Global well-posedness and asymptotic behavior in Besov-Morrey spaces for chemotaxis-Navier-Stokes fluids

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Abstract

In this work we consider the Keller-Segel system coupled with Navier-Stokes equations in \(\mathbb{R}^N\) for \(N \geq 2\). We prove the global well-posedness with small initial data in Besov-Morrey spaces. Our initial data class extends previous ones found in the literature such as that obtained by Kozono-Miura-Sugiyama (J. Funct. Anal. 2016). It allows to consider initial cell density and fluid velocity concentrated on smooth curves or at points depending on the spatial dimension. Self-similar solutions are obtained depending on the homogeneity of the initial data and considering the case of chemical attractant without degradation rate. Moreover, we analyze the asymptotic stability of solutions at infinity and obtain a class of asymptotically self-similar ones.

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1 Introduction

We consider the double chemotaxis-Navier-Stokes equations in the whole space \(\mathbb{R}^N\)

\[
\begin{aligned}
\partial_t n + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c) - \nabla \cdot (n \nabla v), & \text{in } \mathbb{R}^N \times (0, \infty),
\partial_t c + u \cdot \nabla c &= \Delta c - nc, & \text{in } \mathbb{R}^N \times (0, \infty),
\partial_t v + u \cdot \nabla v &= \Delta v - \gamma v + n, & \text{in } \mathbb{R}^N \times (0, \infty),
\partial_t u + (u \cdot \nabla) u &= \Delta u - \nabla \pi - nf, & \text{in } \mathbb{R}^N \times (0, \infty),
\nabla \cdot u &= 0, & \text{in } \mathbb{R}^N \times (0, \infty),
n(x, 0) = n_0(x), \ c(x, 0) = c_0(x), \ v(x, 0) = v_0(x), \ u(x, 0) = u_0(x), & \text{in } \mathbb{R}^N,
\end{aligned}
\]

(1.1)

where \(N \geq 2\) and \(\gamma \geq 0\). The unknown \(n(x, t), c(x, t), v(x, t), u(x, t)\) and \(\pi(x, t)\) stands for cell density, oxygen concentration, chemical-attractant concentration, fluid velocity field, and pressure of the fluid, respectively. The time-independent field \(f\) denotes a force field acting on the motion of the fluid.

The system (1.1) was introduced by Tuval \textit{et al.} in [23] and corresponds to a double chemotaxis model that describes the movement of swimming bacteria living in an incompressible viscous fluid, which swim toward a higher concentration of oxygen and chemical attractant. The fluid movement is modeled by the Navier-Stokes equations under the influence of a force \(-nf\) that can be produced by different mechanisms, e.g., force due to the aggregation of bacteria onto the fluid generating a buoyancy-like force. In turn, the chemical attractant \(v\) is produced by the bacteria themselves that degrades at a constant rate \(\gamma \geq 0\).

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In the case with no chemical-attractant degradation rate (i.e., \( \gamma = 0 \)), system (1.1) has the scaling (see Section 3)
\[
(n, c, v, u) \to (\lambda^2 n(\lambda x, \lambda^2 t), c(\lambda x, \lambda^2 t), v(\lambda x, \lambda^2 t), \lambda u(\lambda x, \lambda^2 t)),
\]
which, by taking \( t = 0 \), induces the initial data scaling
\[
(n_0, c_0, v_0, u_0) \to (\lambda^2 n(\lambda x), c(\lambda x), v(\lambda x), \lambda u(\lambda x)).
\]

For mathematical analysis purposes, the above scaling relations also work well for the case \( \gamma \neq 0 \) and functional spaces invariant by them are called critical ones for (1.1). Throughout the paper, spaces of scalar and vector functions are denoted in the same way, e.g., we write \( u_0 \in L^p(\mathbb{R}^N) \) in place of \( u_0 \in (L^p(\mathbb{R}^N))^N \).

In what follows we give a review of some results about (1.1) and related systems. Firstly we recall the classical Keller-Segel system (without fluid coupling)
\[
\begin{align*}
\partial_t n &= \Delta n - \nabla \cdot (n \nabla v), \\
\varepsilon \partial_t v &= \Delta v - \gamma v + \beta n,
\end{align*}
\]
which has been studied by several authors. It models aggregation of biological species (e.g. amoebae and bacteria) moving towards high concentration of a chemical secreted by themselves or of food molecules (e.g. glucose). We have the two basic cases \( \varepsilon = 0 \) (parabolic-elliptic) and \( \varepsilon = 1 \) (parabolic-parabolic). For the former with \( \gamma = 0 \), it is well known that there is a threshold value \( 8\pi/\beta \) for the initial mass that decides between global existence and the finite time blow-up. For further details, see e.g. Blanchet-Dolbeault-Perthame [1] and Dolbeault-Perthame [7].

Considering the parabolic-parabolic case and a 2D smooth bounded domain \( \Omega \) with Neumann conditions, Nagai-Senba-Yoshida [22] proved global-in-time existence of solutions for \( \varepsilon = 1, \gamma \geq 0, \beta > 0 \) and nonnegative initial data \( n_0, v_0 \in H^{1+\delta}(\Omega) \) with mass \( M = \int_\Omega n_0 < 4\pi/\beta \). Horstman [13] considered (1.4) with \( \beta(n-1) \) in place of \( \beta n \) and showed the existence of blow-up solutions with \( M = |\Omega| > 4\pi/\beta \) and \( \beta |\Omega| \neq 4m\pi, m \in \mathbb{N} \), without assuming any symmetry properties. In the radial setting, the mass \( 8\pi/\beta \) is a threshold value for the existence or blow-up of solutions (see [11, 12]). In the whole plane \( \mathbb{R}^2 \), Calvez-Corrias [4] obtained global solutions for subcritical masses \( M < 8\pi/\beta, \varepsilon = 1, \gamma \geq 0 \) and \( \beta > 0 \), as well as blow-up of solutions for \( \varepsilon = 0 \) and \( M > 8\pi/\beta \).

They also conjectured blow-up of solutions for \( \varepsilon = 1 \) and \( M > 8\pi/\beta \) which, to the best of our knowledge, is still open. For \( N \geq 3 \), Corrias-Perthame [6] proved the existence of weak solution for (1.4) with \( \varepsilon = \alpha = \beta = 1 \) and small data \( n_0 \in L^a(\mathbb{R}^N) \) and \( \nabla v_0 \in L^a(\mathbb{R}^N) \) with \( N/2 < a \leq N \). After, Kozono-Sugiyama [17] considered (1.4) with \( \varepsilon = \beta = 1 \) and showed that if \( \max \{1, N/4\} < a \leq N/2 \) and \( (n_0, v_0) \in H^{N-2a}(\mathbb{R}^N) \times H^{N-a}(\mathbb{R}^N) \) is small enough, then there exists a unique global mild solution. Kozono-Sugiyama [16] extended the results of [17] to \( L^{(N/2,\infty)} \times \text{BMO} \) and \( N \geq 2 \), where \( \text{BMO} \) stands for the space of bounded mean oscillation functions and \( L^{(p,\infty)} \) is the weak-\( L^p \). Results on global mild solutions for (1.4) with small initial data belonging to larger critical spaces can be found in the literature, namely Besov-spaces \( B_{p,\infty}^{-(2-N)/p} \times \dot{B}_{p,\infty}^0 \) [26] (with \( \gamma = 0 \) and \( \varepsilon = \beta = 1 \)), Morrey spaces [2, 24] and Besov-Morrey spaces \( \mathcal{N}^{-(2-N)/p}_{\theta,\infty} \times \dot{B}_{\infty,\infty}^0 \) [10].

In the context of chemotaxis-fluids, Duan-Lorz-Markowich [8] considered the model in 3D
\[
\begin{align*}
\partial_t n + u \cdot \nabla n &= \delta \Delta n - \nabla \cdot (\chi(c) n \nabla c), \\
\partial_t c + u \cdot \nabla c &= \mu \Delta c - \kappa(c) n, \\
\partial_t u + (u \cdot \nabla) u &= \nu \Delta u - \nabla p - n f, \\
\nabla \cdot u &= 0,
\end{align*}
\]
and showed the global existence of classical solutions provided that the initial data \( (n_0, c_0, u_0) \) is a small smooth perturbation of the constant state \( (n_\infty, 0, 0) \) with \( n_\infty \geq 0 \). Lorz [20] considered (1.4) with \( \varepsilon = 0 \) coupled to Stokes equations in 2D, without oxygen concentration (i.e., \( c = 0 \)), and showed global-in-time existence of solutions for small initial data \( u_0 \in L^2(\mathbb{R}^2) \) and \( n_0 \in L^1(\mathbb{R}^2) \). In turn, considering smooth bounded domains, Winkler [25] analyzed (1.1) without chemical attractant (i.e., \( v = 0 \)) and showed the existence and uniqueness of global classical solutions in 2D. In 3D, he considered (1.1) with the evolution Stokes equation (in place of the Navier-Stokes one)
and obtained the existence of a global weak solution for \((n_0, c_0, u_0) \in C^0(\Omega) \times W^{1,q}(\Omega) \times D((-\mathbb{P}\Delta)^s)\) where \(q > n, s \in (3/4, 1)\) and \(\mathbb{P}\) is the Leray-Helmholz projection. Zhang [27] obtained the local well-posedness for (1.1) with \(v = 0\) and initial data in the nonhomogeneous Besov spaces
\[
(n_0, c_0, u_0) \in B^s_{p,r}(\mathbb{R}^N) \times B^{s+1}_{p,r}(\mathbb{R}^N) \times B^{s+1}_{p,r}(\mathbb{R}^N)
\]
with \(1 < p < \infty, 1 \leq r < \infty\) and \(s > \frac{N}{p} + 1\) for \(N = 2, 3\). Considering (1.1) with \(v = 0\), Choe-Lkhagvasuren [5] showed the existence of global mild solution for small initial data \((n_0, c_0, u_0)\) in the critical Besov spaces
\[
\dot{B}^{-2+\frac{q}{r}}_{r,1}(\mathbb{R}^3) \times \dot{B}^{\frac{q}{r}}_{r,1}(\mathbb{R}^3) \times \dot{B}^{-1+\frac{q}{r}}_{r,1}(\mathbb{R}^3) \text{ for } r \in [1, 3).
\]
After, Zhao-Zhou [28] extended the result obtained in [5] to \(r \in [1, 6]\). Finally, for results about chemotaxis-fluid models with logistic terms, we refer the reader to [3],[9],[19] and their references.

In comparison with the aforementioned models, system (1.1) consists in a double chemotaxis-fluid model that includes the effect of both oxygen concentration and chemical attractant. In [15], Kozono-Miura-Sugiyama obtained existence of global mild solutions for (1.1) by considering \(N \geq 3\), small initial data \(n_0 \in L^w_w, c_0 \in L^\infty\) with \(\nabla c_0 \in L^N_w, v_0 \in S'/P\) with \(\nabla v_0 \in L^N_w\) and \(u_0 \in L^w_w\), and small force \(f \in L^w_w\), where \(P\) denotes the set of polynomials with \(N\) variables. In the case \(N = 2\), the condition \(n_0 \in L^1_w\) is replaced by \(n_0 \in L^1\).

The main aim of this paper is to prove the global well-posedness for (1.1) with small initial data in a larger critical framework based on Besov-Morrey spaces \(\mathcal{N}^s_{q,q_1,\infty}(\mathbb{R}^N)\) (see (2.7)). More precisely, we consider the following critical initial data class
\[
\begin{align*}
 n_0 &\in \mathcal{N}^{\frac{N}{q_1}-2}_{q,q_1,\infty}(\mathbb{R}^N), \quad c_0 \in L^{\infty}(\mathbb{R}^N) \quad \text{with} \quad \nabla c_0 \in \mathcal{N}\frac{N}{r_1}-1_{r_1,\infty}(\mathbb{R}^N), \quad (1.6) \\
v_0 &\in S'/P\quad \text{with} \quad \nabla v_0 \in \mathcal{N}^{\frac{N}{r}-1}_{r,r_1,\infty}(\mathbb{R}^N), \quad \text{and} \quad u_0 \in \mathcal{N}^{\frac{N}{p}-1}_{p,p_1,\infty}(\mathbb{R}^N),
\end{align*}
\]
where the exponents \(p, p_1, q, q_1, r, r_1\) and \(N_1\) satisfy suitable conditions (for more details, see Assumption 1 in Section 3). For the above exponents, we have the strict continuous inclusions
\[
\begin{align*}
L^1 &\hookrightarrow \mathcal{M} \hookrightarrow \mathcal{N}^{\frac{N}{q_1}-2}_{q,q_1,\infty}(N = 2), \quad L^N_w \hookrightarrow \mathcal{M}^{\frac{N}{q_1}-1}_{q_1,q_1,\infty} \hookrightarrow \mathcal{N}^{\frac{N}{q_1}-2}_{q,q_1,\infty}(q_1 < q), \quad (1.7) \\
L^N_w &\hookrightarrow \mathcal{M}^{\frac{N}{r}-1}_{r,r_1,\infty} \hookrightarrow \mathcal{N}^{\frac{N}{r_1}-1}_{r_1,\infty}(r_1 < r), \quad \text{and} \quad L^N_w \hookrightarrow \mathcal{N}^{\frac{N}{p}-1}_{p,p_1,\infty}(p_1 < p),
\end{align*}
\]
where \(\mathcal{M}^p\) denotes Morrey spaces and \(\mathcal{M}^1 = \mathcal{M}\) the space of finite signed Radon measures (see (2.1)). In particular, depending on the spatial dimension \(N\) and taking suitable values for the indexes in (1.7), we can consider the initial cell density \(n_0\) and initial fluid velocity \(u_0\) as some measures concentrated on smooth curves and surfaces (manifolds) or at points.

In view of the inclusions above, our initial data class is larger than that of Kozono-Miura-Sugiyama [15]. Moreover, the force field \(f\) is assumed to belong to the Morrey space \(\mathcal{M}^{\frac{N}{r}}_{N_1}(\mathbb{R}^N)\), where \(N_1\) can be taken equal to \(\frac{N}{r_1}\) with \(r_1\) less and close to \(r\) and \(p/p_1 = r/r_1\). Thus, in view of (1.7), our class of forces \(f\) is larger than that of [15]. For \(N = 3\) and \(r \in [1, 6]\), there exist indexes \(p, p_1, q\) and \(q_1\) satisfying Assumption 1 such that
\[
\dot{B}^{-2+\frac{q}{r}}_{r,1} \hookrightarrow \mathcal{N}^{\frac{N}{q_1}-2}_{q,q_1,\infty}, \quad \dot{B}^{\frac{q}{r}}_{r,1} \hookrightarrow \mathcal{N}^{\frac{N}{q_1}-1}_{q_1,q_1,\infty} \hookrightarrow L^\infty, \quad \dot{B}^{-1+\frac{q}{r}}_{r,1} \hookrightarrow \mathcal{N}^{\frac{N}{p_1}-1}_{p,p_1,\infty},
\]
where \(\dot{B}^s_{p,r}\) stands for homogeneous Besov spaces \((\dot{B}^s_{p,r} = \mathcal{N}^s_{p,p,r})\). Then, in the case of (1.1) without chemical attractant \((v = 0)\), our initial data class is larger than those of [5, 28]. It is worth pointing out that Besov-Morrey spaces were introduced in [18] (see also [21]) in order to study Navier-Stokes equations.

The mild solutions are obtained by means of a contraction argument in a time-dependent critical space defined in (3.6). Under additional conditions of homogeneity on the initial data \(n_0, c_0, v_0, u_0\) and the external force \(f\), we can ensure that the solution obtained in Theorem 3.1 is self-similar when \(\gamma = 0\). Finally, we show that solutions
are asymptotically stable under small initial perturbations, as the time goes to infinity. As a byproduct, we obtain a class of asymptotically self-similar solutions when \( \gamma = 0 \).

This paper is organized as follows. In Section 2, we recall the definitions of Morrey and Besov-Morrey spaces and present some properties about these spaces. Our results on well-posedness and asymptotic behavior of solutions are stated in Section 3. In Section 4, we obtain the needed linear and nonlinear estimates and prove our results.

## 2 Preliminaries

This section is devoted to some preliminaries about Morrey and Besov-Morrey spaces. For further details about these spaces, see [14, 18, 21].

**Definition 2.1.** For \( 1 \leq p_1 \leq p < \infty \), the Morrey space \( M_{p_1}^p = M_{p_1}^p(\mathbb{R}^N) \) is defined as the set of all measurable functions \( u \) such that

\[
\| u \|_{M_{p_1}^p} = \sup_{x_0 \in \mathbb{R}^N} \sup_{R > 0} R^N \| u \|_{L^p(D(x_0, R))} < \infty, \tag{2.1}
\]

where \( D(x_0, R) \) denotes the closed ball in \( \mathbb{R}^N \) with center \( x_0 \) and radius \( R \).

The space \( M_{p_1}^p \) endowed with \( \| \cdot \|_{M_{p_1}^p} \) is a Banach space. In the case \( p_1 = 1 \), \( M_1^p \) is a space of signed Radon measures and \( \| u \|_{L^p(D(x_0, R))} \) is meant as the total variation of the measure \( u \) in the ball \( D(x_0, R) \). For \( 1 < p < \infty \) we have that \( M_1^p = L^p \) and \( M_1^1 = M \) where \( M \) stands for the space of signed Radon measures with finite total variation. In the case \( p = p_1 = \infty \), we consider \( M_{\infty}^\infty = L^\infty \).

Next we recall Hölder inequality and heat semigroup estimates in the framework of Morrey spaces.

**Lemma 2.1.** (Hölder inequality) Let \( 1 \leq p_1 \leq p \leq \infty \), \( 1 \leq q_1 \leq q \leq \infty \) and \( 1 \leq r_1 \leq r \leq \infty \). If \( \frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1} \) and \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \), then

\[
\| fg \|_{M_{r_1}^r} \leq \| f \|_{M_{r_1}^{r_1}} \| g \|_{M_{r_1}^{q_1}}, \tag{2.2}
\]

for all \( f \in M_{p_1}^p \) and \( g \in M_{q_1}^q \).

Let \( \{ e^{t\Delta} \}_{t \geq 0} \) denote the heat semigroup. We have the following estimates in the framework of Morrey spaces.

**Lemma 2.2.** Let \( 1 \leq p_1 \leq p < \infty \) and \( 1 \leq q_1 \leq q < \infty \). If \( p \geq q \) and \( \frac{p}{p_1} \geq \frac{q}{q_1} \), then there exists a universal constant \( C > 0 \) such that

\[
\| e^{t\Delta} f \|_{M_{p_1}^p} \leq C t^{-N\left(\frac{1}{p} - \frac{1}{p_1}\right)} \| f \|_{M_{p_1}^{q_1}}, \tag{2.3}
\]

\[
\| \partial_x e^{t\Delta} f \|_{M_{p_1}^p} \leq C t^{-N\left(\frac{1}{p} - \frac{1}{p_1}\right) - \frac{1}{2}} \| f \|_{M_{p_1}^{q_1}}. \tag{2.4}
\]

Furthermore, for \( 1 \leq q_1 \leq q < \infty \), it holds that

\[
\| e^{t\Delta} f \|_{L^\infty} \leq C t^{-N\frac{q}{q_1}}, \tag{2.5}
\]

\[
\| \partial_x e^{t\Delta} f \|_{L^\infty} \leq C t^{-N\frac{q}{q_1} - \frac{1}{2}} \| f \|_{M_{p_1}^{q_1}}, \tag{2.6}
\]

where \( C > 0 \) is a universal constant.

Let us denote by \( S \) and \( S' \) the Schwartz class and the space of tempered distributions, respectively. For \( u \in S' \), we denote the Fourier transform of \( u \) by \( \hat{u} \) and its inverse by \( u^{\vee} \). Let \( \chi(z) \) be a \( C^\infty \)-function on \( [0, \infty) \) such that \( 0 \leq \chi(z) \leq 1 \), \( \chi(z) \equiv 1 \) for \( z \leq 3/2 \) and \( \text{supp} \chi \subset [0, 5/3) \). Then, for all \( j \in \mathbb{Z} \), put \( \varphi_j(\xi) = \chi(2^{-j}|\xi|) - \chi(2^{1-j}|\xi|) \). It follows that \( \varphi_j(\xi) \in C_0^\infty(\mathbb{R}^N) \) and we have the dyadic decomposition

\[
\sum_{j=-\infty}^{\infty} \varphi_j(\xi) = 1, \text{ for all } \xi \neq 0.
\]
Definition 2.2. The homogeneous Besov-Morrey space $\mathcal{N}_{p,p_1,r}^s = \mathcal{N}_{p,p_1,r}^s(\mathbb{R}^N)$ is the set of all $u \in \mathcal{S}'/\mathcal{P}$ such that $\varphi^j \ast u \in \mathcal{M}_{p_1}^\theta$, for each $j$, and

$$
\|u\|_{\mathcal{N}_{p,p_1,r}^s} = \begin{cases}
\left( \sum_{j \in \mathbb{Z}} \left(2^{sj} \|\varphi^j \ast u\|_{\mathcal{M}_{p_1}^\theta}\right)^r \right)^{\frac{1}{r}}, & \text{for } 1 \leq p_1 \leq p \leq \infty, 1 \leq r < \infty, s \in \mathbb{R}, \\
\sup_{j \in \mathbb{Z}} \left(2^{sj} \|\varphi^j \ast u\|_{\mathcal{M}_{p_1}^\theta}\right) < \infty, & \text{for } 1 \leq p_1 \leq p \leq \infty, r = \infty, s \in \mathbb{R},
\end{cases}
$$

for all $\lambda > 0$.

where $\mathcal{P}$ denotes the set of polynomials with $N$ variables.

The space $\mathcal{N}_{p,p_1,r}^s$ is a Banach space with the norm $\| \cdot \|_{\mathcal{N}_{p,p_1,r}^s}$. For all $s \in \mathbb{R}$, $1 \leq p_2 \leq p_1 \leq p < \infty$ and $1 \leq r \leq \tilde{r} \leq \infty$, we have the continuous inclusions (see [18])

$$
\mathcal{N}_{p,p_1,r}^s \hookrightarrow \mathcal{N}_{p_2,p_2,\tilde{r}}^s \quad \text{and} \quad \mathcal{N}_{p,p_1,r}^s \hookrightarrow \mathcal{N}_{p,p_1,\tilde{r}}^{s-\frac{N(1-\theta)}{\tilde{r} - r}}, \quad \text{for all } \theta \in (0,1).
$$

Furthermore, if $s < 0$ we have the following equivalence of norms (see [21])

$$
\|u\|_{\mathcal{N}_{p,p_1,\infty}^s} \equiv \sup_{t>0} t^{-\frac{s}{2}} \|e^{t\Delta} u\|_{\mathcal{M}_{p_1}^\theta}.
$$

3 Results

In this section we present our global well-posedness and asymptotic behavior results for the system (1.1). Before exposing them, we perform a scaling analysis for finding the suitable functional setting.

For $\gamma \neq 0$, the system (1.1) has no scaling relation. However, we can consider for (1.1) the scaling of the case $\gamma = 0$. Assume temporarily that $\gamma = 0$ and $f$ is a homogeneous distribution of degree $-1$. If $(n, c, v, u, p)$ is a classical solution for (1.1), then so does $(n_\lambda, c_\lambda, v_\lambda, u_\lambda, p_\lambda)$ with initial data $(\lambda^2 n_0(\lambda x), c_0(\lambda x), v_0(\lambda x), \lambda u_0(\lambda x))$, for each $\lambda > 0$, where $n_\lambda(x,t) := \lambda^2 n(\lambda x, \lambda^2 t)$, $c_\lambda(x,t) := c(\lambda x, \lambda^2 t)$, $v_\lambda(x,t) := v(\lambda x, \lambda^2 t)$, $u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t)$ and $p_\lambda(x,t) := \lambda^2 p(\lambda x, \lambda^2 t)$. This leads us to consider the scaling maps (1.2) and (1.3).

Solutions invariant by (1.2) are named self-similar ones. Then, for $(n, c, v, u, p)$ to be a self-similar solution, it is necessary that the initial data $n_0, c_0, v_0, u_0$ and force $f$ are homogeneous functions of degree $-2, 0, 0, -1$ and $-1$, respectively. Motivated by the above scaling analysis, we consider the following critical initial data class

$$
n_0 \in \mathcal{N}_{q,q_1,\infty}^{N-2} \left(\mathbb{R}^N\right), \quad c_0 \in L^\infty \left(\mathbb{R}^N\right) \quad \text{with} \quad \nabla c_0 \in \mathcal{N}_{\tilde{r},r_1,\infty}^{-1} \left(\mathbb{R}^N\right),
$$

$$
v_0 \in \mathcal{S}'/\mathcal{P} \quad \text{with} \quad \nabla v_0 \in \mathcal{N}_{\tilde{r},r_1,\infty}^{-1} \left(\mathbb{R}^N\right), \quad \text{and} \quad u_0 \in \mathcal{N}_{p_1,\infty}^{N-1} \left(\mathbb{R}^N\right),
$$

and force $f \in \mathcal{M}_{N_1}^N \left(\mathbb{R}^N\right)$, where the exponents $p, p_1, q, q_1, r, r_1$ and $N_1$ are as in the following assumption.

Assumption 1. Assume that $N \geq 2$ and $\gamma \geq 0$. For $N \geq 3$, suppose that the exponents $p, q$ and $r$ satisfy either (i), (ii) or (iii) where

(i) $\frac{N}{2} < q < N$, $N < p < \frac{Nq}{N-q}$, $N < r < \frac{Nq}{N-q}$;

(ii) $q = N$, $N < p < \infty$, $N < r < \infty$;

(iii) $N < q < 2N$, $N < p < \frac{Nq}{q-N}$, $q \leq r < \frac{Nq}{q-N}$.

In the case $N = 2$ we assume that $p, q$ and $r$ satisfy the condition (iii) above. Moreover, suppose also that $p_1, q_1,$
\( r_1 \) and \( N_1 \) satisfy the following conditions

\[
\begin{align*}
(A) & \quad 1 \leq p_1 \leq p, \quad 1 \leq q_1 \leq q, \quad 1 \leq r_1 \leq r, \quad 1 \leq N_1 \leq N; \\
(B) & \quad \frac{1}{p_1} + \frac{1}{q_1} \leq 1, \quad \frac{1}{r_1} + \frac{1}{q_1} \leq 1, \quad \frac{1}{p_1} + \frac{1}{r_1} \leq 1, \quad \frac{1}{N_1} + \frac{1}{q_1} \leq 1; \\
(C) & \quad \frac{p}{p_1} \leq \frac{q}{q_1} = \frac{r}{r_1}; \\
(D) & \quad p_1 \left( \frac{1}{N_1} + \frac{1}{q_1} \right) \leq p \left( \frac{1}{N} + \frac{1}{q} \right).
\end{align*}
\]

**Remark 3.1.** It is always possible to find indexes \( p_1, q_1, r_1 \) and \( N_1 \) sufficiently close to \( p, q, r \) and \( N \), respectively, satisfying either (i), (ii) or (iii), and such that (A), (B), (C) and (D) hold true. In other words, Assumption 1 is not empty.

Let \( Z \) be a Banach space continuously included in \( S' \) and denote by \( BC_w ((0, \infty); Z) \) the class of bounded functions from \((0, \infty)\) to \( Z \) that are weakly time continuous in the sense of \( \{ \} \). We define the functional spaces

\[
\begin{align*}
X_1 & := \left\{ n : t^{\frac{N}{p_1} + \frac{1}{2}} n \in BC_w ((0, \infty); \mathcal{M}_{q_1}^p) \right\}, \\
X_2 & := \left\{ c : c \in BC_w ((0, \infty); L^\infty) \text{ with } t^{\frac{N}{2p} + \frac{1}{2}} \nabla c \in BC_w ((0, \infty); \mathcal{M}_{r_1}^p) \right\}, \\
X_3 & := \left\{ v : v(t) \in S'/P \text{ for } t > 0 \text{ and } t^{\frac{N}{2q} + \frac{1}{2}} \nabla v \in BC_w ((0, \infty); \mathcal{M}_{r_1}^p) \right\}, \\
X_4 & := \left\{ u : t^{\frac{N}{2p} + \frac{1}{2}} u \in BC_w ((0, \infty); \mathcal{M}_{r_1}^p) \right\},
\end{align*}
\]

which are Banach spaces endowed with the respective norms

\[
\begin{align*}
\| n \|_{X_1} & := \sup_{t > 0} t^{-\frac{N}{p_1} + 1} \| n(t) \|_{\mathcal{M}_{q_1}^p}, \\
\| c \|_{X_2} & := \sup_{t > 0} \| c(t) \|_{L^\infty} + \sup_{t > 0} t^{\frac{N}{2p} + \frac{1}{2}} \| \nabla c(t) \|_{\mathcal{M}_{r_1}^p}, \\
\| v \|_{X_3} & := \sup_{t > 0} t^{\frac{N}{2q} + \frac{1}{2}} \| \nabla v(t) \|_{\mathcal{M}_{r_1}^p}, \\
\| u \|_{X_4} & := \sup_{t > 0} t^{-\frac{N}{2p} + \frac{1}{2}} \| u(t) \|_{\mathcal{M}_{r_1}^p}.
\end{align*}
\]

Next, let us introduce the spaces \( \mathcal{X} \) and \( \mathcal{I} \) as

\[
\mathcal{X} := \{(n, c, v, u) : n \in X_1, c \in X_2, v \in X_3, u \in X_4\}
\]

with the norm

\[
\|(n, c, v, u)\|_\mathcal{X} := \|n\|_{X_1} + \|c\|_{X_2} + \|v\|_{X_3} + \|u\|_{X_4},
\]

and

\[
\mathcal{I} := \{(n_0, c_0, v_0, u_0) : n_0, c_0, v_0 \text{ and } u_0 \text{ are as in (3.1)}\}
\]

with the norm

\[
\|(n_0, c_0, v_0, u_0)\|_\mathcal{I} := \|n_0\|_{\mathcal{N}_{q_1}^{p_1, \infty}} + \|c_0\|_{L^\infty} + \|\nabla c_0\|_{\mathcal{N}_{p_1}^{r_1, \infty}} + \|\nabla v_0\|_{\mathcal{N}_{p_1}^{r_1, \infty}} + \|u_0\|_{\mathcal{N}_{p_1}^{r_1, \infty}}.
\]

Note that \( \mathcal{X} \) and \( \mathcal{I} \) are Banach spaces equipped with the norms \( \| \cdot \|_\mathcal{X} \) and \( \| \cdot \|_\mathcal{I} \), respectively.
Let $\mathbb{P} = I + R \otimes R$ stand for the Leray-Helmholtz projection onto the spaces of solenoidal vector fields, where $I$ is the identity and $R = (R_1, R_2, \ldots, R_N)$ is a vector of operators whose components are the Riesz transforms $R_j$. Applying $\mathbb{P}$ on the fourth equation in (1.1) and using Duhamel’s principle, system (1.1) can be formally converted to the following integral formulation

$$
\begin{cases}
  n(t) &= e^{t\Delta} n_0 - \int_0^t e^{(t-\tau)\Delta} (u \cdot \nabla n)(\tau) d\tau - \int_0^t \nabla \cdot e^{(t-\tau)\Delta} (n\nabla c + n\nabla v)(\tau) d\tau, \\
  c(t) &= e^{t\Delta} c_0 - \int_0^t e^{(t-\tau)\Delta} (u \cdot \nabla c + nc)(\tau) d\tau, \\
  v(t) &= e^{-\gamma t} e^{t\Delta} v_0 - \int_0^t e^{-\gamma(t-\tau)} e^{t\Delta} (u \cdot \nabla v - n)(\tau) d\tau, \\
  u(t) &= e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P}(u \cdot \nabla u)(\tau) d\tau - \int_0^t e^{(t-\tau)\Delta} \mathbb{P}(nf)(\tau) d\tau.
\end{cases} \tag{3.7}
$$

A 4-tuple $(n, c, v, u)$ satisfying (3.7) is called a mild solution of (1.1). In what follows, we state our main result.

**Theorem 3.1.** Let $N \geq 2$, and let the exponents $p, p_1, q, q_1, r, r_1$ and $N_1$ be as in Assumption 1. Suppose that the initial data $(n_0, c_0, v_0, u_0) \in \mathcal{I}$ and the external force $f \in \mathcal{M}^N$ ($\mathbb{R}^N$). There exist positive constants $\varepsilon, \delta$ ($\delta = C\varepsilon$) and $K_1$ such that the system (3.7) has a unique global mild solution $(n, c, v, u) \in \mathcal{X}$ satisfying $\|(n, c, v, u)\|_{\mathcal{X}} \leq 2K_1\varepsilon$ provided that $\|(n_0, c_0, v_0, u_0)\|_{\mathcal{X}} \leq \delta$. Moreover, the data-solution map is locally Lipschitz continuous.

**Remark 3.2.** Let the constants $C_i$’s, $i = 1, \ldots, 7$, be as in Lemma 4.2 and $\alpha, \beta$ as in Lemma 4.3. The constant $\varepsilon$ in Theorem 3.1 can be chosen so that $0 < \varepsilon < \frac{1}{4K_1K_2}$, where $K_1$ and $K_2$ depend on $C_i, \alpha, \beta$ (see (4.30)). We also point out that the mild solution $(n, c, v, u) \rightarrow (n_0, c_0, v_0, u_0)$ in the sense of distributions, as $t \rightarrow 0^+$.

Since the space $\mathcal{X}$ is critical with respect to the scaling of the case $\gamma = 0$, we can obtain self-similar solutions by assuming the right homogeneity on the data and force.

**Corollary 3.1.** (Self-similar solution) Let $N \geq 3$ and $\gamma = 0$. Assume that $(n_0, c_0, v_0, u_0)$ and $f$ are as in Theorem 3.1. Suppose that $n_0, c_0, v_0, u_0$ and $f$ are homogeneous functions with degree $-2, 0, -1$ and $-1$, respectively. Then, the solution $(n, c, v, u)$ obtained through Theorem 3.1 is self-similar; that is, for every $\lambda > 0$ we have that

$$
n(x, t) = \lambda^2 n(\lambda x, \lambda^2 t), \quad c(x, t) = c(\lambda x, \lambda^2 t), \quad v(x, t) = v(\lambda x, \lambda^2 t) \quad \text{and} \quad u(x, t) = \lambda u(\lambda x, \lambda^2 t).
$$

Now we present an asymptotic stability result for solutions of system (1.1).

**Theorem 3.2.** Under the hypotheses of Theorem 3.1. Assume that $(n, c, v, u)$ and $(\tilde{n}, \tilde{c}, \tilde{v}, \tilde{u})$ are two solutions given by Theorem 3.1 corresponding to the initial data $(n_0, c_0, v_0, u_0)$ and $(\tilde{n}_0, \tilde{c}_0, \tilde{v}_0, \tilde{u}_0)$, respectively. We have that

$$
\begin{align*}
  \lim_{t \to \infty} t^{-\frac{N}{2p} + \frac{1}{2}} \|n(\cdot, t) - \tilde{n}(\cdot, t)\|_{\mathcal{M}^{q_1}_1} &= \lim_{t \to \infty} \|c(\cdot, t) - \tilde{c}(\cdot, t)\|_{L^\infty} = \lim_{t \to \infty} t^{-\frac{N}{2p} + \frac{1}{2}} \|\nabla(c(\cdot, t) - \tilde{c}(\cdot, t))\|_{\mathcal{M}^{r_1}_1} \\
  \lim_{t \to \infty} t^{-\frac{N}{2p} + \frac{1}{2}} \|\nabla(v(\cdot, t) - \tilde{v}(\cdot, t))\|_{\mathcal{M}^{r_1}_1} &= \lim_{t \to \infty} t^{-\frac{N}{2p} + \frac{1}{2}} \|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{\mathcal{M}^{p_1}_1} = 0. \quad (3.8)
\end{align*}
$$

if only if

$$
\begin{align*}
  \lim_{t \to \infty} \left(t^{-\frac{N}{2p} + \frac{1}{2}} \|e^{t\Delta}(n_0 - \tilde{n}_0)\|_{\mathcal{M}^{q_1}_1} + \|e^{t\Delta}(c_0 - \tilde{c}_0)\|_{L^\infty} + t^{-\frac{N}{2p} + \frac{1}{2}} \|\nabla e^{t\Delta}(c_0 - \tilde{c}_0)\|_{\mathcal{M}^{r_1}_1} + \\
  t^{-\frac{N}{2p} + \frac{1}{2}} \|\nabla e^{-\gamma t} e^{t\Delta}(v_0 - \tilde{v}_0)\|_{\mathcal{M}^{r_1}_1} + t^{-\frac{N}{2p} + \frac{1}{2}} \|e^{t\Delta}(u_0 - \tilde{u}_0)\|_{\mathcal{M}^{p_1}_1}\right) = 0. \quad (3.9)
\end{align*}
$$

**Remark 3.3.** (Asymptotically self-similar solutions) In the case $\gamma = 0$, Theorem 3.2 together with Corollary 3.1 provide a class of solutions asymptotically self-similar at infinity. Indeed, taking the initial data $(\tilde{n}_0, \tilde{c}_0, \tilde{v}_0, \tilde{u}_0) = (n_0, c_0, v_0, u_0) + (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ with $\varphi_i \in C_0^\infty$ and $n_0, c_0, v_0, u_0$ and $f$ as in Corollary 3.1, we have that the corresponding solution $(\tilde{n}, \tilde{c}, \tilde{v}, \tilde{u})$ is attracted to the self-similar solution $(n, c, v, u)$ in the sense of (3.8).
4 Proofs

In this section we present the proofs of the results stated in Section 3. First we prove an abstract fixed point lemma which will be useful for our ends.

Lemma 4.1. For $1 \leq i \leq 4$, let $X_i$ be a Banach space with the norm $\| \cdot \|_{X_i}$. Consider the Banach space $\mathcal{X} = X_1 \times X_2 \times X_3 \times X_4$ endowed with the norm

$$\|x\|_{\mathcal{X}} = \|x_1\|_{X_1} + \|x_2\|_{X_2} + \|x_3\|_{X_3} + \|x_4\|_{X_4},$$

where $x = (x_1, x_2, x_3, x_4) \in \mathcal{X}$. For $1 \leq i, j, k \leq 4$, assume that $B_{i,j}^k : X_i \times X_j \to X_k$ is a continuous bilinear map, that is, there is a constant $C_{i,j}^k > 0$ such that

$$\|B_{i,j}^k(x_i, x_j)\|_{X_k} \leq C_{i,j}^k \|x_i\|_{X_i} \|x_j\|_{X_j}, \quad \text{for all } (x_i, x_j) \in X_i \times X_j. \quad (4.1)$$

Assume also that $L_3 : X_1 \to X_3$ and $L_4 : X_1 \to X_4$ are continuous linear maps such that $\|L_3\|_{X_1 \to X_3} \leq \alpha$ and $\|L_4\|_{X_1 \to X_4} \leq \beta$. Set the constants

$$K_1 := 1 + \alpha + \beta \quad \text{and} \quad K_2 := (\alpha + \beta) \sum_{i,j=1}^{4} C_{i,j}^1 + \sum_{k,i,j=1}^{4} C_{i,j}^k,$$

and let $0 < \varepsilon < \frac{1}{4K_1K_2}$ and $\mathcal{B}_\varepsilon = \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \leq 2K_1 \varepsilon\}$. If $\|y\|_{\mathcal{X}} \leq \varepsilon$ then there exists a unique solution $x \in \mathcal{B}_\varepsilon$ for the equation $x = y + B(x)$, where $y = (y_1, y_2, y_3, y_4)$, $B(x) = (B_1(x), B_2(x), B_3(x), B_4(x))$ and

- $B_1(x) = \sum_{i,j=1}^{4} B_{i,j}^1(x_i, x_j)$,
- $B_2(x) = \sum_{i,j=1}^{4} B_{i,j}^2(x_i, x_j)$,
- $B_3(x) = \sum_{i,j=1}^{4} B_{i,j}^3(x_i, x_j) + (L_3 \circ (y_1 + B_1))(x)$,
- $B_4(x) = \sum_{i,j=1}^{4} B_{i,j}^4(x_i, x_j) + (L_4 \circ (y_1 + B_1))(x)$.

Proof. For all $x \in \mathcal{X}$, it follows from (4.1) that

$$\|B_1(x)\|_{X_1} \leq \sum_{i,j=1}^{4} \|B_{i,j}^1(x_i, x_j)\|_{X_1} \leq \sum_{i,j=1}^{4} C_{i,j}^1 \|x_i\|_{X_i} \|x_j\|_{X_j} \leq \left(\sum_{i,j=1}^{4} C_{i,j}^1\right) \|x\|_{\mathcal{X}}^2. \quad (4.2)$$

Analogously, we have

$$\|B_2(x)\|_{X_2} \leq \left(\sum_{i,j=1}^{4} C_{i,j}^2\right) \|x\|_{\mathcal{X}}^2.$$
Next, using (4.1) and (4.2), we estimate $B_3$ as follows:

$$
\|B_3(x)\|_{X_3} \leq \sum_{i,j=1}^{4} \|B_{i,j}^3(x_i, x_j)\|_{X_3} + \|(L_3 \circ (y_1 + B_1))(x)\|_{X_3}
$$

$$
\leq \sum_{i,j=1}^{4} C_{i,j}^3 \|x_i\|_{X_i} \|x_j\|_{X_j} + \alpha \|(y_1 + B_1)(x)\|_{X_1}
$$

$$
\leq \left( \sum_{i,j=1}^{4} C_{i,j}^3 \right) \|x\|_X^2 + \alpha \left( \|y\|_X + \left( \sum_{i,j=1}^{4} C_{i,j}^1 \right) \|x\|^2_X \right)
$$

$$
= \left( \sum_{i,j=1}^{4} C_{i,j}^3 + \alpha \sum_{i,j=1}^{4} C_{i,j}^1 \right) \|x\|_X^2 + \alpha \|y\|_X. \tag{4.3}
$$

Similarly, it follows that

$$
\|B_4(x)\|_{X_4} \leq \left( \sum_{i,j=1}^{4} C_{i,j}^4 + \beta \sum_{i,j=1}^{4} C_{i,j}^1 \right) \|x\|_X^2 + \beta \|y\|_X. \tag{4.4}
$$

Now, consider the mapping $F : \mathcal{X} \to \mathcal{X}$ given by $F(x) = y + B(x)$. For $x \in B_z$, from (4.2)-(4.4) we obtain that

$$
\|F(x)\|_X \leq \|y\|_X + \sum_{k=1}^{4} \|B_k(x)\|_{X_k}
$$

$$
\leq (1 + \alpha + \beta)\|y\|_X + \left( \alpha + \beta \sum_{i,j=1}^{4} C_{i,j}^1 + \beta \sum_{k,i,j=1}^{4} C_{i,j}^k \right) \|x\|_X^2
$$

$$
\leq K_1 \epsilon + K_2 4 K_1^2 \epsilon^2 = (1 + 4 K_1 K_2 \epsilon) K_1 \epsilon \leq 2 K_1 \epsilon,
$$

and then $F(B_z) \subset B_z$. Next, we take $x, z \in B_z$ and estimate

$$
\|F(x) - F(z)\|_X = \|B(x) - B(z)\|_X
$$

$$
= \sum_{k=1}^{4} \|B_k(x) - B_k(z)\|_{X_k}
$$

$$
\leq \sum_{k=1}^{4} \sum_{i,j=1}^{4} \|B_{i,j}^k(x_i - z_i, x_j) + B_{i,j}^k(z_i, x_j - z_j)\|_{X_k} + \sum_{l=3}^{4} \|(L_l \circ (B_1(x) - B_1(z))\|_{X_1}
$$

$$
\leq \sum_{k,i,j=1}^{4} C_{i,j}^k \left( \|x_i - z_i\|_{X_i} \|x_j\|_{X_j} + \|z_i\|_{X_i} \|x_j - z_j\|_{X_j} \right) + \alpha \beta \|B_1(x) - B_1(z)\|_{X_1}
$$

$$
\leq \sum_{k,i,j=1}^{4} C_{i,j}^k \|x - z\|_X \left( \|x\|_X + \|z\|_X \right)
$$

$$
+ \alpha \beta \sum_{i,j=1}^{4} C_{i,j}^1 \left( \|x_i - z_i\|_{X_i} \|x_j\|_{X_j} + \|z_i\|_{X_i} \|x_j - z_j\|_{X_j} \right)
$$

$$
\leq \left( \alpha \beta \right) \sum_{i,j=1}^{4} C_{i,j}^1 + \sum_{k,i,j=1}^{4} C_{i,j}^k \|x - z\|_X \left( \|x\|_X + \|z\|_X \right)
$$

$$
\leq K_2 4 K_1 \epsilon \|x - z\|_X.
$$
Since $4K_1K_2\varepsilon < 1$, $\mathcal{F}$ is a contraction in $B_\varepsilon$, and the Banach fixed point theorem concludes the proof.

\[\]

Remark 4.1. Due to the fixed point argument in the proof of Lemma 4.1, we have that the solution $x$ depends continuously on the data $y$. More precisely, the data-solution map is Lipschitz continuous from $\{y \in \mathcal{X}; \|y\| \leq \varepsilon\}$ to $B_\varepsilon$. In addition, the solution obtained through Lemma 4.1 is the limit in $\mathcal{X}$ of the sequence of iterates $x^{(1)} = y$ and $x^{(m+1)} = \mathcal{F}(x^{(m)})$, $m \geq 1$. This fact will be useful in the proof of Corollary 3.1.

Now, for each initial data tuple $(n_0, c_0, v_0, u_0)$ and force $f$, we consider $\mathcal{F}(n, c, v, u) = (N, C, V, U)$, where

\[
\begin{align*}
N(t) &= e^{t\Delta}n_0 - \int_0^t e^{(t-\tau)\Delta}(u \cdot \nabla n)(\tau) \, d\tau - \int_0^t \nabla \cdot e^{(t-\tau)\Delta}(n\nabla c)(\tau) \, d\tau - \int_0^t \nabla \cdot e^{(t-\tau)\Delta}(n\nabla v)(\tau) \, d\tau, \\
C(t) &= e^{t\Delta}c_0 - \int_0^t e^{(t-\tau)\Delta}(u \cdot \nabla c)(\tau) \, d\tau - \int_0^t e^{(t-\tau)\Delta}(nc)(\tau) \, d\tau, \\
V(t) &= e^{-\gamma t}e^{t\Delta}v_0 - \int_0^t e^{-\gamma(t-\tau)}e^{(t-\tau)\Delta}(u \cdot \nabla v)(\tau) \, d\tau + \int_0^t e^{-\gamma(t-\tau)}e^{(t-\tau)\Delta}n(\tau) \, d\tau, \\
U(t) &= e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathcal{P}(u \cdot \nabla u) \, d\tau - \int_0^t e^{(t-\tau)\Delta}\mathcal{P}(nf)(\tau) \, d\tau,
\end{align*}
\]

(4.5)

4.1 Estimates for the bilinear terms in (4.5)

Lemma 4.2. Under the hypotheses of Theorem 3.1. There exist positive constants $C_1, C_2, C_3, C_4, C_5, C_6, C_7$ such that

\[
\begin{align*}
\|B_{1,1}^1(u, n)\|_{X_1} &\leq C_1 \|u\|_{X_4} \|n\|_{X_1}, \\
\|B_{1,2}^1(n, c)\|_{X_1} &\leq C_2 \|n\|_{X_1} \|c\|_{X_2}, \\
\|B_{1,3}^1(n, v)\|_{X_1} &\leq C_3 \|n\|_{X_1} \|v\|_{X_3}, \\
\|B_{2,2}^1(u, c)\|_{X_2} &\leq C_4 \|u\|_{X_4} \|c\|_{X_2}, \\
\|B_{1,2}^2(n, c)\|_{X_2} &\leq C_5 \|n\|_{X_1} \|c\|_{X_2}, \\
\|B_{3,3}^1(u, v)\|_{X_3} &\leq C_6 \|u\|_{X_4} \|v\|_{X_3}, \\
\|B_{4,4}^1(u, \bar{u})\|_{X_4} &\leq C_7 \|u\|_{X_4} \|\bar{u}\|_{X_4},
\end{align*}
\]

(4.6)-(4.12)

for all $n \in X_1, c \in X_2, v \in X_3$ and $u, \bar{u} \in X_4$.

Proof. From the conditions (i), (ii) and (iii) in Assumption 1, we have that

\[
\frac{1}{2} - \frac{N}{2p} > 0, \quad -\frac{1}{2} + \frac{N}{2p} + \frac{N}{2q} > 0.
\]

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Taking \( s_1 = \frac{pq}{p_1+q_1} \), from (A), (B) and (C) in Assumption 1, it follows that

\[
1 \leq s_1 \leq \frac{pq}{p+q} \leq q \quad \text{and} \quad \frac{q}{q_1} \geq \frac{pq}{p+q} \frac{1}{s_1},
\]

and hence we can estimate

\[
\|B_{4,1}^1(u, n)(t)\|_{\mathcal{M}_{q_1}^q} = \left\| \int_0^t e^{(t-\tau)\Delta} (u \cdot \nabla n)(\tau) \, d\tau \right\|_{\mathcal{M}_{q_1}^q} \\
\leq \int_0^t \left\| \nabla \cdot e^{(t-\tau)\Delta} (un)(\tau) \right\|_{\mathcal{M}_{q_1}^p} \, d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} + \frac{1}{2} - \frac{1}{q}} \left\| (un)(\tau) \right\|_{\mathcal{M}_{q_1}^p} \, d\tau \quad \text{(by (2.4))} \\
\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} + \frac{1}{2}} \|u(\tau)\|_{\mathcal{M}_{p_1}^p} \|n(\tau)\|_{\mathcal{M}_{q_1}^q} \, d\tau \quad \text{(by (2.2))} \\
\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} + \frac{1}{2} - \frac{N}{2q}} \|u\|_{X_4} \|n\|_{X_1} \, d\tau \\
= C \frac{t^{\frac{N}{2q} - 1}}{t^{\frac{N}{2q}}} b \left( \frac{1}{2} - \frac{N}{2r}, \frac{1}{2} + \frac{N}{2q} + \frac{N}{2r} \right) \|u\|_{X_4} \|n\|_{X_1},
\]

(4.13)

for all \( t > 0 \), where \( C_1 = C(N, p, p_1, q, q_1) \) and \( b(\cdot, \cdot) \) denotes the beta function.

Taking \( s_2 = \frac{rq}{r_1+q_1} \), we have that

\[
\frac{1}{2} - \frac{N}{2r} > 0, -\frac{1}{2} + \frac{N}{2q} + \frac{N}{2r} > 0, 1 \leq s_2 \leq \frac{rq}{r+q} \leq q \quad \text{and} \quad \frac{q}{q_1} \geq \frac{rq}{r+q} \frac{1}{s_2},
\]

and then

\[
\|B_{4,2}^1(n, c)(t)\|_{\mathcal{M}_{q_1}^q} = \left\| \int_0^t \nabla \cdot e^{(t-\tau)\Delta} (n \nabla c)(\tau) \, d\tau \right\|_{\mathcal{M}_{q_1}^q} \\
\leq \int_0^t \left\| \nabla \cdot e^{(t-\tau)\Delta} (n \nabla c)(\tau) \right\|_{\mathcal{M}_{q_1}^p} \, d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} + \frac{1}{2} - \frac{1}{2}} \left\| (n \nabla c)(\tau) \right\|_{\mathcal{M}_{q_1}^p} \, d\tau \quad \text{(by (2.4))} \\
\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} + \frac{1}{2}} \|n(\tau)\|_{\mathcal{M}_{q_1}^q} \|\nabla c(\tau)\|_{\mathcal{M}_{q_1}^q} \, d\tau \quad \text{(by (2.2))} \\
\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} + \frac{1}{2} - \frac{N}{2q}} \|n\|_{X_1} \|c\|_{X_2} \, d\tau \\
= C \frac{t^{\frac{N}{2q} - 1}}{t^{\frac{N}{2q}}} \|n\|_{X_1} \|c\|_{X_2},
\]

(4.14)

for all \( t > 0 \), where \( C_2 = C(N, q, q_1, r, r_1) \). Similarly,

\[
\|B_{1,3}^1(n, v)(t)\|_{\mathcal{M}_{q_1}^q} \leq C \frac{t^{\frac{N}{2q} - 1}}{t^{\frac{N}{2q}}} \|n\|_{X_1} \|v\|_{X_2} b \left( \frac{1}{2} - \frac{N}{2r}, \frac{1}{2} + \frac{N}{2q} + \frac{N}{2r} \right) \\
= C_3 \frac{t^{\frac{N}{2q} - 1}}{t^{\frac{N}{2q}}} \|n\|_{X_1} \|v\|_{X_2},
\]

(4.15)
for all $t > 0$, where $C_3 = C_3(N, q, q_1, r, r_1)$. Thus, from (4.13)-(4.15), we obtain the inequalities (4.6), (4.7) and (4.8).

Now, from (i), (ii) and (iii) in Assumption 1, we have that

$$\frac{1}{2} - \frac{N}{2p} > 0, 1 - \frac{N}{2q} > 0 \quad \text{and} \quad \frac{1}{2} - \frac{N}{2q} + \frac{N}{2r} > 0.$$ 

Taking $s_3 = \frac{pr}{p + r}$, the conditions (A), (B) and (C) in Assumption 1 gives that

$$1 \leq s_3 \leq \frac{pr}{p + r} \leq r \quad \text{and} \quad \frac{r}{r_1} \geq \frac{pr}{p + r} \frac{1}{s_3}.$$ 

Hence, we have the following estimates for $B^2_{4,2}$ and $\nabla B^2_{4,2}$:

$$\|B^2_{4,2}(u, c)(t)\|_{L^\infty} = \left\| \int_0^t e^{(t-\tau)\Delta} (u \cdot \nabla c)(\tau) \, d\tau \right\|_{L^\infty}$$

$$\leq \int_0^t \left\| \nabla \cdot e^{(t-\tau)\Delta}(uc)(\tau) \right\|_{L^\infty} \, d\tau$$

$$\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} - \frac{1}{2}} \left\| (uc)(\tau) \right\|_{M^p_{\infty}} \, d\tau \quad \text{(by \,(2.6))}$$

$$\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} + \frac{1}{2}} \left\| u(\tau) \right\|_{M^p_{\infty}} \left\| c(\tau) \right\|_{L^\infty} \, d\tau$$

$$\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} + \frac{1}{2}} \tau \frac{N}{2p} + \frac{1}{2} \, d\tau \left\| u \right\|_{X_4} \left\| c \right\|_{X_2}$$

$$= C \|u\|_{X_4} \left\| c \right\|_{X_2} b \left( \frac{1}{2} - \frac{N}{2p}, 1 + \frac{N}{2p} \right)$$

$$= C_{4,1} \|u\|_{X_4} \left\| c \right\|_{X_2},$$

(4.16)

and

$$\|\nabla B^2_{4,2}(u, c)(t)\|_{M^p_{\infty}} = \left\| \nabla \int_0^t e^{(t-\tau)\Delta} (u \cdot \nabla c)(\tau) \, d\tau \right\|_{M^p_{\infty}}$$

$$\leq \int_0^t \left\| \nabla e^{(t-\tau)\Delta} (u \cdot \nabla c)(\tau) \right\|_{M^p_{\infty}} \, d\tau$$

$$\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} + \frac{1}{2}} \left\| (u \cdot \nabla c)(\tau) \right\|_{M^p_{\infty}} \left\| c(\tau) \right\|_{M^p_{\infty}} \, d\tau \quad \text{(by \,(2.4))}$$

$$\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} + \frac{1}{2}} \frac{N}{2p} + \frac{1}{2} \tau \frac{N}{2p} - \frac{1}{2} \, d\tau \left\| u \right\|_{X_4} \left\| c \right\|_{X_2}$$

$$\leq C t^{\frac{N}{2p} - \frac{1}{2}} b \left( \frac{1}{2} - \frac{N}{2p}, \frac{N}{2p} + \frac{N}{2r} \right) \left\| u \right\|_{X_4} \left\| c \right\|_{X_2}$$

$$= C_{4,2} t^{\frac{N}{2p} - \frac{1}{2}} \left\| u \right\|_{X_4} \left\| c \right\|_{X_2}.$$ 

(4.17)
In turn, we can estimate $B_{1,2}^2$ and $\nabla B_{1,2}^2$ as follows:

$$
\|B_{1,2}^2(n, c)(t)\|_{L^\infty} = \left\| \int_0^t e^{(t-\tau)\Delta} (nc)(\tau) \, d\tau \right\|_{L^\infty} 
\leq \int_0^t \left\| e^{(t-\tau)\Delta} (nc)(\tau) \right\|_{L^\infty} \, d\tau 
\leq C \int_0^t (t-\tau)^{-\frac{N}{2q}} \| (nc)(\tau) \|_{\mathcal{M}_{q_1}^3} \, d\tau \quad \text{(by (2.5))} 
\leq C \int_0^t (t-\tau)^{-\frac{N}{2q} + \frac{N}{2q} - \frac{1}{2}} \| n(\tau) \|_{\mathcal{M}_{q_1}^q} \| c(\tau) \|_{L^\infty} \, d\tau 
\leq C \int_0^t (t-\tau)^{-\frac{N}{2q} + \frac{N}{2q} - \frac{1}{2}} \| n(\tau) \|_{\mathcal{M}_{q_1}^q} \| c(\tau) \|_{L^\infty} \, d\tau 
\leq C t^{\frac{N}{2q} - \frac{1}{2}} b \left( \frac{1}{2} - \frac{N}{2q} + \frac{N}{2r} \right) \| n \|_{X_1} \| c \|_{X_2} 
= C_{5,1} \| n \|_{X_1} \| c \|_{X_2}, \quad (4.18)
$$

and

$$
\| \nabla B_{1,2}^2(n, c)(t) \|_{\mathcal{M}_{r_1}^r} = \left\| \nabla \int_0^t e^{(t-\tau)\Delta} (nc)(\tau) \, d\tau \right\|_{\mathcal{M}_{r_1}^r} 
\leq \int_0^t \left\| \nabla e^{(t-\tau)\Delta} (nc)(\tau) \right\|_{\mathcal{M}_{r_1}^r} \, d\tau 
\leq C \int_0^t (t-\tau)^{-\frac{N}{2q} + \frac{N}{2q} - \frac{1}{2}} \| (nc)(\tau) \|_{\mathcal{M}_{q_1}^q} \, d\tau \quad \text{(by (2.4))} 
\leq C \int_0^t (t-\tau)^{-\frac{N}{2q} + \frac{N}{2q} - \frac{1}{2}} \| n(\tau) \|_{\mathcal{M}_{q_1}^q} \| c(\tau) \|_{L^\infty} \, d\tau 
\leq C \int_0^t (t-\tau)^{-\frac{N}{2q} + \frac{N}{2q} - \frac{1}{2}} \| n(\tau) \|_{\mathcal{M}_{q_1}^q} \| c(\tau) \|_{L^\infty} \, d\tau 
\leq C t^{\frac{N}{2q} - \frac{1}{2}} b \left( \frac{1}{2} - \frac{N}{2q} + \frac{N}{2r} \right) \| n \|_{X_1} \| c \|_{X_2} 
= C_{5,2} t^{\frac{N}{2q} - \frac{1}{2}} \| n \|_{X_1} \| c \|_{X_2}, \quad (4.19)
$$

for all $t > 0$, where $C_{4,1} = C_{4,1}(N, p, p_1)$, $C_{4,2} = C_{4,2}(N, p, p_1, r, r_1)$, $C_{5,1} = C_{5,1}(N, q, q_1)$ and $C_{5,2} = C_{5,2}(N, q, q_1, r, r_1)$. Taking $C_4 = C_{4,1} + C_{4,2}$ and $C_5 = C_{5,1} + C_{5,2}$, estimates (4.9) and (4.10) follow from (4.16)-(4.19).

Proceeding similarly to (4.16), we can estimate $\nabla B_{4,3}^3$ in $\mathcal{M}_{r_1}^r$ as

$$
\| \nabla B_{4,3}^3(u, v)(t) \|_{\mathcal{M}_{r_1}^r} = \left\| \nabla \int_0^t e^{-\gamma(t-\tau)} e^{(t-\tau)\Delta} (u \cdot \nabla v)(\tau) \, d\tau \right\|_{\mathcal{M}_{r_1}^r} 
\leq C t^{\frac{N}{2q} - \frac{1}{2}} b \left( \frac{1}{2} - \frac{N}{2q} + \frac{N}{2r} \right) \| u \|_{X_4} \| v \|_{X_3} 
= C_6 t^{\frac{N}{2q} - \frac{1}{2}} \| u \|_{X_4} \| v \|_{X_3}, \quad (4.20)
$$

for all $t > 0$, where $C_6 = C_6(N, p, p_1, r, r_1)$, which gives (4.11).
Finally, since the projection operator $P$ is bounded in $M^p_{p_1}$, we have that

$$\|B_{4,4}^4(u, \tilde{u})(t)\|_{M^p_{p_1}} = \left\| \int_0^t e^{(t-\tau)\Delta} P(u \cdot \nabla \tilde{u}) d\tau \right\|_{M^p_{p_1}}$$

$$\leq \int_0^t \| \nabla e^{(t-\tau)\Delta} (u \otimes \tilde{u})(\tau) \|_{M^p_{p_1}} d\tau$$

$$\leq C \int_0^t \| e^{(t-\tau)\Delta} (u \otimes \tilde{u})(\tau) \|_{M^p_{p_1}} d\tau$$

$$\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} - \frac{1}{4} - \frac{1}{2} \cdot \frac{N}{p}} \| (u \otimes \tilde{u})(\tau) \|_{M^p_{p_1}} d\tau$$

$$\leq C \int_0^t t^{-\frac{N}{4}} \left( \frac{1}{2} - \frac{N}{2p} + \frac{N}{4} \right) \| u \|_{X^4} \| \tilde{u} \|_{X^4}$$

$$= C \int_0^t t^{-\frac{N}{4}} \| u \|_{X^4} \| \tilde{u} \|_{X^4},$$

for all $t > 0$, where $C_7 = C_7(N, p, p_1)$, and then we obtain (4.12).

\[ \square \]

### 4.2 Estimates for the linear terms in (4.5)

**Lemma 4.3.** Under the hypotheses of Theorem 3.1. There exist constants $\alpha, \beta > 0$ such that

$$\|L_3(n)\|_{X_3} \leq \alpha \| n \|_{X_1},$$

$$\|L_4(n)\|_{X_4} \leq \beta \| n \|_{X_1},$$

for all $n \in X_1$.

**Proof.** Using (2.4), we can estimate

$$\| \nabla L_3(n)(t) \|_{M^p_{p_1}} = \left\| \nabla \int_0^t e^{-\gamma(t-\tau)} e^{(t-\tau)\Delta} n(\tau) d\tau \right\|_{M^p_{p_1}}$$

$$\leq \int_0^t \| \nabla e^{(t-\tau)\Delta} n(\tau) \|_{M^p_{p_1}} d\tau$$

$$\leq C \int_0^t (t-\tau)^{-\frac{N}{2p} - \frac{1}{4} - \frac{1}{2} \cdot \frac{N}{p}} \| n(\tau) \|_{M^q_{q_1}} d\tau$$

$$\leq C t^{-\frac{N}{2p} - \frac{1}{2} \cdot \left( \frac{1}{2} - \frac{N}{2q} + \frac{N}{2q} \right) \| n \|_{X_1}}$$

$$= C t^{-\frac{N}{2p} - \frac{1}{2} \cdot \| n \|_{X_1}},$$

for all $t > 0$, where $\alpha = \alpha(N, q, q_1, r, r_1)$, which gives (4.22).

Now, considering $s_4 = \frac{Nq}{N_1 + q}$, from (A), (B) and (D) in Assumption 1, we have that

$$1 \leq s_4 \leq \frac{Nq}{N + q} \leq p$$

and $\frac{Nq}{N + q} \leq \frac{Nq}{N + q s_4}$.
Thus, $L_4(n)$ can be estimated as follows:

$$
\|L_4(n)(t)\|_{\mathcal{M}_p^{\alpha}} = \left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P}(nf)(\tau) \, d\tau \right\|_{\mathcal{M}_p^{\alpha}} \\
\leq \int_0^t \left\| \mathbb{P} e^{(t-\tau)\Delta} (nf)(\tau) \right\|_{\mathcal{M}_p^{\alpha}} \, d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{N}{2p}+\frac{N}{2p}-\frac{1}{2}} \left\| n f(\tau) \right\|_{\mathcal{M}_p^{\alpha}} \, d\tau \quad \text{(by (2.3))} \\
\leq C t^{-\frac{N}{2p}+\frac{N}{2p}-\frac{1}{2}} \left\| f \right\|_{\mathcal{M}_p^{\alpha}} \left\| n(\tau) \right\|_{\mathcal{M}_p^{\alpha}} \, d\tau \quad \text{(by (2.2))} \\
\leq C t^{-\frac{N}{2p}+\frac{N}{2p}-\frac{1}{2}} \left\| f \right\|_{\mathcal{M}_p^{\alpha}} \left\| n \right\|_{X_1} b \left( \frac{1}{2} + \frac{N}{2p} - \frac{N}{2q}, \frac{N}{2q} \right) \\
= \beta t^{-\frac{N}{2p}+\frac{N}{2p}-\frac{1}{2}} \left\| n \right\|_{X_1}, \quad (4.25)
$$

for all $t > 0$, where $\beta = (N_1, p, p_1, q, q_1, f)$, as requested.

\[\blacksquare\]

### 4.3 Proof of Theorem 3.1

Consider $X_1, X_2, X_3$ and $X_4$ as in (3.2)-(3.5) and let $y = (e^{t\Delta}n_0, e^{t\Delta}c_0, e^{-\gamma t}e^{t\Delta}v_0, e^{t\Delta}u_0)$. For $X = X_1 \times X_2 \times X_3 \times X_4$ and $x = (n, c, v, u) \in X$, we denote

$$
B_1(x) := B_{1,1}^1(u, n) + B_{1,2}^1(n, c) + B_{1,3}^1(n, v), \quad (4.26) \\
B_2(x) := B_{2,2}^1(u, c) + B_{1,2}^1(n, c), \quad (4.27) \\
B_3(x) := B_{3,3}^1(u, v) + L_3 \circ (e^{t\Delta}n_0 + B_1)(x), \quad (4.28) \\
B_4(x) := B_{4,4}^1(u, u) + L_4 \circ (e^{t\Delta}n_0 + B_1)(x). \quad (4.29)
$$

From Lemma 4.2, the operators $B_{i,j}^k$ in (4.26)-(4.29) are continuous bilinear maps. Also, from Lemma 4.3, $L_3$ and $L_4$ are continuous linear maps. Moreover, it is not difficult to see that all of them are time-weakly continuous at $t > 0$.

Next, we set

$$
K_1 = 1 + \alpha + \beta \quad \text{and} \quad K_2 = (\alpha + \beta)(C_1 + C_2 + C_3) + \sum_{i=1}^7 C_i. \quad (4.30)
$$

In view of equivalence (2.9), we have that

$$
\|y\|_X = \|e^{t\Delta}n_0\|_{X_1} + \|e^{t\Delta}c_0\|_{X_2} + \|e^{-\gamma t}e^{t\Delta}v_0\|_{X_3} + \|e^{t\Delta}u_0\|_{X_4} \\
= \sup_{t>0} e^{-\frac{N}{2p}+\frac{N}{2p}+\frac{1}{2}} \|e^{t\Delta}n_0\|_{\mathcal{M}_p^{\alpha}} + \sup_{t>0} \|e^{t\Delta}c_0\|_{L^\infty} + \sup_{t>0} e^{-\frac{N}{2q}+\frac{1}{2}} \|\nabla e^{t\Delta}c_0\|_{\mathcal{M}_p^{\alpha}} \\
+ \sup_{t>0} e^{-\frac{N}{2p}+\frac{1}{2}} \|\nabla^{-\gamma t}e^{t\Delta}v_0\|_{\mathcal{M}_p^{\alpha}} + \sup_{t>0} e^{-\frac{N}{2q}+\frac{1}{2}} \|e^{t\Delta}u_0\|_{\mathcal{M}_p^{\alpha}} \\
\leq C_0 \left( \|n_0\|_{\mathcal{N}_p^{\alpha,2}} + \|c_0\|_{L^\infty} + \|\nabla c_0\|_{\mathcal{N}_p^{\alpha,1} \cap L^\infty} + \|\nabla v_0\|_{\mathcal{N}_p^{\alpha,1} \cap L^\infty} + \|u_0\|_{\mathcal{N}_p^{\alpha,1} \cap L^\infty} \right) \\
= C_0 \|(n_0, c_0, v_0, u_0)\|_{X} \leq \varepsilon \quad (4.31)
$$

provided that $\|(n_0, c_0, v_0, u_0)\|_X \leq \delta = \frac{\varepsilon}{C_0}$. If $0 < \varepsilon < \frac{1}{K_1 K_2}$, then Lemma 4.1 implies that there exists a unique solution $(n, c, v, u) \in X$ of (3.7) such that $\|(n, c, v, u)\|_X \leq 2K_1 \varepsilon$. The continuity of the data-solution map follows from Remark 4.1 and estimate (4.31).

\[\blacksquare\]
4.4 Proof of Corollary 3.1

Since we use a fixed point argument to prove Theorem 3.1, the solution \((n, c, v, u)\) is the limit in the space \(X\) of the following Picard sequence (see Remark 4.1):

\[
(n^{(1)}, c^{(1)}, v^{(1)}, u^{(1)}) = (e^{t\Delta}n_0, e^{t\Delta}c_0, e^{-\gamma t}e^{t\Delta}v_0, e^{t\Delta}u_0)
\]

and

\[
(n^{(m+1)}, c^{(m+1)}, v^{(m+1)}, u^{(m+1)}) = (n^{(1)}, c^{(1)}, v^{(1)}, u^{(1)}) + \mathcal{F}(n^{(m)}, c^{(m)}, v^{(m)}, u^{(m)}), \text{ for } m \in \mathbb{N}.
\]

In other words,

\[
\begin{cases}
  n^{(m+1)} = e^{t\Delta}n_0 - \int_0^t e^{(t-\tau)\Delta}(u^{(m)} \cdot \nabla n^{(m)})(\tau) \, d\tau - \int_0^t \nabla \cdot e^{(t-\tau)\Delta}(n^{(m)} \nabla c^{(m)} + n^{(m)} \nabla v^{(m)})(\tau) \, d\tau, \\
  c^{(m+1)} = e^{t\Delta}c_0 - \int_0^t e^{(t-\tau)\Delta}(u^{(m)} \cdot \nabla c^{(m)} + n^{(m)} c^{(m)})(\tau) \, d\tau, \\
  v^{(m+1)} = e^{-\gamma t}e^{t\Delta}v_0 - \int_0^t e^{-\gamma(t-\tau)}e^{(t-\tau)\Delta}(u^{(m)} \cdot \nabla v^{(m)} - n^{(m)})(\tau) \, d\tau, \\
  u^{(m+1)} = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}d\mathbb{P}(u^{(m)} \cdot \nabla u^{(m)})(\tau) \, d\tau - \int_0^t e^{(t-\tau)\Delta}d\mathbb{P}(n^{(m)} f)(\tau) \, d\tau.
\end{cases}
\]

By hypotheses we have that \(n_0, c_0, v_0, u_0\) and \(f\) are homogeneous functions of degree \(-2, 0, 0, -1\) and \(-1\), respectively. Then, through a simple computation we can verify that \((n^{(1)}, c^{(1)}, v^{(1)}, u^{(1)})\) is invariant by (1.2), that is,

\[
\begin{align*}
  n^{(1)}(x,t) &= \lambda^2 n^{(1)}(\lambda x, \lambda^2 t), \\
  c^{(1)}(x,t) &= c^{(1)}(\lambda x, \lambda^2 t), \\
  v^{(1)}(x,t) &= v^{(1)}(\lambda x, \lambda^2 t) \quad \text{and} \quad u^{(1)}(x,t) = \lambda u^{(1)}(\lambda x, \lambda^2 t).
\end{align*}
\]

By means of an induction argument, we can check that \((n^{(m)}, c^{(m)}, v^{(m)}, u^{(m)})\) also satisfies the scaling property (4.32), for all \(m\). Since \((n, c, v, u)\) is the limit in \(X\) of the sequence \(\{ (n^{(m)}, c^{(m)}, v^{(m)}, u^{(m)}) \}_{m \in \mathbb{N}}\) and the norm \(\| \cdot \|_X\) is scaling invariant, we obtain that the solution \((n, c, v, u)\) is self-similar.

\[\blacksquare\]

4.5 Proof of Theorem 3.2

We first show that (3.9) implies (3.8). Let \((n, c, v, u)\) and \((\tilde{n}, \tilde{c}, \tilde{v}, \tilde{u})\) be two mild solutions given by Theorem 3.1 and set

\[
l_q = -\frac{N}{2q} + 1, \quad \mu_r = -\frac{N}{2r} + \frac{1}{2} \quad \text{and} \quad \mu_p = -\frac{N}{2p} + \frac{1}{2}.
\]

Estimating the difference \(n - \tilde{n}\) in the norm \(l_q \| \cdot \|_{\mathcal{M}^q_{21}}\), we obtain

\[
\begin{align*}
  t^{\frac{N}{2q} + 1} \|n(t) - \tilde{n}(t)\|_{\mathcal{M}^q_{21}} &\leq t^q \| e^{t\Delta}(n_0 - \tilde{n}_0)\|_{\mathcal{M}^q_{21}} \\
  & \quad + t^q \int_0^t \| e^{(t-\tau)\Delta}(u \cdot \nabla n - \tilde{u} \cdot \nabla \tilde{n})(\tau)\|_{\mathcal{M}^q_{21}} \, d\tau \\
  & \quad + t^q \int_0^t \| \nabla \cdot e^{(t-\tau)\Delta}(n \nabla c + n \nabla v - \tilde{n} \nabla \tilde{c} - \tilde{n} \nabla \tilde{v})(\tau)\|_{\mathcal{M}^q_{21}} \, d\tau \\
  &\quad := t^q \| e^{t\Delta}(n_0 - \tilde{n}_0)\|_{\mathcal{M}^q_{21}} + J_1(t) + J_2(t).
\end{align*}
\]
The integral $J_1$ is estimated as follows:

\[
J_1(t) \leq C_1 t^{k_1} \int_0^t (t - \tau)^{\mu_1 - 1}(\|u - \tilde{u}(\tau)\|_{\mathcal{M}_{\gamma_1}^p} + \|n(\tau)\|_{\mathcal{M}_{\gamma_1}^q}) d\tau \\
\leq C_1 t^{k_1} \int_0^t (t - \tau)^{\mu_1 - 1} \tau^{\mu_{l_0} - l_0} \tau^{\mu_{l_1}}(\|u - \tilde{u}(\tau)\|_{\mathcal{M}_{\gamma_1}^p} + \|n(\tau)\|_{\mathcal{M}_{\gamma_1}^q}) \, d\tau \\
+ C_1 t^{k_1} \int_0^t (t - \tau)^{\mu_1 - 1} \tau^{\mu_{l_0} - l_0} \tau^{\mu_{l_1}} \|\tilde{u}(\tau)\|_{\mathcal{M}_{\gamma_1}^p} \, d\tau, \text{ taking } \tau = tz \\
= C_1 \int_0^1 (1 - z)^{\mu_1 - 1} z^{-\mu_{l_0} - l_0} (tz)^{\mu_{l_1}} \|u - \tilde{u}(tz)\|_{\mathcal{M}_{\gamma_1}^p} \|n(\tau)\|_{\mathcal{M}_{\gamma_1}^q} \, dz \\
+ C_1 \int_0^1 (1 - z)^{\mu_1 - 1} z^{-\mu_{l_0} - l_0} (tz)^{\mu_{l_1}} \|\tilde{u}(\tau)\|_{\mathcal{M}_{\gamma_1}^p} \, dz. \quad (4.34)
\]

Similarly, from (4.14) and (4.15) we arrive at

\[
J_2(t) \leq C_2 \int_0^1 (1 - z)^{\mu_1 - 1} z^{-\mu_{l_0} - \mu_{l_1}} ((tz)^{l_1} \|n(\tau)\|_{\mathcal{M}_{\gamma_1}^q} \|c\|_{\mathcal{X}_2} \\
+ (tz)^{l_0} \|\nabla(c - \tilde{c})(tz)\|_{\mathcal{M}_{\gamma_1}^r} \|\tilde{n}\|_{\mathcal{X}_1}) \, dz \\
+ C_3 \int_0^1 (1 - z)^{\mu_1 - 1} z^{-\mu_{l_0} - \mu_{l_1}} ((tz)^{l_1} \|n(\tau)\|_{\mathcal{M}_{\gamma_1}^q} \|v\|_{\mathcal{X}_3} \\
+ (tz)^{l_0} \|\nabla(v - \tilde{v})(tz)\|_{\mathcal{M}_{\gamma_1}^r} \|\tilde{n}\|_{\mathcal{X}_1}) \, dz. \quad (4.35)
\]

In the following we estimate the differences $c - \tilde{c}$, $v - \tilde{v}$ and $u - \tilde{u}$ in the norms $\|\cdot\|_{L^\infty} + t^{\mu_{l_1}} \|\nabla\cdot\|_{\mathcal{M}_{\gamma_1}^r}$, $t^{\mu_1} \|\nabla\|_{\mathcal{M}_{\gamma_1}^r}$ and $t^{\mu_1} \|\cdot\|_{\mathcal{M}_{\gamma_1}^p}$, respectively. In this direction, we obtain

\[
\|c - \tilde{c}(t)\|_{L^\infty} \leq \|e^{t\Delta}(c_0 - \tilde{c}_0)\|_{L^\infty} + \int_0^t \|e^{(t - \tau)\Delta}(u \cdot \nabla c + nc - \tilde{u} \cdot \nabla \tilde{c} - \tilde{n}\tilde{c})(\tau)\|_{L^\infty} \, d\tau \\
\leq \|e^{t\Delta}(c_0 - \tilde{c}_0)\|_{L^\infty} + J_3(t), \quad (4.36)
\]

\[
t^{\mu_1} \|\nabla(c - \tilde{c})(t)\|_{\mathcal{M}_{\gamma_1}^r} \leq t^{\mu_1} \|\nabla e^{t\Delta}(c_0 - \tilde{c}_0)\|_{\mathcal{M}_{\gamma_1}^r} \\
+ t^{\mu_1} \int_0^t \|\nabla e^{(t - \tau)\Delta}(u \cdot \nabla c + nc - \tilde{u} \cdot \nabla \tilde{c} - \tilde{n}\tilde{c})(\tau)\|_{\mathcal{M}_{\gamma_1}^r} \, d\tau \\
\leq t^{\mu_1} \|\nabla e^{t\Delta}(c_0 - \tilde{c}_0)\|_{\mathcal{M}_{\gamma_1}^r} + J_4(t), \quad (4.37)
\]

\[
t^{\mu_1} \|\nabla(v - \tilde{v})(t)\|_{\mathcal{M}_{\gamma_1}^r} \leq t^{\mu_1} \|\nabla(-\gamma t e^{t\Delta}(v_0 - \tilde{v}_0))\|_{\mathcal{M}_{\gamma_1}^r} \\
+ t^{\mu_1} \int_0^t \|\nabla(-\gamma(t - \tau)e^{(t - \tau)\Delta}(u \cdot \nabla v + n - \tilde{u} \cdot \nabla \tilde{v} - \tilde{n}\tilde{v})(\tau))\|_{\mathcal{M}_{\gamma_1}^r} \, d\tau \\
\leq t^{\mu_1} \|\nabla(-\gamma t e^{t\Delta}(v_0 - \tilde{v}_0))\|_{\mathcal{M}_{\gamma_1}^r} + J_5(t) \quad (4.38)
\]

and

\[
t^{\mu_1} \|u - \tilde{u}(t)\|_{\mathcal{M}_{\gamma_1}^p} \leq t^{\mu_1} \|e^{t\Delta}(u_0 - \tilde{u}_0)\|_{\mathcal{M}_{\gamma_1}^p} \quad + t^{\mu_1} \int_0^t \|e^{(t - \tau)\Delta}\mathbb{P}(u \cdot \nabla u - \tilde{u} \cdot \nabla \tilde{u})(\tau))\|_{\mathcal{M}_{\gamma_1}^p} \, d\tau \\
+ t^{\mu_1} \int_0^t \|\mathbb{P}(n\tilde{f} - \tilde{n}f)(\tau))\|_{\mathcal{M}_{\gamma_1}^p} \, d\tau \\
\leq t^{\mu_1} \|e^{t\Delta}(u_0 - \tilde{u}_0)\|_{\mathcal{M}_{\gamma_1}^p} + J_6(t) + J_7(t). \quad (4.39)
\]
In view of (4.16), (4.18), (4.17), (4.19), (4.20), (4.21), (4.24) and (4.25), we have the following estimates for the integrals $J_3$, $J_4$, $J_5$, $J_6$ and $J_7$:

$$J_3(t) \leq C_{4,1} \int_0^1 (1 - z)^{\mu_p - 1} z^{-\mu_p} \|(tz)^{\mu_p} \|(u - \tilde{v})(t)\|_{\mathcal{M}_p^\mu} \|c\|_{X_2} + \|\tilde{u}\|_{X_2} \|(c - \tilde{c})(t)\|_{L_\infty} \, dz$$

$$+ \tilde{C}_{5,1} \int_0^1 (1 - z)^{l_q - 1} z^{-l_q} \|(tz)^{l_q} \|(n - \tilde{n})(t)\|_{\mathcal{M}_q^\mu} \|c\|_{X_2} + \|\tilde{n}\|_{X_2} \|(c - \tilde{c})(t)\|_{L_\infty} \, dz,$$

$$J_4(t) \leq C_{4,2} \int_0^1 (1 - z)^{\mu_p - 1} z^{-\mu_p - \mu_r} \|(tz)^{\mu_p} \|(u - \tilde{u})(t)\|_{\mathcal{M}_p^\mu} \|v\|_{X_2} + \|\tilde{u}\|_{X_2} \|(v - \tilde{v})(t)\|_{\mathcal{M}_1^\mu} \, dz$$

$$+ \tilde{C}_{5,2} \int_0^1 (1 - z)^{l_q - \mu_r - 1} z^{-l_q} \|(tz)^{l_q} \|(n - \tilde{n})(t)\|_{\mathcal{M}_q^\mu} \|c\|_{X_2} + \|\tilde{n}\|_{X_2} \|(c - \tilde{c})(t)\|_{L_\infty} \, dz,$$

$$J_5(t) \leq \tilde{C}_6 \int_0^1 (1 - z)^{\mu_p - 1} z^{-\mu_p - \mu_r} \|(tz)^{\mu_p} \|(u - \tilde{u})(t)\|_{\mathcal{M}_p^\mu} \|v\|_{X_2} + \|\tilde{u}\|_{X_2} \|(v - \tilde{v})(t)\|_{\mathcal{M}_1^\mu} \, dz$$

$$+ \tilde{C}_7 \int_0^1 (1 - z)^{l_q - \mu_r - 1} z^{-l_q} \|(tz)^{l_q} \|(n - \tilde{n})(t)\|_{\mathcal{M}_q^\mu} \|c\|_{X_2} + \|\tilde{n}\|_{X_2} \|(c - \tilde{c})(t)\|_{L_\infty} \, dz,$$

and

$$J_7(t) \leq \tilde{b} \int_0^1 (1 - z)^{l_q - \mu_r - 1} z^{-l_q} \|(tz)^{l_q} \|(n - \tilde{n})(t)\|_{\mathcal{M}_q^\mu} \|c\|_{X_2} + \|\tilde{n}\|_{X_2} \|(c - \tilde{c})(t)\|_{L_\infty} \, dz.$$

Now we define

$$A_1 := \limsup_{t \to \infty} t^{l_q} \|n(\cdot, t) - \tilde{n}(\cdot, t)\|_{\mathcal{M}_q^\mu},$$

$$A_2 := \limsup_{t \to \infty} \|c(\cdot, t) - \tilde{c}(\cdot, t)\|_{L_\infty},$$

$$A_3 := \limsup_{t \to \infty} t^{\mu_r} \|\nabla(c(\cdot, t) - \tilde{c}(\cdot, t))\|_{\mathcal{M}_1^\mu},$$

$$A_4 := \limsup_{t \to \infty} t^{\mu_r} \|\nabla(v(\cdot, t) - \tilde{v}(\cdot, t))\|_{\mathcal{M}_1^\mu},$$

$$A_5 := \limsup_{t \to \infty} t^{\mu_r} \|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{\mathcal{M}_p^\mu}.$$
\[ A_1 \leq 0 + \tilde{C}_1 2 K_1 \varepsilon \int_0^1 (1 - z)^{\mu_p - 1} z^{-\mu_p - l_q} \, dz \left( A_5 + A_1 \right) \\
+ \tilde{C}_2 2 K_1 \varepsilon \int_0^1 (1 - z)^{\mu_r - 1} z^{-l_q - \mu_r} \left( A_1 + A_3 \right) \, dz \\
+ \tilde{C}_3 2 K_1 \varepsilon \int_0^1 (1 - z)^{\mu_r - 1} z^{-l_q - \mu_r} \left( A_1 + A_4 \right) \, dz \\
\leq 2 K_1 \varepsilon [C_1 (A_1 + A_5) + C_2 (A_1 + A_3) + C_3 (A_1 + A_4)], \quad (4.45) \]
\[ A_2 \leq 0 + \tilde{C}_{4,1} 2 K_1 \varepsilon \int_0^1 (1 - z)^{\mu_p - 1} z^{-\mu_p} \, dz \left( A_5 + A_2 \right) \\
+ \tilde{C}_{5,1} 2 K_1 \varepsilon \int_0^1 (1 - z)^{l_q - 1} z^{-l_q} \, dz \left( A_1 + A_2 \right) \\
\leq 2 K_1 \varepsilon [C_{4,1} (A_2 + A_5) + C_{5,1} (A_1 + A_2)], \]
\[ A_3 \leq 0 + \tilde{C}_{4,2} 2 K_1 \varepsilon \int_0^1 (1 - z)^{\mu_p - 1} z^{-\mu_p - \mu_r} \, dz \left( A_5 + A_3 \right) \\
+ \tilde{C}_{5,2} 2 K_1 \varepsilon \int_0^1 (1 - z)^{l_q - \mu_r - 1} z^{-l_q} \, dz \left( A_1 + A_2 \right) \\
\leq 2 K_1 \varepsilon [C_{4,2} (A_3 + A_5) + C_{5,2} (A_1 + A_2)], \]
\[ A_4 \leq 0 + \tilde{C}_6 2 K_1 \varepsilon \int_0^1 (1 - z)^{\mu_p - 1} z^{-\mu_p - \mu_r} \, dz \left( A_5 + A_4 \right) \\
+ \tilde{\alpha} \int_0^1 (1 - z)^{l_q - \mu_r - 1} z^{-l_q} \, dz A_1 \\
\leq 2 K_1 \varepsilon [C_6 (A_4 + A_5)] + \alpha A_1, \]
\[ A_5 \leq 0 + \tilde{C}_7 2 K_1 \varepsilon \int_0^1 (1 - z)^{\mu_p - 1} z^{-2\mu_p} \, dz \left( A_5 + A_5 \right) \\
+ \tilde{\beta} \int_0^1 (1 - z)^{l_q - \mu_p - 1} z^{-l_q} \, dz A_1 \\
\leq 2 K_1 \varepsilon [C_7 2 A_5] + \beta A_1, \]

where \{C_1, C_2, C_3, C_4 = C_{4,1} + C_{4,2}, C_5 = C_{5,1} + C_{5,2}, C_6, C_7\} and \{\alpha, \beta\} are as in Lemmas 4.2 and 4.3, respectively.

Recalling that \( K_1 = 1 + \alpha + \beta \) and \( K_2 = (\alpha + \beta)(C_1 + C_2 + C_3) + \sum_{i=1}^7 C_i \) (see (4.30)) and summing all \( A_i \)'s, we arrive at
\begin{align*}
A_1 + A_2 + A_3 + A_4 + A_5 & \leq 2 K_1 \varepsilon \left[ A_1 (C_1 + C_2 + C_3 + C_{5,1} + C_{5,2}) + A_2 (C_{4,1} + C_{5,1} + C_{5,2}) + A_3 (C_2 + C_{4,2}) + A_4 (C_3 + C_6) + A_5 (C_1 + C_{4,1} + C_{4,2} + C_6 + 2C_7) \right] + (\alpha + \beta)A_1 \\
& \leq 2 K_1 \varepsilon \left[ A_1 (C_1 + C_2 + C_3 + C_{5,1} + C_{5,2} + (\alpha + \beta)[C_1 + C_2 + C_3]) + A_2 (C_{4,1} + C_{5,1} + C_{5,2}) + A_3 (C_2 + C_{4,2} + (\alpha + \beta)C_2) + A_4 (C_3 + C_6 + (\alpha + \beta)C_3) + A_5 (C_1 + C_{4,1} + C_{4,2} + C_6 + 2C_7 + (\alpha + \beta)C_1) \right] \text{ (by (4.45))}
\end{align*}

As \( C_4 = C_{4,1} + C_{4,2} \) and \( C_5 = C_{5,1} + C_{5,2} \), note that \( C_1 + C_2 + C_3 + C_{4,1} + C_{5,1} + C_{4,2} + C_{5,2} + C_6 + 2C_7 + (\alpha + \beta)(C_1 + C_2 + C_3) \leq 2 K_2 \), and then

\[ A_1 + A_2 + A_3 + A_4 + A_5 \leq 4 K_1 K_2 \varepsilon (A_1 + A_2 + A_3 + A_4 + A_5). \]

Since \( 4K_1K_2\varepsilon < 1 \), it follows that \( A_1 = A_2 = A_3 = A_4 = A_5 = 0 \).

Now we turn to show that (3.8) implies (3.9). We proceed as in the estimates (4.33) and (4.36)-(4.39) and use the hypothesis \( A_1 = A_2 = A_3 = A_4 = A_5 = 0 \) (see (3.8)) in order to obtain

\begin{align*}
\lim_{t \to \infty} \sup_{\mathcal{M}_{t;1}} t^\mu \|e^{t^\Delta} (n_0 - \tilde{n}_0)\|_{\mathcal{M}_{t;1}} & \leq A_1 + \lim_{t \to \infty} \sup_{t} (J_1(t) + J_2(t)) \\
& \leq A_1 + 2 K_1 \varepsilon C_1 (A_1 + A_5) + 2 K_1 \varepsilon C_2 (A_1 + A_3) + 2 K_1 \varepsilon C_3 (A_1 + A_4) = 0 + 0 + 0 = 0,
\end{align*}

\begin{align*}
\lim_{t \to \infty} \sup_{\mathcal{M}_{t;1}} \|e^{t^\Delta} (c_0 - \tilde{c}_0)\|_{L^\infty} & \leq A_2 + \lim_{t \to \infty} \sup_{t} J_5(t) \\
& \leq A_2 + 2 K_1 \varepsilon C_{4,1} (A_2 + A_5) + 2 K_1 \varepsilon C_{5,1} (A_1 + A_2) = 0 + 0 + 0 = 0,
\end{align*}

\begin{align*}
\lim_{t \to \infty} \sup_{\mathcal{M}_{t;1}} t^{\mu^*} \|\nabla e^{t^\Delta} (c_0 - \tilde{c}_0)\|_{L^\infty} & \leq A_3 + \lim_{t \to \infty} \sup_{t} J_4(t) \\
& \leq A_3 + 2 K_1 \varepsilon C_{4,2} (A_3 + A_5) + 2 K_1 \varepsilon C_{5,2} (A_1 + A_2) = 0 + 0 + 0 = 0,
\end{align*}

\begin{align*}
\lim_{t \to \infty} \sup_{\mathcal{M}_{t;1}} t^{\mu^*} \|\nabla e^{-\gamma t} e^{t^\Delta} (v_0 - \tilde{v}_0)\|_{\mathcal{M}_{t;1}} & \leq A_4 + \lim_{t \to \infty} \sup_{t} J_5(t) \\
& \leq A_4 + 2 K_1 \varepsilon C_6 (A_4 + A_5) + \alpha A_1 = 0 + 0 + 0 = 0
\end{align*}

and

\begin{align*}
\lim_{t \to \infty} \sup_{\mathcal{M}_{t;1}} t^{\mu^*} \|e^{t^\Delta} (u_0 - \tilde{u}_0)\|_{\mathcal{M}_{t;1}} & \leq A_5 + \lim_{t \to \infty} \sup_{t} (J_6(t) + J_7(t)) \\
& \leq A_5 + 2 K_1 \varepsilon C_7 2 A_5 + \beta A_1 = 0 + 0 + 0 = 0,
\end{align*}

and we are done.
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