A LOCALLY QUASI-CONVEX ABELIAN GROUP WITHOUT
MACKEY TOPOLOGY

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Abstract. We give the first example of a locally quasi-convex (even countable reflexive and $k_ω$) abelian group $G$ which does not admit the strongest compatible locally quasi-convex group topology. Our group $G$ is the Graev free abelian group $A_G(s)$ over a convergent sequence $s$.

1. Introduction

Let $(E, τ)$ be a locally convex space. A locally convex vector topology $ν$ on $E$ is called compatible with $τ$ if the spaces $(E, τ)$ and $(E, ν)$ have the same topological dual space. The famous Mackey–Arens theorem states the following

Theorem 1.1 (Mackey–Arens). Let $(E, τ)$ be a locally convex space. Then $(E, τ)$ is a pre-Mackey locally convex space in the sense that there is the finest locally convex vector space topology $µ$ on $E$ compatible with $τ$. Moreover, the topology $µ$ is the topology of uniform convergence on absolutely convex weakly* compact subsets of the topological dual space $E'$ of $E$.

The topology $µ$ is called the Mackey topology on $E$ associated with $τ$, and if $µ = τ$, the space $E$ is called a Mackey space.

For an abelian topological group $(G, τ)$ we denote by $\hat{G}$ the group of all continuous characters of $(G, τ)$. Two topologies $µ$ and $ν$ on an abelian group $G$ are said to be compatible if $(G, µ) = (G, ν)$. Being motivated by the Mackey–Arens Theorem the following notion was introduced and studied in [3] (for all relevant definitions see the next section):

Definition 1.2 ([3]). A locally quasi-convex abelian group $(G, µ)$ is called a Mackey group if for every locally quasi-convex group topology $ν$ on $G$ compatible with $τ$ it follows that $ν ≤ µ$. In this case the topology $µ$ is called a Mackey topology on $G$. A locally quasi-convex abelian group $(G, τ)$ is called a pre-Mackey group and $τ$ is called a pre-Mackey topology on $G$ if there is a Mackey topology $µ$ on $G$ associated with $τ$.

Not every Mackey locally convex space is a Mackey group. Indeed, answering a question posed in [3], we proved in [5] that the metrizable locally convex space $(ℝ^{(N)}, p_0)$ of all finite sequences with the topology $p_0$ induced from the product space $ℝ^N$ is not a Mackey group. In [7] we show that the space $C_p(X)$, which is a Mackey space for every Tychonoff space $X$, is a Mackey group if and only it is barrelled.

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A weaker notion than to be a Mackey group was introduced in [6]. Let \((G, \tau)\) be a locally quasi-convex abelian group. A locally quasi-convex group topology \(\mu\) on \(G\) is called quasi-Mackey if \(\mu\) is compatible with \(\tau\) and there is no locally quasi-convex group topology \(\nu\) on \(G\) compatible with \(\tau\) such that \(\mu < \nu\). The group \((G, \tau)\) is quasi-Mackey if \(\tau\) is a quasi-Mackey topology. Proposition 2.8 of [6] implies that every locally quasi-convex abelian group has quasi-Mackey topologies.

The Mackey–Arens theorem suggests the following general question posed in [3]: Does every locally quasi-convex abelian group is a pre-Mackey group? In the main result of the paper, Theorem 1.3, we answer this question in the negative.

Let \(s = \{0\} \cup \{1/n : n \in \mathbb{N}\}\) be the convergent sequence endowed with the topology induced from \(\mathbb{R}\). Denote by \(A_G(s)\) the (Graev) free abelian group over \(s\). Note that the group \(A_G(s)\) is a countable reflexive and \(k_\omega\)-group, see [5] and [8] respectively. In Question 4.4 of [6] we ask: Is it true that \(A_G(s)\) is a Mackey group? Below we answer this question negatively in a stronger form.

**Theorem 1.3.** The group \(A_G(s)\) is neither a pre-Mackey group nor a quasi-Mackey group.

This result gives the first example of a locally quasi-convex group which is not pre-Mackey additionally showing a big difference between the case of locally quasi-convex groups and the case of locally convex spaces.

2. **Proof of Theorem 1.3**

Set \(\mathbb{N} := \{1, 2, \ldots\}\). Denote by \(S\) the unit circle group and set \(S_+ := \{z \in S : \text{Re}(z) \geq 0\}\). Let \(G\) be an abelian topological group. If \(\chi \in \hat{G}\), it is considered as a homomorphism from \(G\) into \(S\). A subset \(A\) of \(G\) is called quasi-convex if for every \(g \in G \setminus A\) there exists \(\chi \in \hat{G}\) such that \(\chi(x) \notin S_+\) and \(\chi(A) \subseteq S_+\). An abelian topological group \(G\) is called locally quasi-convex if it admits a neighborhood base at the neutral element 0 consisting of quasi-convex sets. It is well known that the class of locally quasi-convex abelian groups is closed under taking products and subgroups. The dual group \(\hat{G}\) of \(G\) endowed with the compact-open topology is denoted by \(G^\wedge\). The homomorphism \(\alpha_G : G \to G^\wedge, g \mapsto (\chi \mapsto \chi(g))\), is called the canonical homomorphism. If \(\alpha_G\) is a topological isomorphism the group \(G\) is called reflexive. Any reflexive group is locally quasi-convex.

Let \(X\) be a Tychonoff space with a distinguished point \(e\). Following [8], an abelian topological group \(A_G(X)\) is called the Graev free abelian topological group over \(X\) if \(A_G(X)\) satisfies the following conditions:

(i) \(X\) is a subspace of \(A_G(X)\);

(ii) any continuous map \(f\) from \(X\) into any abelian topological group \(H\), sending \(e\) to the identity of \(H\), extends uniquely to a continuous homomorphism \(\tilde{f} : A_G(X) \to H\).

For every Tychonoff space \(X\), the Graev free abelian topological group \(A_G(X)\) exists, is unique up to isomorphism of abelian topological groups, and is independent of the choice of \(e\) in \(X\), see [8]. Further, \(A_G(X)\) is algebraically the free abelian group on \(X \setminus \{e\}\).

We denote by \(\tau\) the topology of the group \(A_G(s)\). For every \(n \in \mathbb{N}\), set \(e_n := (0, \ldots, 0, 1, 0, \ldots) \in \mathbb{Z}^{(\mathbb{N})}\),
where 1 is placed in position \( n \) and \( \mathbb{Z}^{(N)} \) is the direct sum \( \bigoplus_{n \in \mathbb{N}} \mathbb{Z} \). Now the map \( i(1/n) := e_n, n \in \mathbb{N} \), defines an algebraic isomorphism of \( A_G(s) \) onto \( \mathbb{Z}^{(N)} \). So we can identify algebraically \( A_G(s) \) and \( \mathbb{Z}^{(N)} \).

Let \( g_n \) be a sequence in \( A_G(s) \) of the form
\[
g_n = (0, \ldots, 0, r^n_{i_n}, r^n_{i_n+1}, r^n_{i_n+2}, \ldots),
\]
where \( i_n \to \infty \) and there is a \( C > 0 \) such that \( \sum_j |r^n_j| \leq C \) for every \( n \in \mathbb{N} \). Since \( e_n \to 0 \) in \( \tau \) we obtain
\[
g_n \to 0 \quad \text{in } \tau.
\]

The following group plays an essential role in the proof of Theorem 2.2. Set
\[
c_0(S) := \{(z_n) \in S^\mathbb{N} : z_n \to 1\},
\]
and denote by \( \mathfrak{S}_0(S) \) the group \( c_0(S) \) endowed with the metric \( d((z_n^1), (z_n^2)) = \sup\{|z_n^1 - z_n^2|, n \in \mathbb{N}\} \). Then \( \mathfrak{S}_0(S) \) is a Polish group, and the sets of the form \( V^\mathbb{N} \cap c_0(S) \), where \( V \) is an open neighborhood at the identity \( 1 \) of \( S \), form a base at 1 in \( \mathfrak{S}_0(S) \). Actually \( \mathfrak{S}_0(S) \) is isomorphic to \( c_0/\mathbb{Z}^{(N)} \) (see [3]). In [5] we proved that the group \( \mathfrak{S}_0(S) \) is reflexive and \( \mathfrak{S}_0(S)/\mathfrak{S}_0(s) = A_G(s) \).

If \( g \) is an element of an abelian group \( G \), we denote by \( \langle g \rangle \) the subgroup of \( G \) generated by \( g \). We need the following lemma.

**Lemma 2.1.** Let \( z, w \in S \) and let \( z \) have infinite order. Let \( V \) be a neighborhood of 1 in \( S \). If \( w^l = 1 \) for every \( l \in \mathbb{N} \) such that \( z^l \in V \), then \( w = 1 \).

**Proof.** The main result of [2] applied to \( \langle z \rangle \) states the following: there exists a sequence \( A = \{a_n\}_{n \in \mathbb{N}} \) in \( \mathbb{N} \) such that if \( v \in S \), then
\[
\lim_n v^{a_n} = 1 \quad \text{if and only if } \quad v \in \langle z \rangle.
\]

Now suppose for a contradiction that \( w \neq 1 \). Since \( \langle z \rangle \) is dense in \( S \), there is an \( l \in \mathbb{N} \) such that \( z^l \in V \). So \( w \) has finite order, say \( q \). Observe that \( w \notin \langle z \rangle \). Then, by assumption, for every \( l \in \mathbb{N} \) such that \( z^l \in V \) we have \( w^l = 1 \), and hence there is a \( c(l) \in \mathbb{N} \) such that \( l = c(l) \cdot q \). Since \( \lim_n z^{a_n} = 1 \), there exists an \( N \in \mathbb{N} \) such that \( z^{a_n} \in V \) for every \( n > N \). So \( a_n = c(a_n) \cdot q \) for every \( n > N \). But in this case we trivially have \( \lim_n w^{a_n} = 1 \) which contradicts the choice of the sequence \( A \) since \( w \notin \langle z \rangle \). Thus \( w = 1 \). \( \square \)

In the proof of Theorem 2.2 we use the following result, see Theorem 2.7 of [6].

**Theorem 2.2 ([6]).** For a locally quasi-convex abelian group \((G, \tau)\) the following assertions are equivalent:

(i) the group \((G, \tau)\) is pre-Mackey;

(ii) \( \tau_1 \lor \tau_2 \) is compatible with \( \tau \) for every locally quasi-convex group topologies \( \tau_1 \) and \( \tau_2 \) on \( G \) compatible with \( \tau \).

Now we are ready to prove Theorem 2.2.

**Proof of Theorem 2.2**. First we construct a family
\[
\{T_z : z \in S \text{ has infinite order}\}
\]
of topologies on \( \mathbb{Z}^{(N)} \) compatible with the topology \( \tau \) of \( A_G(s) \). To this end, we use the idea described in Proposition 4.1 of [6].
Let \( z \in \mathcal{S} \) be of infinite order. For every \( i \in \mathbb{N} \), set
\[
\chi_i := (0, \ldots, 0, z, 0, \ldots) \in \mathfrak{F}_0(\mathcal{S}) = A_G(\mathcal{S})^\wedge,
\]
where \( z \) is placed in position \( i \). For every \( (n_k) \in A_G(\mathcal{S}) \), it is clear that \( \chi_i((n_k)) = 1 \) for all sufficiently large \( i \in \mathbb{N} \) (i.e., \( \chi_i \to 1 \) in the pointwise topology on \( \mathfrak{F}_0(\mathcal{S}) \)). So we can define the following algebraic monomorphism \( T_z : \mathbb{Z}^{(N)} \to A_G(\mathcal{S}) \times \mathfrak{F}_0(\mathcal{S}) \) by
\[
(2.2) \quad T_z((n_k)) := \left((n_k), (\chi_i((n_k)))\right) = \left((n_k), (z^{n_k})\right), \quad \forall (n_k) \in \mathbb{Z}^{(N)}.
\]
Denote by \( \mathcal{T}_z \) the locally quasi-convex topology on \( \mathbb{Z}^{(N)} \) induced from \( A_G(\mathcal{S}) \times \mathfrak{F}_0(\mathcal{S}) \).

**Claim 1.** The topology \( \mathcal{T}_z \) is compatible with \( \tau \). Indeed, set \( G := (\mathbb{Z}^{(N)}, \mathcal{T}_z) \). Since \( \mathcal{T}_z \) is weaker than the discrete topology \( \tau_d \) on \( \mathbb{Z}^{(N)} \), we obtain \( G^\wedge \subseteq (\mathbb{Z}^{(N)}, \tau_d)^\wedge = \mathbb{S}^N \).

Fix arbitrarily \( \chi = (y_n) \in G^\wedge \). To prove the claim we have to show that \( y_n \to 1 \).

Suppose for a contradiction that \( y_n \not\to 1 \). As \( \mathbb{S} \) is compact we can find a sequence \( 0 < m_1 < m_2 < \ldots \) of indices such that \( y_{m_i} \to w \neq 1 \) at \( i \to \infty \). Since \( \chi \) is \( \mathcal{T}_z \)-continuous, there exists a standard neighborhood \( W = T_z^{-1}(U \times \mathbb{V}^N) \) of zero in \( G \), where \( U \) is a \( \tau \)-neighborhood of zero in \( A_G(\mathcal{S}) \) and \( \mathbb{V}^N \) is a neighborhood of 1 in \( \mathcal{S} \), such that \( \chi(W) \subseteq \mathbb{S}^+ \). Observe that, by (2.2), \( (n_k) \in W \) if and only if
\[
(2.3) \quad (n_k) \in U \text{ and } z^{n_k} \in V \text{ for every } k \in \mathbb{N},
\]
and, the inclusion \( \chi(W) \subseteq \mathbb{S}^+ \) means that
\[
(2.4) \quad \chi((n_k)) = \prod_k y_{n_k}^k \in \mathbb{S}^+, \quad \text{for every } (n_k) \in W.
\]
We assume additionally that \( w \not\in \mathbb{V} \). Since \( \langle z \rangle \) is dense in \( \mathbb{S} \), choose arbitrarily an \( l \in \mathbb{N} \) such that \( z^l \in \mathcal{V} \). Fix arbitrarily a \( t \in \mathbb{N} \). Now, by (2.1), there is an \( N(t) \in \mathbb{N} \) such that every \( x_{it} := (n_k) \in \mathbb{Z}^{(N)} \) of the form
\[
(2.5) \quad x_{it} = (0, \ldots, 0, \overbrace{l, 0, \ldots, 0}^{m_{i+1}}, l, 0, \ldots, 0, \overbrace{l, 0, \ldots}^{m_{i+t}}),
\]
belongs to \( W \) for every \( i \geq N(t) \). For every \( x_{it} \in W \) of the form (2.5), (2.4) implies
\[
(2.6) \quad \chi(x_{it}) = \left(y_{m_{i+1}} \cdots y_{m_{i+t}} \right)^i \to w^t, \quad \text{at } i \to \infty.
\]
Now, if \( w^t \neq 1 \) for some \( l \in \mathbb{N} \) such that \( z^l \in \mathcal{V} \), then there exists a \( t \in \mathbb{N} \) such that \( w^t \not\in \mathbb{S}^+ \). Therefore, by (2.3), \( \chi(W) \not\subseteq \mathbb{S}^+ \), a contradiction. Assume that \( w^t = 1 \) for every \( l \in \mathbb{N} \) such that \( z^l \in \mathcal{V} \). Then Lemma (2.1) implies \( w = 1 \) that is impossible. So our assumption that \( y_n \not\to 1 \) is wrong. Therefore \( y_n \to 1 \) and \( \bar{G} = c_0(\mathcal{S}) \). Thus \( \mathcal{T}_z \) is compatible with \( \tau \).

**Claim 2.** For every element \( a \in \mathcal{S} \) of finite order, the topology \( \mathcal{T}_z \lor \mathcal{T}_{az} \) is not compatible with \( \tau \). Let \( r \) be the order of \( a \). Consider standard neighborhoods
\[
W_z = T_z^{-1}(U \times \mathbb{V}^N) \in \mathcal{T}_z \quad \text{and} \quad W_{az} = T_{az}^{-1}(U \times \mathbb{V}^N) \in \mathcal{T}_{az},
\]
where \( U \in \tau \) and a symmetric neighborhood \( V \) of 1 in \( \mathcal{S} \) is chosen such that \( V \cdot V \cap \langle a \rangle = \{1\} \). Then, by (2.3), we have
\[
W_z \cap W_{az} = \left\{(n_k) \in \mathbb{Z}^{(N)} : (n_k) \in U \text{ and } z^{n_k}, (az)^{n_k} \in V \text{ for every } k \in \mathbb{N}\right\}.
\]
In particular, \( a^{n_k} \in V \cdot V \), and hence \( a^{n_k} = 1 \) for every \( k \in \mathbb{N} \). Therefore, for every \( k \in \mathbb{N} \), there is an \( s_k \in \mathbb{N} \) such that \( n_k = s_k \cdot r \). Set \( \eta := (a, a, \ldots) \in \mathbb{S}^N \). Then
As $W_z \cap W_{az} = \{1\}$. As $W_z \cap W_{az} \in T_z \vee T_{az}$ it follows that $\eta$ is $T_z \vee T_{az}$-continuous.

Since $\eta \notin c_0(S)$ we obtain that $T_z \vee T_{az}$ is not compatible with $\tau$.

Claim 3. $\tau < T_z$, so $\tau$ is not quasi-Mackey. By (2.2), it is clear that $\tau \leq T_z$.

To show that $\tau \neq T_z$, suppose for a contradiction that $T_z = \tau$. Then, by Claim 1, $T_z \vee T_{az} = \tau \vee T_{az} = T_{az}$ is compatible with $\tau$. But this contradicts Claim 2.

Claim 4. The group $A_G(s)$ is not pre-Mackey. This immediately follows from Claim 2 and Theorem 2.2.

Remark 2.3. Just before submission of the preprint, Prof. Lydia Außenhofer informed the author that she had also solved the problem posed by me: namely if the group $A_G(s)$ is a Mackey group and proved Theorem 1.3; see [1]. It is worth mentioning that my proof totally differs from hers, being much simpler and shorter.

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