AN UPPER BOUND ON THE NUMBER OF RATIONAL POINTS
OF ARBITRARY PROJECTIVE VARIETIES
OVER FINITE FIELDS

ALAIN COUVREUR

(Communicated by Matthew A. Papanikolas)

ABSTRACT. We give an upper bound on the number of rational points of an
arbitrary Zariski closed subset of a projective space over a finite field $\mathbb{F}_q$.
This bound depends only on the dimensions and degrees of the irreducible
components and holds for very general projective varieties, even reducible and
nonequidimensional. As a consequence, we prove a conjecture of Ghorpade
and Lachaud on the maximal number of rational points of an equidimensional
projective variety.

INTRODUCTION

Counting or finding bounds on the number of rational points of a variety over
a finite field is a classical task which arises naturally in number theory, algebraic
geometry or finite geometry. This problem has motivated the development of very
elegant constructions of cohomology theories called Weil cohomologies. The work of
Deligne [7] on Weil conjectures entails among others an upper bound on the number
of rational points of a smooth projective complete intersection $\mathcal{X}$ depending only
on $q$ and the Betti numbers of $\mathcal{X}$ for the étale cohomology [7, Theorem 8.1].

If $\mathcal{X}$ is a smooth curve, this bound is nothing but the well-known Weil bound,
$$|\mathcal{X}(\mathbb{F}_q)| \leq q + 1 + 2g_{\mathcal{X}} \sqrt{q},$$
where $g_{\mathcal{X}}$ denotes the genus of $\mathcal{X}$. This bound was actually improved by Serre
[18] as
$$|\mathcal{X}(\mathbb{F}_q)| \leq q + 1 + g_{\mathcal{X}} \lfloor 2\sqrt{q} \rfloor.$$ Subsequently, this bound has been extended to the case of singular curves in [1][2].
Another approach due to Stöhr and Voloch [21] is based on the use of the Weierstrass
points of the curve and provides upper bounds on the number of rational points
of smooth projective curves. Finally, other upper bounds on the number of points
of arbitrary plane curves arise from purely combinatorial methods, like Sziklai’s
bound for arbitrary plane curves [22] which was improved by Homma and Kim [12]
using Stöhr–Voloch bounds.

For higher-dimensional varieties, Deligne’s results hold for arbitrary smooth geo-
metrically irreducible projective varieties over a finite field. In addition, a bound
“à la Weil” for irreducible complete intersections in a projective space is given in
[10]. On the other hand, an upper bound for the number of rational points of
an arbitrary projective hypersurface $\mathcal{X} \subseteq \mathbf{P}^n$ depending only on the degree and dimension of $\mathcal{X}$ has been proved by Serre [19] and independently by Sørensen [20],

$$|\mathcal{X}(\mathbf{F}_q)| \leq \delta q^{n-1} + \frac{q^{n-1} - 1}{q - 1},$$

where $\delta$ denotes the degree of the hypersurface $\mathcal{X}$. Serre’s proof is combinatorial and based on a nice double counting argument. Notice that an interesting alternative proof of this bound has been recently proposed in [6]. Further bounds on the number of rational points of hypersurfaces of fixed dimension and degree and based on such combinatorial methods can be found, for instance, in [13, 23].

Generalisations of Serre’s bound to projective subvarieties of codimension larger than or equal to 1 have also been studied or conjectured. Actually, this problem can be considered from two different points of view. One can look for bounds on the number of rational points of a subvariety of $\mathbf{P}^n$ either in terms of the degrees of a family of defining polynomials or in terms of its dimension (provided it is equidimensional) and its degree as a subvariety of $\mathbf{P}^n$.

For the first point of view, a generalisation of Serre’s bound has been conjectured by Tsfasman and Boguslavsky [4]. Given a family $F_1, \ldots, F_r$ of linearly independent homogeneous polynomials of total degree $\delta$ in $\mathbf{F}_q[x_0, \ldots, x_m]$, the number of rational points of the corresponding variety is conjectured to be bounded above by a quantity depending only on $r, m$ and $\delta$. This conjecture has been recently proved to be true for all $r \leq m + 1$ and $\delta < q - 1$ by Datta and Ghorpade [5, 6].

For the second point of view, Ghorpade and Lachaud raised a conjecture [10, Conjecture 12.2] on the maximal number of rational points of an arbitrary complete intersection in $\mathbf{P}^n$ depending only on its dimension, its degree and the dimension of its ambient projective space. This conjectural upper bound coincides with Serre’s bound [19] when the variety is a hypersurface. This conjecture has been discussed more recently in a survey paper of Lachaud and Rolland [16, Conjecture 5.3].

In the present paper, we prove a general upper bound for arbitrary projective subvarieties of $\mathbf{P}^n$ possibly nonequidimensional. This bound depends only on $n$ and the degrees and dimensions of the irreducible components of the variety. From this bound and considering the equidimensional case, we prove Ghorpade and Lachaud’s conjecture [10, Conjecture 12.2]. This proves in particular that Ghorpade and Lachaud’s conjecture holds for equidimensional projective varieties even if they are not complete intersections.

Our bound is proved by purely combinatorial methods inspired by Serre’s proof in [19]. The context is however more difficult since Serre’s proof for hypersurfaces consists in studying an incidence structure involving the intersections of this hypersurface with hyperplanes. The point is that, for a hypersurface $\mathcal{X}$ and a hyperplane $\mathcal{H}$ such that $\mathcal{H} \nsubseteq \mathcal{X}$, then $\mathcal{H} \cap \mathcal{X}$ has codimension 1 in $\mathcal{H}$. In particular, no irreducible component of the hypersurface is contained in the hyperplane. In the case of an arbitrary variety, the situation gets harder since some irreducible components may be contained in a hyperplane. For this reason, our proof treats separately the case of a variety with no irreducible components contained in a hyperplane and varieties having some irreducible components which are contained in hyperplanes.

Finally, we discuss the sharpness of this bound. In particular, we prove that in the equidimensional case, this bound is reached by some arrangements of linear varieties which we call flowers. We also leave as open questions some further possible improvements.
1. Notation and definitions

1.1. Schemes and varieties. In this article, we fix a finite field $\mathbb{F}_q$. We denote respectively by $\mathbb{A}^n$ and $\mathbb{P}^n$ the affine and projective space of dimension $n$ over $\mathbb{F}_q$ defined as

$$\mathbb{A}^n = \text{Spec} \mathbb{F}_q[x_1, \ldots, x_n] \quad \text{and} \quad \mathbb{P}^n = \text{Proj} \mathbb{F}_q[x_0, \ldots, x_n].$$

The dual of $\mathbb{P}^n$, which is the variety of hyperplanes of $\mathbb{P}^n$, is denoted as $\mathbb{P}^n$. Given a closed subscheme $\mathcal{S}$ of $\mathbb{A}^n$ or $\mathbb{P}^n$, we denote by $\mathcal{S}_{\text{red}}$ the reduced scheme supporting $\mathcal{S}$.

In this article, an affine variety (resp. projective variety) denotes a closed reduced subscheme of $\mathbb{A}^n$ (resp. $\mathbb{P}^n$). Hence, such a scheme is always defined over $\mathbb{F}_q$. In particular, in what follows and unless otherwise specified, whenever we speak about a hyperplane or a linear subvariety of $\mathbb{A}^n$ or $\mathbb{P}^n$ it is always defined over $\mathbb{F}_q$. Finally, given two closed subvarieties $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{A}^n$ or $\mathbb{P}^n$, by the intersection $\mathcal{X} \cap \mathcal{Y}$ we always mean the scheme theoretic intersection.

1.2. Irreducibility equidimensionality. An affine (resp. projective) variety is said to be irreducible if it is an integral scheme (see [11, Chapter II.3]), or equivalently if its defining ideal is prime in $\mathbb{F}_q[x_1, \ldots, x_n]$ (resp. $\mathbb{F}_q[x_0, \ldots, x_n]$). We emphasise that, in what follows and unless otherwise specified, “irreducible” means “irreducible over $\mathbb{F}_q$”. In particular, an irreducible variety need not be absolutely irreducible.

A variety is said to be equidimensional if its irreducible components all have the same dimension. A closed subscheme $\mathcal{S}$ of $\mathbb{A}^n$ or $\mathbb{P}^n$ is said to be equidimensional if $\mathcal{S}_{\text{red}}$ is.

1.3. Degree. The degree of a closed equidimensional subscheme of dimension $d$ of $\mathbb{P}^n$ is $d!$ times the leading coefficient of its Hilbert polynomial (for instance, see [8, Chapter III.3]). The degree of a closed subscheme of $\mathbb{A}^n$ is defined as the degree of its projective closure. For a more geometrical point of view, the degree of an equidimensional projective variety $\mathcal{X} \subseteq \mathbb{P}^n$ is the maximum possible number of points of a zero-dimensional intersection of $\mathcal{X} \otimes \mathbb{F}_q$ with a linear subvariety of codimension $\dim \mathcal{X}$ in $\mathbb{P}^n \otimes \mathbb{F}_q$.

1.4. Dimension and degree sequences. A general subvariety $\mathcal{X} \subseteq \mathbb{A}^n$ or $\mathbb{P}^n$ may be reducible and nonequidimensional. Therefore, it has an irredundant decomposition,

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_r,$$

where $\mathcal{X}_1, \ldots, \mathcal{X}_r$ are irreducible varieties and for all $i \neq j$, $\mathcal{X}_i \not\subseteq \mathcal{X}_j$. For all $i \in \{1, \ldots, r\}$, the integers $d_{\mathcal{X}_i}$ and $\delta_{\mathcal{X}_i}$ denote respectively the dimension and the degree of $\mathcal{X}_i$. The sequences $(d_{\mathcal{X}_i})_{i \in \{1, \ldots, r\}}$ and $(\delta_{\mathcal{X}_i})_{i \in \{1, \ldots, r\}}$ are referred to as the dimension sequence and the degree sequence of $\mathcal{X}$. It is worth noting that the $d_{\mathcal{X}_i}$’s need not be distinct. For instance, when $\mathcal{X}$ is equidimensional, the $d_{\mathcal{X}_i}$’s are all equal. Obviously, for all $i$, we have $d_{\mathcal{X}_i} \geq 0$ and $\delta_{\mathcal{X}_i} \geq 1$.

More generally, a closed subscheme $\mathcal{S} \subseteq \mathbb{A}^n$ or $\mathbb{P}^n$ has an irredundant decomposition,

$$\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_r,$$

where $(\mathcal{S}_1)_{\text{red}}, \ldots, (\mathcal{S}_r)_{\text{red}}$ are the irreducible components of the irredundant decomposition of $\mathcal{S}_{\text{red}}$. The dimension and degree sequences of $\mathcal{S}$ are defined in a similar way.
fashion. Notice that \( S \) and \( S_{\text{red}} \) have the same dimension sequence while for all \( i \), 
\[ \deg S_i \geq \deg(S_i)_{\text{red}} \]
with equality if \( S_i \) is reduced.

Finally, we always denote by \( D_S \) the maximum dimension of a component of the irredundant decomposition of \( S \), that is,
\[ D_S \overset{\text{def}}{=} \max \{d_{S_1}, \ldots, d_{S_r} \} . \]

1.5. **Rational points.** A closed point of a scheme over \( \mathbf{F}_q \) is said to be rational or \( \mathbf{F}_q \)-rational if its residue field is \( \mathbf{F}_q \). The set of rational points of a scheme \( \mathcal{Y} \) is denoted as \( \mathcal{Y}(\mathbf{F}_q) \). When \( \mathcal{Y} = \mathbf{P}^n \), we denote its number of points by
\[ \pi_n \overset{\text{def}}{=} |\mathbf{P}^n(\mathbf{F}_q)| = \frac{q^{n+1} - 1}{q - 1} . \]

Moreover, for convenience's sake, we set
\[ \pi_j \overset{\text{def}}{=} 0, \text{ for all } j < 0 . \]

Let us recall that
\[ \forall n \geq 0, \quad \pi_n = q\pi_{n-1} + 1 , \]
which straightforwardly entails
\[ \forall k \geq \ell \geq 0, \quad \pi_k - \pi_\ell = q(\pi_{k-1} - \pi_{\ell-1}) . \]

2. **THE AFFINE CASE**

**Theorem 2.1.** Let \( \mathcal{X} \subseteq \mathbf{A}^n \) be an affine variety. Let \( d_{\mathcal{X}}_1, \ldots, d_{\mathcal{X}}_r \) be its dimension sequence and \( \delta_{\mathcal{X}}_1, \ldots, \delta_{\mathcal{X}}_r \) its degree sequence. Then, we always have
\[ |\mathcal{X}(\mathbf{F}_q)| \leq \sum_{i=1}^r \delta_{\mathcal{X}}_i q^{d_{\mathcal{X}}_i} . \]

**Proof.** The equidimensional case is proved in [16]. It holds in particular for irreducible varieties. Thus, for all \( i \in \{1, \ldots, r\} \), we have
\[ |\mathcal{X}_i(\mathbf{F}_q)| \leq \delta_{\mathcal{X}}_i q^{d_{\mathcal{X}}_i} . \]
The result is obtained by summing up all of these inequalities. \( \square \)

3. **THE PROJECTIVE CASE**

The main result of this article is stated below. Section 4 is devoted to its proof.

**Theorem 3.1.** Let \( \mathcal{X} \subseteq \mathbf{P}^n \) be a projective variety with dimension sequence \( d_{\mathcal{X}}_1, d_{\mathcal{X}}_2, \ldots, d_{\mathcal{X}}_r \) with \( d_{\mathcal{X}}_i < n \) for all \( i \in \{1, \ldots, r\} \), and degree sequence \( \delta_{\mathcal{X}}_1, \ldots, \delta_{\mathcal{X}}_r \). Then,
\[ |\mathcal{X}(\mathbf{F}_q)| \leq \left( \sum_{i=1}^r \delta_{\mathcal{X}}_i (\pi d_{\mathcal{X}}_i - \pi_2 d_{\mathcal{X}}_i - n) \right) + \pi_2 D_{\mathcal{X}} - n , \]
where \( D_{\mathcal{X}} = \max \{d_{\mathcal{X}}_1, \ldots, d_{\mathcal{X}}_r\} \).

**Remark 3.2.** Theorem 3.1 actually holds when replacing \( \mathcal{X} \) by a closed subscheme \( S \) of \( \mathbf{P}^n \). Indeed, the result applies to the variety \( S_{\text{red}} \). Then, notice that
(i) \( S(\mathbf{F}_q) = S_{\text{red}}(\mathbf{F}_q) \);
(ii) for all schemes \( S_i \) of the irredundant decomposition of \( S \), we have \( \deg S_i \geq \deg(S_i)_{\text{red}} \) and \( \dim S_i = \dim(S_i)_{\text{red}} \).
Therefore, the right hand side of (3) is smaller when applied to the irredundant decomposition of $S_{\text{red}}$ than when applied to that of $S$.

Ghorpade and Lachaud’s conjecture [10, Conjecture 12.2] is a straightforward corollary of Theorem 3.1 since it is nothing but the equidimensional case.

**Corollary 3.3.** Let $\mathcal{X} \subseteq \mathbb{P}^n$ be an equidimensional projective variety of dimension $d < n$ and degree $\delta$. Then,

$$|\mathcal{X}(\mathbb{F}_q)| \leq \delta(\pi d - \pi 2d - n) + \pi 2d - n.$$ 

**Remark 3.4.** Actually Ghorpade and Lachaud stated this conjecture under the additional hypothesis “$\mathcal{X}$ is a complete intersection”. The conjecture turns out to be true even without this hypothesis.

4. The proof

We will prove Theorem 3.1 by induction on the dimension $n$ of the ambient space. First, let us introduce some notation.

**Notation 4.1.** Let $S \subseteq \mathbb{P}^n$ be a closed subscheme of $\mathbb{P}^n$ with dimension and degree sequences $d_1, \ldots, d_s$ and $\delta_1, \ldots, \delta_s$. We define $B_n(S)$ as

$$B_n(S) \overset{\text{def}}{=} \left( \sum_{i=1}^{s} \delta_i (\pi d_i - \pi 2d_i - n) \right) + \pi 2D_S - n.$$ 

4.1. A consequence of Bézout’s theorem. The following classical statement is central in the proofs to follow. It is nothing but a corollary of a refined version of Bézout’s theorem (see, for instance, [9]). We outline an ad hoc proof for the comfort of the reader.

**Proposition 4.2.** Let $\mathcal{X} \subseteq \mathbb{P}^n$ be an irreducible projective variety of dimension $d \geq 1$ and degree $\delta$ and let $\mathcal{H}$ be a hyperplane of $\mathbb{P}^n$ which does not contain $\mathcal{X}$. Then, $\mathcal{X} \cap \mathcal{H}$ is an equidimensional scheme of dimension $d - 1$ and degree $\delta$.

**Proof.** Since $\mathcal{X} \not\subseteq \mathcal{H}$, an irreducible component of $(\mathcal{X} \cap \mathcal{H})_{\text{red}}$ has dimension $< d$, while from [11, Theorem 7.2] it has dimension $\geq d - 1$. Thus, $\mathcal{X} \cap \mathcal{H}$ is equidimensional of dimension $d - 1$. For the degree, the exact sequence

$$0 \longrightarrow \mathbb{F}_q[x_0, \ldots, x_n]/I \xrightarrow{\times f} \mathbb{F}_q[x_0, \ldots, x_n]/I \longrightarrow \mathbb{F}_q[x_0, \ldots, x_n]/I + (f) \longrightarrow 0$$

entails a relation on the Hilbert polynomials $P_{\mathcal{X}}$ and $P_{\mathcal{X} \cap \mathcal{H}}$ of $\mathcal{X}$ and $\mathcal{X} \cap \mathcal{H}$, namely:

$$P_{\mathcal{X} \cap \mathcal{H}}(t) = P_{\mathcal{X}}(t) - P_{\mathcal{X}}(t - 1).$$

Since the leading term of $P_{\mathcal{X}}$ is $\frac{\delta}{d!}$, after an easy computation, we see that the leading term of $P_{\mathcal{X} \cap \mathcal{H}}$ equals $\frac{\delta}{(d-1)!}$, which concludes the proof. 

**Corollary 4.3.** Let $\mathcal{X} \subseteq \mathbb{P}^n$ be a projective variety with irredundant decomposition $\mathcal{X}_1 \cup \cdots \cup \mathcal{X}_s$. Let $\mathcal{H} \in \mathbb{P}^n(\mathbb{F}_q)$ be a hyperplane which does not contain any of the $\mathcal{X}_i$’s. Then,

$$B_{n-1}(\mathcal{X} \cap \mathcal{H}) = \left( \sum_{i=1}^{s} \delta_{\mathcal{X}_i} (\pi d_{\mathcal{X}_i} - 1 - \pi 2d_{\mathcal{X}_i} - n - 1) \right) + \pi 2D_{\mathcal{X}} - n - 1.$$
In particular, this quantity is independent from $\mathcal{H}$ and denoted $B_{n-1}(\mathcal{X} \cap \cdot)$.

Proof. For all $i \in \{1, \ldots, s\}$, define $\mathcal{Y}_{i,1}, \ldots, \mathcal{Y}_{i,s_i}$ to be the components of the irredundant decomposition of $\mathcal{X}_i \cap \mathcal{H}$. We have

$$B_{n-1}(\mathcal{X} \cap \cdot) = \left( \sum_{i=1}^{s} \sum_{j=1}^{s_i} \delta_{\mathcal{Y}_{i,j}} \left( \pi d_{\mathcal{Y}_{i,j}} - \pi 2 d_{\mathcal{Y}_{i,j}} - (n-1) \right) \right) + \pi 2 D_{\mathcal{X} \cap \mathcal{H}} - (n-1).$$

From Proposition 4.2 for all $i, j$, we have $d_{\mathcal{Y}_{i,j}} = d_{\mathcal{X}_i,1}$. Hence, $D_{\mathcal{X} \cap \mathcal{H}} = D_{\mathcal{X}} - 1$ and

$$B_{n-1}(\mathcal{X} \cap \cdot) = \left( \sum_{i=1}^{s} \sum_{j=1}^{s_i} \delta_{\mathcal{Y}_{i,j}} \left( \pi d_{\mathcal{X}_i} - \pi 2 d_{\mathcal{X}_i} - (n-1) \right) \right) + \pi 2 (D_{\mathcal{X}} - 1) - (n-1)$$

$$= \left( \sum_{i=1}^{s} \left( \pi d_{\mathcal{X}_i} - \pi 2 d_{\mathcal{X}_i} - n \right) \sum_{j=1}^{s_i} \delta_{\mathcal{Y}_{i,j}} \right) + \pi 2 D_{\mathcal{X}} - n - 1.$$

For all $i$, we have $\sum_{j=1}^{s_i} \delta_{\mathcal{Y}_{i,j}} = \delta_{\mathcal{X}_i \cap \mathcal{H}}$ which equals $\delta_{\mathcal{X}_i}$ from Proposition 4.2. This concludes the proof. \qed

4.2. A remark on the zero-dimensional part. If $\mathcal{X} = \mathcal{Y} \cup \mathcal{Z}$, where $\mathcal{Y}$ is a union of irreducible components of dimension larger than or equal to 1 and $\mathcal{Z}$ has dimension 0, then it is sufficient to prove the upper bound on $\mathcal{Y}$. Indeed, denote by $\delta_\mathcal{X}$ the degree of $\mathcal{X}$ and by $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$ the irreducible components of $\mathcal{Y}$. Then, notice that $D_{\mathcal{X}} = D_{\mathcal{Y}}$. The upper bound we wish to prove becomes

$$|\mathcal{X}(\mathbb{F}_q)| \leq \sum_{i=1}^{r} \delta_{\mathcal{Y}_i} \left( \pi d_{\mathcal{Y}_i} - \pi 2 d_{\mathcal{Y}_i} - n \right) + \delta_\mathcal{X} + \pi 2 D_{\mathcal{Y}} - n.$$

Thus, if we can prove the result for $\mathcal{Y}$, then since

$$|\mathcal{X}(\mathbb{F}_q)| \leq \delta_\mathcal{X} \quad \text{and} \quad |\mathcal{X}(\mathbb{F}_q)| \leq |\mathcal{Y}(\mathbb{F}_q)| + |\mathcal{Z}(\mathbb{F}_q)|,$$

we get the result. Therefore, from now on, we assume that $\mathcal{X}$ has no zero-dimensional component. That is to say, the dimension sequence satisfies

$$\forall i \in \{1, \ldots, r\}, \quad d_{\mathcal{X}_i} > 0.$$

4.3. Initialisation. The case $n = 1$, which corresponds to that of a zero-dimensional subscheme of the projective line is obvious. Actually, the case $n = 2$ is already known. Indeed, as suggested in §4.2, one can assume that $\mathcal{X}$ has no zero-dimensional component and hence is a curve of degree $\delta$. Then, the upper bound

$$|\mathcal{X}(\mathbb{F}_q)| \leq \delta q + 1$$

is a direct consequence of [19].

Remark 4.4. Notice that some refined bounds on the number of points of plane curves are given in [12,21,22].
4.4. The induction step under different assumptions. From now on, we assume that $n \geq 3$. Consider the two following assumptions.

Assumption 1. No irreducible component of $\mathcal{X}$ is contained in a hyperplane.

Assumption 2. Every irreducible component of $\mathcal{X}$ is either linear or is not contained in any hyperplane.

The induction step is proved under Assumption 1 in §4.5, then under Assumption 2 in §4.6. Finally, the general case is treated in §4.7.

4.5. Proof under Assumption 1. First, notice that the upper bound in Theorem 3.1 is obviously true if $\mathcal{X}(\mathbb{F}_q) = \emptyset$. Therefore, assume from now on that $\mathcal{X}(\mathbb{F}_q)$ is nonempty and let $P \in \mathcal{X}(\mathbb{F}_q)$. By Assumption 1, no $H \in \mathcal{H}(\mathbb{F}_q)$ containing $P$ contains an irreducible component of $\mathcal{X}$. Next, let us introduce the bipartite graph $G$ whose first and second vertex sets are $V_1 \overset{\text{def}}{=} \mathcal{X}(\mathbb{F}_q) \setminus \{P\}$, $V_2 \overset{\text{def}}{=} \{H \in \mathcal{H}(\mathbb{F}_q) \mid P \in H\}$, and whose edge set is $E \overset{\text{def}}{=} \{(Q, H) \in V_1 \times V_2 \mid Q \in H\}.$

The heart of the proof consists in counting the set of edges in two distinct manners. In what follows, the number of edges containing a given vertex is referred to as the valency of the vertex.

Remark 4.5. Usually, in graph theory, the number of edges containing a given vertex is referred to as the degree of the vertex. We chose valency to avoid confusion with the notion of degree of a subscheme of $\mathbb{A}^n$ or $\mathbb{P}^n$.

Let us summarise some properties of the graph:

(i) $|V_2| = \pi_{n-1};$
(ii) the valency of a vertex $Q \in V_1$ equals $\pi_{n-2};$
(iii) the valency of a vertex $H \in V_2$ equals $|(\mathcal{X} \cap H)(\mathbb{F}_q) \setminus \{P\}|$ which, by induction, is bounded above by $B_{n-1}(\mathcal{X} \cap H) - 1$ (see Notation 4.1) and hence by $B_{n-1}(\mathcal{X} \cap \cdot ) - 1$ (see Corollary 4.3). From Remark 3.2, this holds even if $\mathcal{X} \cap H$ is nonreduced.

First, assume that $B_{n-1}(\mathcal{X} \cap \cdot ) \geq \pi_{n-1}$. That is, 

$$\left(\sum_{i=1}^{r} \delta_{x,i}(\pi_{d_{x,i}-1} - \pi_{2d_{x,i}-n-1})\right) + \pi_{2d_{x}-n-1} \geq \pi_{n-1};$$

then, multiplying both sides by $q$ yields (thanks to (2))

$$\left(\sum_{i=1}^{r} \delta_{x,i}(\pi_{d_{x,i}} - \pi_{2d_{x,i}-n})\right) + \pi_{2d_{x}-n} \geq \pi_{n}.\tag{4}$$

Since $\mathcal{X} \subseteq \mathbb{P}^n$, we have $|\mathcal{X}(\mathbb{F}_q)| \leq \pi_n$. Therefore, from (4), the result is straightforward.

From now on, we assume that

$$B_{n-1}(\mathcal{X} \cap \cdot ) < \pi_{n-1}.\tag{5}$$
From (ii), we have
\[(6) \quad |E| = |V_1| \pi_{n-2} = (|V(F_q)| - 1)\pi_{n-2}.\]
From (i) and (iii), we have
\[(7) \quad |E| \leq \pi_{n-1} (B_{n-1}(\mathcal{X} \cap \cdot) - 1).\]
This yields
\[(8) \quad |X(F_q)| \leq 1 + \pi_{n-1} \frac{B_{n-1}(\mathcal{X} \cap \cdot) - 1}{\pi_{n-2}}.\]
By the definition of $B_{n-1}(\mathcal{X} \cap \cdot)$ and using (2) we get
\[|X(F_q)| \leq \left( \sum_{i=1}^{r} \delta_{\mathcal{X}_i} (\pi_{d_{\mathcal{X}_i}} - \pi_{2d_{\mathcal{X}_i}} - n) \right) + \pi_{2D_{\mathcal{X}} - n}.\]
From (5), we have $\frac{B_{n-1}(\mathcal{X} \cap \cdot) - 1}{\pi_{n-2}} < \frac{\pi_{n-1} - 1}{\pi_{n-2}}$ and, using (1), we obtain $\frac{B_{n-1}(\mathcal{X} \cap \cdot) - 1}{\pi_{n-2}} < q$, which yields
\[|X(F_q)| \leq \left( \sum_{i=1}^{r} \delta_{\mathcal{X}_i} (\pi_{d_{\mathcal{X}_i}} - \pi_{2d_{\mathcal{X}_i}} - n) \right) + \pi_{2D_{\mathcal{X}} - n}.\]

Remark 4.6. Instead of considering an incidence graph, the proof can be realised using a purely algebraic geometric point of view by defining the incidence variety
\[T \defeq \{(Q, L) \in \mathcal{X} \times \tilde{\mathbb{P}}^n \mid Q \in L\}.\]
Then, the approach consists of counting the number of rational points of $T$ in two different manners by estimating the number of rational points of the fibres of the canonical projections $T \to \mathbb{P}^n$ and $T \to \mathcal{X}$. This point of view is developed in [10, §12] and [16, §2].

4.6. Proof under Assumption 2 The proof under Assumption 2 is done in three steps. Under the assumption that no nonlinear irreducible component of $\mathcal{X}$ is contained in an $F_q$-rational hyperplane:

1. we first treat the case when one of the irreducible components of $\mathcal{X}$ is a hyperplane;
2. then, we treat the case when only one irreducible component $L$ of $\mathcal{X}$ is linear and $L$ is not a hyperplane;
3. finally, we treat the case of multiple linear irreducible components which are not hyperplanes.

The treatment of the first step will require the following lemma.

Lemma 4.7. Let $\mathcal{Y}$ be an irreducible closed subvariety of $\mathbb{P}^n$ of dimension $d$ and degree $\delta$. Let $\mathcal{H} \in \tilde{\mathbb{P}}^n(F_q)$ and let $\mathcal{Y}_{aff}$ be the affine chart $\mathcal{Y} \setminus \mathcal{H}$ of $\mathcal{Y}$. Then, $\mathcal{Y}_{aff}$ has degree $d$ and dimension $\delta$.

Proof. Since $\mathcal{Y}$ is irreducible, its dimension can be defined as the transcendence degree of its function field over $F_q$. Since the function field of every open subset of $\mathcal{Y}$ equals that of $\mathcal{Y}$, we deduce that $\mathcal{Y}$ and $\mathcal{Y}_{aff}$ have the same dimension.

The variety $\mathcal{Y}_{aff}$ is affine, its degree is that of its projective closure (see [11]), which is nothing but $\mathcal{Y}$. \[\square\]
4.6.1. If $\mathcal{X}$ has a hyperplane in its irredundant decomposition. Let $\mathcal{H} \in \mathbf{P}^n(F_q)$ be this hyperplane. Write the irredundant decomposition of $\mathcal{X}$ as a union of irreducible varieties,

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_r,$$

with $\mathcal{X}_1 = \mathcal{H}$,

and such that for all $i \neq j$, $\mathcal{X}_i \not\subseteq \mathcal{X}_j$. Notice that $D_{\mathcal{X}} = d_{\mathcal{X}} = n - 1$. Now, $\mathcal{X}$ is the disjoint union of $\mathcal{H}$ and $\mathcal{X}_{\text{aff}} \overset{\text{def}}{=} \mathcal{X} \setminus \mathcal{H}$. The variety $\mathcal{X}_{\text{aff}}$ is affine and, from Lemma 4.7, its dimension and degree sequences are $d_{\mathcal{X}_2}, \ldots, d_{\mathcal{X}_r}$ and $\delta_{\mathcal{X}_2}, \ldots, \delta_{\mathcal{X}_r}$.

Thanks to Theorem 2.1, we get

$$|\mathcal{X}(F_q)| = |\mathcal{H}(F_q)| + |\mathcal{X}_{\text{aff}}(F_q)| \leq \pi_{n-1} + \sum_{i=2}^{r} \delta_{\mathcal{X}_i} q^{d_{\mathcal{X}_i}}. \quad (9)$$

Next, notice that for all $i$, we have

$$q^{d_{\mathcal{X}_i}} \leq \pi_{d_{\mathcal{X}_i}} - \pi_{2d_{\mathcal{X}_i} - n}. \quad (10)$$

Indeed, since $d_{\mathcal{X}_i} < n$, we have $2d_{\mathcal{X}_i} - n < d_{\mathcal{X}_i}$. On the other hand, since $\delta_{\mathcal{H}} = 1$, we get

$$\pi_{n-1} = \pi_{d_{\mathcal{H}}} = \delta_{\mathcal{H}}(\pi_{d_{\mathcal{H}}} - \pi_{2d_{\mathcal{H}} - n}) + \pi_{2d_{\mathcal{H}} - n}. \quad (11)$$

Putting (9), (10) and (11) together, we get

$$|\mathcal{X}(F_q)| \leq \left( \delta_{\mathcal{H}}(\pi_{d_{\mathcal{H}}} - \pi_{2d_{\mathcal{H}} - n}) + \sum_{i=2}^{r} \delta_{\mathcal{X}_i}(\pi_{d_{\mathcal{X}_i}} - \pi_{2d_{\mathcal{X}_i} - n}) \right) + \pi_{2d_{\mathcal{H}} - n}$$

and, since $d_{\mathcal{H}} = D_{\mathcal{X}}$, this yields the result.

4.6.2. When $\mathcal{X}$ has a single linear subvariety which is not a hyperplane in its irredundant decomposition. Let $\mathcal{L}$ be this linear subvariety of $\mathbf{P}^n$ which is not a hyperplane. That is, its dimension satisfies $d_{\mathcal{L}} < n - 1$. Moreover, we assume that the other irreducible components of $\mathcal{X}$ are nonlinear. Recall that, by assumption, none of the other components is contained in a hyperplane. Here again, we write the irredundant decomposition of $\mathcal{X}$ as a union of irreducible components as

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_r,$$

with $\mathcal{X}_1 = \mathcal{L}$.

As for the proof under Assumption $\mathbf{I}$ we apply a combinatorial proof based on another incidence structure. The incidence graph we consider is obtained as follows. Choose $P \in \mathcal{L}(F_q)$ and set

$$\mathcal{V}_1 \overset{\text{def}}{=} (\mathcal{X} \setminus \mathcal{L})(F_q),$$

$$\mathcal{V}_2 \overset{\text{def}}{=} \{ \mathcal{H} \in \mathbf{P}^n(F_q) \mid P \in \mathcal{H} \quad \text{and} \quad \mathcal{L} \not\subseteq \mathcal{H} \},$$

$$\mathcal{E} \overset{\text{def}}{=} \{ (Q, \mathcal{H}) \in \mathcal{V}_1 \times \mathcal{V}_2 \mid Q \in \mathcal{H} \}.$$

Here, we have

$$|\mathcal{V}_2| = \pi_{n-1} - \pi_{n-d_{\mathcal{X}}-1}.$$
Indeed, it suffices to notice that the set of hyperplanes in $\mathbf{P}^n(F_q)$ containing $\mathcal{L}$ has $\pi_{n-d_{\mathcal{L}}-1}$ elements. Next, notice that in this incidence graph:

- the valency of a vertex of $\mathcal{V}_1$ is $\pi_{n-2} - \pi_{n-d_{\mathcal{L}}-2}$;
- by induction and since no irreducible component of $\mathcal{K}$ but $\mathcal{L}$ is contained in a hyperplane, the valency of a vertex $\mathcal{H}$ of $\mathcal{V}_2$ equals $|(\mathcal{K} \cap \mathcal{H})(F_q)| - |(\mathcal{L} \cap \mathcal{H})(F_q)|$ which, from Corollary 4.3, is bounded above by

$$
\sum_{i=1}^{r} (\delta_{g_i}(\pi_{d_{\mathcal{g}_i}} - \pi_{2d_{\mathcal{g}_i}} - n_1) + \pi_{2D_{\mathcal{g}_i}} - n_1 - \pi_{d_{\mathcal{g}_i}} - 1),
$$

where the “$-\pi_{d_{\mathcal{g}_i}} - 1$” term corresponds to the rational points of $\mathcal{L} \cap \mathcal{H}$, which are not counted.

Now, as in [11,5], by counting the number of edges of the graph in two different manners, we get

$$
(|\mathcal{K}(F_q)| - \pi_{d_{\mathcal{L}}}) (\pi_{n-2} - \pi_{n-d_{\mathcal{L}}-2})
\leq (\pi_{n-1} - \pi_{n-d_{\mathcal{L}}-1}) \left[ \left( \sum_{i=1}^{r} \delta_{g_i}(\pi_{d_{\mathcal{g}_i}} - \pi_{2d_{\mathcal{g}_i}} - n_1) + \pi_{2D_{\mathcal{g}_i}} - n_1 - \pi_{d_{\mathcal{g}_i}} - 1 \right) \right].
$$

Using [2], we get

$$
|\mathcal{K}(F_q)| \leq \pi_{d_{\mathcal{L}}} + q \left[ \left( \sum_{i=1}^{r} \delta_{g_i}(\pi_{d_{\mathcal{g}_i}} - \pi_{2d_{\mathcal{g}_i}} - n_1) + \pi_{2D_{\mathcal{g}_i}} - n_1 - \pi_{d_{\mathcal{g}_i}} - 1 \right) \right]
\leq \pi_{d_{\mathcal{L}}} + \left( \sum_{i=1}^{r} \delta_{g_i}(\pi_{d_{\mathcal{g}_i}} - \pi_{2d_{\mathcal{g}_i}} - n_1) + \pi_{2D_{\mathcal{g}_i}} - n_1 - \pi_{d_{\mathcal{g}_i}} \right)
\leq \left( \sum_{i=1}^{r} \delta_{g_i}(\pi_{d_{\mathcal{g}_i}} - \pi_{2d_{\mathcal{g}_i}} - n_1) + \pi_{2D_{\mathcal{g}_i}} - n_1 \right).
$$

4.6.3. When there are several linear components. Now assume that $\mathcal{K}$ contains more than one linear irreducible component. Its irredundant decomposition is

$$
\mathcal{K} = \mathcal{Y} \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_s,
$$

where $s > 1$ and $\mathcal{Y}$ has no linear irreducible component. Moreover, recall that we are still under Assumption [2] that is to say that none of the irreducible components $\mathcal{Y}_1, \ldots, \mathcal{Y}_s$ of $\mathcal{Y}$ is contained in a hyperplane of $\mathbf{P}^n$. Assume that the linear components $\mathcal{L}_1, \ldots, \mathcal{L}_s$ are sorted by decreasing dimensions $d_{\mathcal{L}_1} \geq d_{\mathcal{L}_2} \geq \cdots \geq d_{\mathcal{L}_s}$. From [14,6,2] we have

$$
|(\mathcal{Y} \cup \mathcal{L}_1)(F_q)| \leq \left( \sum_{i=1}^{r} \delta_{g_i}(\pi_{d_{\mathcal{g}_i}} - \pi_{2d_{\mathcal{g}_i}}) + (\pi_{d_{\mathcal{g}_i}} - \pi_{2d_{\mathcal{g}_i}} - n_1) + \pi_{2D_{\mathcal{g}_i}} - n_1 \right).
$$

Now, notice that

$$
|\mathcal{K}(F_q)| \leq |(\mathcal{Y} \cup \mathcal{L}_1)(F_q)| + \sum_{i=2}^{s} \left( |\mathcal{L}_i(F_q)| - |(\mathcal{L}_1 \cap \mathcal{L}_i)(F_q)| \right).
$$

Moreover, for all $i \in \{2, \ldots, s\}$,

$$
\dim \mathcal{L}_1 \cap \mathcal{L}_i \geq d_{\mathcal{L}_1} + d_{\mathcal{L}_i} - n \geq 2d_{\mathcal{L}_i} - n.
$$
Putting (12) and (13) together, we get
\[ |\mathcal{X}(F_q)| \leq \left( \sum_{i=1}^{r} \delta_{\mathcal{Y}_i}(\pi_{d_{\mathcal{Y}_i}} - \pi_{2d_{\mathcal{Y}_i}}) \right) + (\pi_{d_{\mathcal{Y}_1}} - \pi_{2d_{\mathcal{Y}_1}} - n) + \pi_{2D_{\mathcal{X}}} - n + \sum_{i=2}^{s} (\pi_{d_{\mathcal{X}_i}} - \pi_{2d_{\mathcal{X}_i}} - n), \]
and hence,
\[ |\mathcal{X}(F_q)| \leq \left( \sum_{i=1}^{r} \delta_{\mathcal{Y}_i}(\pi_{d_{\mathcal{Y}_i}} - \pi_{2d_{\mathcal{Y}_i}}) \right) + \left( \sum_{i=1}^{s} (\pi_{d_{\mathcal{X}_i}} - \pi_{2d_{\mathcal{X}_i}} - n) \right) + \pi_{2D_{\mathcal{X}}} - n, \]
which yields the expected upper bound.

4.7. Proof in the general case. Now, assume that
\[ \mathcal{X} = \mathcal{Y} \cup \mathcal{Z} = \mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_s \cup \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_t, \]
where the \(\mathcal{Y}_i\)'s are irreducible varieties which are either linear or are not contained in any hyperplane in \(\mathbf{P}^n(F_q)\) and the \(\mathcal{Z}_i\)'s are nonlinear and each one is contained in at least one hyperplane in \(\mathbf{P}^n(F_q)\). In particular, since the \(\mathcal{Z}_i\)'s are nonlinear, for all \(i \in \{1, \ldots, t\}\), we have \(\delta_{\mathcal{Z}_i} > 1\). From the previous results, and since we clearly have \(D_{\mathcal{Y}} \leq D_{\mathcal{X}}\), we already know that
\[ |\mathcal{Y}(F_q)| \leq \left( \sum_{i=1}^{r} \delta_{\mathcal{Y}_i}(\pi_{d_{\mathcal{Y}_i}} - \pi_{2d_{\mathcal{Y}_i}} - n) \right) + \pi_{2D_{\mathcal{X}}} - n. \]
Next, for all \(i \in \{1, \ldots, t\}\), since \(\mathcal{Z}_i\) is contained in a hyperplane, one can apply the induction hypothesis to this hyperplane and get
\[ |\mathcal{Z}_i(F_q)| \leq B_{n-1}(\mathcal{Z}_i) = \delta_{\mathcal{Z}_i}(\pi_{d_{\mathcal{Z}_i}} - \pi_{2d_{\mathcal{Z}_i}} - n+1) + \pi_{2d_{\mathcal{Z}_i}} - n+1. \]

**Lemma 4.8.** For all integers \(d > 0\) and \(\delta > 1\), we have
\[ \delta(\pi_d - \pi_{2d-n+1}) + \pi_{2d-n+1} \leq \delta(\pi_d - \pi_{2d-n}). \]

**Proof.** Consider the difference,
\[ \delta(\pi_d - \pi_{2d-n}) - \left( \delta(\pi_d - \pi_{2d-n+1}) + \pi_{2d-n+1} \right) = \delta(\pi_{2d-n+1} - \pi_d - \pi_{2d-n+1}) \]
\[ = \delta q^{2d-n+1} - \pi_{2d-n+1} \]
\[ = (\delta - 1)q^{2d-n+1} - \pi_{2d-n}. \]
Since \(\delta > 1\), to prove that this difference is nonnegative, it is sufficient to prove that \(q^{2d-n+1} \geq \pi_{2d-n}\). It is obviously true if \(2d-n < 0\). It also holds true if \(2d-n \geq 0\) since, using that \(q \geq 2\), we have
\[ \pi_{2d-n} = \frac{q^{2d-n+1} - 1}{q - 1} \leq q^{2d-n+1} - 1. \]

From Lemma 4.8 and since by assumption on the \(\mathcal{Z}_i\)'s, we have \(\delta_{\mathcal{Z}_i} > 1\) for all \(i \in \{1, \ldots, t\}\), we see that
\[ \forall i \in \{1, \ldots, t\}, \quad \delta_{\mathcal{Z}_i}(\pi_{d_{\mathcal{Z}_i}} - \pi_{2d_{\mathcal{Z}_i}} - n+1) + \pi_{2d_{\mathcal{Z}_i}} - n+1 \leq \delta_{\mathcal{Z}_i}(\pi_{d_{\mathcal{Z}_i}} - \pi_{2d_{\mathcal{Z}_i}} - n). \]
Next, use the obvious inequality
\[ |\mathcal{X}(F_q)| \leq |\mathcal{Y}(F_q)| + \sum_{j=1}^{\ell} |\mathcal{Z}_j(F_q)|, \]
and put it together with the bounds (14), (15) and (16) to get
\[ |\mathcal{X}(F_q)| \leq \left( \sum_{i=1}^{r} \delta_{\mathcal{Y}_i}(\pi_{d_{\mathcal{Y}_1}} - \pi_{2d_{\mathcal{Y}_1}} - n) \right) + \left( \sum_{i=1}^{r} \delta_{\mathcal{Z}_i}(\pi_{d_{\mathcal{Z}_1}} - \pi_{2d_{\mathcal{Z}_1}} - n) \right) + \pi_{2D_{\mathcal{X}}} - n. \]
This concludes the proof.

5. Is this new bound optimal?

5.1. The bound is optimal for equidimensional varieties. We will show that the bound given by Theorem 3.1 is reached by equidimensional arrangement of linear varieties. This shows that Corollary 3.3 is an optimal upper bound. For that we introduce two objects: partial $d$–spreads and $d$–flowers. The notion of partial $d$–spread is well known and is the subject of intense study in finite geometry. For instance, see (the list is far from being exhaustive) [3,14,15,17]. On the other hand, the terminology of $d$–flower is introduced by the author.

**Definition 5.1** (partial $d$–spreads). Let $d, n$ be two positive integers with $2d < n$. A partial $d$–spread is a disjoint union of linear subvarieties of dimension $d$ of $\mathbb{P}^n$.

**Definition 5.2** ($d$–flowers). Let $d, n$ be two positive integers with $d < n$ and $2d \geq n$. A $d$–flower is a union $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_r$ of linear subvarieties of dimension $d$ of $\mathbb{P}^n$ such that there exists a linear variety $\mathcal{M}$ of dimension $2d - n$ satisfying
\[ \forall i \neq j \in \{1, \ldots, r\}, \quad \mathcal{L}_i \cap \mathcal{L}_j = \mathcal{M}. \]

**Example 5.3.** An $(n-1)$–flower is nothing but a hypersurface obtained as a union of hyperplanes meeting at a common 2–codimensional linear variety. These flowers reach Serre’s bound for the number of points of hypersurfaces [19].

**Example 5.4.** A union of planes of $\mathbb{P}^4$ meeting at a single point is a 2–flower.

**Proposition 5.5.** Let $\mathcal{X}$ be a partial $d$–spread or a $d$–flower of degree $\delta$. Then,
\[ |\mathcal{X}(F_q)| = \delta(\pi_d - \pi_{2d} - n) + \pi_{2d} - n. \]

**Proof.** It is a straightforward consequence of the definition of partial $d$–spreads and $d$–flowers. \qed

5.2. The nonequidimensional case might have a sharper bound. For nonequidimensional varieties, the optimality of our bound is less clear. In particular, the following statement asserts that the upper bound of Theorem 3.1 cannot be reached by nonequidimensional arrangements of linear varieties.

**Proposition 5.6.** Let $\mathcal{X} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_r$ be a union of linear subvarieties of $\mathbb{P}^n$ defined over $F_q$ such that $d_{\mathcal{X}_1} \geq \cdots \geq d_{\mathcal{X}_r}$ and $r \geq 2$. Then,
\[ |\mathcal{X}(F_q)| \leq \pi_{d_{\mathcal{X}_1}} + \left( \sum_{i=2}^{r} (\pi_{d_{\mathcal{X}_1}} - \pi_{d_{\mathcal{X}_i}} + d_{\mathcal{X}_i} - n) \right). \]
Proof. Use the obvious upper bound

\[ |\mathcal{X}(\mathbb{F}_q)| \leq |\mathcal{L}_1(\mathbb{F}_q)| + \sum_{i=2}^{r} |\mathcal{L}_i(\mathbb{F}_q)| - \sum_{i=2}^{r} |(\mathcal{L}_1 \cap \mathcal{L}_i)(\mathbb{F}_q)|, \]

together with the fact that

\[ \dim \mathcal{L}_1 \cap \mathcal{L}_i \geq d_{\mathcal{L}_1} + d_{\mathcal{L}_i} - n. \]

□

Remark 5.7. The construction of arrangements of linear subvarieties reaching this upper bound can be done as follows:

1. choose an arbitrary \( \mathcal{L}_1 \subseteq \mathbb{P}^n \) linear of dimension \( d_{\mathcal{L}_1} \);
2. choose \( \mathcal{L}_2 \) of dimension \( d_{\mathcal{L}_2} \) such that \( \dim \mathcal{L}_1 \cap \mathcal{L}_2 = d_{\mathcal{L}_1} + d_{\mathcal{L}_2} - n \);
3. choose the \( \mathcal{L}_i \)'s so that \( \mathcal{L}_1 \cap \mathcal{L}_i \subseteq \mathcal{L}_1 \cap \mathcal{L}_2 \).

Remark 5.8. Compared to the upper bound of Theorem 3.1, the above upper bound is sharper. The difference between the bound of Theorem 3.1 and Proposition 5.6 is

\[
\left( \sum_{i=1}^{r} (\pi d_{\mathcal{X}_i} - \pi d_{\mathcal{X}_i} + d_{\mathcal{X}_1} - n) \right) + \pi d_{\mathcal{X}_1} - n - \left( \sum_{i=2}^{r} (\pi d_{\mathcal{X}_i} - \pi d_{\mathcal{X}_i} + d_{\mathcal{X}_1} - n) \right)
\]

\[
= \sum_{i=2}^{r} (\pi d_{\mathcal{X}_i} + d_{\mathcal{X}_1} - n - \pi d_{\mathcal{X}_i} - n),
\]

and the above difference is positive as soon as the variety is not equidimensional.

Since in the equidimensional case, the upper bound is reached by arrangement of linear varieties, the previous observations on arrangements of linear varieties suggest a possible sharper upper bound which we leave as an open question.

Question 1. For a projective variety \( \mathcal{X} \) decomposed in a union of irreducible components \( \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_r \) sorted by decreasing dimensions, i.e. \( d_{\mathcal{X}_1} \geq \cdots \geq d_{\mathcal{X}_r} \), do we have

\[
|\mathcal{X}(\mathbb{F}_q)| \leq \left( \sum_{i=1}^{r} \delta_{\mathcal{X}_i} (\pi d_{\mathcal{X}_i} - \pi d_{\mathcal{X}_i} + d_{\mathcal{X}_1} - n) \right) + \pi d_{\mathcal{X}_1} - n ?
\]

Notice that if the degrees \( \delta_{\mathcal{X}_i} \) are all equal to 1 we get the upper bound of Proposition 5.6.

5.3. An open question on complete intersections. First, notice that partial spreads and flowers of dimension \( < n - 1 \) are never complete intersections unless they are irreducible and hence of degree 1. Indeed, from [11, Ex II.8.4, III.5.5], complete intersections are always connected while partial spreads are not. For a flower \( \mathcal{F} \), of dimension \( < n - 1 \) and degree \( > 1 \), one can choose a linear variety \( \mathcal{L} \) which is transverse to \( \mathcal{F} \) and such that the intersection \( \mathcal{L} \cap \mathcal{F} \) is a partial spread, and hence is disconnected. If \( \mathcal{F} \) was a complete intersection, then \( \mathcal{F} \cap \mathcal{L} \) would be a complete intersection too, which is a contradiction.

Second, in [16, §5], the authors study some arrangements of linear varieties which are complete intersections and have a large number of points. They call
these varieties tubular sets. They prove in particular that the number of points of a tubular set $\mathcal{X}$ of degree $\delta$ and dimension $d$ equals

$$|\mathcal{X}(\mathbb{F}_q)| = \delta q^d + \pi_{d-1}.$$  

We observed in §5.1 that, for general equidimensional varieties, the upper bound on the number of rational points is reached by arrangements of linear varieties. That fact was already known for hypersurfaces [19]. If this property holds for complete intersections, one does not know arrangements of linear varieties which are complete intersection and have more points than tubular sets. For this reason one can hope for the existence of a sharper bound for the maximal number of points of complete intersection with respect to their dimension and degree. This problem remains completely open.

**Conclusion**

We obtained in Theorem 3.1 a new upper bound on the number of rational points of an arbitrary closed subset of a projective space. This bound holds even for nonequidimensional varieties. In the equidimensional case, thanks to this upper bound, we proved that Ghorpade and Lachaud’s conjecture [10] is true and is optimal for equidimensional varieties.

**Acknowledgements**

The author expresses his gratitude to the anonymous referee for the careful work and relevant suggestions.

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INRIA & LIX, UMR 7161, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE
E-mail address: alain.couvreur@lix.polytechnique.fr