ARITHMETIC AND INTERMEDIATE JACOBIANS OF SOME RIGID CALABI-YAU THREEFOLDS

ALEXANDER MOLNAR

Abstract. We construct Calabi-Yau threefolds defined over $\mathbb{Q}$ via quotients of abelian threefolds, and re-verify the rigid Calabi-Yau threefolds in this construction are modular by explicitly computing their $L$-series without [2]. We compute the intermediate Jacobians of the rigid Calabi-Yau threefolds as complex tori, then compute a $\mathbb{Q}$-structure for the 1-torus given a $\mathbb{Q}$-structure on the rigid Calabi-Yau threefolds, and verify a conjecture of Yui about the relation between the $L$-series of the rigid Calabi-Yau threefolds and the $L$-series of their intermediate Jacobians.

1. Introduction

Associated to any smooth projective variety, the intermediate Jacobian varieties generalize Jacobian varieties of curves. For $X$ an $n$-dimensional complex variety we define the (Griffiths) intermediate Jacobians of $X$ to be the varieties

$$J^{k+1}(X) := H^{2k+1}(X, \mathbb{C})/(F^{k+1}H^{2k+1}(X, \mathbb{C}) \oplus H^{2k+1}(X, \mathbb{Z}))$$

where the quotient involves only the torsion free part of $H^{2k+1}(X, \mathbb{Z})$ and $F^{k+1}$ is the $(k+1)$-th level in the Hodge filtration, i.e.,

$$F^{k+1}H^{2k+1}(X, \mathbb{C}) = \bigoplus_{p+q=2k+1, q<k+1} H^{p,q}(X).$$

Thus, for a curve $C$, the only intermediate Jacobian is $J^1(C)$, which is the Jacobian variety of the curve. We will be interested mostly in Calabi-Yau varieties, so for the one-dimensional case we have elliptic curves which are known to be isomorphic to their (intermediate) Jacobians, and these are all the possibilities. For two dimensional examples one has K3 surfaces which have trivial first and third cohomology, hence non-trivial intermediate Jacobians, and so our focus will be with Calabi-Yau threefolds. Here the only non-trivial intermediate Jacobian is

$$J(X) := J^2(X) = (H^{0,3}(X) \oplus H^{1,2}(X))/H^3(X, \mathbb{Z}).$$

Not much is known about these varieties, even when focusing only on Calabi-Yau threefolds. As complex varieties one has e.g., [2], [23] and [25], as well as an interest in the physics literature including [3], [7] and [16]. As $X$ is Kähler we have that the intermediate Jacobian is a complex torus of dimension $1 + h^{1,2}(X)$. Thus, when $X$ is rigid, $J^2(X)$ is a 1-torus, possibly with a canonical structure of an elliptic curve.

The work to date studies the geometry of $J(X)$, i.e., the complex structure, but not much is known, including simply how to compute the tori in most cases. As intermediate Jacobians generalize the classical Jacobian variety of a curve as well as the Picard varieties and Albanese varieties of any $n$-dimensional varieties, it is natural to ask if one can study arithmetic on all intermediate Jacobians by finding a canonical $\mathbb{Q}$-structure when $X$ is defined over $\mathbb{Q}$. All of our examples of Calabi-Yau threefolds will be defined over $\mathbb{Q}$, but one can similarly try to associate a canonical $K$-structure for any number field $K$ if $X$ is defined over $K$ instead. We will construct some rigid Calabi-Yau threefolds in which we are able to compute their intermediate Jacobians as complex varieties, and then refine this computation to give a canonical $\mathbb{Q}$-structure as well, given a choice of model over $\mathbb{Q}$ for the Calabi-Yau threefolds.

Shafarevich conjectures that every variety of CM-type (meaning its Hodge group is abelian) has the $L$-function of a Grossencharacter and Borcea shows that a rigid Calabi-Yau threefold with CM-type has a CM elliptic curve as its intermediate Jacobian, which is well known to have a Hecke $L$-function. If the conjecture

Date: February 11, 2015.
is true, it is a natural question to ask if there is any relation between the associated Grossencharacters of a rigid Calabi-Yau threefold and its intermediate Jacobian. The motivation for the current work is to study a precise conjecture of Yui in [25] to this effect.

\textbf{Conjecture 1 (Yui).} Let $X$ be a rigid Calabi-Yau threefold of CM-type defined over a number field $F$. Then the intermediate Jacobian $J^2(X)$ is an elliptic curve with CM by an imaginary quadratic field $K$, and has a model defined over the number field $F$.

If $\chi$ is a Hecke character associated to $J^2(X)$ and

$$L(J^2(X), s) = \begin{cases} L(\chi, s) & \text{if } K \subset F, \\ L(\chi, s) & \text{otherwise} \end{cases}$$

then

$$L(X, s) = \begin{cases} L(\chi^3, s) & \text{if } K \subset F, \\ L(\chi^3, s) & \text{otherwise} \end{cases}$$

Consequently, $X$ is modular.

We will show our rigid Calabi-Yau threefolds of CM-type satisfy this conjecture. After this we show a similar construction of Calabi-Yau $n$-folds for any odd $n$ also satisfies the natural generalization of this conjecture, giving evidence a much stronger result may be true.

We start by constructing our varieties and studying their geometry over $\mathbb{C}$. We use a generalized Borcea construction of Calabi-Yau threefolds using products of elliptic curves, and determine which of our varieties are rigid Calabi-Yau threefolds. We then choose a $\mathbb{Q}$-structure for the Calabi-Yau threefolds, via the underlying elliptic curves, and compute their respective $L$-functions. Once this is done we compute the intermediate Jacobians as complex tori, and then over $\mathbb{Q}$, via the choice of $\mathbb{Q}$-structure given on the threefolds. We are then able to compare the $L$-functions of the intermediate Jacobians and their respective threefolds and verify the conjecture.

Our construction also gives rise to non-rigid Calabi-Yau threefolds, which we leave to future work [14] as the intermediate Jacobians are no longer elliptic curves, and the arithmetic of 2-tori and 4-tori is far more complicated, and the question of modularity (automorphy) is also nowhere near as resolved, as with [12].

\textbf{Acknowledgements.} The author was partially supported by Ontario Graduate Scholarships, as well as partial support and hospitality at the Fields Institute during the thematic program on Calabi-Yau varieties in 2013 and the University of Copenhagen in 2014. We are very grateful for these opportunities. We are also grateful for the patience and many insightful discussions we have had throughout writing this work, in particular with Ian Kiming, Hector Pasten, Simon Rose and our supervisor Noriko Yui, without whom we would likely never have been introduced to the interesting questions we address here.

\section{2. Construction of threefolds}

We will generalize a construction of Calabi-Yau threefolds due to Borcea [5] using elliptic curves with complex multiplication. To start, we first consider our threefolds over $\mathbb{C}$ and determine which ones are rigid as well as compute the dimension of the space of complex deformations for those that are not.

Over $\mathbb{C}$, up to isomorphism there is only one elliptic curve with an automorphism of order 3, and one elliptic curve with an automorphism of order 4. Denote these by $E_3$ and $E_4$ with their respective CM automorphisms $\iota_3$ and $\iota_4$, and note that $\iota_6 := -\iota_3$ is an automorphism of order 6 on $E_6 := E_3$.

On the triple product $E^3_j := E_j \times E_j \times E_j$ we have an action of the group $G_j := \{\iota_j \times \iota_j^{j-1} \times \id, \iota_j \times \id \times \iota_j^{j-1}\}$ for $j = 3, 4$ and 6. As $G_j$ preserves the holomorphic threeform of $E^3_j$ we have $h^{3,0}(E^3_j / G_j) = 1$ and $h^{1,0}(E^3_j / G_j) = 0$, so a crepant resolution of $E^3_j / G_j$ is a Calabi-Yau threefold. The same is true for many subgroups of $G_j$ but the geometry varies widely with the choice of subgroup.

\textbf{Theorem 2.} Consider the following groups of automorphisms acting on $E^3_6$.

$$G_6 = \left\langle \iota_6 \times \iota_6^5 \times \id, \iota_6 \times \id \times \iota_6^2 \right\rangle, \quad H_6 = \left\langle \iota_6^2 \times \iota_6^4 \times \id, \iota_6^2 \times \id \times \iota_6^4 \right\rangle,$$

$$I_6 = \left\langle \iota_6^2 \times \iota_6^4 \times \id, \iota_6^4 \times \id \times \iota_6^2 \right\rangle, \quad J_6 = \left\langle \iota_6 \times \iota_6^5 \times \id, \iota_6^4 \times \iota_6^5 \times \iota_6^3 \right\rangle,$$

$$K_6 = \left\langle \iota_6 \times \iota_6^5 \times \id, \iota_6^2 \times \id \times \iota_6^4 \right\rangle, \quad L_6 = \left\langle \iota_6^2 \times \iota_6^4 \times \id, \iota_6^2 \times \iota_6^4 \times \iota_6 \right\rangle,$$

$$M_6 = \left\langle \iota_6^2 \times \iota_6^2 \times \iota_6^2 \right\rangle, \quad N_6 = \left\langle \iota_6 \times \iota_6^2 \times \iota_6^3 \right\rangle, \quad O_6 = \left\langle \iota_6^4 \times \iota_6 \times \iota_6^2 \right\rangle.$$
Crepant resolutions of the respective quotients are Calabi-Yau threefolds with Hodge numbers

\[
\begin{align*}
\text{h}^{1,1}(E_0^3/\langle \iota_6 \rangle) &= 84, \\
\text{h}^{1,1}(E_0^3/H_6) &= 84, \\
\text{h}^{1,1}(E_0^3/I_6) &= 73, \\
\text{h}^{1,1}(E_0^3/J_6) &= 51, \\
\text{h}^{1,1}(E_0^3/K_6) &= 51, \\
\text{h}^{1,1}(E_0^3/L_6) &= 36, \\
\text{h}^{1,1}(E_0^3/M_6) &= 36, \\
\text{h}^{1,1}(E_0^3/N_6) &= 35, \\
\text{h}^{1,1}(E_0^3/\langle \iota_6 \rangle) &= 29,
\end{align*}
\]

\[\text{h}^{2,1}(E_0^3/\langle \iota_6 \rangle) = 0, \quad \text{h}^{2,1}(E_0^3/H_6) = 0, \quad \text{h}^{2,1}(E_0^3/I_6) = 1, \quad \text{h}^{2,1}(E_0^3/J_6) = 3, \quad \text{h}^{2,1}(E_0^3/K_6) = 3, \quad \text{h}^{2,1}(E_0^3/L_6) = 0, \quad \text{h}^{2,1}(E_0^3/M_6) = 0, \quad \text{h}^{2,1}(E_0^3/N_6) = 11, \quad \text{h}^{2,1}(E_0^3/\langle \iota_6 \rangle) = 5.\]

**Remark.** The example using \(G_6\) was studied in [6], while all of the Hodge numbers using \(G_6, H_6, L_6\) and \(M_6\) can be found in [8] as well as references therein. The pair \((h^{1,1}, h^{2,1}) = (73,1)\) can be found in [10] and [13], and a large set of pairs from a toric construction including \((35,11), (29,5)\) can be found in [13]. Lastly, as mentioned above, the pair \((51,3)\) is the original Borcea construction [5]. These exhaust all the Calabi-Yau threefolds one can obtain from this construction, up to isomorphism, noting that any subgroup of \(G_6 \cong (\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})\) is isomorphic to one of the above subgroups, or the quotient is not a Calabi-Yau threefold.

**Remark.** While the rigid examples cannot have Calabi-Yau mirror partners, all of the non-rigid examples have (topological) mirrors that have been constructed in the literature. All mirror pairs except \((1,73)\) can be found in the toric construction of [13], while the last mirror can be found in [4] which constructs Calabi-Yau varieties and their mirrors via conifold transitions. It would be interesting to see if there is a relationship between the intermediate Jacobians in a mirror pair.

**Proof.** As all the examples are similar, we only look at the cyclic example

\[O_6 = \langle \iota_6^4 \times \iota_6 \times \iota_6 \rangle\]

which contains all the geometry necessary for the resolution of each example.

We must investigate the fixed points under the action of each element of this group, so we break things up into steps to make things more accessible.

**Step 1.** By continuously extending the appropriate automorphisms we have a commutative diagram

\[
\begin{array}{ccc}
E_0^3 & \rightarrow & E_0^3/\langle \iota_6 \rangle \\
\downarrow & & \downarrow \\
E_0^3/\iota_3^3 & \rightarrow & (E_0^3/\iota_3^3) / (\langle \iota_6^4 \times \iota_6 \times \iota_6 \rangle) \\
\downarrow & & \downarrow \\
(E_0^3/\iota_3^3) / (\langle 1 \times \iota_6 \times \iota_6 \rangle) & \rightarrow & (E_0^3/\iota_3^3) / (\langle 1 \times \iota_6 \times \iota_6 \rangle)
\end{array}
\]

The Künneth formula gives

\[\text{h}^{1,1}(E_0^3/\iota_3^3) = 9, \quad \text{h}^{2,1}(E_0^3/\iota_3^3) = 0,\]

and the resolution involves blowing up the 27 fixed points. Using [1] to compute the cohomology of a resolution, or (picking a model, and) looking at an affine patch explicitly and seeing the induced action on
Step 3. The final resolution is now two separate resolutions. We start with the 3 codimension three subvariety groups, or a crepant resolution of the quotient is not Calabi-Yau.

\[
\left\langle \iota_6^3 \times \iota_6 \times \iota_6 \right\rangle = \left\langle \iota_6^3 \times \iota_6 \times \iota_6, \text{id} \times \iota_6^3 \times \iota_6 \right\rangle
\]

Note that in this last resolution the action by \( \text{id} \times \iota_6^3 \times \iota_6 \) also gives rise to \((2,1)\)-classes for each of the new fixed points, so the resolved threefold is not rigid, and we have

\[
h^{2,1}(\overline{E_6^3/\iota_6^3}) = 5
\]
as required. \(\square\)

Step 2. We now quotient this resolution by (continuous extensions of) the remaining elements in \( O_6 \) to see what \((1,1)\)-classes remain. Note that only 5 classes from the K"unneth formula are preserved. Furthermore, the action of \( \text{id} \times \iota_6^3 \times \iota_6^3 \) identifies many of the 27 exceptional divisors from the previous blowup, so that

\[
h^{1,1}(\overline{(E_6^3/\iota_6^3)/O_6}) = 23, \quad \text{and} \quad h^{2,1}(\overline{(E_6^3/\iota_6^3)/O_6}) = 0.
\]

Step 3. The final resolution is now two separate resolutions. We start with the 3 codimension three subvariety under \( \iota_6^3 \times \iota_6 \times \iota_6 \), and then the remaining codimension two subvariety from \( \text{id} \times \iota_6^3 \times \iota_6^3 \). This adds classes above each of the six remaining singular points, so our final tally is

\[
h^{1,1}(\overline{E_6^3/O_6}) = 29.
\]

Note that for each subgroup \( H \) of \( G_3 \), the variety \( E_3^3/H \) is isomorphic to \( E_6^3/J \) for some subgroup \( J \) of \( G_6 \), so the above covers all the threefolds that arise using \( E_3 \) and subgroups of \( G_3 \) as well.

All of these Hodge pairs, except for \((36,0)\), can be found using a generalization of a construction studied by Borcea [5] and Voisin [22]. For example, with the pair \((h^{1,1}, h^{2,1}) = (29, 5)\) we can look at the action of

\[
\left\langle \iota_6^3 \times \iota_6 \times \iota_6 \right\rangle = \left\langle \iota_6^3 \times \iota_6 \times \iota_6, \text{id} \times \iota_6^3 \times \iota_6 \right\rangle
\]

first as a “birational Kummer construction” taking the quotient of \( E_6^3 \) by \( \text{id} \times \iota_6^3 \times \iota_6^3 \), and then noting the induced action of \( \iota_6^3 \times \iota_6 \times \iota_6 \) is simply \( \iota_3 \times \iota_3 \times \iota_3 \) which is a generalized Borcea-Voisin threefold. See [10].

Similarly, one can find the Calabi-Yau threefolds using this construction with \( E_3^3 \).

Theorem 3. Consider the groups of automorphisms

\[
G_4 = \left\langle \iota_4 \times \iota_4^3 \times \text{id}, \iota_4 \times \text{id} \times \iota_4^3 \right\rangle,
\]

\[
H_4 = \left\langle \iota_4 \times \iota_4 \times \iota_4^2, \iota_4 \times \iota_4^2 \times \text{id} \right\rangle,
\]

\[
I_4 = \left\langle \iota_4^2 \times \iota_4^2 \times \text{id}, \iota_4^2 \times \text{id} \times \iota_4^2 \right\rangle,
\]

\[
J_4 = \left\langle \iota_4 \times \iota_4 \times \iota_4^2 \right\rangle,
\]

acting on the abelian threefold \( E_4^3 \). Crepant resolutions of the respective quotients are Calabi-Yau threefolds with Hodge numbers

\[
h^{1,1}(\overline{E_4^3/G_4}) = 90,
\]

\[
h^{1,1}(\overline{E_4^3/H_4}) = 61,
\]

\[
h^{1,1}(\overline{E_4^3/I_4}) = 51,
\]

\[
h^{1,1}(\overline{E_4^3/J_4}) = 25,
\]

\[
h^{2,1}(\overline{E_4^3/G_4}) = 0,
\]

\[
h^{2,1}(\overline{E_4^3/H_4}) = 1,
\]

\[
h^{2,1}(\overline{E_4^3/I_4}) = 3,
\]

\[
h^{2,1}(\overline{E_4^3/J_4}) = 1.
\]

Remark. Again, this exhausts all the possible Calabi-Yau threefolds arising from this generalized Borcea/Kummer construction with \( E_4 \), noting that any subgroup of \( G_4 \simeq (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}) \) is isomorphic to one of the above groups, or a crepant resolution of the quotient is not Calabi-Yau.
for all \( a \) a Hecke character \( \chi \) on fractional ideals of \( K \) where

\[ N(a) \equiv 1 \pmod{m}. \]

We associate to any pair \( G \) the computation, and for the convenience of the reader, we recall some basic facts about these characters.

Set

\[ S = \langle t_4^3 \rangle \subset J_4, \]

and, extending the actions continuously when appropriate, the diagram

\[ \begin{array}{ccc}
E_4^3 & \longrightarrow & E_4^3/J_4 \\
\downarrow & & \downarrow \\
E_4/S & \longrightarrow & \widetilde{E_4}/S \\
\end{array} \]

commutes. We have 16 fixed points on \( E_4 \times E_4 \) under \( t_4^3 \times t_4^2 \) and so along with the fixed K"unneth components we have

\[ h^{1,1}(E_4/S) = 21, \quad h^{2,1}(\widetilde{E_4}/S) = 21. \]

The quotient by \( \langle t_4^3 \rangle \) removes all the \((2,1)\)-classes as each of the corresponding elliptic surfaces is no longer fixed. Only the four 2-torsion points (as well as their exceptional divisors) on four of those divisors remain, and so, noting all but one of the \((2,1)\)-classes from the K"unneth component are no longer fixed, we get another 16 exceptional projective lines, so find the desired Hodge numbers.

\[ \square \]

Remark. The Hodge pairs \((90,0)\) and \((61,1)\) can again be found in \[10\] while \((61,1)\) also arises in a toric construction in \[13\], and the Borcea example with \((51,3)\) completing the list once more.

3. Modularity

We say a variety \( X/\mathbb{Q} \) of dimension \( d \) is modular if its L-function, the L-function of the \( d \)-th \( \ell \)-adic cohomology of \( X \) is (up to the bad primes) the L-function of a newform \( f = \sum a_n q^n \) of weight \( d + 1 \) on \( \Gamma_0(N) \) for some \( N \), where a newform here is taken to mean a normalized \((a_1 = 1)\) eigenform on \( \Gamma_0(N) \) that is not induced by a cusp form on \( \Gamma_0(N') \) for any smaller \( N' \mid N \). We say the newform \( f \) has CM by a quadratic number field \( K \) if \( a_n = \varphi_K(n)a_n \) for all \( n \) where \( \varphi_K \) is the non-trivial Dirichlet character associated to \( K \).

All rigid Calabi-Yau threefolds defined over \( \mathbb{Q} \) are modular by \[12\]. In particular, all of the rigid Calabi-Yau threefolds in our construction. One can use Serre’s criterion to determine the associated newforms (hence L-functions) but we will be able to show this without the a priori knowledge of modularity, by computing the L-functions via a method shown to us by Hector Pasten \[17\], taking advantage of the elliptic curves/abelian threefold structure of our threefolds.

We will need one lemma about Hecke characters of imaginary quadratic number fields before approaching this computation, and for the convenience of the reader, we recall some basic facts about these characters. We associate to any pair \((K, m)\), where \( K \) is a quadratic imaginary number field and \( m \) is an ideal of \( K \), a Hecke character \( \chi \) of modulus \( m \) and infinite type \( c \) given by the homomorphism

\[ \chi : I_m \to \mathbb{C}^\times \]

on fractional ideals of \( K \) relatively prime to \( m \) given by setting

\[ \chi(aO_K) = a^c \]

for all \( a \in K^\times \) with \( a \equiv 1 \pmod{m} \). For any fractional ideal not coprime to \( m \) we define \( \chi \) to be 0.

The L-function of \( \chi \) is given by the product over all prime ideals \( p \) of \( K \)

\[ L(s, \chi) = \prod_p \left( 1 - \chi(p)N(p)^{-s} \right)^{-1} \]

where \( N(p) \) is the norm of \( p \).
Lemma 4 ([9] lemma 4.2). Let $\psi$ be a Hecke character of an imaginary quadratic field $K$ and suppose its associated newform $f_\psi$ has trivial Nebentypus. Suppose that we have Fourier $q$-expansions

$$f_\psi = \sum_{n=1}^{\infty} a_n q^n, \quad f_{\psi^3} = \sum_{n=1}^{\infty} b_n q^n.$$  

Then

$$b_p = a_p^3 - 3pa_p$$

for any prime $p$ that split in $K$. For $p$ inert in $K$ we have $b_p = 0$.

Proof. If $p = pp'$ splits in $K$, then $a_p = \psi(p) + \psi(p')$, so

$$b_p = (\psi(p))^3 + (\psi(p'))^3 = a_p^3 - 3pa_p.$$  

If $p$ is inert, then $f_{\psi^3}$ has CM by $K$, so $b_p = 0$. □

Remark. In general one has a decomposition of the space of weight $k$ cusp forms on $\Gamma_1(N)$ over the Nebentypus characters $\epsilon$ modulo $N$,

$$S_k(\Gamma_1(N)) = \bigoplus_{\epsilon} S_k(\Gamma_0(N), \epsilon),$$

but in our case all our newforms have real coefficients, and we will only be interested in weight 2 or 4 newforms, so by [20] Corollary II.1.2 there is only trivial Nebentypus. Hence $S_k(\Gamma_1(N)) = S_k(\Gamma_0(N))$ in our case and the above lemma will apply.

We are now able to give a nice description of the modular forms associated to our rigid Calabi-Yau threefolds using the arithmetic of the underlying elliptic curves in the construction. For this we need to fix models of our elliptic curves over $Q$, and their respective automorphisms. We fix the affine models

$$E_3 : y^2 = x^3 - 1, \quad \iota_3 : (x,y) \mapsto (\zeta_3 x, y),$$

$$E_4 : y^2 = x^3 - x, \quad \iota_4 : (x,y) \mapsto (-x, iy),$$

where $\zeta_3$ is a fixed primitive third root of unity.

Note that the fixed points on (a projective model of) $E_3$ under $\iota_3$ are $(0 : 1 : 0), (0 : i : 1)$ and $(0 : -i : 1)$ while the fixed points on (a projective model of) $E_4$ under $\iota_4$ are $(0 : 1 : 0), (0 : 0 : 1)$. For the respective involutions we have fixed points

$$E_3 \quad (0 : 1 : 0), (\zeta_3 : 0 : 1), (\zeta_3^2 : 0 : 1), \text{ and } (1 : 0 : 1),$$

$$E_4 \quad (0 : 1 : 0), (0 : 0 : 1), (1 : 0 : 1) \text{ and } (-1 : 0 : 1).$$

Hence, each of the threefolds in this construction is defined over $Q$, as one can check the subgroups in Theorems [2] and [3] all act with $G_Q := \text{Gal}(\overline{Q}/Q)$ invariant orbits. We may thus compute their L-functions and explicitly show modularity.

Theorem 5. Let $H$ be a subgroup of $G_3$ such that $X_3$, a crepant resolution of $E_3^3/H$, is a rigid Calabi-Yau threefold defined over $Q$. We have

$$L(X_3, s) = L(s, \chi_3^3)$$

where $\chi_3$ is the Hecke character of $E_3$, i.e., such that

$$L(E_3, s) = L(s, \chi_3).$$

Proof. As seen above, we have

$$H^k_{\ell}(X_3) = H^k_{\ell}(E_3^3/H),$$

where $H^k_{\ell}(X_3) = H^k_{\ell}(X_3, \mathbb{Q}_\ell)$. This simplifies further, as

$$H_{\ell}(E_3^3/H) = H_{\ell}(E_3^3)^H.$$

The Künneth formula gives

$$H_{\ell}^k(E_3^3)^H = (H_{\ell}^1(E_3) \otimes H_{\ell}^1(E_3) \otimes H_{\ell}^1(E_3))^H,$$

which is 2-dimensional. To be explicit, we will study the Galois representation on

$$\text{Aut}_{\overline{Q}}((V_\ell(E) \otimes V_\ell(E) \otimes V_\ell(E))^H).$$

where \( V_\ell(E_3) := T_\ell(E_3)^\vee \otimes \overline{\mathbb{Q}_\ell} \) is the extended Tate module of \( E_3 \).

To understand the action of Frobenius under the Galois representation, we start by investigating the action on \( E_3 \). Denote the automorphism \( \iota_3 \) by \([\zeta_3]\) for notational convenience. This induces a non-trivial action \([\zeta_3]\), on the Tate module \( V_\ell(E_3) \) with characteristic polynomial \( T^2 + T + 1 \). The eigenvalues of this action are thus the distinct primitive third roots of unity.

For any \( \sigma \in G_\mathbb{Q} \), and any \( (x,y) \in E_3(\mathbb{Q}) \), we have
\[
(1) \quad \sigma([\iota_3](x,y)) = \sigma((\zeta_3 x, y)) = (\sigma(\zeta_3) \sigma(x), \sigma(y)) = [\sigma(\zeta_3)] \sigma((x,y))
\]
and so
\[
\zeta_3 \sigma_*(v) = \sigma_*([\zeta_3]_*(v)) = (\sigma \circ [\zeta_3])_*(v) = ([\sigma(\zeta_3)] \circ \sigma)_*(v) \quad \text{by (1)}
\]
\[
= [\sigma(\zeta_3)] \sigma_*(v).
\]

Hence, \( \sigma_*(v) \) is in the \( \sigma(\zeta_3) \)-eigenspace of \([\zeta_3]\_\_\_\_. \) In particular, if \( c \in G_\mathbb{Q} \) is complex conjugation, then \( w := c_*(v) \) is a \( \zeta_3^2 \)-eigenvector for \([\zeta_3]\_\_\_. \)

Let \( \chi : G_\mathbb{Q} \to \mathbb{Q}_\ell^\times \) be the non-trivial character on \( \mathbb{Q}(\zeta_3) \). Fix a prime \( p \neq 2, 3 \), so that \( E_3 \) has good reduction at \( p \). If \( \chi(\text{Frob}_p) = 1 \), then the above shows that \( (\text{Frob}_p)_*(v) \) is a \( \zeta_3 \)-eigenvector for \([\zeta_3]\_\_\_. \) and \( v \) is an eigenvector of \( (\text{Frob}_p)_* \). Similarly, \( w \) is another eigenvector of \( (\text{Frob}_p)_* \), so that \( v, w \) form a basis. The induced action of Frobenius is thus given by a matrix
\[
\begin{pmatrix}
\alpha_p & 0 \\
0 & \beta_p
\end{pmatrix}
\]
where \( \alpha_p, \beta_p \) are the eigenvalues of the \((\text{Frob}_p)_* \_\_\_. \)

On the other hand, if \( \chi(\text{Frob}_p) = -1 \), then \( (\text{Frob}_p)_*(v) \) is a \( \zeta_3^2 \)-eigenvector, and \( v, w \) form a basis again, so that the action is given by
\[
\begin{pmatrix}
0 & h_p \\
0 & k_p
\end{pmatrix}
\]
for some \( h_p, k_p \) such that \( h_pk_p = -p \).

On \( V_\ell(E) \otimes V_\ell(E) \otimes V_\ell(E) \), we know the pure tensors
\[
v \otimes v \otimes v \quad \text{and} \quad w \otimes w \otimes w
\]
are fixed by \( H \), and span \( V \), hence form a basis. If we denote the Galois representation by
\[
\rho_3 : G_\mathbb{Q} \to \text{Aut}_{\mathbb{Q}_\ell}((V_\ell(E) \otimes V_\ell(E) \otimes V_\ell(E))^H),
\]
then we have two possibilities for \( \rho_3(\text{Frob}_p) \). If \( \chi(\text{Frob}_p) = 1 \), the action of \( \rho(\text{Frob}_p) \) is given by the matrix
\[
\begin{pmatrix}
\alpha_p^3 & 0 \\
0 & \beta_p^3
\end{pmatrix}
\]
while if \( \chi(\text{Frob}_p) = -1 \) the action is given by
\[
\begin{pmatrix}
0 & h_p^3 \\
0 & k_p^3
\end{pmatrix}.
\]
Now, as \( \alpha_p, \beta_p = \pm i \sqrt{p} \) when \( \chi(\text{Frob}_p) = -1 \), we simply have
\[
\text{tr}(\rho(\text{Frob}_p)) = \alpha_p^3 + \beta_p^3 = (\alpha_p + \beta_p)^3 - 3p(\alpha_p + \beta_p),
\]
and Lemma 4 completes the proof. \( \square \)

The main interesting piece of arithmetic that comes from working over \( \mathbb{Q} \) instead of \( \mathbb{C} \) is that our elliptic curves are no longer unique up to isomorphism, and we can investigate what occurs if we pick another model. Using twists of the elliptic curves and proceeding with the construction, we get an appropriate twist of the
The action of Frobenius on $E_3$ where $\chi$ Let

**Theorem 6.** Let $H$ be a subgroup of $G_3$ such that a crepant resolution of $E_3(D_1, D_2, D_3)/H$ is a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. If we denote this resolution by $Y_3$, then

$$L(Y_3, s) = L(s, \chi_3^3)$$

where $\chi_3$ is the Hecke character such that

$$L(E_3(D_1D_2D_3), s) = L(s, \chi_3).$$

In a similar fashion, we define the twist

$$E_4(D) : y^2 = x^3 - D^3 x$$

and the threefolds

$$E_3^3(D_1, D_2, D_3) := (E_3(D_1) \times E_3(D_2) \times E_3(D_3))/H_4.$$ 

**Theorem 7.** Let $I$ be a subgroup of $G_4$ such that a crepant resolution $Y_4$ of $E_4^3(D_1, D_2, D_3)/I$ is a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. We have

$$L(Y_4, s) = L(s, \chi_4^3)$$

where $\chi_4$ is the Hecke character such that

$$L(E_4(D_1D_2D_3), s) = L(s, \chi_4).$$

**Proof.** The induced action on the (extended) Tate module $V_i(E_4)$ is the only difference here. If we denote the action on $E_4$ by $[i]$ we have $[i]^2 = [-1]$, so the eigenvalues are $\pm i$. For any $\sigma \in G_4$ and $(x, y) \in E_4(\mathbb{Q})$ we have

$$\sigma([i](x, y)) = [\sigma(i)]\sigma((x, y))$$

hence

$$i\sigma_* (v) = \sigma_* ([i]_* (v))$$

$$= ([\sigma(i)] \circ [i])_* (v)$$

$$= [\sigma(i) \circ \sigma_* (v)]$$

$$= [\sigma(i)]_* (\sigma_* (v))$$

$$= \chi(\sigma) [\overline{i}]_* (\sigma_* (v))$$

where $\chi$ is the non-trivial Dirichlet character on $\mathbb{Q}(i)$. Let's denote complex conjugation once more. If $\chi(Frob_p) = 1$, then $(Frob_p)_* (v)$ is an $i$-eigenvector of $[i]_*$ and so $v, c_*(v)$ are a basis for the action of $(Frob_p)_*$. Otherwise, if $\chi(Frob_p) = -1$, then $(Frob_p)_* (v)$ is a $(-i)$-eigenvector of $[i]_*$ and $v, c_*(v)$ form a basis once more.

Otherwise, the computation of the action of Frobenius here is as with $E_3$. 

\[\square\]
With a little adjustment to the action on the Tate module of $E_3$ we can find the L-function associated to the rest of the constructions of rigid Calabi-Yau threefolds using the automorphism of order 6 on $E_3$. (Which reproves Theorem 4.)

**Theorem 8.** Let $J$ be a subgroup of $G_6$ such that a crepant resolution of $E_6^3(D_1, D_2, D_3)/J$ is a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. If we denote this resolution by $Y_6$, then

$$L(Y_6, s) = L(s, \chi_6^3)$$

where $\chi_6$ is the Hecke character such that

$$L(E_6(D_1D_2D_3), s) = L(s, \chi_6).$$

4. Intermediate Jacobians

In this section we compute the intermediate Jacobians of the rigid Calabi-Yau threefolds described above. The method is to reconstruct the torus structure from a suitable quotient of cohomology groups, extending Roan’s work on Kummer threefolds in [19]. We then analyze the construction of the intermediate Jacobians as complex varieties more closely to determine a model defined over $\mathbb{Q}$ in each case.

4.1. Complex torus structure. For any $\tau = \alpha + \beta i$ in the upper half plane we have the elliptic curve $E_{\tau} = \mathbb{C}/(1, \tau)$ with uniformizing parameter $z = x + iy$. Thus, $E_1 = E_4$ and $E_3 = E_6$ where $\zeta = e^{2\pi i/3}$.

Translations by $1, \tau$ in $\mathbb{C}$ give rise to a basis $e, f \in H_1(E_{\tau}, \mathbb{Z})$ so that

$$\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

where $\partial_x$ and $\partial_y$ are a basis for $H_1(E, \mathbb{C})$ corresponding to the uniformizing parameter. Taking duals in cohomology then gives a basis $e^*, f^* \in H^1(E_{\tau}, \mathbb{Z})$ such that

$$\begin{pmatrix} e^* \\ f^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} = (dx \quad dy)$$

where $dz = dx + i dy$ is a holomorphic 1-form on $E_{\tau}$.

We can rearrange the expressions above to find that

$$2e^* = \left(1 + \frac{\alpha}{\beta i}\right) dz + \left(1 - \frac{\alpha}{\beta i}\right) d\bar{z},$$

$$2f^* = i\frac{\beta}{\alpha}(d\bar{z} - dz)$$

and so using $d\bar{z}/2$ as a generator for $H^{0,1}(E_{\tau})$ we have

$$H^{0,1}(E_{\tau})/H^1(E_{\tau}, \mathbb{Z}) \cong \mathbb{C}/(\mathbb{C} \oplus (\alpha + i \beta)\mathbb{Z}) = E_{\tau}.$$

Applying the homothety given by multiplication by $\beta/i$ we have

$$H^{0,1}(E_{\tau})/H^1(E_{\tau}, \mathbb{Z}) \cong \mathbb{C}/(\mathbb{Z} \oplus (\alpha + i \beta)\mathbb{Z}) = E_{\tau}.$$

To mimic this procedure for the intermediate Jacobians of our rigid Calabi-Yau threefolds we need to find a basis for the integral cohomology and find a period relation with a basis for the complex cohomology.

As $H^3(E^3/(G, \mathbb{Z}), \mathbb{Z}) \cong H^3(E^3, \mathbb{Z})^{G}$, we may start with the simpler $H^3(E^3, \mathbb{Z})$. As above, let $(z_1, z_2, z_3) \in \mathbb{C}^3$ be uniformizing coordinates for $E^3$, with $z_k = x_k + iy_k$ corresponding to the $k$-th coordinate. Let $\{e_k, f_k\}$ be a basis for $H_1(E_{\tau}, \mathbb{Z})$ for $k = 1, 2, 3$, with dual bases $e_k^*, f_k^*$. This naturally gives bases for both the homology $H_3(E^3, \mathbb{Z})$ and the cohomology $H^3(E^3, \mathbb{Z})$, with the relations

$$\begin{pmatrix} e_k \\ f_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

and

$$\begin{pmatrix} e_k^* \\ f_k^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} = (dx_k \quad dy_k)$$

for each $k$. The holomorphic threeform $\Omega = dz_1 \wedge dz_2 \wedge dz_3$ can be written using the dual basis to give the desired period relation for the intermediate Jacobian. Indeed, as

$$dz_1 \wedge dz_2 \wedge dz_3 = dx_1 \wedge dx_2 \wedge dx_3 + i(dx_1 \wedge dx_2 \wedge dy_3 + dx_1 \wedge dy_2 \wedge dx_3 + dy_1 \wedge dx_2 \wedge dx_3)$$

$$- i(dx_1 \wedge dy_2 \wedge dy_3 + dy_1 \wedge dx_2 \wedge dy_3 + dy_1 \wedge dy_2 \wedge dx_3) - i(dy_1 \wedge dy_2 \wedge dy_3),$$
we may write $\Omega = \text{Re}(\Omega) + i \text{Im}(\Omega)$, and have

$$
\text{Re}(\Omega) = dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge dy_2 \wedge dy_3 - dy_1 \wedge dx_2 \wedge dy_3 - dy_1 \wedge dy_2 \wedge dx_3 \\
= e_1^* \wedge e_2^* \wedge e_3^* + \alpha(e_1^* \wedge e_2^* \wedge f_3^* + e_1^* \wedge f_2^* \wedge e_3^* + f_1^* \wedge e_2^* \wedge e_3^*) \\
+ (\alpha^2 - \beta^2)(e_1^* \wedge f_2^* \wedge f_3^* + f_1^* \wedge e_2^* \wedge f_3^* + f_1^* \wedge f_2^* \wedge e_3^*) + (\alpha^3 - 3\alpha\beta^2)f_1^* \wedge f_2^* \wedge f_3^*,
$$

and

$$
\text{Im}(\Omega) = dx_1 \wedge dx_2 \wedge dy_3 - dx_1 \wedge dy_2 \wedge dx_3 + dy_1 \wedge dx_2 \wedge dx_3 - dy_1 \wedge dy_2 \wedge dy_3 \\
= \beta(e_1^* \wedge e_2^* \wedge f_3^* + e_1^* \wedge f_2^* \wedge e_3^* + e_2^* \wedge e_3^* + f_1^* \wedge e_2^* \wedge e_3^*) + 2\alpha\beta(e_1^* \wedge f_2^* \wedge f_3^* + f_1^* \wedge e_2^* \wedge f_3^* + f_1^* \wedge f_2^* \wedge e_3^*) \\
+ (3\alpha^2 \beta - \beta^3)f_1^* \wedge f_2^* \wedge f_3^*.
$$

We can now compute the intermediate Jacobians of our rigid Calabi-Yau threefolds. We start with the simpler case of $X_4$, a crepant resolution of $E_4^3/G_4$.

To compute the intermediate Jacobian of $X_4$ we have each underlying elliptic curve in the product having complex period $\tau = i$, and so

$$
\text{Re}(\Omega_4) = e_1^* \wedge e_2^* \wedge e_3^* - e_1^* \wedge f_2^* \wedge f_3^* - f_1^* \wedge e_2^* \wedge f_3^* - f_1^* \wedge f_2^* \wedge e_3^*,
$$

$$
\text{Im}(\Omega_4) = e_1^* \wedge e_2^* \wedge f_3^* + e_1^* \wedge f_2^* \wedge e_3^* + f_1^* \wedge e_2^* \wedge e_3^* - f_1^* \wedge f_2^* \wedge f_3^*.
$$

where $\Omega_4 \in H^{3,0}(X_4)$. These give a basis for $H^3(X_4, \mathbb{C})$. For the left hand side of the period relation, we choose classes $A = \text{Re}(\Omega_4)$ and $B = \text{Im}(\Omega_4)$ and note that $A, B \in H^3(X, \mathbb{Z})$. We claim this is a basis for $H^3(X_4, \mathbb{Z})$.

Indeed, suppose we have a basis $C, D$ of $H^3(X_4, \mathbb{Z})$. As $A, B \in H^3(X_4, \mathbb{Z})$ we know there is some matrix $M$ with integral entries so that

$$
\begin{pmatrix}
A \\
B
\end{pmatrix} = M \begin{pmatrix}
C \\
D
\end{pmatrix}.
$$

Write

$$
M = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
$$

If $ad = bc$ then $A$ must be a multiple of $B$, which we know is not true from our explicit expressions above, so $a, b, c, d \in \mathbb{Z}$ and $ad \neq bc$, i.e., $M \in \text{GL}_2(\mathbb{Q})$. Moreover, as $C, D \in H^3(X_4, \mathbb{Z})$ we may write

$$
C = q(e_1^* \wedge e_2^* \wedge e_3^* + \cdots),
$$

$$
D = s(e_1^* \wedge e_2^* \wedge e_3^* + \cdots),
$$

for some $q, r, s, t \in \mathbb{Z}$. Multiplying all this out in $\mathbb{Z}$ then gives

$$
A = (aq + bs)e_1^* \wedge e_2^* \wedge e_3^* + (ar + bt)e_1^* \wedge e_2^* \wedge f_3^* + \cdots,
$$

$$
B = (cq + ds)e_1^* \wedge e_2^* \wedge e_3^* + (cr + dt)e_1^* \wedge e_2^* \wedge f_3^* + \cdots,
$$

and so, comparing with our expressions for $A$ and $B$ above, we have integral matrices such that

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \begin{pmatrix}
q & r \\
s & t
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
$$

i.e., $M \in \text{GL}_2(\mathbb{Z})$. Hence, $A, B$ are a basis for $H^3(X_4, \mathbb{Z})$.

Thus, the period relation for the intermediate Jacobian of $X_4$ is

$$
\begin{pmatrix}
A & B
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = (\text{Re}(\Omega_4) \quad \text{Im}(\Omega_4))
$$

so that

$$
2A = \Omega_4 + \overline{\Omega_4},
$$

$$
2B = i(\Omega_4 - \overline{\Omega_4}),
$$

and using $\Omega_4/2$ as a basis for $H^{0,3}(X_4)$, we have

$$
J(X_4) \simeq \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) = E_4.
$$
4.2. Model for the intermediate Jacobian $J(X_4)$ over $\mathbb{Q}$. As we are not just interested in the complex torus structure of the intermediate Jacobian, we may not work up to homothety anymore, and we must be more careful in how to recover not just the torus structure from the period relation, but the exact model defined over $\mathbb{Q}$. By the Uniformization Theorem we know for any elliptic curve $E = \mathbb{C}/\Lambda$, there is a $\lambda \in \mathbb{C}^\times$ such that any particular model of $E$ corresponds uniquely to the torus $\mathbb{C}/(\lambda \Lambda)$, the correspondence being

$$E : y^2 = x^3 - \lambda^{-4}g_2(\Lambda) x - \lambda^{-6}g_3(\Lambda) \leftrightarrow \mathbb{C}/(\lambda \Lambda).$$

We are interested in using the computation above to recover a particular model, thus suppose we have some $E = \mathbb{C}/\lambda(1, \tau)$. Translation by $\Lambda$ for $\lambda$ and $\lambda \tau$ will give the desired result. On the threefold we have $\mathbb{C}/(\lambda \tau)$. We now get a basis using translation by $\lambda$ and $\lambda \tau$. Hence,

$$(e^* f^*) \begin{pmatrix} \lambda & 0 \\ \lambda \alpha & \lambda \beta \end{pmatrix} = (dx \ dy)$$

and

$$2\lambda e^* = \begin{pmatrix} 1 + \frac{\alpha}{\beta} \\ \frac{1 - \alpha}{\beta} \end{pmatrix} dz + \begin{pmatrix} 1 - \frac{\alpha}{\beta} \\ \frac{1 + \alpha}{\beta} \end{pmatrix} d\bar{z},$$

$$2\lambda f^* = \frac{i}{\beta} (d\bar{z} - dz).$$

Using $d\bar{z}/2$ as a basis for $H^{0,1}(E)$, like above, gives

$$H^{0,1}(E)/H^1(E, \mathbb{Z}) = \mathbb{C}/\left( \frac{1}{\lambda} \begin{pmatrix} 1 - \frac{\alpha}{\beta} \\ \frac{i}{\lambda \beta} \end{pmatrix} \mathbb{Z} \oplus \frac{i}{\lambda \beta} \mathbb{Z} \right)$$

which is not $\mathbb{C}/\lambda \Lambda$, the exact torus we started with. Instead, we must use the basis $d\bar{z}/(2\beta \lambda^2 i)$ so that

$$e^* = \lambda (\alpha - \beta i) \frac{dz}{2\beta \lambda^2 i} - \frac{dz}{\mathbb{Z}} \alpha - \frac{\mathbb{Z}}{\beta} \lambda i,$$

$$f^* = \lambda \frac{dz}{2\beta \lambda^2 i} - \frac{dz}{\mathbb{Z}} \beta i.$$

Then we have

$$H^{0,1}(E)/H^1(E, \mathbb{Z}) = \mathbb{C}/\lambda(\mathbb{Z} \oplus (\alpha + i\beta)\mathbb{Z}) = E.$$

Thus for any $E = \mathbb{C}/\lambda(\mathbb{Z} \oplus \tau \mathbb{Z})$, we can recover the exact model using the period relation and basis $d\bar{z}/(2\beta \lambda^2 i)$ for $H^{0,1}(E)$.

In our case of interest with $E_4 : y^2 = x^3 - x$ we have (see [24]) $E_4 = \mathbb{C}/(1, i)$ with

$$\lambda = \frac{\Gamma(\frac{1}{2})^2}{6\sqrt{2\pi}}.$$

Remark. We note that $\lambda$ is transcendental, which may be used to narrow down the possible uniformizing parameters of the intermediate Jacobian, but this will not be required as we will show any scalar multiple of $\lambda$ will give the desired result.

On the threefold we have

$$(e^*_k f^*_k) \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = (dx_k \ dy_k)$$

for the $k$-th component of $E^3_4$. Writing $\Omega_4 = \text{Re}(\Omega_4) + i \text{Im}(\Omega_4)$ and using the period relations for each of the underlying $E_4$ to write these in terms of the $e^*_k$ and $f^*_k$ we now have

$$\text{Re}(\Omega_4) = \lambda^3 (e^*_1 \wedge e^*_2 \wedge e^*_3 + e^*_1 \wedge f^*_2 \wedge f^*_3 - f^*_1 \wedge e^*_2 \wedge f^*_3 - f^*_1 \wedge f^*_2 \wedge e^*_3) = \lambda^3 A,$$

$$\text{Im}(\Omega_4) = \lambda^3 (e^*_1 \wedge e^*_2 \wedge f^*_3 + e^*_1 \wedge f^*_2 \wedge e^*_3 + f^*_1 \wedge e^*_2 \wedge e^*_3 - f^*_1 \wedge f^*_2 \wedge f^*_3) = \lambda^3 B.$$

These classes are no longer integral, but using the classes $A, B$ above we have

$$(A \quad B) \begin{pmatrix} \lambda^3 & 0 \\ 0 & \lambda^3 \end{pmatrix} = \begin{pmatrix} \text{Re}(\Omega_4) & \text{Im}(\Omega_4) \\ \text{Im}(\Omega_4) & \text{Re}(\Omega_4) \end{pmatrix}$$

To get our model for $J(X_4)$ we will use the period relation

$$(A \quad B) \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \text{Re}(\Omega_4) & \text{Im}(\Omega_4) \\ \text{Im}(\Omega_4) & \text{Re}(\Omega_4) \end{pmatrix}$$
From here, noting that $\tau = \alpha + i\beta = i$, we wish to recover the correct model of $J(X_4)$ by using the method above, i.e., the basis 

$$\frac{\Omega_4}{2\beta \lambda^2 i} = \frac{\Omega_4}{2\lambda^4 i}$$

for $H^{0,3}(X_4)$. Thus we must write 

$$A = i\lambda \left(\frac{\Omega_4}{2\lambda^4 i}\right) + i\lambda \left(\frac{\Omega_4}{2\lambda^4 i}\right)$$

$$B = \lambda \left(\frac{\Omega_4}{2\lambda^4 i}\right) - \lambda \left(\frac{\Omega_4}{2\lambda^4 i}\right),$$

and so 

$$J(X_4) = C/\lambda(Z \oplus iZ) = E_4.$$ 

**Remark.** We can find a basis for $H^{0,3}(X_4)$ up to a multiple, and our choice of uniformizing parameter $\Omega/\lambda^2$ is not necessarily correct, but scaling the uniformizing parameter by $\mathbb{C}^\times$ does not affect the isomorphism class of the elliptic curve over $\mathbb{Q}$, so we may without loss of generality use the uniformizing parameter above. Indeed, going back to the general situation working on an arbitrary elliptic curve will make this clear.

Suppose we have an elliptic curve $E = C/\lambda(1, \tau)$ as above, but consider two different uniformizing parameters $z = x + iy$ and $w = u + iv$ where $w = \gamma z$. Then translations by $\lambda$ and $\lambda \tau$ give the relations 

$$\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \alpha \\ \lambda \beta \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

and 

$$\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \alpha \\ \lambda \beta \end{pmatrix} \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix}$$

where $e, f$ are a basis for $H_1(E, \mathbb{Z})$ and $g, h$ are another basis.

As $w = \gamma z$ we may rewrite the second expression as 

$$\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \alpha \\ \lambda \beta \end{pmatrix} \gamma \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \gamma \begin{pmatrix} \lambda \\ \lambda \alpha \\ \lambda \beta \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \gamma \begin{pmatrix} e \\ f \end{pmatrix}.$$ 

This means $g = \gamma e$ and $h = \gamma f$, and so the second expression gives 

$$\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \alpha \\ \lambda \beta \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix},$$

the same period relation as with uniformizing parameter $z$.

Moreover, for an elliptic curve $E$ with uniformizing parameter $z = x + iy$, its twist $E(D)$ with $D$ square-free has uniformizing parameter (up to a multiple) $z' = x + iDy$ in terms of the coordinates on $E$, and cannot be a scalar multiple unless $D = 1$.

Putting everything together, we have the following.

**Theorem 9.** Let $E_4(D)$ be as in Theorem 5 and let $X_4(D)$ be the corresponding rigid Calabi-Yau threefold. Then 

$$J(X_4(D)) = E_4(D)$$

and hence, if we write 

$$L(J(X_4(D)), s) = L(s, \chi)$$

where $\chi$ is the Hecke character of $E_4(D)$, then 

$$L(X_4(D), s) = L(s, \chi^3).$$

In particular, Conjecture 11 is true for $X_4(D)$. 

12
4.3. Model for the intermediate Jacobian $J(X_6)$ over $\mathbb{Q}$. Similarly for $E_6$, we have our exact model corresponding to the complex torus $\mathbb{C}/\mu\Gamma$ where

$$\mu = \frac{\Gamma(\frac{1}{3})^3}{24^{3/4}3^{1/2}\pi}$$

and $\Gamma = \mathbb{Z} \oplus \zeta_3\mathbb{Z}$ with $\zeta_3 = e^{2\pi i/3}$.

**Remark.** Again, we have that $\mu$ is transcendental, which may narrow the possible uniformizing parameters, but this will not be necessary.

Abusing notation, let $z_k = x_k + iy_k$ be the uniformizing parameter of the $k$-th elliptic curve on $E_6^3$. The period relation for the (complex) intermediate Jacobian of $X_6$ is

$$(A_6 \ B_6) \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = (\text{Re}(\Omega_6) \ \text{Im}(\Omega_6))$$

where $\Omega_6 = dz_1 \wedge dz_2 \wedge dz_3$. This time we find

$$\text{Re}(\Omega_6) = A_6 - \frac{1}{2}B_6,$$

$$\text{Im}(\Omega_6) = \frac{\sqrt{3}}{2}B_6,$$

with integral classes

$$A_6 = e_1^* \wedge e_2^* \wedge e_3^* - e_1^* \wedge f_2^* \wedge f_3^* - f_1^* \wedge e_2^* \wedge f_3^* - f_1^* \wedge f_2^* \wedge e_3^* - f_1^* \wedge f_2^* \wedge f_3^*$$

$$B_6 = e_1^* \wedge e_2^* \wedge f_3^* + e_1^* \wedge f_2^* \wedge e_3^* + f_1^* \wedge e_2^* \wedge e_3^* - e_1^* \wedge f_2^* \wedge f_3^* - f_1^* \wedge e_2^* \wedge f_3^* - f_1^* \wedge f_2^* \wedge e_3^*$$

The method above shows

$$H^{0,3}(X_6)/H^3(X_6, \mathbb{Z}) \simeq E_6$$

as complex tori.

We can apply the same steps above to moreover get an exact model for $J(X_6)$ over $\mathbb{Q}$, and putting everything together we find

**Theorem 10.** Let $E_6(D)$ be as in Theorem 3 and let $X_6(D)$ be one of the corresponding rigid Calabi-Yau threefolds. If $\chi$ is the Hecke character of $E_6(D)$, then

$$L(X_6(D), s) = L(s, \chi^3).$$

In particular, Conjecture 1 is true for $X_6(D)$.

Thus, we have verified Yui’s conjecture in these cases. As L-functions and cohomology of Calabi-Yau threefolds are birational invariants, any rigid Calabi-Yau threefold birational to $X_4(D)$ or $X_6(D)$ for some $D$ also satisfies the conjecture.

This Borcea-type construction does not produce any other rigid Calabi-Yau threefolds of CM-type, so we cannot verify the conjecture further. On the other hand, it is entirely possible these are the only rigid Calabi-Yau threefolds of CM-type defined over $\mathbb{Q}$. The Borcea construction with other CM elliptic curves will produce other Calabi-Yau threefolds, but as the CM endomorphisms are not automorphisms in these cases, we cannot do the generalized Borcea construction and may only using involutions which preserve all the $(1,2)$ and $(2,1)$ classes. Thus, a quotient by only the involutions gives Calabi-Yau threefolds that are not rigid. Otherwise, we do not know of any other examples of rigid Calabi-Yau threefolds of CM-type. In 21 the Borcea construction is used to find non-rigid Calabi-Yau threefolds of CM-type corresponding to the other CM elliptic curves, and in each case one finds the $L$-function of a CM-newform dividing the $L$-function of the non-rigid Calabi-Yau threefold. It may be possible to find a quotient of these non-rigid Calabi-Yau threefolds that is defined over $\mathbb{Q}$, or a rigid Calabi-Yau threefold defined over $\mathbb{Q}$, birational to them which admits a rigid quotient, but this is highly non-trivial.
4.4. **Intermediate Jacobians of higher dimensional Calabi-Yau varieties.** The construction of Calabi-Yau threefolds above generalizes readily to higher dimensional examples, in the spirit of [6]. For $j = 4, 6$ one may consider an $n$-fold product 

$$E_j \times E_j \times \cdots \times E_j$$

and $G_j$, the maximal group of automorphisms given by products of the form 

$$\iota_j^{a_1} \times \iota_j^{a_2} \times \cdots \times \iota_j^{a_n}$$

preserving the holomorphic $n$-form, i.e., such that $\sum a_i \equiv 0 \pmod{j}$. When $n$ is odd, only the holomorphic and anti-holomorphic $n$-forms are preserved by the entire group, we can again find, for any fixed $n$, all subgroups of $G_j$ such that a crepant resolution of the quotient by $G_j$ is a Calabi-Yau $n$-fold $Z_n$ with $h^n(Z_n) = 2$.

We can repeat the computation of the L-functions, and using the notation above one finds the action of Frobenius on $Z_n$, at good primes $p$, given by matrices 

$$
\begin{pmatrix}
\alpha^n & 0 \\
0 & \beta_p
\end{pmatrix}
\begin{pmatrix}
0 & h^n_p \\
k^n_p & 0
\end{pmatrix}
$$

so that 

$$L(Z_n, s) = L(s, \chi^n)$$

where $L(E_j, s) = L(s, \chi)$. The computation of the middle intermediate Jacobian can also be extended. Again, let the $k$-th component in the product $E^n_r$ have uniformizing parameter $z_k = x_k + iy_k$, and period $\tau = \alpha + i\beta$, such that the period relation 

$$
\begin{pmatrix}
e_k^* & f_k^*
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\alpha & \beta
\end{pmatrix}
= (dx_k, dy_k)
$$

holds. The holomorphic $n$-form is 

$$\Omega_n = \bigwedge_{k=1}^n dz_k = \bigwedge_{k=1}^n dx_k + idy_k$$

$$= \bigwedge_{k=1}^n (e_k^* + \alpha f_k^*) + i\beta f_k^*$$

$$= \bigwedge_{k=1}^n e_k^* + \tau f_k^*.$$ 

Hence, we have 

$$\Omega_n = e_1^* \wedge e_2^* \wedge \cdots \wedge e_n^* + \tau(e_1^* \wedge \cdots \wedge e_{n-1}^* \wedge f_2^* + \cdots + f_{n-1}^* \wedge e_2^* \wedge \cdots \wedge e_n^*)$$

$$+ \tau^2(e_1^* \wedge \cdots \wedge e_{n-2}^* \wedge f_{n-1}^* + \cdots + f_1^* \wedge f_2^* \wedge e_2^* \wedge \cdots \wedge e_n^*) + \cdots$$

$$\cdots + \tau^n(f_1^* \wedge f_2^* \wedge \cdots \wedge f_n^*).$$

Our particular choices of $\tau$ are roots of unity of small order, and we have 

$$\zeta^n \in \{1, \zeta, \bar{\zeta}\}$$

so that each of the coefficients of $\Omega_n$ using $E_6$ are appropriate combinations of $\pm 1, \pm \frac{1}{2}$ or $\pm \frac{\sqrt{3}}{2}$, and 

$$i^n \in \{\pm 1, \pm i\}$$

so that each of the coefficients of $\Omega_n$ in this case are the appropriate $\pm 1$ or $\pm i$. For each of our $n$-folds there are integral classes $A_n, B_n \in H^n(Z_n, \mathbb{Z})$ such that 

$$\begin{pmatrix}
A_n & B_n
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\text{Re}(\tau) & \text{Im}(\tau)
\end{pmatrix}
= \begin{pmatrix}
\text{Re}(\Omega_n) & \text{Im}(\Omega_n)
\end{pmatrix}$$

and so the intermediate Jacobian $J^{n-1}(Z_n) \simeq E_r$ as complex varieties.
Remark. We are interested only in $n$ odd, not only because $h^n(Z_n) > 2$ when $n$ is even, so conjecture [1] is not relevant, but because there is no intermediate Jacobian associated to even cohomology, which is the middle cohomology when $n$ is even.

In the arithmetic setting we now have
\[
\begin{align*}
\text{Re}(\Omega_n) &= \lambda^n A_n \\
\text{Im}(\Omega_n) &= \lambda^n B_n
\end{align*}
\]
so that
\[
\begin{pmatrix}
A & B
\end{pmatrix}
\begin{pmatrix}
\lambda \\
\lambda \text{Re}(\tau)
\end{pmatrix}
= \begin{pmatrix}
\text{Re}(\Omega_n) \\
\lambda \text{Im}(\tau)
\end{pmatrix}
\]
and
\[
L(J^{n-1}(Z_n), s) = L(s, \chi) = L(E_\tau, s).
\]
In this manner, one could ask for any $n$ odd, if any rigid CM-type Calabi-Yau $n$-fold $Z$ defined over a number field $F$, having intermediate Jacobian $J^{n-1}(Z)$ with CM by a number field $K$, such that
\[
L(J^{n-1}(Z), s) = \begin{cases}
L(\chi, s)L(\overline{\chi}, s) & \text{if } K \subset F, \\
L(\chi, s) & \text{otherwise}
\end{cases}
\]
must then have
\[
L(Z, s) = \begin{cases}
L(\chi^n, s)L(\overline{\chi}^n, s) & \text{if } K \subset F, \\
L(\chi^n, s) & \text{otherwise}
\end{cases}
\]

5. Remarks on special values of $L$-functions

With a $\mathbb{Q}$-model of the intermediate Jacobians associated to any $\mathbb{Q}$-model of our rigid Calabi-Yau threefolds, we are able to define the $L$-functions of both varieties and investigate their behaviour in their critical strips. In particular, does the non-vanishing of one determine the non-vanishing of the other? Does the vanishing of one determine the vanishing of the other?

If $\chi$ is a Hecke character of conductor $f$, there is a primitive character $\chi_f$ equivalent to $\chi$ in the sense that
\[
L(s, \chi) = L(s, \chi_f).
\]
In this section, we will always assume we are dealing with a primitive Hecke character. If $K$ is imaginary quadratic with discriminant $D = |D|$ and $\chi$ a primitive Hecke character of $K$ of infinite type 1, then the completed $L$-function
\[
\Lambda(s, \chi) = (D\mathcal{N}(f(\chi)))^{s/2} \cdot 2(2\pi)^{-s} \Gamma(s)L(s, \chi)
\]
satisfies the functional equation
\[
\Lambda(s, \chi) = W(\chi)\Lambda(2-s, \overline{\chi}),
\]
where $W(\chi) = \pm 1$ is the root number, and $\Gamma(s)$ is the complex gamma function that analytically continues the factorial. Moreover, this extends to powers of $\chi$. For an integer $n$, let $\chi_n$ denote the primitive character associated to $\chi^n$. The completed $L$-function
\[
\Lambda(s, \chi_n) = \begin{cases}
(D\mathcal{N}(f(\chi)))^{s/2} \cdot 2(2\pi)^{-s} \Gamma(s)L(s, \chi_n) & \text{if } n \text{ is odd}, \\
D^{s/2} \cdot 2(2\pi)^{-s} \Gamma(s)L(s, \chi_n) & \text{otherwise}
\end{cases}
\]
satisfies the functional equation
\[
\Lambda(s, \chi_n) = W(\chi_n)\Lambda(w + 1 - s, \chi_w)
\]
where
\[
W(\chi_n) = \begin{cases}
(-1)^{(n-1)/2}W(\chi) & \text{if } n \text{ is odd}, \\
1 & \text{otherwise}
\end{cases}
\]
This follows by use of the adelic language for Hecke characters and considering local factors as in Tate’s thesis. For full details, see Rohrlich’s work in [18].

As the completed $L$-functions are simply the analytic continuation of the $L(s, \chi_n)$ we abuse notation when computing with them, and are interested in computing the central values in the critical strip, namely
\[
L(n/2, \chi_n).
\]
These can be very difficult to compute exactly, but a method of Waldspurger relates these values to certain half-integral weight modular forms. In this direction, one can show that the critical values $L(X_4(D), s)$ are determined by certain cusp forms

$$f_1 = q - 3q^9 - 4q^{17} + 25q^{25} - 4q^{33} - 48q^{41} + q^{49} + 20q^{57} + 48q^{65} - 4q^{73} - 27q^{81} + 68q^{89} - 76q^{97} + O(q^{105})$$

$$f_2 = -q^3 + 5q^{11} - 7q^{19} + 2q^{35} + q^{43} + 14q^{51} - 13q^{59} + q^{67} - 27q^{75} + 7q^{83} + 26q^{91} + 15q^{99} + O(q^{107})$$

of weight $5/2$ and level 128.

These methods are very different from those in this work, and somewhat involved, so we leave the details to future work in [15], but as an example of what is possible, we find results of the form

**Theorem 11.** Let $E_4(D) : y^2 = x^3 - D^2x$, and $X_4(D)$ a crepant resolution of $E_4(D)^3/G_4$. For any odd square-free $D \in \mathbb{N}$ we have

$$L(X_4(D), 2) = \begin{cases} \frac{a_D^2}{\alpha \sqrt{D^3}} & \text{if } D \equiv 7 \pmod{8}, \\ \frac{b_D^2}{\beta \sqrt{D^3}} & \text{if } D \equiv 5 \pmod{8}, \\ 0 & \text{if } D \equiv 1, 3 \pmod{8}. \end{cases}$$

where $\alpha, \beta \in \mathbb{C}^\times$ are related to the real periods of $X_4(D)$, and

$$f_1 = \sum_{D=1}^{\infty} a_D q^D,$$

$$f_2 = \sum_{D=1}^{\infty} b_D q^D.$$

In particular, we see which twists of the Calabi-Yau threefolds have vanishing central critical values and which do not.

**References**

1. Cohomologie $l$-adique et fonctions $L$, Lecture Notes in Mathematics, Vol. 589, Springer-Verlag, Berlin-New York, 1977, Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966 (SGA 5), Edité par Luc Illusie. MR 0491704 (58 #10907)

2. A. G. Baarsma, The hypermultiplet moduli space of compactified type IIA string theory, 2011.

3. Ling Bao, Axel Kleinschmidt, Bengt E. W. Nilsson, Daniel Persson, and Boris Pioline, Instanton corrections to the universal hypermultiplet and automorphic forms on $SU(2,1)$, Commun. Number Theory Phys. 4 (2010), no. 1, 187–266. MR 2679380 (2012b:81185)

4. Victor Batyrev and Maximilian Kreuzer, Constructing new Calabi-Yau 3-folds and their mirrors via conifold transitions, Adv. Theor. Math. Phys. 14 (2010), no. 3, 879–898. MR 2801412 (2012h:14105)

5. Ciprian Borcea, Calabi-Yau threefolds and complex multiplication, Essays on mirror manifolds, Int. Press, Hong Kong, 1992, pp. 489–503. MR 1191439 (94b:14042)

6. S. Cynk and K. Hulek, Higher-dimensional modular Calabi-Yau manifolds, Canad. Math. Bull. 50 (2007), no. 4, 486–503. MR 2364200 (2008i:14063)

7. D. E. Diaconescu, R. Donagi, and T. Pantev, Intermediate Jacobians and ADE Hitchin systems, Math. Res. Lett. 14 (2007), no. 5, 745–756. MR 2350120 (2008j:14085)

8. Sara Angela Filippini and Alice Garbagnati, A rigid Calabi-Yau three-fold, Adv. Theor. Math. Phys. 15 (2011), no. 6, 1745–1787. MR 2989812

9. Fité Francesc, Kiran S. Kedlaya, and Andrew V. Sutherland, Sato-Tate groups of some weight 3 motives, 2013.
10. Alice Garbagnati, \textit{New families of Calabi-Yau threefolds without maximal unipotent monodromy}, \textit{Manuscripta Math.} \textbf{140} (2013), no. 3-4, 273–294. MR 3019127

11. Fernando Q. Gouvêa, Ian Kiming, and Noriko Yui, \textit{Quadratic twists of rigid Calabi-Yau threefolds over $\mathbb{Q}$}, Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds, Fields Inst. Commun., vol. 67, Springer, New York, 2013, pp. 517–533. MR 3156430

12. Fernando Q. Gouvêa and Noriko Yui, \textit{Rigid Calabi-Yau threefolds over $\mathbb{Q}$ are modular}, \textit{Expo. Math.} \textbf{29} (2011), no. 1, 142–149. MR 2785550 (2012g:14067)

13. Maximilian Kreuzer and Harald Skarke, \textit{Complete classification of reflexive polyhedra in four dimensions}, \textit{Adv. Theor. Math. Phys.} \textbf{4} (2000), no. 6, 1209–1230. MR 1894855 (2003e:52020)

14. Alexander Molnar, \textit{Arithmetic and intermediate Jacobians of some non-rigid Calabi-Yau threefolds}.

15. \textit{Special values of $L$-functions of some rigid Calabi-Yau threefolds}.

16. David R. Morrison, \textit{Mirror symmetry and the type II string}, Nuclear Phys. B Proc. Suppl. \textbf{46} (1996), 146–153, S-duality and mirror symmetry (Trieste, 1995). MR 1411467 (97j:81279)

17. Hector Pasten, \textit{A family of modular CY3 - communication with Alexander Molnar}, 2014.

18. Cristian Popescu, Karl Rubin, and Alice Silverberg (eds.), \textit{Arithmetic of $L$-functions}, IAS/Park City Mathematics Series, vol. 18, American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2011, Lectures from the Graduate Summer School held in Park City, UT, June 29–July 17, 2009. MR 2882750 (2012i:11001)

19. Shi-shyr Roan, \textit{Intermediate Jacobian and some arithmetic properties of Kummer-surface-type CY 3-folds}, 2005.

20. Matthias Schuett, \textit{Hecke eigenforms and the arithmetic of singular $K3$ surfaces}, Doctoral thesis, Universität Hannover (2006), 109.

21. \textit{CM newforms with rational coefficients}, Ramanujan J. \textbf{19} (2009), no. 2, 187–205. MR 2511671 (2010c:11052)

22. Claire Voisin, \textit{Miroirs et involutions sur les surfaces $K3$}, Astérisque (1993), no. 218, 273–323, Journées de Géométrie Algébrique d’Orsay (Orsay, 1992). MR 1265318 (95j:14051)

23. \textit{The Griffiths group of a general Calabi-Yau threefold is not finitely generated}, Duke Math. J. \textbf{102} (2000), no. 1, 151–186. MR 1741781 (2001a:14010)

24. Michel Waldschmidt, \textit{Transcendence of periods: the state of the art}, Pure Appl. Math. Q. \textbf{2} (2006), no. 2, part 2, 435–463. MR 2251476 (2007d:11083)

25. Noriko Yui, \textit{Update on the modularity of Calabi-Yau varieties}, Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001), Fields Inst. Commun., vol. 38, Amer. Math. Soc., Providence, RI, 2003, With an appendix by Helena Verrill, pp. 307–362. MR 2019160 (2005d:11098)

\text{Department of Mathematics and Statistics, Queen’s University, Kingston, Ontario, K7L 3N6}

\textit{E-mail address}: a.molnar@queensu.ca