Positivity and convexity in incomplete cooperative games

Martin Černý, Jan Bok, David Hartman, Milan Hladík

Abstract

Incomplete cooperative games generalise the classical model of cooperative games by omitting the values of some of the coalitions. This allows to incorporate uncertainty into the model and study the underlying games as well as possible payoff distribution based only on the partial information. In this paper we perform a systematic study of incomplete games, focusing on two important classes of cooperative games: positive and convex games.

Regarding positivity, we generalise previous results for a special class of minimal incomplete games to general setting. We characterise non-extendability to a positive game by the existence of a certificate and provide a description of the set of positive extensions using its extreme games. The results are then used to obtain explicit formulas for several classes of incomplete games with special structures.

The second part deals with convexity. We begin with considering the case of non-negative minimal incomplete games. Then we survey existing results in the related theory of set functions, namely providing context to the problem of completing partial functions. We provide a characterisation of extendability and a full description of the set of symmetric convex extensions. The set serves as an approximation of the set of convex extensions.

Finally, we outline an entirely new perspective on a connection between incomplete cooperative games and cooperative interval games.

Keywords: cooperative games, incomplete games, upper game, lower game, positive games, convex games, totally monotonic games

Preprint submitted to Elsevier
1. Introduction and motivation

Developing various approaches to deal with uncertainty is heavily intertwined with decision making and thus also with game theory. It is only natural since inaccuracy in data is an everyday problem in real-world situations, be it a lack of knowledge on the behaviour of others, corrupted data, signal noise or a prediction of outcomes such as voting or auctions. Since the degree of applications is so wide, a lot of models of various complexity and use-case scenarios exist. Among those models most relevant to cooperative game theory, there are fuzzy cooperative games \cite{12, 26, 27}, multi-choice games \cite{12}, cooperative interval games \cite{1, 2, 3, 8}, fuzzy interval games \cite{25}, games under bubbly uncertainty \cite{31}, ellipsoidal games \cite{38}, and games based on grey numbers \cite{30}.

In the theory of classical (transferable utility) cooperative games, we know the precise reward (or payoff) for the cooperation of every group of players, called coalition. In incomplete cooperative games, this is generally no longer true, since only some of the coalition values are known. This models the uncertainty over data and its consistency. The model was first introduced in literature by Willson \cite{39} in 1993. Willson gave the basic notion of incomplete game (called there partially defined games) and he generalised the definition of the Shapley value for such games. Two decades later, Inuiguchi and Masuya revived the research. In \cite{29}, they focused mainly on the class of superadditive games (and also briefly mentioned particular cases of convex and positive games in which precisely the values of singleton coalitions and grand coalition are known). Further, Masuya \cite{28} considered approximations of the Shapley value for incomplete games, and Yu \cite{40} introduced a generalisation of incomplete games to games with coalition structures and studied the proportional Owen value (which is a generalisation of the Shapley value for these games). Apart from that, Bok and Černý considered the property of 1-convexity and related solution concepts (values) for incomplete games \cite{9}.

1.1. Motivation

We would like to mention some of the motivations for our research.

The first one (already mentioned) is that the model can be seen as one of the possible approaches to uncertainty. This is no doubt a widespread problem in real world and thus in applications. Our paper does not discuss specific scenarios but it provides theoretical foundations for such analysis.

The model of incomplete games also has strong connections to other uncertainty models in cooperative game theory. In particular, we discuss some natural connections to cooperative interval games in Section 6.

The uncertainty issue can be actually turned around. The study of incomplete games shows us what kind of information can we infer if we intentionally forget a part of the input. This is especially valuable since in general the size of cooperative games is exponential in the number of players. Thus we can turn
otherwise computationally difficult task into an easier one (in most cases, at some cost of precision).

The set function is a function having as its domain the power set of a given set. We note that incomplete cooperative games can also be viewed as partial set functions. Indeed such structures were already studied, yet without highlighting the relation to the theory of incomplete games. In particular, extension of partial set functions into so called submodular functions was studied e.g. in [5, 4, 34]. We refer to the exhaustive book of Grabisch [21] which discusses in detail connections of various types of set functions to entirely different parts of mathematics, with cooperative games being one of them.

A main goal of cooperative game theory is the study of solution concepts, those are functions assigning a set of payoff vectors to each game. Solution concepts assigning precisely one payoff vector to each game are called values. For symmetric games (also studied in Section 5), tend to behave in a quite simple way. However, general (multi-point) solution concepts, like core, imputations, stable sets, or the Weber set are interesting for symmetric games as they remain to be multi-point even with the symmetry.

We also note that for analysing solution concepts of incomplete games, it is absolutely necessary to first analyse possible sets of extension (belonging to some given class). If a certain set of extensions is difficult to describe, it might be advantageous to describe its proper subsets (e.g. symmetric extensions if the incomplete game is symmetric), which then also serve as approximations of the former set. Our results here can be also regarded in this way.

1.2. Main results and structure of the paper

Our results concern two important classes of games: convex games and their subclass of positive games. Let us highlight the main contributions of this paper and its structure.

- In Section 2 we outline the necessary background of the cooperative game theory and also introduce fundamental definitions of incomplete cooperative games, both needed further in the text.

- In Section 3 we study positivity of incomplete games in general. We tackle questions considering extendability to a positive extension, boundedness of the set of positive extensions and provide a description of the set of positive extensions using its extreme games in case the set is bounded.

- Section 4 is focused on three different classes with special structure of the known values. We analyse these classes as an application of the characterisation of extreme games from the previous section.

- Section 5 is dedicated to convexity and to symmetric convex extensions. We characterise under which conditions an incomplete game is extendable into a symmetric convex extension. We provide the range of each coalition’s worth over all such possible extensions and fully describe the set of symmetric convex extensions as a set of convex combinations of its
extreme games. We also provide a geometrical point of view on the set of symmetric convex extensions.

- Section 6 concludes the paper by providing connections between the theory of incomplete cooperative games and cooperative interval games.

2. Preliminaries

2.1. Classical cooperative games

Comprehensive sources on classical cooperative game theory are for example [12, 16, 20, 32]. For more on applications, see e.g. [6, 15, 24]. Here we present only the necessary background needed for the study of incomplete cooperative games. The crucial definition is that of a classical cooperative game. We note that the following definition assumes transferable utility (shortly TU).

**Definition 1.** A cooperative game is an ordered pair \((N, v)\) where the set \(N = \{1, 2, \ldots, n\}\) and \(v: 2^N \to \mathbb{R}\) is the characteristic function of the cooperative game. Further, \(v(\emptyset) = 0\).

The set of \(n\)-person cooperative games is denoted by \(\Gamma^n\). The subsets of \(N\) are called coalitions and \(N\) itself is called the grand coalition. We often write \(v\) instead of \((N, v)\) whenever there is no confusion over what the player set is. We often associate the characteristic functions \(v: 2^N \to \mathbb{R}\) with vectors \(v \in \mathbb{R}^{2^|N|-1}\).

To avoid cumbersome notation, we use the following abbreviations. We often replace singleton set \(\{i\}\) with \(i\). We use \(\subseteq\) for the relation of “being a subset of” and \(\subset\) for the relation “being a proper subset of”. By \(\emptyset \neq S \subseteq N\), we mean \(S \subseteq N\) and \(S \neq \emptyset\). To denote the sizes of coalitions e.g. \(N, S, T\), we often use \(n, s, t\), respectively.

A cooperative game \((N, v)\) is monotonic if for every \(T \subseteq S \subseteq N\), we have \(v(T) \leq v(S)\), and it is superadditive if for every \(S, T \subseteq N\) such that \(S \cap T = \emptyset\), we have \(v(S) + v(T) \leq v(S \cup T)\). The set of superadditive \(n\)-person games is denoted by \(S^n\).

**Definition 2.** A cooperative game \((N, v)\) is convex if for every \(S, T \subseteq N\),

\[v(T) + v(S) \leq v(S \cup T) + v(S \cap T).\]  (1)

The set of convex \(n\)-person games is denoted by \(C^n\).

The property of the characteristic function of a convex game is called supermodularity. If Conditions (1) hold with opposite inequality, we talk about submodular functions (see [21] for more). Each of the three aforementioned classes incorporates a different approach to formalising the concept of bigger coalitions being stronger. Clearly, convex games are superadditive. The class of convex games is maybe the most prominent class in cooperative game theory since it has many applications and it enjoys both elegant and powerful characterisations. Among them, the following characterisation by Shapley is necessary for proving our results.
Theorem 1. A cooperative game \((N, v)\) is convex if and only if for every \(i \in N\) and every \(S \subseteq T \subseteq N \setminus \{i\}\), it holds that \(v(S \cup i) - v(S) \leq v(T \cup i) - v(T)\).

Positive games (also called totally monotonic games) form a subset of convex games. The concept of total monotonicity generalises the notion of monotonicity in cooperative games. The class was introduced by Shapley by using unanimity games, special positive games, to study his now well-known value.

Definition 3. For every nonempty \(T \subseteq N\), the unanimity game \((N, u_T)\) is defined as

\[
u_T(S) := \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}
\]

The set of all unanimity games forms a basis of the vector space of cooperative \(n\)-person games, i.e. any cooperative \(n\)-person game \(v\) can be expressed as \(v = \sum_{\emptyset \neq T \subseteq N} d_v(T) u_T\). Coordinates \(d_v(T)\) are called Harsanyi dividends. Positive games form a non-negative orthant in this coordinate system. In this paper, we employ a different definition of Harsanyi dividends.

Definition 4. For every nonempty \(T \subseteq N\), the Harsanyi dividend \(d_v(T)\) of a game \((N, v)\) is defined as

\[
d_v(T) := \sum_{S \subseteq T} (-1)^{|T \setminus S|} v(S).
\]

Definition 5. A cooperative game \((N, v)\) is positive if the Harsanyi dividend \(d_v(T)\) is non-negative for every nonempty \(T \subseteq N\). The set of positive \(n\)-person games is denoted by \(P^n\).

The results concerning positive games are relatively sparse and scattered (the games are also called totally monotonic games). We would like to refer the reader to [13, 18, 22, 33] for further resources.

The dividend \(d_v(T)\) has another interesting property; it is also the value corresponding to \(T\) in Möbius transform of value \(v(T)\) (see [21] for further details). Another way to represent \(d_v\) is the following. Let

\[
d_v(T) = \begin{cases} 0, & \text{if } T = \emptyset, \\ v(i), & \text{if } T = \{i\}, \text{for } i \in N, \\ v(T) - \sum_{S \subseteq T} d_v(S), & \text{if } T \subseteq N, |T| > 1. \end{cases}
\]

The general case of both positive and convex games yields very complex situations and so we often restrict ourselves to simpler subclasses of these games. One of the possible approaches is to impose the additional property of symmetry.

Definition 6. A cooperative game \((N, v)\) is said to be symmetric if for every \(S, T \subseteq N\) such that \(|S| = |T|\), it holds that \(v(S) = v(T)\).
We denote the sets of symmetric convex and symmetric positive \( n \)-person games by \( C^n_\sigma \) and \( P^n_\sigma \), respectively. It is easy to observe that symmetric games can be described in a succinct way. This helps in our analysis and provides an interesting view-point on our results; while the set of symmetric convex extensions is easy to describe, it is not the case for symmetric positive extensions. We provide examples showing that trying to describe such extensions is a difficult problem to tackle.

2.2. Incomplete cooperative games

The following definitions are inspired by [29].

**Definition 7. (Incomplete game)** An incomplete game is a tuple \((N, K, v)\) where the set \( N = \{1, \ldots, n\} \), \( K \subseteq 2^N \) is the set of coalitions with known values and \( v: K \to \mathbb{R} \) is the characteristic function of the incomplete game. Further, \( \emptyset \in K \) and \( v(\emptyset) = 0 \).

Compared to definitions in [29], our assumptions are slightly more general. Most importantly, we do not a priori assume that the values of singleton coalitions and the grand coalition are among the known values, i.e. that they are in \( K \).

The fundamental tool to analyse incomplete games are their \( C \)-extensions.

**Definition 8.** Let \( C \subseteq \Gamma^n \) be a class of \( n \)-person games. A cooperative game \((N, w) \in C\) is a \( C \)-extension of an incomplete game \((N, K, v)\) if \( w(S) = v(S) \) for every \( S \in K \).

The set of all \( C \)-extensions of an incomplete game \((N, K, v)\) is denoted by \( C(v) \). We write \( C(v) \)-extension whenever we want to emphasize the game \((N, K, v)\). Also, if there is a \( C(v) \)-extension, we say \((N, K, v)\) is \( C \)-extendable. Finally, the set of all \( C \)-extendable incomplete games with fixed \( K \) is denoted by \( C(K) \).

The sets of \( C \)-extensions studied in this text are always convex. One of the main goals of the model of incomplete cooperative games is to describe these sets using their extreme points and extreme rays whenever the description is possible. We refer to the extreme points as to extreme games. For the sake of completeness, let us recall a formal definition of extreme points. This particular definition is being used later in our proofs.

**Definition 9.** Let \( K \) be a convex set. A point \( x \in K \) is an extreme point (or vertex) of \( K \) if there is no way to express \( x \) as a convex combination \( \lambda y + (1 - \lambda)z \) such that \( y, z \in K \) and \( 0 \leq \lambda \leq 1 \), except for taking \( y = z = x \).

If the structure of \( C(v) \) is too difficult to describe and it is bounded from either above or from below, we introduce the lower and the upper game.
Definition 10. (The lower and the upper game of a set of $C$-extensions) Let $(N,K,v)$ be a $C$-extendable incomplete game. If $C(v)$ is bounded then the lower game $(N,v)$ and the upper game $(N,\overline{v})$ of $C(v)$ are complete games such that for every $(N,w) \in C(v)$ and for every $S \subseteq N$, we have

$$v(S) \leq w(S) \leq \overline{v}(S)$$

and for every $S \subseteq N$, there are $(N,w_1), (N,w_2) \in C(v)$ such that

$$v(S) = w_1(S) \text{ and } \overline{v}(S) = w_2(S).$$

It is important to note that if $C(v)$ is bounded only from above and not from below, the lower game does not exist. The analogous holds for the case of being bounded only from below. These games delimit the area of $\mathbb{R}^2$ that contains the set of $C$-extensions. Even if we know the description of $C(v)$, the lower and the upper game are still useful as they encapsulate the range of possible profits of coalition $S$ across all possible $C$-extensions by the interval $[v(S), \overline{v}(S)]$.

We remark that it is important to distinguish between lower and upper games of different sets of extensions. For example, lower games of superadditive and convex extensions do not coincide in general. There are also examples of incomplete games where the set of convex extensions might be empty and therefore, the lower game of convex extensions might not exist. However, the same incomplete game can have the lower game of superadditive extensions.

As we already mentioned, we are interested in the property of symmetry in games. The following is the generalisation of this property to incomplete games.

Definition 11. An incomplete game $(N,K,v)$ is symmetric if for every $K_1, K_2 \in K$ such that $|K_1| = |K_2|$, the equality $v(K_1) = v(K_2)$ holds.

3. Positive extensions

In [29], Masuya and Inuiguchi studied $P^n$-extensions of incomplete games with special structure, namely $(N,K,v)$ with $K = \{\emptyset, N\} \cup \{\{i\} \mid i \in N\}$ and $v(S) \geq 0$ for $S \in K$. In our text, we refer to these games as non-negative minimal incomplete games. In their work, as a consequence of an approach slightly different from ours, they do not consider the question of $P^n$-extendability. To characterise $P^n$-extendability of non-negative minimal incomplete games, we denote $\Delta := v(N) - \sum_{i \in N} v(i)$ and $N_1 := \{T \subseteq N \mid |T| > 1\}$.

Theorem 2. Let $(N,K,v)$ be a non-negative minimal incomplete game. It is $P^n$-extendable if and only if $\Delta \geq 0$.

Proof. If $\Delta \geq 0$, it immediately follows that game $(N,w^*)$ defined using its dividends as

$$d_{w^*}(S) := \begin{cases} v(i) & \text{if } S = \{i\}, \\ \Delta & \text{if } S = N, \\ 0 & \text{otherwise}, \end{cases}$$

and for every $S \subseteq N$, there are $(N,w_1), (N,w_2) \in C(v)$ such that

$$v(S) = w_1(S) \text{ and } \overline{v}(S) = w_2(S).$$
is $P^n(v)$-extension. If $\Delta < 0$, it follows for any $P^n(v)$-extension $(N, w)$ that

$$\Delta = \sum_{\emptyset \neq S \subseteq N} d_w(S) - \sum_{i \in N} \delta_w(i) = \sum_{S \in N_1} d_w(S) < 0.$$  

As $d_w(S) \geq 0$ for every $S \in N_1$, this leads to a contradiction. □

In [29], they showed the lower and the upper game of $P^n$-extensions coincide with those of $S^n$-extensions. Games $(N, v)$ and $(N, \bar{v})$ are defined as

$$v(S) := \begin{cases} 0, & \text{if } S = \emptyset, \\ \Delta + \sum_{i \in S} v(i), & \text{if } S \notin K, \text{ and } T \subseteq S, \\ \sum_{i \in S} v(i), & \text{if } S \notin K, \text{ and } T \subset S. \end{cases}$$  

(2)

The set of $P^n$-extensions of non-negative minimal incomplete games is described in [29] as a set of all convex combinations of its extreme points. Those correspond to games $(N, v^T)$, parametrised by coalitions $\emptyset \neq T \subseteq N$,

$$v^T(S) = \begin{cases} 0, & S = \emptyset, \\ \Delta + \sum_{i \in S} v(i), & S \notin K \text{ and } T \subseteq S, \\ \sum_{i \in S} v(i), & S \notin K \text{ and } T \subset S. \end{cases}$$  

(3)

Notice similarity between games $(N, v^T)$ and unanimity games $(N, u_T)$.

**Theorem 3.** [29] Let $(N, K, v)$ be a non-negative minimal incomplete game and let $(N, v^T)$ for $T \in N_1$ be games from (3). The set of $P^n$-extensions can be expressed as

$$P^n(v) = \left\{ \sum_{T \in N_1} \alpha_T v^T \mid \sum_{T \in N_1} \alpha_T = 1, \alpha_T \geq 0 \right\}. $$  

(4)

In this section, we generalise the discussed results to general setting. In Subsection 3.1, we provide a characterisation of $P^n$-extendability based on duality of linear programming and give an example of its application in a time-complexity analysis of $P^n$-extendability of incomplete games with special structures. We also give a sufficient and necessary condition for boundedness of $P^n(v)$. In Subsection 3.2, we investigate a description of the set of $P^n(v)$-extensions if the set is bounded. We do so by characterising extreme games of the set by following (and slightly modifying) the proof of the sharp form of Bondareva-Shapley theorem (the theorem was introduced independently by Bondareva in 1963 [10] and Shapley in 1967 [36]).

3.1. $P^n$-extendability and boundedness of $P^n$

To provide a certificate for non-$P^n$-extendability of an incomplete game, we employ duality of linear systems. This approach was motivated by a work by Seshadhri and Vondrak [34] and the so called path certificate for non-extendability of submodular functions (corresponding to convex games). Although its size is
exponential in the number of players in general, for special cases, the solvability of the dual system is polynomial in $n$ and therefore, the $P^n$-extendability is polynomially decidable in $n$ for such cases. In the proof of the characterisation, we use the seminal result of Farkas\textsuperscript{17}.

Lemma 1. (Farkas’ lemma, \textsuperscript{17}) Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$. Then exactly one of the following two statements is true.

1. There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.
2. There exists $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y \leq -1$.

Theorem 4. Let $(N, K, v)$ be an incomplete game. The game is $P^n$-extendable if and only if the following system of linear equations is not solvable:

1. $\forall T \subseteq N, T \neq \emptyset : \sum_{S \in K, T \subseteq S} y(S) \geq 0$,
2. $\sum_{S \in K} v(S) y(S) \leq -1$.

Proof. Let $M := 2^n - 1$ and $U' \in \mathbb{R}^{M \times M}$ be a matrix with characteristic vectors of unanimity games $u_T$ as its columns. Then it holds that $U'd = w$ for every game $(N, w)$ and its vector of Harsanyi dividends $d$.

For an incomplete cooperative game $(N, K, v)$, we reduce matrix $U'$ by deleting the rows corresponding to coalitions with unknown values, reaching a system $Ud = v$. This adjustment eliminates unknowns on the right hand side of the equation, yet no information about the complete game is lost since the vector of Harsanyi dividends carries full information.

Game $(N, K, v)$ is $P^n$-extendable if and only if $Ud = v$ is solvable for $d \geq 0$. By Farkas’ lemma (Lemma\textsuperscript{17}) this happens if and only if the following system has no solution,

$$U^T y \geq 0 \text{ and } v^T y \leq -1. \hspace{1cm} (5)$$

The conditions given by (5) correspond to those from the statement of the theorem.\hfill $\Box$

Notice that even though the number of inequalities $\sum_{S \in K, T \subseteq S} y(S) \geq 0$ is $2^n - 1$ (since we have one inequality for every $\emptyset \neq T \subseteq N$), the actual number of distinct inequalities is not larger than $2^{|K|} - 1$ because each inequality sums over a subset of $K$. Depending on the structure of $K$, the actual number might be even smaller as is shown in the following result.

Theorem 5. Let $(N, K, v)$ be an incomplete game such that sizes of all $S \in K$ are bounded by a fixed constant $c$. Then the problem of $P^n$-extendability is polynomially-time solvable in $n$.

Proof. In $(N, K, v)$, the number of coalitions with a defined value is at most $\sum_{i=1}^c \binom{n}{i}$, which is a polynomial in $n$. Also, if we consider the linear system from Theorem\textsuperscript{4} every $T \subseteq N$ such that $|T| > c$ yields an empty sum in its corresponding inequality. Therefore, the number of unique conditions in the
problem is bounded by the number of coalitions with defined value, that is by the sum $\sum_{i=1}^{n} \binom{n}{i}$. We conclude that the linear system can be solved in polynomial time by means of linear programming.

Now we address the question of boundedness of $P^n(v)$. Notice, the set of $P^n$-extensions is always bounded from below, as for every $P^n$-extension $(N, w)$, $w(S) = \sum_{\emptyset \neq T \subseteq S} d_w(T)$ and $d_w(T) \geq 0$ for every $\emptyset \neq T \subseteq N$. Therefore, 0 serves as a lower bound the value of any coalition $S \subseteq N$. To find the lower bound that is binding the profit of every coalition as well as the binding upper bound (hence the lower and the upper game) remains an open problem.

**Theorem 6.** Let $(N, K, v)$ be a $P^n$-extendable incomplete game. The set of positive extensions $P^n(v)$ is bounded if and only if $N \in K$.

**Proof.** If $N \in K$, then for any $P^n(v)$-extension $(N, w)$, $\sum_{T \subseteq N} d_w(T) = v(N)$, and since for all $\emptyset \neq T \subseteq N$, $d_w(T) \geq 0$, it follows $d_w(T) \in [0, v(N)]$. This yields a bound (possibly an overestimation) for all possible values of $d_w(T)$. Since the dividends are bounded, the set $P^n(v)$ is also bounded.

If $N \notin K$, then the value of coalition $N$ can be arbitrarily large, since there is no upper bound on $d_w(N)$ for a $P^n(v)$-extension $(N, w)$. Thus, $P^n(v)$ is not bounded. □

3.2. Description of the set of $P^n$-extensions

For an incomplete game $(N, K, v)$, the set of $P^n(v)$-extensions can be described as

$$P^n(v) = \left\{ (N, w) \mid \forall S \in K : w(S) = v(S) \text{ and } \forall T \subseteq N : d_w(T) \geq 0 \right\},$$

or equivalently in terms of dividends and $M := 2^n - 1$ as

$$P^n_d(v) := \left\{ d_w \in \mathbb{R}^M \mid \forall S \in K : \sum_{T \subseteq S} d_w(T) = v(S), \forall T \subseteq N : d_w(T) \geq 0 \right\}.$$

Notice that $P^n(v) \neq P^n_d(v)$ since the former is a set of cooperative games and the latter is a set of vectors of dividends.

Both sets are formed by intersections of closed half-space, thus they are (convex) polyhedrons. If we suppose that $(N, K, v)$ is $P^n$-extendable, then both sets are nonempty. Furthermore, the sets are bounded if and only if $N \in K$. Bounded convex polyhedrons are convex hulls of their extreme points.

To be able to freely neglect the distinction between extreme points of both sets, we recall a basic result from linear algebra. For a sake of completeness, we include the proof.

**Lemma 2.** Let $P$ be a convex subset of $\mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ a nonsingular matrix, and $x \in P$ an extreme point of $P$. Then $Ax$ is an extreme point of the convex set $A(P) := \{ Au \mid u \in P \}$. 10
Proof. Suppose that \( x \in P \) is an extreme point of \( P \) and the image \( Ax \) is not an extreme point of \( A(P) \). Therefore, there are \( Au, Av \in A(P) \) and \( \alpha \in (0, 1) \) such that \( \alpha Au + (1-\alpha)Av = Ax \). But then \( \alpha Au + (1-\alpha)Av = A(\alpha u + (1-\alpha)v) = Ax \), and therefore, \( x \) is not an extreme point of \( P \), as it is a nontrivial convex combination of \( u, v \in P \). This is a contradiction.

Let \( U \in \mathbb{R}^{M \times M} \) be a matrix with vectors of unanimity games \( u_T \in \mathbb{R}^M \) as columns. It holds that \( Ud_w = w \) where \( w \in \mathbb{R}^M \) is a characteristic vector of game \( (N, w) \) and \( d_w \in \mathbb{R}^M \) represents a vector of Harsanyi dividends of the game. Since unanimity games form a basis of \( \mathbb{R}^M \), the matrix \( U \) is nonsingular and thus, by Lemma 2, the extreme points of \( P_n^d(v) \) correspond to those of \( P^d_n(v) \), allowing us to further consider those instead of the former ones.

Following the proof of the sharp form of Bondareva-Shapley theorem from [32], we give an insight into the description of extreme games of \( P_n^d(v) \). We show that for these games, the set of coalitions zero dividends is inclusion-wise maximal.

Our result is based on the following characterisation of extreme points of polyhedrons.

Lemma 3. [32] Let \( P \) be a polyhedron given by

\[
P := \left\{ x \in \mathbb{R}^k \left| \sum_{j=1}^{k} a_{tj} x_j \geq b_t, t = 1, \ldots, m \right. \right\}.
\]

For \( x \in P \), let \( S(x) := \{ t \in \{1, \ldots, m\} | \sum_{j=1}^{k} a_{tj} x_j = b_t \} \). The point \( x \in P \) is an extreme point of \( P \) if and only if the system of linear equations

\[
\sum_{j=1}^{k} a_{tj} y_j = b_t \text{ for all } t \in S(x)
\]

has \( x \) as its unique solution.

Applying Lemma [3], \( d_e \in \mathbb{R}^M \) is an extreme game of \( P^d_n(v) \) if and only if there is no \( d_x \neq d_e \) such that \( d_w(T) = 0 \iff d_x(T) = 0 \) for every nonempty \( T \subseteq N \). For any \( P^n(v) \)-extension \( (N, w) \), we denote by \( E(w) \) the set of negligible coalitions defined as \( E(w) := \{ T \subseteq N | d_w(T) = 0 \} \). This set proves itself useful in the following lemma. The lemma states that inclusion-maximality of \( E(e) \) across \( E(x) \) for \( d_x \in P^d_n(v) \) is equivalent with uniqueness of \( E(e) \) across \( E(x) \) for \( d_x \in P^d_n(v) \). Together with Lemma 3, this connects the extremality of games with the inclusion-maximality of sets \( E(e) \).

Lemma 4. Let \( (N, K, v) \) be a \( P^n \)-extendable incomplete game and \( d_e \in P^d_n(v) \). Then the following are equivalent:

1. there is no \( d_x \in P^d_n(v) \) such that \( E(e) \subsetneq E(x) \),
2. there is no \( d_y \in P^d_n(v) \) different from \( d_e \), such that \( E(e) = E(y) \).
PROOF. First, suppose that there is \( d_x \in P_d^n(v) \) such that \( E(e) \subseteq E(x) \). We show that there is not only one, but infinitely many vectors \( d_y \in P_d(v) \) different from \( d_x \) such that \( E(e) = E(y) \). The idea is to take any non-trivial convex combination \( d_y := \alpha d_x + (1 - \alpha)d_z \) for \( 0 < \alpha < 1 \). Such game is clearly positive (a convex combination of non-negative dividends remains non-negative) as it is also an extension of \((N, K, v)\), because for every \( S \in K \),

\[
\sum_{T \subseteq S} d_y(T) = \alpha \sum_{T \subseteq S} d_x(T) + (1 - \alpha) \sum_{T \subseteq S} d_z(T) = \alpha v(S) + (1 - \alpha)v(S) = v(S).
\]

And since \( d_x \neq d_z \), there is \( S \notin K \) such that \( x(S) \neq e(S) \) for which

\[
y^\alpha(S) = \sum_{T \subseteq S} d_y(T) = \alpha \sum_{T \subseteq S} d_x(T) + (1 - \alpha) \sum_{T \subseteq S} d_z(T) = \alpha x(S) + (1 - \alpha)e(S).
\]

Therefore, any two parameters \( \alpha_1, \alpha_2 \) such that \( 0 < \alpha_1 < \alpha_2 < 1 \) yield different values \( y^{\alpha_1}(S) \neq y^{\alpha_2}(S) \), thus \( d_y^{\alpha_1} \neq d_y^{\alpha_2} \).

Now suppose that there is \( d_y \in P_d^n(v) \) different from \( d_x \) such that \( E(e) = E(y) \). We take a combination \( d_z = d_x - \beta(d_y - d_e) \) with \( \beta \) such that for at least one \( S \notin E(e), d_z(S) = 0 \). Thus \( E(e) \subseteq E(z) \) and still, \( d_z \in P_d^n(v) \). For such \( S \), it must hold

\[
d_z(S) = d_e(S) - \beta (d_y(S) - d_e(S)) = 0,
\]

therefore \( \beta = \frac{d_e(S)}{d_y(S) - d_e(S)} \). We have to choose \( S \) such that \( d_y(S) \neq d_e(S) \). Furthermore, we have to secure that for every \( T \notin E(e), d_z(T) \geq 0 \), or equivalently

\[
d_z(T) = d_e(T) - \beta (d_y(T) - d_e(T))
= d_e(T) - \frac{d_e(S)}{d_y(S) - d_e(S)} \cdot (d_y(T) - d_e(T)) \geq 0
\]

This can be done by taking the minimum for \( S \) over all such coalitions \( T \), i.e.

\[
\beta := \min_{T \notin E(e); d_z(T) \neq d_e(T)} \frac{d_e(T)}{d_y(T) - d_e(T)}.
\]

Then for \( T \notin E(e), d_z(T) \geq 0 \), since it is equal to

\[
d_e(T) - \frac{d_e(S)}{d_y(S) - d_e(S)} (d_y(T) - d_e(T)) \geq d_e(T) - \frac{d_e(T)}{d_y(T) - d_e(T)} (d_y(T) - d_e(T)).
\]

Clearly, the last expression is equal to zero. Finally, for \( K \in K \),

\[
z(K) = \sum_{C \subseteq K} d_z(K) = \sum_{C \subseteq K} d_e(K) - \beta \left( \sum_{C \subseteq K} d_y(K) - \sum_{C \subseteq K} d_e(K) \right),
\]

and since all the three sums in the last expression are equal to \( v(K) \), we conclude that \( z(K) = v(K) \) and thus, \( d_z \in P_d^n(v) \). \( \square \)
The following characterisation of extreme points follows as a direct application of Lemma 3 and 4.

**Theorem 7.** For $P^n$-extendable incomplete game $(N, K, v)$, it holds $(N, e)$ is an extreme game of $P^n(v)$ if and only if its set of negligible coalitions $E(e)$ is inclusion-maximal, i.e. there is no $(N, w) \in P^n(v)$ such that $E(e) \subseteq E(w)$.

4. Application to analysis of positive extensions for special cases

This section contains an analysis of $P^n$-extensions of several classes of incomplete games. We show a direct application of Theorem 7 to the description of the set of $P^n$-extensions for three classes of incomplete games. We do not show only a derivation of extreme games but also a derivation of the lower and the upper game together with a characterisation of $P^n$-extendability.

4.1. Pairwise disjoint coalitions of known values

For the first class of incomplete games it holds that the coalitions with known values (excluding $N$) are pairwise-disjoint.

**Theorem 8.** Let $(N, K, v)$ be a $P^n$-extendable incomplete game, where $K = \{S_1, \ldots, S_{k-1}, N\}$ and for all $i, j \in \{1, \ldots, k-1\}$, it holds that $S_i \cap S_j = \emptyset$. Then the extreme games $v^T$, the lower game $v_L$, and the upper game $v_U$ can be described as follows:

$$v^T(S) := \begin{cases} 0, & \text{if } \nexists T \in K : T \subseteq S, \\ \sum_{i : T_i \subseteq S} v(S_i), & \text{if } \exists T \in K : T \subseteq S \text{ and } T_N \nsubseteq S \\ v(N) - \sum_{i : T_i \nsubseteq S} v(S_i), & \text{if } \exists T \in K : T \subseteq S \text{ and } T_N \subseteq S, \end{cases}$$

$$v(L)(S) := v^K(S) = \begin{cases} 0, & \text{if } \nexists T \in K : T \subseteq S, \\ \sum_{i : S_i \subseteq S} v(S_i), & \text{if } \exists T \in K : T \subseteq S \text{ and } N \neq S, \\ v(N), & \text{if } \exists T \in K : T \subseteq S \text{ and } N = S, \end{cases}$$

$$v(U)(S) := \begin{cases} v(S_i), & \text{if } S \subseteq S_i, \\ v(N) - \sum_{i : S_i \nsubseteq S} v(S_i), & \text{otherwise}, \end{cases}$$

where $T := \{T_1, \ldots, T_{k-1}, T_N\}$ such that $T_i \subseteq S_i$, $T_N \subseteq N$ and $T_N \nsubseteq S_i$ for any $i \in \{1, \ldots, k-1\}$. Furthermore, the $P^n$-extendability of $(N, K, v)$ is characterised by a condition

$$v(N) \geq \sum_{i=1}^{k-1} v(S_i).$$

**Proof.** Let $(N, K, v)$ be an incomplete game with the properties above. For any $P^n(v)$-extension $(N, w)$, from the fact that the coalitions in $K \setminus \{N\}$ are disjoint, at least one subcoalition $T_i$ of each coalition $S_i \in K \setminus \{N\}$ must have a nonzero
Let $K$ be a set of known values.

**Theorem 9.** Let $(N, K, v)$ be a $P^n$-extendable incomplete game such that $N \in K$ and for every $S \in K \setminus \{N\}, T \subseteq S \implies T \in K$. Furthermore, for $S \in K$, let $\delta_S$ be defined as $\delta_S = v(\{i\})$ and $\delta_S = v(S) - \sum_{T \subseteq S} \delta_T$. Then the extreme games $v^C$, the lower game $v$, and the upper game $\bar{v}$ can be described as follows:

$$v^C(S) := \begin{cases} \delta_N + \sum_{T \in K, T \subseteq S} \delta_T, & \text{if } C \subseteq S, \\ \sum_{T \in K, T \subseteq S} \delta_T, & \text{otherwise}, \end{cases}$$

for $C \notin K \setminus \{N\}$, and

$$v(S) := \begin{cases} \delta_N + \sum_{T \in K, T \subseteq S} \delta_T, & \text{if } S = N, \\ \sum_{T \in K, T \subseteq S} \delta_T, & \text{otherwise}, \end{cases}$$

and

$$\bar{v}(S) := \sum_{T \subseteq S} v(T).$$
\[ \pi(S) := \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ v^S(S), & \text{otherwise.} \end{cases} \]

Furthermore, \((N, \mathcal{K}, v)\) is \(P^n\)-extendable if and only if \(\delta_S \geq 0\) for all \(S \in \mathcal{K}\).

**Proof.** Let \((N, w) \in P^n(v)\). Thanks to the structure of \(\mathcal{K}\), the dividends \(d_w(S)\) for \(S \in \mathcal{K} \setminus \{N\}\) are the same for any \((N, w) P^n(v)\) and they are equal to \(\delta_S\). As a consequence, for any \(S\) such that \(\delta_S = 0\) it holds \(S \in E(w)\) and this holds for any \(P^n(v)\)-extension. Now if the uniquely defined value \(\delta_N = v(N) - \sum_{S \in \mathcal{K} \setminus \{N\}} \delta_S > 0\), there has to be at least one \(C \notin \mathcal{K} \setminus \{N\}\) such that its dividend \(d_w(C) \neq 0\). By Theorem 7, following a similar argument as in the proof of the previous theorem, \(E(w)\) is maximal if and only if there is only one such \(C\), otherwise if there are \(C_1 \neq C_2\) such that \(d_w(C_1) \neq 0\) and \(d_w(C_2) \neq 0\), by taking \((N, x) \in P^n(v)\) such that \(d_x(C_1) = 0\), \(d_x(C_2) = d_w(C_1) + d_w(C_2)\) we arrive into contradiction with maximality, since \(E(w) \subset E(x)\). Thus choosing \((N, w) \in P^n(v)\), such that \(d_w(C) = \delta_n\) yields an extreme game \(v^C\) of \(P^n(v)\) for any \(C \notin \mathcal{K} \setminus \{N\}\).

For any coalition \(S\), its value in any \(P^n(v)\)-extension has to be larger or equal to \(\sum_{T \in \mathcal{K}, T \subseteq S} \delta_T\). Notice that \(v^N(S)\) is equal to this number for any \(S\), thus being the lower game.

For any coalition \(S\), its maximal value is either \(v(S)\) if \(S \in \mathcal{K}\), or at most \(v(N) - \sum_{T \in \mathcal{K} \setminus \{N\}, T \subseteq S} \delta_T = \delta_N + \sum_{T \in \mathcal{K}, T \subseteq S} \delta_T\), which is equal to \(v^S(S)\) and thus it is the upper game.

For both studied classes of incomplete games, it holds \(v(S) \in P^n(v)\). Also notice that the number of extreme games \(v^C\) equals the number of coalitions \(C\) such that \(C \notin \mathcal{K} \setminus \{N\}\), that is \(2^n - |\mathcal{K}| + 1\) if \(v(N) - \sum_{S \in \mathcal{K} \setminus \{N\}} \delta_S > 0\), otherwise \(P^n(v)\) contains precisely one game (in case \(v(N) - \sum_{S \in \mathcal{K} \setminus \{N\}} \delta_S = 0\) or no game at all (if \(v(N) - \sum_{S \in \mathcal{K} \setminus \{N\}} \delta_S < 0\).

4.3. Symmetric positive extensions

We denote the set of symmetric positive extensions of \((N, \mathcal{K}, v)\) by \(P^*_n(v)\). Analogously to study of \(C^*_n(v)\), we make use of the reduced forms \((N, s)\) and \((N, \mathcal{K}, \sigma)\) of games \((N, v)\) and \((N, \mathcal{K}, v)\), respectively, which are defined in Definition 2. We can easily obtain the following result as a corollary of Theorem 9.

**Theorem 10.** Let \((N, \mathcal{A}, \sigma)\) be the reduced form of a symmetric incomplete game such that \(n \in \mathcal{A}\) and \(i \in N, i \leq k \implies i \in \mathcal{A}\). Then the lower game and the upper game of \(P^*_n(v)\) can be described as

\[ s(i) := \begin{cases} s(i), & \text{for } i \in \mathcal{A}, \\ s(k), & \text{otherwise}, \end{cases} \quad \text{and} \quad \sigma(i) := \begin{cases} s(i), & \text{for } i \in \mathcal{K}, \\ s(n), & \text{otherwise}. \end{cases} \]

The following game illustrates that even in the symmetric scenario, there is \((N, \mathcal{A}, \sigma)\) such that \((N, \mathcal{A}, \sigma) \notin P^*_n(v)\).
Example 1. (The lower game is not necessarily a $P_σ^4$-extension) Let $(N, X, \sigma)$ be the reduced form of a symmetric 4-person incomplete game such that $X = \{2, 4\}$. From the properties of symmetric positive games we know that any $(N, s) \in P_σ^4(v)$ is given by 4 non-negative dividends with corresponding values $d_1, d_2, d_3, d_4$ such that

- $s(1) = d_1$,
- $s(2) = 2d_1 + d_2$,
- $s(3) = 3d_1 + 3d_2 + 3d_3$,
- $s(4) = d_4 + 4d_3 + 6d_2 + 4d_1$.

By setting $d_1 := 0, d_2 := \sigma(2), d_3 := 0,$ and $d_4 := \sigma(4) - 6d_2$ we get a $P_σ^4$-extension where $s(1) = 0$ (clearly the minimum) and it is achieved if and only if $d_1 = 0$. Setting $d_1 = 0$ yields $s(3) = 3\sigma(2)$. However, to minimize $s(3)$, we can choose $d_1 := \frac{3\sigma(2)}{2}, d_2 := 0, d_3 := 0,$ and $d_4 := \sigma(4) - 4d_1$, obtaining $s(3) = 3d_1 = \frac{3}{2}\sigma(2)$. We cannot minimize both values simultaneously and thus $(N, s) \notin P_σ^4(v)$.

It is not difficult to generalise this example for symmetric $n$-person games. For similar reasons, even the lower game of (non-symmetric) $P^n$-extensions of non-symmetric incomplete games is not contained in $P^n(v)$. This is contrary to what we showed for the classes of incomplete games in Theorem 8 and 9.

5. Convex extensions

For non-negative incomplete games with minimal information, sets of $S^n$-extensions and $P^n$-extensions are described in [29]. For the sake of completeness, in this section we derive similar results for the set of $C^n$-extensions.

Theorem 11. Let $(N, K, v)$ be a non-negative incomplete game with minimal information. It is $C^n$-extendable if and only if $\Delta \geq 0$.

Proof. If $\Delta \geq 0$, it immediately follows that game $(N, w^*)$ defined using its dividends as

$$d_w(S) := \begin{cases} v(i) & \text{if } S = \{i\}, \\ \Delta & \text{if } S = N, \\ 0 & \text{otherwise}, \end{cases}$$

is $C^n$-extension. If $\Delta < 0$, it follows $v(N) < \sum_{i \in N} v(i)$, thus any extension of $(N, K, v)$ cannot be convex. $\square$

In [29], they showed the lower and the upper game of $P^n$-extensions coincide with those of $S^n$-extensions, thus they must coincide with the lower and the upper game of $C^n$-extensions as well (see [2]). Finally, we derive a description of the set of $C^n$-extensions. We employ $N_1 := \{T \subseteq N \mid |T| > 1\}$.
Theorem 12. Let $(N,K,v)$ be a non-negative incomplete game with minimal information, and let $(N,v^T)$ for $T \in N_1$ be games from $[5]$. The set of $C^n$-extension can be expressed as
\[
C^n(v) = \left\{ \sum_{T \in N_1} \alpha_T v^T \mid \sum_{T \in N_1} \alpha_T = 1, \forall S_1,S_2 \subseteq N : \sum_{T \in E(S_1,S_2)} \alpha_T \geq 0 \right\},
\]
where $E(S_1,S_2) := \{ T \subseteq S_1 \cup S_2 \mid T \not\subseteq S_1 \text{ and } T \not\subseteq S_2 \}$.

Proof. The proof follows from the proof of Theorem 6 in [29]. The only difference is in the condition for coefficients $\alpha_T$. For the description of the set of $S^n$-extensions, a condition $\sum_{T \in E(S_1,S_2)} \alpha_T \geq 0$ for every pair of conditions $S_1 \cap S_2 = \emptyset$ is enforced. This condition corresponds to the fact that for $S_1,S_2 \subseteq N$ such that $S_1 \cap S_2 = \emptyset$, it holds $v(S_1) + v(S_2) \leq v(S_1 \cup S_2)$. In terms of Harsanyi dividends, it is equivalent to $\sum_{T \in E(S_1,S_2)} \delta_v(T) \geq 0$. For convex games and $S_1,S_2 \subseteq N$ (not necessarily disjoint coalitions), the conditions $v(S_1) + v(S_2) \leq v(S_1 \cap S_2) + v(S_1 \cup S_2)$ can be equivalently expressed in terms of Harsanyi dividends as
\[
\sum_{T \subseteq S_1 \cup S_2, T \not\subseteq S_1, T \not\subseteq S_2} \alpha_T \geq 0.
\]
Notice that coalitions $T$ are exactly those from the set $E(S_1,S_2)$. \qed

In our attempt, to derive similar results for a more general setting, we surveyed existing results regarding submodular set functions (recall a set function $v : 2^N \to \mathbb{R}$ is submodular if and only if $-v$ is supermodular).

The study of extendability of submodular functions initiated Seshadhri and Vondrák in [34]. They introduced path certificate, a combinatorial structure whose existence certifies that a submodular function is not extendable. They also showed an example of a partial function defined on almost all coalitions that is not extendable, but by removing a value for any coalition, the game becomes extendable. Later in 2018, Bhaskar and Kumar [5] studied extendability of several classes of set functions, including submodular functions. Inspired by the results of Seshadhri and Vondrák, they introduced a more natural combinatorial certificate of non-extendability — square certificate. Using this concept, they were able to show that a submodular function is extendable on the entire domain if and only if it is extendable on the lattice closure of the sets with defined values. The lattice closure $LC(K)$ of a set of points $K \subseteq 2^N$ in a partially ordered set $(2^N, \subseteq)$ is the inclusion-minimal subset of $2^N$ that contains $K$ and that is closed under the operation of union and intersection of sets. Following is a modification of Theorem 7 from [5].

Theorem 13. Let $(N,K,v)$ be an incomplete cooperative game and
\[
\mathcal{F} := LC(K) \cap \{ S \subseteq N \mid S \subseteq \overline{S}, S \subseteq \overline{S} \}.
\]
Incomplete game $(N,K,v)$ is $C^n$-extendable if and only if there is supermodular $w : 2^\mathcal{F} \to \mathbb{R}$ such that $w(S) = v(S)$ for $S \in K$.\

17
In 2019, the same authors showed that the problem of extendability for a subclass of submodular functions, so called coverage functions (see [4]), is NP-complete. Thus, the question of \( C^n \)-extendability is in general NP-complete as well.

The rest of questions concerning \( C^n \)-extensions of general incomplete games remain open problems. From now on, we focus on extensions that are both convex and symmetric. The reasons are twofold. First, symmetry yields a simpler analysis of the set of \( C^n \)-extensions. Second, symmetric \( C^n \)-extensions form an important subset of \( C^n \)-extensions and may be considered as an approximation of the set.

5.1. Symmetric convex extensions

Since the set of convex extensions seems to be quite hard to describe in its full generality, we focus on a subset of \( C^n \)-extensions that are symmetric and denote this set by \( C^n_\sigma \). The additional property of symmetry yields compact (and by our opinion elegant) descriptions of the set of \( C^n_\sigma \)-extensions and since \( C^n_\sigma (v) \subseteq C^n (v) \) for symmetric incomplete games, the set of symmetric convex extensions may be regarded as an approximation of set \( C^n (v) \).

The main ingredient for our results is the following characterisation of symmetric convex games. For completeness, the proof of this folklore result is provided in appendix.

**Proposition 1.** Let \((N,v)\) be a symmetric cooperative game. Then for every \( S \subseteq N \setminus j \) and \( i \in S \), it holds that

\[
v(S) \leq \frac{v(S \setminus i) + v(S \cup j)}{2}
\]

if and only if the game is convex.

We note that the characterisation from Proposition 1 does not hold for general convex games. This can be seen in the following example.

**Example 2.** (A convex game not satisfying conditions from Proposition 1) The game \((N,v)\) given in Table 2 is convex, as can be easily checked. However, the inequality

\[
v(\{1, 3\}) \leq \frac{v(\{1\}) + v(\{1, 2, 3\})}{2}
\]

is not satisfied, as \( 6 \not\geq \frac{1+9}{2} \).

| \( S \) | \{1\} | \{2\} | \{3\} | \{1, 2\} | \{1, 3\} | \{2, 3\} | \{1, 2, 3\} |
|---|---|---|---|---|---|---|---|
| \( v(S) \) | 1 | 1 | 1 | 4 | 6 | 4 | 9 |

Table 1: The game \((N,v)\) from Example 2 with its characteristic function given in the table.
For symmetric games, we can denote by $s(k)$ the value of $v(S)$ of any $S \subseteq N$ such that $|S| = k$. This allows us to formulate the following characterisation of symmetric convex games.

**Theorem 14.** A game $(N, v)$ is symmetric convex if and only if for all $k \in \{1, \ldots, n - 1\}$,

$$s(k) \leq \frac{s(k - 1) + s(k + 1)}{2}. \tag{8}$$

Hence we can associate every symmetric convex game $(N, v)$ with a function $s: \{0, \ldots, n\} \rightarrow \mathbb{R}$ having the above property. Similarly, we can apply this to $(N, K, v)$ with a function $\sigma: X \rightarrow \mathbb{R}$ where $X \subseteq \{0, \ldots, n\}$ is constructed from $K$. To formalise these constructions, we define reduced forms of games $(N, v)$ and $(N, K, v)$.

**Definition 12.** Let $(N, v)$ be a symmetric game and $(N, K, v)$ a symmetric incomplete game.

- The reduced form of a game $(N, v)$ is an ordered pair $(N, s)$, where the function $s: \{0, \ldots, n\} \rightarrow \mathbb{R}$ is a reduced characteristic function such that $s(k) := v(S)$ for any $S \subseteq N$ with $|S| = k$.

- The reduced form of an incomplete game $(N, K, v)$ is a tuple $(N, X, \sigma)$ where $X = \{i | i \in \{0, \ldots, n\}, \exists S \in K : |S| = i\}$ and the function $\sigma: X \rightarrow \mathbb{R}$ is defined as $\sigma(k) := v(S)$ for any $S \in K$ such that $|S| = k$.

We also call $(N, s)$ and $(N, X, \sigma)$ the reduced game and the reduced incomplete game, respectively.

Since $\emptyset$ always belongs to $K$, for every reduced incomplete game $(N, X, \sigma)$, it also holds that $0 \in X$ and $\sigma(0) = 0$. When we consider a reduced game $(N, s)$ of a $C^*_\sigma(v)$-extension, we often denote this, for brevity, by $(N, s) \in C^*_\sigma(v)$. By $\overline{X}$, we denote the complement of $X$ in $\{0, \ldots, n\}$, i.e. $\overline{X} := \{0, \ldots, n\} \setminus X$.

Notice that a game $(N, v)$ is symmetric convex if and only if the function $s$ of its reduced form $(N, s)$ satisfies property (8) from Theorem 14.

We can visualize the reduced form $(N, s)$ of a symmetric convex game $(N, v)$ by a graph in $\mathbb{R}^2$. On the $x$-axis we put the coalition sizes and on the $y$-axis the values of $s$. The point $(0, 0)$ is fixed for all reduced games. Now by Theorem 14 the conditions for $k \in \{1, \ldots, n - 1\}$ enforce that for $i \in \{0, \ldots, n\}$, points $(i, s(i))$ lie in a convex position. More precisely, if we connect the neighbouring pairs $(i, s(i)), (i + 1, s(i + 1))$ (where $i \in \{0, \ldots, n - 1\}$) by line segments, we obtain a graph of a convex function. The graph is illustrated with an example in Figure 1. Further in this text, we refer to this function as the line chart of $(N, s)$. Similarly, for $(N, X, \sigma)$, the line chart is obtained by connecting consecutive elements from $X$ by line segments. If $n \in \overline{X}$, the rightmost line segment is extended to end at $x$-coordinate $n$. The values of $s$ are then set to lie on the union of these line segments.
5.1.1. $C^n_\sigma$-extendability

For an incomplete game in reduced form, i.e. $(N, \mathcal{X}, \sigma)$, the first question that arises is that of $C^n_\sigma$-extendability. For $\mathcal{X} = \{0, i\}$ with $i \in \{1, \ldots, n\}$, the game is always $C^n_\sigma$-extendable (a possible $C^n_\sigma$-extension is the one where the values of each coalition size lie on the line coming through $(0, \sigma(0))$ and $(i, \sigma(i))$). Therefore, in the following theorem, we consider $|\mathcal{X}| > 2$.

**Theorem 15.** Let $(N, \mathcal{X}, \sigma)$ be a reduced form of a symmetric incomplete game $(N, K, v)$ where $|\mathcal{X}| > 2$. The game is $C^n_\sigma$-extendable if and only if

$$\sigma(k_2) \leq \sigma(k_1) + (k_2 - k_1) \frac{\sigma(k_3) - \sigma(k_1)}{k_3 - k_1},$$

for all consecutive elements $k_1 < k_2 < k_3$ from $\mathcal{X}$.

**Proof.** If the game is $C^n_\sigma$-extendable, let $(N, s)$ be the reduced form of any of its $C^n_\sigma$-extension. By Theorem 14, the line chart of $(N, s)$ is a convex function that coincides with $\sigma$ on the values of $\mathcal{X}$. Therefore, for any consecutive elements $k_1, k_2, k_3$ from $\mathcal{X}$, the inequality must hold.

For the opposite implication, we construct a $C^n_\sigma(v)$-extension by setting the values of $s$ to lie on the line chart of $(N, \mathcal{X}, \sigma)$. The construction is illustrated in Figure 2.

Notice that $s(k) = \sigma(k)$ for $k \in \mathcal{X}$ and also, because the inequalities for consecutive elements $k_1, k_2, k_3$ from $\mathcal{X}$ hold, the line chart represents a convex function. Thus for all $k \in \{1, \ldots, n - 1\}$, it holds

$$s(k) \leq \frac{s(k - 1) + s(k + 1)}{2},$$

and by Theorem 14 the game $(N, s)$ is in $C^n_\sigma(v)$. \qed
Figure 2: The construction of a $C^\sigma_n$-extension of $(N, X, \sigma)$ where $X = \{x_1, x_2, x_3, x_4\}$, using the line chart of $(N, X, \sigma)$. The value $s(k)$ lies on the line segment connecting $(x_3, \sigma(x_3))$ and $(x_4, \sigma(x_4))$.

As a direct consequence of the previous theorem, the problem of $C^\sigma_n$-extendability of symmetric incomplete games can be decided in linear time with respect to the size of the original game (i.e. the size of the characteristic function).

5.1.2. The lower game and the upper game

The following proposition addresses the boundedness of the set of $C^\sigma_n$-extensions. The restriction to $|N| \geq 3$ is without loss of generality, because for $|N| \leq 2$, when the game $(N, X, \sigma)$ is not complete and is $C^\sigma_n$-extendable, the set of $C^\sigma_n$-extensions is always unbounded.

Proposition 2. Let $(N, X, \sigma)$ be the reduced form of a $C^\sigma_n$-extendable symmetric incomplete game $(N, \mathcal{K}, v)$ with $|N| \geq 3$. The $C^\sigma_n(v)$ is bounded if and only if $|\mathcal{X}| \geq 3$ and $n \in \mathcal{X}$.

Proof. Let $(N, X, \sigma)$ be the reduced form of a $C^\sigma_n$-extendable incomplete game. If $n \in \mathcal{X}$, clearly, from Theorem 14 there is no upper bound on the profit of $n$. Let $n \in \mathcal{X}$ and suppose for a contradiction that there is $k \in N$ such that there is no upper bound on its profit. Choose a $C^\sigma_n(v)$-extension $(N, s)$ such that $s(k) > k \sigma(n)/n$. The line chart of $(N, s)$ is not a convex function (the property is violated for $(0, s(0)), (k, s(k)), (n, s(n)))$, therefore $(N, s) \notin C^\sigma_n(v)$.

If $|\mathcal{X}| \leq 2$, then $\mathcal{X} = \{0, n\}$ (otherwise the set of $C^\sigma_n$-extensions is not bounded from above). Let $\ell$ be a negative value smaller than or equal to $\sigma(n)$. Any game $(N, s_\ell)$ with $s_\ell(k) = \ell$ for $k \in \{1, \ldots, n-1\}$ and $s_\ell(0) = \sigma(0), s_\ell(n) = \sigma(n)$ is a $C^\sigma_n(v)$-extension of $(N, \mathcal{X}, \sigma)$. Thus, there is no lower bound on values of $1, \ldots, n-1$.

If $|\mathcal{X}| \geq 3$, then let $i \in \mathcal{X} \setminus \{0, n\}$. For $k \in \{1, \ldots, i-1\}$, the point $(k, s(k))$ must lie on or above the line coming through points $(i, \sigma(i)), (n, \sigma(n))$, otherwise
the convexity of line chart of \((N,s)\) is violated, leading to a contradiction. Similarly, for any \(k \in \{i+1, \ldots, n-1\}\) the value \(s(k)\) must lie on or above the line coming through points \((0, \sigma(0)), (i, \sigma(i))\), otherwise the convexity is violated, again. The profit of every \(k\) is therefore bounded from below.

\[\text{Theorem 16.} \] Let \((N, \mathcal{X}, \sigma)\) be the reduced form of a \(C^n_{\sigma}\)-extendable symmetric incomplete game. Suppose that \(C^n_{\sigma}(v)\) is bounded. Furthermore, for every \(k \in \overline{\mathcal{X}}\), denote by \(i_1, i_2, j_1, j_2\) the closest distinct elements from \(\mathcal{X}\) such that it holds \(i_1 < i_2 < k < j_1 < j_2\), if they exist. Then the lower game has the following form:

\[
s(k) := \begin{cases} 
\sigma(k), & \text{if } k \in \mathcal{K}, \\
\sigma(i_1) + (k - i_1) \frac{\sigma(i_2) - \sigma(i_1)}{i_2 - i_1}, & \text{if } k \notin \mathcal{K} \text{ and } j_2 \text{ does not exist}, \\
\sigma(j_1) + (k - j_1) \frac{\sigma(j_2) - \sigma(j_1)}{j_2 - j_1}, & \text{if } k \notin \mathcal{K} \text{ and } i_1 \text{ does not exist}, \\
\max \left\{ \sigma(i_1) + (k - i_1) \frac{\sigma(i_2) - \sigma(i_1)}{i_2 - i_1}, \sigma(j_1) + (k - j_1) \frac{\sigma(j_2) - \sigma(j_1)}{j_2 - j_1} \right\}, & \text{if } k \notin \mathcal{K} \text{ and } i_1, i_2, j_1, j_2 \text{ exist}.
\end{cases}
\]

The upper game has the following form:

\[
\overline{s}(k) := \begin{cases} 
\sigma(k), & \text{if } k \in \mathcal{X}, \\
\overline{s}(i_2) + (k - i_2) \frac{\overline{s}(j_1) - \overline{s}(i_2)}{j_1 - i_2}, & \text{otherwise}.
\end{cases}
\]

Proof. To prove that \((N,s)\) is the lower game, we start by showing that for every \(C^n_{\sigma}\)-extension \((N,w)\) and every coalition size \(k \in N\), it holds that \(s(k) \leq w(k)\). If \(k \notin \mathcal{X}\), trivially \(s(k) = \sigma(k) = w(k)\). If \(k \notin \mathcal{X}\), then since any \(C^n_{\sigma}\)-extension must have a convex line chart, the value \(w(k)\) must lie on or above the lines coming through pairs of points \((i_1, \sigma(i_1)), (i_2, \sigma(i_2))\) and \((j_1, \sigma(j_1)), (j_2, \sigma(j_2))\). The three cases in the definition of the lower game capture this fact by setting the value of \(s(k)\) so that it lies on either one of the lines (if the other one does not exist) or on the maximum of both of them.

Now it remains to show that for every \(k \in N\), the value \(s(k)\) is attained for at least one \(C^n_{\sigma}\)-extension. We introduce a \(C^n_{\sigma}\)-extension \((N, s^{(a,b)})\) for consecutive \(a,b \in \mathcal{X}\) such that \(a < b\), described as

\[
s^{(a,b)}(\ell) := \begin{cases} 
\sigma(\ell), & \text{if } \ell \in \mathcal{X}, \\
\overline{s}(\ell), & \text{if } \ell \notin \mathcal{X} \text{ and } a < \ell < b, \\
\overline{s}(\ell), & \text{if } \ell \notin \mathcal{X} \text{ and either } \ell < a, \text{ or } b < \ell.
\end{cases}
\]

Clearly the game is an extension of \((N, \mathcal{X}, \sigma)\). For \(i \in \{2, \ldots, n-1\}\) such that all three values \(s^{(a,b)}(i-1), s^{(a,b)}(i), s^{(a,b)}(i+1)\) coincide with the respective values of the upper game \(\overline{s}\), it holds \(s^{(a,b)}(i) \leq s^{(a,b)}(i-1) + s^{(a,b)}(i+1)\), because \((N, \overline{s})\) is a symmetric convex game (as we show further in this proof) so by Theorem 13 the same inequality holds for values of \(s\). In the rest of the cases, either all the three points \((i-1, s^{(a,b)}(i-1)), (i, s^{(a,b)}(i)), (i+1, s^{(a,b)}(i+1))\) lie on the same line and the inequality holds with the equal sign, or the three points lie on the
maximum of two lines coming through pairs of points \((a_2, \sigma(a_2)), (a, \sigma(a))\) and 
\((b, \sigma(b)), (b_2, \sigma(b_2))\) where \(a_2 < a\) and \(b < b_2\) are consecutive pairs from \(\mathcal{X}\). If
\[
s^{(a,b)}(i) > s^{(a,b)}(i-1) + s^{(a,b)}(i+1)\]
then either \(\sigma(a) > \sigma(a_2) + (a-a_2) \frac{\sigma(b)-\sigma(a_2)}{b-b_2}\) or
\[
\sigma(b) > \sigma(a) + (b-a) \frac{\sigma(b_2)-\sigma(a)}{b_2-a}\]
both resulting, by Theorem 15, in a contradiction with the \(C^n_v\)-extendability of \((N, \mathcal{X}, \sigma)\). Now for \(k \in \mathcal{X}\), we choose \((N, s^{(a,b)})\) such that \(a = k\) and for \(k \notin \mathcal{X}\), we choose \((N, s^{(a,b)})\) such that \(a < k < b\) are the closest coalition sizes with defined value.

For the upper game \((N, \bar{\sigma})\), suppose for a contradiction that there is the reduced form \((N, \bar{s})\) of a \(C^n_v(v)\)-extension such that for \(k \in N\), \(\bar{s}(k) < s(k)\). As for \(k \in \mathcal{X}\), \(\bar{s}(k) = \sigma(k) = s(k)\), it must be that \(k \notin \mathcal{X}\). But if \(k \notin \mathcal{X}\) and \(\bar{s}(k) = \sigma(i_2) + (k-i_2) \frac{\sigma(j)-\sigma(i_2)}{j_1-i_2} < s(k)\), the convexity of the line chart is violated, because \((k, s(k))\) lies above the line segment between points \((i_2, \sigma(i_2)), (j_1, \sigma(j_1))\). This is a contradiction.

Now we prove that \((N, \bar{\sigma})\) is a \(C^n_v\)-extension of \((N, \mathcal{X}, \sigma)\). First, it is clearly an extension. Furthermore, notice that the values of \((N, \bar{\sigma})\) lie on the line chart of \((N, \mathcal{X}, \sigma)\). Since the game is \(C^n_v\)-extendable, the line chart is a convex function, therefore inequalities hold from Theorem 14, meaning \((N, \bar{\sigma}) \in C^n_v(v)\).

The game \((N, \bar{\sigma})\) is always a \(C^n_v\)-extension, however, this is not true for \((N, \sigma)\) in general, as can be seen in the example in Figure 3.

5.1.3. Extreme Games

Games \((N, s^{(a,b)})\) are actually even more important because they are extreme games of \(C^n_v(v)\).

**Proposition 3.** Let \((N, \mathcal{X}, \sigma)\) be the reduced form of a \(C^n_v\)-extendable symmetric incomplete game \((N, \mathcal{K}, v)\). Games \((N, s^{(a,b)})\) for consecutive \(a, b \in \mathcal{X}\), where
a < b, and (N, \pi), are extreme games of C^0_\sigma(v).

Proof. For a contradiction, suppose that for some a, b, (N, s^{(a,b)}) is not an extreme game of C^0_\sigma(v). By Definition [3] there are two C^0_\sigma(v)-extensions (N, s_1) and (N, s_2) such that (N, s^{(a,b)}) is their nontrivial convex combination and without loss of generality, there is i \in \{0, \ldots, n\} such that s_1(i) < s^{(a,b)}(i) < s_2(i). For i \in \mathcal{X}, this is not possible as s_1(i) = s_2(i) = s^{(a,b)}(i). Furthermore, for i \notin \mathcal{X} and a < i < b, this is a contradiction with s_1(i) < s^{(a,b)}(i) = s(i) and finally for i \notin \mathcal{X} and either i < a or b < i, we get again a contradiction because \pi(i) = s^{(a,b)}(i) < s_2(i). Following a similar argument, we conclude that the upper game (N, \pi) is also an extreme game. \qed

In general, (N, \pi) and (N, s^{(a,b)}) are not the only extreme games. In the following theorem, we describe all the extreme games of C^0_\sigma(v).

Theorem 17. Let (N, \mathcal{X}, \sigma) be the reduced form of a C^0_\sigma-extendable symmetric incomplete game such that C^0_\sigma(v) is bounded. For k \in \{0, \ldots, n\} \setminus \mathcal{X} and i, j \in \mathcal{X} closest to k such that i < k < j, games (N, s^k) defined as

\[
s^k(m) := \begin{cases} 
\sigma(m), & \text{if } m \in \mathcal{X}, \\
\pi(m), & \text{if } m \notin \mathcal{X} \text{ and either } m < i \text{ or } j < m, \\
s(m), & \text{if } m = k, \\
\sigma(j) + (m - j)\frac{\sigma(j) - \pi(j)}{j - k}, & \text{if } m \notin \mathcal{X} \text{ and } k < m < j, \\
\sigma(i) + (m - i)\frac{\sigma(i) - \pi(i)}{k - i}, & \text{if } m \notin \mathcal{X} \text{ and } i < m < k 
\end{cases}
\]

together with (N, \pi) form all the extreme games of C^0_\sigma(v).

Proof. We divide the proof into two parts. In the first part, we show that any C^0_\sigma(v)-extension (N, s) is a convex combination of games (N, \pi) and (N, s^k) for k \in \overline{\mathcal{X}}. In the second part, we show that every game (N, s^k) is an extreme game, thus (together with the upper game (N, \pi)) they form all the extreme games.

Before we begin, let us define a gap as an inclusion-wise maximal nonempty sequence of consecutive coalition sizes with undefined profit. In other words, we can say that there is a gap between i and j if i, j \in \mathcal{X}, i < j, j - i > 1, and for every i' such that i < i' < j, it holds i' \notin \overline{\mathcal{X}}. The size of the gap between i and j is defined as j - i - 1, that is the number of coalition sizes with unknown values in the given gap. It is immediate that the size of every gap is at least one.

We now prove the first part of the theorem. First, let us suppose that there is only one gap in (N, \mathcal{X}, \sigma). We prove this case by induction on the size of the gap.

If the size of the gap is 1, there is only one game (N, s^k) that is equal to (N, s^{(k-1,k+1)}). Any C^0_\sigma-extension (N, s) can be expressed as a convex combination of this game and the upper game (N, \pi) as s = \alpha s^k + (1 - \alpha)\pi with

\[
\alpha = \frac{s(k) - \pi(k)}{s^k(k) - \pi(k)} \in [0, 1].
\]
For the induction step, suppose that the size of the gap between \( i \) and \( j \) is \( \ell, \ell > 1 \). Hence there are \( \ell \) games

\[(N, s^{i+1}), (N, s^{i+2}), \ldots, (N, s^{j-1}) \text{ together with } (N, \overline{s}).\]

We construct a new system of \( \ell - 1 \) games

\[(N, (s^{i+2})'), (N, (s^{i+3})'), \ldots, (N, (s^{j-1})') \text{ together with } (N, (s^{i+1})'),\]

where \((s^m)’ := \alpha s^m + (1 - \alpha)\overline{s} \) and \( \alpha = \frac{s(i + 1) - \overline{s}(i + 1)}{s^{i+1}(i + 1) - \overline{s}(i + 1)} \).

These games correspond to the extreme games of an incomplete game \((N, \mathcal{X}', \sigma')\) where \( \mathcal{X}' := \mathcal{X} \cup \{i + 1\} \), and the function \( \sigma' \) is defined as \( \sigma'(m) := \sigma(m) \) for \( m \in \mathcal{X} \) and \( \sigma'(i + 1) := s(i + 1) \). The game \((N, (s^{i+1})')\) represents the upper game of \( C^\sigma_n(v) \). Since the new system of \( \ell \) games forms the extreme games of \( C^\sigma_n\)-extensions of \((N, \mathcal{X}', \sigma')\), the game \((N, s)\) (which is also a \( C^\sigma_n\)-extension of \((N, \mathcal{X}', \sigma')\)) is, by induction hypothesis, their convex combination. And as each game \((N, (s^m)')\) is a convex combination of \((N, \overline{s})\) and \((N, s^m)\), the game \((N, s)\) is also a convex combination of the former system

\[(N, s^{i+1}), (N, s^{i+2}), \ldots, (N, s^{j-1}) \text{ together with } (N, \overline{s}).\]

Notice that if there is more than one gap between the coalition sizes in \( \mathcal{X} \), then we can follow a similar construction as in the situation with precisely one gap. This is because any two extreme games parametrised by two coalition sizes from one gap assign the same profit to any coalition size from a different gap. Thus, we can start our construction by filling in the first gap, after that, taking the extreme games of the extended incomplete game and so on, until there is no gap left.

As for the second part of the proof, suppose for a contradiction that \((N, s^k)\) for \( k \in \overline{\mathcal{X}} \) is not an extreme game of \( C^\sigma_n(v) \). By Definition 9, there are \( C^\sigma_n\)-extensions \((N, s_1), (N, s_2)\) and \( m \in N \) such that \( s_1(m) < s^k(m) < s_2(m) \).

Clearly, \( m \notin \mathcal{X} \) (since \( s_1(m) = s^k(m) = s_2(m) = \sigma(m) \)) and if \( m \) is such that \( s^k(m) = \overline{s}(k) \) or \( s^k(m) = s(m) \), we arrive at a contradiction. Therefore, the only case that remains is \( m \notin \mathcal{X} \) together with \( i < m < j \) and \( m \neq k \). For any such \( m \), the convexity of the line chart is violated either for \((i, s_1(i)), (k, s_1(k)), (m, s_1(m))\) (if \( k < m \)), or for \((i, s_2(i)), (m, s_2(m)), (k, s_2(k))\) (if \( m < k \)). Both cases are depicted in Figure 4.

For a \( C^\sigma_n\)-extendable symmetric incomplete game in a reduced form \((N, \mathcal{X}, \sigma)\) with \( C^\sigma_n(v) \) bounded and \( |C^\sigma_n(v)| > 1 \), the number of extreme games is always \(|\overline{\mathcal{X}}| + 1 = n - |\mathcal{X}| + 2 \), no matter what the values of \( \sigma \) are.

Algebraically, we can describe the set \( C^\sigma_n(v) \) as

\[
C^\sigma_n(v) = \left\{ \left( N, \overline{s} + \sum_{k \in \overline{\mathcal{X}}} \alpha_k s^k \right) \bigg| \overline{s} + \sum_{k \in \overline{\mathcal{X}}} \alpha_k s^k = 1, \overline{s}, \alpha_k \geq 0, k \in \overline{\mathcal{X}} \right\}, \quad (9)
\]
Figure 4: Examples of a violation of convexity of the line chart of both \((N, s_1)\) and \((N, s_2)\). The full lines depict the line chart of \((N, s)\) and the dotted lines depict the line charts of \((N, s_1)\) and \((N, s_2)\). On the left, the situation where \(k < m\) is shown. We have values \(s^k(i) = s_1(k)\) and \(s^k(k) = s_1(k)\), yet \(s_1(m)\) is too small. Similarly, on the right, the situation where \(m < k\) is shown, with \(s^k(i) = s_2(k)\), \(s^k(k) = s_2(k)\). However, in this case, the value \(s_2(m)\) is too big.

namely as the set of convex combinations of extreme games \(\pi\) and \(s^k\) for \(k \in \Xi\).

Geometrically, we can describe the set \(C^n_\sigma(v)\) when we restrict the game \((N, \mathcal{X}, \sigma)\) a little. First, suppose \(\mathcal{X} = \{0, n\}\) and \(\sigma(0) = \sigma(n) = 0\). According to Theorem 14, we can describe \(C^n_\sigma(v)\) by a system of \(n - 1\) inequalities with \(n - 1\) unknowns, \(Ay \leq 0\), where

\[
A = \begin{pmatrix}
2 & -1 & 0 \\
-1 & \ddots & \ddots \\
0 & -1 & 2
\end{pmatrix}.
\]

The matrix \(A\) is an \textit{M-matrix} [23], therefore it is nonsingular and \(A^{-1} \geq 0\). Nonsingularity of \(A\) implies that \(C^n_\sigma(v)\) is a pointed polyhedral cone, which is translated such that its vertex is not necessarily in the origin of the coordinate system. Furthermore, because \(A^{-1} \geq 0\), the \textit{normal cone} \(C^n_\sigma(v)^*\) of \(C^n_\sigma(v)\) (see [11]) contains the whole nonnegative orthant. Thus, the vertex of polyhedral cone \(C^n_\sigma(v)\) is the biggest element of \(C^n_\sigma(v)\) when restricted to each coordinate (this corresponds with the statement that the upper game is a \(C^n_\sigma\)-extension). Therefore, geometrically, the set \(C^n_\sigma(v)\) looks like \textit{squeezed} negative orthant. For an incomplete game \((N, \mathcal{X}', \sigma')\) where \(\{0, n\} \subseteq \mathcal{X}'\) and \(\sigma'(0) = \sigma(n) = 0\), the set of \(C^n_\sigma\)-extensions is \(C^n_\sigma(v)\) with some of the coordinates fixed, i.e.

\[
C^n_\sigma(v) \cap \{s(k) = \sigma(k)\}.
\]

6. Conclusion

We would like to conclude our paper with observations on a connection between incomplete cooperative games and cooperative interval games. This
connection has not been mentioned in literature so far and we think it provides a nice bridge between the two approaches to uncertainty in cooperative game theory.

We remind the reader that a cooperative interval game is a pair $(N, w)$, where $N$ is a finite set of players and $w: 2^N \rightarrow \mathbb{IR}$ is the characteristic function of this game with $\mathbb{IR}$ being the set of all real closed intervals. We further set $v(\emptyset) := [0, 0]$.

**Proposition 4.** For a given incomplete game $(N, K, v)$ and a set of its extensions $E$, the associated lower game and upper game induce a cooperative interval game $(N, w)$ containing the set $E$ of extensions of $(N, K, v)$. Furthermore, this game is inclusion-wise minimal, i.e. for every $S \subseteq N$, there is an extension from $E$ attaining the lower bound of $w(S)$ and an extension from $E$ attaining the upper bound of $w(S)$.

We can also take another view-point. We can generalise the definition of incomplete game to the interval setting by allowing the partial game to be an interval game. We can then ask what are the extensions having some desired property, for example being selection superadditive interval games. Indeed, this aligns with the main motivation behind some of the results in [8] and [7].

We think that this is just the first step towards unifying both theories together. The fact that so far, no connection has been made between these two areas in literature seems surprising to us. Some of the issues raised here are work in progress.

**References**

[1] S. Z. Alparslan Gök. *Cooperative interval games*. PhD thesis, Middle East Technical University, 2009.

[2] S. Z. Alparslan Gök, O. Branzei, R. Branzei, and S. Tijs. Set-valued solution concepts using interval-type payoffs for interval games. *Journal of Mathematical Economics*, 47(4):621–626, 2011.

[3] S. Z. Alparslan Gök, S. Miquel, and S. H. Tijs. Cooperation under interval uncertainty. *Mathematical Methods of Operations Research*, 69(1):99–109, 2009.

[4] U. Bhaskar and G. Kumar. The Complexity of Partial Function Extension for Coverage Functions. In Dimitris Achlioptas and László A. Végh, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2019)*, volume 145 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 30:1–30:21, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. URL: [http://drops.dagstuhl.de/opus/volltexte/2019/11245](http://drops.dagstuhl.de/opus/volltexte/2019/11245), doi:10.4230/LIPIcs.APPROX-RANDOM.2019.30
[5] U. Bhaskar and G. Kumar. Partial Function Extension with Applications to Learning and Property Testing. In Yixin Cao, Siu-Wing Cheng, and Minming Li, editors, 31st International Symposium on Algorithms and Computation (ISAAC 2020), volume 181 of Leibniz International Proceedings in Informatics (LIPIcs), pages 46:1–46:16, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. URL: https://drops.dagstuhl.de/opus/volltexte/2020/13390, doi:10.4230/LIPIcs.ISAAC.2020.46.

[6] J. M. Bilbao. Cooperative Games on Combinatorial Structures, volume 26 of Theory and Decision Library. Springer Science & Business Media, 2012.

[7] J. Bok. On convexity and solution concepts in cooperative interval games. arXiv preprint arXiv:1811.04063, 2018.

[8] J. Bok and M. Hladík. Selection-based approach to cooperative interval games. In Communications in Computer and Information Science, ICORES 2015 - International Conference on Operations Research and Enterprise Systems, Lisbon, Portugal, 10-12 January, 2015, volume 577, pages 40–53, 2015.

[9] J. Bok and M. Černý. 1-convex extensions of incomplete cooperative games and the average value. arXiv preprint arXiv:2107.04679, 2022.

[10] O. N. Bondareva. Some applications of linear programming methods to the theory of cooperative games. Problemy kibernetiki, 10:119–139, 1963.

[11] S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, 2004.

[12] R. Branzei, D. Dimitrov, and S. Tijs. Models in Cooperative Game Theory, volume 556 of Lecture Notes in Economics and Mathematical Systems. Springer, 2008.

[13] A. Chateauneuf and J.-Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. Mathematical Social Sciences, 17:263–283, 1989.

[14] G. Choquet. Theory of capacities. Annales de l’Institut Fourier, 5:131–295, 1954. doi:10.5802/aif.53

[15] I. Curiel. Cooperative Game Theory and Applications: Cooperative Games Arising from Combinatorial Optimization Problems. Springer, Dordrecht, 2013.

[16] T. Driessen. Cooperative Games, Solutions and Applications, volume 3 of Theory and Decision Library C. Kluwer, Dordrecht, 1988.

[17] J. Farkas. Theorie der einfachen ungleichungen. Journal für die reine und angewandte Mathematik, 1902(124):1–27, 1902.
[18] K. Fujimoto and T. Murofushi. Some characterizations of $k$-monotonicity through the bipolar M"obius transform in bi-capacities. *Journal of Advanced Computational Intelligence and Intelligent Informatics*, 9(5):484–495, 205.

[19] I. Gilboa and D. Schmeidler. Additive representations of non-additive measures and the Choquet integral. *Annals of Operations Research*, 52(1):43–65, 1994.

[20] R. P. Gilles. *The Cooperative Game Theory of Networks and Hierarchies*, volume 44 of *Theory and Decision Library C*. Springer, 2010.

[21] M. Grabisch. *Set Functions, Games and Capacities in Decision Making*. Springer, 2016.

[22] M. Grabisch, J. L. Marichal, and M. Roubens. Equivalent representations of set functions. *Mathematics of Operations Research*, 25(2):157–178, 2000.

[23] R. A. Horn and Ch. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, 1991.

[24] J. Lemaire. *Cooperative Game Theory and its Insurance Applications*. Center for Research on Risk and Insurance, Wharton School of the University of Pennsylvania, 1991.

[25] L. Mallozzi, V. Scalzo, and S. Tijs. Fuzzy interval cooperative games. *Fuzzy Sets and Systems*, 165(1):98–105, 2011.

[26] M. Mareš. *Fuzzy Cooperative Games: Cooperation with Vague Expectations*, volume 72. Physica-Verlag Heidelberg, 2001.

[27] M. Mareš and M. Vlach. Fuzzy classes of cooperative games with transferable utility. *Scientiae Mathematicae Japonica*, 2:269–278, 2004.

[28] S. Masuya. An approximated Shapley value for partially defined cooperative games. *Procedia Computer Science*, 192:100–108, 2021.

[29] S. Masuya and M. Inuiguchi. A fundamental study for partially defined cooperative games. *Fuzzy Optimization Decision Making*, 15(1):281–306, 2016.

[30] O. Palancu, S. Z. Alparslan Gök, S Ergün, and G.W. Weber. Cooperative grey games and the grey Shapley value. *Optimization*, 64(8):1657–1668, 2015.

[31] O. Palancu, S. Z. Alparslan Gök, and G. W. Weber. Cooperative games under bubbly uncertainty. *Mathematical Methods of Operations Research*, 80(2):129–137, 2014.

[32] B. Peleg and P. Sudhölter. *Introduction to the Theory of Cooperative Games*, volume 34 of *Theory and Decision Library*. Springer Science & Business Media, 2nd edition, 2007.
Appendix A. Omitted proofs

Proof. (of Proposition 1) If the game is symmetric convex, we consider the characterisation from Theorem 1 for coalitions $S, S \cup j$ and $i \in S$, obtaining
\[ v(S) - v(S \setminus i) \leq v(S \cup j) - v(S \cup j \setminus i). \] (A.1)
Because $|S \cup j \setminus i| = |S|$, we have $v((S \cup j) \setminus i) = v(S)$ by symmetry. By adding $v(S)$ to (A.1) and rearranging the inequality, we get (7)
\[ v(S) \leq \frac{v(S \setminus i) + v(S \cup j)}{2}. \]

For the opposite implication, suppose that conditions (7) hold and $(N, v)$ is not convex. Then there is a player $k \in N$ and coalitions $T_1 \subseteq T_2 \subseteq N \setminus k$ for which the condition from Theorem 1 is violated, i.e.
\[ v(T_1 \cup k) - v(T_1) > v(T_2 \cup k) - v(T_2). \] (A.2)
We choose player $k$ and coalitions $T_1, T_2$ such that the difference $|T_2| - |T_1|$ is minimal. We distinguish two possible cases.
1. If $|T_2| - |T_1| = 1$, then by symmetry of $v$, we have that $v(T_2) = v(T_1 \cup k)$. In that case, we get

$$v(T_2) > \frac{v(T_1) + v(T_2 \cup k)}{2}.$$ 

Furthermore, there exists a unique $\ell \in T_2 \setminus T_1$ such that $T_1 \cup \ell = T_2$. Thus we can write

$$v(T_2) > \frac{v(T_2 \setminus \ell) + v(T_2 \cup k)}{2},$$

which leads to a contradiction with (7).

2. If $|T_2| - |T_1| > 1$, then there is a coalition $T_3$ such that $T_1 \subseteq T_3 \subseteq T_2 \subseteq N \setminus k$. By minimality of $|T_2| - |T_1|$, we know that

$$v(T_1 \cup k) - v(T_1) \leq v(T_3 \cup k) - v(T_3) \quad (A.3)$$

and

$$v(T_3 \cup k) - v(T_3) \leq v(T_2 \cup k) - v(T_2). \quad (A.4)$$

By adding (A.3) and (A.4) together, we get

$$v(T_1 \cup k) - v(T_1) \leq v(T_2 \cup k) - v(T_2),$$

which is a contradiction with (A.2). \qed