Errata to: A New Graph over Semi-Direct Products of Groups

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Abstract

The goal of the paper “A new graph over semi-direct products of groups” is to define a graph $\Gamma(G)$ on a group $G$ when $G$ splits over a normal subgroup. We demonstrate herein that the graph is ill-defined. We also attempt to ascertain causes for the discrepancies.

MSC: Primary 20A99, 05C12

1 Background

In group theory, a familiar theme is the application of group invariants that can be used to demonstrate two groups are not isomorphic. In this vein, placing a graph structure on groups and showing corresponding graph invariants differ shows the graphs are distinct, hence the underlying groups are nonisomorphic. The goal of the paper [3] is such a graph. There however is a huge problem with the paper; the graph it presents is ill-defined. In fact, the paper explicitly presents two isomorphic groups then computes distinct graph invariants thereof. Hence, the purpose of this paper is to demonstrate the graph is in fact ill-defined.

The graph in question is described as being on groups $G$ which are split extensions $G = K \rtimes A$. Letting $[X; r]$ and $[Y; s]$ be respectively presentations for $K$ and $A$ (with $X$ the generators and $r$ the relations), the graph $\Gamma(G)$ has vertex set $G$ and (quoting the paper directly) “... the edge set $E$ is obtained by the following steps:

(I) Each of the vertices in this graph must be adjoined to the vertex $1_G$ (except $1_G$ itself since the graph is assumed to be simple).

(II)(i) For any two vertices $w_1 = x_1^{\varepsilon_1}x_2^{\varepsilon_2}...x_m^{\varepsilon_m}$ and $w_2 = y_1^{\delta_1}y_2^{\delta_2}...y_n^{\delta_n}$ (where $n \geq 2$, $\varepsilon_i$ and $\delta_i$ are integers) and for all $x_i, y_j \in X \cup Y$ ($1 \leq i \leq m, 1 \leq j \leq n$), if $x_i \neq y_j$, then $w_1$ is adjoined to the $w_2$ (shortly, $w_1 \sim w_2$).

(ii) As a consequence of (i), for any two vertices $w_1 = x_i^k$ and $w_2 = x_j^t$ ($1 \leq i, j \leq n$, $i \neq j$, and $k, t$ are integers), we can directly take $w_1 \sim w_2$. However, to adjoin $w_1$ and $w_2$ while $i = j$, it must be $k \neq t$.”

Clearly, whether or not an edge connects two vertices is strongly tied to the presentation. To mitigate this dependence, the article states “... all elements $z_i$ ($i = 1, 2, ..., k$) in the generating set $X \cup Y$ of $G$ will be formed as $z_i \neq z_1^{z_1}z_2^{z_2}...z_k^{z_k}$, where $k \geq 2$ according to the Normal Form Theorem (NFT) (see [11]).” (The article’s reference [11] is our reference [2].)

2 Counterexample

In trying to digest the above definition (wherein 0 exponents may or may not be allowed, wherein order of multiplicands is not specified, wherein (II)(i) claims the adjacency of $x_i^k$ and $x_j^t$ when $i \neq j$ follows from (II)(i) while (II)(i) has a hypothesis that there be at least $n \geq 2$ generators in at least one of the vertices, etc.), this article’s author turned to the examples presented in [3] to see what interpretations were used there; specifically, Examples 2.9 and 2.10. Example 2.9 is given as the dihedral group $D_8$ viewed as the split extension of the cyclic $C_4$ being acted upon by $C_2$ (the action, of course, being inversion).
Example 2.10 is the group $G$ defined as the split extension of the Klein 4-group $V_4$ being acted upon by $C_2$. Since “... the homomorphism $\varphi$ will always be not identity $id_{G}$ unless stated otherwise,” we conclude this is a nontrivial action, in which case $C_2$ exchanges two of the nontrivial members, while fixing the third. The article \cite{3} calculates the respective degree sequences as

$$DS(\Gamma(D_8)) = \{1,1,1,4,4,4,7\} \text{ and } DS(\Gamma(G)) = \{1,2,2,2,4,4,7\}.$$  

In point of fact, $D_8$ and $G = V_4 \rtimes C_2$ are isomorphic. To see this, we invoke (23.4) on page 107 of \cite{1}, which states:

Let $G$ be a nonabelian group of order $p^n$ with a cyclic subgroup of index $p$. Then $G \cong \text{Mod}_{p^n}$, $D_{2^n}$, $SD_{2^n}$, or $Q_{2^n}$.

The paragraph immediately preceding this result states the modular group $\text{Mod}_8 = D_8$, while two paragraphs further back states the semidihedral group $SD_{2^n}$ is only defined for $n \geq 4$. Now, consider an arbitrary nonabelian group $H$ of order 8. Since every group of exponent 2 is necessarily abelian, $H$ must have an element of order 4. It therefore has a cyclic subgroup of index 2 and thus satisfies the theorem. Consequently, there are exactly 2 nonabelian groups of order 8: the dihedral group $D_8$ and the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. The only involution in $Q_8$ is $-1$. Therefore, whenever a nonabelian group $H$ of order 8 has more than one element of order 2, then necessarily $H$ is dihedral. The subgroup $V_4$ of $G = V_4 \rtimes C_2$ already contains 3 involutions; $G$ is therefore not quaternion, and is then necessarily dihedral.

3 The Errors

Given that the same group yielded distinct graphs, where are the logical flaws in \cite{3}? One place where fault can be found is in the quoting of the Normal Form Theorem (NFT). Although not mentioned in the article \cite{3}, the open letter makes explicit that the NFT being referenced is found on page 31 of \cite{2}. This NFT is in regards to an amalgamated free product, not to a semidirect product. Another issue at point, which comes up in the given examples above, is that a semidirect factorization of groups is not unique; a group $G$ can split over nonisomorphic, normal subgroups $N_1, N_2$ even with $|N_1| = |N_2|$. In short, there is no clear way to adjust this graph’s definition to allow the resulting object to reflect the underlying group’s structure.

References

[1] M. Aschbacher, Finite Group Theory, (2nd edition), Cambridge University Press, Cambridge, 2000.

[2] D.E. Cohen, Combinatorial Group Theory: a Topological Approach, Cambridge University Press, Cambridge, 1989.

[3] S. Topkaya , A.S. Cevik, A new graph over semi-direct products of groups, Filomat 30 (2016), 611–619.

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\footnote{Proof: $xxyy = x^2y^2 = 1 = (xy)^2 = xyyx$ implies $xy = yx$ over all $x, y$ in the group.}

\footnote{A look at \cite{2}’s index reveals four NFTs; there was thus need for external verification as to which was being used.}