Online Prediction with Selfish Experts

Tim Roughgarden* Okke Schrijvers†
May 23, 2017

Abstract

We consider the problem of binary prediction with expert advice in settings where experts have agency and seek to maximize their credibility. This paper makes three main contributions. First, it defines a model to reason formally about settings with selfish experts, and demonstrates that “incentive compatible” (IC) algorithms are closely related to the design of proper scoring rules. Designing a good IC algorithm is easy if the designer’s loss function is quadratic, but for other loss functions, novel techniques are required. Second, we design IC algorithms with good performance guarantees for the absolute loss function. Third, we give a formal separation between the power of online prediction with selfish experts and online prediction with honest experts by proving lower bounds for both IC and non-IC algorithms. In particular, with selfish experts and the absolute loss function, there is no (randomized) algorithm for online prediction—IC or otherwise—with asymptotically vanishing regret.

1 Introduction

In the months leading up to elections and referendums, a plethora of pollsters try to figure out how the electorate is going to vote. Different pollsters use different methodologies, reach different people, and may have sources of random errors, so generally the polls don’t fully agree with each other. Aggregators such as Nate Silver’s FiveThirtyEight, The Upshot by the New York Times, and HuffPost Pollster by the Huffington Post consolidate these different reports into a single prediction, and hopefully reduce random errors. FiveThirtyEight in particular has a solid track record for their predictions, and as they are transparent about their methodology we use them as a motivating example in this paper. To a first-order approximation, they operate as follows: first they take the predictions of all the different pollsters, then they assign a weight to each of the pollsters based on past performance (and other factors), and finally they use the weighted average of the pollsters to run simulations and make their own prediction.

But could the presence of an institution that rates pollsters inadvertently create perverse incentives for the pollsters? The FiveThirtyEight pollster ratings are publicly available. The ratings can be interpreted as a reputation, and a low rating can negatively impact future revenue opportunities for a pollster. Moreover, it has been demonstrated in practice that experts do not always report their true beliefs about future events. For example, in weather forecasting there is a known “wet bias,” where consumer-facing weather forecasters deliberately overestimate low chances of rain (e.g. a 5% chance of rain is reported as a 25% chance of rain) because people don’t like to be surprised by rain [Bickel and Kim 2008].

*Department of Computer Science, Stanford University, 474 Gates Building, 353 Serra Mall, Stanford, CA 94305. This research was supported in part by NSF grant CCF-1215965. Email: tim@cs.stanford.edu.
†Department of Computer Science, Stanford University, 482 Gates Building, 353 Serra Mall, Stanford, CA 94305. This research was supported in part by NSF grant CCF-1215965. Email: okkes@cs.stanford.edu.
1FiveThirtyEight: https://fivethirtyeight.com/ The Upshot: https://www.nytimes.com/section/upshot HuffPost Pollster: http://elections.huffingtonpost.com/pollster/
2This is of course a simplification. FiveThirtyEight also uses features like the change in a poll over time, the state of the economy, and correlations between states. See https://fivethirtyeight.com/features/how-fivethirtyeight-calculates-pollster-ratings/ for details. Our goal in this paper is not to accurately model all of the fine details of FiveThirtyEight (which are anyways changing all the time). Rather, it is to formulate a general model of prediction with experts that clearly illustrates why incentives matter.
3https://projects.fivethirtyeight.com/pollster-ratings/
These examples motivate the development of models of aggregating predictions that endow agency to the data sources. While there are multiple models in which we can investigate this issue, a natural candidate is the problem of prediction with expert advice. By focusing on a standard model, we abstract away from the fine details of FiveThirtyEight (which are anyways changing all the time), which allows us to formulate a general model of prediction with experts that clearly illustrates why incentives matter. In the classical model [Littlestone and Warmuth 1994, Freund and Schapire 1997], at each time step, several experts make predictions about an unknown event. An online prediction algorithm aggregates experts’ opinions and makes its own prediction at each time step. After this prediction, the event at this time step is realized and the algorithm incurs a loss as a function of its prediction and the realization. To compare its performance against individual experts, for each expert the algorithm calculates what its loss would have been had it always followed the expert’s prediction. While the problems introduced in this paper are relevant for general online prediction, to focus on the most interesting issues we concentrate on the case of binary events, and real-valued predictions in [0, 1]. For different applications, different notions of loss are appropriate, so we parameterize the model by a loss function $\ell$. Thus our formal model is: at each time step $t = 1, 2, \ldots, T$:

1. Each expert $i$ makes a prediction $p_{i}^{(t)} \in [0, 1]$, with higher values indicating stronger advocacy for the event “1.”
2. The online algorithm commits to a probability distribution over $\{0, 1\}$, with $q^{(t)}$ denoting the probability assigned to “1.”
3. The outcome $r^{(t)} \in \{0, 1\}$ is realized.
4. The algorithm incurs a loss of $\ell(q^{(t)}, r^{(t)})$ and calculates for each expert $i$ a loss of $\ell(p_{i}^{(t)}, r^{(t)})$.

The standard goal in this problem is to design an online prediction algorithm that is guaranteed to have expected loss not much larger than that incurred by the best expert in hindsight. The classical solutions maintain a weight for each expert and make a prediction according to which outcome has more expert weight behind it. An expert’s weight can be interpreted as a measure of its credibility in light of its past performance. The (deterministic) Weighted Majority (WM) algorithm always chooses the outcome with more expert weight. The Randomized Weighted Majority (RWM) algorithm randomizes between the two outcomes with probability proportional to their total expert weights. The most common method of updating experts’ weights is via multiplication by $1 - \eta\ell(p_{i}^{(t)}, r^{(t)})$ after each time step $t$, where $\eta$ is the learning rate. We call this the “standard” or “classical” version of the WM and RWM algorithm.

The classical model instills no agency in the experts. To account for this, in this paper we replace Step 1 of the classical model by:

1a. Each expert $i$ formulates a belief $b_{i}^{(t)} \in [0, 1]$.
1b. Each expert $i$ reports a prediction $p_{i}^{(t)} \in [0, 1]$ to the algorithm.

Each expert now has two types of loss at each time step — the reported loss $\ell(p_{i}^{(t)}, r^{(t)})$ with respect to the reported prediction and the true loss $\ell(b_{i}^{(t)}, r^{(t)})$ with respect to her true beliefs.

When experts care about the weight that they are assigned, and with it their reputation and influence in the algorithm, different loss functions can lead to different expert behaviors. For example, in Section 2 we observe that for the quadratic loss function, in the standard WM and RWM algorithms, experts have no reason to misreport their beliefs. The next example shows that this is not the case for other loss functions,

\[4\] More generally, one can investigate how the presence of machine learning algorithms affects data generating processes, either during learning, e.g. Dekel et al. 2010, Cai et al. 2015, or during deployment, e.g. Hardt et al. 2016, Brückner and Scheffer 2011. We discuss some of this work in the related work section.

\[5\] When we speak of the best expert in hindsight, we are always referring to the true losses. Guarantees with respect to reported losses follow from standard results [Littlestone and Warmuth 1994, Freund and Schapire 1997, Cesa-Bianchi et al. 2007], but are not immediately meaningful.

\[6\] The loss function is often tied to the particular application. For example, in the current FiveThirtyEight pollster rankings, the performance of a pollster is primarily measured according to an absolute loss function and also whether the candidate with the highest polling numbers ended up winning (see https://github.com/fivethirtyeight/data/tree/master/pollster-ratings).
Example 1. Consider the standard WM algorithm, where each expert initially has unit weight, and an expert’s weight is multiplied by $1 - \eta |p^{(t)}_i - r^{(t)}|$ at a time step $t$, where $\eta \in (0, \frac{1}{2})$ is the learning rate. Suppose there are two experts and $T = 1$, and that $b_1^{(1)} = .49$ while $b_2^{(1)} = 1$. Each expert reports to maximize her expected weight. Expanding, for each $i = 1, 2$ we have

$$
\mathbb{E}[w_i^{(1)}] = \Pr(r^{(1)} = 1) \cdot (1 - \eta(1 - p_i^{(1)})) + \Pr(r^{(1)} = 0) \cdot (1 - \eta p_i^{(1)})
$$

$$
= b_i^{(1)} \cdot (1 - \eta(1 - p_i^{(1)})) + (1 - b_i^{(1)}) \cdot (1 - \eta p_i^{(1)})
$$

$$
= b_i^{(1)} - b_i^{(1)} \eta + b_i^{(1)} \eta p_i^{(1)} + 1 - \eta p_i^{(1)} - b_i^{(1)} + b_i^{(1)} \eta p_i^{(1)}
$$

$$
= 2b_i^{(1)} \eta p_i^{(1)} - p_i^{(1)} \eta - b_i^{(1)} \eta + 1,
$$

where all expectations and probabilities are with respect to the true beliefs of agent $i$. To maximize this expected weight over the possible reports $p_i^{(1)} \in [0, 1]$, we can omit the second two terms (which are independent of $p_i^{(1)}$) and divide out by $\eta$ to obtain

$$
\arg\max_{p_i^{(1)} \in [0, 1]} 2b_i^{(1)} \eta p_i^{(1)} - p_i^{(1)} \eta - b_i^{(1)} \eta + 1 = \arg\max_{p_i^{(1)} \in [0, 1]} p_i^{(1)}(2b_i^{(1)} - 1)
$$

$$
= \begin{cases} 
1 & \text{if } b_i^{(1)} \geq \frac{1}{2} \\
0 & \text{otherwise.}
\end{cases}
$$

Thus an expert always reports a point mass on whichever outcome she believes more likely. In our example, the second expert will report her true beliefs ($p_2^{(t)} = 1$) while the first will not ($p_1^{(t)} = 0$). While the combined true beliefs of the experts clearly favor outcome 1, the WM algorithm sees two opposing predictions and must break ties arbitrarily between them.

This conclusion can also be drawn directly from the property elicitation literature. Here, the absolute loss function is known to elicit the median [Bonin 1976, Thomson 1979], and since we have binary realizations, the median is either 0 or 1. Example 1 shows that for the absolute loss function the standard WM algorithm is not “incentive-compatible” in a sense that we formalize in Section 2. There are similar examples for the other commonly studied weight update rules and for the RWM algorithm. We might care about truthful reporting for its own sake, but additionally the worry is that non-truthful reports will impede our ability to get good regret guarantees (with respect to experts’ true losses).

We study several fundamental questions about online prediction with selfish experts:

1. What is the design space of “incentive-compatible” online prediction algorithms, where every expert is incentivized to report her true beliefs?

2. Given a loss function like absolute loss, are there incentive-compatible algorithm that obtain good regret guarantees?

3. Is online prediction with selfish experts strictly harder than in the classical model with honest experts?

1.1 Our Results

The first contribution of this paper is the development of a model for reasoning formally about the design and analysis of weight-based online prediction algorithms when experts are selfish (Section 2), and the definition of an “incentive-compatible” (IC) such algorithm. Intuitively, an IC algorithm is such that each expert wants to report its true belief at each time step. We demonstrate that the design of IC online prediction algorithms is closely related to the design of strictly proper scoring rules. Using this, we show that for the quadratic loss function, the standard WM and RWM algorithms are IC, whereas these algorithms are not generally IC for other loss functions.

However, in 2008 FiveThirtyEight used the notion of “pollster introduced error” or PIE, which is the square root of a difference of squares, as the most important feature in calculating the weights, see [https://fivethirtyeight.com/features/pollster-ratings-v31/](https://fivethirtyeight.com/features/pollster-ratings-v31/)
Our second contribution is the design of IC prediction algorithms for the absolute loss function with non-trivial performance guarantees. For example, our best result for deterministic algorithms is: the WM algorithm, with experts’ weights evolving according to the spherical proper scoring rule (see Section 3), is IC and has loss at most \(2 + \sqrt{2}\) times the loss of best expert in hindsight (in the limit as \(T \to \infty\)). A variant of the RWM algorithm with the Brier scoring rule is IC and has expected loss at most 2.62 times that of the best expert in hindsight (also in the limit, see Section 6).

Our third and most technical contribution is a formal separation between online prediction with selfish experts and the traditional setting with honest experts. Recall that with honest experts, the classical (deterministic) WM algorithm has loss at most twice that of the best expert in hindsight (as \(T \to \infty\)) [1994]. We prove that the worst-case loss of every (deterministic) IC algorithm (Section 4) and every non-IC algorithm satisfying mild technical conditions (Section 5) has worst-case loss bounded away from twice that of the best expert in hindsight (even as \(T \to \infty\)). A consequence of our lower bound is that, with selfish experts, there is no natural (randomized) algorithm for online prediction—IC or otherwise—with asymptotically vanishing regret.

1.2 Related Work

We believe that our model of online prediction over time with selfish experts is novel. We next survey the multiple other ways in which online learning and incentive issues have been blended, and the other efforts to model incentive issues in machine learning.

There is a large literature on prediction and decision markets (e.g. [Chen and Pennock, 2010, Horn et al., 2014]), which also aim to aggregate information over time from multiple parties and make use of proper scoring rules to do it. There are several major differences between our model and prediction markets. First, in our model, the goal is to predict a sequence of events, and there is feedback (i.e., the realization) after each one. In a prediction market, the goal is to aggregate information about a single event, with feedback provided only at the end (subject to secondary objectives, like bounded loss). Second, our goal is to make accurate predictions, while that of a prediction market is to aggregate information. For example, if one expert is consistently incorrect over time, we would like to ignore her reports rather than aggregate them with others’ reports. Finally, while there are strong mathematical connections between cost function-based prediction markets and regularization-based online learning algorithms in the standard (non-IC) model [Abernethy et al., 2013], there does not appear to be analogous connections with online prediction with selfish experts.

There is also an emerging literature on “incentivizing exploration” (as opposed to exploitation) in partial feedback models such as the bandit model (e.g. [Frazier et al., 2014, Mansour et al., 2016]). Here, the incentive issues concern the learning algorithm itself, rather than the experts (or “arms”) that it makes use of.

The question of how an expert should report beliefs has been studied before in the literature on strictly proper scoring rules [Brier, 1950, McCarthy, 1956, Savage, 1971, Gneiting and Raftery, 2007], but this literature typically considers the evaluation of a single prediction, rather than low-regret learning. The work by Bayarri and DeGroot [1989] specifically looks at the question of how an expert should respond to an aggregator who assigns and updates weights based on their predictions. Their work focuses on optimizing relative weight under different objectives and informational assumptions. However, it predates the work on low-regret learning, and it does not include performance guarantees for the aggregator over time. [Boutilier, 2012] discusses a model in which an aggregator wants to take a specific action based on predictions that she elicits from experts. He explores incentive issues where experts have a stake in the action that is taken by the decision maker.

Finally, there are many works that fall under the broader umbrella of incentives in machine learning. Roughly, work in this area can be divided into two genres: incentives during the learning stage, or incentives during the deployment stage. During the learning stage, one of the main considerations is incentivizing data providers to exert effort to generate high-quality data. There are several recent works that propose ways to elicit data in crowdsourcing applications in repeated settings through payments, e.g. [Cai et al., 2015, Shah and Zhou, 2015, Liu and Chen, 2016]. Outside of crowdsourcing, [Dekel et al., 2010] consider a regression task where different experts have their own private data set, and they seek to influence the learner to learn a function such that the loss of their private data set with respect to the function is low.

---

7In the even more distantly related peer prediction scenario [Miller et al., 2005], there is never any realization at all.
During deployment, the concern is that the input is given by agents who have a stake in the result of the classification, e.g. an email spammer wishes to avoid its emails being classified as spam. Brückner and Scheffer [2011] model a learning task as a Stackelberg game. On the other hand Hardt et al. [2016] consider a cost to changing data, e.g. improving your credit score by opening more lines of credit, and give results with respect to different cost functions.

Online learning does not fall neatly into either learning or deployment, as the learning is happening while the system is deployed. Babaioff et al. [2010] consider the problem of no-regret learning with selfish experts in an ad auction setting, where the incentives come from the allocations and payments of the auction, rather than from weights as in our case.

1.3 Organization

Section 2 formally defines weight-update online prediction algorithms and shows a connection between algorithms that are incentive compatible and proper scoring rules. We use the formalization to show that when we care about achieving guarantees for quadratic losses, the standard WM and RWM algorithms work well. Since the standard algorithm fails to work well for absolute losses, we focus in the remainder of the paper on proving guarantees for this case.

Section 3 gives a deterministic weight-update online prediction algorithm that is incentive-compatible and has absolute loss at most $2 + \sqrt{2}$ times that of the best expert in hindsight (in the limit). Additionally we show that the weighted majority algorithm with the standard update rule has a worst-case true loss of at least 4 times the best expert in hindsight.

To show the limitations of online prediction with selfish experts, we break our lower bound results into two parts. In Section 4 we show that any deterministic incentive compatible weight-update online prediction algorithm has worst case loss bounded away from 2, even as $T \to \infty$. Then in Section 5 we show that under mild technical conditions, the same is true for non-IC algorithms.

Section 6 contains our results for randomized algorithms. It shows that the lower bounds for deterministic algorithms imply that under the same conditions randomized algorithms cannot have asymptotically vanishing regret. We do give an IC randomized algorithm that achieves worst-case loss at most 2.62 times that of the best expert in hindsight (in the limit).

Finally, in Section 7 we show simulations that indicate that different IC methods show similar regret behavior, and that their regret is substantially better than that of the non-IC standard algorithms, suggesting that the worst-case characterization we prove holds more generally.

The appendix contains omitted proofs (Appendix A), and a discussion on the selecting appropriate proper scoring rules for good guarantees (Appendix B).

2 Preliminaries and Model

2.1 Standard Model

At each time step $t \in 1, \ldots, T$ we want to predict a binary realization $r^{(t)} \in \{0,1\}$. To help in the prediction, we have access to $n$ experts that for each time step report a prediction $p^{(t)}_i \in [0,1]$ about the realization. The realizations are determined by an oblivious adversary, and the predictions of the experts may or may not be accurate. The goal is to use the predictions of the experts in such a way that the algorithm performs nearly as well as the best expert in hindsight. Most of the algorithms proposed for this problem fall into the following framework.

**Definition 2 (Weight-update Online Prediction Algorithm).** A weight-update online prediction algorithm maintains a weight $w^{(t)}_i$ for each expert and makes its prediction $q^{(t)}$ based on $\sum_{i=1}^n w^{(t)}_i p^{(t)}_i$ and $\sum_{i=1}^n w^{(t)}_i (1 - p^{(t)}_i)$. After the algorithm makes its prediction, the realization $r^{(t)}$ is revealed, and the algorithm updates the weights of experts using the rule

$$w^{(t+1)}_i = f \left(p^{(t)}_i, r^{(t)}\right) \cdot w^{(t)}_i,$$

where $f : [0,1] \times \{0,1\} \to \mathbb{R}^+$ is a positive function on its domain.
The standard WM algorithm has \( f(p_i(t), r(t)) = 1 - \eta \ell(p_i(t), r(t)) \) where \( \eta \in (0, \frac{1}{T}) \) is the learning rate, and predicts \( q(t) = 1 \) if and only if \( \sum w_i(t) p_i(t) \geq \sum w_i(t) (1 - p_i(t)) \). Let the total loss of the algorithm be \( M(T) = \sum_{t=1}^{T} \ell(q(t), r(t)) \) and let the total loss of expert \( i \) be \( m_i(T) = \sum_{t=1}^{T} \ell(p_i(t), r(t)) \). The MW algorithm has the property that \( M(T) \leq 2(1 + \eta)m_i^*(T) + \frac{2m_i}{\eta} \) for each expert \( i \), and RWM — where the algorithm picks 1 with probability proportional to \( \sum w_i(t) p_i(t) \) — satisfies \( M(T) \leq (1 + \eta)m_i^*(T) + \frac{\ln \eta}{\eta} \) for each expert \( i \).

The notion of “no \( \alpha \)-regret” in [Kakade et al., 2009] captures the idea that the per time-step loss of an algorithm is \( \alpha \) times that of the best expert in hindsight, plus a term that goes to 0 as \( T \) grows:

**Definition 3** (\( \alpha \)-regret). An algorithm is said to have no \( \alpha \)-regret if \( M(T) \leq \alpha \min_i m_i^*(T) + o(T) \).

By taking \( \eta = O(1/\sqrt{T}) \), MW is a no 2-regret algorithm, and RWM is a no 1-regret algorithm.

### 2.2 Selfish Model

We consider a model in which experts have agency about the prediction they report, and care about the weight that they are assigned. In the selfish model, at time \( t \) the expert formulates a private belief \( b_i(t) \) about the realization, but she is free to report any prediction \( p_i(t) \) to the algorithm. Let \( \text{Bern}(p) \) be a Bernoulli random variable with parameter \( p \). For any non-negative weight update function \( f \),

\[
\max_p \mathbb{E}_{b_i(t)}[w_i(t+1)] = \max_p \mathbb{E}_{r \sim \text{Bern}(b_i(t))} [f(p, r) w_i(t)] = w_i(t) \cdot \left( \max_p \mathbb{E}_{r \sim \text{Bern}(b_i(t))} [f(p, r)] \right).
\]

So expert \( i \) will report whichever \( p_i(t) \) will maximize the expectation of the weight update function.

Performance of an algorithm with respect to the reported loss of experts follows from the standard analysis in [Littlestone and Warmuth, 1994]. However, the true loss may be worse (in Section 3 we show this for the standard update rule, Section 5 shows it more generally). Unless explicitly stated otherwise, in the remainder of this paper \( m_i^*(T) = \sum_{t=1}^{T} \ell(b_i^*(t), r(t)) \) refers to the true loss of expert \( i \). For now this motivates restricting the weight update rule \( f \) to functions where reporting \( p_i(t) = b_i(t) \) maximizes the expected weight of experts. We call these weight-update rules *Incentive Compatible (IC)*.

**Definition 4** (Incentive Compatibility). A weight-update function \( f \) is incentive compatible (IC) if reporting the true belief \( b_i(t) \) is always a best response for every expert at every time step. It is strictly IC when \( p_i(t) = b_i(t) \) is the only best response.

By a “best response,” we mean an expected utility-maximizing report, where the expectation is with respect to the expert’s beliefs.

**Collusion.** The definition of IC does not rule out the possibility that experts can collude to jointly misreport to improve their weights. We therefore also consider a stronger notion of incentive compatibility for groups with transferable utility.

**Definition 5** (Incentive Compatibility for Groups with Transferable Utility). A weight-update function \( f \) is incentive compatible for groups with transferable utility (TU-GIC) if for every subset \( S \) of players, the total expected weight of the group \( \sum_{i \in S} \mathbb{E}_{b_i(t)}[w_i(t+1)] \) is maximized by each reporting their private belief \( b_i^*(t) \).

Note that TU-GIC is a strictly stronger concept than IC, as for any algorithm that is TU-GIC, the condition needs to hold for groups of size 1, which is the definition of IC. The concept is also strictly stronger than that of GIC with nontransferable utility (NTU-GIC), where for every group \( S \) it only needs to hold that there are no alternative reports that would make no member worse off, and at least one member better off in [Moulin, 1999] and [Jain and Mahdian, 2007].
2.3 Proper Scoring Rules

Incentivizing truthful reporting of beliefs has been studied extensively, and the set of functions that do this is called the set of proper scoring rules. Since we focus on predicting a binary event, we restrict our attention to this class of functions.

**Definition 6** (Binary Proper Scoring Rule, [Schervish, 1989]). A function $f : [0, 1] \times \{0, 1\} \to \mathbb{R} \cup \{\pm\infty\}$ is a binary proper scoring rule if it is finite except possibly on its boundary and whenever for $p \in [0, 1]$

$$p \in \max_{q \in [0,1]} p \cdot f(q, 1) + (1 - p) \cdot f(q, 0).$$

A function $f$ is a strictly proper scoring rule if $p$ is the only value that maximizes the expectation. The first perhaps most well-known proper scoring rule is the Brier scoring rule.

**Example 7** (Brier Scoring Rule, [Brier, 1950]). The Brier score is $Br(p, r) = 2pr - (p^2 + (1 - p)^2)$ where $pr + (1 - p)(1 - r)$ is the report for the event that materialized.

We will use the Brier scoring rule in Section 6 to construct an incentive-compatible randomized algorithm with good guarantees. The following proposition follows directly from Definitions 4 and 6:

**Proposition 8.** A weight-update rule $f$ is (strictly) incentive compatible if and only if $f$ is a (strictly) proper scoring rule.

Surprisingly, this result remains true even when experts can collude. While the realizations are obviously correlated, linearity of expectation causes the sum to be maximized exactly when each expert maximizes their expected weight.

**Proposition 9.** A weight-update rule $f$ is (strictly) incentive compatible for groups with transferable utility if and only if $f$ is a (strictly) proper scoring rule.

Thus, for online prediction with selfish experts, we get TU-GIC “for free.” It is quite uncommon for problems in non-cooperate game theory to admit good TU-GIC solutions. For example, results for auctions (either for revenue or welfare) break down once bidders collude, see [Goldberg and Hartline, 2005] and references therein for more examples from theory and practice. In the remainder of the paper we will simply use IC to refer to both incentive compatibility and incentive compatibility for groups with transferable utility, as strictly proper scoring rules lead to algorithms that satisfy both definitions.

So when considering incentive compatibility in the online prediction with selfish experts setting, we are restricted to considering proper scoring rules as weight-update rules. Moreover, since $f$ needs to be positive, only bounded proper scoring rules can be used. Conversely, any bounded scoring rule can be used, possibly after an affine transformation (which preserves proper-ness). Are there any proper scoring rules that give an online prediction algorithm with a good performance guarantee?

2.4 Online Learning with Quadratic Losses

The first goal of this paper is to describe the class of algorithms that lead incentive compatible learning. Proposition 8 answers this question, so we can move over to our second goal, which is for different loss functions, do there exist incentive compatible algorithms with good performance guarantees? In this subsection we show that a corollary of Proposition 8 is that the standard MW algorithm with the quadratic loss function $\ell(p, r) = (p - r)^2$ is incentive compatible.

**Corollary 10.** The standard WM algorithm with quadratic losses, i.e. $w_{i}^{(t+1)} = (1 - \eta(p_i^{(t)} - r_i^{(t)}))^2 \cdot w_i^{(t)}$ is incentive compatible.

**Proof.** By Proposition 8 it is sufficient to show that $b_i^{(t)} = \max_p b_i^{(t)} \cdot (1 - \eta(p - 1)^2) + (1 - b_i^{(t)}) \cdot (p - 0)^2$. 

7
max \( b_i^{(t)}(1 - \eta(p - 1)^2) + (1 - b_i^{(t)})(1 - \eta(p - 0)^2) \)

\[= \max_p b_i^{(t)} - b_i^{(t)} \eta p^2 + 2b_i^{(t)} \eta p - b_i^{(t)} \eta + 1 - b_i^{(t)} - \eta p^2 + b_i^{(t)} \eta p^2 \]

\[= \max_p 1 - b_i^{(t)} \eta + 2b_i^{(t)} \eta p - \eta p^2 \]

\[= \max_p 1 - b_i^{(t)} \eta + \eta p(2b_i^{(t)} - p) \]

To solve this for \( p \), we take the derivative with respect to \( p \):

\[\frac{d}{dp} 1 - b_i^{(t)} \eta + \eta p(2b_i^{(t)} - p) = \eta(2b_i^{(t)} - 2p). \]

So the maximum expected value is uniquely \( p = b_i^{(t)}. \)

A different way of proving the Corollary is by showing that the standard update rule with quadratic losses can be translated into the Brier strictly proper scoring rule. Either way, for applications with quadratic losses, the standard algorithm already works out of the box. However, as we saw in Example [1], this is not the case with the absolute loss function. As the absolute loss function arises in practice—recall that FiveThirtyEight uses absolute loss to calculate their pollster ratings—in the remainder of this paper we focus on answering questions (2) and (3) from the introduction for the absolute loss function.

3 Deterministic Algorithms for Selfish Experts

This section studies the question if there are good online prediction algorithms for the absolute loss function. We restrict our attention here to deterministic algorithms; Section 6 gives a randomized algorithm with good guarantees.

Proposition [8] tells us that for selfish experts to have a strict incentive to report truthfully, the weight-update rule must be a strictly proper scoring rule. This section gives a deterministic algorithm based on the spherical strictly proper scoring rule that has no \((2 + \sqrt{2})\)-regret (Theorem [12]). Additionally, we consider the question if the non-truthful reports from experts in using the standard (non-IC) WM algorithm are harmful. We show that this is the case by proving that the algorithm is not a no \((4 - O(1))\)-regret algorithm, for any constant smaller than 4 (Proposition [13]). This shows that, when experts are selfish, the IC online prediction algorithm with the spherical rule outperforms the standard WM algorithm (in the worst case).

3.1 Deterministic Online Prediction using a Spherical Rule

We next give an algorithm that uses a strictly proper scoring rule that is based on the spherical rule scoring rule. In the following, let \( s^{(t)}_i = |p_i^{(t)} - r^{(t)}| \) be the absolute loss of expert \( i \).

Consider the following weight-update rule:

\[ f_{sp}(p_i^{(t)}, r^{(t)}) = 1 - \eta \left( 1 - \frac{1 - s^{(t)}_i}{\sqrt{p_i^{(t)}}^2 + (1 - p_i^{(t)})^2} \right). \] (2)

The following proposition establishes that this is in fact a strictly proper scoring rule.

**Proposition 11.** The spherical weight-update rule in (2) is a strictly proper scoring rule.

**Proof.** The standard spherical strictly proper scoring rule is \((1 - s^{(t)}_i)/\sqrt{(p_i^{(t)})^2 + (1 - p_i^{(t)})^2}\). Any positive affine transformation of a strictly proper scoring rule yields another strictly proper scoring rule, see e.g. See Appendix [B] for intuition about why this rule yields better results than other natural candidates, such as the Brier scoring rule.
In a total worst-case loss no better than $\frac{1}{2}$ is also a strictly proper scoring rule. Now we multiply this by $\eta$ and add 1 to obtain

$$1 + \eta \left( \frac{1 - s_i^{(t)}}{\sqrt{(p_i^{(t)})^2 + (1 - p_i^{(t)})^2}} - 1 \right),$$

and rewriting proves the claim.

In addition to incentivizing truthful reporting, the WM algorithm with the update rule $f_{sp}$ does not do much worse than the best expert in hindsight. (See the appendix for the proof.)

**Theorem 12.** The WM algorithm with weight-update rule (2) for $\eta = O(1/\sqrt{T}) < \frac{1}{2}$ has no $(2 + \sqrt{2})$-regret.

### 3.2 True Loss of the Non-IC Standard Rule

It is instructive to compare the guarantee in Theorem 12 with the performance of the standard (non-IC) WM algorithm. With the standard weight update function $f(p_i^{(t)}, r^{(t)}) = 1 - \eta s_i^{(t)}$ for $\eta \in (0, \frac{1}{2})$, the WM algorithm has the guarantee that $M^{(T)} \leq 2 \left((1 + \eta)m_i^{(T)} + \frac{\ln n}{\sqrt{T}} \right)$ with respect to the reported loss of experts.

However, Example 1 demonstrates that this algorithm incentivizes extremal reports, i.e. if $b_i^{(t)} \in [0, \frac{1}{2})$ the expert will report $p_i^{(t)} = 0$ and if $b_i^{(t)} \in (\frac{1}{2}, 1]$ the expert will report 1. The following proposition shows that, in the worst case, this algorithm does no better than a factor 4 times the true loss of the best expert in hindsight. Theorem 12 shows that a suitable IC algorithm can obtain a superior worst-case guarantee.

**Proposition 13.** The standard WM algorithm with weight-update rule $f \left(p_i^{(t)}, r^{(t)} \right) = 1 - \eta |p_i^{(t)} - r^{(t)}|$ results in a total worst-case loss no better than

$$M^{(T)} \geq 4 \cdot \min_i m_i^{(T)} - o(1).$$

**Proof.** Let $A$ be the standard weighted majority algorithm. We create an instance with 2 experts where $M^{(T)} \geq 4 \cdot \min_i m_i^{(T)} - o(1)$. Let the reports $p_1^{(t)} = 0$, and $p_2^{(t)} = 1$ for all $t \in 1,...,T$; we will define $b_i^{(t)}$ shortly. Given the reports, $A$ will choose a sequence of predictions, let $r^{(t)}$ be 1 whenever the algorithm chooses 0 and vice versa, so that $M^{(T)} = T$.

Now for all $t$ such that $r^{(t)} = 1$, set $b_1^{(t)} = \frac{1}{2} - \epsilon$ and $b_2^{(t)} = 1$, and for all $t$ such that $r^{(t)} = 0$ set $b_1^{(t)} = 0$ and $b_2^{(t)} = \frac{1}{2} + \epsilon$, for small $\epsilon > 0$. Note that the beliefs $b_i^{(t)}$ indeed lead to the reports $p_i^{(t)}$ since $A$ incentivizes rounding the reports to the nearest integer.

Since the experts reported opposite outcomes, their combined total number of incorrect reports is $T$, hence the best expert had a reported loss of at most $T/2$. For each incorrect report $p_i^{(t)}$, the real loss of expert is $|r^{(t)} - b_i^{(t)}| = \frac{1}{2} + \epsilon$, hence $\min_i m_i^{(T)} \leq (\frac{1}{2} + \epsilon) T/2$, while $M^{(T)} = T$. Taking $\epsilon = o(T^{-1})$ yields the claim.

### 4 The Cost of Selfish Experts for IC algorithms

We now address the third fundamental question: whether or not online prediction with selfish experts is strictly harder than with honest experts. In this section we restrict our attention to deterministic algorithms; we extend the results to randomized algorithms in Section 3. As there exists a deterministic algorithm for honest experts with no 2-regret, showing a separation between honest and selfish experts boils down to proving that there exists a constant $\delta$ such that the worst-case loss is no better than a factor of $2 + \delta$ (with $\delta$ bounded away from 0 as $T \to \infty$).

In this section we show that such a $\delta$ exists for all incentive compatible algorithms, and that $\delta$ depends on properties of a “normalized” version of the weight-update rule $f$, independent of the learning rate. This
implies that the lower bound also holds for algorithms that, like the classical prediction algorithms, use a
time-varying learning rate. In Section 5 we show that under mild technical conditions the true loss of non-IC
algorithms is also bounded away from 2, and in Section 6 the lower bounds for deterministic algorithms are
used to show that there is no randomized algorithm that achieves vanishing regret.

To prove the lower bound, we have to be specific about which set of algorithms we consider. To cover
algorithms that have a decreasing learning parameter, we first show that any positive proper scoring rule can
be interpreted as having a learning parameter η.

**Proposition 14.** Let f be any strictly proper scoring rule. We can write f as f(p, r) = a + b f′(p, r) with
a ∈ R, b ∈ R+ and f′ a strictly proper scoring rule with min(f′(0, 1), f′(1, 0)) = 0 and max(f′(0, 0), f′(1, 1)) = 1.

**Proof.** Let fmin = min(f(0, 1), f(1, 0)) and fmax = max(f(0, 0), f(1, 1)) = 1. Then define f′(p, r) =
f(p, r)−fmin fmax−fmin, a = fmin and b = fmax − fmin. This is a positive affine translation, hence f′ is a strictly proper
scoring rule.

We call f′ : [0, 1] × {0, 1} → [0, 1] a normalized scoring rule. Using normalized scoring rules, we can define
a family of scoring rules with different learning rates η.

**Definition 15.** Let f be any normalized strictly proper scoring rule. Define F as the following family of
proper scoring rules generated by f:

\[ F = \{ f'(p, r) = a (1 + η(f(p, r) − 1)) : a > 0 \text{ and } η ∈ (0, 1) \} \]

By Proposition 14 the union of families generated by normalized strictly proper scoring rules cover all
strictly proper scoring rules. Using this we can now formulate the class of deterministic algorithms that are
incentive compatible.

**Definition 16** (Deterministic Incentive-Compatible Algorithms). Let \( A_d \) be the set of deterministic algorithms
that update weights by \( w_i^{t+1} = a(1 + η(f(p_i^t, r_i^t) − 1))w_i^t \), for a normalized strictly proper scoring rule
f and η ∈ (0, 1] with η possibly decreasing over time. For \( q = \sum_{i=1}^n w_i^t p_i^t / \sum_{i=1}^n w_i^t \), A picks \( q^t = 0 \)
if \( q < \frac{1}{2} \), \( q^t = 1 \) if \( q > \frac{1}{2} \) and uses any deterministic tie breaking rule for \( q = \frac{1}{2} \).

Using this definition we can now state our main result:

**Theorem 17.** For the absolute loss function, there does not exists a deterministic and incentive-compatible
algorithm A ∈ \( A_d \) with no 2-regret.

To prove Theorem 17 we proceed in two steps. First we consider strictly proper scoring rules that are
symmetric with respect to the outcomes, because they lead to a lower bound that can be naturally interpreted
by looking at the geometry of the scoring rule. We then extend these results to cover weight-update rules
that use any (potentially asymmetric) strictly proper scoring rule.

### 4.1 Symmetric Strictly Proper Scoring Rules

We first focus on symmetric scoring rules in the sense that \( f(p, 0) = f(1 − p, 1) \) for all \( p ∈ [0, 1] \). We can thus
write these as \( f(p) = f(p, 1) = f(1 − p, 0) \). Many common scoring rules are symmetric, including the Brier
rule [Brier, 1950], the family of pseudo-spherical rules (e.g. [Gneiting and Raftery, 2007]), the power family
(e.g. [Jose et al., 2008]), and the beta family [Buja et al., 2005] when \( α = β \). We start by defining the scoring
rule gap for normalized scoring rules, which will determine the lower bound constant.

**Definition 18** (Scoring Rule Gap). The scoring rule gap \( γ \) of family \( F \) with generator \( f \) is \( γ = f(\frac{1}{2}) − \frac{1}{2}(f(0) + f(1)) = f(\frac{1}{2}) − \frac{1}{2} \).

The following proposition shows that for all strictly proper scoring rules, the scoring rule gap must be
strictly positive.

**Proposition 19.** The scoring rule gap \( γ \) of a family generated by a symmetric strictly proper scoring rule \( f \)
is strictly positive.
Proof. Since $f$ is symmetric and a strictly proper scoring rule, we must have that $\frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f\left(\frac{1}{2}\right) > \frac{1}{2} f(0) + \frac{1}{2} f(1)$ (since an expert with belief $\frac{1}{2}$ must have a strict incentive to report $\frac{1}{2}$ instead of 1). The statement follows from rewriting.

We are now ready to prove our lower bound for all symmetric strictly proper scoring rules. The interesting case is where the learning rate $\eta \to 0$, as otherwise it is easy to prove a lower bound bounded away from 2.

The following lemma establishes that the gap parameter is important in proving lower bounds for IC online prediction algorithms. Intuitively, the lower bound instance exploits that experts who report $\frac{1}{2}$ will have a higher weight (due to the scoring rule gap) than an expert who is alternatingly right and wrong with extreme reports. This means that even though the indifferent expert has the same absolute loss, she will have a higher weight and this can lead the algorithm astray. The scoring rule gap is also relevant for the discussion in Appendix [B]. We give partial proof of the lemma below; the full proof appears in Appendix [A].

**Lemma 20.** Let $F$ be a family of scoring rules generated by a symmetric strictly proper scoring rule $f$, and let $\gamma$ be the scoring rule gap of $F$. In the worst case, MW with any scoring rule $f' \in F$ with $\eta \in (0, \frac{1}{2})$, algorithm loss $M(T)$ and expert loss $m_i(T)$, satisfies

$$M(T) \geq \left(2 + \frac{1}{\gamma - 1}\right) \cdot m_i(T).$$

**Proof Sketch.** Let $a$, $\eta$ be the parameters of $f'$ in the family $F$, as in Definition [15]. Fix $T$ sufficiently large and an integer multiple of $2[\gamma - 1] + 1$, and let $e_1$, $e_2$, and $e_3$ be three experts. For $t = 1, \ldots, \alpha \cdot T$ where $\alpha = \frac{2[\gamma - 1] + 1}{\gamma}$ such that $\alpha T$ is an even integer, let $p_1(t) = \frac{1}{3}$, $p_2(t) = 0$, and $p_3(t) = 1$. Fix any tie-breaking rule for the algorithm. Realization $r(t)$ is always the opposite of what the algorithm chooses.

Let $M(t)$ be the loss of the algorithm up to time $t$, and let $m_1(t)$ be the loss of expert $i$. We first show that at $t' = \alpha T$, $m_1(t') = m_2(t') = m_3(t') = \frac{\alpha T}{3}$ and $M(t') = \alpha T$. The latter part is obvious as $r(t)$ is the opposite of what the algorithm chooses. That $m_1(t') = \frac{\alpha T}{3}$ is also obvious as it adds a loss of $\frac{1}{3}$ at each time step. To show that $m_2(t') = m_3(t') = \frac{\alpha T}{2}$ we do induction on the number of time steps, in steps of two. The induction hypothesis is that at an even number of time steps, $m_2(t) = m_3(t)$ and that $w_2(t) = w_3(t)$. Initially, all weights are 1 and both experts have loss of 0, so the base case follows. Consider the algorithm after an even number of time steps. Since $w_2(t) = w_3(t)$, $p_3(t) = 1 - p_2(t)$, and $p_1(t) = \frac{1}{2}$ we have that $\sum_{i=1}^3 w_i(t) = \sum_{i=1}^3 w_i(t)(1 - p_i(t))$ and hence the algorithm will use its tie-breaking rule for its next decision. Thus, either $e_2$ or $e_3$ is wrong. Wlog let’s say that $e_2$ was wrong (the other case is symmetric), so $m_2(t+1) = 1 + m_3(t+1)$. Now at time $t + 1$, $w_2(t+1) = (1 - \eta)w_3(t+1) < w_3(t+1)$. Since $e_1$ does not express a preference, and $e_3$ has a higher weight than $e_2$, the algorithm will follow $e_3$’s advice. Since the realization $r(t+1)$ is the opposite of the algorithms choice, this means that now $e_3$ incurs a loss of one. Thus $m_2(t+2) = m_2(t+1)$ and $w_2(t+2) = w_2(t+1)$ and $m_3(t+2) = 1 + m_3(t+1) = m_3(t+2)$. The weight of expert $e_2$ is $w_2(t+2) = a\alpha(1 - \eta)w_2(t)$ and the weight of expert $e_3$ is $w_3(t+2) = a\alpha(1 - \eta)w_3(t)$. By the induction hypothesis $w_2(t) = w_3(t)$, hence $w_2(t+2) = w_3(t+2)$, and since we already showed that $m_2(t+2) = m_3(t+2)$, this completes the induction.

Now, for $t = \alpha T + 1, \ldots, T$, we let $p_1(t) = 1$, $p_2(t) = 0$, $p_3(t) = \frac{1}{2}$ and $r(t) = 0$. So henceforth $e_3$ does not provide information, $e_1$ is always wrong, and $e_2$ is always right. If we can show that the algorithm will always follow $e_3$, then the best expert is $e_2$ with a loss of $m_2(T) = \frac{\alpha T}{3}$, while the algorithm has a loss of $M(T) = T$. If this holds for $\alpha < 1$ this proves the claim. So what’s left to prove is that the algorithm will always follow $e_1$. Note that since $p_3(t) = \frac{1}{2}$ it contributes equal amounts to $\sum_{i=1}^3 w_i(t)p_i(t)$ and $\sum_{i=1}^3 w_i(t)(1 - p_i(t))$ and is therefore ignored by the algorithm in making its decision. So it suffices to look at $e_1$ and $e_2$. The algorithm will pick 1 iff $\sum_{i=1}^3 w_i(t)(1 - p_i(t)) \leq \sum_{i=1}^3 w_i(t)p_i(t)$ which after simplifying becomes $w_1(t) > w_2(t)$.

At time step $t$, $w_1(t) = (a(1 + \eta(f(\frac{1}{2}) - 1)))^{\alpha T}(a \cdot (1 - \eta))^{-t} - \alpha T$ and $w_2(t) = (a(1 - \eta))^{\alpha T} + a \frac{\alpha T}{2} + t - \alpha T$.

We have that $w_1(t)$ is decreasing faster in $t$ than $w_2(t)$. So if we can show that $w_1(T) \geq w_2(T)$ for some $\alpha < 1$, then $e_2$ will incur a total loss of $\alpha T/2$ while the algorithm incurs a loss of $T$ and the statement is proved. This is shown in the appendix.

□
As a consequence of Lemma 20, we can calculate lower bounds for specific strictly proper scoring rules. For example, the spherical rule used in Section 3.1 is a symmetric strictly proper scoring rule with a gap parameter $\gamma = \frac{\sqrt{2}}{2} - \frac{1}{2}$, and hence $1/(\gamma^{-1}) = \frac{1}{5}$.

Corollary 21. In the worst case, the deterministic algorithm based on the spherical rule in Section 3.1 has
\[ M^{(T)} \geq (2 + \frac{1}{5}) m_i^{(T)}. \]

We revisit the scoring rule gap parameter again in Appendix E when we discuss considerations for selecting different scoring rules.

4.2 Beyond Symmetric Strictly Proper Scoring Rules

We now extend the lower bound example to cover arbitrary strictly proper scoring rules. As in the previous subsection, we consider properties of normalized scoring rules to provide lower bounds that are independent of learning rate, but the properties in this subsection have a less natural interpretation.

For arbitrary strictly proper scoring rule $f'$, let $f$ be the corresponding normalized scoring rule, with parameters $\alpha$ and $\eta$. Since $f$ is normalized, $\max \{ f(0,0), f(1,1) \} = 1$ and $\min \{ f(0,1), f(1,0) \} = 0$. We consider 2 cases, one in which $f(0,0) = f(1,1) = 1$ and $f(0,1) = f(1,0) = 0$ which is locally symmetric, and the case where at least one of those equalities does not hold.

**The semi-symmetric case.** If it is the case that $f$ has $f(0,0) = f(1,1) = 1$ and $f(0,1) = f(1,0) = 0$, then $f$ has enough symmetry to prove a variant of the lower bound instance discussed just before. Define the semi-symmetric scoring rule gap as follows.

**Definition 22** (Semi-symmetric Scoring Rule Gap). The `semi-symmetric` scoring rule gap $\mu$ of family $F$ with normalized generator $f$ is $\mu = \frac{1}{2} (f(\frac{1}{2},0) + f(\frac{1}{2},1)) - \frac{1}{2}$.

Like the symmetric scoring rule gap, $\mu > 0$ by definition, as there needs to be a strict incentive to report $\frac{1}{2}$ for experts with $h_i^{(t)} = \frac{1}{2}$. Next, observe that since $f(\frac{1}{2},0), f(\frac{1}{2},1) \in [0,1]$ and $f(\frac{1}{2},0) + f(\frac{1}{2},1) = 1 + 2\mu$, it must be that $f(\frac{1}{2},0) \cdot f(\frac{1}{2},1) \geq 2\mu$. Using this it follows that:

\[
(1 + \eta(f(\frac{1}{2},0) - 1)) (1 + \eta(f(\frac{1}{2},1) - 1)) \\
= 1 + \eta \cdot (f(\frac{1}{2},0) + f(\frac{1}{2},1) - 2) + \eta^2 \cdot (f(\frac{1}{2},0) \cdot f(\frac{1}{2},1) - f(\frac{1}{2},0) - f(\frac{1}{2},1) + 1) \\
= 1 + \eta \cdot (1 + 2\mu - 2) + \eta^2 \cdot (f(\frac{1}{2},0) \cdot f(\frac{1}{2},1) - 2\mu) \\
\geq 1 - \eta(1-2\mu) + \eta^2(2\mu-2\mu) \\
= 1 - \eta + 2\mu\eta
\] (3)

Now this can be used in the same way as we proved the setting before:

**Lemma 23.** Let $F$ be a family of scoring rules generated by a normalized strictly proper scoring rule $f$, with $f(0,0) = f(1,1)$ and $f(0,1) = f(1,0)$. In the worst case, MW with any scoring rule $f'$ from $F$ with $\eta \in (0, \frac{1}{2})$ can do no better than
\[ M^{(T)} \geq \left(2 + \frac{1}{|\mu^{-1}|}\right) \cdot m_i^{(T)}. \]

**Proof Sketch.** Take the same instance as used in Lemma 20 with $\alpha = \frac{2[\mu^{-1}]}{2[\mu^{-1}] + 1}$. The progression of the algorithm up to $t = \alpha T$ is identical in this case, as expert $e_1$ is indifferent between outcomes, and $f(0,0) = f(1,1)$ and $f(0,1) = f(1,0)$ for experts $e_2$ and $e_3$. What remains to be shown is that the weight of $e_1$ will be higher at time $T$. At time $T$ the weights of $e_1$ and $e_2$ are:

\[
\alpha^{-T} w_1^{(T)} = (1 + \eta(f(\frac{1}{2},0) - 1))^{\alpha T} (1 + \eta(f(\frac{1}{2},1) - 1))^{\alpha T} (1 - \eta)^{(1-\alpha)T} \\
\alpha^{-T} w_2^{(T)} = (1 - \eta)^{\frac{T}{2}}.
\]
Similarly to the symmetric case, we know that \( w_1^{(T)} > w_2^{(T)} \) if we can show that

\[
\left(1 + \eta(f(\frac{1}{2}, 0) - 1)\right)^{[\mu^{-1}]} \left(1 + \eta(f(\frac{1}{2}, 1) - 1)\right)^{[\mu^{-1}]} (1 - \eta) > (1 - \eta)^{[\mu^{-1}]}.\]

By \([3]\), it suffices to show that \((1 - \eta + 2\mu)^{[\mu^{-1}]} (1 - \eta) > (1 - \eta)^{[\mu^{-1}]},\) which holds by following the derivation in the proof of Lemma \([20]\) given in the appendix, starting at \([6]\).

\[\square\]

The asymmetric case. We finally consider the setting where the weight-update rule is not symmetric, nor is it symmetric evaluated only at the extreme reports. The lower bound that we show is based on the amount of asymmetry at these extreme points, and is parametrized as follows.

Definition 24. Let \( c > d \) be parameters of a normalized strictly proper scoring rule \( f, \) such that \( c = 1 - \max\{f(0, 1), f(1, 0)\} \) and \( d = 1 - \min\{f(0, 0), f(1, 1)\}. \)

Scoring rules that are not covered by Lemmas \([20]\) or \([23]\) must have either \( c < 1 \) or \( d > 0 \) or both. The intuition behind the lower bound instance is that two experts who have opposite predictions, and are alternatingly right and wrong, will end up with different weights, even though they have the same loss. We use this to show that eventually one expert will have a lower loss, but also a lower weight, so the algorithm will follow the other expert. This process can be repeated to get the bounds in the Lemma below. The proof of the lemma appears in the appendix.

Lemma 25. Let \( F \) be a family of scoring rules generated by a normalized strictly proper scoring rule \( f, \) with not both \( f(0, 0) = f(1, 1) \) and \( f(0, 1) = f(1, 0) \) and parameters \( c \) and \( d \) as in Definition 24. In the worst case, MW with any scoring rule \( f' \) from \( F \) with \( \eta \in (0, \frac{1}{2}) \) can do no better than

\[
M^{(T)} \geq \left(2 + \max\left\{\frac{1-c}{2}, \frac{d}{4(1-d)}\right\}\right) \cdot m_1^{(T)}.
\]

Theorem \([17]\) now follows from combining the previous three lemmas.

Proof of Theorem \([17]\) follows from combining Lemmas \([20]\) \([23]\) and \([25]\) \[\square\]

5 The Cost of Selfish Experts for Non-IC Algorithms

What about non-incentive-compatible algorithms? Could it be that, even with experts reporting strategically instead of honestly, there is a deterministic no-2-regret algorithm (or a randomized algorithm with vanishing regret), to match the classical results for honest experts? Proposition \([13]\) shows that the standard algorithm fails to achieve such a regret bound, but maybe some other non-IC algorithm does?

Typically, one would show that this is not the case by a “revelation principle” argument: if there exists some (non-IC) algorithm \( A \) with good guarantees, then we can construct an algorithm \( B \) which takes private values as input, and runs algorithm \( A \) on whatever reports a self-interested agent would have provided to \( A. \) It does the strategic thinking for agents, and hence \( B \) is an IC algorithm with the same performance as \( A. \) This means that generally, whatever performance is possible with non-IC algorithms can be achieved by IC algorithms as well, thus lower bounds for IC algorithms translate to lower bounds for non-IC algorithms. In our case however, the reports impact both the weights of experts as well as the decision of the algorithm simultaneously. Even if we insist on keeping the weights in \( A \) and \( B \) the same, the decisions of the algorithms may still be different. Therefore, rather than relying on a simulation argument, we give a direct proof that, under mild technical conditions, non-IC deterministic algorithms cannot be no 2-regret\[9\] As in the previous section, we focus on deterministic algorithms; Section \([8]\) translates these lower bounds to randomized algorithms, where they imply that no vanishing-regret algorithms exist.

The following definition captures how players are incentivized to report differently from their beliefs.

\[9\]Similarly to Price of Anarchy (PoA) bounds, e.g. Roughgarden and Tardos, 2007, the results here show the harm of selfish behavior. Unlike PoA bounds, we sidestep the question of equilibrium concepts and our results are additive rather than multiplicative.
Definition 26 (Rationality Function). For a weight update function \( f \), let \( \rho_f : [0, 1] \to [0, 1] \) be the function from beliefs to predictions, such that reporting \( \rho_f(b) \) is rational for an expert with belief \( b \).

We restrict our attention here on rationality functions that are proper functions, meaning that each belief leads to a single prediction. Note that for incentive compatible weight update functions, the rationality function is simply the identity function.

Under mild technical conditions on the rationality function, we show our main lower bound for (potentially non-IC) algorithms.\(^{10}\)

Theorem 27. For a weight update function \( f \) with continuous or non-strictly increasing rationality function \( \rho_f \), there is no deterministic no 2-regret algorithm.

Note that Theorem 27 covers the standard algorithm, as well as other common update rules such as the Hedge update rule \( f_{\text{Hedge}}(p_i^{(t)}, r^{(t)}) = e^{-\eta p_i^{(t)} - r^{(t)}} \) [Freund and Schapire, 1997], and all IC methods, since they have the identity rationality function (though the bounds in Thm 17 are stronger).

We start with a proof that any algorithm with non-strictly increasing rationality function must have worst-case loss strictly more than twice the best expert in hindsight. Conceptually, the proof is a generalization of the proof for Proposition 13.

Lemma 28. Let \( f \) be a weight update function with a non-strictly increasing rationality function \( \rho_f \), such that there exists \( b_1 < b_2 \) with \( \rho_f(b_1) \geq \rho_f(b_2) \). For every deterministic algorithm, in the worst case

\[
M^{(T)} \geq (2 + |b_2 - b_1|)m_i^{(T)}.
\]

Proof. Fix \( f, b_1, b_2 \) such that \( \rho_f(b_1) \geq \rho_f(b_2) \) with \( b_1 < b_2 \). Let \( \pi_1 = \rho_f(b_1), \pi_2 = \rho_f(b_2), b_0 = 1 - \frac{b_2 + b_1}{2}, \) and \( \pi_0 = \rho_f(b_0). \)

Let there be two experts \( e_0 \) and \( e_1 \). Expert \( e_0 \) always predicts \( \pi_0 \) with belief \( b_0 \). If \( \pi_1 = \pi_2, e_1 \) predicts \( \pi_1 \) (similar to Proposition 13 we first fix the predictions of \( e_1 \), and will give consistent beliefs later). Otherwise \( \pi_1 > \pi_2 \), and expert \( e_1 \) has the following beliefs (and corresponding predictions) at time \( t \):

\[
b_{1}^{(t)} = \begin{cases} 
1 & \text{if } \frac{w_1^{(t)} \pi_0 + w_2^{(t)} \pi_2}{w_1^{(t)} + w_2^{(t)}} \geq \frac{1}{2} \\
2 & \text{otherwise}
\end{cases}
\]

The realizations are opposite of the algorithm’s decisions.

We now fix the beliefs of \( e_1 \) in the case that \( \pi_1 = \pi_2 \). Whenever \( r^{(t)} = 1 \), set expert \( e_1 \)’s belief to \( b_2 \), and whenever \( r^{(t)} = 0 \), set her belief to \( b_1 \). Note that the beliefs indeed lead to the predictions she made, by the fact that \( \pi_1 = \rho_f(b_1) = \rho_f(b_2) \).

For the case where \( \pi_1 > \pi_2 \), if \( (w_0^{(t)} \pi_0 + w_1^{(t)} \pi_2)/(w_0^{(t)} + w_1^{(t)}) \geq \frac{1}{2} \) then \( e_1 \)’s belief will be \( b_1 \) leading to a report of \( \pi_1 \) and as \( \pi_1 > \pi_2 \) it must hold that \( (w_0^{(t)} \pi_0 + w_1^{(t)} \pi_1)/(w_0^{(t)} + w_1^{(t)}) > \frac{1}{2} \), hence the algorithm will certainly choose 1, so the realization is 0. Conversely, if \( (w_0^{(t)} \pi_0 + w_1^{(t)} \pi_2)/(w_0^{(t)} + w_1^{(t)}) < \frac{1}{2} \), then the belief of \( e_1 \) will be \( b_2 \) and her report will lead the algorithm to certainly choose 0, so the realization is 1. So in all cases, if the realization is 1, then the belief of expert \( e_1 \) is \( b_2 \) and otherwise it is \( b_1 \).

The total number of mistakes \( M^{(T)} \) for the algorithm after \( T \) time steps is \( T \) by definition. Every time the realization was 1, \( e_0 \) will incur loss of \( \frac{b_1 + b_2}{2} \) and \( e_1 \) inures a loss of \( 1 - b_2 \), for a total loss of \( 1 - b_2 + \frac{b_1 + b_2}{2} = 1 - \frac{b_2 - b_1}{2} \). Whenever the realization was 0, \( e_0 \) incurs a loss of \( 1 - \frac{b_1 + b_2}{2} \) and \( e_1 \) incurs a loss of \( b_1 \) for a total loss of \( 1 - \frac{b_1 + b_2}{2} + b_1 = 1 - \frac{b_2 - b_1}{2} \).

So the total loss for both of the experts is \( \left(1 - \frac{b_2 - b_1}{2}\right) \cdot T \), so the best expert in hindsight has \( m_i^{(T)} \leq \frac{1}{2} \left(1 - \frac{b_2 - b_1}{2}\right) \cdot T \). Rewriting yields the claim.

For continuous rationality functions, we can generalize the results in Section 4 using a type of simulation argument. First, we address some edge cases.

Proposition 29. For a weight update function \( f \) with continuous strictly increasing rationality function \( \rho_f \),

\(^{10}\) This holds even when the learning rate is parameterized similarly to Definition 15 as the rationality function does not change for different learning rates due to the linearity of the expectation operator.
1. the regret is unbounded unless $\rho_f(0) < \frac{1}{2} < \rho_f(1)$; and

2. if $\rho_f(b) = \frac{1}{2}$ for $b \neq \frac{1}{2}$, the worst-case loss of the algorithm satisfies $M^{(T)} \geq (2 + |b - 1/2|) m_i^{(T)}$.

Proof. First, assume that it does not hold that $\rho_f(0) < \frac{1}{2} < \rho_f(1)$. Since $\rho_f(0) < \rho_f(1)$ by virtue of $\rho_f$ being strictly increasing, it must be that either $\frac{1}{2} \leq \rho_f(0) < \rho_f(1)$ or $\rho_f(0) < \rho_f(1) \leq \frac{1}{2}$. Take two experts with $b_1(t) = 0$ and $b_2(t) = 1$. Realizations are opposite of the algorithm’s predictions. Even though the experts have opposite beliefs, their predictions agree (potentially with one being indifferent), so the algorithm will consistently pick the same prediction, whereas one of the two experts will never make a mistake. Therefore the regret is unbounded.

As for the second statement. Since $\rho_f(0) < \frac{1}{2} < \rho_f(1)$, there is some $b$ such that $\rho_f(b) = \frac{1}{2}$. Wlog, assume $b < \frac{1}{2}$ (the other case is analogous). Since $\rho_f$ is continuous and strictly increasing, $\rho_f\left(\frac{b+1/2}{2}\right) > \frac{1}{2}$ while $\frac{b+1/2}{2} < \frac{1}{2}$. Take one expert $e_1$ with belief $b(t) = \frac{b+1/2}{2} < \frac{1}{2}$, who will predict $p(t) = \rho_f\left(\frac{b+1/2}{2}\right) > \frac{1}{2}$. Realizations are opposite of the algorithms decisions, and the algorithms decision is consistently 1, due to there only being one expert, and that expert putting more weight on 1. However, the absolute loss of the expert is only $\frac{1}{2} - \frac{|b-1/2|}{2}$ at each time step. Summing over the timesteps and rewriting yields the claim.

We are now ready to prove the main result in this section. The proof gives lower bound constants that are similar (though not identical) to the constants given in Lemmas 20, 23 and 25, though due to a reparameterization the factors are not immediately comparable. The proof appears in the appendix.

Theorem 30. For a weight update function $f$ with continuous strictly increasing rationality function $\rho_f$, with $\rho_f(0) < \frac{1}{2} < \rho_f(1)$ and $\rho_f\left(\frac{1}{2}\right) = \frac{1}{2}$, there is no deterministic no $2$-regret algorithm.

Theorem 27 now follows from Lemma 28, Proposition 29 and Theorem 30.

6 Randomized Algorithms: Upper and Lower Bounds

6.1 Imppossibility of Vanishing Regret

We now consider randomized online learning algorithms, which can typically achieve better worst-case guarantees than deterministic algorithms. For example, with honest experts, there are randomized algorithms with worst-case loss $M^{(T)} \leq (1 + O\left(\frac{1}{\sqrt{T}}\right)) m_i^{(T)}$, which means that the regret with respect to the best expert in hindsight is vanishing as $T \to \infty$. Unfortunately, the lower bounds in Sections 4 and 5 below imply that no such result is possible for randomized algorithms.

Corollary 31. Any incentive compatible randomized weight-update algorithm or non-IC randomized algorithm with continuous or non-strictly increasing rationality function cannot be no 1-regret.

Proof. We can use the same instances as for Theorems 17 and 30 and Lemma 28 (whenever the algorithm was indifferent, the realizations were defined using the algorithm’s tie-breaker rule; in the current setting simply pick any realization, say $r^t = 1$).

Whenever the algorithm made a mistake, it was because $\sum_i w^t_i s^t_i \geq \frac{1}{2} \sum_i w^t_i$. Therefore, in the randomized setting, it will still incur an expected loss of at least $\frac{1}{2}$. Therefore the total expected loss of the randomized algorithm is at least half that of the deterministic algorithm. Since the approximation factor for the latter is bounded away from 2 in all cases in Theorems 17 and 30 and Lemma 28 in these cases the worst-case loss of a randomized algorithm satisfies $M^{(T)} \geq (1 + \Omega(1)) m_i^{(T)}$.

6.2 An Incentive-Compatible Randomized Algorithm for Selfish Experts

While we cannot hope to achieve a no-regret algorithm for online prediction with selfish experts, we can do better than the deterministic algorithm from Section 3. We now focus on the more general class of algorithms where the algorithm can make any prediction $q^{(t)} \in [0,1]$ and incurs a loss of $|q^{(t)} - r^{(t)}|$. We give a randomized algorithm based on the Brier strictly proper scoring rule with loss at most 2.62 times that of the best expert as $T \to \infty$. 

15
Perhaps the most natural choice for a randomized algorithm is to simply report a prediction of \( q^{(t)} = \sum_{i=1}^{n} w_i^{(t)} p_i^{(t)} / \sum_{j=1}^{n} w_j^{(t)} \). However, this is problematic when the experts are highly confident and correct in their predictions. By the definition of a (bounded) strictly proper scoring rule, \( \frac{d}{dp_i^{(t)}} f(p_i^{(t)}, 1) = 0 \) at 1 (and similarly the derivative is 0 around 0 for a realization of 0). This means that experts that are almost certain and correct will not have their weight reduced meaningfully, and so the proof that uses the potential function does not go through.

This motivates looking for an algorithm where the sum of weights of experts is guaranteed to decrease significantly whenever the algorithm incurs a loss. Consider the following generalization of RWM that rounds the algorithm chooses \( \frac{i}{t} \) with probability

\[
p^{(t)} = \begin{cases} 
0 & \text{if } b^{(t)} < \theta \\
b^{(t)} & \text{if } \theta < b^{(t)} \leq 1 - \theta \\
1 & \text{otherwise.}
\end{cases}
\]

We call algorithms in \( \mathcal{A}_r \) \( \theta \)-RWM algorithms. We’ll use a \( \theta \)-RWM algorithm with the Brier rule. Recall that \( s_i^{(t)} = |p_i^{(t)} - r^{(t)}| \); the Brier rule is defined as:

\[
f_{Br}(p_i^{(t)}, r^{(t)}) = 1 - \eta \left( \frac{(p_i^{(t)})^2 + (1 - p_i^{(t)})^2 + 1}{2} - (1 - s_i^{(t)}) \right).
\]

**Theorem 33.** Let \( A \in \mathcal{A}_r \) be a \( \theta \)-RWM algorithm with the Brier weight update rule \( f_{Br} \) and \( \theta = 0.382 \) and with \( \eta = O(1/\sqrt{T}) \in (0, \frac{1}{2}) \). \( A \) has no 2.62-regret.

The proof appears in the appendix.

### 7 Simulations

The theoretical results presented so far indicate that when faced with selfish experts, one should use an IC weight update rule, and ones with smaller scoring rule gap are better. Two objections to these conclusions are: first, the presented results are worst-case, and different instances are used to obtain the bounds for different scoring rules. A priori it is not obvious that for an arbitrary (non worst-case) input, the regret of different scoring rules follow the same relative ordering. It is of particular interest to see if the non-IC standard weight-update rule does better or worse than the IC methods proposed in this paper. Second, there is a gap between our upper and lower bounds for IC rules. It is therefore informative to look at different instances for which we expect our algorithms to do badly, to see if the performance is closer to the upper bound or to the lower bound.

#### 7.1 Data-Generating Processes

To address these two concerns, we look at three different data-generating processes.

**Hidden Markov Model.** The experts are represented by a simple two-state hidden Markov model (HMM) with a “good” state and a “bad” state. We first flip a fair coin to determine the realization \( r^{(t)} \). For \( r^{(t)} = 0 \) (otherwise beliefs are reversed), in the good state expert \( i \) believes \( b_i^{(t)} \sim \min\{\exp(1)/5, 1\} \): the belief is exponentially distributed with parameter \( \lambda = 1 \), values are rescaled by \( \frac{1}{5} \) and clamped between 0 and 1. In the bad state, expert \( i \) believes \( b_i^{(t)} \sim U[1/2, 1] \). The transition probabilities to move to the other state are \( \frac{1}{10} \) for both states. This data generating process models that experts that have information about the event are more accurate than experts who lack the information.
Table 1: Comparison of lower bound results with simulation. The simulation is run for \( T = 10,000, \eta = 10^{-4} \) and we report the average of 30 runs. For the lower bounds, the first number is the lower bound from Lemma 20 i.e. \( 2 + \frac{1}{2 + \gamma} \), the second number (in parentheses) is \( 2 + \gamma \).

|                | Beta .1 | Beta .3 | Beta .5 | Beta .7 | Beta .9 | Brier (\( \beta = 1 \)) | Spherical |
|----------------|---------|---------|---------|---------|---------|-------------------------|-----------|
| Greedy LB Sim  | 2.3708  | 2.3833  | 2.3983  | 2.3758  | 2.3584  | 2.2507                  | 2.2071    |
| LB Simulation  | 2.4414  | 2.3657  | 2.3186  | 2.2847  | 2.2599  | 2.2509                  | 2.2070    |
| Lem 20 LB      | 2.33 (2.441) | 2.33 (2.365) | 2.25 (2.318) | 2.25 (2.285) | 2.25 (2.260) | 2.25                  | 2.2 (2.207) |

Lower Bound Instance. The lower bound instance described in the proof of Lemma 20.

Greedy Lower Bound. A greedy version of the lower bound described the proof of Lemma 20. There are 3 experts, one (\( e_0 \)) who is mostly uninformative, and two (\( e_1 \) and \( e_2 \)) who are alternating correct and incorrect. Whenever the weight of \( e_0 \) is “sufficiently” higher than that of \( e_1 \) and \( e_2 \), we have “punish the algorithm” by making \( e_0 \) wrong twice: \( b_0^{(t)} = 0, b_1^{(t)} = 1, b_2^{(t)} = \frac{1}{2}, r^{(t)} = 1 \), and \( b_0^{(t+1)} = 0, b_3^{(t)} = \frac{1}{2}, b_1^{(t)} = 1, r^{(t)} = 1 \). “Sufficiently” here means that weight of \( e_0 \) is high enough for the algorithm to follow its advice during both steps.

7.2 Results

Hidden Markov Model Data. In Figure 1 we show the regret as a function of time for the standard weight-update function, the Brier scoring rule, the spherical scoring rule, and a scoring rule from the Beta family [Buja et al., 2005] with \( \alpha = \beta = \frac{1}{2} \). The expert’s report \( p_i^{(t)} \) for the IC methods correspond to their belief \( b_i^{(t)} \), whereas for the standard weight-update rule, the expert reports \( p_i^{(t)} = 1 \) if \( b_i^{(t)} \geq \frac{1}{2} \) and \( p_i^{(t)} = 0 \) otherwise. The y axis is the ratio of the total loss of each of the algorithms to the performance of the best expert at that time. For clarity, we include the performance of the best expert at each time step, which by definition is 1 everywhere. The plot is for 10 experts, \( T = 10,000, \eta = 10^{-2} \), and the randomized versions of the algorithms (we return to why in a moment), averaged over 30 runs.

From the plot, we can see that each of the IC methods does significantly better than the standard weight-update algorithm. Whereas the standard weight-update rule levels off between 1.15 and 1.2, all of the IC methods dip below a regret of 1.05 at \( T = 2,000 \) and hover around 1.02 at \( T = 10,000 \). This trend continues and at \( T = 200,000 \) (not shown in the graph), the IC methods have a regret of about 1.003, whereas the regret for the standard algorithm is still 1.14. This gives credence to the notion that failing to account for incentive issues is problematic beyond the worst-case bounds presented earlier.

Moreover, the plot shows that while there is a worst-case lower bound for the IC methods that rules out no-regret, for quite natural synthetic data, the loss of all the IC algorithms approaches that of the best expert in hindsight, while the standard algorithm fails to do this. It curious to note that the performance of all IC methods are comparable (at least for this data-generating process). This seems to indicate that eliciting the truthful beliefs of the experts is more important than the exact weight-update rule.

Finally, note that the results shown here are for randomized weighted majority, using the different weight-update rules. For the deterministic version of the algorithms the difference between the non-IC standard weight-update rules and the IC ones is even starker. Different choices for the transition probabilities of the HMM, and different distributions, e.g. the bad state has \( b_i^{(t)} \sim U[0, 1] \), give similar results to the ones presented here.

Comparison of LB Instances. Now let’s focus on the performance of different IC algorithms. First, in Figure 2 we show the regret for different algorithms on the greedy lower bound instance. Note that this instance is different from the one used in the proof of Lemma 20 but the regret is very close to what is obtained there. In fact, when we look at Table 1 we can see that very closely traces \( 2 + \gamma \). In Table 1 we can also see the numerical results for the lower bound from Lemma 20. In fact, for the analysis, we needed to use \( \gamma^{-1} \) when determining the first phase of the instance. When we use \( \gamma \) instead numerically, the regret

\footnote{Here we use the regular RWM algorithm, so in the notation of Section 6 we have \( \theta = 0 \).}
Figure 1: The time-averaged regret for the HMM data-generating process.

Figure 2: Regret for the greedy lower bound instance.
seems to trace $2 + \gamma$ quite closely, rather than the weaker proven lower bound of $2 + \frac{1}{\lfloor \gamma + 1 \rfloor}$. By using two different lower bound constructions, we can see that the analysis of Lemma 20 is essentially tight (up to the rounding of $\gamma$), though this does not exclude the possibility that stronger lower bounds are possible using more properties of the scoring rules (rather than only the scoring rule gap $\gamma$). In these experiments (and others we have performed), the regret of IC methods never exceeds the lower bound we proved in Lemma 20. Closing the gap between the lower and upper bound requires finding a different lower bound instance, or a better analysis for the upper bound.
References

Jacob Abernethy, Yiling Chen, and Jennifer Wortman Vaughan. Efficient market making via convex optimization, and a connection to online learning. *ACM Transactions on Economics and Computation*, 1(2):12, 2013.

Moshe Babaioff, Robert D. Kleinberg, and Aleksandrs Slivkins. Truthful mechanisms with implicit payment computation. In *Proceedings of the 11th ACM Conference on Electronic Commerce*, EC ’10, pages 43–52, New York, NY, USA, 2010. ACM. ISBN 978-1-60558-822-3. doi: 10.1145/1807342.1807349. URL http://doi.acm.org/10.1145/1807342.1807349.

M. J. Bayarri and M. H. DeGroot. Optimal reporting of predictions. *Journal of the American Statistical Association*, 84(405):214–222, 1989. doi: 10.1080/01621459.1989.10478758.

J Eric Bickel and Seong Dae Kim. Verification of the weather channel probability of precipitation forecasts. *Monthly Weather Review*, 136(12):4867–4881, 2008.

John P Bonin. On the design of managerial incentive structures in a decentralized planning environment. *The American Economic Review*, 66(4):682–687, 1976.

Craig Boutilier. Eliciting forecasts from self-interested experts: scoring rules for decision makers. In *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*, pages 737–744. International Foundation for Autonomous Agents and Multiagent Systems, 2012.

Glenn W Brier. Verification of forecasts expressed in terms of probability. *Monthly weather review*, 78(1):1–3, 1950.

Michael Brückner and Tobias Scheffer. Stackelberg games for adversarial prediction problems. In *Proceedings of the 17th ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 547–555. ACM, 2011.

Andreas Buja, Werner Stuetzle, and Yi Shen. Loss functions for binary class probability estimation and classification: Structure and applications. 2005.

Yang Cai, Constantinos Daskalakis, and Christos H Papadimitriou. Optimum statistical estimation with strategic data sources. In *COLT*, pages 280–296, 2015.

Nicolo Cesa-Bianchi, Yishay Mansour, and Gilles Stoltz. Improved second-order bounds for prediction with expert advice. *Machine Learning*, 66(2-3):321–352, 2007.

Yiling Chen and David M Pennock. Designing markets for prediction. *AI Magazine*, 31(4):42–52, 2010.

Ofer Dekel, Felix Fischer, and Ariel D Procaccia. Incentive compatible regression learning. *Journal of Computer and System Sciences*, 76(8):759–777, 2010.

Peter Frazier, David Kempe, Jon Kleinberg, and Robert Kleinberg. Incentivizing exploration. In *Proceedings of the fifteenth ACM conference on Economics and Computation*, pages 5–22. ACM, 2014.

Yoav Freund and Robert E Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. *Journal of computer and system sciences*, 55(1):119–139, 1997.

Tilmann Gneiting and Adrian E Raftery. Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association*, 102(477):359–378, 2007.

Andrew V Goldberg and Jason D Hartline. Collusion-resistant mechanisms for single-parameter agents. In *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 620–629. Society for Industrial and Applied Mathematics, 2005.

Moritz Hardt, Nimrod Megiddo, Christos Papadimitriou, and Mary Wootters. Strategic classification. In *Proceedings of the 2016 ACM Conference on Innovations in Theoretical Computer Science*, pages 111–122. ACM, 2016.
Christian Franz Horn, Bjoern Sven Ivens, Michael Ohneberg, and Alexander Brem. Prediction markets—a literature review 2014. *The Journal of Prediction Markets*, 8(2):89–126, 2014.

Kamal Jain and Mohammad Mahdian. Cost sharing. *Algorithmic game theory*, pages 385–410, 2007.

Victor Richmond R Jose, Robert F Nau, and Robert L Winkler. Scoring rules, generalized entropy, and utility maximization. *Operations research*, 56(5):1146–1157, 2008.

Sham M Kakade, Adam Tauman Kalai, and Katrina Ligett. Playing games with approximation algorithms. *SIAM Journal on Computing*, 39(3):1088–1106, 2009.

Nick Littlestone and Manfred K Warmuth. The weighted majority algorithm. *Information and computation*, 108(2):212–261, 1994.

Yang Liu and Yiling Chen. A bandit framework for strategic regression. In *Advances in Neural Information Processing Systems*, pages 1813–1821, 2016.

Yishay Mansour, Aleksandrs Slivkins, Vasilis Syrgkanis, and Zhiwei Steven Wu. Bayesian exploration: Incentivizing exploration in bayesian games. *arXiv preprint arXiv:1602.07570*, 2016.

John McCarthy. Measures of the value of information. *Proceedings of the National Academy of Sciences of the United States of America*, 42(9):654, 1956.

Edgar C Merkle and Mark Steyvers. Choosing a strictly proper scoring rule. *Decision Analysis*, 10(4):292–304, 2013.

Nolan Miller, Paul Resnick, and Richard Zeckhauser. Eliciting informative feedback: The peer-prediction method. *Management Science*, 51(9):1359–1373, 2005.

Hervé Moulin. Incremental cost sharing: Characterization by coalition strategy-proofness. *Social Choice and Welfare*, 16(2):279–320, 1999.

Tim Roughgarden and Eva Tardos. Introduction to the inefficiency of equilibria. *Algorithmic Game Theory*, 17:443–459, 2007.

Leonard J Savage. Elicitation of personal probabilities and expectations. *Journal of the American Statistical Association*, 66(330):783–801, 1971.

Mark J Schervish. A general method for comparing probability assessors. *The Annals of Statistics*, pages 1856–1879, 1989.

Nihar Bhadresh Shah and Denny Zhou. Double or nothing: Multiplicative incentive mechanisms for crowdsourcing. In *Advances in neural information processing systems*, pages 1–9, 2015.

William Thomson. Eliciting production possibilities from a well-informed manager. *Journal of Economic Theory*, 20(3):360–380, 1979.
A Proofs

A.1 Proof of Theorem 12

Let \( A \) be the WM algorithm that updates weights according to (2) for \( \eta < \frac{1}{2} \). Let \( M^T \) be the total loss of \( A \) and \( m^T_i \) the total loss of expert \( i \). Then for each expert \( i \), we have

\[
M^T \leq (2 + \sqrt{2}) \left( (1 + \eta)m^T_i + \frac{\ln n}{\eta} \right).
\]

**Proof.** We use an intermediate potential function \( \Phi(t) = \sum_i w_i(t) \). Whenever the algorithm incurs a loss, the potential must decrease substantially. For the algorithm incur a loss, it must have picked the wrong outcome. Therefore it loss

\[
|r(t) - t_i| = 1 \quad \text{and} \quad \sum_i w_i(t) s_i(t) \geq \frac{1}{2} \cdot \Phi(t).
\]

We use this to show that in those cases the potential drops significantly:

\[
\begin{align*}
\Phi(t+1) &= \sum_i \left( 1 - \eta \left( \frac{1 - s_i(t)}{\sqrt{\left(p_i(t)^2 + (1 - p_i(t))^2\right)}} \right) \right) \cdot w_i(t) \\
&\leq \sum_i \left( 1 - \eta \left( 1 - \sqrt{2} \left( 1 - s_i(t) \right) \right) \right) \cdot w_i(t) \\
&= (1 - \eta) \Phi(t) + \sqrt{2}\eta \sum_i \left( 1 - s_i(t) \right) w_i(t) \\
&\leq (1 - \eta) \Phi(t) + \frac{\sqrt{2}\eta}{2} \Phi(t) \\
&= \left( 1 - \frac{2 - \sqrt{2}}{2} \eta \right) \Phi(t)
\end{align*}
\]

Since initially \( \Phi^0 = n \), after \( M^T \) mistakes, we have:

\[
\Phi^T \leq n \left( 1 - \frac{2 - \sqrt{2}}{2} \eta \right)^{M^T}.
\]  \hspace{1cm} (5)

Now, let’s bound the final weight of expert \( i \) in terms of the number of mistakes she made:

\[
\begin{align*}
w_i(T) &= \prod_t \left( 1 - \eta \left( \frac{1 - s_i(t)}{\sqrt{\left(p_i(t)^2 + (1 - p_i(t))^2\right)}} \right) \right) \\
&\geq \prod_t \left( 1 - \eta s_i(t) \right) \\
&\geq \prod_t (1 - \eta)^{s_i(t)} \\
&= (1 - \eta)^{\sum_t s_i(t)} \\
&= (1 - \eta)^{m^T_i}
\end{align*}
\]
Combining this with \( w_i^{(t)} \leq \Phi^{(t)} \) and \( [5] \), and taking natural logarithms of both sides we get:

\[
\ln \left( 1 - \eta \right)^{m_i(T)} \leq \ln \left( n \left( 1 - 2 - \frac{\sqrt{2}}{2} \eta \right)^{M(T)} \right)
\]

\[
m_i^{(T)} \cdot \ln(1 - \eta) \leq M(T) \cdot \ln \left( 1 - 2 - \frac{\sqrt{2}}{2} \eta \right) + \ln n
\]

\[
m_i^{(T)} \cdot (-\eta - \eta^2) \leq M(T) \cdot \ln \left( \exp \left( -2 - \frac{\sqrt{2}}{2} \eta \right) \right) + \ln n
\]

\[
m_i^{(T)} \cdot (-\eta - \eta^2) \leq M(T) \cdot -\frac{2}{\sqrt{2}} \eta + \ln n
\]

\[
M(T) \leq \left( \frac{2}{2 - \sqrt{2}} \right) \cdot \left( (1 + \eta)m_i^{(T)} + \ln n \right)
\]

where in the third inequality we used \(-\eta - \eta^2 \leq \ln(1 - \eta)\) for \( \eta \in (0, \frac{1}{2}) \). Rewriting the last statement proves the claim. \( \square \)

### A.2 Proof of Lemma 20

Let \( \mathcal{F} \) be a family of scoring rules generated by a symmetric strictly proper scoring rule \( f \), and let \( \gamma \) be the scoring rule gap of \( \mathcal{F} \). In the worst case, MW with any scoring rule \( f' \in \mathcal{F} \) with \( \eta \in (0, \frac{1}{2}) \), algorithm loss \( M(T) \) and expert loss \( m_i^{(T)} \), satisfies

\[
M(T) \geq \left( 2 + \frac{1}{|\gamma - 1|} \right) \cdot m_i^{(T)}.
\]

**Proof.** Let \( a, \eta \) be the parameters of \( f' \) in the family \( \mathcal{F} \), as in Definition 15. Fix \( T \) sufficiently large and an integer multiple of \( 2[\gamma^{-1}] + 1 \), and let \( e_1, e_2, \) and \( e_3 \) be three experts. For \( t = 1, \ldots, \alpha \cdot T \) where \( \alpha = \frac{2[\gamma^{-1}]}{2[\gamma^{-1}] + 1} \) such that \( \alpha T \) is an even integer, let \( p_1^{(t)} = \frac{1}{2}, p_2^{(t)} = 0, \) and \( p_3^{(t)} = 1 \). Fix any tie-breaking rule for the algorithm. Realization \( r^{(t)} \) is always the opposite of what the algorithm chooses.

Let \( M^{(t)} \) be the loss of the algorithm up to time \( t \), and let \( m_i^{(t)} \) be the loss of expert \( i \). We first show that at \( t' = \alpha T \), \( m_1^{(t')} = m_2^{(t')} = m_3^{(t')} = \frac{\alpha T}{2} \) and \( M^{(t')} = \alpha T \). The latter part is obvious as \( r^{(t)} \) is the opposite of what the algorithm chooses. That \( m_1^{(t')} = \frac{\alpha T}{2} \) is also obvious as it adds a loss of \( \frac{1}{2} \) at each time step. To show that \( m_2^{(t')} = m_3^{(t')} = \frac{\alpha T}{2} \) we do induction on the number of time steps, in steps of two. The induction hypothesis is that after an even number of time steps, \( m_2^{(t)} = m_3^{(t)} \) and that \( w_2^{(t)} = w_3^{(t)} \). Initially, all weights are 1 and both experts have loss of 0, so the base case holds. Consider the algorithm after an even number \( t \) time steps. Since \( w_2^{(t)} = w_3^{(t)} \), \( p_3^{(t)} = 1 - p_2^{(t)} \), and \( p_1^{(t)} = \frac{1}{2} \) we have that \( \sum_{t=1}^{T} w_i^{(t)} p_i^{(t)} = \sum_{t=1}^{T} w_i^{(t)} (1 - p_i^{(t)}) \) and hence the algorithm will use its tie-breaking rule for its next decision. Thus, either \( e_2 \) or \( e_3 \) is wrong. Wlog let’s say that \( e_2 \) was wrong (the other case is symmetric), so \( m_2^{(t+1)} = 1 + m_3^{(t+1)} \). Now at time \( t + 1 \), \( w_2^{(t+1)} = (1 - \eta)w_3^{(t+1)} < w_3^{(t+1)} \). Since \( e_1 \) does not express a preference, and \( e_3 \) has a higher weight than \( e_2 \), the algorithm will follow \( e_3 \)’s advice. Since the realization \( r^{(t+1)} \) is the opposite of the algorithms choice, this means that now \( e_3 \) incurs a loss of one. Thus \( m_2^{(t+2)} = m_2^{(t+1)} = m_2^{(t+1)} = m_2^{(t+2)} \) and \( m_3^{(t+2)} = 1 + m_3^{(t+1)} = m_2^{(t+2)} \). The weight of expert \( e_2 \) is \( w_2^{(t+2)} = \alpha(1 - \eta)w_2^{(t)} \) and the weight of expert \( e_3 \) is \( w_3^{(t+2)} = \alpha(1 - \eta)w_3^{(t)} \). By the induction hypothesis \( w_2^{(t)} = w_3^{(t)} \), hence \( w_2^{(t+2)} = w_3^{(t+2)} \), and since we already showed that \( m_2^{(t+2)} = m_3^{(t+2)} \), this completes the induction.

Now, for \( t = \alpha T + 1, \ldots, T \), we let \( p_1^{(t)} = 1, p_2^{(t)} = 0, p_3^{(t)} = \frac{1}{2} \) and \( r^{(t)} = 0 \). So henceforth \( e_3 \) does not provide information, \( e_1 \) is always wrong, and \( e_2 \) is always right. If we can show that the algorithm will always follow \( e_1 \), then the best expert is \( e_2 \) with a loss of \( m_2 = \frac{\alpha T}{2} \), while the algorithm has a loss of \( M(T) = T \). If this holds for \( \alpha < 1 \) this proves the claim. So what’s left to prove is that the algorithm will always follow
$e_1$. Note that since $p_3^{(t)} = \frac{1}{2}$ it contributes equal amounts to $\sum_{i=1}^{3} w_i^{(t)} p_i^{(t)}$ and $\sum_{i=1}^{3} w_i^{(t)} (1 - p_i^{(t)})$ and is therefore ignored by the algorithm in making its decision. So it suffices to look at $e_1$ and $e_2$. The algorithm will pick 1 iff $\sum_{i=1}^{3} w_i^{(t)} (1 - p_i^{(t)}) \leq \sum_{i=1}^{3} w_i^{(t)} p_i^{(t)}$, which after simplifying becomes $w_1^{(t)} > w_2^{(t)}$.

At time step $t$, $w_1^{(t)} = (a(1 + \eta(f(\frac{1}{2}) - 1)))^\alpha (a \cdot (1 - \eta))^{1-\alpha t}$ and $w_2^{(t)} = (a(1 - \eta))^{\frac{T}{2}} a \cdot (\frac{\alpha T}{2})^{1+t-\alpha t}$.

We have that $w_1^{(t)}$ is decreasing faster in $t$ than $w_2^{(t)}$. So if we can show that $w_1^{(t)} \geq w_2^{(t)}$ for some $\alpha < 1$, then $e_2$ will incur a total loss of $\alpha T/2$ while the algorithm incurs a loss of $T$ and the statement is proved.

We have that $w_1^{(t)}$ is decreasing faster in $t$ than $w_2^{(t)}$. So if we can show that at time $T$, $w_1^{(T)} \geq w_2^{(T)}$ for some $\alpha < 1$, then $e_2$ will incur a total loss of $\alpha T$ while the algorithm incurs a loss of $T$ and the statement is proved. First divide both weights by $\alpha T$ so that we have

$$a^{-T}w_1^{(T)} = (1 + \eta(f(\frac{1}{2}) - 1))^{\alpha T} (1 - \eta)^{(1-\alpha)T}$$

$$a^{-T}w_2^{(T)} = (1 - \eta)^{\frac{T}{2}}$$

Let $\alpha = \frac{2[\gamma^{-1}]}{2[\gamma^{-1}] + 1}$ and recall that $T = k \cdot (2[\gamma^{-1}] + 1)$ for positive integer $k$. Thus we can write

$$a^{-T}w_1^{(T)} = (1 + \eta(f(\frac{1}{2}) - 1))^{k2[\gamma^{-1}]} (1 - \eta)^k$$

$$= \left((1 + \eta(f(\frac{1}{2}) - 1))^{2[\gamma^{-1}]} (1 - \eta)\right)^k$$

$$a^{-T}w_2^{(T)} = (1 - \eta)^{k[\gamma^{-1}]}$$

$$= \left((1 - \eta)^{[\gamma^{-1}]}\right)^k$$

So it holds that $w_1^{(T)} > w_2^{(T)}$ if we can show that $(1 + \eta(f(\frac{1}{2}) - 1))^{2[\gamma^{-1}]} (1 - \eta) > (1 - \eta)^{[\gamma^{-1}]}

\begin{equation}
(1 + \eta(f(\frac{1}{2}) - 1))^{2[\gamma^{-1}]} (1 - \eta) = (1 - (\frac{1}{2} - \gamma) \eta)^{2[\gamma^{-1}]} (1 - \eta) \geq (1 - \eta + 2\gamma \eta)^{[\gamma^{-1}]} (1 - \eta) = \left(\frac{1 - \eta + 2\gamma \eta}{1 - \eta}\right)^{[\gamma^{-1}]} (1 - \eta)^{[\gamma^{-1}]+1} = (1 + 2\gamma \eta)^{[\gamma^{-1}]} (1 - \eta)^{[\gamma^{-1}]+1} \geq (1 + [\gamma^{-1}] 2\gamma \eta) (1 - \eta)^{[\gamma^{-1}]+1} \geq ((1 + 2\eta) (1 - \eta)) (1 - \eta)^{[\gamma^{-1}]} > (1 - \eta)^{[\gamma^{-1}]} \quad \text{(for } \eta < \frac{1}{2})
\end{equation}

Therefore expert $e_2$ will not incur any more loss during the last stage of the instance, so her total loss is $m_i^{(T)} = k[\gamma^{-1}]$ while the loss of the algorithm is $M^{(T)} = T = k \cdot (2[\gamma^{-1}] + 1)$. So

$$\frac{M^{(T)}}{m_i^{(T)}} \geq \frac{k \cdot (2[\gamma^{-1}] + 1) k[\gamma^{-1}]}{k[\gamma^{-1}]} = 2 + \frac{1}{[\gamma^{-1}]}$$

rearranging proves the claim.

\[\square\]

### A.3 Proof of Lemma 25

Let $\mathcal{F}$ be a family of scoring rules generated by a normalized strictly proper scoring rule $f$, with not both $f(0, 0) = f(1, 1)$ and $f(0, 1) = f(1, 0)$ and parameters $c$ and $d$ as in Definition 24. In the worst case, MW with any scoring rule $f'$ from $\mathcal{F}$ with $\eta \in (0, \frac{1}{2})$ can do no better than

$$M^{(T)} \geq \left(2 + \max\{\frac{1-c}{2c}, \frac{d}{4(1-d)}\}\right) m_i^{(T)}.$$
Proof. Fix $f$, and without loss of generality assume that $f(0, 0) = 1$ (since $f$ is normalized, either $f(0, 0)$ or $f(1, 1)$ needs to be 1, rename if necessary). As $f$ is normalized, at least one of $f(0, 1)$ and $f(1, 0)$ needs to be 0. For now, we consider the case where $f(0, 1) = 0$, we treat the other case later. For now we have $f(0, 0) = 1$, $f(0, 1) = 0$, and by definition $f(1, 0) = 1 - c$ and $f(1, 1) = 1 - d$, where $c > d$ (since correctly reporting 1 needs to score higher than reporting 0 when 1 materialized) and $(c = 1 \land d = 0)$ (since that would put us in the semi-symmetric case).

We construct an instance as follows. We have two experts, $e_0$ reports 0 always, and $e_1$ reports 1 always, and as usual, the realizations are opposite of the algorithms decisions. Since the experts have completely opposite predictions, the algorithm will follow whichever expert has the highest weight. We will show that after a constant number of time steps $t$, the weight $w_0^{(t)}$ of $e_0$ will be larger than the weight $w_1^{(t)}$ of $e_1$ even though $e_0$ will have made one more mistake. Note that when this is true for some $t$ independent of $\eta$, this implies that the algorithm cannot do better than $\frac{2}{t-1} > 2 + \frac{2}{t}$.

While it hasn’t been the case that $w_0^{(t)} > w_1^{(t)}$ with $m_0^{(t)} = m_1^{(t)} + 1$, realizations alternate, and the weight of each expert is:

$$w_0^{(2t)} = a^{2t} (1 + \eta(f(0, 0) - 1))^t(1 + \eta(f(0, 1) - 1))^t = a^{2t} (1 + \eta(1 - 1))^t(1 + \eta(1 - c - 1))^t = a^{2t}(1 - c\eta)^t$$

$$w_1^{(2t)} = a^{2t} (1 + \eta(f(1, 1) - 1))^t(1 + \eta(f(1, 0) - 1))^t = a^{2t} (1 + \eta(1 - d - 1))^t(1 + \eta(0 - 1))^t = a^{2t}(1 - d\eta)^t(1 - \eta)^t$$

What remains to be shown is that for some $t$ independent of $\eta$,

$$a^{2t+1}(1 - c\eta)^{t+1} > a^{2t+1}(1 - d\eta)^{t+1}(1 - \eta)^t.$$  

We know that it cannot be the case that simultaneously $c = 1$ and $d = 0$, so let’s first consider the case where $c < 1$. In this case, it is sufficient to prove the above statement assuming $d = 0$, as this implies the inequality for all $d \in [0, c)$. The following derivation shows that $a^{2t+1}(1 - c\eta)^{t+1} > a^{2t+1}(1 - d\eta)^{t+1}(1 - \eta)^t$ whenever $\frac{c}{1-c} < t$.

$$a^{2t+1}(1 - c\eta)^{t+1} > a^{2t+1}(1 - d\eta)^{t+1}(1 - \eta)^t$$

$$(1 - c\eta)^{t+1} > (1 - \eta)^t$$

$$\ln(1 - c\eta) > t \cdot \ln \left( \frac{1 - \eta}{1 - c\eta} \right)$$

$$1 - \frac{1}{1 - c\eta} > t \cdot \left( \frac{1 - \eta}{1 - c\eta} - 1 \right)$$

$$(1 - \frac{1}{1 - c\eta} - 1) > t \cdot \left( \frac{1 - \eta - 1 + c\eta}{1 - c\eta} \right)$$

$$\frac{c\eta}{1 - c\eta} < t \cdot \left( \frac{(1 - c)\eta}{1 - c\eta} \right)$$

$$\frac{c\eta}{(1 - c)\eta} < t$$

$$\frac{c}{(1 - c)} < t$$

25
So after $2t + 1$ time steps for some $t \leq \frac{c}{c-d} + 1$, expert $e_0$ will have one more mistake than expert $e_1$, but still have a higher weight. This means that after at most another $2t + 1$ time steps, she will have two more mistakes, yet still a higher weight. In general, the total loss of the algorithm is at least $2 + \frac{1-c}{c}$ times that of the best expert. Now consider the case where $c = 1$ and therefore $d > 0$. We will show that after $2t + 1$ time steps for some $t \leq 2 \frac{1-d}{d} + 1$ expert $e_0$ will have one more mistake than expert $e_1$.

\[ a^{2t}(1 - c\eta)^{t+1} > a^{2t}(1 - d\eta)^{t}(1 - \eta)^{t}(1 - d\eta) \]
\[ (1 - \eta)^{t+1} > (1 - d\eta)^{t+1}(1 - \eta)^{t} \]
\[ \frac{1 - \eta}{1 - d\eta} > (1 - d\eta)^{t} \]
\[ \ln \left( \frac{1 - \eta}{1 - d\eta} \right) > t \ln(1 - d\eta) \]
\[ 1 - \frac{1 - d\eta}{1 - \eta} > t(1 - d\eta - 1) \]
\[ \frac{1 - \eta - 1 + d\eta}{1 - \eta} > -td\eta \]
\[ \frac{(1 - d\eta)\eta}{1 - \eta} < td\eta \]
\[ \frac{1 - d}{d} \frac{1}{1 - \eta} < t \]
\[ 2 \frac{1 - d}{d} < t \]

(by $\eta < \frac{1}{2}$)

So in any case, after $t \leq 2 \max\{ \frac{c}{1-c}, \frac{1-d}{2(1-d)} \} + 1$ time steps so the loss compared to the best expert is at least

\[ 2 + \max\{ \frac{c}{1-c}, \frac{1-d}{2(1-d)} \}. \]

What remains to be proven is the case where $f(0, 1) > 0$. In this case, it will have to be that $f(1, 0) = 0$, as $f$ is normalized. And similarly to before, by Definition 24 we have $f(0, 1) = 1 - c$ and $f(1, 1) = 1 - d$ for $c > d$ and $-c = 1 \wedge d = 0$. Now, whenever $w(t) > w_1(t)$, $e_0$ will predict 1 and $e_1$ predicts 0, and otherwise $e_0$ predicts 0 and $e_1$ predicts 1. As usual, the realizations are opposite of the algorithm’s decisions. For now assume tie of the algorithm is broken in favor of $e_1$, then the weights will be identical to (7), (8). If the tie is broken in favor of $e_0$ initially, it takes at most twice as long before $e_0$ makes two mistakes in a row. Therefore, the loss with respect to the best expert in hindsight of an algorithm with any asymmetric strictly proper scoring rule is

\[ 2 + \max\{ \frac{c}{1-c}, \frac{d}{2(1-d)} \}. \]

\[ \square \]

A.4 Proof of Theorem 30

For a weight update function $f$ with continuous strictly increasing rationality function $\rho_f$, with $\rho_f(0) < \frac{1}{2} < \rho_f(1)$ and $\rho_f(\frac{1}{2}) = \frac{1}{2}$, there is no deterministic no 2-regret algorithm.

**Proof.** Fix $f$ with $\rho_f(0) < \frac{1}{2} < \rho_f(1)$ and $\rho_f(\frac{1}{2}) = \frac{1}{2}$. Define $p = \max\{ \rho_f(0), 1 - \rho_f(1) \}$, so that $p$ and $1 - p$ are both in the image of $\rho_f$ and the difference between $p$ and $1 - p$ is as large as possible. Let $b_1 = p^{-1}(p)$ and $b_2 = \rho^{-1}(1 - p)$ and observe that $b_1 < \frac{1}{2} < b_2$.

Next, we rewrite the weight-update function $f$ in a similar way as the normalization procedure similar to Definition 15: $f(p, r) = a(1 + \eta(f'(p, r) - 1))$. where $\max\{ f'(p, 0), f'(1 - p, 1) \} = 1$ and $\min\{ f'(p, 1), f'(1 - p, 0) \} = 0$. Again we do this to prove bounds that are not dependent on any learning rate parameter.
Note that the composition of \( \rho_f \) and \( f \), namely \( f(\rho_f(p), r) \) is a strictly proper scoring rule, since it changes the prediction in the same way as the selfish expert would do. Since \( f(\rho_f(p), r) \), it must also be that \( f'(\rho_f(p), r) \) is a strictly proper scoring rule, since it is a positive affine transformation of \( f \circ \rho_f \).

We now continue similarly to the lower bounds in Section 4. We only treat the semi-symmetric and asymmetric cases as the former includes the special case of the symmetric weight-update function.

For the semi-symmetric case, by definition \( f'(\rho_f(b_1), 0) = f'(\rho_f(b_2), 1) = 1 \) and \( f'(p, 0), f'(1 - p, 1) \). And since \( \min\{f'(p, 1), f'(1 - p, 0) = 0 \). Because \( f' \circ \rho_f \) is a strictly proper scoring rule, the following inequality holds:

\[
\frac{1}{2} f'(\rho_f(\frac{1}{2}, 0)) + \frac{1}{2} f'(\rho_f(\frac{1}{2}, 1)) + \mu = \frac{1}{2} f'(\rho_f(b_1), 0) + \frac{1}{2} f'(\rho_f(\frac{1}{2}, 1)) = \frac{1}{2}
\]

for some \( \mu > 0 \), since an expert with belief \( \rho_f(\frac{1}{2}) \) must have a strict incentive to report this. Here \( \mu \) plays the same role as the semi-symmetric scoring rule gap \( \mu \).

We now pitch three experts against each other in a similar lower bound instance as Lemma 23. For the first stage, they have beliefs \( b_0^{(t)} = \frac{1}{2}, b_1^{(t)} = b_1, b_2^{(t)} = b_2 \), so they have predictions \( p_0^{(t)} = \frac{1}{2}, p_1^{(t)} = \rho_f(b_1) = p, p_2^{(t)} = \rho_f(b_2) = 1 - p \). For the second stage, recall that either \( b_1 = 0 \) or \( b_2 = 1 \). In the former case, \( b_0^{(t)} = 1, b_1^{(t)} = 0, b_2^{(t)} = \frac{1}{2} \) and \( r^{(t)} = 0 \) and in the latter case \( b_0^{(t)} = 0, b_1^{(t)} = 1, b_2^{(t)} = \frac{1}{2} \) and \( r^{(t)} = 1 \). We now show a bijection between the instance in Lemma 23 and this instance, which establishes the lower bound for the semi-symmetric non-incentive compatible case.

First of all, note that the weights of each of the experts in the first stage is the same (up to the choice of \( a \) and \( \eta \), and for now assuming that the algorithms choices and thus the realizations are the same):

\[
\begin{align*}
\bar{w}_0^{(2t)} &= a^{2t} \left((1 + \eta(f'((\frac{1}{2}, 0)) - 1))((1 + \eta(f'((\frac{1}{2}, 1)) - 1))\right) \\
&\geq a^{2t}(1 - \eta + 2\mu\eta) \\
\bar{w}_1^{(2t)} &= a^{2t}(1 - \eta) \\
\bar{w}_2^{(2t)} &= a^{2t}(1 - \eta)
\end{align*}
\]

(Follows from 4)

In the second stage expert \( e_0 \) is always wrong and \( e_1 \) is always right, and hence at time \( T \) the weights

Also note that the predictions of \( e_1 \) and \( e_2 \) are opposite, i.e. \( p \) and \( 1 - p \), so the algorithm will follow the expert which highest weight, meaning the algorithms decisions and the realizations are identical to the instance in Lemma 23.

To complete the proof of the lower bound instance, we need to show that the total loss of \( e_1 \) is the same. During the first stage, alternatingly the true absolute loss of \( e_1 \) is \( b_1 \) and \( 1 - b_1 \), so after each 2 steps, her loss is 1. During the last stage, since her belief is certain (i.e. \( b_0 \) if \( b_1 = 0 \) or \( b_2 = 1 \) ans she is correct, she incurs no additional loss. Therefore the loss of the algorithm and the true loss of \( e_1 \) are the same as in Lemma 23 hence the loss of the algorithm is at least \( \frac{1}{\mu+1} \) times that of the best expert in hindsight.

Finally, we consider the asymmetric case. We use a similar instance as Lemma 23 with two experts \( e_0, e_1 \). If \( f'(1 - p, 0) = 0 \) we have \( b_0^{(t)} = b_1 \) and \( b_1^{(t)} = b_2 = 1 - b_2 \), so \( p_0^{(t)} = p \) and \( p_1^{(t)} = 1 - p \), otherwise the beliefs (and thus predictions) alternate. Again, the predictions are opposite of each other, and the weights evolve identically (up to the choice of \( a \) and \( \eta \)) as before. Again the loss up until the moment that the same expert is chosen twice in a row is the same.

Once the same expert is chosen twice (after at most \( 2 \max\{\frac{c}{1-c}, \frac{1-d}{d}\} + 1 \) steps), it is not necessarily the case that the total loss of one expert exceeds the other by 2, as the true beliefs are \( b_1 \) and \( b_2 \), rather than 0 and 1. However, since at least either \( b_1 = 0 \) or \( b_2 = 1 \), and \( b_1 < \frac{1}{2} < b_2 \), the difference in total absolute loss in this non-IC instance is at least half of the IC instance, so we lose at most factor \( \frac{1}{2} \) in the regret bound, hence for the asymmetric case \( M(T) \geq \left(2 + \max\{\frac{c}{4c}, \frac{d}{8(1-d)}\}\right) m_i^{(t)} \), completing the proof of the statement.\footnote{And since \( f' \) is a positive affine transformation of \( f \), the rationality function is unchanged due to the linearity of the expectation operator.}

\footnote{It is defined slightly differently though, as the image of \( \rho_f \) may not be \([0, 1]\).}
A.5 Proof of Theorem 33

Let \( A \in \mathcal{A} \) be a \( \theta \)-RWM algorithm with the Brier weight update rule \( f_{\text{Br}} \) and \( \theta = 0.382 \) and with \( \eta \in (0, \frac{1}{2}) \).

For any expert \( i \) it holds that
\[
M^{(T)} \leq 2.62 \left( (1 + \eta) m_i^{(T)} + \frac{\ln n}{\eta} \right).
\]

**Proof.** The core difference between the proof of this statement, and the proof for Theorem 12 is in giving the upper bound of \( \Phi^{(t+1)} \). Here we will give an upper bound of \( \Phi^{(T)} \leq n \cdot \exp \left( -\frac{\eta}{2.62} M^{(T)} \right) \).

Before giving this bound, observe that this would imply the theorem: since the weight updates are identical to the deterministic algorithm, we can use the same lower bound for \( \Phi^{(T)} \), namely \( \Phi^{(T)} \geq (1 - \eta)^{m_i^{(T)}} \) for each expert. Then taking the log of both sides we get:
\[
\ln n - \frac{\eta}{2.62} M^{(T)} \geq m_i^{(T)} \cdot \ln(1 - \eta)
\]
\[
\ln n - \frac{\eta}{2.62} M^{(T)} \geq m_i^{(T)} \cdot (-\eta - \eta^2)
\]
\[
M^{(T)} \leq 2.62 \left( (1 + \eta) m_i^{(T)} + \frac{\ln n}{\eta} \right)
\]

So all that’s left to prove is that whenever the algorithm incurs a loss \( \ell \), \( \Phi^{(t+1)} \leq \exp \left( -\frac{\eta}{2.62} \ell \right) \). At time \( t \), the output \( q_i^{(t)} \) of a \( \theta \)-RWM algorithm is one of three cases, depending on the weighted expert prediction. The first options is that the algorithm reported the realized event, in which case the \( \ell^{(t)} = 0 \) and the statement holds trivially. We treat the other two cases separately.

Let’s first consider the case where the algorithm reported the incorrect event with certainty: \( \ell^{(t)} = 1 \). The means that \( \sum_{i=1}^{n} w_i^{(t)} s_i^{(t)} \geq (1 - \theta) \Phi^{(t)} \). Since the Brier rule is concave, \( \Phi^{(t+1)} \) is maximized when \( s_i^{(t)} = 1 - \theta \) for all experts \( i \). In this case each we get
\[
\Phi^{(t+1)} \leq \sum_i \left( 1 - \eta \left( \frac{(p_i^{(t)})^2 + (1 - p_i^{(t)})^2 + 1}{2} - (1 - s_i^{(t)}) \right) \right) w_i^{(t)}
\]
\[
\leq \sum_i \left( 1 - \eta \left( \frac{(\theta)^2 + (1 - \theta)^2 + 1}{2} - \theta \right) \right) w_i^{(t)}
\]
\[
\leq \sum_i \left( 1 - \frac{\eta}{2.62} \right) w_i^{(t)}
\]
\[
= \left( 1 - \frac{\eta}{2.62} \right) \Phi^{(t)}
\]

Otherwise the algorithms report is between \( \theta \) and \( 1 - \theta \). Let \( \ell^{(t)} \in [\theta, 1-\theta] \) be the loss of the algorithm. Again, since the Brier rule is concave, \( \Phi^{(t+1)} \) is maximized when \( s_i^{(t)} = \ell^{(t)} \) for all experts \( i \). On \( [\theta, 1-\theta] \) the Brier proper scoring rule can be upper bounded by
\[
1 - \frac{\eta}{f_{\text{Br}}(1-\theta, 1)/\theta} s_i^{(t)} = 1 - \frac{\eta}{2.62} s_i^{(t)}.
\]

This yields
\[
\Phi^{(t+1)} \leq \sum_i \left( 1 - \eta \left( \frac{(p_i^{(t)})^2 + (1 - p_i^{(t)})^2 + 1}{2} - (1 - s_i^{(t)}) \right) \right) w_i^{(t)}
\]
\[
\leq \sum_i \left( 1 - \frac{\eta}{2.62} s_i^{(t)} \right) w_i^{(t)}
\]
\[
\leq \left( 1 - \frac{\eta}{2.62} \ell^{(t)} \right) \Phi^{(t)}
\]

So the potential at time \( T \) can be bounded by \( \Phi^{(T)} \leq n \cdot \prod_t \left( 1 - \frac{\eta}{2.62} \ell^{(t)} \right) \leq n \cdot \exp \left( -\frac{\eta}{2.62} M^{(T)} \right) \), from which the claim follows.
Figure 3: Three different normalized weight-update rules for $r(t) = 1$. The line segment is the standard update rule, the concave curve the Brier rule and the other curve the spherical rule.

B Selecting a Strictly Proper Scoring Rule

When selecting a strictly proper scoring rule for an IC online prediction algorithm, different choices may lead to very different guarantees. Many different scoring rules exist [McCarthy, 1956, Savage, 1971], and for discussion of selecting proper scoring rules in non-online settings, see also [Merkle and Steyvers, 2013]. Figure 3 shows two popular strictly proper scoring rules, the Brier rule and the spherical rule, along with the standard rule as comparison. Note that we have normalized all three rules for easy comparison.

Firstly, we know that for honest experts, the standard rule performs close to optimally. For every $\delta > 0$ we can pick a learning rate $\eta$ such that as $T \to \infty$ the loss of the algorithm $M(T) \leq (2 + \delta)m_i(t)$, while no algorithm could do better than $M(T) < 2m_i(T)$ [Littlestone and Warmuth, 1994; Freund and Schapire, 1997]. This motivates looking at strictly proper scoring rule that are “close” to the standard update rule in some sense. In Figure 3, if we compare the two strictly proper scoring rules, the spherical rule seems to follow the standard rule better than Brier does.

A more formal way of look at this is to look at the scoring rule gap. In Figure 3 we marked the $p = \frac{1}{2}$ location. Visually, the scoring rule gap $\gamma$ is the difference between a scoring rule and the standard rule at $p = \frac{1}{2}$. Since the Brier score has a large scoring rule gap, we’re able to prove a strictly stronger lower bound for it: the scoring rule gap $\gamma = \frac{1}{4}$, hence MW with the Brier scoring rule cannot do better than $M(T) \geq (2 + \frac{1}{4})m_i(T)$ in the worst case, according to Lemma 20. Corollary 21 shows that for the Spherical rule, this factor is $2 + \frac{1}{5}$. The ability to prove stronger lower bounds for scoring rules with larger gap parameter $\gamma$ is an indication that it is probably harder to prove strong upper bounds for those scoring rules.