RANDOM PERMUTATIONS AND RELATED TOPICS

GRIGORI OLSHANSKI

Abstract. We present an overview of selected topics in random permutations and random partitions highlighting analogies with random matrix theory.

Contents

1. Introduction 1
2. The Ewens measures, virtual permutations, and the Poisson–Dirichlet distributions 2
   2.1. The Ewens measures 2
   2.2. Virtual permutations, central measures, and Kingman’s theorem 4
   2.3. Application to representation theory 7
   2.4. Poisson–Dirichlet distributions 7
3. The Plancherel measure 9
   3.1. Definition of the Plancherel measure 9
   3.2. Limit shape and Gaussian fluctuations 9
   3.3. The poissonized Plancherel measure as a determinantal process 12
   3.4. The bulk limit 13
   3.5. The edge limit 13
   3.6. Longest increasing subsequences 14
4. The z-measures and Schur measures 15
   4.1. The z-measures 15
   4.2. Special instances of z-measures 17
   4.3. The Schur measures 18
   4.4. Some generalizations 19
References 20

1. Introduction

An ensemble of random permutations is determined by a probability distribution on $S_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$. Even the simplest instance of the uniform probability distribution is already very interesting and leads to a deep theory. The symmetric group $S_n$ is in many ways linked to classical matrix groups, and ensembles of random permutations should be treated on equal footing
with random matrix ensembles, such as the ensembles of classical compact groups and symmetric spaces of compact type with the normalized invariant measure. The role of matrix eigenvalues is then played by partitions of \( n \) that parameterize the conjugacy classes in \( S_n \). The parallelism with random matrices becomes especially striking in applications to constructing representations of “big groups” — inductive limits of symmetric or classical compact groups.

The theses stated above are developed in Section 2. The two main themes of this section are the space \( \mathfrak{S} \) of virtual permutations (\( \mathfrak{S} \) is a counterpart of the space of Hermitian matrices of infinite size) and the Poisson–Dirichlet distributions (a remarkable family of infinite-dimensional probability distributions). We focus on a special family of probability distributions on \( S_n \) with nice properties, the so-called Ewens measures (they contain the uniform distributions as a particular case). It turns out that the large-\( n \) limits of the Ewens measures can be interpreted as probability measures on \( \mathfrak{S} \). On the other hand, the Ewens measures give rise to ensembles of random partitions, from which one gets, in a limit transition, the Poisson–Dirichlet distributions.

A remarkable discovery of Frobenius, the founder of representation theory, was that partitions of \( n \) not only parameterize conjugacy classes in \( S_n \) but also serve as natural coordinates in the dual space \( \hat{S}_n \) — the set of equivalence classes of irreducible representations of \( S_n \). This fact forms the basis of a “dual” theory of random partitions, which turns out to have many intersections with random matrix theory. This is the subject of Sections 3–4. Here we survey results related to the Plancherel measure on \( \hat{S}_n \) and its consecutive generalizations: the \( z \)-measures and the Schur measures.

Thus, the two-faced nature of partitions gives rise to two kinds of probabilistic models. At first glance, they seem to be weakly related, but under a more general approach one sees a bridge between them. The idea is that the probability measures in the “dual picture” can be further generalized by introducing an additional parameter, which is an exact counterpart of the \( \beta \) parameter in random matrix theory. This parameter interpolates between the group level and the dual space level, in the sense that in the limit as \( \beta \to 0 \), the “beta \( z \)-measures” degenerate to the measures on partitions derived from the Ewens measures, see Section 4.4 below.

2. The Ewens measures, virtual permutations, and the Poisson–Dirichlet distributions

2.1. The Ewens measures. Permutations \( s \in S_n \) can be represented as \( n \times n \) unitary matrices \( [s_{ij}] \), where \( s_{ij} \) equals 1 if \( s(j) = i \), and 0 otherwise. This makes it possible to view random permutations as a very special case of random unitary matrices.

Given a probability distribution on each set \( S_n \), \( n = 1, 2, \ldots \), one may speak of a sequence of ensembles of random permutations and study their asymptotic
properties as \( n \to \infty \). The simplest yet fundamental example of a probability distribution on \( S_n \) is the \textit{uniform distribution} \( P^{(n)} \), which gives equal weights \( 1/n! \) to all permutations \( s \in S_n \); this is also the normalized Haar measure on the symmetric group. However, a more complete picture is achieved by considering a one-parameter family of distributions, \( \{ P^{(n)}_\theta \}_{\theta > 0} \), forming a deformation of \( P^{(n)} \):

\[
P^{(n)}_\theta (s) = \frac{\theta^{\ell(s)}}{\theta(\theta + 1) \ldots (\theta + n - 1)}, \quad s \in S_n,
\]

where \( \ell(s) \) denotes the number of cycles in \( s \) (the uniform distribution corresponds to \( \theta = 1 \)).

For reasons explained below we call \( P^{(n)}_\theta \) the \textit{Ewens measure} on the symmetric group \( S_n \) with parameter \( \theta \). Obviously, the Ewens measures are invariant under the action of \( S_n \) on itself by conjugations.

We propose to think of \( (S_n, P^{(n)}_\theta) \) as of an analogue of CUE\(_N\), Dyson’s circular unitary ensemble \([\text{For10a}]\) formed by the unitary group \( U(N) \) endowed with the normalized Haar measure \( P^{U(N)}_t \). More generally, the Ewens family \( \{ P^{(n)}_\theta \} \) should be viewed as a counterpart of a family of probability distributions on the unitary group \( U(N) \) forming a deformation of the Haar measure:

\[
P^{U(N)}_t (dU) = \text{const} \left| \det((1 + U)^t) \right|^2 P^{U(N)}(dU), \quad U \in U(N),
\]

where \( t \) is a complex parameter, \( \text{Re} \ t > -\frac{1}{2} \) (the Haar measure corresponds to \( t = 0 \)). Using the Cayley transform one can identify the manifold \( U(N) \), within a negligible subset, with the flat space \( H(N) \) of \( N \times N \) Hermitian matrices; then \( P^{U(N)}_t \) turns into the so-called \textit{Hua–Pickrell measure} on \( H(N) \):

\[
P^{H(N)}_t (dX) = \text{const} \left| \det(1 + iX)^{-t-N} \right|^2 dX,
\]

where “\( dX \)” in the right-hand side denotes Lebesgue measure. For more detail, see \([\text{Bor01b}], [\text{Ner02}], [\text{Ols03a}]\). A similarity between \( P^{(n)}_\theta \) and \( P^{U(N)}_t \) (or \( P^{H(N)}_t \)) is exploited in \([\text{Bou07}]\).

A fundamental property of the Ewens measures is their consistency with respect to some natural projections \( S_n \to S_{n-1} \) that we are going to describe:

Given a permutation \( s \in S_n \), the \textit{derived permutation} \( s' \in S_{n-1} \) sends each \( i = 1, \ldots, n-1 \) either to \( s(i) \) or to \( s(n) \), depending on whether \( s(i) \neq n \) or \( s(i) = n \). In other words, \( s' \) is obtained by removing \( n \) from the cycle of \( s \) that contains \( n \). For instance, if \( s = (153)(24) \) (meaning that one cycle in \( s \) is \( 1 \to 5 \to 3 \to 1 \) and the other is \( 2 \to 4 \to 2 \)), then \( s' = (13)(24) \). The map \( S_n \to S_{n-1} \) defined in this way is called the \textit{canonical projection} \([\text{Ker04}]\) and denoted as \( p_{n-1,n} \). For \( n \geq 5 \), it can be characterized as the only map \( S_n \to S_{n-1} \) commuting with the two-sided action of the subgroup \( S_{n-1} \).

The next assertion \([\text{Ker04}]\) is readily verified:
Lemma 2.1. For any \( n = 2, 3, \ldots \), the push-forward of \( P^{(n)}_\theta \) under \( p_{n-1,n} \) coincides with \( P_{\theta}^{(n-1)} \).

The Hua–Pickrell measures enjoy a similar consistency property with respect to natural projections \( p^{H}_{N-1,N} : H(N) \to H(N-1) \) (removal of the \( N \)th row and column from an \( N \times N \) matrix) \([Bor01b],[Ols03a]\). This fact is hidden in the old book by Hua Lokeng \([Hua58]\), in his computation of the matrix integral

\[
\int_{H(N)} \det(1 + X^2)^{-t} dX
\]

by induction on \( N \). (Note that \([Hua58]\) contains a lot of masterly computations of matrix integrals.) Much later, the consistency property was rediscovered and applied to constructing measures on infinite-dimensional spaces by Shimomura \([Shi75]\), Pickrell \([Pic87]\), and Neretin \([Ner02]\). Note that analogues of the projections \( p^{H}_{N-1,N} \) can be defined for other matrix spaces including the three series of compact classical groups and, more generally, the ten series of classical compact symmetric spaces \([Ner02]\).

2.2. Virtual permutations, central measures, and Kingman’s theorem. As we will see, the consistency property of the Ewens measures \( P^{(n)}_\theta \) makes it possible to build an “\( \infty \)” version of these measures. In the particular case \( \theta = 1 \), this leads to a concept of “uniformly distributed infinite permutations”.

Let \( \mathcal{S} = \lim \leftarrow S_n \) be the projective limit of the sets \( S_n \) taken with respect to the canonical projections. By the very definition, each element \( \sigma \in \mathcal{S} \) is a sequence \( \{\sigma_n \in S_n\}_{n=1,2,\ldots} \) such that \( \sigma_{n-1} = p_{n-1,n}(\sigma_n) \) for any \( n = 2, 3, \ldots \). We call \( \mathcal{S} \) the space of virtual permutations \([Ker04]\). It is a compact topological space with respect to the projective limit topology.

By classical Kolmogorov’s theorem, any family \( \{P^{(n)}\} \) of probability measures on the groups \( S_n \), consistent with the canonical projections, gives rise to a probability measure \( P = \lim \leftarrow P^{(n)} \) on the space \( \mathcal{S} \). Taking \( P^{(n)} = P^{(n)}_\theta, \theta > 0 \), we get some measures \( P_\theta \) on \( \mathcal{S} \) with nice properties; we still call them the Ewens measures.

A parallel construction exists in the context of matrix spaces. In particular, by making use of the projections \( p^{H}_{N-1,N} \), one can define a projective limit space \( \mathcal{H} := \lim \leftarrow H(N) \), which is simply the space of all Hermitian matrices of infinite size. This space carries the measures \( P^{\mathcal{H}}_t := \lim \leftarrow P^{H(N)}_t, \Re t > -\frac{1}{2} \). Using the Cayley transform, the measures \( P^{\mathcal{H}}_t \) can be transformed to some measures \( P^{\Pi}_t \) on a projective limit space \( \mathcal{U} := \lim \leftarrow U(N) \).

As is argued in \([Bor01b]\), the \( t = 0 \) case of the probability space \( (\mathcal{H}, P^{\mathcal{H}}_0) \simeq (\mathcal{U}, P^{\Pi}_0) \) may be viewed as an “\( N = \infty \)” version of \( \text{CUE}_N \). Likewise, we regard \( (\mathcal{S}, P_1) \) as an “\( n = \infty \)” version of the finite uniform measure space \( (S_n, P^{(n)}_1) \).

A crucial property of \( \text{CUE}_N \) is its invariance under the action of the unitary group \( U(N) \) on itself by conjugation. This property is shared by the deformed ensemble
Lemma 2.2. Let, as above, \( P = \lim P^{(n)} \) be a projective limit probability measure on \( S \). If \( P^{(n)} \) is central for each \( n \), then so is \( P \).

As a consequence we get that the measures \( P_{\theta} \) are central.

In a variety of random matrix problems, the invariance property under an appropriate group action makes it possible to pass from matrices to their eigenvalues or singular values. For random permutations from \( S_n \) directed by a central measure, a natural substitute of eigenvalues is another invariant — partitions of \( n \) parameterizing the cycle structure of permutations.

In combinatorics, by a partition one means a sequence \( \rho = (\rho_1, \rho_2, \ldots) \) of weakly decreasing nonnegative integers with infinitely many terms and finite sum \( |\rho| := \sum \rho_i \). Of course, the number of nonzero terms \( \rho_i \) in \( \rho \) is always finite; it is denoted by \( \ell(\rho) \). The finite set of all partitions \( \rho \) with \( |\rho| = n \) will be denoted as \( \mathcal{P}(n) \). To a permutation \( s \in S_n \) we assign a partition \( \rho \in \mathcal{P}(n) \) (in words, a partition of \( n \)) comprised by the cycle-lengths of \( s \) written in weakly decreasing order. Obviously, \( \rho \) is a full invariant of the conjugacy class of \( s \). The projection \( s \mapsto \rho \) takes any probability measure on \( S_n \) to a probability measure on \( \mathcal{P}(n) \). Now, the point is that this establishes a one-to-one correspondence \( \mathcal{P}(n) \leftrightarrow \Pi(n) \) between arbitrary central probability measures \( \mathcal{P}(n) \) on \( S_n \) and arbitrary probability measures \( \Pi(n) \) on \( \mathcal{P}(n) \). In this sense, random permutations \( s \in S_n \) (directed by a central measure) may be replaced by random partitions \( \rho \in \mathcal{P}(n) \).

The link between \( \mathcal{P}(n) \) and \( \Pi(n) \) is simple: Given \( \rho \in \mathcal{P}(n) \), let \( C(\rho) \subset S_n \) denote the corresponding conjugacy class in \( S_n \). Then \( \Pi(n)(\rho) = |C(\rho)| \mathcal{P}(n)(s) \) for any \( s \in C(\rho) \). Further, there is an explicit expression for \( |C(\rho)| \): it is equal to \( n!/z_\rho \), where \( z_\rho = \prod k^{m_k} m_k! \) and \( m_k \) stands for the multiplicity of \( k = 1, 2, \ldots \) in \( \rho \).
In this notation, the measure $\Pi^{(n)}_\theta$ corresponding to the Ewens measure $P^{(n)} = P^{(n)}_\theta$ is given by the expression

$$\Pi^{(n)}_\theta(\rho) = \frac{\theta^{|\rho|}}{|\theta(\theta + 1) \ldots (\theta + n - 1)|} \prod_k \frac{1}{k^{m_k} m_k!}, \quad \rho \in P(n),$$

widely known under the name of Ewens sampling formula [Ewe98]. This justifies the name given to the measure (2.1).

The following result provides a highly nontrivial “$n = \infty$” version of the evident correspondence $P^{(n)} \leftrightarrow \Pi^{(n)}$:

**Theorem 2.3.** There exists a natural one-to-one correspondence $\mathcal{P} \leftrightarrow \Pi$ between arbitrary central probability measures $\mathcal{P}$ on the space $\mathcal{S}$ of virtual permutations and arbitrary probability measures $\Pi$ on the space

$$\nabla_\infty := \{(x_1, x_2, \ldots) \in [0, 1]^\infty: x_1 \geq x_2 \geq \ldots, \sum x_i \leq 1\}. \quad \text{(2.6)}$$

In other words, each central measure $\mathcal{P}$ on $\mathcal{S}$ is uniquely representable as a mixture of indecomposable (or ergodic) central measures, which in turn are parameterized by the points of $\nabla_\infty$. The measure $\Pi$ assigned to $\mathcal{P}$ is just the mixing measure.

**Idea of proof.** The theorem is a reformulation of celebrated Kingman’s theorem, see [Kin78b]. Kingman did not deal with virtual permutations but worked with some sequences of random permutations that he called partition structures. Represent $\mathcal{P}$ as a projective limit measure, $\mathcal{P} = \lim \leftarrow \mathcal{P}^{(n)}$. By Lemma 2.2 all measures $\mathcal{P}^{(n)}$ are central. Pass to the corresponding measures $\Pi^{(n)}$ on partitions. The consistency of the family $\{\mathcal{P}^{(n)}\}$ with the canonical projections $S_n \to S_{n-1}$ then translates as the consistency of the family $\{\Pi^{(n)}\}$ with some canonical Markov transition kernels $\mathcal{P}(n) \to \mathcal{P}(n-1)$. In Kingman’s language this just means that $\{\mathcal{P}^{(n)}\}$ is a partition structure. Kingman’s theorem provides a kind of Poisson integral representation of partition structures via probability measures on $\nabla$, which is equivalent to the claim of Theorem 2.3.

Other proofs of Kingman’s theorem can be found in [Ker03, Ker98], where this result is placed in the broader context of potential theory for branching graphs. For our purpose it is worth emphasizing that the claim of Theorem 2.3 has a counterpart in the random matrix context — description of $U(\infty)$-invariant probability measures on $\mathcal{S}$, which in turn is equivalent to classical Schoenberg’s theorem on totally positive functions [Sch51, Pic91, Ols96].

From the proof of Kingman’s theorem it is seen that the space $\nabla_\infty$ arises as a large-$n$ limit of finite sets $\mathcal{P}(n)$, and every measure $\Pi$ can be interpreted as a limit of the corresponding measures $\Pi^{(n)}$. In this picture, the ergodic measures $\Pi(x_1, x_2, \ldots)$ that are parameterized by points $(x_1, x_2, \ldots) \in \nabla_\infty$ arise as limits of uniform distributions.
on conjugacy classes $C(\rho)$ in growing finite symmetric groups. Thus, it is tempting to regard the $\Pi_{(x_1,x_2,\ldots)}$'s as a substitute of those uniform measures.

2.3. **Application to representation theory.** Pickrell’s pioneer work [Pic87] demonstrated how some non-Gaussian measures on infinite-dimensional matrix spaces can be employed in representation theory. The idea of [Pic87] was further developed in [Ols03a]. As shown there, the measures $P_t^\theta$ on $\mathfrak{H}$ (or, equivalently, the measures $P_t^{\mathfrak{U}}$ on $\mathfrak{U}$) meet a lack of the Haar measure in the infinite-dimensional situation and can be applied to the construction of some “generalized (bi)regular representations” of the group $U(\infty) \times U(\infty)$.

Pickrell’s work was also a starting point for a parallel theory for the group $S_\infty$, [Ker93c], [Ker04]. The key point is that the Ewens measures $P_\theta$ on $\mathfrak{S}$ have good transformation properties with respect to a natural action of the group $S_\infty \times S_\infty$ on $\mathfrak{S}$ extending the two-sided action of $S_\infty$ on itself. Namely, the measure $P_1$ is $S_\infty \times S_\infty$-invariant and is the only probability measure on $\mathfrak{S}$ with such a property, so that it may be viewed as a substitute of the uniform distribution on $S_n$. As for the Ewens measures $P_\theta$ with general $\theta > 0$, they turn out to be quasi-invariant with respect to the action of $S_\infty \times S_\infty$. The quasi-invariance property forms the basis of the construction of some “generalized (bi)regular representations” $T_z$ of the group $S_\infty \times S_\infty$. Here $z$ is a parameter ranging over $\mathbb{C}$, and the Hilbert space of $T_z$ is $L^2(\mathfrak{S}, P_{|z|^2})$. We refer to [Ker04] and [Ols03b] for details.

2.4. **Poisson–Dirichlet distributions.** The probability measures on $\nabla_\infty$ assigned by Theorem 2.3 to the Ewens measures $P_\theta$ are known under the name of Poisson–Dirichlet distributions; denote them by $PD(\theta)$. Continuing our juxtaposition of the Ewens measures and the Hua–Pickrell measures one may say that the Poisson–Dirichlet distributions are counterparts of the determinantal point processes directing the decomposition of the measures $P_t^\theta$ into ergodic components (those processes involve, in a slightly disguised form, the sine-kernel process, see [Bor01b]). Although the Poisson–Dirichlet distributions $PD(\theta)$ seem to be much simpler than the sine-kernel process, they are still very interesting objects with a rich structure. Below we list a few equivalent descriptions of the $PD(\theta)$'s:

(a) **Projection of a Poisson process.** Let $P(\theta)$ denote the inhomogeneous Poisson process on the half-line $\mathbb{R}_{>0} := \{ \tau \in \mathbb{R} \mid \tau > 0 \}$ with intensity $\theta \tau^{-1} e^{-\tau}$, and let $y = \{y_i\}$ be the random point configuration on $\mathbb{R}_{>0}$ with law $P(\theta)$. Due to the fast decay of the intensity at $\infty$, the configuration $y$ is almost surely bounded from above, so that we may arrange the $y_i$'s in weakly decreasing order: $y_1 \geq y_2 \geq \cdots > 0$. Furthermore, the sum $r := \sum y_i$ is almost surely finite. Finally, it turns out that $r$ and the normalized vector $x := (y_1/r, y_2/r, \ldots) \in \nabla_\infty$ are independent from each other, the random variable $r$ has the gamma distribution on $\mathbb{R}_{>0}$ with density $(\Gamma(\theta))^{-1} t^{\theta-1} \exp(-t)$, and the random vector $x$ is distributed according to $PD(\theta)$. 
This means, in particular, that $PD(\theta)$ arises as the push-forward of the Poisson process $P(\theta)$ under the projection $y \mapsto x$.

(b) Limit of Dirichlet distributions [Kin75]. Let $D_n(\theta)$ denote the probability distribution on the $(n-1)$-dimensional simplex

$$\Delta_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, \ldots, x_n \geq 0, \sum x_i = 1\}$$

with the density proportional to $\prod x_i^{n-1}\theta^{-1}$ (with respect to Lebesgue measure on $\Delta_n$). Note that $D_n(\theta)$ enters the family of the Dirichlet distributions. Rearranging the coordinates $x_i$ in weakly decreasing order and adding infinitely many 0’s one gets a map $\Delta_n \to \nabla_\infty$; let $\tilde{D}_n(\theta)$ stand for the push-forward of $D_n(\theta)$ under this map. Then $PD(\theta)$ appears as the weak limit of the measures $\tilde{D}_n(\theta)$ as $n \to \infty$.

(c) Projection of a product measure [Ver77a], [Arr03, §4.11]. Consider the infinite-dimensional simplex

$$\Delta_\infty := \{(u_1, u_2, \ldots) \in [0, 1]^\infty \mid u_1 + u_2 + \cdots \leq 1\}.$$ 

The triangular transformation of coordinates $v = (v_1, v_2, \ldots) \mapsto u = (u_1, u_2, \ldots)$ given by

$$u_1 = v_1; \quad u_n = v_n(1 - v_1) \cdots (1 - v_{n-1}), \quad n \geq 2,$$

maps the cube $[0, 1]^\infty$ onto the simplex $\Delta_\infty$. This map is almost one-to-one: it admits the inversion $u \mapsto v$,

$$v_1 = u_1; \quad v_n = \frac{u_n}{1 - u_1 - \cdots - u_{n-1}}, \quad n \geq 2,$$

which is well defined provided that all the partial sums of the series $u_1 + u_2 + \ldots$ are strictly less than 1.

Next, the rearrangement of coordinates in weakly decreasing order determines a projection $\Delta_\infty \to \nabla_\infty$.

Denoting by $B(\theta)$ the probability measure on $[0, 1]^\infty$ obtained as the product of infinitely many copies of the measure $\theta(1 - t)^{\theta-1}dt$ on $[0, 1]$, the Poisson–Dirichlet distribution $PD(\theta)$ coincides with the push-forward of $B(\theta)$ under the composition map $[0, 1]^\infty \to \Delta_\infty \to \nabla_\infty$.

(d) Characterization via correlation functions [Wat76, §3]. Removing possible 0's from a sequence $x \in \nabla_\infty$ one may interpret it as a locally finite point configuration on the semi-open interval $(0, 1]$. This allows one to interpret any probability measure on $\nabla_\infty$ as a random point process on $(0, 1]$ (see [Bor10] for basic definitions). It turns out that the correlation functions of the point process associated to $PD(\theta)$ have a very simple form:

$$\rho_n(u_1, \ldots, u_n) = \begin{cases} 
\theta^n (1 - u_1 - \cdots - u_n)^{\theta-1} 
\frac{u_1 \cdots u_n}{u_1 \cdots u_n}, & \sum_{i=1}^{n} u_i < 1; \\
0, & \text{otherwise.}
\end{cases}$$
This provides one more characterization of $PD(\theta)$.

The literature devoted to the Poisson–Dirichlet distributions and their various connections and applications is very large. The interested reader will find a rich material in [Arr03], [Ver72], [Wat76], [Ver77a], [Ver78], [Ign82], [Pit97], [Hol01] [Kin75].

Note that $PD(\theta)$ describes the asymptotics of the large cycle-lengths of random permutations with law $P^{(n)}_{\theta}$ (namely, the $i$th coordinate $x_i$ on $\nabla_{\infty}$ corresponds to the $i$th largest cycle-length scaled by the factor of $1/n$). The literature also contains results concerning the asymptotics of other statistics on random permutations, for instance, small cycle-lengths and the number of cycles [Arr03].

3. The Plancherel measure

3.1. Definition of the Plancherel measure. Partitions parameterize not only the conjugacy classes in the symmetric groups but also their irreducible representations. So far we focused on the conjugacy classes, but now we will exploit the connection with representations. It is convenient to identify partitions of $n$ with Young diagrams containing $n$ boxes. The set of such diagrams will be denoted as $\mathbb{Y}_n$. Given a diagram $\lambda \in \mathbb{Y}_n$, let $V_{\lambda}$ denote the corresponding irreducible representation of $S_n$ and $\dim \lambda$ its dimension. In particular, the one-row diagram $\lambda = (n)$ and the one-column diagram $\lambda = (1^n)$ correspond to the only one-dimensional representations, the trivial and the sign ones. Note that the symmetry map $\mathbb{Y}_n \to \mathbb{Y}_n$ given by transposition $\lambda \mapsto \lambda'$ amounts to tensoring $V_{\lambda}$ with the sign representation, so that $\dim \lambda' = \dim \lambda$.  

By virtue of Burnside’s theorem,

$$\sum_{\lambda \in \mathbb{Y}_n} (\dim \lambda)^2 = n!.$$  

(3.1)

This suggests the definition of a probability distribution $M^{(n)}$ on $\mathbb{Y}_n$:

$$M^{(n)}(\lambda) := \frac{(\dim \lambda)^2}{n!}, \quad \lambda \in \mathbb{Y}_n.$$  

(3.2)

Following [Ver77b], one calls $M^{(n)}$ the Plancherel measure on $\mathbb{Y}_n$.

In purely combinatorial terms, $\dim \lambda$ equals the number of standard tableaux of shape $\lambda$ [Sag01, Section 2.5]. Several explicit expressions for $\dim \lambda$ are known, see [Sag01 Sections 3.10, 3.11].

3.2. Limit shape and Gaussian fluctuations. We view each $\lambda \in \mathbb{Y}_n$ as a plane shape, of area $n$, in the $(r, s)$ plane, where $r$ is the row coordinate and $s$ the column coordinate. In new coordinates $x = s - r$, $y = r + s$, the boundary $\partial \lambda$ of the shape $\lambda \subset \mathbb{R}^2_+$ may be viewed as the graph of a continuous piecewise linear function, which we denote as $y = \lambda(x)$. Note that $\lambda'(x) = \pm 1$, and $\lambda(x)$ coincides with $|x|$ for

\footnote{In the context of conjugacy classes the operation of transposition has no natural interpretation.}
sufficiently large values of $|x|$. The area of the shape $|x| \leq y \leq \lambda(x)$ equals $2n$. (See Figure [1])

![Diagram](image)

**Figure 1.** The function $y = \lambda(x)$ for the Young diagram $\lambda = (4,2,1)$.

Assuming $\lambda$ to be the random diagram distributed according to the Plancherel measure $M(n)$, we get a random ensemble $\{\lambda(\cdot)\}$ of polygonal lines. We will describe the behavior of this ensemble as $n \to \infty$.

Informally, the result can be stated as follows: Let $y = \bar{\lambda}(x)$ be obtained from $y = \lambda(x)$ by shrinking along both the $x$ and $y$ axes with coefficient $\sqrt{n}$,

$\bar{\lambda}(x) = \frac{1}{\sqrt{n}} \lambda(\sqrt{n}x)$;

then we have

$\bar{\lambda}(x) \approx \Omega(x) + \frac{2}{\sqrt{n}} \Delta(x), \quad n \to \infty, \quad (3.3)$

where $y = \Omega(x)$ is a certain nonrandom curve coinciding with $y = |x|$ outside $[-2,2] \subset \mathbb{R}$, and $\Delta(x)$ is a generalized Gaussian process. Let us explain the exact meaning of (3.3).

First of all, the purpose of the scaling $\lambda(\cdot) \to \bar{\lambda}(\cdot)$ is to put the random ensembles with varying $n$ on the same scale: note that the area of the shape $|x| \leq y \leq \bar{\lambda}(x)$ equals 2 for any $n$.

The function $y = \Omega(x)$ is given by two different expressions depending on whether or not $x$ belongs to the interval $[-2,2] \subset \mathbb{R}$:

$\Omega(x) = \begin{cases} \frac{2}{\pi} (x \arcsin \frac{x}{2} + \sqrt{4-x^2}), & |x| \leq 2 \\ |x|, & |x| \geq 2 \end{cases}$

In the first approximation, the asymptotic relation (3.3) means concentration of the random polygonal lines $y = \bar{\lambda}(x)$ near a limit curve. The exact statement (see [Log77], [Ver77b], and also [Iva02]) is:

**Theorem 3.1** (Law of large numbers). For each $n = 1,2,\ldots$, let $\lambda \in \mathbb{Y}_n$ be the random Plancherel diagram and $\bar{\lambda}(\cdot)$ be the corresponding random curve, as defined
above. As \( n \to \infty \), the distance in the uniform metric between \( \bar{\lambda}(\cdot) \) and the curve \( \Omega(x) \) tends to 0 in probability:

\[
\lim_{n \to \infty} M(n) \left\{ \lambda \in Y_n \mid \sup_{x \in \mathbb{R}} |\bar{\lambda}(x) - \Omega(x)| \leq \varepsilon \right\} = 1, \quad \forall \varepsilon > 0.
\]

The second term in the right-hand side of (3.3) describes the fluctuations around the limit curve. The Gaussian process \( \Delta(x) \) can be defined by a random trigonometric series on the interval \([-2, 2] \subset \mathbb{R}\), as follows. Let \( \xi_2, \xi_3, \ldots \) be independent Gaussian random variables with mean 0 and variance 1, and set \( x = 2 \cos \theta \), where \( 0 \leq \theta \leq \pi \). Then

\[
\Delta(x) = \Delta(2 \cos \theta) = \frac{1}{\pi} \sum_{k=2}^{\infty} \frac{\xi_k}{\sqrt{k}} \sin(k\theta), \quad x \in [-2, 2].
\]

This is a generalized process, meaning that its trajectories are not ordinary functions but generalized ones (i.e., distributions). In other words, it is a Gaussian measure on the space of distributions supported by \([-2, 2]\). For any smooth test function \( \varphi \) on \( \mathbb{R} \), the smoothed series

\[
\frac{1}{\pi} \sum_{k=2}^{\infty} \frac{\xi_k}{\sqrt{k}} \int_{-2}^{2} \sin(k\theta) \varphi(x) dx, \quad \theta = \arccos(x/2),
\]

converges and represents a Gaussian random variable. However, the value of \( \Delta(x) \) at a point \( x \) is not defined.

More precisely, the result about the Gaussian fluctuations looks as follows:

**Theorem 3.2** (Central limit theorem for global fluctuations). Let, as above, \( \{\bar{\lambda}(\cdot)\} \) be the random ensemble governed by the Plancherel measure \( M(n) \), and set

\[
\Delta_n(x) = \frac{\sqrt{n}}{2} (\bar{\lambda}(x) - \Omega(x)), \quad x \in \mathbb{R}.
\]

For any finite collection of polynomials \( \varphi_1(x), \ldots, \varphi_m(x) \), the joint distribution of the random variables

\[
\int_{\mathbb{R}} \varphi_i(x) \Delta_n(x) dx, \quad 1 \leq i \leq m
\]

converges, as \( n \to \infty \), to that of the Gaussian random variables

\[
\int_{\mathbb{R}} \varphi_i(x) \Delta(x) dx, \quad 1 \leq i \leq m.
\]

This result is due to Kerov [Ker93a]; a detailed exposition is given in [Iva02].

Note that for any diagram \( \lambda \), the function \( \bar{\lambda}(x) - \Omega(x) \) vanishes for \(|x|\) large enough, so that the integral in (3.4) makes sense.

The theorem implies that the normalized fluctuations \( \Delta_n(x) \), when appropriately smoothed, are of finite order. This can be rephrased by saying that, in the \((r, s)\)
coordinates, the global fluctuations of the boundary $\partial \lambda$ of the random Plancherel diagram $\lambda \in Y_n$ in the direction parallel to the diagonal $r = s$ have finite order.

A different central limit theorem is stated in [Bog07]: that result describes fluctuations at points (so that there is no smoothing); then an additional scaling of order $\sqrt{\log n}$ along the $y$-axis is required.

Theorem 3.1 should be compared to a similar concentration result for spectra of random matrices (convergence to Wigner’s semicircle law). A similarity between the two pictures becomes especially convincing in view of the fact (discovered in [Ker93b]) that there is a natural transform relating the curve $\Omega$ to the semicircle law. As for Theorem 3.2, it has a strong resemblance to the central limit theorems for random matrix ensembles, established in [Dia94], [Joh98].

Biane [Bia01] considered a modification of the Plancherel measures $M^{(n)}$ related to the Schur-Weyl duality and found a one-parameter family of limit curves forming a deformation of $\Omega$.

### 3.3. The poissonized Plancherel measure as a determinantal process.

Let $Y = Y_0 \cup Y_1 \cup \ldots$ be the countable set of all Young diagrams including the empty diagram $\emptyset$. To each $\lambda \in Y$ we assign an infinite subset $\mathcal{L}(\lambda)$ on the lattice $\mathbb{Z}':=\mathbb{Z}+\frac{1}{2}$ of half-integers, as follows

$$\mathcal{L}(\lambda) = \{ \lambda_i - i + \frac{1}{2} \mid i = 1, 2, \ldots \}.$$  

We interpret $\mathcal{L}(\lambda)$ as a particle configuration on the nodes of the lattice $\mathbb{Z}'$ and regard the unoccupied nodes $\mathbb{Z}' \setminus \mathcal{L}(\lambda)$ as holes. In particular, the configuration $\mathcal{L}(\emptyset)$ is $\mathbb{Z}'_- := \{ \ldots, -\frac{3}{2}, -\frac{1}{2} \}$ and the corresponding holes occupy $\mathbb{Z}'_+ := \{ \frac{1}{2}, \frac{3}{2}, \ldots \}$. In this picture, appending a box to a diagram $\lambda$ results in moving a particle from $\mathcal{L}(\lambda)$ to the neighboring position on the right. Thus, growing $\lambda$ from the empty diagram $\emptyset$ to the configuration $\mathcal{L}(\lambda)$ by moving at each step one of the particles to the right by 1.

The configurations $\mathcal{L}(\lambda)$ are precisely those configurations for which the number of particles on $\mathbb{Z}'_+$ is finite and equal to the number of holes on $\mathbb{Z}'_-$. Note also that transposition $\lambda \rightarrow \lambda'$ translates as replacing particles by holes and vice versa, combined with the reflection map $x \rightarrow -x$ on $\mathbb{Z}'$.

The poissonized Plancherel measure with parameter $\nu > 0$ [Bai99] is a probability measure $M_\nu$ on $Y$, which is obtained by mixing together the measures $M^{(n)}$ (see (3.2)), $n = 0, 1, 2, \ldots$, by means of a Poisson distribution on the set of indices $n$:

$$M_\nu(\lambda) = e^{-\nu} \frac{\nu^{\dim(\lambda)}}{\lambda!} M^{(\lambda)}(\lambda) = e^{-\nu} \frac{\nu^{\dim(\lambda)}}{\lambda!} \left( \frac{\dim(\lambda)}{|\lambda|} \right)^2, \quad \lambda \in Y$$

(see also [Bor10, Sect. 1.6]).
Theorem 3.3. Under the correspondence $\lambda \rightarrow \mathcal{L}(\lambda)$ defined by (3.5), the poissonized Plancherel measure $M_\nu$ turns into the determinantal point process on the lattice $\mathbb{Z}'$ whose correlation kernel is the discrete Bessel kernel.

About determinantal point processes in general, see [Bor10]. The discrete Bessel kernel is written down in [Bor10, Sect. 11.6] (replace there $\theta$ by $\sqrt{\nu}$). Note that it is a projection kernel. Theorem 3.3 was obtained in [Joh01a] and (in a slightly different form) in [Bor00b]. Johansson’s approach [Joh01a] is also discussed in his note [Joh01b] and the expository paper [Joh05].

3.4. The bulk limit. (See [Bor00b].) Fix $a \in (-2, 2)$. Recall that the point $(a, \Omega(a))$ on the limit curve $y = \Omega(x)$ (see Section 4.2) corresponds to the intersection of the boundary $\partial \lambda$ of the typical large Plancherel diagram $\lambda \in \mathcal{Y}_n$ with the line $j - i = a \sqrt{n}$. The next result describes the asymptotic behavior of the boundary $\partial \lambda$ near this point.

Theorem 3.4. Assume that $n \rightarrow \infty$ and $x(n) \in \mathbb{Z}'$ varies together with $n$ in such a way that $x(n)/\sqrt{n} \rightarrow a \in (-2, 2)$. Let $\lambda \in \mathcal{Y}_n$ be the random diagram with law $M^{(n)}$ given by (3.2) and let $X_n$ be the random particle configuration on $\mathbb{Z}$ obtained from the configuration $\mathcal{L}(\lambda)$ defined by (3.3) under the shift $x \mapsto x - x(n)$ mapping $\mathbb{Z}'$ onto $\mathbb{Z}$. Then $X_n$ converges to a translation invariant point process on $\mathbb{Z}$, with the correlation kernel

$$S^a(k, l) = \begin{cases} \frac{\sin(\arccos(a/2)(k - l))}{\pi(k - l)}, & k, l \in \mathbb{Z}, \ k \neq l; \\ \frac{\arccos(a/2)}{\pi}, & k = l. \end{cases}$$

The kernel $S^a(k, l)$ is called the discrete sine kernel. It is a projection kernel and should be viewed as a lattice analogue of the famous sine kernel on $\mathbb{R}$ originated in random matrix theory. Like the sine kernel, the discrete sine kernel possesses a universality property [Bai07].

Theorem 3.4 is derived from Theorem 3.3. Let $J^\nu(x, y)$ denote the discrete Bessel kernel; one shows that

$$\lim_{\nu \rightarrow \infty} J^\nu(x(\nu) + k, x(\nu) + l) = S^a(k, l), \quad x(\nu) \in \mathbb{Z}', \ x(\nu) \sim a \sqrt{\nu},$$

and then one applies a depoissonization argument to check that the large-$n$ limit and the large-$\nu$ limit are equivalent.

3.5. The edge limit.

Theorem 3.5. Let $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{Y}_n$ be distributed according to the $n$th Plancherel measure $M^{(n)}$ given by (3.2). For any fixed $k = 1, 2, \ldots$, introduce real-valued random variables $u_1, \ldots, u_k$ by setting

$$\lambda_i = 2n^{1/2} + u_i n^{1/6}, \quad i = 1, \ldots, k. \quad (3.6)$$
Then, as \( n \to \infty \), the joint distribution of \( u_1, \ldots, u_k \) converges to that of the first \( k \) particles in the Airy point process.

Recall ([Bor10, Sect. 1.9]) that the Airy point process is a determinantal process on \( \mathbb{R} \) living on point configurations \((u_1 > u_2 > \ldots)\) bounded from above; it is determined by the Airy correlation kernel, which is a projection kernel on \( \mathbb{R} \).

The Airy point process arises in the edge limit transition from a large class of random matrix ensembles. It turns out that it also describes the limit distribution of a few (appropriately scaled) largest rows of the random Plancherel diagram.

Due to symmetry of \( M^{(n)} \) under transposition \( \lambda \to \lambda' \), the same result holds for the largest column lengths as well.

Already the simplest case \( k = 1 \) of Theorem 3.5 is very interesting, especially because of its connection to longest increasing subsequences in random permutations (see Section 3.6 below). The claim for \( k = 1 \) was first established by Baik, Deift, and Johansson [Bai99]; then they proved the claim for \( k = 2 \), [Bai00]. Their work completed a long series of investigations and at the same time opened the way to generalizations. The general case of Theorem 3.5 is due to Okounkov [Oko00]; note that his approach is very different from that of [Bai99], [Bai00]. Shortly afterwards, the theorem was obtained by yet another method in independent papers [Bor00b] and [Joh01a], by using Theorem 3.3 as an intermediate step. Note that once one knows Theorem 3.3 the precise form of the scaling (3.6) can be guessed by a simple argument, see [Ols08].

3.6. Longest increasing subsequences. Given a permutation \( s \in S_n \), let \( L_n(s) \) stand for the length of the longest increasing subsequence in the permutation word \( \tilde{s} := s(1)s(2)\ldots s(n) \). Under the uniform distribution on \( S_n \), \( L_n \) becomes a random variable. In the sixties, S. Ulam raised the question about its asymptotic properties as \( n \to \infty \). This seemingly rather particular problem turned out to be surprisingly deep (about the history of the problem and many related results, see [Bai99] and the survey papers [Ald99], [Dei00], [Sta07]). The next claim relates \( L_n \) to the Plancherel measure \( M^{(n)} \):

**Theorem 3.6.** The distribution of \( L_n \) under the uniform measure on \( S_n \) coincides with the distribution of \( \lambda_1 \), the first row length of the random Young diagram \( \lambda \in \mathbb{Y}_n \) with law \( M^{(n)} \).

This result is obtained with the help of the Robinson-Schensted correspondence, which establishes an explicit bijection \( RS : s \leftrightarrow (\mathcal{P}, \mathcal{Q}) \) between permutations \( s \in S_n \) and couples \((\mathcal{P}, \mathcal{Q})\) of standard tableaux of one and the same shape \( \lambda \in \mathbb{Y}_n \). The bijection \( RS \) is described in detail in many textbooks, e.g., [Ful97] and [Sag01]. The latter book also contains an elegant geometric interpretation of \( RS \) due to Viennot.

---

\(^2\)An increasing subsequence in \( \tilde{s} \) is a subword \( s(i_1)\ldots s(i_k) \) such that \( i_1 < \cdots < i_k \) and \( s(i_1) < \cdots < s(i_k) \).
By the very definitions, the push-forward under $RS$ of the uniform measure on $S_n$ is $M^{(n)}$. A nontrivial fact is that under this bijection, $L_n(s) = \lambda_1$.

By virtue of Theorem 3.6, the Ulam problem is completely solved by the $k=1$ case of Theorem 3.5 discussed above: the limit distribution of the scaled random variable $(L_n - 2\sqrt{n})n^{-1/6}$ is the GUE Tracy-Widom distribution $F_2$ [Tra94].

Given a subset $S^n_\ast \subset S_n$, denote by $L^n_\ast$ the random variable $L_n(\cdot)$ directed by the uniform measure on $S^n_\ast$. A modification of the Ulam problem consists in studying the limit distribution of $L^n_\ast$ (suitably centered and scaled) for subsets $S^n_\ast$ determined by certain symmetry conditions imposed on the matrix $[s_{ij}]$ of a permutation $s \in S_n$.

Baik and Rains (see [Bai01] and references therein) showed that in this way one can get two other Tracy-Widom distributions [Tra96], $F_1$ and $F_4$, as well as a large family of allied probability distributions including an interpolation between $F_1$ and $F_4$. These results demonstrate once again a similarity in asymptotic properties of random permutations and random matrices. Here is the simplest example from [Bai01], which shows that involutions $s = s^{-1}$ in $S_n$ (i.e., symmetric permutation matrices) model real symmetric matrices:

**Theorem 3.7.** Take as $S^n_\ast$ the subset of involutions in $S_n$, and let $L^n_\ast$ be the corresponding random variable. Then the limit distribution of $(L^n_\ast - 2\sqrt{n})n^{-1/6}$ is the GOE Tracy-Widom distribution $F_1$.

### 4. The z-measures and Schur measures

#### 4.1. The z-measures.

The identity (3.1) admits an extension depending on two parameters $z, z' \in \mathbb{C}$:

$$\sum_{\lambda \in \mathbb{Y}_n} (z)_\lambda (z')_\lambda (\dim \lambda)^2 = (zz')_n n!,$$

where $(x)_n := x(x+1)\ldots(x+n-1)$ is the Pochhammer symbol and $(x)_\lambda$ is its generalization,

$$(x)_\lambda := \prod_{(i,j) \in \lambda} (x+j-i),$$

the product taken over the boxes $(i,j)$ belonging to $\lambda$, where $i$ and $j$ stand for the row and column number of a box. The (complex-valued) z-measure $M^{(n)}_{z,z'}$ on $\mathbb{Y}_n$ assigns weights

$$M^{(n)}_{z,z'}(\lambda) = \frac{(z)_\lambda (z')_\lambda}{(zz')_n} M^{(n)}(\lambda) = \frac{(z)_\lambda (z')_\lambda}{(zz')_n} \frac{(\dim \lambda)^2}{n!}$$

to diagrams $\lambda \in \mathbb{Y}_n$. This is a deformation of the Plancherel measure $M^{(n)}$ in the sense that $M^{(n)}_{z,z'}(\lambda) \to M^{(n)}(\lambda)$ as $z, z' \to \infty$. In what follows we assume that the parameters take admissible values meaning that $(z)_\lambda (z')_\lambda \geq 0$ for any $\lambda \in \mathbb{Y}$ and $zz' > 0$ (for instance, one may assume $z' = \bar{z} \in \mathbb{C} \setminus \{0\}$). Then $M^{(n)}_{z,z'}$ is a probability measure for every $n$. 

The z-measures first emerged in [Ker93c]; they play an important role in the representation theory of the infinite symmetric group \( S_\infty \). Recall that in Section 2.3 we have mentioned generalized regular representations \( T_z \); it turns out that when \( z' = \bar{z} \), a suitably defined large-\( n \) scaled limit of the z-measures governs the spectral decomposition of \( T_z \) into irreducibles: [Bor01a, §3], [Ols03b].

The mixed z-measure \( M_{z,z',\xi} \) on \( \mathbb{Y} \) with admissible parameters \((z, z')\) and an additional parameter \( \xi \in (0, 1) \) is obtained by mixing up the z-measures with varying superscript \( n \) by means of a negative binomial distribution on \( \mathbb{Z}^+ \):

\[
M_{z,z',\xi}(\lambda) = (1 - \xi)^{zz'}(\frac{\xi|\lambda|}{|\lambda|!})M_{z,z'}(\lambda) = (1 - \xi)^{zz'}\xi^{|\lambda|}(z)(z')^{|\lambda|}(\frac{\dim \lambda}{|\lambda|})^2,
\]

where \( \lambda \) ranges over \( \mathbb{Y} \). This procedure is similar to poissonization of the Plancherel measure and serves the same purpose of facilitating the study of limit transitions. Note that the poissonized Plancherel measure \( M_\nu \) is a degeneration of \( M_{z,z',\xi} \) when \( z, z' \to \infty \) and \( \xi \to 0 \) in such a way that \( zz'\xi \to \nu \).

**Theorem 4.1.** Under the correspondence \( \lambda \to L(\lambda) \) defined by (3.5), the mixed z-measure \( M_{z,z',\xi} \) turns into a determinantal point process on the lattice \( \mathbb{Z}' \) whose correlation kernel can be explicitly expressed through the Gauss hypergeometric function.

This is a generalization of Theorem 3.3. Various proofs have been given in [Bor00a], [Bor00c], [Oko01b], [Bor06].

For the lattice determinantal process from Theorem 4.1 there are three interesting limit regimes, as \( \xi \to 1 \), leading to continuous and discrete determinantal processes:

1. Split \( \mathbb{Z}' \) into positive and negative parts, \( \mathbb{Z}' = \mathbb{Z}'_+ \sqcup \mathbb{Z}'_- \). Given \( \lambda \in \mathbb{Y} \), let \( L^\varepsilon(\lambda) \subset \mathbb{Z}' \) be obtained from \( L(\lambda) \) by switching from particles to holes on \( \mathbb{Z}'_- \); then \( L^\varepsilon(\lambda) \) is finite and contains equally many particles in \( \mathbb{Z}'_+ \) and in \( \mathbb{Z}'_- \). Note that this particle/hole involution does not affect the determinantal property. Next, scale the lattice \( \mathbb{Z}' \) making its mesh equal to small parameter \( \varepsilon = 1 - \xi \). Letting \( \xi \to 1 \), one gets in this way from \( M_{z,z',\xi} \) a determinantal process living on the punctured real line \( \mathbb{R} \setminus \{0\} \). The corresponding correlation kernel is called the Whittaker kernel, because it is expressed through the classical Whittaker function. This limit process is of great interest for harmonic analysis on the infinite symmetric groups. For more detail, see [Bor00a], [Ols03b].

2. No scaling, we remain on the lattice. The limit determinantal process is directed by a diffuse measure on the space \( \{0, 1\}^{\mathbb{Z}'} \) of all lattice point configurations, and the limit correlation kernel is expressed through Euler’s gamma function, see [Bor05c].
An “intermediate” limit regime assuming a scaling. It leads to a stationary limit process whose correlation kernel is expressed through trigonometric functions and is a deformation of the sine kernel, see [Bor05c].

These three different regimes describe the asymptotics of the largest, smallest and intermediate Frobenius coordinates of random Young diagrams, respectively.

Remark 4.2. Note a special role of the quantity \(\dim \lambda/|\lambda|!\) in the expression for \(M_{z,z',\xi}\): this is a Vandermonde-like object, which creates a kind of log-gas pair interaction between particles from the random configuration \(\mathcal{L}(\lambda)\) (about log-gas systems, see [For10a], [For10b]). The particle/hole involution \(\mathcal{L}(\lambda) \rightarrow \mathcal{L}^0(\lambda)\) changes the sign of interaction between particles on the different sides from 0, so that we get two kinds of particles which are oppositely charged. Note that in the first regime, the particle/hole involution is necessary for existence of a limiting point process.

The Whittaker kernel is an instance of a correlation kernel which is symmetric with respect to an indefinite inner product.

4.2. Special instances of \(z\)-measures. (a) Meixner and Laguerre ensembles. Assume \(z = N = 1, 2, \ldots \) and \(z' = N + b - 1\) with \(b > 0\); these are admissible values. Then \(M_{z,z',\xi}\) is supported by the subset \(\mathcal{Y}(N) \subset \mathcal{Y}\) of Young diagrams with at most \(N\) nonzero rows. Under the correspondence

\[\mathcal{Y}(N) \ni \lambda \mapsto (l_1, l_2, \ldots, l_N) = (\lambda_1 + N - 1, \lambda_2 + N - 2, \ldots, \lambda_N) \subset \mathbb{Z}_+,\]

the measure turns into a random-matrix-type object: the \(N\)-particle Meixner orthogonal polynomial ensemble with the discrete weight function \((b)_l \xi^l/l!\), where the argument \(l\) ranges over \(\mathbb{Z}_+\) (for generalities about orthogonal ensembles, see [Kon05]). It follows that for general values of \((z, z')\), the measure \(M_{z,z',\xi}\) may be viewed as the result of analytic continuation of the Meixner ensembles with respect to parameters \(N\) and \(b\). This observation is exploited in [Bor06]. In a scaling limit regime as \(\xi \rightarrow 1\), the \(N\)-particle Meixner ensemble turns into the \(N\)-point Laguerre ensemble; the correlation kernel for the latter ensemble is a degeneration of the Whittaker kernel, see [Bor00a].

(b) Generalized permutations. Recall that the Plancherel measure \(M^{(n)}\) on \(\mathcal{Y}_n\) coincides with the push-forward of the uniform measure on \(S_n\) under the projection \(S_n \rightarrow \mathcal{Y}_n\) afforded by the Robinson–Schensted correspondence \(RS\) (Section 3.6). Here is a generalization:

Fix natural numbers \(N \leq N'\) and replace \(S_n\) by the finite set \(S_{N,N'}^{(n)}\) consisting of all \(N \times N'\) matrices with entries in \(\mathbb{Z}_+\) such that sum of all entries equals \(n\). Elements of \(S_{N,N'}^{(n)}\) are called generalized permutations. Knuth’s generalization of the Robinson–Schensted correspondence (the \(RSK\) correspondence, see, e.g., [Ful97, Section 4.1]) provides a projection of \(S_{N,N'}^{(n)}\) onto \(\mathcal{Y}_n(N) := \mathcal{Y}_n \cap \mathcal{Y}(N)\), the set of Young diagram with \(n\) boxes and at most \(N\) nonzero rows. It turns out that the push-forward of the uniform distribution on \(S_{N,N'}^{(n)}\) coincides with the \(z\)-measure \(M_{N,N'}^{(n)}\), see [Bor01a].
(c) A variation. In the same way one can get the mixed z-measure $M_{N,N',\xi}$ if instead of $S_{N,N'}^{(n)}$, one takes $N \times N'$ matrices whose entries are i.i.d. random variables, the law being the geometric distribution with parameter $\xi$.

(d) Random words. Denote by $S_{N,\infty}^{(n)}$ the set of words of length $n$ in the alphabet $[N] := \{1, \ldots, N\}$. Endowing $S_{N,\infty}^{(n)}$ with the uniform measure we get a model of random words. This model may be viewed as a degeneration of the model of random generalized permutations (item (b) above) in the limit $N' \to \infty$ (this explains the notation $S_{N,\infty}^{(n)}$). The RSK correspondence (or rather its simpler version due to Schensted) provides a projection $S_{N,\infty}^{(n)} \to Y_n(N)$ taking random words to random Young diagrams $\lambda \in Y_n(N)$ with distribution $M_{N,\infty}^{(n)} := \lim_{N' \to \infty} M_{N,N'}^{(n)}$. Asymptotic properties of random words are studied in [Tra01] and [Joh01a]. The model of random words can be generalized by allowing non-uniform probability distributions on the alphabet (see [Its01] and references therein). As explained in [Its01], this more general model is connected to the Schur measure discussed in Section 4.3 below.

(e) The Charlier ensemble and the Plancherel degeneration. Poissonization of the measure $M_{N,\infty}^{(n)}$ with respect to parameter $n$ leads to the $N$-particle Charlier ensemble [Bor01a, §9]. Alternatively, it can be obtained as a limit case of the mixed z-measures $M_{N,N',\xi}^{(n)}$ when $z, z' \to \infty$ and $\xi \to 0$ in such a way that $zz'\xi \to \nu$. This fact prompted the derivation of the discrete Bessel kernel (Theorem 3.3) in [Bor00b]. Alternatively, $M_{\nu}$ can be obtained through a limit transition from the Charlier or Meixner ensembles; this leads to another derivation of the discrete Bessel kernel: [Joh01a], [Joh01b].

4.3. The Schur measures. Let $\Lambda$ denote the graded algebra of symmetric functions. The Schur functions $s_\lambda$, indexed by arbitrary partitions $\lambda \in \mathcal{Y}$, form a distinguished homogeneous basis in $\Lambda$. As a graded algebra, $\Lambda$ is isomorphic to the algebra of polynomials in countably many generators; as these generators, one can take, for instance, the complete homogeneous symmetric functions $h_1, h_2, \ldots$ where $\deg h_k = k$. One has $s_\lambda = \det[h_{\lambda_i-i+j}]$ with the understanding that $h_0 = 1$ and $h_k = 0$ for $k < 0$ (the Jacobi-Trudi formula); here the order of the determinant can be chosen arbitrarily provided it is large enough. For more detail, see, e.g., [Sag01].

Given two multiplicative functionals $\varphi, \psi: \Lambda \to \mathbb{C}$, the corresponding (complex-valued) Schur measure $M_{\varphi,\psi}$ on $\mathcal{Y}$ is defined by

$$M_{\varphi,\psi}(\lambda) = \text{const}^{-1} \varphi(s_\lambda)\psi(s_\lambda), \quad \lambda \in \mathcal{Y}, \quad \text{const} = \sum_{\lambda \in \mathcal{Y}} \varphi(s_\lambda)\psi(s_\lambda),$$

provided that the sum is absolutely convergent (which is a necessary condition on $\varphi, \psi$). This notion, due to Okounkov [Oko01a], provides a broad generalization of the mixed z-measures. Since a multiplicative functional is uniquely determined by its values on the generators $h_k$, the Schur measure has a doubly-infinite collection of
parameters \( \{ \varphi(h_k), \psi(h_k); k = 1, 2, \ldots \} \). In this picture, the z-measures correspond to a very special collection of parameters

\[
\varphi(h_k) = \xi^{k/2}(z)_k/k!, \quad \psi(h_k) = \xi^{k/2}(z')_k/k!, \quad k = 1, 2, \ldots,
\]

and the poissonized Plancherel measure \( M_\nu \) appears when \( \varphi(h_k) = \psi(h_k) = \nu^{k/2}/k! \).

As shown in [Oko01a], Theorem 4.1 extends to Schur measures: if the parameters are such that the measure \( M_{\varphi,\psi} \) is nonnegative (and hence is a probability measure), then it gives rise to a lattice determinantal point process. Moreover, for the corresponding correlation kernel one can write down an explicit contour integral representation [Bor00c]. Such a representation is well suited for asymptotic analysis.

If \( \varphi \) and \( \psi \) are evaluations of symmetric functions at finitely many positive variables, the first row \( \lambda_1 \) can be interpreted as the last passage percolation time in a suitable directed percolation model on the plane, see [Joh05].

4.4. Some generalizations. Kerov [Ker00] generalized the construction of the z-measures \( M_{z,z'}^{(n)} \) by introducing an additional parameter related to Jack polynomials. This new parameter is similar to the \( \beta \) parameter in random matrix ensembles [For10b]. In particular, the Plancherel measure \( M^{(n)} \), which is a limit case of the z-measures, also allows a \( \beta \)-deformation [Ker00], [Oko05], [Oko06]. The ordinary z-measures correspond to the special value \( \beta = 2 \), and in the limit \( \beta \to 0 \) the beta z-measures degenerate to the measures (2.5) derived from the Ewens measures, see [Ols10, Section 1.2]. Thus, the \( \beta \) parameter interpolates between the models of Section 2 and those of Sections 3-4, as has been pointed out in the end of Section 1.

Note also that replacing the Schur functions by the Jack symmetric functions leads to a natural \( \beta \)-deformation of the Schur measures.

As in random matrix theory, the value \( \beta = 2 \) is a distinguished one, while in the general case \( \beta > 0 \) the situation is much more complex. Some results for \( \beta \neq 2 \) can be found in [Bor05b], [Ful04], [Ols10], [Str10a], [Str10b].

In a somewhat different direction, one can define natural analogues of the Plancherel measure and Schur measures for shifted Young diagrams (equivalently, strict partitions): [Tra04], [Mat05]. This theory is related to Schur’s Q-functions (a special case of Hall–Littlewood symmetric functions that appears in the theory of projective representations of the symmetric group). Surprisingly enough, a natural analogue of the z-measures for shifted diagrams, discovered by Borodin and recently studied in [Pet10], seems to be not related to Schur’s Q-functions.

Finally, note that there are many points of contact between the results described in this chapter and Fulman’s work on “random matrix theory over finite fields”, see his survey [Ful01] and references therein.
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Institute for Information Transmission Problems, Bolshoy Karetny 19, Moscow 127994, Russia;
Independent University of Moscow, Russia

E-mail address: ols2007@gmail.com