On the dynamics of unified k-essence cosmologies

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Abstract. We analyze the phase space of a particular unified model of dark matter, dark energy, and inflation that we recently studied in [1] whose Lagrangian is of the form \( L(X, \phi) = F(X) - V(\phi) \). We show that this model possesses a large set of initial conditions consistent with a successful cosmological model in which an inflationary phase is possible, followed by a matter era to end with dark energy domination. In order to understand the success of the model, we study the general features that unified dark matter (UDM) models should comply and then we analyze some particular models and find their constrictions.

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INTRODUCTION

The current standard model of cosmology is based, on the one hand, on the existence of dark matter as a clustering agent to yield both local galactic dynamics and the large scale structure of the Universe and, on the other hand, on dark energy as a substance responsible for the cosmic accelerated expansion, and they both comprise around 96% of the matter-energy content at present. The rest 4% is mainly in the form of baryons that are important to understand what we actually observe. Astonishingly, this cosmological model have received very much support from different cosmological probes, such as type Ia supernovae, CMB anisotropies, measurements of Baryon Acoustic Oscillations, galactic and cluster dynamics, among others, for a short review see ref. [2].

In spite of the above-mentioned success, the standard model of cosmology relies on the existence of the dark components that make possible the desired cosmological dynamics. So far, we do not know, by certain, the origin of dark matter and dark energy, although well-motivated candidates exist for their origin. Given this, one is tempted to look for alternatives to dark matter and/or dark energy in such a way that the known, correct dynamics of the standard model is recovered and, if possible, models having smoking guns to be able to discriminate among them with the observations at hand. One of the interesting possibilities that has appeared in recent years is that different phenomena such as inflation, dark matter, and dark energy could be due to a single scalar field [3, 4, 5, 6, 7, 8, 9]. A practical scheme is to look for toy models that can accomplish the desired dynamics, and in that sense it is important to understand the key elements of the Lagrangian that play a particular cosmic role. Different works [3, 10] have analyzed the features that a unified model should have, and have pointed out the
difficulties to build such a single description of the different phenomena. Recently, a unified k-essence model for a particular Lagrangian has been proposed [11] and later generalized to a whole class of models in Ref. [1]. It was shown there that these models work finely to achieve a cosmological dynamics that emulates that of the standard model of cosmology including inflation. However, an initial conditions analysis is missing and some key features of the success have not yet been investigated. The aim of the present work is therefore to analyze the phase space dynamics of that model to identify the key elements to later generalize our results to other unified models, such as unified dark matter (UDM) models [12, 13].

DYNAMICAL ANALYSIS OF THE UNIFICATION MODEL

\[ \mathcal{L} = F(X) - V(\phi) \]

We consider here a scalar field \( \phi \) with Lagrangian \( \mathcal{L}(X, \phi) = F(X) - V(\phi) \), where the kinetic term \( F \) in this generalized class of models is a function of the canonical kinetic term \( X = -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2 \). The cosmological equations of motion for this field in a flat Friedmann-Robertson-Walker universe are

\[ H^2 = \frac{1}{3M_{Pl}^2} (2X F_X - F + V) \]

and

\[ \frac{d}{dt} (2X F_X - F + V) + 6HF_X = 0, \]

where \( M_{Pl}^2 \equiv 1/8\pi G \) and equation (2) is the continuity equation.

In Refs. [1] and [11] one particular choice is made for these Lagrangians corresponding to

\[ F(X) = \frac{1}{(2\alpha - 1)} \left[ (AX)^\alpha - 2\alpha \alpha_0 \sqrt{AX} \right] + M, \]

\[ V(\phi) = \frac{1}{2} m^2 \phi^2, \]

where \( \alpha = 1 \) corresponds to the model studied in [11]. In those works it is shown that this scalar field has the interesting properties to emulate the dark matter and dark energy, and in the very early Universe to drive inflation. To obtain this behaviour several elections of the constant parameters have to be made, for example for \( \alpha = 1 \) then

\[ 10^{-48} M_{Pl}^2 < \alpha_0 < 10^{-40} M_{Pl}^2, \]

\[ m \sim 10^{-6} M_{Pl}, \]

\[ \alpha_0^2 - M \sim 10^{-120} M_{Pl}^4, \]

\[ A \sim 1. \]

A missing part of the study of these models is the analysis of the phase space of its solutions. This can tell us whether the solutions that have these cosmologically
interesting features are feasible to obtain or not. In other words, it tells us whether the initial conditions of the solutions are generic or have to be fine tuned. In Ref. [14] a study on the general features of the phase space for the models with Lagrangian \( \mathcal{L} = F(X) - V(\phi) \) is presented. Here however, we will carry out a similar analysis adapted to the particular choice (3,4).

The continuity equation, assuming \( \alpha = 1 \) becomes

\[
A\ddot{\phi} + 3HA\dot{\phi} + m^2\phi = 3\sqrt{2A}\alpha_0\text{sign}(\dot{\phi})H,
\]

where \( \text{sign}(\dot{\phi}) = -1 \) if \( \dot{\phi} \) is negative and +1 otherwise. Performing a change of variable to

\[
z \equiv \sqrt{A}\dot{\phi},
\]

and using the Friedmann equation (1) to substitute the value of \( H \)

\[
H = \sqrt{\frac{-2M + z^2 + m^2\phi^2}{6M^2_{\text{Pl}}}},
\]

we obtain an evolution equation for \( z \) in terms of the variables \( \phi \) and \( z \). This equation and the definition of \( z \) together can be treated as a system of first order autonomous equations

\[
\dot{z} = -\frac{m^2\phi}{\sqrt{A}} + \frac{\sqrt{3}}{2M_{\text{Pl}}} \left( -\sqrt{2z} + 2\alpha_0\text{sign}(z) \right) \sqrt{z^2 + m^2\phi^2 - 2M},
\]

\[
\dot{\phi} = \frac{z}{\sqrt{A}}.
\]

With the equation of state of the field \( p_\phi/\rho_\phi \) written in terms of these variables as

\[
\omega_\phi = \frac{2M + z^2 - \sqrt{8}\alpha_0|z| - m^2\phi^2}{-2M + z^2 + m^2\phi^2}.
\]

It can be seen that the system doesn’t have any critical points which means that both variables \( z \) and \( \phi \) are always time dependent. The usual dynamical system analysis based on the fixed points is thus not possible in this case. However the system can be solved numerically to obtain its phase space, shown in Fig. 1. There we have plotted in dotted (red) lines those of constant equation of state, the horizontal lines corresponding to \( \omega_\phi = -1 \) and diagonal lines to \( \omega_\phi = 0 \). As can be seen the sector of initial conditions with big negative \( \phi \) values and positive \( z \) values evolves towards a solution with equation of state near \(-1\) which in the phase space corresponds to the left horizontal branch. This in the unification models is interpreted as the initial period of inflation in which the equation of state of the solution gets close to \(-1\). To see this one can show that equation (9) drives \( z \) to small values when the constant parameters are in the intervals giving in Eq. (5). For the expected value of the field at the beginning of inflation \( \phi_i \sim 15M_{\text{Pl}} \) as obtained in Ref. [1] under slow-roll conditions, the value of \( z \) will be around \( 10^{-7}M_{\text{Pl}}^2 \) corresponding to \( \omega \sim -0.994 \). If the system starts in bigger \( z \), equation (9) will acquire negative values driving the system to this small value corresponding to a potential dominated phase. For more details on the inflationary realization of the model see [1].
This solution later crosses the lines corresponding to a equation of state equal to 0 (diagonal lines) that in the unification models correspond to the matter domination epoch. The time that the system stays in the regime of $\omega_\phi \sim 0$ has to be long in order to represent the Dark Matter. This time will depend on the value of the parameters (5) in the Lagrangian, and in references [1] and [11] it is shown that these parameters can be adjusted in order to obtain this behaviour from a redshift of order $10^{10}$ up until a recent time when the transition to $\omega_\phi < 0$ has to occur; e.g. Eqs. (5) provide the parameters for model $\alpha = 1$. Finally, the solution evolves towards a second period of $\omega_\phi$ close to −1, that in the phase space corresponds to the right horizontal branch. The whole behaviour occurs also for solutions beginning with big positive values of $\phi$ and negative values of $z$, in which solutions go from positive to negative values in $\phi$ and live in the $z < 0$ part of the phase space, as can be seen in Fig. 1.

The analysis of this phase space is important in understanding the dynamics of the cosmological solutions of the system (3, 4). We can conclude that an important sector of the possible initial conditions can give rise to the behavior needed to unify the phenomena of dark matter, dark energy, and inflation. If fact, half of the possible initial conditions give rise to the behaviour needed for unification. As was stated in the previous paragraph, all the solutions that at early times begin with big negative values of $\phi$ and positive values of $z$, or those that begin with big positive values of $\phi$ and negative values of $z$, achieve a successful unified behaviour. A problem that can be seen in Fig. 1 is the crossing of the $\omega_\phi = −1$ line, that has been argued in [15] it presents stability problems for k-essence scalar fields, however this will occur in a future epoch for the case of our Universe, as the current equation of state for the field is expected to be close to
$\omega_\phi = -0.75$ [11].

**Purely kinetic Lagrangian**

The kinetic term in the previous Lagrangian (3) was proposed originally in [16] as a purely kinetic Lagrangian, corresponding in our case to the scenario in which the potential term (4) is small compared to the kinetic term (3). In this case the system poses a shift symmetry $\phi \rightarrow \phi + \phi_0$ which implies that it has only one degree of freedom. This simplification makes it possible to obtain an analytical solution to the system [17]. The energy density and pressure take the form

$$\rho = (AX)^\alpha - M, \quad (12)$$

$$p = \frac{1}{(2\alpha - 1)} \left[ (AX)^\alpha - 2\alpha\alpha_0\sqrt{AX} \right] + M. \quad (13)$$

Thus, the continuity equation can be written as

$$\frac{d(AX)}{dN} = \frac{6}{2\alpha - 1} (AX)(\alpha_0(AX)^{1/2 - \alpha} - 1). \quad (14)$$

And defining $y \equiv (AX)^{\alpha - 1/2}$, this equation can be transformed into:

$$\frac{dy}{dN} = 3(\alpha_0 - y), \quad (15)$$

This differential equation has the analytical solution $y = \alpha_0 + ce^{-3N}$, where $c$ is a constant of integration. After a few e-folds of expansion the constant term dominates. The equation of state can be computed as

$$\omega_\phi = \frac{(AX)^\alpha - 2\alpha\alpha_0\sqrt{AX} + (2\alpha - 1)M}{(2\alpha - 1)[(AX)^\alpha - M]}. \quad (16)$$

From the Eq. (14) we can obtain the critical values of the system as $AX_1 = 0$ and $AX_2 = \alpha_0^{2/(2\alpha - 1)}$, which correspond to systems with equation of state $-1$ in both cases. To study the stability we expand equation (14) around each critical point. For the first one, we expand $AX = 0 + \varepsilon$ with $\varepsilon \ll 1$, then the evolution equation can be approximated as

$$\frac{d(AX)}{dN} \approx \frac{6\varepsilon}{1 - 2\alpha}, \quad (17)$$

for $1 - 2\alpha > 0$, which corresponds to an unstable point. If instead $2\alpha - 1 > 0$ then

$$\frac{d(AX)}{dN} \approx \frac{6\alpha_0\varepsilon^{(3 - 2\alpha)/2}}{2\alpha - 1}, \quad (18)$$

and the critical point is stable for $\alpha_0 < 0$ and unstable for $\alpha_0 > 0$. For example, for the case studied in the equations (5) where $\alpha = 1$ and $\alpha_0 > 0$, the critical point $X = 0$ is
unstable. For the cases in which $X = 0$ is unstable, the value of the late time cosmological constant is not $\rho(X = 0) = M$, that corresponds to the constant added in the Lagrangian.

For the other critical point, we expand as $AX = \alpha^2/(2\alpha - 1) + \varepsilon$, then

$$\frac{d(AX)}{dN} \approx -3\varepsilon,$$

(19)
corresponding to a stable point. This result is important because the dynamical evolution of the system will drive the field to a behaviour similar to a cosmological constant at late times, with $\omega_\phi = -1$ in which the system tends to a value of the field with $AX_2 = \alpha^2/(2\alpha - 1)$ corresponding to a density $\rho(X = X_2) = \alpha^2/(2\alpha - 1) - M$ that is a combination of constants that yields the dark energy of the model, see [1].

**CONDITIONS FOR UDM**

In the first section we studied the model (3, 4), which in the late time Universe can give rise to the phenomena of dark matter and dark energy. However this model is very specific, and therefore the aim of this section is to study the conditions for a more general class of scalar fields to reproduce the same dynamical features. If the Lagrangian has a general form in terms of the field and the kinetic term $L = L(X, \phi)$, its equation of state turns out to be

$$\omega = \frac{L}{2X L_X - L},$$

(20)
and the sound speed

$$c_s^2 = \frac{L_X}{2X L_{XX} + L_X}.$$  

(21)
A sufficient condition for the field to behave as dark matter is that both quantities be close to zero [13], leaving to the conditions

$$\frac{L}{X L_X} \ll 1,$$

(22)
and

$$\frac{L_X}{X L_{XX}} \ll 1.$$

(23)

There are several Lagrangians that accomplish the above conditions and they have been proposed as models for unified dark matter (UDM) meaning that they can behave as dark matter and, adding a constant to the Lagrangian, as a combination of dark matter and dark energy. To proceed testing different Lagrangians we first consider condition (22), to later analyze models with (23).

An example from the literature is the purely kinetic Lagrangian proposed by Scherrer on Ref. [12] corresponding to

$$L = F(X) = F_0 + F_m(X - X_0)^2.$$  

(24)
When the kinetic term is near the minimum \((X - X_0)/X_0 \ll 1\), this Lagrangian is known to behave as UDM. Another similar example of this type is the ghost condensate model \([18]\) that served as an attempt to stabilize k-essence scalar fields when having an equation of state \(\omega \phi\) smaller than \(-1\). These examples show that the behaviour around the minimum is important. Let us analyze the conditions for a general Lagrangian having a minimum. In this case it can be expanded as

\[
\mathcal{L}(X, \phi) = \mathcal{L}_0 + \frac{1}{2} \mathcal{L}_2 \delta^2 + \frac{1}{3!} \mathcal{L}_3 \delta^3 + \cdots, \tag{25}
\]

where \(\mathcal{L}_i\) is the \(i\)th derivative with respect to \(X\) evaluated at the minimum \(X_0\), and \(\delta\) is the deviation from the minimum, \(\delta = X - X_0\). The constant term \(\mathcal{L}_0\) can be dropped from the analysis as can be considered as a cosmological constant. This leave us with the condition (22) written as

\[
\frac{\delta}{2X_0} - \frac{6\mathcal{L}_2 + X_0 \mathcal{L}_3}{12X_0^2 \mathcal{L}_2} \delta^2 + \cdots \ll 1, \tag{26}
\]

that imposes the condition on \(\delta/X_0\) to be small, in other words the deviation from the minimum has to be small. The higher order coefficients in the expansion are close to zero as long as the first one is, except for very particular cases. In ref. [10] it is concluded that for Scherrer’s model this deviation \(\delta/X_0\) has to be smaller than \(10^{-16}\) at the present epoch to avoid discrepancies in the structure formation and CMB power spectrum in comparison with the observations.

In ref. [1] it is shown that for the model (3) the deviation \(\delta/X_0\) is of order \(10^{-13}\) during the equality epoch and at the present epoch of order \(10^{-23}\), resulting in a correct description of the cosmology. Considering our model in more detail, the pressure and density of the purely kinetic part, as given in (12) and (13), turn out to be around the minimum \(AX_0 = \alpha^{2/(2\alpha - 1)}\) as

\[
P \approx M - \alpha_0^{2\alpha/(2\alpha - 1)} + \frac{A^2 \alpha \alpha_0^{(2\alpha - 4)/(2\alpha - 1)}}{4} \delta^2, \tag{27}
\]
\[
\rho \approx M - \alpha_0^{2\alpha/(2\alpha - 1)} + A \alpha \alpha_0^{(2\alpha - 2)/(2\alpha - 1)} \delta. \tag{28}
\]

Initially, the terms with \(\delta\)'s dominate over the constant terms and the effective equation of state of the k-fluid behaves as dark matter:

\[
\omega_\phi = \frac{A \delta}{4 \alpha_0^{2/(2\alpha - 1)}} \approx 0. \tag{29}
\]

Later on, as \(\delta \to 0\), the solution tends to the attractor in which the constant terms act as a cosmological constant, \(\rho_\Lambda = -M + \alpha_0^{2\alpha/(2\alpha - 1)}\).

On the other hand, the condition (23) states that the speed of sound has to be small in order to have the growth in the matter inhomogeneities needed for the structure formation. For the field around a minimum the condition is written as

\[
\frac{\delta}{X_0} - \frac{2\mathcal{L}_2 + \mathcal{L}_3}{2X_0^2 \mathcal{L}_2} \delta^2 + \cdots \ll 1. \tag{30}
\]
This condition is similar to (26) and, except for very particular Lagrangians, the accomplishment of the first one will be enough. In other words the deviation from the minimum $\delta/X_0$ must be small.

A different set of Lagrangians that fulfill (22) without the necessity of being around the minimum, are those with a large derivative $d\mathcal{L}/dX$, for example if $\mathcal{L} = AX^\eta$, and $\eta \gg 1$. The condition (23) for these powerlaw Lagrangians corresponds to $\eta + 1 \gg 1$ that is satisfied once the first condition is.

Another important case is the canonical scalar field $\mathcal{L} = X - V(\phi)$, where the condition (22) becomes $V(\phi)/X \approx 1$. It can be accomplished for different initial conditions of the field. The potential $V(\phi) = \Lambda/2[Cosh^2(\sqrt{3}\phi/2) + 1]$ has been studied in Ref. [13] due to its property of satisfy this condition. However, the condition (23) corresponding to have a small speed of sound is never satisfied as $c_s = 1$, making it a bad UDM model. However see [19] for arguments in favour of the validity of the canonical scalar field as UDM.

CONCLUSIONS

We have presented a phase space analysis for the unified model (3, 4) of dark matter, dark energy, and inflation. We have shown that for a large set of the initial conditions $(\phi, \dot{\phi})$ a viable dynamics occurs in which inflation ($\omega_\phi = -1$) happens first, followed by a period of dark matter domination ($\omega_\phi = 0$), to finish with dark energy ($\omega_\phi = -1$). An intermediate radiation period is possible in this model once it is added an extra radiation component as in the standard model of cosmology.

Once inflation ends, the model is fully described by the purely kinetic Lagrangian $\mathcal{L} = F(X)$ with $F$ as in Eq. (3). We have shown that this system possesses a late time stable solution in which $\omega_\phi = -1$, that is dark energy. In ref. [1] the range of parameters were given to achieve a successful cosmological model, and in the present work the dynamical analysis clearly shows why the system is tenable. A problem however may arise in the form of possible instabilities due to the crossing of the system trough $\omega_\phi = -1$, as observed in Fig. 1, but this crossing will occur in the future as the equation of state should have only moved from $\omega_\phi \sim -1$ in the matter dominated epoch to $\omega_\phi \sim -0.75$ in the present epoch.

In the last part of our work we have presented the general features that are necessary to have a model that behaves as dark matter. If one adds a cosmological constant to this model, one ends with a unified dark matter and dark energy model, called generically UDM. There are two conditions that these models should fulfill, equations (22, 23) playing the role of an effective fluid with small pressure (in comparison to the density) and small speed of sound (in comparison to the speed of light). We have analyzed some models studied in the literature that fulfill these conditions. In particular, $F(X)$ models that possess a minimum, as Sherrer’s model (24) or the model studied in the first section (3), when they are close enough to the minimum, they behave as dark matter. Departures from the minimum cause a change in the transfer function and therefore to a different growth history in comparison to the standard model of cosmology [10]. Additionally, we have analyzed other models of the literature, such as models with large derivative $\mathcal{L}_X$, for example $\mathcal{L} = AX^\eta$ with large $\eta$, and models with canonical
Lagrangian $\mathcal{L} = X - V(\phi)$. The constrictions were given for these models.

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