The average size of the 5-Selmer group of elliptic curves is 6, and the average rank is less than 1

Manjul Bhargava and Arul Shankar

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1 Introduction

The purpose of this article is to show that the average rank of elliptic curves over \( \mathbb{Q} \), when ordered by height, is less than 1 (in fact, less than .885). As a consequence of our methods, we also prove that at least four fifths of all elliptic curves over \( \mathbb{Q} \) have rank either 0 or 1; furthermore, at least one fifth of all elliptic curves in fact have rank 0. The primary ingredient in the proofs of these theorems is a determination of the average size of the 5-Selmer group of elliptic curves over \( \mathbb{Q} \); we prove that this average size is 6. Another key ingredient is a new lower bound on the equidistribution of root numbers of elliptic curves; we prove that there is a family of elliptic curves over \( \mathbb{Q} \) having density at least 55% for which the root number is equidistributed.

We now describe these results in more detail. Recall that any elliptic curve \( E \) over \( \mathbb{Q} \) is isomorphic to one of the form \( E_{A,B} : y^2 = x^3 + Ax + B \). If, for all primes \( p \), we further assume that \( p^6 \nmid B \) whenever \( p^4 \mid A \), then this expression is unique. We define the (naive) height of \( E_{A,B} \) by \( H(E_{A,B}) := \max\{4|A^3|, 27B^2\} \). In previous work ([8], [9], and [10]), we showed that when elliptic curves over \( \mathbb{Q} \) are ordered by height, the average sizes of their 2-, 3-, and 4-Selmer groups are given by 3, 4, and 7, respectively. These results, and their proofs, led us to conjecture in [10] that for all \( n \), the average size of the \( n \)-Selmer group of elliptic curves over \( \mathbb{Q} \), when ordered by height, is the sum of the divisors of \( n \).

In this paper, we prove the following theorem which confirms the conjecture when \( n = 5 \):

**Theorem 1** When elliptic curves \( E/\mathbb{Q} \) are ordered by height, the average size of the 5-Selmer group \( S_5(E) \) is equal to 6.

We note that Theorem 1 also confirms a case of the Poonen–Rains heuristics [37, Conjecture 1.1(b)], which predict in particular that for any prime number \( p \), the average size of the \( p \)-Selmer group of elliptic curves is \( p + 1 \).

We actually prove a stronger version of Theorem 1 where we determine the average size of the 5-Selmer group over elliptic curves whose defining coefficients satisfy any finite set of congruence conditions:

**Theorem 2** When elliptic curves \( E : y^2 = x^3 + Ax + B \) over \( \mathbb{Q} \), in any family defined by finitely many congruence conditions on the coefficients \( A \) and \( B \), are ordered by height, the average size of the 5-Selmer group \( S_5(E) \) is 6.

Thus the average size of the 5-Selmer groups of elliptic curves in any congruence family is independent of the family.
We use Theorem 2 together with some further ingredients to be described below, to obtain a number of results on the distribution of ranks of elliptic curves. The rank distribution conjecture, due to Goldfeld [30] and Katz–Sarnak [34] (see also [3] for a beautiful survey, where it is termed the “minimalist conjecture”), states that the average rank of elliptic curves should be 1/2, with 50% of curves having rank 0 and 50% having rank 1. However, prior to the work [8] giving the average size of the 2-Selmer group of elliptic curves, it was not known unconditionally that the average rank of elliptic curves is even finite. (Conditional on GRH and BSD, a finite upper bound of 2.3 on the average rank was demonstrated by Brumer [13]; improved conditional upper bounds of 2.0 and 1.79 were given by Heath-Brown [32] and more recently by Young [42], respectively.)

In this article, we give the first proof that the average rank of elliptic curves over $\mathbb{Q}$ is less than 1:

**Theorem 3** When elliptic curves over $\mathbb{Q}$ are ordered by height, their average rank is < .885.

The rank distribution/minimalist conjecture predicts that elliptic curves over $\mathbb{Q}$ should tend to have rank either 0 or 1. We prove that this is the case for the vast majority of elliptic curves over $\mathbb{Q}$:

**Theorem 4** When elliptic curves over $\mathbb{Q}$ are ordered by height, a density of at least 83.75% have rank 0 or 1.

In [9], we showed that a positive proportion of elliptic curves have rank 0; however, the proportion that we demonstrated there was quite small. As a consequence of our methods here, we are able to deduce that a fairly significant proportion of elliptic curves have rank 0:

**Theorem 5** When elliptic curves over $\mathbb{Q}$ are ordered by height, a density of at least 20.62% have rank 0.

If the Tate–Shafarevich groups of elliptic curves are finite, then our methods also demonstrate that a proportion of at least 26.12% of elliptic curves have rank 1.

We now describe some of the methods behind the proofs of Theorems 1–2 and 3–5. The key algebraic ingredient in proving Theorems 1 and 2 is a parametrization of elements of the 5-Selmer group of an elliptic curve. Recall that an element in the 5-Selmer group of an elliptic curve $E$ may be viewed as a locally soluble 5-covering of $E$. Given any integer $n \geq 1$, an $n$-covering of an elliptic curve $E/\mathbb{Q}$ is a genus one curve $C/\mathbb{Q}$ equipped with maps $\phi : C \to E$ and $\theta : C \to E$, where $\phi$ is an isomorphism defined over $C$ and $\theta$ is a degree $n^2$ map defined over $\mathbb{Q}$, such that the following diagram commutes:

$$
\begin{array}{ccc}
E & \xrightarrow{[n]} & E \\
\phi \downarrow & & \downarrow \theta \\
C & \rightarrow & \\
\end{array}
$$

An $n$-covering $C$ of $E$ is said to be soluble if it has a rational point, and locally soluble if it has a point over every completion of $\mathbb{Q}$.

Cassels [17] showed that any locally soluble $n$-covering of $E$ admits a rational divisor of degree $n$, yielding a map $C \to \mathbb{P}^{n-1}$ (which gives an embedding once $n \geq 3$). In the case $n = 2$, we obtain double cover $C \to \mathbb{P}^1$ ramified at 4 points, and thus we may describe 2-coverings $C$ of elliptic curves $E$ over $\mathbb{Q}$ via binary quartic forms over $\mathbb{Q}$; this perspective on 2-coverings, which
played an important role in the original rank computations of Birch and Swinnerton-Dyer [11], was also the key to the work in [8]. In the cases $n = 3$ and $n = 4$, one obtains a genus one curve $C$ embedded in $\mathbb{P}^2$ or $\mathbb{P}^3$, respectively, which may be described as the locus of zeros in $\mathbb{P}^2$ of a ternary cubic form or by a complete intersection of two quadrics in $\mathbb{P}^3$, respectively (see, e.g., [11] for an excellent treatment over general fields). These geometric descriptions of genus one normal curves of degrees 3 and 4 indeed played key roles in the works on 3- and 4-Selmer groups in [9] and [10].

In the case $n = 5$, however, a genus one curve in $\mathbb{P}^4$ is not a complete intersection. Nevertheless, a genus one curve in $\mathbb{P}^4$ may be expressed—essentially uniquely—as an intersection of the five quadrics defined by the $4 \times 4$ sub-Pfaffians of a $5 \times 5$ skew-symmetric matrix of linear forms! Conversely, given a generic $5 \times 5$ matrix of linear forms in five variables, the $4 \times 4$ sub-Pfaffians yield five quadrics, whose intersection gives a genus one curve in $\mathbb{P}^4$. These facts are essentially classical over an algebraically closed field; over a general field, they follow from the seminal Buchsbaum–Eisenbud structure theorem (see [15], [16]).

The theory over arithmetic fields, such as $\mathbb{Q}$ and $\mathbb{Q}_p$, has been subsequently developed in a beautiful series of papers by Fisher [25, 26, 27, 28, 29]. In particular, Fisher relates the invariant theory of the representation $V = 5 \otimes \wedge^3(5)$ of the group $G' = SL(5) \times SL(5)$ to explicit formulae for 5-coverings of elliptic curves. The representation of $G'$ on $V$ is classically known to have a remarkable invariant theory; it is one of the first “exotic” examples arising in Vinberg’s theory of $\theta$-groups [38], and arises in the classification of coregular spaces [39], i.e., representations having a free ring of invariants. The invariant ring of the representation of $G_C'$ on $V_C$ is freely generated by two invariants, which we denote by $I$ and $J$, and which are integral polynomials in the entries of $V$. Fisher shows that every locally soluble 5-covering of an elliptic curve $E_{A,B}$ can be represented by a genus one curve in $\mathbb{P}^4$ corresponding to an element $v \in V_\mathbb{Q}$, where the $I$ and $J$ invariants of $v$ agree with $A$ and $B$, respectively (up to bounded powers of 2 and 3); even more remarkably, this element $v \in V_\mathbb{Q}$ can in fact always be taken in $V_\mathbb{Z}$ ([28 Theorem 2.1]).

To obtain an exact parametrization of 5-coverings of elliptic curves, it is necessary to use the action of a slightly different group $G$, as defined in [2]; see [7, §4.4] for a detailed explanation. It then follows that 5-Selmer elements of elliptic curves $E_{A,B}$ having bounded height can be represented by certain $G_\mathbb{Z}$-orbits on $V_\mathbb{Z}$ having bounded invariants $I$ and $J$. Therefore, to count the total number of 5-Selmer elements of elliptic curves of bounded height, it suffices to count the number of corresponding $G_\mathbb{Z}$-orbits on $V_\mathbb{Z}$ having bounded invariants $I$ and $J$. To carry out such a count, we adapt the methods of [5] and [8]. Specifically, we construct fundamental domains for the action of $G_\mathbb{Z}$ on $V_\mathbb{R}$, and then count lattice points having bounded invariants in these domains. As usual, the difficulties lie in the (numerous) cusps of such a fundamental domain. We divide the fundamental domain into two parts, the “main body” and the “cuspidal region”. We show that in the main body, only a negligible number of lattice points correspond to identity 5-Selmer elements. Meanwhile, in the cuspidal region, we show that a negligible number of points correspond to non-identity 5-Selmer elements! The latter is proven by partitioning the cuspidal region into thousands of subregions, on each of which the argument of [5] is then applied. If we actually wrote out the proof for each such subregion as in [5], it would take hundreds of pages! Thus we introduce a new, more uniform method of certifying that the argument works on each of these subregions, allowing us to present a fairly short proof that is checkable by hand; see Section 3. We expect that this method will also be useful in other contexts in handling geometry-of-numbers difficulties in complex cuspidal regions.

We conclude from the above results that, in order to count the number of non-identity 5-Selmer elements of elliptic curves having bounded height, it suffices to count lattice points in the main body of the fundamental region satisfying suitable congruence conditions, so that we are
counting each non-identity 5-Selmer element exactly once. This is also accomplished via geometry-of-numbers arguments, together with a suitable sieve using the results of [6]. The sieve reveals that the average number of non-identity elements in the 5-Selmer group of elliptic curves is the Tamagawa number (= 5) of the group \( G \). We conclude that the average number of elements in the 5-Selmer groups of elliptic curves is \( 5 + 1 = 6 \), proving Theorem 1. An analogous argument, and the latter sieve, also allows us to prove Theorem 2.

We now describe how Theorem 3 is deduced (the deduction of Theorems 4 and 5 being similar). First, we note that Theorem 1 immediately yields an upper bound of 1.05 on the average rank of elliptic curves. Indeed, recall that the 5-Selmer group of an elliptic curve fits into the exact sequence

\[
0 \to E(\mathbb{Q}) \to S_5(E) \to \text{III}_{E}[5] \to 0.
\]

If \( r \) denotes the rank of an elliptic curve \( E \), then the size of the 5-Selmer group of \( E \) is an upper bound for \( 5^r \). Since \( 20r - 15 \leq 5^r \) for any nonnegative integer \( r \), we conclude by Theorem 1 that (the limsup of) the average rank \( \bar{r} \) of elliptic curves, when ordered by height, must satisfy \( 20\bar{r} - 15 \leq 6 \), whence \( \bar{r} \leq 21/20 \).

To improve this bound further, we observe that the bound of 1.05 can be attained only if 95% of elliptic curves have rank 1 and 5% have rank 2. However, it is widely expected that 50% of elliptic curves should have even rank and 50% should have odd rank. This is because the parity conjecture (implied by the Birch and Swinnerton-Dyer conjecture) states that the rank of an elliptic curve is even if and only if its root number is +1; furthermore, one expects that the root number of elliptic curves should be equidistributed. The parity conjecture has not been proven, but we may instead use the remarkable result of Dokchitser and Dokchitser ([24]) which states that the parity of the \( p \)-Selmer rank of an elliptic curve is determined by its root number. This result suffices for our purposes because Theorems 1 and 2 indeed yield bounds on not just the rank but the 5-Selmer rank of elliptic curves.

Any result towards the equidistribution of root numbers of elliptic curves would thus imply a better bound on the average rank. However, no cancellation in the root numbers of elliptic curves has been established. In [11] and [9], it was proved that there exists positive proportion families of elliptic curves having equidistributed root number.

In this article, we prove that a majority of elliptic curves in fact do have equidistributed root number:

**Theorem 6** There exists a family \( F \) of elliptic curves \( E_{A,B} \), having density greater than 55.01% among all elliptic curves when ordered height and defined by congruence conditions on \( A \) and \( B \), such that the root number of elliptic curves in \( F \) is equidistributed.

Specifically, we construct \( F \) so that, for every elliptic curve \( E \in F \), the quadratic twist \( E_{-1} \) of \( E \) is also in \( F \) and, moreover, \( E \) and \( E_{-1} \) have opposite root numbers. We show that we can find such an \( F \) whose density in the family of all elliptic curves over \( \mathbb{Q} \) is > 55.01%. The construction makes key use, in particular, of the work of Rohrlich [39] and Halberstadt [31] on computations of local root numbers.

To the special family \( F \) constructed in Theorem 6 we may then apply Theorem 2 along with the aforementioned theorem of Dokchitser–Dokchitser [24]. (This explains why Theorem 2—the congruence version of Theorem 1—is also critical in the proof of Theorem 3.) Together, they imply that the average rank of elliptic curves in \( F \) is at most .75 (indeed, this upper bound can be attained only if 37.5% of curves in \( F \) have 5-Selmer size 1, 50% have size 5, and 12.5% have
size 25). This yields an upper bound of
\[ 0.5501 \times 0.75 + 0.4499 \times 1.05 < 0.885 \]
on the average rank of elliptic curves, yielding Theorem 3. Similar arguments are used to obtain Theorems 4 and 5; see the last section §6 for details.

This paper is organized as follows. In Section 2, we describe the parametrization of elements of 5-Selmer groups of elliptic curves using quintuples of \( 5 \times 5 \) skew-symmetric matrices. This follows essentially from the work of Fisher, although we must slightly modify the group action so that our counting and sieve methods work more effectively. In Section 3, we then count integral orbits of bounded height in this representation in terms of volumes of certain fundamental domains. In Section 4, we carry out the necessary sieve to count only 5-Selmer elements of elliptic curves, thereby proving Theorem 1; we also obtain Theorem 2.

In Section 5, we then turn to root numbers, and construct the family \( F \) above, thereby proving Theorem 6. Finally, we complete the proofs of Theorems 3, 4, and 5 in Section 6.

## 2 Parametrization of elements in the 5-Selmer groups of elliptic curves

For an elliptic curve \( E : y^2 = x^3 + Ax + B \) over \( \mathbb{Q} \), we define the quantities \( I(E) \) and \( J(E) \) by
\[
I(E) = -3A,  \\
J(E) = -27B.
\]

There invariants are related to the classical invariants \( c_4 \) and \( c_6 \) in the following way: we have \( I = 3^4 c_4 \) and \( J = 2 \cdot 3^6 c_6 \). We denote the elliptic curve with invariants \( I \) and \( J \) by \( E^I,J \). The primary purpose of this section is to describe a method to parametrize elements of the 5-Selmer groups of elliptic curves over \( \mathbb{Q} \). We also describe similar parametrizations for elliptic curves over \( \mathbb{R} \) and \( \mathbb{Q}_p \).

For any ring \( R \), let \( V_R \) denote the space \( R^5 \otimes \Lambda^5(R) \) of quintuples of skew-symmetric \( 5 \times 5 \) matrices with coefficients in \( R \). The group \( \text{GL}_5(R) \times \text{GL}_5(R) \) acts on \( V_R \) via:
\[(g_1, g_2) \cdot (A, B, C, D, E) := (g_1 A g_1^t, g_1 B g_1^t, g_1 C g_1^t, g_1 D g_1^t, g_1 E g_1^t) \cdot g_2^t,
\]
for \( (g_1, g_2) \in \text{GL}_5(R) \times \text{GL}_5(R) \) and \( (A, B, C, D, E) \in V_R \). We define the determinant of an element \( (g_1, g_2) \in \text{GL}_5(R) \times \text{GL}_5(R) \) by \( \det(g_1, g_2) := (\det g_1)^2 \det g_2 \). Let \( G_R \) denote the group
\[
G_R := \{ (g_1, g_2) \in \text{GL}_5(R) \times \text{GL}_5(R) : \det(g_1, g_2) = 1 \}/\{ (\lambda I_5, \lambda^{-2} I_5) \},
\]
where \( I_5 \) denotes the identity element of \( \text{GL}_5(R) \) and \( \lambda \in R^\times \). It is then easy to check that the action of \( \text{GL}_5(R) \times \text{GL}_5(R) \) on \( V_R \) descends to an action of \( G_R \) on \( V_R \).

The ring of invariants for the action of \( G_C \) on \( V_C \) is freely generated by two elements (see, e.g., [38]). In [25], the generators of the ring of invariants of the above action are denoted by \( c_4 \) and \( c_6 \), and they have degrees 20 and 30 on \( V \), respectively. For purposes of convenience, we consider the invariants \( I \) and \( J \) given by \( 3^4 c_4 \) and \( 2 \cdot 3^6 c_6 \), respectively. We then define the discriminant \( \Delta(v) \) of an element \( v \in V_R \) having invariants \( I \) and \( J \) by
\[
\Delta(v) := \Delta(I,J) := (4I^3 - J^2)/27;
\]
one checks that $\Delta(v)$ is an integer polynomial of degree 60 in the 50 entries of $V$.

Let $K$ be a field of characteristic not equal to 2, 3, or 5. Given $v = (A, B, C, D, E) \in V_K$ having nonzero discriminant, let $Q_1, \ldots, Q_5$ be the five $4 \times 4$ sub-Pfaffians of $\cdot (t_1, \ldots, t_5) = \Delta t_1 + B t_2 + C t_3 + D t_4 + E t_5$, i.e., $Q_i$ is the Pfaffian of the $4 \times 4$ matrix obtained by removing the $i$th row and column of $\Delta$. The intersection of the quadrics $Q_i(t_1, t_2, t_3, t_4, t_5) = 0$ in $\mathbb{P}^4$ is generically a genus one curve $C_v$, whose Jacobian is given by the elliptic curve $E: y^2 = x^3 - \frac{1}{3} x - \frac{1}{2}$ (see [25, Proposition 2.3]).

An element $v \in V_K$ is called $K$-soluble if $C_v$ has a $K$-rational point, i.e., $C_v(K) \neq \emptyset$. If $C_v$ is $K$-soluble, then it corresponds naturally to an element of $E(K)/5E(K)$, where $E$ is the Jacobian of $C$. This leads naturally to the following two results, which are essentially due to Fisher (cf. [26, Theorem 6.1]); however, as in [7, §4.4], we use a slightly different group action so that the stabilizer of the group action is given exactly by the 5-torsion subgroup $E(K)[5]$ of $E(K)$; this will be important in our applications.

**Theorem 7** ([26, Thm. 6.1], [7, §4.4]) Let $K$ be a field of characteristic not equal to 2, 3, or 5. Let $E = E^{I,J}: y^2 = x^3 - \frac{1}{3} x - \frac{1}{2}$ be an elliptic curve over $K$. Then there exists a canonical bijection between elements of $E(K)/5E(K)$ and $G_K$-orbits of $K$-soluble elements in $V_K$ having invariants $I$ and $J$. Furthermore, the stabilizer in $G_K$ of any (not necessarily $K$-soluble) element in $V_K$, having invariants $I$ and $J$, is isomorphic to $E(K)[5]$.

An element in $V_Q$ is called locally soluble if it is $\mathbb{R}$-soluble and $Q_p$-soluble for all primes $p$.

**Theorem 8** ([26, 7]) Let $E/\mathbb{Q}$ be an elliptic curve. Then the elements in the 5-Selmer group of $E$ are in bijective correspondence with $G_\mathbb{Q}$-equivalence classes on the set of locally soluble elements in $V_Q$ having invariants equal to $I(E)$ and $J(E)$.

If $v \in V_Q$ is $Q_p$-soluble and has integral invariants, then it is a further result of Fisher [28] that $v$ is $G_\mathbb{Q}_p$-equivalent to an element in $V_{\mathbb{Z}_p}$.

**Theorem 9** ([28, Thm. 2.1]) Let $p$ be a prime and let $v \in V_Q$ be an element having integral invariants that is soluble over $\mathbb{Q}_p$. Then $v$ is $G_\mathbb{Q}_p$-equivalent to an element in $V_{\mathbb{Z}_p}$.

Since $G_\mathbb{Q}$ has class number 1, Theorems 8 and 9 immediately imply:

**Theorem 10** ([28]) Let $E/\mathbb{Q}$ be an elliptic curve. Then the elements in the 5-Selmer group of $E$ are in bijective correspondence with $G_\mathbb{Q}$-equivalence classes on the set of locally soluble elements in $V_\mathbb{Z}$ having invariants equal to $I(E)$ and $J(E)$.

Finally, we will need the following proposition, also due to Fisher [29]. It states that any element $v \in V_{\mathbb{Z}_p}$ for which $p^2$ does not divide the discriminant is automatically $Q_p$-soluble; moreover, the stabilizer of $v$ in $G_{\mathbb{Z}_p}$ is the same as that in $G_{\mathbb{Q}_p}$, and the notion of $G_{\mathbb{Z}_p}$-equivalence of such elements $v$ is the same as that of $G_{\mathbb{Q}_p}$-equivalence:

**Proposition 11** ([29]) Let $p$ be any prime and let $v \in V_{\mathbb{Z}_p}$ be an element such that $p^2 \nmid \Delta(v)$. Then

$v$ is $Q_p$-soluble; \hspace{1cm} $\text{Stab}_{G_{\mathbb{Q}_p}}(v) = \text{Stab}_{G_{\mathbb{Z}_p}}(v)$; \hspace{1cm} $(G_{\mathbb{Q}_p} \cdot v) \cap V_{\mathbb{Z}_p} = G_{\mathbb{Z}_p} \cdot v.$
3 Counting orbits of bounded height

We write elements in $V_\mathbb{R} = \mathbb{R}^5 \otimes \wedge^2 \mathbb{R}^5$ as quintuples $(A, B, C, D, E)$ of skew-symmetric $5 \times 5$ matrices, where the matrices $A, B, C, D,$ and $E$ have entries $a_{ij}, b_{ij}, c_{ij}, d_{ij},$ and $e_{ij},$ respectively, with $1 \leq i < j \leq 5.$ We define the height of an element $v \in V_\mathbb{R}$ having invariants $I$ and $J$ to be

$$H(v) := H(I, J) := \max\{|I|^3, J^2/4\}$$

which is clearly $G_\mathbb{R}$-invariant.

The action of $G_\mathbb{Z}$ preserves the lattice $V_\mathbb{Z} \subset V_\mathbb{R}$ consisting of the quintuples of skew-symmetric $5 \times 5$ matrices whose entries are integral. In fact, it also preserves the two sets $V_\mathbb{Z}^+$ and $V_\mathbb{Z}^-$ consisting of elements in $V_\mathbb{Z}$ having positive and negative discriminant, respectively. We say that an element $v \in V_\mathbb{Z}$ having invariants $I$ and $J$ is strongly irreducible if it has nonzero discriminant and does not correspond to the identity element in the 5-Selmer group of $E^{I,J},$ i.e., the 5-covering $C_v$ corresponding to $v$ is the not the trivial 5-covering of $E^{I,J}.$

In this section, we compute asymptotics for the number of $G_\mathbb{Z}$-orbits on strongly irreducible elements in $V_\mathbb{Z}^\pm$ having bounded height. To state the result, let $N^+(X)$ (resp. $N^-(X)$) denote the number of pairs $(I, J) \in \mathbb{Z} \times \mathbb{Z}$ having height less than $X$ and positive (resp. negative) discriminant. For any $G_\mathbb{Z}$-invariant set $S \subset V_\mathbb{Z},$ let $N(S; X)$ denote the number of $G_\mathbb{Z}$-orbits on strongly irreducible elements in $S$ having height less than $X.$ Finally, throughout this paper, we fix $\omega$ to be a differential that generates the rank 1 module of top-degree left-invariant differential forms of $G$ over $\mathbb{Z}.$

Then we prove the following theorem:

**Theorem 12** There exists a nonzero rational constant $\mathcal{J},$ to be defined in Proposition 24, such that

$$N(V_\mathbb{Z}^\pm; X) = |\mathcal{J}| \cdot \text{Vol}(G_\mathbb{Z}\backslash G_\mathbb{R}) \cdot N^\pm(X) + o(X^{5/6}),$$

where the volume of $G_\mathbb{Z}\backslash G_\mathbb{R}$ is computed with respect to $\omega.$

**Remark 13** It follows from [8, Equations (24),(25)] that, up to an error of $O(X^{1/2}),$ we have $N^+(X) = 8X^{5/6}$ and $N^-(X) = 22X^{5/6}.$ Thus the error term in the equation of the above theorem is indeed smaller than the main term.

### 3.1 Reduction theory

Let $V_\mathbb{R}^+$ and $V_\mathbb{R}^-$ denote the set of elements in $V_\mathbb{R}$ having positive and negative discriminant, respectively. The purpose of this subsection is to construct finite covers of fundamental domains for the action of $G_\mathbb{Z}$ on $V_\mathbb{R}^\pm$. We first construct fundamental sets for the action of $G_\mathbb{R}$ on $V_\mathbb{R}^\pm$.

To apply Theorem 7 in the case of $G_\mathbb{R}$-orbits on $V_\mathbb{R}^\pm,$ we need the following lemma:

**Lemma 14** Let $v \in V_\mathbb{R}$ have nonzero discriminant. Then $v$ is $\mathbb{R}$-soluble.

**Proof:** Let $C_v$ be the curve corresponding to $v$ under the correspondence described in Section 2. It is known (see [13, Remark 1.23]) that $C_v$ has 25 flex points defined over $\mathbb{C}.$ Since $v \in V_\mathbb{R},$ these flex points come in complex conjugate pairs. Thus, at least one of them is defined over $\mathbb{R},$ implying that $C_v(\mathbb{R}) \neq \emptyset.$ \(\square\)

The next proposition follows from Theorem 7, Lemma 14, and the fact that the group $E(\mathbb{R})/5E(\mathbb{R})$ is trivial for every elliptic curve $E$ over $\mathbb{R}$:
Proposition 15 Let \((I, J) \in \mathbb{R} \times \mathbb{R}\) be such that \(\Delta(I, J) \neq 0\). Then the set of elements in \(V_{\mathbb{R}}\) having invariants \(I\) and \(J\) consists of one soluble \(G_{\mathbb{R}}\)-orbit.

Thus we may construct fundamental sets \(R^\pm\) for the action of \(G_{\mathbb{R}}\) on \(V_{\mathbb{R}}^\pm\) by choosing one element \(v \in V_{\mathbb{R}}^\pm\) having invariants \(I\) and \(J\) for each pair \((I, J) \in \mathbb{R} \times \mathbb{R}\) such that \(\Delta(I, J) \in \mathbb{R}^\pm\). We now choose specific sets \(R^\pm\). The element \(v_{I,J}\) defined by

\[
v_{I,J} := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

has invariants equal to \(I\) and \(J\) by [25 §6]. We define \(R^\pm\) by

\[
R^+ := \{ \lambda \cdot v_{I,J} : \lambda \in \mathbb{R}_{>0}, H(I, J) = 1, \Delta(I, J) > 0 \};
\]

\[
R^- := \{ \lambda \cdot v_{I,J} : \lambda \in \mathbb{R}_{>0}, H(I, J) = 1, \Delta(I, J) < 0 \}.
\]

Since \(H(\lambda \cdot v) = \lambda^6 H(v)\) (as \(I(v)\) and \(J(v)\) are polynomials in the coefficients of \(v\) having degrees 20 and 30, respectively), we see that the coefficients of all the elements in \(R^\pm\) having height less than \(X\) are bounded by \(O(X^{1/60})\). Note also that for any \(g \in G_{\mathbb{R}}\), the set \(g \cdot R^\pm\) is also a fundamental set for the action of \(G_{\mathbb{R}}\) on \(V_{\mathbb{R}}^\pm\). Furthermore, for any compact set \(G_0 \subset G_{\mathbb{R}}\), the coefficients of elements in \(g \cdot R^\pm\), with \(g \in G_0\), having height less than \(X\), is bounded by \(O(X^{1/60})\), where the implied constant depends only on \(G_0\).

Let \(F\) denote a fundamental domain for the left action of \(G_{\mathbb{Z}}\) on \(G_{\mathbb{R}}\) that is contained in a standard Siegel set [12 §2]. We may assume that \(F = \{nak : n \in N'(a), a \in A', k \in K\}\), where \(K = \{\text{subgroup of orthogonal transformations } SO_5(\mathbb{R}) \times SO_5(\mathbb{R}) \subset G_{\mathbb{R}}\}\); \(A' = \{a(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8) : s_i, t_i > c\}\),

\[
\text{where } a(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8) = \begin{pmatrix}
\begin{array}{cccc}
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3
\end{array}
\end{pmatrix}, \quad \begin{pmatrix}
\begin{array}{cccc}
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3
\end{array}
\end{pmatrix}, \quad \begin{pmatrix}
\begin{array}{cccc}
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3
\end{array}
\end{pmatrix}, \quad \begin{pmatrix}
\begin{array}{cccc}
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3
\end{array}
\end{pmatrix}, \quad \begin{pmatrix}
\begin{array}{cccc}
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3
\end{array}
\end{pmatrix}, \quad \begin{pmatrix}
\begin{array}{cccc}
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3 \\
1 & -2 & 3 & -3
\end{array}
\end{pmatrix};
\]

\(N' = \{n(u_1, \ldots, u_{20}) : (u_i) \in \nu(a)\}\),

\[
\text{where } n(u) = \begin{pmatrix}
\begin{array}{cccc}
1 & u_1 & u_2 & u_3 \\
1 & u_4 & u_5 & u_6 \\
1 & u_7 & u_8 & u_9 \\
u_{10} & 1 & 1 & 1
\end{array}
\end{pmatrix}, \quad \begin{pmatrix}
\begin{array}{cccc}
1 & u_1 & u_1 & u_2 \\
1 & u_4 & u_5 & u_6 \\
1 & u_7 & u_8 & u_9 \\
u_{12} & u_{13} & u_{14} & u_{15}
\end{array}
\end{pmatrix};
\]

here \(\nu(a)\) is a measurable subset of \([-1/2, 1/2]^{20}\) dependent only on \(a \in A'\), and \(c > 0\) is an absolute constant. We now require the following result that follows from Theorem [7] and the fact that every elliptic curve over \(\mathbb{R}\) has exactly five 5-torsion points defined over \(\mathbb{R}\).

Lemma 16 Let \(v \in V_{\mathbb{R}}\) be any element having nonzero discriminant. Then the size of the stabilizer in \(G_{\mathbb{R}}\) of \(v\) is equal to 5.

For \(h \in G_{\mathbb{R}}\), we regard \(Fh \cdot R^\pm\) as a multiset, where the multiplicity of an element \(v \in V_{\mathbb{R}}\) is equal to \(#\{g \in F : v \in gh \cdot R^\pm\}\). By an argument identical to that in [8 §2.1], it follows that for any \(h \in G_{\mathbb{R}}\) and any \(v \in V_{\mathbb{R}}^\pm\), the \(G_{\mathbb{Z}}\)-orbit of \(v\) is represented \(m(v)\) times in \(Fh \cdot R^\pm\), where

\[
m(v) := \# \text{Stab}_{G_{\mathbb{R}}}(v)/\# \text{Stab}_{G_{\mathbb{Z}}}(v).
\]
That is, the multiplicity of \( v' \) in \( \mathcal{F}h \cdot R^\pm \), summed over all \( v' \) that are \( G_Z \)-equivalent to \( v \), is equal to \( m(v) \).

The set of elements in \( V^\pm_{\mathbb{R}} \) that have a nontrivial stabilizer in \( G_Z \) has measure 0 in \( V^\pm_{\mathbb{R}} \). Thus, by Lemma 10 for any \( h \in G_{\mathbb{R}} \) the multiset \( \mathcal{F}h \cdot R^\pm \) is a 5-fold cover of a fundamental domain for the action of \( G_Z \) on \( V^\pm_{\mathbb{R}} \).

Let \( R^\pm(X) \) denote the set of elements in \( R^\pm \) having height less than \( X \). It then follows, for any \( G_Z \)-invariant subset \( S \subset V_{\mathbb{Z}} \), that \( 5N(S; X) \) is equal to the number of strongly irreducible elements in \( \mathcal{F}h \cdot R^\pm(X) \cap S \), with the slight caveat that the (relatively rare—see Lemma 30) elements with \( G_Z \) stabilizers of size \( r \) \((r > 1)\) are counted with weight \( 1/r \).

Counting strongly irreducible integer points in a single such domain \( \mathcal{F}h \cdot R^\pm(X) \) is difficult because this domain is unbounded (although we will show it has finite volume). As in [8], we simplify the counting by averaging over lots of such domains, i.e., by averaging over a continuous range of elements \( g \) lying in a compact subset of \( G_{\mathbb{R}} \).

### 3.2 Averaging

Let \( G_0 \subset G_{\mathbb{R}} \) be a compact, semialgebraic, left \( K \)-invariant set that is the closure of an open nonempty set. Let \( dh \) be the Haar measure on \( G_{\mathbb{R}} \) normalized as follows: we set \( dh = du \, ds \, da \, dk \), where \( n, a, du, \) and \( ds \) are given by \( n = n(u_1, \ldots, u_{20}) \), \( a = a(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8) \), \( du = du_1 \cdots du_{20} \), and

\[
d^s a = s_1^{-20} s_2^{-30} s_3^{-30} s_4^{-20} s_5^{-20} s_6^{-30} s_7^{-30} s_8^{-20} d^x s_1 d^x s_2 d^x s_3 d^x s_4 d^x s_5 d^x s_6 d^x s_7 d^x s_8, \tag{6}
\]

respectively, and \( dk \) is Haar measure on \( K \) normalized so that \( K \) has volume 1.

For any \( G_Z \)-invariant set \( S \subset V^\pm_{\mathbb{Z}} \), the arguments of §3.1 imply that we have

\[
N(S; X) = \frac{\int_{h \in G_0} \# \{ \mathcal{F}h \cdot R^\pm(X) \cap S^{\text{irr}} \} \, dh}{C_{G_0}}, \tag{7}
\]

where \( S^{\text{irr}} \) denotes the set of strongly irreducible elements in \( S \), and \( C_{G_0} := 5 \int_{h \in G_0} dh \). We take the right hand side of (7) as the definition of \( N(S; X) \) also for sets \( S \) that are not necessarily \( G_Z \)-invariant. By an argument identical to the proof of [8, Theorem 2.5], we see that the right hand side of (7) is equal to

\[
N(S; X) = \frac{1}{C_{G_0}} \int_{na \in \mathcal{F}} \# \{ B^\pm(n, a; X) \cap S^{\text{irr}} \} \, da \, ds \, da \, dk, \tag{8}
\]

where \( B^\pm(n, a; X) \) denotes the multiset \( naG_0 \cdot R^\pm(X) \).

To estimate the number of integral points in \( B^\pm(n, a; X) \), we use the following proposition due to Davenport.

**Proposition 17** Let \( \mathcal{R} \) be a bounded, semi-algebraic multiset in \( \mathbb{R}^n \) having maximum multiplicity \( m \), and that is defined by at most \( k \) polynomial inequalities each having degree at most \( \ell \). Then the number of integer lattice points (counted with multiplicity) contained in the region \( \mathcal{R} \) is

\[
\text{Vol}(\mathcal{R}) + O(\max\{\text{Vol}(\mathcal{R}), 1\}),
\]

where \( \text{Vol}(\mathcal{R}) \) denotes the greatest \( d \)-dimensional volume of any projection of \( \mathcal{R} \) onto a coordinate subspace obtained by equating \( n - d \) coordinates to zero, where \( d \) takes all values from 1 to \( n - 1 \). The implied constant in the second summand depends only on \( n, m, k, \) and \( \ell \).
Proposition 17 yields good estimates on the number of integral points in $B^{\pm}(n, a; X)$, for $a = a(s_1, \ldots, s_8)$, when the $s_i$’s are “small” compared to $X$. However, when any of the $s_i$’s are “large”, the error term in Proposition 17 dominates the main term. To resolve this issue, in §3.3 we divide the fundamental domain into a “main body” and a “cuspidal region”. Proposition 17 will yield good estimates on the number of integral points in the main body. We then bound the number of strongly irreducible elements in the cuspidal region, the volume of the cuspidal region, and the number of reducible elements in the main body. These results together then allow us to deduce that $N(V_{x_i}^{\pm}; X)$ is well-approximated by the volume $F \cdot R^{\pm}(X)$. Finally, in §3.4, we compute the volume of the region $F \cdot R^{\pm}(X)$, thus completing the proof of Theorem 12.

3.3 Conditions on reducibility and cutting off the cusp

Our first aim in the subsection is to prove that the number of strongly irreducible points in the “cuspidal region” of the fundamental domain is negligible:

Proposition 18 Let $V_{x_i}^{\text{irr}}(0)$ denote the set of strongly irreducible points $(A, B, C, D, E) \in V_{x_i}$ satisfying $a_{12} = 0$. Then $N(V_{x_i}^{\text{irr}}(0); X) = O(X^{499/600})$.

We begin by describing sufficient conditions to ensure that points $(A, B, C, D, E) \in V_{x_i}$ are not strongly irreducible.

Lemma 19 Let $v = (A, B, C, D, E)$ be an element in $V_{x_i}$ and let $Q_1, Q_2, Q_3, Q_4,$ and $Q_5$ denote the five $4 \times 4$ sub-Pfaffians of $A t_1 + B t_2 + C t_3 + D t_4 + E t_5$. Then

(a) Let $C_v$ be the curve in $\mathbb{P}^4$ defined by $Q_i(t_1, t_2, t_3, t_4, t_5) = 0$ for all $i \in \{1, \ldots, 5\}$. If $C_v$ is not a smooth genus one curve, then the discriminant of $v$ is 0.

(b) If $Q_1$ is reducible over $\overline{\mathbb{Q}}$ (i.e., $Q_1$ factors into a product of linear forms over $\overline{\mathbb{Q}}$), then the discriminant of $v$ is 0.

(c) Let $Q'_1, Q'_2, Q'_3, Q'_4,$ and $Q'_5$ be the quadratic forms in four variables obtained from $Q_1, Q_2, Q_3, Q_4,$ and $Q_5$, respectively, by setting $t_5 = 0$. If the intersection of the quadrics $Q'_i(t_1, t_2, t_3, t_4) = 0$ in $\mathbb{P}^3(\overline{\mathbb{Q}})$ consists of a single point, then $v$ corresponds to the identity element in the 5-Selmer group of $E^{(v), J(v)}$.

Proof: Parts (a) and (b) follow from [25, Theorem 4.4(ii)] and [25, Theorem 5.10(i)], respectively. For Part (c), note that an element in the 5-Selmer group of an elliptic curve $E$ may viewed as a torsor for $E[5]$ (see, e.g., [18, §1.4]). By [18, Remarks 1.15, 1.20, and 1.23], it follows that given $v \in V_{x_i}$, the corresponding torsor for $E[5]$ is obtained by taking the set of points $P$ on $C_v$ such that $5 \cdot P$ is linearly equivalent to $D$, where $D$ is the hyperplane divisor corresponding to a fixed rational hyperplane section $H$. If the hypotheses of Part (c) are satisfied, then $C_v$ intersects the hyperplane section $H$ given by $t_5 = 0$ in a rational point $P$ with multiplicity 5. Therefore the torsor for $E[5]$ contains a rational point, namely $P$. It follows that $C_v$ is the trivial 5-covering of $E$. □

Proposition 20 Let $v = (A, B, C, D, E) \in V_{x_i}$ be such that all the variables in at least one of the following sets vanish:

1. $\{a_{12}, a_{13}\} \cup \{b_{12}, b_{13}\} \cup \{c_{12}, c_{13}\} \cup \{d_{12}, d_{13}\} \cup \{e_{12}, e_{13}\}$
2. $\{a_{12}, a_{13}, a_{14}\} \cup \{b_{12}, b_{13}, b_{14}\} \cup \{c_{12}, c_{13}, c_{14}\} \cup \{d_{12}, d_{13}, d_{14}\}$

10
In Cases (2) and (3), the quadratic form corresponding to $Q_1$ is of the form

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ * & * & * & * & * \end{pmatrix}.$$ 

Thus, in the first three cases the quadratic form corresponding to $Q_1$ factors into two linear factors, and hence by Lemma 19(b) we have that $\Delta(A, B, C, D, E) = 0$.

In Cases (4) and (5), the $t_1^2$, $t_1t_2$, and $t_2^2$ coefficients of all the $Q_i$'s are equal to zero, and so the Gram matrices of the $Q_i$'s take the form

$$[Q_1, Q_2, Q_3, Q_4, Q_5] = \left[ \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}, \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}, \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}, \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}, \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \right].$$

This implies that the curve $C_v$ defined by the vanishing of the $Q_i$ is not a smooth genus one curve, as it contains a projective line in $\mathbb{P}^4$. Meanwhile, in Case (6) we may further assume (replacing $(A, B, C, D, E)$ with a $G_C$-translate, if necessary) that $a_{34} = b_{34} = 0$. Now the $t_1^2$, $t_1t_2$, and $t_2^2$ coefficients of all the $Q_i$'s are again equal to zero, as before. Thus, Lemma 19(a) implies that the discriminant of $(A, B, C, D, E)$ is 0 in these three cases.

To prove Cases (7) through (12), note that the discriminant of $(A, B, C, D, E)$ is a degree 60 polynomial in the $a_{ij}$'s, $b_{ij}$'s, $c_{ij}$'s, $d_{ij}$'s, and $e_{ij}$'s. Let $m$ be any monomial summand of the discriminant polynomial. We define $a(m)$ to be the number of factors $a_{ij}$ ($i, j \in \{1, 2, 3, 4, 5\}$, $i > j$) that occur in $m$, and $1(m)$ to be the number of factors $x_{1j}$ ($x \in \{a, b, c, d, e\}$, $j \in \{2, 3, 4, 5\}$)
that occur in $m$, counted with multiplicity. We similarly define $b(m)$, $c(m)$, $d(m)$, $e(m)$, $2(m)$, $3(m)$, $4(m)$, and $5(m)$. Classical invariant theory implies that

$$
a(m) = b(m) = c(m) = d(m) = e(m) = 12, \\
1(m) = 2(m) = 3(m) = 4(m) = 5(m) = 24.
$$

(9)

From these observations we see that an element in $V_2$ having all the $a_{ij}$ equal to zero has discriminant zero, since every monomial term in the discriminant polynomial has some $a_{ij}$ as a factor. Thus, Case (7) follows.

Now suppose that there exists some element satisfying the condition of Case (8) and having nonzero discriminant. Then the discriminant polynomial must have a monomial summand $m$ with no factor of the form $a_{1j}$, $b_{1j}$, or $c_{1j}$. Then (9) implies that every factor $d_{ij}$ or $e_{ij}$ of this summand $m$ satisfies $i = 1$. We claim that Case (5) shows that this is impossible. Indeed, the truth of Case (5) implies that the discriminant polynomial cannot have a nonzero monomial summand in which every factor $a_{ij}$ or $b_{ij}$ has $j = 5$. The same argument then applies with $a$, $b$, and 5 replaced by $d$, $e$, and 1, respectively. Thus, Case (8) follows.

In Case (9), a nonzero monomial summand $m$ of the discriminant polynomial cannot have a factor of the form $b_{34}$, $b_{35}$, $b_{45}$, $c_{34}$, $c_{35}$, $c_{45}$, $d_{34}$, $d_{35}$, $d_{45}$, $e_{34}$, $e_{35}$, or $e_{45}$. (Otherwise $1(m) + 2(m)$ would be strictly smaller than 48, contradicting (9).) Case (9) now follows from Case (3) just as Case (8) followed from Case (5). Case (10) follows immediately from Case (9) since any element satisfying the conditions of Case (10) is $G_2$-equivalent to one satisfying the conditions of Case (9).

We turn to Case (11). A nonzero monomial summand $m$ of the discriminant polynomial cannot have a factor of the form $d_{44}$, $e_{44}$, $d_{45}$, or $e_{45}$ (otherwise, $4(m) + 5(m)$ would be greater than 48). Case (4) shows that no such summand exists. Case (12) follows from Case (11) just as Case (10) followed from Case (9).

Finally, in Case (13), the Gram matrices of the $Q_i$’s have the following form:

$$[Q_1, Q_2, Q_3, Q_4, Q_5] = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

It follows that if the element $v = (A, B, C, D, E)$ has nonzero discriminant, then the only possible point of intersection of the quadrics cut out by the $Q_i$’s in the hyperplane section $t_5 = 0$ is at the point $[1 : 0 : 0 : 0] \in \mathbb{P}^3(\mathbb{C})$. By Lemma 19(c), $v$ corresponds to the identity element in the Selmer group of the Jacobian of $C_v$, and so is not strongly irreducible. \(\square\)

We are now ready to prove Proposition 18.

**Proof of Proposition 18.** Consider the set

$$\text{Var} := \{x_{ij} : x \in \{a, b, c, d, e\}, i, j \in \{1, 2, 3, 4, 5\}, i < j\}.$$

We consider each $u \in \text{Var}$ as a $\mathbb{Z}$-valued (resp. $\mathbb{R}$-valued) function on $V_2$ (resp. $V_\mathbb{R}$) in the obvious way. Each variable $u \in \text{Var}$ has a weight $w(u)$ defined by $a(s_1, \ldots, s_8) \cdot u = w(u) u$. The weight $w(u)$ is evidently a rational function in $s_1, \ldots, s_8$. We may define a natural partial order $\preceq$ on $\text{Var}$, where $x_{ij} \preceq y_{kl}$ if $x$ is either lexicographically ahead of $y$, $i \leq k$, and $j \leq l$. Note that for any $u_1, u_2 \in \text{Var}$, we have $u_1 \preceq u_2$ if and only if the exponent of every $s_i$ in $w(u_1)$ is less than or equal to the corresponding exponent in $w(u_2)$.

Now given a subset $Z \subseteq \text{Var}$, we define the set $V_Z(\mathbb{Z}) \subseteq V_2$ to be the set of all elements $v \in V_2$ such that $u(v) = 0$ for each $u \in Z$ and $u(v) \neq 0$ for each $u \in N = N(Z)$, where $N(Z)$ is
the set of minimal elements (under the above partial ordering) in \( \text{Var} \setminus \mathcal{Z} \). Identically to the proof of \([5, \text{Lemma } 11]\), we partition \( V_{\mathbb{Z}} \) into disjoint subsets of \( V_{\mathbb{Z}} \) by the following process. Start with \( S = V_{\mathbb{Z}}(\{a_{12}\}) \). (Thus, \( S \) is the set of all \((A, B, C, D, E) \in V_{\mathbb{Z}} \) with \( a_{12} = 0 \) and \( a_{13}, b_{12} \neq 0 \).) At every step of the process, for each set \( V_{\mathbb{Z}}(\mathcal{Z}) \) generated in the previous step and each \( u \in \mathcal{N} \), we add the set \( V_{\mathbb{Z}}(\mathcal{Z} \cup \{u\}) \) to our list of subsets, provided that \( \mathcal{Z} \cup \{u\} \) does not contain any of the thirteen sets listed in Proposition \([20]\).

Let \( SS \) denote the set of all subsets of \( \text{Var} \) generated by the above process. It is clear that every strongly irreducible element with \( a_{12} = 0 \) is contained in exactly one of the sets in \( SS \). Thus to prove Proposition \([18]\), it suffices to prove the estimate \( N(S; X) = O(X^{499/600}) \) for each \( S \in SS \).

Equation \([8]\) implies that given a fixed set \( S = V_{\mathbb{Z}}(\mathcal{Z}) \in SS \), we have the estimate

\[
N(V_{\mathbb{Z}}(\mathcal{Z}); X) = O\left( \int_{s_1, \ldots, s_8 = c}^{\infty} \sigma(\mathcal{Z}, a) d^* a \right),
\]

where \( d^* a \) is given by \([9]\) and \( \sigma(\mathcal{Z}, a) \) is the number of integer points in the region

\[
B^\pm(0, a; X; \mathcal{Z}) := \{ v \in B^\pm(0, a; X) : u(v) = 0 \text{ for } u \in \mathcal{Z}, \text{ and } |u(v)| \geq 1 \text{ for } u \in \mathcal{N} \}.
\]

An element \( v \in B^\pm(0, a; X; \mathcal{Z}) \) satisfies \( u(v) \ll X^{1/60} w(u) \) for each \( u \in \text{Var} \), and therefore \( \sigma(\mathcal{Z}, a) \) is nonzero only if \( X^{1/60} w(u) \gg 1 \) for each \( u \in \mathcal{N} \). Since \( \mathcal{N} \) was chosen to be the set of minimal elements in \( \text{Var} \setminus \mathcal{Z} \), it follows that \( \sigma(\mathcal{Z}, a) \) is nonzero only if \( X^{1/60} w(u) \gg 1 \) for every \( u \in \text{Var} \setminus \mathcal{Z} \).

If we define the weight \( w(u_{i_1}^{r_1} \cdots u_{k}^{r_k}) \) to be \( w(u_1)^{r_1} \cdots w(u_k)^{r_k} \) for \( u_1, \ldots, u_k \in \text{Var} \) and \( r_1, \ldots, r_k \in \mathbb{R} \), then we have the estimate

\[
N(V_{\mathbb{Z}}(\mathcal{Z}); X) = O\left( \int_{s_1, \ldots, s_8 = c}^{\infty} X^{50 - \# \mathcal{Z}} \cdot w(\prod_{u \in \text{Var} \setminus \mathcal{Z}} u)^{d^* a} \right),
\]

for any \( \pi_{\mathcal{Z}} = \prod u_i^{r_i} \) such that all the \( u_i \)'s are in \( \text{Var} \setminus \mathcal{Z} \) and all the \( r_i \)'s are positive real numbers. The first equality in \([10]\) follows by applying Proposition \([17]\) on the set \( B^\pm(0, a; X; \mathcal{Z}) \), along with the fact that \( \sigma(\mathcal{Z}, a) \) is nonzero only if \( X^{1/60} w(u) \gg 1 \) for each \( u \in \text{Var} \setminus \mathcal{Z} \). The second equality also follows directly from the latter fact. Therefore, if we find \( \pi_{\mathcal{Z}} \) as above such that the exponent of each \( s_i \) in

\[
\prod_{u \in \text{Var} \setminus \mathcal{Z}} u
\]

is negative, then we may conclude that

\[
N(V_{\mathbb{Z}}(\mathcal{Z}); X) = O\left( X^{\frac{50 - \# \mathcal{Z} + \deg(\pi_{\mathcal{Z}})}{60}} \right).
\]

We now describe how, if we have a suitable factor \( \pi_{\mathcal{Z}} \) for the set \( \mathcal{Z} \) in \([10]\)–\([12]\), then we can obtain the estimate on the right hand side of \([12]\) also for \( N(V_{\mathbb{Z}}(\mathcal{Z}'); X) \) for all subsets \( \mathcal{Z}' \subset \mathcal{Z} \). Indeed, if we can construct products \( \pi_{\mathcal{Z},u} \) of nonnegative powers of elements in \( \text{Var} \setminus \mathcal{Z} \), for each \( u \in \mathcal{Z} \setminus \{a_{12}\} \), such that the exponent of every \( s_i \) in \( w(u/\pi_{\mathcal{Z},u}) \) is negative and \( \pi_{\mathcal{Z}}/\prod_{u \in \mathcal{Z} \setminus \{a_{12}\}} \pi_{\mathcal{Z},u} \) is also a product of nonnegative powers of elements in \( \text{Var} \setminus \mathcal{Z} \), then we may similarly conclude that

\[
N(V_{\mathbb{Z}}(\mathcal{Z}'); X) = O\left( X^{\frac{50 - \# \mathcal{Z} + \deg(\pi_{\mathcal{Z}})}{60}} \right).
\]
for every subset $Z'$ of $Z$ with $a_{12} \in Z'$. This follows by using $\pi_{Z'} := \pi_Z/\prod_{u \in Z \setminus Z'} \pi_{Z,u}$ and $Z'$ in place of $\pi_Z$ and $Z$, respectively, in (10).

We now have the following lemma which gives a list of sets $Z$ such that every element of $SS$ is contained in at least one such $Z$.

**Lemma 21** Let $V_\pi(Z)$ be an element of $SS$. Then $Z$ is contained in one of the following sets:

1. $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}, a_{34}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}, b_{23}, b_{24}\} \cup \{c_{12}, c_{13}, c_{14}, c_{23}\} \cup \{d_{12}\}$
2. $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}, a_{34}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}, b_{23}, b_{24}\} \cup \{c_{12}, c_{13}, c_{14}\} \cup \{d_{12}, d_{13}\}$
3. $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}, a_{34}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}, b_{23}, b_{24}\} \cup \{c_{12}, c_{13}, c_{23}\} \cup \{d_{12}, d_{13}\}$
4. $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}, a_{34}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}, b_{23}, b_{24}\} \cup \{c_{12}, c_{13}, c_{14}, c_{23}\} \cup \{d_{12}, d_{13}\}$
5. $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}, a_{34}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}, b_{23}, b_{24}\} \cup \{c_{12}, c_{13}, c_{14}, c_{23}\} \cup \{d_{12}, d_{13}\}$
6. $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}, b_{23}, b_{24}\} \cup \{c_{12}, c_{13}, c_{14}, c_{23}\} \cup \{d_{12}, d_{13}\}$
7. $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{34}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}, b_{23}, b_{24}\} \cup \{c_{12}, c_{13}, c_{14}, c_{23}\} \cup \{d_{12}, d_{13}\}$
8. $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{34}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}, b_{23}, b_{24}\} \cup \{c_{12}, c_{13}, c_{14}, c_{23}\} \cup \{d_{12}, d_{13}\}$
9. $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{34}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}, b_{23}, b_{24}\} \cup \{c_{12}, c_{13}, c_{14}, c_{23}\} \cup \{d_{12}, d_{13}\}$
10. $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{34}, a_{35}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}, b_{23}, b_{24}\} \cup \{c_{12}, c_{13}, c_{14}\} \cup \{d_{12}\}$

Lemma 21 follows immediately from Proposition 20.

For each set $Z$ of Lemma 21 we construct in Table 1 monomials $\pi_{Z,u}$ with $u_i \in \text{Var}\setminus Z$ and $r_i \geq 0$ such that the exponent of each $s_i$ in (10) is negative. In Table 2 for each such set $Z$, we determine $\pi_{Z,u}$'s for $u \in Z \setminus \{a_{12}\}$ such that each $\pi_{Z,u}$ is a product of nonnegative powers.
Table 2: The factors $\pi_u = \pi_{\Sigma, u}$ used in the proof of Proposition 18.
of elements in \(\text{Var}ackslash \mathcal{Z}\), the exponent of every \(s_i\) in \(w(u/\pi_{\mathcal{Z},u})\) is negative, and \(\pi_{\mathcal{Z}}/\prod_{u\in \mathcal{Z} \backslash \{a_{12}\}} \pi_{\mathcal{Z},u}\) is also a product of nonnegative powers of elements in \(\text{Var}ackslash \mathcal{Z}\). We conclude that for every \(\mathcal{Z}' \subset \mathcal{Z}\), we have \(N(V_{\mathcal{Z}'};X) = O(X^{\frac{50-\deg(\pi_{\mathcal{Z}})}{60}}) = O(X^{\frac{50}{60}})\). Since we have constructed \(\pi_{\mathcal{Z}}\) and \(\pi_{\mathcal{Z},u}\)’s for each set \(\mathcal{Z}\) of Lemma \[21\] Proposition \[18\] follows. □

We have proven that the number of irreducible elements in the “cuspidal region” of the fundamental domain is negligible. The next proposition states that the number of reducible elements in the “main body” is also negligible:

**Proposition 22** Let \(V_{\mathcal{Z}}^{\text{red}}\) denote the set of elements in \(V_{\mathcal{Z}}\) that are not strongly irreducible. Then

\[
\int_{na \in \mathcal{F}} \#\{v \in B^\pm(n,a;X) \cap V_{\mathcal{Z}}^{\text{red}} : a_{12}(v) \neq 0\} \, dnd^*a = o(X^{5/6}).
\] (13)

We defer the proof of Proposition \[22\] to §3.6.

Therefore, in order to estimate \(N(V_{\mathcal{Z}};X)\), it suffices to count the number of (not necessarily strongly irreducible) integral points in the main body of the fundamental domain. We do this in the following proposition:

**Proposition 23** We have

\[
\frac{1}{CG_0} \int_{na \in \mathcal{F}} \#\{B^\pm(n,a;X) \cap V_{\mathcal{Z}}^{\text{irr}}\} \, dnd^*a = \frac{1}{5} \text{Vol}(\mathcal{F} \cdot R^\pm(X)) + o(X^{5/6}).
\]

**Proof:** The proof of Proposition \[23\] is very similar to that of [5, Proposition 12]. If \(v \in B^\pm(n,a;X)\), then we know that \(a_{12}(v) = O(X^{1/60}w(a_{12}))\). Thus, by Propositions \[18\] and \[22\] we obtain

\[
\frac{1}{CG_0} \int_{na \in \mathcal{F}} \#\{B^\pm(n,a;X) \cap V_{\mathcal{Z}}^{\text{irr}}\} \, dnd^*a = \frac{1}{CG_0} \int_{na \in \mathcal{F}} \#\{B^\pm(n,a;X) \cap V_{\mathcal{Z}}\} \, dnd^*a = o(X^{5/6}).
\]

(14)

Since \(a_{12}\) has minimal weight, and the projection of \(B^\pm(n,a;X)\) onto \(a_{12}\) has length greater than an absolute positive constant when \(X^{1/60}w(a_{12}) \gg 1\), Proposition \[17\] implies that the main term on the right hand side of (14) is equal to

\[
\frac{1}{CG_0} \int_{X^{1/60}w(a_{12}) \gg 1} \left[ \text{Vol}(B^\pm(n,a;X)) + O\left(\frac{\text{Vol}(B^\pm(n,a;X))}{X^{1/60}w(a_{12})}\right) \right] \, dnd^*a.
\]

(15)

Since the region \(\{nak \in \mathcal{F} : w(a_{12}) \ll X^\epsilon\}\) has volume \(o(1)\) for any \(\epsilon < 1/60\), (15) is equal to

\[
\frac{1}{CG_0} \int_{na \in \mathcal{F}} \text{Vol}(B^\pm(n,a;X)) \, dnd^*a + o(X^{5/6}).
\]

The proposition follows since

\[
\frac{1}{CG_0} \int_{na \in \mathcal{F}} \text{Vol}(B^\pm(n,a;X)) \, dnd^*a = \frac{1}{CG_0} \int_{h \in G_0} \text{Vol}(\mathcal{F}h \cdot R^\pm(X)) \, dh,
\]

and the volume of \(\mathcal{F}h \cdot R^\pm(X)\) is independent of \(h\). □

Propositions \[18\] \[22\] and \[23\] imply that

\[
N(V_{\mathcal{Z}}^\pm,X) = \frac{1}{5} \text{Vol}(\mathcal{F} \cdot R^\pm(X)) + o(X^{5/6}).
\]

Thus, to prove Theorem \[12\] it only remains to compute the volume \(\text{Vol}(\mathcal{F} \cdot R^\pm(X))\).
3.4 Computing the volume

Let $dv$ denote the Euclidean measure on $V_{\mathbb{R}}$ normalized so that $V_{\mathbb{Z}}$ has covolume 1. The sets $R^\pm$ contain exactly one point having invariants $I$ and $J$ for every pair $(I, J) \in \mathbb{R} \times \mathbb{R}$ satisfying $\pm \Delta(I, J) > 0$. Let $dIdJ$ be the measure on these sets $R^\pm$. Recall that we defined $\omega$ to be a differential that generates the rank 1 module of top-degree left-invariant differential forms of $G$ over $\mathbb{Z}$. With these measure normalizations, we have the following proposition whose proof is identical to that of \cite[Proposition 2.10]{[8]}

**Proposition 24** For any measurable function $\phi$ on $V_{\mathbb{R}}$, we have

$$|J| \cdot \int_{p_{I,J} \in R^\pm} \int_{h \in G_{\mathbb{R}}} \phi(h \cdot p_{I,J})) \omega(h) dI dJ = \int_{G_{\mathbb{R}}} \phi(v)dv = 5 \int_{V_{\mathbb{R}}}^+ \phi(v)dv, \quad (16)$$

where $J$ is a nonzero rational constant and $p_{I,J}$ is the point in $R^\pm$ having invariants $I$ and $J$.

We now compute the volume of the multiset $\mathcal{F} \cdot R^\pm(X)$:

$$\int_{\mathcal{F} \cdot R^\pm(X)} dv = |J| \cdot \int_{p_{I,J} \in R^\pm(X)} \int_{\mathcal{F}} \omega(h) dI dJ = |J| \cdot \text{Vol}(\mathcal{F}) \int_{R^\pm(X)} dI dJ. \quad (17)$$

Up to an error of $O(X^{1/2})$, the quantity $\int_{R^\pm(X)} dI dJ$ is equal to $N^\pm(X)$ (see the proof of \cite[Proposition 2.10]{[8]} for details).

We conclude that

$$N(V_{\mathbb{Z}}^\pm; X) = \frac{1}{5} |J| \cdot \text{Vol}(G_{\mathbb{Z}} \setminus G_{\mathbb{R}})N^\pm(X) + o(X^{5/6}). \quad (18)$$

3.5 Congruence conditions and a squarefree sieve

In this subsection, we prove a version of Theorem 12 where we count strongly irreducible $G_{\mathbb{Z}}$-orbits on points $V_{\mathbb{Z}}$ that satisfy any specified finite set of congruence conditions.

For any set $S$ in $V_{\mathbb{Z}}$ that is definable by congruence conditions, denote by $\mu_p(S)$ the $p$-adic density of the $p$-adic closure of $S$ in $V_{\mathbb{Z}_p}$, where we normalize the additive measure $\mu_p$ on $V_{\mathbb{Z}_p}$ so that $\mu_p(V_{\mathbb{Z}_p}) = 1$. We then have the following theorem whose proof is identical to that of \cite[Theorem 2.11]{[8]}

**Theorem 25** Suppose $S$ is a subset of $V_{\mathbb{Z}}^\pm$ defined by finitely many congruence conditions. Then we have

$$N(S \cap V_{\mathbb{Z}}^\pm; X) = N(V_{\mathbb{Z}}^\pm; X) \prod_p \mu_p(S) + o(X^{5/6}), \quad (19)$$

where $\mu_p(S)$ denotes the $p$-adic density of $S$ in $V_{\mathbb{Z}}$, and where the implied constant in $o(X^{5/6})$ depends only on $S$.

We furthermore have the following weighted version of Theorem 25 whose proof is identical to that of \cite[Theorem 2.12]{[8]}

**Theorem 26** Let $p_1, \ldots, p_k$ be distinct prime numbers. For $j = 1, \ldots, k$, let $\phi_{p_j} : V_{\mathbb{Z}} \to \mathbb{R}$ be a $G_{\mathbb{Z}}$-invariant function on $V_{\mathbb{Z}}$ such that $\phi_{p_j}(v)$ depends only on the congruence class of $v$ modulo some power $p_j^{a_j}$ of $p_j$. Let $N_{\phi}(V_{\mathbb{Z}}^\pm; X)$ denote the number of irreducible $G_{\mathbb{Z}}$-orbits in $V_{\mathbb{Z}}^\pm$ having
height less than $X$, where each orbit $G_{\mathbb{Z}} \cdot v$ is counted with weight $\phi(v) := \prod_{j=1}^k \phi_{p_j}(v)$. Then we have

$$N_{\phi}(V_{\mathbb{Z}}^\pm; X) = N(V_{\mathbb{Z}}^\pm; X) \prod_{j=1}^k \int_{v \in V_{\mathbb{Z}_{p_j}}} \tilde{\phi}_{p_j}(v) \, dv + o(X^{5/6}),$$

(20)

where $\tilde{\phi}_{p_j}$ is the natural extension of $\phi_{p_j}$ to $V_{\mathbb{Z}_{p_j}}$ by continuity, $dv$ denotes the additive measure on $V_{\mathbb{Z}_{p_j}}$ normalized so that $\int_{v \in V_{\mathbb{Z}_{p_j}}} \, dv = 1$, and where the implied constant in the error term depends only on the local weight functions $\phi_{p_j}$.

For our applications, we also require a version of Theorem 26 which counts certain weighted $G_{\mathbb{Z}}$-orbits where the weights are defined by congruence conditions modulo infinitely many prime powers. To describe which weights are permissible, we have the following definitions.

A function $\phi : V_{\mathbb{Z}} \to [0,1] \subset \mathbb{R}$ is said to be defined by congruence conditions if, for all primes $p$, there exist functions $\phi_p : V_{p} \to [0,1]$ satisfying the following conditions:

(2) For all $v \in V_{\mathbb{Z}}$, the product $\prod_p \phi_p(v)$ converges to $\phi(v)$.

(3) For each prime $p$, the function $\phi_p$ is locally constant outside some closed set $S_p \subset V_{p}$ of measure zero.

Such a function $\phi$ is called acceptable if, for sufficiently large primes $p$, we have $\phi_p(v) = 1$ whenever $p^2 \nmid \Delta(v)$.

The key ingredient in proving the stronger version of Theorem 26 is the following uniformity/tail estimate:

**Theorem 27** For a prime $p$, let $W_p$ denote the set of elements in $V_{\mathbb{Z}}$ whose discriminants are divisible by $p^2$. Let $\epsilon > 0$ be fixed. Then we have:

$$N(\bigcup_{p > M} W_p, X) = O_\epsilon(X^{5/6} / (M \log M) + X^{49/60}) + O(\epsilon X^{5/6}),$$

(21)

where the implied constant is independent of $M$ and $X$.

**Proof:** Let $W_p^{(1)} \subset V_{\mathbb{Z}}$ be the $G_{\mathbb{Z}}$-invariant subset consisting of elements whose discriminants are strongly divisible by $p^2$, where an element $v$ is said to have discriminant strongly divisible by $p^2$ if for every $w \in V_{\mathbb{Z}}$, we have $p^2 \mid \Delta(v + pw)$. For $\epsilon > 0$, let $F^{(\epsilon)} \subset F$ denote the subset of elements $na(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8)k \in F$ such that the $s_i$ are bounded above by a constant to ensure that $\text{Vol}(F^{(\epsilon)}) = (1-\epsilon)\text{Vol}(F)$. Then $F^{(\epsilon)} \cdot R^\pm(X)$ is a bounded domain in $V_\mathbb{R}$ that expands homogeneously as $X$ grows. From [6] Theorem 3.3, we obtain

$$\#\{F^{(\epsilon)} \cdot R^\pm(X) \cap (\bigcup_{p > M} W_p^{(1)})\} = O(X^{5/6} / (M \log M) + X^{49/60}).$$

(22)

Also, the results of [3.1–3.3] imply that

$$\#\{(F \setminus F^{(\epsilon)}) \cdot R^\pm(X) \cap V_{\mathbb{Z}}^{\text{irr}}\} = O(\epsilon X^{5/6}).$$

(23)

Combining the estimates (22) and (23) yields (21) with $W_p$ replaced with $W_p^{(1)}$.

Therefore, it remains to prove (21) with $W_p$ replaced with $W_p^{(2)} := W_p \setminus W_p^{(1)}$. The set $W_p^{(2)}$ consists of the elements $v \in V_{\mathbb{Z}}$ having discriminant weakly divisible by $p^2$, i.e., $p^2$ divides $\Delta(v)$ but does not strongly divide $\Delta(v)$. Thus an element $v$ has discriminant weakly divisible by $p^2$ precisely
when $p^2 \mid \Delta(v)$ and the genus one curve over $\mathbb{F}_p$ corresponding to the reduction of $v$ modulo $p$ has a single nodal singularity.

Let $v = (A, B, C, D, E) \in \mathcal{W}_p^{(2)}$ be any such element, let $\bar{v} = (\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E}) \in V_{p^2}$ be its reduction modulo $p$, and let $C$ be the curve over $\mathbb{F}_p$ corresponding to $\bar{v}$. We may assume that the nodal singularity of $C$ is at $[1 : 0 : 0 : 0 : 0] \in \mathbb{P}^4_{\mathbb{F}_p}$, which implies that $A$ has rank 2. Therefore, by replacing $\bar{v}$ with a $G_{\mathbb{F}_p}$-translate if necessary, we may assume that $\bar{a}_{12}$ is the only nonzero coefficient of $A$. We next claim that we may replace $v$ with a $G_{\mathbb{F}_p}$-translate to ensure that $\bar{b}_{45} = \bar{c}_{45} = \bar{d}_{45} = \bar{e}_{45} = 0$. Indeed, since $C$ has a double point at $P = [1 : 0 : 0 : 0 : 0]$, the intersection of $C$ and the hyperplane section $t_1 = 0 \subset \mathbb{P}^4_{\mathbb{F}_p}$ contains $P$ with multiplicity at least 2.

As explained in [6], this implies that by replacing $\bar{v}$ with a $G_{\mathbb{F}_p}$-translate, if necessary, the $\mathbb{F}_p$-span of the four $3 \times 3$ matrices

\[
\begin{pmatrix}
0 & b_{34} & b_{35} \\
-b_{34} & 0 & b_{45} \\
-b_{35} & -b_{45} & 0
\end{pmatrix}, \begin{pmatrix}
0 & c_{34} & c_{35} \\
-c_{34} & 0 & c_{45} \\
-c_{35} & -c_{45} & 0
\end{pmatrix}, \begin{pmatrix}
0 & d_{34} & d_{35} \\
-d_{34} & 0 & d_{45} \\
-d_{35} & -d_{45} & 0
\end{pmatrix}, \begin{pmatrix}
0 & e_{34} & e_{35} \\
-e_{34} & 0 & e_{45} \\
-e_{35} & -e_{45} & 0
\end{pmatrix}
\]

has rank at most 2; thus, by again replacing $\bar{v}$ by a suitable $G_{\mathbb{F}_p}$-translate, we may assume that $\bar{b}_{45} = \bar{c}_{45} = \bar{d}_{45} = \bar{e}_{45} = 0$.

Let $Z \subset \text{Var}$ denote the set $\{a_{ij} : (i, j) \neq (1, 2)\} \cup \{b_{45}, c_{45}, d_{45}, e_{45}\}$. Given $v \in \mathcal{W}_p^{(2)}$, we have already proven that there exists $v'$ in the $G_Z$-orbit of $v$ such that $p \mid u(v')$ for every $u \in Z$. By evaluating the discriminant polynomial on such a $v'$, we conclude that if $p^2 \mid \Delta(v')$, then $p^2 \mid a_{45}(v')$.

Let $\gamma \in G_{\mathbb{Q}}$ be

\[
\gamma := \begin{pmatrix}
1 & 1 & p^{-1} \\
1 & p & p
\end{pmatrix}
\]

Then $\gamma \cdot v'$ is an element of $\mathcal{W}_p^{(1)}$, and it has the same discriminant as $v'$. We thus obtain a map $\phi : G_Z \backslash \mathcal{W}_p^{(2)} \rightarrow G_Z \backslash \mathcal{W}_p^{(1)}$ that is discriminant-preserving. We now have the following lemma:

**Lemma 28** The map $\phi$ is at most 2 to 1.

**Proof:** Consider a $G_Z$-orbit in the image of $\phi$ and an element $v \in \mathcal{W}_p^{(1)}$ in this orbit of the form $\gamma \cdot v'$ for some $v' \in \mathcal{W}_p^{(2)}$. Let $Q_1(t_1, t_2, t_3, t_4, t_5), \ldots, Q_5(t_1, t_2, t_3, t_4, t_5)$ be the five quadratic forms corresponding to $v$. It is easy to check that the action of $\gamma_p^{-1}$ acts on the quadratic forms as follows: the forms $Q_1$, $Q_2$, and $Q_3$ are multiplied by $p^2$, the forms $Q_4$ and $Q_5$ are multiplied by $p$, and the variables $t_2, \ldots, t_5$ are divided by $p$. Thus, for $\gamma_p^{-1} \cdot v$ to be integral, it is necessary and sufficient that the bottom right $4 \times 4$ submatrices of $Q_4$ and $Q_5$ be multiples of $p$.

Therefore, preimages of $G_Z \cdot v$ under $\phi$ correspond to 2-dimensional subspaces of quinary quadratic forms over $\mathbb{F}_p$ generated by the reductions modulo $p$ of $Q_1, \ldots, Q_5$, such that the quadratic forms in this subspace contain a common 4-dimensional isotropic hyperplane. Let $\mathcal{O}_1, \ldots, \mathcal{O}_5$ denote the reductions modulo $p$ of $Q_1, \ldots, Q_5$. Then we claim that no nonzero element outside the $\mathbb{F}_p$-span of $\mathcal{O}_4$ and $\mathcal{O}_5$ can have a 4-dimensional isotropic subspace. Indeed, if there was such an element, then we could assume without loss of generality that it was $\mathcal{O}_3$. However, then the action of $\gamma_p^{-1}$ would take $Q_3$ to $Q_3'$, whose reduction modulo $p$ would also contain a 4-dimensional isotropic subspace. This would force $\gamma_p^{-1} \cdot v \in \mathcal{W}_p^{(1)}$, contradicting our assumption that $\gamma_p^{-1} \cdot v \in \mathcal{W}_p^{(2)}$.

Thus, the lemma is true unless $\mathcal{O}_4$ and $\mathcal{O}_5$ possess more than two common isotropic 4-dimensional subspaces. This only happens when $\mathcal{O}_4 = \mathcal{O}_5 = 0$. In this case, the reduction modulo $p$ of the quadratic forms corresponding to $\gamma_p^{-1} \cdot v$ have $t_0^2$, $t_0t_1$, $t_0t_2$, $t_0t_3$, $t_0t_4$, and $t_0t_5$-coefficients equal to zero. This again forces $\gamma_p^{-1} \cdot v \in \mathcal{W}_p^{(1)}$, a contradiction. \$\square$
Therefore
\[ N\left(\cup_{p>M} W_p^{(2)}, X\right) \leq 2N\left(\cup_{p>M} W_p^{(1)}, X\right) = O_\epsilon\left(X^{5/6}/(M \log M) + X^{49/60}\right) + O(\epsilon X^{5/6}), \]
which concludes the proof of Theorem 27. \(\square\)

We thus obtain the following theorem:

**Theorem 29** Let \(\phi : V \to [0, 1]\) be an acceptable function that is defined by congruence conditions via the local functions \(\phi_p : V_p \to [0, 1]\). Then, with notation as in Theorem 26, we have:
\[ N_\phi(V^\pm; X) = N(V^\pm; X) \prod_p \int_{v \in V_p} \phi_p(v) dv + o(X^{5/6}). \]  
(24)

Theorem 29 follows from Theorem 27 just as [8, Theorem 2.21] followed from [8, Theorem 2.13].

### 3.6 The number of reducible points and points with large stabilizers in the main bodies of the fundamental domains is negligible

In this section we first prove Proposition 22, which states that the number of integral elements of bounded height that are not strongly irreducible in the main body of the fundamental domain is negligible. We then also prove, by similar methods, that the number of strongly irreducible \(G_\mathbb{Z}\)-orbits of elements of bounded height having a nontrivial stabilizer in \(G_\mathbb{Q}\) is negligible.

**Proof of Proposition 22**: Let \(v \in V\) have invariants \(I\) and \(J\), and let \(p > 5\) be a prime. If the \(G_\mathbb{Q}\)-orbit of \(v\) corresponds to the identity element in the 5-Selmer group of \(E^{I,J}\), then the \(G_{\mathbb{F}_p}\)-orbit of the reduction of \(v\) modulo \(p\) also corresponds to the identity element of \(E^{I,J}(\mathbb{F}_p)/5E^{I,J}(\mathbb{F}_p)\) under the correspondence of Theorem 17. Thus, if \(\bar{v} \in V_{\mathbb{F}_p}\) is an element having nonzero discriminant that corresponds to a nontrivial element in \(E^{I(v),J(v)}(\mathbb{F}_p)/5E^{I(v),J(v)}(\mathbb{F}_p)\), then every \(v \in V\) that reduces to \(\bar{v}\) modulo \(p\) is strongly irreducible. Denote the set of all such \(\bar{v} \in V_{\mathbb{F}_p}\) by \(V_{\mathbb{F}_p}^{\text{id}}\). We now show that \(#V_{\mathbb{F}_p}^{\text{id}} \gg \#V_{\mathbb{F}_p}/p\) where the implied constant is independent of \(p\). Indeed, by work of Deuring 23, there exists an elliptic curve \(E\) over \(\mathbb{F}_p\) such that \(#E(\mathbb{F}_p)\) is a multiple of 5. Thus, \(E(\mathbb{F}_p)/5E(\mathbb{F}_p)\) is nontrivial, and the nontrivial elements correspond to elements in \(V_{\mathbb{F}_p}^{\text{id}}\). Next, note that the set \(V_{\mathbb{F}_p}^{\text{id}}\) is closed under multiplication by nonzero elements of \(\mathbb{F}_p\) and under the action of \(G_{\mathbb{F}_p}\). Therefore, we have \(#V_{\mathbb{F}_p}^{\text{id}} \gg p\#G_{\mathbb{F}_p} \gg \#V_{\mathbb{F}_p}/p\). It follows that for any \(Y > 0\), we have

\[
\int_{\{v \in B^\pm(n,a;X) \cap V_{\mathbb{Z}}^{\text{red}} : a_{12}(v) \neq 0\}} dn da = O\left(X^{5/6} \prod_{p < Y} \left(1 - \frac{\#V_{\mathbb{F}_p}^{\text{id}}}{\#V_{\mathbb{F}_p}}\right) \right)
= O\left(X^{5/6} \prod_{p < Y} \left(1 - \frac{1}{p}\right)\right).
\]

(25)

The proposition now follows by letting \(Y\) tend to infinity. \(\square\)

**Lemma 30** Let \(V_{\mathbb{Z}}^{\text{bigstab}} \subset V\) be the set of elements that are strongly irreducible and have a nontrivial stabilizer in \(G_\mathbb{Q}\). Then we have
\[ N(V_{\mathbb{Z}}^{\text{bigstab}}; X) = o(X^{5/6}). \]
Proof: First, we note that by Proposition 15, it suffices to prove the estimate (13) with \( V^\text{red}_Z \) replaced by \( V^\text{bigstab}_Z \).

If \( v \in V_Z \) has a nontrivial stabilizer in \( G_Q \), then we see from Theorem 7 that \( E^{(v,J(v))(Q)}[5] \) must be nontrivial. If furthermore \( E^{(v,J(v))} \) has good reduction at \( p > 5 \), then it follows by [40, §VII, Proposition 3.1] that \( E^{(v,J(v))(F_p)}[5] \) must also be nontrivial. Therefore, if \( \tilde{v} \in V_{\overline{F}_p} \) is an element having nonzero discriminant such that \( E^{(\tilde{v},J(\tilde{v}))(F_p)}[5] \) is trivial, then any strongly irreducible \( v \in V_Z \) that reduces to \( \tilde{v} \) modulo \( p \) must have trivial stabilizer in \( G_Q \). Denote the set of all such \( \tilde{v} \in V_{\overline{F}_p} \) by \( V_{\overline{F}_p}^\text{smallstab} \).

The set \( V_{\overline{F}_p}^\text{smallstab} \) is nonempty because there exists an elliptic curve \( E \) over \( F_p \) such that \( \#E(F_p) \) is prime to 5, again by [23]: the identity element of \( E(F_p)/5E(F_p) \) then corresponds to an element in \( V_{\overline{F}_p}^\text{smallstab} \). The rest of the proof now proceeds identically to that of Proposition 22. \( \square \)

4 The average number of elements in the 5-Selmer groups of elliptic curves

Let \( E \) be an elliptic curve over \( \mathbb{Q} \), and define the invariants \( I(E) \) and \( J(E) \) of \( E \) as in (1). Throughout this section, we work with the slightly different height \( H' \) on elliptic curves \( E \), defined by

\[
H'(E) := \max(|I(E)|^3, J(E)^2/4),
\]

so that the height on elliptic curves agrees with the height on \( V_{\mathbb{Z}} \) defined in (3). Note that since the heights \( H \) and \( H' \) on elliptic curves differ only by a constant factor, they induce the same ordering on the set of all (isomorphism classes of) elliptic curves over \( \mathbb{Q} \).

In this section, we prove Theorem (1) by averaging the size of the 5-Selmer group of all elliptic curves over \( \mathbb{Q} \), when these curves are ordered by height. In fact, we prove a generalization of these theorems that allows us to impose certain infinite sets of congruence conditions on the defining equations of the elliptic curves. To state this more general theorem, we need the following definitions.

For each prime \( p \), let \( \Sigma_p \) be a closed subset of \( \mathbb{Z}_p^2 \setminus \{\Delta = 0\} \) whose boundary has measure 0. To this collection \( (\Sigma_p)_p \), we associate the family \( F_\Sigma \) of elliptic curves, such that \( E^{I,J} \in F_\Sigma \) whenever \( (I,J) \in \Sigma_p \) for all \( p \). Such a family of elliptic curves over \( \mathbb{Q} \) is said to be defined by congruence conditions. We may also impose “congruence conditions at infinity” on \( F_\Sigma \) by insisting that an elliptic curve \( E^{I,J} \) belongs to \( F_\Sigma \) if and only if \( (I,J) \) belongs to \( \Sigma_\infty \), where \( \Sigma_\infty \) is equal to \( \{(I,J) \in \mathbb{R}^2 : \Delta(I,J) > 0\}, \{(I,J) \in \mathbb{R}^2 : \Delta(I,J) < 0\}, \) or \( \{(I,J) \in \mathbb{R}^2 : \Delta(I,J) \neq 0\} \).

For a family \( F \) of elliptic curves defined by congruence conditions, let \( \text{Inv}(F) \) denote the set \( \{(I,J) \in \mathbb{Z} \times \mathbb{Z} : E^{I,J} \in F\} \), and \( \text{Inv}_p(F) \) the \( p \)-adic closure of \( \text{Inv}(F) \) in \( \mathbb{Z}_p^2 \setminus \{\Delta = 0\} \). We define \( \text{Inv}_\infty(F) \) to be \( \{(I,J) \in \mathbb{R}^2 : \Delta(I,J) > 0\}, \{(I,J) \in \mathbb{R}^2 : \Delta(I,J) < 0\}, \) or \( \{(I,J) \in \mathbb{R}^2 : \Delta(I,J) \neq 0\} \) in accordance with whether \( F \) contains only curves of positive discriminant, negative discriminant, or both, respectively. Such a family \( F \) of elliptic curves is said to be large if, for all but finitely many primes \( p \), the set \( \text{Inv}_p(F) \) contains at least those pairs \( (I,J) \in \mathbb{Z}_p \times \mathbb{Z}_p \) such that \( p^2 \nmid \Delta(I,J) \). Our purpose in this section is to prove the following theorem, which extends Theorem (1) to more general congruence families of elliptic curves:

**Theorem 31** Let \( F \) be a large family of elliptic curves. When elliptic curves \( E \) in \( F \) are ordered by height, the average size of the 5-Selmer group \( S_5(E) \) is equal to 6.
4.1 Assigning weights to elements in $V_T$, and a local mass computation

Let $F$ be a fixed large family of elliptic curves. Recall that for an elliptic curve $E^I,J$, Proposition\[10\] asserts that nontrivial elements in $S_5(E^I,J)$ are in bijection with $G_Q$-equivalence classes on the set of locally soluble and strongly irreducible elements in $V_T$ having invariants $I$ and $J$. In order to use the counting results of Section 3 to prove Theorem\[31\] we need to define an appropriate weight function on $V_T$.

For $v \in V_T$, (resp. $v \in V_{T_p}$), let $B(v)$ (resp. $B_p(v)$) denote a set of representatives for the action of $G_T$ (resp. $G_{T_p}$) on the $G_Q$-equivalence class of $v$ in $V_T$ (resp. the $G_{T_p}$-equivalence class of $v$ in $V_{T_p}$). We define our weight function $\phi$ via:

$$
\phi(v) := \begin{cases} 
\left( \sum_{v' \in B(v)} \frac{\# \text{Aut}_Q(v')}{\# \text{Aut}_Z(v')} \right)^{-1} & \text{if } v \text{ is locally soluble and } (I(v), J(v)) \in \text{Inv}_p(F) \text{ for all } p; \\
0 & \text{otherwise},
\end{cases}
$$

(27)

where $\text{Aut}_Q(v)$ and $\text{Aut}_Z(v)$ denote the stabilizers of $v \in V_T$ in $G_Q$ and in $G_T$, respectively. Since Lemma\[30\] states that $\text{Aut}_Q(v)$ is trivial for all but a negligible set of elements $v \in V_T$, the function $\phi$ also satisfies the following three conditions at all but a negligible set of $v$:

1. If $v \in V_T$ is not locally soluble, then $\phi(v)$ is zero.
2. If $(I(v), J(v))$ is not in $\text{Inv}(F)$, then $\phi(v)$ is zero.
3. Otherwise, $\phi(v)$ is the reciprocal of the number of $G_T$-orbits in the $G_Q$-equivalence class of $v$ in $V_T$.

For the application of Theorem\[26\] to counting $G_T$-orbits on $V_T$ weighted by $\phi$, we need to define the following local weight functions $\phi_p : V_{T_p} \to \mathbb{R}_{\geq 0}$:

$$
\phi_p(v) := \begin{cases} 
\left( \sum_{v' \in B_p(v)} \frac{\# \text{Aut}_Q(v')}{\# \text{Aut}_{Z_p}(v')} \right)^{-1} & \text{if } v \text{ is } Q_p\text{-soluble and } (I(v), J(v)) \in \text{Inv}_p(F); \\
0 & \text{otherwise},
\end{cases}
$$

(28)

where $\text{Aut}_Q(v)$ and $\text{Aut}_{Z_p}(v)$ denote the stabilizer of $v \in V_{T_p}$ in $G_{Q_p}$ and $G_{Z_p}$, respectively. We then have the following proposition.

**Proposition 32** If $v \in V_T$ has nonzero discriminant, then $\phi(v) = \prod_p \phi_p(v)$.

Noting the fact that the group $G_Q$ has class number one, the proof of the above proposition is identical to that of [8] Proposition 3.6.

We end the subsection with a proposition that evaluates $\int_{V_{T_p}} \phi_p(v)dv$.

**Proposition 33** We have

$$
\int_{v \in V_{T_p}} \phi_p(v)dv = |J|_p \cdot \text{Vol}(G_{Z_p}) \cdot \int_{(I,J) \in \text{Inv}_p(F)} \frac{\#(E^I,J(Q_p)/5E^I,J(Q_p))}{\#(E^I,J(Q_p)[5])}dv
$$

$$
= \begin{cases} 
|J|_p \cdot \text{Vol}(G_{Z_p}) \cdot \text{Vol}(\text{Inv}_p(F)) & \text{if } p \neq 5; \\
5 \cdot |J|_p \cdot \text{Vol}(G_{Z_p}) \cdot \text{Vol}(\text{Inv}_p(F)) & \text{if } p = 5.
\end{cases}
$$

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The first equality in Proposition 33 follows from an argument identical to Proposition 3.9. The second follows from an argument identical to the proof of [14, Lemma 3.1], which shows that 

\[ \#(E^I,J(Q_p)/5E^I,J(Q_p)) = \#(E^I,J(Q_p)[5]) \]

when \( p \neq 5 \) and equal to \( 5\#(E^I,J(Q_p)[5]) \) when \( p = 5 \).

4.2 The proof of Theorem 31

Let \( F \) be a large family of elliptic curves. We start with the following proposition that is proven in [8, Theorem 3.17].

**Proposition 34** Let \( N(F^\pm, X) \) denote the number of elliptic curves in \( F \) such that \( H'(E) \leq X \) and \( \Delta(E) \in \mathbb{R}^\pm \). Then

\[ N(F^\pm; X) = N^\pm(X) \prod_p \text{Vol}(\text{Inv}_p(F)) + o(X^{5/6}). \]

The results of §2 and §4.1 imply that we have

\[ \sum_{E \in F^\pm, H'(E) < X} (#S_5(E) - 1) = N_\phi(V^\pm_Z, X) + o(X^{5/6}). \] (29)

By Propositions [11] and [32] it follows that \( \phi \) is acceptable. Therefore, we may use Theorem 29 and Proposition 33 to estimate \( N_\phi(V^\pm_Z, X) \), obtaining

\[
\lim_{X \to \infty} \frac{\sum_{E \in F^\pm, H'(E) < X} (#S_5(E) - 1)}{\sum_{E \in F^\pm, H'(E) < X} 1} = \lim_{X \to \infty} \frac{\frac{1}{5}|\mathcal{J}| \text{Vol}(G_Z \backslash G_{\mathbb{A}}) N^\pm(X) \cdot 5 \prod_p |\mathcal{J}_p| \text{Vol}(G_{Z_p}) \text{Vol}(\text{Inv}_p(F))}{N^\pm(X) \cdot \prod_p \text{Vol}(\text{Inv}_p(F))} = \tau(G),
\]

where \( \tau(G) = 5 \) denotes the Tamagawa number of \( G \). We have proven Theorem 31 and hence also Theorems [1] and [2].

5 Families of elliptic curves with equidistributed root number

In this section, our aim is to construct a union \( F \) of large families of elliptic curves in which exactly 50% have root number 1, and where the density of \( F \) among all elliptic curves is large (indeed, > 55%).

Recall that the root number \( r(E) \) of an elliptic curve \( E \) can be expressed as a local product \( r(E) = -\prod_p r_p(E) \), where \( r_p(E) \) is the local root number of \( E \) at \( p \). Local root numbers of elliptic curves over \( \mathbb{Q} \) were computed in [31] and [39], and these computations will be key in our constructions. The local root number \( r_p(E) \) of an elliptic curve \( E \) having multiplicative reduction
at \( p \) is 1 or \(-1\) depending on whether the reduction of \( E \) at \( p \) is split or non-split, respectively. When applying sieve methods, it can often be difficult to distinguish between these two cases. For example, it is not known whether the sum \( \sum \mu(-4A^3 - 27B^2) \) of Möbius function values over all pairs \((A,B)\) having height less than \( X \) is \( o(X^{5/6}) \), which has been the traditional approach to this type of problem\[3\]. To circumvent this issue, we take the indirect approach of working with \( d(E) = \prod_{p} d_{p}(E) \) instead of \( r(E) \), where

\[
d(E) := r(E)r(E_{-1}),
\]
\[
d_{p}(E) := r_{p}(E)r_{p}(E_{-1}).
\]

Here \( E_{-1} \) denotes the quadratic twist of \( E \) by \(-1\).

In the rest of this section, we construct a finite union of large families \( F \) of elliptic curves such that every curve \( E \in F \) satisfies \( E_{-1} \in F \) and \( d(E) = -1 \). Since the height of \( E \) is equal to the height of \( E_{-1} \), it follows that exactly 50% of elliptic curves in \( F \) have root number 1. The definitions of \( d(E) \) and \( d_{p}(E) \) imply immediately that \( d(E) = \prod_{p} d_{p}(E) \). Denoting \( d_{2}(E)d_{3}(E) \) and \( \prod_{p>3} d_{p}(E) \) by \( d_{6}(E) \) and \( d_{1/6}(E) \), respectively, it follows that \( d(E) = d_{6}(E)d_{1/6}(E) \).

We use \( \Delta_{p}(E) \) and \( \Delta_{p'}(E) \) to denote \( p^{r_{p}E(\Delta(E))} \) and \( \Delta(E)/\Delta_{p}(E) \), respectively. We further denote \( \Delta(E)/(\Delta_{2}(E) \cdot \Delta_{3}(E)) \) by \( \Delta_{0}(E) \). In the next proposition, we construct two families of elliptic curves \( E \), defined by finitely many congruence conditions modulo powers of 2 and 3, in which we control \( d_{6}(E) \) in terms of \( \Delta_{0}(E) \).

**Proposition 35** There exist two families \( F_{1} \) and \( F_{2} \) of elliptic curves \( E_{A,B} \), defined by finitely many congruence conditions on \( A \) and \( B \) modulo powers of 2 and 3, such that:

1. \( d_{6}(E) \equiv \Delta_{0}(E) \pmod{4} \) for \( E \in F_{1} \);
2. \( d_{6}(E) \not\equiv \Delta_{0}(E) \pmod{4} \) for \( E \in F_{2} \);
3. The density of \( F_{1} \) is greater than 59.179%;
4. The density of \( F_{2} \) is greater than 40.32%;
5. \( F_{1} \) and \( F_{2} \) are closed under twisting by \(-1\).

**Proof:** Using the local root number computations at the prime 2 in \[31\] Table 1, we construct families of elliptic curves \( E \) in Table 3 with prescribed values of \( d_{2}(E) \) and \( \Delta_{2}(E) \pmod{4} \). Similarly, we use the local root number computations at the prime 3 in \[31\] Table 2 to construct families of elliptic curves \( E \) in Table 4 with prescribed values of \( d_{3}(E) \) and \( \Delta_{3}(E) \pmod{4} \). We also compute the densities of these families. Tables 3 and 4 are to be read as follows: each row except the last corresponds to a family of elliptic curves of the form \( y^{2} = x^{3} + Ax + B \) defined by congruence conditions modulo powers of 2 and 3, respectively. These families are disjoint. The first three columns describe the family by specifying the condition that \( A \) and \( B \) must satisfy. The fourth column gives the density of the family. The final four columns give the relative density of elliptic curves \( E \) within the family with prescribed values of \( \Delta(E) \pmod{4} \) and \( d_{p}(E) \). In the final row we simply sum the densities over all the other rows. For example, the first row of Table 3 corresponds to the family of elliptic curves satisfying \( 2 \nmid A \) and \( 2 \nmid B \). This family has density 1/4 among all integer pairs \((A,B)\). The prime to 2 part of the discriminant of such elliptic curves is

---

\[1\] There has been progress on obtaining equidistribution of root numbers of elliptic curves in one-parameter families; see Helfgott \[33\].
Table 3: Densities of elliptic curves having prescribed values of $\Delta_2'$ (mod 4) and $d_2$

| $v_2(A)$ | $v_2(B)$ | Additional Condition | Density | Relative density with given $(\Delta_2' \text{ (mod 4)}, d_2)$ |
|----------|----------|----------------------|---------|--------------------------------------------------------|
| 0        | 0        | -                    | $2^{-2}$ | (1, 1) $\frac{3}{4}$, (1, −1) $\frac{1}{4}$, (3, 1) 0, (3, −1) 0 |
| ≥ 1      | 0        | -                    | $2^{-2}$ | (1, 1) $\frac{3}{4}$, (1, −1) $\frac{1}{4}$, (3, 1) 0, (3, −1) 0 |
| ≥ 1      | 1        | -                    | $2^{-3}$ | (1, 1) $\frac{1}{3}$, (1, −1) $\frac{2}{3}$, (3, 1) 0, (3, −1) 0 |
| 0        | ≥ 2      | -                    | $2^{-3}$ | (1, 1) $\frac{1}{3}$, (1, −1) $\frac{2}{3}$, (3, 1) 0, (3, −1) 0 |
| 1        | 2        | -                    | $2^{-5}$ | (1, 1) $\frac{1}{2}$, (1, −1) $\frac{1}{2}$, (3, 1) 0, (3, −1) 0 |
| 2        | 2        | -                    | $2^{-6}$ | (1, 1) $\frac{1}{2}$, (1, −1) $\frac{1}{2}$, (3, 1) 0, (3, −1) 0 |
| ≥ 3      | 2        | -                    | $2^{-6}$ | (1, 1) 1, (1, −1) 0, (3, 1) 0, (3, −1) 0 |
| ≥ 1      | 3        | -                    | $2^{-5}$ | (1, 1) 0, (1, −1) $\frac{1}{2}$, (3, 1) 0, (3, −1) $\frac{1}{2}$ |
| 1        | ≥ 4      | -                    | $2^{-6}$ | (1, 1) $\frac{1}{2}$, (1, −1) 0, (3, 1) $\frac{1}{2}$, (3, −1) 0 |
| 3        | ≥ 4      | -                    | $2^{-8}$ | (1, 1) $\frac{1}{16}$, (1, −1) $\frac{9}{16}$, (3, 1) $\frac{1}{16}$, (3, −1) $\frac{1}{16}$ |
| ≥ 4      | 4        | -                    | $2^{-9}$ | (1, 1) 0, (1, −1) 1, (3, 1) 0, (3, −1) 0 |
| 2        | ≥ 5      | -                    | $2^{-8}$ | (1, 1) $\frac{1}{2}$, (1, −1) $\frac{1}{2}$, (3, 1) $\frac{1}{2}$, (3, −1) $\frac{1}{2}$ |
| ≥ 4      | 5        | -                    | $2^{-10}$ | (1, 1) 0, (1, −1) 1, (3, 1) 0, (3, −1) 0 |
| 0        | ≥ 1      | $v_2(\Delta) = 7$    | $2^{-4}$ | (1, 1) $\frac{1}{2}$, (1, −1) 0, (3, 1) 0, (3, −1) $\frac{1}{2}$ |
| 0        | ≥ 1      | $v_2(\Delta) = 8$    | $2^{-5}$ | (1, 1) $\frac{1}{2}$, (1, −1) $\frac{1}{2}$, (3, 1) 0, (3, −1) $\frac{1}{2}$ |
| 0        | ≥ 1      | $v_2(\Delta) = 9$    | $2^{-6}$ | (1, 1) $\frac{1}{2}$, (1, −1) 0, (3, 1) 0, (3, −1) $\frac{1}{2}$ |
| 0        | ≥ 1      | $v_2(\Delta) = 10$   | $2^{-7}$ | (1, 1) $\frac{1}{2}$, (1, −1) $\frac{1}{2}$, (3, 1) 0, (3, −1) $\frac{1}{2}$ |
| 0        | ≥ 1      | $v_2(\Delta) = 11$   | $2^{-8}$ | (1, 1) $\frac{1}{2}$, (1, −1) $\frac{1}{2}$, (3, 1) $\frac{1}{2}$, (3, −1) $\frac{1}{2}$ |
| 2        | 4        | $v_2(\Delta) = 13$   | $2^{-9}$ | (1, 1) $\frac{1}{2}$, (1, −1) 0, (3, 1) 0, (3, −1) $\frac{1}{2}$ |
| 2        | 4        | $v_2(\Delta) = 14$   | $2^{-10}$ | (1, 1) 0, (1, −1) $\frac{1}{2}$, (3, 1) $\frac{1}{2}$, (3, −1) $\frac{1}{2}$ |
| 2        | 4        | $v_2(\Delta) = 15$   | $2^{-11}$ | (1, 1) $\frac{1}{2}$, (1, −1) 0, (3, 1) $\frac{1}{2}$, (3, −1) 0 |

Total: $\geq 0.9946$, $\geq 0.5703$, $\geq 0.2814$, $\geq 0.0903$, $\geq 0.0524$

always 1 modulo 4 and exactly three quarters of such elliptic curves satisfy $d_2(E) = 1$. Hence the final four entries of the first row are $3/4$, $1/4$, $0$, and $0$. All the rows (apart from the last one) in both tables can be read similarly.

For $(i, j) \in \{(1, 1), (1, -1), (3, 1), (3, -1)\}$, let $G_2(i, j)$ denote the family of elliptic curves $E$ in Table 3 with $(\Delta_2'(E) \text{ (mod 4)}, d_2(E)) = (i, j)$, and let $G_3(i, j)$ denote the family in Table 4 with $(\Delta_3(E) \text{ (mod 4)}, d_3(E)) = (i, j)$. The families $G_p(i, j)$ are defined by finitely many congruence conditions modulo powers of $p$, and their densities are listed in the final row of Tables 3 and 4.

We now define the families $F_1$ and $F_2$ to be

$$F_1 := \bigcup_{i, j, k, \ell \equiv 1 \text{ (mod 4)}} (G_2(i, j) \cap G_3(k, \ell)), \quad F_2 := \bigcup_{i, j, k, \ell \equiv 3 \text{ (mod 4)}} (G_2(i, j) \cap G_3(k, \ell)). \quad \text{(30)}$$
| $v_3(A)$ | $v_3(B)$ | Additional Condition | Density | Relative density with given $(\Delta_3 \mod 4), d_3)$ |
|-------|-------|---------------------|--------|------------------------------------------|
|       |       |                     | $\frac{3^2}{3}$ | (1, 1) | (1, −1) | (3, 1) | (3, −1) |
| 0     | ≥ 0   | -                   | $\frac{1}{3}$ | 1     | 0      | 0      | 0      |
| ≥ 2   | 0     | -                   | $\frac{2}{3}$ | 0     | 0      | $\frac{1}{3}$ | $\frac{2}{3}$ |
| 1     | 1     | -                   | $\frac{4}{3}$ | 0     | 0      | 0      | 1      |
| ≥ 2   | 1     | -                   | $\frac{2}{3}$ | 0     | 0      | 0      | 1      |
| 1     | ≥ 2   | -                   | $\frac{2}{3}$ | 0     | 0      | 1      | 0      |
| 2     | 2     | -                   | $\frac{4}{3}$ | 1     | 0      | 0      | 0      |
| ≥ 3   | 2     | -                   | $\frac{2}{3}$ | 0     | 0      | 0      | 1      |
| 2     | ≥ 3   | -                   | $\frac{2}{3}$ | 0     | 1      | 0      | 0      |
| ≥ 4   | 3     | -                   | $\frac{2}{3}$ | 0     | 0      | $\frac{1}{3}$ | $\frac{2}{3}$ |
| 3     | 4     | -                   | $\frac{2}{3}$ | 0     | 0      | 0      | 1      |
| ≥ 4   | 4     | -                   | $\frac{2}{3}$ | 0     | 0      | 0      | 1      |
| 3     | ≥ 5   | -                   | $\frac{2}{3}$ | 0     | 0      | 1      | 0      |
| 4     | 5     | -                   | $\frac{4}{3}$ | 1     | 0      | 0      | 0      |
| ≥ 5   | 5     | -                   | $\frac{2}{3}$ | 0     | 0      | 0      | 1      |
| 1     | 0     | $v_3(\Delta) = 3$   | $\frac{2}{3}$ | 0     | 0      | $\frac{5}{6}$ | $\frac{1}{3}$ |
| 1     | 0     | $v_3(\Delta) = 4$   | $\frac{4}{3}$ | 1     | 0      | 0      | 0      |
| 1     | 0     | $v_3(\Delta) = 5$   | $\frac{1}{3}$ | 0     | 0      | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 1     | 0     | $v_3(\Delta) = 2n, n \geq 3$ | $\frac{1}{3}$ | 1     | 0      | 0      | 0      |
| 1     | 0     | $v_3(\Delta) = 2n + 1, n \geq 3$ | $\frac{1}{3}$ | 0     | 0      | 1      | 0      |

Table 4: Densities of elliptic curves having prescribed values of $\Delta_3 \mod 4$ and $d_3$.

Since the first two conditions of the proposition are invariant under twisting by $-1$, the final condition is easily satisfied by replacing (if necessary) $F_1$ and $F_2$ by $F_1 \cup \{E : E_{-1} \in F_1\}$ and $F_2 \cup \{E : E_{-1} \in F_2\}$, respectively. This concludes the proof of the proposition. □

We next prove a result analogous to Proposition 35, but where we now instead control the value of $d_{1/6}$ in terms of the absolute value of $\Delta_6$. To this end, we have the following lemma.

Lemma 36 If $p > 3$ is prime, then $d_p(E) = -1$ if and only if $E$ has multiplicative reduction at $p$ and $p \equiv 3 \pmod{4}$.

Proof: Since $p$ is odd, we know that the reduction type (good, multiplicative, or additive) of $E$ is the same as the reduction type of $E_{-1}$. It follows from [39, Propositions 2 and 3] that if $E$ has good or additive reduction at $p$, then $r_p(E) = r_p(E_{-1})$.

Assume that $E$ has multiplicative reduction at $p$. Then, from [39, Proposition 3], we see that $r_p(E) = 1$ if and only if the reduction of $E$ at $p$ is split. Thus $d_p(E) = 1$ exactly when both $E$ and $E_{-1}$ have split reduction or when both $E$ and $E_{-1}$ have nonsplit reduction at $p$. It can be checked that this happens precisely when $p \equiv 1 \pmod{4}$. The lemma follows. □
Proposition 37 There exist two finite unions of large families $F_3$ and $F_4$ of elliptic curves $E_{A,B}$, defined by congruence conditions on $A$ and $B$ modulo primes greater than 3, such that:

1. $d_{1/6}(E) \equiv |\Delta_{6'}(E)| \pmod{4}$ for $E \in F_3$;
2. $d_{1/6}(E) \not\equiv |\Delta_{6'}(E)| \pmod{4}$ for $E \in F_4$;
3. The density of $F_3$ is greater than 96.689%;
4. The density of $F_4$ is greater than 3.26%;
5. $F_3$ and $F_4$ are closed under twisting by $-1$.

Proof: For an elliptic curve $E = E_{A,B}$ and a prime $p$ greater than 3, define $\alpha_p(E)$ by

$$\alpha_p(E) = \alpha_p(A,B) := d_p(E_{A,B}) \cdot (-1)^{v_p(\Delta(A,B))} \text{ if } p \equiv 3 \pmod{4},$$
$$\alpha_p(E) = \alpha_p(A,B) := 1 \text{ otherwise.}$$

If $p > 3$ is a prime congruent to 3 modulo 4, then $\alpha_p(E)$ is 1 if and only if $E$ has good reduction at $p$, or $E$ has multiplicative reduction at $p$ and $v_p(\Delta(E))$ is odd, or $E$ has additive reduction at $p$ and $v_p(\Delta(E))$ is even. We control the quantity $\alpha_p(A,B)$ by imposing one of the following two conditions:

1. $E$ has good reduction at $p$, or $E$ has multiplicative reduction at $p$ and $v_p(\Delta(E)) \in \{1, 3\}$, or $E$ has additive reduction at $p$ and $v_p(\Delta(E)) \in \{2, 4\}$. In either case, we have $\alpha_p(E) = 1$.
2. $E$ has multiplicative reduction at $p$ and $v_p(\Delta(E)) = 2$ or $E$ has additive reduction at $p$ and $v_p(\Delta(E)) = 3$. In both cases, we have $\alpha_p(E) = -1$.

Define $F_3$ to be the set of elliptic curves $E$ such that $E$ satisfies the first of the above two conditions at all primes congruent to 3 modulo 4, or $E$ satisfies the first of the above two conditions at all but two primes congruent to 3 modulo 4 and satisfies the second condition at these two primes. To ensure that we are constructing a finite union of large families, we further assume that these two primes are smaller than 10000. Similarly, define $F_4$ to be the set of elliptic curves that satisfy the first condition at all but one prime (which is bounded by 10000) and satisfies the second condition at this one prime. These two sets are both clearly finite unions of large families. Furthermore, since Lemma 36 implies that $\prod_{p \geq 3} \alpha_p(E) \equiv d_{1/6}(E) \cdot |\Delta_{6'}(E)| \pmod{4}$, we have $d_{1/6}(E) \equiv |\Delta_{6'}(E)| \pmod{4}$ if $E \in F_3$ and $d_{1/6}(E) \not\equiv |\Delta_{6'}(E)| \pmod{4}$ if $E \in F_4$.

It is easy to compute the densities of $F_3$ and $F_4$. Assume that $p$ is a prime greater than 3. Then it follows from an elementary computation that a density of $1 - \frac{1}{p}$ of elliptic curves have good reduction at $p$, a density of $\frac{1}{p} - \frac{1}{p^2}$ have multiplicative reduction at $p$, and a density of $\frac{1}{p^2}$ have additive reduction at $p$.

Suppose an elliptic curve $E$ has additive reduction at $p$. Then $E$ is given by an equation $y^2 = x^3 + ax + b$, where $p$ divides both $a$ and $b$. This ensures that $p^2 \mid \Delta(E) = 4a^3 - 27b^2$. Clearly $p^3 \mid \Delta(E)$ if and only if $p^2 \mid b$. Hence, of elliptic curves $E$ having additive reduction at $p$, a density of $\frac{p-1}{p}$ satisfy $v_p(\Delta(E)) = 2$. Now if $E : y^2 = x^3 + ax + b$ has additive reduction at $p$ with $p^2 \mid b$ (so that $p^3 \mid \Delta(E)$), then $p^4 \mid \Delta(E)$ if and only if $p^2 \mid a$. Similarly, if we assume that $p^2$ divides both $a$ and $b$, then $p^5 \mid \Delta(E)$ if and only if $p^3 \mid b$. Thus, among elliptic curves $E$ having additive reduction at $p$, a density of $\frac{p-1}{p^2}$ satisfy $v_p(\Delta(E)) = 3$ and a density of $\frac{p-1}{p^3}$ satisfy $v_p(\Delta(E)) = 4$. 

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Finally, suppose that an elliptic curve $E$ has multiplicative reduction at $p$. Then we may assume that $E$ is given by an equation $y^2 = x^3 + cx^2 + ax + b$, where $p$ divides both $a$ and $b$. This ensures that $p | \Delta(E)$. As above, we may verify that among elliptic curves $E$ having multiplicative reduction at $p$, a density of $\frac{2-1}{p^2}$, $\frac{5-1}{p^2}$, and $\frac{2-1}{p^2}$ satisfy $v_p(\Delta(E)) = 1$, $v_p(\Delta(E)) = 2$, and $v_p(\Delta(E)) = 3$, respectively. From this, it is easy to compute the density of elliptic curves having prescribed $3, \alpha$. We may thus compute the densities of elliptic curves having prescribed values of $\alpha_p$. We may thus compute the densities of $F_3$ and $F_4$ using Proposition 33 and verify that they are as claimed by the proposition. Finally, as before, we may replace $E$ by Theorem 31, the limsup $\bar{r}_5$ of the average 5-Selmer rank of elliptic curves, when ordered by height, satisfies $20\bar{r}_5 \leq 5$, while the sets $\bar{r}_5 = 5\%$ have 5-Selmer rank 2. The following remarkable result of Dokchitser and Dokchitser [24] asserts that the parity of the 5-Selmer rank of an elliptic curve is determined by its root number:

\begin{align}
F^+ &:= ((F_1 \cap F_4) \cup (F_2 \cap F_3)) \cap \{E : \Delta(E) > 0\}, \\
F^- &:= ((F_1 \cap F_3) \cup (F_2 \cap F_4)) \cap \{E : \Delta(E) < 0\}.
\end{align}

These sets $F^+$ and $F^-$ are also finite unions of large families and every elliptic curve $E$ in either of them satisfies $d(E) = 1$ by construction. We may compute their densities by Propositions 34, 35, and 37 to be at least 40.914% and 58.534%, respectively. This yields Theorem 6.

6 The average rank of elliptic curves is less than 1

In this section we prove Theorems 3, 4, and 5. First note that using only Theorem 31 we obtain the following result:

**Proposition 38** Let $F$ be a large family of elliptic curves. Then, when elements in $F$ are ordered by height, we have:

(a) The average 5-Selmer rank of elliptic curves in $F$ is bounded by 1.05.

(b) The set of elliptic curves in $F$ with 5-Selmer rank 0 or 1 has density at least 19/24.

**Proof:** As in the introduction, we note that $20r - 15 \leq 5r$ for nonnegative integers $r$. Therefore, by Theorem 31 the limsup $\bar{r}_5$ of the average 5-Selmer rank of elliptic curves, when ordered by height, satisfies $20\bar{r}_5 - 15 \leq 6$, proving (a). This bound is achieved when 95% of elliptic curves have 5-Selmer rank 1 and 5% have 5-Selmer rank 2.

Let $x_{0,0,1}$ be the lower density of elliptic curves in $F$ having 5-Selmer rank 0 or 1. Then, from Theorem 31 we obtain

$$x_{0,0,1} + 25(1 - x_{0,0,1}) \leq 6,$$

and hence $x_{0,0,1} \geq 19/24$, proving (b). The bound is achieved when a proportion of 19/24 of elliptic curves in $F$ have 5-Selmer rank 0, and a proportion of 5/24 have 5-Selmer rank 2. \(\square\)

We now improve Proposition 38 in the case of large families $F$ having equidistributed root number. Recall that the analytic rank of an elliptic curve $E$ is defined to be the order of vanishing at 1/2 of its $L$-function $L(E, s)$. The evenness or oddness of the analytic rank of $E$ is determined by whether the sign of the functional equation of $L(E, s)$—the root number $r(E)$ of $E$—is 1 or $-1$, respectively.

The following remarkable result of Dokchitser and Dokchitser [24] asserts that the parity of the $p$-Selmer rank of an elliptic curve is determined by its root number:
Theorem 39 (Dokchitser–Dokchitser [24]) Let \( E \) be an elliptic curve over \( \mathbb{Q} \) and let \( p \) be any prime. Let \( s_p(E) \) and \( t_p(E) \) denote the rank of the \( p \)-Selmer group of \( E \) and the rank of \( E(\mathbb{Q})[p] \), respectively. Then the quantity \( s_p(E) - t_p(E) \) is even if and only if the root number of \( E \) is \(-1\).

It is widely believed that when elliptic curves are ordered by height, 50% have root number \( 1 \) and 50% have root number \(-1\). The same is believed true in any large family as well. For such families we have the following result whose proof is similar to [9, Theorem 39].

Proposition 40 Let \( F \) be a large family of elliptic curves such that exactly 50% of the curves in \( F \), when ordered by height, have root number \( 1 \). Then we have:

(a) The average 5-Selmer rank of elliptic curves in \( F \) is bounded above by \( .75 \).

(b) The set of elliptic curves in \( F \) with 5-Selmer rank 0 or 1 has density at least \( 7/8 \).

(c) The set of elliptic curves in \( F \) with 5-Selmer rank 0 has density at least \( 3/8 \).

Proof: Note that \( 12n + 1 \leq 5^n \) for \( n \) even and \( 60n - 55 \leq 5^n \) for \( n \) odd. Let \( r_5^{\text{even}} \) (resp. \( r_5^{\text{odd}} \)) denote the lim sup of the average 5-Selmer rank of elliptic curves having even (resp. odd) 5-Selmer rank. Since the root number in \( F \) denote the lim sup of the average 5-Selmer rank of elliptic curves having even (resp. odd) 5-Selmer rank. Since the root number in \( F \), it follows that \( \bar{r}_5 \) is equidistributed, Theorems 31 and 39 imply that

\[
6r_5^{\text{even}} + 30r_5^{\text{odd}} \leq 6 - \frac{1}{2} + \frac{55}{2} = 33. \tag{33}
\]

Under the above constraint, \( r_5^{\text{even}} + r_5^{\text{odd}} \) is clearly maximized when \( r_5^{\text{odd}} \) is minimized, which happens when \( r_5^{\text{odd}} = 1 \). Therefore, we have \( (r_5^{\text{even}} + r_5^{\text{odd}})/2 \leq (1/2 + 1)/2 = .75 \), proving (a). This bound is achieved when 37.5% of elliptic curves have 5-Selmer rank 0, 50% of elliptic curves have 5-Selmer rank 1, and 12.5% of elliptic curves have 5-Selmer rank 2.

Let \( x_{0,1} \) denote the lower density of elliptic curves with 5-Selmer rank 0 or 1. Also, let \( x_0 \) (resp. \( x_1 \)) denote the lower density of elliptic curves with 5-Selmer rank 0 (resp. 1). By Theorems 31 and 39 we have

\[
x_0 + 25(1/2 - x_0) + 5(x_1 + 25(1/2 - x_1)) \leq 6.
\]

Thus, we obtain \( 12x_0 + 120x_1 \geq 69 \). In conjunction with the constraint \( x_1 \leq 1/2 \), it follows that \( x_{0,1} \geq x_0 + x_1 \geq 7/8 \), proving (b). Again, this bound is achieved when 37.5% of elliptic curves have 5-Selmer rank 0, 50% of elliptic curves have 5-Selmer rank 1, and 12.5% of elliptic curves have 5-Selmer rank 2.

Finally, let \( x_0 \) again denote the lower density of elliptic curves with 5-Selmer rank 0. Since 50% of elliptic curves in \( F \) have odd 5-Selmer rank, we obtain from Theorems 31 and 39 that

\[
x_0 + 25(1/2 - x_0) + 5/2 \leq 6.
\]

It follows that \( x_0 + 25(1/2 - x_0) \leq 7/2 \) and thus \( x_0 \geq 3/8 \), proving (c). Once again, this bound is achieved when 37.5% of elliptic curves have 5-Selmer rank 0, 50% of elliptic curves have 5-Selmer rank 1, and 12.5% of elliptic curves have 5-Selmer rank 2. □

Theorems 41, 41, and 5 now follow by applying Proposition 40 on the family \( F \) constructed in Theorem 5, applying Proposition 38 on the complement of \( F \), and noting that the 5-Selmer rank of an elliptic curve \( E \) is an upper bound for its rank \( r(E) \). Since the density of \( F \) is \( \mu(F) \geq .5501 \), we have by Part (a) of Propositions 10 and 38 that the average rank of elliptic curves is at most

\[
.5501 \times .75 + .4499 \times 1.05 = .88497 < .885
\]
which proves Theorem 3. By Part (b) of the propositions, we see that the lower density of elliptic curves with rank 0 or 1 is at least
\[ 0.5501 \times \frac{7}{8} + 0.4499 \times \frac{19}{24} \geq 0.8375 \]
which proves Theorem 4. Finally, Part (c) of Proposition 40 implies that the lower density of elliptic curves with rank 0 is at least
\[ 0.5501 \times \frac{3}{8} \geq 0.2062 \]
which proves Theorem 5.

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