THE 2-ADIC VALUATION OF THE COEFFICIENTS OF A POLYNOMIAL

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Abstract. In this paper we compute the 2-adic valuations of some polynomials associated with the definite integral

\[ \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}. \]

1. Introduction

In this paper we present a study of the coefficients of a polynomial defined in terms of the definite integral

\[ N_{0,4}(a; m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} \]

where \( m \) is a positive integer and \( a > -1 \) is a real number.

Apart from their intrinsic interest, these polynomials form the basis of a new algorithm for the definite integration of rational functions.

An elementary calculation shows that

\[ P_m(a) := \frac{2^{m+3/2}}{\pi} (a + 1)^{m+1/2} N_{0,4}(a; m) \]

is a polynomial of degree \( m \) in \( a \) with rational coefficients. Let

\[ P_m(a) = \sum_{l=0}^{m} d_l(m) a^l. \]

Then it can be shown that \( d_l(m) \) is equal to

\[ \sum_{j=0}^{l} \sum_{s=0}^{m-l} \sum_{k=s+l}^{m} (-1)^{k-l-s} 2^{-3k} \binom{2k}{k} \frac{2m+1}{2(s+j)} \binom{m-s-j}{j} \binom{s+j}{j} \binom{k-s-j}{l-j} \]

from which it follows that \( d_l(m) \) is a rational number with only a power of 2 in its denominator. Extensive calculations have shown that, with rare exceptions, the numerators of \( d_l(m) \) contain a single large prime divisor and its remaining factors are very small. For example

\[ d_6(30) = 2^{12} \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 37 \cdot 639324594880985776531. \]

Similarly, \( d_{10}(200) \) has 197 digits with a prime factor of length 137 and its second largest divisor is 797. This observation lead us to investigate the arithmetic properties of \( d_l(m) \). In this paper we discuss the 2-adic valuation of these \( d_l(m) \).

The fact that the coefficients of \( P_m(a) \) are positive is less elementary. This follows from a hypergeometric representation of \( N_{0,4}(a; m) \) that implies the expression

\[ d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}. \]

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We have produced a proof of (1.4) that is independent of this hypergeometric connection and is based on the Taylor expansion
\[
\sqrt{a + \sqrt{1 + c}} = \sqrt{a + 1} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{P_{k-1}(a)}{2^{k+1} (a + 1)^{k}} c^k\right);
\]
see [2] for details.

The expression (1.4) can be used to efficiently compute the coefficients \(d_l(m)\) when \(l\) is large relative to \(m\). In Section 8 we derive a representation of the form
\[
d_l(m) = \frac{1}{l! m! 2^m} \left(\alpha_l(m) \prod_{k=1}^{m} (4k - 1) - \beta_l(m) \prod_{k=1}^{m} (4k + 1)\right),
\]
where \(\alpha_l(m)\) and \(\beta_l(m)\) are polynomials in \(m\) of degrees \(l\) and \(l - 1\) respectively.

For example
\[
d_1(m) = \frac{1}{m! 2^m} \left((2m + 1) \prod_{k=1}^{m} (4k - 1) - \prod_{k=1}^{m} (4k + 1)\right).
\]
This representation can now be used to efficiently examine the coefficients \(d_l(m)\) when \(l\) is small compared to \(m\). In Section 7 we prove that
\[
\nu_2(d_1(m)) = 1 - 2m + \nu_2\left(\binom{m + 1}{2}\right) + s_2(m)
\]
where \(s_2(m)\) is the sum of the binary digits of \(m\).

2. The polynomial \(P_m(a)\).

Let
\[
N_{0,4}(a; m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}.
\]
Then
\[
P_m(a) = \frac{2^{m+3/2}}{\pi} (a + 1)^{m+1/2} N_{0,4}(a; m)
\]
is a polynomial in \(a\) with positive rational coefficients. The proof is elementary and is presented in [2]. It is based on the change of variables \(x = \tan \theta\) and \(u = 2\theta\) that yields
\[
N_{0,4}(a; m) = 2^{-m-1} \int_0^\pi \frac{(1 + \cos u)^{2m+1}}{((1 + a) + (1 - a) \cos^2 u)^{m+1}} du.
\]
Expanding the numerator and employing the standard substitution \(z = \tan u\) produces
\[
N_{0,4}(a; m) = 2^{-2m-3/2} \sum_{\nu=0}^{m} \frac{2m+1}{2\nu} \frac{(a - 1)^{m-\nu}}{(a + 1)^{m-\nu+1/2}} \sum_{k=0}^{m-\nu} \binom{m-\nu}{k} \frac{2^k}{(a - 1)^k} B(m - k + 1/2, 1/2)
\]
where \(B\) is Euler’s beta function, defined by
\[
B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}.
\]
The expression (2.1) now produces the first formula for \(d_l(m)\) given in the Introduction.
3. The triple sum for $d_l(m)$.

The expression for the coefficients $d_l(m)$ given in the Introduction can be written as

$$
\sum_{j=0}^{l} \sum_{s=0}^{m-l} \sum_{k=s+j}^{m} (-1)^{k-l-s} 2^{-3k} \binom{2k}{k} \binom{2m+1}{2(s+j)} \binom{m-s-j}{m-k} \binom{s+j}{j} \binom{k-s-j}{l-j}.
$$

(3.1)

This expression follows directly from expanding (2.2) and the value $B(j+1/2, 1/2) = \frac{\pi}{2^{2j}} \binom{2j}{j}$.

It follows that $d_l(m)$ is a rational number whose denominator is a power of 2, therefore

Lemma 3.1. Let $p$ be an odd prime. Then

$$\nu_p(d_l(m)) \geq 0.$$ 

The positivity of $d_l(m)$ remains to be seen.

4. The single sum expression for $d_l(m)$.

An alternative form of the coefficients $d_l(m)$ is obtained by recognizing $N_0,4(a; m)$ as a hypergeometric integral. A standard argument shows that

$$N_{0,4}(a; m) = \frac{\pi \binom{2m}{m}}{2^{m+3/2}(a+1)^{m+1/2}} 2F_1[-m, m+1; 1/2 - m; (1 + a)/2]$$

where $2F_1$ is a hypergeometric function, defined by

$$2F_1[a, b, c; z] := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k,$$

where $(r)_k$ is the rising factorial

$$(r)_k = r(r+1)(r+2) \cdots (r+k-1).$$

It follows that $P_m(a)$ is the Jacobi polynomial of degree $m$ with parameters $m+1/2$ and $-(m+1/2)$. Therefore the coefficients are given by

$$d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{l} \binom{k}{l}$$

(4.1)

from which their positivity is obvious. We have obtained a proof of (4.1) that is independent of hypergeometric considerations and is based on the presence of $P_m(a)$ in the Taylor expansion (1.5). See [2] for details.

The formula (4.1) is very efficient for the calculation of the coefficients $d_l(m)$ when $l$ approximately equal to $m$. For instance, we have

$$d_m(m) = 2^{-m} \binom{2m}{m};$$

$$d_{m-1}(m) = 2^{-(m+1)} \binom{2m}{m}.$$
The expression (4.1), rewritten in the form
\[ d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^{k-l} \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{l}, \]
shows that
\[ \nu_2(d_l(m)) \geq l - 2m. \] (4.2)

5. Basics on valuations

Here we describe what is required on valuations.

Given a prime \( p \) and a rational number \( r \), there exist unique integers \( a, b, m \) with \( p \nmid a, b \) such that
\[ r = \frac{a}{b} p^m. \] (5.1)

The integer \( m \) is the \( p \)-adic valuation of \( r \) and we denote it by \( \nu_p(r) \).

Now recall a basic result of number theory which states that
\[ \nu_p(m!) = \sum_{k=1}^{\infty} \left\lfloor \frac{m}{p^k} \right\rfloor. \] (5.2)

Naturally the sum is finite and we can end it at \( k = [\log_p m] \).

There is a famous result of Legendre [3, 5] for the \( p \)-adic valuation of \( m! \). It states that
\[ \nu_p(m!) = \frac{m - s_p(m)}{p - 1}, \] (5.3)
where \( s_p(m) \) is the sum of the base-\( p \) digits of \( m \). In particular
\[ \nu_2(m!) = m - s_2(m). \] (5.4)

6. The constant term.

The calculation of the 2-adic valuation of the coefficients can be made very explicit for the first few. We begin with the case of the constant term. We first compute
\[ N_{0,4}(0; m) = \int_0^\infty \frac{dx}{(x^4 + 1)^{m+1}} \]
via the change of variable \( u = x^4 \), yielding
\[ N_{0,4}(0; m) = \frac{1}{4} B(1/4, m + 3/4) \]
\[ = \frac{\pi}{m! 2^{2m+3/2}} \prod_{k=1}^{m} (4k - 1). \]
Therefore
\[ d_0(m) = \frac{1}{m! 2^m} \prod_{k=1}^{m} (4k - 1). \] (6.1)

**Theorem 6.1.** The 2-adic valuation of the constant term \( d_0(m) \) is given by
\[ \nu_2(d_0(m)) = -(m + \nu_2(m!)) = s_2(m) - 2m. \]

**Proof.** This follows directly from (6.1). The second expression comes from (5.4). \( \square \)
Using the single sum formula for $d_0(m)$ we obtain

**Corollary 6.2.**

$$\nu_2 \left( \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \right) = m - \nu_2(m!) = s_2(m).$$

**Corollary 6.3.** The 2-adic valuation of the constant term $d_0(m)$ satisfies

$$\nu_2(d_0(m)) \geq 1 - 2m$$

with equality if and only if $m$ is a power of 2.

We now present a different proof of Corollary 3 that is based on the expression

$$d_0(m) = \frac{1}{m!2^m} \prod_{k=1}^{m} (4k - 1)$$

and the single sum formula

$$2^{2m}d_0(m) = \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m}$$

$$= \binom{2m}{m} + 2 \sum_{k=1}^{m} 2^{k-1} \binom{2m-2k}{m-k} \binom{m+k}{m}.$$  \hspace{1cm} (6.3)

**Proof.** From (6.3) it follows that

$$\nu_2(d_0(m)) \geq 1 - 2m$$

because the central binomial coefficient is an even number. Now from (6.2) we obtain

$$\nu_2(d_0(m)) = -(m + \nu_2(m!)).$$

From (6.2) we have

$$\nu_2(m!) = \sum_{k=1}^{\infty} \left\lfloor \frac{m}{2^k} \right\rfloor.$$  \hspace{1cm} (6.4)

Thus, from (6.4),

$$\nu_2(d_0(m)) = -\sum_{k=0}^{\infty} \left\lfloor \frac{m}{2^k} \right\rfloor.$$  \hspace{1cm} (6.5)

We know $\nu_2(d_0(m)) \geq 1 - 2m$, so it suffices to determine when equality occurs. Indeed, the equation

$$\sum_{k=0}^{\infty} \left\lfloor \frac{m}{2^k} \right\rfloor = 2m - 1$$

can be solved explicitly. Write $m = 2^e r$ with $r$ odd, and say $2^N < r < 2^{N+1}$. Then

$$\sum_{k=0}^{\infty} \left\lfloor \frac{m}{2^k} \right\rfloor = 2^e \cdot r + 2^{e-1} \cdot r + \cdots + r + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{r}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{r}{2^N} \right\rfloor$$

and (6.5) leads to

$$r - 1 = \sum_{k=1}^{N} \left\lfloor \frac{r}{2^k} \right\rfloor < \sum_{k=1}^{N} \frac{r}{2^k} \leq \sum_{k=1}^{\infty} \frac{r}{2^k} = \frac{r}{2}$$

and we conclude that $r = 1$. The proof is finished. \hfill \Box
7. The linear term.

From the triple sum we obtain
\[
d_1(m) = \sum_{s=0}^{m-1} \sum_{k=s+1}^{m} (-1)^{k-s-1} 2^{-3k} (m-s) \binom{2k}{k} \binom{2m+2}{2s+1} \binom{m-s-1}{m-k}.
\]

Differentiating (2.1) and \(d_1(m) = P_m'(0)\) we produce
\[
d_1(m) = \frac{1}{m!2^{m+1}} \left( (2m+1) \prod_{k=1}^{m} (4k-1) - \prod_{k=1}^{m} (4k+1) \right).
\]

Therefore the linear coefficient is given in terms of
\[
A_1(m) := (2m+1) \prod_{k=1}^{m} (4k-1) - \prod_{k=1}^{m} (4k+1)
\]
so that
\[
d_1(m) = \frac{A_1(m)}{m!2^{m+1}}.
\]

We prove

**Theorem 7.1.** The 2-adic valuation of the linear coefficient \(d_1(m)\) is given by
\[
\nu_2(d_1(m)) = 1 - 2m + \nu_2 \left( \binom{m+1}{2} \right) + s_2(m).
\]

Recall that the inequality \(\nu_2(d_1(m)) \geq 1 - 2m\) follows directly from the single sum expression. The theorem determines the exact value of the correction term.

**Proof.** We prove
\[
\nu_2(A_1(m)) = \nu_2(2m(m+1)) = 2 + \nu_2 \left( \binom{m+1}{2} \right).
\]

The result then follows from \(\nu_2(A_1(m)) \geq 2\) and \(\nu_2(A_1(m)) \leq 3\).

Define
\[
B_m = \prod_{k=1}^{m} (4k+1) - 1
\]
and
\[
C_m = (2m+1) \prod_{k=1}^{m} (4k-1) - 1.
\]

Then evidently \(A_1(m) = B_m - C_m\).

We show
\[
a) \quad \nu_2(B_m) = 2 + \nu_2 \left( \binom{m+1}{2} \right)
b) \quad \nu_2(C_m) \geq 3 + \nu_2 \left( \binom{m+1}{2} \right)
\]
from which the result follows immediately.
a) We have
\[ B_m = \prod_{k=1}^{m} (4k + 1) - 1 \]
\[ = \left( \sum_{j=1}^{m+1} 4^{m+1-j} \left[ \begin{array}{c} m+1 \\ j \end{array} \right] \right) - 1 \]
\[ = \sum_{j=1}^{m} 2^{2(n+1-j)} \left[ \begin{array}{c} m+1 \\ j \end{array} \right] \]
\[ = 2^2 \left[ \begin{array}{c} m+1 \\ m \end{array} \right] + \sum_{k=2}^{m} 2^{2k} \left[ \begin{array}{c} m+1 \\ m+1-k \end{array} \right] \]
\[ = 2^2 \left( \begin{array}{c} m+1 \\ 2 \end{array} \right) + \sum_{k=2}^{m} 2^{2k} \left[ \begin{array}{c} m+1 \\ m+1-k \end{array} \right] \]

where \( \left[ \begin{array}{c} m \\ k \end{array} \right] \) is an (unsigned) Stirling numbers of the first kind, i.e.,
\[ x(x+1) \cdots (x+m-1) = \sum_{k=0}^{m} \left[ \begin{array}{c} m \\ k \end{array} \right] x^k. \]

To prove a), it suffices to show that
\[ \nu_2 \left( 2^2 \left( \begin{array}{c} m+1 \\ 2 \end{array} \right) \right) < \nu_2 \left( 2^{2k} \left[ \begin{array}{c} m+1 \\ m+1-k \end{array} \right] \right) \]
for \( 2 \leq k \leq m \).

To do this we observe that there exist integers \( C_{k,i} \) \( (k \geq 1, \ i \geq 0) \) such that
\[ \left[ \begin{array}{c} m \\ m-k \end{array} \right] = \sum_{i=0}^{k-1} \left( \begin{array}{c} m \\ 2k-i \end{array} \right) C_{k,i} \]
see [4, p. 152]. For example
\[ \left[ \begin{array}{c} m \\ m-1 \end{array} \right] = \left( \begin{array}{c} m \\ 2 \end{array} \right) \]
\[ \left[ \begin{array}{c} m \\ m-2 \end{array} \right] = 3 \left( \begin{array}{c} m \\ 4 \end{array} \right) + 2 \left( \begin{array}{c} m \\ 3 \end{array} \right) \]
\[ \left[ \begin{array}{c} m \\ m-3 \end{array} \right] = 15 \left( \begin{array}{c} m \\ 6 \end{array} \right) + 20 \left( \begin{array}{c} m \\ 5 \end{array} \right) + 6 \left( \begin{array}{c} m \\ 4 \end{array} \right) \]
\[ \left[ \begin{array}{c} m \\ m-4 \end{array} \right] = 105 \left( \begin{array}{c} m \\ 8 \end{array} \right) + 210 \left( \begin{array}{c} m \\ 7 \end{array} \right) + 130 \left( \begin{array}{c} m \\ 6 \end{array} \right) + 24 \left( \begin{array}{c} m \\ 5 \end{array} \right). \]

Hence the rational number
\[ u := \frac{m(m-1) \cdots (m-k)}{(2k)!} \]
divides \( \left[ \begin{array}{c} m \\ m-k \end{array} \right] \) in the sense that the quotient
\[ \left[ \begin{array}{c} m \\ m-k \end{array} \right] \]
is an integer.

It follows that
\[ \nu_2 \left( \left[ \begin{array}{c} m \\ m-k \end{array} \right] \right) \geq \nu_2(m(m-1) \cdots (m-k)) - \nu_2((2k)!) \]
\[ = \nu_2(m(m-1) \cdots (m-k)) - 2k + s_2(k) \]
where we have used (5.3).
Hence, provided $k \geq 3$,
\[
\nu_2\left(\left\lfloor \frac{m + 1}{m + 1 - k} \right\rfloor \right) \geq \nu_2((m + 1)m(m - 1) \cdots (m + 1 - k)) - 2k + s_2(k)
\]
so that
\[
\nu_2\left(2^{2k} \left\lfloor \frac{m + 1}{m + 1 - k} \right\rfloor \right) \geq \nu_2((m + 1)m + \nu_2((m - 1)(m - 2)) + s_2(k)
\geq \nu_2((m + 1)m) + 1 + 1
\geq \nu_2\left(2^2 \left(\frac{m + 1}{2}\right)\right)
\]
provided $m \geq 3$. (For $m = 1$, 2 it is easy to check $\nu_2(B_m) = 2$.)
On the other hand, if $k = 2$, then
\[
\left\lfloor \frac{m}{m - 2} \right\rfloor = 3\left(\frac{m}{4}\right) + 2\left(\frac{m}{3}\right)
= \frac{1}{24}m(m - 1)(m - 2)(3m - 1),
\]
so if $m$ is even, $m \geq 4$, we have
\[
\nu_2\left(\left\lfloor \frac{m}{m - 2} \right\rfloor \right) = \nu_2\left(\frac{m(m - 1)}{2}\right) + \nu_2(m - 2) - \nu_2(12)
\geq \nu_2\left(\frac{m(m - 1)}{2}\right) + 1 - 2
\geq 2\left(\frac{m(m - 1)}{2}\right) - 1
\]
while if $m$ is odd, $m \geq 3$, we have
\[
\nu_2\left(\left\lfloor \frac{m}{m - 2} \right\rfloor \right) = \nu_2\left(\frac{m(m - 1)}{2}\right) + \nu_2(3m - 1) - \nu_2(12)
\geq \nu_2\left(\frac{m(m - 1)}{2}\right) + 1 - 2
\geq \nu_2\left(\frac{m(m - 1)}{2}\right) - 1
\]
so in either event
\[
\nu_2\left(\left\lfloor \frac{m + 1}{m - 1} \right\rfloor \right) \geq \nu_2\left(\left\lfloor \frac{m + 1}{2} \right\rfloor \right) - 1.
\]
Hence
\[
\nu_2\left(2^4 \left\lfloor \frac{m + 1}{m - 1} \right\rfloor \right) \geq \nu_2\left(\left\lfloor \frac{m + 1}{2} \right\rfloor + 3
\geq \nu_2\left(\left\lfloor \frac{m + 1}{2} \right\rfloor \right)
\]
as desired.

We now prove b):
\[
C_m = (2m + 1) \prod_{k=1}^{m} (4k - 1) - 1.
\]
We have
\[
\prod_{k=1}^{m} (4k-1) = 4^m \prod_{k=1}^{m} (k - 1/4)
\]
\[
= -4^{m+1} \sum_{k=0}^{m+1} \left[ \begin{array}{c} m+1 \\ k \end{array} \right] (-1/4)^k
\]
\[
= (-1)^m \sum_{k=1}^{m+1} \left[ \begin{array}{c} m+1 \\ k \end{array} \right] (-4)^{m+1-k}
\]
\[
= (-1)^m \sum_{k=1}^{m+1} \left[ \begin{array}{c} m+1 \\ k \end{array} \right] (-4)^{m+1-k}
\]
thus
\[
C_m = (-1)^m(2m+1) \sum_{k=1}^{m+1} \left[ \begin{array}{c} m+1 \\ k \end{array} \right] (-4)^{m+1-k} - 1.
\]

When \( m \) is even, we have
\[
C_m = (2m+1)(2m+3) - 2m^2(2m+3) + (2m+1) \sum_{k=2}^{m} \left[ \begin{array}{c} m+1 \\ m+1-k \end{array} \right] (-4)^k
\]
so, as in the proof of a), we have
\[
\nu_2(C_m) \geq \min \left( \nu_2(2m^2), \nu_2\left( 4^2 \left[ \begin{array}{c} m+1 \\ m-1 \end{array} \right] \right), \nu_2\left( 4^4 \left[ \begin{array}{c} m+1 \\ m-1 \end{array} \right] \right), \ldots, \nu_2\left( 4^{4m} \left[ \begin{array}{c} m+1 \\ 1 \end{array} \right] \right) \right)
\]
\[
\geq \min \left( 1 + 2\nu_2(m), 3 + \nu_2\left( \left[ \begin{array}{c} m+1 \\ 2 \end{array} \right] \right) \right)
\]
since \( m \) is even.

On the other hand, when \( m \) is odd we observe that
\[
C_m + 1 = (2m+1) \prod_{k=1}^{m} (4k - 1)
\]
and
\[
C_{m+1} + 1 = (2m+3)(4m+3) \prod_{k=1}^{m} (4k - 1)
\]
so
\[
\frac{C_{m+1} + 1}{(2m+3)(4m+3)} = \frac{C_m + 1}{2m+1}
\]
and hence
\[
C_m = \frac{(C_{m+1} + 1)(2m + 1)}{(2m + 3)(2m + 3)} - 1
\]
\[
= \frac{(2m + 1)C_{m+1} - 8(m + 1)^2}{(2m + 3)(4m + 3)}
\]
so
\[
\nu_2(C_m) \geq \min (\nu_2(C_{m+1}), 2\nu_2(m + 1) + 3)
\geq 3 + \nu_2 \left( \binom{m + 1}{2} \right)
\]
since $m$ is odd.
This completes the proof. □

The corresponding question of the 3-adic valuation of $d_1(m)$ seems to be more difficult. We propose

**Problem 7.2.** Prove the existence of a sequence of positive integers $m_j$ such that $
u_3(d_1(m_j)) = 0$. Extensive calculations show that
\[
m_{j+1} - m_j \in \{2, 7, 20, 61, 182, \ldots \}
\]
where the sequence $\{q_j\}$ in (7.4) is defined by $q_1 = 2$ and $q_{j+1} = 3q_j + (-1)^{j+1}$. It would be of interest to know whether $\nu_3(d_1(m))$ is unbounded: the maximum value for $2 \leq m \leq 20000$ is 12, so perhaps $\nu_3(d_1(m)) = O(\log m)$ as $m \to \infty$.

### 8. The general situation.

In this section we prove the existence of polynomials $\alpha_l(x)$ and $\beta_l(x)$ with positive integer coefficients such that
\[
d_l(m) = \frac{1}{l!m^{2m+1}} \left( \alpha_l(m) \prod_{k=1}^{m} (4k - 1) - \beta_l(m) \prod_{k=1}^{m} (4k + 1) \right).
\]
These polynomials are efficient for the calculation of $d_l(m)$ if $l$ is small relative to $m$, so they complement the results of Section 4.

For example
\[
\begin{align*}
\alpha_0(m) &= 1 \\
\alpha_1(m) &= 2m + 1 \\
\alpha_2(m) &= 2(2m^2 + 2m + 1) \\
\alpha_3(m) &= 4(2m + 1)(m^2 + m + 3) \\
\alpha_4(m) &= 8(2m^4 + 4m^3 + 26m^2 + 24m + 9).
\end{align*}
\]
and
\[
\begin{align*}
\beta_0(m) &= 0 \\
\beta_1(m) &= 1 \\
\beta_2(m) &= 2(2m + 1) \\
\beta_3(m) &= 12(m^2 + m + 1) \\
\beta_4(m) &= 8(2m + 1)(2m^2 + 2m + 9).
\end{align*}
\]
The proof consists in computing the expansion of $P_m(a)$ via the Leibnitz rule:
\[
P_m(a) = \frac{2^{m+3/2}}{\pi} \sum_{j=0}^{l} \binom{l}{j} \left( \frac{d}{da} \right)^{l-j} a^{m+1/2} \bigg|_{a=0} \left( \frac{d}{da} \right)^j N_{0,4}(a; m) \bigg|_{a=0}.
\]
We have

\[(8.1) \quad \left(\frac{d}{da}\right)^r (a + 1)^{m+1/2} \bigg|_{a=0} = 2^{-2r} \frac{(2m + 2)!}{(m+1)!} \frac{(m-r+1)!}{(2m-2r+2)!}\]

and

\[(8.2) \quad \left(\frac{d}{da}\right)^r N_{0,4}(a; m) \bigg|_{a=0} = (-1)^r \frac{(m+r)!}{m!} 2^r \int_0^\infty \frac{x^{2r}}{(x^4 + 1)^{m+r+1}} \, dx.\]

The integral is evaluated via the change of variable \( t = x^4 \) as

\[
\int_0^\infty \frac{x^{2r}}{(x^4 + 1)^{m+r+1}} \, dx = \frac{1}{4} B \left( \frac{r}{2} + \frac{1}{4}, m + \frac{r}{2} + \frac{3}{4} \right). \]

This yields

\[(8.3) \quad \left(\frac{d}{da}\right)^r N_{0,4}(a; m) \bigg|_{a=0} = \frac{(-1)^r (2r)!}{2^{2r+2m+3/2}} \frac{\pi}{m!r!} \prod_{t=1}^m (4t - 1 + 2r).\]

Therefore

\[P^{(l)}_m(0) = \frac{l!(2m + 2)!}{2^{m+2l}m!(m+1)!} \sum_{j=0}^l (-1)^j (m - l + j + 1)!(2j)! \prod_{t=1}^m (4(t - 1) + 2j).\]

We now split the sum according to the parity of \( j \). In the case \( j \) is odd \((= 2t - 1)\) we use

\[
\prod_{\nu=1}^m (4\nu - 1 + 2j) = \prod_{\nu=1}^m (4\nu + 1) \left( \prod_{\nu=m+1}^{m+t-1} (4\nu + 1)/ \prod_{\nu=1}^{t-1} (4\nu + 1) \right)
\]

and if \( j \) is even \((= 2t)\) we employ

\[
\prod_{\nu=1}^m (4\nu - 1 + 2j) = \prod_{\nu=1}^m (4\nu - 1) \left( \prod_{\nu=m+1}^{m+t} (4\nu - 1)/ \prod_{\nu=1}^{t} (4\nu - 1) \right).
\]

We conclude that

\[d_t(m) = X(m, l) \prod_{\nu=1}^m (4\nu - 1) - Y(m, l) \prod_{\nu=1}^m (4\nu + 1)\]

with

\[X(m, l) = \frac{(2m + 2)!}{2^{m+2l}m!(m+1)!} \sum_{t=0}^{\lfloor l/2 \rfloor} \frac{(m-l+2t+1)!(4t)!}{(2t)!2(l-2t)!(2m-2l+4t+2)!} \prod_{\nu=m+1}^{m+t+1} (4\nu - 1)/ \prod_{\nu=1}^{t} (4\nu - 1)\]

and

\[Y(m, l) = \frac{(2m + 2)!}{2^{m+2l}m!(m+1)!} \sum_{t=1}^{\lfloor (l+1)/2 \rfloor} \frac{(m-l+2t)!(4t-2)!}{(2t-1)!2(l-2t+1)!(2m-2l+4t)!} \prod_{\nu=m+1}^{m+t-1} (4\nu + 1)/ \prod_{\nu=1}^{t} (4\nu + 1).\]

The quotients of factorials appearing above can be simplified via

\[
\frac{(m+1)!}{(m-l+2t+1)!} = \prod_{j=1}^{l-2t} (j + m - l + 2t + 1)
\]
and
\[
\frac{(2m+2)!}{(2m-2l+4t+2)} = 2^{l-2t} \left( \prod_{i=1}^{l-2t} (i + m - l + 2t + 1) \right) \left( \prod_{i=1}^{l-2t} (2i + 2m - 2l + 4t + 1) \right).
\]
We conclude that
\[
d_l(m) = \frac{1}{l!m!2^{m+l}} \left( \alpha_l(m) \prod_{\nu=1}^{m} (4\nu - 1) - \beta_l(m) \prod_{\nu=1}^{m} (4\nu + 1) \right)
\]
with
\[
\alpha_l(m) = l! \sum_{t=0}^{[l/2]} \frac{(4t)!}{2^{2t}(l-2t)!} \prod_{\nu=m+1}^{m+t} (4\nu - 1) \left( \prod_{\nu=m-(l-2t)}^{m} (4\nu + 1) \right)
\]
and
\[
\beta_l(m) = l! \sum_{t=1}^{[(l+1)/2]} \frac{(4t-2)!}{2^{2t-1}(l-2t+1)!} \left( \prod_{\nu=m+1}^{m+t-1} (4\nu + 1) \right) \left( \prod_{\nu=m-(l-2t)}^{m} (2\nu + 1) \right).
\]
The identity
\[
\prod_{\nu=1}^{t} (4\nu - 1) = \frac{(4t)!}{2^{2t}(2t)!} \left( \prod_{\nu=1}^{t-1} (4\nu + 1) \right)^{-1}
\]
is now employed to produce
\[
\alpha_l(m) = \sum_{t=0}^{[l/2]} \left( \frac{l}{2t} \right) \prod_{\nu=m+1}^{m+t} (4\nu - 1) \prod_{\nu=m-(l-2t)}^{m} (2\nu + 1) \prod_{\nu=1}^{l-1} (4\nu + 1)
\]
and
\[
\beta_l(m) = \sum_{t=1}^{[(l+1)/2]} \left( \frac{l}{2t-1} \right) \prod_{\nu=m+1}^{m+t-1} (4\nu + 1) \prod_{\nu=m-(l-2t)}^{m} (2\nu + 1) \prod_{\nu=1}^{l-1} (4\nu - 1).
\]
We have proven:

**Theorem 8.1.** There exist polynomials \( \alpha_l(x) \) and \( \beta_l(x) \) with integer coefficients such that
\[
d_l(m) = \frac{1}{l!m!2^{m+l}} \left( \alpha_l(m) \prod_{k=1}^{m} (4k - 1) - \beta_l(m) \prod_{k=1}^{m} (4k + 1) \right).
\]
Based on extensive numerical calculations we propose

**Conjecture 8.2.** All the roots of the polynomials \( \alpha_l(m) \) and \( \beta_l(m) \) lie on the line \( \Re(m) = -1/2 \).
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