A new non-Abelian topological superfluid of cold Fermi gases in anisotropic and spin-dependent optical lattices

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Abstract

To realize a non-Abelian s-wave topological superfluid of cold Fermi gases, generally a Zeeman energy splitting larger than the superfluid pairing gap is necessary. In this paper, we find that using an anisotropic and spin-dependent optical lattice to trap gases, a new non-Abelian topological superfluid phase appears, in contrast to an isotropic and spin-independent optical lattice. A characteristic of this new non-Abelian topological superfluid is that the Zeeman energy splitting can be smaller than the superfluid pairing gap. By self-consistently solving the gap equation and considering the competition against the normal state and phase separation, this new phase is also stable. We also investigate edge states and the effects of a harmonic trap potential.

(Some figures may appear in colour only in the online journal)

1. Introduction

In two-dimensional condensed matter physics, the search for non-Abelian Majorana fermions (MFs) \cite{1} attracts much attention to topological superfluids and superconductors (TS) \cite{2,3,4}. Inspired by the fact that MFs exist at the vortex cores of a two-dimensional (2D) $p_x + ip_y$ superconductor \cite{5}, theoretically some practical systems have been proposed, such as the fractional quantum Hall effect at filling $\nu = 5/2$ \cite{5}, superfluid He-3 \cite{6}, non-centrosymmetric superconductors \cite{7,8}, the surface of a three-dimensional topological insulator in proximity to an s-wave superconductor \cite{9,10} and a spin–orbit-coupled (SOC) semiconductor quantum well coupled to an s-wave superconductor \cite{11}. In particular, the latter has been improved by applying an in-plane magnetic field to a (1 1 0)-grown semiconductor \cite{9,10} and a spin–orbit-coupled (SOC) semiconductor quantum well coupled to an s-wave superconductor. This improvement not only enhances the ability to tune into the TS, but also avoids the orbital effect of the external magnetic field \cite{12}.

It is widely known that cold Fermi gases can be used to simulate many other systems owing to their many controllable advantages and operabilities. Certainly the simulations to TS are also possible. To the best of our knowledge, three main ways have been suggested. The first was direct and based on the p-wave superfluid of degenerate Fermi gases by means of p-wave Feshbach resonance \cite{13}. Although this method is very simple, it is challenging due to the short lifetimes of the p-wave pairs. Subsequently, Zhang \textit{et al} \cite{14} proposed to create an effective $p_x + ip_y$ TS from an s-wave interaction making use of an artificially generated SOC. In fact, SOC have been realized in a neutral atomic Bose–Einstein condensate (BEC) and the same technique is also feasible for cold Fermi gases \cite{15,16}. Realizing that in a dual transformation SOC is formally equivalent to a p-wave superfluid gap, Sato \textit{et al} \cite{17} suggested to artificially generate the vortices of SOC by using lasers carrying orbital angular momentum. In terms of the latter two ansatzs, in order to enter into non-Abelian TS, a Zeeman energy splitting (ZES) larger than the superfluid pairing gap is essentially needed.

In this paper, we investigate the effects of an anisotropic and spin-dependent optical lattice (ASDOL) on non-Abelian TS in cold Fermi gases with Rashba SOC. In contrast to a spin-independent optical lattice \cite{17,18}, our model realizes another
new non-Abelian TS phase in which ZES can be smaller than the superfluid pairing gap. Experimentally such a system is also definitely feasible. On the one hand, two requirements for optical lattices are both accessible. The anisotropy of an optical lattice is determined by the intensity of the laser beams in different directions, while spin-dependence is also available in light of the fact that the strength of the optical potential crucially depends on the atomic dipole moment between the internal states involved [19–22]. On the other hand, Rashba SOC can be generated in the same way as that mentioned in [17].

The organization of this paper is as follows. In section 2, we first give our model and analyze the condition of gap closing. Then we calculate the TKNN number $I_{TKNN}$ [23] of occupied bands addressing the topological properties of the model to obtain the topological phase diagram. A new non-Abelian TS is discovered. In addition, we also investigate the properties of edge states to check bulk-boundary correspondence. In section 3, by self-consistently determining the s-wave pairing gap and considering the competition against the normal state and phase separation, the stability of TS and the effects of a harmonic trap potential are discussed. We find that this new non-Abelian TS is stable. A brief conclusion is given in section 4.

2. Hamiltonian and topological phase diagram

In this paper, we consider the s-wave superfluid of cold Fermi gases with Rashba SOC in a two-dimensional square optical lattice which is anisotropic and spin-dependent. For convenience, we regard the lattice constant as the length unit. The Hamiltonian suggested is

$$H = \sum_{k\sigma} \varepsilon_{k\sigma} a_{k\sigma}^\dagger a_{k\sigma} + \sum_k \left[ J_ka_{k\uparrow}^\dagger a_{k\downarrow} - \Delta a_{k\downarrow}^\dagger a_{k\uparrow}^\dagger \right] + \text{H.C.},$$

(1)

where $a_{k\sigma}$ creates a spin $\sigma = \uparrow, \downarrow$ fermion at momentum $k = (k_x, k_y)$. $J_k = 2J \sin (k_x) + i \sin (k_y)$ with $J (J > 0)$ denoting the strength of Rashba SOC, $\mu$, $\Gamma$ and $\Delta$ are the chemical potential, ZES and the s-wave superfluid pairing gap, respectively.

The kinetic energy terms $\varepsilon_{k\sigma}$ come from ASDOLs [24]. Imagine tuning the intensities of lasers so that one spin state prefers to hop along the $x$-axis and the other prefers to hop along the $y$-axis. In this paper, we concentrate on a specific situation where the hopping amplitudes of the two spin states are rotated by 90° with respect to one another, and only consider a near-neighbour hopping Hamiltonian with single particle dispersions:

$$\varepsilon_{k\downarrow} = -2t_0 \cos (k_x) - 2t_0 \cos (k_y),$$

$$\varepsilon_{k\uparrow} = -2t_0 \cos (k_x) - 2t_0 \cos (k_y).$$

(2)

When there are no Zeeman magnetic field and SOC $\Gamma = J = 0$, this model realized a stable paired superfluid state with gapless unpaired fermions [24], similar to the Sarma state in polarized mixtures [25–27]. Moreover in the strong coupling limit, a d-wave pairing superfluid as well as a d-wave density wave state, are also proposed in this system [28].

Generally speaking, the closeness of the bulk gap is a signal of topological phase transitions. Thus, to obtain the topological phase diagram of Hamiltonian (1), we first calculate the bulk spectrum of the system to search parameter regions for which different topological phases are possible. Then for every region we calculate topological invariants to label topological properties. To obtain the bulk spectrum, we rewrite the Hamiltonian as

$$H = \frac{1}{2} \sum_k \psi_k^\dagger M_{4\times4} \psi_k,$$

(3)

where $\psi_k^\dagger = (a_{k\uparrow}^\dagger, a_{k\downarrow}^\dagger, a_{-k\downarrow}^\dagger, a_{-k\uparrow}^\dagger)$ is a row vector and $M_{4\times4}$ is a matrix with $M_{11} = -M_{33} = \epsilon_{k\uparrow} - \mu - \Gamma$, $M_{22} = -M_{44} = \epsilon_{k\downarrow} - \mu + \Gamma$, $M_{12} = M_{43} = M_{24} = M_{32} = -M_{14} = -M_{41} = \Delta$, $M_{13} = M_{31} = M_{24} = M_{42} = 0$.

Diagonalizing the matrix $M_{4\times4}$, we find the energy spectrum

$$E_k^\pm = \sqrt{\xi_k^2 + \xi_k^2} + |J_k|^2 + \Delta^2 \pm 2E_0$$

with $\xi_{k\uparrow} = -(t_0 + t_0) \cos (k_x) + \cos (k_y) - \mu$, $\xi_{k\downarrow} = -(t_0 + t_0) \cos (k_x) - \cos (k_y) - \mu - \Gamma$ and $E_0 = \sqrt{\xi_k^2 + \xi_k^2} + |J_k|^2 + \Delta^2$.

The closeness of the bulk energy gap is possible only if $E_k^- = 0$, in other words

$$\xi_{k\downarrow}^2 + \xi_{k\uparrow}^2 + |J_k|^2 + \Delta^2 = 2E_0,$$

(5)

which is equivalent to

$$\xi_{k\downarrow}^2 - \xi_{k\uparrow}^2 - |J_k|^2 + \Delta^2 = 0, \quad |J_k|^2 \Delta^2 = 0.$$  

(6)

For the s-wave pairing, $\Delta \neq 0$ and the second equation in (6) is satisfied only when $k = (0, 0), (\pi, 0), (0, \pi), (\pi, \pi)$. Substituting these values into the first equation in (6), four different gap closing conditions are obtained:

$$\mu^2 + \Delta^2 = [2(t_0 - t_0) + \Gamma^2],$$

$$\mu^2 + \Delta^2 = [2(t_0 - t_0) - \Gamma^2],$$

$$\Gamma^2 = \Delta^2 + [2(t_0 + t_0) + \mu]^2,$$

$$\Gamma^2 = \Delta^2 + [2(t_0 + t_0) - \mu]^2.$$ 

(7)

By means of these conditions, figure 1(a) shows that there are at least 12 regions, which may be topologically distinct. Below we will explore the topological numbers to classify the topological phases of the model (1).

In terms of our system, it explicitly breaks the time-reversal symmetry even though the Zeeman magnetic field is zero. Thus the TKNN number $I_{TKNN}$ plays a central role in the topological nature of the system [23], consistent with the conclusion in the periodic table of TS that a BdG Hamiltonian only with particle-hole symmetry in two dimensions is classified by an integer group $Z$ [29]. TKNN numbers of 12 regions have been numerically calculated [30] and given in figure 1(a). A total of five non-Abelian TS phases and one Abelian TS phase are found. In order to make a contrast, in figure 1(b), we also show the topological phase diagram of an isotropic and spin-independent optical lattice and there are four
non-Abelian TS and one Abelian TS phases. By comparison, the effects of ASDOL are mainly twofold. On the one hand, it creates another non-Abelian TS phase around the chemical potential \( \mu = 0 \), which can be realized for ZES \( \Gamma \) less than the superfluid pairing gap \( \Delta \). This is the main result in this paper and can be understood from the fact that an ASDOL \( (\Delta_0 \neq \Delta_1) \) supplies an effective ZES \( \pm 2(\Delta_0 - \Delta_1) \), as seen from (7). On the other hand, in the direction of the increasing Zeeman magnetic field it separates two successive non-Abelian TS phases in figure 1(b).

According to the bulk-boundary correspondence, a topologically nontrivial bulk guarantees the existence of topologically stable gapless edge states on the boundary. Cold Fermi gases with sharp edges may be realized along the lines proposed in [31]. In our case, the gapless edge state is a chiral Majorana fermion mode. As a matter of fact, the core of a vortex is topologically equivalent to an edge which has been closed on itself. The topologically protected edge modes we describe here are therefore equivalent to the MFs known to exist in the core of the p-wave superfluid vortices [18].

To study edge states, we transform Hamiltonian (1) into the lattice representation

\[
H = -t_x \sum_i \left[ a_i^\dagger a_{i+\tau x} + a_i^\dagger a_{i+\tau y} + \text{H.C.} \right] \\
-t_y \sum_i \left[ a_i^\dagger a_{i+\tau y} + a_i^\dagger a_{i+\tau y} + \text{H.C.} \right] \\
+ J \sum_i \left[ a_i^\dagger a_{i+\tau x} - a_i^\dagger a_{i+\tau y} \right] \\
- i(\sigma_i^x a_{i+\tau y} + \sigma_i^y a_{i+\tau x}) + \text{H.C.} \\
- \sum_{\sigma \omega} (\mu + \sigma \Gamma) a_{\sigma \omega} a_{\sigma \omega} = \Delta \sum_i (a_i^\dagger a_i^\dagger + \text{H.C.}) 
\]

(8)

where \( a_i^\dagger \) is the creation operator of a fermion with spin \( \sigma \) at the lattice site \( i = (i_x, i_y) \). Without loss of generality, we suppose that the system has two open boundaries in the \( x \)-direction and is periodic in the \( y \)-direction. After performing a Fourier transformation along the \( y \)-direction only, we numerically diagonalize Hamiltonian (8) and correspondingly obtain the excitation spectrum \( E_{n, k_y} \) with the subscript \( n \) labelling different energy bands.

In figure 2, we show the energy spectrum for an ASIOL with edges at \( i_x = 0 \) and \( i_y = 30 \). These 12 figures correspond to 12 regions in figure 1(a), respectively. Figures 2(b), (d), (f), (h) and (j) are for non-Abelian TS phases, (k) for the Abelian TS phase and the other non-TS phases. As stated in [18, 32], in the non-Abelian TS phases, where \( I_{\text{TKNN}} \pm \Gamma \) and \( (-1)^{I_{\text{TKNN}}} \pm \Gamma \), a single pair of gapless edge modes appears. Since on a given boundary there is no state available for backward spin-conserving scattering, these states are topologically protected [18]. In this case, the edge zero mode is a chiral MF. While in the Abelian TS phase \( I_{\text{TKNN}} = \pm \Gamma \) and \( (-1)^{I_{\text{TKNN}}} \pm \Gamma \), the system contains either zero or two pairs of edge states. These edge states are topologically trivial since either of the edge modes can perform spin-conserving backscattering. In addition, it is also found that edge excitations only cross zero energy with a linear dispersion at \( k_y = 0 \) or \( \pi \). Their origin is topological and can be identified by a topological winding number \( I_k \) defined only for \( k_y = 0 \) or \( \pi \) [32]: when \( I_k \) is non-zero for \( k_y = 0 \) or \( \pi \), the energy of the gapless edge mode becomes zero at this value of \( k_y \).

3. Phase diagram with the self-consistent pairing gap

In section 2, we have determined the topological phase diagram for a fixed s-wave superfluid pairing gap. However, for a realistic physical system the pairing gap generally varies when the chemical potential and Zeeman magnetic field change. In particular, the Zeeman magnetic field breaks time-reversal symmetry and weakens the stability of superfluidity. A well-known example is the so-called Chandrasekar–Clogston (CC) limit [33, 34] in superconducting systems without SOC. Hence, it is possible that not all phases in figure 1(a) are accessible. In this section in the frame of BCS mean-field theory and following [18], we self-consistently determine the s-wave superfluid pairing gap and consider the competition from the normal phase and phase separation to investigate the stability of TS.
Figure 2. The energy spectra of an anisotropic and spin-dependent optical lattice with open edges at $\varphi = 0$ and $\varphi = 30$. We have chosen $t_b / t_a = 0.35$, $J / J_a = 0.5$ and $\Delta / t_a = 0.8$. From figures 2(a)–(l), there are one-to-one correspondences with regions labeled by corresponding letters in figure 1(a). (a) $\mu / t_a = -4.0$, $\Gamma / t_a = 1.0$, (b) $\mu / t_a = -4.0$, $\Gamma / t_a = 2.5$, (c) $\mu / t_a = -4.0$, $\Gamma / t_a = 4.0$, (d) $\mu / t_a = -4.0$, $\Gamma / t_a = 6.5$, (e) $\mu / t_a = 0.0$, $\Gamma / t_a = 6.5$, (f) $\mu / t_a = 4.0$, $\Gamma / t_a = 0.0$, (g) $\mu / t_a = 4.0$, $\Gamma / t_a = 6.0$, (h) $\mu / t_a = 4.0$, $\Gamma / t_a = 4.0$, (i) $\mu / t_a = 4.0$, $\Gamma / t_a = 2.0$, (j) $\mu / t_a = 4.0$, $\Gamma / t_a = 4.0$, (k) $\mu / t_a = 0.0$, $\Gamma / t_a = 1.0$, (l) $\mu / t_a = 0.0$, $\Gamma / t_a = 2.5$. 

Let $-U$ ($U > 0$) denote the effective attraction strength between fermions, then the pairing gap $\Delta = U \sum_k < a_{k \downarrow} \ldots a_{k \uparrow} >$ can be obtained from the minimization of the thermodynamic potential $\Omega_s = \sum_k \left[ \xi_{k \uparrow} - \frac{1}{2}(E_{k \uparrow} + E_{k \downarrow}) \right] + N \Delta^2 / U$. The instability of superfluidity against phase separation is signalled by the condition $\Delta \neq 0$ and $\frac{\partial^2 \Omega_s}{\partial \Delta^2} < 0$, while instability against the normal state is $\Omega_n < \Omega_s$, where $\Omega_n$ is the thermodynamic potential of the normal state. From the self-consistent calculation, we find that for the parameters chosen superfluidity is stable against the normal state and phase separation, i.e. $\Omega_n > \Omega_s$ and $\frac{\partial^2 \Omega_s}{\partial \Delta^2} > 0$. The resulting zero-temperature pairing gap and phase diagram are shown in figure 3. Figure 3(b) shows structures similar to those in figure 1(a) except that TS with a large Zeeman magnetic field is not available. Comparing figure 3(a) with (b), it is easily seen that for the small chemical
Figure 3. The $s$-wave superfluid pairing gap (a) and topological phase diagram (b) from self-consistent mean-field solution. In (b), grey, black, blue and purple colours correspond to TKNN numbers 0, $-1$, 1, $-2$, respectively. We have chosen $t_b/t_a = 0.35$, $J/t_a = 0.5$ and $U/t_a = 4.0$.

Figure 4. The space distributions of the pairing gap (square), particle number (triangle) and TKNN number. We have chosen $t_b/t_a = 0.35$, $J/t_a = 0.5$, $\mu/t_a = 3$, $\Delta_1/t_a = 0.01$, $\Gamma/t_a = 1.2$ and $U/t_a = 4.0$.

4. Conclusions
In conclusion, we have investigated the effects of an anisotropic and spin-dependent optical lattice on the non-Abelian topological superfluid and found that a new non-Abelian topological superfluid phase exists steadily, in contrast to an isotropic and spin-independent optical lattice. Moreover, in this new non-Abelian topological superfluid phase, Zeeman energy splitting can be smaller than the superfluid pairing gap. In addition, we also calculated chiral Majorana edge states and investigated the effects of a harmonic trap potential.

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References

[1] Majorana E 1937 *Nuovo Cimento* 5 171
[2] Nayak C, Simon S H, Stern A, Freedman M and Das Sarma S 2008 *Rev. Mod. Phys.* 80 1083
[3] Qi X-L and Zhang S-C 2011 *Rev. Mod. Phys.* 83 1057
[4] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* 82 3045
[5] Read N and Green D 2000 *Phys. Rev. B* 61 10267
[6] Tsutsumi Y, Kawakami T, Mizushima T, Ichioka M and Machida K 2008 *Phys. Rev. Lett.* 101 135302
[7] Sato M and Fujimoto S 2009 *Phys. Rev. B* 79 094504
[8] Lee P A 2009 arXiv:0907.2681
[9] Fu L and Kane C L 2008 *Phys. Rev. Lett.* 100 096407
[10] Linder J, Tanaka Y, Yokoyama T, Sudbo A and Nagaosa N 2010 *Phys. Rev. Lett.* 104 067001
[11] Sato M, Takahashi Y and Fujimoto S 2009 *Phys. Rev. Lett.* 104 040402
[12] Alicea J 2010 *Phys. Rev. B* 81 125318
[13] Gurarie V, Radzihovsky L and Andreev A V 2005 *Phys. Rev. Lett.* 94 230403
[14] Zhang C, Tewari S, Lutchyn R M and Das Sarma S 2008 *Phys. Rev. Lett.* 101 160401
[15] Wang P, Yu Z, Fu Z, Miao J, Huang L, Chai S, Zhai H and Zhang J 2012 *Phys. Rev. Lett.* 109 095301
[16] Cheuk L W, Sommer A T, Hadzibabic Z, Yeşiltaş T, Bakr W S and Zwierlein M W 2012 *Phys. Rev. Lett.* 109 095302
[17] Sato M, Takahashi Y and Fujimoto S 2009 *Phys. Rev. Lett.* 103 020401
[18] Kubasiak A, Massignan P and Lewenstein M 2010 *Europhys. Lett.* 92 46004
[19] Mandel O, Greiner M, Widera A, Rom T, Hänsch T W and Bloch I 2003 *Nature* 425 937
[20] Mandel O, Greiner M, Widera A, Rom T, Hänsch T W and Bloch I 2003 *Phys. Rev. Lett.* 91 010407
[21] Liu W Y, Wilczek F and Zoller P 2004 *Phys. Rev. A* 70 033603
[22] Jaksch D and Zoller P 2005 *Ann. Phys.* 315 52
[23] Thouless D J, Kohmoto M, Nightingale M P and Nijs M 1982 *Phys. Rev. Lett.* 49 405
[24] Feiguin A E and Fisher M P A 2009 *Phys. Rev. Lett.* 103 025303
[25] Sarma G 1963 *J. Phys. Chem. Solids* 24 1029
[26] Liu W Y and Wilczek F 2003 *Phys. Rev. Lett.* 90 047002
[27] Forbes M M, Gubankova E, Liu W Y and Wilczek F 2005 *Phys. Rev. Lett.* 94 017001
[28] Cai Z, Wang L, Li J, Chen S, Xie X C and Wang Y 2009 arXiv:0910.0508
[29] Schnyder A P, Ryu S, Furusaki A and Ludwig A W W 2008 *Phys. Rev. B* 78 195125
[30] Fukui T, Hatsugai Y and Suzuki H 2005 *J. Phys. Soc. Japan* 74 1674
[31] Goldman N, Satija I, Nikolic P, Bermudez A, Martin-Delgado M A, Lewenstein M and Spielman I B 2010 *Phys. Rev. Lett.* 105 255302
[32] Sato M, Takahashi Y and Fujimoto S 2010 *Phys. Rev. B* 82 134521
[33] Clogston A M 1962 *Phys. Rev. Lett.* 9 266
[34] Chandrasekhar B S 1962 *Appl. Phys. Lett.* 1 7