ON THE SCHMIDT AND ANALYTIC RANKS FOR TRILINEAR FORMS

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Abstract. We discuss relations between different notions of ranks for multi-
linear forms. In particular we show that the Schmidt and the analytic ranks
for trilinear forms are essentially proportional.

1. Introduction

In recent years there is growing interest in properties of polynomials which are
independent of the number of variables. To study such properties various notions
were introduced for measuring the complexity of polynomials. In this paper we
compare three notion of rank of polynomials \( P \) over a field \( k \) - the Schmidt rank
\( r_k(P) \), the slice rank \( s_k(P) \) both defined for arbitrary fields \( k \), and the analytic
rank \( a_k(P) \) defined for finite fields \( k = \mathbb{F}_q \).

Definition 1.1. Let \( k \) be a field, \( V \) be a finite-dimensional \( k \)-vector space, \( V = V(k) \), let \( S_k = \bigoplus_{d \geq 0} S_k^d \) be the graded algebra of polynomial functions on \( V \) and
let \( P \in S^d_k \).

(1) The Schmidt rank \( r_k(P) \) is the minimal number \( r \) such that \( P \) can be
written in the form \( P = \sum_{i=1}^{r} Q_i R_i \), where \( Q_i, R_i \in S \) are polynomials on
\( V \) of degrees \( < d \).

(2) The slice rank \( s_k(P) \) is the minimal number \( r \) such that \( P \) can be written
in the form \( P = \sum_{i=1}^{r} Q_i R_i \), where \( \text{deg}(Q_i) = 1 \).

(3) In the case when \( k = \mathbb{F}_q \) is a finite field and \( \psi : k \to \mathbb{C}^* \) a non-trivial ad-
ditive character we write \( A_{k,\psi}(P) := \sum_{v \in V} \psi(P(v)) q^{\dim(V)} \). We define the analytic
rank \( a_{k,\psi}(P) := -\log_q(|A_{k,\psi}(P)|) \).

These three notions of complexity of polynomials play an important role in
many problems in number theory, additive combinatorics, and algebraic geometry.
The Schmidt rank, also called the \( h \)-invariant, was first introduced by Schmidt
in his paper on integer points in varieties defined over the rationals. In his paper
Schmidt showed that over the complex field the Schmidt rank of a polynomials
is proportional to the codimension of the singular locus of the associated variety.
The same notion of complexity was reintroduced in the work of Ananyan and

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Hochster \cite{Hoch} as the \textit{strength} of a polynomial and was used in the proof of the Stillman conjecture. The notion of slice rank played an important role in the arguments for the cap set problem in combinatorics \cite{Gow,ST1,ST2}. Finally the notion of analytic rank for polynomials over finite fields was introduced in Gower and Wolf \cite{GW} as a tool for studying the CS-complexity of systems of linear equations.

\textbf{Remark 1.2.} It is clear that \( r_k(P) \leq s_k(P) \) and \( r_k(P) = s_k(P) \) if \( d \leq 3 \).

In this paper we consider only the case of multilinear polynomials. Namely \( V = V_1 \times \cdots \times V_d \) is a product of \( k \)-vector spaces and \( P : V_1 \times \cdots \times V_d \to k \) is a multilinear polynomial.

We introduce two additional definitions.

\textbf{Definition 1.3.} \begin{enumerate}
\item In the case when \( V = V_1 \times \cdots \times V_d \) is a product of \( k \)-vector spaces and \( P : V_1 \times \cdots \times V_d \to k \) is a multilinear polynomial we define a \( k \)-subvariety \( \mathcal{Z}_P \subset V_2 \times \cdots \times V_d \) by \( \mathcal{Z}_P := \{(v_2, \ldots, v_d) \in V_2 \times \cdots \times V_d | P(v_1, v_2, \ldots, v_d) = 0, \forall v_1 \in V_1\} \).
\item For a subspace \( L \subset (V_1 \otimes \cdots \otimes V_d) \) we define \( r_k(L) \) as the minimum of \( \sum_{i=1}^{d} \text{codim}(W_i) \) over the set of subspaces \( W_i \subset V_i, i = 1, \ldots, m \), such that \( l_{W_1 \otimes \cdots \otimes W_d} \equiv 0 \) for all \( l \in L \).
\end{enumerate}

\textbf{Claim 1.4.} For a multilinear polynomial \( P : V_1 \times \cdots \times V_d \to k \) the slice rank \( s_k(P) \) is equal to the minimum of \( \sum_{i=1}^{d} \text{codim}(W_i) \) where \( W_i \subset V_i, 1 \leq i \leq d \) are subspaces such that \( P_{W_1 \times \cdots \times W_d} \equiv 0 \). It follows that if \( L = kP \) then \( r_k(L) = s_k(P) \).

\textbf{Lemma 1.5.} In the case when \( k = \mathbb{F}_q \) we have \( A_{k,\psi}(P) = \frac{|\mathcal{Z}_P(\mathbb{F}_q)|}{q^{\text{dim}(V_1)}} \).

\textbf{Proof.} For \( (v_2, \ldots, v_n) \in V_2 \times \ldots \times V_d \) we define \( \psi_{v_2,\ldots,v_n} : V_1 \to \mathbb{C}^* \) by \( \psi_{v_2,\ldots,v_n}(v_1) := \psi(P(v_1, v_2, \ldots, v_n)) \).

It is clear that \( \psi_{v_2,\ldots,v_n} \) is an additive character of \( V_1 \) which trivial if and only if \( (v_2, \ldots, v_n) \in \mathcal{Z}_P(\mathbb{F}_q) \). Therefore \( \sum_{v_1 \in V_1} \psi_{v_2,\ldots,v_n}(v_1) = 0 \) if \( v_1 \not\in \mathcal{Z}_P(\mathbb{F}_q) \) and \( \sum_{v_1 \in V_1} \psi_{v_2,\ldots,v_n}(v_1) = q^{\text{dim}(V_1)} \) if \( v_1 \in \mathcal{Z}_P(\mathbb{F}_q) \). \( \square \)

\textbf{Corollary 1.6.} The analytic rank of a multilinear polynomial \( P \) does not depend on the choice of the non-trivial additive character \( \psi \). We will denote it by \( a_k(P) \).

Fro a field \( k \) we denote by \( \overline{k} \) the algebraic closure. Obviously we have \( r_{\overline{k}}(P) \leq r_k(P) \). We conjecture that the reverse inequality is essentially holds as well.

\textbf{Conjecture 1.7} (d). For any \( d \geq 2 \) there exists \( \kappa_d > 0 \) such that \( r_{\overline{k}}(P) \leq \kappa_d r_k(P) \) for any field \( k \) and a multilinear \( k \)-polynomial \( P \) of degree \( d \), where \( \overline{k} \) is the algebraic closure of \( k \).

\textbf{Remark 1.8.} It is easy to see that Conjecture \cite{DR} holds for \( d = 2 \) with \( \kappa_2 = 1 \).
In [4] Derksen introduced the notion of the $G$-stable rank of a multilinear polynomial (denoted $r^G_k(P)$) defined in terms of Geometric Invariant Theory and proved the following:

1. $r^G_k(P) = r^G_k(\bar{P})$ and
2. $\frac{2s_k(P)}{d} \leq r^G_k(P) \leq s_k(P)$

This immediately implies the inequality $s_k(P) \leq \frac{3}{2}s_k(P)$ for trilinear polynomials $P$. Since $s_k(P) = r_k(P)$ for trilinear polynomials $P$ we obtain the following result.

**Theorem 1.9** (Dersken). For trilinear polynomials we have $r_k(P) \leq \frac{3}{2}r_k(\bar{P})$.

Quantitative estimates for the relation between analytic rank and Schmidt rank provide a means for obtaining quantitative bounds for important problems in additive combinatorics, number theory, and algebraic geometry.

We conjecture the following quantitative relation between the Schmidt rank and analytic rank.

**Conjecture 1.10** (d). For any $d \geq 2$ there exists $c_d > 0$ such that $r_{F_q}(P) \leq c_d a_{F_q}(P)$ for any multilinear polynomial of degree $d$.

**Remark 1.11.** (1) The inequality $a_{F_q}(P) \leq r_{F_q}(P)$ is known. See [8, 11].

(2) It is easy to see that Conjecture 1.10 holds for $d = 2$ with $c_2 = 1$.

(3) In [13] it was shown that there exist constants $c_d, e_d$ such that for any multilinear polynomial of degree $d$ we have $r_{F_q}(P) \leq c_d (a_{F_q}(P))^{e_d}$ (earlier work [2] gave ineffective bounds). Conjecture 1.10 states that one can take $e_d = 1$.

In his paper [14], Schmidt proved the following result.

**Theorem 1.12** (Schmidt). For any $d \geq 2$ there exists $D_d$ such that for any complex multilinear polynomial $P : V_1 \times \cdots \times V_d \rightarrow \mathbb{C}$ we have $r_{\mathbb{C}}(P) \leq D_d g$ where $g := \text{codim}_{V_1 \times \cdots \times V_d} \mathbb{Z}_P$.

The main goal of this paper is the proof of the following result.

**Theorem 1.13.** (1) Assuming the validity Theorem 1.12 for multilinear polynomials of degree $d$ over algebraically closed fields of finite characteristic we show that Conjecture 1.10(d) implies the validity of Conjecture 1.10(d) for $k = F_q, q > d$ with $c_d = \frac{D_d g_q}{1 - \log_q(d)}$.

(2) Theorem 1.12 holds for multilinear polynomials of degree 3 over algebraically closed fields of characteristic $\geq 3$ with $D_3 = 2$.

The following statement follows now from Theorem 1.9(2).

**Corollary 1.14.** Conjecture 1.10(3) holds with $c_3 = 3$.

The proof of Theorem 1.13 is based on an extension of the rough bound from [5], and as such the method of proof provided for Conjecture 1.10(3) is completely different that the approach taken in [2], [13].
The conjectured bound in 1.10 can be viewed as a special case of the conjectured bounded for the inverse theorem for the Gowers norms over finite fields. Let $f : V \to \mathbb{C}$. The Gowers uniformity norms of $f$, introduced in the study of arithmetic progressions in subsets of the integers, are defined as follows

$$\|f\|_{\overline{U}_d}^2 = \frac{1}{|V|^{d+1}} \sum_{x, h_1, \ldots, h_d \in V} \Delta_{h_d} \cdots \Delta_{h_1} f(x)$$

where $\Delta_h f(x) = f(x + h)f(x)$.

**Conjecture 1.15.** Let $k = \mathbb{F}_p$, and let $d > p$. There exists a constant $F = F(d, k)$ such that the following holds: for any $\delta > 0$, any $k$-vector space $V$, any $f : V \to \mathbb{C}$ satisfying $\|f\|_{U_d} \geq \delta$, there exists a degree $d - 1$ polynomial $P$ such that $|\frac{1}{|V|} \sum_{x \in V} f(x)\psi(P(x))| \geq \delta^F$.

When $f = \psi(Q)$ where $Q$ is a polynomial of degree $d$, then $\Delta_{h_d} \cdots \Delta_{h_1} f(x) = \psi(Q(h_1, \ldots, h_d))$, where $Q$ is the multilinear symmetric form associated with $Q$. Via Fourier analysis the conjectured bound in 1.10 implies the validity of Conjecture 1.15 in this case.

Finally we present a conjecture relating the rank of a subspace over a field $k$ to its rank over the algebraic closure, and prove it in a special case.

**Conjecture 1.16.** There exists a constant $E_d$ such that $r_k(L) \leq E_d r_{\overline{k}}(L)$ for any field $k$ and a linear subspace $L \subset (V_1 \otimes \ldots \otimes V_d)^{\vee}$.

**Theorem 1.17.** Conjecture 1.10 holds for $d = 2$, with $E_d = 2$.

## 2. Proof of Theorem 1.17

In this section we prove Theorem 1.17. We start the proof of with the following result.

**Proposition 2.1.** For any field $k$ and a linear subspace $L \subset \text{Hom}_k(W, V)$ there exists $k$-subspaces $W' \subset W$ and $V' \subset V$ such that $\dim(W/W') = \dim(V') = r_k(L)$ and $l(W') \subset V'$ for all $l \in L$ where $r(L) := \max_{B \in L} r(B)$.

**Proof.** We start with the follows result of Kronecker (see [9]).

**Definition 2.2.**

1. Let $V, W$ be $k$-vector spaces. A pair $A, B \in \text{Hom}(V, W)$ is irreducible is there is no non-trivial decomposition $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$ such $A(V_i), B(V_i) \subset W_i$, for $i = 1, 2$.
2. For any $n \geq 0$ we denote by $A_{n}, B_{n} : k^{n} \to k^{n+1}$ maps such that $A_{n}(e_{i}) = f_{i}, B_{n}(e_{i}) = f_{i+1}$ for $1 \leq i \leq n$, and denote by $A'_{n}, B'_{n} : k^{n+1} \to k^{n}$ maps such that $A'_{n}(e_{i}) = f_{i}$ for $1 \leq i \leq n$, $A_{n}(e_{n+1}) = 0$ and $B'_{n}(e_{i}) = 0, B'_{n}(e_{i+1}) = f_{i}$ for $2 \leq i \leq n + 1$. 


Claim 2.3. Let $A, B \in \text{Hom}(V, W)$ be an irreducible pair of linear maps between are finite dimensional $k$-vector spaces. Then there exist automorphisms $T \in \text{Aut}(V), S \in \text{Aut}(W)$ such that either

1. $A$ is an isomorphism or
2. $B$ is an isomorphism or
3. there exist automorphisms $T \in \text{Aut}(V), S \in \text{Aut}(W)$ such that $(SAT, SBT) = (A_n, B_n), n = \dim(V)$ such that $(SAT, SBT) = (A_n, B_n)$ or
4. there exist automorphisms $T \in \text{Aut}(V), S \in \text{Aut}(W)$ such that $(SAT, SBT) = (A_n', B_n'), n = \dim(W)$.

Lemma 2.4. If $|k| = \infty$ and $A, B \in \text{Hom}_k(V, W)$ linear maps such that $r(A) \geq r(A + tB)$ for all $t \in k$. Then $B(\text{Ker}(A)) \subset \text{Im}(A)$.

Remark 2.5. The assumption that $|k| = \infty$ is necessary. Indeed in the case when $k = \mathbb{F}_q$ we can take $V = W = k[\mathbb{F}_q]$ with the basis $\{e_x\}_{x \in \mathbb{F}_q}, A(e_x) = xe_x$ and $B = Id$. Then $r(A + tB) = r(A) = q - 1$ for all $t \in k$ but $B(\text{Ker}(A)) \not\subset \text{Im}(A)$.

Proof. Consider first the case when a pair $(A, B)$ is irreducible. If $\text{Im}(A) = W$ or $\text{Ker}(A) = \{0\}$ we obviously have the inclusion $B(\text{Ker}(A)) \subset \text{Im}(A)$. In cases (1) and (4) the map $A$ is onto and in the case (3) the map $A$ is an imbedding. In the case (2) we have $\dim(V) = \dim(W)$ and either $A$ is onto or $r(A) < r(B)$. We claim that the assumption that $r(A) < r(B)$ leads to a contradiction.

Let $S \subset A$ be the subset of $s_0$ such that $r(s_0A + B) < \max_{s \in k} r(sA + B)$. Then $S(k) \subset k$ is finite. So $r(sA + B) \geq r(B)$ outside a finite set of $s$, and since $|k| = \infty$ and there exists $s \in k$ such that $r(A + s^{-1}B) \geq r(B) > r(A)$, which is a contradiction.

Now consider the general case when a pair $(A, B)$ is a finite direct sum of irreducible pairs $(A_i, B_i), i \in I$. Since, as we have seen in the previous paragraph, the condition $r(A_i) \leq r(A_i + tB_i)$ is automatically true outside a finite set of $t \in k$ and $|k| = \infty$ the assumption that $r(A) \geq r(A + tB)$ for all $t \in k$ implies that that $r(A_i) \geq r(A_i + tB_i)$ outside a finite set of $t \in k, i \in I$. So $B_i(\text{Ker}(A_i)) \subset \text{Im}(A_i), i \in I$. Therefore $B(\text{Ker}(A)) \subset \text{Im}(A)$.

Lemma 2.6. Proposition 2.5 holds in the case when $|k| = \infty$.

Proof. Choose $A \in L$ such that $r(A) = \hat{r}(L)$ and define $W' := \text{Ker}(A), V' := \text{Im}(A)$. As follows from Lemma 2.4 we have $l(W') \subset V'$ for all $l \in L$. But $\dim(W/W') = \dim(V') = r(A)$.

Now consider the case when the field $k$ is finite. Let $\text{Gr}$ be the Grassmannian of subspaces $W' \subset W$ of codimension $\hat{r}(L)$, and let $\text{Gr}'$ be the Grassmannian of subspaces $V' \subset V$ of dimension $\hat{r}(L)$ and $X_{r_k(L)} \subset \text{Gr} \times \text{Gr}'$ be the subvariety of pairs $(W', V')$ such that and $l(W') \subset V'$ for all $l \in L$. It is clear that the subvariety $X \subset \text{Gr} \times \text{Gr}'$ is closed. Therefore it is proper.
Let $K = k(t)$. Since the field $K$ is infinite it follows from Claim 2.6 that $X(K) \neq \emptyset$. So there exists a rational $k$-morphism $\hat{f} : P^1 \to X$. Any such morphism is regular outside a finite $k$- subset $S \subset P^1$. So we obtain a regular $k$-morphism $\hat{f} : P^1 \to X$. Since $X$ is proper, $\hat{f}$ extends to a regular $k$- morphism $f : P^1 \to X$. Let $(W'_0, V'_0) := f(0) \in X(k)$. By definition of the variety $X$ we see that $l(W'_0) \subset V'_0$ for all $l \in L$.

Now we can finish the proof of Theorem 1.17. As follows from Lemma 1.17 there exists $k$-subspaces $V' \subset V, W' \subset W$ such that $l(W'_0) \subset V'_0$ for all $l \in L \otimes_k K$ and $\dim(W/W') + \dim(V') = 2r_k(L)$. So $r_k(L) \leq 2r_k(L)$.

### 3. Rough bound

#### 3.1. A lemma on the codimension.

Let $K$ be an infinite field, let $A$ be a finitely generated $K$-algebra of (Krull) dimension $n$ and let $L \subset A$ be linear subspace of $A$. We denote by $J \subset A$ the ideal generated by $L$ and denote by $A_L$ the quotient algebra $A/J$.

**Lemma 3.1.** If $\dim A_L < n$ then there exists a finite collection of subspaces $L_i \subset L, i \in I$ such that the algebra $A_i$ has dimension $< n$ for any $l \in L \setminus \bigcup_{i \in I} L_i$.

**Proof.** Let $X_i, i = 1, \ldots, k$, be irreducible components of $\text{Spec} A$ of dimension $n$.

For every $i$ we denote by $K_i$ the field of rational functions on the component $X_i$, and consider the natural morphism $\nu_i : A \to K_i$.

Since $\dim A_L < n$ the image $\nu_i(L)$ is not zero. Hence the subspace $L_i := L \cap \text{Ker}(\nu_i) \subset L$ is strictly contained in $L$.

If an element $l \in L$ does not belong to any of the spaces $L_i$ then its image in every field $K_i$ is not zero. This implies that the dimension of the algebra $A_i = A/Al$ is less than $n$. \hfill $\Box$

**Corollary 3.2.** There exist elements $l_1, \ldots, l_m \in L, m := \dim(A) - \dim(A/L)$ such that $\dim(A/J) = \dim(A/L)$ where $J \subset A$ is the ideal generated by $l_1, \ldots, l_m$.

**Proof.** Induction in $m$. \hfill $\Box$

#### 3.2. A proof of the rough bound.

In this subsection we present a proof of a generalization of the rough bound from [5].

For any subset $\Theta \subset \mathbb{F}_q[x_1, \ldots, x_n]$ we denote by $X_\Theta \subset A^n$ the subscheme which is the intersection of zeros of $\theta \in \Theta$.

**Proposition 3.3.** Let $M \subset \mathbb{F}_q[x_1, \ldots, x_n]$ be a linear subspace of polynomials of degrees $d$ such that $Y := X_M$ is of dimension $m$. Then $|Y(\mathbb{F}_q)| \leq q^m d^c, c := n - m$.

**Proof.** Let $F$ be the algebraic closure of $\mathbb{F}_q$. 

As follows from Lemma 3.1 there exists \( P_i \in M \otimes_{\mathbb{F}_q} F, 1 \leq i \leq c \) such that \( \dim(Y') = m \) where \( Y' = X_P := \{ P_i \}_{1 \leq i \leq c} \).

It is clear that \( Y(\mathbb{F}_q) \) is the intersection of \( Y \) with hypersurfaces \( S_j, 1 \leq j \leq n \) defined by the equations \( h_j(x_1, \ldots, x_n) = 0 \) where \( h_j(x_1, \ldots, x_n) = x_j^q - x_j \). Since \( Y \subset Y' \) we see that \( Y(\mathbb{F}_q) \) is contained in the intersection of \( Y' \) with hypersurfaces \( S_j, 1 \leq j \leq n \).

For \( j = 1, \ldots, m \) let \( H_j = \sum_{i=1}^n a_{ij} h_i, \ a_{ij} \in F' \) be a linear combination of \( h_j \) such that coefficients \( a_{ij} \) are algebraically independent over \( F \) where \( F'/F \) is a transcendental extension. We denote by \( Z_1, \ldots, Z_m \subset \mathbb{A}^n \) be the corresponding hypersurfaces and define \( B_j := Y' \cap (\cap_{i=1}^j Z_i) \).

Claim 3.4. Each component \( C \) of \( B_j \) is of dimension \( m - j \).

Proof. The proof is by induction in \( j \). The statement obviously is true for \( j = 0 \).

Any component \( C \) of \( B_{j+1} \) is a component of an intersection \( C' \cap Z_{j+1} \) for some component \( C' \) of \( B_j \). By induction \( \dim(C') = m - j \). So not all the functions \( h_j \) vanish on \( C' \). Hence by the genericity of the choice of linear combinations \( \{ H_j \} \) we see that \( H_{j+1} \) does not vanish on \( C' \) and therefore \( C' \cap Z_{j+1} \) is of pure dimension \( m - j - 1 \).

As follows from Claim 3.4 the intersection \( Y' \cap Z_1 \cap \cdots \cap Z_m \) has dimension \( 0 \). Therefore the Bézout’s theorem implies that \( |Y' \cap Z_1 \cap \cdots \cap Z_{n-c}| \leq q^r d^c \). Since \( Y(\mathbb{F}_q) = Y \cap Z_1 \cap \cdots \cap Z_n = X \cap Y_1 \cap \cdots \cap Y_{n-c} \) we see that \( |Y(\mathbb{F}_q)| \leq q^r d^c \).

4. Proof of Theorem 1.13(1)

Proof. Let \( k = \mathbb{F}_q, P : V_1 \times \cdots \times V_d \to k \) be a multilinear polynomial and \( g = \text{codim}_{\mathbb{A}^{2d}} \mathbb{Z}_P \) and \( k \) be the algebraic closure of \( \mathbb{F}_q \).

Since (see Lemma 1.5) \( a_k(P) = \sum_{i=2}^d \dim(V_i) - \log_q(|\mathbb{Z}_P(k)|) \) it follows from Proposition 3.3 that \( |\mathbb{Z}_P(k)| \leq d^n q^{n-g} \) where \( n = \sum_{i=2}^d \dim(V_i) \). So \( \log_q(|\mathbb{Z}_P(k)|) \leq g \log_q(d) + n - g \). We see that \( a_k(P) \geq g(1 - \log_q(d)) \).

Assuming the validity of Theorem 1.12 for multilinear polynomials of degree \( d \) over \( k \) we see that \( g \geq \frac{\text{rank}_k(P)}{\text{rank}_{k_d}D_d} \) and therefore \( a(P) \geq \frac{\text{rank}_k(P)}{\text{rank}_{k_d}D_d} (1 - \log_q(d)) \). Theorem 1.13(1) is proven. \( \square \)

In the next section we prove the second part of Theorem 1.13.

5. The adaptation of the Schmidt’s result for fields of finite characteristic in the case when \( d = 3 \).

We use notations from Definitions 1.11, 1.3. In [14] W. Schmidt proved the following result.

Theorem 5.1 (Schmidt). For any \( d \geq 2 \) there exists \( D_d \) such that for any complex multilinear polynomial \( P : V_1 \times \cdots \times V_d \to \mathbb{C} \) we have \( r_C(P) \leq D_d g \) where \( g := \text{codim}_{\mathbb{A}^{2d}} \mathbb{Z}_P \).
His proof extends to any algebraically closed field, and we sketch it here for the three-dimensional case.

In this section we fix an algebraically closed field \( k \) and write \( r(P) \) instead of \( r_k(P) \). Let \( X \) be an algebraic \( k \)-variety. Since our field \( k \) is algebraically closed any constructible subset \( Y \subset X \) defines uniquely a subset \( Y = Y(k) \). We say that a constructible subset \( Y \subset X \) is open if the subset \( Y \subset X \) is open and we define \( \dim(Y) := \dim(Y) \).

For a trilinear polynomial \( P : U \times V \times W \to k \) we denote by \( Z_P \subset V \times W \) the constructible subset of points \( (v,w) \) such that \( P(u,v,w) = 0 \) for all \( u \in U \).

The main goal of this section is to prove the following theorem:

**Theorem 5.2.** For any trilinear polynomial \( P : U \times V \times W \to k \) we have \( r(P) \leq 2g \) where \( g := \text{codim}_{V \times W} Z_P \).

Before proving Theorem 5.2 we remind the reader of some results from Algebra.

### 5.1. Linear Algebra.

**Definition 5.3.** For a bilinear form \( B : U \times V \to k \) we write

1. \( S_U(B) := \{ u \in U | B(u,v) = 0, \forall v \in V \} \).
2. \( S_V(B) := \{ v \in V | B(u,v) = 0, \forall u \in U \} \).

**Remark 5.4.** \( r(B) = \text{codim}_V S_U(B) = \text{codim}_U S_V(B) \).

Let \( X \) be a smooth curve over \( k \), let \( t_0 \in X \) and let \( B(t) : U \times V \to k, t \in X \) be a family of bilinear forms. We write \( B := B(t_0), S_U := S_U(B) \), \( S_V := S_V(B) \), and \( C := (\partial B(t)/\partial t)|_{t=t_0} \).

**Claim 5.5.** If \( r(B(t)) \leq r(B), t \in X \) then \( C_{|S_U \times S_V} \equiv 0 \).

**Proof.** We show that the assumption that \( C_{|S_U \times S_V} \neq 0 \) leads to a contradiction. To simplify notations we assume that \( X = A \) and \( t_0 = 0 \).

To start with we choose bases \( e_i, f_j \), where \( 1 \leq i \leq \dim(U), 1 \leq j \leq \dim(V) \) of \( U \) and \( V \) such that

1. \( B(e_i, f_j) = \delta_{ij}, 1 \leq i, j \leq r(B) \).
2. \( B(e_i, f_j) = 0 \) if either \( i > r(B) \) or \( j > r(B) \).

Then \( S_U \) is the span of \( \{e_i\} \), where \( r(B) + 1 \leq i \leq \dim(U) \) and \( S_V \) is the span of \( \{f_j\} \), where \( r(B) + 1 \leq j \leq \dim(V) \). Since \( C_{|S_U \times S_V} \neq 0 \) there exist a pair \((i_0, j_0)\) such that \( i_0, j_0 > r(B) \) and \( C(e_{i_0}, f_{j_0}) \neq 0 \).

Let \( U' \subset U, \dim(U') = r(B) + 1 \) be the span of \( \{e_i\}_{1 \leq i \leq r(B)} \) and of \( e_{i_0} \) and \( V' \subset V, \dim(V') = r(B) + 1 \) be the span of \( \{f_j\}_{1 \leq j \leq r(B)} \) and of \( f_{j_0} \).

We denote by \( \Delta(t) \) be the determinant of the bilinear form of \( B(t)_{U' \times V'} \) with respect to chosen bases of \( U' \) and \( V' \). The condition \( r(B(t)) \leq r(B), t \in k \) implies that \( \Delta(t) \equiv 0 \). On the other hand \( \Delta(t) \equiv tC(e_{i_0}, f_{j_0}) \mod (t^2) \).

This contradiction proves Lemma 5.5. □
5.2. Transcendence degree.

**Definition 5.6.** (1) For a field extension $K/k$ we define the transcendence degree $tr(K/k)$ to be the minimal $n \geq 0$ such that there is no imbeddings of $k(u_0, u_1, \ldots, u_n) \rightarrow K$.

(2) For a point $w \in K^N$ we denote by $k(w) \subset K$ the subfield generated by $k$ and the coordinates of $w$ and write $tr_k(w) := tr(k(w)/k)$.

(3) Let $X \subset A^N$ be an irreducible $k$-subvariety. A point $w \in X(K)$ is generic if it is not contained in any proper $k$-subvariety of $X$.

**Claim 5.7.** (1) Let $X \subset A^M$ be an irreducible $k$ variety and $w \in X(K)$ be generic point. Then $tr(k(w)/k) = \dim(X)$.

(2) Let $p : A^M \rightarrow A^N$ be a $k$-linear map, $Z \subset A^M$ an irreducible variety, and $Y = p(Z)$. Let $v \in Z(K)$ be generic point and $w := p(v) \in Y(K)$. Then $w \in Y(K)$ is a generic point, $k(w) \subset k(v)$ and $tr(k(v)/k(w)) = \dim(Z) - \dim(Y)$.

5.3. Proof of Theorem 5.2.

**Proof.** Let $P : U \times V \times W \rightarrow k$ be as in Theorem 5.2 and $Z \subset Z_P$ be an irreducible component of the maximal dimension, $g := \text{codim}_{U \times V \times W} Z$. We have to show that $r(P) \leq 2g$.

Let $z_0 = (v_0, w_0)$ be a generic point of $Z$, and let $Y \subset W$ be the projection of $Z$ on $W$. For $w \in W$ we denote by $P_w$ the bilinear forms on $V$ given by $P_w(u, v) := P(u, v, w)$.

Let $S_U = \{u \in U | P_{w_0}(u, v) = 0, \forall v \in V\}$, and let $S_V = \{v \in V | P_{w_0}(u, v) = 0, \forall u \in U\}$. Since $w_0$ is a generic point of $Y$, we see that $r(P_y) \leq r(P_{w_0})$ for all $y \in Y$. Since $w_0 \in Y$ is a generic point, $Y$ is smooth at $w_0$ and we can define $S_W := T_Y(w_0)$ the tangent space at $w_0$.

It follows from Claim 1.4 that the following statement implies the validity of Theorem 5.2.

**Proposition 5.8.** (1) $P|_{S_U \times S_V \times S_W} \equiv 0$.

(2) $\text{codim}_U(S_U) + \text{codim}_W(S_W) + \text{codim}_V(S_V) \leq 2g$.

**Proof.** To show that $P|_{S_U \times S_V \times S_W} \equiv 0$ we have to show that for any $C \in S_W$ the restriction of the bilinear form $P_C$ on $S_U \times S_V$ is equal to 0. So we fix $C \in S_W$. Since $P_w$ for $w \in W$ is a linear function on $W$ we have $P_C = (\partial P_w/\partial C)|_{w=w_0}$. So we have to show that $(\partial P_w/\partial C)|_{w=w_0}|_{S_U \times S_V} \equiv 0$.

Choose a smooth curve on $Y$ passing through $w_0$ and tangent to $C$. In other words choose a map $\phi : X \rightarrow Y$ of a smooth curve $X$ to $Y$ and a point $t_0 \in X$ such that $\phi(t_0) = w_0$ and $C = (\phi(t)/\partial t)|_{t=t_0}$. Since $r(P_y) \leq r(P_{w_0})$ for $y \in Y$ the family $B(t) := P_{\phi(t)}$ of bilinear forms on $U \times V$ satisfies the assumption of Claim 5.5. Therefore $(\partial P_w/\partial C)|_{w=w_0}|_{S_U \times S_V} \equiv 0$, and thus $P_C|_{S_U \times S_V} \equiv 0$. 

Since $\text{codim}_U(S_U) = \text{codim}_V(S_V) = r(P_{w_0})$ it follows from Claim 5.7 that $\text{codim}_U(S_U) + \text{codim}_W(S_W) = \text{codim}_V(S_V) + \text{codim}_W(S_W) \leq g$. So $\text{codim}_U(S_U) + \text{codim}_W(S_W) + \text{codim}_V(S_V) \leq 2g$. □

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