Mean Square Exponential Stability Analysis of Stochastic Cellular Neural Networks

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Abstract. The stability of stochastic delayed cellular neural networks (SDCNN) is investigated in this paper. The interconnections and delays appeared in such networks are time-variable. A set of novel sufficient conditions on mean square exponential stability is given. A numerical example is also given to illustrate the effectiveness of our results.

1. Introduction

Since the seminal work for Cellular Neural Networks in [1,2], the past nearly two decades have witnessed the successful applications of Cellular Neural Networks in many areas such as combinatorial optimization, signal processing and pattern recognition. Up to now, a great deal of results have been reported in the literature, see e.g. [3] and references therein, where most models discussed are deterministic. These models do not take into account the inherent randomness that is associated with signal transmission. Just as pointed out by Haykin [4], in real nervous systems and in the implementation of artificial neural networks, noise is unavoidable and should be taken into consideration in modeling. Therefore, it is of significant importance to consider stochastic effects to the stability of neural networks. On the other hand, although various stability of neural networks has been extensively investigated by many authors in the past two decades, the problem of stochastic effects on the stability has not studied until very recent years (see [5-10]). In the mentioned papers, the interconnections and delays considered are constants. It is well known that the interconnections and delays in artificial neural networks are usually time-varying, and it is more important to investigate the

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dynamic behavior of neural networks with time-varying interconnections and delays, and this fact motivates our work.

2. Preliminaries
In this paper, we consider the following stochastic delayed cellular neural network (SDCNN) model.

$$dx(t)=\left[-C(t)x(t)+A(t)F(x(t)+B(t)F(x(t-\tau(t))))\right]dt + \sigma(t,x(t),x(t-\tau(t)))d\omega(t)$$  \(1\)

where, \(x(t)\) is the state vector, \(x(t)=(x_1(t),...,x_n(t))\), \(C(t)=\text{diag}(c_1(t),...,c_n(t)),\ A(t)=(a_{ij}(t))\), \(B(t)=(b_{ij}(t))\), \(F(x(t-\tau(t)))=(f_{ij}(x_1(t-\tau_j(t))))\) is the activation function, \(\sigma(t,x(t),x(t-\tau(t)))=(\sigma_{ij}(t,x_1(t),x_1(t-\tau_j(t))))\) is the diffusion coefficient matrix, \(\omega(t)=(\omega_1(t),...,\omega_n(t))\) is an \(n\)-dimensional Brownian motion defined on a complete probability space \((\Omega, F, P)\) with a natural filtration \(\{F_t\}_{t\geq 0}\) (i.e. \(F_t=\sigma\{\omega(s): 0 \leq s \leq t\}\)).

The initial conditions for system (1) are \(x(t)=\varphi(t), \ -\tau \leq t \leq 0, \ \varphi \in L^2_{F_0}([-\tau,0],\mathbb{R}^n)\), here \(L^2_{F_0}([-\tau,0],\mathbb{R}^n)\) is regarded as a \(\mathbb{R}^n\)-valued stochastic process \(\varphi(t), \ -\tau \leq t \leq 0\), moreover, \(\varphi(t)\) is \(F_0\) measurable, \(\int_{-\tau}^0 E|\varphi(t)|^2dt<\infty\). Throughout this paper, we always assume that \(F(x(t))\) and \(\sigma(t,x(t),x(t-\tau(t)))\) satisfy the local Lipschitz condition and the linear growth condition. This implies that (1) has a unique global solution on \(t \geq 0\) for the initial conditions [11]. As usual, we will also assume that \(F(0)=0, \sigma(t,0,0)=0\) for all \(t\) in this paper. So system (1) admits a zero solution or trivial solution \(x(t,0)=0\). We denote \(\|x\|_i=\sum_{i=1}^nx_i^2\) and \(\|x\|_{\infty}=\sum_{i=1}^n \sup_{-\tau \leq s \leq 0}|x_i(t+s)|^2\).

We assume the following conditions are satisfied:

\(H_1\) There exist positive constants \(p_{ij}, i,j=1,...,n,\) such that
\[|f_{ij}(u)-f_{ij}(v)| \leq p_{ij}|u-v|, \ \forall \ u, v \in \mathbb{R}. \quad (2)\]

\(H_2\) There are nonnegative constants \(v_i, u_i\) such that
\[\text{trace}[\sigma^T(t;x,y)\sigma(t;x,y)] \leq \sum_{i=1}^n (v_i x_i^2 + u_i y_i^2), \ \forall \ (t;x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n. \quad (3)\]

\textbf{Definition 2.1} [11] The trivial solution of (1) is said to be exponentially stable in mean square if for all \(x_0 \in \mathbb{R}^n\), there is a pair of positive constants \(\lambda\) and \(C\) such that
\[E\|x(t;t_0,x_0)\|^2 < C\|x_0\|^2 e^{-\lambda(t-t_0)}, \ t \geq t_0. \quad (4)\]

3. Main results
We will apply the Dini derivative of the expectation of Lyapunov function to deal with the mean square exponential stability for system (1). If \(V(t,x)\) is a function defined on \([-\tau, \infty) \times \mathbb{R}^n\), which is continuous twice differentiable in \(x\) and once differentiable in \(t\), as usual, we denote an operator \(LV\) associated with (1) as
\[LV=V_t+V_x[\partial(C(t)x(t)+A(t)F(x(t)+B(t)F(x(t-\tau(t)))))+\frac{1}{2}\text{trace}[\sigma^T V_x \sigma]], \quad (5)\]
where \(V_t=\frac{\partial V(t,x)}{\partial t}, V_x=(\frac{\partial V(t,x)}{\partial x_1},...,\frac{\partial V(t,x)}{\partial x_n}), V_{xx}=(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j})_{n \times n}\).

The main ideas of the Lemma 3.1 due to Luo [12], we list it as follows without proof.
Lemma 3.1 For two positive-valued functions \( a(t), b(t) \) defined on \([t_0, \infty)\), if there exist constant numbers \( a_0, 0 \leq \mu < 1 \) such that \( 0 < a(t) \leq a(t) \leq \mu a(t) \) hold for all \( t \geq t_0 \), and

\[
D^+ y(t) \leq -a(t)y(t)+b(t)y(t), \quad \text{for } t \geq t_0,
\]

then \( y(t) \leq \|y(t_0)\|e^{-\int_{t_0}^{t} (a(t)) \ d t} \), for \( t \geq t_0 \), where \( \lambda^* = \inf_{t \geq t_0} \{ \lambda(t): \lambda(t) = a(t) - b(t)e^{\lambda(t) \tau} \} > 0 \) and

\[
D^+ y(t) = \limsup_{\gamma \to 0} \frac{(y(t+\gamma) - y(t))}{\gamma}.
\]

Theorem 3.2 Under the assumptions \((H_1)-(H_2)\), if there are two constants \( 0 < N_2, 0 \leq \mu < 1 \), and a positive diagonal matrix \( M = \text{diag}(m_1, \ldots, m_n) \) such that

\[
0 < N_2 \leq N_2(t) \leq \mu N_2(t), \quad \text{holds for } t \geq t_0,
\]

where

\[
N_2(t) = \max_{1 \leq i \leq n} \frac{u_i}{m_i} m_i \max_{1 \leq i \leq n} \{ a_i \} + \sum_{j=1}^{n} \frac{m_{ij}}{m_i} \max_{1 \leq i \leq n} \{ b_{ij} \} p_j^2,
\]

then the trivial solution of system (1) is exponentially stable in mean square.

Proof. Consider the following Lyapunov function

\[
V(t, x) = x^T(t)Mx(t).
\]

From Itô formula, then the operator \( LV \) associated with (1) has the form as follows

\[
LV(t,x) = 2x^T(t)M \{-C(t)x(t)+A(t)F(x(t)+B(t)F(x(t-\tau(t)))\} + \text{trace}[\sigma^TM\sigma]
\]

\[
= -2 \sum_{i=1}^{n} m_i x_i(t)c_i(t)x_i(t) + 2 \sum_{i=1}^{n} m_i x_i(t) \sum_{j=1}^{n} a_{ij}(t)f_{ij}(x_j(t))
\]

\[
+ 2 \sum_{i=1}^{n} m_i x_i(t) \sum_{j=1}^{n} b_{ij}(t)f_{ij}(x_j(t-\tau_{ij}(t))) + \text{trace}[\sigma^T M \sigma]
\]

\[
\leq -2 \sum_{i=1}^{n} m_i c_i(t)x_i^2(t) + 2 \sum_{i=1}^{n} m_i |x_i(t)| \sum_{j=1}^{n} |a_{ij}(t)| p_{ij} x_j(t)
\]

\[
+ 2 \sum_{i=1}^{n} m_i |x_i(t)| \sum_{j=1}^{n} |b_{ij}(t)| p_{ij} x_j(t-\tau_{ij}(t))) + m \sum_{i=1}^{n} [v_i x_i^2(t) + u_i x_i^2(t-\tau_{ij}(t))]
\]

\[
\leq -2 \sum_{i=1}^{n} m_i c_i(t)x_i^2(t) + \sum_{i=1}^{n} m_i \sum_{j=1}^{n} |a_{ij}(t)| x_i^2(t) + \sum_{j=1}^{n} |b_{ij}(t)| x_j^2(t-\tau_{ij}(t))]
\]

\[
= -2 \sum_{i=1}^{n} m_i \{2c_i(t) - \sum_{j=1}^{n} a_{ij}(t) \} x_i^2(t) - \frac{1}{m_i} \sum_{j=1}^{n} |a_{ij}(t)| p_{ij} x_j(t)\sum_{j=1}^{n} |b_{ij}(t)| x_j^2(t)
\]

\[
+ \sum_{i=1}^{n} m_i \sum_{j=1}^{n} |b_{ij}(t)| x_j^2(t-\tau_{ij}(t)) + m \sum_{i=1}^{n} u_i x_i^2(t-\tau_{ij}(t))
\]

where \( m = \max_{1 \leq i \leq n} \{ m_i \} \). For \( \delta \geq 0 \), using the result of the \( p_{44} \) in [13], we can get

\[
E\left[\int_{t_0}^{t+\delta} V(s, x(s)) \sigma(s, x(s), x(s-\tau(s)))d\omega(s)\right] = 0.
\]

Denote

\[
y(t) = E[V(t,x)] = E[V(t,x(t_0,x_0))]
\]

and apply the inequality (9), the preceding result leads directly to
Hence, from Lemma 3.1, we have
\[
D^t y(t) \leq -N_1(t)y(t) + N_2(t)\|y_i\|, \tag{10}
\]
That is to say
\[
E\|x(t;\tau_0, x_0)\| \leq C\|x_0\|e^{-\lambda(t)\|t-t_0\|}, \tag{12}
\]
where \( C = \frac{\max \{m_i\}}{\min \{m_i\}} \) and \( \lambda^* = \inf_{t \geq t_0} \{\lambda(t)\} > 0 \), \( \lambda(t) \) is the unique positive function solution of the following equation
\[
\lambda(t) = N_1(t) - N_2(t)e^{\lambda(t)\tau}. \tag{13}
\]
Therefore, the trivial solution of system (1) is exponentially stable in mean square.

**Corollary 3.3** Under the assumptions (H1)\(-(H2)\), if there are two constants \(0 < N_2, \ 0 \leq \mu < 1\), and a positive diagonal matrix \(M = \text{diag}(m_1, ..., m_n)\) such that
\[
0 < N_2 \leq N_2(t) \leq \mu N_1(t), \text{ holds for } t \geq t_0, \tag{14}
\]
then the trivial solution of system (1) is exponentially stable in mean square.

**Proof** Let \(M\) as the identity matrix, from the Theorem 3.2, it is obvious that Corollary 3.3 is true.

**Remark 3.4** To the best of our knowledge, few authors have considered the exponential stability for Stochastic Neural Networks with time-varying delays. We can find the recent papers [5-10] in this direction. However, it is assumed in [5-10] that all delays are constants, the connections in [5-8, 10] are constants, and the activation functions appear in [6, 8] are bounded, obviously, these requirements are relaxed in our results.

### 4. An Illustrative Example
Consider the following stochastic cellular neural networks:
\[
dx(t) = \left[\begin{array}{cc}
4 + \frac{4}{15}t & 0 \\
0 & 1 + \frac{4}{15}t
\end{array}\right] x(t) + \frac{1}{2}t + \frac{1}{2} \left[\begin{array}{c}
tanh(x_1(t)) \\
\tanh(x_2(t))
\end{array}\right] dt + \sigma(t, x(t), x(t-\tau(t)))d\omega(t), \ t > 0, \tag{15}
\]
where \(\tau(t)\) is any bounded positive function and \(\sigma : R_+ \times R^2 \times R^2 \rightarrow R^2 \times R\) satisfies
\[
\text{trace}[\sigma^T(t, x, y)\sigma(t, x, y)] \leq 0.3x_1^2 + 0.4x_2^2 + 0.6y_1^2 + 0.7y_2^2. \tag{16}
\]
By simple computation, we can easily get that \(N_1(t) = 4 + \frac{4}{15}t, \ N_2(t) = \frac{21}{30} + t > \frac{21}{30} > 0\). Let \(\mu = \frac{15}{16}\), we have \(0 < \frac{21}{30} < N_2(t) < \mu N_1(t)\). Thus, it follows Corollary 3.3 that the trivial solution of system (15) is exponentially stable in mean square.

**Remark 4.1** It is obvious that the results in [5-10] and the references therein cannot be applicable to system (15) as the fact that the connections and delays considered in this model are time-variable. This implies that the results of this paper are essentially new.

### 5. Conclusions
In this paper, we have investigated the exponential stability for SDCNN with time-varying interconnections and delays. Under the help of the Dini derivative of the expectation of $V(t, x(t))$ “along” $x(t)$, several sufficient conditions have been derived to ensure the mean square exponential stability of SDCNN. Compared with the recent papers, this method does not resort to the delays and the interconnections are constants, which improve and extend some earlier publications. One illustrative example is given to demonstrate the effectiveness of the proposed results. Furthermore, if we remove noise from the system, the derived conditions for exponential stability of general Cellular Neural Networks can be viewed as the byproducts of our results, and the method of this paper may be extended to some deterministic neural networks.

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