INSTANTANEOUS BETHE–SALPETER EQUATION: IMPROVED ANALYTICAL SOLUTION

Wolfgang LUCHA
Institut für Hochenergiephysik, Österreichische Akademie der Wissenschaften, Nikolsdorfergasse 18, A-1050 Wien, Austria

Franz F. SCHÖBERL
Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Wien, Austria

Abstract

Studying the Bethe–Salpeter formalism for interactions instantaneous in the rest frame of the bound states described, we show that, for bound-state constituents of arbitrary masses, the mass of the ground state of a given spin may be calculated almost entirely analytically with high accuracy, without the (numerical) diagonalization of the matrix representation obtained by expansion of the solutions over a suitable set of basis states.

PACS numbers: 11.10.St, 03.65.Ge

* E-mail address: wolfgang.lucha@oeaw.ac.at
† E-mail address: franz.schoeberl@univie.ac.at
1 Introduction

The Bethe–Salpeter equation in instantaneous approximation for its interaction kernel has proven to represent a very powerful tool for the description of bound states within the framework of relativistic quantum field theories. In contrast to the Bethe–Salpeter equation, the instantaneous Bethe–Salpeter equation (or “Salpeter equation”) may be formulated as an eigenvalue problem for the “Salpeter amplitudes” which describe the bound states under study; its eigenvalues are the masses $M$ of these bound states. In the physical sector of bound states of positive norm, all mass eigenvalues $M$ are real [1].

In a recent series of papers [2, 3, 4], we developed a technique for the (approximate) determination of the mass eigenvalues $M$ and the corresponding Salpeter amplitudes $\chi$ in an almost entirely analytical way, by conversion of the instantaneous Bethe–Salpeter equation into an eigenvalue problem for an explicitly given, analytically known matrix. Here we demonstrate, for our (now almost “standard”) example, how the ground-state mass $M$ for given spin of the bound state may be obtained, with comparable precision, even without the necessity of the numerical diagonalization of a (rather large) matrix.

2 $PC = -1$ instantaneous Bethe–Salpeter equation

In order to facilitate an eventual comparison with the results of previous investigations, we try to imitate the analysis of Refs. [2, 3, 4] to the utmost possible extent. First of all, let us base our formalism on the same two simplifying assumptions as in Refs. [2, 3, 4]:

1. Every (full) fermion propagator entering in the Bethe–Salpeter equation may be approximated by the corresponding free one, involving a mass parameter which then must be interpreted as some effective mass of the bound-state constituents.

2. The particles forming the bound state under consideration have equal masses $m$.

Moreover, let us focus our interest to fermion–antifermion bound states with spin $J$, parity $P = (-1)^{J+1}$ and charge-conjugation quantum number $C = (-1)^J$ (this means $PC = -1$ for all $J$), denoted by $^1J_J$ in the usual spectroscopic notation. The Salpeter amplitude $\chi$ describing these states involves two independent components, $\Psi_1$ and $\Psi_2$. Its momentum-space representation $\chi(k)$ is given, for two fermions of equal masses $m$ and internal momentum $k$, in the center-of-momentum frame of the bound state, by

$$\chi(k) = \left[ \Psi_1(k) \frac{m - \gamma \cdot k}{E(k)} + \Psi_2(k) \gamma^0 \right] \gamma_5,$$

where

$$E(k) \equiv \sqrt{k^2 + m^2}, \quad k \equiv |k|,$$

denotes the energy of a free particle of mass $m$ and momentum $k$. Assume that the Salpeter amplitude $\chi$ describes bound states with the total spin $J$ and its projection $J_3$. Its two independent components $\Psi_1(k)$ and $\Psi_2(k)$ may then be factorized, according to

$$\Psi_i(k) = \Psi_i(k) \mathcal{Y}_J J_3(\Omega), \quad i = 1, 2,$$

into the radial wave functions $\Psi_1(k)$ and $\Psi_2(k)$, and the spherical harmonics $\mathcal{Y}_{\ell m}(\Omega)$ for angular momentum $\ell$ and its projection $m$; the latter depend on the solid angle $\Omega$, which encompasses the angular variables, and satisfy the orthonormalization condition

$$\int d\Omega \mathcal{Y}_{\ell m}^*(\Omega) \mathcal{Y}_\ell m'(\Omega) = \delta_{\ell \ell'} \delta_{mm'}.$$
A crucial point in the construction of any Bethe–Salpeter model for bound states is the determination of the Lorentz structure of the Bethe–Salpeter kernel. According to the analyses presented in Refs. [3, 4, 5], for an interaction potential rising linearly with the distance of the two bound-state constituents—as is, for instance, frequently used in relativistic quark models of hadrons in order to describe the confining quark–antiquark interaction arising from quantum chromodynamics; see, e.g., Refs. [8, 9]—a kernel with a pure time-component Lorentz-scalar Dirac structure certainly does not. For this reason, a pure time-component Lorentz-vector Dirac structure yields stable solutions whereas a kernel with a pure Lorentz-scalar Dirac structure certainly does not. For this reason, we have chosen in Refs. [2, 3, 4] to discuss pure time-component Lorentz-vector kernels.

Finally, merely for notational simplicity, we confine ourselves to the case $J = 0$, i.e., to $^1S_0$ bound states, with the spin-parity-charge conjugation assignment $J^{PC} = 0^+$. For pure time-component Lorentz-vector interactions, that is, for a kernel with the Dirac structure $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$, the instantaneous Bethe–Salpeter equation describing $PC = -1$ fermion–antfermion bound states with total spin $J = 0$ is equivalent to the following set of coupled equations for the radial wave functions $\Psi_1(k)$ and $\Psi_2(k)$ [1, 4]:

$$2 E(k) \Psi_2(k) + \int_0^\infty \frac{dk'}{(2\pi)^2} V_0(k, k') \Psi_2(k') = M \Psi_1(k) ,$$

$$2 E(k) \Psi_1(k) + \int_0^\infty \frac{dk'}{(2\pi)^2} \frac{m^2 V_0(k, k') + k k' V_1(k, k')}{E(k) E(k')} \Psi_1(k') = M \Psi_2(k) ; \quad (1)$$

the interaction between the bound-state constituents, described by the static potential $V(r)$ in configuration space, enters in form of the expressions

$$V_L(k, k') \equiv 8\pi \int_0^\infty dr r^2 V(r) j_L(k r) j_L(k' r) , \quad L = 0, 1 , \quad (2)$$

where $j_n(z)$ ($n = 0, \pm 1, \pm 2, \ldots$) are the spherical Bessel functions of the first kind [11].

3 Ground-state mass eigenvalue

The structure of the $PC = -1$ instantaneous Bethe–Salpeter equation [1] suggests to solve this set of equations by expressing, for bound-state mass $M \neq 0$, from the first of Eqs. (1), the Salpeter component $\Psi_1(k)$ in terms of the Salpeter component $\Psi_2(k)$, and inserting the resulting expression into the second of Eqs. (1). By this procedure the set of equations (1) is reduced to an eigenvalue equation for $\Psi_2(k)$ with $M^2$ as eigenvalue:

$$M^2 \Psi_2(k) = 4 E^2(k) \Psi_2(k) + 2 E(k) \int_0^\infty \frac{dk'}{(2\pi)^2} V_0(k, k') \Psi_2(k')$$

$$+ \int_0^\infty \frac{dk'}{(2\pi)^2} \frac{m^2 V_0(k, k') + k k' V_1(k, k')}{E(k)} \Psi_2(k')$$

$$+ \int_0^\infty \frac{dk''}{(2\pi)^2} \frac{m^2 V_0(k, k') + k k' V_1(k, k')}{E(k) E(k')} \int_0^\infty \frac{dk'''}{(2\pi)^2} V_0(k', k'') \Psi_2(k'') . \quad (3)$$

Our goal is to find an as far as possible analytic albeit approximate characterization of the bound-state masses $M$. To this end, we rely on an approximate description of the analyzed bound states by (trial) states $|\phi\rangle$ which involve a real variational parameter $\mu$. 


Our choice for \( |\phi\rangle \) makes use of the exponential; our \( |\phi\rangle \) is defined in terms of its real configuration-space or momentum-space representation \( \phi(r) \) or \( \phi(p) \), respectively, by

\[
\phi(r) = 2 \mu^{3/2} \exp(-\mu r) \quad \phi(p) = \sqrt{\frac{2}{\pi}} \frac{4 \mu^{5/2}}{(p^2 + \mu^2)^2},
\]

Normalizability of the Hilbert-space states \( |\phi\rangle \) requires \( \mu \) to be strictly positive: \( \mu > 0 \). These trial functions \( \phi(r) \) and \( \phi(p) \) are related by the Fourier–Bessel transformations

\[
\phi(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty dp \, p^2 \, j_0(pr) \, \phi(p) \quad \phi(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty dr \, r^2 \, j_0(pr) \, \phi(r),
\]

and they satisfy the normalization conditions

\[
\int_0^\infty dr \, r^2 \, \phi^2(r) = \int_0^\infty dp \, p^2 \, \phi^2(p) = 1.
\]

The latter feature allows us to extract, from Eq. (3), the eigenvalue \( M^2 \) in closed form:

\[
M^2 = 4 \int_0^\infty dk \, k^2 \, E^2(k) \, \phi^2(k) + \frac{2}{(2\pi)^2} \int_0^\infty dk \, k^2 \, E(k) \, \phi(k) \int_0^\infty dk' \, k'^2 \, V_0(k, k') \, \phi(k')
\]

\[
+ \frac{2 \mu^2}{(2\pi)^2} \int_0^\infty \frac{dk \, k^2}{E(k)} \, \phi(k) \int_0^\infty dk' \, k'^2 \, V_0(k, k') \, \phi(k')
\]

\[
+ \frac{2}{(2\pi)^2} \int_0^\infty \frac{dk \, k^3}{E(k)} \, \phi(k) \int_0^\infty dk' \, k'^3 \, V_1(k, k') \, \phi(k')
\]

\[
+ \frac{m^2}{(2\pi)^4} \int_0^\infty \frac{dk \, k^2}{E(k)} \, \phi(k) \int_0^\infty \frac{dk'}{E(k')} \, V_0(k, k') \int_0^\infty dk'' \, k''^2 \, V_0(k', k'') \, \phi(k'')
\]

\[
+ \frac{1}{(2\pi)^4} \int_0^\infty \frac{dk \, k^3}{E(k)} \, \phi(k) \int_0^\infty \frac{dk' \, k'^3}{E(k')} \, V_1(k, k') \int_0^\infty dk'' \, k''^2 \, V_0(k', k'') \, \phi(k'').
\]

Now, the angular-momentum component \( V_1(k, k') \) of the interaction potential \( V(r) \) involves—according to its definition (3)—the spherical Bessel function \( j_1(z) \) of order 1. Consequently, in order to deal with those terms in Eq. (3) which involve the component \( V_1(k, k') \), it is advisable to introduce, in addition, states \( |\psi\rangle \) that are related to angular momentum 1; the configuration-space and momentum-space representations of \( |\psi\rangle \) are then related by a Fourier–Bessel transformation similar to Eq. (3) but involving \( j_1(z) \):

\[
\psi(r) = \frac{2 \mu^{5/2}}{\sqrt{3}} \, r \exp(-\mu r) \quad \psi(p) = -i \sqrt{\frac{2}{3\pi}} \frac{16 \mu^{7/2} p}{(p^2 + \mu^2)^3},
\]

with

\[
\psi(r) = \frac{2}{\pi} \int_0^\infty dp \, p^2 \, j_1(pr) \, \psi(p) \quad \psi(p) = -i \frac{2}{\pi} \int_0^\infty dr \, r^2 \, j_1(pr) \, \psi(r).
\]

These further functions \( \psi(r) \) and \( \psi(p) \) are also normalized to unity, that is, they satisfy

\[
\int_0^\infty dr \, r^2 \, \psi^2(r) = \int_0^\infty dp \, p^2 \, |\psi(p)|^2 = 1.
\]
The evaluation of the kinetic energy in the expression (5) for \( M^2 \) is straightforward:

\[
\int_0^\infty dk \, k^2 \, E^2(k) \, \phi^2(k) = \int_0^\infty dk \, k^4 \, \phi^2(k) + m^2 = \mu^2 + m^2.
\]

However, in order to evaluate the interaction terms in Eq. (5), we have to approximate the various expressions entering in the integrands of these terms in the following way:

\[
E(k) \, \phi(k) = b \, \phi(k), \\
\frac{k}{E(k)} \, \phi(k) = c \, \psi(k), \\
k \, \phi(k) = d \, \psi(k), \\
\frac{1}{E(k)} \, \phi(k) = e \, \phi(k).
\]

Expressed in terms of the coefficients \( b, c, d, \) and \( e \) defined above, the square (5) of the bound-state mass eigenvalue \( M \) of the instantaneous Bethe–Salpeter equation (1) may then be traced back to the expectation values

\[
V^{(0)} \equiv \langle \phi | V(r) | \phi \rangle = \int_0^\infty dr \, r^2 \, V(r) \, \phi^2(r), \\
V^{(1)} \equiv \langle \psi | V(r) | \psi \rangle = \int_0^\infty dr \, r^2 \, V(r) \, \psi^2(r)
\]

of the interaction potential \( V(r) \) with respect to the approximation functions \( \phi \) and \( \psi \):

\[
M^2 = 4 \left( m^2 + \mu^2 \right) + 2 \left( b + m^2 \, e \right) \, V^{(0)} + 2 \, c^* \, d \, V^{(1)} + m^2 \, e^2 \left( V^{(0)} \right)^2 + |c|^2 \, V^{(0)} \, V^{(1)}. \tag{6}
\]

For the factors \( b, c, d, \) and \( e \) analytic expressions may be found; the latter will obviously involve the mass \( m \) of the bound-state constituents and the variational parameter \( \mu \):

\[
b = \int_0^\infty dk \, k^2 \, E(k) \, \phi^2(k) = \mu \, x^4 \left[ \frac{x^2 - 2}{(x^2 - 1)^{5/2}} + \frac{26}{15 \pi \, x^8} \, F \left( 2, 4; \frac{7}{2}; \frac{1}{x^2} \right) \right],
\]

\[
c = \int_0^\infty \frac{dk \, k^3 \, \psi^*(k) \, \phi(k)}{E(k)} = \frac{i \, x^4}{2 \sqrt{3}} \left[ \frac{3 \, x^4 - 16 \, x^2 + 48}{(x^2 - 1)^{9/2}} - \frac{211}{15 \pi \, x^{10}} \, F \left( 3, 5; \frac{7}{2}; \frac{1}{x^2} \right) \right],
\]

\[
d = \int_0^\infty \frac{dk \, k^3 \, \psi^*(k) \, \phi(k)}{E(k)} = \frac{i \, \sqrt{3}}{2 \, \mu},
\]

\[
e = \int_0^\infty \frac{dk \, k^2 \, \phi^2(k)}{E(k)} = \frac{x^2}{\mu} \left[ \frac{x^4 - 4 \, x^2 + 8}{(x^2 - 1)^{7/2}} - \frac{28}{15 \pi \, x^8} \, F \left( 3, 4; \frac{7}{2}; \frac{1}{x^2} \right) \right],
\]

with the Gauss hypergeometric series \( F \), defined, in terms of the gamma function \( \Gamma \), by

\[
F(u, v; w; z) = \frac{\Gamma(w)}{\Gamma(u) \, \Gamma(v)} \sum_{n=0}^\infty \frac{\Gamma(u + n) \, \Gamma(v + n) \, z^n}{n!},
\]
and the abbreviation

\[ x \equiv \frac{m}{\mu} \]

for the ratio of the mass parameters involved. It is rather easy to convince oneself that these expressions for the coefficients \( b, c, d, \) and \( e \) reduce, in the (ultrarelativistic) limit \( m \to 0 \), to the lowest entries of the corresponding matrices given in Ref. [2], where the instantaneous Bethe–Salpeter equation has been studied for the simpler special case of vanishing masses of all bound-state constituents, and, for the particular value \( \mu = m \), to the lowest entries of the corresponding (similarly defined) matrices given in Ref. [3], where, because of the nonvanishing masses of the bound-state constituents, the desired analytical treatment forced us to fix the value of the variational parameter \( \mu \) to \( \mu = m \).

The necessarily numerical optimization of the right-hand side of the analytic result [3] with respect to the variational parameter \( \mu \) then yields a first approximation to the masses \( M \) of bound states described by the instantaneous Bethe–Salpeter equation (1).

### 4 Accuracy of the approximation

The decisive question clearly is whether it is possible to achieve a satisfactory accuracy of the bound-state mass \( M \) calculated from the approximation represented by Eq. (6). The answer to this question will, of course, depend on the interaction potential \( V(r) \).

The general expressions for the expectation values of \( V(r) \) for arbitrary power-law potentials may be deduced from Refs. [11, 12, 13]. Here we study as a simple but, from the physical point of view, nevertheless relevant example the case of a linear potential:

\[ V(r) = \lambda r, \quad \lambda > 0. \]

Inserting the required expectation values \( V^{(0)} \) and \( V^{(1)} \) of this potential (which may be found, e.g., in Refs. [2, 3]),

\[ V^{(0)} = \frac{3\lambda}{2\mu}, \quad V^{(1)} = \frac{5\lambda}{2\mu}, \]

into our general formula (3) for the mass squared of the lowest \( 1\text{S}_0 \) bound state yields

\[ M^2 = 4 \left( m^2 + \mu^2 \right) + \frac{3\lambda}{\mu} \left( b + m^2 e \right) + \frac{5\lambda}{\mu} c^* d + \frac{3\lambda^2}{4\mu^2} \left( 3m^2 e^2 + 5|c|^2 \right). \] (7)

Minimization of this expression with respect to \( \mu \), for \( m = 0.1 \text{ GeV} \) and \( \lambda = 0.2 \text{ GeV}^2 \), gives \( M = 1.703 \text{ GeV} \); this is only 2.5\% larger than the “exact” value \( M = 1.661 \text{ GeV} \) computed in Ref. [3] by the diagonalization of a \( 25 \times 25 \) matrix (cf. Table 1 of Ref. [3]). Table [1] compares the squared masses predicted by Eq. (7) with the findings presented in Table 1 of Ref. [3]. Again, the relative errors are only of the order of a few percent.

### 5 Summary and conclusion

The demand to derive, also for the case of massive bound-state constituents, analytical matrix representations of the instantaneous Bethe–Salpeter equation led us, in Ref. [3], to identify the variational parameter in the trial states used in Ref. [2] with the mass \( m \) of the bound-state constituents. However, the rate of convergence of the corresponding bound-state masses \( M \) with increasing matrix size is, because of this identification, for small but nonvanishing masses \( m \) of the bound-state constituents not extremely rapid. For a first idea of the location of \( M \), one might want to avoid matrix diagonalizations.
Table 1: Lowest mass eigenvalues $M$ of the instantaneous Bethe–Salpeter equation, with a time-component Lorentz-vector interaction kernel, describing bound states with spin-parity-charge conjugation assignment $J^{PC} = 0^{-+}$ of two spin-$\frac{1}{2}$ fermions of mass $m$, which experience a confining interaction described by a linear potential $V(r) = \lambda r$ with slope $\lambda = 0.29$ GeV$^2$. The values of $M^2$ arising, for various values of the mass $m$ of the bound-state constituents, within the present approach (third column) differ by a few percent (fourth column) from the values of $M^2$ obtained in Ref. [5] by expanding the Salpeter amplitude $\chi$ over a basis of 25 harmonic-oscillator eigenfunctions (second column). Moreover, the relative errors decrease with increasing constituents’ mass $m$.

| $m$ [MeV] | $M^2$ [GeV$^2$] (Ref. [5]) | $M^2$ [GeV$^2$] (present work) | Relative Error of $M^2$ [%] | Relative Error of $M$ [%] |
|-----------|-----------------|-----------------|-----------------|-----------------|
| 300       | 4.357           | 4.568           | 4.8             | 2.4             |
| 500       | 5.255           | 5.478           | 4.2             | 2.1             |
| 900       | 8.248           | 8.515           | 3.2             | 1.6             |

Here, by applying the same variational technique as used in Ref. [2], we have been able to demonstrate that, at least for ground states, the bound-state mass $M$ may be found with reasonable accuracy, certainly sufficient for the initial steps of a fitting procedure.

**References**

[1] J.-F. Lagaë, Phys. Rev. D 45 (1992) 305.
[2] W. Lucha, K. Maung Maung, and F. F. Schöberl, Phys. Rev. D 63 (2001) 056002.
[3] W. Lucha, K. Maung Maung, and F. F. Schöberl, Phys. Rev. D 64 (2001) 036007.
[4] W. Lucha, K. Maung Maung, and F. F. Schöberl, Vienna preprint HEPHY-PUB 733/00 (2000), [hep-ph/0010078](http://arxiv.org/abs/hep-ph/0010078), to appear in: Proceedings of the International Conference on *Quark Confinement and the Hadron Spectrum IV*, edited by W. Lucha and K. Maung Maung (World Scientific).
[5] J. Parramore and J. Piekarewicz, Nucl. Phys. A 585 (1995) 705.
[6] J. Parramore, H.-C. Jean, and J. Piekarewicz, Phys. Rev. C 53 (1996) 2449.
[7] M. G. Olsson, S. Veseli, and K. Williams, Phys. Rev. D 52 (1995) 5141.
[8] W. Lucha, F. F. Schöberl, and D. Gromes, Phys. Rep. 200 (1991) 127.
[9] W. Lucha and F. F. Schöberl, Int. J. Mod. Phys. A 7 (1992) 6431.
[10] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1964).
[11] W. Lucha and F. F. Schöberl, Phys. Rev. A 56 (1997) 139.
[12] W. Lucha and F. F. Schöberl, Int. J. Mod. Phys. A 14 (1999) 2309.
[13] W. Lucha and F. F. Schöberl, Fizika B 8 (1999) 193.