Geometrical representation of the multi-dimensional consistency: 1-form case

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Abstract

A new notion of integrability called the multi-dimensional consistency for the integrable systems with the Lagrangian 1-form structure is captured in the geometrical language both classical and quantum levels. A zero-curvature condition, which implies the multi-dimensional consistency, will be a key relation in various contexts, e.g. Hamiltonian vector fields, Lagrangian vector fields, temporal Lax matrices and Hamiltonian operators. Therefore, the existence of the zero-curvature condition directly leads to the path-independent feature in various maps (which will be expressed in terms of the Wilson line), e.g. a family of maps in \(N\)-parameter groups on cotangent and tangent bundles, time evolution maps in the Lax pair level and unitary multi-time evolution operators in the Schrödinger picture. The main highlights of this work are the following. The new mathematical objects, called Hamilton vector field and Lagrange vector field defined in the space of independent variables, are introduced to alternatively capture integrability of the systems. To ensure the integrability, these new vector fields must be conservative and irrotational. Another important result is the formulation of the continuous multi-time propagator. With this new type of the propagator, a new perspective on summing all possible paths unavoidably arises as not only all possible paths on the space of dependent variables but also on the space of independent variables must be taken into account. The semi-classical approximation is applied to the multi-time propagator expressing in terms of the classical action and the fluctuation around the classical path and, therefore, the integrability of the quantum system can be captured through the path independent feature on the space of independent variables of the multi-time propagator.

Keywords: Integrability, Zero-curvature condition, Multi-dimensional consistency, Wilson loop

1 Introduction

Classically, the standard notion of integrability of the Hamiltonian systems is the Liouville-Arnold theorem \cite{1,2}. In this notion of integrability, the Hamiltonian systems, whose the evolution is given on \(2N\)-dimensional manifold called the cotangent bundle, must possess \(N\) invariances which are independent and in involution. A key feature in this context is the Hamiltonian commuting flows as a direct consequence of the involution. Alternatively, the integrability can be inferred from the existence of the r-matrix, which is equivalent with the involution relation, through the language of the Lax matrices \cite{1}.

In the discrete context, the standard Liouville-Arnold theorem can be constructed \cite{3}. However, the discrete world is quite fascinating in the sense that all variables are treated on the same equal footing. Consequently, there are various notions of integrability, e.g. existence of r-matrix \cite{4}, singularity confinement \cite{5} and algebraic entropy \cite{6}. However, there is one remarkable aspect of integrable multi-dimensional discrete systems known as a multi-dimensional consistency \cite{7}. With this feature, one can consistently express the difference equations in a multi-dimensional lattice, i.e., two dimensional lattice system can be consistently embedded
in a three dimensional lattice such that the quadrilateral equations describing three side-to-side connected surfaces of a cube can be solved for a coincide result with a given initial conditions [8]. This feature is known as the consistency-around-the cube (CAC). Later, Adler, Bobenko and Suris employed this property to classify the quadrilateral equations for two dimensional lattice known as the ABS list [9]. Another important aspect of the multi-dimensional consistency in the discrete level is the Lagrangian multi-form theory. Lobb and Nijhoff first set out to formulate the discrete theory for 2-form and 3-form cases [8,10].

A key relation in this context is the Lagrangian closure relation, which holds on the solution of the system, as a direct result of the variation of the action with respect to independent variables. The existence of the Lagrangian closure relation guarantees the constant value of the action under local deformation of the surface in the 2-form case and the volume in the 3-form case on the space of independent variables. Soon later, the 1-form case was formulated by Yoo-Kong, Lobb and Nijhoff [11] in both discrete and continuous levels through an important model known as the Calogero-Moser system [12,13], see also [14]. Again, the existence of the Lagrangian 1-form closure relation guarantees the constant value of the action under local deform of the curve on the space of independent variables. Indeed, this is nothing but the multi-dimensional consistency feature [1]. After these pioneer works, a series of papers has been producing and pushing further in various aspects as well as various systems [15–30].

In quantum realm, the notion of integrability is not well established. Naively, one can follow the canonical quantisation by promoting a set of invariances or a set of Hamiltonians to be a set of Hamiltonian operators. Therefore, the integrability demands commutator of the Hamiltonian operators to be zero [3]. However, Weigert [31] provided an encounter example, which is non-integrable system, satisfying the vanishing commutator condition. However, many attempts have been put further to investigate quantum integrability on demanding a quantum correction terms $\hbar^2$ [32,33], promoted from the invariances of the counterpart of classical system and commutations of them, see in [35]. In discrete level, a key tool to study quantum integrable system is the quantum mapping was established in [36] and was applied in the integrability context in [37,38]. Alternatively, Feynman approach on quantising the system might be a better choice [39]. The pioneer works on this direction were investigated by Field and Nijhoff [40], see also [41] in the discrete systems. Recently, King and Nijhoff set out to formulate quantum path integration incorporated with the quadratic Lagrangian multi-form structure in the discrete level [42]. What they did is to impose the periodic reduction on linearised discrete KdV in one particular direction, resulting in the discrete harmonic oscillator. Imposing on another discrete direction, one obtains another discrete harmonic oscillator. Therefore, the discrete Lagrangians for these harmonic oscillators can be explicitly written. Consequently, the explicit form of the multi-discrete propagator can be obtained, since the Gaussian integral can be used in this case. An intriguing feature of this multi-discrete propagator is path-independent feature on the space of independent discrete variables. In other words, the multi-discrete propagator remains the same under local deformation of the discrete paths on the space of independent discrete variables. Then, in this quantum scenario, one might have to take all possible discrete paths not only on the space of dependent variables, but also on the space of independent variables into account. This new conceptual view on propagator was first proposed by Nijhoff [43] in the continuous case to capture the multi-dimensional consistency in language of the path integrals. However, the explicit connection between the discrete set up and the Nijhoff’s continuous proposal for the Lagrangian multi-form of the propagator is still missing. In this contribution, the multi-dimensional consistency will be geometrically captured through the zero-curvature condition in various contexts, e.g. Hamiltonian vector fields, Lagrangian vector fields, temporal Lax matrices and Hamiltonian operators. A main result is the formulation for the multi-time propagator for the case of arbitrary Lagrangian 1-forms, satisfying the closure relation.

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1 One needs to include the variation with respect to dependent variables resulting in the generalised Euler-Lagrange equations and constraints. These equations all together give us a compatible system of equations describing the multi-time evolution of the system.

2 As we mentioned earlier that the multi-dimensional consistency was first formulated on the level of discrete equations of motion.

3 This can be viewed as the quantum analogue of the involution.
The structure of this paper is as follows. In section 2.1, the standard notion of the classical integrability for the Hamiltonian systems will be briefly discussed together with a key feature called the Hamiltonian commuting flows. The multi-dimensional consistency will be geometrically translated through the zero-curvature condition of the Hamiltonian vector fields. The structure of the N-parameter group will be explained and the family of composite maps will be expressed in terms of the Wilson loop. These composite maps are consistent, path-independent feature, as a consequence of the zero-curvature condition. In section 2.2, the Lagrangian analogue of the zero-curvature conditions in section 2.1 will be presented. In section 2.3, the compatibility of a set of Hamilton-Jacobi equations is discussed. In section 2.4, the zero-curvature condition of the temporal Lax matrices is given. The multi-time evolution of the spatial Lax matrix will be also discussed. In section 3.1, a set of multi-time Schrödinger equations is given and the consistency of these equation will be derived resulting in the zero-curvature condition for the Hamiltonian operators. Then the unitary multi-time operator is expressed in terms of the Wilson line. Again, this unitary multi-time operator is path-independent as a direct result of the zero-curvature condition of the Hamiltonian operators. In section 3.2, the multi-time propagator is systematically derived. The properties, such that path-independent feature and sum all possible paths on spaces of both dependent and independent variables, of this propagator will be discussed. In section 4, the results in the previous sections will be graphically illustrated through the generalised Stokes theorem. Here, new mathematical objects, called Hamilton vector fields and Lagrange vector fields defined on the space of time variables, are introduced. The properties of these vector fields will be discussed to ensure the integrability of the systems. In section 5, the summary together with important remarks will be given.

2 MULTI-DIMENSIONAL CONSISTENCY - CLASSICAL CASE

In this section, the integrability for the classical Hamiltonian and Lagrangian systems will be geometrically described. A main feature called the multi-dimensional consistency will be captured in terms of the zero-curvature condition through geometrical objects, e.g. Hamiltonian vector fields, Lagrangian vector fields, temporal Lax matrices.

2.1 Hamiltonian systems

We shall first give a standard definition of integrability for the Hamiltonian system \( H \), defined on the cotangent bundle \( T^{*}Q \) with a symplectic matrix \( \Omega \), through the existence of the invariances or integrals of motion defined in a set of all possible functions on the cotangent bundle \( \mathscr{F}(T^{*}Q) \).

**Theorem 2.1.** Liouville-Arnold theorem: Let \( ((T^{*}Q, \Omega), H) \) be an integrable Hamiltonian system with integrals of motion \( \{ T_{1}, T_{2}, ..., T_{N} \} \) and define \( \mathcal{F}: T^{*}Q \rightarrow \mathbb{R} \), where \( \mathcal{F} = \{ T_{1}, T_{2}, ..., T_{N} \} \in \mathscr{F}(T^{*}Q) \). These integrals of motion are independent and involution: \( \{ T_{m}, T_{n} \} = 0 \), where \( m \neq n = 1, 2, ..., N \). Then there exists a canonical transformation to action-angle variables \( (I, \theta) = (I_{1}, I_{2}, ..., I_{N}, \theta_{1}, \theta_{2}, ..., \theta_{N}) \), where \( I_{k} = I_{k}(H) \). The transformed Hamiltonian depends only the action variables and angle variables evolve linearly in time. Therefore, the equation of motion can be solved by quadratures: \( \dot{I}_{k} = 0, \dot{\theta}_{k} = \omega_{k}(I) \).

We will not provide a proof here, since there exists in the literature \[124\]. However, we shall explain an important feature called the Hamiltonian commuting flows. Traditionally, the set of integrals of motion will be treated as a set of Hamiltonians: \( \{ T_{1}, T_{2}, ..., T_{N} \} = \{ H_{1}, H_{2}, ..., H_{N} \} \). For any function \( F(\xi) \in \mathcal{F}(T^{*}Q) \), where \( \xi = (p, q) = (p_{1}, p_{2}, ..., p_{N}, q_{1}, q_{2}, ..., q_{N}) \), we find that

\[
\frac{\partial F}{\partial t_{k}} = \{ F, H_{k} \}, \tag{2.1}
\]

which gives the time evolution of the function \( F \) with respect to time \( t_{k} \). Moreover, we also have

\[
\frac{\partial F}{\partial t_{l}} = \{ F, H_{l} \}, \tag{2.2}
\]
which again gives the time evolution of the function $F$ with respect to time $t_l$. An interesting question is as follows: Does the order of the time evolutions of the function $F$ matter? To answer this question, we consider

$$
\frac{\partial}{\partial t_l} \frac{\partial F}{\partial t_k} - \frac{\partial}{\partial t_k} \frac{\partial F}{\partial t_l} = \frac{\partial}{\partial t_l} \{F, H_k\} - \frac{\partial}{\partial t_k} \{F, H_l\} = \{\{F, H_k\}, H_l\} - \{\{F, H_l\}, H_k\} = \{H_l, H_k\}, F\} = 0 ,
$$

where the last line is obtained by applying the Jacobi identity. Since $\{H_l, H_k\} = 0$, where $j \neq k = 1, 2, ..., N$, therefore, the order of the time evolutions does not matter. This feature is known as the Hamiltonian commuting flows which shall be referred as the multi-dimensional consistency on the Hamiltonian level.

### 2.1.1 Zero-curvature condition of the Hamiltonian vector fields

In this section, we will capture the integrability condition in terms of the Hamiltonian vector fields defined in $\mathcal{X}(T^*Q)$ which is a set of all possible vector fields on the cotangent bundle.

**Theorem 2.2.** Let $((T^*Q, \Omega), H)$ be a $2N$-dimensional manifold, equipped with $N$ linearly independent vector fields $X_{H_k} \in \mathcal{X}(T^*Q)$ satisfying

$$
\frac{\partial X_{H_k}}{\partial t_l} - \frac{\partial X_{H_l}}{\partial t_k} + [X_{H_k}, X_{H_l}] = 0 ,
$$

where $k \neq l = 1, 2, ..., N$. Therefore, the Hamiltonian commuting flows can be equivalently expressed in terms of the zero-curvature of the Hamiltonian vector fields and equation (2.4) implies the multi-dimensional consistency on the level of the Hamiltonian vector fields.

**Proof.** On $2N$-dimensional cotangent bundle $T^*Q$, the Hamiltonian vector field, $X_H \in \mathcal{X}(T^*Q)$, is treated as an operator which maps a function $F(\xi(0))$ to some later time $F(\xi(t))$ as follows

$$
\frac{dF(\xi)}{dt} = X_H(F(\xi)) .
$$

We then write equations (2.1) and (2.2) in alternative expressions

$$
\frac{\partial F(\xi)}{\partial t_k} = X_{H_k}(F(\xi)) , \text{ where } k = 1, 2, ..., N.
$$

Imposing the compatibility (2.3), which is a direct consequence of the involution, we obtain

$$
\left( \frac{\partial}{\partial t_l} X_{H_k}(F(\xi)) \right) = \frac{\partial}{\partial t_k} \left( X_{H_l}(F(\xi)) \right) = \left( \frac{\partial}{\partial t_l} X_{H_k} + X_{H_k} X_{H_l} \right) (F(\xi)) = \left( \frac{\partial}{\partial t_k} X_{H_l} + X_{H_l} X_{H_k} \right) (F(\xi)) = 0 \quad \left( \frac{\partial X_{H_k}}{\partial t_l} - \frac{\partial X_{H_l}}{\partial t_k} + [X_{H_k}, X_{H_l}] \right) (F(\xi)) ,
$$

where $l \neq k = 1, 2, ..., N$. Equation (2.8) holds if

$$
\frac{\partial X_{H_k}}{\partial t_l} - \frac{\partial X_{H_l}}{\partial t_k} + [X_{H_k}, X_{H_l}] = 0 ,
$$

which is the zero-curvature condition in terms of the Hamiltonian vector fields. Alternatively, the quantities $\partial_{t_k} X_{H_k}$, where $k = 1, 2, ..., N$, can be treated as the covariant derivative \[ ^4 \] and the system of equations

\[ ^4 \text{In general, all Hamiltonian vector fields } X_{H_t} \text{ are function of } t = (t_1, t_2, \cdots, t_N). \]
(2.6) is overdetermined. Thus, \( X_{H_k} \), where \( k = 1, 2, \ldots, N \), must satisfy a compatible condition leading to the zero-curvature condition (2.9). Another way to interpret the process in equations (2.7) and (2.9) is something to do with the parallel transport on the space of time(independent) variables. This means that the compatibility of two different flows will complete a parallelogram. We remark that the equation (2.9) can be simply reduced to \([X_{H_k}, X_{H_l}] = 0\), but the expression in equation (2.9) is better for further consideration.

2.1.2 \( N \)-parameter groups

In the standard language, the dynamics \( X_H \) would provide a family of maps \( \phi_X^H : T^*Q \to T^*Q \), known as a one-parameter group of transformations. A point \( \xi(t) \) is mapped to a new point \( \xi(t + \Delta t) \) along the vector field \( X_H \), see the figure 1. The mapping is defined as follows

\[
\xi(t + \Delta t) = \phi_{\Delta t}^H \xi(t),
\]

satisfying the composition rule as

\[
\phi_{\Delta s + \Delta t}^H = \phi_{\Delta s}^H \circ \phi_{\Delta t}^H.
\]

Figure 1: The action of the mapping \( \phi_X^H \) on \( T^*Q \).

The mapping \( \phi_{\Delta t}^H \) can be expressed in terms of the Wilson line

\[
\phi_{\Delta t}^H =: T e^{\int_0^{\Delta t} X_H dt},
\]

where \( T \) is the time ordering operator and the Hamiltonian vector field plays a role of gauge variable. Thus, the composition condition will be also expressed in terms of the Wilson line [51] as follows

\[
\phi_{\Delta s + \Delta t}^H = T e^{\int_0^{\Delta s + \Delta t} X_H dt} = T e^{\int_0^{\Delta t} X_H dt} T e^{\int_0^{\Delta s} X_H dt} =: \phi_{\Delta s}^H \circ \phi_{\Delta t}^H.
\]

**Theorem 2.3.** Let \(((T^*Q, \Omega), H)\) be a \( 2N \)-dimensional manifold, equipped with \( N \) linearly independent vector fields \( X_{H_k} \in \mathcal{X}(T^*Q) \). There exists an \( N \)-parameter group structure with composition rules

\[
\phi_{\Delta t_k}^H \circ \phi_{\Delta t_l}^H = \phi_{\Delta t_l}^H \circ \phi_{\Delta t_k}^H,
\]

where \( l \neq k = 1, 2, \ldots, N \). Therefore, the equation (2.14) implies the multidimensional consistency in terms of mapping on the cotangent bundle.

**Proof.** For simplicity, we shall consider here only 2-time situation. Since the Hamiltonian vector fields are actually commute, the composition map between 2 directions of times is given by, see appendix [A]

\[
\phi_X^{H_2} \circ \phi_X^{H_1} = T e^{\int_0^{\Delta t_2} X_{H_2} dt_2} T e^{\int_0^{\Delta t_1} X_{H_1} dt_1} = T e^{\int_0^{\sum_{j=1}^2 X_{H_j} dt_j}},
\]
where $\Gamma$ represents the trajectory of system on the cotangent bundle, see figure 2. Reordering of the maps, we obtain
\[
\phi_{\Delta t_1} \circ \phi_{\Delta t_2} = Te^{\int_{(0,\Delta t_2)} X_{H_1} dt_1} Te^{\int_{(0,\Delta t_2)} X_{H_2} dt_2} = Te^{\int_{\gamma} \sum_{j=1}^N X_{H_j} dt_j},
\]
where $\Gamma'$ represents the another trajectory of system on the cotangent bundle, see figure 2.

The results in equations (2.15) and (2.16) can be directly extended to the $N$-time variables resulting in
\[
\phi_{\Delta t_1} \circ \phi_{\Delta t_2} \circ \cdots \circ \phi_{\Delta t_N} = Te^{\int_{\gamma} \sum_{j=1}^N X_{H_j} dt_j},
\]
where $\gamma$ is the path defined on the cotangent bundle. Equation (2.17) is now expressed in terms of the Wilson line. Here is a thing. There are $N!$ possible composite maps as a result of permutation on the left hand side of (2.17). These mappings are consistent if the hyper-parallelogram is completed or, equivalently, the mapping for a loop yields identity: $\phi^H = I$, where the superscript $H$ refers to the Hamiltonian dynamics.

To see this argument, we consider the closed curve $C$ bounding the surface $S$ on cotangent bundle. The mapping along curve $C$ in multi-time structure can be expressed as follows
\[
\phi^H \bigcirc = Te^{\oint_C \sum_{j=1}^N X_{H_j} dt_j},
\]
Equation (2.18) is nothing but the Wilson loop representation of the mapping. Applying non-abelian Stokes theorem, we obtain
\[
\phi^H \bigcirc = Pe^{\int_S \sum_{h,j} \sum_{l=1}^N Z^H_{lk} dt_l \wedge dt_k},
\]
where $P$ is a surface ordering operator [52] and
\[
Z^H_{lk} = (\phi^H)^{-1} \left( \frac{\partial X_{H_k}}{\partial H_l} - \frac{\partial X_{H_l}}{\partial H_k} + [X_{H_k}, X_{H_l}] \right) \phi^H
\]
is the twisted curvature where $\gamma$ represents an arbitrary path connecting between the origin and $t$. Recalling equation (2.19), we obtain $Z^H_{lk} = 0$ which implies identity for the loop mapping.

2.2 Lagrangian systems

In this section, we consider the zero-curvature condition or integrability for the Lagrangian system $L$ defined on the tangent bundle $TQ$ with a set of coordinates $\eta = (\partial q/\partial t; j = 1, 2, ..., N, \mathbf{q})$, equipped with $N$ linearly independent vector fields $X_{L_k} \in \mathfrak{X}(TQ)$. We note here that a set of Lagrangians $\{L_1, L_2, ..., L_N\}$, where $L_j = L_j(\eta; \mathbf{t} = (t_1, t_2, ..., t_N))$, can be directly obtained from the set of Hamiltonians $\{H_1, H_2, ..., H_N\}$ by applying the Legendre transformation [29, 46].

![Figure 2: The action of the composite mapping.](image)
2.2.1 Lagrangian closure relation or Lagrangian commuting flows

In section 2.1, the key relation for integrability is the Hamiltonian commuting flows which is a direct result of the involution \( \{ H_k, H_l \} = 0 \), where \( k \neq l = 1, 2, ..., N \). In the Lagrangian context, there exists an equivalent relation known as the Lagrangian closure relation.

**Theorem 2.4.** The relation

\[
\frac{\partial L_k}{\partial t_l} = \frac{\partial L_l}{\partial t_k}, \quad \text{where} \quad k \neq l = 1, 2, ..., N,
\]

(2.21)

hold on solutions of the Euler-Lagrange equations. This relation implies that the value of the action \( S_\Gamma \) is constant with different choice of curve \( \Gamma \) connecting two given points on the space of time variables. In other words, the value of the action does not depend on path (path-independent) connecting two given points on the space of time variables.

\[
\text{(a) The action is invariant under local deformation of path: } \Gamma \mapsto \Gamma'.
\]

\[
\text{(b) The closed loop } C \text{ bounds the surface } S.
\]

**Figure 3:** The deformation of the path on the space of time variables.

**Proof.** The standard way to prove this theorem can be employed the variational principle, see [11, 29]. However, here we shall proceed with alternative approach. The action functional is give by

\[
S_\Gamma[q(t)] = \int_\Gamma \mathcal{L},
\]

(2.22)

where \( \mathcal{L} = \sum_{j=1}^{N} L_j dt_j \) is the Lagrangian 1-form and the curve \( \Gamma \) is given in figure 3a. If the action \( S_\Gamma = S_{\Gamma'} \) is invariant under local deformation of the curve: \( \Gamma \rightarrow \Gamma' \), it is equivalent to consider the action of a closed loop \( C \) bounding the surface \( S \) on the space of time variables, see figure 3b. What we have now is

\[
\oint_{C=\partial S} \sum_{j=1}^{N} L_j dt_j = \iint_{S} \sum_{k \neq l}^{N} \sum_{l=1}^{N} \left( \frac{\partial L_l}{\partial t_k} - \frac{\partial L_k}{\partial t_l} \right) dt_k \wedge dt_l = 0,
\]

(2.23)

after applying the Stokes theorem. This loop integral vanishes since any loop \( C \) on the space of time variables can be contracted to a point (of course, this is a direct result of path-independent feature). Therefore, the right hand side of equation (2.23) holds if

\[
\frac{\partial L_l}{\partial t_k} - \frac{\partial L_k}{\partial t_l} = 0, \quad \text{where} \quad k \neq l = 1, 2, ..., N,
\]

(2.24)

which shall be referred as multi-dimensional consistency on the Lagrangian level.
Next, we will show that the Hamiltonian involution can be directly obtained from equation (2.23). Employing the Legendre transformation, we obtain
\[
\oint_{C=\partial S} \sum_{j=1}^{N} L_j dt_j = \oint_{C=\partial S} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} \partial S \frac{\partial q_i}{\partial t_j} - H_j \right) dt_j = 0 .
\] (2.25)

Since \( \oint_{C=\partial S'} \sum_{j=1}^{N} \frac{\partial S}{\partial q_i} dq_i = 0 \) for any contractible closed curve \( C' \) bounding the surface \( S' \) on invariant torus according to the Liouville-Arnold theorem, equation (2.25) becomes
\[
\oint_{C=\partial S} \sum_{j=1}^{N} L_j dt_j = - \oint_{C'=\partial S'} \sum_{j=1}^{N} H_j dt_j = - \oint_{S'} \sum_{k>i} \sum_{l=1}^{N} \left( \frac{\partial H_i}{\partial t_k} - \frac{\partial H_k}{\partial t_i} \right) dt_k \wedge dt_l = 0 .
\] (2.26)

Therefore, one obtains
\[
\frac{\partial H_i}{\partial t_k} - \frac{\partial H_k}{\partial t_i} = 0 ,
\] (2.27)
which leads to \( \{ H_k, H_l \} = 0 \).

**Remark.** In the case of Lagrangian 1-forms, there are another two systems of equations, generalised Euler-Lagrange equations and constraints. Together with the closure relations, they will form a set of compatible equations. In other words, these three systems of equations will give the compatible evolution of the systems both on the space of dependent and independent variables, see [46]. Another intriguing point is that one can view the Lagrangian as the solution of this set of equations. This means that one can use this set of equations in searching for integrable systems, see [7]. In the case of Hamiltonian, there are also another set of equations known as Hamilton’s equations, see [20, 46]. Again, together with the Hamiltonian commuting flows (2.27), they will form a set of compatible equations and therefore the evolution of the system will be compatible both on the space of dependent and independent variables.

### 2.2.2 Zero-curvature condition of the Lagrangian vector fields

Here, the multi-dimensional consistency will be captured in terms of the Lagrangian vector fields defined in \( \mathcal{X}(TQ) \) which is a set of all possible vector fields on the tangent bundle.

**Theorem 2.5.** Let \((TQ, L)\) be a \(2N\)-dimensional manifold, equipped with \(N\) linearly independent vector fields \(X_{L_k} \in \mathcal{X}(TQ)\) satisfying
\[
\frac{\partial X_{L_k}}{\partial t_l} - \frac{\partial X_{L_l}}{\partial t_k} + [X_{L_k}, X_{L_l}] = 0 ,
\] (2.28)
where \( k \neq l = 1, 2, ..., N \). Therefore, equation (2.28) implies the multi-dimensional consistency in terms of the zero-curvature condition of the Lagrangian vector fields.

**Proof.** On 2N-dimensional tangent bundle \( TQ \) with any function \( F(\eta) \in \mathcal{F}(TQ) \), the Lagrangian vector field, \( X_L \in \mathcal{X}(TQ) \), is treated as an operator which maps a function \( F(\eta(0)) \) to some later time \( F(\eta(t)) \) as follows
\[
\frac{dF(\eta)}{dt} = X_L(\eta) .
\] (2.29)
With existence of the Lagrangian hierarchy \( \{ L_1, L_2, ..., L_N \} \), it is natural to introduce the evolutions
\[
\frac{\partial F(\eta)}{\partial t_k} = X_{L_k}(\eta) , \text{ where } k = 1, 2, ..., N .
\] (2.30)
The quantities \( \partial_{t_k} - X_{L_k} \), where \( k = 1, 2, ..., N \), can be treated as the covariant derivative and the system of equations (2.30) is overdetermined. Thus, \( X_{L_k} \), where \( k = 1, 2, ..., N \), must satisfy a compatible condition.
such that
\[
\frac{\partial}{\partial t_l} \left( X_{L_k}(F(\eta)) \right) = \frac{\partial}{\partial t_k} \left( X_{L_l}(F(\eta)) \right)
\]
\[
\left( \frac{\partial}{\partial t_l} X_{L_k} + X_{L_k} X_{L_l} \right)(F(\eta)) = \left( \frac{\partial}{\partial t_k} X_{L_l} + X_{L_l} X_{L_k} \right)(F(\eta))
\]
\[
0 = \left( \frac{\partial X_{L_k}}{\partial t_l} - \frac{\partial X_{L_l}}{\partial t_k} + [X_{L_k}, X_{L_l}] \right)(F(\eta)) ,
\]
where \( l \neq k = 1, 2, \ldots, N \). Equation (2.32) holds if
\[
\frac{\partial X_{L_k}}{\partial t_l} - \frac{\partial X_{L_l}}{\partial t_k} + [X_{L_k}, X_{L_l}] = 0 ,
\]
which is the zero-curvature condition in terms of the Lagrangian vector fields.

\[\square\]

2.2.3 \( N \)-parameter groups

Figure 4: The action of the mapping \( \phi_{\Delta t}^{X_L} \) on \( TQ \).

Again, in the standard language, the dynamics \( X_L \) would provide a family of maps \( \phi_{\Delta t}^{X_L} : TQ \to TQ \), known a one-parameter group of transformations. A point \( \eta(t) \) is mapped to a new point \( \eta(t+\Delta t) \) along the vector field \( X_L \), see the figure 4. The mapping is defined as follows
\[
\eta(t + \Delta t) = \phi_{\Delta t}^{X_L} \eta(t) ,
\]
satisfying the composition rule as
\[
\phi_{\Delta s+\Delta t}^{X_L} = \phi_{\Delta s}^{X_L} \circ \phi_{\Delta t}^{X_L} .
\]
The mapping \( \phi_{\Delta t}^{X_L} \) can also be expressed in terms of the Wilson line
\[
\phi_{\Delta t}^{X_L} = T e^{\int_{0}^{\Delta t} X_L dt} ,
\]
where \( T \) is the time ordering operator and here the Lagrangian vector field plays a role of gauge variable. Moreover, the composite map gives
\[
\phi_{\Delta s+\Delta t}^{X_L} = T e^{\int_{0}^{\Delta s+\Delta t} X_L dt} = T e^{\int_{0}^{\Delta s} X_L dt} T e^{\int_{0}^{\Delta t} X_L dt} =: \phi_{\Delta s}^{X_L} \circ \phi_{\Delta t}^{X_L} .
\]
\textbf{Theorem 2.6.} Let \((T\mathcal{Q}, L)\) be a \(2N\)-dimensional manifold, equipped with \(N\) linearly independent vector fields \(X_{L_l} \in \mathcal{X}(T\mathcal{Q})\). There exists an \(N\)-parameter group structure with composition rules
\[ X_{L_k} \circ X_{L_l} = \phi_{\Delta_{k_l}} \circ \phi_{\Delta_{k_l}}, \]
where \(l \neq k = 1, 2, \ldots, N\). Therefore, equation (2.38) implies the multidimensional consistency in terms of mapping on the tangent bundle.

\textit{Proof.} The proof is the same with theorem 2.3, then we shall not repeat it here. However, we give a key relation as follows
\[ \phi^L = T_e \exp_{\theta A} \sum_{j=1}^N X_{L_j} dt_j = P e^{-t A} \sum_{k=1}^N \sum_{l=1}^N \dot{Z}^L_{lk} dt_l dt_k, \]
where
\[ \dot{Z}^L_{lk} = (\phi^L)^{-1} \left( \frac{\partial X_{L_k}}{\partial t_l} - \frac{\partial X_{L_l}}{\partial t_k} + [X_{L_k}, X_{L_l}] \right) \phi^L, \]
is the twisted curvature in this description. Recalling equation (2.33), \( \dot{Z}^l_k \equiv 0 \) and therefore, \( \phi^L = I \). \( \square \)

\subsection{2.3 Hamilton-Jacobi equations}

Since we have a set of Hamiltonians \( \{H_1, H_2, \ldots, H_N\} \), it therefore exists a set of Hamilton-Jacobi equations
\[ \frac{\partial S(q, t)}{\partial t_k} + H_k \left( q, \frac{\partial S}{\partial q}; t \right) = 0, \quad k = 1, 2, \ldots, N, \]
where \( S(q, t) \) is the multi-time Hamilton principle function and \( \frac{\partial S}{\partial q} = \left\{ \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \ldots, \frac{\partial S}{\partial q_N} \right\} \).

\textbf{Theorem 2.7.} A set of partial differential equations (2.41) will be simultaneously satisfied if
\[ \frac{\partial H_k}{\partial t_l} - \frac{\partial H_l}{\partial t_k} + \{H_k, H_l\} = 0, \quad k \neq l = 1, 2, \ldots, N, \]
hold.

\textit{Proof.} If a set of equations (2.41) will have a nontrivial common solution, Hamiltonians must satisfy a certain condition. To see this, we consider a compatibility between the flow \( t_k \) and flow \( t_l \) in the following. We first compute
\[ \frac{\partial^2 S}{\partial t_k \partial t_l} = \frac{\partial H_l}{\partial t_k} + \sum_{m=1}^N \frac{\partial H_l}{\partial q_m} \left( \frac{\partial S}{\partial q_m} \right) \left( \frac{\partial S}{\partial q_n} \right) \left( \frac{\partial S}{\partial q_n} \right), \]
\[ \frac{\partial^2 S}{\partial t_l \partial t_k} = -\frac{\partial H_k}{\partial t_l} + \sum_{m=1}^N \frac{\partial H_k}{\partial q_m} \left( \frac{\partial S}{\partial q_m} \right) \left( \frac{\partial S}{\partial q_n} \right) \left( \frac{\partial S}{\partial q_n} \right). \]
The compatibility
\[ \frac{\partial^2 S}{\partial t_k \partial t_l} = \frac{\partial^2 S}{\partial t_l \partial t_k}, \]
gives
\[ \frac{\partial H_k}{\partial t_l} - \frac{\partial H_l}{\partial t_k} + \{H_k, H_l\} = 0, \]
where \( k \neq l = 1, 2, \ldots, N \). Therefore, this result is consistent with what we have in theorem 2.4, see the equation (2.27). \( \square \)

\subsection{2.4 Lax pairs}

In previous sections, the zero-curvature condition is captured in various contexts e.g. Hamiltonian vector fields and Lagrangian vector fields. Here in this section, the zero-curvature condition will be expressed in terms of the temporal Lax matrices.
2.4.1 Zero-curvature condition of the temporal part of Lax matrix

In multi-time structure, the Lax pair consists of the $N$-pair matrices $(L_i, M_i)$, where $l = 1, 2, ..., N$, satisfying

\[L\Psi = \lambda \Psi, \quad M_i \Psi = \frac{\partial \Psi}{\partial t_i},\tag{2.47}\]

where $\Psi$ is an auxiliary function, $\lambda$ is a constant.

**Theorem 2.8.** A set of Lax equations

\[\frac{\partial L}{\partial t_l} = -[L, M_l], \quad l = 1, 2, ..., N,\tag{2.49}\]

will be simultaneously satisfied if

\[\frac{\partial M_k}{\partial t_l} - \frac{\partial M_l}{\partial t_k} + [M_k, M_l] = 0, \quad k \neq l = 1, 2, ..., N,\tag{2.50}\]

hold.

**Proof.** From equations (2.48), the quantities $\partial_t k - M_k$, where $k = 1, 2, ..., N$, can be treated as the covariant derivative and the system of equations (2.48) is over determined. Thus, $M_k$, where $k = 1, 2, ..., N$, must satisfy a compatible condition

\[\frac{\partial^2 \Psi}{\partial t_k \partial t_l} = \frac{\partial^2 \Psi}{\partial t_l \partial t_k},\tag{2.51}\]

to ensure the existence of the function $\Psi$, resulting in

\[\frac{\partial M_k}{\partial t_l} - \frac{\partial M_l}{\partial t_k} + [M_k, M_l] = 0,\tag{2.52}\]

where $k \neq l = 1, 2, ..., N$. Indeed, equation (2.52) is nothing but the zero-curvature condition.

Alternatively, given an invertible matrix $G(t)$, the spatial and temporal matrices can be expressed as

\[L(t) = G(t)L(0)G^{-1}(t),\tag{2.53}\]
\[M_l(t) = \frac{\partial G(t)}{\partial t_l}G^{-1}(t).\tag{2.54}\]

We note here that, in the Liouville-Arnold notion of integrability, the integrals of motion can be obtained from $T_k = \frac{1}{k} \text{Tr} L^k(t)$. The invertible matrix $G$ can be expanded in terms of the temporal Lax matrices. From equation (2.54), one obtains

\[G_l(t) = G(0, 0, ..., \Delta t_l, ..., 0) = T e^{\int_{0}^{\Delta t_l} M_l dt_l},\tag{2.55}\]

which is a Wilson line with the temporal Lax matrices playing a role of the gauge variable. With the structure of equations (2.53) and (2.54), the invertible matrix $G$ behaves as a time evolution operator. Therefore, on the space of time variables, the spatial Lax matrix $L$ can be mapped in many different ways

\[L(t) = G_{\Gamma}(t)L(0)G_{\Gamma}^{-1}(t),\tag{2.56}\]
\[L(t) = G_{\Gamma'}(t)L(0)G_{\Gamma'}^{-1}(t),\tag{2.57}\]

where $\Gamma$ and $\Gamma'$ are given in figure 3a and

\[G_{\Gamma}(t) = T e^{\int_{t_0}^{t} \sum_{k=1}^{N} M_k dt_k}.\tag{2.58}\]

This mapping will be consistent if $G_{\Gamma} = G_{\Gamma'}$, which is again nothing but the path-independent feature on the space of time variables or, equivalently, the loop evolution is identity: $G_{\Gamma}(t) = I$. To see this, we write

\[G_{\Gamma}(t) = T e^{\int_{t_0}^{t} \sum_{k=1}^{N} M_k dt_k} = P e^{\int_{t_0}^{t} \sum_{k=1}^{N} M_k dt_k} \sum_{i=1}^{N} Z_{i}^{M} dt_i \wedge dt_k,\tag{2.59}\]

where

\[Z_{i}^{M} = G_{\gamma}^{-1} \left(\frac{\partial M_k}{\partial t_l} - \frac{\partial M_l}{\partial t_k} + [M_k, M_l]\right) G_{\gamma},\tag{2.60}\]

is the twisted curvature in this description. Using equation (2.52), $Z_{i}^{M} = 0$ and, therefore, the loop mapping is identity as we expected.
3 MULTI-DIMENSIONAL CONSISTENCY - QUANTUM CASE

In section 2, the multi-dimensional consistency in the classical level is captured in terms of the zero-curvature condition both Hamiltonian and Lagrangian systems. Moreover, all compatible evolution mappings can be expressed in the language of the Wilson line, resulting in identity for loop evolution. At this point, it is very natural to extend the idea to the quantum level. In the standard manner, with the existence of the Hamiltonian, one can lift to the quantum case with the Schrödinger approach. With the existence of the Hamiltonian hierarchy, one expects to have a set of Schrödinger equations and therefore, the compatible multi-time evolution of the wave function is what we are interested. Alternatively, one can prefer to quantise the system with Feynman approach based on the action and Lagrangian. In the same analogy, here there exists the Lagrangian hierarchy. Then, one also expects to have multi-time propagators with a compatible multi-time evolution.

3.1 Schrödinger picture and zero-curvature condition

In this section, the multi-time evolution of the wave function will be studied. To derive a set of Schrödinger equations, we shall use (2.41) and take
\[ S = -i\hbar \ln \Psi(q, t) = -i\hbar \ln \langle q | \Psi(t) \rangle, \]
resulting in
\[ i\hbar \frac{\partial}{\partial t_l} | \Psi(t) \rangle = \hat{H}_l | \Psi(t) \rangle, \quad l = 1, 2, ..., N, \]
(3.1)
where \( \hat{H}_l \) are Hamiltonian operators associated with time variable \( t_l \) and \( | \Psi(t) \rangle \) is the eigenstate for \( \hat{H}_l, l = 1, 2, ..., N \).

**Theorem 3.1.** Let \( | \Psi(t) \rangle \) be a multi-time normalised vector in a Hilbert space \( \mathcal{H} \) and \( \{ \hat{H}_l : \mathcal{H} \rightarrow \mathcal{H}, l = 1, 2, ..., N \} \) be a set of Hamiltonian operators. All Schrödinger equations in (3.1) will be consistent if
\[ \frac{\partial \hat{H}_k}{\partial t_l} - \frac{\partial \hat{H}_l}{\partial t_k} - \frac{i}{\hbar} [\hat{H}_k, \hat{H}_l] = 0, \quad k \neq l = 1, 2, ..., N, \]
(3.2)
hold.

**Proof.** The compatibility of (3.1), where the same argument with theorems 2.2, 2.5 and 2.8 is applied,
\[ \frac{\partial^2}{\partial t_l \partial t_k} | \Psi(t) \rangle = \frac{\partial^2}{\partial t_l \partial t_k} | \Psi(t) \rangle, \]
(3.3)
will directly give
\[ \frac{\partial \hat{H}_k}{\partial t_l} - \frac{\partial \hat{H}_l}{\partial t_k} - \frac{i}{\hbar} [\hat{H}_k, \hat{H}_l] = 0, \quad k \neq l = 1, 2, ..., N, \]
(3.4)
which is therefore the zero-curvature condition for the Hamiltonian operators.

Alternatively, if we define a unitary operator for the multi-time evolution \( \hat{U}(t) \) such that
\[ | \Psi(t) \rangle = \hat{U}(t) | \Psi(0) \rangle, \]
(3.5)
where \( \hat{U}^\dagger(t) \hat{U}(t) = 1 \) results in \( \langle \Psi(t) | \Psi(t) \rangle = \langle \Psi(0) | \Psi(0) \rangle \), the equations (3.1) give a set of equations
\[ i\hbar \frac{\partial}{\partial t_l} \hat{U}(t) = \hat{H}_l \hat{U}(t), \quad l = 1, 2, ..., N. \]
(3.6)
If we are interested in the unitary evolution associated with time variable \( t_l \), we obtain
\[ \hat{U}_l(t) = \hat{U}_l(0, 0, \ldots, \Delta t_l, \ldots, 0) = T e^{-\frac{i}{\hbar} \int_0^{\Delta t_l} \hat{H}_l dt_l}. \]
(3.7)
Therefore, on the space of time variables, the unitary multi-time evolution operator, mapping the state along the path $\Gamma$ shown in figure 3a, is given by

$$\hat{U}_\Gamma(t) = T e^{-\frac{i}{\hbar} \int_{\gamma} \sum_{j=1}^{N} \hat{H}_j dt_j}.$$  \hspace{1cm} (3.8)

Equation (3.8) is nothing but the Wilson line representation of the operator $\hat{U}$ and the Hamiltonian operators will be treated as the gauge variables in this situation. Under the local deformation: $\Gamma \rightarrow \Gamma'$, the evolution will consistent if $\hat{U}_\Gamma = \hat{U}_{\Gamma'}$. Therefore, this relation implies the path-independent feature of the multi-time evolution of the state $|\Psi\rangle$. Equivalently, this path-independent feature gives identity for the loop evolution such that

$$\hat{U}_\gamma(t) = T e^{-\frac{i}{\hbar} \int_{C=0\gamma} \sum_{j=1}^{N} \hat{H}_j dt_j} = P e^{-\frac{i}{\hbar} \int_{\gamma} \sum_{k,l=1}^{N} Z_{ik} \hat{H}_k dt_l \wedge dt_k} = I,$$  \hspace{1cm} (3.9)

where

$$Z_{ik}^\gamma = U^{-1} \left( \frac{\partial \hat{H}_k}{\partial t_l} - \frac{\partial \hat{H}_l}{\partial t_k} - \frac{i}{\hbar} [\hat{H}_k, \hat{H}_l] \right) U = 0 \hspace{1cm} (3.10)$$

is the twisted curvature in the Schrödinger picture. The vanishing curvature is nothing but the equation (3.4). We finally note that this compatible multi-time evolution is actually commuting of the multi-time unitary operators: $[\hat{U}_l, \hat{U}_k] = 0$.

**Remark.** The integrable quantum systems must possess the zero-curvature condition, but the converse is not necessary true [31].

### 3.2 Feynman picture and compatible multi-time propagators

Let us first briefly outline what we are going to do in this section. We shall first give derivation with a general setup and later fit our results with a recent development on quantum multi-dimensional consistency [42]. Therefore, we would like to provide some key points of King and Nijhoff work. What they did is looking on the two dimensional periodic reduction of the lattice KdV, resulting in two different discrete-time harmonic oscillators. However, by doing redefinition of discrete parameters, one obtains two different continuous harmonic oscillators

$$\frac{d^2 q}{dt_1^2} + \omega_1^2 q = 0 \hspace{1cm} (3.11)$$

$$\frac{d^2 q}{dt_2^2} + \omega_2^2 q = 0 \hspace{1cm} (3.12)$$

where $q(t_1, t_2)$ is position variables and $(t_1, t_2)$ are two different time variables. Here the mass of the system is set to be one. The variables $\omega_1$ and $\omega_2$ play the role of frequency for the first and second systems, respectively. Then, they study properties of the multi-time propagator (in the discrete setup) for the quadratic Lagrangian 1-form case. The King-Nijhoff formula for multi-discrete propagator

$$K(q_b(M, N); q_a(0, 0)) = \sum_{\Gamma \in \mathcal{P}} \mathcal{M}_\Gamma \mathcal{K}_\Gamma(q_b(M, N); q_a(0, 0)) \hspace{1cm} (3.13)$$

possesses the path independent feature. Then loops will not contribute to the propagator as a direct result of Lagrangian closure relation. Here $\mathcal{M}_\Gamma$ is a normalising factor and $\mathcal{P}$ is a set of all possible paths in the discrete space of independent variables $(n, m)$, see figure 5.

This section aims to extend their result to the continuous case. We first would like to provide some basic ingredients for the standard propagator (the single-time propagator). Performing time-slicing process,
we obtain
\[
\langle \mathbf{q}'' | \Psi(t'') \rangle = \int_{-\infty}^{\infty} d^N \mathbf{q}' \int_{-\infty}^{\infty} d^N \mathbf{q}_1 \langle \mathbf{q}' | \hat{U}(t'' - t') | \mathbf{q}' \rangle \langle \mathbf{q}'' | \hat{U}(t'' - t') | \mathbf{q}' \rangle
\]
\[
= \int_{-\infty}^{\infty} d^N \mathbf{q}' \int_{-\infty}^{\infty} d^N \mathbf{q}_1 \langle \mathbf{q}' | \hat{U}(t'' - t_1) | \mathbf{q}_1 \rangle \langle \mathbf{q}_1 | \hat{U}(t_1 - t') | \mathbf{q}' \rangle \langle \mathbf{q}'' | \Psi(t') \rangle
\]
\[
\vdots
\]
\[
\langle \mathbf{q}_N | \Psi(t_N) \rangle = \lim_{N \to \infty} \left( \prod_{j=0}^{N-1} \int_{-\infty}^{\infty} d^N \mathbf{q}_j \right) \left( \prod_{j=0}^{N-1} \langle \mathbf{q}_j+1 | \hat{U}(t_{j+1} - t_j) | \mathbf{q}_j \rangle \right) \langle \mathbf{q}_0 | \Psi(t_0) \rangle .
\] (3.14)

Now we set \( \langle \mathbf{q}'' | \Psi(t'') \rangle = \langle \mathbf{q}_N | \Psi(t_N) \rangle, \) \( \langle \mathbf{q}' | \Psi(t') \rangle = \langle \mathbf{q}_0 | \Psi(t_0) \rangle \) and \( \epsilon = t_{j+1} - t_j \) with \( t_N > t_{N-1} > \cdots > t_{j+1} > t_j \cdots > t_1 > t_0. \) We then define a discrete propagator:
\[
K(\mathbf{q}_{j+1}, t_{j+1}; \mathbf{q}_j, t_j) =: \langle \mathbf{q}_{j+1} | \hat{U}(\epsilon) | \mathbf{q}_j \rangle = \mathcal{N}_j e^{i S(\mathbf{q}_j, \mathbf{q}_{j+1})},
\] (3.15)

where \( \mathcal{N}_j \) is a normalising factor and \( L(\mathbf{q}_j, \mathbf{q}_{j+1}) \) is a discrete Lagrangian. Taking the continuum limit on equation (3.14), one obtains
\[
\langle \mathbf{q}_N | \Psi(t_N) \rangle = \int_{-\infty}^{\infty} d^N \mathbf{q}_0 K(\mathbf{q}_N, t_N; \mathbf{q}_0, t_0) \langle \mathbf{q}_0 | \Psi(t_0) \rangle ,
\] (3.16)

where \( K(\mathbf{q}_N, t_N; \mathbf{q}_0, t_0) \) is the propagator given by
\[
K(\mathbf{q}_N, t_N; \mathbf{q}_0, t_0) =: \int_{\mathbf{q}_0}^{\mathbf{q}_N} \mathcal{D}[\mathbf{q}(t)] e^{i \frac{\hbar}{\epsilon} S(\mathbf{q}(t))} .
\] (3.17)

Here \( S[\mathbf{q}(t)] = \int_{t_0}^{t_N} L(\mathbf{q}, \dot{\mathbf{q}}; t) dt \) is the action functional and \( L(\mathbf{q}, \dot{\mathbf{q}}; t) \) is the standard Lagrangian. The notation
\[
\int_{\mathbf{q}_0}^{\mathbf{q}_N} \mathcal{D}[\mathbf{q}(t)] =: \lim_{N \to \infty} \prod_{j=1}^{N-1} \mathcal{N}_j \int_{-\infty}^{\infty} d^N \mathbf{q}_j
\] (3.18)
plays the role of the functional measure over the configuration space of the paths (spatial paths). The role of the propagator is to map the state from \( \langle \mathbf{q}_0 | \Psi(t_0) \rangle \) to \( \langle \mathbf{q}_N | \Psi(t_N) \rangle, \) see (3.16).
In practice, the propagator (3.17) is not handy in the calculation. Therefore, the semi-classical approximation method is applied by considering all constructive contributions around the classical path. To proceed such method, we write \( q(t) = q_c(t) + y(t) \), see figure 6 where \( q_c(t) \) is the classical path and \( y(t) \) is a fluctuation with the boundary conditions:

\[
y(t'') = y(t') = 0 . \tag{3.19}
\]

![Figure 6: The fluctuation around the classical path \( q_c(t) \).](image)

The action can be written as

\[
S[q(t)] = S[q_c(t) + y(t)] = S[q_c(t)] + \int_{t'}^{t''} dt \frac{\delta S[q_c(t)]}{\delta q(\tau)} y(\tau) + \frac{1}{2!} \int_{t'}^{t''} dt \int_{t'}^{t''} d\tau \frac{\delta^2 S[q_c(t)]}{\delta q(\tau) \delta q(\sigma)} y(\tau)y(\sigma) + \frac{1}{3!} \int_{t'}^{t''} dt \int_{t'}^{t''} d\tau \int_{t'}^{t''} d\sigma \frac{\delta^3 S[q_c(t)]}{\delta q(\tau) \delta q(\sigma) \delta q(\zeta)} y(\tau)y(\sigma)y(\zeta) + O(y^4) ,
\]

where

\[
\int_{t'}^{t''} dt \frac{\delta S[q_c(t)]}{\delta q(\tau)} y(\tau) = \int_{t'}^{t''} dt y(t) \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \bigg|_{q = q_c} = 0 , \tag{3.21}
\]

\[
\int_{t'}^{t''} dt \int_{t'}^{t''} d\sigma \frac{\delta^2 S[q_c(t)]}{\delta q(\tau) \delta q(\sigma)} y(\tau)y(\sigma) = \int_{t'}^{t''} dt \left[ y^2 \left( \frac{\partial^2 L}{\partial q^2} - \frac{d}{dt} \left( \frac{\partial^2 L}{\partial q \partial \dot{q}} \right) \right) + y^3 \frac{\partial^2 L}{\partial q^3} \right] \bigg|_{q = q_c} , \tag{3.22}
\]

\[
\int_{t'}^{t''} dt \int_{t'}^{t''} d\tau \int_{t'}^{t''} d\sigma \frac{\delta^3 S[q_c(t)]}{\delta q(\tau) \delta q(\sigma) \delta q(\zeta)} y(\tau)y(\sigma)y(\zeta) = \int_{t'}^{t''} dt \left[ y^3 \left( \frac{\partial^2 L}{\partial q^3} - \frac{d}{dt} \left( \frac{\partial^3 L}{\partial q^2 \partial \dot{q}} \right) \right) + 3y^2 \frac{\partial^2 L}{\partial q^2 \partial \dot{q}} + y^3 \frac{\partial^3 L}{\partial q^3} \right] \bigg|_{q = q_c} . \tag{3.23}
\]

Applying (3.21) (Euler-Lagrange equation) and rewriting the variables \( y = \sqrt{\beta} \tilde{y} \) with \( \tilde{y} \sim O(1) \), the propagator takes the form

\[
K \left( q'' , t'' ; q' , t' \right) = e^{i S[q_c(t)]} \int_{q'}^{q''} \mathcal{D}[y(t)] e^{i \int_{t'}^{t''} dt \int_{t'}^{t''} d\tau \left( \tilde{y}(\tau) \frac{\delta^2 S[q_c(t)]}{\delta q(\tau) \delta q(\sigma)} \tilde{y}(\sigma) \right) + O(\sqrt{\beta}^3) . \tag{3.24}
\]
Since \( y \) is extremely small, the expansion of the propagator gives
\[
K(q'', t''; q', t') = e^\frac{iS[q(t)]}{\hbar} \int_{q'}^{q''} \mathcal{D}[y(t)] e^{\frac{i}{\hbar} \int_{t'}^{t''} ds \left( \frac{\delta^2 S[q(t)]}{\delta q(t) \delta q(t')} y(t) y(t') \delta^2 y(t) \delta^2 y(t') \right)}
\]
\[
\times \left\{ 1 + \frac{i\sqrt{\hbar}}{3!} \int_{t'}^{t''} d\tau \int_{t'}^{t''} d\sigma \int_{t'}^{t''} d\zeta \frac{\delta^3 S[q(t)]}{\delta q(t) \delta q(\tau) \delta q(\sigma)} \delta^3 y(\tau) \delta^3 y(\sigma) \right\}.
\]
(3.25)

We see that the integrand of order \( \hbar^{1/2} \) is odd under the interchange \( y \rightarrow -y \) resulting in
\[
K(q'', t''; q', t') = e^\frac{iS[q(t)]}{\hbar} \mathcal{Q}(q'', t'', q', t') [1 + O(\hbar)],
\]
(3.26)
where
\[
\mathcal{Q}(q'', t'', q', t') = \int_{q'}^{q''} \mathcal{D}[y(t)] e^{\frac{i}{\hbar} \int_{t'}^{t''} ds \left( \frac{\delta^2 S[q(t)]}{\delta q(t) \delta q(t')} y(t) \delta^2 y(t) \right)}.
\]
(3.27)

Here \( \mathcal{Q}(q'', t'', q', t') \) is a smooth function of end points since the variable \( y \), is integrated out. The explicit form of the function \( \mathcal{Q}(q'', t'', q', t') \) can be obtained by
\[
\mathcal{Q}(q'', t'', q', t') = \det \left( \frac{1}{2\pi \hbar} \frac{\partial^2 S[q(t)]}{\partial q(t') \partial q(t'')} \right)^{1/2}.
\]
(3.28)

Next, we would like to extend the notion of the propagator (3.17) into the case of multi-time Lagrangian 1-forms.

**Definition 3.1** (A multi-time propagator). Let \( \mathcal{L} = \sum_{j=1}^{N} L_j dt_j \) be Lagrangian 1-form, where \( L_j = L_j(\eta) \).

On the space of independent variables (time variables) parametrised by a variable \( s \) such that \( t(s) \), where \( s' < s < s'' \), multi-time propagator is given by
\[
K(q(s''), s''; q(s'), s') = \int_{q(s')}^{q(s'')} \mathcal{D}[q(s); \Gamma] e^{\frac{i}{\hbar} \int_{s'}^{s''} ds \mathcal{L}},
\]
(3.29)
where \( \mathcal{L} = ds \sum_{j=1}^{N} L_j dt_j / ds \) and \( \int \mathcal{D}[q(s); \Gamma] \) is the functional measure over all possible spatial-temporal paths. Here \( \Gamma \) is any curve connecting the point \( t(s') \) with the point \( t(s'') \) on space of time variables.

To construct the multi-time propagator (3.29), we shall first start with the case of 2-time variables \( (t_1, t_2) \) for simplicity. We therefore partition space into \( \mathbb{N} \times \mathbb{N} \), see figure 4 and keep in mind that the continuum limit is already taken into account. Moreover, we will consider only the forward time steps (as we did in the standard single time case) and we will also employ the symmetry of the lattice by considering the first evolution in \( t_1 \) together with all possible deformations and later \( t_2 \) together with all possible deformations, see figure 8. To illustrate the idea, let’s consider first the simplest curve, shown in figure 8a. The propagator is given by
\[
K^{(1)} = \int_{(0,0)}^{(e_1 N,0)} \mathcal{D}[q(t_1, 0)] \int_{-\infty}^{\infty} dN q(e_1 N, 0) \int_{(e_1 N,0)}^{(e_1 N,e_2 N)} \mathcal{D}[q(e_1 N, t_2)] e^{\frac{i}{\hbar} \left( \int_{(0,0)}^{(e_1 N,0)} L_1(t_1, 0) dt_1 + \int_{(e_1 N,0)}^{(e_1 N,e_2 N)} L_2(e_1 N, t_2) dt_2 \right)},
\]
(3.30)
where the superscript (1) denotes the first simplest path. The propagator in (3.30) contains all spatial paths along \((0,0) - (N,0)\) and all spatial paths along \((N,0) - (N,N)\). Both sections are glued by the completeness term \( \int_{-\infty}^{\infty} dN q \) at the corner \((N,0)\). We note that, in equation (3.30), the normalising factor is dropped out for our convenient.

\(^5\)One can consider the space without symmetry. However, at the end, we are going to consider the limit \( N \rightarrow \infty \). Therefore, it is simpler in terms of formulation with the symmetric case.
Next, for the paths given in 8a and 8b, the propagator can be written as

\[
K^{(2)} = \int_{0,0}^{(N,0)} \mathcal{D}[q(t_1, 0)] \int_{-\infty}^{\infty} d^N q(0, 0) \int_{0,0}^{(N,N)} \mathcal{D}[q(N, t_2)] e^{i \int_{0,0}^{(N,0)} L_1(t_1,0)dt_1 + f(N,N)} L_2(N, t_2)dt_2 \\
+ \sum_{n_1=1}^{N-1} \int_{0,0}^{(n_1,0)} \mathcal{D}[q(t_1, 0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{0,0}^{(n_1,N)} \mathcal{D}[q(n_1, t_2)] \int_{-\infty}^{\infty} d^N q(n_1, N) \int_{0,0}^{(n_1,N)} \mathcal{D}[q(t_1, N)]
\]

\[
\times e^{\int_{0,0}^{(n_1,0)} L_1(t_1,0)dt_1 + \int_{(n_1,0)}^{(n_1,N)} L_2(n_1,t_2)dt_2 + \int_{(n_1,N)}^{(N,N)} L_1(t_1,N)dt_1},
\]

(3.31)

where (2) denotes the first two possible simple paths. Here we note again that, for simplicity, the width parameters \(\epsilon_1\) and \(\epsilon_2\) of the plaquette are dropped out and also further consideration. What we see is that the exponential terms in (3.31) are not the same for the first term and the second term. However, if we introduce a time parametrised variable \(s' < s < s''\) such that \(t_1(s)\) and \(t_2(s)\), therefore, we have \(\mathcal{L} = (L_1dt_1/ds + L_2dt_2/ds)ds\). Here, if we replace the \(n_1\) of the second term in (3.31) by \(N\), one would obtain

\[
\int_{0,0}^{(N,0)} \mathcal{D}[q(t_1, 0)] \int_{-\infty}^{\infty} d^N q(0, 0) \int_{0,0}^{(N,N)} \mathcal{D}[q(N, t_2)] \int_{-\infty}^{\infty} d^N q(N, N) \int_{0,0}^{(N,N)} \mathcal{D}[q(t_1, N)] e^{i \int_{0,0}^{(N,0)} L_1(t_1,0)dt_1 + \int_{(N,0)}^{(N,N)} L_2(N, t_2)dt_2 + \int_{(N,N)}^{(N,N)} L_1(t_1,N)dt_1}
\]

\[
\times e^{\int_{0,0}^{(N,0)} L_1(t_1,0)dt_1 + \int_{(N,0)}^{(N,N)} L_2(N, t_2)dt_2 + \int_{(N,N)}^{(N,N)} L_1(t_1,N)dt_1},
\]

which is the propagator in the equation (3.30). Then the propagator \(K^{(2)}\) can be simply reduced to

\[
K^{(2)} = \sum_{n_1=1}^{N} \int_{0,0}^{(n_1,0)} \mathcal{D}[q(t_1, 0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{0,0}^{(n_1,N)} \mathcal{D}[q(n_1, t_2)] \int_{-\infty}^{\infty} d^N q(n_1, N) \int_{0,0}^{(n_1,N)} \mathcal{D}[q(t_1, N)] e^{i \int_{0,0}^{(n_1,0)} L_1(t_1,0)dt_1 + \int_{(n_1,0)}^{(n_1,N)} L_2(n_1,t_2)dt_2 + \int_{(n_1,N)}^{(N,N)} L_1(t_1,N)dt_1}
\]

Diagrammatically, what we do in equation (3.33) is just shifting vertical line from \((1,0)\) to \((N,0)\) - \((N,N)\).
Next, we include the third possible path, see figure 8c, into the calculation. Now the propagator in this case becomes

\[
K^{(3)} = \int \mathcal{D}[\mathbf{q}(t_1, 0)] \int_{-\infty}^{\infty} d^N \mathbf{q}(N, 0) \int \mathcal{D}[\mathbf{q}(N, t_2)] e^{\frac{i}{\hbar} \int_{(0,0)}^{(N,0)} L_1 dt_1 + \int_{(N,0)}^{(N,N)} L_2 dt_2} \\
+ \sum_{n_1=1}^{N-1} \int_{(0,0)}^{(n_1,0)} \mathcal{D}[\mathbf{q}(t_1, 0)] \int_{-\infty}^{\infty} d^N \mathbf{q}(n_1, 0) \int \mathcal{D}[\mathbf{q}(n_1, t_2)] \int_{-\infty}^{\infty} d^N \mathbf{q}(n_1, N) \\
\times \int \mathcal{D}[\mathbf{q}(t_1, N)] e^{\frac{i}{\hbar} \int_{(0,0)}^{(n_1,0)} L_1 dt_1 + \int_{(n_1,0)}^{(n_1,N)} L_2 dt_2 + \int_{(n_1,N)}^{(N,N)} L_1 dt_1} \\
+ \sum_{m_1=1}^{N-1} \sum_{n_1=1}^{N-1} \int_{(0,0)}^{(n_1,0)} \mathcal{D}[\mathbf{q}(t_1, 0)] \int_{-\infty}^{\infty} d^N \mathbf{q}(n_1, 0) \int \mathcal{D}[\mathbf{q}(n_1, t_2)] \\
\times \int \mathcal{D}[\mathbf{q}(t_1, m_1)] \int_{(n_1,m_1)}^{(N,m_1)} \mathcal{D}[\mathbf{q}(t_1, m_1)] \int_{-\infty}^{\infty} d^N \mathbf{q}(N, m_1) \\
\times \int \mathcal{D}[\mathbf{q}(N, t_2)] e^{\frac{i}{\hbar} \int_{(0,0)}^{(n_1,0)} L_1 dt_1 + \int_{(n_1,0)}^{(n_1,m_1)} L_2 dt_2 + \int_{(n_1,m_1)}^{(N,m_1)} L_1 dt_1 + \int_{(N,m_1)}^{(N,N)} L_2 dt_2}.
\]

(3.34)

Here the labeled variables activating on the exponent term are omitted. We notice that the third term

Figure 8: The 6 first formats of possible paths.
will be identical with the second term if \( m_1 = N \) for every single \( n_1 \) and will be the first term if we let \( m_1 = 0 \). Here comes to a crucial point. The order of summation is matter since one might need to avoid the repetition of arbitrary \( n_1 \) at \( m_1 = 0 \), see appendix [B]. Then, the sum over possible \( m_1 \) comes first and sum over \( n_1 \) comes second. The propagator \( K^{(3)} \) becomes

\[
K^{(3)} = \sum_{n_1=1}^{N-1} \sum_{m_1=0}^{N} \int_{(0,0)}^{(n_1,0)} \mathcal{D}[q(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{(n_1,0)}^{(n_1,m_1)} \mathcal{D}[q(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,m_1) \times \int_{(n_1,m_1)}^{(N,m_1)} \mathcal{D}[q(t_1,m_1)] \int_{-\infty}^{\infty} d^N q(N,m_1) \int_{(N,m_1)}^{(N,N)} \mathcal{D}[q(N,t_2)] e^{\frac{i}{\hbar} \int_{\mathcal{L}} L}. \tag{3.35}
\]

Diagrammatically, what we do in the equation (3.35) is shifting the horizontal line from \((n_1,0) - (N,0)\) to \((n_1,N) - (N,N)\) for \( n_1 = [1,N] \).

With the structure what we proceed so far, it is now not difficult to see that the propagator \( K^{(5)} \), included figures [Sc] can be expressed in the form

\[
K^{(5)} = \left( \int_{(0,0)}^{(N,0)} \mathcal{D}[q(t_1,0)] \int_{-\infty}^{\infty} d^N q(N,0) \int_{(N,0)}^{(N,N)} \mathcal{D}[q(N,t_2)] \right) + \sum_{n_1=1}^{N-1} \int_{(0,0)}^{(n_1,0)} \mathcal{D}[q(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{(n_1,0)}^{(n_1,N)} \mathcal{D}[q(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,N) \int_{(n_1,N)}^{(N,N)} \mathcal{D}[q(t_1,N)] \times \int_{(n_1,m_1)}^{(N,m_1)} \mathcal{D}[q(t_1,m_1)] \int_{-\infty}^{\infty} d^N q(N,m_1) \int_{(N,m_1)}^{(N,N)} \mathcal{D}[q(N,t_2)]
\]

\[
+ \sum_{m_1=1}^{N-1} \int_{(0,0)}^{(n_1,m_1)} \mathcal{D}[q(t_1,m_1)] \int_{-\infty}^{\infty} d^N q(n_1,m_1) \int_{(n_1,m_1)}^{(n_1,N)} \mathcal{D}[q(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,N) \int_{(n_1,N)}^{(N,N)} \mathcal{D}[q(t_1,N)].
\]

The forth path, see figure [Sc] is nothing but the fifth one, see figure [Sc] in case of \( m_2 = N \) for every single \( m_1 \) together with \( m_1 = N - 1 \) and \( m_2 = N \). Moreover, the fifth path can be reduced to be the third path in the case of \( m_2 = m_1 \) for every single \( m_1 \) and the second one in the case of \( m_1 = m_2 = N \). For the first path can be obtained from the fifth path by letting \( m_1 = m_2 = 0 \), but the case of \( m_1 = 0 \) with any \( m_2 \neq 0 \) is the same with the case of \( m_2 = m_1 \) (third path). This over-counted problem could be settled by fixing \( n_1 = 1 \), see appendix [C]. The propagator, therefore, can be further simplified to

\[
K^{(5)} = \sum_{n_2=2}^{N-1} \sum_{m_2=n_2+1}^{N-1} \int_{(0,0)}^{(n_2,0)} \mathcal{D}[q(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{(n_1,0)}^{(n_1,m_1)} \mathcal{D}[q(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,m_1) \int_{(n_1,m_1)}^{(n_2,m_1)} \mathcal{D}[q(n_2,m_1)] \int_{-\infty}^{\infty} d^N q(n_2,m_2) \int_{(n_2,m_2)}^{(N,m_2)} \mathcal{D}[q(n_1,m_1)] \int_{-\infty}^{\infty} d^N q(n_1,m_1) \int_{(n_1,m_1)}^{(N,m_1)} \mathcal{D}[q(t_1,m_1)]
\]

\[
+ \sum_{m_2=m_1+1}^{N-1} \sum_{n_2=n_2+1}^{N-2} \int_{(0,0)}^{(m_2,0)} \mathcal{D}[q(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_2,0) \int_{(n_2,0)}^{(n_2,m_2)} \mathcal{D}[q(n_2,t_2)] \int_{-\infty}^{\infty} d^N q(n_2,m_2) \int_{(n_2,m_2)}^{(N,m_2)} \mathcal{D}[q(n_1,m_1)] \int_{-\infty}^{\infty} d^N q(n_1,m_1) \int_{(n_1,m_1)}^{(N,m_1)} \mathcal{D}[q(t_1,m_1)].
\]
Here the order of summation is still crucial since we would have the repeated paths where $n_2$ is arbitrary at $m_2 = m_1$. Diagrammatically, what we do in the equation (3.37) is first shifting the horizontal line from $(1, 0) - (n_2, 0)$ to $(1, N) - (n_2, N)$ and second shifting the another horizontal line from $(n_2, 0) - (N, 0)$ to $(n_2, N) - (N, N)$, where the second horizontal line must be always above the first one, for $n_2 = [2, N]$.

By proceeding, this deformation of the curve, we could account for all configurations of the path. The propagator, including the normalising factors\(^7\), could be expressed as

\[
K^{(\text{All})} = \sum_{m_{N-1} \geq \cdots \geq m_2 \geq m_1 \geq 0} N_{m_I} \left( \prod_{i=1}^{N-1} \int_{(0,0)}^{(1,0)} \mathcal{D}[q(t_1, 0)] \int_{-\infty}^{\infty} d^N q(i, m_{i-1}) \mathcal{D}[q(i, t_2)] \int_{-\infty}^{\infty} d^N q(i, m_i) \right) \times \int_{(i,m_i)}^{(i+1,m_i)} \mathcal{D}[q(t_1, m_{i+1})] \int_{-\infty}^{\infty} d^N q(N, m_{N-1}) \mathcal{D}[q(N, t_2)]) \right) e^{\int \mathcal{L}''} \mathcal{L},
\]

where $m_I = m_1 m_2 \cdots m_{N-1}$ is multi-indices and $m_0 \equiv 0$. Alternatively, in terms of diagram, we could imagine that we slice the horizontal line as $N$ pieces ($n_I = 1$, \forall $l = 0, 1, 2, \ldots, N - 1$) then shift the second piece onwards at $m_1, m_2, \ldots, m_{N-1}$, respectively, with conditions: $m_i \geq m_j$, where $i > j$.

We know that the equation (3.38) is just only the propagator starting in the $t_1$-direction and all possible types of deformation. However, there is a path starting in the $t_2$-direction as well. We therefore employ the symmetry of the lattice structure under interchange $t_1$ and $t_2$. Hence, the propagator can be presented as follows

\[
K(q(s''), s''; q(s'), s') = \int \mathcal{D}[q(s); \Gamma] e^{\int \mathcal{L}''} \mathcal{L},
\]

where

\[
\int \mathcal{D}[q(s''); \Gamma] = \int \mathcal{D}[q(t(s'))] \left( \prod_{i=1}^{N-1} \int_{(0,0)}^{(1,0)} \mathcal{D}[q(t_1, 0)] \int_{-\infty}^{\infty} d^N q(i, m_{i-1}; e_2) \mathcal{D}[q(i, t_2)] \times \int_{(i,m_i)}^{(i+1,m_i)} \mathcal{D}[q(t_1, m_{i+1})] \int_{-\infty}^{\infty} d^N q(N, m_{N-1}) \mathcal{D}[q(N, t_2)]) \right) e^{\int \mathcal{L}''} \mathcal{L},
\]

(3.39)

Here, $\epsilon_1$ and $\epsilon_2$ are put back into the formula. The new notation $\int \mathcal{D}[q(t(s)); \Gamma]$ is a extended definition of the standard one $\int \mathcal{D}[q(t)]$. Then the propagator in (3.39) represents sum all possible paths not only on the configuration space (standard one), but also all possible paths $\Gamma$ on the time space, see figure 9.

What we obtain in (3.39) is just the case of two times. Therefore, the process can be directly extended to the case of arbitrary $N$ times by using a diagrammatic method. Here, we will sketch the idea in the case

\(^7\)This normalising factors are different from the normalising factor in equation (3.18), but they are specified to keep a preserved norm of quantum states.
of three times. The propagator in this case will get a contribution from all possible types of deformation starting in $t_1$-direction, see figure 10. Mathematically, we just simply shift each step in the $t_1$-direction,
resulting in

\[
\int_{q(s')} \mathbb{D}[q(s); \Gamma] = \int_{q(t(s'))} \mathbb{D}[q(t(s')); \Gamma]
\]

\[
= \lim_{N \to \infty} \left( \sum_{i=1}^{N} \sum_{t_{i-1} \geq 2} \int_{t_{i-1} \geq 2} d^m q(t_{i-1}) \int_{t_{i-1} \geq 2} d^m q(t_{i}) \mathcal{N} \right) \int_{t_{i-1} \geq 2} d^N q(0, 0, 0) \]

\[
\times \left\{ \int_{(t_{i-1}, t_{i})} \mathbb{D}[q(t_{i-1}, t_{i})] \int_{(t_{i-1}, t_{i})} \mathbb{D}[q(t_{i-1}, t_{i})] \right\}
\]

\[
\times \int_{-\infty}^\infty d^N q(t_{i-1}, t_{i}) \int_{(t_{i-1}, t_{i})} \mathbb{D}[q(t_{i-1}, t_{i})] \int_{(t_{i-1}, t_{i})} \mathbb{D}[q(t_{i-1}, t_{i})]
\]

\[
\cdot \left\{ \int_{(t_{i-1}, t_{i})} \mathbb{D}[q(t_{i-1}, t_{i})] \int_{(t_{i-1}, t_{i})} \mathbb{D}[q(t_{i-1}, t_{i})] \right\}
\]

\[
+ \left( \text{the } t_2\text{-symmetric term} \right) + \left( \text{the } t_3\text{-symmetric term} \right), \quad (3.41)
\]

where \( \epsilon_3 \) is a width of step-evolution in \( t_3 \)-direction.

\[\text{Figure 10: The deformation of paths starting in } t_1\text{-direction on the three-dimensional space of time variables.}\]
Here, we introduce
\[
\int \mathcal{D}[q(t_2)], \mathcal{D}[q(t_3)] 
\]
where \( q(t_i) \) means that \( t_i \) is active but \( t_j \neq i \) are fixed. With \( n_0 = m_0 = l_0 = 0 \), hence equation (3.41) will be simply expressed as
\[
\int_{\mathcal{D}[q(t_2)]} q(t_2) d[q(t_2)] = \lim_{N \to \infty} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \mathcal{M} \int_{(0,0,0)}^{(\epsilon_1,0,0)} \mathcal{D}[q(t_1), 0, 0])
\]
\[
\times \prod_{i=1}^{N-1} \int_{-\infty}^{\infty} d^N q(i \epsilon_1, m_i \epsilon_2, l_i \epsilon_3) \int_{(i \epsilon_1, m_i \epsilon_2, l_i \epsilon_3)}^{(l \epsilon_1, m_i \epsilon_2, l_i \epsilon_3)} \mathcal{D}[q(t_1), 0, 0])
\]
\[
\times \int_{-\infty}^{\infty} d^N q(N_{-1} \epsilon_1, m_{-1} \epsilon_2, l_{-1} \epsilon_3) \int_{(N_{-1} \epsilon_1, m_{-1} \epsilon_2, l_{-1} \epsilon_3)}^{(N_{-1} \epsilon_1, m_{-1} \epsilon_2, l_{-1} \epsilon_3)} \mathcal{D}[q(t_1), 0, 0])
\]
\[
\times \int_{-\infty}^{\infty} d^N q(N_{-1} \epsilon_1, m_{-1} \epsilon_2, l_{-1} \epsilon_3) \int_{(N_{-1} \epsilon_1, m_{-1} \epsilon_2, l_{-1} \epsilon_3)}^{(N_{-1} \epsilon_1, m_{-1} \epsilon_2, l_{-1} \epsilon_3)} \mathcal{D}[q(t_1), 0, 0])
\]
\[
+(\text{the } t_2\text{-symmetric term}) + (\text{the } t_3\text{-symmetric term})
\]
\[
(3.43)
\]
For \( N \)-dimensional of time space, the functional measure over all possible spatial-temporal paths could be presented as
\[
\int_{\mathcal{D}[q(t')]} q(t') d[q(t')]; \Gamma = \int_{\mathcal{D}[q(t')]} q(t') d[q(t')]; \Gamma
\]
\[
= \lim_{N \to \infty} \left( \sum_{N_1=1}^{N} \alpha_1^{N_1} \alpha_2^{N_2} \alpha_3^{N_3} \cdots \right) \mathcal{M} \int_{(0,0,0)}^{(\epsilon_1,0,0)} \mathcal{D}[q(t_1)]
\]
\[
\times \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} d^N q(j \epsilon_1, \alpha_j^{N_1} \epsilon_1, \alpha_j^{N_2} \epsilon_2, \alpha_j^{N_3} \epsilon_3)
\]
\[
\times \int_{(j \epsilon_1, \alpha_j^{N_1} \epsilon_1, \alpha_j^{N_2} \epsilon_2, \alpha_j^{N_3} \epsilon_3)}^{(l \epsilon_1, \alpha_j^{N_1} \epsilon_1, \alpha_j^{N_2} \epsilon_2, \alpha_j^{N_3} \epsilon_3)} \mathcal{D}[q(t_j)], \mathcal{D}[q(t_3)], \cdots \mathcal{D}[q(t_{N-1})], \mathcal{D}[q(t_N)]
\]
\[
\times \int_{-\infty}^{\infty} d^N q(N_{-1} \epsilon_1, \alpha_j^{N_1} \epsilon_1, \alpha_j^{N_2} \epsilon_2, \alpha_j^{N_3} \epsilon_3)
\]
\[
\times \int_{(N_{-1} \epsilon_1, \alpha_j^{N_1} \epsilon_1, \alpha_j^{N_2} \epsilon_2, \alpha_j^{N_3} \epsilon_3)}^{(N \epsilon_1, \alpha_j^{N_1} \epsilon_1, \alpha_j^{N_2} \epsilon_2, \alpha_j^{N_3} \epsilon_3)} \mathcal{D}[q(t_1)]
\]
\[
\times \int_{-\infty}^{\infty} d^N q(N_{-1} \epsilon_1, \alpha_j^{N_1} \epsilon_1, \alpha_j^{N_2} \epsilon_2, \alpha_j^{N_3} \epsilon_3)
\]
\[
\times \int_{(N_{-1} \epsilon_1, \alpha_j^{N_1} \epsilon_1, \alpha_j^{N_2} \epsilon_2, \alpha_j^{N_3} \epsilon_3)}^{(N \epsilon_1, \alpha_j^{N_1} \epsilon_1, \alpha_j^{N_2} \epsilon_2, \alpha_j^{N_3} \epsilon_3)} \mathcal{D}[q(t_1)]
\]
\[
+(\text{all symmetric terms})
\]
\[
(3.44)
\]
where \( \alpha_j \) are \( j \)-th step-evolution in \( t_i \)-direction and \( \alpha_0 = 0 \).
What we have now for the propagator in the multi-time case is

\[ K(q(s''), s''; q(s'), s') = \int_{q(s')}^{q(s'')} D[q(s); \Gamma] e^{\frac{i}{\hbar} \int_{s'}^{s''} L} , \] (3.45)

where \( \int D[q(s); \Gamma] \) measures the contribution of all possible paths \( q(s) \) and all possible paths \( \Gamma \in \mathcal{B} \) between \( t(s') \) and \( t(s'') \) on the space of time variables.

Again, the propagator (3.45) is not appropriate for further consideration and, therefore, we shall apply the semi-classical approximation. Since we work with the parametrised time variable \( s \), the action can be expanded in the same fashion with the single-time case yielding

\[
S[q(s)] = S[q_e(s) + y(s)] = S[q_e(s)] + \int_{s'}^{s''} dt \frac{\delta S[q_e(s)]}{\delta y(\tau)} y(\tau) + \frac{1}{2!} \int_{s'}^{s''} dt \int_{s'}^{s''} da \frac{\delta^2 S[q_e(s)]}{\delta y(\tau) \delta y(\sigma)} y(\tau)y(\sigma) \\
+ \frac{1}{3!} \int_{s'}^{s''} dt \int_{s'}^{s''} ds \int_{s'}^{s''} dc \frac{\delta^3 S[q_e(s)]}{\delta y(\tau) \delta y(\sigma) \delta y(\xi)} y(\tau)y(\sigma)y(\xi) + O(y^4) .
\] (3.46)

The first order in the expansion vanishes, since it is nothing but the Euler-Lagrange equations and constraints. Employing the same trick with the single time case, we can again redefine the variable \( y = \sqrt{\hbar} \tilde{y} \) with \( \tilde{y} \sim O(1) \). Therefore, the multi-time propagator now becomes

\[
K(q'', s''; q', s') = e^{\frac{i}{\hbar} S[q_e(s)]} \mathcal{Q}(q'', s'', q', s') \left[ 1 + O(\hbar) \right] ,
\] (3.47)

where

\[
\mathcal{Q}(q'', s'', q', s') = \int_{y(s'')=0}^{y(s'')=0} D[y(s); \Gamma] e^{\frac{i}{\hbar} \int_{s'}^{s''} L} e^{\int_{s'}^{s''} dy \int_{s'}^{s''} dy \left( \tilde{y}(\tau) \frac{\delta^2 S[q_e(s)]}{\delta y(\tau) \delta y(\sigma)} \tilde{y}(\sigma) \right)} ,
\] (3.48)

where \( \mathcal{Q}(q'', s'', q', s') \) is a smooth function of end points.

We now consider the multi-time propagator along an arbitrary path \( \Gamma \) connecting between end points \( t'' \) and \( t' \) on space of time variables as follows

\[
K_{\Gamma}(q(t''), t''; q(t'), t') = e^{\frac{i}{\hbar} S_{\Gamma}[q_e(t)]} \mathcal{Q}_{\Gamma}(q'', t'', q', t') \left[ 1 + O(\hbar) \right] ,
\] (3.49)

where

\[
\mathcal{Q}_{\Gamma}(q'', t'', q', t') = \int_{y(t'')=0}^{y(t'')=0} \mathcal{D}_{\Gamma}[y(t)] e^{\frac{i}{\hbar} \sum_{j=1}^{N} \int_{t_j} \int_{t_j} \frac{\delta^2 S_{\Gamma}[q_e(t)]}{\delta y(\tau) \delta y(\sigma)} \tilde{y}(\tau)} ,
\] (3.50)

Here \( \mathcal{D}_{\Gamma}[y] \) is a functional measure all possible fluctuations \( y \) along path \( \Gamma \) on \( N \)-dimensional time-space and \( S_{\Gamma}[q(t)] = \int_{\Gamma} L dt j \). Therefore, the function \( \mathcal{Q}_{\Gamma} \) in (3.50) can be expressed, see appendix \( \mathcal{D} \) in the form

\[
\mathcal{Q}_{\Gamma}(q'', q', t'', t') = \text{det} \left( \frac{i}{2\pi \hbar} \frac{\delta^2 S_{\Gamma}[q_e(t)]}{\delta y(\tau) \delta y(\sigma)} \right)^{\frac{1}{2}} ,
\] (3.51)

where \( S_{\Gamma}[q(t)] = \int_{\Gamma} L \).

Here is an interesting point. We knew that the Lagrangian is not unique since different Lagrangian would produce an identical equation of motion. Moreover, in the context of integrable systems, a different set of Lagrangians, producing the same equations of motion, may not all satisfy the closure relation, but only a special set of Lagrangians does. This structure would provide the space of possible Lagrangians. Consequently, in multi-time quantum systems, the propagator possessed the path independent feature on the space of time variables is the one that comes with a special set of Lagrangians satisfying the closure relation. This structure gives us an on top feature of the classical variational principle in the sense that this special set of Lagrangians plays a role of critical point resulting path independent propagator on the space of independent variables coined as the quantum variation [22].
Theorem 3.2. Let \( \{L_1, L_2, \ldots, L_N\} \) be a set of Lagrangians satisfying the Lagrangian closure relation (2.21) and \( \mathcal{L} = \sum_{j=1}^{N} L_j dt_j \) be the Lagrangian 1-form, where \( L_j = L_j(\eta) \). On the space of independent variables (time variables), the multi-time propagator for any \( \Gamma \in \mathcal{B} \) gives equally contribution leading to

\[
\oint_{C=\partial S} \mathcal{D}_C \rho[\mathbf{q}(t)] e^{i\oint_{C=\partial S} \mathcal{L}} = \mathbb{I},
\]

where \( S \) is an arbitrary surface bounded by a contractible loop \( C \) on the space of time variables, and therefore the multi-time quantum system is integrable.

Proof. According to the equations (3.49)-(3.51), the propagator along path \( \Gamma \) on space of time variables in the semi-classical limit reads

\[
K_{\Gamma}(\mathbf{q}(t''), t'''; \mathbf{q}(t'), t') = e^{\pm S_{\Gamma}[\mathbf{q}_c(t)] [1 + \mathcal{O}(\hbar)]},
\]

\[
\Omega_{\Gamma} = \text{det} \left( \frac{i}{2\pi\hbar} \frac{\partial^2 S_{\Gamma}[\mathbf{q}_c(t)]}{\partial t_i \partial t_j} \right)^{\frac{1}{2}},
\]

and, for the path \( \Gamma' \) with the same end points, the propagator reads

\[
K_{\Gamma'}(\mathbf{q}(t''), t'''; \mathbf{q}(t'), t') = e^{\pm S_{\Gamma'}[\mathbf{q}_c(t)] [1 + \mathcal{O}(\hbar)]},
\]

\[
\Omega_{\Gamma'} = \text{det} \left( \frac{i}{2\pi\hbar} \frac{\partial^2 S_{\Gamma'}[\mathbf{q}_c(t)]}{\partial t_i \partial t_j} \right)^{\frac{1}{2}}.
\]

The Lagrangian closure relation provides

\[
S_{\Gamma}[\mathbf{q}_c(t)] - S_{\Gamma'}[\mathbf{q}_c(t)] = \left( \int_{\Gamma} - \int_{\Gamma'} \right) \mathcal{L} = \oint_{C=\partial S} \mathcal{L} = \int_{t_i}^{t_f} \sum_{k \geq 1} \sum_{l=1}^{N} \left( \frac{\partial L_i}{\partial t_k} - \frac{\partial L_l}{\partial t_i} \right) dt_k \wedge dt_l = 0,
\]

which is noting but the path independent feature, see figure [3]. Here \( S \) is an arbitrary surface bounded by a contractible loop \( C \) on the space of time variables. Therefore, \( \Omega_{\Gamma} = \Omega_{\Gamma'} \) and consequently we have

\[
K_{\Gamma}(\mathbf{q}(t''), t'''; \mathbf{q}(t'), t') = K_{\Gamma'}(\mathbf{q}(t''), t'''; \mathbf{q}(t'), t'),
\]

where the \( \mathcal{O}(\hbar) \) is ignored since there is extremely tiny contribution to the propagator in the semi-classical limit.

For a contractible loop \( C \), the propagator can be captured as

\[
K_{C=\partial S} = \lim_{(t''-t') \to 0} \int_{t_i}^{t_f} d^N q'' \int_{t_i}^{t_f} d^N q' \det \left( \frac{i}{2\pi\hbar} \right)^2 \frac{\partial^2 S_{\Gamma}[\mathbf{q}_c(t)]}{\partial q(t'') \partial q(t')} \frac{\partial^2 S_{\Gamma'}[\mathbf{q}_c(t)]}{\partial q(t'') \partial q(t)} \right)^{\frac{1}{2}} e^{\pm \left( \int_{t_i}^{t_f} \mathcal{L} - \int_{t_i}^{t_f} \mathcal{L} \right)} \mathcal{L},
\]

where the subscript \( c \) refers to the classical action. Then we write

\[
S_{c}[\mathbf{q}', \mathbf{q}''] =: \int_{t_i}^{t_f} \mathcal{L},
\]

\[
S_{c}[\mathbf{q}, \mathbf{q}'] =: \int_{t_i}^{t_f} \mathcal{L}.
\]

Dropping out the subscripts \( \Gamma \) and \( \Gamma' \) because of path independent feature, we obtain

\[
\lim_{(t''-t') \to 0} \left( \int_{t_i}^{t_f} - \int_{t_i}^{t_f} \right) \mathcal{L} = \lim_{(t''-t') \to 0} \sum_{i=1}^{N} \frac{S_{c}[\mathbf{q}', \mathbf{q}''] - S_{c}[\mathbf{q}, \mathbf{q}']}{\dot{q}_i - q_i} (\dot{q}_i - q_i) = -\frac{\partial S_{c}}{\partial \mathbf{q}} \cdot (\dot{\mathbf{q}} - \mathbf{q}').
\]

Substituting (3.62) into (3.59), the propagator (3.59) becomes

\[
K_{C=\partial S} = \left( \frac{1}{2\pi} \right)^N \int_{-\infty}^{\infty} d^N \mathbf{q} \int_{-\infty}^{\infty} d^N \frac{1}{\hbar} \frac{\partial S_{c}}{\partial \mathbf{q}} e^{-\frac{i}{\hbar} \frac{\partial S_{c}}{\partial \mathbf{q}} \cdot (\dot{\mathbf{q}} - \mathbf{q})} = \int_{-\infty}^{\infty} \frac{d^N \mathbf{q}^c}{\delta^N (\mathbf{q} - \mathbf{q}')} = \mathbb{I}.
\]

\( \square \)**See appendix E** for the case of quadratic Lagrangians
We shall point out a final feature of the multi-time propagator. From equation (3.6), it is not difficult to see that we could have a set of equations

$$i\hbar \frac{\partial}{\partial t_j} K(q(t''), t''; q(t'), t') = \hat{H}_j K(q(t''), t''; q(t'), t'),$$

where \(j = 1, 2, ..., N\) and \(t'' > t'\). \(3.64\)

Again, the quantity \(\frac{\partial}{\partial t_j} - \frac{1}{i\hbar}\hat{H}_j\), where \(j = 1, 2, ..., N\), can be treated as the covariant derivative and the system of equations (3.64) is overdetermined. Thus, a common nontrivial solution \(K(q(t''), t''; q(t'), t')\) exists simultaneously if

$$\frac{\partial}{\partial t_k} \frac{\partial}{\partial t_j} K(q(t''), t''; q(t'), t') = \frac{\partial}{\partial t_j} \frac{\partial}{\partial t_k} K(q(t''), t''; q(t'), t')$$

holds. This compatibility (3.65) gives again directly to the zero-curvature condition of the Hamiltonian operators (3.4).

4 Geometrical interpretation

In this section, we will give a visualisation of the integrability in the 1-form case and key features of new mathematical objects called the Lagrange vector field and Hamilton vector field defined on the space of time variables.

Proposition 1: For integrable systems that possess a set of Hamiltonians \(\{H_1, H_2, \cdots, H_N\}\) satisfying the Hamiltonian commuting flows and a set of Lagrangians \(\{L_1, L_2, \cdots, L_N\}\) satisfying the closure relation, on space of time variables, there exist Hamilton vector field and Lagrange vector field which must be conservative and irrotational.

Proof. Here we will recall the relation, see figure 3b,

$$\delta S \hookrightarrow = \oint _{C = \partial S} L = \iint _S dL = 0,$$

where

$$L = \sum _{j=1} ^N L_j dt_j,$$

$$dL = \sum _{k \geq 1} ^N \sum _{l=1} ^N \left( \frac{\partial L_l}{\partial t_k} - \frac{\partial L_k}{\partial t_l} \right) dt_k \wedge dt_l,$$

are the Lagrangian 1-form and the corresponding 2-form, respectively. In terms of the Hamiltonian, we have

$$\delta S \hookrightarrow = - \oint _{C = \partial S} H = - \iint _S dH = 0,$$

where

$$H = \sum _{j=1} ^N H_j dt_j,$$

$$dH = \sum _{k \geq 1} ^N \sum _{l=1} ^N \left( \frac{\partial H_l}{\partial t_k} - \frac{\partial H_k}{\partial t_l} \right) dt_k \wedge dt_l,$$

are the Hamiltonian 1-form and the corresponding 2-form, respectively. At this point, we emphasise again that these relations give a key feature called the path-independent on the space of time variables.
Next, let us recall the gradient theorem: For \( \Phi \in \mathcal{U} \subseteq \mathbb{R}^N \mapsto \mathbb{R} \) as a differentiable function and any curve \( \Gamma \) in \( \mathcal{U} \) which starts at a point \( a \) and ends at a point \( b \), then
\[
\int_{\Gamma} \nabla_r \Phi(r) \cdot dr = \Phi(b) - \Phi(a) ,
\]
where \( \nabla_r \Phi \) is the gradient associated with the vector field \( A \) such that \( A = -\nabla_r \Phi \), \( r = (x_1 \ x_2 \cdots x_N)^T \) and \( \nabla_r \equiv (\partial/\partial x_1 \ \partial/\partial x_2 \cdots \partial/\partial x_N)^T \). Therefore, the gradient theorem implies nothing but the path-independent feature of the line integrals through the vector field \( A \). Furthermore, this means that the loop integrals will give zero
\[
\oint_{C=\partial S} A(r) \cdot dr = -\oint_{C=\partial S} \nabla_r \Phi(r) \cdot dr = 0 ,
\]
where \( C \) bounds the hyper-surface \( S \) in \( \mathcal{U} \). If we now write \( A = (A_1 \ A_2 \cdots A_N)^T \) and \( dr = (dx_1 \ dx_2 \cdots dx_N)^T \), the left hand side of equation (4.8) becomes
\[
\oint_{C=\partial S} \mathcal{A} = 0 ,
\]
where \( \mathcal{A} = \sum_{j=1}^N A_j dx_j \) is the 1-form.

Now, given a \( N \)-dimensional time space: \( T^N = \mathbb{R}^N \), we define
\[
\mathbf{L} = (L_1 \ L_2 \cdots L_N)^T ,
\]
\[
\mathbf{H} = (H_1 \ H_2 \cdots H_N)^T ,
\]
as the Lagrange vector field and Hamilton vector field, respectively. Then equations (4.11) and (4.12) can be expressed as
\[
\delta S_{\mathcal{A}} = \oint_{C=\partial S} \mathbf{L} \cdot dt = 0 ,
\]
\[
\delta S_{\mathcal{A}} = -\oint_{C=\partial S} \mathbf{H} \cdot dt = 0 ,
\]
where \( t = (t_1 \ t_2 \cdots t_N)^T \). According to the gradient theorem, there must be
\[
\mathbf{L} = -\nabla_t \Phi_L(t) ,
\]
\[
\mathbf{H} = +\nabla_t \Phi_H(t) ,
\]
where \( \Phi_L \) and \( \Phi_H \) are the scalar functions associated with the Lagrange vector field and Hamilton vector field, respectively. Here the gradient operator is defined as \( \nabla_t \equiv (\partial/\partial t_1 \ \partial/\partial t_2 \cdots \partial/\partial t_N)^T \).

In order to see the properties of the Lagrange vector field and Hamilton vector field, we shall consider the case of two time variables \( t = (t_1 \ t_2)^T \) and let \( A \) be either \( \mathbf{L} \) or \( \mathbf{H} \). Applying the Stokes theorem, we obtain
\[
\oint_{C=\partial S} A \cdot dt = -\int_{C=\partial S} \nabla_t \Phi_A \cdot dt = \int_S \left( \frac{\partial A_2}{\partial t_1} - \frac{\partial A_1}{\partial t_2} \right) dt_1 \wedge dt_2 = 0 .
\]
This is very trivial that \( A \) must be conservative and irrotational the vector field, see figure [11]

**Proposition 2:** The integrability condition could be interpreted that the loop \( \partial S \) gets through the layers of \( \mathcal{A} \) (one-form) twice (back and forth), resulting zero number of pierce. Similarly, the flux \( d\mathcal{A} \) that gets through any oriented surface \( S \) bounded by \( \partial S \) is zero (number of flux getting in is the same with the number of flux getting out.)
A = A_1 \dot{t}_1 + A_2 \dot{t}_2

Stack of surfaces: \Phi_A

Figure 11: The circulation of a vector field \( A = -\nabla_t \Phi_A(t_1, t_2) \) around every closed loop \( C = \partial A \) on the space of time variable is zero.

\[ \Phi_A(t_1) \Rightarrow \Phi_A(t_2) \]

Proof. We again recall all relations for the loop mapping of the Hamiltonian vector fields, Lagrangian vector fields, temporal Lax matrices and Hamiltonian operators

\[ \phi_A^\gamma = T e^{\oint_{C=0} \mathcal{A} = P e^{\int_S d\mathcal{A}} = I} , \] (4.17)

where

\[ \mathcal{A} = \sum_{j=1}^N A_j dt_j = \left\{ \sum_{j=1}^N X_{Hj} dt_j, \sum_{j=1}^N X_{Lj} dt_j, \sum_{j=1}^N M_j dt_j, \sum_{j=1}^N \hat{H}_j dt_j \right\} , \] (4.18)

\[ d\mathcal{A} = \sum_{k=1}^N \sum_{l=1}^N (\phi_A^\gamma)^{-1} \left( \frac{\partial A_k}{\partial \tau_j} - \frac{\partial A_l}{\partial \tau_k} + [A_k, A_l] \right) (\phi_A^\gamma) dt_l \wedge dt_k , \] (4.19)

with \( \phi_A^\gamma \) is a mapping along an arbitrary path \( \gamma \) connecting between the origin and \( t \). What we can see is that the loop mapping will be identity if

\[ \oint_{\partial S} \mathcal{A} = 0 , \] (4.20)

which means that any closed path \( \partial S \) must go through the stack of surfaces \( \mathcal{A} \) twice (back and forth) resulting in zero number of pierce, see figure [12a]. Equivalently, the loop mapping will be identity if

\[ \iint_S d\mathcal{A} = 0 , \] (4.21)

which means that the total flux \( d\mathcal{A} \), the exterior derivative of the differential 1-form \( \mathcal{A} \), through the surface \( S \) bounded by \( \partial S \) is zero, see figure [12b].

Remark. We note that the structure in equation [4.16] can be perfectly explained by figures [12a] and [12b].

5 Concluding discussion

In the classical level, the multi-dimensional consistency is successfully captured through various geometrical objects. On the cotangent bundle, the multi-dimensional consistency can be represented by the existence of the zero-curvature condition of the Hamiltonian vector fields. Moreover, there exists the \( N \)-parameter group
structure. A family of composite maps can be expressed in terms of the Wilson line and the Hamiltonian vector fields can be treated as the gauge variables in this case. The zero-curvature condition implies the path-independent feature of the composite maps. Therefore, for the loop evolution, the mapping, expressing in terms of the Wilson loop, will be identity. The same analogy can be applied on the tangent bundle by replacing Hamiltonian vector fields with Lagrangian vector fields. In the level of Lax matrices, the set of partial differential equations for the temporal Lax matrices will possess a non-trivial solution if there exists a zero-curvature condition in terms of the temporal Lax matrices. The multi-time evolution of the spatial Lax matrix will also possess the path-independent feature as a consequence of the zero-curvature condition of the temporal Lax matrices. We first conclude here that these three different perspectives, namely Hamiltonian vector fields, Lagrangian vector fields and Lax matrices, do share the same geometric property to capture the multi-dimensional consistency and therefore integrability. Another interesting point is that the integrable systems with a set of Hamiltonians and Lagrangians must exhibit the important features called the Hamiltonian commuting flows and Lagrangian closure relations, respectively. Here, in this work, we go further to study properties of these functions in an alternative perspective. We introduce new objects called the Hamilton vector fields and Lagrange vector fields defined on the space of time variables. It turns out that, for the integrable systems, these vector fields must be conservative and irrotational. Equivalently, one could say that, for the integrable systems, line integral of Lagrangian 1-form on the space of time variables is path-independent and of course the loop integral vanishes. This means that, according to the gradient theorem, there must exist stack of hyper-surfaces, 1-form object, defined by scalar functions, associated with the Hamilton and Lagrange vector fields, as the gradient vector fields. Then the vanishing loop integral means that the loop gets through each of the hyper-surface twice, back and forth. Applying the Stokes theorem, it turns out that the flux of the 2-form objects, which are nothing but the Hamiltonian commuting flows and Lagrangian closure relations, gets through any oriented surface bounded by the loop is zero.

In the quantum level, the multi-dimensional consistency is also geometrically described. In Schrödinger picture, one can promote the set of Hamiltonians in the classical integrable systems to be a set of Hamiltonian operators and the set of Schrödinger equations are obtained from the multi-time Hamilton-Jacobi equations through the connection between the Hamilton principle function and multi-time wave function. This set of Schrödinger equations is overdetermined and therefore a common non-trivial solution, wave function, exists if the Hamiltonian operators must satisfy the zero-curvature condition. The unitary multi-time evolution operator can be expressed in terms of the Wilson line and possesses the path-independent feature. This means that, for the loop evolution, the unitary map in terms of the Wilson loop is identity. At this point, we may state that, for integrable quantum systems, the Hamiltonian operators must follow the zero-curvature condition, but the inverse is not necessary true, see [31]. In Feynman picture, the continuous multi-time propagator is derived. The interesting point is that this multi-time propagator comes with a new feature on sum over all possible paths. One needs to take into account not only all possible paths on the space of dependent variables, but also on the space of independent variables(time variables). Of course,
this idea is not new and it was first introduced by Nijhoff [43] in 2013. We point at this stage that what we come up for the formula of the continuous multi-time propagator in the 1-form case is not the same with Nijhoff’s proposal. However, they do share the exactly the same interpretation. Another point is that, as we mention earlier on taking all possible path both dependent and independent variables, the propagator contains also non-classical paths which do not satisfy the closure relation. Then this new beauty beast must be tamed. Therefore, the semi-classical approximation is applied to the continuous multi-time propagator. The propagator is then written in terms of the classical action together with the fluctuation (prefactor). With this new form of the continuous multi-time propagator (approximated one), the integrability criterion can be constructed. A major intriguing feature in this context is that there exists a space of Lagrangians. All Lagrangians produce the same equations of motion, but only a special set of Lagrangians satisfies the closure relation. With this special set of Lagrangians, the continuous multi-time propagator is extremum yielding path-independent feature on the space of independent variables. This new feature is known as the quantum variation [42]. The last point that we would like to mention is that our set up on deriving the continuous multi-time propagator is not only restricted to the quadratic Lagrangian cases. However, the result from King and Nijhoff [42] for the quadratic Lagrangians, namely harmonic oscillators, provides a solid verification of our model as the special case, see appendix E. However, further investigation is needed for non-quadratic Lagrangians.

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8 This new perspective of treating the dependent and independent variables on the same equal footing was suggested in many places [49, 50], see further discussion in [48].
A The exponential maps for \( N \)-parameter group

In this appendix, the full derivation of (2.15) will be presented. For simplicity, we will first consider the 2-time variables: \( t = (t, \tau) \). The composite map \( \phi^X_{t_2} \circ \phi^X_{t_1} \) could be written as:

\[
\phi^X_{t_2} \circ \phi^X_{t_1} = T e^{\int dt X_{H_2}(t)} T e^{\int dt X_{H_1}(t)} = \left\{ T \left[ I + \sum_{m=1}^{\infty} \frac{1}{m!} \left( \prod_{j=1}^{n} \int dt_j \left( \prod_{i=1}^{n} X_{H_2}(t_i) \right) \right) \right] \right\} \left\{ T \left[ I + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \prod_{j=1}^{n} \int dt_j \left( \prod_{i=1}^{n} X_{H_1}(t_i) \right) \right) \right] \right\} .
\]

The first two terms in equation (A.1) would give

\[
\int d\tau_1 X_{H_2}(t_1) + \int dt_1 X_{H_1}(t_1) = \int_{\Gamma} dt_1 \cdot X_H(t_1) , \tag{A.2}
\]

where \( X_H(t_1) = (X_{H_1}(t_1), X_{H_2}(t_1)) \). The next terms in the expansion will be

\[
\frac{1}{2!} \left( \int d\tau_1 \int d\tau_2 T [X_{H_2}(t_1)X_{H_2}(t_2)] + \int d\tau_2 \int dt_1 [X_{H_2}(t_2)X_{H_1}(t_1)] + \int dt_1 \int dt_2 [X_{H_2}(t_1)X_{H_1}(t_2)] \right) . \tag{A.3}
\]

Since the time variables of the second and third terms of the equation (A.3) have been ordered, we can insert the time ordering operator into both of them. Using the fact that \([X_{H_1}, X_{H_2}] = 0\), the equation (A.3) would become

\[
\frac{1}{2!} T \left( \int d\tau_1 \int d\tau_2 [X_{H_2}(t_1)X_{H_2}(t_2)] + \int d\tau_2 \int dt_1 [X_{H_2}(t_2)X_{H_1}(t_1)] + \int dt_1 \int dt_2 [X_{H_2}(t_1)X_{H_1}(t_2)] \right) \]

\[
= \frac{1}{2!} T \left( \int d\tau_1 \int d\tau_2 [X_{H_2}(t_1)X_{H_2}(t_2)] + \int d\tau_2 \int dt_1 [X_{H_2}(t_2)X_{H_1}(t_1)] + \int dt_1 \int dt_2 [X_{H_2}(t_1)X_{H_1}(t_2)] \right) \]

\[
= \frac{1}{2!} T \left( \int d\tau_1 \int d\tau_2 [X_{H_2}(t_1)X_{H_2}(t_2)] + \int dt_1 \int d\tau_2 [X_{H_1}(t_1)X_{H_2}(t_2)] + \int dt_1 \int dt_2 [X_{H_1}(t_1)X_{H_1}(t_2)] \right) \]

\[
= \frac{1}{2!} T \left( \int d\tau_1 \int d\tau_2 [X_{H_2}(t_1)X_{H_2}(t_2)] + \int dt_1 \int d\tau_2 [X_{H_1}(t_1)X_{H_2}(t_2)] \right) \]

\[
= \frac{1}{2!} T \left( \int d\tau_1 X_{H_2}(t_1) + \int dt_1 X_{H_1}(t_1) \right) \left( \int d\tau_2 X_{H_2}(t_2) + \int dt_2 X_{H_1}(t_2) \right) \]

\[
= \frac{1}{2!} T \left\{ \int_{\Gamma} dt_1 \cdot X_H(t_1) \int_{\Gamma} dt_2 \cdot X_H(t_2) \right\} . \tag{A.4}
\]

For further terms in the expansion of (A.1), we apply the same trick. Finally, we obtain

\[
\phi^X_{t_2} \circ \phi^X_{t_1} = T \left[ I + \sum_{m=1}^{\infty} \frac{1}{m!} \left( \prod_{j=1}^{n} \int_{\Gamma} dt_j \cdot X_H(t_j) \right) \right] \]

\[
= T e^{\int_{\Gamma} dt X_H(t)} . \tag{A.5}
\]
B To avoid redundant paths of $K^{(3)}$

In equation (3.30), the order the summation is crucial to avoid redundant paths. To see this, we first consider a point $m_1 = 0$. The variable $n_1$ will not contribute because of collapsing of the integration as follows

$$K^{(3)} = \sum_{n_1=1}^{N-1} \int_{(0,0)}^{(n_1,0)} \mathcal{D}[q(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{(n_1,t_2)}^{(n_1,0)} \mathcal{D}[q(n_1,t_2)] \int_{-\infty}^{\infty} d^N q(n_1,0) \int_{(n_1,0)}^{(N,0)} \mathcal{D}[q(t_1,0)] \int_{-\infty}^{\infty} d^N q(N,0)$$

$$\times \int_{(N,0)}^{(N,N)} \mathcal{D}[q(N,t_2)] e^{\frac{i}{\hbar} \int_{t'}^{t''} L} \mathcal{L}$$

$$= \langle q(N,N) | \hat{U}(N,N;N,0) \hat{U}(N,0;n_1,0) \hat{U}(n_1,0;n_1,0) \hat{U}(n_1,0;0,0) | q(0,0) \rangle$$

$$= \langle q(N,N) | \hat{U}(N,N;N,0) \hat{U}(N,0;0,0) | q(0,0) \rangle$$

$$= \int_{(0,0)}^{(N,0)} \mathcal{D}[q(t_1,0)] \int_{-\infty}^{\infty} d^N q(N,0) \int_{(N,0)}^{(N,N)} \mathcal{D}[q(N,t_2)] e^{\frac{i}{\hbar} \int_{t'}^{t''} L} \mathcal{L},$$

(B.1)

where is the propagator in the equation (3.30). Hence, the variable $n_1$ in the first line of equation (B.1) is arbitrary. On the other hand, if the summation over $n_1$ is considered first, there are redundant paths for every $n_1$ at $m_1 = 0$.

C Eliminating the redundant paths of $K^{(5)}$

For $m_1 = 0$ and $m_2 = m_1$ of $K^{(5)}$, we obtain the repeated paths such as the figure 13.

Figure 13: The example of repeated paths between cases of $m_1 = 0$ with $m_2 \in [1, N-1]$ and $m_1 = m_2 \neq 0$.

In figure 13a, $n_1$ is in the range $[1, n_2]$ and therefore $n_2 \geq 2$. We obtain

$$K^{(5)}_a = \sum_{n_2=2}^{N} \sum_{m_2=1}^{N} \int_{(0,0)}^{(n_2,0)} \mathcal{D}[q(t_1,0)] \int_{-\infty}^{\infty} d^N q(n_2,0) \int_{(n_2,t_2)}^{(n_2,m_2)} \mathcal{D}[q(n_2,t_2)] \int_{-\infty}^{\infty} d^N q(n_2,m_2) \int_{(n_2,m_2)}^{(N,m_2)} \mathcal{D}[q(t_1,m_2)]$$

$$\times \int_{-\infty}^{\infty} d^N q(N,m_2) \int_{(N,m_2)}^{(N,N)} \mathcal{D}[q(N,t_2)] e^{\frac{i}{\hbar} \int_{t'}^{t''} L} \mathcal{L}.$$ (C.1)
In figure 13b, the path must be collapsed at \((n_2, m_2)\) and \(n_1 \geq 2\) will give exactly the same path with figure 13a. Therefore, the propagator is

\[
K_b^{(5)} = \sum_{n_2=2}^{N-1} \sum_{m_1=1}^{N} \int_{(0,0)}^{(n_1,0)} D[q(t_1,0)] \int_{-\infty}^{(n_1,m_1)} d^N q(n_1,0) \int_{(n_1,0)}^{(n_1,t_2)} D[q(n_1,t_2)] \int_{(n_1,m_1)}^{(N,m_1)} d^N q(n_1,m_1) \int_{(n_1,m_1)}^{(N,m_1)} D[q(t_1,m_1)]
\]

\[
\times \int_{-\infty}^{\infty} d^N q(N,m_1) \int_{(N,m_1)}^{(N,N)} D[q(N,t_2)] e^{\frac{i}{\hbar} \int_{t_2}^{t} \not{S}^\gamma \cdot \mathbf{L}}.
\]

To avoid this problem, we have to set \(n_1 = 1\) then the propagator (C.2) would vanish.

For \(K^{(7)}\), we would fix \(n_1 = 1, n_2 = 2\) and \(n_3 = [3, N - 1]\) to avoid redundant paths. And, for \(K^{(9)}\), we would fix \(n_1 = 1, n_2 = 2, n_3 = 3\) and \(n_4 = [4, N - 1]\) to avoid redundant paths. Finally, for \(K^{(\text{All})}\), every single \(n_i\) will be fixed, where \(i = 1, 2, \ldots, N - 1\) to avoid redundant paths then the sum over \(n_i\) will disappear, resulting in the propagator (3.38).

**D The explicit form of \(Q_\gamma\) in the equation (3.50)**

Recalling the equation (3.50)

\[
Q_\gamma(q''', q', t'', t') = \int_{y(t'')}^0 \frac{\partial^2 S_\gamma[q(t)\, \not{q}]}{\partial \not{q}^2} y(v) \, y(u) =: y(u)^2 (u - v) \not{O}_\gamma y(v).
\]

we then treat

\[
y(u) \frac{\partial^2 S_\gamma[q(t)\, \not{q}]}{\partial \not{q}^2} y(v) =: y(u) \left( -\delta(u - v) \not{O}_\gamma \right) y(v).
\]

We write \(y(t) = \sum_n a_n y_{n,\gamma}(t)\), where \(a_n\) now represent fluctuation and \(y_{n,\gamma}\) are eigenbases associated with the path \(\gamma\) satisfying

\[
\int_\gamma dt_j y_{n,\gamma}(t) y_{m,\gamma}(t) |_{t_j \neq t_i} = \delta_{nm},
\]

and the boundary conditions

\[
y_{n,\gamma}(t'') = y_{n,\gamma}(t') = 0.
\]

The equation (3.50) becomes

\[
Q_\gamma(q''', q', t'', t') = \int D[a_n] e^{-\frac{i}{\hbar} \sum_{j=1}^N \left( \sum_{n,\gamma} (\lambda_j)_{n,\gamma} |a_n|^2 \right)}
\]

where \((\lambda_j)_{n,\gamma}\) are eigenvalues associated with the eigenbases \(y_{n,\gamma}\) for the operator \(\not{O}_\gamma\). Inserting the equation (D.2) into (D.4), we obtain

\[
Q_\gamma(q''', q', t'', t') = \int D[a_n] e^{-\frac{i}{\hbar} \sum_{j=1}^N \left( \sum_{n,\gamma} (\lambda_j)_{n,\gamma} |a_n|^2 \right)}
\]

\[
= \int D[y(t)] e^{-\frac{i}{\hbar} \sum_{j=1}^N \left( \sum_{n,\gamma} (\lambda_j)_{n,\gamma} \not{O}_j \right) y(t)}
\]

\[
= \int D[y(t)] e^{-\frac{i}{\hbar} \sum_{j=1}^N \left( \sum_{n,\gamma} (\lambda_j)_{n,\gamma} y(t) \int_{\Sigma_{j=1}^N \not{S}_j} \frac{\partial^2 S_{\gamma}}{\partial \not{q}^2} \right) y(v)}
\]

\[
= \int D[y(t)] e^{-\frac{i}{\hbar} \sum_{j=1}^N \left( \sum_{n,\gamma} (\lambda_j)_{n,\gamma} y(t) \int \left( \frac{\partial^2 S_{\gamma}}{\partial \not{q}^2} \right) y(v) \right)}
\]

which is identical with the single-time case, see (3.27). Therefore, we obtain

\[
Q_\gamma(q''', q', t'', t') = \det \left( \frac{i}{2\pi \hbar} \frac{\partial^2 S_\gamma[q(t)]}{\partial \not{q}(t'') \partial \not{q}(t')} \right)^{\frac{1}{2}}
\]

We note that the existence of bases \(y_{n,\gamma}(t)\), satisfying the equation (D.2), is nontrivial. However, an explicit example is illustrated in appendix E.
E The path-independent feature of the propagator for the two harmonic oscillators

Here we will show that, with a special set of quadratic Lagrangians satisfying the closure relation, the multi-time propagator possesses the path independent feature on the space of independent variables. We first write the set of Lagrangians \( \{L_1, L_2\} \), giving the equations of motion in (3.11) and (3.12), as

\[
L_1 = \frac{1}{2} \left( \frac{\partial q_1}{\partial t_1} \right)^2 - \frac{\omega_1^2 q_1^2}{2}, \quad (E.1)
\]

\[
L_2 = \frac{1}{2} \left( \frac{\partial q_2}{\partial t_2} \right)^2 - \frac{\omega_2^2 q_2^2}{2}, \quad (E.2)
\]

where \( \omega_{1,2} \) are constants and \( q(t_1, t_2) = (q_1(t_1, t_2), q_2(t_1, t_2)) \). We first show that a set of Lagrangians satisfying Lagrangian closure relation as follows:

\[
\frac{\partial L_1}{\partial t_2} \big|_{q=q_e} - \frac{\partial L_2}{\partial t_1} \big|_{q=q_e} = 2 \frac{\partial q_e}{\partial t_1} \left( \frac{\partial^2 q_e}{\partial t_2 \partial t_1} \right) - 2 \omega_1^2 q_e \frac{\partial q_e}{\partial t_2} - 2 \frac{\partial q_e}{\partial t_2} \left( \frac{\partial^2 q_e}{\partial t_1 \partial t_2} \right) + 2 \omega_2^2 q_e \frac{\partial q_e}{\partial t_1} = 0.
\]

\[
(E.3)
\]

Imposing the constraint equation \([11]\)

\[
\frac{\partial L_1}{\partial \left( \frac{\partial q_e}{\partial t_2} \right)} \big|_{q=q_e} \left( \frac{dt_1}{ds} \right)^2 + \left( \frac{\partial L_2}{\partial \left( \frac{\partial q_e}{\partial t_2} \right)} \big|_{q=q_e} - \frac{\partial L_1}{\partial \left( \frac{\partial q_e}{\partial t_1} \right)} \big|_{q=q_e} \right) \frac{dt_1}{ds} \frac{dt_2}{ds} - \frac{\partial L_2}{\partial \left( \frac{\partial q_e}{\partial t_1} \right)} \big|_{q=q_e} \left( \frac{dt_2}{ds} \right)^2 = 0,
\]

we obtain

\[
\frac{\partial q_e}{\partial t_1} = \frac{\partial q_e}{\partial t_2}.
\]

The equation \((E.3)\)

\[
2 \frac{\partial q_e}{\partial t_1} \left( \frac{\partial^2 q_e}{\partial t_2^2} + \omega_2^2 q_e \right) - 2 \frac{\partial q_e}{\partial t_2} \left( \frac{\partial^2 q_e}{\partial t_1^2} + \omega_1^2 q_e \right) = 0
\]

vanishes with the help of equations of motion \((3.11)\) and \((3.12)\).

Then, in order to verify path independent feature, we consider the exponent term in the equation \((3.50)\), taken in a system of two harmonic oscillators, becoming

\[
-\frac{i}{2} \sum_{j=1}^{2} \int \Gamma du_j \int \Gamma dv_j \left( \tilde{y}(u) \frac{\delta^2 S_j}{\delta q_e(t) \delta q_e(v)} \tilde{y}(v) \right)
\]

\[
= \frac{i}{2\hbar} \left( \int \Gamma dt_1 \left( \frac{\partial y}{\partial t_1} \right)^2 - \omega_1^2 y^2 \right) + \int \Gamma dt_2 \left( \frac{\partial y}{\partial t_2} \right)^2 - \omega_2^2 y^2 \right) \right).
\]

\[
(E.7)
\]

In figure[14a] the multi-time propagator reads

\[
K_A(q(T_1, T_2), (T_1, T_2); q(0, 0), (0, 0)) = \mathcal{Q}_A \mathcal{K}^S_A[q_e],
\]

where

\[
\mathcal{Q}_A = \int \gamma(T_1, T_2) = 0 \quad \mathfrak{D}_A[y(t_1, t_2)] e^{\frac{i}{\hbar} \left( f^{(T_1, 0)} dt_1 \left( \frac{\partial y}{\partial t_1} \right)^2 - \omega_1^2 y^2 \right) + f^{(T_2, T_1)} dt_2 \left( \frac{\partial y}{\partial t_2} \right)^2 - \omega_2^2 y^2 \right)}.
\]

\[
(E.9)
\]

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Therefore, the equation (E.9) becomes

\[ \int_{(0,0)}^{(T_1,0)} dt_1 \left[ \left( \frac{\partial y}{\partial t_1} \right)^2 - \omega_1^2 y^2 \right] + \int_{(T_1,0)}^{(T_1,T_2)} dt_2 \left[ \left( \frac{\partial y}{\partial t_2} \right)^2 - \omega_2^2 y^2 \right] \]

\[ = \int_{(0,0)}^{(T_1,0)} dt_1 y \left[ - \left( \frac{\partial y}{\partial t_1} \right)^2 - \omega_1^2 \right] + y(T_1,0) \frac{\partial y(T_1,0)}{\partial t_1} \]

\[ + \int_{(T_1,0)}^{(T_1,T_2)} dt_2 y \left[ - \left( \frac{\partial y}{\partial t_2} \right)^2 - \omega_2^2 \right] - y(T_1,0) \frac{\partial y(T_1,0)}{\partial t_2} . \]  

(E.10)

For this particular path, the fluctuation \( y \) can be expressed in the form

\[ y(t_1,t_2) = \sum_n a_n y_{n,A}(t_1,t_2) = \sum_n a_n \left( \int_{0}^{T_1} \sin \left( \frac{n\pi}{T_1} t_1 \right) \cos \left( \frac{n\pi}{T_2} t_2 \right) + \int_{0}^{T_2} \sin \left( \frac{n\pi}{T_2} t_2 \right) \cos \left( \frac{n\pi}{T_1} t_1 \right) \right) , \]

where \( 0 \leq t_1,t_2 \leq T_{1,2} \) and it is not difficult to show that the orthonormality condition holds

\[ \int_{(0,0)}^{(T_1,0)} dt_1 y_{n,A} y_{m,A} = \int_{(T_1,0)}^{(T_1,T_2)} dt_2 y_{n,A} y_{m,A} = \delta_{nm} . \]  

(E.11)

Therefore, the equation (E.9) becomes

\[ Q_A = \int \mathcal{D}[a_n] e^{\frac{i}{\hbar} \sum_n |a_n|^2 \left( -\omega_1^2 - \omega_2^2 + \left( \frac{\omega_1}{T_1} \right)^2 + \left( \frac{\omega_2}{T_2} \right)^2 \right)} . \]  

(E.13)

Next, we will repeat the same process with the path in figure 14B and we find that

\[ Q_B = \int_{y(0,0)=0}^{y(T_1,T_2)=0} \mathcal{D}[y(t_1,t_2)] e^{\frac{i}{\hbar} \int_{(0,0)}^{(T_1,T_2)} dt \left[ \left( \frac{\partial y}{\partial t_1} \right)^2 - \omega_2^2 y^2 \right] + f(T_1,T_2) \int_{(0,0)}^{(T_1,T_2)} dt \left[ \left( \frac{\partial y}{\partial t_2} \right)^2 - \omega_1^2 y^2 \right]} . \]  

(E.14)

Fortunately, the eigenbases in the equation (E.11) are still applicable. Thus, the equation (E.11) can be figured out as

\[ Q_B = \int \mathcal{D}[a_n] e^{\frac{i}{\hbar} \sum_n |a_n|^2 \left( -\omega_1^2 - \omega_2^2 + \left( \frac{\omega_1}{T_1} \right)^2 + \left( \frac{\omega_2}{T_2} \right)^2 \right)} . \]  

(E.15)

For the path in figure 14C the Q-factor for multi-time propagator is

\[ Q_C = \int_{y(0,0)=0}^{y(T_1,T_2)=0} \mathcal{D}[y(t_1,t_2)] e^{\frac{i}{\hbar} \int_{(0,0)}^{(T_1,T_2)} dt_1 \left[ \left( \frac{\partial y}{\partial t_1} \right)^2 - \omega_1^2 y^2 \right] + f(T_1,T_2) \int_{(0,0)}^{(T_1,T_2)} dt_2 \left[ \left( \frac{\partial y}{\partial t_2} \right)^2 - \omega_2^2 y^2 \right]} . \]  

(E.16)
The fluctuation \( y \) must be expressed in a new set of eigenbases as

\[
y(t_1, t_2) = \sum_n a_n y_{n,C}(t_1, t_2) = \begin{cases} 
\sum_n a_n \sqrt{\frac{2}{T_1}} \sin \left( \frac{n\pi}{T_1} t_1 \right) \cos \left( \frac{n\pi}{T_2} t_2 \right) ; & (t_1 \leq \tau, t_2 = 0) \cup (t_1 \geq \tau, t_2 = T_2) \\
\sum_n a_n \sqrt{\frac{2}{T_1}} \cos \left( \frac{n\pi}{T_1} t_1 \right) \sin \left( \frac{n\pi}{T_2} t_2 \right) ; & (t_1 = \tau, t_2 \in [0, T_2])
\end{cases}
\]

(E.17)

We finally obtain the equation (E.16) in the form

\[
Q_C = \int D[a_n] e^{\frac{i}{\hbar} \sum_n |a_n|^2 \left( -\omega_1^2 - \omega_2^2 + \left( \frac{n\pi}{T_1} \right)^2 + \left( \frac{n\pi}{T_2} \right)^2 \right)}.
\]

(E.18)

Here we notice that \( Q_A = Q_B = Q_C \) and therefore

\[
K(\mathbf{q}(T_1, T_2), (T_1, T_2); \mathbf{q}(0, 0), (0, 0)) = Q_A e^{\hat{S}_A[\mathbf{q}_c]} = Q_B e^{\hat{S}_B[\mathbf{q}_c]} = Q_C e^{\hat{S}_C[\mathbf{q}_c]},
\]

(E.19)

which is nothing but the path independent feature of the propagator for the integrable system.
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