In this paper, a Sturm–Liouville boundary value problem which includes conformable fractional derivatives of order $\alpha$, $0 < \alpha \leq 1$, is considered. We give some uniqueness theorems for the solutions of inverse problems according to the Weyl function, two given spectra and classical spectral data. We also study the half-inverse problem and prove a Hochstadt–Lieberman-type theorem.

**Keywords:** Inverse problem, conformable fractional derivatives, Weyl function, Hochstadt–Lieberman theorem

**MSC 2010:** 31B20, 34A55, 34B24, 26A33

**1 Introduction**

The fractional calculus is one of the most important mathematical fields. Over the last decade, different types of fractional derivatives have been tried to be given because of the benefits provided by this concept for modelling applied problems (see [26, 30] and the references therein). In 2014, Khalil, Al Horani, Yousef and Sababheh introduced conformable fractional derivatives [18]. One year later, Abdeljawad gave the conformable fractional versions of some important concepts, e.g., the chain rule, exponential functions, Gronwall’s inequality, integration by parts, Taylor power series expansions etc. [1]. Also, other basic properties of this type of derivation can be found in [5]. It seems to satisfy all the requirements of the standard derivative. Because of its effectiveness and applicability, conformable derivatives have received a lot of attention and have been applied quickly in various areas.

Fractional Sturm–Liouville problems are different from those defined classically. This kind of problems include derivatives of fractional order instead of the ordinary derivatives in a traditional problem. In recent years, some new fractional Sturm–Liouville problems have been studied (see [2, 3, 19, 20, 33]). These problems appear in applied sciences (see [6, 23, 28, 32, 35]).

Inverse spectral problems for Sturm–Liouville operators generated with ordinary derivation have been studied for about ninety years (see [4, 7–17, 21, 22, 24, 25, 27, 31, 34, 36] and the references therein). This kind of problems consists in recovering the coefficients of an operator from some given data; for example, the Weyl function, spectral function, nodal points and some special sequences which consist of some spectral values. However, we can mention only one study about inverse spectral problems for conformable fractional Sturm–Liouville (CFSL) operators. Mortazaasl and Akbarfam give a solution of inverse nodal problems for conformable fractional Sturm–Liouville operators in [29].
In the present paper, we consider a conformable fractional Sturm–Liouville boundary value problem and give uniqueness theorems for the solution of the inverse problem according to the Weyl function, two sets of eigenvalues and the sequences which consist of eigenvalues and norming constants. We also study an half-inverse problem and prove a Hochstadt and Lieberman-type theorem.

2 Preliminaries

Before presenting our main results, we recall some important concepts of the conformable fractional calculus theory.

Definition 2.1. Let \( f : [0, \infty) \to \mathbb{R} \) be a given function. Then the conformable fractional derivative of order \( 0 < \alpha \leq 1 \) of \( f \) at \( x > 0 \) is defined by
\[
D^\alpha f(x) = \lim_{h \to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h},
\]
and the fractional derivative at \( 0 \) is defined by \( D^\alpha f(0) = \lim_{x \to 0^+} D^\alpha f(x) \).

Definition 2.2. Let \( f : [0, \infty) \to \mathbb{R} \) be a given function. The conformable fractional integral of \( f \) of order \( \alpha \) is defined by
\[
I^\alpha f(x) = \int_0^x f(t) \, dt = \int_0^x t^{\alpha-1} f(t) \, dt
\]
for all \( x > 0 \).

We collect some necessary relations in the following lemma.

Lemma 2.3. Let \( f, g \) be \( \alpha \)-differentiable at \( x, x > 0 \). Then the following assertions hold:
(i) \( D^\alpha (af + bg) = aD^\alpha f + bD^\alpha g \) for all \( a, b \in \mathbb{R} \).
(ii) \( D^\alpha (x^\alpha) = ax^{\alpha-1} \) for all \( a \in \mathbb{R} \).
(iii) \( D^\alpha (c) = 0 \) (\( c \) is a constant).
(iv) \( D^\alpha (fg) = D^\alpha f g + fD^\alpha g \).
(v) One has
\[
D^\alpha (fg) = \frac{D^\alpha f g - fD^\alpha g}{g^2}.
\]
(vi) If \( f \) is a continuous function, then for all \( x > 0 \) we have \( D^\alpha I^\alpha f(x) = f(x) \).
(vii) If \( f \) is a differentiable function, then we have \( D^\alpha f(x) = x^{1-\alpha} f'(x) \).

Lemma 2.4. Let \( f, g : (0, \infty) \to \mathbb{R} \) be \( \alpha \)-differentiable functions and let \( h(x) = f(g(x)) \). Then \( h(x) \) is \( \alpha \)-differentiable and for all \( x \neq 0 \) and \( g(x) \neq 0 \),
\[
(D^\alpha h)(x) = (D^\alpha f)(g(x))(D^\alpha g)(x)g^{\alpha-1}(x),
\]
if \( x = 0 \), then \( (D^\alpha h)(0) = \lim_{x \to 0^+}(D^\alpha f)(g(x))(D^\alpha g)(x)g^{\alpha-1}(x) \).

For further knowledge about the conformable fractional derivative, the reader is referred to [1, 5, 18].

Let us consider the following boundary value problem \( L_\alpha(q(x), h, H) \):
\[
\begin{align*}
\hat{e}y &:= -D^\alpha D_x^\alpha y + q(x)y = Hy, \quad 0 < x < \pi, \\
U(y) &:= D^\alpha_y y(0) - hy(0) = 0, \\
V(y) &:= D^\alpha_y y(\pi) + Hy(\pi) = 0,
\end{align*}
\]
where \( D^\alpha \) is the conformable fractional (CF) derivative of order \( \alpha \), \( 0 < \alpha \leq 1 \), \( q(t) \) is a real-valued continuous function on \([0, \pi]\), \( h, H \in \mathbb{R} \) and \( \lambda \) is the spectral parameter.
Let the functions \(\varphi(x, \lambda)\) and \(\psi(x, \lambda)\) be the solutions of (2.1) under the initial conditions
\[
\varphi(0, \lambda) = 1, \quad D_x^\delta \varphi(0, \lambda) = h, \quad \psi(\pi, \lambda) = 1, \quad D_x^\delta \psi(\pi, \lambda) = -H,
\] respectively. These solutions are entire according to \(\lambda\) for each fixed \(x\) in \([0, \pi]\) and they satisfy the following asymptotic formulas [29]:
\[
\begin{align*}
\varphi(x, \lambda) &= \cos \left( \frac{\sqrt{\lambda} x^\alpha}{\alpha} \right) + O \left( \frac{1}{|\sqrt{\lambda}|} \exp \left( \frac{|\tau|}{\alpha} x^\alpha \right) \right), \\
D_x^\delta \varphi(x, \lambda) &= -\sqrt{\lambda} \sin \left( \frac{\sqrt{\lambda} x^\alpha}{\alpha} \right) + O \left( \exp \left( \frac{|\tau|}{\alpha} x^\alpha \right) \right), \\
\psi(x, \lambda) &= \cos \left( \frac{\sqrt{\lambda} (\pi^\alpha - x^\alpha)}{\alpha} \right) + O \left( \frac{1}{|\sqrt{\lambda}|} \exp \left( \frac{|\tau|}{\alpha} (\pi^\alpha - x^\alpha) \right) \right), \\
D_x^\delta \psi(x, \lambda) &= \sqrt{\lambda} \sin \left( \frac{\sqrt{\lambda} (\pi^\alpha - x^\alpha)}{\alpha} \right) + O \left( \exp \left( \frac{|\tau|}{\alpha} (\pi^\alpha - x^\alpha) \right) \right),
\end{align*}
\] (2.5)
(2.6)
(2.7)
(2.8)
where \(\tau := \text{Im} \sqrt{\lambda}\). The function
\[
W_\alpha[\psi(x, \lambda), \varphi(x, \lambda)] = \psi(x, \lambda) D_x^\delta \varphi(x, \lambda) - \varphi(x, \lambda) D_x^\delta \psi(x, \lambda)
\]
is called the fractional Wronskian of \(\psi\) and \(\varphi\). It is proven in [29] that \(W_\alpha\) does not depend on \(x\) and it can be written as \(W_\alpha[\psi(x, \lambda), \varphi(x, \lambda)] = \Delta(\lambda) = V(\varphi) = -U(\psi)\). Put
\[
G_\delta := \left\{ \sqrt{\lambda} : |\sqrt{\lambda} - \frac{\alpha}{\pi^\alpha - k}| \geq \delta, \ k = 0, 1, 2, \ldots \right\},
\]
where \(\delta\) is a sufficiently small positive number. The function \(\Delta(\lambda)\) satisfies the inequality
\[
|\Delta(\lambda)| \geq C_\delta |\sqrt{\lambda}| \exp \left( \frac{|\tau|}{\alpha} \pi^\alpha \right), \quad \sqrt{\lambda} \in G_\delta, \ |\sqrt{\lambda}| \geq \rho^*,
\] (2.9)
for sufficiently large \(\rho^* = \rho^*(\delta)\).

Let \(\lambda_n\) be the eigenvalue-set of \(L_0(q(x), h, H)\). The numbers \(\lambda_n\) are real, simple and satisfy the following asymptotic estimate:
\[
\sqrt{\lambda_n} = \left( \frac{\alpha}{\pi^\alpha - 1} \right) n + \frac{\omega_n}{\pi n} + \frac{\kappa_n}{n}, \quad \kappa_n \in l_2^0,
\]
where (see [29])
\[
\omega_n = h + H + \frac{1}{2} \int_0^1 q(t)d_a t.
\]

3 Uniqueness theorems

Together with \(L_0\), we consider a boundary value problem \(L_\alpha = L_0(q(x), \hat{h}, H)\) of the same form but with different coefficients. We assume that if a certain symbol \(s\) denotes an object related to \(L_0\), then \(s\) will denote an analogous object related to \(L_\alpha\).

3.1 According to the Weyl function

Let \(S(x, \lambda)\) be a solution of (2.1) that satisfies the conditions \(S(0, \lambda) = 1\) and \(D_x^\delta S(0, \lambda) = 1\). It is clear that \(W_\alpha[\varphi(x, \lambda), S(x, \lambda)] = 1\) and the function \(\psi(x, \lambda)\) can be represented by
\[
\Phi(x, \lambda) := -\frac{\psi(x, \lambda)}{\Delta(\lambda)} = S(x, \lambda) + M(\lambda) \varphi(x, \lambda),
\] (3.1)
where \(M(\lambda) = -\frac{\psi(0, \lambda)}{\Delta(\lambda)}\) is called the Weyl function.
Theorem 3.1. If \( M(\lambda) = \overline{M}(\lambda) \), then \( q(x) = \overline{q}(x) \) a.e. in \([0, \pi]\), \( h = \overline{h} \) and \( H = \overline{H} \).

Proof. Let us consider the functions \( P_1(x, \lambda) \) and \( P_2(x, \lambda) \) which are defined by the following formulas:

\[
P_1(x, \lambda) = \varphi(x, \lambda)D^s_\lambda\overline{\Phi}(x, \lambda) - \Phi(x, \lambda)D^s_\lambda\overline{\varphi}(x, \lambda),
\]

\[
P_2(x, \lambda) = \Phi(x, \lambda)\overline{\varphi}(x, \lambda) - \varphi(x, \lambda)\overline{\Phi}(x, \lambda),
\]

where \( \Phi(x, \lambda) = -\frac{\psi(x, \lambda)}{\Delta(\lambda)} \). It is easy to see that the functions \( P_1(x, \lambda) \) and \( P_2(x, \lambda) \) are meromorphic with respect to \( \lambda \). Moreover, according to (3.1),

\[
P_1(x, \lambda) = \varphi(x, \lambda)D^s_\lambda\overline{\Phi}(x, \lambda) - \Phi(x, \lambda)D^s_\lambda\overline{\varphi}(x, \lambda) + (\overline{M}(\lambda) - M(\lambda))\varphi(x, \lambda)D^s_\lambda\overline{\varphi}(x, \lambda),
\]

\[
P_2(x, \lambda) = S(x, \lambda)\overline{\varphi}(x, \lambda) - \varphi(x, \lambda)\overline{\Phi}(x, \lambda) + (M(\lambda) - \overline{M}(\lambda))\varphi(x, \lambda)\overline{\varphi}(x, \lambda).
\]

Since \( M(\lambda) = \overline{M}(\lambda) \), we obtain that \( P_1(x, \lambda) \) and \( P_2(x, \lambda) \) are entire in \( \lambda \). On the other hand, one can calculate from

\[
W_\alpha[\varphi(x, \lambda), \Phi(x, \lambda)] = \frac{W_\alpha[\varphi(x, \lambda), \psi(x, \lambda)]}{\Delta(\lambda)} = 1
\]

that

\[
P_1(x, \lambda) = 1 + \varphi(x, \lambda)[D^s_\lambda\overline{\Phi}(x, \lambda) - D^s_\lambda\overline{\varphi}(x, \lambda)] + \Phi(x, \lambda)[D^s_\lambda\varphi(x, \lambda) - D^s_\lambda\overline{\varphi}(x, \lambda)].
\]

It follows from the asymptotic formulas (2.5)–(2.9) that

\[
P_1(x, \lambda) - 1 = O\left(\frac{1}{\sqrt{\lambda}}\right), \quad P_2(x, \lambda) = O\left(\frac{1}{\sqrt{\lambda}}\right), \quad \sqrt{\lambda} \in G_\delta, |\sqrt{\lambda}| \geq \rho^*.
\]

Therefore, we obtain \( P_1(x, \lambda) = 1 \) and \( P_2(x, \lambda) = 0 \) by the well-known Liouville theorem. Hence by using (3.2) and (3.3), we get \( \varphi(x, \lambda) = \overline{\varphi}(x, \lambda) \), \( \Phi(x, \lambda) = \overline{\Phi}(x, \lambda) \), and so \( q(x) = \overline{q}(x) \) a.e. in \([0, \pi]\), \( h = \overline{h} \), \( H = \overline{H} \). \( \square \)

3.2 According to two given spectra or a spectrum and norming constants

We consider the boundary value problem \( L_{a,1} \) with the condition \( y(0, \lambda) = 0 \) instead of (2.2) in \( L_a \). Let \( \{\xi_n\}_{n \geq 0} \) be the eigenvalues of the problem \( L_{a,1} \). It is obvious that \( \xi_n \) are zeros of \( \Delta_1(\xi) := \psi(0, \xi) \).

Theorem 3.2. If \( \{\lambda_n, \xi_n\}_{n \geq 0} = \{\overline{\lambda}_n, \overline{\xi}_n\}_{n \geq 0} \), then \( q(x) = \overline{q}(x) \), a.e. in \([0, \pi]\), \( h = \overline{h} \), \( H = \overline{H} \).

Proof. According to [29, Theorem 3.11], the function \( \Delta(\lambda) \) can be represented as follows:

\[
\Delta(\lambda) = \frac{n^{2n-2}}{\alpha^3} (\lambda_0 - \lambda) \prod_{n=0}^{\infty} \left( \frac{\lambda_n - \lambda}{n^2} \right).
\]

Therefore, \( \Delta(\lambda) \equiv \overline{\Delta}(\lambda) \) (resp. \( \Delta_1(\xi) \equiv \overline{\Delta}_1(\xi) \)), when \( \lambda_n = \overline{\lambda}_n \) (resp. \( \xi_n = \overline{\xi}_n \)) for all \( n \). Consequently, \( M(\lambda) = \overline{M}(\lambda) \), and so the proof is completed by Theorem 3.1. \( \square \)

Lemma 3.3 ([29, Lemma 3.5]). Set

\[
a_n = \|\varphi(x, \lambda_n)\|^2_{L^2_a} = \int_0^\pi \varphi^2(x, \lambda_n) d_a x.
\]

Then we have \( \beta_n a_n = -\Delta'(\lambda_n) \), where \( \beta_n = \psi(0, \lambda_n) \).

The numbers \( a_n \) in Lemma 3.3 are called norming constants.

Theorem 3.4. If \( \{\lambda_n, a_n\}_{n \geq 0} = \{\overline{\lambda}_n, \overline{a}_n\}_{n \geq 0} \), then \( q(x) = \overline{q}(x) \) a.e. in \([0, \pi]\), \( h = \overline{h} \), \( H = \overline{H} \).

Proof. Since \( \lambda_n = \overline{\lambda}_n \), we have \( \Delta(\lambda) \equiv \overline{\Delta}(\lambda) \). Therefore, by using Lemma 3.3, we obtain that \( \beta_n = \overline{\beta}_n \), and so \( \psi(0, \lambda_n) = \overline{\psi}(0, \lambda_n) \). Hence the function

\[
G(\lambda) := \frac{\psi(0, \lambda) - \overline{\psi}(0, \lambda)}{\Delta(\lambda)}
\]

is entire on \( \lambda \). Moreover, we can obtain from (2.7) and (2.9) that \( G(\lambda) = O(\frac{1}{\lambda}) \) for \( |\lambda| \to \infty \). Thus \( G(\lambda) \equiv 0 \) and \( \psi(0, \lambda) \equiv \overline{\psi}(0, \lambda) \). Finally, we get \( M(\lambda) = \overline{M}(\lambda) \), and so we obtain our desired result by Theorem 3.1. \( \square \)
Remark 3.5. It follows from Lemma 3.3 that

\[ M(\lambda) = \sum_{n=0}^{\infty} \frac{1}{a_n(\lambda - \lambda_n)}. \]

Indeed, consider the contour integral

\[ F_n(\lambda) = \int_{\Gamma_n} \frac{M(\mu)}{\lambda - \mu} d\mu, \quad \lambda \in \text{int} \Gamma_n, \]

on

\[ \Gamma_n = \left\{ \mu : |\mu| = \left( \frac{an}{\pi n^{\alpha + 1}} + \varepsilon \right)^{2} \right\}, \]

where \( \varepsilon \) is a sufficiently small number. It can be calculated from (2.7) and (2.9) that \( |M(\mu)| \leq C|\mu|^{-1/2} \), and so \( \lim_{n \to \infty} F_n(\lambda) = 0 \). From the residue theorem, we have

\[ -M(\lambda) + \sum_{n=0}^{\infty} \text{Res} \left[ \frac{M(\mu)}{\lambda - \mu}, A_n \right] = 0. \]

According to Lemma 3.3, we obtain

\[ M(\lambda) = \sum_{n=0}^{\infty} \frac{\psi(0, \lambda_n)}{(\lambda_n - \lambda) \Delta'(\lambda_n)} = \sum_{n=0}^{\infty} \frac{1}{a_n(\lambda - \lambda_n)}. \]

Consequently, the function \( M(\lambda) \) is uniquely determined by \( \{\lambda_n, a_n\}_{n \geq 0} \). This relation gives another proof of Theorem 3.4.

### 3.3 According to the mixed data

The next theorem is an adaptation of Hochstadt–Lieberman theorem in the classical Sturm–Liouville theory to the CFSL operator.

**Theorem 3.6.** If \( \{\lambda_n\}_{n \geq 0} = \{\tilde{\lambda}_n\}_{n \geq 0} \), \( H = \overline{H} \) and \( q(x) = \tilde{q}(x) \) on \((c_a, \pi]\), then \( q(x) = \tilde{q}(x) \) a.e. in \([0, \pi]\) and \( h = \tilde{h} \), where \( c_a = \pi 2^{-1/a} \).

**Proof.** It is clear that the following equality holds:

\[ D_{\alpha}^{\alpha} [\overline{\varphi}(x, \lambda) D_{\alpha}^{\alpha} \varphi(x, \lambda) - \varphi(x, \lambda) D_{\alpha}^{\alpha} \overline{\varphi}(x, \lambda)] = \int [q(t) - \tilde{q}(t)] \varphi(t, \lambda) \overline{\varphi}(t, \lambda) dt. \]

By integrating (in the conformable fractional integral) both sides of this equality on \([0, \pi]\), we obtain

\[ [\overline{\varphi}(x, \lambda) D_{\alpha}^{\alpha} \varphi(x, \lambda) - \varphi(x, \lambda) D_{\alpha}^{\alpha} \overline{\varphi}(x, \lambda)]_{x=0}^{\pi} = \int_{0}^{\pi} [q(t) - \tilde{q}(t)] \varphi(t, \lambda) \overline{\varphi}(t, \lambda) dt. \]

Since \( q(x) = \tilde{q}(x) \) on \((c_a, \pi]\) and from (2.4), it is obvious that

\[ \overline{\varphi}(\pi, \lambda) D_{\alpha}^{\alpha} \varphi(\pi, \lambda) - \varphi(\pi, \lambda) D_{\alpha}^{\alpha} \overline{\varphi}(\pi, \lambda) = h - \tilde{h} + \int_{0}^{c_a} [q(t) - \tilde{q}(t)] \varphi(t, \lambda) \overline{\varphi}(t, \lambda) dt. \]

Let

\[ H(\lambda) := h - \tilde{h} + \int_{0}^{c_a} [q(t) - \tilde{q}(t)] \varphi(t, \lambda) \overline{\varphi}(t, \lambda) dt. \]

Since

\[ \overline{\varphi}(\pi, \lambda_n) D_{\alpha}^{\alpha} \varphi(\pi, \lambda_n) - \varphi(\pi, \lambda_n) D_{\alpha}^{\alpha} \overline{\varphi}(\pi, \lambda_n) = 0, \]
we obtain $H(\lambda_n) = 0$ for all $n$, and so $\chi(\lambda) := \frac{H(\lambda)}{M(\lambda)}$ is entire on $\lambda$. It can be calculated from the asymptotic expressions of $\varphi(x, \lambda)$ and $\overline{\varphi}(x, \lambda)$ that

$$H(\lambda) = O\left(\exp\left(\frac{|r|}{a}\right)\right).$$

Therefore, taking into account (2.9), we obtain that $|\chi(\lambda)| \leq C/|\sqrt{\lambda}|$ for $\sqrt{\lambda} \in G_\delta$, $|\sqrt{\lambda}| \geq \rho^*$. By Liouville’s Theorem, $\chi(\lambda) = 0$ for all $\lambda$, and so $H(\lambda) \equiv 0$.

By integrating again both sides of equality (3.4) on $(0, c_a)$, we get

$$\overline{\varphi}(c_a, \lambda)D_x^\alpha \varphi(c_a, \lambda) = \varphi(c_a, \lambda)D_x^\alpha \overline{\varphi}(c_a, \lambda). \quad (3.5)$$

Put

$$u(x) := \left(\frac{\pi^a}{2} - x^a\right)^{1/a} \quad \text{and} \quad \psi(x, \lambda) := \varphi(u(x), \lambda).$$

Obviously, $x \in [0, c_a]$ if and only if $u \in [0, c_a]$. From Lemma 2.4 it can be proven that

$$D_x^\alpha \psi(x, \lambda) := -(D_x^\alpha \varphi)(u(x), \lambda) \quad \text{and} \quad D_x^\alpha D_x^\alpha \psi(x, \lambda) := (D_x^\alpha D_x^\alpha \varphi)(u(x), \lambda).$$

Therefore it is clear that $\psi(x, \lambda)$ is the solution of the equation

$$-D_x^\alpha D_x^\alpha \psi(x) + q\left(\left(\frac{\pi^a}{2} - x^a\right)^{1/a}\right)\psi(x) = \lambda \psi(x), \quad x \in (0, c_a), \quad (3.6)$$

under the initial conditions $\psi(c_a, \lambda) = 1$ and $D_x^\alpha \psi(c_a, \lambda) = -h$. It follows from (3.5) that

$$\overline{\psi}(0, \lambda)D_x^\alpha \psi(0, \lambda) = \psi(0, \lambda)D_x^\alpha \overline{\psi}(0, \lambda),$$

and so

$$\frac{D_x^\alpha \psi(0, \lambda) - H\psi(0, \lambda)}{\psi(0, \lambda)} = \frac{D_x^\alpha \overline{\psi}(0, \lambda) - H\overline{\psi}(0, \lambda)}{\overline{\psi}(0, \lambda)} \quad (3.7)$$

by the hypothesis $H = \overline{H}$. Let $M(\lambda)$ be the Weyl function of the boundary value problem, generated by (3.6) and boundary conditions $D_x^\alpha \gamma(0) - H\gamma(0) = D_x^\alpha \gamma(c_a) + h\gamma(c_a) = 0$. Thus from (3.7) we get $M(\lambda) = \overline{M(\lambda)}$. Taking into account Theorem 3.1, it is concluded that $q(x) = \overline{q}(x)$ a.e. on $[0, c_a]$, $h = \overline{h}$. This completes the proof.

### 3.4 Conclusion

The aim of this paper is to give some uniqueness theorems for an inverse Sturm–Liouville problem with conformable derivation. We prove that the coefficients of the boundary value problem (2.1)–(2.3) can be uniquely determined by each of the following:

(i) The Weyl function $M(\lambda)$.
(ii) Two given spectra: $\{\lambda_n, \xi_n\}$.
(iii) Eigenvalues and norming constants of the problem: $\{\lambda_n, \alpha_n\}$.
(iv) The eigenvalue-sequence $\{\lambda_n\}$ and the potential function $q(x)$ on $(c_a, \pi^a]$.

These results are adaptations of the classical uniqueness theorems of Borg [7], Gelfand and Levitan [11], Hochstadt and Lieberman [16] and Marchenko [25] to the conformable fractional Sturm–Liouville operator. After this, some reconstruction-procedures for the coefficients of the above problems can be studied. Moreover, each of our results can be attempted for CFSL operators with parameter-dependent boundary conditions or transmission conditions.

**Acknowledgment:** The authors thanks the reviewers for constructive comments and recommendations which helped to improve the readability and quality of the paper.
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