Optimal characterization of Gaussian channels using photon-number-resolving detector

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We present optimal schemes, based on photon number measurements, for Gaussian state tomography and for Gaussian process tomography. An $n$-mode Gaussian state is completely specified by $2n^2 + 3n$ parameters. Our scheme requires exactly $2n^2 + 3n$ distinct photon number measurements to tomograph the state and is therefore optimal. Further, we describe an optimal scheme to characterize Gaussian processes by using coherent state probes and photon number measurements. With much recent progress in photon number measurement experimental techniques, we hope that our scheme will be useful in various quantum information processing protocols including entanglement detection, quantum computation, quantum key distribution and quantum teleportation. This work builds upon the works of Parthasarathy et al. [Infin. Dimens. Anal. Quantum Probab. Relat. Top., 18(4): 1550023, 21, 2015].

I. INTRODUCTION

Continuous variable (CV) systems are ubiquitous in quantum information and communication protocols. Most of the CV quantum information protocols are based on Gaussian states as they are easy to prepare, manipulate and measure [1, 2]. One of the central tasks in quantum information processing is the estimation of quantum states which is formally called quantum state tomography (QST) [3–5]. Generally, homodyne and heterodyne measurements are employed in CV QST, which measure quadrature operators of a given state [6–8]. However, with the recent development of experimental techniques in photon-number-resolving-detector (PNRD) [9, 10], the possibility of carrying out QST via photon number measurements has opened up. Cerf et al. devised a scheme using beam splitters and on-off detectors, where one can obtain the trace and determinant of the covariance matrix of a Gaussian state [11, 12]. In a similar endeavor, Parthasarathy et al. have developed a theoretical scheme to determine the Gaussian state by estimating its mean and covariance matrix [13].

Another important task in quantum information processing is quantum process tomography (QPT), where we wish to characterize quantum processes which in general are completely positive maps. For CV systems, theoretical as well as experimental studies for QPT have been undertaken by several authors [14–23]. Lobino et al. used coherent state probes along with homodyne measurements to characterize quantum processes [14]. Similarly, Ghalaii et al. have developed a coherent state based QPT scheme via the measurement of normally ordered moments that are measured using homodyne detection [21]. In this direction, Parthasarathy et al. have utilized QST schemes based on photon number measurements for Gaussian states, to characterize the Gaussian channel [13].

In this paper, we simplify the scheme given by Parthasarathy et al. [13] and describe an optimal scheme which involves a minimum number of measurements and utilizes smaller number of optical elements for the QST of Gaussian states based on PNRD. We employ this scheme to devise an optimal scheme for Gaussian channel characterization. An $n$-mode Gaussian state is completely specified by their $2n$ first moments and second order moments arranged in the form of a covariance matrix which has $2n^2 + n$ parameters. Therefore, we require a total of $2n^2 + 3n$ parameters to completely determine an $n$-mode Gaussian state. The QST based on photon number measurements is optimal in the sense that we require exactly $2n^2 + 3n$ measurements to determine all the $2n^2 + 3n$ parameters of the state. Next we deploy the QST scheme that we develop, to estimate the output, with coherent state probes as inputs for the Gaussian channel characterization. An $n$-mode Gaussian channel is described by a pair of $2n \times 2n$ real matrices $A$ and $B$ with $B = B^T \geq 0$ which satisfy certain complete positivity and trace preserving conditions [24–26]. The matrices $A$ and $B$ together can be described by a total of $6n^2 + n$ parameters. We show that we can characterize a Gaussian quantum channel optimally, i.e., we require exactly $6n^2 + n$ measurements to determine all the $6n^2 + n$ parameters of the Gaussian channel. We compare the variance of transformed number operators arising in the aforementioned QST scheme which provides an insight into the efficiency of the scheme. Finally, we relate the variance of transformed number operators to the variance of quadrature operators.

The paper is organized as follows. In Sec. II we give a detailed mathematical background about CV systems. In Sec. III we provide our optimal QST scheme based
II. CV SYSTEM

An n-mode continuous variable quantum system is represented by n pairs of Hermitian quadrature operators $\hat{q}_i, \hat{p}_i$ ($i = 1, \ldots, n$) which can be arranged in a column vector as $\vec{\xi} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)^T$, $i = 1, 2, \ldots, 2n$. 

The bosonic commutation relation between them in a compact form read as $(\hbar=1)$

$$[\hat{\xi}_i, \hat{\xi}_j] = i\Omega_{ij}, \quad (i, j = 1, 2, \ldots, 2n),$$

where $\Omega_{ij}$ is the $2n \times 2n$ matrix given by

$$\Omega = \sum_{k=1}^{n} \omega = \begin{pmatrix} \omega & \cdots & \omega \\ \vdots & \ddots & \vdots \\ \omega & \cdots & \omega \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3)$$

The operators $\hat{q}_i$ and $\hat{p}_i$ have real continuous eigenvalues $q_i$ and $p_i$ and satisfy the eigenvalue equations:

$$\langle q_i | q_i \rangle = \delta(q_i - q_i), \quad \langle p_i | p_i \rangle = \delta(p_i - p_i),$$

$$\langle q_i | p_i \rangle = (2\pi)^{-1/2}e^{iq_i p_i},$$

$$\int dq_i |q_i\rangle \langle q_i| = \int dp_i |p_i\rangle \langle p_i| = 1. \quad (4)$$

The field annihilation and creation operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ ($i = 1, 2, \ldots, n$) are related to the quadrature operators as

$$\hat{a}_i = \frac{1}{\sqrt{2}}(\hat{q}_i + i\hat{p}_i), \quad \hat{a}_i^\dagger = \frac{1}{\sqrt{2}}(\hat{q}_i - i\hat{p}_i). \quad (5)$$

They can be arranged into a column vector as

$$\vec{\xi}^{(c)} = (\xi_i^{(c)}) = (\hat{a}_1, \hat{a}_1^\dagger, \ldots, \hat{a}_n, \hat{a}_n^\dagger)^T \quad i = 1, 2, \ldots, 2n, \quad (6)$$

and the bosonic commutation relation can be written compactly as

$$[\hat{\xi}_i^{(c)}, \hat{\xi}_j^{(c)}] = \Omega_{ij}. \quad (7)$$

The number operator for the $i$th mode and total number operator for $n$-mode system can be expressed as

$$\hat{N}_i = \hat{a}_i^\dagger \hat{a}_i = \frac{1}{2} (\hat{q}_i^2 + \hat{p}_i^2 - 1), \quad (8a)$$

$$\hat{N} = \sum_{i=1}^{n} \hat{N}_i. \quad (8b)$$

The state space known as Hilbert space $\mathcal{H}_i$ for $i$th mode is spanned by the eigen vectors $|n_i\rangle$, $\{n_i = 0, 1, \ldots, \infty\}$ of $\hat{N}_i = a_i^\dagger a_i$. The combined Hilbert space $\mathcal{H}_n = \bigotimes_{i=1}^{n} \mathcal{H}_i$ of the $n$-mode state is spanned by the product basis vector $|n_1, n_2, \ldots, n_n\rangle$ with $\{n_1, n_2, \ldots, n_n = 0, 1, \ldots, \infty\}$. The numbers $n_i$ correspond to photon number in the $i$th mode. The irreducible action of the field operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ on $\mathcal{H}_i$ is dictated by the commutation relation Eq. (7) and is given by

$$\hat{a}_i|n_i\rangle = \sqrt{n_i}|n_i - 1\rangle \quad n_i \geq 1, \quad \hat{a}_i^\dagger|0\rangle = 0,$$

$$\hat{a}_i^\dagger|n_i\rangle = \sqrt{n_i + 1}|n_i + 1\rangle \quad n_i \geq 0. \quad (9)$$

We define displacement operator acting on the $i$th mode and the corresponding coherent states as:

$$\hat{D}_i(q_i, p_i) = e^{i(q_i \hat{a}_i - p_i \hat{a}_i^\dagger)}.$$  

$$|q_i, p_i\rangle = \hat{D}_i(q_i, p_i)|0\rangle. \quad (10)$$

Here $q_i$ corresponds to displacement along $\hat{q}$-quadrature while $p_i$ corresponds to displacement along $\hat{p}$-quadrature.

A. Symplectic transformations

The group $Sp(2n, \mathcal{R})$ is defined as the group of linear homogeneous transformations $S$ specified by real $2n \times 2n$ matrices $S$ acting on the quadrature variables and preserving the canonical commutation relation Eq. (2):

$$\hat{\xi}_i \rightarrow S\hat{\xi}_i S^{-1} = \Sigma_{ij}^0 \hat{\xi}_j, \quad S\Omega S^T = \Omega. \quad (11)$$

The unitary representation of this group turns out to be infinite dimensional where we have $U(S)$ for each $S \in Sp(2n, \mathcal{R})$ acting on a Hilbert space and is known as the metaplectic representation. These unitary transformations are generated by Hamiltonian which are quadratic functions of quadrature and field operators. Further, any symplectic matrix $S \in Sp(2n, \mathcal{R})$ can be decomposed as

$$S = PK(X, Y), \quad (12)$$

$P \in \Pi(n)$ is a subset of $Sp(2n, \mathcal{R})$ defined as

$$\Pi(n) = \{S \in Sp(2n, \mathcal{R}) | S^T = S, \quad S > 0\}, \quad (13)$$

and $K(X, Y)$ is the maximal compact subgroup of $Sp(2n, \mathcal{R})$ which is isomorphic to the unitary group $U(n) = X + iY$ in $n$-dimensions. The action of $U(n)$ transformation on the annihilation and creation operators is given as

$$\hat{a} \rightarrow U \hat{a}, \quad \hat{a}^\dagger \rightarrow U^* \hat{a}^\dagger, \quad (14)$$

where $\hat{a} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)^T$ and $\hat{a}^\dagger = (\hat{a}_1^\dagger, \hat{a}_2^\dagger, \ldots, \hat{a}_n^\dagger)^T$. The $2n \times 2n$ dimensional symplectic transformation matrix $K(X, Y)$ acting on the Hermitian quadrature operators can be easily obtained using Eqs. (5) and (14).

Now we discuss three basic symplectic operations which will be used later.
Phase change operation: The symplectic transformation for phase change operation acting on the quadrature operators $\hat{q}_i, \hat{p}_i$ is

$$R_i(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.$$  \hfill (15)

Phase change operation for a single mode system corresponds to $U(1)$ subgroup of $Sp(2, R)$. This operation can be generated by Hamiltonian of the form $H = \hat{a}_i^\dagger \hat{a}_i$ and the corresponding metaplectic representation is

$$U(R_i(\theta)) = \exp(-i\theta \hat{a}_i^\dagger \hat{a}_i).$$ \hfill (16)

Action of phase change operation on the annihilation operator is

$$U(R_i(\theta))^\dagger \hat{a}_i U(R_i(\theta)) = e^{-i\theta} \hat{a}_i.$$ \hfill (17)

Single mode squeezing operation: Symplectic transformation for the single mode squeezing operation acting on quadrature operators $\hat{q}_i$ and $\hat{p}_i$ is written as

$$S_i(r) = \begin{pmatrix} e^{-r} & 0 \\ 0 & e^{r} \end{pmatrix}.$$ \hfill (18)

The corresponding unitary operator is given by

$$U(S_i(r)) = \exp[r (a_i^2 - a_i^\dagger 2^\dagger) / 2].$$ \hfill (19)

Annihilation operator $\hat{a}_i$ transforms as following under the action of $U(S(r))$:

$$U(S_i(r))^\dagger \hat{a}_i U(S_i(r)) = (\cosh r) \hat{a}_i - (\sinh r) \hat{a}_i^\dagger.$$ \hfill (20)

Beam splitter operation: For two-mode systems with quadrature operators $\hat{\xi} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2)^T$ the beam splitter transformation $B_{ij}(\theta)$ acts as follows

$$B_{ij}(\theta) = \begin{pmatrix} \cos \theta & 0 \\ 0 & \cosh \theta \end{pmatrix},$$ \hfill (21)

where $\hat{1}_2$ represents $2 \times 2$ identity matrix and transmittivity is specified through $\gamma$ via the relation $\gamma = \cos^2 \theta$. For a balanced (50:50) beam splitter, $\theta = \pi/4$. Beam splitter transformation acting on field operators is an element of the compact group $U(2)$:

$$\begin{pmatrix} \hat{a}_i \\ \hat{a}_j \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{a}_i \\ \hat{a}_j \end{pmatrix},$$ \hfill (22)

and the corresponding metaplectic representation is

$$U(B_{ij}(\theta)) = \exp[\theta (\hat{a}_i^\dagger \hat{a}_j - \hat{q}_i \hat{a}_j^\dagger)].$$ \hfill (23)

It is to be noted that the expressions involved in the Eqs. (16) and (23) are quadratic in field operators and are photon number conserving, while the quadratic expression involved in Eq. (19) is photon non-conserving.

B. Phase space description

For a density operator $\hat{\rho}$ of a quantum system the corresponding Wigner distribution is defined as

$$W(\xi) = \frac{1}{(2\pi)^n} \int d^n\xi' \langle \xi - \frac{1}{2} \xi' | \hat{\rho} | \xi + \frac{1}{2} \xi' \rangle \exp(iq'^T \xi),$$ \hfill (24)

where $\xi = (q_1, p_1, \ldots, q_n, p_n)^T$, $\xi' \in \mathbb{R}^n$ and $q = (q_1, q_2, \ldots, q_n)^T$, $p = (p_1, p_2, \ldots, p_n)^T$. Therefore, $W(\xi)$ depends upon $2n$ real phase space variables.

For an $n$-mode system, the first order moments are defined as

$$d = \langle \hat{\xi} \rangle = \text{Tr}[\hat{\rho} \hat{\xi}],$$ \hfill (25)

and the second order moments are best represented by the real symmetric $2n \times 2n$ covariance matrix defined as

$$V = (V_{ij}) = \frac{1}{2} \langle \{ \Delta \hat{\xi}_i, \Delta \hat{\xi}_j \} \rangle,$$ \hfill (26)

where $\Delta \hat{\xi}_i = \hat{\xi}_i - \langle \hat{\xi}_i \rangle$, and $\{ , \}$ denotes anti-commutator. The number of independent real parameters required to specify the covariance matrix is $n(2n + 1)$. The uncertainty principle in terms of covariance matrix reads $V + \frac{1}{4} \Omega \geq 0$ which implies that the covariance matrix is positive definite i.e., $V > 0$.

A state is called a Gaussian state if the corresponding Wigner distribution is a Gaussian. Gaussian states are completely determined by their first and second order moments and thus we require a total of $2n + n(2n + 1) = 2n^2 + 3n$ parameters to completely determine an $n$-mode Gaussian state. For the special case of Gaussian states, Eq. (24) can be written as [1]

$$W(\xi) = \exp[-(1/2)(\xi - d)^T V^{-1}(\xi - d)] \cdot (2\pi)^n \sqrt{\det V},$$ \hfill (27)

where $V$ is the covariance matrix and $d$ denotes the displacement of the Gaussian state in phase space.

We now compute averages of a few quantities that will be required later, using the phase space representation.

$$\langle \hat{N} \rangle = \sum_{j=1}^{n} \hat{N}_j = \frac{1}{2} \sum_{j=1}^{n} (\hat{q}_j^2 + \hat{p}_j^2 - 1)$$ \hfill (28)

is symmetrically ordered in $\hat{q}$ and $\hat{p}$ operators, therefore, average number of photons $\langle \hat{N} \rangle$ for an $n$-mode Gaussian state can be readily computed using the Wigner distribution as follows [13, 30]:

$$\langle \hat{N} \rangle = \frac{1}{2} \sum_{j=1}^{n} \int d^2\xi (q_j^2 + p_j^2 - 1) W(\xi),$$ \hfill (29)

$$= \frac{1}{2} \left[ \text{Tr} \left( V - \frac{1}{2} \hat{1}_{2n} \right) + ||d||^2 \right].$$
in Heisenberg representation the number operator transforms as, \( N \rightarrow \mathcal{U}^\dagger N \mathcal{U} \). Specifically for a phase space displacement \( D(r) \), we have
\[
\langle \hat{D}(r)^\dagger \hat{N} \hat{D}(r) \rangle = \frac{1}{2} \left[ \text{Tr} \left( V - \frac{1}{2} \mathbb{I}_{2n} \right) + \|d + r\|^2 \right],
\]
(30)
which simplifies by using Eq. (29) to
\[
\langle \hat{D}(r)^\dagger \hat{N} \hat{D}(r) \rangle - \langle \hat{N} \rangle = \frac{1}{2} \left( \|d + r\|^2 - \|d\|^2 \right).
\]
(31)
For a homogeneous symplectic transformation \( S \), the density operator follows the metaplectic representation \( \mathcal{U}(S) \) as \( \rho \rightarrow \mathcal{U}(S) \rho \mathcal{U}(S)^\dagger \). The corresponding transformation of the displacement vector \( d \) and covariance matrix \( V \) is given by [27]
\[
d \rightarrow Sd, \quad \text{and} \quad V \rightarrow SVS^T.
\]
(32)
Thus, we can easily evaluate the average of the number operator after the state has undergone a metaplectic transformation using the Eqs. (29) & (32) as
\[
\langle \hat{\mathcal{U}}(S)^\dagger \hat{N} \hat{\mathcal{U}}(S) \rangle = \frac{1}{2} \text{Tr} \left( V S^T S - \frac{1}{2} \mathbb{I}_{2n} \right) + \frac{1}{2} d^T S^T S d.
\]
(33)
Therefore,
\[
\langle \hat{\mathcal{U}}(S)^\dagger \hat{N} \hat{\mathcal{U}}(S) \rangle - \langle \hat{N} \rangle = \frac{1}{2} \text{Tr} \left[ V (S^T S - \frac{1}{2} \mathbb{I}_{2n}) \right] + \frac{1}{2} d^T (S^T S - \frac{1}{2} \mathbb{I}_{2n}) d.
\]
(34)
More mathematical details are available in [27].

III. ESTIMATION OF GAUSSIAN STATES USING PHOTON NUMBER MEASUREMENTS

![Diagram showing photon number measurements](image)

FIG. 1. To estimate the mean of an \( n \)-mode Gaussian state, the state is displaced along one of the \( 2n \) phase space variables before performing photon number measurement on each of the modes. In the figure, displacement gate \( \hat{D}_i(1,0) \) is applied on the state which displaces the \( \hat{q}_i \)-quadrature of the \( i \)-th mode by an unit amount.

In this section, we present a variant of the scheme developed in [13] where the authors have devised a scheme to estimate the mean and covariance matrix of Gaussian state using PNRD. In our scheme which is optimal and uses minimum optical elements, photon number measurement is performed on the original Gaussian state as well as transformed Gaussian state. These transformations or gates consist of displacement, phase rotation, single mode squeezing and beam splitter operation denoted by \( \hat{D}_i(q,p), \mathcal{U}(R_i(\theta)), \mathcal{U}(S_i(r)), \) and \( \mathcal{U}(B_{ij}(\theta)) \), respectively.

A. Mean estimation

We first perform photon number measurement on the original \( n \)-mode Gaussian state giving us \( \langle \hat{N} \rangle \). Then we consider two different photon number measurements after displacing one of the quadratures \( \hat{q}_i \) or \( \hat{p}_i \) of the \( i \)-th mode by an unit amount giving us \( \langle \hat{D}_i(1,0)^\dagger \hat{N} \hat{D}_i(1,0) \rangle \) and \( \langle \hat{D}_i(0,1)^\dagger \hat{N} \hat{D}_i(0,1) \rangle \). (Figure 1 depicts displacement gate \( \hat{D}_i(1,0) \) acting on the \( i \)-th mode of the state.) We therefore have by using Eq. (31):
\[
\langle \hat{D}_i(1,0)^\dagger \hat{N} \hat{D}_i(1,0) \rangle - \langle \hat{N} \rangle = \frac{1}{2} (1 + 2d_{q_i}),
\]
\[
\langle \hat{D}_i(0,1)^\dagger \hat{N} \hat{D}_i(0,1) \rangle - \langle \hat{N} \rangle = \frac{1}{2} (1 + 2d_{p_i}),
\]
(35)
which can be rewritten as
\[
d_{q_i} = \langle \hat{D}_i(1,0)^\dagger \hat{N} \hat{D}_i(1,0) \rangle - \langle \hat{N} \rangle - \frac{1}{2},
\]
\[
d_{p_i} = \langle \hat{D}_i(0,1)^\dagger \hat{N} \hat{D}_i(0,1) \rangle - \langle \hat{N} \rangle - \frac{1}{2}.
\]
(36)
Thus, we can obtain the mean values of \( \hat{q}_i \) and \( \hat{p}_i \)-quadratures once the values of \( \langle D_i(1,0)^\dagger \hat{N} D_i(1,0) \rangle, \langle D_i(0,1)^\dagger \hat{N} D_i(0,1) \rangle, \) and \( \langle \hat{N} \rangle \) have been obtained. Therefore, to obtain all the \( 2n \) elements of mean \( d \) of the Gaussian state, we need to perform \( 2n \) photon number measurements after displacing the state by an unit amount along \( 2n \) different phase spaces variables along with photon number measurement on the original state. We also note that \( \text{Tr}(V) \) can be obtained using Eq. (29) once mean \( d \) of the Gaussian state has been obtained.
\[
\text{Tr}(V) = 2\langle \hat{N} \rangle - \|d\|^2 + n.
\]
(37)
Thus, we are able to estimate \( 2n \) elements of mean \( d \) of the Gaussian state and trace of the covariance matrix \( \text{Tr}(V) \) using a total of \( 2n + 1 \) photon number measurements.

B. Estimation of intra-mode covariance matrix

For convenience in representation, we express the covariance matrix of the \( n \)-mode Gaussian state as follows:
\[
V = \begin{pmatrix}
V_{1,1} & V_{1,2} & \cdots & V_{1,n} \\
V_{2,1} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
V_{n,1} & \cdots & V_{n,n-1} & V_{n,n}
\end{pmatrix},
\]
(38)
Gaussian state: To estimate the intra-mode covariance matrix, consider where
squeezing U
\( \hat{S}(r) \)
and phase shifter U(R(\( \phi \))) followed by a squeezer U(S(r)) is applied on the i\( ^{th} \) mode of the state.

where \( V_{i,j} \) is a \( 2 \times 2 \) matrix. Further, we represent the mean and covariance matrix of the marginal state of mode \( i \) (or intra-mode covariance matrix for mode \( i \)) as

\[
d_i = \begin{pmatrix} d_{q_i} \\ d_{p_i} \end{pmatrix}, \quad V_{i,i} = \begin{pmatrix} \sigma_{qq} & \sigma_{qp} \\ \sigma_{pq} & \sigma_{pp} \end{pmatrix}.
\]

(39)

To estimate the intra-mode covariance matrix, consider the single-mode symplectic gate \( P_i(r, \phi) \) consisting of a squeezer and phase shifter acting on the \( i^{th} \) mode of the Gaussian state:

\[
P_i(r, \phi) = S_i(r)R_i(\phi) = \begin{pmatrix} e^{-r} & 0 \\ 0 & e^{r} \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.
\]

(40)

The schematic representation of \( P_i(r, \phi) \) is shown in Fig. 2. When \( P_i(r, \phi) \) acts on the \( i^{th} \) mode of the Gaussian state, Eq. (34) reduces to

\[
\langle \hat{U}(P_i)^\dagger \hat{N} \hat{U}(P_i) \rangle - \langle \hat{N} \rangle = \frac{1}{2} \text{Tr} [V_{i,i}(P_i^T P_i - I_2)] + \frac{1}{2} d_i^T (P_i^T P_i - I_2) d_i.
\]

(41)

Here

\[
P_i^T P_i = \begin{pmatrix} e^{-2r} \cos^2 \phi + e^{2r} \sin^2 \phi & -\sin 2r \sin 2\phi \\ -\sinh 2r \sin 2\phi & e^{-2r} \sin^2 \phi + e^{2r} \cos^2 \phi \end{pmatrix}.
\]

(42)

For brevity, we assume

\[
P_i^T P_i - I_2 = \begin{pmatrix} k_1 & k_3 \\ k_3 & k_2 \end{pmatrix},
\]

(43)

and thus Eq. (41) simplifies as

\[
\langle \hat{U}(P_i)^\dagger \hat{N} \hat{U}(P_i) \rangle - \langle \hat{N} \rangle = \frac{1}{2} \left[ k_1 \sigma_{qq} + k_2 \sigma_{pp} + 2k_3 \sigma_{qp} + \right. \\
+ k_1 d_{q_i}^2 + k_2 d_{p_i}^2 + 2k_3 d_{q_i} d_{p_i} \right].
\]

(44)

Rearranging the above equation, we obtain

\[
k_1 \sigma_{qq} + k_2 \sigma_{pp} + 2k_3 \sigma_{qp} = 2 \left( \langle \hat{U}(P_i)^\dagger \hat{N} \hat{U}(P_i) \rangle - \langle \hat{N} \rangle \right) \\
- (k_1 d_{q_i}^2 + k_2 d_{p_i}^2 + 2k_3 d_{q_i} d_{p_i}).
\]

(45)

Since \( d_{q_i} \) and \( d_{p_i} \) have already been obtained in Sec. III A (Eq. (36)), the above equation contains three unknown parameters \( \sigma_{qq}, \sigma_{pp}, \) and \( \sigma_{qp} \). We can determine these three unknowns by performing three distinct photon number measurement for appropriate combinations of squeezing parameter \( r \) and phase rotation angle \( \phi \), as follows:

(i) For \( e^r = \sqrt{2} \) and \( \phi = 0 \), we obtain

\[
- \frac{1}{2} (\sigma_{qq} - 2\sigma_{pp}) = c_1.
\]

(46)

(ii) For \( e^r = \sqrt{3} \) and \( \phi = 0 \), we obtain

\[
- \frac{2}{3} (\sigma_{qq} - 3\sigma_{pp}) = c_2.
\]

(47)

(iii) For \( e^r = \sqrt{2} \) and \( \phi = \pi/4 \), we obtain

\[
\frac{1}{4} (\sigma_{qq} + \sigma_{pp} - 6\sigma_{qp}) = c_3.
\]

(48)

Here \( c_1, c_2, \) and \( c_3 \) correspond to right-hand side (RHS) of Eq. (45) which can be easily determined once the photon number measurements have been performed. Equations (46) and (47) can be solved to yield value of \( \sigma_{qq} \) and \( \sigma_{pp} \), which can be put in Eq. (48) to obtain value of \( \sigma_{qp} \). Thus \( V_{i,i} \) can be completely determined by performing three photon number measurements after applying the three distinct single mode symplectic gates Eqs. (46)-(48). To determine all \( V_{i,i} \) (\( 1 \leq i \leq n - 1 \)), we require \( 3(n-1) \) measurements. For \( V_{n,n} \), we need to determine \( \sigma_{qq} \) and one of \( \sigma_{qq} \) or \( \sigma_{pp} \). As \( \text{Tr}(V) \) is already known. Thus, a total of \( 3(n-1) + 2 = 3n-1 \) distinct photon number measurements are required to determine all the parameters of the intra-mode covariance matrix of a Gaussian state.

C. Estimation of inter-mode correlations matrix

To estimate the inter-mode correlations matrix, we perform two-mode symplectic operations on the Gaussian state before measuring photon number distribution. We write the covariance matrix of the reduced state of the \( i \) \( j \) mode in accord with Eq. (38) as

\[
\begin{pmatrix} V_{i,i} & V_{i,j} \\ V_{i,j}^T & V_{j,j} \end{pmatrix}.
\]

(49)

Here \( i < j \) need not be successive modes. Since \( V_{i,i} \) and \( V_{j,j} \) has already been determined in Sec. III B, we need to
FIG. 3. To estimate the inter-mode correlations matrix, we apply a two-mode symplectic gate on the state before performing photon number measurement on each of the modes. As shown in the figure, first a phase shifter \( U(R_i(\phi)) \) is applied on the \( i \)th mode of the state. This is followed by a balanced beam splitter \( U(B_{ij}(\frac{\pi}{4})) \) acting on \( i \ j \) modes and finally a squeezer \( U(S_i(r)) \) is applied on the \( i \)th mode of the state.

We determine only \( V_{i,j} \). We further take the matrix elements of \( V_{i,j} \) to be

\[
V_{i,j} = \begin{pmatrix} \gamma_{qq} & \gamma_{qp} \\ \gamma_{pq} & \gamma_{pp} \end{pmatrix}.
\]  

(50)

The two-mode symplectic gate is comprised of phase shifter acting on the \( i \)th mode followed by a balanced beam splitter acting on modes \( i \ j \) and finally a squeezer acting on mode \( i \). We represent this mathematically as

\[
Q_{ij}(r, \phi) = (S_i(r) \oplus \mathbb{I}_2) B_{ij}(\frac{\pi}{4}) (R_i(\phi) \oplus \mathbb{I}_2),
\]

\[
= \begin{pmatrix} S_i(r) & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \sqrt{2} & -1 \sqrt{2} \\ 1 & -1 \sqrt{2} & 1 \sqrt{2} \end{pmatrix} \begin{pmatrix} R_i(\phi) & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}.
\]

(51)

The schematic representation of \( Q_{ij}(r, \phi) \) is illustrated in Fig. 3. When the aforementioned gate \( Q_{ij}(r, \phi) \) acts on the modes \( i \ j \) of the Gaussian state, Eq. (34) reduces to

\[
\text{Tr} \left[ U(\hat{Q}_{ij}) \hat{N} U(\hat{Q}_{ij})^\dagger \right] - \langle \hat{N} \rangle = \frac{1}{2} \text{Tr} \left[ \begin{pmatrix} V_{i,i} & V_{i,j} \\ V_{i,j}^T & V_{j,j} \end{pmatrix} \begin{pmatrix} K - \mathbb{I}_2 & M \\ M^T & L - \mathbb{I}_2 \end{pmatrix} \right] + \frac{1}{2} \begin{pmatrix} d_{qi} \\ d_{pi} \end{pmatrix}^T \begin{pmatrix} K - \mathbb{I}_2 & M \\ M^T & L - \mathbb{I}_2 \end{pmatrix} \begin{pmatrix} d_{qi} \\ d_{pj} \end{pmatrix},
\]

(52)

where we have used

\[
Q_{ij}^T Q_{ij} = \begin{pmatrix} K & M \\ M^T & L \end{pmatrix}.
\]

(53)

Using the following simplification for trace

\[
\text{Tr} \left[ \begin{pmatrix} V_{i,i} & V_{i,j} \\ V_{i,j}^T & V_{j,j} \end{pmatrix} \begin{pmatrix} K - \mathbb{I}_2 & M \\ M^T & L - \mathbb{I}_2 \end{pmatrix} \right] = \text{Tr} [V_{i,i}(K - \mathbb{I}_2) + V_{j,j}(L - \mathbb{I}_2)] + 2 \text{Tr} [V_{i,j}M^T],
\]

(54)

Eq. (52) can be rearranged as

\[
\text{Tr} \left[ V_{i,j}M^T \right] = \langle \hat{N} \rangle - \frac{1}{2} \text{Tr} [V_{i,i}(K - \mathbb{I}_2) + V_{j,j}(L - \mathbb{I}_2)] - \frac{1}{2} \left( \begin{pmatrix} d_{qi} \\ d_{pi} \end{pmatrix} \right)^T \begin{pmatrix} K - \mathbb{I}_2 & M \\ M^T & L - \mathbb{I}_2 \end{pmatrix} \begin{pmatrix} d_{qi} \\ d_{pj} \end{pmatrix}.
\]

(55)

Various terms appearing in the RHS of the above equation, for instance \( V_{i,i}, V_{j,j}, d_{qi}, d_{pi}, d_{qi}, d_{pj} \) have already been determined. Thus the four unknowns \( \gamma_{qq}, \gamma_{pp}, \gamma_{qp}, \gamma_{pq} \) appearing on the LHS of the above equation can be determined by performing four different photon number measurements for appropriate combinations of squeezing parameter \( r \) and phase rotation angle \( \phi \). Further, LHS of Eq. (55) can be expressed as following:

\[
\text{Tr} \left[ V_{i,j}M^T \right] = \frac{1}{2} \left[ \begin{pmatrix} d_{qi} \\ d_{pi} \end{pmatrix} \right]^T \begin{pmatrix} \gamma_{qq} & \gamma_{qp} \\ \gamma_{pq} & \gamma_{pp} \end{pmatrix} \begin{pmatrix} d_{qi} \\ d_{pj} \end{pmatrix} + \left( 1 - e^{2r} \right) \sin \gamma_{qp} + \left( e^{2r} - 1 \right) \sin \gamma_{pp}
\]

(56)

We take these four different combinations of squeezing parameter \( r \) and phase rotation angle \( \phi \) to determine the four unknowns:

(i) For \( e^r = \sqrt{2} \) and \( \phi = 0 \), we obtain

\[
- \frac{1}{4} (\gamma_{qq} - 2\gamma_{pp}) = d_1.
\]

(57)

(ii) For \( e^r = \sqrt{3} \) and \( \phi = 0 \), we obtain

\[
- \frac{1}{3} (\gamma_{qq} - 3\gamma_{pp}) = d_2.
\]

(58)

(iii) For \( e^r = \sqrt{2} \) and \( \phi = \pi/2 \), we obtain

\[
- \frac{1}{4} (2\gamma_{qp} + \gamma_{pp}) = d_3.
\]

(59)

(iv) For \( e^r = \sqrt{3} \) and \( \phi = \pi/2 \), we obtain

\[
- \frac{1}{3} (3\gamma_{qp} + \gamma_{pp}) = d_4.
\]

(60)

Here \( d_1, d_2, d_3, \) and \( d_4 \) are the RHS of Eq. (55) which can be easily determined once the photon number measurements have been performed. Equations (57) and (58) can be solved to yield values of \( \gamma_{qq} \) and \( \gamma_{pp} \), whereas Eqs. (59) and (60) can be solved to yield value of \( \gamma_{qp} \) and \( \gamma_{pq} \). Thus, we have used four distinct measurements to determine the four parameters of \( V_{i,j} \). The inter-mode correlations of the Gaussian states thus require \( 4 \times n(n-1)/2 = 2n(n-1) \) measurements. So the total number of measurements required to determine all the \( 2n^2 + 3n \) parameters of the \( n \)-mode Gaussian state adds...
up to $2n^2 + 3n$. The results are summarized in Table I. Thus, our tomography scheme for Gaussian state using photon number measurement is optimal in the sense that we require exactly the same number of distinct measurements as the number of independent real parameters of the Gaussian state.

| TABLE I. Tomography of an n-mode Gaussian state by photon number measurements |
|-------------------------|-----------------|-----------------|-----------------|
| Estimate type           | Parameters number | Gaussian Operations | Measurement number |
| Mean ($d$)              | $2n$             | Displacement     | $2n + 1$         |
| Intra-mode covariance ($V_{ii}$) | $3n$             | Phase shifter, squeezer | $3n - 1$         |
| Inter-mode correlations ($V_{ij}$) | $2n(n - 1)$      | Phase shifter, squeezer | $2n(n - 1)$      |

Total $2n^2 + 3n$ $2n^2 + 3n$

IV. CHARACTERIZATION OF GAUSSIAN CHANNELS

![Schematic diagram of the scheme](image)

FIG. 4. Scheme for a complete characterization of an n-mode Gaussian channel. A total of $2n$ coherent state probes are sent through the channel and full or partial state tomography is carried out on the output state. In the figure, displacement operator $D_i(1, 0)$ displaces the $q$-quadrature of the $i$th mode by an unit amount of an $n$-mode vacuum state to give one of the required probe state. Single and two mode gate operations involved in state tomography and described in Section III are indicated as “Gates”.

In this section, we move on to the characterization of a Gaussian channel using coherent state probes [14, 15, 17] by employing the tomography techniques developed in Section III. Gaussian channels are defined as those channels which transform Gaussian states into Gaussian states [25, 26]. An n-mode Gaussian channel is specified by a pair of $2n \times 2n$ real matrices $A$ and $B$ with $B = B^T \geq 0$ [24]. The matrices $A$ and $B$ are described by a total of $4n^2 + 2n(2n + 1)/2 = 6n^2 + n$ real parameters and satisfy complete positivity and trace preserving condition

$$B + i\Omega - iA\Omega A^T \geq 0.$$  \hspace{1cm} (61)

The action of the Gaussian channel on mean $d$ and covariance matrix $V$ of a Gaussian state is given by

$$d \rightarrow Ad, \quad V \rightarrow AVA^T + \frac{1}{2}B.$$  \hspace{1cm} (62)

Here again we follow the scheme proposed in [13]. Schematic diagram of the scheme is shown in Fig. 4. We prepare $2n$ coherent state probes by displacing $n$-mode vacuum state by an unit amount along any of the $2n$ different phase space variables. These coherent state probes are sent through the channel and full or partial state tomography using photon number measurement is carried out on the output state. The information about the output state parameters enable us to characterize the Gaussian channel. Now we describe the exact scheme in detail. For convenience, we define a $2n$ dimensional column vector as

$$e_j = (0, 0, \ldots, 1, \ldots, 0)^T,$$  \hspace{1cm} (63)

with 1 present at the $j^{th}$ position. First set of $n$ coherent state probes are prepared by displacing $n$-mode vacuum state ($d = 0, V = I_{2n}/2$) by an unit amount along $n$ different $q$-quadratures. For instance, application of displacement operator $D_i(1, 0)$ on the $j^{th}$ mode of the $n$-mode vacuum state yields the coherent state

$$|e_{2j-1}\rangle = D_{j}(1, 0)|0\rangle,$$  \hspace{1cm} (64)

where $|0\rangle$ denotes $n$-mode vacuum state. The mean and covariance matrix of the coherent state $|e_{2j-1}\rangle$ is given by

$$d = e_{2j-1}, \quad V = \frac{1}{2}I_{2n}.$$  \hspace{1cm} (65)

This coherent state is sent through the Gaussian channel and the mean and covariance matrix of the probe state transforms according to Eq. (62):

$$d_G = Ae_{2j-1}, \quad V_G = \frac{1}{2}(AA^T + B).$$  \hspace{1cm} (66)

Now we perform full state tomography on the output state $\rho_G$ ($j = 1$) which requires $2n^2 + 3n$ measurements. This provides us the matrix $AA^T + B$ and the first column of matrix $A$. For the rest $n - 1$ probe states ($2 \leq j \leq n$), we measure only the mean of the output state $\rho_G$ which enables us to determine all the odd columns of matrix $A$.

However, as we noticed in Sec. III.A, we need to perform $2n + 1$ measurements to obtain the $2n$ elements of mean vector $d_G$. But in our case, all coherent state probes have been displaced by the same (unit) amount, and thus $\langle N \rangle$ is same for all probe states (from Eq. (29)), so we can use the $\langle N \rangle$ estimation obtained from the full state tomography on the first coherent state probe. Thus for other output states $\rho_G$, only $2n$ measurements are required to determine the mean vector $d_G$.

In case, we had used coherent state probes with different mean, then $\langle N \rangle$ would have been different for each
of them. However, $\text{Tr}(V) = \text{Tr}(AA^T + B)/2$ is same for all probe states as all the output states have the same covariance matrix (66) and has already been obtained in the process of tomography of the first output state ($j = 1$). Now we show how this fact can be exploited to obtain the value of $\langle \hat{N} \rangle$ for the other coherent state probes. We perform $2n$ measurements after displacing the output state $\rho_G$ corresponding to second coherent state probe and obtain $2n$ equations as follows:

$$d_q = (\hat{D}_i(1,0)^\dagger \hat{N} \hat{D}_i(1,0)) - \langle \hat{N} \rangle - \frac{1}{2}, \quad 1 \leq i \leq n,$$

$$d_p = (\hat{D}_i(0,1)^\dagger \hat{N} \hat{D}_i(0,1)) - \langle \hat{N} \rangle - \frac{1}{2}, \quad 1 \leq i \leq n.$$  

(67)

We substitute $d_q$ and $d_p$ ($1 \leq i \leq n$) in Eq. (37) and obtain a quadratic equation in $\langle \hat{N} \rangle$. After solving for $\langle \hat{N} \rangle$, we put its value in Eq. (67) to obtain $d_q$ and $d_p$ ($1 \leq i \leq n$). Thus, we require only $2n$ measurements even for the case of coherent state probes with different means and no additional measurements are required.

The other set of $n$ coherent state probes are prepared by displacing $n$-mode vacuum state by an unit amount along $n$ different $\hat{p}$-quadratures. For instance, application of displacement operator $\hat{D}_j(0,1)$ on the $j^{th}$ mode of the $n$-mode vacuum state yields the coherent state

$$|e_{2j}\rangle = \hat{D}_j(0,1)|0\rangle.$$  

(68)

The mean and covariance matrix of the coherent state $|e_{2j}\rangle$ is given by

$$\mathbf{d} = e_{2j}, \quad V = \frac{1}{2} \mathbf{1}_{2n}.$$  

(69)

This coherent state is sent through the Gaussian channel and the mean and covariance matrix of the probe state transforms according to Eq. (61):

$$\mathbf{d}_G = A e_{2j}, \quad V_G = \frac{1}{2} (AA^T + B).$$  

(70)

For all these $n$ output states $\rho_G$ ($1 \leq j \leq n$), we measure only the mean which enables us to determine all the even columns of matrix $A$. This information completely specifies matrix $A$ as odd columns had already been determined using the first set of $\hat{q}$-displaced $n$ coherent state probes. This also enables us to obtain matrix $B$ as matrix $AA^T + B$ was already known from the full state tomography on the first coherent state probe. Thus, the total number of measurements required sum up to $6n^2 + n$ as shown in Table II which exactly coincides with the parameters specifying a Gaussian channel. In the scheme of Parthasarathy et al. [13], $2n - 1$ additional measurements were required which we do not need, leading to the optimality of our scheme.

| Coherent state probe | Information obtained |
|----------------------|----------------------|
| $\hat{q}$-displaced  | Odd columns of $A$ $2n^2 + 3n + (n-1) \times 2n$ & $(AA^T + B)$ |
| $\hat{p}$-displaced  | Even columns of $A$ $n \times 2n$ |
| **Total**            | $6n^2 + n$ |

V. VARIANCE IN PHOTON NUMBER MEASUREMENTS

In this section, we analyze the variance of photon number distribution of the original state and gate-transformed states which we used towards state and process estimation in Sections III & IV. This study will provide us with an idea of the quality of our estimates of the Gaussian states and channels.

To evaluate the variance of photon number we note that the square of the number operator can be easily put in symmetrically ordered form as follows:

$$\hat{N}^2 = \frac{1}{4} \sum_{i,j=1}^{n} (\hat{q}_i^2 + \hat{p}_i^2 - 1) (\hat{q}_j^2 + \hat{p}_j^2 - 1)$$

$$\{\hat{N}^2\}_{\text{sym}} = f(\hat{q}, \hat{p}) = \frac{1}{4} \sum_{i,j=1}^{n} (\hat{q}_i^2 + \hat{p}_i^2 - 1) (\hat{q}_j^2 + \hat{p}_j^2 - 1)$$

$$+ \frac{1}{4} \sum_{i=1}^{n} \left[ \hat{q}_i^4 + \hat{p}_i^4 - 2\hat{q}_i^2 - 2\hat{p}_i^2 + \frac{1}{3} (\hat{q}_i^2 \hat{p}_i^2 + \hat{q}_i \hat{p}_i \hat{q}_i \hat{p}_i + \hat{q}_i \hat{p}_i \hat{q}_i \hat{p}_i) \right].$$

(71)

Thus the average of $\hat{N}^2$ can be readily evaluated as

$$\langle \hat{N}^2 \rangle = \int d^{2n} \xi \, f(q,p) W(\xi).$$

(72)

Using the above equation and Eq. (29), variance of number operator can be written in an elegant form as [13, 30, 31]

$$\text{Var}(\hat{N}) = \langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2$$

$$= \frac{1}{2} \text{Tr} \left[ \left( V - \frac{1}{2} \mathbf{1}_{2n} \right) \left( V + \frac{1}{2} \mathbf{1}_{2n} \right) \right] + \mathbf{d}^T V \mathbf{d}.$$  

(73)

We first explore the mean and variance of photon number of a single mode system to get some insights. We consider a single mode Gaussian state with mean $\mathbf{d} = (u, u)^T$ and covariance matrix

$$V(\beta) = \frac{1}{2} (2N + 1) R(\beta) S(2s) R(\beta)^T,$$  

(74)

where $N$ is the thermal noise parameter, $s$ is the squeezing and $\beta$ is the phase shift angle. The mean and variance
of the number operator for the above state reads

\[ \langle \hat{N} \rangle = N \cosh 2s + \sinh^2 s + u^2, \]
\[ \text{Var}(\hat{N}) = \left( N + \frac{1}{2} \right)^2 \cosh 4s - \frac{1}{4} \]
\[ + 2u^2 \left( N + \frac{1}{2} \right) (\cosh 2s + \sin 2\beta \sinh 2s). \quad (75) \]

Here both mean and variance depend on displacement parameter \( u \) and squeezing parameter \( s \) of the state. However, the mean photon number is independent of the phase shift angle \( \beta \) while variance of photon number depends on \( \beta \). The variance of displaced number operator is given by

\[ \text{Var}(\hat{D}(r)\hat{N}\hat{D}(r)) = (d + r)^T V (d + r) \]
\[ + \frac{1}{2} \text{Tr} \left[ \left( V - \frac{1}{2} I_{2n} \right) \left( V + \frac{1}{2} I_{2n} \right) \right]. \quad (76) \]

Using this expression we first compare the variance of number operator under the action of \( P_1(r, \phi) \) gate (Eqn. (40)) for different values of the parameters \( r \) and \( \phi \). In Fig. 6(a), we plot the variance of different \( P_1(r, \phi) \) gate transformed number operators as a function of displacement parameter \( u \) for single-mode squeezed coherent thermal state (74). We can see that the variance of different \( P_1(r, \phi) \) gate transformed number operators increase with displacement parameter \( u \). While the variance of \( U^T(P)\hat{N}U(P) \) with \( e^r = \sqrt{2}, \phi = \pi/4 \) is always lower than the variance of \( \hat{N} \) and variance of \( U^T(P)\hat{N}U(P) \) with \( e^r = \sqrt{3}, \phi = 0 \) is always higher than the variance of \( \hat{N} \), variance of \( U^T(P)\hat{N}U(P) \) with \( e^r = \sqrt{2}, \phi = 0 \) crosses the variance of \( \hat{N} \) at a certain value of displacement parameter \( u \). We show the variance of the photon number as a function of squeezing parameter \( s \) in Fig. 6(b). As we can see, variance of different \( P_1(r, \phi) \) gate transformed number operators show a similar dependence on squeezing parameter \( s \) as that of displacement parameter \( u \).

Now to compare the variance of photon number under the action of two mode gates \( Q_{ij}(r, \phi) \) (Eqn. (51)), we consider a two mode Gaussian state with mean \( \alpha = (u, u, u, u)^T \) and covariance matrix \( V \)

\[ V = B_{12} \left( \frac{\pi}{4} \right) V(\beta_1) \oplus V(\beta_2) \]
\[ \cdot B_{12} \left( \frac{\pi}{4} \right)^T, \quad (78) \]

where \( V(\beta) \) is defined in Eq. (74). We use Eq. (77) to compute the variance of \( Q_{ij}(r, \phi) \) gate transformed number operator corresponding to the above state.

In Fig. 6(c), we plot the variance of different \( Q_{ij}(r, \phi) \) gate transformed number operators as a function of displacement parameter \( u \) for two-mode squeezed coherent thermal state (78). We can see that variance of different \( Q_{ij}(r, \phi) \) gate transformed number operators increase with displacement parameter \( u \). While the variance of \( U^T(Q)\hat{N}U(Q) \) with \( e^r = \sqrt{2}, \phi = 0 \) and \( e^r = \sqrt{3}, \phi = 0 \) always remain higher than the variance of \( \hat{N} \), variance of \( U^T(Q)\hat{N}U(Q) \) with \( e^r = \sqrt{2}, \phi = \pi/2 \) and \( e^r = \sqrt{3}, \phi = \pi/2 \) crosses the variance of \( \hat{N} \) at a certain value of the displacement parameter \( u \). Variance of different \( Q_{ij}(r, \phi) \) gate transformed number operators as a function of squeezing parameter \( s \) is shown in Fig. 6(d). As we can see, the squeezing dependence of different variances exhibits a similar trend as that of dependence on displacement.
Thus the variance of $\hat{q}_i$ becomes

$$\text{Var}(\hat{q}_i) = \text{Var}(\hat{\mathcal{D}}_i(1,0)\hat{N}\hat{\mathcal{D}}_i(1,0)) + \text{Var}(\hat{N}),$$

and parameter $\beta = \pi/3$. (c) Variance of photon number as a function of displacement $u$ for two-mode squeezed thermal state (78). (d) Variance of photon number as a function of squeezing $s$ for two-mode squeezed thermal state (78).

For both panel (c) and (d), various curves correspond to $\text{Var}(\hat{U}(Q)\hat{N}\hat{U}(Q))$ with $\epsilon' = \sqrt{2}$, $\phi = 0$ (Red dashed), $\epsilon' = \sqrt{3}$, $\phi = 0$ (Orange dotted), $\epsilon' = \sqrt{2}$, $\phi = \pi/4$ (Purple dot dashed), while Black solid curve represents $\text{Var}(\hat{N})$, and parameter $\beta = \pi/3$.

Thus the variance of $\hat{q}_i^2$ can be expressed as

$$\text{Var}(\hat{q}_i^2) = 6 \left[ 2 \text{Var}(\hat{U}(P_1)^\dagger \hat{N}\hat{U}(P_1)) - 2 \text{Var}(\hat{U}(P_1)^{1/2}) \hat{N}\hat{U}(P_1)^{1/2} - \hat{N} \right].$$

We see from the above analysis that the variance of $\hat{q}_i^2$ also depends on both displacement $u$ and squeezing $s$. In this case too, a proper study of the optimization of $P_1(r, \phi)$ gate parameters for the minimization of $\text{Var}(\hat{q}_i^2)$ is needed. Such an analysis will be useful for the best estimation of Gaussian state parameters. Similarly, various intra-mode correlation terms such as $\text{Var}(\hat{p}_j^2)$ and $\text{Var}(\hat{q}_i\hat{p}_j)$, as well as various inter-mode correlation terms such as $\text{Var}(\hat{q}_i\hat{q}_j)$ and $\text{Var}(\hat{q}_i\hat{p}_j)$, can be expressed in terms of the variances of different transformed number operators.

VI. CONCLUDING REMARKS

In this work we presented a Gaussian state tomography and Gaussian process tomography scheme based on photon number measurements. While the work builds upon the proposal given in [13], the current proposal offers an optimal solution to the problem, with smaller number of optical elements which renders the scheme more accessible to experimentalists. After describing our optimal scheme for Gaussian state tomography, we use it for estimation of a Gaussian channel in an optimal way, where a total number of $6n^2 + n$ measurements are required to determine $6n^2 + n$ parameters specifying a Gaussian channel. This in some sense completes the problem of finding an optimal solution of the Gaussian channel characterization posed in [13].

For the Gaussian channel tomography, we provide two methods depending on whether we use coherent state probes with same or different mean values. For the coherent state probes with the same mean, we used the estimation of $\langle \hat{N} \rangle$ available from the full state tomography of the first coherent state probe in the partial tomography of the remaining coherent state probes to make the scheme optimal. In the case when coherent state probes have different mean values, we exploited the fact that $\text{Tr}(V)$ is the same for all the coherent state probes. Full state tomography of the first coherent state probe yields an estimation of $\text{Tr}(V)$ which can be used to estimate $\langle \hat{N} \rangle$ for each of the remaining coherent state probes, thus making the scheme optimal again.

The analysis of variance in photon number measurements of the original and transformed states shows that

\begin{align*}
\nonumber 
(80)
\end{align*}
the variance increases with the mean of the state and with the squeezing parameter. Thus, this scheme is well suited for state with small mean values or small displacements and small values of squeezing. Extending the scheme for states with large mean value but better estimation performance is under consideration and will be reported elsewhere. While we have chosen certain specific values of gate parameters (see Eq. (46)), to extract information about the parameters of the state, the effect of different values of gate parameters on the quality of estimates and determination of optimal parameters that maximize the performance of the scheme needs further investigation. The optimality of the procedure may have a relationship with mutually unbiased basis for the CV systems. Further analysis of this aspect will require us to go beyond Gaussian states and will be taken up elsewhere.

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