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Finite-horizon optimal tracking control for constrained-input nonlinear interconnected system using aperiodic distributed nonzero-sum games

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Abstract
This paper proposes a distributed adaptive dynamic programming scheme to investigate the optimal tracking control problem for finite-horizon non-linear interconnected systems with constraint inputs under aperiodic sampling. A N-player nonzero-sum differential game system is constructed with the presented non-linear interconnected system and the tracking error system by introducing the augment vectors. To address the problems of constrained-input and finite-horizon control, a non-quadratic utility function and a finite-horizon cost function are utilized which will arise in the time-varying Hamilton–Jacobi (HJ) equation. Then, a periodic event-triggered scheme is designed to realize aperiodic sampling, where the consumption of communication resources is reduced and the Zeno behavior is avoided. Under the designed periodic event-triggered scheme, the time-varying HJ equation is almost impossible to get an analytical solution due to its hybrid properties and non-linearity. Therefore, the critic neural networks are used to estimate the optimal solution of the HJ equation, and the weight update law is constructed to guarantee the uniformly ultimate bounded of approximated errors. Further, the hybrid nonzero-sum differential game is confirmed to be uniformly ultimate bounded by using the Lyapunov theory. Finally, the obtained distributed PET control strategy is successfully applied to dispose the missile-target intercepter problem.

1 | INTRODUCTION

As a class of complex systems, the missile guidance systems can be modelled as the large-scale interconnected system that consists of several subsystems with interconnections. Note that the interconnection terms among subsystems propose a challenge in designing the stabilizing controller for interconnected systems, since the interconnection terms affect stability and performance of subsystems or even the entire interconnected system. Therefore, various control methods have been proposed to analysis and control the interconnected system, which included decentralized controller and distributed controller. As in [1, 2], the decentralized control strategy enabled interconnected systems to guarantee stability only when the interconnection terms among subsystems are weak. To overcome the limitation of the decentralized control approach, a distributed control strategy was proposed to guarantee the stability and transient performance of interconnected systems with strong interconnections, where the controller was designed by using the system information of local and neighboring subsystems [3]. Although plenty of theoretical results for interconnected systems have been obtained, those results focused on stability analysis of the systems, with few on the optimal tracking control problem.

Note that the optimal tracking controller for non-linear systems is hard to construct through exact mathematical derivation, since the corresponding Hamiltonian function is non-linear and coupled. To approximate the performance index function and the feedback controller, the adaptive dynamic programming (ADP) technique was proposed by using two neural networks (NNs) [4]. The ADP technique has been applied successfully to strict-feedback non-linear systems [5, 6], discrete-time non-linear systems [7, 8] and nonlinear switched
systems [9, 10]. Very recently, the ADP technique was extended to derive the decentralised tracking controller for non-linear interconnected systems [11, 12], [11] used an observer–critic structure-based to reconstruct unknown system dynamics and solve the coupled Hamilton–Jacobi–Bellman (HJB) equation, respectively. In contrast to [11], the optimality was considered in [12], and tracking error subsystems were guaranteed to be asymptotically stable by using a model-free ADP algorithm. Later, the authors in [13] put forward a distributed optimal tracking controller to ensure that large-scale systems with disturbances and saturating actuators were uniformly ultimately bounded (UUB). Nevertheless, the above ADP algorithms were designed without considering the limitation of network bandwidth and computing resources.

Along with the improvement of guidance systems, the modern guidance laws with high precision demand more network bandwidth and data storage space. By and large, the time-triggered communication scheme used in literatures [14, 15] will generate excessive redundant signals and even influence system stability. To reduce the consumption of communication resources, some aperiodic sampling schemes (ASSs), such as the continuous event-triggered (CET) scheme, the periodic event-triggered (PET) scheme, the self-triggered (ST) scheme and so on, were presented in [16–18]. Using these ASSs, the control strategies were updated only when the designed triggering conditions were breached, where the triggering condition was monitored continuously under the CET scheme [16], while the triggering condition was verified only periodically under the PET scheme [17]. Further, as in [18], under the ST scheme, the next triggering instant was precomputed based on previously received data and system dynamic knowledge. Thus, the ASSs can greatly reduce the consumption of communication resources as well as avoid unnecessary information transmission between system controllers and actuators. At present, the ASSs have been widely used to study problems of optimal control and tracking control. In [19], an event-triggered optimal control law for uncertain non-linear systems was constructed in terms of the ADP algorithm, where the actor–critic framework was used to approximate the optimal value function and control inputs. To simplify the ADP algorithm, critic NNs were used to approximate value functions, then the distributed optimal control strategies were proposed for non-linear interconnected systems [20]. Using the same framework as [20, 21] developed an optimal tracking controller for constrained non-linear systems under the ASSs, but until now, there are few reports concerning the optimal tracking control problem for interconnected systems under aperiodic sampling, especially under the PET scheme.

In fact, the interconnected system with strong interconnections can be converted into a NZS differential game by designing augment vectors, where the controller design for each sub-system is regarded as a player in NZS differential game. Based on the NZS differential game framework, the authors in [22] proposed a distributed control strategy to guarantee the stability and transient performance of interconnected systems with strong interconnections, where the controller was designed by using the system information of local and neighboring sub-

| TABLE 1 | Some related notations |
|---|---|
| Notations | Meaning |
| \( \mathbb{R} \) | The set of real numbers |
| \( \mathbb{R}^{n \times m} \) | The set of real \( n \times m \) matrices |
| \( \mathbb{R}^m \) | The \( m \)-dimensional Euclidean space |
| \( \tau \) | The transpose operation |
| \( \lambda_{\min}(\cdot) \) | The maximal eigenvalue of a matrix |
| \( \lambda_{\max}(\cdot) \) | The minimal eigenvalue of a matrix |

The remainder of this paper is organized as follows: The problem descriptions and transformations are given in Section 2. In Section 3, a PET scheme is proposed to save communication resources. Under this scheme, the distributed optimal tracking problem for the finite-horizon NZS differential game with input constraints is analyzed, then stability criteria are derived to insure the UUB of the corresponding closed-loop system. Section 4 proposes the distributed optimal tracking control strategy by critic NNs, meanwhile the tracking error and approximation error are proved to be UUB. In Section 5, the multi-missile cooperative guidance system is applied to verify this paper designed algorithm. Finally, conclusions are formulated in Section 6, and notations utilized in this paper are shown in Tables 1 and 2.
TABLE 2  Some related notations

| Notations | Meaning |
|-----------|---------|
| $V_{iM}$ | The velocity of the $i$th missile |
| $V_T$ | The target velocity |
| $u_i$ | The control vector perpendicular to the velocity vector of the $i$th missile |
| $r$ | The target control vector perpendicular to the velocity vector |
| $\alpha_i$ | The flight-path angle (FPA) of the $i$th missile |
| $a_{iM}$ | The lateral acceleration of the $i$th missile |
| $a_T$ | The lateral acceleration of the target |
| $\theta_i$ | The line-of-sight (LOS) angle |
| $\theta_j$ | The LOS angle rate |
| $r_i$ | The relative distance between the $i$th missile and the target |
| $\tau_T$ | The time constant of the target autopilot |
| $\tau_{iM}$ | The time constant of the $i$th missile autopilot |
| $v_i$ | The LOS rate along the LOS |
| $\beta$ | The target FPA |

2 | PROBLEM FORMULATION AND PRELIMINARIES

2.1 | Problem formulation

We consider the non-linear interconnected system, which is composed of $N$ input-constraints subsystems as

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i(t) + \sum_{j=1,j\neq i}^{N} \Gamma_{ij}(x_i,x_j),$$  
\hspace{2cm} \text{for } i, j \in \mathbb{I} = \{1, 2, ..., N\}, \quad (1)

where $x_i(t) \in \mathbb{R}^{n_i}$, $f_i(x_i) \in \mathbb{R}^{n_i}$, $g_i(x_i) \in \mathbb{R}^{n_i \times p_i}$, $u_i(t) = [u_{i1}, u_{i2}, ..., u_{ip_i}]^T \in \mathbb{R}^{p_i}$ with $u_{ij}(t) \leq \lambda_i$, $i \in \mathbb{I}$, $j = 1, 2, ..., p_i$ denote the state vector, the internal dynamic, the input gain function and the constrained control input for the $i$th subsystem. $\Gamma_{ij}(x_i,x_j) : \mathbb{R}^{n_i} \times \mathbb{R}^{n_j}$ is the interconnection term between the $i$th and $j$th subsystems.

Assumption 1. The functions $f(\cdot), g(\cdot), \Gamma_{ij}(\cdot), i, j \in \mathbb{I}, i \neq j$ are locally Lipschitz continuous on $\Omega_i \subset \mathbb{R}^{n_i}$. Further, $g(\cdot)$, $\forall i \in \mathbb{I}$ are bounded and satisfy $\|g(\cdot)\| \leq g_{iM}$, $i \in \mathbb{I}$.

For non-linear interconnected system (1), the target of optimal tracking control is to design $N$ optimal controllers $(u_{i1}^*, u_{i2}^*, ..., u_{ip_i}^*)$, ensure that the system states track the desired trajectories $\tilde{y}_i$ of every subsystem with finite time successfully, and the tracking error of every subsystem is defined by

$$\tilde{z}_i(t) = x_i(t) - \tilde{y}_i(t), \quad \text{for } i \in \mathbb{I}, \quad (2)$$

where the designed signal $\tilde{y}_i(t) \in \mathbb{R}^{n_i}$ is bounded and Lipschitz continuous, and satisfy

$$\dot{\tilde{y}}_i(t) = d_i(\tilde{y}_i), \quad d_i(0) = 0, \quad i \in \mathbb{I}.$$

Taking the time derivative of (2) yields,

$$\ddot{z}_i(t) = \dot{z}_i(t) - \dot{\tilde{y}}_i(t),$$

$$= f_i(x_i) + g_i(x_i)u_i(t) + \sum_{j=1,j\neq i}^{N} \Gamma_{ij}(x_i,x_j) - d_i(\tilde{y}_i), \quad \forall i \in \mathbb{I}. \quad (3)$$

2.2 | Problem transformation

In this section, a $N$-player differential game system is derived by defining augmented vectors $\tilde{z}_i(t) = [\tilde{y}_i^T(t), \tilde{y}_j^T(t)]^T \in \mathbb{R}^{2n_i}$,

$$Z(t) = [\tilde{z}_1^T(t), ..., \tilde{z}_N^T(t)]^T \in \mathbb{R}^{2N}, \quad i, j \in \mathbb{I}, \quad n = n_1 + \cdots + n_N,$$

$$\dot{Z}(t) = F(Z) + \sum_{j=1}^{N} G_j(\tilde{z}_j)u_j, \quad j \in \mathbb{I}, \quad (4)$$

where

$$F(Z) = [\tilde{y}_1^T(\tilde{y}_1), ..., \tilde{y}_N^T(\tilde{y}_N)] \in \mathbb{R}^{2N},$$

$$G_j(\tilde{z}_j) = [0, ..., 0, g_j(\tilde{z}_j + \tilde{y}_j), 0, ..., 0]^T \in \mathbb{R}^{2^{\mathbb{I}\setminus j}},$$

$$\dot{\tilde{y}}_j(\tilde{z}_j) = f_j(\tilde{z}_j + \tilde{y}_j), \quad d_i(\tilde{y}_i) = \sum_{j=1,j\neq i}^{N} \Gamma_{ij}(\tilde{z}_i,\tilde{z}_j) + d_i(\tilde{y}_i), \quad i, j \in \mathbb{I}. \quad (3)$$

Remark 1. Combining with the definition of $G_i, \forall i \in \mathbb{I}$ and Assumption 1, we know that the augmented vector $\tilde{z}_i$ is bounded, and satisfy $\|G_i\| \leq G_{Mi}$, where $G_{Mi}, i \in \mathbb{I}$ are positive constants.

To realize the target of optimal tracking control, the finite-horizon value function of NZS differential game (4) is defined as

$$V_i(Z(t),u_1,...,u_N) = \int_{t_i}^{t_f} Z^T(\tau)Q_i Z(\tau) + U_i(u_1,...,u_N) d\tau + \Psi_i(Z(t_f),t_f), \quad i \in \mathbb{I}, \quad (5)$$

where $Q_i = \text{diag}(Q_{i1}, Q_{i2}, ..., Q_{iN}) \in \mathbb{R}^{2n_i \times 2n_i}$, $Q_{ij} = \text{diag}(q_{ij1}, ..., q_{ijn})$ and $q_{ij1} \cdots q_{ijN}$ are the positive definite matrix and the zero matrix, respectively. $\Psi_i(Z(t_f),t_f), \forall i \in \mathbb{I}$ denote the terminal cost associated with the $i$th player, where $t_f$ is the fixed final time. As in [25], we use generalized non-quadratic functions to deal with the input constraint problems,

$$U_i(u_1,...,u_N) = 2 \sum_{j=1}^{N} \int_{t_i}^{t_f} \lambda_j \tanh \left( \frac{v_j}{\lambda_j} \right) R_{ij} d\tau, \quad (6)$$

$$\lambda_j = \text{diag}(0,1,0,1,0,1) \in \mathbb{R}^{2 \times 2}, \quad Q_{ij} = \text{diag}(q_{ij1}, ..., q_{ijn})$$

and $q_{ij1} \cdots q_{ijN}$ are the positive definite matrix and the zero matrix, respectively.
where \( \tanh(\cdot) \) is a strictly monotonic odd function, and it satisfy \( |\tanh(\cdot)| < 1 \). Note that \( U_i(u_1, \ldots, u_N) \) is positive definite since \( \Phi^{-1}(\cdot) \) is a monotonic odd function and \( R_{ij} = \text{diag}\{r_{1j}, \ldots, r_{pj}\} > 0 \).

The infinitesimal equivalent to (5) can be derived if \( V'(Z, t) \in C^1 \),

\[
-V_i = V_{Zi}^T\left(F(Z(t)) + \sum_{j=1}^{N} G_{ij}u_j\right) + Z_i^T(t)Q_iZ(t) + U_i(u_1, \ldots, u_N),
\]

(7)

where \( V_{Zi} = \frac{\partial V(Z(t))}{\partial Z} \) and \( V_{zi} = \frac{\partial V(Z(t))}{\partial Z} \) are bounded by \( B_{zi} \) and \( B_{Zi}, \) respectively, and the terminal cost for the finite-horizon value function (5) is

\[
V_i(Z(t), t_f) = \Psi_i(Z(t_f), t_f), i \in I.
\]

Therefore, the Hamiltonian function of system (4) is defined by

\[
H_i(Z, t, U_i, \ldots, U_N) = V_{Zi}^T\left(F(Z) + \sum_{j=1}^{N} G_{ij}u_j\right) + Z_i^T(t)Q_iZ(t) + U_i(u_1, \ldots, u_N),
\]

According to the definition of Nash equilibrium [26], assume that the following conditions hold for all \( u_i^*, i \in I \)

\[
V_i^*(Z(t), t) = \min_{u_i} \left\{ V_i(Z(t), t) + \sum_{j=1}^{N} u_j \sum_{j=1}^{N} R_{ij} + \lambda_j^2 \ln \left( 1 - \left( \frac{u^*_j}{\lambda_j} \right)^2 \right) \right\}, i \in I.
\]

(9)

Then, the optimal value function \( V_i^*(Z, t), \forall i \in I \) is obtained

\[
V_i^*(Z(t), t) = \min_{u_i} \left\{ \Psi_i(Z(t), t_f) + \int_{t_f}^{t} \sum_{j=1}^{N} \left( Z_j^T(t)Q_jZ(t) + U_j(u_1, \ldots, u_N) \right) dt \right\}.
\]

(10)

Applying the stationarity conditions, the optimal tracking strategy and the semi-positive definite functional associated with each player of system (4) are calculated, respectively

\[
\frac{\partial H_i}{\partial u_i} = 0 \Leftrightarrow u_i^* = -\lambda_i \tanh\left( \frac{1}{2\lambda_i} R_{ii}^{-1} G_i^T(z_i) V_i^* \right),
\]

\[
\frac{\partial H_i}{\partial \lambda_i} = 0 \Leftrightarrow \lambda_i = \frac{1}{2} R_{ii}^{-1} G_i^T(z_i) V_i^*,
\]

(11)

\[
U_i(u_1^*, \ldots, u_N^*) = \sum_{j=1}^{N} \left[ 2\lambda_j \tanh^{-1} \left( \frac{u^*_j}{\lambda_j} \right), R_{ij} u_j^* + \lambda_j^2 \ln \left( 1 - \left( \frac{u^*_j}{\lambda_j} \right)^2 \right) \right], \forall i, j \in I.
\]

(12)

where \( D_i = \frac{1}{2\lambda_i} R_{ii}^{-1} G_i^T(z_i) V_i^* T, \quad 1 = [1, \ldots, 1]^T, \quad \text{and} \quad R_{ij} = [r_{1j} \ldots r_{pj}] \in \mathbb{R}^{N \times P}, \forall i, j \in I. \)

Inserting equations (10) and (11) into the Hamiltonian function, the time-varying HJ equation can be derived

\[
-(V_i^*)^T = (V_i^*)^TF(Z) - \sum_{j=1}^{N} G_{ij} \lambda_j \tanh(D_i)
\]

\[
+ Z_i^T(t)Q_iZ(t) + U_i(u_1, \ldots, u_N), \forall i, j \in I.
\]

(13)

Remark 2.

(1) Under the finite-horizon scenario, the cost function (5) is the time-dependent solution for the coupled HJ equation (13), which is harder to solve than under the infinite-horizon case.

(2) Different from existing literatures, a additional cost is added to the \( i \)th subsystem. Therefore, the influence of the optimal control strategy for the neighboring subsystems is considered in the \( i \)th subsystem optimization, which ensures that finite-horizon cost function (5) is the optimal solution for both all subsystems and the entire system.

In generally, the optimal solution of the time-varying HJ equation for every subsystem is almost impossible to calculated since its a coupled partial differential equation. Therefore, the ADP technique is used to approximate the optimal solution through the use of critic NNs, then the distributed optimal tracking controller is derived approximately. However, the time-triggered controllers always inevitably waste computing resources and communication costs as in [14, 15]. To settle this issues, the PET scheme is used to design the distributed optimal tracking control strategy for non-linear interconnected system (1).

3 | DISTRIBUTED OPTIMAL TRACKING STRATEGIES VIA PET SCHEME

First, we design a PET scheme to reduce the frequency of information transmission and controller update while ensuring the tracking performance of error system (3). Then, event-based HJ equations and control strategies for \( N \)-player NZS differential game system (4) are derived, in which the Zeno behaviour is avoid since the time-triggered periodic is \( \bar{\tau} > 0 \). Further, a Theorem is proposed to ensure that the stability of the corresponding closed-loop NZS differential game.

3.1 | Periodic event-triggered scheme

As shown in Figure 1, the sampling rate of system states for the PET control system is a constant \( \bar{\tau} > 0 \), and sampling signals
The diagram of the PET control scheme

Define system states at triggering instant as

$$\zeta_i(t) = G_j(\tilde{z}_i(t)), r = 1, 2, ..., N, i \in I,$$  \hspace{1cm} (14)

where $t, \tilde{t}, r = 1, 2, ..., N$ denotes the triggered instance.

Combing with the definition of the augmented vector $Z(t)$, state vectors for the NZS differential game system at triggering instants are defined as $Z_i = [\tilde{z}_i(t), z_{i-N}^T] \in R^{2n}$. For the $i$th subsystem, the gap between the last triggered state $\tilde{z}_i$ and the current sampled state $\tilde{z}_i(t)$ is

$$e_i(t) = \tilde{z}_i(t) - \tilde{z}_i(t+\tilde{t}), \forall i \in I, r \in N, i \in I,$$

and the next triggered instant is denoted as

$$t_{r+1} = t_{r} + \min[\tilde{t} | e_i(t) \geq \varepsilon_f, i \in I, \forall i \in I],$$

where $\varepsilon_f$ is the trigger threshold.

Note that the triggering gap $e_i(t)$ is reset to zero when an event is triggered at $t = f_i$. Meanwhile, the optimal tracking controller and the positive definition functional are updated,

$$u_i^*(\tilde{z}_i) = -\lambda_i \tanh\left(\frac{1}{2\lambda_i} R_{ij}^{-1} \Gamma_j^T(\tilde{z}_i) \tilde{G}_j \tilde{V}_i^*\right),$$

$$V_i \in [t, t_{r+1}], \forall i \in I,$$  \hspace{1cm} (15)

where $\tilde{G}_j(\tilde{z}_i) = \tilde{z}_j(t)$, $\tilde{G}_j(\tilde{z}_i) = \tilde{G}_j$, $D_j = \frac{1}{2\lambda_j^2} R_{ij}^{-1} \tilde{G}_j^T \tilde{G}_j \tilde{V}_i^*$. Then, the hybrid NZS differential game can be obtained

$$Z(t) = F(Z) + \sum_{j=1}^{N} G_j(\tilde{z}_i) u^*_j(\tilde{z}_i), \hspace{1cm} j \in I,$$

Combining (13) and (15), the PET optimal tracking HJ equation is derived

$$H_i(Z(t), V_i, \mu_{Nj}^*, ..., \mu_{Nj}^*)$$

$$= \frac{\partial V_i^*}{\partial t} + \left(\frac{\partial V_i^*}{\partial Z}\right)^T F(Z) + \sum_{j=1}^{N} G_j(\mu_{Nj}^*)$$

$$+ Z^T(t) Q_i Z(t) + U_i(\mu_{1j}^*, ..., \mu_{Nj}^*),$$

$$= \frac{\partial V_i^*}{\partial t} + \left(\frac{\partial V_i^*}{\partial Z}\right)^T F(Z) - \left(\frac{\partial V_i^*}{\partial Z}\right)^T \sum_{j=1}^{N} \alpha_j G_j \tanh(D_j)$$

$$+ Z^T(t) Q_i Z(t) + U_i(\mu_{1j}^*, ..., \mu_{Nj}^*).$$

Assumption 2. The optimal tracking strategy of every subsystem is Lipschitz continuous with respect to $e_i(t)$, that is, $||u_i^*(\tilde{z}_i(t)) - u_i^*(\tilde{z}_i(t'))|| \leq \mathcal{L}_u \|e_i(t)||$. Further, $||u_i^*(\cdot)|| \leq \mathcal{B}_u$ is satisfied, where $\mathcal{L}_u$ and $\mathcal{B}_u$ are positive constants.

Remark 4. As in [16, 18], the designed controllers under the mechanisms of the CTE and the ST may be occur Zeno behaviours (the occurrence of an infinite number of events in finite time), which will degrade system performance and destroy the system stability. Compared with the above schemes, triggering conditions only need to be verified periodically under the PET scheme as shown in Figure 1, which reduces the resource consumption during the detection process and avoids the Zeno phenomenon.
3.2 Distributed optimal tracking control strategy via PET scheme

Theorem 1. Consider the augmented N2S differential game system (4). Let Assumptions 1-2 hold, and suppose \( V_{i}^{*}(Z,t), i \in \mathbb{I} \) be a solution of the HJ equation associated with the ith player. Then, the UUB of the corresponding closed-loop system can be guaranteed under the PET optimal controller, if triggering conditions below are satisfied

\[
\|e_{i}(t)\| \leq \sqrt{\frac{1 - \eta_{i}^{2}}{\lambda_{2}}\lambda_{\min}(Q)}\|Z(t)\|^{2} + U_{i}^{*}(\mathbf{u}_{i}^{1}, ..., \mathbf{u}_{N}^{N}) \lambda_{2}, \quad i \in \mathbb{I}, \tag{21}
\]

where \( 0 < \eta_{i} < 1, i \in \mathbb{I} \) are designed parameters, \( \lambda_{\min} = \frac{N}{2}B_{V}^{2} \), and \( \lambda_{2} = \frac{N}{2}G_{m}L_{e}^{2} \).

Proof. Choosing the Lyapunov function as \( L_{0}(Z,t) = \sum_{i=1}^{N} V_{i}^{*}(Z,t) \), where \( V_{i}^{*}(Z,t) \) is the solution of the HJ equation (19). Taking the time derivative of \( L_{0}(Z,t) \), we have

\[
L_{0}(Z,t) = \sum_{i=1}^{N} \left[ V_{i}^{*} + (V_{i}^{*})^{T} \left( F(Z) + \sum_{j=1}^{N} G_{ij}\mu_{jr}^{*} \right) \right].
\]

From (13), we can obtain

\[
\sum_{i=1}^{N} (V_{i}^{*})^{T} F(Z) = -\sum_{i=1}^{N} \left( (V_{i}^{*})^{T} \sum_{j=1}^{N} G_{ij}\mathbf{u}_{jr}^{*} \right) + Z^{T}(t)Q_{i}Z(t) + U_{i}(\mathbf{u}_{i}^{1}, ..., \mathbf{u}_{N}^{N}).
\]

Therefore, \( L_{0}(Z,t) \) can be rewritten as

\[
L_{0}(Z,t) = \sum_{i=1}^{N} \left[ (V_{i}^{*})^{T} \sum_{j=1}^{N} G_{ij}\mu_{jr}^{*} - \mathbf{u}_{i}^{*} \right] + Z^{T}(t)Q_{i}Z(t) - U_{i}(\mathbf{u}_{i}^{1}, ..., \mathbf{u}_{N}^{N}).
\]

Note that

\[
\sum_{i=1}^{N} (V_{i}^{*})^{T} \sum_{j=1}^{N} G_{ij}\mu_{jr}^{*} - \mathbf{u}_{i}^{*} \leq \frac{1}{2} \left[ \sum_{i=1}^{N} (V_{i}^{*})^{T} \sum_{j=1}^{N} V_{j}^{*} + \sum_{i=1}^{N} G_{rj}^{T} \sum_{j=1}^{N} G_{ij}(\mu_{jr}^{*} - \mathbf{u}_{i}^{*}) \right], i \in \mathbb{I}.
\]

Therefore, \( L_{0}(Z(t),t) \) becomes

\[
L_{0}(Z(t),t) \leq \sum_{i=1}^{N} \left( \lambda_{ai} + \lambda_{bi} \|e_{i}(t)\|^{2} \right)
- \eta_{i}^{2}\lambda_{\min}(Q)\|Z(t)\|^{2} - U_{i}(\mathbf{u}_{i}^{1}, ..., \mathbf{u}_{N}^{N})
- (1 - \eta_{i}^{2})\lambda_{\min}(Q)\|Z(t)\|^{2}.
\]

It is worth pointing out that (22) yields (23), if the triggering condition (21) holds

\[
L_{0}(Z(t),t) \leq \sum_{i=1}^{N} \left( \lambda_{ai} - \eta_{i}^{2}\lambda_{\min}(Q)\|Z(t)\|^{2} \right).
\]

Therefore, \( L_{0}(Z(t),t) < 0 \) only if \( Z(t) \) is out of the following compact set

\[
\Omega_{Z} = \left\{ Z : \|Z\| \leq \min_{i} \left\{ \sqrt{\frac{\lambda_{ai}}{\eta_{i}^{2}\lambda_{\min}(Q)}}, i \in \mathbb{I} \right\} \right\}.
\]

According to the Lyapunov extension theorem, the UUB of the tracking error trajectory of N-player differential game system (4) is proved under the PET optimal control strategy \( \mu_{jr}^{*} \) and the triggering condition (21).

4 DISTRIBUTED OPTIMAL TRACKING CONTROLLER DESIGN VIA APERIODIC CRITIC NNS

In this section, a group of approximate solution for the event-based tracking error system is obtained by using adaptive dynamic programming technique, where critic NNs are used to reconstruct the optimal value function and the optimal control strategy.

4.1 The aperiodic critic NNs design

According to the universal approximation quality of NNs [27–29], the finite-horizon cost function and its terminal cost are approximated through the utilization of the following time-dependent active function and ideal weights:

\[
V_{i}(Z(t),t) = \hat{\sigma}_{i}^{T}(Z,t) - t + \varepsilon_{i}(Z,t) - t, i \in \mathbb{I}, \tag{24}
\]

where \( \hat{\sigma}_{i} \in \mathbb{R}^{L_{\hat{\sigma}}}, \sigma_{i} \in \mathbb{R}^{L_{\sigma}}, \forall i \in \mathbb{I} \) are the ideal weights and the activation functions for critic NNs, respectively. \( L_{\hat{\sigma}} \) denotes the number of hidden-layer neurons and \( \varepsilon_{i}(Z,t) - t \) represents the approximation errors. In addition, the terminal cost function for
the $i$th subsystem is approximated by

$$ V_i(Z(t_j), t_j) = \partial_i^T \sigma(Z(t_j), 0) + \varepsilon_i(Z(t_j), 0), i \in \mathbb{I}. \quad (25) $$

Note that the solution of the HJ equation for every subsystem can be approximated by using critic NNs, and the partial derivative of $V_i(Z,t)$ can be calculated as

$$ V_{Zi} = \nabla_Z^T \sigma(Z, t_j - t) \vartheta_i + \nabla_Z \varepsilon_i(Z, t_j - t), \quad (26) $$

$$ V_{\vartheta i} = \nabla_{\vartheta}^T \sigma(Z, t_j - t) \vartheta_i + \nabla_{\vartheta} \varepsilon_i(Z, t_j - t), \quad (27) $$

where $\nabla_Z \sigma(Z, t_j - t) = \frac{\partial \sigma(Z(t_j - t))}{\partial Z}$, $\nabla_{\vartheta} \sigma(Z, t_j - t) = \frac{\partial \sigma(Z(t_j - t))}{\partial \vartheta_i}$.

Combining with (11) and (15), the optimal tracking control strategy under the time-triggered and the event-triggered scheme can be written as

$$ u^*_{i} = -\lambda_i \tanh \left( \frac{1}{2 \lambda_i} R^{-1}_{ji} \hat{C}_{ji} \nabla_Z \sigma(Z, t_j - t) \vartheta_i \right) + \nabla_Z \varepsilon_i(Z, t_j - t) \vartheta_i, \quad (28) $$

$$ \mu^*_{ji} = -\lambda_i \tanh \left( \frac{1}{2 \lambda_i} R^{-1}_{ji} \hat{C}_{ji} \nabla_Z \sigma(Z, t_j - t) \vartheta_i \right) + \nabla_Z \varepsilon_i(Z, t_j - t) \vartheta_i, \quad (29) $$

$$ U_i(\mu^*_{1i}, ..., \mu^*_{Ni}) = \sum_{j=1}^{N} 2\lambda_j \tanh^{-1} \left( \frac{\mu^*_{ji}}{\lambda_j} \right) R_{ji} \mu^*_{ji} + \lambda_j^2 R_{ji} \ln \left( 1 - \left( \frac{\mu^*_{ji}}{\lambda_j} \right)^2 \right), \quad (30) $$

where

$$ D^*_j = \frac{1}{2 \lambda_j} R^{-1}_{ji} \hat{C}_{ji} \nabla_Z^T \sigma(Z, t_j - t) \vartheta_i, $$

$$ \varepsilon^*_j = -\frac{1}{2} (1 - \tanh^2 (\xi)) R_{ji} \hat{C}_{ji} \nabla_Z \varepsilon_i(Z, t), $$

$$ \varepsilon^*_ji = -\frac{1}{2} (1 - \tanh^2 (\xi)) R_{ji} \hat{C}_{ji} \nabla_Z \varepsilon_i(Z, t). $$

Inserting (26) and (27) into (20), the Hamiltonian function becomes

$$ H_i(Z, \mu^*_1, ..., \mu^*_N) = \partial_i^T \nabla_Z \sigma(Z, t_j - t) \left( F(Z) + \sum_{j=1}^{N} G_j \mu^*_j \right) + Z^T Q Z(t) + \nabla_i^T \sigma(Z, t_j - t) \vartheta_i + U_i(\mu^*_1, ..., \mu^*_N), \quad (31) $$

$$ = e_{id} \forall i \in \mathbb{I}, \quad (32) $$

where $e_{id} = -\nabla_i \varepsilon_i(Z, t) - \nabla_i^T \varepsilon_i(Z, t) (F(Z) + \sum_{j=1}^{N} G_j \mu^*_j)$ denotes the residual error of the $i$th subsystem.

Due to the ideal weights for all subsystems are unknown, critic NNs are constructed to estimate the cost function,

$$ V_i(Z(t), t) = \partial_i^T \sigma(Z, t_j - t), \quad (33) $$

$$ \dot{V}_i(Z(t_j), t_j) = \partial_i^T \sigma(Z, t_j, 0), \quad (34) $$

where $\hat{\theta}_i \in \mathbb{R}^{l \times n}$ is the estimate value of $\theta_i, \forall i \in \mathbb{I}$, and $Z(t_j)$ can be derived based on system dynamics and current states.

Similarly, the partial derivative of $\dot{V}_i(Z,t)$ with respect to $Z$ and $t$ are derived,

$$ \frac{\partial \dot{V}_i}{\partial Z} = \nabla_Z \sigma(Z, t_j - t) \dot{\theta}_i, \quad (35) $$

$$ \frac{\partial \dot{V}_i}{\partial t} = \nabla_t \sigma(Z, t_j - t) \dot{\theta}_i. \quad (36) $$

Using (11) and (12), the approximated optimal control strategy and positive definition function are obtained

$$ \hat{u}_i = -\lambda_i \tanh(D_i), \quad (37) $$

$$ U_i(\hat{u}_1, ..., \hat{u}_N) = \sum_{j=1}^{N} \left[ 2\lambda_j \tanh^{-1} \left( \frac{\hat{u}_j}{\lambda_j} \right) R_{ji} \hat{u}_j + \lambda_j^2 R_{ji} \ln \left( 1 - \left( \frac{\hat{u}_j}{\lambda_j} \right)^2 \right) \right], \quad (38) $$

where

$$ \hat{u}_j = \frac{1}{2 \lambda_j} R^{-1}_{ji} \hat{C}_{ji} \nabla_Z^T \sigma(Z, t_j - t) \vartheta_i + \lambda_j^2 R_{ji} \ln \left( 1 - \tanh^2 (\xi) \right), \quad (39) $$
where $\tilde{D}_i = \frac{1}{\lambda_i} R_j^{-1} C_j \nabla_j \sigma(Z, t_j - t) \hat{\theta}_i, i = [1, \ldots, I]^T \in \mathbb{R}^N$ and $R_{ji} = [r_{ji1}, \ldots, r_{jin}], \forall i \in I$.

Under the PET scheme, (37)-(39) becomes

$$\mu_{jr} = -\lambda_i \tanh(\tilde{D}_i),$$

$$U_i(\mu_{1r}, \ldots, \mu_{Nr}) = \sum_{j=1}^{N} \left[ 2\lambda_i \tanh^2 \left( \frac{\mu_{jr}}{\lambda_i} \right) R_j \tilde{\mu}_{jr} \right] + \lambda_i^2 R_{ij} \ln \left( 1 - \tanh^2 (\tilde{D}_j) \right),$$

$$\sum_{j=1}^{N} \left[ \lambda_i \tilde{\mu}_{jr} \nabla_j \sigma(Z, t_j - t) \tilde{G}_j R_j^{-1} R_{ij} \tanh(\tilde{D}_j) \right] = \sum_{j=1}^{N} \lambda_i [2\lambda_i \tanh^2 (\frac{\mu_{jr}}{\lambda_i}) R_j \tilde{\mu}_{jr} + \lambda_i^2 R_{ij} \ln (1 - \tanh^2 (\tilde{D}_j))].$$

To deal with the optimal game system, both the time-varying cost function and its terminal constraints should be minimized along the system states. Inspired by [14, 19, 20], the following total squared error is defined

$$E_{total} = \sum_{j=1}^{N} e_{\theta j}^T e_{\theta j} + e_{\phi j}^T e_{\phi j}.$$
norm bounded, and their upper bound are $\Delta_{M\mu}$, $B_{\varepsilon_Z}$ and $B_{\varepsilon_{\mu\mu}}$, respectively. The gradient of $\sigma(Z, t)$ and $\varepsilon(Z, t)$ are also norm bounded, and satisfy $\|\nabla_2\sigma(Z, t)\| \leq B_{\sigma_Z}$, $\|\nabla_2\varepsilon(Z, t)\| \leq B_{\varepsilon_Z}$.

Based on the above assumptions, it is clear that there exist two constants $B_{\varepsilon_w}$ and $B_{\varepsilon_{\mu\mu}}$ such that $\|\varepsilon_{\mu\varepsilon}^*\| \leq B_{\varepsilon_w}$ and $\|\varepsilon_{\mu\mu}^*\| \leq B_{\varepsilon_{\mu\mu}}$.

**Theorem 2.** Under Assumptions 1–3 and the PET scheme, consider the distributed optimal tracking control strategy (40) and the critic NNs tuning law (47). The corresponding closed-loop system and $\tilde{\theta}_i, \forall i \in I$ are guaranteed to be UUB, if inequalities below hold

$$
\|\tilde{\theta}_i\| > \sqrt{\frac{\lambda_{\mu\mu}}{\lambda_{ii}}},
$$

(48)

$$
\|\varepsilon_{\mu\varepsilon}(t)\| \leq \sqrt{\frac{\eta \lambda_{\min}(Q_i)}{\lambda_{ii}}} \|Z(t)\|,
$$

(49)

$$
\|Z(t)\| > \sqrt{\frac{\lambda_{ii}}{\eta \lambda_{\min}(Q_i)}}, \forall i \in I.
$$

(50)

where

$$
\lambda_{jj} = \lambda_{j} B_{\sigma_Z} G_{Mj} \|R_{j}^{-1}\| R_{j} + \frac{1}{\lambda_{j}} \|R_{j}\| \|G_{Mj} B_{\sigma_Z}\|,
$$

$$
\lambda_{ij} = \frac{1}{\lambda_{j}} \|R_{j}\| \|G_{Mj} B_{\varepsilon_Z}\| R_{j} B_{\sigma_Z} \|R_{j}\| R_{j} + 2 \lambda_{j} \|\tilde{\theta}_j\| B_{\sigma_Z} \|R_{j}\| R_{j} B_{\mu\mu}\|
$$

$$
+ 2 \lambda_{ij} + \|\tilde{\theta}_j\| B_{\sigma_Z} \|R_{j}\| R_{j} B_{\mu\mu} + \lambda_{j} \lambda_{ij} B_{\varepsilon_Z} B_{\sigma_Z} R_{j},
$$

$$
\lambda_{ii} = \frac{N}{2} \left( \|\tilde{\theta}_i\|^2 B_{\varepsilon_Z} + 2 \|\tilde{\theta}_i\|^2 B_{\varepsilon_{\mu\mu}} + 2 \|\tilde{\theta}_i\| \|\tilde{\theta}_i\|^2 \right)
$$

$$
+ G_{M\mu} \|R_{j}\| \|R_{j}\| \|B_{\sigma_Z}\| + 2 \lambda_{\mu\mu}^2,
$$

$$
\lambda_{ii} = 4 G_{M\mu} A_{\lambda}^2 + 2 G_{M\mu} \epsilon_{\mu\varepsilon}^2 + 2 \|\tilde{\theta}_i\| \|\varepsilon_{\mu\varepsilon}^*\|^2
$$

$$
+ 2 \|\tilde{\theta}_i\| B_{\sigma_Z} \|\varepsilon_{\mu\varepsilon}^*\|^2 + N \lambda_{\mu\mu}^2,
$$

$$
\lambda_{ii} = \frac{N}{2}\lambda_{\min}(\tilde{\theta}_i^T \tilde{\theta}_i) + \frac{5}{8} \lambda_{\min}(\tilde{\theta}_i^T \tilde{\theta}_i) - \lambda_{ii}.
$$

**Proof.** Choosing the Lyapunov function as follows:

$$
L(t) = L_1(t) + L_2(t) + L_3(t),
$$

where

$$
L_1(t) = \sum_{i=1}^{N} V^*_i(Z(t)), \quad L_2(t) = \sum_{i=1}^{N} V^*_i(Z_i(t)),
$$

$$
L_3(t) = \frac{1}{2} \tilde{\theta}_i^T(T) \tilde{\theta}_i(T).
$$

Case 1:

When $t \in [t, t+\tilde{\tau}]$, that is, events are not triggered. It is clear that $V^*_i(Z(t), t) = 0, \forall i \in I$.

Taking the time derivative of $L_1(t), t \in [t, t+\tilde{\tau}]$,

$$
\dot{L}_1(t) = \sum_{i=1}^{N} \left\{ \frac{\partial V^*_i}{\partial t} + \left( \frac{\partial V^*_i}{\partial Z} \right)^T F(Z) + \sum_{j=1}^{N} G_j \mu_{j} \right\}.
$$

(51)

From Equation (13), we obtain

$$
\frac{\partial V^*_i}{\partial t} + \left( \frac{\partial V^*_i}{\partial Z} \right)^T F(Z) = -Z^T(t)Q_i(t) - U_i(u_i^*, ..., u_N^*)
$$

$$
- \left( \frac{\partial V^*_i}{\partial Z} \right)^T \sum_{j=1}^{N} G_j u_j, \forall i, j \in I.
$$

(52)

Due to $U_i(u_i^*, ..., u_N^*)$ being positive semi-definite, and substituting Equation (52), $\dot{L}_1(t)$ becomes

$$
\dot{L}_1(t) \leq \sum_{i=1}^{N} \left\{ -Z^T(t)Q_i(t) + \left( \frac{\partial V^*_i}{\partial Z} \right)^T \sum_{j=1}^{N} G_j (\mu_{j}^* - u_j^*) \right\}
$$

$$
+ \left( \frac{\partial V^*_i}{\partial Z} \right)^T \sum_{j=1}^{N} G_j (\mu_{j}^* - u_j^*) \right\},
$$

$$
\leq \sum_{i=1}^{N} -Z^T(t)Q_i(t) + \left( \frac{\partial V^*_i}{\partial Z} \right)^T \sum_{j=1}^{N} G_j (\mu_{j}^* - u_j^*) \right\},
$$

$$
\leq \sum_{i=1}^{N} -Z^T(t)Q_i(t) + N \|\tilde{\theta}_i^T \nabla_2 \sigma(Z(t), t) - t\| \nabla_2 \varepsilon(Z(t), t) \| \|V_2 \varepsilon(Z(t), t) \| + \left( \frac{\partial V^*_i}{\partial Z} \right)^T \sum_{j=1}^{N} G_j (\mu_{j}^* - u_j^*) \right\},
$$

$$
+ \|G_i(\varepsilon_{\mu\varepsilon}(Z(t), t) - t)\| \|\varepsilon_{\mu\varepsilon}(Z(t), t)\| + \left( \frac{\partial V^*_i}{\partial Z} \right)^T \sum_{j=1}^{N} G_j (\mu_{j}^* - u_j^*) \right\},
$$

$$
\leq \sum_{i=1}^{N} -Z^T(t)Q_i(t) + 2N \|\tilde{\theta}_i^T \nabla_2 \sigma(Z(t), t) - t\| \|\varepsilon_{\mu\varepsilon}(Z(t), t)\| + \left( \frac{\partial V^*_i}{\partial Z} \right)^T \sum_{j=1}^{N} G_j (\mu_{j}^* - u_j^*) \right\},
$$

$$
\leq \sum_{i=1}^{N} -Z^T(t)Q_i(t) + 2N \|\tilde{\theta}_i^T \nabla_2 \sigma(Z(t), t) - t\| \|\varepsilon_{\mu\varepsilon}(Z(t), t)\|^2 + \frac{N}{2} C_{M\mu}^2 I_{\mu\mu} \|\varepsilon_{\mu\varepsilon}(t)\|^2
$$

$$
+ \|G_i(\varepsilon_{\mu\varepsilon}(Z(t), t) - t)\| \|\varepsilon_{\mu\varepsilon}(Z(t), t)\| + \left( \frac{\partial V^*_i}{\partial Z} \right)^T \sum_{j=1}^{N} G_j (\mu_{j}^* - u_j^*) \right\},
$$

$$
\leq \sum_{i=1}^{N} -Z^T(t)Q_i(t) + 2N \|\tilde{\theta}_i^T \nabla_2 \sigma(Z(t), t) - t\| \|\varepsilon_{\mu\varepsilon}(Z(t), t)\|^2
$$

$$
+ 8 \lambda_{\mu\mu}^2 C_{M\mu}^2 + 2 G_{M\mu} \epsilon_{\mu\varepsilon}^2 + N \lambda_{\mu\mu}^2,
$$

$$
\leq \sum_{i=1}^{N} -Z^T(t)Q_i(t) + \lambda_{\mu\mu} \|\varepsilon_{\mu\varepsilon}(t)\|^2 + \lambda_{\mu\mu}.
$$

\(\Delta\sum_{i=1}^{N} -Z^T(t)Q_i(t) + \lambda_{\mu\mu} \|\varepsilon_{\mu\varepsilon}(t)\|^2 + \lambda_{\mu\mu}.

Using (47), the time derivative of $L_\lambda(t)$ is obtained

$$
L_\lambda(t) = \sum_{i=1}^{N} \frac{1}{\beta_i} \dot{\theta}_i^T \dot{\theta}_i
$$

$$
= \sum_{i=1}^{N} \ddot{\theta}_i^T \ddot{\phi}_i \dot{\phi}_i (\theta_i^T \nabla_Z \sigma(Z, t_j - t) \sum_{j=1}^{N} G_j (\mu_j^r - \mu^r_p))
$$

$$
- \ddot{\theta}_i^T \ddot{\phi}_i \ddot{\phi}_i \ddot{\theta}_i - \ddot{\theta}_i - \ddot{\phi}_i \ddot{\phi}_i \dot{\theta}_i + \ddot{\theta}_i^T \ddot{\phi}_i \ddot{\phi}_i \dot{\theta}_i + U_i (\mu_j^r, ..., \mu_N^r) + U_i (\mu_j^r, ..., \mu_N^r) + \ddot{\phi}_i \ddot{\phi}_i \dot{\theta}_i + \ddot{\phi}_i \ddot{\phi}_i \sigma' (Z(t_j), 0) \dot{\theta}_i
$$

The following inequality is derived by means of the formula of Tailor

$$
\| \tanh(D^p_j) - \tanh(D_j) \|^2
$$

$$
= \| \frac{\partial \tanh(D_j)}{\partial D_j} (D^p_j - D_j) + o((D^p_j - D_j)^2) \|^2
$$

$$
= \| \frac{1}{2\lambda_j} (1 - \tanh^2(D_j)) R_{lj} \sum_{i=1}^{N} G_j (\mu_j^r - \mu^r_p) + o((D^p_j - D_j)^2) \|^2
$$

$$
\leq \| \frac{1}{2\lambda_j} (1 - \tanh^2(D_j)) R_{lj} \sum_{i=1}^{N} G_j (\mu_j^r - \mu^r_p) \|^2 + 2\| o((D^p_j - D_j)^2) \|^2
$$

$$
\leq \| \frac{1}{2\lambda_j} R_{lj} \| \| R_{lj} \| \| R_{lj} \| \| \dot{\theta}_i \|^2 + 2\| \dot{\theta}_i \|^2
$$

(54)

where \(\| o((D^p_j - D_j)^2) \| \) \(\leq \tilde{l}, \forall: i \in 1\).

Based on (29), (40) and (54), the first term in (53) is

$$
\leq \sum_{i=1}^{N} \frac{1}{2\lambda_j} \| R_{lj} \| \| R_{lj} \| \| R_{lj} \| \| \dot{\theta}_i \|^2 + 2\| \dot{\theta}_i \|^2
$$

(55)

According to (30) and (42), we get

$$
\leq \sum_{i=1}^{N} \frac{1}{2\lambda_j} \| R_{lj} \| \| R_{lj} \| \| R_{lj} \| \| \dot{\theta}_i \|^2 + 2\| \dot{\theta}_i \|^2
$$

where it is assumed that \(\| R_{lj} \| \leq R_j, \forall: i, j \in 1\) to simplify the analysis.

According to the first order approximation of Taylor expansion, we can get

$$
\| \ln \left( 1 - \left( \frac{\mu_j^r}{\lambda_j} \right)^2 \right) - \ln \left( 1 - \left( \frac{\mu_j^r}{\lambda_j} \right)^2 \right) \|
$$

$$
= \| \ln (1 - \tanh^2(D_j)) - \ln (1 - \tanh^2(D_j)) \|
$$

$$
= \| 2 \ln \left( \frac{1}{\cosh(D_j)} \right) - 2 \ln \left( \frac{1}{\cosh(D_j)} \right) \|
$$

$$
= \| \frac{1}{\cosh(D_j)} - \frac{1}{\cosh(D_j)} \|
$$

$$
= \| \frac{1}{\cosh(D_j)} - \frac{1}{\cosh(D_j)} \|
$$
\[ \|2(\ln(\epsilon_{D_j}) + \epsilon_{-D_j}) - \ln(\epsilon_{D_j} + \epsilon_{-D_j})\|, \]

\[ = \|2(\dot{D}_j - \dot{D}_j + \ln(1 + \epsilon_{-2D_j}) - \ln(1 + \epsilon_{-2D_j}))\|, \]

\[ \leq \|2(\dot{D}_j - \dot{D}_j + \epsilon(\epsilon_{-4D_j}) - \epsilon(\epsilon_{-4D_j})) + \epsilon(-6D_j) - \epsilon(-6D_j)\|, \]

\[ = \|2 \left( \frac{1}{2\lambda_j} R_{jj}^{-1} \sigma_j(\dot{\mathbf{Z}}_R, t_j - t) \right) \]

\[ - \frac{1}{2\lambda_j} R_{jj}^{-1} \sigma_j(\dot{\mathbf{Z}}_R, t_j - t) \]

\[ + \epsilon(\epsilon_{-4D_j}) + \epsilon(-6D_j) - \epsilon(-6D_j) \right\|, \]

\[ \leq \frac{1}{\lambda_j} \| R_{jj}^{-1} \| G_{Mj} \| B_{2x} \| \| \dot{\mathbf{Z}}_R \| + \frac{1}{\lambda_j} \| R_{jj}^{-1} \| G_{Mj} \| B_{2x} \| + 2\delta_{||t \|}. \] (57)

where \( \| \epsilon(\epsilon_{-4D_j}) + \epsilon(-6D_j) - \epsilon(-6D_j) \| \leq \delta_{||t \|}. \)

Inserting (57) into (56), we obtain

\[ \sum_{i=1}^{N} \| \dot{\mathbf{Z}}_R \| \| \dot{\mathbf{Z}}_R \| \| \delta_{i} \| + \lambda_{\delta_i}, \]

\[ \leq \sum_{i=1}^{N} \| \dot{\mathbf{Z}}_R \| \| \dot{\mathbf{Z}}_R \| + \lambda_{\delta_i}, \]

\[ \leq \left( \frac{N}{2} \| \dot{\mathbf{Z}}_R \|^2 \| \dot{\mathbf{Z}}_R \| + N\alpha_{2} \right) \| \dot{\mathbf{Z}}_R \|^2 + N\alpha_{2}. \] (58)

Inserting (56) and (55) into inequality (53), \( \Delta_{3} \) becomes

\[ \Delta_{3} \leq \sum_{i=1}^{N} -\| \dot{\mathbf{Z}}_R \| \| \dot{\mathbf{Z}}_R \| + \| \dot{\mathbf{Z}}_R \| + 2\| \dot{\mathbf{Z}}_R \| \right\| \]

\[ + 4G_{Mj} \| \dot{\mathbf{Z}}_R \| + 4G_{Mj} \| \dot{\mathbf{Z}}_R \| + \lambda_{\delta_i} \]

\[ + \lambda_{\delta_i} \]

\[ \leq \sum_{i=1}^{N} -\| \dot{\mathbf{Z}}_R \| \| \dot{\mathbf{Z}}_R \| + \lambda_{\delta_i}. \] (59)

Combing with \( \Delta_{1}(t), \Delta_{2}(t) \) and \( \Delta_{3}(t), \Delta_{4}(t) \) is derived

\[ \Delta_{4}(t) \leq \sum_{i=1}^{N} -\| \dot{\mathbf{Z}}_R \| \| \dot{\mathbf{Z}}_R \| \]

\[ + (1 - \eta_{2}) \lambda_{\min} \left( \dot{\mathbf{Z}}_R \right) \| \dot{\mathbf{Z}}_R \|^2 \]

\[ - \lambda_{\delta_i} \| \dot{\mathbf{Z}}_R \|^2 + \lambda_{\delta_i}, \]

\[ \Delta_{2}(t) \leq \Delta_{2}(t) + \Delta_{2}(t) + \Delta_{3}(t). \]

Since system states and optimal value functions are continuous, thus inequalities below can be obtained

\[ \Delta_{1}(t) = \| \dot{\mathbf{Z}}_R \| \| \dot{\mathbf{Z}}_R \| \leq 0, \]

\[ \Delta_{2}(t) = \| \dot{\mathbf{Z}}_R \| \| \dot{\mathbf{Z}}_R \| \leq \kappa_{i} \| \dot{\mathbf{Z}}_R \|, \]

\[ \Delta_{3}(t) = \frac{1}{2} \dot{\mathbf{Z}}_R \| \dot{\mathbf{Z}}_R \| \| \dot{\mathbf{Z}}_R \| \| \dot{\mathbf{Z}}_R \| \leq 0, \]

where \( \kappa_{i} \|, i = 1, 2 \) are class-\( \kappa \) functions, and \( \delta_{||t \|} = \dot{\mathbf{Z}}_R - \dot{\mathbf{Z}}_R \). This indicates that \( \Delta_{4}(t) \leq 0 \) when events are triggered.

By analyzing these two cases, we can derive that the UUB of system states and approximation errors of critic NNs can be ensured according to the Lyapunov extension theorem [33], as long as inequality conditions (48)–(50) hold.

5 | SIMULATION

5.1 Application to the cooperative guidance system

In this subsection, a multi-missile cooperative guidance system borrowed from [34] is applied to verify the effectiveness of this paper proposed optimal tracking control algorithm. As shown in Figure 2, we consider the case of two missiles attacking the
same target from different directions. Assume that the velocity vector between missiles and the target is constant during the engagement, then the relative kinematics equation can be expressed as below, and related some notations are displayed at Table 2.

\[
\begin{aligned}
\dot{r}_i &= V_T \cos(\bar{\beta} - \theta_i) - V_{Mi} \cos(\alpha_i - \theta_i), \\
\dot{r}_i \dot{\theta}_i &= V_T \sin(\bar{\beta} - \theta_i) - V_{Mi} \sin(\alpha_i - \theta_i), \\
\dot{\alpha}_i &= u_i / V_{Mi}, \\
\dot{\beta} &= t / V_T.
\end{aligned}
\] (60)

Inspired by [34, 35], a new time variable \( t_{goi} = -r_i / \dot{r}_i \) is introduced to satisfy with the finite-horizon constraint condition, and new variables are defined as \( \bar{x} = [x_{i1}, x_{i2}]^T = [\theta_i, \bar{\beta}, t_{goi}]^T \), \( \bar{r}_i = \ln(r_i(0)) - \ln(r_i(t)) \).

Therefore, dynamics equation (60) can be rewritten as

\[
\begin{aligned}
\dot{x}_i &= \begin{bmatrix} x_{i2} \\ x_{i2} + x_{i1}^2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{r_i} \cos(\alpha_i - x_{i1}) + x_{i2} \sin(\alpha_i - x_{i1}) \end{bmatrix} \bar{r}_i \\
&+ \begin{bmatrix} -1 \\ \frac{0}{r_i} \cos(\bar{\beta} - x_{i1}) + x_{i2} \sin(\bar{\beta} - x_{i1}) \end{bmatrix} \bar{r}_i, \\
\dot{\alpha}_i &= a_i / V_{Mi}, \\
\dot{\beta} &= a_T / V_T, \quad \dot{\alpha}_i = (v - a_T) / \tau_T,
\end{aligned}
\] (61)

where \( \bar{r}_i = t_{goi} u_i (|u_i| < 100), i = 1, 2 \) and \( \bar{r}_i = t_{goi} v_i \). As in [34], dynamics equation (61) indicates that the conditions for the missile to successfully capture the target aircraft during the terminal guidance phase are

\[
\dot{\alpha}_i \rightarrow 0, \dot{r}_i < 0, i = 1, 2.
\] (62)

Further, the autopilot dynamics of missiles and the maneuvering target are given as below,

\[
\begin{aligned}
\dot{x}_{Mi} &= V_{Mi} \cos \alpha_i, \\
\dot{y}_{Mi} &= V_{Mi} \sin \alpha_i, \\
\dot{\alpha}_i &= a_m / V_{Mi}, \\
\dot{z}_{Mi} &= (v_i - a_m) / \tau_{Mi},
\end{aligned}
\] (63)

where \((x_{T}, y_{T})\), \((x_{Mi}, y_{Mi})\), \(i = 1, 2\) are coordinates of missiles and target. The finite-horizon performance index and optimal tracking controller are respectively designed by (5) and (40), where related parameters are set as \( Q_1 = 0.001, Q_2 = 0.01, R_{11} = 20, R_{12} = 1, R_{21} = 15, R_{22} = 100, A_1 = 100, A_2 = 100 \). The control target is to enable Missile1 and Missile2 successfully intercept the aircraft within finite time, in which missiles run in accordance with their desired trajectory \( \bar{r}_1 = [10 \sin(0.01t), 2.2 \sin(0.05t)]^T \) and \( \bar{r}_2 = [0.16 \cos(0.15t + \pi), 0.14 \cos(0.15t + \pi)]^T \), respectively. Further, the terminal time and designed parameters are chosen as \( t_f = 20s, \eta_1 = 0.8 \) and \( \eta_2 = 0.3 \), respectively.

In the terminal guidance phase, initial value of parameters in (61)-(63) are chosen as \((x_{T}, y_{T}) = (5000, 0)m, V_T = 400m/s, \beta = 60^\circ, (x_{Mi1}, y_{Mi1}) = (500, 200)m, V_{Mi1} = 580m/s, \alpha_1 = 34.6^\circ, (x_{Mi2}, y_{Mi2}) = (0, 0)m, V_{Mi2} = 620m/s, \alpha_2 = 34^\circ, \tau_T = \tau_{Mi} = 0.1s, i = 1, 2 \). In the critic NNs approximation process, the active function with 20 nodes is employed, in which \( \tau = t_f - t \).

\[
\sigma(Z, t_f - t) = \begin{bmatrix} \chi_1 \exp(-\tau); \chi_2 \exp(-1.3\tau); \chi_3 \chi_2 \tau; \\
\chi_4 \chi_1 \tau; \chi_5 \chi_2 \tau; \chi_6 \chi_2 \tau; \chi_7 \chi_1 \tau; \\
\chi_8 \chi_1 \tau; \chi_9 \chi_2 \tau; \chi_{10} \chi_2 \tau; \chi_{11} \chi_1 \tau; \\
\chi_{12} \chi_1 \tau; \chi_{13} \chi_2 \tau; \chi_{14} \chi_2 \tau; \chi_{15} \chi_1 \tau; \\
\chi_{16} \chi_1 \tau; \chi_{17} \chi_2 \tau; \chi_{18} \chi_2 \tau; \chi_{19} \chi_2 \tau; \chi_{20} \chi_2 \tau; \end{bmatrix},
\]

and the initial weight value of the CNN are given as

\[
\hat{\beta}_{1, initial} = \hat{\beta}_{2, initial} = [10; 10; 10; 10; 10; 10; 10; 10; 10; 10].
\]

With respect to the updating laws (47), the learning rate are set as \( \hat{\beta}_1 = 0.01 \) and \( \hat{\beta}_2 = 0.02 \). As in [34, 35], a probing noise is introduced in the first 15 seconds to excite the system.

**Remark 5.** The learning rate is an important hyperparameter in neural network weight training. To show the effect on the convergence speed of critic NN weights when changing the learning rate, several values of \( \hat{\beta}_i, i = 1, 2 \) are adopted, and the relevant simulation results are given in the Table 3.

### 5.2 Simulated results and analysis

Simulated results are depicted in Figures 3–14, which show that the distributed control method has a good control effect with stabilization compared with the centralized control method. In the meantime, a mass of communication resources are saved by using the designed PET control scheme.
In contrast to the single missile attack method used in [35], this paper proposed distributed guidance law realized the cooperative operation of several missiles, where different missiles use different guidance laws. As shown in Figure 3, the target aircraft was successfully intercepted by two missiles fired from different locations simultaneously. Obviously, the proposed distributed optimal tracking control strategies improved missile penetration capability. Further, the relative distance between missiles and the target are trend to zero in 16 seconds, which satisfied with the finite-horizon constraints. In Figure 4, tracking errors converging to a small region around zero, and the control signals of two subsystems are described in Figure 5. Internal dynamics of the multi-missile interceptor system are depicted in Figures 6–11, and Figures 6 and 8 reveal that the LOS angular rate \( \dot{\theta}_i \) and the range rate \( \dot{r}_i \) always satisfy capture conditions in (62), where the LOS angular rate \( \dot{\theta}_i \) are always in the neighborhood of zero and the range rate \( \dot{r}_i \) are always less than zero.
The convergence process of weight vectors of critic NNs are plotted in Figure 12, and sampling numbers of two subsystems under the proposed PET scheme are depicted in Figure 13, where the number of PET-based samples for subsystem1 and subsystem2 are 107 and 146, respectively. On the contrary, two subsystems both need to sample 800 times under the traditional periodic sampling (PS) scheme. This indicates that this paper presented sampling scheme saved 86.63% and 81.75% of the communication resources for two subsystems, respectively. Compared to the schemes of the CET and ST, the avoidance of the Zeno behaviour and the reduction of communication costs could be actualized at the same time under the PET scheme. As shown in Figure 13, the minimum sampling interval under the PET scheme is greater than or equal to the sampling interval under the PS scheme, that is, $\bar{\tau} = 0.02\, s$, which proves that the PET optimal tracking control strategy is able to exclude the Zeno behaviour.

6 | CONCLUSION

In this paper, an aperiodic distributed optimal tracking control strategy has been developed for finite-horizon interconnected systems with constraint inputs. The large-scale interconnected system has been transmitted into a N-player NZS differential game. Then, the issues of distributed optimal tracking and finite-horizon constraints have been converted to deal with the time-varying HJ equations. In order to approximate the optimal solution effectively, critic NNs and the PET scheme have been employed. Based on the above scheme, system structures have been simplified and the consumption of communication resources has been saved, in the meantime, the Zeno phenomenon has been avoided. Finally, simulation results have shown that the update frequency of the distributed optimal tracking controller could be greatly saved and the target aircraft can be intercepted by missiles successfully. Next, we will extend the proposed control strategy to deal with the dynamic event-triggered control problem of unknown non-linear systems.
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REFERENCES
1. Liu, D., Wang, D., Li, H.: Decentralized stabilization for a class of continuous-time nonlinear interconnected systems using online learning optimal control approach. IEEE Trans. Neural Netw. Learn. Syst. 25(2), 418–428 (2013)
2. Liu, D., et al.: Neural-network-based decentralized control of continuous-time nonlinear interconnected systems with unknown dynamics. Neurocomputing 165, 90–98 (2015)
3. Narayanan, V., Jagannathan, S.: Distributed adaptive optimal control of uncertain large-scale interconnected systems using hybrid q-learning approach. IET Control Theory Appl. 10(12), 1448–1457 (2016)
4. Zhang, H., et al.: Data-driven robust approximate optimal tracking control for unknown general nonlinear systems using adaptive dynamic programming method. IEEE Trans. Neural. Netw. 22(12), 2226–2236 (2011)
5. Gao, W., Jiang, Z.-P.: Learning-based adaptive optimal tracking control of switched nonlinear systems. IEEE Trans. Neural Netw. Learn. Syst. 29(6), 2614–2624 (2017)
6. Lai, N.T.: Distributed cooperative $H_{\infty}$ optimal tracking control of mimo nonlinear multi-agent systems in strict-feedback form via adaptive dynamic programming. Int. J. Control 91(4), 952–968 (2018)
7. Liu, Y., et al.: Data-driven optimal tracking control for discrete-time systems with delays using adaptive dynamic programming. J. Franklin Inst. 355(13), 5649–5666 (2018)
8. Wei, Q., Liu, D.: Neural-network-based adaptive optimal tracking control scheme for discrete-time nonlinear systems with approximation errors. Neurocomputing 149, 106–115 (2015)
9. Qin, C., Zhang, H., Luo, Y.: Optimal tracking control of a class of nonlinear discrete-time switched systems using adaptive dynamic programming. Neural Comput. Appl. 24(3–4), 531–538 (2014)
10. Qin, C., et al.: Approximate optimal tracking control for nonlinear discrete-time switched systems via approximate dynamic programming. In: 2019 Chinese Control And Decision Conference, Nanchang, China, pp. 1456–1461 (2019)
11. Zhao, B., Liu, D., Yang, X.: Observer-critic structure-based adaptive dynamic programming for decentralized tracking control of unknown large-scale nonlinear systems. Int. J. Syst. Sci. 48(9), 1978–1989 (2017)
12. Zhao, B., Li, Y.: Model-free adaptive dynamic programming based near-optimal decentralized tracking control of reconfigurable manipulators. Int. J. Control. Auton. Syst. 16(2), 478–490 (2018)
13. Tan, L.: Distributed $H_{\infty}$ optimal tracking control for strict-feedback nonlinear large-scale systems with disturbances and saturating actuators. IEEE Trans. Syst. Man Cybern.: Syst. 50, 4719–4731 (2020) https://doi.org/10.1109/TSMC.2018.2861470
14. Sun, J., Liu, C.: Decentralised zero-sum differential game for a class of large-scale interconnected systems via adaptive dynamic programming. Int. J. Control. 92(12), 2917–2927 (2019)
15. Niu, B., et al.: Multiple lyapunov functions for adaptive neural tracking control of switched nonlinear nonlinear triangular systems. IEEE Trans. Cybern. 50(5), 1877–1886 (2020)
16. Li, Y., Tong, S., Yang, G.: Observer-based adaptive fuzzy decentralized event-triggered control of interconnected nonlinear system. IEEE Trans. Cybern. 50(7), 3104–3112 (2020) https://doi.org/10.1109/TCYB.2019.2894024
17. Linsenmayer, S., Dimarogonas, D.V., Allgower, F.: Periodic event-triggered control for networked control systems based on non-monotonic Lyapunov functions. Automatica 106, 35–46 (2019)
18. Yang, H., et al.: Dynamic self-triggered output-feedback control for nonlinear stochastic systems with time delays. Int. J. Robust Nonlinear Control, https://doi.org/10.1002/rnc.4963, pp. 1–15 (2020)
19. Wang, A., Liao, X., Dong, T.: Event-driven optimal control for uncertain nonlinear systems with external disturbance via adaptive dynamic programming. Neurocomputing 281, 188–195 (2018)
20. Narayanan, V., Jagannathan, S.: Event-triggered distributed approximate optimal state and output control of affine nonlinear interconnected systems. IEEE Trans. Neural Netw. Learn. Syst. 29(7), 2846–2856 (2017)
21. Cui, L., et al.: Event-triggered single-network ADP method for constrained optimal tracking control of continuous-time non-linear systems. Appl Math Comput. 352, 220–234 (2019)
22. Narayanan, V., et al.: Approximate optimal distributed control of nonlinear interconnected systems using event-triggered nonzero-sum games. IEEE Trans. Neural Netw. Learn. Syst. 30(5), 1512–1522 (2018)
23. Narayanan, V., Sahoo, A., Jagannathan, S.: Approximate optimal distributed control of nonlinear interconnected systems using nonzero-sum games. Int. J. IEEE Conf. Decision and Control, Miami Beach, FL, pp. 2872–2877 (2018)
24. Zou, W., Xiang, Z.: Event-triggered leader-following consensus of nonlinear multi-agent systems with switched dynamics. IET Control Theory Appl. 13(9), 1222–1228 (2019)
25. Liu, C., et al.: Integral reinforcement learning based decentralized optimal tracking control of unknown nonlinear large-scale interconnected systems with constrained-input. Neurocomputing 323, 1–11 (2019)
26. Su, H., et al.: Online event-triggered adaptive critic design for non-zero-sum games of partially unknown networked systems. Neurocomputing 368, 84–98 (2019)
27. Liu, Y., et al.: Finite-time synchronization of complex-valued neural networks with multiple time-varying delays and infinite distributed delays. Neural Process Lett. 50(2), 1773–1787 (2019)
28. Liu, Y., et al.: Adaptive neural network control for active suspension systems with time-varying vertical displacement and speed constraints. IEEE Trans. Ind. Electron. 66, 9458–9466 (2019)
29. Shao, S., Chen, M., Zhang, Y.: Adaptive discrete-time flight control using disturbance observer and neural networks. IEEE Trans. Neural Netw. Learn. Syst. 30(12), 3708–3721 (2019)
30. Zhang, Q., Zhao, D., Wang, D.: Event-based robust control for uncertain nonlinear systems using adaptive dynamic programming. IEEE Trans. Neural Netw. Learn. Syst. 29(1), 37–50 (2018)
31. Zhang, H., et al.: Event-triggered adaptive dynamic programming algorithm for non-zero-sum games of unknown nonlinear systems via generalized fuzzy hyperbolic models. IEEE Trans. Fuzzy Syst. 27(11), 2202–2214 (2019)
32. Mu, C., Wang, K.: Single-network adp for near optimal control of continuous-time zero-sum games without using initial stabilising control laws. IET Control Theory Appl. 12(18), 2449–2458 (2018)
33. Lewis, F., Jagannathan, S., Yesildirak, A.: Neural Network Control of Robot Manipulators and Nonlinear Systems. Taylor and Francis, London (1999)
34. Sun, J., Liu, C.: Distributed fuzzy adaptive backstepping optimal control for nonlinear multilinear guidance systems with input saturation. IEEE Trans. Fuzzy Syst. 27(3), 447–461 (2018)
35. Sun, J., Liu, C.: Finite-horizon differential games for missile-target interception system using adaptive dynamic programming with input constraints. Int. J. Syst. Sci. 49(2), 264–283 (2018)

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