6-Cycle Double Covers of Cubic Graphs

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Abstract

A cycle double cover (CDC) of an undirected graph is a collection of the graph’s cycles such that every edge of the graph belongs to exactly two cycles. We describe a constructive method for generating all the cubic graphs that have a 6-CDC (a CDC in which every cycle has length 6). As an application of the method, we prove that all such graphs have a Hamiltonian cycle. A sense of direction is an edge labeling on graphs that follows a globally consistent scheme and is known to considerably reduce the complexity of several distributed problems. In [9], a particular instance of sense of direction, called a chordal sense of direction (CSD), is studied and the class of k-regular graphs that admit a CSD with exactly k labels (a minimal CSD) is analyzed. We now show that nearly all the cubic graphs in this class have a 6-CDC, the only exception being $K_4$.

Keywords: Cycle double covers, Cubic graphs, Chordal sense of direction, Circulant graphs, Hexagonal tilings, Fullerenes, Polyhexes.

1 Introduction

In this paper we consider connected undirected graphs having no multiple edges or self-loops. For terminology or notation not defined here we refer the reader to [3]. A cycle double cover (CDC) of a graph $G$ is a collection of cycles in $G$ such that every edge of $G$ belongs to exactly two of the cycles. It can be easily seen that a necessary condition for a graph to have a CDC is that the graph be 2-edge-connected. It has been conjectured that this condition is also sufficient [13] [12], but the conjecture has remained unsettled and constitutes one of the classic unsolved problems in graph theory.

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A \( k \)-cycle double cover (\( k \)-CDC), for \( k \geq 3 \), is a CDC whose every cycle has length \( k \). Previous results on \( k \)-CDC’s are the ones in \([1, 14, 15]\), all motivated by the relationship between \( k \)-CDC’s and embeddings on surfaces. In \([1]\), the 6-regular graphs that have a 3-CDC are studied and some results are shown to carry over, by duality, to the class of girth-6 cubic graphs that have a 6-CDC. This latter class is characterized in \([14]\), and in \([15]\) deciding whether a graph has a 3-CDC is proven NP-complete.

In this paper we introduce a constructive method for generating all the cubic graphs that have a 6-CDC, and prove in addition that all such graphs are Hamiltonian. For the particular case of girth-6 cubic graphs, these contributions provide both an alternative to the method of \([14]\) and an answer to the question raised in \([1]\) regarding the graphs’ Hamiltonicity. Graphs in this case can also be viewed as hexagonal tilings of the torus (the orientable surface of genus 1) or of the Klein bottle (the non-orientable surface of cross-cap number 2; cf. \([14]\)) and have many applications in chemistry, where they are also called toroidal and Klein-bottle fullerenes \([4]\) or polyhexes \([8]\).

Our initial motivation, though, has been the relationship between 6-CDC’s and the chordal sense of direction (CSD) \([7]\) of a graph. A sense of direction is an edge labeling on graphs that follows a globally consistent scheme and is known to considerably reduce the complexity of several distributed problems \([5]\).

In the particular case of a CSD, to be defined precisely in Section 6, we have in another study characterized the \( k \)-regular graphs that admit a CSD with exactly \( k \) labels \([9]\), also called a minimal CSD (MCSD). A further contribution of the present study is to demonstrate that, except for \( K_4 \), every cubic graph that has an MCSD also has a 6-CDC. Since in \([9]\) we also prove that the class of regular graphs that have an MCSD is equivalent to that of circulant graphs, this contribution also holds for cubic circulant graphs. We note that circulant graphs have great practical relevance due to their connectivity properties (small diameter, high symmetry, etc.), which render them excellent topologies for network interconnection, VLSI, and distributed systems \([2]\).

The following is how we organize the remainder of the paper. We start in Section 2 with preliminary results on 6-CDC’s and their cycles. Then we move in Section 3 to the introduction of our method to generate all cubic graphs that have a 6-CDC. In Section 4 we describe the method’s details by explaining how it is applied for each possible girth value. Our method never generates duplicates or misses a graph, as we explain in Section 5 along with a demonstration that all cubic graphs that have a 6-CDC are Hamiltonian. The relationship to MCSD’s is discussed in Section 6 and then we close in Section 7 with concluding remarks.

### 2 Preliminaries

We henceforth assume that \( G \) is a cubic graph on \( n \) vertices and \( m \) edges and that it has a 6-CDC. Clearly, \( m = 3n/2 \) and \( n \) is necessarily even. We say that
a cycle $C$ of such a 6-CDC *covers* a certain edge whenever that edge belongs to $C$. Let us initially establish some properties of the cycles of a 6-CDC of $G$.

**Lemma 1.** Every 6-CDC of $G$ has $n/2$ cycles.

*Proof.* Let $t$ be the number of cycles in a 6-CDC of $G$. As each edge belongs to two of the $t$ cycles, we have $6t = 2m$. And because $m = 3n/2$, it follows that $6t = 3n$, whence $t = n/2$. \hfill $\square$

**Lemma 2.** No two cycles of a 6-CDC of $G$ share a path containing more than one edge.

*Proof.* Suppose, contrary to our aim, that $uvw$ is a path of length 2 in $G$ belonging to two cycles, say $C$ and $C'$, of the 6-CDC. Let $z$ be the other vertex adjacent to $v$. The two cycles of the 6-CDC that cover the edge $vz$ must also cover either $uv$ or $vw$. However, both $uv$ and $vw$ are already covered by $C$ and $C'$, a contradiction. \hfill $\square$

**Lemma 3.** Every vertex of $G$ belongs to exactly three cycles of a 6-CDC of $G$.

*Proof.* By Lemma 2, each of the cycles going through a vertex $v$ must cover a distinct pair of edges incident to $v$. The result follows from recognizing that there exist $(\binom{3}{2}) = 3$ such pairs. \hfill $\square$

Now, for $C$ and $C'$ any two cycles of a 6-CDC of $G$, let $\mu(C, C')$ be the number of edges covered by both $C$ and $C'$, and $\sigma(C)$ the number of cycles of the 6-CDC, excluding $C$, that have at least one edge in common with $C$. We can bound these numbers as follows.

**Lemma 4.** If $C$ and $C'$ are cycles in a 6-CDC of $G$, then $0 \leq \mu(C, C') \leq 3$.

*Proof.* The lower bound is trivial and corresponds to $C$ and $C'$ being edge-disjoint. As for the upper bound, it follows directly from Lemma 2 since $\mu(C, C') > 3$ requires $C$ and $C'$ to share a path containing more than one edge. \hfill $\square$

**Lemma 5.** If $C$ is a cycle in a 6-CDC of $G$, then $2 \leq \sigma(C) \leq 6$.

*Proof.* The lower bound follows from Lemma 2 and corresponds to $C$ having all three edges in each of its two possible sets of noncontiguous edges in common with a same cycle of the 6-CDC. The upper bound is trivial and corresponds to the case in which $C$ shares each of its edges with a different cycle of the 6-CDC. \hfill $\square$

We can also characterize the graphs that effectively attain the upper bound of Lemma 4. As we see in Lemma 6 below, each such graph has a three-cycle 6-CDC whose cycles all attain the lower bound of Lemma 4 as well. We first review some definitions. A *chord* is an edge interconnecting two noncontiguous vertices of a cycle. Let $C$ be a cycle of even length. A *Möbius ladder* on $n$ vertices, denoted by $M_n$, is the graph obtained by adding chords between all
Figure 1: The only cubic graphs that attain both the upper bound of Lemma 4 and the lower bound of Lemma 5. Each is shown with a three-cycle 6-CDC highlighted (thin solid edges belong to one cycle, thick solid edges to another, dashed edges to both, and the external hexagon is the third cycle). The graph in part (a) is isomorphic to $M_6$ and the one in part (b) to $T_{6,2}$.

vertex pairs that are $n/2$ edges apart on $C$ ($M_6$ is shown in Figure 1(a)). For $l$ a divisor of $n$, an $l$-layer torus on $n$ vertices, denoted by $T_{n,l}$, is an $l \times n/l$ mesh in which maximally distant vertices on the same row or column are connected to each other ($T_{6,2}$ is shown in Figure 1(b)).

**Lemma 6.** If $C$ and $C'$ are cycles in a 6-CDC of $G$ such that $\mu(C, C') = 3$, then $G$ is isomorphic to either $M_6$ or $T_{6,2}$.

**Proof.** By Lemma 2 it suffices to identify a perfect matching on one of $C$ or $C'$, say $C$, as the set of edges shared by the two cycles. The remaining edges of $C'$ can only be arranged so that $G$ is either isomorphic to $M_6$ or to $T_{6,2}$, as illustrated in Figure 1.

In the remainder of the paper, for any graph $H$ we use $V(H)$ to denote its vertex set and $E(H)$ its edge set. Furthermore, we call a cycle fragment any subgraph of a 6-CDC’s cycle, and a cycle configuration any collection of cycles or cycle fragments of a 6-CDC.

### 3 A recursive method

Having established some restrictions on the cycles that form a 6-CDC, we are now in position to describe a constructive method for generating cubic graphs that have a 6-CDC. Let $H$ be a proper subgraph of $G$ whose vertices have degree 1 or 3. When we consider the intersection of the cycles of a 6-CDC of $G$ with $H$, we obtain a cycle configuration in $H$ such as the one illustrated in Figure 2, where an edge is labeled $i, j$ to indicate that it belongs to the cycles $C_i$ and $C_j$ of the 6-CDC.

We say that a vertex is deficient in a certain graph if its degree is less than 3 in that graph. The deficiency of graph $H$ is given by $3|V(H)| - \sum_{v \in V(H)} d_H(v)$, where $d_H(v)$ is the degree of $v$ in $H$. Note that the deficient vertices in $H$ are exactly the ones that are also deficient in $G - E(H)$, where for $E \subset E(G)$ we
use $G - E$ to denote the graph obtained from $G$ by removing the edges in $E$ and the vertices that become isolated after the edge removal. For $G$ and $H$ as in Figure 2, $G - E(H)$ is the external triangle in part (a) of the figure.

Now suppose that there exists another graph $H'$ that can replace $H$ in $G$ in such a way that the resulting graph, call it $G'$, is cubic and has a 6-CDC with the same cycle configuration on the edges of $G' - E(H')$ as on the edges of $G - E(H)$. In other words, suppose that $H'$ leads to a cubic $G' = H' \cup [G - E(H)]$ such that the cycle configuration in $G - E(H)$ remains unchanged from the 6-CDC of $G$ to that of $G'$. If such is the case, then $H'$ must have the same deficiency as $H$ (so the resulting $G'$ is cubic) and also a cycle configuration that completes the one in $G - E(H)$ as needed to yield the 6-CDC of $G'$. For $G$ and $H$ as in Figure 2, $H'$ and $G'$ are as illustrated in Figures 3(a) and 3(b), respectively.

For the latter condition to be satisfied, the following two properties must hold. First, if $v_1, \ldots, v_y$ are the deficient vertices of $H$ (all of degree 1, by
definition), then there has to exist a partition \(\{U_1, \ldots, U_y\}\) of the deficient vertices of \(H'\) such that \(v_x\) and \(U_x\) have the same deficiency\(^1\) \(x = 1, \ldots, y\). Clearly, \(|U_x| \leq 2\) necessarily, so \(U_x\) has either one degree-1 vertex or two degree-2 vertices. By Lemma 3, each degree-1 vertex in \(H\) or \(H'\), or degree-2 vertex in \(H'\), has exactly two cycles of the 6-CDC of \(G\) going through it along edges of both \(H\) and \(G - E(H)\), or of \(H'\) and \(G' - E(H')\), as the case may be. For \(v_x\) and \(U_x\), \(x = 1, \ldots, y\), let \(C_i\) and \(C_j\) be the two cycles in \(H\), \(C_i'\) and \(C_j'\) the two cycles in \(H'\). The second property is that the fragment of \(C_i\) in \(H\) and the fragment of \(C_i'\) in \(H'\) both have the same length, and similarly for \(C_j\) and \(C_j'\).

In the case of Figures 2 and 3, no degree-2 vertices exist in \(H'\) and the above holds with \(C_i = C_i' = C_1\) and \(C_j = C_j' = C_2\), for example. The fragments of \(C_1\) and \(C_2\) in \(H\) are dabcf and dace, respectively, while in \(H'\) they are dglkf and dghie.

We say that cycle configurations such as the ones of \(H\) and \(H'\) are equivalent to each other. When it is the case, in addition, that \(H'\) has at least one subgraph that is isomorphic to \(H\) and all such subgraphs have cycle configurations that are equivalent to that of \(H\) (hence to that of \(H'\) also), then we say that \(H'\) is self-similar with respect to \(H\). This is certainly the case of the \(H\) and \(H'\) of Figures 2(b) and 3(a), since the triangle of Figure 3(a), when augmented by vertices \(h, j, l\) and the edges that lead to them from the triangle, is isomorphic to the graph in Figure 2(b) with equivalent cycle configuration.

Self-similarity is a property of positive-deficiency graphs and constitutes the core of our method. Before proceeding to a description of the method, we let \(g(G)\) denote the girth of \(G\). In our present case of graphs that have a 6-CDC, \(3 \leq g(G) \leq 6\) necessarily, and we use the value of \(g(G)\) to divide our approach into cases, as presented in Section 4.

For a fixed value of \(g\) in \(\{3, \ldots, 6\}\), let \(S_g\) be the deficiency-2\(g\) graph on \(2g\) vertices that comprises a length-\(g\) cycle and \(g\) additional vertices, each of them connected to a distinct vertex of the cycle. For \(g = 3\), \(S_3\) is the \(H\) of Figure 2(b). In general, it is easy to see that every girth-\(g\) cubic graph having a 6-CDC has a subgraph isomorphic to \(S_g\) with a cycle configuration that is consistent with the 6-CDC, even though the \(g\) off-cycle vertices of this subgraph are not always all distinct\(^2\). For a fixed cycle configuration of \(S_g\), let also \(I_g\) be a minimal girth-\(g\) proper supergraph of \(S_g\) which, along with a cycle configuration of its own, is self-similar with respect to \(S_g\)\(^3\). In the \(g = 3\) example, \(I_g\) is the \(H'\) of Figure 3(a).

Now, a very important observation is that, for \(g = 6\), it may be impossible for \(I_g\) to exist as defined. The reason is that the cycle configuration of \(S_g\) does not necessarily include a complete cycle of the 6-CDC, while it may

\(^1\)We extend the definition of a graph’s deficiency to that of a vertex or vertex set in the obvious way.

\(^2\)In fact, vertex distinctness holds for all but one single case, specifically one of the cycle configurations of \(S_4\), as we discuss in Section 4.2.

\(^3\)Notwithstanding the formal generality of this definition, what happens is that, as we show in Sections 4.1 through 4.4 for every valid cycle configuration of \(S_g\) there exists only one \(I_g\) instance.
happen that every girth-6 proper supergraph of $S_6$ whose cycle configuration is equivalent to that of $S_6$, is also a supergraph of an isomorph of $S_6$ whose cycle configuration does include a complete 6-CDC cycle. We then see that the definition of self-similarity must be modified in the girth-6 case when the cycle configuration of $S_6$ does not contain a complete 6-CDC cycle. The modification is that not all subgraphs of $I_6$ that are isomorphic to $S_6$ are required to have cycle configurations equivalent to that of $S_6$, but rather only those whose cycle configurations do not contain a complete 6-CDC cycle. That the girth-6 case should require such an exceptional treatment is not really a surprise, since we are throughout dealing with graphs that have a 6-CDC, and thence it is only natural that 6-cycles that are in the 6-CDC be distinguished from those that are not.

Given the notion of self-similarity, the definitions of $S_g$ and $I_g$ imply that $I_g$ can substitute indefinitely for any isomorph of $S_g$ that has a cycle configuration equivalent to that of $I_g$, thus generating an infinite sequence of deficiency-2g, girth-$g$ graphs whose first graph is $S_g$ itself. For $g < 6$, such an isomorph is any of the $S_g$-isomorphs that the current graph has as subgraphs; for $g = 6$, isomorphs whose cycle configuration includes a complete 6-CDC cycle are excluded if the cycle configuration of $S_g$ does not itself contain a complete 6-CDC cycle. Turning the resulting graphs into girth-$g$ cubic graphs that have a 6-CDC requires that we define yet another graph based on $S_g$. This graph is denoted by $B_g$ and its definition, too, depends on what happens in the $g = 6$ case.

$B_g$ is in all cases defined to be a girth-$g$ supergraph of $S_g$ that has a 6-CDC. If either $g < 6$ or else $g = 6$ but the cycle configuration of $S_g$ does not include a complete 6-CDC cycle, then $B_g$ is furthermore of one of two types:

(i) $B_g$ is not a supergraph of $I_g$.

(ii) $B_g$ is a supergraph of $I_g$ such that substituting $S_g$ for $I_g$ causes the girth of $B_g$ to be reduced.

The remaining case is that of $g = 6$ when the cycle configuration of $S_g$ does include a complete cycle of the 6-CDC. In this case, $B_g$ has the following property, in addition to being a girth-$g$ supergraph of $S_g$ that has a 6-CDC:

(iii) $B_g$ is a supergraph of $I_g$ and all its 6-cycles are cycles of the 6-CDC. In addition, it is such that substituting $S_g$ for $I_g$ causes the appearance of 6-cycles that are not in the 6-CDC.

In any of cases (i)–(iii), the 6-CDC of $B_g$ is assumed to coincide with the cycle configuration of $S_g$ or $I_g$, depending respectively on whether $B_g$ is a supergraph of $S_g$ only or of $I_g$ as well. It is also curious to note that, if $B_g$ is of type (iii), then in $I_g$ it automatically holds that every 6-cycle is a cycle of the 6-CDC; but this already follows from the very definition of $I_g$, since in this case $S_g$ itself contains a complete 6-CDC cycle in its cycle configuration.

The reason for making a distinction between these three types is immaterial at this point and will only become clear in Section 4.4 in which we handle
the girth-6 case, and in Section 5 when we argue for the completeness of our method. For the girth-3 example we have been using as illustration, notice that the $G$ of Figure 2(a) is a type-(i) instance of $B_3$.

One crucial property emerging from the definitions of $S_g$, $I_g$, and $B_g$ is that, except for type-(i) instances of $B_g$, every girth-$g$ cubic graph that has a 6-CDC also has a subgraph isomorphic to $I_g$ with a cycle configuration that renders it self-similar with respect to $S_g$. So not only is the indefinite substitutability of $I_g$ for $S_g$ true, but it can be used to generate all girth-$g$ cubic graphs that have a 6-CDC, as follows. For each possible cycle configuration of $S_g$, we identify $I_g$ and all pertinent $B_g$ instances. By starting at each such instance and substituting $I_g$ for $S_g$ indefinitely, ever larger girth-$g$ cubic graphs are generated having a 6-CDC.

4 Applying the method

For each pertinent girth value $g$, in this section we start with $S_g$ and identify all its possible cycle configurations. For each of these cycle configurations, we then expand $S_g$ (along with its cycle configuration, by adding vertices and edges) without disrupting the 6-CDC nature of its cycle configuration or altering the girth. We do this until $I_g$ and all instances of $B_g$ are obtained.

While expanding $S_g$ we first attempt to generate type-(i) instances of $B_g$, that is, those that are not supergraphs of $I_g$. Then we proceed to generating $I_g$ itself and from there we move to expanding $I_g$ towards obtaining the instances of $B_g$ that are supergraphs of $I_g$, that is, type-(ii) or (iii) instances. It is important to realize that, since $S_g$ and $I_g$ have cycle configurations that are equivalent, carrying the expansion beyond $I_g$ need not attempt the same expansion steps that generated type-(i) $B_g$ instances: doing this would only lead to graphs that already belong to the sequence of graphs generated by substituting $I_g$ for $S_g$ recursively from a type-(i) $B_g$ instance onward. What must be attempted, rather, are expansion steps that failed previously but may now succeed (like those that somehow disrupt the girth or the 6-CDC when attempted on $S_g$).

4.1 The girth-3 case

We start with the graph of Figure 2(b) as $S_3$ (that is, the core cycle of $S_3$ is $abca$). It is easy to see that the cycle configuration given in Figure 2(b) is the only one that does not violate the restrictions discussed in Section 2. Furthermore, note that the vertices $d$, $e$, and $f$ must all remain distinct as we expand $S_3$, otherwise either the resulting graph would be isomorphic to $K_4$ (which is too small to have a 6-CDC) or the resulting cycle configuration would be inconsistent with the requirements of a 6-CDC.

Given the unique cycle configuration for $S_3$ in Figure 2(b), we proceed with the expansion. This is done by completing the cycles $C_1$, $C_2$, and $C_3$. We have two ways of completing $C_1$: either using an existing vertex ($e$) or including a new one. While the former option leads unavoidably to $B_3$ and its cycle
configuration shown in Figure 2(a) when applied to all three cycles, the latter results, after a suitable renaming of vertices and cycles, and also unavoidably, in the $I_3$ of Figure 3(a) and its cycle configuration. As noted in Section 3, $B_3$ is of type (i); also, for the reasons given above in the introduction to Section 4, seeking type-(ii) instances of $B_3$ any further is in this case meaningless.

Notice that we can now replace $S_3$ by $I_3$ in $B_3$, thus obtaining a larger cubic graph of girth 3 (the one in Figure 3(b)) that has a 6-CDC. This substitution process can proceed recursively, always generating cubic graphs of girth 3 with a 6-CDC. The graphs resulting from the second and third iterations are shown in Figure 4.

### 4.2 The girth-4 case

We start by analyzing all the possible cycle configurations of $S_4$. In Figures 5(a)–(c), the cycle configurations of $S_4$ that infringe neither Lemma 2 nor Lemma 3, and also do not lead to the existence of a cycle with length smaller than 6 in the 6-CDC, are presented. Note that vertices $e$, $f$, $g$, and $h$ are all distinct in these graphs. The cases in which these vertices may coincide will be treated later.

For each cycle configuration shown in Figure 5, we must identify $I_4$ and $B_4$. We denote by $S_{4a}$ the graph with the cycle configuration of Figure 5(a), and likewise $I_{4a}$ and $B_{4a}$ refer to the expansions of $S_{4a}$. We proceed similarly in the cases of Figures 5(b) and 5(c).

The only possibilities for $I_4$ are the $I_{4a}$, $I_{4b}$, and $I_{4c}$ of Figure 6. As for $B_4$, the only possibilities that Lemmas 2 and 3 allow are the $B_{4a}$, $B_{4b}$, $B_{4b'}$, and $B_{4c}$ of Figure 7. Notice that each of $B_{4a}$, $B_{4b}$, and $B_{4c}$ is isomorphic to $T_{8,2}$, and that $B_{4b'}$ is isomorphic to $M_{8}$. Also, they are all type-(i) instances of $B_4$.

It is illustrative to notice also that the cycle configurations of $S_{4a}$ and $I_{4a}$ are in fact equivalent to each other, and also that every subgraph of $I_{4a}$ that

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Figure 4: The graphs that result from the second (a) and third (b) substitutions of $I_3$ for $S_3$. 

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4The expansion of $S_{4b}$ into a $B_4$ instance has two possible outcomes, which we denote by $B_{4b}$ and $B_{4b'}$. 

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is isomorphic to $S_{4a}$ has a cycle configuration that is equivalent to that of $S_{4a}$. That is, $I_{4a}$ is indeed self-similar with respect to $S_{4a}$ and does as such allow for recursive substitutions of $I_{4a}$ for $S_{4a}$ starting at $B_{4a}$. Except for the initial $B_{4a}$, since it is of type (i), all $S_{4a}$-isomorphic subgraphs of the resulting graphs have cycle configurations equivalent to that of $S_{4a}$.<sup>5</sup> The cases of $S_{4b}$ and $I_{4b}$ with $B_{4b}$ (or $B_{4b'}$) and of $S_{4c}$ and $I_{4c}$ with $B_{4c}$ are entirely analogous.

Now let us consider the cases in which vertices $e$, $f$, $g$, and $h$ are not necessarily distinct. It is easy to see that the only way for this to happen without altering the girth is to let $e = g$ or $f = h$. If either $e = g$ or $f = h$, then clearly the cycle configurations of Figures 5(a) and 5(c) acquire a cycle of length 4, which is inconsistent with the nature of a 6-CDC, while in the cycle configuration of Figure 5(b) either vertex $e$ or vertex $f$ becomes part of four distinct cycles, which infringes Lemma 3.

Letting both $e = g$ and $f = h$, similarly, violates the 6-CDC in the cases of Figures 5(a) and 5(c). However, the cycle configuration of Figure 5(b) remains valid, and by simply adding edge $ef$ and letting $C_2 = C_4$ we obtain $M_6$ with a consistent 6-CDC, as in Figure 5. It is interesting to note that the first replacement of $S_{4b}$ by $I_{4b}$ in $M_6$ generates $M_8$, which is isomorphic to $B_{4b'}$, so we may actually let $B_{4b'}$ be $M_6$ instead (and thus avoid creating another type-(i) instance of $B_4$).

### 4.3 The girth-5 case

In Figures 9(a)–(c), the cycle configurations of $S_5$ that are consistent with Lemmas 2 and 3 and do not disrupt the nature of the 6-CDC are depicted. Notice that vertices $f$, $g$, $h$, $i$, and $j$ must necessarily be distinct in order for the girth not to fall below 5. As in the girth-4 case, these graphs along with their cycle configurations are denoted by $S_{5a}$, $S_{5b}$, and $S_{5c}$, respectively.

Now, as we try to expand $S_{5a}$, we invariably generate the $I_{5a}$ of Figure 10 before we get to a $B_5$ instance, and do so without ever turning down an edge

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<sup>5</sup>In Section 5 we use this property to argue for the uniqueness of each graph generated in the process.
Figure 6: $I_{4a}$ (a), $I_{4b}$ (b), and $I_{4c}$ (c).
Figure 7: $B_{4a}$ (a), $B_{4b}$ (b), $B_{4b'}$ (c), and $B_{4c}$ (d).
addition exclusively on account that the graph’s girth would be thus reduced. One consequence of this is that any $B_5$ instance we may come to generate as we proceed with the expansion will have $I_{5a}$ as a subgraph and therefore not be of type (i). Furthermore, as we consider that the cycle configurations of $S_{5a}$ and $I_{5a}$ are equivalent to each other, we realize that expanding beyond $I_{5a}$ is almost completely constrained to repeating the same steps that initially led from $S_{5a}$ to $I_{5a}$. The only exception is that now we may be precluded from adding a certain edge solely because such an addition would reduce the graph’s girth.\footnote{The fact that adding an edge $uv$ to $I_g$ creates a cycle with length smaller than $g$ does not necessarily imply that an edge $u'v'$ in $S_g$ will form a cycle with length smaller than $g$ as well, where $u'$ and $v'$ in $I_g$ correspond to $u$ and $v$ in $S_g$, respectively.}

But since nothing of this sort happens in the expansion from $S_{5a}$ to $I_{5a}$, we see in any event that substituting $S_{5a}$ for $I_{5a}$ in any deficiency-$0$ graph resulting from expanding beyond $I_{5a}$ preserves the girth, and then that graph is not a type-(ii) $B_5$ instance. So it turns out that no $B_5$ instance can be generated, and then the cycle configuration of $S_{5a}$ is invalid. As for $S_{5b}$, it is relatively easy to see that its expansion cannot proceed without infringing Lemma 2 or reducing the graph’s girth. This cycle configuration is therefore also invalid.

The expansion of $S_{5c}$, on the other hand, leads to the $I_{5c}$ of Figure 11. And even though we arrive at $I_{5c}$ before obtaining $B_{5c}$, this expansion does refrain from adding edges that would reduce the graph’s girth. So we may proceed with the expansion of $I_{5c}$ until we generate the $B_{5c}$ of Figure 12 which is a type-(ii) instance of $B_5$. As before, it is important to note that the two subgraphs of $B_{5c}$ isomorphic to $S_{5c}$ have the same cycle configuration as $S_{5c}$.

### 4.4 The girth-6 case

Following our development so far, we present in Figures 13(a)–(e) the consistent cycle configurations of $S_6$, namely $S_{6a}$ through $S_{6e}$. Of these, $S_{6c}$ is the only one to include a complete 6-CDC cycle (cycle $C_1$) in its cycle configuration. The $I_6$ and $B_6$ instances for $S_{6a}$, …, $S_{6d}$ are given in Figures 14 and 15. Notice that, consistently with our comments in Section 3, every one of $I_{6a}$ through

![Figure 8: Graph, with a 6-CDC, obtained from $S_{4b}$ by letting $e = g$, $f = h$, and $C_2 = C_4$ while $e$ is connected to $f$.](image)
Figure 9: \( S_{5a} \) (a), \( S_{5b} \) (b), and \( S_{5c} \) (c).
$I_{6d}$ has subgraphs that are isomorphic to $S_6$ but do not have the same cycle configuration as, respectively, $S_{6a}$ through $S_{6d}$ (having, as those subgraphs do, a complete 6-CDC cycle in their cycle configurations). Notice also that $B_{6b}$ (the Heawood graph [3]) is the only type-(i) instance of $B_6$ in the group; the others are all of type (ii). The case of $S_{6e}$, however, embodies peculiarities we have not yet encountered, and does as such require further elaboration.

As we noted above, $S_{6e}$ contains a complete cycle from the 6-CDC ($C_1$ in Figure 13(e)). Also, it represents the only possible cycle configuration for a 6-CDC cycle in a girth-6 graph. It then follows that $S_{6e}$ is contained in every girth-6 graph that has a 6-CDC. However, there exist graphs that have a 6-CDC and $S_{6e}$ as a subgraph (including its cycle configuration) but do not have $S_{6a}$, $S_{6b}$, $S_{6c}$, or $S_{6d}$ as subgraphs: they are the graphs in which every 6-cycle belongs to the 6-CDC. So, analogously to our strategy thus far, let us look for corresponding $I_6$ and $B_6$ instances. From Section 3, we know that the desired $B_6$ instances are of type (iii).

In order to facilitate the task of searching for $I_{6e}$ and also for $B_{6e}$, we first look into some properties related to the graph’s girth.

**Lemma 7.** If $C$ is a cycle in a 6-CDC of $G$ such that $\sigma(C) < 6$, then $g(G) < 5$.

*Proof.* If $C$ has a chord, then the lemma holds trivially. Let us then assume that $C$ is chordless. In this case, there has to exist another 6-CDC cycle, say $C'$, such that $\mu(C, C') > 1$, since $\sigma(C) < 6$. By Lemma 4, $\mu(C, C') \leq 3$; by Lemma 6 if $\mu(C, C') = 3$ then $G$ is isomorphic to either $M_6$ or $T_{6,2}$ and, consequently, every 6-CDC cycle has a chord, which cannot be by assumption. Thus, $\mu(C, C') = 2$.

Let $e$ and $f$ be edges of $G$ belonging to both $C$ and $C'$. Because $C'$ is chordless, $C'$ contains two distinct paths, call them $P_1$ and $P_2$, of length 2, both
Figure 11: $I_{5e}$.

Figure 12: $B_{5e}$.
Figure 13: $S_{6a}$ (a), $S_{6b}$ (b), $S_{6c}$ (c), $S_{6d}$ (d), and $S_{6e}$ (e).
Figure 14: $I_{6a}$ (a), $B_{6a}$ (b), $I_{6b}$ (c), $B_{6b}$ (d), $B_{6b'}$ (e).
Figure 15: $I_{6c}$ (a), $B_{6c}$ (b), $I_{6d}$ (c), $B_{6d}$ (d), and $B_{6d'}$ (e).
interconnecting an end vertex of $e$ with an end vertex of $f$, as in Figure 16. Let $u$ and $v$ be end vertices of $e$ and $f$, respectively, such that $P_1$ interconnects $u$ and $v$. It is easy to see that there exists another path in $C$, call it $P_3$, that interconnects $u$ and $v$ in such a way that $P_3$ has length less than 4. Hence, there exists a cycle in $G$ containing $u$ and $v$ whose length is less than 6 and whose edges are those of $P_1$ and $P_3$.

However, there is only one way, shown in Figure 16, of interconnecting the end vertices of $e$ and $f$ such that the subgraph of $G$ induced by the edges of $C$ and $C'$ has girth 5. And there exists only one possible cycle configuration for this subgraph, considering the already given cycles $C$ and $C'$. This cycle configuration contains $S_{6a}$, which from Section 4.3 we know is not valid. So the interconnection pattern of Figure 16 is invalid as well, hence $g(G) < 5$.

Theorem 8. $g(G) \geq 5$ if and only if every cycle $C$ in a 6-CDC of $G$ is such that $\sigma(C) = 6$.

Proof. If $\sigma(C) < 6$ for some 6-CDC cycle $C$, then $g(G) < 5$ by Lemma 7. In order to prove the converse statement, we assume $g(G) = 3$ or $g(G) = 4$ and simply verify, respectively from Figure 2(b) or Figures 5(a)–(c), that 6-CDC cycles $C$ and $C'$ always exist such that $\mu(C, C') > 1$. It then follows that a 6-CDC cycle $C$ always exists for which $\sigma(C) < 6$.

Let us first see how Theorem 8 simplifies the search for $B_{6e}$. We begin by defining a graph $D_{6e}$ such that $V(D_{6e}) = \{C_1, \ldots, C_t\}$, where $C_1, \ldots, C_t$ are the 6-CDC cycles of $B_{6e}$, and $C_iC_j \in E(D_{6e})$ if and only if $C_i$ and $C_j$ share an edge in $B_{6e}$. Since $S_{6e}$ is a subgraph of $B_{6e}$, it induces the subgraph of $D_{6e}$ shown in Figure 17(a) with dashed edges. It is easy to see that $D_{6e}$ is 6-regular and that, by Lemma 8, every vertex in $B_{6e}$ corresponds to a triangle in $D_{6e}$ (and conversely).

Since $B_{6e}$ is cubic, each triangle in $D_{6e}$ shares an edge with exactly three other triangles and each edge in $D_{6e}$ is shared by two triangles. These properties restrict the way in which the degree-3 vertices of $D_{6e}$ in Figure 17(a) may be connected to other vertices. In particular, they must not be connected among themselves, meaning that the six incomplete cycles of the subgraph of $B_{6e}$ in

Although this definition of $D_{6e}$ is very similar to that of a dual graph, we refrain from using this denomination because we do not assume that $B_{6e}$ is planar.
Figure 17: Subgraphs of $B_{6e}$ with subgraph of $D_{6e}$ in the background (with dashed edges).

Figure 18: Smallest girth-6 graphs, $B_{6e}$ (a) and $B_{6e'}$ (b), that have a 6-CDC which contains every 6-cycle.

It is relatively easy to see that the only expansions of the graph of Figure 17(b) that qualify as type-(iii) instances of $B_{6}$ and moreover comply with Theorem 8 are the ones in Figure 18. We denote them by $B_{6e}$ (Figure 18(a)) and $B_{6e'}$ (Figure 18(b)).

The $I_{6e}$ shown in Figure 19(a) can be obtained straightforwardly from $S_{6e}$ using the same restrictions as the ones used for obtaining $B_{6e}$. Notice that the deficiency of $I_{6e}$ is equal to that of $S_{6e}$ (12 in both cases). Larger graphs in this class can be obtained in the same way as for the previous cases; the outcome of the first replacement of $S_{6e}$ by $I_{6e}$ in $B_{6e}$ is shown in Figure 19(b).

Unlike most of our previous illustrations, Figures 18 and 19 contain no annotation for vertex or cycle identification. They are omitted for clarity and are furthermore needless, since in these figures all 6-cycles are in the 6-CDC.
Completeness of the method and Hamiltonian cycles

It is possible to prove that the generation method discussed in Sections 3 and 4 never outputs the same graph twice, and also that all cubic graphs having a 6-CDC are generated. In what follows, we separate the $g < 6$ case from that of $g = 6$.

We first explain the absence of duplicates during generation. Notice first that the method always keeps the girth constant as $I_g$ substitutes for $S_g$, so there is no interference between distinct-girth instances. In order to see that outputs are unique also for fixed $g$, consider first the $g < 6$ case. It then suffices to recall that all subgraphs isomorphic to $S_g$ in $I_g$ or in $B_g - E(S_g)$ have the same cycle configuration, which forbids any hybrid cycle configuration to be generated (i.e., a cycle configuration with remnants from more than one $S_g$ instance).

For $g = 6$, what might prevent the same simple argument from holding is that there is, of course, the issue related to girth-6 graphs that we discussed in Sections 3 and 4.4. In this case, the occurrence of more than one cycle configuration for $S_g$-isomorphs is verified in all type-(i) and (ii) instances of $B_g$. However, the extra cycle configurations always contain a complete 6-CDC cycle and our method never replaces them in the process of generating new graphs from a type-(i) or (ii) $B_g$ instance. They only get replaced when the method starts at a type-(iii) instance, so the no-duplicity argument remains essentially unaltered.

Let us now demonstrate that no cubic graph $G$ having a 6-CDC is missed during generation. The overall strategy here is to start from $G$ itself and to repeatedly substitute $S_g$ for a subgraph of the current graph that is isomorphic to $I_g$ until a $B_g$ instance is reached. If for every cubic $G$ that has a 6-CDC we can argue that this “reversal” of the generation process is possible, then we have proven that the method is complete.
Let us consider the $g < 6$ case first. Let $G$ have girth $g$ and a 6-CDC, and recall that both $I_g$ and all instances of $B_g$ are girth-$g$ supergraphs of $S_g$. The difference between them is that each $B_g$ is a cubic graph having a 6-CDC, while $I_g$ has nonzero deficiency and a cycle configuration with the important property of being self-similar with respect to $S_g$. Because our method relies on the explicit knowledge of every possible cycle configuration of $S_g$, there are only two possibilities for $G$: either it is isomorphic to a type-(i) instance of $B_g$ or it has a subgraph that is isomorphic to $I_g$. While in the former case $G$ is obviously generated by the method, in the latter it is possible to recursively substitute $S_g$ for $I_g$ through a sequence of ever smaller graphs until either a type-(i) instance of $B_g$ is finally obtained or else a substitution yields a graph that has less-than-$g$ girth. If it is not the case that the process ends at a type-(i) instance of $B_g$, then by definition the last substitution must have been applied on a type-(ii) instance of $B_g$. We then conclude that, in any case, $G$ is output by the method.

The case of $g = 6$ is analogous, but the possibilities for ending the sequence of substitutions are more varied. The sequence may end when a type-(i) instance of $B_g$ is reached, or when a graph is obtained whose girth is less than $g$ (if $G$ has 6-cycles that are not in the 6-CDC), or yet when a graph is obtained that has acquired 6-cycles that are not in the 6-CDC (if all of $G$'s 6-cycles are in the 6-CDC). Similarly to the case of $g < 6$, if the process does not end at a type-(i) instance of $B_g$, then by definition the last substitution must have been applied respectively on a type-(ii) or (iii) instance of $B_g$. Once again, $G$ is in any case seen to be output by the method.

It is important to note that this argument for the method’s completeness relies crucially on the fact that every possible instance of $B_g$ is known. For $g = 3, \ldots, 6$, this is part of what we established in Sections 4.1 through 4.4. A key observation related to our exhaustive enumeration of $B_g$ instances in those sections is that in none of those instances is more than one cycle configuration of $S_g$ present, with the important exception in the girth-6 case we noted in Section 3. For this reason, in the above completeness argument we need not concern ourselves with the presence of multiple cycle configurations for $S_g$: in girth-$g$ cubic graphs that have a 6-CDC, such multiplicity never occurs, unless $g = 6$ and the cycle configuration of $S_g$ does not include a complete 6-CDC cycle—in this case, the argument is already split into finishing at a type-(ii) or a type-(iii) instance of $B_g$.

It is also possible to identify a Hamiltonian cycle in every cubic graph that has a 6-CDC. We first find a Hamiltonian cycle $C$ in $B_g$. Then we take paths in $I_g$ that are equivalent to the path used by $C$ in the $S_g$-isomorphic subgraph of $B_g$. For example, in $B_3$ we have the Hamiltonian cycle $dachefd$, whose intersection with $S_3$ is the path $dache$ (cf. Figure 2), and an equivalent path in $I_3$ (cf. Figure 3(a)) is $dylkjabchie$. Successive substitutions of $I_3$ for $S_3$ will then always ensure the presence of a Hamiltonian cycle. In Figures 20 through 26 we show (as thick edges) Hamiltonian cycles in $B_g$ and equivalent paths in $I_g$ for all the remaining pertinent values of $g$. 
6 6-CDC’s and the minimal chordal sense of direction

In this section, $G$ is no longer assumed to have a 6-CDC, but rather to be such that every one of its edges has two labels, each corresponding to one of its end vertices. In [11], a property of this edge labeling has been introduced which can considerably reduce the complexity of many problems in distributed computing [5]. This property refers to the ability of a vertex to distinguish among its incident edges according to a globally consistent scheme and is formally described in [6]. An edge labeling for which the property holds is called a sense of direction. It is symmetric if for every edge one label can be inferred from the other. While the full-fledged definition of sense of direction is irrelevant to our present discussion, the special sense of direction that we describe next is closely related to a cubic graph’s having a 6-CDC.

We say that a sense of direction is minimal if it requires exactly $\Delta(G)$ distinct labels, where $\Delta(G)$ is the maximum degree in $G$. A particular instance of symmetric sense of direction, called a chordal sense of direction, can be constructed on any graph by fixing an arbitrary cyclic ordering of the vertices and, for each edge $uv$, selecting the difference (modulo $n$) from the rank of $u$ in the ordering to that of $v$ as the label of $uv$ that corresponds to $u$ (likewise, the label that corresponds to $v$ is the rank difference from $v$ to $u$). In Figure 27, an example is given of a minimal chordal sense of direction (MCSD). For a survey on sense of direction, we refer the reader to [7].

Before proceeding to our result in this section, we pause briefly to review some relevant definitions. Given a finite group $A$ and a set of generators $S \subseteq A$, a Cayley graph is a graph $H$ whose vertices are the elements of the group $(V(H) = A)$ and whose edges correspond to the action of the generators ($uv \in E(H) \iff \exists s \in S : v = s \ast u$, where $\ast$ is the operation defined for $A$). We assume that the set of generators is closed under inversion, so $H$ is an undirected
Figure 21: Hamiltonian cycle in $B_{5c}$ (a) and equivalent path in $I_{5c}$ (b).

Figure 22: Hamiltonian cycle in $B_{6a}$ (a) and equivalent paths in $I_{6a}$ (b).

Figure 23: Hamiltonian cycle in $B_{6b}$ (a) and in $B_{6b'}$ (b), and equivalent paths in $I_{6b}$ (c).
Figure 24: Hamiltonian cycle in $B_{6c}$ (a) and equivalent paths in $I_{6c}$ (b).

Figure 25: Hamiltonian cycle in $B_{6d}$ (a) and in $B_{6d'}$ (b), and equivalent path in $I_{6d}$ (c).
Figure 26: Hamiltonian cycle in $B_{6e}$ (a) and in $B_{6e'}$ (b), and equivalent paths in $I_{ae}$ (c).
Figure 27: A graph with an edge labeling that is an MCSD. Vertices are ordered clockwise.

graph. A circulant graph (also known as a chordal ring) is a Cayley graph over $\mathbb{Z}_n$, the cyclic group of order $n$ under the addition operation.

In [9], we have analyzed the class of regular graphs that admit an MCSD, and proved that this class is equivalent to that of circulant graphs. In this section, we prove that nearly all the cubic graphs in this class have a 6-CDC, the only exception being $K_4$. We start by noting that a characterization of such cubic graphs follows directly from the results of [9] (specifically, Theorem 5, Lemma 6, and Lemma 7) and can be stated as follows.

**Lemma 9.** $G$ admits an MCSD (or, equivalently, $G$ is a circulant graph) if and only if $G$ is isomorphic to either $M_n$, with $n \geq 4$, or to $T_{n,2}$, with $n \geq 6$.

We now present the main result of this section.

**Theorem 10.** Except for $K_4$, every cubic graph that admits an MCSD has a 6-CDC.

**Proof.** By Lemma 9 it suffices to consider instances of $M_n$, with $n \geq 4$, and of $T_{n,2}$, with $n \geq 6$.

First, notice that all instances of $M_n$ or $T_{n,2}$ have girth less than 5. Also, the only instances of girth 3 are $M_4$ and $T_{6,2}$. The former of these is isomorphic to $K_4$ and obviously does not have a 6-CDC. As for the latter, a 6-CDC is shown in Figure 2(a).

Let us then consider the girth-4 instances; we do this by resorting to the material of Section 4.2. First notice that $B_{4b'}$ is isomorphic to $M_6$ and that all instances of $M_n$, for $n \geq 6$, can be generated by successive replacements of $S_{4b'}$ by $I_{4b'}$ in $B_{4b'}$. Similarly, $B_{4b}$ is isomorphic to $T_{8,2}$ and all instances of $T_{n,2}$, for $n > 8$, can be generated by successive replacements of $S_{4b}$ by $I_{4b}$ in $B_{4b}$. It then follows that every instance of $M_n$ or $T_{n,2}$ having $n \geq 6$ also has a 6-CDC. □
7 Conclusions

We have in this paper demonstrated how to generate all the cubic graphs that have a 6-CDC in a constructive manner. For an arbitrary cubic graph $G$ with $n$ vertices and girth $g$, our method provides, at least in principle, a mechanism for checking whether $G$ has a 6-CDC: one simply generates all girth-$g$ cubic graphs on $n$ vertices that have a 6-CDC and checks each one against $G$ for isomorphism. This check, we recall, can be performed polynomially for cubic graphs [10].

Our method also provides a mechanism for pinpointing a Hamiltonian cycle in any cubic graph that has a 6-CDC. That all such graphs are Hamiltonian is a result consistent with the one in [9], given our further demonstration, in this paper, that all non-$K_4$ cubic graphs that have an MCSD also have a 6-CDC. The alluded result in [9] is that all regular graphs that have an MCSD are Hamiltonian.

Our results relating 6-CDC’s to MCSD’s in cubic graphs create a bridge connecting these two concepts and also, by consequence, the notion of a circulant graph in the cubic case. The obvious implication of this is that results obtained within one context can now be extended directly to any other.

There are several open problems that may be addressed to expand on the results we have presented. Some of them come from generalizing the vertices’ fixed degree or the constant length of a CDC’s cycles, or yet from relaxing at least one of the two constraints by letting vertices have different degrees or CDC cycles different lengths. Relaxing both is really tantamount to addressing the Szekeres-Seymour conjecture [13][12], according to which every 2-edge-connected graph has a CDC. This conjecture has stood for over thirty years, so perhaps an easier (though by no means trivial) starting problem for further research is to characterize the $k$-regular graphs that have a $2k$-CDC, $k \geq 3$.

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