Robust estimation of dependent competing risk model under interval monitoring and determining optimal inspection intervals

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Abstract

Recently, a growing amount interest is quite evident in modelling dependent competing risks in life time prognosis problem. In this work, we propose to model the dependent competing risks by Marshal-Olkin bivariate exponential distribution. The observable data consists of number of failures due to different causes across different time intervals. The failure count data is common in instances like one shot devices where state of the subjects are inspected at different inspection times rather than the exact failure times. The point estimation of the life time distribution in presence of competing risk has been studied through divergence based robust estimation method called minimum density power divergence estimation (MDPDE). The testing of hypothesis is performed based on a Wald type test statistic. The influence function is derived both for the point estimator and the test statistic, which reflects the degree of robustness. Another, key contribution of this work is to determine the optimal set of inspection times based on some predefined objectives. This article presents determination of multi criteria based optimal design. Population based heuristic algorithm non-dominated sorting-based multiobjective Genetic algorithm is exploited to solve this optimization problem.

Key Words and Phrases: Divergence Based Robust Estimation, Competing Risk, Multi Objective Optimization, Marshal Olkin Bivariate Exponential Distribution, Influence function.

AMS Subject Classifications: 62F10, 62F03, 62H12.

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1 Introduction

Competing risk data arises when an event takes place due to different simultaneously effective causes. The occurrence of an event due to one specific cause precludes one from observing the occurrence of events due to the other causes. In the literature, a significant amount of work has been done in competing risk problem. Crowder [15] provided a monograph on the analysis of different competing risk models. Prentice et al. [33] analysed failure time data in the competing risk environment. Austin et al. [1] proposed the analysis of survival data in the presence of competing risks. Balakrishnan et al. [7, 8] studied estimation of different lifetime distributions in presence of competing risks. Balakrishnan et al. [9] provided Bayesian inference under competing risk setup. Wang et al. [35] studied competing risk failure time data for a frailty-copula model. Dutta and Kayal [20] conducted inferential study under censoring scheme on competing risk data.

Most of those articles present stochastically independent competing risks in action. Recently, a growing interest is quite evident in modelling dependent competing risks in life time prognosis problem. Justification of such modelling lies in instances like shock model originally found in Marshall and Olkin [30]. Suppose, in a system with two components, shock 1 is responsible for failure of component 1, shock 2 is for component 2 while shock 3 results in failure of both the components. In such case, the system fails if any one component fails, it is indeed an example of dependent competing risk set up. In the literature, we find the study on dependent competing risk in Bai et al. [2], Cai et al. [13], Feizjavdian and Hashemi [23], Kundu [26], Kundu and Mondal [25], Shen and Xu [34], Lyu et al. [29] and references therein.

In this work, we explore the study of statistical inference of the life time distribution under dependent competing risk set up. It is assumed that life time under two dependent competing risks follows a bivariate Marshall Olkin distribution. The subjects of interest are put on life testing experiment which continues until a pre-specified time point. The observable data consists of number of failures due to different causes across different time intervals. The failure count data is common in instances like one shot devices where state of the subjects are inspected at different inspection times rather than the exact failure times, readers may see Blakrishnan and Ling [4, 5],
Balakrishnan et al. [6] for references.

In inference study, conventional point estimation method is the maximum likelihood estimation (MLE) which is quite popular because of its well-known properties such as asymptotic efficiency, consistency, sufficiency, invariance transformation. But in presence of outliers, MLE can not perform well. Basu et al [10] proposed divergence based robust estimation method called minimum density power divergence estimator (MDPDE) by incorporating a tuning parameter which brings a trade-off between robustness and efficiency. In this work, along with the MLE, we develop the MDPDE in dependent competing risk set up based on the failure count data.

Along with point estimation, testing of hypothesis is an essential component in inference study. In this article, we present hypothesis testing based on the robust MDPDE. The null hypothesis is constituted based on the equality of the scale parameters of the competing risks. In this regard a Wald type test statistic is developed based on the asymptotic distribution of the MDPDE. An approximation method is applied for the power calculation.

The robustness of any statistic can be assessed by its influence function. In the context of dependent competing risk set-up, the influence functions are computed both for the MDPDE and the Wald type test statistic. Through numerical experiment also, we depict the robustness of the MDPDE compared to MLE.

Apart from inference study, in interval monitoring set-up, it is essential to set the inspection times such that the experiment serves different goals of the experimenter adequately. In this context, we desire the precision of the estimator to be as high as possible along with minimum budget for the experiment. We try to achieve both the goals through multi-objective optimization. Population based heuristic algorithm, Genetic Algorithm (GA) is implemented which returns a set of Pareto optimal solutions. In the literature, Genetic algorithm has been successfully implemented in different situations. Readers may refer to Faraz [22], Liu et al. [28], Parkinson [32], Yang et al. [36]. In this work, we exploit a version of non-dominated sorting GA called NSGA-II proposed by Deb et al. [16].
The rest of the article goes as follows. In Section 2, we put down the description of the model along with the study of likelihood function and the maximum likelihood estimators. We derive the robust density power divergence estimator in section 3. Section 4 provides the study of the testing of hypothesis based on the robust estimator. In Section 5, we study the influence functions for both point estimator and the test statistic. Determination of optimal inspection times is studied in Section 6. In Section 7, an extensive numerical experiment along with a real data analysis for illustration purposes are presented for the performance evaluation of the developed methods.

2 Model Description

In this section, we briefly describe the Marshall-Olkin Bivariate Exponential (MOBE) distribution as the life time model followed by description of the model layout.

The cumulative distribution function (cdf) of an exponential distribution with scale parameter $\lambda$ is defined as

$$F_{Exp}(x) = 1 - e^{-\lambda x}$$

and the pdf is derived as

$$f_{Exp}(x; \alpha, \lambda) = \lambda e^{-\lambda x}, \quad \text{where } x > 0, \quad \lambda > 0,$$

and it will be denoted by $Exp(\lambda)$. Suppose, $U_0 \sim Exp(\lambda_0), U_1 \sim Exp(\lambda_1), U_2 \sim Exp(\lambda_2)$ and they are independently distributed. Define, $X_1 = \min\{U_0, U_1\}$ and $X_2 = \min\{U_0, U_2\}$. The bivariate random vector $(X_1, X_2)$ is said to follow Marshall-Olkin bivariate Exponential distribution denoted by $MOBE(\lambda_0, \lambda_1, \lambda_2)$. The joint survival function of $(X_1, X_2)$ can be derived as,

$$S_{MOBE}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) = P(U_0 > z, U_1 > x_1, U_2 > x_2) = e^{-(\lambda_0 z + \lambda_1 x_1 + \lambda_2 x_2)}$$

where $z = \max\{x_1, x_2\}$. Therefore, the joint probability density function (PDF) of $(X_1, X_2)$ can be obtained as

$$f_{MOBE}(x_1, x_2) = \begin{cases} 
\lambda_1 (\lambda_0 + \lambda_2) e^{-\lambda_1 x_1 - (\lambda_0 + \lambda_2) x_2} & 0 < x_1 < x_2 < \infty \\
\lambda_2 (\lambda_0 + \lambda_1) e^{-(\lambda_0 + \lambda_1) x_1 - \lambda_2 x_2} & 0 < x_2 < x_1 < \infty \\
\lambda_0 e^{-\lambda x} & 0 < x_1 = x_2 = x < \infty.
\end{cases}$$

(1)
where, $\lambda = \lambda_0 + \lambda_1 + \lambda_2$.

Suppose $n$ units are put on the life testing experiment and each unit is subject to two competing risks. Let $T_1$ denote the failure time due to risk 1 and $T_2$ denote the same for risk 2. Here, we assume that $(T_1, T_2) \sim MOBE(\lambda_0, \lambda_1, \lambda_2)$. Under these competing risk set-up, the observable failure time is $T = \min(T_1, T_2)$. In the life testing experiment, at different inspection times say $\tau_1, \ldots, \tau_K$, the experimenter will observe the number of failures in each interval due to the competing causes and the experiment is terminated at $\tau_K$ time point. Let $N_i$ be the number of failures which take place in $(\tau_{i-1}, \tau_i]$ interval for $i = 1, \ldots, K$ where $\tau_0 = 0$. $N_i$ can be decomposed as $N_i = N_{i0} + N_{i1} + N_{i2}$, where $N_{i1}(N_{i2})$ is the number of failures due to cause 1 (cause 2) and $N_{i0}$ is the number of failure due to both the causes. Let $N_s$ be the censored units at the time point $\tau_K$, therefore, $N_s = n - \sum_{i=1}^{K} \sum_{l=0}^{2} N_{il}$.

It is evident that, $(N_{11}, N_{12}, N_{10}, \ldots, N_{K1}, N_{K2}, N_{K0}, N_s) \sim Multinomial(n, p)$, with the probability vector $p = (p_{11}, p_{12}, p_{10}, \ldots, p_{K1}, p_{K2}, p_{K0}, p_s)$, where for $i = 1, \ldots, K$,

$$p_{i1} = P(\tau_{i-1} < T_1 \leq \tau_i, T_2 > T_1)$$
$$= \int_{\tau_{i-1}}^{\tau_i} \int_{x_2}^{\infty} f_{MOBE}(x_1, x_2) I(x_1 < x_2) \, dx_1 \, dx_2$$
$$= \frac{\lambda_1}{\lambda} (e^{-\lambda \tau_{i-1}} - e^{-\lambda \tau_i})$$

$$p_{i2} = P(\tau_{i-1} < T_2 \leq \tau_i, T_1 > T_2)$$
$$= \frac{\lambda_2}{\lambda} (e^{-\lambda \tau_{i-1}} - e^{-\lambda \tau_i})$$

$$p_{i0} = P(\tau_{i-1} < T_1 = T_2 \leq \tau_i)$$
$$= \frac{\lambda_0}{\lambda} (e^{-\lambda \tau_{i-1}} - e^{-\lambda \tau_i}), \text{ and}$$

$$p_s = P(\min(T_1, T_2) > \tau_K)$$
$$= e^{-\lambda \tau_K}.$$

Based on the failure count data across the intervals, the likelihood function can be written as

$$L(\theta) \propto \left( \prod_{i=1}^{K} \prod_{l=0}^{2} p_{il}^{N_{il}} \right) \times p_s^{N_s}$$
$$= \frac{\lambda_1^{\sum_{i=1}^{K} N_{i1}} \lambda_2^{\sum_{i=1}^{K} N_{i2}} \lambda_0^{\sum_{i=1}^{K} N_{i0}}}{\lambda^{\sum_{i=1}^{K} N_i}} \times \prod_{i=1}^{K} \left( e^{-\lambda \tau_{i-1}} - e^{-\lambda \tau_i} \right)^{N_i} \times e^{-\lambda N_s \tau_K}.$$
where \( \theta = (\lambda_0, \lambda_1, \lambda_2)^T \).

Therefore, the log-likelihood can be written as
\[
    \begin{align*}
        l(\theta) &= \sum_{i=1}^{K} N_{i1} \log \lambda_1 + \sum_{i=1}^{K} N_{i2} \log \lambda_2 + \sum_{i=1}^{K} N_{i0} \log \lambda_0 - \sum_{i=1}^{K} N_i \log \lambda \\
        &\quad + \sum_{i=1}^{K} N_i \log \left( e^{-\lambda \tau_i - 1} - e^{-\lambda \tau_i} \right) - \lambda N_s \tau K.
    \end{align*}
\]

The estimating equations are
\[
    \sum_{i=1}^{K} N_{ij} \frac{1}{\lambda_j} - \sum_{i=1}^{K} N_i \frac{1}{\lambda} + \sum_{i=1}^{K} \frac{N_i (\tau_i e^{-\lambda \tau_i} - \tau_{i-1} e^{-\lambda \tau_{i-1}})}{(e^{-\lambda \tau_{i-1}} - e^{-\lambda \tau_i})} - N_s \tau K = 0
\]
for \( j = 0, 1, 2 \).

Though MLE is a very popular estimator due to its several properties like consistency, efficiency, it is not able to perform the analysis well in presence of outliers in the dataset. In the following section we will study a robust estimation method to obtain the estimates of the unknown parameter \( \theta \).

### 3 The Method of Density Power Divergence

Basu et.al [10] first developed the density power divergence method for robust estimation. They considered a parametric family of models with densities \( \{f_t\} \) with respect to Lebesgue measure where unknown parameter \( t \in \Omega \), which is the parameter space. With respect to the same measure, let \( G \) be the class of all distributions having densities \( g \). Under these assumptions they define the divergence between density functions \( g \) and \( f_t \) as
\[
    d_\beta(g, f_t) = \int \left\{ f_t^{1+\beta}(u) - (1 + \frac{1}{\beta})g(u) f_t^\beta(u) + \frac{1}{\beta} g^{1+\beta}(u) \right\} du, \quad \beta > 0. \tag{2}
\]

Note that when \( \beta \) tends to 0, \( d_\beta(g, f_t) \) tends to become the Kullback-Leibler divergence between \( g \) and \( f_t \). In case of having a random sample \( X_1, \ldots, X_n \), from \( G \), the true distribution \( G \) can be replaced by the empirical distribution and the minimum density power divergence estimate (MDPDE) is the value of the parameter \( t \), which will minimize
\[
    d_\beta(g, f_t) = \int f_t^{1+\beta}(u) du - (1 + \frac{1}{\beta}) \sum_{i=1}^{n} f_t^\beta(X_i).
\]
In our context, the density power divergence between the theoretical probability vector \( p = (p_{11}, p_{12}, p_{10}, \ldots, p_{K1}, p_{K2}, p_{K0}, p_s) \) and the empirical measure \( (N_{11}/n, N_{12}/n, \ldots, N_{K1}/n, N_{K2}/n, N_{K0}/n, N_{s}/n) \) can be obtained as,

\[
d_\beta = \sum_{i=1}^{K} \sum_{j=0}^{2} p_{ij}^{1+\beta} + p_s^{1+\beta} - \frac{1 + \beta}{\beta} \left[ \sum_{i=1}^{K} \sum_{j=0}^{2} \left( \frac{N_{ij}}{n} p_{ij}^\beta \right) + \frac{N_s}{n} p_s^\beta \right] + \frac{1}{\beta} \left[ \sum_{i=1}^{K} \sum_{j=0}^{2} \left( \frac{N_{ij}}{n} \right)^{1+\beta} + \left( \frac{N_s}{n} \right)^{1+\beta} \right].
\] (3)

Minimizing \( d_\beta \) with respect to the parameters \( \lambda_0, \lambda_1, \lambda_2 \) is equivalent as minimizing \( H_n(\beta) \) where,

\[
H_n(\beta) = \sum_{i=1}^{K} \sum_{j=0}^{2} p_{ij}^{1+\beta} + p_s^{1+\beta} - \frac{1 + \beta}{\beta} \left[ \sum_{i=1}^{K} \sum_{j=0}^{2} \left( \frac{N_{ij}}{n} p_{ij}^\beta \right) + \frac{N_s}{n} p_s^\beta \right].
\] (4)

Based on (3) and (4), the minimum density power divergence estimator of \( \theta = (\lambda_0, \lambda_1, \lambda_2)^T \) can be derived as

\[
\hat{\theta}_\beta = \arg\min_{\theta} H_n(\beta); \quad \beta > 0.
\]

The set of estimating equations can be obtained as

\[
\sum_{i=1}^{K} \sum_{j=0}^{2} p_{ij}^{1+\beta} \frac{\partial \log p_{ij}}{\partial \theta} + p_s^{1+\beta} \frac{\partial \log p_s}{\partial \theta} - \left[ \sum_{i=1}^{K} \sum_{j=0}^{2} \left( \frac{N_{ij}}{n} p_{ij}^\beta \frac{\partial \log p_{ij}}{\partial \theta} \right) + \frac{N_s}{n} p_s^\beta \frac{\partial \log p_s}{\partial \theta} \right] = 0.
\]

The estimating equations are unbiased and the estimator is Fisher consistent.

In the following result, the asymptotic distribution of the MDPD estimator is presented for Marshal-Olkin bivariate exponential distribution under competing risk set up.

**Result 1:** Let \( \theta_0 \) be the true value of parameter \( \theta \). The asymptotic distribution of the MDPD estimator \( \hat{\theta}_\beta \) is given by

\[
\sqrt{n}(\hat{\theta}_\beta - \theta_0) \sim N(0_3, J_\beta(\theta_0)^{-1} K_\beta(\theta_0) J_\beta(\theta_0)^{-1})
\]

where, \( J_\beta(\theta_0) \) and \( K_\beta(\theta_0) \) are defined in Appendix.

Proof: See in Appendix.

### 4 Robust test statistics for hypothesis testing:

In this section, a testing of hypothesis is developed based on the asymptotic distribution of the MDPDE on the same way as Wald test. Suppose, the null hypothesis is set as \( H_0 : \lambda_1 = \lambda_2 \) where
the alternative hypothesis $H_1 : \lambda_1 \neq \lambda_2$. Therefore, it can be rewritten as $H_0 : a_0^T \theta = 0$ where $a_0 = (0, 1, -1)^T$ or equivalently, $H_0 : \theta \in \Theta_0$ where $\Theta_0 = \{ \theta : a_0^T \theta = 0 \}$ and $H_1 : \theta \notin \Theta_0$.

When $\theta_0$ is the true value of $\theta$, $\sqrt{n}(\hat{\theta}_0 - \theta_0) \sim N(0, \Sigma(\theta_0))$ where $\Sigma(\theta_0) = J_\beta(\theta_0)^{-1}K_\beta(\theta_0)J_\beta(\theta_0)^{-1}$, and under $H_0$

$$\sqrt{n}a_0^T \hat{\theta}_0 \sim N\left(0, a_0^T \Sigma(\theta_0) a_0\right).$$

Therefore, the test statistic can be defined as,

$$M_n(\hat{\theta}_0) = n(a_0^T \hat{\theta}_0)^T\left(a_0^T \Sigma(\theta_0) a_0\right)^{-1}(a_0^T \hat{\theta}_0).$$

When $H_0$ is true, $M_n(\hat{\theta}_0) \sim \chi^2_1$ and at level $\alpha$, the rejection region can be obtained as $M_n(\hat{\theta}_0) \geq \chi^2_{1, \alpha}$ where $\chi^2_{1, \alpha}$ is the upper $\alpha$ percentile of $\chi^2_1$.

In this testing, the power function cannot be calculated explicitly. Therefore, an approximation method is being implemented as suggested by Basu et al [11]. Define,

$$m(\theta_1, \theta_2) = (a_0^T \theta_1)^T\left(a_0^T \Sigma(\theta_2) a_0\right)^{-1}(a_0^T \theta_1).$$

where $\theta_1, \theta_2$ are some values of $\theta$.

Taylor series expansion of $m(\hat{\theta}_0, \theta^*)$ at $\hat{\theta}_0$ around $\theta^*$ is given as

$$m(\hat{\theta}_0, \theta^*) = m(\theta^*, \theta^*) + \frac{\partial m(\theta_0, \theta^*)}{\partial \theta} \bigg|_{\theta = \theta^*} (\hat{\theta}_0 - \theta^*) + o_p(||\hat{\theta}_0 - \theta^*||).$$

When, $\theta = \theta^* \notin \Theta_0$,

$$\sqrt{n}\left(m(\hat{\theta}_0, \theta^*) - m(\theta^*, \theta^*)\right) \sim N(0, \sigma^2(\theta^*))$$

where, $\sigma^2(\theta^*) = \frac{\partial m(\theta, \theta^*)}{\partial \theta} \bigg|_{\theta = \theta^*} \Sigma(\theta^*) \frac{\partial m(\theta, \theta^*)}{\partial \theta} \bigg|_{\theta = \theta^*}$.

Therefore, the power of the test at $\theta = \theta^*$ can be obtained as

$$P\left(M_n(\hat{\theta}_0) \geq \chi^2_{1, \alpha}\right) = P\left(n\left(m(\hat{\theta}_0, \hat{\theta}_0) - m(\theta^*, \theta^*)\right) \geq \chi^2_{1, \alpha} - nm(\theta^*, \theta^*)\right)$$

$$= P\left(\frac{\sqrt{n}\left(m(\hat{\theta}_0, \hat{\theta}_0) - m(\theta^*, \theta^*)\right)}{\sigma(\theta^*)} \geq \frac{1}{\sigma(\theta^*)}\left(\frac{\chi^2_{1, \alpha}}{\sqrt{n}} - \sqrt{n}m(\theta^*, \theta^*)\right)\right)$$

$$= 1 - \Phi\left(\frac{1}{\sigma(\theta^*)}\left(\frac{\chi^2_{1, \alpha}}{\sqrt{n}} - \sqrt{n}m(\theta^*, \theta^*)\right)\right).$$
5 Robustness Property:

Robustness of any estimator can be expressed through its influence function (IF). This section presents the influence function of the MDPD point estimator and the Wald type test statistic.

5.1 Influence Function of MDPDE:

Suppose \( G \) is the true distribution from where data have been generated. If \( T_\beta(G) \) denotes the statistical functional of the MDPD \( \hat{\theta}_\beta \), then \( T_\beta(G) \) be the value of \( \theta \) which will minimize

\[
\sum_{i=1}^{k} \sum_{j=0}^{2} p_{ij}^{1+\beta} + p_{s}^{1+\beta} - \frac{1+\beta}{\beta} \left[ \sum_{i=1}^{k} \sum_{j=0}^{2} \left( \int_{I_{ij}} dG \right) p_{ij}^{\beta} + \int_{I_{s}} dG p_{s}^{\beta} \right].
\]

where \((x_1, x_2) \in I_{i1} \implies (\tau_{i-1} < x_1 \leq \tau_i, x_2 > x_1), (x_1, x_2) \in I_{i2} \implies (\tau_{i-1} < x_2 \leq \tau_i, x_1 > x_2), (x_1, x_2) \in I_{i0} \implies (\tau_{i-1} < x_1 = x_2 \leq \tau_i), \) for \( i = 1, \ldots, K \) and \((x_1, x_2) \in I_{s} \implies (x_1 > \tau_K, x_2 > \tau_K)\).

Therefore, \( T_\beta(G) \) will satisfy

\[
\sum_{i=1}^{k} \sum_{j=0}^{2} p_{ij}^{1+\beta} \frac{\partial \log p_{ij}}{\partial \theta} + p_{s}^{1+\beta} \frac{\partial \log p_{s}}{\partial \theta} - \left[ \sum_{i=1}^{k} \sum_{j=0}^{2} \left( \int_{I_{ij}} dG \right) p_{ij}^{\beta} \frac{\partial \log p_{ij}}{\partial \theta} + \int_{I_{s}} dG p_{s}^{\beta} \frac{\partial \log p_{s}}{\partial \theta} \right] = 0.\tag{5}
\]

Then the influence function is obtained as

\[
IF(x, T_\beta, G) = \lim_{\epsilon \to 0} \frac{T_\beta(G_\epsilon) - T_\beta(G)}{\epsilon} = \frac{\partial T_\beta(G_\epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0}.
\]

where \( x = (x_1, x_2), x_1, x_2 \in (0, \infty), G_\epsilon = (1-\epsilon)G + \epsilon \Delta_x \) and \( \Delta_x \) is the degenerate distribution with point mass 1 on \( x \). In the following result, the influence function is derived under the assumed set-ups.

Result 2: The IF of \( \hat{\theta}_\beta \), for the Marshal-Olkin bivariate exponential distribution under competing risk set up is given by

\[
IF(x, T_\beta, F_\theta) = J_\beta(\theta)^{-1} \left[ \sum_{i=1}^{k} \sum_{j=0}^{2} (\delta_{I_{ij}}(x) - p_{ij}) p_{ij}^{\beta} \frac{\partial \log p_{ij}}{\partial \theta} + (\delta_{I_s}(x) - p_{s}) p_{s}^{\beta} \frac{\partial \log p_{s}}{\partial \theta} \right].
\]

where \( \delta_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases} \)

Proof: See in Appendix.

The maximum of this Influence function over \( x \) indicates the extent of bias due to contamination. Therefore, smaller value of IF will indicate the estimator as more robust.
5.2 Influence function of Wald type test statistics:

The statistical functional of $M_n(\hat{\theta}_\beta)$ can be obtained as $M_n(T_\beta(G))$. Therefore the influence function of $M_n()$ is given by $IF(x, M_n, F_{\theta_0}) = \lim_{\epsilon \to 0} \frac{\partial M_n(T_\beta(G))}{\partial \epsilon}$ where, $G = (1 - \epsilon)F_{\theta_0} + \epsilon \Delta x$.

$$\frac{\partial M_n(T_\beta(G_\epsilon))}{\partial \epsilon} = \frac{\partial T_\beta^T(G_\epsilon)}{\partial \epsilon} a_0 \left( a_0^T \Sigma(\theta_0) a_0 \right)^{-1} \left( a_0^T T_\beta(G_\epsilon) \right)$$

$$+ (a_0^T T_\beta(G_\epsilon))^T \frac{\partial \left( a_0^T \Sigma(\theta_0) a_0 \right)}{\partial \epsilon} \left( a_0^T T_\beta(G_\epsilon) \right)$$

$$+ (a_0^T T_\beta(G_\epsilon))^T \left( a_0^T \Sigma(\theta_0) a_0 \right)^{-1} a_0^T \frac{\partial T_\beta(G_\epsilon)}{\partial \epsilon}$$

As, $T(F_{\theta_0}) = \theta_0 \in \Theta_0$ and $a_0^T \theta_0 = 0$, the first order influence function $IF(x, M_n, F_{\theta_0}) = 0$. The second order influence function is derived as $IF_2(x, M_n, F_{\theta_0}) = \lim_{\epsilon \to 0} \frac{\partial^2 M_n(T_\beta(G))}{\partial \epsilon^2}$.

$$\frac{\partial^2 M_n(T_\beta(G_\epsilon))}{\partial \epsilon^2} = \frac{\partial^2 T_\beta^T(G_\epsilon)}{\partial \epsilon^2} a_0 \left( a_0^T \Sigma(\theta_0) a_0 \right)^{-1} \left( a_0^T T_\beta(G_\epsilon) \right)$$

$$+ (a_0^T T_\beta(G_\epsilon))^T \frac{\partial^2 \left( a_0^T \Sigma(\theta_0) a_0 \right)}{\partial \epsilon^2} \left( a_0^T T_\beta(G_\epsilon) \right)$$

$$+ (a_0^T T_\beta(G_\epsilon))^T \left( a_0^T \Sigma(\theta_0) a_0 \right)^{-1} a_0^T \frac{\partial^2 T_\beta(G_\epsilon)}{\partial \epsilon^2}$$

$$+ 2 \frac{\partial T_\beta^T(G_\epsilon)}{\partial \epsilon} \frac{\partial \left( a_0^T \Sigma(\theta_0) a_0 \right)}{\partial \epsilon} \left( a_0^T T_\beta(G_\epsilon) \right)$$

$$+ 2 (a_0^T T_\beta(G_\epsilon))^T \frac{\partial \left( a_0^T \Sigma(\theta_0) a_0 \right)}{\partial \epsilon} a_0^T \frac{\partial T_\beta(G_\epsilon)}{\partial \epsilon}$$

$$+ 2 \frac{\partial^2 T_\beta^T(G_\epsilon)}{\partial \epsilon^2} a_0 \left( a_0^T \Sigma(\theta_0) a_0 \right)^{-1} a_0^T \frac{\partial T_\beta(G_\epsilon)}{\partial \epsilon}$$

Therefore, $IF_2(x, M_n, F_{\theta_0}) = 2 IF^T(x, T_\beta, F_{\theta_0}) a_0 \left( a_0^T \Sigma(\theta_0) a_0 \right)^{-1} a_0^T IF(x, T_\beta, F_{\theta_0})$.

6 Optimal inspection times

In this experimental set-up, the design parameter of the life testing experiment consists of $\mathcal{D} = (\tau_1, \ldots, \tau_K)$ for fixed sample size. To the experimenters, the concerned aspects influenced by the design are the cost of the experiment and the precision of the estimators of the model parameters. Increasing the precision of the estimators is equivalent to minimizing trace or determinant of the covariance matrix of the estimators. With low precision, the estimated values of the model
parameters are highly unreliable which may result in wrong prediction of the life time distribution. Hence, it is desirable to set the inspection times such that those concerned issues can be overcome. The optimal design is the set of time points which will determine the best experimental set-up based on some objectives defined by the experimenter.

In the literature, determination of optimal design or plan is found in wide spectrum of applications. In Balakrishnan and Han [3], the optimal plan was studied in application of accelerated life testing. In reliability analysis of one-shot devices optimal inspection times was determined in Ling [27]. Ng et al. [31] studied optimal plan in application of progressive censoring schemes. Recently, Bhattacharya [12] et al. studied multi-criteria based optimal life testing plan. This article presents determination of multi criteria based optimal design.

In this work, both the experiment cost and determinant of covariance matrix of the estimators will be minimized simultaneously. The experiment cost is defined in the following objective function as

$$\Phi_1 = C_0 + C_n n + C_f E(n - N_s)$$

where $C_n$ is the cost per unit put on in the experiment, $C_f$ indicates cost per failure in the experiment and $C_0$ is the unavoidable additive cost for the entire experiment. The second objective function is defined as

$$\Phi_2 = det \left( J_\beta(\theta_0)^{-1} K_\beta(\theta_0) J_\beta(\theta_0)^{-1} \right)$$

Therefore the overall optimization problem can be framed as

$$\min \left( \Phi_1, \Phi_2 \right)$$

subject to $\Phi_1 < C_1$, $\Phi_2 < C_2$, $\tau_1 < \tau_2 < \cdots < \tau_K$, and $\tau_K < \tau^*$,

where $C_1$ is the prefixed maximum budget that can be expended and $C_2$ is the pre-defined upper bound of the determinant of the covariance matrix, $\tau^*$ is the maximum permitted time length of the experiment.

This is a multi-objective optimization (MOO) problem. In MOO there is usually no single solution that is optimal with respect to all objectives. Consequently there are a set of optimal solutions,
known as Pareto optimal solutions. The Pareto optimal solution refers to a solution, around which there is no way of improving any objective without degrading at least one other objective. Without additional information, all these solutions are equally satisfactory. In this work, we apply Pareto Genetic Algorithm (GA). A Pareto GA returns a population with many solutions on the Pareto front which is the set of Pareto optimal solutions. The population is ordered based on dominance. Solution $D_1$ dominates solution $D_2$, if

$$\Phi_1(D_1) < \Phi_1(D_2) \text{ and } \Phi_2(D_1) \leq \Phi_2(D_2)$$

or $$\Phi_1(D_1) \leq \Phi_1(D_2) \text{ and } \Phi_2(D_1) < \Phi_2(D_2).$$

A solution is non-dominated if no solution can be found that dominates it.

In this work, we exploit a version of non-dominated sorting GA called NSGA-II proposed by Deb et. al [16]. Along with the determination of level of non-dominance, NSGA-II incorporates the density calculation to maintain a good spread of the solutions in the Pareto optimal set. The tools and the steps of the algorithm are discussed as follows.

**Non-dominance rank:** Let us define non-dominance rank say $i_{rank}$. At first stage each solution of the population can be compared with all the other solutions to check it is non-dominated or not. All the solutions which are non-dominated are assigned $i_{rank} = 1$ and temporarily removed from the population. Next, from the reduced population, we find all the non-dominated solutions and assign $i_{rank} = 2$. The process continues until all the solutions are assigned non-dominance rank.

**Rank based on crowding distance:** Let us define crowding distance rank say $i_{distance}$. The crowding-distance reflects the density of solutions surrounding a particular solution in the population. The crowding-distance computation requires sorting the population according to each objective function value in ascending order of magnitude. Let $I$ be any non-dominated set. For objective function $j$, $i_{j, distance}$ for solution $D_i$ is assigned as follows.

$$l = |I|$$

for each $D_i \in I$, $i_{j, distance} = 0$

Sort the solutions of $I$ according to the ascending order of magnitude of objective function $j$.

$$I = sort(I, \Phi_j).$$
In the sorted $\mathcal{I}$, define, $l_{\text{distance}} = l_{\text{distance}} = \infty$

for $i = 2, \ldots, l - 1$

$$i_{j, \text{distance}} = i_{j, \text{distance}} + \frac{\Phi_j(D_i - 1) - \Phi_j(D_i - 1)}{\Phi_j^{\text{max}} - \Phi_j^{\text{min}}}$$

where $\Phi_j^{\text{max}} (\Phi_j^{\text{min}})$ is the maximum (minimum) value of $\Phi_j$ in $\mathcal{I}$.

Based on all the objective functions for any solution $i_{\text{distance}} = \sum_{j=1}^{2} i_{j, \text{distance}}$

Based on these $i_{\text{rank}}$ and $i_{\text{distance}}$, Deb et al. [15], define partial order as $<_n$ such that

$$\mathcal{P}_{i_1} <_n \mathcal{P}_{i_2}$$

if $i_{1\text{rank}} < i_{2\text{rank}}$

or $i_{1\text{rank}} = i_{2\text{rank}}$ and $i_{1\text{distance}} > i_{2\text{distance}}$.

This indicates a lower non-dominance rank is preferred. Otherwise if non-dominance ranks are same, solution in lesser crowded region is preferred.

**Main Loop:** Initially, a population of size $N$, say $P_0$ is generated. Next through binary tournament selection, crossover, and mutation operations off-spring population say $Q_0$ is generated. Elitism is incorporated through $i_{\text{rank}}$ and $i_{\text{distance}}$ on the combined population.

At any $t$-th generation of the algorithm, off-spring $Q_t$ is generated from $P_t$. Next population $R_t$ is formed where $R_t = P_t \cup Q_t$ which is of size $2N$. In $R_t$ find out all the non-dominated solutions and store them in $F_1$. If $|F_1| < N$, next from $R_t - F_1$ find out all the non-dominated solutions and store them in $F_2$. Here, $|\cdot|$ denotes size of the set. If $|F_1 \cup F_2| < N$ we continue the process. Let $l$ be the minimum integer such that $|F_1 \cup F_2 \cup \ldots \cup F_{l-1} \cup F_l| > N$. Then, we sort the solutions of $F_l$ through the operator $<_n$ and choose the best $M = N - |F_1 \cup F_2 \cup \ldots \cup F_{l-1}|$ solutions. Based on the elitism, thus the next generation will be formed as $P_{t+1} = F_1 \cup F_2 \cup F_{l-1} \cup \{\text{best } M \text{ solutions of } F_l\}$.

**Binary Tournament selection:** In this algorithm, binary tournament selection is executed through the operator $<_n$. In presence of constraints, a solution can be feasible or infeasible. When two solutions are feasible, they can be compared with the partial order $<_n$. For one feasible and other one infeasible solution, the feasible one will be chosen. In presence of two infeasible solutions, the solution having small constraint violation will be chosen. In this context, the main constraint
is set as \( \tau_1 \leq \ldots \leq \tau_K \). For an infeasible solution this inequality will not be satisfied and the constraint violation is measured through \( \sum_{i=1}^{K} \sum_{j=1}^{i-1} \delta(\tau_i - \tau_{ij}) \). where, \( \delta(u) = \begin{cases} 1 & u < 0 \\ 0 & \text{otherwise} \end{cases} \).

**Crossover:** In this algorithm simulated binary crossover is exploited. The steps are given as follows.

Select two parents say \( D_{l1} \) and \( D_{l2} \).

Generate a random number \( u \sim U(0, 1) \). If \( u < P_c \), generate \( r \sim U(0, 1) \)

Compute \( \beta \) such that \( \beta = \begin{cases} (2r)^{\frac{1}{\eta_c+1}} & \text{if } r \leq 0.5 \\ \left(\frac{1}{2(1-r)}\right)^{\frac{1}{\eta_c+1}} & \text{otherwise} \end{cases} \)

where \( P_c \) is the crossover probability and \( \eta_c \) is called the distribution index. Large \( \eta_c \) tends to generate children closer to the parents and small \( \eta_c \) allows the children to be far from the parents.

Generated off-springs are

\[
D_{new}^{l1} = 0.5[(1 + \beta)D_{l1} + (1 - \beta)D_{l2}], \\
D_{new}^{l2} = 0.5[(1 - \beta)D_{l1} + (1 + \beta)D_{l2}].
\]

For detail study reader may refer to Deb and Agrawal [17].

**Mutation :** A polynomial mutation is implemented in the mutation operation. For a solution \( D = (\tau_1, \ldots, \tau_k) \), the mutation is operated as follows.

Set \( i=1 \).

Generate \( u \sim U(0, 1) \). If \( u < P_m \) set, \( \delta = \frac{\min(\tau_{i}^{upper} - \tau_{i}, \tau_{i} - \tau_{i}^{lower})}{\tau_{i}^{upper} - \tau_{i}^{lower}} \).

Generate \( r \sim U(0, 1) \).

Compute \( \delta_q = \begin{cases} [2r + (1 - 2r)(1 - \delta)(\eta_m+1)]^{\frac{1}{\eta_m+1}} - 1 & \text{if } r \leq 0.5 \\ 1 - [(2(1-r) + 2(r - 0.5)(1 - \delta)(\eta_m+1)]^{\frac{1}{\eta_m+1}} & \text{otherwise} \end{cases} \)

\( \tau_i = \tau_i + \delta_q(\tau_i^{upper} - \tau_i^{lower}) \).

Until \( i++ = K \).

Here \( P_m \) is the mutation probability and \( \tau_i^{upper} (\tau_i^{lower}) \) is the pre-defined upper (lower) bound of \( \tau_i \). For detailed study, the references are Deb et al [16], Hamdan [24].
7 NUMERICAL EXPERIMENT AND REAL DATA ANALYSIS:

7.1 NUMERICAL EXPERIMENT:

In this section a simulation study has been conducted using the Markov Chain Monte Carlo (MCMC) simulation based on 1000 generations to assess the performances of the developed methods. Under two dependent competing causes of failure following MOBE life time model, 20 subjects of interest are put on life testing experiment. Across different inspection time intervals taken as (0,0.2], (0.2,0.3], (0.3,0.4], the number of failures due to cause 1, cause 2 and both causes are recorded. The three different sets of model parameters are taken for the study and these model parameters are contaminated to study the robustness of DPDEs. Those sets are given in the Table [1].

Table 1: Model Parameters (Pure and Contaminated Data)

| S.No. | Pure Data  | Contaminated Data |
|-------|------------|-------------------|
|       | $\lambda_0$ | $\tilde{\lambda}_0$ |       |
| $\theta_1$ | 4.5 2.5 3.5 | $\lambda_0$-0.5 $\lambda_1$-0.6 $\lambda_2$-0.4 |       |
| $\theta_2$ | 6.3 2.1 4.2 | $\lambda_0$-0.8 $\lambda_1$-0.5 $\lambda_2$-0.6 |       |
| $\theta_3$ | 2.0 3.0 4.0 | $\lambda_0$-0.2 $\lambda_1$-0.1 $\lambda_2$-0.3 |       |

To obtain DPDEs and MLEs, Coordinate-Descent method is implemented using following steps.

- Start iteration process with the initial values $\theta^{(0)} = (\lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)})$ where at the $m + 1^{th}$ iteration, the estimate of the parameters can be derived as,

  $\lambda_0^{(m+1)} = \lambda_0^{(m)} - h \frac{\partial H(\lambda_0^{(m)}, \lambda_1^{(m)}, \lambda_2^{(m)})}{\partial \lambda_0}$

  $\lambda_1^{(m+1)} = \lambda_1^{(m)} - h \frac{\partial H(\lambda_0^{(m+1)}, \lambda_1^{(m)}, \lambda_2^{(m)})}{\partial \lambda_1}$

  $\lambda_2^{(m+1)} = \lambda_2^{(m)} - h \frac{\partial H(\lambda_0^{(m+1)}, \lambda_1^{(m+1)}, \lambda_2^{(m)})}{\partial \lambda_2}$
where $H = -l(\theta)$ for MLEs and $H = H_n(\beta)$ for DPDEs and $h$ is the learning rate taken here as $h = 0.01$.

- The process continues until $\{(\max |\theta^{(m+1)}_j - \theta^{(m)}_j|, \max |H(\theta^{(m+1)}) - H(\theta^{(m)})| ; j = 0, 1, 2) < c\}$ where $c$ is the threshold value chosen here as $0.0001$.

The Bias of MLEs and DPDEs are given in the Table 2 for the pure data and contaminated data scheme. It is observed that MLEs are highly affected by contamination as biases of MLEs are increased in contaminated data setting compared to pure data setting. But observing the bias of the DPDEs, it is evident that the DPDEs are unaffected by the contamination. The overall behaviour of the bias is that if tuning parameter $\beta$ increases, bias decreases. Therefore, higher value of $\beta ; 0 \leq \beta \leq 1$ is preferred for robustness of the DPDEs.

Table 2: Bias of MLE and DPDE (Pure Data and Contaminated Data)

| $\theta_1$ | Pure Data | Contaminated Data |
|------------|-----------|-------------------|
|            | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ |
| MLE        | -0.00055843 | -0.00064128 | -0.00082983 | -0.01949653 | -0.01623247 | -0.01194419 |
| $\beta = 0.2$ | -0.00031438 | -0.00091560 | -0.00086597 | -0.00091536 | -0.00013561 | -0.00052607 |
| $\beta = 0.4$ | 0.00000369 | -0.00001053 | 0.00001073 | -0.00012021 | -0.00002257 | -0.00000226 |
| $\beta = 0.6$ | -0.00002044 | 0.00000736 | 0.00001134 | -0.00007559 | -0.00001391 | 0.00000589 |
| $\beta = 0.8$ | 0.0000448 | -0.00000914 | 0.00000694 | -0.00005951 | 0.00000260 | 0.00000970 |
| $\beta = 1.0$ | -0.0000273 | 0.00000454 | 0.00000358 | -0.00004392 | 0.00000707 | 0.0000160 |

| $\theta_2$ | Pure Data | Contaminated Data |
|------------|-----------|-------------------|
|            | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ |
| MLE        | -0.00087963 | -0.00073572 | -0.00041731 | -0.01227007 | -0.01675230 | -0.01325183 |
| $\beta = 0.2$ | -0.0001251 | -0.0001281 | 0.00001422 | -0.00010689 | -0.00004566 | -0.00003516 |
| $\beta = 0.4$ | 0.00001188 | -0.00001086 | 0.00000451 | -0.00007924 | -0.00001869 | -0.00001065 |
| $\beta = 0.6$ | 0.0000152 | 0.00000194 | 0.00000558 | -0.00005309 | -0.00000735 | -0.00000435 |
| $\beta = 0.8$ | -0.0000394 | -0.0000406 | 0.00000857 | -0.00003254 | -0.0000457 | -0.0000088 |
| $\beta = 1.0$ | -0.0000047 | 0.00000138 | 0.00000177 | -0.00002687 | -0.00000164 | 0.00000051 |

| $\theta_3$ | Pure Data | Contaminated Data |
|------------|-----------|-------------------|
|            | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ |
| MLE        | 0.00045100 | -0.00453577 | -0.00632121 | -0.01077694 | -0.01012153 | -0.01448080 |
| $\beta = 0.2$ | -0.00041102 | -0.00003321 | -0.00025326 | -0.00019593 | -0.00004566 | -0.00003516 |
| $\beta = 0.4$ | 0.00001188 | -0.00001086 | 0.00000451 | -0.00007924 | -0.00001869 | -0.00001065 |
| $\beta = 0.6$ | 0.0000152 | 0.00000194 | 0.00000558 | -0.00005309 | -0.00000735 | -0.00000435 |
| $\beta = 0.8$ | 0.00000349 | -0.0000406 | 0.00000857 | -0.00003254 | -0.0000457 | -0.0000088 |
| $\beta = 1.0$ | 0.00000407 | 0.00000138 | 0.00000177 | -0.00002687 | -0.00000164 | 0.00000051 |
The approximated power of the Wald type test using various set of parameters are calculated which are given in the Table (3). From this table it can be observed that as the difference between \( \lambda_1 \) and \( \lambda_2 \) increases, power of the test increases. It is also observed that as the value of tuning parameter increases, power of the test gradually decreases.

Table 3: Power of the test

| Parameters | \( \beta = 0.2 \) | \( \beta = 0.4 \) | \( \beta = 0.6 \) | \( \beta = 0.8 \) | \( \beta = 1.0 \) |
|------------|------------------|------------------|------------------|------------------|------------------|
| \( \lambda_0 \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_0 \) | \( \lambda_1 \) | \( \lambda_2 \) |
| 4.5 | 2.5 | 3.0 | 0.6024 | 0.6012 | 0.5999 | 0.5983 | 0.5967 |
| 6.3 | 2.0 | 3.5 | 0.7405 | 0.7376 | 0.7345 | 0.7315 | 0.7286 |
| 4.5 | 2.5 | 4.0 | 0.7634 | 0.7615 | 0.7591 | 0.7563 | 0.7533 |
| 4.5 | 2.5 | 5.5 | 0.8969 | 0.8959 | 0.8942 | 0.8922 | 0.8899 |
| 6.3 | 2.0 | 5.5 | 0.9025 | 0.9004 | 0.8979 | 0.8953 | 0.8927 |

To obtain the optimum inspection time points which would determine the best experimental set-up based on objectives discussed in Section 6, we have set \( n = 20 \), \( P_c = 0.9 \), \( \eta_c = 20 \), \( P_m = \frac{1}{K} \), \( K=3 \) and \( \eta_m = 20 \). The parameter values are chosen as \( \lambda_0 = 0.15, \lambda_1 = 0.02, \lambda_2 = 0.07 \) with tuning parameter \( \beta = 0.5 \) and the population size is set as 50. Here, we set \( \tau_{i}^{upper} = 70 \) and \( \tau_{i}^{lower} = 0 \) for \( i = 1, 2, 3 \). The Pareto optimal solutions for the initial population and after 100 generations are given in the Table (4). For the initial population, 4 Pareto optimal solutions are obtained where for the 100\(^{th} \) generation, number Pareto optimal solutions is increased to 42.

7.2 Real Data Analysis:

For the real life implementation of the results obtained in the previous sections, an analysis has been performed on the bivariate data taken from Ebrahimi [21]. For the inspection time intervals (0,0.032], (0.032,0.12], (0.12,0.23], failure time data due to cause 1, cause 2 and both causes are recorded for first 30 observations from the data set found in Ebrahimi [21]. The failure time values are divided by 10 for the ease of computation. The description of failure time data is given in the Table (5). Failure due to cause 1, cause 2 and both causes are indicated as (1,2,0), respectively.

To check whether Marshall-Olkin Bivariate Exponential distribution fits the data, a bootstrap
### Table 4: Optimal Time Points

| S.No. | Optimal Time Points          | S.No. | Optimal Time Points          |
|-------|-----------------------------|-------|-----------------------------|
|       | $\tau_1$ | $\tau_2$ | $\tau_3$ |       | $\tau_1$ | $\tau_2$ | $\tau_3$ |
| 1     | 6.428751 | 42.670939 | 65.692024 | 2     | 9.211211 | 15.385411 | 29.028185 |
| 2     | 5.390008 | 19.465359 | 57.243752 | 4     | 8.516763 | 34.996492 | 41.205904 |

**Pareto front of Initial Population**

| S.No. | Optimal Time Points          |
|-------|-----------------------------|
| 1     | 8.037446 | 19.707918 | 19.817200 |
| 3     | 4.919766 | 45.716619 | 45.805101 |
| 5     | 4.932801 | 50.605646 | 50.634229 |
| 7     | 4.932801 | 50.605646 | 50.634229 |
| 9     | 4.906357 | 45.962833 | 45.805101 |
| 11    | 8.036706 | 19.707918 | 19.817200 |
| 13    | 8.056663 | 19.513107 | 19.015796 |
| 15    | 6.232382 | 36.185958 | 37.218926 |
| 17    | 4.932801 | 50.605646 | 50.634229 |
| 19    | 4.922787 | 43.050969 | 43.106004 |
| 21    | 4.93795  | 42.96136  | 42.913691 |
| 23    | 4.933422 | 50.882763 | 50.907122 |
| 25    | 4.948874 | 44.928987 | 45.045212 |
| 27    | 6.205449 | 37.715003 | 37.990882 |
| 29    | 4.927668 | 43.027484 | 43.082879 |
| 31    | 4.972067 | 45.83462  | 45.944780 |
| 33    | 6.284095 | 35.913353 | 36.985382 |
| 35    | 6.22404  | 36.11990  | 37.90714  |
| 37    | 5.071818 | 42.023778 | 42.166894 |
| 39    | 5.02793  | 42.07590  | 42.34212  |
| 41    | 5.069286 | 42.054582 | 42.301416 |

**Pareto front of 100th Generation**

| S.No. | Optimal Time Points          |
|-------|-----------------------------|
| 1     | 8.037446 | 19.707918 | 19.817200 |
| 3     | 4.919766 | 45.716619 | 45.805101 |
| 5     | 4.932801 | 50.605646 | 50.634229 |
| 7     | 4.932801 | 50.605646 | 50.634229 |
| 9     | 4.906357 | 45.962833 | 45.805101 |
| 11    | 8.036706 | 19.707918 | 19.817200 |
| 13    | 8.056663 | 19.513107 | 19.015796 |
| 15    | 6.232382 | 36.185958 | 37.218926 |
| 17    | 4.932801 | 50.605646 | 50.634229 |
| 19    | 4.922787 | 43.050969 | 43.106004 |
| 21    | 4.93795  | 42.96136  | 42.913691 |
| 23    | 4.933422 | 50.882763 | 50.907122 |
| 25    | 4.948874 | 44.928987 | 45.045212 |
| 27    | 6.205449 | 37.715003 | 37.990882 |
| 29    | 4.927668 | 43.027484 | 43.082879 |
| 31    | 4.972067 | 45.83462  | 45.944780 |
| 33    | 6.284095 | 35.913353 | 36.985382 |
| 35    | 6.22404  | 36.11990  | 37.90714  |
| 37    | 5.071818 | 42.023778 | 42.166894 |
| 39    | 5.02793  | 42.07590  | 42.34212  |
| 41    | 5.069286 | 42.054582 | 42.301416 |

Based testing has been conducted. The test statistic is defined as $S = (\sum_{i=1}^{K} |N_{ij} - E_{ij}| + |N_s - E_s|)$ where $N_{ij}$'s are the number of observed failures and $E_{ij}$'s are the number of expected failures in time interval $(\tau_{i-1}, \tau_i]$ due to cause $j$ for $i = 1, \ldots, K$ and $j = 0, 1, 2$. $N_s$ and $E_s$ respectively, are the number of observed and expected survived units at the time point $\tau_K$. The MLE or DPDE method can be used to estimate $E_{ij} = n \times p_{ij}; j = 0, 1, 2$ and $E_s = n \times p_s$.

The MLEs and DPDEs of the model parameters based on the real data set are given in Table (7). Based on those estimated values of the model parameters, 10000 bootstrap samples are generated and in each bootstrap sample, we compute the test statistic. The count of the bootstrapped test statistics greater than the real data based test statistic $S$ divided by the number of bootstrap
Table 5: Failure Time Data

| Failure Time | Cause | Failure Time | Cause | Failure Time | Cause |
|--------------|-------|--------------|-------|--------------|-------|
| 0.610        | 1     | 0.150        | 2     | 0.170        | 0     |
| 0.017        | 2     | 0.180        | 0     | 0.034        | 1     |
| 0.105        | 1     | 0.042        | 2     | 0.030        | 2     |
| 0.223        | 1     | 0.250        | 2     | 0.130        | 0     |
| 0.397        | 0     | 0.010        | 1     | 0.080        | 2     |
| 0.047        | 1     | 0.036        | 1     | 0.080        | 0     |
| 0.004        | 0     | 0.006        | 2     | 0.250        | 2     |
| 0.016        | 0     | 0.070        | 2     | 0.092        | 2     |
| 0.046        | 0     | 0.030        | 2     | 0.027        | 0     |
| 0.047        | 1     | 0.002        | 1     | 0.106        | 2     |

The sample is the approximate p-value. The values of the real data based test statistics and the corresponding approximate p-values using MLE and DPDE for different tuning parameters ($\beta$) are given in Table (6). The obtained approximated p-values indicate that MOBE can be applied as the lifetime distribution for this real data set.

Table 6: Approximate p-value Calculation

| $\theta$   | Test Statistic | Approximate p-value |
|------------|----------------|---------------------|
| MLE        | 15.03554       | 0.2213              |
| $\beta = 0.2$ | 15.03560   | 0.2084              |
| $\beta = 0.4$ | 15.03565   | 0.2178              |
| $\beta = 0.6$ | 15.03570   | 0.2118              |
| $\beta = 0.8$ | 15.03573   | 0.2168              |
| $\beta = 1.0$ | 15.03576   | 0.2235              |

The MLEs and DPDEs are calculated using the Coordinate Descent algorithm. In this algorithm, the initial values of the model parameters ($\lambda_0 = 3.5, \lambda_1 = 1.5, \lambda_2 = 2.5$) are obtained through grid-search procedure. Also bootstrap estimates of the bias (BT Bias) are computed for each of the estimators which are reported in Table (7).
Table 7: MLEs and DPDEs and Bootstrap Estimates of Bias

| θ   | λ_0 Estimate | BT Bias | λ_1 Estimate | BT Bias | λ_2 Estimate | BT Bias |
|-----|--------------|---------|--------------|---------|--------------|---------|
| MLE | 3.500992     | -0.027849 | 1.500634     | -0.017526 | 2.499711     | -0.007305 |
| β = 0.2 | 3.500711 | -0.021686 | 1.500497     | -0.012530 | 2.499757     | -0.001851 |
| β = 0.4 | 3.500498 | -0.024127 | 1.500381     | -0.013548 | 2.499802     | -0.000306 |
| β = 0.6 | 3.500344 | -0.019039 | 1.500287     | -0.008495 | 2.499841     | -0.001805 |
| β = 0.8 | 3.500236 | -0.019323 | 1.500214     | -0.013260 | 2.499874     | -0.000929 |
| β = 1.0 | 3.500161 | -0.017740 | 1.500158     | -0.010565 | 2.499902     | -0.003314 |

8 Conclusion

In this work, a robust estimation method has been developed to estimate the lifetime distribution under two dependent competing risks which is modelled by Marshall-Olkin bivariate exponential distribution. The point estimation has been studied through robust minimum density power divergence estimator (MDPDE) and also we have computed maximum likelihood estimators (MLE). Testing of hypothesis has been performed through Wald type test statistic based on asymptotic distribution of MDPDE. It is observed through simulation study that MLEs provide misleading results in presence of contamination while MDPDEs remain unaffected by contamination. The influence function of the MDPDE and the test statistic have also been derived which measures the robustness analytically. In the study of power of wald type test, it is observed that for $H_0 : \lambda_1 = \lambda_2$, power would be high if difference between $\lambda_1$ and $\lambda_2$ is high and value of tuning parameter is small.

In determination of optimal inspection times, Pareto Genetic Algorithm has been applied and it is observed that by increasing the number of generations, number of Pareto optimal solutions are also increased. Real data analysis has also been conducted to assess the performances of the theoretical results in practical situations.

The model analysed here can be studied by incorporating covariates and can also be extended to the analysis of missing information on covariates, masked cause of failures etc. Same study can be conducted taking other lifetime distributions and model can be applied on some particular situations like reliability analysis of one-shot devices. Efforts in this direction is under way and we
would report these findings as soon as possible.

**Conflict of interest**

The authors declare that they have no conflict of interest.

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Appendix:

Proof of Result 1:

Here, true value of $\theta$ is denoted by $\theta_0 = (\lambda_{00}, \lambda_{10}, \lambda_{20})^T$ and MDPDE $\hat{\theta}_\beta = (\lambda_{00}, \lambda_{31}, \lambda_{32})^T$.

Define $M = 3 \times K + 1$ and $p_t = p_{3(i-1)+j+1}$ and $N_i = N_{3(i-1)+j+1}$ for $j = 0, 1, 2$ and $i = 1, \ldots, M - 1$, $p_M = p_s$, $N_M = N_s$.

Therefore, $H_n(\beta)$ can be expressed as

$$H_n(\beta) = \sum_{i=1}^{M} p_i^{\beta+1} - \frac{1 + \beta}{\beta} \sum_{i=1}^{M} \left( \frac{N_i}{n} p_i^\beta \right).$$

Define, $X_s = (X_{s1}, X_{s2}, \ldots, X_{sM}) \sim Multinomial(1, p_1, p_2, \ldots, p_M)$. Therefore $N_i$ can be expressed as $N_i = \sum_{s=1}^{n} X_{sl}$ and $H_n(\beta)$ can be re-written as

$$H_n(\beta) = \frac{1}{n} \sum_{s=1}^{n} \left( \sum_{l=1}^{M} p_l^{\beta+1} - \frac{1 + \beta}{\beta} \sum_{l=1}^{M} X_{sl} p_l^\beta \right)$$

where, $V_\beta(X_s, \theta) = \sum_{l=1}^{M} p_l^{\beta+1} - \frac{1 + \beta}{\beta} \sum_{l=1}^{M} X_{sl} p_l^\beta$.

Denote, $H_{nj} = \frac{\partial H_n(\beta)}{\partial \lambda_j} = \frac{1}{n} \sum_{s=1}^{n} \frac{\partial V_\beta(X_s, \theta)}{\partial \lambda_j}$ for $j = 0, 1, 2$. Here, we get $E(\frac{\partial V_\beta(X_s, \theta)}{\partial \lambda_j}) = 0$ and

$$Var\left(\frac{\partial V_\beta(X_s, \theta)}{\partial \lambda_j}\right) = (\beta + 1)^2 Var\left(\sum_{l=1}^{M} X_{sl} p_l^{\beta-1} \frac{\partial p_l}{\partial \lambda_j}\right)$$

$$= (\beta + 1)^2 \left( \sum_{l=1}^{M} p_l^{2(\beta-1)} p_l (1 - p_l) \frac{\partial p_l}{\partial \lambda_j} \right)^2 - 2 \sum_{1 \leq l_1 < l_2 \leq M} p_l^{(\beta-1)} p_{l_2}^{(\beta-1)} p_{l_1} p_{l_2} \frac{\partial p_{l_1}}{\partial \lambda_{j_1}} \frac{\partial p_{l_2}}{\partial \lambda_{j_2}}$$

for $j = 0, 1, 2$.

$$Cov\left(\frac{\partial V_\beta(X_s, \theta)}{\partial \lambda_{j_1}}, \frac{\partial V_\beta(X_s, \theta)}{\partial \lambda_{j_2}}\right) = (\beta + 1)^2 \left( \sum_{l=1}^{M} p_l^{2(\beta-1)} p_l (1 - p_l) \frac{\partial p_l}{\partial \lambda_{j_1}} \frac{\partial p_l}{\partial \lambda_{j_2}} \right)$$

$$- 2 \sum_{1 \leq l_1 < l_2 \leq M} p_l^{(\beta-1)} p_{l_2}^{(\beta-1)} p_{l_1} p_{l_2} \frac{\partial p_{l_1}}{\partial \lambda_{j_1}} \frac{\partial p_{l_2}}{\partial \lambda_{j_2}}$$
for $j_1, j_2 = 0, 1, 2$ and $j_1 \neq j_2$.

Define matrix, $K_\beta(\theta)$ where $K_\beta(\theta)_{jj} = \frac{1}{(\beta+1)^2} Var\left(\frac{\partial V_\beta(X, \theta)}{\partial \lambda_j}\right)$ and 

$$K_\beta(\theta)_{j_1 j_2} = \frac{1}{(\beta+1)^2} Cov\left(\frac{\partial V_\beta(X, \theta)}{\partial \lambda_{j_1}}, \frac{\partial V_\beta(X, \theta)}{\partial \lambda_{j_2}}\right)$$

for $j = 0, 1, 2$ and $j_1, j_2 = 0, 1, 2; j_1 \neq j_2$.

Define, $T_{n\beta} = (T_{0n\beta}, T_{1n\beta}, T_{2n\beta})$ where,

$$T_{0n\beta} = -\sqrt{n} H_{n0\beta}(\theta_0) = -\sqrt{n} \frac{\partial H_{n0\beta}(\theta)}{\partial \lambda_0} |_{\theta = \theta_0},$$

$$T_{1n\beta} = -\sqrt{n} H_{n1\beta}(\theta_0) = -\sqrt{n} \frac{\partial H_{n1\beta}(\theta)}{\partial \lambda_1} |_{\theta = \theta_0},$$

$$T_{2n\beta} = -\sqrt{n} H_{n2\beta}(\theta_0) = -\sqrt{n} \frac{\partial H_{n2\beta}(\theta)}{\partial \lambda_2} |_{\theta = \theta_0}.$$

Applying Central Limit Theorem, $T_{n\beta} \sim N(0, (\beta+1)^2 K_\beta(\theta_0))$.

Next, we get,

$$\frac{\partial^2 V_\beta(X, \theta)}{\partial \lambda_{j_1} \partial \lambda_{j_2}} = \sum_{l=1}^{M} \left[ (\beta + 1) \beta p_l^{\beta-1} \frac{\partial p_l}{\partial \lambda_{j_1}} \frac{\partial p_l}{\partial \lambda_{j_2}} + (\beta + 1) p_l^\beta \frac{\partial^2 p_l}{\partial \lambda_{j_1} \partial \lambda_{j_2}} \right]$$

$$- \sum_{l=1}^{M} \left[ (\beta + 1)(\beta - 1) X_s p_l^{\beta-2} \frac{\partial p_l}{\partial \lambda_{j_1}} \frac{\partial p_l}{\partial \lambda_{j_2}} + (\beta + 1) X_s p_l^{\beta-1} \frac{\partial^2 p_l}{\partial \lambda_{j_1} \partial \lambda_{j_2}} \right]$$

and $\frac{1}{n} \sum_{s=1}^{n} X_s \to^P p_l$.

Therefore, it is evident that, $\frac{\partial H_{n1\beta}}{\partial \lambda_{j_2}} \to^P (\beta + 1) \sum_{l=1}^{M} \left( p_l^{\beta-1} \frac{\partial p_l}{\partial \lambda_{j_1}} \frac{\partial p_l}{\partial \lambda_{j_2}} \right)$.

Taylor series expansion of $H_{n\beta}(\theta)$ around $\theta_0$ gives

$$H_{n\beta}(\theta) = H_{n\beta}(\theta_0) + \sum_{k=0}^{2} \frac{\partial H_{n\beta}(\theta)}{\partial \lambda_k} |_{\theta = \theta_0} (\lambda_k - \lambda_{k0})$$

$$+ \frac{1}{2} \sum_{j_1=0}^{2} \sum_{j_2=0}^{2} \frac{\partial^2 H_{n\beta}(\theta)}{\partial \lambda_{j_1} \partial \lambda_{j_2}} |_{\theta = \theta_0} (\lambda_{j_1} - \lambda_{j_10})(\lambda_{j_2} - \lambda_{j_20}).$$

As, $H_{n\beta}(\hat{\theta}_\beta) = 0$, it can be written that

$$-\sqrt{n} H_{n\beta}(\theta_0) = \sqrt{n} \sum_{k=0}^{2} \left[ \frac{\partial H_{n\beta}(\theta)}{\partial \lambda_k} |_{\theta = \theta_0} + \frac{1}{2} \sum_{j_1=0}^{2} \frac{\partial^2 H_{n\beta}(\theta)}{\partial \lambda_k \partial \lambda_{j_1}} |_{\theta = \theta_0} (\hat{\lambda}_{j_1} - \lambda_{j_10}) \right] (\lambda_{k} - \lambda_{k0}).$$

Define, $A_{jkn\beta} = \frac{\partial H_{n\beta}(\theta)}{\partial \lambda_k} |_{\theta = \theta_0} + \frac{1}{2} \sum_{j_1=0}^{2} \frac{\partial^2 H_{n\beta}(\theta)}{\partial \lambda_k \partial \lambda_{j_1}} |_{\theta = \theta_0} (\hat{\lambda}_{j_1} - \lambda_{j_10})$ and it is easy to show that $A_{jkn\beta} \to^P (\beta + 1) \sum_{l=1}^{M} p_l^{\beta-1} \frac{\partial p_l}{\partial \lambda_{j_1}} \frac{\partial p_l}{\partial \lambda_k}$. 

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Define, 3x3 matrix $A_{n}\beta$ with $j$, $k$th element $A_{jkn}\beta$. $A_{n}\beta \overset{P}{\rightarrow} (\beta + 1)J_{\beta}(\theta_{0})$ where

$$J_{\beta}(\theta_{0}) = ((\sum_{l=1}^{M} p_{l}^{\beta-1} \frac{\partial p_{l}}{\partial \lambda_{j1}} \frac{\partial p_{l}}{\partial \lambda_{j2}})_{j, j}). \quad (6)$$

Define, $Z_{kn}\beta = \sqrt{n}(\hat{\lambda}_{3k} - \lambda_{k0})$, for $k = 0, 1, 2$. Therefore, $T_{jn}\beta$ can be expressed as $T_{jn}\beta = \sum_{k=0}^{2} A_{jkn}\beta Z_{kn}\beta$. Denote, $Z_{n}\beta = (Z_{0n}\beta, Z_{1n}\beta, Z_{2n}\beta)$ and therefore, we obtain, $T_{n}\beta = A_{n}\beta Z_{n}\beta$.

It can be expressed that $Z_{n}\beta = A_{n}\beta^{-1}T_{n}\beta \implies \sqrt{n}(\hat{\theta}_{\beta} - \theta_{0}) = Z_{n}\beta \sim N(0, J_{\beta}^{-1}(\theta_{0})K_{\beta}(\theta_{0})J_{\beta}^{-1}(\theta_{0}))$.

Proof of Result 2:

In (5), replacing $G$ by $G_{\epsilon} = (1 - \epsilon)F_{\theta} + \epsilon \Delta x$, differentiating with respect to $\epsilon$, and taking $\epsilon \rightarrow 0$, we obtain

$$\left[A_{\beta}(\theta) + (\beta + 1)J_{\beta}(\theta)\right]IF - \left[A_{\beta}(\theta) + \beta J_{\beta}(\theta)\right]IF = \sum_{i=1}^{k} \sum_{j=0}^{2} \left(\int_{I_{ij}} d\Delta x - P_{ij}\right) \frac{\partial \log p_{ij}}{\partial \theta} + \left(\int_{I_{s}} d\Delta x - P_{s}\right) \frac{\partial \log p_{s}}{\partial \theta}$$

$$\implies IF = J_{\beta}^{-1}(\theta) \sum_{i=1}^{k} \sum_{j=0}^{2} (\delta_{I_{ij}}(x) - P_{ij}) \frac{\partial \log p_{ij}}{\partial \theta} + (\delta_{I_{s}}(x) - P_{s}) \frac{\partial \log p_{s}}{\partial \theta}.$$ 

Here, $A_{\beta}(\theta) = ((\sum_{i=1}^{k} \sum_{j=0}^{2} P_{ij}^{\beta+1} \frac{\partial^{2} \log p_{ij}}{\partial \theta_{1} \partial \theta_{1}} + P_{s}^{\beta+1} \frac{\partial^{2} \log p_{s}}{\partial \theta_{1} \partial \theta_{1}})_{l, l}, J_{\beta}(\theta)$ as defined in (6) and $\delta_{A}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$