TREES, GRAPHS AND AGGREGATES: 
A CATEGORICAL PERSPECTIVE ON COMBINATORIAL SURFACE 
TOPOLOGY, GEOMETRY, AND ALGEBRA 

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Dedicated to Dennis Sullivan on the occasion of his 80th birthday 

Abstract. Taking a Feynman categorical perspective, several key aspects of the geometry of surfaces are deduced from combinatorial constructions with graphs. This provides a direct route from combinatorics of graphs to string topology operations via topology, geometry and algebra. In particular, the inclusion of trees into graphs and the dissection of graphs into aggregates yield a concise formalism for cyclic and modular operads as well as their polycyclic and surface type generalizations. The latter occur prominently in two-dimensional topological field theory and in string topology. The categorical viewpoint allows us to use left Kan extensions of Feynman operations as an efficient computational tool. The computations involve the study of certain categories of structured graphs which are expected to be of independent interest.

Introduction 

Graphs are an ubiquitous tool in mathematics. In geometry, for instance, they arise in the description of surfaces, and in algebra via flow–charts of compositions. The latter point of view is what is formalized with operads [BV68, May72, Mar08]. Graph theoretically operads deal with rooted trees or forests. Forgetting the root, and with it direction, one considers trees and, dropping the condition of being simply connected, graphs in general. Operadically this corresponds to cyclic operads [GK95] and types of modular operads [Sch98, GK98, KWZ15]. Adding a cyclic ordering for each vertex-corolla yields the notion of ribbon graph, which is central to the theory of Riemann surfaces. Special ribbon graphs, the Sullivan graphs, underlie string topology operations [CS99, TZ07, Kau07, Kau08a]. Adding further data or forgetting some of the data leads to a host of other graphical structures, which appear and are useful in specific contexts.

Beyond the notion of a graph, the notion of a graph morphism is of prime importance. The graph morphisms of Borisov-Manin [BM08] are adapted to capture all relevant aspects. Their level of sophistication allows to compute the automorphisms correctly and formalizes the operations of contracting, grafting and merging. Importantly, such a graph morphism defines an underlying graph which will allow us to define graph insertion in a precise way. Indeed, the first result of this article realizes these graph morphisms as the two–morphisms of a double category in which horizontal composition is graph insertion, while vertical composition is the usual composition of graph morphisms restricted to aggregates, where throughout the text an aggregate is a disjoint union of corollas.

**Theorem A.** Each Borisov-Manin graph morphism functorially defines source and target morphisms of aggregates obtained by cutting, respectively contracting, all edges. This is part of a category internal to the Feynman categories whose horizontal composition corresponds to graph insertion. If suitably restricted this double category has holonomy and connections in the sense of [BM09].

In particular, we show Borisov-Manin’s category of graphs \( \mathcal{Gr} \) yields a Feynman category \( \mathcal{F}^\mathcal{Gr} \) whose monoidal structure of is disjoint union. We will call the morphisms between Feynman categories Feynman functors and a strong monoidal functor out of a Feynman category will be called a Feynman operation. The property of being a Feynman category allows one to use several key
results, notably the existence of pull−backs and push−forwards of Feynman operations, generalizing Frobenius reciprocity for group operations, and a factorization system of Feynman functors between Feynman categories into connected Feynman functors and coverings [KW17, KL17, BK17].

All Feynman categories relevant for this article are graphical in the sense that they are obtained from \( \mathcal{S}^{Gr} \) either as Feynman subcategories or as coverings of such. Restricting the objects to aggregates, we recover the Feynman category \( \mathcal{S}^{nc-\text{ng-mod}} \) — called \( \mathcal{S} \) in [KW17] — central to operad−like theories. Restriction of the type of the underlying graphs of basic morphisms, called ghost graphs, defines subcategories while decorations of graphs with additional data are handled by coverings. The relevant categories and their operations are listed in Table 1. The approach presented here is a bootstrap, whose ingredients are only the adding/forgetting of roots, inclusion of trees into graphs and the existence of cyclic orders which provides the Feynman operation \( \mathcal{O}_{\text{cycass}} \) for \( \mathcal{S}^{\text{cyc}} \). Denoting the terminal Set valued Feynman operation for \( \mathcal{S}^{\text{cyc}} \) by \( \mathcal{O}_{\text{cyc}} \), these graphical Feynman categories are related by structure preserving functors as summarized below.

**Theorem B.** There is a commutative diagram of Feynman categories and Feynman functors,

\[
\begin{array}{ccc}
\mathcal{S}^{-\Sigma-\text{opd}} & \xrightarrow{j^r} & \mathcal{S}^{\text{pl-cyc}} \\
\pi_1 & \downarrow & \pi_2 \\
\mathcal{S}^{\text{opd}} & \xrightarrow{j} & \mathcal{S}^{\text{mod}} \\
\pi_3 & \downarrow & \pi \\
\mathcal{S}^{\text{surf-mod}} & \xrightarrow{k} & \mathcal{S}^{\text{ng-mod}}
\end{array}
\]  

(0.1)

in which \( i, i' \) correspond to forgetting the root and \( j, j' \) are defined by the inclusion of trees into graphs. The vertical functors are coverings and \( j, j' \) are connected and \( j\pi \) is the unique factorization of \( k \) into a connected morphisms and a covering. In particular, there are equivalences of Feynman categories:

\[
\begin{array}{ll}
\mathcal{S}^{\text{opd}}(\mathcal{O}_{\text{ass}}) \simeq \mathcal{S}^{-\Sigma-\text{opd}} \\
\mathcal{S}^{\text{cyc}}(\mathcal{O}_{\text{cycass}}) \simeq \mathcal{S}^{\text{pl-cyc}} \\
\mathcal{S}^{\text{ng-mod}}(\mathcal{O}_{\text{genus}}) \simeq \mathcal{S}^{\text{mod}} \\
\mathcal{S}^{\text{surf-mod}} \simeq \mathcal{S}^{\text{surf-mod}}
\end{array}
\]

(0.2)

where the subscript \( \text{dec} \) indicates a covering obtained by a decorations with the indicated Feynman operation. We have the following identifications of Feynman operations

\[
\begin{array}{ll}
i^*(\mathcal{O}_{\text{cycass}}) \simeq \mathcal{O}_{\text{ass}}, & k_1(\mathcal{O}_{\text{cyc}}) \simeq \mathcal{O}_{\text{genus}}, \\
k_1(\mathcal{O}_{\text{cycass}}) \simeq \mathcal{O}_{\text{surf}}
\end{array}
\]

(0.3)
There are several intermediate coverings that arise naturally on the modular side, which allow us to address different constructions that have appeared in the literature, cf. Table 2.

In this framework, everything boils down to the computation of left Kan extensions. This is possible as soon as the relevant slice categories are well understood. Interestingly, these slice categories are often equivalent to certain categories of structured graphs, e.g. categories of ribbon graphs with subforest contractions as morphisms, as they appear in the theory of moduli spaces and in physics. Taking a more topological approach, the same categories can also be represented by surfaces with extra structure, often explicitly given in form of a system of arcs or curves. This is what lends the theory to applications in topology and geometry.

For instance, the central computation of the pushforward \( k_! \mathcal{O}_{\text{cycass}} \) can be done using several different but equivalent combinatorial objects. The calculation of the push-forward can be done in graphs, where the calculation involves the category of spanning forest contractions of ribbon graphs as they appear in the work of Igusa [Igu02]. This is novel and important in the relationship to moduli spaces. We present a computation based on cyclic words, closely related to the classification of oriented surfaces, see e.g. [Mun75]. Other presentations are in [KP06,CL07,Mar16,Dou17]. We outline also the relationship with chord diagrams thereby obtaining a link with string topology [Kau05] and knot theory [BN95].

Algebraically, the adjunction between induction and restriction functors (aka Frobenius reciprocity) gives new insight into well known results linking 1+1 d Topological Quantum Field Theory, resp. open/closed TQFT to commutative, resp. symmetric Frobenius algebras. To obtain these results, we generalize the notion of an algebra by introducing reference functors. We show that for undirected graphical Feynman categories the natural reference functors are given by pairs consisting of an object of the target category and a propagator; see Table 3 for examples. This also formalizes the correlation functions of [Kau08a] with values in twisted hom operads, which are necessary to formulate Deligne’s conjecture. Let \( \mathcal{O}^F_1 \) denote the trivial Feynman operation for \( F \).

**Theorem C.** Unital algebras over \( \mathcal{O}^\text{cyc}_1 \) are commutative Frobenius algebras. Unital algebras over \( \mathcal{O}^{\text{pl-cyc}}_1 \) are symmetric Frobenius algebras. Algebras over \( \mathcal{O}^{\text{cyc}}_1 \) (resp. \( \mathcal{O}^{\text{pl-cyc}}_1 \)) are commutative (resp. symmetric) Frobenius objects with a trace and a propagator.

By adjunction, that is Frobenius reciprocity, unital algebras over \( \mathcal{O}^{\text{mod}}_1 \), i.e. closed 1+1 d TQFTs are equivalent to commutative Frobenius algebras. Unital algebras over \( \mathcal{O}^{\text{surf-mod}}_1 \), equivalently over \( \mathcal{O}_1 \), i.e. open 1+1 d TQFTs are equivalent to symmetric Frobenius algebras. Without the unit assumption these are commutative, resp. symmetric Frobenius objects with a trace and a propagator.

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**Table 2.** Decorated Feynman categories and their ghost graphs for the intermediate covers, cf. (4.7)

| Feynman category | Feynman category | description |
|------------------|------------------|-------------|
| \( \Sigma \)-opd \( \mathcal{O}_{\text{ass}} \) | \( \Sigma \)-opd | planar rooted tree |
| \( \Sigma \)-opd \( \mathcal{O}_{\text{cycass}} \) | \( \text{pl-cyc} \) | planar tree |
| \( \Sigma \)-opd \( \mathcal{O}_{\text{genus}} \) | \( \text{mod} \) | genus labelled tree |
| \( \Sigma \)-opd \( \mathcal{O}_{\text{surf}} \) | \( \text{surf-mod} \) | genus/puncture labelled polycyclic graph |
| \( \Sigma \)-opd \( \mathcal{O}_{\text{N,poly}} \) | \( \text{genpoly} \) | genus labelled polycyclic graph |
| \( \Sigma \)-opd \( \mathcal{O}_{\text{poly}} \) | \( \text{poly} \) | polycyclic graph |
In this formalism, the correlation functions underlying the algebraic string topology operations of [Kau08a,Kau18] become the pullback to graphs. This completely characterizes them in terms of the bootstrap from graphs and cyclic orders.

**Theorem D.** *The correlation functions for a symmetric Frobenius algebra $A$ are given by a natural transformation $s^*(Y) \in \text{Nat}[s^*\mathcal{O}_{\text{surf}}, \text{Cor}_{A,P}]$, where $s$ is the source functor of Theorem A.*

Suitably interpreted, the correlations functions furthermore induce actions on the Hochschild chain and cochain complexes as well as on the Tate–Hochschild complex, cf. [KRW21].

This approach allows the results to transfer to other areas in further work. One, [BK22b], will deal with PROP actions, such as the one of string topology, cf. [Kau07,Kau08a] and its generalization. The compositions are intricate, as they are along cycles, not along tails. The theorems and importantly the computations also allow us to construct moduli spaces [BK22a] using the $W$–construction of [KW17]. This is a generalization of the theorem of Igusa [Igu02] that the moduli spaces can be constructed as the nerve of categories of ribbon graph with subforest contractions.

Using the diagram (0.1) and denoting a surface type of a surface of genus $g$ with $b$ boundaries marked by points sets $S_1, \ldots, S_b$ and $p$ unmarked boundaries by $(g,p,S_1 \ldots S_b)$.

**Theorem E.**

1. We have the following chain of inclusions

\[
W_{j!}(\mathcal{O}_{\text{cycass}})(g,n) = \text{Cone}(\bar{M}_{g,n}^{K/P}) \supset \bar{M}_{g,n}^{K/P} \supset M_{g,n}
\]  

identified as spaces of metric surface marked graphs where the cone point is the corolla of the given type.

2. We have the identification $j'_!(W\mathcal{O}_*(g,p,S_1\ldots S_b)) \simeq M_{g,S_1\ldots S_b}$. where $\bar{M}_{g,n}^{K/P}$ is the Kontsevich/Penner/combinatorial compactification of moduli space.

The text is organized as follows:

- In §1 we introduce the relevant notion of graphs including structured graphs such as ribbon graphs. Additionally, an interpretation of these graphs in terms of surfaces with extra data is furnished.
- In §2 we discuss Borisov-Manin graph morphisms and organise all data into a double category. This section also contains explicit presentations in terms of generators and relations needed for later computations.
- In §3 we present the correspondence between Feynman operations and coverings. We furthermore discuss a commutative hexagon of coverings relating our approach to others occurring in literature.
- In §4 key aspects of graphical Feynman categories are established including Theorem A.
- In §5 Theorem B is proved. One key issue is the computation of the pushforward $k_!(\mathcal{O}_{\text{cycass}})$. One central technical result is the equivalence of a comma category needed to compute the push–forward and a category of Ribbon graphs with spanning forest contractions. The section is closed by the generalization to non–connected structures such as disconnected surfaces.
In §6 Frobenius algebras and open/closed TFT are linked via Theorems C and D.

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1. Graphs

1.1. Basic definitions. A graph \( \Gamma = (F, V, \partial, i) \) is given by the following data: a set of flags \( F \), a set of vertices \( V \), a boundary map \( \partial : F \to V \) indicating the incidence of a flag to a vertex, and an involution \( i : F \to F \) whose two-element orbits are the edges of \( \Gamma \). Each edge \( e = \{f, i(f)\} \) is thus formed by two flags, also called half-edges or inner flags of the graph. The fixpoints of the involution are the tails or outer flags of the graph (aka legs, hairs, leads, external flags).

We let \( E \) be the set of edges. An edge is called a loop if its two flags are incident to the same vertex. The flags incident to \( v \) form the set \( F_v := \partial^{-1}(v) \). The cardinality of \( F_v \) is called the valency of the vertex \( v \).

The disjoint union of two graphs is given by taking the disjoint unions of the flag and vertex sets and extending boundary map and involution accordingly. A graph is connected if it is not the disjoint union of two non-empty subgraphs. Any graph decomposes into a disjoint union of connected components \( \Gamma = \bigsqcup_{v \in V} \Gamma_v \) where \( \Gamma_v \) is the maximal connected subgraph containing \( v \) and \( V = V/\sim \) where \( v \sim w \) if there is an edge path from \( v \) to \( w \).

A graph is said to be a corolla if it has a single vertex and no edges. We denote such a corolla by \( v_F = (\{v\}, F, \partial_F, id_F) \) where \( \partial_F \) is the unique map \( F \to \{v\} \) and \( id_F \) is the identity. An aggregate is a disjoint union of corollas. A rose is a one vertex graph, which is not necessarily a corolla.

A subgraph \( (V', F', \partial', i') \) of a graph \( (V, F, \partial, i) \) is given by subsets of vertices and flags such that the edges of the subgraph form a subset of the edges of the ambient graph. Formally, \( V' \subset V \), \( F' = \bigsqcup_{v \in V'} F'_v \), \( \partial' = \partial|_{F'} \), and \( i'(f) = i(f) \) or \( i'(f) = f \). Each vertex \( v \in V \) defines a subgraph \( v_{F_v} \), the so-called vertex-corolla. A subgraph is spanning, if contains all vertices. A spanning tree/forest is a spanning subgraph that is a tree/forest. Any subgraph \( \Gamma' \) of a graph \( \Gamma \) can be completed to the spanning subgraph \( \Gamma' \sqcup \bigsqcup_{v \in V', v_{F_v}} \).

The contraction of a graph along a spanning subgraph \( \Gamma \) of \( \Gamma' \) is the graph \( \Gamma/\Gamma' := (V'_t, F, \bar{i}) \), where \( V'_t \) denotes the connected components of \( \Gamma' \), \( \bar{F} \) are the flags not belonging to edges of \( \Gamma' \) and \( \bar{i} \) is the restriction of \( i \) to \( \bar{F} \). The disjoint union of the vertex-corollas \( agg(\Gamma) = \bigsqcup_{v \in V} v_{F_v} \) is a spanning subgraph of \( \Gamma \), which in general is distinct from \( \Gamma \) because its involution is the identity.

The Euler characteristic of a graph is defined by \( \chi(\Gamma) = |V| - |E| \). Let \( b_0 \) be the number of connected components of \( G \) and \( b_1 \) be the loop number of \( \Gamma \), i.e. the number of edges in the complement of a spanning forest of \( G \). Then \( \chi(\Gamma) = b_0 - b_1 \). We call the pair \( (b_0, b_1) \) the topological type of \( \Gamma \). A graph is a forest if and only if \( b_1 = 0 \) and a tree if moreover \( b_0 = 1 \).

There are two inclusion chains

Trees \( \subset \) Connected Graphs \( \subset \) Graphs \hspace{1cm} \text{and} \hspace{1cm} \text{Trees} \( \subset \) Forests \( \subset \) Graphs \hspace{1cm} (1.1)
If $\Gamma$ is connected and $T \subset \Gamma$ is a spanning tree, then $\Gamma/T$ is a rose with $b_1(\Gamma)$ loops. More generally, for a spanning forest $F \subset \Gamma$ the topological types of $\Gamma$ and $\Gamma/F$ are the same.

**Remark 1.1.** The two orderings of the pair $\{f, i(f)\}$ forming an edge can also be identified with the two ways of directing the edge, i.e. $(f, i(f))$ and $(i(f), f)$. This gives a precise meaning to the orientation of a loop.

### 1.2. Topological realization

Each graph can be realized as a one-dimensional topological space: this space is defined by attaching to the (discrete) set of vertices one closed interval for each edge and one semi-open interval for each outer flag, with attaching maps induced by $\partial$. Observe that outer flags are attached on one side only so that the topological realization of a graph with outer flags is not a CW-complex. Retracting the outer flags, one does obtain a CW complex. For the topological realization of a graph, $b_0$ and $b_1$ are the Betti numbers. The topological realization of a subforest contraction is a deformation retraction and does not change the topological type.

The distinction between edges and inner/outer flags is crucial for graphical Feynman categories.

At some places in literature, outer flags are represented by edges having a univalent vertex on one side. Such a choice yields a CW-structure on the topological realization and is understandable from a geometric point of view but is source of confusion from a combinatorial point of view. One is then compelled to distinguish between “inner” and “outer” vertices, and “inner” and “outer” edges, while in our setting these distinctions are built into the structure of a graph via its flag involution.

### 1.3. Ribbon, polycyclic graphs and genus/puncture labeling

A **cyclic ordering** of a finite set $S$ is given by a permutation $\sigma : S \to S$ such that the order of $\sigma$ equals the cardinality $\#S$ of $S$. Starting with an element $s \in S$, we obtain a cycle $s, \sigma(s), \ldots, \sigma^{\#S-1}(s), \sigma^0(s) = s$ which we shall represent as a cyclic word $(s \sigma(s) \cdots \sigma^{\#S-1}(s))$.

A **polycyclic ordering** of a finite set $S$ is given by a general permutation $\sigma : S \to S$. In this case there might be several orbits of $\sigma$ decomposing $S$. A polycyclic ordering of $S$ is then equivalent to an unordered partition of $S$, $S = C_1 \sqcup \cdots \sqcup C_b$, together with a cyclic ordering of each of the pieces $C_i$ ($i = 1, \ldots, b$)—whence the terminology “polycyclic”. If $S = \{1, \ldots, n\}$ then this decomposition is the cycle decomposition of the permutation $\sigma$. Observe that each cycle is cyclically ordered and the cycles commute with each other so that the coproduct of the permutations of the $F_v$ making up the polycyclic structure.

A **Sullivan graph** is a ribbon graph such that its boundary cycles are distinguished into “in–” and “out–” cycles and there are no edges both of whose flags belong to in–cycles.

A **surface-marked graph** is a polycyclic graph together with a genus and a puncture labeling.

**Example 1.3.** We consider a graph $\Gamma$ with one vertex and two loops, i.e. $\{1, 2, 3, 4\} \xrightarrow{\delta} \{\ast\}$ with $i(1) = 2, i(3) = 4$. There are six ribbon structures on $\Gamma$ represented respectively by the cyclic permutations $(1234), (1243), (1324), (1342), (1423), (1432)$. These fall into two isomorphism classes. The first isomorphism class has 3 boundary cycles, while the second isomorphism class has a single boundary cycle, cf. Figure 1.

Likewise, we could define polycyclic structures. Up to isomorphism, there is the trivial polycyclic structure $(1)(2)(3)(4)$, in which the number of boundary cycles is 2. Furthermore, we have the permutations $(12)(3)(4), (13)(2)(4), (123)(4), (12)(34), (13)(24)$. The first has 3, the second 1, the third 2, the fourth 4 and the last 2 boundary cycles.
Figure 1. Two non-isomorphic ribbon structures on the same graph. The blue arrow indicates the cyclic order at the vertex. The respective surfaces are the sphere with three holes and a torus with one boundary.

The automorphism group, defined below, for the \((1)(2)(3)(4)\) polycyclic structure has the full automorphism group of the underlying graph while \((123)(4)\) has trivial automorphism group.

Ribbon graphs are ubiquitous in the theory of moduli spaces [Pen87, Str84, Har85, Kon92], as is genus labeling [DM69, Knu83], see [Mon09] for a survey. The polycyclic structure and the puncture labeling are needed for combinatorial compactifications [Kon92, Pen87, Loo95, Zn15, Kau09]. Sullivan graphs are relevant for string topology [CS99, TZ07, Kau07].

The genus labeling also arises naturally from non–forest contractions as $$b_1(\Gamma/\Gamma') = b_1(\Gamma) - b_1(\Gamma')$$.

In particular, considering a rose $$r$$, $$r$$ is an one vertex aggregate without flags. To keep track of the rose structure one can simply label the aggregate by the loop number $$b_1(p_r q)$$. This will be formalized in §3.1.

1.4. Surface realizations. There are several surface realizations associated to structured graphs.

1.4.1. Surface with curve system associated to a graph. Each graph defines a curve system $$\alpha(\Gamma)$$, i.e an element in the curve complex [Har85] of a topological oriented surface $$\Sigma(\Gamma)$$ with labelled boundary, such that the curve system cuts the surface into topological spheres with boundaries. For this replace each $$k$$–valent vertex $$v$$ by a 2–sphere $$S^2_v$$ with $$k$$ discs removed. Label the resulting boundaries by the flag set $$F_v$$. For each edge $$\{f, v(f)\}$$ glue these spheres together at the corresponding boundary components labelled by $$f$$ and $$v(f)$$ and let $$C_e$$ be the image curve of the glued boundary. The remaining boundary components are labelled by the outer flags. We have $$b_0(\Sigma(\Gamma)) = b_0(\Gamma)$$ and $$b_1(\Sigma(\Gamma)) = 2b_1(\Gamma)$$. This readily implies that $$\chi(\Sigma(\Gamma)) = 2\chi(\Gamma)$$. An example is given in Figure.

The gluing along boundaries can be thought of as a connected sum operation. Indeed, the curve system $$\alpha(\Gamma)$$ induces a connected sum decomposition of $$\Sigma(\Gamma)$$ (whose boundary $$\partial(\Sigma(\Gamma))$$ is labelled by the set of outer flags of $$\Gamma$$) according to the formula $$\Sigma(\Gamma) \#_{e \in E} \sqcup_{v \in V} S^2_v$$. This is a higher generalisation of a pair of pants decomposition where surfaces with a higher number of boundaries, but not with higher genus, are allowed.

To obtain a graph from such a curve system, one takes a vertex for each component of the surface obtained by cutting along the curve system. The flags are the boundary components of the cut surface. The original boundary components are the outer flags and are labelled. The remaining boundary components are the two sides of a cut curve interchanged by the involution $$v$$. An example is given in Figure 2. An alternative way is to use an appropriate height function and to take its Reeb graph.

1.4.2. Surface with arc system associated to a ribbon graph. Each ribbon graph $$R$$ defines an arc system, i.e. an element of the arc complex, $$\alpha(R)$$ on a surface $$\Sigma(R)$$ with boundaries [Str84, Pen87], where now arcs run in between boundary components and cut the surface into polygons. This property is called quasi–filling. For each vertex $$v$$ of valence $$n_v$$ take a $$2n_v$$-gon and mark the sides of this polygon with the elements of $$F_v$$ in the given cyclic order, marking
Figure 2. The result of gluing a two- and a threevalent vertex as a decomposition of the torus with one boundary component with two cut curves.

Figure 3. The result of gluing two threevalent cyclic vertices together to the Θ graph and its thickening it to a sphere with three holes and its decomposition into two hexagons. The blue graphs are original graphs embedded as a spine in the surface, s, t, u, ˘s, ˘t, ˘u are the flags of the graph and equivalently the markings of the marked intervals of the polygons.

only each second side. Glue these polygons together according to i by gluing (for each flag f) the f-marked side to the i(f)-marked side. Then identify the glued sides as an arc on the resulting surface Σ(R). The boundary components of this surface correspond one-to-one to the boundary cycles of the ribbon graph R. In particular, they are polygonal circles. The outer flags give rise to marked intervals on the boundary. The surface Σ(R) has thus two types of boundary components. Those containing marked intervals, and those not containing any. We shall call the former boundary components marked and the latter unmarked. If R has no outer flags then all boundary components of Σ(R) are unmarked. Note that regardless of any marking all boundary components are hit by at least one arc. See Figure 3 for an example.

Conversely, given a quasi-filling arc system α on an oriented surface Σ with boundary, the ribbon graph R(α) is constructed as follows. Cut Σ along the arcs into 2nυ-gons. The centers of these 2nυ-gons define the set of vertices υ. The alternating sides of the polygons are the flags which inherit a cyclic ordering from the orientation of the surface. They are labelled either by intervals from the boundary, corresponding to outer flags, or by the edges of an arc. Choose a point on each marked boundary interval and on each arc. Insert an arc from the central vertex to each marked point of the boundary intervals or marked point of an arc, such that these inserted arcs do not intersect except at vertices. This yields a ribbon graph R(α) with a topological realization on the given surface Σ. Note that Σ deformation retracts onto R(α) which is therefore often called the spine of Σ. It is unique up to isotopy and combinatorially transverse to the given arc system α.

If n(R) is the number of boundary cycles of the ribbon graph R, then the Euler characteristic of the closed surface Σ(R) obtained by gluing in discs to the boundary components equals χ(Σ(R)) = χ(R) + n(R). For a connected ribbon graph R, the surface Σ(R) is connected, and genus and
Euler characteristic of the aforementioned closed surface determine each other by the formula \(\chi(\Sigma(R)) = 2 - 2g(\Sigma(R))\). For a connected ribbon graph \(R\) we set \(g(R) = 1 - \frac{1}{2}(\chi(R) + n)\). We call \((b_0(R), b_1(R), n(R))\) the topological type of the ribbon graph \(R\). If \(R\) is connected, then the pair \((g(R), n(R))\) is an alternative way of representing the topological type of \(R\). Note that \(n(R)\) depends on the ribbon structure of \(R\) while \((b_0(R), b_1(R))\) only depends on the graph underlying \(R\).

This construction is related to the previous one by doubling. One can double each 2–gon and glue together corresponding marked sides. This gives a sphere with labelled boundaries. The gluing is the gluing of this doubled graph. This explains the factor of 2 in the formula for the Euler characteristic.

1.4.3. **Surface with arc system associated to a genus/puncture labelled polycyclic graph.** Finally each genus and puncture marked polycyclic graph \(P\) defines a surface with boundaries, punctures and an arc system, [Kau07, Kau09]. There is now no filling constraint on the arc system. For each vertex \(v\) with genus/puncture labeling \((g_v, p_v)\) and orbit decomposition \(S_1\Omega \cdots \Omega S_b\) take a topological surface \(\Sigma_v\) of genus \(g_v\) with \(p_v\) unmarked boundary components —which can topologically be considered as being equivalent to punctures in the interior— and \(b\) marked boundary components, of which the \(i\)–boundary is a 2\(|S_i|\)–gon whose alternating sides are intervals marked by the elements of \(S_i\) in the cyclic order. Glue these surfaces together as above using \(i\) and keep the glued intervals as arcs as above. The result is an arc system \(\alpha(P)\) on a surface \(\Sigma(P)\) whose boundaries of this surface are again either marked or unmarked.

The converse construction again takes a vertex for each region cut out by the arc system and a flag for each marked boundary interval and each side of an arc. The incidence relations \(\partial\) and \(\iota\) as are as above. Since the surface is oriented, this gives a polycyclic decomposition of the flags at each vertex.

There are now several ways to realize the dual graph on the surface. Since the arc system is not quasi–filling, a dual graph can only be constructed by adding arcs until one reaches a quasi–filling arc system. For this one can quasi–triangulate the surfaces \(\Sigma_v\). This is a choice, however, and to undo this choice one should either consider an equivalence relation on the vertices [Kon92] or take equivalence classes under Whitehead moves on the triangulation of the \(\Sigma_v\) [Pen04]. Here a triangulation is given by a system of arcs running from boundary to boundary cutting the surface into hexagons. Shrinking the boundaries to points one obtains a system of arcs running between marked points decomposing the surface into triangles, whence the name. A Whitehead move replaces one edge by another edge as shown in Figure 4.

**Proposition 1.4.** The following notions are equivalent.

1. A genus and puncture marked polycyclic graph.
2. A ribbon graph with an equivalence relation on the vertex–set and a genus marking for each equivalence class.
3. Equivalence classes of pairs consisting of a ribbon graph and a trivalent spanning ribbon subgraph \(\Gamma_{ph} \subset \Gamma\), where two pairs are equivalent if and only if they transform into each other by Whitehead moves on the “phantom” part \(\Gamma_{ph}\).

The first notion is what is used in this article, cf. [Bar07, Kau09, Kau07, Kau08a].

**Proof.** (1) \(\iff\) (2) For a polycyclic order on \(F_v\) with \(b\) orbits and puncture marking \(p_v\), replace the vertex \(v\) by \(b + p_v\) distinct vertices \(v_1, \ldots, v_b, w_1, \ldots, w_{p_v}\) and attach to each \(v_i\) the flags of \(F_v\) belonging to the \(i\)-th orbit, keeping the same cyclic order. The \(w_i\) will have no flags. This defines a ribbon graph together with an equivalence relation on its vertices, and a genus for each equivalence class.

Conversely, given such an equivalence relation on the vertices of a ribbon graph, we can define in a straightforward way a polycyclic structure on the graph obtained by identifying the vertices in
the same equivalence class. The resulting polycyclic graph inherits the genus-labeling, the puncture marking is the number of 0-valenced vertices. These processes are inverses to each other.

(1) $\iff$ (3). Given a polycyclic vertex with a genus marking and puncture marking, again decompose $F_v = S_1 \amalg \cdots \amalg S_b$ replace each vertex $v$ with a trivalent connected ribbon graph $\Gamma_v$ whose associated surface $\Sigma(\Gamma_v)$ has genus $g_v$ and $b + p_v$ boundaries, such that the orbit-decomposition $S_1 \amalg \cdots \amalg S_b$ of $F_v$ corresponds one-to-one to the outer flags of the boundary cycles of $G_v$. This choice is not unique, but Whitehead moves act transitively on the set of trivalent ribbon graphs of a given topological type and the polycyclic structure of the outer flags is an invariant, so the process is well defined on equivalence classes.

Conversely, consider $\Gamma_{\text{vir}} \subset G$, the quotient $\Gamma/\Gamma_{\text{ph}}$ is a polycyclic graph. The genus and puncture marking of its vertices are those of the corresponding connected components of $\Gamma_{\text{vir}}$. As the topological type of the connected components and the polycyclic structure is invariant under Whitehead moves, this construction passes to the equivalence classes. These processes are again inverses to each other. □

Remark 1.5.

(1) Note that if the genus and puncture marked polycyclic graph is connected, its representation in the other two points of view need not be.

(2) In these equivalences unmarked boundary components are treated as punctures. It is possible to treat both unmarked boundary components and extra internal punctures, this becomes necessary if one considers open/closed theories, cf. [KP06, Kau10].

(3) The mapping class group acts on the curve and arc complexes. The underlying graphs are invariant under the mapping class group action. This means that the graphs without additional markings such as a fixed embedding into a surface, can be only be used to reconstruct moduli spaces as opposed to Teichmüller spaces.

2. Categories of Graphs

2.1. Graph morphisms and compositions. A graph morphism $\phi : (V, F, \partial, \iota) \to (V', F', \partial', \iota')$ is given by a triple $(\phi_V, \phi^F, \iota_\phi)$, consisting of a covariant surjection of vertices $\phi_V : V \to V'$, a contravariant injection of flags $\phi^F : F' \hookrightarrow F$ and a fixed point–free involution $\iota_\phi$ on the set $F \setminus \phi^F(F')$ of flags not contained in the image of $\phi^F$. The following constraints have to be satisfied:
(1) $\phi_V \circ \partial \circ \phi^F = \partial'$, and on the complement of image of $\phi^F$: $\phi_V \circ \partial = \phi_V \circ \partial \circ \iota_\phi$.

(2) If a flag $f$ does not belong to the image of $\phi^F$ then either $\{f, t(f)\}$ is an edge of $\Gamma$ (in which case $\phi$ is said to contract the edge), or both $f$ and $t(f)$ are outer flags (in which case $\{f, t_\iota(f)\}$ is called a ghost edge virtually contracted by $\phi$).

(3) Edges of $\Gamma$ that are not contracted are preserved. That is if $\{f, t(f)\}$ form an edge of $\Gamma$ and $f$ is in the image of $\phi^F$ then so is $t(f)$, and $t'(\phi^F)^{-1}(f) = (\phi^F)^{-1}(t(f))$.

The information about the involution $\iota_\phi$ is encoded in the ghost graph $\Gamma(\phi)$ of $\phi$, which is defined by $\Gamma(\phi) = (V, F, \tilde{\iota}_\phi)$ where $\tilde{\iota}_\phi$ is the extension of $\iota_\phi$ to all of $F$ by the identity.

For two graph morphisms $\phi = (\phi_V, \phi^F, \iota_\phi) : \Gamma \rightarrow \Gamma'$ and $\psi = (\psi_V, \psi^F, \iota_\psi) : \Gamma' \rightarrow \Gamma''$, the composition $\psi \phi : \Gamma \rightarrow \Gamma''$ is defined by setting $(\psi \phi)_V = \psi_V \phi_V$ and $(\psi \phi)^F = \phi^F \psi^F$. The involution $\iota_{\psi \phi}$ pairs two flags of the source graph if they are either paired by $\iota_\phi$ or they belong to the image of $\phi^F$ and their preimages are paired by $\iota_\psi$. Graphs and their morphisms form the category $\text{Gr}$.

The composition product of two ghost graphs is defined to be $\Gamma(\psi) \circ \Gamma(\phi) := \Gamma(\psi \phi)$. The composition product has the following description. The vertices and flags of $\Gamma(\psi) \circ \Gamma(\phi)$ are those of $\Gamma(\phi)$. The definition of $\iota_{\psi \phi}$ says that the edges of $\Gamma(\psi \phi)$ are the disjoint union of those of $\Gamma(\phi)$ and those of $\Gamma(\psi)$ pulled back along $\psi^F$. Starting from $\Gamma(\psi)$, the composition product expands the vertices $v' \in V'$ into the graphs $\Gamma(\phi_{v'})$. An example can be seen in Figure 5.

**Figure 5.** A composition of graph morphisms and ghost graphs. The graph morphisms are virtual edge contractions. The composition either adds the edges $\{e, m\}$ and $\{i, j\}$ to $\Gamma(\phi)$ or inserts the left triangular graph with vertices $u, v, w$ into the vertex $r$ of the right triangular graph respecting flag identifications.

The disjoint union endows the category $\text{Gr}$ with a monoidal structure. Every morphisms decomposes according to the connected components of its target. Given $\phi : \Gamma \rightarrow \Gamma'$ the fiber $\Gamma_{\phi^{-1}}$ over a
connected component \( \overline{v'} \) of \( \Gamma' \) is the subgraph given by the vertices \( \phi_{\overline{v'}}^{-1}(\overline{v'}) \) —thinking of \( \overline{v'} \) as the set of vertices in the equivalence class— together with all their flags and the restriction of \( \iota \). This restriction is possible, since all edges of \( \Gamma \) are either preserved or contracted. In this notation:

\[
\phi = \bigcup_{v' \in V'} \phi_{\overline{v'}}
\]

(2.1)

Notice that the preimages \( \Gamma_{\overline{v'}} \) need not be connected.

2.2. **Special types of morphisms.** From the definition it follows that graph isomorphisms are given by triples \( (\phi_V, \phi_F, \iota_\emptyset) \) such that \( \phi_V \) and \( \phi_F \) are bijections, \( \iota_\emptyset \) is the identity of the empty set, and \( \phi_F \) induces a bijection on the edges of the target graph to edges of the source graph. In particular, graph automorphisms may permute edges, flags and vertices as long as the incidence relations and the flag involutions are preserved.

Graftings are graph morphisms for which \( \phi_V \) and \( \phi_F \) are bijections, but \( \phi_F \) is not necessarily edge-preserving, i.e. source and target may have different flag involutions. The source edges are preserved, but the target may contain additional edges comprised of outer flags of the source which are called grafted edges.

Mergers are graph morphisms, for which \( \phi_F \) is an edge-preserving bijection while \( \phi_V \) may be arbitrary.

Contractions are morphisms, in which \( F \setminus \phi_F(F') \) is a collection of edges of \( \Gamma \) and two vertices are in the same fiber of \( \phi_V \) only if they are joined by a path of these edges.

It is readily verified that graph isomorphisms belong to all of the three classes, and that the three classes are closed under composition. We denote by \( \Gr^{\text{graft}}, \Gr^{\text{merge}}, \Gr^{\text{contr}} \) the wide subcategories of \( \Gr \) generated by graftings, mergers, and contractions respectively, cf. Corollary 2.7. Moreover, we will see below that graftings and contractions together also generate a wide subcategory which we shall denote \( \Gr^{\text{ctd}} \) because the morphisms in \( \Gr^{\text{ctd}} \) may be characterized as those graph morphisms whose fibres are connected subgraphs of the source graph.

A loop contraction means that all contracted edges are loops and a forest contraction is a contraction none of whose ghost edges form a cycle. This condition is equivalent to the condition that the ghost graph is a forest. Forest contractions do not change the topological type.

**Example 2.1.** The single rose graph given by \( V = \{v\}, F = \{s, t\} \) and \( \iota(s) = t \) has automorphism group \( \mathbb{Z}/2\mathbb{Z} \) with the generator given by \( \phi_F(s) = t \). The automorphism group of the \( n \)-fold rose, given by \( V = \{v\}, F = \{s_1, t_1, \ldots, s_n, t_n\} \) and \( \iota(s_i) = t_i \), is \((\mathbb{Z}/2\mathbb{Z})^n \wr \Sigma_n\), where the first factor switches the \( s_i \) and \( t_i \) and the \( \Sigma_n \) action permutes the edges \( \{s_i, t_i\} \).

**Lemma 2.2.** A tree automorphism is determined by its action on outer flags and on univalent vertices. The only tree automorphism fixing outer flags and univalent vertices is the identity.

**Proof.** By hypothesis, the automorphism is determined on univalent vertices and on outer flags. It follows that the automorphism is also determined on inner flags attached to univalent vertices and on vertices attached to outer flags. Therefore, deleting all univalent vertices and outer flags from the tree leaves us with a strictly smaller tree restricted to which the automorphism satisfies the same hypothesis. An easy induction allows us to conclude the statements. \( \square \)

The inclusion of a spanning subgraph \( \iota_{\Gamma'}: \Gamma' \rightarrow \Gamma \) is a grafting of the edges not in \( \Gamma' \). Dually, the dissection along such a subgraph is given by cutting the edges of \( \Gamma' \), that is the graph \((V_\Gamma, F_\Gamma, \partial_\Gamma, \iota_\Gamma')\) where the new flag involution is the identity on the flags of \( \Gamma' \) and equal to \( \iota_\Gamma \) otherwise. If \( \Gamma' = \Gamma \) then we get the total dissection (or underlying aggregate) \( \text{agg}(\Gamma) \) of \( \Gamma \) which comes equipped with a canonical grafting \( \iota_\Gamma: \text{agg}(\Gamma) \rightarrow \Gamma \).

The quotient \( \Gamma/\Gamma' \) is defined to be the graph whose vertices are the connected components of \( \Gamma' \). The flags of \( \Gamma/\Gamma' \) are the outer flags of \( \Gamma' \). The flag involution on \( \Gamma/\Gamma' \) is given as restriction of \( \iota_\Gamma \). The total contraction is the quotient \( c_\Gamma: \Gamma \rightarrow \Gamma/\Gamma \) where \( \Gamma/\Gamma \) is the aggregate whose vertices
are the connected components of $\Gamma$ and whose flags are the outer flags of $\Gamma$, each outer flag being attached to its connected component. It follows from a simple computation that $c_p\varphi$ preserves the number of connected components, and if $\Gamma'$ has loop number $b_1(\Gamma')$ then $b_1(\Gamma) = b_1(\Gamma/\Gamma') + b_1(\Gamma')$. Since $b_1(\Gamma') = b_1(\Gamma(c_p\varphi))$, we see that the drop in loop number is encoded in the morphism and kept track of by the ghost graph. If $\Gamma'$ is a spanning subforest, then the contraction does not change the topological type.

The total dissection $agg(\Gamma)$ and total contraction $\Gamma/\Gamma'$ are aggregates. As the graftings $i_\Gamma : agg(\Gamma) \to \Gamma$ and the contractions $c_\Gamma : \Gamma \to \Gamma/\Gamma'$ are natural in $\Gamma$, these constructions actually define functors $Gr \to Agg$.

2.3. Simple generators for graph morphisms. Although we have a global presentation of the category $Gr$, for further analysis and to perform calculations, it is useful to give a presentation of the morphisms in terms of generators and relations. To this end, we provide a structural theorem which refines and concretizes statements about generators and decompositions made in [BM08]. This also allows us to analyze several wide subcategories.

There are the following three standard simple morphisms which together with the isomorphisms generate all graph morphisms:

(i) A simple grafting is the grafting of two flags $s, t$ of a graph $\Gamma$ into an edge. As a morphism $s \varnothing t : \Gamma \to \Gamma'$ is given by $(id_V, id_F, i)$ where $\Gamma' = (V, F, t')$ with $t'(s) = t$ and $t'(f) = i(f)$ if $f \neq s, t$.

(ii) A simple contraction $c_e = c_{\{s, t\}}$ of an edge $e = \{s, t\}$ of a graph $\Gamma$ is given by $\Gamma \to \Gamma'$ where $V' = V/\sim$ where $\hat{\partial}(s) \sim \hat{\partial}(t)$, $F' = F\setminus\{s, t\}$, $t' = i|_{F'}$, and the morphism is given by the quotient map $\phi_V : V \to V'$, the inclusion $\phi^F(F') \hookrightarrow F$ and $i_\phi(s) = t$. Note that if $\{s, t\}$ is a loop then $V' = V$.

(iii) A simple merger is the merging of two vertices $v$ and $w$. As a morphism it is given by $v \Box w : \Gamma \to \Gamma'$, where $V' = V/\sim$ where $v \sim w$, $F' = F$ and $t' = i$ with the morphism given by $i : V \to V'$ the projection, $id_F$ and the empty map $i_{\Box}$.

A morphism which is the composition of simple morphisms is called pure. Both the inclusions $i_\Gamma$ and the total contractions $c_\Gamma$ are pure.

**Proposition 2.3.** The following relations hold.

1. Generators commute among themselves. Given two pairs of outer flags $s, t$ and $s', t'$, two edges $e_1 = \{s_1, t_1\}$ and $e_2 = \{s_2, t_2\}$, two pairs of vertices $v, w$ and $v', w'$

$$s \varnothing t s' \varnothing t' = s' \varnothing t s \varnothing t' \quad c_{e_1} c_{e_2} = c_{e_2} c_{e_1} \quad v \Box w v' \Box w' = v' \Box w' v \Box w$$

2. Mixed relations. Graftings commute with the other generators.

$$s' \varnothing t' c_e = c_{es} \varnothing t' \quad s \varnothing t v \Box w = v \Box ws \varnothing t$$

3. Isomorphisms. Isomorphisms and simple morphisms are crossed in the following sense. Given an isomorphism $\sigma$ there exists a unique isomorphism $\sigma'$ such that

$$s \varnothing t \sigma = \sigma_{\varnothing F(s)} \varnothing t \sigma_{\varnothing F(t)} \quad \sigma_{V(v) \Box V(w)} \sigma = \sigma'_{v \Box w} \quad c_{\{s, t\}} \sigma = \sigma' c_{\{\varnothing F(s), \varnothing F(t)\}}$$
In the case of a merger $\sigma^{1F} = \sigma^F$, $\sigma'_V(w) = \sigma(w)$ for $w \neq u, v$ and $\sigma'([u, v]) = \{\sigma_V(u), \sigma_V(v)\}$ to conform with the conventions of the simple mergers. In the case of a contraction $\sigma^{1F} = \sigma^F|_{F \setminus (\sigma^F(s), \sigma^F(t))}$ and in the case of a loop contraction $\sigma'_V = \sigma_V$ while for a non-loop contraction $\sigma'_V(w) = \sigma(w)$ for $w \neq u, v$ and $\sigma'([u, v]) = \{\sigma_V(u), \sigma_V(v)\}$.

**Proof.** These are straightforward computations. \qed

An example of a grafting composed with an edge contraction is given in Figure 6.

![Figure 6. A composition of morphisms with ghost graphs, the first is a grafting, the second an edge contraction, the composition is a virtual edge contraction.](image)

**Theorem 2.4** (Structure Theorem I). Every morphism in $\text{Gr}$ can be uniquely factored in two ways

$$\phi = \sigma \phi_{\text{con}} \phi_{gr} \phi_{m} = \sigma \phi_{m} \phi_{\text{con}} \phi_{gr} \tag{2.6}$$

where $\sigma$ is an isomorphism, $\phi_{m}$ is a pure merger, $\phi_{gr}$ is a pure grafting and $\phi_{\text{con}}$ is a pure contraction. In the second decomposition $\phi_{\text{con}}$ can be further decomposed non-uniquely as $\phi_{\text{con-t}} \phi_{\text{con-f}}$ where $\phi_{\text{con-t}}$ is a pure loop contraction and $\phi_{\text{con-f}}$ is a pure forest contraction.

**Proof.** For $\phi : \Gamma \to \Gamma'$ factor $\phi_V : V \to V'$ as $\phi_V = \sigma_V \pi$ where $\pi$ is the quotient map $V \to \tilde{V}$ identifying the fibers and set $\phi_m = (\pi, \text{id}_F, t_{\emptyset}) : (V, F, t) \to \Gamma_m := (\tilde{V}, F, t)$, which is a composition of simple mergers. Next, factor the inclusion $\phi^F : F' \to F$ as $\sigma^F i$ where $i : \phi^F(F') \to F$ is the inclusion of the image. Define $\Gamma_{gr}$ to be the graph $(\tilde{V}, F, t_{gr})$ which has the edges of $\Gamma'$ and the ghost edges of $\phi$. That is $t_{gr}(\phi^F(f')) = \phi^F(i'(f'))$ and $t_{gr}(f) = i_{\phi}(f) \text{ if } f \notin \phi^F(F')$ and let $\phi_{gr} : \Gamma_m \to \Gamma_{gr}$ be given by the map $(id_{\emptyset}, \text{id}_F, t_{\emptyset})$. In the next step, let $\Gamma = (\tilde{V}, \phi^F(F'), \phi^F i')$ and define $\phi_{\text{con}} : \Gamma_{gr} \to \Gamma$ by $(id_{\emptyset}, i, t_{\phi})$. Finally define $\sigma = (\sigma_V, \sigma^F, t_{\emptyset})$. It is clear that this is a decomposition. The uniqueness follows from the construction.

The rest of the statement follows from Proposition 2.3. \qed

**Remark 2.5.** Note that postcomposing with an isomorphism leaves the ghost graph invariant $\Gamma(\phi) = \Gamma(\sigma \phi)$ while precomposing with an isomorphisms yields an isomorphic ghost graph with the isomorphism of graphs induced by $\sigma$: $\sigma : \Gamma(\phi) \xrightarrow{\sim} \Gamma(\sigma \phi)$ where by abuse of notation $\sigma$ is the morphism having the same components, but having a different source and target. The non-uniqueness of the decomposition of $\phi_{\text{con}}$ into loop and subforest contractions is precisely given by choosing one edge in each cycle of $\Gamma(\phi)$ which when contracted last is the loop contraction. Equivalently, this decomposition is fixed by a spanning tree for each component of $\Gamma(\phi)$.

The following two corollaries are immediate.
Corollary 2.6. The isomorphisms together with $v \Box w, s \Theta t, c_e$ generate $\text{Gr}$ with quadratic and triangular relations. The isomorphisms together with $v \Box w, s \Theta t$ and simple loop contractions generate a subcategory with quadratic relations.

For any $\phi$ and any precomposable isomorphism $\sigma$ there are unique $\phi'$ and $\sigma'$ with $\Gamma(\phi \sigma) = \Gamma(\phi')$ such that

$$\phi \sigma = \sigma' \phi'$$

(2.7)

and, moreover, this makes $\text{Gr}$ into a crossed product of pure morphisms and isomorphisms. □

Corollary 2.7. Restricting the types of generators yield wide subcategories:

(i) Isomorphisms, graftings and contractions define a wide subcategory $\text{Gr}^{\text{ctd}}$.

Every morphism in $\text{Gr}^{\text{ctd}}$ can be uniquely factorized as $\phi = \sigma \text{con} \phi \text{gr}$.

(ii) Isomorphisms and contractions define a wide subcategory $\text{Gr}^{\text{contr}}$ of $\text{Gr}^{\text{ctd}}$.

(iii) Isomorphisms and forest contractions define a wide subcategory $\text{Gr}^{\text{forest}}$ of $\text{Gr}^{\text{contr}}$.

(iv) Isomorphisms and graftings define a wide subcategory $\text{Gr}^{\text{graft}}$ of $\text{Gr}^{\text{ctd}}$.

2.4. Simple generators for morphisms between aggregates. An important role is played by the full subcategory $\text{Agg}$ of $\text{Gr}$ whose objects are aggregates. It underlies the categories relevant for operadic structures including those in algebra and geometry. In this category, general graphs appear as ghost graphs of the morphisms.

Graftings do not preserve aggregates and aggregates have no edges which can be contracted. Instead the following morphism takes over their role as generators.

(iv) A simple virtual edge/loop contraction of two flags is $s \circ t := c_{\{s,t\}} s \Theta t$. If $\{s, t\}$ is a loop, we write $\circ_{st} = c_{\{s,t\}} s \Theta t$. (This notation is in accordance with the standard notation for modular operads.)

An example of a virtual edge contraction as a grafting followed by a merger is depicted in Figure 6. The generating morphisms are given in Figure 7.

![Diagram](image)

**Figure 7.** Generating morphisms of $\text{Agg}$, virtual edge contractions $s \circ t$, virtual loop contractions $\circ_{st}$ and mergers $v \Box w$ and their ghost graphs. The shaded region for the merger indicates the data $\phi_V$

Corollary 2.8. The relations among the virtual edge/loop contractions are

$$s \circ_{t'} s' \circ t' = s' \circ t' s \circ t \text{ if } \{\partial s, \partial t\} \neq \{\partial s', \partial t'\}$$

$$\circ_{s't'} s \circ t = \circ_{st} s' \circ t' \text{ if } \{\partial s, \partial t\} = \{\partial s', \partial t'\}$$

(2.8)
with the mixed relations
\[ s \circ_t v \square_w = v \square_w s \circ_t \text{ if } \{ \partial s, \partial t \} \neq \{ v, w \} \]
\[ s \circ_t = \circ st v \square_w \text{ if } \{ \partial s, \partial t \} = \{ v, w \} \]
(2.9)

The relation with isomorphisms are
\[ s \circ_t \sigma = \sigma' \sigma F(s) \circ \sigma F(t) \quad \circ st \sigma = \sigma' \circ \sigma F(s) \sigma F(t) \]
(2.10)

with \( \sigma' \) as in (2.5).

Proof. These follow directly from Proposition 2.3.

This description was obtained directly in [KW17, §5]. A pure morphism is again a morphism that is a composition of theses \( v \square_w, \circ st \) and \( s \circ_t \). Note that any graph \( \Gamma \) appears as the ghost graph of the pure morphism \( v_\Gamma = c_\Gamma i_\Gamma \) (called the virtual contraction of \( \Gamma \)) whose source aggregate is the total dissection of \( \Gamma \) and whose target aggregate is its total contraction. As the ghost graph is invariant under post-composing with isomorphisms or mergers this is not the only morphism whose ghost graph is \( \Gamma \), in general.

Let \( \text{Agg}^{ctd} \) be the full subcategory of \( \text{Gr}^{ctd} \) whose objects are aggregates. The morphisms in \( \text{Agg}^{ctd} \) are precisely those for which the connected components of the ghost graphs are connected graphs. Note that this property fails for the category \( \text{Agg} \) because of the existence of mergers. Define the subcategory \( \text{Agg}^{\text{forest}} \) by restricting to forests as ghost graphs.

Corollary 2.9. For a pure morphism \( \phi : X \to Y \) in \( \text{Agg}^{ctd} \), the source may be identified with the total dissection \( \text{agg}(\Gamma(\phi)) \), the target with the total contraction \( \bar{\Gamma}(\phi)/\bar{\Gamma}(\phi) \), and the morphism itself with \( \bar{\Gamma}(\phi) = c_{\bar{\Gamma}(\phi)} i_{\bar{\Gamma}(\phi)} \). Therefore, any pure morphism in \( \text{Agg}^{ctd} \) is uniquely determined by its ghost graph.

Proof. By the relations (2.2) one can preform all the grafting before the contractions. By definition of virtual contraction, all glued edges are contracted.

Theorem 2.10 (Structure Theorem II). Every morphism in \( \text{Agg} \) can be uniquely factored as \( \sigma \phi_p \) with \( \phi_p \) is a pure morphism and this decomposition can be further uniquely factored in two ways
\[ \phi = \sigma \phi_{\text{con}} \phi_m = \sigma \phi_m \phi_{\text{con}} \]
(2.11)
where \( \sigma \) is an isomorphism, \( \phi_m \) is a pure merger, and \( \phi_{\text{con}} \) is a pure contraction, which in the first decomposition is a loop contraction. In the second decomposition, \( \phi_{\text{con}} \) can be further decomposed non-uniquely as \( \phi_{\text{con}} = \phi_{\text{con}-l} \phi_{\text{con}-f} \) where \( \phi_{\text{con}-l} \) is a pure loop contraction and \( \phi_{\text{con}-f} \) is a pure forest contraction.

Morphisms in \( \text{Agg}^{ctd} \) uniquely decompose as \( \phi = \sigma \phi_{\text{con}} \) and in \( \text{Agg}^{\text{forest}} \) as \( \phi = \sigma \phi_{\text{con}-f} \).

Proof. Since an aggregate has no edges, there are no contractions of actual edges and any grafted edge has to be contracted. The claims then follow from Theorem 2.3 and Proposition 2.3.

Remark 2.11. This theorem was obtained in [Kau18, §4] and formalizes the observations in [KW17, §2.1.1]. The ghost graph captures \( \phi_{\text{con}} \) while the additional data to determine \( \phi \) is the isomorphism \( \sigma \), and in the presence of mergers, the information which components of the ghost graph belong to the same fibers of \( \phi \).

Corollary 2.12. The category \( \text{Agg}^{ctd} \) is a crossed product of pure morphisms and isomorphisms.

Proof. Immediate from Corollary 2.6.

Remark 2.13. The category of graphs of [BM08] has a slightly different definition of \( \iota_\phi \) which is only defined on the outer flags of \( F \setminus \phi^F(F') \). This is equivalent to the current definition, the canonical extension to the inner flags that are not in the image of \( \phi^F \) being given by \( \iota_\Gamma \). By restricting to
outer flags this keep track only of virtually contracted edges, while not doing this restriction keeps track of all contracted edges, virtual and actual. The ghost graph for the subcategory Agg and the composition of ghost graphs was defined in [KW17] and elaborated upon in [Kau18], which also contains the tweak to the definitions of [BM08] used here.

2.5. **Double category relating graphs and aggregates.** The two categories Gr and Agg are related by three functors. The first is the inclusion functor \( i : \text{Agg} \to \text{Gr} \). The other two functors are given by total dissection \( s = \text{agg} : \text{Gr} \to \text{Agg} \) and total contraction \( t = \{-/-\} : \text{Gr} \to \text{Agg} \).

From now on, we shall call \( sp\Gamma q, \) resp. \( tp\Gamma q\), source and target aggregate of \( \Gamma \). We have the relations

\[
s \circ i = t \circ i \quad \text{id}_{\text{Agg}}.
\]

The values of the source/target aggregate functors on simple generators are as follows:

\[
s(p\sigma q) = \sigma \quad t(p\sigma q) = \sigma \quad s(p\varphi q) = \varphi \quad t(p\varphi q) = \varphi
\]

and

\[
E_{\Gamma} (\phi_R \phi) = E_{\Gamma}(\phi) \cup E_{\Gamma}(\phi_R) = E_{\Gamma}(\phi_L) \cup E_{\Gamma}(\phi') = E_{\Gamma}(\phi' \phi).
\]
We now define such “decorating” functors on the categories of the preceding chapter. In Gr they with the respective projection functors. There is a projection functor \( \pi \) of construction defines a new category \( F \) sense of Grothendieck. Given a category \( F \) \( \phi \) isomorphisms and the data of the source and target aggregates. To capture the whole picture one would need to add the encode all the morphisms of \( \text{Agg} \), but only the pure connected ones. This is due to the definition Remark 2.18. Connections and holonomy and thin structure were introduced in [BM99]. The Proof. Straightforward from the definitions. Restricting the vertical morphisms to be pure, this yields a thin structure. Corollary 2.16. For a morphism \( \phi \) in \( \text{Agg}^{\text{ctd}} \) there is an isomorphism \( \text{Aut}(\phi) \) in the category of arrows with the automorphism group of \( \Gamma(\phi) \). Proof. Every such morphism \( \phi \) is isomorphic to a pure morphism. For pure morphisms this follows from the proposition above. □ Proposition 2.17. Restricting to pure morphisms in \( \text{Gr}^{\text{ctd}} \) and \( \text{Agg}^{\text{ctd}} \) the ghost graph is a holonomy and the restricted double category has a pair of connections \( s(\phi) = c_{\Gamma(\phi)} \) and \( t(\phi) = \text{gr}_{\Gamma(\Gamma)} \) and is thin.

\[
\begin{array}{ccc}
X \xrightarrow{\Gamma(\phi_p)} & Y & X \\
\phi_p & \downarrow c_{\Gamma(\phi_p)} & \downarrow \text{id} & \downarrow \text{id} & \downarrow \text{id} & \downarrow \phi_p \\
Y & \xrightarrow{\text{id}} & Y & X & \xrightarrow{\text{id}} & Y
\end{array}
\]

(2.16) Proof. Straightforward from the definitions. □

Remark 2.18. Connections and holonomy and thin structure were introduced in [BM99]. The second part of Proposition 2.17 shows that in a sense the category \( \text{Gr} \) is deficient, since it does not encode all the morphisms of \( \text{Agg} \), but only the pure connected ones. This is due to the definition of the source and target aggregates. To capture the whole picture one would need to add the isomorphisms and the data of \( \phi_V \), see Remark 2.11.

3. Decorating functors

Decorations are best understood in terms of functors and their categories of elements in the sense of Grothendieck. Given a category \( \mathcal{F} \) and a set-valued functor \( \mathcal{O} : \mathcal{F} \to \text{Set} \) Set the Grothendieck construction defines a new category \( \mathcal{F}_{\text{dec}}(\mathcal{O}) \) whose objects are pairs \((X, a_x)\), consisting of an object \( X \) of \( \mathcal{F} \) and a “decorating” element \( a_x \) of \( \mathcal{O}(X) \), and whose morphisms are given by

\[
\mathcal{F}_{\text{dec}}(\mathcal{O})(((X, a_x), (Y, a_y)) = \{ \phi \in \mathcal{F}(X, Y) : \mathcal{O}(\phi)(a_x) = a_y \}.
\]

There is a projection functor \( \pi : \mathcal{F}_{\text{dec}}(\mathcal{O}) \to \mathcal{F} \) taking \((X, a_x)\) to \( X \). Each natural transformation \( \eta : \mathcal{O} \to \mathcal{O}' \) induces a functor \( \mathcal{F}_{\text{dec}}(\mathcal{O}) \to \mathcal{F}_{\text{dec}}(\mathcal{O}') \) taking \((X, a_x)\) to \((X, \eta_X(a_x))\), and compatible with the respective projection functors.

In order to apply Grothendieck’s construction, we thus have to promote decorations to functors. We now define such “decorating” functors on the categories of the preceding chapter. In Gr they
decorate the graphs directly, in Agg they decorate aggregates and thereby the ghost graphs by pullback with respect to the source aggregate.

3.1. Genus labeling as a decoration. The genus labeling on the vertices can be promoted to a monoidal functor $O_{\text{genus}} : (\text{Gr}, \sqcup) \to (\text{Set}, \times)$. On objects it is given by $O_{\text{genus}}(\Gamma) = \mathbb{N}_0^{\Gamma}$ and on generators as follows: for $g : V_{\Gamma} \to \mathbb{N}_0$

$$O_{\text{genus}}(\sigma)(g)(v') = g(\sigma^{-1}(v')) \quad O_{\text{genus}}(v \sqcup w)(g)(\{v, w\}) = g(v) + g(w) - 1$$

where all other non-indicated values of $g$ are unchanged.

A global formula is given by considering the fibers of the ghost graphs $\bar{\Gamma}(\phi)_{v'}$ given by the vertices $\phi^{-1}(v')$, outer flags, $\phi^F(F_{v'})$ and the ghost edges whose vertices lie in $\phi^{-1}(v')$.

$$\left( O_{\text{genus}}(\phi)(g) \right)(v') = \sum_{v \in \phi^{-1}(v')} \left[ g(v) + (1 - \chi(\phi_v)) \right]$$

(3.2)

One easily computes that this is a functor using the relation or that composition of morphisms corresponds to the composition product of the ghost graphs.

This restricts to Agg along the inclusion $i : \text{Agg} \to \text{Gr}$ and gives the following values on the simple generators of Agg:

$$O_{\text{genus}}(s \circ t)|_{\{\hat{v}, \hat{t}\}} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0, \quad O_{\text{genus}}(s \circ t)|_{\hat{v}} : \mathbb{N}_0 \to \mathbb{N}_0 : n \mapsto n + 1,$$

$$O_{\text{genus}}(v \sqcup w)|_{\{v, w\}} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 : (m, n) \mapsto m + n - 1.$$  

(3.3)

with objects of the element category $\text{Gr}_{\text{dec}}(O_{\text{genus}})$ are genus labelled graphs. The restriction to $\text{Agg}_{\text{ctd}}$ is related to modular operads via $\text{Agg}_{\text{ctd}}(O_{\text{genus}})$, see §4.4. On mergers this functor is determined by (2.9). This extension was introduced in [KWZ15, §7.A.3] to define the non-connected (nc) version of modular operads. The grading in loc. cit. is as in the open gluing of [KP06, Appendix A] and corresponds to $1 - \chi$.

Remark 3.1. The name genus stems from the connection to modular operads and the moduli spaces $M_{g,n}$. A more appropriate interpretation and the right combinatorial notion is $1 - \chi$, this is the negative reduced Euler characteristic. For a connected graph $1 - \chi(\Gamma) = b_1$ is the loop number. This is closely related to the geometric genus of associated surface representations, cf. §1.4 and Lemma 3.7 below.

The extension to Gr above is the pull–back along $s$: $O_{\text{genus}} = s^*(O_{\text{genus}i}) = O_{\text{genus}is}$. There is also the pull–back by $t$ that is $O_{\text{genus}it}$. The objects of the element category $\text{Gr}_{\text{dec}}(O_{\text{genus}it})$ are graphs whose components are marked by a genus.

3.2. Polycyclic orders as decoration. For each graph $\Gamma$ define

$$O_{\text{poly}}(\Gamma) = \{\text{Polycyclic orders on each of the corollas } F_v \} \cong \prod_{v \in V} \text{Aut}(F_v)$$

(3.4)

This extends to a monoidal functor with respect to disjoint union of graphs. We define the action of the functor $O_{\text{poly}}$ on simple generators of Gr. On graph isomorphisms $O_{\text{poly}}$ acts by relabeling, i.e. via conjugation by $\phi^F$. On graftings $O_{\text{poly}}$ is the identity and on mergers $O_{\text{poly}}$ is just the union of the polycyclic structures. For simple contractions, if $\{s, t\}$ is not a loop, i.e. $\hat{s} \neq \hat{t}$
then $O_{\text{poly}}(c_{s,t})(\sigma_S, \sigma_T)$ is the polycyclic order induced by $\sigma_S\sigma_T\tau_{st}$ on $S\backslash\{s, s, \sigma^{-1}_T(t)\}$, where $\tau_{st}$ interchanges $s$ and $t$. Explicitly,

$$O_{\text{poly}}(c_{s,t})(\sigma_S, \sigma_T)(x) = \begin{cases} 
\sigma_S(x) & \text{if } x \in S\backslash\{s, \sigma_S^{-1}(s)\} \\
\sigma_T(x) & \text{if } x \in S\backslash\{t, \sigma_T^{-1}(t)\} \\
\sigma_T(t) & \text{if } x = \sigma_S^{-1}(s) \\
\sigma_S(s) & \text{if } x = \sigma_T^{-1}(t)
\end{cases} \quad (3.5)$$

If $\{s, s'\}$ is a loop, i.e. $\partial s = \partial s'$ then $O_{\text{poly}}(c_{s,s})(\sigma_S)$ is the polycyclic order on $S\backslash\{s, s'\}$ given by:

$$O_{\text{poly}}(c_{s,s})(\sigma_S)(x) = \begin{cases} 
\sigma_S(x) & \text{if } x \in S\backslash\{s, s', \sigma^{-1}_S(s), \sigma^{-1}_S(s')\} \\
\sigma_S(s') & \text{if } x = \sigma^{-1}_S(s) \\
\sigma_S(s) & \text{if } x = \sigma^{-1}_S(s')
\end{cases} \quad (3.6)$$

There is no simple formula for general graph morphisms $\phi$, since the action does not only depend on the local structures.

Objects of the element category $\text{Gr}_{\text{dec}}(O_{\text{poly}})$ are polycyclic graphs. Polycyclic graphs have also been called almost ribbon graphs [Kau09] or stable ribbon graphs [Kon92, Bar07] and occur in the combinatorial compactification of moduli spaces, cf. [Pen87, Kau09]. Notice that there are several distinct notions of stability in this context, certain involving the condition of having at least trivalent vertices and/or the condition of having negative Euler characteristic, which we currently do not impose.

**Remark 3.2.** Starting with a cyclic order $\sigma$ on a set $S$, and performing a loop contraction produces a polycyclic order on $S\backslash\{s, s'\}$ and similarly for mergers. This explains why polycyclic orders are unavoidable in modular situations. It is a tedious and futile, but sobering, exercise to try to find a well-defined self-composition for cyclic structures. As test case the reader should consider the mutation of Figure 8 for any definition that is proposed and see that the putative structure will not be well defined.

**Lemma 3.3.** *Ribbon graphs form a full subcategory* $\text{Rib}_{\text{forest}}$ of $\text{Gr}_{\text{dec}}(O_{\text{poly}})$ and $\text{Rib}_{\text{graft}}$ of $\text{Gr}_{\text{dec}}(O_{\text{poly}})$, *but are not stable with respect to all of the operations of* $\text{Gr}_{\text{dec}}(O_{\text{poly}})$.

*Proof.* Isomorphisms preserve the number of orbits. A non-loop contraction of two cyclic orders $\sigma_S$ and $\sigma_T$ again produces a cyclic order $O_{\text{poly}}(c_{s,t})(\sigma_S, \sigma_T)$. Graftings act as identities. Both loop contractions and mergers, however, produce polycyclic orders if the input consists of cyclic orders only. \hfill $\square$

The full ribbon graph subcategory of $\text{Gr}_{\text{dec}}(O_{\text{poly}})$ has been considered in [WW16] while the full ribbon graph subcategory of $\text{Gr}_{\text{dec}}(O_{\text{poly}})$ has been used by [Igu02] and is crucial for us, cf. §5.2.

**Remark 3.4.** If mergers are present, a polycyclic order is always the result of merging cyclic orders. This relates to the definition of stable ribbon graph of Kontsevich [Kon92], cf. Proposition 1.4.

The functor $O_{\text{poly}}$ restricts to aggregates and corollas as follows:

$$O_{\text{poly}}(*S) = \{\text{Polycyclic orders } \sigma_S \text{ on } S \} \cong Aut(S) \quad (3.7)$$
On isomorphisms the $O_{\text{poly}}$ acts via conjugation with $\phi^F$ as before. Polycyclic orders compose under mergers as before: $O_{\text{poly}}(\ast)O((\sigma_S, \sigma_T)) = \sigma_S \times \sigma_T$. On virtual contractions:

$$O_{\text{poly}}(\ast)(\sigma_S, \sigma_T)(x) = \begin{cases} 
\sigma_S(x) & \text{if } x \in S\setminus\{s, \sigma_S^{-1}(s)\} \\
\sigma_T(x) & \text{if } x \in S\setminus\{t, \sigma_T^{-1}(t)\} \\
\sigma_T(t) & \text{if } x = \sigma_S^{-1}(s) \\
\sigma_S(s) & \text{if } x = \sigma_T^{-1}(t)
\end{cases} \quad (3.8)$$

$$O_{\text{poly}}(\circ)(\sigma_S)(x) = \begin{cases} 
\sigma_S(x) & \text{if } x \in S\setminus\{s, s', \sigma_S^{-1}(s), \sigma_S^{-1}(s')\} \\
\sigma_S(s') & \text{if } x = \sigma_S^{-1}(s) \\
\sigma_S(s) & \text{if } x = \sigma_S^{-1}(s')
\end{cases} \quad (3.9)$$

Note that $\sigma_S \circ \sigma_T$ corresponds to the usual block composition of permutations. We see that for aggregates only $\circ$ preserves cyclic orders. The restriction to the wide subcategory generated by these defines $\text{Agg}^\text{pforest} \subset \text{Agg}^{\text{dec}}(O_{\text{poly}})$, where $\text{pforest}$ stands for planar forests, because the cyclic orders on the vertices induces a planar embedding of the forest. This embedding is unique up to isotopy.

Pulling back along $t : \text{Gr} \to \text{Agg}$ the restriction $i^*O_{\text{poly}} : \text{Agg} \to \text{Set}$ defines graphs equipped with a polycyclic order of the set of the outer flags. In other words, the objects of the element category $\text{Gr}_{\text{dec}}(t^*i^*O_{\text{poly}})$ can be thought of as graphs with a polycyclically ordered set of outer flags. Cyclically ordered sets of outer flags are only stable under grafting and subforest contraction.

**Remark 3.5.** The construction of the monoidal functor $O_{\text{poly}}$ can performed analogously for the Feynman category $\mathfrak{F}^{\text{cyc}}$ using a commutative monoid. Adding a unit and a distinguished element allows one to extend the construction to Feynman operations of $\mathfrak{F}^{\text{ng-mod}}$, and, whenever the distinguished element is invertible, further to Feynman operations of $\mathfrak{F}^{\text{nc-ng-mod}}$, cf. [KL17].

### 3.3. Oriented surface types as decoration

In order to handle all combinatorial data appearing in 1+1 d open TQFT and in the compactification of moduli spaces with punctures and marked boundaries, one needs to consider a decoration induced by a monoidal functor $O_{\text{surf}} : \text{Agg} \to \text{Set}$. This functor has a rather intricate combinatorial description. Geometrically, it corresponds to gluing surfaces with marked points on the boundary, and taking disjoint unions.

We will define the functor on the category of aggregates. The homonymous functor on the category of graphs will be defined by pullback along the source aggregate $s : \text{Gr} \to \text{Agg}$. Categorically the monoidal functor $O_{\text{surf}}$ arises as a pushforward of the functor $O_{\text{dec}}$ as we will see in §5.2.

Since $O_{\text{surf}}$ is monoidal, it suffices to define it on corollas $\ast_S$ where $S$ is the flag set. We put

$$O_{\text{surf}}(\ast) = N_0 \times N_0 \times \text{Aut}(S) \quad (3.10)$$

An element of $O_{\text{surf}}(\ast)$ is thus a triple $(g, p, \sigma_S)$ where $g$ and $p$ are natural numbers and $\sigma_S$ is a polycyclic order on the flag set $S$. Note that, due to the specific action of the morphisms of $\text{Agg}$, the functor $O_{\text{surf}}$ is not a direct product of functors.

Isomorphisms act as identity on $g, p$ and by conjugation on the polycyclic order $\sigma_S$ as above. The action of a simple gluing is given by:

$$O_{\text{surf}}(\ast)((g, p, \sigma_S), (g', p', \sigma_T)) = \begin{cases} 
(g + g', p + p' + 1, \sigma_S \circ \sigma_T) & \text{if } \sigma_S(s) = s \text{ and } \sigma_T(t) = t \\
(g + g', p + p', \sigma_S \circ \sigma_T) & \text{otherwise}
\end{cases} \quad (3.11)$$
The self gluing (i.e. the action of a simple loop contraction) is given by

\[
O_{\text{surf}}(\circ_{st})(g,p,\sigma_S) = \begin{cases} 
(g + 1,p,\circ_{st}(\sigma_S)) & \text{if } st \text{ are not in the same } \sigma \text{ orbit and both not fixed by } \sigma \\
(g + 1, p + 1, \circ_{st}(\sigma_S)) & \text{if } \sigma_S(s) = s \text{ and } \sigma_S(t) = t \\
(g,p,\circ_{st}(\sigma_S)) & \text{if } s \text{ and } t \text{ are in the same } \sigma \text{ orbit but neither } \sigma_S(s) = t \text{ nor } \sigma_S(t) = s \\
(g,p + 1, \circ_{st}(\sigma_S)) & \text{if either } \sigma_S(s) = t \text{ or } \sigma_S(t) = s \\
(g,p + 2, \circ_{st}(\sigma_S)) & \text{if } \sigma_S(s) = t \text{ and } \sigma_S(t) = s 
\end{cases}
\tag{3.12}
\]

The action of a simple merger is given by:

\[
O_{\text{surf}}(\vee_{\text{mer}})((g,p,\sigma_S),(g',p',\sigma_T)) := (g + g',p + p',\sigma_S \cup \sigma_T)
\tag{3.13}
\]

In order to understand the role of the second factor \(p\) it is convenient to view \(\sigma_S\) as a polycyclic order on \(S\) represented by the set of orbits of the permutation \(\sigma\). The number \(p\) is then the number of “empty” orbits so that the pair \((p,\sigma_S)\) may be viewed as a polycyclic order on \(S\) with \(p\) empty orbits. This may illuminate certain of the formulas above.

**Remark 3.6** (Topological type). The geometric interpretation of the decoration \((g,p,\sigma_S)\) is a topological type of a bordered oriented surface, i.e. of an oriented surface \(\Sigma\) with boundary \(\partial \Sigma\) having \(p\) unmarked boundary components and as many marked boundary components as \(\sigma_S\) has orbits. The marking of an individual boundary component consists of a choice of marked points (or subintervals) in one-to-one correspondence with the elements of the corresponding orbit of \(\sigma_S\) respecting the cyclic order coming from the orientation of the surface \(\Sigma\). Moreover, the associated closed surface should have topological genus \(g\), cf. §1.4.2 and [KP06, Figure 5].

This topological interpretation of the functor \(O_{\text{surf}}\) appears in [KP06] as the connected components of the open part of a c/o structure, see [KP06, Appendix A]\(^1\). The gluing operations of the functor \(O_{\text{surf}}\) correspond then to the gluing of cobordisms for open two-dimensional topological field theory, see e.g. [KP06, LP08, Dou17, Kau08a, Kau18].

**Lemma 3.7.** There is a natural transformation of monoidal functors \(O_{\text{surf}} \to O_{\text{genus}}\) taking a connected topological type \((g,p,\sigma_S)\) to \(1 - \chi(\Sigma(g,p,\Sigma_S)) = 2g + p - b - 1\), where \(b\) is the number of cycles of \(\sigma_S\).

**Proof.** Note first that the natural transformation is already determined on connected objects because of the monoidality of the functors \(O_{\text{surf}}\) and \(O_{\text{genus}}\). For a surface \(\Sigma\) of type \((g,p,\sigma_S)\), the associated closed surface \(\overline{\Sigma}\) has Euler characteristic \(\chi(\overline{\Sigma}) = 2 - 2g + \chi(\Sigma) + (p + b)\) and \(1 - \chi(\overline{\Sigma}) = 2g + p - b - 1\). We now check naturality with respect to the action of simple generators of \(\text{Agg}\). We go through this case by case.

For \(\circ_{st}\) we get \(2g_1 + 2g_2 + p + q + n + m - 2 = 2(g_1 + g_2) + (p + q + 1) + (m + n) - 1\) and hence there is no change in \(1 - \chi\). In the second case, we have \(2g_1 + 2g_2 + p + q + n + m - 2 = 2(g_1 + g_2) + (p + q) + (m + n - 1) - 1\) and again no change.

For the self–gluing \(\circ_{ss'}\), \(1 - \chi\) needs to increase by one. The output needs to be \([2g + p + s - 1] + 1 = 2g + p + s\). Indeed we get in the first case \(2\left((g + 1) + p + (s - 1) - 1\right) - 1\), in the second case \(2((g + 1) + p + 1 + s - 2 - 1) - 1\), in the third case \(2g + (p + 1) + s - 1\), in the fourth case \(2g + p + 1 + s - 1\) and in the final case \(2g + p + 2 + (s - 1) - 1\) which are all equal to \(2g + p + s\).

For a merger, the output should be \(2(g + g') + b + b' - 2 + 1\) which it indeed is. \(\square\)

The natural transformation \(O_{\text{surf}} \to O_{\text{genus}}\) belongs to a hexagon of natural transformations between monoidal functors which have appeared at various places in literature. Namely, one can

---

\(^1\) [KP06] is more general, since it allows for general D–brane labels. Here we have the case of only one such label.
retain only part of the decoration \((g,p,\sigma_S)\). This results in the following diagram of monoidal functors and natural transformations between them.

**Proposition 3.8.** There is a hexagon of monoidal functors \(\text{Agg} \rightarrow \text{Set} \) and natural transformations

\[
\begin{array}{ccc}
\mathcal{O}_\text{Euler,poly} & \rightarrow & \mathcal{O}_\text{genus} \\
\downarrow & & \downarrow \\
\mathcal{O}_\text{surf} & \rightarrow & \mathcal{O}_\text{poly} \end{array}
\]

(3.14)

where

- \(\mathcal{O}_\text{Euler,poly}(*=S) = \mathbb{N}_0 \times \text{Aut}(S)\) and \(\mathcal{O}_\text{surf} \rightarrow \mathcal{O}_\text{Euler,poly}\) takes \((g,p,\sigma_S)\) to \((2g + p - 1, \sigma_S)\).
- The action of simple generators is induced accordingly by summing the first two entries as above to the first two entries in (3.11), (3.12), (3.13).
- \(\mathcal{O}_\text{Euler,poly} \rightarrow \mathcal{O}_\text{genus}\) is given by \((l, \sigma_S) \mapsto l + b\) where \(b\) is the number of orbits of \(\sigma_S\).
- \(\mathcal{O}_\text{N,poly}(*=S) = \mathbb{N}_0 \times \text{Aut}(S)\) and \(\mathcal{O}_\text{surf} \rightarrow \mathcal{O}_\text{N,poly}\) takes \((g,p,\sigma_S)\) to \((p,\sigma_S)\). The action of simple generators is given by projecting to the two last components in (3.11), (3.12), (3.13).
- \(\mathcal{O}_N\text{poly} \rightarrow \mathcal{O}_\text{poly}\) takes \((p,\sigma_S)\) to \(\sigma_S\).

**Proof.** The naturality follows from a computation similar to the proof of Lemma 3.7. \(\square\)

Geometrically, the monoidal functor \(\mathcal{O}_\text{Euler,poly}\) decorates the corollas by surfaces with boundary and no punctures/unmarked boundaries, using \(1 - \chi\) to summarily keep track of the puncture/genus labeling. The number \(2g + p - 1\) is 1 minus the Euler characteristic of the surface, where the boundaries have been filled, but the punctures are still there. On the other hand \(\mathcal{O}_\text{N,poly}\) only keeps track of the punctures, but forgets the genus. Note that neither \(\mathcal{O}_\text{N,poly}\) nor \(\mathcal{O}_\text{poly}\) maps to \(\mathcal{O}_\text{genus}\).

**Remark 3.9.** The hexagon pulls back along \(s\) or \(t\) to functors \(\text{Gr} \rightarrow \text{Set}\). There are also associated hexagons for the associated element categories over \(\text{Gr}\) and \(\text{Agg}\).

### 3.4. Directed graphs and rooted forests.

A graph with in/outputs is a graph equipped with a map \(i/o : F \rightarrow \mathbb{Z}/2\). A flag \(f\) is called input (resp. output) flag if \((i/o)(f) = 1\) (resp. \((i/o)(f) = 0\)).

A graph is said to have directed edges if each edge contains one input and one output flag.

The map \(i/o\) can be promoted to a decorating functor \(\mathcal{O}_{i/o}(\Gamma) = F^2/\mathbb{Z}\). The action of a graph morphism \(\phi\) is precomposition by \(\phi^\mathbb{F}\). We will consider the subcategory \(\text{Gr}^{\text{dir}}\) of the category of elements \(\text{Gr}_{\text{dec}}(\mathcal{O}_{i/o})\) whose objects are graphs with directed edges and whose morphisms satisfy that only outer flags of opposite orientation are grafted. This induces a full subcategory \(\text{Agg}^{\text{dir}}\) of directed aggregates with the property that its corollas have in/output flags and the ghost graphs are directed graphs. We further restrict to the subcategory \(\text{Agg}^{\text{dir,forest}}\) with the property that each corolla has a single output flag (its root) and the morphisms have forests as ghost graphs.

The projection \(\pi : \text{Gr}^{\text{dir}} \rightarrow \text{Gr}\) restricts to these subcategories and induces a functor \(i : \text{Agg}^{\text{dir,forest}} \rightarrow \text{Agg}^{\text{forest}}\). The decorating functor \(\mathcal{O}_{\text{ass}} : \text{Agg}^{\text{dir,forest}} \rightarrow \text{Set}\) is then related to the decorating functor \(\mathcal{O}_{\text{cycass}} : \text{Agg}^{\text{forest}} \rightarrow \text{Set}\) via pullback, namely \(i^*\mathcal{O}_{\text{cycass}} = \mathcal{O}_{\text{ass}}\).

Indeed, \(\mathcal{O}_{\text{cycass}}(v_S)\) represents the set of cyclic orders on \(S\). If \(v_S\) is in the image of the functor \(i\), then cyclic orders on \(S\) are in one-to-one correspondence with total orders on the set of input flags of \(S\) which is \(S\) minus the root flag. This set of total orders is precisely the set of automorphisms of \(v_S\) when viewed as an object of \(\text{Agg}^{\text{forest}}\), which by definition is \(\mathcal{O}_{\text{ass}}(v_S)\).

### 3.5. History.

The functor \(\mathcal{O}_{\text{surf}} : \text{Agg}^{\text{ctd}} \rightarrow \text{Set}\) first appeared in the gluing description with the operations \(s \circ t\) and \(\circ_{ss'}\) in [KP06] as the open part of a \(c/o\) structure given by connected components of the closed/open arc structure. The list in §3.3 corresponds to [KP06, §3, Figure 5]. In a non-obviously equivalent version it also appears in [CL91, LP08, Dou17]. The category \(\text{Gr}_{\text{dec}}(s^*\mathcal{O}_{\text{poly}})\) is what is taken as open gluing in [WW16] in lieu of the OTFT-gluing induced by the functor \(\mathcal{O}_{\text{surf}}\).
The categories of the last section are monoidal categories of a special type, they are **Feynman categories**. Set-valued monoidal functors like in the preceding section are then their operations which often can be identified with operad-like structures. Our formalism permits a uniform treatment of these structures, which is the basis of the further analysis.

4.1. **Basic definitions.** To each category \( \mathcal{V} \) we associate the free symmetric monoidal category \( \mathcal{V}^\otimes \) generated by \( \mathcal{V} \). For any functor \( i : \mathcal{V} \to \mathcal{C} \) with symmetric monoidal target category \( \mathcal{C} \), there exists a unique symmetric monoidal functor \( i^\otimes : \mathcal{V}^\otimes \to \mathcal{C} \). For any category \( \mathcal{F} \) we denote by \( \text{Iso}(\mathcal{F}) \) the maximal groupoid contained in \( \mathcal{F} \), i.e. the objects of \( \mathcal{F} \) together with their isomorphisms.

**Definition 4.1** ([KW17]). Let \( \mathcal{F} \) be a symmetric monoidal category and \( i : \mathcal{V} \hookrightarrow \mathcal{F} \) be the inclusion of a groupoid. The triple \( \mathfrak{F} = (\mathcal{F}, \mathcal{V}, i) \) is called a Feynman category if

(i) (Isomorphism condition) The functor \( i^\otimes \) induces an equivalence of symmetric monoidal categories between \( \mathcal{V}^\otimes \) and \( \text{Iso}(\mathcal{F}) \).

(ii) (Hereditary condition) The functor \( i^\otimes \) induces an equivalence of symmetric monoidal categories between \( \text{Iso}(\mathcal{F} \downarrow \mathcal{V})^\otimes \) and \( \text{Iso}(\mathcal{F} \downarrow \mathcal{F}) \).

(iii) (Size condition) For every \( v \in \mathcal{V} \), the comma category \( \mathcal{F} \downarrow v \) is essentially small.

A Feynman functor \( \mathfrak{f} : (\mathcal{F}, \mathcal{V}, i) \to (\mathcal{F}', \mathcal{V}', i') \) is given by a pair of functors \( (f : \mathcal{F} \to \mathcal{F}', g : \mathcal{V} \to \mathcal{V}') \) such that \( f \) is strong symmetric monoidal and \( i'g = fi \). We will usually suppress \( g \) from notation and identify notationally \( \mathfrak{f} \) and \( f \).

Due to conditions (i) and (ii) every morphism in a Feynman category can be written essentially uniquely as a tensor product of morphisms with target in \( i(\mathcal{V}) \). These morphisms are said to be the basic morphisms of the Feynman category \( \mathfrak{F} \). For more details on the general theory of Feynman categories we refer the reader to the book [KW17], a short introduction is contained in [Kau18].

For a Feynman category \( \mathfrak{F} = (\mathcal{F}, \mathcal{V}, i) \) an operation in a symmetric monoidal category \( \mathcal{C} \) is a strong symmetric monoidal functor \( (\mathcal{F}, \otimes_{\mathcal{F}}) \to (\mathcal{C}, \otimes_{\mathcal{C}}) \). The category of such strong symmetric monoidal functors and symmetric monoidal natural transformations will be denoted \( \mathfrak{F}-\text{Ops}_{\mathcal{C}} \). If \( (\mathcal{C}, \otimes_{\mathcal{C}}) = (\text{Set}, \times) \) we suppress it from the notation. There is a monoidal structure on \( \mathfrak{F}-\text{Ops}_{\mathcal{C}} \) given by pointwise tensor product. The unit for this monoidal structure is the trivial operation \( \mathcal{O}_1^\mathfrak{F} \) defined by \( \mathcal{O}_1^\mathcal{F}(X) = \mathcal{O}_{\mathcal{C}}(X) \) and \( \mathcal{O}_1^\mathcal{F}(\phi) = \text{id}_{\mathcal{O}_1^\mathcal{C}} \) where \( \mathcal{O}_1^\mathcal{C} \) is the monoidal unit of \( \mathcal{C} \). Whenever \( \mathcal{O}_1^\mathcal{C} \) is terminal in \( \mathcal{C} \) (for instance if \( \mathcal{C} = \text{Set} \)), the trivial operation \( \mathcal{O}_1^\mathfrak{F} \) is terminal in \( \mathfrak{F}-\text{Ops}_{\mathcal{C}} \). To indicate this we will write \( \mathcal{O}_* \).

**Remark 4.2.** \( \mathfrak{F} \)-operations or ops for short, depending on \( \mathfrak{F} \), can be operads, algebras, algebras over operads, crossed simplicial objects and so on, see [KW17,Kau21].

The categories of the last section are Feynman categories and they are related by Feynman functors. The first set of examples is related to the category of aggregates, cf. [KW17, §2]: let \( \text{Crl} \) be the subcategory of corollas together with their isomorphisms and let \( i : \text{Crl} \to \text{Agg} \) be the inclusion, then \( \mathfrak{F}^{\text{ncg-mod}} = (\text{Crl}, \text{Agg}, i) \) is a Feynman category. By restriction, we obtain the Feynman categories \( \mathfrak{F}^{\text{ng-mod}} = (\text{Crl}, \text{Agg}^{\text{ctd}}, i) \) and \( \mathfrak{F}^{\text{cyc}} = (\text{Crl}, \text{Agg}^{\text{forest}}, i) \) whose basic morphisms have connected graphs, respectively trees as ghost graphs. Decorations (§4.3) yield further Feynman categories. The corresponding operations are operad-like, see §4.4 and Table 1.

The second set of examples are Feynman categories of graphs, which have thus far not been considered. Let \( \text{Ctd} \) be the subgroupoid of \( \text{Gr} \) spanned by connected graphs and their isomorphisms and let \( i \) be the inclusion functor, then \( \mathfrak{F}_t = (\text{Ctd}, \text{Gr}, i) \) is a Feynman category. Indeed, every graph decomposes into a disjoint union of its connected components and isomorphisms respect this decomposition. Furthermore, every morphism \( \phi \) decomposes essentially uniquely into a disjoint union of the \( \phi_{\text{ctd}} \) according to (2.1). Finally the slice categories are essentially small.
The functors $s,t,i$ of §2.5 extend naturally to Feynman functors and $\mathcal{G}t$ and $\mathfrak{S}^{\text{eng-mod}}$ form a double Feynman category, that is a Feynman category internal to Feynman categories, using graph insertion as the horizontal morphisms. By restriction we obtain the Feynman categories $\mathfrak{S}^{\text{ctd}} = (\text{Ctd}, \text{Gr}^{\text{ctd}}, t)$, $\mathfrak{S}^{\text{graft}} = (\text{Ctd}, \text{Gr}^{\text{graft}}, t)$ and $\mathfrak{S}^{\text{forest}} = (\text{Ctd}, \text{Gr}^{\text{forest}}, t)$.

### 4.2. Pullback, pushforward and Frobenius reciprocity

One of the main features of Feynman categories is that restriction functors $f^*$ have computable left adjoints $f_!$.

For each Feynman functor $f : (\mathcal{F}, V, i) \to (\mathcal{F}', V', i')$, precomposition with $f$ defines a restriction functor $f^* : \mathfrak{S}'-\text{Ops} \to \mathfrak{S}-\text{Ops}$. Its left adjoint pushforward functor $f_! : \mathfrak{S}-\text{Ops} \to \mathfrak{S}'-\text{Ops}$ can be computed like in ordinary category theory as pointwise left Kan extensions, the symmetric monoidal structure being guaranteed by the axioms of a Feynman category, cf. [KW17].

**Theorem 4.3** ([KW17]). Any Feynman functor $f : \mathfrak{S} \to \mathfrak{S}'$ induces a “induction-restriction” Frobenius-reciprocity adjunction $f_! : \mathfrak{S}-\text{Ops} \subseteq \mathfrak{S}'-\text{Ops} : f^*$ with left adjoint given by pointwise left Kan extension

$$
(f_! \mathcal{O})(X) = \text{colim}_{f(-) \downarrow X} \mathcal{O}(-) \quad (4.1)
$$

Indeed these are even adjoint symmetric monoidal functors.

There are two possible notations for the push–forwards. We adopt here the categorical notation $f_!$ which is commonly used for left Kan extensions. In [KW17, War19] the notation $f_*$ was used instead, in order to avoid confusion with extension by zero.

**Remark 4.4.** For $\mathcal{C} = \text{Set}$ (or more generally if the monoidal unit is terminal in $\mathcal{C}$), an extension along $f$ preserves trivial operations if and only if the comma categories $(f \downarrow X)$ are non-empty and connected for all objects $X$ of $\mathfrak{S}$. Feynman functors with this property will be called connected.

These extensions are computable if the comma categories $(f \downarrow X)$ are sufficiently well understood. In a category with coproducts and coequalisers any colimit over a small category is a coequaliser. In the special case $\mathcal{C} = \text{Set}$, for a functor $f : I \to \text{Set}$, the colimit can be computed as

$$
(\text{colim}_I f)(x) = \left( \bigsqcup_{i \in I} f(i) \right) / \sim \quad (4.2)
$$

where $x \in f(i)$ is identified with $y \in f(j)$ in the colimit if there is a morphism $\phi : i \to j$ in $I$ such that $f(\phi)(x) = y$. In other words, the colimit colim$_I f$ may be identified with $\pi_0(\text{Id}_{\text{dec}}(f))$, the set of connected components of the category of elements of $f$, cf. Lemma 5.1. In particular for the Kan extension, $I = (f(-) \downarrow i(*)$ has elements $(X, \phi : f(X) \to i(*))$ with $X \in \mathcal{F}$ and morphisms induced by $\psi \in \mathcal{F}(X,Y))$. That is $\psi : (X, \phi) \to (Y, \phi \circ f(\psi))$.

**Example 4.5.** For a Feynman category $\mathfrak{S} = (\mathcal{F}, \mathcal{V}, i)$ and functor $\mathcal{O} : \mathcal{V} \to \text{Set}$, let $\mathfrak{S}^\mathcal{V} = (\mathcal{V}, \mathcal{V}^\otimes, i)$ be the free Feynman category on $\mathcal{V}$ and let $i^\otimes : \mathcal{V}^\otimes \to \mathfrak{S}$ be the induced inclusion of Feynman categories. Note that $\mathcal{O}$ extends canonically to a operation $\mathcal{O}^\otimes$ of $\mathcal{V}^\otimes$. Then $(i^\otimes)_!(\mathcal{O}^\otimes)$ is the free $\mathfrak{S}$-operation generated by $\mathcal{O}$. This construction is left adjoint to the obvious forgetful functor, see [KW17, Example 1.6.3].

**Example 4.6.** There are Feynman functors $i : \mathfrak{S}^{\text{opd}} \to \mathfrak{S}^{\text{cyc}}$ and $j : \mathfrak{S}^{\text{cyc}} \to \mathfrak{S}^{\text{mod}}$. The pushforwards $i^*$ and $j^*$ correspond respectively to the cyclic envelope of a symmetric operad, and to the modular envelope of a cyclic operad. While $i^*$ is the restriction of a cyclic operad to its underlying pseudo–operad and $j^*$ the restriction of a modular operad to its underlying cyclic operad. Note that all operads are not required to be unital.

### 4.3. Decorated Feynman categories

The essential ingredient in the constructions at hand is the notion of a decorated Feynman category as introduced in [KL17]. Decorated Feynman categories
are the “Feynman analogs” categories of elements, cf. [BK17] for a parallel treatment of both constructions. More precisely, we have

\[ \mathcal{F}_{dec}(\mathcal{O}) = (\mathcal{F}_{dec}(\mathcal{O}), V_{dec}(\mathcal{O}), t_{dec}(\mathcal{O})) \]

where the functor \( t_{dec}(\mathcal{O}) : V_{dec}(\mathcal{O}) \to \mathcal{F}_{dec}(\mathcal{O}) \) takes \((v, a)\) to \((\nu(v), a)\). With slight modifications, the decoration also exists for non–Cartesian \( \mathcal{C} \), see [KL17]. If \( \mathcal{O} \) is Set valued, we call the projection \( \mathcal{F}_{dec}(\mathcal{O}) \to \mathfrak{F} \) a covering of Feynman categories following the terminology of [BK17]. Among category theorists such coverings are usually called discrete opfibrations.

**Theorem 4.7** ([KL17, BK17]). \( \mathcal{F}_{dec}(\mathcal{O}) \) is indeed a Feynman category. Projecting to the first factor is a canonical Feynman functor \( \mathcal{F}_{dec}(\mathcal{O}) \to \mathfrak{F} \). Decorations are functorial with respect to Feynman functors and natural transformations of algebras \( \sigma : \mathcal{O} \to \mathcal{P} \), that is the following squares exist and commute

\[
\begin{array}{ccc}
\mathcal{F}_{dec}(\mathcal{O}) & \xrightarrow{t_{\mathcal{O}}} & \mathcal{F}_{dec}(f_!(\mathcal{O})) \\
p \downarrow & & \downarrow p' \\
\mathfrak{F} & \xrightarrow{\mathfrak{F}} & \mathfrak{F}'
\end{array}
\]

(4.3)

and a diagram of adjoint functors for categories of Feynman operations

\[
\begin{array}{ccc}
\mathcal{F}_{dec}(\mathcal{O})-\mathcal{O}ps & \xrightarrow{f_!} & \mathcal{F}_{dec}(f_!(\mathcal{O}))-\mathcal{O}ps \\
p_! \downarrow & & \downarrow p'_! \\
\mathfrak{F}-\mathcal{O}ps & \xrightarrow{f_*} & \mathfrak{F}'-\mathcal{O}ps
\end{array}
\]

(4.4)

such that the square of left adjoints and the square of right adjoints commute.

**Remark 4.8.** A Feynman functor \( f : (\mathcal{F}, \mathcal{V}, i) \to (\mathcal{F}', \mathcal{V}', i') \) is a covering if and only if the underlying functor \( f : \mathcal{F} \to \mathcal{F}' \) is a covering. The characterisation of coverings of categories is well-known: for each \( \phi' : X' \to Y' \) in \( \mathcal{F}' \) and each \( X \) in \( \mathcal{F} \) such that \( f(X) = Y \) there exists one and only one \( \phi : X \to Y \) such that \( f(\phi) = \phi' \). In particular, \( f \) is a full functor. If \( f \) is a covering then the Feynman category \( \mathfrak{F} \) may be identified with the decorated Feynman category \( \mathfrak{F}'_{dec}(f_!(\mathcal{O}^{\mathfrak{F}}_s)) \), cf. [BK17].

**Example 4.9.** The functors \( s, t : \mathfrak{O}^{ctd} \to \mathfrak{F}^{ng-mod} \) satisfy the characteristic property of a covering, cf. Remark 4.8. The decorating functors are thus given by \( s_!(\mathcal{O}_s) \) and \( t_!(\mathcal{O}_s) \).

Lemma 5.1 shows that the second decorating functor \( t_!(\mathcal{O}_s)(*)_S \) is the set of isomorphism classes of connected graphs \( \Gamma \) such that the total contraction \( \Gamma/\Gamma \) is \(*_S \). This is the set of isomorphism classes of connected graphs with outer flag set \( S \). Picking representatives we get \( (t_!(\mathcal{O}_s))_{\circ_{ct}} = \circ_{ct} \) and \( (t_!(\mathcal{O}_s))_{\circ_{st}} = \circ_{st} \). The trivial operation \( \mathcal{O}_s \) has a surface interpretation via a cutting curve system, cf. §1.4.2. The pushforward by \( t \) forgets the cutting curves. The morphisms \( \circ_{ct} \) and \( \circ_{st} \) glue the boundaries while the morphism \( \circ_{st} \) glues the boundaries and remembers the boundaries as new cutting curves.

**Proposition 4.10.** Let \( \mathcal{O} \) be a set-valued operation of a Feynman category \( \mathfrak{F}' \). Each Feynman functor \( f : \mathfrak{F} \to \mathfrak{F}' \) induces a commutative diagram of Feynman functors

\[
\begin{array}{ccc}
\mathfrak{F}_{dec}(i^!(\mathcal{O})) & \xrightarrow{i'} & \mathfrak{F}'_{dec}(\mathcal{O}) \\
\downarrow & & \downarrow \\
\mathfrak{F} & \xrightarrow{i} & \mathfrak{F}'
\end{array}
\]

(4.5)
Proof. Follows from Theorem 4.7 and the natural transformation $i_! \circ \mathcal{O} \rightarrow \mathcal{O}$. □

The next result is the precise analog for Feynman functors of the comprehensive factorisation of an ordinary functor (into initial functor followed by discrete opfibration), first established by Street-Walters [SW73]. The proof is mutatis mutandis the same.

**Theorem 4.11 (BK17).** Every Feynman functor $f : \mathcal{F} \rightarrow \mathcal{F}'$ factors essentially uniquely as a connected Feynman functor $\mathcal{F} \rightarrow \mathcal{F}'_{dec}(f_!(\mathcal{O}_{\mathcal{F}}))$ followed by a covering $\mathcal{F}'_{dec}(f_!(\mathcal{O}_{\mathcal{F}})) \rightarrow \mathcal{F}'$.

### 4.4. Operad-like structures as Feynman operations

The usual operad-like structures can be recovered in the formalism of Feynman categories and their operations. For reference, we briefly review the main characters here, which are summarized in Table 1; cf. [KW17, §2] and [Kau18, §4] for more examples and details.

Cyclic operads have been introduced by Getzler-Kapranov [GK95]. A operation $\mathcal{O}$ of $\mathcal{F}_{cyc}$ is equivalent to a non–unital cyclic operad [KW17, §2.3.1]. The correspondence in the usual unbiased notation is given by $\mathcal{O}(S) = (S)$ and $\mathcal{O}(s_{\nu}) = s_{\nu} : \mathcal{O}(S) \otimes \mathcal{O}(T) \rightarrow \mathcal{O}((S \setminus \{s\} \sqcup T \setminus \{t\}))$, cf. [GK95]. This is the action of a virtual edge contraction. See [Kau18, §4] for more details.

Non–unital symmetric operads (aka pseudo–operads, cf. [Mar08]) are equivalent to Feynman operations of $\mathcal{F}^{opd}$ where the Feynman category $\mathcal{F}^{opd} = (\text{Crl}^\text{rt}, \text{Agg}_{\text{forest}}, \iota)$ has been introduced in [KW17, §2.2.1]. For its groupoid $\text{Crl}^\text{rt}$ of rooted corollas $v_{S,S}$, see §3.4.

The correspondence in unbiased notation is given by $\mathcal{O}(s_{S,s}) = \mathcal{O}(S \setminus s)$ and in biased notation by $\mathcal{O}(s_{\{0,\ldots,n]\},0) = \mathcal{O}(n)$, cf. [MSS02,KW17]. Forgetting the distinction of the root flags defines a Feynman functor $i : \mathcal{F}^{opd} \rightarrow \mathcal{F}_{cyc} : v_{S,S} \mapsto v_S$.

The operations of the Feynman category $\mathcal{F}^{ng-mod}$ have been introduced in [KL17] under the name unmarked modular operads. They additionally come equipped with operations $\mathcal{O}(s_{st}) = o_{st} : \mathcal{O}(S) \rightarrow \mathcal{O}(S \setminus \{s,t\})$ induced by virtual loop contractions. There is a Feynman category inclusion $k : \mathcal{F}_{cyc} \rightarrow \mathcal{F}^{ng-mod}$.

To obtain the modular operads of Getzler-Kapranov [GK98] one has to add genus labeling. The category $\mathcal{F}^{mod}$ for modular operads has as groupoid genus labelled corollas $v_{S,G}$ and their auto-isomorphisms. The morphisms of $\mathcal{F}^{mod}$ are those of the subcategory $\text{Agg}^{cld}$ with the constraint that $o_{st} : *_{S,G} \rightarrow *_{S \setminus \{s,t\},G+1}$ and $s_{ct} : *_{S,G} \sqcup *_{T,G'} \rightarrow *_{S',G+G'+G'}$, see [KW17]. The correspondence is via $\mathcal{O}(s_{S,G}) = \mathcal{O}(S,g)$ and $\mathcal{O}(s_{ct}) = o_{ct}$ and $\mathcal{O}(o_{st}) = o_{st}$ in the standard notation (cf. [MSS02]; see [Kau18, §4] for more details.

Forgetting genus labeling yields a Feynman functor $\pi : \mathcal{F}^{mod} \rightarrow \mathcal{F}^{ng-mod}$ which is a covering. There is also a Feynman category inclusion $j : \mathcal{F}^{cyc} \rightarrow \mathcal{F}^{mod}$ taking $*$ to $*_{S,0}$ and $s_{ct}$ to $s_{ct}$. This Feynman functor is connected. The composite Feynman functor $\pi \circ j$ is precisely $k : \mathcal{F}_{cyc} \rightarrow \mathcal{F}^{ng-mod}$. According to Theorem 4.11, the Feynman category $\mathcal{F}^{mod}$ for modular operads can thus be formally deduced from the Feynman category $\mathcal{F}^{ng-mod}$ for unmarked modular operads by comprehensive factorisation. This is a decoration by $\mathcal{O}_{\text{genus}}$ [KL17, §6.4.2] and the forgetful functor forgetting the genus marking $\pi : \mathcal{F}^{mod} \rightarrow \mathcal{F}^{ng-mod}$ is a covering. These facts can also be derived from Lemma 4.8 and Theorem 4.7.

The genus gives a grading to objects and morphisms additive under composition and monoidal structure. With $\deg(s,t) = l$, $\deg(\phi) = b_1(\mathcal{F}(\phi))$. For a basic morphisms this is $1 - \chi(\mathcal{F}(\phi)) = -\tilde{\chi}$ where $\tilde{\chi}$ is the reduced Euler characteristic and for a general morphism $\deg(\phi) = \deg(\bigcup_0 \phi_v) = -\sum_v \tilde{\chi}(\mathcal{F}(\phi_v))$. Thus any morphism $\phi = \phi_0 \circ \phi_r \circ \phi_p : X \rightarrow Y$ satisfies

$$b_1(\mathcal{F}(\phi)) = |\ell| = \deg(X) - \deg(Y) \quad (4.6)$$

The version of modular operads considered by Schwarz [Sch98], called MOs, amounts to operations of the Feynman category $\mathcal{F}^{nc,ng-mod}$ by Theorem 2.10. His $\nu_{m,m}$ are induced by mergers and $\sigma^{(m)}$ by virtual loop contractions. He also considers genus labeling as an additional grading. The category of these graded MOs is equivalent to the category of operations of $\mathcal{F}^{nc-mod}$.
Definition 4.12. Define $\mathfrak{F}^{\Sigma\text{-}opd} = \mathfrak{F}^{\text{opd}}(\mathcal{O}_{\text{ass}})$, $\mathfrak{F}^{\text{pl-cyc}} = \mathcal{F}_{\text{cyc}}^{\text{dec}}(\mathcal{O}_{\text{cycass}})$ and $\mathfrak{F}^{\text{surf-mod}} := \mathfrak{F}^{\text{dec}}(\mathcal{O}_{\text{surf}})$.

Proposition 4.13. The category of operations of $\mathfrak{F}^{\Sigma\text{-}opd}$, resp. $\mathfrak{F}^{\text{pl-cyc}}$, resp. $\mathfrak{F}^{\text{surf-mod}}$ is equivalent to the category of non-symmetric, resp. non-$\Sigma$-cyclic, resp. non-$\Sigma$-modular operads of Markl [Mar16].

Proof. This is contained in [KL17] and is straightforward from the definitions in the first two cases. In the last case, it follows from comparing the results of [Mar16] with the reinterpretation of the pair $(p,\sigma_S)$ as giving a partition of $S$ into $p+b$ subsets such that $p$ are empty and $b$ are non-empty, each equipped with a cyclic order. □

Remark 4.14. Note that in $\mathfrak{F}^{\Sigma\text{-}opd}$ the automorphism group of $v_{S,s}$ is trivial, while in $\mathfrak{F}^{\text{pl-cyc}}$ the automorphism group of $v_{S,\sigma_S}$ is cyclic of order the cardinality of $S$, and in $\mathfrak{F}^{\text{surf-mod}}$ the automorphism group of $v_{g,p,\sigma_S}$ is $p\mathbb{Z}\{n_1\}\ldots\mathbb{Z}\{n_b\}\wr S_b$ whenever $\sigma_S$ has $b$ orbits of length $n_i$.

Therefore, the terminology non-$\Sigma$ may be confusing. We call operations of $\mathfrak{F}^{\text{pl-cyc}}$ planar-cyclic operads and operations of $\mathfrak{F}^{\text{surf-mod}}$ surface-modular operads. The aforementioned automorphism groups are important for structures on the coinvariants, such as Gerstenhaber brackets, Lie brackets and BV structures, see [KWZ15].

Via Theorem 4.7, the natural transformations of Proposition 3.8 yield two hexagons of coverings.

Proposition 4.15. There is a hexagon of coverings:

$$\mathfrak{F}_{\text{dec}}(\mathcal{O}_{\text{Euler,poly}}) \to \mathfrak{F}_{\text{dec}}(\mathcal{O}_{\text{genus}}) \to \mathfrak{F}_{\text{dec}}(\mathcal{O}_{\text{surf}})$$

Restriction to $\mathfrak{F}^{\text{ng-mod}}$ yields the hexagon of coverings:

$$\mathfrak{F}_{\text{dec}}(\mathcal{O}_{\text{Euler,poly}}) \to \mathfrak{F}_{\text{dec}}(\mathcal{O}_{\text{poly}}) \to \mathfrak{F}_{\text{dec}}(\mathcal{O}_{\text{surf}})$$

The types of ghost graphs are given in Table 2.

4.5. History. The interpretation of the genus labelling $g$ as $1 - \chi$ occurs in the open modular part of the c/o structure in [KP06, Appendix A.3]. The use of the grading $1 - \chi$ in the presence of mergers is in [KWZ15, VII A 3]. An extension of the operations of $\mathcal{O}_{\text{poly}}$ is given in the form of brane-labeling in [KLP03, Appendix A.6]. Restricting to a single brane label defines $\mathcal{O}_{\text{poly}}$.

Implicitly $\mathfrak{F}^{\text{surf-mod}}$ and its operations occur as specialisations of algebras over the c/o structure $\pi_0(\text{Arc}(S,T))$, cf. [KP06, §5]. The morphism $\mathcal{O}_{\text{surf}} \to \mathcal{O}_{\text{Euler,poly}}$ is used in [Kau08a] to define the correlation functions. This is explicit in the formula (4.3) of [Kau08a]. The necessity to work with the full decorating functor $\mathcal{O}_{\text{surf}}$, i.e. the full indexing by topological surface types appears when the Hochschild cochain complex is viewed as an algebra in the open/closed case [Kau10]. The fact that the book keeping must include the internal punctures in the open case is explicitly stated there. Furthermore, the generalisation to the associative case given in [Kau18] shows that $\mathcal{O}_{\text{surf}}$ and $\mathfrak{F}^{\text{surf-mod}}$ are needed to provide compatible correlation functions.
The relation to modular operads was outlined and clarified in [Mar16], especially the role of empty cycles, viz. punctures or unmarked boundaries. The description of stable ribbon graphs using $O_{\text{poly}}$ is in [Bar07, Kau09]. The necessity of internal punctures in the open/closed case was discussed in [Kau10]. Although the correlation functions exist without punctures [TZ07], the gluing introduces them in the open case.

On the chain level, even in the closed case, punctures appear due to the differential. It is possible to factor these contributions out using a filtration [Kau07] or a stabilisation [Kau09]. For actions on Hochschild complexes, the Euler class has to be the unit for the stabilisation to act. In this case, one obtains an $E_8$-structure on the Hochschild cochain complex [Kau08b]. The suppression of punctures works on the chain level by setting the respective components to zero, which has been exploited by [Bar07]. This can now also be understood via a right Kan extension.

5. Computing pushforwards

5.1. Main diagram. Consider the morphisms $i: \mathcal{O}_{\text{ass}} \to \mathcal{O}_{\text{cyc}}$, and $k: \mathcal{O}_{\text{cyc}} \to \mathcal{O}_{\text{genus}}$. Then from Theorems 4.7 and 4.11 and Proposition 4.10 have the following commutative diagram:

\[
\begin{array}{cccc}
\mathcal{O}_{\text{ass}} & \xrightarrow{i^*} & \mathcal{O}_{\text{cyc}} & \xrightarrow{\mathcal{F}_{\text{cyc}}} & \mathcal{O}_{\text{genus}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{\text{ass}} & \xrightarrow{k^*} & \mathcal{O}_{\text{cyc}} & \xrightarrow{\mathcal{F}_{\text{cyc}}} & \mathcal{O}_{\text{surf}}
\end{array}
\]

(5.1)

the vertical functors are coverings and $k \cdot f$ is the comprehensive factorization into a connected morphism and a covering and $j'$ are connected. The only input op for the construction besides $O_*$ is the cyclic operad $O_{\text{cyc}} \in \mathcal{F}_{\text{cyc}}-O_{\text{ops}}$, that is cyclic orders, the existence of the other categories, functors and operad types is now a consequence of push–forward and decorations.

In view of Definition 4.12 to finish the proof of Theorem B of the introduction it remains to establish the following identifications of Feynman operations:

\[
i^* (O_{\text{ass}}) \cong O_{\text{cyc}}, \quad k^* (O_{\text{cyc}}) \cong O_{\text{genus}}, \quad k^! (O_{\text{cyc}}) \cong O_{\text{surf}}
\]

(5.2)

The first identification has been described in §3.4. The other two identifications will be established in Propositions 5.6 and 5.11 respectively. The following lemma will be most useful.

**Lemma 5.1.** The colimit of a functor $F : I \to \text{Set}$ on a small category $I$ can be identified with the set of connected components $\pi_0(I_{\text{dec}}(F))$ of the category of elements $I_{\text{dec}}(F)$ of $F$, cf. §3.

**Proof.** This follows from combining the following three facts: (1) the projection $\pi : I_{\text{dec}}(F) \to I$ is a covering, (2) $\pi_0(I_{\text{dec}}(F)) = F$ and (3) left Kan extensions compose. Alternatively, the set $\pi_0(I_{\text{dec}}(F))$ can directly be identified with the canonical coequaliser presentation of the colimit colim$_{F}$, cf. formula (4.2) in Remark 4.4. □

In order to compute left Kan extensions it suffices thus to determine the connected components of certain well defined categories. In our case, these categories will be categories of structured graphs, and one of the main issues consists of describing them explicitly. The way to proceed is to identify complete graphical invariants of the connected components above.
5.2. Genus labeling $\mathcal{O}_{\text{genus}}$ as pushforward.

**Lemma 5.2.** The the slice category $(\mathfrak{S}^{\text{nc-\text{ng-mod}}} \downarrow \ast \mathcal{S})$ is equivalent to the subcategory of $\text{Gr}^{\text{contr}}$ whose objects are connected graphs whose set of tails is $\mathcal{S}$ and morphisms given by contracting spanning sub–graphs and isomorphisms.

Note that the slice categories $(\mathfrak{S}^{\text{nc-\text{ng-mod}}} \downarrow \ast \mathcal{S})$ are also the essential fibres of $t : \text{Gr}^{\text{ctd}} \to \text{Agg}^{\text{ctd}}$.

**Proof.** Consider with $\mathfrak{S}^{\text{nc-\text{ng-mod}}}$ and $X = \ast \mathcal{S}$. Up to isomorphism, we can restrict to pure morphisms $\phi_p : X \to \ast \mathcal{S}$ up to isomorphism. In this situation Proposition 2.15 applies with $\hat{\phi}_R = \text{id}$ and provides the identification via the ghost graphs. Since there are no mergers, the respective morphisms on the ghost graphs are contractions. \[\square\]

For later computations, we need the following precise version of Corollary 2.12 for contractions with one-vertex target.

**Lemma 5.3.** Any morphism $\phi : X \to \ast \mathcal{S}$ in $\mathfrak{S}^{\text{ng-mod}}$ decomposes as $\phi = \sigma_v \phi_p \sigma$ where $\phi_p : X' \to \check{v}_S$ is the total pure contraction, and $\sigma : X \to X'$ and $\sigma_v : \check{v}_S \to \ast \mathcal{S}$ are isomorphisms with $\sigma_v$ fixing the outer flag set $\mathcal{S}$.

A canonical choice is provided by the unique decomposition $\phi = \sigma \circ \phi_p$ of Theorem 2.10, which can be rewritten as $\phi = \sigma_v \phi_p \hat{\sigma}$ where $\hat{\sigma}$ fixes all vertices and extends $\sigma_F$ by the identity on inner flags.

Given two such decompositions $\sigma_v \phi_p \sigma$ and $\sigma_v' \phi_p' \sigma'$ of $\phi$ the pair $(\sigma' \sigma^{-1}, \sigma_v'^{-1} \sigma_v)$ defines an isomorphism from $\phi_p$ to $\phi'_p$ in the arrow category.

This implies that $\sigma_2 \sigma_1^{-1}$ as a morphism $\Gamma(\phi_p) \to \Gamma(\phi'_p)$ is an isomorphism. In particular fixing $\phi_p$ in the decomposition $\sigma$ is fixed up to a graph automorphism fixing the set of outer flags.

**Proof.** By Theorem 2.10 there is a unique decomposition $\phi = \sigma \phi_p$ where $\phi_p$ is pure and $\sigma$ is an isomorphism. Besides the identification of the single vertex of $\Gamma(\phi)/\Gamma(\phi)$ with the single vertex of $\ast \mathcal{S}$, $\sigma$ induces a bijection between $\mathcal{S}$ and the outer flags of $\Gamma(\phi)$. We define $X'$ to be the aggregate obtained by replacing the flags of $X$ corresponding to $\mathcal{S}$ with the flags $S$. It is then obvious that there is an isomorphism $\hat{\sigma} : X \to X'$ giving rise to the asserted decomposition. The last claim is immediate. \[\square\]

To make reduce the categories one can use the following standard labeling of vertices of reduced graphs, i.e. admitting at most one vertex without leaves. A reduced graph has a standard vertex set $\mathcal{S}$. Every reduced graph is isomorphic to a graph with standard vertex labeling by an isomorphism with $\phi_F = \text{id}$, that is $\sigma^n : \Gamma \to \Gamma^n$ where $(\sigma^n)_F = \text{id}$ and $\sigma^n(v) = F_v$ is the isomorphism that assigns to each vertex its standard name.

A standardized pure morphism is defined to be $\phi_p^n = \sigma^n \phi_p$, we will assume that $t(\phi_p)$ has standard vertex labeling. This yields a unique standard decomposition of a morphisms as $\phi = \sigma \phi^n_p$ via Theorem 2.10 $\phi = \sigma \phi_p = \sigma(\sigma^{-1}) \phi_p^n$.

If the graph is not reduced, there may be several one vertex components without flags and these vertices would need different names. This can be achieved by introducing a skeletal labeling in terms of a number.

Let $k : \mathfrak{S}^{\text{cyc}} \to \mathfrak{S}^{\text{nc-\text{ng-mod}}}$ denote the canonical inclusion. We will now describe the comma categories $k \downarrow \ast \mathcal{S}$. An object is a pair $(X, \phi_X)$ consisting of an aggregate $X$ in $\mathcal{F}^{\text{cyc}}$ and a morphism $\phi_X : k(X) \to \ast \mathcal{S}$ in $\mathfrak{S}^{\text{nc-\text{ng-mod}}}$. A morphism in the comma category is given by a morphism of aggregates $\psi : X \to Y$ in $\mathcal{F}^{\text{cyc}}$ rendering commutative the following triangle:
Lemma 5.4. The comma category \((k \downarrow \ast_S)\) is equivalent to the subcategory whose objects are standard contractions \(v^n_{\phi} : X^n \to \ast_S\) and whose morphisms are generated by standard spanning forest contractions \(v^n_{\phi} : v^n_{\phi} \to v^n_{\phi'}\) and isomorphisms \(\sigma : v^n_{\phi} \to v^n_{\phi'} = \sigma v^n_{\phi}\) given by isomorphisms of the underlying source aggregate fixing the image \(\phi(E)(S)\).

Proof. We show that the inclusion is an equivalence. First, we show the essential surjectivity. Given \(\phi : X \to \ast_S\), we can can decompose \(\phi = \sigma v^n_{\phi(\phi)}\). Furthermore by Lemma 5.3 this is equal to \(v^n_{\phi(\phi)'}\) which is isomorphic in the comma category to \(v^n_{\phi'(\psi)}\) by precomposition with \(\sigma'\). The functor is clearly faithful. To show that it is full consider any \(\psi : v^n_{\phi} \to v^n_{\phi'}\), then \(\psi = \sigma v^n_{\psi}\) with \(v^n_{\psi} = v^n_{\phi}\) a forest contraction. Considering the diagram

\[
\begin{array}{ccc}
Y^n & \xrightarrow{v^n_{\phi}} & \ast_S \\
\phi & \downarrow & \phi \\
X^n & \xrightarrow{v^n_{\phi}} & \ast_S
\end{array}
\]

We can identify \(f \subset \Gamma\) as a spanning forest since \(\Gamma(v^n_{\phi} \circ v^n_{\phi}) = \Gamma(v^n_{\phi'}) \circ \Gamma(v^n_{\phi}) = \Gamma(v^n_{\phi})\) and thus \(E^n_{\phi} = v^n_{\phi}(E^n_{\phi}) \sqcup E^n_{\phi}\).

We consider the following category \(I_S\), where \(I\) stands for an Igusa type category. This is the subcategory of \(\text{Gr}_{ctd}\) whose objects are standard connected graphs \(\Gamma\), with outer flag set \(S\) and standard vertex set, and whose morphisms are generated by standard subforest contractions and isomorphisms leaving the outer flag set fixed.

Proposition 5.5. The comma category \((k \downarrow \ast_S)\) is equivalent to \(I_S\).

Proof. The equivalence is given by combining Lemmas 5.2 and 5.4.

Proposition 5.6. The pushforward of the trivial operation \(O^\text{cyc}_\text{a} \downarrow X\) along \(k : \mathfrak{S}^\text{cyc} \to \mathfrak{S}^\text{ng-mod}\) yields genus labeling \(O_{\text{genus}}\).

Proof. In virtue of Theorem 4.3 and Lemma 5.1, the pushforward of the trivial operation is given for any aggregate \(X\) by the set of connected components of the comma category \((k \downarrow X)\). Since this is a Feynman operation of \(\mathfrak{S}^\text{ng-mod}\) (i.e. a monoidal functor), it suffices to compute the connected components of \((k \downarrow \ast_S)\). By Proposition 5.5 these correspond to those of \(I_S\). Thus, we have to compute the connected components of the category of connected graphs and standard subforest contractions fixing the outer flag set.

The loop number \(b_1\) of a connected graph remains unchanged under subforest contraction. Moreover, any connected graph lies in the same connected component as the one-vertex graph obtained by contracting a spanning subtree. Since any two one-vertex graphs with same loop number are isomorphic, the loop number is a complete invariant, and we can identify \(k_0(O^\text{cyc}_\text{a})(\ast_S) = \pi_0(k \downarrow \ast_S)\) with the set \(N_0 = O_{\text{genus}}(\ast_S)\).

In order to identify this \(\mathfrak{S}^\text{ng-mod}\)-operation with \(O_{\text{genus}}\) we have to compare the actions of the morphisms of \(\mathfrak{S}^\text{ng-mod}\). On the comma categories, the morphisms act by postcomposition. It suffices to check on the generators. Now \(b_1(\Gamma_{s,t}(\phi)) = b_1(\Gamma_{s,t})\) since a non-loop gluing do not change the loop number, and \(b_1(\Gamma_{s,t}(\phi)) = b_1(\Gamma_{s,t}) + 1\) as the loop number is increased, just as under the action of \(O_{\text{genus}}\).
5.3. Surface type labeling $\mathcal{O}_{\text{surf}}$ as pushforward. We begin by describing some inherent structural difficulties of the comma categories $(k \downarrow \ast_S)$.

Two spanning trees of the same connected graph $\Gamma$ are called adjacent if they share all but one edge. Passing from one spanning tree to an adjacent one is called a mutation. The spanning trees of a connected graph $\Gamma$ form again a graph $T(\Gamma)$, the so-called spanning tree graph of $\Gamma$. The set of vertices of $T(\Gamma)$ is the set of spanning trees of $\Gamma$ with an edge between any two adjacent spanning trees. The following theorem refines the connectivity result of Proposition 5.6.

**Theorem 5.7** ([Cum66]). For any connected graph $\Gamma$, the spanning tree graph $T(G)$ is connected.

**Remark 5.8.** There is even a Hamiltonian cycle, i.e., a cycle passing through all edges. Such a Hamiltonian cycle can be determined algorithmically [Kam67]. Observe that the edge which has been removed from the first and the edge which has been added to the second of two adjacent spanning trees belong to a common Hamiltonian cycle of $T(G)$.

We have seen in the proof of Proposition 5.6 that any object of $(k \downarrow \ast_S)$ maps to an object with a one-vertex ghost graph. Since objects of $(k \downarrow \ast_S)$ with one-vertex ghost graph have non-trivial automorphism groups, they are not terminal, and there are parallel morphisms into any such object. Such a parallel pair $X \Rightarrow Y$ is related by an elementary mutation if there exists a diagram $X \to Y' \Rightarrow Y$ composing to the given parallel pair such that the ghost-graph of $Y'$ has two vertices and the parallel pair $Y' \Rightarrow Y$ represents contraction to each of the two vertices. This corresponds to a mutation of the underlying spanning trees.

**Proposition 5.9.** In the comma category $(k \downarrow \ast_S)$ parallel morphisms into objects with one-vertex ghost graph are connected by a finite sequence of elementary mutations.

**Proof.** Any two parallel morphisms correspond to two spanning trees of the corresponding graph in $I_S$. By Theorem 5.7 these two spanning trees are related by a finite sequence of mutations. It suffices thus to show that any mutation of spanning trees factors through an elementary mutation.

Consider a standard morphisms $v_T$ and choose two different spanning trees $\tau$ and $\tau'$. Then then there are two decompositions $v_T \circ q_1 = \phi \circ q_1 = v_T' \circ q_1$. Where $b_1(\Gamma(v_T)) = b_1(\Gamma(v_T')) = b_1(\Gamma)$ Furthermore, there is an isomorphism $\tau$ given by any $\sigma_T, \tau$ that preserves the incidence conditions with $\phi_T' = \phi_T \circ \sigma_T$. That is, there is a diagram

\[
\begin{array}{ccc}
v_1 & \rightarrow & v_1' \\
\ast S & \rightarrow & \ast S \\
\ast \text{STT} & \rightarrow & \ast \text{STT}' \\
\sigma \nearrow & \sigma \nearrow & \sigma \nearrow \\
v_r & \rightarrow & v_{r'} \\
X & \rightarrow & X \\
diagram \end{array}
\]

where the upper triangle commutes, but the lower does not in general. The choice of $\sigma$ and hence the diagram is unique up to unique automorphism of $v_1$. Recall that $T$ and $T'$ are the sets of flags that are not in the spanning tree, and these are different (not only by name). This yields the two parallel morphisms $v_{e_1}$ and $\sigma v_{e_2}$ on the right. The mutation is depicted in Figure 8, where $e_i = \{t_i, \tilde{t}_i\}$, $Y = \ast S_{1 \cup T_1 \cup (t_1, t_2)} \amalg \ast S_{1 \cup T_2 \cup (\tilde{t}_1, \tilde{t}_2)}$ with $S_1 \cup S_2 = S$ and $T_1 \cup T_2 \cup e_2 = T$ and $T_1 \cup T_2 \cup e_2 = T'$.

Suppose $\tau_1$ and $\tau_2$ are adjacent in the spanning tree graph, let $\tau$ be their common subtree and let $\tau_i$ have an additional edge $e_i$, then the we can factor $v_{\tau_i} = v_{e_i} \circ v_{\tau}$, where $v_{e_i}$ is a simple edge...
contraction. Then two parallel morphisms $v_{\tau_1}$ and $\sigma v_{\tau_2}$ factor through an elementary mutation:

$$\begin{align*}
*_{R_1} & \xrightarrow{v_1} *_S \\
v_{e_1} & \xleftarrow{\sigma v_{e_2}} X \\
v_{e} & \xrightarrow{v_{1'}} Y
\end{align*}$$  \quad (5.6)

**Proposition 5.10.** The comma category $(k\pi_2 \downarrow *_S)$ is equivalent to $RI_S \subset R_{\text{for}}$. The objects of $RI_S$ are standard connected ribbon graphs with outer flag set $S$. Morphisms are standard subforest contractions fixing $S$.

**Proof.** This follows from Proposition 5.5, Proposition 5.6, via Proposition 4.15 and Theorem 4.7 and the fact that a $\mathcal{O}_{\text{cycass}}$-decoration is a ribbon structure. \hfill \Box

**Proposition 5.11.** $k_!(\mathcal{O}_{\text{cycass}}) = \mathcal{O}_{\text{surf}}$.

**Proof.** Combining Proposition 5.10 and Lemma 5.1 implies that $k_!(\mathcal{O}_{\text{cycass}})(*_S) = (k\pi_2)_!(\mathcal{O}_{\text{surf}}^{\text{cyl-cyc}})(*_S)$ can be identified with $\pi_0(RI_S)$. It thus remains to be shown that the nc–modular operad $\mathcal{O}_{\text{surf}}$ of oriented surface types may be identified with the connected components of the categories $RI_S$.

Each connected ribbon graph contracts to a one-vertex ribbon graph by contraction of a spanning tree. We will show that each one-vertex ribbon graph is equivalent to a one-vertex ribbon graph in normal form, and that each connected component of $RI_S$ contains a single one-vertex ribbon graph in normal form. Then we describe a one-to-one correspondence between one-vertex ribbon graphs in normal form and connected topological types respecting the modular operad structures.

One extra-information of our proof is the fact that two one-vertex ribbon graphs are in the same connected component if and only if they are mutation-equivalent, i.e. transformable into each other by a finite sequence of elementary mutations where an elementary mutation between one-vertex ribbon graphs is defined to be a two-vertex ribbon graph which contracts to both of them.
We represent one-vertex ribbon graphs as cyclic words of their flags where two inner flags making up a loop are denoted \(t, \bar{t}\) and outer flags get capital letters. A one-vertex ribbon graph is in normal form if the representing cyclic word is of the form

\[
(S_1l_1S_2l_2s_1l_2 \cdots l_{b-1}s_bl_{b-1}c_1 \cdots c_p\bar{c}_p a_1b_1\bar{a}_1\bar{b}_1 \cdots a_gb_g\bar{a}_g\bar{b}_g)
\]  

(5.7)

where \(S_i\) denotes a cyclic flag set of cardinality \(p_i\) and \(0 < p_1 \leq \cdots \leq p_b\) so that \(S = (S_1, \ldots, S_b)\) is a polycyclic set \((S, \sigma_S)\) with \(b\) cycles. It follows from Lemma 5.12 and Corollary 5.13 below that every cycle word is mutation-equivalent to a cyclic word in normal form. The normal form is determined by and determines the triple \((g, p, \sigma_S)\).

To understand the functor \(k_!(\mathcal{O}_{\text{cycass}})\) on morphisms, we only have to consider post–composition with the generators. For isomorphisms post–composition is the usual action by isomorphisms. A virtual loop contraction \(\circ_{st}\) adds a loop, by renaming \(t\) to \(\bar{s}\) to the respective one–vertex ribbon graph. The polycyclic structure on the \(S_i\)’s is the one given by \(\circ_{st}\). If the new pair is adjacent in the normal form, then it produces an empty partition, that is \(p\) increases by 1. This is exactly (3.12).

For the operation \(e_{ij}\), two standard words are concatenated and \(t\) is renamed \(\bar{s}\) providing a new pair. The genus \(g\) and the number of pairs \(e_{ij}\) is additive. The effect on the polycyclic structure of the \(S_i\) is \(e_{ij}\). The relative position of this corresponds exactly to the cases in (3.11). If they are from different sets \(S_i, S_j\), where we can assume that \(S_i = S_1\), they introduce a new interleaved pair which increases the genus and if they are additionally both the only element in their set, then \(p\) also increases by one. If \(s, t\) are both in the same \(S_i\), we can assume that this is \(S_1\). If \(S_2 = \{s, t\}\) then this pair is empty and \(p\) increases by 2. If \(s, t\) are adjacent, but are not the only two elements of \(S_i\), \(p\) only increases by 1. If they are not adjacent, then then \(p\) stays constant.

**Lemma 5.12.** There is an elementary mutation to the effect \((\text{AtBCiD}) \leftrightarrow (\text{DtCBiA})\)

Thus we may cyclically permute the letters between an occurrence of \(t\) and \(\bar{t}\) as well inside (between \(t\) and \(\bar{t}\)) as well as outside (between \(t\) and \(\bar{t}\)).

**Proof.** Given a cyclic word \((\text{AtBiC})\) we split the unique vertex of \(\Gamma\) into two vertices joined by two parallel edges one being \((tt)\), the other \((ss)\), in such a way that contraction of \((ss)\) yields \(\Gamma\). This implies that the outer flags \(B\) and \(C\) sit inside the circle defined by \((ttss)\). Contracting the edge \((tt)\) then produces a one-vertex ribbon graph \(\Gamma’\) represented by the cyclic word \((\text{AsCBiD})\). Up to renaming \(s\) by \(t\) this yields the desired result, see Figure 8. □

**Corollary 5.13.**

(i) There is a mutation to the effect \((\text{AtBiC}) \leftrightarrow (\text{ACTBi})\).

(ii) There is a mutation to the effect \((s\text{AtBiC}s) \leftrightarrow (s\text{ACTsBi})\).

(iii) There is a mutation to the effect \((\text{AsBtCsDiE}) \leftrightarrow (\text{ADCBEstti})\).

In particular, every cyclic word is mutation-equivalent to one in normal form.

**Proof.** For (i) this is the special case of Lemma 5.12 where \(C = \emptyset\). For (ii) we may apply (i) to move the letters \(Cs\) across the loop \(tBl\). For (iii) we need a sequence of mutations of the previous types \((\ldots \text{AsBtCsDiE} \cdots) \leftrightarrow (\ldots \text{ADsEtBsCi} \cdots) \leftrightarrow (\ldots \text{ADCstEsBi} \cdots) \leftrightarrow (\ldots \text{ADCBEstti} \cdots)\) where in the first mutation we moved \(D\) left over the \(s, \bar{s}\) loop, \(E\) left over the \(t, \bar{t}\) loop and moved \(B\) and \(C\) to the right inside the loops \(s, \bar{s}\) and \(t, \bar{t}\). The next step iterates this process until everything is moved out to the left.

Now using (ii) we unnest, using (iii) we isolate interleaved pairs, and in a final step, we move all the remaining letters that are not in between \(t\) and \(\bar{t}\) to the left using (i). □
5.4. **Combinatorial realizations of cyclic words.** There are other combinatorial presentations of one-vertex ribbon graphs which can be used for an alternative proof of the existence and uniqueness of normal forms, see Figures 9, 11 and 12. For Lemma 5.12 in the respective formalism, see Figures 13, 14 and 15.

5.4.1. **Labelled polygons and oriented surfaces.** The flags of a one-vertex ribbon graph correspond one-to-one to the sides of a polygon, the loops are realized by self-gluing. The resulting bordered oriented surface has the same homotopy type as the one constructed in section §1.4. We refer the reader to [Mun75] where this kind of structure is been used for a complete classification of bordered oriented surfaces following [Mas67]. The nc–modular operad structure is visible on this level.

One can blow up the vertices of the polygons to intervals and thereby obtain $2n$-gons with alternating sides that are labelled. In this way a triangle turns into a planar pair of pants. This point of view is common for open TFT [CL91, LP08]. It also corresponds to looking at $\pi_0$ in the arc picture [KLP03, KP06] and basically goes back to triangulations of surfaces with boundary and hyperbolic geometry [Tra79]. It has later been used under the name of cogwheels or tabs [CL07, Mar16].

The composition $s \circ t$ of planar corollas is called mating spiders in [CV03].

5.4.2. **Chord/rainbow diagrams.** The flags of a one-vertex ribbon graph are represented by points on a circle, the loops are realized by segments between the two points representing the internal flags of the loop. One obtains in this way a chord diagram. Cutting the circle at one point,
the chord diagram becomes a rainbow diagram. The composition now is given by cutting open the chord diagram at the marked vertices and connecting the outer circles according to the orientation.

The gluing in terms of chord diagrams is related to Kontsevich’s coproduct on chord diagrams [BN95]. More precisely, if one considers the Feynman category of one vertex ribbon graphs in Rib_for,con, the coproduct dual to the composition [GCKT20] is indeed the Kontsevich coproduct.

5.5. **Pushforwards to \( \mathfrak{F}_{\text{nc ng-mod}} \).** One can furthermore study pushforward along the inclusion \( l : \mathfrak{F}_{\text{ng-mod}} \to \mathfrak{F}_{\text{nc ng-mod}} \). These pushforwards carry more structure and allow us to keep track of several components at a time.

For a partition \( P = S_1 \sqcup \cdots \sqcup S_k \) of \( S \), we set \( \text{Aut}(S/P) = \text{Aut}(S)/(\text{Aut}(S_1) \times \cdots \times \text{Aut}(S_n) \setminus S_n) \).
Lemma 5.14. The connected components \( \pi_0(l \downarrow \ast_S) \) are given by pairs consisting of a partition \( S = S_1 \sqcup \cdots \sqcup S_n \) into possibly empty sets and an element \( \sigma \in \text{Aut}(S/P) \).

Proof. Using Theorem 2.10 we can factor any \( \phi \) uniquely as \( \phi = \sigma \phi_m \phi_{v-con} \). Precomposing with an isomorphisms of the source, we stay in the same fiber, but can assume that \( \Gamma(\phi) \) has \( S \) as outer flag set. Precomposing with \( \phi_{v-con} \) this decomposition receives a map from \( \sigma \phi_m \) where \( \phi_m \) is a merger \( t(\Gamma(\phi)) \to \ast_S \) and \( \sigma : \ast_S \to \ast_S \) is an isomorphism. Here the \( S_i \) are the outer flags of the component of \( \Gamma(\phi) \) indexed by \( \psi \). The image of \( \text{Aut}(t(\Gamma(\phi)) \in \text{Aut}(\ast_S) \) is precisely \( \text{Aut}(S_1) \times \cdots \times \text{Aut}(S_n) \ast_S \) under the crossed structure of Corollary 2.12, see equation (2.7). This also shows that the partition together with an element completely classifies the fibre.

\[ \square \]

Corollary 5.15. \( l(\mathcal{O}_{\text{genus}}) \) is given by

\[ \mathcal{O}_{\text{genus}}^\text{nc}(\ast_S) = \{ \text{genus labelled partitions of } S \} \times \text{Aut}(S)/\text{Aut}(P) \]

A typical element is an unordered tuple \( [(g_1, S_1), \ldots, (g_n, S_n)]] \) where the \( S_i \) are a partition of \( S \) by possibly empty subsets and \( g_i \in \mathbb{N}_0 \). This is an unordered tuple, the order of the entries does not matter and we may have repetitions. We have the following behaviour under morphisms: Isomorphisms act naturally on the partition and \( \text{Aut}(P/S) \).

For the compositions, say \( s \in S_i, t \in T_j \):

\[ \mathcal{O}_{\text{genus}}^\text{nc}(s \circ_t)[[(g_1, S_1), \ldots, (g_n, S_n)]] = [(g_1, S_1), \ldots, (g_i, S_i), \ldots, (g_n, S_n), (g_1, T_1), \ldots, (g_i, T_i), \ldots, (g_n, T_n), (g_i + g_j, S_i \circ_t T_j)] \] \((5.8)\)

and the elements of the automorphisms groups are given by the restriction along \( S \setminus \{s\} \sqcup T \setminus \{t\} \to S \sqcup T \). If \( s \in S_i \) and \( s' \in S_j \neq S_i \) then

\[ \mathcal{O}_{\text{genus}}^\text{nc}(s \circ_{ss'}^\ast)[[(g_1, S_1), \ldots, (g_n, S_n)]] = [(g_1, S_1), \ldots, (g_i, S_i), \ldots, (g_j, S_j), \ldots, (g_n, S_n), (g_i + g_j, S_i \circ_{ss'} S_j)] \] \((5.9)\)

and if \( s, s' \in S_i \).

\[ \mathcal{O}_{\text{genus}}^\text{nc}(s \circ_{ss'}^\ast)[[(g_1, S_1), \ldots, (g_n, S_n)]] = [(g_1, S_1), \ldots, (g_i, S_i), \ldots, (g_n, S_n), (g_i + g_j, \circ_{ss'} S_i)] \] \((5.10)\)

with the elements of the automorphisms groups again given by restriction. Finally, mergers just merge lists.

\[ \mathcal{O}_{\text{genus}}^\text{nc}(\ast_w \circ_{ww}) : [[(g_1, S_1), \ldots, (g_n, S_n)]] \to [((g_1, S_1), \ldots, (g_n, S_n), (g_1', T_1), \ldots, (g_n', T_n))] \] \((5.11)\)
with the elements of the automorphisms given by inclusion $\text{Aut}(S/P) \times \text{Aut}(T/P) \to \text{Aut}((S \sqcup T)/(P \sqcup P'))$.

Proof. This is a straightforward computation following from Lemma 5.1 and Lemma 5.14. \qed

There is a natural transformation $\mathcal{O}^{nc}_{\text{genus}} \to \mathcal{O}_{\text{genus}}$ given by

$$\left(\left[(S_1, g_1), \ldots, (S_n, g_n)], \sigma \right) \mapsto 1 - \chi = 1 - n + \sum_i g_i \right) \quad (5.12)$$

Remark 5.16. The surface interpretation is a disconnected surface. Note that the automorphisms groups cannot mix boundary components of the different components of the surface. To get the action on all of them, one has to induce up the automorphisms groups, which is what $\sigma$ keeps track of. The natural transformation is what is used in [Zwi93, Sch98, HVZ10, KWZ15] to forget the internal disconnected structure. The upshot of including the nc case is a BV structure vs just a track of. The natural transformation is what is used in [Zwi93, Sch98, HVZ10, KWZ15] to forget

A polycyclic partition $(P, \sigma_P)$ of $S$ is a partition $S = S_1 \sqcup \cdots \sqcup S_n$ with individual polycyclic structures $\sigma_i$, i.e. $\sigma_P \in \prod_i \text{Aut}(S_i)$. We set $\text{Aut}(S/(P, \sigma_P)) := \text{Aut}(S)/\text{Stab}(\sigma_P)$. A general element is given by $\left(\left[(g_1, p_1, S_1, \sigma_1), \ldots, (g_1, p_1, S_1, \sigma_1)], \sigma_P \right)$. We will write $S^{\sigma}_n$ for $(S_i, \sigma_i)$.

Corollary 5.17. $\mathcal{O}_{\text{surf}}$ is given by the set $\mathcal{O}^{nc}_{\text{surf}}(s,s)$ consisting of polycyclic partitions $(P, \sigma_P)$ of $S$ together with two natural numbers for each element in the partition.

The action of isomorphisms is via pullback, the composition for mergers is joining of lists as above. For the morphisms $s \circ t$, the composition is that of $\mathcal{O}_{\text{surf}}$ on the two entries

$$\mathcal{O}_{\text{surf}}(s \circ t)((g_i, p_i, S^{\sigma}_i, (g_j, p_j, T_j)))$$

while the others are unchanged. Similarly if $s \in S_i, s' \in S_j, i \neq j$ then the action on the only changed entries is $\mathcal{O}_{\text{surf}}(s \circ s')(\left[(g_i, p_i, S^{\sigma}_i), (g_j, p_j, T_j)\right])$, while if $s, s' \in S_i$ only one entry changes $\mathcal{O}_{\text{surf}}(s \circ s')(g_i, p_i, S^{\sigma}_i)$. The poly-polyccyclic structures $\sigma_P$ compose via inclusion as above.

Proof. This again follows from Lemma 5.1 and Lemma 5.14. With the addition that the automorphisms group in the decorated index category has to fix the polycyclic structure. \qed

There is a natural transformation $\mathcal{O}^{nc}_{\text{surf}} \to \mathcal{O}_{\text{surf}}$ given by

$$\left(\left[(g_1, p_1, S^{\sigma}_1), \ldots, (g_n, p_n, S^{\sigma}_n)], \sigma_P \right) \mapsto (1 - n + \sum_i g_i, \sum_i p_i, \{S^{\sigma}_1, \ldots, S^{\sigma}_n, T^{\sigma}_1, \ldots, T^{\sigma}_m\}\right) \quad (5.13)$$

5.6. Connected sum as a $B_+$ operator. There is another operation which we can perform, and this is to take two tuples and simply merge them. This is how the polycyclic structures arise in Kontsevich’s description, see Proposition 1.4.

$$B_+ : \left(\left[(g_1, p_1, S^{\sigma}_1), \ldots, (g_n, p_n, S^{\sigma}_n)], \sigma_P \right) = \left(\sum_i g_i, \sum_i p_i, \{S^{\sigma}_1, \ldots, S^{\sigma}_n, T^{\sigma}_1, \ldots, T^{\sigma}_m\}\right) \right) \quad (5.14)$$

This is not a natural transformation of $3^{\text{nc-ang-mod}}$ operations, as the equation 2.9 does not hold. It does however define a new Feynman category. The relationship is as in [KW17, §3.2.1] in terms of surfaces $F_1$ and $F_2$, this corresponds to the connected sum $F_1 \# F_2$ and in terms of physics it is a $B_+$ operator in the sense of Connes and Kreimer [CK98]. This also plays a role in string topology, which will be explained in [BK22b].

Remark 5.18. Geometrically the $B_+$ is the connected sum operation. This means that the boundary components of the different components are now boundary components of the same connected component.
6. Actions

The structure of the category of aggregates, in particular the adjunction between pushforward and pullback functors, has a direct application to 1+1 dimensional TFTs. Beyond this there is an interpretation for the correlators [Kau08a] giving rise to algebraic string topology operations as well as to operations on the Tate–Hochschild complex [Kau18, KRW21].

6.1. Algebras via reference functors. For operads, the usual definition of an algebra in a closed symmetric monoidal category \( \mathcal{C} \) is an object \( A \) of \( \mathcal{C} \) together with a morphism of operads \( \rho : \mathcal{O} \to \text{End}_A \), where \( \text{End}_A(n) = \text{Hom}(A^ \otimes n, A) \) denotes the endomorphism operad of \( A \) and \( \text{Hom} \) denotes the internal hom of \( \mathcal{C} \). Likewise, for a PROP \( \mathcal{P} \), an algebra is a pair consisting of an object \( A \) and a morphism of PROPs \( \rho : \mathcal{P} \to \text{End}_A \) where now the endomorphism PROP of \( A \) is \( \text{End}_A(n, m) = \text{Hom}(A^ \otimes n, A^ \otimes m) \).

In order to generalize these notions, we define a reference functor for \( \mathfrak{F} \) to be a monoidal functor \( \mathcal{E} \in [\mathcal{C}, [\mathcal{F}, \mathcal{C}]_\otimes]_\otimes \).

Definition 6.1. Given a reference functor \( \mathcal{E} \) and a \( \mathcal{F} \)-operation \( \mathcal{O} \in [\mathcal{F}, \mathcal{C}]_\otimes \), an algebra over \( \mathcal{O} \) with values in \( \mathcal{E} \) is a pair \( (X, \rho) \) consisting of an object \( X \) of \( \mathcal{C} \) and a natural transformation \( \mathcal{O} \to \mathcal{E}(X) \).

This is functorial in all variables when regarded as elements of the functor \( [\mathcal{C}, [\mathcal{F}, \mathcal{C}]_\otimes] \times \mathcal{C} \to \text{Set} \), given by \( (\mathcal{O}, X, \mathcal{E}) \to \text{Nat}(\mathcal{O}, \mathcal{E}(X)) \), i.e. evaluation and application the hom–functor in the functor category \( [\mathcal{F}, \mathcal{C}]_\otimes \). Reference functors transfer between Feynman categories via pullback.

6.2. Reference functors for \( \mathfrak{F} \)-\( \text{ng-mod} \) and correlation functions. Consider any functor \( \mathcal{O} : \mathfrak{F} \to \mathcal{C} \). First, \( \mathcal{O}(\ast_\mathfrak{F}) \) forms a monoid in \( \mathcal{C} \) under \( \mathcal{O}(\ast_\mathfrak{F} \sqcup \ast_\mathfrak{F}) \) (cf. [KW17, §2.9.1]) and by changing \( \mathcal{C} \) if necessary to objects over \( \mathcal{O}(\ast_\mathfrak{F}) \), we may assume that \( \mathcal{O}(\ast_\mathfrak{F}) = \mathbb{1}_\mathcal{C} \). Second, there is an operation \( \mathcal{O}(\ast_0) : \mathcal{O}(\ast_0) \otimes \mathcal{O}(\ast_0) \to \mathcal{O}(\ast_\mathfrak{F}) = \mathbb{1} \). Setting \( W = \mathcal{O}(\ast_0) \) makes \( P := \mathcal{O}(\ast_0) \in \text{Hom}(W^{\otimes 2},1) \) into a pairing. The pairing is symmetric, as there is only one morphism \( \ast_0 \sqcup \ast_0 \to \ast_\mathfrak{F} \) whose automorphism group is given by interchanging the two factors.

The existence of a pair \( (W, P) \) is thus common to all functors on \( \mathfrak{F} \)-\( \text{ng-mod} \). This motivates the construction of a particular reference functor. Let \( \mathcal{P}_\text{Cor} \) be the category of pairs \( (W, P) \) with \( W \in \text{Obj}(\mathcal{C}) \) and \( P \) a symmetric pairing on \( W \). Morphisms are the subsets \( \text{Hom}_\mathcal{P}_\text{Cor}((W, P), (W', Q)) \subset \text{Hom}_\mathcal{C}(W, W') \) given by those morphisms \( \phi : W \to W' \) which respect the pairings under pullback: \( \phi^*(Q) = Q \circ (\phi \otimes \phi) = P \).

Definition 6.2. Each pair \( (W, P) \) in \( \mathcal{P}_\text{Cor} \) defines a \( \mathfrak{F} \)-\( \text{ng-mod} \)-operation \( \text{Cor}(W, P) \) called the universal \( W \)-correlation functions with pairing \( P \) defined as follows: \( \text{Cor}_W(\ast_\mathfrak{F}) = W^{\otimes 2} \).

For any morphism \( \phi : X \to Y \), the correlation functions \( \text{Cor}(W, P)(\phi) : W^{\otimes F(X)} \to W^{\otimes F(Y)} \) are given by contracting the along the ghost edges using \( P \):

\[
S^{\otimes F(X)} = W^{\otimes (\phi^F)^{-1}(F(Y))} \otimes (W \otimes W)^{E(\Gamma(\phi))} = W^{\otimes F(Y)} \otimes (W \otimes W)^{E(\Gamma(\phi))} \xrightarrow{id \otimes F^{\otimes E(\Gamma(\phi))}} W^{\otimes F(Y)}
\]

where two tensors factors of \( W \) indexed \( f \) and \( f' \) for each ghost edge \( e = \{f, f' = i(f)\} \) of \( \Gamma(\phi) \) are contracted with \( P \), which is well defined as \( P \) is symmetric, and we used the bijection of \( \phi^F \) onto its image. The action by isomorphisms is by permutations and relabelling of factors. The action of mergers is the multiplication in the tensor algebra.

Lemma 6.3. \( \text{Cor}(V, P) : \mathcal{P}_\text{Cor} \to \mathfrak{F} \)-\( \text{ng-mod} \)-\( \text{Ops}_\mathcal{C} \) is functorial and provides a reference functor of \( \mathfrak{F} \)-\( \text{ng-mod} \)-\( \mathcal{O} \text{Ps}_\mathcal{C} \).

Proof. Straightforward.
Define $\triangledown : \mathcal{C} \to \mathcal{C}^{op}$ as usual by $V \to \tilde{V} = \text{Hom}(V, \mathbb{C})$. $P$ is non-degenerate, if $\triangledown_P$ is an isomorphism.

Example 6.4 (Correlations functions from propagators.). Often, for instance in physical and geometric applications, $W = \tilde{V}$ and the pairing on $W$ is given by a propagator or Casimir element, that is a symmetric element $C \in V \otimes V$ which yields a pairing $P \in \text{Hom}(\tilde{V} \otimes \tilde{V}, \mathbb{C})$ by evaluation. Physically, if $V$ is a space of fields, then an element in $V^{\otimes}S$ thought of as a morphism $V^{\otimes}S \to k$ is a correlation function, whence the name. A geometric example is furnished by $V = H^*(M)$ for $M$ a compact manifold and $P$ is the class of the diagonal in $H_*(M) \otimes H_*(M)$, cf. [Kau18]. Thus the present formalism is the most general. In the non-degenerate case, these formulations are equivalent and are induced via the isomorphism $\triangledown_P$.

Remark 6.5. For the special case of $\mathcal{C} = k\text{-}Vect$ the notion of an algebra over a cyclic and modular operad was defined in [GK95,GK98] where it is assumed that $P$ is non-degenerate. The even/odd distinction was stressed in [CV03] and pairings of different degrees were treated in [Bar07], see also [KWZ15]. Without the assumption of non-degeneracy this treatment also yields the notion of abstract correlation functions of [Kau08a] where also the values were taken in twisted $\text{Hom}$ functors—a necessary step for Deligne’s conjecture. The formalism of contracting tensors goes back to [Ger63] and is used in Gromov–Witten theory [KM94,Man99].

If $\mathcal{F}$ has a functor to $\mathcal{F}_{\text{ncng-mod}}$, then let $\mathcal{B}$ be the underlying functor $\mathcal{F} \to \text{Agg}$. We define $\text{Cor}_{\mathcal{F}}(V,P) = \mathcal{B}^*(\text{Cor}_{\mathcal{F}}(V,P)) = \text{Cor}_{\mathcal{F}}(V,P) \circ \mathcal{B}$. If it is obvious from the context, we will omit the superscript $\mathcal{F}$. An algebra over an $\mathcal{F}$-operation $\mathcal{O}$ in $\mathcal{C}$ is defined to be an algebra over $\mathcal{O}$ with values in $\text{Cor}_{\mathcal{F}}$. These are given by an object $(V,P) \in \mathcal{C}_P$ and a natural transformation $N$ from $\mathcal{O}$ to $\mathcal{B}^*(\text{Cor}_{\mathcal{F}}(V,P))$. An algebra is hence a tuple $((V,P),\mathcal{O},N)$. In the non-degenerate case, we use the usual notation for $\text{Cor}_{\mathcal{F}}(V,P)$ is $\text{End}_{\mathcal{F}}(V,P)$.

Example 6.6. For $\mathcal{F}_{\text{cyc}}$, it is common to work with a skeleton of $\text{FinSet}$, cf. [GK95]. This means that one uses a standard set of correlas, $\ast\{n\}$ with vertex $\ast$ and with flag sets $\{0,\ldots,n\}$. For $n = -1$ the flag set is empty by convention.

In this setting, one also defines $\text{End}_{\mathcal{F}}((\ast\{n\}) = \text{Hom}(V^{\otimes n},V) \simeq \tilde{V} \otimes V^{\otimes n}$ with the first factors of $V$ called inputs and the last factor of $V$ the output. This is the dualisation of $\text{Cor}_{\mathcal{F}}(V,P)$ in the target variable using $\triangledown_P$. The compositions are given by contracting the “out” $V$ with an “in” $V$. The condition of non-degeneracy then implies that under $\triangledown_P$, this corresponds precisely to contracting with $P$. The equivariance is harder to formulate in this framework and is not as natural, see e.g. [Kau04,KW17].

Table 3 contains algebras over given operations. The first two are well known and establishing the remaining entries is the goal of this section.

Remark 6.7. There is a directed version of Feynman categories indexed over the directed version of aggregates of §3.4, cf. [KW17, §2.2], which has a simpler reference functor $\mathcal{E}$ given by $\mathcal{E}_W(*_{\text{in}}*_{\text{out}}) = \text{Hom}(V^{\otimes \text{in}},V^{\otimes \text{out}})$ and the functor uses evaluation on each of the ghost edges, which have one $V$ (“out”) and one $\tilde{V}$ (“in”) associated to them. This explains why there is no need to choose a pairing or propagator for algebras over operads or PROPs. Additionally, there is a generalisation to the coloured context [KW17, §2.5], where now there is a set of objects in $\mathcal{C}_P$, one for each color.

An algebra $(W,P,Y)$ over the trivial operation $\mathcal{O}_1$ yields elements in each $\text{Cor}_{(W,P)}(X)$ via $Y_X : \mathcal{O}_1(X) = 1 \to \text{Cor}_{(W,P)}(X)$. These are called correlation functions. Since $Y$ is a natural transformation, these correlators are not independent, but have to satisfy compatibilities.

Lemma 6.8. Given a set of elements $Y_S = Y_{*S}$ the condition for being a correlation function corresponding to the different generators of $\mathcal{F}_{\text{ncng-mod}}$ are:
(1) For an isomorphism given by the bijection $\sigma^F : T \to S$, the compatibility is equivariance $Y_S = \sigma^F_* Y_T$.
(2) The compatibility for $s \bullet t$ is $\iota_s P_t Y_S \otimes T_T = Y_{(s \setminus \{s\} \cup T \setminus \{t\})}$ where $\iota_t P_t$ contracts the tensors in positions $s$ and $t$, then $Y_S$ is a set of correlation functions for $\mathfrak{F}^{\text{cyc}}$.
(3) The compatibility with $u \circ_{s,s'}$ is $\iota_t P_t Y_S = Y_{S_{\{s,s'\}}}$.
(4) Finally, the correlation functions are compatible with $v \boxtimes w$ if $Y_S \otimes Y_T = Y_{S \cup T}$.

For being correlation functions on $\mathfrak{F}^{\text{cyc}}$ (1) is necessary and sufficient, (1) and (2) are for $\mathfrak{F}^{\text{cyc}}$, and all are for $\mathfrak{F}^{\text{cyc}}$. For being correlation functions on $\mathfrak{F}^{\text{cyc}}$ (1) is necessary and sufficient, (1) and (2) are for $\mathfrak{F}^{\text{cyc}}$, and all are for $\mathfrak{F}^{\text{cyc}}$.

Proof. This is an application of naturality. Since $N$ is a natural transformation, the diagram below commutes and gives the equality for (1).

$$
\begin{array}{ccc}
\mathcal{O}_2(\ast_S \amalg \ast_T) = 1 \otimes 1 & \overset{\mathcal{O}_2(s \circ_t) = l_4}{\longrightarrow} & \mathcal{O}_2(\ast_{S \setminus \{s\} \cup T \setminus \{t\}}) = 1 \\
N_{s \mid t} = Y_S \otimes Y_T & & N_{S_{\{s\} \cup T \setminus \{t\}}} \\
\end{array}
$$

where $l_4$ is the unit constraint and the morphism $\mathcal{O}_2(s \circ_t)$ is the contraction with $P$ in the positions $s$ and $t$. The rest is analogous for isomorphisms $s \circ_t$, $s \circ s'$ and $v \boxtimes w$. Since the 4 classes of morphisms generate, we get the necessary part. For the sufficient part, one has to check the relations, but this is straightforward, since the edges of the ghost graph are contracted with $P$ and it does not matter in which order this is done.

Traditionally, many calculations are done in a skeletal version. Here the standard notation for $Y_{\{1,\ldots,n\}}$ is $Y_n$. In the case of $\mathfrak{F}^{\text{pl-cyc}}$ the corolla $\ast_{\{1,\ldots,n\}}$ is taken to have the standard cyclic order on $\{1,\ldots,n\}$ and $Y_n$ denotes the respective correlation function.

Definition 6.9. An $\mathfrak{F}^{\text{cyc}}$ or $\mathfrak{F}^{\text{pl-cyc}}$ algebra over $\mathcal{O}_2$ given by $((W, P), Y)$ is unital if $Y_2$ and $P$ are inverse to each other, i.e. the image of $Y_2 \otimes P \in W \otimes W$ under the canonical pairing on the second and third factor is $id_W$.

This implies that $Y_2$ is non-degenerate and $W$ and $W$ are dual and $P$ is the Casimir element for the form pulled back to $V = W$. For elements of $V$, using Sweedler notation for $P$, this is equivalent to the familiar $\sum (a, P^{(1)})P^{(2)} = a$.

Note that the property of being unital is natural in $\mathcal{C}_P$.

Proposition 6.10. $\mathfrak{F}^{\text{cyc}}$ resp. $\mathfrak{F}^{\text{pl-cyc}}$ algebras $(W, P), Y)$ over $\mathcal{O}_2$ are classified up to isomorphism by pairs $(Y_1, Y_3)$ consisting of an element $Y_1 \in W$ and symmetric resp. cyclic invariant tensor $Y_3 \in W \otimes 3$, which satisfy the three compatibility equations

$$
\iota_1 P_1 (Y_1 \otimes Y_1 Y_3) = Y_1, \quad \iota_2 P_1 (Y_1 \otimes Y_3) = Y_3, \quad \iota_3 P_1 (Y_3 \otimes Y_3) \text{ is cyclically invariant (6.3)}
$$

Proof. By Proposition 6.8 part (1), picking skeletal objects, we can reduce to the $Y_n$. From part (2) $Y_n$, $Y_0$, and $Y_2$ satisfy:

$$
Y_n = \iota_{n-1} P_1 Y_{n-1} \otimes Y_3, \quad Y_2 = \iota_1 P_1 Y_1 \otimes Y_3, \quad Y_0 = \iota_2 P_2 Y_2 \in 1
$$

this allows to reduce to $Y_3$ and $Y_1$ and explains the necessity of the first two compatibility equation. The third compatibility concerns two different virtual edge contractions that both result in $\ast_{\{1,2,3,4\}}$, see Figure 4. Notice that these are all cyclic relations and they thus lift to $\mathfrak{F}^{\text{pl-cyc}}$.

The fact that these relations generate all relations, follows from standard arguments, see e.g. [Dij89, KP06]. Geometrically, this is the fact that Whitehead moves act transitively on pairs of pants decompositions or diagonal compositions of polygons. Combinatorially this is the case, since the space of (planar) trees with fixed tails is connected by edge contractions and expansions, which amount to mutations. A purely algebraic proof is in e.g. in [Kau18].
6.3. Commutative and symmetric (aka closed and open) Frobenius algebras. There are several equivalent characterizations for symmetric Frobenius algebras, cf. e.g. [Man99,Kau18,PK09]. We will discuss two convenient forms using the standard notation. This is \( W = \tilde{V}, Y_n = \langle , \ldots , \rangle. \) The element \( \langle \rangle_1 \) is usually denoted by \( \epsilon \) or \( \int \) and the element \( \langle , \rangle_2 \) simply by \( \langle , \rangle. \)

**Definition 6.11.** A symmetric (aka open) Frobenius algebra is a unital associative algebra with a symmetric non-degenerate bilinear form \( \langle , \rangle \) which is invariant \( \langle a, bc \rangle = \langle ab, c \rangle. \)

A commutative (aka closed) Frobenius algebra is a symmetric Frobenius algebra which is also commutative.

**Proposition 6.12.** The following is an equivalent definition of symmetric, resp. commutative Frobenius algebras, namely a quadruple \( (V, \epsilon, \langle , \rangle, \langle , , \rangle) \) where

1. \( V \) is a vector space,
2. \( \epsilon : V \to 1 \) a so-called counit
3. \( \langle , \rangle \) is a symmetric non-degenerate bilinear product on \( V \)
4. \( \langle , , \rangle : V^{\otimes 3} \to 1 \), is a 3-tensor which is cyclically invariant in the symmetric case and \( S_3 \) invariant in the commutative case.

which satisfies the compatibility equations (6.3). Where in these equations \( P \in V \otimes V \) is dual to the metric \( \langle , \rangle \in V \otimes V. \)

**Proof.** A Frobenius algebra furnishes the data satisfying the axioms: Set \( \epsilon(a) = \langle 1, a \rangle \) and \( \langle a, b, c \rangle_3 = \langle ab, c \rangle. \) The cyclicity of \( \langle , \rangle_3 \) then follows from the symmetry and invariance of the metric: \( \langle b, c, a \rangle = \langle bc, a \rangle = \langle a, bc \rangle = \langle ab, c \rangle = \langle a, b, c \rangle. \)

For the first compatibility equations one calculates:

\[
\sum \epsilon(P^{(1)})\langle P^{(2)}, a \rangle = \sum \langle 1, P^{(1)} \rangle \langle P^{(2)}, a \rangle = \langle 1, a \rangle = \epsilon(a)
\]

(6.5)

As \( \langle , \rangle \) is non-degenerate, this also shows that \( \sum \epsilon(P^{(1)})P^{(2)} = 1. \) Using this the second equation follows immediately:

\[
\sum \epsilon(P_1^{(1)})\langle P_2^{(1)}, P_2^{(2)}, b, c \rangle = \langle a, bc \rangle = \langle 1a, b \rangle = \langle a, b, c \rangle
\]

(6.6)

Finally, for the third condition:

\[
\sum \langle a, b, P^{(1)} \rangle \langle P^{(2)}, c, d \rangle = \sum \langle ab, P^{(1)} \rangle \langle P^{(2)}, c, d \rangle \sum \langle ab, P^{(1)} \rangle \langle P^{(2)}, cd \rangle
\]

\[
= \langle ab, cd \rangle = \langle a, b(cd) \rangle = \langle b(cd), a \rangle = \langle bc, da \rangle
\]

\[
= \langle bc, P^{(1)} \rangle \langle P^{(2)}, d, a \rangle = \langle bc, P^{(1)} \rangle \langle P^{(2)}, d, a \rangle
\]

(6.7)

In the commutative case, \( ab = ba \), thus \( \langle a, b, c \rangle = \langle ab, c \rangle = \langle ba, c \rangle = \langle b, c, a \rangle \) which together with the cyclic symmetry implies the full \( S_3 \) symmetry.

The data and axioms define a Frobenius algebra: Set \( 1 = \sum \epsilon(P^{(1)})P^{(2)} \), and define the multiplication via \( \langle ab, c \rangle = \langle a, b, c \rangle. \) The invariance of the metric follows from the cyclicity of \( \langle , \rangle_3 \) and symmetry of \( \langle , , \rangle: \) \( \langle ab, c \rangle = \langle a, b, c \rangle = \langle b, c, a \rangle = \langle bc, a \rangle = \langle a, bc \rangle. \) The first and second equations of (6.3) guarantees that 1 is indeed a unit, see (6.6). The associativity follows from the third equation.

\[
\langle (ab)c, d \rangle = \langle ab, cd \rangle = \sum \langle ab, P^{(1)} \rangle \langle P^{(2)}, cd \rangle = \sum \langle a, b, P^{(1)} \rangle \langle P^{(2)}, c, d \rangle
\]

\[
= \sum \langle b, c, P^{(1)} \rangle \langle P^{(2)}, d, a \rangle = \sum \langle bc, P^{(1)} \rangle \langle P^{(2)}, d, a \rangle = \langle bc, d, a \rangle = \langle a, bc, d \rangle
\]

(6.8)

Furthermore, a full \( S_3 \) symmetry of \( \langle , \rangle_3 \) implies that the multiplication is commutative: \( \langle ab, c \rangle = \langle a, b, c \rangle = \langle b, a, c \rangle = \langle ba, c \rangle. \)
Remark 6.13. A Frobenius algebra also gives rise to a comultiplication. Using the non–degenerate form $\langle \cdot , \cdot \rangle = \langle \cdot \rangle \otimes \langle \cdot \rangle \otimes \langle \cdot \rangle \otimes \langle \cdot \rangle \otimes \langle \cdot \rangle (23)$ on $V \otimes V$, one defines $\Delta := \mu^T$ that is $\langle \Delta(a), b \otimes c \rangle = \langle a, bc \rangle$. The dual of the unit $\nu : 1 \rightarrow V$ is a counit $\epsilon : V \rightarrow 1$ and the algebra and coalgebra structure satisfy the compatibility
\[(\mu \otimes \text{id}) \circ (\text{id} \otimes \Delta) = \Delta \circ \mu = (\text{id} \otimes \mu) \circ (\Delta \otimes \text{id}) \tag{6.9}\]
as maps $V \otimes V \rightarrow V \otimes V$. The counit is again given by $\epsilon(a) = \langle a, 1 \rangle$ and is indeed a counit for $\Delta$:
$$\langle (\text{id} \otimes \epsilon) \Delta(a), b \rangle = \sum \langle a^{(1)}, \epsilon(a^{(2)}), b \rangle = \sum \langle a^{(1)}, b \rangle \langle a^{(2)}, 1 \rangle = \langle \Delta(a), b \otimes 1 \rangle = \langle a, b \rangle \tag{6.10}$$
the equation for $\epsilon \otimes \text{id}$ is analogous.

This allows one to define weaker structures which naturally occur for instance in the setting of $K$–theory, cf. e.g. [KP09, §3.1–3.3] and string topology [CG04, Sul05, Kau08a, Kau18].

Definition 6.14. A Frobenius object in a symmetric monoidal category $\mathcal{C}$ is an object $V$, together with an associative multiplication $\mu : V \otimes V \rightarrow V$ and a coassociative comultiplication $\Delta : A \rightarrow V \otimes V$ which satisfy the compatibility equation (6.9). A Frobenius algebra object in a symmetric monoidal category $\mathcal{C}$ is a Frobenius object together with a unit for the multiplication and a counit for the comultiplication.

Remark 6.15. Having a multiplication and a morphism $\epsilon : V \rightarrow 1$ produces a form $\langle a, b \rangle = \epsilon(ab)$. An element $u$ and a co–multiplication gives a propagator $P = \Delta(u)$. Requiring both $\epsilon$ to be a co–unit and $u$ to be a unit, makes the bi–linear form non–degenerate as the contraction of $\langle \cdot , \cdot \rangle$ with $P$ in one variable yields the map $a \mapsto (\epsilon \otimes \text{id})(\mu \otimes \text{id})(\Delta \otimes \text{id})(1 \otimes a) = (\epsilon \otimes \text{id}) \Delta \mu(1 \otimes a) = a$ which is the identity map. Note, by (6.10), if $u$ is indeed a unit, then $\epsilon$ is automatically a co–unit.

By Theorem 4.7 algebras over $\mathcal{O}_{\text{cyc}}$ are in one-to-one correspondence with algebras over the trivial operation of $\mathfrak{S}^\text{bl-cyc}$, and an algebra over $\mathcal{O}_{\text{cycass}}$ is unital, if its corresponding $\mathfrak{S}^\text{bl-cyc}$ algebra is unital. The following in different guises is part of folklore, for detailed examples on the needed algebraic manipulations, see e.g. [Kau18], but the presentation in this framework is new as well as the treatment of the non–unital case.

Theorem 6.16.

1. Unital algebras over $\mathcal{O}_{\text{cyc}}^\text{cyc}$ are commutative Frobenius algebras;
2. Unital algebras over $\mathcal{O}_{\text{cyc}}^\text{pl-cyc}$ (resp. unital algebras over $\mathcal{O}_{\text{cycass}}$) are symmetric Frobenius algebras;
3. Algebras over $\mathcal{O}_{\text{cyc}}^\text{cyc}$ are commutative Frobenius objects, with a trace $\epsilon$ and a propagator;
4. Algebras over $\mathcal{O}_{\text{cyc}}^\text{pl-cyc}$ are symmetric Frobenius objects, with a trace $\epsilon$ and a propagator.

Proof. (1) and (2) follow immediately from Propositions 6.12 and 6.10. Without the non–degeneracy assumption, we can define a multiplication by dualising $\langle \cdot \rangle_3$ in the last variable using $\vee_P$ and a comultiplication by dualising in the last two variables. The Frobenius equation is then a straightforward check using (6.3). The trace $\epsilon = Y_1$ and $P$ give the extra structures. Conversely, these dually allow to recover the $Y_n$ from the multiplication and comultiplication. \[\square\]

Remark 6.17.

1. If one sets $u = (\text{id} \otimes \epsilon)(P)$, then one obtains a second propagator $Q = \Delta(u)$. These two propagators agree if the form is non–degenerate.
2. $u$ plays the role of a unit in the sense that $\langle a_1, \ldots, u, \ldots, a_n \rangle_{n+1} = \langle a_1, \ldots, a_n \rangle_n$.
3. Dualizing $\langle \cdot , \cdot \rangle$ defined via $P$ in one variable gives a morphism $p : V \rightarrow V$. It is easy to check that this is a projection $p^2 = p$.
4. The quantity $\mu \Delta(1) = e$, (here $e$ stands for the Euler element, cf. [Kau18]), is important, see also Remark 6.20 below. For instance if $A = H^*(M)$ for a compact oriented manifold $M$, with cup product and evaluation at the fundamental class, then $e$ is the Euler–class in
top degree and $\epsilon(e) = \chi(M)$. It is the obstruction for the lift to a $O^\text{mod}_1$ algebra, viz. by Lemma 6.8 the lift is possible if and only if $e = 1$. This corresponds to the possibility to pass to a stabilization cf. [Kau09,MM21], which moreover appears in the theory of Steenrod operations [KMM21].

(5) We see that $\langle \rangle_0 = \epsilon(u)$ and in the non–degenerate case this is $\epsilon(1)$. This quantity is sensitive to nilpotent vs. semisimple Frobenius algebras, cf. [Man99,Kau08b,Kau18]. In the geometric case above $A = H^*(M)$, one sees that unless $\dim(M) = 0$, $\epsilon(1) = 0$.

6.4. Adjunction and 1+1 d QTFTs. The functor $j: g^\text{cyc} \to g^\text{mod}$ provides interesting adjunctions. Unital algebras over $j_!(O^\text{cyc}_2) = O^\text{mod}_2$ are known as 1+1 d TQFTs since they associate a correlation function to each $*_{g,p}$, which can be thought of as an oriented surface of genus $g$ with $S$ boundaries. Similarly, unital algebras over $j_!(O^\text{cycass}_2) = O^\text{surf-mod}_2$ are 1+1 d open TQFTs since they associate a correlation function to each $*_{g,p,S_1,...,S_b}$, which can be viewed as an oriented surface of genus $g$ with $p$ marked points in the interior, $b$ boundary components, or equivalently unmarked boundaries, and $S_i$ marked points on boundary $i$. In both cases, the composition along a graph corresponds to sewing together the surfaces along the respective boundaries, thus realizing a version of a cobordism category.

Part of the following is folklore and has been proven several times in the literature [Dij89,Man99, Abr96] in different settings. We add the novel feature is that everything follows from adjunctions. Our presentation also makes the constructions of [Cos04] clear.

**Theorem 6.18.**

1. Algebras over $O^\text{mod}_1$ are equivalent to algebras over $O^\text{cyc}_1$.
2. Unital algebras over $O^\text{surf-mod}_1$, i.e. 1+1 d closed TQFTs, are equivalent to commutative Frobenius algebras.
3. The following are equivalent:
   a. Algebras over $O_{1,\text{surf-mod}}^\text{mod}$;
   b. Algebras over $O_{\text{surf}} = j_!(O_{\text{cycass}})$, i.e. the modular envelope of $O_{\text{cycass}}$.
   c. Algebras over $O_{1,\text{pl-cyc}}^\text{mod}$;
   d. Algebras over $O_{\text{cycass}}$.
4. Unital algebras for any of the 4 equivalent cases (a)-(d), i.e. 1+1 d open TQFTs, are equivalent to symmetric Frobenius algebras.
5. Without the assumption of being unital, the algebras are commutative, resp. symmetric, Frobenius objects with trace and propagator.

**Proof.** Using the main diagram (0.1), Propositions 5.6 and 5.11, Theorem 4.7 and Theorem 4.3, we obtain adjunctions from which the first two statements follow:

$$Nat(O_{1,\text{cyc}}^\text{cyc}, \text{Cor}_{V,P}^\text{cyc}) = Nat(O_{1,\text{cyc}}^\text{cyc}, j^* \text{Cor}_{V,P}^\text{mod}) \leftrightarrow Nat(j_!(O_{1,\text{cyc}}^\text{cyc}), \text{Cor}_{V,P}^\text{mod}) = Nat(O_{2,\text{surf}}^\text{mod}, \text{Cor}_{V,P}^\text{mod}) \tag{6.11}$$

The third and fourth statement follow from the adjunctions:

$$Nat(O_{\text{cycass}}, \text{Cor}_{V,P}^\text{cyc}) = Nat((\pi_2)_!(O_{1,\text{pl-cyc}}^\text{pl-cyc}), \text{Cor}_{V,P}^\text{cyc}) \leftrightarrow Nat(O_{1,\text{pl-cyc}}^\text{pl-cyc}, \text{Cor}_{V,P}^\text{pl-cyc})$$

$$= Nat(O_{1,\text{pl-cyc}}^\text{pl-cyc}, j^* \text{Cor}_{V,P}^\text{surf-mod}) \leftrightarrow Nat(j_!(O_{1,\text{pl-cyc}}^\text{pl-cyc}), \text{Cor}_{V,P}^\text{mod}) = Nat(O_{2,\text{surf-mod}}^\text{surf-mod}, \text{Cor}_{V,P}^\text{surf-mod})$$

$$= Nat(O_{1,\text{surf-mod}}^\text{surf-mod}, \pi_3^* \text{Cor}_{V,P}^\text{mod}) \leftrightarrow Nat((\pi_3!)_!(O_{1,\text{surf-mod}}^\text{surf-mod}), \text{Cor}_{V,P}^\text{mod}) = Nat(O_{\text{surf}}^\text{surf}, \text{Cor}_{V,P}^\text{mod}) \tag{6.12}$$

6.5. Algebraic string topology operations. The framework also naturally yields the correlation functions of [Kau08a,Kau18] which underly the algebraic string topology operations. For this we
have to pull back the correlation functions graphs using the source functor \( s : \text{Gr} \to \text{Agg} \) promoted to a Feynman functor \( \mathcal{F}^{\text{Gr}} \to \mathcal{F}^{\text{ncng-mod}} \).

**Theorem 6.19.** The correlation functions of \([\text{Kau08a}]\) in the general setting for symmetric Frobenius algebra \( A \) \([\text{Kau18}]\) are given by the natural transformation \( s^*(Y) \in \text{Nat}[s^* \mathcal{O}_{\text{surf}}, \text{Cor}_{A,P}] \).

**Proof.** Pulling back along \( s \) using Theorem 4.7 one has \( s^*(Y) \in \text{Nat}[s^* \mathcal{O}_{\text{surf}}, s^* \text{Cor}_{A,P}] \). For a given surface decorated graph \( \Gamma \) we have that \( s^*(\Gamma) \) is the underlying corolla set. As we are dealing with monoidal functors, we obtain \( s^*(Y)(\Gamma, a \in s^* \mathcal{O}_{\text{surf}}(X)) = \bigotimes_{e \in V_1} Y(v_{S,g,\sigma,f_e}) \) which is the formula (3.1) of \([\text{Kau08a}]\) generalized to surface marked graphs as detailed in Corollary 5.2 of \([\text{Kau18}]\), see equation (5.10), where \( \Gamma \) is dual to the surface with arcs as explained in §1.4. \( \square \)

**Remark 6.20.**

1. The value \( Y_{s_1} \in \mathcal{F} \) is \( \text{Tr}(P) = \langle , \rangle \circ P =: e \), that is the quantum dimension. In the unital case this is \( \mu \Delta(1) \). (This follows from the morphism \( \circ_{0,1} : *_{0,[1]} \to *_{1,*} \).)

2. If \( V \) is commutative then \( Y(*_{g,p,S_1,\ldots,S_n})(\bigotimes_{s \in S}(a_s)) \) is \( \prod_{s \in S} a_s e^{-\chi(\Sigma)}+1 \) where \( \Sigma \) is the corresponding surface, cf. e.g. \([\text{Kau08a,Kau18}]\). For the general formula in the non-commutative case, which is an algebraic analog of the chord diagrams used in the computations, see \([\text{Kau18}]\). It is essentially given by the normal form (5.7).

3. For string topology \( A = H^*(M) \) or using a propagator given by the diagonal the correlation functions lift to \( C^*(M) \), cf. \([\text{Kau08a,Kau18}]\).

**Remark 6.21.** Note that gluing on outer flags is not the PROP structure for string topology neither closed nor open, cf. \([\text{Kau10}]\), which involves gluing on the boundary components of the polycyclic graph \( \Gamma \) as in \([\text{Kau07,Kau08a,Kau18}]\). This will be treated in \([\text{BK22b}]\).

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