NONLINEAR GREY BERNOULLI MODEL NGBM (1, 1)’S PARAMETER OPTIMISATION METHOD AND MODEL APPLICATION

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Abstract. In the grey prediction, the nonlinear Grey Bernoulli model NGBM (1, 1) is an important type. The NGBM (1, 1) has good adaptability to data fitting and then small prediction errors, and thus has been applied widely. However, if we improve the modelling method, the prediction precision shall be improved to some extent. The important factors of prediction error are the approximation of background value and the approximation of power exponent. Therefore, the paper tries to combine the optimisation of background value with the optimisation of the power exponent of NGBM (1, 1) model and then improves the model from parameter estimation. The paper gives three methods for the following three cases respectively: the background value in the form of exponential curve, the background value in the form of the polynomial curve and the background value in the form of interpolation function, to combine background value optimisation with power exponent optimisation for parameter optimisation. The final section of the paper builds the NGBM (1, 1) models of China’s GDP and energy consumption with three improvement methods. The simulation and prediction results show the three improvement methods all have high precision. The methods given offer good approaches for the in-depth study on nonlinear grey Bernoulli model, enrich the method system of grey modelling and can be applied to the studies on other grey models to promote the study and wide application of the grey model.

1. Introduction. In recent years, many new prediction methods emerge continuously [19, 20, 21], and the study on grey prediction method is very popular. The grey prediction has been applied widely in many fields [5, 6, 10, 25, 28]. Currently, the grey prediction models used widely include the GM (1, 1) model, the GM (1, N) model, the GM(N, 1) model and the GM(1, 1) power model [3, 4, 17, 22, 26]. In these grey prediction models, the GM (1, 1) power model is an important type, which is also called the nonlinear grey Bernoulli model NGBM(1, 1) and has been applied widely due to its strong adaptability to original data. The nonlinear grey
Bernoulli model was first proposed by Chen et al. [1]. Chen et al. [1] proposed a new grey model based on the concept of Bernoulli differential equation in an ordinary differential equation. In the study, researchers named the new model the nonlinear grey Bernoulli model (NGBM). NGBM is the nonlinear differential equation with \( m \) as the power exponent. Researchers can fit the once accumulated generating sequence of original data by adjusting the curvature of the curve. Then, many scholars have made various studies on NGBM (1, 1). Zeng [29], basing on current grey model GM\((1, 1|\sin)\), proposed a GM\((1, 1|\sin)\) Bernoulli model specific for the oscillation sequence prediction problem common in reality, and gave a parameter calculation formula following the least-squares criterion. He built a nonlinear optimisation model aiming for average simulation relative error minimization and got the optimal parameter using the PSO (Particle Swarm Optimisation). To remedy the defect of prediction precision decline of conventional NGBM (1, 1) power model, Ma and Wang [16] made a coefficient modification using the PSO. Through a comparative study of cases, they found the improved NGBM (1, 1) power model improved prediction precision significantly compared with the conventional GM (1, 1) model. To predict the port handling capacity with oscillation property, Huang et al. [9] proposed an NGBM (1, 1) model based on sine sum. They first built an NGBM (1, 1) model with optimised exponent through the original sequence to describe the overall trend, and then described the periodic oscillation laws contained in residual errors using the sine sum to build a sine sum correction model. Ding et al. [7] proposed a multivariable discrete NGBM (1, 1) model specific for the multivariable small-sample nonlinear system modelling problem and explored the solving method for parameters. They introduced a driving control function to construct the optimisation model of multivariable discrete NGBM (1, 1) model. Hu [8] built an improved NGBM (1, 1) model through a deterministic function translating background values into adjacent sequence points, gave relative properties of background value sequence and verified the model’s superiority in fitting and predictions using the calculating examples in existing reference documents. Li et al. [12] introduced an unequal-interval NGBM (1, 1) model specific for the incompleteness of gas concentration data collection and small-sample characteristic of transformer fault gas chromatographic analyses, and made a collaborative optimisation of background value and power exponent based on genetic algorithm, and then built grey prediction models for solubilities of various gases in the transformer. To improve the fitting precision of NGBM (1, 1), Wang [18] improved the background value of NGBM (1, 1) model and built a type of improved NGBM (1, 1) model, and then gave the parameter optimisation of improved NGBM (1, 1) model using the PSO. Li et al. [11] proposed an unequal-interval NGBM (1, 1) model based on the grey Verhulst model and the equal-interval NGBM (1, 1) model and solved the model. They considered the average relative error as a function of power exponent, determined the range of power exponent according to the shape of sequence and then got the power exponent using the PSO. According to NGBM (1, 1) model’s defect of mismatch between the grey differential equation and whitening equation, Li and Chen [13] built an unbiased NGBM (1, 1) model based on the reconstruction of the grey differential equation. The method offered a better consistency between the parameters in difference equation and the corresponding parameters in the differential equation. Wang et al. [24] gave an estimation method for the power exponent of NGBM (1, 1) model according to the basic theory of information overlaying of grey system. They explored the influences of different values of power
exponent on the model’s solution, supplemented the theorem of the solution of the whitening differential equation of the model and gave an optimisation method for whitening differential equation’s solution. To further enhance the grey prediction model’s adaptability to original data, Wang [23] proposed a time-varying parameter NGBM (1, 1) model and described the dynamic variation law of NGBM (1, 1) model’s structure parameter over time by introducing a polynomial function. Considering the size of modelling samples, he gave the model’s parameter identification equations for three cases and the analytical solution of whitening equation of time-varying-parameter NGBM (1, 1) model. Considering the inherent error in NGBM (1, 1) model’s time response mode from discrete estimation to continuous predictions, Yang [27] built a discrete NGBM (1, 1) model and extended the model to be a fractional-order discrete NGBM (1, 1) model. Ma et al. [15] built a more general nonlinear grey prediction model called the nonlinear grey Bernoulli multivariable model, which can be considered as the extension of NGB (1, 1) model. Chen et al. [2] also proposed an extended version of NGBM (1, 1) model. The model considered various factors such as power exponent, the smooth factor of background value, the selection of original conditions, the scale factor of residual correction and so on, and then solved the problems of hard artificial parameter selection and overfitting by making parameter optimisation and selection using the genetic algorithm. The model obtained was called the self-adaptive NGBM (1, 1) model. Lu et al. [14] proposed an optimised NGBM(1,1) (ONGBM) which considered the \( n^{th} \) component of the 1-AGO sequence as the original condition and optimised it using the boundary value correction method. Meanwhile, ONGBM calculated the unknown parameter using a synchronous optimisation method to realise the minimum of average simulation relative error. Zhou et al. [31] proposed a new NGBM parameter optimisation scheme based on PSO. Considering the Bernoulli differential equation’s generation coefficients of power exponent and background value as decision variables and prediction error as the optimisation objective, researchers adopted PSO for the solution. They applied the parameter optimisation algorithm to the long-term prediction of power load. The studies above focus on the model’s modification, extension and objective function optimisation, and improve the model’s precision to some extent. The key factors affecting NGBM (1, 1) model’s prediction error are the approximation of background value and the approximation of power exponent of the model. Therefore, improving the estimation method of background value and power exponent can improve the model’s precision. The paper tries to combine the background value optimisation with the power exponent optimisation of NGBM (1, 1) model to improve the model’s parameter estimation method, and gives the following three methods: (1) optimising the parameter of the model with the background value in the form of the exponential curve; (2) optimising the parameter of the model with the background value in the form of the polynomial curve; (3) optimising the parameter of the model with the background value in the form of the interpolation function. The paper builds the NGBM (1, 1) models of China’s GDP and energy consumption with the three improved methods respectively. The simulation and prediction results show that these methods all improve the prediction precision of the model significantly.

2. Nonlinear grey bernoulli model NGBM (1, 1) and its common parameter estimation method. Suppose the original time sequence is \( x^{(0)}(t) = \)
\{x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(N)\}$, and make an accumulated generating operation, i.e.

$$x^{(1)}(t) = \sum_{i=0}^{t} x^{(0)}(i), \quad t = 1, 2, \cdots, N.$$ 

Then, generate column $x^{(1)}(t) = \{x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(N)\}$.

Write the background value as $z(k) = \frac{1}{2}(x^{(1)}(k) + x^{(1)}(k-1)), \quad (k = 1, 2, \cdots, N)$.

Then, call $x^{(0)}(k) + az(k) = b(z(k))^m$ the NGBM (1, 1) model’s grey differential equation, in which $-a$ is the development coefficient, $b$ is the grey action and $m$ is the model’s power exponent.

The white differential (whitening) equation is

$$\frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b(x^{(1)}(t))^m.$$ 

In the original condition $\hat{x}^{(1)}(t)|_{t=1} = x^{(1)}(1) = x^{(0)}(1)$, the time response function is

$$\hat{x}^{(1)}(t) = \left\{x^{(1)}(1)^{(1-m)} - \frac{b}{a}e^{a(m-1)t-1} + \frac{b}{a}\right\}^{\frac{1}{1-m}},$$ 

where $a, b$ and $m$ are the parameters to be estimated.

Get the common power exponent $m$ from the following formula [30]:

$$m = \frac{\sum_{k=2}^{n-1} (x^{(0)}(k+1) - x^{(0)}(k))z(k+1)z(k)x^{(0)}(k) - (x^{(0)}(k) - x^{(0)}(k-1))z(k) + z(k)x^{(0)}(k+1)}{(n-2)((x^{(0)}(k+1))^2z(k)x^{(0)}(k) - (z(k))^2z(k+1)x^{(0)}(k+1))}.$$ 

Get parameters $a$ and $b$ from NGBM (1, 1) model’s grey differential equation using the least square method:

$$\begin{bmatrix} a \\ b \end{bmatrix} = (B' B)^{-1} B' Y,$$

where

$$B = \begin{bmatrix} -z(2) & (z(2))^m \\ -z(3) & (z(3))^m \\ \vdots & \vdots \\ -z(N) & (z(N))^m \end{bmatrix} , Y = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(N) \end{bmatrix}.$$ 

Substitute $a, b$ and $m$ into the following time response function sequence:

$$\hat{x}^{(1)}(k) = \left\{x^{(1)}(1)^{(1-m)} - \frac{b}{a}e^{a(m-1)(k-1)} + \frac{b}{a}\right\}^{\frac{1}{1-m}}.$$ 

Get the simulation value of the original sequence from $\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)$ $k = 2, 3, \cdots, N$, and get the prediction value at the $q^{th}$ step of the original sequence from $\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)(k = N + 1, \cdots, N + q)$. 
3. Nonlinear grey bernoulli model NGBM (1, 1) ’s parameter optimisation methods.

3.1. Parameter optimisation method 1 of NGBM (1, 1) model. For sequence \( x^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(N)) \), make the accumulation of \( x^{(1)}(t) = \sum_{i=1}^{t} x^{(0)}(i) | i = 1, 2, \cdots, N \), and suppose \( x^{(1)} = (x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(N)) \), the sequence after accumulation, satisfies the dynamic differential equation (whitening equation): 

\[
\frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b(x^{(1)}(t))^m.
\]

Make the integration operation in both sides of the equation in the interval \([k-1, k]\), and then get

\[
\int_{k-1}^{k} \frac{dx^{(1)}(t)}{dt} dt + a \int_{k-1}^{k} x^{(1)}(t) dt = b \int_{k-1}^{k} (x^{(1)}(t))^m dt.
\]

Have \( z^{(1)}(k) = \int_{k-1}^{k} x^{(1)}(t) dt \), \( z^{(2)}(k) = \int_{k-1}^{k} (x^{(1)}(t))^m dt \).

Apparently,

\[
z^{(1)}(k) = \int_{k-1}^{k} x^{(1)}(t) dt = x^{(1)}(\xi), \quad (k - 1 \leq \xi \leq k),
\]

\[
z^{(2)}(k) = \int_{k-1}^{k} (x^{(1)}(t))^m dt = (x^{(1)}(\eta))^m, \quad (k - 1 \leq \eta \leq k).
\]

This is the case that the background value function is an exponential curve. That is, the background value function is

\[
x^{(1)}(t) = x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{t - (k - 1)}, \quad (k - 1 \leq t \leq k).
\]

Apparently, it satisfies:

when \( t = k - 1 \), \( x^{(1)}(t) = x^{(1)}(k - 1) \); when \( t = k \), \( x^{(1)}(t) = x^{(1)}(k) \).

Because \( \xi \in (k - 1, k) \), \( \xi = \alpha(k - 1) + (1 - \alpha)k = k - \alpha \), \( 0 \leq \alpha \leq 1 \); because \( \eta \in (k - 1, k) \), \( \eta = \beta(k - 1) + (1 - \beta)k = k - \beta \), \( 0 \leq \beta \leq 1 \).

Then,

\[
z^{(1)}(k) = x^{(1)}(\xi) = x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{k - \alpha - (k - 1)} = x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{1 - \alpha},
\]

\[
z^{(2)}(k) = (x^{(1)}(\eta))^m = \left\{ x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{k - \beta - (k - 1)} \right\}^m = \left\{ x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{1 - \beta} \right\}^m.
\]

From \( x^{(0)}(k) + a z^{(1)}(k) = b z^{(2)}(k) \), get

\[
x^{(0)}(k) + a \left\{ x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{1 - \alpha} \right\} = b \left\{ x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{1 - \beta} \right\}^m.
\]

For the given \( \alpha, \beta \) and \( m \), get the estimate of the parameter \( (a, b) \) with the least square method:

\[
\begin{bmatrix}
a \\
b
\end{bmatrix} = (G'H)^{-1}G'H,
\]
where

\[ G = \begin{bmatrix}
-2^{(1)}(1)\left(\frac{z^{(1)}(2)}{z^{(1)}(1)}\right)^{1-\alpha} & \left\{ x^{(1)}(1)\left(\frac{z^{(1)}(2)}{z^{(1)}(1)}\right)^{1-\beta}\right\}^m \\
-2^{(1)}(2)\left(\frac{z^{(1)}(3)}{z^{(1)}(2)}\right)^{1-\alpha} & \left\{ x^{(1)}(2)\left(\frac{z^{(1)}(3)}{z^{(1)}(2)}\right)^{1-\beta}\right\}^m \\
\vdots & \vdots \\
-x^{(1)}(N-1)\left(\frac{z^{(1)}(N)}{z^{(1)}(N-1)}\right)^{1-\alpha} & \left\{ x^{(1)}(N)\left(\frac{z^{(1)}(N)}{z^{(1)}(N-1)}\right)^{1-\beta}\right\}^m
\end{bmatrix},
\]

\[ H = \begin{bmatrix}
x^{(0)}(2) \\
x^{(0)}(3) \\
\vdots \\
x^{(0)}(N)
\end{bmatrix}.
\]

In fact, the values of \(\alpha, \beta\) and \(m\) required can be obtained using an optimisation method as follows:

\[
\min_{\alpha, \beta, m} MAPE = \frac{1}{N-1} \sum_{i=2}^{N} \left| \frac{x^{(0)}(i) - \hat{x}^{(0)}(i)}{x^{(0)}(i)} \right| \times 100%,
\]

s.t. 
\[
\hat{x}^{(0)}(i) = \hat{x}^{(1)}(i) - \hat{x}^{(1)}(i-1),
\]

\[
\hat{x}^{(1)}(i) = \begin{cases} 
[x^{(1)}(1)(1-m) - \frac{b}{a}e^{a(m-1)(i-1)} + \frac{b}{a}]^{\frac{1}{m-1}}, \\
(a,b) = (G'H)^{-1}G'H, \\
\end{cases}
\]

where \(0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, m \geq 0, m \neq 1\).

Then, get the sequence of the time response function:

\[ \hat{x}^{(1)}(k) = \begin{cases} 
[x^{(1)}(1)(1-m) - \frac{b}{a}e^{a(m-1)(k-1)} + \frac{b}{a}]^{\frac{1}{m-1}}, \\
\end{cases} \]

From \(\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)\), get the simulation value and prediction value of the original sequence.

3.2. Parameter optimisation method 2 of NGBM (1, 1) model. For sequence \(x^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(N))\), make the accumulation of \(x^{(1)}(t) = \sum_{i=1}^{t} x^{(0)}(i)\) \((t = 1, 2, \cdots, N)\), and get \(x^{(1)} = (x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(N))\), the sequence after accumulation.

Suppose the grey differential equation is

\[ x^{(0)}(k) + az^{(1)}(k) = bz^{(2)}(k), \]

where

\[ z^{(1)}(k) = \int_{k-1}^{k} x^{(1)}(t)dt = x^{(1)}(\xi), \quad (k-1 \leq \xi \leq k), \]

\[ z^{(2)}(k) = \int_{k-1}^{k} (x^{(1)}(t))^m dt = (x^{(1)}(\eta))^m, \quad (k-1 \leq \eta \leq k). \]

This is the case that the background value function is a polynomial curve, and we choose a quadratic polynomial. That is, the background value function is

\[ x^{(1)}(t) = x^{(1)}(k-1) \]
\[ + \frac{x^{(1)}(k-1)}{2} \left[ (t-(k-1)) + (t-(k-1))^2 \right], \quad (k-1 \leq t \leq k). \]

Apparently, it satisfies:
when \( t = k - 1 \), \( x^{(1)}(t) = x^{(1)}(k - 1) \), and when \( t = k \), \( x^{(1)}(t) = x^{(1)}(k) \). Then,
\[
z^{(1)}(k) = x^{(1)}(\xi) = x^{(1)}(k - 1) + \frac{\varphi^{(1)}(k - x^{(1)}(k - 1))}{2} \left[ (k - \alpha - (k - 1)) + (k - \alpha - (k - 1))^2 \right]
= x^{(1)}(k - 1) + \frac{x^{(1)}(k - x^{(1)}(k - 1))}{2} \left[ (1 - \alpha) + (1 - \alpha)^2 \right],
\]
\[
z^{(2)}(k) = \left\{ x^{(1)}(\eta)^m \right\}^m
= \left\{ x^{(1)}(k - 1) + \frac{x^{(1)}(k - x^{(1)}(k - 1))}{2} \left[ (k - \beta - (k - 1)) + (k - \beta - (k - 1))^2 \right]\right\}^m
= \left\{ x^{(1)}(k - 1) + \frac{x^{(1)}(k - x^{(1)}(k - 1))}{2} \left[ (1 - \beta) + (1 - \beta)^2 \right]\right\}^m.
\]
From \( x^{(0)}(k) + a z^{(1)}(k) = b z^{(2)}(k) \), get
\[
\begin{align*}
x^{(0)}(k) + a \left\{ x^{(1)}(k - 1) + \frac{x^{(1)}(k - x^{(1)}(k - 1))}{2} \left[ (1 - \alpha) + (1 - \alpha)^2 \right]\right\}
&= b \left\{ x^{(1)}(k - 1) + \frac{x^{(1)}(k - x^{(1)}(k - 1))}{2} \left[ (1 - \beta) + (1 - \beta)^2 \right]\right\}^m.
\end{align*}
\]
For the given \( \alpha, \beta \) and \( m \) get the estimate of the parameter \((a, b)\) with the least square method:
\[
\begin{bmatrix} a \\ b \end{bmatrix} = (S'D)^{-1} S'D,
\]

where
\[
S = \begin{bmatrix}
-\frac{x^{(1)}(1) - \frac{1}{(\alpha + 1)\alpha}}{1} \\
-\frac{x^{(1)}(2) - \frac{1}{(\alpha + 1)\alpha}}{1} \\
\vdots \\
-\frac{x^{(1)}(N-1) - \frac{1}{(\alpha + 1)\alpha}}{1}
\end{bmatrix}
\]
\[
D = \begin{bmatrix}
x^{(0)}(2) \\
x^{(0)}(3) \\
\vdots \\
x^{(0)}(N)
\end{bmatrix}.
\]

In fact, the values of \( \alpha, \beta \) and \( m \) required can be obtained using an optimisation method as follows:
\[
\min_{\alpha, \beta, m} MAPE = \frac{1}{N-1} \sum_{t=2}^{N} \left| \frac{x^{(0)}(t) - \hat{x}^{(0)}(t)}{x^{(0)}(t)} \right| \times 100%,
\]
\[
\begin{align*}
\hat{x}^{(0)}(t) &= \hat{x}^{(1)}(t) - \hat{x}^{(1)}(t - 1), \\
\hat{x}^{(1)}(t) &= \left\{ x^{(1)}(1)^{(1-m)} - \frac{b}{a} e^a (m-1)(t-1) + \frac{b}{a} \right\}^{\frac{1}{m-1}},
\end{align*}
\]
\[
(a, b)^{'} = (S'D)^{-1} S'D,
\]
\[
0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, m \geq 0, m \neq 1.
\]

Get the time sequence response function:
\[
\hat{x}^{(1)}(k) = \left\{ x^{(1)}(1)^{(1-m)} - \frac{b}{a} e^a (m-1)(k-1) + \frac{b}{a} \right\}^{\frac{1}{m-1}}.
\]
Then, get the simulation value and prediction value of the original sequence from 
\( \hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1) \).

### 3.3. Parameter optimisation method 3 of NGBM (1, 1) model

For sequence \( x^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(N)) \), make the accumulation of \( x^{(1)}(t) = \sum_{i=1}^{t} x^{(0)}(i)(t = 1, 2, \ldots, N) \), and get \( x^{(1)} = (x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(N)) \), the sequence after accumulation.

Suppose the grey differential equation is
\[
x^{(0)}(k) + a z^{(1)}(k) = b z^{(2)}(k),
\]
where
\[
z^{(1)}(k) = \int_{k-1}^{k} x^{(1)}(t) dt = x^{(1)}(\xi), \quad (k - 1 \leq \xi \leq k),
\]
\[
z^{(2)}(k) = \int_{k-1}^{k} (x^{(1)}(t))^m dt = (x^{(1)}(\eta))^m, \quad (k - 1 \leq \eta \leq k).
\]

This is the case that the background value function is an interpolation function, and we choose the cubic spline interpolation function. That is, the background value function is
\[
x^{(1)}(t) = \begin{cases} x^{(1)}_k(t), & t \in [k - 1, k], k = 2, 3, \ldots, N - 1 \end{cases},
\]
where
\[
x^{(1)}_k(t) = a_k t - (k - 1)^3 + b_k t - (k - 1)^2 + c_k t - (k - 1) + d_k.
\]

Then,
\[
z^{(1)}(k) = x^{(1)}(\xi) = a_k [\xi - (k - 1)]^3 + b_k [\xi - (k - 1)]^2 + c_k [\xi - (k - 1)] + d_k
\]
\[
= a_k [k - \alpha - (k - 1)]^3 + b_k [k - \alpha - (k - 1)]^2 + c_k [k - \alpha - (k - 1)] + d_k
\]
\[
= a_k [1 - \alpha]^3 + b_k [1 - \alpha]^2 + c_k [1 - \alpha] + d_k,
\]
\[
z^{(2)}(k) = \left( x^{(1)}(\eta) \right)^m = a_k [\eta - (k - 1)]^3 + b_k [\eta - (k - 1)]^2 + c_k [\eta - (k - 1)] + d_k
\]
\[
= a_k [k - \beta - (k - 1)]^3 + b_k [k - \beta - (k - 1)]^2 + c_k [k - \beta - (k - 1)] + d_k
\]
\[
= a_k [1 - \beta]^3 + b_k [1 - \beta]^2 + c_k [1 - \beta] + d_k.
\]

From \( x^{(0)}(k) + a z^{(1)}(k) = b z^{(2)}(k) \), get
\[
x^{(0)}(k) + a \left\{ a_k [1 - \alpha]^3 + b_k [1 - \alpha]^2 + c_k [1 - \alpha] + d_k \right\}
\]
\[
= b \cdot \left\{ a_k [1 - \beta]^3 + b_k [1 - \beta]^2 + c_k [1 - \beta] + d_k \right\}.
\]

For the given \( \alpha, \beta \) and \( m \), get the estimate of the parameter \( (a, b) \) with the least square method:
\[
\begin{bmatrix} a \\ b \end{bmatrix} = (R^T)^{-1} R^T,
\]

where
\[
R = \begin{pmatrix}
-\{a_2 [1 - \alpha]^3 + b_2 [1 - \alpha]^2 + c_2 [1 - \alpha] + d_2 \} & \{a_2 [1 - \beta]^3 + b_2 [1 - \beta]^2 + c_2 [1 - \beta] + d_2 \}^m \\
-\{a_3 [1 - \alpha]^3 + b_3 [1 - \alpha]^2 + c_3 [1 - \alpha] + d_3 \} & \{a_3 [1 - \beta]^3 + b_3 [1 - \beta]^2 + c_3 [1 - \beta] + d_3 \}^m \\
\vdots & \vdots \\
-\{a_N [1 - \alpha]^3 + b_N [1 - \alpha]^2 + c_N [1 - \alpha] + d_N \} & \{a_N [1 - \beta]^3 + b_N [1 - \beta]^2 + c_N [1 - \beta] + d_N \}^m
\end{pmatrix}.
\]
In fact, the values of $\alpha, \beta$ and $m$ required can be obtained using an optimisation method as follows:

$$\min_{\alpha, \beta, m} MAPE = \frac{1}{N - 1} \sum_{i=2}^{N} \left| \frac{x(0)(t) - \hat{x}(0)(t)}{x(0)(t)} \right| \times 100\%,$$

s.t.\n
$$\hat{x}(0)(t) = \hat{x}(1)(t) - \hat{x}(1)(t - 1),$$

$$\hat{x}(1)(t) = \left\{ [x(1)(1)(1-m) - \frac{b}{a}]e^{a(m-1)(t-1)} + \frac{b}{a} \right\}^{\frac{1}{m-1}},$$

$$(a, b)' = (R'T)^{-1}R'T,$$

$$0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, m \geq 0, m \neq 1.$$

Then, get the time sequence response function

$$\hat{x}(1)(k) = \left\{ [x(1)(1)(1-m) - \frac{b}{a}]e^{a(m-1)(k-1)} + \frac{b}{a} \right\}^{\frac{1}{m-1}}.$$

Get the simulation value and prediction value of the original sequence from

$$\hat{x}(0)(k) = \hat{x}(1)(k) - \hat{x}(1)(k - 1).$$

4. Application example of grey modelling.

4.1. Grey modelling for China’s GDP. Since China began to implement the reform and opening-up policy, China’s economy has been growing rapidly. However, China’s pressure of economic downturn increased significantly since 2018. GDP’s growth rate declined from 6.8% in the first quarter in 2018 to 6% in the third quarter in 2019. Currently, China is in a critical period tackling key problems of development mode transformation, economic structure optimisation and economic stimulus measure adjustment. In the period dealing with these tasks simultaneously, structural, systematic and periodic problems have a cumulative influence on the economy. The economic growth of China is concerned by everyone, so predicting China’s GDP has great significance. The paper tries to build the NGBM (1, 1) model for China’s GDP using the methods given.

Build a conventional NGBM (1, 1) model for the original sequence and then get the parameter estimate:

$$a = -0.082842, b = 25705.202, m = 0.1681.$$ 

The time response equation is

$$\hat{x}(1)(k) = \left\{ [x(1)(1)(1-m) - \frac{b}{a}]e^{a(m-1)(k-1)} + \frac{b}{a} \right\}^{\frac{1}{m-1}} = (334619.88e^{0.0689(k-1)} - 310293.42)^{1.2021}.$$

Calculate and get the simulation values and prediction values of the original sequence in these periods from $\hat{x}(0)(t) = \hat{x}(1)(t) - \hat{x}(1)(t - 1)$. Calculate and get the relative errors of simulation values and prediction values of the original sequence in the periods with

$$RE(t) = \left| \frac{x(0)(t) - \hat{x}(0)(t)}{x(0)(t)} \right| \times 100\%.$$

Calculate and get average relative errors with

$$MAPE = \frac{1}{N-1} \sum_{t=2}^{N} \left| \frac{x(0)(t) - \hat{x}(0)(t)}{x(0)(t)} \right| \times 100\%.$$ 

See Table 1 for the results.
Table 1. Calculation Results of GreyModelling for China’s GDP

| Year | No. | \(x^{(0)}(t)\) | Conventional Method of NGBM (1, 1) Model | Improvement Method 1 |
|------|-----|-----------------|----------------------------------------|----------------------|
|      |     |                 | Simulation Value | Relative Error % | Simulation Value | Relative Error % |
| 2005 | 1   | 187318.         | -               | -                | -                | -                |
| 2006 | 2   | 219438.5        | 238832.77       | 8.84             | 217434.79       | 0.913            |
| 2007 | 3   | 270092.3        | 284734.09       | 5.42             | 270086.82       | 0.0023           |
| 2008 | 4   | 319244.6        | 328292.04       | 2.83             | 319272.6        | 0.00877          |
| 2009 | 5   | 348517.7        | 372350.46       | 6.84             | 367555.16       | 5.46             |
| 2010 | 6   | 412119.3        | 418204.34       | 1.48             | 416204.97       | 0.991            |
| 2011 | 7   | 487940.2        | 466656.87       | 4.36             | 465997.29       | 4.5              |
| 2012 | 8   | 538580.0        | 518315.16       | 3.76             | 517476.83       | 3.92             |
| 2013 | 9   | 592963.2        | 573702.33       | 3.25             | 571068.67       | 3.69             |
| 2014 | 10  | 641280.6        | 633309.02       | 1.24             | 627132.03       | 2.21             |
| 2015 | 11  | 685992.9        | 697620.56       | 5.18e-4          | 685989.35       | 5.18e-4          |
| 2016 | 12  | 740060.8        | 767133.24       | 3.66             | 747943.29       | 1.07             |

Build the NGBM (1, 1) model using method 1 given, and then get
\(\alpha = 0.5059, \beta = 0.3056, m = 0.2816\).

In this case, get
\[z^{(1)}(k) = x^{(1)}(\xi) = x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{0.4941},\]
\[z^{(2)}(k) = (x^{(1)}(\eta))^m = \left\{ x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{0.6944} \right\}^{0.2816} \]

And then, get the parameter \(a = -0.059876, b = 5811.734\), so the time response equation is
\[\hat{x}^{(1)}(k) = \left\{ [x^{(1)}(1-m) - \frac{b}{a}]e^{a(m-1)(k-1)} + \frac{b}{a} \right\}^{\frac{1}{m-1}} = (103198.62^{0.0430(k-1)} - 97063.508)^{1.3920}.\]

Calculate and get the simulation values and prediction values of the original sequence in these periods with \(\hat{x}^{(0)}(t) = \hat{x}^{(1)}(t) - \hat{x}^{(1)}(t-1)\) and the results are shown in Table 1. See Table 1 for the relative errors and average relative errors in the periods.

Build the NGBM (1, 1) model using method 2 given, and then get
\(\alpha = 0.3723, \beta = 0.2942, m = 0.2811\).

In this case,
\[z^{(1)}(k) = x^{(1)}(\xi) = x^{(1)}(k - 1) + \frac{\hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)}{2} [(1 - \alpha) + (1 - \alpha)^2]
= 0.4891x^{(1)}(k - 1) + 0.5109z^{(1)}(k),\]
Table 2. Calculation Results of Grey Modelling for China’s GDP
(Continued Table of Table 1)

| Year | No. | \(x^{(0)}(t)\) | Improvement Method 2 | Improvement Method 3 | |
|------|-----|----------------|----------------------|----------------------|------|
|      |     |                | Simulation Value | Relative Error % | Simulation Value | Relative Error % |
| 2005 | 1   | 187318         | -                  | -                    | -                | -                |
| 2006 | 2   | 219438.5       | 217479.83          | 0.893                | 217472.44        | 0.896            |
| 2007 | 3   | 270092.3       | 270092.3           | 8.83e-8              | 270087.67        | 0.00171          |
| 2008 | 4   | 319244.6       | 319244.64          | 1.31e-5              | 319242.61        | 6.22e-4          |
| 2009 | 5   | 348517.7       | 367502.09          | 5.45                 | 367502.19        | 5.45             |
| 2010 | 6   | 412119.3       | 416135.74          | 0.975                | 416137.41        | 0.975            |
| 2011 | 7   | 487940.2       | 465921.3           | 4.51                 | 465923.92        | 4.51             |
| 2012 | 8   | 538580.0       | 517404.02          | 3.93                 | 517401.89        | 3.93             |
| 2013 | 9   | 592963.2       | 571009.63          | 3.7                  | 571012.09        | 3.7              |
| 2014 | 10  | 641280.6       | 627098.13          | 2.21                 | 627099.39        | 2.21             |
| 2015 | 11  | 685992.9       | 685992.9           | 1.59e-7              | 685992.1         | 1.16e-4          |
| 2016 | 12  | 740060.8       | 747997.67          | 1.07                 | 747993.9         | 1.07             |

| Year | No. | \(x^{(0)}(t)\) | Improvement Method 2 | Improvement Method 3 | |
|------|-----|----------------|----------------------|----------------------|------|
|      |     |                | Prediction Value | Relative Error % | Prediction Value | Relative Error % |
| 2017 | 13  | 820754.3       | 813497.26          | 0.895                | 813399.51        | 0.896            |
| 2018 | 14  | 900309.5       | 882514.71          | 1.98                 | 882501.89        | 1.98             |

Average Simulation Relative Error (2005-2016): -2.07 -2.07
Average Prediction Relative Error (2017-2018): -1.44 -1.44
Average Relative Error (2005-2018): -1.97 -1.97

\[ z^{(2)}(k) = \left\{ x^{(1)}(\eta) \right\}^m \]
\[ = \left\{ x^{(1)}(k-1) + \frac{\left(x^{(1)}(k)-x^{(1)}(k-1) \right)}{2} \right\}^m \]
\[ = \left\{ 0.3980x^{(1)}(k-1) + 0.6020x^{(1)}(k) \right\}^{0.2811}. \]

Then, get the estimate of the parameter
\[ a = -0.060035, b = 5851.7145, \]
and then the time response equation is
\[ \dot{x}^{(1)}(k) = \left\{ x^{(1)}(1)^{1-m} - \frac{m}{a} \right\} e^{a(m-1)(k-1)} + \frac{b}{a} \]
\[ = \left\{ 103647.95 e^{0.0432(k-1)} - 97471.854 \right\}^{1.3909}. \]

Calculate and get the simulation values and prediction values of the original sequence in these periods with \( \dot{x}^{(0)}(t) = \dot{x}^{(1)}(t) - \dot{x}^{(1)}(t-1) \) and the results are shown in Table 1. See Table 1 for the relative errors and average relative errors in the periods.

Build the NGBM (1, 1) model using method 3 given, and then get
\[ \alpha = 0.4913, \beta = 0.3637, m = 0.2811. \]

Then,
\[ z^{(1)}(k) = \left\{ x^{(1)}(\xi) \right\} \]
\[ = a_k[1-\alpha]^3 + b_k[1-\alpha]^2 + c_k[1-\alpha] + d \]
\[ = 0.1316a_k + 0.2588b_k + 0.5087c_k + d, \]
\[ z^{(2)}(k) = \left\{ x^{(1)}(\eta) \right\}^{0.2811} \]
\[ = \left\{ a_k[1-\beta]^3 + b_k[1-\beta]^2 + c_k[1-\beta] + d \right\}^{0.2811}. \]

See Table 3 for the coefficients of the cubic spline interpolation function.
Table 3. Coefficients of the Cubic Spline Interpolation Function

| k | a_k  | b_k  | c_k  | d_k  |
|---|------|------|------|------|
| 2 | -946.81174 | 26774.212 | 193611.1 | 187318.9 |
| 3 | 1839.4352   | 23933.777 | 244319.09 | 406757.4 |
| 4 | -7912.4291  | 29452.082 | 297704.95 | 676849.7 |
| 5 | 9931.0812   | 5714.7949 | 332871.82 | 996094.3 |
| 6 | 2516.6044   | 35508.038 | 374094.66 | 1344612.0 |
| 7 | -7778.1987  | 43057.852 | 452660.55 | 1756731.3 |
| 8 | 3415.0903   | 19723.256 | 515441.65 | 2244671.5 |
| 9 | -2138.7625  | 29968.526 | 565133.44 | 2783251.5 |
| 10| 2237.0231   | 20774.718 | 662981.16 | 4017495.3 |
| 11| 1333.3479   | 27485.788 | 711241.66 | 4703488.2 |

Then, get the estimate of the parameter 

\[ a = -0.060023, b = 5848.0671, \]

and then, the time response equation is

\[ \hat{x}^{(1)}(k) = [x^{(1)}(1) - b] e^{-a(k-1)} + \frac{b}{a}. \]

Calculate and get the simulation values and prediction values of the original sequence in these periods with \( \hat{x}^{(0)}(t) = \hat{x}^{(1)}(t) - \hat{x}^{(1)}(t-1) \) and the results are shown in Table 1. See Table 2 for the relative errors and average relative errors in the periods.

From Table 1 and Table 2 we can see the NGBM (1, 1) models built using the three methods given all have small simulation and prediction errors and significantly improve simulation precision and prediction precision compared with the conventional model, indicating the methods given have high reliability and effectiveness.

4.2. Grey modelling for China’s energy consumption. China has become the biggest energy consumer and importer in the world in the recent ten years because of rapid economic development, especially the demand for industrial development. The rapid growth of economy shall cause the gradual growth of total energy consumption in China, so the future trend of energy consumption has been an issue of public concern. In this case, building the prediction model for China’s energy consumption has important significance. The paper builds NGBM (1, 1) models for China’s energy consumption using the methods given. China’s energy consumption is represented by \( x^{(0)}(t) \) in the unit of 10,000 tons of standard coal. The data came from the China Statistical Yearbook. See Table 4 for the data.

Build a grey GM (1, 1) model for the original sequence and then get the parameter estimate

\[ a = -0.041398, b = 284492.77. \]

The time response equation is

\[ \hat{x}^{(1)}(k) = [x^{(1)}(1) - b] e^{-a(k-1)} + \frac{b}{a}. \]

Calculate and get the simulation values and prediction values of the original sequence in these periods from \( \hat{x}^{(0)}(t) = \hat{x}^{(1)}(t) - \hat{x}^{(1)}(t-1) \). See Table 4 for the results. See Table 4 for the relative errors and average relative errors in the periods.

Build an NGBM (1, 1) model with method 3 and then get

\[ (a, b, m) = (-0.0194, 61730.10, 0.1170). \]
Table 4. Calculation Results of Grey Modelling for China's Energy Consumption

| Year | No. | \(x^{(0)}(t)\) | GM (1, 1) Model Simulation Value | GM (1, 1) Model Relative Error % | Improvement Method 3 Simulation Value | Improvement Method 3 Relative Error % |
|------|-----|----------------|---------------------------------|--------------------------------|-------------------------------------|---------------------------------|
| 2005 | 1   | 261369.0       | 261369.0                        | -                               | -                                   | -                               |
| 2006 | 2   | 286467.0       | 301511.04                      | 5.25                            | 286467.0                           | 2.49e-10                        |
| 2007 | 3   | 311442.0       | 314255.12                      | 0.903                           | 311442.0                           | 1.74e-8                         |
| 2008 | 4   | 320611.0       | 327537.85                      | 2.16                            | 331239.51                          | 3.32                            |
| 2009 | 5   | 336126.0       | 341382.0                       | 1.56                            | 348511.62                          | 3.69                            |
| 2010 | 6   | 360648.0       | 355811.32                      | 1.34                            | 364321.66                          | 1.02                            |
| 2011 | 7   | 387043.0       | 370850.52                      | 4.18                            | 379210.19                          | 2.02                            |
| 2012 | 8   | 402138.0       | 386525.39                      | 3.88                            | 393491.8                           | 2.15                            |
| 2013 | 9   | 416913.0       | 402862.8                       | 3.37                            | 407366.8                           | 2.29                            |
| 2014 | 10  | 425806.0       | 419890.74                      | 1.39                            | 420971.29                          | 1.14                            |
| 2015 | 11  | 429905.0       | 437638.41                      | 4.18                            | 434402.48                          | 2.02                            |
| 2016 | 12  | 435819.0       | 456136.2                       | 4.66                            | 447732.61                          | 2.73                            |
| 2017 | 13  | 448529.0       | 475415.9                       | 5.99                            | 461017.19                          | 2.78                            |
| 2018 | 14  | 464000.0       | 495510.47                      | 6.79                            | 474300.09                          | 2.22                            |
| 2019 | 15  | 479312.0       | 516454.39                      | 7.75                            | 487616.83                          | 1.73                            |

Average Simulation Relative Error (2005-2016) - 2.77 - 1.76
Average Prediction Relative Error (2017-2019) - 6.84 - 2.35
Average Relative Error (2005-2019) - 3.64 - 1.88

The time response equation is

\[ \hat{x}^{(1)}(k) = \left\{ \frac{b}{a} + \left[ x^{(1)}(1)^{(1-m)} - \frac{b}{a} \right] e^{a(m-1)(k-1)} + \frac{b}{a} \right\} \frac{1}{m-1} \]

Calculate and get the simulation values and prediction values of the original sequence in these periods from \( \hat{x}^{(0)}(t) = \hat{x}^{(1)}(t) - \hat{x}^{(1)}(t-1) \). See Table 4 for the results. See Table 4 for the relative errors and average relative errors in the periods.

To compare the methods proposed with the NGBM (1, 1) model improvement methods offered by other scholars in terms of modelling precision, the paper makes a calculation. We build a model using the improved method of NGBM (1, 1) model proposed by Zhang and Chen [30] and get the following estimate of the parameter:

\[(a, b, m) = (0.00728338, 13559.64, 0.23259)\]

Then, the time response equation is

\[ \hat{x}^{(1)}(k) = \left\{ \frac{b}{a} + \left[ x^{(1)}(1)^{(1-m)} - \frac{b}{a} \right] e^{a(m-1)(k-1)} + \frac{b}{a} \right\} \frac{1}{m-1} \]

Calculate and get the simulation value and prediction value of the original sequence with \( \hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1) \), and the results are shown in Table 5. See Table 5 for the relative errors and average relative error in the periods.

We build a model using the improvement method of NGBM (1, 1) model given by Ma and Wang [16] and get the following estimate of the parameter:

\[(a, b, m) = (0.012376, 10669.89, 0.25104)\]

Then, the time response equation is

\[ \hat{x}^{(1)}(k) = \left\{ \frac{b}{a} + \left[ x^{(1)}(1)^{(1-m)} - \frac{b}{a} \right] e^{a(m-1)(k-1)} + \frac{b}{a} \right\} \frac{1}{m-1} \]

Calculate and get the simulation value and prediction value of the original sequence with \( \hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1) \), and the results are shown in Table 5. See Table 5 for the relative errors and average relative error in the periods.
Table 5. Calculation Results of Grey Modelling for China’s Energy Consumption

| Year | No. | \(x^{(0)}(t)\) | Improvement Method Proposed by Zhang and Chen \[30\] | Improvement Method Proposed by Ma and Wang \[16\] |
|------|-----|----------------|---------------------------------|---------------------------------|
|      |     |                | Simulation Value | Relative Error % | Simulation Value | Relative Error % |
| 2005 | 1   | 261369.0       | -                | -                 | -                |
| 2006 | 2   | 286467.0       | 267260.08        | 6.7               | 264410.21        | 7.7               |
| 2007 | 3   | 311442.0       | 302546.19        | 2.86              | 301168.24        | 3.3               |
| 2008 | 4   | 320611.0       | 329138.11        | 2.66              | 328840.42        | 2.57              |
| 2009 | 5   | 336126.0       | 350612.31        | 4.31              | 351024.11        | 4.43              |
| 2010 | 6   | 360648.0       | 368636.25        | 2.21              | 369441.76        | 2.44              |
| 2011 | 7   | 387043.0       | 384142.61        | 0.749             | 383073.85        | 0.509             |
| 2012 | 8   | 402138.0       | 397713.35        | 1.1               | 395454.77        | 0.894             |
| 2013 | 9   | 416913.0       | 409739.41        | 1.72              | 407274.19        | 1.59              |
| 2014 | 10  | 425806.0       | 420497.92        | 1.25              | 420565.97        | 1.23              |
| 2015 | 11  | 429905.0       | 430193.48        | 0.0671            | 429645.9        | 0.0663             |
| 2016 | 12  | 435819.0       | 438982.1         | 0.726             | 437687.7        | 0.429             |

| Year | No. | \(x^{(0)}(t)\) | Improvement Method Proposed by Zhang and Chen \[30\] | Improvement Method Proposed by Ma and Wang \[16\] |
|------|-----|----------------|---------------------------------|---------------------------------|
|      |     |                | Simulation Value | Relative Error % | Simulation Value | Relative Error % |
| 2017 | 13  | 448529.0       | 446985.87        | 0.344             | 444828.45        | 0.825             |
| 2018 | 14  | 464000.0       | 454302.42        | 2.09              | 451178.59        | 2.76              |
| 2019 | 15  | 475512.0       | 461011.22        | 3.82              | 456828.7        | 4.69              |

Average Simulation Relative Error (2005-2016) - 2.21 - - 2.29
Average Prediction Relative Error (2017-2019) - 2.08 - - 2.76
Average Relative Error (2005-2019) - 2.18 - - 2.39

Calculate and get the simulation value and prediction value of the original sequence with \(\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k - 1)\), and the results are shown in Table 5. See Table 5 for the relative errors and average relative error in the periods.

From the calculation results above we can see the NGBM (1, 1) models built using the methods given have the simulation precision and prediction precision significantly higher than that of conventional grey GM (1, 1) model and conventional grey GM (1, 1) power model, and also have the modelling precision higher than the improved NGBM (1, 1) models proposed by Zhang & Chen \[30\] and Ma & Wang \[16\]. It indicates that the methods proposed have high reliability and effectiveness.

5. Conclusion. The paper combines NGBM (1, 1) model’s background value optimisation with power exponent optimisation, and improves the model from parameter estimation. The paper gives the following three improvement methods: (1) optimising the parameter of the model with the background value in the form of the exponential curve; (2) optimising the parameter of the model with the background value in the form of the polynomial curve; (3) optimising the parameter of the model with the background value in the form of the interpolation function. The paper builds the NGBM (1, 1) models of China’s GDP with the three improved methods respectively. The optimised background values and power exponents obtained with the three methods are as follows:

Method 1:

\[
z^{(1)}(k) = x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{0.4941},
\]

\[
z^{(2)}(k) = \left\{ x^{(1)}(k - 1) \left( \frac{x^{(1)}(k)}{x^{(1)}(k - 1)} \right)^{0.6944} \right\}^{0.2816},
\]
and power exponential $m = 0.2816$;

Method 2:

$$z^{(1)}(k) = 0.4891x^{(1)}(k - 1) + 0.5109x^{(1)}(k),$$

$$z^{(2)}(k) = \left\{0.3980x^{(1)}(k - 1) + 0.6020x^{(1)}(k)\right\}^{0.2811},$$

and power exponential $m = 0.2811$;

Method 3:

$$z^{(1)}(k) = 0.1316a_k + 0.2588b_k + 0.5087c_k + d,$$

$$z^{(2)}(k) = \left\{0.2576a_k + 0.4049b_k + 0.6363c_k + d\right\}^{0.2811},$$

and power exponential $m = 0.2811$.

From Table 1 to Table 5, we can see the three improvement methods given have similar precision. The average simulation relative errors are all less than 2.10%. The average prediction relative errors are all less than 2.50%. The average overall relative errors are all less than 2.00%. All errors are small.

The calculation results of models show that the models built with the methods given have high simulation and prediction precision and thus are superior to the conventional grey GM (1, 1) model, and the conventional NGBM (1, 1) model and the methods given are superior to the methods proposed by Zhang & Chen [30] and Ma & Wang [16]. Different from my previous research purely focusing on background value optimisation, the paper combines background value optimisation with power exponent optimisation and thus has better modelling results. Besides, the background value function given is pretty new and can be adapted to the changes of various data sequences and thus can be promoted to the modelling of other grey models.

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