Filling Area Conjecture, Optimal Systolic Inequalities, and the Fiber Class in Abelian Covers

Mikhail G. Katz* and Christine Lescop†

Dedicated to the memory of Robert Brooks, a colleague and friend.

Abstract. We investigate the filling area conjecture, optimal systolic inequalities, and the related problem of the nonvanishing of certain linking numbers in 3-manifolds.

Contents

1. Filling radius and systole 2
2. Inequalities of C. Loewner and P. Pu 4
3. Filling area conjecture 5
4. Lattices and successive minima 6
5. Precise calculation of filling invariants 8
6. Stable and conformal systoles 11
7. Gromov’s optimal inequality for n-tori 11
8. Inequalities combining dimension and codimension 1 13
9. Inequalities combining varieties of 1-systoles 15
10. Fiber class, linking number, and the generalized Casson invariant λ 17

2000 Mathematics Subject Classification. Primary 53C23; Secondary 57N65, 57M27, 52C07.

Key words and phrases. Abel-Jacobi map, Casson invariant, filling radius, filling area conjecture, Hermite constant, linking number, Loewner inequality, Pu’s inequality, stable norm, systole.

*Supported by the Israel Science Foundation (grants no. 620/00-10.0 and 84/03).

†Supported by CNRS (UMR 5582).
1. Filling radius and systole

The filling radius of a simple loop $C$ in the plane is defined as the largest radius, $R > 0$, of a circle that fits inside $C$, see Figure 1.1:

$$\text{FillRad}(C \subset \mathbb{R}^2) = R.$$ 

![Figure 1.1. Largest inscribed circle has radius R](image)

There is a kind of a dual point of view that allows one to generalize this notion in an extremely fruitful way, as shown in [14]. Namely, we consider the $\epsilon$-neighborhoods of the loop $C$, denoted $U_\epsilon C \subset \mathbb{R}^2$, see Figure 1.2.

As $\epsilon > 0$ increases, the $\epsilon$-neighborhood $U_\epsilon C$ swallows up more and more of the interior of the loop. The “last” point to be swallowed up is precisely the center of a largest inscribed circle. Therefore we can reformulate the above definition by setting

$$\text{FillRad}(C \subset \mathbb{R}^2) = \inf \{ \epsilon > 0 \mid \text{loop } C \text{ contracts to a point in } U_\epsilon C \}.$$ 

Given a compact manifold $X$ imbedded in, say, Euclidean space $E$, we could define the filling radius \textit{relative} to the imbedding, by minimizing the size of the neighborhood $U_\epsilon X \subset E$ in which $X$ could be
homotoped to something smaller dimensional, e.g., to a lower dimensional polyhedron. Technically it is more convenient to work with a homological definition.

Denote by $A$ the coefficient ring $\mathbb{Z}$ or $\mathbb{Z}_2$, depending on whether or not $X$ is oriented. Then the fundamental class, denoted $[X]$, of a compact $n$-dimensional manifold $X$, is a generator of $H_n(X; A) = A$, and we set

$$\text{FillRad}(X \subset E) = \inf \left\{ \epsilon > 0 \mid \iota_{\epsilon}([X]) = 0 \in H_n(U_{\epsilon}X) \right\}, \quad (1.1)$$

where $\iota_{\epsilon}$ is the inclusion homomorphism.

To define an “absolute” filling radius in a situation where $X$ is equipped with a Riemannian metric $g$, M. Gromov proceeds as follows, cf. (1.3). One exploits an imbedding due to C. Kuratowski [31] (while the spelling of the author’s first name used in [31] starts with a “C”, the correct spelling seems to be Kazimierz). One imbeds $X$ in the Banach space $L^\infty(X)$ of bounded Borel functions on $X$, equipped with the sup norm $\| \|$. Namely, we map a point $x \in X$ to the function $f_x \in L^\infty(X)$ defined by the formula $f_x(y) = d(x, y)$ for all $y \in X$, where $d$ is the distance function defined by the metric. By the triangle inequality we have

$$d(x, y) = \| f_x - f_y \|, \quad (1.2)$$

and therefore the imbedding is strongly isometric, in the precise sense of formula (1.2) which says that internal distance and ambient distance coincide. Such a strongly isometric imbedding is impossible if the ambient space is a Hilbert space, even when $X$ is the Riemannian circle (the distance between opposite points must be $\pi$, not $2\pi$). We then
set $E = L^\infty(X)$ in formula (1.1) and define

$$\text{FillRad}(X) = \text{FillRad} (X \subset L^\infty(X)).$$

(1.3)

Exact calculation of the filling radius is discussed in Section 5. The related invariant called filling volume is defined in Section 3. For the relation between filling radius and systole, see (5.1). The defining text for this material is [16], with more details in [14, 15]. See also the recent survey [11].

The systole, $\text{sys} \pi_1(g)$, of a compact non simply connected Riemannian manifold $(X, g)$ is the least length of a noncontractible loop $\gamma \subset X$:

$$\text{sys} \pi_1(g) = \min_{[\gamma] \neq 0 \in \pi_1(X)} \text{length}(\gamma).$$

(1.4)

This notion of systole is apparently unrelated to the systolic arrays of [30]. We will be concerned with comparing this Riemannian invariant to the total volume of the metric, as in Loewner’s inequality (2.2).

In Section 2, we recall the classical Loewner inequality, as well as Pu’s inequality. In Section 3, we define an invariant called the filling volume, and state Gromov’s filling volume (e.g., area) conjecture. The notion of the successive minima of a lattice in Banach space is recalled in Section 4. The results on the precise calculation of filling invariants (both the filling radius and the filling area) are reviewed in Section 5. Stable and conformal systoles are defined in Section 6.

In Section 7, we recall the definition of the Abel-Jacobi map to the torus, and present M. Gromov’s optimal stable systolic inequality for $n$-tori. Higher dimensional optimal generalisations of Loewner’s inequality are discussed in Section 8 and Section 9. Thus, in Section 8, we recall J. Hebda’s inequality combining systoles of dimension and codimension 1 in the case of unit Betti number, and present a generalisation to arbitrary Betti number.

Optimal inequalities combining varieties of 1-systoles appear in Section 9. Here a necessary topological condition is the nonvanishing of the homology class of the lift to the maximal free abelian cover, of the typical fiber of the Abel-Jacobi map. The relation of this condition to the generalized Casson invariant $\lambda$ of [33] is examined in Section 10. The example of nilmanifolds is discussed in Section 11, and the general case treated in Section 12.

## 2. Inequalities of C. Loewner and P. Pu

The Hermite constant, denoted $\gamma_n$, can be defined as the optimal constant in the inequality

$$\text{sys} \pi_1(\mathbb{T}^n)^2 \leq \gamma_n \text{vol}(\mathbb{T}^n)^{2/n},$$

(2.1)
over the class of all flat tori $\mathbb{T}^n$, cf. Section 4. Here $\gamma_n$ is asymptotically linear in $n$, cf. [32, pp. 334, 337]. The precise value is known for small $n$, e.g., $\gamma_2 = \frac{2}{\sqrt{3}}$ (see Lemma 4.5), $\gamma_3 = 2^\frac{4}{3}, \ldots$. It can be shown via the filling invariants that an inequality of type (2.1) remains valid in the class of all metrics, but with a nonsharp constant on the order of $n^{4n}$ [14]. M. Gromov’s proof uses inequality (5.1) as a starting point.

Around 1949, Charles Loewner proved the first systolic inequality, cf. [39]. He showed that every Riemannian metric $g$ on the torus $\mathbb{T}^2$ satisfies the inequality

$$\text{sys}\pi_1(g)^2 \leq \gamma_2 \text{area}(g),$$

while a metric satisfying the boundary case of equality in (2.2) is necessarily flat, and is homothetic to the quotient of $\mathbb{C}$ by the lattice spanned by the cube roots of unity.

The following estimate is found in [14, Corollary 5.2.B]. Namely, every aspherical compact surface $(\Sigma, g)$ admits a metric ball $B = B_p (\frac{1}{2} \text{sys}\pi_1(g)) \subset \Sigma$ of radius $\frac{1}{2} \text{sys}\pi_1(g)$, which satisfies

$$\text{sys}\pi_1(g)^2 \leq \frac{4}{3} \text{area}(B).$$

P. Pu’s inequality [39] admits an immediate generalisation via Gromov’s inequality (2.3). Namely, every surface $(X, g)$ which is not a 2-sphere satisfies the inequality

$$\text{sys}\pi_1(g)^2 \leq \frac{\pi}{2} \text{area}(g),$$

where the boundary case of equality in (2.4) is attained precisely when, on the one hand, the surface $X$ is a real projective plane, and on the other, the metric $g$ is of constant Gaussian curvature.

### 3. Filling area conjecture

Consider a compact manifold $N$ of dimension $n \geq 1$ with a distance function $d$. Here $d$ could be induced by a Riemannian metric $g$, but could also be more general. The notion of the Filling Volume, $\text{FillVol}(N^n, d)$, of $N$ was introduced in [14], where it is shown that when $n \geq 2$,

$$\text{FillVol}(N^n, d) = \inf_g \text{vol}_{n+1}(X^{n+1}, g)$$

where $X$ is any fixed manifold such that $\partial X = N$. Here one can even take a cylinder $X = N \times [0, \infty)$. The infimum is taken over all complete Riemannian metrics $g$ on $X$ for which the boundary distance function is $\geq d$, i.e., the length of the shortest path in $X$ between boundary points $p$ and $q$ is $\geq d(p, q)$. In the case $n = 1$, the topology of the
2-dimensional filling can affect the infimum, as is shown by example in [14, Counterexamples 2.2.B].

The precise value of the filling volume is not known for any Riemannian metric. However, the following values for the canonical metrics on the spheres (of sectional curvature +1) is conjectured in [14], immediately after Proposition 2.2.A.

**Conjecture 3.1.** \( \text{FillVol}(S^n, \text{can}) = \frac{1}{2} \omega_{n+1} \), where \( \omega_{n+1} \) denotes the volume of the unit \((n+1)\)-sphere.

This conjecture is still open in all dimensions. The case \( n = 1 \) can be broken into separate problems depending on the genus \( s \) of the filling, cf. (5.3).

### 4. Lattices and successive minima

Let \( b > 0 \). By a lattice \( L \) in Euclidean space \( \mathbb{R}^b \), we mean a discrete subgroup isomorphic to \( \mathbb{Z}^b \), i.e., the integer span of a linearly independent set of \( b \) vectors.

More generally, let \( B \) be a finite-dimensional Banach space, i.e., a vector space together with a norm \( \| \| \). Let \( L \subset (B, \| \|) \) be a lattice of maximal rank \( \text{rank}(L) = \dim(B) \).

**Definition 4.1.** For each \( k = 1, 2, \ldots, \text{rank}(L) \), define the \( k \)-th successive minimum of the lattice \( L \) by

\[
\lambda_k(L, \| \|) = \inf \left\{ \lambda \in \mathbb{R} \left| \exists \text{ lin. indep. } v_1, \ldots, v_k \in L \text{ with } \|v_i\| \leq \lambda \right. \right\}. \quad (4.1)
\]

Thus, the first successive minimum, \( \lambda_1(L, \| \|) \) is the least length of a nonzero vector in \( L \). If \( \text{rank}(L) \geq 2 \), then given a pair of vectors \( S = \{v, w\} \) in \( L \), define the length \( |S| \) of \( S \) by setting \( |S| = \max(\|v\|, \|w\|) \). Then the second successive minimum, \( \lambda_2(L, \| \|) \) is the least length of a pair of non-proportional vectors in \( L \): \( \lambda_2(L) = \inf_S |S| \), where \( S \) runs over all linearly independent pairs \( \{v, w\} \subset L \).

**Example 4.2.** The standard lattice \( \mathbb{Z}^b \subset \mathbb{R}^b \) satisfies

\[
\lambda_1(\mathbb{Z}^b) = \ldots = \lambda_b(\mathbb{Z}^b) = 1.
\]

Note that the torus \( \mathbb{T}^b = \mathbb{R}^b/\mathbb{Z}^b \) satisfies \( \text{vol}(\mathbb{T}^b) = 1 \) as it has the unit cube as a fundamental domain.

Here the volume of the torus \( \mathbb{R}^b/L \) (also called the covolume of the lattice \( L \)) is by definition the volume of a fundamental domain for \( L \), e.g., a parallelepiped spanned by a \( \mathbb{Z} \)-basis for \( L \). The volume can be calculated as the square root of the determinant of the Gram matrix of such a basis. Given a finite set \( S = \{v_i\} \in \mathbb{R}^b \), we define its Gram
matrix as the matrix of inner products $\text{Gram}(S) = \langle v_i, v_j \rangle$. Then the parallelepiped $P$ spanned by the vectors $\{v_i\}$ satisfies $\text{vol}(P) = \text{det}(\text{Gram}(S))^{1/2}$.

**Example 4.3.** Consider the lattice $L_\zeta \subset \mathbb{R}^2 = \mathbb{C}$ spanned by $1 \in \mathbb{C}$ and the sixth root of unity $\zeta = e^{\frac{2\pi i}{6}} \in \mathbb{C}$. Then $L_\zeta = \mathbb{Z}_\zeta + \mathbb{Z}1$ satisfies $\lambda_1(L_\zeta) = \lambda_2(L_\zeta) = 1$. Meanwhile, the torus $T^2 = \mathbb{R}^2/L_\zeta$ satisfies $\text{area}(T^2) = \frac{\sqrt{3}}{2}$.

**Definition 4.4.** The Hermite constant $\gamma_b$ is defined in one of the following two equivalent ways:

(a) $\gamma_b$ is the square of the maximal first successive minimum, among all lattices of unit covolume;

(b) $\gamma_b$ is defined by the formula

$$\sqrt{\gamma_b} = \sup \left\{ \frac{\lambda_1(L)}{\text{vol}(\mathbb{R}^b/L)^{1/b}} \left| L \subseteq (\mathbb{R}^b, \| \|) \right\}, \right.$$  \hspace{1cm} (4.2)

where the supremum is extended over all lattices $L$ in $\mathbb{R}^b$ with a Euclidean norm $\| \|$.

A lattice realizing the supremum is called critical. Such a lattice may be thought of as realizing a densest packing in $\mathbb{R}^b$, when we place balls of radius $\frac{1}{2} \lambda_1(L)$ at the points of $L$.

**Lemma 4.5.** Let $b = 2$. Then we have the following value for the Hermite constant: $\gamma_2 = \frac{2}{\sqrt{3}} = 1.1547...$. The corresponding optimal lattice is homothetic to the $\mathbb{Z}$-span of cube roots of unity in $\mathbb{C}$.

**Proof.** Consider a lattice $L \subset \mathbb{C} = \mathbb{R}^2$. Clearly, multiplying $L$ by nonzero complex numbers does not change the value of the quotient $\frac{\lambda_1(L)^2}{\text{area}(\mathbb{C}/L)}$. Choose a “shortest” vector $z \in L$, i.e., we have $|z| = \lambda_1(L)$. By replacing $L$ by the lattice $z^{-1}L$, we may assume that the complex number $+1 \in \mathbb{C}$ is a shortest element in the lattice. We will denote the new lattice by the same letter $L$, so that now $\lambda_1(L) = 1$. Now complete $+1$ to a $\mathbb{Z}$-basis $\{\tau, 1\}$, so that

$$L = \mathbb{Z}\tau + \mathbb{Z}1.$$ 

Thus $|\tau| \geq \lambda_1(L) = 1$. Consider the real part $\Re(\tau)$. We can adjust the basis by adding or subtracting a suitable integer to $\tau$, so as to satisfy the condition $-\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2}$. Thus, the second basis vector $\tau$ may be assumed to lie in the closure of the standard fundamental domain, cf. [43, p. 78]

$$D = \left\{ z \in \mathbb{C} \left| |z| > 1, \Re(z) < \frac{1}{2}, \Im(z) > 0 \right. \right\}.$$
(for the action of $\text{PSL}(2, \mathbb{Z})$ in the upper half-plane of $\mathbb{C}$). Then the imaginary part satisfies $\Im(\tau) \geq \frac{\sqrt{3}}{2}$, with equality possible precisely for $\tau = e^{i\frac{\pi}{3}}$ or $\tau = e^{i\frac{2\pi}{3}}$. The proof is concluded by calculating the area of the parallelogram in $\mathbb{C}$ spanned by $\tau$ and $+1$ as follows: $\frac{l_1(L)^2}{\text{area}(\mathbb{C}/L)} = \frac{1}{\Im(\tau)} \leq \frac{2}{\sqrt{3}}$.

**Example 4.6.** In dimensions $b \geq 3$, the Hermite constants are harder to compute, but explicit values (as well as the associated critical lattices) are known for small dimensions, e.g., $\gamma_3 = 2^{\frac{1}{3}} = 1.2599...$, while in dimension 4, one has $\gamma_4 = \sqrt{2} = 1.4142...$ [10].

The lattice $L^*$ dual to $L$ may be described as follows. If $L$ is the $\mathbb{Z}$-span of vectors $(x_i)$, then $L^*$ is the $\mathbb{Z}$-span of a dual basis $(y_j)$ satisfying $\langle x_i, y_j \rangle = \delta_{ij}$. (4.3)

**Definition 4.7.** The Bergé-Martinet constant, denoted $\gamma_b'$, is defined as follows:

$$\gamma_b' = \sup \left\{ \lambda_1(L)\lambda_1(L^*) \mid L \subseteq \mathbb{R}^b \right\},$$

where the supremum is extended over all lattices $L$ in $\mathbb{R}^b$ with its Euclidean norm. A lattice attaining the supremum is called dual-critical.

Like the Hermite constant $\gamma_b$, the Bergé-Martinet constant $\gamma_b'$ is asymptotically linear in $b$. Its value is known in dimensions up to 4.

**Example 4.8.** In dimension 3, the value of the Bergé-Martinet constant,

$$\gamma_3' = \sqrt{\frac{3}{2}} = 1.2247...,$$

is slightly below the Hermite constant $\gamma_3 = 2^{\frac{1}{3}} = 1.2599...$. It is attained by the face-centered cubic lattice, which is not isodual [36, p. 31], [5, Proposition 2.13(iii)], [10].

5. **Precise calculation of filling invariants**

Unlike the filling area and volume, the filling radius lends itself somewhat easier to precise calculation. The following result was proved in [19].

**Theorem 5.1.** Let $X$ be a length space with distance function $d$. Let $Y \subset X$ be a subset. Let $R > 0$. Assume that the following two conditions are satisfied:

(a) $\text{diam}(Y) \leq 2R$;
(b) $d(x, Y) \leq 2R$ for all $x \in X$.

Then $\text{FillRad}(X) \leq R$. 
Corollary 5.2. Let $S^1$ be the Riemannian circle. Then

$$\text{FillRad}(S^1) = \frac{1}{6} \text{length}(S^1).$$

More precisely, the neighborhood $U_\epsilon S^1 \subset L^\infty$ has the homotopy type of $S^1$ for $\epsilon < \frac{1}{2} d_1$, and the homotopy type of the sphere $S^3$ when $\frac{1}{2} d_1 < \epsilon < \frac{1}{2} d_2$, where $d_i = \frac{i}{2i+1} \text{length}(S^1)$.

Proof. Using the set of vertices of a regular inscribed triangle as our subset $Y \subset S^1$ in Theorem 5.1 proves the upper bound. The lower bound follows from the general inequality

$$\text{sys}\pi_1(X) \leq 6 \text{FillRad}(X)$$

valid for an arbitrary essential manifold $X$ [14]. The more precise result on the homotopy type of the neighborhood was proved in [21] using a kind of Morse theory on the space of subsets of $S^1$, for which regular odd polygons are the extrema, as discussed below. $\Box$

Let $d_i(X)$ be the $i$-th extremal value of the diameter functional on the space of finite subsets of $X$, with the convention that $d_0 = 0$. Here a subset $Y \subset X$ is extremal if small perturbations of size $\epsilon > 0$ can decrease the diameter of $Y$ at most quadratically in $\epsilon$. For the Riemannian circle we have a complete classification of the extrema, namely sets of vertices of odd regular polygons inscribed in the circle. Thus, we obtain the following extremal values:

$$d_i(S^1) = \frac{i}{2i+1} \text{length}(S^1).$$

(5.2)

For the $n$-sphere, the first extremum is the set of vertices of a regular inscribed $n+1$-simplex, but the higher ones are difficult to classify even on $S^2$ [20].

The precise calculation of the filling radius is possible in the following cases [19, 21, 22, 23].

Theorem 5.3. The filling radius of the following two-point homogeneous spaces $X$ equals half the first extremal value $d_1(X)$:

(a) $X = S^n$, where $d_1(S^n) = \arccos \left( -\frac{1}{n+1} \right) K^{-1/2}$, where $K$ denotes the (constant) value of sectional curvature;

(b) $X = \mathbb{R}P^n$, where $d_1(\mathbb{R}P^n) = \frac{\pi}{3} K^{-1/2}$;

(c) $X = \mathbb{C}P^2$, where $d_1(\mathbb{C}P^2) = \arccos \left( -\frac{1}{3} \right) K^{-1/2}$, where $K$ denotes the maximal value of sectional curvatures.

Meanwhile, for $\mathbb{C}P^3$ we have a strict inequality

$$\text{FillRad}(\mathbb{C}P^3) > \frac{1}{2} d_1(\mathbb{C}P^3) = \frac{1}{2} \arccos \left( -\frac{1}{3} \right) K^{-1/2}.$$
Remark 5.4. In general, understanding the homotopy type of neighborhoods of $X$ in $L^\infty$ depends on the study of the extremal values of the diameter functional, which is an interesting open question. For example, the value of $d_2(\mathbb{C}P^n)$ is unknown for $n \geq 2$, but progress was made in [23]. Successful calculation of these extrema depends on analyzing certain semialgebraic sets, defined in terms of the distance functions among points, in the Cartesian (or symmetric) powers of the space.

Recently, L. Liu [35] undertook the study of the mapping properties of the filling radius.

We now turn to the related invariant of the filling area. We report on the recent progress on calculating the optimal ratio $\sigma_{2,s}$ of (5.3), in the case of hyperelliptic fillings, in the following sense. We introduce the constant

$$\sigma_{2,s} = \sup_{\Sigma_s} \frac{\text{length}(S^1)^2}{\text{area}(\Sigma_s)}, \quad (5.3)$$

where $\Sigma_s$ denotes an orientable filling, of genus $s$, of a Riemannian circle $S^1$, meaning that the inclusion of the boundary $S^1 \subset \Sigma_s$ is strongly isometric, cf. formula (1.2). A related invariant appears in the systolic inequality (9.4).

For simply connected fillings (i.e., by a disk $D$), we have $\sigma_{2,0} = 2\pi$, by Pu’s inequality (2.4) applied to the real projective plane obtained from $\Sigma_0 = D$ by identifying pairs of opposite points of the boundary circle. The case of genus 1 fillings has recently been solved, as well [1]. Thus,

$$\sigma_{2,s} = 2\pi \text{ for } s = 0 \text{ or } 1. \quad (5.4)$$

Given a filling by an orientable surface $X_1$ of genus $s$, one identifies pairs of antipodal points of the boundary, so to form a nonorientable closed surface $X_2$. We construct the orientable double cover $X_3 \to X_2$ with deck transformation $\tau$. The pair $(X, \tau)$ is thus a real Riemann surface of genus $2s$, which can be called ovalless since $\tau$ is fixed point free, cf. [1]. Assuming that $X_3$ is in addition hyperelliptic [37], one can prove that it satisfies a bound equivalent to the formula $\sigma_{2,s} = 2\pi$. The result follows from a combination of two ingredients. On the one hand, one exploits integral geometric comparison with orbifold metrics of constant positive curvature on real surfaces of even positive genus, where the conical singularities correspond to Weierstrass points on the hyperelliptic surface. On the other hand, one exploits an analysis of the combinatorics on unions of closed curves, arising as geodesics of such orbifold metrics.
6. Stable and conformal systoles

In a Riemannian manifold \((X, g)\), we define the volume \(\text{vol}_k(\sigma)\) of a Lipschitz \(k\)-simplex \(\sigma : \Delta^k \to X\) to be the integral over the \(k\)-simplex \(\Delta^k\) of the “volume form” of the pullback \(\sigma^*(g)\). The stable norm \(\|h\|\) of an element \(h \in H_k(X; \mathbb{R})\) is the infimum of the volumes \(\text{vol}_k(c) = \sum |r_i| \text{vol}_k(\sigma_i)\) over all real Lipschitz cycles \(c = \sum r_i \sigma_i\) representing \(h\). We define the stable \(k\)-systole of the metric \(g\) by setting

\[
\text{stsys}_k(g) = \lambda_1 \left( H_k(X; \mathbb{Z})_\mathbb{R}, \| \| \right),
\]

where \(\| \|\) is the stable norm in homology associated with the metric \(g\), while \(H_k(X; \mathbb{Z})_\mathbb{R}\) denotes the integral lattice in real homology, and \(\lambda_1\) is the first successive minimum, cf. Definition 4.1.

The conformally invariant norm \(\| \|_n\) in \(H_1(X^n; \mathbb{R})\) is by definition dual to the conformally invariant \(L^n\)-norm in de Rham cohomology. The latter norm on \(H^1(X, \mathbb{R})\) is the quotient norm of the corresponding norm on closed one-forms. Thus, given \(\alpha \in H^1(X, \mathbb{R})\), we set

\[
\|\alpha\|_n = \inf_{\omega} \left\{ \left( \int_X |\omega|^n dvol(x) \right)^{\frac{1}{n}} \mid \omega \in \alpha \right\},
\]

where \(\omega\) runs over closed one-forms representing \(\alpha\). The conformal 1-systole of \((X^n, g)\) is the quantity

\[
\text{confsys}_1(g) = \lambda_1 \left( H_1(X; \mathbb{Z})_\mathbb{R}, \| \|_n \right),
\]

satisfying \(\text{stsys}_1(g) \leq \text{confsys}_1(g) \text{vol}(g)^{\frac{1}{n}}\) on an \(n\)-manifold \((X, g)\).

7. Gromov’s optimal inequality for \(n\)-tori

Let \(X\) be a smooth compact manifold with positive first Betti number. We define the Jacobi variety \(J_1(X)\) of \(X\) by setting

\[
J_1(X) = H_1(X; \mathbb{R})/H_1(X; \mathbb{Z})_\mathbb{R}.
\]

The homotopy class of the Abel-Jacobi map can be defined as follows. Consider the abelianisation homomorphism on the fundamental group of \(X\), followed by quotienting by torsion. Consider a map \(K(\pi_1(X), 1) \to \mathbb{T}^{b_1(X)} = J_1(X)\) associated with the resulting homomorphism. The composition \(X \to K(\pi_1(X), 1) \to \mathbb{T}^{b_1(X)}\) represents the desired homotopy class. When \(X\) is equipped with a metric, the Abel-Jacobi map

\[
\mathcal{A}_X : X \to J_1(X),
\]

can be induced by the harmonic one-forms on \(X\), originally introduced by A. Lichnerowicz [34], cf. [16, 4.21]. More precisely, let \(E\) be the space of harmonic 1-forms on \(X\), with dual \(E^*\) canonically identified
with $H_1(X, \mathbb{R})$. By integrating an integral harmonic 1-form along paths from a basepoint $x_0 \in X$, we obtain a map to $\mathbb{R}/\mathbb{Z} = S^1$. In order to define a map $X \to J_1(X)$ without choosing a basis for cohomology, we argue as follows. Let $x$ be a point in the universal cover $\widetilde{X}$ of $X$. Thus $x$ is represented by a point of $X$ together with a path $c$ from $x_0$ to it. By integrating along the path $c$, we obtain a linear form, $h \mapsto \int_c h$, on $E$. We thus obtain a map $\widetilde{X} \to E^* = H_1(X, \mathbb{R})$, which, furthermore, descends to a map

$$\overline{A}_X : \overline{X} \to E^*, \ c \mapsto \left( h \mapsto \int_c h \right), \tag{7.3}$$

where $\overline{X}$ is the universal free abelian cover. By passing to quotients, this map descends to the Abel-Jacobi map (7.2). The Abel-Jacobi map defined using harmonic forms is unique up to translations of the Jacobi torus $J_1(X)$.

An optimal higher-dimensional generalisation of Loewner’s inequality (2.2) is due to M. Gromov [16, pp. 259-260] (cf. [11, inequality (5.14)]) based on the techniques of D. Burago and S. Ivanov [7, 8]. We give below a slight generalisation of Gromov’s statement.

**Definition 7.1.** Given a map $f : X \to Y$ between closed manifolds of the same dimension, we denote by $\deg(f)$, either the algebraic degree of $f$ when both manifolds are orientable, or the absolute degree [6], otherwise.

We denote by $A_X$ the Abel-Jacobi map of $X$, cf. (7.2); by $\gamma_n$, the Hermite constant, cf. Definition 4.4; and by $\text{stsys}_1(g)$, the stable 1-systole of a metric $g$, cf. formula (6.1).

**Theorem 7.2 (M. Gromov).** Let $X^n$ be a compact manifold of equal dimension and first Betti number: $\dim(X) = b_1(X) = n$. Then every metric $g$ on $X$ satisfies the following optimal inequality:

$$\deg(A_X) \text{stsys}_1(g)^n \leq (\gamma_n)^{\frac{n}{2}} \text{vol}_n(g). \tag{7.4}$$

The boundary case of equality in inequality (7.4) is attained by flat tori whose group of deck transformations is a critical lattice in $\mathbb{R}^n$.

Note that the inequality is nonvacuous in the orientable case, only if the real cuplength of $X$ is $n$, i.e., the Abel-Jacobi map $A_X$ is of nonzero algebraic degree.

The method of proof is to construct a suitable map $A$ from $X$ to its Jacobi torus,

$$A : X \to J_1(X). \tag{7.5}$$
Here the torus is endowed with the flat Riemannian metric of least volume which dominates the stable norm. The map $A$ has the following properties:

(a) $A$ belongs to the homotopy class of the Abel-Jacobi map $A_X$;
(b) the derivative of $A$ is volume-decreasing at every point.

The existence of such a map $A$ follows from the techniques of [7, 8], as explained in [16, pp. 259-260]. Inequality (7.4) now follows from the definition of the Hermite constant. This approach was systematized and generalized in [18, 2], as explained in Section 9.

8. Inequalities combining dimension and codimension 1

The material treated here is in [3, 4]. An optimal stable systolic inequality, for $n$-manifolds $X$ with $b_1(X, \mathbb{R}) = 1$, is due to J. Hebda [17]:

$$\text{stsys}_1(g) \text{sys}_{n-1}(g) \leq \text{vol}_n(g), \quad (8.1)$$

with equality if and only if $(X, g)$ admits a Riemannian submersion with connected minimal fibers onto a circle.

Here the systole, $\text{sys}_k$, and the stable systole, $\text{stsys}_k$, coincide in codimension 1 (i.e., when $k = n - 1$) for orientable manifolds. Thus $\text{sys}_{n-1} = \text{stsys}_{n-1}$. In general, the $k$-systole is defined by minimizing the $k$-dimensional area of integral $k$-cycles which are not nullhomologous.

PROOF OF (8.1). Let $\omega \in H^1(X, \mathbb{Z})_{\mathbb{R}}$ be a primitive element in the integer lattice in cohomology, and similarly $\alpha \in H_1(X, \mathbb{Z})_{\mathbb{R}}$ in homology. Let $\eta \in \omega$ be the harmonic 1-form for the metric $g$. Then there exists a map $f$ to the circle such that

$$f : X \to S^1 = \mathbb{R}/\mathbb{Z}, \quad df = \eta.$$

Using the Cauchy-Schwartz inequality, we obtain

$$\|\omega\|^2_2(\text{vol}_n(g))^{1/2} = |\eta|^2_2(\text{vol}_n(g))^{1/2} \geq \int_X |df|dvol_n. \quad (8.2)$$

Using the coarea formula, cf. [13, 3.2.11], [9, p. 267], we obtain

$$\|\omega\|^2_2(\text{vol}_n(g))^{1/2} \geq \int_{S^1} \text{vol}_{n-1}(f^{-1}(t)) \, dt \geq \|\text{PD}(\omega)\|_{\infty} \geq \text{sys}_{n-1}(g), \quad (8.3)$$
since the hypersurface \( f^{-1}(t) \subset X \) is Poincaré dual to \( \omega \) for every regular value \( t \) of \( f \). Let us normalize the metric to unit total volume to simplify the calculations. In the case \( b_1(X) = 1 \), we have

\[
\| \alpha \|_2 \| \omega \|_2^* = 1.
\] (8.4)

Therefore

\[
\text{stsys}_1(g) \text{sys}_{n-1}(g) = \| \alpha \|_\infty \text{sys}_{n-1}(g) \\
\leq \| \alpha \|_\infty \| \omega \|_2^* \\
\leq \| \alpha \|_2 \| \omega \|_2^* \\
= 1,
\]

proving Hebda’s inequality (8.1).

An optimal inequality, involving the conformal 1-systole (6.2), is proved in [4], namely inequality (8.5) below. This inequality generalizes simultaneously Loewner’s inequality (2.2), Hebda’s inequality (8.1), the inequality [3, Corollary 2.3], as well as certain results of G. Paternain [38]. Let \( X \) be a compact, oriented, \( n \)-dimensional manifold with positive first Betti number \( b_1(X) \geq 1 \). Then every metric \( g \) on \( X \) satisfies

\[
\text{confsys}_1(g) \text{sys}_{n-1}(g) \leq \gamma'_{b_1(X)} \text{vol}_n(g) \frac{n-1}{n},
\] (8.5)

where equality occurs if and only if the following three conditions are satisfied:

(a) the stable norm in \( H_1(X; \mathbb{R}) \) is Euclidean;
(b) the deck transformations of the Jacobi torus (7.1) form a dual-critical lattice (4.4);
(c) the Abel-Jacobi map (7.2) is a Riemannian submersion with connected minimal fibers.

The proof of inequality (8.5) is similar to the calculation (8.3). To conclude the argument, identity (8.4) is replaced by the inequality

\[
\| \alpha \|_2 \| \omega \|_2^* \leq \gamma'_{b_1(X)}
\] (8.6)

for a pair of minimizing elements

\[
\alpha \in H_1(X, \mathbb{Z})_\mathbb{R} \setminus \{ 0 \}, \quad \omega \in H^1(X, \mathbb{Z})_\mathbb{R} \setminus \{ 0 \}.
\]

Note that inequality (8.6) results from the definition of the Bergé-Martinet’s constant \( \gamma'_b \) combined with the fact that the \( L^2 \) norm in cohomology is actually a Euclidean norm since harmonic forms form a vector space.
9. Inequalities combining varieties of 1-systoles

A generalisation of Gromov’s inequality (7.4) was obtained in [18], with a number of further generalisations in [2]. The following two theorems were proved in [18] (the second one is an immediate corollary of the first one). Denote by \([F_X] \in H_{n-b}(X; A)\) the class of a typical fiber of the map \(\overline{\mathcal{A}}\) of (7.3), where \(A = \mathbb{Z}\) or \(\mathbb{Z}_2\) according as \(X\) is orientable or not. Following Gromov [14], denote by \(\deg(A_X)\) the infimum of \((n-b)\)-volumes of cycles representing the class \([F_X]\). Note that this quantity is a topological invariant only if \(n = b\).

**Theorem 9.1 ([18])**. Let \(X\) be a closed Riemannian manifold of dimension \(n = \dim(X)\). Let \(b = b_1(X)\), and assume \(n \geq b \geq 1\). Then every metric \(g\) on \(X\) satisfies the inequality

\[
\deg(A_X) \stsys_1(g)^b \leq (\gamma_b)^{\frac{b}{2}} \vol_n(g). \tag{9.1}
\]

**Theorem 9.2 ([18])**. Let \(X\) be a closed orientable manifold. Let \(b = b_1(X)\). Assume that \(\dim(X) = b + 1\), and \([F_X] \neq 0\). Then every metric \(g\) on \(X\) satisfies the following optimal inequality:

\[
\stsys_1(g)^b \sys_\pi_1(g) \leq (\gamma_b)^{\frac{b}{2}} \vol_{b+1}(g). \tag{9.2}
\]

The main tool in the proof of Theorems 9.1 and 9.2 is a generalisation of the construction of an area-decreasing map (7.5) in a situation where the dimension is bigger than the Betti number. A suitable version of the coarea formula is then applied to this map to conclude the proof.

The paper [2] contains a number of generalisations of such stable inequalities as well as the study of the boundary case of equality, with conclusions along the closing lines of Section 8. In particular, the Abel-Jacobi map is a submersion.

Now let \(X\) be a closed 3-manifold. Let \(\lambda\) be the extension of the Casson-Walker invariant defined in [33]. The general definition is a bit involved, but in our case the invariant can be expressed in terms of the self-linking number of a typical fiber of the Abel-Jacobi map, cf. (10.2). We show in Section 10 that the nonvanishing of the \(\lambda\) invariant is a sufficient condition for the nonvanishing of the class \([F_X]\).

The argument runs roughly as follows. Consider the map \(\overline{\mathcal{A}}: \overline{X} \to \mathbb{R}^2\) of (7.3). Let \(a, b \in \mathbb{R}^2\). Let \(\overline{F_a} = \overline{\mathcal{A}}^{-1}(a)\) and \(\overline{F_b} = \overline{\mathcal{A}}^{-1}(b)\) be lifts of the corresponding fibers \(F_a, F_b \subset X\). Choose a properly imbedded ray \(r_b \subset \mathbb{R}^2\) joining \(b\) to infinity while avoiding \(a\) (as well as its translates), and consider the complete surface \(S = \overline{\mathcal{A}}^{-1}(r_b) \subset \overline{X}\) with \(\partial S = \overline{F_b}\). Furthermore, \(S \cap g.\overline{F_a} = \emptyset\) for
all \( g \in G \), where \( G \) denotes the group of deck transformations of the cover \( p_G : \overline{X} \rightarrow X \). If \( F_a \) is zero-homologous in \( \overline{X} \), choose a compact surface \( M \subset \overline{X} \) with

\[
\partial M = \overline{F}_a.
\]

The linking number \( \ell_X(F_a, F_b) \) in \( X \) can therefore be computed as the algebraic intersection

\[
\ell_X(F_a, F_b) = p_G(M) \cap F_b = \sum_{g \in G} g.M \cap \overline{F}_b = \sum_{g \in G} \partial (g.M \cap S) = 0,
\]

where all sums and intersections are algebraic, and the curve \( M \cap S \) is compact by construction. The details of a rigorous proof are found in Section 12.

Therefore we obtain the following immediate corollary of Theorem 9.2.

**Corollary 9.3.** Suppose \( X \) is a three-manifold such that \( b_1(X) = 2 \), with nonvanishing \( \lambda \)-invariant. Then every metric \( g \) on \( X \) satisfies the inequality

\[
\text{stsyst}_1(g)^2 \text{sys}_1(g) \leq \gamma_2 \text{vol}(g). \tag{9.3}
\]

In the boundary case of equality, the manifold must be one of the bundles of Example 11.1.

Analyzing the condition \( \left[ F_X \right] \neq 0 \) leads to interesting questions about 3-manifolds, see Section 10.

**Remark 9.4 (Relation to LS).** In dimension 3, there is a connection between systoles and the Lusternik-Schnirelmann category, denoted \( \text{cat}_{\text{LS}} \). A new invariant \( \text{cat}_{\text{sys}} \), called systolic category, is defined in terms of the existence of systolic inequalities of a suitable type [24]. It turns out that the two categories coincide in dimension three: \( \text{cat}_{\text{LS}} = \text{cat}_{\text{sys}} \). This and other phenomena relating the two categories are discussed in [24, 25].

**Remark 9.5 (Codimension 2).** In the case \( n - b = 2 \), inequality (9.1) is related to the filling area conjecture. For simplicity, let \( X = \mathbb{R}P^2 \times \mathbb{T}^2 \). It was proved in [18] that every metric \( g \) on \( \mathbb{T}^2 \times \mathbb{R}P^2 \)
satisfies a “Pu times Loewner” inequality
\[ \text{sys} \pi_1(g)^2 \text{stsys}_1(g)^2 \leq \frac{\tilde{\sigma}_2}{4} \gamma_2 \text{vol}_4(g), \]
(9.4)
where \(\tilde{\sigma}_2\) is defined similarly to (5.3) by allowing all possible topology of fillings.

All the inequalities discussed so far depend on the rather restrictive hypothesis \(\dim(X) \geq b_1(X)\). In the case \(\dim(X) < b_1(X)\) it is harder to obtain optimal inequalities, and typically the extremal metrics are not smooth, see [11]. Recently, progress was achieved toward removing the topological assumption in Loewner’s inequality (2.2) (similarly to the generalisation (2.4) of Pu’s inequality), which involves the study of surfaces of genus \(s \geq 2\). Namely, all hyperelliptic surfaces satisfy Loewner’s inequality
\[ \text{sys} \pi_1(X)^2 \leq \gamma_2 \text{area}(X) \text{ if } X \text{ hyperelliptic,} \]
(9.5)
and in particular [12] all metrics in genus 2 [26]. Moreover, all surfaces of genus at least 20 satisfy Loewner’s inequality, as well [27].

10. Fiber class, linking number, and the generalized Casson invariant \(\lambda\)

The results of this section tie in with Corollary 9.3. Namely, Proposition 10.1 provides a necessary condition so that the invariant \(\lambda\) does not vanish. The condition is the nonvanishing of the fiber class in the maximal free abelian cover. Thus the nonvanishing of \(\lambda\) is a sufficient condition for the inequality (9.3) to be satisfied.

From now on, unless otherwise mentioned, all the manifolds considered are smooth, compact, and oriented. Boundaries are oriented according to the outward normal first convention.

In any (oriented) 3-manifold \(X\), the linking number of two disjoint closed curves \(\gamma\) and \(d\) with null homology classes in \(H_1(X; \mathbb{Q})\) is defined as follows. Let \(N(\gamma)\) be a tubular neighborhood of \(\gamma\) disjoint from \(d\). There exists a curve \(C(k\gamma)\) in \(N(\gamma)\) that is homologous to \(k\gamma\) in \(N(\gamma)\) for some integer \(k > 0\), and that bounds a surface \(\Sigma(C(k\gamma))\) transverse to \(d\). Then the linking number of \(\gamma\) and \(d\) in \(X\) is defined as
\[ \ell_X(\gamma, d) = \frac{\langle \Sigma(C(k\gamma)), d \rangle_X}{k} \]
where \(\langle , \rangle_X\) denotes the algebraic intersection number in \(X\).

**Proposition 10.1.** Let \(X\) be a connected closed three-manifold whose first Betti number is two. Let \(S_1\) and \(S_2\) be two surfaces imbedded in \(X\) whose homology classes generate \(H_2(X; \mathbb{Z})\). The intersection
of $S_1$ and $S_2$ is denoted by $\gamma$. Let $\gamma'$ be a parallel of $\gamma$ in $S_1$. Let $\overline{X}$ be the maximal free abelian cover of $X$, and let $\overline{\gamma}$ be a lift of $\gamma$ in $\overline{X}$. If the homology class of $\overline{\gamma}$ vanishes in $H_1(\overline{X}; \mathbb{Q})$, then the linking number
\[ \ell_X(\gamma, \gamma') \] (10.1)
of $\gamma$ and $\gamma'$ in $X$ vanishes, as well.

This proposition will be proved in Section 12 as a particular case of a more general result, Proposition 12.1, as suggested by Alexis Marin.

Remark 10.2. The two parallels of $\gamma$ in $S_1$ are homotopic to each other in the complement of $\gamma$ in $X$ and they are also homotopic to the two parallels of $\gamma$ in $S_2$. We could equivalently define $\gamma'$ as the parallel of $\gamma$ with respect to the framing of $\gamma$ induced by $S_1$ or $S_2$.

Remark 10.3. By Poincaré duality, under the assumptions of the proposition, there exists a basis $(a_1, a_2)$ of $H_1(X; \mathbb{Q})$ such that the homology class $[d]$ of any closed curve $d$ of $X$ reads
\[ [d] = \langle d, S_1 \rangle_X a_1 + \langle d, S_2 \rangle_X a_2, \]
cf. (4.3). In particular, we have $[\gamma] = 0$ in $H_1(X; \mathbb{Q})$. Therefore, the number $\ell_X(\gamma, \gamma')$ is well-defined. In fact, if $\lambda$ denotes the extension of the Casson invariant defined in [33], and if $|\text{Torsion}(H_1(X))|$ is the cardinality of the torsion part of $H_1(X; \mathbb{Z})$, then
\[ \ell_X(\gamma, \gamma') = -\frac{\lambda(X)}{|\text{Torsion}(H_1(X))|}, \] (10.2)
cf. (11.2). Thus the linking number does not depend on the choice of the transverse surfaces $S_1$ and $S_2$ whose homology classes generate $H_2(X; \mathbb{Z})$. An easy direct proof of this fact is given in [33, pp. 93-94].

Definition 10.4. Let $X$ be a connected topological space, and let $g : \pi_1(X) \to G$ be a homomorphism. The connected cover
\[ \hat{X}(\text{Ker}(g)) \] (10.3)
of $X$ associated to $g$ is the connected cover of $X$ where a loop of $X$ lifts homeomorphically if and only if its homotopy class is in the kernel of $g$.

In particular, we have $\pi_1(\hat{X}(\text{Ker}(g))) = \text{Ker}(g)$. The covering group is the image of $g$. For example, the maximal free abelian cover $\overline{X}$ of $X$ is associated with the natural composition
\[ \pi_1(X) \to H_1(X) \to H_1(X)/\text{Torsion}(H_1(X)). \]
Since \([\gamma]=0\) in \(H_1(X;\mathbb{Q})\), the covering map \(\rho\) from \(\overline{X}\) to \(X\) maps \(\overline{\gamma}\) to \(\gamma\), homeomorphically. In particular, \(\overline{\gamma}\) is a closed curve.

11. Fiber class in NIL geometry

Proposition 10.1 may be illustrated by the following example of the non-trivial circle bundles \(N_e\) over the torus. For them, we shall directly see that

- the roles of \(\gamma\) and \(\gamma'\) can be played by two disjoint fibers \(F\) and \(F'\),
- \(\ell_{N_e}(F, F')\) does not vanish, and
- \(H_1(N_e;\mathbb{Z}) = \mathbb{Z}[F]\).

Example 11.1. Let \(p_e : N_e \rightarrow S^1 \times S^1\) be the circle bundle over the torus with Euler number \(e \neq 0\). Denote by \(x = S^1 \times \{1\}\) and \(y = \{1\} \times S^1\) the two \(S^1\)-factors of the base \(S^1 \times S^1 = x \cup y\) of \(N_e\).

Let \(D \subset x \times y\) be a disk disjoint from the union \(x \cup y\). The bundle is trivial both over \(D\) and over its complement. Thus we have

\[ N_e = S^1 \times s \left(\left(\frac{x \times y}{D}\right) \cup_{S^1 \times \partial D} S^1 \times D\right), \]

where \(s : (x \times y) \setminus D \rightarrow N_e\) is a section, and the gluing map reads

\[(u, s(v)) \in S^1 \times (\partial D \cong S^1) \mapsto (uv^{\pm e}, v) \quad (11.1)\]

(up to signs). Let \(F = p_e^{-1}(x \cap y)\) be the fiber. Then

\[ H_1(N_e) = \mathbb{Z}[s(x)] \oplus \mathbb{Z}[s(y)] \oplus \mathbb{Z}[F]/|e|\mathbb{Z}[F]. \]

The surfaces \(S_1\) and \(S_2\) of Proposition 10.1 may be chosen to be \(S_1 = p_e^{-1}(x)\) and \(S_2 = p_e^{-1}(y)\), since these surfaces are dual to the basis

\([s(y)]; [s(x)]\)

of \(H_1(N_e)/\text{Torsion}\) with respect to the algebraic intersection. Then the intersection \(S_1 \cap S_2\) is precisely the fiber \(F\), and its parallel induced by the surfaces is another fiber \(F'\).

In this case, if the loop \(s(\partial D)\) bounds a section of the bundle over \(s((x \times y) \setminus D)\), then the loop \((s(\partial D) \pm eF)\) bounds a section of the bundle over \(D\). This allows us to see that \(|e|F\) bounds a surface that is pierced once by \(F'\), so that the linking number satisfies

\[ \ell_{N_e}(F, F') = \pm \frac{1}{e}, \quad (11.2)\]

cf. (10.2). Now, let us study the maximal free abelian covering

\[ \rho : \overline{N_e} \rightarrow N_e \quad (11.3)\]
associated with the map
\[ \pi_1(N_e) \to (H_1(N_e)/\text{Torsion}) = \mathbb{Z}[s(x)] \oplus \mathbb{Z}[s(y)] = G. \]

The free abelian group \( G \) acts on \( N_e \) freely as the covering group. Therefore, \( H_1(N_e) \) becomes a \( \mathbb{Z}[G] \)-module. The ring \( \mathbb{Z}[G] \) is denoted by \( R \) and is identified with \( \mathbb{Z}[t^\pm_1, t^\pm_2] \) by mapping \([s(x)]\) and \([s(y)]\) to \( t_x \) and \( t_y \), respectively. Consider the standard product covering
\[ \rho : \mathbb{R}^2 \to (S^1)^2 \]
\[ (u, v) \mapsto (\exp(2i\pi u), \exp(2i\pi v)). \]

Then \( N_e \) is obtained from the trivialisation of \( \rho \) of (11.3) over the complement of \( D \),
\[ \rho^{-1}\left( p^{-1}_e \left( (S^1)^2 \setminus D \right) \right) = F \times (\mathbb{R}^2 \setminus \rho^{-1}(D)), \]
by filling in the \( \mathbb{Z}^2 \) holes (with boundaries the lifts of \( F \times \partial D \)) by \( \mathbb{Z}^2 \) copies of \( S^1 \times D \). Here we use the same gluing map (11.1) as before, \( \mathbb{Z}^2 \) times, equivariantly. Then
\[ H_1(N_e) = \frac{R[F] \oplus R[\partial D]}{(t_x - 1)R[F] + (t_y - 1)R[F] + R\left( [\partial D] \pm c[F] \right)} \]
\[ = \frac{R}{(t_x - 1)R + (t_y - 1)R[F]} \]
\[ = \mathbb{Z}\left[ F \right]. \]

12. A nonvanishing result (following A. Marin)

As it will be shown in Remark 12.2 below, Proposition 10.1 is a particular case of Proposition 12.1 that has been suggested and proved by Alexis Marin.

**Proposition 12.1.** Let \( X \) be a connected closed three-manifold. Let \( S_1 \) and \( S_2 \) be two surfaces imbedded in \( X \) that intersect transversely in \( X \) along a curve \( \gamma \). Let \( \gamma' \) be a parallel of \( \gamma \) in \( S_1 \). Let \( \hat{X}_2 \) be the connected cover of \( X \) associated with the composition
\[ f_{2*} : \pi_1(X) \to H_1(X) \to \mathbb{Z} \]
\[ \langle d, S_2 \rangle_X \] (12.1)
where the first map is the Hurewicz homomorphism, cf. (10.3). Let \( \hat{\gamma} \) be a lift of \( \gamma \) in \( \hat{X}_2 \). If the homology class of \( \hat{\gamma} \) vanishes in \( H_1(\hat{X}_2; \mathbb{Q}) \), then the linking number \( \ell_X(\gamma, \gamma') \) of \( \gamma \) and \( \gamma' \) in \( X \) is well-defined and vanishes.
Remark 12.2. Let $\hat{X}$ be the connected cover of $X$ associated with the composition
\[ A_* : \pi_1(X) \to H_1(X) \to \mathbb{Z} \oplus \mathbb{Z} \] (12.2)
where the first map is the Hurewicz homomorphism. Then $\hat{X}$ is a cover of $\hat{X}_2$. Suppose the homology class of a lift of $\gamma$ in $\hat{X}$ vanishes in $H_1(\hat{X}; \mathbb{Q})$. Then its image in $\hat{X}_2$ under the covering map, that is a lift of $\gamma$ in $\hat{X}_2$, vanishes, as well. Therefore, assuming that Proposition 12.1 is true, replacing $\hat{X}_2$ by $\hat{X}$ in its statement yields another true proposition. In particular, Proposition 12.1 implies Proposition 10.1 because in this case the above $\hat{X}$ is precisely the maximal free abelian cover $X$.

Remark 12.3. In Proposition 12.1 that applies to 3-manifolds with arbitrarily large Betti numbers, the covering group is the subgroup $f_2(\pi_1(X))$ of $\mathbb{Z}$ that is isomorphic to $\{0\}$ or $\mathbb{Z}$.

Proof of Proposition 12.1. Let $\rho_2 : \hat{X}_2 \to X$ be the covering map. Assume that the homology class of $\hat{\gamma}$ vanishes in $H_1(\hat{X}_2; \mathbb{Q})$. Note that this implies that $\gamma$ is rationally null-homologous in $X$ and that $\ell_X(\gamma, \gamma')$ is well-defined.

There exists a surface $\Sigma$ in $\hat{X}_2$ whose boundary $\partial \Sigma$ lies in a small tubular neighborhood $N(\hat{\gamma})$ of $\hat{\gamma}$ such that

- $\partial \Sigma$ is homologous to $k[\gamma]$ in $N(\hat{\gamma})$, for some $k > 0$,
- $N(\hat{\gamma})$ does not meet $\rho_2^{-1}(\gamma')$,
- the restriction of $\rho_2$ to $N(\hat{\gamma})$ is injective.

Then
\[ k\ell_X(\gamma, \gamma') = \langle \rho_2(\Sigma), \gamma' \rangle_X = \langle \Sigma, \rho_2^{-1}(\gamma') \rangle_{\hat{X}_2} \]
and $k\ell_X(\gamma, \gamma')$ is the sum over the lifts of $\gamma'$ of their algebraic intersections with $\Sigma$.

We now apply the following Lemma 12.4 to complete the proof of Proposition 12.1.

□

Lemma 12.4. For any lift $\hat{\gamma}'$ of $\gamma'$, the algebraic intersection of $\Sigma$ and $\hat{\gamma}'$ vanishes.

Proof of Lemma 12.4. The proof consists in seeing $\langle \Sigma, \hat{\gamma}' \rangle_{\hat{X}_2}$ as the algebraic boundary of the compact oriented intersection of $\Sigma$ with a noncompact surface $\hat{S}_1^*$ that is bounded by $\hat{\gamma}'$ and that does not meet $N(\hat{\gamma})$.

Let us first construct $\hat{S}_1^*$. 
Consider a tubular neighborhood $S_2 \times [-1, 1]$ of $(S_2 = S_2 \times 0)$ such that $S_1 \cap (S_2 \times [-1, 1])$ reads $\gamma \times [-1, 1]$ in $S_2 \times [-1, 1]$ (up to orientations).

Let $f_2 : X \to S^1$ be the map to the unit circle in $\mathbb{C}$, defined as follows on the subset $S_2 \times [-1, 1] \subset X$:

$$f_2 : S_2 \times [-1, 1] \to S^1 \quad (x, t) \mapsto \exp(i \pi t),$$

while $f_2$ maps the complement of $S_2 \times [-1, 1]$ in $X$ to the point $(-1)$.

Note that the map $f_2$ induces the morphism $f_{2*}$ of (12.1). Thus the composition $f_2 \circ \rho_2$ induces the trivial map from $\pi_1(\hat{X}_2)$ to $\pi_1(S^1)$. Therefore, it factors through the covering $\text{Exp} : \mathbb{R} \to S^1$,

$$u \mapsto \exp(2i \pi u)$$

yielding a map $\hat{f}_2 : \hat{X}_2 \to \mathbb{R}$ such that $\text{Exp} \circ \hat{f}_2 = f_2 \circ \rho_2$.

Without loss assume that

$$\gamma' = S_1 \cap f^{-1}_2(i)$$

and that

$$N(\hat{\gamma}) \subset \hat{f}_2^{-1}
\left[\begin{array}{c}
-\frac{1}{8} \\
\frac{1}{8}
\end{array}
\right].$$

Now, any lift $\hat{\gamma}'$ of $\gamma'$ reads

$$\rho^{-1}_2(S_1) \cap \hat{f}_2^{-1}\{n + 1/4\}$$

for some $n \in \mathbb{Z}$, and splits $\rho^{-1}_2(S_1)$ as the union of

$$\hat{S}_1^+ = \rho^{-1}_2(S_1) \cap \hat{f}_2^{-1}\left(n + \frac{1}{4} + [0, +\infty]\right)$$

and

$$\hat{S}_1^- = \rho^{-1}_2(S_1) \cap \hat{f}_2^{-1}\left(n + \frac{1}{4} + ]-\infty, 0]\right).$$

One of the non-compact subsurfaces $\hat{S}_1^+$ and $\hat{S}_1^-$ does not meet $N(\hat{\gamma})$. This will be our surface $\hat{S}_1^*$.

We may assume that the surface $\Sigma$ is transverse to $\hat{S}_1^*$. Then these two surfaces intersect along an oriented curve $\Sigma \cap \hat{S}_1^*$ that is oriented so that the triple

(a) tangent vector $\vec{t}(\Sigma \cap \hat{S}_1^*)$ to the intersection curve,

(b) positive normal vector $\vec{n}(\Sigma)$ to $\Sigma$,

(c) positive normal vector $\vec{n}(\hat{S}_1^*)$ to $\hat{S}_1^*$

is direct. The curve $\Sigma \cap \hat{S}_1^*$ is a collection of arcs and closed curves properly imbedded in $\hat{S}_1^*$ since $\hat{S}_1^*$ does not meet $\partial \Sigma$. Since the oriented boundary of $\Sigma \cap \hat{S}_1^*$ (that obviously vanishes in $H_0(X)$) represents the algebraic intersection of $(\partial \hat{S}_1^* = \pm \hat{\gamma}')$ with $\Sigma$, the lemma is proved.
Let us be slightly more explicit about this last argument. A point $x$ of the oriented boundary of $\Sigma \cap \hat S^*_1$ gets the sign $\varepsilon = \pm 1$ such that the tangent vector $\vec t_x(\Sigma \cap \hat S^*_1)$ of $\Sigma \cap \hat S^*_1$ at $x$ is oriented as $\varepsilon \vec n_x(\Sigma \cap \hat S^*_1)$ where $\vec n_x(\Sigma \cap \hat S^*_1)$ is the outward normal vector to the curve at $x$. (In other words, final points of arcs get a positive sign while initial points get a minus sign.) On the other hand, $\vec n_x(\Sigma \cap \hat S^*_1)$ may be identified to the outward normal of $\hat S^*_1$ and the triple
\[
\left( \vec n_x \left( \Sigma \cap \hat S^*_1 \right), \vec t_x \left( \partial \hat S^*_1 \right), \vec n_x \left( \hat S^*_1 \right) \right)
\]
is direct. Furthermore, $\vec n_x(\Sigma)$ may be identified with $\text{sign}(x)\vec t_x(\partial \hat S^*_1)$ where $\text{sign}(x)$ is the sign of the intersection of $\partial \hat S^*_1$ and $\Sigma$ at $x$. Thus, the triple
\[
\left( \varepsilon \vec n_x \left( \Sigma \cap \hat S^*_1 \right), \text{sign}(x)\vec t_x \left( \partial \hat S^*_1 \right), \vec n_x \left( \hat S^*_1 \right) \right)
\]
is direct, too. This shows that $\varepsilon = \text{sign}(x)$. Then the sum of the $\varepsilon$ vanishes because there are as many initial points of arcs in $\Sigma \cap \hat S^*_1$ as there are final points, and that makes the algebraic intersection of $\hat \gamma'$ and $\Sigma$ that is the sum of the $\text{sign}(x)$ vanish, too. This concludes the proof of Lemma 12.4 and the proof of the proposition. \hfill \Box

Note that Lemma 12.4 yields the following corollary.

**Corollary 12.5.** Let $X$ be a 3-manifold with Betti number 2. Suppose the lift $L$ of a typical fiber of the Abel-Jacobi map, to an infinite cyclic cover $C$ of $X$ is rationally zero-homologous. Equip $L$ with the framing induced by the Abel-Jacobi map. Then the self-linking of $L \subset C$ is zero.

This corollary is obviously not true when $C$ is replaced by the compact manifold $X$ itself, or by any finite cyclic cover of $X$.

**Acknowledgments**

We are grateful to A. Marin for suggesting and proving Proposition 12.1.

**References**

1. V. Bangert, C. Croke, S. Ivanov, and M. Katz, *Filling area conjecture and ovalless real hyperelliptic surfaces*, Geometric and Functional Analysis (GAFA) 15 (2005), no. 3, 577-597. See arXiv:math.DG/0405583
2. , *Boundary case of equality in optimal Loewner-type inequalities*, Trans. Amer. Math. Soc. 358 (2006). See arXiv:math.DG/0406008
3. V. Bangert and M. Katz, *Stable systolic inequalities and cohomology products*, Comm. Pure Appl. Math. 56 (2003), 979-997. math.DG/0204181
4. ______, An optimal Loewner-type systolic inequality and harmonic one-forms of constant norm, Comm. Anal. Geom. 12 (2004), no. 3, 703-732. See arXiv:math.DG/0304494
5. A.-M. Bergé and J. Martinet, Sur un problème de dualité lié aux sphères en géométrie des nombres, J. Number Theory 32 (1989), 14-42.
6. R. B. S. Brooks, R. F. Brown, and H. Schirmer, The absolute degree and the Nielsen root number of compositions and Cartesian products of maps, Theory of Fixed Points and its Applications (São Paulo, 1999), Topology Appl. 116 (2001), no. 1, 5-27.
7. D. Burago and S. Ivanov, Riemannian tori without conjugate points are flat, Geom. Funct. Anal. 4 (1994), no. 3, 259-269.
8. ______, On asymptotic volume of tori, Geom. Funct. Anal. 5 (1995), no. 5, 800-808.
9. I. Chavel, Riemannian Geometry–A Modern Introduction, Cambridge Tracts in Mathematics, 108, Cambridge University Press, Cambridge, 1993.
10. J. H. Conway, and N. J. A. Sloane, On lattices equivalent to their duals, J. Number Theory 48 (1994), no. 3, 373-382.
11. C. Croke and M. Katz, Universal volume bounds in Riemannian manifolds, Surveys in Differential Geometry VIII (2003), 109-137. Available at arXiv:math.DG/0302248
12. H. M. Farkas and I. Kra, Riemann Surfaces, Second edition, Graduate Texts in Mathematics 71, Springer-Verlag, New York, 1992.
13. H. Federer, Geometric Measure Theory, Springer 1969.
14. M. Gromov, Filling Riemannian manifolds, J. Diff. Geom. 18 (1983), 1-147.
15. ______, Systoles and intersystolic inequalities, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 291-362. Available at the following site: www.emis.de/journals/SC/1996/1/ps/smf_sem-cong_1_291-362.ps.gz
16. ______, Metric structures for Riemannian and non-Riemannian spaces, Progr. in Mathematics 152, Birkhäuser, Boston, 1999.
17. J. Hebda, The collars of a Riemannian manifold and stable isosystolic inequalities, Pacific J. Math. 121 (1986), 339-356.
18. S. Ivanov and M. Katz, Generalized degree and optimal Loewner-type inequalities, Israel J. Math. 141 (2004), 221-233. arXiv:math.DG/0405019
19. M. Katz, The filling radius of two-point homogeneous spaces, J. Diff. Geom. 18 (1983), 505-511.
20. ______, Diameter-extremal subsets of spheres, Discrete Comput. Geom. 4 (1989), 117-137.
21. ______, On neighborhoods of the Kuratowski imbedding beyond the first extremum of the diameter functional, Fundamenta Math. 137 (1991), 15-29.
22. ______, The rational filling radius of complex projective space, Topology and its Appl. 42 (1991), 201-215.
23. ______, Pyramids in the complex projective plane, Geom. Dedicata 40 (1991), 171-190.
24. M. Katz and Y. Rudyak, Lusternik-Schnirelmann category and systolic category of low dimensional manifolds, Communications on Pure and Applied Mathematics, 59 (2006). See arXiv:math.DG/0410456
25. M. Katz and Y. Rudyak, *Bounding volume by systoles of 3-manifolds*. Available at the site arXiv:math.DG/0504008

26. M. Katz and S. Sabourau, *Hyperelliptic surfaces are Loewner*, Proc. Amer. Math. Soc., to appear. See arXiv:math.DG/0407009

27. _____, *Entropy of systolically extremal surfaces and asymptotic bounds*, Ergodic Theory and Dynamical Systems **25** (2005), no. 4, 1209-1220. See arXiv:math.DG/0410312

28. _____, *An optimal systolic inequality for CAT(0) metrics in genus two*. See arXiv:math.DG/0501017

29. S. Kodani, *On two-dimensional isosystolic inequalities*, Kodai Math. J. **10** (1987), no. 3, 314-327.

30. H. T. Kung and C. E. Leiserson, *Systolic arrays (for VLSI)*, Sparse Matrix Proceedings 1978 (Sympos. Sparse Matrix Comput., Knoxville, Tenn., 1978), SIAM, Philadelphia, Pa., 1979, pp. 256-282.

31. C. Kuratowski, *Quelques problèmes concernant les espaces métriques non-séparables*, Fund. Math. **25** (1935), 534-545.

32. J. C. Lagarias, H. W. Lenstra, Jr. and C.P. Schnorr, *Bounds for Korkin-Zolotarev reduced bases and successive minima of a lattice and its reciprocal lattice*, Combinatorica **10** (1990), 343-358.

33. C. Lescop, *Global surgery formula for the Casson-Walker invariant*, Annals of Mathematics Studies **140**, Princeton University Press, Princeton 1996.

34. A. Lichnerowicz, *Applications harmoniques dans un tore*, C.R. Acad. Sci., Sér. A, **269** (1969), 912-916.

35. L. Liu, *The mapping properties of filling radius and packing radius and their applications*, Differential Geom. Appl. **22** (2005), no. 1, 69-79.

36. J. Milnor and D. Husemoller, *Symmetric Bilinear Forms*, Springer, 1973.

37. R. Miranda, *Algebraic Curves and Riemann Surfaces*, Graduate Studies in Mathematics **5**, American Mathematical Society, Providence, RI, 1995.

38. G. P. Paternain, *Schrödinger operators with magnetic fields and minimal action functionals*, Israel J. Math. **123** (2001), 1-27.

39. P.M. Pu, *Some inequalities in certain nonorientable Riemannian manifolds*, Pacific J. Math. **2** (1952), 55-71.

40. S. Sabourau, *Systoles des surfaces plates singulières de genre deux*, Math. Zeitschrift **247** (2004), no. 4, 693–709.

41. _____, *Entropy and systoles on surfaces*, preprint.

42. _____, *Systolic volume and minimal entropy of aspherical manifolds*, preprint.

43. J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics **7**, Springer-Verlag, New York-Heidelberg, 1973.