Restricted Causal Inference

Mieczysław A. Klopotek
Institute of Computer Science, Polish Academy of Sciences
e-mail: klopotek@ipipan.waw.pl

1 Introduction

Causally insufficient structures (models with latent or hidden variables, or with confounding etc.) of joint probability distributions have been subject of intense study not only in statistics (e.g. [4], [1]) but also in various AI systems [2], [10]. In AI, belief networks, being representations of joint probability distribution with an underlying directed acyclic graph (dag), [3]) structure, are paid special attention due to the fact that efficient reasoning (uncertainty propagation) methods have been developed for them [7], [9]. Algorithms have been therefore elaborated to acquire the belief network structure from data [2], [7]. As artifacts due to variable hiding negatively influence the performance of derived belief networks, models with latent variables have been studied and several algorithms for learning belief network structure under causal insufficiency have also been developed [2], [10]. Regrettably, some of them are known already to be erroneous (e.g. IC algorithm of [8] - compare Discussion in chapter 6 of [10], FCI algorithm of [10] - compare [5]) and other not tractable for more than 10 variables (e.g. CI [10] - compare chapter 6 therein). This paper proposes a new algorithm for recovery of belief network structure from data handling hidden variables. It consists essentially in an extension of the CI algorithm of [10] by restricting the number of conditional dependencies checked up to k variables and in an extension of the original CI by additional steps transforming so called partial including path graph into a belief network. Its correctness is demonstrated.

2 Basic Definitions

Please refer to [10] (chapter 6) to recall definitions of including path graph, partially oriented including path graph, directed path, definite discriminating path, collider. Refer to [3] for definitions of active and passive trail, d-separation.

A partially oriented including path graph \(\pi\) contains the following types of edges unidirectional: \(A \rightarrow B\), bidirectional \(A < \rightarrow B\), partially oriented \(Ao \rightarrow B\) and non-oriented
A-oB, as well as some local constraint information $A \ast \rightarrow B \ast \rightarrow C$ meaning that edges between A and B and between B and C cannot meet head to head at B. (Subsequently an asterisk (*) means any orientation of an edge end: e.g. $A \ast \rightarrow B$ means either $A \rightarrow B$ or $A \rightarrow > B$ or $A < \rightarrow > B$).

Let us introduce some notions specific for r(k)CIA:

(i) A is r(k)-separated from B given set S ($\text{card}(S) \leq k$) iff A and B are conditionally independent given S

(ii) In a partially oriented including path graph $\pi$, a node A is called legally removable iff there exists no local constraint information $B \ast \rightarrow \overline{A} \ast \rightarrow C$ for any nodes B and C and there exists no edge of the form $A \ast \rightarrow B$ for any node B.

### 3 The Algorithm

The Restricted-to-k-Variables Causal Inference Algorithm (r(k)CIA):

**Input:** Empirical joint probability distribution

**Output:** Belief network.

A) Form the complete undirected graph $Q$ on the vertex set $V$.

B) if A and B are r(k)-separated given any subset S of $V$, remove the edge between A and B, and record S in Sepset(A,B) and Sepset(B,A).

C) Let $F$ be the graph resulting from step B). Orient each edge o-o. For each triple of vertices A,B,C such that the pair A,B and the pair B,C are each adjacent in F, but the pair A,C are not adjacent in F, orient $A \ast \rightarrow B \ast \rightarrow C$ as $A \ast \rightarrow > B < \ast \rightarrow C$ if and only if B is not in Sepset(A,C), and orient $A \ast \rightarrow B \ast \rightarrow C$ as $A \ast \rightarrow \overline{B} \ast \rightarrow C$ if and only if B is in Sepset(A,C).

D) Repeat

(D1) if there is a directed path from A to B, and an edge $A \ast \rightarrow B$, orient $A \ast \rightarrow B$ as $A \ast \rightarrow > B$,

(D2) else if B is a collider along $< A, B, C >$ in $\pi$, B is adjacent to D, A and C are not adjacent, and there exists local constraint $A \ast \rightarrow \overline{D} \ast \rightarrow C$, then orient $B \ast \rightarrow \overline{D}$ as $B \ast \rightarrow D$,

(D3) else if U is a definite discriminating path between A and B for M in $\pi$ and P and R are adjacent to M on U, and P-M-R is a triangle, then if M is in Sepset(A,B) then M is marked as non-collider on subpath $P \ast \rightarrow \overline{M} \ast \rightarrow R$ else $P \ast \rightarrow \overline{A} \ast \rightarrow \overline{M} \ast \rightarrow \overline{R}$ is oriented as $P \ast \rightarrow M \ast \rightarrow R$,
(D4) else if \( P \rightarrow M \rightarrow R \) then orient as \( P \rightarrow M \rightarrow R \).

until no more edges can be oriented.

E) Orient every edge \( A \rightarrow B \) as \( A \rightarrow B \).

F) Copy the partially oriented including path graph \( \pi \) onto \( \pi' \).

Repeat:

In \( \pi' \) identify a legally removable node \( A \). Remove it from \( \pi' \) together with every edge \( A \rightarrow B \) and every constraint with \( A \) involved in it. Whenever an edge \( A \rightarrow oB \) is removed from \( \pi' \), orient edge \( A \rightarrow oB \) in \( \pi \) as \( A < B \).

Until no more node is left in \( \pi' \).

End of r(k)CIA

4 Differences to Spirtes et al. CI Algorithm

Steps E) and F) constitute an extension of the original CI algorithm of [10], bridging the gap between partial including path graph and the belief network. Their properties are subject of [6] with respect to CI. Their correctness for r(k)CIA may be proven in the same way.

Step B) was modified by substituting the term ”d-separation” with ”r(k)-separation”. This means that not all possible subsets \( S \) of the set of all nodes \( V \) (with card(\( S \)) up to card(\( V \))-2) are tested on rendering nodes \( A \) and \( B \) independent, but only those with cardinality 0,1,2,...,k. If one takes into account that higher order conditional independences require larger amounts of data to remain stable, superior stability of this step in r(k)CIA becomes obvious.

Step D2) has been modified in that the term ”not d-connected” of CI was substituted by reference to local constraints. In this way results of step B) are exploited more thoroughly and in step D) no more reference is made to original body of data (which clearly accelerates the algorithm). This modification is legitimate since all the other cases covered by the concept of ”not d-connected” of CI would have resulted in orientation of \( D \rightarrow B \) already in step C). Hence the generality of step D2) of the original CI algorithm is not needed here.

5 Properties of the Algorithm

Obviously, the algorithm r(k)CIA will leave some edges actually not present in original data. As demonstrated in [6], superfluous edges may lead to incorrect belief network recovery. We shall show therefore that this is not the case with r(k)CIA.

In [6] it has been proven that the original CI extended by above-mentioned steps E) and F) will produce a dag compatible with the original data. Preliminaries for that result are that given the ”real” dag \( G \) with visible variables \( V_s \) and hidden ones \( V_h \) one can define
an "intrinsic" dag $F$ in $V_s$ indistinguishable from $G$ with respect to dependencies and independences within set $V_s$ such that the modified CI algorithm produces a dag statistically indistinguishable from $F$. (This dag $F$ is called "including path graph" in [10]). Below we show possibility of defining such a dag $F$ for the r(k)CIA algorithm.

Let us define the r(k)-including path graph: $G$ be a dag with a set of hidden variables $V_h$ and of visible variables $V_s$. A graph $\pi$ be a r(k)-including path graph for $G$ iff its set of nodes is $V_s$, and an edge between $A$ and $B$ from $V_s$ exists in $\pi$ iff no subset $S$ of $V_s$ with cardinality not exceeding $k$ does not d-separate nodes $A$ and $B$ in $G$. This edge is out of $A$ iff there exists such a subset $S'$ of $V_s$ with cardinality not exceeding $k-1$ that no trail in $G$ from $B$ to $A$ into $S'$ is active with respect to $S'$. Otherwise this edge is ingoing into $A$. It is easily demonstrated that every edge in $\pi$ is either unidirectional or bidirectional (no edge is left unoriented). Because there exists never a trail outgoing from $A$ and outgoing from $B$ which is active with respect to an empty set $S$ ($\text{card}(S) = 0 \leq k$). Furthermore, if there is an edge $A \rightarrow B$ in $\pi$, then there exists a directed path from $A$ to $B$ in $G$. This is easily seen: Let $S'$ be a subset of $V_s$ with cardinality not exceeding $k-1$ that no path in $G$ from $B$ to $A$ into $S'$ is active with respect to $S'$. (1) Then clearly there must exist a trail in $G$ outgoing out of $A$ towards $B$ which is active with respect to $S'$ (otherwise edge $AB$ would be absent from $\pi$ as $S'$ would d-separate $A$ and $B$). (2) Let us go along this trail in $G$ as long as edges along it passing edges from tail to head. In this way we either reach $B$ (which would complete the proof) or stop at a collider along this trail. This collider must either be in $S'$ or have a successor in $S'$ (as active trail definition requires). Let us continue the journey towards the blocking node in $S'$. The node is either not necessary for $S'$ to block all ingoing trails from $B$ to $A$ (in this case we remove it from $S'$ and start the procedure from the beginning - that is from point (1)) or it is necessary for that purpose. (3) In the latter case there is a trail between $B$ and $A$ ingoing into $A$ this node is blocking. Let us continue our journey along this trail now in the direction where we pass edges from tail to head (at least one such direction exists). We continue at point (2). As the graph is a dag and the set $S'$ is finite, the procedure is granted to terminate on reaching node $B$. This proves our claim.

Furthermore, in $\pi$ if we have two edges $A \rightarrow B < -C$ with $A$ and $C$ not adjacent in $\pi$ then no subset $S$ of $V_s$ with cardinality not greater than $k$ containing $B$ such that $S$ d-separates $A$ and $C$ in $G$. Because if such a set $S$ existed then the set $S-\{B\}$ with cardinality not greater than $k-1$ would have to block in $G$ all trails from $A$ to $B$ into $B$ or all trails from $C$ to $B$ into $B$ (as this is required by definition of d-separation). But then the aforementioned definition of $\pi$ would require that either edge $BA$ or $BC$ resp. would be out of $B$.

Also in $\pi$ if $A,B$ are adjacent, $B,C$ are adjacent, but $A,C$ are not adjacent in $\pi$ and on the trail $A-B-C$ node $B$ is non-collider then there exists no such subset $S$ of $V_s$ with cardinality not greater than $k$ not containing $B$ that $S$ d-separates $A$ and $C$ in $G$. Otherwise if such a set $S$ existed then (without restriction of generality let us assume $A < -B$) there exists a directed path from $A$ to $B$ in $G$. The set $S$ would either block it or not. If not, then $S$ would have to block all the trails from $C$ to $B$ which is a contradiction because then edge $BC$ could
not exist in \( \pi \). Hence it must block it. But then S would have to block every trail ingoing into B either from direction of A or of C. Should it block those from direction of A (C) then there would exist an active trail outgoing from B towards A (C) and an active trail between B and C (A). But this is a contradiction as then there would exist an active trail connecting A and C (via B). This proves our claim.

Last not least it may be demonstrated that if there exists in \( \pi \) a bidirectional edge between A and B, and if there existed an oriented path from A to B and if there exists edge \( C \rightarrow A \) in \( \pi \), then in \( \pi \) there exists also the edge \( C \rightarrow B \). This means that a bidirectional edge \( A \leftarrow B \) in \( \pi \) can be treated as a unidirectional edges \( A \leftarrow H \rightarrow B \), with H being a parentless hidden variable and A and B being not adjacent in the graph.

The above-mentioned statements indicate that for a faithful graph G for edge pair A-B and B-C with A and C not adjacent a statistical test of independence of A and C relatively to sets S containing B with cardinality not greater than \( k \) will correctly decide about orientation of edges with respect to the \( r(k) \)-including path graph \( \pi \).

In this sense we can prove for \( \pi \) that it is a directed acyclic graph. Because graph G is a dag, bidirectional edges in \( \pi \) have no impact on ordering, and unidirectional edges in \( \pi \) must have orientation in accordance with G as existence of an oriented path in G between nodes at both ends of an edge of \( \pi \) is assumed.

With these prerequisites correctness proof of a modified version of CI algorithm from [6] extends straight forward to \( r(k) \)CIA algorithm

\section{Discussion and Concluding Remarks}

Within this paper a new algorithm of recovery of belief network structure from data has been presented and its correctness demonstrated. It relies essentially on ”acceleration” of the known CI algorithm of Spirtes, Glymour and Scheines [10] by restricting the number of conditional dependencies checked up to \( k \) variables and it extends CI by additional steps transforming so called partial including path graph into a belief network. Sample outputs of CI, \( r(1) \)CIA and \( r(2) \)CIA are shown in Fig.1. Though \( r(k) \)CIA introduces redundant edges (e.g. AC and FC in Fig.1b), indicating dependencies not present in the original data, it actually avoids pitfalls of the FCI algorithm, another CI ”accelerator” proposed by Spirtes, Glymour and Scheines [10], as visible from section 5 and [6].

\section*{References}

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Figure 1: a) Original dag, b) CI output, c) r(1)CIA output, d) r(2)CIA output

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