Some remarks on formality in families

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To my first math teacher Victor Ginzburg, on the occasion of his 50-th birthday

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Introduction

The notion of a formal differential graded algebra – that is, a DG algebra $A^*$ quasiisomorphic to its cohomology algebra $H^*(A^*)$ – is by now a familiar sight in many areas of mathematics; we can quote, for instance, the classic paper [DGMS], where formality was established for the de Rham cohomology algebra of a compact Kähler manifold $X$, which had numerous applications to the topology of compact Kähler manifolds. A well-known series of obstructions to formality is given by the so-called Massey products. It would be very convenient to know that these give the only obstruction – a DG algebra with Massey products is formal. Unfortunately, this is not true (for a counterexample, see e.g. [HS]). Therefore in works such as [DGMS] Massey products play only a marginal role, and the main technical tool is the notion of a minimal model introduced by D. Sullivan. However, this brings about some problems, of which the most obvious one is that minimal

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models usually do not exist for families of DG algebras over a sufficiently non-trivial base.

In this paper, we construct a certain refinement of the Massey products which does characterise formality uniquely, and moreover, behaves well for families of DG algebras. As an application, we prove two results on formality in families.

Of course, at least morally, and at least in some cases, both results are not new. However, it seems that accurate and complete proofs are not available in the existing literature, which precludes applications in non-standard setting. The goal of this paper is to provide such a proof. To save space, we only sketch those proofs that deal with DG algebras over a field, – this material is quite standard, – and conversely, we try to be really precise when it comes to families of DG algebras over a base. Our approach to formality is motivated by and partially follows the paper [H]: we treat formality of a DG algebra \( A^* \) as triviality of the normal cone deformation associated to the canonical filtration on \( A^* \), and we use deformation theory methods to find criteria for this triviality.

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1 Kodaira-Spencer classes.

Let \( A \) be an associative, not necessarily commutative algebra. The embedding of the diagonal defines a canonical map \( A \otimes A \to A \) of \( A \)-bimodules. Denote its kernel by \( \mathcal{I}_A \subset A \otimes A \). For any \( A \)-bimodule \( M \), the groups

\[
HH^{*+1}(A, M) = \text{Ext}_{A-bimod}^{*}(\mathcal{I}_A, M)
\]

are called the (reduced) Hochschild cohomology groups of the algebra \( A \) with coefficients in \( M \). (One can show that alternatively, \( HH^*(A, M) = \text{Ext}_{A-bimod}^*(A, M) \), which explains the shift in the index; we will not need this.) If \( M = A \) is \( A \) itself considered as an \( A \)-bimodule, than the groups \( HH^*(A, A) \) are denoted simply by \( HH^*(A) \).

A square-zero extension \( \tilde{A} \) of the algebra \( A \) by the bimodule \( M \) is by definition an associative algebra \( \tilde{A} \) equipped with a two-sided ideal \( N \subset \tilde{A} \).
such that we are given an isomorphism $\tilde{A}/N \cong A$, the $\tilde{A}$-bimodule structure on $N$ factors through an $A$-bimodule structure, and we are given an $A$-bimodule isomorphism $N \cong M$. Every square-zero extension $\tilde{A}$ defines a Hochschild cohomology class $\theta_{\tilde{A}} \in HH^2(A, M)$ by means of the following procedure. Denote by $I_M \subset M \otimes A$ the kernel of the natural $A$-bimodule map $M \otimes A \to M$. Consider the $\tilde{A}$-$A$-bimodule $\tilde{A} \otimes A$ ($\tilde{A}$ acts by left multiplication, $A$ acts by right multiplication). We have a short exact sequence of $\tilde{A}$-$A$-bimodules

$$0 \longrightarrow M \otimes A \longrightarrow \tilde{A} \otimes A \longrightarrow A \otimes A \longrightarrow 0.$$  

In particular, we have an embedding $I_M \to \tilde{A} \otimes A$ and a surjection $\tilde{A} \otimes A \to A$. The middle cohomology $\tilde{I}_{A,M}$ of the complex $I_M \to \tilde{A} \otimes A \to A$ appears as the middle term of a short exact sequence

$$(1.2) \quad 0 \longrightarrow M \longrightarrow \tilde{I}_{A,M} \longrightarrow I_A \longrightarrow 0$$

of $\tilde{A}$-$A$-bimodules. One checks easily that this sequence is in fact a sequence of $A$-bimodules. We take $\theta_{\tilde{A}} \in HH^2(A, M)$ to be the Yoneda class of the extension (1.2).

A first-order deformation $\tilde{A}$ of the algebra $A$ is by definition a square-zero extension of $A$ by $A$; equivalently, it is an associative algebra $\tilde{A}$ equipped with a $\tilde{A}$-bimodule map $\varepsilon : \tilde{A} \to \tilde{A}$ such that $\ker \varepsilon = \text{im} \varepsilon \subset \tilde{A}$, and an algebra isomorphism $\tilde{A}/\ker \varepsilon \cong A$. Any first-order deformation $\tilde{A}$ defines a class $\theta_{\tilde{A}} \in HH^2(A) = HH^2(A, A)$.

There are many ways to present this construction. The one we have chosen has the following advantage: it works without any changes for a flat algebra $A$ in an arbitrary abelian tensor category $\mathcal{C}$.

As a first application of this additional degree of freedom, we show that the same definition can be used to study higher-order deformations. Namely, by a formal deformation $\tilde{A}$ of an algebra $A$ in a symmetric tensor category $\mathcal{C}$ we will understand an associative algebra $\tilde{A}$ equipped with an injective algebra map $h : \tilde{A} \to \tilde{A}$ and an isomorphism $\tilde{A}/h(\tilde{A}) \cong A$. A formal deformation $\tilde{A}$ is an algebra in the tensor category $\mathcal{C}[h]$ of objects in $\mathcal{C}$ equipped with an endomorphism $h$; since $h : \tilde{A} \to \tilde{A}$ is injective, $\tilde{A}$ is flat in $\mathcal{C}[h]$ if $A$ is flat in $\mathcal{C}$. Given such a formal deformation $\tilde{A}$, we can consider a trivial first-order deformation $\tilde{\mathcal{A}} = \tilde{A}(\varepsilon) = \tilde{A} \oplus \tilde{A} \cdot \varepsilon$ of the algebra $\tilde{A}$ in $\mathcal{C}[h]$, and redefine the endomorphism $h : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ by setting $h_{\text{new}} = h_{\text{old}} + \varepsilon$. Since $\varepsilon^2 = 0$ and $h_{\text{old}}$ is injective, $h_{\text{new}}$ is also injective – $\ker h_{\text{new}} \cap \ker \varepsilon \subset \ker h_{\text{old}}$ must be trivial, by induction, $\ker h_{\text{new}} \cap \ker \varepsilon^l$ is then trivial for every $l \geq 1$. 


but already $\text{Ker } h_{\text{new}} \cap \text{Ker } \varepsilon^2 = \text{Ker } h_{\text{new}}$. Thus algebra $\widetilde{A}$ with the new endomorphism $h$ is still a first-order deformation of the algebra $\tilde{A}$ in $C[h]$. If $\tilde{A} \cong A[h]$ were a trivial formal deformation of $A$, then this first order deformation is trivial; in general, however, it might be non-trivial and defines a cohomology class

$$\Theta_{\tilde{A}} \in HH^2(\tilde{A})$$

called the Kodaira-Spencer class of the deformation $\tilde{A}$. This describes the non-triviality of the deformation $\tilde{A}$ (not completely – we do not claim that in the general case, $\Theta_{\tilde{A}} = 0$ implies $\tilde{A} \cong A[[h]]$).

We note that the groups $HH^*(\tilde{A})$ are equipped with a natural endomorphism $h$; we have a natural map $HH^2(\tilde{A})/h \rightarrow HH^2(A)$, and the image of the class $\Theta_{\tilde{A}}$ under this map is the cohomology class corresponding to the first-order deformation $\tilde{A}/h^2(\tilde{A})$ of the algebra $A$.

2 Explicit cocycles.

Let us compare the formalism of Section 1 with the more standard approaches to the Kodaira-Spencer class. Firstly, assume that the $A$-bimodule $M$ is injective as an object of the category $C$ (for example, this is always true if $C$ is the category of vector spaces over a field $k$, so that $A$ is a $k$-algebra in the usual sense). In this case, for any free $A$-bimodule $N = A \otimes V \otimes A$, $V \in C$, we have

$$\text{Ext}^l_{A-\text{bimod}}(N, M) = \text{Ext}^l_C(V, M) = 0, \quad l \geq 1,$$

and one can compute the Hochschild cohomology $HH^*(A, M)$ by using the bar-resolution of the $A$-bimodule $\mathcal{I}_A$. This results in the Hochschild cochain complex $C^* (A, M)$, where

$$C^l(A, M) = \text{Hom}_k(A^\otimes l, M)$$

for any integer $l \geq 1$. If $M = A$ and $C$ is the category of $k$-vector spaces, one can describe the differential $\delta$ in this complex as follows: interpret $C^*(A)$ as the graded Lie algebra of coderivations of the free coalgebra $T^*(A) = A^\otimes [1]$ generated by the vector space $A$ placed in degree $-1$ (the bracket $[-, -]$ is given by the graded commutator of coderivations). Then the multiplication $m : A \otimes A \rightarrow A$ gives an element $\delta \in C^2(A)$, and it is easy to check that $m$ is associative if and only if $\delta^2 = [\delta, \delta] = 0$. We assume that this is the case, and the Hochschild differential is then given by $a \mapsto [\delta, a]$ (for details, and for a description of the differential for a general $M$, we refer the reader for instance to [GK, Appendix]).
Lemma 2.1. Assume that an $A$-bimodule $M$ is injective as an object in $\mathcal{C}$, and that a square-zero extension $\tilde{A}$ of $A$ by $M$ is identified with $A \oplus M$ as an object in $\mathcal{C}$. Assume that under this identification, the multiplication in $\tilde{A}$ is expressed as

$$a \ast b = ab + \gamma(a, b)$$

for some $\gamma \in C^2(A, M)$. Then $\gamma$ is a Hochschild cocycle, and it represents the class $\theta_{\tilde{A}} \in HH^2(A, M)$.

Proof. The first claim is completely standard; we will prove that $\theta_{\tilde{A}}$ is represented by the cocycle $\gamma$. Fixing an identification $\tilde{A} \cong A \oplus M$ is equivalent to fixing a map $P_0 : A \to \tilde{A}$ in $\mathcal{C}$ which splits the projection $\tilde{A} \to A$. To compute $\theta_{\tilde{A}}$, consider the free $\tilde{A}$-$A$-bimodule $\tilde{A} \otimes A \otimes A$ which maps surjectively onto $I_A \subset A \otimes A$ by $a \otimes b \otimes c \mapsto ab \otimes c - a \otimes bc$ (this is the first term of the bar-resolution of the bimodule $I_A$). The map $\tilde{A} \otimes A \otimes A \to I_A$ obviously lifts to a map $\tilde{P} : \tilde{A} \otimes A \otimes A \to \ker \tilde{T} \subset \tilde{A} \otimes A$, where $\tilde{T} : \tilde{A} \otimes A \to I_A$ is the natural projection: we take $\tilde{P}(a \otimes b \otimes c) = (a \ast P_0(b)) \otimes c - a \otimes bc$.

To represent $\theta_{\tilde{A}}$ by a cocycle, one has to compose $\tilde{P}$ with the projection $\ker \tilde{T} \to \tilde{I}_A$, notice that it factors through a map $P' : A^{\otimes 3} \to \tilde{I}_A$, then compose $P'$ with the bar-resolution differential $\delta : A^{\otimes 4} \to A^{\otimes 3}$, and notice that $P' \circ \delta : A^{\otimes 4} \to \tilde{I}_A$ factors through an $A$-bimodule map $P'' : A^{\otimes 4} \to A$. Explicitly, $\delta$ is given by $\delta(a \otimes b \otimes c \otimes d) = ab \otimes c \otimes d - a \otimes bc \otimes d + a \otimes b \otimes cd$; therefore, since $\gamma$ is a Hochschild cocycle, we have

$$P''(a \otimes b \otimes c \otimes d) = P'(ab \otimes c \otimes d - a \otimes bc \otimes d + a \otimes b \otimes cd)$$

$$= \gamma(ab, c)d - \gamma(a, bc)d + \gamma(a, b)cd$$

$$= a\gamma(b, c)d,$$

as required. □

If $M$ is not injective – for instance, if we want to study first-order deformations of an algebra $A$ which is not injective as an object in $\mathcal{C}$ – then Lemma 2.1 no longer applies. When $\mathcal{C}$ admits enough injectives, one can circumvent this problem by replacing $M$ with an injective resolution $\mathcal{I}^*$. Then as before, the Hochschild cohomology groups $HH^* (A, M)$ can be computed by the bar-resolution, and this resulting complex is the complex

$$C^* (A, \mathcal{I}^*) = \text{Hom}_\mathcal{C}(A^{\otimes *}, \mathcal{I}^*)$$
of Hochschild cochains with values in $I$. On the other hand, for any square-zero extension $\tilde{A}$ of $A$ by $M$, the algebra $\tilde{A}' = (\tilde{A} \oplus I^0)/M$ is a square-zero extension of $A$ by $I^0$; moreover, we have an exact sequence

\begin{equation}
0 \longrightarrow \tilde{A} \longrightarrow \tilde{A}' \longrightarrow \tau \longrightarrow I^0/M \subset I^1.
\end{equation}

Since $I^0$ is injective, we in fact can fix an isomorphism $\tilde{A}' \cong A \oplus I^0$ as objects in $C$, so that the multiplication in $\tilde{A}'$ is given by (2.1) for some $\gamma \in C^2(A, I^0)$. Moreover, composing the splitting map $P_0 : A \to \tilde{A}'$ with the map $\tau : \tilde{A}' \to I^0/M \subset I^1$, we obtain a map $A \to I^1$, which we can treat as an element $\gamma' \in C^1(A, I^1) = \text{Hom}_C(A, I^1)$. Then Lemma 2.1 can be easily generalized to show that under these identifications, the Hochschild cohomology class $\theta_{\tilde{A}} \in HH^2(A, M)$ is represented by the cocycle

\begin{equation}
\gamma + \gamma' \in C^2(A, I^0) \oplus C^1(A, I^1)
\end{equation}

in the double complex $C^\bullet(A, I^\bullet)$. We leave the proof to the reader.

3 DG algebras.

The particular situation where we will use (2.3) is when $C$ is the tensor category of complexes of vector spaces over the field $k$, so that algebras in $C$ are DG algebras over $k$. We will only need complexes $K^\bullet$ which are bounded below ($K^p = 0$ for $p \ll 0$). A complex is injective if and only if it is acyclic. Every complex $K^\bullet$ can be canonically embedded into the acyclic complex $C(K)^\bullet$, the cone of the identity map $K^\bullet \to K^\bullet$. The quotient $C(K)^\bullet/K^\bullet$ is by definition identified with the shifted complex $K^\bullet[1]$. The construction can be iterated, so that every complex $K^\bullet$ admits a functorial injective resolution $I^\bullet$ with $I^p = C(K)^\bullet[p]$. Then for any DG algebra $A^\bullet$ and DG-bimodule $M^\bullet$ over $A^\bullet$, the complex of Hochschild cochains is given by

\begin{equation}
C^p(A^\bullet, M^\bullet) = \bigoplus_{0 \leq l \leq p-1} C^{p-l}(A^\bullet, M^\bullet) = \bigoplus_{0 \leq l \leq p-1} \text{Hom}^l((A^\bullet)^{\otimes p-l}, M^\bullet),
\end{equation}

where $\text{Hom}^l$ is the space of vector space maps of degree $l$. The differential in this complex is the sum of the usual Hochschild differential which comes from the bar construction, and the differential which comes from the differentials in the complexes $A^\bullet, M^\bullet$. In the case $M^\bullet = A^\bullet$, one can again interpret $C^\bullet(A^\bullet)$ as the space of positive-degree coderivations of the free coassociative
coalgebra $T^*(A^*)$ generated by the graded vector space $A^*[1]$; the sum of the differential $d \in C^1(A^*)$ and the multiplication $m \in C^2(A^*)$ extends to a coderivation $\delta : T^*(A^*) \to T^*(A^*)$ of degree 1 satisfying $\delta^2 = 0$. Then the differential in $C^*(A^*)$ is given by $a \mapsto [\delta, a]$. If we are given a first-order deformation $\tilde{A}^*$ of a DG algebra $A^*$ over $k$, then splitting the corresponding square-zero extension $\tilde{A}'$ in (2.2) is equivalent to fixing an isomorphism $\tilde{A}' \cong A^*[\varepsilon]$ of graded vector spaces. Then the multiplication and the differential $\tilde{d}$ in $\tilde{A}'$ are given by

$$a \ast b = ab + \gamma_2(a, b)\varepsilon,$$

$$\tilde{d}(a) = d(a) + \gamma_1(a)\varepsilon$$

for some $\gamma_1 \in C^{1,1}(A^*)$, $\gamma_2 \in C^{2,0}(A)$, and by (2.3), $\gamma = \gamma_1 + \gamma_2$ is a Hochschild cocycle representing the class $\theta_{\tilde{A}} \in HH^2(A)$.

However, we will also need a variation of the Hochschild cohomology construction specific to DG algebras. Namely, given a flat DG algebra $A^*$ in some tensor category $\mathcal{C}$, one can invert quasiisomorphism in the category of $A^*$-bimodules and obtain the derived category $\mathcal{D}(A^*)$ of DG $A^*$-bimodules. This gives a DG version

$$HH_D^*(A^*) = R\text{Hom}_{\mathcal{D}(A^*)}(I_{A^*}, A)$$

of the Hochschild cohomology. We have a canonical map

$$HH^*(A^*) \to HH_D^*(A^*).$$

If $\mathcal{C}$ is the category of $k$-vector spaces, the groups $HH_D^*(A^*)$ can be computed by the same complex (3.1), but without the condition $l \geq 0$.

The point of introducing the groups $HH_D^*(A^*)$ is that they control deformations “up to quasiisomorphism”. Namely, recall that the category of DG algebras up to a quasiisomorphism is obtained from the category of DG algebras by formally inverting all algebra maps which are quasiisomorphisms – in other words, DG algebras $A^*$, $B^*$ are quasiisomorphic if there exists a chain of quasiisomorphisms

$$A^* \leftarrow A_1^* \longrightarrow A_2^* \leftarrow \ldots \leftarrow A_n^* \longrightarrow B^*.$$ 

Then we have the following fact.

**Lemma 3.1.** Assume that $\mathcal{C}$ is the category of vector spaces of a field $k$ of characteristic $\text{char} \ k = 0$. Assume given a DG algebra $A^*$ in $\mathcal{C}$ and
Then for any integer $p \geq 1$, the DG algebra $\widetilde{A}^*/h^{p+1}$ is quasiisomorphic to the DG algebra $A^*[h]/h^{p+1}$ if and only if the Kodaira-Spencer class $\Theta$ of the deformation $\widetilde{A}$ vanishes after projection to $HH_D^0(\widetilde{A}^*)/h^p$.

**Proof.** This is, in a sense, a DG version of Z. Ran’s $T_1$-lifting principle [R]. To control quasiisomorphisms, it is convenient to use the notion of an $A_{\infty}$-morphism. Recall (see e.g. [Ke]) that an $A_{\infty}$-morphism $\iota$ between DG algebras $A$ and $B$ is by definition a map $\iota : T^*(A^*) \to T^*(B^*)$ between the free coassociative DG coalgebras $T^*(A^*)$, $T^*(B^*)$ generated by $A[1]$ and $B[1]$ such that $\delta_B \circ \iota = \iota \circ \delta_A$, where $\delta_A$ and $\delta_B$ are Hochschild differentials on $T^*(A^*)$, $T^*(B^*)$. Every DG algebra map obviously induces an $A_{\infty}$-morphism. However, if a DG algebra map is a quasiisomorphism, then the corresponding $A_{\infty}$-map $\iota$ is invertible (that is, there exists an $A_{\infty}$-map $\iota^{-1}$ such that $\iota \circ \iota^{-1}$ and $\iota^{-1} \circ \iota$ are identical on cohomology). Therefore if two DG algebras $A^*$, $B^*$ are quasiisomorphic, not only there exists a chain of quasiisomorphisms $\ldots$ – there in fact exists a single $A_{\infty}$-quasiisomorphism $\iota : T^*(A^*) \to T^*(B^*)$.

Now, fix a graded vector space isomorphism $\widetilde{A}^* \cong A^* \otimes_k h$; under this isomorphism, the multiplication and the differential in $\widetilde{A}$ are given by

$$d = d_0 + \sum_{l \geq 1} h^l d_l$$

$$m = m_0 + \sum_{l \geq 1} h^l m_l,$$

where $m_0$ and $d_0$ are the multiplication and the differential in $A^*$. The Hochschild cohomology $HH_D^*(\widetilde{A}^*)$ of the DG $k[h]$-algebra $\widetilde{A}^*$ can be computed by the same Hochschild complex $C^*(A^*)[h]$ as $HH_D^*(A^* \otimes_k h)$, but with a different differential: the differentials $\delta$ and $\delta_0$ computing $HH_D^*(\widetilde{A}^*)$ and $HH_D^*(A^*)[h]$ are given by $\delta(a) = [d + m, a]$, $\delta_0(a) = [d_0 + m_0, a]$. The quotient $\widetilde{A}^*/h^{p+1}$ is quasiisomorphic to $A^*[h]/h^{p+1}$ if and only if there exists a $k[h]/h^{p+1}$-linear $A_{\infty}$-quasiisomorphism between them – in other words, a coalgebra map $\iota_{p+1} : T^*(A^*)[h] \to T^*(A^*)[h]$ such that $\delta \circ \iota_{p+1} = \iota_{p+1} \circ \delta_0$ mod $h^{p+1}$.

To compute the cocycle $\Theta = \Theta_{\widetilde{A}}$, we can use (3.3). Namely, we replace $h$ with $h+\varepsilon$ in (3.4), and we conclude that the image of the Kodaira-Spencer class $\Theta$ in the group $H_D^2(\widetilde{A}^*)$ is represented by the cocycle

$$Q = \sum_{l \geq 1} l h^l (m_l + d_l) \in C^*(A^*)[h]$$
of total degree 2. To prove the claim of the Lemma, use induction on $p$. Assume by induction that $\Theta = 0 \mod h^p$ and that there exists a map $\iota_p : T^*(A^*)[h] \to T^*(A^*)[h]$ such that $\delta \circ \iota_p = \iota_p \circ \delta_0 \mod h^p$. Then $\iota_p(Q)$ is divisible by $h^p$, and it represents the class $\Theta$. Thus $\Theta = 0 \mod h^{p+1}$ if and only if $\iota_p(Q) = h^p \delta(\gamma) \mod h^{p+1}$ for some Hochschild cochain $\gamma \in C^*(A^*)$ of total degree 1. Since $\delta = \delta_0 \mod h$, this can be rewritten as

$$\iota_p(Q) = h^p[\delta, \gamma] = h^p[\delta_0, \gamma] \mod h^{p+1},$$

which is in turn equivalent to $\delta \circ \iota_{p+1} = \iota_{p+1} \circ \delta_0 \mod h^{p+1}$, where we set $\iota_{p+1} = \iota_p + \frac{1}{p} h^p \gamma$. This proves the claim. □

**Remark 3.2.** The point in the above proof where we do need to consider $A_\infty$-morphisms is in the construction of the correction term $\gamma$: the cochain $\gamma \in C^2(A^*)$ in the Hochschild complex [3.1] can have non-trivial components in $C^{l,2-l}(A^*)$ with $l < 0$.

When $C$ is a general tensor category, the relation between the Kodaira-Spencer class and the triviality of deformations of DG algebras is a difficult subject better left untouched in the present paper. However, the Kodaira-Spencer class itself is perfectly well defined.

### 4 Obstructions to formality.

We can now proceed to our objective – the study of formality. Assume given a DG algebra $A^*$ in a tensor abelian category $C$. We want to study whether $A^*$ is formal – that is, quasiisomorphic to the cohomology DG algebra $H^*(A^*)$ (with trivial differential). To do this, consider the canonical filtration $F$, on $A^*$ – that is, set

$$F_k A^p = \begin{cases} 0, & \text{if } p > k, \\ \text{Ker } d : A^p \to A^{p+1}, & \text{if } p = k, \\ A^p, & \text{if } p < k, \end{cases}$$

and denote by $B^* \cong \text{gr}^F A^*$ the associated graded quotient. The canonical filtration induces a filtration $F_*$ on the Hochschild cohomology complex $HH^*_D(A^*)$; for any two integers $p \leq q$ we denote

$$HH^*_{(p,q)}(A^*; D) = F_p HH^*_{D}(A^*)/F_q HH^*_{D}(A^*).$$
The associated graded quotient $\text{gr}_F^p \HH^*_D(A^*)$ is naturally quasiisomorphic to $\HH^*_D(B^*)$. Denote the induced grading on the complex $\HH^*_D(B^*)$ by $\HH^*_D(B^*)_*$, so that we have

\begin{equation}
\text{gr}_F^p \HH^*_D(A^*) = \HH^*_D(A^*)_{(p,p)} \cong \HH^*_D(B^*)_p
\end{equation}

for any integer $p$. If $\mathcal{C}$ is the category of vector spaces over $k$, one can compute $\HH^*_D(B^*)$ by means of the bar-construction; then the complex computing the component $\HH^*_D(B^*)_p$ of degree $p$ consists of spaces

\begin{equation}
\text{Hom}^q-p((B^*)^q, B^*).
\end{equation}

Now, the natural quasiisomorphism $\text{gr}_F^p A^* \to H^*(A)$ is compatible with the multiplication. Therefore the question of formality of the DG algebra $A^*$ is equivalent to the existence of a quasiisomorphism between the DG algebras $A^*$ and $B^* = \text{gr}_F^p A^*$. To refine this, consider the Rees algebra

$$\hat{A}^* = \bigoplus_p F_p A^*;$$

this is a graded DG-algebra in $\mathcal{C}[h]$, with $h$ of degree 1 given by the natural embedding $h : F_* A^* \to F_{*+1} A^*$. Then $\hat{A}^*/h\hat{A}^* \cong B^*$, so that $\hat{A}^*$ is a formal deformation of the DG algebra $B^*$.

**Definition 4.1.** The DG algebra $A^*$ is $p$-formal for some integer $p \geq 1$ if $\hat{A}^*/h^{p+1} \hat{A}^*$ is quasiisomorphic to $B^*[h]/h^{p+1}$.

**Lemma 4.2.** A DG algebra $A^*$ in the category of vector spaces over a field $k$ is formal if and only if it is $p$-formal for every $p \geq 1$.

**Proof.** The “only if” part is obvious: if $A^*$ is formal, then its deformation $\hat{A}^*$ is trivial, so that for every $p \geq 1$, the truncation $\hat{A}^*/h^{p+1} \hat{A}^*$ is quasiisomorphic to $B^*[h]/h^{p+1}$. Conversely, if $A^*$ is $p$-formal, then we can choose an $A_\infty$-map $s : B^* \to \hat{A}^*/h^{p+1}$ which splits the natural projection $\hat{A}^*/h^{p+1} \to B^*$. Moreover, if $A^*$ is $p$-formal for every $p$, then we can choose these splitting maps in a compatible way and obtain an $A_\infty$-map $s : B^* \to \hat{A}^*$ from $B^*$ to the completion $\hat{A}^* = \lim_{l \to -\infty} \hat{A}^*/h^{p+1}$ of the algebra $\hat{A}^*$ with respect to the $h$-adic topology. Explicitly, this completion $\hat{A}^*$ is a graded DG algebra, whose component of degree $m$ is equal to

$$\hat{A}^*_m = \lim_{l \to -\infty} F_m A^*/F_l A^*.$$
However, for any complex $V^\ast$ with the canonical filtration $F, V^\ast$, the inverse system $V^\ast/F_lV^\ast$ obviously stabilizes in any degree at a finite step ($V^m/F_lV^m$ stops depending on $l$ when $l < m - 1$). Therefore the completion $\hat{A}^\ast$ is actually isomorphic to $\hat{A}^\ast$, so that we have an $A_\infty$-map $B^\ast \to \hat{A}^\ast$ which splits the projection $\hat{A}^\ast \to B^\ast$. Evaluating at $h = 1$, we get an $A_\infty$-quasiisomorphism $B^\ast \cong A^\ast$. □

To measure $p$-formality for all $p \geq$, we use Lemma 3.1. Denote by $Q_A^\ast \in HH^2_D(\hat{A}^\ast)$ the image of the Kodaira-Spencer class $\Theta_{\hat{A}^\ast}$ under the natural map $[3.2]$. The grading on the Rees algebra induces a natural grading on $HH^q_D(\hat{A}^\ast)$, with the component of degree $p$ canonically identified with $F_pHH^2_D(A^\ast)$. Since $h$ is of degree 1, the class $Q_A^\ast$ is of degree $-1$, so that in fact we have the canonical class $Q_A^\ast \in F_{-1}HH^2_D(A^\ast)$. This class is a version of the so-called higher Massey products in the DG algebra $A^\ast$ – all of them in one package. Modulo $h^p$, this class restricts to a class in the degree $-1$-component of $HH^2_D(A^\ast)/h^p$, which is identified with $HH^2_D(A^\ast)_{(-p,-1)}$.

Let now $\mathcal{C}$ be the category of sheaves of $\mathcal{O}_X$-modules on a scheme $X$ over a field of characteristic 0. Then in addition to tensor structure, the category $\mathcal{C} = \text{Coh}(X)$ has inner Hom and its derived functors, which we denote by $R\mathcal{H}om^\ast$. The same is true for $C^\ast(\mathcal{C})$ and for the category of $A^\ast$-bimodules for some DG algebra $A^\ast$ in $\mathcal{C}$. This allows to refine the construction of Hochschild cohomology: we can define the Hochschild cohomology complex

$$\mathcal{H}H^\ast_D(A^\ast) = R\mathcal{H}om^\ast_D(A^\ast)(I_{A^\ast}, A^\ast)$$

of sheaves of $\mathcal{O}_X$-modules on $X$, and we have $HH^2_D(A^\ast) \cong \mathcal{H}H^2_D(A^\ast)$. The Rees algebra, the canonical filtration on $A^\ast$, and $\mathcal{H}H^\ast_D(A^\ast)$ are also well-defined on the level of sheaves, and so is the grading $[4.1]$.

We note that if $X$ is Noetherian, then the inner Hom between two coherent sheaves of $\mathcal{O}_X$-modules is also coherent, and $R\mathcal{H}om^\ast$ is a complex with coherent homology sheaves.

**Theorem 4.3.** Let $A^\ast$ be a DG algebra of flat sheaves of $\mathcal{O}_X$-modules on a Noetherian reduced irreducible scheme $X$ over a field of characteristic 0. Let $B^\ast$ be the homology algebra of the DG algebra $A^\ast$. Assume that the sheaves
\(B^*\) are flat and coherent on \(X\). Assume also that for every integers \(l, i\), the component \(HH^i_D(B^*)_{-l}\) of degree \((-l)\) of \(i\)-th Hochschild cohomology sheaf \(HH^*_D(B^*)\) with respect to the grading \((4.1)\) is coherent and flat.

(i) If the fiber \(A^*\) is formal for a generic point \(x \in X\), then it is formal for an arbitrary point \(x \in X\).

(ii) Assume in addition that for every integer \(l \geq 1\), we have

\[
H^0(X, (HH^2_D(B^*))_{-l}) = 0.
\]

Then the DG algebra \(A^*_x\) is formal for every point \(x \in X\).

**Proof.** Use induction and Lemma 3.1. Assume that the fiber \(A^*_x\) is \(p\)-formal for some integer \(p \geq 1\) and every point \(x \in X\). Consider the spectral sequence in the category of sheaves of \(\mathcal{O}_X\)-modules on \(X\) associated to the filtration \(F_*\) on the complex \(HH^*_D(A^*)_{-p-1}\). In terms of the Rees algebra, this is the \(h\)-adic filtration on the \((-1)\)-component of \(HH^*_D(\tilde{A}^*/h^p)\). The terms of this spectral sequence are the Hochschild cohomology sheaves \(HH^i_D(B^*)_{-l}\), \(1 \leq l \leq p\), and by assumption, these are flat coherent sheaves on \(X\). The fiber of the spectral sequence at any point \(x \in X\) gives the corresponding spectral sequence for the fiber \(A^*_x\). Since by assumption the algebra \(A^*_x\) is \(p\)-formal for every \(x \in X\), the differential in the spectral sequence vanishes at every point. Therefore the spectral sequence degenerates, and the homology sheaves of the complex \(HH^*_D(A^*)_{-p-1}\) are iterated extensions of the homology sheaves of the complexes \(HH^*_D(B^*)_{-l}\), \(1 \leq l \leq p\). In particular, the second homology sheaf \(HH^2_D(A^*)_{-p-1}\) is flat and coherent. Thus if the reduction \(\overline{Q\mathcal{A}^*} = Q\mathcal{A}^* \mod h^p \in H^0(X, HH^2_D(A^*)_{-p-1})\) vanishes at the generic point of \(X\), it vanishes everywhere, and \(A^*\) is \((p + 1)\)-formal everywhere by Lemma 3.1. This proves (i). Moreover, in the assumptions of (ii) we in fact have \(H^0(X, (HH^2_D(A^*))_{-p-1}) = 0\), which by Lemma 3.1 again proves that \(A^*\) is \((p + 1)\)-formal at every point \(x \in X\). \(\square\)

**References**

[DGMS] P. Deligne, Ph. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. 29 (1975), 245–274.

[GK] V. Ginzburg and D. Kaledin, *Poisson deformations of symplectic quotient singularities*, Adv. in Math. 186 (2004), 1-57.
[HS] S. Halperin and J. Stasheff, *Obstructions to homotopy equivalences*, Adv. in Math. 32 (1979), 233–279.

[H] V. Hinich, *Tamarkin’s proof of Kontsevich formality theorem*, Forum Math. 15 (2003), 591–614.

[Ke] B. Keller, *Introduction to A-infinity algebras and modules*, Homology, Homotopy and Applications, 3 (2001), 1-35.

[R] Z. Ran, *Deformations of manifolds with torsion or negative canonical bundles*, J. Alg. Geom. 1 (1992), 279–291.

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