Rational maps from products of curves to surfaces with $p_g = q = 0$

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Abstract
We study dominant rational maps from a product of two curves to surfaces with $p_g = q = 0$. Given two curves which satisfy a mild genericity assumption and have large genus relative to their gonality, we show that the degree of irrationality of their product is equal to the product of their gonalities. Moreover, we prove that the degree of irrationality of a product of two hyperelliptic curves is 4.

Introduction
Dominant rational maps from a product of two curves $C_1$ and $C_2$ to a projective surface $S$ have been studied by Bastianelli, Lee, and Pirola in [5, 9]. Lee and Pirola prove that if $g(C_1) \geq 7$, $g(C_2) \geq 3$, and $C_1, C_2$ are very general then $S$ must be ruled. Instead of describing restrictions on the geometry of $S$, we assume $p_g(S) = q(S) = 0$ and give sharp lower bounds on the degrees of such maps when the gonality maps of $C_1$ and $C_2$ do not factor and these curves have large genus relative to their gonality. Recall that a gonality map for a curve $C$ is a map $C \rightarrow \mathbb{P}^1$ of minimal degree.

Our motivation stems in part from the desire to understand the degree of irrationality of a product of two curves. Recall that the degree of irrationality of a projective variety $X$ is defined as:

$$\text{irr}(X) = \text{def} \min \left\{ \delta > 0 \mid \exists \text{ dominant rational map } X \rightarrow \mathbb{P}^\delta \text{ of degree } \delta \right\}.$$ 

This birational invariant generalizes the classical notion of gonality of an algebraic curve to varieties of higher dimension and is equal to 1 exactly when $X$ is rational. A considerable

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amount of recent work has been devoted to estimating and calculating this invariant for various classes of algebraic varieties—see [4] and [10, §2.2] for a survey. Most relevant to the subject at hand are bounds for measures of irrationality of the symmetric square of a curve given by Bastianelli [3]. For products of curves of genus ≥ 2, there are naive bounds

\[ \text{gon}(C_1) \cdot \text{gon}(C_2) \geq \text{irr}(C_1 \times C_2) \geq \max\{\text{gon}(C_1), \text{gon}(C_2)\} \]

coming from the product of the gonality maps and [6, Theorem 2].

We first prove that this upper bound is achieved for products of hyperelliptic curves. 1

**Theorem A** Consider a dominant rational map \( \varphi : C_1 \times C_2 \to S \) of degree 3 from a product of curves to a surface with \( p_g(S) = q(S) = 0 \) and suppose that \( g(C_1) \leq g(C_2) \). Then \( g(C_1) \leq 1 \). In particular, the degree of irrationality of a product of two hyperelliptic curves is 4.

Theorem A should be contrasted with the situation for products of elliptic curves. Yoshihara [14] has provided examples of products of elliptic curves which admit dominant rational maps of degree 3 to \( \mathbb{P}^2 \). While it is believed that the degree of irrationality of the product of two very general elliptic curves is 4, this question remains open.

Our main result is that this upper bound for \( \text{irr}(C_1 \times C_2) \) is achieved for most pairs of curves whose genera are sufficiently large relative to their gonality.

**Theorem B** Let \( C_1 \) and \( C_2 \) be curves of gonality \( a_1 \) and \( a_2 \) with gonal maps \( f_1 \) and \( f_2 \) which do not factor, and let \( \varphi : C_1 \times C_2 \to S \) be a dominant rational map of degree \( k \) to a surface \( S \) with \( p_g(S) = 0 \). Suppose that \( g(C_i) > (a_i - 1)(\max\{k, a_i\} - 1) \) for \( i = 1, 2 \). Then

(i) \( \varphi \) factors through one of the maps:

\[ f_1 \times \text{id}_{C_2} : C_1 \times C_2 \to \mathbb{P}^1 \times C_2, \quad \text{id}_{C_1} \times f_2 : C_1 \times C_2 \to C_1 \times \mathbb{P}^1. \]

(ii) Moreover, if \( q(S) = 0 \) then \( k \geq a_1 a_2 \). In particular, \( \text{irr}(C_1 \times C_2) = a_1 a_2 \).

The technical assumption that the gonal maps \( f_1 \) and \( f_2 \) do not factor is necessary (see Example 1.2).

**Remark 0.1** The gonal map of a very general \( a \)-gonal curve of genus \( g > (a - 1)^2 \) does not factor. Indeed, consider the Hurwitz scheme

\[ \mathcal{H}_{g,a} = \{[C : \mathbb{P}^1] : g(C) = g, \deg \varphi = a \text{ and } \varphi \text{ is simply ramified}\}. \]

A simply ramified map \( C \to \mathbb{P}^1 \) of degree \( a \) cannot factor nontrivially, and if \( g(C) > (a - 1)^2 \) then the Castelnuovo–Severi inequality (Lemma 1.4) implies that there is a unique such map \( C \to \mathbb{P}^1 \) of degree \( \leq a \). This allows us to define a generically injective morphism \( \mathcal{H}_{g,a} \to \mathcal{M}^{\text{gon} \leq a}_g \), where

\[ \mathcal{M}^{\text{gon} \leq a}_g = \{[C] : \mathcal{M}_g : \text{gon}(C) \leq a \} \subseteq \mathcal{M}_g. \]

It then suffices to observe that \( \mathcal{M}^{\text{gon} \leq a}_g \) is irreducible [1] and of the same dimension as \( \mathcal{H}_{g,a} \).

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1 By hyperelliptic curve, we mean a curve of genus at least 2 with a \( g_2 \).
There are some natural follow-up questions to ask:

**Question 0.2** Let $C_1, \ldots, C_n$ be very general curves:

1. Under similar hypotheses, can one extend Theorem B to a product of $n$ curves?
2. Is the degree of irrationality of $C_1 \times \cdots \times C_n$ equal to $\text{gon}(C_1) \cdots \text{gon}(C_n)$?
3. Is every rational map $C_1 \times \cdots \times C_n \to \mathbb{P}^n$ of degree $\text{gon}(C_1) \cdots \text{gon}(C_n)$ birationally equivalent to a product of rational maps $C_i \to \mathbb{P}^1$?
4. If $g(C_1) = 1$ and $g(C_2) \geq 1$, does $C_1 \times C_2$ admit a dominant rational map of degree 3 to a surface with $p_g = 0$?

**Outline**

In §1 we collect some examples and elementary results regarding dominant rational maps to varieties without 1-forms or 2-forms and present several tools which will be used in the proofs of Theorems A and B. The proofs of these theorems will occupy §2. Finally, in §3 we propose some questions and open problems. Throughout the paper, we work over $\mathbb{C}$ and take the liberty to identify effective zero-cycles of degree $k$ on $X$ with elements of $\text{Sym}_k(X)$. Likewise, we identify effective zero-cycles whose coefficients are all equal to 1 with their support.

**1 Background**

**1.1 Miscellaneous results about rational maps from products of curves**

In this subsection we gather some results about rational maps from products of curves which complement Theorems A and B. First, given a product of curves $C_1 \times \cdots \times C_n$, it is well-known (see for instance [6, Theorem 2]) that

$$\text{irr}(C_1 \times \cdots \times C_n) \geq \max\{\text{gon}(C_1), \ldots, \text{gon}(C_n)\}. \quad (1.1)$$

Moreover, products of curves do not have interesting dominant rational maps of degree 2 to varieties without 1-forms or 2-forms.

**Lemma 1.1** Consider curves $C_1, \ldots, C_n$ of genera $g(C_1) \leq \cdots \leq g(C_n)$ and suppose that

$$\varphi : C_1 \times \cdots \times C_n \to Y$$

is a dominant rational map of degree 2 to a variety with $h^{2,0}(Y) = h^{1,1}(Y) = 0$. Then $g(C_1) = \cdots = g(C_{n-1}) = 0$ and $C_n$ is rational, elliptic, or hyperelliptic.

**Proof** The map $\varphi$ gives a birational involution

$$\iota : C_1 \times \cdots \times C_n \to C_1 \times \cdots \times C_n$$

and thus an involution $\iota^*$ of $V = \text{def} \ H^1(C_1 \times \cdots \times C_n, \mathbb{C})$. Since $H^1(Y) = 0$, the 1-eigenspace for $\iota^*$ must be trivial and thus the $(-1)$ generalized eigenspace for $\iota^*$ is all of $V$. Accordingly, $\iota^*$ acts on $\wedge^2 V$ with eigenvalue 1. By the K"unneth formula, we have the following surjective map which commutes with $\iota^*$:

$$\wedge^2 V \longrightarrow H^2(C_1 \times \cdots \times C_n, \mathbb{C}).$$
This ensures that 1 is the only eigenvalue of \( t^* \) on \( H^2(C_1 \times \cdots \times C_n, \mathbb{C}) \). Since \( H^{2,0}(Y) = 0 \), the space \( H^2(C_1 \times \cdots \times C_n, \mathbb{C}) \) must vanish. It follows that \( g(C_1) = \cdots = g(C_{n-1}) = 0 \) and the result is then a consequence of (1.1).

The following example provides some instances in which the inequality

\[
\text{irr}(C_1 \times C_2) \leq \text{gon}(C_1) \cdot \text{gon}(C_2)
\]

is strict, and highlights the importance of the non-factoring assumptions on the gonal maps of \( C_1 \) and \( C_2 \) in Theorem B.

**Example 1.2** (Product of curves with low degree of irrationality) Yoshihara [14] has provided examples of products of elliptic curves \( E_1 \times E_2 \) with degree of irrationality 3. Hence, a product of curves can have degree of irrationality strictly less than the product of the gonality of the factors. For examples involving curves of higher genus, one can use this example in conjunction with the results of Kato–Martens [7] and Keem–Martens [8]. They give covers \( C_i \rightarrow E_i \) of degree \( d_i \), genus \( g(C_i) \gg d_i \), and gonality \( 2d_i \) for \( i = 1, 2 \). Then

\[
\text{irr}(C_1 \times C_2) \leq 3d_1 d_2 < 4d_1 d_2 = \text{gon}(C_1) \cdot \text{gon}(C_2).
\]

### 1.2 Tools used in the proofs of Theorems A and B

A key tool that we will use to prove part (1) of Theorem B is the following lemma which states that an appropriate partition of the fibers of a rational map gives rise to a factorization.

**Lemma 1.3** Consider generically finite dominant rational maps \( f : X \rightarrow Y \) and \( g : X \rightarrow W \). Suppose that for a general element \( y \in Y \), the fiber \( f^{-1}(y) \) decomposes into a disjoint union

\[
f^{-1}(y) = \bigsqcup_i g_i^{-1}(w_i)
\]

for some finite collection of points \( w_i \in W \). Then there exists a generically finite rational map \( h : W \rightarrow Y \) making the following diagram commute:

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow^f & & \downarrow^g \\
W & \rightarrow & Y \ \\
\downarrow^h & & \\
& & \\
\end{array}
\]

**Proof** After replacing \( X, Y, W \) with open subsets, we may assume that both \( f \) and \( g \) are étale morphisms and that the hypothesis holds for all \( y \in Y \). Consider the incidence variety

\[
I = \text{def} \{(w, y) \in W \times Y : g^{-1}(w) \subseteq f^{-1}(y)\}.
\]

Note that this is an algebraic subvariety of \( W \times Y \). The projection of \( I \) to \( W \) is an isomorphism and \( I \) dominates \( Y \). Therefore we may define \( h \) as the composition of this isomorphism with the second projection

\[
h : W \cong I \xrightarrow{\text{pr}_2} Y
\]

and one easily verifies that \( f = h \circ g \).

The following lemma will be used frequently in order to rule out the existence of a \( g^i_k \) which does not factor through the gonal map of a curve \( C \), when \( k \) is small relative to \( g(C) \).
Lemma 1.4 (Castelnuovo–Severi inequality [2, VIII, Ex. C-1]) Let \( X, C_1, C_2 \) be curves of respective genera \( g, g_1, g_2 \). Assume that \( f_i : X \to C_i \) is a degree \( a_i \) cover (for \( i = 1, 2 \)) such that the morphism \( f_1 \times f_2 : X \to C_1 \times C_2 \) is birational onto its image. Then
\[
g \leq a_1 g_1 + a_2 g_2 + (a_1 - 1)(a_2 - 1).
\]

As an application, we have:

Proposition 1.5 Let \( C_1, C_2 \) be curves of positive genus such that \( \text{gon}(C_1) = d \) and \( g(C_1) > (d - 1)(e - 1) \), where \( e \geq 2 \) is a positive integer which is relatively prime to \( d \). Then there are no dominant rational maps of degree \( e \) from \( C_1 \times C_2 \) to a surface \( S \) with \( q(S) = 0 \).

Proof Consider a dominant rational map \( \varphi : C_1 \times C_2 \to S \) of degree \( e \), where \( S \) is a smooth projective surface with \( q(S) = 0 \). The rational map \( \varphi \) gives rise to a rational map
\[
S \to \text{Sym}^e(C_1 \times C_2) \to \text{Sym}^e(C_1)
\]
where \( \varphi^{-1}(t) = \{ p_1, \ldots, p_e \} \). Moreover, the composition of this map with the Abel–Jacobi map \( \text{Sym}^e(C_1) \to \text{Alb}(C_1) \) is constant, as it factors through \( \text{Alb}(S) = 0 \). By Abel’s theorem the image of \( S \) in \( \text{Sym}^e(C_1) \) is contained in a basepoint-free \( g^r_e \) for some positive integer \( r \). A general line in this basepoint-free \( g^r_e \) gives a basepoint-free \( g^1_e \) on \( C_1 \). Lemma 1.4 then implies that this \( g^1_e \) and a gonal map for \( C_1 \) factor non-trivially through the same curve, so \( \gcd(d, e) \neq 1 \). \( \square \)

Remark 1.6 Given a projective variety \( X \) of dimension \( n \), one can consider the threshold
\[
\delta_0(X) = \min \{ \delta > 0 \mid \exists \text{ dominant rational map } X \to \mathbb{P}^n \text{ of degree } \delta \}.
\]

It is straightforward to show that \( \delta_0(X) \) exists (cf. [4, Example 4.6]). The previous proposition illustrates that there is no upper bound on \( \delta_0(C_1 \times C_2) \) which depends only on the gonality of \( C_1 \) and the gonality of \( C_2 \).

Next, let us consider a dominant rational map \( \varphi : X \to Y \) between two smooth projective \( n \)-folds. In this setting, Mumford [12] constructs a trace map
\[
\text{Tr}_\varphi : H^0(X, K_X) \to H^0(Y, K_Y).
\]
For \( \eta \in H^0(X, K_X) \) and a general point \( y \in Y \), \( \text{Tr}_\varphi(\eta) \) is determined by summing over the fibers of \( \varphi \):
\[
\text{Tr}_\varphi(\eta)(y) = \sum_{x \in \varphi^{-1}(y)} \eta(x).
\]
For more details, see [13, §2.3].

Lemma 1.7 Let \( m \) be a positive integer and \( C \) be a curve of gonality \( a > 2 \) and genus
\[
g(C) > (ma - 1)(a - 1).
\]
Assume that \( C \) has a gonal map which does not factor. Given elements \( D_1, \ldots, D_m \) of the (unique) \( g^1_a \) on \( C \) and \( p, q \in \text{Supp}(D_m) \), there exists a 1-form on \( C \) vanishing at all of the points of \( \bigcup_{i=1}^m \text{Supp}(D_i) \) except for \( p \) and \( q \).
Proof By our hypothesis, the gonal map $C \longrightarrow \mathbb{P}^1$ of degree $a$ does not factor so Lemma 1.4 together with the bound $g(C) > (ma - 1)(a - 1)$ implies that (1) there is a unique line bundle $L$ of degree $a$ such that $|L|$ is a base-point free $g^1_a$, and (2) any base-point free pencil of degree $\leq ma$ must factor through the gonal map. Since $|mL|$ is base-point free, given any element $D \in |mL|$ we can find a base-point free pencil $\delta$ passing through $D$. By (2), $\delta$ factors as

$$C \xrightarrow{|L|} \mathbb{P}^1 \xrightarrow{\alpha} \mathbb{P}^1 \xrightarrow{\delta} \mathbb{P}^1,$$

where $\alpha$ is a map of degree $m$. Since $D$ was arbitrary, this implies that the addition map

$$|L|^m \longrightarrow |mL|$$

is surjective. Because this map is finite onto its image, $\dim |mL| = m$ and hence $\dim |mL| - \dim |(m - 1)L| = 1$.

With $D_1, \ldots, D_m$ in the $g^1_a$ and $p, q \in \text{Supp}(D_m)$ as above, it suffices to show that

$$h^0(K_C - (mL - p - q)) - h^0(K_C - mL) \geq 1.$$

This is because any form which vanishes at all points in $D_m - p$ must also vanish at $p$, and similarly for $q$ (by tracing along the $g^1_a$). By Serre duality and Riemann–Roch, we may rewrite the inequality as $h^0(mL) - h^0(mL - p - q) \leq 1$. But this follows immediately from the fact that $|mL| - |(m - 1)L| = 1$ and $(m - 1)L \leq mL - p - q$. \qed

2 Proofs of Theorems A and B

2.1 Proof of Theorem A

By Lemma 1.1, we know that any dominant rational map

$$\varphi : C_1 \times C_2 \longrightarrow S$$

from a product of curves with $g(C_1), g(C_2) \geq 1$ to a surface $S$ satisfying $p_g(S) = q(S) = 0$ must have degree at least 3. Suppose for contradiction that $g(C_1), g(C_2) \geq 2$ and that there exists such a map $\varphi$ with $\deg(\varphi) = 3$. The induced map

$$F_\varphi : S \longrightarrow \text{Sym}^3(C_1 \times C_2) \longrightarrow \text{Sym}^3(C_i) \longrightarrow \text{Alb}(C_i)$$

factors through $\text{Alb}(S) = 0$. Hence, the image of $S$ in $\text{Sym}^3(C_i)$ is a surface or a rational curve in a fiber of the Albanese map, which is a linear system $|L_i|$ for some line bundle $L_i$ of degree 3.

Fix a general $s \in S$ and consider $\varphi^{-1}(s) = \{p_1, p_2, p_3\}$. By Lemma 2.2 below, the subsets $\text{pr}_i(\varphi^{-1}(s)) \subseteq C_i (i = 1, 2)$ both consist of three distinct points. For each $i$, $\text{pr}_i(\varphi^{-1}(s))$ cannot contain a pair of points which are images of one another under the hyperelliptic involution. Indeed, the hyperelliptic pencil is unique and such a pair would single out the third point, thereby violating the irreducibility of $C_1 \times C_2$. Assuming $g(C_1), g(C_2) \geq 2$, we may therefore choose a 1-form $\omega_1$ on $C_1$ which vanishes at $\text{pr}_1(p_1)$ but is nonzero at $\text{pr}_1(p_2)$ and $\text{pr}_1(p_3)$. Similarly, we may choose a 1-form $\omega_2$ on $C_2$ which vanishes at $\text{pr}_2(p_2)$ but is nonzero at $\text{pr}_2(p_1)$ and $\text{pr}_2(p_3)$. The 2-form $\text{pr}_1^* \omega_1 \wedge \text{pr}_2^* \omega_2$ then vanishes at $p_1$ and $p_2$ but not $p_3$. By applying Mumford’s trace map (see §1), we have

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Let $s \in S$. The restrictions of the projection maps $\operatorname{pr}_i : C_1 \times C_2 \rightarrow C_i$ to $\varphi^{-1}(s)$ are coverings of their images. In other words, the number

$$d_{i,x} \overset{\text{def}}{=} \# \{ z \in \varphi^{-1}(s) : \operatorname{pr}_i(z) = \operatorname{pr}_i(x) \}$$

is independent of $x \in \varphi^{-1}(s)$ for a general $s \in S$.

**Proof** Let $U \subseteq C_1 \times C_2$ be an open subset on which $\varphi$ restricts to an étale morphism. Fix $i \in \{1, 2\}$. For each $r \in \mathbb{Z}_{>0}$, consider the quasi-projective subvariety

$$U^i_r : = \{ x \in U : d_{i,x} = r \} \subseteq U.$$

Since $U$ is irreducible and $\bigcup_{r>0} U^i_r = U$, there is an integer $r_0$ such that $U^i_{r_0}$ is open in $U$. If $Z \subseteq U$ is the complement of $U_{r_0}$ then $\varphi^{-1}(\varphi(Z)) \subseteq U$ is a proper closed subset of $U$. Given any $s \in U_{r_0} \setminus \varphi^{-1}(\varphi(Z))$ and any $x \in \varphi^{-1}(s)$, we have $d_{i,x} = r_0$.

Throughout, we will let $d_1$ be the value of $d_{1,x}$ for $x \in \varphi^{-1}(s)$ and for a general $s \in S$. There are several cases to consider:

**Lemma 2.3** Under the assumptions of Theorem B, for a general $s \in S$ either:

1. The image of the set $\operatorname{pr}_1(\varphi^{-1}(s))$ under the canonical map to $\mathbb{P}^{g(C_1)-1}$ spans a linear subspace of dimension $k/d_1 - 1 = \#\operatorname{pr}_1(\varphi^{-1}(s)) - 1$, or
2. The set $\operatorname{pr}_1(\varphi^{-1}(s))$ is uniquely partitioned into elements of $|L_1|$. 
Moreover, case (2) breaks up into two subcases. For each subset $D \subseteq \text{pr}_1(\varphi^{-1}(s))$ which is the support of an element of $|L_1|$, the image of $(D \times C_2) \cap \varphi^{-1}(s)$ under the projection to $C_2$ either

(2a) does not contain an element of $|L_2|$, or

(2b) is a union of elements of $|L_2|$.

In the setting of case (2), let us introduce some additional notation which will help streamline our arguments. Consider the open subset $U \subseteq C_1 \times C_2$ such that $\varphi|_U$ is an étale morphism and write

$$F_x := \text{def} \varphi^{-1}(\varphi(x)) \quad \text{for } x \in U.$$ 

By shrinking $U$ if needed, we may assume that $\text{pr}_i(F_x)$ is uniquely partitioned into elements of $|L_i|$ for all $x \in U$ and $i = 1, 2$. Given $x \in U$, let $D_{i,x}$ be the element of $|L_i|$ satisfying

$$\text{pr}_i(x) \subseteq D_{i,x} \subseteq \text{pr}_i(F_x).$$

**Proof of Lemma 2.3** By an irreducibility argument similar to the one used in Lemma 2.2, if $\text{pr}_1(\varphi^{-1}(s))$ contains an element of $|L_1|$ then every point of $\text{pr}_1(\varphi^{-1}(s))$ is contained in an element of $|L_1|$. Since elements of $|L_1|$ are fibers of the gonal map, they have empty intersection. Hence $\text{pr}_1(\varphi^{-1}(s))$ is uniquely partitioned into elements of $|L_1|$.

Now suppose that $\text{pr}_1(\varphi^{-1}(s))$ does not contain an element of $|L_1|$. We claim that these points span a linear subspace of dimension

$$k/d_1 - 1 = \#\text{pr}_1(\varphi^{-1}(s)) - 1$$

in canonical space. Indeed if these points spanned a smaller linear subspace, then by geometric Riemann–Roch they belong to a $g^1_{k/d_1}$ on $C_1$ for some $r \geq 1$. Any divisor in this $g^1_{k/d_1}$ belongs to a $g^1_{k/d_1}$. If the genus of $C_1$ is larger than $(a_1 - 1)(k/d_1 - 1)$, then any element of a $g^1_{k/d_1}$ must contain an element of $|L_1|$, which provides the desired contradiction.

As for the second claim, suppose that we are in case (2). Consider the closed subvariety

$$V := \text{def} \left\{ x \in U : \text{pr}_1(\varphi^{-1}(D_{1,x})) \text{ contains an element of } |L_2| \right\} \subseteq U.$$ 

Either $V = U$, in which case we are in situation (2b), or $V$ is a proper closed subset of $U$, in which case we are in situation (2a). □

The proof of Theorem B is easier in cases (1) and (2a), so we first single out these cases:

**Proposition 2.4** Theorem B holds if $\varphi : C_1 \times C_2 \dashrightarrow S$ falls under case (1) or case (2a) of Lemma 2.3.

**Proof** Recall from Lemma 2.2 that the restriction of $\text{pr}_1$ to $\varphi^{-1}(s)$ is a covering onto its image. Let $l$ denote the degree of this covering.

We will first take care of case (1). Given a generic $s \in S$ and any point $x \in \text{pr}_1(\varphi^{-1}(s))$, if $g(C_1) > k/d_1 - 1$ then we can find a 1-form $\omega_1$ on $C_1$ which vanishes at all the points in $\text{pr}_1(\varphi^{-1}(s))$ aside from $x$. Indeed, the points of $\text{pr}_1(\varphi^{-1}(s))$ are linearly independent so $\text{pr}_1(\varphi^{-1}(s)) \setminus \{x\}$ spans a subspace of dimension $k/d_1 - 2$ in canonical space. Now a generic hyperplane in canonical space containing this subspace will not contain $\text{pr}_1(\varphi^{-1}(s))$.

By an irreducibility argument similar to the one used in the proof of Lemma 2.2, for a general $s \in S$ either the set

$$\Phi_x(p_1) := \text{def} \left\{ p_2 \in C_2 : (p_1, p_2) \in \varphi^{-1}(s) \right\} \subseteq C_2$$

is an étale morphism, Let $l$ denote the degree of this covering.
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**Fig. 1** The diagram above depicts case (2b) of Lemma 2.3

**Fig. 2** Case (2a) when 
\( \Phi_s(p_1) = \Phi_s(q_1) \) for all 
\( x \in \text{pr}_1(\varphi^{-1}(s)) \) and 
\( p_1, q_1 \in D_{1,x} \)

(i) does not contain elements of \( |L_2| \) for all \( p_1 \in \text{pr}_1(\varphi^{-1}(s)) \), or (ii) is uniquely partitioned into elements of \( |L_2| \) for all \( p_1 \in \text{pr}_1(\varphi^{-1}(s)) \). In case (i), since

\[ g(C_2) \geq \max\{ (a_2 - 1)(l - 1), l \} \]

there is a 1-form \( \omega_2 \) on \( C_2 \) vanishing at all points of \( \Phi_s(x) \) except for one. In particular, the 2-form

\[ \text{pr}_1^* \omega_1 \land \text{pr}_2^* \omega_2 \in H^0(C_1 \times C_2, \Omega^2) \]

vanishes at all but one point of \( \varphi^{-1}(s) \). By tracing to \( S \), it follows that \( 0 \neq \text{Tr}_\varphi(\text{pr}_1^* \omega_1 \land \text{pr}_2^* \omega_2) \in H^0(S, \Omega^2) \), contradicting the assumption that \( p_g(S) = 0 \). In case (ii), we can use Lemma 1.3 to deduce that the rational map \( \varphi \) factors through

\[ \text{id}_{C_1} \times f_2: C_1 \times C_2 \rightarrow C_1 \times \mathbb{P}^1. \]

Let us now consider case (2a). For \( x \in C_1 \times C_2 \), let \( D_{1,x} \) be the element of \( |L_1| \) which contains \( \text{pr}_1(x) \). Consider a generic \( s \in S \). If for all \( x \in \varphi^{-1}(s) \) and all \( p_1, q_1 \in \text{Supp}(D_{1,x}) \) we have

\[ \Phi_s(p_1) = \Phi_s(q_1), \]

then \( \varphi^{-1}(s) \) is partitioned into subsets of the form \( D \times \{z\} \) for some \( D \in |L_1| \) and some \( z \in C_2 \). This is depicted in Fig. 2.
By Lemma 1.3, the map $\varphi$ must factor through

$$f_1 \times \text{id}_{C_2} : C_1 \times C_2 \longrightarrow \mathbb{P}^1 \times C_2.$$  

Otherwise, there exist $x \in \varphi^{-1}(s)$ and $p_1, q_1 \in D_{1,x}$ such that $\Phi_s(p_1) \neq \Phi_s(q_1)$. Pick some

$$z_2 \in \Phi_s(p_1) \setminus \Phi_s(q_1).$$

Since $g(C_2) \geq 2l + 1$ and $\Phi_s(p)$ does not contain an element of $|L_2|$, we can find a 1-form $\omega_2$ on $C_2$ which vanishes everywhere on $\Phi_s(p_1) \cup \Phi_s(q_1)$ aside from $z_2$. Then the 2-form

$$\text{pr}^*_1 \omega_1 \wedge \text{pr}^*_2 \omega_2 \in H^0(C_1 \times C_2, \Omega^2)$$

vanishes at all but one point of $\varphi^{-1}(s)$, and tracing it to $S$ contradicts the fact that $p_\Sigma(S) = 0$. \hfill \Box  

This leaves the remaining case (2b). In this setting, we will prove a series of lemmas that describe some strong constraints on the fibers of such a map $\varphi : C_1 \times C_2 \longrightarrow S$.

**Lemma 2.5** Suppose that $\varphi : C_1 \times C_2 \longrightarrow S$ falls under case (2b). Then for $s \in S$ generic, the set $\varphi^{-1}(s)$ is canonically partitioned into subsets $T_\alpha$ which are indexed by a set $J(s)$ satisfying the following properties.

- $|T_\alpha| = |T_\beta|$ for all $\alpha, \beta \in J(s)$.
- For any $\alpha \in J(s)$, there is an $x \in \varphi^{-1}(s)$ such that
  $$T_\alpha = (D_{1,x} \times D_{2,x}) \cap \varphi^{-1}(s) \quad \text{and} \quad \text{pr}_1(T_\alpha) = D_{i,x}.$$  
- The map $\text{pr}_i : T_\alpha \longrightarrow D_{i,x}$ is a covering whose degree is independent of $\alpha$.

**Proof** For any point $x \in \varphi^{-1}(s)$, the $T_\alpha$ containing $x$ is $(D_{1,x} \times D_{2,x}) \cap \varphi^{-1}(s) \subseteq C_1 \times C_2$. The first claim follows from the fact that we would otherwise be able to canonically single out the union of the $T_\alpha$ with the largest cardinality, thereby violating irreducibility of $C_1 \times C_2$. The second claim follows from the definition of the $T_\alpha$ and the fact that we are in case (2b). For the third claim, consider the projection

$$\text{pr}_i : T_\alpha \longrightarrow D_{i,x}$$

and collect the union of the largest cardinality fibers in $T_\alpha$ for all $\alpha \in J$. By an irreducibility argument, this must be all of $\varphi^{-1}(s)$ and thus $\text{pr}_i |_{T_\alpha}$ is a covering of its image. \hfill \Box  

**Definition 2.6** A subset $T$ of the Cartesian product of two sets $A$ and $B$ is called checkered if there are partitions $A = A_1 \sqcup A_2$ and $B = B_1 \sqcup B_2$ such that $\#A_1 = \#A_2$, $\#B_1 = \#B_2$, and

$$T = (A_1 \times B_1) \sqcup (A_2 \times B_2) \subseteq A \times B.$$  

**Lemma 2.7** Suppose that $\varphi$ falls under case (2b) and let $T_\alpha, \alpha \in J(s)$ be as defined in Lemma 2.5 for a generic fiber $\varphi^{-1}(s)$. Then either

- $T_\alpha = \text{pr}_1(T_\alpha) \times \text{pr}_2(T_\alpha)$ for all $\alpha \in J(s)$, or
- $T_\alpha$ is a checkered subset of $\text{pr}_1(T_\alpha) \times \text{pr}_2(T_\alpha)$ for all $\alpha \in J(s)$.  

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\textbf{Proof} Fix \(\alpha \in J(s)\). In order to describe which points of \(\text{pr}_1(T_\alpha) \times \text{pr}_2(T_\alpha)\) are in \(T_\alpha\) and which points are not, we will represent \(T_\alpha\) as a rectangular box with a lattice of circles. The rows corresponds to points of \(\text{pr}_2(T_\alpha)\) and the columns to points of \(\text{pr}_1(T_\alpha)\). If the circle in the entry corresponding to \((p_1, p_2) \in \text{pr}_1(T_\alpha) \times \text{pr}_2(T_\alpha)\) is filled in then \((p_1, p_2) \in T_\alpha\). For instance, the diagram

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
 & p_1 & q_1 & r_1 & s_1 \\
\hline
p_2 & \bullet & \bullet & \circ & \circ \\
q_2 & \circ & \bullet & \circ & \circ \\
\end{array}
\]

corresponds to \(T_\alpha = \{(p_1, p_2), (q_1, q_2), (r_1, p_2), (s_1, q_2)\}\), and \(T_\alpha\) is an example of a checkered subset of \(\text{pr}_1(T_\alpha) \times \text{pr}_2(T_\alpha)\), with \(A_1 = \{p_1, r_1\}, A_2 = \{q_1, s_1\}, B_1 = \{p_2\}, B_2 = \{q_2\}\).

We will mostly avoid labelling the columns and rows and allow ourselves to permutes rows and columns as we please.

Consider any two points \(p_i, q_i \in \text{pr}_i(T_\alpha)\) for \(i = 1, 2\). Suppose that the set \(T_\alpha \cap \{(p_1, q_1) \times \{p_2, q_2\}\}\) has the following configuration:

\[
\bullet \circ \circ \circ
\]

(2.1)

For \(i = 1, 2\), by Lemma 1.7 we can find a form \(\omega_i \in H^0(C_i, \Omega^1_{C_i})\) which vanishes everywhere on \(\text{pr}_i(\varphi^{-1}(s))\) except at \(p_i\) and \(q_i\). The 2-form \(\text{pr}_1^* \omega_1 \wedge \text{pr}_2^* \omega_2\) then vanishes everywhere on \(\varphi^{-1}(s)\) except at the point which corresponds to the filled in circle in diagram (2.1), thereby contradicting the assumption \(p_g(S) = 0\). Similarly, suppose that the set \(T_\alpha \cap \{(p_1, q_1) \times \{p_2, q_2\}\}\) has the following configuration:

\[
\circ \bullet \bullet
\]

(2.2)

If \((p_1, p_2)\) is the point corresponding to the empty circle in (2.2) then the first configuration in (2.1) must appear in \(T_\beta\) for some \(\beta \in J(\varphi(p_1, p_2))\). Again, we are able to find a form which vanishes everywhere on \(\varphi^{-1}(\varphi(p_1, p_2))\) aside from a point, which contradicts the assumption \(p_g(S) = 0\). Indeed, given \(s \in S\) generic, any \(\alpha \in J(s)\), and any choice of \(p_1, q_1 \in \text{pr}_1 T_\alpha\) and \(p_2, q_2 \in \text{pr}_2 T_\alpha\) giving rise to configuration (2.2), the point \(\varphi(p_1, p_2) \in S\) is also generic.

Thus, we may assume that for any choice of \(p_1, q_1 \in \text{pr}_1 T_\alpha\) and \(p_2, q_2 \in \text{pr}_2 T_\alpha\) the set

\[
T_\alpha \cap \{(p_1, q_1) \times \{p_2, q_2\}\}
\]

consists of 0, 2, or 4 points. After permuting the rows and columns of the diagram of \(T_\alpha\) we can assume that both the first row and first column consist of a series of \(\bullet\) followed only by \(\circ\) as pictured in the first diagram of Fig. 3. The fact that the configurations (2.1) and (2.2) do not arise in this diagram places constraints on what the rest of this diagram can look like.

It is useful to distinguish two cases. In the first case the entire first row consists only of \(\bullet\) or only of \(\circ\). It is easy to see that the fact that configurations of the form (2.1) and (2.2) do not arise forces the entire diagram to consist of a series of rows filled with \(\bullet\) followed by a series of rows filled with \(\circ\). Since by assumption \(T_\alpha \neq \emptyset\) and the projection of \(T_\alpha\) to the second factor must surject onto \(\text{pr}_2(T_\alpha)\), there can be no rows consisting entirely of \(\circ\) and thus \(T_\alpha = \text{pr}_1(T_\alpha) \times \text{pr}_2(T_\alpha)\).

The second case where the first row consists of a positive number of \(\bullet\) followed by a positive number of \(\circ\) is more interesting:
(1) Consider the second row of the diagram from the left. If the second entry of this row from the top were $\circ$ then the $2 \times 2$ square made up of the first two entries of the first two columns would be

```
••
• ○
```

This would contradict the parity statement about the cardinality of sets of the form $T_\alpha \cap (\{p_1, q_1\} \times \{p_2, q_2\})$. The second entry of the second row must therefore be $\bullet$. Moving down the second column, the same reasoning shows that every row containing a $\bullet$ in the first column contains a $\bullet$ in the second column. If the first row from the top containing a $\circ$ in the first column also contained a $\bullet$ in the second column we would get the following forbidden configuration:

```
••
○•
```

Going down the second column further, the same reasoning then shows that every row containing a $\circ$ in the first column also contains a $\circ$ in the second.

(2) We can repeat the argument used in (1) to show that every column starting with a $\bullet$ consists of a series of $\bullet$ followed by a series of $\circ$. Moreover, the number of $\bullet$ is constant. The third diagram in Fig. 3 illustrates what we know about the diagram at this point of the argument.

(3) Now consider the first column from the left which starts with a $\circ$. This is also the first column which we have yet to describe. If the first entry of the second row were a $\bullet$ and the second entry of this column were a $\bullet$ we would get a forbidden configuration of the form

```
• ○
• •
```

Repeating this argument, we can check that every row which starts with a $\bullet$ contains a $\circ$ in this column and every row that starts with a $\circ$ contains a $\bullet$ in that column. Moving to the next column and repeating the same argument shows that $T_\alpha$ must be as pictured in the last diagram of Fig. 3. It follows that $T_\alpha$ is a checkered subset of $\text{pr}_1(T_\alpha) \times \text{pr}_2(T_\alpha)$.

We are finally ready to resolve the last case of Theorem B.

**Proposition 2.8** Theorem B holds if $\varphi : C_1 \times C_2 \rightarrow S$ falls under case (2b) of Lemma 2.3.

**Proof** Following the notation above, consider a general $s \in S$ and $\alpha \in J(s)$. By Lemma 2.7, there are two possibilities for the structure of $T_\alpha$. If $T_\alpha = \text{pr}_1(T_\alpha) \times \text{pr}_2(T_\alpha)$ for all $\alpha \in J$, then $\varphi$ factors through $f_1 \times f_2 : C_1 \times C_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by Lemma 1.3. Hence, we may assume that $T_\alpha$ is checkered as a subset of $\text{pr}_1(T_\alpha) \times \text{pr}_2(T_\alpha)$.

Since $\text{pr}_1(T_\alpha)$ is the support of an element of $|L_1|$, this allows us to write $a_1 = 2a'_1$ for some positive integer $a'_1$. The idea is that this checkered structure on $T_\alpha$ can be used to find an algebraically defined partition of the support of elements of $|L_1|$ into two subsets of equal sizes. Such a partition gives rise to a factorization of the gonal map $f_1$ associated to the pencil $|L_1|$.

Let $U \subseteq C_1 \times C_2$ be an open subset on which $\varphi$ is étale, and fix a point $c \in \text{pr}_2(U)$. Since the subset $T_\alpha$ containing a point $(x_1, c) \in U$ is checkered for a general $x_1 \in C_1$, there exist two partitions $\text{pr}_1(T_\alpha) = A_1^{x_1} \sqcup A_2^{x_1}$ and $\text{pr}_2(T_\alpha) = B_1^{x_1} \sqcup B_2^{x_1}$ as in Definition 2.6,
with $\#A_1^{x_1} = \#A_2^{x_1} = a'_1$. By identifying the subsets $A_1^{x_1}, A_2^{x_1} \subseteq C_1$ with the corresponding points in $\text{Sym}^{d_i}(C_1)$, we may then define the subvariety

$$Z' = \{ A_1^{x_1} + A_2^{x_1} \mid x_1 \in U \cap (C_1 \times c) \} \subseteq \text{Sym}^2(\text{Sym}^{d_i}(C_1)).$$

The following lemma then implies that the gonal map $C_1 \to \mathbb{P}^1$ must factor, which provides the desired contradiction. \hfill $\square$

**Lemma 2.9** Consider a curve $C$ with a morphism $f : C \to \mathbb{P}^1$ of degree $k$. Write $Z \subseteq \text{Sym}^k(C)$ for the corresponding linear system. Suppose that $k = dk'$ for some integers $d$ and $k'$ and that there exists a subvariety $Z' \subseteq \text{Sym}^{d}(\text{Sym}^{k'}(C))$ along with a birational map $v : Z' \to Z$ which fit into the commutative diagram

$$
\begin{array}{ccc}
Z' & \hookrightarrow & \text{Sym}^{d}(\text{Sym}^{k'}(C)) \\
v \downarrow & & \downarrow \\
Z & \twoheadrightarrow & \text{Sym}^{k}(C),
\end{array}
$$

where the vertical map on the right hand side is the summation map

$$
\text{Sym}^{d}(\text{Sym}^{k'}(C)) \to \text{Sym}^{k}(C)
$$

Then the map $f$ factors through a map $C \to C'$ of degree $k'$, for some curve $C'$. 

---

**Fig. 3** A picture of the diagram for $T_\alpha \subseteq \text{pr}_1(T_\alpha) \times \text{pr}_2(T_\alpha)$
Proof Consider the incidence correspondence
\[ I = \{ (c, x) \mid c \in \text{supp}(x) \text{ and } \exists y_1, \ldots, y_{d-1} \in \text{Sym}^{k'}(C) \text{ such that } x + y_1 + \ldots + y_{d-1} \in Z' \} \subseteq C \times \text{Sym}^{k'}(C). \]

Note that \( I \) maps birationally onto \( C \) under the first projection and generically finitely onto its image \( C' \) under the second projection. The projection \( C \to C' \) has degree \( k' \) and the fibers of \( C \to \mathbb{P}^1 \) can be partitioned by the fibers of this map. By Lemma 1.3, we have a factorization as required.

This completes the proof of Theorem B.

Remark 2.10 In the proofs of Theorems A and B we only trace decomposable 2-forms on \( S \), namely forms \( pr_1^*\omega_1 \wedge pr_2^*\omega_2 \), where \( \omega_i \) is a 1-form on \( C_i \). Such 2-forms make up a \((g(C_1) + g(C_2))\)-parameter family, whereas
\[ \dim H^0(C_1 \times C_2, \Omega_2^2) = g(C_1) \cdot g(C_2). \]

We expect that a lot more information can be captured by tracing arbitrary 2-forms on \( C_1 \times C_2 \).

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