text{if } \phi \in C^\infty(\mathcal{M}; \mathcal{A}^{p,q} \mathcal{M})
\]
for general \( (p,q) \). Let \( \pi_{p,q} : \mathcal{A}^k \mathcal{M} \to \mathcal{A}^{p,q} \mathcal{M} \) be the projection according to the decomposition \( (2.9) \), and define
\[
\mathcal{b} \partial = \pi_{p+1,q} \mathcal{b} d, \quad \overline{\mathcal{b} \partial} = \pi_{q,p+1} \mathcal{b} d,
\]
so \( \mathcal{b} d = \mathcal{b} \partial + \overline{\mathcal{b} \partial} \). The operators \( \overline{\mathcal{b} \partial} \) are identical to the \( \overline{\mathcal{b} d} \)-operators over the interior of \( \mathcal{M} \) and with the previously defined \( \overline{\mathcal{b} \partial} \) operators on \( (0,q) \)-forms, and give a complex
\[
(2.10) \quad \cdots \to C^\infty(\mathcal{M}; \mathcal{A}^{p,q} \mathcal{M}) \xrightarrow{\overline{\mathcal{b} \partial}} C^\infty(\mathcal{M}; \mathcal{A}^{p,q+1} \mathcal{M}) \to \cdots
\]
for each \( p \). On functions \( f : \mathcal{M} \to \mathbb{C} \),
\[
(2.11) \quad \overline{\mathcal{b} \partial} f = \pi_{0,1} \mathcal{b} d f.
\]
The formula
\[
(2.7) \quad \overline{\mathcal{b} \partial} f \phi = \overline{\mathcal{b} \partial} f \wedge \phi + f \overline{\mathcal{b} \partial} \phi, \quad f \in C^\infty(\mathcal{M}), \phi \in C^\infty(\mathcal{M}; \mathcal{A}^{p,q} \mathcal{M}),
\]
a consequence of \( (2.7) \), implies that \( \overline{\mathcal{b} \partial} \) is a first order operator. As a consequence of \( (2.8) \),
\[
(2.8) \quad \overline{\mathcal{b} \partial} f \text{ vanishes on } \partial \mathcal{M} \text{ if } f \text{ does.}
\]
The operators of the \( b \)-de Rham complex are first order operators because of \( (2.7) \), and \( (2.8) \) implies that these are \( b \)-operators, see \( (A.1) \). Likewise, \( (2.7) \) and
(2.8′) imply that in any bidegree, the operator \( \phi \mapsto r^{-1} \overline{\partial} r \phi \) has coefficients smooth up to the boundary, so

\[
\overline{\partial} \in \text{Diff}^{1}_{b}(M; b\Lambda^{p,q}M, b\Lambda^{p,q+1}M),
\]

see (A.1). We also get from these formulas that the \( b \)-symbol of \( \overline{\partial} \)

\[
b\sigma(\overline{\partial})(\phi) = i \pi_{0,1}(\xi) \land \phi, \quad x \in M, \, \xi \in bT_{x}^{*}M, \, \phi \in b\Lambda^{p,q}M,
\]

see (A.2). Since \( \pi_{0,1} \) is injective on the real \( b \)-cotangent bundle (this follows from (2.2)), the complex (2.10) is \( b \)-elliptic.

3. Holomorphic vector bundles

The notion of holomorphic vector bundle in the \( b \)-category is a translation of the standard one using connections. Let \( \rho : F \to M \) be a complex vector bundle. Recall from [6] that a \( b \)-connection on \( F \) is a linear operator \( b\nabla : C^{\infty}(M; F) \to C^{\infty}(M; b\Lambda^{1}M \otimes F) \) such that

\[
b\nabla f \phi = f b\nabla \phi + bdf \otimes \phi
\]

for each \( \phi \in C^{\infty}(M; F) \) and \( f \in C^{\infty}(M) \). This property automatically makes \( b\nabla \) a \( b \)-operator.

A standard connection \( \nabla : C^{\infty}(M; F) \to C^{\infty}(M; \Lambda^{1}M \otimes F) \) determines a \( b \)-connection by composition with

\[
ev^{*} \otimes I : \Lambda^{1}M \otimes F \to b\Lambda^{1}M \otimes F,
\]

but \( b \)-connections are more general than standard connections. Indeed, the difference between the latter and the former can be any smooth section of the bundle \( \text{Hom}(F, b\Lambda^{1}M \otimes F) \). A \( b \)-connection \( b\nabla \) on \( F \) arises from a standard connection if and only if \( b\nabla_{\partial} = 0 \) along \( \partial M \).

As in the standard situation, the \( b \)-connection \( b\nabla \) determines operators

\[
b\nabla : C^{\infty}(M; b\Lambda^{k}M \otimes F) \to C^{\infty}(M; b\Lambda^{k+1}M \otimes F)
\]

by way of the usual formula translated to the \( b \) setting:

\[
b\nabla(\alpha \otimes \phi) = (-1)^{k}\alpha \land b\nabla \phi + b\nabla \alpha \land \phi, \quad \phi \in C^{\infty}(M; F), \, \alpha \in b\Lambda^{k}M.
\]

Since

\[
b\nabla \tau \alpha \otimes \phi = \tau b\nabla(\alpha \otimes \phi) + b\nabla \tau \alpha \otimes \phi
\]

is smooth and vanishes on \( \partial M \), also

\[
b\nabla \in \text{Diff}^{1}_{b}(M; b\Lambda^{k}M \otimes F, b\Lambda^{k+1}M \otimes F).
\]

The principal \( b \)-symbol of \( b\nabla \), easily computed using (3.3) and

\[
b\sigma(b\nabla)(bdf)(\phi) = \lim_{\tau \to \infty} e^{-i\tau f} b\nabla e^{i\tau f} \phi
\]

for \( f \in C^{\infty}(M; \mathbb{R}) \) and \( \phi \in C^{\infty}(M; b\Lambda^{k}M \otimes F) \), is

\[
b\sigma(b\nabla)(\xi)(\phi) = i\xi \land \phi, \quad \xi \in bT_{x}^{*}M, \, \phi \in b\Lambda^{k}_{x}M \otimes F_{x}, \, x \in M.
\]
As expected, the connection is called holomorphic if the component in $b\Lambda^{0,2}\mathcal{M} \otimes F$ of the curvature operator
\[ \Omega = b\nabla^2 : C^\infty(\mathcal{M}; F) \to C^\infty(\mathcal{M}; b\Lambda^{2}\mathcal{M} \otimes F), \]
vanishes. Such a connection gives $F$ the structure of a complex $b$-manifold. Its complex $b$-structure can be described locally as in the standard situation, as follows.

Fix a frame $\eta_\mu$ for $F$ and let the $\omega_\mu^\nu$ be the local sections of $b\Lambda^{0,1}\mathcal{M}$ such that
\[ \overline{\partial} \eta_\mu = \sum_\nu \omega_\mu^\nu \otimes \eta_\nu. \]

Denote by $\zeta^\mu$ the fiber coordinates determined by the frame $\eta_\mu$. Let $V_1, \ldots, V_{n+1}$ be a frame of $\mathcal{T}^{0,1}\mathcal{M}$ over $U$, denote by $\tilde{V}_j$ the sections of $\mathbb{C}\mathcal{T}F$ over $\rho^{-1}(U)$ which project on the $V_j$ and satisfy $\tilde{V}_j \zeta^\mu = \tilde{V}_j \zeta^\mu, \eta_\nu = 0$ for all $\mu$, and by $\partial_{\zeta^\mu}$ the vertical vector fields such that $\partial_{\zeta^\mu} \zeta^\nu = \delta^\nu_\mu$ and $\partial_{\zeta^\mu} \eta_\nu = 0$. Then the sections
\[ \tilde{V}_j - \sum_{\mu, \nu} \zeta^\mu (\omega_\mu^\nu, V_j) \partial_{\zeta^\nu}, \ j = 1, \ldots, n + 1, \ \partial_{\zeta^\nu}, \nu = 1, \ldots, k \]
of $\mathbb{C}\mathcal{T}F$ over $\rho^{-1}(U)$ form a frame of $\mathcal{T}^{0,1}F$. As in the standard situation, the involutivity of this subbundle of $\mathbb{C}\mathcal{T}F$ is equivalent to the condition on the vanishing of the $(0,2)$ component of the curvature of $b\nabla$. A vector bundle $F \to \mathcal{M}$ together with the complex $b$-structure determined by a choice of holomorphic $b$-connection (if one exists at all) is a holomorphic vector bundle.

The $\overline{\partial}$ operator of a holomorphic vector bundle is
\[ \overline{\partial} = (\pi_{0, q+1} \otimes I) \circ b\nabla : C^\infty(\mathcal{M}; b\Lambda^{0,q}\mathcal{M} \otimes F) \to C^\infty(\mathcal{M}; b\Lambda^{0,q+1}\mathcal{M} \otimes F). \]
As is the case for standard complex structures, the condition on the curvature of $b\nabla$ implies that these operators form a complex, $b$-elliptic since
\[ b\sigma(\overline{\partial})(\xi)(\phi) = i\pi_{0, 1}(\xi) \wedge \phi, \ \xi \in \mathcal{T}^*_x \mathcal{M}, \ \phi \in b\Lambda^k \mathcal{M} \otimes F_x, \ x \in \mathcal{M} \]
and $\pi_{0, 1}(\xi) = 0$ for $\xi \in \mathcal{T}^* \mathcal{M}$ if and only if $\xi = 0$.

Also as usual, a $b$-connection $b\nabla$ on a Hermitian vector bundle $F \to \mathcal{M}$ with Hermitian form $h$ is Hermitian if
\[ b\overline{d}h(\phi, \psi) = h(b\nabla \phi, \psi) + h(\phi, b\nabla \psi) \]
for every pair of smooth sections $\phi, \psi$ of $F$. In view of the definition of $b\overline{d}$ this means that for every $v \in \mathbb{C}\mathcal{T}\mathcal{M}$ and sections as above,
\[ e(v) h(\phi, \psi) = h(b\nabla_v \phi, \psi) + h(\phi, b\nabla \psi) \]
on a complex $b$-manifold $\mathcal{M}$, if an arbitrary connection $b\nabla'$ and the Hermitian form $h$ are given for a vector bundle $F$, holomorphic or not, then there is a unique Hermitian $b$-connection $b\nabla$ such that $\pi_{0, 1} b\nabla = \pi_{0, 1} b\nabla'$. Namely, let $\eta_\mu$ be a local orthonormal frame of $F$, let
\[ (\pi_{0, 1} \otimes I) \circ b\nabla' \eta_\mu = \sum_\nu \omega_\mu^\nu \otimes \eta_\nu, \]
and let $b\nabla$ be the connection defined in the domain of the frame by
\[ b\nabla \eta_\mu = (\omega_\mu^\nu - \overline{\omega}_\nu^\mu) \otimes \eta_\nu. \]

(3.5)
If the matrix of functions $Q = [q^p_\lambda]$ is unitary and $\tilde{\eta}_\lambda = \sum_\mu q^p_\lambda \eta_\mu$, then

$$(\pi_{0,1} \otimes I)\circ \nabla \tilde{\eta}_\lambda = \sum_\nu \omega^\nu_\lambda \otimes \tilde{\eta}_\nu$$

with

$$\tilde{\omega}^\nu_\lambda = \sum_\mu \overline{\eta}_\sigma \overline{q}^\nu_\mu + \sum_{\mu,\nu} \overline{\eta}_\sigma q^\nu_\mu \omega_\nu^\mu,$$

using (3.1), that $Q^{-1} = [\overline{q}^p_\lambda]$, and that $\pi_{0,1} d\overline{f} = \overline{\partial} f$. Thus

$$\tilde{\omega}^\nu_\lambda - \overline{\omega}^\nu_\lambda = \sum_\mu \overline{\eta}_\sigma (\overline{\partial} q^\nu_\mu - q^\nu_\mu \overline{\partial} \eta_\sigma) + \sum_{\mu,\nu} (\overline{\eta}_\sigma q^\nu_\mu \omega_\nu^\mu - q^\nu_\mu \overline{\eta}_\sigma \omega_\nu^\mu)$$

$$= \sum_\mu (\overline{\partial} q^\nu_\mu + b \partial q^\nu_\mu) \overline{\eta}_\sigma + \sum_{\mu,\nu} q^\nu_\mu (\partial \omega_\nu^\mu - \overline{\omega}^\nu_\mu) \overline{\eta}_\sigma$$

$$= \sum_\mu b \partial q^\nu_\mu + \overline{\eta}_\sigma + \sum_{\mu,\nu} q^\nu_\mu (\partial \omega_\nu^\mu - \overline{\omega}^\nu_\mu) \overline{\eta}_\sigma$$

using that $\overline{\partial} f = b \partial f$ and that $\sum_\mu q^\nu_\mu \overline{\partial} \eta_\sigma = - \sum_\mu b \partial q^\nu_\mu \eta_\sigma$ because $\sum_\mu q^\nu_\mu \overline{\eta}_\sigma$ is constant, and that $\overline{\partial} q^\nu_\mu + b \partial q^\nu_\mu = b \partial q^\nu_\mu$. Thus there is a globally defined Hermitian connection locally given by (3.3). We leave to the reader to verify that this connection is Hermitian. Clearly $\nabla'$ is the unique Hermitian connection such that $\pi_{0,1} \nabla = \pi_{0,1} \nabla'$. When $\nabla'$ is a holomorphic connection, $\nabla'$ is the unique Hermitian holomorphic connection.

**Lemma 3.6.** The vector bundles $^b\Lambda^{p,0} M$ are holomorphic.

We prove this by exhibiting a holomorphic $b$-connection. Fix an auxiliary Hermitian metric on $^b\Lambda^{p,0} M$ and pick an orthonormal frame $(\eta_\mu)$ of $^b\Lambda^{p,0} M$ over some open set $U$. Let $\omega_\mu^\nu$ be the unique sections of $^b\Lambda^{0,1} M$ such that

$$\overline{\partial} \eta_\mu = \sum_\nu \omega_\mu^\nu \wedge \eta_\nu,$$

and let $\nabla$ be the $b$-connection defined on $U$ by the formula (3.3). As in the previous paragraph, this gives a globally defined $b$-connection. That it is holomorphic follows from

$$\overline{\partial} \omega_\mu^\nu + \sum_\lambda \omega_\mu^\nu \wedge \omega_\lambda^\lambda = 0,$$

a consequence of $\overline{\partial}^2 = 0$. Evidently, with the identifications $^b\Lambda^{0,0} M \otimes \Lambda^{p,0} M = ^b\Lambda^{p,0} M$, $\pi_{p,q+1} \nabla$ is the $\overline{\partial}$ operator in (2.12).

4. The boundary a complex $b$-manifold

Suppose that $\mathcal{M}$ is a complex $b$-manifold and $\mathcal{N}$ is a component of its boundary. We shall assume $\mathcal{N}$ compact, although for the most part this is not necessary.

The homomorphism

$$\text{ev} : \mathcal{C}^\bullet T \mathcal{M} \to \mathcal{C}T \mathcal{M}$$

is an isomorphism over the interior of $\mathcal{M}$, and its restriction to $\mathcal{N}$ maps onto $\mathcal{C}T \mathcal{N}$ with kernel spanned by $\text{td}_r$. Write

$$\text{ev}_\mathcal{N} : \mathcal{C}^\bullet T_\mathcal{N} \mathcal{M} \to \mathcal{C}T \mathcal{N}$$
for this restriction and
\[ (4.1) \quad \Phi : bT_{\mathcal{N}}^{0,1} \mathcal{M} \to \overline{\mathcal{V}} \]
for of the restriction of ev_\mathcal{N} to bT_{\mathcal{N}}^{0,1} \mathcal{M}. From (2.1) and the fact that the kernel of ev_\mathcal{N} is spanned by the real section r\partial_r one obtains that \( \Phi \) is injective, so its image,
\[ \overline{\mathcal{V}} = \Phi(bT_{\mathcal{N}}^{0,1} \mathcal{M}) \]
is a subbundle of CT\mathcal{N}.

Since bT_{\mathcal{N}}^{0,1} \mathcal{M} is involutive, so is \( \overline{\mathcal{V}} \), see [7, Proposition 3.12]. From (2.2) and the fact that ev_\mathcal{N} maps onto CT\mathcal{N}, one obtains that
\[ (4.2) \quad \mathcal{V} + \overline{\mathcal{V}} = CT\mathcal{N}, \]
see [7, Lemma 3.13]. Thus

**Lemma 4.3.** \( \overline{\mathcal{V}} \) is an elliptic structure.

This just means what we just said: \( \overline{\mathcal{V}} \) is involutive and (4.2) holds, see Treves [16, 17]; the sum need not be direct. All elliptic structures are locally of the same kind, depending only on the dimension of \( \mathcal{V} \cap \overline{\mathcal{V}} \). This is a result of Nirenberg [15] (see also Hörmander [4]) extending the Newlander-Nirenberg theorem. In the case at hand, \( \overline{\mathcal{V}} \cap \mathcal{V} \) has rank 1 because of the relation
\[ \text{rank}_\mathbb{C}(\mathcal{V} \cap \overline{\mathcal{V}}) = 2 \text{rank}_\mathbb{C} \overline{\mathcal{V}} - \dim \mathcal{N} \]
which holds whenever (4.2) holds.

Every \( p_0 \in \mathcal{N} \) has a neighborhood in which there coordinates \( x^1, \ldots, x^{2n}, t \) such that with \( z^j = x^j + mx^{j+n} \), the vector fields
\[ (4.4) \quad \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial t} \]
span \( \overline{\mathcal{V}} \) near \( p_0 \). The function \( (z^1, \ldots, z^n, t) \) is called a hypoanalytic chart (Baouendi, Chang, and Treves [1], Treves [17]).

The intersection \( \overline{\mathcal{V}} \cap \mathcal{V} \) is, in the case we are discussing, spanned by a canonical globally defined real vector field. Namely, let \( r\partial_t \) be the canonical section of \( bT\mathcal{M} \) along \( \mathcal{N} \). There is a unique section \( Jr\partial_t \) of \( bT\mathcal{M} \) along \( \mathcal{N} \) such that \( r\partial_t + iJr\partial_t \) is a section of \( bT_{\mathcal{N}}^{0,1} \mathcal{M} \) along \( \mathcal{N} \). Then
\[ \mathcal{T} = \text{ev}_{\mathcal{N}}(Jr\partial_t) \]
is a nonvanishing real vector field in \( \mathcal{V} \cap \overline{\mathcal{V}} \), (see [8, Lemma 2.1]). Using the isomorphism (4.1) we have
\[ \mathcal{T} = \Phi(J(r\partial_t) - it\partial_t). \]

Because \( \overline{\mathcal{V}} \) is involutive, there is yet another complex, this time associated with the exterior powers of the dual of \( \overline{\mathcal{V}} \):
\[ (4.5) \quad \cdots \to C^\infty(\mathcal{N}; \wedge^n \overline{\mathcal{V}}^*) \xrightarrow{i^*} C^\infty(\mathcal{N}; \wedge^{n+1} \overline{\mathcal{V}}^*) \to \cdots, \]
where \( i^* \) is defined by the formula (2.4) where now the \( V_j \) are sections of \( \overline{\mathcal{V}} \). The complex (4.5) is elliptic because of (4.2). For a function \( f \) we have \( i^* f = i^* df \), where \( i^*: CT^* \mathcal{N} \to \overline{\mathcal{V}}^* \) is the dual of the inclusion homomorphism \( i: \overline{\mathcal{V}} \to CT\mathcal{N} \).

For later use we show:

**Lemma 4.6.** Suppose that \( \mathcal{N} \) is compact and connected. If \( \zeta: \mathcal{N} \to \mathbb{C} \) solves \( \overline{i^* \zeta} = 0 \), then \( \zeta \) is constant.
Proof. Let $p_0$ be an extremal point of $|\zeta|$. Fix a hypoanalytic chart $(z, t)$ for $\overline{\nu}$ centered at $p_0$. Since $\overline{\nu}\zeta = 0$, $\zeta(z, t)$ is independent of $t$ and $\partial\nu\zeta = 0$. So there is a holomorphic function $Z$ defined in a neighborhood of 0 in $\mathbb{C}^n$ such that $\zeta = Z \circ z$. Then $|Z|$ has a maximum at 0, so $Z$ is constant near 0. Therefore $\zeta$ is constant, say $\zeta(p) = c$, near $p_0$. Let $C = \{ p : \zeta(p) = c \}$, a closed set. Let $p_1 \in C$. Since $p_1$ is also an extremal point of $\zeta$, the above argument gives that $\zeta$ is constant near $p_1$, therefore equal to $c$. Thus $C$ is open, and consequently $\zeta$ is constant on $\mathcal{N}$.

Since the operators $\overline{\nu} : C^\infty(\mathcal{M}, b\Lambda^\cdot \mathcal{M}) \to C^\infty(\mathcal{M}, b\Lambda^\cdot + 1 \mathcal{M})$ are totally characteristic, they induce operators

$$
\overline{\nu}_b : C^\infty(\mathcal{N}, b\Lambda^\cdot \mathcal{M}) \to C^\infty(\mathcal{M}, b\Lambda^\cdot + 1 \mathcal{M}),
$$

see [A.3]; these boundary operators define a complex because of (A.4). By way of the dual

$$(4.7) \Phi^* : \overline{\nu}_b \to b\Lambda^\cdot + 1 \mathcal{M}$$

of the isomorphism (4.1) the operators $\overline{\nu}_b$ become identical to the operators of the $\overline{\nu}$-complex (4.5): The diagram

$$
\cdots \longrightarrow C^\infty(\mathcal{N}; \Lambda^\cdot \overline{\nu}_b^\ast) \overset{\overline{\nu}}{\longrightarrow} C^\infty(\mathcal{N}; \Lambda^\cdot + 1 \overline{\nu}_b^\ast) \longrightarrow \cdots \bigg|_{\Phi^*} \bigg|_{\Phi^*}
$$

$$
\cdots \longrightarrow C^\infty(\mathcal{N}, b\Lambda^\cdot \mathcal{M}) \overset{\overline{\nu}_b}{\longrightarrow} C^\infty(\mathcal{M}, b\Lambda^\cdot + 1 \mathcal{M}) \longrightarrow \cdots
$$

is commutative and the vertical arrows are isomorphisms. This can be proved by writing the $\overline{\nu}$ operators using Cartan’s formula (2.5) for $\overline{\nu}$ and $\overline{\nu}$ and comparing the resulting expressions.

Let $r : \mathcal{M} \to \mathbb{R}$ be a smooth defining function for $\partial \mathcal{M}$, $r > 0$ in the interior of $\mathcal{M}$. Then $\overline{\nu}r$ is smooth and vanishes on $\partial \mathcal{M}$, so $\overline{\nu}r$ is also a smooth $\overline{\nu}$-closed section of $b\Lambda^\cdot + 1 \mathcal{M}$. Thus we get a $\overline{\nu}$-closed element

$$(4.8) \beta_r = [\Phi^*]^{-1} \frac{\overline{\nu} r}{r} \in C^\infty(\partial \mathcal{M}; \overline{\nu}^\ast).$$

By definition,

$$\langle \beta_r, T \rangle = \left( \overline{\nu} \frac{r}{r}, J(r \partial_t) - i t \partial_t \right).$$

Extend the section $r \partial_t$ to a section of $bT \mathcal{M}$ over a neighborhood $U$ of $\mathcal{N}$ in $\mathcal{M}$ with the property that $r \partial_t \tau = r$. In $U$ we have

$$\langle \overline{\nu} r, J(r \partial_t) - i t \partial_t \rangle = (J(r \partial_t) - i t \partial_t) r = J(r \partial_t) r - i t r.$$

The function $J(r \partial_t) r$ is smooth, real-valued, and vanishes along the boundary. So $r^{-1} J(r \partial_t) r$ is smooth, real-valued. Thus

$$\langle \beta_r, T \rangle = a_r - i t$$

on $\mathcal{N}$ for some smooth function $a_r : \mathcal{N} \to \mathbb{R}$, see [8] Lemma 2.5].

If $r'$ is another defining function for $\partial \mathcal{M}$, then $r' = r e^a$ for some smooth function $u : \mathcal{M} \to \mathbb{R}$. Then

$$\overline{\nu} r' = e^u \overline{\nu} r + e^u r \overline{\nu} u$$

and it follows that

$$\beta r' = \beta_r + \overline{\nu} u.$$
In particular, 
\[ a_{\epsilon'} = a_\epsilon + T u. \]
Let \( a_\epsilon \) denote the one-parameter group of diffeomorphisms generated by \( T \).

**Proposition 4.9.** The functions \( a_{\sup}^\epsilon, a_{\inf}^\epsilon : \mathcal{N} \to \mathbb{R} \) defined by
\[
a_{\sup}^\epsilon(p) = \limsup_{t \to \infty} \frac{1}{2t} \int_{-t}^t a_\epsilon(a_s(p)) \, ds, \quad a_{\inf}^\epsilon(p) = \liminf_{t \to \infty} \frac{1}{2t} \int_{-t}^t a_\epsilon(a_s(p)) \, ds
\]
are invariants of the complex b-structure, that is, they are independent of the defining function \( \tau \). The equality \( a_{\sup}^\epsilon = a_{\inf}^\epsilon \) holds for some \( \tau \) if and only if it holds for all \( \tau \).

Indeed,
\[
\lim_{t \to \infty} \left( \frac{1}{2t} \int_{-t}^t a_\epsilon(a_s(p)) \, ds - \frac{1}{2t} \int_{-t}^t a_\epsilon(a_s(p)) \, ds \right) = \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^t \frac{d}{ds} u(a_s(p)) \, ds = 0
\]
because \( u \) is bounded (since \( \mathcal{N} \) is compact).

The functions \( a_{\sup}^\epsilon, a_{\inf}^\epsilon \) are constant on orbits of \( T \), but they may not be smooth.

**Example 4.10.** Let \( \mathcal{N} \) be the unit sphere in \( \mathbb{C}^{n+1} \) centered at the origin. Write \( (z^1, \ldots, z^{n+1}) \) for the standard coordinates in \( \mathbb{C}^{n+1} \). Fix \( \tau_1, \ldots, \tau_{n+1} \in \mathbb{R}\{0\} \), all of the same sign, and let
\[
T = i \sum_{j=1}^{n+1} \tau_j (z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j}).
\]
This vector field is real and tangent to \( \mathcal{N} \). Let \( \mathcal{K} \) be the standard CR structure of \( \mathcal{N} \) as a submanifold of \( \mathbb{C}^{n+1} \) (the part of \( T^{0,1} \mathbb{C}^{n+1} \) tangential to \( \mathcal{N} \)). The condition that the \( \tau_j \) are different from 0 and have the same sign ensures that \( T \) is never in \( \mathcal{K} \oplus \overline{\mathcal{K}} \). Indeed, the latter subbundle of \( CT \mathcal{N} \) is the annihilator of the pullback to \( \mathcal{N} \) of \( i\partial \sum_{j=1}^{n+1} |z|^2 \). The pairing of this form with \( T \) is
\[
\langle i \sum_{j=1}^{n+1} \tau_j (z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j}) \rangle = \sum_{j=1}^{n+1} \tau_j |z^j|^2,
\]
a function that vanishes nowhere if and only if all \( \tau_j \) are different from zero and have the same sign. Thus \( \overline{\mathcal{V}} = \overline{\mathcal{K}} \oplus \text{span}_\mathbb{C} T \) is a subbundle of \( CT \mathcal{N} \) of rank \( n+1 \) with the property that \( \mathcal{V} \) is involutive. To show that \( \overline{\mathcal{V}} \) is involutive we first note that \( \mathcal{K} \) is the annihilator of the pullback to \( \mathcal{N} \) of the span of the differentials \( dz^1, \ldots, dz^{n+1} \). Let \( L_T \) denote the Lie derivative with respect to \( T \). Then \( L_T dz^j = i \tau_j dz^j \), so if \( L \) is a CR vector field, then so is \( [L, T] \). Since in addition \( \mathcal{K} \) and \( \text{span}_\mathbb{C} T \) are themselves involutive, \( \overline{\mathcal{V}} \) is involutive. Thus \( \overline{\mathcal{V}} \) is an elliptic structure with \( \text{span}_\mathbb{C} T \), and \( \overline{\mathcal{V}} \) is involutive. Let \( \beta \) be the section of \( \overline{\mathcal{V}} \) which vanishes on \( \mathcal{K} \) and satisfies \( \langle \beta, T \rangle = -i \). Let \( \overline{\mathcal{W}} \) denote the operators of the associated differential complex. Then \( \overline{\mathcal{W}} \beta = 0 \), since \( \beta \) vanishes on commutators of sections of \( \mathcal{K} \) (since \( \mathcal{K} \) is involutive) and on commutators of \( T \) with sections of \( \mathcal{K} \) (since such commutators are in \( \mathcal{K} \)).

If the \( \tau_j \) are positive (negative), this example may be viewed as the boundary of a blowup (compactification) of \( \mathbb{C}^{n+1} \), see [9].
Let now $\rho : F \to M$ be a holomorphic vector bundle. Its $\overline{\partial}$-complex also determines a complex along $\mathcal{N}$,

\begin{equation}
\cdots \to C^\infty(\mathcal{N}; \bigwedge^q \mathbf{V}^* \otimes F_N) \overset{\overline{\partial}}{\to} C^\infty(\mathcal{N}; \bigwedge^{q+1} \mathbf{V}^* \otimes F_N) \to \cdots,
\end{equation}

where $\overline{\partial}$ is defined using the boundary operators $\overline{\partial}_b$ and the isomorphism (4.7):

\begin{equation}
\overline{\partial}(\phi \otimes \eta) = (\Phi^*)^{-1} b[\Phi^*(\phi \otimes \eta)]
\end{equation}

where $\Phi^*$ means $\Phi^* \otimes I$. These operators can be expressed locally in terms of the operators of the complex (4.5). Fix a smooth frame $\eta_\mu$, $\mu = 1, \ldots, k$, of $F$ in a neighborhood $U \subset M$ of $p_0 \in \mathcal{N}$, and suppose

$$\overline{\partial}_b \eta_\mu = \sum_{\nu} \omega^\nu_\mu \otimes \eta_\nu.$$ 

The $\omega^\nu_\mu$ are local sections of $\bigwedge^{0,1} \mathcal{N}$, and if $\sum_\mu \phi^\mu \otimes \eta_\mu$ is a section of $\bigwedge^{0,q} \mathcal{N} \otimes F$ over $U$, then

$$\overline{\partial} \left( \sum_\mu \phi^\mu \otimes \eta_\mu \right) = \sum_\nu \left( \overline{\partial} \phi^{\nu} + \sum_\mu \omega^\nu_\mu \wedge \phi^\mu \right) \otimes \eta_\nu.$$ 

Therefore, using the identification (4.7), the boundary operator $\overline{\partial}_b$ is the operator given locally by

\begin{equation}
\overline{\partial} \sum_\mu \phi^\mu \otimes \eta_\mu = \overline{\partial} \left( \sum_\nu (\overline{\partial} \phi^{\nu} + \sum_\mu \omega^\nu_\mu \wedge \phi^\mu) \otimes \eta_\nu \right)
\end{equation}

where now the $\phi^\mu$ are sections of $\bigwedge^q \mathbf{V}^*$, the $\omega^\nu_\mu$ are the sections of $\mathbf{V}^*$ corresponding to the original $\omega^\nu_\mu$ via $\Phi^*$, and $\overline{\partial}$ on the right hand side of the formula is the operator associated with $\nabla$.

The structure bundle $\bigwedge^{0,1} F$ is locally given as the span of the sections $[5.3]$. Applying the evaluation homomorphism $\mathbb{C}^q \mathbf{T}_B F \to \mathbb{C} \mathbf{T} \partial F$ (over $\mathcal{N}$) to these sections gives vector fields on $F_N$ forming a frame for the elliptic structure $\nabla_F$ inherited by $F_N$. Writing $V_j^0 = ev V_j$, this frame is just

\begin{equation}
\tilde{V}_j^0 = \sum_{\mu, \nu} \zeta^{\mu}_\nu (\omega^\nu_\mu, V_j^0) \partial_{\zeta^{\nu}_\mu}, \quad j = 1, \ldots, n + 1, \quad \partial_{\zeta^{\nu}_\mu}, \quad \nu = 1, \ldots, k,
\end{equation}

where now the $\omega^\nu_\mu$ are the forms associated to the $\overline{\partial}$-operator of $F_N$. Alternatively, one may take the $\overline{\partial}$ operators of $F_N$ and use the formula (4.13) to define a subbundle of $\mathbb{C} \mathbf{T} F$ locally as the span of the vector fields (4.14), a fortiori an elliptic structure on $F_N$, involutive because

$$\overline{\partial} \omega^\nu + \sum_{\lambda} \omega^\lambda_\mu \wedge \omega^\nu_\mu = 0.$$ 

To obtain a formula for the canonical real vector field $\mathcal{F}_F$ in $\nabla_F$, let $J_F$ be the almost complex $b$-structure of $\mathbf{T} F$ and consider again the sections (4.4); they are defined in an open set $\rho^{-1}(U)$, $U$ a neighborhood in $M$ of a point of $\mathcal{N}$. Since the elements $\partial_{\zeta^{\nu}_\mu}$ are sections of $\bigwedge^{0,1} F$,

\begin{equation}
J_F \mathfrak{R} \partial_{\zeta^{\nu}_\mu} = \mathfrak{J} \partial_{\zeta^{\nu}_\mu}.
\end{equation}

Pick a defining function $\tau$ for $\mathcal{N}$. Then $\tilde{\tau} = \rho^* \tau$ is a defining function for $F_N$. We may take $V_{n+1} = \tau \partial_\tau + iJ \tau \partial_\tau$ along $U \cap \mathcal{N}$. Then $\tilde{V}_{n+1} = \tilde{\tau} \partial_\tilde{\tau} + i\tilde{J} \tilde{\tau} \partial_\tilde{\tau}$ along
\[ \rho^{-1}(U) \cap F_N \] and so
\[
J_F \Re(\overline{\partial} \xi + iJ \overline{\partial} \xi - \sum_{\mu, \nu} \zeta^\mu \langle \omega^\nu_{\mu}, \overline{\partial} \xi + iJ \overline{\partial} \xi \rangle) = \\
\Im(\overline{\partial} \xi + iJ \overline{\partial} \xi - \sum_{\mu, \nu} \zeta^\mu \langle \omega^\nu_{\mu}, \overline{\partial} \xi + iJ \overline{\partial} \xi \rangle)
\]
along \( \rho^{-1}(U) \cap F_N \). Using (4.15) this gives
\[
J_F \overline{\partial} \xi = \overline{\partial} \xi - 2\Im \sum_{\mu, \nu} \zeta^\mu \langle \omega^\nu_{\mu}, \overline{\partial} \xi + iJ \overline{\partial} \xi \rangle \partial^\nu.
\]
Applying the evaluation homomorphism gives
\[ T_F = \overline{T} - 2\Im \sum_{\mu, \nu} \zeta^\mu \langle \omega^\nu_{\mu}, \overline{\partial} \xi + iJ \overline{\partial} \xi \rangle \partial^\nu \]
where \( \overline{T} \) is the real vector field on \( \rho^{-1}(U \cap N) = \rho^{-1}(U) \cap F_N \) which projects on \( T \) and satisfies \( \overline{T} \zeta^\mu = 0 \) for all \( \mu \).

Let \( h \) be a Hermitian metric on \( F \), and suppose that the frame \( \eta^\mu \) is orthonormal. Applying \( T_E \) as given in (4.16) to the function \( |\zeta|^2 = \sum |\zeta^\mu|^2 \) we get that \( T_F \) is tangent to the unit sphere bundle of \( F \) if and only if
\[ \langle \omega^\nu_{\mu}, \overline{\partial} \xi + iJ \overline{\partial} \xi \rangle - \langle \omega^\mu_{\nu}, \overline{\partial} \xi + iJ \overline{\partial} \xi \rangle = 0 \]
for all \( \mu, \nu \). Equivalently, in terms of the isomorphism (4.17),
\[ \langle (\Phi^*)^{-1} \omega^\nu_{\mu}, T \rangle + \langle (\Phi^*)^{-1} \omega^\mu_{\nu}, \overline{T} \rangle = 0 \]
for all \( \mu, \nu \).

**Definition 4.18.** The Hermitian metric \( h \) will be called exact if (4.17) holds.

The terminology in Definition 4.18 is taken from the notion of exact Riemannian \( b \)-metric of Melrose [6, pg. 31]. For such metrics, the Levi-Civita \( b \)-connection has the property that \( b \nabla_{\overline{\partial} \xi} = 0 \) [op. cit., pg. 58]. We proceed to show that the Hermitian holomorphic connection of an exact Hermitian metric on \( F \) also has this property. Namely, suppose that \( h \) is an exact Hermitian metric, and let \( \eta^\mu \) be an orthonormal frame of \( F \). Then for the Hermitian holomorphic connection we have
\[ \langle \omega^\nu_{\mu} - \overline{\omega}^\mu_{\nu}, \overline{\partial} \xi \rangle = \langle \omega^\nu_{\mu}, \overline{\partial} \xi \rangle - \langle \overline{\omega}^\mu_{\nu}, \overline{\partial} \xi \rangle = \frac{1}{2} \left( \langle \omega^\nu_{\mu}, \overline{\partial} \xi + iJ \overline{\partial} \xi \rangle - \langle \overline{\omega}^\mu_{\nu}, \overline{\partial} \xi + iJ \overline{\partial} \xi \rangle \right) \]
using that the \( \omega^\nu_{\mu} \) are of type \((0, 1)\). Thus \( b \nabla_{\overline{\partial} \xi} = 0 \).

5. **Local invariants**

Complex structures have no local invariants: every point of a complex \( n \)-manifold has a neighborhood biholomorphic to a ball in \( \mathbb{C}^n \). It is natural to ask the same question about complex \( b \)-structures, namely,

is there a local model depending only on dimension for every complex \( b \)-structure?

In lieu of a Newlander-Nirenberg theorem, we show that complex \( b \)-structures have no local formal invariants at the boundary. More precisely:
Lemma 5.5. Let $f_0$ be smooth in $V \setminus U$ and suppose that $\overline{\partial}(f_0 \circ f)$ vanishes to infinite order on $U$. Then there is $f : V \to \mathbb{C}$ smooth vanishing at $U$ such that $\overline{\partial}(f_0 \circ f)$ vanishes to infinite order on $U$.

Proof. Suppose that $f_1, \ldots, f_{N-1}$ are defined on $V$ and that

$$\overline{\partial} \sum_{k=0}^{N-1} (te^{ia})^k f_k = (te^{ia})^N \psi_N$$

for some smooth section $\psi$ of $b\Lambda^{0,1}M$ over $V$. The following lemma will be applied to $f_0$ equal to log $\tau$ $+ i\alpha$ or each of the functions $z^j$.

Lemma 5.3. On such $U$, the problem

$$\overline{\partial} \phi = \psi, \quad \psi \in C^\infty(U; \bigwedge^{q+1} \overline{T}|_{U})$$

and $\overline{\partial} \psi = 0$ has a solution in $C^\infty(U; \bigwedge^{q} \overline{T}|_{U})$.

Extend the functions $z^j$ and $t$ to a neighborhood of $p_0$ in $M$. Shrinking $U$ if necessary, we may assume that in some neighborhood $V$ of $p_0$ in $M$ with $V \cap \partial M = U$, $(z, t, \tau)$ maps $V$ diffeomorphically onto $B \times (-\delta, \delta) \times [0, \varepsilon)$ for some $\delta, \varepsilon > 0$. Since the form $\beta_t$ defined in (4.3) is $\overline{\partial}$-closed, there is $\alpha \in C^\infty(U)$ such that

$$-i\overline{\partial} \alpha = \beta_t.$$

Extend $\alpha$ to $V$ as a smooth function. The section

$$(5.4) \quad \overline{\partial}(\log \tau + i\alpha) = \frac{\overline{\partial} \tau}{\tau} + i\overline{\partial} \alpha$$

of $b\Lambda^{0,1}M$ over $V$ vanishes on $U$, since $\beta_t + i\overline{\partial} \alpha = 0$. So there is a smooth section $\phi$ of $b\Lambda^{0,1}M$ over $V$ such that

$$\overline{\partial}(\log \tau + i\alpha) = \tau e^{ia} \phi.$$
holds with $\psi_N$ smooth in $V$; by the hypothesis, (5.6) holds when $N = 1$. Using (5.4) we get that $\overline{D}(e^{i\alpha}) = (e^{i\alpha})^2\phi$, therefore
\[
0 = \overline{D}((e^{i\alpha})^N\psi_N) = (e^{i\alpha})^N(\overline{D}\psi_N + N e^{i\alpha} \psi_N),
\]
which implies that $\overline{D}\psi_N = 0$ on $U$. With arbitrary $f_N$ we have
\[
\overline{D}\sum_{k=0}^N (e^{i\alpha})^k f_k = (e^{i\alpha})^N(\psi_N + \overline{D}f_N + N e^{i\alpha} f_N \phi).
\]
Since $\overline{D}\psi_N = 0$ and $H^1_U(U) = 0$ by Lemma 5.3 there is a smooth function $f_N$ defined in $U$ such that $\overline{D}f_N = -\psi_N$ in $U$. So there is $\chi_N$ such that $\psi_N + \overline{D}f_N = e^{i\alpha}\chi_N$. With such $f_N$, (5.6) holds with $N + 1$ in place of $N$ and some $\psi_{N+1}$. Thus there is a sequence $\{f_j\}_{j=1}^\infty$ such that (5.6) holds for each $N$. Borel's lemma then gives $f$ smooth with
\[
f \sim \sum_{k=1}^\infty (e^{i\alpha})^k f_k \quad \text{on } U
\]
such that $\overline{D}(f_0 + f)$ vanishes to infinite order on $U$.

Proof of Proposition 5.7

Apply the lemma with $f_0 = \log r + i\alpha$ to get a function $f$ such that $\overline{D}(f_0 + f)$ vanishes to infinite order at $U$. Let
\[
x^{n+1} = e^{-3\alpha + \Re f}, \quad x^{2n+2} = \Re \alpha + \Im f.
\]
These functions are smooth up to $U$.

Applying the lemma to each of the functions $f_0 = z^j$, $j = 1, \ldots, n$ gives smooth functions $\zeta^j$ such that $\zeta^j = z^j$ on $U$ and $\overline{D}\zeta^j = 0$ to infinite order at $U$. Define
\[
x^j = \Re \zeta^j, \quad x^{j+n+1} = \Im \zeta^j, \quad j = 1, \ldots, n.
\]
The functions $x^j$, $j = 1, \ldots, 2n + 2$ are independent, and the forms
\[
\eta^j = b d\zeta^j, j = 1, \ldots, n, \quad \eta^{n+1} = \frac{1}{x^{n+1} e^{ix^{2n+2}} b^d [x^{n+1} e^{ix^{2n+2}}]}
\]
together with their conjugates form a frame for $\mathcal{C}^0 \mathcal{T} \mathcal{M}$ near $p_0$. Let $\eta^j_{0,1}$ and $\eta^j_{0,1}$ be the $(1,0)$ and $(0,1)$ components of $\eta^j$ according to the complex $b$-structure of $\mathcal{M}$. Then
\[
\eta^j_{0,1} = \sum_k p^j_k \eta^k + q^j_k \overline{\eta}^k.
\]
Since $\eta^j_{0,1} = \overline{D}\zeta^j$ vanishes to infinite order at $U$, the coefficients $p^j_k$ and $q^j_k$ vanish to infinite order at $U$. Replacing this formula for $\eta^j_{0,1}$ in $\eta^j = \eta^j_{1,0} + \eta^j_{0,1}$ get
\[
\sum_k (\delta^j_k - p^j_k) \eta^k - \sum_k q^j_k \overline{\eta}^k = \eta^j_{1,0}.
\]
The matrix $I - [p^j_k]$ is invertible with inverse of the form $I + [P^j_k]$ with $P^j_k$ vanishing to infinite order at $U$. So
\[
(5.7) \quad \eta^j - \sum_k \gamma^j_k \overline{\eta}^k = \sum_k (\delta^j_k + P^j_k) \eta^j_{1,0}
\]
with suitable $\gamma_j^k$ vanishing to infinite order on $U$. Define the vector fields $\mathcal{T}_j^0$ as in (5.2). The vector fields

$$\mathcal{T}_j = \mathcal{T}_j^0 + \sum_k \gamma_j^k L_k^0, \quad j = 1, \ldots, n + 1$$

are independent and since $\langle \mathcal{T}_j^0, \eta^k \rangle = 0$ and $\langle L_j^0, \eta^k \rangle = \delta^k_j$, they annihilate each of the forms on the left hand side of (5.7). So they annihilate the forms $\eta_{1,0}^k$, which proves that the $\mathcal{T}_j$ form a frame of $b\mathcal{T}^{0,1}$. \qed

6. Indicial complexes

Throughout this section we assume that $N$ is a connected component of the boundary of a compact manifold $\mathcal{M}$. Let

$$\cdots \to C^\infty(\mathcal{M}; E^q) \xrightarrow{A_q} C^\infty(\mathcal{M}; E^{q+1}) \to \cdots$$

be a $b$-elliptic complex of operators $A_q \in \text{Diff}^1_b(\mathcal{M}; E^q, E^{q+1})$; the $E^q$, $q = 0, \ldots, r$, are vector bundles over $\mathcal{M}$.

Note that since $A_q$ is a first order operator,

$$A_q(f \phi) = f A_q \phi - i b \sigma(\sigma(A_q))(\mathcal{b} \mathcal{d} r) \phi,$$

This formula follows from the analogous formula for the standard principal symbol and the definition of principal $b$-symbol. It follows from (6.2) and (2.8) that $A_q$ defines an operator $A_{b,q} : \text{Diff}^1_b(N; E^q_N, E^{q+1}_N)$.

Fix a smooth defining function $r : \mathcal{M} \to \mathbb{R}$ for $\partial \mathcal{M}$, $r > 0$ in the interior of $\mathcal{M}$, let

$$A_q(\sigma) : \text{Diff}^1_b(N; E^q_N, E^{q+1}_N), \quad \sigma \in \mathbb{C}$$

denote the indicial family of $A_q$ with respect to $r$, see (A.5). Using (6.2) and defining

$$A_{\sigma, q} = b \sigma(A_q)(\mathcal{b} \mathcal{d} r),$$

the indicial family of $A_q$ with respect to $\tau$ is

$$A_q(\sigma) = A_{b,q} + \sigma A_{\sigma,q} : C^\infty(N; E^q_N) \to C^\infty(N; E^{q+1}_N).$$

Because of (A.4), these operators form an elliptic complex

$$\cdots \to C^\infty(N; E^q_N) \xrightarrow{A_q(\sigma)} C^\infty(N; E^{q+1}_N) \to \cdots$$

for each $\sigma$ and each connected component $N$ of $\partial \mathcal{M}$. The operators depend on $\tau$, but the cohomology groups at a given $\sigma$ for different defining functions $\tau$ are isomorphic. Indeed, if $\tau'$ is another defining function for $\partial \mathcal{M}$, then $\tau' = e^u \tau$ for some smooth real-valued function $u$, and a simple calculation gives

$$(A_{b,q} + \sigma A_{\sigma,q})(e^{i\sigma u} \phi) = e^{i\sigma u}(A_{b,q} + \sigma A_{\sigma,q})\phi.$$
Proposition 6.7. For each \( q \), \( \text{spec}^q_{b,N}(A) \subset \text{spec}_{b,N}(\square q) \).

Note that the set \( \text{spec}_{b,N}(\square q) \) depends on the choice of Hermitian metrics and \( b \)-density used to construct the Laplacian, but that the subset \( \text{spec}^q_{b,N}(A) \) is independent of such choices. For a general \( b \)-elliptic complex (6.1), it may occur that \( \text{spec}^q_{b,N}(A) \neq \text{spec}_{b,N}(\square q) \). In Example 6.18 we show that \( \text{spec}^q_{b,N}(\square) \subset \{0\} \). As is well known, \( \text{spec}_{b,N}(\Delta q) \) is an infinite set if \( \dim M > 1 \). At the end of this
section we will give an example where \( \text{spec}_{b,N}^q(\mathbb{T}) \) is an infinite set. A full discussion of \( \text{spec}_{b,N}^q(\mathbb{T}) \) for any \( q \) and other aspects of the indicial complex of complex \( b \)-structures is given in Section 9.

**Proof of Proposition 6.7.** Since \( \Box_q \) is \( b \)-elliptic, the set \( \text{spec}_{b,N}(\Box_q) \) is closed and discrete. Let \( H^2(\mathcal{N}; E_N^q) \) be the \( L^2 \)-based Sobolev space of order 2. For \( \sigma \notin \text{spec}_{b,N}(\Box_q) \) let

\[
G_q(\sigma) : L^2(\mathcal{N}; E_N^q) \rightarrow H^2(\mathcal{N}; E_N^q)
\]

be the inverse of \( \Box_q(\sigma) \). The map \( \sigma \mapsto G_q(\sigma) \) is meromorphic with poles in \( \text{spec}_{b,N}(\Box_q) \). Since

\[
A_q^\ast(\sigma) = [A_q(\Box)]^\ast
\]

the operators \( \Box_q(\sigma) \) are the Laplacians of the complex \( (6.34) \) when \( \sigma \) is real. Thus for \( \sigma \in \mathbb{R}\setminus(\text{spec}_{b,N}(\Box_q) \cup \text{spec}_{b,N}(\Box_{q+1})) \) we have

\[
A_q(\sigma)G_q(\sigma) = G_{q+1}(\sigma)A_q(\sigma), \quad A_q(\sigma)^\ast G_{q+1}(\sigma) = G_q(\sigma)A_q^\ast(\sigma)
\]

by standard Hodge theory. Since all operators depend holomorphically on \( \sigma \), the same equalities hold for any \( \sigma \in \mathbb{R} = \mathbb{C}\setminus(\text{spec}_{b,N}(\Box_q) \cup \text{spec}_{b,N}(\Box_{q+1})) \). It follows that

\[
A_q^\ast(\sigma)A_q(\sigma)G_q(\sigma) = G_q(\sigma)A_q^\ast(\sigma)A_q(\sigma)
\]

in \( \mathbb{R} \). By analytic continuation the equality holds on all of \( \mathbb{C}\setminus\text{spec}_{b,N}(\Box_q) \). Thus if \( \sigma_0 \notin \text{spec}_{b,N}(\Box_q) \) and \( \phi \) is a \( A_q(\sigma_0) \)-closed section, \( A_q(\sigma_0)\phi = 0 \), then the formula

\[
\phi = [A_q^\ast(\sigma_0)A_q(\sigma_0) + A_{q-1}(\sigma_0)A_{q-1}^\ast(\sigma_0)]G_q(\sigma_0)\phi
\]

leads to

\[
\phi = A_{q-1}(\sigma_0)[A_{q-1}^\ast(\sigma_0)G_q(\sigma_0)\phi].
\]

Therefore \( \sigma_0 \notin \text{spec}_{b,N}^q(A) \). \( \square \)

Since \( \Box_q \) is \( b \)-elliptic, the set \( \text{spec}_{b,N}(\Box_q) \) is discrete and intersects each horizontal strip \( a \leq \Im \sigma \leq b \) in a finite set (Melrose [3]). Consequently:

**Corollary 6.8.** The sets \( \text{spec}_{b,N}^q(A) \), \( q = 0, 1, \ldots \), are closed, discrete, and intersect each horizontal strip \( a \leq \Im \sigma \leq b \) in a finite set.

We note in passing that the Euler characteristic of the complex \( (6.34) \) vanishes for each \( \sigma \). Indeed, let \( \sigma_0 \in \mathbb{C} \). The Euler characteristic of the \( A(\sigma_0) \)-complex is the index of

\[
A(\sigma_0) + A(\sigma_0)^\ast : \bigoplus_{q \text{ even}} C^\infty(\mathcal{N}; E^q) \rightarrow \bigoplus_{q \text{ odd}} C^\infty(\mathcal{N}; E^q).
\]

The operator \( A_q(\sigma) \) is equal to \( A_{b,q} + \sigma \Lambda_{r,q} \), see \( (6.3) \). Thus \( A_q(\sigma)^\ast = A_{b,q}^\ast + \sigma \Lambda_{r,q}^\ast \), and it follows that for any \( \sigma \),

\[
A(\sigma) + A(\sigma)^\ast = A(\sigma_0) + A(\sigma_0)^\ast + (\sigma - \sigma_0)\Lambda_r + (\overline{\sigma} - \overline{\sigma_0})\Lambda_r^\ast
\]

is a compact perturbation of \( A(\sigma_0) + A(\sigma_0)^\ast \). Therefore, since the index is invariant under compact perturbations, the index of \( A(\sigma) + A(\sigma)^\ast \) is independent of \( \sigma \). Then it vanishes, since it vanishes when \( \sigma \notin \bigcup_q \text{spec}_{b,N}^q(A) \).

Let \( \mathfrak{M}_{\text{mer}}^q(\mathcal{N}) \) be the sheaf of germs of \( C^\infty(\mathcal{N}; E^q) \)-valued meromorphic functions on \( \mathbb{C} \) and let \( \mathfrak{H}_{\text{hol}}^q(\mathcal{N}) \) be the subsheaf of germs of holomorphic functions. Let \( \mathcal{E}_q(\mathcal{N}) = \mathfrak{M}_{\text{mer}}^q(\mathcal{N})/\mathfrak{H}_{\text{hol}}^q(\mathcal{N}) \). The holomorphic family \( \sigma \mapsto A_q(\sigma) \) gives a
sheaf homomorphism \( \mathcal{A}_q : \mathcal{M} \rightarrow \mathcal{M} \) such that \( \mathcal{A}_q(\mathcal{M}(N)) \subset \mathcal{M}(N) \) and \( \mathcal{A}_{q+1} \circ \mathcal{A}_q = 0 \), we have a complex

\[
\cdots \rightarrow \mathcal{S}^q(N) \xrightarrow{\Delta \psi} \mathcal{S}^{q+1}(N) \rightarrow \cdots.
\]

The cohomology sheaves \( \mathcal{S}_A^q(N) \) of this complex contain more refined information about the cohomology of the complex \( A \).

**Proposition 6.10.** The sheaf \( \mathcal{S}_A^q(N) \) is supported on \( \text{spec}^q_{\partial N}(A) \).

**Proof.** Let \( \sigma_0 \in C \) be such that \( H^q_{\mathcal{A}(\sigma_0)}(N) = 0 \) and let

\[
\phi(\sigma) = \sum_{k=1}^{\mu} \frac{\phi_k}{(\sigma - \sigma_0)^k},
\]

\( \mu > 0, \phi_k \in C^\infty(N; \Lambda^q \mathcal{T}^\sigma) \), represent the \( A \)-closed element \( [\phi] \) of the stalk of \( \mathcal{S}^q(N) \) over \( \sigma_0 \). The condition that \( \mathcal{A}_q[\phi] = 0 \) means that \( \mathcal{A}_q(\sigma)\phi(\sigma) \) is holomorphic, that is,

\[
\mathcal{A}_q(\sigma_0)\phi_\mu = 0.
\]

In particular \( \mathcal{A}_q(\sigma_0)\phi_\mu = 0 \). Since \( H^q_{\mathcal{A}(\sigma_0)}(N) = 0 \), there is \( \psi_\mu \in C^\infty(N; \Lambda^q \mathcal{T}^\sigma) \) such that \( \mathcal{A}_{q-1}(\sigma_0)\psi_\mu = \phi_\mu \). This shows that if \( \mu = 1 \), then \( [\phi] \) is exact, and that if \( \mu > 1 \), then letting \( \phi'(\sigma) = \phi(\sigma) - \mathcal{A}_{q-1}(\sigma)\psi_\mu/(\sigma - \sigma_0)^\mu \), that \( \phi \) is cohomologous to an element \( [\phi'] \) represented by a sum as in (6.11) with \( \mu - 1 \) instead of \( \mu \). By induction, \( [\phi] \) is exact. \( \square \)

**Definition 6.12.** The cohomology sheaves \( \mathcal{S}_A^q(N) \) of the complex (6.9) will be referred to as the indicial cohomology sheaves of the complex \( A \). If \( [\phi] \in \mathcal{S}_A^q(N) \) is a nonzero element of the stalk over \( \sigma_0 \), the smallest \( \mu \) such that there is a meromorphic function (6.11) representing \( [\phi] \) will be called the order of the pole of \( [\phi] \).

The relevancy of this notion of pole lies in that it predicts, for any given cohomology class of the complex \( A \), the existence of a representative with the most regular leading term (the smallest power of log that must appear in the expansion at the boundary). We will see later (Proposition 9.5) that for the \( b \)-Dolbeault complex, under certain geometric assumption, the order of the pole of \( [\phi] \in \mathcal{S}_A^q(N) \setminus 0 \) is 1.

**Example 6.13.** For the \( b \)-de Rham complex one has \( \text{spec}_{b,N}(\mathcal{M}) \subset \{ 0 \} \) and

\[
H^q_{\partial N}(N) = H^q_{dR}(N) \oplus H^q_{dR}(-1)(N)
\]

for each component \( N \) of \( \partial \mathcal{M} \), and that every element of the stalk of \( \mathcal{S}_q(N) \) over 0 has a representative with a simple pole. By way of the residue we get an isomorphism from the stalk over 0 onto \( H^q_{dR}(N) \).

Since the map (2.1) is surjective with kernel spanned by \( \partial \mathcal{M} \), the dual map

\[
ev^*: T^*N \rightarrow bT^*_N \mathcal{M}
\]

is injective with image the annihilator, \( \mathcal{H} \), of \( \partial \mathcal{M} \). Let \( \mathcal{I} \mathcal{H} : b\Lambda^{\leq q}_{N; \mathcal{M}} \rightarrow b\Lambda^{q-1}_{N; \mathcal{M}} \mathcal{M} \) denote interior multiplication by \( \partial \mathcal{M} \). Then \( \Lambda^{\leq q}_{N; \mathcal{M}} = \ker(1_{\partial \mathcal{M}} : b\Lambda^{q}_{N; \mathcal{M}} \rightarrow b\Lambda^{q-1}_{N; \mathcal{M}} \mathcal{M}) \). The isomorphism (6.14) gives isomorphisms

\[
ev^*: \Lambda^{\leq q}_{N; \mathcal{M}} \rightarrow \mathcal{H}^q
\]
for each $q$. Fix a defining function $r$ for $\mathcal{N}$ and let $\Pi : b\wedge^q_N M \to b\wedge^q_N M$ be the projection on $H^q$ according to the decomposition
\[ b\wedge^q_N M = H^q \oplus \frac{b}{r} \wedge H^{q-1}, \]
that is,
\[ \Pi \phi = \phi - \frac{b}{r} \wedge i_{\partial r} \phi. \]
If $\phi^0 \in C^\infty(\mathcal{N}, H^q)$ and $\phi^1 \in C^\infty(\mathcal{N}, H^{q-1})$, then
\[ b(d(\phi^0 + \frac{b}{r} \wedge \phi^1) = \Pi b d \phi^0 + \frac{b}{r} \wedge (-\Pi b d \phi^1). \]
Since
\[ r^{-i\sigma} b d i_{\sigma} \phi = b d \phi + i\sigma \frac{b}{r} \wedge \phi, \]
the indicial operator $\mathcal{D}(\sigma)$ of $b d$ is
\[ \mathcal{D}(\sigma)(\phi^0 + \frac{b}{r} \wedge \phi^1) = \Pi b d \phi^0 + \frac{b}{r} \wedge (i\sigma \phi^0 - \Pi b d \phi^1). \]
If $\mathcal{D}(\sigma)(\phi^0 + \frac{b}{r} \wedge \phi^1) = 0$, then of course $\Pi b d \phi^0 = 0$ and $i\sigma \phi^0 = \Pi b d \phi^1$, and it follows that if $\sigma \neq 0$, then
\[ (\phi^0 + \frac{b}{r} \wedge \phi^1) = \mathcal{D}(\sigma) \frac{1}{i\sigma} \phi^1. \]
Thus all cohomology groups of the complex $\mathcal{D}(\sigma)$ vanish if $\sigma \neq 0$, i.e., $\text{spec}_{b,\mathcal{N}}(b d) \subseteq \{0\}$.

It is not hard to verify that
\[ \Pi b d \text{ev}_\mathcal{N}^* = \text{ev}_\mathcal{N}^* d. \]
Since
\[ r^{-i\sigma} b d i_{\sigma} \phi = b d \phi + i\sigma \frac{b}{r} \wedge \phi, \]
the indicial operator of $b d$ at $\sigma = 0$ can be viewed as the operator
\[ \begin{bmatrix} d & 0 \\ 0 & -d \end{bmatrix} : \wedge^q \mathcal{N} \oplus \wedge^{q-1} \mathcal{N} \to \wedge^q \mathcal{N} \oplus \wedge^{q-1} \mathcal{N}. \]
From this we get the cohomology groups of $\mathcal{D}(0)$ in terms of the de Rham cohomology of $\mathcal{N}$:
\[ H^q_{\mathcal{D}(0)}(\mathcal{N}) = H^q_{\text{dR}}(\mathcal{N}) \oplus H^{q-1}_{\text{dR}}(\mathcal{N}). \]
Thus the groups $H^q_{\mathcal{D}(0)}(\mathcal{N})$ do not vanish for $q = 0, 1, \dim \mathcal{M} - 1, \dim \mathcal{M}$ but may vanish for other values of $q$.

We now show that every element of the stalk of $\mathcal{H}^q_{b,\mathcal{N}}(\mathcal{N})$ over 0 has a representative with a simple pole at 0. Suppose that
\[ (6.15) \phi(\sigma) = \sum_{k=1}^\mu \frac{1}{\sigma^k} \left( \phi^0_k + \frac{b}{r} \wedge \phi^1_k \right). \]
is such that $D(\sigma)\phi(\sigma)$ is holomorphic. Then

$$\sum_{k=1}^{\mu} \frac{1}{\sigma^k} \left( d\phi_k^0 - \frac{b_{1\tau}}{r} \wedge d\phi_k^1 \right) + \frac{b_{1\tau}}{r} \wedge \left( \sum_{k=1}^{\mu-1} \frac{i}{\sigma^{k+1}} \phi_k^0 \right) = 0,$$

hence $d\phi_k^0 = 0$, $d\phi_k^1 = 0$ and $\phi_k^0 = -id\phi_k^1$, $k = 2, \ldots, \mu$. Let

$$\psi(\sigma) = -i \sum_{k=2}^{\mu+1} \frac{1}{\sigma^k} \phi_k^1.$$

Then

$$D(\sigma)\psi(\sigma) = -i \sum_{k=2}^{\mu+1} \frac{1}{\sigma^k} d\phi_k^1 + \frac{b_{1\tau}}{r} \wedge \sum_{k=2}^{\mu+1} \frac{1}{\sigma^{k-1}} \phi_k^1$$

$$= \sum_{k=2}^{\mu+1} \frac{1}{\sigma^k} \phi_k^1 + \frac{b_{1\tau}}{r} \wedge \sum_{k=2}^{\mu} \frac{1}{\sigma^k} \phi_k^1$$

so

$$\phi(\sigma) - D(\sigma)\psi(\sigma) = \frac{1}{\sigma} \phi_1^0.$$

The map that sends the class of the $D(\sigma)$-closed element (6.15) to the class of $\phi_1^0$ in $H^0_{dR}(\mathcal{N})$ is an isomorphism.

**Example 6.16.** As we just saw, the boundary spectrum of the $\bar{\partial}$ complex in degree 0 is just $\{0\}$. In contrast, $\mathrm{spec}^0_{b,\mathcal{N}}(\overline{\partial})$ may be an infinite set. We illustrate this in the context of Example 4.10. The functions

$$z^\alpha = (z^1)^{\alpha_1} \cdots (z^{n+1})^{\alpha_{n+1}},$$

where the $\alpha_j$ are nonnegative integers, are CR functions that satisfy

$$Tz^\alpha = i(\sum \tau_j \alpha_j) z^\alpha.$$

This implies that

$$\overline{T}z^\alpha + i(-\sum \tau_j \alpha_j) z^\alpha = 0$$

with $\beta$ as in Example 4.10 so the numbers $\sigma_\alpha = (-\sum \tau_j \alpha_j)$ belong to $\mathrm{spec}^0_{b,\mathcal{N}}(\overline{\partial})$.

For the sake of completeness we also show that if $\sigma \in \mathrm{spec}^0_{b,\mathcal{N}}(\overline{\partial})$, then $\sigma = \sigma_\alpha$ for some $\alpha$ as above. To see this, suppose that $\zeta : S^{2n+1} \to \mathbb{C}$ is not identically zero and satisfies

$$\overline{T}\zeta + i\sigma \zeta = 0$$

for some $\sigma \neq 0$. Then $\zeta$ is smooth, because the principal symbol of $\overline{T}$ on functions is injective. Since $\langle \beta, T \rangle = -i$,

$$T\zeta + \sigma \zeta = 0.$$

Thus $\zeta(a_t(p)) = e^{-\tau t} \zeta(p)$ for any $p$. Since $|\zeta(a_t(p))|$ is bounded as a function of $t$ and $\zeta$ is not identically 0, $\sigma$ must be purely imaginary. Since $\zeta$ is a CR function, it extends uniquely to a holomorphic function $\hat{\zeta}$ on $B = \{ z \in \mathbb{C}^{n+1} : ||z|| < 1 \}$, necessarily smooth up to the boundary. Let $\zeta_t = \zeta \circ a_t$. This is also a smooth CR function, so it has a unique holomorphic extension $\hat{\zeta}_t$ to $B$. The integral curve through $z_0 = (z_0^1, \ldots, z_0^{n+1})$ of the vector field $T$ is

$$t \mapsto a_t(z_0) = (e^{it} z_0^1, \ldots, e^{it} z_0^{n+1})$$
Extending the definition of $\alpha_t$ to allow arbitrary $z \in \mathbb{C}^{n+1}$ as argument we then have that $\tilde{\zeta}_t = \zeta \circ \alpha_t$. Then

$$\partial_t \tilde{\zeta}_t + \sigma \tilde{\zeta}_t = 0$$

gives

$$\tilde{\zeta}(z) = \sum_{\alpha \cdot r = i \sigma} c_{\alpha} z^\alpha$$

for $|z| < 1$, where $r = (r_1, \ldots, r_{n+1})$. Thus $\sigma = -i \sum r_j \alpha_j$ as claimed. Note that $\Re \sigma$ is negative (positive) if the $r_j$ are positive (negative) and $\alpha \neq 0$.

7. UNDERLYING CR COMPLEXES

Again let $a : \mathbb{R} \times \mathcal{N} \to \mathcal{N}$ be the flow of $\mathcal{T}$. Let $\mathcal{L}_T$ denote the Lie derivative with respect to $\mathcal{T}$ on de Rham $q$-forms or vector fields and let $i_T$ denote interior multiplication by $\mathcal{T}$ of de Rham $q$-forms or of elements of $\bigwedge q^* \mathcal{V}$.

The proofs of the following two lemmas are elementary.

**Lemma 7.1.** If $\alpha$ is a smooth section of the annihilator of $\nabla$ in $\mathcal{C}T^* \mathcal{N}$, then $(\mathcal{L}_T \alpha)|_{\mathcal{V}} = 0$. Consequently, for each $p \in \mathcal{N}$ and $t \in \mathbb{R}$, $da_t : \mathcal{C}T_p \mathcal{N} \to \mathcal{C}T_{a_t(p)} \mathcal{N}$ maps $\nabla_p$ onto $\nabla_{a_t(p)}$.

It follows that there is a well defined smooth bundle homomorphism $a_t^* : \bigwedge q^* \mathcal{V} \to \bigwedge q^* \mathcal{V}$ covering $a_{-t}$. In particular, one can define the Lie derivative $\mathcal{L}_T \phi$ with respect to $\mathcal{T}$ of an element in $\phi \in \mathcal{C}^\infty(\mathcal{N}; \bigwedge q^* \mathcal{V})$. The usual formula holds:

**Lemma 7.2.** If $\phi \in \mathcal{C}^\infty(\mathcal{N}; \bigwedge q^* \mathcal{V})$, then $\mathcal{L}_T \phi = i_T \nabla \phi + \nabla i_T \phi$. Consequently, for each $t$ and $\phi \in \mathcal{C}^\infty(\mathcal{N}; \bigwedge q^* \mathcal{V})$, $\mathcal{L}^* \phi = a_t^* \nabla \phi$.

For any defining function $r$ of $\mathcal{N}$ in $\mathcal{M}$, $\mathcal{K}_r = \ker \beta_r$ is a CR structure of CR codimension 1: indeed, $\mathcal{K}_r \cap \mathcal{K}_r \subset \text{span}_\mathcal{C} \mathcal{T}$ but since $\langle \beta_r, \mathcal{T} \rangle$ vanishes nowhere, we must have $\mathcal{K}_r \cap \mathcal{K} = 0$. Since $\mathcal{K} \oplus \mathcal{K} \oplus \text{span}_\mathcal{C} \mathcal{T} = \mathcal{C} \mathcal{T} \mathcal{N}$, the CR codimension 1 is finally, if $V, W \in \mathcal{C}^\infty(\mathcal{N}; \mathcal{K}_r)$, then

$$\langle \beta_r, [V, W] \rangle = V \langle \beta_r, W \rangle - W \langle \beta_r, V \rangle - 2i \beta(V, W),$$

Since the right hand side vanishes, $[V, W]$ is again a section of $\mathcal{K}_r$.

Since $\nabla = \mathcal{K}_r \oplus \text{span}_\mathcal{C} \mathcal{T}$, the dual of $\mathcal{K}_r$ is canonically isomorphic to the kernel of $i_T : \nabla^* \to \mathcal{C}$. We will write $\mathcal{K}^*$ for this kernel. More generally, $\bigwedge q^\mathcal{K}^*$ and the kernel, $\bigwedge q^\mathcal{K}$, of $i_T : \bigwedge q^\mathcal{V} \to \bigwedge q^{-1} \mathcal{V}$ are canonically isomorphic. The vector bundles $\bigwedge q^\mathcal{K}$ are independent of the defining function $r$. We regard the $\partial_h$-operators of the CR structure as operators

$$\mathcal{C}^\infty(\mathcal{N}; \bigwedge q^\mathcal{K}) \to \mathcal{C}^\infty(\mathcal{N}; \bigwedge q+1 \mathcal{K}).$$

They do depend on $r$ but we will not indicate this in the notation.

To get a formula for $\partial_h$, let

$$\tilde{\beta}_t = \frac{i}{i - \alpha} \beta_t$$

(so that $\langle i \tilde{\beta}_t, \mathcal{T} \rangle = 1$). The projection $\Pi_r : \bigwedge q^\mathcal{V} \to \bigwedge q^r \mathcal{V}$ on $\bigwedge q^\mathcal{K}$ according to the decomposition

$$(7.3) \quad \bigwedge q^\mathcal{V} = \bigwedge q^\mathcal{K} \oplus i \tilde{\beta}_t \wedge \bigwedge q^{-1} \mathcal{K}^*$$

...
Lemma 7.5. With the identification of $\bigwedge^q\mathcal{K}_e$ with $\bigwedge^q\mathcal{K}^{\ast}$ described above, the $\overline{\partial}_b$-operators of the CR structure $\mathcal{K}_e$ are given by

\begin{equation}
\overline{\partial}_b \phi = \Pi_z \overline{\partial} \phi \quad \text{if } \phi \in C^\infty(\mathcal{N} \setminus \bigwedge^q\mathcal{K}^{\ast}),
\end{equation}

Proof. Suppose that $(z,t)$ is a hypoanalytic chart for $\overline{\mathcal{V}}$ on some open set $U$, with $\mathcal{T}t = 1$. So $\partial_{\omega_{\mu}}$, $\mu = 1, \ldots, n$, $\mathcal{T} = \partial_{\xi}$ is a frame for $\overline{\mathcal{V}}$ over $U$ with dual frame $\overline{\mathcal{V}}^\ast$, $\overline{\partial} t$. If

$$
\beta_t = \sum_{\mu=1}^n \beta_{\mu} \overline{\mathcal{V}}^\mu + \beta_0 \overline{\partial} t,
$$

then

$$
\overline{\mathcal{L}}^\mu = \partial_{\omega_{\mu}} - \frac{\beta_{\mu}}{\beta_0} \partial_{\xi}, \quad \mu = 1, \ldots, n
$$

is a frame for $\overline{\mathcal{K}}_e$ over $U$. Let $\overline{\eta}^\mu$ denote the dual frame (for $\overline{\mathcal{K}}_e^\ast$). Since the $\overline{\mathcal{L}}^\mu$ commute, $\overline{\partial}_b \overline{\eta}^\mu = 0$, so if $\phi = \sum_j \phi_j \overline{\eta}^j$, then (with the notation as in eg. Folland and Kohn [2])

\begin{equation}
\overline{\partial}_b \phi = \sum_j \phi_j \overline{\mathcal{V}}^j.
\end{equation}

On the other hand, the frame of $\overline{\mathcal{V}}^\mu$ dual to the frame $\overline{\mathcal{L}}^\mu$, $\mu = 1, \ldots, n$, $\mathcal{T}$ of $\overline{\mathcal{V}}$ is $\overline{\partial} \overline{\mathcal{V}}^\mu$, $i\beta_t$, and the identification of $\overline{\mathcal{K}}_e$ with $\overline{\mathcal{K}}^{\ast}$ maps the $\eta^\mu$ to the $\overline{\partial} \overline{\mathcal{V}}^\mu$. So, as a section of $\bigwedge^q \overline{\mathcal{V}}^\ast$,

$$
\phi = \sum_j \phi_j \overline{\mathcal{V}}^j
$$

and

$$
\overline{\mathcal{D}} \phi = \sum_j \phi_j \overline{\mathcal{L}}^\mu \mu_j \overline{\mathcal{V}}^j + i\beta_t \wedge \sum_j \mathcal{T} \phi_j \overline{\mathcal{V}}^j.
$$

Thus $\Pi_z \overline{\partial} \phi$ is the section of $\bigwedge^{q+1} \mathcal{K}^{\ast}$ associated with $\overline{\partial}_b \phi$ by the identifying map. \(\square\)

Using (7.4) in (7.6) and the fact that $i_{\mathcal{T}} \overline{\mathcal{D}} \phi = \mathcal{L}_{\mathcal{T}} \phi$ if $\phi \in C^\infty(\mathcal{N} \setminus \bigwedge^q\mathcal{K}^{\ast})$ we get

\begin{equation}
\overline{\partial}_b \phi = \overline{\partial} \phi - i\beta_t \wedge \mathcal{L}_{\mathcal{T}} \phi \quad \text{if } \phi \in C^\infty(\mathcal{N} \setminus \bigwedge^q\mathcal{K}^{\ast}).
\end{equation}

The $\overline{\mathcal{D}}$ operators can be expressed in terms of the $\overline{\partial}_b$ operators. Suppose $\phi \in C^\infty(\mathcal{N} \setminus \bigwedge^q\mathcal{V}^{\ast})$. Then $\phi = \phi^0 + i\beta_t \wedge \phi^1$ with unique $\phi^0 \in C^\infty(\mathcal{N} \setminus \bigwedge^q\mathcal{K}^{\ast})$ and $\phi^1 \in C^\infty(\mathcal{N} \setminus \bigwedge^{q-1}\mathcal{K}^{\ast})$, and

$$
\overline{\mathcal{D}} \phi^0 = \overline{\partial}_b \phi^0 + i\beta_t \wedge \mathcal{L}_{\mathcal{T}} \phi^0,
$$

see (7.7). Using

$$
\overline{\mathcal{D}} \beta_t = \frac{\overline{\mathcal{D}} \phi^1}{i - \alpha_t} \wedge \beta_t
$$

and (7.7) again we get

$$
\overline{\mathcal{D}}(i\beta_t \wedge \phi^1) = i\beta_t \wedge \left( - \frac{\overline{\mathcal{D}} \phi^1}{i - \alpha_t} \wedge \phi^1 - \overline{\mathcal{D}} \phi^1 \right) = i\beta_t \wedge \left( - \frac{\overline{\partial}_b \phi^1}{i - \alpha_t} \wedge \phi^1 - \overline{\partial}_b \phi^1 \right).
$$
This gives
\[
(7.8) \quad \overline{\nabla} = \begin{bmatrix} \overline{\partial}_b & 0 & \mathcal{L}_T \\
-\overline{\partial}_b - \overline{\partial}_b a_t & i - a_t & 0 \\
\mathcal{L}_T & -\overline{\partial}_b & 0 \end{bmatrix} : \begin{array}{c} C^\infty(N; \wedge^q \mathcal{K}^*) \oplus C^\infty(N; \wedge^q+1 \mathcal{K}^*) \\ \oplus C^\infty(N; \wedge^q-1 \mathcal{K}^*) \end{array} \to \begin{array}{c} C^\infty(N; \wedge^q \mathcal{K}^*) \\ \oplus C^\infty(N; \wedge^q+1 \mathcal{K}^*) \end{array}.
\]

Since $\mathcal{T}$ itself is $\mathcal{T}$-invariant, $i_\mathcal{T} a_t^* = a_t^* i_\mathcal{T}$: the subbundle $\mathcal{K}^*$ of $\mathcal{V}^*$ is invariant under $a_t^*$ for each $t$. This need not be true of $\mathcal{K}_t$, i.e., the statement that for all $t$, $da_t(\mathcal{K}_t) \subset \mathcal{K}_t$, equivalently,
\[
L \in C^\infty(\mathcal{M}; \mathcal{K}_t) \implies [\mathcal{T}, L] \in C^\infty(\mathcal{M}; \mathcal{K}_t),
\]
may fail to hold. Since $\overline{\nabla} a_t = 0$, the formula
\[
0 = \mathcal{T}(\beta_t, L) - L(\beta_t, \mathcal{T}) - \langle \beta_t, [\mathcal{T}, L] \rangle
\]
with $L \in C^\infty(\mathcal{N}; \mathcal{K}_t)$ gives that $\mathcal{K}_t$ is invariant under $da_t$ if and only if $L a_t = 0$ for each CR vector field, that is, if and only if $a_t$ is a CR function. This proves the equivalence between the first and last statements in the following lemma. The third statement is the most useful.

**Lemma 7.9.** Let $\tau$ be a defining function for $\mathcal{N}$ in $\mathcal{M}$ and let $\overline{\partial}_b$ denote the operators of the associated CR complex. The following are equivalent:

1. The function $a_t$ is CR;
2. $\mathcal{L}_T \hat{\beta}_t = 0$;
3. $\mathcal{L}_T \overline{\partial}_b - \overline{\partial}_b \mathcal{L}_T = 0$;
4. $\mathcal{K}_t$ is $\mathcal{T}$-invariant.

**Proof.** From $\beta_t = (a_t - i) i \hat{\beta}_t$ and $\mathcal{L}_T \beta_t = \overline{\nabla} a_t$ we obtain
\[
\overline{\nabla} a_t = (\mathcal{L}_T a_t) i \hat{\beta}_t + (a_t - i) i \mathcal{L}_T \hat{\beta}_t,
\]
so
\[
\overline{\partial}_b a_t = \overline{\nabla} a_t - (\mathcal{L}_T a_t) i \hat{\beta}_t = (a_t - i) i \mathcal{L}_T \hat{\beta}_t.
\]

Thus $a_t$ is CR if and only if $\mathcal{L}_T \hat{\beta}_t = 0$.

Using $\mathcal{L}_T \overline{\nabla} = \overline{\nabla} \mathcal{L}_T$ and the definition of $\overline{\partial}_b$ we get
\[
\mathcal{L}_T \overline{\partial}_b \phi = \mathcal{L}_T (\overline{\nabla} \phi - i \hat{\beta}_t \wedge \mathcal{L}_T \phi) = \overline{\partial}_b \mathcal{L}_T \phi - i (\mathcal{L}_T \hat{\beta}_t) \wedge \mathcal{L}_T \phi
\]
for $\phi \in C^\infty(\mathcal{N}; \wedge^q \mathcal{K}^*)$. Thus $\mathcal{L}_T \overline{\partial}_b - \overline{\partial}_b \mathcal{L}_T = 0$ if and only if $\mathcal{L}_T \hat{\beta}_t = 0$. 

**Lemma 7.10.** Suppose that $\overline{\nabla}$ admits a $\mathcal{T}$-invariant metric. Then there is a defining function $\tau$ for $\mathcal{N}$ in $\mathcal{M}$ such that $a_t$ is constant. If $\tau$ and $\tau'$ are defining functions such that $a_t$ and $a_{t'}$ are constant, then $a_t = a_{t'}$. This constant will be denoted $a_{\text{av}}$.

**Proof.** Let $h$ be a metric as stated. Let $\mathcal{H}^{0,1}$ be the subbundle of $\mathcal{V}$ orthogonal to $\mathcal{T}$. This is $\mathcal{T}$-invariant, and since the metric is $\mathcal{T}$-invariant, $\mathcal{H}^{0,1}$ has a $\mathcal{T}$-invariant metric. This metric gives canonically a metric on $\mathcal{H}^{1,0} = \mathcal{H}^{0,1}$. Using the decomposition $\mathcal{C} \mathcal{T} \mathcal{N} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1} \oplus \text{span}_\mathbb{C} \mathcal{T}$ we get a $\mathcal{T}$-invariant metric on $\mathcal{C} \mathcal{T} \mathcal{N}$ for which the decomposition is orthogonal. This metric is induced by a Riemannian metric $g$. Let $m_0$ be the corresponding Riemannian density, which is $\mathcal{T}$-invariant because $g$ is. Since $\overline{\nabla}$, $\overline{\partial}$, $h$, and $m_0$ are $\mathcal{T}$-invariant, so are the formal adjoint $\overline{\nabla}$ of $\overline{\nabla}$ and the Laplacians of the $\overline{\nabla}$-complex, and if $G$ denotes the Green’s operators for these Laplacians, then $G$ is also $\mathcal{T}$-invariant, as is the orthogonal projection $\Pi$ on
the space of $\overline{\nabla}$-harmonic forms. Arbitrarily pick a defining function $r$ for $N$ in $M$. Then
\[ a_t - G\overline{\nabla} a_t = \Pi a_t \]
where $\Pi a_t$ is a constant function by Lemma 4.3. Since $\beta_t$ is $\overline{\nabla}$-closed, $\overline{\nabla} a_t = L_T \beta_t$. Thus $G\overline{\nabla} a_t = T G\overline{\nabla} \beta_t$, and since $a_t$ is real valued and $T$ is a real vector field,
\[ a_t - T R G\overline{\nabla} \beta_t = \Re \Pi a_t. \]
Extend the function $u = \Re G\overline{\nabla} \beta_t$ to $M$ as a smooth real-valued function. Then $r' = e^{-u}r$ has the required property.

Suppose that $r$, $r'$ are defining functions for $N$ in $M$ such that $a_t$ and $a_{t'}$ are constant. Then these functions are equal by Proposition 4.9.

Note that if for some $r$, the subbundle $\overline{K}_r$ is $T$-invariant and admits a $T$-invariant Hermitian metric, then there is a $T$-invariant metric on $\overline{\nabla}$.

Suppose now that $\rho : F \to M$ is a holomorphic vector bundle over $M$. Using the operators
\[ \overline{\nabla} : C^\infty(N; \Lambda^q \overline{\nabla}^* \otimes F_N) \to C^\infty(N; \Lambda^{q+1} \overline{\nabla}^* \otimes F_N), \]
see (4.12), define operators
\begin{equation}
\cdots \to C^\infty(N; \Lambda^q \overline{\nabla}^* \otimes F_N) \xrightarrow{\overline{\nabla}_b} C^\infty(N; \Lambda^{q+1} \overline{\nabla}^* \otimes F_N) \to \cdots
\end{equation}
by
\[ \overline{\nabla}_b \phi = \Pi_r \overline{\nabla} \phi, \quad \phi \in C^\infty(N; \Lambda^q \overline{\nabla}^* \otimes F_N) \]
where $\Pi_r$ means $\Pi_r \otimes I$ with $\Pi_r$ defined by (7.4). The operators (7.11) form a complex. Define also
\[ L_T = i_T \overline{\nabla} + \overline{\nabla} i_T \]
where $i_T$ stands for $i_T \otimes I$. Then
\[ i_T L_T = L_T i_T, \quad L_T \overline{\nabla} = \overline{\nabla} L_T. \]
The first of these identities implies that the image of $C^\infty(N; \Lambda^q \overline{\nabla}^* \otimes F_N)$ by $L_T$ is contained in $C^\infty(N; \Lambda^{q+1} \overline{\nabla}^* \otimes F_N)$. With these definitions, $\overline{\nabla}$ as an operator
\[ \overline{\nabla} : C^\infty(N; \Lambda^q \overline{\nabla}^* \otimes F_N) \oplus C^\infty(N; \Lambda^{q+1} \overline{\nabla}^* \otimes F_N) \to C^\infty(N; \Lambda^{q+1} \overline{\nabla}^* \otimes F_N) \]
is given by the matrix in (7.8) with the new meanings for $\overline{\nabla}_b$ and $L_T$.

Assume that there is a $T$-invariant Riemannian metric on $N$, that $\tau$ has been chosen so that $a_t$ is constant, that $\overline{K}_r$ is orthogonal to $T$, and that $T$ has unit length. Then the term involving $\partial_b a_t$ in the matrix (7.8) is absent, and since $\overline{\nabla}^2 = 0$,
\[ L_T \overline{\nabla}_b = \overline{\nabla}_b L_T. \]
Write $\eta_{\mu}$ for the metric induced on the bundles $\Lambda^q \overline{\nabla}^*$ or $\Lambda^q \overline{\nabla}^*$. If $\eta_{\mu}$, $\mu = 1, \ldots, k$ is a local frame of $F_N$ over an open set $U \subset N$ and $\phi$ is a local section of $\Lambda^q \overline{\nabla}^* \otimes F_N$ over $U$, then for some smooth sections $\phi^\nu$ of $\Lambda^q \overline{\nabla}^*$ and $\omega^\nu_\mu$ of $\overline{\nabla}^*$ over $U$,
\[ \phi = \sum_{\mu} \phi^\mu \otimes \eta_{\mu}, \quad \overline{\nabla} \sum_{\mu} \phi^\mu \otimes \eta_{\mu} = \sum_{\nu} (\overline{\nabla} \phi^\nu + \sum_{\mu} \omega^\nu_\mu \wedge \phi^\mu) \otimes \eta_{\nu}. \]
This gives
\[
\delta_b \sum_\mu \phi^\mu \otimes \eta_\mu = \sum_\nu (\overline{\partial}_b \phi^\nu + \sum_\mu \Pi_4 \omega^\mu_{\mu, \nu} \wedge \phi^\nu) \otimes \eta_\nu
\]
and
\[
\mathcal{L}_T \sum_\mu \phi^\mu \otimes \eta_\mu = \sum_\nu (\mathcal{L}_T \phi^\nu + \sum_\mu \langle \omega^\nu_{\mu, T}, T \rangle \phi^\mu) \otimes \eta_\nu.
\]

Suppose now that \( h_F \) is a Hermitian metric on \( F \). With this metric and the metric \( h_{\overline{V}} \), we get Hermitian metrics \( h \) on each of the bundles \( \Lambda^q V^* \otimes F_N \). If \( \eta_\mu \) is an orthonormal frame of \( F_N \) and \( \phi = \sum \phi^\mu \otimes \eta_\mu, \psi = \sum \psi^\mu \otimes \eta_\mu \) are sections of \( \Lambda^q V^* \otimes F_N \), then
\[
h(\phi, \psi) = \sum_\nu h_{\overline{V}}(\phi^\nu, \psi^\nu).
\]

Therefore
\[
h(\mathcal{L}_T \phi, \psi) + h(\phi, \mathcal{L}_T \psi)
\]
\[= \sum_\nu h_{\overline{V}}(\mathcal{L}_T \phi^\nu + \sum_\mu \langle \omega^\nu_{\mu, T}, T \rangle \phi^\mu, \psi^\nu) + \sum_\mu h_{\overline{V}}(\phi^\mu, \mathcal{L}_T \psi^\mu + \langle \omega^\nu_{\mu, T}, T \rangle \psi^\nu)
\]
\[= \sum_\nu (\omega^\nu_{\mu, T} + \langle \omega^\nu_{\mu, T}, T \rangle) h_{\overline{V}}(\phi^\mu, \psi^\nu) + \sum_\mu \langle \omega^\nu_{\mu, T}, T \rangle h_{\overline{V}}(\phi^\mu, \psi^\nu)
\]
\[= T h(\phi, \psi) + \sum_\mu (\omega^\mu_{\mu, T} + \langle \omega^\nu_{\mu, T}, T \rangle) h_{\overline{V}}(\phi^\mu, \psi^\nu).
\]
Thus \( T h(\phi, \psi) = h(\mathcal{L}_T \phi, \psi) + h(\phi, \mathcal{L}_T \psi) \) if and only if
\[
(7.12) \quad \langle \omega^\nu_{\mu, T}, T \rangle + \langle \omega^\nu_{\mu, T}, T \rangle = 0 \text{ for all } \mu, \nu.
\]
This condition is (4.17); just note that by the definition of \( \overline{\partial}_b \), the forms \( \Phi^* \omega^\mu_{\mu, T} \) in (4.17) are the forms that we are denoting \( \omega^\mu_{\mu, T} \) here. Thus (7.12) holds if and only if \( h_F \) is an exact Hermitian metric, see Definition (4.18).

Consequently,

Lemma 7.13. The statement
\[
(7.14) \quad T h(\phi, \psi) = h(\mathcal{L}_T \phi, \psi) + h(\phi, \mathcal{L}_T \psi) \quad \forall \phi, \psi \in C^\infty(N; \Lambda^q V^* \otimes F_N)
\]
holds if and only the Hermitian metric \( h_F \) is exact.

8. Spectrum

Suppose that \( \overline{V} \) admits an invariant Hermitian metric. Let \( r \) be a defining function of \( N \) in \( M \) such that \( a_\epsilon \) is constant. By Lemma (7.10), \( \overline{\mathcal{K}}_q \) is \( \mathcal{T} \)-invariant, so the restriction of the metric to this subbundle gives a \( \mathcal{T} \)-invariant metric; we use the induced metric on the bundles \( \Lambda^q \overline{\mathcal{K}}^* \) in the following. As in the proof of Lemma (7.10) there is a \( \mathcal{T} \)-invariant density \( m_0 \) on \( N \).

Let \( \rho: F \to M \) be a Hermitian holomorphic vector bundle, assume that the Hermitian metric of \( F \) is exact, so with the induced metric \( h \) on the vector bundles \( \Lambda^q \overline{\mathcal{K}}^* \otimes F_N \), (7.14) holds. We will write \( F \) in place of \( F_N \).

Let \( \overline{\partial}_b \) be the formal adjoint of the \( \overline{\partial}_b \) operator (7.11) with respect to the inner on the bundles \( \Lambda^q \overline{\mathcal{K}}^* \otimes F \) and the density \( m_0 \), and let \( \square_{b,q} = \overline{\partial}_b \overline{\partial}_b + \overline{\partial}_b \overline{\partial}_b \) be the
formal \( \overline{\partial}_b \)-Laplacian. Since \( -iL_\mathcal{T} \) is formally selfadjoint and commutes with \( \overline{\partial}_b \), \( L_\mathcal{T} \) commutes with \( \Box_{b,q} \). Let 

\[
\mathcal{H}_\partial^q(N; F) = \ker \Box_{b,q} = \{ \phi \in L^2(N; \Lambda^q \overline{\mathbb{C}}^r \otimes F) : \Box_{b,q} \phi = 0 \}
\]

and let 

\[
\text{Dom}_q(L_\mathcal{T}) = \{ \phi \in \mathcal{H}_\partial^q(N; F) \text{ and } L_\mathcal{T} \phi \in \mathcal{H}_\partial^q(N; F) \}. 
\]

The spaces \( \mathcal{H}_\partial^q(N; F) \) may be of infinite dimension, but in any case they are closed subspaces of \( L^2(N; \Lambda^q \overline{\mathbb{C}}^r \otimes F) \), so they may be regarded as Hilbert spaces on their own right. If \( \phi \in \mathcal{H}_\partial^q(N; F) \), the condition \( L_\mathcal{T} \phi \in \mathcal{H}_\partial^q(N; F) \) is equivalent to the condition 

\[
L_\mathcal{T} \phi \in L^2(N; \Lambda^q \overline{\mathbb{C}}^r \otimes F). 
\]

So we have a closed operator

\[
(8.1) \quad -iL_\mathcal{T} : \text{Dom}_q(L_\mathcal{T}) \subset \mathcal{H}_\partial^q(N; F) \to \mathcal{H}_\partial^q(N; F).
\]

The fact that \( \Box_{b,q} - L_\mathcal{T}^2 \) is elliptic, symmetric, and commutes with \( L_\mathcal{T} \) implies that \( (8.1) \) is a selfadjoint Fredholm operator with discrete spectrum (see [13, Theorem 2.5]).

**Definition 8.2.** Let \( \text{spec}_0^q(-iL_\mathcal{T}) \) be the spectrum of the operator \( (8.1) \), and let \( \mathcal{H}_\partial^q(N; F) \) be the eigenspace of \( -iL_\mathcal{T} \) in \( \mathcal{H}_\partial^q(N; F) \) corresponding to the eigenvalue \( \tau \).

Let \( \tau \) denote the principal symbol of \( -i\mathcal{T} \). Then the principal symbol of \( L_\mathcal{T} \) acting on sections of \( \Lambda^q \overline{\mathbb{C}}^r \) is \( \tau I \). Because \( \Box_{b,q} - L_\mathcal{T}^2 \) is elliptic, \( \text{Char}(\Box_{b,q}) \), the characteristic variety of \( \Box_{b,q} \), lies in \( \tau \neq 0 \). Let 

\[
\text{Char}^\pm(\Box_{b,q}) = \{ \nu \in \text{Char}(\Box_{b,q}) : \tau(\nu) \geq 0 \}.
\]

By [13, Theorem 4.1], if \( \Box_{b,q} \) is microlocally hypoelliptic on \( \text{Char}^+(\Box_{b,q}) \), then 

\[
\{ \tau \in \text{spec}_0^q(-iL_\mathcal{T}) : \tau \geq 0 \}
\]

is finite. We should perhaps point out that \( \text{Char}(\Box_{b,q}) \) is equal to the characteristic variety, \( \text{Char}(\overline{\mathbb{C}}^r) \), of the CR structure.

As a special case consider the situation where \( F \) is the trivial line bundle. Let \( \theta_\tau \) be the real 1-form on \( N \) which vanishes on \( \overline{\mathbb{C}}^r \) and satisfies \( (\theta_\tau, \mathcal{T}) = 1 \); thus \( \theta_\tau \) is smooth, spans \( \text{Char}(\overline{\mathbb{C}}^r) \), and has values in \( \text{Char}^+(\overline{\mathbb{C}}^r) \). The Levi form of the structure is

\[
\text{Levi}_{\theta_\tau}(v, w) = -i\theta_\tau(v, w), \quad v, w \in \mathbb{K}_{c, p}, \quad p \in N.
\]

Suppose that \( \text{Levi}_{\theta_\tau} \) is nondegenerate, with \( k \) positive and \( n-k \) negative eigenvalues. It is well known that then \( \Box_{b,q} \) is microlocally hypoelliptic at \( \nu \in \text{Char}(\mathbb{K}_{c,p}) \) for all \( q \) except if \( q = k \) and \( \tau(\nu) < 0 \) or if \( q = n-k \) and \( \tau(\nu) > 0 \).

Then the already mentioned Theorem 4.1 of [13] gives:

**Theorem 8.3** ([13, Theorem 6.1]). Suppose that \( \nabla \) admits a Hermitian metric and that for some defining function \( \mathfrak{r} \) such that \( \mathfrak{a}_{\mathfrak{r}} \) is constant, \( \text{Levi}_{\theta_{\mathfrak{r}}} \) is nondegenerate with \( k \) positive and \( n-k \) negative eigenvalues. Then

1. \( \text{spec}_0^q(-iL_\mathcal{T}) \) is finite if \( q \neq k, n-k \);
2. \( \text{spec}_{n-k}^q(-iL_\mathcal{T}) \) contains only finitely many positive elements, and
3. \( \text{spec}_{n-k}^q(-iL_\mathcal{T}) \) contains only finitely many negative elements.
9. Indicial Cohomology

Suppose that there is a $T$-invariant Hermitian metric $\hat{h}$ on $\mathcal{V}$. By Lemma [7.10] there is a defining function $r$ such that $\langle \beta_r, T \rangle$ is constant, equal to $a_{av}-i$. Therefore $K_r$ is $T$-invariant. Let $h$ be the metric on $\mathcal{V}$ which coincides with $\hat{h}$ on $K_r$, makes the decomposition $\mathcal{V} = K_r \oplus \text{span}_c T$ orthogonal, and for which $T$ has unit length. The metric $h$ is $T$-invariant. We fix $r$ and such a metric, and let $m_0$ be the Riemannian measure associated with $h$. The decomposition (9.1) of $\wedge^q \mathcal{V}$ is an orthogonal decomposition.

Recall that $\mathcal{D}(\sigma)\phi = \Box \phi + i\sigma \beta \wedge \phi$. Since $a_t = a_{av}$ is constant (in particular CR),

$$
\mathcal{D}(\sigma)(\phi^0 + i\beta_\tau \wedge \phi^1) = \overline{\partial}_h \phi^0 + i\beta_\tau \wedge \left[ (\mathcal{L}_T + (1 + ia_{av})\sigma)\phi^0 - \overline{\partial}_h \phi^1 \right]
$$

if $\phi^0 \in C^\infty(N; \wedge^q K_r)$ and $\phi^1 \in C^\infty(N; \wedge^{q-1} K_r)$. So $\mathcal{D}(\sigma)$ can be regarded as the operator

$$
(9.1) \quad \mathcal{D}(\sigma) = \begin{bmatrix} \overline{\partial}_h & 0 \\ -\mathcal{L}_T + (1 + ia_{av})\sigma & -\overline{\partial}_h \end{bmatrix} : C^\infty(N; \wedge^q K_r) \oplus C^\infty(N; \wedge^{q-1} K_r) \rightarrow C^\infty(N; \wedge^q K_r) \oplus C^\infty(N; \wedge^{q-1} K_r).
$$

Since the subbundles $\wedge^q K_r$ and $\wedge^{q-1} K_r$ are orthogonal with respect to the metric induced by $h$ on $\wedge^q \mathcal{V}$, the formal adjoint of $\mathcal{D}(\sigma)$ with respect to this metric and the density $m_0$ is

$$
\mathcal{D}(\sigma)^* = \begin{bmatrix} \overline{\partial}_h & -\mathcal{L}_T + (1 - ia_{av})\sigma \\ 0 & -\overline{\partial}_h \end{bmatrix} : C^\infty(N; \wedge^q K_r) \oplus C^\infty(N; \wedge^{q-1} K_r) \rightarrow C^\infty(N; \wedge^{q-1} K_r) \oplus C^\infty(N; \wedge^q K_r),
$$

where $\overline{\partial}_h$ is the formal adjoint of $\overline{\partial}_h$. So the Laplacian, $\Box_{(\sigma), q}$, of the $\mathcal{D}(\sigma)$-complex is the diagonal operator with diagonal entries $P_q(\sigma), P_{q-1}(\sigma)$ where

$$
P_q(\sigma) = \Box_{b,q} + (\mathcal{L}_T + (1 + ia_{av})\sigma)(-\mathcal{L}_T + (1 - ia_{av})\sigma)
$$

acting on $C^\infty(N; \wedge^q K_r)$ and $P_{q-1}(\sigma)$ is the “same” operator, acting on sections of $\wedge^{q-1} K_r$; recall that $\mathcal{L}_T$ commutes with $\overline{\partial}_h$ and since $\mathcal{L}_{T}^2 = -\mathcal{L}_T$, also with $\overline{\partial}_h$, and that $a_{av}$ is constant. Note that $P_q(\sigma)$ is an elliptic operator.

Suppose that $\phi \in C^\infty(N; \wedge^q K_r)$ is a nonzero element of $\ker P_q(\sigma)$; the complex number $\sigma$ is fixed. Since $P_q(\sigma)$ is elliptic, $\ker P_q(\sigma)$ is a finite dimensional space, invariant under $-\mathcal{L}_T$ since the latter operator commutes with $P_q(\sigma)$. As an operator on $\ker P_q(\sigma)$, $-\mathcal{L}_T$ is selfadjoint, so there is a decomposition of $\ker P_q(\sigma)$ into eigenspaces of $-\mathcal{L}_T$. Thus

$$
\phi = \sum_{j=1}^{N} \phi_j, \quad -\mathcal{L}_T \phi_j = \tau_j \phi_j
$$

where the $\tau_j$ are distinct real numbers and $\phi_j \in \ker P_q(\sigma), \phi_j \neq 0$. In particular,

$$
\Box_{b,q} \phi_j + (\mathcal{L}_T + (1 + ia_{av})\sigma)(-\mathcal{L}_T + (1 - ia_{av})\sigma)\phi_j = 0,
$$

for each $j$, that is,

$$
\Box_{b,q} \phi_j + |i\tau_j + (1 + ia_{av})\sigma|^2 \phi_j = 0.
$$

Since $\Box_{b,q}$ is a nonnegative operator and $\phi_j \neq 0$, $i\tau_j + (1 + ia_{av})\sigma = 0$ and $\phi_j \in \ker \Box_{b,q}$. Since $\sigma$ is fixed, all $\tau_j$ are equal, which means that $N = 1$. Conversely, if
\( \phi \in C^\infty(\mathcal{N}; \Lambda^n \mathcal{E}) \) belongs to \( \ker \Box_{b,q} \) and \( -iL_T \phi = \tau \phi \), then \( P_q(\sigma)\phi = 0 \) with \( \sigma \) such that \( \tau = (i - a_{av})\sigma \).

Let \( \mathcal{H}^q_{\Box(\sigma)}(\mathcal{N}) \) be the kernel of \( \Box_{\mathcal{H}(\sigma),q} \).

**Theorem 9.2.** Suppose that \( \nabla \) admits a \( T \)-invariant metric and let \( \tau \) be a defining function for \( \mathcal{N} \) in \( \mathcal{M} \) such that \( (\beta, T) = a_{av} - i \) is constant. Then

\[ \text{spec}^q_{b,N}(\nabla) = (i - a_{av})^{-1} \text{spec}^0_{b,N}(-iL_T) \cup (i - a_{av})^{-1} \text{spec}^{-1}_{0,-1}(-iL_T), \]

and if \( \sigma \in \text{spec}^q_{b,N}(\nabla) \), then, with the notation in Definition 8.2

\[ \mathcal{H}^q_{\Box(\sigma)}(\mathcal{N}) = \mathcal{H}^q_{\tau_{\mathcal{N}}(\sigma)}(\mathcal{N}) \oplus \mathcal{H}^{q-1}_{\partial_{\mathcal{N}}(\tau_{\mathcal{N}})}(\mathcal{N}) \]

with \( \tau(\sigma) = (i - a_{av})\sigma \).

If the CR structure \( \mathcal{E} \) is nondegenerate, Proposition 9.3 gives more specific information on \( \text{spec}^0_{b,N}(\nabla) \). In particular,

**Proposition 9.3.** With the hypotheses of Theorem 9.2, suppose that Levi-\( b \) is nondegenerate with \( k \) positive and \( n - k \) negative eigenvalues. If \( k > 0 \), then \( \text{spec}^0_{b,N} \subset \{ \sigma \in \mathbb{C} : 3\sigma \leq 0 \} \), and if \( n - k > 0 \), then \( \text{spec}^0_{b,N}(\nabla) \subset \{ \sigma \in \mathbb{C} : 3\sigma \geq 0 \} \).

**Remark 9.4.** The \( b \)-spectrum of the Laplacian of the \( \nabla \)-complex in any degree can be described explicitly in terms of the joint spectra \( \text{spec}(-iL_T, \Box_{b,q}) \). We briefly indicate how. With the metric \( h \) and defining function \( \tau \) as in the first paragraph of this section, suppose that \( h \) is extended to a metric on \( T^{0,1} \mathcal{M} \). This gives a Riemannian \( b \)-metric on \( \mathcal{M} \) that in turn gives a \( b \)-density \( m \) on \( \mathcal{M} \). With these we get formal adjoints \( \overline{\nabla}^* \) whose indicial families \( \overline{\nabla}^*(\sigma) \) are related to those of \( \nabla \) by

\[ \overline{\nabla}^*(\sigma) = \nabla^*(\sigma) = (\nabla^*(\tau))^* = \nabla^*(\sigma). \]

By 9.31,

\[ \overline{\nabla}^*(\sigma) = \begin{bmatrix} \nabla \sigma - L_T + (1 - i a_{av})\sigma \\ 0 \\ -\nabla \sigma \end{bmatrix}. \]

Using this one obtains that the indicial family of the Laplacian \( \Box_{b,q} \) of the \( \nabla \)-complex in degree \( q \) is a diagonal operator with diagonal entries \( P'_q(\sigma), P'_{q-1}(\sigma) \) with

\[ P'_q(\sigma) = \Box_{b,q} + (L_T + (1 + i a_{av})\sigma)(-L_T + (1 - i a_{av})\sigma) \]

and the analogous operator in degree \( q - 1 \). The set \( \text{spec}_{b,q}(\Box_{b,q}) \) is the set of values of \( \sigma \) for which either \( P'_q(\sigma) \) or \( P'_{q-1}(\sigma) \) is not injective. These points can be written in terms of the points \( \text{spec}(-iL_T, \Box_{b,q}) \) as asserted. In particular one gets

\[ \text{spec}_{b,N}(\Box_{b,q}) \subset \{ \sigma : |\Re\sigma| \leq |a_{av}|(3|\Im\sigma|) \} \]

with \( \text{spec}^q_{b,N}(\nabla) \) being a subset of the boundary of the set on the right.

We now discuss the indicial cohomology sheaf of \( \nabla \), see Definition 6.12. We will show:

**Proposition 9.5.** Let \( \sigma_0 \in \text{spec}^q_{b,N}(\nabla) \). Every element of the stalk of the sheaf \( \mathcal{H}^q_{\nabla}(\mathcal{N}) \) at \( \sigma_0 \) has a representative of the form

\[ \frac{1}{\sigma - \sigma_0} \begin{bmatrix} \phi^0 \\ 0 \end{bmatrix} \]

where \( \phi^0 \in \mathcal{H}^q_{\partial_{\mathcal{N}},\tau_0}(\mathcal{N}), \tau_0 = (i - a_{av})\sigma_0. \)
Proof. Let

\[ \phi(\sigma) = \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \left[ \phi^0_k \phi^1_k \right] \]

represent an element in the stalk at \( \sigma_0 \) of the sheaf of germs of \( C^\infty(\mathcal{N}; \bigwedge^* \mathcal{V} \otimes F) \)-valued meromorphic functions on \( \mathbb{C} \) modulo the subsheaf of holomorphic elements. Letting \( \alpha = 1 + ia_{\text{av}} \) we have

\[ \overline{D}(\sigma)\phi(\sigma) = \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \left[ (L_T + \alpha \sigma_0) \phi^0_k - \overline{\partial}_b \phi^1_k \right] + \sum_{k=0}^{\mu-1} \frac{\alpha}{(\sigma - \sigma_0)^k} \left[ \phi^0_{k+1} \right], \]

so the condition that \( \overline{D}(\sigma)\phi(\sigma) \) is holomorphic is equivalent to

\[ \overline{D}_b \phi^0_k = 0, \quad k = 1, \ldots, \mu \]

and

\[ L_T + \alpha \sigma_0 \phi^0_k - \overline{\partial}_b \phi^1_k = 0, \quad k = 1, \ldots, \mu - 1. \]

Let \( P_{q'} = \Box_{b,q'} - L^2 \) in any degree \( q' \). For any \( (\tau, \lambda) \in \mathbb{R}^2 \) and \( q' \) let

\[ \mathcal{E}^q_{\tau, \lambda} = \{ \psi \in C^\infty(\mathcal{N}; \bigwedge^q \mathcal{V} \otimes F) : P_{q'} \psi = \lambda \psi, \quad -iL_T \psi = \tau \psi \}. \]

This space is zero if \( (\tau, \lambda) \) is not in the joint spectrum \( \Sigma^{q'} = \text{spec}^{q'}(-iL_T, P_{q'}) \). Each \( \phi^i_k \) decomposes as a sum of elements in the spaces \( \mathcal{E}^{q-i}_{\tau, \lambda} \), \( (\tau, \lambda) \in \Sigma^{q-i} \). Suppose that already \( \phi^i_k \in \mathcal{E}^{q-i}_{\tau, \lambda} \):

\[ P_{q-i} \phi^i_k = \lambda \phi^i_k, \quad -iL_T \phi^i_k = \tau \phi^i_k, \quad i = 0, 1, k = 1, \ldots, \mu. \]

Then \( \text{(9.8)} \) becomes

\[ \begin{align*}
(\tau + \alpha \sigma_0) \phi^0_k - \overline{\partial}_b \phi^1_k &= 0, \\
(\tau + \alpha \sigma_0) \phi^0_k - \overline{\partial}_b \phi^1_k + \alpha \phi^0_{k+1} &= 0, \quad k = 1, \ldots, \mu - 1.
\end{align*} \]

If \( \tau \neq \tau_0 \), then \( \tau + \alpha \sigma_0 \neq 0 \), and we get \( \phi^0_k = \overline{\partial}_b \psi^0_k \) for all \( k \) with

\[ \psi^0_k = \sum_{j=0}^{\mu-k} \frac{(-\alpha)^j}{i(\tau + \alpha \sigma_0)^{j+1}} \phi^1_{k+j}. \]

Trivially

\[ (L_T + \alpha \sigma_0) \psi^0_k = \phi^1_k \]

and also

\[ (L_T + \alpha \sigma_0) \psi^0_k + \alpha \psi^0_{k+1} = \phi^1_k, \quad k = 1, \ldots, \mu - 1, \]

so

\[ \phi(\sigma) - \overline{D}(\sigma) \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \left[ \psi^0_k \right] 0 = 0 \]

modulo an entire element.

Suppose now that the \( \phi^i_k \) are arbitrary and satisfy \( \text{(9.7)}-\text{(9.8)} \). The sum

\[ \phi^i_k = \sum_{(\tau, \lambda) \in \Sigma^{q-i}} \phi^i_{k,\tau,\lambda}, \quad \phi^i_{k,\tau,\lambda} \in \mathcal{E}^{q-i}_{\tau, \lambda} \]
converges in $C^\infty$, indeed for each $N$ there is $C_{i,k,N}$ such that
\begin{equation}
(9.11) \quad \sup_{p \in N} \|\phi^i_{k,\tau,\lambda}(p)\| \leq C_{i,k,N} (1 + \lambda)^{-N} \quad \text{for all } \tau, \lambda.
\end{equation}

Since $D(\sigma)$ preserves the spaces $E^q_{\tau,\lambda} \oplus E^{q-1}_{\tau,\lambda}$, the relations \((9.9)\) hold for the $\phi^i_{k,\tau,\lambda}$ for each $(\tau, \lambda)$. Therefore, with
\begin{equation}
(9.12) \quad \psi^0_k = \sum\limits_{(\tau, \lambda) \in \Sigma^{q-1}} \int_{\tau \neq \tau_0} \frac{(-\alpha)^j}{(i\tau + \alpha \sigma_0)^{j+1}} \phi^1_{k,j+\tau,\lambda}
\end{equation}
we have formally that
\begin{equation}
\phi(\sigma) - D(\sigma) \sum\limits_{k=1}^\mu \frac{1}{(\sigma - \sigma_0)^k} \left[ \psi^0_k \right]_0 = \sum\limits_{k=1}^\mu \frac{1}{(\sigma - \sigma_0)^k} \left[ \phi^0_k \right]
\end{equation}
with
\begin{equation}
(9.13) \quad \tilde{\phi}^i_k = \sum\limits_{(\tau, \lambda) \in \Sigma^{q-1}} \phi^i_{k,\tau,\lambda}, \quad \phi^i_{k,\tau,\lambda} \in E^{q-i}_{\tau,\lambda}.
\end{equation}

However, the convergence in $C^\infty$ of the series \((9.12)\) is questionable since there may be a sequence $\{(\tau_\ell, \lambda_\ell)\}_{\ell=1}^\infty \subset \text{spec}(-iL_T, P_{q-1})$ of distinct points such that $\tau_\ell \to \tau_0$ as $\ell \to \infty$, so that the denominators $i\tau_\ell + \alpha \sigma_0$ in the formula for $\psi^0_k$ tend to zero so fast that for some nonnegative $N$, $\lambda^{(N)}_{\ell} = [i\tau_\ell + \alpha \sigma_0]$ is unbounded. To resolve this difficulty we will first show that $\phi(\sigma)$ is $D(\sigma)$-cohomologous (modulo holomorphic terms) to an element of the same form as $\phi(\sigma)$ for which in the series \((9.11)\) the terms $\phi^i_{k,\tau,\lambda}$ vanish if $\lambda - \tau^2 > \varepsilon$; the number $\varepsilon > 0$ is chosen so that
\begin{equation}
(9.14) \quad (\tau_0, \lambda) \in \Sigma^q \cup \Sigma^{q-1} \implies \lambda = \tau_0^2 \text{ or } \lambda \geq \tau_0^2 + \varepsilon.
\end{equation}

Recall that $\text{spec}q(-iL_T, P_{q'}) \subset \{(\tau, \lambda) : \lambda \geq \tau^2\}$.

For any $V \subset \bigcup_{q'} \Sigma^q$ let
\begin{equation}
\Pi_{q'}^V : L^2(N; \Lambda^q \mathcal{V}^* \otimes F) \to L^2(N; \Lambda^q \mathcal{V}^* \otimes F)
\end{equation}
be the orthogonal projection on $\bigoplus_{(\tau, \lambda) \in V} E^q_{\tau,\lambda}$. If $\psi \in C^\infty(N; \Lambda^q \mathcal{V}^* \otimes F)$, then the series
\begin{equation}
\Pi_{q'}^V \psi = \sum\limits_{(\tau, \lambda) \in V} \psi_{\tau,\lambda}, \quad \psi_{\tau,\lambda} \in E^q_{\tau,\lambda}
\end{equation}
converges in $C^\infty$. It follows that $\Box_{b,q'}$ and $L_T$ commute with $\Pi_{q'}^V$ and that $\Box_{b} \Pi_{q'}^V = \Pi_{q'}^{V+1} \Box_{b}$. Since the $\Pi_{q'}^V$ are selfadjoint, also $\Box_{b} \Pi_{q'}^{V+1} = \Pi_{q'}^{V+1} \Box_{b}$.

Let
\begin{equation}
U = \{(\tau, \lambda) \in \Sigma^q \cup \Sigma^{q-1} : \lambda < \tau^2 + \varepsilon\}, \quad U^c = \Sigma^q \cup \Sigma^{q-1} \setminus U.
\end{equation}
Then, for any sequence $\{(\tau_\ell, \lambda_\ell)\} \subset U$ of distinct points we have $|\tau_\ell| \to \infty$ as $\ell \to \infty$. Define
\begin{equation}
G_{U^c}^{q'} \psi = \sum\limits_{(\tau, \lambda) \in U^c} \frac{1}{\lambda - \tau^2} \psi_{\tau,\lambda}
\end{equation}
In this definition the denominators $\lambda - \tau^2$ are bounded from below by $\varepsilon$, so $G_{U^c}^{q'}$ is a bounded operator in $L^2$ and maps smooth sections to smooth sections because
the components of such sections satisfy estimates as in (9.11). The operators are analogous to Green operators: we have

\[ \square_{b,q'} G_{U_\sigma}^{q'} = G_{U_\sigma}^{q'} \square_{b,q'} = I - \Pi_{U_\sigma}^{q'} \]

so if \( \overline{\partial}_b \psi = 0 \), then

\[ \square_{b,q'} G_{U_\sigma}^{q'} \psi = \overline{\partial}_b \partial_b G_{U_\sigma}^{q'} \psi \]

since \( \overline{\partial}_b G_{U_\sigma}^{q'} = G_{U_\sigma}^{q'+1} \overline{\partial}_b \).

Write \( \phi(\sigma) \) in (9.6) as

\[ \phi(\sigma) = \Pi_{U_\sigma} \phi(\sigma) + \Pi_U \phi(\sigma) \]

where

\[ \Pi_{U_\sigma} \phi(\sigma) = \sum_{k=1}^\mu \frac{1}{(\sigma - \sigma_0)^k} \left[ \frac{\Pi_{U_\sigma}^0 \phi_k^0}{\Pi_{U_\sigma}^{-1} \phi_k} \right], \quad \Pi_U \phi(\sigma) = \sum_{k=1}^\mu \frac{1}{(\sigma - \sigma_0)^k} \left[ \frac{\Pi_U^0 \phi_k^0}{\Pi_U^{-1} \phi_k} \right]. \]

Since \( \overline{\partial}(\sigma) \phi(\sigma) \) is holomorphic, so are \( \overline{\partial}(\sigma) \Pi_{U_\sigma} \phi(\sigma) \) and \( \overline{\partial}(\sigma) \Pi_U \phi(\sigma) \).

We show that \( \Pi_{U_\sigma} \phi(\sigma) \) is exact modulo holomorphic functions. Using (9.7), (9.15), and (9.16), \( \Pi_{U_\sigma}^0 \phi_k^0 = \overline{\partial}_b \partial_b \Pi_U \phi_k^0 \).

Then

\[ \Pi_{U_\sigma} \phi(\sigma) - \overline{\partial}(\sigma) \sum_{k=1}^\mu \frac{1}{(\sigma - \sigma_0)^k} \left[ \overline{\partial}_b G_{U_\sigma}^{q'} \Pi_U^0 \phi_k^0 \right] = \sum_{k=1}^\mu \frac{1}{(\sigma - \sigma_0)^k} \left[ \frac{0}{\phi_k^1} \right] \]

modulo a holomorphic term for some \( \hat{\phi}_k^j \) with \( \Pi_{U_\sigma}^{q'-1} \hat{\phi}_k^j = \hat{\phi}_k^j \). The element on the right is \( \overline{\partial}(\sigma) \)-closed modulo a holomorphic function, so its components satisfy (9.7), (9.8), which give that the \( \hat{\phi}_k^j \) are \( \overline{\partial}_b \)-closed. Using again (9.15) and (9.16) we see that \( \Pi_{U_\sigma} \phi(\sigma) \) represent an exact element.

We may thus assume that \( \Pi_{U_\sigma}^0 \phi(\sigma) = 0 \). If this is the case, then the series (9.12) converges in \( C^\infty \), so \( \phi(\sigma) \) is cohomologous to the element

\[ \hat{\phi}(\sigma) = \sum_{k=1}^\mu \frac{1}{(\sigma - \sigma_0)^k} \left[ \hat{\phi}_k^0 \hat{\phi}_k^1 \right] \]

where the \( \hat{\phi}_k^j \) are given by (9.13) and satisfy \( \Pi_{U_\sigma}^{q'-1} \hat{\phi}_k^j = 0 \). By (9.14), \( \hat{\phi}_k^j \in \mathcal{E}_{\tau_0,\tau_0}^{q'-i} \).

In particular, \( \square_{b,q-i} \phi_k^j = 0 \).

Assuming now that already \( \phi_k^j \in \mathcal{E}_{\tau_0,\tau_0}^{q'-i} \), the formulas (9.19) give (since \( \tau = \tau_0 \) and \( i\tau_0 + \alpha \sigma_0 = 0 \))

\[ \overline{\partial}_b \phi_k^j = 0, \quad \phi_k^j = \overline{\partial}_b \frac{1}{\alpha} \phi_k^{j-1}, \quad k = 2, \ldots, \mu. \]

Then

\[ \phi(\sigma) - \frac{1}{\alpha} \overline{\partial}(\sigma) \sum_{k=2}^{\mu+1} \frac{1}{(\sigma - \sigma_0)^k} \left[ \frac{\phi_k^{j-1}}{\alpha} \right] = \frac{1}{\sigma - \sigma_0} \left[ \phi_0^0 \right] \]

with \( \square_{b,q} \phi_0^0 = 0. \)

\( \square \)
Appendix A. Totally characteristic differential operators

We review here some basic definitions and notation concerning totally characteristic differential operators.

Let \( E, F \to M \) be vector bundles and let \( \text{Diff}^m(M; E, F) \) be the space of differential operators \( \mathcal{C}^\infty(M; E) \to \mathcal{C}^\infty(M; F) \) of order \( m \). Then

\[
\text{Diff}^m_b(M; E, F), \quad \text{the space of totally characteristic differential operators of order } m,
\]

(A.1)

consists of those elements \( P \in \text{Diff}^m(M; E, F) \) with the property

\[
\tau^{-\nu}P\tau^\nu \in \text{Diff}^m(M; E, F), \quad \nu = 1, \ldots, m
\]
i.e., \( \tau^{-\nu}P\tau^\nu \) has coefficients smooth up to the boundary.

Let \( \pi : T^*M \to M \) and \( b\pi : bT^*M \to M \) be the canonical projections. Suppose \( P \in \text{Diff}^m_b(M; E, F) \). Since \( P \) is in particular a differential operator, it has a principal symbol

\[
\sigma(P) \in \mathcal{C}^\infty(T^*M; \text{Hom}(\pi^*E, \pi^*F)).
\]

The fact that \( P \) is totally characteristic implies that \( \sigma(P) \) lifts to a section

\[
b\sigma(P) \in \mathcal{C}^\infty(bT^*M; \text{Hom}(b\pi^*E, b\pi^*F)),
\]
the principal \( b \)-symbol of \( P \), characterized by

(A.2)

\[
b\sigma(P)(ev^*\xi) = \sigma(P)(\xi).
\]

If \( P \in \text{Diff}^m_b(M; E, F) \), then \( P \) induces a differential operator

(A.3)

\[
P_b \in \text{Diff}^m_b(M; E_{\partial M}, F_{\partial M}),
\]
as follows. If \( \phi \in \mathcal{C}^\infty(\partial M; E_{\partial M}) \), let \( \tilde{\phi} \in \mathcal{C}^\infty(M; E) \) be an extension of \( \phi \) and let

\[
P_b\phi = (P\tilde{\phi})|_{\partial M}.
\]

The condition (A.1) ensures that \( P\tilde{\phi}|_{\partial M} \) is independent of the extension of \( \phi \) used. Clearly if \( P \) and \( Q \) are totally characteristic differential operators, then so is \( PQ \), and

(A.4)

\[
(PQ)_b = P_bQ_b.
\]

The indicial family of \( P \in \text{Diff}^m_b(M; E, F) \) is defined as follows. Fix a defining function \( \tau \) for \( \partial M \). Then for any \( \sigma \in \mathbb{C} \),

\[
P(\sigma) = \tau^{-i\sigma}P\tau^{i\sigma} \in \text{Diff}^m_b(M; E, F).
\]

Let

(A.5)

\[
\hat{P}(\sigma) = P(\sigma)_b.
\]

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