Critical site percolation in high dimension

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Abstract

We use the lace expansion to prove an infra-red bound for site percolation on the hypercubic lattice in high dimension. This implies the triangle condition and allows us to derive several critical exponents that characterize mean-field behavior in high dimensions.

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1 Introduction

1.1 Site percolation on the hypercubic lattice

We consider site percolation on the hypercubic lattice $\mathbb{Z}^d$, where sites are independently occupied with probability $p \in [0, 1]$, and otherwise vacant. More formally, for $p \in [0, 1]$, we consider the probability space $(\Omega, F, \mathbb{P}_p)$, where $\Omega = \{0, 1\}^{\mathbb{Z}^d}$, the $\sigma$-algebra $F$ is generated by the cylinder sets, and $\mathbb{P}_p = \bigotimes_{x \in \mathbb{Z}^d} \text{Ber}(p)$ is a product-Bernoulli measure. We call $\omega \in \Omega$ a configuration and say that a site $x \in \mathbb{Z}^d$ is occupied in $\omega$ if $\omega(x) = 1$. If $\omega(x) = 0$, we say that the site $x$ is vacant. For convenience, we identify $\omega$ with the set of occupied sites $\{x \in \mathbb{Z}^d : \omega(x) = 1\}$.

Given a configuration $\omega$, we say that two points $x \neq y \in \mathbb{Z}^d$ are connected and write $x \leftrightarrow y$ if there is an occupied path between $x$ and $y$—that is, there are points $x = v_0, \ldots, v_k = y$ in $\mathbb{Z}^d$ with $k \in \mathbb{N}_0$ such that $|v_i - v_{i-1}| = 1$ (with $|y| = \sum_{i=1}^k |y_i|$ the 1-norm) for all $1 \leq i \leq k$, and $v_i \in \omega$ for $1 \leq i \leq k - 1$ (i.e., all internal sites are occupied). We adopt the convention that $\{x \leftrightarrow x\} = \emptyset$, that is, $x$ is not connected to itself. Mind that two neighbors are automatically connected (i.e., $\{x \leftrightarrow y\} = \Omega$ for all $x, y$ with $|x - y| = 1$).

We define the cluster of $x$ to be $\mathcal{C}(x) := \{x\} \cup \{y \in \omega : x \leftrightarrow y\}$. Note that apart from $x$ itself, points in $\mathcal{C}(x)$ need to be occupied. We also define the expected cluster size (or susceptibility) $\chi(p) = \mathbb{E}_p[|\mathcal{C}(0)|]$, where for a set $A \subseteq \mathbb{Z}^d$, we let $|A|$ denote the cardinality of $A$, and $0$ denotes the origin in $\mathbb{Z}^d$.

We define the two-point function $\tau_p : \mathbb{Z}^d \to [0, 1]$ by $\tau_p(x) := \mathbb{P}_p(0 \leftrightarrow x)$. The percolation probability is defined as $\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty)$. We note that $p \mapsto \theta(p)$ is increasing and define the critical point for $\theta$ as

$$p_c = \inf\{p > 0 : \theta(p) > 0\}.$$ 

The critical point $p_c = p_c(G)$ depends on the underlying graph $G$.

1.2 Main result

The triangle condition is a versatile criterion for several critical exponents to exist and to take on their mean-field value. In order to introduce this condition, we define the open triangle diagram as

$$\Delta_p(x) = p^2(\tau_p \ast \tau_p \ast \tau_p)(x)$$

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and the triangle diagram as $\Delta_p = \sup_{x \in \mathbb{Z}^d} \Delta_p(x)$. In the above, the convolution ‘*’ is defined as $(f \ast g)(x) = \sum_{y \in \mathbb{Z}^d} f(y)g(x-y)$. We also set $f^{\oplus} = f^{\otimes -1} \ast f$ and $f^{\otimes 1} \equiv f$. The triangle condition is the condition that $\Delta_{p_c} < \infty$. To state Theorem 1.1 we recall that the discrete Fourier transform of an absolutely summable function $f : \mathbb{Z}^d \to \mathbb{R}$ is defined as $\hat{f} : [\pi,\pi]^d \to \mathbb{C}$ with

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} f(x),$$

where $k \cdot x = \sum_{j=1}^d k_j x_j$ denotes the scalar product. Letting $D(x) = \frac{1}{2\pi} \mathbf{1}_{|x|=1}$ for $x \in \mathbb{Z}^d$ be the step distribution of simple random walk, we can formulate our main theorem:

**Theorem 1.1** (The triangle condition and the infra-red bound). There exist $d_0 \geq 6$ and a constant $C = C(d_0)$ such that, for all $d > d_0$,

$$p|\tau_p(k)| \leq \frac{[\hat{D}(k)] + C/d}{1 - D(k)} \quad (1.1)$$

for all $k \in (-\pi,\pi]^d$ uniformly in $p \in [0,p_c]$ (we interpret the right-hand side of (1.1) as $\infty$ for $k = 0$). Additionally, $\Delta_p \leq C/d$ uniformly in $[0,\min(p_c)]$, and the triangle condition holds.

We also point to Proposition 4.2 proving the convergence of the lace expansion for $p < p_c$. The resulting equation for $\tau_p$ is the Ornstein-Zernike equation, which is

$$\tau_p(x) = C(x) + p(C \ast \tau_p)(x) \quad (1.2)$$

for some appropriately defined function $C(\cdot) = 2dD(\cdot) + \Pi_p(\cdot)$. An approximate version of this relation is proved in Proposition 2.9 and we refrain from giving a definition of $\Pi_p$ at this point.

### 1.3 Consequences of the infra-red bound

The triangle condition is the classical criterion for mean-field behavior in percolation models. It is immediate that the triangle condition implies readily that $\theta(p_c) = 0$ (since otherwise $\Delta_{p_c}$ could not be finite), a problem that is still open in smaller dimension (except $d = 2$).

Moreover, the triangle condition implies that a number of critical exponents takes on their mean-field values. Indeed, using results by Aizenman and Newman [1] Section 7.7], the triangle condition implies that the critical exponent $\gamma$ exists and takes its mean-field value 1, that is

$$\frac{c}{p_c - p} \leq \chi(p) \leq \frac{C}{p_c - p} \quad (1.3)$$

for $p < p_c$ and constants $0 < c < C$. We write $\chi(p) \sim (p_c - p)^{-\gamma}$ as $p \nearrow p_c$ for the behavior of $\chi$ as in (1.3).

There are several other critical exponents that are predicted to exists. For example, $\theta(p) \sim (p - p_c)^\delta$ as $p \searrow p_c$, and $\mathbb{P}_p([S(0)] \geq n) \sim n^{-\beta}$ as $n \to \infty$.

Aizenman and Barsky [3] show that under the triangle condition,

$$\delta = 2 \quad \text{and} \quad \beta = 1. \quad (1.4)$$

Their results are stated for a class of percolation models including site percolation. Hence, Theorem 1.1 implies (1.4). However, “for simplicity of presentation”, the presentation of the proofs is restricted to bond percolation models.

Moreover, as shown by Nguyen [2], Theorem 1.1 implies that $\Delta = 2$, where $\Delta$ is the gap exponent.

### 1.4 Discussion of literature and results

Percolation theory is a fundamental part of contemporary probability theory and its foundations are generally attributed to a 1957 paper of Broadbent and Hammersley [7]. Meanwhile, a number of textbooks appeared, and we refer to Grimmett [11] for a comprehensive treatment of the subject, as well as Bollobás and Riordan [5], Werner [24] and Beffara and Duminil-Copin [4] for extra emphasis on the exciting recent progress in two-dimensional percolation.
The investigation of percolation in high dimensions was started by the seminal 1990 paper of Hara and Slade [12], who applied the lace expansion to prove the triangle condition for bond percolation in high dimension. A number of modifications and extensions of the lace expansion method for bond percolation has appeared in the meantime. The expansion itself is presented in Slade’s Saint Flour notes [21]. A detailed account of the full lace expansion proof for bond percolation (including convergence of the expansion and related results) is given in a recent textbook by the first author and van der Hofstad [16].

Despite the fantastic understanding of bond percolation in high dimensions, site percolation is not yet analyzed with this method, and the present paper aims to remedy this situation. Together with van der Hofstad and Last [17], we recently applied the lace expansion to the random connection model, which can be viewed as a continuum site percolation model. The aim of this paper is to give a rigorous exhibition of the lace expansion applied to one of the simplest site percolation lattice models. Given the many parallels to papers that apply the lace expansion to bond percolation models—in particular, to the paper by Borgs et. al [6] and the book by Heydenreich and van der Hofstad [16], which inspired the outline of the technique as it is applied in this paper—, a second aim is to highlight the novelties and differences that show up for site percolation. Most of them are visible in the treatment of the random connection model [17] as well, but site percolation on \( \mathbb{Z}^d \) allows for a much cleaner presentation and better comparison to other lattice models.

To point out one difference between bond and site percolation, define the random walk Green’s function as \( G_\lambda(x) = \sum_{m \geq 0} \lambda^m D^m(x) \) for \( \lambda \in (0,1] \). Consequently,

\[
\hat{G}_\lambda(k) = \frac{1}{1 - \lambda \hat{D}(k)}.
\]

One of the key ideas behind the lace expansion for bond percolation is to show that the two-point function is close to \( G_\lambda \) in an appropriate sense (this includes an appropriate parametrization of \( \lambda \)). In site percolation, (1.1) already hints at the fact that \( p \tau_p \) should be close to \( D \ast G_\lambda \), and therefore \( p \tau_p \) should be close to \( G_\lambda \hat{D} \). The care that has to be put into an “exact account” for factors of \( p \) is a guiding thread throughout this paper, particularly in Section 3. Furthermore, one may already observe that the number of factors of \( p \) and two-point functions in \( \triangle_p(x) \) does not match, and resolving this issue poses one of the novel tasks of Section 4. The reason that this problem shows up prominently here and not in [17] is that in continuum space, we can apply a re-scaling argument.

Theorem 1.1 proves the triangle condition in dimension \( d > d_0 \) for sufficiently large \( d_0 \). It is folklore in the physics literature that \( d_0 = 6 \) suffices (6 is the “upper critical dimension”) but the perturbative nature of our argument does not allow us to derive that. Instead, we only get the result for some \( d_0 \geq 6 \). For bond percolation, already the original paper by Hara and Slade [12] treated a second, spread-out version of bond percolation, and they proved that for this model, \( d_0 = 6 \) suffices (under suitable assumption on the spread-out nature). For ordinary bond percolation, it was announced that \( d_0 = 19 \) suffices for the triangle condition in [13], and the number 19 circulated for many years in the community. Finally, Fitzner and van der Hofstad [19] devised involved numerical methods to verify rigorously that an adaptation of the method is applicable for \( d > d_0 = 10 \). It is clear that an analogous result of Theorem 1.1 would hold for “spread-out site percolation” in suitable form (see e.g. [16] Section 5.2), but such a model appeared somewhat artificial to us in the site percolation context, so that we decided not to pursue this direction.

A classical question for high-dimensional percolation is an expansion of the critical threshold \( p_c(d) \) when \( d \to \infty \). It is known in the physics literature that

\[
p_c(d) = (2d)^{-1} + \frac{5}{2}(2d)^{-2} + \frac{31}{4}(2d)^{-3} + \frac{75}{4}(2d)^{-4} + \frac{11977}{48}(2d)^{-5} + \frac{209183}{96}(2d)^{-6} + \cdots.
\]

The first four terms are due to Gaunt, Ruskin and Sykes [10], the latter two were found recently by Mertens and Moore [19] by exploiting involved numerical methods.

The lace expansion devised in this paper enables us to give a rigorous proof of the first terms of (1.5). Indeed, we use the representation obtained in this paper to show that

\[
p_c(d) = (2d)^{-1} + \frac{5}{2}(2d)^{-2} + \frac{31}{4}(2d)^{-3} + O((2d)^{-4}) \quad \text{as } d \to \infty.
\]

This is the content of a forthcoming paper [15]. The idea to derive \( p_c \) expansions from lace expansion coefficients has been earlier achieved for bond percolation by Hara and Slade [14] and van der Hofstad and Slade [23].
1.5 Outline of the paper

The paper is organized as follows. The aim of Section 2 is to establish a lace-expansion identity for \( \tau_p \), which is formulated in Proposition 2.9. To this end, we use Section 2.1 to state some known results that we are going to make use of in Section 2 as well as in later sections. We then introduce a lot of the language and quantities needed to state Proposition 2.9 in Section 2.2, followed by the actual derivation of the identity in Section 2.3.

Section 3 bounds the lace-expansion coefficients derived in Section 2.3 in terms of simpler diagrams, which are large sums over products of two-point (and related) functions. Section 4 finishes the argument of the identity in Section 2.3.

2 The expansion

2.1 The standard tools

We require two standard tools of percolation theory, namely Russo’s formula and the BK inequality, both for increasing events. Recall that \( A \) is called increasing if \( \omega \in A \) and \( \omega \subseteq \omega' \) implies \( \omega' \in A \). Given \( \omega \) and an increasing event \( A \), we introduce

\[
\Pi_\omega(A) = \{ y \in \mathbb{Z}^d : \omega \cup \{ y \} \in A, (\omega \setminus \{ y \}) \notin A \}.
\]

If \( A \) is an increasing event determined by sites in \( \Lambda \subset \mathbb{Z}^d \) with \( |A| < \infty \), then Russo’s formula [20], proved independently by Margulis [18], tells us that

\[
\frac{d}{dp} \mathbb{P}_p(A) = \mathbb{E}[|\Pi_\omega(A)|] = \sum_{y \in \Lambda} \mathbb{P}_p(y \in \Pi_\omega(A)). \tag{2.1}
\]

To state the BK inequality, let \( \Lambda \subset \mathbb{Z}^d \) be finite and, given \( \omega \in \Omega \), let

\[
[\omega]_\Lambda = \{ \omega' \in \Omega : \omega'(x) = \omega(x) \text{ for all } x \in \Lambda \}
\]

be the cylinder event of the restriction of \( \omega \) to \( \Lambda \). For two events \( A, B \), we can define the disjoint occurrence as

\[
A \circ B = \{ \omega : \exists K, L \subseteq \mathbb{Z}^d : K \cap L = \emptyset, [\omega]_K \subseteq A, [\omega]_L \subseteq B \}.
\]

The BK inequality, proved by van den Berg and Kesten [22] for increasing events, states that, given two increasing events \( A \) and \( B \),

\[
\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B). \tag{2.2}
\]

The following proposition about simple random walk will be of importance later:

**Proposition 2.1** (Random walk triangle, [16], Proposition 5.5). Let \( m \in \mathbb{N}_0, n \geq 0 \) and \( \lambda \in [0,1] \). Then there exists a constant \( c_{2m,n}^{(RW)} \) independent of \( d \) such that, for \( d > 2n \),

\[
\int_{(-\pi,\pi]^d} \frac{\hat{D}(k)^{2m}}{[1 - \lambda \hat{D}(k)]^n} \frac{dk}{(2\pi)^d} \leq c_{2m,n}^{(RW)} d^{-m}.
\]

In [16], \( d > 4n \) is required; however, more careful analysis shows that \( d > 2n \) suffices (see [16] (2.19)).

We will also need the following related result:

**Proposition 2.2** (Related random walk bounds, [16], Exercise 5.4). Let \( m \in \{0,1\} \), \( \lambda \in [0,1] \), and \( r, n \geq 0 \) such that \( d > 2(n + r) \). Then, uniformly in \( k \in (-\pi, \pi]^d \),

\[
\int_{(-\pi,\pi]^d} \hat{D}(l)^{2m} \hat{G}_\lambda(l)^{r} \frac{1}{2} [\hat{G}_\lambda(l + k) + \hat{G}_\lambda(l - k)]^r \frac{dl}{(2\pi)^d} \leq c_{2m,n+r}^{(RW)} d^{-m},
\]
\[ \int_{(-\pi,\pi)^d} \bar{D}(l)^{2m} \bar{G}_\lambda(l)^{n-1} \left[ \bar{G}_\lambda(l+k) \bar{G}_\lambda(l-k) \right]^{r/2} \frac{dl}{(2\pi)^d} \leq c_{2m,n-1+r/2} d^{-m}, \]

where the constants \(c^{(RW)}\) are from Proposition [2.4].

The following differential inequality is an application of Russo’s formula and the BK inequality. It applies them to events which are not determined by a finite set of sites. We refer to the literature [16, Lemma 4.4] for arguments justifying this and for a more detailed proof. Observation [2.3] will be of use in Section [4].

**Observation 2.3.** Let \( p < p_c \). Then
\[
\frac{d}{dp} \hat{\tau}_p(0) \leq \hat{\tau}_p(0)^2, \quad \frac{d}{dp} \chi(p) \leq \chi(p) \hat{\tau}_p(0).
\]

As a proof sketch, note that
\[
\frac{d}{dp} \hat{\tau}_p(0) = \sum_{x \in \mathbb{Z}^d} \frac{d}{dp} \hat{\tau}_p(x) = \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \mathbb{P}_p(y \in \text{Piv}(0 \leftrightarrow x)) \leq \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \mathbb{P}_p(\{0 \leftrightarrow y\} \circ \{y \leftrightarrow x\})
\]
\[
\leq \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \tau_p(y) \tau_p(x-y) = \hat{\tau}_p(0)^2.
\]

The inequality for \( \chi(p) \) follows from the identity \( \chi(p) = 1 + p \hat{\tau}_p(0) \).

### 2.2 Definitions and preparatory statements

We need the following definitions:

**Definition 2.4 (Elementary definitions).** Let \( x, u \in \mathbb{Z}^d \) and \( A \subseteq \mathbb{Z}^d \).

1. We set \( \omega^x := \omega \cup \{x\} \) and \( \omega^{u,x} := \omega \cup \{u, x\} \).
2. We define \( J(x) := 1_{|x| = 1} = 1_{0 \sim x} = 2dD(x) \).
3. Let \( \{u \leftrightarrow x \in A\} \) be the event that \( \{u \leftrightarrow x\} \), and there is a path from \( u \) to \( x \), all of whose internal vertices are elements of \( \omega \cap A \). Moreover, write \( \{u \leftrightarrow x \circ \text{off} A\} := \{u \leftrightarrow x \in \mathbb{Z}^d \setminus A\} \).
4. We define \( \{u \leftrightarrow x\} := \{u \leftrightarrow x\} \circ \{u \leftrightarrow x\} \) and say that \( u \) and \( x \) are **doubly connected**.
5. We define the modified cluster of \( x \) with a designated vertex \( u \) as
\[
\hat{C}^u(x) := \{x\} \cup \{y \in \omega \setminus \{u\} : x \leftrightarrow y \in \mathbb{Z}^d \setminus \{u\}\}.
\]
6. For a set \( A \subseteq \mathbb{Z}^d \), define \( \langle A \rangle := A \cup \{y \in \mathbb{Z}^d : \exists x \in A : |x-y| = 1\} \).

**Definition 2.4** allows us to speak of events like \( \{a \leftrightarrow b \in \omega^x\} \) for \( a, b \in \mathbb{Z}^d \), which is the event that \( a \) is connected to \( b \) in the configuration where \( x \) is fixed to be occupied. We remark that \( \{x \leftrightarrow y \in \mathbb{Z}^d\} = \{x \leftrightarrow y \in \omega\} \) and that \( \{u \leftrightarrow x\} = \Omega \) for \( |u-x| = 1 \). Similarly, \( \{u \leftrightarrow x\} = \emptyset \) for \( u = x \). The following, more specific definitions are important for the expansion:

**Definition 2.5 (Extended connection probabilities and events).** Let \( v, u, x \in \mathbb{Z}^d \) and \( A \subseteq \mathbb{Z}^d \).

1. Define
\[
\{u \leftrightarrow^A x\} := \{u \leftrightarrow x\} \cap \left( \{u \leftrightarrow x \circ \text{off} \langle A \rangle\} \cup \{x \in \langle A \rangle\} \right).
\]

In words, this is the event that \( u \) is connected to \( x \), but either any path from \( u \) to \( x \) has an interior vertex in \( \langle A \rangle \), or \( x \) itself lies in \( \langle A \rangle \).
2. Define
\[
\tau^A_p(u, x) := \mathbb{P}_p(u \leftrightarrow x \circ \text{off} \langle A \rangle, x \notin \langle A \rangle).
\]
3. We introduce $Piv(u, x) := Piv(u \leftrightarrow x)$ as the set of pivotal points for $\{u \leftrightarrow x\}$. That is, $v \in Piv(u, x)$ if the event $\{u \leftrightarrow x \text{ in } \omega^v\}$ holds but $\{u \leftrightarrow x \text{ in } \omega \setminus \{v\}\}$ does not.

4. Define the events

\begin{align*}
E'(v, u; A) &:= \{ v \leftrightarrow u \} \cap \{ \exists u' \in Piv(v, u) : v \leftrightarrow u' \}, \\
E(v, u, x; A) &:= E'(v, u; A) \cap \{ u \in \omega \cap Piv(v, x) \}.
\end{align*}

First, we remark that $\{u \overset{\omega^v}{\leftrightarrow} x\} = \{u \leftrightarrow x\}$. Secondly, note that we have the relation

$$
\tau_p(x - u) = \tau_p^A(u, x) + P_p(u \overset{A}{\leftrightarrow} x). \quad (2.3)
$$

We next state a partitioning lemma (whose proof is left to the reader; see [17, Lemma 3.5]) relating the events $E$ and $E'$ to the connection event $\{u \overset{A}{\leftrightarrow} x\}$:

**Lemma 2.6** (Partitioning connection events). Let $v, x \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$. Then

$$
\{v \overset{A}{\leftrightarrow} x\} = E'(v, x; A) \cup \bigcup_{u \in \mathbb{Z}^d} E(v, u, x; A),
$$

and the appearing unions are disjoint.

The next lemma, titled the Cutting-point lemma, is at the heart of the expansion:

**Lemma 2.7** (Cutting-point lemma). Let $v, u, x \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$. Then

$$
P_p(E(v, u, x; A)) = P_p \left[ \mathbf{1}_{E'(v, u; A)} \mathbf{1}_{\tilde{\gamma}^u(v)}(u, x) \right].
$$

**Proof.** The proof is a special case of the general setting of [17]. Since it is essential, we present it here. We abbreviate $\tilde{\gamma} = \tilde{\gamma}^u(v)$ and observe that

$$
\{u \in Piv(v, x)\} = \{v \leftrightarrow u\} \cap \{u \leftrightarrow x \text{ off } \tilde{\gamma}\} \cap \{x \notin \langle \tilde{\gamma} \rangle\}
$$

$$
= \{v \leftrightarrow u\} \cap \{u \leftrightarrow x \text{ off } \tilde{\gamma}\} \cap \{x \notin \langle \tilde{\gamma} \rangle\}.
$$

In the above, we can replace $\tilde{\gamma}$ by $\langle \tilde{\gamma} \rangle$, as, by definition, we know that, apart from $u$, any site in $\langle \tilde{\gamma} \rangle \setminus \tilde{\gamma}$ must be vacant. Now, since $E'(v, u; A) \subseteq \{v \leftrightarrow u\}$, we get

$$
E(v, u, x; A) = E'(v, x; A) \cap \{u \leftrightarrow x \text{ off } \langle \tilde{\gamma} \rangle\} \cap \{x \notin \langle \tilde{\gamma} \rangle\}.
$$

Taking probabilities, conditioning on $\tilde{\gamma}$, and observing that the status of $u$ is independent of all other events, we see

$$
P_p(E(v, u, x; A)) = P_p \mathbf{1}_{E'(v, u; A)} P_p \mathbf{1}_{\{x \notin \langle \tilde{\gamma} \rangle\}} \mathbf{1}_{\{u \leftrightarrow x \text{ off } \langle \tilde{\gamma} \rangle\}} \mathbf{1}_{\tilde{\gamma}^u(v)}(u, x),
$$

making use of the fact the the first two events are measurable w.r.t. $\tilde{\gamma}$. The proof is complete with the observation that under $\mathbb{E}_p$, almost surely,

$$
\mathbf{1}_{\{x \notin \langle \tilde{\gamma} \rangle\}} \mathbf{1}_{\{u \leftrightarrow x \text{ off } \langle \tilde{\gamma} \rangle\}} \mathbf{1}_{\tilde{\gamma}^u(v)}(u, x) = \tau_p^u(u, x).
$$

**2.3 Derivation of the expansion**

We introduce a sequence $(\omega_i)_{i \in \mathbb{N}_0}$ of independent site percolation configurations. For an event $E$ taking place on $\omega_i$, we highlight this by writing $E_i$. We also stress the dependence of random variables on the particular configuration they depend on. For example, we write $\mathcal{C}(u; \omega_i)$ to denote the cluster of $u$ in configuration $i$. 

6
Definition 2.8 (Lace-expansion coefficients). Let $m \in \mathbb{N}, n \in \mathbb{N}_0$ and $x \in \mathbb{Z}^d$. We define

\[
\Pi_p^{(0)}(x) := \mathbb{P}_p(0 \leftrightarrow x) - J(x),
\]

\[
\Pi_p^{(m)}(x) := p^m \sum_{u_0, \ldots, u_{m-1}} \mathbb{P}_p(0 \Leftrightarrow u_0) \cap \prod_{i=1}^{m} E'(u_{i-1}, u_i; \mathcal{C}_{i-1}),
\]

where $u_{-1} = 0, u_m = x$ and $\mathcal{C}_i = \mathcal{G}^m(u_{i-1}; \omega_i)$. Let furthermore

\[
R_{p,n}(x) := (-p)^{n+1} \sum_{u_0, \ldots, u_n} \mathbb{P}_p(0 \Leftrightarrow u_0) \cap \prod_{i=1}^{n} E'(u_{i-1}, u_i; \mathcal{C}_{i-1}) \cap \{u_n \leftrightarrow \mathcal{G}_n \rightarrow x\}_{n+1}.
\]

Finally, set

\[
\Pi_{p,n}(x) = \sum_{m=0}^{n} (-1)^m \Pi_p^{(m)}(x).
\]

It should be noted that the events $E'(u_{i-1}, u_i; \mathcal{C}_{i-1})$, appearing in Definition 2.8, take place on configuration $i$ only if $\mathcal{C}_{i-1}$ is taken to be a fixed set—otherwise, they are events determined by configurations $i = 0$ and $i$.

Proposition 2.9 (The lace expansion). Let $p < p_c, x \in \mathbb{Z}^d$, and $n \in \mathbb{N}_0$. Then

\[
\tau_p(x) = J(x) + \Pi_{p,n}(x) + p((J + \Pi_{p,n}) \ast \tau_p)(x) + R_{p,n}(x).
\]

Proof. We have

\[
\tau_p(x) = J(x) + \Pi_p^{(0)}(x) + \mathbb{P}_p(0 \leftrightarrow x, 0 \leftrightarrow x).
\]

We can partition the last summand via the first pivotal point. Pointing out that $\{0 \leftrightarrow u\} = E'(0, u; \mathbb{Z}^d)$, we obtain

\[
\mathbb{P}_p(0 \leftrightarrow x, 0 \leftrightarrow x) = \sum_{u \in \mathbb{Z}^d} \mathbb{P}_p(0 \leftrightarrow u, u \in \omega, u \in \text{Piv}(0, x)) = \sum_u \mathbb{P}_p(E(0, u, x; \mathbb{Z}^d))
\]

\[
= p \sum_u E_p \left[ \mathbf{1}_{(0 \leftrightarrow u)} \cdot \tau_p(u, x) \right]
\]

via the Cutting-site lemma 2.7. Using 2.3 for $A = \mathcal{C}_0$, we have

\[
\tau_p(x) = J(x) + \Pi_p^{(0)}(x) + p \sum_u \left(J(u) + \Pi_p^{(0)}(u)\right) \tau_p(x - u) - p \sum_u E_p \left[ \mathbf{1}_{(0 \leftrightarrow u)} \cdot \mathbb{P}_p(u \leftrightarrow \mathcal{G}_0 \rightarrow x) \right]. \tag{2.4}
\]

This proves the expansion identity for $n = 0$. Next, Lemma 2.6, together with the fact that $E(u, u_1, x; A)$ is independent of the occupation status of $u_1$, yields

\[
\mathbb{P}_p(u \leftrightarrow A \rightarrow x) = \mathbb{P}_p(E'(u, x; A)) + \sum_{u_1 \in \mathbb{Z}^d} \mathbb{P}_p(E(u, u_1, x; A))
\]

\[
= \mathbb{P}_p(E'(u, x; A)) + p \sum_{u_1 \in \mathbb{Z}^d} E_p \left[ \mathbf{1}_{E'(u, u_1; A)} \cdot \tau_p^m(u_1, x) \right].
\]

Plugging this into 2.4, we use the inclusion-exclusion formula $\tau^A_p(u_1, x) = \tau_p(x - u_1) - \mathbb{P}_p(u_1 \leftrightarrow A \rightarrow x)$ for $A = \mathcal{G}^m(u_1)$ to extract $\Pi_p^{(1)}$ and get

\[
\tau_p(x) = J(x) + \Pi_p^{(0)}(x) - \Pi_p^{(1)}(x) + p((J + \Pi_p^{(0)}) \ast \tau_p)(x)
\]

\[
+ p^2 \sum_{u_1} \tau_p(x - u_1) \sum_{u_2} E_p \left(0 \leftrightarrow u_0 \right) \cap E'(u_0, u_1; \mathcal{G}_0) + R_1(x)
\]

\[
= J(x) + \Pi_p^{(0)}(x) - \Pi_p^{(1)}(x) + p((J + \Pi_p^{(0)} - \Pi_p^{(1)}) \ast \tau_p)(x) + R_1(x).
\]

The expansion for general $n$ is an induction on $n$ where the step is analogous to the step $n = 1$ (but heavier on notation). Note that all appearing sums are bounded by $\sum_y \tau_p(y)$. This sum is finite for $p < p_c$, justifying the above changes in order of summation. \[\square\]
3 Diagrammatic bounds

3.1 Setup, bounds for \( n = 0 \)

We use this section to state Lemma 3.1 and state bounds on \( \Pi_p^{(0)} \), which are rather simple to prove. The more involved bounds on \( \Pi_p^{(n)} \) for \( n \geq 1 \) are dealt with in Section 3.2. Note that if \( f(-x) = f(x) \), then \( \hat{f}(k) = \sum_{x \in \mathbb{Z}^d} \cos(k \cdot x) f(x) \). We furthermore have the following tool at our disposal:

**Lemma 3.1** (Split of cosines, [8], Lemma 2.13). Let \( t \in \mathbb{R} \) and \( t_i \in \mathbb{R} \) for \( i = 1, \ldots, n \) such that \( t = \sum_{i=1}^n t_i \). Then

\[
1 - \cos(t) \leq n \sum_{i=1}^n [1 - \cos(t_i)].
\]

We begin by treating the coefficient for \( n = 0 \), giving a glimpse into the nature of the bounds to follow in Sections 3.2 and 3.3. To this end, we define the two displacement quantities

\[
J_k(x) := [1 - \cos(k \cdot x)] J(x) \quad \text{and} \quad \tau_{p,k}(x) = [1 - \cos(k \cdot x)] \tau_p(x).
\]

**Proposition 3.2** ( Bounds for \( n = 0 \)). For \( k \in (-\pi, \pi]^d \),

\[
|\hat{\Pi}_p^{(0)}(k)| \leq p^2 (J \circ \tau_p^2)(0),
\]

\[
|\hat{\Pi}_p^{(0)}(0) - \hat{\Pi}_p^{(0)}(k)| \leq 2p^2 \left( (J_k \ast J \ast \tau_p^2)(0) + (J^* \ast \tau_{p,k} \ast \tau_{p})(0) \right).
\]

**Proof.** Note that \( |x| \leq 1 \) implies \( \Pi_p^{(0)}(0) = 0 \) by definition. For \( |x| \geq 2 \), we have

\[
\Pi_p^{(0)}(x) \leq \mathbb{E} \left[ \sum_{y \neq z \in \mathbb{Z}^d} 1_{\{|y| = |z| = 1\}} 1_{\{y \rightarrow x \} \cap \{z \rightarrow x \}} \right] \leq p^2 \left( \sum_{y \in \mathbb{Z}^d} J(y) \tau_p(x-y) \right)^2 = p^2 (J \ast \tau_p)(x)^2.
\]

Summation over \( x \) gives the first bound. The last bound is obtained by applying Lemma 3.1 to the bounds derived for \( \Pi_p^{(0)}(x) \):

\[
|\hat{\Pi}_p^{(0)}(0) - \hat{\Pi}_p^{(0)}(k)| = \sum_x [1 - \cos(k \cdot x)] \Pi_p^{(0)}(x)
\]

\[
\leq 2p^2 \sum_x (J \ast \tau_p(x)) \sum_y \left( (1 - \cos(k \cdot y)] J(y) \tau_p(x-y) + J(y) [1 - \cos(k \cdot (x-y))] \tau_p(x-y) \right).
\]

Resolving the sums gives the claimed convolution. \( \square \)

3.2 Bounds in terms of diagrams

The main result of this section is Proposition 3.5, providing bounds on the lace-expansion coefficients in terms of so-called diagrams, which are sums over products of two-point (and related) functions. To state it, we introduce some functions related to \( \tau_p \) as well as several “modified triangles” closely related to \( \triangle_p \).

**Definition 3.3** (Modified two-point functions). Let \( x \in \mathbb{Z}^d \) and define

\[
\tau_p^\circ(x) := \delta_0,x + \tau_p(x), \quad \tau_p^\bullet(x) = \delta_0,x + p \tau_p(x), \quad \gamma_p(x) = \tau_p(x) - J(x).
\]

**Definition 3.4** (Modified triangles). We let

\[
\triangle_p^\circ(x) = p^2 (\tau_p^\circ \ast \tau_p \ast \tau_p)(x), \quad \triangle_p^\bullet(x) = p(\tau_p^\bullet \ast \tau_p \ast \tau_p)(x), \quad \triangle_p^{\bullet\circ}(x) = p(\tau_p^\bullet \ast \tau_p^\circ \ast \tau_p)(x), \quad \triangle_p^{\bullet\bullet\circ}(x) = (\tau_p^\bullet \ast \tau_p^\bullet \ast \tau_p^\circ)(x).
\]

We also set

\[
\triangle_p^\circ := \sup_{x \in \mathbb{Z}^d} \triangle_p^\circ(x), \quad \triangle_p^\bullet := \sup_{x \in \mathbb{Z}^d} \triangle_p^\bullet(x), \quad \delta_p^{\bullet\circ} := \sup_{x \in \mathbb{Z}^d} \triangle_p^{\bullet\circ}(x), \quad \triangle_p^{\bullet\bullet\circ} := \sup_{x \in \mathbb{Z}^d} \triangle_p^{\bullet\bullet\circ}(x), \quad T_p := (1 + \triangle_p)^{\bullet\bullet\circ} + \triangle_p \triangle_p^{\bullet\bullet\circ}.
\]
Definitions 3.3 and 3.4 allow us to properly keep track of factors of \( p \), which turns out to be important throughout Section 3.

**Proposition 3.5** (Triangle bounds on the lace-expansion coefficients). For \( n \geq 0 \),

\[
p \sum_{x \in \mathbb{Z}^d} \Pi_{p}^{(n)}(x) \leq \Delta_{p}^{*}(0)(T_{p})^{n}.
\]

The proof of Proposition 3.5 relies on two intermediate steps, successively giving bounds on \( \sum \Pi_{p}^{(n)} \). These two steps are captured in Lemmas 3.7 and 3.10 respectively. We first state the former lemma.

Recall that \( \Pi_{p}^{(n)} \) is defined on independent percolation configurations \( \omega_{0}, \ldots, \omega_{n} \). A crucial step in proving Proposition 3.5 is to group events taking place on the percolation configuration \( i \), and then to use the independence of the different configurations. To this end, note that event \( E'(u_{i-1}, u_{i}; \mathcal{E}_{i-1}) \) takes place on configuration \( i \) only if \( \mathcal{E}_{i-1} \) is considered to be a fixed set. Otherwise, it is a product event made up of the connection events of configuration \( i \) as well as a connection event in configuration \( i - 1 \), preventing a direct use of the independence of the \( \omega_{i} \). Resolving this issue is the one of the goals of Lemma 3.7; another is to give bounds in terms of the simpler events (amenable to application of the BK inequality) introduced below in Definition 3.6.

**Definition 3.6** (Bounding events). Let \( x, y \in \mathbb{Z}^d \). We define

\[
\{x \sim y\} := \{x \leftrightarrow y\} \cup \{x = y\}.
\]

Let now \( i \in \{1, \ldots, n\} \) and set \( \vec{v}_{i} = (u_{i-1}, t_{i}, w_{i}, z_{i}, u_{i}, z_{i+1}) \). We define

\[
\begin{align*}
F_0(a, w_0, u_0, z_1) & = \left( \{w_0 = a, |w_0 - a| = 1\} \right) \cap \{a \sim w_1\} \\
& \cap \left( \{|u_0 - a| > 1\} \cap \{a \sim w_0\} \cap \{a \leftrightarrow u_0\} \cap \{w_0 \leftrightarrow u_0\} \cap \{w_0 \sim z_1\}\right), \\
F_n(a_{n-1}, t_n, z_n, x) & = \left( \{|t_n, z_n, x| \neq 2\} \cap \{u_{n-1} \sim t_n\} \right) \\
& \cap \{t_n \sim x\} \cap \{t_n \sim z_n\} \cap \{z_n \sim x\}, \\
F^{(1)}(\vec{v}_{i}) & = \left( \{|w_i, t_i, z_i, u_i| = 4\} \cap \{u_{i-1} \sim t_i\} \cap \{t_i \leftrightarrow u_i\} \cap \{t_i \sim z_i\}\right) \\
& \cap \{w_i \leftrightarrow u_i\} \cap \{z_i \leftrightarrow u_i\} \cap \{w_i \sim z_{i+1}\}, \\
F^{(2)}(\vec{v}_{i}) & = \left( \{|w_i \notin \{z_i, u_i\}, |t_i, z_i, u_i| \neq 2\} \cap \{u_{i-1} \sim w_i\} \cap \{w_i \sim z_{i+1}\} \right) \\
& \cap \{w_i \sim t_i\} \cap \{t_i \sim u_i\} \cap \{t_i \sim z_i\} \cap \{z_i \sim u_i\}.
\end{align*}
\]

The coincidence requirements in \( F^{(2)} \) means that among the points \( t_i, w_i, z_i, u_i \), the point \( w_i \) may coincide only with \( t_i \); and additionally, the triple \( \{t_i, z_i, u_i\} \) are either all distinct, or collapsed into a single point. The above events are depicted in Figure 1.

We use the notation \( (\mathbb{Z}^d)^{(m, 1)} \) to denote the set of vectors \( \{\vec{v}_{[1, m]} \in (\mathbb{Z}^d)^m, v_i \neq v_{i+1}, \forall 1 \leq i < m\} \).
Lemma 3.7 (Coefficient bounds in terms of $F$ events). For $n \geq 1$ and $(u_0, \ldots, u_{n-1}, x) \in (\mathbb{Z}^d)^{(n+1,1)}$,
\[
\{0 \iff u_0\} \cap \bigcap_{i=1}^{n} E'(u_{i-1}, u_i; \mathcal{G}_{i-1}) \subseteq \bigcup_{z \in [1,n]} F_0(0, u_0, z_1) \cap \left( \bigcup_{t_i \in \mathcal{G}_{i-1}} F(1)(\tilde{u}_i) \cup \bigcup_{t_i \in \mathcal{G}_{i-1}} F(2)(\tilde{u}_i) \right) \cap F_n(u_{n-1}, t_n, z_n, x).
\]

The proof is analogous to the one in [17, Lemma 4.12] and we do not perform it here. The second important lemma is Lemma 3.10 and its bounds are phrased in terms of the following functions.

Definition 3.8 (The $\psi$ and $\phi$ functions). Let $n \in \mathbb{N}$ and $a_1, a_2, b, w, t, u, z \in \mathbb{Z}^d$. We define
\[
\psi_0(b, w, u) := \delta_{b,w} p J(u - b) + p \tau_p^*(w - b) \gamma_p(u - b) \tau_p(w - u),
\]
\[
\tilde{\psi}_0(b, w, u) := p \tau_p^*(w - b) \tau_p(u - b) \tau_p(w - u),
\]
\[
\psi_n(a_1, a_2, t, z, x) := \Pi_{1 \leq i \leq n} \tau_p^*(z - a_i) \tau_p(t - a_2) \tau_p(x - t) \tau_p(x - z).
\]
Moreover, we define
\[
\psi^{(1)}(a_1, a_2, t, w, z, u) := p^3 \Pi_{1 \leq i \leq n} \tau_p^*(z - a_i) \tau_p(t - a_2) \tau_p(w - t) \tau_p(z - t) \tau_p(u - w) \tau_p(u - z),
\]
\[
\psi^{(2)}(a_1, a_2, t, w, z, u) := \Pi_{1 \leq i \leq n} \tau_p^*(z - a_i) \tau_p(t - a_2) \tau_p(w - t) \tau_p(z - t) \tau_p(u - w) \tau_p(u - z),
\]
and $\psi := \psi^{(1)} + \psi^{(2)}$. Furthermore, for $j \in \{1, 2\}$, let
\[
\phi_0(b, w, u, z) := \delta_{b,w} p J(u - b) + p \tau_p^*(w - b) \gamma_p(u - b) \tau_p(z - w),
\]
\[
\phi_n(a_1, a_2, t, z, x) := \Pi_{1 \leq i \leq n} \tau_p^*(z - a_i) \tau_p(t - a_2) \tau_p(z - t) \tau_p(x - t) \tau_p(x - z),
\]
\[
\phi^{(j)}(a_1, a_2, t, w, z, u, b) := \frac{\tau_p^*(b - w)}{\tau_p^*(z)} \psi^{(j)}(0, a_2, t, w, z, u),
\]
and $\tilde{\phi}_0(b, w, u, z) := \tilde{\psi}_0(b, w, u) \tau_p^*(z - w)$ as well as $\phi := \phi^{(1)} + \phi^{(2)}$.

We remark that $\psi_0 \leq \tilde{\psi}_0$ as well as $\phi_0 \leq \tilde{\phi}_0$, and we are going to use this fact later on. In the definition of $\phi^{(j)}$, the factor $\tau_p^*(z)$ cancels out. In that sense, $\phi(j)$ differs from $\psi(j)$ by "replacing" the factor $\tau_p^*(z - a_1)$ with the factor $\tau_p^*(b - w)$, and the two functions are closely related.

We first obtain a bound on $\Pi_0^{(n)}$ in terms of the $F$ events (this is Lemma 3.7). Bounding those with the BK inequality, we will naturally observe the $\phi$ functions (Lemma 3.10). To decompose them further, we would like to apply induction; for this purpose, the $\psi$ functions are much better-suited. Introducing both $\phi$ and $\psi$ functions thus increases the readability throughout this section (and later ones).

Definition 3.9 (The $\Psi$ function). Let $w_n, u_n \in \mathbb{Z}^d$ and define
\[
\Psi^{(n)}(w_n, u_n) := \sum_{\tilde{t}, \tilde{w}, \tilde{z}, \tilde{u} : u_n \neq u_n} \psi_0(0, w_0, u_0) \prod_{i=1}^{n} \psi(w_{i-1}, u_{i-1}, \tilde{t}_i, w_i, \tilde{z}_i, u_i),
\]
where $\tilde{t}_{[1,n]}, \tilde{z}_{[1,n]}, \tilde{w}_{[0,n-1]} \in (\mathbb{Z}^d)^n$ and $\tilde{u}_{[0,n-1]} \in (\mathbb{Z}^d)^{(n+1,1)}$.

Lemma 3.10 (Bound in terms of $\psi$ functions). For $n \geq 0$,
\[
p \sum_{x \in \mathbb{Z}^d} \Pi_p^{(n)}(x) \leq \sum_{w, u \in \mathbb{Z}^d} \Psi^{(n-1)}(w, u) \psi_n(w, u, t, z, x) \leq \sum_{w, u \in \mathbb{Z}^d} \Psi^{(n)}(w, u).
\]
Proof. Definition 2.8 and Lemma 3.7 yield a bound on $\Pi_p^{(n)}$ of the form

$$p\Pi_p^{(n)}(u_n) \leq p^{n+1} \sum_{d} \mathbb{P}_p \left( \bigcup_{w_0, t_0, \vec{z}} F_0(0, w_0, u_0, z_1) \cap \left( \bigcap_{i=1}^{n-1} \bigcup_{t_i, w_i} F^{(1)}(\vec{v}_i) + \bigcup_{t_i, w_i} F^{(2)}(\vec{v}_i) \right) \right) \bigcup F_n(u_{n-1}, t_n, z_n, x_n)$$

$$= \sum_{\vec{t}, \vec{w}, \vec{z}, \vec{u}} p^{\{0, u_0\}|\mathbb{P}_p(F_0(0, w_0, u_0, z_1))} \times \prod_{i=1}^{n-1} \left( p^{\{u_{i-1}, t_{i}\} + 2\mathbb{P}_p(F^{(1)}(\vec{v}_i)) + p^{\{u_{i-1}, t_{i}\} + \{t_{i}, w_{i}\} + \{t_{i}, z_{i}, u_{i}\} - 3\mathbb{P}_p(F^{(2)}(\vec{v}_i))} \right) \times \prod_{\vec{u}} \left( p^{\{u_{n-1}, t_{n}\} + \{t_{n}, z_{n}, u_{n}\} - 2\mathbb{P}_p(F_n(u_{n-1}, t_n, z_n, u_n)) \right) \right). \quad (3.2)$$

In the first line, $\vec{t}, \vec{w}, \vec{z}$ are occupied points as in Lemma 3.7. In both lines, $\vec{u}_{\{0, n\}} \in (\mathbb{Z}^d)^{(n+1, 1)}$, and in the second line, $\vec{t}, \vec{w}, \vec{z} \in (\mathbb{Z}^d)^n$. Crucially, the identity in (3.2) holds due to the independence of the different percolation configurations. Moreover, it is crucial here that the number of factors of $p$ (appearing when we switch from a sum over points in $\omega$ to a sum over points in $\mathbb{Z}^d$) depends on the number of coinciding points.

We can now decompose the $F$ events by heavy use of the BK inequality, producing bounds in terms of the $\phi$ functions introduced in Definition 3.8. We start by bounding

$$p^{\{a, w\}|\mathbb{P}_p(F_0(a, w, u, z)) \leq \phi_0(a, w, u, z),$$

$$p^{\{a, t\} + \{t, z, u\} - 3\mathbb{P}_p(F^{(2)}(a, t, w, z, u, b)) \leq \phi(a, t, w, z, u, b).$$

We continue to bound

$$p^{\{a, t\} + 2\mathbb{P}_p(F^{(1)}(a, t, w, z, u, b))} + p^{\{a, w\} + \{w, t\} + \{t, z, u\} - 3\mathbb{P}_p(F^{(2)}(a, t, w, z, u, b)) \leq \phi(a, t, w, z, u, b).$$

Plugging these bounds into (3.2), we obtain the new bound

$$p\Pi_p^{(n)}(u_n) \leq \sum_{\{\vec{t}, \vec{w}, \vec{z}\} \in (\mathbb{Z}^d)^n} \phi_0(0, w_0, u_0, z_1) \phi_n(u_{n-1}, t_n, z_n, x_n) \prod_{i=1}^{n-1} \phi(u_{i-1}, t_i, w_i, z_i, u_i, z_{i+1}), \quad (3.3)$$

where $\vec{t}, \vec{w}, \vec{z} \in (\mathbb{Z}^d)^n$, and $\vec{u}_{\{0, n\}} \in (\mathbb{Z}^d)^{(n+1, 1)}$. We rewrite the right-hand side of (3.3) by replacing the $\phi_0, \phi_n$ and $\phi$ functions by $\psi_0, \psi_n$ and $\psi$ functions. As the additional factors arising from this replacement exactly cancel out, this gives the first bound in Lemma 3.10. The observation

$$\psi_n(a_1, a_2, t, z, u) \leq \psi(a_1, a_2, t, a_2, z, u) \quad (4.4)$$

gives the second bound and finishes the proof. \[ \square \]

We can now prove Proposition 3.5.

Proof of Proposition 3.5. We show that

$$\sum_{w, u \in \mathbb{Z}^d} \Psi^{(n)}(w, u) \leq \Delta_p^*(0)(T_p)^n, \quad (3.5)$$

which is sufficient due to (3.1). The proof of (3.5) is an induction on $n$. For the base case, we need to bound

$$\sum_{w, u} \psi_0(0, w, u) \leq \sum_{w, u} \tilde{\psi}_0(0, w, u) = p \sum_{w, u} \tau_p^*(w) \tau_p(u) \tau_p(u - w) = \Delta_p^*(0).$$

Let now $n \geq 1$. Then

$$\sum_{w, u} \Psi^{(n)}(w, u) = \sum_{w', u'} \Psi^{(n-1)}(w', u') \sum_{z, t, w, u \neq u'} \psi(w', u', t, w, z, u)$$

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The second term contains an indicator. Resolving it splits this term into two further terms. We first consider the term arising from \( \psi \) (1) and \( \psi \) (2), we start with the first one and obtain

\[
\sum_{t,z} p^3 (\tau_p(t-a)\tau_p(z-t)(z)(\sum_{u,w} \tau_p(w-t)\tau_p(u-w)\tau_p(z-u))) \\
\leq p \sum_{t,z} (\tau_p(t-a)\tau_p(z-t)(z)(\sup_{u,w} p^2 \sum_{u,w} \tau_p(w-t)\tau_p(u-w)\tau_p(z-u))) \leq \Delta_p \Delta_p^\circ(a). \tag{3.7}
\]

Before treating the second term, we show how to obtain the bound from (3.7) pictorially, using diagrams very similar to the ones introduced in Figure 2. In particular, factors of \( \tau_p \) are represented by lines, factors of \( \tau_p^\circ \) and \( \tau_p^\bullet \) by lines with an added ‘\*’ or ‘\◦’, respectively. Points summed over are represented by squares, and points which we take the supremum over (point \( a \) in our case) are represented by colored disks. We interpret the factor \( \tau_p^\circ(z) \) as a \((\circ\text{-decorated})\) line between \( 0 \) and \( z \); the origin is represented by lack of decorating the incident line. Finally, we indicate the distinctness of a pair of points (in our case \( 0 \neq u \)) by a disrupted two-headed arrow. With this notation, (3.7) becomes

\[
p^3 \sum_{t,z} \sum_{\square} \leq p \sum \left( \sum_{\square} \left( \sup_{\square} p^7 \sum \right) \right) \leq \Delta_p \Delta_p^\circ(a).
\]

The fact that \( \Psi_n(x, x) = 0 \) for any \( n \) and \( x \) allows us to assume \( w' \neq u' \) in the supremum in the second line of (3.6). After applying the induction hypothesis, it remains to bound the second factor for \( w' \neq u' \), which we rewrite as \( \sup_{w',u'} \sum_{t,w,z,u \neq a} \psi(0, a, t, w, z, u) \) by translation invariance. As it is a sum of two terms (originating from \( \psi(1) \) and \( \psi(2) \)), we start with the first one and obtain

\[
\sum_{w',u'} \psi^{(n-1)}(w', u') \left( \sup_{w',u'} \sum_{t,w,z,u} \psi(w', u', t, w, z, u) \right). \tag{3.6}
\]
The bound on (3.8) thus becomes
\[ p^2 \sum_{t,z} \vec{\Delta} = p^2 \sum_{t,z} \vec{\Delta} \leq p^2 \sum_{t,z} \left( \sup_{t,z} \vec{\Delta} \right) \leq \Delta_p^{\ast \ast} \Delta_p, \]
where we point out that we did not use \( a \neq 0 \) for the bound \( \Delta_p^{\ast \ast} \), and so it was not indicated in the diagram.

The following corollary will be needed later to show that the limit \( \Pi_{p,n} \) for \( n \to \infty \) exists:

**Corollary 3.11.** For \( n \geq 1 \),
\[ \sup_{x \in \mathbb{Z}^d} \Pi_p^{(n)}(x) \leq \Delta_p^{\ast}(0)(1 + \Delta_p^{\ast \ast})(T_p)^{n-1}. \]

**Proof.** Note that
\[ \Pi_p^{(n)}(x) \leq \left( \sup_{w \neq u} \psi_n(w, u, t, z, x) \right) \sum_{w,u} \Psi^{(n-1)}(w, u). \]
The claim follows from
\[ \sum_{t,z} \psi_n(w, u, t, z, x) \leq \tau_p^{\ast}(x-u)\tau_p^{\ast}(x-w) + \Delta_p^{\ast \ast}(u-w) \leq 1 + \Delta_p^{\ast \ast}(u-w) \]
and Proposition 3.3.

### 3.3 Displacement bounds

The aim of this section is to give bounds on \( p \sum_x [1 - \cos(k \cdot x)]\Pi_p^{(n)}(x) \). Such bounds are important in the analysis in Section 4. We regard \([1 - \cos(k \cdot x)] \) as a “displacement factor”. To state the main results, Propositions 3.13 and 3.14, we introduce some displacement quantities:

**Definition 3.12** (Diagrammatic displacement quantities). Let \( x \in \mathbb{Z}^d \) and \( k \in (-\pi, \pi)^d \).

- \( W_p(x; k) := p(\tau_{p, k} \ast \tau_p^{\ast})(x) \)
- \( W_p(k) = \max_{x \in \mathbb{Z}^d} W_p(x; k) \)
- \( H_p(b_1, b_2; k) := p^5 \sum_{t,w,z,u,v} \tau_p(z)\tau_p(t-u)\tau_p(t-z)\tau_{p,k}(u-z)\tau_p(t-w)\tau_p(w-b_1)\tau_p(v-w)\tau_p(v+b_2-u) \)
- \( H_p(k) = \max_{b_1 \neq 0, b_2 \in \mathbb{Z}^d} H_p(b_1, b_2; k) \)

**Proposition 3.13** (Displacement bounds for \( n \geq 2 \)). For \( n \geq 2 \) and \( x \in \mathbb{Z}^d \),
\[ p \sum_x [1 - \cos(k \cdot x)]\Pi_p^{(n)}(x) \leq 11(n + 1)(T_p)^{1 + (n-2)}(\Delta_p^{\ast \ast})^3 W_p(k) \left( 1 + \Delta_p^{\ast} + T_p + \frac{H_p(k)}{W_p(k)} \right). \]

**Proposition 3.14** (Displacement bounds for \( n = 1 \)). For \( x \in \mathbb{Z}^d \),
\[ p \sum_x [1 - \cos(k \cdot x)]\Pi_p^{(1)}(x) \leq 9 W_p(k) \left( \Delta_p^{\ast}(0)(\Delta_p^{\ast \ast} + \Delta_p^{\ast}) + \Delta_p^{\ast \ast} + \Delta_p^{\ast} \right) + p^2 (J \ast \tau_{p, k} \ast \tau_p^{\ast})(0). \]

In preparation for the proofs, we define a function \( \tilde{\Psi}^{(n)} \), similar to \( \tilde{\Psi}^{(n)} \), and prove an almost identical bound the the one in Proposition 3.5. Let \( \tilde{\Psi}^{(0)}(t, z) = \phi_n(0, t, z, 0) / \tau_p^{\ast}(t) \). For \( i \geq 1 \), define
\[ \tilde{\Psi}^{(i)}(t, z) = \sum_{w, u, t', z'} \tilde{\Psi}^{(i-1)}(t', z') \left[ \phi^{(1)}(0, t, w, z, u, z') + \phi^{(2)}(0, t, w, z, u, z') \right] \tau_p^{\ast}(t') / \tau_p^{\ast}(t). \]

Note that in \( \phi^{(2)} \), the points \( t \) and \( w \) swap roles, so that in both \( \phi^{(1)} \) and \( \phi^{(2)} \), \( u \) is adjacent to \( t' \) and \( t \) is the point adjacent to \( 0 \)—and in particular, the factor \( \tau_p^{\ast}(t) \) cancels out. The following lemma, in combination with Lemma 3.10, is analogous to the bound 3.3, and so is its proof, which is omitted.
Lemma 3.15. For $n \geq 0$,
\[
\sum_{t,z \in \mathbb{Z}^d} \Phi^{(n)}(t,z) \leq \Delta_p^{\bullet \bullet \bullet}(T_p)^n.
\]

Proof of Proposition 3.13 Setting $\vec{v}_i = (w_{i-1}, u_{i-1}, t_i, w_i, z_i, u_i)$, we use the bound
\[
p \sum_x [1 - \cos(k \cdot x)] \Pi_p^{(n)}(x) \leq \sum_{x, t, z} [1 - \cos(k \cdot x)] \psi_0(0, w_0, 0) \psi_n(w_{n-1}, u_{n-1} - t_n, z_n, x) \prod_{i=1}^{n-1} \psi(\vec{v}_i),
\]
which is, in essence, the first bound of Lemma 3.10. The next step is to distribute the displacement factor $1 - \cos(k \cdot x)$ over the $n + 1$ segments. To this end, we write $x = \sum_{i=0}^n d_i$, where $d_i = w_i - u_{i-1}$ for even $i$ and $d_i = u_i - w_{i-1}$ for odd $i$ (with the convention $u_{-1} = 0$ and $w_n = u_n = x$).

Using the Cosine-split lemma 3.1 we obtain
\[
p \sum_x [1 - \cos(k \cdot x)] \Pi_p^{(n)}(x) \leq (n + 1) \sum_{i=0}^n \sum_{x, t, z} [1 - \cos(k \cdot d_i)] \psi_0(0, w_0, 0) \psi_n(w_{n-1}, u_{n-1} - t_n, z_n, x) \prod_{j=1}^{n-1} \psi(\vec{v}_j),
\]
with $d_i$ as introduced above. We now handle these terms for different $i$.

Case (a): $i \in \{0, n\}$. Let us start with $i = n$, so that $d_n \in \{x - u_{n-1}, x - w_{n-1}\}$. We see that
\[
p \sum_x [1 - \cos(k \cdot d_n)] \Pi_p^{(n)}(x) \leq \sum_{x, t, z} [1 - \cos(k \cdot d)] \psi_n(w, u, t, z, x) \leq \Delta_p^{\bullet \bullet \bullet}(0)(T_p)^{n-1} \max_{d \in \{x - w, x - u\}} \sum_{x, t, z} [1 - \cos(k \cdot d)] \psi_n(0, w, t, z, x),
\]
where $d' \in \{x - w, x - u\}$ and $d \in \{x - w, x - u\}$. We expand the indicator in $\psi_n$ into two cases. If $t = z = x$, we can bound the maximum in (3.9) by $p \sum_x [1 - \cos(k \cdot d)] \tau_p(x) \tau_p(x - u)$, which is bounded by $W_p(k)$ for both values of $d$. If $t, z, x$ are distinct points, then for $d = x$, the maximum in (3.9) becomes
\[
p^2 \max_{u \neq 0} \sum_{t, z} [1 - \cos(k \cdot d)] \tau_p^2(t - u) \tau_p(t - z) \tau_p(x - z) \tau_p(x - t) = p^2 \sum \square + \square + \square,
\]
Note that in the pictorial representation, we represent the factor $[1 - \cos(k \cdot (x - 0))]$ with a line from 0 to $x$ carrying a ‘×’ symbol. We use the Cosine-split lemma again to bound
\[
[1 - \cos(k \cdot x) \leq 2([1 - \cos(k \cdot z)] + [1 - \cos(k \cdot (x - z))]),
\]
which results in
\[
p^2 \sum \square + \square + \square \leq 2p^2 \left( \sum \square + \sum \square + \sum \square \right) \leq 2p^2 \left( \sum \square + \sum \square \right) + 2p \left( \sum \square \right) \lesssim p^2 W_p(k)
\]
It is not hard to see that a displacement $d = x - u$ yields the same bound. Similar computations show that the case $i = 0$ yields a contribution of at most
\[
\Delta_p^{\bullet \bullet \bullet}(T_p)^{n-1} \Delta_p^{\bullet \bullet \bullet} W_p(k).
\]
We want to apply both the bound (3.5) and Lemma 3.15. To this end, we rewrite
\[ p \sum_x [1 - \cos(k \cdot d)] \Pi_p^{(n)}(x) \leq \sum_{a_1, a_2, b_1, b_2} \left( \Psi^{i-1}(a_1, a_2) \bar{\Psi}^{n-i-1}(b_1 - x, b_2 - x) \right. \]
\[ \times \sum_{t, w, z, u} \phi(a_2, t, w, z, u, b_2) [1 - \cos(k \cdot d)] \tau_p^0(z - a_1) \phi(b_1 - u) \]
\[ = \phi(a_1, a_2, t, w, z, u, b_1, b_2, k, d) \]
\[ \leq (\Delta_p^*(\phi)(T_p)^{-1} \sum_{b_1', b_2'} (\Psi^{n-i-1}(b_1', b_2') \max_{a_1 \neq a_2} \sum_{t, w, z, u, x} \tilde{\phi}(a_1, a_2, t, w, z, u, b_1' + x, b_2' + x; k, d)) \]
\[ \leq (\Delta_p^*(\phi)\Delta_p^{**}) (T_p)^{n-2} \max_{a_1 \neq a_2, b_1 \neq b_2} \sum_{t, w, z, u, x} \tilde{\phi}(a_1, a_2, t, w, z, u, b_1 + x, b_2 + x; k, d) \]
\[ \leq (\Delta_p^{**})^2 (T_p)^{n-2} \max_{a_1 \neq a_2, b_1 \neq b_2} \sum_{t, w, z, u, x} \tilde{\phi}(0, a, t, w, z, u, b + x; k, d), \]

where we use the substitution \( b_1' = x - b_1 \) in the second line and the bound \( \Delta_p^*(\phi) \leq \Delta_p^{**} \) in the last line. It remains to bound the sum over \( \tilde{\phi} \). We first handle the term due to \( \phi^{(1)} \), and we call it \( \tilde{\phi}^{(1)} \). Depending on the orientation of the diagram (i.e., the parity of \( i \)), the displacement \( d \) is either \( d = w - a = (w - t) + (t - a) \) or \( d = u = (u - z) + z \). We perform the bound for \( d = u \) and use the Cosine-splint lemma once, so that we now have a displacement on an actual edge. In pictorial bounds, abbreviating \( \bar{v} = (0, a, t, w, z, u, b + x; x, k, u) \),

\[
\sum_{t, w, z, u, x} \tilde{\phi}^{(1)}(\bar{v}) = 2p^3 \sum \text{[pictorial bounds]} \leq 2p^3 \left[ \sum \text{[pictorial bounds]} + \sum \text{[pictorial bounds]} \right] \]
\[ = 2p^3 \left[ \sum \text{[pictorial bounds]} + \text{[sum over } \tilde{\phi}^{(1)} \text{]} \right] \]
\[ \leq 2p^3 \Delta_p^{**} \Delta_p W_p(k), \] (3.10)

The bound in (3.10) consists of three summands. The first is
\[ 2p^3 \sum \text{[pictorial bounds]} \leq 2p \left( \sum \text{[sum over } \tilde{\phi}^{(1)} \text{]} \right) \leq 2p^3 \Delta_p^{**} \Delta_p W_p(k), \]

the second is
\[ 2p^4 \sum \sum \text{[pictorial bounds]} \leq 2p \sum \left( \sum \text{[pictorial bounds]} \right) \leq 2p \Delta_p^{**} \Delta_p W_p(k), \]

and the third is
\[ 2p^3 \sum \left( \sum \text{[pictorial bounds]} \right) \leq 2p^3 \Delta_p^{**} \Delta_p W_p(k), \]
\[ \leq 2p^2 \Delta_p^{**} \Delta_p W_p(k), \]

The displacement \( d = w - a \) satisfies the same bound. In total, the contribution in \( \tilde{\phi} \) due to \( \phi^{(1)} \) is at most
\[ 4(\Delta_p^{**})^3 (T_p)^{n-2} (\Delta_p^{**} + \Delta_p) W_p(k). \]

Let us now tend to \( \tilde{\phi}^{(2)} \). To this end, we first write \( \tilde{\phi}^{(2)} = \sum_{j=3}^5 \tilde{\phi}^{(j)} \), where
\[
\tilde{\phi}^{(3)}(0, a, t, w, z, u, b + x; k, d) = \tilde{\phi}^{(2)}(0, a, t, w, z, u, b + x; x, k; d) \mathbb{I}_{\{t = \{t, z, u\} = \{t\}\}}, \\
\tilde{\phi}^{(4)}(0, a, t, w, z, u, b + x; k, d) = [1 - \cos(k \cdot d)] \delta_{z, u} \delta_{t, u} \tau_p^0(u) \tau_p(w - u) \tau_p^0(a - w) \tau_p(b - w - x), \\
\tilde{\phi}^{(5)}(0, a, t, w, z, u, b + x; k, d) = [1 - \cos(k \cdot d)] \delta_{z, u} \delta_{t, u} \delta_{a, w} \tau_p^0(u) \tau_p(a - u) \tau_p^0(u + x) \tau_p^0(b - a - x). 
\]

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Again, we set \(\vec{v} = (0, a, t, w, z, u, b + x, x; k, u)\). Then

\[
\sum_{t, w, z, u, x} \vec{\phi}^{(3)}(\vec{v}) = p^2 \sum_{t, w, z, u, x} \begin{array}{c}
\text{\includegraphics{diagram1.png}}
\end{array}
\leq p^2 \sum_{t, w, z, u, x} \begin{array}{c}
\text{\includegraphics{diagram2.png}}
\end{array} + 2p^3 \left[ \sum_{t, w, z, u, x} \begin{array}{c}
\text{\includegraphics{diagram3.png}}
\end{array} + \sum_{t, w, z, u, x} \begin{array}{c}
\text{\includegraphics{diagram4.png}}
\end{array} \right].
\] (3.11)

The first term in (3.11) is

\[
p^2 \sum_{t, w, z, u, x} \begin{array}{c}
\text{\includegraphics{diagram2.png}}
\end{array} \leq p^2 \sum_{t, w, z, u, x} \begin{array}{c}
\text{\includegraphics{diagram5.png}}
\end{array} \leq \Delta^\bullet W_p(k)
\]

and the substitution \(y' = y - u\) for \(y \in \{w, z, t\}\), we obtain

\[
2p^3 \sum_{t, w, z, u, x} \begin{array}{c}
\text{\includegraphics{diagram6.png}}
\end{array} \leq 2p^3 \sum_{t, w, z, u, x} \begin{array}{c}
\text{\includegraphics{diagram7.png}}
\end{array} \leq 2\Delta^\bullet W_p(k),
\] (3.12)

We are left to handle the last diagram appearing in the last bound of (3.12), which contains one factor \(\tau_p\) and one factor \(\tau_p^\bullet\). We distinguish the case where neither collapses (this leads to the diagram \(H_p(k)\)) and the case where are least one of the factors collapses. Using \(\tau_p^\bullet \leq \tau_p^\circ\) and the substitution \(u' = u - j\) for \(y \in \{w, z, t\}\), we obtain

\[
2p^4 \sum_{t, w, z, u, x} \begin{array}{c}
\text{\includegraphics{diagram8.png}}
\end{array} \leq 2H_p(k) + 4p^4 \sum_{t, w, z, u} \begin{array}{c}
\text{\includegraphics{diagram9.png}}
\end{array} \leq 2H_p(k) + 4p^4 \sum_{t, w, z} \begin{array}{c}
\text{\includegraphics{diagram10.png}}
\end{array} \leq 2H_p(k) + 4\Delta^\bullet (0)\Delta^\circ W_p(k).
\]

In total, this yields an upper bound on (3.11) of the form

\[
6(\Delta_p^{\bullet \circ})^3(T_p)^{n-2}[(\Delta_p^{\bullet \circ} + \Delta_p + \Delta_p^\circ)W_p(k) + H_p(k)].
\]
The same bound is good enough for the displacement $d = w - a$. Turning to $j = 4$, we consider the displacement $d = u$ and see that

$$\sum_{t, w, z, u, x} \tilde{\phi}^{(4)}(v) = p^2 \sum_{t, w, z, u, x} = p^2 \sum_{t, w, z, u, x} \leq p \sum \left( \left( \frac{\psi}{\sum p} \sum_{t, w, z, u, x} \right) \right) \leq \Delta_p^* W_p(k),$$

which is also satisfied for $d = w - a$. Finally, $j = 5$ forces $d = u$, and we have

$$\sum_{t, w, z, u, x} \tilde{\phi}^{(5)}(v) = p^2 \sum_{t, w, z, u, x} \leq p \sum \left( \left( \frac{\psi}{\sum p} \sum_{t, w, z, u, x} \right) \right) \leq \Delta_p^* W_p(k),$$

and we see that this bound is not good enough for $n = 2$. To get a better bound for $n = 2$, we aim to control

$$p \sum_{w, u, s, f, z, x} \psi_0(0, w, u) \tau_p, \kappa(s - w) \tau_p(s - u) \psi_n(u, s, t, z, x) \leq \left( p^2 \sum_{t, w, z, u, x} \psi_n(u, s, t, z, x) \right) \leq \Delta_p^* W_p(k) T_p.$$

The above bound is due to the fact that, thanks to (3.4), the supremum over the sum over $\psi_n$ is bounded by the supremum in (3.6).

**Proof of Proposition 3.14.** We let $n = 1$; our goal is to obtain a bound on

$$p \sum_{x} \left[ 1 - \cos(k \cdot x) \right] \Pi_p^{(1)}(x) \leq p \sum \left[ 1 - \cos(k \cdot x) \right] \phi_0(0, w, u, z) \phi_n(u, t, z, x) \leq p^2 \sum \left[ 1 - \cos(k \cdot x) \right] \phi_0(0, w, u, z) \tau_p(t - u) \tau_p(z - t) \tau_p(z - x) \tau_p(t - x) + p \sum \left[ 1 - \cos(k \cdot x) \right] \phi_0(0, w, u, z) \tau_p(x - u) = p^3 \sum \left[ 1 - \cos(k \cdot x) \right] \tau_p^*(w) \tau_p(u - w) \tau_p(z - w) \tau_p(t - u) \tau_p(z - t) \tau_p(x - z) \tau_p(x - t) + p^2 \sum J(u) \tau_p, \kappa(x) \tau_p(x - u) + p^2 \sum \left[ 1 - \cos(k \cdot x) \right] \tau_p^*(w) \gamma_p(u) \tau_p(u - w) \tau_p(x - u).$$

We use the Cosine-split lemma on the first term to decompose $x = u + (z - u) + (x - z)$, which gives

$$p^3 \sum_{w, u, t, z, x} \left[ 1 - \cos(k \cdot x) \right] \tau_p^*(w) \tau_p(u - w) \tau_p(z - w) \tau_p(t - u) \tau_p(z - t) \tau_p(x - u),$$

which gives

$$p^3 \sum_{w, u, t, z, x} \left[ 1 - \cos(k \cdot x) \right] \tau_p^*(w) \tau_p(u - w) \tau_p(z - w) \tau_p(t - u) \tau_p(z - t) \tau_p(x - u) = 3 p^2 \left( \sum_{t, w, z, u, x} \frac{\psi}{\sum p} \sum_{t, w, z, u, x} \phi_0(0, w, u, z) \phi_n(u, t, z, x) \right) \leq 3 p^2 \sum \left( \left( \frac{\psi}{\sum p} \sum_{t, w, z, u, x} \right) \right) \leq 3 W_p(k) \Delta_p^* \Delta_p + 3 p^2 \sum \left( \left( \frac{\psi}{\sum p} \sum_{t, w, z, u, x} \right) \right) \leq 3 W_p(k) \Delta_p^* \Delta_p + 3 W_p(k) \Delta_p^* \Delta_p(0) \leq 3 W_p(k) \Delta_p^* (3 \Delta_p^* + 3 p^2 \Delta_p^* \sum \left( \left( \frac{\psi}{\sum p} \sum_{t, w, z, u, x} \right) \right) \leq 3 W_p(k) \Delta_p^* (3 \Delta_p^* + \Delta_p).$$
The second term is \( p^2(J \ast \tau_{p,k} \ast \tau_p)(0) \). Depicting the factor \( \gamma_p \) as a disrupted line, the third term is

\[
p^2 \sum_{w,u,x} |1 - \cos(k \cdot x)| \tau_p^p(w) \gamma_p(u) \tau_p(u - w) \tau_p^p(x - w) \tau_p(x - u) = p^2 \sum \left| \frac{\gamma_p}{2} \right|
\]

\[
\leq 2p^3 \sum \left| \frac{\gamma_p}{2} \right| + 2p^2 \sum \left| \frac{\gamma_p}{2} \right| \leq 2W_p(k) \left[ T^p(0) + p(\delta_0 + \tau_p \ast \gamma_p)(0) \right]
\]

\[
\leq 2W_p(k) \left[ T^p(0) + \gamma_p + \tau_p \ast \gamma_p(0) + p^2(\tau_p^2 \ast \gamma_p)(0) \right] \leq 2W_p(k) \left[ T^p + 2\Delta_p \right].
\]

In the above, we used that \( \gamma_p(x) \leq \tau_p(x) \) as well as \( \gamma_p(x) \leq p(J \ast \tau_p)(x) \leq p\tau_p^p(x) \).

\[
\square
\]

4 Bootstrap analysis

4.1 Introduction of the bootstrap functions

This section brings the previous results together to prove Proposition 4.2, from which Theorem 1.1 follows with little extra effort. The remaining strategy of proof is standard and described in detail in [16]. In short, it is the following: We introduce \( f \) in (4.1). In Sect. 4.2 and in particular in Proposition 4.2, we prove several bounds in terms of \( f \), including bounds uniform in \( p \in [0, p_c) \) under the additional assumption that \( f \) is uniformly bounded.

In Section 4.3, we show that \( f(0) \leq 3 \) and that \( f \) is continuous on \([0, p_c)\). Lastly, we show that on \([0, p_c)\), the bound \( f \leq 4 \) implies \( f \leq 3 \). This is called the improvement of the bounds, and it is shown by employing the implications from Section 4.2. As a consequence of this, the results from Section 4.2 indeed hold uniformly in \( p \in [0, p_c) \), and we may extend them to \( p_c \) by a limiting argument.

Let us recall the notation \( \tau_{p,k}(x) = 1 - \cos(k \cdot x) \tau_p(x) \), \( J_k(x) = 1 - \cos(k \cdot x) \). We extend this to \( D_k(x) = 1 - \cos(k \cdot x) \). We note that \( \chi(p) \) was defined as \( \chi(p) = E[\gamma^i(0)] \) and that \( \chi(p) = 1 + p \sum_{x \in \mathbb{Z}^d} \tau_p(x) \). We define

\[
\lambda_p = 1 - \frac{1}{\chi(p)} = 1 - \frac{1}{1 + p\tau_p(0)}.
\]

We define the bootstrap function \( f = f_1 \lor f_2 \lor f_3 \) with

\[
f_1(p) = 2dp, \quad f_2(p) = \sup_{k \in (-\pi, \pi)^d} \left\{ \frac{|\tilde{\tau}_p(k)|}{\tilde{G}_{\lambda_p}(k)} \right\}, \quad f_3(p) = \sup_{k \in (-\pi, \pi)^d} \left\{ \frac{|\tilde{\tau}_p,k(l)|}{\tilde{U}_{\lambda_p}(k,l)} \right\},
\]

where \( \tilde{U}_{\lambda_p}(k, l) \) is defined as

\[
\tilde{U}_{\lambda_p}(k, l) := 3000[1 - \tilde{D}(k)] \left( \tilde{G}_{\lambda}(l - k) + \tilde{G}_{\lambda}(l) \tilde{G}_{\lambda}(l + k) + \tilde{G}_{\lambda}(l - k) \tilde{G}_{\lambda}(l + k) \right).
\]

We note that \( \tilde{\tau}_{p,k} \) relates to \( \Delta_k \tilde{\tau}_p \), the discretized second derivative of \( \tilde{\tau}_p \), as follows:

\[
\Delta_k \tilde{\tau}_p(l) := \tilde{\tau}_p(l - k) + \tilde{\tau}_p(l + k) - 2\tilde{\tau}_p(l) = -2\tilde{\tau}_{p,k}(l).
\]

The following result bounds the discretized second derivative of the random walk Green’s function.

**Lemma 4.1** (Bounds on \( \Delta_k \tilde{G}_{\lambda}(l) \)). Let \( a(x) = a(-x) \) for all \( x \in \mathbb{Z}^d \), set \( \tilde{A}(k) = (1 - \tilde{a}(k))^{-1} \), and let \( k, l \in (-\pi, \pi)^d \). Then

\[
|\Delta_k \tilde{A}(l)| \leq \left| [\tilde{a}(0) - \tilde{a}(l)] \times \left( [\tilde{A}(l - k) + \tilde{A}(l + k)] \tilde{A}(l) 
\right.
\]

\[
\left. + 8\tilde{A}(l - k) \tilde{A}(l + k) \tilde{A}(l) [\tilde{a}(0) - \tilde{a}(l)] \right|.
\]

In particular,

\[
|\Delta_k \tilde{G}_{\lambda}(l)| \leq 1 - \tilde{D}(k) \left( \tilde{G}_{\lambda}(l) \tilde{G}_{\lambda}(l - k) + \tilde{G}_{\lambda}(l) \tilde{G}_{\lambda}(l + k) + 8\tilde{G}_{\lambda}(l - k) \tilde{G}_{\lambda}(l + k) \right).
\]

A natural first guess for \( f_3 \) might have been \( \sup \| \Delta_k \tilde{\tau}_p(l)/\| \Delta_k \tilde{G}_{\lambda_p}(l) \|. \) However, \( \Delta_k \tilde{G}_{\lambda_p}(l) \) may have roots, which makes this guess an inconvenient choice for \( f_3 \). In contrast, \( \tilde{U}_{\lambda_p}(k, l) > 0 \) for \( k \neq 0 \). Hence, the bound in Lemma 4.1 supports the idea that \( f_3 \) is a reasonable definition.
4.2 Consequences of the bootstrap

The main result of this section, and a crucial result in this paper, is Proposition 4.2. Proposition 4.2 proves (in high dimension) the convergence of the lace expansion derived in Proposition 2.9 by giving bounds on the lace-expansion coefficients. Under the additional assumption that \( f \leq 4 \) on \([0, p_c)\), these bounds are shown to be uniform in \( p \in [0, p_c) \).

**Proposition 4.2** (Convergence of the lace expansion and Ornstein-Zernike equation).

1. Let \( n \in \mathbb{N}_0 \) and \( p \in [0, p_c) \). Then there is \( d_0 \geq 6 \) and a constant \( c_f = c(f(p)) \) (increasing in \( f \) and independent of \( d \)) such that, for all \( d > d_0 \),

\[
\sum_{x \in \mathbb{Z}^d} p|x| \Pi_{p,n}(x) \leq c_f/d, \quad \sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] p|x| \Pi_{p,n}(x) \leq [1 - \tilde{D}(k)] c_f/d, \tag{4.2}
\]

\[
\sup_{x \in \mathbb{Z}^d} p \sum_{m=0}^{n} |\Pi_{p}^m(x)| \leq c_f, \tag{4.3}
\]

and

\[
\sum_{x \in \mathbb{Z}^d} |R_{p,n}(x)| \leq c_f(c_f/d)^n \tau_p(0). \tag{4.4}
\]

Consequently, \( \Pi_p := \lim_{n \to \infty} \Pi_{p,n} \) is well defined and \( \tau_p \) satisfies the Ornstein-Zernike equation, taking the form

\[
\tau_p(x) = J(x) + \Pi_p(x) + p(J + \Pi_p) \tau_p(x). \tag{4.5}
\]

2. Let \( f \leq 4 \) on \([0, p_c) \). Then there is a constant \( c \) and \( d_0 \geq 6 \) such that the bounds \( \Pi \), \( \Pi^m \), \( R \), and \( \tau \) hold for all \( d > d_0 \) with \( c_f \) replaced by \( c \) for all \( p \in [0, p_c) \). Moreover, the OZE \( \Pi \) holds.

We recall that in Section 4.3 we prove that \( f \leq 3 \) and so the second part of Proposition 4.2 applies. We now show that with this result at hand, we can extend the OZE to the critical point as well as prove the main theorem.

**Corollary 4.3** (OZE at \( p_c \)). There is \( d_0 \) such that for all \( d > d_0 \), the limit \( \Pi_p = \lim_{p \to p_c} \Pi_p \) exists and is given by \( \Pi_p = \sum_{n \geq 0} (-1)^n \Pi_p^{(n)} \), where \( \Pi_p^{(n)} \) is the extension of Definition 2.8 at \( p = p_c \). Consequently, the bounds in Proposition 4.2 and the OZE \( (4.5) \) extend to \( p_c \).

The corollary as well as the main theorem now follow from Proposition 4.2 in conjunction with Proposition 4.12, which is proven in Section 4.3 below:

**Proof of Corollary 4.3 and Theorem 1.1.** Proposition 4.12 implies that indeed \( f(p) \leq 3 \) for all \( p \in [0, p_c) \), and therefore, the consequences in the second part of Proposition 4.2 are valid. Lemma 4.14 together with monotone convergence then implies the triangle condition.

The remaining arguments are analogous to the proofs of [17] Corollary 6.1 and [17] Theorem 1.1. The rough idea for the proof of Corollary 4.3 is to use \( \theta(p_c) = 0 \) (which follows from the triangle condition) to couple the model at \( p_c \) with the model at \( p < p_c \), and then show that, as \( p \uparrow p_c \), the (a.s.) finite cluster of the origin is eventually the same. For the full argument and the proof of the infra-red bound, we refer to [17].

Proposition 4.2 follows without too much effort as a sequence of Lemmas 4.7, 4.8, 4.9, and 4.10. The following observations turn out to be helpful in proving them:

**Observation 4.4** (Convolutions of \( J \)). Let \( m \in \mathbb{N} \) and \( x \in \mathbb{Z}^d \) with \( m \geq |x| \). Then there is a constant \( c = c(m, x) \) with \( c \leq m! \) such that

\[
J^m(x) = c \mathbb{1}_{|m-x| \text{ is even}} (2d)^{|m-x|}/2.
\]

**Proof.** This is an elementary matter of counting the number of \( m \)-step walks from 0 to \( x \). If \( m - |x| \) is odd, then there is no way of getting from 0 to \( x \) in \( m \) steps.

So assume \( m - |x| \) is even. To get from 0 to \( x \), \( |x| \) steps must be chosen to reach \( x \). Only taking these \( |x| \) steps (in any order) would amount to a shortest 0-\( x \)-path. Out of the remaining steps, half can be chosen freely (each producing a factor of 2d), and the other half must compensate them. In counting the different walks, we have to respect the at most \( m! \) unique ways of ordering the steps.
We remark that this also shows that the maximum is attained for \( x = 0 \) when \( m \) is even and for \( x \) being a neighbor of 0 when \( m \) is odd.

**Observation 4.5 (Elementary bounds on \( \tau_{p}^{*n} \)).** Let \( n, m \in \mathbb{N} \). Then there is \( c = c(m, n) \) such that, for all \( p \in [0, 1] \) and \( x \in \mathbb{Z}^d \),

\[
\tau_{p}^{*n}(x) \leq c \sum_{l=0}^{m-n} p^l J^{*(l+n)}(x) + c \sum_{j=1}^{n} p^{m+j-n} (J^{*m} \ast \tau_{p}^{*j})(x),
\]

where we use the convention that \( \sum_{l=0}^{m-n} \) vanishes for \( n > m \).

**Proof.** The observation heavily relies on the bound

\[
\tau_{p}(x) \leq J(x) + \mathbb{E} \left[ \sum_{y \in \omega} \mathbb{E}(0 \leq y) \right] = J(x) + p(J \ast \tau_{p})(x). \tag{4.6}
\]

We prove the statement by induction on \( m - n \); for the base case, let \( m \leq n \). Then we apply (4.6) to \( m - n \) \( \tau_{p} \) terms to obtain

\[
\tau_{p}^{*n}(x) \leq \left( \tau_{p}^{*(m-n)} \ast (J + p(J \ast \tau_{p}))^{*m} \right)(x) = \sum_{l=0}^{m-n} \binom{m}{l} p^l (J^{*m} \ast \tau_{p}^{*(m-l)})(x) = \sum_{l=0}^{m-n} \binom{m}{l} p^l J^{*(m-l)-n} \ast \tau_{p}^{*(m-l)}(x).
\]

Let now \( m - n > 0 \). Applying (4.6) once yields a sum of two terms, namely

\[
\tau_{p}^{*n}(x) \leq (J \ast \tau_{p}^{*(n-1)})(x) + (J \ast \tau_{p}^{*n})(x). \tag{4.7}
\]

We can apply the induction hypothesis on the second term with \( \tilde{m} = m - 1 \) and \( \tilde{n} = n \), producing terms of the sought-after form. Now, observe that application of (4.6) yields

\[
(J^{*j} \ast \tau_{p}^{*(n-j)})(x) \leq (J^{*j+1} \ast \tau_{p}^{*(n-j-1)})(x) + (J^{*j+1} \ast \tau_{p}^{*(n-j)})(x) \tag{4.8}
\]

for \( 1 \leq j < n \). For every \( j \), the second term can be bounded by the induction hypothesis for \( \tilde{m} = m - j - 1 \) and \( \tilde{n} = n - j \) (so that \( \tilde{m} - \tilde{n} < m - n \)) with suitable \( c(m, n) \). Hence, we can iteratively break down (4.7); after applying (4.8) for \( j = n - 1 \), we are left with the term \( J^{*n}(x) \), finishing the proof. \( \square \)

We now define

\[
V_{p}^{(m,n)}(a) := (J^{*m} \ast \tau_{p}^{*n})(a), \quad W_{p}^{(m,n)}(a; k) := (\tau_{p,k} \ast V_{p}^{(m,n)})(a), \quad \tilde{W}_{p}^{(m,n)}(a; k) := (J_{k} \ast V_{p}^{(m,n)})(a).
\]

Note that \( W_{p} \) from Definition 3.12 relates to the above definition via \( W_{p} = pW_{p}^{(0,0)} + pW_{p}^{(0,1)} \).

**Lemma 4.6 (Bounds on \( V_{p}^{(m,n)}, W_{p}^{(m,n)}, \tilde{W}_{p}^{(m,n)} \)).** Let \( p \in [0, p_{c}) \) and \( m, n \in \mathbb{N}_0 \) with \( d > \frac{20}{3}n \). Then there is a constant \( c_f = c(m, n, f(p)) \) (increasing in \( f \)) such that the following hold true:

1. For \( m + n \geq 2 \),

\[
p^{m+n-1} V_{p}^{(m,n)}(a) \leq \begin{cases} c_f & \text{if } m + n = 2 \text{ and } a = 0, \\ c_f \frac{d}{d} & \text{else.} \end{cases}
\]

2. For \( m + n \geq 1 \), and under the additional assumption \( d > 2n + 4 \) for the below bound on \( W_{p}^{(m,n)} \),

\[
p^{m+n} \max \left\{ \sup_{a \in \mathbb{Z}^d} \tilde{W}_{p}^{(m,n)}(a; k), \sup_{a \in \mathbb{Z}^d} W_{p}^{(m,n)}(a; k) \right\} \leq [1 - \hat{D}(k)] \times \begin{cases} c_f & \text{if } m + n \leq 2, \\ c_f \frac{d}{d} & \text{if } m + n \geq 3. \end{cases}
\]
Proof. Bound on $V_p$. We start with the case $m \geq 4$ where we can rewrite the left-hand side via Fourier transform and apply Hölder’s inequality to obtain
\[
p^{m+j-1}(J^{*m} \ast \tau^*_p)(a) = p^{m+j-1} \int_{(-\pi,\pi]^d} e^{-ik \cdot a} \tilde{\mathcal{F}}(k)^m \mathcal{F}(k)^j \frac{dk}{(2\pi)^d}
\]
\[
\leq \left( p^{10(m-1)} \int_{(-\pi,\pi]^d} \tilde{\mathcal{F}}(k)^{10m} \frac{dk}{(2\pi)^d} \right)^{1/10} \left( \int_{(-\pi,\pi]^d} (p\tau^*_p(k))^{10j/9} \frac{dk}{(2\pi)^d} \right)^{9/10}
\]
\[
\leq \left( p^{10(m-1)} J^{*10m}(0) \right)^{1/10} \times f_2(p)^j \left( \int_{(-\pi,\pi]^d} \tilde{\mathcal{G}}_p(k)^{10j/9} \frac{dk}{(2\pi)^d} \right)^{9/10}.
\]
The first factor in (4.9) is handled by Observation 4.4 as
\[
\left( p^{10(m-1)} J^{*10m}(0) \right)^{1/10} \leq \left( c p^{10(m-1)} (2d)^5m \right)^{1/10} \leq \left( c(2dp)^{10(m-1)} (2d)^{-5m+10} \right)^{1/10}
\]
and $m \geq 4$. Regarding the second factor in (4.9), note that $10j/9 \leq 10n/9 < d/2$ and so Proposition 2.1 gives a uniform upper bound.

If $m < 4$, then we first use Observation 4.5 with $\tilde{m} = 4 - m$ to get that $p^{m+n-1}V_p^{(m,n)}$ is bounded by a sum of terms of two types, which are constant multiples of
\[
p^{l+m+n-1} J^{*(l+m+n)}(a) = p^{s-1} J^{*(s)}(a) \quad \text{and} \quad p^{4+j-1} \tau^*_p(a),
\]
where $0 \leq l \leq 4 - m - 1$ (and therefore $s \geq 2$) and $1 \leq j \leq n$. If $s$ is odd, we can write $s = 2r + 1$ for some $r \geq 1$, and Observation 4.4 gives
\[
p^{2r} J^{*(2r+1)}(a) \leq cp^{2r}(2d)^r = c(2dp)^{2r}(2d)^r \leq c(f_1(p))^{2r}(2d)^r \leq cf/d.
\]
Similarly, if $s$ is even and $a \neq 0$ or $s \geq 4$, then $p^{s-1} J^{*(s)}(a) \leq cf/d$. Finally, if $s = 2$ and $a = 0$, then $p(J \ast J)(0) \leq cf$. This shows that the terms of the first type in (4.10) are of the correct order. The second type is of the form $p^{4+j-1}V_p^{(4,1)}(a)$ and included in the previous considerations. Together this proves the claimed bound on $V_p^{(m,n)}$.

**Bound on $\tilde{W}_p$.** Let first $m + n \geq 3$. Then
\[
p^{m+n} \tilde{W}_p^{(m,n)}(a;k) = \sum_{y \in \mathbb{Z}^d} J_k(y) V_p^{(m,n)}(a-y)
\]
\[
\leq p^{m+n-1} \left( \sup_{a \in \mathbb{Z}^d} V_p^{(m,n)}(a) \right) (2dp) \sum_{y \in \mathbb{Z}^d} [1 - \cos(k \cdot y)] D(y)
\]
\[
\leq cf/d \times f_1(p)[1 - \tilde{D}(k)],
\]
applying the bound on $V_p$.

Consider now $m + n = 2$. Using first that $J \leq \tau_p$ and then (4.6),
\[
p^2 \tilde{W}_p^{(m,n)}(a;k) \leq p^2 \tilde{W}_p^{(10,2)}(a;k) \leq p^2 \tilde{W}_p^{(2,0)}(a;k) + p^3 \tilde{W}_p^{(2,1)}(a;k) + p^3 \tilde{W}_p^{(1,2)}(a;k).
\]
The second and third summand right-hand side of (4.11) can be dealt with as before, we only have to deal with first summand. Indeed,
\[
p^2 \tilde{W}_p^{(2,0)}(a;k) = p^2 \sum_y J_k(y) J^{*2}(a-y) \leq 2dp^2 J^{*2}(0) \sum_y D_k(y) = f_1(p)^2[1 - \tilde{D}(k)],
\]
and we can choose $c_f = f_1(p)^2$.

Finally, for $m + n = 1$, we have $p \tilde{W}_p^{(m,n)}(a;k) \leq p \tilde{W}_p^{(1,0)}(a;k) + p^2 \tilde{W}_p^{(1,1)}(a;k)$. The second term was already bounded, the first is
\[
p(J_k \ast J)(a) \leq p \sum_y J_k(y) = f_1(p)[1 - \tilde{D}(k)].
\]
Bound on $W_p$. We note that a combination of (4.6) and the Cosine-split lemma 3.1 yields
\[ \tau_{p,k}(x) \leq J_k(x) + 2p(J_k \ast \tau_p)(x) + 2p(J \ast \tau_{p,k})(x). \] (4.12)
Applying this repeatedly, we can bound $p^{m+n}W_p^{(m,n)}(a; k)$ by a sum of quantities of the form $p^{s+t}W_p^{(s,t)}$ (where $s + t \geq 1$) plus $c(m, n)p^{m+n}W_p^{(m,n)}$, where we can now assume $m \geq 4$ w.l.o.g. The terms of the form $W_p^{(s,t)}$ were bounded above already. Similarly to how we obtained the bound (4.9), we bound the last term by applying Hölder’s inequality, and so
\[
p^{m+n}W_p^{(m,n)}(a; k) \leq p^{m+n} \int_{(-\pi,\pi]^d} |\hat{J}(l)|^{|m|} |\hat{\tau}_p(l)|^{|n|} |\hat{\tau}_{p,k}(l)| \frac{dl}{(2\pi)^d}
\leq \left(p^{10(m-1)} \int_{(-\pi,\pi]^d} |\hat{J}(l)|^{10m} \frac{dl}{(2\pi)^d}\right)^{1/10} \left(\int_{(-\pi,\pi]^d} (p|\hat{\tau}_p(l)|)^{10n/9} (p|\hat{\tau}_{p,k}(l)|)^{10/9} \frac{dl}{(2\pi)^d}\right)^{9/10}
\leq c_f/d \times 3000 f(p)^{n+1} \left(\int_{(-\pi,\pi]^d} \hat{G}_{\lambda_p}(l)^{10n/9} \left[\hat{G}_{\lambda_p}(l)(l-k) + \hat{G}_{\lambda_p}(l+k)\right] + \hat{G}_{\lambda_p}(l-k)\hat{G}_{\lambda_p}(l+k)\right)^{10/9} \frac{dl}{(2\pi)^d}\right)^{9/10}
\leq (c_f)^2/d,
(4.13)
where the last bound is due to Proposition 2.2.

The proofs of the following lemmas, bounding the quantities appearing in Section 3, are direct consequences of Lemma 4.6.

**Lemma 4.7** (Bounds on various triangles). Let $p \in [0, p_c)$ and $d > 6$. Then there is $c_f = c(f(p))$ (increasing in $f$) such that
\[
\max\{\Delta_p, \Delta_p^\circ, \Delta_p^*, \Delta_p^{**}\} \leq c_f/d, \quad \max\{\Delta_p^*(0), \Delta_p^{**}(0), \Delta_p^{***}\} \leq c_f.
\]
**Proof.** Note that
\[
\Delta_p(x) = p^2 V_p^{(0,3)}(x), \quad \Delta_p^*(x) = p^2 V_p^{(0,2)}(x) + \Delta_p(x), \quad \Delta_p^*(x) = p V_p^{(0,2)}(x) + \Delta_p(x),
\Delta_p^{**}(x) = \delta_{0,x} + \Delta_p^*(x).
\]
For the bound on $p\tau_p \leq p$, we use that $p \leq f_1(p)/d$. For all remaining quantities, we use Lemma 4.6, which is applicable since $n \leq 3$ and $\frac{20}{p}n \leq \frac{90}{p} < 7 \leq 7$.

**Lemma 4.8** (Bound on $W_p$). Let $p \in [0, p_c)$ and $d > 6$. Then there is a constant $c_f = c(f(p))$ (increasing in $f$) such that
\[ W_p(k) \leq [1 - \hat{D}(k)]c_f. \]
**Proof.** By (4.12),
\[
W_p(x; k) = pW_p^{(0,1)}(x; k) + p\tau_{p,k}(x)
\leq pW_p^{(0,1)}(x; k) + 2p^2 \tilde{W}_p^{(0,1)}(x; k) + 2p^2 W_p^{(1,0)}(x; k) + pJ_k(x).
\]
The proof follows from Lemma 4.6 together with the observation that
\[ pJ_k(x) = (2dp)D_k(x) \leq f_1(p) \sum_{x \in \mathbb{Z}^d} D_k(x) = f_1(p)[1 - \hat{D}(k)]. \]

**Lemma 4.9** (Displacement bounds on $\Pi_p^{(0)}$ and $\Pi_p^{(1)}$). Let $p \in [0, p_c)$, $i \in \{0, 1\}$, and $d > 6$. Then there is a constant $c_f = c(f(p))$ (increasing in $f$) such that
\[ p \sum_x \Pi_p^{(0)}(x) \leq c_f/d, \quad p \sum_x [1 - \cos(k \cdot x)]\Pi_p^{(1)}(x) \leq [1 - \hat{D}(k)]c_f/d. \]
Proof. We recall the two bounds obtained in Proposition \[3.2\]. The first one yields \( p|\hat{H}_p^{(0)}(k)| \leq p^3 V_p^{(2,2)}(0) \), the second one yields
\[
p(\hat{H}_p^{(0)}(k) - \hat{H}_p^{(0)}(0)) \leq 2p^3 \tilde{W}_p^{(1,2)}(0; k) + 2p^3 W_p^{(2,1)}(0; k).
\]
All of these bounds are handled directly by Lemma \[4.6\]. Similarly, the only quantity in the bound of Proposition \[3.14\] that was not bounded already is \( p^2 W_p^{(1,1)}(0; k) \). By a combination of \[4.6\] and \[4.12\], we can bound
\[
p^2 W_p^{(1,1)}(0; k) \leq p^2 \left( W_p^{(2,0)}(0; k) + W_p^{(2,1)}(0; k) \right)
\leq p^2 \left( \tilde{W}_p^{(2,0)}(0; k) + 2pW_p^{(2,1)}(0; k) + 2pW_p^{(3,0)}(0; k) + pW_p^{(2,1)}(0; k) \right).
\]
But \( 0 \leq \tilde{W}_p^{(2,0)}(0; k) \leq 2J^3(0) = 0 \) by Observation \[4.4\]. The other three terms are bounded by Lemma \[4.6\].

**Lemma 4.10** (Displacement bounds on \( H_p \)). Let \( p \in [0, p_c) \) and \( d > 6 \). Then there is a constant \( c_f = c(f(p)) \) (increasing in \( f \)) such that
\[
H_p(k) \leq [1 - \tilde{D}(k)]c_f/d.
\]

Proof. We recall that
\[
H_p(b_1, b_2; k) = p^5 \sum_{t, u, z, v} \tau_p(t) \tau_p(t - u) \tau_p(t - z) \tau_p(u - z) \tau_p(t - w) \tau_p(w - b_1) \tau_p(v - w) \tau_p(v + b_2 - u).
\]
We bound the factor \( \tau_p(z) \leq J(z) + p(J \ast \tau_p)(z) \), splitting \( H_p \) into a sum of two. The first term is easy to bound. Indeed,
\[
p^5 \sum_{t, u, z} J(z) \tau_p(t - z) \tau_p(w - t) \tau_p(t - z) \tau_p(u - z) \tau_p(t - u) \tau_p(k) (b_1 - w) \tau_p(k - u) \tau_p(t - u) \tau_p(k - t - z)
\leq \Delta_p(0) p^4 \sum_{t, u, z, v} J(z) \tau_p(t - z) \tau_p(w - t) \tau_p(b_1 - w) \tau_p(k - t - z)
\leq \Delta_p(0) W_p(k) p^3 V_p^{(1,3)}(b_1) \leq [1 - \tilde{D}(k)]c_f/d
\]
by the previous Lemmas \[4.6\], \[4.8\], \[4.7\]. We can thus focus on bounding
\[
p^6 \sum_{t, u, z, v} (J \ast \tau_p)(z) \tau_p(t - u) \tau_p(t - z) \tau_p(u - z) \tau_p(t - w) \tau_p(w - b_1) \tau_p(v - w) \tau_p(v + b_2 - u). \tag{4.14}
\]
To prove such a bound (and thus the lemma), we need to recycle some ideas from the proof of Lemma \[4.6\] in a more involved fashion. To this end, let
\[
\sigma(x) := p^4 (J^{*u} \ast \tau_p)(x) + \sum_{j=1}^4 p^{j-1} J^{*j}(x)
\]
and note that \( \tau_p(x) \leq \sigma(x) \) by application of \[4.6\]. Consequently, \[4.14\] is bounded by \( \tilde{H}_p(a_1, a_2; k) \), where we define
\[
\tilde{H}_p(a_1, a_2; k) = p^6 \sum_{t, u, z, v} (J \ast \sigma)(z) \sigma(t - u) \sigma(t - z) \sigma(u - z) \sigma(t - w) \sigma(w - a_1) \sigma(v - w) \sigma(v + a_2 - u).
\]
In terms of the Fourier transform, we can write
\[
\tilde{H}_p(a_1, a_2; k) = p^6 \int_{(-\pi, \pi)^d} e^{-i\xi_1 \cdot l_1 - i\xi_2 \cdot l_2} |\tilde{J}(l_1)| \tilde{\sigma}(l_1)^2 \tilde{\sigma}(l_2)^2 \tilde{\tau}_p(k)(l_3) \tilde{\sigma}(l_1 - l_2) \tilde{\sigma}(l_1 - l_3) \tilde{\sigma}(l_2 - l_3) \frac{d(l_1, l_2, l_3)}{(2\pi)^d}.\]
(For details on the above identity, see [17, Lemma 5.7] and the corresponding bounds on $H_\lambda$ therein.)

We bound

$$
\bar{H}_p(a_1, a_2; k) \leq 3000 f_3(p)[1 - \bar{D}(k)] p^5 \int_{(-\pi, \pi]^d} |\bar{J}(l_1)| |\bar{\sigma}(l_1)|^2 |\bar{\sigma}(l_2)|^2 |\bar{\sigma}(l_1 - l_2)||\bar{\sigma}(l_1 - l_3)||\bar{\sigma}(l_2 - l_3)|
$$

$$
\times \left( \bar{G}_{\lambda_\nu}(l_3) \bar{G}_{\lambda_\nu}(l_3 - k) + \bar{G}_{\lambda_\nu}(l_3) \bar{G}_{\lambda_\nu}(l_3 + k) + \bar{G}_{\lambda_\nu}(l_3 - k) \bar{G}_{\lambda_\nu}(l_3 + k) \right) \frac{d(l_1, l_2, l_3)}{(2\pi)^d}.
$$

(4.15)

Opening the brackets in (4.15) gives rise to three summands. We show how to treat the third one. Applying Cauchy-Schwarz, we obtain

$$
\left( \int_{(-\pi, \pi]^d} [p^2 |\bar{J}(l_1)||\bar{\sigma}(l_1)|^2] [p^2 |\bar{\sigma}(l_2 - l_1)|^2 |\bar{\sigma}(l_2)|] [p |\bar{G}_{\lambda_\nu}(l_3 + k)|^2 |\bar{\sigma}(l_3 - l_2)|] \frac{d(l_1, l_2, l_3)}{(2\pi)^d} \right)^{1/2}
$$

(4.16)

$$
\times \left( \int_{(-\pi, \pi]^d} [p^2 |\bar{\sigma}(l_2)|^2] [p^2 |\bar{\sigma}(l_1 - l_2)|^2 |\bar{J}(l_1)||\bar{\sigma}(l_1)|] [p |\bar{G}_{\lambda_\nu}(l_3 - k)|^2 |\bar{\sigma}(l_3 - l_2)|] \frac{d(l_1, l_2, l_3)}{(2\pi)^d} \right)^{1/2}
$$

(4.17)

The square brackets indicate how we want to decompose the integrals. We first bound (4.16), and we start with the integral over $l_3$. We intend to treat the five summands constituting $\bar{\sigma}(l_3 - l_2)$ simultaneously. Indeed, note that with our bound on $f_2$,

$$
|\bar{\sigma}(l)| \leq \sum_{j=0}^{4} p^{j-1} |\bar{J}(l)|^j + p^5 \bar{J}(l)^4 \bar{G}_{\lambda_\nu}(l) \leq 5 \max_{n \in \{0, 1, \ldots, 4\}} p^{(j \vee 4)n - 1} |\bar{J}(l)|^{(j \vee 4)n} \bar{G}_{\lambda_\nu}(l)^n.
$$

(4.18)

With this,

$$
p^{j \vee 4n} \int_{(-\pi, \pi]^d} |\bar{J}(l_1 - l_2)|(j \vee 4n) \bar{G}_{\lambda_\nu}(l_3 - l_2)^n \bar{G}_{\lambda_\nu}(l_3 + k)^2 \frac{d l_3}{(2\pi)^d}
$$

$$
\leq \left(p^{10(j \vee 4n)} \int_{(-\pi, \pi]^d} \bar{J}(l_3)^{10(j \vee 4n)} \frac{d l_3}{(2\pi)^d} \right)^{1/10}
$$

$$
\times \left( \int_{(-\pi, \pi]^d} \bar{G}_{\lambda_\nu}(l_3 + k)^{20/9} \bar{G}_{\lambda_\nu}(l_3 - l_2) \frac{d l_3}{(2\pi)^d} \right)^{10/9}
$$

$$
\leq (c_f/d)^{1/2}
$$

by the same considerations that were performed in (4.13). We use the same approach to treat the integral over $l_2$ in (4.16). Applying (4.18) to all three factors of $\bar{\sigma}$ gives rise to tuples $(j, n_i)$ for $i \in [3]$, and so

$$
p^{-1 + \sum_{i=1}^{3} (j_i \vee 4n_i)} \int_{(-\pi, \pi]^d} |\bar{J}(l_1 - l_2)|^{\sum_{i=1}^{3} (j_i \vee 4n_i)} |\bar{J}(l_2)|^{j_3 \vee 4n_3} \bar{G}_{\lambda_\nu}(l_2 - l_1)^{n_1 + n_2} \frac{d l_2}{(2\pi)^d}
$$

$$
\leq \left(p^{10(-1 + \sum_{i=1}^{3} (j_i \vee 4n_i))} \int_{(-\pi, \pi]^d} \bar{J}(l_2 - l_1)^{10 \sum_{i=1}^{3} (j_i \vee 4n_i)} \bar{J}(l_2)^{10(j_3 \vee 4n_3)} \frac{d l_2}{(2\pi)^d} \right)^{1/10}
$$

$$
\times \left( \int_{(-\pi, \pi]^d} \bar{G}_{\lambda_\nu}(l_2 - l_1)^{10(n_1 + n_2)/9} \bar{G}_{\lambda_\nu}(l_2 - 2l_1)^{10n_3/9} \frac{d l_2}{(2\pi)^d} \right)^{9/10}
$$

$$
\leq c_f \left(p^{20(-1 + \sum_{i=1}^{3} (j_i \vee 4n_i))} \int_{(-\pi, \pi]^d} \bar{J}(l_2 - l_1)^{20 \sum_{i=1}^{3} (j_i \vee 4n_i)} \frac{d l_2}{(2\pi)^d} \right)^{1/20}
$$

$$
\times \left(p^{20(j_3 \vee 4n_3)} \int_{(-\pi, \pi]^d} \bar{J}(l_2)^{20(j_3 \vee 4n_3)} \frac{d l_2}{(2\pi)^d} \right)^{1/20}
$$

$$
\leq (c_f^2/d)^{1/2}.
$$

We finish by proving that the integral over $l_1$ in (4.16) is bounded by a constant. Indeed,

$$
p^{-1 + \sum_{i=1}^{3} (j_i \vee 4n_i)} \int_{(-\pi, \pi]^d} |\bar{J}(l_1)|^{\sum_{i=1}^{3} (j_i \vee 4n_i)} \bar{G}_{\lambda_\nu}(l_1)^{n_1 + n_2 + n_3} \frac{d l_1}{(2\pi)^d}
$$

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This proves that (4.16) is bounded by \( (c_f/d)^{1/2} \). Note that (4.17) is very similar to (4.16), and the same bounds can be applied to get a bound of \( (c_f/d)^{1/2} \). Since the other two terms in (4.15) are handled analogously, we obtain the bound \( \tilde{H}_p(b_1, b_2; k) \leq [1 - \tilde{D}(k)]c_f/d \), which is what was claimed.

**Proof of Proposition 4.2.** Recalling the bounds on \( |\Pi_p^{(m)}(k)| \) obtained in Propositions 3.2 and 3.5, and the bounds on \( \|\Pi_p^{(m)}(k) - \Pi_p^{(m)}(0)\| \) obtained in Propositions 3.2, 3.14, and 3.13, we can combine them with the bounds just obtained in Lemmas 4.7, 4.8, 4.9, and 4.10. This gives

\[
p(\tilde{\Pi}_p^{(m)}(k)) \leq p \sum_{x \in \mathbb{Z}^d} \Pi_p^{(m)}(x) \leq c_f(c_f/d)^{n\nu_1},
\]

(4.19)

\[
p(\tilde{\Pi}_p^{(m)}(k) - \tilde{\Pi}_p^{(m)}(0)) = p \sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)]\Pi_p^{(m)}(x) \leq c_f(1 - \tilde{D}(k))(c_f/d)^{1\nu(m-2)}.
\]

We recognize the geometric series in the above bounds. The series converges for sufficiently large \( d \). If \( f \leq 4 \) on \([0, p_c)\), we can replace \( c_f \) by \( c = c_4 \) in the above, so that the bounds are uniform in \( p \in [0, p_c) \), which means that the value of \( d \) above which the series converges is independent of \( p \). Hence,

\[
p(\tilde{\Pi}_n(k)) \leq \sum_{m=0}^{\infty} p\Pi_p^{(m)}(x) \leq c_f/d, \quad p(\tilde{\Pi}_n(k) - \tilde{\Pi}_n(0)) \leq [1 - \cos(k \cdot x)]c_f/d.
\]

The bound (4.3) follows from Corollary 3.11 by analogous arguments. Recalling the definition of the remainder term yields the straight-forward bound

\[
\sum_x R_{p,n}(x) = \sum_x \sum_u p(\Pi_p^{(n)}(u)\tau_p(x - u) - u) \leq \tilde{p}(\Pi_p^{(n)}(0)\tilde{\tau}_p(0) \leq c_f(c_f/d)^{n\tilde{\tau}_p}(0),
\]

(4.20)

applying (4.19). Hence, if \( c_f/d < 1 \), then \( \sum_x R_{p,n}(x) \to 0 \) as \( n \to \infty \). Again, if \( f \leq 4 \), we can replace \( c_f \) by \( c = c_4 \) in (4.20) and the smallness of \( (c_f/d) \) does not depend on the value of \( p \).

The existence of the limit \( \Pi_p \) follows by dominated convergence with the bound (4.3). Together with (4.20), this implies that the lace expansion identity in Proposition 2.9 converges as \( n \to \infty \) and satisfies the OZE.

**4.3 Completing the bootstrap argument**

It remains to prove that \( f \leq 4 \) on \([0, p_c)\) and so that we can apply the second part of Proposition 4.2. This is achieved by Proposition 4.12 where three claims are made: First, \( f(0) \leq 4 \), secondly, \( f \) is continuous in the subcritical regime, and thirdly, \( f \) does not take values in \([3, 4)\) on \([0, p_c)\). This implies the desired boundedness of \( f \). The following observation will be needed to prove the third part of Proposition 4.12.

**Observation 4.11.** Suppose \( a(x) = a(-x) \) for all \( x \in \mathbb{Z}^d \). Then

\[
\frac{1}{2} |\Delta_k \tilde{a}(l)| \leq |a(0)| - |\tilde{a}(k)|
\]

for all \( k, l \in (-\pi, \pi]^d \) (where \( \tilde{a} \) denotes the Fourier transform of \( |a| \)). As a consequence, \( |\tilde{D}_k(l)| \leq 1 - \tilde{D}(k) \). Moreover, there is \( d_0 \geq 6 \) a constant \( c_f = c(f(p)) \) (increasing in \( f \)) such that, for all \( d > d_0 \),

\[
|\Delta_k p\Pi_p(l)| \leq |1 - \tilde{D}(k)|c_f/d.
\]

**Proof.** The statement for general \( a \) can be found, for example, in [16] (8.2.29). For convenience, we give the proof. Setting \( a_k(x) = [1 - \cos(k \cdot x)]a(x) \), we have

\[
\frac{1}{2} |\Delta_k a(x)| = |\tilde{a}_k(l)| \leq \sum_{x \in \mathbb{Z}^d} |a(x)| \cos(l \cdot x)|1 - \cos(k \cdot x)| \leq \sum_{x \in \mathbb{Z}^d} |a(x)||1 - \cos(k \cdot x)|
\]
The consequence about $\hat{D}$ now follows from $D(x) \geq 0$ for all $x$. Moreover, the statement for $\Delta_k p \hat{H}_p(l)$ follows applying the observation to $a = \Pi_p$, together with the bounds in (4.2).

**Proposition 4.12.** The following are true:

1. The function $f$ satisfies $f(0) \leq 3$.
2. The function $f$ is continuous on $[0, p_c]$.
3. Let $d$ be sufficiently large; then assuming $f(p) \leq 4$ implies $f(p) \leq 3$ for all $p \in [0, p_c]$.

Consequently, there is some $d_0$ such that $f(p) \leq 3$ uniformly for all $p \in [0, p_c]$ and $d > d_0$.

As a remark, in the third step of Proposition 4.12 we prove the stronger statement $f_i(p) \leq 1 + \text{const} / d$ for $i \in \{1, 2\}$. In the remainder of the paper, we prove this proposition and thereby finish the proof the main theorem. We prove each of the three assertions separately.

1. **Bounds on $f(0)$**. This one is straightforward. Clearly, $f_1(0) = 0$. Further, note that $\lambda_0 = 0$ and $\hat{G}_0(k) = 1$ for all $k \in (-\pi, \pi]^d$. Since $\tau_0(x) = J(x)$, we conclude that $f_2(0) = f_3(0) = 0$, and so we actually get $f(0) = 0$.

2. **Continuity of $f$**. The continuity of $f_1$ is obvious. For the continuity of $f_2$ and $f_3$, we proceed as in [16], that is, we prove continuity on $[0, p_c - \varepsilon]$ for every $0 < \varepsilon < p_c$. This again is done by taking derivatives and bounding them uniformly in $k$ and in $p \in [0, p_c - \varepsilon]$. To this end, we calculate

$$
\frac{d}{dp} \hat{\tau}_p(k) = \frac{1}{\hat{G}_{\lambda_p}(k)^2} \left[ \hat{G}_{\lambda_p}(k) \frac{d \hat{\tau}_p(k)}{dp} - \hat{\tau}_p(k) \frac{d \hat{G}_{\lambda_p}(k)}{d \lambda} \right] \bigg|_{\lambda = \lambda_p} \frac{d \lambda_p}{dp}.
$$

(4.21)

Recall that $\lambda_p = 1 - 1/\chi(p)$, and so

$$
\frac{1}{2} \leq \frac{1}{1 - \lambda_p D} = \hat{G}_{\lambda_p}(k) \leq \hat{G}_{\lambda_p}(0) = \chi(p) \leq \chi(p_c - \varepsilon).
$$

(4.22)

Further,

$$
\hat{\tau}_p(k) \leq \hat{\tau}_p(0) = \frac{\chi(p) - 1}{p} \leq \frac{\chi(p_c - \varepsilon) - 1}{p_c - \varepsilon}.
$$

(4.23)

We use Observation 2.3 to obtain

$$
\left| \frac{d}{dp} \hat{\tau}_p(k) \right| = \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} \frac{d}{dp} \hat{\tau}_p(k) \leq \sum_{x \in \mathbb{Z}^d} \frac{d}{dp} \tau_p(x) = \frac{d}{dp} \sum_{x \in \mathbb{Z}^d} \tau_p(x) \leq \hat{\tau}_p(0)^2,
$$

and the same bound as in (4.23) applies. The exchange of sum and derivative is justified as both sums are absolutely summable. Note that $d \hat{G}_{\lambda_p}(k) / dp = \hat{D}(k) \hat{G}_{\lambda_p}(k)^2$, and this is bounded by $\chi(p_c - \varepsilon)^2$ by (4.22). Finally, by Observation 2.3

$$
\frac{d \lambda_p}{dp} = \frac{d}{dp} \frac{\chi(p)}{\chi(p)^2} \leq \hat{\tau}_p(0),
$$

which is bounded by (4.23) again. In conclusion, all terms in (4.21) are bounded uniformly in $k$ and $p$, which proves the continuity of $f_2$. We can treat $f_3$ in the exactly same manner, as it is composed of terms of the same type as the ones we just bounded.
3. The forbidden region \((3,4)\). Note that we assume \(f(p) \leq 4\) in the following, and so the second part of Proposition \([4,2]\) applies with \(c = c_4\).

Improvement of \(f_1\). Recalling the definition of \(\lambda_p \in [0,1]\), this implies
\[
f_1(p) = \lambda_p - p\hat{\Pi}_p(0) \leq 1 + c_4/d.
\]

Improvement of \(f_2\). We introduce \(a = p(J + \Pi_p)\), and moreover
\[
\hat{N}(k) = \frac{\hat{a}(k)}{1 + p\hat{\Pi}_p(0)}, \quad \hat{F}(k) = \frac{1 - \hat{a}(k)}{1 + p\hat{\Pi}_p(0)}.
\]

By adapting the analogous argument from \([17]\) proof of Theorem 1.1, we can show that \(1 - \hat{a}(k) > 0\). Therefore, under the assumption that \(f(p) \leq 4\), we have \(p\hat{\tau}_p(k) = \hat{N}(k)\hat{F}(k) = \hat{a}(k)/(1 - \hat{a}(k))\), and furthermore \(\lambda_p = \hat{a}(0)\). In the following lines, \(M\) and \(M'\) denote constants (typically multiples of \(c_4\)) whose value may change from line to line. An important observation is that
\[
\frac{1}{1 + p\hat{\Pi}_p(0)} \leq 1 + M/d, \quad |\hat{a}(k)| \leq 1 + M/d, \quad |p\hat{\Pi}_p(0)| \leq M/d.
\]

We are now ready to treat \(f_2\) and observe that
\[
\left|\frac{p\hat{\tau}_p(k)}{G_{\lambda_p}(k)}\right| = \left|\hat{N}(k) + p\hat{\tau}_p(k)[1 - \lambda_p\hat{D}(k) - \hat{F}(k)]\right|
\leq 1 + M/d + |p\hat{\tau}_p(k)|\left|1 - \lambda_p\hat{D}(k) - \hat{F}(k)\right|,
\]
and
\[
|1 - \lambda_p\hat{D}(k) - \hat{F}(k)| = \left|\frac{1 + p\hat{\Pi}_p(0) - (2dp + p\hat{\Pi}_p(0))(1 + p\hat{\Pi}_p(0))\hat{D}(k) - 1 + 2dp\hat{D}(k) + p\hat{\Pi}_p(k)}{1 + p\hat{\Pi}_p(0)}\right|
= \left|\frac{1 - \hat{D}(k)\hat{\Pi}_p(0)}{1 + p\hat{\Pi}_p(0)}\right| + \left|\hat{a}(0)p\hat{\Pi}_p(0)\hat{D}(k) + p\hat{\Pi}_p(k)\right|.
\]

The first term in the right-hand side of \((4.25)\) is bounded by \(1 - \hat{D}(k)/M/d\). In the second term, recalling that \(\hat{a}(0) = \lambda_p\), we add and subtract \(p\hat{\Pi}_p(0)\) in the numerator, and so
\[
|1 - \lambda_p\hat{D}(k) - \hat{F}(k)| \leq |1 - \hat{D}(k)/M/d + \left|1 - \lambda_p\hat{D}(k)p\hat{\Pi}_p(0) + p(\hat{\Pi}_p(k) - \hat{\Pi}_p(0))\right|/1 + p\hat{\Pi}_p(0)\]
\leq |1 - \hat{D}(k)/M'/d + [1 - \lambda_p\hat{D}(k)/M']/d| \leq 3[1 - \lambda_p\hat{D}(k)/M']/d
\]
for some constant \(M'\). In the second to last bound, we used that \(p\hat{\Pi}_p(0) - p\hat{\Pi}_p(k) \leq |1 - \hat{D}(k)/M'|. For the last, we used that \(1 - \hat{D}(k) \leq 2(1 - \lambda_p\hat{D}(k)) = 2\hat{G}_{\lambda_p}(k)^{-1}\). Putting this back into \((4.24)\), we obtain
\[
\left|\frac{p\hat{\tau}_p(k)}{G_{\lambda_p}(k)}\right| \leq 1 + M/d + |p\hat{\tau}_p(k)|/\hat{G}_{\lambda_p}(k)/M/d \leq 1 + 4M/d.
\]

This concludes the improvement on \(f_2\). Before dealing with \(f_3\), we make an important observation:

**Observation 4.13.** Given the improved bounds on \(f_1\) and \(f_2\),
\[
\sup_{k \in (-\pi,\pi)} \left|\frac{1 - \lambda_p\hat{D}(k)}{1 - \hat{a}(k)}\right| \leq 3.
\]

**Proof.** Consider first those \(p\) such that \(2dp \leq 3/7\). Then
\[
\left|\frac{1 - \lambda_p\hat{D}(k)}{1 - \hat{a}(k)}\right| = \left|\frac{1 - 2dp\hat{D}(k) - p\hat{\Pi}_p(0)\hat{D}(k)}{1 - 2dp\hat{D}(k) - p\hat{\Pi}_p(k)}\right| \leq \frac{\frac{10}{7} + M/d}{\frac{7}{2} + M/d} \leq \frac{(1 + M'/d)^2}{2} \leq 3
\]
for $d$ sufficiently large. Next, consider those $k \in (-\pi, \pi]^d$ such that $|\tilde{D}(k)| \leq 7/8$. Then

$$\frac{1 - \lambda_p \tilde{D}(k)}{1 - \tilde{a}(k)} = 1 - \frac{p \tilde{A}_p(0) \tilde{D}(k) - p \tilde{A}_p(k)}{1 - \tilde{a}(k)} \leq 1 + \frac{2M/d}{1 - (1 + M/d)^\frac{d}{2} - M/d} \leq 1 + 16M'/d$$

for $d$ sufficiently large. Let now $p$ such that $2dp > 3/7$ and $k$ such that $|\tilde{D}(k)| > 7/8$. Then

$$\frac{1 - \lambda_p \tilde{D}(k)}{1 - \tilde{a}(k)} \leq \frac{8 \tilde{a}(k)}{G_{\lambda_p}(k)} \left( \frac{2dp \tilde{D}(k)}{|\tilde{a}(k)|} \right) \leq \frac{8}{3} \frac{1 + M/d}{1 + \frac{M/d}{\frac{d}{2} - M/d}} \leq 3$$

for $d$ sufficiently large.

Improvement of $f_{3, l}$. Elementary calculations give

$$\Delta_k \tilde{p}_l = \frac{\Delta_k \tilde{a}(l)}{1 - \tilde{a}(l)} + \sum_{\sigma \in \pm \{1\}} \frac{(\tilde{a}(l + \sigma k) - \tilde{a}(l))^2}{(1 - \tilde{a}(l))(1 - \tilde{a}(l + \sigma k))} + \tilde{a}(l) \Delta_k \left( \frac{1}{1 - \tilde{a}(l)} \right).$$

We bound each of the three terms (I)-(III) separately. For the first term,

$$||l|| = \left| \Delta_k \tilde{a}(l) \right| \left( \frac{1 - \lambda_p \tilde{D}(l)}{1 - \tilde{a}(l)} \right) \leq 3 \tilde{G}_{\lambda_p}(l) |2dp \tilde{D}(l) + \Delta_k p \tilde{A}_p(l)|$$

$$\leq (3 + M/d)|1 - \tilde{D}(k)| \tilde{G}_{\lambda_p}(l) \tilde{G}_{\lambda_p}(l + k).$$

In the above, we used first Observation 4.13, then Observation 4.11, and finally the fact that $\tilde{G}_{\lambda_p}(l + k) \geq 1/2$. Note that if we obtained similar bounds on (II) and (III), we could prove a bound of the form $|\Delta_k \tilde{p}_l| \leq c \tilde{U}_{\lambda_p}(k, l)$ for the right constant $c$ and the improvement of $f_3$ would be complete.

To deal with (II), we need a bound on $\tilde{a}(l + \sigma k) - \tilde{a}(l)$ for $\sigma \in \{\pm 1\}$. The following calculations can be found in [16] (8.4.19)-(8.4.21), and so we skip some details. We have

$$|\tilde{D}(l \pm k) - \tilde{D}(l)| \leq \sum_k \left( \sin(k \cdot x) |D(x) + [1 - \cos(k \cdot x)]D(x) \right) = 1 - \tilde{D}(k) + \sum_k |\sin(k \cdot x)|D(x)$$

$$\leq 1 - \tilde{D}(k) + \left( \sum_k D(x) \right)^{1/2} \left( \sum_k |\sin(k \cdot x)|^2 D(x) \right)^{1/2}$$

$$\leq 1 - \tilde{D}(k) + 2 \left( \sum_k |1 - \cos(k \cdot x)|D(x) \right)^{1/2}$$

$$\leq 4|1 - \tilde{D}(k)|^{1/2},$$

and similarly

$$p|\tilde{A}_p(l \pm k) - \tilde{A}_p(l)| \leq \left( p \sum_x |\Pi_p(x)| \right)^{1/2} \left( 2p \sum_x |1 - \cos(k \cdot x)||\Pi_p(x)| \right)^{1/2} + p \sum_x |1 - \cos(k \cdot x)||\Pi_p(x)|$$

$$\leq M |1 - \tilde{D}(k)|^{1/2}/d.$$

Putting this together yields

$$|\tilde{a}(l \pm k) - \tilde{a}(l)| \leq 2dp|\tilde{D}(l \pm k) - \tilde{D}(l)| + p|\tilde{A}_p(l \pm k) - \tilde{A}_p(l)| \leq (16 + M/d)|1 - \tilde{D}(k)|^{1/2}. \quad (4.26)$$

Combining (4.26) with Observation 4.13 yields

$$(II) \leq (16 + M/d)^2 |1 - \tilde{D}(k)| \left| \frac{1 - \lambda_p \tilde{D}(l)}{1 - \tilde{a}(l)} \right| \left| \frac{1 - \lambda_p \tilde{D}(l + \sigma k)}{1 - \tilde{a}(l + \sigma k)} \right| \tilde{G}_{\lambda_p}(l) \tilde{G}_{\lambda_p}(l + \sigma k)$$

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To bound (III), we want to use Lemma 4.1. We first provide bounds for the three types of quantities arising in the use of the lemma. First, note that $|\hat{a}(l)| \leq 4 + M/d$. Next, we observe

$$|\hat{a}(0) - \hat{a}(k)| = \sum x(1 - \cos(k \cdot x)) |2dpD(x) + p\Pi_p(x)| \leq (4 + M/d)[1 - \hat{D}(k)].$$

The third ingredient we need is Observation 4.13, which produces $|1 - \hat{a}(l)|^{-1} \leq 3\hat{G}_{\lambda_p}(l)$. Putting all this together and applying Lemma 4.1 gives

$$\Delta_k \left( \frac{1}{1 - \hat{a}(l)} \right) \leq (4 + M/d)[1 - \hat{D}(k)] \left( 9 \hat{G}_{\lambda_p}(l - k) + \hat{G}_{\lambda_p}(l + k) \right) \hat{G}_{\lambda_p}(l)
+ 216(4 + M/d)\hat{G}_{\lambda_p}(l - k)\hat{G}_{\lambda_p}(l + k)\hat{G}_{\lambda_p}(l)[1 - \hat{D}(l)]
\leq (6912 + M/d)[1 - \hat{D}(k)] \left( \hat{G}_{\lambda_p}(l - k) + \hat{G}_{\lambda_p}(l + k) \hat{G}_{\lambda_p}(l) \hat{G}_{\lambda_p}(l - k) \hat{G}_{\lambda_p}(l + k) \right),$$

noting that $\hat{G}_{\lambda_p}(l)[1 - \hat{D}(l)] \leq 2$. In summary, (I) + (II) + (III) \leq 3\hat{G}_{\lambda_p}(k,l), which finishes the improvement on $f_3$. This finishes the proof of Proposition 4.12 and therewith also the proof of Theorem 1.1.

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