Vanishing of the Kontsevich integrals of the wheels

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Abstract

We prove that the Kontsevich integrals (in the sense of the formality theorem [K]) of all even wheels are equal to zero. These integrals appear in the approach to the Duflo formula via the formality theorem. The result means that for any finite-dimensional Lie algebra $g$, and for invariant polynomials $f, g \in [S(g)]^g$ one has $f \cdot g = f \ast g$, where $\ast$ is the Kontsevich star-product, corresponding to the Kirillov-Poisson structure on $g^*$. We deduce this theorem from the result of [FSh] on the deformation quantization with traces.

1 Introduction

1.1

First of all, let us recall what the Duflo formula is.

Let $g$ be a finite-dimensional Lie algebra, we denote by $S(g)$ and $U(g)$ the symmetric and the universal enveloping algebras of the Lie algebra $g$, correspondingly. There is the adjoint action of $g$ on both spaces $S(g)$ and $U(g)$, and the classical Poincaré-Birkhoff-Witt map $\varphi_{PBW} : S(g) \rightarrow U(g)$

$$\varphi_{PBW}(g_1 \cdot \ldots \cdot g_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} g_{\sigma(1)} \otimes \ldots \otimes g_{\sigma(k)}$$

(1)

is an isomorphism of the $g$-modules. In particular, it defines a map of invariants

$$[\varphi_{PBW}] : [S(g)]^g \rightarrow [U(g)]^g \cong Z(U(g))$$

(2)

where $Z(U(g))$ is the center of the universal enveloping algebra.

The Duflo theorem states that $[S(g)]^g$ and $Z(U(g))$ are isomorphic as algebras, and gives an explicit formula for the isomorphism.
For each $k \geq 1$ there exists a canonical invariant element $\text{Tr}_k \in [S^k(g)]^\ast$. It is just the trace of $k$-th power of the adjoint action, i.e. the symmetrization of the following map $\text{Tr}_k$:

\[ \text{Tr}_k(g) = \text{Tr}_g(ad(g))^k. \]  

(3)

It is easy to show that for semisimple Lie algebras $\text{Tr}_k = 0$ for odd $k$. Hence, we will consider the elements $\text{Tr}_k$ only for even $k$.

One can consider any element of $[S^k(g)]^\ast$ as a differential operator of $k$-th order with constant coefficients, acting on $S^\cdot(g)$. Let us note that for a fixed element $\theta \in S^\ell(g)$ the values $\text{Tr}_k(\theta)$ are not equal to 0 only for $k \leq \ell$.

Finally, define the map $\varphi_{\text{strange}} : S^\cdot(g) \to S^\cdot(g)$ by the formula

\[ \varphi_{\text{strange}} = \exp \left( \sum_{k \geq 1} \alpha_{2k} \text{Tr}_{2k} \right) \]  

(4)

where the rational numbers $\alpha_{2k}$ are defined from the formula

\[ \sum_{k \geq 0} \alpha_{2k} q^{2k} = \log \sqrt{e^{q/2} - e^{-q/2}}. \]  

(5)

**Theorem.** (Duflo) For a finite-dimensional Lie algebra $g$, the restriction of the map $\varphi_{\text{PBW}} \circ \varphi_{\text{strange}} : S^\cdot(g) \to U(g)$ to the space $[S^\cdot(g)]^g$ defines an isomorphism of the algebras

\[ [\varphi_{\text{PBW}} \circ \varphi_{\text{strange}}] : [S^\cdot(g)]^g \simeq Z(U(g)). \]

1.2

Here we outline the Kontsevich’s approach to the Duflo formula via the formality theorem ([K], Sect. 8).

For any Poisson structure on a finite-dimensional vector space $V$, i.e. for a bivector field $\alpha$ on $V$ such that $[\alpha, \alpha] = 0$, M. Kontsevich defined a deformation quantization of the algebra structure on functions $C^\infty(V)$.

Any Lie algebra $g$ defines the Kirillov-Poisson structure on $g^\ast$. The Poisson bracket of two linear functions on $g^\ast$, i.e. of two elements of $g$, is equal to their bracket: $\{g_1, g_2\} := [g_1, g_2]$. This bracket can be extended to $S^\cdot(g)$ by the Leibniz rule. The corresponding bivector field in coordinates $\{x_i\}$ on $g$ is

\[ \alpha = \sum_{i,j,k} C^k_{ij} x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \]  

(6)

where $\{C^k_{ij}\}$ is the structure constants of the Lie algebra $g$ in the basis $\{x_i\}$. Finally, the bracket of any two functions is $\{f, g\} = \alpha(df \wedge dg)$. 

2
The Kontsevich deformation quantization of this structure defines a star-product on $S'(\mathfrak{g})$, and the deformed algebra $(S'(\mathfrak{g}), \ast)$ is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$.

**Theorem.** (Kontsevich \cite{K}) There exist numbers $W_{2k}$ and $\alpha'_{2k}$ such that:

(i) for any finite-dimensional Lie algebra $\mathfrak{g}$ the map

$$\varphi_W = \exp \left( \sum_{k \geq 1} W_{2k} \text{Tr}_{2k} \right) : S'(\mathfrak{g}) \to S'(\mathfrak{g})$$

defines the isomorphism of the algebras $[\varphi_W] : [S'(\mathfrak{g})]^{\mathfrak{g}} \simto [S'(\mathfrak{g}), \ast]^{\mathfrak{g}}$ where $\ast$ is the Kontsevich star-product;

(ii) the canonical map $\varphi_{\alpha'} : (S'(\mathfrak{g}), \ast) \to U(\mathfrak{g})$,

$$\varphi_{\alpha'}(g_1 \ast \ldots \ast g_k) = g_1 \otimes \ldots \otimes g_k$$

is equal to

$$\varphi_{\alpha'} = \varphi_{\text{PBW}} \circ \exp \left( \sum_{k \geq 1} \alpha'_{2k} \text{Tr}_{2k} \right).$$

As a consequence, we obtain that the coefficients $\alpha_{2k}$ in the Duflo formula are equal to the sum

$$\alpha_{2k} = W_{2k} + \alpha'_{2k}. \quad (7)$$

The numbers $W_{2k}$ and $\alpha'_{2k}$ are defined as integrals over configuration spaces. They were not computed in \cite{K}. The main result is that $W_{2k}$ and $\alpha'_{2k}$ do not depend on the Lie algebra $\mathfrak{g}$.

The number $W_{2k}$ is the Kontsevich integral corresponding to the wheel with $2k$ vertices, see Figure 1.
The wheel $W_6$

Remarks.

1. The Kontsevich star-product on $S^*(\mathfrak{g})$ is well-defined only for finite-dimensional Lie algebras $\mathfrak{g}$, while the algebra $\mathcal{U}(\mathfrak{g})$ (as well as some other quantizations) is well-defined for any (maybe infinite-dimensional) Lie algebra.

2. In [K] the Duflo isomorphism was extended from the invariants (i.e. zero degree cohomology) to the whole algebras of cohomology. The result is that the map $\varphi_{\text{PBW}} \circ \varphi_{\text{strange}} : S^*(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ defines an isomorphism

$$[\varphi_{\text{PBW}} \circ \varphi_{\text{strange}}] : H^*(\mathfrak{g}; S^*(\mathfrak{g})) \to H^*(\mathfrak{g}; \mathcal{U}(\mathfrak{g})).$$

In this cohomological setting it seems that an isomorphism should exist for infinite-dimensional Lie algebras as well. The Duflo formula can not be applied because the traces $\text{Tr}_{2k}$ are ill-defined for infinite-dimensional Lie algebras. One of our goals in this work was to understand the nature of Duflo isomorphism for infinite-dimensional Lie algebras.

1.3

The main result of the present paper was conjectured by Alberto S. Cattaneo and Giovanni Felder:

**Theorem.** All numbers $W_{2k}$ ($k \geq 1$) are equal to 0.

In an equivalent form, $f \cdot g = f \ast g$ for any two invariant polynomials $f, g \in [S^*(\mathfrak{g})]^\mathfrak{g}$ for arbitrary finite-dimensional Lie algebra $\mathfrak{g}$ ($\ast$ is the Kontsevich star-product). Also we obtain $\alpha'_{2k} = \alpha_{2k}$.
A. S. Cattaneo and G. Felder had computed the number $W_2$ as a four-dimensional integral and had found that it is equal to zero. In this paper we prove that all the numbers $W_{2k}$ are equal to 0 using the deformation quantization with traces \cite{FS}. Let us recall the main result of \cite{FS}.

Consider a vector space $V$ equipped with a Poisson structure $\alpha$ and a volume form $\Omega$ compatible in the following way. For any manifold $M$ a volume form on $M$ allows to identify polyvector fields with differential forms. Then the de Rham operator on differential forms defines an operator of degree $-1$ on polyvector fields. Such an operator is called the divergence operator, corresponding to the volume form $\Omega$, we denote it by $\div_\Omega$. It is a second order operator with respect to the wedge product of polyvector fields, and for any volume form $\Omega$, the defect for the Leibniz rule

$$[\eta_1, \eta_2] = \pm [\div_\Omega (\eta_1 \wedge \eta_2) - (\div_\Omega \eta_1) \wedge \eta_2 \pm \eta_1 \wedge \div_\Omega \eta_2]$$

(8)

is equal to the Schouten-Nijenhuis bracket of the polyvector fields $\eta_1$ and $\eta_2$ and does not depend on $\Omega$. As a consequence, we obtain

$$\div_\Omega [\eta_1, \eta_2] = [\div_\Omega \eta_1, \eta_2] \pm [\eta_1, \div_\Omega \eta_2]$$

(9)

for any volume form $\Omega$.

1.3.1 Theorem \cite{FS}

Let $V$ be a finite-dimensional vector space, $\Omega$ be a constant volume form on $V$ (with respect to an affine coordinate system), and $\alpha$ be a Poisson bivector field on $V$ such that $\div_\Omega \alpha = 0$. Then

$$\int_V f \cdot g \cdot \Omega = \int_V (f \ast g) \cdot \Omega$$

for any two functions $f, g \in C^\infty(V)$ one of which has a compact support. (Here $\ast$ is the Kontsevich star-product with the harmonic angle function see \cite{K}, Sect. 6.2).

The identity

$$\int_V f \cdot g \cdot \Omega = \int_V (f \ast g) \cdot \Omega$$

holds for all functions, and we want to remove the integral sign when $V = g^*$ and $f, g$ are invariant.

2 Deformation quantization with traces for semi-simple Lie algebras
2.1 Theorem.

Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\alpha$ be the Poisson-Kirillov structure on $\mathfrak{g}^*$. Then a constant volume form $\Omega$ on $\mathfrak{g}^*$ satisfies the equation $\text{div}_\Omega \alpha = 0$.

Proof. For a bivector field

$$\alpha = \sum_{i,j} a_{ij} (x) \partial_i \wedge \partial_j \quad (a_{ij} = -a_{ji}),$$

its divergence with respect to the constant volume form $\Omega = dx_1 \wedge \ldots \wedge dx_n$ is equal to

$$\text{div}_\Omega \alpha = 2 \sum_{i,j} \partial_i (a_{ij} (x)) \partial_j . \quad (10)$$

Suppose now that $\alpha$ is the Poisson-Kirillov bivector field on $\mathfrak{g}^*$. Then, by formula (6), one has:

$$a_{ij} (x) = \sum_k C_{ij}^k x_k$$

where $C_{ij}^k$ are structure constants of the Lie algebra $\mathfrak{g}$ in the basis $\{x_i\}$. Then, by (10), the condition $\text{div}_\Omega \alpha = 0$ is equivalent to

$$\sum_{i,j} C_{ij}^i \partial_j = 0 \quad (11)$$

or

$$\text{for any } j \sum_i C_{ij}^i = 0 . \quad (12)$$

Let us suppose that the Lie algebra $\mathfrak{g}$ is semisimple, and the basis $\{x_i\}$ is chosen in a way compatible with the triangular decomposition $\mathfrak{g} = \mathcal{N}_- \oplus \mathfrak{h} \oplus \mathcal{N}_+$. The condition (12) is nontrivial only when $x_j \in \mathfrak{h}$. Let $e_1, \ldots, e_\ell$ be the positive root elements in $\mathcal{N}_+$, and let $f_1, \ldots, f_\ell$ be the dual root elements in $\mathcal{N}_-$.

For any element $h \in \mathfrak{h}$ we set

$$[h, e_k] = \alpha_k (h) e_k$$

$$[h, f_k] = \beta_k (h) f_k$$

where $\alpha_k, \beta_k \in \mathfrak{h}^*$ are the roots. Then (12) holds because $\beta_k = -\alpha_k$. □

Remark. It follows from the proof that the theorem is true also for any unipotent Lie algebra.
2.2 The vanishing of the wheels.

From the previous result and from the result of \[FSK\] (see Sect. 1.3) it follows that for a semisimple Lie algebra \( \mathfrak{g} \) one has

\[
\int_{\mathfrak{g}^*} f \cdot g \cdot \Omega = \int_{\mathfrak{g}^*} (f \ast g) \cdot \Omega
\]

where \( \Omega \) is a constant volume form and \( f \ast g \) is the Kontsevich star-product with the harmonic angle function.

We have

\[
f \ast g = f \cdot g + \sum_{k \geq 1} \hbar^k B_k(f, g)
\]

and

\[
B_k(f, g) = (-1)^k B_k(g, f).
\]

When \( f, g \) are invariant, \( f \ast g = g \ast f \), and, therefore, \( B_k(f, g) \equiv 0 \) for odd \( k \).

Hence,

\[
f \ast g = f \cdot g + \sum_{k \geq 1} \hbar^{2k} B_{2k}(f, g).
\]

Then (13) means that

\[
\int_{\mathfrak{g}^*} B_{2k}(f, g) \cdot \Omega = 0
\]

for any \( k \geq 1 \).

From now on we will work with a real compact semisimple Lie algebra. For instance, we replace the complex Lie algebra \( \mathfrak{sl}_n \) to the real Lie algebra \( \mathfrak{su}_n \). We want to apply (16) to invariant \( f, g \), that is, to \( f \) and \( g \) constant on symplectic leaves. In the compact case \( \mathfrak{g} = \text{Lie} G \), where \( G \) is a compact Lie group. Therefore, the symplectic leaves, being the orbits of the coadjoint action of \( G \) on \( \mathfrak{g}^* \), are compact. Hence, we then have some freedom in manipulations with invariant functions with compact support.

We will prove that \( W_{2l} = 0 \) by induction on \( k \). Let us suppose that it is proven for \( k < l \). Then, by Theorem 1.2(i), we have for invariant \( f, g \):

\[
f \ast g = f \cdot g + \hbar^{2l}(W_{2l} \cdot \text{Tr}_{2l}(f \cdot g) - W_{2l} \cdot \text{Tr}_{2l}(f) \cdot g - W_{2l} \cdot f \cdot \text{Tr}_{2l}(g)) + O(\hbar^{2l+2})
\]

Formulas (17) and (18) give:

\[
W_{2l} \cdot \int_{\mathfrak{g}^*} \left( \text{Tr}_{2l}(f \cdot g) - \text{Tr}_{2l}(f) \cdot g - f \cdot \text{Tr}_{2l}(g) \right) \cdot \Omega = 0
\]

Now we want to prove that the integral in (19) does not vanish for some \( f \) and \( g \), and, therefore, \( W_{2l} = 0 \).

We have:

\[
\text{Tr}_{2l}(f) = \sum_{i} a_{i_1 \cdots i_{2l}}(f) \partial_{i_1} \cdots \partial_{i_{2l}}(f)
\]
where $a_{i_1 \ldots i_2l}$ are constants (depending on the structure constants $C_{ij}^k$). It is clear from (20) that
\[
\int_{\mathfrak{g}^*} \Tr_{2l}(f \cdot g) \cdot \Omega = 0 \tag{21}
\]
for any $f$ and $g$ with compact support.

Furthermore, it follows from (20) that
\[
\int_{\mathfrak{g}^*} \Tr_{2l}(f) \cdot g \cdot \Omega = \int_{\mathfrak{g}^*} f \cdot \Tr_{2l}(g) \cdot \Omega \tag{22}
\]

Finally, (19) is equivalent to
\[
2 \cdot W_{2l} \cdot \int_{\mathfrak{g}^*} f \cdot \Tr_{2l}(g) \cdot \Omega = 0 \tag{23}
\]
which is satisfied for any invariant $f$ and $g$ with compact support.

For the Lie algebra $\mathfrak{su}_n$, $n \gg 0$, one can choose an invariant $g$ with a compact support such that $\Tr_{2l}(g) \neq 0$. It follows from the general description of the algebra of invariants $[\mathcal{S}(\mathfrak{g})]^\mathfrak{g}$ for $\mathfrak{g} = \mathfrak{su}_n$. Then in (23) $f$ is arbitrary, and we obtain $W_{2l} = 0$.

2.3 Remarks.

2.3.1 Remark.

It is interesting to note that the integrals of the wheels with the opposite direction of the central arrows are not equal to 0. Let us denote by $W_{k}^\vee$ such a wheel.
The wheel $W_6^\vee$

Then $W_{2k+1}^\vee = 0, k \geq 1$, and $W_{2k}^\vee = \alpha_{2k}$ (see formula (5)), $k \geq 1$. It follows from an alternative approach to the Duflo formula via the formality theorem, developed in an (unpublished) joint paper of the author with M.Kontsevich.

2.3.2 Non-linear Poisson structures.

**Conjecture** For any Poisson structure $\alpha$ on a vector space $V$ and for any two functions $f, g \in Z(C^\infty(V), \ast)$ in the center of the deformed algebra one has

$$f \ast g = f \cdot g$$

Here $\ast$ is the Kontsevich deformation quantization with the harmonic angle function.

2.3.3 Tangential deformation quantization.

Note that our theorem, $f \ast g = f \cdot g$ for invariant $f, g$, seems to be very closed to the condition of the tangential deformation quantization [CGR], which is $f \ast g = f \cdot g$ for invariant (=constant on leaves) $f$ and for any $g$. The last condition means geometrically that we quantize each symplectic leaf separately and then “glue” all the quantized leaves. More algebraically, it means, that in formula (14) all the differential operators $B_k(f, g)$ are formed by a composition of vector fields tangential to the leaves.
It turns out, however, that this condition is much stronger; in particular, it is proven in [CGR] that any such a quantization does not exist for semisimple Lie algebras.

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