The Complex Bateman Equation in a space of arbitrary dimension

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Abstract

A general solution to the Complex Bateman equation in a space of arbitrary dimensions is constructed.
1 Introduction

We define the Complex Bateman equation in \( n \)-dimensional space by analogy with the so-called Universal Field Equation [1] as

\[
\begin{vmatrix}
0 & \frac{\partial \phi}{\partial y_1} & \cdots & \frac{\partial \phi}{\partial y_n} \\
\frac{\partial \phi}{\partial y_1} & \frac{\partial^2 \phi}{\partial y_1 \partial y_1} & \cdots & \frac{\partial^2 \phi}{\partial y_1 \partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \phi}{\partial y_n} & \frac{\partial^2 \phi}{\partial y_n \partial y_1} & \cdots & \frac{\partial^2 \phi}{\partial y_n \partial y_n}
\end{vmatrix} = 0.
\] (1)

Recently it was shown that the general solution to this equation, in the case where \( n = 2 \) is given implicitly, by equating two arbitrary functions of three variables, \( F(\phi, y_1, y_2) \) and \( G(\phi, \bar{y}_1, \bar{y}_2) \) and solving the resulting equation

\[ F(\phi, y_1, y_2) = G(\phi, \bar{y}_1, \bar{y}_2). \] (2)

for \( \phi(y_1, y_2, \bar{y}_1, \bar{y}_2) \). This assertion may be readily verified by partial differentiation. The corresponding result for (1) is the subject of this article.

2 Equivalent First Order Equations

The complex Bateman equation, (1) is the eliminant of \( (n + 1) \) linear equations which may be written as:

\[
\sum_{i=1}^{n} \alpha_i \phi_{\bar{y}_i} = 0, \quad \phi_{y_s} = \sum_{i=1}^{n} \alpha_{y_s} \phi_{y_i}
\] (3)

where \( \phi_{y_i} \) denotes \( \frac{\partial \phi}{\partial y_i} \) etc. Similarly it is also the eliminant of the equations

\[
\sum_{i=1}^{n} \beta_i \phi_{y_i} = 0, \quad \phi_{y_s} = \sum_{i=1}^{n} \beta_{y_s} \phi_{y_i}.
\] (4)

From (3) and (4) it follows that:

\[
\sum_s \alpha_{y_s} \beta_{\bar{y}_s}^k = \delta^k_i,
\] (5)
or in other words the Jacobian matrices $\alpha_{yi}$ and $\beta_{yi}$ introduced above are inverses of one another. Let us multiply and sum each equation of the first system (3) by $\beta_i$ (and do the same for system (4)). We obtain:

$$\sum_{i=1}^{n} \beta_i \alpha_{yi} \phi_{yi} = 0, \quad \sum_{i=1}^{n} \alpha_i \beta_{yi} \phi_{yi} = 0$$ (6)

Now we come to a crucial step; the equations (6) cannot contain any new information, but will only repeat those equations already quoted. In other words

$$\sum_i \beta_i \alpha_{y_i} = \theta \alpha_s$$ (7)

for some $\theta$ from which follows immediately (taking into account the inverse properties of the matrices introduced above) in symmetrical fashion,

$$\sum_i \alpha_i \beta_{y_i} = \theta^{-1} \beta_s$$ (8)

which is consistent with the other set of equations. The direct proof of this proposal will be given in the next section.

Dividing these last equations respectively by $\alpha_s, \beta_s$ and introducing the notation $v^\nu = \frac{\alpha^\nu}{\alpha_n}, u^\mu = \frac{\beta^\mu}{\beta_n}$ with the convention that Greek indices take values from 1 up to $n - 1$ we can eliminate $\theta$ by subtracting the last equation to arrive at the following system:

$$-v_{y_n}^\nu = \sum u^\mu v_{y_n}^\mu, \quad -u_{y_n}^\mu = \sum v^\nu u_{y_n}^\nu.$$ (9)

In the familiar case of two dimensional space this system takes the form

$$-v_{y_1} = u_{y_2}, \quad -u_{y_1} = v_{y_2}$$ (10)

the general solution of which is connected with the equation

$$G(\phi; y_1, y_2) = F(\phi; \bar{y}_1, \bar{y}_2)$$ (11)

We therefore expect that in the general case of arbitrary dimensions the general solution of the Complex Bateman equation (1) is connected in some a way with a system of $n - 1$ equations for $n - 1$ unknown functions $\psi^\mu$:

$$Q^\nu(\psi^\mu; y_1, ..y_n) = P^\nu(\psi^\mu; \bar{y}_1, ..\bar{y}_n) \quad \nu = 1 \ldots n - 1.$$ (12)
3 Conditions of selfconsistency

As a direct corollary of (3), (4) it follows that the function $\phi$ can be thought of as either depending upon the set of variables $(u^\mu; y_i)$, or else $(v^\nu; \bar{y}_j)$

$$\phi(v^\nu; \bar{y}_j) = \phi(u^\mu; y_j)$$

This result is equivalent to using two equations from (3) and (4). The remaining $2n$ equations, taking into account (13) may be transformed in the following way:

$$\phi_{y_s} = \sum \alpha_i \phi_{u^\mu} u^\mu_{y_i} = \sum \alpha_i \phi_{u^\mu} \left( \frac{\beta^\mu_{\bar{y}_j}}{\beta^n} - \frac{\beta^\mu\beta^n_{\bar{y}_j}}{(\beta^n)^2} \right)$$

$$= \frac{1}{\beta^n} \sum \phi_{u^\mu} (\delta_{sp} - u^\mu \delta_{sn})$$

(14)

Or finally:

$$\phi_{y_\mu} = \frac{1}{\beta^n} \phi_{u^\mu} (u^\mu; y_i), \quad \phi_{\bar{y}_n} = -\frac{1}{\beta^n} \sum u^\mu \phi_{u^\mu}$$

$$\phi_{\bar{y}_\nu} = \frac{1}{\alpha^n} \phi_{v^\nu} (v^\nu; \bar{y}_i), \quad \phi_{y_n} = -\frac{1}{\alpha^n} \sum v^\nu \phi_{v^\nu}$$

(15)

(16)

(The reader can compare these equations with the analogous equations in the paper on the real Universal equation). Now let us use the integrability conditions of selfconsistency of the second order mixed derivatives. Let us assume that conditions of self consistency for Greek barred and unbarred indices are satisfied (details will be given later) :

$$\left( \frac{\phi_{v^\nu}}{\alpha^n} \right)_{y_\mu} = \left( \frac{\phi_{u^\mu}}{\beta^n} \right)_{\bar{y}_\nu}$$

(17)

and consider what follows from conditions of selfconsistency for the other pairs $(y_\mu, \bar{y}_n), (\bar{y}_\nu, y_n)$ and $y_n, \bar{y}_n$. We have in consequence, (for the first pair of variables):

$$\frac{\partial}{\partial y_\mu} \frac{\partial \phi}{\partial \bar{y}_n} = -\frac{\partial}{\partial y_\mu} \sum v^\nu \phi_{v^\nu} = -\frac{\phi_{y_\mu}}{\alpha^n} - \sum v^\nu \left( \frac{\phi_{u^\mu}}{\alpha^n} \right)_{y_\mu} =$$

$$-\frac{\phi_{y_\mu}}{\alpha^n} - \sum v^\nu \left( \frac{\phi_{u^\mu}}{\beta^n} \right)_{\bar{y}_\nu} = \frac{\partial}{\partial \bar{y}_n} \left( \frac{\phi_{u^\mu}}{\beta^n} \right)$$

(18)
The last row of this equality can be transformed into:

\[ \sum_{\alpha} \phi_{\alpha\mu} (u_{\alpha y}^\mu + \sum v^\nu u_{\nu y}^\alpha) + (\phi_{\alpha n}^\mu) (1 - \sum_i \alpha^i \beta_{\gamma n}^i) = 0 \]  

(19)

Rewriting the equality:

\[ \phi_{\gamma n} + \sum v^\nu \phi_{\gamma v} = 0 \]  

(20)

in terms of variables \((u, y)\):

\[ \sum \phi_{\alpha n}^\mu (u_{\alpha y}^\mu + \sum v^\nu u_{\nu y}^\alpha) = 0 \]  

(21)

we arrive at a linear system of \(n\) equations for the \(n\) unknowns: \((u_{\alpha y}^\mu + \sum v^\nu u_{\nu y}^\alpha), \alpha^n(1 - \sum_i \alpha^i \beta_{\gamma n}^i)\), and assuming that the determinant of the corresponding universal equation in the \((n - 1)\) dimensional space \(u^\alpha\) is different from zero (the degenerate case, when it is equal to zero demands special consideration):

\[ \sum_i \alpha^i \beta_{\gamma n}^i = \beta_n, \quad u_{\gamma y}^\mu + \sum v^\nu u_{\nu y}^\mu = 0 \]  

(22)

\[ \sum_i \beta^i \alpha_{\gamma n}^i = \alpha^n, \quad v^\nu + \sum u^\mu v^\nu = 0 \]  

(23)

which proves the previously assumed equations (8), (9) and shows that the hydrodynamical system (9) is the direct corollary of the main equations. The calculations of the second mixed derivatives \((\phi_{\gamma y})_{\gamma n}\) and in the opposite order leads to equivalent expressions. They are not essential for what follows and their proof we omit here.

4 The system of hydrodynamic type

We understand by a system of hydrodynamic type the system (9) rewritten below:

\[ v^\nu_{\gamma n} + \sum u^\mu v^\nu_{\gamma y} = 0, \quad u_{\gamma y}^\mu + \sum v^\nu u_{\nu y}^\mu = 0 \]  

(24)

Two propositions with respect to this system will be required in what follows.

Proposition 1.

The pair of operators:

\[ D = \frac{\partial}{\partial y_n} + \sum u^\mu \frac{\partial}{\partial y_\mu}, \quad \bar{D} = \frac{\partial}{\partial \bar{y}_n} + \sum v^\nu \frac{\partial}{\partial \bar{y}_\nu} \]  

(25)
are mutually commutative if \((u^\mu, v^\nu)\) are solutions of the system (24).

Acting with the help of operators \((D, \bar{D})\) on the second and the first equations of (24) respectively we come to the conclusion that \(2(n - 1)\) functions:

\[
\bar{D}(v^\nu) = v^\nu_{yn} + \sum v^\mu v^\nu_{\bar{y}_\mu}, \quad D(u^\mu) = u^\mu_{yn} + \sum u^\nu u^\mu_{y^\nu}
\] (26)

are also solutions of the first and the second system of equations (24).

As a corollary we obtain the following

**Proposition 2**

\[
v^\nu_{yn} + \sum v^\mu v^\nu_{\bar{y}_\mu} = Q^\nu(v; \bar{y}), \quad u^\mu_{yn} + \sum u^\nu u^\mu_{y^\nu} = P^\mu(u; y)
\] (27)

Indeed the \(n\) sets of variables \((1, u)\), and \((1, v)\) respectively satisfy a linear system of algebraic equations of \(n\) equations, the matrix of which coincides with the Jacobian matrix

\[
J = \det_n \begin{vmatrix} v^1 & \ldots & v^{n-1} & Q^\nu \\ y_1 & \ldots & y_{n-1} & y_n \end{vmatrix}
\] (28)

which in the case of a non-zero solution of the linear system must vanish. So Proposition 2 is proved.

Compared with (24) (27) is an inhomogeneous system of hydrodynamic equations separated into functions \((u, v)\).

Now we are able to find solutions of the primary equations (22),(23). To this end, let us rewrite them in the terms of operators \(D_{1,2}:\)

\[
\bar{D}(\frac{1}{\beta^n}) = -\frac{1}{\alpha^n \beta^n} = D(\frac{1}{\alpha^n})
\] (29)

We have consequently:

\[
\frac{1}{\beta^n} = D\Theta, \quad \frac{1}{\alpha^n} = \bar{D}\Theta, \quad (\bar{D}\Theta)(D\Theta) = -\bar{D}D\Theta
\] (30)

The solution of the last equation is obvious:

\[
\exp \Theta = -c(u; y) + \bar{c}(v; \bar{y})
\] (31)
5 General solution of hydrodynamic system

Let us have the following system of equations defining implicitly \((n - 1)\) unknown functions \((\psi)\) in \((2n)\) dimensional space \((y, \bar{y})\):

\[
Q^\nu(\psi; y) = P^\nu(\psi; \bar{y})
\]

(32)

with the convention that all Greek indices take values between 1 and \((n - 1)\). The number of equations in (32) coincides with the number of unknown functions \(\phi^\alpha\).

With the help of the usual rules of differentiation of implicit functions we find from (32):

\[
\psi_y = (P_\psi - Q_\psi)^{-1}Q_y, \quad \psi_{\bar{y}} = -(P_\psi - Q_\psi)^{-1}P_{\bar{y}}
\]

(33)

Let us assume, that between \(n\) derivatives with respect to barred and unbarred variables there exists the linear dependences:

\[
\sum_{i=1}^{n} c_i \psi^{\alpha}_{y_i} = 0, \quad \sum_{i=1}^{n} d_i \psi^{\alpha}_{\bar{y}_i} = 0
\]

(34)

and analyse the consequences of these facts.

Assuming that \(c_n \neq 0, d_n \neq 0\), dividing them into each equation of the left and right systems respectively and introducing the notation \(u^\alpha = \frac{c_\alpha}{c_n}, v^\alpha = \frac{d_\alpha}{d_n}\) we may rewrite the last set in the form:

\[
\psi^{\alpha}_{y_n} + \sum_{\nu=1}^{n-1} u^\nu \psi^{\alpha}_{y_{\nu}} = 0, \quad \psi^{\alpha}_{\bar{y}_n} + \sum_{\nu=1}^{n-1} v^\nu \psi^{\alpha}_{\bar{y}_{\nu}} = 0
\]

(35)

Substituting values of the derivatives from (33) and multiplying the result by the matrix \((P_\phi - Q_\phi)\) from the left we obtain:

\[
Q^\alpha_{y_n} + \sum_{\nu=1}^{n-1} u^\nu Q^\alpha_{y_{\nu}} = 0, \quad P^\alpha_{\bar{y}_n} + \sum_{\nu=1}^{n-1} v^\nu P^\alpha_{\bar{y}_{\nu}} = 0
\]

(36)

From these last equations it immediately follows:

\[
u^\nu = -(Q_y)^{-1}Q_{y_n}, \quad v^\nu = -(P_{\bar{y}})^{-1}P_{\bar{y}_n}
\]

(37)
We see that if we increase the initial system (32), by \((n - 1)\) vector functions \((u, v)\) defined by (37) then the differential operators \(D, \bar{D}\) defined by (4) in connection with (35) annihilate each \(\psi\) either as a \(Q\) or a \(P\) function:

\[
D\psi = \bar{D}\psi = DQ = DP = \bar{D}Q = \bar{D}P = 0 \quad (38)
\]

This means that \(D\bar{f}(\phi, \bar{y}) = \bar{D}f(\phi, y) = 0\). And as a direct corollary of this fact \(Dv = \bar{D}u = 0\) and the generators \(D, \bar{D}\) constructed in this way mutually commute. Thus we have found the general solution of the hydrodynamic system and a concrete realisation of the manifold with the properties of the previous section.

With respect to the generators \(D, \bar{D}\) all functions of \(2n\) dimensional space may be divided into the following subclasses: functions of general position \(F, DF \neq 0, \bar{D}F \neq 0\), holomorphic functions \(f, \bar{D}f = 0, \bar{D}f \neq 0\), antiholomorphic ones \(\bar{f}, D\bar{f} = 0, \bar{D}\bar{f} \neq 0\) and \(f^0\) ”central" functions, both holomorphic and antiholomorphic simultaneously; \(Df^0 = D\bar{f}^0 = 0\). Each central function may be represented in the form:

\[
f^0 = f^0(Q) = f^0(P) = g^0(\phi)
\]

6 Equations following from the other restrictions

Formulae (37) together with (32) give the general solution of the hydrodynamic system (24). Indeed this solution depends upon \(2(n - 1)\) arbitrary functions (32) each of \((2n - 1)\) independent arguments, which are sufficient for the statement of Cauchy or Gursat initial value problems. The general solution of the complex Bateman equation (1) depends upon only two arbitrary functions each of \((2n - 1)\) arguments. Thus all other restrictions arising on the way must reduce the \(2(n - 1)\) arbitrary functions of (32) to only two.

For this purpose it is necessary to calculate derivatives of the functions \(u, v\) defined by (37). We have in consequence:

\[
u_{\gamma\alpha} = -Q_y^{-1}(Q_{y\gamma}, y_\alpha) + \sum Q_{y\gamma, \phi^\alpha} \phi^\alpha_{y_\alpha} - Q_{y\gamma, y_\alpha} Q_y^{-1} Q_{y\gamma} - \sum Q_{y\gamma, \phi^\alpha} \phi^\alpha_{y_\alpha} Q_y^{-1} Q_{y\gamma}
\]

\[
\equiv -Q_y^{-1}(DQ_{y\gamma}) + Q_y^{-1}(DQ_{\phi})(P_\phi - Q_\phi)^{-1} Q_{y\gamma} \quad (39)
\]

7
By the same technique we can calculate $u_y, v_y$ using:

$$u_y = Q^{-1}_y (DQ_\phi (P_\phi - Q_\phi)^{-1} P_y, \quad v_y = -P^{-1}_y (DP_\phi (P_\phi - Q_\phi)^{-1} Q_y \quad (40)$$

Comparing (13) with results of the previous section we conclude that the function $\phi$ is central and so depends only on $(n-1)$ arguments $\phi = \phi(\psi)$. For us it will be more convenient to go back directly to linear systems (3),(4) and investigate their properties. We have in consequence:

$$\sum \phi_\psi \alpha \psi_i = \sum \phi_\psi \sum \psi_\alpha \alpha_i \psi_y$$

Further evaluation of the last equality is connected with the substitution of the explicit expressions for the derivatives of the functions $\psi$ (33). The results of the further calculations we present in the form of multiplication of the row $\phi_\psi$ on the corresponding matrix:

$$\phi_\psi (P\psi - Q\psi)^{-1} (Q_{\psi_y} + \sum \alpha_i \psi Q_{\psi_i}) = \phi_\psi (P\psi - Q\psi)^{-1} (Q_{\psi_y} + \sum \nu_\psi \psi Q_{\psi_i})$$

$$= \phi_\psi (I - \alpha^n (P\psi - Q\psi)^{-1} DP_{\psi}) (P\psi - Q\psi)^{-1} Q_{\psi_y} = 0 \quad (41)$$

In the process of the above evaluation we have used the equalities $\bar{D}Q = 0$ and the explicit expression for the derivatives of the functions $\psi$ with respect to the unbarred coordinates (40).

Equation (11) for $s = n$ is a direct corollary of the equations with Greek indices as consequence of the equality $\bar{D}Q = 0$. Assuming that $\det Q^\psi_{\mu} \neq 0$, $\det (P\psi - Q\psi) \neq 0$ we may rewrite the equations (11) with the Greek indices in the final form:

$$\sum \phi_\psi \alpha (I - \alpha^n (P\psi - Q\psi)^{-1} DP_{\psi})^\alpha_\nu = 0 \quad (42)$$

A similar equation follows from (3):

$$\sum \phi_\psi \alpha (I - \beta^n (P\psi - Q\psi)^{-1} DQ_{\psi})^\alpha_\nu = 0 \quad (43)$$

Now we assume that only one from the set of the $\psi$ functions satisfies the complex Bateman equation. Suppose it is $\psi^1$ and that the solution may be chosen in the form $\phi(\psi^1)$. Of course this suggestion must be confirmed by the detailed investigations of all results following from (12) and (13). We omit here this consideration, replacing it by checking the final result. Under
this assumption the equations (42) and (43) are equivalent to the following $2(n-1)$ equalities:

$$(I - \beta^n(P_\psi - Q_\psi)^{-1}DQ_\psi)_{\nu} = 0, \quad (I - \alpha^n(P_\psi - Q_\psi)^{-1}\bar{D}P_\psi)_{\nu} = 0$$

which after substituting into them the explicit expressions for $\frac{1}{\alpha^n} = \bar{D}\Theta, \frac{1}{\beta^n} = D\Theta$ from (31) will be convenient to rewrite in the form of multiplication of the row on matrix $(P_\psi - Q_\psi) = \delta$:

$$(\bar{D}\Theta, 0...0) = (1, 0...0)\delta^{-1}\bar{D}\delta$$

multiplying the last equality with the matrix $\delta^{-1}$ on the right we obtain:

$$(\bar{D}\Theta, 0...0)\delta^{-1} = -(1, 0...0)\bar{D}\delta^{-1}, \quad (\bar{D}\Theta, 0...0)\delta^{-1} = -(1, 0...0)D\delta^{-1} \quad (44)$$

(and also a similar system with unbarred differentiation). The integration of the last system is straightforward with the result:

$$(\delta^{-1})_{1,\beta} = \nu(\psi)_{\beta} \exp \Theta = \frac{\nu(\psi)_{\beta}}{(\bar{c} - c)} \quad (45)$$

Below we consider the simplest examples of solutions of the last system for functions $Q, P$ from which the situation in the general case of arbitrary $n$ will be clarified.

### 6.1 The case $n = 2$

In this case there is only one Greek index and two scalar equations (44) lead to the result:

$$\exp \Theta = \nu(\psi^1)(P_{\psi^1} - Q_{\psi^1}) = \bar{c} - c$$

The last equality may be considered as the definition of the functions $c, \bar{c}$ in terms of $P, Q$:

$$\bar{c} = \nu(\psi^1)P_{\psi^1}(\psi^1; \bar{y}_1, \bar{y}_2), \quad c = \nu(\psi^1)Q_{\psi^1}(\psi^1, y_1, y_2)$$

with the correct dependence upon their independent arguments.
6.2 The case $n = 3$

This case is a more crucial one. Using the explicit form of the matrix $\delta$

$$\delta = \begin{pmatrix} \Delta_{\psi_1} & \Delta_{\psi_2} \\ \Delta_{\psi_2} & \Delta_{\psi_1} \end{pmatrix}$$

we without any difficulties can obtain the explicit form of the matrix elements of the inverse matrix, which after substitution into (45) lead to:

$$(P^1_{\psi_1} - Q^1_{\psi_1}) + \frac{\nu_2}{\nu_1} (P^2_{\psi_1} - Q^2_{\psi_1}) = 0$$

$$(P^1_{\psi_2} - Q^1_{\psi_2}) + \frac{\nu_2}{\nu_1} (P^2_{\psi_1} - Q^2_{\psi_1}) = \bar{c} - c. \quad (46)$$

Taking into account the explicit dependence of $P, Q$ functions upon their arguments we separate the last system into two systems of equations for $Q, P$ respectively ($\frac{\nu_2}{\nu_1} = \nu$):

$$Q^1_{\psi_1} + \nu Q^2_{\psi_1} = c, \quad Q^1_{\psi_2} + \nu Q^2_{\psi_2} = 0$$

The condition of selfconsistency of the last two equations (equality of second mixed derivatives of the function $Q^1$) leads to

$$\nu Q^1_{\psi_1} + Q^2_{\psi_1} = c$$

a single equation for the determination of the function $Q^2$. Let us consider in the last equation $Q^1$ as an arbitrary given function $p = p(\psi; y)$. Then the equation for it may be considered as a definition of the function $c$, which for what follows it is better to rewrite in the form:

$$\nu p_{\psi_1} - (\int d\psi^2 \nu p_{\psi_2})_{\psi_1} = c$$

Substituting this expression into the first initial equation we obtain for $Q^1$:

$$Q^1 = \int d\psi^2 \nu p_{\psi_2} = \nu p - \int d\psi^2 \nu p_{\psi_2}$$

Introducing the new function $F = \int d\psi^2 \nu p_{\psi_2}$ we now are able with its help to represent both functions $Q^1, Q^2$ in a local form:

$$Q^2 = \frac{F_{\psi_2}}{\nu_{\psi_2}}, \quad Q^1 = \nu Q^2 - F$$
The same procedure may to be done with a similar result with the functions $P^1, P^2$:

$$P^2 = \frac{F_{\psi^2}}{\nu_{\psi^2}}, \quad P^1 = \nu P^2 - \bar{F}$$

Equating $P^{1,2} = Q^{1,2}$ and taking into account that all factors depending upon functions $\psi$ may be cancelled we arrive at the following system

$$F = \bar{F}, \quad F_{\psi^2} = \bar{F}_{\psi^2}$$

which determine implicitly two functions $\psi$, one of which, $\psi^1$ is the solution of the complex Bateman equation in three dimensions.

### 6.3 The general case of arbitrary $n$

Let us denote the minors of $(n - 2)$th order of the first column of the matrix $\delta, \Delta_{\psi^\beta}$ by $M^\beta$ and introduce the notation $\nu^\beta = \frac{M^\beta}{M^1}$. Then the systems (44) and (45) may be solved in the following way:

$$\sum \nu^\alpha \Delta_{\psi^1}^\alpha = \bar{c} - c, \quad \nu^\alpha = \nu^\alpha(\psi)$$

and rewrite the definition of the functions $\nu^\alpha$ in the form:

$$M^\alpha = \nu^\alpha M^1$$

Multiplying the last equalities by elements of each (given) column (except for the first one) with further summation of the results we always obtain zero on the left hand sides of the equalities arising (determinants with equal columns). Thus we may rewrite (48) in the equivalent form:

$$\sum \nu^\alpha \Delta_{\psi^1}^\alpha = \bar{c} - c, \quad \sum \nu^\alpha \Delta_{\psi^A}^\alpha = 0, \quad 2 \neq A \neq (n - 1)$$

Keeping in mind that $\Delta^\alpha_{\psi^\beta} = P^\alpha_{\psi^\beta} - Q^\alpha_{\psi^\beta}$ and recalling the definition of (anti) holomorphic functions of section 5, we separate the last system into two independent ones:

$$\sum \nu^\alpha Q^\alpha_{\psi^1} = c, \quad \sum \nu^\alpha Q^\alpha_{\psi^A} = 0, \quad \sum \nu^\alpha P^\alpha_{\psi^1} = \bar{c}, \quad \sum \nu^\alpha P^\alpha_{\psi^A} = 0$$

Further transformations of both barred and unbarred systems are similar and so we will follow through the evaluation of the first one.
Introducing new function $F = \sum \nu^\alpha Q^\alpha$ we obtain:

$$\sum \nu^\alpha_{\psi^1} Q^\alpha = -c + F_{\psi^A}, \quad \sum \nu^\alpha_{\psi^A} Q^\alpha = F_{\psi^A}$$

(49)

In particular we recall that $\nu^1 = 1$ and so each equation in its left hand side contains only $(n - 2)$ terms. Thus between $(n - 1)$ equations above at least one linear dependence exists. Denoting the coefficients of it by $d^A = D^A(\psi)$ we rewrite the condition of self consistency of the last system in the form of a single equation relating the function $F$ in terms of the function $c$:

$$c = F_{\psi^1} + \sum d^A F_{\psi^A}$$

Inverting the problem we can consider the last equality as a definition of the function $c$ in terms of the given $F$. Solving the last $(n - 2)$ equations of the system (49) and adding to them $Q^1$ obtained from definition of the function $F$, we obtain finally:

$$Q^B = \sum (\nu^1)^{-1}_BT^A F_{\psi^A}, \quad Q^1 = F - \sum \nu^A Q^A$$

Completely similar calculations lead to the following expressions for the functions $P$:

$$P^B = \sum (\nu^1)^{-1}_BT^A \tilde{F}_{\psi^A}, \quad P^1 = \tilde{F} - \sum \nu^A P^A$$

We specially emphasize that all coefficient functions in the expressions above determining functions $Q, P$ in terms of two arbitrary functions $F, \tilde{F}$ are functions only of the arguments $\psi$ (central functions) and are the same in both cases.

Equating $P^\beta = Q^\beta$ after obvious cancellation of all (scalar, matrix) factors depending only upon the functions $\psi$ we reach the final system of equations implicitly determining all functions $\psi$:

$$F = \tilde{F}, \quad F_{\psi^A} = \tilde{F}_{\psi^A}$$

(50)

The function $\psi^1$ is the solution of the Complex Bateman equation (1) in the space of $n$ dimension.

7 The main theorem

Theorem
The general solution of the Complex Bateman equation \((\Pi)\) is defined by the function \(\psi^1\), which is implicitly determined from the following system of \((n - 1)\) equations for \((n - 1)\) functions \(\psi^\nu\):

\[
F(\psi; y) = \bar{F}(\psi; \bar{y}), \quad F_{\psi^\nu} = \bar{F}_{\psi^\nu}, \quad 2 \leq A \leq (n - 1)
\]

8 Outlook

The principal concrete results of the present paper are concentrated in the Theorem of the previous section, giving the explicit general solution (in implicit form) of the complex Bateman equation \((\Pi)\) in \(n\) dimensional space.

But by the way a no less important a hydrodynamic type system with two times \((y_n, \bar{y}_n)\) was discovered and solved. Definite reduction of its general solution leads to the general solution of Complex Bateman equation. We can’t exclude the possibility that there are exist other reductions leading to no less interesting systems and equations.

Nevertheless the majority of these results were obtained more on the basis of intuitive calculations. We think that the algebraic-geometrical form of the answer tells us of the necessity to consider the group of motion (the symmetry structure) of the manifold \((\Pi)\) determining the general solution of hydrodynamic system \((\Pi)\). The properties of the group of inner symmetry of it must explain more precisely and directly the proposed way of integration of the systems under consideration (hydrodynamic and Complex Bateman) as the uniquely possible one.

We have a feeling that in these problems algebraic-geometrical methods are more appropriate and effective. Unfortunately they are not within our area of expertise.

References

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