Cohomologies of superalgebra of pointwise superproduct

S.E. Konstein* and I.V. Tyutin†‡

I.E. Tamm Department of Theoretical Physics,
P. N. Lebedev Physical Institute,
119991, Leninsky Prospect 53, Moscow, Russia.

Abstract

We consider associative superalgebra realized on the smooth Grassmann-valued functions with compact supports in \( \mathbb{R}^n \). The lower Hochschild cohomologies of this superalgebra are found.

1 Introduction

The hope to construct the quantum mechanics on nontrivial manifolds is connected with geometrical or deformation quantization [1] - [4]. The functions on the phase space are associated with the operators, and the product and the commutator of the operators are described by associative \( \ast \)-product and \( \ast \)-commutator of the functions. These \( \ast \)-product and \( \ast \)-commutator are the deformations of usual product and of usual Poisson bracket.

The gauge theories on the noncommutative spaces, the so-called noncommutative gauge theories, are formulated in terms of the \( \ast \)-product (see [5] and [6] and references wherein).

The structure and the properties of the \( \ast \)-product on the usual (even) manifolds are investigated in details [2], [3] and [4]. On the other hand, the \( \ast \)-product on supermanifolds is not investigated sufficiently.

It is easy to extend formally the noncommutative product proposed in [7] to the supercase [8], however, there is a problem of the uniqueness of the \( \ast \)-product (the uniqueness of the "pointwise" product deformation). In [9], the general form of the associative \( \ast \)-product, treated as a deformation of the "pointwise" product, on the Grassman algebra of a finite number of generators, is found and uniqueness is proved. It is interesting to solve analogous problem for functions of even and odd variables.

Each \( \ast \)-product generates \( \ast \)-commutators, but converse is not true (see discussion in [10]). It seems that non Moyal deformations of Poisson superbracket found in [11] are not generated by \( \ast \)-product.

*E-mail: konstein@lpi.ru
†E-mail: tyutin@lpi.ru
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The problem of finding *-products is connected with the problem of calculating Hochschild cohomologies. The Hochschild cohomologies of the algebra of smooth functions on \( \mathbb{R}^n \) are calculated in [12].

Below we calculate lower Hochschild cohomologies (up to 2-nd) of the algebra of smooth Grassmann valued functions with compact supports on \( \mathbb{R}^n \). Because we need these cohomologies for finding *-product, which is even 2-form, we consider only even cocycles in this paper.

## 2 Notation

Let \( \mathbb{K} \) be either \( \mathbb{R} \) or \( \mathbb{C} \). We denote by \( \mathcal{D}(\mathbb{R}^n) \) the space of smooth \( \mathbb{K} \)-valued functions with compact support on \( \mathbb{R}^n \). This space is endowed with its standard topology: by definition, a sequence \( \varphi_k \in \mathcal{D}(\mathbb{R}^n) \) converges to \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) if the supports of all \( \varphi_k \) are contained in a fixed compact set, and \( \partial^\lambda \varphi_k \) converge uniformly to \( \partial^\lambda \varphi \) for every multi-index \( \lambda \). We set

\[
\mathbf{D}^m_{n+} = \mathcal{D}(\mathbb{R}^{n+}) \otimes \mathcal{G}^m_{n-}, \quad \mathbf{D}'_{n+} = \mathcal{D}'(\mathbb{R}^{n+}) \otimes \mathcal{G}^m_{n-},
\]

where \( \mathcal{G}^m_{n-} \) is the Grassmann algebra with \( n_- \) generators and \( \mathcal{D}'(\mathbb{R}^{n+}) \) is the space of continuous linear functionals on \( \mathcal{D}(\mathbb{R}^{n+}) \). The generators of the Grassmann algebra (resp., the coordinates of the space \( \mathbb{R}^{n+} \)) are denoted by \( \xi^\alpha, \alpha = 1, \ldots, n_- \) (resp., \( x^i, i = 1, \ldots, n_+ \)). We shall also use collective variables \( z^A \) which are equal to \( x^A \) for \( A = 1, \ldots, n_+ \) and are equal to \( \xi^{A-n_+} \) for \( A = n_+ + 1, \ldots, n_+ + n_- \). The spaces \( \mathbf{D}^m_{n+} \) possess a natural grading which is determined by that of the Grassmann algebra. The parity of an element \( f \) of these spaces is denoted by \( \varepsilon(f) \). We also set \( \varepsilon_A = 0 \) for \( A = 1, \ldots, n_+ \) and \( \varepsilon_A = 1 \) for \( A = n_+ + 1, \ldots, n_+ + n_- \).

The integral on \( \mathbf{D}^m_{n+} \) is defined by the relation \( \int dz f(z) = \int_{\mathcal{G}^m_{n+}} dx \int d\xi f(z) \), where the integral on the Grassmann algebra is normed by the condition \( \int d\xi \xi^1 \cdots \xi^{n_-} = 1 \). We identify \( \mathcal{G}^m_{n-} \) with its dual space \( \mathcal{G}^{m-} \) setting \( f(g) = \int d\xi f(\xi)g(\xi), f, g \in \mathcal{G}^{m-} \). Correspondingly, the space \( \mathbf{D}'_{n+} \) of continuous linear functionals on \( \mathbf{D}^m_{n+} \) is identified with the space \( \mathcal{D}'(\mathbb{R}^{n+}) \otimes \mathcal{G}^{m-} \). As a rule, the value \( m(f) \) of a functional \( m \in \mathbf{D}'_{n+} \) on a test function \( f \in \mathbf{D}^m_{n+} \) will be written in the “integral” form: \( m(f) = \int dz m(z)f(z) \).

Let \( L \) be a superalgebra of functions \( f(z) \in \mathbf{D}^m_{n+} \) with the usual “pointwise” product.

Consider an algebra \( \mathcal{A} = \bigoplus \mathcal{A}_k \), where \( \mathcal{A}_k \) is the vector space of even \( k \)-linear separately continuous forms \( \Phi_k(z|f_1, \ldots, f_k) \in \mathcal{A}_k, k = 0, 1, 2, \ldots \) taking value in \( \mathbf{D}^m_{n+} \), \( \varepsilon(\Phi_k(z|f_1, \ldots, f_k)) = \sum_{i=1}^k \varepsilon(f_i), \mathcal{A}_0 \subset \mathbf{D}^m_{n+}, \Phi_0(z) = \phi(z) \in \mathbf{D}^m_{n+}, \varepsilon_{\phi_0} = \varepsilon(\phi) = 0 \), with (noncommutative) associative product

\[
(\Phi_k \circ \Psi_p)(z|f_1, \ldots, f_{k+p}) = \Phi_k(z|f_1, \ldots, f_k)\Psi_p(z|f_{k+1}, \ldots, f_{k+p}).
\]

This algebra has a natural grading \( g: g(\Phi_k) = k, g(\mathcal{A}_k) = k \), and the Hochschild differential \( d_H: \mathcal{A}_k \to \mathcal{A}_{k+1}, g(d_H) = 1 \), which acts by the rule:

\[
d_k \Phi_k(z|f_1, \ldots, f_{k+1}) = f_1(z)\Phi_k(z|f_2, \ldots, f_{k+1}) + \sum_{i=1}^k (-1)^i \Phi_k(z|f_1, \ldots, f_{i-1}, f_if_{i+1}, f_{i+2}, \ldots, f_{k+1}) + (-1)^k \Phi_k(z|f_1, \ldots, f_k)f_{k+1}(z),
\]

(2.1)
Here $d_k \overset{\text{def}}{=} d_H|_{A_k}$. Differential $d_H$ has the following evident properties

\begin{align*}
    d_0\Phi_0(z|f) &= f(z)\phi(z) - \phi(z)f(z) = 0, \\
    d_{k+p}(\Phi_k\Psi_p) &= (d_k\Phi_k)\Psi_p + (-1)^{\sigma(\Phi_k)}\Phi_k d_p\Psi_p, \\
    d_{k+1}d_k &= 0, \quad k = 0, 1, \ldots. \tag{2.2}
\end{align*}

Zeroth, first and second Hochschild cohomologies are found in successive sections.

3 \quad H^0

The cohomological equation

\[ d_0\Phi_0(z|f) = 0 \]

is satisfied identically for any forms $\Phi_0(z)$, such that we have $H^0 = A_0$.

4 \quad H^1

In this case the cohomological equation has the form

\[ f(z)\Phi_1(z|g) - \Phi_1(z|fg) + \Phi_1(z|f)g(z) = 0. \tag{4.1} \]

Let

\[ [z \cup \text{supp}(g)] \cap \text{supp}(f) = \emptyset. \tag{4.2} \]

We obtain from Eq. (4.1)

\[ \hat{\Phi}_1(z|f) = 0, \]

from what it follows

\[ \Phi_1(z|f) = \sum_{q=0}^{Q} m(z)^{(A)_q} (\partial_A)^q f(z). \]

Here $\hat{\Phi}_1(z|f)$ means the restriction of the linear form $\Phi_1(z|f)$ on the domain (4.2) or, equivalently, on the domain $z \cap \text{supp}(f) = \emptyset$,

Analogously to method used in [14], [13] let us choose $f(z) = e^{zp}$, $g(z) = e^{zk}$ in some neighborhood of $x$. We obtain from Eq. (4.1)

\[ F(z|k) - F(z|p+k) + F(z|p) = 0, \tag{4.3} \]

where

\[ F(z|p) = \sum_{q=0}^{Q} m(z)^{(A)_q} (p_A)^q. \]

Eq. (4.3) gives

\[ F(z|p) = m(z)^{A} p_A \]

\[ ^1 \text{Let } z = (x, \xi). \text{ Here and below we use the notation } z \cup U (z \cap U) \text{ instead of } V_z \cup U (V_z \cap U) \text{ where } V_z \text{ is some neighborhood of } x. \]

\[ ^2 \text{details can be found in [13].} \]
or
\[ \Phi_1(z|f) = m(z)^A \partial_A f(z). \]
It is obvious that the cohomologies with different \( m(z)^A \) are independent.

5 \( H^2 \)

The cohomological equation has the form
\[
f(z)\Phi_2(z|g, h) - \Phi_2(z|fg, h) + \Phi_2(z|f, gh) - \Phi_2(z|f, g)h(z) = 0.
\]

(5.1)

Let \( \mathcal{L}_k \subset \mathcal{A}_k \) be the spaces of local forms. The local form is such \( \Phi_k(z|f, ..., f_k) \) that if \( z \cap \text{supp}(f_i) = \emptyset \) for some \( 1 \leq i \leq k \) then \( \Phi_k(z|f_1, ..., f_k) = 0. \)

5.1 Nonlocal part of cocycle

Let
\[
[z \cup \text{supp}(h)] \cap \text{supp}(f) = [z \cup \text{supp}(h)] \cap \text{supp}(g) = \text{supp}(f) \cap \text{supp}(g) = \emptyset.
\]

(5.2)

We obtain from Eq. (5.1)
\[ \hat{\Phi}_2(z|f, g) = 0, \]
which yields
\[
\Phi_2(z|f, g) = \sum_{q=0}^Q \{m_1^{(A)q}(z|f)(\partial_A)^q g(z) + [(\partial_A)^q f(z)]m_2^{(A)q}(z|g) + m_3^{(A)q}(z|f(\partial_A)^q g)\}. \]

(5.3)

Here notation \( \hat{\Phi}_2(z|f, g) \) is used for nondiagonal part of \( \Phi_2(z|f, g) \), i.e. for the restriction of \( \Phi \) on the domain \( \text{5.2} \) or on the domain \( z \cap \text{supp}(f) = z \cap \text{supp}(g) = \text{supp}(f) \cap \text{supp}(g) = \emptyset. \)

Let us note that all the forms \( m_1^{(A)q} \), and the form \( m_1^{(A)0} \) particularly, are globally defined distributions.

Consider the domain
\[
[z \cup \text{supp}(f) \cup \text{supp}(h)] \cap \text{supp}(g) = \emptyset.
\]

(5.4)

We obtain from Eq. (5.1) the equation for the restriction of the form \( \Phi \) on the domain \( \text{5.4} \)
\[ f(z)\hat{\Phi}_2(z|g, h) - \hat{\Phi}_2(z|f, g)h(z) = 0. \]

(5.5)

Substituting representation (5.3) in Eq. (5.5), we find
\[
\sum_{q=0}^Q \{f(z)m_1^{(A)q}(z|g)(\partial_A)^q h(z) - [(\partial_A)^q f(z)]m_2^{(A)q}(z|g)h(z)\} = 0,
\]
from what it follows
\[
m_1^{(A)q}(z|g) \in \mathcal{L}_1, \quad m_2^{(A)q}(z|g) \in \mathcal{L}_1, \quad q \geq 1,
m_2^{(A)0}(z|g) - m_1^{(A)0}(z|g) \in \mathcal{L}_1.
\]
So, the form \( \Phi_2(z|f, g) \) can be represented in the form

\[
\Phi_2(z|f, g) = \sum_{q=0}^{Q} m_3^{(A)q}(z|f(\partial_A)^q g) + m_1^{(A)0}(z|fg) + [f(z)m_1^{(A)0}(z|fg) - m_1^{(A)0}(z|fg) + m_1^{(A)0}(z|f)g(z)] + \Phi_{loc}(z|f, g),
\]

or, redefining \( m_3^{(A)0}(z|f) \),

\[
\Phi_2(z|f, g) = \Phi_{2|3}(z|f, g) + d_1 \Phi_{1|1}(z|f, g) + \Phi_{loc}(z|f, g), \quad \Phi_{1|1}(z|f) = m_1^{(A)0}(z|f),
\]

\[
\Phi_{2|3}(z|f, g) = \sum_{q=0}^{Q} m_3^{(A)q}(z|f(\partial_A)^q g).
\]

(5.7)

To specify \( \Phi_{2|3} \), consider the domain

\[
z \cap [\text{supp}(f) \cup \text{supp}(g) \cup \text{supp}(h)] = \emptyset.
\]

(5.8)

Using representation (5.6), we obtain from Eq. (5.1)

\[
\hat{\Phi}_{2|3}(z|fg, h) - \hat{\Phi}_{2|3}(z|f, gh) = 0.
\]

(5.9)

Substituting representation (5.7) in Eq. (5.9), we find

\[
\sum_{q=0}^{Q} \hat{m}_3^{(A)q}(z\{|fg(\partial_A)^q h - f(\partial_A)^q (gh)\}) = 0,
\]

Choosing \( g(z) = e^{-zp} \), \( h(z) = e^{zp} \) on \( \text{supp}(f) \), we obtain

\[
\hat{F}(z|f; p) - \hat{F}(z|f; 0) = 0, \quad F(z|f; p) = \sum_{q=0}^{Q} \hat{m}_3^{(A)q}(z|f)(pA)^q,
\]

and then

\[
\hat{m}_3^{(A)q}(z|f) = 0 \Rightarrow m_3^{(A)q}(z|f) \in \mathcal{L}_1, \quad q \geq 1.
\]

So, the form \( \Phi_2(z|f, g) \) can be represented in the form

\[
\Phi_2(z|f, g) = m(z|fg) + d_1 \Phi_{1|1}(z|f, g) + \Phi_{loc}(z|f, g), \quad m(z|f) = m_3^{(A)0}(z|f).
\]

(5.10)

At last, consider the domain

\[
[z \cup \text{supp}(h)] \cap [\text{supp}(f) \cup \text{supp}(g)] = \emptyset.
\]

(5.11)

Using representation (5.10), we obtain from Eq. (5.1)

\[
m(z|fg) + m(z|fg)h(z) = 0
\]
and

\[ m(z|f) = 0. \]

Finally, we have obtained

\[ \Phi_2(z|f, g) = \Phi_{2|\text{loc}}(z|f, g) + d_1 \Phi_{1|1}(z|f, g), \quad (5.12) \]

\[ \Phi_{2|\text{loc}}(z|f, g) = \sum_{k,l=0}^{Q} f(z) \left( \hat{\partial}_A^k m^{(A)k|(B)_l}(z)(\partial_B)^l h(z) \right) - f(z) g(x) \left( \hat{\partial}_A^k m^{(A)k|(B)_l}(z)(\partial_B)^l h(z) \right) + \]

\[ + f(z) \left( \hat{\partial}_A^k m^{(A)k|(B)_l}(z)(\partial_B)^l \right) \left[ g(z) h(z) - f(z) \left( \hat{\partial}_A^k m^{(A)k|(B)_l}(z)(\partial_B)^l \right) h(z) = 0 \right]. \quad (5.13) \]

Let \( f(z) = e^{\theta z}, \ g(z) = e^{\theta z}, \ h(z) = e^{\theta z} \) in some neighborhood of \( x \). Then the cohomological equation transforms to the form

\[ F(z|q, \tilde{r}) - F(z|p + q, \tilde{r}) + F(z|p, \tilde{q} + \tilde{r}) - F(z|p, \tilde{q}) = 0, \quad (5.14) \]

\[ F(z|p, q) = \sum_{k,l=0}^{Q} (p_A)^k m^{(A)k|(B)_l}(z)(q_B)^l, \quad \tilde{p}_A = (-1)^{\varepsilon_A} p_A. \]

Let us apply the operator \( \partial/\partial p_A|_{p=0} \) to Eq. (5.14). We obtain

\[ \frac{\partial}{\partial p_A} F(z|q, \tilde{r}) = \Psi^A(z|q + r) - \Psi^A(z|q), \quad (5.15) \]

\[ \Psi^A(z|q) = \frac{\partial}{\partial p_A} F(z|p, \tilde{q}) \bigg|_{p=0}. \]

It follows from Eq. (5.15)

\[ \frac{\partial}{\partial q_A} \Psi^B(z|q + r) - (-1)^{\varepsilon_A \varepsilon_B} \frac{\partial}{\partial q_B} \Psi^A(z|q + r) = \frac{\partial}{\partial q_A} \Psi^B(z|q) - (-1)^{\varepsilon_A \varepsilon_B} \frac{\partial}{\partial q_B} \Psi^A(z|q) \quad (5.16) \]

and then

\[ \frac{\partial}{\partial r_A} \Psi^B(z|r) - (-1)^{\varepsilon_A \varepsilon_B} \frac{\partial}{\partial r_B} \Psi^A(z|r) = -(-1)^{\varepsilon_A \varepsilon_B} \omega^{AB}(z), \quad (5.17) \]

\[ \omega^{AB}(z) = \frac{\partial}{\partial q_B} \Psi^A(z|q) - (-1)^{\varepsilon_A \varepsilon_B} \frac{\partial}{\partial q_A} \Psi^B(z|q) \bigg|_{q=0} = -(-1)^{\varepsilon_A \varepsilon_B} \omega^{BA}(z). \]
A general solution of Eq. (5.17) is

\[
\Psi^A(z|r) = \frac{1}{2} \omega^{AB}(z)r_B + \frac{\partial}{\partial r_A} \phi(z|r). \tag{5.18}
\]

Function (5.18) satisfies Eq. (5.16) also. Substituting Exp. (5.18) in Eq. (5.15), we obtain

\[
\frac{\partial}{\partial q_A} \left( F(z|q, r) - \frac{1}{2} q_A \omega^{AB}(z)r_B - \phi(z|q + r) + \phi(z|q) \right) = 0
\]

and as a result

\[
F(z|q, r) = \frac{1}{2} q_A \omega^{AB}(z)r_B + \phi(z|q + r) - \phi(z|q) + \varphi(z|r). \tag{5.19}
\]

Substituting Exp. (5.19) in Eq. (5.14) we find that function (5.19) satisfies Eq. (5.14) if function \( \varphi(z|r) \) is equal to \( \varphi(z|r) = -\phi(z|r) - \varphi_1(z) \). So, we obtain

\[
F(z|p, q) = \frac{1}{2} p_A \omega^{AB}(z)\tilde{q}_B + \phi(z|p + \tilde{q}) - \phi(z|p) - \phi(z|\tilde{q}),
\]

where a redefinition \( \phi(z|p) \rightarrow \phi(z|p) + \varphi_1(z) \) was made, or

\[
\Phi_{2loc}(z|f, g) = \frac{1}{2} f(z) \overset{\leftarrow}{\partial_A} \omega^{AB}(z) \partial_B g(z) + d_1 \Phi_{1|2}(z|f, g),
\]

\[
\Phi_{1|2}(z|f) = -f(z) \sum_{k=0}^{K} (\overset{\leftarrow}{\partial_A})^k \phi^{(A)k}(z),
\]

where \( \phi^{(A)k}(z) \) are coefficients of the polynomial \( \phi(z|p) = \sum_{k=0}^{K} (p_A)^k \phi^{(A)k}(z) \).

Finally, with (5.12) taken into account, we find that general solution of cohomological equation (5.1) has the form

\[
\Phi_2(z|f, g) = m_\omega(z|f, g) + d_1 \Phi_1(z|f, g), \tag{5.20}
\]

\[
m_\omega(z|f, g) = \frac{1}{2} f(z) \overset{\leftarrow}{\partial_A} \omega^{AB}(z) \partial_B g(z) = -(-1)^{\varepsilon(f)\varepsilon(g)} m_\omega(z|g, f), \quad \varepsilon(\omega^{AB}) = \varepsilon_A + \varepsilon_B,
\]

\[
\Phi_1(z|f) = \Phi_{1|1}(z|f, g) + \Phi_{1|2}(z|f).
\]

The cohomologies with different \( \omega^{AB}(z) \) are independent. Indeed, consider an equation

\[
f(z) \overset{\leftarrow}{\partial_A} \omega^{AB}(z) \partial_B g(z) = d_1 \Psi_1(z|f, g) = f(z) \Psi_1(z|g) - \Psi_1(z|fg) + \Psi_1(z|f)g(z). \tag{5.21}
\]

Let

\[
[z \cup \text{supp}(g)] \cap \text{supp}(f) = \emptyset.
\]

We find from Eq. (5.21)

\[
\overset{\leftarrow}{\Psi_1}(z|f) = 0,
\]

i.e. the form \( \Psi_1(z|f) \) is local:

\[
\Psi_1(z|f) = f(z) \sum_{k=0}^{K} (\overset{\leftarrow}{\partial_A})^k \psi^{(A)k}(z) \equiv f(z) \psi(z|\overset{\leftarrow}{\partial}).
\]
Let \( f(z) = e^{pz} \), \( g(z) = e^{qz} \). It follows from Eq. (5.21)

\[
p_A \omega^{AB}(z)(-1)^{\varepsilon_B} q_B = \psi(z|q) - \psi(z|p + q) + \psi(z|p).
\]

R.h.s. of this equation is symmetric under exchange \( p \leftrightarrow q \), such that we have

\[
p_A \omega^{AB}(z)(-1)^{\varepsilon_B} q_B = q_B \omega^{BA}(z)(-1)^{\varepsilon_A} p_A = -p_A \omega^{AB}(z)(-1)^{\varepsilon_B} q_B
\]

from what it follows \( \omega^{AB}(z) = 0 \), that is, Eq. (5.21) has solutions only for \( \omega^{AB}(z) = 0 \).

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