Quantum group structure of the $q$-deformed $W$ algebra $W_q$

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Abstract. In this paper the $q$-deformed $W$ algebra $W_q$ is constructed, whose nontrivial quantum group structure is presented.

Key words: Quantum group, $q$-deformed $W$ algebra $W_q$, Hopf algebra.

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§1. Introduction

The deformation theory acts important roles in many fields such as mathematics and physics, which is closely related to quantum groups, originally introduced by Drinfeld in [3]. From the day that the conception of the quantum groups was born, there appear many papers on this relatively new object, so does the deformation theory (cf. [1, 2, 4], [15]–[19]). The quantum group structure on the $q$-deformed Virasoro algebra and $q$-deformed Kac-Moody algebra had been investigated by many authors (cf. [15]–[19]), and beautiful results were presented therein.

Now let’s introduce the object algebra concerned with in the present paper. The algebra $W$-algebra $W(2,2)$, denoted by $W$ for convenience and introduced by Zhang and Dong in [20], is an infinite-dimensional Lie algebra, possessing a $\mathbb{C}$-basis \{ $L_n, W_n, C \mid n \in \mathbb{Z}$ \} and admitting the following Lie brackets (other components vanishing):

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C, \tag{1.1}
\]

\[
[L_m, W_n] = (m - n)W_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C. \tag{1.2}
\]

There appeared some papers investigating the structures and representations on such $W$ algebra recently. In [20], Zhang and Dong produced a new class of irrational vertex operator algebras by studying its highest weight modules, while [7] and [8] classified its irreducible weight modules and indecomposable modules and [6] determined its derivations, central extensions and automorphisms. Afterwards, the Lie bialgebra structures on $W$ (centerless form) were proved to triangular coboundary in [10], which were quantized in [11]. The generalized Verma modules for the generalized $W$-algebra $W(2,2)$ are also being under

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consideration in [3]. However, the existence of a $q$-deformation of the $W$-algebra $W(2,2)$ and its quantum group structure is still an open problem, which may be interesting to physicists. That is what our paper shall focus on. In other words, we shall construct a $q$-deformation of the $W$ algebra, i.e., $\mathcal{W}_q$, which admits a nontrivial Hopf structure. The Harish-Chandra modules, Verma modules and also Unitary representations for the $q$-deformed $W$-algebra $\mathcal{W}_q$ have been investigated and shall be presented in a series of papers (c.f. [12]–[14]) by the authors maybe in the near future.

Let’s formulate our main results below. The following definition can be found in many references (e.g. [15]).

**Definition 1.1** A vector space $V$ over $\mathbb{C}$, with an bilinear operation $V \times V \to V$, denoted $(x, y) \to [x, y]_q$ and called the $q$-bracket or $q$-commutator of $x$ and $y$, and meanwhile with an endomorphism of $V$, denoted $f_q$, is called a $q$-deformed Lie algebra over $\mathbb{C}$ if the following axioms are satisfied:

$$[u, v]_q = -[v, u]_q, \quad (1.3)$$

$$[f_q(u), [v, w]_q]_q + [f_q(w), [u, v]_q]_q + [f_q(v), [w, u]_q] = 0, \quad (1.4)$$

for any $u, v, w \in V$.

As the usual definition of 2-cocycle, we also can introduce the corresponding one of $q$-deformed 2-cocycle on the centerless $q$-deformed Lie algebra $V$ defined in Definition 1.1.

**Definition 1.2** A bilinear $\mathbb{C}$-value function $\psi_q : V \times V \to \mathbb{C}$ is called $q$-deformed 2-cocycle on $V$ if the following conditions are satisfied

$$\psi_q(u, v) = -\psi_q(v, u), \quad (1.5)$$

$$\psi_q(f_q(u), [v, w]_q) + \psi_q(f_q(w), [u, v]_q) + \psi_q(f_q(v), [w, u]_q) = 0, \quad (1.6)$$

for any $u, v, w \in V$.

Denote by $C^2_q(V, \mathbb{C})$ the vector space of $q$-deformed 2-cocycles on $V$. For any linear $\mathbb{C}$-value function $\chi_q : V \to \mathbb{C}$, the 2-cocycle $\psi_{\chi_q}$ defined by

$$\psi_{\chi_q}(u, v) = \chi_q([u, v]_q), \quad \forall \ u, v \in V, \quad (1.7)$$

is called 2-coboundary on $V$. Denote by $B^2(V, \mathbb{C})$ the vector space of 2-coboundaries on $V$. The quotient space $H^2(V, \mathbb{C}) := C^2(V, \mathbb{C})/B^2(V, \mathbb{C})$ is called the second cohomology group of $V$.

**Theorem 1.3** The algebra $\mathcal{U}_q$ is a noncommutative but cocommutative Hopf algebra under the comultiplication $\Delta$, the counity $\epsilon$ and the antipode $S$ defined by (2.15)–(2.17).
§2. Proof of the main result

Firstly, we shall construct a $q$-deformation of $\mathcal{W}$, denoted $\mathcal{W}_q$, by using the technique developed in [15]. In fact, the Witt algebra can be recognized as the Lie algebra of derivations on $\mathbb{C}[t^{\pm 1}]$, i.e., the Lie algebra of its linear operators $\Omega$ satisfying

$$\Omega(xy) = \Omega(x)y + x\Omega(y),$$

whose Lie bracket also can be obtained by simple computations. Fix some generic $q \in \mathbb{C}^*$, and $\delta \in \text{End}(\mathbb{C}[t^{\pm 1}])$ such that $\delta(t) = qt$. Define a $q$-derivation $D$ as

$$D(f(t)) = -(q - 1)^{-1}(Id - \delta)f(t), \ \forall \ f(t) \in \mathbb{C}[t^{\pm 1}].$$

It is easy to see that $\delta(t^n) = q^n t^n$ and $D(t^n) = \frac{q^n - t^n}{q(t^n - t^n)} = [n]_q t^n$, where $[n]_q = \frac{q^n - 1}{q - 1}$ for some $n \in \mathbb{Z}$. The following way of defining $q$-deformed Virasoro algebra can be found in many references (e.g., [15]), on which our construction is based

$$[L_m, L_n]_q = ([m]_q - [n]_q)L_{m+n} + \frac{q^{-m}[m-1]_q[m]_q[m+1]_q}{6(1+q^m)}\delta_{m,-n}C. \quad (2.1)$$

**Definition 2.1** The 2-cocycle on the $q$-deformed Virasoro algebra given in (2.1) is called the $q$-deformed Virasoro 2-cocycle.

Combining the structures of the algebra $\mathcal{W}$ listed in (1.1)–(1.2) and the $q$-deformed Virasoro Lie algebras given in (2.1), we introduce the centerless $q$-deformed $W$ algebra $\mathcal{W}_q$, which possesses a $\mathbb{C}$-basis $\{L_m, W_m | m \in \mathbb{Z}\}$ with the following relations

$$[L_m, L_n]_q = ([m]_q - [n]_q)L_{m+n}, \ [L_m, W_n]_q = ([m]_q - [n]_q)W_{m+n}, \ [W_m, W_n]_q = 0. \quad (2.2)$$

Observing (1.4), (1.6), (2.1) and (2.2), one can take

$$f_q(L_m) = (q^m + 1)L_m, \ f_q(W_m) = (q^m + 1)W_m, \ \forall \ m \in \mathbb{Z}, \quad (2.3)$$

where $f_q$ is that defined in Definition 1.1. By simple computations, one can see that the algebra $\mathcal{W}_q$ defined by (2.2) with the $f_q$ defined by (2.3) is indeed a $q$-deformed Lie algebra.

Using (2.1), in order to obtain the $q$-deformed algebra $\mathcal{W}_q$, we have to determine the $q$-deformed 2-cocycle $\psi_q(L_m, W_n)$ determined by the following identity

$$[L_m, W_n]_q = ([m]_q - [n]_q)W_{m+n} + \psi_q(L_m, W_n)C. \quad (2.4)$$
Using (2.3) and respectively, replacing \((u, v)\) by \((L_i, W_j)\) \((\forall i, j \in \mathbb{Z})\) in (1.5) and the triple \((u, v, w)\) by \((L_i, L_j, W_k)\) \((\forall i, j, k \in \mathbb{Z})\) in (1.6), one has
\[
\psi_q(L_i, W_j) = -\psi_q(W_j, L_i), \quad \text{(2.5)}
\]
\[
(q^i + 1)([j]_q - [k]_q)\psi_q(L_i, W_{j+k}) = (q^k + 1)([i]_q - [j]_q)\psi_q(L_{i+j}, W_k) + (q^j + 1)([i]_q - [k]_q)\psi_q(L_j, W_{k+i}). \quad \text{(2.6)}
\]
Let \(i = 0\) in (2.6), one has
\[
(q^j - q^k)\psi_q(L_0, W_{j+k}) = (q^j+k - 1)\psi_q(L_j, W_k),
\]
which together with our assumption on \(q\), forces
\[
\psi_q(L_0, W_0) = 0. \quad \text{(2.7)}
\]
According to the second bracket in (2.2), we can write
\[
L_0 = (1 + q^{-1})[L_1, L_{-1}]_q, \quad W_0 = (1 + q^{-1})[L_1, W_{-1}]_q,
\]
\[
L_m = ([m]_q)^{-1}[L_0, L_m]_q, \quad W_m = ([m]_q)^{-1}[L_0, W_m]_q \quad \text{if} \quad m \in \mathbb{Z}^*.
\]
Define a \(\mathbb{C}\)-linear function \(\chi_q : \mathcal{W}_q \rightarrow \mathbb{C}\) as follows
\[
\chi_q(L_0) = (1 + q^{-1})\psi_q(L_1, L_{-1}), \quad \chi_q(W_0) = (1 + q^{-1})\psi_q(L_1, W_{-1}),
\]
\[
\chi_q(L_m) = ([m]_q)^{-1}\psi_q(L_0, L_m), \quad \chi_q(W_m) = ([m]_q)^{-1}\psi_q(L_0, W_m) \quad \text{if} \quad m \in \mathbb{Z}^*.
\]
Let \(\varphi_q = \psi_q - \psi_q\chi_q\) where \(\psi_q\chi_q\) is defined in (1.7). One has
\[
\varphi_q(L_1, L_{-1}) = \varphi_q(L_1, W_{-1}) = \varphi_q(L_0, L_m) = \varphi_q(L_0, W_m) = 0 \quad \text{if} \quad m \in \mathbb{Z}^*. \quad \text{(2.8)}
\]
Denote by \(\mathfrak{W}_q\) the \(q\)-deformed Witt subalgebra of \(\mathcal{W}_q\) spanned by \(\{L_m | m \in \mathbb{Z}\}\). The by simple discussion or cite the result given in [15], one can suppose that \(\varphi_q|_{\mathfrak{W}_q}\) is exactly the \(q\)-deformed Virasoro 2-cocycle (up to a constant factor).

Recalling (2.7) and (2.9), one can deduce \(\varphi_q(L_m, W_n) = 0\) if \(m + n \neq 0\). Thus, the left components we have to compute are
\[
\varphi_q(L_m, W_{-m}), \quad \forall \ m \in \mathbb{Z}^*. \quad \text{(2.9)}
\]
By employing the same techniques developed in [15], we obtain (up to a constant factor)
\[
\varphi_q(L_m, W_{-m}) = \frac{q^{-m}[m - 1]_q[m]_q[m + 1]_q}{6(1 + q^m)}, \quad \forall \ m \in \mathbb{Z}^*. \quad \text{(2.10)}
\]
Then we have

\[ [L_m, W_n]_q = ([m]_q - [n]_q)W_{m+n} + \frac{q^{-m}[m-1]_q [m]_q [m+1]_q}{6(1+q^m)} \delta_{m,-n} C. \] (2.11)

Now we can safely present the following lemma.

**Lemma 2.2** The algebra \( W_q \) with a \( \mathbb{C} \)-basis \( \{L_m, W_m, C \mid m \in \mathbb{Z} \} \) satisfying the following relations (while other components vanishing) is a \( q \)-deformation of the algebra \( W \).

\[ [L_m, L_n]_q = q^m L_m L_n - q^n L_n L_m, \quad [L_m, W_n]_q = q^m L_m W_n - q^n W_n L_m, \] (2.12)

where the \( q \)-deformed brackets are respectively given in (2.1) and (2.11).

Next we shall proceed with our construction of the Hopf algebra structure based on the \( q \)-deformed algebra \( W_q \) given in Lemma 2.2. Firstly, for convenience to express, we shall recall the definition of a Hopf algebra, which can be found in many books and also references.

**Definition 2.3** A tuple \( (\mathcal{A}, \nabla, \varepsilon, \Delta, \delta, S) \), \( \mathcal{A} \) being a \( \mathbb{C} \)-vector space, \( \nabla : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \) a multiplication map, \( \varepsilon : \mathbb{C} \rightarrow \mathcal{A} \) a unit map, \( \Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \) a comultiplication map, \( \delta : \mathcal{A} \rightarrow \mathbb{C} \) a counit map, \( S : \mathcal{A} \rightarrow \mathcal{A} \) an antipode map, is called a Hopf algebra over \( \mathbb{C} \) if the following axioms are satisfied

1. the map \( \nabla \) gives an associative algebra structure on \( \mathcal{A} \) with the unit \( \varepsilon(1) \),
2. \( \Delta \) and \( \delta \) give a coassociative coalgebra structure on \( \mathcal{A} \),

\[ (1 \otimes \Delta)\Delta(x) = (\Delta \otimes 1)\Delta(x), \quad (1 \otimes \delta)\Delta(x) = (\delta \otimes 1)\Delta(x), \] (2.13)

3. both \( \Delta \) and \( \delta \) are algebra homomorphisms,
4. \( S \) is an automorphism with the following relations

\[ \nabla(1 \otimes S)\Delta(x) = \nabla(S \otimes 1)\Delta(x) = \delta(\delta(x)). \] (2.14)

We say the Hopf algebra \( \mathcal{A} \) is cocommutative if \( \Delta = \Delta^{op} \). A vector space \( \mathcal{L} \) over \( \mathbb{C} \), is called a bialgebra if it admits the maps \( \nabla, \varepsilon, \Delta, \delta \) with the axioms (1)–(3) given in Definition 2.3.

Denote \( \mathcal{U}_q \) to be the \( q \)-deformed enveloping algebra of \( W_q \). Then \( \mathcal{U}_q \) allows the Hopf algebra structure given below

\[ \varepsilon(L_m) = \varepsilon(W_m) = \varepsilon(C) = 0, \quad \Delta(C) = C \otimes 1 + 1 \otimes C, \] (2.15)

\[ \Delta(L_m) = L_m \otimes T^m + T^m \otimes L_m, \quad \Delta(W_m) = W_m \otimes T^m + T^m \otimes W_m, \] (2.16)

\[ S(L_m) = -T^{-m} L_m T^{-m}, \quad S(W_m) = -T^{-m} W_m T^{-m}, \quad S(C) = -C. \] (2.17)
where the operators \( \{ T, T^{-1} \} \) are given by

\[
\Delta(T) = T \otimes T, \quad \epsilon(T) = 1, \quad S(T) = T^{-1}.
\]

The following relations also can be obtained by simple computations:

\[
\begin{align*}
T^m L_n &= q^{-(n+1)m} L_n T^m, \quad T^m W_n = q^{-(n+1)m} W_n T^m, \\
T^m L_n &= q^{-(n+1)m} L_n T^m, \quad T^m W_n = q^{-(n+1)m} W_n T^m,
\end{align*}
\]

\[
\Delta(T T^{-1}) = T T^{-1} = 1, \quad q^m T^m C = CT^m, \quad q^m T^m C = C T^m.
\]

**Proof of Theorem 1.3** We shall follow some techniques developed in [9]. It is not difficult to see that the coassociativity and cocommutative of \( \Delta \) hold in \( \mathcal{U}_q \) and, \( \epsilon \) is an algebra homomorphism, also \( (1 \otimes \epsilon) \Delta = (\epsilon \otimes 1) \Delta = 1 \). Firstly, We shall ensure that \( \Delta \) is an algebra homomorphism while \( S \) is an algebra anti-homomorphism of \( \mathcal{U}_q \). Using the relations obtained above, we can present the following computations:

\[
\begin{align*}
q^m \Delta(L_m) \Delta(W_n) - q^n \Delta(W_n) \Delta(L_m) \\
&= (q^m L_m W_n - q^n W_n L_m) \otimes T^{m+n} + T^{m+n} \otimes (q^m L_m W_n - q^n W_n L_m) \\
&= [L_m, W_n]_q \otimes T^{m+n} + T^{m+n} \otimes [L_m, W_n]_q \\
&= ([m]_q - [n]_q) \Delta(W_{m+n}) + \frac{q^{-m}[m-1]_q[m]_q[m+1]_q}{6(1 + q^m)} \delta_{m,-n} \Delta(C).
\end{align*}
\]

Other formulate also can be proved to be preserved by the map \( \Delta \), which together implies that \( \Delta \) is an algebra homomorphism. Thus, \( \mathcal{U}_q \) indeed a bialgebra. We also have the following computations:

\[
\begin{align*}
S(L_m W_n) = S(W_n) S(L_m) &= T^{-n} W_n T^{-n} T^{-m} L_m T^{-m} = q^{n-m} T^{-m-n} L_n L_m T^{-m-n},
\end{align*}
\]

which further gives

\[
\begin{align*}
q^m S(L_m W_n) - q^n S(W_n L_m) \\
&= q^n T^{-m-n} W_n L_m T^{-m-n} - q^m T^{-m-n} L_m W_n T^{-m-n} \\
&= -T^{-m-n} (q^m L_m W_n - q^n W_n L_m) T^{-m-n} \\
&= -T^{-m-n} [L_m, W_n]_q T^{-m-n} \\
&= -([m]_q - [n]_q) S(W_{m+n}) + \frac{q^{-m}[m-1]_q[m]_q[m+1]_q}{6(1 + q^m)} \delta_{m,-n} S(C),
\end{align*}
\]

and which actually implies the fact that \( S \) preserves the second identity of (2.12). Other formulate also can be proved to be preserved by the antipode map \( S \). Thus, \( \mathcal{U}_q \) admits the referred Hopf algebra structure. \( \square \)
Before ending this short note, employing the main techniques developed in [9], one can easily obtain the following corresponding corollary.

**Corollary 2.4** As vector spaces,

\[ U_q \cong \mathbb{C}[T, T^{-1}] \otimes \mathcal{U}(\mathcal{W}_q), \]

where \( \mathcal{U}(\mathcal{W}_q) \) is the universal enveloping algebra of \( \mathcal{W}_q \) generated by \( \{L_m, W_m, C \mid m \in \mathbb{Z}\} \) with the relations presented in (2.12).

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