About the effects of rotation on the Landau levels in an elastic medium with a spiral dislocation

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Abstract
In this Paper we analyze a model proposed recently with the purpose of studying the effects of rotation on the interaction of a point charge with a uniform magnetic field in an elastic medium with a spiral dislocation. In particular we focus on the approximation proposed by the authors that consists of changing the left boundary condition in order to obtain analytical results. We show that this approximation leads to quantitative and qualitative errors, the most relevant one being a wrong prediction of the level spacing.

1 Introduction
In a recent paper Maia and Bakke analyzed the effects of rotation on the interaction of a point charge with a uniform magnetic field in an elastic medium with
a spiral dislocation. In order to obtain analytical solutions to the Schrödinger equation in a rotating frame with a constant angular velocity Maia and Bakke [1] changed the boundary conditions of the model. Based on this approximation they concluded that both the topology of the defect and rotation modify the degeneracy of the Landau levels.

The purpose of this paper is to analyze the effect of the change of the left boundary condition on the results because Maia and Bakke [1] did not discuss this point in detail and their weak argument leaves much to be desired. In section 2 we derive the analytical results by means of the Frobenius method because it is clearer than the approach based on the confluent hypergeometric function followed by Maia and Bakke [1]. In section 3 we solve the eigenvalue equation with the correct left boundary condition and compare the numerical results thus obtained with the analytical ones. Finally, in section 4 we summarize the main results and draw conclusions.

2 Analytical results

In this section we outline some of the results derived by Maia and Bakke [1]. Our starting point is the radial differential equation

\[
\left(1 + \frac{\beta^2}{r^2}\right) h'' + \left(\frac{1}{r} - \frac{\beta^2}{r^3}\right) h' - \frac{l^2 + m\omega l \beta^2}{r^2 + \beta^2} h - \frac{m^2 \delta^2 r^4}{4 (r^2 + \beta^2)} h
\]

\[
- \frac{m^2 \beta^2 (\delta^2 - \omega^2) r^2}{4 (r^2 + \beta^2)} h + \left[2m (E + \Omega l) - k^2 + m\omega l\right] h = 0,
\]

(1)

derived by those authors. By means of the change of variables \( y = m\delta \left(r^2 + \beta^2\right)/2 \) they obtained the simpler equation

\[
y^2 g'' + yg' - \frac{\gamma^2}{4} g - \frac{y^2}{4} g + \tau yg = 0,
\]

\[
\gamma = l + \frac{m\omega \beta^2}{2},
\]

\[
\tau = \frac{1}{2m\delta} \left[2m (E + \Omega l) - k^2 + m\omega l + \frac{m^2 (\delta^2 + \omega^2) \beta^2}{4}\right].
\]

(2)
Maia and Bakke \[1\] resorted to the same function \( h \) for equations (1) and (2). However, we decided to use a different name because \( h(z) \) and \( g(z) \) are obviously different because \( h(r) = (g \circ y)(r) = g(y(r)) \). The authors stated that “Henceforth, let us impose that \( h(y) \to 0 \) when \( y \to \infty \) and \( y \to 0 \). Note that, since \( 0 < \beta < 1 \), then, we can assume that \( \beta^2 << 1 \). Thus, when \( r \to 0 \), we can consider \( y \to 0 \) without loss of generality [13].” In what follows we estimate the effect of such drastic change of the left boundary condition.

Maia and Bakke \[1\] derived the allowed values of \( \tau \) by writing \( g(y) \) in terms of the confluent hypergeometric function. Here, we resort to the Frobenius method and write

\[
g(y) = y^{\gamma/2} e^{-y/2} \sum_{j=0}^{\infty} c_j y^j,
\]

that leads to the following recurrence relation for the coefficients:

\[
c_{j+1} = A_j c_j, \quad A_j = \frac{\gamma + 2j - 2\tau + 1}{2(j+1)(\gamma + j + 1)}.
\]

In order to obtain solutions with the correct behaviour when \( y \to \infty \) we have to choose \( \tau \) so that the infinite series in equation (3) terminates. The requirement \( c_n \neq 0 \) and \( c_{n+1} = 0 \), \( n = 0, 1, \ldots \), leads to \( c_j = c_{jn} = 0 \) for all \( j > n \) and the exact solutions

\[
g_n(y) = y^{\gamma/2} e^{-y/2} \sum_{j=0}^{n} c_{jn} y^j,
\]

for

\[
\tau = \tau_n = \frac{1}{2} \left( 2n + 1 + |\gamma| \right),
\]

that agrees with the one derived by Maia and Bakke \[1\]. The factor \( A_j \) takes the simpler form

\[
A_j = \frac{j - n}{(j+1)(\gamma + j + 1)}.
\]

3 Exact boundary conditions

When \( r = 0 \) then \( y = y_0 = m \delta \beta^2 / 2 \) and the correct boundary condition for the differential equation (2) is \( g(y_0) = 0 \). The choice \( y_0 = 0 \) is rather unphysical
because it leads to $r^2 = -\beta^2$; however, Maia and Bakke [1] stated that they could resort to this approximation without loss of generality because $\beta^2 \ll 1$ as mentioned above. In what follows we analyze to which extent this approximation is reasonable. To begin with, note that the behaviour at origin of the solution to the differential equation (1) is $h \sim r^2$ while, on the other hand, the behaviour at origin of the solution to (2) chosen by Maia and Bakke [1] is $g \sim y^{\gamma}/2$. Under the approximation $\beta \sim 0$ it leads to $g \sim r^{\gamma}$ that is consistent with the correct asymptotic behaviour at origin only when $|\gamma| = 2$. Besides, $g'(y)$ diverges at $y = 0$ when $0 < |\gamma| < 2$.

In order to solve the differential equation (2) with the proper boundary condition we resort to the shooting method. The choice of suitable values for the model parameters is rather difficult because the equations reported by Maia and Bakke [1] do not exhibit unit consistency. They did not indicate the chosen units explicitly but we assume that they followed early papers in which they stated that “we shall use the units $\hbar = 1, c = 1$” [2]. Unfortunately, this expression does not mean anything because we do not know what are actually the units of length, energy, etc. (see [4] for a clear pedagogical discussion of the subject and an earlier criticism of this undesirable practice [5]). The fact is that all their equations lack of unit consistency. Consider, for example, the expression for $\gamma$ in equation (2) where we appreciate that $l$ is an integer and $m\omega\beta^2/2$ exhibits units of energy. It is not clear how the authors converted the latter term into a dimensionless one because they did not explain it. As a result the expression for the energy $E_{n,l,k}$ shown by Maia and Bakke [1] is extremely inconsistent. For example, the term $\omega l$ should be multiplied by $\hbar$ in order to have units of energy; on the other hand $k^2/(2m)$ should be multiplied by $\hbar^2$ in order to have the same units. What should we do with the term $m\omega\Omega\beta^2/2$ that exhibits units of frequency×energy? Since it is not possible to estimate physically reasonable values of $y_0$ and $\gamma$ we choose them arbitrarily.

Figure 1 shows that the eigenvalues $\tau_n$ increase monotonously with $y_0$ and that this behaviour increases with the radial quantum number. This effect has not been taken into account by Maia and Bakke [1] because they only considered
Figure 2 shows that the eigenvalues $\tau_n$ increase monotonously with $\gamma$, exactly as the analytical expressions (6). The discrepancy produced by the boundary condition at $y_0$ seems to be less relevant as $\gamma$ increases. We will discuss this point in more detail below.

The difference between two analytical eigenvalues (6) is independent of $\gamma$: $\tau_n - \tau_j = n - j$. Figure 3 shows that if we choose the correct left boundary condition $\tau_{n+1} - \tau_n$, $n = 0, 1, 2, 3$, change with $\gamma$. The unphysical approximation proposed by Maia and Bakke [1] does not take into account this fact. Note that the purpose of their paper was to determine the effect of the dislocation given by the model parameter $\beta$ on the Landau levels and they failed to do it properly.

The variation of $\Delta\tau$ with $\gamma$ may appear to be weak at first sight but it should be taken into account that we have chosen a relatively small value of $y_0 = 0.1$. We are unable to estimate reasonable physical values of $y_0 = m\delta\beta^2/2$ because Maia and Bakke [1] did not show suitable values for the model parameters and also because their equations are inconsistent with respect to units.

Figure 4 shows $g_0(y)$ for $\gamma = 0.5$, $\gamma = 1$ and $\gamma = 2$ with the boundary conditions $g(0) = 0$ and $g(0.1) = 0$. We appreciate that the agreement is better when $\gamma = 2$ as argued above (only in this case the maxima are reasonably close). As $\gamma$ increases both kind of eigenfunctions become increasingly small in a neighbourhood of the origin; consequently, if $y_0$ is small enough then the effect of the Dirichlet boundary condition at $y_0$ is expected to be less noticeable. Obviously, the discrepancy between both kinds of solutions increases as $y_0$ increases.

4 Conclusions

The purpose of this Paper is to analyze the effect of changing the left boundary condition in the model proposed by Maia and Bakke [1]. Our results clearly show that the errors are not only quantitative but also qualitative. If $y_0$ is small enough the discrepancy between the solutions with Dirichlet boundary conditions at $y = 0$ and $y = y_0 > 0$ decreases as $\gamma$ decreases. The reason is
that the larger the value of $\gamma$ the smaller the magnitude of the eigenfunctions in a neighbourhood of the origin. In all our calculations we have chosen $y_0$ small because it is the condition under which the approach of Maia and Bakke should work better. However, the main drawback of their approximation is a wrong prediction of the level spacing. Under the unphysical boundary condition at $y = 0$ the level spacing does not depend on $\gamma$ while the use of the correct boundary condition at $y_0 > 0$ predicts that the level spacing depends on that model parameter. Another relevant point is that the equations derived by Maia and Bakke lack unit consistency. For this reason they are completely useless for any physical application. These authors carried out a similar mistake in an earlier paper but in that case it was easier to derive the equations properly.

References

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Figure 1: Numerical eigenvalues $\tau_n$ vs $y_0$ for $\gamma = 1$

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Figure 2: Eigenvalues $\tau_n$ vs $\gamma$ for $y_0 = 0.1$. The continuous and dashed lines indicate numerical and analytical ($y_0 = 0$) results, respectively.

Figure 3: $\tau_{n+1} - \tau_n$ vs $\gamma$ for $n = 0$ (blue, continuous line), $n = 1$ (red, dashed line), $n = 2$ (orange, dash-point line), $n = 3$ (green, point line) and $y_0 = 0.1$.
Figure 4: Analytical (red dashed line, $y_0 = 0$) and numerical (blue continuous line, $y_0 = 0.1$) values of $g_0(y)$ for $\gamma = 0.5$, $\gamma = 1$ and $\gamma = 2$ (top to bottom)