Bifurcation analysis of a stochastically driven limit cycle

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Abstract

We establish the existence of a bifurcation from an attractive random equilibrium to shear-induced chaos for a stochastically driven limit cycle, indicated by a change of sign of the first Lyapunov exponent. This addresses an open problem posed by Kevin Lin and Lai-Sang Young in [21, 30], extending results by Qiudong Wang and Lai-Sang Young [28] on periodically kicked limit cycles to the stochastic context.

Key words. Furstenberg–Khasminskii formula, Lyapunov exponent, Random dynamical system, shear-induced chaos, stochastic bifurcation

Mathematics Subject Classification (2010). 37D45, 37G35, 37H10, 37H15.

1 Introduction

We consider the following model of a stochastically driven limit cycle

\[ \begin{align*}
\frac{dy}{dt} &= -\alpha y + \sigma \sum_{i=1}^{m} f_i(\vartheta) \circ dW^i_t, \\
\frac{d\vartheta}{dt} &= (1 + by) dt,
\end{align*} \]

where \((y, \vartheta) \in \mathbb{R} \times \mathbb{S}^1\) are cylindrical amplitude-phase coordinates, \(m \in \mathbb{N}\), and \(W^i_t\) for \(i \in \{1, \ldots, m\}\), denote independent one-dimensional Brownian motions entering the equation as noise of Stratonovich type. In the absence of noise (\(\sigma = 0\)), the ODE (1.1) has a globally attracting limit cycle at \(y = 0\) if \(\alpha > 0\). In the presence of noise (\(\sigma \neq 0\)), the amplitude is driven by phase-dependent noise. The real parameter \(b\) induces shear: if \(b \neq 0\), the phase velocity \(\frac{d\vartheta}{dt}\) depends on the amplitude \(y\).

The stable limit cycle turns into a random attractor if \(\sigma \neq 0\). The main question we address in this paper concerns the nature of this random attractor. The crucial quantity is the sign of the first Lyapunov exponent \(\lambda_1\) with respect to the invariant measure associated to the random attractor. In essence, \(\lambda_1\) is the dominant infinitesimal asymptotic expansion rate of almost all trajectories.

To facilitate the analysis, we choose \(f_i: \mathbb{S}^1 \simeq [0, 1) \rightarrow \mathbb{R}\) such that

\[ \sum_{i=1}^{m} f_i'(\vartheta)^2 = 1 \quad \text{for all } \vartheta \in \mathbb{S}^1. \]

If \(m \geq 2\), then the functions are assumed to be smooth. The simplest example is given by

\[ m = 2, \quad f_1(\vartheta) = \cos(\vartheta), \quad f_2(\vartheta) = \sin(\vartheta). \]

For \(m = 1\), condition (1.2) cannot be satisfied for all \(\vartheta \in \mathbb{S}^1\). Hence, we choose \(f := f_1: \mathbb{S}^1 \simeq [0, 1) \rightarrow \mathbb{R}\) to be continuous and piecewise linear with constant absolute value of the derivative almost everywhere. The simplest example is given by

\[ f(\vartheta) = \begin{cases} 
\vartheta & \text{if } \vartheta \leq \frac{1}{2}, \\
(1 - \vartheta) & \text{if } \vartheta \geq \frac{1}{2}.
\end{cases} \]

With such choices of the amplitude-phase coupling we obtain the following bifurcation result.

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**Theorem 1.1.** Consider the SDE (1.1) with \( f_i, i = 1, \ldots, m \), satisfying condition (1.2). Then there is 
\( c_0 \approx 0.2823 \) such that for all \( \alpha > 0 \) and \( b \neq 0 \) the number \( \sigma_0(\alpha, b) = \frac{c_0^{3/2}}{|b|^{1/2}} > 0 \) is the unique value of \( \sigma \) where the top Lyapunov exponent \( \lambda_1(\alpha, b, \sigma) \) of (1.1) changes its sign. In more detail, we have

\[
\lambda_1(\alpha, b, \sigma) \begin{cases} 
< 0 & \text{if } 0 < \sigma < \sigma_0(\alpha, b), \\
= 0 & \text{if } \sigma = \sigma_0(\alpha, b), \\
> 0 & \text{if } \sigma > \sigma_0(\alpha, b).
\end{cases}
\]

As long as \( b, \sigma \neq 0 \), the amplitude variable \( y \) can be rescaled so that the shear parameter becomes equal to 1 and the effective noise-amplitude becomes \( \sigma b \). Hence, the above result also holds with the roles of \( \sigma \) and \( b \) interchanged. The fact that \( \sigma_0(\alpha, b) \) is an increasing function of \( \alpha \) is illustrated in Figure 1. Figure 1a depicts \( \lambda_1 \) as a function of \( \alpha \) and \( \sigma \) for fixed \( b = 2 \). Figure 1b displays the corresponding areas of positive and negative \( \lambda_1 \) in the \((\sigma, \alpha)\)-parameter space being separated by the curve \( \{ (\sigma_0(\alpha, 2), \alpha) \} \). The picture doesn’t display \( \sigma = 0 \): in this case, \( \lambda_1 = 0 \) for all \( \alpha > 0 \).

If the top Lyapunov exponent is negative, it turns out that the (weak) random point attractor is an attracting random equilibrium, i.e. its fibers are singletons almost surely. Properties of random attractors with positive top Lyapunov exponents are not yet well understood, apart from the fact that such attractors are not random equilibria. They are sometimes referred to as random strange attractors \([19, 29]\). Theorem 1.1 confirms numerical results by Lin & Young \([21]\) for a very similar model. The mechanism, whereby a combination of shear and noise causes stretching and folding leading to a positive Lyapunov exponent, was coined with the term shear-induced chaos by Lin & Young. Wang & Young \([27, 28]\) and Ott & Stenlund \([24]\) have demonstrated analytically the validity of this mechanism in the case of periodically kicked limit cycles, including probabilistic characterizations of the dynamics. An analytical proof of shear-induced chaos in the stochastic setting, as presented in this paper, had remained an open problem.

The results of this paper are part of a larger effort to develop a bifurcation theory of random dynamical systems. Earlier attempts to develop such a theory (notably by Ludwig Arnold, Peter Baxendale and coworkers \([2, 3, 5, 25]\) in the 1990s) led to notions of so-called phenomenological (or \"P\") bifurcations and dynamical (or \"D\") bifurcations, but there is growing evidence that these paradigms do not suffice to capture the intricacies of bifurcation in random dynamical systems \([1, 8, 17, 31]\). In the absence of a consensus on useful characterisations of the dynamics of random systems and bifurcations in this context, much of the
current research inevitably focusses on the detailed analysis of relatively elementary examples, to generate insights and guidance towards the further development of a more general theory.

In one-dimensional SDEs, negative Lyapunov exponents and attractive random equilibria prevail [9]. Random strange attractors can only arise in dimension two and higher and up to now, little research has been devoted to such attractors. In contrast, the existence of attractive random equilibria (also referred to as *synchronization*, with reference to the corresponding dynamics of sets of initial conditions) has been studied well, also in higher dimensions [4, 12, 18, 22, 23].

The main technical challenge addressed in this paper is to establish the existence of positive top Lyapunov exponents. Most rigorous results on Lyapunov exponents (and random dynamical systems) are obtained for one-dimensional SDEs, in which case the analysis of Lyapunov exponents significantly simplifies due to the fact that all derivatives commute. It is difficult in general to obtain lower bounds for the top Lyapunov exponent in higher dimensions due to the subadditivity property of matrices, cf. [30]. Thus, the analytical demonstration of positive Lyapunov exponents for noisy systems has been achieved only in certain special cases, like for equilibria [13], simple time-discrete models as in [20] or under special circumstances that allow for the use of stochastic averaging [6, 7]. In our setting, condition (1.2) is crucial to establish rigorous lower bounds on the top Lyapunov exponent $\lambda_1$.

Another prototypical open problem in dimension two is the *stochastic Hopf bifurcation*, concerning the characterisation of dynamics and bifurcations in parametrized families of SDEs that in the deterministic (noise-free) limit display a Hopf bifurcation. A (deterministic) Hopf bifurcation occurs if, by the variation of a model parameter, an asymptotically stable equilibrium loses stability under the emission of a small attracting limit cycle. Numerical studies [29] suggest that the mechanism of shear-induced chaos is at play also in stochastic Hopf bifurcations, but while analytical proofs of parameter regimes with negative top Lyapunov exponents are within reach [10, 11], until now, there are no rigorous results concerning the existence of parameter regimes with positive top Lyapunov exponents in this context. The results of this paper may well be relevant to shed more light on this problem.

The remainder of the paper is organized as follows. Section 2 provides the analysis of Lyapunov exponents for our model: Subsection 2.1 introduces the model on the cylinder within the framework of random dynamical systems and establishes the necessary theoretical concepts. Subsection 2.2 introduces the Furstenberg–Khasminskii formula for the top Lyapunov exponent and in Subsection 2.3, we derive a formula for the top Lyapunov exponent $\lambda_1$. The main result concerning the change of sign of $\lambda_1$ is proven in Section 3 and its consequences are discussed. We give illustrations of $\lambda_1$ in dependence on the parameters and confirm a scaling conjecture by Lin and Young. We conclude with a short summary in Section 4.

## 2 Analysis of the top Lyapunov exponent

### 2.1 Lyapunov exponents for random dynamical systems

Consider the stochastic differential equation of Stratonovich type (1.1). We assume that $f_i : [0,1] \to \mathbb{R}$, $i = 1,\ldots,m$, are Lipschitz continuous functions with $f(0) = f(1)$ (smooth if $m \geq 2$ and piecewise linear if $m = 1$), and the three parameters fulfill $\alpha > 0$, $\sigma > 0$ and $b \in \mathbb{R}$. Note that the equation reads the same in Itô form according to the Itô–Stratonovich conversion formula.

Since the drift and diffusion coefficients are Lipschitz continuous and satisfy linear growth conditions, the SDE (1.1) generates a continuous random dynamical system $(\theta, \varphi)$ [2, Definition 1.1.2] consisting of the following:

(i) A model of the noise on the probability space $\Omega := C_0(\mathbb{R}, \mathbb{R}) = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ with Borel $\sigma$-algebra $\mathcal{F}$ and two-sided Wiener measure $\mathbb{P}$, formalized as the family $(\theta_t)_{t \in \mathbb{R}}$ of $\mathbb{P}$-preserving shift maps given by $(\theta_t)(s) = \omega(s + t) - \omega(t)$.

(ii) A model of the system perturbed by noise formalized as a *cocycle* $\varphi$ over $\theta$ of mappings of $\mathbb{R} \times \mathbb{S}^1$, i.e. $\varphi$ is a $\mathcal{B}(\mathbb{R}_0^+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R} \times \mathbb{S}^1)$-measurable mapping

$$
\varphi : \mathbb{R}_0^+ \times \mathbb{R} \times (\mathbb{R} \times \mathbb{S}^1) \to \mathbb{R} \times \mathbb{S}^1, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x,
$$

such that $(t, x) \mapsto \varphi(t, \omega)x$ is continuous for every $\omega \in \Omega$ and which satisfies

$$
\varphi(0, \omega) = \text{id} \quad \text{and} \quad \varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega) \quad \text{for all} \ \omega \in \Omega \ \text{and} \ \ t, s \in \mathbb{R}_0^+.
$$
The random dynamical system \((\theta, \varphi)\) induced by (1.1) is also a skew product flow \(\Theta = (\theta, \varphi)\), which is a measurable dynamical system on the extended phase space \(\Omega \times X\). The skew product flow \(\Theta\) possesses an ergodic invariant Markov measure \(\mu\) which is associated to the unique invariant measure (also called stationary measure) for the corresponding Markov semigroup. Their existence follows from the same considerations as in [21].

Fundamental for stochastic bifurcation theory is Oseledets’ Multiplicative Ergodic Theorem, which implies the existence of Lyapunov exponents describing stability properties of a differentiable random dynamical system. The random dynamical system \((\theta, \varphi)\) is called \(C^k\) if \(\varphi(t, \omega) \in C^k\) for all \(t \in \mathbb{R}^+_0\) and \(\omega \in \Omega\). In the situation of the Stratonovich SDE

\[
\mathrm{d}X_t = f_0(X_t) \mathrm{d}t + \sum_{i=1}^m f_j(X_t) \circ \mathrm{d}W^j_t
\]
on a smooth manifold \(X\), the Jacobian \(D\varphi(t, \omega, x)\) with respect to the third variable of the cocycle \(\varphi(t, \omega)\) is a linear cocycle over the skew product flow \(\Theta = (\theta, \varphi)\). The Jacobian \(D\varphi(t, \omega, x)\) applied to an initial condition \(x_0 \in T_x X\) solves uniquely the variational equation on \(T_x X \cong \mathbb{R}^d\), given by

\[
\mathrm{d}v = Df_0(\varphi(t, \omega)x)v \, \mathrm{d}t + \sum_{j=1}^m Df_j(\varphi(t, \omega)x)v \circ \mathrm{d}W^j_t, \quad \text{where } v \in T_x X. \tag{2.1}
\]

Suppose the one-sided \(C^1\)-random dynamical system \((\varphi, \theta)\) has an ergodic invariant measure \(\nu\) and satisfies the integrability condition

\[
\sup_{0 \leq t \leq 1} \log^+ \|D\varphi(t, \omega, x)\| \in L^1(\nu).
\]

Then the Multiplicative Ergodic Theorem for differentiable random dynamical systems [2, Theorem 3.4.1, Theorem 4.2.6] guarantees the existence of a \(\Theta\)-forward invariant set \(\Delta \subset \Omega \times X\) with \(\nu(\Delta) = 1\) and the Lyapunov exponents \(\lambda_1 > \cdots > \lambda_p\) with respect to \(\nu\). The tangent space \(T_x X \cong \mathbb{R}^d\) admits a filtration

\[
\mathbb{R}^d = V_1(\omega, x) \supseteq V_2(\omega, x) \supseteq \cdots \supseteq V_p(\omega, x) \supseteq V_{p+1}(\omega, x) = \{0\},
\]
such that for all \(0 \neq v \in T_x X \cong \mathbb{R}^d\), the Lyapunov exponent \(\lambda(\omega, x, v)\) defined by

\[
\lambda(\omega, x, v) = \lim_{t \to \infty} \frac{1}{t} \log \|D\varphi(t, \omega, x)v\|
\]
exists and

\[
\lambda(\omega, x, v) = \lambda_i \iff v \in V_i(\omega, x) \setminus V_{i+1}(\omega, x) \quad \text{for all } i \in \{1, \ldots, p\}.
\]

### 2.2 The Furstenberg–Khasminskii formula

In the following, we calculate the top Lyapunov exponent \(\lambda_1\) for the random dynamical system induced by (1.1). We consider the corresponding variational equation describing the flow on the tangent space \(T_x(\mathbb{R} \times S^1) \cong \mathbb{R}^2\) along trajectories of (1.1). The variational equation reads as

\[
\mathrm{d}v = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v \, \mathrm{d}t + \sigma \sum_{i=1}^m \begin{pmatrix} 0 & f_i^1(\theta) \\ 0 & 0 \end{pmatrix} v \circ \mathrm{d}W^i_t. \tag{2.2}
\]

Note that we omit the \((t, \omega)\)-dependence of \(\theta\) and \(B\). Because of the linearity of (2.2), we introduce the change of variables \(r = \|v\|\) and \(s = v/r\), so that \(s\) lies on the unit circle. Its dynamics are given by

\[
\begin{aligned}
ds &= (As - (s, As)s) \, \mathrm{d}t + \sum_{i=1}^m (B_is - (s, B_is)s) \circ \mathrm{d}W^i_t \\
&= \begin{pmatrix} -\alpha s_1 - s_1(-\alpha s_1^2 + bs_1s_2) \\ bs_1 - s_2(-\alpha s_1^2 + bs_1s_2) \end{pmatrix} \, \mathrm{d}t + \sigma \sum_{i=1}^m \begin{pmatrix} f_i^1(\theta)s_2 - s_1f_i^1(\theta)s_1s_2 \\ -s_2f_i^1(\theta)s_1s_2 \end{pmatrix} \circ \mathrm{d}W^i_t.
\end{aligned}
\]


The Furstenberg–Khasminskii formula for the top Lyapunov exponent [13] is given by

\[ \lambda_1 = \int_{\mathbb{R}} \int_{[0,1]} \int_{\mathbb{S}^1} (h_A(s) + \sum_{i=1}^{m} k_{B_i}(s)) \rho(ds, d\vartheta, dy), \]

(2.3)

where \( \rho \) is the joint invariant measure for the diffusion \( s \) on the unit circle and the processes \( \vartheta \) and \( y \) induced by (1.1); the functions \( h_A \) and \( k_{B_i} \), \( i = 1, \ldots, m \), are given by

\[ h_A(s) = \langle s, As \rangle = -\alpha s_1^2 + bs_1s_2, \]

\[ k_{B_i}(s) = \frac{1}{2} \left( (B_i + B_i^*) s, B_is \right) - (s, B_is)^2 = \frac{1}{2} \sigma f_i'(\vartheta)^2 s_2^2 - \sigma^2 f_i'(\vartheta)^2 s_1^2 s_2^2. \]

Similarly to the calculations in [13], we change variables to \( s = (\cos \varphi, \sin \varphi) \). Note that the functions \( h_A \) and \( k_{B_i} \) are \( \pi \)-periodic, which implies that the formula (2.3) for the top Lyapunov exponent reads as

\[ \lambda_1 = \int_{\mathbb{R} \times [0,1] \times [0,\pi]} \left( -\alpha \cos^2 \varphi + b \cos \varphi \sin \varphi + \sum_{i=1}^{m} f_i'(\vartheta)^2 \left[ \frac{1}{2} \sigma^2 (1 - 2 \cos^2 \varphi) \sin^2 \varphi \right] \right) \tilde{\rho}(d\varphi, d\vartheta, dy), \]

(2.4)

where \( \tilde{\rho} \) denotes the corresponding image measure of \( \rho \). The SDE determining the dynamics of \( \varphi \in [0, \pi] \) reads as

\[ d\varphi = -\frac{1}{\sin \varphi} ds_1 = (\alpha \cos \varphi \sin \varphi + b \cos^2 \varphi)dt - \sum_{i=1}^{m} \sigma f_i'(\vartheta) \sin^2 \varphi \circ dW^i_t, \]

(2.5)

where we denote

\[ c_i(\varphi, \vartheta) = \sigma f_i'(\vartheta) \sin^2 \varphi \quad \text{and} \quad d(\varphi) = \alpha \cos \varphi \sin \varphi + b \cos^2 \varphi. \]

(2.6)

In the Fokker–Planck equation for \( \varphi \), the dependence on \( \vartheta \) is restricted to \( \sum_{i=1}^{m} f_i'(\vartheta)^2 \), and in addition to that, the integrand of (2.4) only depends on \( \varphi \) and not on \( \vartheta \) and \( y \). This means that the calculation of \( \lambda_1 \) becomes much simpler if \( \sum_{i=1}^{m} f_i'(\vartheta)^2 \) is constant, an observation that we exploit in the following.

### 2.3 Explicit formula for the top Lyapunov exponent

Firstly, we have to justify the analysis of the top Lyapunov exponent for the case \( m = 1 \). Let \( f := f_1 : [0,1] \rightarrow \mathbb{R} \) be given by (1.4). Importantly, \( f'(\vartheta)^2 \) is constant in this special case and our results hold in fact for every continuous and piecewise linear \( f \) with constant absolute value of the derivative almost everywhere.

The map is not differentiable at \( \frac{\pi}{2} \) and \( 0 \), and we verify that does not cause any problems. We need the following results to justify the variational equation defining \( D\varphi \):

**Lemma 2.1.** Let \( W : \mathbb{R}^+_0 \times \Omega \rightarrow \mathbb{R} \) denote the canonical real-valued Wiener process, and let \( X : \mathbb{R}^+_0 \times \Omega \rightarrow [0,1] \) be a stochastic process adapted to the natural filtration of the Wiener process. Furthermore, suppose there exists a measurable set \( A \subset [0,1] \) such that

\[ \mathbb{P}\left\{ \omega \in \Omega : \int_0^t 1_{\{X_u \in A\}} \, du = 0 \right\} = 1 \quad \text{for all } t > 0, \]

(2.7)

i.e. \( A \) is visited only on a measure zero set with full probability. Consider a measurable function \( g : [0,1] \rightarrow [0,1] \) such that \( g = 0 \) on \([0,1] \setminus A\). Then

\[ \int_0^t g(X_u) \, dW_u = 0 \quad \text{almost surely for all } t > 0. \]

**Proof.** The statement follows directly from Itô’s isometry

\[ \mathbb{E} \left[ \left( \int_0^t g(X_u) \, dW_u \right)^2 \right] = \mathbb{E} \left[ \int_0^t g(X_u)^2 \, du \right] = \mathbb{E} \left[ \int_0^t \left( g(X_u)^2 1_{\{X_u \in A\}} + g(X_u)^2 1_{\{X_u \in [0,1] \setminus A\}} \right) \, du \right] = 0, \]
where the last equality follows immediately from (2.7) and \( g = 0 \) on \([0, 1] \setminus A\). We conclude

\[
\left( \int_0^t g(X_u) \, dW_u \right)^2 = 0 \quad \text{almost surely}
\]
due to nonnegativity, and the claim follows.

\begin{proof}
First, we show that \( \varrho \) by assuming the contrary to obtain a contradiction. As \( \varrho \) is a continuously differentiable process, this implies that \( \varrho_u = \frac{1}{2} \) for \( u \in [t^*, t^* + \varepsilon] \) for some \( t^* \in (0, t) \) and \( \varepsilon > 0 \) with positive probability. This leads to \( y(u) = -\frac{1}{b} \) mod 1 for \( u \in (t^*, t^* + \varepsilon) \) with positive probability. However, this implies that the continuous process \( y_u \) for \( u \in (t^*, t^* + \varepsilon) \) given by

\[
dy = -\alpha y \, du + \sigma \, dW_u
\]
is constant with positive probability. This contradicts its definition as an Ornstein–Uhlenbeck process. The same reasoning obviously holds for \( \theta = 0 \).

Let \( f_1' = f_2' = f' \) on \((0, 1) \setminus \left\{ \frac{1}{2} \right\} \) and assign arbitrary values at \( \frac{1}{2} \) and 0. Define

\[
dv = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v \, dt + \begin{pmatrix} 0 & \sigma f_1'(\varrho) \\ 0 & 0 \end{pmatrix} v \circ dW^1_t,
\]
\[
dw = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} w \, dt + \begin{pmatrix} 0 & \sigma f_2'(\varrho) \\ 0 & 0 \end{pmatrix} w \circ dW^1_t.
\]

We apply Lemma 2.1 by choosing \( X_u = \varrho_u \) and \( g(\varrho_u) = f_1'(\varrho_u) - f_2'(\varrho_u) \) to conclude that

\[
\int_0^t f_1'(\varrho_u) \, dW_u = \int_0^t f_2'(\varrho_u) \, dW_u \quad \text{almost surely.}
\]

As we do not have an Itô–Stratonovich correction in this case, we can infer that \( v_t = w_t \) almost surely for all \( t > 0 \).

We view \( f' \) in the weak sense, disregarding the points \( \frac{1}{2} \) and 0, and we define \( f'(\varrho) = \text{sign}(\frac{1}{2} - \varrho) \), where

\[
\text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}
\]

By Proposition 2.2, \( D\varphi(t, \omega, x) \) does not depend on the choice of \( f'(\frac{1}{2}) \), so the variational equation (2.2) becomes

\[
dv = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v \, dt + \begin{pmatrix} 0 & \sigma \text{sign}(\frac{1}{2} - \varrho_t) \\ 0 & 0 \end{pmatrix} v \circ dW^1_t. \tag{2.8}
\]

We can now derive the following formula for the first Lyapunov exponent under assumption (1.2), including \( m = 1 \) with \( f \) given by (1.4):

\begin{proposition}
The top Lyapunov exponent of system (1.1) with \( \sum_{i=1}^m f_i'(\varrho)^2 = 1 \) is given by

\[
\lambda_1 = \int_0^\pi q(\phi) p(\phi) \, d\phi, \tag{2.9}
\]

where \( q(\phi) := -\alpha \cos^2 \phi + b \cos \phi \sin \phi + \frac{1}{2} \sigma^2 (1 - 2 \cos^2 \phi) \sin^2 \phi \), and \( p(\phi) \) is the solution of the stationary Fokker–Planck equation \( \mathcal{L}^* p = 0 \). \( \mathcal{L}^* \) is the formal \( L^2 \)-adjoint of the generator \( \mathcal{L} \), which is given by

\[
\mathcal{L} g(\phi) = \left( d(\phi) + \frac{1}{2} \tilde{c}(\phi) c'(\phi) \right) g'(\phi) + \frac{1}{2} \tilde{c}^2(\phi) g''(\phi), \tag{2.10}
\]
\end{proposition}
where \( d = d(\phi) \) is defined as in (2.6), and \( \tilde{c}(\phi) := \sigma \sin^2 \phi \). Hence, \( \lambda_1 \) is identical to the top Lyapunov exponent of the linear system

\[
dv = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v \, dt + \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix} v \circ dW^1_t.
\] (2.11)

**Proof.** Consider the SDE for the process \( \phi(t) \) in Itô form

\[
d\phi = r(\phi) \, dt + \sum_{i=1}^{m} c_i(\phi, \vartheta) \, dW^i_t,
\]

where

\[
r(\phi) = d(\phi) + \frac{1}{2} \sum_{i=1}^{m} c_i(\phi, \vartheta) c_i(\phi, \vartheta) = d(\phi) + \frac{1}{2} \sum_{i=1}^{m} f_i(\vartheta)^2 c_i(\phi) c_i(\phi) = d(\phi) + \frac{1}{2} \tilde{c}(\phi) \tilde{c}(\phi).
\]

Furthermore, recall that by assumption (1.2)

\[
\sum_{i=1}^{m} c_i^2(\phi, \vartheta) = \tilde{c}^2(\phi).
\]

As the coefficients of the SDE are smooth in \( \phi \), we consider the kinetic equation for the probability density function of the process \( \phi(t) \) (cf. [26])

\[
\frac{\partial p(\psi, t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \psi^n} [a_n(\psi, t)p(\psi, t)],
\]

where

\[
a_n(\psi, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E} [(\phi(t + \Delta t) - \phi(t))^n | \phi(t) = \psi]
\]

for all \( n \in \mathbb{N} \).

Pick \( \Delta t \) small, denote \( \Delta W^i_t = W^i(t + \Delta) - W^i(t) \) and recall that \( \mathbb{E}[\Delta W^i_t] = 0 \) and \( \mathbb{E}[(\Delta W^i_t)^2] = \Delta t \) for all \( i = 1, \ldots, m \). Observe that

\[
\phi(t + \Delta t) - \phi(t) = r(\phi(t)) \Delta t + \sum_{i=1}^{m} c_i(\phi(t), \vartheta(t)) \Delta W^i_t + o(\Delta t),
\]

and

\[
(\phi(t + \Delta t) - \phi(t))^2 = r^2(\phi(t))(\Delta t)^2 + 2 \sum_{i=1}^{m} r(\phi(t)) c_i(\phi(t), \vartheta(t)) \Delta W^i_t \Delta t
\]

\[
+ \sum_{i=1}^{m} c_i^2(\phi(t), \vartheta(t))(\Delta W^i_t)^2 + \sum_{i,j=1, i \neq j}^{m} c_i(\phi(t), \vartheta(t)) c_j(\phi(t), \vartheta(t)) [(\Delta W^i_t)(\Delta W^j_t)] + o(\Delta t).
\]

Since the \( \Delta W^i_t \) and \( \Delta W^j_t \) are independent from each other for \( i \neq j \) and are all independent from \( \phi(t) \) and \( \vartheta(t) \), we obtain that

\[
a_1(\psi, t) = r(\psi) \quad \text{and} \quad a_2(\psi, t) = \tilde{c}^2(\psi).
\]

We can see immediately from above that \( a_n(\psi, t) = 0 \) for \( n \geq 3 \). This proves (2.10), and (2.9) follows from (2.4). It follows from the calculations that formula (2.9) also gives the top Lyapunov exponent for system (2.11).

The following statement is now a direct corollary of [14, Theorem 3].
Theorem 2.4. Consider the stochastic differential equation (1.1), where the function $f$ is of the form (1.4). Then the two Lyapunov exponents are given by

$$
\lambda_1(\alpha, b, \sigma) = -\frac{\alpha}{2} + \frac{|b\sigma|}{2} \int_0^\infty v \, m_{\sigma, b, \alpha}(v) \, dv,
$$

$$
\lambda_2(\alpha, b, \sigma) = -\frac{\alpha}{2} - \frac{|b\sigma|}{2} \int_0^\infty v \, m_{\sigma, b, \alpha}(v) \, dv.
$$

where

$$
m_{\sigma, b, \alpha}(v) = \frac{1}{\sqrt{\alpha}} \exp \left(-\frac{|b\sigma|}{6} v^3 + \frac{\alpha^2}{2|b\sigma|} v^2 \right) \frac{1}{\sqrt{v}} \exp \left(-\frac{|b|}{\alpha} v^3 + \frac{\alpha^2}{2|b|} v^2 \right) \, dv.
$$

Proof. Replacing $v = \left(\begin{array}{c} v_1 \\ v_2 \end{array}\right)$ by $\hat{v} = \left(\begin{array}{c} v_2 \\ \frac{v_2}{\sigma} \end{array}\right)$ leaves the Lyapunov exponents invariant and transforms (2.11) into the equation

$$
dv = \left(\begin{array}{cc} 0 & -\sigma b \\ 0 & -\alpha \end{array}\right) v \, dt + \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) v \circ dW_t.
$$

The matrices in this equation satisfy the assumptions of [14, Theorem 3] which gives formulas (2.12) and (2.13).

3 Bifurcation from negative to positive top Lyapunov exponent

We now use Theorem 2.4 to prove Theorem 1.1, which asserts that there is a bifurcation from negative to positive top Lyapunov exponent for the stochastic differential equation (1.1).

Theorem 1.1. Consider the SDE (1.1) with $f_i$, $i = 1, \ldots, m$, satisfying condition (1.2). Then there is $c_0 \approx 0.2823$ such that for all $\alpha > 0$ and $b \neq 0$ the number $\sigma_0(\alpha, b) = \frac{\alpha^{3/2}}{c_0^{1/2}|b|} > 0$ is the unique value of $\sigma$ where the top Lyapunov exponent $\lambda_1(\alpha, b, \sigma)$ of (1.1) changes its sign. In more detail, we have

$$
\lambda_1(\alpha, b, \sigma) \begin{cases} < 0 & \text{if } 0 < \sigma < \sigma_0(\alpha, b), \\ = 0 & \text{if } \sigma = \sigma_0(\alpha, b), \\ > 0 & \text{if } \sigma > \sigma_0(\alpha, b). \end{cases}
$$

Proof. We fix $\alpha > 0$ and $b \neq 0$. Introducing the change of variables $v = \frac{\alpha}{|\sigma|^3} u$ in (2.14), we obtain

$$
\lambda_1(\alpha, b, \sigma) = \frac{\alpha}{2} \left( \int_0^\infty u \, \tilde{m}_{\sigma, b, \alpha}(u) \, du - 1 \right),
$$

where

$$
\tilde{m}_{\sigma, b, \alpha}(u) = \frac{1}{\sqrt{\alpha}} \exp \left(-\frac{\alpha^3}{2|\sigma|^2} \frac{1}{6} u^3 - \frac{1}{2} u \right) \frac{1}{\sqrt{\alpha}} \exp \left(-\frac{\alpha^3}{2|\sigma|^2} \frac{1}{6} |\sigma|^3 - \frac{1}{2} |\sigma|^3 \right) \, dw.
$$

Defining $c := \frac{\alpha^3}{2|\sigma|^2}$, we observe that $\lambda_1(\alpha, b, \sigma)$ has the same sign as the function $G : (0, \infty) \to \mathbb{R}$ given by

$$
G(c) := \int_0^\infty \left( \sqrt{u} - \frac{1}{\sqrt{u}} \right) \exp \left(-c \frac{1}{6} u^3 - \frac{1}{2} u \right) \, du.
$$

Using dominated convergence, we may interchange the order of differentiation and integration and consider

$$
G'(c) = \int_0^\infty h_1(u) h_2(u) \exp (c h_2(u)) \, du, \quad h_1(u) = \left( \sqrt{u} - \frac{1}{\sqrt{u}} \right), \quad h_2(u) = -\frac{1}{6} u^3 - \frac{1}{2} u.
$$
Note that $h_1h_2$, and thereby the integrand, has positive sign on the interval $(1, \sqrt{3})$ and negative sign on $(0, 1)$ and $(\sqrt{3}, \infty)$. Basic calculations show that we have $|h_1(1-\delta)| > h_1(1+\delta)$ and $h_2(1-\delta) > h_2(1+\delta)$ for all $\delta \in (0, \sqrt{3} - 1)$. It follows that

$$G'(c) < \int_{2-\sqrt{3}}^{\sqrt{3}} h_1(u)h_2(u) \exp(c h_2(u)) \, du < 0 \quad \text{for all } c \in (0, \infty).$$

Hence, $G$ is strictly decreasing. Furthermore, we observe that $G(c) \to \infty$ as $c \searrow 0$ (using monotone convergence on $[\sqrt{3}, \infty)$) and that $G(c) \to -\infty$ as $c \to \infty$ (using similar arguments as for $G'$ and monotone convergence on $(0, 2 - \sqrt{3})$).

Combining these observations, we may conclude that there is a unique $c_0$ such that $G(c_0) = 0$, $G(c) > 0$ for all $c \in (0, c_0)$ and $G(c) < 0$ for all $c \in (c_0, \infty)$. This proves the claim with $\sigma_0(\alpha, b) = \frac{2^{3/2}}{c_0^{3/2}|b|}$. Numerical integration gives $c_0 \approx 0.2823$.

\textbf{Remark 3.1.} As explained in the Introduction, the same result holds if we interchange the roles of $\sigma$ and $b$. This can be seen also directly from the proof above.

\textbf{Remark 3.2.} The random dynamical system induced by (1.1) has a random set attractor $\{\hat{A}(\omega)\}_{\omega \in \Omega}$ (see [16, Definition 14.3] for a formal definition) for all parameter values, as can be seen similarly to [15]. The disintegrations of the ergodic invariant measure $\mu$ are supported on the fibers $\hat{A}(\omega)$. In fact, the measurable random compact set $\{A(\omega)\}_{\omega \in \Omega}$ with fibers $A(\omega) = \text{supp}(\mu_\omega) \subset \hat{A}(\omega)$ is a minimal (weak) random point attractor of (1.1) by [12][Proposition 2.20 (1)].

The fact that $\{A(\omega)\}_{\omega \in \Omega}$ is a singleton almost surely if $\lambda_1 < 0$, follows from a slightly modified reasoning alongside [12, Theorem 2.23] and its proof. In the case of $\lambda_1 > 0$, we deduce that $\mu_\omega$ is atomless almost surely as in the proof of [4, Remark 4.12]. Hence, Theorem 1.1 implies the bifurcation from an attractive random point attractor of (1.1) by [12][Proposition 2.20 (1)].

The positive top Lyapunov exponent is the only characterization of chaos we can give in this case as an analysis in the sense of [28] seems not feasible for white noise. However, the geometric mechanism of shear-induced chaos can still be understood along the same lines: the white noise drives some points on the limit cycle up and some down. Due to the phase amplitude coupling $b$, the points with larger $y$-coordinates move faster in the $\vartheta$-direction. At the same time, the dissipation force with strength $\alpha$ attracts the curve back to the limit cycles. This provides a mechanism for stretching and folding characteristic of chaos. The transition to chaos in the continuous time stochastic forcing is much faster than in the case of periodic kicks due to the effect of large deviations [21]. This is due to the fact that points end up in areas with arbitrarily large values of $y$ with positive probability. Hence, not so much shear is needed to generate the described stretching and folding due to phase amplitude coupling. However, for very small shear and noise, the dissipation leads to sinks being formed between these large deviation events, and the attractor ends up to be a singleton.

In Figure 2, we show the top Lyapunov exponent as a function of $\sigma$ for fixed $b$ and $\alpha$ according to formula (2.12). We have used numerical integration up to machine precision to calculate $\lambda_1$. The bifurcations of the sign of $\lambda_1$ at $\sigma_0(\alpha, b)$ is clearly seen in Figures 2a-2c. Furthermore, note that $\lambda_1 \to 0$ from below for $\sigma \to 0$. The figures illustrate that $\sigma_0(\alpha, b)$ is an increasing function of $\alpha$ and a decreasing function of $b$, or differently phrased: the larger the proportion of shear to dissipation, $b/\alpha$, the smaller the bifurcation point $\sigma_0(\alpha, b)$. In Figure 2d, we choose small values of $b$ and $\alpha$, but $b/\alpha$ large. We see no negative values of $\lambda_1$ as we would have to take values of $\sigma$ too small for the numerical integration.

We have chosen the same parameter regimes as in [21], where Lin and Young investigate numerically the Lyapunov exponents of the system given by

$$\begin{align*}
dy &= -\alpha \sqrt{t} + \sigma \sin(2\pi \vartheta) \circ dW_t, \\
d\vartheta &= (1 + by) \, dt,
\end{align*}$$

(3.3)
taking $\vartheta \in [0, 2\pi]$. Their numerical results show exactly the same qualitative behaviour apart from a slightly different scaling due to the factor $2\pi$. Note that our setting contains two approximations of model (3.3). The first option is to take also one Brownian motion, i.e. $m = 1$ in (1.1), and $f_1$ continuous and piecewise linear with constant absolute value of the derivative almost everywhere. The accordance of the numerics
for (3.3) and our results show that the simpler choice of the diffusion coefficient in our case does not change the qualitative behaviour. This is not a surprise, since we can derive formula (2.12) for \( \lambda_1 \) if we choose \( f \) to be piecewise linear on the intervals \( \left[ \frac{i}{4}, \frac{i+1}{4} \right] \) for \( i = 0, 1, 2, 3 \) with \( |f'| \) constant such that it represents a linear approximation of the sine function.

![Figure 2: The top Lyapunov exponent \( \lambda_1 \) as a function of \( \sigma \) for fixed \( b \) and \( \alpha \). The dots indicate the values of \( \lambda_1 \) that were calculated according to (2.12) using numerical integration. Figures 2a-2c illustrate that \( \sigma_0(\alpha, b) \) increases monotonously in \( \alpha \). In Figure 2d, \( b \) and \( \alpha \) are small, but \( b/\alpha \) is large. We don’t see the transition to \( \lambda_1 < 0 \) since we would have to take values of \( \sigma \) too small for the numerical integration.](image)

The second option is given by (1.3) which adds an additional Brownian motion with the cosine function as diffusion coefficient. As we have seen, this slightly extended model does not change the qualitative behaviour and validates analytically Lin and Young’s numerical investigations for model (3.3).

Furthermore, we confirm Lin and Young’s conjecture concerning a scaling property of \( \lambda_1 \) with respect to the parameters. For model (3.3) they observed numerically that, under the transformations \( \alpha \mapsto k\alpha \), \( b \mapsto kb \) and \( \sigma \mapsto \sqrt{k}\sigma \), \( \lambda_1 \) transforms approximately as \( \lambda_1 \mapsto k\lambda_1 \). This scaling property holds exactly for our model (1.1).

**Proposition 3.3.** Consider the stochastic differential equation (1.1), where the function \( f \) is of the form (1.4). Then the top Lyapunov exponent \( \lambda_1 \) as given by (2.12) satisfies

\[
\lambda_1(k\alpha, kb, \sqrt{k}\sigma) = k\lambda_1(\alpha, b, \sigma) \quad \text{for all } k \in \mathbb{R}^+ \setminus \{0\}.
\]

**Proof.** Recall from Proposition 2.3 that \( \lambda_1 \) can be calculated as the top Lyapunov exponent of the linear system (2.11). Then the claim follows immediately if we conduct the time change \( t \mapsto kt \) in this equation. \( \square \)
4 Summary and outlook

We have investigated systems with limit cycles on a cylinder perturbed by white noise. We were able to show a transition from negative to positive top Lyapunov exponents for fixed dissipation parameter $\alpha$ and big enough noise $\sigma$ and/or shear $b$. This implies a bifurcation of the random attractor from a random equilibrium to random strange attractor.

In the case of positive Lyapunov exponents, it remains an open problem to describe the attractor using concepts from ergodic theory, as entropy and SRB measures [19, 28], in order to have a more rigorous notion of chaos.

The results of this paper may well be relevant to shed more light on the problem of stochastic Hopf bifurcation, where numerical studies indicate a transition from negative to positive Lyapunov exponent as explained in the Introduction.

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