Deciding trigonality of algebraic curves (extended abstract)

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Let $C$ be an algebraic curve of genus $g \geq 3$. Let us assume that $C$ is not hyperelliptic, so that it is isomorphic to its image by the canonical map $\varphi : C \to \mathbb{P}^{g-1}$. Enriques proved in [1] that $\varphi(C)$ is the intersection of the quadrics that contain it, except when $C$ is trigonal (that is, it has a $g_1^3$) or $C$ is isomorphic to a plane quintic ($g = 6$). The proof was completed by Babbage [2], and later Petri proved [3] that in those two cases the ideal is generated by the quadrics and cubics that contain the canonical curve. In this context, we present an implementation in Magma of a method to decide whether a given algebraic curve is trigonal, and in the affirmative case to compute a map $C : 3:1 \to \mathbb{P}^1$ whose fibers cut out a $g_1^3$. Our algorithm is part of a larger effort to determine whether a given algebraic curve admits a radical parametrization.

1 Classical results on trigonality

The following theorem [4, p. 535] classifies canonical curves according to the intersection of the quadric hypersurfaces that contain them.

**Theorem 1.** For $C \subset \mathbb{P}^n$ any canonical curve, either

(i) $C$ is entirely cut out by quadric hypersurfaces; or

(ii) $C$ is trigonal, in which case the intersection of all quadrics containing $C$ is the rational normal scroll swept out by the trichords of $C$; or

(iii) $C$ is a plane quintic, in which case the intersection of the quadrics containing $C$ is the Veronese surface in $\mathbb{P}^5$, swept out by the conic curves through five coplanar points of $C$.

We can use this to sketch an algorithm to detect trigonality.

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Algorithm 1: Sketch of algorithm to detect trigonality

**Input:** a non-hyperelliptic curve $C$ of genus $g \geq 3$

**Output:** true if $C$ is trigonal, false otherwise

Compute the canonical map $\varphi : C \to \mathbb{P}^{g-1}$ and its image $\varphi(C)$;
Compute the intersection $D$ of all the quadrics that contain $\varphi(C)$;
if $D = C$ then
    return false;
else
    Determine which type of surface is $D$;
    if $D = \mathbb{P}^2$ then
        return true // $g = 3$;
    else if $D$ is a rational normal scroll then
        return true;
    else
        return false // Veronese;
end
end

For the computation of the canonical map and the computation of the space of forms of fixed degree containing the image of a polynomial curve, there exist efficient implementations in Magma.

For the identification of the surface $D$, we use the Lie algebra method, which has been introduced in [6] (see also [5]) for parametrizing certain classes of Del Pezzo surfaces.

2 The Lie algebra method

The Lie algebra of a projective variety is an algebraic invariant which is relatively easy to calculate (it is often cheaper than a Gröbner basis of the defining ideal, if only generators are given).

Let $X \subset \mathbb{P}^n$ be an embedded projective variety. Let $\text{PGL}_{n+1}(X)$ be the group of all projective transformations that map $X$ to itself (this is always an algebraic group). The Lie algebra $L(X)$ of $X$ is defined as the tangent space of $\text{PGL}_{n+1}(X)$ at the identity, together with its natural Lie product. It is a subalgebra of $\mathfrak{sl}_{n+1}$, the Lie algebra of $\mathbb{P}^n$.

For varieties of general type (in particular curves of genus at least 2), the group $\text{PGL}_{n+1}(X)$ is finite and therefore the Lie algebra is zero. On the other hand, the Veronese surface and the rational scrolls have a Lie algebra of positive dimension. This allows us to reduce the recognition problem for Veronese surfaces/rational scrolls to the computations with Lie algebras and their representations.

If $S$ is a rational normal scroll which is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then $L(S)$ has a Levi subalgebra isomorphic to $\mathfrak{sl}_2$. By decomposing the Lie module given
by the representation $\mathfrak{sl}_2 \hookrightarrow \mathfrak{gl}_{n+1}$, we can construct a $2 \times (n-1)$ matrix $A$, such that the $2 \times 2$ minors of $A$ generate the ideal of $S$. The ratio of the two entries of any column defines a map $\rho : S \to \mathbb{P}^1$ whose fibers are lines. Then the map $C \to \mathbb{P}^1$ is constructed as

$$C \xrightarrow{\phi} \varphi(C) \xrightarrow{i} S \xrightarrow{\rho} \mathbb{P}^1. \quad (2.1)$$

If $S$ is a Veronese surface, then we can construct by similar methods an isomorphism of $C$ with a planar quintic.

### 3 Trigonality algorithm

We describe in more detail the algorithm to detect and compute the trigonality of a curve. In particular, we explain now the computation of a threefold map from $C$ to $\mathbb{P}^1$. Let $S$ be the surface intersection of quadrics containing $\varphi(C)$ and $S \to \mathbb{P}^2$ the parametrization obtained with the Lie algebra method. In all cases we will obtain a threefold map from $\varphi(C)$ to $\mathbb{P}^1$ which can be pulled back to $C$.

- If $g = 3$, the canonical curve is a smooth quartic in $\mathbb{P}^2$. In this case, one can easily compute a $g_1^3$ on $\varphi(C)$, at least in theory: it suffices to take a point $p$ on it and consider the pencil of lines through it, since each one intersects $\varphi(C)$ in $p$ and three more points. In practice, finding a point with coefficients in the base field is problematic, unless one accepts working on algebraic extensions.

- If $S$ is a rational normal scroll not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, we can compute the map in (2.1) explicitly.

- If $S$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, we compute a map $\rho : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and form the composition

  $$C \xrightarrow{\phi} \varphi(C) \xrightarrow{i} S \xrightarrow{\rho} \mathbb{P}^1 \times \mathbb{P}^1 \rho \mathbb{P}^1.$$

- If $C$ is a plane quintic, it is not trigonal.

### 4 Computational experiences

We have tested our Magma V2.14-7 [7] implementation against many examples of trigonal curves. The computer used is a 64 Bit, Dual AMD Opteron Processor 250 (2.4 GHZ) with 8 GB RAM. We have generated trigonal curves in the following two ways:

(i) Let $C : f(x, y, z) = 0$ with $\deg_y f = 3$. Then the projection $(x : y : z) \mapsto (x : z)$ is a $3 : 1$ map to $\mathbb{P}^1$. The genus of a polynomial of degree 3 in $y$ and degree $d$ in $x$ is $2(d-1)$ generically. The size of the coefficients is controlled directly.
(ii) Let $C$ be defined by the affine equation $\text{Resultant}_u(F, G) = 0$ where
\[
0 = x^3 - a_1(u)x - a_2(u) =: F,
0 = y - a_3(u) - a_4(u)x - a_5(u)x^2 =: G
\]
for some polynomials $a_1, \ldots, a_5$. This clearly gives a field extension of degree 3, thus there is a $3:1$ map from $C$ to the affine line. The degree and coefficient size for a given genus are significantly larger than for the previous construction.

These are our timed results\(^1\) for samples of ten random polynomials of different degrees, genera and coefficient sizes.

| deg | bit height | genus | deg | seconds |
|-----|------------|-------|-----|---------|
| 3   | 5          | 4     | 6   | 0.5 – 0.65 |
| 3   | 50         | 4     | 6   | 2.09 – 2.27 |
| 6   | 5          | 10    | 9   | 14 – 17 |
| 6   | 50         | 10    | 9   | 54 – 61 |
| 10  | 5          | 18    | 13  | 271 – 342 |
| 10  | 50         | 18    | 13  | 1059 – 1193 |
| 15  | 5          | 28    | 18  | 3477 – 5317 |

For the second method, we choose $a_1, \ldots, a_5$ randomly of degree $d$ and maximum coefficient size $e$. These are the time results\(^1\) for samples of ten random polynomials, for different values of $d, e$.

| $(d, e)$ | genus | deg | bit height | seconds |
|---------|-------|-----|------------|---------|
| (4, 2)  | 4     | 15  | 20 – 24    | 18 – 62 |
| (4, 10) | 4     | 20  | 26 – 35    | 87 – 191 |
| (5, 2)  | 4 – 6 | 20  | 25 – 17    | 162 – 2353 |
| (5, 10) | 4 – 6 | 23  | 25 – 17    | 1334 – 7940 |
| (6, 2)  | 6 – 7 | 25  | 30 – 24    | 2992 – 22650 |

Our Magma implementation can be obtained by contacting us directly: josef.schicho@oeaw.ac.at and david.sevilla@oeaw.ac.at.

References

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\(^{1}\)Last minute improvements in our implementation have reduced the running times by a factor of about 5, for many of the entries in the tables.
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