Tokamak equilibria with non-parallel flow in a triangularity-deformed axisymmetric toroidal coordinate system

Ap Kuiroukidis\textsuperscript{a}, G.N. Throumoulopoulos\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} Technological Education Institute of Central Macedonia, GR 62124 Serres, Greece
\textsuperscript{b} University of Ioannina, Department of Physics, GR 451 10 Ioannina, Greece

\textsuperscript{*} Corresponding author.
E-mail addresses: kouirouki@astro.auth.gr (A. Kuiroukidis), gthroum@cc.uoi.gr (G.N. Throumoulopoulos).

Abstract

We consider a generalized Grad–Shafranov equation (GGSE) in a triangularity-deformed axisymmetric toroidal coordinate system and solve it numerically for the generic case of ITER-like and JET-like equilibria with non-parallel flow. It turns out that increase of the triangularity improves confinement by leading to larger values of the toroidal beta and the safety factor. This result is supported by the application of a criterion for linear stability valid for equilibria with flow parallel to the magnetic field. Also, the parallel flow has a weaker stabilizing effect.

Keywords: Applied Mathematics, Plasma Physics

1. Introduction

The axisymmetric magnetohydrodynamic equilibrium is governed by the well known Grad–Shafranov equation [1]. Since plasma flow plays a role in the transitions to improved confinement regimes as the L-H transition and the formation of internal transport barriers (ITBs), generalized Grad–Shafranov equations (GGSEs) have also been derived for flowing plasmas, e.g. Eq. (1) below valid for incompressible flows
In addition to the development of equilibrium codes to solve the above mentioned equations, analytic solutions have been constructed primarily in two ways: either by the method of separation of variables for linearized forms of the equations, valid for arbitrary aspect ratio, or on the basis of inverse-aspect-ratio expansions usually indented for equilibria with small toroidicity. An example of the first method is the Solovév solution [2]. The conventional way of employing the second method solves the Grad–Shafranov equation order by order in \( \epsilon \), the zeroth order equation corresponding to the cylindrical approximation and the toroidicity being introduced in the higher-order equations, e.g. [1].

In the framework of the second method a coordinate system \((r, \theta, \phi)\) is usually employed, called pseudo-toroidal or Shafranov coordinates, such that the magnetic surfaces have circular cross-section. Here, \( \phi \) is the toroidal coordinate, and \((r, \theta)\) are polar coordinates on the poloidal cross-section originated either on the magnetic axis or on the geometric centre (cf. Figure 6.1 of Ref. [1]). Since elongation and triangularity (see Eq. (12) below) play an important role on confinement, this coordinate system was generalised for describing configurations with D-shaped magnetic surfaces, e.g. ([3, 4, 5]).

Aim of the present study is to introduce an alternative coordinate system for D-shaped magnetic surfaces containing a parameter, \( b \), which determines in a simple manner the triangularity. Also, alternative to the conventional way of constructing order by order in \( \epsilon \) the equilibrium, the proposed coordinate system permits this construction by keeping any desirable order in \( \epsilon \). As applications the system is employed to construct ITER-like and JET-like equilibria with flow under a Solovév choice of the free functions involved in Eq. (4). The equilibrium characteristics are also examined, in particular the impact of the triangularity in terms of \( b \) on the safety factor, \( q \), and the toroidal beta, \( \beta_t \). Interestingly, it turns out that increase of the triangularity results in an increase of both quantities, \( q \) and \( \beta_t \), thus indicating an improvement of the confinement. This result was also supported by applying a condition for linear stability for the equilibria with parallel flow constructed.

In section 2 the GGSE equation is briefly reviewed and the proposed coordinate system is presented. Two classes of equilibria with ITER-like and JET-like shaping and safety factor are constructed in subsection 3.1 and the equilibrium characteristics are examined including the impact of triangularity on \( q \) and \( \beta_t \). Then in subsection 3.2 we apply the stability condition for the subclass of the steady states with flow parallel to the magnetic field. Section 4 summarises the Conclusions.
2. Theory/calculation

The study is based on the following GGSE [6, 7]

\[
(1 - M_P^2)\Delta^* \psi - \frac{1}{2} (M_P^2)' |\nabla \psi|^2 + \frac{1}{2} \left( \frac{X^2}{1 - M_P^2} \right)'
+ \mu_0 R^2 P_s' + \mu_0 R^4 \frac{\rho (\Phi')^2}{1 - M_P^2} = 0
\] (1)

Here, the poloidal magnetic flux function \( \psi(R, z) \) labels the magnetic surfaces, where \((R, \phi, z)\) are cylindrical coordinates with \(z\) corresponding to the axis of symmetry; \(M_p(\psi)\) is the Mach function of the poloidal fluid velocity with respect to the poloidal Alfvén velocity; \(X(\psi)\) relates to the toroidal magnetic field, \(B_\phi = I / R\), through \(J = X / (1 - M_P^2)\); \(\Phi(\psi)\) is the electrostatic potential; for vanishing flow the surface function \(P_s(\psi)\) coincides with the pressure; \(B\) is the magnetic field modulus which can be expressed in terms of surface functions and \(R\); \(\Delta^* = R^2 \nabla \cdot (\nabla / R^2)\); and the prime denotes derivatives with respect to \(\psi\). Because of incompressibility the density \(\rho(\psi)\) is also a surface quantity and the Bernoulli equation for the pressure decouples from (1):

\[
P = P_s(\psi) - \rho \left[ \frac{v^2}{2} - \frac{R^2 (\Phi')^2}{1 - M_P^2} \right]
\] (2)

where \(v\) is the velocity modulus. The quantities \(M_p(\psi), X(\psi), P_s(\psi), \rho(\psi)\) and \(\Phi(\psi)\) are free functions. Derivation of (1) and (2) is provided in [6, 7].

Eq. (1) can be simplified by the transformation

\[
u(\psi) = \int_0^\psi \left[ 1 - M_P^2(f) \right]^{1/2} df
\] (3)

under which (1) becomes

\[
\Delta^* \nu + \frac{1}{2} \frac{d}{du} \left( \frac{X^2}{1 - M_P^2} \right) + \mu_0 R^2 \frac{dP_s}{du}
+ \mu_0 R^4 \frac{d}{du} \left[ \rho \left( \frac{\Phi'}{\nu} \right)^2 \right] = 0
\] (4)

Note that no quadratic term as \(|\nabla u|^2\) appears any more in (4). It is noted that once a solution of (4) is obtained, the equilibrium can be completely constructed with calculations in the \(u\)-space

Subsequently we make the Solovév-like ansatz for the free functions

\[
\frac{1}{2} \left[ \frac{X^2}{1 - M_P^2} \right] := X_{00} + X_0 u
\]
\[ \mu_0 P_s(u) := P_{00} + P_0 u \]
\[ \mu_0 \left[ \frac{\partial (\Phi')^2}{1 - M_p^2} \right] := G_{00} + G_0 u \] (5)

under which Eq. (4) reduces to
\[ u_{RR} - (1/R)u_R + u_{zz} + X_0 + R^2 P_0 + \frac{1}{2} G_0 R^4 = 0 \] (6)
The values of the various constants appearing in Eq. (5) will be specified below. We also note that the solutions to be constructed in the next section hold for arbitrary poloidal Mach functions, \( M_p^2(u) \), and densities, \( \varrho \).

Now we introduce the new coordinate system that we propose in this paper expressed in terms of the \((r, \theta, \phi)\)-coordinates described in section 1 via
\[ R = R_0 + r \left[ 1 + b \cos \theta \right] \cos \theta \]
\[ z = k r \sin \theta, \quad (0 \leq b < b_{max}) \] (7)

Here, the parameters \( k \) and \( b \) determine the elongation and triangularity of the magnetic surfaces, with \( b_{max} \) the maximum value for which the \( u \)-contours remain convex. For \( b = 0 \) and \( k = 1 \) it reduces to the usual coordinate system describing magnetic surfaces of circular cross-section. The coordinate systems (7) is simpler than others describing D-shaped magnetic surfaces [3, 4, 5] because it does not involve highly non-linear composite trigonometric functions of the form \( \cos(\sin(f(\delta, \theta))) \), where the parameter \( \delta \) is connected with the triangularity. Also, the form of \( R(r, \theta) \) in (7) permits the construction of equilibria in a Fourier-decomposition-like way, that is by keeping in the \( e \) expansions terms \( e^n \) up to any desirable order \( n \). Thus the usual order by order in \( e \) construction can be avoided.

Furthermore, we normalize Eqs. (6)–(7), converting them to completely dimensionless form, by defining \( \rho := R/R_0, \ \zeta := z/R_0, \ (0 \leq w := (r/a) \leq 1) \). Also we normalize the flux function as \( \bar{u} := u/u_0, \ u_0^2 := \mu_0 P_0 R_0^4 \) and drop the bars for simplicity. Then we obtain
\[ u_{\rho\rho} - (1/\rho)u_\rho + u_{\zeta\zeta} + X_0 + \rho^2 P_0 + \frac{1}{2} G_0 \rho^4 = 0 \] (8)
\[ \rho = 1 + c \rho \left[ 1 + b \cos \theta \right] \cos \theta \]
\[ \zeta = k e w \sin \theta, \quad (0 \leq b < b_{max}) \] (9)

We substitute Eq. (9) into Eq. (8), following e.g. [8] and pursue a solution of the form
\[ u := u_1(w) + u_2(w) \cos \theta \] (10)
The details of a lengthy calculation are given in the Appendix entitled “Derivation of Eqs. (11)".
Figure 1. Equilibrium configurations with ITER-like characteristics (left) and JET-like characteristics (right) as described in the text for \( b = 0.375 \).

In this calculation we have kept up to second order terms in \( \epsilon \) and \( \beta \). The result is the following couple of differential equations for the functions \( u_1 \) and \( u_2 \)

\[
\begin{align*}
    u_1''(w) &= \frac{1}{DD} \left[ -\frac{1}{2}(2b^2 + k^2 + 1)F + bG \right] \\
    u_2''(w) &= \frac{1}{DD} \left[ 2bF - \frac{1}{2}(2b^2 + k^2 + 1)G \right]
\end{align*}
\]

\( (11) \)

where \( DD := \frac{1}{4}[2b^2 + k^2 + 1]^2 - 2b^2 \). The form of the functions \( F, G \) is given in Eqs. (S3), (S4) and (S5) of the Appendix.

3. Results and discussion

3.1. Equilibrium

We have numerically integrated Eqs. (11) using the fourth order Runge–Kutta method with adaptive step size. The values of the constants in Eq. (5) were chosen to be \( X_0 = 20.1 \), \( X_{00} = 255 \), \( P_0 = -80.32 \), \( P_{00} = 2 \), \( G_0 = -20 \), \( G_{00} = 12 \). We obtained the ITER-like equilibrium of Figure 1-left and the JET-like equilibrium of Figure 1-right. For the ITER-like configuration we used the parametric values [9] \( R_0 = 0.2m \) for the major radius and \( a = 2.1m \) for the minor radius of the torus, so that the inverse aspect ratio is \( \epsilon := a/R_0 \approx 0.3443 \). The respective values for the JET-like configuration are [10] \( R_0 = 2.96m \), \( a = 2.1m \) and \( \epsilon = 0.32 \). The connection of the parameter \( b \) with the triangularity is also shown in Figure 2. The left equilibrium with circular magnetic-surface cross-sections therein corresponds to \( b = 0 \), while the right one to the maximum value, \( b_{\text{max}} \), for which the magnetic surfaces remain convex.

We want to stress that, for a quite vast range of values for the above parameters, one obtains similar equilibria to those of Figure 1, and the numerical integration is stable. The initial conditions for the integration of Eqs. (11) were chosen to be \( u_1(0) = 0.0 \), \( u_1'(0) = -0.5 \), \( u_2(0) = 0.0 \), \( u_2'(0) = -0.5 \). The functions \( u_1, u_2 \) and \( u \) of Eq. (10) for
Figure 2. Equilibrium configurations corresponding to that of Figure 1-left showing that the parameter $b$ determines the triangularity. The left equilibrium corresponds to the value of $b = 0$ while the right one to the value $b = 0.575$.

Figure 3. The flux functions $u_1$, $u_2$ and $u$ of Eq. (10) for the ITER-like equilibrium of Figure 1-left.

the ITER-like equilibrium of Figure 1-left are shown in Figure 3. For this equilibrium the flux on the magnetic axis, located at $(R_a, z_a) = (6.4, 0)m$, is $u_a = 0.0$, while on the boundary, shown in blue, is $u_b = 0.6$. Also we found that $R_{max} = 7.6m$, $R_{min} = 5.1m$, $R_u = 5.9m$, $z_u = 2.25m$. Thus for the actual elongation and triangularity of this equilibrium we have

\[ \bar{k} := \frac{2z_u}{R_{max} - R_{min}} = 1.8 \]

\[ \bar{\delta} := \frac{R_u - R_a}{R_u - R_{min}} = 0.3846 \] (12)
Figure 4. The toroidal magnetic field, pressure and toroidal current density for the ITER-like equilibrium of Figure 1-left.

For the same values of $b = 0.375$, $u_b = 0.6$ and $u_a = 0$ the respective parametric values of the JET-like configuration of Figure 1-right are $R_{\text{max}} = 3.96m$, $R_{\text{min}} = 2m$, $R_u = 2.85m$, $z_u = 1.1m$, $R_a = 3.1$, $z_a = 0$, $\bar{k} = 1.7$ and $\bar{\delta} = 0.33$.

To calculate the various flow-dependent equilibrium quantities we now make a peaked-on-axis choice for the Mach function, of Eq. (3), and the matter density as

$$M_p^2 := M_0^2 \left( 1 - \frac{u}{u_b} \right)^{2m}$$

$$\rho := \rho_0 \left( 1 - \frac{u}{u_b} \right)$$

where $M_0^2$ is the maximum value on the magnetic axis and $m$ is a profile-shaping parameter. We have used throughout the value $M_0^2 = 0.01$ and $m = 1$ in Eq. (13). Also $\rho_0 = 1$ (in units of $4 \times 10^{-7} Kg/m^3$). The toroidal magnetic field, the pressure and the toroidal current density of the ITR-like equilibrium of Figure 1-left are shown in Figure 4. The values involved are quite close to typical ITER-like values.

The toroidal beta is defined by

$$\beta_t := \frac{2\mu_0 \langle P \rangle}{B_{0\text{vac}}^2}$$

(14)
The toroidal beta, as given by Eqs. (14)–(15), for the equilibria of Figure 1 as a function of the parameter $b$. The left and right curves are associated with the ITER-like and JET-like configurations, respectively. The red circles correspond to the numerically determined values, while the blue curves to a quadratic fitting.

Figure 6. The safety factor, $q(u)$, for the equilibria of Figure 1, as a function of the parameter $b$. The left and right curves are associated with the ITER-like and JET-like configurations, respectively.

where $B_{0\text{vac}}$ is the vacuum magnetic field at the magnetic axis. We have numerically calculated $\beta_t$ for various values of the triangularity parameter $b$. The results are presented in Figure 5-left for the ITER-like equilibrium and in Figure 5-right for the JET-like equilibrium of Figure 1. The numerical values are shown with red circles versus a quadratic fitting, in blue. The respective vacuum magnetic field employed is $B_{0\text{vac}} \approx 7T$. For the ITER-like equilibrium $\beta_t$ was found to be given by

$$\beta_t \approx 0.059b^2 + 0.075b + 0.0239$$

A similar quadratic fitting is possible for the JET-like equilibrium. Finally, the safety factor, $q(u)$, for the equilibria of Figure 1, is given in Figure 6 for various values of the parameter $b$. The function $q(u)$ monotonically increasing from the magnetic axis to the plasma boundary is consistent with the peaked on axis current density of Figure 4 associated with usual confinement modes in tokamaks. In particular the minimum values of $q_{\text{min}} > 2$ on axis and $q_{\text{max}} \approx 3$ on the boundary are consistent with the respective values of $q$ provided in Refs. [9] and [10]
It turns out that both \( \beta_i \) and \( q \) increase noticeably as \( b \) takes larger value, as can be seen in Figures 8 and 9, thus indicating a favourable effect of triangularity on confinement.

### 3.2. Stability consideration

We now consider the important issue of stability for the equilibria constructed with respect to small linear MHD perturbations by applying a sufficient condition \cite{ref}. For this reason we computed again all the relevant quantities for the case where \( G_0 = G_{00} = 0.0 \) in Eq. (5) and \( \phi = \phi_0 = 1 \) for which the sufficient condition applies. This condition states that a general steady state of a plasma of constant density and incompressible flow parallel to \( \mathbf{B} \) is linearly stable to small three-dimensional perturbations if the flow is sub-Alfvenic \( (M_p^2 < 1) \) and \( A \geq 0 \), where \( A \) is given below. In fact, if the density is uniform at equilibrium, it remains so at the perturbed state because of incompressibility. Also, note that here the flows are inherently sub-Alfvenic because of the transformation (3). In the \( u \)-space for axisymmetric equilibria, \( A \) assumes the form

\[
A = A_1 + A_2 + A_3 + A_4
\]

\[
A_1 = -\left(\mu_0 j \times \nabla u\right)^2
\]

\[
A_2 = (\mu_0 j \times \nabla u) \cdot (\nabla u \cdot \nabla) B
\]

\[
A_3 = -\frac{1}{2} \frac{d(M_p^2)}{du} (1 - M_p^2)^{-1} |\nabla u|^2 \nabla u \cdot \nabla B^2/2
\]

\[
A_4 = -\frac{1}{2} \frac{d(M_p^2)}{du} (1 - M_p^2)^{-3/2} |\nabla u|^4 g
\]

\[
g := (1 - M_p^2)^{-1/2} \left( \frac{d(M_p^2)}{du} - \frac{d(M_p^2)}{du} \cdot \frac{B^2}{2} \right)
\]

The quantity \( A_1 \) being always negative consists a destabilizing contribution potentially related to current driven modes. The other terms can be either stabilizing or destabilizing. Specifically, the term \( A_2 \) relates to the current density and the variation of the magnetic field perpendicular to the magnetic surfaces. The term \( A_3 \) involves the shear and magnitude of the flow in conjunction with the variation of the magnitude of the magnetic field perpendicular to the magnetic surfaces. \( A_4 \) is mostly a flow term depending on the magnitude and the shear of the flow.

We calculated the quantity \( A \), of Eq. (16), for several equilibria within a broad range of the free parameters. It turns out that as the triangularity parameter \( b \) gets larger the region of the poloidal cross-section on which the condition \( A \geq 0 \) is satisfied enhances. For the equilibrium of Figure 1-left (in the case of \( G_0 = G_{00} = 0.0 \)) this result is shown in Figure 7. In addition, in Figure 8 we have plotted the minimum of the function \( A, A_{\text{min}} \), as a function of \( b \), which remains nearly unaffected of \( b \).
Figure 7. The stability function of Eq. (16), for the equilibrium of Figure 1-left for two different values of the triangularity-parameter $b$. The left corresponds to $b = 0.35$ while the right to $b = 0.55$. The red regions correspond to the satisfaction of the criterion, namely $A \geq 0$. It is evident that the increased triangularity causes the satisfaction of the criterion in a slightly vaster equilibrium region.

Figure 8. The minimum value of the stability function of Eq. (16), as a function of the $b$–parameter, for the equilibrium of Figure 1-left.

Figure 9. The average value of the stability function of Eq. (16), as a function of the $b$–parameter, for the equilibrium of Figure 1-left.

Furthermore, we calculated the average of the function $A$, $A_{av}$, defined as $A_{av} := \int A dV / \int dV$ where the integral is taken inside the equilibrium region bounded by the blue boundary curve. As it is shown in Figure 9 the effect of the $b$-parameter is to increase this average. In addition, we found by increasing the parameter $M_0$ that the parallel flow has a weaker additional stabilizing effect. This is shown in Figure 10.
Finally, it is noted that the white coloured regions of Figure 7 where $A > 0$ does not imply instability because the condition is sufficient.

4. Conclusions

We have considered a GGSE with flow of arbitrary direction [Eq. (6)] by introducing a coordinate system prescribing D-shaped magnetic surfaces. This system contains a parameter $b$ which determines the triangularity of the magnetic surfaces and is appropriate to construct equilibria on the basis of inverse aspect ratio ($e$) expansions by retaining terms $e^n$ of any desirable order $n$. Specifically, the problem is reduced to a set of two ODEs which are solved numerically to construct two classes of equilibria the one with ITER-like and the other with JET-like characteristics. In particular our equilibria resemble those of equilibria of these tokamaks in terms of aspect ratio, triangularity, elongation and the safety factor. Also it is found that as $b$ increases both the safety factor and the toroidal beta take larger values. In particular, the beta-parameter is related through fitting to the toroidal beta of the equilibria by a quadratic relation [of Eq. (15)]. Also, application of a sufficient condition for linear stability implies that the triangularity plays a stabilizing role, in that the higher $b$ is the larger the part of the plasma volume at which that condition is satisfied. This effect is slightly strengthened by the parallel flow.

It is interesting to generalize the coordinate system proposed here by introducing a radially modulated triangularity ($b = b(r)$), which could be exploited in order that this modulation further optimize confinement. Also, alternative choices of the free functions can be made in order to construct more realistic D-shaped equilibria, e.g. equilibria with peaked toroidal current density profiles vanishing on the boundary or with hollow current density profiles associated with the advanced confinement modes in tokamaks.
Declarations

Author contribution statement

Apostolos Kuiroukidis, George Throumoulopoulos: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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Competing interest statement

The authors declare no conflict of interest.

Additional information

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References

[1] J.P. Freidberg, Ideal Magnetohydrodynamics, Plenum Press, 1987, p. 108.
[2] L.S. Soloiov, The theory of hydromagnetic stability of toroidal plasma configurations, Sov. Phys. JETP 26 (1968) 400.
[3] C. Mercier, N. Luc, Report No. EUR-5127e 140 Commission of the European Communities, Brussels, 1974.
[4] R.L. Miller, M.S. Chu, J.M. Greene, Y.R. Lin-Liu, R.E. Waltz, Noncircular, finite aspect ratio, local equilibrium model, Phys. Plasmas 5 (1998) 973.
[5] T.G. Collarta, W.M. Stacey, Improved analytical flux surface representation and calculation models for poloidal asymmetries, Phys. Plasmas 23 (2016) 052505.
[6] H. Tasso, G.N. Throumoulopoulos, Axisymmetric ideal magnetohydrodynamic equilibria with incompressible flows, Phys. Plasmas 5 (1998) 2378.
[7] Ch. Simintzis, G.N. Throumoulopoulos, G. Pantis, H. Tasso, Analytic magnetohydrodynamic equilibria of a magnetically confined plasma with sheared flows, Phys. Plasmas 8 (2001) 2641.

[8] R.B. White, The Theory of Toroidally Confined Plasmas, Imperial College Press, London, 1989.

[9] ITER Physics Expert Groups on Confinement and Transport and Confinement Modelling and Database, Nucl. Fusion 39 (1999) 2175.

[10] P.H. Rebut, R.J. Bickerton, B.E. Keen, The Joint European Torus: installation, first results and prospects, Nucl. Fusion 25 (1985) 1011.

[11] G.N. Throumoulopoulos, H. Tasso, A sufficient condition for the linear stability of magnetohydrodynamic equilibria with field aligned incompressible flows, Phys. Plasmas 14 (2007) 122104.