Calculations for Extended Thermodynamics of dense gases up to whatever order and with only some symmetries

S. Pennisi
Dipartimento di Matematica ed Informatica, Università di Cagliari, Cagliari, Italy
spennisi@unica.it

Abstract

The 14 moments model for dense gases, introduced in the last years by Ruggeri, Sugiyama and collaborators, is here considered. They have found the closure of the balance equations up to second order with respect to equilibrium; subsequently, Carrisi has found the closure up to whatever order with respect to equilibrium, but for a more constrained system where more symmetry conditions are imposed. Here the closure is obtained up to whatever order and without imposing the supplementary conditions. It comes out that the first non symmetric parts appear only at third order with respect to equilibrium, even if Ruggeri and Sugiyama found a non symmetric part proportional to an arbitrary constant also at first order with respect to equilibrium. Consequently, this constant must be zero, as Ruggeri, Sugiyama assumed in the applications and on an intuitive ground.

1 Introduction

Starting point of this research is the article [1] which belongs to the framework of Extended Thermodynamics. Some of the original papers on this subject are [2], [3] while more recent papers are [4]-[18] and the theory has the advantage to furnish hyperbolic field equations, with finite speeds of propagation of shock waves and very interesting analytical properties. It starts from a given set of balance equations where some arbitrary functions appear; restrictions on these arbitrariness are obtained by imposing the entropy principle and the relativity principle. However, these restrictions were so strong to allow only particular state functions; for example, the function $p = p(\rho, T)$ relating the pressure $p$ with the mass density $\rho$ and the absolute temperature $T$, was determined except for a single variable function so that it was adapt to describe only particular gases or continuum.

This drawback has been overcome in [1] and other articles such as [19]-[34] by considering two blocks
of balance equations, for example, in the 14 moments case treated in [1], they are

\[
\partial_t F^N + \partial_k F^{kN} = P^N, \quad \partial_t G^E + \partial_k G^{kE} = Q^E, \tag{1}
\]

where

\[
F^N = (F, F^i, F^{ij}), \quad G^E = (G, G^i),
\]

\[
F^{kN} = (F^k, F^{ki}, F^{kij}), \quad G^{kE} = (G^k, G^{ki}),
\]

\[
P^N = (0, 0, P^{ij}), \quad Q^E = (0, Q^i). \tag{4}
\]

The first 2 components of \(P^N\) are zero because the first 2 components of equations (1) are the conservation laws of mass and momentum; the first component of \(Q^E\) is zero because the first component of equations (1) is the conservation laws of energy. The whole block (1) can be considered an "Energy Block".

The equations (1) can be written in a more compact form as

\[
\partial_t F^A + \partial_k F^{kA} = P^A, \tag{2}
\]

where

\[
F^A = (F^N, G^E), \quad F^{kA} = (F^{kN}, G^{kE}), \quad P^A = (P^N, Q^E). \tag{4}
\]

In the whole set (2), \(F^A\) are the independent variables, while \(F^{kij}, G^{ki}, P^{ij}, Q^i\) are constitutive functions. Restrictions on their generalities are obtained by imposing

1. **The Entropy Principle** which guarantees the existence of an entropy density \(h\) and an entropy flux \(h^k\) such that the equation

\[
\partial_t h + \partial_k h^k = \sigma \geq 0, \tag{3}
\]

holds whatever solution of the equations (2).

Thanks to Liu’s Theorem [35], [36], this is equivalent to assuming the existence of Lagrange Multipliers \(\mu_A\) such that

\[
d h = \mu_A d F^A, \quad d h^k = \mu_A d F^{kA}, \quad \sigma = \mu_A P^A. \tag{4}
\]

An idea conceived by Ruggeri is to define the 4-potentials \(h', h'^k\) as

\[
h' = \mu_A F^A - h, \quad h'^k = \mu_A F^{kA} - h^k, \tag{5}
\]

so that eqs. (1)1,2 become

\[
d h' = F^A d \mu_A, \quad d h'^k = F^{kA} d \mu_A, \tag{6}
\]

which are equivalent to

\[
F^A = \frac{\partial h'}{\partial \mu_A}, \quad F^{kA} = \frac{\partial h'^k}{\partial \mu_A}, \tag{6}
\]

if the Lagrange Multipliers are taken as independent variables. A nice consequence of eqs. (6) is that the field equations assume the symmetric form.

Other restrictions are given by
2. **The symmetry conditions**, that is the second component of \( F^N \) is equal to the first component of \( F^{kN} \), the third component of \( F^N \) is equal to the second component of \( F^{kN} \), the second component of \( G^E \) is equal to the first component of \( G^{kE} \). Moreover, \( F^{ij} \) is a symmetric tensor.

Thanks to eqs. (6) these conditions assume the form
\[
\frac{\partial h'}{\partial \mu_i} = \frac{\partial h'}{\partial \mu} , \quad \frac{\partial h'}{\partial \mu_{ij}} = \frac{\partial h'}{\partial \mu_j} , \quad \frac{\partial h'}{\partial \lambda_i} = \frac{\partial h'}{\partial \lambda} ,
\]
where we have assumed the decomposition \( \mu_A = (\mu, \mu_i, \mu_{ij}, \lambda, \lambda_i) \) for the Lagrange Multipliers.

Moreover \( \mu_{ij} \) is a symmetric tensor.

Eventual supplementary symmetry conditions are those imposing the symmetry of the tensors \( F^{kij} \) and \( G^{ki} \) and are motivated by the kinetic counterpart of this theory. Thanks to eqs. (6) these conditions may be expressed as
\[
\frac{\partial h'}{\partial \mu_{ij}} = 0 , \quad \frac{\partial h'}{\partial \lambda_{ij}} = 0 .
\]

However, if the kinetic approach is used only to give suggestions on the form of the balance equations, one may think to obtain a more general macroscopic theory if these supplementary symmetry conditions are not imposed. For this reason they have been not considered in [1] and in the present article.

The next conditions come from

3. **The Galilean Relativity Principle**.

There are two ways to impose this principle. One of these is to decompose the variables \( F^A \), \( F^{kA} \), \( P^A \), \( \mu_A \) in their corresponding non convective parts \( \hat{F}^A \), \( \hat{F}^{kA} \), \( \hat{P}^A \), \( \hat{\mu}_A \) and in velocity dependent parts, where the velocity is defined by
\[
v^i = F^{-1} F^i .
\]

This decomposition can be written as
\[
F^A = X^A_B(\bar{v}) \hat{F}^B , \quad F^{kA} - v^k F^A = X^A_B(\bar{v}) \hat{F}^{kB} , \quad P^A = X^A_B(\bar{v}) \hat{P}^B ,
\]

\[
h' = \hat{h}' , \quad h'^k - v^k h' = \hat{h}'^k , \quad \hat{\mu}_A = \mu_B X^B_A(\bar{v}) ,
\]

where
\[
X^A_B(\bar{v}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
v^i & \delta_a^i & 0 & 0 & 0 \\
v^i v^j & 2v^i \delta_a^j & \delta_a^i \delta_b^j & 0 & 0 \\
v^2 & 2v^a & 0 & 1 & 0 \\
v^2 v^i & v^2 \delta_a^i + 2v^i v^a & 2\delta_a^i v_b & v^i & \delta_a^i
\end{pmatrix}
\]

(11)

After that, all the conditions are expressed in terms of the non convective parts of the variables. This procedure is described in [2], [36] for the case considering only the block (1) and is followed in [1] for the whole set (1).

Another way to impose this principle leads to easier calculations; it is described in [37] for the case considering only the the block (1) and here we show how it is adapt also for the whole set (1).
Now we can use a property of the matrix independent from the reference frame.

First of all, we need to know the transformation law of the variables between two reference frames moving one with respect to the other with a translational motion with constant translational velocity \( \vec{v}_r \). To know it, we may rewrite (10) in both frames, that is

\[
F^A_a = X^A_B(\vec{v}_a)\hat{F}^B , \quad F^k_a - v^k_a F^A_a = X^A_B(\vec{v}_a)\hat{F}^k_B ,
\]

where the index \( a \) denotes quantities in the absolute reference frame and index \( r \) denotes quantities in the relative one; \( \hat{F}^B, \hat{F}^k_B, \hat{h}^k, \hat{\mu}_B \) haven’t the index \( a \), nor the index \( r \) because they are independent from the reference frame.

Now we can use a property of the matrix \( X^A_B(\vec{v}) \) which is a consequence of its definition (11) and reads

\[
X^C_A(\vec{-v})X^A_B(\vec{v}) = \delta^C_B .
\]

So we may contract \( 12, 6, 7 \) with \( X^C_A(\vec{-v}_r) \) so obtaining

\[
\hat{F}^C = X^C_A(\vec{-v}_r)F^A_r , \quad \hat{F}^k_C = X^C_A(\vec{-v}_r)(F^k_r - v^k_r F^A_r)
\]

which can be substituted in (12)_{1,2}. The result is

\[
F^A_a = X^A_B(\vec{v}_a)X^B_C(\vec{-v}_r)F^C_r , \quad F^k_a - v^k_a F^A_a = X^A_B(\vec{v}_a)X^B_C(\vec{-v}_r)(F^k_r - v^k_r F^C_r).
\]

Now we use another property of the matrix \( X^A_B(\vec{v}) \) which is a consequence of its definition (11) and reads

\[
X^A_B(\vec{u})X^B_C(\vec{w}) = X^A_C(\vec{u} + \vec{w}) .
\]

Moreover, we use the well known property

\[
\vec{v}_a = \vec{v}_r + \vec{v}_r .
\]

In this way the equations (14) become

\[
F^A_a = X^A_C(\vec{v}_r)F^C_r , \quad F^k_a - v^k_a F^A_a = X^A_C(\vec{v}_r)F^k_C - v^k_r X^A_C(\vec{v}_r)F^C_r,
\]

In eq. (17)\_2 we can substitute \( X^A_C(\vec{v}_r)F^C_r \) from eq. (17)\_1 so that it becomes

\[
F^k_a - v^k_a F^A_a = X^A_C(\vec{v}_r)F^k_C .
\]

Finally, we deduce \( \hat{h}' , \hat{h}^k, \) and \( \hat{\mu}_A \) from \( 12, 8, 9, 10 \) and substitute them in \( 12, 3, 4, 5 \) so obtaining

\[
\hat{h}'_a = h'_a , \quad \hat{h}^k_a - v^k_a h'_a = h^k_a , \quad \hat{\mu}_C = \mu^2_B X^B_C(\vec{v}_r) ,
\]

where for the last one we have also used a contraction with \( X^A_C(\vec{v}_r) \).

Well, eqs. (17)\_1, (18) and (19) give the requested transformation law between the two reference frames and it is very interesting that it looks like eqs. (10).
Now, if the Lagrange Multipliers are taken as independent variables, eqs. (19) are only a change of independent variables from $\mu^a_B$ to $\mu^a_C$, while (17), (18), (19) are conditions because they involve constitutive functions

$$F^A_a = F^A(\mu^a_B), \quad F^{AK}_a = F^{AK}(\mu^a_B), \quad h'_a = h'_a(\mu^a_B), \quad h'^{ak}(\mu^a_B),$$

$$F^A_r = F^A(\mu^r_B), \quad F^{AK}_r = F^{AK}(\mu^r_B), \quad h'_r = h'_r(\mu^r_B), \quad h'^{rk}(\mu^r_B),$$

where the form of the functions $F^A_a$, $F^{AK}_a$, $h'_a$, $h'^{ak}$ don’t depend on the reference frame for the Galilean Relativity Principle. If we substitute $\mu^r_B$ from eq. (19) in (20) and then substitute the result in (17), (18), we obtain

$$F^A(\mu^r_C X_C B(−vτ)) = X^A C(−vτ) F^C_r,$$

$$F^{AK}(\mu^r_C X_C B(−vτ)) - v_k^C X^A C(−vτ) F^C_r = X^A C(−vτ) F^{kC}_r,$$

$$h'_r(\mu^r_C X_C B(−vτ)) = h'_r,$$

$$h'^{rk}(\mu^r_C X_C B(−vτ)) - h'^{rk} = h'^{rk}.$$

Well, these expressions calculated in $v^i_τ = 0$ are nothing more than eqs. (20) as we expected. But, for the Galilean Relativity Principle they must be coincident for whatever value of $v^i_τ$; this amounts to say that the derivatives of (21) with respect to $v^i_τ$ must hold.

This constraint can be written explicitly more easily if we take into account that $\mu^r_C X_C B(−vτ) = \mu^r_B$ which can be written explicitly by use of (11) and reads

$$\mu^a = \mu^a - \mu^r v^i_τ + \mu^r v^j_τ v^i_τ + \lambda^r v^2_τ - \lambda^r v^i_τ v^2_τ,$$

$$\mu^h_k = \mu^h_k - 2\mu^h_k v^i_τ - 2\lambda^h v^r v^h + \lambda^r (v^2_τ δ^h_k + 2v^i_τ v^r_k),$$

$$\mu^h_{kk} = \mu^h_{kk} - 2\lambda^h v^r (h δ^k_k),$$

$$\lambda^a = \lambda^a - \lambda^r v^i_τ,$$

$$\lambda^h = \lambda^h,$$

from which

$$\frac{\partial \mu^a}{\partial v^i_τ} = -\mu^a, \quad \frac{\partial \mu^a_h}{\partial v^i_τ} = -2\mu^a_h - 2\lambda^a δ_{hi},$$

$$\frac{\partial \mu^a_h k}{\partial v^i_τ} = -2\lambda^a_h δ_{ki}, \quad \frac{\partial \lambda^a}{\partial v^i_τ} = -\lambda^a, \quad \frac{\partial \lambda^h}{\partial v^i_τ} = 0.$$

Consequently, the derivatives of (21) with respect to $v^i_τ$ become

$$\frac{\partial h^l}{\partial \mu} \mu^l_i + \frac{\partial h^l}{\partial \mu_h} (2\mu^h_i + 2\lambda^h_i) + 2\frac{\partial h^l}{\partial \mu_h} \lambda_h + \frac{\partial h^l}{\partial \lambda} \lambda_i = 0,$$

$$\frac{\partial h^{lk}}{\partial \mu} \mu^l_i + \frac{\partial h^{lk}}{\partial \mu_h} (2\mu^h_i + 2\lambda^h_i) + 2\frac{\partial h^{lk}}{\partial \mu_h} \lambda_h + \frac{\partial h^{lk}}{\partial \lambda} \lambda_i + h^{lki} = 0,$$

where we have omitted the index $a$ denoting variables in the absolute reference frame because they remain unchanged if we change $v^i_τ$ with $-v^i_τ$, that is, if we exchange the absolute and the relative reference frames.

It is not necessary to impose the derivatives of (21) because they are consequences of (24) and (9).

Consequently, the Galilean Relativity Principle amounts simply in the 2 equations (24).
So we have to find the most general functions satisfying (7) and (24). After that, we have to use eqs. (6) to obtain the Lagrange Multipliers in terms of the variables $F^A$. By substituting them in $\hat{F} = h'(\mu_A)$, $h^{k} = h^{k}(\mu_A)$ we obtain the constitutive functions in terms of the variables $F^A$. If we want the non convective parts of our expressions, it suffices to calculate the left hand side of eqs. (6) in $\vec{v} = 0$ so that they become

$$
\vec{v} = \frac{\partial h^i}{\partial \mu_A}.
$$

(25)

From this equation we obtain the Lagrange Multipliers in terms of $\hat{F}^A$ (Obviously, they will be $\hat{\mu}_A$) and after that substitute them in $h' = h'(\mu_A)$, $h^{k} = h^{k}(\mu_A)$ (the last of which will in effect be $\hat{h}^{k}$) and into $\vec{F}^A = \hat{F}^A$, that is eq. (6) calculated in $\vec{v} = 0$.

It has to be noted that from (9) it follows $\vec{v} = 0$, so that one of the equations (25) is $0 = \frac{\partial h'}{\partial \mu_i}$; this doesn’t mean that $h'$ doesn’t depend on $\mu_i$, but this is simply an implicit function defining jointly with the other equations (25) the quantities $\hat{\mu}_A$ in terms of $\hat{F}^A$.

By using a procedure similar to that of the paper [37], we can prove that we obtain the same results of the firstly described approach.

Now, from (7) it follows $\frac{\partial h^i}{\partial \mu_j} = 0$; this equation, together with (7) are equivalent to assuming the existence of a scalar function $H$ such that

$$
h' = \frac{\partial H}{\partial \mu}, \quad h^{k} = \frac{\partial H}{\partial \mu_i}.
$$

(26)

In fact, the integrability conditions for (26) are exactly (7) and $\frac{\partial h^i}{\partial \mu_j} = 0$.

Thanks to (26), we can rewrite (7) and (24) as

$$
\frac{\partial^2 H}{\partial \mu \partial \mu_i} = \frac{\partial^2 H}{\partial \mu_i \partial \mu_j}, \quad \frac{\partial^2 H}{\partial \lambda \partial \mu_i} = \frac{\partial^2 H}{\partial \lambda \partial \mu_j}.
$$

(27)

$$
\frac{\partial^2 H}{\partial \mu \partial \mu_i} + \frac{\partial^2 H}{\partial \mu \partial \mu_j}(2\mu_{2i} + 2\mu_{2j}) + \frac{\partial^2 H}{\partial \mu \partial \mu_{2i}}\lambda_{2i} + \frac{\partial^2 H}{\partial \mu \partial \mu_{2j}}\lambda_{2j} = 0,
$$

(28)

$$
\frac{\partial^2 H}{\partial \mu \partial \mu_i} + \frac{\partial^2 H}{\partial \mu \partial \mu_j}(2\mu_{2i} + 2\mu_{2j}) + \frac{\partial^2 H}{\partial \mu \partial \mu_k}\lambda_{2k} + \frac{\partial^2 H}{\partial \mu \partial \mu_k}\lambda_{2j} + \frac{\partial H}{\partial \mu} \delta_{2i} = 0.
$$

(29)

We note now that the derivative of (28) with respect to $\mu_k$ is equal to the derivative of (28) with respect to $\mu$; similarly, the derivative of (28) with respect to $\lambda_k$ is equal to the derivative of (28) with respect to $\lambda$, as it can be seen by using also eqs. (27).

Consequently, the left hand side of eq. (28) is a vectorial function depending only on two scalars $\mu$, $\lambda$ and on a symmetric tensor $\mu_{ij}$. For the Representation Theorems [38]-[46], it can be only zero. In other words, eq. (28) is a consequence of (27) and (28), so that it has not to be imposed. By using eqs. (27) we can rewrite eq. (28) as

$$
\frac{\partial^2 H}{\partial \mu \partial \mu_i} + \frac{\partial^2 H}{\partial \mu \partial \mu_j}(2\mu_{2i} + 2\mu_{2j}) + \frac{\partial^2 H}{\partial \mu \partial \mu_k}\lambda_{2k} + \frac{\partial^2 H}{\partial \mu \partial \mu_k}\lambda_{2j} + \frac{\partial H}{\partial \mu} \delta_{2i} = 0.
$$

(29)

In an article which still needs to be written, Carrisi has found the general solution up to whatever order with respect to equilibrium, of the conditions (27), (29) and also of (8) which now, by use of eqs. (26) can be written as

$$
\frac{\partial^2 H}{\partial \mu \partial \mu_i} = 0, \quad \frac{\partial^2 H}{\partial \mu \partial \lambda_i} = 0.
$$

(30)
For a more agreement with the article [1] we want now to do the same thing without imposing (30). However, although it may seem strange, with less conditions the calculations become heavier! In fact, if it was possible to use the conditions (30), we see that the function $\frac{\partial H}{\partial \mu_k}$ has all the derivatives with respect to $\mu_i$, $\mu_{ij}$, $\lambda_i$ which are symmetric tensors, so that its Taylor’s expansion around equilibrium is

$$\frac{\partial H}{\partial \mu_k} = \sum_{p,q,r}^{0 \to \infty} \frac{1}{p! q! r!} H_{p,q,r}^{k_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r} \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}, \quad (31)$$

where $H_{p,q,r}^{k_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r}$ is a symmetric tensor depending only on the scalars $\mu$ and $\lambda$, so that it has the form

$$H_{p,q,r}^{k_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r} = \begin{cases} H_{p,q,r}(\mu, \lambda) \delta^{(k_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r)} & \text{if } p + r + 1 \text{ is even} \\ 0 & \text{if } p + r + 1 \text{ is odd} \end{cases}$$

where $\delta^{(a_1 \cdots a_{2n})}$ denotes $\delta^{(a_1 a_2 \cdots a_{2n-1} a_{2n})}$.

By integrating (31) we obtain

$$H = \sum_{p,q,r} \sum_{r \in I_p+1}^{0 \to \infty} \frac{1}{(p+1)! q! r!} H_{p,q,r}(\mu, \lambda) \delta^{(k_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r)} \mu_{i_1} \cdots \mu_{i_p+1} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} + \sum_{p,q,r}^{0 \to \infty} \frac{1}{(p+1)! q! r!} H_{p,q,r}(\mu, \lambda, \lambda) \delta^{(k_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r)} \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}$$

where $I_p$ denotes the set of all non negative integers $r$ such that $r + p$ is even.

But also $\frac{\partial H}{\partial \mu}$ has all the derivatives which are symmetric tensors; in fact, the derivatives of (30) with respect to $\mu$ are

$$\frac{\partial^3 H}{\partial \mu \partial \mu_i \partial \mu_{ij}} = 0, \quad \frac{\partial^3 H}{\partial \mu \partial \mu_{|k} \partial \mu_{ij}} = 0.$$ \quad (33)

The derivatives of (27) with respect to $\mu_{ab}$ are

$$\frac{\partial^3 H}{\partial \mu_{ab} \partial \mu_{ij}} = \frac{\partial^3 H}{\partial \mu_{ab} \partial \mu_{ij}}, \quad \frac{\partial^3 H}{\partial \mu_{ab} \partial \mu \partial \mu_i} = \frac{\partial^3 H}{\partial \mu_{ab} \partial \mu \partial \mu_i},$$

whose skew-symmetric parts with respect to $b$ and $i$ are

$$\frac{\partial^3 H}{\partial \mu \partial \mu_{a[b} \partial \mu_{ij]}} = 0, \quad \frac{\partial^3 H}{\partial \mu \partial \mu_{a[b} \partial \mu_{ij]}} = 0,$$ \quad (34)

thanks to eq. (30).

By using (32) we obtain that also $\frac{\partial H}{\partial \mu}$ has all the derivatives which are symmetric tensors, so that its expansion around equilibrium is

$$\frac{\partial H}{\partial \mu} = \sum_{q \in I_0} \sum_{r \in I_0}^{0 \to \infty} \frac{1}{q! r!} \frac{\partial H}{\partial \mu}(\mu, \lambda) \delta^{(h_1 k_1 \cdots h_q k_q j_1 \cdots j_r)} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}, \quad (35)$$
where the derivative of $\bar{H}_{q,r}$ has been introduced for later convenience and without loss of generality. By integrating (35) we obtain

$$
\bar{H}_{q,r} = \sum_{q} \sum_{r=0}^{\infty} \frac{1}{q! \, r!} \bar{H}_{q,r}(\mu, \lambda) \delta(\mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}}) + H(\mu_{ab}, \lambda, \lambda_{c}) \quad \text{.} 
$$

But the function $H$ is present in (27), (28), (29) only through its derivatives with respect to $\mu$ and $\mu_{k}$ so that the function $\bar{H}$ doesn’t effect the results and, consequently, it is not restrictive to assume that its derivatives are symmetric tensors. By substituting (36) in (32) we see that the function $H$ has derivatives which are symmetric tensors and we have also its expansion.

Now, in the present article, we cannot use this property because we don’t have the constraints (30). To overcome this difficulty we proceed as follows. Firstly,

1. We show a particular solution of (27) and (29).

Let $\psi_{n}(\mu, \lambda)$ be a family of functions constrained by

$$
\frac{\partial}{\partial \mu} \psi_{n+1} = \psi_{n} \quad \text{for} \quad n \geq 0 \quad . \tag{37}
$$

Let us define the function

$$
H_{1} = \sum_{p,q} \sum_{r=0}^{\infty} \frac{1}{p! \, q! \, r!} \sum_{l_{p}} \left( p + 2q + r + 1 \right)! \prod_{i=1}^{p+q+r} \frac{\partial^{p+q+r}}{\partial \mu^{p} \partial \lambda^{q} \partial \mu_{l_{i}}} \left( \left( -\frac{1}{2\lambda} \right) \psi_{p+q+r} \right) \cdot \delta^{(i_{1} \cdots i_{p} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}})} \mu_{i_{1}} \cdots \mu_{i_{p}} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} \quad . \tag{38}
$$

In Appendix 1 we will show that eqs. (27) and (29) are satisfied if $H$ is replaced by $H_{1}$; in other words, $H = H_{1}$ is a particular solution of our conditions. Moreover, $(H_{1})_{eq.} = \psi_{0}(\mu, \lambda)$ which is an arbitrary two-variables function, such as $H_{eq.}$ So we can identify

$$
H_{0}(\mu, \lambda) = H_{eq.} \quad . \tag{39}
$$

and define

$$
\Delta H = H - H_{1} \quad . \tag{40}
$$

In this way the conditions (27) and (29) become

$$
\frac{\partial^{2} \Delta H}{\partial \mu \partial \mu_{ij}} = \frac{\partial^{2} \Delta H}{\partial \mu \partial \mu_{ij}} \quad , \quad \frac{\partial^{2} \Delta H}{\partial \mu_{i} \partial \mu_{j}} = \frac{\partial^{2} \Delta H}{\partial \mu_{i} \partial \mu_{j}} \quad , \quad \frac{\partial^{2} \Delta H}{\partial \mu_{i} \partial \mu_{j}} + \frac{\partial^{2} \Delta H}{\partial \mu \partial \mu_{k}} = 0 \quad , \tag{41}
$$

and we have also $(\Delta H)_{eq.} = 0$.

An interesting consequence of (41) and (42) is the following

PROPERTY 1: “ The expansion of $\Delta H$ up to order $n \geq 1$ with respect to equilibrium is a polynomial of degree $n - 1$ in the variable $\mu$. ”

Let us prove this with the iterative procedure and let $\Delta H^{n}$ denote the homogeneous part of $\Delta H$ of order $n$ with respect to equilibrium. We have,
• Case $n = 1$: The equation (41) at equilibrium, thanks to (42) becomes $2\frac{\partial^2 H^1}{\partial \mu \partial \mu_i} \lambda = 0$ from which we have that $\frac{\partial H^1}{\partial \mu}$ can depend only on $\mu, \mu_i, \lambda, \lambda_c$; but the representation theorems show that no scalar function of order 1 with respect to equilibrium can depend only on these variables. It follows that $\frac{\partial H^1}{\partial \mu} = 0$ so that $H^1$ is of degree zero with respect to $\mu$ and the property is verified for this case.

• Case $n \geq 2$: Let us suppose, for the iterative hypothesis that $\Delta H$ up to order $n \geq 1$ with respect to equilibrium is a polynomial of degree $n - 1$ in the variable $\mu$; we proceed now to prove that this property holds also with $n + 1$ instead of $n$.

In fact, eq. (41) up to order $n - 1$ gives

$$\frac{\partial^2 H^n}{\partial \mu \partial \mu_{ij}} = \frac{\partial^2 H^{n+1}}{\partial \mu \partial \mu_{ij}}$$

from which we have

$$\Delta H^{n+1} = P_{n-2} + \Delta H^{n+1}_1(\mu_{ab}, \lambda, \lambda_c)\mu^i + \Delta H^{n+1}_0(\mu, \mu_{ab}, \lambda, \lambda_c)$$  \hspace{1cm} (43)

where $P_{n-2}$ is a polynomial of degree $n - 2$ in $\mu$ and which is at least quadratic in $\mu_j$.

After that, eq. (42) up to order $n$ gives

$$\frac{\partial^2 H^{n+1}}{\partial \mu \partial \lambda_i} = \frac{\partial^2 H^{n+1}}{\partial \lambda \partial \mu_i}$$

which, thanks to (43) becomes

$$\frac{\partial^2 P_{n-2}}{\partial \mu \partial \lambda_i} + \frac{\partial^2 \Delta H^{n+1}_1}{\partial \mu \partial \lambda_i} + \frac{\partial^2 \Delta H^{n+1}_0}{\partial \mu \partial \lambda_i} = \frac{\partial^2 P_{n-2}}{\partial \lambda \partial \mu_i} + \frac{\partial \Delta H^{n+1}_1}{\partial \lambda}.$$  \hspace{1cm} (44)

This relation, calculated in $\mu_j = 0$ gives

$$\frac{\partial^2 \Delta H^{n+1}_0}{\partial \mu \partial \lambda_i} = \frac{\partial \Delta H^{n+1}_1}{\partial \lambda}$$  \hspace{1cm} (45)

because $P_{n-2}$ is at least quadratic in $\mu_j$.

The derivative of (44) with respect to $\mu_j$, calculated then in $\mu_j = 0$, is

$$\frac{\partial^2 \Delta H^{n+1}_0}{\partial \mu \partial \lambda_i} = \left( \frac{\partial^3 P_{n-2}}{\partial \mu_j \partial \lambda_i \partial \mu_i} \right)_{\mu_j = 0}$$

from which

$$\frac{\partial \Delta H^{n+1}_0}{\partial \lambda_i} = P^{ij}_{n-1}$$

with $P^{ij}_{n-1}$ a polynomial of degree $n - 1$ in $\mu$. By integrating this relation, we obtain

$$\Delta H^{n+1}_i = P^{ij}_{n-1} + f^i_{n-1}(\mu, \mu_{ab}, \lambda_j)$$

where $P^{ij}_{n-1}$ is a polynomial of degree $n - 1$ in $\mu$. But, for the Representation Theorems, a vectorial function such as $f^i_{n-1}$ is zero because it depends only on scalars and on a second order tensor. It follows that

$$\Delta H^{n+1}_i = P^{ij}_{n-1}.$$  \hspace{1cm} (46)

By using this result, eq. (45) can be integrated and gives

$$\frac{\partial \Delta H^{n+1}_0}{\partial \lambda_i} = P^i_n$$  \hspace{1cm} (47)

with $P^i_n$ a polynomial of degree $n$ in $\mu$.

Now we impose eq. (41) at order $n$ and see that its first, second, fifth and sixth terms are of degree $n - 2$ in $\mu$ so that we have

$$2\frac{\partial^2 \Delta H^{n+1}_0}{\partial \mu \partial \mu_{ki}} \lambda + \gamma \frac{\partial^2 \Delta H^{n+1}_0}{\partial \mu \partial \mu_{ij}} \lambda_j = Q_{n-2}$$

with $Q_{n-2}$ a polynomial of degree $n - 2$ in $\mu$. This relation, thanks to (43) becomes
\[ 2\lambda \frac{\partial^2 \Delta H^{n+1}_{0}}{\partial \mu_{ki} \partial \mu_{k}} \mu^{a} + 2\lambda \frac{\partial^2 \Delta H^{n+1}_{0}}{\partial \mu_{ki} \partial \mu_{j}} + 2\lambda \frac{\partial^2 P_{n-2}}{\partial \mu_{ij} \partial \mu_{j}} + 2\lambda \frac{\partial \Delta H^{n+1}}{\partial \mu_{ij}} = Z_{n-2} \]

with \( Z_{n-2} \) a polynomial of degree \( n - 2 \) in \( \mu \).

This relation, calculated in \( \mu_{j} = 0 \), thanks to (46) and to the fact that \( P_{n-2} \) is at least quadratic in \( \mu_{j} \), gives

\[ 2\lambda \frac{\partial^2 \Delta H^{n+1}_{0}}{\partial \mu_{ki} \partial \mu_{k}} = \overline{Q}_{ki}^{n-1} \]

with \( \overline{Q}_{ki}^{n-1} \) a polynomial of degree \( n - 1 \) in \( \mu \). It follows that

\[ \frac{\partial \Delta H^{n+1}_{0}}{\partial \mu_{ki}} = \overline{P}_{ki}^{n} \]

with \( \overline{P}_{ki}^{n} \) a polynomial of degree \( n \) in \( \mu \). This result, jointly with (47) gives that

\[ \Delta H^{n+1}_{0} = \tilde{P}_{n} + f(\mu, \lambda) . \]  

(48)

But a function depending only on \( \mu \) and \( \lambda \) cannot be of order \( n + 1 \) with respect to equilibrium; it follows that \( f(\mu, \lambda) = 0 \).

Consequently, (48), (46) and (43) give that \( \Delta H^{n+1} \) is a polynomial of degree \( n \) in \( \mu \) and this completes the proof.

Thanks to this property, it is not restrictive to assume for \( \Delta H \) a polynomial expansion of infinity degree in the variable \( \mu \); we simply expect that the equations will stop by itself the terms with higher degree.

So, even if \( \mu \) is not zero at equilibrium, for what concerns \( \Delta H \), we can do an expansion also around \( \mu = 0 \); obviously, the situation is different for the particular solution \( H_{1} \) reported in eq. (38). The next step with which we proceed is the following one

2. We note that \( \frac{\partial^2 \Delta H}{\partial \mu^2} \) has symmetric tensors as derivatives.

In fact, from the derivatives of (41) with respect to \( \mu_{k} \) we can take the skew-symmetric part with respect to \( i \) and \( k \) so obtaining

\[ \frac{\partial^3 \Delta H}{\partial \mu \partial \mu \partial \mu_{ij}} = 0 , \quad \frac{\partial^3 \Delta H}{\partial \mu \partial \mu \partial \lambda_{ij}} = 0 . \]  

(49)

From the second derivatives of (41) with respect to \( \mu \) and \( \mu_{ab} \), we can take the skew-symmetric part with respect to \( i \) and \( b \) so obtaining

\[ \frac{\partial^4 \Delta H}{\partial \mu^2 \partial \mu_{a} \partial \mu_{ij}} = \frac{\partial^4 \Delta H}{\partial \mu^2 \partial \mu_{a} \partial \mu_{ij}} = 0 , \quad \frac{\partial^4 \Delta H}{\partial \mu^2 \partial \mu_{a} \partial \lambda_{ij}} = \frac{\partial^4 \Delta H}{\partial \mu \partial \lambda \partial \mu_{a} \partial \mu_{ij}} = 0 \]  

(50)

where (49) has been used in the second passage.

Well, (50) and the derivatives of (49), with the derivatives of (49), with respect to \( \mu \), prove our property.

3. We now prove that \( \frac{\partial \Delta H}{\partial \mu} \) is sum of \( H^{0}(\mu_{ab}, \lambda, \lambda_{c}) \) and of a scalar function whose derivatives are all symmetric tensors.

In fact, from (49) we deduce that \( \frac{\partial^2 \Delta H}{\partial \mu \partial \mu} \) has all symmetric derivatives so that its expansion around equilibrium is of the type

\[ \frac{\partial^2 \Delta H}{\partial \mu \partial \mu} = \sum_{p, q} \sum_{r \in \ell_{p}} \frac{1}{1!} H^{*}_{p, q, r} \delta^{(k_{1}i_{p}h_{k_{1}}j_{1}) \cdots (k_{r}i_{p}h_{k_{r}}j_{r})} \mu_{i_{1}} \cdots \mu_{i_{p}} \mu_{k_{1}j_{1}} \cdots \mu_{k_{r}j_{r}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} . \]
By integrating this expression with respect to $\mu_k$, we obtain

$$
\frac{\partial \Delta H}{\partial \mu} = \sum_{p,q} \sum_{r \in I_p} \frac{1}{(p+1)!} \frac{1}{q! r!} H_{p,q,r}^* \delta^{(i_1,\ldots,i_{p+1},h_1k_1,\ldots,h_qk_1,\ldots,j_r)}
$$

$$+ \sum_{r \in I_p} \frac{1}{(p+1)!} \frac{1}{q! r!} \frac{\partial H_{p,q,r}}{\partial \mu} \delta^{(i_1,\ldots,i_{p+1},h_1k_1,\ldots,h_qk_1,\ldots,j_r)} \mu_{i_1} \cdots \mu_{i_{p+1}} H_{h_1k_1} \cdots \mu_{h_qk_1} \lambda_{j_1} \cdots \lambda_{j_r} + H^*(\mu, \mu_\alpha, \lambda, \lambda_c). \tag{51}
$$

The derivative of $H_{p,q,r}^*$ with respect to $\mu$ is

$$
\frac{\partial^2 \Delta H}{\partial \mu^2} = \sum_{p,q} \sum_{r \in I_p} \frac{1}{(p+1)!} \frac{1}{q! r!} \frac{\partial \Delta H_{p,q,r}}{\partial \mu} \delta^{(i_1,\ldots,i_{p+1},h_1k_1,\ldots,h_qk_1,\ldots,j_r)} \mu_{i_1} \cdots \mu_{i_{p+1}} H_{h_1k_1} \cdots \mu_{h_qk_1} \lambda_{j_1} \cdots \lambda_{j_r} \tag{52}
$$

$$+ \frac{\partial H^*}{\partial \mu} \mu_{i_1} \cdots \mu_{i_{p+1}} \mu_{h_1k_1} \cdots \mu_{h_qk_1} \lambda_{j_1} \cdots \lambda_{j_r}.$$

But also $\frac{\partial^2 \Delta H}{\partial \mu^2}$ has all symmetric derivatives so that its expansion is

$$
\frac{\partial^2 \Delta H}{\partial \mu^2} = \sum_{p,q} \sum_{r \in I_p} \frac{1}{(p+1)!} \frac{1}{q! r!} \frac{\partial \Delta H_{p,q,r}}{\partial \mu} \delta^{(i_1,\ldots,i_{p+1},h_1k_1,\ldots,h_qk_1,\ldots,j_r)} \mu_{i_1} \cdots \mu_{i_{p+1}} H_{h_1k_1} \cdots \mu_{h_qk_1} \lambda_{j_1} \cdots \lambda_{j_r} \tag{53}
$$

where $\tilde{H}_{p,q,r}$ appears through its derivative with respect to $\mu$ for later convenience and without loss of generality.

By substituting $\tilde{H}_{p,q,r}$ in $\tilde{H}_{p,q,r}$ we find an expression from which we deduce $\frac{\partial H^*}{\partial \mu}$; by integrating it we obtain

$$
H^* = \sum_{p,q} \sum_{r \in I_p} \frac{1}{(p+1)!} \frac{1}{q! r!} \tilde{H}_{p,q,r} \delta^{(i_1,\ldots,i_{p+1},h_1k_1,\ldots,h_qk_1,\ldots,j_r)} \mu_{i_1} \cdots \mu_{i_{p+1}} H_{h_1k_1} \cdots \mu_{h_qk_1} \lambda_{j_1} \cdots \lambda_{j_r} +
$$

$$- \sum_{p,q} \sum_{r \in I_p} \frac{1}{(p+1)!} \frac{1}{q! r!} \tilde{H}_{p,q,r} \delta^{(i_1,\ldots,i_{p+1},h_1k_1,\ldots,h_qk_1,\ldots,j_r)} \mu_{i_1} \cdots \mu_{i_{p+1}} H_{h_1k_1} \cdots \mu_{h_qk_1} \lambda_{j_1} \cdots \lambda_{j_r} +
$$

$$+ H^{*0}(\mu_{ab}, \lambda, \lambda_c),$$

where $H^{*0}$ arises from an integration with respect to $\mu$ so that it doesn’t depend on $\mu$; moreover, it doesn’t depend on $\mu_i$ because $H^*$ doesn’t depend on $\mu_i$.

By substituting this expression in $\tilde{H}_{p,q,r}$ we find that $\frac{\partial \Delta H}{\partial \mu}$ is the sum of $H^{*0}(\mu_{ab}, \lambda, \lambda_c)$ and of a function whose derivatives are all symmetric tensors; consequently, it can be written in the form

$$
\frac{\partial \Delta H}{\partial \mu} = \sum_{p,q} \sum_{r \in I_p} \frac{1}{(p+1)!} \frac{1}{q! r!} \tilde{H}_{p,q,r} \delta^{(i_1,\ldots,i_{p+1},h_1k_1,\ldots,h_qk_1,\ldots,j_r)} \mu_{i_1} \cdots \mu_{i_{p+1}} H_{h_1k_1} \cdots \mu_{h_qk_1} \lambda_{j_1} \cdots \lambda_{j_r} + \tag{54}
$$

$$+ H^{*0}(\mu_{ab}, \lambda, \lambda_c).$$

If we take into account the result of Property 1, we see that also $\frac{\partial \Delta H}{\partial \mu}$ can be expressed as a polynomial of infinity degree in $\mu$, so that $\tilde{H}_{p,q,r}$ can be written as

$$
\frac{\partial \Delta H}{\partial \mu} = \sum_{p,q,s} \sum_{r \in I_p} \frac{1}{(p+1)!} \frac{1}{q! r!} \tilde{H}_{p,q,r,s} \delta^{(i_1,\ldots,i_{p+1},h_1k_1,\ldots,h_qk_1,\ldots,j_r)} \mu_{i_1} \cdots \mu_{i_{p+1}} H_{h_1k_1} \cdots \mu_{h_qk_1} \lambda_{j_1} \cdots \lambda_{j_r} + \tag{55}
$$

$$+ H^{*0}(\mu_{ab}, \lambda, \lambda_c).$$
Now, if we substitute in (55) \( H^{*0} \) with 
\[
H^{*0N} = \sum_{q=0}^{N} \sum_{r \in I_0 \frac{1}{q^2}} \psi_{0,q,r,0}^N(\lambda) \delta^{\mu_1 k_i \cdots \mu_q k_q} \lambda_1 \cdots \lambda_j ,
\]
we note that (55) remains unchanged, except that now we have \( H^{*0N} \) instead of \( H^{*0} \) and zero instead of \( \psi_{0,q,r,0} \). We conclude that we may still use eq. (55) and assume, without loss of generality that
\[
\psi_{0,q,r,0}(\lambda) = 0. \tag{56}
\]

If we calculate (55) at equilibrium, and take into account (42), we obtain
\[
0 = \sum_{s=0}^{\infty} \frac{1}{s!} \psi_{0,0,0,s}(\lambda) \mu^s + H^{*0}(0_{ab}, \lambda, 0_c).
\]
Consequently we have \( \psi_{0,0,0,s}(\lambda) = 0 \) for \( s \geq 1 \), from which and from eq. (56) it follows
\[
\psi_{0,0,0,s}(\lambda) = 0 \quad \text{for} \quad s \geq 0. \tag{57}
\]
Moreover, (42) will give
\[
H^{*0}(0_{ab}, \lambda, 0_c) = 0. \tag{58}
\]

### 1.1 Further restrictions

For the sequel, it will be useful to consider some consequences of eq. (41). They are
\[
\frac{\partial^2 \Delta H}{\partial \mu_j \partial \lambda_i \partial \mu} = \frac{\partial^3 \Delta H}{\partial \lambda \partial \mu_j \partial \mu}, \tag{59}
\]
\[
\left[ \frac{\partial^2 \Delta H}{\partial \mu \partial \mu_k} \right]_{\mu_i} + 2 \frac{\partial^2 \Delta H}{\partial \mu \partial \mu_{k_j}} \mu_{j_i} + 2 \frac{\partial^2 \Delta H}{\partial \mu \partial \lambda_i} \lambda + \frac{\partial \Delta H}{\partial \mu} \delta^{k_i} = 0,
\]
\[
\frac{\partial^3 \Delta H}{\partial \mu_a \partial \mu_k \partial \mu} \mu_i + 2 \frac{\partial^2 \Delta H}{\partial \mu_a \partial \mu_{k_j}} \mu_{j_i} + 2 \frac{\partial^2 \Delta H}{\partial \mu_a \partial \lambda_i} \lambda + 2 \frac{\partial \Delta H}{\partial \mu_a} \mu_{k_j} \lambda_j = 0,
\]
\[
\frac{\partial^3 \Delta H}{\partial \mu_{k_i} \partial \mu \partial \lambda} \mu_i + 2 \frac{\partial^2 \Delta H}{\partial \mu_{k_j} \partial \mu \partial \lambda} \mu_{j_i} + 2 \frac{\partial^2 \Delta H}{\partial \mu_{k_i} \partial \mu \partial \lambda} \lambda + 2 \frac{\partial \Delta H}{\partial \mu_{k_i}} \mu_{k_j} \lambda_j = 0.
\]

The first one of these equations is obtained by taking the derivatives of (41) with respect to \( \mu_j \) and by substituting in its right hand side \( \frac{\partial^2 \Delta H}{\partial \mu \partial \mu_j} \) from (41); the second one is obtained by simply calculating (41) in \( \lambda_j = 0 \); similarly, (41) is obtained by taking the derivative of (41) with respect to \( \mu_a \) and, subsequently, by substituting in its fourth term \( \frac{\partial^2 \Delta H}{\partial \mu_k \partial \mu_a} \) from (41). Finally, in the derivative of (41) with respect to \( \lambda \) we can substitute \( \frac{\partial^2 \Delta H}{\partial \lambda \partial \mu_k} \) from (41) in its fourth term; in this way (59) is obtained.

We see that (59) are conditions on \( \frac{\partial \Delta H}{\partial \mu} \) so that they may be considered a sort of integrability conditions on \( \Delta H \), if \( \frac{\partial \Delta H}{\partial \mu} \) would be known.

In the next section, restrictions will be found to the scalar functions appearing in (55), by analyzing eqs. (41) and (59).
2 The expression for $\frac{\partial \Delta H}{\partial \mu}$.

If we substitute (55) in the derivative of (111)$_{1,2}$ with respect to $\mu$, we obtain

$$\partial_{p,q+1,r,s+1} = \partial_{p+2,q,r,s} \quad \partial_{p,q,r+1,s+1} = \frac{\partial}{\partial \lambda} \partial_{p+1,q,r,s}. \quad (60)$$

From (61)$_1$ we now obtain

$$\partial_{p,q,r,s} = \begin{cases} \partial_{0,q} + \frac{\partial}{\partial \mu} & \text{if } p \text{ is even} \\ \partial_{1,q} + \frac{\partial}{\partial \mu} & \text{if } p \text{ is odd}. \end{cases} \quad (61)$$

After that, we see that (60)$_1$ is satisfied as a consequence of (61). Let us focus now our attention to eq. (60)$_2$; for $p = 0, 1$ it becomes

$$\partial_{0,q,r+1,s+1} = \frac{\partial}{\partial \lambda} \partial_{1,q,r,s} \quad \partial_{1,q,r+1,s+1} = \frac{\partial}{\partial \lambda} \partial_{0,q+1,r,s+1}, \quad (62)$$

where, for (62)$_2$ we have used (60) with $p = 2$.

After that, eq. (60)$_2$ with use of eq. (61)

- in the case with $p$ even, gives (62)$_1$ with $(q + \frac{p}{2}, s + \frac{p}{2})$ instead of $(q, s)$,
- in the case with $p$ odd, gives (62)$_2$ with $(q + \frac{p-1}{2}, s + \frac{p-1}{2})$ instead of $(q, s)$.

But we have now to impose the derivative of (111)$_3$ with respect to $\mu$, that is

$$\frac{\partial^3 \Delta H}{\partial \mu^2 \partial \mu_k} \mu_i + 2 \frac{\partial^3 \Delta H}{\partial \mu^2 \partial \mu_i \partial \mu_k} \lambda_j + 2 \frac{\partial^3 \Delta H}{\partial \mu \partial \mu_k \partial \mu_i} \lambda_j + \frac{\partial^3 \Delta H}{\partial \mu^2 \partial \lambda_k} \lambda_i + \frac{\partial^2 \Delta H}{\partial \mu^2} \delta^{ki} = 0. \quad \text{(63)}$$

To impose this relation, let us take its derivatives with respect to $\mu_{i_1}, \cdots, \mu_{i_p}, \mu_{h_{1k_1}}, \cdots, \mu_{h_{Qk_Q}}, \lambda_{j_1}, \cdots, \lambda_{j_R}$ and let us calculate the result at equilibrium; in this way we obtain

$$0 = P \delta^{i_1 \cdots i_p} \delta_{Q_{j_1} \cdots j_R} \partial_{P, Q_{R, s+1}} + 2 Q \delta^{j_1 \cdots j_R} \delta_{Q_{i_1} \cdots i_p k_1 \cdots k_Q} \partial_{P, Q_{R, s+1}} + 2 Q \delta^{j_1 \cdots j_R} \delta_{Q_{i_1} \cdots i_p k_1 \cdots k_Q} \partial_{P, Q_{R, s+1}} + 2 R \delta^{j_1 \cdots j_R} \delta_{Q_{i_1} \cdots i_p k_1 \cdots k_Q} \partial_{P, Q_{R, s+1}} +$$

$$+ R \delta^{j_1 \cdots j_R} \delta_{Q_{i_1} \cdots i_p k_1 \cdots k_Q} \partial_{P, Q_{R, s+1}} + \delta^{j_1 \cdots j_R} \delta_{Q_{i_1} \cdots i_p k_1 \cdots k_Q} \partial_{P, Q_{R, s+1}}, \quad \text{(63)}$$

where overlined indexes denote symmetrization over those indexes, after that the other one (round brackets around indexes) has been taken. (Note that, in the fourth term the index $R - 1$ appears; despite this fact, the equations holds also for $R = 0$ but in this case, this fourth term is not present as it is remembered also by the factor $R$).

Now, the first, second, fifth and sixth term can be put together so that the above expression becomes

$$0 = (P + 2Q + R + 1) \partial_{P, Q_{R, s+1}} + 2 \lambda \partial_{Q_{P, Q_{R, s+1}}} + 2 R \partial_{P, Q_{R, s+1}} +$$

$$+ (2 \lambda \partial_{Q_{P, Q_{R, s+1}}} + 2 R \partial_{P, Q_{R, s+1}}) \delta^{j_1 \cdots j_R} \delta_{Q_{i_1} \cdots i_p k_1 \cdots k_Q} \partial_{P, Q_{R, s+1}},$$

that is

$$0 = (P + 2Q + R + 1) \partial_{P, Q, R, s+1} + 2 \lambda \partial_{Q_{P, Q, R, s+1}} + 2 R \partial_{P, Q, R, s+1}, \quad (63)$$
This relation, for $P = 0, 1$ reads

$$0 = (2Q + R + 1)\vartheta_{0,Q,R,s+1} + 2\lambda \vartheta_{0,Q+1,R,s+1} + 2R \vartheta_{1,Q+1,R,s+1},$$

$$0 = (2Q + R + 2)\vartheta_{1,Q,R,s+1} + 2\lambda \vartheta_{1,Q+1,R,s+1} + 2R \vartheta_{0,Q+2,R,s+1},$$

with the agreement that the last terms are not present in the case $R = 0$. (For eq. (64) we have used (61) with $p = 2$).

For the other values of $P$, eq. (63) with use of eq. (61)

- in the case with $p$ even, gives (63) with $(q + \frac{p}{2}, s + \frac{p}{2})$ instead of $(q, s)$,
- in the case with $p$ odd, gives (63) with $(q + \frac{p-1}{2}, s + \frac{p-1}{2})$ instead of $(q, s)$.

Summarizing the results, we have that (61) gives $\vartheta_{p,Q,R,s}$ in terms of $\vartheta_{0,Q,R,s}$ and $\vartheta_{1,Q,R,s}$, while eqs. (62) and (64) give restrictions on $\vartheta_{0,Q,R,s}$ and $\vartheta_{1,Q,R,s}$.

### 2.1 Consequences of the further restrictions

- We want now to impose eqs. (59). By substituting eq. (55) in (59)1, we find

$$\frac{\partial^2 H^{*0}}{\partial \lambda \partial \mu_{ij}} = \sum_{p,q,r} \sum_{r \in I_p} \frac{1}{p! q! r! s!} (\partial_{p+1,q,r+1,s} - \partial_{p,q+1,r,s}) \delta^{(ijij_1\cdots i_p h_1 k_1\cdots h_q k_j j_1\cdots j_r)} \mu_{i_1 j_1} \cdots \mu_{i_p j_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}.$$  

But, from eq. (61) and (62) it follows that $\vartheta_{p+1,q,r+1,s} - \frac{\partial}{\partial \lambda} \vartheta_{p,q+1,r,s} = 0$ for $p \geq 1$, so that in (65) only the term with $p = 0$ survives. Moreover, from eq. (62) it follows that $\vartheta_{1,q,r+1,s} - \frac{\partial}{\partial \lambda} \vartheta_{0,q+1,r,s} = 0$ for $s \geq 1$, so that in (65) only the term with $s = 0$ survives. Consequently, (65) becomes

$$\frac{\partial^2 H^{*0}}{\partial \lambda \partial \mu_{ij}} = \sum_{q} \sum_{r \in I_0} \frac{1}{q! r!} \partial_{1,q,r+1,0} \delta^{(ij j_1 k_1 \cdots h_q k_j j_1 \cdots j_r)} \mu_{h k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}.$$  

- By substituting now eq. (55) in (59)2, we find

$$\mu_i \sum_{q,s} \sum_{p \in I_1} \frac{1}{p! q! s!} \vartheta_{p+1,q+1,s} \delta^{(ki ij_1 \cdots i_p h_1 k_1 \cdots h_q k_q)} \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} +$$

$$+ 2\mu_{ij} \left\{ \sum_{q,s} \sum_{p \in I_0} \frac{1}{p! q! s!} \vartheta_{p+1,q+1,s} \delta^{(ki j_1 k_1 \cdots h_q k_j j_1 \cdots j_r)} \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \right\} +$$

$$+ 2\lambda \left\{ \sum_{q,s} \sum_{p \in I_0} \frac{1}{p! q! s!} \vartheta_{p+1,q+1,s} \delta^{(ki j_1 \cdots i_h h_1 k_1 \cdots h_q k_q)} \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \right\} +$$

$$+ \delta^{k_1} \left\{ \sum_{q,s} \sum_{p \in I_0} \frac{1}{p! q! s!} \vartheta_{p+1,q+1,s} \delta^{(i_1 j_1 h_1 k_1 \cdots h_q k_q)} \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \left[ H^{*0} \right]_{\lambda_j = 0} = 0 ;

$$
this relation calculated in $\mu_j = 0$ becomes

\[
2\mu_{ji} \left\{ \sum_{q,s}^{0:\infty} \frac{1}{q!} \frac{1}{s!} \partial_{0,q+1,0,s}\mu^s \delta^{(k_1\cdots h_{q+kq})} \mu_{h_1k_1}\cdots \mu_{h_qk_q} + \left[ \frac{\partial H^0}{\partial \mu_{kj}} \right]_{\lambda_j=0} \right\} +
+2\lambda \left\{ \sum_{q,s}^{0:\infty} \frac{1}{q!} \frac{1}{s!} \partial_{0,q+1,0,s}\mu^s \delta^{(k_1\cdots h_{q+kq})} \mu_{h_1k_1}\cdots \mu_{h_qk_q} + \left[ \frac{\partial H^0}{\partial \mu_{ki}} \right]_{\lambda_j=0} \right\} +
+\delta^{ki} \left\{ \sum_{q,s}^{0:\infty} \frac{1}{q!} \frac{1}{s!} \partial_{0,q,0,s}\mu^s \delta^{(h_{1+kq})} \mu_{h_1k_1}\cdots \mu_{h_qk_q} + \left[ H^0 \right]_{\lambda_j=0} \right\} = 0
\]

whose derivative with respect to $\mu_{h_1k_1}, \cdots, \mu_{h_qk_q}$, calculated at equilibrium is

\[
0 = \sum_s^\infty \frac{\mu^s}{s!} \left[ 2Q\delta^{(k_1h_2k_3\cdots h_qkq)} \partial_{0,Q,0,s} + 2\lambda \delta^{(k_1h_2k_3\cdots h_qkq)} \partial_{0,Q+1,0,s} + \delta^{ki} \delta^{(h_1\cdots h_qkq)} \partial_{0,Q,0,s} \right] + \frac{\partial^Q}{\partial \mu_{h_1k_1} \cdots \partial \mu_{h_qk_q}} \left[ 2\mu_{ji} \frac{\partial H^0}{\partial \mu_{kj}} + 2\lambda \frac{\partial H^0}{\partial \mu_{ki}} + H^0 \delta^{ki} \right]_{\lambda_j=0, \mu_{ia}=0}. \tag{68}
\]

where overlined indexes denote symmetrization over those indexes, after that the other one (round brackets around indexes) has been taken. Now, the first and third term can be written as $\sum_s^\infty \frac{\mu^s}{s!} (2Q + 1) \partial_{0,Q,0,s} \delta^{(k_1h_2k_3\cdots h_qkq)}$ so that the above expression becomes

\[
0 = \sum_s^\infty \frac{\mu^s}{s!} \left[ (2Q + 1) \partial_{0,Q,0,s} + 2\lambda \partial_{0,Q+1,0,s} \right] \delta^{(k_1h_2k_3\cdots h_qkq)} + \frac{\partial^Q}{\partial \mu_{h_1k_1} \cdots \partial \mu_{h_qk_q}} \left[ 2\mu_{ji} \frac{\partial H^0}{\partial \mu_{kj}} + 2\lambda \frac{\partial H^0}{\partial \mu_{ki}} + H^0 \delta^{ki} \right]_{\lambda_j=0, \mu_{ia}=0}. \tag{68}
\]

But the terms with $s \geq 1$ are zero for eq. (64) with $R=0$; also the terms with $s=0$ are zero for eq. (56), so that from (68) there remains

\[
0 = \frac{\partial^Q}{\partial \mu_{h_1k_1} \cdots \partial \mu_{h_qk_q}} \left[ 2\mu_{ji} \frac{\partial H^0}{\partial \mu_{kj}} + 2\lambda \frac{\partial H^0}{\partial \mu_{ki}} + H^0 \delta^{ki} \right]_{\lambda_j=0, \mu_{ia}=0}. \tag{69}
\]

But we have still to impose the derivative of (67) with respect to $\mu_{i_1}, \cdots, \mu_{i_p}, \mu_{h_1k_1}, \cdots, \mu_{h_qk_q}$ for $P \geq 1$ and calculated at equilibrium. We obtain

\[
0 = P \sum_{s=0}^\infty \frac{\mu^s}{s!} \delta^{(i_1\cdots i_p h_1k_1\cdots h_qkq)} \partial_{P,Q,0,s} + 2Q \sum_{s=0}^\infty \frac{\mu^s}{s!} \delta^{(i_1\cdots i_p h_1k_1\cdots h_qkq)} \partial_{P,Q+1,0,s} + 2\lambda \sum_{s=0}^\infty \frac{\mu^s}{s!} \delta^{(i_1\cdots i_p h_1k_1\cdots h_qkq)} \partial_{P,Q,0,s} +
+2\lambda \sum_{s=0}^\infty \frac{\mu^s}{s!} \delta^{(i_1\cdots i_p h_1k_1\cdots h_qkq)} \partial_{P,Q+1,0,s} + \delta^{ki} \sum_{s=0}^\infty \frac{\mu^s}{s!} \delta^{(i_1\cdots i_p h_1k_1\cdots h_qkq)} \partial_{P,Q,0,s},
\]

that is,

\[
0 = (P + 2Q + 1) \partial_{P,Q,0,s} + 2\lambda \partial_{P,Q+1,0,s}, \forall P \geq 1, s \geq 0. \tag{70}
\]
Now $P$ must be even because $R = 0$, so that this equation must hold $\forall P \geq 2, s \geq 0$; by using (61) it becomes

$$0 = (P + 2Q + 1)\theta_{0,Q+\frac{s}{2},0,s+\frac{P}{2}} + 2\lambda\theta_{0,Q+1+\frac{s}{2},0,s+\frac{P}{2}}$$

which is already satisfied because it is nothing more than (61) with $R = 0$ and with $(Q + \frac{P}{2}, s + \frac{P}{2})$ instead of $(Q, s)$.

- Let us impose now eq. (59). By using eq. (55) it becomes

\[
0 = \mu_i \sum_{p,q,s} \sum_{r \in I_p} \frac{1}{p!} \frac{1}{q!} \frac{1}{s!} \theta_{p+2,q,r,s}\mu^s \delta(ak_1...ip_{h_1}k_1...hq_{q,j_1}...j_r) \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} + (71)
\]

\[
+2 \sum_{p,q,s} \sum_{r \in I_{p+1}} \frac{1}{p!} \frac{1}{q!} \frac{1}{r!} \frac{1}{s!} \theta_{p+1,q,r,s}\mu^s \delta(i(\delta(k)i_1...ip_{h_1}k_1...hq_{q,j_1}...j_r)) \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}
\]

\[
+2 \sum_{p,q,s} \sum_{r \in I_{p+1}} \frac{1}{p!} \frac{1}{q!} \frac{1}{r!} \frac{1}{s!} \theta_{p+1,q,r,s}\mu^s \delta(ak_1...ip_{h_1}k_1...hq_{q,j_1}...j_r) \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}
\]

\[
+2 \sum_{p,q,s} \sum_{r \in I_{p+1}} \frac{1}{p!} \frac{1}{q!} \frac{1}{r!} \frac{1}{s!} \theta_{p+1,q,r,s+1}\mu^s \delta(akh_1...ip_{h_1}k_1...hq_{q,j_1}...j_r) \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}
\]

\[
+2 \sum_{p,q,s} \sum_{r \in I_{p+1}} \frac{1}{p!} \frac{1}{q!} \frac{1}{r!} \frac{1}{s!} \theta_{p+1,q,r,s}\mu^s \delta(i(\delta(k)i_1...ip_{h_1}k_1...hq_{q,j_1}...j_r)) \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}
\]

This relation, calculated in $\mu_j = 0$ gives

\[
0 = 2 \sum_{q,s} \sum_{r \in I_1} \frac{1}{q!} \frac{1}{r!} \frac{1}{s!} \theta_{q,r,s}\mu \delta(i(\delta(k)h_1k_1...hq_{q,j_1}...j_r)) \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} + (72)
\]

\[
+2 \mu_{ij} \sum_{q,s} \sum_{r \in I_1} \frac{1}{q!} \frac{1}{r!} \frac{1}{s!} \theta_{q+1,r,s}\mu \delta(ajh_1k_1...hq_{q,j_1}...j_r) \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}
\]

\[
+2 \lambda \sum_{q,s} \sum_{r \in I_1} \frac{1}{q!} \frac{1}{r!} \frac{1}{s!} \theta_{q+1,r,s+1}\mu \delta(akh_1k_1...hq_{q,j_1}...j_r) \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}
\]

\[
+2 \lambda \sum_{q,s} \sum_{r \in I_1} \frac{1}{q!} \frac{1}{r!} \frac{1}{s!} \theta_{q,r+1,s}\mu \delta(bkh_1k_1...hq_{q,j_1}...j_r) \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}
\]

\[
+2 \lambda \sum_{q,s} \sum_{r \in I_1} \frac{1}{q!} \frac{1}{r!} \frac{1}{s!} \theta_{q,r,s+1}\mu \delta(ijah_1k_1...hq_{q,j_1}...j_r) \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r}
\]

\[
+2 \lambda \theta^2 H^0(\mu_{ab}, \lambda, \lambda_c)
\]

\[
\frac{\partial^2 H^0(\mu_{ab}, \lambda, \lambda_c)}{\partial \mu_{ij} \partial \mu_{ka}}.
\]
This relation, calculated in \( \mu = 0 \) becomes

\[
0 = 2 \sum_{q} \sum_{r \in I_1} \frac{1}{q! r!} \theta_{1,q,r,0} \delta^{(a \delta(k)h_1k_1 \cdots h_qk_qj_1 \cdots j_r)} \mu_{h_1k_1} \cdots \mu_{h_qk_q} \lambda_{j_1} \cdots \lambda_{j_r} + \tag{73}
\]

\[
+ 2\mu_{ji} \sum_{q} \sum_{r \in I_1} \frac{1}{q! r!} \theta_{1,q+1,r,0} \delta^{(akj_{i}h_1k_1 \cdots h_qk_qj_1 \cdots j_r)} \mu_{h_1k_1} \cdots \mu_{h_qk_q} \lambda_{j_1} \cdots \lambda_{j_r} +
\]

\[
+ 2\lambda \sum_{q} \sum_{r \in I_0} \frac{1}{q! r!} \theta_{1,q+1,r,0} \delta^{(akah_{1}h_1k_1 \cdots h_qk_qj_1 \cdots j_r)} \mu_{h_1k_1} \cdots \mu_{h_qk_q} \lambda_{j_1} \cdots \lambda_{j_r} +
\]

\[
+ \lambda_i \sum_{q} \sum_{r \in I_0} \frac{1}{q! r!} \theta_{1,q+2,r,0} \delta^{(ijk_{ah_{1}h_1k_1 \cdots h_qk_qj_1 \cdots j_r})} \mu_{h_1k_1} \cdots \mu_{h_qk_q} \lambda_{j_1} \cdots \lambda_{j_r} +
\]

\[
+ 2\lambda_j \sum_{q} \sum_{r \in I_0} \frac{1}{q! r!} \theta_{0,q+2,r,0} \delta^{(ijk_{ah_{1}h_1k_1 \cdots h_qk_qj_1 \cdots j_r})} \mu_{h_1k_1} \cdots \mu_{h_qk_q} \lambda_{j_1} \cdots \lambda_{j_r} +
\]

\[
+ 2\lambda_j \partial^2 H^{s0}(\mu_{ab}, \lambda, \lambda_c) \frac{\partial}{\partial \mu_{ij} \partial \mu_{ka}} .
\]

Now, there remains to consider the terms with \( s \geq 1 \) in (72). Taking of this part the derivative with respect to \( \mu_{h_1k_1}, \cdots, \mu_{h_qk_q}, \lambda_{j_1}, \cdots, \lambda_{j_r} \) and calculating the result at equilibrium, it becomes

\[
0 = 2 \delta^{(a \delta(k)h_1k_1 \cdots h_qk_qj_1 \cdots j_r)} \eta_{1,Q,R,s} + 2Q \delta^{(k_1h_1k_2 \cdots h_qk_qk_j \cdots j_R)} \delta_{1,Q,R,s} +
\]

\[
+ 2\lambda \delta^{(kah_{1}h_1k_1 \cdots h_qk_qj_1 \cdots j_r)} \delta_{1,Q+1,R,s} + R \delta^{(kah_{1}h_1k_1 \cdots h_qk_qk_j \cdots j_R)} \delta_{1,Q,R,s} +
\]

\[
+ 2R \delta^{(kah_{1}h_1k_1 \cdots h_qk_qk_j \cdots j_R)} \delta_{0,Q+2,R-1,s} ,
\]

where overlined indexes denote symmetrization over those indexes, after that the other one (round brackets around indexes) has been taken. (Note that, in the last term the index \( R - 1 \) appears; despite this fact, the equations holds also for \( R = 0 \) but in this case, this last term is not present as it is remembered also by the factor \( R \).)

Now, the first, second, and fourth term can be put together so that the above expression becomes

\[
0 = (2Q + R + 2) \delta_{1,Q,R,s} \delta^{(k_1h_1k_2 \cdots h_qk_qk_j \cdots j_R)} +
\]

\[
+ (2\lambda \delta_{1,Q+1,R,s} + 2R \delta_{0,Q+2,R-1,s}) \delta^{(kah_{1}h_1k_1 \cdots h_qk_qj_1 \cdots j_R)} .
\]

Obviously, also for the terms with \( s = 0 \) the corresponding elements can be put together, so that the expression (73) can be written also as

\[
0 = \sum_{q} \sum_{r \in I_1} \frac{1}{q! r!} [(2q + r + 2) \theta_{1,q,r,0} + 2\lambda \theta_{1,q+1,r,0}] \delta^{(kah_{1}h_1k_1 \cdots h_qk_qj_1 \cdots j_r)} \mu_{h_1k_1} \cdots \mu_{h_qk_q} \lambda_{j_1} \cdots \lambda_{j_r} + \tag{74}
\]

\[
+ 2\lambda_j \partial^2 H^{s0}(\mu_{ab}, \lambda, \lambda_c) \frac{\partial}{\partial \mu_{ij} \partial \mu_{ka}} ,
\]

where (56) has been used. After that, (74) becomes

\[
0 = (2Q + R + 2) \delta_{1,Q,R,s} + 2\lambda \delta_{1,Q+1,R,s} + 2R \delta_{0,Q+2,R-1,s} \quad \text{for} \quad s \geq 1 ,
\]
which is nothing more than \((64)_2\).

There remains now to take the derivatives of \((71)\) with respect to \(\mu_1, \cdots, \mu_{i_P}, \mu_{h_1k_1}, \cdots, \mu_{h_Qk_Q}, \lambda_{j_1}, \cdots, \lambda_{j_R}\) and calculate the result at equilibrium, but only for \(P \geq 1; \) we obtain

\[
0 = P\delta_{i_1} \delta_{i_2 \cdots i_P h_1 k_1 \cdots h_Q k_Q j_1 \cdots j_R} \partial_{P+1, Q, R, s} + 2\delta_{i_1} \delta_{i_2 \cdots i_P h_1 k_1 \cdots h_Q k_Q j_1 \cdots j_R} \partial_{P+1, Q, R, s} + 2Q\delta_{i_1} \delta_{i_2 \cdots i_P h_1 k_1 \cdots h_Q k_Q j_1 \cdots j_R} \partial_{P+1, Q, R, s} + 2\delta_{i_1} \delta_{i_2 \cdots i_P h_1 k_1 \cdots h_Q k_Q j_1 \cdots j_R} \partial_{P+1, Q, R, s} +
\]

or,

\[
0 = (P + 2Q + R + 2)\partial_{P+1, Q, R, s} + 2\lambda \partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} +
\]

that is

\[
0 = (P + 2Q + R + 2)\partial_{P+1, Q, R, s} + 2\lambda \partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} +
\]

- Now, in the case with \(P\) even and \(P \geq 2,\) this relation by using \((61)\) becomes
  \[
  0 = (P + 2Q + R + 2)\partial_{P+1, Q, R, s} + 2\lambda \partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} +
  \]
  which is satisfied as a consequence of \((64)_2\) with \((Q + \frac{P}{2}, s + \frac{P-2}{2})\) instead of \((Q, s).\)
- Instead of this, in the case with \(P\) odd and \(P \geq 1,\) eq. \((75)\) by using \((61)\) becomes
  \[
  0 = (P + 2Q + R + 2)\partial_{P+1, Q, R, s} + 2\lambda \partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} + 2R\partial_{P+1, Q, R, s} +
  \]
  which is satisfied as a consequence of \((64)_1\) with \((Q + \frac{P+1}{2}, s + \frac{P-1}{2})\) instead of \((Q, s).\)

We may conclude that the only consequence of eq. \((59)_3\) is given by \((74).\)

- Let us conclude by imposing eq. \((59)_4.\) By using eq. \((55)\) it becomes
\[ 0 = \mu_i \sum_{p,q,r} \sum_{r \in I_{p+1}} \frac{1}{p! q! r! s!} \frac{\partial}{\partial \lambda} \delta_{p+1,q,r,s} \partial^s (k_{i1} \cdots i_p h_{1k1} \cdots h_{qk} j_1 \cdots j_r) \]

\[ + 2 \mu_j \left\{ \sum_{p,q,r} \sum_{r \in I_p} \frac{1}{p! q! r! s!} \frac{\partial}{\partial \lambda} \delta_{p+1,q,r,s} \partial^s (j_{k1} \cdots i_p h_{1k1} \cdots h_{qk} j_1 \cdots j_r) \right\} \]

\[ + \lambda_i \left\{ \sum_{p,q,r} \sum_{r \in I_{p+1}} \frac{1}{p! q! r! s!} \frac{\partial}{\partial \lambda} \delta_{p+1,q,r,s} \partial^s (i_{k1} \cdots i_p h_{1k1} \cdots h_{qk} j_1 \cdots j_r) \right\} \]

\[ + \delta_k \left\{ \sum_{p,q,r} \sum_{r \in I_p} \frac{1}{p! q! r! s!} \frac{\partial}{\partial \lambda} \delta_{p+1,q,r,s} \partial^s (j_{k1} \cdots i_p h_{1k1} \cdots h_{qk} j_1 \cdots j_r) \right\} \]

\[ + 2 \lambda_j \left\{ \sum_{p,q,r} \sum_{r \in I_{p+1}} \frac{1}{p! q! r! s!} \frac{\partial}{\partial \lambda} \delta_{p+1,q,r,s} \partial^s (j_{k1} \cdots i_p h_{1k1} \cdots h_{qk} j_1 \cdots j_r) \right\} \]

Let us consider the part of this expression which doesn't involve \( H^{s0} \) and let us take its derivatives with respect to \( \mu_i \), \( \delta_k \), \( \lambda_j \), \( \lambda_{Rj} \) calculated at equilibrium; it is equal to

\[ 0 = \sum_{s=0}^{\infty} \frac{\mu^s}{s!} \left[ P \delta_{k1} \delta_{k2} \cdots \delta_{kP} h_{1k1} \cdots h_{Qk} j_1 \cdots j_{R} \right] \]

\[ + 2Q \delta_{k1} \delta_{k2} \cdots \delta_{kQ} h_{1k1} \cdots h_{Qk} j_1 \cdots j_{R} \]

\[ + \left( 2\lambda \frac{\partial}{\partial \lambda} \delta_{P,Q+1,R,s} \right) \delta_{k1} \delta_{k2} \cdots \delta_{kQ} h_{1k1} \cdots h_{Qk} j_1 \cdots j_{R} \]

\[ + R \delta_{k1} \delta_{k2} \cdots \delta_{kR} h_{1k1} \cdots h_{Qk} j_1 \cdots j_{R} \]
Moreover, if \( p \) is even and \( s \geq 1 \) are zero, thanks to derivative of (63) with respect to \( \lambda \) with an aid from eq. (60). Consequently, there remains

\[
\begin{align*}
0 = & \sum_{s=0}^{\infty} \frac{\mu^s}{s!} \left[ (P + 2Q + R + 1) \delta^{(i_1 \cdots i_p h_1 \cdots h_Q k_1 \cdots j_R)} \frac{\partial}{\partial \lambda} \partial_{P,Q,R,s} + \\
+ & \left( 2 \lambda \frac{\partial}{\partial \lambda} \partial_{P,Q+1,R,s} + 2 \partial_{P,Q+1,R,s} + 2 R \partial_{P,Q+1,R,s} \right) \delta^{(k_{i_1} \cdots i_p h_1 \cdots h_Q k_Q j_1 \cdots j_R)} \right] = \\
= & \sum_{s=0}^{\infty} \frac{\mu^s}{s!} \left[ (P + 2Q + R + 1) \frac{\partial}{\partial \lambda} \partial_{P,Q,R,s} + 2 \lambda \frac{\partial}{\partial \lambda} \partial_{P,Q+1,R,s} + 2 (R + 1) \partial_{P,Q+1,R,s} \right] \delta^{(k_{i_1} \cdots i_p h_1 \cdots h_Q k_Q j_1 \cdots j_R)}.
\end{align*}
\]

Now, in this expression, the terms with \( s \geq 1 \) are zero, thanks to derivative of (63) with respect to \( \lambda \) with an aid from eq. (60). Consequently, there remains

\[
\left[ (P + 2Q + R + 1) \frac{\partial}{\partial \lambda} \partial_{P,Q+1,R,s} + 2 \lambda \frac{\partial}{\partial \lambda} \partial_{P,Q+1,R,s} + 2 (R + 1) \partial_{P,Q+1,R,s} \right] \delta^{(k_{i_1} \cdots i_p h_1 \cdots h_Q k_Q j_1 \cdots j_R)}.
\]

This result allows to rewrite eq. (77) as

\[
0 = 2 \mu_{ji} \frac{\partial^2 H^0}{\partial \mu_{kj} \partial \lambda} + 2 \lambda \frac{\partial^2 H^0}{\partial \mu_{kj} \partial \lambda} + 2 \partial^2 H^0 + 2 \lambda \frac{\partial^2 H^0}{\partial \lambda \partial \lambda} + \lambda_j \frac{\partial^2 H^0}{\partial \lambda \partial \lambda} + 2 \lambda_j \frac{\partial^2 H^0}{\partial \mu_{kj} \partial \lambda} + (77)
\]

\[
+ \sum_{p,q} \sum_{r \in I_p} \frac{1}{p! q! r!} \left[ (p + 2q + r + 1) \frac{\partial}{\partial \lambda} \partial_{p,q+1,r+1} + 2 \lambda \frac{\partial}{\partial \lambda} \partial_{p,q+1,r+1} + 2 (r + 1) \partial_{p,q+1,r+1} \right] \mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_Q k_Q} \delta^{k_{i_1} \cdots i_p h_1 \cdots h_Q k_Q j_1 \cdots j_R}.
\]

Now we note that, if \( p = 0 \), thanks to (50), the term of (77) inside the square brackets is zero. Moreover, if \( p \) is even and \( p \geq 2 \), the term of (77) inside the square brackets can be written with use of eq. (61) as

\[
(p + 2q + r + 1) \frac{\partial}{\partial \lambda} \partial_{0,q+1,r+1} + 2 \lambda \frac{\partial}{\partial \lambda} \partial_{0,q+1,r+1} + 2 (r + 1) \partial_{0,q+1,r+1}
\]

which is zero, thanks to derivative of (63) with respect to \( \lambda \) with an aid from eq. (60). Similarly, if \( p \) is odd and \( p \geq 3 \), the term of (77) inside the square brackets can be written with use of eq. (61) as

\[
(p + 2q + r + 1) \frac{\partial}{\partial \lambda} \partial_{1,q+1,r+1} + 2 \lambda \frac{\partial}{\partial \lambda} \partial_{1,q+1,r+1} + 2 (r + 1) \partial_{1,q+1,r+1}
\]

which is zero, thanks to derivative of (63) with respect to \( \lambda \) with an aid from eq. (60).

Finally, if \( p = 1 \) the term of (77) inside the square brackets is

\[
(2q + r + 2) \frac{\partial}{\partial \lambda} \partial_{1,q+1,r+1} + 2 \lambda \frac{\partial}{\partial \lambda} \partial_{1,q+1,r+1} + 2 (r + 1) \partial_{1,q+1,r+1} =
\]

\[
= (2q + r + 2) \partial_{0,q+1,r+1} + 2 \lambda \partial_{0,q+1,r+1} + 2 (r + 1) \partial_{0,q+1,r+1},
\]

where, in the second passage (62) has been used. The result is zero, thanks to (64) with \( Q = q, R = r + 1, s = 0 \). Thanks to these results, eq. (77) becomes

\[
0 = 2 \mu_{ji} \frac{\partial^2 H^0}{\partial \mu_{kj} \partial \lambda} + 2 \lambda \frac{\partial^2 H^0}{\partial \mu_{kj} \partial \lambda} + 2 \partial^2 H^0 + 2 \lambda \frac{\partial^2 H^0}{\partial \lambda \partial \lambda} + \lambda_j \frac{\partial^2 H^0}{\partial \lambda \partial \lambda} + 2 \lambda_j \frac{\partial^2 H^0}{\partial \mu_{kj} \partial \lambda}.
\]

Summarizing the results of this subsection, we have found the restrictions (60), (62), (74) and (78) on \( H^0 \).
3 The expression for $\Delta H$.

By integrating (55) with respect to $\mu$, we obtain

$$\Delta H = \sum_{p,q,s \in I_p} \sum_{r \in I_p} \frac{1}{p! q! r!} \frac{1}{(s + 1)!} \frac{1}{\delta_{p,q,r,s}(\lambda)} \mu^{s+1} \delta^{(i_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r)}$$

(79)

$$\mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} + \mu H^* \left( \mu_{ab}, \lambda, \lambda \right) + \tilde{H} \left( \mu, \mu_{bc}, \lambda, \lambda \right).$$

By substituting $\Delta H$ from here into (41), we find

$$\sum_{p,q,s \in I_p} \sum_{r \in I_p} \frac{1}{p! q! r!} \frac{1}{s!} \frac{1}{\delta_{p,q+1,r,s}(\lambda)} \mu^{s+1} \delta^{(i_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r)}$$

$$\mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} + \frac{\partial H^*}{\partial \mu_{ij}} =$$

$$= \sum_{p,q,s \in I_p} \sum_{r \in I_p} \frac{1}{p! q! r!} \frac{1}{s!} \frac{1}{\delta_{p+2,q,r,s-1}(\lambda)} \mu^{s+1} \delta^{(i_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r)}$$

$$\mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} + \frac{\partial^2 \tilde{H}}{\partial \mu_{ij} \partial \mu_{ij}},$$

which can be rewritten by taking into account that, from (61) it follows $\partial_{p+2,q,r,s-1} = \partial_{p,q+1,r,s}$ for $s \geq 1$; so it becomes

$$\frac{\partial^2 \tilde{H}}{\partial \mu_{ij} \partial \mu_{ij}} = \sum_{p,q} \sum_{r \in I_p} \frac{1}{p! q! r!} \frac{1}{(s + 1)!} \frac{1}{\delta_{p,q+1,r,s}} \mu^{s+1} \delta^{(i_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r)}$$

$$\mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} + \frac{\partial H^*}{\partial \mu_{ij}},$$

which can be integrated and gives

$$\tilde{H} = \sum_{p,q} \sum_{r \in I_p} \frac{1}{p! q! r!} \frac{1}{(s + 1)!} \frac{1}{\delta_{p,q+1,r,s}} \delta^{(i_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r)}$$

(80)

$$\mu_{i_1} \cdots \mu_{i_p+2} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} + \frac{1}{2} \mu_{i} \mu_{j} \frac{\partial H^*}{\partial \mu_{ij}} + \tilde{H} \left( \mu_{ab}, \lambda, \lambda \right) \mu_{i} + \tilde{H} \left( \mu_{ab}, \lambda, \lambda \right),$$

By substituting $\Delta H$ from here into (41) we find

$$\sum_{p,q,s \in I_p} \sum_{r \in I_p} \frac{1}{p! q! r!} \frac{1}{s!} \frac{1}{\delta_{p,q+1,r,s}} \mu^{s+1} \delta^{(i_1 \cdots i_p h_1 k_1 \cdots h_q k_q j_1 \cdots j_r)}$$

$$\mu_{i_1} \cdots \mu_{i_p} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} + \mu H^* \left( \mu_{ab}, \lambda, \lambda \right) + \tilde{H} \left( \mu, \mu_{bc}, \lambda, \lambda \right).$$
where $\tilde{H}^i$ and $\tilde{H}^0$ arise from the integration.

By substituting $\Delta H$ from (79) into (11)_2, and by taking into account (80), we find

\[
0 \cdot \infty \sum_{p,q,s \in I_{p+1}} \frac{1}{p!} \frac{1}{q!} \frac{1}{s!} \partial_{p,q,r} \mu^{s} \delta^{(ii \ldots iph_{1}k_{1} \ldots h_{q}k_{j}1 \ldots j_{r})} 
\]

\[
\mu_{i_{1}} \cdots \mu_{ip} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} + \frac{\partial H^{*0}}{\partial \lambda_{i}} = \]

\[
0 \cdot \infty \sum_{p,q,s \in I_{p+1}} \frac{1}{p!} \frac{1}{q!} \frac{1}{r!} \left( s + 1 \right) ! \frac{\partial}{\partial \lambda} \partial_{p,q,r} \mu^{s+1} \delta^{(ii \ldots iph_{1}k_{1} \ldots h_{q}k_{j}1 \ldots j_{r})} 
\]

\[
\mu_{i_{1}} \cdots \mu_{ip} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} + \frac{\partial^2 H^{*0}}{\partial \mu_{ij} \partial \lambda} + \frac{\partial \tilde{H}_{i}}{\partial \lambda}.
\]

Now, thanks to (60)_2, the terms with degree greater than zero in $\mu$ cancel each other so that (81) becomes

\[
0 \cdot \infty \sum_{p,q,s \in I_{p+1}} \frac{1}{p!} \frac{1}{q!} \frac{1}{r!} \left( s + 1 \right) ! \frac{\partial}{\partial \lambda} \partial_{p,q,r+1,0} \delta^{(ii \ldots iph_{1}k_{1} \ldots h_{q}k_{j}1 \ldots j_{r})} 
\]

\[
\mu_{i_{1}} \cdots \mu_{ip} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} + \frac{\partial H^{*0}}{\partial \lambda_{i}} = \]

\[
0 \cdot \infty \sum_{p,q,s \in I_{p}} \frac{1}{(p + 1)!} \left( q + 1 \right)! \frac{\partial}{\partial \lambda} \partial_{p,q+1,r,0} \delta^{(ii \ldots iph_{1}k_{1} \ldots h_{q}k_{j}1 \ldots j_{r})} 
\]

\[
\mu_{i_{1}} \cdots \mu_{ip+1} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} + \mu_{j} \frac{\partial^2 H^{*0}}{\partial \mu_{ij} \partial \lambda} + \frac{\partial \tilde{H}_{i}}{\partial \lambda}.
\]

This relation, calculated in $\mu_{j} = 0$, thanks to (56), becomes

\[
\frac{\partial H^{*0}}{\partial \lambda_{i}} = \frac{\partial \tilde{H}_{i}}{\partial \lambda}.
\]

The terms in (82) which are linear in $\mu_{j}$ cancel each other thanks to (60) and (56). Moreover, from eq. (61) and (62) it follows that $\partial_{p+2,q,r+1,0} = \frac{\partial}{\partial \lambda} \partial_{p+1,q+1,r,0} = 0$ for $p \geq 0$, so that in (82) all the term of degree greater than 1 in $\mu_{j}$ cancel each other. We may conclude that (11)_2 has only (83) as a consequence.

Let us substitute now $\Delta H$ from (79) into (11)_3 and take into account (80); we obtain a relation whose derivative with respect to $\mu_{0}$ has already been imposed because this was the way in which we obtained (59)_3. Consequently, there remains now to impose only its value in $\mu_{0} = 0$, which is

\[
2\mu_{ji} \left\{ 0 \cdot \infty \sum_{q,s \in I_{0}} \frac{1}{q!} \frac{1}{r!} \left( s + 1 \right) ! \frac{\partial_{q,s}}{\partial_{0,q+r+1,s}} \delta^{(kjh_{1}k_{1} \ldots h_{q}k_{j}1 \ldots j_{r})} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} + \frac{\partial H^{*0}}{\partial \mu_{kj}} \right\} + \frac{\partial \tilde{H}_{j}}{\partial \mu_{ij}}
\]

\[
2\mu_{ji} \left\{ 0 \cdot \infty \sum_{q,s \in I_{0}} \frac{1}{q!} \frac{1}{r!} \left( s + 1 \right) ! \frac{\partial_{q,s}}{\partial_{0,q+r+1,s}} \delta^{(kjh_{1}k_{1} \ldots h_{q}k_{j}1 \ldots j_{r})} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} + \frac{\partial H^{*0}}{\partial \mu_{kj}} \right\} + \frac{\partial \tilde{H}_{j}}{\partial \mu_{ij}}
\]
Is there some integrability conditions on (83) and (85) to determine \( \tilde{\lambda} \) are zero. We note also that eq. (69) is a particular case of (85), when this last one is calculated in \( \lambda \) (85) with respect to equilibrium; it is equal to

\[
H_{\mu} \mu \]  

But the polynomial terms in \( \mu \) which have been already taken into account and that is (78).

Now the first, third and fourth term of this expression can be put together so that it becomes

\[
\sum_{s=0}^{\infty} \frac{\mu}{s!} \left[ (2Q + R + 1) \delta_{ij} \delta_{k_1 k_2 \cdots k_{Q+1} R} \partial_{0, Q, R, s} + 2\lambda \partial_{0, Q+1, R, s} \delta_{(k_1 k_2 \cdots k_{Q+1} R)} + \delta^{k_1 k_2 \cdots k_{Q+1} R} \partial_{0, Q, R, s} \right] + \sum_{s=0}^{\infty} \frac{\mu}{(s+1)!} 2R \partial_{0, Q+1, R-1, s} \delta_{(k_1 k_2 \cdots k_{Q+1} R)} \partial_{1, Q+1, R, s} = 0.
\]

But the polynomial terms in \( \mu \) with degree greater than zero elide one another, thanks to eq. (64); the remaining part is zero, thanks to eq. (56). Consequently, from (54) there remains

\[
2\mu_{ji} \frac{\partial H^0}{\partial \mu_{k_j}} + 2\lambda \frac{\partial H^0}{\partial \mu_{ki}} + \lambda_i \frac{\partial H^0}{\partial \lambda_k} + \delta^{k_1 k_2 \cdots k_{Q+1} R} H^0 + 2\lambda_j \frac{\partial \tilde{H}^k}{\partial \mu_{ij}} = 0.
\]  

So the situation is now that eqs. (41) are equivalent to (80) (which gives \( \tilde{H} \) in terms of \( \tilde{H}^0(\mu_{ab}, \lambda, \lambda_c) \) and of \( \tilde{H}^0(\mu_{ab}, \lambda, \lambda_c) \)) and to the conditions (83) and (85) on \( \tilde{H}^k \), while \( \tilde{H}^0 \) remains arbitrary, as it was obvious because in (41) it appears only through its derivatives with respect to \( \mu \) and \( \mu_k \) which are zero. We note also that eq. (60) is a particular case of (85), when this last one is calculated in \( \lambda_j = 0 \).

Is there some integrability conditions on (83) and (85) to determine \( \tilde{H}^k \)? Well, in the derivative of (85) with respect to \( \lambda \) we can substitute \( \frac{\partial \tilde{H}^k}{\partial \lambda} \) from (83); but we obtain an integrability condition which have been already taken into account and that is (78).

Another type of integrability condition can be obtained in the following way: Let us take the
derivative of (85) with respect to \( \mu_{ab} \), let us contract the result with \( \lambda_b \) and let us take from the resulting equation the skew-symmetric part with respect to \( i \) and \( a \). In this way we obtain

\[
0 = 2\mu_{ij} \frac{\partial^2 H^{*0}}{\partial \mu_{ab}[\partial \mu_{ij}]} \lambda_b + 2\lambda \frac{\partial^2 H^{*0}}{\partial \mu_{k[i} [\partial \mu_{a]b]} \lambda_b + \lambda_i \frac{\partial^2 H^{*0}}{\partial \mu_{ab}[\partial \lambda_k]} \lambda_b + \delta^k[i] \frac{\partial H^{*0}}{\partial \mu_{ij}} \lambda_b .
\] (86)

To conclude this section, we can say that we have to impose the conditions (66), (69), (74), (78) and (86) on \( H^{*0}(\mu_{ab}, \lambda, \lambda_c) \).

After that, (83) and (85) will give \( \tilde{H}^i \); we will see that a further little integrability condition will be necessary to this end.

## 4 Solution of the conditions on \( H^{*0} \).

To solve (66), it is useful to define \( \psi_{0,q+1,r,0} \) and \( \tilde{H}^{*0} \) from

\[
\frac{\partial}{\partial \lambda} \psi_{0,q+1,r,0} = \vartheta_{1,q,0} = \frac{\partial}{\partial \lambda} \psi_{0,q+1,r,0} ,
\] (87)

\[
H^{*0} = \tilde{H}^{*0} + \sum_{q=0}^{\infty} \sum_{r=1}^\infty \frac{1}{(q + 1)!} r! \psi_{0,q+1,r,0}(\lambda) \delta(\lambda_1 \cdots \lambda_q \lambda_{q+1} \cdots \lambda_r) \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} .
\] (88)

By using them, we see that (66) becomes

\[
\frac{\partial^2 \tilde{H}^{*0}}{\partial \mu_{ij} \partial \lambda_k} = 0 ,
\]

that is, \( \tilde{H}^{*0} \) is sum of a function not depending on \( \lambda \) and of a function not depending on \( \mu_{ij} \), that is

\[
\tilde{H}^{*0} = \tilde{H}^{*01} (\lambda, \lambda_j) + \tilde{H}^{*02} (\mu_{ij}, \lambda_k) = \sum_{r=1}^\infty \frac{1}{r!} \psi_{0,0,r,0}\lambda(\lambda) \delta(\lambda_1 \cdots \lambda_r) \lambda_{j_1} \cdots \lambda_{j_r} + \tilde{H}^{*02} (\mu_{ij}, \lambda_k) ,
\]

where we have introduced the expansion of \( \tilde{H}^{*01} \) around equilibrium and called \( \psi_{0,0,r,0} \) the coefficients. We may also assume, without loss of generality, that

\[
\tilde{H}^{*02}(0, \lambda_k) = 0 ,
\] (89)

because its eventual non zero value can be enclosed in \( \tilde{H}^{*01} \). By substituting the result in (88) this is transformed in

\[
H^{*0} = \tilde{H}^{*02} (\mu_{ij}, \lambda_k) + \sum_{q=0}^{\infty} \sum_{r=1}^\infty \frac{1}{q! r!} \psi_{0,q,r,0}(\lambda) \delta(\lambda_1 \cdots \lambda_q \lambda_{q+1} \cdots \lambda_r) \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} .
\] (90)

This is the solution of (66). By substituting it in (69), we obtain

\[
0 = [(2Q + 1) \psi_{0,0,0,0}(\lambda) + 2\lambda \psi_{0,0,1,0,0}(\lambda)] \delta(\lambda_1 \cdots \lambda_k) + \left\{ \frac{\partial \mu_{ab}}{\partial \mu_{ab}} \right\}_{\lambda_j = 0, \mu_{ij} = 0} .
\] (91)

The derivative of this relation with respect to \( \lambda \) gives the information

\[
(2Q + 1) \psi_{0,0,0,0}(\lambda) + 2\lambda \psi_{0,0,1,0,0}(\lambda) = \phi_{0,0,1,0,0} ,
\] (92)
with $\phi_{0,Q+1,0,0}$ constant. After that, (91) becomes

$$0 = \left\{ \frac{\partial^Q}{\partial \mu_{h_1 k_1} \cdots \partial \mu_{h_Q k_Q}} \left[ 2 \mu_{i_j} \frac{\partial \tilde{H}^{s02}}{\partial \mu_{j_k}} + 2 \lambda \frac{\partial \tilde{H}^{s02}}{\partial \mu_{k_i}} + \tilde{H}^{s02} \delta^{k_i} \right] \right\} \Big|_{\lambda_j = 0, \mu_{ia} = 0} + \phi_{0,Q+1,0,0} \delta^{(k_1 h_1 \cdots h_Q k_Q)}.$$  

Let us consider now (74); thanks to (90) it is transformed in

$$0 = 2 \lambda_j \frac{\partial^2 \tilde{H}^{s02}}{\partial \mu_{j_i} \partial \mu_{k_a}} + \sum_{Q=0}^{\infty} \sum_{R \in I_1} \frac{1}{Q!} R! \left[ 2 R \psi_{0,Q+2,R-1,0} + (2Q + R + 2) \psi_{1,Q,R,0} + 2 \lambda \psi_{1,Q+1,R,0} \right] \delta^{(k_1 h_1 \cdots h_Q k_Q j_1 \cdots j_R)} m_{h_1 k_1} \cdots m_{h_Q k_Q} \lambda_{j_1} \cdots \lambda_{j_R},$$

which is equivalent to

$$2 R \psi_{0,Q+2,R-1,0} + (2Q + R + 2) \psi_{1,Q,R,0} + 2 \lambda \psi_{1,Q+1,R,0} = \phi_{1,Q,R,0} \quad \text{for } R \geq 1$$  

and to

$$0 = 2 \lambda_j \frac{\partial^2 \tilde{H}^{s02}}{\partial \mu_{j_i} \partial \mu_{k_a}} + \sum_{Q=0}^{\infty} \sum_{R \in I_1} \frac{1}{Q!} R! \phi_{1,Q,R,0} \delta^{(k_1 h_1 \cdots h_Q k_Q j_1 \cdots j_R)} m_{h_1 k_1} \cdots m_{h_Q k_Q} \lambda_{j_1} \cdots \lambda_{j_R},$$

with $\phi_{1,Q,R,0}$ constant.

We want now to see how (78) is transformed by use of (90); to this end, let us firstly substitute (90) except for the term $\tilde{H}^{s02}$. Of the resulting expression, let us take the derivatives with respect to $\mu_{h_1 k_1}, \cdots, m_{h_Q k_Q}, \lambda_{j_1}, \cdots, \lambda_{j_R}$ calculated at equilibrium; it is equal to

$$2 Q \delta^{(k_1 h_1 \cdots h_Q k_Q j_1 \cdots j_R)} \frac{\partial}{\partial \lambda} \psi_{0,Q,R,0} +$$

$$+ \left( 2 \lambda \frac{\partial}{\partial \lambda} \psi_{0,Q+1,R,0} + 2 \psi_{0,Q+1,R,0} \right) \delta^{(k_1 h_1 \cdots h_Q k_Q j_1 \cdots j_R)} +$$

$$+ R \delta^{(k_1 h_1 \cdots h_Q k_Q k)} \frac{\partial}{\partial \lambda} \psi_{0,Q,R,0} + \delta^{k_i} \delta^{(k_1 h_1 \cdots h_Q k_Q j_1 \cdots j_R)} \frac{\partial}{\partial \lambda} \psi_{0,Q,R,0} +$$

$$+ 2 R \delta^{(k_1 h_1 \cdots h_Q k_Q j_1 \cdots j_R)} \psi_{0,Q+1,R,0} =$$

$$= (2Q + R + 1) \delta^{(k_1 h_1 \cdots h_Q k_Q j_1 \cdots j_R)} \frac{\partial}{\partial \lambda} \psi_{0,Q,R,0} +$$

$$+ \left( 2 \lambda \frac{\partial}{\partial \lambda} \psi_{0,Q+1,R,0} + 2 \psi_{0,Q+1,R,0} + 2 R \psi_{0,Q+1,R,0} \right) \delta^{(k_1 h_1 \cdots h_Q k_Q j_1 \cdots j_R)}.$$
which is equivalent to
\[
(2q + r + 1) \frac{\partial}{\partial \lambda} \psi_{0,q,r,0} + 2\lambda \frac{\partial}{\partial \lambda} \psi_{0,q+1,r,0} + 2(r + 1)\psi_{0,q+1,r,0} = \phi'_0, \tag{96}
\]
and to
\[
0 = 2 \frac{\partial \tilde{H}^{*02}}{\partial \mu_{ki}} + 2\lambda_j \frac{\partial^2 \tilde{H}^{*02}}{\partial \mu_j \partial \lambda_k} + \sum_{q=0}^{\infty} \sum_{r=0}^{q} \frac{1}{q! r!} \frac{1}{q! r!} \psi_{0,q+1,r,0} \delta^{(\bar{k}h_1 k_1 \cdots h_q k_q \bar{j}_1 \cdots j_r)} \mu_{h_1 k_1} \cdots \mu_{h_q k_q} \lambda_{j_1} \cdots \lambda_{j_r} \cdot \tag{97}
\]
with \( \phi'_0, \ldots, r \) constant.

Finally, let us substitute (94) in (96), so obtaining
\[
0 = 2 \mu_{ji} \frac{\partial^2 \tilde{H}^{*02}}{\partial \mu_{j|i} \partial \mu_{k,j}} \lambda_b + 2\lambda \frac{\partial^2 \tilde{H}^{*02}}{\partial \mu_{k,i} \partial \mu_{a}^{(j)} \partial \lambda_k} \lambda_b + \lambda_j \frac{\partial^2 \tilde{H}^{*02}}{\partial \mu_{a}^{(j)} \partial \lambda_k} \lambda_b + \delta^{ki} \frac{\partial \tilde{H}^{*02}}{\partial \mu_{a}^{(j)} \partial \lambda_k} \lambda_b + \tag{98}
\]
\[
+2\mu_{ji} \sum_{q=0}^{\infty} \sum_{r=0}^{q} \frac{1}{q! r!} \psi_{0,q+1,r,0} \delta^{(a)h_{j}h_{1}k_{1} \cdots h_{q}k_{q} j_{1} \cdots j_{r})} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} \lambda_{b} +
\]
\[
+\lambda_j \sum_{q=0}^{\infty} \sum_{r=1}^{q} \frac{1}{q! r!} \psi_{0,q+1,r,0} \delta^{(a)bh_{1}k_{1} \cdots h_{q}k_{q} j_{1} \cdots j_{r})} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} \lambda_{b} +
\]
\[
+\delta^{ki} \sum_{q=0}^{\infty} \sum_{r=0}^{q} \frac{1}{q! r!} \psi_{0,q+1,r,0} \delta^{(a)h_{1}k_{1} \cdots h_{q}k_{q} j_{1} \cdots j_{r})} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{j_{1}} \cdots \lambda_{j_{r}} \lambda_{b}.
\]

Of the expression in the right hand side, let us consider the part not involving \( \tilde{H}^{*02} \) and without skew-symmetrization; its part linear in \( \lambda_j \) is
\[
2\mu_{ji} \sum_{q=0}^{\infty} \frac{1}{q!} \psi_{0,q+1,0,0} \delta^{(a)h_{j}h_{1}k_{1} \cdots h_{q}k_{q})} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{b} +
\]
\[
+\delta^{ki} \sum_{q=0}^{\infty} \frac{1}{q!} \psi_{0,q+1,0,0} \delta^{(a)h_{1}k_{1} \cdots h_{q}k_{q})} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{b}.
\]
whose derivatives with respect to \( \mu_{h_{1}k_{1}}, \ldots, \mu_{h_{q}k_{q}}, \lambda_{b} \) calculated at equilibrium is equal to
\[
2Q \delta^{h_{1}k_{1}} \delta^{(h_{1}k_{2}k_{2} \cdots h_{q}k_{q})(abk)} \psi_{0,Q+1,0,0} + \delta^{ki} \delta^{(a)h_{1}k_{1} \cdots h_{q}k_{q})} \psi_{0,Q+1,0,0} =
\]
\[
= (2Q + 3) \delta^{h_{1}k_{1}} \delta^{(h_{1}k_{2}k_{2} \cdots h_{q}k_{q})(abk)} \psi_{0,Q+1,0,0} - \psi_{0,Q+1,0,0} \left[ \delta^{ia} \delta^{(h_{1}k_{1} \cdots h_{q}k_{q} b)} + \delta^{ib} \delta^{(h_{1}k_{1} \cdots h_{q}k_{q} a)} \right].
\]
So we can write this linear part in \( \lambda_j \) as
\[
\sum_{q=0}^{\infty} \frac{1}{q!} \psi_{0,q+1,0,0} \left[ (2q + 3) \delta^{(i h_{1}k_{1}k_{2} \cdots h_{q}k_{q} a)} - \delta^{ia} \delta^{(h_{1}k_{1} \cdots h_{q}k_{q} b)} - \delta^{ib} \delta^{(h_{1}k_{1} \cdots h_{q}k_{q} a)} \right] \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \lambda_{b}
\]
whose skew-symmetric part with respect to \( i \) and \( a \) is
\[
- \sum_{q=0}^{\infty} \frac{1}{q!} \psi_{0,q+1,0,0} \lambda^{[i} \delta^{(a)h_{1}k_{1} \cdots h_{q}k_{q})} \mu_{h_{1}k_{1}} \cdots \mu_{h_{q}k_{q}} \cdot \tag{99}
\]
This is the linear part in $\lambda_j$ of the expression (98) without the terms in $\tilde{H}^{02}$. In order to consider the terms with degree in $\lambda_j$ greater than 1, let us change index in the summations of the parts with $\mu_j[\cdots$ and $\delta^{k[i}\cdots$; the change of index is $r = R + 1$. In this way we obtain the skew-symmetric part with respect to $i$ and $a$ of

$$
2 \mu_{ji} \sum_{q=0}^{\infty} \sum_{R \in I_1} \frac{1}{q!} \frac{1}{(R+1)!} \psi_{0,q+2,R+1,0} \delta^{(akj_1k_2\cdots h_1h_2\cdots j_{R+2})} \mu_{h_1k_1} \cdots \mu_{h_qk_q} \lambda_{j_1} \cdots \lambda_{j_{R+2}} + 
$$

\[+ \delta^{j_{R+2}} \sum_{q=0}^{\infty} \sum_{R \in I_1} \frac{1}{q!} \frac{1}{R!} \psi_{0,q+1,R+1,0} \delta^{(akj_1k_2\cdots h_1h_2\cdots j_{R+1})} \mu_{h_1k_1} \cdots \mu_{h_qk_q} \lambda_{j_1} \cdots \lambda_{j_{R+2}} +
$$

\[+ \delta^{k[i} \sum_{q=0}^{\infty} \sum_{R \in I_1} \frac{1}{q!} \frac{1}{(R+1)!} \psi_{0,q+1,R+1,0} \delta^{(akj_1k_2\cdots h_1h_2\cdots j_{R+2})} \mu_{h_1k_1} \cdots \mu_{h_qk_q} \lambda_{j_1} \cdots \lambda_{j_{R+2}},
$$

where we have put also $b = j_{R+2}$. The derivatives of this expression with respect to $\mu_{h_1k_1}, \cdots, \mu_{h_qk_q}$, calculated in $\mu_{ab} = 0$ is equal to

$$
\sum_{R \in I_1} \frac{1}{(R+1)!} \psi_{0,Q+1,R+1,0} \left[ 2Q \delta^{h_1} \delta^{(k_1h_2k_2\cdots h_Qk_Q)akj_1\cdots j_{R+2})} \mu_{0,Q+1,R+1,0} + 
$$

\[+(R+1) \delta^{j_{j_{R+2}} \delta^{j_{j_{R+2}}} \delta^{(akj_1k_2\cdots h_1h_2\cdots j_{R+2})} \mu_{0,Q+1,R+1,0} +
$$

\[+ \delta^{k[i} \delta^{(akj_1k_2\cdots h_1h_2\cdots j_{R+2})} \mu_{0,Q+1,R+1,0} \right] \lambda_{j_1} \cdots \lambda_{j_{R+2}} =
$$

\[= \sum_{R \in I_1} \frac{1}{(R+1)!} \psi_{0,Q+1,R+1,0} \left[ (2Q + R + 4) \delta^{h_1} \delta^{(k_1h_2k_2\cdots h_Qk_Q)akj_1\cdots j_{R+2})} \mu_{0,Q+1,R+1,0} +
$$

\[+ \delta^{j_{j_{R+2}}} \delta^{(akj_1k_2\cdots h_1h_2\cdots j_{R+2})} \mu_{0,Q+1,R+1,0} + 
$$

\[+ \delta^{k[i} \delta^{(akj_1k_2\cdots h_1h_2\cdots j_{R+2})} \mu_{0,Q+1,R+1,0} \right] \lambda_{j_1} \cdots \lambda_{j_{R+2}},
$$

so that (100) is equal to

$$
\sum_{q=0}^{\infty} \sum_{R \in I_1} \frac{1}{q!} \frac{1}{(R+1)!} \psi_{0,q+1,R+1,0} \left[ (2q + R + 4) \delta^{(ih_1j_1k_1k_2\cdots h_qk_qk_1\cdots j_{R+2})} \mu_{0,q+1,R+1,0} +
$$

\[- \delta^{j_{j_{R+2}}} \delta^{(akj_1k_2\cdots h_1h_2\cdots j_{R+2})} \mu_{0,q+1,R+1,0} + 
$$

\[- \delta^{k[i} \delta^{(akj_1k_2\cdots h_1h_2\cdots j_{R+2})} \mu_{0,q+1,R+1,0} \right] \lambda_{j_1} \cdots \lambda_{j_{R+2}+\mu_{h_1k_1} \cdots \mu_{h_qk_q}}
$$

whose skew-symmetric part with respect to $i$ and $a$ is

$$
- \sum_{q=0}^{\infty} \sum_{R \in I_1} \frac{1}{q!} \frac{1}{(R+1)!} \psi_{0,q+1,R+1,0} \lambda^{[i} \delta^{[a[j_1k_1j_1k_1\cdots j_{R+2}]j_1} \cdots \lambda_{j_{R+2}+\mu_{h_1k_1} \cdots \mu_{h_qk_q}}
$$

This result, jointly with (99), allows to rewrite (98) as

$$
0 = 2 \mu_{ji} \cdots \mu_{ab} \partial^2 \tilde{H}^{02} \lambda_b + 2 \lambda_i \cdots \lambda_b \partial^2 \tilde{H}^{02} \lambda_b + \lambda_i \cdots \lambda_b \frac{\partial^2 \tilde{H}^{02}}{\partial \mu_{ab}^2} \lambda_b + 
$$

\[- \sum_{q=0}^{\infty} \frac{1}{q!} \psi_{0,q+1,R+1,0} \lambda^{[i} \delta^{[a[j_1k_1j_1k_1\cdots j_{R+2}]j_1} \cdots \lambda_{j_{R+2}+\mu_{h_1k_1} \cdots \mu_{h_qk_q}} +
$$

\[- \sum_{q=0}^{\infty} \frac{1}{q!} \psi_{0,q+1,R+1,0} \lambda^{[i} \delta^{[a[j_1k_1j_1k_1\cdots j_{R+2}]j_1} \cdots \lambda_{j_{R+2}+\mu_{h_1k_1} \cdots \mu_{h_qk_q}}.
$$
But the last 2 terms of this expression can be written in a more compact form, so that the whole expression becomes

$$0 = 2\mu_j \frac{\partial^2 \tilde{H}^{*02}}{\partial \mu_{a_j} \partial \mu_{a_b}} \lambda_b + 2\lambda_i \frac{\partial^2 \tilde{H}^{*02}}{\partial \mu_{a_i} \partial \mu_{a_b}} \lambda_b + \lambda_i \frac{\partial^2 \tilde{H}^{*02}}{\partial \mu_{a_i} \partial \lambda_k} \lambda_b + \delta^k \frac{\partial \tilde{H}^{*02}}{\partial \mu_{a_j}} \lambda_b +$$

$$- \sum_{q=0}^{\infty} \sum_{R \subseteq I_0} \frac{1}{q! R!} \psi_{0,q+1,R,0} \lambda_i \delta^{[a_i k_1 k_2 \cdots k_q]} \lambda_j_1 \cdots \lambda_j_r \mu \lambda_1 k_1 \cdots \mu \lambda_q k_q,$$

whose derivative with respect to \( \lambda \) gives

$$\frac{\partial}{\partial \lambda} \psi_{0,q+1,R,0} = 0 .$$

(102)

So there remains now to impose the conditions (93), (95), (97) and (101) on \( \tilde{H}^{*02}(\mu_{ab}, \lambda_c) \).

### 4.1 The conditions on \( \tilde{H}^{*02} \)

- Let us begin with (95).

Let us define \( \tilde{H}^{*03} \) from

$$\tilde{H}^{*02} = \tilde{H}^{*03} - \frac{1}{2} \sum_{q=0}^{\infty} \sum_{r \subseteq I_0} \frac{1}{(q+2)! (r+1)!} \phi_{1,q,r+1,0} \delta^{[h_1 k_1 \cdots h_q k_q]} \mu \lambda_1 k_1 \cdots \mu \lambda_q k_q \lambda_{j_1} \cdots \lambda_{j_r} \lambda_{j_r}.$$

(103)

After that, eq. (95) becomes

$$0 = 2\mu_j \frac{\partial^2 \tilde{H}^{*03}}{\partial \mu_{a_j} \partial \mu_{a_k}} .$$

(104)

- Instead of this, (97) (by writing explicitly the term with \( q = 0 \) and changing index in the remaining part according to \( q = Q + 1 \)) becomes

$$0 = 2\frac{\partial \tilde{H}^{*03}}{\partial \mu_{a_k}} + 2\lambda_i \frac{\partial^2 \tilde{H}^{*03}}{\partial \mu_{a_i} \partial \mu_{a_k}} + \sum_{r \subseteq I_0} \frac{1}{r!} \phi_{0,1,r,0} \delta^{[k_{i1} \cdots k_{i_r}]} \lambda_{j_1} \cdots \lambda_{j_r} +$$

$$- \sum_{q=0}^{\infty} \sum_{r \subseteq I_0} \frac{1}{(q+1)! (r+1)!} \left( \phi_{1,q,r+1,0} + r \phi_{1,q,r,0} - (r+1) \phi_{0,q+2,r,0} \right) \delta^{[k_{i1} k_1 \cdots h_q k_q]} \mu \lambda_1 k_1 \cdots \mu \lambda_q k_q \lambda_{j_1} \cdots \lambda_{j_r} .$$

(105)

Now, if we contract this expression with \( \lambda_i \), we will obtain another expression whose first 2 terms don’t depend on \( \mu_{ab} \) for (104), the third term explicitly doesn’t depend on \( \mu_{ab} \), the last one will be at least linear in \( \mu_{ab} \), except if the coefficients are zero. In other words, we have

$$\phi_{1,q,r+1,0} + r \phi_{1,q,r,0} - (r+1) \phi_{0,q+2,r,0} = 0 .$$

(106)

After this result, what remains of (105) shows that \( \frac{\partial \tilde{H}^{*03}}{\partial \mu_{a_k}} \) depends only on \( \lambda_j \); consequently,

$$\frac{\partial \tilde{H}^{*03}}{\partial \mu_{a_k}} + \frac{1}{2} \sum_{r \subseteq I_0} \frac{1}{(r+1)!} \phi_{0,1,r,0} \delta^{[k_{i1} \cdots k_{i_r}]} \lambda_{j_1} \cdots \lambda_{j_r}$$
is a second order symmetric tensor depending only on $\lambda_j$. By applying the Representation Theorems, we can say that it is a linear combination, through scalar coefficients, of $\delta^{ij}$ and of $\lambda^i \lambda^j$; writing the expansion of the coefficients around $\lambda_j = 0$, we obtain that

$$\frac{\partial \tilde{H}^{03}}{\partial \mu_{ki}} = -\frac{1}{2} \sum_{r \in I_0} \frac{1}{(r + 1)!} \phi_0^{r,0,0} \delta^{(kij_1\ldots j_r)} \lambda_{j_1} \cdots \lambda_{j_r} + \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda_a \lambda^a)^r (\alpha_r \delta^{ik} + \beta_r \lambda^i \lambda^k),$$

(107)

with $\alpha_r$ and $\beta_r$ constants. By using this, (105) becomes

$$\alpha_0 = 0 \quad \Rightarrow \quad \alpha_{r+1} = -(r + 1) \beta_r$$

(108)

and (104) is an identity.

Now, from (89) and (103) it follows $\tilde{H}^{03}(0, \lambda_k) = 0$. This result, jointly with (107) allows us to obtain

$$\hat{H}^{03} = \mu_{ki} \left[ -\frac{1}{2} \sum_{r \in I_0} \frac{1}{(r + 1)!} \phi_0^{r,0,0} \delta^{(kij_1\ldots j_r)} \lambda_{j_1} \cdots \lambda_{j_r} + \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda_a \lambda^a)^r \beta_r (\lambda^i \lambda^k - \lambda_a \lambda^a \delta^{ik}) \right],$$

(109)

so that (103) now becomes

$$\hat{H}^{02} = \mu_{ki} \left[ -\frac{1}{2} \sum_{r \in I_0} \frac{1}{(r + 1)!} \phi_0^{r,0,0} \delta^{(kij_1\ldots j_r)} \lambda_{j_1} \cdots \lambda_{j_r} + \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda_a \lambda^a)^r \beta_r (\lambda^i \lambda^k - \lambda_a \lambda^a \delta^{ik}) \right] + (110)$$

$$- \frac{1}{2} \sum_{q=0}^{\infty} \sum_{r \in I_0} \frac{1}{(q + 2)! (r + 1)!} \phi_1^{r+1,0,0} \delta^{(h_{1k_1} h_{2k_2} \cdots j_1 \ldots j_r)} \mu_{h_{1k_1}} \cdots \mu_{h_{2k_2}} \lambda_{j_1} \cdots \lambda_{j_r}.$$
\[ + \lambda_i \left[ - \frac{1}{2} \sum_{r \in I_0} \frac{r}{(r+1)!} \phi'_{0,1,r,0} \delta(a)[k_{1j_r\cdots j_{r-1}}] \lambda_b \lambda_{j_1} \cdots \lambda_{j_{r-1}} + \right. \\
-\frac{1}{2} \sum_{q=0}^{\infty} \sum_{r \in I_0} \frac{1}{(q+1)! (r+1)!} \phi_{1,q,r+1,0} \delta(a)[k_{1j_1\cdots j_qk_{q+1} j_{r+1} \cdots j_{r-1}}] \mu_{h_{1}k_1} \cdots \mu_{h_{q+1}k_{q+1}} \lambda_b \lambda_{j_1} \cdots \lambda_{j_{r-1}} + \\
+ \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda^c \lambda^c)^r \beta_r \left( \delta^a[k] \lambda_b - \lambda^a[\lambda^k] \right) + \\
+ \delta^{ki} \left[ - \frac{1}{2} \sum_{r \in I_0} \frac{1}{(r+1)!} \phi'_{0,1,r,0} \delta(a)[k_{1j_r\cdots j_{r-1}}] \lambda_b \lambda_{j_1} \cdots \lambda_{j_r} + \\
- \sum_{q=0}^{\infty} \sum_{r \in I_0} \frac{1}{q! r!} \psi_{0,q+1,r,0} \lambda^i \delta(a)[k_{1j_1\cdots j_qk_{q+1} j_{r+1} \cdots j_{r-1}}] \lambda_{j_1} \cdots \lambda_{j_r} \mu_{h_{1}k_1} \cdots \mu_{h_{q}k_{q}} \right]. \\
\]

But now we have
\[ r \lambda_i \delta(abk_{j_1\cdots j_{r-1}}) \lambda_b \lambda_{j_1} \cdots \lambda_{j_{r-1}} + \delta^{ki} \delta(abj_{1\cdots j_r}) \lambda_b \lambda_{j_1} \cdots \lambda_{j_r} = \\
= \lambda_b \lambda_{j_1} \cdots \lambda_{j_r} \left[ r \delta^i \delta(abk_{j_1\cdots j_{r-1}}) + \delta^{ki} \delta(abj_{1\cdots j_r}) \right] = \\
= \lambda_b \lambda_{j_1} \cdots \lambda_{j_r} \left[ (r+3) \delta^i \delta(abj_{1\cdots j_{r-1}+1}) - \delta^i \delta(abj_{1\cdots j_{r-1}+1}) \right] = \\
= \lambda_b \lambda_{j_1} \cdots \lambda_{j_r} \left[ (r+3) \delta^i \delta(abj_{1\cdots j_{r-1}+1}) - \delta^i \delta(abj_{1\cdots j_{r-1}+1}) \right]. \\
\]

By using this identity in (13) calculated in \( \mu_{ab} = 0 \), we find
\[ 0 = \frac{1}{2} \sum_{r \in I_0} \frac{1}{(r+1)!} \phi'_{0,1,r,0} \lambda^i \delta(a)[k_{1j_1\cdots j_r}] \lambda_{j_1} \cdots \lambda_{j_r} + \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda^c \lambda^c)^{r+1} \beta_r \lambda^i \delta^a[k] - \psi_{0,1,0,0} \lambda^i \delta^a[k] + \\
- \sum_{R \in I_1} \frac{1}{(R+1)!} \psi_{0,1,R+1,0} \lambda^i \delta(a)[k_{1j_1\cdots j_{R+1}}] \lambda_{j_1} \cdots \lambda_{j_{R+1}}. \\
\]

This relation, by changing index in the last term according to \( R = r-1 \), transforms itself in
\[ 0 = \frac{1}{2} \sum_{r \in I_0} \frac{1}{(r+1)!} \left[ \phi'_{0,1,r,0} - 2(r+1) \psi_{0,1,r,0} \right] \lambda^i \delta(a)[k_{1j_1\cdots j_r}] \lambda_{j_1} \cdots \lambda_{j_r} + \\
+ \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda^c \lambda^c)^{r+1} \beta_r \lambda^i \delta^a[k] - \psi_{0,1,0,0} \lambda^i \delta^a[k]. \\
\]

The linear part in \( \lambda^j \) of this equation gives
\[ \phi'_{0,1,0,0} = 2 \psi_{0,1,0,0}. \]

For the remaining part, we use the identity
\[ \delta(a[k_{1j_1\cdots j_r}] = \frac{1}{r+1} (\delta^a[k_{1j_1\cdots j_r}] + r \delta^a[j_1\cdots j_r]k) \]
that is, \( \phi_{0,1,2} \) says that the skew-symmetric part of this expression, with respect to \( i \) and \( a \), is zero, that is,

\[
0 = \sum_{q=0}^{\infty} \sum_{r \in I_0} \left( \frac{1}{(q+1)! (r+1)!} \phi_{1,q,r+1,0} \right) \lambda_{j_1} \cdots \lambda_{j_r}.
\]

By using this identity, eq. (114) becomes

\[
0 = \frac{1}{2} \sum_{r \in I_0} \frac{1}{(r+1)!} \left[ \phi_{0,1,r,0} - 2(r+1) \lambda_{j_1} \cdots \lambda_{j_r} \right] + \sum_{R=0}^{\infty} \frac{1}{R!} (\lambda^k \lambda_c)^R \beta_R \lambda^{[i] \delta^{[j^k]}}.
\]

or, equivalently,

\[
\phi_{0,1,2R+2} = 2(2R + 3) \psi_{0,1,2R+2} - \frac{(2R + 3)!}{R!} 2(2R + 3) \beta_R.
\]

After this result, what remains in the right hand side of eq. (113), without skew-symmetrization and by putting \( b = j_{r+1} \), becomes

\[
- \sum_{Q=0}^{\infty} \sum_{r \in I_0} \frac{1}{(Q+1)! r!} \phi_{0,Q+2,r,0} \lambda^{[i] \delta^{[j]}} k \lambda_{j_1} \cdots \lambda_{j_r} = \sum_{Q=0}^{\infty} \sum_{r \in I_0} \frac{1}{(Q+1)! r!} \phi_{0,Q+2,r,0} \left( \frac{Q+2}{r} \right) \delta^{[j_1 \cdots j_r] k} \lambda_{j_1} \cdots \lambda_{j_r}.
\]

and

\[
\phi_{1,q,r+1,0} = \frac{1}{2} \left( \phi_{0,q+2,r,0} - \frac{Q+2}{r} \right) \lambda_{j_1} \cdots \lambda_{j_r} \lambda_{j_{r+1}}.
\]

Now (113) says that the skew-symmetric part of this expression, with respect to \( i \) and \( a \), is zero, that is,

\[
0 = \sum_{q=0}^{\infty} \sum_{r \in I_0} \frac{1}{(q+1)! (r+1)!} \phi_{0,q+2,r,0} \lambda^{[i] \delta^{[j]}} k \lambda_{j_1} \cdots \lambda_{j_r}
\]

from which

\[
\phi_{1,q,r+1,0} = \frac{1}{2} \phi_{0,q+2,r,0}.
\]
But \( \delta^{(ij_j \ldots j_{2r+2})} = \delta^{(ij_j \ldots j_{2r+2})} = \frac{1}{2^{r+3}} \left( \delta^{ij} \delta^{(j_1 \ldots j_{2r+2})} + (2r + 2) \delta^{(j_1 \ldots j_{2r+2})} \right) \) so that the last 2 terms in the above expression are equal to \( \sum_{r=0}^{\infty} \frac{2r+3}{r!} (\lambda_a^a)^r \beta_r \mu ik \lambda^i \lambda^k \). This allows to rewrite the total expression as

\[
\bar{H}^{02} = - \sum_{r \in I_0} \frac{1}{r!} \psi_{0,0,0,0} \delta^{(ij_j \ldots j_r)} \mu_{ij} \lambda_j \ldots \lambda_j + 
- \sum_{q=0}^{\infty} \sum_{r \in I_0} \frac{1}{(q+2)!} \psi_{0,q+2,r,0} \delta^{(h_1 k_1 \ldots h_{q+2} k_{q+2})} \mu_{h_1 k_1} \ldots \mu_{h_{q+2} k_{q+2}} \lambda_j \ldots \lambda_j + 
+ \sum_{r=0}^{\infty} \frac{2r+3}{r!} (\lambda_a^a)^r \beta_r \mu ik \lambda^i \lambda^k.
\]  

(118)

Thanks to this result, eq. (80) now becomes

\[
H^{*0} = \sum_{r \in I_0} \frac{1}{r!} \psi_{0,0,0,0} \delta^{(ij_j \ldots j_r)} \lambda_j \ldots \lambda_j + \sum_{r=0}^{\infty} \frac{2r+3}{r!} (\lambda_a^a)^r \beta_r \mu ik \lambda^i \lambda^k.
\]  

(119)

Let us resume now some of the conditions found on the coefficients.

Eq. (117) says that

\[
\phi_{1,q,r+1,0} = 2(r+1) \psi_{0,q+2,r,0}.
\]  

(120)

Eqs. (115) and Eqs. (116) give

\[
\phi_{0,1,0} = 2 \psi_{0,1,0} \quad \text{and} \quad \phi_{0,1,2R+2,0} = 2(2R+3) \psi_{0,1,2R+2,0} - \frac{(2R+3)!}{R!} 2(2R+3) \beta_R.
\]  

(121)

Eq. (112), thanks also to Eq. (120) and Eq. (121) says that

\[
\psi_{0,q+2,0,0} = 0; \psi_{0,1,0,0} = 0 \text{ (which can be compacted in } \psi_{0,q+1,0,0} = 0) ; \psi_{0,0,1,0,0} = 0.
\]  

(122)

Eq. (106)

for \( r = 0 \) gives

\[
2 \psi_{0,q+2,0,0} = \phi_{0,q+2,0,0}.
\]  

(123)

with \( r+1 \) instead of \( r \), gives

\[
2(r+2) \psi_{0,q+2,r+1,0} + 2(r+1)^2 \psi_{0,q+2,r,0} - (r+1) \phi_{0,q+2,r+1,0} = 0.
\]

(124)

Eq. (102) is

\[
\frac{\partial}{\partial \lambda} \psi_{0,q+1,0,0} = 0.
\]  

(124)

Eq. (124) with \( R = r+1 \) and with aid by (120), is

\[
(2q+r+3) \phi_{1,q,r+1,0} + 2 \lambda \phi_{1,q+1,r+1,0} = 0.
\]  

(125)

Eq. (87), thanks to (124), says that

\[
\phi_{1,q,r+1,0} = 0,
\]  

(126)
which implies (125) as a particular case.

Eq. (92) is a consequence of (122)\textsubscript{3,4}, except for \(Q = 0\); in this case it gives

\[
\psi_{0,0,0,0} = 0 .
\]  

(127)

Eq. (96) with \(q + 1\) instead of \(q\) gives

\[
2(r + 1)\psi_{0,q+2,r,0} = \phi'_0,q+2,r,0 ,
\]  

(128)

where (124) has been used.

Eq. (96) with \(q = 0\), thanks to (124) and (121)\textsubscript{1} gives

\[
(r + 1) \frac{\partial}{\partial \lambda} \psi_{0,0,r,0} + 2(r + 1)\psi_{0,1,r,0} = \phi'_0,1,r,0 ,
\]

which for \(r = 0\) gives again (127), while written with \(2r + 2\) instead of \(r\) gives

\[
\frac{\partial}{\partial \lambda} \psi_{0,0,2r+2,0} = -2\frac{(2r + 3)!}{r!} \beta_r ,
\]

where we have used (121)\textsubscript{2}. By integrating the result, we obtain

\[
\psi_{0,0,2r+2,0} = -2\lambda \frac{(2r + 3)!}{r!} \beta_r + \psi_{0,0,2r+2,0,0} ,
\]  

(129)

where \(\psi_{0,0,2r+2,0,0}\) is a constant arising from integration.

Eq. (119), thanks to (127) and (129) becomes

\[
H^* = \sum_{r \in I_0} \frac{1}{(r + 2)!} \psi_{0,0,r,0,0} \delta^{(j_1 \ldots j_{r+2})} \lambda_{j_1} \ldots \lambda_{j_{r+2}} - \sum_{r=0}^{\infty} 2\lambda \frac{2r + 3}{r!} \left(\lambda a \lambda^a\right)^{r+1} \beta_r + \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \left(\lambda a \lambda^a\right)^r \beta_r \mu_{ik} \lambda^i \lambda^k .
\]  

(130)

It is interesting that this expressions determines \(H^*\) in terms of two arbitrary sets of constants \(\psi_{0,0,r,0,0}\) and \(\beta_r\) and that it satisfies eq. (58).

The doubt may arise that the other equations on the coefficients \(\phi, \phi', \psi, \ldots\) are too much restrictive. The doubt may be eliminate by direct analysis or, equivalently, noting that they appeared only in the present section; only the \(\vartheta, \ldots\) were present also in previous section and here we have found that they are further restricted by (126). This can be rewritten also as

\[
\vartheta_{1,q,r,0} = 0 ,
\]  

(131)

because the sum of the first and third index must be an even number, so that obviously we must have \(r \geq 1\) in \(\vartheta_{1,q,r,0}\).

Well, we may eliminate the above mentioned doubt by verifying directly that \(H^*\) given by (130) satisfies, for whatever value of the constants \(\psi_{0,0,r,0,0}\) and \(\beta_r\), the conditions described at the end of the previous section, that is (66), (69), (74), (78) and (86). To this end we may use only the restriction (131) on \(\vartheta, \ldots\), jointly with those found in the previous sections.
4.2 Verifying the conditions on $H^{s0}$.

- It is to verify equation (66), because $H^{s0}$ given by (130) is sum of a function not depending on $\lambda$ and of a function not depending on $\mu_{ij}$; moreover the right hand side of eq. (66) is zero, thanks to (131).

- It is to verify equation (69), because $H^{s0}$ given by (130) becomes zero when calculated in $\lambda_i = 0$.

- It is to verify equation (74), thanks to (131) and because $H^{s0}$ given by (130) is linear in $\mu_{ij} = 0$.

- Let us verify equation (78). By a substitution of $H^{s0}$ from (130) it becomes

$$0 = 2 \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r \lambda^k \lambda^i + \lambda_i \frac{\partial}{\partial \lambda_k} \left[ -\sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r + 1 \right] +$$

$$+ \delta^{ki} \left[ -\sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r + 1 \right] + 2 \lambda_j \frac{\partial}{\partial \lambda_k} \left[ \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r \lambda^i \lambda^j \right],$$

which is true because the sum of $1^{\text{th}}$ and $4^{\text{th}}$ term is equal to $\frac{\partial}{\partial \lambda_k} \left[ 2 \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r \lambda^i \lambda^j \right]$, while the sum of $2^{\text{nd}}$ and $3^{\text{rd}}$ term is equal to $\frac{\partial}{\partial \lambda_k} \left[ -\lambda_i \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r + 1 \right]$.

- Let us verify equation (86). By a substitution of $H^{s0}$ from (130) it becomes

$$0 = \lambda_b \frac{\partial \lambda^i}{\partial \lambda_k} \left[ \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda c \lambda^c)^r \lambda^a \lambda^b \right] + \lambda_b \delta^{ij} \left[ \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda c \lambda^c)^r \lambda^a \lambda^b \right].$$

In the first term, when we don’t take the derivative of $\lambda^a$ with respect to $\lambda_k$, we obtain zero for the identity $\lambda^{[i} \lambda^{a]} = 0$; when we take the derivative of $\lambda^a$ with respect to $\lambda_k$, we obtain $\lambda_b \delta^{[i} \delta^{a]} \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda c \lambda^c)^r \lambda^b$ which is the opposite of the second term! This completes our verification.

5 Solution of the conditions on $\tilde{H}^i$.

Let us firstly change unknown function, from $\tilde{H}^k$ to $\tilde{H}^{*k}$ defined by

$$\tilde{H}^k = \tilde{H}^{*k} + \sum_{r \in I_o} \frac{1}{(r + 1)!} \left[ \lambda^{(k \{j_1 \ldots j_r + 1\} - \frac{1}{2} \sum_{r=2}^{\infty} \frac{(2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r \mu_{ij} \right] \psi_{0,0,0,0} \lambda_{j_1} \ldots \lambda_{j_{r+1}} +$$

$$+ \frac{\partial}{\partial \lambda_k} \left[ \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r \mu_{bc} \lambda^b \lambda^c - \lambda^2 \lambda_b \lambda^b \right] +$$

$$- \mu^{kd} \lambda_d (\mu_{bc} \lambda^c) \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r - 1 - \frac{1}{4} \lambda^k (\mu_{bc} \lambda^b \lambda^c)^2 \sum_{r=2}^{\infty} (2r - 3) \frac{2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r +$$

$$- \lambda^k (\mu_{bd} \mu_{dc} \lambda^b \lambda^c) \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda a \lambda^a)^r - 1.$$
By substituting $\tilde{H}^k$ from (132) and $H^{r0}$ from (130) in (83) and (85) these equations are transformed respectively in

$$
\frac{\partial}{\partial \lambda} \tilde{H}^{ri} = 0, \quad (133)
$$

$$
0 = 2\mu_{ji} \left\{ \sum_{r=0}^{\infty} \frac{2r+3}{r!} \beta_r (\lambda_a \lambda^a)^r \lambda^i \lambda^j \right\} + 2\lambda \left\{ \sum_{r=0}^{\infty} \frac{2r+3}{r!} \beta_r (\lambda_a \lambda^a)^r \lambda^i \lambda^j \right\} + 
$$

$$
+ \lambda_i \left\{ \sum_{r=0}^{\infty} \frac{1}{(r+1)!} \psi_{0,0,r,0,0} \delta^{(k_{j1} \cdots j_{r+1})} \lambda_{j_1} \cdots \lambda_{j_{r+1}} + 
$$

$$
+ \frac{\partial}{\partial \lambda_k} \left\{ \sum_{r=0}^{\infty} \frac{2r+3}{r!} \beta_r (\lambda_a \lambda^a)^r \left( \mu_{bc} \lambda^b \lambda^c - 2\lambda \lambda_b \lambda_b \right) \right\} \right\} + 
$$

$$
+ \delta^{ki} \left\{ \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \psi_{0,0,r,0,0} \delta^{(j_{1} \cdots j_{r+2})} \lambda_{j_1} \cdots \lambda_{j_{r+2}} + \sum_{r=0}^{\infty} \frac{2r+3}{r!} \beta_r (\lambda_a \lambda^a)^r \left( \mu_{bc} \lambda^b \lambda^c - 2\lambda \lambda_b \lambda_b \right) \right\} + 
$$

$$
+ 2\lambda_j \left\{ \frac{\partial \tilde{H}^{rk}}{\partial \mu_{ij}} - \frac{1}{2} \sum_{r=0}^{\infty} \frac{r+3}{(r+2)!} \psi_{0,0,r,0,0} \delta^{(k_{j1} \cdots j_{r+1})} \lambda_{j_1} \cdots \lambda_{j_{r+1}} + 
$$

$$
+ \frac{\partial}{\partial \lambda_k} \left\{ \sum_{r=0}^{\infty} \frac{2r+3}{r!} \beta_r (\lambda_a \lambda^a)^r \lambda^i \lambda^j \right\} \right\} 
$$

$$
- \sum_{r=2}^{\infty} \frac{2r+3}{r!} \beta_r (\lambda_a \lambda^a)^{r-1} \left[ \lambda^i \lambda^j \mu^{kd} \lambda_d + (\mu_{bc} \lambda^b \lambda^c) \delta^{(i \lambda^j)} \right] + 
$$

$$
- \frac{1}{4} \sum_{r=2}^{\infty} (2r-3) \frac{2r+3}{r!} \beta_r (\lambda_a \lambda^a)^{r-2} 2(\mu_{bc} \lambda^b \lambda^c) \lambda^i \lambda^j \lambda^k + 
$$

$$
- \sum_{r=2}^{\infty} \frac{2r+3}{r!} \beta_r (\lambda_a \lambda^a)^{r-1} \lambda^k (\lambda^i \mu^{jb} \lambda_b + \lambda^j \mu^{ib} \lambda_b) \right\} .
$$

Now, in eq. (134), the second and third term containing $\lambda$ can be written together as

$$
\frac{\partial}{\partial \lambda} \left\{ \sum_{r=0}^{\infty} \frac{2r+3}{r!} \beta_r (\lambda_a \lambda^a)^r (-2\lambda) \lambda_i (\mu^{bc} \lambda^b \lambda_b) \right\},
$$

while the first and fourth term can be written together as

$$
2 \frac{\partial}{\partial \lambda} \left\{ \sum_{r=0}^{\infty} \frac{2r+3}{r!} \beta_r (\lambda_a \lambda^a)^r \lambda^i \lambda^j \lambda^k \right\},
$$

consequently the terms containing $\lambda$ elide each other. This result could be expected for eq. (133).

Furthermore, the coefficient of $\psi_{0,0,r,0,0}$ in the third term containing it, can be written as

$$
- \sum_{r=0}^{\infty} \frac{r+3}{(r+2)!} \delta^{(k_{j1} \cdots j_{r+1})} \lambda_j \lambda_{j_1} \cdots \lambda_{j_{r+1}} = - \sum_{r=0}^{\infty} \frac{r+3}{(r+2)!} \delta^{(k_{j1} \cdots j_{r+2})} \lambda_j \lambda_{j_1} \cdots \lambda_{j_{r+2}} =
$$

$$
= - \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \left[ \delta^{ik} \delta^{(j_{1} \cdots j_{r+2})} + (r+2) \delta^{(j_{1} \cdots j_{r+2})} \right] \lambda_{j_1} \cdots \lambda_{j_{r+2}} =
$$

$$
= - \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \left[ \delta^{ik} \delta^{(j_{1} \cdots j_{r+2})} \lambda_{j_1} \cdots \lambda_{j_{r+2}} + (r+2) \lambda^i \delta^{(j_{1} \cdots j_{r+1})} \lambda_j \lambda_{j_{r+2}} \right],
$$

35
so that it and the other terms in eq. (134) containing \( \psi_{0,0,r,0} \) elide each other. After that, of eq. (134) there remains

\[
0 = 2\mu_j \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^r \lambda^j \lambda^i + \lambda_i \frac{\partial}{\partial \lambda_i} \left[ \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^r (\mu_{bc} \lambda^b \lambda^c) \right] + \\
+ \delta^{ki} \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^r (\mu_{bc} \lambda^b \lambda^c) + 2\lambda_j \frac{\partial \tilde{H}^{*k}}{\partial \mu_{ij}} + \\
- \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^{r-1} \left[ 2(\lambda_c \lambda^c) \lambda^i \mu^{kd} \lambda_d + (\mu_{bc} \lambda^b \lambda^c)(\lambda_d \lambda^d) \delta^{ki} + (\mu_{bc} \lambda^b \lambda^c) \lambda^k \lambda^i \right] + \\
- \sum_{r=2}^{\infty} (2r - 3) \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^{r-2} (\mu_{bc} \lambda^b \lambda^c)(\lambda_d \lambda^d) \lambda^i \lambda^k + \\
- 2 \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^{r-1} \lambda^k \left[ \lambda^i (\mu_{bc} \lambda^b \lambda^c) + (\lambda_d \lambda^d) \mu^{ib} \lambda_b \right].
\]

or,

\[
0 = 2\lambda_i \frac{\partial \tilde{H}^{*k}}{\partial \mu_{ij}} + \\
+ \delta^{ki} \left[ \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^r (\mu_{bc} \lambda^b \lambda^c) - \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^r (\mu_{bc} \lambda^b \lambda^c) \right] + \\
+ \lambda^i \mu^{ij} \lambda_j \left[ 2 \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^r - 2 \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^r \right] + \\
+ \lambda^i \mu^{kj} \lambda_j \left[ 2 \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^r - 2 \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^r \right] + \\
+ \lambda^k \lambda^i \left[ 2 \sum_{r=0}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^{r-1} (\mu_{bc} \lambda^b \lambda^c) - \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^{r-1} (\mu_{bc} \lambda^b \lambda^c) + \\
- \sum_{r=2}^{\infty} (2r - 3) \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^{r-1} (\mu_{bc} \lambda^b \lambda^c) - 2 \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda^a)^{r-1} (\mu_{bc} \lambda^b \lambda^c) \right].
\]

But in this expression, all the terms with \( r \geq 2 \) elide each other, so that there remain only the terms with \( r = 0, 1 \), that is

\[
0 = 2\lambda_i \frac{\partial \tilde{H}^{*k}}{\partial \mu_{ij}} + \left[ \delta^{ki} (\mu_{bc} \lambda^b \lambda^c) + 4\lambda^i (\mu^{ij} \lambda_j) \right] (3\beta_0 + 5\beta_1 \lambda_a \lambda^a) + 10\lambda^i \lambda^k (\mu_{bc} \lambda^b \lambda^c) \beta_1. \tag{135}
\]

A further refinement of the situation can be obtained with another change of unknown function, from \( \tilde{H}^{*k} \) to \( \tilde{H}^{**k} \) defined by

\[
\tilde{H}^{**k} = \tilde{H}^{*k} - \frac{5}{4} \beta_1 \left\{ 4(\mu_{bc} \lambda^b \lambda^c) \mu^{kd} \lambda_d + \lambda^k \left[ (\mu_{bc} \mu^{bc})(\lambda_a \lambda^a) + 2\lambda^a \lambda^b \mu_{ac} \mu_{cb} \right] \right\}. \tag{136}
\]

By using this, eq. (135) becomes

\[
0 = 2\lambda_i \frac{\partial \tilde{H}^{**k}}{\partial \mu_{ij}} + 3\beta_0 \left[ \delta^{ki} (\mu_{bc} \lambda^b \lambda^c) + 4\lambda^i (\mu^{ij} \lambda_j) \right]. \tag{137}
\]
We can now prove that, as a consequence of this equation, we have

$$\beta_0 = 0.$$  \hspace{1cm} (138)

5.1 Solution of the conditions on $\tilde{H}^{**k}$.

Let us consider the Taylor expansion of $\tilde{H}^{**k}$ around the state with $\mu^{ij} = 0$; eq. (137) at the order 1 with respect to this state is

$$0 = 2\lambda_j \frac{\partial \tilde{H}^{**k}}{\partial \mu_{ij}} + 3\beta_0 \left[ \delta^{ki}(\mu_{bc} \lambda^b \lambda^c) + 4\lambda^{(k)ij}\lambda_j \right],$$  \hspace{1cm} (139)

where $\tilde{H}^{**k}$ is the homogeneous part of $\tilde{H}^{**k}$ of second degree with respect to $\mu^{ij}$. Thanks to the Representation Theorems, it has the form

$$\tilde{H}^{**k} = f_1(G_0)\mu^{ka} \lambda_b + f_2(G_0)\mu^{hl} \mu^{ka} \lambda_a + f_3(G_0)(\mu_{bc} \lambda^b \lambda^c)\mu^{ka} \lambda_a +$$

$$+ \lambda^k \left[ f_4(G_0)(\mu^{hl})^2 + f_5(G_0)(\mu_{bc} \lambda^b \lambda^c)^2 + f_6(G_0)(\mu_{bc} \lambda^b \lambda^c)\mu^{hl} + f_7(G_0)\mu_{bc} \lambda_{bc} + f_8(G_0)\mu_{ab} \mu_{bc} \lambda^b \lambda^c \right],$$

where $G_0 = \lambda_a \lambda^a$. By substituting this in (139), we obtain

$$0 = 2\lambda_j \left\{ f_1 \delta^{ki}(\mu_{bc} \lambda^b \lambda^c) + f_1 \lambda^{(k)ij} \lambda_j + f_1 \lambda^{(k)ij} \lambda_j + f_2 \delta^{ij} \mu^{kb} \lambda_b + f_2 \mu^{hl} \delta^{(k)ij} + f_3 \lambda^{(k)ij} \mu^{kb} \lambda_b + f_3(\mu_{bc} \lambda^b \lambda^c)\delta^{(k)ij} +$$

$$+ \lambda^k \left[ 2f_4 \mu^{hl} \delta^{ij} + 2f_5(\mu_{bc} \lambda^b \lambda^c)\lambda^i \lambda^j + 2f_6(\mu_{bc} \lambda^b \lambda^c)\lambda^i \mu^{hl} + 2f_6(\mu_{bc} \lambda^b \lambda^c)\delta^{ij} + 2f_7 \mu^{ij} + 2f_8 \lambda^{(k)ij} \lambda_j \right] +$$

$$+ 3\beta_0 \left[ \delta^{ki}(\mu_{bc} \lambda^b \lambda^c) + 4\lambda^{(k)ij} \lambda_j \right],$$

that is,

$$0 = f_1 \delta^{ki}(\mu_{bc} \lambda^b \lambda^c) + 2f_1 \lambda^{(k)ij} \lambda_j + 2f_1 \lambda^{(k)ij} \lambda_j + f_2 \delta^{ij} \mu^{kb} \lambda_b + f_2 \mu^{hl} \delta^{(k)ij} + f_3 \lambda^{(k)ij} \mu^{kb} \lambda_b + f_3(\mu_{bc} \lambda^b \lambda^c)\delta^{(k)ij} +$$

$$+ 2f_4 \mu^{hl} \lambda^i + 2f_5(\mu_{bc} \lambda^b \lambda^c)G_0 \lambda^i + 2f_6 G_0 \lambda^i \mu^{hl} + 2f_6(\mu_{bc} \lambda^b \lambda^c)\lambda^i + 2f_7 \mu^{ij} \lambda_j + 2f_8 \lambda^{(k)ij} \lambda_j +$$

$$+ 2f_8 \mu^{hl} \lambda_b + 3\beta_0 \left[ \delta^{ki}(\mu_{bc} \lambda^b \lambda^c) + 4\lambda^{(k)ij} \lambda_j \right].$$

The skew-symmetric part of this relation, with respect to $i$ and $k$ is

$$0 = \lambda^{[k} \mu^{ij]} \lambda_b (-2f_2 - 2f_3 G_0 + 4f_7 + 2f_8 G_0),$$

from which we obtain

$$4f_7 = 2f_2 + 2f_3 G_0 + 2f_8 G_0.$$  \hspace{1cm} (140)

By taking into account this value of $f_7$, the remaining part of our condition becomes

$$0 = \delta^{ki} \left[ f_2 \mu^{hl} G_0 + (f_1 + f_3 G_0 + 3\beta_0)(\mu_{bc} \lambda^b \lambda^c) \right] + f_1 \mu^{ki} G_0 +$$

$$+ \lambda^{(k)ij} \lambda_b (2f_1 + 4f_2 + 4f_3 G_0 + 12\beta_0) +$$

$$+ \lambda^k \left[ \mu^{hl}(f_2 + 4f_4 + 2f_6 G_0) + (\mu_{bc} \lambda^b \lambda^c)(f_3 + 4f_5 G_0 + 2f_6 + 2f_8) \right].$$
In this relation, the coefficients of $\mu^{ki}$ and $\delta^{ki}\mu^l$ give respectively $f_1 = 0$ and $f_2 = 0$. After that, the coefficient of $\delta^{ki}(\mu_{bc}\lambda^b\lambda^c)$ gives $f_3 G_0 + 3\beta_0 = 0$, which calculated in $\lambda_j = 0$ gives the above mentioned eq. (138).

This result transforms eq. (137) into

$$0 = 2\lambda_j \frac{\partial \tilde{H}^{**k}}{\partial \mu_{ij}}. \tag{141}$$

Now we proceed to find the general solution of this last equation and we prove that it is

$$\tilde{H}^{**k} = \lambda^k F(G_0, G_1, G_2), \tag{142}$$

where

$$G_1 = G_0 \delta^{bc} \mu_{bc} - \mu_{bc} \lambda^b \lambda^c, \quad G_2 = G_0 \mu^{bc} \mu_{bc} - 2\mu_{bc} \mu_{ca}\lambda^b \lambda^c + 2(\delta^{bc}\mu_{bc})(\mu_{bc}\lambda^b \lambda^c) - G_0(\delta^{bc}\mu_{bc})^2$$

and $F$ is an arbitrary function of its variables.

In fact, if $\lambda_j = 0$, from the Representation Theorems we know that $\tilde{H}^{**k} = 0$, just as in (142) and (141) is an identity.

If $\lambda_j \neq 0$, we can define the projector into the subspace orthogonal to $\lambda_j$, that is

$$h^{ij} = \delta^{ij} - \frac{1}{G_0} \lambda^i \lambda^j, \tag{143}$$

from which it follows $h^{ij} \lambda_j = 0$, as obvious. By taking as independent variables $\lambda^i$, $\tilde{\mu} = \mu_{bc} \lambda^b \lambda^c$, $\tilde{\mu}^i = h^{ij} \mu_{ja} \lambda^a$, $\tilde{\mu}^{ij} = h^{ia} \mu_{ab} h^{bj}$, eq. (141) becomes

$$0 = 2\lambda_j \left( \frac{\partial \tilde{H}^{**k}}{\partial \tilde{\mu}} \lambda^i \lambda^j + \frac{\partial \tilde{H}^{**k}}{\partial \tilde{\mu}^b} h^{b(i} \lambda^{j)} + \frac{\partial \tilde{H}^{**k}}{\partial \tilde{\mu}^{ab}} h^{a(i} \mu^{j)b} \right) =$$

$$= 2G_0 \frac{\partial \tilde{H}^{**k}}{\partial \tilde{\mu}} \lambda^i + \frac{\partial \tilde{H}^{**k}}{\partial \tilde{\mu}^b} G_0 h^{bi}.$$

By contracting this relation with $\lambda^i$ and with $h_{ia}$, we obtain respectively

$$\frac{\partial \tilde{H}^{**k}}{\partial \tilde{\mu}} = 0, \quad \frac{\partial \tilde{H}^{**k}}{\partial \tilde{\mu}^a} = 0. \tag{144}$$

It follows that $\tilde{H}^{**k}$ may depend only on $\lambda^i$ and $\tilde{\mu}^{ij}$. But $\tilde{\mu}^{ij} \lambda_j = 0$ so that, for the Representation Theorems we have that $\tilde{H}^{**k}$ is proportional to $\lambda^k$ as in eq. (142); moreover, the coefficient $F$ can be a scalar function of $G_0$, $Q_1 = \delta_{ij} \tilde{\mu}^{ij}$, $Q_2 = \delta_{ij} \tilde{\mu}^{ia} \tilde{\mu}^{aj}$.

Now we have

$$Q_1 = \delta_{ij} \tilde{\mu}^{ij} = h_{ij} \mu^{ij} = \delta_{ij} \mu^{ij} - \frac{1}{G_0} \lambda_j \lambda_j \mu^{ij},$$

$$Q_2 = \delta_{ij} \mu^{ib} h_{ca} h^{ad} \mu_{de} h^{ej} = h_{be} \mu^{bc} h_{cd} \mu^{de} = \delta_{be} \mu^{bc} \delta_{cd} \mu^{de} - 2 \frac{G_0}{G_0} \delta_{be} \mu^{bc} \lambda_d \lambda_c \mu^{de} + \left( \frac{1}{G_0} \right)^2 (\mu_{bc} \lambda^b \lambda^c)^2.$$

But an arbitrary function of $G_0$, $Q_1$, $Q_2$ is also an arbitrary function of $G_0$, $Q_1$ and of

$$Q_2 - (Q_1)^2 = \mu^{ee} \mu_{ee} - 2 \frac{G_0}{G_0} \mu^{de} \mu^{ec} \lambda_d \lambda_c + 2 \frac{G_0}{G_0} (\delta_{ij} \mu^{ij})(\mu_{ab} \lambda^a \lambda^b) - (\delta_{ij} \mu^{ij})^2$$
and an arbitrary function of $G_0$, $Q_1$, $Q_2 - (Q_1)^2$ is also an arbitrary function of $G_0$, $Q_1G_0$, $[Q_2 - (Q_1)^2]G_0$ and this completes the proof of eq. (142)\textsubscript{2,3}. These last passages have been done with the end to have a function defined also in $\lambda^j = 0$, without going too much far from equilibrium.

6 Conclusions.

We can collect now all our results. By substituting $\tilde{H}^{*k}$ from (142) into (139) we obtain $\tilde{H}^{*k}$; by substituting this and (138) into (132), we obtain

$$
\tilde{H}^k = \lambda^k F(G_0, G_1, G_2) - \frac{5}{4} \beta_1 \left\{ 4(\mu_{bc}\lambda^b \lambda^c)\mu^{kd}\lambda_d + \lambda^k \left[ (\mu_{bc}\mu^{bc})(\lambda_a \lambda^a) + 2\lambda^a \lambda^b \mu_{ac}\mu_{cb}\right] \right\} + (145)
$$

$$
+ \sum_{r\in I_0} \frac{1}{(r + 1)!} \left[ \lambda\delta^{(j_1\ldots j_{r+1})} - \frac{r + 3}{2r + 2} \delta^{(j_1\ldots j_{r+1})} \mu_{ij} \right] \psi_{0,0,r,0,0} \lambda_{j_1} \ldots \lambda_{j_{r+1}} +
$$

$$
- \mu^{kd}\lambda_d(\mu_{bc}\lambda^b \lambda^c) \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r(\lambda_a \lambda^a)^{r-1} - \frac{1}{4} \lambda^k(\mu_{bc}\lambda^b \lambda^c)^2 \sum_{r=2}^{\infty} (2r - 3) \frac{2r + 3}{r!} \beta_r(\lambda_a \lambda^a)^{r-2} +
$$

$$
- \lambda^k(\mu_{bd}\mu_{dc}\lambda^b \lambda^c) \sum_{r=2}^{\infty} \frac{2r + 3}{r!} \beta_r(\lambda_a \lambda^a)^{r-1} .
$$

Thanks to this expression and of (130), taking also into account (138), we can rewrite the expression for $\tilde{H}$ in (80); finally, we can substitute this new expression and that of (130) for $H^{*d}$ in (79). In this way we obtain

$$
\Delta H = \sum_{p,q,s} \sum_{r\in I_p} \frac{1}{p!} \frac{1}{q!} \frac{1}{r!} \frac{1}{(s + 1)!} \vartheta_{p,q,r,s}(\lambda) \mu^{x+1} \delta^{(i_1\ldots i_p h_{1k_1} \ldots h_{qk_1} j_1 \ldots j_r)} \mu_{i_1} \ldots \mu_{i_p} \mu_{h_{1k_1}} \ldots \mu_{h_{qk_1}} \lambda_{j_1} \ldots \lambda_{j_r} +
$$

$$
+ \mu \left\{ \sum_{r\in I_0} \frac{1}{(r + 2)!} \psi_{0,0,r,0,0} \delta^{(j_1\ldots j_{r+2})} \lambda_{j_1} \ldots \lambda_{j_{r+2}} - \sum_{r=1}^{\infty} 2\lambda \frac{2r + 3}{r!} (\lambda_a \lambda^a)^{r+1} \beta_r +
$$

$$
+ \sum_{r=1}^{\infty} \frac{2r + 3}{r!} (\lambda_a \lambda^a)^r \beta_r \mu_{ik} \lambda^i \lambda^k \right\} +
$$

$$
+ \sum_{p,q} \sum_{r\in I_p} \frac{1}{(p + 2)!} \frac{1}{q!} \frac{1}{r!} \vartheta_{p,q+1,r,0,0} \delta^{(i_1\ldots i_{p+2} h_{1k_1} \ldots h_{qk_1} j_1 \ldots j_r)} \mu_{i_1} \ldots \mu_{i_{p+2}} \mu_{h_{1k_1}} \ldots \mu_{h_{qk_1}} \lambda_{j_1} \ldots \lambda_{j_r} + \frac{1}{2} \mu_i \mu_j \sum_{r=1}^{\infty} 2\lambda \frac{2r + 3}{r!} (\lambda_a \lambda^a)^r \beta_r \lambda^i \lambda^j +
$$

$$
+ \mu \left\{ \lambda^i F(G_0, G_1, G_2) - \frac{5}{4} \beta_1 \left[ 4(\mu_{bc}\lambda^b \lambda^c)\mu^{id}\lambda_d + \lambda^i \left[ (\mu_{bc}\mu^{bc})(\lambda_a \lambda^a) + 2\lambda^a \lambda^b \mu_{ac}\mu_{cb}\right] \right] \right\} +
$$

$$
+ \mu_i \left\{ \mu^{ik} F(G_0, G_1, G_2) - \frac{5}{4} \beta_1 \left[ 4(\mu_{bc}\lambda^b \lambda^c)\mu^{ik}\lambda_k + \lambda^i \left[ (\mu_{bc}\mu^{bc})(\lambda_a \lambda^a) + 2\lambda^a \lambda^b \mu_{ac}\mu_{cb}\right] \right] \right\} +
$$

$$
+ \mu_{ik} \left\{ \mu^{ik} F(G_0, G_1, G_2) - \frac{5}{4} \beta_1 \left[ 4(\mu_{bc}\lambda^b \lambda^c)\mu^{ik}\lambda_k + \lambda^i \left[ (\mu_{bc}\mu^{bc})(\lambda_a \lambda^a) + 2\lambda^a \lambda^b \mu_{ac}\mu_{cb}\right] \right] \right\} +
$$

$$
+ \mu_{ik} \left\{ \mu^{ik} F(G_0, G_1, G_2) - \frac{5}{4} \beta_1 \left[ 4(\mu_{bc}\lambda^b \lambda^c)\mu^{ik}\lambda_k + \lambda^i \left[ (\mu_{bc}\mu^{bc})(\lambda_a \lambda^a) + 2\lambda^a \lambda^b \mu_{ac}\mu_{cb}\right] \right] \right\} .
$$
\[ + \sum_{r \in I_0} \frac{1}{(r + 1)!} \left[ \lambda^b_{(ij_1 \ldots j_{r+1})} - \frac{1}{2} r + 3 \frac{\delta(ij_1 \ldots j_{r+1})}{2} \mu_{kj} \right] \psi_{0,0,r,0,0} \lambda_{j_1} \cdots \lambda_{j_{r+1}} + \]
\[ + \frac{\partial}{\partial \lambda_i} \left[ \sum_{r = 1}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda_a)^r \left( \lambda \mu_{bc} \lambda^b \lambda^c - \lambda^2 \lambda_b \lambda^b \right) \right] + \]
\[ - \mu^i d \lambda_d (\mu_{bc} \lambda^b \lambda^c) \sum_{r = 2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda_a)^{r-1} - \frac{1}{4} \lambda^i (\mu_{bc} \lambda^b \lambda^c)^2 \sum_{r = 2}^{\infty} (2r - 3) \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda_a)^{r-2} + \]
\[ - \lambda^i (\mu_{bd} \mu_{dc} \lambda^b \lambda^c) \sum_{r = 2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda_a)^{r-1} \right) \right] + \tilde{H}^0 (\mu_{ab}, \lambda, \lambda_c). \]

We recall that in this expression, \( F(G_0, G_1, G_2) \) is an arbitrary function, \( \psi_{0,0,r,0,0} \) and \( \beta_r \) are arbitrary constants, while \( \partial_{\rho,q,r,s} (\lambda) \) are constrained by (61), (62), (63), (64), (50) and (57). The presence of the arbitrary function \( \tilde{H}^0 (\mu_{ab}, \lambda, \lambda_c) \) is obvious because it is not constrained by (41) because it doesn’t depend on \( \mu \), nor on \( \mu_k \). Consequently, it is not necessary to impose the condition \( \tilde{H}^0 (0_{ab}, \lambda, 0_c) = 0 \) which comes out from eqs. (42), (57) and (146).

The sum of the expression (146) for \( \Delta H \) and of the expression (88) for \( H_1 \) give the general solution for the unknown function \( H \). Let us substitute it in the equations
\[ F^{kij} = \frac{\partial^2 H}{\partial \mu_k \partial \mu_j}, \quad G^{ki} = \frac{\partial^2 H}{\partial \mu_k \partial \lambda_i} \]
which are a subset of the equations (6). We obtain that \( F^{kij} \) is sum of a symmetric tensor and of
\[ \Delta F^{kij} = \lambda^k \frac{\partial F(G_0, G_1, G_2)}{\partial \mu_{ij}} - \frac{5}{4} \beta_1 \left[ 4 (\mu_{bc} \lambda^b \lambda^c) \delta^{k(i} \lambda^{j)} + \lambda^k \left( 2 \mu^{ij} (\lambda_a \lambda_a) - 4 \lambda^{(i} \mu^{j)bc} \lambda_b \right) \right] + (148) \]
\[ + 2 \sum_{r = 1}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda_a)^r \lambda \delta^{k(i} \lambda^{j)} - \delta^{k(i} \lambda^{j)} (\mu_{bc} \lambda^b \lambda^c) \sum_{r = 2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda_a)^{r-1}. \]
Similarly, \( G^{ki} \) is sum of a symmetric tensor and of
\[ \Delta G^{ki} = \sum_{r = 1}^{\infty} \frac{2r + 3}{r!} (\lambda_a \lambda_a)^r \beta_r \lambda^k \mu^i + \lambda^k \frac{\partial F}{\partial G_1} \frac{\partial G_1}{\partial \lambda_i} + \lambda^k \frac{\partial F}{\partial G_2} \frac{\partial G_2}{\partial \lambda_i} - 5 \beta_1 \lambda^k \mu^{i} \mu^{cb} \lambda_b + (149) \]
\[ - 2 \lambda^i \mu^{kd} \lambda_d (\mu_{bc} \lambda^b \lambda^c) \sum_{r = 2}^{\infty} \frac{2r + 3}{r!} (r - 1) \beta_r (\lambda_a \lambda_a)^{r-2} + \]
\[ - \lambda^i \mu^{id} \lambda_d (\mu_{bc} \lambda^b \lambda^c) \sum_{r = 2}^{\infty} (2r - 3) \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda_a)^{r-2} - 2 \lambda^k \mu^{id} \mu^{dc} \lambda_c \sum_{r = 2}^{\infty} \frac{2r + 3}{r!} \beta_r (\lambda_a \lambda_a)^{r-1}. \]
But, from (142) we see that \( \frac{\partial G_1}{\partial \mu_{ij}} \) is a tensor at least of second order with respect to equilibrium and \( \frac{\partial G_2}{\partial \mu_{ij}} \) is a tensor at least of third order with respect to equilibrium. Consequently, from (148) it is clear that \( F^{kij} \) up to second order with respect to equilibrium is a symmetric tensor; its eventual non symmetric parts may appear only from the third order with respect to equilibrium. This result is different from its counterpart in (11) where a non symmetric part appeared also at first order with
respect to equilibrium. We shall see in Appendix 2 that from the equations of that paper it follows that this non symmetric part is proportional to a constant; consequently, here we have proved that this constant is zero and that this further constraint follows by imposing the equations up to order higher than one with respect to equilibrium. This is not a problem, because the authors of paper [1] assumed (for example in the first 3 lines of subsection 7.2) that the integration constants vanish and furnished reasons for this assumption based on the kinetic theory approach.

Similarly, from (142) we see that \( \frac{\partial G_1}{\partial \lambda_i} \) is a tensor of second order with respect to equilibrium and \( \frac{\partial G_1}{\partial \lambda_i} \) is a tensor at least of third order with respect to equilibrium. Consequently, from (149) it is clear that \( G_{ki} \) up to second order with respect to equilibrium is a symmetric tensor; its eventual non symmetric parts may appear only from the third order with respect to equilibrium. This result agrees with its counterpart in [1].

A last question that can be considered is the following: It is possible to write (146) in a form close to (55), that is

\[
\Delta H = \sum_{p,q,s} \sum_{r \in I_p} \frac{1}{p! q! r! s!} \chi_{p,q,r,s}(\lambda) \mu^s \delta^{(i_1 \ldots i_p h_1 \ldots h_q j_1 \ldots j_r)} \mu_{i_1} \ldots \mu_{i_p} h_{k_1} \ldots h_{k_q} \lambda_{j_1} \ldots \lambda_{j_r} + (150)
\]

\[+ \omega(\mu_{ab}, \lambda, \lambda_c).\]

where \( \chi_{p,q,r,s}(\lambda) \) are the counterparts of the \( \vartheta_{p,q,r,s}(\lambda) \) and \( \omega(\mu_{ab}, \lambda, \lambda_c) \) is the counterpart of \( H^{s0}(\mu_{ab}, \lambda, \lambda_c) \)? Obviously, they must also satisfy the counterparts of eqs. (55), (57), (58), (61), (62) and (64) because in this case we would have all the symmetries (In fact we obtained the expression for \( \frac{\partial \Delta H}{\partial \mu} \), just imposing all the symmetries), that is

\[
\chi_{0,q,0,0}(\lambda) = 0.
\]

\[
\chi_{0,0,0,s}(\lambda) = 0 \quad \text{for} \quad s \geq 0.
\]

\[
\omega(0_{ab}, \lambda, 0_c) = 0.
\]

\[
\chi_{p,q,r,s} = \begin{cases} 
\chi_{0,q,0,0} + \frac{p}{2} r, s + \frac{q}{2} & \text{if } p \text{ is even} \\
\chi_{1,q,1,0} + \frac{p-1}{2} r, s + \frac{q-1}{2} & \text{if } p \text{ is odd}.
\end{cases}
\]

\[
\chi_{0,q,r+1,s+1} = \frac{\partial}{\partial \lambda} \chi_{1,q,r,s}, \quad \chi_{1,q,r+1,s+1} = \frac{\partial}{\partial \lambda} \chi_{0,q+1,r,s+1},
\]

\[
0 = (2Q + R + 1) \chi_{0,Q,R,s+1} + 2\lambda \chi_{0,Q+1,R,s+1} + 2R \chi_{1,Q+1,R-1,s},
\]

\[
0 = (2Q + R + 2) \chi_{1,Q,R,s+1} + 2\lambda \chi_{1,Q+1,R,s+1} + 2R \chi_{0,Q+2,R-1,s+1},
\]

A Appendix 1: The particular solution \( H = H_1 \).

Let us prove that \( H = H_1 \), with \( H_1 \) given by (38) and \( \psi_n \) constrained by (37), is a particular solution of (27) and (29).
In fact, by substituting (38) in (27)1, we obtain
\[
\frac{\partial^{r+p+1}}{\partial \lambda^r \partial \mu^{p+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right] = \frac{\partial^{r+p+2}}{\partial \lambda^r \partial \mu^{p+2}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right]
\]
which surely holds because \( \psi_{\frac{P+R}{2}} = \frac{\partial}{\partial \lambda} \psi_{\frac{P+R}{2}+1} \), thanks to (37).

By substituting (38) in (27)2, we obtain
\[
\frac{\partial^{r+p+2}}{\partial \lambda^r+1 \partial \mu^{p+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right] = \frac{\partial^{r+p+2}}{\partial \lambda^r+1 \partial \mu^{p+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right]
\]
which is an evident identity.

It is more delicate to verify (29). To do it, let us substitute (29) with its derivatives with respect to \( \mu_{i_1}, \cdots, \mu_{i_p}, \mu_{h_1k_1}, \cdots, \mu_{h_Qk_Q}, \lambda_{j_1}, \cdots, \lambda_{j_R} \); let us substitute (38) in the resulting equation and let us calculate the last form at equilibrium. We obtain
\[
0 = P \delta^i_{i_1} \delta_{(i_2 \cdots i_p k_1 \cdots h_Q k_Q j_1 \cdots j_R)} \left( P + 2Q + R + 1 \right)!! \frac{\partial^{R+P+1}}{\partial \lambda^R \partial \mu^{P+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right] + \\
+ 2Q \delta^h_{h_1} \delta_{(k_1 h_2 \cdots h_Q k_Q k_1 \cdots i_p j_1 \cdots j_R)} \left( P + 2Q + R + 1 \right)!! \frac{\partial^{R+P+1}}{\partial \lambda^R \partial \mu^{P+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right] + \\
+ 2\lambda \delta^i_{(i_1 h_1 k_1 \cdots h_Q k_Q i_1 \cdots i_p j_1 \cdots j_R)} \left( P + 2Q + R + 1 \right)!! \frac{\partial^{R+P+1}}{\partial \lambda^R \partial \mu^{P+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right] + \\
+ 2R \delta^i_{(k_1 h_1 \cdots h_Q k_Q i_1 \cdots i_p j_1 \cdots j_R)} \left( P + 2Q + R + 1 \right)!! \frac{\partial^{R+P}}{\partial \lambda^{R-1} \partial \mu^{P+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right] + \\
+ R \delta^i_{(i_1 \cdots i_p h_1 k_1 \cdots h_Q k_Q j_1 \cdots j_R)} \left( P + 2Q + R + 1 \right)!! \frac{\partial^{R+P+1}}{\partial \lambda^R \partial \mu^{P+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right],
\]
where overlined indexes denote symmetrization over those indexes, after that the other one (round brackets around indexes) has been taken.

Now, the first, second, fifth and sixth term can be put together so that the above expression becomes
\[
0 = \delta^i_{i_1} \delta_{(i_2 \cdots i_p k_1 \cdots h_Q k_Q j_1 \cdots j_R)} \left( P + 2Q + R + 1 \right)!! \frac{\partial^{R+P+1}}{\partial \lambda^R \partial \mu^{P+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right] + \\
+ (P + 2Q + R + 1)!! \delta_{(k_1 h_1 k_1 \cdots h_Q k_Q i_1 \cdots i_p j_1 \cdots j_R)} \left\{ 2\lambda \frac{\partial^{R+P+1}}{\partial \lambda^R \partial \mu^{P+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right] + \\
+ 2R \frac{\partial^{R+P}}{\partial \lambda^{R-1} \partial \mu^{P+1}} \left[ \left( -1 \right)^{q+1+\frac{P+R}{2}} \psi_{\frac{P+R}{2}} \right] \right\},
\]
which is satisfied as a consequence of the property

\[ \delta^{k_{1}k_{2}\cdots k_{h}Q_{1}\cdots Q_{j_{1}}\cdots j_{R}} = \delta^{k_{1}k_{2}\cdots k_{h}Q_{1}\cdots Q_{j_{1}}\cdots j_{R}} \]

and of the identity

\[ \partial \frac{R}{\partial \lambda} \left[ \left( -\frac{1}{2\lambda} \right)^{Q+\psi_{P_R}} \right] = \partial \frac{R}{\partial \lambda} \left[ -2\lambda \left( -\frac{1}{2\lambda} \right)^{Q+\psi_{P_R}} \right] = \]

\[ = -2\lambda \partial \frac{R}{\partial \lambda} \left[ \left( -\frac{1}{2\lambda} \right)^{Q+\psi_{P_R}} \right] - 2R \frac{R}{\partial \lambda} \frac{R}{\partial \lambda} \left[ \left( -\frac{1}{2\lambda} \right)^{Q+\psi_{P_R}} \right]. \]

This completes the proof that \( H = H_1 \) is a particular solution of (27) and (29).

**B Appendix 2: A further integration in the framework of the initial article.**

A further integration is possible for one combination of eqs. (44) of the paper [1]. In fact, the integrability condition on (44) of that paper allows us to obtain

\[ \partial \frac{R}{\partial \rho} h_4 = -2T^2 \partial \frac{\varepsilon}{\partial T} \partial \rho - 2T^3 \left( \frac{\partial \rho}{\partial \varepsilon} \right)^2 \left( 2T^2 \frac{h_2 + 5T^2 p}{3p} \right). \]  

(157)

(Here and in the sequel, we use the notation of [1]. For example, the scalars \( \beta_2, \beta_3 \) are different from the constants with the same name the present paper). After that, by using (44) and the present eq. (157), we obtain

\[ \frac{\partial}{\partial \rho} \left[ \beta_2 - \frac{5}{6} \beta_3 - \left( 4h_2 + \frac{10}{3} pT \right) \left( \varepsilon + \frac{p}{\rho} \right) \right] = 0. \]  

(158)

Consequently, \( \beta_2 - \frac{5}{6} \beta_3 - \left( 4h_2 + \frac{10}{3} pT \right) \left( \varepsilon + \frac{p}{\rho} \right) \) may depend only on temperature. Similarly, from (44) and the present eq. (157), we obtain

\[ \frac{\partial}{\partial T} \left[ \beta_2 - \frac{5}{6} \beta_3 - \left( 4h_2 + \frac{10}{3} pT \right) \left( \varepsilon + \frac{p}{\rho} \right) \right] = -\frac{1}{T} \left[ \beta_2 - \frac{5}{6} \beta_3 - \left( 4h_2 + \frac{10}{3} pT \right) \left( \varepsilon + \frac{p}{\rho} \right) \right], \]  

(159)

which is a differential equation for the unknown \( \beta_2 - \frac{5}{6} \beta_3 - \left( 4h_2 + \frac{10}{3} pT \right) \left( \varepsilon + \frac{p}{\rho} \right) \) whose solution is

\[ \beta_2 - \frac{5}{6} \beta_3 - \left( 4h_2 + \frac{10}{3} pT \right) \left( \varepsilon + \frac{p}{\rho} \right) = \frac{\text{constant}}{T}. \]  

(160)

Consequently, we have the symmetry of \( M_{ijk} \) at first order, if and only if this constant arising from integration is zero! On the other hand \( m_{ppik} \) at first order is already symmetric; eventual its skew-symmetric parts may appear at higher orders with respect to equilibrium.

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