Non-diagonal reflection for the non-critical $XXZ$ model

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Abstract
The most general physical boundary $S$-matrix for the open $XXZ$ spin chain in the non-critical regime ($\cosh(\eta) > 1$) is derived starting from the bare Bethe ansatz equations. The boundary $S$-matrix as expected is expressed in terms of $\Gamma_1q$-functions. In the isotropic limit the corresponding results for the open $XXX$ chain are also reproduced.

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1. Introduction
The open $XXZ$ model is considered as one of the prototype models in describing a plethora of interesting boundary phenomena, and as such has attracted much attention especially after the derivation of the spectrum in the generic case where non-diagonal boundary magnetic fields are applied [1–5]. Our objective in the present study is to derive from first principles the most general physical boundary $S$-matrix for the $XXZ$ chain in the non-critical (massive) regime.

Diagonal boundary $S$-matrices for the open $XXZ$ model in the non-critical regime were extracted in [6] using vertex-operator techniques, while parallel results were obtained in [7] from the Bethe ansatz point of view (see, e.g., [8–10]). Similarly, diagonal reflection matrices were derived in [11, 12] for the critical $XXZ$ model corresponding to the sine-Gordon boundary $S$-matrix for ‘fixed’ boundary conditions [13]. After the derivation of the exact spectrum and Bethe equations for the $XXZ$ chain with non-diagonal boundaries the generic boundary $S$-matrix for the critical $XXZ$ chain was computed in [14, 15] corresponding to the boundary $S$-matrix of sine-Gordon model [13] for ‘free’ boundary conditions. A relevant discussion on the generic breather boundary $S$-matrix within the $XXZ$ framework may also be found in [14]. Note also that analogous results regarding diagonal and non-diagonal solitonic boundary $S$-matrices were formulated in [16, 17] using the so-called nonlinear integral equation (NLIE) method [18].

To extract the generic boundary $S$-matrix for the non-critical $XXZ$ model we follow the logic of [14], i.e. we focus on the open chain with a trivial left boundary and a generic non-diagonal right boundary associated with the full $K$-matrix [13, 19]. As also noted in [14]...
the main advantage of the approach adopted—considering special boundary conditions—is that one eventually deals with a simple set of Bethe ansatz equations similar to those of the XXZ chain with two diagonal boundaries. Thus all relevant computations are drastically simplified (see also [14]), and one may follow the logic described in [7, 12, 20, 21] for purely diagonal boundary magnetic fields. Ultimately, the boundary $S$-matrix eigenvalues are extracted directly from the Bethe equations and are expressed in terms of $\Gamma_q$-functions ($q = e^{-\eta}$) [22] as in [6, 7], where only diagonal boundaries are assumed. In the isotropic limit $q \to 1$, the corresponding rational boundary $S$-matrix for the XXX open chain is also recovered [23].

2. Bethe ansatz and boundary $S$-matrix

Before we proceed with the Bethe ansatz analysis it will be useful for our purposes here to give the explicit expressions of the right and left boundary $K$-matrices that give rise to the open Hamiltonian under consideration:

$$
\mathcal{H} = -\frac{1}{4} \sum_{i=1}^{N-1} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh(\eta) \sigma_i^z \sigma_{i+1}^z \right) - \frac{N}{4} \cosh(\eta) + \frac{\sinh(\eta)}{4} \sigma_N^z
$$

where in the non-critical regime we are focusing here $\cosh(\eta) > 1$, also $\sigma^x,y,z$ are the $2 \times 2$ Pauli matrices, and the boundary parameters $\xi, \kappa, \theta$ are the free parameters of the generic $K$-matrix [13, 19], which will be introduced subsequently.

To obtain such a Hamiltonian we consider the open chain constructed using Sklyanin’s formalism [24], with left boundary $K^+ \propto I$ and right boundary associated with the general solution of the reflection equation [25] given in [13, 19], i.e.,

$$
K^-(\lambda) = \begin{pmatrix} \sin[\eta(-\lambda + i\xi)] e^{i\eta \lambda} & \kappa e^{i\eta \lambda} \sin(2\eta \lambda) \\ \kappa e^{-i\eta \lambda} \sin[\eta(\lambda + i\xi)] & \sin[\eta(\lambda + i\xi)] e^{-i\eta \lambda} \end{pmatrix}.
$$

The latter $K$-matrix has two eigenvalues given as follows:

$$
\varepsilon_1(\lambda) = 2\kappa \sin[\eta(\lambda + ip^+)] \sin[\eta(\lambda + ip^-)]
$$

$$
\varepsilon_2(\lambda) = 2\kappa \sin[\eta(\lambda - ip^+)] \sin[\eta(\lambda - ip^-)]
$$

where the parameters $p^\pm$ are defined as:

$$
e^{i\eta \xi} = \frac{1}{2\kappa} i \sin[\eta(p^+ \pm p^-)].
$$

Note that we assume here the parametrization used in [13] in the sine-Gordon context, (see also [14] and the references therein). Such a parametrization is also quite practical within the Temberley–Lieb algebra framework [26]. The parameter $\theta$ appearing in (2.2) may be removed by means of a simple gauge transformation, that leaves the XXZ $R$-matrix invariant, and henceforth we consider it for simplicity to be zero (see also [13, 14]). The $K$-matrix (2.2) may be easily diagonalized by virtue of a constant ($\lambda$-independent) gauge transformation:

$$
\text{diag}(\varepsilon_1(\lambda), \varepsilon_2(\lambda)) = M^{-1}(p^+, p^-) K(\lambda) M(p^+, p^-)
$$

where $M$ is defined as

$$
M(p^+, p^-) = \begin{pmatrix} 1 & \frac{1}{i e^{\eta(p^+ + p^-)}} \\ i e^{\eta(p^+ - p^-)} & 1 \end{pmatrix}.
$$

Note that $M^{-1}(p^+, p^-) K(p^+, p^-)$ is a constant matrix (see also [13, 14]).
Note that the above transformation modifies dramatically the XXZ $R$-matrix, so it is not possible to simply implement a global gauge transformation changing the basis in order to diagonalize the open transfer matrix as in, e.g., [27]. It is also worth pointing out the similarity between the matrix $\mathcal{M}(p^+, p^-)$ and the local gauge transformation employed for the diagonalization of the open XXZ transfer matrix with non-diagonal boundaries [1, 28, 29].

We recall now the exact Bethe ansatz equations for the open XXZ chain in the case of a right non-diagonal boundary and a left trivial diagonal. The Bethe equations in this case reduce to the following simple form (see also [14]):

$$
\sin\left[\eta(\lambda_i - \frac{i}{2}(2p^+ + 1))\right] \sin\left[\eta(\lambda_i - \frac{i}{2}(2p^+ - 1))\right] \cos\left[\eta(\lambda_i + \frac{i}{2})\right] \left(\frac{\sin\left[\eta(\lambda_i + \frac{i}{2})\right]}{\sin\left[\eta(\lambda_i - \frac{i}{2})\right]}\right)^{2N+1}
$$

$$
= -\prod_{j=1}^{M} \frac{\sin[\eta(\lambda_i - \lambda_j + i)] \sin[\eta(\lambda_i + \lambda_j + i)]}{\sin[\eta(\lambda_i - \lambda_j - i)] \sin[\eta(\lambda_i + \lambda_j - i)]}.
$$

(2.7)

We consider here, without loss of generality $\eta > 0$, $p^\pm > \frac{1}{2}$ and $\text{Re} (\lambda_0) \in [0, \frac{\pi}{2}] \lambda_0 \neq 0, \frac{\pi}{2}$ (see, e.g., [20] for details on this restriction). For relevant results on various representations of $U_h(sl_2)$ see [28–30].

As pointed out in [14] the integer $M$ is associated with a non-local conserved quantity $S$, which has the same spectrum as $S^z$ (for more details we refer the interested reader to [14, 28] and references therein), i.e.,

$$
M = \frac{N}{2} - S_z,
$$

(2.8)

the subscript $\varepsilon$ stands for the eigenvalue.

Our objective now is to explicitly derive the physical boundary $S$-matrix, and in particular the relevant overall physical factor, which provides in general significant information on the existence of boundary bound states. We define the boundary $S$-matrices $k^\pm$ by the quantization condition [10, 20]

$$
(e^{i\eta(\lambda)N} k^+ k^- - 1 |\tilde{\lambda}\rangle = 0.
$$

(2.9)

Here $\tilde{\lambda}$ is the rapidity of the ‘hole’ –particle-like excitation and $p(\tilde{\lambda})$ is the momentum of the hole.

The density of a state is obtained in a standard way from the Bethe ansatz equations after taking the log and the derivative [7, 9, 10, 20, 21]. More precisely, the Fourier transform of the density for the one-hole state turns out to be

$$
\hat{\sigma}_1(\omega) = 2\varepsilon(\omega) + \frac{1}{N} \frac{\hat{a}_2(\omega)}{1 + \hat{a}_2(\omega)} (e^{i\hat{\omega} \tilde{\lambda}} + e^{-i\hat{\omega} \tilde{\lambda}})
$$

$$
+ \frac{1}{N} \frac{1}{1 + \hat{a}_2(\omega)} [\hat{a}_1(\omega) + \hat{a}_2(\omega) + \hat{b}_1(\omega) - \hat{a}_{2p^-+1}(\omega) - \hat{a}_{2p^+}(\omega)]
$$

(2.10)

where we define the following Fourier transforms

$$
\hat{a}_n(\omega) = e^{-\eta n |\omega|}, \quad \hat{b}_n(\omega) = (-)^n \hat{a}_n(\omega), \quad \varepsilon(\omega) = \frac{\hat{a}_1(\omega)}{1 + \hat{a}_2(\omega)} = \frac{1}{2 \cosh (\omega/2)}
$$

(2.11)

where $\varepsilon(\tilde{\lambda})$ corresponds also to the energy of the particle-like excitation. The similarity of the latter formula (2.10) with the one obtained in the case of two diagonal boundaries [7, 20] is indeed noticeable. This is a crucial point enabling a simplified derivation of the boundary $S$-matrix. In our case both terms depending on $p^\pm$ are assigned to the right boundary, otherwise one follows the logic of the fully diagonal case (see e.g. [7, 20]).
The boundary matrix $k^-$, of the generic form (2.2) has two eigenvalues $k_{1,2}$ whereas
the left boundary matrix is trivial $k^+(\lambda) = k_{0}(\lambda)I$. From the density (2.10), the quantization
condition (2.9) and recalling that $\epsilon(\lambda) = \frac{1}{2} \frac{d\ln(\epsilon_0)}{dx}$ we can explicitly derive the quantities $k_0, k_{1,2}$
(see, e.g. [7, 20, 21] for more details). Actually, the eigenvalues $k_1(\lambda, p^\pm)$ and $k_2(\lambda, p^\pm)$ may
be seen as the boundary scattering amplitudes for the one-particle-like excitation with $S = +\frac{1}{2}$
and $S = -\frac{1}{2}$, respectively (see also [14] for more details).

We first compute the eigenvalue $k_1$, which is expressed in terms of the $\Gamma_q(x)$-function,
—the $q$-analogue of the Euler gamma function— ($q = e^{-\eta}$) defined [22] as

$$\Gamma_q(x) = (1 - q)^{-x} \prod_{j=0}^{\infty} \left( \frac{(1 - q^{j+1})}{(1 - q^{j+1})^4} \right), \quad 0 < q < 1. \quad (2.12)$$

Using also the $q$-analogue of the duplication formula [22]

$$\Gamma_q(2x)\Gamma_q\left(\frac{x}{2}\right) = (1 + q)^{2x-1}\Gamma_q(x)\Gamma_q(x + \frac{1}{2}), \quad (2.13)$$

we obtain the following result for the first eigenvalue $k_1(\lambda, p^+, p^-)$ (up to a constant phase factor):

$$k_1(\lambda, p^+, p^-) = 2\kappa \left[ \eta(\lambda + \frac{i}{2}(2p^+ - 1)) \right] \left[ \eta(\lambda + \frac{i}{2}(2p^- - 1)) \right]$$

$$\times k_0(\lambda)k_1(\lambda, p^+)k_1(\lambda, p^-) \quad (2.14)$$

where we define:

$$k_0(\lambda) = q^{-4i\lambda} \frac{\Gamma_q\left(\frac{\lambda}{2} + \frac{1}{2}\right) \Gamma_q\left(\frac{\lambda}{2} + 1\right)}{\Gamma_q\left(\frac{\lambda}{2} + \frac{1}{2}\right) \Gamma_q(\frac{\lambda}{2} + 1)} \quad (2.15)$$

$$k_1(\lambda, x) = \frac{(2\kappa)^{\frac{1}{2}} \Gamma_q\left(\frac{\lambda}{2} + \frac{1}{2}(2x - 1)\right) \Gamma_q\left(\frac{\lambda}{2} + \frac{1}{2}(2x + 1)\right)}{\sin[\eta(\lambda - \frac{i}{2}(2x - 1))] \Gamma_q\left(\frac{\lambda}{2} + \frac{1}{2}(2x - 1)\right) \Gamma_q\left(\frac{\lambda}{2} + \frac{1}{2}(2x + 1)\right)}. \quad (2.16)$$

We turn now to the computation of the second eigenvalue $k_2(\lambda, p^\pm)$, corresponding to a one-
hole state with $S = -\frac{1}{2}$. We implement the ‘duality’ transformation [14, 24], which modifies
the boundary parameters $p^\pm \rightarrow -p^\pm$ in the Bethe ansatz equations (2.7). This transformation
is the equivalent of deriving the Bethe ansatz equations starting from the second reference state
(the analogue of the ‘spin down’ state) [30, 31]. Then we conclude for the second eigenvalue:

$$\frac{k_1(\lambda, p^+, p^-)}{k_2(\lambda, p^+, p^-)} = \frac{\sin[\eta(\lambda + \frac{i}{2}(2p^+ - 1))] \sin[\eta(\lambda + \frac{i}{2}(2p^- - 1))]}{\sin[\eta(\lambda - \frac{i}{2}(2p^+ - 1))] \sin[\eta(\lambda - \frac{i}{2}(2p^- - 1))]} \quad (2.17)$$

An alternative way to extract the second eigenvalue is instead of the ‘kink’ state with $S = -\frac{1}{2}$,
—after implementing the duality transformation—to consider the anti-kink state consisting
of a hole and a two-string state. Such configurations have been utilized in deriving the
kink–antikink scattering amplitudes in the bulk XXZ model (see, e.g., [7]) as well as in open
XXZ chain with the most general boundary conditions [15], where the ‘duality’ $p^\pm \rightarrow -p^\pm$
cannot be implemented for the derivation of the second eigenvalue of the boundary $S$-matrix.
Note that the term depending on the boundary parameters (2.16) is ‘double’ compared to the
diagonal case studied in [6, 7]. Analogous phenomenon occurs in the open critical XXZ chain
[14, 15] and the sine-Gordon model [13]. It is straightforward to see that in the diagonal limit
we recover the results of [6, 7]. Also, in the isotropic limit $q \rightarrow 1$, $\Gamma_q(x) \rightarrow \Gamma(x)$ and the
trigonometric functions turn to rational, hence the generic rational reflection matrix for the
open XXX spin chain is easily recovered (see also [23]).
It is finally convenient to rewrite the two eigenvalues in terms of ‘renormalized’ boundary parameters $\tilde{p}^\pm$ defined as
\[ \tilde{p}^\pm = p^\pm - \frac{1}{2 \eta} \left( \frac{i \pi}{\eta} \right) \] (2.18)
then the similarity between (2.17) and the ratio of the ‘bare’ eigenvalues (2.3) becomes apparent. We have actually derived the physical boundary $S$-matrix up to a gauge transformation; indeed the $S$-matrix of the generic form (2.2) may be reproduced by:
\[ k(\lambda, \tilde{p}^+, \tilde{p}^-) = M(\tilde{p}^+, \tilde{p}^-) \text{diag}(k_1(\lambda), k_2(\lambda)) M^{-1}(\tilde{p}^+, \tilde{p}^-). \] (2.19)
$M$ is defined in (2.5). This concludes our derivation of the general boundary $S$-matrix for the open XXZ chain in the non-critical regime.

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References

[1] Cao J, Lin H-Q, Shi K-j and Wang Y 2003 Nucl. Phys. B 663 487
[2] Nepomechie R I 2003 J. Stat. Phys. 111 1363
[3] Murgan R, Nepomechie R I and Shi C 2006 J. Stat. Mech. 0608 P006
[4] Baseilhac P and Koizumi K 2007 Exact spectrum of the XXZ open spin chain from the q-Onsager algebra representation theory Preprint hep-th/0703106
[5] Galleas W 2007 Functional relations from the Yang–Baxter algebra: eigenvalues of the XXZ model with non-diagonal twisted and open boundary conditions Preprint 0708.0009
[6] Jimbo M, Kedem R, Kojima T, Konno H and Miwa T 1995 Nucl. Phys. B 441 437
[7] Doikou A, Mezincescu L and Nepomechie R I 1998 J. Phys. A: Math. Gen. 31 53
[8] Faddeev L D and Takhtajan L A 1984 J. Sov. Math. 24 241
[9] Faddeev L D and Takhtajan L A 1981 Phys. Lett. 85A 375
[10] Korepin V E 1980 Theor. Math. Phys. 76 165
[11] Korepin V E, Izergin G and Bogoliubov N M 1993 Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz (Cambridge: Cambridge University Press)
[12] Andrei N and Destri C 1984 Nucl. Phys. B 231 445
[13] Fendley P and Saleur H 1994 Nucl. Phys. B 428 881
[14] Doikou A and Nepomechie R I 1999 J. Phys. A: Math. Gen. 32 3663
[15] Ghoshal S and Zamolodchikov A B 1994 Int. J. Mod. Phys. A 9 3841
[16] Ghoshal S and Zamolodchikov A B 1994 Int. J. Mod. Phys. A 9 4353
[17] Doikou A 2007 Generic boundary scattering in the open XXZ chain Preprint 0711.0716
[18] Murgan R 2007 Boundary $S$ matrix of an open XXZ spin chain with nondiagonal boundary terms Preprint 0711.1631
[19] LeClair A, Mussardo G, Saleur H and Skorik S 1995 Nucl. Phys. B 453 581
[20] Ahn C, Bellacosa M and Ravanini F 2004 Phys. Lett. B 595 537
[21] Ahn C and Nepomechie R I 2004 Nucl. Phys. B 676 637
[22] Ahn C, Bajnok Z, Nepomechie R I, Palla L and Takacs G 2005 Nucl. Phys. B 714 307
[23] Klumper A, Batchelor M T and Pearce P A 1991 J. Phys. A: Math. Gen. 24 3111
[24] Destri C and Vega H J de 1992 Phys. Rev. Lett. 69 2313
[25] Vega H J de and Gonzalez-Ruiz A 1994 J. Phys. A: Math. Gen. 27 6129
[26] Grisaru M T, Mezincescu L and Nepomechie R I 1995 J. Phys. A: Math. Gen. 28 1027
[27] Doikou A, Mezincescu L and Nepomechie R I 1997 J. Phys. A: Math. Gen. 30 L507
[28] Gasper G and Rahman M 1990 Basic Hypergeometric Series (Cambridge: Cambridge University Press)
[29] MacKay N J and Short B J 2003 Commun. Math. Phys. 233 313
[30] MacKay N J and Short B J 2004 Commun. Math. Phys. 245 425 (erratum)
[31] Sklyanin E K 1988 J. Phys. A: Math. Gen. 21 2375
[32] Cherednik I V 1984 Theor. Math. Phys. 61 977

5
[26] Doikou A and Martin P P 2003 J. Phys. A: Math. Gen. 36 2203
[27] Arnaudon D, Avan J, Crampe N, Doikou A, Frappat L and Ragoucy E 2004 J. Stat. Mech. 0408 P005
[28] Doikou A 2006 J. Stat. Mech. 0605 P010
[29] Doikou A 2003 Nucl. Phys. B 668 447
Doikou A 2007 Phys. Lett. A 366 556
[30] Frappat L, Nepomechie R I and Ragoucy E 2007 J. Stat. Mech. 09 P09008
[31] Yang W-L and Zhang Y-Z 2007 J. High Energy Phys. JHEP04(2007)044