Hochschild Lefschetz class for $\mathcal{D}$-modules

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Abstract We introduce a notion of Hochschild Lefschetz class for a good coherent $\mathcal{D}$-module on a compact complex manifold, and prove that this class is compatible with the direct image functor. We prove an orbifold Riemann–Roch formula for a $\mathcal{D}$-module on a compact complex orbifold.

Keywords Equivariant Hochschild class · $D$-module

1 Introduction

In [9], Kashiwara and Schapira systematically studied the Hochschild class for deformation quantization algebroids. As an application, they obtained a new way to define the Euler class of a good coherent $\mathcal{D}$-module on a complex manifold, introduced by Schapira and Schneider [16]. In this paper, we aim to generalize the notion of Hochschild class to the equivariant setting.

Let $M$ be a complex manifold and $\mathcal{D}_M$ the sheaf of holomorphic differential operators on $M$. A coherent $\mathcal{D}_M$-module $\mathcal{M}$ is called “good” if for any compact subset of $M$ there is a neighborhood in which $\mathcal{M}$ admits a finite filtration $(\mathcal{M}_k)$ by coherent $\mathcal{D}_M$-submodules
such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ can be endowed with a good filtration. We denote by $D^b(\mathcal{D}_M)$ the bounded derived category of $\mathcal{D}_M$-modules, and $D^b_{\text{coh}}(\mathcal{D}_M)$ the full triangulated subcategory of $D^b(\mathcal{D}_M)$ consisting of objects with coherent cohomologies. Let $X := T^*M$ be the cotangent bundle of $M$. Following [9], we consider the sheaf $\widehat{H}_X$ of formal microdifferential operators on $X$. Let $\pi_M : X = T^*M \to M$ be the canonical projection. There is a natural flat embedding map $\pi_M^{-1}\mathcal{D}_M \hookrightarrow \widehat{H}_X$. This gives a natural functor from $D^b_{\text{coh}}(\mathcal{D}_M)$ to $D^b_{\text{coh}}(\widehat{H}_X)$. Such a functor allows the use of microlocal techniques to study $\mathcal{D}_M$-modules. Let $\mathcal{H}(\widehat{H}_X, \widehat{H}_X)$ be the $\mathbb{C}$-sheaf of Hochschild homologies of $\widehat{H}_X$ on $X$. For $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{D}_M)$ and an element $u \in \text{Hom}_{\mathcal{D}_M}(\mathcal{M}, \mathcal{M})$, Kashiwara and Schapira [9] introduced a Hochschild class $\text{hh}(\mathcal{M}, u) \in H^0_{\text{supp}(\mathcal{M})}(X; \mathcal{H}(\widehat{H}_X, \widehat{H}_X))$.

The image of the the Hochschild class $\text{hh}(\mathcal{M}, u)$ under the quasi-isomorphism

$$\mathcal{H}(\widehat{H}_X, \widehat{H}_X) \to \mathbb{C}_X[\dim_X]$$

is called the Euler class $\gamma$ of $(\mathcal{M}, u)$, where $\dim_X$ is the complex dimension of $X$. As Hochschild homology behaves well under direct images, the Hochschild class $\text{hh}(\mathcal{M}, u)$ satisfies a nice formula [9, Theorem 4.3.5] under the direct image functor. This formula is analogous to the direct image property of the Euler class of $(\mathcal{M}, u)$, which was proved by Schapira and Schneider [16].

In this paper we consider a holomorphic diffeomorphism $\gamma$ on $M$. Let $\mathcal{M}$ be a $\mathcal{D}_M$-module. Then, the sheaf $\gamma_*\mathcal{M}$ of $\mathbb{C}$-vector spaces on $M$ has a natural $\mathcal{D}_M$-module structure. This in turn gives a natural functor $\gamma_* : D^b_{\text{coh}}(\mathcal{D}_M) \to D^b_{\text{coh}}(\mathcal{D}_M)$. A similar construction and functor can be introduced for $\widehat{H}_X$-modules. Given an element $u \in \text{Hom}_{\mathcal{D}_M}(\mathcal{M}, \gamma_*(\mathcal{M}))$, we introduce a Hochschild Lefschetz class

$$\text{hh}^\gamma(\mathcal{M}, u) \in H^0(X, \mathcal{H}(\widehat{H}_X, \widehat{H}_X))$$

for $X = T^*M$. Our construction coincides with the Kashiwara-Schapira Hochschild class $\text{hh}(\mathcal{M}, u)$ when $\gamma = \text{id}$. The Lefschetz class of $u$ is defined to be the image of $\text{hh}^\gamma(\mathcal{M}, u)$ under the quasi-isomorphism

$$\mathcal{H}(\widehat{H}_X, \widehat{H}_X) \to \gamma_!(\mathbb{C}_X[\dim(X^\gamma)])$$

Like the Hochschild class, we prove that the Hochschild Lefschetz class satisfies nice formulas under the direct image functor. We expect that this approach will provide a relatively easy route to results about the Lefschetz class introduced by Guillermou [7]. Let $\Gamma$ be a finite group acting on $M$ by holomorphic diffeomorphisms. We apply our developments to study the Hochschild class and the Euler class of a good $\Gamma$-equivariant coherent $\mathcal{D}_M$-module $\mathcal{M}$. In this situation, every $\gamma \in \Gamma$ naturally defines an element $\gamma$ in $\text{Hom}_{\mathcal{D}_M}(\mathcal{M}, \gamma_*(\mathcal{M}))$. We can use the expression

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \text{hh}^\gamma(\mathcal{M}, \gamma)$$

to introduce the orbifold Hochschild class of $\mathcal{M}$ on the quotient orbifold $Q_X := X/\Gamma = T^*M/\Gamma$. We prove that this description of the orbifold Hochschild class for $\mathcal{M}$ is equivalent

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1 This is actually a slight abuse of terminology. In fact, $\mathcal{H}(\widehat{H}_X, \widehat{H}_X)$ is an object in the derived category of sheaves of $\mathbb{C}$-vector spaces on $X$.

2 See Eq. (1) for the definition of $\mathcal{H}(\widehat{H}_X, \widehat{H}_X)$.
to the more abstract definition that arises by working with the sheaf of algebras $\mathcal{D}_M \rtimes \Gamma$ (and $\hat{\mathcal{E}}_X \rtimes \Gamma$) over $Q_M := M / \Gamma$ (and $Q_X$) using techniques developed by Bressler et al. [1].

The main result of this paper is a Riemann–Roch formula for the Euler class of a good $\Gamma$-equivariant coherent $\mathcal{D}_M$-module $\mathcal{M}$ on $M$, when $M$ is a compact complex manifold. We prove that (see Theorem 4.2)

$$\text{eu}_Q(\mathcal{M}) = \left( \frac{1}{m} \text{ch}_Q(\sigma_{\text{char}(\mathcal{M})(\mathcal{M})}) \wedge \text{eu}_Q(N) \wedge \pi_M^*Td(I_Q M) \right) \dim(I_Q M).$$

Hereby, $IQ_X$ (and $IQ_M$) is the inertia orbifold associated to the orbifold $Q_X$ (and $Q_M$); $\text{ch}_Q$ is the orbifold Chern character for the orbifold K-theory element $\sigma_{\text{ch}(\mathcal{M})}(\mathcal{M})$; $\text{eu}_Q(N)$ is a characteristic class associated to the normal bundle $N$ of the local embedding $IQ_X \to Q_X$; $Td(I_Q M)$ is the Todd class of the orbifold bundle $TI_Q M$ over $I_Q M$; $\pi_M$ is the canonical projection from $I_Q X$ to $I_Q M$; and $m$ is the locally constant function on $IQ_X$ measuring the size of the isotropy group at each point. The proof generalizes the original idea of Bressler et al. [1] along the developments in [11] and [12].

The paper is organized as follows. We start with fixing some basic notations in Sect. 2. In Sect. 3, we introduce the construction of the Hochschild Lefschetz class and orbifold Euler class for a good coherent $\mathcal{D}_M$-module, and discuss their properties. In Sect. 4, we explain the computation of the orbifold Euler (Chern) class.

## 2 Basic notations

Throughout this paper, we closely follow the terminologies and conventions introduced in [9]. Let $M$ be a complex manifold of complex dimension $d_M$. In this paper, the dimension of a complex manifold/orbifold always refers to its complex dimension. Consider $\mathcal{D}_M$ the sheaf of holomorphic differential operators on $M$ and $\mathcal{D}_M^+$ the sheaf of holomorphic differential operators on $M$ with the opposite algebra structure from $\mathcal{D}_M$. A coherent $\mathcal{D}_M$-module $\mathcal{M}$ is called “good” if in a neighborhood of any compact subset of $M$, $\mathcal{M}$ admits a finite filtration $(\mathcal{M}_i)$ by coherent $\mathcal{D}_M$-submodules such that each quotient $\mathcal{M}_i / \mathcal{M}_{i-1}$ can be endowed with a good filtration (see [8, Definition 4.24]). We denote by $D^b(\mathcal{D}_M)$ the bounded derived category of $\mathcal{D}_M$ modules, and $D^b_{\text{coh}}(\mathcal{D}_M)$ the full triangulated subcategory of $D^b(\mathcal{D}_M)$ consisting of objects with coherent cohomologies.

Let $X = T^*M$ be the cotangent bundle with dimension $d_X = 2d_M$ with the projection $\pi_M : X = T^*M \to M$. Denote by $\Omega^i_X$ the sheaf of holomorphic $i$-forms on $X$. On $X = T^*M$, consider the filtered sheaf $\hat{\mathcal{E}}_X$ of $\mathbb{C}$-algebras of formal microdifferential operators, and the subsheaf $\hat{\mathcal{E}}(0)_X$ of operators of order $\leq 0$. Let $\pi_M^{-1}\mathcal{D}_M$ be the pullback of $\mathcal{D}_M$ on $X$. Denote by $D^b(\hat{\mathcal{E}}_X)$ (and $D^b_{\text{coh}}(\hat{\mathcal{E}}_X)$) the bounded derived category of $\hat{\mathcal{E}}_X$-modules (and the subcategory of objects with coherent cohomologies.) There is a natural morphism $\pi_M^{-1}\mathcal{D}_M \hookrightarrow \hat{\mathcal{E}}_X$ such that $\hat{\mathcal{E}}_X$ is flat over $\pi_M^{-1}\mathcal{D}_M$. Given a coherent $\mathcal{D}_M$-module $\mathcal{M}$,

$$\hat{\mathcal{M}} := \hat{\mathcal{E}}_X \otimes_{\pi_M^{-1}\mathcal{D}_M} \pi_M^{-1}\mathcal{M}$$

defines a coherent $\hat{\mathcal{E}}_X$-module. In this paper, we will mainly work with the $\hat{\mathcal{E}}_X$-module $\hat{\mathcal{M}}$ associated to $\mathcal{M}$. The support of $\hat{\mathcal{M}}$ is called the characteristic variety of $\mathcal{M}$ and denoted by $\text{char}(\mathcal{M})$. Denote by $\hat{\mathcal{E}}_X^+$ the sheaf of formal microdifferential operators on $X = T^*M$ with the opposite algebra structure from $\hat{\mathcal{E}}_X$. 
Define $\omega := \Omega^d_X [d_X]$, and define the duality functor $D^{\prime}_{\mathcal{E}^0}$ by

$$D^{\prime}_{\mathcal{E}^0}(\mathcal{M}) := R\text{Hom}_{\mathcal{E}^0}(\mathcal{M}, \mathcal{E}^0) \in D^b(\mathcal{E}^0),$$

for an $\mathcal{E}^0$-module $\mathcal{M}$.

Let $\Gamma$ be a finite group acting on $M$ holomorphically and also on $X = T^* M$. Note that for any $\gamma \in \Gamma$, one has a natural isomorphism $\gamma^{-1} \mathcal{E}^0 \to \mathcal{E}^0$ of sheaves of $\mathbb{C}$-algebras on $X$ defined by the $\gamma$ action on $\mathcal{E}^0$. Hence, for any $\gamma \in \Gamma$, any $\mathcal{E}^0$-module $\mathcal{M}$ has the natural structure of a $\gamma^{-1} \mathcal{E}^0$-module. Consequently, the pushforward $\gamma_* \mathcal{M}$ (in the category of sheaves of $\mathbb{C}$-vector spaces on $X$) has the natural structure of a $\mathcal{E}^0$-module. It is easy to verify that the $\gamma$-action gives rise to a functor from $D^b(\mathcal{E}^0)$ (resp., $D^b_{\text{coh}}(\mathcal{E}^0)$) to itself.

We denote by $\delta : X \to X \times X$ the diagonal embedding. For $\gamma \in \Gamma$, let

$$\delta^\gamma_X : X \to X \times X, \quad \delta^\gamma_X(x) := (\gamma(x), x)$$

be the graph of the action of $\gamma$. Let $\mathcal{C}_X$ be the $\mathcal{E}^0_{\times X^0}$-module $\delta^\gamma_X \mathcal{E}^0_X$, and let $\mathcal{C}^\gamma_X$ be the $\mathcal{E}^0_{\times X^0}$-module $\delta_X^{\gamma \gamma} \mathcal{E}^0_X$. The sheaf of Hochschild homologies $\mathcal{H}\mathcal{H}(\mathcal{E}^0_X)$ (resp., $\gamma$-twisted Hochschild homologies $\mathcal{H}\mathcal{H}(\mathcal{E}^0_X)$) is defined to be the object

$$\mathcal{H}\mathcal{H}_X(\mathcal{E}^0_X) := \delta_X^{-1}(\mathcal{C}_X \otimes_{\mathcal{E}^0_X} \mathcal{C}_X) \quad \text{(resp., } \mathcal{H}\mathcal{H}_X(\mathcal{E}^0_X) := \delta_X^{-1}(\mathcal{C}_X \otimes_{\mathcal{E}^0_X} \mathcal{C}^\gamma_X))$$

of the derived category of sheaves of $\mathbb{C}$-vector spaces on $X$. For any object $\mathcal{F}$ in the derived category of sheaves of $\mathbb{C}$-vector spaces on $X$, $H^\bullet(X; \mathcal{F})$ shall denote the cohomology of $X$ with coefficients in $\mathcal{F}$.

The completed tensor products $\otimes, \boxtimes, \ast$ etc. have exactly the same meaning as in [9].

## 3 Lefschetz class

### 3.1 Definition of Lefschetz class

Let $\mathcal{M}$ be a good coherent $\mathcal{D}_M$ module on $M$. We apply the functor

$$\mathcal{M} \mapsto \hat{\mathcal{M}} := \mathcal{E}^0 \otimes_{\pi^{-1}_M \mathcal{D}_M} \mathcal{M}$$

and work with the corresponding $\mathcal{E}^0$-module $\hat{\mathcal{M}}$. Let $\gamma$ act on $M$ holomorphically. Lift the action of $\gamma$ to an action on $X := T^* M$. An element $u$ in $\text{Hom}_{\mathcal{D}_M}(\mathcal{M}, \gamma^*(\mathcal{M}))$ naturally defines an element $\hat{u}$ in $\text{Hom}_{\mathcal{E}^0}(\hat{\mathcal{M}}, \gamma^*(\hat{\mathcal{M}}))$. In what follows, we will introduce a Lefschetz class for $u$ by studying $\hat{u}$ in $\text{Hom}_{\mathcal{E}^0}(\hat{\mathcal{M}}, \gamma^*(\hat{\mathcal{M}}))$. Our construction generalizes analogous constructions in [9].

**Lemma 3.1** Let $\hat{\mathcal{M}} \in D^b_{\text{coh}}(\mathcal{E}^0_X)$. There is a natural morphism in $D^b_{\text{coh}}(\mathcal{E}^0_{X \times X^0})$:

$$\gamma_*(\hat{\mathcal{M}}) \boxtimes D^\prime_{\mathcal{E}^0}(\hat{\mathcal{M}}) \to C^\gamma_{\mathcal{E}^0}.$$

**Proof** By [9, Lemma 4.1.1], there is a natural morphism

$$\hat{\mathcal{M}} \boxtimes D^\prime_{\mathcal{E}^0}(\hat{\mathcal{M}}) \to C_{\mathcal{E}^0}.$$

---

$^3$ This is a minor abuse of terminology.
Applying the functor \((\gamma \times 1)_*\) to the above morphism, we obtain the desired morphism
\[
\gamma_* (\widehat{\mathcal{M}}) \boxtimes D'_{\mathcal{E}_X} (\widehat{\mathcal{M}}) \to C'_X.
\]

Let \(u \in \text{Hom}_{\widehat{\mathcal{E}_X}} (\widehat{\mathcal{M}}, \gamma_* (\widehat{\mathcal{M}}))\). Consider the morphism
\[
\text{RHom}_{\widehat{\mathcal{E}_X}} (\widehat{\mathcal{M}}, \gamma_* (\widehat{\mathcal{M}})) \leftarrow D'_{\mathcal{E}_X} (\widehat{\mathcal{M}}) \boxtimes \mathcal{E}_X \gamma_* (\widehat{\mathcal{M}})
\]
\[
\cong C_X^\gamma (\gamma_* (\widehat{\mathcal{M}}), D'_{\mathcal{E}_X} (\widehat{\mathcal{M}}))
\]
\[
\rightarrow C_X^\gamma (\gamma_* (\mathcal{M}) \boxtimes D'_{\mathcal{E}_X} (\mathcal{M})).
\]
This defines a natural map
\[
\text{Hom}_{\widehat{\mathcal{E}_X}} (\widehat{\mathcal{M}}, \gamma_* (\widehat{\mathcal{M}})) \rightarrow H^0_{\text{supp}(\mathcal{M})} (X; \mathcal{H}\mathcal{H}(\widehat{\mathcal{E}_X}, \widehat{\mathcal{E}'_X})). \tag{3}
\]

**Definition 3.2** For an element \(u \in \text{Hom}_{\mathcal{D}(\mathcal{M})} (\mathcal{M}, \gamma_* (\mathcal{M}))\), define the Hochschild Lefschetz class \(hh^\gamma (\mathcal{M}, u) \in H^0_{\text{supp}(\mathcal{M})} (X; \mathcal{H}\mathcal{H}(\widehat{\mathcal{E}_X}, \widehat{\mathcal{E}'_X}))\) to be the image of \(\hat{u} \in \text{Hom}_{\widehat{\mathcal{E}_X}} (\widehat{\mathcal{M}}, \gamma_* (\widehat{\mathcal{M}}))\) under the morphism (3).

Recall that \(\mathcal{H}\mathcal{H}(\widehat{\mathcal{E}_X}, \widehat{\mathcal{E}'_X})\) can be naturally identified with \(\text{RHom}_{\mathcal{E}_X \times X^a} (\omega_X^{-1}, C_X^\gamma)\) using the duality functor \(D'_{\mathcal{E}_X \times X^a}\) and [9, Theorem 2.5.7]. The following analog of [9, Lemma 4.1.4] holds. We leave its proof to the interested reader.

**Lemma 3.3** Under the natural identification of \(\mathcal{H}\mathcal{H}(\widehat{\mathcal{E}_X}, \widehat{\mathcal{E}'_X})\) with \(\text{RHom}_{\mathcal{E}_X \times X^a} (\omega_X^{-1}, C_X^\gamma)\), the Hochschild Lefschetz class \(hh^\gamma (\mathcal{M}, u)\) coincides with the following composite of morphisms
\[
\omega_X^{-1} \xrightarrow{[\text{KS, Lemma 4.1.1(i)}]} \widehat{\mathcal{M}} \boxtimes D'_{\mathcal{E}_X} (\widehat{\mathcal{M}}) \xrightarrow{\hat{u} \boxtimes \text{id}} \gamma_* (\widehat{\mathcal{M}}) \boxtimes D'_{\mathcal{E}_X} (\mathcal{M}) \xrightarrow{\text{Lemma 3.1}} C_X^\gamma.
\]

Let \(X^\gamma\) be the submanifold\(^4\) of \(X\) consisting of \(\gamma\)-fixed points and let \(\iota : X^\gamma \hookrightarrow X\) be the inclusion. Let \(\Omega^*_X\) be the (smooth) de Rham complex on \(X^\gamma\) (viewed as a complex of sheaves on \(X^\gamma\)). As in [12, Section 3], one can construct a quasi-isomorphism
\[
\mathcal{H}\mathcal{H}(\widehat{\mathcal{E}_X}, \widehat{\mathcal{E}'_X}) \xrightarrow{\Omega^{(\text{dim}(X^\gamma))}} \bullet
\]
following a construction in [5, Section 4] and [3, Section 2] (see also [6]). Therefore,

**Proposition 3.4** The Hochschild homology \(\mathcal{H}\mathcal{H}(\widehat{\mathcal{E}_X}, \widehat{\mathcal{E}'_X})\) is quasi-isomorphic to \(\iota_!(C_{X^\gamma} [\text{dim}(X^\gamma)])\).

Thus one can define the (microlocal) Lefschetz class \(\mu^\gamma u\) of \(u\) to be the image of \(hh^\gamma (\mathcal{M}, u)\) under the quasi-isomorphism
\[
\mathcal{H}\mathcal{H}(\widehat{\mathcal{E}_X}, \widehat{\mathcal{E}'_X}) \rightarrow \iota_!(C_{X^\gamma} [\text{dim}(X^\gamma)]).
\]

When \(M\) is a point, \(\gamma\) acts on \(M\) trivially. Then \(\mu^\gamma u (\mathcal{M}, u)\) is equal to the trace of \(u\) as an endomorphism of \(\mathcal{M}\).

\(^4\) \(X^\gamma\) is a disjoint union of embedded submanifolds possibly of different dimensions.
Remark 3.5 One also has a class $\text{eu}(\mathcal{M}, u) \in H^2 \dim M^\gamma (M^\gamma, \mathbb{C})$. Its construction is completely analogous to that of $\mu \text{eu}^\gamma (\mathcal{M}, u)$. When $\gamma = \text{Id}$ and when $\mathcal{M} = D_M \otimes_{O_M} \mathcal{E}$ for some holomorphic vector bundle $\mathcal{E}$ on $M$, then eu$(\mathcal{M}, u)$ is equal to the trace of $u$ as an endomorphism of $O_M \otimes_{D_M} \mathcal{M}$ (see [3, 6, 14]).

3.2 Composition of Hochschild Lefschetz classes

Consider three complex manifolds $M_i$, $i = 1, 2, 3$, and $X_i = T^* M_i$, $i = 1, 2, 3$. Let $\mathcal{E}_i = \mathcal{O}_i \otimes_{\mathcal{O}_i} \mathcal{E}$ and $\mathcal{E}_{ij}$ be the sheaf of formal microdifferential operators on $X_i \times X_j$, $i = 1, 2$. Assume that the group $\Gamma$ acts on $M_i$ and $X_i$ holomorphically. Let $p_{ij}$ be the canonical projection from $X_1 \times X_2 \times X_3$ to $X_i \times X_j$ for $1 \leq i < j \leq 3$. Also, let $d_i$, $i = 1, 2, 3$ denote the complex dimensions of the $X_i$. In this subsection, as in [9], we implicitly identify $X = T^* M$ with its image in $X \times X$ under the embedding $\delta_X^\gamma$ whenever required. Also\(^5\), to simplify notations, we sometimes denote $\mathcal{E}_i$ by $\mathcal{E}_{ij}$: for example, $\otimes_{2, 2}$ actually stands for $\otimes_{\mathcal{E}_{2, 2}}$.

Proposition 3.6 There is a natural morphism
\[
oindent \circ : \text{Rp}_{13} \left( p_{12}^{-1} \mathcal{H} \left( \mathcal{E}_{1, 2}, \mathcal{E}_{1, 2} \right) \otimes p_{23}^{-1} \mathcal{H} \left( \mathcal{E}_{2, 3}, \mathcal{E}_{2, 3} \right) \right) \\
\rightarrow \mathcal{H} \left( \mathcal{E}_{1, 2}, \mathcal{E}_{1, 2} \right).
\]

Proof Following [9], we will denote by $\mathcal{E}_i$, the complex manifold $\mathcal{E}_{1, 2}$, and identify the Hochschild homology $\mathcal{H} \left( \mathcal{E}_{1, 2}, \mathcal{E}_{1, 2} \right)$ as follows:

\[
\mathcal{H} \left( \mathcal{E}_{1, 2}, \mathcal{E}_{1, 2} \right) \cong \left( \mathcal{C}_{\mathcal{E}_{1, 2}} \boxtimes \mathcal{C}_{\mathcal{E}_{1, 2}} \right) \otimes_{\mathcal{E}_{1, 2} \boxtimes \mathcal{E}_{1, 2}} \left( \mathcal{C}_{\mathcal{E}_{1, 2}} \boxtimes \mathcal{C}_{\mathcal{E}_{1, 2}} \right)
\]

\[
\cong \mathcal{RHom}_{\mathcal{E}_{1, 2} \boxtimes \mathcal{E}_{1, 2}} \left( \omega_{\mathcal{E}_{1, 2}} \boxtimes \omega_{\mathcal{E}_{1, 2}}, \mathcal{C}_{\mathcal{E}_{1, 2}} \boxtimes \mathcal{C}_{\mathcal{E}_{1, 2}} \right)
\]

\[
\cong \mathcal{RHom}_{\mathcal{E}_{1, 2} \boxtimes \mathcal{E}_{1, 2}} \left( \omega_{\mathcal{E}_{1, 2}} \boxtimes \omega_{\mathcal{E}_{1, 2}}, \mathcal{C}_{\mathcal{E}_{1, 2}} \boxtimes \mathcal{C}_{\mathcal{E}_{1, 2}} \right)
\]

\[
= \mathcal{RHom}_{\mathcal{E}_{1, 2} \boxtimes \mathcal{E}_{1, 2}} \left( \omega_{\mathcal{E}_{1, 2}} \boxtimes \omega_{\mathcal{E}_{1, 2}}, \mathcal{C}_{\mathcal{E}_{1, 2}} \boxtimes \mathcal{C}_{\mathcal{E}_{1, 2}} \right).
\]

As in [9], let $S_{ij} := \omega_{\mathcal{E}_{1, 2}} \boxtimes \omega_{\mathcal{E}_{1, 2}}$, and let $K_{ij} := \mathcal{C}_{\mathcal{E}_{1, 2}} \boxtimes \omega_{\mathcal{E}_{1, 2}}$, where $\omega_{\mathcal{E}_{1, 2}}$ is $\gamma_{\omega_{\mathcal{E}_{1, 2}}}$. The above computation can be summarized as

\[
\mathcal{H} \left( \mathcal{E}_{1, 2}, \mathcal{E}_{1, 2} \right) \cong \mathcal{RHom}_{\mathcal{E}_{1, 2} \boxtimes \mathcal{E}_{1, 2}} \left( S_{ij}, K_{ij} \right).
\]

We obtain the morphism

\[
K_{12} \boxtimes K_{23} \rightarrow p_{13}^{-1} (\mathcal{C}_{\mathcal{E}_{1, 2}} \boxtimes \omega_{\mathcal{E}_{1, 2}})(2d_2) = p_{13}^{-1} (K_{13})(2d_2).
\]

\(^5\) As in [9].
For the last arrow in the above composition, note that \( \omega_{X_2}^\gamma \overset{L}{\otimes} \mathcal{E}_{Z_2} \mathcal{C}_{X_2}^\gamma \) is naturally isomorphic to \((\gamma \times 1)_*(\omega_{X_2}^\gamma \overset{L}{\otimes} \mathcal{E}_{Z_2} \mathcal{C}_X)\). Also recall that the morphism \( \omega_{X_2}^\gamma \overset{L}{\otimes} \mathcal{E}_{Z_2} \mathcal{C}_X \rightarrow \delta^\gamma_* \mathcal{C}_X[2d_2] \) is defined by [9, Theorem 2.5.7]. Hence, one obtains a morphism \( \omega_{X_2}^\gamma \overset{L}{\otimes} \mathcal{E}_{Z_2} \mathcal{C}_X \rightarrow \delta^\gamma_* \mathcal{C}_X[2d_2] \) which induces the last arrow in the above composition. The morphism (5) induces, by adjunction, a morphism

\[
Rp_{13!}(K_{12}^\gamma \overset{L}{\otimes} \mathcal{E}_{Z_2} \mathcal{K}^\gamma_{23}) \rightarrow K_{13}^\gamma.
\]

(6)

As explained in the proof of [9, Proposition 4.2.1], there is a natural morphism

\[
S_{13} \rightarrow Rp_{13*}(S_{12} \overset{L}{\otimes} \mathcal{E}_{Z_2} S_{23}).
\]

With the above two morphisms, we have natural morphisms

\[
Rp_{13!}(p^{-1}_{12} \mathcal{H}(\mathcal{E}_X_{1 \times X_3}) \otimes p^{-1}_{23} \mathcal{H}(\mathcal{E}_X_{2 \times X_3}))
\]

\[
\rightarrow Rp_{13!}R\text{Hom}_{\mathcal{E}_{Z_1 \times Z_3}^\gamma}(S_{12} \overset{L}{\otimes} \mathcal{E}_{Z_2} S_{23}, K^\gamma_{12} \overset{L}{\otimes} \mathcal{E}_{Z_2} K^\gamma_{23})
\]

\[
\rightarrow R\text{Hom}_{\mathcal{E}_{Z_1 \times Z_3}^\gamma}(S_{13}, K^\gamma_{13}) \rightarrow \mathcal{H}(\mathcal{E}_X_{1 \times X_3}, \mathcal{E}_X^\gamma_{1 \times X_3}).
\]

This proves the desired proposition. \(\square\)

As a corollary, if \(X_1 = X_3 = \text{pt} \) and \(X_2 = X\), then Proposition 3.6 defines a morphism

\[
Ra_t \left( \mathcal{H}(\mathcal{E}_X, \mathcal{E}_X^\gamma) \right) \rightarrow \mathcal{C}_{\text{pt}},
\]

where \(a : X \rightarrow \text{pt}\) is the natural map. By the adjunction formula, we have

**Corollary 3.7** Let \(X\) be the underlying real manifold of \(X\). There is a canonical morphism

\[
\mathcal{H}(\mathcal{E}_X, \mathcal{E}_X^\gamma) \otimes \mathcal{H}(\mathcal{E}_X, \mathcal{E}_X^\gamma) \rightarrow \omega_{X\mathbb{R}}^{\text{top}},
\]

where \(\omega_{X\mathbb{R}}^{\text{top}}\) is the topological dualizing complex on \(X\) with coefficients in \(\mathbb{C}\).

**Remark 3.8** Let \(H\mathcal{H}_*(\mathcal{E}_X, \mathcal{E}_X^\gamma)\) denote the cohomology \(H^\ast(X, \mathcal{H}(\mathcal{E}_X, \mathcal{E}_X^\gamma))\). We remark that by integration, Corollary 3.7 defines a pairing on \(H\mathcal{H}_*(\mathcal{E}_X, \mathcal{E}_X^\gamma)\), which is a \(\gamma\)-equivariant generalization of the Mukai pairing. We hope to discuss more about this pairing in a future publication.

We recall that [9, Definition 3.1.3] that for \(K_i \in D^b(\mathcal{E}_X_{i \times X_3}) (i = 1, 2), \)

\[
K_1 \circ_{X_2} K_2 = Rp_{13!}(K_1 \mathcal{E}_{Z_2} K_2) \in D^b(\mathcal{E}_X_{1 \times X_3}),
\]

\[
K_1 \ast_{X_2} K_2 = Rp_{13*}(K_1 \mathcal{E}_{Z_2} K_2) \in D^b(\mathcal{E}_X_{1 \times X_3}).
\]

In what follows, we often simplify notations by writing \(\circ_{X_2}\) for \(\circ_{X_2}\) and \(\ast_{X_2}\) for \(\ast_{X_2}\).

We have the following generalization of [9, Lemma 4.3.3].
Lemma 3.9 Let $\gamma \in \Gamma$. Let $K$ be a $\gamma$-equivariant element in $D^{b}_{\text{coh}}(\mathcal{E}_{X_1 \times X_2})$. There is a natural morphism in $D^{b}(\mathcal{E}_{X_1 \times X_2})$,

$$K o_2 \omega^\gamma o_2 D^\gamma_{\mathcal{E}}(K) \longrightarrow C^\gamma_{X_1}.$$ 

Proof By Lemma 3.1, we have a morphism in $D^{b}(\mathcal{E}_{X_1 \times X_2})$

$$\gamma_{*}(K) \boxtimes D_{\mathcal{E}}^\gamma(K) \longrightarrow C_{X_1 \times X_2}^\gamma.$$ 

Applying the functor $(-) \boxtimes_{X_2 \times X_2} \omega_{X_2}^\gamma$, we obtain

$$\left( K \boxtimes D_{\mathcal{E}}^\gamma(K) \right) \boxtimes_{X_2 \times X_2} \omega_{X_2}^\gamma \overset{(1)}{\longrightarrow} \left( \gamma_{*}(K) \boxtimes D_{\mathcal{E}}^\gamma(K) \right) \boxtimes_{X_2 \times X_2} \omega_{X_2}^\gamma \overset{(2)}{\longrightarrow} C_{X_1 \times X_2}^\gamma \boxtimes_{X_1 \times X_2} \omega_{X_2}^\gamma.$$ 

Here, in arrow (1), we use the assumption that $K$ is $\gamma$-equivariant, i.e. $\hat{\gamma}$ is a natural element in $\text{Hom}(K, \gamma_{*}(K))$; in arrow (2), we have used the morphism in Lemma 3.1; in arrow (3), we have used the natural isomorphism between $C_{X_1 \times X_2}^\gamma$ and $\delta_{X_2}^\gamma C_{X_2} \{2d_2\}$; this morphism is obtained by applying the functor $(\gamma \times 1)_{*}$ to the morphism from [9, Theorem 2.5.7]. The desired morphism is induced by the above composition of morphisms via adjunction. 

For $\Lambda$ a closed subset of $X$, let

$$H H_{\Lambda}(\mathcal{E}_{X}, \mathcal{E}_{X_1}^\gamma) := H^{0}(R\Gamma_{\Lambda}(X; \mathcal{H}H(\mathcal{E}_{X}, \mathcal{E}_{X_1}^\gamma))).$$

Let $\Lambda_{12}$ and $\Lambda_2$ be closed subsets of $X_1 \times X_2$ and $X_2$. Define $\Lambda_{12} \times X_2, \Lambda_{2} \subset X_1 \times X_2$ to be the fiber product of $\Lambda_{12}$ and $\Lambda_2$ over $X_2$, and $\Lambda_{12} \circ \Lambda$ to be $p_1(\Lambda_{12} \times X_2, \Lambda_{2}) \subset X_1$. Given a $\gamma$-equivariant kernel $K \in D^{b}_{\text{coh}}(\mathcal{E}_{X_1 \times X_2})$ with support $\Lambda_{12}$, we define the following map

$$\Phi_{K} : H H_{\Lambda_{12}}(\mathcal{E}_{X_2}, \mathcal{E}_{X_1}^\gamma) \rightarrow H H_{\Lambda_{12} \circ \Lambda}(\mathcal{E}_{X_1}, \mathcal{E}_{X_1}^\gamma)$$

via a sequence of compositions,

$$H H_{\Lambda_{2}}(\mathcal{E}_{X_2}, \mathcal{E}_{X_1}^\gamma) \cong H^{0}(R\Gamma_{\Lambda_{2}} \text{Hom}_{X_2 \times X_2}^{D^{b}_{\text{coh}}} (\omega_{2}^{\otimes -1}, C_{2}^\gamma)) \rightarrow H^{0} \left( R\Gamma_{\Lambda_{12} \times X_2, \Lambda_{2}} \text{Hom}_{X_1 \times X_1}^{D^{b}_{\text{coh}}} (\mathcal{K} \boxtimes_{X_2} \omega_{2}^{\otimes -1} \circ o_2 \omega_2 \circ o_2 D_{\mathcal{E}}^\gamma \mathcal{K}, \mathcal{K} \boxtimes_{X_2} C_{2}^\gamma \circ o_2 \omega_2 \circ o_2 D_{\mathcal{E}}^\gamma \mathcal{K}) \right) \rightarrow H^{0} \left( R\Gamma_{\Lambda_{12} \circ \Lambda} \text{Hom}_{X_1 \times X_1}^{D^{b}_{\text{coh}}} (R\pi_{1}^{L}(\mathcal{K} \boxtimes_{X_2} \omega_{2}^{\otimes -1} \circ o_2 \omega_2 \circ o_2 D_{\mathcal{E}}^\gamma \mathcal{K})), \right)$$

$$\cong H^{0}(\Gamma_{\Lambda_{12} \circ \Lambda} \text{Hom}_{X_1 \times X_1}^{D^{b}_{\text{coh}}} (\mathcal{K} \ast_{2} D_{\mathcal{E}}^\gamma \mathcal{K}, \mathcal{K} \circ o_2 \omega_2 \circ o_2 D_{\mathcal{E}}^\gamma \mathcal{K})) \rightarrow H^{0}(R\Gamma_{\Lambda_{12} \circ \Lambda} \text{Hom}_{X_1 \times X_1}^{D^{b}_{\text{coh}}} (\omega_{2}^{\otimes -1}, C_{1}^\gamma)) \cong H H_{\Lambda_{12} \circ \Lambda}(\mathcal{E}_{X_1}; \mathcal{E}_{X_1}^\gamma),$$

where in the first arrow, we have applied the functor $\mathcal{L} \mapsto \mathcal{K} \boxtimes_{X_2} (\mathcal{L} \circ o_2 \omega_2 \circ o_2 D_{\mathcal{E}}^\gamma \mathcal{K})$, and in the last arrow we have used Lemma 3.9, and [9, Lemma 4.3.3].

---

6 By a $\gamma$-equivariant element in $D^{b}_{\text{coh}}(X)$, we mean an element $K$ in $D^{b}_{\text{coh}}(X)$ together with a morphism from $K$ to $\gamma_{*}K$, which is denoted by $\hat{\gamma}$. 

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Let \( f : X_2 \to X_1 \) be a \( \gamma \)-equivariant symplectic map. The graph \( \Gamma_f \) of \( f \) in \( X_1 \times X'_2 \) is a Lagrangian submanifold. Denote by \( \mathcal{B}_f \) the holonomic \( D \)-module supported at \( \Gamma_f \). It is easy to check from the property of \( \mathcal{B}_f \) that \( \mathcal{B}_f \) is \( \gamma \)-equivariant. By Definition 3.2, we can define \( hh(f, \gamma) = hh'(\mathcal{B}_f, \gamma) \in H^0_{\Gamma_f}(X_1 \times X_2; \mathcal{H}(\mathcal{E}_{X_1 \times X'_2}, \mathcal{E}'_{X_1 \times X'_2})) \).

The proof of [9, Lemma 4.3.4] may be generalized word for word to give the following result.

**Proposition 3.10** The following morphisms are equal,

\[
\Phi_{\mathcal{B}_f} = hh(f, \gamma) : HH_{\Lambda_2}(\mathcal{E}_X, \mathcal{E}'_X) \to HH_{\Lambda_2 \otimes \Lambda_2}(\mathcal{E}_{X_1}, \mathcal{E}'_{X_1}).
\]

**Proof** Let \( \alpha_2 \) be a class in \( HH(\mathcal{E}_X, \mathcal{E}'_X) \). By the isomorphism

\[
\mathcal{H}(\mathcal{E}_X, \mathcal{E}'_X) \cong \delta_X^{-1} RHom_{X \times X}(D'_{X \times X}(C_X), C_X) \cong \delta_X^{-1} RHom_{X \times X}(\omega_X^\otimes, C_X),
\]

we can regard \( \alpha_2 \) as a morphism \( \alpha_2 : \omega_X^\otimes \to C_{X_2} \) in the derived category of sheaves of \( \mathbb{C} \)-vector spaces on \( X_{22a} := X_2 \times X'_2 \). Similarly, we can regard the element \( \alpha = hh(f, \gamma) \) in \( HH(\mathcal{E}_{X_1 \times X'_2}, \mathcal{E}'_{X_1 \times X'_2}) \) as a morphism \( \alpha : \omega_{X_1 \times X'_2} \to C_{X_1 \times X'_2} \) in the derived category of sheaves on \( X_{11 \times 22a} := X_1 \times X'_1 \times X_2 \times X'_2 \). By Lemma 3.3, \( \alpha \) is given by a composite of morphisms

\[
\omega_{12}^\otimes \xrightarrow{\text{Lemma 4.1.1(i)}} K^{\otimes} D' \xrightarrow{\alpha} C_{12a}
\]

in the derived category of sheaves on \( X_{11 \times 22a} \), where \( \mathcal{B}_{\Gamma_f} \) is denoted by \( \mathcal{K} \). The element \( \Phi_{\mathcal{B}_f}(\alpha) \) is an element represented by the morphism

\[
\omega_{12}^\otimes \to K \otimes D' \to R\rho_1(\left( K^{\otimes} \omega_{12}^\otimes \to \omega_2 \otimes D' \right)) \xrightarrow{\alpha} R\rho_1(\left( K \otimes \omega_2 \otimes D' \right)) \xrightarrow{\text{Lemma 3.9}} C_{X_1}^\gamma
\]

in the derived category of sheaves on \( X_1 \). The following commutative diagram in the category \( D^b(\mathcal{E}_{11a} \otimes \mathcal{C}_{22a}) \) directly generalizes a subdiagram of a diagram appearing in the one in the proof of [9, Lemma 4.3] (see [9, p. 111]). The only genuine change in following diagram from the one in the proof of [9, Lemma 4.3] is to change \( \otimes \) to \( \otimes \). We also point out to the reader that the last row in the diagram below is written in a different (though equivalent) way than the corresponding row in [9, p. 111] (modulo the above mentioned change from \( \otimes \) to \( \otimes \)).
By adjunction, the map from \( p_{11}^{-1} \omega_1^{\otimes -1} \) to \( c^f_1 \otimes \mathbb{C}_{X_2}[2\dim(X_2)] \) via the composition of the upper row with the right column is \( \alpha \circ \alpha_2 \) while the map via the composition of the left column with the lower row is \( \Phi_{B_f}(\alpha_2) \). This gives the desired equality of morphisms. □

**Remark 3.11** It is interesting to compare Proposition 3.10 with [7, Theorem 5.4], which is the direct image theorem for the Lefschetz class constructed in [7]. The integral transform \( \Phi_{B_f} \) in Proposition 3.10 corresponds to an honest morphism \( f : X_1 \to X_2 \) of complex manifolds in [7]. On the other hand, the holomorphic diffeomorphisms \( \gamma_{X_1} \) and \( \gamma_{X_2} \) that appear\(^7\) in Proposition 3.10 correspond to a pair of integral transforms, one on \( X_1 \) and the other on \( X_2 \) satisfying certain compatibility criteria with respect to \( f \).

It would be interesting to generalize the material in this and the previous subsection (Proposition 3.10 in particular) to the case when \( \gamma \) acts on \( X_1 \) as well as \( X_2 \) by integral transforms rather than holomorphic diffeomorphisms. The approach here seriously utilizes the fact that \( \gamma \) acts by holomorphic diffeomorphisms, making such a generalization non-trivial. Such a generalization would yield a more general direct image theorem for the Hochschild Lefschetz class than [7, Theorem 5.4]. Further, when combined with an understanding of trace densities, such a result would yield a (possibly simpler) approach to generalizations of [7, Theorem 5.4] itself.

### 3.3 Orbifold Hochschild and Chern class

Let \( QX \) (and \( QM \)) be the orbifold defined by the quotient \( X/\Gamma \) (and \( M/\Gamma \)) for \( X = T^*M \) and let \( q : X \to QX \) be the canonical quotient map. Define a coherent sheaf of algebras \( A \) on \( QX \) by

\[
A(U) := \hat{\mathcal{E}}_X(q^{-1}(U)) \rtimes \Gamma,
\]

for any open subset \( U \subset QX \). In the above definition, \( \Gamma \) acts on \( q^{-1}(U) \), and therefore acts on the algebra \( \hat{\mathcal{E}}_X(q^{-1}(U)) \). \( \hat{\mathcal{E}}_X(q^{-1}(U)) \rtimes \Gamma \) is the associated crossed product algebra. A good coherent \( A \)-module is defined in the same way as a good coherent \( D_M \)-module.

Let \( M \) be a good \( \Gamma \)-equivariant coherent \( D_M \)-module and \( \hat{\mathcal{M}} \) the corresponding \( \Gamma \)-equivariant \( \hat{\mathcal{E}}_X \)-module. Define \( \mathfrak{M} \) to be a sheaf on \( QX \) by

\[
\mathfrak{M}(U) := \hat{\mathcal{M}}(q^{-1}(U)),
\]

for any open subset \( U \subset QX \). On an open subset \( U \subset QX \), both \( \gamma \) and \( \hat{\mathcal{E}}_X(q^{-1}(U)) \) naturally act on \( \hat{\mathcal{M}}(q^{-1}(U)) \) with the appropriate commutation relation between these actions. This equips \( \mathfrak{M} \) with a natural \( A \)-module structure. It is not difficult to check that if \( M \) is a good coherent \( D_M \)-module, \( \mathfrak{M} \) is a good coherent \( A \)-module. We apply the following theorem to construct the Hochschild class \( hh_A^{\mathcal{M}}(\supp(M)/\Gamma,i)(\mathfrak{M}) \) and the cyclic class \( ch_A^{\mathcal{M}}(\supp(M)/\Gamma,i)(\mathfrak{M}) \) of a perfect complex \( \mathfrak{M} \) of \( A \)-modules, where \( \supp(M)/\Gamma \) is the support of \( \mathfrak{M} \) in \( QX \).

**Theorem 3.12** [1, Theorem 2.1.1.] Let \( Q \) be a topological space and \( Z \) a closed subset of \( Q \). Let \( A \) be a sheaf of algebras on \( Q \) such that there is a global section \( 1 \in \Gamma(Q;A) \) which restricts to \( 1_A \), for all \( x \in Q \). Let \( \mathcal{H}^{-}(A) \) (resp., \( \mathcal{H}(A) \)) be the sheaf of negative cyclic (resp., Hochschild) homologies\(^8\) of \( A \). Denote by \( K^Z(A) \) the \( i \)-th \( K \)-group of the category of perfect complexes of \( A \)-modules which are acyclic outside \( Z \). There exists the cyclic

\(^7\) \( \gamma_{X_1} \) (resp., \( \gamma_{X_2} \)) denotes the holomorphic diffeomorphism \( \gamma \) acting on \( X \) (resp., \( X_2 \)).

\(^8\) As in Sect. 2, we abuse terminology here: \( \mathcal{H}^{-}(A) \) and \( \mathcal{H}(A) \) are objects in the derived category of sheaves of \( \mathbb{C} \)-vector spaces on \( Q \). Also, when \( Q = X := T^*M \) as in Sect. 2 and when \( A = \hat{\mathcal{E}}_X \), \( \mathcal{H}(A) \) as defined in [1] is isomorphic to \( \mathcal{H}(A) \) as defined in Sect. 2.
class $\text{ch}_{Z,i}^A : K^i_Z(A) \to H^{-i}_Z(Q; \mathcal{H}C^-(A))$ and the Hochschild class $h^A_{Z,i} : K^i_Z(A) \to H^{-i}_Z(Q; \mathcal{H}H(A))$ such that

- the composition

$$K^i_Z(A) \xrightarrow{\text{ch}_{Z,i}^A} H^{-i}_Z(Q; \mathcal{H}C^-(A)) \rightarrow H^{-i}_Z(Q; \mathcal{H}H(A))$$

coincides with $h^A_{Z,i}$;
- for a perfect complex $\mathcal{F}^\bullet$ of $A$-modules supported on $Z$ the Hochschild class

$$hh^A_{Z,0}(\mathcal{F}^\bullet) \in H^0_Z(Q; \mathcal{H}H(A))$$

coincides with the composition

$$k \xrightarrow{1 \cdot \text{id}} \mathcal{R}\text{Hom}_A(\mathcal{F}^\bullet, \mathcal{F}^\bullet) \xrightarrow{\cong} (\mathcal{R}\text{Hom}_A(\mathcal{F}^\bullet, A) \otimes \mathcal{F}^\bullet) \otimes_{A \otimes A^{\text{op}}} L \xrightarrow{\text{ev} \otimes \text{id}} A \otimes_{A \otimes A^{\text{op}}} A.$$

Applying Theorem 3.12, for a good coherent $\Gamma$-equivariant $\mathcal{D}_M$-module $\mathcal{M}$, we have a well defined Hochschild class $hh^A_{Z,0}(\mathcal{M}) \in H^0_Z(Q_X; \mathcal{H}H(A))$ and cyclic class $\text{ch}_{Z,0}^A(\mathcal{M}) \in H^0_Z(Q_X; \mathcal{H}C^-(A))$ where $Z = \text{supp}(\mathcal{M})/\Gamma$ and $A$ is the sheaf of crossed product algebras defined by $A(U) := \hat{\mathcal{E}}_X(q^{-1}(U)) \rtimes \Gamma$ (for open sets $U$ in $Q_X$).

On $Q_X$, we can also consider the sheaf of algebras $\hat{\mathcal{E}}_{Q_X}$ defined by

$$\hat{\mathcal{E}}_{Q_X}(U) := \hat{\mathcal{E}}_X(q^{-1}(U))^\Gamma.$$

Here, $\hat{\mathcal{E}}_X(q^{-1}(U))^\Gamma$ is the space of $\Gamma$-invariant sections of $\hat{\mathcal{E}}_X(q^{-1}(U))$. Similarly, we consider the good coherent $\hat{\mathcal{E}}_{Q_X}$-module $\hat{\mathcal{M}}_{Q_X}$ defined by

$$\hat{\mathcal{M}}_{Q_X}(U) := \hat{\mathcal{M}}(q^{-1}(U))^\Gamma.$$

Applying Theorem 3.12 to $\hat{\mathcal{E}}_{Q_X}$ and $\hat{\mathcal{M}}_{Q_X}$, we obtain the Hochschild and cyclic classes

$$hh_{Z,0}^{\hat{\mathcal{E}}_{Q_X}}(\hat{\mathcal{M}}_{Q_X}) \in H^0_Z(Q_X; \mathcal{H}H(\hat{\mathcal{E}}_{Q_X}))$$

and $\text{ch}_{Z,0}^{\hat{\mathcal{E}}_{Q_X}}(\hat{\mathcal{M}}_{Q_X}) \in H^0_Z(Q_X; \mathcal{H}C(\hat{\mathcal{E}}_{Q_X}))$, where $Z = \text{supp}(\mathcal{M})/\Gamma$.

Consider the global section

$$e = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \in \Gamma(Q_X; A)$$

of the sheaf $A$. It is easy to check that $e$ is a projection. Define a sheaf $\mathcal{V}$ of $A$-$\hat{\mathcal{E}}_{Q_X}$-bimodules by

$$\mathcal{V}(U) := \hat{\mathcal{E}}_X(q^{-1}(U)) \rtimes \Gamma e|_U,$$

and a sheaf $\mathcal{W}$ of $\hat{\mathcal{E}}_{Q_X}$-$A$-bimodules by

$$\mathcal{W}(U) := e|_U \hat{\mathcal{E}}_X(q^{-1}(U)) \rtimes \Gamma.$$

$\mathcal{V}$ and $\mathcal{W}$ are Morita equivalence bimodules between $A$ and $\hat{\mathcal{E}}_{Q_X}$. Under this Morita equivalence, $\hat{\mathcal{M}}_{Q_X}$ corresponds to the sheaf $\mathcal{M}$. With the explicit bimodules $\mathcal{V}$ and $\mathcal{W}$, we can easily check that under the Morita isomorphism between the Hochschild and cyclic homologies of
\( \mathcal{E}_{Q_X} \) and those of \( \mathcal{A} \), the Hochschild and cyclic classes of \( \mathcal{M}_{Q_X} \) are identified with those of \( \mathcal{M} \).

\[
\begin{align*}
hh_{Z,0}^A(\mathcal{M}) &= hh_{Z,0}(\mathcal{M}_{Q_X}) \in H_2^0(Q_X; \mathcal{H}(\mathcal{A})) \cong H_2^0(Q_X; \mathcal{H}(\mathcal{E}_{Q_X})), \\
ch_{Z,0}^A(\mathcal{M}) &= ch_{Z,0}(\mathcal{M}_{Q_X}) \in H_2^0(Q_X; \mathcal{H}^-)(\mathcal{A}) \cong H_2^0(Q_X; \mathcal{H}^-(\mathcal{E}_{Q_X})).
\end{align*}
\]

The Hochschild and cyclic homology of \( \mathcal{A} \) is computed in [2] and [10]

\[
\begin{align*}
\mu^A : \mathcal{H}^-(\mathcal{A}) &\cong (\oplus_{\gamma} \mathcal{C}_X^{\gamma} \cdot [\dim(X^\gamma)])^\Gamma, \\
\mu^A : \mathcal{H}^+(\mathcal{A}) &\cong (\oplus_{\gamma} \mathcal{C}_X^{\gamma} \cdot [\dim(X^\gamma) - 2(\bullet)])^\Gamma,
\end{align*}
\]

where \( \gamma \in \Gamma \) acts on \( \oplus_{\gamma} \mathcal{C}_X^{\gamma} \cdot [\dim(X^\gamma)] \) mapping the \( \alpha \)-component to the \( \gamma \alpha \gamma^{-1} \)-component.

Let \( I_QX \) be the inertia orbifold associated to \( Q_X \), defined by

\[ I_QX := (\sqcup_{\gamma} X^\gamma)/\Gamma, \]

where \( \gamma \in \Gamma \) acts on \( \sqcup_{\gamma} X^\gamma \) by mapping (\( \alpha, x \)) with \( \alpha(x) = x \) to \( (\gamma \alpha \gamma^{-1}, \gamma(x)) \). Let \( t_{IQX} : I_QX \to Q_X \) be the natural map defined by forgetting the group element. Thus, we have

\[
(\oplus_{\gamma} \mathcal{C}_X^{\gamma} \cdot [\dim(X^\gamma)])^\Gamma = t_{IQX,*} \mathcal{C}[\dim(I_QX)].
\]

**Definition 3.13** The orbifold Euler class \( eu_{Q_X}(\mathcal{M}) \) (resp., the orbifold Chern class \( ch_{Q_X}(\mathcal{M}) \)) of a good \( \Gamma \)-equivariant coherent \( D_M \)-module \( \mathcal{M} \) is defined to be the images of \( hh_{Z,0}^A(\mathcal{M}) \) (resp., \( ch_{Z,0}^A(\mathcal{M}) \)) in \( H_2^0(I_QX; \mathcal{C}[\dim(I_QX)]) \) (resp., \( \oplus_{n \geq 0} H_2^0(I_QX; \mathcal{C}[\dim(I_QX) - 2n]) \)).

**Remark 3.14** In [1], the classes \( hh_{Z,0}^A(\mathcal{M}) \) and \( ch_{Z,0}^A(\mathcal{M}) \) are called the Euler and the Chern class respectively. Here, we distinguish them from their images in the \( H_2^0(I_QX; \mathcal{C}[\dim(I_QX)]) \), which are closer to the classical Euler and Chern characters.

In the remaining part of this section, we will explain the relation between the Hochschild Lefschetz class in Definition 3.2 and orbifold Hochschild class in Theorem 3.12.

We observe that \( \sum_{\gamma} hh^A(\mathcal{M}, \gamma) \in \oplus_{\gamma} H_0^{supp(\mathcal{M})}(X; \mathcal{H}(\mathcal{E}_X^\gamma)) \) is invariant under the action of \( \Gamma \) on \( \oplus_{\gamma} H_0^{supp(\mathcal{M})}(X; \mathcal{H}(\mathcal{E}_X^\gamma)) \) induced by the conjugation action \( \alpha \mapsto \gamma \alpha \gamma^{-1} \) of \( \Gamma \) on itself. Consider

\[
\begin{align*}
\widehat{hh}_{Z,0}^{Q_X}(\mathcal{M}) &:= \frac{1}{|\Gamma|} \sum_{\gamma} hh^A(\mathcal{M}, \gamma) \\
&\cong H_0^{supp(\mathcal{M})} \left( Q_X; \mathcal{H}(\mathcal{E}_X^\gamma) \right)^\Gamma.
\end{align*}
\]

Here, by abuse of notation, we also use the symbol \( \mathcal{E}_X \) to denote the sheaf \( U \mapsto \mathcal{E}_X(q^{-1}(U)) \) of algebras on the orbifold \( Q_X \). Note that the sheaf \( \mathcal{E}_X \) is a sheaf of algebras on \( Q_X \) with \( \mathcal{A} \) (local) \( \Gamma \)-action and that \( \mathcal{E}_{Q_X} = \mathcal{E}_X \). The Hochschild homology \( H_2^0(Q_X, \mathcal{H}(\mathcal{A})) \) is naturally isomorphic to \( (\oplus_{\gamma} H_0^{supp(\mathcal{M})}(X; \mathcal{H}(\mathcal{E}_X^\gamma)))^\Gamma \) (see e.g. [2]). Identifying \( (\oplus_{\gamma} H_0^{supp(\mathcal{M})}(X; \mathcal{H}(\mathcal{E}_X^\gamma)))^\Gamma \) with \( H_2^0(Q_X, \mathcal{H}(\mathcal{A})) \) using this isomorphism, the following equality holds.
Theorem 3.15

\[ \overline{hh}_Z^0(M) = hh_Z^A(M). \]

We shall now sketch the proof of Theorem 3.15, leaving details to the interested reader.

Sketch of Proof In what follows, \( \hat{E} := \hat{E}_X \) is thought of as a sheaf of algebras on \( Q_X \). Let \( \gamma_*(M) \) denote the sheaf \( M \) on \( Q_X \), whose \( \hat{E} \)-module structure is twisted by \( \gamma \) like \( \gamma_*(M) \).

Define \( L: \text{Hom}_{\hat{E} \times \Gamma}(M, M) \to (\bigoplus_{\gamma \in \Gamma} \text{Hom}_\hat{E}(M, \gamma_*(M)))^\Gamma \) by

\[ L(F) := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \circ F. \]

Here, \( \gamma \circ F(m) = \gamma^{-1}(F(\gamma(m))) \) for \( m \in \Gamma(U, \hat{E}_X) \). Define

\[ \tilde{L}: \text{Hom}_{\hat{E} \times \Gamma}(M, \hat{E} \times \Gamma) \otimes_{\hat{E} \times \Gamma} M \to \left( \bigoplus_{\gamma \in \Gamma} \text{Hom}_\hat{E}(M, \hat{E} \otimes \gamma_*(M)) \right)^\Gamma \]

by

\[ \tilde{L}(F \otimes \gamma \otimes m) := \frac{1}{|\Gamma|} \sum_{\gamma} \sum_{\alpha} F_{\alpha} \otimes \gamma^{-1}(\alpha(m)), \]

where \( F_{\alpha} \in \text{Hom}_\hat{E}(M, \hat{E}) \) is defined by \( F = \sum_{\alpha} F_{\alpha} \otimes \alpha \in \text{Hom}_\hat{E}(M, \hat{E} \otimes (\hat{E} \times \Gamma)) \) and \( \gamma^{-1}(\alpha(m)) \) is viewed as a section of \( \gamma_*(M) \). Also recall that the action of an element \( g \in \Gamma \) on \( \bigoplus_{\gamma \in \Gamma} \text{Hom}_\hat{E}(M, \hat{E}) \otimes \gamma_*(M) \) takes a section of the form \( F(-) \otimes m \) to \( g(F(g^{-1}(-))) \otimes g.m \).

The morphisms denoted by \( \mu \) in the diagram below are the obvious “evaluation” maps:

\[
\begin{array}{c}
\text{Hom}_{\hat{E} \times \Gamma}(M, M) \xleftarrow{\mu} \text{Hom}_{\hat{E} \times \Gamma}(M, \hat{E} \times \Gamma) \otimes_{\hat{E} \times \Gamma} M \\
L \downarrow \hspace{2cm} \downarrow \tilde{L} \\
\left( \bigoplus_{\gamma \in \Gamma} \text{Hom}_\hat{E}(M, \gamma_*(M)) \right)^\Gamma \hspace{2cm} \left( \bigoplus_{\gamma \in \Gamma} \text{Hom}_\hat{E}(M, \hat{E}) \otimes \gamma_*(M) \right)^\Gamma
\end{array}
\]

It is straightforward to check that the following diagram commutes.

Define \( \tilde{L}: (\bigoplus_{\gamma} \text{Hom}_\hat{E}(M, \hat{E}) \otimes \gamma_*(M)) \otimes A \otimes Q \to (\bigoplus_{\gamma} \text{Hom}_\hat{E}(M, \hat{E}) \otimes \gamma_*(M)) \otimes A \otimes Q \) by

\[ \tilde{L}(((F_{\alpha} \otimes \alpha) \otimes m) \otimes (d \otimes \beta)) := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} (F_{\alpha} \otimes \gamma^{-1}(\alpha)(m)) \otimes \alpha(d). \]

Here, \( \gamma^{-1}(\alpha)(m) \) is viewed as a section of \( \gamma_*(M) \). Also recall that the action of an element \( g \in \Gamma \) on \( \bigoplus_{\gamma} \text{Hom}_\hat{E}(M, \hat{E}) \otimes \gamma_*(M) \) takes a section \( F(-) \otimes m \) to \( g(F(g^{-1}(-))) \otimes g.m \).
\[ g.m \otimes g.f. \text{ The following diagram commutes:} \]

\[
\begin{array}{cccc}
\text{\( L \)} & \text{\( \Hom_{\tilde E \times \Gamma}(M, \tilde E \times \Gamma) \otimes_{\tilde E \times \Gamma} M \)} & \text{\( \cong \)} & \text{\( (\Hom_{\tilde E \times \Gamma}(M, \tilde E \times \Gamma) \otimes M) \otimes_{A_{\tilde E \times \Gamma}} (\tilde E \times \Gamma) \)} \\
\text{\( L \)} & & & \\
\end{array}
\]

Define \( \hat{L} : (\tilde E \times \Gamma) \otimes_{A_{\tilde E \times \Gamma}} (\tilde E \times \Gamma) \rightarrow (\bigoplus_{\gamma \in \Gamma} \gamma_*(\tilde E) \otimes \tilde E) \Gamma \) by

\[
\hat{L}((e_0 \otimes \alpha) \otimes (e_1 \otimes \beta)) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma(e_0) \otimes \gamma(\alpha(e_1)).
\]

Here, \( \gamma(e_0) \) is viewed as a section of \( (\gamma \alpha \beta \gamma^{-1})_*(\tilde E) \) and \( \gamma(\alpha(e_1)) \) is viewed as a section of \( E \). Also recall that the action of an element \( g \in \Gamma \) maps a section \( e \otimes f \) of \( \gamma_*(\tilde E) \otimes E \) to the section \( ge \otimes gf \) of \( (g \gamma^{-1})_* (\tilde E) \otimes E \). We further have the following commutative diagram:

\[
\begin{array}{cccc}
\text{\( L \)} & \text{\( \Hom_{\tilde E \times \Gamma}(M, \tilde E \times \Gamma) \otimes_{\tilde E \times \Gamma} M \)} & \text{\( \ev \otimes \id \)} & \text{\( (\tilde E \times \Gamma) \otimes_{A_{\tilde E \times \Gamma}} (\tilde E \times \Gamma) \)} \\
\text{\( L \)} & & & \\
\end{array}
\]

Combining the above three commutative diagrams gives us the desired theorem: indeed, the image of \( \id \in \Hom_{\tilde E \times \Gamma}(M, M) \) in \( H^0(Q_X, (\tilde E \times \Gamma) \otimes_{A_{\tilde E \times \Gamma}} (\tilde E \times \Gamma)) \) under the morphism induced by the upper horizontal arrows in the above three diagrams is \( hh^A_{\tilde E \times \Gamma}(M) \), while the image of \( \gamma \in \Hom_{\tilde E}(M, \gamma_*(M)) \) in \( H^0(Q_X; \mathcal{H}(\tilde E, \tilde E \gamma)) \) under the morphism induced by the lower arrows in the above three diagrams is \( hh^\gamma(M, \gamma) \).

**Remark 3.16** The orbifold Euler class in Definition 3.13 has a direct generalization to general orbifolds other than global quotient orbifolds, i.e. orbifolds of the form \( M / \Gamma \). This generalization is obtained by working with the sheaf of invariant differential operators as explained in [6]. Our orbifold Riemann–Roch theorem in the next section also generalizes to this setting. We will leave the details of this generalization to the reader.
4 Euler class on an orbifold

In this section, we prove an orbifold Riemann–Roch theorem for the orbifold Euler class \( e\text{u}_{QX}(\mathcal{M}) \) of a good \( \Gamma \)-equivariant coherent \( D_{\mathcal{M}} \)-module introduced in Definition 3.13. Our main strategy is to generalize the method developed by Bressler et al. [1] to orbifold setting.

4.1 Deformation quantization

Our strategy to compute the \( e\text{u}_{QX}(\mathcal{M}) \) is to transfer the computation to a more flexible context: that of deformation quantization modules. Closely related to the deformation quantization algebra(s) \( \hat{\mathcal{V}}_X(0) \) on \( \mathcal{X} = T^*\mathcal{M} \) over the ring \( \mathbb{C}[[\hbar]] \) constructed in [13] modeled on (the sheaf of) negative order formal microdifferential operators. Let \( \hat{\mathcal{V}}_X \) be the localization of \( \hat{\mathcal{V}}(0) \) defined by
\[
\hat{\mathcal{V}}_X := \hat{\mathcal{V}}(0) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}(\hbar).
\]
The sheaves of algebras \( D_{\mathcal{M}}, \hat{\mathcal{E}}_X, \) and \( \hat{\mathcal{V}}_X \) are naturally related to one another by the inclusions

\[
\pi_{\mathcal{M}}^{-1}D_{\mathcal{M}} \hookrightarrow \hat{\mathcal{E}}_X \hookrightarrow \hat{\mathcal{V}}_X.
\]

Following [9] we consider the functor
\[
(\cdot)^W : Mod(D_{\mathcal{M}}) \to Mod(\hat{\mathcal{V}}_X),
\]

\[
\mathcal{M} \mapsto \mathcal{M}^W := \hat{\mathcal{V}}_X \otimes_{\pi_{\mathcal{M}}^{-1}D_{\mathcal{M}}} \pi_{\mathcal{M}}^{-1}\mathcal{M}.
\]

By [9, Proposition 6.4.1], this functor is exact, faithful, and preserves properties such as coherence and goodness.

Note that the Lefschetz class of a good coherent \( D_{\mathcal{M}} \)-module with values in the Hochschild homology of \( \hat{\mathcal{V}}_X \) can be defined in exactly the same way as how the Lefschetz class of a good coherent \( D_{\mathcal{M}} \)-module defined in Sect. 3.

To be precise, given a good coherent \( D_{\mathcal{M}} \)-module \( \mathcal{M} \), we consider the associated good coherent \( \hat{\mathcal{V}}_X \)-module \( \mathcal{M}^W \). As is explained in Sect. 2, \( \gamma_* \) defines a natural functor from \( D^b(\hat{\mathcal{V}}_X) \) and \( D^b_{\text{coh}}(\hat{\mathcal{V}}_X) \) to itself. Let \( C^W_X := \delta_{X,*}\hat{\mathcal{V}}_X \) viewed as a \( \hat{\mathcal{V}}_{X \times X^a} \)-module. Similarly, let \( \hat{C}^W_X := \delta_{X^a,*}\hat{\mathcal{V}}_X \). We have a natural morphism analogous to that of Lemma 3.1:
\[
\gamma_*(\mathcal{M}^W) \boxtimes_{\hat{\mathcal{V}}_X} \hat{\mathcal{V}}_X^W (\mathcal{M}^W) \to \hat{C}^W_X.
\]

For \( u \in \text{Hom}_{D_{\mathcal{M}}}(\mathcal{M}, \gamma_*(\mathcal{M})) \), the definition of the Hochschild Lefschetz class \( hh^{\gamma,W}(\mathcal{M}, u) \) of a good coherent \( D_{\mathcal{M}} \) module \( \mathcal{M} \) is completely analogous to Definition 3.2. Indeed, \( hh^{\gamma,W}(\mathcal{M}, u) \) is defined to be the image of \( \hat{u}^W \in \text{Hom}_{\hat{\mathcal{V}}_X}(\mathcal{M}^W, \gamma_*(\mathcal{M}^W)) \) under the morphism induced on cohomologies by the following composite of morphisms:

\[
\text{RHom}_{\hat{\mathcal{V}}_X}(\mathcal{M}^W, \gamma_*(\mathcal{M}^W)) \leftarrow D^l_{\hat{\mathcal{V}}_X}(\mathcal{M}^W) \boxtimes_{\hat{\mathcal{V}}_X} \gamma_*(\mathcal{M}^W) \cong C^W_X \boxtimes_{\hat{\mathcal{V}}_X} \gamma_*(\mathcal{M}^W) \to \hat{C}^W_X = \mathcal{H}L(\hat{\mathcal{V}}_X, \hat{\mathcal{V}}_X^W).
\]

One can similarly provide definitions of \( \mu_{\text{eu}}^{\gamma,W}(\mathcal{M}, u) \), \( \text{eu}^{W}_{QX}(\mathcal{M}) \), and \( \text{ch}^{W}_{QX}(\mathcal{M}) \) that are completely analogous to the corresponding definitions in Sect. 3.
Recall that the support of $\tilde{\mathcal{M}} := \hat{\mathcal{E}} \otimes_{\pi\pi_M^{-1} D_M} \pi^{-1} \mathcal{M}$ in $X$ is called the characteristic variety of $\mathcal{M}$ and denoted by $\text{char}(\mathcal{M})$. The following Lemma is a direct generalization of [9, Lemma 6.5.1] to the $\gamma$ twisted setting. Let $\iota : X^\gamma \to X$ be as in Sect. 3.

**Proposition 4.1** There is a natural trace density isomorphism

$$\mathcal{H}(\hat{\mathcal{E}}_X, \hat{\mathcal{E}}^\gamma_X) \xrightarrow{\mu_{\hat{\mathcal{E}}}^\gamma} \iota_! C_{X^\gamma} ([\dim(X^\gamma)])$$

in the derived category of sheaves of $\mathbb{C}((\hbar))$-vector spaces on $X$ such that the diagram following commutes:

$$\mathcal{H}(\hat{\mathcal{E}}_X, \hat{\mathcal{E}}^\gamma_X) \xrightarrow{\mu_{\hat{\mathcal{E}}}^\gamma} \iota_! C_{X^\gamma} [\dim(X^\gamma)] \quad .$$

Therefore, using the natural map from $H^{\dim(X^\gamma)}(X; C_{X^\gamma})$ to $H^{\dim(X^\gamma)}(X; C_{X^\gamma}((\hbar)))$ to identify $H^{\dim(X^\gamma)}(X; C_{X^\gamma})$ with its image in $H^{\dim(X^\gamma)}(X; C_{X^\gamma}((\hbar)))$, one obtains the following identities for a good $\Gamma$-equivariant coherent $D_M$ module $\mathcal{M}$.

$$hh^\gamma(\mathcal{M}, \gamma) = hh^\gamma,\mathbb{W}(\mathcal{M}, \gamma), \quad \text{eu}_{Q_X}(\mathcal{M}) = \text{eu}_{Q_X}^W(\mathcal{M}),$$

$$\mu \text{eu}^\gamma(\mathcal{M}, \gamma) = \mu \text{eu}^\gamma,\mathbb{W}(\mathcal{M}, \gamma), \quad \text{ch}_{Q_X}(\mathcal{M}) = \text{ch}_{Q_X}^W(\mathcal{M}).$$

### 4.2 Orbifold Riemann–Roch theorem

In this subsection, we describe the geometric formula for the orbifold Euler class $\text{Eu}_Q(\mathcal{M})$ of a $\Gamma$-equivariant good coherent $D_M$ module $\mathcal{M}$.

We recall some geometry of the orbifold $Q_X = X/\Gamma$. Note that $X^\gamma$ may have several components with different dimensions, but each component of $X^\gamma$ is a submanifold of $X$. Consider a vector bundle $V$ on $X^\gamma$ equipped with a $\gamma$ action on each fiber. Let $R^\gamma$ be the curvature of a connection on $V$. Define $\text{ch}_{\gamma}^\gamma(V)$ to be

$$\text{ch}_{\gamma}^\gamma(V) := \text{tr} \left( \gamma \exp \left( \frac{R^\gamma}{2\pi \sqrt{-1}} \right) \right) \in H^{\text{even}}(X^\gamma; \mathbb{C}).$$

Over each component of $X^\gamma$, let $N^\gamma$ be the normal bundle of $X^\gamma$ to $X$. Observe that $\gamma$ acts on fibers of $N^\gamma$ and $\wedge\cdot N^\gamma$. Define

$$\text{eu}_{\gamma}^\gamma(N^\gamma) := \sum (-1)^\cdot \text{ch}_{\gamma}(\wedge\cdot N^\gamma) = \det \left( 1 - \gamma^{-1} \exp \left( \frac{-R^\perp}{2\pi \sqrt{-1}} \right) \right),$$

where $R^\perp$ is the curvature of a connection on $N^\gamma$. The group $\Gamma$ naturally acts on $\sqcup_{\gamma} X^\gamma$: $\gamma \in \Gamma$ maps $x \in X^\alpha$ to $\gamma(x) \in X^{\gamma\alpha \gamma^{-1}}$. It is straightforward to see that

$$\text{eu}_Q(N) := \sum_{\gamma \in \Gamma} \text{eu}_{\gamma}^\gamma(N^\gamma) \in \bigoplus_{\gamma \in \Gamma} H^{\text{even}}_{\text{char}(\mathcal{M})^\gamma}(X^\gamma; \mathbb{C}),$$

is invariant under the $\Gamma$ action on the above direct sum of the cohomology groups.
Given a $\Gamma$-equivariant good coherent $\mathcal{D}_M$-module $\mathcal{M}$, we consider
\[ \hat{\mathcal{M}} := \hat{\mathcal{E}}_X \otimes_{\pi^{-1}_M \mathcal{D}_M} \pi^{-1}_M \mathcal{M}, \]
equipped with a natural filtration, whose support is denoted by $\text{char}(\mathcal{M})$. The sheaf $\hat{\mathcal{E}}_X$ has a natural filtration $\{\mathcal{F}^n \hat{\mathcal{E}}_X\}$ by the order $n$ of an operator. Let $\text{Gr}(\hat{\mathcal{E}}_X)$ (resp., $\hat{\mathcal{M}}$) denote the associated graded algebra and module of $\hat{\mathcal{E}}_X$ (resp., $\mathcal{M}$). Define
\[ \hat{\text{Gr}}(\mathcal{M}) := \mathcal{O}_X \otimes_{\text{Gr}(\hat{\mathcal{E}}_X)} \text{Gr}(\hat{\mathcal{M}}). \]

The symbol $\sigma \text{char}(\mathcal{M})(M)$ is the element in the $\Gamma$-equivariant K-theory $K_{\text{top}}(\Gamma)\text{char}(\mathcal{M})(M)$ defined by $\hat{\text{Gr}}(\mathcal{M})$. Restricted to $X^\gamma$, $\sigma \text{char}(\mathcal{M})(M|_{X^\gamma})$ defines a K-theory element on $X^\gamma$ inheriting a $\gamma$-action. Applying $\chi_\gamma(-)$ to this element, one defines an element
\[ \chi_\gamma(M) := \chi_\gamma(M|_{X^\gamma}) \in \bigoplus_{\gamma \in \Gamma} H_{\text{even}}^{\text{char}(\mathcal{M})}(X^\gamma; \mathbb{C}) \]
is invariant under the $\Gamma$ action on the above direct sum of the cohomology groups.

Define $\text{char}_Q(M) \subset I Q_X$ to be the quotient $(\sqcup_{\gamma \in \Gamma} \text{char}(M|_{X^\gamma}))/\Gamma$, where $\Gamma$ acts on $\sqcup_{\gamma \in \Gamma} \text{char}(M|_{X^\gamma})$ via its action on $\sqcup_{\gamma \in \Gamma} X^\gamma$. We are now ready to state the main theorem of this paper.

**Theorem 4.2** (Orbifold Riemann–Roch) Let $Q_M$ be the quotient of $M$ by $\Gamma$. For a $\Gamma$-equivariant good coherent $\mathcal{D}_M$ module $\mathcal{M}$, we have
\[ \text{eu}_Q(M) = \frac{1}{m} \left( \chi_\gamma(M) \wedge \text{eu}_Q(N) \wedge \pi^* T d_I Q_M \right)_{\text{dim}(I Q_X)} \]
as a cohomology class in $H_{\text{dim}(I Q_X)}^{\text{char}(\mathcal{M})}(I Q_X; \mathbb{C})$. Here, $\pi^* T d_I Q_M$ is the Todd class of the orbifold $I Q_M$ defined by
\[ T d_I Q_M := \text{tr} \left( \frac{R}{2 \pi \sqrt{-1}} \frac{R}{1 - \exp \frac{R}{2 \pi \sqrt{-1}}} \right), \]
where $R$ is the curvature of a connection on the tangent bundle $T I Q_M$, $\pi : I Q_X \to I Q_M$ is the natural projection, and $m$ denotes the locally constant function on $I Q$ measuring the size of the isotropy group.

**Remark 4.3** The wedge product of differential forms on $I Q_X$ used in Theorem 4.2 is the wedge product on each component of $I Q_X$.

The proof of Theorem 4.2 occupies the next two subsections. Our basic idea is to generalize the Bressler–Nest–Tsygan proof in [1] to the $\Gamma$-equivariant setting.

**4.3 Rees construction**

We consider the Rees ring $\mathcal{R} \hat{\mathcal{E}}_X$ associated to the filtration $\{\mathcal{F}^n \hat{\mathcal{E}}_X\}$ of $\hat{\mathcal{E}}_X$:
\[ \mathcal{R} \hat{\mathcal{E}}_X := \bigoplus_p \mathcal{H}^p \mathcal{F}^p \hat{\mathcal{E}}_X. \]

We list a few well-known properties of $\mathcal{R} \hat{\mathcal{E}}_X$ without proofs.
Proposition 4.4 (cf. [1])

(1) Let $\text{Gr} \hat{\mathcal{E}}_X$ be the associated graded ring of $\hat{\mathcal{E}}_X$ with respect to the filtration $\mathcal{F}^\bullet \hat{\mathcal{E}}_X$. There are natural algebra homomorphisms

$$\sigma^{\text{Gr} \hat{\mathcal{E}}} : \mathcal{R} \hat{\mathcal{E}}_X \xrightarrow{\text{Gr}} \text{Gr} \hat{\mathcal{E}}_X \xrightarrow{i} \mathcal{O}_X.$$ 

(2) ([1, I, Proposition 4.5.1]) There is a natural flat embedding by mapping $h \xi$ to $\xi$ along the fiber direction of $T^*M$,

$$i^{\mathcal{R} \hat{\mathcal{E}}} : \mathcal{R} \hat{\mathcal{E}}_X \rightarrow \hat{\mathcal{W}}_X(0).$$

(3) Define $\mathcal{R} \hat{\mathcal{E}}_X[h^{-1}] := \mathcal{R} \hat{\mathcal{E}}_X \otimes \mathbb{C}[[h]] \mathbb{C}((h))$. There is a natural identification

$$\mathcal{R} \hat{\mathcal{E}}_X[h^{-1}] \cong \hat{\mathcal{E}}_X((h)) := \hat{\mathcal{E}}_X \otimes \mathbb{C} \mathbb{C}((h)).$$

(4) Given a $\hat{\mathcal{E}}_X$-module $\hat{\mathcal{M}}$ with a filtration $\mathcal{F}^\bullet \hat{\mathcal{M}}$, define an $\mathcal{R} \hat{\mathcal{E}}_X$-module by

$$\mathcal{R} \hat{\mathcal{M}} := \bigoplus_p h^p \mathcal{F}^p \hat{\mathcal{M}}.$$ 

One has the following natural isomorphisms of $\mathcal{O}_X$-modules,

$$\mathcal{R} \hat{\mathcal{M}} \otimes_{\mathcal{R} \hat{\mathcal{E}}_X} \mathcal{R} \hat{\mathcal{E}}_X[h^{-1}] \cong \hat{\mathcal{M}} \otimes_{\hat{\mathcal{E}}_X} \hat{\mathcal{E}}_X[h^{-1}, h], \quad (7)$$

$$\mathcal{R} \hat{\mathcal{M}} \otimes_{\mathcal{R} \hat{\mathcal{E}}_X} \mathcal{O}_X \cong \text{Gr} \hat{\mathcal{M}} \otimes_{\text{Gr} \hat{\mathcal{E}}_X} \mathcal{O}_X. \quad (8)$$

In addition, when a finite group $\Gamma$ acts on $M$, the same results hold for $\hat{\mathcal{E}}_X \times \Gamma$, $\hat{\mathcal{W}}_X(0) \times \Gamma$, $\mathcal{R} \hat{\mathcal{E}}_X \times \Gamma$, $\text{Gr} \hat{\mathcal{E}}_X \times \Gamma$, $\mathcal{O}_X \times \Gamma$ and a $\Gamma$-equivariant $\hat{\mathcal{E}}_X$-module $\hat{\mathcal{M}}$.

By Proposition 4.4 (3), we have natural morphisms

$$i^{h^{-1}, \mathcal{R} \hat{\mathcal{E}}} : \mathcal{R} \hat{\mathcal{E}}_X \rightarrow \mathcal{R} \hat{\mathcal{E}}_X[h^{-1}, h] \leftarrow \mathcal{E}_X : i^{h^{-1}, \mathcal{E}}. \quad (9)$$

Following the idea developed in Sect. 3.3, when a finite group $\Gamma$ group acts on a manifold $M$ and therefore also $X = T^*M$, we view $\hat{\mathcal{E}}_{Q_X} := \hat{\mathcal{E}}_X \times \Gamma$, $\mathcal{R} \hat{\mathcal{E}}_{Q_X} := \mathcal{R} \hat{\mathcal{E}}_X \times \Gamma$, $\text{Gr} \hat{\mathcal{E}}_{Q_X} := \text{Gr} \hat{\mathcal{E}}_X \times \Gamma$, $\mathcal{O}_{Q_X} := \mathcal{O}_X \times \Gamma$, $\hat{\mathcal{W}}(0)_{Q_X} := \hat{\mathcal{W}}(0) \times \Gamma$, and $\hat{\mathcal{W}}_{Q_X} := \hat{\mathcal{W}}_X \times \Gamma$ as sheaves of algebras over the orbifold $Q_X = X/\Gamma$. And similarly a $\Gamma$-equivariant $\hat{\mathcal{E}}_X$-module $\hat{\mathcal{M}}$ is viewed as a sheaf of $\hat{\mathcal{E}}_{Q_X} := \hat{\mathcal{E}}_X \times \Gamma$-module $\hat{\mathcal{M}}_{Q_X}$ over $Q_X$.

A crucial observation is that Theorem 3.12 applies to the sheaves of algebras introduced above and defines Chern character maps on the corresponding $K$-groups of perfect complexes of modules. These Chern characters are denoted by $\text{ch}^A(-)$ with $A$ being relevant sheaves of algebras. We apply Proposition 4.4 to study the Chern character of $\hat{\mathcal{M}}_{Q_X}$:

**Proposition 4.5**

$$\sigma^\mathcal{R} \hat{\mathcal{E}}_x : \mathcal{R} \hat{\mathcal{E}}_{Q_X} \cong i_* \text{ch}^\mathcal{O} (\mathcal{M}_{Q_X} \otimes \hat{\mathcal{E}}_{Q_X} \mathcal{O}_{Q_X}).$$

**Proof** This is a direct corollary of Proposition 4.4 (1), and Eqs. (8). \qed

**Proposition 4.6**

$$i^{h^{-1}, \mathcal{R} \hat{\mathcal{E}}} : \mathcal{R} \hat{\mathcal{E}}_{Q_X} \cong \text{ch}^{h^{-1}} (\mathcal{R} \hat{\mathcal{M}}_{Q_X} \otimes \mathcal{R} \hat{\mathcal{E}}_{Q_X} \mathcal{R} \hat{\mathcal{E}}_{Q_X}[h^{-1}]) = i^{h^{-1}, \mathcal{E}} (\text{ch}^{\mathcal{O}} (\mathcal{M}_{Q_X})).$$

**Proof** This is a direct corollary of Eq. (7), (9), and Proposition 4.4, (3). \qed
There is a natural map $\sigma^\hat{\psi} : \hat{\psi}_X(0) \to O_X$ defined by taking the quotient by the two sided (sheaf of) ideal(s) generated by $\hbar$. It is easy to check that
\[ \sigma^\hat{\psi} \circ \hat{\psi} = \sigma^\hat{\psi} : \hat{\psi}_{Q_X} \to O_{Q_X}. \] (10)

**Proposition 4.7**
\[ \sigma^\hat{\psi} (\text{ch}^\hat{\psi} (\hat{\psi}_{Q_X})) = \sigma^\hat{\psi} \circ \hat{\psi} (\text{ch}^\hat{\psi} (\hat{\psi}_{Q_X})) = \text{ch}^\hat{\psi} (\hat{\psi}_{Q_X} \hat{\psi} O_{Q_X}). \]

**Proof** This is a direct corollary of Eq. (10).

**Proposition 4.8** The following diagram commutes
\[
\begin{array}{ccc}
\hat{\psi}_X(0) & \xrightarrow{\psi^{-1} \hat{\psi}} & \hat{\psi}_X \\
\downarrow \hat{\psi} & & \downarrow \hat{\psi} \\
\hat{\psi}_{Q_X} & \xrightarrow{i} & \hat{\psi}_{Q_X} \\
\end{array}
\]

where $i^{\psi^{-1}} \hat{\psi}$ is the natural inclusion map $\hat{\psi}_X(0) \hookrightarrow \hat{\psi}_X$, and $i^{\hat{\psi}}$ is the natural extension of $i^\psi$.

**Proof** This is a straightforward verification using the definitions.

We denote the hypercohomology $H^{-\bullet} (\psi Q_X ; \psi C^\text{per} (\hat{\psi}_{Q_X}))$ by $HC^{-\bullet} (\hat{\psi}_{Q_X})$, where $\psi C^\text{per}$ is the sheaf of periodic cyclic homology. Similar notation is used for other sheaves of algebras on $Q_X$ (and for other versions of cyclic homology). Note that for any sheaf $A$ of algebras on $Q_X$ (or on any topological space for that matter), there is a natural map $\psi C^{-} (A) \to \psi C^\text{per} (A)$ in the derived category of sheaves of $\mathbb{C}$-vector spaces on $Q_X$. Hence, one may view $\text{ch}_{Z,i}$ (see Theorem 3.12) as a map to $HC^0_{\text{per}} (A)$. The following proposition is a direct corollary of Proposition 4.8.

**Proposition 4.9** The following diagram commutes.

\[
\begin{array}{ccc}
HC^0_{\text{per}} (\hat{\psi}_{Q_X}) & \xrightarrow{i^\psi} & HC^0_{\text{per}} (\hat{\psi}_{Q_X}) \\
\downarrow \hat{\psi} & & \downarrow \hat{\psi} \\
HC^0_{\text{per}} (\hat{\psi}_{Q_X}) & \xrightarrow{i^\psi} & HC^0_{\text{per}} (\hat{\psi}_{Q_X}) \\
\end{array}
\]
4.4 Proof of Theorem 4.2

The following is a reformulation of [12, Theorem 5.13].

**Theorem 4.10** ([12, Theorem 5.13]) Let $u$ be the parameter in the definition of cyclic homology, and $Q_M$ (resp., $Q_X$) be the quotient of $M$ (resp., $X$) by $\Gamma$. The following diagram commutes:

$$
\begin{array}{ccc}
HC_0^{\text{per}}(\hat{\mathcal{V}}_{Q_M}(0)) & \overset{\sigma^*_x(\hat{\mathcal{V}}(0))}{\longrightarrow} & H^{-\bullet}(\Omega^*_I Q_M((u)), d) \\
\hat{\mathcal{V}}_{Q_M} & \overset{\mu(\hat{\mathcal{V}}_{Q_M})}{\longrightarrow} & H^{-\bullet}(\Omega^*_I Q_M((\hat{h}))), d)
\end{array}
$$

Proof Let $i^Q : IQ \rightarrow Q_X$ be the natural forgetful map. The key observation is that the quasi-isomorphisms $\sigma^*_x(\hat{\mathcal{V}}(0))$ and $\mu(\hat{\mathcal{V}}_{Q_M})$ constructed in [12, Theorem 5.13] are morphisms of $i^Q_{\text{per}}(\mathcal{Q}_X))$-modules on $Q_X$ (where $\mathcal{Q}_X$ denotes the (locally) constant sheaf whose space of sections over any connected open subset of $IQ_X$ is $\mathbb{C}(u)$).

For an element $x$ of $H^{-\bullet}(IQ_X ; \mathcal{C}(u))$. Let $1$ denote the trivial $\hat{\mathcal{V}}_{Q_M}(0)$-module. Then, we notice that $\sigma^*_x(\hat{\mathcal{V}}(0))$ maps $\sigma^*_x(\hat{\mathcal{V}}(0)) = \chi(\hat{\mathcal{V}}(0))$, as an element in $HC_0^{\text{per}}(\hat{\mathcal{V}}_{Q_M}(0))$, also to $\sigma^*_x(\hat{\mathcal{V}}(0))$ in $HC_0^{\text{per}}(\hat{\mathcal{V}}_{Q_M}(0))$, where $\cup$ is the cup product

$$
\cup : H^{-\bullet}(IQ_X ; \mathcal{C}(u)) \otimes H^{-\bullet}(IQ_X ; HC^{\text{per}}(\hat{\mathcal{V}}_{Q_M}(0))) \rightarrow H^{-\bullet}(IQ_X ; HC^{\text{per}}(\hat{\mathcal{V}}_{Q_M}(0))).
$$

As $\sigma^*_x(\hat{\mathcal{V}}(0))$ is a quasi-isomorphism, we conclude that $x = [\sigma^*_x(\hat{\mathcal{V}}(0))] \cup [\chi(\hat{\mathcal{V}}(0))]$ is in $H^{-\bullet}(IQ_X ; HC^{\text{per}}(\hat{\mathcal{V}}_{Q_M}(0)))$. Since $\mu(\hat{\mathcal{V}}_{Q_M})$ is a $H^{-\bullet}(IQ_X , \mathcal{C}(u))$-module map, we have

$$
\mu(\hat{\mathcal{V}}_{Q_M}((x) \cup \chi(\hat{\mathcal{V}}(0))) = \sigma^*_x(\hat{\mathcal{V}}(0)) \cup \mu(\hat{\mathcal{V}}_{Q_M}(h^{-1},(\chi(\hat{\mathcal{V}}(0))))).
$$

This reduces the proof to computing $\mu(\hat{\mathcal{V}}_{Q_M}(h^{-1},(\chi(\hat{\mathcal{V}}(0))))).$ This computation is done by the same proof as that of [12, Theorem 5.13], but in the holomorphic setting. As is computed in [1, 4.5.1], the characteristic class of the quantization $\hat{\mathcal{V}}_{Q_M}(0)$ is equal to $\frac{1}{\pi} \omega + \frac{1}{2} \pi_M c_1(TX)$ with $\omega$ the symplectic form on $T^* M$. Substituting this characteristic class into [12, Theorem 5.13] yields the desired identity. \qed
Proof of Theorem 4.2 The Euler class is the top degree component of the Chern character, which is computed by the following steps.

\[
\text{ch}_Q(M) = \mu \hat{\chi}^E \circ \text{ch}^E(M) \quad \text{Definition 3.13}
\]

\[
= \mu \hat{\chi}^E \circ i^*_s \text{ch}^E(M) \quad \text{Proposition 4.1}
\]

\[
= \mu \hat{\chi}^E \circ i^*_s \hat{\chi}^{[h^{-1}]} \circ I^*_s \text{ch}^E(M) \quad \text{right front triangle of Proposition 4.9}
\]

\[
= \mu \hat{\chi}^E \circ i^*_s \hat{\chi}^{[h^{-1}]} \circ \text{ch}^E(\hat{M}, Q_X) \quad \text{Proposition 4.6}
\]

\[
= \frac{1}{m} \sigma^*_s \hat{\chi}^{(0)} \circ I^*_s \hat{\chi}^{E} \circ \text{ch}^E(\hat{M}, Q_X) \wedge \text{eu}_Q(N) \wedge \pi^{-1} Td_{Q,M} \quad \text{Theorem 4.10}
\]

\[
= \frac{1}{m} \text{ch}^O(\hat{M}, Q_X) \otimes \text{ch}^E(\hat{M}, Q_X) \wedge \text{eu}_Q(N) \wedge \pi^{-1} Td_{Q,M} \quad \text{Proposition 4.7}
\]

\[
= \frac{1}{m} \text{ch}^O(\hat{M}, Q_X) \otimes \text{ch}^E(\hat{M}, Q_X) \wedge \text{eu}_Q(N) \wedge \pi^{-1} Td_{Q,M} \quad \text{Eq. (8)}
\]

\[
= \frac{1}{m} \text{ch}_Q(\sigma_{\text{char}}(M))(\mathcal{M}) \wedge \text{eu}_Q(N) \wedge \pi^{-1} Td_{Q,M}.
\]

\[\square\]

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