MA CKEY FORMULA FOR BISETS OVER GROUPOIDS

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Abstract. In this paper we establish the Mackey formula for groupoids, extending the well known formula in abstract groups context. This formula involves the notion of groupoid-biset, its orbit set and the tensor product over groupoids, as well as cosets by subgroupoids.

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1. Introduction

We will describe the motivations behind our work and how the Mackey formula for groupoids fits into the contemporary mathematical framework. Thereafter, we will briefly describe our main result.

1.1. Motivations and overview. The classical Mackey formula, which deals with linear representations of finite groups, appeared for the first time in [18, Theorem 1]. Roughly speaking, this formula involves simultaneously the restriction and the induction functors (with respect to two different subgroups) and gives a decomposition, as direct sum, of their composition, although in a non canonical way. As was explained in [23, Section 7.4], the Mackey formula is a key tool in proving Mackey irreducibility criterion, which gives necessary and sufficient conditions for the irreducibility of an induced representation, and proves to be useful to study linear representations of a semidirect product by an abelian group, see [23, Proposition 25]. Another formulation of the classical Mackey formula, using modules over groups algebras, was stated in [8, Theorem 44.2]. Successively, in [19] Theorems 7.1 and 12.1, the Mackey formula was extended to the context of locally compact groups (with opportune hypotheses), and used to prove a generalization of the Frobenius Reciprocity Theorem, see [19] Theorems 8.1, 8.2 and 13.1. Later on, many variants and different formulations of the Mackey formula have been investigated. For example, in [24], [1] and [2] Taylor and Bonnafé proved opportune versions of this formula for algebraic groups. The importance of Mackey formula version in this context had already been made clear in [9] and previous work had been done in [10, Theorem 6.8], [17, Lemma 2.5] and [11, Theorem 7].

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Apparently, classical Mackey formula is so intuitive that can be applied, somehow in a non trivial way, in more general and different contexts. In this direction, motivated by the study of the structure of biset functors over finite groups (see [3] Definitions 3.1.1, 3.2.2 ] for pertinent definitions), Serge Bouc proved in [3] Lemma 2.3.24] a different kind of the classical Mackey formula in the framework of group-bisets. The gist is that, given two groups $H$ and $G$ and a field $\mathbb{F}$, an $(H, G)$-biset (of groups) is a left $H$-invariant and right $G$-invariant $\mathbb{F}$-basis of an $(\mathbb{F}H, \mathbb{F}G)$-bimodule. Since the classical Mackey formula on linear representations can be rephrased using bimodules, and bimodules induce bisets, the Mackey formula can be reformulated using an isomorphism of group-bisets (see the end of [3] Section 1.1.5]) which is furthermore reformulated in [3] Lemma 2.3.24]. We have to mention that, in [4], Bouc himself proved an additional version of the Mackey formula, which is expressed using bimodules and group-bisets.

Groupoids are natural generalization of groups and prove to be useful in different branches of mathematics, see [5] and [6] (and the references therein) for a brief survey. Specifically, a groupoid is defined as a small category whose every morphism is an isomorphism and can be thought as a “group with many objects”. In the same way, a group can be seen as a groupoid with only one object. As explained in [5], while a groupoid in its very simple facet can be seen as a disjoint union of groups, this forces unnatural choices of base points and obscures the overall structure of the situation. Besides, even under this simplicity, structured groupoids, like topological or differential groupoids, cannot even be thought like a disjoint union of topological or differential groups, respectively. Different specialists realized (see for instance [5] and [7] page 6-7]) in fact that the passage from groups to groupoids is not a trivial research and have its own difficulties to overcome.

This paper, which fits in this line of research, is a long term project on groupoids representation theory that aims, among other things, to extends the results of [3] to the context of groupoid-bisets. Our main aim here, is to extended the formula in [3] Lemma 2.3.24] to groupoid-bisets. The paper is written in very elementary language, in order to make its content accessible to all kinds of readers.

1.2. Description of the main result. Let $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{K}$ three groupoids and consider $M$ and $L$ two sub-groupoids of $\mathcal{K} \times \mathcal{H}$ and $\mathcal{H} \times \mathcal{G}$, respectively, with $M_0 = \mathcal{K} \times \mathcal{H}_0$ and $L_0 = \mathcal{H}_0 \times \mathcal{G}_0$. Let $(\mathcal{K} \circ \mathcal{H})^\times$ and $(\mathcal{H} \circ \mathcal{G})^\times$ be the corresponding left cosets of $M$ and $L$, respectively (see the left version of Definition 3.5). Consider the following $(M, L)$-biset, see Proposition 4.7

$$X = \left\{ (w, u, h, v, a) \in \mathcal{K} \times \mathcal{H}_0 \times \mathcal{H} \times \mathcal{H}_0 \times \mathcal{G}_0 \mid (w, u) \in M_0, (v, a) \in L_0, u = \alpha(h), v = \beta(h) \right\}.$$ 

Denote by $\text{rep}_{M,L}(X)$ the set of representatives of the orbits of $X$ as $(M, L)$-biset. For each element $(w, u, h, v, a) \in \text{rep}_{M,L}(X)$, we define as in Definition 4.3 the subgroupoid $M^{a,v} \ast \left((h, a), \mathcal{L}_0^{u-1}\right)$ (with only one object) of the groupoid $\mathcal{K} \times \mathcal{G}$. Our main result stated as Theorem 4.8 below says:

**Theorem A** (Mackey Formula for groupoid-bisets). There is a (non canonical) isomorphism of bisets

$$\left(\mathcal{K} \times \mathcal{H}\right)_{\delta0} \otimes_{\mathcal{M}} \left(\mathcal{H} \times \mathcal{G}\right)_{\delta0} \cong \bigoplus_{(w, u, h, v, a) \in \text{rep}_{M,L}(X)} \left(\mathcal{K} \times \mathcal{G}\right)_{\delta0} \otimes_{\mathcal{M}^{a,v} \ast \left((h, a), \mathcal{L}_0^{u-1}\right)}^\times,$$

where the symbol $-\otimes_{\mathcal{M}}-$ stand for the tensor product over $\mathcal{H}$, and where the right hand-side term in the formula is a direct sum in the category of $(M, L)$-biset of left cosets.

2. Abstract groupoids: General notions and basic properties

After providing the basic definitions about groupoids, we will demonstrate, with many examples, the relevance of the notion. After that, we will expound the main concepts regarding groupoid actions: equivariant maps, orbit sets and stabilizers.

2.1. Notations, basic notions and examples. A groupoid is a small category where each morphism is an isomorphism. That is, a pair of two sets $\mathcal{G} := (\mathcal{G}, \mathcal{G}_0)$ with diagram of sets

$$\mathcal{G} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{G}_0.$$
where \( s \) and \( t \) are resp. the source and the target of a given arrow, and \( t \) assigns to each object its identity arrow; together with an associative and unital multiplication \( G_1 := G_1 \times, G_1 \rightarrow G_1 \), as well as a map \( G_1 \rightarrow G_1 \), which associated to each arrow its inverse. Notice, that \( t \) is an injective map, and so \( G_1 \) is identified with a subset of \( G_1 \). A groupoid is then a (small) category with more structure, namely, the map which send any arrow to its inverse. We implicitly identify a groupoid with its underlying category. Interchanging the source and the target will lead to the opposite groupoid which we denote by \( G^\op \).

Given a groupoid \( G \), consider two objects \( x, y \in G_0 \). We denote by \( G(x, y) \) the set of all arrows with source \( x \) and target \( y \). The isotropy group of \( G \) at \( x \), is then the group:

\[
G^x := G(x, x) = \{ g \in G_1 \mid s(g) = t(g) = x \}.
\]

(2) Clearly each of the sets \( G(x, y) \) is, by the groupoid multiplication, a left \( G^x \)-set and right \( G^y \)-set. In fact, each of the \( G(x, y) \)'s is a \((G^x, G^y)\)-biset, in the sense of [3].

A morphism of groupoids \( \phi : \mathcal{H} \rightarrow G \) is a functor between the underlying categories. Obviously any such a morphism induces homomorphisms of groups between the isotropy groups: \( \phi^y : \mathcal{H}^y \rightarrow G^{y\circ \phi} \), for every \( y \in H_0 \). The family of homomorphisms \( \{\phi^y\}_{y \in H_0} \) is referred to as the isotropy maps of \( \phi \). For a fixed object \( x \in G_0 \), its fibre \( \phi^{-1}_x((x)) \), if not empty, leads to the following "star" of homomorphisms of groups:

\[
\xymatrix{ H \ar[rd] & \mathcal{H} \ar[ld] \ar[d] \ar[rd] \ar[d] \ar[r] & G \ar[ld] \ar[rd] \ar[d] \ar[r] & \mathcal{H} \ar[ld] \ar[d] \ar[r] & H \ar[l] \ar[rrd] & \mathcal{H} \ar[l] \ar[rru] & G \ar[l] } \nonumber
\]

where \( y \) runs in the fibre \( \phi^{-1}_x((x)) \).

**Example 2.1 (Trivial and product groupoids).** Let \( X \) be a set. Then the pair \( (X, X) \) is obviously a groupoid (in fact a small discrete category, i.e., with only identities as arrows) with trivial structure. This is known as the trivial groupoid.

The **product** \( G \times \mathcal{H} \) **of two** groupoids \( G \) and \( \mathcal{H} \) is the groupoid whose set of objects is the Cartesian product \( G_0 \times \mathcal{H}_0 \) and set of arrows \( G_1 \times \mathcal{H}_1 \). The multiplication, inverse and units arrows are canonically given as follows:

\[
(g, h)(g', h') = (gg', hh'), \quad (g, h)^{-1} = (g^{-1}, h^{-1}), \quad (g, e) = (g, i_s).
\]

**Example 2.2 (Action groupoid).** Any group \( G \) can be considered as a groupoid by taking \( G_1 = G \) and \( G_0 = \{*\} \) (a set with one element). Now if \( X \) is a right \( G \)-set with action \( \rho : X \times G \rightarrow X \), then one can define the so called **the action groupoid** \( \mathcal{G} \) whose set of objects is \( G_0 = X \) and its set of arrows is \( G_1 = X \times G \); the source and the target are \( s = \rho \) and \( t = pr_1 \), the identity map sends \( x \mapsto (x, e) = \iota_x \), where \( e \) is the identity element of \( G \). The multiplication is given by \( (x, g)(x', g') = (xg, gg') \), whenever \( xg = x' \), and the inverse is defined by \( (x, g)^{-1} = (xg, g^{-1}) \). Clearly the pair of maps \( \{ pr_2, \iota \} : \mathcal{G} = (G_1, G_0) \rightarrow (G, \{\iota\}) \) defines a morphism of groupoids. For a given \( x \in X \), the isotropy group \( \mathcal{G} \) is clearly identified with the stabilizer \( \text{Stab}_G(x) = \{ g \in G \mid gx = x \} \) subgroup of \( G \).

**Example 2.3 (Equivalence relation groupoid).** We expound here several examples ordered by inclusion.

1. One can associated to a given set \( X \) the so called the **groupoid of pairs** (called **fine groupoid in** [5] and **simplicial groupoid in** [13]), its set of arrows is defined by \( G_1 = X \times X \) and the set of objects by \( G_0 = X \); the source and the target are \( s = pr_1 \) and \( t = pr_2 \), the second and the first projections, and the map of identity arrows \( \iota \) is the diagonal map \( x \mapsto (x, x) \). The multiplication and the inverse maps are given by

\[
(x, x')(x', x'') = (x, x'') \quad \text{and} \quad (x, x')^{-1} = (x', x).
\]

2. Let \( v : X \rightarrow Y \) be a map. Consider the fibre product \( X \times X \), \( X \cong X \), as a set of arrows of the groupoid \( X \times X \), \( X \times X = \{ (x, x') \mid x, x' \in X \} \), where as before \( s = pr_1 \) and \( t = pr_2 \), and the map of identity arrows \( \iota \) is the diagonal map. The multiplication and the inverse are clear.

3. Assume that \( R \subseteq X \times X \) is an equivalence relation on the set \( X \). One can construct a groupoid \( \mathcal{R} \), with structure maps as before. This is an important class of groupoids known as the **groupoid of equivalence relation** (or **equivalence relation groupoid**). Obviously \( (R, X) \mapsto (X \times X, X) \) is a morphism of groupoid, see for instance [13] Exemple 1.4, page 301].
Notice, that in all these examples each of the isotropy groups is the trivial group.

Example 2.4 (Induced groupoid). Let \( \mathcal{G} = (\mathcal{G}_r, \mathcal{G}_t) \) be a groupoid and \( \varsigma : X \to \mathcal{G}_0 \) a map. Consider the following pair of sets:

\[
\mathcal{G}^r := X \times X, \quad \mathcal{G}_t \times X = \{(x, g, x') \in X \times G_r \times X \mid \varsigma(x) = \rho(g), \varsigma(x') = \varsigma(g)\}, \quad \mathcal{G}^t := X.
\]

Then \( \mathcal{G}^r = (\mathcal{G}_r, \mathcal{G}^r_0) \) is a groupoid, with structure maps: \( \varsigma = pr_r, \quad t = pr_t, \quad \varsigma^{-1} = (\varsigma(x), \varsigma'(x)) \), \( x \in X \).

The multiplication is defined by \( (x, g, y)(x', g', y') = (x, gg', y') \), whenever \( y = x' \), and the inverse is given by \( (x, g, y)^{-1} = (y, g^{-1}, x) \). The groupoid \( \mathcal{G}^r \) is known as the induced groupoid of \( \mathcal{G} \) by the map \( \varsigma \), (or the pull-back groupoid of \( \mathcal{G} \) along \( \varsigma \), see [13] for dual notion). Clearly, there is a canonical morphism \( \phi^r := (pr_r, \varsigma) : \mathcal{G}^r \to \mathcal{G} \) of groupoids. A particular instance of an induced groupoid, is the one when \( \mathcal{G} = G \) is a groupoid with one object. Thus for any group \( G \) one can consider the Cartesian product \( X \times G \times X \) as a groupoid with set of objects \( X \).

2.2. Groupoid actions and equivariant maps. The following definition is a natural generalization to the context of groupoids, of the usual notion of group-set, see for instance [3]. It is an abstract formulation of that given in [20] Definition 1.6.1 for Lie groupoids, and essentially the same definition based on the Sets-bundles notion given in [22] Definition 1.11.

Definition 2.5. Let \( \mathcal{G} \) be a groupoid and \( \varsigma : X \to \mathcal{G}_0 \) a map. We say that \( (X, \varsigma) \) is a right \( \mathcal{G} \)-set (with a structure map \( \varsigma \)), if there is a map (the action) \( \rho : X \times \mathcal{G}_r \to X \) sending \((x, g) \mapsto xg \) and satisfying the following conditions:

1. \( \varsigma(g) = \varsigma(xg) \), for any \( x \in X \) and \( g \in \mathcal{G}_r \), with \( \varsigma(x) = \rho(g) \).
2. \( x \varsigma (x) = x \), for every \( x \in X \).
3. \( \varsigma(xg)h = \varsigma(xgh) \), for every \( x \in X, g, h \in \mathcal{G}_r \), with \( \varsigma(x) = \rho(g) \) and \( \varsigma(h) = \rho(s(g)) \).

In order to simplify the notation, the action map of a given right \( \mathcal{G} \)-set \((X, \varsigma)\) will be omitted from the notation. A left action is analogously defined by interchanging the source with the target and similar notations will be employed. In general a set \( X \) with a (right or left) groupoid action is called a groupoid-set, we also employ the terminology: a set \( X \) with a left (or right) \( \mathcal{G} \)-action.

Obviously, any groupoid \( \mathcal{G} \) acts over itself on both sides by using the regular action, that is, the multiplication \( \mathcal{G}_r \times \mathcal{G}_r \to \mathcal{G}_r \). This means that \((\mathcal{G}_r, \times)\) is a right \( \mathcal{G} \)-set and \((\mathcal{G}_t, \times)\) is a left \( \mathcal{G} \)-set with this action.

Given a groupoid \( \mathcal{G} \), let \((X, \varsigma)\) be a right \( \mathcal{G} \)-set with action map \( \rho \). Then the pair of sets

\[
X \times \mathcal{G} := \{ (x, x, \mathcal{G}_r, X) \}
\]

is a groupoid with structure maps \( \varsigma^\rho = \rho, \quad \varsigma^t = pr_r, \quad \varsigma = (x, \varsigma(x)) \), for each \( x \in X \). The multiplication and the inverse maps are defined as follows: For each \((x, g), (y, h) \in X \times \mathcal{G} \), such that \( \varsigma^t(y, h) = \varsigma^\rho(x, g) \) the multiplication is given by

\[
(x, g)(y, h) = (x, gh).
\]

That is, for any the pairs of elements in \( X \times \mathcal{G}_r \) as before, we have

\[
\varsigma(x) = \varsigma(g), \quad \varsigma^\rho(x, g) = \rho(x, g) = xg, \quad \varsigma^t(x, g) = pr_t(x, g) = x,
\]

and

\[
\varsigma(y) = \varsigma(h), \quad \varsigma^\rho(y, h) = \rho(y, h) = yh, \quad \varsigma^t(y, h) = pr_t(y, h) = y,
\]

So, the multiplication is explicitly given by

\[
y = \varsigma^t(y, h) = \varsigma^\rho(x, g) = xg,
\]

\[
\varsigma(x) \xrightarrow{g} \varsigma(xg) = \varsigma(y) \xrightarrow{h} \varsigma(yh) = \varsigma(xgh).
\]
and schematically can be presented by

\[
\begin{array}{c}
  x \\
  \Downarrow{(x,g)} \\
  xg = y \\
  \Downarrow{(y,h)} \\
  yh = xgh.
\end{array}
\]

For each \((x, g) \in X \times G\), the inverse arrow is defined by \((x, g)^{-1} = (xg, g^{-1})\). The groupoid \(X \rtimes G\) is called the right translation groupoid of \(X\) by \(G\). Furthermore, there is a canonical morphism of groupoids \(\sigma : X \rtimes G \to \mathcal{G}\), given by the pair of maps \(\sigma = (\xi, pr)\).

The left translation groupoids are similarly defined and denoted by \(G \ltimes Z\) whenever \((Z, \partial)\) is a left \(G\)-set.

A morphism of right \(G\)-sets (or \(G\)-equivariant map) \(F : (X, \sigma) \to (X', \sigma')\) is a map \(F : X \to X'\) such that the diagrams

\[
\begin{array}{ccc}
  X & \xrightarrow{F} & X' \\
  \downarrow{r} & & \downarrow{r'} \\
  X & \xrightarrow{F_1} & X'
\end{array}
\]

commute. We denote by \(\text{Hom}_{\text{Reto}}(X, X')\) the set of all \(G\)-equivariant maps from \((X, \xi)\) to \((X', \xi')\). Clearly any such a \(G\)-equivariant map induces a morphism of groupoids \(F : X \rtimes G \to X' \rtimes G\). A subset \(Y \subseteq X\) of a right \(G\)-set \((X, \xi)\) is said to be \(G\)-invariant whenever the inclusion \(Y \subseteq X\) is a \(G\)-equivariant map.

**Example 2.6.** Let \(\phi : \mathcal{H} \to \mathcal{G}\) be a morphism of groupoids. Consider the triple \((\mathcal{H}_{0, \phi} \times \mathcal{H}, \phi_{1}, \xi)\), where \(\xi : \mathcal{H}_{0} \times X, \mathcal{G}_{1} \to \mathcal{G}_{0}\) sends \((u, a) \mapsto s(a)\), and let \(pr_{1}\) be the first projection. Then the following maps

\[
\begin{array}{c}
  (u, a, g) \\
  \Downarrow{(u, ag) = (u, a) - g} \\
  (h, (u, a)) \\
  \Downarrow{(h, (u, a)) = (t(h), \phi(h)a) = h - (u, a)}
\end{array}
\]

define, respectively, a structure of right \(G\)-sets and that of left \(H\)-set. Analogously, the maps

\[
\begin{array}{c}
  (a, u, h) \\
  \Downarrow{(a \phi(h), s(h)) = (a, u) - h} \\
  (g, (a, u)) \\
  \Downarrow{(ga, u) = g - (a, u)}
\end{array}
\]

where \(\theta : \mathcal{G}_{1} \times_{\phi_{0}} \mathcal{H}_{0} \to \mathcal{G}_{0}\) sends \((a, u) \mapsto t(a)\), define a left \(H\)-set and right \(G\)-set structures on \(\mathcal{G}_{1} \times_{\phi_{0}} \mathcal{H}_{0}\), respectively. This in particular can be applied to any morphism of groupoids of the form \((X, X) \to (Y \times Y, Y)\), \((x, x') \mapsto ((f(x), f(x)), f(x'))\), where \(f : X \to Y\) is any map. On the other hand, if \(f\) is a \(G\)-equivariant map, for some group \(G\) acting on both \(X\) and \(Y\), then the above construction applies to the morphism \((G \times X, X) \to (G \times Y, Y)\) sending \(((g, x), x') \mapsto ((g, f(x)), f(x'))\) of action groupoids, as well.

The proofs of the following useful Lemmas are immediate.

**Lemma 2.7.** Given a groupoid \(\mathcal{G}\), let \((X, \xi)\) be a right \(G\)-set with action \(\rho\) and let be \((X', \xi')\) be a right \(G\)-set with action \(\rho'\). Let \(F : (X, \xi) \to (X', \xi')\) be a \(G\)-equivariant map with bijective underlying map. Then \(F^{-1} : (X', \xi') \to (X, \xi)\) is also \(G\)-equivariant.

**Lemma 2.8.** Given a groupoid \(\mathcal{G}\), let \((X, \xi)\) be a right \(G\)-set with action \(\rho\) and let be \(Y \subseteq X\). We define

\[
\begin{array}{c}
  \xi' = \xi|_{Y} : Y \to G, \quad \text{and} \quad \rho'|_{Y} : Y \times_{\xi} G \to X
\end{array}
\]

and let’s suppose that for each \((a, g) \in Y \times_{\xi} G\), we have \(\rho(a, g) \in Y\). Then \((Y, \xi')\) is a right \(G\)-set with action map \(\rho'\). In particular \(Y\) is a \(\mathcal{G}\)-invariant subset of \(X\).

### 2.3. Orbit sets and stabilizers

Next we recall the notion of the orbit set attached to a right groupoid-set. This notion is a generalization of the orbit set in the context of group-sets. Here we use the (right) translation groupoid to introduce this set. First we recall the notion of the orbit set of a given groupoid.

**The orbit set of a groupoid \(\mathcal{G}\)** is the quotient set of \(G\) by the following equivalence relation: take an object \(x \in G_{0}\), define

\[
\Theta_{x} := t(s^{-1}(\{x\})) = t(\text{Star}(x)) = \{y \in G_{0} \mid \exists g \in G_{1} \text{ such that } s(g) = x, t(g) = y\}.
\]


which is equal to the set $s(t^{-1}(x))$. This is a non empty set, since $x \in \mathcal{O}$. Two objects $x, x' \in \mathcal{G}$, are said to be equivalent if and only if $\mathcal{O}_x = \mathcal{O}_{x'}$, or equivalently, two objects are equivalent if and only if there is an arrow connecting them. This in fact defines an equivalence relation whose quotient set is denoted by $\mathcal{G}/\mathcal{G}$.

In others words, this is the set of all connected components of $\mathcal{G}$, which we denote by $\pi_0(\mathcal{G}) := \mathcal{G}/\mathcal{G}$.

Given a right $\mathcal{G}$-set $(X, \zeta)$, the orbit set $X/\mathcal{G}$ of $(X, \zeta)$ is the orbit set of the (right) translation groupoid $X \rtimes \mathcal{G}$, that is, $X/\mathcal{G} = \pi_0(X \rtimes \mathcal{G})$. For an element $x \in X$, the equivalence class of $x$, called the orbit of $x$, is denoted by

$$\text{Orb}_{\mathcal{G}}(x) = \left\{ y \in X \middle| \exists (x, g) \in (X \rtimes \mathcal{G}), \begin{array}{l} x = t^g(x, g) \\ y = s^g(x, g) = xg \end{array} \right\} = \left\{ xg \in X | (g) = \zeta(x) \right\} := [x] \mathcal{G}.$$ 

The representative set of the orbit set $X/\mathcal{G}$ is denoted by $\text{rep}_\mathcal{G}(X)$ or by $[X/\mathcal{G}]$.

If $\mathcal{G} = (X \times G, X)$ is an action groupoid as in Example 2.2, then obviously the orbit set of this groupoid coincides with the classical set of orbits $X/G$. Of course, the orbit set of an equivalence relation groupoid $(\mathcal{R}, X)$, see Example 2.3, is precisely the quotient set $X/\mathcal{R}$.

The left orbits sets for left groupoids sets are analogously defined by using the left translation groupoids. We will use the following notations for left orbits sets: Given a left $\mathcal{G}$-set $(Z, \vartheta)$ its orbit set will be denoted by $\mathcal{G} \backslash Z$ and the orbit of an element $z \in Z$ by $[\mathcal{G}]z$.

Let $(X, \zeta)$ be a right $\mathcal{G}$-set with action $\rho : X \times \mathcal{G} \rightarrow X$. The stabilizer $\text{Stab}_\mathcal{G}(x)$ of $x$ in $\mathcal{G}$ is the groupoid with arrows

$$\text{Stab}_\mathcal{G}(x) = \{ g \in \mathcal{G} \mid \zeta(x) = t(g) \text{ and } xg = x \}$$

and objects

$$\text{Stab}_\mathcal{G}(x)_i = \{ u \in \mathcal{G}_o \mid \exists g \in \mathcal{G}_r(\zeta(x), u) : xg = x \} \subseteq \mathcal{G}_{o,o}.$$ 

Note that $x_{\zeta(x)} = x$ so $\zeta(x) \in (\text{Stab}_\mathcal{G}(x))_i$. Besides that, using the first condition of the right $\mathcal{G}$-set, if $t(g) = \zeta(x)$ and $g \in (\text{Stab}_\mathcal{G}(x))_i$, then $s(g) = \zeta(xg) = \zeta(x)$. Therefore

$$\text{Stab}_\mathcal{G}(x)_o = \{ \zeta(x) \} \text{, } \text{Stab}_\mathcal{G}(x)^{\zeta(x)} \leq \mathcal{G}^{\zeta(x)}$$

and as a groupoid with only one object $\zeta(x)$, the set of arrow is:

$$\text{Stab}_\mathcal{G}(x)_i = \left\{ g \in \mathcal{G} \mid s(g) = t(g) = \zeta(xg) \text{ and } xg = x \right\}.$$ 

Henceforth, the stabilizer of an element $x \in X$ is the subgroup of the isotropy group $\mathcal{G}_{x}$ consisting of those loops $g$ which satisfy $xg = x$. The following lemma is then an immediate consequence of this observation:

**Lemma 2.9.** Let $(X, \zeta)$ be a right $\mathcal{G}$-set and consider its associated morphism of groupoids $\sigma : X \rtimes \mathcal{G} \rightarrow \mathcal{G}$, given by the pair of maps $\sigma = (\zeta, pr)$. Then, for any $x \in X$, the stabilizer $\text{Stab}_\mathcal{G}(x)$ is the image by $\sigma$ of the isotropy group $(X \rtimes \mathcal{G})^x$.

The left stabilizer for elements of left groupoid sets are similarly defined and enjoy analogues properties as in Lemma 2.9.

3. Groupoid-bisets, translation groupoids, orbits and cosets

We will define the notions of groupoid-biset, two sided translation groupoid, coset by a subgroupoid and tensor product of bisets. After that, we will discuss the decomposition of a set, with a groupoid acting over it, into disjoint orbits. Moreover, we will prove the bijective correspondence between groupoid-bisets and left sets over the product of the involved groupoids.

3.1. Bisets, two sided translation groupoids and (left) cosets. Let $\mathcal{G}$ and $\mathcal{H}$ be two groupoids and $(X, \vartheta, \zeta)$ a triple consisting of a set $X$ and two maps $\zeta : X \rightarrow \mathcal{G}_o$, $\vartheta : X \rightarrow \mathcal{H}_o$. The following definitions are abstract formulations of those given in [14, 21] for topological and Lie groupoids. In this regard, see also [15].

**Definition 3.1.** The triple $(X, \vartheta, \zeta)$ is said to be a right $\mathcal{H}$-action $\lambda : \mathcal{H} \times X \rightarrow X$ and a right $\mathcal{G}$-action $\rho : X \times \mathcal{G} \rightarrow X$ such that

1. For any $x \in X$, $h \in \mathcal{H}$, $g \in \mathcal{G}$, with $\vartheta(h) = s(h)$ and $\zeta(x) = t(g)$, we have
   $$\vartheta(hg) = \vartheta(h) \text{ and } \zeta(xh) = \zeta(x).$$

   (1) For any $x \in X$, $h \in \mathcal{H}$, $g \in \mathcal{G}$, with $\vartheta(h) = s(h)$ and $\zeta(x) = t(g)$, we have
   $$\vartheta(hg) = \vartheta(h) \text{ and } \zeta(xh) = \zeta(x).$$
For simplicity the actions maps of a groupoid-biset are omitted in the notions. The two sided translation groupoid associated to a given \((H, G)-\)biset \((X, \zeta, \theta)\) is defined to be the groupoid \(H \ltimes X \rtimes G\) whose set of objects is \(X\) and set of arrows is
\[
H \times X \times G, \quad \mathcal{G}_1 = \{ (h, x, g) \in H \times X \times G \mid s(h) = \theta(x), \ s(g) = \zeta(x) \}.
\]
The structure maps are:
\[
s(h, x, g) = x, \quad t(h, x, g) = h x g^{-1} \quad \text{and} \quad \iota_x = (t_0(x), x, t_0(x)).
\]
The multiplication and the inverse are given by:
\[(h, x, g)(h', x', g') = (hh', x', gg'), \quad (h, x, g)^{-1} = (h^{-1}, hxg^{-1}, g^{-1}).\]

The orbit space of the two translation groupoid is the quotient set \(X/\{(H, G)\}\) defined using the equivalence relation \(x \sim x', \) if and only if, there exist \(h \in H\) and \(g \in G\) with \(s(h) = \theta(x)\) and \(t(g) = \zeta(x')\), such that \(hx = x'g\). We will also employ the notation \(X/\{(H, G)\} := H\backslash X/G\) and denote sometimes by \([H\backslash X/G] := \text{rep}_{H\ltimes G}(X)\) its representative set.

**Example 3.2.** Let \(\phi : H \to G\) be a morphism of groupoids. Consider, as in Example 2.6 the associated triples \((H_{\phi} \times X, \mathcal{G}_1, \zeta, \theta)\) and \((G \times \mathcal{G}_1, \zeta, \theta)\) with actions defined as in equations (4) and (5). Then these triples are an \((H, G)\)-biset and a \((G, H)\)-biset, respectively.

**Proposition 3.3.** Let \((X, \zeta, \theta)\) be an \((H, G)\)-biset with actions \(\lambda : H \times X \to X\) and \(\rho : X \times G \to X\). Then \(H\backslash X\) is a right \(G\)-set with structure map and action as follows:
\[
\overline{\zeta} : H\backslash X \to G, \quad \overline{\rho} : (H\backslash X) \times G \to H\backslash X
\]
\[
H[x] \to \overline{\zeta}(H[x]) = \zeta(x) \quad \text{and} \quad (H[x], g) \to \overline{\rho}(H[x], g) = H[xg] = H[\rho(x, g)].
\]

**Proof.** Let be \(x_1, x_2 \in X\) such that \(H[x_1] = H[x_2]\). Then by definition of orbit there is \(h \in H\) such that \(x_1 = hx_2\) and \(\theta(x_2) = s(h)\). One of the biset conditions says that \(\zeta(x_1) = \zeta(hx_2) = \zeta(x_2)\). This shows that \(\overline{\rho}\) is well defined. Now let be \(g \in G\), \(x_1, x_2 \in X\) such that \(H[x_1] = H[x_2]\), \(\overline{\zeta}(H[x_1]) = t(g)\) and \(\overline{\zeta}(H[x_2]) = t(g)\). We have
\[
\zeta(x_1) = \overline{\zeta}(H[x_1]) = t(g) = \overline{\zeta}(H[x_2]) = \zeta(x_2)
\]
and
\[
\rho(x_1, g) = x_1g = (hx_2)g = h(x_2g) = h\rho(x_2, g)
\]
so \(H[x_1g] = H[x_2g]\) which shows that \(\overline{\rho}\) is well defined. There remain to be verified the axioms of right \(G\)-set.

1. **For each** \(H[x] \in H\backslash X\) and \(g \in G\), such that \(\overline{\zeta}(H[x]) = t(g)\), we have \(\zeta(x) = \overline{\zeta}(H[x]) = t(g)\) so
\[
\overline{\zeta}(H[x]) = \overline{\zeta}(H[xg]) = \zeta(xg) = \zeta(x) = t(g) = \overline{\zeta}(H[x])
\]
2. **For each** \(H[x] \in H\backslash X\) we have
\[
(H[x])_{t_0(x)} = (H[x])_{t_0(x)} = H[x_{t_0(x)}] = H[x].
\]
3. **For each** \(H[x] \in H\backslash X\) and \(g_1, g_2 \in G\), such that \(\overline{\zeta}(H[x]) = t(g_1)\) and \(\zeta(x_1) = t(g_2)\) we have
\[
(\overline{\zeta}(H[x])g_1) = (H[xg_1])g_2 = H[(xg_1)g_2] = H[H[xg_1g_2]] = (H[x])_{(g_1g_2)}.
\]
As a consequence we have proved that \(H \backslash \text{X}\) is a right \(G\)-set as stated. \(\square\)

The left version of Proposition 3.3 also holds true. Precisely, given an \((H, G)\)-biset \((X, \zeta, \theta)\); since \(X\) is obviously a \((G^\circ, H^\circ)\)-biset, applying Proposition 3.3, we obtain that \(G^\circ \backslash X\) is right \(H^\circ\)-set, that is, \(X/G\) becomes a left \(H\)-set.

Next we deal with the left and right cosets attached to a morphism of groupoids. So let us assume that a morphism of groupoids \(\phi : H \to G\) is given and denote by \(\text{rep}_H(G) = H_{\phi} \times X, \mathcal{G}_1\), the underlying set of the \((H, G)\)-biset of Example 3.2. Then the left translation groupoid is given by
\[
H \ltimes \text{rep}_H(G) = H \ltimes \left( H_{\phi} \times X, \mathcal{G}_1 \right) = \left( H_{\phi} \times X, \mathcal{G}_1 \right), (H_{\phi} \times X, \mathcal{G}_1).
\]
where the source $s^r$ is the action $\sim$ described in equation (4) and the target $r^s$ is the second projection on $X$. The multiplication of two objects $(h_1, a_1, g_1), (h_2, a_2, g_2) \in H \times H \times G$, such that $s^r((h_1, a_1, g_1) = t^s((h_2, a_2, g_2)$, is given as follows: First we have

$t((h_1), \phi(h_1)g_1) = h_1 \rightarrow (a_1, g_1) = s^r(h_1, a_1, g_1) = t^s(h_2, a_2, g_2)$

and

$s((h_2)) = p_G(a_2, g_2) = a_2 = \tau(h_1)$ so we can write $h_2 \rightarrow h_1$. Second we have $t^s(h_1, a_1, g_1) = (a_1, g_1)$ and $s^r(h_2, a_2, g_2) = h_2 \rightarrow (a_2, g_2) = (t(h_2), \phi(h_2)a_2)$, and so

\[(a_1, g_1) \xrightarrow{h_1 \rightarrow (a_1, g_1)} (t(h_1), \phi(h_1)g_1) \xrightarrow{(h_2, a_2, g_2)} h_2 \rightarrow (a_2, g_2)\]

Thus,

\[(h_1, a_1, g_1)(h_2, a_2, g_2) = (h_2h_1, a_1, g_1).\]

**Definition 3.4.** Given a morphism of groupoids $\phi : H \rightarrow G$, we define

\[(G/H)^\phi := \pi_i[H \rtimes \phi(G)]\]

the orbit set $H \rtimes \phi(G)$ and, for each $(a, g) \in H \times G$, we set

\[\phi^\circ(a, g) = [h \rightarrow (a, g) \in \phi(G) : h \in H, \ s(h) = a].\]

If $H$ is a subgroupoid of $G$, that is, $\phi := \tau : H \rightarrow G$ is the inclusion functor, we use the notations

\[(G/H)^\phi = H \setminus (H_e \times G)^\phi = H^\phi(G)\]

and

\[H[(a, g)] = \{h \rightarrow (a, g) \in H(G) : h \in H, \ s(h) = a\}.\]

(7)

where $(a, g) \in H \times G$. On the other hand, for each $(h, a, g) \in H \times H \times G$, we have

\[h \rightarrow (a, g) = (t(h), \phi(h)g) = (t(h), hg)\]

and

\[s(h) = a = \phi(h) = t(g)\]

so, for each $(a, g) \in H \times G$, we have

\[H[(a, g)] = \{t(h), hg) \in H(G) : h \in H, \ s(h) = t(g)\}.\]

**Definition 3.5.** Let $H$ be a subgroupoid of $G$ via the injection $\tau : H \rightarrow G$. The right coset of $G$ by $H$ is defined as:

\[(G/H)^\phi = \{H[(a, g)] : (a, g) \in H \times G\}.\]

where each class $H[(a, g)]$ is as in equation (7).

Keeping the notation of the previous definition we can state:

**Lemma 3.6.** Given $(a_1, g_1), (a_2, g_2) \in H \times G$, we have $H[(a_1, g_1)] = H[(a_2, g_2)]$ if and only if there is $h \in H$ such that $h = g_1g_2^{-1}$.

**Proof.** We have $H[(a_1, g_1)] = H[(a_2, g_2)]$ if and only if $(a_1, g_1) \in H[(a_2, g_2)]$, if and only if there is $h \in H$, such that $s(h) = t(g_2)$ and $(a_1, g_1) = (t(h), hg_2)$, if and only if there is $h \in H$, such that

\[(t(g_2) = a_1 = t(h)\]

\[g_1 = hg_2.\]

if and only if there is $h \in H$, such that $s(h) = t(g_2)$ and $t(h) = t(g_1) = a_1$ and $s(h) = g_1g_2^{-1}$. If and only if there is $h \in H$, such that $s(h) = g_1g_2^{-1}$.

The left coset of $G$ by $H$ is defined using the $(G, H)$-biset $H(G)^\tau := G \times H \times G$, of Example 5.2 with actions maps as in equation (5). If $\tau : H \rightarrow G$ is the inclusion functor, then we use the notations

\[(G/H)^\tau = H(G)^\tau / H = \{(a, u) : (a, u) \in H(G)^\tau\}\]

(9)

and, for each $(a, u) \in H(G)^\tau$,

\[[(a, u)]H = \{(ah, s(h)) : (ah, s(h)) \in H(G)^\tau : h \in H\}.\]

The following is an analogue of Lemma 3.6.
**Lemma 3.7.** If \( \tau \colon \mathcal{H} \rightarrow \mathcal{G} \) is an inclusion functor, then for each \((a_1, u_1), (a_2, u_2) \in \mathcal{G}_1 \times \mathcal{H}_0 \) we have \([a_1, u_1] \mathcal{H} = [a_2, u_2] \mathcal{H} \) if and only if \( a_2^{-1} a_1 \in \mathcal{H} \).

**Proof.** Is similar to that of Lemma 3.6 \( \square \)

As a corollary of Proposition 3.3, we obtain:

**Corollary 3.8.** Let \( \mathcal{H} \) be a subgroupoid of \( \mathcal{G} \) via the injection \( \tau \colon \mathcal{H} \hookrightarrow \mathcal{G} \). Then \( (\mathcal{G}/\mathcal{H})^\# \) becomes a right \( \mathcal{G} \)-set with structure map and action given as follows:

\[
\bar{\tau} : (\mathcal{G}/\mathcal{H})^\# \rightarrow \mathcal{G}, \quad \bar{\rho} : (\mathcal{G}/\mathcal{H})^\# \times \mathcal{G}_1 \rightarrow (\mathcal{G}/\mathcal{H})^\#
\]

\((\mathcal{H}((a, g_1), g_2) \rightarrow \mathcal{H}((a, g_1) g_2)) \)

The following crucial proposition characterizes, as in the classical case of groups, the right cosets by the stabilizer subgroupoid:

**Proposition 3.9.** Let \( (X, \zeta) \) be a right \( \mathcal{G} \)-set with action map \( \rho : X \times \mathcal{G}_1 \rightarrow X \). Given \( x \in X \), let us consider \( \mathcal{H} = \text{Stab}_\rho(x) \) its stabilizer as a subgroupoid of \( \mathcal{G} \) (see Subsection 2.3). Then the following map

\[
\varphi : (\mathcal{G}/\mathcal{H})^\# \rightarrow [x] \mathcal{G}
\]

establishes an isomorphism of right \( \mathcal{G} \)-sets.

**Proof.** Given \( \mathcal{H}((a, g)) \in (\mathcal{G}/\mathcal{H})^\# \), we have \( a \in \mathcal{H}_0 = \{ \zeta(x) \} \) and, since \((a, g) \in (\mathcal{H}_0 \times \mathcal{G}_1)_1 \), where \( \tau \colon \mathcal{H} \rightarrow \mathcal{G} \) is the inclusion functor, we have \( t(g) = a = \zeta(x) \) and we can write \( xg \). Now consider \((a_1, g_1), (a_2, g_2) \in (\mathcal{H}_0 \times \mathcal{G}_1)_1 \) (that is, \( a_1 = t(g_1) \) and \( a_2 = t(g_2) \)) such that \( \mathcal{H}((a_1, g_1)) = \mathcal{H}((a_2, g_2)) \). Then, by Lemma 3.6 there exists \( h \in \mathcal{H}_0 \) such that \( s(h) = a_2, t(h) = a_1 \) and \( h = g_1 g_2^{-1} \). Since \( \mathcal{H} = \text{Stab}_\rho(x) \) we have \( a_1 = a_2 = \zeta(x) \) and \( xh = x \) so \( x = xh = xg_1 g_2^{-1} \), whence \( xg_2 = xg_1 \). This shows that \( \varphi \) is well defined. Now let \( \mathcal{H}((a_1, g_1)), \mathcal{H}((a_2, g_2)) \in (\mathcal{G}/\mathcal{H})^\# \) such that \( \varphi(\mathcal{H}((a_1, g_1))) = \varphi(\mathcal{H}((a_2, g_2))) \). Then we have

\[
xg_1 = \varphi(\mathcal{H}((a_1, g_1))) = \varphi(\mathcal{H}((a_2, g_2))) = xg_2.
\]

so \( xg_1 g_2^{-1} = x \), which means that \( g_1 g_2^{-1} \in \text{Stab}_\rho(x) = \mathcal{H} \). Therefore \( \mathcal{H}((a_1, g_1)) = \mathcal{H}((a_2, g_2)) \) and \( \varphi \) is then injective. Now consider an element \( xg \in [x] \mathcal{G} \), by definition we have \( \varphi(\mathcal{H}((\zeta(x), g))) = xg \), so \( \varphi \) is a surjective map. Therefore, \( \varphi \) is bijective.

By Corollary 3.8 and Lemma 2.7 it follows that \( (\mathcal{G}/\mathcal{H})^\# \) and \([x] \mathcal{G}\) are right \( \mathcal{G} \)-sets. We denote by \( \zeta \) and \( \rho \), the structure and the action maps of \([x] \mathcal{G}\), respectively. To prove that \( \varphi \) is a morphism of right \( \mathcal{G} \)-sets we have to prove that the following two diagrams are commutative:

\[
\begin{array}{ccc}
(\mathcal{G}/\mathcal{H})^\# \times \mathcal{G}_1 & \xrightarrow{\bar{\rho}} & (\mathcal{G}/\mathcal{H})^# \\
(\varphi \times \text{Id}_{\mathcal{G}_1}) & & (\varphi \times \text{Id}_{\mathcal{G}_1}) \\
([x] \mathcal{G})_1 \times \mathcal{G}_1 & \xrightarrow{\varphi \times \text{Id}_{\mathcal{G}_1}} & [x] \mathcal{G}
\end{array}
\]

Let us check the commutativity of the triangle. So take \( \mathcal{H}((a, g)) \in (\mathcal{G}/\mathcal{H})^\# \), using the definition of right action, we have

\[
\zeta, \varphi(\mathcal{H}((a, g))) = \zeta, (xg) = \zeta, (xg) = s, (g) = \bar{\tau}(\mathcal{H}((a, g)))
\]

since \( t(g) = a = \zeta(x) \). As for the rectangle, take an arbitrary element \( \mathcal{H}((a, g_1), g_1) \in (\mathcal{G}/\mathcal{H})^\# \times \mathcal{G}_1 \), we have

\[
\rho, (\varphi \times \text{Id}_{\mathcal{G}_1})(\mathcal{H}((a, g)), g_1) = \rho, (\varphi(\mathcal{H}((a, g)))) g_1
\]

\[
= \rho, (xg, g_1) = \rho, (xg, g_1) = (xg) g_1 = x(g_1) = \varphi(\mathcal{H}((a, g_1))) = \bar{\tau}(\mathcal{H}((a, g)), g_1).
\]

Therefore, \( \varphi \) is compatible with the action and by using Lemma 2.7 we conclude that \( \varphi \) is an isomorphism of right \( \mathcal{G} \)-sets as desired. \( \square \)

**Corollary 3.10.** Let \( (X, \zeta) \) be a right \( \mathcal{G} \)-set. Then there is an isomorphism of right \( \mathcal{G} \)-sets:

\[
X \cong \bigoplus_{x \in \text{rep}_\rho(X)} \left( (\mathcal{G}/\text{Stab}_\rho(x))^\# \right).
\]
where the right hand side is the direct sum in the category of right $G$-sets and with $\text{rep}_G(X)$ we indicate a set of representatives of the orbits of the right $G$-set $X$.

**Proof.** Immediate from Proposition [3.9] considering the fact that a right $G$-set is the disjoint union of its orbits. \hfill \square

### 3.2. Groupoid-bisets versus (left) sets.

In this subsection we give the complete proof of the fact that the category of groupoids $(\mathcal{H}, \mathcal{G})$-bisets is isomorphic to the category of left groupoids $(\mathcal{H} \times \mathcal{G}^op)$-sets (equivalently right $(\mathcal{H}^op \times \mathcal{G})$-sets). Here the structure groupoid of the (cartesian) product of two groupoids is the one given by the product of the underlying categories as described in Example [2.11].

**Proposition 3.11.** Given a set $X$ and two groupoids $(\mathcal{H}, \mathcal{G})$. Then there is a bijective correspondence between structures of $(\mathcal{H}, \mathcal{G})$-bisets on $X$ and structures of left $(\mathcal{H} \times \mathcal{G}^op)$-sets on $X$.

**Proof.** Let $X$ be an $(\mathcal{H}, \mathcal{G})$-biset with actions and structures map as follows:

- $\vartheta: X \to \mathcal{H}_0$
- $\lambda: \mathcal{H}_0 \times G \to X$
- $\rho: X \times \mathcal{G}_1 \to X$

We define the structure map and action as follows:

- $\alpha: X \to (\mathcal{H} \times \mathcal{G}^op)_b$
- $\beta: (\mathcal{H} \times \mathcal{G}^op)_b \times X \to X$
- $x \mapsto ((\vartheta(x), \lambda(x))$ and $(\beta(g, x) \mapsto h(xg)$.

We have to check the axioms of a left action.

1. For each $x \in X$ and $(h, g) \in (\mathcal{H} \times \mathcal{G}^op)_b$, such that $\alpha(x) = s(h, g)$ we have

   $$(\vartheta(x), \lambda(x)) = \alpha(x) = s(h, g) = (s(h), t(g))$$

   so

   $$\alpha(h, g) x = (\vartheta(hx), \lambda(xg)) = (\vartheta(h), \lambda(s(g))) = t(h, g).$$

2. For each $x \in X$ we have

   $$x_{(h, g, x)} = x_{(h, x, g, x)} = x_{(h, 1)} = t_{(h, x)}(x_{(h, x)}) = t_{h, x}x = x.$$  

3. For each $x \in X$ and $(h_1, g_1), (h_2, g_2) \in (\mathcal{H} \times \mathcal{G}^op)_b$ such that $\alpha(x) = s(h_2, g_2)$ and $s(h_1, g_1) = t(h_2, g_2)$ we have

   $$(\vartheta(x), \lambda(x)) = \alpha(x) = s(h_2, g_2) = (s(h_2), t(g_2))$$

   and

   $$\beta(h_1, g_1) = s(h_1, g_1) = t(h_2, g_2) = (t(h_2), s(g_2)).$$

In this way we have

$$\begin{align*}
(h_1, g_1)((h_2, g_2)x) &= (h_1, g_1)(h_2 (xg_2)) \\
&= h_1 ((h_2 (xg_2)) g_1) \\
&= h_1 (((h_2x) g_2) g_1) \\
&= h_1 ((h_2x) (g_2 g_1)) \\
&= h_1 (h_2x) (g_2 g_1)) \\
&= (h_1 h_2x) (g_1 g_2) \\
&= (h_1 h_2 (g_1 g_2)) x \\
&= ((h_1, g_1)(h_2, g_2)) x.
\end{align*}$$

Therefore $X$ becomes an $(\mathcal{H} \times \mathcal{G}^op)$-set.

Conversely, let $X$ be an $(\mathcal{H} \times \mathcal{G}^op)_b$-left set with

- $\alpha: X \to (\mathcal{H} \times \mathcal{G}^op)_b$ and $\beta: (\mathcal{H} \times \mathcal{G}^op)_b \times X \to X$

as structure map and action. Let be $p_1$ and $p_2$ the canonical projection

$$p_1: (\mathcal{H} \times \mathcal{G}^op)_b \to \mathcal{H}_0$$
$$p_2: (\mathcal{H} \times \mathcal{G}^op)_b \to \mathcal{G}_0$$

$$(h, g) \mapsto h$$ and $$(h, g) \mapsto g.$$
We define a structure of left $H$-set as follows

$$\theta : X \rightarrow H$$

$$x \mapsto p_1 \alpha (x)$$

and a structure of right $G$-set as follows:

$$\zeta : X \rightarrow G$$

$$x \mapsto p_2 \alpha (x)$$

and $\lambda : H \times X \rightarrow X$

$$(x, g) \mapsto \beta ((h, t_{\alpha}), x)$$

For each $(h, x) \in H \times X$ we have

$$\alpha (x) = (\theta (x), \zeta (x)) = (s(h), t(l_{\alpha})) = s(h, l_{\alpha})$$

so $\lambda$ is well defined. We have to verify the axioms of left action.

1. For each $x \in X$ and $h \in H$ such that $\theta (x) = s(h)$ we have

$$\alpha (x) = (\theta (x), \zeta (x)) = (s(h), t(l_{\alpha})) = s(h, l_{\alpha})$$

so

$$\theta (hx) = p_1 \alpha (hx) = p_1 \alpha ((h(l_{\alpha})) x) = p_1 t(h(l_{\alpha})) = t(h).$$

2. For each $x \in X$ we have

$$t_{\alpha} x = (t(h), l_{\alpha}) x = x.$$  

3. For each $x \in X$ and $g, h \in H$ such that $s(g) = t(h)$ and $s(h) = \theta (x)$ we have

$$g(hx) = g((h(l_{\alpha})) x) = (g, t(h(l_{\alpha}))) ((h, t_{\alpha})) x.$$  

Since

$$\zeta ((h, l_{\alpha})) x = p_2 \alpha ((h(l_{\alpha})) x) = p_2 t(h(l_{\alpha})) = s(l_{\alpha}) = \zeta (x)$$

we have

$$g(hx) = ((g, t_{\alpha}))((h, t_{\alpha})) x = (gh, t_{\alpha}) x = (gh) x.$$  

As for the right action, for each $(x, g) \in X \times G$, we have

$$\alpha (x) = (\theta (x), \zeta (x)) = (s(l_{\alpha}), t(g)) = s(l_{\alpha}, g)$$

so $\rho$ is well defined and satisfies the required axioms as follows:

1. For each $x \in X$ and $g, h \in G$ such that $t(g) = \zeta (x)$ we have

$$\alpha (x) = (\theta (x), \zeta (x)) = (s(l_{\alpha}), t(g)) = s(l_{\alpha}, g)$$

so

$$\zeta (xg) = p_2 \alpha (xg) = p_2 \alpha ((l_{\alpha})) x = p_2 t(l_{\alpha}, g) = s(g).$$

2. For each $x \in X$ we have

$$x l_{\alpha} = (l_{\alpha}, h_{\alpha}) x = x.$$  

3. For each $x \in X$ and $h, g \in G$ such that $\zeta (x) = t(g)$ and $s(g) = t(h)$ we have

$$(xg) h = ((l_{\alpha}, t_{\alpha}), g) x h = (l_{\alpha}, h_{\alpha}(t_{\alpha})) h ((l_{\alpha}, t_{\alpha})) x.$$  

Since

$$\zeta ((l_{\alpha}, t_{\alpha})) x = p_1 \alpha ((l_{\alpha}, t_{\alpha})) x = p_1 t(l_{\alpha}, g) = \zeta (x)$$

we have

$$(xg) h = ((l_{\alpha}, h_{\alpha}(t_{\alpha})), g) x = (l_{\alpha}, t_{\alpha}) x = x (gh).$$  

Let us check the properties of a biset. For each $x \in X$, $g \in G$, and $h \in H$, such that $\theta (x) = s(h)$ and $\zeta (x) = t(g)$ we have

$$\alpha (x) = (\theta (x), \zeta (x)) = (s(l_{\alpha}), t(g)) = s(l_{\alpha}, g)$$

so

$$\theta (xg) = p_1 \alpha (xg) = p_1 \alpha ((l_{\alpha}, t_{\alpha})) x = p_1 t(l_{\alpha}, g) = \theta (x)$$

and

$$\zeta (hx) = p_2 \alpha (hx) = p_2 \alpha ((l_{\alpha}, t_{\alpha})) x = p_2 t(h, l_{\alpha}) = \zeta (x).$$
This gives the first compatibility condition. As for the second one, we know that
\[ h(xg) = h((t_{\varepsilon_0}, g)x) = \left(h, t_{\varepsilon_0}(t_{\varepsilon_0}, g)\right)(t_{\varepsilon_0}, g)x \]
and, since
\[ \varsigma((t_{\varepsilon_0}, g)x) = p_2\varsigma((t_{\varepsilon_0}, g)x) = p_2t_{(\varepsilon_0,\varepsilon_0)} = s(g), \]
we get
\[ h(xg) = \left(h, t_{\varepsilon_0}(t_{\varepsilon_0}, g)\right)x = \left(h, g_{(t_{\varepsilon_0})}\right) = (h, g)x. \] (10)

On the other hand, we have
\[ (hg)x = ((h, t_{\varepsilon_0})x)g = \left(t_{\varepsilon_0}(t_{\varepsilon_0}, g)\right)= ((h, t_{\varepsilon_0})x) \]
and, since
\[ \vartheta((h, t_{\varepsilon_0})x) = p_1\vartheta((h, t_{\varepsilon_0})x) = p_1(t(h, t_{\varepsilon_0})) = t(h), \]
we get
\[ (hx)g = \left((t_{\varepsilon_0}, h)g\right)(t_{\varepsilon_0}, g)x = (h, g_{(t_{\varepsilon_0})}) = (h, g)x. \] (11)

Comparing equations (10) and (11), we obtain the equality \( h(xg) = (hx)g \), which shows that \( X \) is an \( (\mathcal{H}, \mathcal{G}) \)-biset. Lastly, it is clear that these two constructions are mutually inverses and this completes the proof. \( \square \)

A similar proof to that of Proposition 3.11 works to show that there is a one-to-one correspondence between right \( (\mathcal{H}^r \times \mathcal{G}) \)-sets structures and \( (\mathcal{H}, \mathcal{G}) \)-sets structures. Furthermore, any \( (\mathcal{H}, \mathcal{G}) \)-equivariant map (i.e., any morphism of \( (\mathcal{H}, \mathcal{G}) \)-biset) is canonically transformed, under this correspondence, to a left \( (\mathcal{H} \times \mathcal{G}^l) \)-equivariant map. In this way, we have the following corollary.

**Corollary 3.12.** Let \( \mathcal{H} \) and \( \mathcal{G} \) be two groupoids. Then there are canonical isomorphisms of categories between the category of \((\mathcal{H}, \mathcal{G})\)-biset, left \((\mathcal{H} \times \mathcal{G}^l)\)-sets and right \((\mathcal{H}^r \times \mathcal{G})\)-sets.

### 3.3. Orbits and stabilizers of biset and double cosets.

We will use the notations of the proof of Proposition 3.11. Let \( X \) be an \((\mathcal{H}, \mathcal{G})\)-biset: it becomes an \((\mathcal{H} \times \mathcal{G}^l)\)-left set, so we have

\[ (\text{Stab}_{(\mathcal{H}, \mathcal{G})}(x))_h = \left(\text{Stab}_{(\mathcal{H} \times \mathcal{G}^l)}(x)\right)_h = \{(\vartheta(x), \varsigma(x))\} \]

and

\[ (\text{Stab}_{(\mathcal{H}, \mathcal{G})}(x))_h = \left(\text{Stab}_{(\mathcal{H} \times \mathcal{G}^l)}(x)\right)_h = \left\{(h, g) \in \mathcal{H} \times \mathcal{G}^l \mid \begin{array}{c} s(h, g) = t(h, g) = \vartheta(x) \\ (h, g)x = x \end{array} \right\} \]

For a given elements \( x \in X \), the orbit set of \( x \) is given by

\[ \text{Orb}_{(\mathcal{H}, \mathcal{G})}(x) = \mathcal{H}[x]_{\mathcal{G}} = \left\{hxg = (h, g)x \in X \mid s(h) = \vartheta(x), \varsigma(x) = t(g)\right\}. \]

**Proposition 3.13.** Given a groupoid \( \mathcal{H} \), let \( \mathcal{A} \) and \( \mathcal{B} \) be subgroupoid of \( \mathcal{H} \). We define

\[ X = \mathcal{A}_0 \times \mathcal{H} \times \mathcal{B}_0 = \left\{(a, h, b) \in \mathcal{A}_0 \times \mathcal{H} \times \mathcal{B}_0 \mid a = t(h), s(h) = b\right\}. \]

Then \( X \) is an \((\mathcal{A}, \mathcal{B})\)-biset with structure maps

\[ \vartheta: X \rightarrow \mathcal{A}_0 \quad \text{and} \quad \varsigma: X \rightarrow \mathcal{B}_0 \]

and action maps

\[ \lambda: \mathcal{A}_0 \times \mathcal{H} \rightarrow X \quad \text{and} \quad \rho: \mathcal{H} \times \mathcal{B}_0 \rightarrow X \]

\[ (r, (a, h, b)) \mapsto (t(r), rh, b) \quad \text{and} \quad ((a, h, b), q) \mapsto (a, hq, s(q)). \]

**Proof.** We have to check the properties of a groupoid right action.

1. For each \((a, h, b) \in X \) and \( q \in \mathcal{B} \), such that \( \varsigma(a, h, b) = t(q) \) we have

\[ \varsigma((a, h, b)d) = \varsigma(a, hq, s(q)) = s(q). \]
For each \((a, h, b) \in X\) we have

\[(a, h, b)_{\vartheta_1, b_{\vartheta}} = (a, h, b)_{\vartheta} = (a, h_{\vartheta} S(\vartheta)) = (a, h, b).\]

(3) For each \((a, h, b) \in X\) and \(q', q \in B\), such that \(\varphi(a, h, b) = t(q)\) and \(S(q) = t(q')\) we have

\[(a, h, b) q' = (a, h q, S(q)) q' = (a, h q q', S(q)) = (a, h, b) q q'.\]

The properties of the left action are similarly proved. Now, we have to check the compatibility conditions of a biset. For each \((a, h, b) \in X\), \(r \in \mathcal{A}\), and \(q \in B\), such that \(\vartheta(a, h, b) = S(a)\) and \(\varphi(a, h, b) = t(q)\) we have

\[\vartheta((a, h, b) q) = \vartheta(a, h q, S(q)) = a = \vartheta(a, h, b),\]

\[\varphi((a, h, b) q) = \varphi(a, h q q', S(q)) = a = \varphi(a, h, b),\]

and

\[r((a, h, b) q) = r(a, h q, S(q)) = (t(r), r q, S(q)) = (t(r), r h, b) q = (r(a, h, b)) q,\]

and this finishes the proof. 

### 3.4. The tensor product of groupoid-bisets

Next we recall the definition of the tensor product of two groupoid-bisets and show it universal property. Fix three groupoids \(G, H\) and \(K\). Given \((Y, x, q)\) a \((G, K)\)-biset and \((X, \vartheta, \varsigma)\) an \((H, G)\)-biset. Considering the triple \((X, X, G, X, Y, G, G)\), where

\[
\overline{\mathbf{b}} : X \times X, G, X, Y \to \mathcal{H}, \quad ((x, g, y) \mapsto \vartheta(x)); \quad \overline{\mathbf{b}} : X \times X, G, X, Y \to \mathcal{K}, \quad ((x, g, y) \mapsto \varphi(y)),
\]

we have that \((X, X, G, X, Y, G, G)\) is an \((H, K)\)-biset with actions:

\[
\mathcal{H}, (X, X, G, X, Y) \times X, G, X, Y, \quad ((x, g, y), k) \mapsto (x, g, y k)
\]

On the other hand, consider the map \(\omega : X, X, Y \to G, Y\) sending \((x, y) \mapsto \kappa(y) = \varsigma(x)\). Then the pair \((X, X, Y, \omega)\) admits a structure of right \(G\)-set with action

\[
(x, x, Y) \times X, G, \to (X, X, Y), \quad ((x, y), g) \mapsto (x g^{-1}, y))
\]

Following the notation and terminology of [15] Remark 2.12, we denote by \((X, X, Y) / G := X \otimes_G Y\) the orbit set of the right \(G\)-set \((X, X, Y, \omega)\). We refer to \(X \otimes_G Y\) as the tensor product over \(X\) and \(Y\). It turns out that \(X \otimes_G Y\) admits a structure of \((H, K)\)-biset whose structure maps are given as follows. First, denote by \(x \otimes_G y\) the equivalence class of an element \((x, y) \in X \times X, Y\). That is, we have

\[
x g \otimes_G y = x \otimes_G g y, \quad \text{for every } \ g \in G, \ \text{with } \kappa(y) = t(h) = \varsigma(x).
\]

Second, one can easily check that the maps

\[
\overline{\mathbf{b}} : X \otimes_G Y \to \mathcal{K}, \quad (x \otimes_G y \mapsto \varphi(y)); \quad \overline{\mathbf{b}} : X \otimes_G Y \to \mathcal{H}, \quad (x \otimes_G y \mapsto \vartheta(x))
\]

are well defined, in such a way that the following maps

\[
(X \otimes_G Y) \times X, G, \to X \otimes_G Y, \quad ((x \otimes_G y, k) \mapsto x \otimes_G y k)
\]

induce a structure of \((H, K)\)-biset on \(X \otimes_G Y\).

The following describes the universal property of the tensor product between groupoid-bisets, see also [16] Remark 2.2.

**Lemma 3.14.** Let \(G, H\) and \(K\) be three groupoids and \((X, \vartheta, \varsigma)\), \((Y, x, q)\) the above groupoid-bisets. Then the following diagram

\[
\begin{array}{ccc}
X \times X, G, X, Y & \xrightarrow{\rho \times 1_X} & X \times X, Y \\
1 \times \lambda & \xrightarrow{1 \times \lambda} & 1 \times \lambda \\
\end{array}
\]

is the co-equalizer, in the category of \((H, K)\)-biset, of the pair of morphisms \((\rho \times 1_x, 1 \times \lambda)\).

**Proof.** Straightforward. \(\square\)
4. Mackey Formula for Groupoids

Now we will explain and prove the main result of this work. Before doing this, however, we have to introduce some particular groupoid-bisets and prove their properties. We will also have to define a specific kind of product of two subgroupoids.

4.1. Orbits of products and cosets. Let $G$ and $H$ be two groupoids and $L$ a subgroupoid of the product $H \times G$. Consider the set of equivalence classes $\left(\frac{H \times G}{L}\right)$. An element in this set is an equivalence class of a fourfold element $(h, g, u, v) \in L(H \times G)$, where $\tau : L \to H \times G$ is the inclusion functor, that is,

$$ [(h, g, u, v)]L = \{(hh_1, gg_1, s(g_1), s(g_1)) \in L(H \times G)| (h_1, g_1) \in L\}, $$

where $t(h_1) = s(h), t(g_1) = s(g)$.

**Lemma 4.1.** Given two groupoids $G$ and $H$, let $L$ be a subgroupoid of $H \times G$. Then the left $L$-coset

$$ X = \left(\frac{H \times G}{L}\right) $$

is an $(H, G)$-biset with structure maps

$$ \vartheta : X \to H, \quad [(h, g, u, v)]L \mapsto t(h) $$

and actions

$$ \lambda : H \times X \to X, \quad (h, [(h, g, u, v)]L) \mapsto [(h_1h, g, u, v)]L $$

and

$$ \rho : G \times X \to X, \quad [(h, g, u, v)]L \mapsto [(h_1g, g', u, v)]L. $$

**Proof.** Let us first check that $\vartheta$ and $\varsigma$ are well defined maps. So given two representatives of the same equivalence class $[(h, g, u, v)]L = [(h', g', u', v')]L$, by Lemma 3.7 we know that $(h', g')^{-1}(h, g) \in L$. From which one obtains $t(h') = t(h)$ and $t(g') = t(g)$. As for $\lambda$ and $\rho$, take $h_1 \in H$, and $g_1 \in G$, such that $s(h_1) = \vartheta([(h, g, u, v)]L)$, $t(g_1) = \varsigma([(h, g, u, v)]L) = t(g)$ and $[(h, g, u, v)]L = [(h', g', u', v')]L$. Then, as before, we have

$$ L \ni (h', g')^{-1}(h, g) = \left(h^{-1}h_1g_1^{-1}g\right) = \left(h'^{-1}h_1h_1g_1^{-1}g_1^{-1}g\right) = \left(h_1h_1g_1^{-1}g_1^{-1}g\right)^{-1}(h_1h_1g_1^{-1}g). $$

which also shows that

$$ \left(h_1h_1g_1^{-1}g, u, v\right) \in L. $$

Therefore, $\lambda$ and $\rho$ are well defined. We have to verify the axioms of left $H$-action.

1. For each $h' \in H$ and $y = [(h, g, u, v)]L \in X$ such that $s(h') = \vartheta([(h', g', u', v')]L)$, we have

$$ \vartheta(h'y) = \vartheta((h'h', g', u', v')]L) = t(h'h) = t(h'). $$

2. For each $y = [(h, g, u, v)]L \in X$ we have

$$ y_{u_{s_1}} = y_{u_{s_2}} = [(h_{u_{s_1}}, g, u, v)]L = y. $$

3. For each $y = [(h, g, u, v)]L \in X$ and $h', h'' \in H$, such that $s(h') = t(h'')$ and $s(h'') = \vartheta(y)$ we have

$$ h'(h''y) = h'((h''h, g, u, v)]L = [(h'h''h, g, u, v)]L = (h'h'')y. $$

We have to verify the axioms of right $G$-action.

1. For each $y = [(h, g, u, v)]L \in X$ and $g' \in G$, such that $\varsigma(y) = t(g')$ we have

$$ \varsigma(yg') = \varsigma([(h, g^{-1}g, u, v)]L) = t(g^{-1}g) = s(g'). $$

2. For each $y = [(h, g, u, v)]L \in X$ we have

$$ y_{u_{s_1}} = y_{u_{s_2}} = [(h, u_{s_1}g, u, v)]L = y. $$

3. For each $y = [(h, g, u, v)]L \in X$ and $g', g'' \in G$, such that $\varsigma(y) = t(g')$ and $s(g') = t(g'')$ we have

$$ (yg')g'' = ([(h, g^{-1}g, u, v)]L)g'' = [(h, g^{-1}g^{-1}g, u, v)]L = y(g'g''). $$
To conclude we have to verify the properties of biset. For each \( y = [(h, g, u, v)]L \in X, h' \in \mathcal{H}, \) and \( g' \in \mathcal{G} \) such that \( s(h') = \vartheta(y) \) and \( \varsigma(y) = t(g') \) we have

\[
\vartheta(yg') = \vartheta([(h, g^{-1}g, u, v)]L) = t(h) = \vartheta(y), \quad \varsigma(h'y) = \varsigma([(h'h, g, u, v)]L) = t(g) = \varsigma(y)
\]

and

\[
h'(yg') = h([(h, g^{-1}g, u, v)]L) = [(h'h, g^{-1}g, u, v)]L = [(h'h, g, u, v)]L \hspace{1cm} g' = (h'y)g',
\]

which finishes the proof. □

**Lemma 4.2.** Given two groupoids \( \mathcal{H} \) and \( \mathcal{G} \), let \( (X, \vartheta, \varsigma) \) be an \((\mathcal{H}, \mathcal{G})\)-biset and take \( x \in X \). We define

\[
(\mathcal{L}_x)_h = \left\{ (h, g) \in \mathcal{H} \times \mathcal{G} \mid s(h) = \vartheta(x), \ t(g) = \varsigma(x), \ hx = xg \right\}
\]

and

\[
(\mathcal{L}_x)_o = \left\{ (\vartheta(x), \varsigma(x)) \right\}.
\]

Then \( \mathcal{L}_x \) is a subgroupoid of the groupoid \( \mathcal{H} \times \mathcal{G} \).

**Proof.** It is immediate, since by using the first axiom of a biset, we know that, for every \((h, g) \in (\mathcal{L}_x)_h, \) we have

\[
\vartheta(x) = \vartheta(xg) = \vartheta(hx) = t(h) \quad \text{and} \quad \varsigma(x) = \varsigma(hx) = \varsigma(xg) = s(g).
\]

Thus, \( \mathcal{L}_x \) is a subgroupoid of the groupoid \( \mathcal{H} \times \mathcal{G} \).

**Proposition 4.3.** Given two groupoids \( \mathcal{H} \) and \( \mathcal{G} \), let \( X \) be an \((\mathcal{H}, \mathcal{G})\)-biset and take \( x \in X \). We define:

\[
(\mathcal{L}_x)_h = (\mathcal{K}_x)_h = \left\{ (\vartheta(x), \varsigma(x)) \right\},
\]

\[
(\mathcal{L}_x)_o = \left\{ (h, g) \in \mathcal{H} \times \mathcal{G} \mid hx = xg, \ \vartheta(x) = s(h), \ t(g) = \varsigma(x) \right\},
\]

\[
(\mathcal{K}_x)_o = \left\{ (h, g) \in \mathcal{H} \times \mathcal{G} \mid hx = x, \ \vartheta(x) = s(h), \ t(g) = \varsigma(x) \right\},
\]

that is, \( \mathcal{K} = \text{Stab}_{(\mathcal{H} \times \mathcal{G})}(x) \). Then

\[
\varphi : \left( \frac{\mathcal{H} \times \mathcal{G}}{\mathcal{K}_x} \right) \rightarrow \left( \frac{\mathcal{H} \times \mathcal{G}}{\mathcal{L}_x} \right), \quad ((h, g, \vartheta(x), \varsigma(x)))[\mathcal{K}_x] \mapsto [(h, g^{-1}, \vartheta(x), \varsigma(x))[\mathcal{L}_x]
\]

is a well-defined isomorphism of \((\mathcal{H}, \mathcal{G})\)-biset with structure given as in Lemma 4.4.

**Proof.** For each \( h', h \in \mathcal{H}, \) and \( g', g \in \mathcal{G}, \) with \( s(h) = \vartheta(x) = s(h') \) and \( s(g) = \varsigma(x) = s(g') \), we have

\[
[(h', g', \vartheta(x), \varsigma(x))][\mathcal{K}] = [(h, g^{-1}, \vartheta(x), \varsigma(x))][\mathcal{K}]
\]

if and only if

\[
(h^{-1}h', g^{-1}g') = (h^{-1}h', g^{-1}g') \in (\mathcal{K}_x),
\]

if only if \( h^{-1}hxg^{-1} = x \), if and only if \( h^{-1}hx = xg^{-1} \), if and only if

\[
(h, g^{-1})^{-1}(h', g^{-1}) \in \mathcal{L}_x,
\]

if and only if

\[
[(h', g^{-1}, \vartheta(x), \varsigma(x))][\mathcal{K}] = [(h, g^{-1}, \vartheta(x), \varsigma(x))][\mathcal{K}].
\]

Therefore \( \varphi \) is well defined and injective. For each \( h' \in \mathcal{H}, \) and \( g' \in \mathcal{G}, \) we have

\[
\varphi((h', g^{-1}, \vartheta(x), \varsigma(x)))[\mathcal{K}] = [(h', g^{-1}, \vartheta(x), \varsigma(x))][\mathcal{L}_x],
\]

hence \( \varphi \) is surjective.

Now given \( y = [(h', g', \vartheta(x), \varsigma(x))][\mathcal{K}] \in \left( \frac{\mathcal{H} \times \mathcal{G}}{\mathcal{K}_x} \right), h \in \mathcal{H}, \) and \( g \in \mathcal{G}, \) such that \( s(h) = \vartheta(y) \) and \( \varsigma(y) = t(g) \) we have

\[
\varphi(hyg) = \varphi([(hh'h', g'^{-1}g', \vartheta(x), \varsigma(x))][\mathcal{K}]) = \varphi([(hh'h', g', \vartheta(x), \varsigma(x))][\mathcal{K}])
\]

\[
= [(hh'h^{-1}, g^{-1}g^{-1}, \vartheta(x), \varsigma(x))][\mathcal{L}].
\]

Thus \( \varphi \) is an isomorphism of \((\mathcal{H}, \mathcal{G})\)-biset as stated. □
DEFINITION 4.4. Given groupoids $G$, $H$ and $K$, let $L$ be a subgroupoid of $H \times G$ and $M$ be a subgroupoid of $K \times H$. We define

$$(M \ast L)_h = \{(k, g) \in K \times G | \exists h \in H \text{ such that } (k, h) \in M, \ (h, g) \in L\}$$

and

$$(M \ast L)_h = \{(v, a) \in K \times G | \exists u \in H \text{ such that } (v, u) \in M, \ (u, a) \in L\}.$$  

Notice that, if $pr_1(M) \cap pr_1(L) = \emptyset$, where $pr_1$ and $pr_2$ are the first and second projections, then $(M \ast L)_h$ is obviously an empty set.

LEMMA 4.5. Given groupoids $G$, $H$ and $K$ such that $H$ has only one object, let $L$ be a subgroupoid of $H \times G$ and $M$ be a subgroupoid of $K \times H$. Then $M \ast L$, as defined in Definition 4.4, is a subgroupoid of $K \times G$.

Proof. Given $(k, g) \in (M \ast L)$, then there is $h \in H$ such that $(k, h) \in M$ and $(h, g) \in L$, so $(k^{-1}, h^{-1}) \in M$, and $(h^{-1}, g^{-1}) \in L$ for all $(k, g) \in (M \ast L)$. Now let be $(k_1, g_1), (k_2, g_2) \in (M \ast L)$, such that $s(k_1, g_1) = t(k_2, g_2)$. There are $h_1, h_2 \in H$, such that $(k_i, h_i) \in M$, and $(h_i, g_i) \in L$, for each $i \in \{1, 2\}$. Since $H$ has only one object we have $s(h_1) = t(h_2)$ thus we can write $h_1 h_2$ and we have

$$(k_1 h_1, h_2) = (k_1 k_2, h_1 h_2) \in M, \quad (h_1, g_1) (h_2, g_2) = (h_1 h_2, g_1 g_2) \in L.$$\

Therefore $(k_1, g_1) (k_2, g_2) \in (M \ast L)_h$, which completes the proof. \hfill \Box

EXAMPLE 4.6. Given groupoids $K$, $H$ and $G$ such that $H$ has only one object $\omega$, we consider subgroupoids $D \subseteq K, C \subseteq H, B \subseteq H, \text{ and } A \subseteq G$ where $C$ and $B$ are not empty, that is, have exactly the object $\omega$. Then $M = D \times C$ is a subgroupoid of $K \times H$ and $L = B \times A$ is a subgroupoid of $H \times G$. For each $d_0 \in D_0$ and $a_0 \in A_0$, we have $(d_0, \omega) \in M_0$ and $(\omega, a_0) \in L_0$, thus $(d_0, a_0) \in (M \ast L)_0$. We have $t_i \in C_i \cap B_i$ for each $d_i \in D_i$ and $a_i \in A_i$, we have $(d_i, t_i) \in M_i$ and $(t_i, a_i) \in L_i$, therefore $(d_i, a_i) \in (M \ast L)_i$. As a consequence we have $D_i \times A_i \subseteq (M \ast L)_i$ for $i = 0$ and $i = 1$. For each $i \in \{0, 1\}$ and for each $(k_i, g_i) \in (M \ast L)_i$, there is $h_i \in H$ such that $(k_i, h_i) \in M_i = D_i \times C$ and $(h_i, g_i) \in L_i = B_i \times A_i$, thus $k_i \in D_i$ and $g_i \in A_i$, therefore $(k_i, g_i) \in D_i \times A_i$ and $(M \ast L)_i \subseteq D_i \times A_i$. This shows that $M \ast L = D \times A$ is not an empty groupoid if both $D$ and $A$ are not so.

PROPOSITION 4.7. Given groupoids $K$, $H$ and $G$, let $M$ be a subgroupoid of $K \times H$ and $L$ be a subgroupoid of $H \times G$. Let be

$$X = \left\{(w, u, h, v, a) \in K \times H \times H \times G \ | \ (w, u) \in M_0, \ (v, a) \in L_0, \ u = t(h), \ v = s(h) \right\}. \quad (15)$$

Then $X$ is a $(M, L)$-biset with structure maps

$$\vartheta : X \rightarrow M_0 \quad \text{ and } \quad \varsigma : X \rightarrow L_0 \quad \text{ and } \quad \varphi : X \rightarrow M_0$$



left action

$$\lambda : M_0 \times X \rightarrow X$$

$$(k, h', (w, u, h, v, a)) \mapsto (t(k), t(h'), h'h, v, a)$$

and right action

$$\rho : X \times L_0 \rightarrow X$$

$$((w, u, h, v, a), h'', g) \mapsto (w, u, h'h'', s(h''), s(g)).$$

Proof. We only check the properties of a right action, since a similar proof shows the left action properties.

1. For each $y = (w, u, h, v, a) \in X$ and $(h'', g) \in L_0$, such that $\varsigma(y) = t(h'', g)$ we have

$$\varsigma(y(h'', g)) = \varsigma(w, u, h'h'', s(h''), s(g)) = (s(h''), s(g)) = s(h'', g)$$

2. For each $y = (w, u, h, v, a) \in X$ we have

$$\gamma_{t(h)}(y) = \gamma_{t(h)}(y, u, a) = (w, u, h, v, a) = (w, u, h, v, a).$$
(3) For each \( y = (w, u, h, v, a) \in X \) and \((h_1, g), (h_2, g') \in \mathcal{L}_e \) such that \( \varsigma(y) = t(h_1, g) \) and \( s(h_1, g) = t(h_2, g') \) we have
\[
(y(h_1, g)(h_2, g') = (w, u, hh_1, s(h_1), s(g))(h_2, g') = (w, u, hh_1h_2, s(h_2), s(g'))
\]
\[
= (w, u, hh_1h_2, s(h_1h_2), s(gg')) = y(h_1h_2, g, g') = y((h_1, g), (h_2, g')).
\]

Now we have to check the properties of a biset on \( X \), that is, condition (2) in Definition 3.1 for the stated actions \( \lambda \) and \( \rho \). So, for each \( y = (w, u, h, v, a) \in X \), \((h', g) \in M \) and \((h'', g') \in L \), such that \( s(k, h') = \varnothing(y) \) and \( \varsigma(y) = t(h'', g) \) we have
\[
\vartheta((k, h')y) = \vartheta((w, u, h', g', s(h'), s(g))) = (w, u) = \vartheta(y)
\]
\[
\varsigma((k, h')y) = \varsigma(t(k), t(h'), h'h, v, a) = (v, a) = \varsigma(y)
\]
and
\[
((k, h')y)(h'', g) = ((k, h')((w, u, hh'', s(h''), s(g))) = (t(k), t(h'), h''h, v, a) = (h'', g)
\]
\[
= ((k, h')y(h'', g).
\]
and this gives the desired properties, which finishes the proof. \( \square \)

4.2. The main result. Let’s keep the notations of Proposition 4.7 and let’s assume we are given \( w \in \mathcal{K}_e \), \( u \in \mathcal{H}_e \) and \( a \in \mathcal{G}_a \) such that \((w, u) \in M_a \) and \((u, a) \in L_a \). Under these assumptions, we can apply Lemma 4.5 to the groupoids \( \mathcal{K}^{w,a} \), \( \mathcal{H}^{a,w} \) and \( \mathcal{G}^{a,a} \) by taking the subgroupoids \( M^{w,a} \) of \( \mathcal{K}^{w,a} \times \mathcal{H}^{a,w} \) and \( L^{a,a} \) of \( \mathcal{H}^{a,a} \times \mathcal{G}^{a,a} \). Of course, here we are identifying the isotropy groups \( M^{w,a} \) and \( L^{a,a} \) with groupoids having only one object \((w, u) \) and \((u, a) \), respectively. In this way, we obtain that
\[
M^{w,a} = \left\{(h, w, a) \in \mathcal{L}^{w,a} \right\}
\]
(16)
is a subgroupoid of \( \mathcal{K}^{w,a} \times \mathcal{G}^{a,a} \) for every \((h, w, a) \in \mathcal{L}_e \), where \( s(h) = t(h) = u \), and where we have used the notation \( gG = gGg^{-1} \) to denote the conjugation class of a given element \( g \) in a group \( G \). Since we know that \( \mathcal{K}^{w,a} \times \mathcal{G}^{a,a} \) is a subgroupoid of \( \mathcal{K} \times \mathcal{G} \), we have that \( M^{w,a} \) is a subgroupoid of \( \mathcal{K} \times \mathcal{G} \). This will be used implicitly in the sequel.

Our main result is the following.

Theorem 4.8 (Mackey formula for bisets). Let \( \mathcal{K}, \mathcal{H}, \mathcal{G}, M \) and \( L \) be as in Proposition 4.7. Consider the biset \( X \) defined in equation (15) and the subgroupoids \( M^{w,a} = \left\{(h, w, a) \right\} \) of equation (16). Assume that \( \mathcal{M}_e = \mathcal{K}_e \times \mathcal{H}_e \) and \( \mathcal{L}_e = \mathcal{H}_e \times \mathcal{G}_e \), then there is a (non canonical) isomorphism of bisets
\[
\mathcal{M}^{w,a} \cong \mathcal{H}^{a,w} \cong \mathcal{G}^{a,a}
\]
where \( \text{rep}_{(\mathcal{M}, \mathcal{L})}(X) \) is a set of representatives of the orbits of \( X \) as \((\mathcal{M}, \mathcal{L})\)-biset.

Proof. Notice that under assumptions the denominator in the right hand side of Formula (17) is a well defined subgroupoid of \( \mathcal{K} \times \mathcal{G} \) and thus the right hand side of this formula is well defined as well. For simplicity let us denote
\[
\mathcal{V} := \left\{ \mathcal{K} \times \mathcal{H} \right\}^{w,a} \quad \text{and} \quad \mathcal{U} := \left\{ \mathcal{H} \times \mathcal{G} \right\}^{a,w}
\]
As expounded in Lemma 4.1 \( \mathcal{V} \) is a \((\mathcal{K}, \mathcal{H})\)-biset with structure maps
\[
\Theta: \mathcal{V} \to \mathcal{K}_e \quad \text{and} \quad \Xi: \mathcal{V} \to \mathcal{H}_e
\]
\[
[(k, h, w, u)]M \mapsto t(k) \quad \text{and} \quad [(k, h, w, u)]M \mapsto t(h)
\]
and \( \mathcal{U} \) is an \((\mathcal{H}, \mathcal{G})\)-biset with structure maps
\[
\Theta: \mathcal{U} \to \mathcal{H}_e \quad \text{and} \quad \Xi: \mathcal{U} \to \mathcal{G}_e
\]
\[
[(h, g, v, a)]L \mapsto t(h) \quad \text{and} \quad [(h, g, v, a)]L \mapsto t(g)
\]
Therefore, following subsection 3.3 the tensor product \( \mathcal{V} \otimes \mathcal{U} \) in the left hand side of (17) makes sense and it is a \((\mathcal{K}, \mathcal{G})\)-biset by Lemma 3.14. The orbit of a given element \([k, h, w, u]M \otimes \mathcal{H}^{a,w} \mathcal{G}^{a,a}L \) in \( \mathcal{V} \otimes \mathcal{U} \) will be denoted by \( \mathcal{K}^{w,a} \left\{ [(k, h, w, u)]M \otimes \mathcal{H}^{a,w} \mathcal{G}^{a,a}L \right\} \mathcal{G} \). If \( y \in \mathcal{K} \setminus (\mathcal{V} \otimes \mathcal{U})/\mathcal{G} \) is an element
We have to check that the orbits set of \( \begin{array}{c}
\{ \Theta (y) \cdot \Lambda (y) \} \end{array} \),
\((\mathcal{K}, \mathcal{G})_g = \{(k, g) \in \mathcal{K} \times \mathcal{G} \mid kg \equiv yg \}, \Theta (y) = s (k), t (g) = \Lambda (y)\),
\((\mathcal{K}, \mathcal{G})_g = \{(k, g) \in \mathcal{K} \times \mathcal{G} \mid kg \equiv yg \}, \Theta (y) = s (k), t (g) = \Lambda (y)\).

Since, by Lemma 3.10 and Proposition 5.11, every biset is the disjoint union of its orbits. By Proposition 4.3, we obtain the following isomorphisms of \((\mathcal{K}, \mathcal{G})\)-biset:

\[ \text{is well defined. So let us choose two representatives for the same orbit, that is, let } k_1 \in \mathcal{K}\text{, and } g_1 \in \mathcal{G}, \text{ such that } s (k_1) = t (l), t (l) = t (g_1) \text{ and } \]

\[ [(k_1, h, w, u)]M \otimes_{\mathcal{G}} [(h', g, v, a)] \mathcal{L} \]  

\[ = [(k_1, l, e, r, n)]M \otimes_{\mathcal{G}} [(e', f, m, b)] \mathcal{L}, \]

\[ [(k_1, h, w, u)]M \otimes_{\mathcal{G}} [(h', g, v, a)] \mathcal{L} = [(k_1, h_1^{-1} e, r, n)]M \otimes_{\mathcal{G}} [(h_1^{-1} e', f, m, b)] \mathcal{L} \]

\[ \in \mathcal{V} \times \mathcal{U}. \]

This means that

\[ \left\{ \begin{array}{c}
[(k, h, w, u)]M = [(k_1, h_1^{-1} e, r, n)]M \\
[(h', g, v, a)] \mathcal{L} = [(h_1^{-1} e', g_1^{-1} f, m, b)] \mathcal{L}
\end{array} \right. \]

As a consequence, from one hand, there is \((k_2, h_2) \in M\), such that \(s (k_2, h_2) = (w, u), t (k_2, h_2) = (r, n)\) and

\[ (k, h, w, u) = (k_1 l k_2, h_1^{-1} e h_2, s (k_2), s (h_2)). \]

On the other hand, there is \((h_3, g_2) \in \mathcal{L}\), such that \(s (h_3, g_2) = (v, a), t (h_3, g_2) = (m, b)\) and

\[ (h', g, v, a) = (h_1^{-1} e', g_1 f, m, b), (h_3, g_2) = (h_1^{-1} e' h_3, g_1 f g_2, s (h_3), s (g_2)). \]

Therefore we obtain the following equalities from equations (18) and (19):

\[ k_2 = l^{-1} k_1^{-1} k, \quad h_2 = e^{-1} h_1 h, \quad h_3 = e^{-1} l h_1 h', \quad g_2 = f^{-1} g_1^{-1} g. \]

Thus

\[ (k_2, h_2) (w, u, h_1^{-1} h', v, a) (h_3, g_2)^{-1} = (s (k_2), t (h_2), h_3 h_1^{-1} h' h_3, t (h_3), t (g_2)) = (w, n, e^{-1} h_1 h h^{-1} h' h_1^{-1} e', m, b) = (w, n, e^{-1} e', m, b), \]

which shows that \(M[(w, u, h^{-1} h', v, a)] \mathcal{L} = M[(w, n, e^{-1} e', m, b)] \mathcal{L}\) in the orbits set \(M/X, \mathcal{L}\). Henceforth, \(\varphi\) is a well defined map.

In other direction, we have a well defined map given by

\[ \psi: M/X, \mathcal{L} \longrightarrow \mathcal{K} \setminus \left( (\mathcal{V} \otimes_{\mathcal{G}} \mathcal{U}) / \mathcal{G} \right), \]

\[ M[(w, u, h, v, a)] \mathcal{L} \longrightarrow \mathcal{K} \left( [(t, t, w, u)]M \otimes_{\mathcal{G}} [(h, v, a)] \mathcal{L} \right) \mathcal{G}. \]
Let us check that \( \varphi \) and \( \psi \) are one the inverse of the other. So, for each orbit

\[
\mathcal{K}[(k, h, w, u)] M \otimes \mathcal{L}[\psi] \in \mathcal{K}(\mathcal{V} \otimes \mathcal{U}) / \mathcal{L}
\]

we have

\[
\psi \circ \varphi(\mathcal{K}[(k, h, w, u)] M \otimes \mathcal{L}[\psi]) = \mathcal{K}[(\iota_1, \iota_1, w, u)] M \otimes \mathcal{L}[\psi] \mathcal{L}
\]

\[
\mathcal{K}[(k, h, w, u)] M \otimes \mathcal{L}[\psi(g^{-1})] \mathcal{G}
\]

\[
\mathcal{K}[(k, h, w, u)] M \otimes \mathcal{L}[\psi(g^{-1})] \mathcal{G}
\]

which shows that \( \psi \circ \varphi = id \).

Conversely, for each element \( \varphi([(w, u, h, v, a)] M) \mathcal{L} \in \mathcal{M}(\mathcal{X} / \mathcal{L}, \mathcal{G}) \), we have

\[
\varphi \circ \psi(M[(w, u, h, v, a)] L) = \varphi(\mathcal{K}[(\iota_1, \iota_1, w, u)] M \otimes \mathcal{L}[\psi] \mathcal{G})
\]

\[
\mathcal{M}[(w, u, c^{-1}, h, u, a)] \mathcal{L}
\]

\[
\mathcal{M}[(w, u, h, u, a)] \mathcal{L}
\]

whence \( \varphi \circ \psi = id \). Therefore \( \varphi \) is bijective with inverse \( \psi \).

Let us check that, for every element of the form

\[
y = [(\iota_1, \iota_1, w, u)] M \otimes \mathcal{L} \in [\mathcal{K}(\mathcal{V} \otimes \mathcal{U}) / \mathcal{L}],
\]

there is the following equality of subgroupoids

\[
(\mathcal{K}, \mathcal{G}) = \mathcal{M}^\alpha \ast \mathcal{L}^{\alpha_{\mathcal{G}}}
\]

So, taking \( (k_1, g_1) \in \mathcal{K} \times \mathcal{G} \), such that \( s(k_1) = \iota(k_1) = w \) and \( s(g_1) = \iota(g_1) = a \) we have \( k_3 y = y g_3 \) if and only if

\[
y = [(k_3, \iota_1, w, u)] M \otimes \mathcal{L} \in [\mathcal{K}(\mathcal{V} \otimes \mathcal{U}) / \mathcal{L}]
\]

if and only if there exists \( h_4 \in \mathcal{H} \), such that \( s(h_4) = t(h_4) = a \) and

\[
[(\iota_1, \iota_1, w, u)] M \otimes \mathcal{L} \in [\mathcal{H} \times \mathcal{G}]
\]

This holds true, if and only if, there exists \( h_4 \in \mathcal{H}^\alpha \otimes \mathcal{G} \) such that

\[
[(\iota_1, \iota_1, w, u)] M \otimes \mathcal{L} \in [\mathcal{H} \times \mathcal{G}]
\]

if and only if there exists \( h_4 \in \mathcal{H}^\alpha \) such that

\[
\left\{ (k_3, h_4^{-1}) \in \mathcal{M}^\alpha \right\}
\]

if and only if there exists \( h_4 \in \mathcal{H}^\alpha \) such that

\[
\left\{ (k_3, h_4^{-1}) \in \mathcal{M}^\alpha \right\}
\]

if and only if

\[
\left\{ (k_3, g_3) \in \mathcal{H}^\alpha \right\}
\]

As a consequence we get the following isomorphisms of \( (\mathcal{K}, \mathcal{G}) \)-bisets:

\[
\mathcal{V} \otimes \mathcal{U} \cong \left( \mathcal{M}^\alpha \otimes \mathcal{G} \right) \cong \mathcal{K} \times \mathcal{G}
\]

\[
\mathcal{V} \otimes \mathcal{U} \cong \left( \mathcal{M}^\alpha \otimes \mathcal{G} \right) \cong \mathcal{K} \times \mathcal{G}
\]
which depends on the choice of a representatives set of the orbits of the biset $X$, and this finishes the proof. □

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