Monads and Vector Bundles on Quadrics

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Abstract

We improve Ottaviani’s splitting criterion for vector bundles on a quadric hypersurface and obtain the equivalent of the result by Rao, Mohan Kumar and Peterson. Then we give the classification of rank 2 bundles without ”inner” cohomology on $Q_n$ ($n > 3$). It surprisingly exactly agrees with the classification by Ancona, Peternell and Wisniewski of rank 2 Fano bundles.

Introduction

A monad on $P^n$ or, more generally, on a projective variety $X$, is a complex of three vector bundles

$$0 \to \mathcal{A} \overset{\alpha}{\to} \mathcal{B} \overset{\beta}{\to} \mathcal{C} \to 0$$

such that $\alpha$ is injective and $\beta$ is surjective. Monads have been studied by Horrocks, who proved (see [Ho] or [BH]) that every vector bundle on $P^n$ is the homology of a suitable minimal monad. Throughout the paper we often use the Horrocks correspondence between a bundle $\mathcal{E}$ on $P^n$ ($n \geq 3$) and the corresponding minimal monad

$$0 \to \mathcal{A} \overset{\alpha}{\to} \mathcal{B} \overset{\beta}{\to} \mathcal{C} \to 0,$$

where $\mathcal{A}$ and $\mathcal{C}$ are sums of line bundles and $\mathcal{B}$ satisfies:

1. $H^i_{\ast}(\mathcal{B}) = H^{n-1}_{\ast}(\mathcal{B}) = 0$
2. $H^i_{\ast}(\mathcal{B}) = H^i_{\ast}(\mathcal{E}) \quad \forall i, 1 < i < n - 1.$

This correspondence holds also on $X$ ($\dim X \geq 3$). Indeed the proof of the result in ([BH] proposition 3) can be easily extended to $X$ (see [MI] theorem 2.1.6.).

Rao, Mohan Kumar and Peterson have successfully used this tool to investigate the intermediate cohomology modules of a vector bundle on $P^n$ and give cohomological splitting conditions (see [KPR1]).

The first aim of the present paper is to extend to smooth quadric hypersurfaces the above result by Rao, Mohan Kumar and Peterson. In $Q_n$, the Horrocks criterion does not work, but there is a theorem that classifies all the ACM bundles (see [Kn]) as direct sums of line bundles.

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bundles and spinor bundles (up to a twist - for generalities about spinor bundles see [Ot2]).

In the first section we prove some necessary conditions that a minimal monad associated to a bundle \( E \) must satisfy.

The second aim of this paper is the improvement of Ottaviani’s splitting criterion (see [Ot1] and [Ot3]): we obtain the equivalent of the result by Rao, Mohan Kumar and Peterson on a quadric hypersurface. In the last section we focus our interest on rank two vector bundles on \( Q_4 \) and prove the following theorem, which is our main result:

For an indecomposable rank 2 bundle \( E \) on \( Q_4 \) with \( H^1(E) \neq 0 \) and \( H^2(E) = 0 \), the only possible minimal monad, such that both \( A \) and \( C \) do not vanish, is (up to a twist)

\[
0 \rightarrow O \rightarrow S'(1) \oplus S''(1) \rightarrow O(1) \rightarrow 0,
\]

(1)

and such a monad exists.

This means that the two spinor bundles and the bundle corresponding to this monad are the only rank 2 bundles without “inner” cohomology (i.e. \( H^2(E) = ... = H^{n-2}(E) = 0 \)). By using monads again we can also understand the behavior of rank two bundles on \( Q_5 \) and also on \( Q_n, n > 5 \). More precisely we can prove that:

1. For an indecomposable rank 2 bundle \( E \) on \( Q_5 \) with \( H^2(E) = 0 \) and \( H^3(E) = 0 \), the only possible minimal monad, such that both \( A \) and \( C \) do not vanish, is (up to a twist)

\[
0 \rightarrow O \rightarrow S_5(1) \rightarrow O(1) \rightarrow 0,
\]

and such a monad exists.

2. For \( n > 5 \), there is no indecomposable bundle of rank 2 on \( Q_n \) with \( H^2(E) = ... = H^{n-2}(E) = 0 \).

It is surprising that this classification of rank 2 bundle on \( P^n \) and \( Q_n (n > 3) \) exactly agrees with the classification by Ancona, Peternell and Wisniewski of rank 2 Fano bundles (see [APW]).

We can say that if \( E \) is a rank 2 bundle on \( P^n \) and \( Q_n (n > 3) \), then

\( E \) is a Fano bundle \( \iff \) \( E \) is without inner cohomology.

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1 Monads for Bundles without inner cohomology

In this section \( X \) denotes a nonsingular subcanonical, irreducible ACM projective variety.

If \( M \) is a finitely generated module over the homogeneous coordinate ring of \( X \), we denote by \( \beta_i(M) \) the total Betti numbers of \( M \).

We say that a bundle is indecomposable if it does not split as a direct sum of line bundles.
Definition 1.1. We will call bundle without inner cohomology a bundle $E$ on $X$ with
$$H^2_*(E) = \cdots = H^{n-2}_*(E) = 0,$$
where $n = \dim X$.

In $\mathbb{P}^n$ Kumar Peterson and Rao showed that, if $n$ is even and $\text{rank}(E) < n$ (or if $n$ is odd and $\text{rank}(E) < n - 1$), and
$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$$
is a minimal monad for $E$ such that $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are not zero, then $\mathcal{B}$ cannot split.
This means that $E$ splits if and only if it is without inner cohomology.

On $X$ we are able to prove the first part of the theorem about monads:

Theorem 1.2. Let $E$ be a vector bundle on $X$ of dimension $n$, with $n > 3$.

1. If $n$ is even and if
$$\text{rank}(E) < n$$
then no minimal monad for $E$ exists such that $\mathcal{A}$ or $\mathcal{C}$ are not zero and $\mathcal{B}$ is split.

2. If $n$ is odd and if
$$\text{rank}(E) < n - 1$$
then no minimal monad for $E$ exists such that $\mathcal{A}$ or $\mathcal{C}$ are not zero and $\mathcal{B}$ is split.

First of all we prove a simple and useful lemma:

Lemma 1.3. Let $E$ be a bundle on $X$ with $H^2_*(E) = H^{n-2}_*(E) = 0$ where $n = \dim X > 3$ and let $H$ be a hyperplane such that $X' = X \cap H$ is a subcanonical, irreducible, ACM, nonsingular projective variety. (use Bertini’s theorem for irreducibility).

If
$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$
is a minimal monad for $E$, then a minimal monad for the restriction $E|_{X'}$ is just the restriction
$$0 \to \mathcal{A}|_{X'} \to \mathcal{B}|_{X'} \to \mathcal{C}|_{X'} \to 0.$$ 

Proof. From the sequence
$$0 \to E(-1) \to E \to E|_{X'} \to 0,$$
and the corresponding sequence in cohomology
$$H^2_*(E(-1)) \to H^1_*(E) \xrightarrow{\gamma} H^1_*(E|_{X'}) \to H^2_*(E(-1)) = 0$$
we see that the map $\gamma$ is surjective. Then the module $H^1_*(E|_{X'})$ has the same generators of $H^1_*(E)$ of the same degrees restricted to $X'$ and this means that, if
$$0 \to \mathcal{A'} \xrightarrow{\alpha} \mathcal{B'} \xrightarrow{\beta} \mathcal{C'} \to 0$$
is a minimal monad for $E|_{X'}$, then
$$\mathcal{C'} \cong \mathcal{C}_{X'}.$$
In the same way, by using the fact that $H^{n-2}_*(E) = 0$, we see that
\[ A' \simeq A|_{X'} \].
Then, by construction we see that also
\[ B' \simeq B|_{X'} \].

Proof. (of theorem 1.2)
Let us suppose that we know the result of the theorem for $n$ even. Let $E$ be a bundle on $X$ with
\[ \text{rank}(E) < n - 1, \]
n > 3, $n$ odd. Let us also suppose that we have a minimal monad
\[ 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0, \]
where $A$ and $C$ not zero and $B$ splits.
Let $H$ be any hyperplane such that $X' = X \cap H$ is a subcanonical, irreducible, ACM projective variety. By (1.3) we have that
\[ 0 \to A|_{X'} \to B|_{X'} \to C|_{X'} \to 0 \]
is the minimal monad for $E|_{X'}$, where $E|_{X'}$ is a bundle of rank $< n - 1$, and $n - 1 = \dim X$ is even.
Now if $B$ splits also $B|_{X'}$ has to split and this is against our assumption of the result of the theorem for $n$ even.
Thus, establishing the result of the theorem for the case of $n$ even will also establish the result for $n$ odd.

Now if one of $A$ or $C$ is zero, and $B$ splits, then either $E$ or its dual is a first syzygy module. In this case, $E$ must have rank at least $n$ by the following argument.
Assume that $C$ is zero and let $r$ be the rank of $E$. From the short exact sequence
\[ 0 \to A \to B \to E \to 0 \]
we get the exact sequence
\[ 0 \to S^r A \xrightarrow{\gamma_r} S^{r-1} A \otimes B \xrightarrow{\gamma_{r-1}} \ldots \xrightarrow{\gamma_1} S^1 A \otimes B \xrightarrow{\gamma_0} B \to \bigwedge^r E \to 0. \]
If we put
\[ \Gamma_i = ker \gamma_i, \]
we see that
\[ H^r_*(S^r A) = H^r_*(\Gamma_{r-1}) = \ldots = H^1_*(\Gamma_{i-1}) = \ldots = H^1_*(\Gamma_0) \]
and $\forall 0 < j < r,$
\[ H^{r-j}_*(S^r A) = H^{r-j}_*(\Gamma_{r-1}) = \ldots = H^{r-j}_*(\Gamma_{i-1}) = \ldots = H^{r-j}_*(\Gamma_j). \]
When \( r < n \),
\[ H^r_\ast(S^r \mathcal{A}) = 0, \]
so
\[ H^1_\ast(\Gamma_0) = 0 \]
and
\[ H^0_\ast(\wedge^r \mathcal{B}) \to H^0_\ast(\wedge^r \mathcal{E}) \]
is a surjective map between free modules.
This means that the map \( \gamma_0 \) splits and the bundle \( \Gamma_0 \) is a direct sum of line bundles.
Now, since also
\[ H^{r-1}_\ast(S^r \mathcal{A}) = \cdots = H^1_\ast(S^r \mathcal{A}) = 0, \]
we have
\[ H^1_\ast(\Gamma_1) = \cdots = H^1_\ast(\Gamma_{r-1}) = 0. \]
We consider, then, the short exact sequence
\[ 0 \to \Gamma_1 \to S^1 \mathcal{A} \otimes \wedge^{r-1} \mathcal{B} \to \Gamma_0 \to 0. \]
Since \( \Gamma_0 \) is free and
\[ H^1_\ast(\Gamma_1) = 0 \]
we have that also this sequence splits and, hence, the map \( \gamma_1 \) splits and the bundle \( \Gamma_1 \) is a sum of line bundles.
By iterating this argument we can conclude that the long exact sequence is split at each place.
In particular, the map
\[ S^r \mathcal{A} \to S^{r-1} \mathcal{A} \otimes \wedge^1 \mathcal{B}, \]
which is obtained from \( \alpha \) as
\[ a_1 a_2 \ldots a_r \to \sum (\pm a_1 a_2 \ldots \hat{a}_i \ldots a_r \otimes \alpha(a_i)), \]
is split.
This goes against the minimality of the monad.
Suppose now that \( \mathcal{A} \) and \( \mathcal{C} \) are both not zero and \( n \) is even with \( n = 2k \).
Let \( \mathcal{E} \) be a bundle on \( X \) with
\[ \text{rank}(\mathcal{E}) \leq n - 1. \]
By adding line bundles to \( \mathcal{E} \) (if necessary), we may suppose that
\[ \text{rank}(\mathcal{E}) = n - 1. \]
Now we can follow the proof in \((\text{KPR1}) \) pages 7-8] and see that such a monad
\[ 0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0 \]
cannot exist. \( \square \)
Remark 1.4. The Kumar-Peterson-Rao theorem tells us that on $\mathbb{P}^n$ there is no indecomposable bundle without inner cohomology with small rank.

In a more general space $X$ we cannot say that because the Horrocks theorem fails. But is still true the following:

In a minimal monad

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

for a bundle without inner cohomology on $X$, the bundle $B$ must be ACM and indecomposable.

Now we prove a theorem about minimal monads for rank 2 bundles:

**Theorem 1.5.** Let $X$ be of dimension $n > 3$, and $E$ a rank 2 bundle with $H^2_*(E) = 0$. Then any minimal monad

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

for $E$, such that $A, B$ and $C$ are not zero, must satisfy the following conditions:

1. $H^1_*(\wedge^2 B) \neq 0$ and $\beta_0(H^1_*(\wedge^2 B)) \geq \beta_0(H^0_*(S^2C))$.
2. $H^2_*(\wedge^2 B) = 0$

**Proof.** First of all, since $X$ is ACM, the sheaf $\mathcal{O}_X$ does not have intermediate cohomology. The same is true for $A$ and $C$ that are free $\mathcal{O}_X$-modules.

Let us now assume the existence of a minimal monad with $H^1_*(\wedge^2 B) = 0$

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0.$$

Then, if $G = \ker \beta$, from the sequence

$$0 \to S^2A \to (A \otimes G) \to \wedge^2G \to \wedge^2E \to 0,$$

we have

$$H^2_*(\wedge^2 G) = H^2_*(A \otimes G) = 0,$$

since $H^2_*(B) = H^2_*(G) = H^2_*(E) = 0$.

Moreover, from the sequence

$$0 \to \wedge^2G \to \wedge^2B \to B \otimes C \to S^2C \to 0,$$

passing to the exact sequence of maps on cohomology groups, since $H^1_*(\wedge^2 B) = H^2_*(\wedge^2 G) = 0$, we get

$$H^0_*(B \otimes C) \to H^0_*(S^2C) \to 0.$$

Now, if we call $S_X$ the coordinate ring, we can say that $H^0_*(S^2C)$ is a free $S_X$-module, hence projective; then there exists a map

$$H^0_*(B \otimes C) \leftarrow H^0_*(S^2C)$$

and this means that

$$B \otimes C \to S^2C \to 0$$

splits.

But this map is obtained from $\beta$ as $b \otimes c \mapsto \beta(b)c$, so if it splits also $\beta$ has to split and this
violates the minimality of the monad. We can say something stronger.

From the sequence
$$0 \to \wedge^2 \mathcal{G} \to \wedge^2 \mathcal{B} \to \mathcal{B} \otimes \mathcal{C} \xrightarrow{\gamma} S_2 \mathcal{C} \to 0,$$
since $H^2_*(\wedge^2 \mathcal{G}) = 0$, we have a surjective map
$$H^1_*(\wedge^2 \mathcal{B}) \to H^1_*(\Gamma) \to 0$$
where $\Gamma = \ker \gamma$, and then
$$\beta_0(H^1_*(\wedge^2 \mathcal{B})) \geq \beta_0(H^0_*(\Gamma)).$$

On the other hand we have the sequence
$$H^0_*(\mathcal{B} \otimes \mathcal{C}) \xrightarrow{\gamma} H^0_*(S_2 \mathcal{C}) \to H^1_*(\Gamma) \to 0;$$
so, if
$$\beta_0(H^1_*(\wedge^2 \mathcal{B})) < \beta_0(H^0_*(S_2 \mathcal{C})), $$
also
$$\beta_0(H^1_*(\Gamma)) < \beta_0(H^0_*(S_2 \mathcal{C})), $$
and some of the generators of $H^0_*(S_2 \mathcal{C})$ must be in the image of $\gamma$.

But $\gamma$ is obtained from $\beta$ as $b \otimes c \mapsto \beta(b)c$, so also some generators of $\mathcal{C}$ must be in the image of $\beta$ and this contradicts the minimality of the monad.

We conclude that not just $H^1_*(\wedge^2 \mathcal{B})$ has to be non zero but also
$$\beta_0(H^1_*(\wedge^2 \mathcal{B})) \geq \beta_0(H^0_*(S_2 \mathcal{C})).$$

The second condition comes from the sequence
$$0 \to \wedge^2 \mathcal{G} \to \wedge^2 \mathcal{B} \to \mathcal{B} \otimes \mathcal{C} \to S_2 \mathcal{C} \to 0,$$
since $H^2_*(\wedge^2 \mathcal{G}) = H^2_*(\mathcal{B} \otimes \mathcal{C}) = 0.

\[\square\]

2 \ Splitting Criteria on $Q_n$

In this section we apply our results to a smooth quadric hypersurface $Q_n$ in $\mathbb{P}^{n+1}$. Let us notice that $Q_n$ is a nonsingular, ACM, irreducible projective variety and, if $n > 3$, we also have
$$\text{Pic}(Q_n) = \mathbb{Z},$$
so it satisfies all the conditions of $X$.

First of all we need a useful remark about spinor bundles:

\textbf{Remark 2.1.} By applying ([OT2] Lemma 2.7. and Theorem 2.8) we have that if $n = 2m + 1$,
$$h^0(Q_n, S(1) \otimes \mathcal{S}) = 1.$$ 
So from the sequence
$$0 \to \mathcal{S} \otimes \mathcal{S} \to \mathcal{O}_{Q_n}^{2m+1} \otimes \mathcal{S} \to S(1) \otimes \mathcal{S} \to 0,$$
and the sequence in cohomology

\[ 0 = H^0(O_{\mathbb{P}^n}^{2m+1} \otimes S) \to H^0(S(1) \otimes S) \to H^1(S \otimes S) \to 0 \]

we see that

\[ h^0(Q_n, S(1) \otimes S) = h^1(Q_n, S \otimes S) = 1. \]

Moreover, if \( n = 4m \), we have

\[ h^0(Q_n, S'(1) \otimes S') = h^0(Q_n, S''(1) \otimes S'') = 1 \]

and

\[ h^0(Q_n, S'(1) \otimes S'') = h^0(Q_n, S''(1) \otimes S') = 0. \]

Then

\[ h^1(Q_n, S'' \otimes S') = 1 \]

and

\[ h^1(Q_n, S'' \otimes S'') = h^1(Q_n, S' \otimes S') = 0 \]

while, if \( n = 4m + 2 \),

\[ h^0(Q_n, S'(1) \otimes S') = h^0(Q_n, S''(1) \otimes S'') = 0 \]

and

\[ h^0(Q_n, S'(1) \otimes S'') = h^0(Q_n, S''(1) \otimes S') = 1. \]

Then

\[ h^1(Q_n, S'' \otimes S') = 0 \]

and

\[ h^1(Q_n, S'' \otimes S'') = h^1(Q_n, S' \otimes S') = 1. \]

Our starting point is the splitting criterion of Ottaviani (see [Ot1] or [Ot3]). By using monads we can improve this criterion in the case of bundle with a small rank:

**Theorem 2.2.** Let \( E \) a vector bundle on \( Q_n \) (\( n > 3 \)). If \( n \) is odd, \( S \) the spinor bundle and rank \( E < n - 1 \), then \( E \) splits if and only if

1. \( H^i_*(Q_n, E) = 0 \) for \( 2 \leq i \leq n-2 \)
2. \( H^1_*(Q_n, E \otimes S) = 0. \)

If \( n \) is even, \( S' \) and \( S'' \) are the two spinor bundles and rank \( E < n \), then \( E \) splits if and only if

1. \( H^i_*(Q_n, E) = 0 \) for \( 2 \leq i \leq n-2 \)
2. \( H^1_*(Q_n, E \otimes S') = H^1_*(E \otimes S'') = 0. \)
Proof. Let us assume that $E$ does not split and let us consider a minimal monad for $E$,

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0.$$ 

Since $H^i_*(Q_n, E) = 0$ for $2 \leq i \leq n - 2$, by (1.4), $B$ is an ACM bundle on $Q_n$ and, it has to be isomorphic to a direct sum of line bundles and spinor bundles twisted by some $O(t)$. If $S$ is a spinor bundle and $H^1_*(E \otimes S) = 0$, from the two sequences

$$0 \to G \otimes S \to B \otimes S \to C \otimes S \to 0$$

and

$$0 \to A \otimes S \to G \otimes S \to E \otimes S \to 0,$$

we can see that also $H^1_*(B \otimes S) = 0$.

Now, in the odd case, since $H^1_*(S \otimes S) \neq 0$ see (2.1), we can say that no spinor bundle can appear in $B$. So $B$ has to split and this is a contradiction.

In the even case, since, according with (2.1), when $n \equiv 2 \pmod{4}$,

$$H^1_*(S' \otimes S') \neq 0$$

and

$$H^1_*(S'' \otimes S'') \neq 0,$$

or, when $n \equiv 0 \pmod{4}$,

$$H^1_*(S' \otimes S'') \neq 0,$$

we can say that no spinor bundles can appear in $B$. So $B$ has to split and this is a contradiction.

This theorem is the equivalent in $Q_n$ of the result by Kumar, Peterson and Rao.

Remark 2.3. The techniques of this proof are similar to those used by Arrondo and Graña on the Grassmannian $G(1, 4)$ (see [AG]).

3 Rank 2 Bundles without Inner Cohomology

Let us study more carefully the rank 2 bundles in $Q_n$ ($n > 3$).

In $Q_4$ by (2.1) we have that

$$H^1_*(S' \otimes S') = H^1_*(S'' \otimes S'') = 0$$

and

$$H^1_*(S' \otimes S'') = C.$$

So from the sequence (see [Or2])

$$0 \to S' \to O_{Q_n}^{\oplus 4} \to S''(1) \to 0,$$

and his dual we see that

$$H^2_*(S' \otimes S') = H^2_*(S'' \otimes S'') = C,$$

It is then possible to prove the following theorem:
Theorem 3.1. For an indecomposable rank 2 bundle $\mathcal{E}$ on $\mathbb{Q}_4$ with $H^1_1(\mathcal{E}) \neq 0$ and $H^2_2(\mathcal{E}) = 0$, the only possible minimal monad with $A$ or $C$ different from zero is (up to a twist)

$$0 \to O \to S'(1) \oplus S''(1) \to O(1) \to 0,$$

and such a monad exists.

Proof. First of all in a minimal monad for $\mathcal{E}$,

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0,$$

$B$ is an ACM bundle on $\mathbb{Q}_4$; then it has to be isomorphic to a direct sum of line bundles and spinor bundles twisted by some $O(t)$.

Since $B$ cannot split at least a spinor bundle must appear.

Assume that just one copy of $S'$ or one copy of $S''$ it appears in $B$. Since

$$\text{rank } S'' = \text{rank } S' = 2$$

and then $\wedge^2 S'$ and $\wedge^2 S''$ are line bundles, also the bundle $\wedge^2 B$ is ACM and the condition

$$H^1_1(\wedge^2 B) \neq 0,$$

in (1.5), is not satisfied.

Assume that more than one copy of $S'$ or more than one copy of $S''$ appears in $B$. Then in the bundle $\wedge^2 B$, $(S' \otimes S')(t)$ or $(S'' \otimes S'')(t)$ appears and, since

$$H^2_2(S' \otimes S') = H^2_2(S'' \otimes S'') = C,$$

the condition

$$H^2_2(\wedge^2 B) = 0$$

in (1.5), fails to be satisfied. So $B$ must contain both $S'$ and $S''$ with some twist and only one copy of each. We can conclude that $B$ has to be of the form

$$(\bigoplus_i O(a_i)) \oplus (S'(b)) \oplus (S''(c)).$$

Let us notice furthermore that if $H^1_1(\mathcal{E})$ has more than 1 generator, rank $C > 1$ and $H^0_0(S_2 C)$ has at least 3 generators.

But

$$H^1_1(\wedge^2 B) \simeq H^1_1(S' \otimes S'') = C$$

has just 1 generator and this is a contradiction because by (1.5)

$$\beta_0(H^1_1(\wedge^2 B)) \geq \beta_0(H^0_0(S_2 C)).$$

This means that rank $A = \text{rank } C = 1$.

At this point the only possible minimal monads are like

$$0 \to O(-a + c_1(\mathcal{E})) \to S'(b) \oplus S''(c) \to O(a) \to 0.$$
where $a, b$ and $c$ are integer numbers.

Since $\mathcal{B}$ must be isomorphic to $B^\vee(c_1(\mathcal{E}))$ and $S^\vee \cong S(1)$ and $S'^\vee \cong S''(1)$, we have that

$$b = c = \frac{1 + c_1(\mathcal{E})}{2};$$

this means that $c_1(\mathcal{E})$ must be odd so we can assume $c_1(\mathcal{E}) = -1$ and $b = c = 0$. Now our monad, twisted by $\mathcal{O}(a + 1)$ looks like

$$0 \to \mathcal{O} \xrightarrow{\alpha} S'(a + 1) \oplus S''(a + 1) \to \mathcal{O}(2a + 1) \to 0$$

and we can assume $a \geq 0$ because both $S'(l)$ and $S''(l)$ do have sections only if $l \geq 1$.

It is possible to have an injective map $\alpha$ at level of bundles only if

$$c_4(S'(a + 1) \oplus S''(a + 1)) = c_4(S'^\vee(a) \oplus S''^\vee(a)) = 0.$$

Our goal now is to find the values of $a$ such that this condition is satisfied.

We know (see [Fr]) the intersection ring of $\mathcal{Q}_4$:

$$A^\vee(\mathcal{Q}_4) = \mathbb{Z}e_1 \oplus (\mathbb{Z}e_2 \oplus \mathbb{Z}e_2') \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4.$$  

We also know that $c_1(S'^\vee) = c_1(S''^\vee) = 1$, $c_2(S'^\vee) = (1, 0) = e_2$ and $c_2(S''^\vee) = (0, 1) = e_2'$. Then

$$c_2(S'^\vee(a)) = e_2 + ae_1 * (1)e_1 + ae_1 * ae_1 = (1 + a + a^2)e_2 + (a + a^2)e_2'$$

and

$$c_2(S''^\vee(a)) = e_2' + ae_1 * (1)e_1 + ae_1 * ae_1 = (a + a^2)e_2 + (1 + a + a^2)e_2';$$

so

$$c_4(S'^\vee(a) \oplus S''^\vee(a)) = c_2(S'^\vee(a)) \ast c_2(S''^\vee(a)) =$$

$$(1 + a + a^2)(a + a^2)e_4 + (a + a^2)(1 + a + a^2)e_4 = 2(1 + a + a^2)(a + a^2)e_4$$

This is zero if and only if $a = 0$ or $a = -1$ and we can not accept the last case. For $a = 0$ we have the claimed monad

$$0 \to \mathcal{O} \xrightarrow{\alpha} S'(1) \oplus S''(1) \xrightarrow{\beta} \mathcal{O}(1) \to 0.$$  

We finally want to prove that such a monad exists.

We denote by $\mathcal{Z}_4(1)$ the homology of our monad. We compute $c_1(\mathcal{Z}_4) = -1$, $c_2(\mathcal{Z}_4) = (1, 1)$ and $H^0(\mathcal{Z}_4) = 0$ and by ([AS] Proposition p. 205) we can conclude that the bundle $\mathcal{Z}_4$ lies in a sequence

$$0 \to \mathcal{O} \to \mathcal{Z}_4(1) \to \mathcal{I}_Y(1) \to 0$$

where $Y$ is the disjoint union of a plane in $\Lambda$ and a plane in $\Lambda'$, the two families of planes in $\mathcal{Q}_4$.

We can hence conclude that our monad exists because it is the homology of a well known bundle. $\square$

**Remark 3.2.** We can say then that there exist only three rank 2 bundles without inner cohomology in $\mathcal{Q}_4$. They are $S$, $S'$ and $\mathcal{Z}_4$ that is associated, by the Serre correspondence, to two disjoint planes, one in $\Lambda$ and one in $\Lambda'$. 

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Corollary 3.3. In higher dimension we have:

1. For an indecomposable rank 2 bundle $E$ on $Q_5$ with $H^2_*(E) = 0$ and $H^3_*(E) = 0$, the only possible minimal monad with $A$ or $C$ not zero is (up to a twist)

$$0 \rightarrow O \rightarrow S_5(1) \rightarrow O(1) \rightarrow 0.$$ and such a monad exists.

2. For $n > 5$, no indecomposable bundle of rank 2 in $Q_n$ exists with $H^2_*(E) = \ldots = H^{n-2}_*(E) = 0$.

Proof. First of all let us notice that for $n > 4$ there is no indecomposable ACM rank 2 bundle since the spinor bundles have rank greater than 2.

Let us then assume that $H^1_*(E) \neq 0$ and let us see how many minimal monads it is possible to find:

1. In a minimal monad for $E$ in $Q_5$,

$$0 \rightarrow A \overset{\alpha}{\rightarrow} B \overset{\beta}{\rightarrow} C \rightarrow 0,$$

$B$ is an ACM bundle on $Q_5$; then it has to be isomorphic to a direct sum of line bundles and spinor bundles twisted by some $O(t)$.

Moreover, since $H^2_*(E) = 0$ and $H^3_*(E) = 0$, $E|_{Q_4} = F$ is a bundle with $H^2_*(F) = 0$ and for (1.3) his minimal monad is just the restriction of the minimal monad for $E$

$$0 \rightarrow A \overset{\alpha}{\rightarrow} B \overset{\beta}{\rightarrow} C \rightarrow 0.$$

For the theorem above, hence, this minimal monad must be

$$0 \rightarrow O \rightarrow S'(1) \oplus S''(1) \rightarrow O(1) \rightarrow 0.$$ 

Now, since

$$S_5|_{Q_4} \simeq S' \oplus S'',$$

the only bundle of the form

$$(\bigoplus_i O(a_i)) \oplus (\bigoplus_j S_5(b_j))$$

having $S'(1) \oplus S''(1)$ as restriction on $Q_4$ is $S_5(1)$ and then the claimed monad

$$0 \rightarrow O \overset{\alpha}{\rightarrow} S_5(1) \overset{\beta}{\rightarrow} O(1) \rightarrow 0$$

is the only possible.

We finally want to prove that such a monad exists.

We denote by $Z_5(1)$ the homology of our monad.

We compute $c_1(Z_5) = -1$, $c_2(Z_5) = 1$ and $H^0(Z_5) = 0$ and by ([Ot4] Main Theorem p. 88) we can conclude that the bundle $Z_5$ is a Cayley bundle (see [Ot4] for generalities on Cayley bundles).

The bundle $Z_5$ appear also in [13] and [KPR2].

We can hence conclude that our monad exists because it is the homology of a well known bundle.
2. In \( Q_6 \) we use the same argument but, since \( S'_6 \mid Q_5 \simeq S'_5 \) and also \( S''_6 \mid Q_5 \simeq S''_5 \), we have two possible minimal monads:

\[ 0 \to O \to S'_6(1) \to O(1) \to 0 \]

and

\[ 0 \to O \to S''_6(1) \to O(1) \to 0. \]

In both sequences the condition

\[ B \simeq B^\vee(c_1) \]

is not satisfied, since \( S'_6 \mid Q_5 \simeq S''_6(1) \) and \( S''_6 \mid Q_5 \simeq S'_6(1) \).

So they cannot be the minimal monads of a rank 2 bundles.

We can conclude that no indecomposable bundle of rank 2 in \( Q_6 \) exists with \( H^2(E) = \cdots = H^4(E) = 0 \) and clearly also in higher dimension it is not possible to find any bundle without inner cohomology.

\[ \square \]

As a conclusion, the Kumar-Peterson-Rao theorem tells us that in \( \mathbb{P}^n \) with \( n > 3 \) there are no rank 2 bundles without inner cohomology while in \( Q_n \) with \( n > 3 \) there are 4 of them: precisely 3 in \( Q_4 \) and 1 in \( Q_5 \).

It is surprising that this classification of rank 2 bundle on \( \mathbb{P}^n \) and \( Q_n \) (\( n > 3 \)) exactly agrees with the classification by Ancona, Peternell and Wisniewski of rank 2 Fano bundles (see [APW]).

**Theorem 3.4** (Ancona, Peternell and Wisniewski). Let \( E \) be a rank 2 Fano bundle on \( \mathbb{P}^n \) (\( n > 3 \)). Then \( E \) splits.

Let \( E \) be a rank 2 Fano bundle on \( Q_n \) (\( n > 3 \)). Then either \( E \) splits or:

1. \( n = 4 \) and \( E \) is (up to twist) a spinor bundle or the bundle \( Z_4 \).
2. \( n = 5 \) and \( E \) is (up to twist) a Cayley bundle.

**Corollary 3.5.** If \( E \) is a rank 2 bundle on \( \mathbb{P}^n \) and \( Q_n \) (\( n > 3 \)), then

\[ E \text{ is a Fano bundle } \iff \text{E is without inner cohomology.} \]

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