Comments on Truncation Errors for Polynomial Chaos Expansions

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Abstract—Polynomial chaos expansion methods allow to approximate the behavior of uncertain stochastic systems by deterministic dynamics. These methods are used in a wide range of applications, spanning from simulation of uncertain systems to estimation and control. For practical purposes the exploited spectral series expansion is typically truncated to allow efficient computation, which leads to approximation errors. Despite the Hilbert space nature of polynomial chaos, there are only a few results in the available literature that explicitly discuss and quantify approximation errors. This work derives simple error bounds for polynomial chaos approximations of polynomial and non-polynomial mappings. Sufficient conditions for zero truncation errors are established, which allow to investigate the question if zero truncation errors can be achieved and which series order is required to achieve this. Furthermore, convex quadratic programs, whose solution operator is a special case of a polynomial mapping, are studied in detail due to their relevance in predictive control. Several simulation examples illustrate our findings.

Index Terms—Polynomial chaos expansion, stochastic systems, stochastic uncertainties, model predictive control

I. INTRODUCTION

Uncertainty is inherent to many applications. Considering and counteracting disturbances is becoming ever more important as systems are pushed to the boundaries of operation, for economic reasons or for increased interoperability. By now, many strategies have been developed to predict and counteract disturbances and uncertainties. With respect to systems and control they span robust control [26], stochastic, and robust model predictive control approaches [10, 20].

Various methods for uncertainty description, prediction and decision making under uncertainties exist [9, 24]. Besides stochastic uncertainty descriptions, often deterministic uncertainty descriptions and bounds are used. In the deterministic setting, uncertainties are typically described by bounded sets, leading to worst-case assumptions and worst-case predictions of the future system behavior.

Probabilistic approaches instead treat the uncertainty as a realization of a random variable (often continuous, second-order) that has to be propagated through given mappings, e.g. system dynamics, to obtain insights on the influence on the variable of interest, or for control. Recently, polynomial chaos expansion (PCE) has gained popularity in the field of systems and control to propagate probabilistic uncertainty descriptions and to quantify their influence [15, 16, 18]. PCE originates in the works of Norbert Wiener [23]. In PCE the stochastic variables are replaced by an (infinite) sum of weighted orthogonal polynomials [24]. The approximated system is deterministic but of larger dimension than the original system. This expanded system has been used, for example, to design linear controllers [5, 8, 21], and has been exploited in model predictive control [6, 11, 16, 17, 19].

For sake of computational tractability, it is necessary to truncate the infinite polynomial chaos expansion to finite order. While the use of PCE in the field of systems and control is steadily increasing it is commonly and frequently assumed that: (i) the input uncertainty x can be exactly described using finitely many PCE coefficients; (ii) the nonlinear function—denoted in the following by \( f(\cdot) \)—that maps x to the desired output \( y = f(x) \) is analytically known; and (iii) the output y can be exactly realized by a finite number of PCE coefficients. Moreover, whereas under these conditions PCE is exact in the limit (in the \( L^2 \)-sense), its truncation is often a trade-off between approximation accuracy and computational tractability. For stability and performance guarantees, however, bounding the approximation error is important. As such, stability and performance guarantees derived for the approximated system do not necessarily apply to the original stochastic system.

To the best of the authors’ knowledge, there are only a limited number of results that consider the PCE truncation errors directly: In [7] several illustrative examples are given that evaluate the accuracy of PCE. Yet, the errors are not computed rigorously. Rather, they are studied via extensive simulations. Similarly, [4] list several numerical challenges when using PCE including the potential need for large PCE dimensions.

For MPC-specific applications of PCE, error bounds on the first- and second-order moments, which are functions of the zero- and first-order PCE coefficients, are established in [12]. These results provide a deep insight, however, no bounds on the error of the underlying projections in Hilbert spaces are given. The authors of [1] provide an upper bound on the truncation error using a univariate Hermitian basis based on differentiability assumptions of \( f(\cdot) \). These results do not carry over to other bases easily.

The main contribution of the present paper is to leverage well-established Hilbert space theory [13] to the end

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1Here, \( f(\cdot) \) represents a generic mapping, e.g. the state transition map of a system of ordinary differential equations, an LTI system in discrete or continuous time, a system of nonlinear algebraic equations, or the argmin-operator of a suitable convex optimization.
of quantifying truncation errors for $y$ in the $L^2$-sense for multivariate uncertainties for applications in the field of systems and control. We provide exact error descriptions instead of error bounds. Considering polynomial and non-polynomial mappings $f(\cdot)$, we tackle the question of how to choose the output PCE dimension such that the truncation error vanishes. Moreover, we establish bounds that allow the computation of minimum PCE dimensions such that a user-specified error tolerance is met. Furthermore, we study truncation errors for convex quadratic programs due to their relevance in stochastic model predictive control. Illustrative examples accompany the findings.

The remainder is organized as follows: Section II introduces PCE and the tackled research questions. Section III establishes results on PCE truncation errors for polynomial mappings. Furthermore, truncation errors for convex quadratic programs are derived as they typically appear in model predictive control. Section IV derives error bounds for the non-polynomial case.

II. PROBLEM FORMULATION

We consider random variables $y$ that are the image of random variables $x$ via the square-integrable mapping $f$:

$$y = f(x).$$

We assume that $x$ and $y$ are real-valued second-order random variables with multivariate $\mathbb{R}^{n_x}$-valued stochastic germ $\xi$, cf. [22, 24]. The mapping $f: x \mapsto y$ can, for example, describe the state transition map for continuous-time and discrete-time systems subject to uncertainties in the system matrix. Also, it can describe the influence of a set of uncertain parameters/initial conditions on the output. Describing the dependence of the random variable $y$ on the properties of the random variable $x$ is in general challenging. One way to do this is via PCE, which allows the representation of any random variable with finite second-order moment as a possibly infinite, series of weighted orthogonal polynomials [16, 24]. The main question of interest in the present paper is to quantify the approximation error made if the series expansion is terminated early.

A. Polynomial Chaos Expansion

We first focus on describing real-valued random variables $x$ from the Hilbert space $L^2(\Omega; \mu; \mathbb{R})$ of equivalence classes of univariate real-valued second-order random variables given the probability space $(\Omega, \mathcal{F}, \mu)$.

First note that the Hilbert space over the product probability space is (under mild technical assumptions) equivalent to the Hilbert space tensor product and given by

$$L^2(\Omega; \mu; \mathbb{R}) := \bigotimes_{i=1}^{n_x} L^2(\Omega_i; \mu_i; \mathbb{R}),$$

where $\Omega = \Omega_1 \times \cdots \times \Omega_{n_x}$, and $\mu = \mu_1 \otimes \cdots \otimes \mu_{n_x}$, cf. [22].

Assume that the set of $n_x$-variate polynomials $\{\phi_i\}_{i=0}^d$ spans the space $L^2(\Omega, \mu; \mathbb{R})$ and satisfies the orthogonality relation for all $i, j \in \mathbb{N}_0$

$$\langle \phi_i, \phi_j \rangle := \int_{\Omega} \phi_i(\tau)\phi_j(\tau)d\mu(\tau) = \delta_{ij}\|\phi_i\|^2,$$

with Kronecker-delta $\delta_{ij}$, and the induced norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. This allows to define the PCE of $x$:

Definition 1 (Polynomial chaos expansion): The polynomial chaos expansion of a real-valued random variable $x \in L^2(\Omega, \mu; \mathbb{R})$ is

$$x = \sum_{i=0}^{\infty} x_i \phi_i,$$

where $x_i \in \mathbb{R}$ is called the $i$th PCE coefficient [22, 24].

In practice, the truncated PCE of $x$ is often considered to allow for more efficient calculations:

Definition 2 (Truncated Polynomial chaos expansion): The truncated PCE of $x$ is

$$P_\ell x := \sum_{i=0}^{\ell} x_i \phi_i,$$

where $\ell + 1$ is the PCE dimension of the subspace $X \subseteq L^2(\Omega, \mu; \mathbb{R})$ spanned by $\{\phi_i\}_{i=0}^\ell$; PCE dimension in short. The basis is chosen to contain all $n_\xi$-variate polynomials $\phi_i$ of degree at most $d$, yielding

$$\ell + 1 = \frac{(n_\xi + d)!}{n_\xi!d!}.$$

If the PCE coefficients are computed using the Fourier quotient (3), the truncated PCE $P_\ell x$ from (4) is the orthogonal projection of $x$ onto $X$.

B. Truncation Error and Mappings

First note that it can be shown that the truncation error $e_\ell := x - P_\ell x$ is orthogonal to the PCE $P_\ell x$, i.e. $x - P_\ell x \perp P_\ell x$, and satisfies

$$\lim_{\ell \to \infty} \|e_\ell\| = 0,$$

cf. [3, 22, 24]. Furthermore, if the weight to which the polynomials are orthogonal matches the (product) measure $\mu$—as given by the Askey scheme—convergence of (6) is known to be exponential [3, 23, 25]. Besides exponential convergence several questions with respect to the truncation error are immediate: Firstly, is it possible to describe a random variable and its mapping precisely by a finite expansion? Secondly, if the PCE is truncated early, is it possible to establish an error bound on $y$? To answer the above questions we define the minimum degree of a PCE as follows:

Definition 3 (Minimum expansion degree): The minimum degree of $x \in L^2(\Omega, \mu; \mathbb{R})$ is the natural number $d \in \mathbb{N}_0$ that all PCE coefficients associated with higher-degree basis polynomials are zero, i.e. $x_i=0$ for all $i$ with $\deg \phi_i > d$.

Assumption 1 (Exact PCE input): The PCE of the real-valued random variable $x \in L^2(\Omega, \mu; \mathbb{R})$ has the known and
finite minimum degree $d_x \in \mathbb{N}_0$, and $\ell_x + 1$ PCE coefficients, cf. (5), for an orthogonal polynomial basis $\{\phi_i\}_{i=0}^{\ell}$. In other words, a finite number of known PCE coefficients yields a vanishing truncation error for the input uncertainty,

$$\forall \ell \geq \ell_x \text{ from Assumption 1: } \|x - P_\ell x\| = 0.$$  \hfill (7)

We now turn back to the main question of quantifying the approximation error when describing the random variable $y$ as a function of $x$ as given by (1). For the case of univariate Gaussian $\xi$ the following result is known.

**Theorem 1 (Bound in univariate Hermite basis [1]):**
Let $n_x = 1$, $n_\xi = 1$ and let the stochastic germ $\xi$ be a standard Gaussian random variable. Consider $x, y \in H = L^2(\mathbb{R}, \mu_{\text{Gauss}}; \mathbb{R})$, where $y = f(x)$ with $f : H \to H$ square-integrable. Furthermore, let $y$ be $k \in \mathbb{N}$ times differentiable with $y^{(k)} = f(x^{(k)}) \in H$. Then, for $k \leq n + 1$ it holds that

$$\left\| y - \sum_{i=0}^n y_i H\xi_i \right\| \leq \frac{\|f(x^{(k)})\|}{\sum_{i=0}^n \sqrt{n - i + 1}} = : \tilde{e}_n^k,$$  \hfill (8)

where $H\xi_i$ is the $i$th probabilists’ Hermite polynomial. □

Theorem 1 provides an error bound specifically tailored to univariate Gaussian uncertainties. However, obtaining sharp bounds for the general case is especially important for many applications in systems and control, specifically w.r.t. performance and stability results. Two questions arise:

Q1 Choosing the PCE dimension equal to the PCE input dimension, i.e. $n + 1 = \ell_x + 1$, what truncation error is made given a square-integrable nonlinear mapping $f(.)$?

Q2 What is the minimum PCE dimension $n + 1$ such that no truncation error is made for $y$?

### III. TRUNCATION ERRORS FOR POLYNOMIAL MAPS

Let $x_i, y \in H = L^2(\Omega; \mu; \mathbb{R})$ with $i = 1, \ldots, n_x$ be real-valued random variables. Moreover, let $X \subset H$ be a complete subspace of dimension $n + 1$ generated by the orthogonal polynomial basis functions $\{\phi_i\}_{i=0}^{n}$. For ease of presentation we consider that $y$ and all $x_i$ have the same PCE dimension.

**Theorem 2 (Error under polynomial mapping):** Suppose that all $x_i$ satisfy Assumption 1 with minimum degree $d_x$, and let $f : H^{n_x} \to H$ be a square-integrable polynomial mapping of degree $d_f$ such that $y = f(x_1, \ldots, x_{n_x})$. Then, the magnitude of the truncation error $e_n = y - P_n y$ is

$$e_n := \|e_n\| = \left\{ \begin{array}{ll} \sum_{i=n+1}^{\ell_x} y_i^2\|\phi_i\|^2, & n < \ell, \\
0, & n \geq \ell, \end{array} \right.$$  \hfill (9)

where

$$\ell + 1 = \frac{(n_x + d_x d_f)!}{(n_\xi^2 d_x d_f)!},$$  \hfill (10)

and $y_i$ are the PCE coefficients of $y$.

**Proof:** The PCE for $x_i$ is given by

$$x_i = \sum_{j=0}^{\ell_x} x_{i,j} \phi_j,$$  \hfill (11)

**TABLE I**

| $d_x$ | 1     | 2     | 3     |
|-------|-------|-------|-------|
| $e_{d_x}^2$ | $2x_1^4$ | $24x_1^2(x_2^2 + x_3^2)$ | $480x_1^2x_2^2 + 24(x_2^4 + 9x_3^4 + 2x_1x_3x_4^2) + 720x_4^4$ |
| $(e_{d_x+1}^2)^2$ | $2x_1^4$ | $24x_1^2(x_2^2 + 4x_3^2)$ | $2400x_1^2x_2^2 + 24(x_2^4 + 9x_3^4 + 2x_1x_3x_4^2) + 10800x_4^4$ |

where $\ell_x + 1 = (n_x + d_x)!/(n_\xi^2 d_x d_f)!$. Substituting this into the polynomial mapping $f(.)$, one obtains

$$y = f\left(\sum_{j=0}^{\ell_x} x_{1,j} \phi_j, \ldots, \sum_{j=0}^{\ell_x} x_{n_x,j} \phi_j\right)$$

$$= \sum_{j=0}^{\ell_x} \sum_{k=0}^{d_x} \alpha_{ij} \xi_i^k = \sum_{i=0}^{\ell_x} y_i \phi_i,$$

The highest-degree polynomial term of $y$ has degree $d_x d_f$, thus enlarging the basis by $\ell - \ell_x$ elements. The number of basis elements is given by (5) with $d = \ell_x d_f$. Projecting $y$ onto $X$ yields $P_n y = \sum_{i=0}^{\ell_x} y_i \phi_i$. Consequently, the truncation error $e_n$ becomes $e_n = \sum_{i=n+1}^{\ell_x} y_i \phi_i$, which is zero in case of $n \geq \ell$. For $n < \ell$, apply Parseval’s identity to obtain \(\|e_n\|\), cf. [2]. □

In light of Theorem 2, the answers to questions Q1 and Q2 are summarized.

**Corollary 1 (Error/minimum degree – polynomial):**

A1 Given a polynomial mapping $f(.)$ such that $y = f(x)$ with $x, y \in L^2(\Omega; \mu; \mathbb{R})$, and choosing the PCE dimension $\ell + 1$ equal to the PCE input dimension $\ell_x + 1$, the truncation error is given by $\tilde{e}_n$ from (9). □

A2 Furthermore, the minimum dimension is $\ell + 1 = (n_x + d_x d_f)!/(n_\xi^2 d_x d_f)!$.

In view of Theorem 2, it is fair to ask for a comparison with respect to the error bound from Theorem 1. This is examined by means of a numerical example.

**Example 1:** Let $n_x = 1$, $n_\xi = 1$. $x$ is a Gaussian random variable with mean $\mu$ and standard deviation $\sigma > 0$. Consider the mapping $y = x^2$. If $X = \text{span}\{H_0, H_1\}$, i.e. the subspace $X$ is spanned by the first two Hermite polynomials, then the error becomes $e = \sigma^2 H_2$ with norm $e = \sqrt{2}\sigma^2$. For derivates $k \in \{1, 2\}$ the respective error (8) becomes $\tilde{e}_k = \sqrt{2}\sigma \sqrt{\mu^2 + \sigma^2} \geq \sqrt{2}\sigma^2 = e_1$. The minimum exact PCE degree for $y$ is $d_x d_f = 2$. Adding another basis function span$\{H_3\}_{i=0}^n \supset X$, the projection error becomes zero. Table 1 shows the squared norm of $e_{d_x} = y - P_{d_x} y$ for ascending input degree $d_x$ and symbolic PCE input coefficients $x_0, \ldots, x_{d_x}$. Exactness of the error $e_{d_x+1}^2$ can only be ensured in the case of $d_x = 1$. In the other cases shown, $\tilde{e}_{d_x+1}^2 > e$ holds. □
state transition map is polynomial in the uncertainty. In the following, we focus on uncertain convex quadratic programs (QPs) due to their important role in systems and control. For example, model predictive control (MPC) for discrete-time LTI systems with convex constraints and a convex quadratic cost function is well-known to be equivalent to solving a QP repeatedly online at each time instant [14, 20]. Also, QPs are the basis for sequential quadratic programming methods for solving nonlinear programs that are encountered in nonlinear MPC. In many cases, however, the problem data of the QP is uncertain—in these cases PCE is of advantage.

**Problem 1 (QP with uncertain data):** Let $h$ be an $\mathbb{R}^{n_x}$-valued random vector with elements $h_1, \ldots, h_{n_x} \in H = L^2(\Omega, \mu; \mathbb{R})$. Let $X \subset \mathbb{R}^{n_x}$ be a complete subspace of dimension $n + 1$ generated by the orthogonal polynomial basis functions $\{\phi_i\}_{i=0}^n$. Consider

$$y := \arg\min_{\chi \in \mathbb{R}^{n_x}} \frac{1}{2} \chi^\top H \chi + x_1^\top \chi$$

for positive definite $H \in \mathbb{R}^{n \times n}$ and a non-empty feasible set $\{\chi \in \mathbb{R}^{n_x} : Ax + x_2 \leq 0\}$. The entries of the vectors $x_1 \in \mathbb{R}^{n_x}$ and $x_2 \in \mathbb{R}^{n_{con}}$ are realizations of the vector-valued random variables $x_1$ and $x_2$, respectively. Then, the problem is to find the $\mathbb{R}^{n_x}$-valued random variable $y$ and quantify the element-wise truncation error $\|y_i - P_n y_i\|$ for all $i = 1, \ldots, n_x$.

**Remark 1 (QPs and MPC):** In case of linear-quadratic MPC, Problem 1 is equivalent to considering uncertainty with respect to the initial condition $x_0$ at every time instant [14]. The uncertainty of $x_0$ may be due to state estimation, or a lack of measurement precision/availability.

PCE allows to specify the influence of $x$ on $y$ as follows.

**Theorem 3 (Uncertainty quantification for convex QPs):** For all realizations of $x$, let the active constraints in Problem 1 satisfy the linear inequality constraint qualification (LICQ) at the optimal solution $y$.

(i) If the PCE dimension is chosen according to $n \geq d$, then the element-wise truncation error of $y$ becomes zero, i.e.

$$\|y_i - P_n y_i\| = 0, \quad i = 1, \ldots, n_x.$$  \hspace{2cm} (14)

(ii) If the dimension is chosen according to $n < d$, and if the set of active constraints $A = \{a_1, \ldots, a_{n_{act}}\} \subseteq \{1, \ldots, n_{con}\}$ is the same for all realizations of $x = [h^\top, b^\top]^\top$, then the element-wise truncation error becomes

$$\|y_i - P_n y_i\| = \sqrt{\sum_{j=n+1}^{d} (w_i^h h_j + w_i^b M_A b_j)^2\|\phi_j\|^2},$$  \hspace{2cm} (15)

where $w_i^h$, $w_i^b$ are the $i$th rows with $i = 1, \ldots, n_x$ of the matrices $W^h$, $W^b$ that satisfy

$$\begin{bmatrix} W^h & W^b \\ V^h & V^b \end{bmatrix} = - \begin{bmatrix} H & A^\top M_A^\top \\ M_A A & 0 \end{bmatrix}^{-1}. \hspace{2cm} (16)$$

The active constraint selection matrix $M_A \in \mathbb{N}_{0,n_{act}}^{n_{con} \times n_{con}}$ is constructed from the active set $A$ and has elements $(M_A)_{ia} = 1$ for $i = 1, \ldots, n_{act}$, zero elsewhere.

**Proof:** Part (i)—Regardless of the realizations of $x$ there always exists a (possibly empty) set of active constraints $A$ for the optimal solution $y = \chi^\star$. Rewrite (13) as

$$\begin{bmatrix} H & A^\top M_A^\top \\ M_A A & 0 \end{bmatrix} \begin{bmatrix} y \\ \chi^\star \end{bmatrix} = - \begin{bmatrix} x_1 \\ M_A x_2 \end{bmatrix}. \hspace{2cm} (17)$$

Due to LICQ the coefficient matrix is invertible, yielding exactly one solution. The argmin-operator maps the realizations $x$ linearly to $y$. Consequently, it maps the random variable $x$ linearly to the random variable $y$. Theorem 2 is applicable with $d_f = 1$.

Part (ii)—Because the set of active constraints is supposed to be $A$ for all realizations, the KKT conditions (17) hold in terms of a function of random variables

$$\begin{bmatrix} y \\ \lambda^\star \end{bmatrix} = \begin{bmatrix} W^h & W^b \\ V^h & V^b \end{bmatrix} \begin{bmatrix} h \\ M_A b \end{bmatrix}, \hspace{2cm} (18)$$

where (15) and $x = [x_1^\top, x_2^\top]^\top = [h^\top, b^\top]^\top$ are used. Invertibility follows again from LICQ. Consequently for all $i = 1, \ldots, n_x$,

$$y_i = w_i^h h + w_i^b M_A b = \sum_{j=0}^{d} (w_i^h h_j + w_i^b M_A b_j) \phi_j,$$

and the result follows from Theorem 2 with $d_f = 1$. $\blacksquare$

**Remark 2 (Extension to changes in the active set):** Note that even if the active set changes, part i) of Theorem 3 still holds. Furthermore, the error description from part ii) can be turned into an upper bound by considering the worst case active set, which maximizes $\|y_i - P_n y_i\|$. Due to space limitations, we leave the details to future work.

**Example 2:** Consider linear-quadratic MPC for an LTI discrete-time model $\chi(k+1) = A x(k) + B u(k)$ of an aircraft. The open-loop optimal control problem can be cast as a QP [14]. The numerical values for the nominal system $(A, B)$ and weights $Q, R$ are taken from [11]; the horizon length is $N = 35$. The input is the rate of change of the
elevator angle, which introduces discrete-time integral action and an additional state. Uncertainty is introduced via the initial condition \( \chi(0) = x \) for the altitude: it is modeled by the random variable \( x_4 \) that follows a \( \beta \)-distribution on \([-402, -381] \) with shape parameters \( \alpha = 2, \beta = 5 \), yielding the uncertain initial condition \( x = [0 \ 0 \ 0 \ x_4 \ 0]^T \). Assumption 1 is satisfied with \( d_x = \ell_x = 1 \) and the PCE coefficients are \( x_{4,0} = -396, x_{4,1} = 3 \) for a Jacobi polynomial basis. Following Theorem 3, a Jacobi polynomial basis with \( n \geq 1 \) allows for a zero PCE truncation error in the decision variable \( y \). Figure 1 shows the evolution of the 6σ-interval of the optimal input over time—note that the realization \( y \) of the optimal random variable \( y(t) \) resembles the input \( u \) to the system. For all realizations of the initial condition \( x \) the constraints for the second state are active on the interval \([0.5, 7.5] \) s. The corresponding optimal input trajectory over \([0.0, 7.0] \) s is deterministic, as shown in Figure 1a. In terms of PCE coefficients, this is equivalent to all PCE coefficients of order greater than zero being zero, yielding a Dirac-\( \delta \)-distribution. It is after the constraints become inactive that uncertainty plays a role; depicted in Figure 1b for \( t \in [7.5, 17] \). Because the closed-loop system is asymptotically stable, the input uncertainty eventually fades out, resulting again in Dirac-\( \delta \)-distributions.

IV. Truncation Errors for Non-Polynomial Maps

We now turn to the question of PCE truncation errors for non-polynomial mappings \( f(\cdot) \). Consider the real-valued random variables \( x_i, y \in H = L^2(\Omega, \mu; \mathbb{R}) \) with \( i = 1, \ldots, n_x \). Let \( X \subset H \) be a complete subspace of dimension \( n+1 \) generated by the orthogonal polynomial basis functions \( \{\phi_n\}^\infty_{n=0} \).

**Theorem 4 (Error for non-polynomial mapping):** Let all \( x_i \) satisfy Assumption 1 with minimum degree \( d_{x_i} \), and let \( f : H^{n+1} \to H \) be a square-integrable mapping such that \( y = f(x_1, \ldots, x_{n_x}) \). Then, the magnitude of the truncation error \( e_n = y - P_n y \perp P_n y \) is

\[
e_n := ||e_n|| = \sqrt{||y||^2 - g^T Q g}, \tag{19}\]

where \( Q = \text{diag}(1/||\phi_0||^2, \ldots, 1/||\phi_n||^2) \in \mathbb{R}^{(n+1) \times (n+1)} \) is positive definite, and \( g = [g_1, \ldots, g_{n+1}]^T \in \mathbb{R}^{n+1} \) with \( g_{i+1} = \langle y, \phi_i \rangle \) for all \( i = 0, \ldots, n \).

**Proof:** The PCE coefficients of \( P_n y \) satisfy the normal equations

\[
\langle \phi_k, \phi_k \rangle y_k = \langle y, \phi_k \rangle, \quad k = 0, \ldots, n \iff Q^{-1} y = g, \tag{20}\]

which follows from orthogonality of the basis spanning \( X \). The vector of PCE coefficients \( y \in \mathbb{R}^{n+1} \) contains all PCE coefficients \( y = [y_0, \ldots, y_n]^T \). The truncation error satisfies

\[
||e_n||^2 = ||y - P_n y||^2 - g^T y = ||y||^2 - g^T y, \tag{21}\]

because \( y - P_n y \perp P_n y \). Using (20), result (19) follows.

In light of Theorem 4, the answers to questions Q1 and Q2 are summarized.

**Corollary 2 (Error/minimum degree – non-polynomial):**

A3 Given a non-polynomial mapping \( f(\cdot) \) such that \( y = f(x) \) with \( x, y \in L^2(\Omega, \mu; \mathbb{R}) \), and choosing the PCE dimension \( \ell + 1 \) equal to the PCE input dimension \( \ell_x + 1 \), the truncation error is given by \( e_{\ell_x} \) from (19).

A4 No general statement is possible. However, for a user-specified error threshold the according minimum PCE dimension is obtained from Theorem 4.

The error (19) can be computed efficiently using Gauss quadrature. We illustrate Theorem 4 for a continuous-time LTI example with LQR and an uncertain system matrix.

**Example 3:** Consider the continuous-time LTI dynamics \( \dot{\chi} = A(x) \chi + B u \) for a modified aircraft model from [14]. The initial condition is \( \chi(0) = [0 \ 0 \ 0 \ 40]^T \), and

\[
A = \begin{bmatrix}
-1.2822 + 0.4 x & 0 & 0.98 & 0 \\
0 & 0 & 1 & 0 \\
-5.4293 & 0 & -1.8366 & 0 \\
-128.2 & 128.2 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
-0.3 \\
0 \\
0 \\
-17
\end{bmatrix},
\]

where \( x \sim U[-1, 1] \). The realization \( x = 0 \) corresponds to the nominal system matrix \( A \). The control \( u(t) = -R \chi(t) \) is computed via LQR using the weights \( Q = 0.001 I_4, R = 100 \) for the nominal system \((A, B)\). Now apply the above feedback to the uncertain system matrix. The closed-loop altitude trajectories \( \chi_A(t) \) are given in Figure 2 (left) for best case and worst case realizations, clearly showing the performance degradation under uncertainty. The uncertainty \( x \) is mapped to the state \( \chi(t) \) via the state transition map \( \chi(t) = \exp((A(x) - BK)t) \chi_0 \). Figure 2 shows the altitude...
truncation error $e_{4,n}(t)$ from (19) over time for increasing highest-degree $n \in \{2, 3, 4\}$. The basis consists of Legendre polynomials. The closed-loop system is asymptotically stable for all realizations of $x$, hence the truncation error decays to zero. However, it is clearly non-monotonic over time. Note how over- and undershooting of the deterministic solution, Figure 2 left, carry over to the PCE error, Figure 2 right.

As to be expected, in case of a polynomial $f(\cdot)$, result (9) is recovered and computationally cheaper.

**Lemma 1**: Let $f : H^m \rightarrow H$ be a polynomial of degree $d_f$. Then (19) is equivalent to (9).

**Proof**: For polynomial $f$, the exact PCE for $y$ is given from (12). Consequently, for $\ell+1=(n_\ell+d_f d_f)/(n_\ell!(d_2 d_f)!)$ we have

$$e^2_n = \|y\|^2 - g^T Q g = \sum_{i=0}^\ell g^2_i \|\phi_i\|^2 - \sum_{i=0}^n g^2_{i+1} \|\phi_i\|^2. \quad (22)$$

For all $i=0, 1, \ldots, n$ with $n \leq \ell$, the numerators $g^2_{i+1}$ become

$$g^2_{i+1} = \langle y, \phi_i \rangle^2 = \left(\sum_{j=0}^\ell g_j \phi_j, \phi_i \right)^2 = \left( g_i \|\phi_i\|^2 \right)^2. \quad (23)$$

For $n > \ell$, use (23) in (22) to obtain $e^2_n = \sum_{i=n+1}^\ell g^2_i \|\phi_i\|^2$, from which (9) follows.

**V. Conclusion & Outlook**

Polynomial chaos is an increasingly popular method for uncertainty propagation in systems and control. Thus, the quantification of truncation errors stemming from the spectral expansions used in PCE is important. For the case of polynomial and non-polynomial mappings—which might be state propagation maps of dynamics systems, algebraic equations or argmin-operators of convex problems—this work derived error bounds based on Hilbert space methods. Specifically, the presented results provide an answer to the question of how to choose the PCE order such that the truncation error vanishes. Several simulation results underpin the accuracy of the provided bounds and demonstrate how they can be used. Future work will focus on non-convex optimization and optimal control problems.

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