A Generalized Composition of Quadratic Forms based on Quadratic Pairs

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Abstract

For quadratic spaces which represent 1 there is a characterization of hermitian compositions in the language of algebras-with-involutions using the even Clifford algebra. We extend this notion to define a generalized composition based on quadratic pairs and determine the degrees of minimal compositions for any given quadratic pair.

0 Notation

We introduce the following notations: For an algebra $A$ over a field $F$, we denote by $\mathbb{A}$ the underlying $F$-vector-space. If $A$ is central simple $\text{Trd}_A(x)$ denotes the reduced trace of an element $x \in A$. For an algebra $A$ over a commutative ring $R$ we denote by $A^{\text{op}} = \{a^{\text{op}} \mid a \in A\}$ the opposite algebra, endowed with the same vector-space structure as $A$ but the reversed multiplication. For a vector space $V$, the notation $T(V)$ refers to the tensor algebra on $V$.

If $(A, \sigma)$ is a central simple algebra with an involution of the first kind, the sets of symmetric, skew-symmetric, symmetrized and alternating elements are defined by

\[
\begin{align*}
\text{Sym}(A, \sigma) &= \{a \in A \mid \sigma(a) = a\} \\
\text{Skew}(A, \sigma) &= \{a \in A \mid \sigma(a) = -a\} \\
\text{Symd}(A, \sigma) &= \{a + \sigma(a) \mid a \in A\} \\
\text{Alt}(A, \sigma) &= \{a - \sigma(a) \mid a \in A\}
\end{align*}
\]

If $A$ is an $F$-algebra and $B \subset A$ a sub-algebra, the centralizer of $B$ in $A$ is denoted by $C_A(B) = \{a \in A \mid ab = ba \forall b \in B\}$. The center of $A$, $C_A(A)$ is denoted by $Z(A)$. 
Let $A$ be Azumaya with center $Z(A) \cong F \times F$. Fix an isomorphism $\theta : F \times F \to Z(A)$ and let $e = \theta((1, 0))$. We will write $A = A^+ \times A^-$, where $A^+ = eA$ and $A^- = (1 - e)A$ are central simple $F$-algebras.

The standard involution on a separable quadratic extension $S/F$ is denoted by $\iota$. If $B$ is $S$-Azumaya $B$ denotes the algebra with scalar multiplication twisted through $\iota$. For a commutative ring $R$ the notation $\text{Br}(R)$ refers to its Brauer group and $[A]$ indicates the class of an Azumaya $R$-algebra $A$ in the Brauer group.

1 Introduction

Throughout this paper we fix a base field $F$ of arbitrary characteristic. All quadratic and hermitian spaces are supposed to be finite dimensional. A $m$-dimensional quadratic space is a couple $(V, q)$, where $V$ is an $m$-dimensional vector space and $q$ is a regular quadratic form on $V$ if $m$ is even, and a semi-regular quadratic form on $V$ if $m$ is odd, see [Knu91].

Let $R$ be a commutative ring, $\varepsilon = \pm 1$ and $A$ be an $R$-algebra with involution $a \mapsto \bar{a}$. An $\varepsilon$-hermitian space over $A$ is a couple $(E, h)$, where $E$ is a faithfully projective, finitely generated $A$-right-module and $h : E \to A$ is a regular, with respect to the bar-involution $\varepsilon$-hermitian form on $E$. For $A = R$ with the identity as involution, $\varepsilon$-hermitian spaces are called symmetric bilinear spaces ($\varepsilon = 1$) and antisymmetric bilinear spaces ($\varepsilon = -1$).

For a $\varepsilon$-hermitian space $(E, h)$ we denote by $\sigma_h$ the adjoint involution with respect to $h$, i.e. the involution of $\text{End}_A(E)$ subject to the condition

$$h(f(x), y) = h(x, \sigma_h(f)(y)),$$

for any $x, y \in E, f \in \text{End}_A(E)$.

Let now $A$ be $R$-Azumaya with an involution $a \mapsto \bar{a}$ and $E$ a faithfully projective $A$-module. Let $\tau$ be an involution (of the first or second kind) on $\text{End}_A(E)$. By a generalized Skolem-Noether theorem, there exists an invertible $R$-module $I$, such that $\tau$ is adjoint to a nonsingular $\varepsilon$-hermitian form on $E$ with values in $A \otimes I$, see [Knu70].

For $A$ as above there exists a faithfully flat $R$-algebra $S$ such that $A \otimes S \cong \text{End}_S(E)$, where $E$ is a faithfully projective $S$-module. Assume that $\tau$ is of the first kind. The induced involution $\tau \otimes \text{id}_S$ is then adjoint to an $\varepsilon$-symmetric bilinear form $b$ on $E$. In convention with [KMR198] we call $\tau$ of symplectic type if $b$ is alternating, i.e. $b(y, y) = 0$ for $y \in E$, and of orthogonal type otherwise.

Let now $(V, q)$ and $(E, p)$ be quadratic spaces. A bilinear map

$$\phi : V \times E \to E$$
satisfying
\[ p(\phi(x, y)) = q(x)p(y), \forall x, \forall y. \] (1)
is called a quadratic composition of \((V, q)\) with \((E, p)\). Let \((E, h)\) be an \(\epsilon\)-hermitian space over an \(F\)-algebra \(A\) with involution. A map \(\phi: V \times E \to E\) which is \(R\)-linear in the first variable and \(A\)-linear in the second variable is called \(\epsilon\)-hermitian composition of \((V, q)\) with \((E, h)\), if it satisfies the equation
\[ h(\phi(x, y_1), \phi(x, y_2)) = q(x)h(y_1, y_2), \forall x \in V, \forall y_1, y_2 \in E. \] (2)

A quadratic composition \(\phi: V \times E \to E\) of spaces \((V, q)\) and \((E, p)\) induces a composition of \((V, q)\) with the polar form of \(p\). Equation (2) with \(h = b_p\) results from linearizing equation (1) in the second variable. If \(\text{char } F \neq 2\), then the two equations are really equivalent. For fields of characteristic 2 (and more generally for rings, in which 2 is a zero-divisor) the situation is more complicated. There are examples, in which a composition of a quadratic space with an even symmetric bilinear form is not induced by a quadratic composition. However, that only happens for quadratic spaces of dimension \(\leq 5\), see the following theorem from [Lot06, Theorem 4]:

**Theorem 1** Let \(\phi: V \times E \to E\) be a composition of a quadratic space \((V, q)\) with a symmetric bilinear space \((E, b)\). Assume \(\dim V \geq 6\) and the existence of \(z \in V\) with \(q(z) = 1\). If \(\dim V\) is even, then there exists a quadratic form \(p\) on \(E\) with polar \(b\) and a quadratic composition of \((V, q)\) with \((E, p)\). If \(\dim V\) is odd, then the same statement holds, if we assume that \(z\) is contained in a regular subspace of \((V, q)\).

Compositions of quadratic spaces with \(\epsilon\)-hermitian spaces can be characterized in terms of algebras-with-involution. For a quadratic space \((V, q)\) let \(C_0 = C_0(V, q)\) denote its even Clifford algebra with canonical involution \(\tau_0\). For the proof of the next theorem, see [Lot06, Theorem 3].

**Theorem 2** Let \((V, q)\) be a quadratic space containing an element \(z \in V\) with \(q(z) = 1\) and let \((E, h)\) be a \(\epsilon\)-hermitian space. There exists a composition \(\phi: V \times E \to E\) of \((V, q)\) with \((E, h)\) iff there exists a homomorphism \(\alpha: (C_0(V, q), \tau_0) \to (\text{End}_A(E), \sigma_h)\) of algebras-with-involution.

**Example** Assume \(\text{char } F \neq 2\) and let
\[(V, q) = (F^5, \langle 1, -a, -b, -1, 1 \rangle)\]
for some \(a, b \in F^\times\). We take \(z = (1, 0, 0, 0, 0)\) and decompose \((V, q)\) as \((Fz \oplus V', \langle 1 \rangle) \perp -q'\). Sending \(x \in V'\) to \(zx \in C_0(V, q)\) yields, by the universal property of \(C(V, q)\), an isomorphism
\[(C_0(V, q), \tau_0) \cong (C(V', q'), \sigma),\]
where \( \sigma \) is the standard involution on \( C(V', q') \). Let \( Q = (a, b)_F \) be the quaternions with generators \( i, j \) and relations \( i^2 = a, j^2 = b, ij + ji = 0 \) and let \( \{e_i\}_{i=1...4} \) be the canonical basis of \( V' = F^4 \). We get an isomorphism \( C(V', q') \cong M_2(Q) \) by sending

\[
e_1 \mapsto \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \quad e_3 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e_4 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Under that isomorphism the canonical involution on \( C(V', q') \) corresponds to the involution

\[
\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \mapsto \begin{pmatrix} \bar{m}_{22} & -\bar{m}_{12} \\ -\bar{m}_{21} & \bar{m}_{11} \end{pmatrix},
\]

where \( \bar{a} = k\bar{a}k^{-1} \) (for \( a \in Q \) with \( k = ij \) is the involution of \( Q \) which fixes \( i, j \) and sends \( k \) to \(-k\). The above involution on \( M_2(Q) \cong \End_Q(Q \oplus Q) \) is adjoint to a \( \varepsilon \)-hermitian form with respect to any fixed involution on \( \overline{Q} \).

For the involution \( a \mapsto \bar{a} \) on \( Q \) the above involution is adjoint to the anti-hermitian form

\[
h_1((y_1, y_2), (y'_1, y'_2)) = \bar{y}_1y'_2 - \bar{y}_2y'_1
\]
on \( Q \oplus Q \). For the canonical involution \( a \mapsto \bar{a} \) on \( Q \) the above involution is adjoint to the hermitian form

\[
h_2((y_1, y_2), (y'_1, y'_2)) = \bar{y}_1ky'_2 - \bar{y}_2ky'_1.
\]

The resulting homomorphism

\[
\alpha: (C_0(V, q), \tau_0) \rightarrow (\End_Q(Q \oplus Q), \sigma_{h_i})
\]
corresponds to an anti-hermitian composition of \( (V, q) \) with \( (Q \oplus Q, h_1) \) (with respect to the tilde-involution) and to a hermitian composition of \( (V, q) \) with \( (Q \oplus Q, h_2) \) (with respect to the bar-involution), respectively. It is obtained by \( \phi(x, y) = \alpha(\tau(x))(y) \) for \( x = (x_0, x_1, x_2, x_3, x_4) \in V \) and \( y = (y_1, y_2) \in Q \oplus Q \).

Explicitly, we have

\[
\phi(x, y) = \begin{pmatrix} x_0 + x_1i + x_2j \\ x_3 - x_4 \end{pmatrix} \cdot y_1 + \begin{pmatrix} x_3 + x_4 \end{pmatrix} \cdot y_2
\]
and one may check, that \( h_i(\phi(x, y), \phi(x, y')) = q(x)h_i(y, y') \) for \( i = 1, 2 \).

In the more general setting of quadratic forms over a commutative ring \( R \) Ziger [Züg99] has studied in detail \( \varepsilon \)-hermitian compositions over \( R \), over a separable quadratic extension \( S/R \) or over a quaternion algebra \( Q/R \). In the present paper, we only consider quadratic spaces over fields. However,
we consider hermitian and anti-hermitian spaces over a large class of central simple $F$-algebras and over $S$-Azumaya algebras, where $S/F$ is a separable quadratic extension. Moreover, we generalize on the side of $(V, q)$, which we replace by a so-called quadratic pair.

Quadratic pairs were introduced in [KMRT98]. The notion of an even Clifford algebras is generalized in their setting. We briefly recall the definition of a quadratic pair and its Clifford algebra, following [KMRT98].

Let $A$ be a central simple $F$-algebra of degree $n$. A quadratic pair on $A$ is a couple $(\sigma, f)$, where

\begin{align*}
&\sigma \text{ is an involution of the first kind, that is of orthogonal type if } \text{char } F \neq 2 \text{ and of symplectic type if } \text{char } F = 2, \text{ respectively, and} \\
&f : \text{Sym}(A, \sigma) \to F \text{ is a } F\text{-linear map subject to the condition:}
\end{align*}

\[ f(x + \sigma(x)) = \text{Trd}_A(x) \text{ for all } x \in A. \] (3)

If $A$ is not known from the context, we shall write $(A, \sigma, f)$ for a quadratic pair $(\sigma, f)$ on $A$.

If char $F \neq 2$, the map $f$ is uniquely determined by (3), $f(x) = \frac{1}{2} \text{Trd}_A(x)$. If char $F = 2$, the map $f$ is only determined on $\text{Sym}_d(A, \sigma) \subset \text{Sym}(A, \sigma)$ and in general there exist several maps, with which the involution $\sigma$ forms a quadratic pair on $A$.

In any case, for a given quadratic pair $(A, \sigma, f)$ there exists an element $\ell \in A$ with

\[ f(s) = \text{Trd}_A(\ell s) \text{ for all } s \in \text{Sym}(A, \sigma) \quad \text{and} \quad \ell + \sigma(\ell) = 1, \]

The element $\ell$ is unique up to additivity of an element in $\text{Alt}(A, \sigma)$. If char $F \neq 2$, it can be taken to be $\ell = \frac{1}{2}$.

The (generalized even) Clifford algebra is defined as a quotient of the tensor algebra $T(A)$ of the underlying vector space $A$ of $A$:

\[ C(A, \sigma, f) = \frac{T(A)}{J_1(\sigma, f) + J_2(\sigma, f)}, \]

where the ideals $J_1(\sigma, f)$, $J_2(\sigma, f) \subset T(A)$ are given as follows:

- $J_1(\sigma, f)$ is generated by all elements of the form $s - f(s)$, for $s \in A$ with $\sigma(s) = s$.

- $J_2(\sigma, f)$ is generated by all elements of the form $u - \text{Sand}(u)(\ell)$, for $u \in A \otimes A$ with $\text{Sand}(u)(x) = \text{Sand}(u)(\sigma(x))$ for all $x$.

The Clifford algebra is equipped with an involution $\sigma$, which is induced by the involution $\sigma$ on $A$. By construction:

\[ \sigma(a_1 \otimes \cdots \otimes a_k) = \sigma(a_k) \otimes \cdots \otimes \sigma(a_1) \text{ for all } k \in \mathbb{N}, a_1, \ldots, a_k \in A. \]
considered as elements of the Clifford algebra.

Quadratic pairs on trivial algebras, i.e. of the form $A = \text{End}_F(V)$, can be identified with quadratic forms modulo a factor in $F^\times$ as follows: For a regular quadratic space $(V, q)$ consider the map

$$V \otimes V \to \text{End}_F(V), \quad \varphi_q(v \otimes w)(x) = vb_q(w, x).$$

The map $\varphi_q$ is a linear bijection and induces an isomorphism of algebras-with-involution:

$$\varphi_q: (V \otimes V, \sigma_{sw}) \cong (\text{End}_F(V), \sigma_q),$$

where $\sigma_{sw}(v \otimes w) = w \otimes v$ and the multiplication on $V \otimes V$ is suitably defined.

There exists a unique linear map $f_q: \text{Sym}(\text{End}_F(V), \sigma_q) \to F$ subject to the condition:

$$f_q \circ \varphi_q(v \otimes v) = q(v) \text{ for all } x, y \in V.$$  

The couple $(\sigma_q, f_q)$ is a quadratic pair on the algebra $\text{End}_F(V)$. Conversely, every quadratic pair $(\sigma, f)$ on $\text{End}_F(V)$ is of the form $(\sigma_q, f_q)$ for a regular quadratic form $q$, which is unique up to a factor in $F^\times$.

The Clifford algebra of a quadratic pair $(\sigma_q, f_q)$ on $\text{End}_F(V)$ is isomorphic to the even Clifford algebra of the quadratic space $(V, q)$ by a canonical isomorphism, under which the involution $\sigma$ corresponds to the canonical involution $\tau_0$ on $C_0(V, q)$.

The Clifford algebra construction commutes with scalar extension. If extending scalars to a splitting field $L$, a quadratic pair becomes a quadratic form up to a factor in $L^\times$ and its Clifford algebra becomes the usual even Clifford algebra of a quadratic space, which contributes to derive the basic structure properties of the Clifford algebra of a quadratic pair. The following two theorems are taken from [KMR98, Theorems 8.10, 9.14, 9.16; 8.12].

**Theorem 3 (Structure Theorem)** Let $(A, \sigma, f)$ be a quadratic pair of degree $n$ and let $C = C(A, \sigma, f)$ be its Clifford algebra.

1. $n = 2k$: $Z = Z(C)$ is a separable quadratic $F$-Algebra and $C$ is $Z$-Azumaya of degree $2^{k-1}$. If $Z \cong F \times F$, then $C \cong C^+ \times C^-$ for Azumaya $F$-algebras $C^\pm$ of degree $2^{k-1}$ and we have $[C^+][C^-][A] = 1$ in $\text{Br}(F)$ if $k$ is even, and $[C^+][C^-] = 1$ if $k$ is odd, respectively.

2. $n = 2k + 1$: $(A, \sigma, f) \cong (\text{End}_F(V), \sigma_q, f_q)$ for a regular quadratic space $(V, q)$; we have $C \cong C_0(V, q)$ which is $F$-Azumaya.

**Theorem 4** Let $(A, \sigma, f)$ be a quadratic pair of degree $n$ or a semi-regular quadratic space of dimension $n$ (if char $F = 2$ and $n$ is odd) and let $\sigma$ be the canonical involution of the corresponding (generalized even) Clifford algebra.
1. \( n = 2k \): The involution \( \sigma \) is unitary if \( k \) is odd, and orthogonal if \( \text{char } F \neq 2 \), \( k \equiv 0 \, (4) \), and symplectic if \( \text{char } F = 2 \) or \( k \equiv 2 \, (4) \), respectively.

2. \( n = 2k+1 \): The involution \( \sigma \) is orthogonal if \( \text{char } F \neq 2 \), \( n \equiv \pm 1 \, (8) \), and symplectic if \( \text{char } F = 2 \), \( n > 1 \) or \( n \equiv \pm 3 \, (8) \), respectively. The only exception is \( n = 1 \), \( \text{char } F = 2 \), where \( \sigma \) is orthogonal.

2 Generalization of the Composition Problem

We replace the usual Brauer group \( \text{Br}(R) \) defined for a commutative ring \( R \) by restricting the equivalence classes to Azumaya algebras of constant rank.

Definition 5 Let \( R \) be an arbitrary commutative ring. Recall that two \( R \)-Azumaya algebras \( A \) and \( B \) are called Brauer-equivalent, if there exist faithfully projective \( R \)-modules \( P_1, P_2 \) with \( A \otimes \text{End}_R(P_1) \cong B \otimes \text{End}_R(P_2) \). The quotient of the monoid of Azumaya \( R \)-algebras of constant rank modulo Brauer-equivalence is called constant Brauer group, denoted by \( \text{Br}_c(R) \). The equivalence class of an Azumaya \( R \)-algebra \( A \) of constant rank is denoted by \([A]_c\).

Clearly, the above definition yields nothing new for rings with connected spectrum, in particular for fields. In general the inclusion

\[ j: \text{Br}_c(R) \hookrightarrow \text{Br}(R), \quad [A]_c \mapsto [A]. \]

is actually an isomorphism:

Lemma 6 Any equivalence class \( c \in \text{Br}(R) \) contains an Azumaya-algebra of constant rank.

Proof Let \( c = [A] \) for an Azumaya-algebra \( A \) of not necessarily constant rank. We decompose \( R \) into a finite product \( R = R_1 \times \cdots \times R_l \) and write \( A \) as \( A = A_1 \times \cdots \times A_l \) such that \( A_i \) is \( R_i \)-Azumaya of constant rank \( n^2_i \). Let \( n = n_1 \cdots n_l \) and \( P_i = R_i^{n_i}, i = 1, \ldots, l \). Then \( P_1 \oplus \cdots \oplus P_l \) is a faithfully projective \( R \)-module and the Azumaya \( R \)-algebra \( A' = A \otimes \text{End}_R(P) \) satisfies \([A'] = [A] \) and has constant rank \( n^2 \). \( \square \)
Definition 7 Let \((A, \sigma, f)\) be a quadratic pair over a field \(F\) and let \(B\) be an \(F\)-algebra with involution \(\tau\). We call a homomorphism

\[
\alpha: (C(A, \sigma, f), \sigma) \rightarrow (B, \tau)
\]

of algebras-with-involutions a composition of type \((c, t) \in \text{Br}(F) \times \mu_2(F)\) if \(B\) is central simple over \(F\) and \(\tau\) is of the the first kind, with \(c = [B]\) and \(t = -1\) if \(\tau\) is symplectic, otherwise \(t = 1\).

Let \(S\) be a separable quadratic extension of \(F\) and assume \(B\) to be a \(S\)-Azumaya of constant rank. We call \(\alpha: (C(A, \sigma, f), \sigma) \rightarrow (B, \tau)\) a composition of type \((c', 0) \in \text{Br}_c(S) \times \{0\}\), if \(\tau\) is an involution of the second kind (with \(\tau|_F = \text{id}_F\)), where \(c' = [B]_c\).

In both cases we talk about compositions of the quadratic pair \((A, \sigma, f)\) and we call the couple \((c, t)\) or \((c', 0)\), respectively, the type of the composition, denoted by \(\text{type} \alpha\).

The motivation for the preceding definition comes from Theorem \(2\). Recall from the introduction that involutions on Azumaya algebras with involution are adjoint to \(\varepsilon\)-hermitian forms. Thus, we have:

Proposition 8 Let \((\sigma_q, f_q)\) be a quadratic pair on \(A = \text{End}_F(V)\). Scaling \(q\), we may assume, that it represents \(1\). Then compositions of \((\sigma_q, f_q)\) correspond to compositions of \((V, q)\) with nonsingular \(\varepsilon\)-hermitian spaces \(\phi: (V, q) \times (E, h) \rightarrow (E, h)\), where \((B, \tau) \simeq (\text{End}_A(E), \sigma_h)\) and \(h\) is \(\varepsilon\)-hermitian with respect to some involution on \(A\) which restricts to the same involution as \(\tau\) on \(Z(B) = Z(D)\).

Moreover, from Theorem \(1\) we get that quadratic compositions of spaces \((V, q)\) and \((E, p)\) correspond to compositions of the quadratic pair \((\sigma_q, f_q)\) with an orthogonal (resp. symplectic, if \(\text{char}\ F = 2\)) involution on the trivial algebra \(B = \text{End}_F(E)\), assuming \(\dim V \geq 6\).

If \((\sigma, f)\) is a quadratic pair on a non-trivial central simple algebra \(A\), we may extend scalars to a splitting field \(L\) to get a \(\varepsilon\)-hermitian composition of a quadratic space (over \(L\)) again. If choosing \(L\) large enough such that \([B \otimes L]_c = 1 \in \text{Br}_c(Z(B) \otimes L)\), scalar extension yields compositions of quadratic spaces with \(\varepsilon\)-symmetric bilinear forms if \(Z(B) = F\), and with \(\varepsilon\)-hermitian forms with respect to the standard involution of \(Z(B) \otimes L\) over \(L\) if \(Z(B)/F\) is a separable quadratic extension, respectively.

Examples 1. If \(\alpha: (C(A, \sigma, f), \sigma) \rightarrow (B, \tau)\) is a composition of type \((c, t)\) and \((B', \tau')\) is a central simple algebra over \(F\) with involution of the first kind, then \(C(A, \sigma, f) \rightarrow (B, \tau) \leftarrow (B \otimes B', \tau \otimes \tau')\) is a composition of \((A, \sigma, f)\) with \((B \otimes B', \tau \otimes \tau')\) of type \((c \cdot c', t \cdot t')\).
2. Let \((Q_1, \gamma_1)\) and \((Q_2, \gamma_2)\) be quaternion algebras over \(F\) with standard involutions. There exists a canonical quadratic pair \((A, \sigma, f)\) with \(A = Q \otimes Q_2\), \(\sigma = \gamma_1 \otimes \gamma_2\) and its Clifford algebra is \((Q_1, \gamma_1) \times (Q_2, \gamma_2)\), see [KMRT98, Example (8.19)]. Projection to the first factor yields a homomorphism: 

\((C(A, \sigma, f), \sigma) \rightarrow (Q_1, \gamma_1)\), hence a composition of \((A, \sigma, f)\) with \((Q_1, \gamma_1)\). Including \((Q_1, \gamma_1)\) in \((A, \sigma)\) gives a composition of \((A, \sigma, f)\) with \((A, \sigma)\).

**Remark** For trivial algebras \(A = \text{End}_F(V)\) there exists a composition of \((A, \sigma_q, f_q)\) with \((A, \sigma_q)\) whenever \((V, \lambda \cdot q)\) is a unitary composition algebra for some \(\lambda \in F^\times\), i.e. a quadratic space (representing 1) that admits a composition with itself. If \(\dim V \geq 6\) or \(\text{char } F = 2\) the converse is also true, which follows from Theorem [1].

It would be interesting to generalize the notion of composition of two quadratic spaces to quadratic pairs. A composition of two quadratic pairs \((A, \sigma, f)\) and \((B, \tau, g)\) should be defined such that it induces a composition of \((A, \sigma, f)\) with \((B, \tau)\) like introduced in the present paper. For fields of characteristic 2 the notions should show to be equivalent (presumably also if \(\text{deg } A \geq 6\), like in Theorem [1]). Also it should be stable under scalar extension, so that a composition of two quadratic pairs yields a composition of quadratic spaces over a common splitting field. Interestingly, for trivial algebras a composition \(\phi: (V, q) \times (E, p) \rightarrow (E, p)\) induces a bilinear map \(\psi: \text{Sym}(\text{End}_F(V), \sigma_q) \times \text{Sym}(\text{End}_F(E), \sigma_p) \rightarrow \text{Sym}(\text{End}_F(E), \sigma_p)\) such that \(f_q(s)f_p(t) = f_p(\psi(s,t))\). It is given on generators \(s = \varphi_q(x \otimes x)\), \(t = \varphi_p(y \otimes y)\) as \(\psi(s,t) = \varphi_p(\phi(x, y) \otimes \phi(x, y))\). Note that \(\phi\) can be expressed through the composition homomorphism \(\alpha: (C_0(V, q), \tau_0) \rightarrow (\text{End}_R(E), \sigma_b)\) if choosing an element \(z \in V\) with \(q(z) = 1\). Therefore we suggest to define a composition of two quadratic pairs \((A, \sigma, f)\) and \((B, \tau, g)\) via a bilinear map \(\psi: \text{Sym}(A, \sigma) \times \text{Sym}(B, \tau) \rightarrow \text{Sym}(B, \tau)\) subject to the condition \(f(s)g(t) = g(\psi(s,t))\) and some further restrictions. More restrictions are needful since the bilinear map \(\psi(s,t) = f(s)t\) should not be allowed. Unfortunately we did not manage to define \(\psi\) in an explicit way nor to find suitable restrictions.

A composition pair \((A, \sigma, f)\) could then be defined as a quadratic pair which admits a composition with itself. Much of the classical results follow directly from the existence of a composition \((A, \sigma, f) \rightarrow (A, \sigma)\). For example \(\text{deg } A\) must be 1, 2, 4 or 8. If \(\text{deg } A = 8\) it is necessary that the center of the (even) Clifford algebra splits and one factors is trivial (hence the other factor is isomorphic to \(A\)). See [Eld02, Corollary 2] for the situation on trivial algebras, where those two conditions are proved to be sufficient as well.
In characteristic 2 there do not exist quadratic pairs of odd degree. In that case we consider semi-regular quadratic spaces instead of quadratic pairs. Let us introduce the following notation:

**Notation** We denote by the letter $P$ a quadratic pair $(A, \sigma, f)$ or (in odd dimension, char $F = 2$) a semi-regular quadratic space $(V, q)$. We call $P$ an extended quadratic pair. Furthermore we denote by $(C(P), \sigma)$ the (even) Clifford algebra $C(A, \sigma, f)$ and $C_0(V, q)$, respectively, with canonical involution. We write $\deg P$ for $\deg A$ and $\dim V$, respectively.

**Remark** For any extended quadratic pair $P$ and any given type $(c, t) \in \text{Br}(F) \times \mu_2(F)$ with $c^2 = 1 \in \text{Br}(F)$ there do exist compositions. A composition of such type can be constructed as follows: Consider the embedding $C(P) \otimes C(P)^{op} \simeq \text{End}_F(E) \subset \text{End}_F(E)$. By Lemma 13 and Lemma 14 the involution $\sigma$ on $C(P)$ can be extended to an involution of the first kind $\rho$ on $\text{End}_F(E)$. Secondly, let $D$ be central simple of Brauer class $c$. Since $c^2 = 1 \in \text{Br}(F)$, there exist an involution of the first kind $\tau$ on $D$. We take the tensor product with $(D, \tau)$. The type of $\rho \otimes \tau$ may not be as required. Taking the tensor product with $M_2(F)$ equipped with the standard involution finds a remedy for that.

The same holds for compositions with unitary involutions: Let $S/F$ be a separable quadratic extension and $c' \in \text{Br}_c(S)$ satisfying $N_{S/F}(c') = 1 \in \text{Br}(F)$, i.e. there exists a unitary involution $\tau'$ on $D'$. Taking the tensor product $(C(P), \sigma) \subset (\text{End}_F(E), \rho) \subset (\text{End}_F(E) \otimes D', \rho \otimes \tau')$ yields a composition of $P$ of type $(c', 0)$. Thus, one always gets a composition of given type. The difficulty is just to find and to reach the minimal degree.

### 3 Minimal Compositions

We say, that a composition $\alpha : (C(P), \sigma) \rightarrow (B, \tau)$ is minimal, if there exists no composition $\beta : (C(P), \sigma) \rightarrow (B', \tau')$ with type $\alpha = \text{type} \beta$ and $\deg B' < \deg B$. Note, that in contrast to most other works on compositions we fix $P$ itself, not only the degree of $P$. We are interested to find a good composition for every $P$ (and every suitable type of composition). By the minimal composition degree of given composition type $\alpha$ for $P$ we mean $\deg B'$ for any minimal composition $\beta : (C(P), \sigma) \rightarrow (B', \tau')$ of that type. We denote the minimal composition degree by $\text{mcd}(P, c, t)$ for compositions of type $(c, t)$.

We will express the minimal composition degrees $\text{mcd}(P, \alpha)$ with the help of a metric on the constant Brauer group. In order to introduce the metric, we need some preparations. For an Azumaya $R$-algebra of constant rank,
the minimum degree of \( B \) taken over all Azumaya \( R \)-algebras \( B \) of constant rank with \( [B]_c = [A]_c \) is called \textit{Index} of \( A \), denoted by \( \text{ind} A \). We have \( \text{ind} M_r(D) = r \) if \( D \) is a division algebra with center \( F \). If \( A \) is Azumaya over \( R = F \times F \), then \( A \) is of the form \( A_1 \times A_2 \) with \( A_1, A_2 \) central simple over \( F \) and \( \text{deg} A_1 = \text{deg} A_2 = \text{deg} A \). It is easy to see then, that \( \text{ind} A = \text{lcm}(\text{ind} A_1, \text{ind} A_2) \). In both cases the index of \( A \) divides the degree of \( A \).

**Lemma 9** Let \( R \) be a commutative ring and \( \text{Br}_c(R) \) the constant Brauer group. The map

\[
d : \text{Br}_c(R) \times \text{Br}_c(R) \rightarrow \mathbb{R}, \quad d([B_1]_c, [B_2]_c) = \log_2 \text{ind}(B_1 \otimes B_2^{op})
\]

is well-defined and satisfies the axioms of a metric. We omit the brackets and write e.g. \( d(B_1, B_2) \) instead of \( d([B_1]_c, [B_2]_c) \) for short. The above metric has the following properties:

\[
d(A \otimes B, A \otimes C) = d(B, C) \quad \text{and} \quad d(B^{op}, C^{op}) = d(B, C)
\]

for \( R \)-Azumaya-algebras \( A, B, C \) of constant rank.

**Proof** By construction, the index of an Azumaya-algebra of constant rank only depends on its class in the constant Brauer group. Using the properties

1) \([A]_c^{-1} = [A^{op}]_c \), 2) \( \text{ind} A = \text{ind} A^{op} \) und 3) \( \text{ind}(A_1 \otimes A_2) \leq \text{ind}(A_1) \text{ind}(A_2) \)

the assertions can easily be shown. \( \square \)

**Remark** If \( R = F \), the induced metric on the subgroup of classes of central simple algebras with involutions of the first kind maps to \( \mathbb{N} \).

The relevance of the metric to solve the composition problem shows up in the following two lemmas:

**Lemma 10** Let \( C \) be \( R \)-Azumaya of constant rank. The minimal rank of an Azumaya \( R \)-algebra \( A \) of constant rank of given class \( [B]_c \in \text{Br}_c(R) \) for which there exists a homomorphism \( f : C \rightarrow A \) of \( R \)-algebras is exactly \( \text{deg} A = \text{deg} C \cdot 2^{d(B, C)} \). If \( R = F \) or \( R = F \times F \) then the degree of any (not necessarily minimal) such \( A \) is a multiple of \( \text{deg} C \cdot 2^{d(B, C)} \).

**Proof** For a homomorphism \( f : C \rightarrow A \) of Azumaya algebras of constant rank we may write \( A \simeq C \otimes G \), where \( G = C_A(f(C)) \) is Azumaya of constant rank with \( [G]_c = [B \otimes C^{op}]_c \). Thus, the inequality \( \text{deg} A = \text{deg} C \text{deg} G \geq \text{deg} C \text{ind} G = \text{deg} C \text{ind}(B \otimes C^{op}) = \text{deg} C \cdot 2^{d(B, C)} \) holds. If \( R = F \) or \( R = F \times F \) then the degree of \( G \) is a multiple of the index of \( G \). Furthermore, there exists an Azumaya algebra \( G \) with \( 2^{d(B, C)} = \text{deg} G \) and \( [B \otimes C^{op}]_c = [G]_c \). Let \( f : C \hookrightarrow A = C \otimes G \) be the inclusion. We have \( \text{deg} A = \text{deg} C \text{deg} G = \text{deg} C \cdot 2^{d(B, C)} \) and \( [A]_c = [C]_c[G]_c = [C]_c[B \otimes C^{op}]_c = [B]_c \). \( \square \)
Lemma 11 Let $F$ be a field and $Z,S/F$ separable extensions of degree 1 or 2. Let $B$ and $C$ be two Azumaya algebras with center $S$ and $Z$, respectively. If there exists a homomorphism of $F$-algebras $\alpha : C \to B$, then the degree of $B$ is of the following form:

1. If $Z = F$ then $\deg B = n \deg C \cdot 2^{d(B,C \otimes S)}$ with $n \geq 1$.

2. If $Z \cong F \times F$ and $\alpha$ is not injective, then $\deg B = n \deg C \cdot 2^{d(B,C^+ \otimes S)}$ with $n \geq 1$.

3. If $Z \cong F \times F$ and $S$ is a field, then $\deg B = \deg C(n_1 2^{d(B,C^+ \otimes S)} + n_2 2^{d(B,C^- \otimes S)})$ with $n_1 + n_2 \geq 1$. Moreover, if $\alpha$ is injective and $S = F$ then $n_1, n_2 \geq 1$.

4. If $Z$ and $S$ are isomorphic quadratic field extensions of $F$, then $\deg B = \deg C(n_1 2^{d(B,C)} + n_2 2^{d(B,C)})$ with $n_1 + n_2 \geq 1$.

5. If $Z \cong F \times F \cong S$, then $\deg B = \deg C(n_1 2^{d(B,C^+)} + n_2 2^{d(B^+,C^-)}) = \deg C(m_1 2^{d(B^-,C^+)} + m_2 2^{d(B^-,C^-)})$ with $n_1 + n_2 \geq 1, m_1 + m_2 \geq 1$.

6. If $Z$ is a quadratic field extensions of $F$ and $Z \not\cong S$, then $\deg B = n \deg C \cdot 2^{d(B \otimes Z,C \otimes S) + 1}$ with $n \geq 1$.

Proof 1. The homomorphism $\alpha : C \to B$ together with the inclusion $S \hookrightarrow B$ induces a homomorphism of $S$-algebras $C \otimes S \to B$, thus the claim follows from the preceding lemma.

2. Since $\alpha : C^+ \times C^- \to B$ is non-injective, its kernel must be one of its nontrivial ideals $C^+ \times \{0\}$ or $\{0\} \times C^-$. Thus $\alpha$ factors through $C^-$ or $C^+$ and then the preceding argument shows the claim.

3. The homomorphism $C \otimes S \to B$ from $\mathbb{1}$ together with the identity on $B^{op}$ induces a $S$-linear homomorphism $\gamma : C^+ \otimes B^{op} \times C^- \otimes B^{op} \to B \otimes_S B^{op} \cong \text{End}_S(E)$, where $E$ is the underlying $S$-vector space of $B$. Let $E_1 = \gamma(1,0)E, E_2 = \gamma(0,1)E$. Then $\gamma$ restricts to homomorphisms $C^+ \otimes B^{op} \to \text{End}_S(E_1)$ and $C^- \otimes B^{op} \to \text{End}_S(E_2)$. Observe that $\dim E_1, \dim E_2 > 0$ if $\alpha$ is injective and $S = F$ (and therewith $\gamma$ is injective). Since $E = E_1 \oplus E_2$, in particular $\dim E = \dim E_1 + \dim E_2$, the claim follows with $\mathbb{1}$.

4. As above we get a homomorphism $C \otimes_F B^{op} \to \text{End}_S(E)$. Let $K = Z \cong S$. The map $C \otimes_F B^{op} \to C \otimes_K B^{op} \times C \otimes_K B^{op}$ given by $c \otimes b^{op} \mapsto (c \otimes_K b^{op}, c \otimes_K b^{op})$ is an isomorphism $F$-algebras, yielding
a homomorphism $\gamma: C \otimes_k B^{\text{op}} \times C \otimes_k B^{\text{op}} \to \text{End}_S(E)$, which is $S$-linear, if we view $E$ as a $S$-vector space through $\gamma$. Now the claim follows from the argument in $\S$.

5. The homomorphism $C \to B = B^+ \times B^-$ yields homomorphisms $C \to B^\pm$. The claim follows from the argument in $\S$ and the equation $\deg B = \deg B_1 = \deg B_2$.

6. We distinguish the cases of $S$ being a field and of $S \simeq F \times F$. In the first case, since $Z \not\cong S$ are both quadratic extensions, the tensor product $Z \otimes S$ is a field as well. The homomorphism $\gamma: C \otimes B^{\text{op}} \to \text{End}_S(E)$ as in $\S$ takes values in $\text{End}_{Z \otimes S} E$ and is $Z \otimes S$-linear if viewing $E$ as a $Z \otimes S$-vector space through $\gamma$. Hence $(\deg B)^2 = \dim_S E = 2 \dim_{Z \otimes S} E = 2 \deg C \deg B \cdot 2^{d(C \otimes B^{\text{op}},1)}$. By the isomorphism $(C \otimes S) \otimes_{Z \otimes S} (B \otimes Z)^{\text{op}} \simeq (C \otimes S) \otimes_{Z \otimes S} (Z \otimes B^{\text{op}}) \simeq C \otimes B^{\text{op}}$, the claim follows. In the second case, there exist homomorphisms $C \to B^\pm$. As above we conclude $\deg B^\pm = 2n_\pm \deg C 2^{d(B^\pm \otimes Z,C)}$ for some $n_\pm \geq 1$, hence $\deg B = 2n \deg C \text{lcm} (2^{d(B^+ \otimes Z,C)},2^{d(B^- \otimes Z,C)}) = n \deg C \cdot 2^{d(B\otimes Z,C \otimes S) + 1}$, where $n \geq 1$.

$\square$

**Theorem 12 (Degree of minimal compositions)** Let $P$ be an extended quadratic pair of degree $n > 1$. Let $C = C(P)$ the associated Clifford algebra and $Z = Z(C)$ be its center.

1. Compositions with unitary involutions:

   Let $S/F$ be a separable quadratic extension and let $c' \in \text{Br}_c(S)$ with $N_{S/F}(c') = 1$ in $\text{Br}(F)$. The degrees of minimal compositions are given as follows:

   (a) $n = 2k + 1$: We have $\text{mcd}(P,c',0) = 2^{k+d(c',C \otimes S)}$. For any composition $\alpha: (C(P),\sigma) \to (B,\tau)$ of type $(c',0)$, the degree of $B$ is a multiple of $\text{mcd}(P,c,t)$.

   (b) $n = 4k$: If $Z \simeq F \times F$, $C \simeq C^+ \times C^-$ then $\text{mcd}(P,c',0) = 2^{k-1+\min\{d(c',C^+ \otimes S),d(c',C^- \otimes S)\}}$.

   If $Z$ is a field and $Z \not\cong S$ then $\text{mcd}(P,[D]_c,0) = 2^{2k+d(D \otimes Z,C \otimes S)}$ and in general a multiple of $\text{mcd}(P,[D]_c,0)$.

   (c) $n = 4k + 2$: If $Z \simeq S$ then $\text{mcd}(P,c',0) = 2^{2k+\min\{d(c',C),d(c',C^{\text{op}})\}}$.

   If $Z$ is a field and $Z \not\cong S$ then $\text{mcd}(P,[D]_c,0) = 2^{2k+1+d(D \otimes Z,C \otimes S)}$ and in general a multiple of $\text{mcd}(P,[D]_c,0)$.
2. Compositions with involutions of the first kind:

Let $c \in \text{Br}(F)$ with $c^2 = 1$ in $\text{Br}(F)$, let $t \in \mu_2(F)$ and let $\varepsilon = 0$ if the canonical involution of $C$ is of type $t$, otherwise $\varepsilon = 1$. Then the degrees of minimal compositions are given as follows:

(a) $n = 2k + 1$: We have $\text{mcd}(P,c,t) = 2^{k+d(c,C)+\delta}$, where $\delta = 1$ if $[C] = c$ and $\varepsilon = 1$, otherwise $\delta = 0$. For any composition $\alpha: (C(P),\sigma) \longrightarrow (B,\tau)$ of type $(c,t)$, the degree of $B$ is a multiple of $\text{mcd}(P,c,t)$.

(b) $n = 4k$: If $Z$ is a field, then $\text{mcd}(P,[D],t) = 2^{2k+d(D \otimes Z,C)+\delta}$ and in general a multiple of the minimal degree; if $Z \simeq F \times F$, $C \simeq C^+ \times C^-$, then the minimal composition degree is $\text{mcd}(P,c,t) = 2^{2k-1+\min\{d(c,C^+),d(c,C^-)\}+\delta}$, where in the two cases $\delta = 1$ if $[C] = [D \otimes Z]$ and $([C^+] = c$ or $[C^-] = c)$, respectively, and $\varepsilon = 1$, otherwise $\delta = 0$.

(c) $n = 4k + 2$: We have $\text{mcd}(P,[D],t) = 2^{2k+1+d(D \otimes Z,C)}$. In general $\deg B = n \cdot 2^{2k+d(D \otimes Z,C)}$ with $n \geq 2$ and $n$ even if $Z$ is a field.

The proof of Theorem (12) consists of 2 parts: existence and minimality. For both in addition to Lemma (11) the following Lemma about the extension of involutions is needed. It can be found in [KMRT98, p. 52] and is reproduced here, as far as necessary:

Lemma 13 Let $B$ be a simple sub-algebra of a central simple algebra $A$ over a field $K$. Suppose, that $A$ and $B$ have involutions $\sigma$ and $\tau$ respectively with the same restriction to $K$. Then $A$ has an involution $\sigma'$ whose restriction to $B$ is $\tau$.

If $\sigma$ is of the first kind, the types $\sigma'$ and $\tau$ are related as follows:

- If $\text{char} K \neq 2$, then $\sigma'$ can be arbitrarily chosen of orthogonal or symplectic type, except if $\tau$ is of the first kind and $\deg C_A(B)$ is odd. In that case, every extension $\sigma'$ of $\tau$ is of the same type as $\tau$.

- Suppose $\text{char} K = 2$: If $\tau$ is symplectic or unitary, then every extension $\sigma'$ of $\tau$ is symplectic.

For unitary involutions on Azumaya-algebras over $F \times F$ we use the following construction:

Lemma 14 Let $R = F \times F$. 

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1. Let $B$ be $R$-Azumaya and sub-algebra of an Azumaya $R$-algebra $A$. Assume, that $A$ and $B$ possess unitary involutions $\sigma$ and $\tau$, respectively. Then $A$ has an involution $\sigma'$, which restricts to $\tau$ on $B$.

2. Let $D$ be a central simple division-algebra with an involution of the first kind $d \mapsto \overline{d}$ and let $B$ be $R$-Azumaya of the form $B = \text{End}_D(E) \otimes_R (E)$ for a faithfully projective $D \otimes_R$-module $E$. Let $\tau$ be a unitary involution on $B$. Then there exists an involution $\sigma'$ of the first kind on $\text{End}_D(E) \supset \text{End}_{D\otimes_R}(E)$, which extends $\tau$ and $\sigma'$ can be arbitrarily chosen of orthogonal or symplectic type if $\text{char} F \neq 2$ and of symplectic type if $\text{char} F = 2$, respectively.

Proof

1. Because of the unitary involutions $A$ and $B$ have constant rank. There exists a Azumaya $R$-algebra $C$ of constant rank and unitary involution $\rho$ with $A = B \otimes C$. The involution $\sigma' = \tau \otimes \rho$ extends $\tau$ to $A$.

2. Let $k = \deg B$. Let $*$ be the involution on $M_k(D)$ given by $(d_{ij})^* = (\overline{d}_{ij})^t$. We identify $\text{End}_D(E)$ with $M_{2k}(D)$ and $(\text{End}_{D\otimes_R}(E), \tau)$ with $(M_k(D) \times M_k(D), \rho)$, $\rho(m_1, m_2) = (m_2^\ast, m_1^\ast)$. Let $\sigma_\pm$ be the involution on $M_2(F)$ defined by $\sigma_\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & \pm b \\ \pm c & a \end{pmatrix}$. The involution $\sigma_-$ is symplectic, $\sigma_+$ is orthogonal if $\text{char} F \neq 2$. Define $\sigma'$ as the tensor product $* \otimes \sigma_\pm$ on $M_{2k}(D) = M_2(M_k(D))$, where $\sigma_+$ or $\sigma_-$ are chosen appropriately such that $\sigma'$ has the required type. By construction $\sigma'|_{M_k(D) \times M_k(D)} = \rho$.

Construction of Compositions

For an extended quadratic pair $P$ we construct compositions with involutions of the first kind, i.e. of type $(P, c, t)$ with $c \in \text{Br}(F)$, $t \in \mu_2(F)$ and compositions with unitary involutions, i.e. of type $(P, c', 0)$ with $c' \in \text{Br}_c(S)$ for some separable quadratic extension $S/F$.

1. Compositions with unitary involutions:

   (a) $\deg P = 2k + 1$: There exists a homomorphism $C(P) \otimes S \rightarrow B$, where $B$ is of Brauer class $c'$ and degree $2^{k+d(c', C(P) \otimes S)}$. Since $B$ has a unitary involution and the involution $\sigma \otimes t$ on $C(P) \otimes S$ extending $\sigma$ is unitary, there exists a unitary involution $\tau$ on $B$ which extends the transport of $\sigma$.
(b) $\deg P = 4k$: The involution on $C(P)$ restricts to the identity on $Z = Z(C(P))$. First consider the case $Z \simeq F \times F$. The projections onto $C^+$ and $C^-$ yield two homomorphisms of algebras-with-involutions $(C(P), \sigma) \to (C^\pm, \overline{\sigma}^\pm)$. Proceeding as above yields compositions of degree $2^{2k-1+d(c', C^\pm \otimes S)}$. The composition with smaller degree gives the asserted minimal degree.

Now assume $Z$ is a field. Consider a homomorphism $C \to C \otimes S \to \text{End}_{D' \otimes Z}(E)$, where $\tilde{B} = \text{End}_{D' \otimes Z}(E)$ is $Z \otimes S$-Azumaya of degree $2^{2k-1+d(D' \otimes Z, C \otimes S)}$. By Lemma 13 or 14 there exists an involution $\tilde{\tau}$ on $B$ which extends $\sigma \otimes \iota$, in particular $\tilde{\tau}|_{Z \otimes S} = \text{id}_Z \otimes \iota_S$. Now if $S$ is a field, $S \not\cong Z$ Lemma 13 shows, that $\tilde{\tau}$ can be extended to an unitary involution on $\text{End}_{D'}(E)$. If $S \simeq F \times F$ we use an explicit construction: There exists a central simple $F$-algebra $G$ and an isomorphism $(\tilde{B}, \tilde{\tau}) \cong (G \otimes Z \times G^{\text{op}} \otimes Z, \rho)$ where $\rho$ is the involution $\rho(g_1 \otimes z_1, g_2^{\text{op}} \otimes z_2) = (g_2 \otimes z_2, g_1^{\text{op}} \otimes z_1)$. Clearly $\rho$ leaves $G \times G^{\text{op}} \subset (G \times G^{\text{op}}) \otimes Z$ invariant. By extending the identity on $Z$ to an involution of the first kind on $M_2(F)$ we get an involution $\sigma$ on $M_2(G \times G^{\text{op}}) \cong \text{End}_{D'}(E)$ extending $\tilde{\tau}$.

(c) $\deg P = 4k + 2$: The involution on $C(P)$ restricts to the standard involution on $Z$. First consider the case $Z \simeq S$. Observe that $(C(P), \sigma)$ and $(C(P)^{\text{op}}, \overline{\sigma}^{\text{op}})$ are isomorphic as $F$-algebras-with-involutions. There exist homomorphisms $C(P) \to B$ and $C(P) \to B'$ where $[B]_c = [B']_c = c'$ and $\deg B = 2^{2k+d(c', C(P))}$, $\deg B' = 2^{2k+d(c', C(P)^{\text{op}})}$. The unitary involutions on $C(P)$ and $C(P)^{\text{op}}$ can be extended to unitary involutions on $B$ and $B'$, respectively. The composition with smaller degree gives the asserted minimal degree.

Now assume $Z$ is a field and $Z \not\cong S$. Let $\tilde{B} = \text{End}_{D' \otimes Z}(E)$ be of minimal degree such that there exists a homomorphism $C \otimes S \to \tilde{B}$. By Lemma 13 $\deg \tilde{B} = 2^{2k+d(D' \otimes Z, C \otimes S)}$. The involution $\sigma \otimes \iota$ on $C(P) \otimes S$ extends to an involution $\tilde{\tau}$ on $\tilde{B}$ with $\tilde{\tau}|_{Z \otimes S} = \iota_Z \otimes \iota_S$. If $S$ is a field then by Lemma 13 $\tilde{\tau}$ extends to a unitary involution $\tau$ on $B = \text{End}_{D'}(E)$. If $S \simeq F \times F$ we use the following construction: There exists a central simple $F$-algebra $G$ and an isomorphism of $F$-algebras $(\tilde{B}, \tilde{\tau}) \cong (G \otimes Z \times G^{\text{op}} \otimes Z, \rho)$ where $\rho(g_1 \otimes z_1, g_2^{\text{op}} \otimes z_2) = (g_2 \otimes z_2, g_1^{\text{op}} \otimes z_1)$. Extending the standard involution on $Z$ to an involution of the first kind on $M_2(F)$ yields an involution $\tau$ on $M_2(G \times G^{\text{op}}) \cong \text{End}_{D'}(E)$ extending $\tilde{\tau}$.

2. Compositions with involutions of the first kind:
(a) \( \deg P = 2k + 1 \), there exists a homomorphism \( C(P) \to B \), where \( B \) is of Brauer class \( c \) and degree \( 2^{k+d(c, C(P))} \). If \( [C(P)] \neq c \), then \( \deg C_B(C(P)) \) is of the form \( 2^l, l \geq 1 \) since there exist an involution of the first kind on \( C_B(C(P)) \), in particular even. Thus, if \( [C(P)] \neq c \) or if \( \sigma \) is already of the required type, the involution \( \sigma \) on \( C(P) \) can be extended to an involution of type \( t \) on \( B \). Otherwise, embedding \( B \) into \( M_2(F) \otimes B \) with symplectic involution on \( M_2(F) \) yields a composition of type \( t \) and degree \( 2^{k+d(c, C(P))}+1 \). In both cases we get compositions of the required type and degree.

(b) \( \deg P = 4k \): If \( Z(C(P)) \simeq F \times F \), projection to \( C^\pm \) and arguing as for odd degree of \( P \) yields a composition of the required degree and type. If \( Z(C(P)) = K \) is a field, there exists a homomorphism \( C(P) \to \tilde{B} = End_{D \otimes K}(E) \subset End_D(E) \), where \( E \) is a faithfully projective \( D \otimes K \)-module and \( \deg \tilde{B} = 2^{2k+d(D \otimes K, C(P))} \). Let \( B = End_D(E) \). If \( [C(P)] \neq [D] \), then \( \deg C_B(C(P)) = \deg C_B(C(P)) \) is even. Thus, if \( [C(P)] \neq [D] \) or if \( \sigma \) is already of the required type, the involution \( \sigma \) on \( C(P) \) can be extended to an involution of type \( t \) on \( B \). Otherwise, after embedding \( B \) in \( M_2(B) \) the required type can be obtained.

(c) \( \deg P = 4k + 2 \): There exists a homomorphism \( C(P) \to \tilde{B} = End_{D \otimes Z}(E) \), where \( E \) is a faithfully projective \( D \otimes Z \)-module of constant rank and \( \deg \tilde{B} = 2^{2k+d(D \otimes Z, C(P))} \). The unitary involution \( \sigma \) on \( C(P) \) can be extended to a unitary involution on \( \tilde{B} \). Embedding \( \tilde{B} = End_{D \otimes Z}(E) \) in \( B = End_D(E) \) and extending the involution of \( \tilde{B} \) to a involution of type \( t \) on \( B \) yields a composition of required type and degree \( 2^{2k+d(D \otimes Z, C(P))}+1 \).

\[ \square \]

Minimality of the Constructed Compositions

1. Compositions with unitary involutions:

(a) \( \deg P = 2k + 1 \): Let \( \alpha : C(P) \to B \) be a composition homomorphism. Minimality as well as the assertion, that \( \deg B \) is a multiple of \( 2^{k+d(C(P), B)} \) follow from Lemma \[ \square \]

(b) \( \deg P = 4k \): Let \( \alpha : C(P) \to B \) be a composition homomorphism. First consider the case \( Z = Z(C(P)) \simeq F \times F \). If \( \alpha \) is non-injective, then \( C(P) = C^+ \times C^- \) and \( \alpha \) factors through either \( C^+ \) or \( C^- \). Lemma \[ \square \] gives \( \deg B \geq 2^{2k-1+d(C^+ \otimes S, \sigma')} \) or \( \deg B \geq 2^{2k-1+d(C^- \otimes S, \sigma')} \), respectively, showing the claim. If \( S = Z(B) \) is
a field, then by (11.3) there exist $n_1, n_2 \in \mathbb{N}$, $n_1 + n_2 \geq 1$ with $\deg B = 2^{2k-1}(n_12^{d(B,C^+ \otimes S)} + n_22^{d(B,C^- \otimes S)}) \geq 2^{2k-1+d(B,C^+ \otimes S)}$. If $S \simeq F \times F$, then by (11.3) there exist $n_1, n_2 \in \mathbb{N}$ with $n_1 + n_2 \geq 1$ and $\deg B = \deg C(n_12^{d(B^+,C^+) + n_22^{d(B^+,C^-)})} \geq \deg C2^{d(B^+,C^+)}. Since there exists a unitary involution on $B = B^+ \times B^-$ we have $B^- \simeq B^{+\text{op}}$ and moreover $C^{+\text{op}} \simeq C^+$, hence $d(B^-,C^+) = d(B^+,C^{+\text{op}})$. Thus $2^{d(B,C^+ \otimes S)} = \text{lcm}(2^{d(B^+,C^+)}, 2^{d(B^+,C^-)}) = 2^{d(B^+,C^+)}$ shows the assertion.

Now assume that $Z$ is a field, $Z \neq S$. In that case minimality follows straightly from (11.6).

(c) $\deg P = 4k + 2$: Let $\alpha: C(P) \to B$ be a composition homomorphism. First consider the case $Z \simeq S$. Then by (11.4) there exist $n_1, n_2 \in \mathbb{N}$ with $n_1 + n_2 \geq 1$ such that $\deg B = 2^{2k}(n_12^{d(B,C^+)} + n_22^{d(B,C^--)}) \geq 2^{2k+\text{min}(d(B,C^+),d(B,C^-))}$, and since $d(B,C(P)) = d(B^{\text{op}},C(P)) = d(B,C(P)^{\text{op}})$ the claim follows.

Now assume that $Z$ is a field and $Z \neq S$. In that case minimality follows straightly from (11.6).

2. Compositions with involutions of the first kind:

(a) $\deg P = 2k+1$: Lemma (11) gives $\deg B = n2^{k+d(c,C(P))}$. If $[C(P)] \neq c$ or the involution $\sigma$ on $C(P)$ is of the required type, the claim follows. Otherwise $\text{char } F \neq 2$ ($\deg P > 1$) and we cannot have odd $n$, since $\text{deg } C_B(C(P)) = n$ and by Lemma (13) the involution $\tau$ extending $\sigma$ would then have to be of the same type. Thus, $\deg P = m2^{k+d(c,C(P))}+1$ with $m = \frac{n}{2} \in \mathbb{N}$.

(b) $\deg P = 4k$: First let $Z = Z(C(P))$ be a field. By (11.6) there exists $n \in \mathbb{N}$ with $\deg B = n2^{2k+d(B \otimes Z,C(P))}$. If $[C(P)] \neq [B \otimes Z]$ or if the involution $\sigma$ on $C(P)$ is of the required type, the claim follows. Otherwise by Lemma (13) $\deg C_B(C(P)) = n$ must be even and thus $\deg B = m2^{2k+1}$ with $m = \frac{n}{2} \in \mathbb{N}$.

Let now $Z \simeq F \times F$. By (11.3) there exist $n_1, n_2 \in \mathbb{N}$, $n_1 + n_2 \geq 1$ with $\deg B = 2^{2k-1}(n_12^{d(B,C^+)} + n_22^{d(B,C^-)})$. If $[C^+] \neq [B]$, $[C^-] \neq [B]$ or if the involution $\sigma$ on $C(P)$ is of the required type, the claim follows. Otherwise, since $\sigma$ is not of the right type $\deg B \neq \deg C = 2^{2k-1}$ and thus $\deg B \geq 2^{2k}$, showing minimality.

(c) $\deg P = 4k + 2$: In that case $\sigma$ is unitary and every composition $\alpha: (C(P),\sigma) \to (B,\tau)$ is injective. Otherwise there would exist a nontrivial central idempotent $e \in \ker \alpha$ and the equation
0 = \tau(\alpha(e)) = \alpha(\sigma(e)) = \alpha(1 - e) = 1 would lead to a contradiction. If \( Z = Z(C(P)) \) is a field, then by Lemma 11.10 there exists \( n \in \mathbb{N} \) with \( \deg B = n2^{2k+1+d(B \otimes Z, C(P))} \) showing the claim. If \( Z \simeq F \times F \), Lemma 11.8 yields \( n_1, n_2 \in \mathbb{N} \) with \( \deg B = (n_12^{2k+d(B, C^+)}) + \frac{1}{2}(n_22^{2k+d(B, C^-)}) \) which is equal to \( (n_1 + n_2)2^{2k+d(B \otimes Z, C(P))} \) since \( 2^{d(B \otimes Z, C(P))} = \text{lcm}(2^{d(B, C^+)}, 2^{d(B, C^-)}) \) and furthermore \( d(B, C^+) = \frac{d(B, C^+ \text{op})}{C^+ \text{op}} = d(B, C^-) \).

\[ \Box \]

**Example** For \( \deg P \) even we have computed the minimal composition degree for compositions with unitary involutions only under restriction to the center \( Z \) of \( C(P) \), namely we have excluded the case \( Z \simeq S \) being a field if \( \deg P \equiv 0 \) (4) and \( Z \simeq F \times F \) where \( S \) is a field if \( \deg P \equiv 2 \) (4), respectively. Also in those cases Lemma 11.11 shows how to reach the minimal degree of an algebra \( B \) of given class \( c \in Br_c(S) \) such that there exist a homomorphism \( C(P) \to B \). The problem however is, that the canonical involution on \( C(P) \) cannot be extended to \( B \) in general. Consider e.g. compositions of type \((c', D) \) for quadratic pairs of degree 4, where \( Z = Z(C(P)) \) and \( S = Z(D') \) are isomorphic fields and \( D' \) is some division \( S \)-algebra. Lemma 11.3 gives \( \deg B = 2^{2k-1}(n_12^{2d(B, C)} + n_22^{d(B, C)}) \) for some \( n_1, n_2 \in \mathbb{N} \) with \( n_1 + n_2 \geq 1 \). A composition \( \alpha : (C(P), D) \to (B, \tau) \) cannot be \( Z \)-linear, since the involutions on \( C(P) \) and \( B \) are not of the same kind. Thus it follows \( n_1 \geq 1 \) and \( n_2 \geq 1 \) as can be seen from the proof of Lemma 11.3. It is in fact possible to construct algebra-homomorphisms \( C(P) \to B \) where \( \deg B \) is of the above form. For that we consider \( C(P) \hookrightarrow C(P) \otimes S \simeq C(P) \times C(P) \). Choose \( B_1 = \text{End}'_{D}(E_1) \) and \( B_2 = \text{End}'_{D}(E_2) \) of minimal degree such that there exist homomorphisms \( C(P) \to B_1 \) and \( C(P) \to B_2 \). That yields a homomorphism \( C(P) \to \text{End}'_{D}(E_1) \times \text{End}'_{D}(E_2) \to B = \text{End}'_{D}(E_1 \oplus E_2) \) or more generally \( C(P) \to B = \text{End}'_{D}(n_1E_1 \oplus n_2E_2) \) for \( n_1 \geq 1, n_2 \geq 1 \). Still it is not possible to extend the canonical involution on \( C(P) \), since under the above homomorphism the center of \( C(P) \) maps onto the center of \( B \). It becomes all much easier if we take only quadratic pairs over trivial algebras, i.e. quadratic forms. Then the Clifford algebra \( \langle C(P), \sigma \rangle = (C_0(V, q), \tau_0) \) can be embedded into the full Clifford algebra \( C(V, q) \) (with canonical or the standard involution) and the same construction like for quadratic pairs of odd degree yields a composition of degree \( 2^{2k+d(C(V, q) \otimes S, c')} = 2^{2k+d(C_0(V, q), c')} \).

By 11.3 (where \( n_1n_2 = 0 \) is excluded) the degree of that composition is minimal.

In Theorem 12 a bound for \( \deg B \) which is independent of the structure
of the Clifford algebra $C(P)$ can be read off. The Theorem indicates as well, when that bound is reached:

**Corollary 15** Let $P$ be an extended quadratic pair of degree $n > 1$ and let $\alpha: (C(P), \sigma) \longrightarrow (B, \tau)$ a composition.

1. Compositions with unitary involutions:
   
   - (a) $n = 2k + 1$: We have $\deg B \geq 2^k$ with equality if and only if $[C(P) \otimes Z(B)]_c = c'$.
   
   - (b) $n = 4k$: We have $\deg B \geq 2^{2k-1}$ with equality if and only if $Z \simeq F \times F$ and $([C^+ \otimes Z(B)]_c = c'$ or $[C^- \otimes Z(B)]_c = c'$).
   
   - (c) $n = 4k + 2$: We have $\deg B \geq 2^{2k}$ with equality if and only if $Z \simeq S$ and $[C(P)]_c = c'$.

2. Compositions with involutions of the first kind: Let $t \in \mu_2(F)$ be the type of $\tau$.
   
   - (a) $n = 2k + 1$: We have $\deg B \geq 2^{k+\delta}$, where $\delta = 1$ if $\sigma$ is of type $t$, otherwise $\delta = 0$.
     
     In case of $\delta = 0$ equality holds if and only if $[C(P)] = c$. In case of $\delta = 1$ equality holds if and only if $[C(P) \otimes Q] = c$ for a quaternion algebra $Q$.
   
   - (b) $n = 4k$: We have $\deg B \geq 2^{2k-1+\delta}$, where $\delta = 1$ if $\sigma$ is of type $t$, otherwise $\delta = 0$.
     
     In case of $\delta = 0$ equality holds if and only if $Z \simeq F \times F$ and $([C^+] = c$ or $[C^-] = c)$. In case of $\delta = 1$ equality holds if and only if $([C^+ \otimes Q] = c$ or $[C^- \otimes Q] = c)$ for a quaternion algebra $Q$.
   
   - (c) $n = 4k + 2$: We have $\deg B \geq 2^{2k+1}$ with equality if and only if $[C(P)]_c = [D \otimes Z]_c$.

**Proof** Everything follows straightly from Theorem (12), except the assertions about the center of the Clifford algebra in (13) and (14). In both cases, the degree of a minimal $B$ coincides with the degree of the Clifford algebra. Since a composition of an extended quadratic pair of degree $\deg P = 4k + 2$ must be injective, both algebras are isomorphic and in particular their centers are isomorphic. On the other hand if $\deg P = 4k$, then a minimal composition cannot be an isomorphism, because the involutions on the centers are not of the same kind. Hence, the kernel is nontrivial and in particular, $Z \simeq F \times F$. ∎
The reader may apply Corollary 15 to compare with the results of [Züg99] in the special cases of (anti-)symmetric compositions and hermitian compositions over quadratic and quaternion algebras. We are doing that only for quadratic spaces \((V, q)\) of dimension \(4k\) and compositions with symplectic involutions. That corresponds to compositions with antisymmetric forms (resp. hermitian forms over a quaternion algebra with respect to the standard involution), see [Züg99, Sections 4, 6]. According to the corollary, the minimal degree of \(B\) for compositions of the form \(\alpha : (C(\text{End}_F(V), \sigma_q, f_q), \sigma_q) \rightarrow (B, \tau)\), \(\tau\) symplectic is given by \(2^{2k}\) if \(\dim V \equiv 0 \ (8)\) and \(\text{char} \ F \neq 2\), and \(2^{2k-1}\) if \(\dim V \equiv 4 \ (8)\) or \(\text{char} \ F = 2\), respectively. Moreover equality holds if and only if \(Z \cong F \times F\) and \([C^\pm] = [B]\) (resp. \([C^\pm] = [B \otimes Q']\) for a quaternion algebra \(Q'\)). In both cases \([C^\pm] = [C(V, q)]\). It remains to remark, that \(\deg B = \dim E\) if \(B = \text{End}_F(E)\) and \(\deg B = \frac{1}{2} \dim E\) if \(B = \text{End}_Q(E)\).

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