THE NORM OF SUMS OF INDEPENDENT
NONCOMMUTATIVE RANDOM VARIABLES IN $L_p(\ell_1)$

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Abstract. We investigate the norm of sums of independent vector-valued random variables in noncommutative $L_p$ spaces. This allows us to obtain a uniform family of complete embeddings of the Schatten class $S^p_m$ in $S_p(\ell^m_q)$ with optimal order $m \sim n^2$. Using these embeddings we show the surprising fact that the sharp type (cotype) index in the sense of operator spaces for $L_p[0,1]$ is $\min(p, p') (\max(p, p'))$. Similar techniques are used to show that the operator space notions of B-convexity and K-convexity are equivalent.

Introduction

Sums of independent random variables have a long tradition both in probability theory and Banach space geometry. More recently, the noncommutative analogs of these probabilistic results have been developed [9, 11, 25] and applied to operator space theory [8, 10, 24]. In this paper, we follow this line of research in studying type and cotype in the sense of operator spaces [19]. This theory is closely connected to the notions of B-convexity and K-convexity. Using embedding results we show that these notions remain equivalent in the category of operator spaces.

We recall from [16] that a Banach space $X$ is called K-convex whenever the Gauss projection

$$P_G : f \in L_2(\Omega; X) \mapsto \sum_{k=1}^{\infty} \left( \int_{\Omega} f(\omega) g_k(\omega) \, d\mu(\omega) \right) g_k \in L_2(\Omega; X)$$

is bounded. Here $g_1, g_2, \ldots$ are independent standard complex-valued Gaussian random variables defined over a probability space $(\Omega, A, \mu)$. An operator space $X$ is called OK-convex if $P_G$ is completely bounded or equivalently $S^2(X)$ is K-convex as a Banach space. Using standard tools from Banach space theory, we know that $S^2(X)$ is K-convex if and only if $S^p_p(X)$ is K-convex for some (any) $1 < p < \infty$.

Therefore this notion does not depend on the parameter $p$. According to a deep theorem of Pisier [21] K-convexity is equivalent to B-convexity. Following Beck [1], a Banach space $X$ is called B-convex if there exists $n \geq 1$ and $0 < \delta \leq 1$ such that

$$\frac{1}{n} \inf_{|\alpha_k|=1} \left\| \sum_{k=1}^{n} \alpha_k x_k \right\| \leq (1 - \delta) \max_{1 \leq k \leq n} \|x_k\|$$

holds for any family $x_1, x_2, \ldots, x_n$ of vectors in $X$. Giesy proved in [5] that a Banach space $X$ is B-convex if and only if $X$ does not contain $\ell^m_n$'s uniformly. In the context

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of operator spaces, the noncommutative analogue of $\ell_1^n$ is the Schatten class $S_1^n$, the dual of $B(\ell_2^n)$. More generally, one might consider arbitrary dual spaces

$$L_1(A) = S_1^{n_1} \oplus S_1^{n_2} \oplus \cdots \oplus S_1^{n_m}$$

of finite dimensional C*-algebras. A priori, it is unclear which analogue of the notion of B-convexity is the right one for operator spaces. Namely, we could only exclude the $\ell_1^n$’s or all the $L_1(A)$’s. The following result clarifies this question.

**Theorem 1.** Let $X$ be an operator space and let $(A_n)$ be a sequence of pairwise different finite dimensional C*-algebras. The following are equivalent:

i) $X$ is OK-convex.

ii) $S_p(X)$ does not contain $\ell_1^n$’s uniformly for some (any) $1 < p < \infty$.

iii) $S_p(X)$ does not contain $L_1(A_n)$’s uniformly for some (any) $1 < p < \infty$.

In fact, we prove a stronger result. We shall say that the spaces $L_1(A_n)$’s embed semi-completely uniformly in $S_p(X)$ when there exists a family of embeddings

$$\Lambda_n : L_1(A_n) \to S_p(X)$$

satisfying $\|\Lambda_n\|_{cb} \|\Lambda_n^{-1}\| \leq c$ for some universal constant $c > 1$. This notion came out naturally in the paper [18]. We refer to [17] for further applications of this concept. For the equivalence of ii) and iii), we prove that if $\ell_1^n$ embeds into $S_p(X)$ with constant $c_n$, then there is a map $u : S_1^n \to S_p(X)$ such that

$$\|u : S_1^n \to S_p(X)\|_{cb}\|u^{-1} : u(S_1^n) \to S_1^n\| \leq C c_n.$$

This map is constructed using (noncommutative) probabilistic tools. Usually, estimates for sums of noncommutative random variables are motivated by classical probabilistic inequalities. Our probabilistic motivation here is given by the following result. Let us consider a finite collection $f_1, f_2, \ldots, f_n$ of independent random variables on a probability space $(\Omega, A, \mu)$. Then, given $1 \leq p < \infty$, the following equivalence of norms holds

$$\langle \Sigma_p \rangle \left( \int_{\Omega} \left[ \sum_{k=1}^{n} |f_k(\omega)| r \right]^p d\mu(\omega) \right)^{1/p} \sim \max_{r \in [1, p]} \left\{ \left( \sum_{k=1}^{n} \int_{\Omega} |f_k(\omega)| r d\mu(\omega) \right)^{1/r} \right\}.$$

We shall provide in this paper the natural analog of $\langle \Sigma_p \rangle$ for noncommutative random variables. This result requires the use of the so-called asymmetric $L_p$ spaces, which will be defined below. Now, going back to the construction of the map $u : S_1^n \to S_p(X)$, we consider positive integers $m, n \geq 1$. Then, if $1 \leq k \leq m$ and $\tau$ stands for the normalized trace, let $\pi_k : L_p(\tau_n) \to L_p(\tau_{n^m})$ be the mapping defined by the relation $\pi_k(x) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$, where $x$ is located at the $k$-th position and 1 stands for the identity of $M_n$. Then, the embedding $u$ can be easily constructed by combining condition ii) with the following result, which might be of independent interest.

**Theorem 2.** Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Then, given $n \geq 1$ and $m \geq n^2$, the following map is a complete isomorphism onto a completely complemented subspace

$$x \in S_q^n \mapsto \frac{1}{n^{1/q}} \sum_{k=1}^{m} \delta_k \otimes \pi_k(x) \in L_p(\tau_{n^m}; \ell_q^m).$$

Moreover, the $cb$-distance constants are uniformly bounded on the dimensions.
Our proof requires \( m \geq n^2 \) which is different from the well-known commutative order \( m \sim n \). Using type/cotype estimates we show that the order \( m \sim n^2 \) is best possible. The cotype for operator spaces is motivated by the Hausdorff-Young inequality for non-abelian compact groups. Let \( G \) be a noncommutative compact group and \( \hat{G} \) its dual object. That is, a list of inequivalent irreducible unitary representations. Given \( 1 \leq p \leq 2 \), an operator space \( X \) has Fourier type \( p \) with respect to \( G \) if the \( X \)-valued Hausdorff-Young inequality

\[
\left( \sum_{k=1}^{n} d_k \|A_k\|_{s_{p'}}^{p' \prime} (X) \right)^{1/p'} \leq cb \mathcal{K}_p (X, \hat{G}) \left( \int_{G} \left\| \sum_{k=1}^{n} d_k \tr (A_k \pi_k (g)) \right\|_X^p d \mu (g) \right)^{1/p}
\]

holds for all finite sequence of matrices \( A_1, A_2, \ldots, A_n \) with \( A_k \in M_{d_{\pi_k}} \otimes X \). Here \( \mu \) is the normalized Haar measure and \( d_k \) denotes the degree of the irreducible representation \( \pi_k : G \to U (d_k) \). Moreover, here and in the following, the symbol \( \leq cb \) is used to indicate the corresponding linear map is indeed completely bounded. The notion of Fourier cotype is dual to the notion of Fourier type stated above. Following \[22\], we notice that this inequality forces us to consider an operator space structure on the vector space where we are taking values. In other words, we need to take values in operator spaces rather than Banach spaces.

Note that the span of the functions of the form \( \tr (A_k \pi_k (g)) \) is dense in \( L_2 (G) \). In the classical notion of cotype the right hand side is replaced by a suitable subset of characters. Since it is not entirely clear which will be such a canonical subset for arbitrary groups, we follow the approach of Marcus/Pisier \[15\] and consider random Fourier series of the form

\[
\sum_{k=1}^{n} d_k \tr (A_k \pi_k (g) U_{\pi_k})
\]

where the \( U_{\pi}'s \) are random unitaries. However, the contraction principle allows us to eliminate the coefficients \( \pi(g) \). Therefore, given such a family of random unitaries over a probability space \( (\Omega, A, \mu) \), a possible notion of cotype for operator spaces is given by the inequality

\[
\left( \sum_{k=1}^{n} d_k \|A_k\|_{s_{p'}}^{p' \prime} (X) \right)^{1/p'} \leq cb \mathcal{K}_p (X, \hat{G}) \left( \int_{\Omega} \left\| \sum_{k=1}^{n} d_k \tr (A_k U_{\pi_k} (\omega)) \right\|_X^p d \mu (\omega) \right)^{1/p}.
\]

Although this definition originated from compact groups, in this formulation only the degrees of the representations of the dual object and their multiplicity are kept. We may therefore consider this notion of cotype for arbitrary collections of random unitaries \( (U_\sigma) \) indexed by \( \sigma \in \Sigma \) and where \( d_\sigma \) represents the dimension of \( U_\sigma \). Examples for \( \Sigma \)'s coming from groups are the commutative set of parameters \( (\Sigma_0 = N \text{ and } d_k = 1 \text{ for all } k \geq 1) \) which arises from any non-finite abelian compact group and the set \( \Sigma_1 = N \) with \( d_k = k \) for \( k \geq 1 \), which comes from the classical Lie group \( SU(2) \). As we shall see in this paper, these two sets of parameters are the most relevant ones in the theory. Let us mention that random unitaries can also be understood as a higher dimensional version of random signs or independent Steinhaus variables. It is rather surprising that in disproving cotype \( q \) larger matrices are not necessarily easier. In part because the sequence \((d_\sigma)_{\sigma \in \Sigma}\) provides a new normalization. Let us also note that Khintchine-Kahane inequalities are not available in the operator space setting because they even fail in the level.
of scalars $[13, 14, 22]$. Indeed, type (see Section 4 for a definition) and cotype turn out to be closer notions to Fourier type and cotype than in the classical theory.

**Theorem 3.** Any infinite dimensional $L_p$ space has:

i) Sharp $\Sigma$-type $\min(p, p')$.

ii) Sharp $\Sigma$-cotype $\max(p, p')$.

The organization of the paper is as follows. Section 1 is devoted to describe the operator space structure and some basic properties of the asymmetric $L_p$ spaces. These spaces provide an important tool in this paper. Section 2 contains some preliminary estimates that will be used in Section 3 to prove the analog of $\Sigma^p$ for noncommutative random variables. In Section 4 we construct the embedding of $S^p_n$ into $S^p_{(q^m)}$ described in Theorem 2. Section 5 is devoted to prove the operator space version of Pisier’s characterization of $K$-convexity. Finally, in Section 6 we find the sharp operator space type and cotype indices of $L_p$ spaces.

1. **Asymmetric $L_p$ spaces**

Throughout this paper, some basic notions of noncommutative $L_p$ spaces and operator space theory will be assumed, see $[22, 23]$ for a systematic treatment. We begin by studying some basic properties of the asymmetric $L_p$ spaces, defined as follows. Let $E$ be an operator space and let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a n.s.f. trace $\varphi$. Given a pair of exponents $2 \leq r, s \leq \infty$ such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$, we define the asymmetric $L_p$ space $L_{(r,s)}(\mathcal{M}, \varphi; E)$ as the completion of $L_p(\mathcal{M}, \varphi) \otimes E$ with respect to the following norm

$$||x||_{L_{(r,s)}(\mathcal{M}, \varphi; E)} = \inf_{x = \alpha y \beta} \left\{ ||\alpha||_{L_r(\mathcal{M}, \varphi)} ||y||_{L_{\infty}(\mathcal{M}, \varphi; E)} ||\beta||_{L_s(\mathcal{M}, \varphi)} \right\},$$

where the infimum runs over all decompositions $x = \alpha y \beta$ with $\alpha \in L_r(\mathcal{M}, \varphi)$, $\beta \in L_s(\mathcal{M}, \varphi)$ and $y \in \mathcal{M} \otimes_{\min} E$. Recall that any noncommutative $L_p$ space can be realized as $L_p(\mathcal{M}, \varphi; E) = L_{(2p,2p)}(\mathcal{M}, \varphi; E)$. In this paper, the von Neumann algebra $\mathcal{M}$ will always be a finite matrix algebra $M_n$ so that the trace $\varphi$ is unique up to a constant factor. In fact, we shall only work with the usual trace $\text{tr}_n$ of $M_n$ and its normalization $\tau_n = \frac{1}{n} \text{tr}_n$. The spaces

$$S^n_{(r,s)}(E) = L_{(r,s)}(\text{tr}_n; E)$$

can be regarded as the asymmetric version of Pisier’s vector-valued Schatten classes. If $R$ and $C$ stand for the row and column operator Hilbert spaces, we shall denote by $C_p$ and $R_p$ the interpolation spaces $[C,R]_{1/p}$ and $[R,C]_{1/p}$. The superscript $n$ will indicate the $n$-dimensional version. By using elementary properties of the Haagerup tensor product, it is not difficult to check that

$$S^n_{(r,s)}(E) = C^m_{r/2} \otimes_h E \otimes_h R^m_{s/2}$$

isometrically. This provides a natural operator space structure for $L_{(r,s)}(\tau_n; E)$.

**Lemma 1.1.** Given $1 \leq r, s, t \leq \infty$ such that $\frac{1}{s} = \frac{1}{r} + \frac{1}{t}$, we have

$$||\alpha||_{CB(C^2_p, C^2_t)} = ||\alpha||_{S^2_n},$$

$$||\beta||_{CB(R^2_p, R^2_t)} = ||\beta||_{S^2_n}.$$
Since the Haagerup tensor product commutes with complex interpolation, we obtain

\[ CB(C^n_s, C^n_s) = R^n_s \otimes_h C^n_s = C^n_s \otimes_h R^n_s. \]

The asymmetric Schatten classes satisfy the following properties:

\begin{equation}
\alpha, \beta \in \mathcal{M}_n \quad \Rightarrow \quad \| \alpha \|_{S^p_{\{r,s\}}(E)} \leq \| \alpha \|_{S^p_{\{r,s\}}(E)} \| \beta \|_{S^p_{\{r,s\}}(E)}.
\end{equation}

Proof. Since the row case can be treated similarly, we just prove the first equality. When \( r = s \) the result is trivial. Assume now that \( r = \infty \). We begin by recalling the following well-known complete isometries

\[ CB(C^n_s, C^n_s) = R^n_s \otimes \min C^n_s = C^n_s \otimes h R^n_s. \]

Since the Haagerup tensor product commutes with complex interpolation, we obtain

\[ C^n_s \otimes h R^n_s = [C^n_s \otimes h R^n_s, R^n_s \otimes h R^n_s]^{1/s} = [S^n_s, S^n_s]^{1/s} = S^n_{2^n}. \]

Since \( s = t \) when \( r = \infty \), the identity holds. Now we take \( s < r < \infty \). In that case, we use the fact that the complex interpolation space \( S_{2^n} = [S_{2^n}, s/r \otimes h \leq \infty \) is contractively included in \( CB(C^n_s, C^n_s) \). This gives \( \| \alpha \|_{CB(C^n, C^n)} \leq \| \alpha \|_{S^n_{2^n}} \). For the lower estimate, we consider the bilinear form

\[ CB(C^n_s, C^n_s) \times CB(C^n_s, C^n_s) \rightarrow CB(C^n_s, C^n_s), \]

defined by \( (\alpha, \beta) \mapsto \alpha \circ \beta \). Then, recalling that the Banach space \( CB(C^n_s, C^n_s) \) is isometrically isomorphic to the Schatten class \( S^{2p}_{2^n} \) (see above), we obtain the following inequality

\[ \| \alpha \|_{S^{2p}_{2^n}} \leq \| \alpha \|_{CB(C^n_s, C^n_s)} \| \beta \|_{S^{2p}_{2^n}}. \]

Taking the supremum over \( \beta \in M_n \), we obtain \( \| \alpha \|_{CB(C^n_s, C^n_s)} \geq \| \alpha \|_{S^{2p}_{2^n}} \).

In the following lemma, we state some basic properties of the asymmetric \( L_p \) spaces which naturally generalize some Pisier’s results in Chapter 1 of [22].

Lemma 1.2. The asymmetric Schatten classes satisfy the following properties:

i) Given \( 2 \leq p, q, r, s, u, v \leq \infty \) such that \( \frac{1}{p} = \frac{1}{r} + \frac{1}{s} \) and \( \frac{1}{q} = \frac{1}{u} + \frac{1}{v} \), we have
\[
\| \alpha \beta \|_{S_{\{r,s\}}(E)} \leq \| \alpha \|_{S_{\{u,v\}}(E)} \| \beta \|_{S_{\{u,v\}}(E)}.
\]

ii) Given \( x \in M_n \otimes E \) and \( 2 \leq p, q \leq \infty \), we have
\[
\| x \|_{M_n(E)} = \sup \left\{ \| \alpha \beta \|_{S_{\{p,q\}}(E)} : \| \alpha \|_{S_{\{p,q\}}(E)} \leq 1 \right\}.
\]

Therefore, any linear map \( u : E \rightarrow F \) between operator spaces satisfies
\[
\| u \|_{cb} = \sup_{n \geq 1} \left\| \text{id} \otimes u : S_{\{p,q\}}(E) \rightarrow S_{\{p,q\}}(F) \right\|.
\]

iii) Any block-diagonal matrix \( D_n(x) \in M_{nn} \otimes E \), with blocks \( x_1, x_2, \ldots, x_n \) in \( M_n \otimes E \), satisfies the following identity
\[
\| D_n(x) \|_{S_{\{r,s\}}(E)} = \left( \sum_{k=1}^{n} \| x_k \|_{S_{\{r,s\}}(E)}^p \right)^{1/p}, \quad \text{where} \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{s}.
\]

Proof. Let us define
\[
\alpha \otimes_h \text{id}_E \otimes_h \beta^t : C^n_{r/2} \otimes_h E \otimes_h R^n_{s/2} \rightarrow C^n_{p/2} \otimes_h E \otimes_h R^n_{q/2}
\]
to be the mapping \( x \mapsto \alpha x \beta \). Then the inequality stated in (i) follows by Lemma [24] and the injectivity of the Haagerup tensor product. Let us prove the first identity in (ii). We point out that
\[
\| x \|_{M_n(E)} \geq \sup \left\{ \| \alpha x \beta \|_{S_{\{p,q\}}(E)} : \| \alpha \|_{S_{\{p,q\}}(E)} \leq 1 \right\}.\]
follows immediately from (i). On the other hand, following Lemma 1.7 of [22], we can write \( \|x\|_{M_n(E)} = \|\alpha x\beta_0\|_{S_p^n(E)} \) for some \( \alpha_0, \beta_0 \) in the unit ball of \( S_p^n \). Then, we consider decompositions \( \alpha_0 = \alpha_1 \alpha \) and \( \beta_0 = \beta_1 \beta \) so that
\[
\|\alpha_1\|_p = \|\alpha\|_p = 1 = \|\beta_1\|_q = \|\beta\|_q,
\]
with \( \frac{1}{p} + \frac{1}{r} = \frac{1}{q} = \frac{1}{s} + \frac{1}{t} \). Applying (i) one more time, we obtain
\[
\|x\|_{M_n(E)} \leq \sup \left\{ \|\alpha x\beta\|_{S_p^n(E)} : \|\alpha\|_{S_p^n}, \|\beta\|_{S_p^n} \leq 1 \right\}.
\]
The second identity in (ii) is immediate. Finally, we prove (iii). Let \( E_k \) be the subspace of \( E \) spanned by the entries of \( x_k \). Since \( E_k \) is finite dimensional, we can find \( \alpha_k, \beta_k \in M_m \) and \( y_k \in M_m \otimes E_k \) satisfying \( x_k = \alpha_k y_k \beta_k \) and
\[
\|x_k\|_{S_{(r,s)}^m(E)} = \|\alpha_k\|_{S_p^m} \|\beta_k\|_{S_p^m}.
\]
By homogeneity, we may assume that
\[
\sum_{k=1}^n \|x_k\|_{S_{(r,s)}^m(E)}^p = \sum_{k=1}^n \|\alpha_k\|_{S_p^m}^p = \sum_{k=1}^n \|\beta_k\|_{S_p^m}^p.
\]
Let us consider the block-diagonal matrices \( D_n(\alpha), D_n(y) \) and \( D_n(\beta) \), made up with blocks \( \alpha_k, y_k \) and \( \beta_k \) \((1 \leq k \leq n)\) respectively. The upper estimate follows by considering the decomposition \( D_n(x) = D_n(\alpha)D_n(y)D_n(\beta) \). In a similar way, taking \( \rho = 2r/(r-2) \) and \( \sigma = 2s/(s-2) \), the upper estimate also holds for the dual space \( S_{(r,s)}^m(E)^* = S_{(\rho,\sigma)}^m(E^*) \). Therefore, the lower estimate follows by duality. \( \square \)

2. Preliminary estimates

In this section we prove the main probabilistic estimates to study the norm of sums of independent noncommutative random variables. First a word of notation. Throughout this paper, \( e_{ij} \), and \( \delta_k \) will denote the generic elements of the canonical basis of \( M_n \) and \( \mathbb{C}^n \) respectively.

**Lemma 2.1.** Let \( E \) be an operator space and let \( D : S_1^n(\ell_\infty^n(E)) \rightarrow \ell_1^n(E) \) be the mapping defined by
\[
D\left( \sum_{i,j=1}^n e_{ij} \otimes x_{ij} \right) = \sum_{k=1}^n \delta_k \otimes x_{kk},
\]
where \( x_{ij} \) stands for the \( k \)-th entry of \( x_{ij} \). Then \( D \) is a complete contraction.

**Proof.** Let us consider the map \( d : \ell_\infty^n \rightarrow M_n \), defined by
\[
d\left( \sum_{i,j=1}^n \lambda_{ij} (\delta_i \otimes \delta_j) \right) = \sum_{k=1}^n \lambda_{kk} e_{kk}.
\]
Since the diagonal projection of \( \ell_\infty^n \) onto \( \ell_\infty^n \) is a complete contraction, we can use the completely isometric embedding of \( \ell_\infty^n \) into the subspace of diagonal matrices of \( M_n \) to deduce that \( d \) is a complete contraction. If \( \mathcal{H} \) stands for a Hilbert space such that \( E \) embeds in \( \mathcal{B}(\mathcal{H}) \) completely isometrically, then we consider the mapping
\[
w : \mathcal{C}B(\mathcal{B}(\mathcal{H}), \ell_\infty^n) \rightarrow \mathcal{C}B(\ell_\infty^n(\mathcal{B}(\mathcal{H})), \ell_\infty^n(\ell_\infty^n))
\]
defined by \( w(T) = id_{\ell_\infty^n} \otimes T \). Clearly, \( w \) is a complete contraction. Therefore the linear map \( v : \mathcal{C}B(\mathcal{B}(\mathcal{H}), \ell_\infty^n) \rightarrow \mathcal{C}B(\ell_\infty^n(\mathcal{B}(\mathcal{H})), M_n) \), given by \( v(T) = d \circ w(T) \), is a
complete contraction. Let $S^1_H$ denote the predual of $B(\mathcal{H})$. Recalling the completely isometric embeddings
\[
\ell_\infty^n(S^1_H) \rightarrow CB(B(\mathcal{H}), \ell_\infty^n),
\]
\[
M_n(\ell_\infty^n(S^1_H)) \rightarrow CB(\ell_\infty^n(B(\mathcal{H})), M_n),
\]
we deduce that $u : \ell_\infty^n(S^1_H) \rightarrow M_n(\ell_\infty^n(S^1_H))$, defined by the relation
\[
u\left(\sum_{k=1}^n \delta_k \otimes a_k\right) = \sum_{k=1}^n \epsilon_{kk} \otimes (\delta_k \otimes a_k),
\]
is a complete contraction. Namely, given $a \in \ell_\infty^n(S^1_H)$ let
\[
T_a(x) = \sum_{k=1}^n \mathrm{tr}(a_k x) \delta_k \in CB(B(\mathcal{H}), \ell_\infty^n).
\]
Then, it can be easily checked that $u(a) = v(T_a)$. Hence, $u$ can be regarded as the restriction of $v$ to $\ell_\infty^n(S^1_H)$. This proves that $u$ is a complete contraction. On the other hand, the original map $D$ is now given by the restriction of the adjoint map $u^* : S^0_\infty(\ell_\infty^n(B(\mathcal{H}))) \rightarrow \ell_\infty^n(B(\mathcal{H}))$ to the subspace $S^0_\infty(\ell_\infty^n(E))$.

In the following result we shall need the following description of the Haagerup tensor norm. Given an operator space $E$, we denote by $M_{p,q}(E)$ the space of $p \times q$ matrices with entries in $E$. The norm in $M_{p,q}(E)$ is given by embedding it into the upper left corner of $S_\infty^n(E)$ with $n = \max(p,q)$. Now, for any pair $E_1, E_2$ of operator spaces, let $x_1 \in M_{p,m}(E_1)$ and $x_2 \in M_{m,q}(E_2)$. We will denote by $x_1 \otimes x_2$ the matrix $x$ in $M_{p,q}(E_1 \otimes E_2)$ defined by
\[
x(i,j) = \sum_{k=1}^m x_1(i,k) \otimes x_2(k,j).
\]
Then, given a family $E_1, E_2, \ldots, E_n$ of operator spaces and given
\[
x \in M_m \otimes (E_1 \otimes E_2 \otimes \cdots \otimes E_n),
\]
we define the norm of $x$ in the space $S_\infty^n(E_1 \otimes_h E_2 \otimes_h \cdots \otimes_h E_n)$ as follows
\[
\|x\|_m = \inf \left\{ \prod_{k=1}^{n+1} \|x_k\|_{M_{p_k,p_{k+1}}(E_k)} \mid p_1 = m = p_{n+1} \right\},
\]
where the infimum runs over all possible decompositions
\[
x = x_1 \otimes x_2 \otimes \cdots \otimes x_n \quad \text{with} \quad x_k \in M_{p_k,p_{k+1}}(E_k).
\]

Lemma 2.2. Let $E$ be an operator space and $x_1, x_2, \ldots, x_n \in M_m \otimes E$. If there are elements $a_k(r), b_k(s) \in M_m$ and $y_k(r,s) \in M_m \otimes E$ with $1 \leq r \leq \rho$ and $1 \leq s \leq \sigma$ such that
\[
x_k = \sum_{r,s} a_k(r)y_k(r,s)b_k(s)
\]
holds for $1 \leq k \leq n$, then we have
\[
\left\| \sum_{k=1}^n \delta_k \otimes x_k \right\|_{L_p(\tau_n; \ell_1^\infty(E))} \leq \left\| \sum_{k,r} a_k(r)a_k(r)^* \right\|_p^{1/2} \left( \sup_k \left\| y_k(r,s) \right\|_\infty \right) \left\| \sum_{k,s} b_k(s)^*b_k(s) \right\|_p^{1/2}.
\]
Proof. Let us consider the positive matrices \( a, b \in M_m \) defined by
\[
a = \left( \sum_{k,r} a_k(r) a_k(r)^* \right)^{1/2} \quad \text{and} \quad b = \left( \sum_{k,s} b_k(s)^* b_k(s) \right)^{1/2}.
\]
Then, we can find matrices \( a_k(r) \) and \( b_k(s) \) in \( M_m \) satisfying
\[
a_k(r) = a \alpha_k(r), \quad b_k(s) = \beta_k(s) b.
\]
and such that
\[
\left\| \sum_{k,r} a_k(r) \alpha_k(r)^* \right\|_\infty \leq 1,
\]
\[
\left\| \sum_{k,s} \beta_k(s)^* \beta_k(s) \right\|_\infty \leq 1.
\]
Let us define, for \( 1 \leq k \leq n \), the matrices
\[
z_k = \sum_{r,s} \alpha_k(r) y_k(r,s) \beta_k(s).
\]
Then, by the definition of \( L_p(\tau_m; \ell^n_1(E)) \), we have
\[
\left\| \sum_{k=1}^n \delta_k \otimes x_k \right\|_p \leq \left\| \sum_{k,r} a_k(r) a_k(r)^* \right\|_p \left\| \sum_{k=1}^n \delta_k \otimes z_k \right\| \left\| \sum_{k,s} b_k(s)^* b_k(s) \right\|_p^{1/2},
\]
where \( \left\| \cdot \right\|_\infty \) denotes here the norm on \( S^n_m(\ell^n_1(E)) \). Therefore, it suffices to estimate the middle term on the right. To that aim, we consider the matrices
\[
\alpha = (\cdots, \alpha_k(r) \otimes e_{1k}, \cdots) \in M_m, mnp(R^n_\infty),
\]
\[
\beta = (\cdots, \beta_k(s) \otimes e_{k1}, \cdots)^t \in M_{mn, m}(C^n_\infty).
\]
Moreover, if \( y_k = (y_k(r,s)) \in M_{m,p, m\sigma} \otimes E \), we also consider the matrix
\[
y = \sum_{k=1}^n e_{kk} \otimes (\delta_k \otimes y_k) \in M_{mnp, m\sigma}(\ell^n_1(E)).
\]
Finally, we notice that
\[
\sum_{k=1}^n \delta_k \otimes z_k = (id \otimes D)(\alpha \otimes y \otimes \beta).
\]
In particular, Lemma 2.1 gives
\[
\left\| \sum_{k=1}^n \delta_k \otimes z_k \right\|_\infty \leq \left\| \sum_{k,r} a_k(r) \alpha_k(r)^* \right\|_\infty^{1/2} \sup_k \left\| \left( y_k(r,s) \right) \right\|_\infty \left\| \sum_{k,s} \beta_k(s)^* \beta_k(s) \right\|_\infty^{1/2}.
\]
This yields the assertion since the first and third terms on the right are \( \leq 1 \). \( \square \)

Let us consider two positive integers \( l \) and \( n \). Then, given \( 1 \leq p \leq \infty \), we define \( \pi_k : L_p(\tau) \to L_p(\tau^{n_1}) \) for each \( 1 \leq k \leq n \) to be the mapping defined by the relation
\[
\pi_k(x) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1,
\]
where \( x \) is located at the \( k \)-th position and 1 stands for the identity of \( M_l \). These operators will appear quite frequently throughout this paper. In the following
lemma, we give an upper estimate of the $L_p$ norm of certain sums of positive matrices constructed by means of the mappings $\pi_k$.

**Lemma 2.3.** Let $a^1, a^2, \ldots, a^n \in M_m$ be a collection of positive matrices and let $1 \leq p < \infty$. Then, we have

$$\left\| \sum_{k=1}^{n} \pi_k(a^k) \right\|_{L_p(\tau_{m^n})} \leq c \max \left\{ \left( \sum_{k=1}^{n} \sum_{i=1}^{l} a^k_{ii} \right)^{1/p} \right\} \left( \sum_{k=1}^{n} \left\| a^k \right\|_{L_p(\tau_{m^n})}^{1/p} \right).$$

Here $c$ denotes an absolute constant not depending on the dimensions.

**Proof.** By homogeneity, we may assume that the maximum on the right is 1. Let $\mathcal{E}$ be the conditional expectation onto $L_p(\tau_m)$, regarded as a subspace of $L_p(\tau_{m^n})$.

Let $b_k$ stand for $\pi_k(a^k)$, by the triangle inequality

$$\left\| \sum_{k=1}^{n} \pi_k(a^k) \right\|_p \leq \sum_{k=1}^{n} \mathcal{E}(b_k) \left\| L_p(\tau_m) \right\| + \left\| b_k - \mathcal{E}(b_k) \right\|_p = A + B.$$

Let us observe that

$$\mathcal{E}(b_k) = \tau_l(a^k) = \frac{1}{l} \sum_{i=1}^{l} a^k_{ii}.$$

Therefore, by assumption, the term $A$ may be estimated by 1. To estimate $B$, we first assume that $p > 2$. Then, applying the Burkholder inequality given in [11] for noncommutative martingales, we obtain

$$B \leq c \max \left\{ \left( \sum_{k=1}^{n} \mathcal{E}(b_k^2) - \mathcal{E}(b_k)^2 \right)^{1/2} \right\} \left( \sum_{k=1}^{n} \left\| b_k - \mathcal{E}(b_k) \right\|_p^{1/p} \right).$$

On the other hand, since $q = p/2 > 1$, we invoke Lemma 5.2 of [11] to obtain

$$\left\| \sum_{k=1}^{n} \mathcal{E}(b_k^2) \right\|_{L_{p/2}(\tau_m)} \leq \left( \sum_{k=1}^{n} \mathcal{E}(b_k) \right)^{2(q-1)/2q} \left( \sum_{k=1}^{n} \left\| b_k \right\|_{L_{2q}(\tau_m)} \right)^{1/2}$$

The first factor on the right is a power of $A$. By assumption, the second factor may be estimated by 1. When $1 \leq p \leq 2$, we proceed in a different way. Given a subset $\Gamma$ of $\{1, 2, \ldots, n\}$ with cardinal $|\Gamma|$, let us consider the conditional expectation $\mathcal{E}_\Gamma$ onto $L_p(\tau_{m^{\{\Gamma\}}})$ given by

$$\mathcal{E}_\Gamma \left( z_0 \otimes \bigotimes_{k \in \Gamma} z_k \right) = \prod_{k \notin \Gamma} \tau_l(z_k) \left( z_0 \otimes \bigotimes_{k \in \Gamma} z_k \right),$$

with $z_0 \in M_m$ and $z_k \in M_l$ for $1 \leq k \leq n$. Therefore, since $x_k = b_k - \mathcal{E}(b_k)$ are independent mean 0 random variables, we can estimate $B$ as follows. For any family of signs $\varepsilon_k = \pm 1$ with $1 \leq k \leq n$, let $\Gamma = \{ k : \varepsilon_k = 1 \}$. Then we have

$$\left\| \sum_{k=1}^{n} x_k \right\|_p \leq \left\| \mathcal{E}_\Gamma \left( \sum_{k=1}^{n} \varepsilon_k x_k \right) \right\|_p + \left\| \mathcal{E}_{\Gamma^c} \left( \sum_{k=1}^{n} \varepsilon_k x_k \right) \right\|_p \leq 2 \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_p.$$
Then, if we write $r_1, r_2, \ldots$ for the classical Rademacher variables, we use the fact that $L_p(\tau_m)$ has Rademacher type $p$ to obtain

$$B = \left\| \sum_{k=1}^{n} x_k \right\|_p \leq 2 \left( \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) x_k \right\|_p^2 dt \right)^{1/2} \leq 2 \left( \sum_{k=1}^{n} \| x_k \|_p^p \right)^{1/p} \leq 4 \left( \sum_{k=1}^{n} \| e_k^p \|_{L_p(\tau_m)}^p \right)^{1/p}.$$ 

This yields the assertion for $1 \leq p \leq 2$. Therefore, the proof is completed. \hfill \Box

3. Proof of noncommutative \([\Sigma_n]\)

In this section, we prove the noncommutative analog of the equivalence of norms \([\Sigma_n]\) described in the Introduction. However, before that we need to set some notation. Let $E$ and $F$ be operator spaces such that $(E, F)$ is a compatible pair for interpolation. In what follows, we shall denote by $J_t(E, F)$ and $K_t(E, F)$ the $J$ and $K$ functionals on $(E, F)$ endowed with their natural operator space structures as defined in \([27]\). Moreover, given $2 \leq r, s \leq \infty$, we shall write $\ell_p^{n}(E)$ to denote the linear space $E^n$ endowed with the operator space structure which arises from the natural identification with the diagonal matrices of $S^n_{(r,s)}$. Then, given $1 \leq p, q \leq \infty$, we use these spaces to define the operator space $J_{p,q}^n(M_1; E) = \bigcap_{r,s \in (2p, 2q)} \ell_p^n(L_{(r,s)}(\tau_1; E)).$

**Lemma 3.1.** Let $1 \leq p, q \leq \infty$ and $\lambda_p = t^{-1/p}$. Then, given $t = \frac{1}{\lambda_1} - \frac{1}{\lambda_2}$, we consider the mapping $u : J_{p,q}^n(M_1; E) \rightarrow J_t(C_{p}^{m_1}, C_{q}^{m_1}) \otimes_h E \otimes_h J_t(R_{p}^{m_1}, R_{q}^{m_1})$, defined by the relation

$$u \left( \sum_{k=1}^{n} \delta_k \otimes x_k \right) = \sum_{k=1}^{n} e_{kk} \otimes x_k.$$ 

Then, $u$ is a complete isometry with completely contractively complemented image.

**Proof.** By the injectivity of the Haagerup tensor product, it can be checked that

$$J_t(C_{p}^{m_1}, C_{q}^{m_1}) \otimes_h E \otimes_h J_t(R_{p}^{m_1}, R_{q}^{m_1}) = \bigcap_{r,s \in (2p, 2q)} \lambda^{-1}_p L_{(r,s)}(\tau_n \otimes \tau_1; E).$$

Taking diagonals at both sides, the first assertion follows. In order to see that the diagonal is completely contractively complemented, we use the standard diagonal projection

$$P((x_{ij})) = \int_{\{-1,1\}^n} (\varepsilon_i x_{ij} \varepsilon_j) d\mu(\varepsilon),$$

where $\mu$ is the normalized counting measure on $\{-1,1\}^n$. Here $x = (x_{ij})$ is a $n \times n$ matrix with entries in $M_1 \otimes E$. By means of the second identity of Lemma \([12]\) (ii), it clearly suffices to check that $P$ is a contraction on $L_{(r,s)}(\tau_n \otimes \tau_1; E)$. For each $\varepsilon \in \{-1,1\}^n$, we consider the matrix $a_\varepsilon = (\varepsilon_i \delta_{ij})$. Then we have

$$\| P(x) \|_{(r,s)} = \left\| 2^{-n} \sum_{\varepsilon} a_\varepsilon x a_\varepsilon \right\|_{(r,s)} \leq 2^{-n} \sum_{\varepsilon} \| a_\varepsilon x a_\varepsilon \|_{(r,s)} = \| x \|_{(r,s)},$$

where $x = (x_{ij})$ is a $n \times n$ matrix.
since \( a_\varepsilon \) is unitary for any \( \varepsilon \in \{-1, 1\}^n \). This completes the proof. \( \square \)

**Proposition 3.2.** Let \( E \) be an operator space and let \( 1 \leq p < \infty \). Then

\[
\Lambda_{p1} : \sum_{k=1}^{n} \delta_k \otimes x_k \in J^n_{p1}(M_1; E) \mapsto \sum_{k=1}^{n} \delta_k \otimes \pi_k(x_k) \in L_p(\tau_1; \ell_1^n(E))
\]

is a completely bounded map with \( \|\Lambda_{p1}\|_{cb} \leq cp \), where \( c \) is independent of \( l \) and \( n \).

**Proof.** Given \( t = t^{\frac{1}{p}-\frac{1}{2}} \), if we regard \( J_l(C^n_{\infty}, R^n_{\infty}) \otimes_h E \otimes_h J_l(R^n_{\infty}, C^n_{\infty}) \) as a space of \( n \times n \) matrices with entries in \( M_1 \otimes E \), then we consider its diagonal subspace \( J^n_{p1}(M_1; E) \). By Lemma 3.1, it suffices to check that the mapping

\[
\tilde{\Lambda}_{p1} : \sum_{k=1}^{n} e_{kk} \otimes x_k \in J^n_{p1}(M_1; E) \mapsto \sum_{k=1}^{n} \delta_k \otimes \pi_k(x_k) \in L_p(\tau_1; \ell_1^n(E)),
\]

satisfies \( \|\tilde{\Lambda}_{p1}\|_{cb} \leq cp \lambda_p \). Given \( m \geq 1 \), let us consider \( x \in S^m_{p}(J^n_{p1}(M_1; E)) \) of norm less than one. Let us consider the spaces

\[
F_1 = C^n_{p} \otimes_h J_l(C^n_{\infty}, R^n_{\infty})
\]

\[
F_2 = J_l(R^n_{\infty}, C^n_{\infty}) \otimes_h R^n_{p}.
\]

Since the space \( S^m_{p}(J^n_{p1}(M_1; E)) \) embeds completely isometrically in \( F_1 \otimes_h E \otimes_h F_2 \), we can write \( x = a \circ y \otimes b \) with \( a \in M_{m n l}(F_1), y \in M_{m l n}(E), b \in M_{m l n, 1}(F_2) \) so that \( \|y\|_{M_{m l n}(E)} < 1 \) and

\[
\max \left\{ \|a\|_{S^m_{2p}}, t \|a\|_{C^n_{p} \otimes_h R^n_{\infty}^{2m^n}} \right\} < 1
\]

\[
\max \left\{ \|b\|_{S^m_{2p}}, t \|b\|_{C^n_{\infty}^{2m^n} \otimes_h R^n_{p}} \right\} < 1.
\]

Now, if we write \( a = (a_{ij}), y = (y_{ij}) \) and \( b = (b_{ij}) \) as \( n \times n \) matrices of \( ml \times ml \) matrices, we have

\[
x_k = \sum_{i,j=1}^{n} a_{ki} y_{ij} b_{jk} \quad \text{where} \quad x = \sum_{k=1}^{n} e_{kk} \otimes x_k.
\]

Therefore, we have

\[
\tilde{\Lambda}_{p1}(x) = \sum_{k=1}^{n} \sum_{i,j=1}^{n} \delta_k \otimes \pi_k(a_{ki}) \pi_k(y_{ij}) \pi_k(b_{jk}).
\]

According to Lemma 3.1, we deduce

\[
\|\tilde{\Lambda}_{p1}(x)\|_p \leq \left[ \sum_{k,i} \left\| \pi_k(a_{ki}) \right\|_{\ell_1^n}^{1/2} \sup_k \left( \left\| \pi_k(y_{ij}) \right\|_{\ell_1^n} \right) \right]^{1/2} \left\| \sum_{k,j} \pi_k(b_{jk}) \right\|_{\ell_1^n}^{1/2},
\]

where \( \|\tilde{\Lambda}_{p1}(x)\|_p \) stands for the norm of \( \tilde{\Lambda}_{p1}(x) \) in \( S^m_{p}(L_p(\tau_1; \ell_1^n(E))) \). As we know, the middle term on the right is bounded above by 1. On the other hand, Lemma 2.2 allows us to write

\[
\left\| \sum_{k,i} \pi_k(a_{ki}) \right\|_{L_p(\tau_1 \otimes \tau_1^n)} \leq cp \max \left\{ \left\| \sum_{k,t} E(a_{ki}) \right\|_{S^m_{p}}, \left( \sum_{k=1}^{n} \left\| a_{ki} a_{ki}^* \right\|_{L_p(\tau_1 \otimes \tau_1^n)} \right)^{1/p} \right\}
\]

Thus, we have

\[
\|\tilde{\Lambda}_{p1}(x)\|_p \leq cp \max \left\{ \left\| \sum_{k,t} E(a_{ki}) \right\|_{S^m_{p}}, \left( \sum_{k=1}^{n} \left\| a_{ki} a_{ki}^* \right\|_{L_p(\tau_1 \otimes \tau_1^n)} \right)^{1/p} \right\}
\]

where \( E(a_{ki}) \) represents the expectation of \( a_{ki}^* a_{ki} \). This completes the proof.
\[ \leq cp \max \left\{ \frac{1}{l} \left| \sum_{r,s=1}^{l} \sum_{k,i} a_{ki}(r,s) a_{ki}(r,s)^* \right|_{S_p^n}, \lambda_p \|aa^*\|_{S_p^{n+1}} \right\}. \]

The last inequality follows from the fact that the projection onto block diagonal matrices is completely contractive, see Corollary 1.3 of [22]. Now, recalling that
\[ \| \sum_{r,s=1}^{l} \sum_{k,i} a_{ki}(r,s) a_{ki}(r,s)^* \|_{S_p^n} = \|a\|^2_{C_p^\infty \otimes_h R_{n+2}^{12}}, \]
we obtain
\[ \left\| \sum_{k,i} \pi_k(a_{ki}a_{ki}^*) \right\|_{L_p(\tau_m \otimes \tau_n)} \leq cp\lambda_p. \]

Since the same estimate holds for \(b\), we get the desired estimate for \(\|\tilde{\Lambda}_{p1}\|_{cb}. \]

Proposition 3.2 provides an upper estimate for the norm of sums of independent noncommutative random variables in \(L_p(\ell^1_q(E))\). Now, we are interested on the dual version of this result. Hence, it is natural to consider the operator spaces \(K_{p,q}^n(M; E)\), which arise when replacing intersections by sums in \(J_{p,q}^n(M; E)\). That is, we define
\[ K_{p,q}^n(M; E) = \sum_{r,s \in \{2p, 2q\}} \ell_{(r,s)}^n(L_{(r,s)}(\tau; E)). \]

**Remark 3.3.** Arguing as in Lemma 3.1, we can regard \(K_{p,q}^n(M; E)\) as the diagonal in the Haagerup tensor product \(K_{1}((C_{p}^{unl}) \otimes_h E \otimes_h K_{1}(R_{p}^{unl} \otimes C_{q}^{unl}))\) normalized by \(\lambda_p = l^{-1/p}\). The projection \(P\) onto the diagonal is also a complete contraction.

**Lemma 3.4.** Let \(1 \leq p \leq q \leq \infty\) and let \(E\) be an operator space. Then, given a positive integer \(n \geq 1\), the following identity maps are complete contractions
\[ \text{id} : \ell_{(2p,2q)}^n(E) \longrightarrow \ell_{q}^n(E) \]
\[ \text{id} : \ell_{(2q,2p)}^n(E) \longrightarrow \ell_{q}^n(E). \]

**Proof.** Given \(2 \leq r, s \leq \infty\), we can argue as inLemma 3.1 to see that the diagonal projection \(P : S_{(r,s)}^n(E) \rightarrow \ell_{(r,s)}^n(E)\) is a complete contraction. Therefore, the complex interpolation space between the diagonals of two asymmetric Schatten classes is the diagonal of the interpolated asymmetric Schatten class. That is, we have the following complete isometries
\[ \ell_{(2p,2q)}^n(E) = \left[ \ell_q^n(E), \ell_{(2q,2p)}^n(E) \right]_{\theta} \]
\[ \ell_{(2q,2p)}^n(E) = \left[ \ell_q^n(E), \ell_{(2q,2p)}^n(E) \right]_{\theta} \]
completely isometrically for \(\frac{1}{2p} = \frac{1-\theta}{2q} + \frac{\theta}{2}\) and \(\gamma = 1/q\). Hence, it suffices to show the result for \(p = 1\) and \(q = \infty\). That is, we have to see that the identity mappings
\[ \text{id} : \ell_{(2,\infty)}^n(E) \longrightarrow \ell_{\infty}^n(E) \]
\[ \text{id} : \ell_{(\infty,2)}^n(E) \longrightarrow \ell_{\infty}^n(E), \]
are complete contractions. In other words, we have to consider the diagonals of \(R_{\infty}^n \otimes_h E \otimes_h R_{\infty}^n\) and \(C_{\infty}^n \otimes_h E \otimes_h C_{\infty}^n\). However, we recall the completely isometric isomorphisms \(E \otimes_h R_{\infty}^n = E \otimes_{\min} R_{\infty}^n\) and \(C_{\infty}^n \otimes_h E = C_{\infty}^n \otimes_{\min} E\) and the complete contractions
\[ R_{\infty}^n \otimes_h (E \otimes_{\min} R_{\infty}^n) \longrightarrow R_{\infty}^n \otimes_{\min} (E \otimes_{\min} R_{\infty}^n), \]
\[ (C_{\infty}^n \otimes_{\min} E) \otimes_h C_{\infty}^n \longrightarrow (C_{\infty}^n \otimes_{\min} E) \otimes_{\min} C_{\infty}^n. \]
Hence, it suffices to show our claim for the diagonals of $R_\infty^m \otimes_{\min} E$ and $C_\infty^m \otimes_{\min} E$. In the first case the diagonal is $R_\infty^m \otimes_{\min} E$ while in the second case is $C_\infty^m \otimes_{\min} E$. By the injectivity of the minimal tensor product and since $\ell^p_n$ carries the minimal operator space structure, the assertion follows. This completes the proof. □

The following result can be regarded as the dual version of Proposition 3.2 where the spaces $J_{p,q}^n(M_t; E)$ are replaced by the spaces $K_{p,q}^n(M_t; E)$. Here we skip the assumption that $q = 1$ and we work in the range $1 \leq p \leq q \leq \infty$.

**Proposition 3.5.** Let $E$ be an operator space and let $1 \leq p \leq q \leq \infty$. Then, the following map is a complete contraction

$$\bar{\Lambda}_{pq} : \sum_{k=1}^n \delta_k \otimes x_k \in K_{p,q}^n(M_t; E) \longrightarrow \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \in L_p(\tau^n; \ell^p_q(E)).$$

**Proof.** Let $t = t^{\frac{1}{p}} - t^{\frac{1}{q}}$, regarding again $K_t(C_{p,q}^n; E \otimes_h K_t(R_{p,q}^n, R_{p,q}^n)$ as a space of $n \times n$ matrices with entries in $M_t \otimes E$, we consider its diagonal subspace $K_{p,q}^n(M_t; E)$. By Remark 3.3 it suffices to check that the mapping

$$\bar{\Lambda}_{pq} : \sum_{k=1}^n e_{kk} \otimes x_k \in K_{p,q}^n(M_t; E) \longrightarrow \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \in L_p(\tau^n; \ell^p_q(E)),$$

satisfies $\|\bar{\Lambda}_{pq}\|_{cb} \leq \lambda_p$. Since the diagonal projection $P$ is a complete contraction, it suffices to prove this estimate for the diagonal in each of the following spaces

$C_{p,q}^n \otimes_h E \otimes_h R_{p,q}^n$, $t C_{p,q}^n \otimes_h E \otimes_h R_{p,q}^n$, $t C_{p,q}^n \otimes_h E \otimes_h R_{p,q}^n$, $t^2 C_{p,q}^n \otimes_h E \otimes_h R_{p,q}^n$.

Notice that, given a scalar $\gamma$ and an operator space $F$, we denote by $\gamma F$ the operator space with operator space structure given by

$$\|f\|_{M_m \otimes_{\min} \gamma F} = \gamma \|f\|_{M_m \otimes_{\min} F}.$$  

The first one is

$$\|\bar{\Lambda}_{pq} : \ell^p_n(S_{p,q}(E)) \rightarrow L_p(\tau^n; \ell^p_q(E))\|_{cb} \leq \lambda_p.$$  

This estimate obviously holds for $p = q$ and, since the identity map $\ell^p_n(E) \rightarrow \ell^p_q(E)$ is a complete contraction, the desired estimate follows. For the last one, we note that

$$\|\bar{\Lambda}_{pq} : t^2 \ell^p_n(S_{p,q}(E)) \rightarrow L_q(\tau^n; \ell^p_q(E))\|_{cb} \leq \lambda_p.$$  

Moreover, since we are using a probability measure, we know that the identity map $L_q(\tau^n; F) \rightarrow L_p(\tau^n; F)$ is a complete contraction. Therefore, the desired estimate for the last case holds. For the second and third terms, we use a similar trick. We claim that the identity mappings

$$L_{(2p,2q)}(\tau_m; F_1) \longrightarrow L_p(\tau_m; F_1)$$

are complete contractions. Namely, by complex interpolation it reduces to the case $p = 1$ and $q = \infty$. However, if we rescale these mappings to replace $\tau_m$ by $\tau_m$, this case follows easily by the injectivity of the Haagerup tensor product and the well-known estimates

$$\|id : R_\infty^m \rightarrow C_\infty^m\|_{cb} \leq \sqrt{m} \quad \text{and} \quad \|id : C_\infty^m \rightarrow R_\infty^m\|_{cb} \leq \sqrt{m}.$$
We take \( m = l^n \) and the operator spaces \( F_1 = \ell_{(2p,2q)}^n(E) \) and \( F_2 = \ell_{(2q,2p)}^n(E) \). According to Lemma 3.2, it suffices to prove the following estimates

\[
\|\widetilde{\Lambda}_{pq} : \ell_{(2p,2q)}^n(S_{(2p,2q)}^r(E)) \rightarrow L_{(2p,2q)}(\tau_{n^*}; \ell_{(2p,2q)}^n(E))\|_{cb} \leq \lambda_p
\]

\[
\|\widetilde{\Lambda}_{pq} : \ell_{(2q,2p)}^n(S_{(2q,2p)}^r(E)) \rightarrow L_{(2q,2p)}(\tau_{n^*}; \ell_{(2q,2p)}^n(E))\|_{cb} \leq \lambda_p.
\]

Since both estimates can be treated in a similar way, we just prove the first one. Given a positive integer \( m \), let us consider a diagonal matrix

\[ x = \sum_{k=1}^n e_{kk} \otimes x_k \in M_n \otimes S_{(2p,2q)}^m(E). \]

According to Lemma 1.2 (iii), the following identities hold for \( \frac{1}{r} = \frac{1}{p^n} + \frac{1}{q^n} \)

\[
\|\widetilde{\Lambda}_{pq}(x)\|_{(2p,2q)} = \left( \sum_{k=1}^n \|\pi_k(x_k)\|_{L_{(2p,2q)}(\text{tr}_m \otimes \tau_{n^*}; E)}^r \right)^{1/r} = \lambda_p \; \ell\|x\|_{\ell_{(2p,2q)}^n(S_{(2p,2q)}^m(E))},
\]

where \( \| \|_{(2p,2q)} \) denotes the norm on the space \( L_{(2p,2q)}(\text{tr}_m \otimes \tau_{n^*}; \ell_{(2p,2q)}^n(E)) \). Thus, applying the second identity of Lemma 1.2 (ii), the assertion follows.

Once we have seen the estimates for intersections and sums given in Propositions 3.2 and 3.5, we are in position to prove the complete equivalence of norms \( S_p \) for sums of independent noncommutative random variables in \( L_p(\ell_1(E)) \).

**Theorem 3.6.** Let \( E \) be an operator space and let \( 1 \leq p < \infty \). Then, the map

\[
\Lambda_{p1} : \sum_{k=1}^n \delta_k \otimes x_k \in J_{p,1}^n(M_1; E) \mapsto \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \in L_p(\tau_{n^*}; \ell_1^n(E))
\]

is a complete isomorphism onto a completely complemented subspace. Similarly, the same holds for the map

\[
\Lambda_{p',\infty} : \sum_{k=1}^n \delta_k \otimes x_k \in K_{p',\infty}^n(M_1; E) \mapsto \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \in L_{p'}(\tau_{n^*}; \ell_1^n(E))
\]

for \( 1 < p' \leq \infty \). Moreover, the \( cb \)-distance constants are independent of \( l \) and \( n \).

**Proof.** It is clear that we can assume \( E \) to be a finite-dimensional operator space. In particular, all the spaces we shall consider along the proof will be of finite dimension and hence reflexive. Now the duality theory for the Haagerup tensor product, see for instance the Chapter 5 of [23], provides a complete isometry

\[
S : (J_l(C_{p^n}^n, C_{q^n}^n) \otimes_h E \otimes_h J_l(P_{p^n}^n, R_{q^n}^n))^* \rightarrow K_{l^{-1}}(C_{p^n}^{m^*}, C_{q^n}^{m^*}) \otimes_h E^* \otimes_h K_{l^{-1}}(R_{p^n}^m, R_{q^n}^m).
\]

On the other hand, according to Lemma 3.4 and Remark 3.3, the projection onto the diagonal is always a complete contraction. Therefore, we obtain the following completely isometric isomorphism

\[ J_{p,1}^n(M_1; E)^* = K_{p,\infty}^n(M_1; E^*). \]
Indeed, if $P$ denotes the diagonal projection and $T = u^{-1} \circ P$ where $u$ stands for the linear mapping considered in Lemma 3.1, then the mapping 

$$T \circ S \circ T^* : J^n_{p,1}(M_1; E)^* \rightarrow K^n_{p',\infty}(M_1; E^*)$$

is a completely isometric isomorphism. Here, the duality is given by 

$$\langle a, b \rangle = \sum_{k=1}^n \delta_k \otimes (a_k \otimes e_k^*) \sum_{k=1}^n \delta_k \otimes (b_k \otimes e_k^*) = \sum_{k=1}^n \tau_k \delta_k^* \langle e_k, e_k^* \rangle.$$ 

Thus, we obviously have 

$$\langle A_{p1}(a), A_{p'\infty}(b) \rangle = \langle a, b \rangle \quad \forall \ a \in J^n_{p,1}(M_1; E), \ b \in K^n_{p',\infty}(M_1; E^*).$$

Consequently, the map $A_{p'\infty}A_{p1}$ is the identity on $J^n_{p,1}(M_1; E)$. In particular, by Propositions 5.2 and 8.3, $A_{p1}$ becomes a complete isometry with constants not depending on the dimensions. Moreover, its image is a completely complemented subspace since $A_{p1}A_{p'\infty}$ is a completely bounded projection with $\|A_{p1}A_{p'\infty}\|_{cb} \leq cp$.

This proves the assertions for $A_{p1}$, but the arguments for $A_{p'\infty}$ are similar. □

**Remark 3.7.** Let us state Theorem 3.6 in a more explicit way. To that aim, we introduce some notation. If $\frac{1}{q} = \frac{1}{p} + \frac{1}{q}$, we define 

$$\|x\|_{p,q} = \max_{r,s \in \{2p, 2q\}} \left\{ \sum_{k=1}^n \left\| \frac{1}{\tau_k} \langle x_k, \pi_k(x_k) \rangle \right\|_{L_{r,s}(\tau_k; E)}^{1/\gamma_{r,s}} \right\},$$

$$\|x\|_{p,q}^E = \inf_{r,s \in \{2p, 2q\}} \left\{ \sum_{k=1}^n \left\| \frac{1}{\tau_k} \langle x_k, \pi_k(x_k) \rangle \right\|_{L_{r,s}(\tau_k; E)}^{1/\gamma_{r,s}} \right\}.$$

Then, recalling the meaning of $\leq_{cb}$ from the Introduction, we have

- Given $1 \leq p < \infty$, we have 

$$\|x\|_{p,1} \leq_{cb} \left\| \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \right\|_{L_p(\ell^\infty_1(E))} \leq_{cb} cp \|x\|_{p,1}.$$

- Given $1 < p' \leq \infty$, we have 

$$\frac{1}{cp} \|x\|_{p',\infty} \leq_{cb} \left\| \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \right\|_{L_{p'}(\ell^\infty_1(E))} \leq_{cb} \|x\|_{p',\infty}.$$

4. A $cb$ embedding of $S^n_q$ into $S_p(\ell^m_q)$

We begin by stating a complementation result for the subspace of $J^n_{p,q}(M_1; E)$ given by constant diagonal matrices. As we shall see immediately, this result plays a relevant role in the embeddings we want to consider.

**Lemma 4.1.** Let $1 \leq p, q \leq \infty$ and let $t = \left( \frac{n}{q} \right)^{1/p} - \frac{1}{q}$ with $l$ and $n$ positive integers. Then, the map 

$$T : J_l(C^n_q, C_p^l) \otimes_h E \otimes_h J_l(R^n_q, R_p^l) \rightarrow J^n_{p,q}(M_1; E)$$

defined by 

$$T(x) = \left( \frac{n}{l} \right)^{-1/q} \left( \sum_{k=1}^n c_{kk} \otimes x \right)$$

is a complete isometry. The image of $T$ is completely contractively complemented.
Proof. To see that the image of $T$ is completely contractively complemented in $J_{p,q}^n(M;E)$, we consider the following projection

$$P(x_1, x_2, \ldots, x_n) = \left( \frac{1}{n} \sum_{k=1}^n x_k, \frac{1}{n} \sum_{k=1}^n x_k, \ldots, \frac{1}{n} \sum_{k=1}^n x_k \right).$$

Then, it suffices to see that $P$ is a complete contraction in

$$\ell^{n}_{(r,s)}(L_{(r,s)}(\tau;E))$$

whenever $r, s \in \{2p, 2q\}$. It is clear that, given any operator space $E$, the projection $P$ is contractive in these four spaces. Then, the complete contractivity follows easily from Lemma 1.2 (ii) and the obvious Fubini type results. Now, given $r, s \in \{2p, 2q\}$, let $\xi_{r,s} = \delta_{r,2p} + \delta_{s,2p}$. To see that $T$ is a complete isometry, it suffices to check that

$$T : \ell^{n}_{(r,s)}S^1_{(r,s)}(E) \rightarrow \ell^{n}_{(r,s)}(L_{(r,s)}(\tau;E))$$

is a complete isometry for any $r, s \in \{2p, 2q\}$. However, this follows one more time as a consequence of Lemma 1.2 (ii) and (iii).

The following theorem provides an embedding of the Schatten class $S^n_q(E)$ into $L_p(M, \tau; \ell^n_q(E))$ with uniformly bounded cb-distance constants.

**Theorem 4.2.** Let $1 \leq q \leq p < \infty$. Then, given any positive integer $n \geq 1$ and any operator space $E$, the following mapping is a complete isomorphism onto a completely complemented subspace

$$\Phi_{pq} : x \in S^n_q(E) \rightarrow \frac{1}{n^{1/q}} \sum_{k=1}^{n^2} \delta_k \otimes \pi_k(x) \in L_p(\tau_{n^{1/2}}; \ell^q_n(E)).$$

Moreover, $\|\Phi_{pq}\|_{cb} \leq cp$ while the inverse mapping $\Phi_{pq}^{-1}$ is completely contractive.

Proof. By Lemma 3.1 $J_t(C^1_n, C^n_p) = R^n_{\infty}$ and $J_t(R^n_q, R^n_p) = C^n_{\infty}$ for $t = n^{1/p} - \frac{1}{2}$. In particular, we can write

$$S^n_q(E) = J_t(C^1_n, C^n_p) \otimes_h E \otimes_h J_t(R^n_q, R^n_p).$$

Then, Proposition 3.2 and Lemma 4.7 give that

$$\Phi_{p1} : S^n_p(E) \rightarrow L_p(\tau_{n^{1/2}}; \ell^q_n(E))$$

is a cb embedding with $\|\Phi_{p1}\|_{cb} \leq cp$. That is, the upper estimate holds for $q = 1$. On the other hand, the map

$$\Phi_{pp} : S^n_p(E) \rightarrow L_p(\tau_{n^{1/2}}; \ell^q_n(E))$$

is clearly a complete isometry. Hence, for the general case, the upper estimate follows by complex interpolation. In order to see that the image of the mapping $\Phi_{pq}$ is completely complemented and $\Phi_{pq}^{-1}$ is completely contractive, we observe again that, by elementary properties of the local theory, we have

$$S^n_q(E^*) = \left( J_t(C^1_q, C^n_p) \otimes_h E \otimes_h J_t(R^n_q, R^n_p) \right)^*$$

for $t = n^{1/p} - \frac{1}{2}$. Thus, if $C^n_{p', q'}(M_n; E^*)$ stands for the subspace of $K^n_{p', q'}(M_n; E^*)$ of constant diagonals, Lemma 4.7 and duality give

$$S^n_q(E^*) = \frac{1}{n^{1/q}} C^n_{p', q'}(M_n; E^*).$$
In particular, Proposition 5.3 gives that
\[ \Phi_{p'q'} : S^m_q(E^*) \longrightarrow L_{p'}(\tau_{n;\epsilon}; \ell^q_{p'}) \]
is completely contractive. Finally, we observe that
\[ \langle \Phi_{pq}(a \otimes e), \Phi_{p'q'}(b \otimes e^*) \rangle = \frac{1}{n} \sum_{k=1}^n \tau_n(a^*b)(e, e^*) = \langle a \otimes e, b \otimes e^* \rangle. \]
Hence, since \( \Phi_{pq}^* \Phi_{pq} \) is the identity and \( \Phi_{pq} \Phi_{pq}^* \) is a projection, we are done. □

**Remark 4.3.** By simple dual arguments, it is not difficult to check that Theorem 4.2 holds for \( 1 < p \leq q \leq \infty \), with \( \Phi_{pq} \) completely contractive and \( \| \Phi_{pq}^{-1} \|_{cb} \leq cp \).

Namely, we first recall that
\[ S^m_q(E) = K_t(C^0_{q,c}, C^0_p) \otimes_h E \otimes_h K_t(R^{m}_{q,c}, R^{m}_p) \quad \text{for} \quad t = n^{\frac{1}{pq}}. \]
Then, by Theorem 3.3 and the dual version of Lemma 4.1 for the \( K \) functional, the complete contractivity of \( \Phi_{pq} \) holds. Finally, we end by interpolation and duality.

**Remark 4.4.** Rescaling Theorem 4.2 we get an embedding \( \Psi_{pq} : S^m_q \rightarrow S^m_{pq} \). In fact, we have taken \( m \) to be \( n^2 \). As we shall see in Section 3 when seeking for \( cb \)-embeddings with uniformly bounded constants, the choice \( m = n^2 \) is optimal.

### 5. K-convex operator spaces

The theory of type and cotype is essential to study some geometric properties of Banach spaces. The operator space analog of that theory has been recently initiated in some works summarized in [19]. The aim of this section is to explore the relation between B-convexity and K-convexity in the category of operator spaces.

#### 5.1. A variant of the embedding theorem

In this paragraph, we study the inverse of \( \Phi_{pq} \) when we impose on \( \ell_q \) its minimal operator space structure. The resulting mapping will be the key in the operator space analog of Pisier’s equivalence between B-convex and K-convex spaces.

**Lemma 5.1.** The following map extends to an anti-linear isometry
\[ T : \sum_{k=1}^n a_k \otimes e_k \in L_p(\tau_{n;\min(E)}) \longrightarrow \sum_{k=1}^n a_k^* \otimes \tau_k \in L_p(\tau_{n;\min(E)}). \]
Here, \( \min(E) \) stands for the complex conjugate operator space as defined in [29].

**Proof.** Since \( \min(E) \) embeds completely isometrically in \( \ell_{\infty} \), we take \( E \) to be \( \ell_{\infty} \). Under this assumption, the result is clear for \( p = \infty \). Namely, given \( x = (x_n)_{n \geq 1} \), in \( L_{\infty}(\tau_{n}; \ell_{\infty}) \), we have
\[ \| T(x) \| = \sup_{n \geq 1} \| x_n^* \| = \| x \|. \]
Now, if \( x \in L_p(\tau_n; \ell_{\infty}) \), there exist \( a, b \in L_{2p}(\tau_n) \) and \( y \in L_{\infty}(\tau_n; \ell_{\infty}) \) such that \( x = ayb \) and
\[ \| a \|_{2p} \| y \|_{\infty} \| b \|_{2p} < (1 + \varepsilon) \| x \|. \]
Therefore
\[ \| T(x) \| \leq \| b^* \|_{2p} \| y \|_{\infty} \| a^* \|_{2p} < (1 + \varepsilon) \| x \|. \]
Since \( \varepsilon > 0 \) is arbitrary, the assertion follows easily. This completes the proof. □
Let us consider the operator space $\tilde{F}_{pq}^n$ defined as the image of $S_q^n$ under the map $\Phi_{pq}$, with the operator space structure inherited from $L_p(\tau_n^{n\times n}; \min(F_q^n))$.

**Proposition 5.2.** The estimate $\|\Phi_{pq}^{-1}\|_{B(\tilde{F}_{pq}^n, S_q^n)} \leq 2$ holds for any $n \geq 1$.

**Proof.** We first consider a self-adjoint matrix $x$ in $S_q^n$. Then taking $m = n^2$, the sequence $\pi_1(x), \pi_2(x), \ldots, \pi_m(x)$ lies in a commutative subalgebra of $M_{nm}$. In fact, using the spectral theorem, we can write $x = u^*d_xu$ where $d_x$ stands for the matrix of eigenvalues of $x$ and $u$ is unitary. In particular, after multiplication by $u^{\otimes m}$ from the left and by $(u^*)^{\otimes m}$ from the right, we may assume that

$$\sum_{k=1}^m \delta_k \otimes \pi_k(x)$$

is a diagonal matrix. In that case, we may apply Corollary 1.3 of [22] to obtain

$$\left\| \sum_{k=1}^m \delta_k \otimes \pi_k(x) \right\|_{L_p(\tau_n^{n\times n}; \min(F_q^n))} = \left\| \sum_{k=1}^m \delta_k \otimes \pi_k(x) \right\|_{L_p(\tau_n^{n\times n}; \tilde{F}_{pq}^n)}.$$

Therefore, Theorem [12] gives

$$\|x\|_{S_q^n} \leq \|\Phi_{pq}(x)\|_{\tilde{F}_{pq}^n}.$$

For arbitrary $x$, we consider its decomposition into self-adjoint elements

$$a = \frac{1}{2}(x + x^*) \quad \text{and} \quad b = \frac{1}{2i}(x - x^*).$$

Then, we deduce from Lemma 5.1 that

$$\|\Phi_{pq}(a)\|_{\tilde{F}_{pq}^n} \leq \frac{1}{2}\|\Phi_{pq}(x)\|_{\tilde{F}_{pq}^n} + \frac{1}{2}\|\Phi_{pq}(x)^*\|_{\tilde{F}_{pq}^n} \leq \|\Phi_{pq}(x)\|_{\tilde{F}_{pq}^n}.$$ 

Obviously, the same estimate holds for $b$. Thus, we obtain the desired estimate. □

Let us consider an infinite dimensional operator space $E$ and a family of finite dimensional operator spaces $A = \{A_n \mid n \geq 1\}$. We shall say that the family $A$ embeds semi-completely uniformly in $E$, and we shall write $A \prec E$, when there exists a constant $c$ and embeddings $\Lambda_n : A_n \to E$ such that

$$\|\Lambda_n\|_{cb}\|\Lambda_n^{-1}\| \leq c \quad \text{for all} \quad n \geq 1.$$

**Corollary 5.3.** Let $1 < p < \infty$ and $1 \leq q < \infty$. Then, we have

$$\left\{ \ell_q^n \mid n \geq 1 \right\} \prec S_p(E) \Rightarrow \left\{ S_q^n \mid n \geq 1 \right\} \prec S_p(E).$$

**Proof.** By hypothesis, there exist $c_1 > 1$ and embeddings $\Lambda_n : \ell_q^n \to S_p(E)$ such that $\|\Lambda_n\|_{cb}\|\Lambda_n^{-1}\| \leq c_1$ for each positive integer $n$. Let $F_n$ denote the image $\Lambda_n(\ell_q^n)$ of $\Lambda_n$ in $S_q(E)$. On the other hand, according to Theorem [12] we know how to construct linear isomorphisms

$$\Phi_n : S_q^n \to F_{pq}^n \subset S_p(\ell_q^{n^2})$$

such that $\|\Phi_n\|_{cb} \leq c_2$ for some constant $c_2$ independent of $n$. Moreover, let $\tilde{F}_{pq}^n$ be the image of $\Phi_n$ endowed with the operator space structure inherited from $S_p(\min(\ell_q^{n^2}))$. 

Then, if $\Psi_n : F_{pq} \rightarrow S^n_q$ stands for $\Phi_n^{-1}$, Proposition $\Ref{proposition}$ gives $\|\Psi_n\| \leq c_3$ for some constant $c_3$ independent of $n$. Let us define
\[
\tilde{\Lambda}_n : S^n_q \rightarrow S_p(E) \quad \text{by} \quad \tilde{\Lambda}_n = (id \otimes \Lambda_{n^2}) \circ \Phi_n.
\]
Then we have
\[
\|\tilde{\Lambda}_n\|_{cb} \|\tilde{\Lambda}_n^{-1}\| \leq \|\Lambda_{n^2}\|_{cb} \|\Phi_n\| \|\Psi_n\| \|\Lambda_{n^2}^{-1}\|_{CB(F_{1,2}, \min(E_{1/2}))} = \|\Lambda_{n^2}\|_{cb} \|\Phi_n\| \|\Psi_n\| \|\Lambda_{n^2}^{-1}\| \leq c_1 c_2 c_3.
\]
Since the constant $c_1 c_2 c_3$ does not depend on $n$, the assertion follows. \hfill \Box

5.2. OB-convexity and OK-convexity. Let us start by defining the notion of OB-convex operator space. Following $\Ref{book}$, let us fix a family $d_\Sigma = \{d_\sigma : \sigma \in \Sigma\}$ of positive integers indexed by an infinite set $\Sigma$ and, given a finite subset $\Gamma$ of $\Sigma$, let
\[
\Delta_{\Gamma} = \sum_{\sigma \in \Gamma} d_\sigma^2.
\]
An operator space $E$ is called OB$_2$-convex if there exists a finite subset $\Gamma$ of $\Sigma$ and certain $0 < \delta \leq 1$ such that, for any family
\[
\left\{ A^\sigma \in M_{d_\sigma} \otimes S_2(E) \right\}_{\sigma \in \Gamma},
\]
we have
\[
\frac{1}{\Delta_{\Gamma}} \inf_{B^\sigma_{\text{unitary}}} \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(A^\sigma B^\sigma) \right\|_{S_2(E)} \leq (1 - \delta) \max_{\sigma \in \Gamma} \|A^\sigma\|_{M_{d_\sigma}(S_2(E))}.
\]

If we replace the Schatten class $S_2(E)$ above by $S_p(E)$ we get an equivalent notion whenever $1 < p < \infty$, see $\Ref{book}$. This definition is inspired by Beck’s original notion for Banach spaces, which corresponds to the commutative set of parameters $\Sigma_0 = \mathbb{N}$ with $d_\sigma = 1$ for all $\sigma \in \Sigma_0$. Our definition depends a priori on the set of parameters $(\Sigma, d_\Sigma)$. However, we shall see below that there is no dependence on $\Sigma$. On the other hand, we also need to provide an operator space analog of the property of containing (uniformly) finite dimensional $L_1$ spaces. However, this time we need to allow the noncommutative $L_1$’s to appear in the definition. Given an operator space $E$, a set of parameters $(\Sigma, d_\Sigma)$ and $1 < p < \infty$, we define the spaces
\[
\mathcal{L}_p(\Sigma; E) = \left\{ A \in \prod_{\sigma \in \Sigma} M_{d_\sigma} \otimes E : \left( \sum_{\sigma \in \Sigma} d_\sigma \|A^\sigma\|^p_{S_p(E)} \right)^{1/p} < \infty \right\}.
\]
We impose on $\mathcal{L}_p(\Sigma; E)$ its natural operator space structure, see Chapter 2 of $\Ref{book}$ for the details. We shall write $\mathcal{L}_p(\Sigma)$ for the scalar-valued case. We shall say that $S_p(E)$ contains $\mathcal{L}_1(\Gamma)$’s semi-completely $\lambda$-uniformly if, for each finite subset $\Gamma$ of $\Sigma$, there exists a linear embedding $\Lambda_{\Gamma} : \mathcal{L}_1(\Gamma) \rightarrow S_p(E)$ such that
\[
\|\Lambda_{\Gamma}\|_{cb} \|\Lambda_{\Gamma}^{-1}\| \leq \lambda.
\]
In other words, if
\[
\left\{ \mathcal{L}_1(\Gamma) \mid \Gamma \text{ finite} \right\} \prec S_p(E),
\]
The following is the analog of a well-known result for Banach spaces, see $\Ref{book}$.

Remark 5.4. Given an operator space $E$, the following are equivalent:

i) $S_p(E)$ contains $\mathcal{L}_1(\Gamma)$’s semi-completely $\lambda$-uniformly for any $\lambda > 1$.

ii) $S_p(E)$ contains $\mathcal{L}_1(\Gamma)$’s semi-completely $\lambda$-uniformly for some $\lambda > 1$. 
Finally we recall, as have already done in the Introduction, that an operator space $E$ will be considered OK-convex whenever the vector-valued Schatten class $S_2(E)$ is K-convex when regarded as a Banach space.

**Remark 5.5.** The given definition of OK-convexity is a bit more flexible. Indeed, an operator space $E$ is OK-convex if and only if $S_p(E)$ is a K-convex Banach space for some (any) $1 < p < \infty$. This follows from the fact that, given $1 < p < \infty$, the Schatten class $S_p(E)$ is K-convex if and only $S_2(E)$ is K-convex. Indeed, it follows from [20, 21] that Banach space K-convexity is stable by complex interpolation assuming only that one of the endpoint spaces is K-convex. Now assume that $S_2(E)$ is K-convex and let $1 < p < \infty$. If $p < 2$ (resp. $p > 2$) we have

$$S_p(E) = [S_2(E), S_1(E)]_{\theta} \quad \text{(resp. } S_p(E) = [S_2(E), S_\infty(E)]_{\theta} \text{)}$$

for some $0 < \theta < 1$. Therefore, we find by complex interpolation that $S_p(E)$ is also a K-convex Banach space. A similar argument shows that $S_2(E)$ is a K-convex Banach space whenever $S_p(E)$ is also K-convex. Thus our claim follows.

**Remark 5.6.** In [18] it was given an a priori more general notion of K-convexity for operator spaces. Namely, let $(\Omega, \mathcal{A}, \mu)$ be a probability space with no atoms. Then, following [15] we define the quantized Gauss system associated to $(\Sigma, d_\Sigma)$ as a collection of matrix-valued functions

$$G_\Sigma = \left\{ g^\sigma : \Omega \to M_{d_\sigma} \right\}_{\sigma \in \Sigma} \quad \text{where} \quad g^\sigma = \frac{1}{\sqrt{d_\sigma}} \begin{pmatrix} g_{ij}^\sigma \end{pmatrix}.$$ 

Here, the functions $g_{ij}^\sigma : \Omega \to \mathbb{C}$ form a family, indexed by $1 \leq i, j \leq d_\sigma$ and $\sigma \in \Sigma$, of independent standard complex-valued gaussian random variables. Given a function $f \in L_2(\Omega; E)$, we can consider the Fourier coefficients of $f$ with respect to the quantized Gauss system

$$\hat{f}_G(\sigma) = \int_\Omega f(\omega) g^\sigma(\omega)^* d\mu(\omega).$$

This gives rise to the Gauss projection defined below

$$P_G : f \in L_2(\Omega; E) \mapsto \sum_{\sigma \in \Sigma} d_\sigma \text{tr}(\hat{f}_G(\sigma) g^\sigma) \in L_2(\Omega; E).$$

An operator space $E$ is called OK$_\Sigma$-convex if the Gauss projection associated to the parameters $(\Sigma, d_\Sigma)$ is a completely bounded map. However, recalling the definition of the quantized Gauss system, we can write

$$\sum_{\sigma \in \Sigma} d_\sigma \text{tr}(\hat{f}_G(\sigma) g^\sigma) = \sum_{\sigma \in \Sigma} \sum_{i,j=1}^{d_\sigma} \int_\Omega f(\omega) g^\sigma_{ij}(\omega) d\mu(\omega) g^\sigma_{ij}.$$ 

Therefore, since now the right hand side can be regarded as the classical Gauss projection, it turns out that the notion of OK$_\Sigma$-convexity does not depend on the set $(\Sigma, d_\Sigma)$, so that we shall simply use in the sequel the term OK-convex, without any explicit reference to the set of parameters $(\Sigma, d_\Sigma)$.

**Remark 5.7.** We can replace $L_2(\Omega; E)$ above by $L_p(\Omega; E)$ for any $1 < p < \infty$.

**Theorem 5.8.** Given an operator space $E$, the following are equivalent:

i) $E$ is OK-convex.

ii) $E$ is OB$_\Sigma$-convex for some (any) set of parameters $(\Sigma, d_\Sigma)$.

iii) $S_p(E)$ does not contain $\ell_1^n$’s uniformly for some (any) $1 < p < \infty$. 

iv) $S_p(E)$ does not contain $L_1(\Gamma)$’s semi-completely for some (any) $1 < p < \infty$.

**Proof.** By definition, $E$ is OK-convex if and only if $S_p(E)$ is a K-convex Banach space for some (any) $1 < p < \infty$, see Remark 5.5 above. Then, applying Pisier’s characterization [21] of K-convexity, conditions i) and iii) are equivalent. Now we prove the equivalence between iii) and iv). To that aim we can fix $1 < p < \infty$ without lost of generality (note that iii) is independent of the index $p \in (1,\infty)$ by its equivalence with i) and Remark 5.5). The implication iii) $\Rightarrow$ iv) is trivial. Reciprocally, let us assume that $S_p(E)$ contains $\ell_1^n$’s uniformly. Note that, since $\ell_1^n$ carries the maximal operator space structure, any Banach space embedding of $\ell_1^n$ is automatically a semi-complete embedding with the same constants. Then, Corollary 5.3 claims that the family

$$\{S_1^n \mid n \geq 1\}$$

also embeds semi-completely uniformly in $S_p(E)$. That is, there exists $c > 1$ and embeddings $\Lambda_n : S_1^n \rightarrow S_p(E)$ such that

$$\|\Lambda_n\|_{cb}\|\Lambda_n^{-1}\| \leq c.$$

Now, given a finite subset $\Gamma$ of $\Sigma$, we also consider the map

$$S_\Gamma : A \in L_1(\Gamma) \mapsto \bigoplus_{\sigma \in \Gamma} d_\sigma A^\sigma \in S_1^N \quad \text{for} \quad N = \sum_{\sigma \in \Gamma} d_\sigma.$$

Finally, let $R_\Gamma : L_1(\Gamma) \rightarrow S_p(E)$ stand for $\Lambda_N \circ S_\Gamma$. Then we have

$$\|R_\Gamma\|_{cb}\|R_\Gamma^{-1}\| \leq \|\Lambda_N\|_{cb}\|\Lambda_N^{-1}\| \leq c,$$

since $S_\Gamma$ is a complete isometry. In summary, the $L_1(\Gamma)$’s embed semi-completely uniformly in $S_p(E)$. This proves the implication iv) $\Rightarrow$ iii). It remains to see that ii) is equivalent to some (any) of the other conditions. As in the commutative case, the implication ii) $\Rightarrow$ iv) follows from Remark 5.4 and by plugging in the ‘right unit vectors’, for details see [18]. The converse implication iv) $\Rightarrow$ ii) (a bit more technical) is the main result in [18]. This completes the proof. $\square$

**Remark 5.9.** Theorem 5.8 implies the $\Sigma$-independence of OB$\Sigma$-convexity.

**Remark 5.10.** We have already mentioned that semi-complete and Banach space embeddings of $\ell_1^n$’s are the same since $\ell_1^n$ carries the maximal o.s.s. It is worthy of mention that, although $L_1(\Gamma)$’s are not longer equipped with the maximal operator space structure, a similar property holds for the latter spaces. Indeed, it is clear that if $L_1(\Gamma)$’s are uniformly contained in $S_p(E)$ in the Banach space sense, then $S_p(E)$ also contains $\ell_1^n$’s uniformly. Finally, by Theorem 5.8 we see that $L_1(\Gamma)$’s embed semicompletely uniformly in $S_p(E)$. The converse is trivial.

### 6. Operator space type and cotype

The notions of Fourier type and cotype of an operator space with respect to a noncommutative compact group were already defined in the Introduction. These are particular cases of a more general notion of type and cotype for operator spaces introduced in [4]. In that paper, the (uniformly bounded) quantized orthonormal systems play the same role of the uniformly bounded orthonormal systems in the classical theory. Some relevant examples of this notion are the dual object of a noncommutative compact group and the quantized analog of the Steinhaus system.
introduced in [15]. Before introducing the notions of type and cotype for operator spaces, let us recover the classical notions. Let \( \varepsilon_1, \varepsilon_2, \ldots \) be a sequence of random signs or independent Steinhaus variables over a probability space \((\Omega, A, \mu)\). Given \( 1 \leq p \leq 2 \), a Banach space \( X \) is called of type \( p \) when there exists a constant \( T_p(X) \) such that
\[
\left( \int_\Omega \left( \sum_{k=1}^n x_k \varepsilon_k(\omega) \right)^p \frac{d\mu(\omega)}{X} \right)^{1/p} \leq T_p(X) \left( \sum_{k=1}^n \| x_k \|_X^p \right)^{1/p}
\]
for any finite family \( x_1, x_2, \ldots, x_n \) in \( X \). As we mentioned in the Introduction, the basic idea is to replace the random variables \((\varepsilon_k)\) by a sequence \( U_1, U_2, \ldots \) of independent random unitaries. That is, each \( U_k : \Omega \to U(d_k) \) is a random unitary \( d_k \times d_k \) matrix uniformly distributed in the unitary group \( U(d_k) \) with respect to the normalized Haar measure. In this setting, we might define the following notion of type
\[
\left( \int_\Omega \left( \sum_{k=1}^n d_k A_k(i,j) U_k(j,i) \right)^p \frac{d\mu(\omega)}{X} \right)^{1/p} \leq T_p(X) \left( \sum_{k=1}^n d_k \| A_k \|_{S_{d_k}(X)}^{p'} \right)^{1/p}.
\]
We want to point out that the right hand side is only well-defined for operator spaces. Moreover, this notion depends on the dimension \( d_k \) and their multiplicity. Note that the presence of \( d_k \)'s in the inequality stated above is quite natural in view of the Peter-Weyl theorem and the connection (explained in the Introduction) with the Hausdorff-Young inequality for non-abelian compact groups. Let us give the precise definitions. The quantized Steinhaus system associated to \((\Sigma, d_\Sigma)\) is defined as a collection
\[
\mathcal{S}_\Sigma = \left\{ \zeta^\sigma : \Omega \to U(d_\sigma) \right\}_{\sigma \in \Sigma}
\]
of independent uniformly distributed random unitaries with respect to the set of parameters \((\Sigma, d_\Sigma)\). Given an operator space \( E \) and a function \( f \in L_2(\Omega; E) \), we can consider the Fourier coefficients of \( f \) with respect to the quantized Steinhaus system
\[
\hat{f}_\Sigma(\sigma) = \int_\Omega f(\omega) \zeta^\sigma(\omega)^* d\mu(\omega).
\]
Let \( \text{St}_p(\Sigma; E) \) be the closure in \( L_p(\Omega; E) \) of the subspace given by functions
\[
f_\Gamma = \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(A^\sigma \zeta^\sigma)
\]
and \( \Gamma \) a finite subset of \( \Sigma \). We shall write \( \text{St}_p(\Sigma) \) for the scalar-valued case. Then, given \( 1 \leq p \leq 2 \), we say that the operator space \( E \) has \( \Sigma \)-type \( p \) when the following inequality holds for any function \( f \in \text{St}_{p'}(\Sigma; E) \)
\[
\left( \int_\Omega \| f(\omega) \|_{E}^{p'} \frac{d\mu(\omega)}{X} \right)^{1/p'} \leq c_b \mathcal{K}_{p'}^p(E, \mathcal{S}_\Sigma) \left( \sum_{\sigma \in \Sigma} d_\sigma \| \hat{f}_\Sigma(\sigma) \|_{p_{d_\sigma}(E)}^p \right)^{1/p}.
\]
In a similar way, \( \Sigma \)-cotype \( p' \) means that any \( f \in \text{St}_p(\Sigma; E) \) satisfies
\[
\left( \sum_{\sigma \in \Sigma} d_\sigma \| \hat{f}_\Sigma(\sigma) \|_{p_{d_\sigma}(E)}^p \right)^{1/p} \leq c_b \mathcal{K}_p^p(E, \mathcal{S}_\Sigma) \left( \int_\Omega \| f(\omega) \|_{E}^{p'} \frac{d\mu(\omega)}{X} \right)^{1/p'}.
\]
Recall that the symbol \( \leq c_b \) means the complete boundedness of the corresponding linear map. The best constants \( \mathcal{K}_{p'}^p(E, \mathcal{S}_\Sigma) \) and \( \mathcal{K}_p^p(E, \mathcal{S}_\Sigma) \) in the inequalities stated above are called the \( \Sigma \)-type \( p \) and \( \Sigma \)-cotype \( p' \) constants of \( E \). More concretely, using
the spaces $L_p(\Sigma; E)$ introduced in Section 5, the given definitions of $\Sigma$-type and $\Sigma$-cotype can be rephrased by requiring the complete boundedness of the following operators:

$$
T_p : A \in L_p(\Sigma; E) \mapsto \sum_{\sigma \in \Sigma} d_{\sigma} \text{tr}(A^\sigma \zeta^\sigma) \in St_p(\Sigma; E),
$$

$$
C_{p'} : \sum_{\sigma \in \Sigma} d_{\sigma} \text{tr}(A^\sigma \zeta^\sigma) \in St_p(\Sigma; E) \mapsto A \in L_{p'}(\Sigma; E).
$$

Remark 6.1. Let us recall that $\Sigma_0$ stands for the commutative set of parameters defined in Section 5. The classical Khintchine inequalities can be rephrased by saying that the norm of $St_p(\Sigma_0)$, regarded as a Banach space, is equivalent to that of $St_q(\Sigma_0)$ whenever $1 \leq p \neq q < \infty$. On the other hand, by means of the noncommutative Khintchine inequalities [12, 14], it turns out that the norm of $St_p(\Sigma_0)$ is not completely equivalent to that of $St_q(\Sigma_0)$. That is, the operator spaces $St_p(\Sigma_0)$ and $St_q(\Sigma_0)$ are isomorphic but not completely isomorphic. More generally, $St_p(\Sigma)$ is Banach isomorphic but not completely isomorphic to $St_q(\Sigma)$, see [13] for the details. Therefore, each space $St_q(\Sigma)$ in the definition of $\Sigma$-type $p$ and $\Sigma$-cotype $p'$ gives a priori a different notion!

Remark 6.2. As in the classical theory, every operator space has $\Sigma$-type 1 and $\Sigma$-cotype $\infty$. An operator space $E$ has non-trivial $\Sigma$-type whenever it has $\Sigma$-type $p$ for some $1 < p \leq 2$. According to [13] and in contrast with the commutative theory, OK-convexity is not equivalent to having non-trivial $\Sigma$-type. Indeed, the operator Hilbert spaces $R$ and $C$ fail this equivalence since both are OK-convex operator spaces but do not have $\Sigma$-type for any $1 < p \leq 2$. This constitutes an important difference between the classical and the noncommutative contexts. Namely, it turns out that we can not expect an operator space version of the Maurey-Pisier theorem [16] since the simplest form of this result asserts that the property of having non-trivial type is equivalent to K-convexity.

Remark 6.3. It is not clear whether or not the notions of $\Sigma$-type and $\Sigma$-cotype depend on $(\Sigma, d_\Sigma)$. Moreover, if we replace the quantized Steinhaus system by the dual object of a noncommutative compact group $G$, we can ask ourselves the same question for the notions of Fourier type and cotype. Note that this group independence is an open problem even in the commutative theory. The reader is referred to the paper [6] for more information on this problem.

The $\Sigma$-type (resp. $\Sigma$-cotype) becomes a stronger condition on any operator space as the exponent $p$ (resp. $p'$) approaches 2. In particular, given an operator space $E$ we consider (as in the Banach space context) the notions of sharp $\Sigma$-type of $E$ (i.e. the supremum over all $1 \leq p \leq 2$ for which $E$ has $\Sigma$-type $p$) as well as sharp $\Sigma$-cotype of $E$ (i.e. the infimum over all $2 \leq p' \leq \infty$ for which $E$ has $\Sigma$-type $p'$). The aim of this section is to investigate the sharp $\Sigma$-type and $\Sigma$-cotype indices of Lebesgue spaces, either commutative or not. However, as we shall see below, some other related problems will be solved with the same techniques.

6.1. Sharp $\Sigma$-type of $L_p$ for $1 \leq p \leq 2$. We begin with the finite dimensional $\Sigma$-type constants for any bounded set of parameters $(\Sigma, d_\Sigma)$. More concretely, let us consider a set of parameters $(\Sigma, d_\Sigma)$ with $d_\Sigma$ bounded. Then, given a finite subset $\Gamma$ of $\Sigma$, we shall write $\ell_p(\Gamma)$ to denote the space of functions $\xi : \Gamma \to \mathbb{C}$ endowed
with the customary norm

\[ \|\xi\|_{\ell_p(\Gamma)} = \left( \sum_{\sigma \in \Gamma} |\xi(\sigma)|^p \right)^{1/p}. \]

Let us consider the function \( f : \Omega \to \ell_p(\Gamma) \) defined by

\[ f = \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(\hat{f}_S(\sigma)\zeta^\sigma) \quad \text{with} \quad \hat{f}_S(\sigma) = e_{11} \otimes \delta_\sigma \in M_{d_\sigma} \otimes \ell_p(\Gamma). \]

Then we recall that

\[ \left( \int_\Omega | \sqrt{d_\sigma \zeta^\sigma_{11}} |^q d\mu \right)^{\frac{1}{q}} \sim \left( \int_\Omega | \sqrt{d_\sigma \zeta^\sigma_{11}} |^2 d\mu \right)^{\frac{1}{2}} = 1, \quad \text{for any} \quad 1 \leq q < \infty. \]

Indeed, the norm equivalence follows from the analog of the Khintchine-Kahane inequalities for the quantized Steinhaus system, proved in \cite{15}. The last equality follows from the definition of \( S_{\Sigma} \). In particular, if we use the symbol \( \lesssim \) to denote an inequality up to a universal positive constant, then we have the following estimate for any \( 1 \leq p < q \leq 2 \)

\[ |\Gamma|^{1/p} \lesssim \left( \int_\Omega \sum_{\sigma \in \Gamma} |d_\sigma \zeta^\sigma_{11}|^p d\mu \right)^{1/p} \]
\[ \leq \left( \int_\Omega \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(\hat{f}_S(\sigma)\zeta^\sigma) \right\|_{\ell_p(\Gamma)}^{q'} d\mu \right)^{1/q'} \]
\[ \leq K^1_q(\ell_p(\Gamma), S_{\Sigma}) \left( \sum_{\sigma \in \Gamma} d_\sigma \| \hat{f}_S(\sigma)\|_{S_q^{d_\sigma}(\ell_p(\Gamma))}^q \right)^{1/q} \lesssim K^1_q(\ell_p(\Gamma), S_{\Sigma}) |\Gamma|^{1/q}. \]

In other words

\[ c |\Gamma|^{1/p-1/q} \leq K^1_q(\ell_p(\Gamma), S_{\Sigma}) \leq |\Gamma|^{1/p-1/q}, \]

for some constant \( 0 < c \leq 1 \). The upper estimate is much simpler and it can be found in \cite{3}. Therefore, since any infinite dimensional (either commutative or noncommutative) \( L_p \) space contains completely isometric copies of \( \ell_p(\Gamma) \) for any finite subset \( \Gamma \) of \( \Sigma \), we deduce that any infinite dimensional \( L_p \) space has sharp \( \Sigma \)-type \( p \) for any bounded set of parameters \( (\Sigma, d_{\Sigma}) \). However, it is evident that our argument doesn’t work for unbounded sets of parameters. This case requires to find the right matrices which give the optimal constants. In the following theorem we compute the finite dimensional constants for the Schatten classes.

**Theorem 6.4.** If \( 1 \leq p < q \leq 2 \), the estimate

\[ K_q^1(S_{d_p}^d, S_{\Sigma}) \geq d_p^{(2(1/p-1/q))} \]

holds for any unbounded set of parameters \( (\Sigma, d_{\Sigma}) \) and any element \( \sigma \) of \( \Sigma \).

**Proof.** Let us take \( f : \Omega \to S_{d_p}^d \) so that \( \hat{f}_S(\xi) = 0 \) if \( \xi \in \Sigma \setminus \{\sigma\} \) and

\[ \hat{f}_S(\sigma) = \left( \sum_{i=1}^{d_\sigma} e_{i1} \otimes e_{i1} \right) \otimes \left( \sum_{j=1}^{d_\sigma} e_{1j} \otimes e_{1j} \right) \in C_{d_p}^{d_\sigma} \otimes_h C_{d_\sigma}^{d_p} \otimes_h R_{d_\sigma}^{d_p} \otimes_h R_{d_p}^{d_\sigma} = S_q^{d_\sigma}(S_p^{d_\sigma}). \]

Then, the following estimate holds by definition of \( \Sigma \)-type

\[ \left( \int_\Omega \left\| d_\sigma \text{tr}(\hat{f}_S(\sigma)\zeta^\sigma) \right\|_{S_q^{d_\sigma}(\ell_p(\Gamma))}^{q'} d\mu \right)^{1/q'} \leq K^1_q(S_{d_p}^{d_\sigma}, S_{\Sigma}) d_p^{1/q} \| \hat{f}_S(\sigma)\|_{S_q^{d_\sigma}(S_p^{d_\sigma})}, \]
Note that we have
\[ \text{tr}(\hat{f}_S(\sigma)\zeta^\sigma) = \sum_{i,j=1}^{d_x} e_{ij} \otimes \zeta_{ji} = (\zeta^\sigma)^t. \]

Thus, since the \( \zeta^\sigma \)'s are unitary, the left hand side of the inequality above is \( d_\sigma^{1+\frac{1}{q'}}. \)

On the other hand, we need to compute the norm of \( \hat{f}_S(\sigma) \) in \( S^d_{q_r}(S^d_{q_l}) \). Since the Haagerup tensor product commutes with complex interpolation, it is not difficult to check that the following natural identifications are Banach space isometries
\[ C^d_{q_r} \otimes_h C^d_{q_l} = S^d_{r} = R^d_{p} \otimes_h R^d_{q} \quad \text{with} \quad \frac{1}{r} = \frac{1}{2} \left( 1 - \frac{1}{p} + \frac{1}{q} \right). \]

For instance,
\[ C^d_{q_r} \otimes_h C^d_{q_l} = [C^d_{\infty} \otimes_h C^d_{\infty}, C^d_{\infty} \otimes_h C^d_{1}]_{1/p} = [S^d_{2}\otimes S^d_{\infty}]_{1/p} = S^d_{2}\otimes S^d_{\infty}. \]

In particular, due to our choice of \( \hat{f}_S(\sigma) \), we can write
\[ \|\hat{f}_S(\sigma)\|_{S^d_{q_r}(S^d_{q_l})} = \|1_{M_{d_x}}\|^2_{S^d_{q_r}} = d_\sigma^{2/r}. \]

Combining our previous results, we obtain the desired estimate. \( \square \)

**Corollary 6.5.** If \( 1 \leq p < q \leq 2 \), the estimate
\[ K_1^1(L^d_{p^*}, S_\Sigma) \gtrsim d_\sigma^{2(1/p-1/q)} \]
holds for any unbounded set of parameters \((\Sigma, d_\Sigma)\) and any element \( \sigma \) of \( \Sigma \).

**Proof.** By Theorem 4.2 we have
\[ K_1^1(S^d_{q_r}, S_\Sigma) \lesssim K_1^1(S^d_{q}(L^d_{p}), S_\Sigma) \lesssim K_1^1(L^d_{p^*}, S_\Sigma). \]
The last inequality follows by Minkowski inequality for operator spaces, see [3]. \( \square \)

**Remark 6.6.** The arguments applied up to now also provide the finite dimensional estimates for the \( \Sigma \)-cotype constants when \( 2 \leq q' < p' \leq \infty \). Namely, the following estimates hold
\[ K_2^2(S^d_{q_r}, S_\Sigma) \gtrsim d_\sigma^{2(1/q'-1/p')} \quad \text{and} \quad K_2^2(L^d_{p^*}, S_\Sigma) \gtrsim d_\sigma^{2(1/q'-1/p')}. \]

**Remark 6.7.** By a simple result of [3], we have \( K_1^1(S^d_{q_r}, S_\Sigma) \leq d_{cb}(S^d_{p^*}, S^d_{q_r}). \) In particular, in Theorem 6.4 we actually have equality
\[ K_1^1(S^d_{p^*}, S_\Sigma) = d_{cb}^{2(1/p-1/q)}. \]

A similar argument applies to Corollary 6.5. In summary, our estimates provide the exact order of growth of the \( \Sigma \)-type (resp. \( \Sigma \)-cotype by Remark 6.6) constants of the corresponding finite dimensional Lebesgue spaces considered above. Moreover, now we can prove the claim given in Remark 4.3. Namely, let us consider the set \( \Sigma = \mathbb{N} \) with \( d_k = k \) for all \( k \geq 1 \). Then, if \( \Psi_{pq} : S^n_p \rightarrow S_p(L^m_q) \) is a \( cb \) embedding with constants not depending on the dimensions \( n \) and \( m \), Corollary 6.5 provides the following estimate
\[ K_1^1(S^n_p, S_\Sigma) \leq \|\Psi_{pq}\|_{cb} \|\Psi_{pq}^{-1}\|_{cb} K_1^1(S^d_{q}(L^d_{p}), S_\Sigma) \leq \|\Psi_{pq}\|_{cb} \|\Psi_{pq}^{-1}\|_{cb} K_1^1(L^d_{p^*}, S_\Sigma) \leq \|\Psi_{pq}\|_{cb} \|\Psi_{pq}^{-1}\|_{cb} m^{1/p-1/q}. \]
Therefore, since $K_q^1(S_p^n, S_\Sigma) = n^{2(1/p - 1/q)}$, we conclude by taking $n$ arbitrary large.

**Remark 6.8.** The main topic of [2] is the sharp Fourier type and cotype of $L_p$ spaces. Given $1 \leq p \leq 2$, it is showed that $L_p$ has sharp Fourier type $p$ with respect to any compact semisimple Lie group. The arguments employed are very different. Namely, the key point is a Hausdorff-Young type inequality for functions defined on a compact semisimple Lie group with arbitrary small support. However, the sharp Fourier cotype of $L_p$ for $1 \leq p \leq 2$ is left open in [2]. Now we can solve it by using Corollary 6.3 and the following inequality

$$K_q^2(L_p, \tilde{\mathcal{G}}) \geq K_q^1(L_p, S_\tilde{\mathcal{G}}).$$

Here $K_q^2(L_p, \tilde{\mathcal{G}})$ denotes the Fourier cotype $q'$ constant of $L_p$ with respect to $\mathcal{G}$ and $S_\tilde{\mathcal{G}}$ stands for the quantized Steinhaus system with the parameters given by the degrees of the irreducible representations of $\mathcal{G}$. That inequality is a particular case of the noncommutative version of the contraction principle given in [15]. This solves the problem posed in [2] not only for compact semisimple Lie groups, but for any non-finite topological compact group.

6.2. Sharp $\Sigma$-cotype of $L_p$ for $1 \leq p \leq 2$. Given any $\sigma$-finite measure space $(\tilde{\Omega}, \mathcal{B}, \nu)$, any set of parameters $(\Sigma, d_\Sigma)$ and any finite subset $\Gamma$ of $\Sigma$, let us consider a family of matrices

$$A = \left\{ A^\sigma \in M_{d_\sigma} \otimes L_p(\tilde{\Omega}) \right\}_{\sigma \in \Gamma}.$$ 

Then, Minkowski inequality and Plancherel theorem give

$$\left( \sum_{\sigma \in \Gamma} d_\sigma \| A^\sigma \|^2_{S_\Sigma^2(L_p(\tilde{\Omega}))} \right)^{1/2} \leq \left( \int_{\tilde{\Omega}} \left[ \sum_{\sigma \in \Gamma} d_\sigma \| A^\sigma(x) \|^2_{S_\Sigma^2} \right]^{p/2} d\nu(x) \right)^{1/p} = \left( \int_{\tilde{\Omega}} \left[ \int_{\tilde{\Omega}} \left[ \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(A^\sigma \zeta^\sigma) \right]^2 d\mu \right]^{p/2} d\nu \right)^{1/p}.$$ 

Second, by the analog given in [15] of Khintchine-Kahane inequalities for $S_\Sigma$,

$$\left( \int_{\tilde{\Omega}} \left[ \int_{\tilde{\Omega}} \left[ \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(A^\sigma \zeta^\sigma) \right] d\mu \right]^{2} d\nu \right)^{1/2} \sim \left( \int_{\tilde{\Omega}} \left[ \int_{\tilde{\Omega}} \left[ \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(A^\sigma \zeta^\sigma) \right] d\nu \right]^{2} d\mu \right)^{1/2}.$$ 

Therefore, there exists some constant $c$ such that

$$\left( \sum_{\sigma \in \Gamma} d_\sigma \| A^\sigma \|^2_{S_\Sigma^2(L_p(\tilde{\Omega}))} \right)^{1/2} \leq c \left( \int_{\tilde{\Omega}} \left[ \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(A^\sigma \zeta^\sigma) \right] d\mu(\omega) \right)^{1/2}.$$ 

for any family of matrices $A$. In other words, we have proved that the mapping $C_2$ defined above is bounded when we take values in $L_p(\tilde{\Omega})$. However, we can not claim $\Sigma$-cotype 2 unless we prove that the same operator $C_2$ is not only bounded, but completely bounded. Now, looking at Remark 6.1 we realize that our arguments do not work to show the complete boundedness. In this paragraph we study this problem. We begin by computing the sharp cotype of $S_q(S_p)$ as a Banach space. This will be the key to find the sharp $\Sigma$-cotype indices of $L_p$ spaces. We want to point out that this fact was independently discovered by Lee in [12].
Lemma 6.9. The Schatten class $S_q(S_p)$ has sharp Banach cotype $r$ with
\[
\frac{1}{r} = \frac{1}{2} \left( 1 - \frac{1}{p} + \frac{1}{q} \right) \quad \text{whenever} \quad 1 \leq p \leq 2 \quad \text{and} \quad p \leq q \leq p'.
\]

Proof. First, we show that $S_q(S_p)$ has cotype $r$. The case $p > 1$ is simple. Indeed, we just need to check that the predual $S_q'(S_p')$ has Banach type $r'$. To that aim we observe that
\[
S_q'(S_p') = [S_p(S_p'), S_p'(S_p')]_\theta \quad \text{with} \quad 1 - \frac{1}{q} = \frac{1 - \theta}{p} + \theta \left( 1 - \frac{1}{p} \right).
\]
Moreover, we have
\[
S_p(S_p') = [S_2(S_2), S_1(S_\infty)]_\eta \quad \text{with} \quad \frac{1}{p} = \frac{1 - \eta}{2} + \eta.
\]
Hence $S_p(S_p')$ has type $p$ and, since $S_p'(S_p')$ has type $2$, $S_q'(S_p')$ has type $s$ with
\[
\frac{1}{s} = \frac{1 - \theta}{p} + \frac{\theta}{2} = 1 - \frac{1}{q} + \theta \left( \frac{1}{p} - \frac{1}{2} \right) = 1 - \frac{1}{q} + \frac{\theta}{2} \left( \frac{1}{p} - \frac{1}{2} + \frac{1}{q} \right) = 1 - \frac{1}{r}.
\]
It remains to see that $S_q(S_1)$ has cotype $2q$. Let us denote by $R_p$ the subspace generated in $L_p(\Omega)$ by the sequence $r_1, r_2, \ldots$ of Rademacher functions. Then, if $R_p(E)$ stands for the closure of the tensor product $R_p \otimes E$ in $L_p(\Omega; E)$, we need to see that the following mapping is bounded
\[
C_{2q} : \sum_{k=1}^n r_k \otimes x_k \in R_2(S_q(S_1)) \mapsto \sum_{k=1}^n \delta_k \otimes x_k \in \ell_{2q}(S_q(S_1)).
\]
First we recall that, according to Khintchine-Kahane and Minkowski inequalities, the following natural map is contractive
\[
R_2(S_q(S_1)) \simeq S_q(R_q(S_1)) \rightarrow S_q(R_1(S_1)).
\]
By the well-known complete isomorphism $R_1 \simeq R + C$, which follows from the noncommutative Khintchine inequalities (see [14, 22]), we can write $S_q(R_q(S_1))$ as the sum $S_q(S_1(R)) + S_q(S_1(C))$. Therefore, it suffices to see that the following natural mappings
\[
S : \quad S_q(S_1(R)) \rightarrow \ell_{2q}(S_q(S_1))
\]
\[
T : \quad S_q(S_1(C)) \rightarrow \ell_{2q}(S_q(S_1)),
\]
which send the canonical basis of $R$ or $C$ to the canonical basis of $\ell_{2q}$, are bounded. Since both cases are similar, we only prove the boundedness of $T$. To that aim recall that, since $S_q(S_1(C)) = [S_\infty(S_1(C)), S_1(S_1(C))]_{1/q}$, it suffices to prove the boundedness of
\[
T_0 : \quad S_\infty(S_1(C)) \rightarrow \ell_\infty(S_\infty(S_1))
\]
\[
T_1 : \quad S_1(S_1(C)) \rightarrow \ell_2(S_1(S_1)).
\]
If we observe that $T_0$ factors through $S_\infty(S_1(\ell_\infty))$, it is clear that $T_0$ is even contractive. To show that $T_1$ is bounded, let us consider a finite family $x_1, x_2, \ldots, x_n$ of elements in $S_1(S_1)$. Then, since $S_1(S_1(C))$ embeds completely isometrically in $S_1(N^3)$, we know from [28] that it has Banach cotype $2$ so that we get
\[
\left\| \sum_{k=1}^n \delta_k \otimes x_k \right\|_{\ell_2(S_1(S_1))} \leq c \int_0^1 \left\| \sum_{k=1}^n r_k(t)(x_k \otimes e_k) \right\|_{S_1(S_1(C))} dt
\]
\[
\leq \left( \sum_{k=1}^n \left\| x_k \otimes e_k \right\|^2_{S_1(S_1(C))} \right)^{1/2}
\]
\[
= \left( \sum_{k=1}^n \left\| x_k \otimes e_k \right\|^2_{S_1(S_1(C))} \right)^{1/2}
\]
\[
\begin{align*}
&= c \left\| \sum_{k=1}^{n} x_k \otimes e_k \right\|_{S_1(S_1(C))}.
\end{align*}
\]

The last equality follows since
\[
\left\| \sum_{k=1}^{n} r_k(t)(x_k \otimes e_k) \right\|_{S_1(S_1(C))} = \left\| \left( \sum_{k=1}^{n} (r_k(t)x_k)^* (r_k(t)x_k) \right)^{1/2} \right\|_{S_1(S_1)}.
\]

This gives the boundedness of \( T_1 \) and consequently the map \( C_{2q} \) is also bounded. In summary, we have seen that \( S_q(S_p) \) has Banach cotype \( r \) in the range of parameters considered. To complete the proof, we need to see that this exponent is sharp. However, recalling that
\[
S_q(S_p) = C_q \otimes_h C_p \otimes_h R_p \otimes_h R_q,
\]
we can regard \( C_q \otimes_h C_p \) as a subspace of \( S_q(S_p) \). Now, since the Haagerup tensor product commutes with complex interpolation, we obtain the following Banach space isometries
\[
C_q \otimes_h C_p = [C_{q'} \otimes_h C_{p'}, C_p \otimes_h C_p]_{\theta} = [S_{p'}, S_2]_{\theta} \quad \text{with} \quad \frac{1}{q'} = 1 - \frac{1}{q} + \theta \left( \frac{2}{p} - 1 \right).
\]

This gives that \( C_q \otimes_h C_p = S_r \), as a Banach space, we leave the details to the reader. Therefore, \( S_q(S_p) \) can not have better cotype than \( r \). This completes the proof. \( \Box \)

Let us recall that the commutative set of parameters \((\Sigma_0, d_{\Sigma_0})\) is the given by \( \Sigma_0 = N \) where we take \( d_\sigma = 1 \) for all \( \sigma \in \Sigma_0 \). In the following result we show that, in contrast with the Banach space situation, any infinite dimensional (commutative or noncommutative) \( L_p \) space with \( p \neq 2 \) fails to have \( \Sigma \)-cotype 2.

**Theorem 6.10.** Any infinite dimensional \( L_p \) space has sharp \( \Sigma \)-cotype \( \max(p, p') \).

**Proof for \( d_{\Sigma} \) bounded.** As it was pointed out in \[4\], it is obvious the any \( L_p \) space has \( \Sigma \)-cotype \( \max(p, p') \) with respect to any set of parameters \( \Sigma \). Let us see that this exponent is sharp when \( d_{\Sigma} \) is bounded. In this particular case, it clearly suffices to consider the commutative set of parameters \( \Sigma_0 \). We also assume that \( 1 \leq p \leq 2 \) since the case \( 2 \leq p \leq \infty \) has been considered in Remark \[66\]. Moreover, since any infinite dimensional \( L_p \) space contains a completely isometric copy of \( \ell_p \), it suffices to check it for \( \ell_p \). Now, let us assume that \( \ell_p \) has \( \Sigma_0 \)-cotype \( q' \), for some \( q' < p' \). Then we can argue as in Corollary \[66\]. Namely, combining Theorem \[12\] with Minkowski inequality for operator spaces, we have
\[
K^2_{q'}(S_{q'}(S_p); S_{\Sigma_0}) \leq K^2_{q'}(S_{q'}(\mathbb{N}^2; \ell_p); S_{\Sigma_0}) \leq K^2_{q'}(\ell_p; S_{\Sigma_0}).
\]

Now, by Lemma \[69\] the best \( \Sigma_0 \)-cotype we can expect to have is \( r \) where
\[
\frac{1}{r} = \frac{1}{2p'} + \frac{1}{2q'} < \frac{1}{q'}.
\]
Therefore, we deduce that \( r > q' \) and the result follows by contradiction. \( \Box \)

**Proof for \( d_{\Sigma} \) unbounded.** Arguing as in the previous case, it suffices to see that \( \ell_p \) has sharp \( \Sigma \)-cotype \( p' \) for \( 1 \leq p \leq 2 \). Let us assume that \( \ell_p \) has \( \Sigma \)-cotype \( q' \) for some \( q' < p' \). Then, again by Theorem \[12\] and Minkowski inequality, the space \( S_{q'}(S_p) \) should have \( \Sigma \)-cotype \( q' \). However, recalling that
\[
S_{q'}(S_p) = C_{q'} \otimes_h C_{p'} \otimes_h R_p \otimes_h R_{q'},
\]
we conclude that the subspace \( C(p, q) = C_{p'} \otimes_h C_p = C_{q'} \otimes_h R_{p'} \) of \( S_{p'}(S_p) \) must also have \( \Sigma \)-cotype \( q' \). Then, we proceed as in Theorem 6.4. Namely, let us consider a function \( f : \Omega \to C(p, q) \) so that

\[
\hat{f}_S(\xi) = 0 \quad \text{for} \quad \xi \in \Sigma \setminus \{\sigma\}
\]

and such that

\[
\hat{f}_S(\sigma) = \sum_{i,j=1}^{d_{\sigma}} e_{i1} \otimes e_{1j} \otimes e_{1j} \otimes e_{1j} \in C_{p'} \otimes_h C(p, q) \otimes_h R_{q'}^{d_{\sigma}}.
\]

By the definition of \( \Sigma \)-cotype we have

\[
\|\hat{f}_S(\sigma)\|_{S_{q'}^{d_{\sigma}}(C(p, q))} \leq K_{q'}^2(C(p, q), S_{\Sigma})(\int_\Omega \left\|d_{1/q}\text{tr}(\hat{f}_S(\sigma)\zeta^\sigma)\right\|^q_c d\mu)^{1/q}.
\]

Now using the Banach space isometry

\[
S^u_s = C^u_v \otimes_h R^u_v \quad \text{for} \quad \frac{1}{s} = \frac{1}{2u} + \frac{1}{2v},
\]

which follows easily by complex interpolation, we obtain

\[
\|\hat{f}_S(\sigma)\|_{S_{q'}^{d_{\sigma}}(C(p, q))} = \left\|\sum_{k=1}^{n} e_{kk}\right\|_{C_{p'}^{d_{\sigma}} \otimes_h C_{p'}^{d_{\sigma}}} \left\|\sum_{k=1}^{n} e_{kk}\right\|_{C_{p}^{d_{\sigma}} \otimes_h R_{q'}^{d_{\sigma}}} = d_{\sigma}^{1/2 + 1/2u}.
\]

Moreover, since \( \text{tr}(\hat{f}_S(\sigma)\zeta^\sigma(\omega)) = \sum_{i,j=1}^{d_{\sigma}} e_{ij} \otimes \zeta^\sigma_i(\omega) = (\zeta^\sigma(\omega))^k \), we have

\[
\left(\int_\Omega \left\|d_{1/q}\text{tr}(\hat{f}_S(\sigma)\zeta^\sigma(\omega))\right\|^q_c d\mu(\omega)^{1/q} = d_{\sigma}^{1/2 + 1/2u}.
\]

Combining our previous results, we obtain \( q' \geq p' \). This completes the proof. \( \Box \)

**Remark 6.11.** By duality, the sharp \( \Sigma \)-type index of \( L_p \) is \( \min(p, p') \).

**Remark 6.12.** By a standard argument using the contraction principle, our results for sharp \( \Sigma \)-type and \( \Sigma \)-cotype also hold for any uniformly bounded quantized orthonormal system. The reader is referred to [1] for further details.

**Remark 6.13.** As it was recalled in Remark 6.12, it seems that there is no analog of the Maurey-Pisier theorem for operator spaces. Theorem 6.4 clearly reinforces that idea. Finally, the reader is referred to Section 4.2 of [7] for an unrelated notion of operator space cotype 2 for which \( L_p \) has cotype 2 whenever \( 1 \leq p \leq 2 \).

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