MEAN-VALUE OF PRODUCT OF SHIFTED MULTIPLICATIVE FUNCTIONS AND AVERAGE NUMBER OF POINTS ON ELLIPTIC CURVES.

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ABSTRACT. In this paper, we consider the mean value of the product of two real valued multiplicative functions with shifted arguments. The functions \( F \) and \( G \) under consideration are close to two nicely behaved functions \( A \) and \( B \), such that the average value of \( A(n-h)B(n) \) over any arithmetic progression is only dependent on the common difference of the progression. We use this method on the problem of finding mean value of \( K(N) \), where \( K(N)/\log N \) is the expected number of primes such that a random elliptic curve over rationals has \( N \) points when reduced over those primes.

1. INTRODUCTION

Let \( F \) and \( G : \mathbb{N} \rightarrow \mathbb{C} \) be non zero multiplicative functions (a function \( F \) is multiplicative if \( F(mn) = F(m)F(n) \) for \( (m, n) = 1 \)). In this paper we are interested in finding the mean value of \( F(n-h)G(n) \) for a fixed integer \( h \). More precisely the sum of the form
\[
M_x, h(F, G) = \frac{1}{x} \sum_{n \leq x} F(n-h)G(n).
\]

A lot of work has been done to find the asymptotic behavior of \( M_x, h(F, G) \) under various conditions, (see for example [17], [12], [18], [19], [5], [20]). In many of those cases, the functions are required to be close to 1 on the set of primes. In some cases (for example [12]) convergence of suitable series involving \( F \) and \( G \) has been assumed.

When the functions grow faster, the problem becomes more difficult. In [8], divisor function and other faster growing functions are discussed. The Euler totient function \( \phi(n) \) has been studied in [11] and [16].

In the first theorem of this paper we consider this problem for a wide class of functions with more general growth conditions. The type of functions that we consider in Theorem 1 need not necessarily be multiplicative. But they can be written as
\[
F(n) = A(n) \sum_{d \mid n} f(d) \quad \text{and} \quad G(n) = B(n) \sum_{d \mid n} g(d),
\]
where
\[
\sum_{d=1}^{\infty} \frac{|f(d)|}{d} < +\infty, \quad \sum_{d=1}^{\infty} \frac{|g(d)|}{d} < +\infty.
\]
Further we assume the existence of two function \( M(x) \) and \( E_1(x) \) such that for any positive integers \( a \) and \( m \),
\[
\sum_{n \leq x \atop n \equiv a (\text{mod} \, m)} A(n-h)B(n) = \frac{1}{m} M(x) + O(E_1(x)).
\]

In the first theorem we show that under the above conditions one can prove an asymptotic estimate of \( M_{x, h}(F, G) \). Further in order to write the error term explicitly, we introduce two suitable monotonic functions \( E_1(x) \) and \( E_2(x) \) such that
\[
\sum_{d \leq x} |f(d)| = O(E_2(x)), \quad \sum_{d \leq x} |g(d)| = O(E_3(x)).
\]
Then the first result of this paper is as follows
Theorem 1. Let $F$ and $G$ be two arithmetic functions, satisfying (2), (3), (4) and (5) where $f$ and $g$ are multiplicative. Let $0 < c < 2$, such that for any large positive real number $y$, $E_i(2y) \leq cE_i(y)$ for $(i = 2, 3)$. Then for any fixed positive integer $h$,\[
abla F(n-h)G(n) = C_hM(x) + O(|E_1(x)|E_2(x)|E_3(x)| + \frac{M(x)}{x}(|E_2(x)| + |E_3(x)|)),
\]with\[
C_h = \prod_p \left(1 + \sum_{j=1}^{\nu_p(h)} \frac{f(p^j) + g(p^j)}{p^j}\right) \prod_{p|h} \left(1 + \sum_{i=1}^{\nu_p(h)} \frac{p^iS_p(p^i)}{S_p(1)}\right).
\]where $S_p(p^i) := \sum_{i \leq \nu_p(h)} \frac{\varphi(p^i)\varphi(p^j)}{p^i+p^j}$, for $i \geq 0$.

Remark 1. The additional condition of $f$ and $g$ being multiplicative in Theorem 1 is only required to get an Euler product form of the constant $C_h$. Also note that if $F$ is multiplicative, then by Möbius inversion formula, $f$ is uniquely determined. Also if $F$ is ‘sufficiently’ close to 1 on primes, then (3) is satisfied for $f$. Similarly for multiplicative functions the idea is to choose ‘smooth’ functions $A$ and $B$ such that for $F$ and $G$ are close to 1. Also $A(n-h)B(n)$ should be nicely summable on every arithmetic progression.

Before proceeding with the proof of Theorem 1 we shall note down some application of the above theorem. One can directly apply it on classical Euler’s totient function $\phi$ and Jordan’s totient function $J_k$. See [9] and [1] for more on the error term related to $\phi$ and $J_k$. Also see [16] for the mean value of the $k$-fold shifted product of $\phi$.

Corollary 1. (a) If $\phi(n)$ is the Euler totient function, i.e. $\phi(n) = n\prod_{p|n}(1 - 1/p)$, then for any fixed integer $h$\[
\sum_{n \leq x} \phi(n)\phi(n-h) = \frac{1}{3}x^3 \prod_{p}(1 - \frac{2}{p^2}) \prod_{p|h}(1 + \frac{1}{p(p^2 - 2)}) + O(x^2(\log x)^2).
\]

(b) If $J_k(n)$ is the Jordan’s totient function defined as $J_k(n) = n^k\prod_{p|n}(1 - 1/p^k)$, then for $k \geq 2$ and fixed integer $h$\[
\sum_{n \leq x} J_k(n)J_k(n-h) = \frac{x^{2k+1}}{2k+1} \prod_{p}(1 - \frac{2}{p^k+1}) \prod_{p|h}(1 + \frac{1}{p^k(p^k+1)}) + O(x^k).
\]

Proof of Corollary 1 follows directly from Theorem 1. In case of (a), $A(n) = B(n) = n$, while for Jordan totient function $J_k(n)$, one takes $A(n) = B(n) = n^k$. For both the cases $f$ and $g$ can be computed using Möbius inversion.

In the next part, we discuss an application of Theorem 1 in computing the mean value of the function $K(N)$ as defined in [5]. Before stating the result we explain the background of this problem.

Let $E$ be an elliptic curve defined over the field of rationals $\mathbb{Q}$. For a primes $p$ where $E$ has good reduction, we denote by $E_p$ the reduction of $E$ modulo $p$. Let $\mathbb{F}_p$ be the finite field with $p$ elements. Define $M_E(N)$ as\[
M_E(N) := \# \{ p \text{ prime} : E \text{ has good reduction over } p \text{ and } |E_p(\mathbb{F}_p)| = N\}.
\]

Using Hasse bound and upper bound sieve one can show that\[
M_E(N) \ll \frac{\sqrt{N}}{\log N}.
\]
If $E$ has complex multiplication (CM), then Kowalski\cite{13} has shown that
\[ M_E(N) \ll N^\varepsilon \]
for any $\varepsilon > 0$.

No stronger bound is known when $E$ is non-CM. A naive probabilistic model suggests $M_E(N) \sim \frac{1}{\log N}$. See \cite{6} for details. Any estimate of $M_E(N)$ for a fixed $E$ is not possible. In fact using Chinese Reminder Theorem it can be shown that for given integer $N$, the bound in \cite{9} is attained for some $E$. In \cite{13}, Kowalski has shown that
\[ \sum_{N \leq x} M_E(N) = \pi(x) + O(\sqrt{x}). \] (10)

In \cite{6} David and Smith introduced an arithmetic function $K(N)$. Later they made a correction\cite{7} in the expression of $K(N)$. The corrected formula is as follows
\[ K(N) := \prod_{p \mid N} \left( 1 - \frac{(N-1)^2}{(p-1)^2} \frac{1}{p} \right) \prod_{p \mid N} \left( 1 - \frac{1}{p^\nu_p(N)(p-1)} \right) \] (11)
where $\nu_p$ denotes the usual $p$-adic valuation.

Now let $K^*(N) = K(N)/\phi(N)$, where $\phi(N)$ is the Euler totient function. In \cite{6}, David and Smith proved an asymptotic estimate for average value of $M_E(N)$ when $E$ varies over a family of curves. But their result was not unconditional. It depends on the following conjecture

**Conjecture 1 (Barban–Davenport–Halberstam).** Let $\theta(x; q, a) = \sum_{\nu \leq x, p \equiv a \pmod{q}} \log p$. Let $0 < \eta \leq 1$ and $\beta > 0$ be real numbers. Suppose that $X, Y, \text{ and } Q$ are positive real numbers satisfying $X^\eta \leq Y \leq X$ and $Y/(\log X)^\beta \leq Q \leq Y$. Then
\[ \sum_{q \leq Q} \sum_{1 \leq a \leq q \atop (a, q) = 1} |\theta(X + Y; q, a) - \theta(X; q, a) - \frac{Y}{\phi(q)}| \ll_{\eta, \beta} YQ\log X. \]

**Remark 2.** For $\eta = 1$, this is the classical Barban–Davenport–Halberstam theorem. Languasco, Perelli, and Zaccagnini \cite{14} have proved the Conjecture for $\eta = \frac{7}{12} + \varepsilon$, which is the best known result. Also under generalized Riemann hypothesis they could prove the conjecture for $\eta = \frac{1}{2} + \varepsilon$.

Given integers $a$ and $b$, let $E_{a, b}$ be the elliptic curve defined by the Weierstrass equation
\[ E_{a, b} : y^2 = x^3 + ax + b. \]

For $A, B > 0$, we define a set of Weierstrass equations by
\[ \mathcal{E}(A, B) := \{ E_{a, b} : |a| \leq A, |b| \leq B, \Delta(E_{a, b}) \neq 0 \}. \]

In \cite{6}, \cite{7}, \cite{4}], the following conditional result has been proved.

**Theorem A.** Assume Conjecture \cite{7} holds for some $\eta < \frac{1}{2}$. Let $\varepsilon > 0$ and $A, B > N^{\frac{1}{2} + \varepsilon}$ such that $AB > N^{\frac{3}{2} + \varepsilon}$. Then for any positive integer $R$,
\[ \frac{1}{\#\mathcal{E}(A, B)} \sum_{E \in \mathcal{E}(A, B)} M_E(N) = \frac{K^*(N)}{\log N} + O_{\eta, \varepsilon, R}\left( \frac{1}{(\log N)^{\varepsilon}} \right). \]

In order to verify the consistency of Theorem A with unconditional results such as \cite{10}, one need to compute the mean value of $K^*(N)$ where $N \leq x$ satisfies congruence conditions. For more details see \cite{15}.

In \cite{15}, Smith, Martin and Pollack have addressed this aspect. They proved that

**Theorem B.** For $x \geq 2$,
\[ \sum_{N \leq x} K^*(N) = x + O\left( \frac{x}{(\log x)} \right) \quad \text{and} \quad \sum_{N \leq x \atop n \text{ odd}} K^*(N) = \frac{x}{3} + O\left( \frac{x}{(\log x)} \right). \]
Using Theorem \[\text{[B]}\] and Abel’s partial summation one can verify that
\[\frac{1}{\#E(A, B)} \sum_{E \in E(A, B), N \leq x} \sum_{N} M_{E}(N) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{\gamma}}\right).\]

So Theorem \[\text{[A]}\] consistent with (10) if one consider \(\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{\gamma}}\right)\).

But it is well known that \(li(x) = \int_{2}^{x} \frac{1}{\log t} dt\) is a better approximation of \(\pi(x)\) compared to \(\frac{x}{\log x}\). So in order to check the consistency of Theorem \[\text{[A]}\] and (10), where main term of \(\pi(x)\) is taken as \(li(x)\), we need significantly better bound for the error terms in Theorem \[\text{[B]}\] In this paper we prove that. We prove

**Theorem 2.** For \(x \geq 2\),

(a) \[
\sum_{N \leq x} K^{*}(N) = x + O(\log x)
\]

(b) \[
\sum_{N \leq x} K^{*}(N) = \frac{x}{3} + O(\log x).
\]

Then Theorem \[\text{[A]}\] and Theorem \[\text{[B]}\] together implies

\[
\frac{1}{\#E(A, B)} \sum_{E \in E(A, B), N \leq x} \sum_{N} M_{E}(N) = li(x) + O\left(\frac{x}{(\log x)^{\gamma}}\right).
\]

This provides further support to the Barban–Davenport–Halberstam conjecture.

Although the function \(K^{*}(N)\) looks like a multiplicative function it is far from it. In fact

\[
K^{*}(N) = C^{*}_{2} F^{*}(N-1) G^{*}(N)
\]

where

\[
C^{*}_{2} = \prod_{p > 2} \left(1 - \frac{1}{(p - 1)^{2}}\right)
\]

\[
F^{*}(N) = \prod_{p | N, p > 2} \left(1 - \frac{1}{(p - 1)^{2}}\right)^{-1} \prod_{p | N} \left(1 - \frac{1}{(p - 1)^{2}(p + 1)}\right)
\]

\[
G^{*}(N) = \frac{N}{\phi(N)} \prod_{p | N, p > 2} \left(1 - \frac{1}{(p - 1)^{2}}\right)^{-1} \prod_{p | N} \left(1 - \frac{1}{p \nu_{p}(N)(p - 1)}\right).
\]

Note that, both \(F^{*}\) and \(G^{*}\) are multiplicative functions.

In the last section of this paper we discuss the original expression of \(K(N)\) as defined in \[Theorem 3 ; \text{[B]}\]. We denote it by \(\hat{K}(N)\). It was defined as follows

\[
\hat{K}(N) : = \prod_{p | N} \left(1 - \frac{(N-1)^{2}p + 1}{(p - 1)^{2}(p + 1)}\right) \prod_{p | N, 2^{2}p(N)} \left(1 - \frac{1}{p \nu_{p}(N)(p - 1)}\right) \prod_{p | N, 2^{2}p(N) + 1} \left(1 - \frac{p - (N_{p}^{2}p)}{p \nu_{p}(N)^{2}(p - 1)}\right)
\]

where \(\nu_{p}\) denotes the usual \(p\)–adic valuation, and \(N_{p} := \frac{N}{p \nu_{p}(N)}\) denotes the \(p\)–free part of \(N\).

This function cannot be written as product of two shifted multiplicative function. In \[\text{[15]}\], it is claimed that the mean of \(K^{*}(N)\) is also equals to 1.

But we show that is not true. The average turns out to be equal to \(\frac{31}{30}\). Also we make improvement on the error term in the average of \(\hat{K}(N)\). We prove that

**Theorem 3.** For \(x \geq 2\),

\[
\sum_{N \leq x} \hat{K}(N) = \frac{31}{30} x + O(\log x).
\]
The main reason behind proving this theorem separately is to show that Theorem 1 can be useful in some cases where one of the shifted multiplicative functions is not multiplicative. Under suitable conditions those non-multiplicative functions can be changed to expected multiplicative form. That way Theorem 1 can also be useful in computing mean value of function.

In the next sections we give give proofs of the above three theorem.

2. Proof of Theorem 1

We have

\[
\sum_{n \leq x} F(n-h)G(n) = \sum_{n \leq x} G(n)A(n-h) \sum_{d|n-h} f(d)
\]

\[
= \sum_{d \leq x-h} f(d) \sum_{n \leq x \mod n-h} G(n)A(n-h)
\]

\[
= \sum_{d \leq x-h} f(d) \sum_{n \leq x \mod n-h} A(n-h)B(n) \sum_{d|n} g(d_1)
\]

\[
= \sum_{d \leq x-h} f(d) \sum_{d_1 \leq x \mod (d_1|h)} \left( \frac{M(x)}{d_1} + O(E_1) \right), \quad \text{where} \ [d,d_1] := \text{lcm}\{d,d_1\}
\]

\[
= M(x) \sum_{d \leq x-h} f(d) \sum_{d_1 \leq x \mod (d_1|h)} \frac{g(d_1)(d,d_1)}{d_1} + O(E_1(x)E_2(x)E_3(x)). \quad (17)
\]

Now, the \(d\)-sum and \(d_1\)-sum can be extended to \(\infty\) to get

\[
M(x) \sum_{d=1}^{\infty} f(d) \sum_{d_1 \leq x \mod (d_1|h)} \frac{g(d_1)(d,d_1)}{d_1}
\]

with an error term

\[
O(M(x) \sum_{1 \leq d < \infty} \frac{f(d)}{d} \sum_{d_1 \leq x \mod (d_1|h)} \frac{g(d_1)(d,d_1)}{d_1}) + O(M(x) \sum_{d > x-h} \frac{f(d)}{d} \sum_{d_1 \leq x \mod (d_1|h)} \frac{g(d_1)(d,d_1)}{d_1}).
\]

Now note that

\[
\sum_{d > x} \frac{|f(d)|}{d} = \sum_{x < d \leq 2x} \frac{|f(d)|}{d} + \sum_{2x < d \leq 4x} \frac{|f(d)|}{d} + \sum_{4x < d \leq 8x} \frac{|f(d)|}{d} + \ldots
\]

\[
\leq \frac{E_2(2x)}{x} + \frac{E_2(4x)}{2x} + \frac{E_2(8x)}{4x} + \ldots
\]

\[
\leq \frac{E_2(x)}{x} \left( c + c^2/2 + c^3/4 + c^4/8 + \ldots \right)
\]

\[
\leq \frac{2c}{2-c} \frac{E_2(x)}{x}.
\]

Thus \(\sum_{d > x} \frac{|f(d)|}{d} = O\left(\frac{E_2(x)}{x}\right)\). Similarly \(\sum_{d_1 > x} \frac{|g(d_1)|}{d_1} = O\left(\frac{E_2(x)}{x}\right)\).

Then by (17)

\[
\sum_{n \leq x} F(n-h)G(n) = M(x) \sum_{d,d_1 \mod (d_1|h)} f(d)g(d_1)(d,d_1) + O(\left|E_1(x)E_2(x)E_3(x)\right| + \frac{M(x)}{x}(\left|E_2(x)\right| + \left|E_3(x)\right|)). \quad (18)
\]
Only thing that remains to complete the proof is to express \( \sum_{d|d_1} \frac{f(d)g(d_1)}{dd_1} \) as an Euler product.

To do that define the following notations

\[
T(p^k) := \frac{S_p(p^k)}{S_p(1)} = \sum_{\min\{e_1, e_2\} = k} \frac{f(p^{e_1})g(p^{e_2})}{p^{e_1+e_2}} \quad \text{and} \quad T(h) := \prod_{p|h} T(p^{\nu_p(h)}).
\]

Then one can verify that

\[
\sum_{(d,d_1) = h_1} \frac{f(d)g(d_1)}{dd_1} = T(h_1) \sum_{(d,d_1) = 1} \frac{f(d)g(d_1)}{dd_1}.
\]

Now

\[
\sum_{d,d_1} \frac{f(d)g(d_1)(d,d_1)}{dd_1} = \sum_{h_1|h} h_1 T(h_1) \sum_{(d,d_1) = 1} \frac{f(d)g(d_1)}{dd_1}
\]

\[
= \left( \sum_{(d,d_1) = 1} \frac{f(d)g(d_1)}{dd_1} \right) \prod_{p|h} (1 + pT(p) + \cdots + p^{\nu_p(h)}T(p^{\nu_p(h)}))
\]

\[
= \prod_p \left( 1 + \frac{f(p) + g(p)}{p} + \frac{f(p^2) + g(p^2)}{p^2} + \cdots \right) \prod_{p|h} (1 + pT(p) + \cdots + p^{\nu_p(h)}T(p^{\nu_p(h)}))
\]

which proves the result.

3. Proof of Theorem

Recall that,

\[ K^*(N) = C_2^*F^*(N - 1)G^*(N) \]

where \( C_2^* \), \( F^* \) and \( G^* \) are given as in (13), (14), (15).

Now in this case \( A(n) = B(n) = 1 \), hence \( M(x) = x \). Also if we set

\[ f^*(m) = \sum_{d|m} \mu(d)F^*(m/d) \quad \text{(19)} \]

and

\[ g^*(m) = \sum_{d|m} \mu(d)G^*(m/d), \quad \text{(20)} \]

then they are multiplicative functions. So it is enough to compute the values on prime powers. It is straight forward to check that

\[
f^*(p^k) = \begin{cases} 1, & \text{if } k = 0 \\ 1/(p + 1)(p - 2), & \text{if } k = 1 \\ 0, & \text{else.} \end{cases} \quad \text{(21)}
\]

\[
g^*(p^k) = \begin{cases} 1, & \text{if } k = 0 \\ (p - 1)/p^k(p - 2), & \text{if } k \geq 1 \end{cases} \quad \text{(22)}
\]

for primes \( p > 2 \). Also

\[
f^*(2^k) = \begin{cases} -1/3, & \text{if } k = 1 \\ 0, & \text{if } k \geq 2 \end{cases} \quad \text{(23)}
\]

\[
g^*(2^k) = \begin{cases} 0, & \text{for } k = 1 \\ 1/2^{k-1}, & \text{if } k \geq 2. \end{cases} \quad \text{(24)}
\]
First we shall compute the error terms. In order to do that it is enough to compute \( E_1(x) \), \( E_2(x) \) and \( E_3(x) \) as in Theorem [1].

It is easy to see that \( E_1(x) = O(1) \).

Now

\[
E_2(x) = \sum_{d \leq x} |f^*(d)| \\
\ll \prod_{p \leq x} (1 + f^*(p) + f^*(p^2) + \cdots) \\
\ll \prod_{2 < p \leq x} \left(1 + \frac{1}{(p+1)(p-2)}\right) \\
= O(1).
\]

Also

\[
E_3(x) = \sum_{d_1 \leq x} |g^*(d_1)| \\
\leq \prod_{p \leq x} (1 + g^*(p) + g^*(p^2) + \cdots) \\
\leq \prod_{2 < p \leq x} \left(1 + \frac{1}{p-2}\right) \\
\ll \exp \left( \sum_{2 < p \leq x} \frac{1}{p-2} \right) \\
\ll \log x.
\]

Now only thing that remains is to compute the constant in the main term. To do that, we use the formula of \( C_1 \) from Theorem [1].

To prove (a), we use the expressions of \( f^*(p^k) \) and \( g^*(p^k) \) from (21), (22), (23) and (24).

If \( p \neq 2 \)

\[
1 + \sum_{i=1}^{\infty} \frac{f^*(p^i) + g^*(p^i)}{p^i} = 1 + \frac{1}{(p+1)(p-2)} + \frac{(p-1)/p(p-2)}{p-2} + \frac{1}{p-2} \sum_{i \geq 2} \frac{1}{p^{2i}} \]

\[
= 1 + \frac{1}{p(p+1)(p-2)} + \frac{p-1}{p-2} \frac{1}{p^2 - 1} \\
= \frac{(p-1)^2}{p(p-2)} \\
= \left(1 - \frac{1}{(p-1)^2}\right)^{-1}. \quad (25)
\]

Also

\[
1 + \sum_{i=1}^{\infty} \frac{f^*(2^i) + g^*(2^i)}{2^i} = 1 + \frac{(-1/3)}{2} + \sum_{j \geq 2} \frac{1}{2^{2j-1}} = 1. \quad (26)
\]

Since \( C_2^* = \prod_{p \geq 2} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \) this completes the proof of (a).

To prove (b), we may assume that \( G \) is supported on odd integers only. Hence \( G(2^k) = 0 \) for all \( k \geq 1 \). In that case

\[
g^*(2^k) = \begin{cases} 
-1, & \text{if } k = 1 \\
0, & \text{if } k \geq 2.
\end{cases} \quad (27)
\]
This gives
\[ 1 + \sum_{i=1}^{\infty} \frac{f^*(2^i) + g^*(2^i)}{2^i} = 1 + \frac{(-1/3) + (-1)}{2} = \frac{1}{3}. \]
This proves (b).

4. PROOF OF THEOREM

Recall that
\[ \hat{K}(N) = C^*_2 F^*(N - 1) G^*_1(N), \]
where
\[ F^*(N) = \prod_{p \mid N} \left(1 - \frac{1}{(p-1)^2(p+1)}\right) \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \]
and
\[ G^*_1(N) = \frac{N}{\phi(N)} \prod_{p \mid N} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \prod_{p^a \mid N, p > 2} \left(1 - \frac{1}{p^a(p-1)}\right) \prod_{p^x \mid N, p > 2} \left(1 - \frac{p - \left(\frac{-N_0}{p}\right)}{p^{a+1}(p-1)}\right). \]

We write \( G^*_1(N) = G^*_2(N) G^*_3(N) \), where
\[ G^*_2(N) = \frac{N}{\phi(N)} \prod_{p \mid N} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \prod_{p^a \mid N, p > 2} \left(1 - \frac{1}{p^a(p-1)}\right) \]
and
\[ G^*_3(N) = \prod_{p^a \mid N} \left(1 - \frac{p - \left(\frac{-N_0}{p}\right)}{p^{2a+1}(p-1)}\right). \]

Then \( G^*_2 \) is multiplicative but \( G^*_3 \) is not. Write
\[ G^*_2(N) = \sum_{l \mid N} \hat{g}(l). \]

Then, if \( p \neq 2 \),
\[ \hat{g}(p^k) = \begin{cases} 1, & \text{if } k = 0 \\ \frac{(p-1)^2}{pp^{a-1}(p-2)}, & \text{if } k = 1 \\ \frac{(p-1)^2}{pp^{a-1}(p-2)}, & \text{if } k = 2s, s \geq 1 \\ \frac{(p-1)^2}{pp^{a+1}(p-2)}, & \text{if } k = 2s + 1, s \geq 1 \end{cases} \]
and
\[ \hat{g}(2^k) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k = 1 \\ \frac{1}{2}, & \text{if } k = 2s, s \geq 1 \\ \frac{1}{2}, & \text{if } k = 2s + 1, s \geq 1. \end{cases} \]

Our claim, which is motivated from a similar idea in [15], is that the whole computation of \( \sum_{N \leq x} F^*(N - 1) G^*_1(N) \) remains the same even if we replace \( \left(\frac{-N_0}{p}\right) \) in \( G^*_3(N) \) by its expected value 0 for every the prime
other than 2 and in case of \( p = 2 \), we replace it by 1. To make this rigorous, define

\[
G_4^*(N) = \prod_{p^n \mid N, \quad p > 2} \left( 1 - \frac{1}{p^{2\alpha}(p - 1)} \right) \prod_{2^m \mid N} \left( 1 - \frac{1}{2^{2\alpha + 1}} \right).
\]

For any \( d, l \) with \((d, l) = 1\), we claim that

\[
\sum_{N \leq x} G_3(N) = \sum_{N \leq x} G_4^*(N) + O(1). \tag{31}
\]

To prove that

\[
\sum_{N \leq x} G_3^*(N) = \sum_{N \leq x} \prod_{p^m \mid N, \quad p \neq 2} \left( 1 - \frac{p - \left( \frac{-N}{p} \right)}{p^{2\alpha + 1}(p - 1)} \right).
\]

From now on \( l_1, l_2, l_3 \) are mutually co-prime positive integers. We define the following notations

\[
\psi(l_i) = \prod_{p \mid l_i} p^{B_i}(p - 1),
\]

\[
A(m, l_i) = \prod_{p \mid l_i} \frac{1 - \left( \frac{-m}{p} \right)}{p},
\]

and

\[
l_3' = \prod_{p \mid n} p.
\]

Now if \( \omega(m) \) denote the number of distinct prime divisors of \( m \), then with these notations, (32) is equal to

\[
\sum_{l_1 l_2 l_3 \leq x} (-1)^{\omega(l_3)} \frac{A(l_1 l_2 l_3)}{\psi(l_1^2 l_2^2 l_3)} = \sum_{l_1 l_2 l_3 \leq x} (-1)^{\omega(l_3)} \frac{A(l_1 l_2 l_3)}{\psi(l_1^2 l_2^2 l_3)} \sum_{l_1 l_2 l_3 \leq x} \frac{-l_1}{l_3'} . \tag{33}
\]

Since \((l_1, l_3) = 1\), \((\frac{-l_1}{l_3})\) can be replaced by 1, for \( l_3' = 1, 2 \), in the last summation. Also in case of other \( l_3' \), the condition \((l_1, l_3) = 1\) is taken care of by \((\frac{-l_1}{l_3})\).

Hence (33) can be broken into two parts, namely \( S(x, 1, d) \) and \( E_5(x) \), where

\[
S(x, 1, d) = \sum_{l_1 l_2 \leq x} \frac{(-1)^{\omega(l_2)}}{\psi(l_2^2 l_3)} \sum_{l_1 l_2^2 l_3 \leq x} \frac{(-1)^{\omega(l_2)}}{2 \psi(l_2^2 l_3)} \sum_{l_1 l_2^2 l_3 \leq x} \frac{-l_1}{l_3'}
\]

and

\[
E_5(x) = \sum_{l_1 l_2 l_3 \leq x} \frac{(-1)^{\omega(l_3)}}{\psi(l_1^2 l_2^2 l_3)} \sum_{l_1 l_2^2 l_3 \leq x} \frac{-l_1}{l_3'} .
\]
If we rewrite $G_4^*$ as

$$G_4^*(n) = \prod_{p \neq 2} \left(1 - \frac{1}{p^{2\alpha}(p - 1)}\right) \prod_{2^a || N} \left(1 - \frac{1}{2^{2\alpha} + \frac{1}{2^{2\alpha+1}}}\right),$$

then it is easy to check that

$$\sum_{N \leq x \atop N \equiv 0 \pmod{d}} G_4^*(N) = S(x, l, d).$$

For $E_5$, note that the congruence relations in the last summation has no solution unless $(l_2 l_3, d) = 1$. So if solutions exists, then there exists $a_1, a_2, \ldots, a_{\phi(l_2)}$, such that the congruence conditions along with the condition $(l_1, l_2) = 1$ is equivalent to any one of the following

$$l_1 \equiv a_i \pmod{M_{d, l_1, l_2, l_3}}, \quad i = 1, 2, \ldots, \phi(l_2)$$

with $(M_{d, l_1, l_2, l_3}, l_3') = 1$.

Then for each fixed $a_i$, the set $\{a_i, a_i + M_{d, l_1, l_2, l_3}, a_i + 2M_{d, l_1, l_2, l_3}, \ldots, a_i + (l_3' - 1)M_{d, l_1, l_2, l_3}\}$ runs over all possible residue class module $l_3'$ exactly once. Hence using the fact that

$$\sum_{a = 1}^{l_3'} \left(\frac{a}{l_3'}\right) = 0 \quad \text{for } l_3' \geq 3,$

we get

$$E_5(x) = \sum_{l_3 \leq \sqrt{x}} (-1)^{\phi(l_3)} l_3^{\psi(l_3)} \left[0 + O(\phi(l_3)l_3')\right]
= O\left( \sum_{l_3 \leq \sqrt{x}} \frac{l_2}{\psi(l_3^2)} \right)
= O\left( \sum_{l_3 \leq \sqrt{x}} \frac{l_2}{\psi(l_3^3)} \right)
= O\left( \sum_{l_3 \leq \sqrt{x}} \frac{1}{\psi(l_3)} \right)
= O(1).$$

Which proves the claim.

Now with these notations, where $f^*(d)$ is as in (19), we have

$$\sum_{N \leq x} F^*(N - 1)G_1^*(N) = \sum_{N \leq x} G_1^*(N) \sum_{d|N-1} f^*(d)
= \sum_{d \leq x-1} f^*(d) \sum_{N \leq x \atop N \equiv 0 \pmod{d}} G_1^*(N)
= \sum_{d \leq x-1} f^*(d) \sum_{N \leq x \atop N \equiv 0 \pmod{d}} G_2^*(N)G_3^*(N)
= \sum_{d \leq x-1} f^*(d) \sum_{N \leq x \atop N \equiv 0 \pmod{d}} G_3^*(N) \sum_{l|N} \hat{g}(l)
= \sum_{d \leq x-1} f^*(d) \sum_{l \leq x \atop (l,d)=1} \hat{g}(l) \sum_{N \leq x \atop N \equiv 0 \pmod{d}} G_3^*(N).$$
Now, using (31) we get
\[
\sum_{N \leq x} F^*(N-1)G_1^*(N) = \sum_{d \leq x-1} f^*(d) \sum_{l \leq x} \hat{g}(l) \sum_{N \leq x \atop N \equiv (l,d) \equiv 1} G_4^*(N) + O(\sum_{d \leq x-1} |f^*(d)| \sum_{l \leq x} |\hat{g}(l)|)
\]
\[= \sum_{d \leq x-1} f^*(d) \sum_{N \leq x \atop N \equiv (l,d) \equiv 1} G_2^*(N)G_4^*(N) + O(\log x)
\]
\[= \sum_{N \leq x} F^*(N-1)G_2^*(N)G_4^*(N) + O(\log x). \quad (34)
\]

To compute the main term, note that if \( G_2^*(N)G_4^*(N) = \sum_{l|N} g_1^*(l) \), then
\[g_1^*(p^k) = \begin{cases} 1, & \text{if } k = 0 \\ \frac{(p-1)}{p^k(p-2)}, & \text{if } k \geq 1 \end{cases}
\]
and
\[g_1^*(2^k) = \begin{cases} 0, & \text{if } k = 2s-1, s \geq 1 \\ \frac{1}{2^k}, & \text{if } k = 2s, s \geq 1. \end{cases}
\]

So in order to compute the constant in the main term it is enough to compute \((1 + \frac{f^*(2^2)+g_1^*(2^2)}{2} + f^*(2^2)+g_1^*(2^2) + \cdots)\), because other factors corresponding to the primes \( p(\neq 2) \) cancels out with the constant \( C_2^* = \prod_{p \geq 2} \left(1 - \frac{1}{(p-1)^2}\right)\).

Now
\[ (1 + \frac{f^*(2)}{2} + \frac{f^*(2^2)+g_1^*(2^2)}{2} + \cdots) = (1 + \frac{-1/3}{2} + \frac{3/2^2}{2^2} + \frac{3/2^4}{2^4} + \frac{3/2^6}{2^6} + \cdots) = (1 - \frac{1}{6} + \frac{1}{5}) = \frac{31}{30}. \]

Remark 3. In [16], Mirsky gave a proof of part (a) of Corollary 1. In that same paper he discussed how to approach the problem of k-fold product. More precisely summation of the form \( \sum_{n \leq x} f_1(n-h_1)f_2(n-h_2) \cdots f_k(n-h_k) \), where each of \( f_i \) are multiplicative.

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