EXTREME EIGENVALUES OF LARGE DIMENSIONAL QUATERNION SAMPLE COVARIANCE MATRIX

HUIQIN LI, ZHIDONG BAI

ABSTRACT. In this paper, we shall investigate the almost sure limits of the largest and smallest eigenvalues of a quaternion sample covariance matrix. Suppose that $X_n$ is a $p \times n$ matrix whose elements are independent quaternion variables with mean zero, variance 1 and uniformly bounded fourth moments. Denote $S_n = \frac{1}{n} X_n X_n^*$. In this paper, we shall show that $s_{\text{max}}(S_n) = s_p(S_n) \rightarrow (1 + \sqrt{y})^2$, a.s. and $s_{\text{min}}(S_n) \rightarrow (1 - \sqrt{y})^2$, a.s. as $n \rightarrow \infty$, where $y = \lim p/n$, $s_1(S_n) \leq \cdots \leq s_p(S_n)$ are the eigenvalues of $S_n$, $s_{\text{min}}(S_n) = s_{p-n+1}(S_n)$ when $p > n$ and $s_{\text{min}}(S_n) = s_1(S_n)$ when $p \leq n$. We also prove that the set of conditions are necessary for $s_{\text{max}}(S_n) \rightarrow (1 + \sqrt{y})^2$, a.s. when the entries of $X_n$ are i. i. d.

Keywords: Extreme eigenvalues, Large dimension, Quaternion matrices, Random matrix theory, Sample covariance matrix.

1. Introduction.

Let $A$ be a $p \times p$ Hermitian matrix with eigenvalues $s_j(A), j = 1, 2, \cdots, p$ arranged ascendedly, i.e., $s_1(A) \leq \cdots \leq s_p(A)$. Then the empirical spectral distribution (ESD) of the matrix $A$ is defined by

$$F^A(x) = \frac{1}{p} \max \{j : s_j(A) \leq x\}.$$ 

If there is a sequence of random matrices whose ESD weakly converges to a limit, then the limit is said to be the LSD (Limiting Spectral Distribution) of the sequence of random matrices.

Eigenvalues of random matrix are often used in multivariate statistical analysis, such as the principal component analysis, multiple discriminant analysis, and canonical correlation analysis, etc. For example, many important statistics in multivariate statistical analysis are constructed by the eigenvalues of sample covariance matrices or those of multivariate $F$ matrices. Moreover,
they can be written as functions of integrals with respect to the ESD of sample covariance matrices or multivariate $F$ matrices. When LSD is known, the corresponding functionals with respect to the LSD can be viewed as the population parameters and those respect to the ESD can be considered as the parameter estimators. Therefore, one may want to apply the Helly-Bray theorem to find the approximation of the statistics to their estimand. Unfortunately, the integrands are usually unbounded which leads to the failure of the application of the Helly-Bray theorem. Thus the limiting behavior of the extreme eigenvalues of sample covariance matrices or multivariate $F$ matrices is of special interest.

When the underlying random variables are real and/or complex, intensive work has been done in the literature (see [10] [17] [5] [14] [2] [3] [7], among others). It is well known that the ESD of a sample covariance matrix $W_n = \frac{1}{n}Y_nY_n^*$ (the entries of $Y_n = (y_{ij})_{p \times n}$ are i.i.d. real random variables with mean zero and variance $\sigma^2$) converges to the M-P (Marčenko-Pastur) law $F_y(x)$ with density

$$f_y(x) = \frac{1}{2\pi xy\sigma^2}\sqrt{(b-x)(x-a)}I_{[a,b]}(x) + I_{(1,\infty)}(y)(1-y^{-1})\delta(x)$$

where $a = \sigma^2(1 - \sqrt{y})^2$, $b = \sigma^2(1 + \sqrt{y})^2$ and $y = \lim p/n \in (0, \infty)$. Here $\delta(x)$ denotes the Dirac delta function and $I_{[a,b]}(x)$ denotes the indicator function of the interval $[a,b]$. Denote the eigenvalues of $W_n$ by $s_1(W_n), \ldots, s_p(W_n)$, arranged in ascending order. For the convergence of $s_p(W_n)$, Yin, Bai and Krishnaiah (1988) [17] proved that $s_p(W_n) \to \sigma^2(1 + \sqrt{y})^2$, a.s. under the condition that

$$E|y_{11}|^4 < \infty.$$  

Moreover, Bai, Silverstein and Yin (1988) [5] showed that finite fourth moment is also necessary for the strong convergence of the largest eigenvalue. Therefore, we obtain the sufficient and necessary conditions of the strong convergence of the largest eigenvalue of $W_n$. For the convergence of the smallest eigenvalue, we need to make the following declaration:

$$s_{\min}(W_n) = \begin{cases} s_{\min}(W_n) = s_1(W_n) & p \leq n, \\ s_{\min}(W_n) = s_{p-n+1}(W_n) & p > n. \end{cases}$$

Bai and Yin (1993) [6] proved that

$$s_{\min}(W_n) \to \sigma^2(1 - \sqrt{y})^2, \text{ a.s.}$$

where the underlying distribution has a zero mean and finite fourth moment. The results above were extended to the complex case in [2]. In this paper, we shall show that the conclusions are still true for the quaternion sample covariance matrix.
EXTREME EIGENVALUES OF QUATERNION SAMPLE COVARIANCE MATRIX

Next we introduce some notations and some basic properties about quaternions. The quaternion base can be represented by four $2 \times 2$ matrices as

\[
e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]

where $i = \sqrt{-1}$ denotes the imaginary unit. Thus, a quaternion can be represented by a $2 \times 2$ complex matrix as

\[
x = a \cdot e + b \cdot i + c \cdot j + d \cdot k = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} = \begin{pmatrix} \lambda & \omega \\ -\overline{\omega} & \overline{\lambda} \end{pmatrix}
\]

where the coefficients $a, b, c, d$ are real and $\lambda = a + bi, \omega = c + di$. The conjugate of $x$ is defined as

\[
\bar{x} = a \cdot e - b \cdot i - c \cdot j - d \cdot k = \begin{pmatrix} a - bi & -c - di \\ c - di & a + bi \end{pmatrix} = \begin{pmatrix} \overline{\lambda} & -\overline{\omega} \\ \overline{\omega} & \overline{\lambda} \end{pmatrix}
\]

and its norm as

\[
\|x\| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{|\lambda|^2 + |\omega|^2}.
\]

By the property of quaternions, one has

\[
(1.1) \quad \det(x) = \|x\|^2.
\]

Furthermore, let $I_p^Q$ denote $p \times p$ quaternion identity matrix, i.e.,

\[
I_p^Q = \text{diag}\left(\overbrace{e, \cdots, e}^{p}\right).
\]

More details can be found in [1, 9, 13, 11, 13, 19, 15]. It is worth mentioning that any $n \times n$ quaternion matrix $Y$ can be represented by a $2n \times 2n$ complex matrix $\psi(Y)$. Consequently, we can deal with quaternion matrices as complex matrices for convenience. It is known (see [19]) that the multiplicities of all the eigenvalues (obviously they are all real) of $\psi(Y)$ are even. Taking one from each of the $n$ pairs of eigenvalues of $\psi(Y)$, the $n$ values are defined as the eigenvalues of $Y$.

This paper is organized as follows. The main theorems are stated in Section 2. In Section 3, we outline some knowledges of graph theory and introduce an operation called “Diamond product” which will be used in Section 4. Section 4, Section 5, and Section 6 give the proofs of the main theorems, respectively. Some technical lemmas are postponed to Section 7.
2. Main Theorem.

In this paper, we consider the strong limits of the largest and smallest eigenvalues of quaternion sample covariance matrices. Let
\[ S_n = \frac{1}{n} X_n X_n^* \]
where \( X_n \) is defined in Theorem 2.1 and denote the eigenvalues of \( S_n \) by \( s_1(S_n), \cdots, s_p(S_n) \), arranged in ascending order. Firstly, we give the upper and lower bounds of extreme eigenvalues in Theorem 2.1 when \( y = \lim p/n \in (0, 1) \). Combining Theorem 1.1 \( (F^{S_n} \to F_y, \text{a.s.}) \) in [12], we can get Theorem 2.2 about the limits of the largest and smallest eigenvalues while \( y \in (0, 1) \). Considering that \( X_n X_n^* \) and \( X_n^* X_n \) have the same set of nonzero eigenvalues, Theorem 2.2 is still true for \( y \in (1, \infty) \). Finally, we present sufficient and necessary conditions for the existence of the strong limit of the largest eigenvalue of \( S_n \). These theorems can be stated as the following:

**Theorem 2.1.** Let \( S_n = \frac{1}{n} X_n X_n^* \) where \( X_n = (x_{jl}, j = 1, \cdots, p, l = 1, \cdots, n) \) and \( x_{jl} \) are quaternion variables. Assume that the following conditions hold:

1. \( x_{jl} \) are independent,
2. \( \mathbb{E}x_{jl} = 0 \) and \( \text{Var}x_{jl} = \sigma^2 \), for all \( j, l \),
3. \( \sup_{jl} \mathbb{E}\|x_{jl}\|^4 \leq M, M \) is a positive constant,
4. there exists a random variable \( \xi \) with finite 4th moment and a constant \( L \) such that for any \( \delta > 0 \),

\[
\frac{1}{np} \sum_{jl} \mathbb{P} (\|x_{jl}\| > \delta \sqrt{n}) < LP (|\xi| > \delta \sqrt{n}).
\]

(2.1)

Then we have
\[
-2\sqrt{y}\sigma^2 \leq \lim \inf s_1(S_n - \sigma^2 (1 + y) I_p^Q) \leq \lim \sup s_p(S_n - \sigma^2 (1 + y) I_p^Q) \leq 2\sqrt{y}\sigma^2 \quad \text{a.s.}
\]
as \( n \to \infty, p \to \infty, p/n \to y \in (0, 1) \).

From Theorem 2.1, one can easily get the following theorem:

**Theorem 2.2.** Under the conditions of Theorem 2.1, we have

\[
\lim s_1(S_n) = (1 - \sqrt{y})^2 \sigma^2 \quad \text{a.s.}
\]
(2.2)
\[
\lim s_p(S_n) = (1 + \sqrt{y})^2 \sigma^2 \quad \text{a.s.}
\]
(2.3)
as \( n \to \infty, p \to \infty, p/n \to y \in (0, 1) \).
Remark 2.3. (2.2) and (2.3) are trivially true for \( y = 1 \). If \( p > n \), one has that the \( p - n \) smallest eigenvalues of \( S_n \) must be zero. Define

\[
   s_{\min}(S_n) = \begin{cases} 
   s_{p-n+1}(S_n) & p > n, \\
   s_1(S_n) & p \leq n.
   \end{cases}
\]

We assert that Theorem 2.2 is still true for \( y \in (1, \infty) \). In fact, when \( y > 1 \), let \( \tilde{S}_n = \frac{1}{p} X_n^* X_n \) and \( \tilde{y} = 1/y \in (0, 1) \). Applying Theorem 2.2, we have

\[
   \lim s_{\min}(\tilde{S}_n) = \left(1 - \sqrt{\tilde{y}}\right)^2 \sigma^2 = \frac{1}{y} (1 - \sqrt{y})^2 \sigma^2 \quad \text{a.s.}
\]

\[
   \lim s_p(\tilde{S}_n) = \left(1 + \sqrt{\tilde{y}}\right)^2 \sigma^2 = \frac{1}{y} (1 + \sqrt{y})^2 \sigma^2 \quad \text{a.s.}
\]

which implies that

\[
   \lim s_{\min}(S_n) = \frac{p}{n} \lim s_{\min}(\tilde{S}_n) = (1 - \sqrt{y})^2 \sigma^2 \quad \text{a.s.}
\]

\[
   \lim s_p(S_n) = \frac{p}{n} \lim s_p(\tilde{S}_n) = (1 + \sqrt{y})^2 \sigma^2 \quad \text{a.s.}
\]

Theorem 2.4. Suppose that the entries of \( Z_n \) are i.i.d. quaternion random variables and the ratio of dimension to sample size \( p/n \to y \), then the largest eigenvalue of \( \Lambda_n = \frac{1}{n} Z_n Z_n^* \) tends to \( \mu \) with probability 1 if and only if the following conditions are true:

(i) \( E z_{11} = 0; \)

(ii) \( \text{Var} z_{11} = \sigma^2; \)

(iii) \( E \|z_{11}\|^4 < \infty; \)

(iv) \( \mu = (1 + \sqrt{y})^2 \sigma^2. \)

3. Preliminaries.

In this section, we will recall some basic knowledges of the graph theory (see Section 3.1.2 or Section 5.2 in [4]) and introduce an operation of matrices.

3.1. Some knowledges of Graph Theory. Suppose that \( i_1, \ldots, i_k \) are \( k \) positive integers (not necessarily distinct) not greater than \( p \) and \( j_1, \ldots, j_k \) are \( k \) positive integers (not necessarily distinct) not larger than \( n \). For a sequence \( (i_1, j_1, \ldots, i_k, j_k) \), draw two parallel lines, referring to the \( I \) line and the \( J \) line. Plot \( i_1, \ldots, i_k \) on the \( I \) line and \( j_1, \ldots, j_k \) on the \( J \) line, and draw \( k \) (down) edges from \( i_u \) to \( j_u \), \( u = 1, \ldots, k \) and \( k \) (up) edges from \( j_u \) to \( i_{u+1} \), \( u = 1, \ldots, k \) (with the convention that \( i_{k+1} = i_1 \)). The graph is denoted by \( G(\mathcal{I}, \mathcal{J}) \), where \( \mathcal{I} = (i_1, \ldots, i_k) \) and \( \mathcal{J} = (j_1, \ldots, j_k) \).

Suppose the number of noncoincident \( I \) -vertices is \( r + 1 \) and the number of noncoincident \( J \) -vertices is \( s \). A canonical graph can be defined as follows:
Definition 3.1. A canonical $\Delta (k, r, s)$ can be directly defined in the following way:
1. Its vertex set $V = V_I + V_J$, where $V_I = \{1, \cdots, r + 1\}$, called the $I$-vertices, and $V_J = \{1, \cdots, s\}$, called the $J$-vertices.
2. There are two functions, $f : \{1, \cdots, k\} \to \{1, \cdots, r + 1\}$ and $g : \{1, \cdots, k\} \to \{1, \cdots, s\}$, satisfying
   \[ f(1) = 1 = g(1) = f(k + 1), \]
   \[ f(j) \leq \max \{f(1), \cdots, f(j - 1)\} + 1, \]
   \[ g(j) \leq \max \{g(1), \cdots, g(j - 1)\} + 1. \]
3. Its edge set $E = \{e_{1d}, e_{1u}, \cdots, e_{kd}, e_{ku}\}$, where $e_{1d}, \cdots, e_{kd}$ are called the down edges and $e_{1u}, \cdots, e_{ku}$ are called the up edges.
4. $F(e_{jd}) = (f(j), g(j))$ and $F(e_{ju}) = (g(j), f(j + 1))$ for $j = 1, \cdots, k$.

Remark 3.2. Two graphs are said to be isomorphic if one becomes the other by a suitable permutation on $(1, 2, \cdots, p)$ and a permutation on $(1, 2, \cdots, n)$. By Definition 3.1, we can easily obtain that there is only one canonical graph for each isomorphic class.

Remark 3.3. By Definition 3.1, the number of graphs in the isomorphic class associated with the canonical $\Delta (k, r, s)$-graph is
\[ p(p - 1) \cdots (p - r)n(n - 1) \cdots (n - s + 1) = p^{r+1}s^n \left[ 1 + O( n^{-1} ) \right]. \]

If two edges have the same vertex sets, we say that the two edges coincide. We call that an edge $e_a$ is single up to $e_b$, $b \geq a$, when the edge $e_a$ does not coincide with any one among $e_1, \cdots, e_b$ other than itself.

Definition 3.4. For a canonical graph, classify the edges into several types:
1. If $f(j + 1) = \max \{f(1), \cdots, f(j)\} + 1$, the edge $e_{ju} = (g(j), f(j + 1))$ is called an up innovation. And if $g(j) = \max \{g(1), \cdots, g(j - 1)\} + 1$, the edge $e_{jd} = (f(j), g(j))$ is called a down innovation. The two cases are both called a $T_1$ edge which leads to a new vertex.
2. An edge is called a $T_3$ edge if it coincides with an innovation that is single until the $T_3$ edge appears. A $T_3$ edge $(g(j), f(j + 1))$ (or $(f(j), g(j))$) is said to be irregular if there is only one innovation single up to $g(j)$ (or $f(j)$). All other $T_3$ edges are called regular $T_3$ edges.
3. All other edges are called $T_4$ edges.
4. The first appearance of a $T_4$ edge is called a $T_2$ edge. There are two cases: the first is the first appearance of a single noninnovation, and the second is the first appearance of an edge that coincides with a $T_3$ edge.

A chain is a consecutive segment of $G(I, J)$, i.e. $i_1j_1 \cdots i_r$ or $i_1j_1 \cdots i_rj_r$. 
Lemma 3.5. Let \( t \) denote the number of \( T_2 \) edges and \( l \) denote the number of innovations in the chain \( i_1 j_1 \cdots i_\tau \) (or \( i_1 j_1 \cdots i_\tau j_\tau \)) that are single up to \( \tau \) and have a vertex coincident with \( f(\tau) \) (or \( g(\tau) \)). Then \( l \leq t + 1 \).

Lemma 3.6. The number of regular \( T_3 \) edges is not greater than twice the number of \( T_2 \) edges.

3.2. Diamond product.

Definition 3.7. Let \( A \) and \( B \) be two \( p \times n \) quaternion matrices. Then, \( A \star B = (a_{ji}b_{ji}) \), called Hadamard product for quaternion matrices.

Lemma 3.8. Let \( A \) and \( B \) be two \( p \times n \) quaternion matrices. Then,

\[
\|A \star B\|_2 \leq \|A\|_2 \|B\|_2
\]

where \( \|\cdot\|_2 \) denotes the 2-norm of a matrix, i.e. \( \|\cdot\|_2 \) is equal to the maximum singular value of this matrix.

Proof. Applying Definition 3.7 and Lemma 7.2, we have

\[
\|A \star B\|_2 = \sup_{e^*e = f^*f = 1} |e^* (A \star B) f|
\]

\[
\leq \sup_{e^*e = f^*f = 1} \sum_{l=1}^{p} \sum_{k=1}^{n} |e^*_l (a_{lk}b_{lk}) f_k|
\]

\[
\leq \sup_{e^*e = f^*f = 1} \left( \sum_{l=1}^{p} \sum_{k=1}^{n} e^*_l a_{lk}a^*_k e_l \right)^{1/2} \sup_{f^*f = 1} \left( \sum_{l=1}^{p} \sum_{k=1}^{n} f^*_k b_{lk}b^*_k f_k \right)^{1/2}
\]

\[
\leq \sup_{e^*e = f^*f = 1} \left( \max_{l} \sum_{k=1}^{n} |a_{lk}|^2 \sum_{l=1}^{p} e^*_l e_l \right)^{1/2} \sup_{f^*f = 1} \left( \max_{k} \sum_{l=1}^{p} |b_{lk}|^2 \sum_{k=1}^{n} f^*_k f_k \right)^{1/2}
\]

\[
\leq \|A\|_2 \|B\|_2
\]

where \( e = (e_1', \cdots, e_p') \), \( e_l \) is a vector of order 2 and \( f = (f_1', \cdots, f_n') \), \( f_k \) is a vector of order 2. \( \square \)

Definition 3.9. Let \( H_j = (h^{(j)}_{\alpha\beta}) \), \( j = 1, 2, \cdots, k \), be \( k \) quaternion matrices with dimensions \( n_j \times n_{j+1} \), respectively. Define the Diamond product of the \( k \) matrices by

\[
H_1 \diamond \cdots \diamond H_k = \left( \sum_{t_2} h^{(1)}_{\alpha t_2} h^{(2)}_{t_2 t_3} \cdots h^{(k-1)}_{t_{k-1} t_k} h^{(k)}_{t_k \beta} \right)
\]

where the summation runs for \( t_1 = 1, 2, \cdots, n_j \), \( j = 2, \cdots, k \), subject to restrictions \( \alpha \neq t_3, t_2 \neq t_4, t_3 \neq t_5, \cdots, t_{k-2} \neq t_k \) and \( t_{k-1} \neq \beta \).
Lemma 3.10. Let $H_j = \left( h^{(j)}_{\alpha\beta} \right)$, $j = 1, 2, \cdots, k$, be $k$ quaternion matrices with dimensions $n_j \times n_j+1$, respectively. Then, we have

$$\|H_1 \diamond \cdots \diamond H_k\|_2 \leq 3^{k-1} \|H_1\|_2 \cdots \|H_k\|_2.$$ 

**Proof.** We shall use induction to prove this lemma.

- i) When $k = 1$, the conclusion is trivially true. When $k = 2$, denote $H_1 H_2 = (Q_{jl})$ where $Q_{jl} = \left( \begin{array}{cc} \lambda_j & \omega_j \\ -\omega_j & \lambda_j \end{array} \right)$. It follows that

$$\|H_1 \diamond H_2\|_2 = \|H_1 H_2 - \text{diag} \left( Q_{11}, \cdots, Q_{pp} \right)\|_2$$

$$\leq \|H_1 H_2\|_2 + \|\text{diag} \left( Q_{11}, \cdots, Q_{pp} \right)\|_2$$

$$\leq \|H_1 H_2\|_2 + \|\text{diag} \left( H_1 H_2 \right)\|_2 + \max_j |w_{jj}|$$

$$\leq 3 \|H_1\|_2 \|H_2\|_2.$$

- ii) Let $k > 1$. Note that

$$H_1 \diamond \cdots \diamond H_k$$

$$= H_1 (H_2 \diamond \cdots \diamond H_k) - \text{diag} \left( Q_{11}, \cdots, Q_{pp} \right) (H_3 \diamond \cdots \diamond H_k)$$

$$+ \left( h^{(1)}_{jl} h^{(2)}_{lj} h^{(3)}_{lj} \right) \diamond H_4 \diamond \cdots \diamond H_k.$$

Here, the $(j, l)$ entry of the matrix $\left( h^{(1)}_{jl} h^{(2)}_{lj} h^{(3)}_{lj} \right)$ is zero if $l > n_2$ or $j > n_3$. Using Lemma 3.8, one has

$$\left\| \left( h^{(1)}_{jl} h^{(2)}_{lj} h^{(3)}_{lj} \right) \right\|_2 \leq \|H_1\|_2 \|H_2\|_2 \|H_3\|_2 = \|H_1\|_2 \|H_2\|_2 \|H_3\|_2.$$

By induction, we complete the proof of Lemma 3.10. \qed

4. **Proof of Theorem 2.1**

By Definition 3.9, we denote

$$R_{n\cdot} (l) = n^{-l} \bigg( X_n \diamond X^*_n \diamond \cdots \diamond X_n \diamond X^*_n \bigg)$$

which implies that $R_{n\cdot} (1) = S_n - n^{-1} \text{diag} \left( X_n X^*_n \right)$. Thus, we shall complete the proof by the following two steps:

- a) Firstly, we derive the estimate of the norm of $R_{n\cdot} (l)$. The aim of subsequent lemmas is to estimate of the norm of $\left( R_n - y \sigma^2 I_p^Q \right)^k$ by using the estimate on $R_{n\cdot} (l)$ (see Section 4.1);

- b) Applying these lemmas, we can easily get the bound of $\|R_n - y \sigma^2 I_p^Q\|_2$. Together with $\|S_n - R_n - \sigma^2 I_p^Q\|_2$, we obtain the bound of $\|S_n - \sigma^2 (1+y) I_p^Q\|_2$ (see Section 4.2).
4.1. Some lemmas.

Lemma 4.1. Under the conditions of Theorem 2.1, we have
\[ \limsup_{n \to \infty} \| R_n (l) \|_2 \leq (2l + 1) (l + 1) y^{(l-1)/2} \sigma^2 l \text{ a.s..} \quad (4.2) \]

Proof. By (4.1),
\[ R_n (l) = n^{-l} \sum x_{uv1} x_{v1} x_{uv2} x_{v2} \cdots x_{uvl-1} x_{vl} \]
where the summation \( \sum \) runs over for \( v_1, \cdots, v_l = 1, \cdots, n \) and \( u_1, \cdots, u_{l-1} = 1, \cdots, p \) subject to the restriction \( a \neq u_1, u_1 \neq u_2, \cdots, u_{l-1} \neq b \) and \( v_1 \neq v_2, v_2 \neq v_3, \cdots, v_{l-1} \neq v_l \).

Without loss of generality, we assume \( \sigma = 1 \). At first, we will truncate and centralize the quaternion random variables without changing the bound of \( \| R_n (l) \|_2 \). Since \( E|\xi|^4 < \infty \), for any \( \delta > 0 \), we have
\[ \sum_{k=1}^{\infty} \delta^{-4} 2^{2k} P (|\xi| > \delta 2^{k/2}) < \infty. \]

Then, we can select a slowly decreasing sequence of constants \( \delta_{2k} \to 0, 2^{k/2} \delta_{2k} \uparrow \infty \), and such that
\[ \sum_{k=1}^{\infty} \delta_{2k}^{-4} 2^{2k} P (|\xi| > \delta_{2k} 2^{k/2}) < \infty. \quad (4.3) \]

Let \( \delta_n = \delta_{2k} \) for \( 2^k < n \leq 2^{k+1} \) and let \( \hat{x}_{uv} = x_{uv} I (\|x_{uv}\| \leq \delta_n \sqrt{n}) (\delta_n \sqrt{n} \uparrow \infty) \), \( \hat{X}_n = (\hat{x}_{uv}) \), and
\[ \hat{R}_n (l) = n^{-l} \hat{X}_n \odot \hat{X}_n^* \odot \cdots \hat{X}_n \odot \hat{X}_n^*. \]
Together with (2.1) and (4.3), one has
\[ P \left( \hat{R}_n \neq R_n, \text{i.o.} \right) = \lim_{K \to \infty} \sum_{k=K}^{\infty} P \left( \bigcup_{2^k < n \leq 2^{k+1}} \bigcup_{u \leq \delta_n \sqrt{n}} \|x_{uv}\| > \delta_n \sqrt{n} \right) \]
\[ \leq \lim_{K \to \infty} \sum_{k=K}^{\infty} P \left( \bigcup_{2^k < n \leq 2^{k+1}} \bigcup_{u \leq (y+1)2^k, v \leq 2^{k+1}} \|x_{uv}\| > \delta_{2k} 2^{k/2} \right) \]
\[ = \lim_{K \to \infty} \sum_{k=K}^{\infty} P \left( \bigcup_{u \leq (y+1)2^k, v \leq 2^{k+1}} \|x_{uv}\| > \delta_{2k} 2^{k/2} \right) \]
\[
\leq \lim_{K \to \infty} L(y + 1) \sum_{k=K}^{\infty} 2^{2k+2} P \left( |\xi| > \delta_2 2^{k/2} \right) \to 0.
\]

Thus we only need to show that (1.2) holds for the matrix \( \hat{R}_n (l) \).

Let \( \tilde{x}_{uv} = \hat{x}_{uv} - E(\hat{x}_{uv}) \), \( \hat{X}_n = (\tilde{x}_{uv}) \), and

\[
\tilde{R}_n (l) = n^{-1} \tilde{X}_n \circ \tilde{X}_n^* \circ \cdots \circ \tilde{X}_n \circ \tilde{X}_n^*.
\]

Suppose (4.2) is true for the matrix \( \tilde{R}_n (l) \), then we assert that, for all \( l \geq 0 \),

\[
(4.4) \quad \left\| \tilde{R}_n (l) - \tilde{R}_n (l) \right\|_2 = 0 \quad a.s..
\]

In fact, \( \tilde{R}_n (l) - \hat{R}_n (l) \) can be written as a sum of \( \circ \) products of matrices \( \frac{1}{\sqrt{n}} \tilde{X}_n, E\left( \frac{1}{\sqrt{n}} \hat{X}_n \right) \) or their complex conjugate transpose. In each product, at least one of them is \( E\left( \frac{1}{\sqrt{n}} \hat{X}_n \right) \) or its complex conjugate transpose. Next, we estimate the bounds of

\[
\left\| \frac{1}{n} \tilde{X}_n \right\|_2 \quad \text{and} \quad \left\| E\left( \frac{1}{\sqrt{n}} \hat{X}_n \right) \right\|_2.
\]

If (4.2) is true for the matrix \( \tilde{R}_n (l) \), we have

\[
\lim \sup \left\| n^{-1/2} \tilde{X}_n \right\|_2^2 = \lim \sup \left\| \tilde{R}_n (1) + \text{diag} \left( \frac{1}{n} \sum_v \| \tilde{x}_{uv} - E(\tilde{x}_{uv}) \|_2^2 I_2, u \leq p \right) \right\|_2
\]

\[
(4.5) \quad \leq 6 + \lim \sup \frac{1}{n} \max_{u \leq p} \sum_{v=1}^{n} \left[ \| \tilde{x}_{uv} \|^2 + 2 \| \tilde{x}_{uv} \| \| E\tilde{x}_{uv} \| + \| E\tilde{x}_{uv} \| ^2 \right]
\]

Denoting \( h_v = \| \tilde{x}_{uv} \|^2 - E \| \tilde{x}_{uv} \|^2 \) and using the fact that for all \( k \geq 1, k! \geq (k/3)^k \), we have

\[
E \left( \frac{1}{n} \sum_{v=1}^{n} h_v \right)^{2m} = \frac{1}{n^{2m}} \sum_{m_1+\cdots+m_n=2m} \frac{2m!}{m_1! \cdots m_n!} E h_{m_1}^{m_1} \cdots E h_{m_n}^{m_n}
\]

\[
\leq \frac{1}{n^{2m}} \sum_{k=1}^{m} \sum_{m_1+\cdots+m_k=2m} \frac{2m!}{k! m_1! \cdots m_k!} \prod_{j=1}^{k} \left( \sum_{v=1}^{n} E h_{v}^{m_j} \right)
\]

\[
\leq \sum_{k=1}^{m} n^{-2m} k^{-m} (k!)^{-1} (2 \delta_4 n) 2^{m-k} M^k n^k
\]

\[
\leq \sum_{k=1}^{m} \left( \frac{3Mk}{n} \right)^k (4 \delta_4 n^2)^{m-k}
\]
\[ (4.6) \quad \leq \sum_{k=1}^{m} \left( \frac{3Mm}{n} \right)^{k} (4\delta_{n}^{4}k^{2})^{m-k}. \]

Select \( m = \lfloor \log n \rfloor \) and let \( f(k) = k \log (3Mm/n) + (m - k) \log (4\delta_{n}^{4}k^{2}) \), then \( f'(k) \) (the derivative of \( f(k) \)) is
\[
\log (3Mm/n) - \log (4\delta_{n}^{4}n^{2}) + 2 (m - k) / k
\]
\[
= \log (3Mm / (4n\delta_{n}^{4})) - 2 \log k + 2 (m - k) / k
\]
\[
\leq - \frac{3}{4} \log n - 2 \log k + 2 (m - k) / k.
\]

We claim that the maximum term on the right hand side of (4.6) can only be \( k = 1 \) or 2. In fact, when \( k > 2 \),
\[
f'(k) \leq - \frac{3}{4} \log n - 2 \log k + 2 (m - k) / 3
\]
\[
\leq - \frac{1}{12} \log n - 2 \log k < 0.
\]

Thus, we obtain for any fixed \( t > 0 \)
\[
E \left( \frac{1}{n} \sum_{v=1}^{n} h_{v} \right)^{2m} \leq m \left( \frac{3M}{n} (4\delta_{n}^{4})^{m-1} + \left( \frac{6M}{n} \right)^{2} (16\delta_{n}^{4})^{m-2} \right)
\]
\[
\leq \delta_{n}^{4m} = o \left( n^{-t} \right).
\]

From the inequality above with \( t > 2 \) and Borel-Cantelli lemma, we have
\[ (4.7) \quad \max_{u \leq p} \left| \frac{1}{n} \sum_{v} \left( \| \hat{x}_{uv} \|^{2} - E \| \hat{x}_{uv} \|^{2} \right) \right| \rightarrow 0, \text{ a.s.} \]

Thus, (4.5) turns into
\[
\limsup \left\| n^{-1/2} \hat{X}_{n} \right\|_{2}^{2}
\]
\[
\leq 6 + \limsup \frac{1}{n} \max_{u \leq p} \sum_{v=1}^{n} \left[ E \| x_{uv} \|^{2} + 2\delta_{n} \sqrt{n} \| E \hat{x}_{uv} \| \right]
\]
\[
\leq 6 + \limsup \frac{1}{n} \max_{u \leq p} \sum_{v=1}^{n} \left[ E \| x_{uv} \|^{2} + 2E \| x_{uv} \|^{2} \right]
\]
\[
\leq 6 + 3 = 9.
\]

And
\[
\left\| E \left( \frac{1}{\sqrt{n}} \hat{X}_{n} \right) \right\|_{2} \leq \sqrt{\frac{1}{n} \max_{u \leq p} \sum_{v} \| E \hat{x}_{uv} \|^{2}} \leq \frac{M}{\delta_{n} \sqrt{n}} = o (1).
\]
Combining the above with Lemma 3.10, the proof of (4.4) is complete. Therefore it suffices to show that (4.2), for the matrix $\tilde{R}_n(l)$ is true.

For brevity, we still use $R_n(l)$ and $x_{uv}$ to denote the matrix and variables after truncation and recentralization. We further assume that:

\begin{equation}
\begin{align*}
(i) & \quad \mathbb{E}(x_{uv}) = 0, \mathbb{E}\|x_{uv}\|^2 \leq 1, \\
(ii) & \quad \mathbb{E}\|x_{uv}\|^l \leq (\delta_n\sqrt{n})^{l-3}, \text{ for all } l \geq 3.
\end{align*}
\end{equation}

(4.8)

We will complete the proof under the additional conditions (4.8). Select a sequence of even integers $m$ with the properties $m/\log n \to \infty$ and $m\delta^{1/3}/\log n \to 0$. For any $\eta > (2l + 1)(l + 1)g^{(l-1)/2}$, we have

\begin{equation}
P\left(\|R_n(l)\|_2 \geq \eta\right) \leq \eta^{-2m}\mathbb{E}\left(\|R_n(l)\|_2^{2m}\right) \leq \eta^{-2m}\mathbb{E}\text{tr}R_n^{2m}(l).
\end{equation}

(4.9)

We only need to estimate

$$\mathbb{E}\text{tr}R_n^{2m}(l) = n^{-2ml} \sum_{G} \mathbb{E}\text{tr} \left( x_{i_1j_1}x_{i_2j_2} \cdots x_{i_{2ml}j_{2ml}} \right)$$

where the summation runs over all integers $i_1, \ldots, i_{2ml}$ from $\{1, 2, \ldots, p\}$ and $j_1, \ldots, j_{2ml}$ from $\{1, 2, \ldots, n\}$ subject to the conditions that, for any $\alpha = 0, 1, \ldots, 2m - 1$,

\begin{equation}
i_{\alpha+1} \neq i_{\alpha+2}, i_{\alpha+2} \neq i_{\alpha+3}, \ldots, i_{(\alpha+1)l} \neq i_{(\alpha+1)l+1};
\end{equation}

(4.10)

$$j_{\alpha+1} \neq j_{\alpha+2}, j_{\alpha+2} \neq j_{\alpha+3}, \ldots, j_{(\alpha+1)l-1} \neq j_{(\alpha+1)l}.$$ 

Defining graphs in accordance with the last section, the equality above can be rewritten as

\begin{equation}
\begin{align*}
\mathbb{E}\left( \text{tr}R_n^{2m}(l) \right) &= n^{-2ml} \sum_{G} \sum_{I,J} \mathbb{E}\text{tr} \left( x_{i_1j_1}x_{i_2j_2} \cdots x_{i_{2ml}j_{2ml}} \right)
\end{align*}
\end{equation}

(4.11)

where $G$ runs over all canonical graphs and $G(I,J)$ runs over the given isomorphic class. Obviously, if $G$ has a single edge, the terms corresponding to this graph are zero. Thus, we need only to estimate the sum of all those terms whose $G$ has no single edge.

Now, we begin to estimate the right-hand side of (4.11). Noticing that

$$x_{i_1j_1}x_{i_2j_2} \cdots x_{i_{2ml}j_{2ml}}$$

can be written as

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \tilde{\alpha} \end{pmatrix},$$

and according to (1.1), one has

\begin{equation}
\begin{align*}
|\text{tr} \left( x_{i_1j_1}x_{i_2j_2} \cdots x_{i_{2ml}j_{2ml}} \right) |& = |\alpha + \tilde{\alpha}| \leq 2 \left( |\alpha|^2 + |\beta|^2 \right)^{1/2} \\
& = 2 \left\{ \det \begin{pmatrix} \alpha & \beta \\ -\beta & \tilde{\alpha} \end{pmatrix} \right\}^{1/2} \\
& \leq 2 \left\| \det (x_{i_1j_1}) \cdots \det (x_{i_{2ml}j_{2ml}}) \right\| \\
& \leq 2 \left\| x_{i_1j_1} \right\| \cdots \left\| x_{i_{2ml}j_{2ml}} \right\| \left\| x_{i_{2ml}j_{2ml}} \right\| \left\| x_{i_1j_1} \right\| \\
\end{align*}
\end{equation}

(4.11)
which is similar to Lemma 3.6 in [16]. Assume $r$ and $s$ be the number of up and down innovations, respectively. Let $k = r + s$ denote the total number of innovations and $t$ denote the number of $T_2$ edges. Due to the inequality above, we get

$$E \left| \text{tr} \left( x_{i1j1}x_{i2j1}x_{i3j2} \cdots x_{i2mlj2ml}x_{i1j2ml} \right) \right| \leq 2E \left| x_{i1j1} \right| \left| x_{i2j1} \right| \cdots \left| x_{i2mlj2ml} \right| \left| x_{i1j2ml} \right| \leq 2 \left( \delta_n \sqrt{n} \right)^{4ml - 2k - t}.$$  

(4.12)

By Remark 3.3, we know that the number of graphs of each isomorphic class is less than $n^{s_p r + 1}$. Thus, (4.11) can be estimated by

$$E \left( \text{tr} R^2_n (l) \right) \leq 2n^{s_p r + 1} \sum_G n^s p^{r+1} \left( \delta_n \sqrt{n} \right)^{4ml - 2k - t}.$$  

(4.13)

In the following, we only consider the number of canonical graphs without single edges. Due to condition (4.10), we split the graph $G$ into $2m$ subgraphs $G_1, \ldots, G_{2m}$. Within each subgraph, except the first and the last edges, all edges do not coincide with their adjacent (prior to or behind) edges, that means, every $T_1$ edge must be followed by a $T_4$ or $T_1$ edge, unless it is the last edge of $G_j$. Let $a_j$ denote the number of pairs of consecutive edges $(t_1, t_4)$ in the subgraph $G_j$ in which $t_1$ is a $T_1$ edge and $t_4$ is a $T_4$ edge. Then the number of consecutive innovations in $G_j$ is not more than $a_j$ or $a_j + 1$ (the latter happens when the last edge of $G_j$ is an innovation). Hence, the number of ways to arrange the consecutive innovation sequences is not more than

$$\binom{2l}{2a_j} + \binom{2l}{2a_j + 1} = \binom{2l + 1}{2a_j + 1}.$$  

The number of the ways to select positions of edges (including $T_1$, $T_3$ and $T_4$) is

$$\prod_{j=1}^{2m} \binom{2l + 1}{2a_j + 1} \binom{4ml - k}{k}.$$  

After fixing the positions of edges, we need to know the selections to plot an edge of the given type. For an innovation or an irregular $T_3$ edge, there is only one way to plot once the subgraph prior to this edge is plotted. By Lemma 3.5, there are at most $t + 1$ single innovations to be matched by a regular $T_3$ edge. By Lemma 3.6, there are at most $2t$ regular $T_3$ edges. Hence, there are at most $\left( t + 1 \right)^{2t} \leq \left( t + 1 \right)^{2(4ml - 2k)}$ ways to plot the regular $T_3$-edges. For each $T_4$ edge, there are at most $\left( k + 1 \right)^2$ ways to determine its two vertices. Therefore, there are at most $\binom{(k + 1)^2}{t}$ ways to plot the $t$ $T_2$ edges. And,
there are at most \( t^{4ml-2k} < (t + 1)^{4ml-2k} \) ways to distribute the \( 4ml - 2k \) \( T_4 \) edges.

Together with the analysis above and (4.12), (4.13) can be estimated by

\[
E(\text{tr} R_n^{2m}(l)) \leq 2n^{-2ml} \sum_{k=1}^{2m} \sum_{t=0}^{4ml-2k} (2l + 1)^{2m} (l + 1)^{2m} \binom{4ml - k}{k} \left( \frac{(k + 1)^2}{\delta_n \sqrt{n}} \right)^t (t + 1)^{3(4ml-2k)} n^s p^{r+1} \left( \delta_n \sqrt{n} \right)^{4ml-2k-t} \]

where the summation is taken subject to restrictions \( 1 \leq k \leq 2ml, 0 \leq t \leq 2ml, \) and \( 0 \leq a_j \leq l \). Applying (5.2.16) and (5.2.17) in [4], i.e.

\[
y_{n}^{r+1} \leq y_{n}^{(k-t-2m)/2} \]

we have

\[
E(\text{tr} R_n^{2m}(l)) \leq 2n^{2ml} \sum_{k=1}^{2m} \sum_{t=0}^{4ml-2k} (2l + 1)^{2m} (l + 1)^{2m} y_{n}^{-m} \sum_{k=1}^{2ml} \binom{4ml - k}{k} \left( \frac{3 (4ml - 2k) \delta_n^{1/3}}{\log \delta_n \sqrt{y_n n / (k + 1)^2}} \right)^{3(4ml-2k)} y_n^{k/2} \]

\[
\leq 2n^2 (2l + 1)^{2m} (l + 1)^{2m} y_{n}^{m} \left[ y_n^{1/4} + \frac{24ml \delta_n^{1/3}}{\frac{1}{3} \log n} \right]^{4ml} \]

where the second inequality follows from the elementary inequality

\[
\alpha^{-(t+1)} (t + 1)^{\beta} \leq \left( \frac{\beta}{\log(\alpha)} \right)^{\beta}, \beta > 0, \alpha > 1. \]
Thus, combining the inequalities above, \( m/\log n \to \infty \) with (4.9), we have
\[
P \left( \| R_n^{2m}(l) \|_2 \geq \eta \right) \leq 2n^2 \left( \frac{(2l+1)(l+1)y_n^{(l-1)}}{\eta} \right)^{2m} [1 + o(1)]^{4ml}
\]
which is summable. Therefore, by Borel-Cantelli lemma, one has
\[
\limsup_{n \to \infty} \| R_n(l) \|_2 \leq (2l+1)(l+1)y^{(l-1)/2} \quad \text{a.s.}
\]
\( \square \)

In the following, we say that a matrix is \( o(1) \) if its 2-norm tends to 0.

**Lemma 4.2.** Under the conditions of Theorem 2.1, we have
\[
R_nR_n(k) = R_n(k + 1) + y\sigma^2R_n(k) + y\sigma^4R_n(k - 1) + o(1) \quad \text{a.s.}
\]  
(4.14)

**Proof.** We only need to show that (4.14) is true for \( \sigma = 1 \). Define
\[
X_n^{(3)} = n^{-\frac{3}{2}} (\| x_{uv} \|_2 x_{uv}) \quad \text{then, by (4.15)}, we get}
\[
\| X_n^{(3)} \|_2^2 \leq n^{-3} \max_{u \leq p} \sum_v \| x_{uv} \|^6 \leq \delta_n^4 \max_{u \leq p} \sum_v \| x_{uv} \|^2
\]
(4.15)
\[
\leq \delta_n^4 \max_{u \leq p} \sum_v \mathbb{E} \| x_{uv} \|^2 \leq \delta_n^4 \to 0 \quad \text{a.s.}
\]  
(4.16)

According to Definition 3.9 and (4.15),
\[
R_n(k) = n^{-k} \underbrace{X_n \diamond X_n \diamond \cdots \diamond X_n \diamond X_n}_k
\]
\[
= n^{-k}X_n \left( \underbrace{X_n^* \diamond \cdots \diamond X_n^* \diamond X_n^*}_k \right) - n^{-1} [\text{diag } (X_nX_n^*)]R_n(k - 1)
\]
\[
+ X_n^{(3)} \diamond \left( n^{-\frac{3}{2}} \underbrace{X_n^* \diamond \cdots \diamond X_n^* \diamond X_n^*}_k \right)
\]
(4.17)
\[
= n^{-k}X_n \left( \underbrace{X_n^* \diamond \cdots \diamond X_n^* \diamond X_n^*}_k \right) - R_n(k - 1) + o(1) \quad \text{a.s.}
\]

where the last equality follows from Lemma 3.10. Similarly
\[
R_n(k + 1) = n^{-k-1}X_n \left( \underbrace{X_n^* \diamond \cdots \diamond X_n^* \diamond X_n^*}_{k+1} \right) - n^{-1} [\text{diag } (X_nX_n^*)]R_n(k)
\]
\[
+ o(1) \quad \text{a.s.}
\]
\[= n^{-1} X_n X_n^* R_n (k) - n^{-k-1} X_n \text{diag} (X_n^* X_n) \left( \frac{k}{X_n^*} \otimes \cdots \otimes X_n \otimes X_n^* \right) \]

\[- n^{-1} \text{diag} (X_n X_n^*) [R_n (k) + o (1) \ a.s.] \]

\[= R_n R_n (k) - n^{-k} X_n \left( \frac{k}{X_n^*} \otimes \cdots \otimes X_n \otimes X_n^* \right) + o (1) \ a.s. \]

(4.18)

The proof is complete.

\[\square\]

**Lemma 4.3.** Under the conditions of Theorem 2.1, we have

\[(R_n - y \sigma I_p^Q)^k = \sum_{r=0}^{k} (-1)^{r+1} \sigma^{2(k-r)} R_n (r) \sum_{j=0}^{[\frac{(k-r)}{2}]} C_j (k, r) y^{k-r-j} + o (1) \]

where the constants \(|C_j (k, r)| \leq 2^k\).

**Proof.** We shall prove this lemma by induction on \(k\).

- 1. When \(k = 1\),
  \[R_n - y \sigma I_p^Q = R_n (1) C_0 (1, 1) - y \sigma^2 R_n (0) C_0 (1, 0)\]
  where \(C_0 (1, 1) = 1\) and \(C_0 (1, 0) = 1\).

- 2. Suppose the lemma is true for \(k\). By Lemma 4.2 we have
  \[(R_n - y \sigma I_p^Q)^{k+1} = (R_n - y \sigma I_p^Q)^k \left( \sum_{r=0}^{k} (-1)^{r+1} \sigma^{2(k-r)} R_n (r) \right) \]
  \[\times \sum_{j=0}^{[\frac{(k-r)}{2}]} C_j (k, r) y^{k-r-j} + o (1) \]
  \[= R_n \left( \sum_{r=1}^{k} (-1)^{r+1} \sigma^{2(k-r)} R_n (r) \sum_{j=0}^{[\frac{(k-r)}{2}]} C_j (k, r) y^{k-r-j} \right) \]
  \[- \sigma^{2k} \sum_{j=0}^{[\frac{k}{2}]} C_j (k, 0) y^{k-j} I_p^Q \]
  \[- \sum_{r=0}^{k} (-1)^{r+1} \sigma^{2(k-r+1)} R_n (r) \sum_{j=0}^{[\frac{(k-r)}{2}]} C_j (k, r) y^{k-r-j+1} + o (1) \]
\[
\sum_{r=1}^{k} (-1)^{r+1} \sigma^{2(k-r)} (R_n (r+1) + y\sigma^2 R_n (r) + y\sigma^4 R_n (r-1))
\]

\[
\times \sum_{j=0}^{[(k-r)/2]} C_j (k, r) y^{k-r-j} - \sigma^{2k} \sum_{j=0}^{[k/2]} C_j (k, 0) y^{k-j} R_n
\]

\[
- \sum_{r=0}^{k+1} (-1)^{r+1} \sigma^{2(k-r+1)} R_n (r) \sum_{j=0}^{[(k+1-r)/2]} [C_j (k, r-1)] y^{k+1-r-j} + o (1)
\]

\[
+ \sum_{r=0}^{k-1} (-1)^{r+1} \sigma^{2(k+1-r)} R_n (r) \sum_{j=1}^{[(k+1-r)/2]} [C_j (k, r+1)] y^{k+1-r-j} + o (1) a.s.
\]

where \( C_j (k+1, r) \) is a sum of one or two terms of the form \(-C_j (k, r-1)\) and \(-C_j (k, r+1)\).

\( \bullet \) 3. By induction, we conclude that (4.19) is true for all fixed \( k \).

Thus, the proof of this lemma is complete. \( \square \)

4.2. Proof of Theorem 2.1 By Lemma 4.1 and Lemma 4.3, for any fixed \( k \), we have

\[
\|R_n - y\sigma^2 I_p^Q\|_2^k \leq \sum_{r=0}^{k} \sigma^{2(k-r)} \|R_n (r)\|_2 \sum_{j=0}^{[(k-r)/2]} C_j (k, r) y^{k-r-j}
\]

\[
\leq \sum_{r=0}^{k} \sigma^{2k} (2r + 1) (r + 1) y^{r-1} [(k-r)/2] 2^k y^{k-r}
\]

\[
\leq C\sigma^{2k} k \cdot 2^k y^{k-1}.
\]

Therefore,

\[
\|R_n - y\sigma^2 I_p^Q\|_2 \leq C^{1/k} \sigma^2 k^{4/k} 2y^{k-1}.
\]

Letting \( k \to \infty \), we obtain

\[
(4.20) \limsup \|R_n - y\sigma^2 I_p^Q\|_2 \leq 2\sigma^2 \sqrt{y} a.s..
\]
By (4.7), we have
\begin{equation}
\| S_n - \sigma^2 I_p^Q - R_n \|_2 = \| \text{diag}(S_n) - \sigma^2 I_p^Q \|_2 \\
\leq \max_{u \leq p} \left| \frac{1}{n} \sum_{v=1}^{n} (\| x_{uv} \|^2 - \sigma^2) \right| \to 0 \text{ a.s.}.
\end{equation}
(4.21)

Together with (4.20) and (4.21), one has
\[ \| S_n - \sigma^2 (1 + y) I_p^Q \|_2 \leq \| S_n - \sigma^2 I_p^Q - R_n \|_2 + \| R_n - y \sigma^2 I_p^Q \|_2 \leq 2 \sigma^2 \sqrt{y} \text{ a.s.} \]

The proof of Theorem 2.1 is complete.

5. PROOF OF THEOREM 2.2

Due to Theorem 1.1 in Li, Bai and Hu [12], with probability 1, we have
\[ \limsup s_{\min} (S_n) \leq \sigma^2 (1 - \sqrt{y})^2 \]
and
\[ \liminf s_p (S_n) \geq \sigma^2 (1 + \sqrt{y})^2. \]

Then, by Theorem 2.1
\[ \limsup s_p (S_n) = \sigma^2 (1 + y) + \limsup s_p (S_n - \sigma^2 (1 + y) I_p^Q) \]
\[ \leq \sigma^2 (1 + y) + 2 \sigma^2 \sqrt{y} = \sigma^2 (1 + \sqrt{y})^2 \]
and
\[ \liminf s_{\min} (S_n) \geq \sigma^2 (1 + y) + \liminf s_{\min} (S_n - \sigma^2 (1 + y) I_p^Q) \]
\[ \geq \sigma^2 (1 + y) - 2 \sigma^2 \sqrt{y} = \sigma^2 (1 - \sqrt{y})^2. \]

Combining the above inequalities, we get a.s.
\[ \lim s_{\min} (S_n) = (1 - \sqrt{y})^2 \sigma^2 \]
and
\[ \lim s_p (S_n) = (1 + \sqrt{y})^2 \sigma^2. \]
Therefore, we conclude the proof of Theorem 2.2.

6. PROOF OF THEOREM 2.4

By Remark 2.3, we only need to prove the necessity of the conditions.
6.1. **Condition** (i). Define a unit vector \( e_{2u} = (0, \cdots, 0, 1, 0, \cdots, 0)' \), \( u = 1, \cdots, p \), then,

\[
sp(Z_n) = s_{2p}(\psi(Z_n)) \geq e_{2u}'\psi(Z_n)e_{2u} = \frac{1}{n} \sum_{v=1}^{n} \|z_{uv}\|^2
\]

which implies that

\[
sp(Z_n) \geq \max_{u \leq p} \frac{1}{n} \sum_{v=1}^{n} \|z_{uv}\|^2.
\]

By Lemma 7.1, if \( \E \|z_{11}\|^4 = \infty \), we obtain

\[
\limsup_{n \to \infty} \max_{u \leq p} \frac{1}{n} \sum_{v=1}^{n} \|z_{uv}\|^2 \to \infty \text{ a.s.}
\]

This contradicts the assumptions. The condition (i) is proved.

6.2. **Condition** (iii). Suppose \( \E \|z_{11}\|^4 < \infty \) but \( \E(z_{11}) = \mathbf{h} \neq 0 \) (\( \mathbf{h} \) is a quaternion). Then

\[
\left\| \frac{1}{\sqrt{n}} Z_n \right\|_2 \geq \left\| \frac{1}{\sqrt{n}} (EZ_n) \right\|_2 - \left\| \frac{1}{\sqrt{n}} (Z_n - EZ_n) \right\|_2 \\
\geq \left\| \mathbf{h} \right\| \sqrt{\frac{p}{n}} - \left\| \frac{1}{\sqrt{n}} (Z_n - EZ_n) \right\|_2 \to \infty, \text{ a.s.}
\]

This contradicts the assumptions. The condition (iii) is proved.

6.3. **The completion of Theorem 2.4**. Conditions (ii) and (iv) follow from Theorem 2.1. Thus, the proof of the theorem is complete.

7. **APPENDIX**

In this section, we list some lemmas for readers convenience.

**Lemma 7.1** (Lemma B.25 in [4]). Let \( \{z_{jk}, j, k = 1, 2, \cdots \} \) be a double array of i.i.d. complex random variables and let \( \alpha > \frac{1}{2}, \beta \geq 0, \) and \( M > 0 \) be constants. Then, as \( n \to \infty \),

\[
\max_{j \leq Mn^\beta} \left| n^{-\alpha} \sum_{k=1}^{n} (z_{jk} - c) \right| \to 0, \text{ a.s.}
\]

if and only if the following hold:

(i) \( \E |z_{11}|^{(1+\beta)/\alpha} < \infty \);

(ii) \( c = \left\{ \begin{array}{ll}
\E(z_{11}), & \text{if } \alpha \leq 1, \\
\text{any number}, & \text{if } \alpha > 1.
\end{array} \right. \)
Lemma 7.2 (Lemma A.11 in [4]). Let $A$ be an $m \times n$ matrix with singular values $s_j(A), j = 1, 2, \ldots, q = \min\{m, n\}$, arranged in decreasing order. Then, for any integer $k (1 \leq k \leq q)$,

$$\sum_{j=1}^{k} s_j(A) = \sup_{E^*E = F^*F = I_k} \left| \text{tr}(E^*AF) \right|,$$

where the orders of $E$ are $m \times k$ and those of $F$ are $n \times k$.

References

[1] S. L. Adler. Quaternionic quantum mechanics and quantum fields, volume 1. Oxford University Press Oxford, 1995.
[2] Z. D. Bai. Methodologies in spectral analysis of large-dimensional random matrices, a review. Statist. Sinica, 9(3):611–677, 1999.
[3] Z. D. Bai and J. W. Silverstein. No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. The Annals of Probability, 26(1):316–345, 1998.
[4] Z. D. Bai and J. W. Silverstein. Spectral analysis of large dimensional random matrices. Springer, 2010.
[5] Z. D. Bai, J. W. Silverstein, and Y. Yin. A note on the largest eigenvalue of a large dimensional sample covariance matrix. Journal of Multivariate Analysis, 26(2):166 – 168, 1988.
[6] Z. D. Bai and Y. Q. Yin. Limit of the smallest eigenvalue of a large dimensional sample covariance matrix. The Annals of Probability, 21(3):pp. 1275–1294, 1993.
[7] Z. D. Bai, Y. Q. Yin, and P. R. Krishnaiah. On the limiting empirical distribution function of the eigenvalues of a multivariate f matrix. Theory of Probability & Its Applications, 32(3):490–500, 1987.
[8] D. L. Burkholder. Distribution function inequalities for martingales. the Annals of Probability, 1(1):19–42, 1973.
[9] D. Finkelstein, J. M. Jauch, S. Schiminovich, and D. Speiser. Foundations of quaternion quantum mechanics. Journal of mathematical physics, 3(2):207, 1962.
[10] S. Geman. A limit theorem for the norm of random matrices. The Annals of Probability, 8(2):252–261, 1980.
[11] J. B. Kuipers. Quaternions and rotation sequences. Princeton university press Princeton, 1999.
[12] H. Q. Li, Z. D. Bai, and J. Hu. Convergence of empirical spectral distributions of large dimensional quaternion sample covariance matrices. arXiv preprint arXiv:1310.5428, 2013.
[13] M. L. Mehta. Random matrices, volume 142. Access Online via Elsevier, 2004.
[14] J. W. Silverstein. The smallest eigenvalue of a large dimensional wishart matrix. The Annals of Probability, 13(4):1364–1368, 1985.
[15] W. So, R. C. Thompson, and F. Zhang. The numerical range of normal matrices with quaternion entries. Linear and Multilinear Algebra, 37(1-3):175–195, 1994.
[16] Y. Yin, Z. D. Bai, and J. Hu. On the limit of extreme eigenvalues of large dimensional random quaternion matrices. arXiv preprint arXiv:1312.1433, 2013.
[17] Y. Q. Yin, Z. D. Bai, and P. Krishnaiah. On the limit of the largest eigenvalue of the large dimensional sample covariance matrix. *Probability Theory and Related Fields*, 78(4):pp. 509–521, 1988.

[18] F. Zhang. On numerical range of normal matrices of quaternions. *J. Math. Physical Sci*, 29(6):235–251, 1995.

[19] F. Zhang. Quaternions and matrices of quaternions. *Linear algebra and its applications*, 251:21–57, 1997.

KLASMOE and School of Mathematics & Statistics, Northeast Normal University, Changchun, P.R.C., 130024.

*E-mail address: lihq118@nenu.edu.cn*

KLASMOE and School of Mathematics & Statistics, Northeast Normal University, Changchun, P.R.C., 130024.

*E-mail address: baizd@nenu.edu.cn*