Research Article

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Sharper existence and uniqueness results for solutions to fourth-order boundary value problems and elastic beam analysis

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Abstract: We examine the existence and uniqueness of solutions to two-point boundary value problems involving fourth-order, ordinary differential equations. Such problems have interesting applications to modelling the deflections of beams. We sharpen traditional results by showing that a larger class of problems admit a unique solution. We achieve this by drawing on fixed-point theory in an interesting and alternative way via an application of Rus’s contraction mapping theorem. The idea is to utilize two metrics on a metric space, where one pair is complete. Our theoretical results are applied to the area of elastic beam deflections when the beam is subjected to a loading force and the ends of the beam are either both clamped or one end is clamped and the other end is free. The existence and uniqueness of solutions to the models are guaranteed for certain classes of linear and nonlinear loading forces.

Keywords: fourth-order boundary value problem, existence and uniqueness of solutions, beam analysis, Rus’s contraction mapping theorem, two metrics

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1 Introduction

This paper studies the existence and uniqueness of solutions to the following nonlinear, fourth-order differential equation:

$$y^{(iv)} = f(x, y, y', y'', y'''), \quad x \in [0, 1],$$  \hspace{1cm} (1.1)

where $f : \Omega \subseteq [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ is assumed to be continuous, and (1.1) is subject to either of the two-point boundary conditions:

$$y(0) = a, \quad y'(0) = c, \quad y(1) = b, \quad y'(1) = d;$$  \hspace{1cm} (1.2)

$$y(0) = a, \quad y'(0) = c, \quad y''(1) = e, \quad y'''(1) = h;$$  \hspace{1cm} (1.3)

and $a, b, c, d, e, h$ are given constants in $\mathbb{R}$.

A natural motivation for the investigation of fourth-order boundary value problems (BVPs) arises in the analysis of elastic beam deflections. Consider a beam occupying the interval $[0, 1]$ with $x$ denoting the position along the beam. The beam is subjected to certain forces, and if $y = y(x)$ represents the resultant
deflection of the beam at position $x$, then the equation of motion leads to differential Eq. (1.1). In simplified situations, the problem is subjected to either of the boundary conditions:

$$y(0) = 0, \ y'(0) = 0; \quad y(1) = 0, \ y'(1) = 0; \quad (1.4)$$

$$y(0) = 0, \ y'(0) = 0; \quad y''(1) = 0, \ y'''(1) = 0. \quad (1.5)$$

The boundary condition (1.4) may be interpreted in a physical sense as the beam having clamped ends at $x = 0$ and at $x = 1$, while (1.5) may be interpreted in a material sense as the beam having a clamped end at $x = 0$ and a free end at $x = 1$.

Fourth-order BVPs and their application to elastic beam deflections have been studied by many researchers. Indeed, entire monographs have appeared regarding the field such as [1,2]. While an exhaustive literature survey is beyond the scope of the present paper due to space constraints, let us situate our current work within the field of research by discussing its differences and connections with recent and noteworthy publications in the area. We analyse the following four dimensions: the type of problem under consideration, the assumptions imposed, the methods employed, and the nature of the results obtained.

Observe that the differential equation under consideration (1.1) in our present paper features a scalar-valued, fully nonlinear right-hand side that depends on each of the lower-order derivatives. This is in contrast to such works as [3–6], where either $f$ does not depend in a nonlinear way on each of the derivatives $y, y', y''$ and $y'''$ or a system of equations were considered. The two sets of boundary conditions (1.2) and (1.3) that we consider herein differ in form from those considered by [4–10]. Thus, the problem under consideration in our present paper is distinct from the aforementioned works.

On the other hand, the fully nonlinear problem (1.1), (1.5) was analysed in [11–13]. The assumptions on $f$ in [11,12] are of a local nature, that is, the domain of $f$ is restricted to closed and bounded sets. While these types of assumptions are quite wide-ranging, the very nature of localized assumptions means that only limited, localized information about solutions can be necessarily obtained. For instance, in the context of localized assumptions, nothing can be concluded about the existence and uniqueness of solutions that may lie outside of the closed and bounded set that is under consideration. This is especially important as multiple solutions to fourth-order BVPs have been shown to exist [14].

Herein, we provide an analysis in both global and local settings. In doing so, we obtain more complete and balanced knowledge in our conclusions regarding the existence and uniqueness of solutions.

In [13], the approach involved obtaining the existence of solutions via the application of fixed point index theory in cones. In contrast, our methods herein involve a suitable application of the Rus fixed point theorem [15] via two metrics. Thus, our assumptions and methods herein are different from the aforementioned works.

Indeed, a range of authors have pursued a spectrum of approaches to the existence and/or uniqueness of solution to fourth-order BVPs. This includes methods such as Schauder fixed-point theorem [4,9] and topological degree [16], monotone iteration [6], reduction to second order systems [11], lower and upper solutions [5], fixed point theorems in cones [3,6,7,13,17], fixed points of general $\alpha$-concave operators, and fixed point theorems in partially ordered metric spaces [18]. Thus, we can see that our approach of applying the Rus fixed point theorem appears to occupy a unique position within the literature as a strategy to ensure the existence and uniqueness of solutions to fourth-order BVPs.

The results herein form an advancement over traditional approaches such as applications of Banach’s fixed point theorem. This is achieved through the use of two metrics and the Rus fixed point theorem. As we will discover, this enables a greater class of problems to be better understood regarding the existence and uniqueness of solutions. This includes sharpening the Lipschitz constants involved within a global (unbounded) context and within closed and bounded domains.

The reader is referred to [14,19–50] for some additional developments in the field of fourth-order BVPs and applications to beam analysis. In some of these works, the fourth-order BVP under consideration is reducible to a larger system of second-order BVPs and thus the underlying analysis for the fourth-order problem is closely linked with that of second-order problems. Herein, we make no such reduction, preferring to work directly on the original form of the problems.
As flagged in [29, p. 108], no reduction of order (to become a second-order problem) is available for BVP (1.1), (1.4) due to the nature of the boundary conditions. This realization may partially explain Yao’s position [48, p. 237] regarding reasons for the slow progress of research into (1.1), (1.2) and (1.1), (1.4). Thus, new methods and perspectives are needed [51,52] to advance the associated existence and uniqueness theory and its application to beam deflection analysis.

Sufficiently motivated by the above discussion and drawing on the position that “knowing an equation has a unique solution is important from both a modelling and theoretical point of view” [53, p. 794], the purpose of the present work then is to address the aforementioned challenges by examining the existence and uniqueness of solutions to BVPs (1.1), (1.2) and (1.1), (1.3). We sharpen traditional results by showing that a larger class of problems admit a unique solution and illustrate the nature of the advancement through exemplification. We achieve this by drawing on the fixed-point theory in an interesting and alternative way via an application of Rus’s contraction mapping theorem [15] which utilizes two metrics on a metric space. Our existence and uniqueness results are then applied to the area of elastic beam deflections when the beam is subjected to a loading force and the ends of the beam are either both clamped or one end is clamped and the other end is free. The existence and uniqueness of solutions to the models are guaranteed under linear and nonlinear loading forces – that is, the models are identified to be well-posed.

By a solution to (1.1), (1.2) we mean a function \( y : [0, 1] \to \mathbb{R} \) such that \( y \) is four times differentiable, with a continuous fourth-order derivative on \([0, 1]\), which we denote by \( y \in C^4([0, 1]) \), and our \( y \) satisfies both (1.1) and (1.2). A solution to (1.1), (1.3) is defined similarly.

An important result that we shall draw on is the following contraction mapping theorem of Rus.

**Theorem 1.1.** [15] Let \( Y \) be a nonempty set and let \( \rho \) and \( \delta \) be two metrics on \( Y \) such that \((Y, \rho)\) forms a complete metric space. If the mapping \( T : Y \to Y \) is continuous with respect to \( \rho \) on \( Y \) and

\[
\rho(Ty, Tz) \leq \kappa \delta(y, z), \quad \text{for some } \kappa > 0 \text{ and all } y, z \in Y;
\]

\[
\delta(Ty, Tz) \leq a \delta(y, z), \quad \text{for some } 0 < a < 1 \text{ and all } y, z \in Y;
\]

then there exists a unique \( y \in Y \) such that \( Ty = y \).

Theorem 1.1 relies on the existence of two metrics (which are not necessarily equivalent). The space \( Y \) is assumed to be complete with respect to \( \rho \), but \( Y \) is not necessarily complete under \( \delta \). The operator \( T \) is assumed to be continuous with respect to the metric \( \rho \) and contractive with respect to \( \delta \), but not necessarily contractive with respect to \( \rho \). As we shall see, these kinds of assumptions can be applied to operators associated with BVPs, sharpening traditional results by showing that a larger class of problems admit a unique solution. Recently, Theorem 1.1 and its variants have been successfully applied to the study of third-order BVPs [55] and nonlinear initial value problems [56]. For recent extensions of Theorem 1.1 see [57].

Our paper is organized as follows.

In Section 2, we present our main existence and uniqueness results for solutions to our BVPs via applications of Rus’s contraction mapping theorem through two metrics.

In Section 3, the new results of Section 2 are applied to the area of elastic beam deflections with either the ends of the beam both being clamped or when one end is clamped and the other end is free. Through discussions of simple applications, we illustrate the nature of the advancements over traditional results and approaches such as Banach’s fixed point theorem in bounded and unbounded settings.

Finally, we include some discussion in Section 4 and identify some possible areas for future research.

## 2 Existence and uniqueness

In this section, we formulate our main results regarding the existence and uniqueness of solutions via fixed point methods under Rus’s contraction mapping theorem.
Let us construct a range of metrics in an appropriate metric space. Consider the set of real-valued functions that are defined on \([0, 1]\) and are thrice continuously differentiable therein. Denote this space by \(C^3([0, 1])\). For functions \(y, z \in C^3([0, 1])\) and appropriate non-negative constants \(L_i\) and \(W_i\) to be determined a little later, consider the following metrics on \(C^3([0, 1])\):

\[
d_{co}(y, z) := \sum_{i=0}^{3} \max_{x \in [0, 1]} |y^{(i)}(x) - z^{(i)}(x)|;
\]

\[
\delta_p(y, z) := \sum_{i=0}^{3} \left[ L_i \int_0^1 |y^{(i)}(x) - z^{(i)}(x)|^p \, dx \right]^{1/p}, \quad p > 1;
\]

\[
d(y, z) := \max_{i \in \{0, 1, 2, 3\}} \left\{ W_i \max_{x \in [0, 1]} |y^{(i)}(x) - z^{(i)}(x)| \right\}.
\]

It is well known that each of the pairs \((C^3([0, 1]), d_{co})\) and \((C^3([0, 1]), d)\) form a complete metric space. The pair \((C^3([0, 1]), \delta_p)\) also forms a metric space; however, it is not complete. A helpful relationship between our metrics on \(C^3([0, 1])\) that we will draw on involves the inequalities

\[
\delta_p(y, z) \leq \max_{i \in \{0, 1, 2, 3\}} \{ L_i \} d_{co}(y, z);
\]

\[
\delta_p(y, z) \leq \sum_{i=0}^{3} \frac{L_i}{W_i} d(y, z).
\]

2.1 On (1.1) and (1.2)

In order to construct an appropriate operator \(T\) and the corresponding fixed-point problem for (1.1), (1.2) we note that BVP (1.1), (1.2) is equivalent to the integral equation [48, p. 238], [40]

\[
y(x) = \int_0^1 G(x, s)f(s, y(s), y'(s), y''(s), y'''(s)) \, ds + \phi(x), \quad x \in [0, 1],
\]

where \(G(x, s)\) is the Green function for the following BVP:

\[
y^{(iv)} = 0, \quad x \in [0, 1],
\]

\[
y(0) = 0, y'(0) = 0, \quad y(1) = 0, y'(1) = 0,
\]

and is given explicitly by

\[
0 \leq G(x, s) = \frac{1}{6} \left[ s^3(1-x)^2[(x-s) + 2(1-s)x], \quad \text{for } 0 \leq s \leq x \leq 1, \right.
\]

\[
\left. \frac{1}{6} x^3(1-s)^2[(s-x) + 2(1-x)s], \quad \text{for } 0 \leq x \leq s \leq 1; \right.
\]

and \(\phi\) is the unique solution to the BVP

\[
y^{(iv)} = 0, \quad x \in [0, 1],
\]

\[
y(0) = a, y'(0) = c, \quad y(1) = b, y'(1) = d,
\]

which is given explicitly by

\[
\phi(x) = a(2x^3 - 3x^2 + 1) + b(-2x^3 + 3x^2) + c(x^3 - 2x^2 + x) + d(x^3 - x^2).
\]

To avoid the repeated use of complicated expressions, we define the following constants to simplify certain notation.
For $i = 0, 1, 2, 3$, define positive constants $\beta_i$ via
\[
\beta_i \geq \max_{x \in [0, 1]} \int_0^1 \left| \frac{\partial^i}{\partial x^i} G(x, s) \right| \, ds.
\] (2.7)
Such choices are always possible due to the smoothness of $G$.

Let $p > 1$ and $q > 1$ be constants such that $1/p + 1/q = 1$. For $i = 0, 1, 2, 3$, define:
\[
c_i = \max_{x \in [0, 1]} \left[ \int_0^1 \left| \frac{\partial^i}{\partial x^i} G(x, s) \right|^q \, ds \right]^{1/q};
\] (2.8)
and
\[
\gamma_i = \left( \int_0^1 \int_0^1 \left| \frac{\partial^i}{\partial x^i} G(x, s) \right|^{p/q} \, ds \right)^{1/p} \right) \int_0^1 \, dx.
\] (2.9)

We are now in a position to state and prove our first result for the existence and uniqueness of solutions to (1.1), (1.2).

**Theorem 2.1.** Let $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous and let $L_i$ be non-negative constants for $i = 0, 1, 2, 3$ (not all zero) such that
\[
|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, v_3)| \leq \sum_{i=0}^3 L_i |u_i - v_i|,
\] (2.10)
for all $(x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in [0, 1] \times \mathbb{R}^4$.

If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\gamma_i$ defined in (2.9) such that
\[
\sum_{i=0}^3 L_i \gamma_i < 1.
\] (2.11)
then BVP (1.1), (1.2) has a unique solution in $C^1([0, 1])$.

**Proof.** Based on form (2.4), we define the operator $T : C^1([0, 1]) \to C^1([0, 1])$ by
\[
(Ty)(x) = \int_0^1 G(x, s)f(s, y(s), y'(s), y''(s), y'''(s)) \, ds + \phi(x), \quad x \in [0, 1].
\]
We wish to show that there exists a unique $y \in C^1([0, 1])$ such that
\[
Ty = y,
\]
which is equivalent to proving that BVP (1.1), (1.2) has a unique solution. (Any solutions lying in $C^1([0, 1])$ will also lie in $C^1([0, 1])$ as repeatedly differentiating (2.4) will show.)

To prove that our $T$ has a unique fixed point, we show that the assumptions of Theorem 2.1 ensure that the conditions of Theorem 1.1 hold.

Consider the pair $(Y, \rho) = (C^1([0, 1]), d_{\infty})$ to form a complete metric space and consider the metric $\delta = \delta_p$ on $Y$, where $p > 1$.

For $y, z \in C^1([0, 1])$ and $x \in [0, 1]$, consider
\[
|(Ty)(x) - (Tz)(x)| \leq \int_0^1 |G(x, s)||f(s, y(s), y'(s), y''(s), y'''(s)) - f(s, z(s), z'(s), z''(s), z'''(s))| \, ds
\] (2.12)
where we have invoked our assumption (2.10) and Hölder’s inequality for $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ and $c_0$ is defined in (2.8).

By repeating the aforementioned approach on derivatives we can obtain

$$\max_{x \in [0,1]} |(Ty)^{(i)}(x) - (Tz)^{(i)}(x)| \leq c_i \delta_p(y, z)$$

for $i = 0, 1, 2, 3$. Thus, defining

$$\kappa := \sum_{i=0}^{3} c_i$$

we see that

$$d_{\infty}(Ty, Tz) \leq \kappa \delta_p(y, z), \quad \text{for some } \kappa > 0 \text{ and all } y, z \in C^3([0, 1]).$$

Now, from (2.2), for all $y, z \in C^3([0, 1])$, we have

$$d_{\infty}(Ty, Tz) \leq \kappa \delta_p(y, z) \leq \kappa \max_{i \in \{0, 1, 2, 3\}} \{L_i d_{\infty}(y, z)\}.$$

Thus, given any $\varepsilon > 0$ we can choose $\Delta = \varepsilon / \kappa \max_{i \in \{0, 1, 2, 3\}} \{L_i\}$ so that $d_{\infty}(Ty, Tz) < \varepsilon$ whenever $d_{\infty}(y, z) < \Delta$. Hence, $T$ is continuous on $C^3([0, 1])$ with respect to the $d_{\infty}$ metric.

Finally, we show that $T$ is contractive on $C^3([0, 1])$ with respect to the $\delta_p$ metric. From (2.12), for each $y, z \in C^3([0, 1])$ and $i = 0, 1, 2, 3$, we have

$$L_i \left( \int_0^1 |(Ty)^{(i)}(x) - (Tz)^{(i)}(x)|^p \, dx \right)^{1/p} \leq L_i \gamma_i \delta_p(y, z),$$

where $\gamma_i$ are defined in (2.9). Summing both sides of the previous inequality over $i$ we obtain

$$\delta_p(Ty, Tz) \leq \sum_{i=0}^{3} L_i \gamma_i \delta_p(y, z)$$

for all $y, z \in C^3([0, 1])$. From our assumption (2.11), we have ensured

$$\delta_p(Ty, Tz) \leq a \delta_p(y, z)$$

for some $a < 1$ and all $y, z \in C^3([0, 1])$.

Thus, Theorem 1.1 is applicable and yields the existence of a unique fixed point to $T$ that lies in $C^3([0, 1])$. This solution is also in $C^4([0, 1])$ as can be verified by differentiating the integral Eq. (2.4). Thus, we have equivalently shown that BVP (1.1), (1.2) has a unique solution. \hfill \blacksquare

**Remark 2.1.** Lipschitz condition (2.10) will be satisfied if, for example, our

$$f(x, u_0, u_1, u_2, u_3)$$

has partial derivatives $\partial f / \partial u_i$ that are uniformly bounded and continuous on $[0, 1] \times \mathbb{R}^4$ for each corresponding $i = 0, 1, 2, 3$. In this case, each bound $M_i$ can form $L_i$ for $i = 0, 1, 2, 3$.

For example, for an $f$ such as

$$f(x, u_0, u_1, u_2, u_3) = [\sin(u_0) + \cos(u_1) + \sin(u_2) + \cos(u_3)] / 10$$

we could choose $M_i = L_i = 1/10$. 

Although Lipschitz condition (2.10) is more difficult to be satisfied on the unbounded domain $[0, 1] \times \mathbb{R}^4$ when compared with a closed and bounded subset of this region, the conclusion of Theorem 2.1 provides robust, global information about the solutions to the problem under consideration.

In comparison, the very nature of localized assumptions means that only limited, localized information about solutions can be necessarily obtained. For instance, in the context of localized assumptions, nothing can be concluded about the existence and uniqueness of solutions that may lie outside of the closed and bounded set that is under consideration. This is especially important as multiple solutions to fourth-order BVPs have been shown to exist [14].

Motivated by the above discussion let us now provide some balance by examining questions of existence and uniqueness of solutions to (1.1), (1.2) within subsets of $[0, 1] \times \mathbb{R}^4$. Essentially, we impose less restrictive conditions on $f$ in exchange for obtaining less information regarding the solutions.

**Theorem 2.2.** Let $f : B \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set

$$B = \{(x, u, v, w, z) \in \mathbb{R}^5 : x \in [0, 1], |u - \phi(x)| \leq R, |v - \phi'(x)| \leq \frac{\beta_1}{\beta_0} R, |w - \phi''(x)| \leq \frac{\beta_2}{\beta_0} R, |z - \phi'''(x)| \leq \frac{\beta_3}{\beta_0} R\},$$

where $R > 0$, $\phi$ is defined in (2.6) and $\beta_i$ are defined in (2.7). Assume $M\beta_0 \leq R$. For $i = 0, 1, 2, 3$, let $L_i$ be non-negative constants (not all zero) such that

$$|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, v_3)| \leq \sum_{i=0}^{3} L_i|u_i - v_i|, \text{ for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in B. \quad (2.13)$$

If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $y_i$ defined in (2.9) and

$$\sum_{i=0}^{3} L_i y_i < 1, \quad (2.14)$$

then BVP (1.1), (1.2) has a unique solution in $C^3([0, 1])$ such that

$$(x, y(x), y'(x), y''(x), y'''(x)) \in B, \text{ for all } x \in [0, 1].$$

**Proof.** Consider the pair $(C^3([a, b]), d)$ where the constants $W_i$ in our $d$ in (2.1) are chosen such that $W_0 = 1$, $W_1 = \beta_1/\beta_0$, $W_2 = \beta_2/\beta_0$, and $W_3 = \beta_3/\beta_0$. Our pair forms a complete metric space. Now, for the constant $R > 0$ and function $\phi$ in the definition of $B$ consider the following set $\mathcal{B}_R \subset C^3([0, 1])$ defined via

$$\mathcal{B}_R = \{y \in C^3([0, 1]) : d(y, \phi) \leq R\}.$$

Since $\mathcal{B}_R$ is a closed subspace of $C^3([0, 1])$, the pair $(\mathcal{B}_R, d)$ forms a complete metric space.

Consider the operator $T : \mathcal{B}_R \to C^3([0, 1])$ defined as in the proof of Theorem 2.1.

To establish the existence and uniqueness to $Tx = x$, we show that the conditions of Theorem 1.1 hold with $Y = \mathcal{B}_R$.

Let us show $T : \mathcal{B}_R \to \mathcal{B}_R$. For $y \in \mathcal{B}_R$ and $x \in [0, 1]$, consider

$$|(Ty)(x) - \phi(x)| \leq \int_{0}^{1} |G(x, s)| |f(s, y(s), y'(s), y''(s), y'''(s))| ds \leq M \int_{0}^{1} |G(x, s)| ds \leq M\beta_0.$$

Similarly,

$$|(Ty)'(x) - \phi'(x)| \leq \int_{0}^{1} \left| \frac{\partial}{\partial x} G(x, s) \right| |f(s, y(s), y'(s), y''(s), y'''(s))| ds \leq M \int_{0}^{1} \left| \frac{\partial}{\partial x} G(x, s) \right| ds \leq M\beta_1.$$

Thus, $\beta_0 |(Ty)'(x) - \phi'(x)|/\beta_1 \leq M\beta_0$. 

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In addition, via similar arguments, we obtain
\[ |(Ty)^n(x) - \phi^n(x)| \leq M_2, \quad |(Ty)^m(x) - \phi^m(x)| \leq M_3, \]
so that \( \beta_0 |(Ty)^n(x) - \phi^n(x)| / \beta_2 \leq M_0 \) and \( \beta_0 |(Ty)^m(x) - \phi^m(x)| / \beta_3 \leq M_0. \)
Thus, for all \( y \in \mathcal{B}_R \) we have
\[ d(Ty, \phi) \leq \max\{M_0, M_0, M_0, M_0\} = M_0 \leq R, \]
where the final inequality holds by assumption. Thus, for all \( y \in \mathcal{B}_R \) we have \( Ty \in \mathcal{B}_R \) so that \( T : \mathcal{B}_R \rightarrow \mathcal{B}_R. \)

Similar to the proof of Theorem 2.1 we have, for \( i = 0, 1, 2, 3 \), and all \( x \in [0, 1] \), that
\[ |(Ty)^{(i)}(x) - (Tz)^{(i)}(x)| \leq c_i d_p(y, z) \]
via applications of Hölder’s inequality.

Combining the aforementioned inequalities we obtain
\[ d(Ty, Tz) \leq c \delta_p(y, z), \quad \text{for some } c > 0 \text{ and all } y, z \in \mathcal{B}_R, \tag{2.15} \]
where
\[ c = \max \left\{ c_0, \frac{\beta_0}{\beta_1}, \frac{\beta_0}{\beta_2}, \frac{\beta_0}{\beta_3} \right\}. \]
Thus, the first inequality of Theorem 1.1 holds.

Furthermore, \( T \) is continuous on \( \mathcal{B}_R \) with respect to the \( d \) metric as can be shown from the following arguments. For all \( y, z \in \mathcal{B}_R \), we may apply (2.3) to (2.15) to obtain
\[ d(Ty, Tz) \leq c \delta_p(y, z) \leq c \left( L_0 + L_1 \frac{\beta_1}{\beta_0} + L_2 \frac{\beta_2}{\beta_0} + L_3 \frac{\beta_3}{\beta_0} \right) d(x, y). \]
Thus, given any \( \varepsilon > 0 \) we can choose
\[ \Delta = \frac{\varepsilon}{L_0 + L_1 \frac{\beta_1}{\beta_0} + L_2 \frac{\beta_2}{\beta_0} + L_3 \frac{\beta_3}{\beta_0}} \]
so that \( d(Tx, Ty) < \varepsilon \) whenever \( d(x, y) < \Delta. \) Hence, \( T \) is continuous on \( \mathcal{B}_R \) with respect to the \( d \) metric.

Finally, the contraction condition of \( T \) on \( \mathcal{B}_R \) follows from essentially the same arguments as those used in the proof of Theorem 2.1 and so we do not repeat it here.

Thus, we conclude that all the conditions of Theorem 1.1 hold for \( T \) on \( \mathcal{B}_R \) and so the unique fixed point of \( T \) in \( \mathcal{B}_R \) is guaranteed to exist. \( \square \)

The following three new results are variations on the theme of Theorem 2.2.

**Theorem 2.3.** Let \( f : C \rightarrow \mathbb{R} \) be continuous and uniformly bounded by \( M > 0 \) on the set
\[ C = \left\{ (x, u, v, w, z) \in \mathbb{R}^5 : x \in [0, 1], |u - \phi(x)| \leq \frac{\beta_0}{\beta_1} R, |v - \phi'(x)| \leq \frac{\beta_2}{\beta_1} R, |w - \phi''(x)| \leq \frac{\beta_3}{\beta_1} R, |z - \phi'''(x)| \leq \frac{\beta_4}{\beta_1} R \right\}, \]
where \( R > 0, \phi \) is defined in (2.6) and \( \beta_i \) are defined in (2.7). Assume \( M_0 \beta_1 \leq R. \) For \( i = 0, 1, 2, 3 \), let \( L_i \) be non-negative constants (not all zero) such that
\[ |f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, v_3)| \leq \sum_{i=0}^{3} L_i |u_i - v_i|, \quad \text{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in C. \tag{2.16} \]
If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\gamma_i$ defined in (2.9) such that
\[ \sum_{i=0}^{3} L_i \gamma_i < 1, \]  
then BVP (1.1), (1.2) has a unique solution in $C^3([0, 1])$ such that
\[ (x, y(x), y'(x), y''(x), y'''(x)) \in C, \quad \text{for all } x \in [0, 1]. \]

**Proof.** The proof follows similar lines of argument to that of Theorem 2.2 and so we just sketch the basic idea.

Consider the pair $(C^3([a, b]), d)$ where the constants $W_i$ in our $d$ in (2.1) are chosen such that $W_0 = \beta_i / \beta_0$, $W_1 = 1$, $W_2 = \beta_i / \beta_1$, and $W_3 = \beta_i / \beta_2$. Our pair forms a complete metric space. Now, for the constant $R > 0$ and function $\phi$ in the definition of $C$, consider the following set $L_0 \lambda_0 = 8(11/1,680)^{1/2} < 1$ defined via
\[ C_R = \{ y \in C^3([0, 1]) : d(y, \phi) \leq R \}. \]

Since $C_R$ is a closed subspace of $C^3([0, 1])$, the pair $(C_R, d)$ forms a complete metric space.

Following similar steps to that of Theorem 2.2, the condition $M \beta_1 \leq R$ ensures $T : C_R \rightarrow C_R$. Furthermore, (2.16) and (2.17) guarantee that $T$ is contractive on $C_R$ with respect to $\delta_p$. \hfill $\square$

**Theorem 2.4.** Let $f : D \rightarrow \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set
\[ D = \{ (x, u, v, w, z) \in \mathbb{R}^5 : x \in [0, 1], |u - \phi(x)| \leq \frac{\beta_0}{\beta_2} R, |v - \phi'(x)| \leq \frac{\beta_1}{\beta_2} R, |w - \phi''(x)| \leq R, |z - \phi'''(x)| \leq \frac{\beta_3}{\beta_2} R \}, \]
where $R > 0$, $\phi$ is defined in (2.6) and $\beta_i$ are defined in (2.7). Assume $M \beta_2 \leq R$. For $i = 0, 1, 2, 3$, let $L_i$ be non-negative constants (not all zero) such that
\[ |f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, v_3)| \leq \sum_{i=0}^{3} L_i |u_i - v_i|, \quad \text{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in D. \]  

If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\gamma_i$ defined in (2.9) such that
\[ \sum_{i=0}^{3} L_i \gamma_i < 1, \]  
then BVP (1.1), (1.2) has a unique solution in $C^3([0, 1])$ such that
\[ (x, y(x), y'(x), y''(x), y'''(x)) \in D, \quad \text{for all } x \in [0, 1]. \]

**Proof.** The proof follows similar lines of argument to that of Theorem 2.2 and so we just sketch the basic idea.

Consider the pair $(C^3([a, b]), d)$ where the constants $W_i$ in our $d$ in (2.1) are chosen such that $W_0 = \beta_i / \beta_0$, $W_1 = 1$, $W_2 = \beta_i / \beta_1$, and $W_3 = \beta_i / \beta_2$. Our pair forms a complete metric space. Now, for the constant $R > 0$ and function $\phi$ in the definition of $D$, consider the following set $\mathcal{D}_R \subset C^3([0, 1])$ defined via
\[ \mathcal{D}_R = \{ y \in C^3([0, 1]) : d(y, \phi) \leq R \}. \]

Since $\mathcal{D}_R$ is a closed subspace of $C^3([0, 1])$, the pair $(\mathcal{D}_R, d)$ forms a complete metric space.

Following similar steps to that of Theorem 2.2, the condition $M \beta_2 \leq R$ ensures $T : \mathcal{D}_R \rightarrow \mathcal{D}_R$. Furthermore, (2.18) and (2.19) guarantee that $T$ is contractive on $\mathcal{D}_R$ with respect to $\delta_p$. \hfill $\square$

**Theorem 2.5.** Let $f : E \rightarrow \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set
\[ E = \{ (x, u, v, w, z) \in \mathbb{R}^5 : x \in [0, 1], |u - \phi(x)| \leq \frac{\beta_0}{\beta_1} R, |v - \phi'(x)| \leq \frac{\beta_1}{\beta_2} R, |w - \phi''(x)| \leq \frac{\beta_2}{\beta_3} R, |z - \phi'''(x)| \leq R \}, \]
where \( R > 0 \), \( \phi \) is defined in (2.6) and \( \beta_i \) are defined in (2.7). Assume \( M \beta \leq R \). For \( i = 0, 1, 2, 3 \), let \( L_i \) be non-negative constants (not all zero) such that
\[
|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, v_3)| \leq \sum_{i=0}^{3} L_i|u_i - v_i|, \text{ for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in E. \tag{2.20}
\]

If there are constants \( p > 1 \) and \( q > 1 \) such that \( 1/p + 1/q = 1 \) with \( \gamma_i \) defined in (2.9) such that
\[
\sum_{i=0}^{3} L_i \gamma_i < 1, \tag{2.21}
\]
then BVP (1.1), (1.2) has a unique solution in \( C^4([0, 1]) \) such that
\[
(x, y(x), y'(x), y''(x), y'''(x)) \in E, \text{ for all } x \in [0, 1].
\]

**Proof.** Omitted due to brevity. \( \square \)

### 2.2 On (1.1), (1.3)

BVP (1.1), (1.3) is equivalent to the integral equation [48, p. 238], [40]
\[
y(x) = \int_{0}^{1} G(x, s)f(s, y(s), y'(s), y''(s), y'''(s)) \, ds + \psi(x), \quad x \in [0, 1], \tag{2.22}
\]
where \( G(x, s) \) is the Green function [46, p. 2] for the following BVP:
\[
y^{(iv)} = 0, \quad x \in [0, 1],
\]
\[
y(0) = 0, \quad y'(0) = 0, \quad y''(1) = 0, \quad y'''(1) = 0,
\]
and is given explicitly by
\[
0 \leq G(x, s) = \frac{1}{6} \begin{cases} s^3(3x - s), & \text{for } 0 \leq s \leq x \leq 1; \\ x^3(3s - x), & \text{for } 0 \leq x \leq s \leq 1, \end{cases} \tag{2.23}
\]
and \( \psi \) is the unique solution to the BVP
\[
y^{(iv)} = 0, \quad x \in [0, 1],
\]
\[
y(0) = a, \quad y'(0) = c, \quad y''(1) = e, \quad y'''(1) = h,
\]
which is given explicitly by
\[
\psi(x) = hx^3/6 + (e - h)x^2/2 + cx + a. \tag{2.24}
\]

Once again, to avoid the repeated use of complicated expressions, we define the following constants to simplify certain notation.

For \( i = 0, 1, 2, 3 \), define positive constants \( \theta_i \) via
\[
\theta_i \geq \max_{x \in [0, 1]} \int_{0}^{1} \left| \frac{\partial^i}{\partial x^i} G(x, s) \right| \, ds. \tag{2.25}
\]
Such choices are always possible due to the smoothness of \( G \).
Let \( p > 1 \) and \( q > 1 \) be constants such that \( 1/p + 1/q = 1 \). For \( i = 0, 1, 2, 3 \), define:

\[
\eta_i = \max_{x \in [0,1]} \left[ \int_0^1 \left| \frac{\partial^i}{\partial x^i} G(x, s) \right|^q \, ds \right]^{1/q};
\]

(2.26)

and

\[
\lambda_i = \left( \int_0^1 \int_0^1 \left| \frac{\partial^i}{\partial x^i} G(x, s) \right|^q \, ds \, dx \right)^{p/q} \right)^{1/p}.
\]

(2.27)

Let us now state and prove a new result for the existence and uniqueness of solutions to (1.1), (1.3) on the set \([0, 1] \times \mathbb{R}^6\).

**Theorem 2.6.** Let \( f : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R} \) be continuous and let \( L_i \) be non-negative constants for \( i = 0, 1, 2, 3 \) (not all zero) such that

\[
|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, v_3)| \leq \sum_{i=0}^3 L_i |u_i - v_i|,
\]

(2.28)

for all \((x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in [0, 1] \times \mathbb{R}^4\).

If there are constants \( p > 1 \) and \( f(x, y) \) such that \( 1/p + 1/q = 1 \) with \( \lambda_i \) defined in (2.27) such that

\[
\sum_{i=0}^3 L_i \lambda_i < 1,
\]

(2.29)

then BVP (1.1), (1.3) has a unique solution in \( C^4([0, 1]) \).

**Proof.** Consider the operator \( T : C^4([0, 1]) \rightarrow C^4([0, 1]) \) constructed from the form (2.22), namely

\[
(Ty)(x) = \int_0^1 G(x, s)f(s, y(s), y'(s), y''(s), y'''(s)) \, ds + \psi(x), \quad x \in [0, 1].
\]

We want to show that there exists a unique \( y \in C^4([0, 1]) \) such that

\[
Ty = y.
\]

To prove this, we shall show that the conditions of Theorem 1.1 hold. Consider the pair \((Y, \rho) = (C^4([0, 1]), d_{\infty})\) which forms a complete metric space. In addition, consider the metric \( \delta = \delta_p \) on \( Y \), where \( p > 1 \).

For \( y, z \in C^4([0, 1]) \) and \( x \in [0, 1] \), consider

\[
|(Ty)(x) - (Tz)(x)| \leq \int_0^1 |G(x, s)||f(s, y(s), y'(s), y''(s), y'''(s)) - f(s, z(s), z'(s), z''(s), z'''(s))| \, ds
\]

\[
\leq \int_0^1 |G(x, s)| \sum_{i=0}^3 (L_i |y^{(i)}(s) - z^{(i)}(s)|) \, ds
\]

\[
\leq \int_0^1 |G(x, s)|^{q/p} \left[ \sum_{i=0}^3 (L_i \int_0^1 |y^{(i)}(s) - z^{(i)}(s)|^p \, ds) \right]^{1/p}
\]

\[
\leq \max_{x \in [0,1]} \left[ \int_0^1 |G(x, s)|^{q/p} \right]^{1/q} \delta_p(y, z) = \eta_0 \delta_p(y, z),
\]

where

\[
\eta_0 = \max_{x \in [0,1]} \left[ \int_0^1 |G(x, s)|^{q/p} \right]^{1/q}.
\]
where $\eta_0$ is defined in (2.26). Repeating the aforementioned argument to derivatives of the operator $\mathcal{T}$ yields

$$
|(\mathcal{T}y)^{(i)}(x) - (\mathcal{T}z)^{(i)}(x)| \leq \eta_i \delta_p(y, z),
$$

for $i = 0, 1, 2, 3$. Thus, defining

$$
\kappa_i := \sum_{i=0}^{3} \eta_i
$$

we see that

$$
d_{\infty}(\mathcal{T}y, \mathcal{T}z) \leq \kappa_i \delta_p(y, z), \quad \text{for some } \kappa_i > 0 \text{ and all } y, z \in Y.
$$

Now, for all $y, z \in C^\lambda([0, 1])$ consider

$$
d_{\infty}(\mathcal{T}y, \mathcal{T}z) \leq \kappa_i \delta_p(y, z) \leq \kappa_i \max_{i \in [0,1,2,3]} |L_i| d_{\infty}(y, z).
$$

Thus, given any $\varepsilon > 0$ we can choose $\Delta = e/\kappa_i \max_{i \in [0,1,2,3]} |L_i|$ so that $d_{\infty}(\mathcal{T}y, \mathcal{T}z) < \varepsilon$ whenever $d_{\infty}(y, z) < \Delta$. Hence, $\mathcal{T}$ is continuous on $C^\lambda([0, 1])$ with respect to the $d_{\infty}$ metric.

Finally, we show that $\mathcal{T}$ is contractive on $C^\lambda([0, 1])$ with respect to the $\delta_p$ metric. For each $y, z \in C^\lambda([0, 1])$ and $i = 0, 1, 2, 3$, we have

$$
L_i \left( \int_0^1 |(\mathcal{T}y)^{(i)}(x) - (\mathcal{T}z)^{(i)}(x)|^p \, dx \right)^{1/p} \leq L_i \left( \int_0^1 \left( \int_0^1 \left| \frac{\partial^i G(x, s)}{\partial x^i} \right|^q \, ds \right)^{p/q} \, dx \right)^{1/p} \delta_p(y, z)
$$

and so summing the previous inequality over $i$ we obtain

$$
\delta_p(\mathcal{T}y, \mathcal{T}z) \leq \left( \sum_{i=0}^{3} L_i \lambda_i \right) \delta_p(y, z).
$$

Thus, by (2.29), we have

$$
\delta_p(\mathcal{T}y, \mathcal{T}z) \leq a_i \delta_p(y, z)
$$

for some $a_i < 1$ and all $y, z \in C^\lambda([0, 1])$.

Thus, Theorem 1.1 is applicable and the operator $\mathcal{T}$ has a unique fixed point in $C^\lambda([0, 1])$. This solution is also in $C^\lambda([0, 1])$ as differentiating the integral Eq. (2.22) shows. Our conclusion is equivalent to showing that BVP (1.1), (1.3) has a unique solution.

Now, let us now explore the existence and uniqueness of solutions to (1.1), (1.3) on subsets of $[0, 1] \times \mathbb{R}^4$.

**Theorem 2.7.** Let $f : F \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set

$$
F = \left\{ (x, u, v, w, z) \in \mathbb{R}^5 : x \in [0, 1], |u - \psi(x)| \leq R, \ |v - \psi'(x)| \leq \frac{\theta_1}{\theta_0} R, \ |w - \psi''(x)| \leq \frac{\theta_2}{\theta_0} R, \ |z - \psi'''(x)| \leq \frac{\theta_3}{\theta_0} R \right\},
$$

where $R > 0$, $\psi$ is defined in (2.24) and $\theta_i$ are defined in (2.25). Assume $M \theta_0 \leq R$. For $i = 0, 1, 2, 3$, let $L_i$ be non-negative constants (not all zero) such that

$$
|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, v_3)| \leq \sum_{i=0}^{3} L_i |u_i - v_i|, \quad \text{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in F. \tag{2.30}
$$

If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\lambda_i$ defined in (2.27) such that

$$
\sum_{i=0}^{3} L_i \lambda_i < 1, \tag{2.31}
$$

then BVP (1.1), (1.3) has a unique solution on $[0, 1] \times \mathbb{R}^4$. 

□
then BVP (1.1), (1.3) has a unique (nontrivial) solution in \( C^4([0, 1]) \) such that
\[
(x, y(x), y'(x), y''(x), y'''(x)) \in F, \quad \text{for all } x \in [0, 1].
\]

**Proof.** The proof is very similar to that of the proof of Theorem 2.2 by making the appropriate modifications (e.g., \( G \) instead of \( G \), etc.). Thus, we omit the proof for brevity. \( \square \)

Similarly, we have the following three results which we state without proof due to brevity and concerns of repetition.

**Theorem 2.8.** Let \( f : H \to \mathbb{R} \) be continuous and uniformly bounded by \( M > 0 \) on the set
\[
H = \left\{ (x, u, v, w, z) \in \mathbb{R}^5 : x \in [0, 1], |u - \psi(x)| \leq \frac{\theta_0}{\theta_1} R, |v - \psi'(x)| \leq \frac{\theta_1}{\theta_2} R, |w - \psi''(x)| \leq \frac{\theta_2}{\theta_3} R, |z - \psi'''(x)| \leq \frac{\theta_3}{\theta_4} R \right\},
\]
where \( R > 0, \psi \) is defined in (2.24) and \( \theta_i \) are defined in (2.25). Assume \( M\theta_1 \leq R \). For \( i = 0, 1, 2, 3 \), let \( L_i \) be non-negative constants (not all zero) such that
\[
|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, v_3)| \leq \sum_{i=0}^{3} L_i |u_i - v_i|, \quad \text{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in H. \quad (2.32)
\]
If there are constants \( p > 1 \) and \( q > 1 \) such that \( 1/p + 1/q = 1 \) with \( \gamma_i \) defined in (2.9) such that
\[
\sum_{i=0}^{3} L_i \gamma_i < 1, \quad (2.33)
\]
then BVP (1.1), (1.3) has a unique (nontrivial) solution in \( C^4([0, 1]) \) such that
\[
(x, y(x), y'(x), y''(x), y'''(x)) \in H, \quad \text{for all } x \in [0, 1].
\]

**Theorem 2.9.** Let \( f : J \to \mathbb{R} \) be continuous and uniformly bounded by \( M > 0 \) on the set
\[
J = \left\{ (x, u, v, w, z) \in \mathbb{R}^5 : x \in [0, 1], |u - \psi(x)| \leq \frac{\theta_0}{\theta_1} R, |v - \psi'(x)| \leq \frac{\theta_1}{\theta_2} R, |w - \psi''(x)| \leq \frac{\theta_2}{\theta_3} R, |z - \psi'''(x)| \leq \frac{\theta_3}{\theta_4} R \right\},
\]
where \( R > 0, \psi \) is defined in (2.24) and \( \theta_i \) are defined in (2.25). Assume \( M\theta_2 \leq R \). For \( i = 0, 1, 2, 3 \), let \( L_i \) be non-negative constants (not all zero) such that
\[
|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, v_3)| \leq \sum_{i=0}^{3} L_i |u_i - v_i|, \quad \text{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in J. \quad (2.34)
\]
If there are constants \( p > 1 \) and \( q > 1 \) such that \( 1/p + 1/q = 1 \) with \( \gamma_i \) defined in (2.9) such that
\[
\sum_{i=0}^{3} L_i \gamma_i < 1, \quad (2.35)
\]
then BVP (1.1), (1.3) has a unique (nontrivial) solution in \( C^4([0, 1]) \) such that
\[
(x, y(x), y'(x), y''(x), y'''(x)) \in J, \quad \text{for all } x \in [0, 1].
\]

**Theorem 2.10.** Let \( f : K \to \mathbb{R} \) be continuous and uniformly bounded by \( M > 0 \) on the set
\[
K = \left\{ (x, u, v, w, z) \in \mathbb{R}^5 : x \in [0, 1], |u - \psi(x)| \leq \frac{\theta_0}{\theta_1} R, |v - \psi'(x)| \leq \frac{\theta_1}{\theta_2} R, |w - \psi''(x)| \leq \frac{\theta_2}{\theta_3} R, |z - \psi'''(x)| \leq \frac{\theta_3}{\theta_4} R \right\},
\]
where \( R > 0, \psi \) is defined in (2.24) and \( \theta_i \) are defined in (2.25). Assume \( M\theta_3 \leq R \). For \( i = 0, 1, 2, 3 \), let \( L_i \) be non-negative constants (not all zero) such that
\[
|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, v_3)| \leq \sum_{i=0}^{3} L_i |u_i - v_i|, \quad \text{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in K. \quad (2.36)
\]
If there are constants \( p > 1 \) and \( q > 1 \) such that \( 1/p + 1/q = 1 \) with \( \lambda_i \) defined in (2.27) such that
\[
\sum_{i=0}^{3} L_i \lambda_i < 1,
\]
then BVP (1.1), (1.3) has a unique (nontrivial) solution in \( C^4([0, 1]) \) such that
\[
(x, y(x), y'(x), y''(x), y'''(x)) \in K, \quad \text{for all } x \in [0, 1].
\]

3 Application to beam deflections

In this section, we analyse fourth-order simplified BVPs arising from the deflection of elastic beams subject to linear and nonlinear loading forces. Through this discussion, we illustrate the nature of the advancement of this paper when compared with traditional approaches.

3.1 Nonlinear loading force

If we consider a loading force on the beam given by \( f(x, y) \), which may be nonlinear, then we obtain the following fourth-order differential equation:
\[
y^{(iv)} = f(x, y), \quad x \in [0, 1],
\]
where \( y = y(x) \) represents the resultant deflection of the beam at position \( x \). This form is similar to that of [14].

In this case, the Lipschitz condition of our theorems reduce to
\[
|f(x, u) - f(x, v)| \leq L_0|u - v|
\]
on \([0, 1] \times \mathbb{R}\) or on suitable subsets.

A standard approach to problem (3.1) with either fully clamped ends (1.4) or clamped/free ends (1.5) could employ Banach’s contraction mapping theorem within the space of continuous functions \( C([0, 1]) \) coupled with the maximum metric
\[
\rho(y, z) = \max_{x \in [0, 1]} |y(x) - z(x)|, \quad \text{for all } y, z \in C([0, 1]).
\]
To obtain a contraction for the operator \( T \) with respect to our \( d \) as per Banach’s contraction mapping theorem, the standard condition for clamped ends takes the form:
\[
L_0 \max_{x \in [0, 1]} \int_{0}^{1} |G(x, s)| ds = L_0/384 < 1
\]
see [18, Theorem 3.1 Condition c] or [12].

For clamped/free ends, to obtain a contraction for the operator \( \mathcal{T} \) with respect to our \( \rho \) as per Banach’s contraction mapping theorem, the condition
\[
L_0 \max_{x \in [0, 1]} \int_{0}^{1} |G(x, s)| ds = L_0/8 < 1
\]
is involved, see, for example, [11, p. 58] or [58, Remark 1].

In comparison, the simplified version of Theorem 2.1 (or related theorems) with \( p = q = 2 \) involves (2.11) taking the form:
\[
L_0 \left( \int_{0}^{1} \left( \int_{0}^{1} |G(x, s)|^2 ds \right)^{1/2} dx \right) = L_0(71/17,463,600)^{1/2} < 1.
\]
The left-hand side of (3.5) can be verified by direct integration or through use of a suitable computing package, via
\[
\int_0^1 |G(x, s)|^2 \, ds = \int_0^x \left[ \frac{\sin^2(1-x^2)(x-s) + 2(1-s)x}{6} \right]^2 \, ds + \int_x^1 \left[ \frac{x^2(1-s)^2[(s-x) + 2(1-s)s]}{6} \right]^2 \, ds
\]
\[
= [(x-1)^4x^2(20x^2 - 50x + 33)/1,260] - x^4(20x^2 + 10x + 3)(x-1)^7/1,260
\]
and another integration yields
\[
\int_0^1 \left( \int_0^1 |G(x, s)|^2 \, ds \right) \, dx = \int_0^1 [(x-1)^4x^2(20x^2 - 50x + 33)/1,260] - x^4(20x^2 + 10x + 3)(x-1)^7/1,260 \, dx
\]
\[
= 71/17,463,600.
\]

One can observe that our condition (3.5) is sharper than condition (3.3) and this illustrates one aspect of how the new results of this paper represent an advancement over traditional approaches, and how they are applicable to a wider class of problems.

The simplified version of Theorem 2.6 (or related theorems) with \( p = q = 2 \) involves (2.29) taking the form:
\[
L \left( \int_0^1 \left( \int_0^1 |G(x, s)|^2 \, ds \right) \, dx \right)^{1/2} = L(11/1,680)^{1/2} < 1. \tag{3.6}
\]

The left-hand side of (3.6) can be verified via the following steps.
\[
\int_0^1 |G(x, s)|^2 \, ds = \int_0^x \left[ \frac{\sin^2(3x-s)}{6} \right]^2 \, ds + \int_x^1 \left[ \frac{x^2(3s-x)}{6} \right]^2 \, ds
\]
\[
= [11x^2/420] + x^4/12 - x^6/36 - x^7/36
\]
and another integration yields
\[
\int_0^1 \left( \int_0^1 |G(x, s)|^2 \, ds \right) \, dx = \int_0^1 [11x^2/420] + x^4/12 - x^6/12 + x^6/36 - x^7/36 \, dx
\]
\[
= 11/1,680.
\]

Again, observe that our condition (3.6) is sharper than condition (3.4) and this illustrates another element of our advancement.

Let us further demonstrate the more aspects of our sharpened assumptions above through the discussion of some concrete cases.

**Remark 3.1.** If we consider the special case of \( f \) in (3.1) in the form
\[
f(y) = 10 \sin y + 1, \tag{3.7}
\]
then we see that the smallest Lipschitz constant that we can calculate so that (3.2) holds on \([0, 1] \times \mathbb{R}\) is \( L_0 = 10 \), which is formulated from the bound on |\( f'(y) \)|. Such an \( L_0 \) satisfies (3.6) but not (3.4). Thus, our \( f \) satisfies the conditions of Theorem 2.6 (with appropriate boundary conditions) and we can obtain information in unbounded sets regarding the existence, uniqueness and approximation of solutions.

On the other hand, if we consider \( f \) on a closed and bounded domain \([-R, R]\), for any constant \( R > 0 \), then we still cannot get (3.4) to hold for the above choice of \( L_0 = 10 \). Hence, the results of [11] do not apply to the \( f \) in this example.
Furthermore, we can rerun the same argument as above for the example
\[ f(y) = 385 \sin y + 1 \]
to show that (3.5) holds but (3.3) does not. Hence, the results of [12] do not apply to this \( f \) with \( L_0 = 385 \).

**Remark 3.2.** Furthermore, we consider the special case
\[ f(x, y) = x + 1 + 8y^2, \tag{3.8} \]
subject to clamped/free ends (1.5).

Choose \( R = 1/2 \) to form the set \( F \) in the statement of Theorem 2.7. Here, choose \( \theta_0 = 1/8 \) (with the remaining \( \theta_i \) not coming in to play) and a bound \( M \) on \( f \) over \( F \) can be chosen to be \( M = 4 \).

In addition, we see \( \partial f / \partial y \) is continuous and bounded on \( F \) by 8. Thus, we can choose our Lipschitz constant to be \( L_0 = 8 \). (In the notation of Theorem 2.7, we have \( L_0 = 8 \) and the remaining \( L_i \) are zero.)

Choosing \( p = q = 2 \), our \( \lambda_0 \) is contained in (3.6) (with the other \( \lambda_i \) not coming in to play). Thus, we have
\[ L_0 \lambda_0 = 8(1/1,680)^{1/2} < 1. \]

Hence, all of the conditions of Theorem 2.7 are satisfied and its conclusion may be applied to our problem.

On the other hand, if we try to verify or apply the conditions in [11] to our problem, then we run into an impossibility. For all \( x \in [0, 1] \) and \( |y| \leq M/8 \), where \( M > 0 \), the assumption in [11] becomes
\[ |f(x, y)| \leq 2 + 8(M/8)^2 \leq M, \]
which has only the solution \( M = 4 \). However, (3.4) takes the form
\[ L_0/8 = 8/8 < 1, \]
which is clearly impossible. Thus, the results in [11] do not apply to this example.

The aforementioned examples and discussion illustrate how the new results of this paper represent an advancement over traditional approaches, and how they are applicable to a wider class of problems.

### 3.2 Linear loading force

If the loading force on the beam is linear and given by \( f(x, y) = h(x)y + j(x) \), then the following fourth-order ordinary differential equation is obtained by:
\[ y^{(iv)} = h(x)y + j(x), \quad x \in [0, 1] \tag{3.9} \]
and we can form the following corollaries.

**Corollary 3.1.** Let \( h \) and \( j \) be continuous. If \( |h(x)| < (1,74,63,600/71)^{1/2} \) for all \( x \in [0, 1] \), then the elastic beam deflection BVP (3.9), (1.4) with linear loading force has a unique solution in \( C^4([0, 1]) \).

**Proof.** This is a special case of Theorem 2.1 with \( p = q = 2 \) and \( f(x, y) = h(x)y + j(x) \).

**Corollary 3.2.** Let \( h \) and \( j \) be continuous. If \( |h(x)| < (1,680/11)^{1/2} \) for all \( x \in [0, 1] \), then the elastic beam deflection BVP (3.9), (1.5) with linear loading force has a unique solution in \( C^4([0, 1]) \).

**Proof.** This is a special case of Theorem 2.6 with \( p = q = 2 \).
4 Discussion and conclusion

Let us briefly discuss our results and also identify some potential open problems for further research.

This paper generated new advancements into the existence and uniqueness of solutions to BVPs involving nonlinear, fourth-order differential equations. We sharpened traditional results by showing that a larger class of problems admit a unique solution and achieved this by drawing on the fixed-point theory in an interesting and alternative way via an application of Rus’s contraction mapping theorem. Our results add to the recent literature on BVPs [54,59–61] and collectively move towards a more complete understanding of their underlying theory and application.

There are many potential ways forward regarding the ideas of the present paper in both pure and applied forms. The application of Theorem 1.1 to all kinds of problems still appears to be underutilized and so there are opportunities to improve our understanding of solutions to a range of differential, integral and difference equations via Theorem 1.1. For example, important work into the area of nanoscale beams has been carried out in [62,63], where nonlocal and higher order theories have been used to capture effect sizes with nonlocal forms of boundary conditions. It is unknown if Theorem 1.1 can be used in these kinds of problems and this remains an open question.

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