Some remarks on analytical solutions for a damped quantum parametric oscillator

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Abstract. The time-dependent Schrödinger equation for quadratic Hamiltonians has Gaussian wave packets as exact solutions. For the parametric oscillator with frequency $\omega(t)$, the width of these wave packets must be time-dependent. This time-dependence can be determined by solving a complex nonlinear Riccati equation or an equivalent real nonlinear Ermakov equation. All quantum dynamical properties of the system can easily be constructed from these solutions, e.g., uncertainties of position and momentum, their correlations, ground state energies, etc. In addition, the link to the corresponding classical dynamics is supplied by linearizing the Riccati equation to a complex Newtonian equation, actually representing two equations of the same kind: one for the real and one for the imaginary part. If the solution of one part is known, the missing (linear independent) solution of the other can be obtained via a conservation law for the motion in the complex plane. Knowing these two solutions, the solution of the Ermakov equation can be determined immediately plus the explicit expressions for all the quantum dynamical properties mentioned above. The effect of a dissipative, linear velocity dependent friction force on these systems is discussed.

1. Introduction
A parametric oscillator, i.e., an oscillator with time-dependent (TD) frequency $\omega(t)$ is a non-trivial problem even in classical mechanics. Whether or not there are analytical solutions strongly depends on the time-dependence of the frequency. However, in most cases, at least a numerical solution should be possible.

Extending the problem to quantum dynamics adds an additional aspect, the dynamics of the typical quantum fluctuations, i.e., the uncertainties of position, momentum and their correlation.

For the harmonic oscillator (HO) (or any other at-most-quadratic Hamiltonian operator), exact analytic solutions in the form of Gaussian wave packets (WPs) exist that are uniquely determined by the maximum and the width of the Gaussian function. The maximum follows the classical trajectory and so its time-evolution is determined by the solution of the corresponding Newtonian equation of motion. This also applies to the parametric oscillator.

The dynamics of the WP-width is determined by a complex nonlinear Riccati equation and is proportional to the time-dependence of the position uncertainty. Once this is known, also the momentum uncertainty and the correlation between these uncertainties can be determined explicitly. In principle, this is possible not only for the HO, but also for the parametric oscillator.

For the HO it can be shown that the solution for the WP-width and, consequently, the solution of the complex Riccati equation, can also be obtained by solving the equation for the
WP-maximum.

In Section 2, the relationship between the classical equations of motion and the dynamics of the WP-width will be given in an overview.

In Section 3, analytical solutions for the classical parametric oscillator with frequency inversely proportional to time are examined, including an extension that incorporates the quantum mechanical uncertainty aspect.

The extension to effective descriptions of dissipative systems and the consequences for the afore-mentioned relations are discussed in Section 4.

A generalization from the damped HO to the damped parametric oscillator is considered and specific differences and corresponding difficulties are pointed out in Section 5.

Finally, a summary and perspectives of future developments are given.

2. Interrelations between the classical and quantum aspects of the Gaussian wave packet dynamics

For Hamiltonians that are at most quadratic in position and momentum variables, the Schrödinger equation (SE) possesses exact analytical solutions in the form of Gaussian WPs. In one dimension, to which the following discussion will be restricted, such a solution can be written in the form \[ \Psi_{WP}(x,t) = N(t) \exp \left\{ i \left[ y(t) \tilde{x}^2 + \langle p \rangle \bar{h} \tilde{x} + K(t) \right] \right\} \] (1)

with the purely TD normalization factor \( N(t) \) (and the TD function \( K(t) \) that will not be relevant to the following discussion). The variable \( \tilde{x} = x - \langle x \rangle = x - \eta(t) \) is so defined that the maximum of the Gaussian function is at \( \langle x \rangle \), the mean value of position, that (according to Ehrenfest) coincides with the classical trajectory \( \eta(t) \) and is calculated via \( \langle x \rangle = \int_{-\infty}^{+\infty} dx \Psi^* x \Psi = \eta(t) \); similarly, \( \langle p \rangle = m \dot{\eta} \) is the mean value of momentum. Important for the quantum mechanical aspect of the system’s dynamics is the generally complex coefficient \( y(t) \) of \( \tilde{x}^2 \) in the exponent, \( y(t) = y_R(t) + iy_I(t) \). The imaginary part is related to the position uncertainty via \( y_I = \frac{1}{4\langle \tilde{x}^2 \rangle} \) with the mean square deviation of position \( \langle \tilde{x}^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 \). The square root of \( \langle \tilde{x}^2 \rangle \) essentially determines the width of the Gaussian function. This function is completely determined by two parameters, the width and the maximum of the bell-shaped curve. In this quantum mechanical context, both parameters can be TD. The equations of motion that determine this time-dependence can be obtained by inserting the WP(1) into the TDSE,

\[ i\hbar \frac{\partial}{\partial t} \Psi_{WP}(x,t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) \right\} \Psi_{WP}(x,t), \] (2)

in the following for \( V(x,t) = \frac{m}{2} \omega^2 x^2 \) with the general case \( \omega = \omega(t) \) and the particular cases \( \omega = \omega_0 = const. \) (HO) and \( \omega = 0 \) (free motion). For the motion of the maximum one obtains, as expected, the Newtonian equation of motion (divided by mass \( m \))

\[ \dot{\eta} + \omega^2(t) \eta = 0. \] (3)

Defining the complex variable \( C(t) = \frac{2\hbar}{m} y(t) \), the terms proportional to \( \tilde{x}^2 \) yield the complex nonlinear Riccati equation

\[ \dot{C} + C^2 + \omega^2(t) = 0 \] (4)

determines the time-evolution of the WP-width, as \( C_1 = \frac{2\hbar}{m} y_1 = \frac{\hbar}{2m\langle \tilde{x}^2 \rangle} \). Therefore, in order to obtain the information about the typical quantum mechanical aspects of the system’s
dynamics, this Riccati equation must be solved. Possible ways of doing this have been discussed, e.g., in [1, 2, 3, 4, 5].

For our discussion, two possible treatments of the Riccati equation (4) will be of relevance. First, one can introduce a new variable $\alpha(t)$ via the definition of $C_I(t)$ and consequently, from the imaginary part of Eq. (4), directly follows the real part $C_R(t)$ as

$$C_I(t) = \frac{1}{\alpha^2}, \quad C_R = \frac{\dot{\alpha}}{\alpha}.$$  \hspace{1cm} (5)

According to $\alpha = \sqrt{\frac{2m}{\hbar}} \langle \tilde{x}^2 \rangle$, $\alpha(t)$ is directly proportional to the WP-width; the real part $C_R$ vanishes only for constant width, i.e., $\dot{\alpha} = 0$. In all other cases, particularly for the parametric oscillator, the Riccati variable $C(t)$ is complex. From the real part of Eq. (4), one finally obtains the equation for $\alpha(t)$,

$$\ddot{\alpha} + \omega^2(t) \alpha = \frac{1}{\alpha^3},$$ \hspace{1cm} (6)

known in the literature as Ermakov equation [6].

In a second treatment, the Riccati equation (4) can be linearized using the logarithmic derivative of a new complex variable $\lambda(t)$ via $C = \frac{1}{\lambda}$ to yield the complex Newtonian equation

$$\ddot{\lambda} + \omega^2(t) \lambda = 0$$  \hspace{1cm} (7)

with $\lambda(t) = u + i z = \alpha e^{i\varphi}$.

Real and imaginary parts, or phase and amplitude, of the complex quantity $\lambda$ are not independent of each other but coupled via the conservation law

$$\dot{z}u - \dot{u}z = \alpha^2 \varphi = 1$$ \hspace{1cm} (8)

which is equivalent to the Wronskian determinant of two linear independent solutions $u$ and $z$ of the complex Newtonian equation (7).

Knowing one solution (e.g., $z(t)$), it is possible to calculate the other via

$$u = -z \int^t \frac{1}{z^2(t)} \, dt'.$$  \hspace{1cm} (9)

Knowing both solutions of the classical Newtonian equation immediately provides the solution for the dynamics of the WP width via $\alpha^2 = u^2 + z^2$. If $\alpha$ is known, all typical quantum mechanical properties like uncertainties and ground state energies can be determined as they can be expressed in terms of this quantity in the form

$$\langle \tilde{x}^2 \rangle = \frac{\hbar}{2m} \alpha^2, \quad \langle \tilde{p}^2 \rangle = \frac{m \hbar}{2} \left[ \alpha^2 + \frac{1}{\alpha^2} \right], \quad \langle [\tilde{x}, \tilde{p}]_+ \rangle = \hbar \alpha \dot{\alpha} ,$$

$$U = \langle \tilde{x}^2 \rangle \langle \tilde{p}^2 \rangle = \frac{\hbar^2}{4} \left( 1 + (\alpha \dot{\alpha})^2 \right), \quad \dot{E} = \frac{\langle \tilde{p}^2 \rangle}{2m} + \frac{m \omega^2}{2} \langle \tilde{p}^2 \rangle = \frac{\hbar}{4} \left( \alpha^2 + \frac{1}{\alpha^2} + \omega^2 \alpha^2 \right).$$ \hspace{1cm} (10)

Further, with the help of the TD Green function (also called Feynman kernel), it can be shown that the imaginary part of $\lambda(t)$ is (up to a constant factor) identical to the classical trajectory, i.e.,

$$z(t) = \frac{m}{\alpha_0 p_0} \eta(t)$$  \hspace{1cm} (11)
with the initial WP-width $\alpha_0$ and the initial classical momentum $p_0 = m\dot{\eta}(t_0)$ (for details, see e.g. [1, 2, 3, 4, 5]).

Therefore, with the solutions of the classical Newtonian equation (analytical or numerical) it is possible to construct the solution of the Ermakov equation that describes the dynamics of the uncertainties and thus the quantum mechanical aspect. In other words, if the dynamics of the WP-maximum is known, the dynamics of the WP-width (for given initial width $\alpha_0$) can be determined in the way shown above.

3. Analytical solution for a parametric oscillator with frequency inversely proportional to time

Comparing the WP with the Gaussian solution of the diffusion equation it has been shown in [7] that an analytical solution of the Newtonian equation

$$\ddot{z} + \omega^2(t) z = 0 \quad (12)$$

can be found for

$$\omega(t) = \frac{1}{2 (t + b)} \quad (13)$$

with $b = \text{constant} \neq 0$, having the dimension “time”.

In this case, the two linear-independent solutions of the (complex) Newtonian equation (7) are

$$z = (t + b)^{\frac{3}{2}} \quad \rightarrow \quad u = -(t + b)^{\frac{3}{2}} \ln(t + b), \quad (14)$$

and, consequently, the solution of the Ermakov equation is obtained from

$$\alpha^2(t) = (t + b) \left[ 1 + \ln^2(t + b) \right]. \quad (15)$$

Using relations (10), the analytic form of the quantum mechanical properties can be given explicitly as

$$\langle \tilde{x}^2 \rangle = \frac{\hbar^2}{2m} (t + b) \left[ 1 + \ln^2(t + b) \right], \quad \langle \tilde{p}^2 \rangle = \frac{\hbar m}{2} \left\{ \frac{1}{4} \left[ 1 + \ln(t + b) \right]^2 + \frac{1}{2} \ln(t + b) + 1 \right\},$$

$$\langle [\tilde{x}, \tilde{p}]_+ \rangle = \frac{\hbar^2}{2} \left[ 1 + \ln(t + b) \right]^2, \quad U(t) = \frac{\hbar^2}{4} \left\{ 1 + \frac{1}{4} \left[ 1 + \ln(t + b) \right]^4 \right\},$$

$$\tilde{E} = \frac{\hbar^2}{4} \left\{ \frac{1}{(t + b)} \left[ \frac{1}{2} \left( 1 + \ln(t + b) \right)^2 + 1 \right] \right\}. \quad (16)$$

Obviously, the position uncertainty and thus the WP width are diverging for $t \to \infty$, as in the case of the free motion, which is reasonable as in this limit $\omega(t)$, and thus the potential, vanishes. The momentum uncertainty decreases and approaches zero, whereas the uncertainty product is going to infinity for $t \to \infty$. The energy contribution $\tilde{E}(t)$ is decreasing and approaches zero, similar to a (classical) dissipative system (without fluctuating force).

According to (11), $z(t)$ is proportional to the classical trajectory $\eta(t)$. Therefore, using Eq. (12), the solution of the classical Newtonian equation (3) for the frequency (13) and consequently the corresponding velocity can immediately be given in the form

$$\eta(t) = v_0 \tau^{\frac{1}{2}} (t + b)^{\frac{1}{2}}, \quad \dot{\eta}(t) = \frac{1}{2} v_0 \tau^{\frac{1}{2}} (t + b)^{-\frac{1}{2}} = v_0 \frac{\tau^{\frac{1}{2}}}{2 (t + b)^{\frac{1}{2}}} \quad (17)$$

with $\eta_0 = v_0 \tau^{\frac{1}{2}} b^{\frac{1}{2}}$ and $\dot{\eta}_0 = v_0 \frac{\tau^{\frac{1}{2}}}{2 b^{\frac{1}{2}}}$ where the constant $\tau$ with dimension time has been introduced to get the correct dimension for $\eta(t)$. From $\dot{\eta}_0 = v_0$ follows $b = \frac{\tau}{4}$. 

The classical energy of the system then acquires the form

$$E = T + V = \frac{m}{2} \dot{\eta}^2 + \frac{m}{2} \left( \frac{1}{2(t+b)} \right)^2 \eta^2 = \frac{m}{2} v_0^2 \frac{\tau}{2(t+b)} \propto \frac{1}{t} \quad (18)$$

and decreases inversely proportional to time $t$. The initial energy $E_0$ is given by $E_0 = \frac{m}{2} v_0^2 \frac{\tau}{2b} = m v_0^2$, the final energy for $t \to \infty$ approaches zero, $E_\infty \to 0$, like in a dissipative system.

To compare this parametric oscillator with decreasing energy with the damped HO with a friction force linear proportional to velocity, obeying the equation of motion

$$\ddot{\eta} + \gamma \dot{\eta} + \omega_0^2 \eta = 0 \quad (19)$$

with friction coefficient $\gamma$ and constant frequency $\omega_0$, the decrease in energy of both systems is compared.

For the parametric oscillator this decrease is given by

$$\dot{E} = -\frac{m}{2} v_0^2 \frac{\tau}{2} \left( \frac{1}{t+b} \right)^2 = -2 \omega(t) E = -4 \omega(t) T < 0 \quad , (20)$$

for the damped HO it is

$$\dot{E} = -2 \gamma T = - \frac{\gamma}{2} T \quad . (21)$$

So, the TD frequency would correspond to half of the friction parameter $\gamma$, $\omega(t) \approx \frac{\gamma}{2}$.

It can also be shown (details will be provided in a forthcoming work) that analytical solutions are possible for a more general parametric oscillator, namely one with the frequency

$$\omega(t) = \frac{1}{a(t+b)} \quad . (22)$$

However, qualitatively different behaviour of dynamics and energetics is obtained for different values of $a$. As for the damped HO, three different regimes can be distinguished. The case of $a = 2$ (discussed above) corresponds to the aperiodic limit, i.e., $\omega_0 = \frac{\gamma}{2}$; the case $a > 2$ to overcritical damping, i.e., $\omega_0 > \frac{\gamma}{2}$; and the case $a < 2$ to undercritical damping, $\omega_0 < \frac{\gamma}{2}$. Only in the last case the damped HO, as well as the corresponding parametric oscillator provide oscillatory solutions.

Looking into the combined effect of TD frequency and damping force on the WP dynamics should be interesting. For this purpose, different approaches for the description of dissipative quantum systems will be presented subsequently that lead to the same equations of motion for the quantum mechanical properties of interest.

4. Effective descriptions of dissipative quantum systems

There are several approaches in the literature to describe open quantum systems in an effective way, i.e., without taking into account the environmental degrees of freedom explicitly, but only include the effect of the environment via modification of the TDSE. Essentially two different types of approaches can be distinguished that shall be called canonical and physical.

In the canonical methods (based on modifications of classical Hamiltonian mechanics) the corresponding Hamiltonian operator (after canonical quantization) might be explicitly TD, but the resulting TDSE still remains linear. However, the connection between the canonical variables of these open Hamiltonian systems and the physical position and momentum is given classically by a non-canonical transformation, corresponding to a non-unitary transformation [8] for the respective WP-solution of the SE (but still allowing normalization of the WPs).
The physical methods keep the definitions of position- and momentum-operators unchanged, as in conventional quantum mechanics, but modify the Hamiltonian operator in a way that, according to Ehrenfest’s theorem, the equation of motion for the mean value of position contains a linear velocity-dependent friction force as given in Eq. (19). The price to be paid is the occurrence of nonlinear terms in the corresponding Hamiltonian operators. Another similar approach [9], changing the reversible continuity equation for the probability density $\varrho = \Psi^* \Psi$ to an irreversible Fokker–Planck-type equation with diffusion term, also leads, via separation, to a NLSE.

Four of these methods are now presented that can be transformed into each other in a unique way and all possess Gaussian WP solutions of the corresponding TDSE where the maximum and width of these WPs obey the same set of equations of motion.

On the canonical level (where canonical quantities are characterized by a hat) the approach of Caldirola [10] and Kanai [11] leads to a TDSE with explicitly TD Hamiltonian (where $\hat{x} = x$ is still the physical position variable)

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(x,t) = \left\{ -\frac{\hbar^2}{2m} e^{-\gamma t} \frac{\partial^2}{\partial x^2} + \frac{m}{2} \omega^2 x^2 e^{\gamma t} \right\} \hat{\Psi}(x,t). \quad (23)$$

This time-dependence of the Hamiltonian can be eliminated by changing to an exponentially expanding position variable via $\hat{Q} = x e^{\gamma t}$, leading to the TDSE

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\hat{Q},t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \hat{Q}^2} + \frac{m}{2} \Omega^2 \hat{Q}^2 \right\} \hat{\Psi}(\hat{Q},t) \quad (24)$$

with $\Omega^2 = \omega^2 - \frac{\gamma^2}{4}$, looking like an undamped HO with frequency $\Omega$. However, to avoid unphysical results, the important point is the difference between the canonical wave function $\hat{\Psi}$ and the physical one, $\Psi$. The connection between both is given by the non-unitary transformation [8]

$$\ln \hat{\Psi} = e^{\gamma t} \ln \Psi. \quad (25)$$

On the physical level, based on Ehrenfest’s theorem, a modification of Hasse’s NLSE [12] yields

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \left\{ H_L + \frac{\gamma}{4} \left( [\hat{x}, p + \langle p \rangle]_+ - [\hat{x}, p + \langle p \rangle]_+ \right) \right\} \Psi(x,t) \quad (26)$$

where $H_L$ is the conventional Hamiltonian of the the linear theory, while the TDSE (26) is nonlinear due to the occurrence of the mean values $\langle \hat{p} \rangle$ and $\langle \hat{x} \rangle$ (in $\hat{x} = x - \langle x \rangle$).

Finally, via separation of a Fokker–Planck-type equation for the probability density [5, 9, 13], one obtains the logarithmic NLSE

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \left\{ H_L + \gamma \frac{\hbar}{4} (\ln \Psi - \langle \ln \Psi \rangle) \right\} \Psi(x,t). \quad (27)$$

In all cases, the two parameters of the Gaussian WP solution obey the equations of motion

$$\text{Maximum : } \ddot{\eta} + \gamma \dot{\eta} + \omega^2(t)\eta = 0, \quad \text{Width : } \ddot{\alpha} + \left( \omega^2(t) - \frac{\gamma^2}{4} \right) \alpha = \frac{1}{\alpha^3}, \quad (28)$$

and, expressed in terms of the expanding coordinates, the maximum obeys

$$\ddot{\hat{Q}} + \left( \omega^2(t) - \frac{\gamma^2}{4} \right) \hat{Q} = 0 \quad (29)$$
with $\eta(t) = \hat{Q} \ e^{-\hat{z}t}$, i.e., the same as for the conservative HO, only $\omega(t)$ is replaced by $
abla(t) = \left( \omega^2(t) - \frac{\gamma^2}{4} \right)^{\frac{1}{2}}$.

The complex Riccati equation corresponding to the Ermakov equation in (28) is $\hat{C} + \gamma \hat{C} + C^2 + \omega^2(t)\hat{C} = 0$ with $C_1 = \frac{1}{\omega}$, $C_2 = \frac{1}{\omega} - \frac{\gamma}{2}$. Via $C = \frac{1}{\lambda}$, this equation can be linearized to $\dot{\lambda} + \gamma \dot{\lambda} + \omega^2(t)\lambda = 0$ with $\lambda = e^{-\hat{z}t}\lambda$, in particular $\hat{z} = e^{-\hat{z}t}z \propto \eta(t)$.

5. Damped parametric oscillator
Considering the damped HO obeying Eqs. (28) and (29), it appears that the transition from the undamped system to the damped one can simply be achieved by:

1) replacing $\omega(t)$ in $\eta$ by $\Omega(t) = \sqrt{\omega^2(t) - \frac{\gamma^2}{4}} \rightarrow \hat{Q}(t)$;
2) multiplying $\hat{Q}(t)$ by $e^{-\hat{z}t}$ to dissipative trajectory $\hat{Q}_{\text{dis}}$;
3) dissipative velocity: $\dot{\eta}_{\text{dis}} \rightarrow$ classical dissipation of energy;
4) $\hat{Q} \propto z \rightarrow u$ in analogy with (9), leading to $\alpha^2 = u^2 + z^2 \rightarrow$ quantum mechanical properties.

However, this procedure is too naive, because $\eta$ changes qualitatively as time proceeds. This is caused by the time-dependence of

$$\Omega^2(t) = \left( \frac{1}{4(t + b)^2} - \frac{\gamma^2}{4} \right) \quad (30)$$

that is positive for $t < \frac{1}{\gamma} - b$ (like for the undercritically damped HO), vanishes for $t = \frac{1}{\gamma} - b$ (like for the aperiodic limit) and becomes negative for $t > \frac{1}{\gamma} - b$ (like for overcritical damping).

That means, the attractive potential in Eq. (29) turns into a repulsive one for $t > \frac{1}{\gamma} - b$, thus changing the solution of Eq. (29) qualitatively and likewise the behaviour of $\eta(t)$ at and after this time.

Therefore, the proper procedure is to solve directly the equation of motion for the expanding coordinate $\hat{Q}$,

$$\frac{d^2}{dt^2} \hat{Q} + \left( \frac{\gamma^2}{4} + \frac{1}{4(t + b)^2} \right) \hat{Q} = 0 \quad (31)$$

which is formally equivalent to Whittaker’s equation

$$\frac{d}{dt^2} W + \left( -\frac{1}{4} + \frac{k}{t} + \frac{1}{4} - \frac{m^2}{i^2} \right) W = 0 \quad (32)$$

for particular choices of parameters $k, m$ (e.g., $k = 0$ leads to Bessel functions).

Knowing the solution of (31) enables one to obtain the classical trajectory of the system via $\eta_{\text{dis}}(t) = \hat{Q} e^{-\hat{z}t}$. However, the particular form of the solution strongly depends on the parameters. If one solution $\hat{Q}_1$ is known, the second linear independent one, $\hat{Q}_2$, can be obtained using $\hat{Q}_1 \dot{\hat{Q}}_2 - \hat{Q}_2 \dot{\hat{Q}}_1 = 1$ in analogy with (9). With both solutions, $\alpha^2 = \hat{Q}_1^2 + \hat{Q}_2^2$ is known and thus all quantum mechanical properties of the system can be determined via

$$\langle \hat{x}^2 \rangle(t) = \frac{h}{2m} \alpha^2, \ \langle \hat{p}^2 \rangle(t) = \frac{m \hbar}{2} \left[ \left( \alpha - \frac{\gamma}{2} \right)^2 + \frac{1}{\alpha^2} \right], \ \langle \hat{x}, \hat{p} \rangle(t) = \hbar \left( \alpha \ \hat{\alpha} - \frac{\gamma}{2} \alpha^2 \right),$$

$$U = \frac{\hbar^2}{4} \left( 1 + \left( \alpha \ \hat{\alpha} - \frac{\gamma}{2} \alpha^2 \right)^2 \right), \ \tilde{E} = \frac{\hbar^2}{4} \left( \left( \alpha - \frac{\gamma}{2} \right)^2 + \frac{1}{\alpha^2} + \omega^2(t) \alpha^2 \right). \quad (33)$$

Alternatively, $\alpha(t)$, and thus all quantities in (33), can also be obtained by solving the Ermakov equation (6) for frequency (30).
6. Conclusions
The TDSE for potentials that are at most quadratic in position variable possesses exact solutions that have the form of Gaussian WPs. These functions are uniquely determined by two parameters, the maximum and the width. In this case, both parameters are TD. The time-dependence of the maximum obeys the classical Newtonian equation of motion for the given potential; the time-evolution of the width is determined by a complex nonlinear Riccati equation or, equivalently, by a real nonlinear Ermakov equation. For TD frequency \( \omega = \omega(t) \), it is the explicit time-dependence of \( \omega(t) \) that determines whether analytic solutions exist or not.

Independent of the particular \( \omega(t) \), the solutions of the Riccati or Ermakov equations provide the entire information about the quantum dynamics of the system.

Furthermore, the complex Riccati equation can also be linearized to a complex Newtonian equation where real and imaginary parts of the complex variable \( \lambda(t) \) fulfilling this equation are two linear-independent solutions of the second-order Newtonian differential equation. However, these two solutions are coupled via a kind of conservation law. Essentially, the solution of the classical Newtonian equation makes it possible to obtain a second linear-independent solution with both providing the solution of the Ermakov equation and thus all information about the quantum aspect.

The TD frequency \( \omega(t) = \frac{1}{2(t+b)} \) enables analytical solutions of the WP maximum and width. Comparison of the (classical) energy decrease of this oscillator with that of the damped HO shows similarity with the aperiodic limit in the damped case. A more general frequency, \( \omega(t) = \frac{1}{a(t+b)} \), also allowing analytical solutions, resembles for \( a < 2 \) the undercritical and for \( a > 2 \) the overcritical damped HO. A first attempt at solving the classical equation (29) by simply comparing it with the equations describing the damped HO fails because the damped frequency \( \Omega(t) \) changes sign as time passes, thus changing the potential from an attractive to a repulsive one. Finally, a Whittaker-type equation must be solved that depends on the values of the occurring parameters. Explicit cases shall be investigated in future work.

References
[1] Schuch D 1989 Connections between Newton- and Schrödinger-type equations in the description of reversible and irreversible dynamics. Int. J. Quant. Chem., Quant. Chem. Symp. 36 S23 59-72
[2] Schuch D and Moshinsky M 2006 Connection between quantum-mechanical and classical time evolution via a dynamical invariant. Phys. Rev. A 73 063111
[3] Schuch D 2008 Riccati and Ermakov Equations in Time-Dependent and Time-Independent Quantum Systems. SIGMA 4 043
[4] Cruz H, Schuch D, Castaños O and Rosas-Ortiz O 2015 Time-evolution of quantum systems via a complex nonlinear Riccati equation I. Conservative systems with time-independent Hamiltonians. Ann. Physics 360 44-60
[5] Schuch D 2018 Quantum Theory from a Nonlinear Perspective – Riccati Equations in Fundamental Physics (Heidelberg: Springer)
[6] Ermakov V P 1880 Second-Order Differential Equations: Conditions of Complete Integrability. Univ. Jev. Kiev 20 No 9 1-25; Harin A O (English translation) 2008 Appl. Anal. Discrete Math. 2 123-145
[7] Schuch D 2018 Analytical solutions for the quantum parametric oscillator from corresponding classical dynamics via a complex Riccati equation J. Phys.: Conf. Ser. 1071 012020
[8] Schuch D 1997 Nonunitary connection between explicitly time-dependent and nonlinear approaches for the description of dissipative quantum systems. Phys. Rev. A 55 935
[9] Schuch D, Chung K-M and Hartmann H 1983 Nonlinear Schrödinger-type field equation for the description of dissipative systems. I. Derivation of the nonlinear field equation and one-dimensional example. J. Math. Phys. 24 1652
[10] Caldirola P 1941 Forze non conservative nella meccanica quantistica. Nuovo Cimento 18 393-400
[11] Kanai E 1948 On the Quantization of the Dissipative Systems. Progr. Theor. Phys. 3 440-442
[12] Hasse R W 1975 On the quantum mechanical treatment of dissipative systems. J. Math. Phys. 16 2005
[13] Cruz H, Schuch D, Castaños O and Rosas-Ortiz O 2016 Time-evolution of quantum systems via a complex nonlinear Riccati equation. II. Dissipative systems. Ann. Physics 375 609-630