NORI’S FUNDAMENTAL GROUP OVER A NON-ALGEBRAICALLY CLOSED FIELD

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Abstract. Let $X$ be a connected reduced scheme over a field $k$, $x \in X(k)$ be a $k$-rational point. M.Nori defined a fundamental group scheme $\pi^N(X, x)$ which generalizes A.Grothendieck’s étale fundamental group by including infinitesimal coverings. However, because $x$ is taken to be a rational point the fundamental group is always trivial when $X = \text{Spec}(k)$, while the étale fundamental group $\pi^\text{ét}_1(\text{Spec}(k), x) = \text{Gal}(\bar{k}/k)$. In this note we generalize Nori’s definition to an arbitrary base point $x$. In this way we could get a non-trivial definition of $\pi^N(X, x)$ when $X$ is taken to be a non-algebraically closed field $k$. We then study various properties of this generalized version of Nori’s fundamental group scheme.

0. Introduction

Let $X$ be a locally Noetherian connected scheme, $x \in X(\bar{k})$ be a geometric point. Let $\text{E Cov}(X)$ be the category of finite étale coverings of $X$. Then we have a fibre functor $F$ from $\text{E Cov}(X)$ to the category of finite sets by sending any finite étale covering $f : Y \to X$ to its fibres $f^{-1}(x)$. In [SGAI Exposé V] A.Grothendieck proved that $\text{E Cov}(X)$ together with the fibre functor $F$ forms a Galois category. Then he defined the étale fundamental group $\pi^\text{ét}_1(X, x) := \text{Aut}(F)$ to be the group of automorphisms of $F$. $\pi^\text{ét}_1(X, x)$ is a profinite group, i.e. a topological group of the form $\lim \leftarrow_{i \in I} G_i$ where $I$ is a small cofiltered category and $G_i$ is a finite group for each $i \in I$. This profinite group classifies all torsors under finite groups, in particular when $G$ is a finite abelian group we have $H^1(\text{ét}, G) = \text{Hom}_{\text{cont}}(\pi^\text{ét}_1(X, x), G)$. If $X = \text{Spec}(k)$, $x \in X(k)$ be the field extension $k \subseteq \bar{k}$, then $I$ can be chosen as the set of finite Galois sub-extensions of $k \subseteq \bar{k}$, and for each $i = (k \subseteq K) \in I$, $G_i := \text{Gal}(K/k)$, so $\pi^\text{ét}_1(X, x) = \lim \leftarrow_{i \in I} G_i$ is just the absolute Galois group $\text{Gal}(\bar{k}/k)$.

Let $X$ be a proper reduced connected scheme over a perfect field $k$, $x \in X(k)$ be a rational point. In [Nori, Part I, Chapter I] M.V.Nori constructed a full subcategory $\text{Ess}(X) \subseteq \text{Vec}(X)$ of the category of vector bundles on $X$. Objects in $\text{Ess}(X)$ are called essentially finite vector bundles. He proved that $\text{Ess}(X)$ with the fibre functor $\omega$ from $\text{Ess}(X)$ to the category of finite dimensional vector spaces sending $V \mapsto V|_x$ is a Tannakian category over $k$. Then he defined $\pi^N(X, x) := \text{Aut}^\otimes(\omega)$ to be the group of tensor automorphisms of $\omega$. $\pi^N(X, x)$ is a profinite $k$-group scheme which classifies all $k$-pointed torsors over $X$ under finite $k$-group schemes. In particular when $G$ is a finite abelian $k$-group scheme we...
have $H^1_{fppf}(X, G) = \text{Hom}_{\text{grp.sch}}(\pi_1^N(X, x), G)$. However, in this construction the properness assumption is vital, so the construction does not apply to non-proper schemes. To remedy this M.V.Nori introduced in [Nori, Part I, Chapter II] another construction of $\pi_1^N(X, x)$ which works for any connected reduced scheme $X$ equipped with a rational point $k$. However, as a generalization of $\pi_1^\text{et}(X, x)$, there is still a problem that when $X/k$ is taken to be $k/k$, $\pi_1^N(X, x)$ is trivial. In [EH] H.Esnault and A.Hogadi constructed a Tannakian category $\text{Strat}(X, \infty)$, the category of generalized vector bundles, for any smooth variety $X$ over a perfect field $k$. If $x \in X(k)$, then they define $\pi^\text{alg,\infty}(X, x)$ to be the Tannakian group of $\text{Strat}(X, \infty)$, and they proved that $\pi^N(X, x)$ is the profinite quotient of $\pi^\text{alg,\infty}(X, x)$. But still $\pi^\text{alg,\infty}(\text{Spec}(k), x) = \{1\}$.

In this note we would like to slightly generalize Nori’s second idea, the non-proper one, with a hope to obtain a profinite group scheme which is more comparable to $\text{Gal}(\overline{k}/k)$ in the case when $X/k$ is $k/k$.

Let $X$ be a reduced connected scheme over a field $k$, $x : S \to X$ be a non-empty morphism of schemes. Let $N(X/k, x)$ be the category of torsors under finite $k$-group schemes with fixed $S$-point lying over $x$. In the first part of this note we construct a neutral Tannakian category $\mathcal{C}(X/k, x)$ with a fibre functor $\omega$ from $\mathcal{C}(X/k, x)$ to the category of finite dimensional $k$-vector spaces. We then define $\pi^N(X/k, x) := \text{Aut}^\omega(\omega)$ to be the Tannakian group of $\mathcal{C}(X/k, x)$. By some purely abstract nonsense we have $\pi^N(X/k, x) = \lim_{i \in N(X/k, x)} G_i$, where $G_i$ are finite group schemes over $k$. Thus $\pi^N(X/k, x)$ is a profinite group scheme which classifies $S$-pointed torsors over $X$ under a finite group scheme $\{2.2\}$. If $x \in X(k)$ is taken to be a $k$-rational point, then this coincides with Nori’s original definition. If $X/k = k/k$, $x : \text{Spec}(\overline{k}) \to \text{Spec}(k)$ is taken to be the natural inclusion $k \subseteq \overline{k}$, then $\pi^N(X/k, x)$ is a profinite $k$-group scheme which admits $\text{Gal}(\overline{k}/k)$ (viewed as a $k$-group scheme) as a quotient. The structure of the category $\mathcal{C}(X/k, x)$ we are considering here is rather crude. It is tautologically equivalent to the direct limit $\lim_{i \in N(X/k, x)} \text{Rep}_k(G_i)$.

This construction also works when we replace $N(X/k, x)$ by certain sub-categories $I(X/k, x) \subseteq N(X/k, x)$. When $I(X/k, x)$ is taken to be those pointed torsors whose group schemes are étale (resp. constant, local), the fundamental group is denoted by $\pi^E(X/k, x)$ (resp. $\pi^G(X/k, x), \pi^L(X/k, x)$). We have the following canonical surjections $\{2.7\}$

$$
\pi^E(X/k, x) \longrightarrow \pi^G(X/k, x) \longrightarrow \pi^L(X/k, x)
$$

In fact, $\pi^G(X/k, x)$ is nothing but a ”group scheme version” of $\pi^\text{et}(X, x)\{2.6\}$. Although $\pi^E(X/k, x)$ and $\pi^\text{et}(X, x)$ are all fundamental groups classifying étale coverings they are indeed largely different. For example, we know $\pi^\text{et}(\text{Spec}(\mathbb{R}), x) = \text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ and the universal covering of $\text{Spec}(\mathbb{R})$ under $\pi^\text{et}(\text{Spec}(\mathbb{R}), x)$ is $\text{Spec}(\mathbb{C})$, while we have
Proposition 0.1. Let $\mathbb{R}$ be the field of real numbers, $\bar{x} : \text{Spec } (\mathbb{C}) \to \text{Spec } (\mathbb{R})$ be the morphism corresponding to the natural inclusion $\mathbb{R} \subset \mathbb{C}$. Then

$$\pi^E(\mathbb{R}/\mathbb{R}, \bar{x}) = \lim_{n \in \mathbb{N}^+} \mu_{n, \mathbb{R}}$$

is an infinite $\mathbb{R}$-group scheme, and the universal covering corresponding to $\pi^E(\mathbb{R}/\mathbb{R}, \bar{x})$ is a non-Noetherian affine scheme with infinitely many connected components.

We can also prove that $\pi^E(F_p/F_p, \bar{x})$ is non-commutative (2.16). For abelian varieties $X/k$ we know $\pi^E_1(X/k, 0) = \lim_{n \in \mathbb{N}^+} X[n](k)$, but we have an example (2.18) to show that $\pi^E(X/\mathbb{R}, 0)$ is non-commutative for $X$ an abelian variety over $\mathbb{C}$.

We then do some trivial computations. The results are

(i) $\pi^E(k/k, \bar{x}) = \{1\}$, $\pi^N(k/k, \bar{x}) = \pi^E(k/k, \bar{x})$,

when $k$ is a perfect field and $x$ is $k \subseteq \bar{k}$;

(ii) $\pi^E(k/k, \bar{x}) = \{1\}$, $\pi^N(k/k, \bar{x}) = \pi^L(k/k, \bar{x})$,

when $k$ is a separably closed field and $x$ is $k \subseteq \bar{k}$;

(iii) $\pi^N(X/k, \bar{x}) = \pi^E(X/k, \bar{x}) = \pi^L(X/k, \bar{x}) = \{1\}$,

when $X$ is $\mathbb{A}^n_k$ with $k$ a field of characteristic 0 or $X$ is $\mathbb{P}^n_k$ with $k$ a field of arbitrary characteristic.

Next we are trying to find the analogue of the fundamental exact sequence of $\pi^E_1$ defined in [SGA1][Exposé IX, Théorème 6.1]. We find two analogues, one is the following.

Proposition 0.2. Let $X$ be a scheme geometrically connected separable over a field $k$, $\bar{x} \in X(k)$ be a geometric point, then there is a complex of $k$-group schemes

$$1 \to \pi^I(X/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \to \pi^L(k/k, \bar{x}) \to 1$$

where $I = N, L$ or $E$. This sequence is always surjective on the right, not always injective on the left, and is exact in the middle if and only if the following two conditions are satisfied.

(i) For any $I$-saturated object $(P, G, p) \in I(X/k, \bar{x})$, the image of the composition of the natural homomorphisms

$$\pi^I(X/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \to G$$

is a normal subgroup of $G$.

(ii) Whenever there is an object $(P, G, p) \in I(X/k, \bar{x})$ whose pull-back along $\bar{X} \to X$ is trivial then there is an object $(Q, H, q) \in I(k/k, \bar{x})$ whose pull-back along $\bar{X} \to \text{Spec } (k)$ is isomorphic to $(P, G, p)$.

We prove that (i) holds for triples $(P, G, p)$ where $G$ is étale and $P$ is connected (3.7) or $G$ is local and $k$ is perfect (3.12). We give an example (3.9) to show that (i) fails when $P$ is not connected while $G$ is still étale. We prove that (ii) holds when $k$ perfect (3.6) or $G$ is étale (3.11) but fails (3.5) when $k$ is not perfect and $G$ is not étale.

The other analogue sequence is
Proposition 0.3. Let $X$ be a scheme geometrically connected separable over a field $k$, $\bar{x} \in X(\bar{k})$ be a geometric point, then there is a natural sequence of $\bar{k}$-group schemes

\[ 1 \to \pi^f(\bar{x}/\bar{k}, \bar{x}) \to \pi^f(X/k, \bar{x}) \times_k \bar{k} \to \pi^f(k/k, \bar{x}) \times_k \bar{k} \to 1. \]

It is a complex, surjective on the right, injective on the left when $k$ is perfect, but it is in general not exact in the middle for $I = N, E, L$.

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Notation and Conventions

(i) We always use $k$ to denote a field, $\bar{k}$ to denote its algebraic closure.

(ii) Let $f : S' \to S$ be a morphism of schemes, $X'$ be a scheme over $S'$. We say $X'$ possess an $S$-form if there is a scheme $X$ over $S$ whose pull-back along $f$ is isomorphic to $X'$.

(iii) When $X$ is a scheme over $k$, we use $\bar{X}$ to denote $X \times_k \bar{k}$. If $\bar{k} \subseteq \bar{K}$ is a field extension we use $X_K$ to denote $X \times_k K$. Sometimes we also use $\bar{X}$ (resp.$X_K$) to denote something over $\bar{k}$ (resp. $K$) which does not necessarily possess a $k$-form $X$. This depends on the situation we are in.

(iv) Let $X \times_S Y$ be a fibred product of schemes. We use $pr_1$ to denote the first projection $X \times_S Y \to X$ and $pr_2$ to denote the second projection.

(v) Let $G$ be a group scheme over $k$. In this note, a $G$-torsor over $X$ is an $X$-scheme $P$ equipped with a right action $\rho : P \times_k G \to P$, where $\rho$ is a morphism of $X$-schemes which induces an isomorphism $pr_1 \times \rho : P \times_k G \to P \times_X P$. Moreover we require that the structure map $P \to X$ of the $X$-scheme $P$ is faithfully flat and quasi-compact.

(vi) Let $X$ be a scheme. We use $X_{\text{red}}$ to denote the reduced closed subscheme structure of $X$.

(vii) Let $f : X \to S$ be a morphism of schemes. We call $f$ separable [SGAI] [Exposé X, Définition 1.1] if it is flat and all its geometric fibres are reduced.

(viii) Let $G$ be a group scheme over $k$, $H \subseteq G$ be a subgroup scheme. We say $H \subseteq G$ is a normal subgroup scheme if for any $k$-scheme $T$, $H(T) \subseteq G(T)$ is a normal subgroup. Note that $H \subseteq G$ is normal if and only if $\bar{H} \subseteq \bar{G}$ is normal.

(ix) Let $S' \to S$ be a Galois covering, i.e. a connected finite étale covering which is a torsor under its own automorphism group $\text{Aut}_S(S')$. Let $\pi' : X' \to S'$ be a morphism of schemes. A twisted action of $\text{Aut}_S(S')$ on $X'$ is a group homomorphism $f : \text{Aut}_S(S') \to \text{Aut}(X')$, where $\text{Aut}(X')$ is the group of scheme automorphisms of $X'$, such that for any $\sigma \in \text{Aut}_S(S')$ the following diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f(\sigma)} & X' \\
\pi \downarrow & & \pi' \downarrow \\
S' & \xrightarrow{\sigma} & S'
\end{array}
\]
is commutative. By Grothendieck’s general descent theory [BLR, 6.2, Example B, pp. 139], there is an equivalence of categories between the category of $S'$-schemes equipped with a twisted action from $\text{Aut}_S(S')$ and the category of $S$-schemes. We often refer to this as Galois descent.

Here is another version of Galois descent. Let $k \subseteq K$ be a finite Galois extension. There is an equivalence of categories between the category of finite abstract groups equipped with a continuous action from $\text{Gal}(\overline{k}/k)$ (resp. an action from $\text{Gal}(K/k)$) via group automorphisms and the category of finite étale $k$-group schemes (resp. $k$-group schemes whose pull-back to $K$ is finite constant). [AV, 3.25-3.26].

1. The Category $\mathcal{C}(X/k, x, I)$

Let $X$ be a reduced connected scheme over a field $k$, $x : S \to X$ be a non-empty morphism of $k$-schemes.

**Definition 1.** Let $(P, G, p)$ be a triple where $G$ is a finite group scheme over $k$, $P$ is a $G$-torsor over $X$, $p : S \to P$ be a $k$-morphism lifting $x : S \to X$. A morphism from $(P_1, G_1, p_1)$ to $(P_2, G_2, p_2)$ is a pair $(s, t)$ where $t : G_1 \to G_2$ is a $k$-group scheme homomorphism, $s : P_1 \to P_2$ is an $X$-scheme morphism which intertwines the group action and sends $p_1 \to p_2$. We denote the category consisting of such triples by $N(X/k, x, I)$.

**Definition 2.** [SGA4, Exposé I, Définition 2.7] A category $I$ is called cofiltered if it satisfies the following three conditions:

(i) it is non-empty;
(ii) for any objects $i, j \in I$, there exists an object $k \in I$ and two arrows $k \to i$, $k \to j$;
(iii) for any two morphisms

$$
\begin{array}{c}
\begin{array}{c}
j \\
\downarrow b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow a
\end{array}
\begin{array}{c}
i
\end{array}
\end{array}
$$

there exists a morphism $c : k \to j$ satisfying $c \circ a = c \circ b$.

**Remark 1.1.** The category $N(X/k, x)$ has finite fibred products and a final object $(X, \{1\}, x)$, so in particular it is cofiltered. The proof is due to M.V. Nori. Considering the importance of the fact to our construction, we would like to reproduce his proof in our settings.

**Proposition 1.2.** [Nori][Chapter II, Proposition 1 and Proposition 2] Fibred products exist in $N(X/k, x)$.

**Proof.** We have to show that given any two morphisms

$$(\phi_i, h_i) : (P_i, G_i, p_i) \to (Q, G, q) \in N(X/k, x)$$

where $i = 1, 2$, the triple $(P_1 \times_Q P_2, G_1 \times_G G_2, p_1 \times_Q p_2)$ is again an object in $N(X/k, x)$.

The action of $G_1$ on $P_1$ (resp. $G_2$ on $P_2$) induces a morphism of $k$-schemes

$$\lambda : (P_1 \times_Q P_2) \times_k (G_1 \times_G G_2) \to (P_1 \times_Q P_2) \times (P_1 \times_Q P_2)$$

$$(x_1, x_2) \times (g_1, g_2) \mapsto (x_1, x_2) \times (x_1g_1, x_2g_2).$$
By a purely abstract nonsense argument, we see that the induced morphism is an isomorphism. Now the problem is to show that the projection $\phi : P_1 \times_Q P_2 \to X$ is FPQC.

Let $Y$ be the quotient of $P_1 \times Q P_2$ by $G_1 \times_G G_2$,

$$\varphi : P_1 \times_Q P_2 \to Y$$

be the quotient map. Then there is a unique morphism of schemes $i : Y \to X$ through which the projection $\phi$ factors. Consider the following commutative diagram:

$$
\begin{array}{c}
(P_1 \times Q P_2) \times_k (G_1 \times_G G_2) \\
\downarrow \varphi \circ pr_1 \\
Y \\
\Delta \\
Y \times X Y
\end{array}
\begin{array}{c}
\cong \\
\varphi \times \varphi \\
\varphi \circ pr_1 \\
\varphi \circ pr_1
\end{array}
\begin{array}{c}
(P_1 \times Q P_2) \times (P_1 \times Q P_2) \\
\Delta \\
Y \times X Y
\end{array}
\begin{array}{c}
\lambda \\
\varphi \circ pr_1 \\
\varphi \circ pr_1 \\
\varphi \circ pr_1
\end{array}
$$

As $i : Y \to X$ is finite, $\Delta$ is of finite presentation [EGA 1.4.3.1, pp.231]. Since $\varphi$ is finite faithfully flat [AV 4.16 (iii)], $\varphi \circ pr_1$, $\varphi \times \varphi$, $\lambda$ are all finite and faithfully flat. So $\Delta$ is also faithfully flat. But $\Delta$ is already a closed immersion, so it has to be an isomorphism. Hence the finite morphism $i : Y \to X$ is a monomorphism [EGA 17.2.6] in the category of schemes. Thus it has to be a closed immersion [EGA 18.12.6]. Now look at the following diagram

$$
\begin{array}{c}
P_1 \times Q P_2 \\
\downarrow \varphi \\
Y \\
i \\
X
\end{array}
\begin{array}{c}
\cong \\
\psi \\
\varphi \\
\varphi
\end{array}
\begin{array}{c}
P_1 \times_X P_2 \\
\varphi \\
\varphi \\
\varphi
\end{array}
$$

Since $P_1 \times Q P_2$ is the fibre of the neutral element of $G$ under the following map

$$P_1 \times Q P_2 \xrightarrow{(\phi_1 \times \phi_2)} Q \times Q \xrightarrow{\Delta} Q \times_k G \xrightarrow{pr_2} G,$$

$P_1 \times Q P_2 \subseteq P_1 \times_X P_2$ must be both open and closed as a sub topological space (but not as a scheme). The map $\psi$ is finite flat and of finite presentation, so the underlying topological space of the scheme $Y$, as the image of $P_1 \times Q P_2$ under $\psi$, is both open and closed in $X$. Since $P_1 \times Q P_2$ admits a morphism from a non-empty scheme $S$, it must be non-empty as well. Thus $Y \neq \emptyset$. Combining this with the condition that $X$ is connected and reduced we conclude that $i : Y \to X$ is an isomorphism. Now $\phi = i \circ \varphi$ is finite locally free and surjective, so in particular FPQC.

Remark 1.3. (i) Proposition 1.2 implies that $N(X/k, x)$ is cofiltered. Indeed, conditions (i), (ii) of Definition 2 are directly checked. For (iii), suppose we have two maps $a, b : j \to i$ as in 2 then we could make the following cartesian diagram

$$
\begin{array}{c}
k \\
\downarrow c \\
\Delta \\
i \\
a \times b \\
\downarrow \Delta \\
i \times i
\end{array}
\begin{array}{c}
X \\
\cong \\
X \\
\cong \\
X
\end{array}
$$

\footnote{This was suggested to us by Jilong Tong. We used a non-standard notion of cofilteredness in the earlier version. We thank him for this suggestion.}
where $\Delta$ stands for the diagonal map. The map $c$ in the diagram is precisely what we are looking for.

(ii) It is rather important that we require a point on the torsor, otherwise the category is not cofiltered. For example, let’s take $X = \text{Spec}(k)$ to be a field, $Q = (\mathbb{Z}/2\mathbb{Z})_k$ be the trivial torsor under the constant group scheme $(\mathbb{Z}/2\mathbb{Z})_k$, $P_1 = P_2 = \text{Spec}(k)$ be the trivial torsor under the trivial $k$-group scheme $\{1\}$, $\phi_i : P_i \to Q$ ($i = 1, 2$) be two maps sending $P_i$ to the two different points of $Q$. If the category was cofiltered, then there should be two morphisms of torsors $\psi_i : P \to P_i$ ($i = 1, 2$) which equalize $\phi_1$ and $\phi_2$. But $P_1 \times_Q P_2 = \emptyset$, so this can’t happen.

(iii) In [12] if $G_1$ and $G_2$ are étale then $\phi$ is automatically FPQC ([SGAI Exposé I, Corollaire 4.8., pp.4]) even when $X$ is non-reduced. However, connectedness is still vital.

**Definition 3.** Let $I(X/k, x) \subseteq N(X/k, x)$ be a cofiltered full subcategory. We use $\mathfrak{C}(X/k, x, I)$ to denote the category whose objects are couples $(V, (P, G_P, p))$, where $(P, G_P, p) \in I(X/k, x), V \in \text{Rep}_k(G_P)$, $\text{Rep}_k(G_P) :=$ the category finite $G_P$-representations.

By abuse of notations we simply write

$$P := (P, G_P, p), \quad (V, P) := (V, (P, G_P, p)), \quad I := I(X/k, x)$$

instead. The morphisms between two triples $(V, P)$ and $(W, Q)$ are defined by the set

$$\text{Hom}_{\mathfrak{C}(X/k, x, I)}((V, P), (W, Q)) := \lim_{i \in I_P} \text{Hom}((V, i), (W, Q)),$$

where $I_P$ denotes the category morphisms $X \to P \in I$, and $\text{Hom}((V, i), (W, Q))$ is the set of "honest morphisms" which is defined by the set of pairs $(\lambda, f)$, where $f : i \to Q$ is a morphism in $I$ and $\lambda : V \to f^*W$ is a morphism in $\text{Rep}_k(G_i)$. If we have another object $(U, O) \in \mathfrak{C}(X/k, x, I)$, then the composition

$$\text{Hom}_{\mathfrak{C}(X/k, x, I)}((U, O), (V, P)) \times \text{Hom}_{\mathfrak{C}(X/k, x, I)}((V, P), (W, Q)) \to \text{Hom}_{\mathfrak{C}(X/k, x, I)}((U, O), (W, Q))$$

is defined in the following way: suppose that we have two honest morphisms

$$(\lambda_1, f_1) \in \text{Hom}((U, O'), (V, P)) \quad \text{and} \quad (\lambda_2, f_2) \in \text{Hom}((V, P'), (W, Q))$$

which represent the two morphisms

$$a \in \text{Hom}_{\mathfrak{C}(X/k, x, I)}((U, O), (V, P)) \quad \text{and} \quad b \in \text{Hom}_{\mathfrak{C}(X/k, x, I)}((V, P), (W, Q))$$

respectively. Then we have $O'' \in I_P$ which dominates both $P'$ and $O'$. Let $f_3 : O'' \to Q$ be the composition of $O'' \to P' \xrightarrow{f_2} Q$. Let $\lambda_3 : U \to f_3^*W$ by composing $\lambda_1$ with $\lambda_2$. So we can define $b \circ a$ to be the morphism which is represented by the honest morphism

$$(\lambda_3, f_3) \in \text{Hom}((U, O''), (W, Q))$$

It is easy to check that this composition is independent of the choice of the representatives. If $I = N(X/k, x)$ we will just write

$$\mathfrak{C}(X/k, x) := \mathfrak{C}(X/k, x, I).$$
Remark 1.4. If we have a morphism $f : i \to j \in I \subseteq N(X/k, x)$ and an object $(V, j) \in \mathcal{C}(X/k, x, I)$, then we get another object $(V, i)$ with $V$ being the pull-back of the $G_j$-representation to $G_i$. Directly from our definition we see that the canonical map $(id, f) : (V, i) \to (V, j)$ is an isomorphism.

Proposition 1.5. Let $Vec_k$ be the category of finite dimensional $k$-vector spaces. The category $\mathcal{C}(X/k, x, I)$ with the forgetful functor

$$\omega : \mathcal{C}(X/k, x, I) \to Vec_k$$

sending $(V, P) \mapsto V$ forms a neutral Tannakian category.

Proof. (i). $\mathcal{C}(X/k, x, I)$ is a $k$-linear category. We define for any two elements

$$a, b \in \text{Hom}_{\mathcal{C}(X/k, x, I)}((V, P), (W, Q))$$

the sum $a + b$ in the following way.

Suppose $a$ is represented by an honest morphism $(\lambda, f) \in \text{Hom}((V, P_1), (W, Q))$ for some $P_1 \in I_P$ and $b$ is represented by $(\mu, g) \in \text{Hom}((V, P_2), (W, Q))$ for some $P_2 \in I_P$, then using the fact that $I$ is filtered we can find $P' \in I_P$ and two maps $s : P' \to P_1$, $t : P' \to P_2$ such that the compositions $P' \xrightarrow{s} P_1 \xrightarrow{f} Q$ and $P' \xrightarrow{t} P_2 \xrightarrow{g} Q$ are the same. By Remark 1.4 we may assume $P' = P_1 = P_2$ and $f' = f = g$. Now we just define $a + b$ to be the morphism represented by $(\lambda + \mu, f') \in \text{Hom}((V, P'), (W, Q))$.

For any morphism

$$a \in \text{Hom}_{\mathcal{C}(X/k, x, I)}((V, P), (W, Q))$$

by using the fact that $I$ is filtered and Remark 1.4 we may assume $P = Q = P'$ for some $P' \in I \subseteq N(X/k, x)$ and $a$ is represented by an honest morphism

$$a = (\lambda, id) \in \text{Hom}((V, P'), (W, P')).$$

We define $a$ to be zero by requiring that $\lambda : V \to W$ is the 0 map. For any $s \in k$ we define the $k$-linear multiplication $sa := (s\lambda, id)$. Under these definition

$$\text{Hom}_{\mathcal{C}(X/k, x, I)}((V, P), (W, Q))$$

becomes a $k$-vector space.

We also have a canonical map

$$\theta : \text{Hom}_{\mathcal{C}(X/k, x, I)}((V, P), (W, Q)) \to \text{Hom}_{Vec_k}(V, W)$$

sending the pair $(\lambda, id) \mapsto \lambda$. One can check easily that this is well defined and that $\theta$ is $k$-linear. It is clear that $\theta$ is injective. This implies that $\text{Hom}_{\mathcal{C}(X/k, x, I)}((V, P), (W, Q))$ is a finite dimensional $k$-vector space.

We can define the direct sum of two objects $(V, P)$ and $(W, Q)$ as

$$(V \oplus W, P')$$

for some $P'$ which dominates both $P$ and $Q$. 
With the above definition $\mathcal{C}(X/k, x, I)$ becomes $k$-linear category.

(ii). $\mathcal{C}(X/k, x, I)$ is an abelian category. Given any morphism

$$a : (V, P) \to (W, Q) \in \mathcal{C}(X/k, x, I),$$

as usual, we could choose some $P'$ which dominates both $P$ and $Q$ and make $a$ a morphism $(\lambda, id) \in \text{Hom}((V, P'), (W, P'))$. Then we can define $\text{Ker}(a)$ to be $(\text{Ker}(\lambda), P')$ and $\text{Coker}(a)$ to be $(\text{Coker}(\lambda), P')$.

(iii). $\mathcal{C}(X/k, x, I)$ is a tensor category. For two objects

$$(V, P), (W, Q) \in \mathcal{C}(X/k, x, I)$$

we can define

$$(V, P) \otimes (W, Q) := (V \otimes W, P').$$

The distinguished element $(k, X)$ serves as the identity object.

(iv). $\mathcal{C}(X/k, x, I)$ is a rigid tensor category. We define the internal Hom

$$\text{Hom}_{\mathcal{C}}((V, P), (W, Q)) := (\text{Hom}_{\text{Rep}_k(G_p)}(V, W), P').$$

One checks that for any $(U, O)$, there is a canonical isomorphism of sets between

$$\text{Hom}_{\mathcal{C}}((U, O) \otimes (V, P), (W, Q))$$

and

$$\text{Hom}_{\mathcal{C}}((U, O), \text{Hom}_{\mathcal{C}}(V, P), (W, Q))).$$

Using the canonical imbedding

$$\theta : \text{Hom}_{\mathcal{C}}((V, P), (W, Q)) \hookrightarrow \text{Hom}_{\text{Vec}_k}(V, W)$$

and the fact that a morphism is invertible in $\text{Hom}_{\mathcal{C}}((V, P), (W, Q))$ if and only if it is invertible in $\text{Hom}_{\text{Vec}_k}(V, W)$ one can check readily that we have the following canonical isomorphisms

$$\text{Hom}(X_1, Y_1) \otimes \text{Hom}(X_2, Y_2) \to \text{Hom}(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

$$X \to \text{Hom}(\text{Hom}(X, U), U),$$

where $X_1, X_2, Y_1, Y_2, X \in \mathcal{C}(X/k, x, I)$.

(v). The forgetful functor

$$\mathcal{C}(X/k, x, I) \to \text{Vec}_k$$

sending $(V, P) \to V$ is clearly a $k$-linear exact faithful tensor functor. This finishes the proof.

\begin{proof}
\end{proof}

**Definition 4.** Let $X$ be a connected reduced scheme over a field $k$, $x : S \to X$ be a non-empty morphism of $k$-schemes, $I(X/k, x) \subseteq N(X/k, x)$ be a cofiltered full subcategory. We define the $I$-fundamental group scheme $\pi^I(X/k, x)$ as the Tannakian group $\text{Aut}^\otimes(\omega)$ of the neutral Tannakian category $(\mathcal{C}(X/k, x, I), \omega)$. 
Definition 5. Now for each $i \in I(X/k, x)$ we have a canonical functor

$$F_i : \text{Rep}_k(G_i) \rightarrow \mathcal{C}(X/k, x, I)$$

sending $V \rightarrow (V, i)$ and $\phi : V \rightarrow W \in \text{Rep}_k(G_i)$ to the map $(\phi, id) : (V, i) \rightarrow (W, i)$. If we have a morphism $\phi_{ij} : i \rightarrow j \in I(X/k, x)$, then the map of $k$-group schemes $G_i \rightarrow G_j$ induces a functor

$$\phi_{ij}^* : \text{Rep}_k(G_j) \rightarrow \text{Rep}_k(G_i)$$

with an isomorphism of $k$-linear tensor functors $\alpha_{ij} : F_j \cong F_i \circ \phi_{ij}^*$ making the following diagram of functors 2-commutative.

Let $\omega_i : \text{Rep}_k(G_i) \rightarrow \text{Vec}_k$ be the forgetful functor. Then we have $\omega_i \circ \phi_{ij}^* = \omega_j$ and $\omega \circ F_i = \omega_i$.

Proposition 1.6. If $\mathcal{D}$ is a category and for each $i \in I(X/k, x)$ we have a functor

$$T_i : \text{Rep}_k(G_i) \rightarrow \mathcal{D}$$

and for each $\phi_{ij} : i \rightarrow j \in I(X/k, x)$ we have an isomorphism of functors $\beta_{ij} : T_j \cong T_i \circ \phi_{ij}^*$ making the following diagram of functors 2-commutative

which satisfies

$$\phi_{jk}^*(\beta_{ij}) \cdot \beta_{jk} = \beta_{ik} : \quad T_k \cong T_j \circ \phi_{jk}^* \cong T_i \circ \phi_{ij}^* \circ \phi_{jk}^* = T_i \circ \phi_{ik}^*,$$
then there is a functor $\iota : \mathfrak{C}(X/k, x, I) \to \mathfrak{D}$ with isomorphisms $a_i : T_i \to \iota \circ F_i$, unique up to a unique isomorphism, making the following diagram of functors 2-commutative

\[
\begin{array}{ccc}
\Rep_k(G_j) & \xrightarrow{\phi_{ij}^*} & \mathfrak{C}(X/k, x, I) \\
\downarrow F_j & & \downarrow T_j \\
\Rep_k(G_i) & \xrightarrow{F_i} & \mathfrak{D}
\end{array}
\]

in the sense that the composition

\[
T_j \xrightarrow{a_j} \iota \circ F_j \xrightarrow{\alpha_{ij}} \iota \circ F_i \circ \phi_{ij}^* \xrightarrow{\alpha_i^{-1}} T_i \circ \phi_{ij}^*
\]

is equal to the prescribed isomorphism $\beta_{ij} : T_j \cong T_i \circ \phi_{ij}^*$. If $\mathfrak{D}$ is a Tannakian category with a neutral fibre functor $o : \mathfrak{D} \to \text{Vec}_k$ such that $T_i$ are all $k$-linear tensor functors with tensor isomorphisms $\xi_i : o \circ T_i \cong \omega_i$, and if the isomorphisms $\beta_{ij} : T_j \xrightarrow{\cong} T_i \circ \phi_{ij}^*$ are tensor isomorphisms with the property that $\phi_{ij}^*(\xi_i) \cdot o(\beta_{ij}) = \xi_j$ (where $o(\beta_{ij}) : o \circ T_j \xrightarrow{\cong} o \circ T_i \circ \phi_{ij}^*$ and $\phi_{ij}^*(\xi_i) : o \circ T_i \circ \phi_{ij}^* \xrightarrow{\cong} \omega_i \circ \phi_{ij}^* = \omega_j$ are the natural compositions induced from $\beta_{ij}$ and $\xi_i$ respectively), then $\iota$ can be chosen to be a $k$-linear tensor functor with $o \circ \iota \cong \omega$ such that $a_i$ is a tensor isomorphism for each $i \in I(X/k, x)$, and such $\iota$ (with $a_i$) is unique up to a unique tensor isomorphism.

\textbf{Proof.} We first define $\iota$. If $(V, i) \in \mathfrak{C}(X/k, x, I)$, we define $\iota(V, i) := T_i(V)$. If

$\sigma : (V, i) \to (W, j) \in \mathfrak{C}(X/k, x, I)$

is a morphism then we choose $i' \in I(X/k, x)$ which dominates $i, j \in I(X/k, x)$ and an honest morphism $\varsigma : (V, i') \to (W, i') \in \text{Hom}((V, i'), (W, i'))$ which represents $\sigma$. Now we define

$\iota(\sigma) := T_i(V) \xrightarrow{\beta_{i'}^{-1}} T_{i'}(V) \xrightarrow{T_{i'}(\varsigma)} T_{i'}(W) \xrightarrow{\beta_{j'}} T_j(W)$.

It is easily seen that $\iota$ is well-defined and we can take $a_i$ to be the identities so that $\iota$ commutes with $T_i$ and $F_i$ in the sense stated in the proposition. If $\iota'$ is another functor with isomorphisms $a_i'$ which satisfies our conditions, then we have

$\delta(V, i) : \iota(V, i) = T_i(V) \xrightarrow{a_i'} \iota'(F_i(V)) = \iota'(V, i)$

for each $(V, i) \in \mathfrak{C}(X/k, x)$. This establishes for each $i \in I(X/k, x)$ an isomorphism of functors $\delta : \iota \to \iota'$ which is compatible with $a_i$. Clearly $\delta$ is uniquely determined by this compatibility.

If $\mathfrak{D}$ is a neutral Tannakian category and $T_i$ are all $k$-linear tensor functors which commute with the fibre functors, then for each two objects $(V, i), (W, j) \in \mathfrak{C}(X/k, x, I)$, we
define an isomorphism in $D$ as follows:

$$\iota((V, i) \otimes (W, j)) \cong \iota((V, i') \otimes (W, i')) = \iota((V \otimes W, i')) = T_{i'}(V \otimes W) \cong T_{i'}(V) \otimes T_{i'}(W) = \iota(V, i') \otimes \iota(W, i') \cong \iota(V, i) \otimes \iota(W, j).$$

This isomorphism is functorial and establishes $\iota$ as a tensor functor. The $k$-linearity of $\iota$ can be checked easily. If $i'$ is another $k$-linear tensor functor with $\alpha_i'$ being tensor isomorphisms, then by construction the isomorphism $\delta : \iota \cong \iota'$ is a tensor isomorphism. Finally for each $(V, i) \in \mathcal{C}(X/k, x, I)$ we define the isomorphism $o \circ \iota(V, i) \cong \omega(V)$ as the following morphism

$$\xi_i(V) : o \circ \iota(V, i) = o \circ T_i(V) \cong \omega_i(V) = \omega(V).$$

This defines an isomorphism of functors $o \circ \iota \cong \omega$ because of the compatibility $\phi_{ij}^p(\xi_i) \cdot o(\beta_{ij}) = \xi_j$. □

**Remark 1.7.** Let $\mathcal{T}$ be the 2-category of neutral Tannakian categories over $k$ whose 1-morphisms between $(\Xi_1, \omega_1)$ and $(\Xi_2, \omega_2)$ are pairs $(f, \sigma)$ where $f : \Xi_1 \to \Xi_2$ is a $k$-linear tensor functor and $\sigma : \omega_2 \circ f \to \omega_1$ is a tensor isomorphism, whose 2-morphisms between $(f, \sigma)$ and $(f', \sigma')$ are tensor natural transformations $f \Rightarrow f'$ which are compatible with $\sigma$ and $\sigma'$. [EGA3] says that $\mathcal{C}(X/k, x, I)$ is the "2-direct limit" of the 2-functor $i \mapsto \text{Rep}_k(G_i)$ not only in the 2-category of categories but also in the 2-category of neutral Tannakian categories over $k$. In fact, in [SGA1, Exposé VI, §6], Grothendieck has already defined the notion of 2-direct limit for a fibred category $\mathcal{F}$ as the localization category of $\mathcal{F}$ at all its cartesian morphisms. In our case we could also take the 2-functor $I(X/k, x) \to \mathcal{T}$, $i \mapsto \text{Rep}_k(G_i)$ as the fibred category and localize it at $I(X/k, x)$.

### 2. First Properties of $\pi^l(X/k, x)$

#### 2.1. The Universal Covering.

As in [Nori, Chapter II, Proposition 2] we can define the universal covering for our fundamental group scheme.

**Theorem 2.1.** Let $X$ be a connected reduced scheme over a field $k$, $x : S \to X$ be a non-empty morphism of $k$-schemes, $I(X/k, x) \subseteq N(X/k, x)$ be a cofiltered full subcategory. Then there exists a triple $(\tilde{X}_x, \pi^l(X/k, x), \tilde{x})$, where $\tilde{X}_x$ is a $\pi^l(X/k, x)$-torsor over $X$, $\tilde{x} : S \to \tilde{X}_x$ is an $S$-point of $\tilde{X}_x$ lying above $x$, which satisfies that for any $(P, G, p) \in I$ there exists a unique morphism

$$(\phi, h) : (\tilde{X}_x, \pi^l(X/k, x), \tilde{x}) \to (P, G, p),$$

where $h : \pi^l(X/k, x) \to G$ is homomorphism of $k$-group schemes and $\phi : \tilde{X}_x \to P$ is a morphism of $X$-schemes which sends $\tilde{x}$ to $p$ and intertwines the group actions.

**Proof.** Consider the following functors

$$F_X : I(X/k, x) \to \text{Aff}(X), \quad (P, G, p) \mapsto P$$

$$F_k : I(X/k, x) \to \text{Grsch}(k), \quad (P, G, p) \mapsto G$$


where \( \text{Aff}(X) \) denotes the category of affine schemes over \( X \), and \( \text{Grsch}(k) \) denotes the category of finite group schemes over \( k \). We have by \((2.4)\) that

\[
\pi^I(X/k, x) := \lim_{i \in I(X/k, x)} F_k(i)
\]

Now let

\[
\widetilde{X}_x := \lim_{i \in I(X/k, x)} F_X(i).
\]

Then \( \widetilde{X}_x \) is an affine scheme over \( X \) which admits a point \( \tilde{x} : S \to \widetilde{X}_x \) lying above \( x \).

Now we get a triple \( (\widetilde{X}_x, \pi^I(X/k, x), \tilde{x}) \) which has the property that for any \( i := (P, G, p) \in I(X/k, x) \) there is a morphism

\[
(\phi_i, h_i) : (\widetilde{X}_x, \pi^I(X/k, x), \tilde{x}) \to (P, G, p)
\]

defined by the projection to the index \( i \in I(X/k, x) \). Let \( H \) be the image of \( h_i \), then we get a factorization of \( h_i \)

\[
\pi^I(X/k, x) \xrightarrow{f} H \xrightarrow{g} G
\]

and a commutative diagram

\[
\xymatrix{(Q, H, q) \ar[rr]^{(\phi, g)} & & (P, G, p) \\
(Q, H, q) \ar[rr]^{(\phi, h)} & & (P, G, p) \
(Q, H, q) \ar[urr]_{(\phi, f)} & & (P, G, p) \
\xymatrix{(Q, H, q) \ar[rr]^{(\phi, g)} & & (P, G, p) \\
(Q, H, q) \ar[rr]^{(\phi, h)} & & (P, G, p) \
(Q, H, q) \ar[urr]_{(\phi, f)} & & (P, G, p) \}}
\]

where \( Q := \widetilde{X}_x \times_{f(X/k, x)} H \) is the contracted product along \( f \), and \( \psi, \phi \) are canonical maps induced by the contracted product. Let \( j := (Q, H, q) \). There is a projection map

\[
(\phi_j, h_j) : (\widetilde{X}_x, \pi^I(X/k, x), \tilde{x}) \to (Q, H, q)
\]

which also makes the above diagram commutative after replacing \( (\psi, f) \) by \( (\phi_j, h_j) \). Since \( \varphi \) and \( g \) are closed imbeddings, we must have \( (\psi, f) = (\phi_j, h_j) \). Hence the affine ring of \( \pi^I(X/k, x) \) is in fact a filtered inductive limit of its sub Hopf-algebras which are induced by those \( j \in I(X/k, x) \) whose \( h_j \) are surjective, and the same thing happens for \( \widetilde{X}_x \). This implies that if there is another morphism

\[
(\phi, h) : (\widetilde{X}_x, \pi^I(X/k, x), \tilde{x}) \to (P, G, p)
\]

then we can find an index \( i' := (P', G', p') \in I \) such that \( \phi, h \) factor through the projection morphisms

\[
\phi_{i'} : \widetilde{X}_x \to P' \quad \text{and} \quad h_{i'} : \pi^I(X/k, x) \to G',
\]
in other words, we have a commutative diagram

\[(\widehat{X}_x, \pi^f(X/k, x), \tilde{x}) \quad \xrightarrow{\langle \phi, h \rangle} \quad (P', G', p') \quad \xrightarrow{\langle \phi, g \rangle} \quad (P, G, p)\]

But by the very definition of a projective limit, we know that \((\varphi, g) \circ (\phi', h') = (\phi, h)\). Thus \((\phi, h) = (\phi, h)\). This completes the proof.

**Corollary 2.2.** Let \(\text{Hom}_{\text{grp.sch}}(\pi^N(X/k, x), -)\) be the category whose objects are finite \(k\)-group schemes equipped with \(k\)-group scheme homomorphisms from \(\pi^N(X/k, x)\), whose morphisms are \(k\)-group scheme homomorphisms which are compatible with the homomorphisms from \(\pi^N(X/k, x)\). Then there is an equivalence of categories

\[\text{Hom}_{\text{grp.sch}}(\pi^N(X/k, x), -) \cong N(X/k, x).\]

Similar statement holds if one replace \(N(X/k, x)\) by some smaller cofiltered subcategory.

**Proof.** Given a \(k\)-group scheme homomorphism \(f : \pi^N(X/k, x) \to G\), we get a contracted product

\[(\widehat{X}_x \times_f \pi^N(X/k, x) \times G, G, \tilde{x}) \in N(X/k, x),\]

and given a morphism in \(\text{Hom}_{\text{grp.sch}}(\pi^N(X/k, x), -)\), we get a morphism in \(N(X/k, x)\) defined by the universal property of the contracted product. This defines a functor

\[\text{Hom}_{\text{grp.sch}}(\pi^N(X/k, x), -) \cong N(X/k, x).\]

The quasi-inverse of this functor is given by 2.1.

**Definition 6.** Let \(X\) be a reduced connected scheme over a field \(k\), \(x : S \to X\) be a non-empty morphism, \(I(X/k, x) \subseteq N(X/k, x)\) be a cofiltered full subcategory. We call a triple \((P, G, p) \in I(X/k, x)\) an \(I\)-saturated object if the corresponding projection map \(\pi^f(X/k, x) \to G\) is surjective.

**Lemma 2.3.** Let \(X\) be a reduced connected scheme over a field \(k\), \(x : S \to X\) be a non-empty morphism, \(I(X/k, x) \subseteq N(X/k, x)\) be a cofiltered full subcategory. Then the full subcategory of \(I(X/k, x)\) consisting of \(I\)-saturated objects is cofinal in \(I(X/k, x)\), i.e. for any object \((P, G, p) \in I(X/k, x)\) there is a morphism

\[(Q, H, q) \to (P, G, p) \in I(X/k, x)\]

where \((Q, H, q)\) is an \(I\)-saturated object. So when we study projective limits indexed by \(I(X/k, x)\) we can restrict ourselves to this smaller category of \(I\)-saturated objects.

---

The terminology saturated is taken from [EHV]. We also used it in [Zh]. In [Nori] such objects are called reduced.
Definition 7. There are various choices of Relations among 2.2. them which will be frequently used in the rest of this note.

Proposition 2.4. Let $X$ be a reduced connected scheme over a field $k$, $x : S \to X$ be a non-empty morphism, $I(X/k, x) \subseteq N(X/k, x)$ be a cofiltered full subcategory. There is an isomorphism of $k$-group schemes

$$\pi^I(X/k, x) \cong \lim_{i \in I(X/k, x)} G_i.$$ 

Proof. This follows from 1.6 and the fact that the category $\mathcal{T}$ of neutral Tannakian categories over $k$ is anti-equivalent to the category of affine group schemes over $k$ [Del]. □

2.2. Relations among $\pi^N$, $\pi^L$, $\pi^E$, $\pi^G$.

Definition 7. There are various choices of $I(X/k, x) \subseteq N(X/k, x)$. We will list some of them which will be frequently used in the rest of this note.

(i) $\pi^N(X/k, x) := \pi^I(X/k, x)$ when $I(X/k, x) = N(X/k, x)$;

(ii) $\pi^E(X/k, x) := \pi^I(X/k, x)$ when $I(X/k, x) = I_{\text{et}}(X/k, x)$ is the subcategory consisting of triples $(P, G, p)$ where $G$ is an étale group scheme over $k$;

(iii) $\pi^G(X/k, x) := \pi^I(X/k, x)$ when $I(X/k, x) = I_{\text{co}}(X/k, x)$ is the subcategory consisting of triples $(P, G, p)$ where $G$ is a constant group scheme over $k$;

(iv) $\pi^L(X/k, x) := \pi^I(X/k, x)$ when $I(X/k, x) = I_{\text{lc}}(X/k, x)$ is the subcategory consisting of triples $(P, G, p)$ where $G$ is a local (i.e. connected) group scheme.

Remark 2.5. (i) As we have seen in 1.3 (iii), $\pi^E(X/k, x)$ can be defined without the assumption that $X$ is reduced.

(ii) When $x : S \to X$ is taken to be a geometric point in $X(\bar{k})$, $\pi^G(X/k, x)$ is a profinite affine group scheme whose group of $k$-points is just Grothendieck’s étale fundamental group $\pi^\text{et}_1(X, x)$. The only difference between $\pi^\text{et}_1(X, x)$ and $\pi^G(X/k, x)$ is that $\pi^\text{et}_1(X, x)$ is a projective limit of finite groups, where the limit is taken in the category of topological groups in which each finite group has the discrete topology while $\pi^G(X/k, x)$ is a projective limit of finite groups, where the limit is taken in the category of affine group group schemes in which each finite group is regarded as a constant group scheme over $k$. In other words, $\pi^G(X/k, x)$ is none other than a linearization of $\pi^\text{et}_1(X, x)$.

Lemma 2.6. Let $I_1 \subseteq I_2 \subseteq N(X/k, x)$ be two cofiltered full subcategories and $(P, G, p)$ be an object in $I_1$. If for any imbedding $(Q, H, q) \hookrightarrow (P, G, p) \in I_2$
(i.e. $H \subseteq G$ is a subgroup), $(P,G,p) \in I_1$ implies $(Q,H,q) \in I_1$, then we have a surjection
\[ \pi^{I_2}(X/k,x) \to \pi^{I_1}(X/k,x). \]

Proof. Let $(P,G,p) \in I_1$ be an $I_1$-saturated object. Then we can take the image of the composition
\[ \pi^{I_2}(X/k,x) \to \pi^{I_1}(X/k,x) \to G \]
and denote it by $H$. By 2.2 we get an inclusion $(Q,H,q) \to (P,G,p) \in I_2$. So by the assumption this inclusion lives in $I_1$. This implies that the surjection $\pi^{I_1}(X/k,x) \to G$ factors through $H \hookrightarrow G$. Thus $H = G$. This concludes the proof. \hfill $\square$

Proposition 2.7. The following natural $k$-group scheme homomorphisms
(i) $\pi^N(X/k,x) \to \pi^E(X/k,x) \to \pi^G(X/k,x)$
(ii) $\pi^N(X/k,x) \to \pi^L(X/k,x)$
(iii) $\pi^N(X/k,x) \to \pi^E(X/k,x) \times_k \pi^L(X/k,x)$
are all surjections.

Proof. In the view of 2.6 only the last statement needs to be explained. For this, we take in 2.6 $I_2 := N(X/k,x)$ and $I_1$ to be the triples $(P,G,p)$ whose group $G$ is isomorphic to a direct product of an étale $k$-group scheme and a local $k$-group scheme, i.e. $G = G^0 \times_k G_{\text{ét}}$. Now suppose $H \subseteq G$ be a subgroup scheme. Then the connected-étale sequence for $H$ splits because $H_{\text{red}} \subseteq G_{\text{red}} = G_{\text{ét}} \Rightarrow H_{\text{red}} = H_{\text{ét}}$. But since $G_{\text{ét}}$ acts trivially on $G^0$ and the action of $H_{\text{ét}}$ on $H^0$ is compatible with that of $G_{\text{ét}}$ on $G^0$, $H_{\text{ét}}$ must act trivially on $H^0$, or in other words, $H = H^0 \times_k H_{\text{ét}}$. \hfill $\square$

Example 2.8. Here we want to point out that all the above surjections are, in general, not isomorphisms.
(i). For $\pi^E_G : \pi^E(X/k,x) \to \pi^G(X/k,x)$. Let’s take $X = \text{Spec}(k) = \text{Spec}(\mathbb{Q})$, $\bar{x} : \text{Spec}(\overline{\mathbb{Q}}) \to \text{Spec}(\mathbb{Q})$ be the natural field extension. Let $\alpha \in \mathbb{Q}$, $n \in \mathbb{N}\setminus\{0\}$, and suppose $x^n - \alpha$ has no root in $\mathbb{Q}$, then $P := \text{Spec}(\mathbb{Q}[\bar{x}]/(x^n - \alpha))$ is a non-trivial $\mu_n$-torsor over $\mathbb{Q}$. Choosing any point $p \in P(\mathbb{Q})$, we get a triple $(P,\mu_n,p) \in N(X/k,\bar{x})$. Let $\varphi : \pi^E(X/k,\bar{x}) \to \mu_n$ be the homomorphism corresponding to $(P,\mu_n,p)$ as in 2.2. If the map $\pi^E_G$ was an isomorphism then there should be a $k$-group scheme homomorphism $\phi : \pi^G(X/k,\bar{x}) \to \mu_n$ satisfying $\phi \circ \pi^E_G = \varphi$. But since $\pi^G(X/k,\bar{x})$ is a cofiltered projective limit of finite constant group schemes, there must be a factorization

\[
\begin{array}{ccc}
\pi^G(X/k,\bar{x}) & \to & \mu_n \\
\downarrow \phi & & \downarrow \mu_n \\
H & \uparrow \lambda & \\
\end{array}
\]

where $H$ is a constant group scheme. But when $n$ is a prime number, $\mu_n$ is a $\mathbb{Q}$-scheme of two connected components. Thus the fact that $P$ is a non-trivial torsor would imply that $\varphi$ is surjective, and so is $\phi$. Therefore, the map $\lambda : H \to \mu_n$ should also be surjective, and
hence $\mu_n$ has to be a constant group scheme. But this is quite not the case when $n > 2$.

(ii). For $\pi_N^E : \pi^N(X/k, x) \to \pi^E(X/k, x)$. If it was an isomorphism then any torsor with local group scheme will be dominated by an étale torsor, then the local torsor has to be trivial. Hence any non-trivial local torsor gives a counterexample. Yet we would like to point out that if $k \to \tilde{k}$ where $\tilde{k}$ is perfect, $\tilde{x} : \text{Spec}(k) \to \text{Spec}(\tilde{k})$ be the natural field extension, then $\pi_N^E$ is an isomorphism (see [2.9]). But if $k$ is not perfect and $\text{char}(k) = p$, one can choose $\alpha \in k$ such that $\alpha \notin k$ but $\alpha^p \in k$. Thus the field extension $k \subseteq k(\alpha)$ is a non-trivial $\mu_p$-torsor.

(iii). For $\pi_N^L : \pi^N(X/k, x) \to \pi^L(X/k, x)$. As in (ii) any non-trivial étale torsor provides a counterexample. And also (iii) is implied by (iv).

(iv). For $\pi^N(X/k, x) \to \pi^E(X/k, x) \times_k \pi^L(X/k, x)$. There is a perfect counterexample in [EPS][Remark 4.3].

2.3. A Little Calculation.

**Proposition 2.9.** Let $k$ be a perfect field, $\tilde{x} : \text{Spec}(\tilde{k}) \to \text{Spec}(k)$ be the natural field extension $k \subseteq \tilde{k}$. Then the canonical surjection

$$\pi_N^N : \pi^N(k/k, \tilde{x}) \to \pi^N(k/k, \tilde{x})$$

is an isomorphism.

**Proof.** Let $(P, G, p) \in N(k/k, \tilde{x})$ be an object. Then there is a canonical isomorphism $P \times_k G \cong P \times_k P$. Let $P_{\text{red}}$ be the reduced closed subscheme of $P$ and $G_{\text{red}}$ be the reduced closed subscheme of $G$. As $k$ is perfect,

$$P_{\text{red}} \times_k G_{\text{red}} \subseteq P \times_k G \quad \text{and} \quad P_{\text{red}} \times_k P_{\text{red}} \subseteq P \times_k P$$

are the unique reduced closed subschemes of the underlying spaces. This induces a diagram

$$\begin{array}{c}
P_{\text{red}} \times_k G_{\text{red}} \longrightarrow P_{\text{red}} \times_k P_{\text{red}} \\
\downarrow \quad \downarrow \\
P \times_k G \cong P \times_k P
\end{array}$$

in which the upper horizontal arrow is an isomorphism. But $G_{\text{red}}$ is étale, as $k$ is perfect. Therefore, we get a morphism

$$(P_{\text{red}}, G_{\text{red}}, p) \subseteq (P, G, p) \in N(k/k, \tilde{x})$$

where $(P_{\text{red}}, G_{\text{red}}, p) \in I_{\text{ét}}(k/k, x)$. Hence $I_{\text{ét}}(k/k, x)$ is cofinal inside $N(k/k, x)$. It follows from this and [1.4] that the canonical inclusion

$$\mathcal{C}(X/k, x, I_{\text{ét}}) \subseteq \mathcal{C}(X/k, x)$$

is an equivalence, or in other words, $\pi_N^N$ is an isomorphism. \qed

**Corollary 2.10.** Assumptions and notations being as in [2.9] we have

$$\pi^L(k/k, \tilde{x}) = \{1\}.$$
Proof. Let \((P, G, p) \in I_k(k/k, x)\) be an object. Then, as in the proof of [2.9] we see that there is an imbedding
\[
(P_{\text{red}}, G_{\text{red}}, p) \subseteq (P, G, p) \in N(k/k, \bar{x}).
\]
But since \(G\) is connected, \((P_{\text{red}}, G_{\text{red}}, p)\) is just the trivial triple. This finishes the proof. \(\Box\)

**Proposition 2.11.** Let \(k\) be a separably closed field, \(\bar{x} : \text{Spec} (\bar{k}) \to \text{Spec} (k)\) be the natural field extension. Then we have
\[
\pi^E(k/k, \bar{x}) = \{1\},
\]
and that the canonical surjection
\[
\pi^N_L : \pi^N(k/k, \bar{x}) \longrightarrow \pi^L(k/k, \bar{x})
\]
is an isomorphism.

**Proof.** Let \((P, G, p) \in N(k/k, \bar{x})\) be an object, \(G_{\text{et}}\) be the maximal étale quotient of \(G\). Then the quotient map \(h : G \twoheadrightarrow G_{\text{et}}\) induces, by [2.2] a triple \((P_{\text{et}}, G_{\text{et}}, p) \in I_{\text{et}}(k/k, \bar{x})\) and a morphism
\[
(\phi, h) : (P, G, p) \twoheadrightarrow (P_{\text{et}}, G_{\text{et}}, p) \in N(k/k, \bar{x}).
\]
Since \(P_{\text{et}}\) is an étale scheme over a separably closed field, every point of \(P_{\text{et}}\) is a \(k\)-rational point. This means that \(P_{\text{et}}\) is a trivial \(G_{\text{et}}\)-torsor, and hence \(\pi^E(k/k, \bar{x}) = \{1\}\). Now we can pull back the map \(\phi : P \to P_{\text{et}}\) along the \(k\)-rational point \(p \in P_{\text{et}}(k)\). Then we get a triple \((P^0, G^0, p) \in I_k(k/k, \bar{x})\) and a morphism
\[
(P^0, G^0, p) \hookrightarrow (P, G, p) \in N(k/k, \bar{x}).
\]
This means that \(I_k(k/k, \bar{x})\) is cofinal inside \(N(k/k, \bar{x})\). By the same argument as in [2.9] we see that \(\pi^N_L\) is an isomorphism. \(\Box\)

**Proposition 2.12.** Let \(k\) be a field, \(X := \mathbb{P}^n_k, n \in \mathbb{N}^+, x : S \to X\) be any morphism with \(S\) connected and non-empty. Then we have
\[
\pi^N(X/k, x) = \pi^E(X/k, x) = \pi^L(X/k, x) = \pi^G(X/k, x) = \{1\}.
\]

**Proof.** Let \((P, G, p) \in N(X/k, x)\) be an object. Then by [Nori Chapter II, lemma, pp.92] plus Künneth formula [MS Theorem 2.3], \(P\) is a trivial \(G\)-torsor, i.e. \(P \cong X \times_k G\). Since \(S\) is connected, it is mapped to a connected component \(Q\) of \(X \times_k G\) via \(p : S \to P\). As \(X \times_k G \cong X \times_k \bar{G}\), the composition
\[
Q_{\text{red}} \subseteq Q \subseteq X \times_k G \twoheadrightarrow X
\]
must be an isomorphism, thus the map \(p\) factors through a section of the structure map \(P \twoheadrightarrow X\). This means that there is a unique morphism
\[
(X, \{1\}, x) \to (P, G, p) \in N(X/k, x).
\]
Therefore \((X, \{1\}, x)\) is a cofinal object in \(N(X/k, x)\). By [2.4] \(\pi^N(X/k, x) = \{1\}\). \(\Box\)
Remark 2.13. The connectedness assumption on $S$ in the above proposition is quite important. Otherwise, we could take $x : P \to X$ to be the natural projection, $P := \mathbb{P}^1_k \coprod \mathbb{P}^1_k$ to be the trivial torsor under $\mathbb{Z}/2\mathbb{Z}$, and $p : P \to P$ to be the identity. In this way, there is no morphism $(X, \{1\}, x) \to (P, G, p) \in N(X/k, x)$. Thus the homomorphism $\pi^N(X/k, x) \to (\mathbb{Z}/2\mathbb{Z})_k$ corresponding to $(P, \mathbb{Z}/2\mathbb{Z}, p)$ is not trivial, but surjective. Therefore, $\pi^N(X/k, x)$ is not trivial.

Proposition 2.14. Let $k$ be a field of characteristic 0, $X := \mathbb{A}^n_k$, $n \in \mathbb{N}^+$, $x : S \to X$ be any morphism with $S$ connected and non-empty. Then we have

$$\pi^N(X/k, x) = \pi^E(X/k, x) = \pi^G(X/k, x) = \{1\}.$$

Proof. The point is that in this case any finite torsor over $X$ is étale and any étale torsor over $X$ is trivial. Then we do 2.12 again. $\square$

2.4. Suprising Phenomena.

Proposition 2.15. Let $\mathbb{R}$ be the field of real numbers, $\bar{x} : \text{Spec} (\mathbb{C}) \to \text{Spec} (\mathbb{R})$ be the morphism corresponding to the natural inclusion $\mathbb{R} \subset \mathbb{C}$. Then

$$\pi^E(\mathbb{R}/\mathbb{R}, \bar{x}) = \lim_{n \in \mathbb{N}^+} \mu_{n, \mathbb{R}}$$

is an infinite $\mathbb{R}$–group scheme, and the universal covering corresponding to $\pi^E(\mathbb{R}/\mathbb{R}, \bar{x})$ is a non-Noetherian affine scheme with infinitely many connected components.

Proof. Let $(P, G, p) \in I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x})$. Then $P(\mathbb{C})$ is a principal homogenous space under $G(\mathbb{C})$. By Galois descent, there is an action of $\text{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle$ on $P(\mathbb{C})$ via set-theoretical automorphisms and an action of $\text{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle$ on $G(\mathbb{C})$ via group automorphisms such that these two actions are compatible. Let $\sigma(p) = pa$ for some $a \in G(\mathbb{C})$, $n \in \mathbb{N}^+$ denote the order of $a$. Then for any $b \in G(\mathbb{C})$, $\sigma(pb) = \sigma(p)\sigma(b) = pa\sigma(b)$. But $\sigma^2 = \text{id}$ is trivial, so $pb = \sigma^2(pb) = \sigma(pa\sigma(b)) = \sigma(p)a\sigma(b) = pa\sigma(a)b$. Thus $a\sigma(a) = e$ is trivial, so $\sigma(a) = a^{-1}$. Let $Q_n(\mathbb{C}) \subseteq P(\mathbb{C})$ be the subset $\{pa^i | i \in \mathbb{N}\}$, $H_n(\mathbb{C}) \subseteq G(\mathbb{C})$ be the subgroup $\langle a \rangle$. These substructures are clearly stable under the $\text{Gal}(\mathbb{C}/\mathbb{R})$-actions, so they descend to $\mathbb{R}$, i.e. we have a subobject

$$(Q_n, H_n, p) \subseteq (P, G, p) \in I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x}),$$

where the set of $\mathbb{C}$-points of $Q_n$ is $Q_n(\mathbb{C})$ and the group of $\mathbb{C}$-points of $H_n$ is $H_n(\mathbb{C})$. Let

$$(P_n, \mu_{n,R}, p_n) := (\text{Spec} (\mathbb{R}[x]/(x^n + 1)), \text{Spec} (\mathbb{R}[x]/(x^n - 1)), e^{(2n-1)\pi i \over n}) \in I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x})$$

where the action of $\mu_{n,R}$ on $P_n$ is defined simply by multiplying a $n$–th root of unity on a root of $x^n + 1 = 0$ in $\mathbb{C}$ and $e^{(2n-1)\pi i \over n}$ is the $n$-th root $\cos((2n-1)\pi i \over n) + i\sin((2n-1)\pi i \over n)$. By sending $a \mapsto e^{2\pi i \over n}$ we get an isomorphism $h : H_n \cong \mu_{n,R} = \text{Spec} (\mathbb{R}[x]/(x^n - 1))$. By sending $p \mapsto e^{2\pi i \over n}$ we get an isomorphism of $\mathbb{R}$-schemes $\phi : Q_n \cong \text{Spec} (\mathbb{R}[x]/(x^n + 1))$ which is compatible with $h$ under the actions. This means that the full subcategory of $I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x})$ consisting of objects of the form $(P_n, \mu_{n,R}, p_n)$ is cofinal.
On the other hand, the triple \((\overline{P}_n, \mu_{n, \mathbb{R}}, p_n)\) is \(I_{\delta l}\)-saturated. If we have a subobject
\[(Q, H, p) \subseteq (\overline{P}_n, \mu_{n, \mathbb{R}}, p_n) \in I_{\delta l}(\mathbb{R}/\mathbb{R}, \bar{x})\]
then \(p_n = p \in Q(\mathbb{C})\) implies \(e^{\frac{2\pi i}{n}} \in Q(\mathbb{C})\) for \(Q\) should always contain the connected component of \(p = p_n = e^{\frac{(2n-1)\pi i}{n}}\). Therefore, by the equation
\[p_n e^{\frac{2\pi i}{n}} = e^{\frac{(2n-1)\pi i}{n}} \cdot e^{\frac{2\pi i}{n}} = e^{\frac{\pi i}{n}}\]
we have \(e^{\frac{2\pi i}{n}} \in H(\mathbb{C})\). Since \(H(\mathbb{C})\) contains the generator of the \(n\)-th cyclic group \(\mu_{n, \mathbb{R}}(\mathbb{C})\), we have \(H(\mathbb{C}) = \mu_{n, \mathbb{R}}(\mathbb{C})\). Or equivalently, \(H = \mu_{n, \mathbb{R}}\) and \(Q = P_n\). Thus \((\overline{P}_n, \mu_{n, \mathbb{R}}, p_n)\) is an \(I_{\delta l}\)-saturated object.

Now if \(m, n \in \mathbb{N}^+\) and \(m|n\), then we can define a "raise to \(\frac{n}{m}\)-power" map
\[(P_n, \mu_{n, \mathbb{R}}, p_n) \mapsto (P_m, \mu_{m, \mathbb{R}}, p_m)\]
for by sending \(x \mapsto x^{\frac{n}{m}}\) in the affine coordinate ring. This defines a projective system in \(I_{\delta l}(\mathbb{R}/\mathbb{R}, \bar{x})\). By taking projective limit in the category of affine schemes (resp. group schemes) over \(\mathbb{R}\), we get a triple
\[
\left(\lim_{n \in \mathbb{N}^+} P_n, \lim_{n \in \mathbb{N}^+} \mu_{n, \mathbb{R}}, \bar{p}\right).
\]

Let \((\tilde{X}_\bar{x}, \pi^E(\mathbb{R}/\mathbb{R}, \bar{x}), \bar{x})\) be the universal triple defined in 2.1. Then by the universality, we get a morphism
\[
(\tilde{X}_\bar{x}, \pi^E(\mathbb{R}/\mathbb{R}, \bar{x}), \bar{x}) \longrightarrow \left(\lim_{n \in \mathbb{N}^+} P_n, \lim_{n \in \mathbb{N}^+} \mu_{n, \mathbb{R}}, \bar{p}\right)
\]
which is indeed an isomorphism because of the fact that \((P_n, \mu_{n, \mathbb{R}}, p_n)\) is cofinal and saturated in \(I_{\delta l}(\mathbb{R}/\mathbb{R}, \bar{x})\). This proves that \(\pi^E(\mathbb{R}/\mathbb{R}, \bar{x})\) is infinite and also that \(\tilde{X}_\bar{x}\) has infinitely many connected components. Since \(\tilde{X}_\bar{x}\) is affine, it must be quasi-compact. But then the connected components of \(\tilde{X}_\bar{x}\) can not be open, otherwise there should be finitely many of them. Therefore \(\tilde{X}_\bar{x}\) is not Noetherian.

\[\square\]

**Proposition 2.16.** Let \(k\) be field whose Galois group \(\text{Gal}(\bar{k}/k)\) admits \(\mathbb{Z}/l\mathbb{Z}\) as a quotient for some prime \(l > 3\). Let \(X = \text{Spec}(k), \bar{x} : \text{Spec}(\bar{k}) \rightarrow X\) be a geometric point. Then \(\pi^E(k/k, \bar{x})\) is a non-commutative \(k\)-group scheme.

**Proof.** Let \(k \subseteq K \subseteq \bar{k}\) a finite Galois subextension so that \(\text{Gal}(K/k) = \langle \sigma \rangle \cong \mathbb{Z}/l\mathbb{Z}\). Let \(G_K := (\mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}) \rtimes \langle b \rangle\), where \(\langle b \rangle \cong \mathbb{Z}/l\mathbb{Z}\) acts on \(\mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}\) by
\[
b \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]
We define an action of \(\text{Gal}(K/k)\) on \(G_K\) by letting \(\sigma(z) = z\) for all \(z \in \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}\) and
\[
\sigma(b) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} b.
\]
This action corresponds, by Galois descent, to a \(k\)-group scheme \(G\) which is a \(k\)-form of the \(K\)-group scheme \(G_K\).
The constant $K$-group scheme $G_K$ can be written as

$$G_K := \bigsqcup_{i \in G_K} Y_i$$

where $Y_i = \text{Spec}(K)$. $G_K$ acts on itself by right translations, i.e. for any $j \in G_K$, $j$ acts on $Y_i$ by the identity map $\text{Spec}(K) = Y_i \to Y_{ij} = \text{Spec}(K)$. Now we define a twisted action of $\text{Gal}(K/k)$ on the $K$-scheme $G_K$. We define the action of $\sigma$ on $Y_i$ to be the morphism $\tau$ in the following commutative diagram

$$
\begin{array}{ccc}
Y_i & \xrightarrow{\tau} & Y_{b\sigma(i)} \\
\downarrow \downarrow & & \downarrow \downarrow \\
\text{Spec}(K) & \xrightarrow{t_\sigma} & \text{Spec}(K)
\end{array}
$$

where $t_\sigma$ is the map obtained by applying the functor $\text{Spec}(-)$ to the field automorphism $\sigma : K \to K$. We have the following compatibility between the action of $\text{Gal}(K/k)$ on the $K$-scheme $G_K$ and that on the $K$-scheme $G_K$, i.e. the diagram

$$
\begin{array}{ccc}
Y_i & \xrightarrow{j} & Y_{ij} \\
\downarrow \tau & & \downarrow \tau \\
Y_{b\sigma(i)} & \xrightarrow{\sigma(j)} & Y_{b\sigma(ij)}
\end{array}
$$

is commutative for any $j \in G_K$. By Galois descent, the $K$-scheme $G_K$ descends to a $k$-scheme $P$ and there is an action of $G$ on $P$ which makes $P$ a $G$-torsor over $k$. Picking $p \in Y_e(k)$ to be the inclusion $K \subseteq \bar{k}$, we get an object $(P, G, p) \in \text{I}_{\text{et}}(k/k, \bar{k})$. This object induces a $k$-homomorphism

$$\lambda : \pi^E(k/k, \bar{x}) \longrightarrow G.$$ 

Let $N \subseteq G$ be the image. Then we get a subobject $(Q, N, p) \subseteq (P, G, p)$. As $p \in Q$, $Y_e \subseteq Q_K \Rightarrow Y_b = \sigma(Y_e) \subseteq Q_K \Rightarrow b \in N_K \subseteq G_K$. But $N_K \subseteq G_K$ is stable under the Galois action, so $\sigma(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in N_K \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in N_K \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = b^{l-1} \in N_K$.

Thus $N_K = G_K$ for $\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, b \}$ generates $G_K$. Therefore $\lambda$ is surjective. Then $\pi^E(k/k, \bar{x})$ must be non-commutative for $G$ is not.

**Remark 2.17.** The point of the assumption $l > 3$ is that one needs the action of $\sigma^l$ on $G_K$ to be trivial, i.e. one needs that $b\sigma(b)\sigma^2(b) \cdots \sigma^{l-1}(b)$ to be trivial in $G_K$. For this one needs

$$1 \ast 0 + 2 \ast 1 + 3 \ast 2 + \cdots + (l-1) \ast (l-2) = \frac{1}{6}(l-1)(2l-1) - \frac{1}{2}l(l-1)$$

to be divisible by $l$. This is OK only when $l > 3$. 


**Exemple 2.18.** In this part we would like to give an example to show that, unlike the classical case [Nori2], the fundamental group $\pi_E$ is in general not commutative for an abelian variety over an algebraically closed field.

Let $A$ be an abelian variety over $k := \mathbb{R}$, $K := \mathbb{C}$, $\bar{x} \in A_K(K)$. Take any Galois covering $Y \to A_K$ with Galois group $\mathbb{Z}/2\mathbb{Z}$. Let $G_K := \langle a \rangle \times \langle b \rangle$, where $\langle a \rangle \cong \mathbb{Z}/n\mathbb{Z}$ for $n \geq 3 \in \mathbb{N}^+$ and $\langle b \rangle \cong \mathbb{Z}/2\mathbb{Z}$ acts on $\langle a \rangle$ by $b(z) = z^{-1}$ for all $z \in \langle a \rangle$. We define an action of $\text{Gal}(K/k)$ on $G_K$ by letting $\sigma(z) = z^{-1}$ for all $z \in \langle a \rangle$ and $\sigma(b) = ab$. Then there is a $k$-form $G$ of the $K$-group scheme $G_K$ which corresponds to this action.

Let $H_K \subset G_K$ denote the subgroup $\langle a \rangle$, and let

$$P_K := \prod_{i \in H_K} Y_i$$

where $Y_i = Y$. Now we define an action of $G_K$ on $P_K$. Take any $g \in G_K$, we can write it uniquely as $g = b^r j$, where $r \in \{0, 1\}$ and $j \in H_K$, then the action of $g$ on $Y_i$ is defined to be the morphism $\tau$ in the following commutative diagram

$$
\begin{array}{ccc}
Y_i & \xrightarrow{\tau} & Y_{b^r - r(i)j} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{b^r} & Y
\end{array}
$$

where $b^r$ is the map defined by the non-trivial $A_K$-automorphism of $Y$ if $r = 1$, the identity if $r = 0$. In this way, $P_K$ becomes a $G_K$-torsor over $A_K$. Now viewing $G_K$ as a constant $K$-group scheme we get a morphism

$$\rho : P_K \times_{\text{Spec}(K)} G_K \to P_K$$

defined by the above action. Composing $\rho$ with the following isomorphism

$$P_K \times_{\text{Spec}(K)} (\text{Spec}(K) \times_{\text{Spec}(k)} G) \cong P_K \times_{\text{Spec}(K)} G_K$$

we get an action of $G$ on $P_K$ which makes $P_K$ a $G$-torsor over $A_K$. Picking any $k$-morphism $p : \text{Spec}(K) \to Y_{\bar{e}} = Y$ (where $\bar{e} \in H_K$ is the identity element) over $\bar{x}$, we get an object $(P_K, G, p) \in I_{at}(A_K/k, \bar{x})$. This object induces a $k$-homomorphism

$$\lambda : \pi_E(A_K/k, \bar{x}) \to G.$$

Let $N \subseteq G$ be the image. Then we get a subobject $(Q, N, p) \subseteq (P_K, G, p)$. As $p \in Q$, $Y_e \subseteq Q$. Thus $b \in N_K \subseteq G_K$ (because $Y_e \subseteq P_K$ is a torsor under $\langle b \rangle \subseteq G_K$). But $N_K \subseteq G_K$ is stable under the Galois action, so $\sigma(b) = ab \in N_K \implies a \in N_K \implies N_K = G_K$. Therefore $\lambda$ is surjective. Then $\pi_E(A_K/k, \bar{x})$ must be non-commutative for $G$ is not.

2.5. **Comparisons with the Geometric Fundamental Groups.** Let $X$ be a separable geometrically connected scheme over a field $k$, $\bar{X} := X \times_k \bar{k}$, $\bar{x} : \text{Spec}(\bar{k}) \to \bar{X}$ be a geometric point. We would like to understand the relation between the two fundamental groups $\pi^N(\bar{X}/k, \bar{x}) \times_k \bar{k}$ and $\pi^N(\bar{X}/\bar{k}, \bar{x})$, the later being the "classical" Nori’s fundamental group. It turns out that they are not equal, or in other words, the new fundamental group is not a $k$-form of the classical one. In fact it is much larger.
Proposition 2.19. Notations and assumptions being as above, if $X$ is moreover quasi-compact and $k$ is perfect, then we have an imbedding
\[
\chi_{X/k}^N : \pi^N(X/k, \bar{x}) \hookrightarrow \pi^N(\bar{X}/k, \bar{x}) \times_k \bar{k}
\]
of $\bar{k}$-group schemes. Similar statement holds if one replaces $N$ by $E, G, L$.

Proof. Given $(P, G, p) \in N(\bar{X}/k, \bar{x})$, $(P, G \times_k \bar{k}, p)$ is naturally an object in $N(\bar{X}/k, \bar{x})$. In this way we get a functor $F$ which makes the following diagram 2-commutative
\[
\begin{array}{ccc}
N(\bar{X}/k, \bar{x}) & \xrightarrow{F} & N(\bar{X}/k, \bar{x}) \\
\varphi \downarrow & & \downarrow \psi \\
\text{Grsch}(\bar{k}) & & \\
\end{array}
\]
where $\varphi$ is the functor sending $(P, G, p) \in N(\bar{X}/k, \bar{x})$ to $G \times_k \bar{k}$, and $\psi$ is the forgetful functor sending $(Q, H, q) \in N(\bar{X}/k, \bar{x})$ to $H$. Since base change is compatible with projective limit, we have
\[
\lim_{i \in N(\bar{X}/k, \bar{x})} \varphi(i) = \pi^N(\bar{X}/k, \bar{x}) \times_k \bar{k}.
\]
Therefore, we get the homomorphism
\[
\chi_{X/k}^N : \pi^N(\bar{X}/k, \bar{x}) \longrightarrow \pi^N(\bar{X}/k, \bar{x}) \times_k \bar{k}
\]
by passing to the limit. The injectivity of $\chi_{X/k}^N$ is proved in [1.1].

Proposition 2.20. If $X$ is a connected scheme over any field $k$, $\bar{x} \in X(\bar{k})$ is a geometric point, and if the projection $X \to \text{Spec } (k)$ factors through the map $\text{Spec } (\bar{k}) \to \text{Spec } (k)$ induced by the field extension $k \subseteq \bar{k}$, then the imbedding
\[
\chi_{X/k}^E : \pi^E(X/k, \bar{x}) \times_k \bar{k} = \pi^E(X/\bar{k}, \bar{x}) = \pi^E(\bar{X}/k, \bar{x}) \hookrightarrow \pi^E(X/k, \bar{x}) \times_k \bar{k}
\]
of $\bar{k}$-group schemes is a section of the map
\[
\pi^E_G \times_k \bar{k} : \pi^E(X/k, \bar{x}) \times_k \bar{k} \to \pi^E(X/k, \bar{x}) \times_k \bar{k}.
\]

Proof. Let’s first redo the construction in [2.19]. Given $(P, G, p) \in I_{\text{et}}(X/k, \bar{x})$, let $G'$ be the abstract group associated to $G \times_k \bar{k}$. Viewing $G'$ as a constant group scheme over $k$, we get an object $(P, G', p) \in I_{\text{co}}(X/k, \bar{x})$. In this way we get a functor which makes the following diagram 2-commutative
\[
\begin{array}{ccc}
I_{\text{et}}(X/k, \bar{x}) & \xrightarrow{F} & I_{\text{co}}(X/k, \bar{x}) \\
\varphi \downarrow & & \downarrow \psi \\
\text{Grsch}(\bar{k}) & & \\
\end{array}
\]
where $\varphi$ is the functor sending $(P, G, p) \in I_{\text{et}}(X/k, \bar{x})$ to $G \times_k \bar{k}$, and $\psi$ is the functor sending $(P, G, p) \in I_{\text{co}}(X/k, \bar{x})$ to the abstract group $G$ regarded as a group scheme over
Since base change is compatible with projective limit, we have
\[ \lim_{i \in I_{co}(X/k, \bar{x})} \varphi(i) = \pi^E(X/k, \bar{x}) \times_k \bar{k} \quad \text{and} \quad \lim_{i \in I_{co}(X/k, \bar{x})} \psi(i) = \pi^G(X/k, \bar{x}) \times_k \bar{k} \]
This defines the homomorphism \( \chi^E_{X/k/k} \) which is then easily seen as a section of \( \pi^E \), because the (right) composition of \( F \) with the inclusion
\[ i : I_{co}(X/k, \bar{x}) \rightarrow I_{et}(X/k, \bar{x}) \]
is isomorphic to the identity functor on \( I_{co}(X/k, \bar{x}) \). \( \square \)

**Example 2.21.** Assumptions and notations are as in [2.20]. Here we would like to give an example to show that the canonical imbedding \( \pi^E(X/k, \bar{x}) \hookrightarrow \pi^E(X/k, \bar{x}) \times_k \bar{k} \) is not, in general, an isomorphism.

Let’s take an abelian variety \( A \) over a perfect field \( k \) of characteristic \( p \geq 0 \) and a positive integer \( n \in \mathbb{N}^+ \) which is prime to \( p \) and which has the property that the \( n \)-torsion points \( A[n] \) of \( A \) is not a constant group scheme over \( k \).

For example, we know from Mordell-Weil that \( A[n](\mathbb{Q}) \subseteq A(\mathbb{Q}) \) is finite while \( A[n](\bar{\mathbb{Q}}) \) is isomorphic to \( (\mathbb{Z}/n\mathbb{Z})^{2 \dim(A)} \), so \( A[n](\mathbb{Q}) \) is way larger than \( A[n](\bar{\mathbb{Q}}) \) for \( n \rightarrow \infty \). Thus for big \( n \), \( A[n] \) is not a constant group scheme over \( \mathbb{Q} \).

Now let \( X := A \times_k \bar{k} \), \( G := X[n](\bar{k}) \). Viewing \( G \) as a constant group scheme over \( k \) we have an isomorphism \( \bar{\lambda} : X[n] \xrightarrow{\sim} G \times_k \bar{k} \). Thus the multiplication by \( n \), \( [n]_X : X \rightarrow X \), is a \( G \)-torsor. Then the triple \( (X, G, 0) \in I_{et}(X/k, 0) \) defines a \( k \)-homomorphism \( \beta : \pi^E(X/k, 0) \rightarrow G \). Now consider the trivial isomorphism
\[ (id, \bar{\lambda}) : (X, X[n], 0) \rightarrow (X, G \times_k \bar{k}, 0) \in I_{et}(X/k, 0). \]
Let \( \alpha : \pi^E(X/k, 0) \rightarrow A[m] \) be the homomorphism corresponding to the triple \( (X, A[m], 0) \).

Then there is a not necessarily commutative diagram

\[ \begin{array}{ccc}
\pi^E(X/k, 0) & \xrightarrow{\chi^E_{X/k/k}} & \pi^E(X/k, 0) \\
\alpha' \downarrow & & \beta' \downarrow \\
X[n] & \xrightarrow{\lambda} & G \times_k \bar{k}
\end{array} \]

in which \( \alpha' \) is the composition of \( \chi^E_{X/k/k} \) with \( \alpha \times id \) and similarly for \( \beta' \). Now by \( (id, \bar{\lambda}) \) we have \( \bar{\lambda} \circ \alpha' = \beta' \). If the inclusion \( \chi^E_{X/k/k} \) was an isomorphism, then we should get a commutative diagram

\[ \begin{array}{ccc}
\pi^E(X/k, 0) \times_k \bar{k} & & \\
\alpha \times id \downarrow & & \beta \times id \downarrow \\
X[n] & \xrightarrow{\bar{\lambda}} & G \times_k \bar{k}
\end{array} \]
Note that the map $\alpha \times id$ is surjective, for $X \xrightarrow{\alpha} X$ is a Galois cover. This implies that the isomorphism $\lambda$ descends to an isomorphism $\lambda_k : A[n] \to G$ over $k$. This contradicts to our assumption that $A[n]$ is non-constant.

2.6. Base Change.

**Proposition 2.22.** Let $X$ be a scheme geometrically connected proper separable over a field $k$, $k \leq l \leq l'$ be a sequence of field extensions, where $l$ and $l'$ are algebraically closed fields. Let $\bar{x} : \text{Spec}(l') \to X$ be a geometric point. Then the following natural map

$$\pi^{\text{et}}_l : \pi^E(X \times_k l'/k, \bar{x}) \to \pi^E(X \times_k l/k, \bar{x})$$

is an isomorphism of $k$-group schemes.

**Proof.** Let $Y' \to X \times_k l'$ be a $G'$-torsor with a fixed point $\text{Spec}(l') \to Y'$ lying over $\bar{x}$. By [SGA1, Exposé X, Corollaire 1.7],

$$\pi^{\text{et}}_l(X \times_k l', \bar{x}) \cong \pi^{\text{et}}_l(X \times_k l, \bar{x}).$$

Thus by [SGA1, Exposé V, Théorème 4.1], the base change functor $- \times l'$ induces an equivalence of categories between the categories of finite étale coverings $\text{ECov}(X \times_k l)$ and $\text{ECov}(X \times_k l')$. Thus there is a finite étale covering $Y \to X \times_k l$ such that $Y \times_k l' = Y'$. Now from the fully faithfulness of $- \times l'$ and the fact that $G \times_k l$ and $G \times_k l'$ are constant group schemes, the action

$$(Y \times_k l') \times (G \times_k l') = Y \times_k l' \times_k G = Y' \times_k G \to Y' = Y \times_k l'$$

descends to an action $Y \times_k G \to Y$ and makes $Y$ a $G$-torsor. This means that the pull back functor

$$F^l_{l'} : N(X \times_k l'/k, \bar{x}) \to N(X \times_k l'/k, \bar{x})$$

is essentially surjective. But by the fully faithfulness of $- \times l'$ the pull back functor $F^l_{l'}$ is also fully faithful. Hence $F^l_{l'}$ is an equivalence, and therefore the canonical morphism $\pi^{\text{et}}_{l'}$ is an isomorphism. \hfill $\square$

2.7. The Étale Universal Covering. In this subsection we want to emphasize a big difference between $\pi^E(X/k, x)$ and $\pi^G(X/k, x)$ (or $\pi^{\text{et}}_l(X, x)$) via comparing their universal coverings. The following statement shall be well known in the literature.

**Statement 2.23.** Let $X$ be a connected Noetherian scheme, $x \in X(\text{Spec}(\bar{k}))$ be any geometric point. Then the universal covering $\tilde{X}_x$ corresponding to $\pi^{\text{et}}_l(X, x)$ is connected.

The major reason behind this phenomenon is the following:

**Fact.** If $X$ is a locally Noetherian connected scheme, $x \in X(\text{Spec}(\bar{k}))$ be any geometric point, then for any triple $(P, G, p) \in I_{\text{co}}(X, x)$ the corresponding map $\pi^{\text{et}}_l(X, x) \to G$ is surjective if and only if $P$ is connected.

But for universal coverings under $\pi^E$, they are usually highly non-connected. We have seen some examples in [2,4] which are caused by complicated structures of the étale group schemes. Here is another example which is caused by the choice of the point on the torsor.
Exemple 2.24. Let $X = \text{Spec} \left( \mathbb{Q} \right)$, $\bar{x} : \mathbb{Q} \subseteq \bar{\mathbb{Q}}$. Consider a prime number $p > 2$. Then $\mu_p \cong \text{Spec} \left( \mathbb{Q} \right) \coprod \text{Spec} \left( K \right)$ as a scheme, where $K$ is the $p$-th cyclotomic field. Let $(\mu_p, \mu_p, q)$ be the trivial $\mu_p$-torsor equipped with the point $q : \text{Spec} \left( \bar{\mathbb{Q}} \right) \to \text{Spec} \left( K \right)$. Obviously $\mu_p$ is not connected, but the unique map

$$(\phi, h) : (\widetilde{X}_x, \pi^E \left( X/k, \bar{x} \right)) \to (\mu_p, \mu_p, q)$$

can not be trivial on $h$, for otherwise $(\phi, h)$ would factor through the trivial triple $(X, \{1\}, \bar{x})$, and then $q$ has to be the trivial point $\text{Spec} \left( \bar{\mathbb{Q}} \right) \to \text{Spec} \left( \mathbb{Q} \right)$. But if $h$ is non-trivial then it has to be surjective. Therefore $(\mu_p, \mu_p, q)$ is saturated but not connected. Since $h$ is surjective, $\phi$ must be faithfully flat. But $\mu_p$ is not connected so $\widetilde{X}_x$ cannot be either.

Proof of the fact. "$\Rightarrow"$ If $P$ was not connected then we can take the connected component $Q \subseteq P$ containing $p$. Let $H \subseteq G$ be the stabilizer of $Q$, then $(Q, H, p) \subseteq (P, G, p)$. Therefore we have a factorization $\pi^E_1(X, x) \to H \subseteq G$ which contradicts to the assumption that $\pi^E_1(X, x) \to G$ is surjective. "$\Leftarrow"$ Suppose $\pi^E_1(X, x) \to G$ factorizes as $\pi^E_1(X, x) \to H \subseteq G$. Then we would have an imbedding $(Q, H, p) \subseteq (P, G, p)$. But $Q \subseteq P$ is finite étale, so it’s both open and closed. Therefore $Q = P$ for $Q \neq \emptyset$. Hence $H = G$. \qed

Proof of the statement. Let $I \subseteq I_{co}(X, x)$ be the full subcategory consisting of saturated objects. Then $\widetilde{X}_x = \lim_{\leftarrow i \in I} P_i$, where $i = (P_i, G_i, p_i) \in I$. Because of the above Fact, these $P_i$ are connected. The scheme $\widetilde{X}_x$ is connected if and only if $H^0(\widetilde{X}_x, O_{\widetilde{X}_x})$ has no non-trivial idempotents. Since $X$ is quasi-compact and $\lim_{\leftarrow i}$ is an exact functor we know that

$$H^0(\widetilde{X}_x, O_{\widetilde{X}_x}) = \lim_{\leftarrow i} H^0(P_i, O_{P_i}).$$

As each $P_i$ is connected, there is no non-trivial idempotents in $H^0(P_i, O_{P_i}) \subseteq H^0(\widetilde{X}_x, O_{\widetilde{X}_x})$, hence there is no non-trivial idempotents in $H^0(\widetilde{X}_x, O_{\widetilde{X}_x})$. \qed

3. The First Fundamental Sequence

3.1. The General Case.

Proposition 3.1. Let $X$ be a scheme geometrically connected separable over a field $k$, $\bar{x} : \text{Spec} \left( \bar{k} \right) \hookrightarrow X$ be a geometric point, then the natural $k$-group scheme homomorphism

$$\pi^l(X/k, \bar{x}) \to \pi^l(k/k, \bar{x})$$

is surjective for $I = E, G, N, L$.

Proof. Suppose that we have an object $(l, G, t) \in I(k/k)$ and that we have a morphism

$$(\lambda, i) : (Q, H, s) \to (l \times_k X, G, t) \in I(X/k),$$

where the group homomorphism $i : H \to G$ is a closed imbedding. Then we have a section in the category of $X$-schemes

$$X = Q/H \hookrightarrow (l \times_k X)/H = (l/H) \times_k X.$$
As $l/H$ is finite over $k$, its connected components are single points. Let $x \in l/H$ be the image of $t \in l$ under the projection $l \to l/H$. Since $X$ is connected, reduced and $\lambda$ sends $s \mapsto t$, the map

$$X \to (l/H) \times_k X \xrightarrow{\text{pr}_1} l/H$$

factors through $x : \text{Spec} (\kappa(x)) \hookrightarrow l/H$ where $\kappa(x)$ is the residue field of $x$. Hence $X$ is a scheme over $\kappa(x)$. But $X$ is geometrically connected and geometrically reduced over $k$, so the extension $k \subseteq \kappa(x)$ has to be trivial, i.e. $k = \kappa(x)$. In other words, $x$ is a $k$-rational point of $l/H$. Now pull back the projection map $l \to l/H$ along $x : \text{Spec} (k) \to l/H$, we get a map $(q, H, t) \to (l, G, t) \in I(k/k)$ in which the group homomorphism is the imbedding $i : H \hookrightarrow G$. In particular if the map $\pi^I(k/k, \bar{x}) \to G$ corresponding to $(l, G, t)$ is surjective, then the composition

$$\pi^I(X/k, \bar{x}) \to \pi^I(k/k, \bar{x}) \to G$$

has to be surjective too. This means precisely that $\pi^I(X/k, \bar{x}) \to \pi^I(k/k, \bar{x})$ is surjective.

\[ \square \]

**Proposition 3.2.** Let $X$ be a scheme geometrically connected separable over a field $k$, $\bar{x} : \text{Spec} (\bar{k}) \hookrightarrow X$ be a geometric point, then the natural sequence of $k$-group schemes

\begin{equation}
1 \to \pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \to \pi^I(k/k, \bar{x}) \to 1
\end{equation}

is a complex, and it is exact in the middle if and only if the following two conditions are satisfied.

(i) For any $I$-saturated object $(P, G, p) \in I(X/k, \bar{x})$, the image of the composition of the natural homomorphisms

$$\pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \to G$$

is a normal subgroup of $G$.

(ii) Whenever there is an object $(P, G, p) \in I(X/k, \bar{x})$ whose pull-back along $\bar{X} \to X$ is trivial then there is an object $(Q, H, q) \in I(k/k, \bar{x})$ whose pull-back along $X \to \text{Spec} (k)$ is isomorphic to $(P, G, p)$.

**Proof.** For the first statement it is enough to see that the pull-back functor

$$\mathcal{C}(k/k, \bar{x}, I) \to \mathcal{C}(\bar{X}/k, \bar{x}, I)$$

sends any object in $\mathcal{C}(k/k, \bar{x}, I)$ to a trivial object in $\mathcal{C}(\bar{X}/k, \bar{x}, I)$. But this is indeed the case, for the pull-back functor $I(k/k, \bar{x}) \to I(\bar{X}/k, \bar{x})$ is trivial.

Now for the second statement. "⇒" (i) is clear, because a normal subgroup is still normal in any quotient. (ii) follows directly from 2.2. Indeed, given $(P, G, p) \in I(X/k, \bar{x})$, there is a unique morphism $\phi : \pi^I(X/k, \bar{x}) \to G$ corresponding to $(P, G, p)$. The pull-back of $(P, G, p)$ is trivial means that the composition

$$\pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \xrightarrow{\phi} G$$

is trivial.
is trivial. By the exactness there is a unique map \( \varphi : \pi^I(k/k, \bar{x}) \to G \) making the diagram

\[
\begin{array}{c}
\pi^I(X/k, \bar{x}) \ar[rr]^\varphi && \pi^I(k/k, \bar{x}) \ar[dl]_\varphi \\
& \ar[rr]_G & &
\end{array}
\]

commutative. Therefore, \( \varphi \) defines an object in \( I(k/k, \bar{x}) \) whose pull-back is isomorphic to \( (P, G, p) \).

“\( \Leftarrow \)” Let \((P, G, p) \in I(X/k, \bar{x})\) be an \( I \)-saturated object. By \( \text{[2.2]} \) it corresponds to a \( k \)-homomorphism \( \phi : \pi^I(X/k, \bar{x}) \to G \). Let \( H \) be the image of the composition

\[
\pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \to G \to G/H
\]

is trivial by definition, the pull-back of \((P/H, G/H, p) \in I(X/k, \bar{x})\). By (ii), \((P/H, G/H, p) \) descends to an object in \( I(k/k, \bar{x}) \), or equivalently, there is a commutative diagram

\[
\begin{array}{c}
\pi^I(\bar{X}/k, \bar{x}) \ar[rr] && \pi^I(X/k, \bar{x}) \\
& \ar[rr]_G & & G/H
\end{array}
\]

Let \( N \) be the image of the kernel of \( \pi^I(X/k, \bar{x}) \to \pi^I(k/k, \bar{x}) \) under the map

\[
\phi : \pi^I(X/k, \bar{x}) \to G.
\]

The above diagram implies that \( N \subseteq H \) and the first statement of this proposition implies that \( H \subseteq N \). Therefore \( H = N \). But since this is valid for all \( I \)-saturated objects, we can conclude the middle exactness. \( \square \)

**Remark 3.3.** The injectivity on the left is in general not true. See \([3.15]\) for an example when \( k \) is perfect \( X = \mathbb{A}^1_k \) and \( I = L \). The example does not work for \( \pi^E \). However, one can not use the injectivity for \( \pi^G \) (or \( \pi^I_1 \)) to conclude the injectivity for \( \pi^E \) either. The injectivity for \( \pi^G \) (or \( \pi^I_1 \)) was deduced from the theory of \textit{Galois closure} for Galois coverings \([\text{SZ}, \text{Proposition 5.3.9, pp. 169}]\). But we can not find an analogue for \( \pi^E \).

**Corollary 3.4.** If \( X = \mathbb{A}^n_k \) where \( k \) is a field of characteristic 0 or \( X = \mathbb{P}^n_k \) for \( n \in \mathbb{N}^+ \), \( \bar{x} \in X(k) \), then there is a canonical isomorphism

\[
\pi^E(X/k, \bar{x}) \to \pi^E(k/k, \bar{x}).
\]

**Proof.** By \( \text{[2.12]} \) and \( \text{[2.14]} \) \( \pi^E(\bar{X}/k, \bar{x}) = \{1\} \). Then the corollary follows from \( \text{[3.11]} \) and \( \text{[3.2]} \). \( \square \)
Exemple 3.5. In this part we would like to give an example to show that the condition (ii) of \[3.2\] is not always satisfied.

Let’s just take \(k = \mathbb{F}_p(s, t)\) (the function field in two variables over \(\mathbb{F}_p\)), \(X = \mathbb{A}^1_k \setminus \{a\}\),

\[P = \text{Spec}(A[T]/(T^p - (s + tx^p)))\]

be the \(\mu_p\)-torsor over \(X\) in a natural way, where \(A := O_X(X)\) and \(a \in \mathbb{A}^1_k\) is the closed point determined by the polynomial \(s + tx^p \in k[x]\). Since \(P\) is a local torsor the base point plays no role. For this reason we are going to omit the base point in the following discussion. Clearly, the equation

\[T^p - (s + tx^p) = 0\]

has no solution in \(A\), thus \(P\) is a non-trivial \(\mu_p\)-torsor. Furthermore, \(P \times_k \bar{k}\) is a trivial torsor over \(\bar{X}\), the section being given by the solution of the above equation in \(\bar{k}[x]\). But \(P \to X\) can not descent to a \(\mu_p\)-torsor over Spec \((k)\). In fact, the two \(\mu_p\)-torsors

\[P_0 = \text{Spec}(k[T]/(T^p - s)) \quad \text{and} \quad P_1 = \text{Spec}(k[T]/(T^p - s - t))\]

which are fibres of \(P \to X\) at \(x = 0\) and \(x = 1\) respectively, can not be isomorphic. Suppose there was an isomorphism of torsors

\[f : k[T]/(T^p - s) \to k[T]/(T^p - s - t)\]

sending \(T \mapsto f(T)\), where \(f(T) \in k[T]\) is a polynomial of degree less than \(p\). Let \(\mu_p = k[Y]/(Y^p - 1)\), then \(\text{Aut}_{k-gp.sch}(\mu_p) = (\mathbb{Z}/p\mathbb{Z})^*\), where \(m \in (\mathbb{Z}/p\mathbb{Z})^*\) stands for \(Y \mapsto Y^m\). Thus we should have a commutative diagram

\[
\begin{array}{ccc}
k[T]/(T^p - s) & \xrightarrow{f} & k[T]/(T^p - s - t) \\
\rho_0 \downarrow & & \rho_1 \downarrow \\
(k[T]/(T^p - s)) \otimes_k k[Y]/(Y^p - 1) & \xrightarrow{f \otimes m} & (k[T]/(T^p - s - t)) \otimes_k k[Y]/(Y^p - 1)
\end{array}
\]

where \(\rho_0\) and \(\rho_1\) are defined by the action of \(\mu_p\) on \(P_0\) and \(P_1\) respectively. Tracing the image of \(T\) in the above diagram, we get \(f(T) \otimes Y^m = f(T \otimes Y)\). This implies that \(f(T)\) is a polynomial of the form \(\alpha T^m\) with \(\alpha \in k\). Then we should have

\[f : T^p - s \mapsto f(T)^p - s = \alpha^p T^{mp} - s = 0 \in k[T]/(T^p - s - t).\]

But \(T^p = s + t \in k[T]/(T^p - s - t)\). Hence we should have \(\alpha^p(s + t)^m - s = 0 \in k \subset k[T]/(T^p - s - t)\). However, this equation can not happen in \(k\) because \(\mathbb{F}_p[s, t]\) is a UFD.

However, when \(k\) is a perfect field condition (ii) actually holds.

Proposition 3.6. If the field \(k\) in \[3.3\] is assumed to be perfect and \(X\) is, in addition, quasi-compact, then condition (ii) holds for \(N(X/k, \bar{x})\).

Proof. Let \((P, G, p) \in N(X/k, \bar{x})\) be an object whose pull-back to \(N(\bar{X}/k, \bar{x})\) is trivial, i.e. there is a morphism

\[\lambda : (\bar{X}, \{1\}, \bar{x}) \to (\bar{P}, G, p) \in N(\bar{X}/k, \bar{x}).\]

Let \((P_{\text{ét}}, G_{\text{ét}}, p) \in I_{\text{ét}}(X/k, \bar{x})\) be the étale quotient of \((P, G, p)\). Then \((\bar{P}_{\text{ét}}, G_{\text{ét}}, p)\) is also trivial. Thus by \[3.1\] there is a triple \((Q, H, q) \in I_{\text{ét}}(k/k, \bar{x}) \subseteq N(k/k, \bar{x})\) such that the
pull-back of \((Q, H, q)\) to \(X\) is isomorphic to \((\tilde{P}_{\text{ét}}, G_{\text{ét}}, p)\). Let \(n\) be the order of the \(k\)-group scheme \(G_{\text{ét}}\). Then \(\tilde{P}_{\text{ét}}\) can be written as \(n\)-copies of \(\tilde{X}\):

\[
\tilde{P}_{\text{ét}} = \coprod_{i=1,\ldots,n} \tilde{X}_i
\]

where \(\tilde{X}_i = \tilde{X}\). The quotient \(\pi : P \to P_{\text{ét}}\) makes \(P\) as \(G^0\)-torsor over \(P_{\text{ét}}\), and we have a decomposition

\[
\bar{P} = \coprod_{i=1,\ldots,n} \bar{P}_i
\]

where \(\bar{P}_i\) is just \((\pi \times_k \tilde{k})^{-1}(\tilde{X}_i)\). Suppose \(p \in \bar{P}_i(\tilde{k})\). Then the map \(\lambda\) makes \(\bar{P}_i\) a trivial \(G^0\)-torsor over \(\bar{X}_i\). Since \(G^0\) is local and \(\bar{X}_i\) is reduced, the closed imbedding \((\tilde{P}_i)_{\text{red}} \hookrightarrow \bar{P}_i\) compositing with the projection \(\bar{P}_i \to \bar{X}_i\) has to be an isomorphism. As \(G_{\text{ét}}(\tilde{k})\) acts transitively on the components \(\bar{X}_i\), for any \(1 \leq i \leq n\) there is an element \(g \in G_{\text{ét}}(\tilde{k}) = G(\tilde{k})\) making the diagram

\[
\begin{array}{ccc}
\tilde{P}_0 & \xrightarrow{g} & \bar{P}_i \\
\downarrow & & \downarrow \\
\bar{X}_0 & \xrightarrow{g} & \bar{X}_i
\end{array}
\]

commutative. Hence the closed imbedding \((\tilde{P}_i)_{\text{red}} \hookrightarrow \bar{P}_i\) compositing with the projection \(\bar{P}_i \to \bar{X}_i\) is also an isomorphism for each \(i\). Therefore the composition \(P_{\text{red}} \hookrightarrow \bar{P} \to \bar{P}_{\text{ét}}\) is an isomorphism too. This defines a section \(s : P_{\text{ét}} \hookrightarrow P\) for the projection \(\pi : P \to P_{\text{ét}}\). The universality of the reduced closed subscheme structure \(P_{\text{red}} \subseteq P\) tells us that there is a unique arrow \(P_{\text{ét}} \times_k G_{\text{ét}} \to P_{\text{ét}}\) making the following diagram

\[
\begin{array}{ccc}
P_{\text{ét}} \times_k G_{\text{ét}} & \xrightarrow{s} & P_{\text{ét}} \\
\downarrow{s \times i} & & \downarrow{s} \\
P \times_k G & \xrightarrow{\rho} & P \\
\downarrow{\pi \times o} & & \downarrow{\pi} \\
P_{\text{ét}} \times_k G_{\text{ét}} & \xrightarrow{\rho_{\text{ét}}} & P_{\text{ét}}
\end{array}
\]

commutative, where \(i : G_{\text{ét}} \subseteq G\) is the inclusion of the reduced subscheme structure of \(G\), \(\rho_{\text{ét}}\) is action of \(G_{\text{ét}}\) on \(P_{\text{ét}}\) induced by \(\rho\), \(o : G \to G_{\text{ét}}\) is the étale quotient map. Therefore we obtain a morphism

\[
(P_{\text{ét}}, G_{\text{ét}}, p) \to (P, G, p) \in N(X/k, \bar{x})
\]

In view of the isomorphism \((P_{\text{ét}}, G_{\text{ét}}, p) \cong (Q \times_k X, H, q)\), we can equip the \(k\)-scheme \(G\) with a left action from \(H\) via \(H \cong G_{\text{ét}} \to G\), then the contracted product \(Q \times H G\) provides a \(k\)-form for the \(G\)-torsor \(P\) over \(X\). This finishes the proof. \(\square\)
3.2. The Étale Case.

Proposition 3.7. Let $X$ be a Noetherian scheme geometrically connected over a field $k$, $\bar{x} : \text{Spec}(\bar{k}) \to X$, $(P, G, p) \in I_{\text{ét}}(X/k, \bar{x})$ be a saturated object. Let $N$ be the image of the following composition

$$\pi^E(\bar{X}/k, \bar{x}) \to \pi^E(X/k, \bar{x}) \hookrightarrow G.$$ 

Then we get an imbedding $(\bar{P}', N, p) \subseteq (\bar{P}, G, p) \in I_{\text{ét}}(\bar{X}/k, \bar{x})$. If one of the following conditions is satisfied, then $N \subseteq G$ is a normal subgroup scheme.

(i) $P$, as a scheme, is connected.
(ii) $\bar{P}'$, as a scheme, is connected.

Proof. By [SGA1, Exposé IX, Théorème 4.10] we may assume that $k$ is a perfect field. There is a finite Galois subextension $\bar{k} \subseteq K$ of $k \subseteq \bar{k}$ such that $G_K$ is constant over $K$ and the number of connected components of $\bar{P}_K$ is the same as that of $\bar{P}$. In this case the image of $\pi^E(\bar{X}/k, \bar{x})$ and $\pi^E(X/k, \bar{x})$ are the same in $G$. Thus replacing $\bar{k}$ by $K$, we may assume that $k \subseteq \bar{k}$ is a finite Galois extension.

Suppose that we have an $I_{\text{ét}}$-saturated object $(P, G, p) \in I_{\text{ét}}(X/k, \bar{x})$. Let $\bar{G} := G \times_k \bar{k}$, $\bar{P} := P \times_k \bar{k}$, $\bar{P}_0$ be the connected component of $\bar{P}$ containing $p$. Let $\bar{H} \subseteq \bar{G}$ be the subgroup which fixes $\bar{P}_0$, i.e.

$$\bar{H} := \{ g \in \bar{G} \mid \bar{P}_0 g = \bar{P}_0 \}.$$ 

Then $\bar{N} \subseteq \bar{G}$ is the smallest subgroup which contains the subset

$$\bigcup_{\sigma \in \text{Gal}(\bar{k}/k)} \sigma(\bar{H}) \subseteq \bar{G}.$$ 

Now let $T$ be the subset of $\bar{G}$ whose elements are those $t_\sigma$ which send $p \mapsto \sigma(p)$ for some $\sigma \in \text{Gal}(\bar{k}/k)$. Let $\bar{M}$ be the smallest subgroup of $\bar{G}$ containing $T$. Since for any $\sigma \in \text{Gal}(\bar{k}/k)$ and $t_\tau \in T$ sending $p \mapsto \tau(p)$ we have $t_\sigma \circ \sigma(t_\tau) = t_{\sigma \tau}$. So $\sigma(t_\tau) = t_\sigma^{-1} \circ t_{\sigma \tau} \in \bar{M}$, then it follows that $\sigma(\bar{M}) \subseteq \bar{M}$. Let $\bar{MN} \subseteq \bar{G}$ be the smallest subgroup of $\bar{G}$ containing $\bar{M}$ and $\bar{N}$. Let

$$\bar{P}'' := \bigcup_{g \in \bar{MN}} (\bar{P}_0) g \subseteq \bar{P},$$ 

i.e. the $\bar{MN}$-obits of $\bar{P}_0$ in $\bar{P}$. Then $\bar{P}''$ is a torsor under $\bar{MN}$. Since both $\bar{P}''$ and $\bar{MN}$ are stable under the induced Galois action, they both descend to $k$, i.e. there exists a $k$-form $\bar{MN} \subseteq G$ of $\bar{MN}$ such that $\bar{P}''$ descends to an $MN$-torsor $P'' \subseteq P$ over $X$. Then there is a homomorphism

$$\pi^E(X/k, \bar{x}) \to \bar{MN} \subseteq \bar{G}.$$ 

But $\pi^E(X/k, \bar{x}) \to G$ is already surjective by the assumption, this immediately implies that $\bar{MN} = G$ or equivalently $\bar{MN} = \bar{G}$.

Next we show that $\bar{N}$ is a normal subgroup of $\bar{G}$. From the above discussion it is enough to check $m^{-1}\bar{H}m \subseteq \bar{N}$ for $\forall m \in \bar{M}$. If (i) is satisfied, then $\text{Gal}(\bar{k}/k)$ acts transitively on the connected components of $\bar{P}$, so any element $g \in \bar{G}$ can be written as $h \circ t_\sigma$, where
h ∈ \bar{H}, t_σ ∈ T. If (ii) is satisfied, then \bar{H} = \bar{N}. In either case it is already enough to check \( t^{-1}_σ H t_σ \subseteq \bar{N} \) for \( \forall t_σ ∈ T \). From the very definition of \( t_σ \) we have
\[
σ(p) t^{-1}_σ h t_σ = (ph) t_σ = σ(p) h'
\]
where \( h' \) is contained in the stabilizer of \( σ(P_0) \), i.e. \( σ(H) \). Thus \( t_σ h t^{-1}_σ ∈ σ(H) ⊆ \bar{N} \).

**Remark 3.8.** The two conditions in 3.7 are already satisfied by most interesting étale torsors. For example it is known in the literature 2.7 that any triple \((P, G, p) \in I_{co}(X/k, \bar{x})\) is \( I_{co} \)-saturated if and only if it is connected. Thus in view of 3.11 and 3.2 this proposition can be seen as a generalization of the fundamental exact sequence of the étale fundamental group \([SGA1][Exposé IX,Théorème 6.1]\).

**Exemple 3.9.** In this part, we would like to construct an example showing that for a saturated object \((P, G, p) \in I_{ét}(X/k, \bar{x})\) which does not satisfy any of the conditions in 3.7 the image of the composition
\[
\pi^E(\bar{X}/k, \bar{x}) → \pi^E(X/k, \bar{x}) → G
\]
needs not to be a normal subgroup of \( G \).

Let \( X \) be any scheme geometrically connected over a field \( k \), \( Y → X \) be a torsor under the constant group scheme \((\mathbb{Z}/2\mathbb{Z})_k\) with \( Y \) geometrically connected over \( k \). Now we are going to construct a finite Galois field extension \( K/k \) with Galois group \( M \), a torsor \( P'_K \) over \( X_K \) under an abstract group \( G'_K \) which contains the \( X_K \)-scheme \( Y_K \) as a connected component. We will also construct a twisted action of \( M \) on the \( X_K \)-scheme \( P'_K \) and an action of \( M \) on the abstract group \( G'_K \) in such a way that these two actions are compatible.

Let \( n ∈ \mathbb{N}^+ \) be an even number which is equal to or larger than 2. Let
\[
G'_K = \langle \langle a_1 \rangle × \langle a_2 \rangle × \langle a_3 \rangle × \langle a_4 \rangle \rangle × \langle \langle b_1 \rangle × \langle b_2 \rangle \rangle \rangle × \langle ξ \rangle,
\]
where \( \langle a_1 \rangle = \langle a_2 \rangle = \langle a_3 \rangle = \langle a_4 \rangle ≅ \mathbb{Z}/2n\mathbb{Z} \), \( \langle b_1 \rangle = \langle b_2 \rangle = \langle ξ \rangle ≅ \mathbb{Z}/2\mathbb{Z} \). The actions are defined by the following relations.

\[
\begin{align*}
b_1 a_1 &= a_2 b_1 & b_1 a_2 &= a_1 b_1 & b_1 a_3 &= a_3 b_1 & b_1 a_4 &= a_4 b_1 \\
b_2 a_1 &= a_1 b_2 & b_2 a_2 &= a_2 b_2 & b_2 a_3 &= a_3 b_2 & b_2 a_4 &= a_4 b_2 \\
ξ b_1 &= b_1 ξ & ξ b_2 &= a_3^n a_1^n b_2 ξ & ξ a_i &= a_i^{n+1} ξ & i = 1, 2, 3, 4.
\end{align*}
\]

In addition we can define an action of \( \mathbb{Z}/2\mathbb{Z} = \{e, σ\} \) on \( G'_K \) (via group automorphisms). The action is given by the following equations.

\[
\begin{align*}
σ(a_1) &= a_3 & σ(a_2) &= a_4 & σ(a_3) &= a_1 & σ(a_4) &= a_2 \\
σ(b_1) &= b_2 & σ(b_2) &= b_1 & σ(ξ) &= a_3^n a_1^n ξ.
\end{align*}
\]

Next we construct the \( G'_K \)-torsor \( P'_K \). Let \( H'_K ⊆ G'_K \) be the subgroup
\[
\langle \langle a_1 \rangle × \langle a_2 \rangle × \langle a_3 \rangle × \langle a_4 \rangle \rangle × \langle \langle b_1 \rangle × \langle b_2 \rangle \rangle ⊆ G'_K
\]
and let

\[ P' := \coprod_{i \in H'_K} Y_i \]

be the disjoint union of copies of \( Y \) (i.e. \( Y_i = Y \)). We define a right action of \( G'_K \) on \( P' \) in the following way. If \( j \in H'_K \) then the action of \( j \) on \( Y_i \) is defined by the identity morphism \( Y = Y_i \to Y_{ij} = Y \). If \( j \notin H'_K \), then \( ij \) is uniquely written as a product \( ij = \xi k \) with \( k \in H'_K \). Then the action of \( j \) on \( Y_i \) will be the morphism \( Y = Y_i \to Y_k = Y \) given by the action of the non-trivial element of \( \mathbb{Z}/2\mathbb{Z} \) (remember that \( Y \) is a \( \mathbb{Z}/2\mathbb{Z} \)-torsor over \( X \)). Viewing \( G'_K \) as a constant group scheme over \( k \), one gets a morphism

\[ \rho : P' \times_k G'_K \to P' \]

over \( X \) defining the right action. This action actually defines \( P' \) as a \( G'_K \)-torsor over \( X \). Indeed, one can take a geometric point \( \bar{x} \in X \). Then the fibre of \( \bar{x} \) under the projection \( Y_{\bar{z}} = Y \to X \) consists of two points. We pick any point in the fibre and denote it by \( p \). Then the other point is \( p\xi \). We can also translate \( p \) and \( p\xi \) by the group \( H'_K \). In this way we get all the fibres of \( \bar{x} \) under the projection \( P' \to X \). Each fibre can be written uniquely as \( pi \) or \( p\xi i \) for some \( i \in H'_K \). By the very definition of the action of \( G'_K \) on the set of fibres of \( \bar{x} \), one sees that the set of all fibres of \( \bar{x} \) is a principal homogeneous space under \( G'_K \). Hence the \( X \)-morphism

\[ P' \times_k G'_K \xrightarrow{id \times \rho} P' \times_X P' \]

induces an isomorphism at the fibre of \( \bar{x} \in X \). Since \( P' \times_k G'_K \) and \( P' \times_X P' \) are all finite \( \text{étale} \) \( X \)-schemes and \( X \) is connected, the morphism \( id \times \rho \) is an isomorphism by [SGA1, Exposé V, Théorème 4.1]. Therefore \( P' \) is a \( G'_K \)-torsor over \( X \). Moreover, we would like to introduce two actions on \( P' \) by \( \mathbb{Z}/2n\mathbb{Z} = \langle u \rangle \) and \( \mathbb{Z}/2\mathbb{Z} = \langle v \rangle \) respectively. The action of \( u \) on a component \( Y_i \) is defined to be the identity morphism \( Y = Y_i \to Y_{a_1a_3\sigma(i)} = Y \). Notice that we have a commutative diagram

\[ \begin{array}{c}
\begin{array}{ccc}
P' & \xrightarrow{g} & P' \\
\downarrow u & & \downarrow u \\
P' & \xrightarrow{\sigma(g)} & P'
\end{array}
\end{array} \]

for all \( g \in G'_K \), even when \( g \notin H'_K \). The action of \( v \) on a component \( Y_i \) is defined to be the identity morphism \( Y = Y_i \to Y_{b_i} = Y \). Similarly, we have a commutative diagram

\[ \begin{array}{c}
\begin{array}{ccc}
P' & \xrightarrow{g} & P' \\
\downarrow v & & \downarrow v \\
P' & \xrightarrow{g} & P'
\end{array}
\end{array} \]

for all \( g \in G'_K \).

Next we would like to construct a finite group \( M \) generated by two elements \( \{x, y\} \) and two group homomorphisms

\[ f : M \to \text{Aut}_X(P') \quad \text{and} \quad g : M \to \text{Aut}(G'_K) \]
such that \( f(x) = u, f(y) = v \) and \( g(x) = \sigma, g(y) = id \), where \( P' \) is considered as an object in the category of \( X \)-schemes and \( G'_K \) is considered as an object in the category of abstract groups. There should be some general procedure to obtain such \( M \) and \( f, g \), because all the automorphism groups are finite. But unfortunately the author has to rely on some brutal computational methods. For simplicity we treat only the case when \( n = 2 \). In this case, we first consider the following equations.

(i)
\[
xyx^2yx = yxyx^2xy = yx^3yx^2yx
\]

(ii)
\[
yx^2yx^2 = x^2yx^2y
\]

(iii)
\[
x^3yxy = yx^3yx
\]
\[
yyx^3 = yx^3y
\]

(iv)
\[
x^3yx^3yx^2yy = yx^3yx^3yx^3y = yxyx^2yx^3y
\]
\[
xyxyx^2yx^3y = yx^3yx^2yx^3yx
\]

(v)
\[
x^3yx^2yx^2y = yx^3yx^2yx^2y = yx^2yx^3yx^2yx = yx^2yxxy^2yx^3
\]

(vi)
\[
yxyxyxy = x^3yx^3yx^3yx
\]

(vii)
\[
x^4 = 1 \quad y^2 = 1
\]

One observes first that the above equations hold in \( \text{Aut}(G'_K) \) when one replaces \( x \) by \( \sigma \), \( y \) by \( id \), and the same hold in \( \text{Aut}_X(P') \) when one replaces \( x \) by \( u \), \( y \) by \( v \). The former is somewhat clear, the latter needs some computation. To verify the above equations for \( u, v \), we choose a geometric point \( \bar{x} \in X \) and a fibre \( p \) of \( \bar{x} \) under \( Y_\sigma = Y \to X \), then check whether the actions from both sides of the equation are agree on \( p \). If so, one could then use [SGA1, Exposée V, Théorème 4.1]. Here is the result of the calculations.

(i)
\[
vvuvu^2vuv(p) = vu^2u^2vuv(p) = vuv^2vuv^3v(p) = p(a_3a_4)^2
\]

(ii)
\[
vuv^2u(p) = u^2vu^2v(p) = p(a_1a_2)^2
\]

(iii)
\[
u^3vuv = vu^3vuv(p) = pa_3^3a_4b_1b_2
\]
\[
vvuv^3(p) = uu^3v(p) = pa_3^3a_4b_1b_2
\]
(iv) 
\[u^3vu^2vuv(p) = vu^3vu^3vu(p) = vuvu^2vuvu^3(p) = pa_1a_2a_3a_4^2\]
\[uvuvu^2vuv(p) = vuvuvuv(p) = vu^3vu^2vuvu(p) = pa_1a_2a_3a_4^2\]

(v) 
\[vuvuvuvu(p) = u^3vu^3vu^3vu(p) = p(a_1a_2a_3a_4)^2\]

(vi) 
\[u^3vu^2vuvu^2v(p) = uvu^2vuvu^2v(p) = vu^2vuvu^2v(p) = vuvu^2vuvu^2v(p) = p(a_1a_2a_3a_4)^2\]

(vii) 
\[u^4(p) = p, \quad v^2(p) = p\]

Let \(M\) be the free group generated by \(x, y\) modulo the relations (i)-(vii). One can see without too much difficulty that \(M\) is a finite group generated by \(x, y\). Clearly there are group homomorphisms \(f : x \mapsto u, y \mapsto v\) and \(g : x \mapsto \sigma, y \mapsto \text{id}\).

Now let \(L\) be any field of any characteristic. Choose an imbedding \(M \subseteq S_m\) for some \(m \in \mathbb{N}^+\), we get a faithful action of \(M\) on \(L(X_1, X_2, \cdots, X_m)\). Let
\[K := L(X_1, X_2, \cdots, X_m) \quad k := K^M\]
where \(K^M \subseteq K\) denotes the subfield of invariant elements under the action of \(M\). Then \(K/k\) is a finite Galois extension with Galois group \(M\).

Let \(P'_K := P' \times_k K\). Then \(P'_K\) is a \(G'_K\)-torsor over \(X_K\). We also have an imbedding \(Y_K = Y_e \times_k K \subseteq P' \times_k K = P'_K\). Since the connected covering \(Y_K \rightarrow X_K\) comes, via base change, from a Galois covering \(Y \rightarrow X\) over \(k\), it has to be again a Galois covering [SGA1, Exposé V, §4, f), (ii)]. Thus the inclusion \(\text{Aut}_X(Y) \subseteq \text{Aut}_{X_K}(Y_K)\) has to be an isomorphism (because \(Y/X\) and \(Y_K/X_K\) are of the same degree).

Now for each element \(\alpha \in M\) we can define a twisted action on \(P'_K\) via \(P'_K = P' \times_k K\). 

By \(\text{and} \quad \text{this twisted action is compatible with the action of} \ M\) on \(G'_K\). 

Viewing \(G'_K\) as a constant group scheme over \(K\) and applying Galois descent we get a \(k\)-group scheme \(G'_k\) and a right \(G'_k\)-torsor \(P'_k\) over \(X\) such that the pull-back of \((P'_K, G'_K)\) to \(K\) is \((P'_k, G'_k)\). Choosing a geometric point \(\bar{x} : \text{Spec}(\bar{K}) \rightarrow X_K\) and a lifting \(p : \text{Spec}(\bar{K}) \rightarrow Y_e \times_k K \subseteq P'_K\), we get a triple \((P'_k, G'_k, p) \in I_{\text{ét}}(X/k, \bar{x})\). This triple corresponds to a homomorphism \(\pi^E(X/k, \bar{x}) \rightarrow G'_k\). Let \(G \subseteq G'_k\) be the image, \((P, G, p) \subseteq (P'_k, G'_k, p)\) be the triple in \(I_{\text{ét}}(X/k, \bar{x})\) corresponding to \(\pi^E(X/k, \bar{x}) \rightarrow G \subseteq G'_k\).

In this case, \((P, G, p)\) is a saturated object by definition, and the pull-back \(G_K \subseteq G'_K\) is a subgroup stable under the action of \(M\). Since \(P_K \subseteq P'_K\) is a subscheme containing \(p \in Y_e \times_k K\) and is stable under the action of \(M\), \(P_K\) contains \(Y_e \times_k K, \quad Y_{b_1} \times_k K = y(Y_e \times_k K), \quad Y_{a_1a_3} \times_k K = x(Y_e \times_k K)\).
As \( Y_{b_1} = Y_e b_1 \) and \( Y_{a_1 a_3} = Y_e a_1 a_3 \), \( G_K \) contains \( \xi, b_1, a_1 a_3 \). Just like in 3.7, we denote by \( N \) the image of the homomorphism

\[
\pi^N(\bar{X}/k, \bar{x}) \to G
\]
corresponding to the triple \((\bar{P}, G, p)\). Then \( \bar{N} \) is generated by the \( \text{Gal} (\bar{k}/k) \) orbit of \( \{ e, \xi \} \subseteq G_K = \bar{G} \), or equivalently by the \( M \) orbit of \( \{ e, \xi \} \subseteq G_K \), i.e.

\[
\bar{N} := \{ e, \xi, (a_1 a_3)^n, (a_1 a_3)^n \xi \}.
\]

However, \( \bar{N} \subseteq \bar{G} \) is not a normal subgroup for \( (a_1 a_3)^n b_1^{-1} = (a_2 a_3)^n \notin \bar{N} \).

**Remark 3.10.** (i) In the above example, we could take \( X := \mathbb{A}^1_k \) and \( Y \to X \) to be the Artin-Schreier covering under \( \mathbb{F}_2 \) if \( k \) is of characteristic 2.

(ii) We could also take \( A \) to be an abelian variety over a field \( l \) of characteristic \( \neq 2 \), then \( D := A[2] \times_\bar{\mathbb{I}} \mathbb{I} \) is a constant group scheme of order \( 2^{\dim(A)} \). Suppose \( A[2] \) stays as a constant group scheme over \( l \subseteq L \subseteq \bar{\mathbb{I}} \) and \( D \to \mathbb{Z}/2\mathbb{Z} \) is a surjective homomorphism (the existence of which is guaranteed by the fundamental theorem of finitely generated abelian groups and the fact that the order of \( D \) is even). Let \( k := L(X_1, X_2, \ldots, X_m)^M \) be as in the example, \( X := A \times_\mathbb{I} k \). Then we could define the \( \mathbb{Z}/2\mathbb{Z} \) torsor \( Y \to X \) to be the one obtained by taking the contracted product of the \( D \) torsor \( X \to X \) along \( D \to \mathbb{Z}/2\mathbb{Z} \). Clearly \( Y \) is geometrically connected over \( k \). In this example \( k \) is allowed to be of any characteristic \( \neq 2 \).

**Lemma 3.11.** Let \( X \) be a geometrically connected quasi-compact scheme over a field \( k \), \( G \) be a finite étale group scheme over \( k \), \( P \) be a \( G \) torsor over \( X \). If \( \bar{P} \) is a trivial \( G \) torsor over \( \bar{X} \), then there is a \( G \) torsor \( Q \) over \( k \) whose pull-back along \( X \to k \) is \( P \).

**Proof.** Let \( K \) be an intermediate finite Galois extension of \( k \subseteq \bar{k} \) over which \( G_K \) becomes a constant group scheme and \( P_K \) remains a trivial torsor. Since \( P_K \) is a trivial \( G \) torsor over \( X_K \), by choosing an \( X_K \) section for the projection \( \pi : P_K \to X_K \) we get isomorphisms (in the category of \( X_K \) schemes)

\[
P_K \cong X_K \times_k G = X_K \times_k G_K = \bigsqcup_{\xi \in G_K} X_K.
\]

By Galois descent, giving the \( X \) scheme \( P \) is equivalent to giving a twisted action of \( \text{Gal}(K/k) \) on \( P_K \), i.e. a homomorphism

\[
f : \text{Gal}(K/k) \to \text{Aut}_X(P_K)
\]

such that the following diagram

\[
\begin{array}{ccc}
P_K = \bigsqcup_{\xi \in G_K} X_K & \xrightarrow{f(\sigma)} & \bigsqcup_{\xi \in G_K} X_K = P_K \\
\downarrow{\pi} & & \downarrow{\pi} \\
X_K & \xrightarrow{id \times \sigma} & X_K \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
K & \to & K
\end{array}
\]

(\*)
is commutative for all \( \sigma \in \text{Gal}(K/k) \). One observes that, as \( X_K \) is connected, such a twisted action on \( P_K \cong \bigsqcup_{i \in G_K} X_i \) is none other than a permutation of the connected components in a twisted manner. The observation can be written more formally.

Let \( n \in \mathbb{N}^+ \) be the order of the \( k \)-group scheme \( G \), \( S_n \) be the \( n \)-th permutation group. Then there is a unique group homomorphism

\[
\lambda_X : S_n \times \text{Gal}(K/k) \rightarrow \text{Aut}_X(P_K)
\]

whose restriction to \( S_n \) is the permutation of the connected components of \( P_K \) and whose restriction to \( \text{Gal}(K/k) \) is

\[
\sigma \mapsto P_K \cong \left( \bigsqcup_{i \in G_K} X \right) \times_k K \xrightarrow{\text{id} \times \sigma} \left( \bigsqcup_{i \in G_K} X \right) \times_k K \cong P_K.
\]

The observation means that there is a group homomorphism \( \theta : \text{Gal}(K/k) \rightarrow S_n \) making the following diagram

\[
\begin{array}{ccc}
\text{Gal}(K/k) & \xrightarrow{f} & \text{Aut}_X(P_K) \\
\downarrow{\theta \times \text{id}} & & \downarrow{\lambda_X} \\
S_n \times \text{Gal}(K/k) & & \\
\end{array}
\]

commutative.

In the above, we could replace \( P_K \) by the \( k \)-scheme \( G_K = \bigsqcup_{i \in G_K} \text{Spec}(K) \) to obtain a homomorphism

\[
\lambda_k : S_n \times \text{Gal}(K/k) \rightarrow \text{Aut}_k(G_K)
\]

where \( G_K \) is regarded as an object in the category of \( k \)-schemes. In this way, we get a homomorphism

\[
g : \text{Gal}(K/k) \xrightarrow{\theta \times \text{id}} S_n \times \text{Gal}(K/k) \xrightarrow{\lambda_k} \text{Aut}_k(G_K)
\]

making the following diagram

\[
\begin{array}{ccc}
G_K = \bigsqcup_{i \in G_K} \text{Spec}(K) & \xrightarrow{g(\sigma)} & \bigsqcup_{i \in G_K} \text{Spec}(K) = G_K \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
K & \xrightarrow{\sigma} & K \\
\end{array}
\]

commutative for each \( \sigma \in \text{Gal}(K/k) \). In other words, we get a twisted action of \( \text{Gal}(K/k) \) on \( G_K \). This defines a \( k \)-form \( Q \) for the \( K \)-scheme \( G_K \).

The \( X \)-scheme \( X \times_k Q \) is an \( X \)-form of the \( X_K \)-scheme \( P_K \cong X_K \times_k G_K = X \times_k G_K \). From the very definition of \( Q \) we see that the twisted action of \( \text{Gal}(K/k) \) on \( P_K \) corresponding to the two \( X \)-forms \( X \times_k Q \) and \( P \) are the same. Therefore, by Galois descent \( P \cong X \times_k Q \) as \( X \)-schemes. On the other hand since \( G \) is a \( k \)-form of the \( K \)-group scheme \( G_K \), there is an action via group automorphisms

\[
\phi : \text{Gal}(K/k) \rightarrow \text{Aut}_{\text{grp}}(G_K)
\]
corresponding to $G$. As $P$ is a $G$-torsor we have the following commutative diagram

$$
\begin{array}{ccc}
P_K & \xrightarrow{a} & P_K \\
\downarrow{f(\sigma)} & & \downarrow{f(\sigma)} \\
\phi(\sigma)(a) & & \phi(\sigma)(a)
\end{array}
$$

for all $\sigma \in \text{Gal}(K/k)$ and $a \in G_K$. Since the identification $P_K \cong X_K \times_K G_K$ is equivariant under the right actions and the action of $G_K$ on $P_K$ is also just a permutation of connected components, we have another commutative diagram

$$
\begin{array}{ccc}
G_K & \xrightarrow{a} & G_K \\
g(\sigma) & & g(\sigma) \\
\phi(\sigma)(a) & & \phi(\sigma)(a)
\end{array}
$$

for all $\sigma \in \text{Gal}(K/k)$ and $a \in G_K$. Therefore, by Galois descent there is an action of $G$ on $Q$ over $k$ which makes the isomorphism $P \cong X \times_k Q$ equivariant under $G$. Hence $Q$ is a $G$-torsor over $k$ whose pull-back to $X$ is the $G$-torsor $P$. □

3.3. The Infinitesimal Case.

**Proposition 3.12.** Let $X$ be a scheme geometrically connected separable over a perfect field $k$. Let $\bar{x} : \text{Spec}(\bar{k}) \to X$ be a geometric point. Then the canonical map

$$
\pi^k : \pi^L(\bar{X}/k, \bar{x}) \to \pi^L(X/k, \bar{x})
$$

is surjective, but not, in general, an isomorphism.

**Proof.** Suppose we have a saturated object $(P, G, p) \in I_k(X/k, \bar{x})$. We take the image of the composition

$$
\pi^L(\bar{X}/k, \bar{x}) \to \pi^L(X/k, \bar{x}) \to G,
$$

and denote it by $H$. By 2.2 there is $(\bar{Q}, H, q) \in N(\bar{X}/k, \bar{x})$ with a morphism

$$(\bar{Q}, H, q) \hookrightarrow (\bar{P}, G, p).$$

As $H \subseteq G$ is an infinitesimal closed imbedding (closed imbedding with nilpotent ideal sheaf), so is $\bar{Q} \subseteq \bar{P}$. Now take quotient by $H$ on both sides. We get a section:

$$
\bar{s} : \bar{X} \cong \bar{Q}/H \to \bar{P}/H.
$$

Since the projection $\bar{P} \to \bar{P}/H$ is faithfully flat, the ideal sheaf of the section $\bar{s}$ is contained in the ideal sheaf of the infinitesimal imbedding $\bar{Q} \subseteq \bar{P}$. Hence imbedding $\bar{s} : \bar{X} = \bar{Q}/H \hookrightarrow \bar{P}/H$ is also infinitesimal. As $\bar{X}$ is reduced, $\bar{X} = (\bar{P}/H)_{\text{red}}$ is the unique reduced closed subscheme of $\bar{P}/H$. Because $k$ is perfect, we have

$$(\bar{P}/H)_{\text{red}} = (P/H)_{\text{red}} \times_k \bar{k} \hookrightarrow (P/H) \times_k \bar{k} = \bar{P}/H.$$

We have known that the composition

$$
\bar{X} = (\bar{P}/H)_{\text{red}} \hookrightarrow \bar{P}/H \to \bar{X}
$$
is an isomorphism, so \((P/H)_{\text{red}} \to X\) is also an isomorphism. In this way we get a section \(s\) for the \(X\)-scheme \(P/H\). Now we pull back the \(H\)-torsor \(P \to P/H\) via \(s\).

\[
\begin{array}{ccc}
Q & \to & X \\
\downarrow & & \downarrow s \\
P & \to & P/H
\end{array}
\]

Then we get a triple \((Q, H, q) \in I_{k}(X/k, \bar{x})\) which dominates \((P, G, p)\). In other words, the map \(\pi^{L}(X/k, \bar{x}) \to G\) factors through the imbedding \(H \subseteq G\). Hence if \((P, G, p)\) is saturated in \(I_{k}(X/k, \bar{x})\) then \((\bar{P}, G, \bar{p})\) is saturated in \(I_{k}(\bar{X}/k, \bar{x})\). This means that \(\pi_{k}^{\bar{X}}\) is surjective. For the failure of the injectivity see 3.15. \(\square\)

**Corollary 3.13.** Let \(X\) be a scheme geometrically connected separable over a perfect field \(k\). Let \(\bar{x} : \text{Spec}(\bar{k}) \to X\) be a geometric point. The canonical map

\[
\bar{\pi}^{\bar{k}} : \pi^{L}(\bar{X}/k, \bar{x}) \to \pi^{L}(X/k, \bar{x})
\]

is an isomorphism if and only if for any \(G\)-torsor \(Y \to \bar{X}\) with \(G\) a finite local \(k\)-group scheme, there exists a \(G\)-torsor \(P\) over \(X\) whose pull-back is isomorphic to \(Y\) as a \(G\)-torsor.

**Proof.** This is an immediate consequence of 3.12 and 3.15. \(\square\)

**Lemma 3.14.** Let \(X\) be a scheme over a perfect field \(k\) of characteristic \(p\). If there is a reduced \(X\)-scheme \(Y\) whose pull-back \(\bar{Y}\) is a torsor over \(\bar{X}\) under an infinitesimal \(k\)-group scheme \(G\), and if \(\bar{Y}'\) is an \(X\)-scheme, then any \(\bar{X}\)-isomorphism \(\bar{\phi} : \bar{Y}' \cong \bar{Y}\) descends to \(X\).

**Proof.** The claim is true if only if there is a map \(\phi : Y \to Y'\) fitting into the following diagram

\[
\begin{array}{ccc}
\bar{Y} & \xrightarrow{\bar{\phi}} & \bar{Y}' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\phi} & Y'
\end{array}
\]

The problem being local on \(X\), we could assume \(X = \text{Spec}(A)\), \(Y = \text{Spec}(B)\), \(Y' = \text{Spec}(B')\). We have to show that the image of the composition \(i : B' \to B' \otimes_{k} \bar{k} \to B \otimes_{k} \bar{k}\) lands on \(B\).

Since \(\bar{Y}\) is a torsor over \(\bar{X}\) under an infinitesimal group scheme, for any \(x \in B \otimes_{k} \bar{k}\), \(x^{p^{n}} \in A \otimes_{k} \bar{k}\) for \(n \in \mathbb{N}\) sufficiently large. This implies that for any \(x \in B\), \(x^{p^{n}} \in A\) for \(n \in \mathbb{N}\) sufficiently large, because \(A \otimes_{k} \bar{k} \cap B = A\) inside \(B \otimes_{k} \bar{k}\). Conversely, if \(x \in B \otimes_{k} \bar{k}\) and \(x^{p^{n}} \in A\) for some \(n \in \mathbb{N}\), then \(x \in B\). Indeed, as \(k\) is perfect, we can assume \(x \in B \otimes_{k} l\) for some finite separable extension \(l/k\) of degree \(m\). Let \(l = k(\alpha)\) for some primitive element \(\alpha \in l\). Then \(x\) can be uniquely written as \(x = s_{0} + s_{1} \otimes \alpha + s_{2} \otimes \alpha^{2} + \cdots + s_{m-1} \otimes \alpha^{m-1}\) with \(s_{i} \in B\). Since \(\alpha^{p^{n}}\) is still a primitive element in \(l\), i.e. \(l = k(\alpha) = k(\alpha^{p^{n}})\), \(x^{p^{n}} \in A\) implies that \(s_{i}^{p^{n}} = 0\) for all \(i > 0\). As \(B\) is reduced, \(s_{i} = 0\) for all \(i > 0\), hence \(x \in B\). Thus \(B \subseteq B \otimes_{k} \bar{k}\) is the subset consisting of elements whose \(p^{n}\)-th power is in \(A\).

By the same argument as above, any element \(x \in B'\) has \(p^{n}\)-th power in \(A\). Hence \(i(B') \subseteq B \otimes_{k} \bar{k}\) is contained in \(B\). This completes the proof. \(\square\)
Corollary 3.15. Let $k$ be a perfect but not separably closed field of characteristic $p$. If $X$ is $\mathbb{A}^1_k$ or an elliptic curve such that $X(F) = \alpha_{p,k}$ (where $F$ is the relative Frobenius), then the canonical map

$$\pi^L(\bar{X}/k, \bar{x}) \rightarrow \pi^L(X/k, \bar{x})$$

is not an isomorphism.

Proof. In any case, we have a non-trivial $\alpha_{p,k}$-torsor $F : X \rightarrow X$ defined by the relative Frobenius. Taking any $a \in \bar{k}/k$ we can define a $\bar{k}$-automorphism of $\alpha_{p,k}$ by the following map of Hopf-algebras: $\bar{k}[x]/x^p \rightarrow \bar{k}[x]/x^p$ sending $x \mapsto ax$. This automorphism of $\alpha_{p,k}$ defines a new action of $\alpha_{p,k}$ on $F : \bar{X} \rightarrow \bar{X}$ which makes $\bar{X}$ an $\alpha_{p,k}$-torsor over itself. But the new action $\bar{X} \times_k \alpha_{p,k} \rightarrow \bar{X}$ certainly does not descent to $X \times_k \alpha_{p,k} \rightarrow X$. However, if $\pi^L(\bar{X}/k, \bar{x}) \rightarrow \pi^L(X/k, \bar{x})$ was an isomorphism, then by 3.13 and 3.14 the morphism $\bar{X} \times_k \alpha_{p,k} \rightarrow \bar{X}$ descends to $X \times_k \alpha_{p,k} \rightarrow X$, a contradiction! □

4. The Second Fundamental Sequence

Proposition 4.1. Let $X$ be a scheme geometrically connected separable over a field $k$, $\bar{x} : \text{Spec}(\bar{k}) \hookrightarrow X$ be a geometric point, then there is a natural sequence of $\bar{k}$-group schemes

$$1 \rightarrow \pi^I(\bar{X}/\bar{k}, \bar{x}) \rightarrow \pi^I(X/k, \bar{x}) \times_k \bar{k} \rightarrow \pi^I(k/k, \bar{x}) \times_k \bar{k} \rightarrow 1. \tag{2}$$

It is a complex, surjective on the right, injective on the left if $k$ is perfect and if $X$ is quasi-compact and quasi-separated, but it is in general not exact in the middle for $I = N, E, L$.

Proof. The homomorphism $\theta : \pi^I(X/k, \bar{x}) \rightarrow \pi^I(X/k, \bar{x}) \times_k \bar{k}$ is obtained via composing $\chi^I_{k/k}$ (cf. [2.19]) with the canonical morphism

$$\delta : \pi^I(\bar{X}/\bar{k}, \bar{x}) \times_k \bar{k} \rightarrow \pi^I(X/k, \bar{x}) \times_k \bar{k}$$

obtained by base-change from the morphism $\pi^I(\bar{X}/\bar{k}, \bar{x}) \rightarrow \pi^I(X/k, \bar{x})$ in the first fundamental sequence (1). The fact that (2) is a complex and that the right map follows from 3.2. As the image of $\theta$ is contained in the image of $\delta$ the failure of exactness of (2) follows from that of (1). Now we show the left injectivity assuming $k$ perfect and $X$ q.c. and q.s.

Let $\mathfrak{c}(X/k, \bar{x}, I) \otimes_k \bar{k}$ be the category of pairs $(V, (P, G, p))$, where $(P, G, p) \in I(X/k, \bar{x})$, $V$ is a $G \times_k \bar{k}$-representation, and the morphisms in $\mathfrak{c}(X/k, \bar{x}, I) \otimes_k \bar{k}$ are defined as in 3. Then $\mathfrak{c}(X/k, \bar{x}, I) \otimes_k \bar{k}$ equipped with the forgetful fibre functor $\omega : (V, (P, G, p)) \mapsto V$ is the Tannakian dual of $\pi^I(X/k, \bar{x}) \times_k \bar{k}$. Now the homomorphism $\theta$ corresponds, by Tannakian duality, to the $\bar{k}$-linear tensor functor

$$\theta^* : \mathfrak{c}(X/k, \bar{x}, I) \otimes_k \bar{k} \longrightarrow \mathfrak{c}(\bar{X}/\bar{k}, \bar{x}, I)$$

$$(V, (P, G, p)) \mapsto (V, (P \times_k \bar{k}, G \times_k \bar{k}, p)).$$

Since $X$ is q.c. and q.s. and $k$ is perfect, any saturated triple $(P, G, p) \in I(X/k, \bar{x})$ is defined over some $X \times_k l$, where $l$ is a finite separable extension of $k$. The Weil restriction $\text{Res}_{X \times_k l/X}(P)$ is then a torsor under $\text{Res}_{l/k}(G)$ over $X$ [BLR][7.6, Theorem 4 and Proposition 5], and there are canonical adjunction maps

$$\phi : \text{Res}_{X \times_k l/X}(P) \times_k \bar{k} \rightarrow P \quad \text{and} \quad h : \text{Res}_{l/k}(G) \times_k \bar{k} \rightarrow G$$
where \( \phi \) has to be surjective for \( P \) is a connected scheme. By choosing a \( \bar{k} \)-point \( q \) in the fibre of \( p \in P(\bar{k}) \) we get a morphism

\[
(\phi, h) : (\text{Res}_{X \times \bar{k}/X}(P) \times_k \bar{k}, \text{Res}_{I/k}(G) \times_k \bar{k}) \to (P, G, p) \in I(\bar{X}/\bar{k}, \bar{x}).
\]

Thus by [44] any object \((V, (P, G, p)) \in \mathcal{C}(\bar{X}/\bar{k}, \bar{x}, I)\) is isomorphic to

\[
(V, (\text{Res}_{X \times \bar{k}/X}(P) \times_k \bar{k}, \text{Res}_{I/k}(G) \times_k \bar{k}, q))
\]

which is contained in the essential image of \( \theta^* \). Therefore, by [DM, Proposition 2.21], \( \theta \) is injective.

\[ \square \]

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