Granularity of wagers in games and the (im)possibility of savings *

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Abstract. In a casino where arbitrarily small bets are admissible, any betting strategy $M$ can be modified into a savings strategy that, not only is successful on each casino sequence where $M$ is (thus accumulating unbounded wealth inside the casino) but also saves an unbounded capital, by permanently and gradually withdrawing it from the game. Teutsch showed that this is no longer the case when a minimum wager is imposed by the casino, thus exemplifying a savings paradox where a player can win unbounded wealth inside the casino, but upon withdrawing a sufficiently large amount out of the game, he is forced into bankruptcy. We characterize the rate at which a variable minimum wager should shrink in order for saving strategies to succeed, subject to successful betting: if the minimum wager at stage $s$ shrinks faster than $1/s$, then savings are possible; otherwise Teutsch’s savings paradox occurs.

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1 Introduction

In a casino where a fixed minimum wager is imposed on all the bets, a player may be forced to quit the game, i.e. stop placing bets, either because (s)he ran out of capital or because (s)he chooses to leave the game, while still in the possession of a certain capital. This basic fact was exploited by (Bienvenu et al., 2010) in an investigation of the strength of effective betting strategies that have restrictions on the admissible wagers. This work motivated further studies on the power of restricted wager strategies, beyond the original algorithmic framework, in the case where the restriction is fixed throughout the game. Given a set of reals \( X \), an \( X \)-valued strategy is one that is restricted on wagers in \( X \). Given two finite sets \( A, B \) of rationals, by (Chalcraft et al., 2012), \( A \)-valued strategies can successfully replace any \( B \)-valued strategy, if and only if there exists \( r \geq 0 \) such that \( B \subseteq r \cdot A \) (where \( r \cdot A \) denotes the multiples of the elements of \( A \) with \( r \)). In particular, subject to the given condition, given any strategy restricted to bets in \( B \), we can produce a strategy that only bets values in \( A \) and succeeds (producing unbounded wealth) on any casino outcome sequence where the \( B \)-restricted strategy succeeds. This characterization was extended to infinite sets, with some additional conditions, in (Peretz and Bavly, 2015). Quite remarkably, (Teutsch, 2014) (also see (Peretz, 2015) for the corrected argument) constructed a casino which allows integer-wager strategies to succeed, producing unbounded wealth inside the casino, but any player who attempts to save an unbounded amount by removing it from the casino, is forced to bankruptcy.

**Our main result.** In the present work we consider casinos which impose a *variable* minimum wager on the players, which may change at each stage of the game according to some rule. Typically, one may think of a casino that allows smaller bets as the game evolves, thus permitting more sophisticated strategies from the players. Our main result, stated informally, is that

\[
\text{in a casino with a shrinking minimum wager, savings are possible for successful strategies if and only if the minimum wager imposed on each stage } s \text{ shrinks at a rate faster than } 1/s. \quad (1)
\]

At this point we make our discussion more precise through some terminology, while a formalization of the underlying notions is given in §2.1. By a (potentially biased) *casino* we mean a set of infinite binary sequences which represent the sequences of possible binary outcomes in a repeated betting game, along with possible restrictions on the admissible wagers at each stage. A betting strategy is a function that, given an initial capital, determines the wager and the favorable outcome, given any position (represented by the binary string of the previous outcomes) in the game. A strategy is successful along a casino sequence if along the game its capital is unbounded. It is customary and natural to restrict the choice of strategies amongst a countable collection, typically representing the feasible or implementable strategies. From an algorithmic perspective, as did many of the authors cited above, we may consider strategies that are computable, in the sense that they can be simulated by a Turing machine. However, as explicitly noted by (Peretz and Bavly, 2015), many of the arguments in this area (including ours) holds for any fixed countable collection of admissible strategies.

A *savings strategy* is a strategy along with a non-decreasing *savings function* which indicates the part of the capital at each position of the game which is saved, hence permanently removed from the active capital of the strategy that can be used for betting. A savings strategy is successful along a casino sequence if its savings function is unbounded. A strategy (or savings strategy) is successful in a casino if its wagers meet the restriction of the casino and it is successful in all outcome sequences of the casino.\(^1\)

\(^1\)Such casinos with restricted possible outcome-sequences were termed ‘probability-free’ in (Chalcraft et al., 2012).
Given a non-decreasing function $g : \mathbb{N} \to \mathbb{N}$, we say that a betting strategy is $g$-granular if its wagers at stage $s$ of the game are (integer) multiples of $2^{-g(s)}$. A casino is $g$-granular if it only accepts wagers that are (integer) multiples of $2^{-g(s)}$ at stage $s$. Consider the sum

$$s_g := \sum_{s \in \mathbb{N}} 2^{-g(s)}$$

and note that for $g(s) = \log(s + 1)$ the sum diverges, while for any slightly more fast-growing order like $g(s) = \epsilon \cdot \log(s + 1)$ with $\epsilon > 1$, the sum converges. Then the following is a more precise version of (1):

Given non-decreasing $g : \mathbb{N} \to \mathbb{N}$ with $s_g < \infty$, for any $g$-granular casino with a winning strategy there exists a savings strategy that succeeds in it. Conversely if $s_g = \infty$, there exists a $g$-granular casino with a winning strategy, such that no savings strategy succeeds in it.

We note that the integer-valued strategies (where wagers are required to be integers) have been studied in many articles, including (Bienvenu et al., 2010; Barmpalias et al., 2015; Herbert, 2016), and are a very special case of $g$-granular strategies, where $g$ is the zero function. Hence (2) implies and generalizes the savings paradox of (Teutsch, 2014) (which was shown for integer-valued strategies), and in fact characterizes the granularity of the casino that is needed in order for this paradox to occur. Furthermore, granular strategies have played a crucial role in the analysis of restricted oracle computations from algorithmically random sources (Barmpalias et al., 2016). Roughly speaking, oracle-computations with oracle-use $n \mapsto n + g(n)$ correspond to $g$-granular strategies. Hence, understanding how the granularity of a betting strategy restricts its power can be used in order to study the impact that restrictions on the access to an oracle can have on an oracle-computation. It is also interesting to note that some existing arguments regarding restricted wager strategies (in the sense of $X$-valued strategies that we discussed in the beginning of this section) actually use wagers of gradually finer granularity; e.g. see the proof of (Peretz, 2015, Theorem 14).

Outline of the presentation. The casino interpretation of our results as we discussed them is rather standard, but was discussed in greater detail and in relation to savings and restricted wagers, in (Chalcraft et al., 2012; Teutsch, 2014; Peretz, 2015) and (Peretz and Bavly, 2015). In §2 we give the required background on granular saving strategies in terms of martingales and supermartingales, that is required for the formal statement and proof of our main result (2). Our results are combinatorial in essence, and do not require any algorithmic background. However martingales, as strategies, are well-known as one of the foundational approaches to algorithmic information theory, as pioneered by (Schnorr, 1971a,b). Hence our results admit a natural interpretation in algorithmic information theory, which we briefly discuss in §2.3.

In §3 we start by formally stating of our main result (2) in the form of two theorems in the framework that is laid out in §2. We give the proof of the positive part of this equivalence in §3.1, and in §3.2, §3.3 we give an introductory analysis that will be used for the proof of the negative and more challenging part of the equivalence. We apply this analysis in §3.4 in order to show a finite version of the second part of (2) where the list of possible strategies considered is finite. Finally §4 is devoted to the proof of the full infinite version of the second part of (2), which is built upon the argument of §3.4 with additional tools and ideas, which we overview and justify in §4.1 and §4.2. The main construction appears in §4.5 and is verified in §4.6, based on facts that are proved in the technical sections §4.3 and §4.4.
2 Background and simple facts about granular saving strategies

In §2.1 we formalize strategies and savings strategies in terms of (super)martingales in a rather standard way, including a simplified version of some terminology in (Peretz, 2015). Then in §2.2 we introduce granular strategies and show some simple properties, especially in relation to saving, which will be used in the main part of this article in §3 and §4.

2.1 Betting and saving strategies as martingales and supermartingales

Betting strategies are formalized by martingales, expressing the capital after each betting stage and each casino outcome. Formally, a martingale is a function $M : 2^{<\omega} \to \mathbb{R}^+$ with the property that

$$2M(\sigma) = M(\sigma \ast 0) + M(\sigma \ast 1)$$

for all $\sigma$. These deterministic (as opposed to probabilistic) martingales provide a formalization of betting strategies on an infinite coin-tossing game: at stage $|\sigma|$ our capital is $M(\sigma)$ and our wager for the next bet is $M(\sigma \ast 1) - M(\sigma)$, which can also be written as

$$w_M(\sigma) = \frac{M(\sigma \ast 1) - M(\sigma \ast 0)}{2}.$$ 

If $w_M(\sigma) > 0$ then at position $\sigma$ we bet on outcome 1, capital $|w_M(\sigma)|$; otherwise we bet the same capital on outcome 0. Hence $M(\sigma \ast j) = M(\sigma) + w_M(\sigma)$ reflects the updated capital with respect to either outcome $j \in \{0, 1\}$. The following multiplicative form of a martingale $M$ is also useful:

$$M(\sigma) = M(\lambda) \cdot \prod_{i<|\sigma|} \left( 1 + \frac{w_M(\sigma \upharpoonright i+1)}{M(\sigma \upharpoonright i)} \right)$$

Saving strategies are formalized by supermartingales, which are functions $M : 2^{<\omega} \to \mathbb{R}^+$ such that

$$2M(\sigma) \geq M(\sigma \ast 0) + M(\sigma \ast 1)$$

for all $\sigma$. Supermartingales can be thought of as strategies (i.e. martingales) with the difference that after each bet there is a certain loss of liquid capital, i.e. capital that can be used for betting. The marginal savings of a supermartingale $M$ at $\sigma$ is defined as:

$$M^*(\sigma) = M^*(\sigma) - \frac{M(\sigma \ast 0) + M(\sigma \ast 1)}{2}$$

i.e. the amount that is lost from position $\sigma$ to the next bet, which is the amount by which $M$ fails to satisfy the martingale inequality (3) at $\sigma$. Given a supermartingale $M$ define the cover of $M$ to be the unique martingale whose initial capital and wagers are the same as those of $M$; we denote the cover of $M$ by $\bar{M}$. The (accumulated) savings of $M$ at the various positions are defined by

$$S_M(\sigma) = \bar{M}(\sigma) - M(\sigma).$$

Clearly $S_M(\sigma)$ is simply the sum of the marginal savings of $M$ on the initial segments of $\sigma$. The wager of a supermartingale $M$ is also given by (4) but now note that it may differ from $M(\sigma \ast 1) - M(\sigma)$, which is the standard expression of wagers in the case of martingales. A (super)martingale $M$ is called history-independent if the wager at any position only depends on the stage of the game, and not the previous outcomes, i.e. if $w_M(\sigma) = w_M(\tau)$ when $|\sigma| = |\tau|$.
Definition 2.1 (Success for strategies and savings). We say that a (super)martingale \( M \) is successful along \( X \) if \( \lim \sup_n M(X \upharpoonright n) = \infty \); we say that \( M \) successfully saves if \( S_M(X \upharpoonright n) \to \infty \) as \( n \to \infty \).

A folklore and useful fact regarding the success of strategies is the following:

Savings trick: Given any betting strategy \( M \), there exists a savings strategy \( N \) which is successful on every sequence on which \( M \) is successful. Formally, given any supermartingale \( M \) there exists a supermartingale \( N \) such that \( \lim_n S_N(X \upharpoonright n) = \infty \) for each sequence \( X \) such that \( \lim \sup_n M(X \upharpoonright n) = \infty \); moreover \( N \) is computable from \( M \).

The saving strategy \( N \) in (7) is rather intuitive. Without loss of generality we may assume that \( M(\lambda) > 1 \). At the beginning, \( N \) bets identically to \( M \), until some position of the game is reached where \( M(\sigma) \) is double the initial capital \( M(\lambda) \). At such a position \( \sigma_1 \) strategy \( N \) saves 1 (making the difference between the \( N \) and \( M \) capital equal to 1), and proceeds with the subsequent bets proportionally adjusted, where the proportion is \( (M(\sigma_1) - 1)/M(\sigma_1) \). At the next position \( \sigma_2 \) where \( M \) doubles with respect to the previous marked value \( M(\sigma_1) \), we repeat the same action, letting \( N \) save another 1, and adjusting the subsequent bets proportionally with respect to the ratio \( (M(\sigma_2) - 2)/M(\sigma_2) \) and so on.

Note that by the proportionality of bets and the multiplicative form (5) of strategies, between positions \( \sigma_1 \) and \( \sigma_2 \) the ratio \( N(\sigma)/M(\sigma) \) remains equal to \( (M(\sigma_1) - 1)/M(\sigma_1) \). In particular, at position \( \sigma_2 \) where \( M \) doubles its capital compared to position \( \sigma_1 \), the same happens to \( N \), compared to \( N(\sigma_1) \). Hence, given that \( N(\sigma_1) = M(\sigma_1) - 1 > 1 \), inductively we have \( N(\sigma_n) \geq 2N(\sigma_{n-1}) - 1 > 1 \) and \( S_N(\sigma_n) = n \) for each \( n > 1 \) where \( \sigma_n \) is defined. Then it is clear that along any \( X \) where \( M \) is successful, the sequence \( (\sigma_i) \) of initial segments of \( X \) is totally defined, hence showing the success of the savings strategy \( N \) along \( X \).

The savings trick implies that the standard success condition \( \lim \sup_n M(X \upharpoonright n) = \infty \) for a supermartingale \( M \) is essentially equivalent to \( \lim_n M(X \upharpoonright n) = \infty \) in the following sense:

\[
\text{Given any supermartingale } M \text{ there exists a supermartingale } T \text{ such that } \lim_n T(X \upharpoonright n) = \infty \text{ for each } X \text{ where } M \text{ is successful; moreover } T \text{ is computable from } M.
\]

(8)

In order to consider realistic strategies it is natural to require that the martingales we consider are definable or have some effectivity properties, for example they are programmable in a Turing machine. These are called computable martingales. However, following the approach in (Peretz and Bavly, 2015), our results can be expressed in terms of any fixed a countable class of martingales that we regard as ‘admissible’.

2.2 Granularity of strategies

Generally speaking, the ‘granularity’ of a function \( f : 2^{<\omega} \to \mathbb{Q} \) measures how far the values of \( f \) are from being integers. For example, informally speaking, we may say that the granularity of \( f \) is the function \( g : \mathbb{N} \to \mathbb{N} \) such that \( g(|\sigma|) \) is the minimum non-negative integer such that \( f(\sigma) \) is an integer multiple of \( 2^{-g(|\sigma|)} \). If we apply this notion to the wagers of a strategy, we can model strategies that adhere to a minimum-bet policy that is a variable of the current round in the game.

Definition 2.2 (Granular martingales). Given a non-decreasing \( g : \mathbb{N} \to \mathbb{N} \), we say that a (super)martingale \( M \) is \( g \)-granular if for every string \( \sigma \) the wager \( w_M(\sigma) \) is a integer multiple of \( 2^{-g(|\sigma|)+1} \).

A function \( f : 2^{<\omega} \to \mathbb{Q}^+ \) is \( g \)-granular if for each string \( \sigma \) the value \( f(\sigma) \) is an integer multiple of \( 2^{-g(|\sigma|)} \). One may also consider to apply the notion of granularity to the capital function \( \sigma \mapsto M(\sigma) \) of a strategy.
instead of its wagers, thus obtaining a stronger notion. However, as we observe below, such a distinction is not consequential in the present work. By the above definitions of granularity it follows that

\[ g : \mathbb{N} \to \mathbb{N} \text{ and a } g\text{-granular martingale } M, \text{ the function } \sigma \mapsto M(\sigma) \text{ is } g\text{-granular if and only if } M(\lambda) \text{ is an integer multiple of } 2^{-g(0)}. \]

Hence, for example, given a non-uniform version of \( N(\sigma) \), the function \( \sigma \mapsto N(\sigma) \) is \( g\)-granular and \( |M(\sigma) - N(\sigma)| = O(1) \). More generally, we show that any \( g\)-granular (super)martingale \( M \) can be easily transformed into a (super)martingale which differs by at most a constant from \( M \), it is \( g\)-granular as a function and its savings function takes integer values.

**Lemma 2.3.** Given non-decreasing \( g : \mathbb{N} \to \mathbb{N} \) and a \( g\)-granular supermartingale \( M \), there exists a supermartingale \( N \) such that \( |M(\sigma) - N(\sigma)| = O(1) \), the function \( \sigma \mapsto N(\sigma) \) is \( g\)-granular and \( S_N(\sigma) \in \mathbb{N} \) for each \( \sigma \). Moreover \( N \) is computable from \( M \) and \( g \), and if \( M \) is a martingale then \( N \) is also a martingale.

**Proof.** Let \( T \) be the unique martingale which has the same wagers as \( M \) and \( T(\lambda) = [M(\lambda)] \). Then clearly the function \( \sigma \mapsto T(\sigma) \) is \( g\)-granular, \( \overline{M}(\sigma) \leq T(\sigma) \) and \( |\overline{M}(\sigma) - T(\sigma)| = O(1) \). Define \( N(\sigma) = T(\sigma) - [S_M(\sigma)] \) and note that since \( |M(\sigma) - T(\sigma)| = O(1) \) we have \( |N(\sigma) - M(\sigma)| = O(1) \). Since \( \overline{M}(\sigma) \leq T(\sigma) \) we also have \( 0 \leq M(\sigma) \leq N(\sigma) \). By the properties of \( T \), the function \( \sigma \mapsto N(\sigma) \) is \( g\)-granular. Finally, note that \( N \) is computable from \( M \) and \( g \), and in the case when \( M \) is a martingale we have \( S_M(\sigma) = 0 \) for all \( \sigma \), so \( N = T \) and \( N \) is a martingale. \( \square \)

The reader may have noticed that the argument justifying the savings trick is no longer applicable if wagers are required to have a certain granularity, since such a condition would prevent us from using proportional strategies. Despite this fact, a non-uniform version of (8) continues to hold for granular strategies.

**Proposition 2.4** (Success notions for granular strategies). Suppose that \( g : \mathbb{N} \to \mathbb{N} \) is nondecreasing and \( M \) is a \( g\)-granular supermartingale which is successful on some sequence \( X \). Then there exists a \( g\)-granular supermartingale \( T \) which is computable from \( M \) and \( \lim_n T(X \upharpoonright n) = \infty \).

**Proof.** In the case where \( \lim_n M(X \upharpoonright n) = \infty \) we can simply let \( T := M \). Otherwise let \( q \) be a positive rational upper bound of \( r := \lim \inf n M(X \upharpoonright n) \) and let \( T \) have initial capital a rational \( T(\lambda) > q - r \). Let \( m_n \) be such that for each \( n \geq m_n \) we have \( M(X \upharpoonright n) \geq r \). Then let \( T \) produce part of the bets of \( M \) along an arbitrary sequence \( Y \) as follows: wait until some \( n_0 \geq m_n \) such that \( M(Y \upharpoonright n_0) < q \), and then let \( m_0 \) be the least \( m > n_0 \) such that \( M(Y \upharpoonright m) > q + 1 \) (if such number does not exist, let \( m_0 = \infty \)). In the interval \( [m_n, m_0] \) the strategy \( T \) does not place any bets, while in \( [n_0, m_0] \) it places the same bets that \( M \) does, along \( Y \). Hence \( T(Y \upharpoonright n) - T(Y \upharpoonright n_0) = M(Y \upharpoonright n) - M(Y \upharpoonright n_0) \) and since \( r \geq M(X \upharpoonright n_0) \), we have \( M(Y \upharpoonright n) - M(Y \upharpoonright n_0) > r - q \) for each \( n \geq m_n \). Hence \( T(Y \upharpoonright n) > T(Y \upharpoonright n_0) + (r - q) > 0 \) for each \( n \in [n_0, m_0] \). Moreover, in the case that \( m_0 < \infty \), \( M(Y \upharpoonright m_0) - M(Y \upharpoonright n_0) > 1 \), so \( T(Y \upharpoonright m_0) > T(Y \upharpoonright n_0) + 1 \).

This process repeats in the same way, defining the intervals \( [m_n, n_i] \) where \( T \) does not bet, and the adjacent intervals \( [m_{n-1}, m_i] \) where \( T \) copies the bets of \( M \). If for some \( i \) we have \( m_i = \infty \) then after position \( Y \upharpoonright n_i \) strategy \( T \) copies the bets of \( M \) along \( Y \). The argument that we used above to show that \( T \) is nonnegative, inductively shows that for each \( i \geq 1 \) such that \( m_i < \infty \) and each \( n > m_i \) we have \( T(Y \upharpoonright n) > i + 1 \). Moreover clearly \( T \) is \( g\)-granular and computable from \( M \). Finally, in the case where \( \limsup_n M(Y \upharpoonright n) = \infty \), the endpoints \( n_i, m_i \) are defined for all \( i \in \mathbb{N} \), which means that \( \lim_n T(Y \upharpoonright n) = \infty \). \( \square \)
The careful reader may notice that in the second case of the proof of Proposition 2.4, the case where \( \liminf_n M(X \restriction n) < \infty \), we essentially make a savings strategy. In particular, in this case \( T \) can be easily modified into a supermartingale \( N \) such that \( \lim_n S_N(X \restriction n) = \infty \). Since the definition of \( N \) only depends on \( M \) and an rational upper bound of \( \liminf_n M(X \restriction n) \), we have the following more detailed fact.

\[ \text{Suppose that } g: \mathbb{N} \rightarrow \mathbb{N} \text{ is nondecreasing and } M \text{ is a } g\text{-granular supermartingale. Then there exists a sequence } (N_i) \text{ of } g\text{-granular supermartingales such that for each } X \text{ where } M \text{ is successful and } \liminf_n M(X \restriction n) < \infty \text{ there exists } i \text{ such that } \lim_n S_{N_i}(X \restriction n) = \infty. \]

Moreover \( (N_i) \) is uniformly computable from \( M \) and \( g \).

However our main result in §3, Theorem 3.2, shows that the condition \( \liminf_n M(X \restriction n) < \infty \) in (9) is necessary. In the same way, condition \( \lim_n T(X \restriction n) = \infty \) in Proposition 2.4 cannot be replaced by \( \lim_n S_T(X \restriction n) = \infty \). We may conclude that

any casino sequence \( X \) where a computable \( g \)-granular strategy succeeds but no computable \( g \)-granular savings strategy succeeds, must also have the property that any successful computable \( g \)-granular strategy \( M \) on \( X \) has \( \lim_n M(X \restriction n) = \infty \)

where \( g \) here is assumed to be computable.

### 2.3 Algorithmic randomness and granular strategies

One approach for the formalization of algorithmic randomness of infinite binary sequences is based on betting strategies and was pioneered in (Schnorr, 1971a,b). The intuitive idea here is that algorithmically random sequences should be sequences of binary outcomes on which no ‘effective’ betting strategy can succeed. This approach is essentially equivalent to earlier formalizations in terms of statistical tests in (Martin-Löf, 1966) or compression in (Levin, 1973). In general, for each choice of a countable collection of strategies as the effective betting strategies, we get a corresponding randomness notion. Standard choices for ‘effective’ are strategies computable by a Turing machine or computably enumerable strategies (which can also be seen as infinite mixtures of computable strategies). We can also interpret granularity as a feasibility condition on the strategies, thus obtaining a more restricted randomness against granular strategies. Considering saving strategies as opposed to betting strategies can also potentially weaken the corresponding randomness notion.

In this sense, our main result can be seen as a characterization of when randomness based on saving strategies differs from the randomness based on betting strategies, in terms of the granularity imposed on the strategies. The first part of (2) says that in the case of fine granularity the two randomness notions are equivalent, while the second part says that for coarse granularity the randomness based on saving strategies is weaker than the randomness based on betting strategies.

### 3 Granularity and savings of strategies

We formally state our main result (2) in the form of the following two theorems.

**Theorem 3.1** (Fine granularity and savings). Let \( g: \mathbb{N} \rightarrow \mathbb{N} \) be nondecreasing and such that \( \sum_n 2^{-g(n)} \) is finite. Given any supermartingale \( M \), there exists a \( g \)-granular supermartingale \( N \) which has unbounded
savings along any casino sequence where \( M \) is successful; formally \( \lim_n S_N(X \uparrow n) = \infty \) for each \( X \) such that \( \limsup_n M(X \uparrow n) = \infty \). Moreover \( N \) is computable from \( M \) and \( g \).

**Theorem 3.2** (Coarse granularity and savings). Let \( g : \mathbb{N} \to \mathbb{N} \) be nondecreasing and such that \( \sum_n 2^{-g(n)} \) diverges. There exists a \( g \)-granular martingale \( M \) such that for every list of \( g \)-granular supermartingales \((N_i)\) there exists a casino sequence \( X \) on which \( M \) is successful and \( \lim_n S_N(X \uparrow n) < \infty \) for each \( i \). Moreover \( M \) is computable in \( g \).

The proof of Theorem 3.2 in the latter part of this section and mainly §4 makes heavy use of **divisions** that are not Euclidian in the strict sense as the numbers involved may not be integers. Given any real numbers \( t, m \) such that \( t \geq 0 \) and \( m > 0 \) we define the quotient \( q \) of the division of \( t \) by \( m \) as the largest integer such that \( q \cdot m \leq t \), and define the remainder \( r \) of this division as \( t - q \cdot m \), so that \( 0 \leq r < m \).

### 3.1 Proof of Theorem 3.1

Let \( G \) be an integer strict upper bound of \( 2 + \sum_{n \in \mathbb{N}} 2^{-g(n)} \) and let \( M \) be a supermartingale. Without loss of generality, we assume \( M(\sigma) \geq 1 \) for all \( \sigma \in 2^{\omega} \), because otherwise we may use \( M + 1 \) instead of \( M \) in the argument. We define the required supermartingale \( N \) and its cover \( \tilde{N} \) simultaneously, following a granular version of the savings argument we used to justify (7).

For any \( \sigma \in 2^{\omega} \), consider the finite sequence \( n_i, i < k \) defined inductively as follows: \( n_0 = 0 \), and for each \( i \) let \( n_{i+1} \) be the least number (if such exists) such that \( n_i < n_{i+1} \) and \( M(\sigma \uparrow n_{i+1}) \geq 2M(\sigma \uparrow n_i) \) for each \( i \). Then \( k \) is the least number \( i \) such that \( n_i \) is undefined, and we may let \( l(\sigma) := \{ n_i \mid i < k \} \) and \( l(\sigma) := |l(\sigma)| - 1 \).

We define \( N, \tilde{N} \) by induction as follows.

First, let \( \tilde{N}(\lambda) = N(\lambda) = [G \cdot M(\lambda)] + 1 \). Then for each \( \sigma \in 2^{\omega} \), and each real \( x \) let \( \text{Int}_g(\sigma, x) \) be the largest integer multiple of \( 2^{-|\sigma| - 1} \) which is at most \( |x| \), multiplied by the sign of \( x \). It follows that

\[
| \text{Int}_g(\sigma, x) - x | < 2^{-|\sigma| + 1} \quad \text{for each } \sigma, x. \tag{11}
\]

We define the wager for \( N \) on \( \sigma \), based on the wager of \( M \), but scaled by the fraction \( (\tilde{N}(\sigma) - l(\sigma))/M(\sigma) \) and rounded to the nearest granular value:

\[
w_N(\sigma) = \text{Int}_g\left( \sigma, \frac{w_M(\sigma) \cdot (\tilde{N}(\sigma) - l(\sigma))}{M(\sigma)} \right) \tag{12}
\]

as well as the values of \( \tilde{N}(\sigma \ast i), N(\sigma \ast i) \) recursively, in terms of the values at \( \sigma \):

\[
\tilde{N}(\sigma \ast 1) = \tilde{N}(\sigma) + w_N(\sigma) \quad \text{and} \quad \tilde{N}(\sigma \ast 0) = \tilde{N}(\sigma) - w_N(\sigma) \]

\[
N(\sigma \ast 1) = \tilde{N}(\sigma \ast 1) - l(\sigma) \quad \text{and} \quad N(\sigma \ast 0) = \tilde{N}(\sigma \ast 0) - l(\sigma). \tag{13}
\]

Clearly \( N \) is \( g \)-granular and computable from \( M \). The intuition for the definition of \( N \) is the same as the intuition in the argument for (7) that we discussed above, but adapted to \( g \)-granular values. It remains to show that \( \tilde{N} \) is the cover of \( N \) and \( \lim_n S_N(X \uparrow n) = \infty \) for each \( X \) such that \( \limsup_n M(X \uparrow n) = \infty \).

**Lemma 3.3** (Growth of \( \tilde{N} \)). For all \( \sigma \in 2^{\omega} \), \( M(\sigma) + l(\sigma) < \tilde{N}(\sigma) \).
Proof. By (11) and the definition of \( w_N(\sigma) \) we have that for any \( \sigma \in 2^{<\omega} \):
\[
\left| w_N(\sigma) - w_M(\sigma) \cdot \frac{\bar{N}(\sigma) - l(\sigma)}{M(\sigma)} \right| \leq 2^{-g(|\sigma|+1)}.
\]

Then
\[
\frac{\bar{N}(\sigma*1) - l(\sigma)}{M(\sigma*1)} \geq \frac{\bar{N}(\sigma) + w_M(\sigma) \cdot \frac{\bar{N}(\sigma) - l(\sigma)}{M(\sigma)} - 2^{-g(|\sigma|+1)} - l(\sigma)}{M(\sigma) + w_M(\sigma)} = \frac{\bar{N}(\sigma) - l(\sigma)}{M(\sigma)} - \frac{2^{-g(|\sigma|+1)}}{M(\sigma*1)}
\]
so
\[
\frac{\bar{N}(\sigma * i) - l(\sigma)}{M(\sigma * 1)} \geq \frac{\bar{N}(\sigma) - l(\sigma)}{M(\sigma)} - 2^{-g(|\sigma|+1)}.
\]
(14)

for \( i = 1 \). Similarly, under outcome 0 we have:
\[
\frac{\bar{N}(\sigma * 0) - l(\sigma)}{M(\sigma * 0)} \geq \frac{\bar{N}(\sigma) - w_M(\sigma) \cdot \frac{\bar{N}(\sigma) - l(\sigma)}{M(\sigma)} - 2^{-g(|\sigma|+1)} - l(\sigma)}{M(\sigma) + w_M(\sigma)} = \frac{\bar{N}(\sigma) - l(\sigma)}{M(\sigma)} - \frac{2^{-g(|\sigma|+1)}}{M(\sigma * 0)}
\]
so (14) also holds for \( i = 0 \). For \( i \in \{0,1\} \), if \( I(\sigma * i) = I(\sigma) \), then we have \( l(\sigma * i) = l(\sigma) \) and (14) gives:
\[
\frac{\bar{N}(\sigma * i) - l(\sigma * i)}{M(\sigma * i)} \geq \frac{\bar{N}(\sigma) - l(\sigma)}{M(\sigma)} - 2^{-g(|\sigma * i|)}.
\]
(15)

If \( I(\sigma * i) \neq I(\sigma) \), then \( |\sigma * i| \leq I(\sigma * i) \), \( l(\sigma * i) = l(\sigma) + 1 \) and \( M(\sigma * i) \geq 2^{l(\sigma * i)} \cdot M(\lambda) \). Combining these facts with (14), we get:
\[
\frac{\bar{N}(\sigma * i) - l(\sigma * i)}{M(\sigma * i)} = \frac{\bar{N}(\sigma * i) - l(\sigma)}{M(\sigma * i)} - \frac{1}{M(\sigma * i)} \geq \frac{\bar{N}(\sigma) - l(\sigma)}{M(\sigma)} - \frac{2^{-g(|\sigma * i|)}}{M(\sigma)} - 2^{-l(\sigma)-1}.
\]
(16)

Inductively applying (15) and (16) for the cases \( I(\sigma * i) = I(\sigma) \) or \( I(\sigma * i) \neq I(\sigma) \) respectively, we get:
\[
\frac{\bar{N}(\sigma) - l(\sigma)}{M(\sigma)} \geq \frac{\bar{N}(\lambda) - l(\lambda)}{M(\lambda)} - \sum_{n=1}^{l(\sigma)} 2^{-g(n)} - \sum_{n=1}^{I(\sigma)} 2^{-n} \geq \sum_{n \in \mathbb{N}} 2^{-g(n)} - 1 > 1
\]

which gives the required inequality. \( \square \)

Lemma 3.4 (Properties of \( N \) and \( \bar{N} \)). The function \( \bar{N} \) is a g-granular martingale and \( N \) is a g-granular supermartingale; moreover \( \bar{N} \) is the cover of \( N \).

Proof. By the equations (13) in the definition of \( N, \bar{N} \) and since \( l(\sigma * i) \geq l(\sigma) \) for \( i \in \{0,1\} \), we have
\[
N(\sigma * i) = \bar{N}(\sigma * i) - l(\sigma) \geq \bar{N}(\sigma * i) - l(\sigma * i) > 0.
\]

which also shows that \( \bar{N}(\sigma) > 0 \) for all \( \sigma \). Given this fact, the equations (13) and the definition of the wager of a (super)martingale from §2.1 we get that:

(a) \( \bar{N} \) is a martingale and \( N(\sigma) \leq \bar{N}(\sigma) \) for all \( \sigma \);

(b) \( N(\sigma * 0) + N(\sigma * 0) \leq 2N(\sigma) \) so \( N \) is a supermartingale;

(c) both \( N \) and \( \bar{N} \) have the same wager \( w_N \) given by (12);
It follows from (a)-(c) that $\tilde{N}$ is the cover of $N$. Moreover by (12) the function $w_N$ is $g$-granular, so both $\tilde{N}$ and $N$ are $g$-granular.

Finally we verify that $N$ has the desired property. Suppose that $\lim sup_n M(X \rvert_n) = \infty$. It follows that $\lim_n l(X \rvert_n) = \infty$. On the other hand $N(X \rvert_{n+1}) = \tilde{N}(X \rvert_{n+1}) - l(X \rvert_n)$ and $N(X \rvert_{n+1}) > 0$ for each $n$, so

$$\lim_n S_N(X \rvert_{n+1}) = \lim_n \left( \tilde{N}(X \rvert_{n+1}) - N(X \rvert_{n+1}) \right) = \lim_n l(X \rvert_n) = \infty$$

which concludes the proof of Theorem 3.1.

### 3.2 Abstract notation, parameters, and the winning martingale for Theorem 3.2

It will be beneficial to view each saving strategy $T$ or supermartingale, as a stochastic process on the underlying product space of binary outcomes. Furthermore, without loss of generality, we distinguish each stage into a savings step, where $T$ simply possibly reduces its capital, thus producing a marginal saving, and a subsequent betting step, when $T$ places its bet and the outcome is revealed (modifying the capital of $T$ accordingly). In this fashion, formally, $T$ is a function from $2^{<\omega} \times [0, 1]$ to $\mathbb{Q}$ such that for each $\sigma \in 2^{<\omega}$ we have $T(\sigma, 1) \leq T(\sigma, 0)$ and $2T(\sigma, 1) = T(\sigma * 0, 0) + T(\sigma * 1, 0)$ (a standard betting step gives the capitals under the two possible outcomes at the first step of stage $|\sigma| + 1$ based on the initial capital at the second step of stage $|\sigma|$).\footnote{This formulation does not play a role in the proof of the finite case of Theorem 3.2, but is helpful for the infinite case which is presented in §4.} Alternatively, we may define the wager on $1$ at betting position $\sigma$ as

$$w_T(\sigma) = \frac{T(\sigma * 1, 0) - T(\sigma * 0, 0)}{2}$$

and define $T(\sigma * 0, 0) = T(\sigma, 1) - w_T(\sigma)$ and $T(\sigma * 1, 0) = T(\sigma, 1) + w_T(\sigma)$, while the marginal saving at saving position $\sigma$ is $T(\sigma, 0) - T(\sigma, 1)$. We will simplify the notation and often suppress the position $\sigma$ or betting and saving positions $(\sigma, 0), (\sigma, 1)$ respectively. Instead, we will specify a stage in the process and one of its steps (saving or betting step) and talk about the process $t$ (corresponding to $T$) at the beginning of the stage and step in question, and denote $t'$ the capital after the step has been completed (which is also the capital at the beginning of the next step).

We also have the stochastic processes $g^*, g^+$ which we call the current granule and the next granule of a stage in the game. Formally, these are functions from the positions $2^{<\omega}$ of the game (the history of outcomes) to $\mathbb{Q}$ so that $g^*(\sigma) = 2^{-\|\sigma\|}, g^+(\sigma) = 2^{-\|\sigma\|+1}$. As before, we often suppress the argument $\sigma$ and treat these as series of random variables that take values according to the stage of the game. Similarly, we let $w$ be the process corresponding to $\sigma \mapsto w_T(\sigma)$. We also let $q, r$ be the quotient and remainder of the division of $t$ by $m$ and let $q', r'$ denote their values after a step in the process has been completed. Hence

$$t = q \cdot m + r \quad \text{where} \quad r < m$$

and $q$ is an non-negative integer. The notation we just discussed is summarized in Table 1 and the first entry of Table 2, where $\mathbb{Z}^+$ denotes the set of non-negative integers and $g^* \cdot \mathbb{Z}^+$ the set of non-negative integer multiples of $g^*$. Note that these considerations require that $m$ is positive, which is something that we will verify along the series of transitions that is relevant for the proof of Theorem 3.2.
For Theorem 3.2 consider the $g$-granular martingale that starts with capital $2^{-g(0)}$ and at each stage $n + 1$ bets the least possible amount $2^{-g(n+1)}$ on outcome 1 (unless its current capital is less than this, in which case it does not bet). This can be written as

$$M(σ ∗ j) = \begin{cases} M(σ) + (-1)^{j+1} \cdot 2^{-g(|σ|+1)} & \text{if } M(σ) \geq 2^{-g(|σ|+1)}; \\ M(σ) & \text{if } M(σ) < 2^{-g(|σ|+1)}. \end{cases}$$

but it can also be written in our abstract notation as a stochastic process $m$, so that at any betting step,

$$m' = \begin{cases} m + g^+ & \text{if outcome is 1 and } m \geq g^+; \\ m - g^+ & \text{if outcome is 0 and } m \geq g^+; \\ m & \text{if } m' < g^+. \end{cases}$$

### 3.3 Analysis of a transition for Theorem 3.2

For Theorem 3.2 we wish to construct an infinite sequence of outcomes $X$ along which $m$ diverges to infinity (i.e. $M(X \upharpoonright n) \to \infty$ as $n \to \infty$) while the given $t$ accumulates a finite amount of savings along $X$ (i.e. $\lim_n S_T(X \upharpoonright n)$ is finite). The key to achieving this, as we will see, is to ensure that along $X$ the ratio $t/m$ of capitals is non-increasing, i.e.

$$\frac{t'}{m'} \leq \frac{t}{m} \quad \text{at each saving or betting step along } X. \quad (17)$$

For the case of a saving step, this inequality follows from the fact that $m' = m$ and $t' \leq t$ at saving step of any stage. In order to ensure (17) for the case of a betting step along $X$, we have to choose the outcomes appropriately.

Under the 1-outcome we have $t' = t + w$ and $m' = m + g^+$, so

$$\frac{t'}{m'} \leq \frac{t}{m} \iff \frac{t + w}{m + g^+} \leq \frac{t}{m} \iff \frac{w}{m + g^+} \leq \frac{t}{m} \iff w \leq g^+ \cdot \frac{r}{m} \iff w \leq g^+ \cdot q$$

where the last equivalence follows from the fact that $r < m$, $q$ is an integer and $w$ is an integer multiple of $g^+$. Under 0-outcome we have $t' = t - w$ and $m' = m - g^+$, so

$$\frac{t'}{m'} < \frac{t}{m} \iff \frac{t - w}{m - g^+} < \frac{t}{m} \iff \frac{w}{m - g^+} > \frac{t}{m} \iff w > g^+ \cdot \frac{r}{m} \iff w > g^+ \cdot q$$

Table 1: Parameters in the analysis of the atomic version of Theorem 3.2 in §3.2 and §3.3
and note that we need strict inequality in the 0-outcome in order to get the equivalence. This means that

\[
\text{if we follow the rule 1-outcome if } w \leq g^+ \cdot q' \text{ and 0-outcome if } w > g^+ \cdot q' \text{ then } t'/m' \leq t/m
\]
at each step and stage; moreover, in the case of 0-outcome in a betting step, \( t'/m' < t/m \). \hfill (18)

For this reason, if we let \( x \) be the outcome chosen at the betting step of each stage (where \( t, m, g^+, q \) have their values) we define:

\[
x = \begin{cases} 
1 & \text{if } w \leq g^+ \cdot q \\
0 & \text{if } w > g^+ \cdot q 
\end{cases}
\]

This recursive equation defines a sequence \( X \) of outcomes that can be used for the proof of the finite case of Theorem 3.2, as we show in §3.4. Note that this definition of \( X \) as well as property (18) are both conditional on the hypothesis that \( m \) remains positive during the previous steps and stages, along the previous choice of outcomes (i.e. along the previous bits of \( X \)), which is a property we will verify.

### 3.4 Proof of the finite case of Theorem 3.2

We prove Theorem 3.2 in the case where the list of supermartingales \((N_i)\) is finite. Since the sum of \( g \)-granular supermartingales is also a \( g \)-granular supermartingale, it suffices to show the following.

Let \( g : \mathbb{N} \mapsto \mathbb{N} \) be nondecreasing and such that \( \sum_n 2^{-g(n)} \) diverges. There exists a \( g \)-granular martingale \( M \) which is computable in \( g \) and such that for every \( g \)-granular supermartingale \( T \) there exists a casino sequence \( X \) on which \( M \) is successful and \( \lim_n S_T(X \upharpoonright_n) < \infty \). \hfill (20)

By Lemma 2.3, without loss of generality we may assume that the function \( \sigma \mapsto T(\sigma) \) is \( g \)-granular. Consider the martingale \( M \) and the stochastic processes \( t, m, q, r \) (along with their successors \( t', q', m', r' \)) that were defined in §3.2. Define the chosen outcome \( x \) at each stage by (19), which recursively defines \( X \).

It remains to show that \( X, M \) satisfy the requirements of (20).

First we show that at each stage along \( X \) the divisions \( t = q \cdot m + r, t' = q' \cdot m' + r' \), are well defined, i.e. \( m > 0 \).

**Lemma 3.5** (\( M \) is never bankrupt along \( X \)). *At all stages along \( X \) we have \( m \geq g^+ \); in the case of a saving step we have \( m' = m \geq g^+ \) and in the case of a betting step \( m' \geq g^+ \).*

**Proof.** Recall that at each stage and step, \( m \) denotes the value of \( M \) along \( X \) at the beginning of the given step, while \( m' \) denotes the value of \( M \) at the end of the given step. At each betting step \( m' \) is also the value of \( M \) along \( X \) at the beginning of the next stage, so it suffices to prove by induction on the stages and steps that \( m \geq g^+ \) at each step.

At stage 0 we have \( m = 2^{-g(0)} = g^+ \). Inductively suppose that \( m \geq g^+ \) at the start of some stage, so \( m' = m \geq g^+ \) at the saving step. In order to complete the induction step, it suffices to show that at the next betting step we have \( m' \geq g^+ \) (which is equivalent to \( m \geq g^+ \) referenced at the next stage).

First we consider the case where \( g^+ < g^+ \). In this case, by the definition of \( M \) we have \( m' \geq m - g^+ > m - g^+ \geq 0 \) so \( m' > 0 \) and since \( m' \) is \( g \)-granular, we have that \( m' \geq g^+ \) as required. In the case that \( g^+ = g^+ \), if \( m > g^+ \), by the same argument we get \( m' \geq g^+ \) as required. So it remains to examine the case where \( m = g^+ \) at the betting step in question. In this case it suffices to show that the outcome chosen by \( X \) will by 1, so that \( M \) increases its capital. Since \( m = g^+ \) it follows that \( m \) divides \( t \), i.e. \( r = 0 \), so \( t = q \cdot m = q \cdot g^+ = q \cdot g^+ \).
Moreover $|w| \leq t$, since $T$ cannot bet more than its current capital, so $w \leq q \cdot g^+$. According to the definition of $x$ in (19) it follows that the chosen outcome is 1, as required, which concludes the induction step and the proof of the lemma.

Lemma 3.6 (Monotonicity of ratios). At each stage and step along $X$ we have $q' \leq q$. In the case where $q' = q$ we have $r' \leq r$. Finally, in the case of a betting step where the 0-outcome is chosen and $q' = q$, we have $r' < r$.

**Proof.** By (19) and (18) we have $t'/m' \leq t/m$ so

$$q' + \frac{r'}{m'} \leq q + \frac{r}{m} \Rightarrow q' \leq q$$  \hspace{1cm} (21)

since $q, q'$ are integers, $r' < m'$ and $r < m$. For the second part of the lemma, assume that $q' = q$. In the case of outcome 1 we have $m' = m + g^+, t' = t + w, w \leq g^+ \cdot q$ so

$$r' = t' - q \cdot (m + g^+) = t + w - qm - qg^+ \leq t - qm = r.$$  

In the case of outcome 0, we have $m' = m - g^+, t' = t - w, w > g^+ \cdot q$ so

$$r' = t' - q \cdot (m - g^+) = t - w - qm + qg^+ < t - qm = r$$

which concludes the proof of the lemma. \qed

Lemma 3.7 (Marginal savings and 0-outcomes). Along $X$, after some stage $q, q'$ remain constant and $r' \leq r$. The marginal savings of $T$ at such a sufficiently large saving step along $X$ is $r - r'$ and in the case of a sufficiently large betting step where the 0-outcome is chosen we have $r - r' \geq g^+$.

**Proof.** The first statement of the lemma follows from Lemma 3.6 since $q, q'$ are non-negative integers. By the divisions $t = q \cdot m + r, t' = q' \cdot m' + r'$ and the fact that $q' = q, m' = m$ at saving steps of any sufficiently large stage, it follows that $t - t'$, i.e. the marginal savings of $t$ at this stage, equals $r - r'$. Finally $r - r' \geq g^+$ for the case of a 0-outcome in a sufficiently large betting step follows from Lemma 3.6 and the fact that $r, r'$ are integer multiples of $g^+$. \qed

Lemma 3.8 (Total savings of $T$ and growth of $M$). Consider a stage after which $q, q'$ remain constant along $X$, and let $r_0$ be the value of $r$ at the start of that stage along $X$. Then the remaining savings of $T$ along $X$ are at most $r_0$. Moreover $m \to \infty$ along $X$.

**Proof.** The total savings of $T$ along $X$ equals the sum of its marginal savings along $X$. Hence the first part of the lemma follows from Lemma 3.7 and the fact that $r$ remains non-negative throughout the process, as well as non-decreasing after a stage where $q, q'$ have reached a limit.

For the second part, recall that $m' - m = g^+$ at betting steps where outcome 1 is chosen. At betting steps where outcome 0 is chosen we have $m' < m$ and by Lemma 3.7, $-m' + m = g^+ \leq r - r'$. At saving steps, $m' = m$. Hence if $s_0$ is the stage mentioned in the statement of the present lemma, and $r_0$ is the value of $r$ at that stage, we have $M(X \upharpoonright s) \geq \sum_{s \geq s_0} 2^{-g^+(s+1)} - r_0$, which shows that $m \to \infty$ along $X$. \qed
4 Proof of the infinite case of Theorem 3.2

We are given $g$ as in Theorem 3.2, a countable family of $g$-granular saving strategies $(T_i)$ (as supermartingales) and we wish to construct a single sequence of outcomes $X$ along which some $g$-granular betting strategy $M$ succeeds, yet each $T_i$ can only save a finite capital. As in the analysis of the finite version of Theorem 3.2 in §3.2, §3.3 and §3.4 we consider the various strategies and parameters as a two-step stochastic process, in which each stage consists of a savings step and a betting step, and where the outcome choice and transition to the next stage happens during the betting step.

4.1 Challenges toward a uniform solution for Theorem 3.2

Let us examine the argument of §3.3 and §3.4. An important quantity at each betting step is

$$q \cdot (m' - m) - (t' - t) = \begin{cases} a & \text{for outcome 1;} \\ -a & \text{for outcome 0;} \end{cases}$$

where $a := q \cdot g^+ - w$.

If $a \geq 0$ we choose outcome 1; otherwise we choose outcome 0. As a result, by (18) and Lemma 3.6 we achieve monotonicity on the ratios $t/m$, and moreover beyond some point the decrease on $r' - r$ is exactly $|a|$. By this choice of outcomes $X$, we eventually force strategy $t$ to bet (almost) proportionally to our strategy $M$. In the special case of integer-valued strategies (the case considered in (Teutsch, 2014)) this choice of outcomes eventually forces $t$ to (a) bet exactly proportionally to $M$ and (b) stop saving altogether, inside a cone of outcomes (i.e. any extension of some prefix of $X$) on the condition that $M$ is not bankrupt. This strong property was used in (Teutsch, 2014) in order to combine this argument for any countable collection of integer-valued strategies, in a rather simple way: each strategy succeeds by the time a finite amount of bits of $X$ as been determined, and remains successful (meaning that savings have ceased) regardless of the following choices for the bits of $X$ (provided that these choices do not bankrupt $M$). This allows for the strategies to be dealt with almost independently, each time choosing the next outcome on $X$ with respect to a single strategy $t$.

In our case where the strategies are not integer-valued, such an argument is no-longer relevant. Strategies cannot be satisfied in a finite way, i.e. on a cone, which means that each time the outcome of $X$ must be decided by collectively taking into account many different strategies. If $a \geq 0$ for a particular strategy, this indicates its preference for outcome 1, as above; while $a < 0$ indicates its preference for outcome 0. In our argument we will typically choose an outcome which will be the preference of some strategies, but not others. An outcome that is against the preference of a strategy may potentially give leverage for future additional savings on the part of the strategy. Although each strategy may save infinitely often, our choice of outcome must be done in such a way that each strategy can only save in total a finite capital (although it may save infinitely often). This is the main challenge in establishing Theorem 3.2.

Before we dive into our proof, we examine why a straightforward application of the argument of §3.4 is not sufficient for the establishment of Theorem 3.2. One approach may be to construct a weighted mixture of all the strategies, and apply (20) to the mixture, bounding the savings of the mixture strategy, and thus the savings of all strategies. The problem here is that the mixture is no-longer $g$-granular, so (20) does not apply. One can also obtain an unweighted mixture of the strategies, with an additional assumption on their initial capital (so that the sum of all initial capitals converges). In this case the mixture would be $g$-granular; however it is not hard to show that such assumptions about the size of the initial capital cannot be made
4.2 Outline of the uniform solution for Theorem 3.2

We will describe a way to nest the argument of §3.3 and §3.4 for the various $T_i$ in the given list of saving strategies, in such a way that the upper bound given for the savings of $T_i$ depends only on $T_j, j < i$, at least beyond some stage. We will use the abstract notation of §3.2 and introduce a number of stochastic processes (represented by lower-case letters and corresponding subscripts) related to $T_i, M$. Here, however, $t_i$ at each stage and state of the process will be scaled version (i.e. a rational multiple) of $T_i$. Moreover instead of $m$ directly representing the values of $M$, we will have $m_i, i \in \mathbb{N}$, where each $m_i$ is a function of $T_j, j < i$ and $M$. Similar dependences will hold for the parameters $q_i, r_i, i \in \mathbb{N}$ which are the quotients and remainders respectively, of the divisions of $t_i$ by $m_i$. Hence for each $i$ we have $t_i = q_i \cdot m_i + r_i$, $q_i$ is integer and $r_i < m_i$. As before, we will ensure that $m_i > 0$ for all $i$ along the transitions of interest, so that the above parameters are well defined. We let $m$ denote that capital of $M$ at the beginning of a stage. We will also use the notation $t_i, t'_i$ to denote the values of $t_i$ at the beginning and the end of a saving or betting step respectively. As before, similar notation applies to the rest of the parameters $q_i, r_i, m_i, m$, while $w_i$ refers to the wager on outcome 1 by $t_i$ at the present step. We also introduce an additional parameter $n$, which is the number of the strategies $T_i, j < n$ that are considered at a particular transition. Similarly with most of the other parameters, during the steps of a stage the value $n$ may change, so typically $n$ refers to the value at the beginning of a step and $n'$ refers to the value after the end of the step.

The solution is based on two ingredients:
(a) nested divisions \( t_i = q_i \cdot m_i + r_i \) and (monotone) shrinking

(b) the marginal saving of each \( T_i \) at stage \( s \) is \( \Theta (|R_s - R_{s+1}|) \), where \( (R_s) \) is a sequence of finite variation.

Recall that the variation of a sequence \( (R_s) \) is defined as \( \sum |R_s - R_{s+1}| \); if we only take into account the summands where \( R_s < R_{s+1} \) then we have the upward variation of \( (R_s) \), while the downward variation is defined analogously.

Before we give the formal definitions of the main parameters \( t_i, m_i, r_i \) in §4.3, we give a high-level overview of the argument. At each stage saving strategies \( T_i, i < n \) are considered and we observe the nested divisions of \( t_i \) by \( m_i \) (with integer quotient \( q_i \) and remainder \( r_i \)). At each stage we may decide to shrink \( t_i, m_i \) by some dyadic factor, which also means that the remainder \( r_i \) will also be shrunk by the same factor, while the quotient \( q_i \) remains the same. The sequence \( s \mapsto R_s \) in clause (b) is called the potential and will be the sum of the remainders \( r_i, i < n \), which will grow as the stages progress and \( n \) increases, including more and more remainders. The shrinking operation we just described is done in order to ensure that the potential converges in the strong way that is spelled out in clause (b).

Ingredient (b) already occurred in a simple form in the argument of §3.4 where the potential was simply the remainder of the division \( t/m \) after \( q \) reached a limit. In this general case the potential \( s \mapsto R_s \) will be a dynamic mixture of the remainders where at any stage some remainders may be eliminated from \( R_s \) and others may be added, by adjusting the value of \( n \), while any remainder may be shrunk according to clause (a) above, effectively scaling down its contribution in the potential. In §3.4 the potential, i.e. the remainder \( r \), was non-increasing after some stage. In the general case, due to the introduction of more and more remainders \( r_i \) in \( R_s \), it may increase infinitely often. The main purpose of shrinking (i.e. scaling the nested divisions) is to ensure that the upward variation of the potential is finite. Since \( R_s \geq 0 \) at all stages \( s \), it follows that \( \sum_i |R_s - R_{s+1}| < \infty \). Then it remains to show that for each \( i \) and each large enough stage \( s \), the marginal saving of \( T_i \) is at most \( \Theta (|R_s - R_{s+1}|) \). This can also be seen as a generalization of the property of Lemma 3.7 in the argument of §3.4. The main difference in the present general case there is a multiplicative factor hidden in \( \Theta (|R_s - R_{s+1}|) \), which will actually be due to the shrinking that has been applied on \( t_i, m_i \). For this reason, in our argument we will need to make sure that each \( t_i \) is shrunk at most finitely often. Table 3 summarizes the main parameters occurring in the argument of the following sections.

### 4.3 A tower of nested divisions

Consider any fixed list \( T_i, i \in \mathbb{N} \) of \( g \)-granular supermartingales. Also consider the current and the next granule variables \( g^*, g^+ \) respectively, which are as introduced in §3.2 and reported in Table 1. The shrinking operations that were informally described in §4.2 will be implemented via the integer variables \( (c_i) \). Informally speaking, \( 2^{-c_i} \) is the factor for the scaling that is done from level \( i - 1 \) to level \( i \). This means that one unit of capital (whatever that may be) at level \( i \) is worth \( 2^{c_i} \) units in the currency of level \( i - 1 \). The actual supermartingales \( T_i \) are located at the ground level and for each \( i \) we define

\[
t_i = 2^{-k_i} \cdot T_i \quad \text{and} \quad g_i^* = 2^{-k_i} \cdot g^* \quad \text{and} \quad g_i^+ = 2^{-k_i} \cdot g^+ \quad \text{where} \quad k_i = \sum_{j \leq i} c_j
\]

and we often refer to \( g_i^*, g_i^+ \) as the current and next granules respectively, at level \( i \). In the following we start with the given list \( T_i, i \in \mathbb{N} \), and additional \( g \)-granular martingale \( M \) (denoted as the process \( m \)) and the variables \( (c_i) \) as processes that may change value at the various transitions depending on the state they
are in (i.e. the chain of the previous binary outcomes). We are also given a variable \( n \) which determines the length of the list \( T_i, i < n \) of strategies that we consider at the particular state and stage. Our description of the tower of divisions which will define \( m_i, q_i, r_i \) does not require any assumptions on \( M, (c_i) \), \( n \), but in the following sections, for the sake of proving Theorem 3.2, we will fix them.\(^3\)

Let \( m_0 = 2^{-c_0} \cdot M \) and let \( r_0 \) be the reminder of the division \((2^{-c_0} \cdot T_i)/(2^{-c_0} \cdot M)\). Assuming that each \( m_j, r_j, j \leq i \) has been defined and \( i < n - 1 \), let

\[
m_{i+1} = 2^{-c_{i+1}} \cdot (m_i - r_i)
\]

and let \( q_{i+1}, r_{i+1} \) the quotient and remainder of the division \( t_{i+1}/m_{i+1} \) respectively, i.e.

\[
q_{i+1} = \lfloor t_{i+1}/m_{i+1} \rfloor \quad \text{and} \quad r_{i+1} = t_{i+1} - q_{i+1} \cdot m_{i+1}.
\]

The tower of divisions at some stage refers to the nested divisions

\[
t_i = q_i \cdot m_i + r_i \quad \text{and for } i < n.
\]

The reader may consider the simple case when no scaling takes place, i.e. when \( c_i = 0 \) for all \( i < n \). The properties of the tower of divisions that we are interested in are essentially independent of the scaling aspect (i.e. any non-zero values of the \( c_i \)). Scaling is only used in the construction of §4.5 in order to ensure certain convergence properties of the potential which is defined as the process \( R := R_n \), where

\[
R_i = \sum_{j<i} r_j, \quad \text{for each } i \leq n.
\]

In the case where \( c_i = 0, i < n \) it follows from the definitions that \( m_i = m - R_i \), so \( m_i \leq m_{i+1} \). In the general case we can consider the ground level values \( \hat{t}_i, \hat{m}_i, \hat{r}_i \) of the parameters \( t_i, m_i, r_i \):

**Definition 4.1** (Ground-level values). Define the ground-level values of the parameters as follows:

\[
\hat{t}_i = 2^{k_i} \cdot t_i \quad \text{and} \quad \hat{w}_i = 2^{k_i} \cdot w_i \quad \text{and} \quad \hat{m}_i = 2^{k_i} \cdot m_i \quad \text{and} \quad \hat{r}_i = 2^{k_i} \cdot r_i \quad \text{and} \quad \hat{R}_i = \sum_{j<i} \hat{r}_j
\]

where \( w_i \) denotes the wager of \( t_i \) (which is a scaled version of the wager \( \hat{w}_i \) of \( T_i \)).

It follows inductively from the definitions that \( \hat{m}_i = m - \hat{R}_i \), \( \hat{t}_i = q_i \cdot \hat{m}_i + \hat{r}_i \) and the relationships between the variables carries through to the ground level; for example \( \hat{r}_i \) is the remainder of the division of \( \hat{t}_i \) by \( \hat{m}_i \).

By the \( g \)-granularity of \( M, T_i, i \in \mathbb{N} \) it follows that for each \( i \) and each stage of the process,

(a) \( t_i \) is an integer multiple of \( g_i^* \) and \( q_i \) is an integer;

(b) \( m_i, r_i \) are integer multiples of \( g_i^* \) and \( r_i < m_i \); hence \( m_i - r_i \geq g_i^* \).

Note that (a) follows from the definitions of \( q_i, T_i, t_i \) while the first part of (b) follows by induction on the recursive definition of \( m_i, r_i \) in combination with (a). The second part of (b) follows from the first part of (b) and the properties of division. The following fact will play a role in the argument of §4.4.

**Lemma 4.2.** For each \( i \leq j \) we have \( m_j + (R_j - R_i) \leq m_i - (g_i^* - g_j^*) \).

\(^3\)In particular, \( M \) will be the simple martingale introduced in §3.2 and \( (c_i) \), \( n \) will be specified along with the outcome sequence \( X \) in the construction of §4.5.
Table 3: The first table displays the parameters that tune the scaling in the nested construction, while the ground-level values of §4.3 are in the second table. The third table displays the ‘advance’ parameters of §4.3 and §4.4 which, along with their aggregates $A_i, B_i$, describe the changes in the nested remainders and the potential under certain stability conditions on the quotients.

**Proof.** First, we show that for each $k$ we have

\[
m_{k+1} + r_k \leq m_k - (g_k^+ - g_{k+1}^+). \tag{22}
\]

Recall that $m_{k+1} = 2^{-c_{k+1}} \cdot (m_k - r_k)$ so $(m_k - r_k) - m_{k+1} = (1 - 2^{-c_{k+1}}) \cdot (m_k - r_k)$. On the other hand, since $m_k, r_k$ are both integer multiples of $g_k^+$ and $r_k < m_k$, we have $m_k - r_k \geq g_k^+ \geq g_k^+$. By definition we also have $g_{k+1}^+ = 2^{-c_{k+1}} \cdot g_k^+$. Hence $(m_k - r_k) - m_{k+1} \geq (1 - 2^{-c_{k+1}}) \cdot g_k^+ = g_k^+ - g_{k+1}^+$ which equivalent to (22).

Now we may use (22) in order to prove the lemma by induction on $j \geq i$. For $j = i$ the lemma clearly holds, so assume it holds for $k \geq i$. This induction hypothesis can be written as $m_k + (R_k - R_i) \leq m_i - (g_i^+ - g_k^+)$. It remains to complete the induction step, showing the lemma for $i = k + 1$. Using (22) we have

\[
m_{k+1} + (R_{k+1} - R_i) \leq m_k - (g_k^+ - g_{k+1}^+) - r_k + (R_{k+1} - R_i) = m_k + (R_k - R_i) - (g_k^+ - g_{k+1}^+)
\]

Using the induction hypothesis on the right-hand-side of the above relation we get

\[
m_{k+1} + (R_{k+1} - R_i) \leq m_i - (g_i^+ - g_k^+) - (g_i^+ - g_{k+1}^+) = m_i - (g_i^+ - g_{k+1}^+).
\]

which completes the induction step and the proof of the lemma. \hfill \Box

Recall the discussion at the start of §4.1, and in particular the role of the parameter $a$ in the argument of §3.3 and §3.4. In this atomic case we concluded that $q \cdot (m' - m) - (t' - t)$ is $a$ or $-a$ for outcome 1 or 0 respectively, so

\[
r \text{ decreases by } a \text{ or } -a \text{ under outcomes 1 or 0 respectively, in the event that } q' = q \tag{23}
\]

(note that a decrease by a negative amount is actually an increase, which is why we chose the outcomes $X$ the way we did in that argument). We need a similar parameter $a_i$ for each $i$, which expresses the preference of outcome with respect to $t_i$, and where the goal is non-increase of $t_i/m_i$. We define $a_i, i \in \mathbb{N}$ inductively on $i$, as processes just like the previous parameters. Assuming that $a_j, j < i$ have been defined, let

\[
A_i = g_i^+ + \sum_{j<i} a_j \quad \text{and} \quad B_i = -A_i \quad \text{and} \quad a_i := q_i \cdot A_i - w_i \quad \text{and} \quad b_i := q_i \cdot B_i + w_i \tag{24}
\]

and note that $a_i = -b_i$ for each $i$. Also note that since $w_i$ is an integer multiple of $g_i^+$, inductively $A_i, a_i, B_i, b_i$ are all integer multiples of $g_i^+$.

**Definition 4.3** (Advance and stability of strategies). We call $a_i, b_i$ the *advance* of $t_i$ under outcomes 1, 0 respectively. We say that $t_i$ is stable at a certain step if $q_i' = q_i$ and $c_i' = c_i$. 18
For each \( i \) define the following nested version of the quantity discussed in §4.1,

\[
D_i = q_i \cdot (m'_i - m_i) - (t'_i - t_i) = (t_i - q_i \cdot m_i) - (t'_i - q_i \cdot m'_i)
\]

which is viewed as a process, just like the rest of the parameters. Also define the ground-level value \( \hat{D}_i \) of \( D_i \) by \( \hat{D}_i := q_i \cdot (\hat{m}'_i - \hat{m}_i) - (\hat{t}'_i - \hat{t}_i) \). Note that

\[
\text{if } t_i \text{ is stable, then } D_i = r_i - r'_i.
\]

so the following lemma is an analogue of (23) for the tower of divisions.

**Lemma 4.4** (Advance conditional to stability). Suppose that during a betting step \( t_j, j < i \) are stable. Then \( D_i \) equals \( a_i \) or \( -a_i \) depending on whether the chosen outcome is 1 or 0 respectively.

**Proof.** Fix \( i \geq 0 \), consider \( m_j, r_j, t_j, a_j, q_j, j \leq i \) at a state of the stochastic process and consider the forthcoming transition. We claim that it suffices to show the statement at the ground level, i.e. for the parameters \( \hat{t}_j, \hat{r}_j, \hat{a}_j \) and \( \hat{D}_i \). Indeed supposing that \( t_j, j < i \) are stable, we also get that \( \hat{t}_j, j < i \) are stable, i.e. the quotients in the divisions with \( \hat{m}_j \) respectively remain constant under the transition. Then given the ground level statement we get that \( \hat{D}_i \) equals \( \hat{a}_i \) or \( -\hat{a}_i \) on outcome 1 or 0 respectively. Then since \( \hat{D}_i = 2^{k_i} \cdot D_i \) and \( \hat{a}_i = 2^{k_i} \cdot a_i \), we get the desired conclusion.

It remains to show the statement at ground level, so recall the definitions and properties of the ground-values of the parameters from §4.3, and in particular that \( \hat{r}_i = 2^{k_i} \cdot r_i, \hat{m}_i = m_i \cdot 2^{k_i} \) and

\[
\hat{m}_i = \hat{m}_0 - \sum_{j<i} \hat{r}_j \quad \text{where } \hat{r}_j \text{ is the remainder of } T_j \text{ divided by } \hat{m}_j.
\]

Consider the divisions \( \hat{t}_j = q_j \cdot \hat{m}_j + \hat{r}_j, \hat{t}'_j = q'_j \cdot \hat{m}_j + \hat{r}'_j \) for \( j < i \) and note that by the hypothesis we have \( q'_j = q_j \). If the outcome is 1, then \( \hat{r}_j = \hat{t}_j - w_j, \hat{m}'_0 = \hat{m}_0 + g^+, \hat{r}'_j = \hat{r}_j - \hat{a}_j \), for each \( j < i \). If the outcome is 0, then \( \hat{r}'_j = \hat{t}_j - \hat{w}_j, \hat{m}'_0 = \hat{m}_0 - g^+, \hat{r}'_j = \hat{r}_j + \hat{a}_j \), for each \( j < i \). Since, by the hypothesis, we have \( q'_j = q_j \) for \( j < i \), we get

\[
\hat{D}_i = q_i \cdot (\hat{m}'_i - \hat{m}_i) - (\hat{t}'_i - \hat{t}_i) = \begin{cases} 
q_i \cdot (\sum_{j<i}(\hat{r}_j - \hat{r}'_j) + g^+) - \hat{w}_i = \hat{a}_i & \text{for outcome 1}; \\
q_i \cdot (\sum_{j<i}(\hat{r}_j - \hat{r}'_j) - g^+) + \hat{w}_i = -\hat{a}_i & \text{for outcome 0};
\end{cases}
\]

where the last two equalities follow by (26), the induction hypothesis and the definition of \( \hat{a}_i \).

**Corollary 4.5** (Transitions and changes in \( m_i \)). If during a betting step, \( t_j, j < i \) are stable, then \( m'_i = m_i + A_i \) and \( D_i = a_i \) in the case of chosen outcome 1, and \( m'_i = m_i + B_i, D_i = b_i \) in the case of outcome 0.

**Proof.** This follows directly from (25), Lemma 4.4, and the definitions of \( r_i, m_i, A_i, B_i \).

According to the analysis in §3.3 and §3.4, \( q'_j > q_j \) indicates a bad (i.e. undesirable) transition for \( t_i \). The following lemma gives conditions for avoiding a bad transition for \( t_i \), assuming that each \( t_j, j < i \) are stable, and will be used in choosing the outcomes in the construction of §4.5.

**Lemma 4.6** (Inductive stability condition). Suppose that during a betting step, \( t_j, j < i \) are stable. Then

- if outcome 1 is chosen: \( q'_i \not\geq q_i \iff -A_i - a_i < m_i - r_i \)
• if outcome 0 is chosen: \( q'_i \neq q_i \iff -B_i - b_i < m_i - r_i \).

**Proof.** Consider the division \( t_i = q_i \cdot m_i + r_i \) and the values after the betting step, \( t'_i, m'_i \). By (25) we get \( t'_i = q_i \cdot m'_i - D_i + r_i \), so \( q'_i > q_i \) if and only if \( r_i - D_i \geq m'_i \).

First assume the case where outcome 1 is chosen, so by Corollary 4.5, \( m'_i = m_i + A_i \). Hence \( q'_i > q_i \) is equivalent to \( r_i - D_i \geq m_i + A_i \). Since \( t_j, j < i \) are stable, by Corollary 4.5 we have \( D_i = a_i \) so the last inequality is equivalent to \( -A_i - a_i \geq m_i - r_i \) as required.

Second, outcome 0 is chosen, so by Corollary 4.5, \( m'_i = m_i + B_i \). Hence \( q'_i > q_i \) is equivalent to \( r_i - D_i \geq m_i + B_i \). Since \( t_j, j < i \) are stable, by Corollary 4.5 we have \( D_i = b_i \) so the last inequality is equivalent to \( -B_i - b_i \geq m_i - r_i \) as required. \( \square \)

### 4.4 Key combinatorial property of the tower of divisions

The tower of divisions process is measurable in, i.e. definable in, the processes \( M, T_i, i \in \mathbb{N} \). In the following proof we use the fact that \( |w_i| \leq t_i \) (equivalently that \( |\hat{w}_i| \leq \hat{t}_i \)) at all stages of the process, which means that each strategy cannot bet more than its present capital. Recall that \( n \) is an arbitrary process (also viewed as a function from strings, representing the previous outcomes, to the positive integers) that will be fixed later during the construction.

**Lemma 4.7** (Tower of divisions property). For any \( i \) it holds that

\[
A_{i+1} + m_{i+1} \leq (1 + q_i) \cdot (A_i + m_i) \tag{27}
\]

\[
B_{i+1} + m_{i+1} \leq (1 + q_i) \cdot (B_i + m_i). \tag{28}
\]

**Proof.** Since \( a_i = q_i \cdot A_i - w_i \) we have

\[
A_{i+1} + m_{i+1} = A_i + a_i + m_{i+1} + (g^+_i - g^+_i) = A_i + q_i A_i - w_i + (g^+_i - g^+_i) + m_{i+1}
\]

By Lemma 4.2, \( t_i = q_i \cdot m_i + r_i \) and \( |w_i| \leq t_i \), we have

\[
A_{i+1} + m_{i+1} \leq A_i + q_i A_i + t_i + m_i - r_i = (1 + q_i) \cdot (A_i + m_i)
\]

Similarly, since \( B_i = -A_i \) and \( b_i := q_i \cdot B_i + w_i \), we get

\[
B_{i+1} + m_{i+1} = B_i + q_i B_i + w_i - (g^+_i - g^+_i) + m_{i+1} \leq B_i + q_i B_i + t_i + m_i - r_i = (1 + q_i) \cdot (B_i + m_i)
\]

which concludes the proof of the lemma. \( \square \)

We can now use Lemma 4.6 in order to choose the outcomes in such a way that bad transitions (where \( q'_i > q_i \)) happen in a prioritized way. We need the following technical consequence of Lemma 4.6.

**Lemma 4.8.** At any betting step of the tower of divisions process,

(a) if \( \sum_{i<n} a_i \geq 0 \) then \( -A_k - a_k < m_k - r_k \) for each \( k \in [0, n) \);

(b) if \( \sum_{i<n} a_i < 0 \) then \( A_k + a_k < m_k - r_k \) for each \( k \in [0, n) \).
Proof. If \( \sum_{i<n} a_i \geq 0 \) then \( A_n + m_n = \sum_{i<n} a_i + m_n + g^+_n > 0 \). By (27), inductively for each \( k \in [0, n) \) we have \( A_{k+1} + m_{k+1} > 0 \). Since \( g^+_k \leq g^+_n \), then

\[
-A_k - a_k \leq -A_{k+1} < m_{k+1} \leq m_k - r_k
\]

where the last inequality is a simple case of Lemma 4.2.

On the other hand, if \( \sum_{i<n} a_i < 0 \) then \( B_n + m_n = -\sum_{i<n} a_i + m_n - g^+_n > 0 \). The last inequality here comes from the fact that \( m_n \) is positive and an integer multiple of \( g^+_n \), and \( g^+_n \leq g^+_n \). By (28), inductively for each \( k \in [0, n) \) we have \( B_{k+1} + m_{k+1} > 0 \), i.e., \( A_{k+1} < m_{k+1} \). Hence

\[
A_k + a_k = A_{k+1} + (g^+_k - g^+_k) < m_{k+1} + (g^+_k - g^+_k) \leq m_k - r_k
\]

where the last inequality is an application of Lemma 4.2. \( \square \)

By Lemma 4.6 and Lemma 4.8 we get the following item which will be used in the construction in §4.5.

**Corollary 4.9** (Outcome choice). Consider a betting step in the tower of divisions process where its length is \( n \) and \( m > 0 \). In each of the following cases

- \( \sum_{i<n} a_i \geq 0 \) and the chosen outcome is 1;
- \( \sum_{i<n} a_i < 0 \) and the chosen outcome is 0;

if some strategy \( t_j \) is not-stable in this step and \( j \) is the least such index, then \( q'_j \neq q_j \).

In other words, if we choose the outcome according to the rule prescribed in Corollary 4.9, the least (in terms of its index) non-stable strategy \( t_j \) will not have a bad transition during the betting step. This is the key priority property that makes the construction of §4.5 work, through an appropriate dynamic definition of the length parameter \( n \).

### 4.5 Nested construction with overflow and shrinking thresholds

We introduce two more parameters \( s_i, p_i \), also viewed as processes, and focus on their values along the outcome sequence \( X \) that we are constructing. Every increase in the potential \( R \) will be charged on an index, on one of two possible ways, corresponding to the scaling thresholds \( s_i \) and the overflow thresholds \( p_i \), both of which start with \( s_i[0] = p_i[0] = i \) and are updated dynamically during the construction. Then \( 2^{-n} \) or \( 2^{-m} \) is the maximum charge that \( t_i \) can accept at a given stage based on \( s_i, p_i \) respectively. Upon accepting a charge based on \( s_i \) or \( p_i \), strategy \( t_i \) will increase the corresponding parameter by 1. The scaling threshold \( s_i \) corresponds to the action of introducing \( r_i \) into the potential (by defining \( n' > i \), and appropriately shrinking \( t_i \)) while the overflow threshold is related to the size that \( |A_i| \) is permitted to have.

**Definition 4.10** (Overflows and shrinking). At any state of the process we say that there is an \( i \)-overflow if \( |A_i - g^+_i| > 2^{-n} \). By shrinking \( t_i \) at some stage we mean increasing \( c_i \) the least integer amount so that \( r'_i := 2^{-k} \cdot r_i \leq 2^{-n} \), and increase \( s_i \) by 1, i.e. set \( s'_i = s_i + 1 \).

We start with \( n = 0 \), so \( R = 0 \). Recall from §3.2 that each stage consists of a savings step and a betting step where the outcome is chosen and the transition to the next stage takes place. In the betting step of the construction below we shrink \( t_i \) if we either introduce it into the potential, setting \( n' > i \), or \( q'_i < q_i \). The parameters may change value in each step, so the notations \( t'_i, m'_i, n' \ldots \) typically indicate the values in the
beginning of next step. If a parameter is not redefined, it retains its present value in the next step. The betting step consists of four sub-steps during which the length-parameter \( n \) may be redefined a number of times.

**Nested construction.** On a savings step, if there is a least \( i < n \) such that \( q'_i < q_i \), let \( n' = i \) for the least such \( i \). On a betting step do the following:

1. **Overflow-check:** If there exists an \( i \)-overflow for some \( i < n \) let \( n' = i \) for the least such \( i \)
2. **Action:** If \( \sum_{j < n'} a_j \geq 0 \) choose outcome 1; otherwise choose outcome 0.
3. **Length-calibration:** If there is a least \( i < n' \) such that \( q'_i < q_i \), redefine \( n' = i \) and let \( p'_i = p_i + 1 \).
4. **Length-increase:** If \( m' > n' + 1 \), shrink strategy \( t_{n'} \), let \( s'_{n'} = s_{n'} + 1 \) and increase \( n' \) by 1.

Let \( X \) be the concatenation of all outcomes chosen by the above routine, run indefinitely.

### 4.6 Verification of the nested construction

We first need to show that \( m > 0 \) at all steps of the construction along \( X \), since the tower of divisions and the associated facts that we obtained all rest on this hypothesis.

**Lemma 4.11.** At all steps and stages along \( X \) we have \( m_i \geq g_i^* \) for each \( i < n \); in the case of a saving step we have \( m'_i = m_i \geq g_i^* \) and in the case of a betting step \( m'_i \geq g_i^* \).

**Proof.** By the definition of \( m_i \), it suffices to show that at saving steps we have \( m' = m \geq g^* \) and at betting steps \( m' \geq g^* \). Recall that at each stage and step, \( m \) denotes the value of \( M \) along \( X \) at the beginning of the given step, while \( m' \) denotes the value of \( M \) at the end of the given step. At each betting step \( m' \) is also the value of \( M \) along \( X \) at the beginning of the next stage, so it suffices to prove by induction on the stages and steps that \( m \geq g^* \) at each step.

At stage 0 we have \( m = 2^{-g(0)} = g^* \). Inductively suppose that \( m \geq g^* \) at some saving step. Since \( M \) is not a savings strategy, we have \( m' = m \geq g^* \) at this step. In order to complete the induction step it suffices to show that at the following betting step we get \( m' \geq g^* \) (which is equivalent to \( m \geq g^* \) referenced at the next stage).

First we consider the case where \( g^+ < g^* \). In this case, by the definition of \( M \) we have \( m' \geq m - g^+ > m - g^* \geq 0 \) so \( m' > 0 \) and since \( m' \) is \( g \)-granular, we have that \( m' \geq g^+ \) as required. In the case that \( g^+ = g^* \), if \( m > g^* \), by the same argument we get \( m' \geq g^+ \) as required. So it remains to examine the case where \( m = g^* \) at the stage in question. In this case it suffices to show that the outcome chosen by \( X \) will by 1, so that \( M \) increases its capital. Since \( m = g^* \) we have that \( m_0 \) divides \( t_0 \), so \( r_0 = 0 \), which also implies that \( m_1 = 2^{-k_1} \cdot m_0 \) divides \( t_1 \) and so on, inductively getting that \( m_i \) divides \( t_i \), so \( r_i = 0 \), for each \( i < n \). Since each \( T_i \) can only bet at most its current capital, we also have \( |w_i| \leq t_i \) for all \( i < n \), so \( w_i \leq t_i = q_i \cdot g_i^+ \) which means that

\[
q_i \cdot g_i^+ - w_i \geq 0 \quad \text{for each } i < n.
\]

Then by the definition of \( a_i, A_i \) in (24) it follows inductively that \( a_i \geq 0 \) and \( A_i \geq 0 \) for each \( i < n \). Hence \( \sum_{j < n} a_j \geq 0 \) and by sub-step (2) of the betting step of the nested construction it follows that in this case outcome 1 will be chosen. This concludes the inductive proof that \( m' \geq g^+ \) at all steps and stages along \( X \), and the proof of the lemma. \( \square \)
**Lemma 4.12** (Saving steps). *At saving steps the potential does not increase, i.e. \( R' \leq R \).

**Proof.** During a saving step each \( t_i \) cannot increase while \( m \) remains constant. Let \( i \leq n \) be the least number such that for all \( j < i \) we have \( q'_j = q_j \). Note that \( m_0 \) cannot increase, so \( m_1 \) cannot decrease and so on. Inductively, we have that \( t_j, r_j \) do not increase and \( m_j \) does not decrease, for each \( j < i \). By the construction we get \( n' = i \), so since \( r'_j \leq r_j \) for all \( j < n' \) and \( r_x \geq 0 \) for all \( x < n \) we have that \( R' \leq R \) at this saving step. \( \square \)

**Lemma 4.13** (Potential limit). *Along \( X \), the variation of the potential \( R \) is finite, hence \( R \) reaches a limit.*

**Proof.** It suffices to show the first claim, since any sequence with bounded variation is Cauchy, hence convergent. At any stage along \( X \), the value of \( R \) is equal to its upward variation minus its downward variation up to that point. Since at all stages \( R \geq 0 \), the downward variation is always bounded by the upward variation, so it suffices to show that the total upward variation of \( R \) is finite. Our calculation of the variation of \( R \) is with respect to the values of \( R \) at the end of each construction saving or betting step. By Lemma 4.12, in our calculation of an upper bound on the upward variation of the potential, it suffices to only consider the betting steps of the construction.

Consider a betting step of the nested construction and observe that sub-steps (1), (3) decrease the current (intermediate) value of \( R \), while sub-steps (2), (4) may increase it. We consider two possible cases for this transition from \( R \) to \( R' \). In the case that after the outcome choice of step (2) all \( t_i, i < n' \) remain stable (i.e. \( q_i \) remains constant), by Lemma 4.4, the choice of outcome in sub-step (2) and (26) it follows that sub-step (2) will not increase \( R \) and sub-step (3) will not apply. Hence in this case any increase from \( R \) to \( R' \) will be due to sub-step (4), hence \( R' - R \leq 2^{-s_{i}} \) (where \( s'_{i} = s_{i} + 1 \)).

Now consider the case where after sub-step (2), there exists some \( i < n \) such that \( q'_i < q_i \), so that sub-step (3) applies. If we choose the least such \( i \), according to Lemma 4.9 we have \( q'_j = q_j \) for all \( j < i \). Then according to sub-step (3), the potential can increase by at most \( |A_i - g^*_i| \) at sub-step (3), plus at most \( 2^{-s_{i}} \) at sub-step (4). But by sub-step (1) we have \( |A_i - g^*_i| \leq 2^{-p_i} \), so in this case \( R \) can increase by at most \( 2^{-s_{i}} + 2^{-p_i} \), and in this case we also get \( s'_{i} = s_{i} + 1 \) and \( p'_{i} = p_{i} + 1 \).

Hence at any betting step where \( R \) increases, there exists some \( i \) such that either the increase is at most \( 2 \cdot 2^{-s_{i}} \) and \( s'_{i} = s_{i} + 1 \), or the increase is at most \( 2^{-s_{i}} + 2^{-p_{i}} \), and \( s'_{i} = s_{i} + 1 \) and \( p'_{i} = p_{i} + 1 \). We can conclude that the upward variation of \( R \) is bounded above by

\[
\sum_{i} \left( \sum_{j} 2^{-s_{i}[0] - j} \right) + \sum_{i} \left( \sum_{j} 2^{-p_{i}[0] - j} \right) = 4 + 4 = 8.
\]

Intuitively, every increase is blamed on some \( i \) in one of two ways, and each time this happens the parameter \( s_{i} \) or both \( s_{i}, p_{i} \), depending on the case, increase by one, making the next increase blamed on the same index upper-bounded considerably more tightly than the previous one. This concludes the proof of the lemma. \( \square \)

In the following lemma, \( \limsup \) denotes \( \limsup_{s} M(X \uparrow s) \) and could be infinite or finite.

**Lemma 4.14** (Stability of strategies). *Along \( X \), for each non-negative integer \( d < \limsup \) \( m \), there exists some stage in the nested construction after which the parameters \( q_{i}, c_{i}, s_{i}, p_{i}, i < d \) remain constant and \( n \geq d \), even during the steps and sub-steps of each stage.*
**Proof.** For \( d = 0 \) it is trivial, so suppose inductively that the statement holds for an arbitrary integer \( d \geq 0 \) such that \( d + 1 < \limsup m \). Upon a \( j \)-overflow we have that \( n \) drops to at most \( j \), so by the inductive hypothesis after a certain stage \( x_d \):

- for each \( j < d \), there will be no \( j \)-overflows and \( q_j, c_j, s_j, p_j \) remain constant;
- the outcome at betting steps will be chosen at sub-step (2) according to the sign of \( \sum_{j < n'} a_j \), where \( n' \) here refers to the current value of the length parameter at sub-step (2) and \( n' \geq d \).

By Lemma 4.9 after this stage \( x_d \), the quotient \( q_d \) cannot increase. Hence after some stage \( y_d > x_d \), the quotient \( q_d \) will remain constant. By the construction, \( p_d \) can only change value in step (3), provided that \( q_d \) changes value. Hence

after some stage, both \( q_d \) and \( p_d \) remain constant. \hfill (29)

It remains to show that

beyond some stage we always have \( n \geq d + 1 \), even during the steps and sub-steps of the stages.

because in this case it also follows that \( s_d, c_d \) reach a final value.

For a contradiction, suppose otherwise, i.e. that there are infinitely many stages and steps such that \( n < d + 1 \). By the induction hypothesis we have that \( n \geq d \) for all but finitely many stages. Hence, since \( d + 1 < \limsup m \), by the construction it follows that there are infinitely many stages and steps such that \( n = d + 1 \) and also infinitely many steps where \( n = d \). Then by (29) it is necessary that there are infinitely many \( d \)-overflows during the construction. In particular, there are infinitely many betting steps such that \( n' = d \) at the completion of sub-step (1) and there is a \( d \)-overflow, so \( |A_d - g_d^i| \geq 2^{-p_d} \). Let as call these stages **bad**, if they are also sufficiently large so that (29) holds. Hence, for the required contradiction, it remains to show that there are only finitely many **bad** stages.

During the sub-steps of a betting step of a **bad** stage, the potential \( R \) can only decrease by step (1). On the other hand since all \( q_i, c_i, i < d \) remain constant, by Lemma 4.4 the potential \( R \) incurs a decrease of \( |A_d - g_d^i| > 2^{-p_d} \) after step (2). Since (29) holds, sub-step (3) does not apply in these special stages, so \( n' = d \) immediately after sub-step (3). Since there are infinitely many bad stages, it follows by sub-step (4) that \( s_d \to \infty \). Moreover, at sub-step (4) of the betting step of each **bad** stage there is an increase on \( R \) of at most \( 2^{-s_d} \). We can conclude that \( R \) incurs total decrease of more than \( 2^{-p_d} - 2^{-s_d} \) during each **bad** stage, where \( 2^{-s_d} \) is an upper bound on the increase that occurs at sub-step (4). But since \( p_d \) remains constant and \( s_d \to \infty \), we can conclude that the downward variation of \( R \) restricted on the **bad** stages is infinite. This contradicts Lemma 4.13.

From this contradiction we may conclude that eventually the length parameter \( n \) is always at least \( d + 1 \), even during the steps and sub-steps of the stages, which concludes the proof of the lemma. \hfill \( \Box \)

Note that by Lemma 4.14 and the nested construction of §4.5 it follows that for each \( i \) along \( X \) there are only finitely many \( i \)-overflows.

**Lemma 4.15** (Upper bound on the savings). *For each \( i < \limsup X \) we have \( \lim_{\gamma} S_i(X \upharpoonright_\gamma) < \infty. \)*

**Proof.** Pick any \( i < \limsup \mathcal{M}(X \upharpoonright_\gamma) - 1 \) and by Lemma 4.14 consider a stage \( x_0 \) after which we always have \( n \geq i \), even during the steps and the sub-steps of each stage. After stage \( x_0 \) the parameters \( q_i, s_i \) will
remain constant, even during the steps and the sub-steps of each stage. Hence no more \( i \)-shrinking will take place in the construction, and \( k_i \) will reach a final value. At any such stage, any marginal saving \( \gamma \) of \( T_i \) will occur at a saving step and will reduce \( r_i \) by \( 2^{-k_i} \cdot \gamma \) during that step. Hence by Lemma 4.12, for each stage \( y > x_0 \) we have that \( S_i(X \upharpoonright y) - S_i(X \upharpoonright x_0) \) is bounded above by the downward variation of \( R \) between stages \( x_0 \) and \( y \), multiplied by \( 2^{-k_i} \). Hence if \( b \) is an upper bound on the total variation of \( R \) by Lemma 4.13, we have that \( S_i(X \upharpoonright y) \leq 2^{-k_i} \cdot b + S_i(X \upharpoonright x_0) \) for all \( y > x_0 \), which concludes the proof. □

Lemma 4.16 (Success of \( M \) on \( X \)). We have \( \lim \sup \_M(X \upharpoonright s) = \infty \).

**Proof.** By (10) it suffices to show that \( \lim \sup \_M(X \upharpoonright s) = \infty \). For a contradiction, assume otherwise, and let \( c \) be the largest non-negative integer which is smaller than \( \lim \sup \_M(X \upharpoonright s) \). Then by sub-step (4) of the nested construction and Lemma 4.14 it follows that along \( X \) beyond some stage the parameter \( n \) equals \( c \). Again by Lemma 4.14 it follows that along \( X \), eventually the potential \( R \) will be non-increasing and an integer multiple of \( g_{c-1}^* \). Furthermore, by sub-step (2) of the nested construction it follows that when outcome 0 is chosen at a betting step, we have \( R' < R \) so \( R - R' \geq g_{c-1}^* \). Hence beyond some betting step along \( X \), the sum of all granules \( g^* \) of the remaining stages where a 0-outcome is chosen is bounded above by some positive integer \( C = O(R_0) \), where \( R_0 \) is the value of \( R \) at the betting step in question. Since \( \sum d 2^{-g(i)} = \infty \), it follows that \( \lim \sup \_M(X \upharpoonright s) \geq \sum 2^{-g(i)} - C = \infty \) which concludes the proof. □

By the combination of Lemma 4.16 and Lemma 4.15 it follows that no saving strategy succeeds along \( X \), but a betting strategy \( M \) succeeds on \( X \). This completes the proof of the infinite case of Theorem 3.2.

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