Fixed-Confidence Best Arm Identification in Multi-Armed Bandits: Simple Sequential Elimination Algorithms

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Abstract—We consider the problem of finding, through adaptive sampling, which of \( n \) arms (arms) has the largest mean. Our objective is to determine a rule which identifies the best arm with a fixed minimum confidence using as few observations as possible, i.e. fixed-confidence (FC) best arm identification (BAI) in multi-armed bandits. We study such problems under the Bayesian setting with both Bernoulli and Gaussian arms. We propose to use the classical vector at a time (VT) rule, which samples each remaining arm once in each round. We show how VT can be implemented and analyzed in our Bayesian setting and be improved by early elimination. We also propose and analyze a variant of the classical play the winner (PW) algorithm. Numerical results show that these rules compare favorably with state-of-art algorithms.

1. Introduction and Formulation

Let \( F_\theta(x|\theta) \) be a family of distributions indexed by its mean \( \theta \). Suppose there are \( n \) arms, and each new observation from arm \( i \) is a random variable independent of previous observations with distribution \( F_{\theta_i} \), where \( \theta_1, \ldots, \theta_n \) are unknown. We want to find which arm has the largest mean. A decision is made at each stage as to which arm to sample next from (sampling rule), when to stop (stopping rule) and declare which arm has the largest mean (recommendation rule). The objective is to minimize number of samples, \( N \), subject to the condition that the probability of correct choice is at least \( \alpha \). This setting is mostly known as Fixed confidence best arm identification in the literature. We study such models both when the arm distributions are Bernoulli and Gaussian with a fixed variance.

Best arm identification has many applications. Foremost is probably in clinical trials to determine which of several medical approaches (e.g., drugs, treatments, procedures, and various vaccines) yields the best performance. A arm here refers to a particular approach, with its use resulting in either a success (suitably defined) or not. Online advertising is another application \([1, 2]\), where a decision maker is trying to decide which of \( n \) different advertisements to utilize. For instance, the advertisements might relate to a recruitment, and a success might refer to a subsequent clicking on the advertisement. Another application is to choose among different methods for teaching a particular skill. Each day, a method can be used on a group of students, with the students being tested at the end of the day with each test resulting in a score which would be pass (1) or fail (0) in the Bernoulli case, and numerical in the Gaussian case.

We note that this problems has been studied for quite some time. However, the early work, was done under the assumption that the difference between the largest and second largest arm mean was at least some known positive value \([3, 4, 5, 6, 7, 8, 9]\). More recent works such as \([10, 11, 12]\) do not make this assumption, while others like \([13, 14, 15]\) keep this assumption. What primarily distinguishes our models from others considered in the literature is that we take a Bayesian approach that supposes the unknown means are the respective values of independent and identically distributed (i.i.d) random variables \( X_1, \ldots, X_n \), with \( X_i \) having a specified prior distribution \( F \). In addition, we show that even though we assume a Bayesian setting, our rules can be applied in the Bernoulli case without this assumption, by using a Bayesian rule when the prior is uniform \((0,1)\), as in the well known Thompson sampling rule. The main motivation for our work is the observation that many recent algorithms for FC BAI are too conservative in their stopping rule \([16, 17, 18]\). This is to say their accuracy is much bigger than \( \alpha \) which causes \( E[N] \) to be too large. This is because they are designed based on a worst-case scenario \([17]\). The worst cases happen with a very small probability so these algorithms are over conservative. Fig. 1 and Fig. 2 shows a sample of these algorithms where their accuracy is almost 1 however they have large sampling complexity. Also we use Markov chain properties, specially the Gambler’s ruin problem \([19]\) to do the analysis. To the best of our knowledge this is the first time BAI is treated directly using these techniques.

In Section 2 we consider the Bernoulli case. We consider the classical ”vector at a time” (VT) rule, with critical value \( k \) as defined in Algorithm 1. The appropriate value of \( k \) that results in confidence of at least \( \alpha \) was determined in \([8, 9]\) under the assumption that the optimality gap of the second best arm, call it \( d \), is strictly positive. We show how this rule can be implemented and analyzed in our Bayesian setting. We also present the numerical experiments indicating that our algorithm outperform recently proposed algorithms in this setting.

We develop a modification of the VT rule by allowing
Algorithm 1 Vector at a time (VT)

Input: all the arms.
repeat
- Sample each remaining arm once.
- For each arm $i$ if $\exists j \neq i$ s.t. $s_i(t) \leq s_j(t) - k$ eliminate arm $i$.
until there is one arm remaining

for early elimination in Section 2.1. We show how to determine the probability that the best arm is eliminated early, as well as the mean number of the non-best arms that are eliminated early.

In Section 3 we consider a variant of another classical rule: "play the winner" (PW). This variant differs from the classical model of [8] which for instance eliminated a arm if at some point - not necessarily at the end of a round - it has fewer successes than another arm for Bernoulli distribution.

We show how to analyze this rule in the Bayesian setting. Finally, in Section 4 we consider the case of Normal arms, with fixed variance $\sigma^2$ with standard Normal for the prior distribution of the means.

2. VT Rule, Bernoulli Case

Suppose there are $n$ Bernoulli arms with respective means $p_1, \ldots, p_n$. Let $s_i(t)$ show the cumulative number of success for arm $i$ at time $t$. We sample arms by the sampling rule and stop at some point to make a decision as to which arm has the largest mean. As noted earlier, we suppose that $p_1, \ldots, p_n$ are the respective values of iid random variables having a specified distribution $F$. The vector at a time (VT) rule, introduced in [3], for a given critical value $k \in \mathbb{Z}^+$, is given in Algorithm 1 where $A$ is the set of remaining arms.

Remark 1. Note that VT algorithm is similar to Action Elimination in [20] while it is slightly different in that it is non-adaptive and more suitable for our setting. In our paper we specifically analyzed VT for Bernoulli or Gaussian distribution and did not use a general estimator of the mean, despite action elimination paper.

Let $p_1 > p_2 > \ldots > p_n$ be the ordered values of the unknown means $p_1, \ldots, p_n$, and let $C$ be the event that the correct choice is made. We now show how to determine $k$ such that $\mathbb{P}(C) \geq \alpha$ when $p_1, \ldots, p_n$ are the values of independent and identically distributed random variables $P_1, \ldots, P_n$ having distribution $F$. We let $I\{A\}$ be the indicator of the event $A$, and $X = \sum Y$ indicates that $X$ and $Y$ have the same distribution. Let $U, U_1, \ldots, U_n$ be independent uniform $(0, 1)$ random variables then.

Lemma 1. $\max_i P_i =_{st} F^{-1}(U^{1/n})$

Proof.

Now suppose that in each round we take an observation from each arm, even those that are eliminated. Let $0$ be the best arm, namely the one with largest mean, and randomly number the others as arm $1, \ldots, n-1$. Imagine that the best arm is playing a "Gambler's Ruin Game" with each of the others, with the best one beating arm $i$ if the difference $S_0 - S_i$ hits the value $k$ before $-k$. Let $B_i$ be the event that the best arm beats arm $i$ and note that the best arm will be chosen if it wins all of its games. That is, if we let $B = B_1B_2 \cdots B_{n-1}$ then $B < C$, thus $\mathbb{P}(C) \geq \mathbb{P}(B)$

Lemma 2. $\mathbb{P}(B) \geq (\mathbb{P}(B_1))^{n-1}$

Proof. Let $U_0, U_1, \ldots, U_{n-1}$ and $U_{i,j}, i = 0, 1, \ldots, n-1, j \geq 1$ all be independent uniform $(0,1)$ random variables. We set $P_0 = F^{-1}(U_0^{1/n})$, $P_i = F^{-1}((1-U_i)U_0^{1/n}), i > 0$. We know that conditional on the maximum of $n$ independent uniform $(0,1)$ random variables, call it $U_{\max}$, the other $n-1$ of them are distributed as independent uniform $(0, U_{\max})$. Now using Lemma 1 it follows that the joint distribution of $P_0, P_1, \ldots, P_{n-1}$ is exactly that of the mean of the best arm, followed by the means of the other $n-1$ arms in a random order.

Let $I_{0,j} = I\{1-U_{0,j} < P_0\}, j \geq 1$, and $I_{i,j} = I\{U_{i,j} < P_i\}, i < n, j \geq 1$. Note that $I_{i,j}$ has the distribution of the $j^{th}$ observation of arm $i$ and also that $I_{0,j}$ is increasing in $U_{0,j}$ whereas $I_{i,j}$ is decreasing in $U_{i,j}, j \geq 1$. Because $P_i$ is decreasing in $U_i$ for $i > 0$, it consequently follows that, conditional on $U_0$, the indicator variables $I\{B_1\}, \ldots, I\{B_{n-1}\}$ are all increasing functions of the independent random variables $U_1, \ldots, U_{n-1}, U_{i,j}, i \leq n-1, j \geq 1$. Consequently, given $U_0$, the indicators $I\{B_1\}, \ldots, I\{B_{n-1}\}$ are associated, implying that

$$\mathbb{P}(B|U_0) \geq \prod_{i=1}^{n-1} \mathbb{P}(B_i|U_0)$$

by symmetry. Taking expectations gives

$$\mathbb{P}(B) \geq E\left[E\left[\mathbb{P}(B_i|U_0)\right]^{n-1}\right] \geq (E\mathbb{P}(B_i|U_0))^{n-1} = (E\mathbb{P}(B_1))^{n-1}$$

where the last inequality follows from Jensen's inequality.

To obtain an upper bound on $\mathbb{P}(B)$, let $B^*$ be the event that the best arm beats the best of arms in $\{1, \ldots, n-1\}$. That is, that arm $0$ beats the one having mean
max_{1 \leq i \leq n-1} X_i$. Because \( B \subset B^* \) we know \( \mathbb{P}(B) \leq \mathbb{P}(B^*) \).

**Remark 2.** It is possible for the best arm to be chosen even if it does not win all its games. Indeed, this will happen if the best arm loses to an arm that at an earlier time was eliminated. However, it is intuitive that this event has a very small probability of occurring. Consequently, \( \mathbb{P}(C) \approx \mathbb{P}(B) \).

Now if we want to compute \( \mathbb{P}(B_1) \) and \( \mathbb{P}(B^*) \) we can use simulation with a conditional expectation estimator. This estimation is used to choose \( k \) for a given \( \alpha \). Note that if arms with known probabilities \( x \) and \( y \) play a game that ends when one has \( k \) more wins than the other, then the probability that the first one wins is the probability that a gambler, starting with fortune \( k \) reaches a fortune of \( 2k \) before 0. This gambler wins each game with probability

\[
p = \frac{x(1-y)}{x(1-y) + y(1-x)}
\]

and if we set \( r = \frac{1-p}{p} = \frac{y(1-x)}{x(1-y)} \) we have

\[
\mathbb{P}(x \text{ wins}) = \frac{1-r^k}{1-r^{2k}} = \frac{1}{1+r^k}
\]

Also, using known results from the gambler’s ruin problem along with Wald’s equation (to account for the fact that not every round leads to a gain or a loss) it follows that the mean number of plays before stopping is

\[
\mathbb{E} [\text{number of plays}] = \frac{k(1-r^k)}{(r^k + 1)(x-y)}
\]

(1)

We can approximate \( \mathbb{E} [N] \) by mean number of plays in the corresponding games using Eq. (1). Summing up, we proved the following:

**Proposition 1.** Let \( U \) and \( V \) be independent uniform \((0,1)\) random variables, and let \( X = F^{-1}(U^{1/n}) \) and \( Y = F^{-1}(U^{1/n}V) \), \( W = F^{-1}(U^{1/n}V^{1/(n-1)}) \)

\[
R = \frac{Y(1-X)}{X(1-Y)}, S = \frac{W(1-X)}{X(1-W)}
\]

with \( C \) and \( B \) as previously defined, then \( \mathbb{P}(B) \leq \mathbb{E} \left[ \frac{1}{1+S^2} \right] \) and

\[
\mathbb{P}(C) \geq \mathbb{P}(B) \geq (\mathbb{E} \left[ \frac{1}{1+R^2} \right])^{n-1}
\]

\[
\mathbb{E} [N] \approx A = (n-1)\mathbb{E} \left[ \frac{k(1-R^k)}{(R^k + 1)(X-Y)} \right] + \mathbb{E} \left[ \frac{k(1-S^k)}{(S^k + 1)(X-W)} \right]
\]

**Remark 3** (Determine \( k \) for VT). Combining Remark 2 and Proposition 1 we get an interval which includes \( \mathbb{P}(C) \) for a given \( k \) with a high probability. Now we can do an independent simulation using only uniform \((0,1)\) random variables to find the smallest \( k \) such that the interval includes the desired \( \alpha \). We show an example of this in Section 5.

### 2.1. Early Elimination

In an improved modification of VT rule we use early elimination (EE) where if an arm is \( j \) behind after the first \( j \) rounds (that is, if the arm had all failures in the first \( j \) rounds while another arm had all successes) then that arm is eliminated. To see by how much that can reduce the accuracy of VT, let us compute \( \mathbb{P}(L) \), where \( L \) is the event that the best arm is eliminated early. Let \( 0 \) be the best arm, and let \( 1, \ldots, n-1 \) be the other arms in random order. With \( U, U_1, \ldots, U_{n-1} \) being i.i.d uniform \((0,1)\) random variables, we know

\[
(P_0, \ldots, P_{n-1}) = st (W,WU_1, \ldots, WU_{n-1})
\]

where \( W = U^{1/n} \). Consequently, letting

\[
(P_0, \ldots, P_{n-1}) = (W,WU_1, \ldots, WU_{n-1})
\]

yields

\[
\mathbb{P}(s_i = j|W) = \mathbb{E} [(U_iW)^j|W] = \frac{W^j}{j+1}
\]

\[
\mathbb{P}(L|W) = (1 - W)^j \left( 1 - \frac{W^j}{j+1} \right)^{n-1}
\]

Taking expectations gives

\[
\mathbb{P}(L) = \mathbb{E} \left[ (1 - U^{1/n})^j \left( 1 - \frac{U^{1/n}}{j+1} \right)^{n-1} \right]
\]

Let us now consider the expected number of non best arms that are eliminated early. We know \( s_0, \ldots, s_{n-1} \) are conditionally independent given \( W \), then

\[
\mathbb{P}(s_{n-1} = 0, \max_{i \in \{0, \ldots, n-2\}} s_i = j|W) = \mathbb{P}(s_{n-1} = 0|W)\mathbb{P}\left( \max_{i \in \{0, \ldots, n-2\}} s_i = j|W \right)
\]

\[
= \mathbb{P}(s_{n-1} = 0|W) \left( 1 - \prod_{i=0}^{n-2} \mathbb{P}(s_i < j|W) \right)
\]

and

\[
\mathbb{P}(s_{n-1} = 0|W) = \mathbb{E} \left[ (1 - U_{n-1}W)^j |W \right]
\]

\[
= \int_0^1 (1 - xW)^j dx
\]

\[
= \frac{1}{W} \int_{1-W}^1 y^j dy = \frac{1 - (1 - W)^{j+1}}{(j + 1)W}
\]

and

\[
\prod_{i=0}^{n-2} \mathbb{P}(s_i < j|W) = (1 - W^j)^{n-2} \mathbb{P}(s_1 < j|W)
\]

\[
= (1 - W^j)^{n-2} \left( 1 - (U_1W)^j |W \right)
\]

\[
= (1 - W^j)^{n-2} \left( 1 - \frac{W^j}{j+1} \right)^{n-2}
\]
Algorithm 2 Play the winner (PW)

set \( a_i = 1 \), and \( s_i(0) = 0 \) for all \( i \in \{1, \ldots, n\} \)

repeat
  set \( l_i = 0 \) for all \( i : a_i = 1 \)
  repeat
    - Sample all arms in \( \{i : l_i = 0\} \)
    - Set \( l_i = 1 \) for the arms with failure
    - If \( \exists j \) s.t. \( l_j = 0 \) and \( l_i = 1 \) \( \forall i \neq j \), and \( s_j(s) \geq \max_{i \neq j} s_i(s) + k \), then stop and return \( j \)
  until \( \{|i : l_i = 0\} = 1\) or \( \{i : a_i = 1\} = 1 \)

Hence, with \( N^* \) being the number of non-best arms that are eliminated early, we have

\[
E[N^*] = (n-1)E\left[\frac{1 - (1 - W)^{j+1}}{j + 1}W\right] \left(1 - (1 - W)^j(1 - \frac{W^j}{j+1})^{n-2}\right)
\]

which could be computed via simulation.

We can also use a randomized \( k \) as follows. Let \( P_k(C) \) be the probability of a correct choice when using VT with critical value \( k \), and suppose

\[
P_{k-1}(C) < \alpha < P_k(C).
\]

The randomized rule that chooses VT with critical value \( k \) with probability \( P = \frac{1}{P_k(C) - P_{k-1}(C)} \), and VT with critical value \( k \) with probability \( 1 - P \) will yield the correct choice with probability \( \alpha \). Another possibility is to use VT along with EE parameter \( j^* \), where \( j^* \) is the smallest value for which using VT with critical value \( k \) along with EE of \( j \) success results in a correct choice with probability at least \( \alpha \). (Of course, we could use a policy that randomizes between VT with critical value \( k \) and EE at \( j^* - 1 \) and VT with critical value \( k \) and EE at \( j^* \)). Example 2 in Section 5 is an instance where randomizing among VT rules results in a smaller \( E[N] \) than does VT with EE.

3. Play the Winner Rule

Another proposed algorithm in the Bayesian setting is play the winner (PW) rule, which in each but the last round continues to sample from each alive (not eliminated) arm until it has a failure. Our proposed PW is described concisely in Algorithm 2. In this algorithm \( a_i \) is the indicator of alive arms and \( l_i \) is the indicator of arms with loss/failure. We should note that:

1. If we define a round by saying that each alive arm is observed until it has a failure, then when \( F \) is the uniform \((0, 1)\) distribution, the expected number of plays until the first arm has a failure could be infinite. Therefore we define rounds using subrounds so that the mean number of plays is finite. For instance, suppose \( F \) is uniform \((0, 1)\) and \( P_1 > P_2 > \ldots > P_n \). Let \( N_i \) denote the number of plays in the first round of the arm with probability \( P_i \). Then, as the density of \( F \) is

\[
f_{F}(p) = \frac{np^{n-1}(1-p)^{i-1}}{(i-1)!(n-i)!}dp, \quad 0 < p < 1
\]

it follows that \( E[N_i] \leq E\left[\frac{1}{1 - F(p)}\right] < \infty \) when \( i > 1 \). In addition,

\[
N_1 \leq k + \max_{2 \leq i \leq n} N_i \leq k + \sum_{i=2}^{n} N_i
\]

\[
\Rightarrow E[N_1] \leq k + \sum_{i=2}^{n} E[N_i] < \infty
\]

2. The PW rule as defined in [3] and [9] was such that the arms are initially randomly ordered. In each round, the alive arms were observed in that order, with each arm being observed until it had a failure. If at any time one of the arms had \( k \) fewer successes than another arm, then the former is no longer alive. The process ends when only a single arm is alive which is declared as the best.

3.1. Analysis of PW

To begin, suppose there are only 2 arms, and that their success probabilities are \( p_1 > p_2 \). Also, suppose we are going to choose an arm by using the procedure which in each round plays each arm until it has a failure. We stop at the end of a round if one of the arms, say \( l \), has \( s_l \geq s_r + k \), then \( l \) is chosen as the best. Let \( q_l = 1 - p_l \), \( i = 1, 2 \), and let \( X_{i,r} = 1, 2, r \geq 1 \), be independent with \( P(X_{i,r} = j) = q_l, j \geq 0 \). We know \( X_{i,r} \) is the number of successes of arm \( i \) in round \( r \). Letting \( Y_r = X_{1,r} - X_{2,r}, r \geq 1 \) we get

\[
E[e^{\theta Y_r}] = \frac{q_1}{1 - p_1 e^{\theta}} \frac{q_2}{1 - p_2 e^{\theta}}
\]

It is now easy to check that \( E[e^{\theta Y_r}] = 1 \) if \( e^\theta = p_2 / p_1 \). That is, \( E[(p_2/p_1) Y_r] = 1 \). If we now let \( S_m = \sum_{i=1}^{m} Y_i \) then \( (p_2/p_1) S_m, m \geq 1 \) is a martingale with mean 1. Letting \( \tau = \min\{m : S_m \geq k \text{ or } S_m \leq -k\} \) it follows by the martingale stopping theorem [19] that \( E[(p_2/p_1)^{S_\tau}] = 1 \). Let \( p = P(S_{\tau} \geq k) \) be the probability that arm 1 is chosen. Then

\[
1 = E[(p_2/p_1)^{S_\tau}] = E[(p_2/p_1)^{S_\tau} \mathbb{1}_{S_{\tau} \geq k}] \cdot p + E[(p_2/p_1)^{S_\tau} \mathbb{1}_{S_{\tau} \leq -k}] \cdot (1 - p)
\]

Letting \( X_i, i = 1, 2 \), have same distribution as \( X_{i,r} \), it follows, by the lack of memory of \( X_i \), that

\[
E\left[\left(\frac{p_2}{p_1}\right)^{S_{\tau}} \mathbb{1}_{S_{\tau} \geq k}\right] = \left(\frac{p_2}{p_1}\right)^k E\left[\left(\frac{p_2}{p_1}\right)^{X_1}\right] = \left(\frac{p_2}{p_1}\right)^k \left(\frac{q_2}{q_1}\right)
\]

\[
E\left[\left(\frac{p_2}{p_1}\right)^{S_{\tau}} \mathbb{1}_{S_{\tau} \leq -k}\right] = \left(\frac{p_2}{p_1}\right)^{-k} E\left[\left(\frac{p_2}{p_1}\right)^{-X_2}\right] = \left(\frac{p_2}{p_1}\right)^k \left(\frac{q_2}{q_1}\right)
\]

Substituting back yields that

\[
p = \frac{1 - (p_1/p_2)^k(q_2/q_1)}{(p_2/p_1)^k(q_2/q_1) - (p_1/p_2)^k(q_2/q_1)}
\]
Conditioning on which arm wins yields that

\[ E[S_r] = (k + E[X_1])p + (-k - E[X_2])(1 - p) \]

Letting \( m_r = E[X_i] = 1/q_i - 1 = p_i / q_i \), the preceding gives \( E[S_r] = p(m_1 + m_2 + 2k) - (m_2 + k) \). Now Wald's equation yields

\[ E[\tau] = \frac{p(m_1 + m_2 + 2k) - m_2 - k}{m_1 - m_2} \tag{3} \]

Because \( X_{1,r} + X_{2,r} + 2 \) is the number of plays in round \( r \), it follows that the total number of plays in this setting, call it \( T \), is \( \sum_{r=1}^T (X_{1,r} + X_{2,r} + 2) \). Applying Wald's equation and using Eq. (3) gives

\[ E[T] = \frac{(m_1 + m_2 + 2k) - m_2 - k}{m_1 - m_2} \]

Now we want to calculate the probability of choosing the best arm correctly under the PW algorithm, along with the number of samples. Let \( B(p_1, p_2) \) and \( N(p_1, p_2) \) be, respectively, the probability that the arm with value \( p_1 \) is chosen and the mean number of plays before stopping. From Eq. (2) we have

\[ B(p_1, p_2) = \frac{1 - (p_1 / p_2)^k(q_2 / q_1)}{(p_2 / p_1)^k(q_1 / q_2) - (p_1 / p_2)^k(q_2 / q_1)} \tag{4} \]

Because PW would stop play once the winning arm is ahead by \( k \), whereas \( E[T] \) is the mean number of plays we continue until a failure occurs, we obtain by conditioning on which arm wins that

\[ N(p_1, p_2) = E[T] - \frac{B(p_1, p_2)}{q_1} - \frac{1 - B(p_1, p_2)}{q_2} \]

\[ = \frac{(B(p_1, p_2)(m_1 + m_2 + 2k) - m_2 - k)}{m_1 - m_2} - \frac{B(p_1, p_2)}{q_1} - \frac{1 - B(p_1, p_2)}{q_2} \]

where \( m_i = p_i / q_i \).

Now suppose there are \( n \) arms with prior distribution \( F \). Akin to our VT analysis, suppose that all arms participate in each round and let arm 0 be the best arm, and randomly number the other arms as \( 1, \ldots, n - 1 \). Then we have

**Lemma 3.** For the PW algorithm, with \( B \equiv B_1 B_2 \cdots B_{n-1} \) we have

\[ P(C) \geq P(B) \geq (P(B_1))^{n-1} \]

Also, \( P(B) \leq P(B^*) \), where \( B^* \) is the event that 0 beats the best of arms \( 1, \ldots, n - 1 \).

Our preceding analysis yields the following corollary.

**Corollary 1.** With \( U \) and \( V \) being independent uniform \((0, 1)\) random variables, we define

\[ X = F^{-1}(U^{1/n}), \]
\[ Y = F^{-1}(U^{1/n} V), \]
\[ W = F^{-1}(U^{1/n} V^{1/(n-1)}) \]

**Algorithm 3** VT for Gaussian rewards

| Input: all the arms. |
| repeat |
| - Sample each remaining arm once. |
| - Eliminate any arm \( i \), if \( 3j \neq i \) s.t. \( S_i(k) < S_j(k) - c \). |
| until there is one arm remaining arm |

then

\[ P(B_1) = E[B(X, Y)] \]
\[ P(B^*) = E[B(X, W)] \]

and

\[ N \approx A = (n - 1)E[N(X, Y)] + E[N(X, W)] \]

**Remark 4** (Determine \( k \) for PW). Based on Lemma 3 and a similar argument to Remark 2 we can derive a simple procedure like in Remark 3 to find appropriate \( k \) for PW algorithm.

We also analysed the early elimination version of PW, PW-EE. Since there is no obvious improvement using PW-EE based on our numerical studies, we defer this to the appendix. Section 5 includes a numerical comparison of VT and PW.

4. The Normal Case

In this section we suppose that rewards of arm \( i \) are independent Normal random variables with mean \( \mu_i \) and variance \( \sigma_i^2 \) where \( \sigma_i^2 \) is known and \( \mu_1, \ldots, \mu_n \) are the unknown values of \( n \) independent standard Normal random variables, i.e. \( F \) is the standard Normal. The VT rule with parameter \( c > 0 \) for this case is given in Algorithm 3.

Before elaborating on how to choose a proper \( c \) we present some preliminaries concerning Normal partial sums.

4.1. Preliminaries

Let \( \Phi \) be the standard Normal distribution function. Define \( R(a) = \Phi(a) - \phi(a) \), then we have the following Lemma for a Normal random variable.

**Lemma 4.** If \( Z \) is a Normal random variable with mean \( \mu \) and variance 1, then

\[ E[e^{-2\mu Z} | Z > 0] = R(-\mu) \]
\[ E[|Z| Z > 0] = \mu + e^{-\mu^2/2} / (\sqrt{2\pi} \Phi(\mu)) \]
\[ E[e^{-2\mu Z} | Z < 0] = R(\mu) \]
\[ E[|Z| Z < 0] = \mu - e^{-\mu^2/2} / (\sqrt{2\pi} (1 - \Phi(\mu)) \]

We prove this Lemma in the appendix. We also prove the following Lemma for the partial sum and its stopping time.
Lemma 5. Let $S_m = \sum_{i=1}^{m} Z_i$, $m \geq 1$, where $Z_i$, $i \geq 1$ are independent Normal random variables with mean $\mu > 0$ and variance 1. For given $b > 0$, let

$$\tau = \min\{m : S_m < -b \text{ or } S_m > b\}.$$ 

then

$$R(-\mu)e^{-2\mu b} < E\left[e^{-2\mu S_{\tau}} | S_{\tau} > b\right] < e^{-\mu b}$$

$$e^{2\mu b} < E\left[e^{2\mu S_{\tau}} | S_{\tau} < -b\right] < e^{\mu b} R(\mu)$$

Now using these Lemmas we can derive a tail bound for the partial sum in the following proposition.

Proposition 2. Let’s have same definitions as in Lemma 5 then

$$\frac{e^{2\mu b} - 1}{e^{2\mu b} - R(-\mu)e^{-2\mu b}} < P(S_{\tau} > b) < \frac{e^{2\mu b} R(\mu) - 1}{e^{2\mu b} R(\mu) - e^{-2\mu b}}$$

(5)

Proof. Let $p = P(S_{\tau} > b)$. Because $E[e^{-2\mu Z}] = 1$, it follows that $\{e^{-2\mu S_{\tau}}\}_{t \geq 1}$ is a martingale with mean 1. Hence, by the martingale stopping theorem

$$1 = E[e^{-2\mu S_{\tau}}]$$

$$= E[e^{-2\mu S_{\tau}} | S_{\tau} > b] p + E[e^{-2\mu S_{\tau}} | S_{\tau} < -b] (1 - p)$$

Hence,

$$p = \frac{E[e^{-2\mu S_{\tau}} | S_{\tau} < -b] - 1}{E[e^{-2\mu S_{\tau}} | S_{\tau} < -b] - E[e^{-2\mu S_{\tau}} | S_{\tau} > b]}$$

(6)

Because $\frac{1 - e^{-\mu}}{\mu}$, $0 < y < 1 < x$, increases in both $x$ and $y$, the inequalities in Eq. (5) now follow from Lemma 5.

We can also bound the expectation of $\tau$ as follows.

Proposition 3. With $\tau$ as previously defined in Lemma 5 we have

$$E[\tau] \leq \frac{e^{2\mu b} R(\mu) - 1}{e^{2\mu b} R(\mu) - e^{-2\mu b}} \left(\frac{2b}{\mu} + \frac{e^{-\mu^2/2}}{\mu \sqrt{2\pi \Phi(\mu)}} + 1\right) - b$$

(7)

$$E[\tau] \geq \frac{e^{2\mu b} - 1}{e^{2\mu b} - R(-\mu)e^{-2\mu b}} \left(\frac{2b}{\mu} + \Psi(\mu)\right) - b + \Psi(\mu)$$

(8)

where $\Psi(\mu) = 1 - \frac{e^{-\mu^2/2}}{\mu \sqrt{2\pi (1 - \Phi(\mu))}}$.

Proof. Wald’s equation gives

$$E[\tau] \mu = E[S_{\tau} | S_{\tau} > b] p + E[S_{\tau} | S_{\tau} < -b] (1 - p)$$

$$\leq (b + E[W | W > 0])p - b(1 - p)$$

$$= p(2b + \frac{e^{-\mu^2/2}}{\sqrt{2\pi \Phi(\mu)}} + \mu) - b$$

where the first inequality used, as shown in Lemma 5, that $S_{\tau} | \{S_{\tau} > b\}$ is stochastically larger than $b + W | \{W > 0\}$.

Inequality (7) now follows from Proposition 2. The lower bound follows from writing $E[\tau]$ as follows

$$E[S_{\tau} | S_{\tau} > b] p + E[S_{\tau} | S_{\tau} < -b] (1 - p)$$

$$\geq bp + (-b + E[W | W < 0])(1 - p)$$

$$= \left(\frac{2b - \mu + \frac{e^{-\mu^2/2}}{\sqrt{2\pi (1 - \Phi(\mu))}}}{2b - \mu + \frac{e^{-\mu^2/2}}{\sqrt{2\pi (1 - \Phi(\mu))}}}\right)p - b + \mu - \frac{e^{-\mu^2/2}}{\sqrt{2\pi (1 - \Phi(\mu))}}$$

where the inequality used that $S_{\tau} | \{S_{\tau} < -b\}$ is stochastically larger than $-b + W | \{W < 0\}$.

Inequality (12) now follows from Proposition 2.

Now we can approximate $p = P(S_{\tau} > b)$ and $E[\tau]$ by “neglecting the excess” and assuming $(S_{\tau} | S_{\tau} > b) \approx_{st} b$ and $(S_{\tau} | S_{\tau} < -b) \approx_{st} -b$. From Eq. (6) this gives that

$$p = e^{2\mu b} - 1 / (e^{2\mu b} - e^{-2\mu b})$$

(9)

Also, $\mu E[\tau] \approx bp - b(1 - p)$, and so Eq. (9) gives that

$$E[\tau] \approx 2b(e^{2\mu b} - 1) / (\mu(e^{2\mu b} - e^{-2\mu b}) - b / \mu)$$

(10)

Example 1. If $b = 3$, $\mu = 1$, then Eq. (7), Eq. (8), and Eq. (10) yield that $2.9838 \leq E[\tau] \leq 4.2842$, and $E[\tau] \approx 2.9852$.

It is easy to generalize the above results to the case where variance is not 1 as follows.

Corollary 2. Let $S_m = \sum_{i=1}^{n} V_i$, $n \geq 1$, where $V_i$, $i \geq 1$ are independent Normal random variables with mean $\mu > 0$ and variance $2\sigma^2$. For given $c > 0$, let $\tau$ be as before, then

$$\frac{e^{\mu c / \sigma^2} - 1}{e^{\mu c / \sigma^2} - R(\frac{\mu}{\sigma \sqrt{2}})e^{-\mu c / \sigma^2}} < P(S_{\tau} > c)$$

$$e^{\mu c / \sigma^2} R(\frac{\mu}{\sigma \sqrt{2}}) - 1$$

(9)

Furthermore, $E[\tau] \approx \frac{2c(e^{\mu c / \sigma^2} - 1)}{\mu(e^{\mu c / \sigma^2} - e^{-\mu c / \sigma^2})} - c / \mu$.

Proof. Let $Z_i = \frac{V_i}{\sigma \sqrt{2}}$, note that $E[Z_i] = \frac{\mu}{\sigma \sqrt{2}}$. Now, using $b = \frac{c}{\sigma \sqrt{2}}$, apply Proposition 2 and Eq. (10) to get the desired result.

4.2. Analyzing the VT Rule in the Normal Case

In this section, we derive a lower bound and an effective approximation of $P(C)$, by a similar argument as in the Bernoulli case. With similar indexing of arms we imagine a “gambler’s ruin” game between arms 0 and i where the goal is c. We can again show exactly as before that

$$P(C) \geq P(B) \geq (P(B_i))^{n-1}$$

and $P(B) \leq P(B^*)$. Given the values $\mu_0, \mu_1, \ldots, \mu_{n-1}$, the difference between the sample from arm 0 and one from a different arm is a Normal random variable with variance $2\sigma^2$. Letting $LB(\mu)$ and $UB(\mu)$ be the lower and upper
bounds on $\Pr(S > c)$ in Corollary 2 it yields the following proposition.

**Proposition 4.** Let $U$ and $V$ be independent uniform $(0, 1)$ random variables, and let

$$X = \Phi^{-1}(U^{1/n}) - \Phi^{-1}(V^{1/n}),$$

$$Y = \Phi^{-1}(U^{1/n}) - \Phi^{-1}(V^{1/n}V^{1/(n-1)}).$$

Then

$$\Pr(C) \geq \Pr(B) \geq (\mathbb{E}[LB(X)])^{n-1}$$

and $\Pr(B) \leq \mathbb{E}[UB(Y)]$.

Now we can approximate $\mathbb{E}[N]$ by approximating the mean number of plays of each of the non best arms by the mean number of plays in their game against the best arm, and also approximating the mean number of plays of the best arm by the mean number of plays in its game against the second best one. Hence, using Eq. \(10\) we have

$$\mathbb{E}[N] \approx A \equiv (n-1)\mathbb{E}[M(X)] + \mathbb{E}[M(Y)]$$

where

$$M(\mu) = \frac{2c(e^{\mu c}/\sigma^2 - 1)}{\mu(e^{\mu c}/\sigma^2 - e^{-\mu c}/\sigma^2)} - \frac{c}{\mu}$$

5. Experimental Results

In this section we present experiments that establish the efficiency of our algorithms and help us evaluate our approximations of $N$ and $\Pr(C)$.

5.1. VT, Bernoulli

Here we assume $F$ is the uniform $(0, 1)$ distribution and compare the estimations with the simulation results. Let $\sigma$ be the standard deviation of estimate of $A$ and $m$ be the number of simulation runs. Table I shows the results of the algorithm and Table II shows the estimation for the same cases with $m = 10^9$. For instance, Table II shows that to obtain 99 percent accuracy with $n = 10$ arms, it seems that the elimination number has to be 57 or 58. Also the results confirm that the quality of the estimations is very high.

| $n$ | $k$ | $\Pr(C)$ | $\mathbb{E}[N]$ | $\sigma$ |
|-----|-----|----------|----------------|--------|
| 10  | 50  | 0.9886   | 5466.318       | 17.34  |
| 5   | 10  | 0.9540   | 358.3993       | 1.156  |

**TABLE 1. VT EXPERIMENTAL RESULTS.**

| $n$ | $k$ | $\Pr(B_1)$ | $\Pr(B^*)$ | $A$ | $\sigma$ |
|-----|-----|------------|------------|-----|--------|
| 10  | 50  | 0.9885     | 0.9889     | 5460.539 | 26.6991 |
| 57  | 0.9896 | 0.9901     | 6462.372   | 32.9701 |
| 58  | 0.9900 | 0.9903     | 6545.46    | 32.4571 |
| 5   | 10  | 0.9523     | 0.9561     | 358.3983 | 0.8256  |

**TABLE 2. VT ESTIMATION RESULTS.**

We compare the VT rule with recent algorithms in the literature. We use Track and Stop (TaSC for TaS with C tracking, TaSD for D tracking [14], Chernoff Racing (ChR) compares VT with VT-EE to show its effectiveness.

**[14]**. Kallback-Leibler Racing (KL-R), and KL-LUCB [21] algorithms. Table III is borrowed from [14]. The table consists of the results for two cases: the first case having $n = 4$ with probabilities: $(0.5, 0.45, 0.43, 0.4)$, and the second having $n = 5$ with probabilities $(0.3, 0.21, 0.20, 0.19, 0.18)$. The parameters of these algorithms are chosen to guarantee at least 90 percent accuracy.

| Case | TaSC | TaSD | ChR | KL-LUCB | KL-R |
|------|------|------|-----|---------|------|
| 1    | 3968 | 4052 | 4516| 8437    | 9590 |
| 2    | 1370 | 1406 | 3078| 2716    | 3334 |

**TABLE 3. RECENT ALGORITHMS RESULTS.**

Because our algorithm assumes knowledge of a prior distribution, in cases where there is no reason to assume that we know what the prior is, it seems reasonable to assume a uniform $(0, 1)$ prior and choose a larger accuracy than is actually desired. So suppose we do so and require 99 and 97 percent accuracy. Table IV shows the results with proper $k$’s for VT. As we can see, the VT algorithm significantly outperforms the newer algorithms even with fixed probabilities. Although, to be fair we should mention that, under the uniform $(0, 1)$ prior, $k = 5$ is sufficient for both $n = 4$ or $n = 5$ to obtain 90 percent accuracy based on Proposition 4 estimation. In case 1, this yields $\mathbb{E}[N] = 166$, but $\Pr(C) = 0.601$. In case 2 it has $\mathbb{E}[N] = 213.2$, with $\Pr(C) = 0.81$.

5.2. VT with EE, Chernoff

First we evaluate the estimations of $\Pr(L)$ and $\mathbb{E}[N^*]$ to illustrate how efficient EE could be. Table V shows the values of $\Pr(L)$, and $\mathbb{E}[N^*]$, for a variety of values of $n$ and $j$ when $F$ is the Uniform $(0, 1)$ distribution. Next example

| $n$ | $j$ | $\mathbb{E}[L|$ | $\mathbb{E}[N^*]$ |
|-----|-----|----------------|----------------|
| 10  | 2   | 0.02053        | 1.317          |
| 3   | 0.00336 | 0.851      | 0.590          |
| 4   | 0.00059 | 0.9325       | 0.432          |
| 5   | 0.00011 | 0.432        | 0.3432         |
| 20  | 2   | 0.0004929     | 6.659          |
| 3   | 0.00053 | 4.978       | 3.942          |
| 4   | 0.00007 | 3.942        | 5.00001        |

**TABLE 5. NUMERICAL EXAMPLE OF VT WITH EARLY ELIMINATION.**
Example 2. Suppose $n = 5$ and $\alpha = 0.95$. Table 6 shows the simulated results based on 500,000 runs. Here $\bar{\sigma}$ is standard deviation of the estimator of $\mathbb{E} [N]$."

| Algorithm | $k$ | $j$ | $P(C)$ | $\mathbb{E} [N]$ | $\bar{\sigma}$ |
|-----------|-----|-----|--------|------------------|----------------|
| VT        | 9   | -   | 0.948  | 313.64           | 2.11           |
|           | 10  | -   | 0.954  | 358.40           | 1.156          |
| VT-EE     | 10  | 2   | 0.9385 | 335.52           | 8.29           |
|           | 10  | 3   | 0.9523 | 348.27           | 2.70           |

TABLE 6. RESULTS FOR VT AND VT WITH EARLY ELIMINATION, VT-EE

Based on these results, randomizing among VT with $k = 9$ and $k = 10$ to obtain $P(C) = 0.95$ has mean $(2/3) 313.64 + (1/3) 358.40 = 328.56$, which is smaller than what can be obtained with VT with EE. (It is also better than the recently proposed algorithms. Of these, the Chernoff-Racing bound algorithm performs the best between others in the literature, giving an average number of 423.4 with accuracy 0.953.)

5.3. VT versus PW, Bernoulli

Based on numerical experiments, VT and PW have roughly similar performances when $F(x) = x$. When $n = 5$, simulation yields the following results for PW in Table 7.

The results show that choosing PW with $k = 42$ with probability 0.25 and $k = 43$ with probability 0.75 results in $P(C) = 0.95$, and requires, on average, 325.795 observations, which is slightly less than the average of 328.56 which, as shown in Example 2, can be obtained by a randomization of VT rules to obtain $P(C) = 0.95$. On the other hand if we wanted $\alpha = 0.954$, then both VT with $k = 10$ and PW with $k = 48$ achieve that, with VT having a mean of 358.4 observations, compared to 375.4 for PW. (Because the average number of trials needed for PW with $k = 47$ is 367.05, randomizing between PW(47) and PW(48) still would not be as good as VT(10).)

5.4. Experimental Result for Normal rewards

This section includes experiments that aim at comparing our algorithms with the literature for Normal rewards with standard Normal prior. In Fig. 1 and Fig. 2 we use simulation to compare the performance of VT with the most quoted algorithms of the recent literature: lilUCB, TrackAndStop, and Chernoff-Racing. The lilUCB algorithm [11] uses upper confidence bounds that are based on the law of the iterated logarithm for the expected reward of the arms. At each stage it uses the arm with the largest upper bound. We use a heuristic variation of the lilUCB, called lilUCB-H, which performs somewhat better than the original [11]. The TrackAndStop algorithm in [4] tracks the lower bounds on the optimal proportions of the arm rewards and uses a stopping rule based on Chernoff’s Generalized Likelihood Ratio statistic. The Chernoff algorithm is similar to TrackAndStop, but rather than track the optimal proportions it instead chooses between the empirical best and second-best. The results below are based on $10^4$ simulation runs, with each run beginning by sampling $\mu_1, \ldots, \mu_n$ from $F$. Table 8 shows the proper $k$ for each problem instance. As the results show, recent algorithms are too conservative, i.e. they stop too late and their $P(C)$ is extra large which makes their $\mathbb{E} [N]$ too large. This is aligned with the observation in [22] that their stopping rule is conservative and could be improved a lot.

6. Conclusion

We study the problem of best arm identification in multi-armed bandit for fixed confidence setting under the Bayesian setting with both Bernoulli and Normal arms. We use the classical vector at a time and play the winner algorithms and analyze them in a novel way using the Gambler’s ruin problem and martingales stopping theorem. We also derive easy estimations for these algorithms. Numerical experiments show that these rules compete favorably with recently proposed algorithms in terms of having higher accuracy and smaller sample complexity.
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Appendix

1. Proofs

Proof of Lemma 4

\[ E[e^{-2\mu W} | W > 0] = \int_{-\infty}^{\infty} e^{-x^2/2} \phi(x) \, dx = \frac{1}{\sqrt{2\pi}} \phi(\mu) \]

Let \( Z = W - \mu \).

\[ E[W | W > 0] = \mu + E[Z | Z > -\mu] = \mu + \frac{1}{\sqrt{2\pi} \phi(\mu)} \int_{-\mu}^{\infty} x e^{-x^2/2} \, dx = \mu + \frac{e^{-\mu^2/2}}{\sqrt{2\pi} \phi(\mu)} \]

Because \( E[e^{-2\mu W}] = 1 \), the third equality follows from the first upon using the identity

\[ 1 = E[e^{-2\mu W} | W > 0] \phi(\mu) + E[e^{-2\mu W} | W < 0] (1 - \phi(\mu)) \]

Similarly, the fourth equality follows from the second since \( \mu = E[W | W > 0] \phi(\mu) + E[W | W < 0] (1 - \phi(\mu)). \)

**Proof Lemma 5** The right hand inequality of (a) is immediate since \( \mu > 0 \). To prove the left side of (a), note that conditional on \( S_N > b \) and on the value \( S_{N-1} \), that \( S_N \) is distributed as \( b \) plus the amount by which a normal with mean \( \mu \) and variance 1 exceeds the positive amount \( b - S_{N-1} \) given that it does exceed that amount. But a normal conditioned to be positive is known to have strict increasing failure rate (see [23]), implying that \( S_N | \{ S_N > b, S_{N-1} \} \) is stochastically smaller than \( b + W_i | \{ W_i > 0 \} \). As this is true no matter what the value of \( S_{N-1} \), it follows that \( S_N \) is stochastically smaller than \( b + W_i | \{ W_i > 0 \} \), implying that \( E[e^{-2\mu S_N} | S_N > b] > e^{-2\mu b} E[e^{-2\mu W_i} | W_i > 0] \). The result now follows from Lemma 4.

The left hand inequality of (b) is immediate. To prove the right hand inequality, note that the same argument as used in part (a) shows that \( S_N | \{ S_N < -b \} \rightarrow -b + W_i | \{ W_i < 0 \} \), implying that \( E[e^{-2\mu S_N} | S_N < -b] < e^{2\mu b} E[e^{-2\mu W_i} | W_i < 0] \). Thus, the result follows from Lemma 4.

**2. A remark on Variance Reduction**

In our experiments for the VT rule, we observe that the estimator of \( E[N] \) has a large variance. In the case where \( F \) is the uniform \((0,1)\) distribution, we can reduce the variance of \( E[N] \) estimator by using \( Y = \frac{1}{P_1(1-P_2)} \) as a control variable, where \( P_1 \) and \( P_2 \) are the random variables representing the means of the best and second best arm. That is, if let \( T \) denote the raw estimator, then the new estimator is \( T + c(Y - E[Y]) \) where the variance is minimized when \( c = -\text{Cov}(T,Y)/\text{Var}(Y) \). To obtain the mean value of the control variable, we condition on \( P_2 \),

\[ E\left[ \frac{1}{P_1(1-P_2)} \right] = E\left[ \frac{1}{P_1} \right] \cdot E\left[ \frac{1}{1-P_2} \right] = E\left[ \frac{-\log(P_2)}{(1-P_2)^2} \right] = n(n-1) \int_0^1 -x^{n-2} \log(x) (1-x) \, dx \approx n(n-1) \sum_{i=1}^r h \left( \frac{i}{r} - 0.5 \right) \]

where \( r \) is a large integer, and \( h(x) = \frac{-x^{n-2} \log(x)}{(1-x)^2} \). The third equality holds because \( P_1 | P_2 \sim \text{unif}(P_2,1) \). The values of \( \text{Cov}(T,Y) \) and \( \text{Var}(Y) \) can be estimated from the simulation, and these can then be used to determine \( c \). In our numerical examples, we observe that the variance is reduced by up to 60 percent using this technique.

3. PW with Early Elimination

Suppose we use PW and add an early elimination on any population whose first \( j \) observations are all failures. Let \( B_e \) be the event that the best population is eliminated early. Because the mean of the best population has density function \( f(p) = np^{a-1} \), \( 0 < p < 1 \), it follows that

\[ P(B_e) = \int_0^1 (1-p)^j np^{a-1} \, dp = \frac{n!j!}{(n+j)!} \]

Let \( N_{nb} \) be the number of nonbest populations that are eliminated early. To compute \( E[N_{nb}] \), note that the probability a randomly chosen population is eliminated early is \( 1/(j+1) \), giving that

\[ \frac{n}{j+1} = E[\text{number eliminated early}] = E[N_{nb}] + \frac{n!j!}{(n+j)!} \]

Hence, \( E[N_{nb}] = \frac{n}{j+1} - \frac{n!j!}{(n+j)!} \). For instance, if \( n = 10, j = 5 \) then \( P(B_e) = 0.000333 \) and \( E[N_{nb}] = 1.666 \). Remarkably, early elimination sometimes has almost no effect on either \( P(C) \) or the average number of needed trials.

**Example 3.** Suppose \( n = 5 \) and \( k = 48 \). Then, a simulation with 20,000,000 runs yielded Table 9. Thus it seems impossible to tell in this example whether early elimination increases either accuracy or efficiency. (In particular, since the mean number of non-best populations that are eliminated early is 0.7121 it seems very surprising that early elimination does not decrease the average number of trials needed.)

| Rule     | \( P(C) \) | \( E[N] \) | sd  |
|----------|------------|------------|-----|
| PW       | 0.9543137  | 375.3552   | 0.1410 |
| PW-EE    | 0.9544769  | 375.4235   | 0.1411 |

**Table 9. Results of PW and PW with Early Elimination, PW-EE**