Chen invariants for Riemannian submersions and their applications in meteorology

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Abstract. In this paper, an optimal inequality involving the delta curvature is exposed. An application of Riemannian submersions dealing meteorology is presented. Some characterizations about the vertical motion and the horizontal divergence are obtained.

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1 Introduction

In 1993, B.-Y. Chen [9] initially introduced a new invariant the so-called delta curvature δ for an n-dimensional Riemannian manifold $M$ by

$$
\delta^k(p) = \tau(p) - (\inf \tau(\Pi_k))(p),
$$

where $2 \leq k \leq n - 1$, $\tau(p)$ is the scalar curvature at $p \in M$ and

$$(\inf \tau(\Pi_k))(p) = \inf \{\tau(\Pi_k) \mid \Pi_k \text{ is a } k\text{-plane section } \subset T_pM\}.
$$

Furthermore, he gave a relation involving the delta curvature, the main intrinsic and extrinsic invariants of submanifolds in a real space form (cf. Lemma 3.2 in [9]). Then, this curvature drew attention of many authors and the notion of discovering simple basic relationships between intrinsic and extrinsic invariants of a submanifold becomes one of the most fundamental problems in submanifold theory (cf. [1], [10], [13], [11], [12], [16], [18], [24], [28], etc.).

Apart from isometric immersions and submanifolds theory, Riemannian submersions have played a substantial role in differential geometry since this frame of maps also makes possible to compare geometrical properties between smooth manifolds. Besides the mathematical significance, Riemannian submersions have important physical and engineering aspects. There exist very nice applications of these mappings in the Kaluza-Klein theory [14][20][29], in the statical machine learning process [30], in the medical imaging cf. [23], in the statical analysis [5], in the robotic theory [2][26].
Motivated by these facts, we firstly establish an optimal inequality involving the delta curvature for Riemannian manifolds admitting a Riemannian submersion. Then, we investigate this inequality for some special cases. Finally, we discuss this inequality in meteorology and obtain some results dealing the vertical motion and horizontal divergence.

2 Riemannian submersions

Let \((M, g)\) and \((B, \tilde{g})\) be \(m\) and \(n\) dimensional Riemannian manifolds with Riemannian metrics \(g\) and \(\tilde{g}\), respectively. A smooth map \(\pi : (M, g) \to (B, \tilde{g})\) is called a Riemannian submersion if

i) \(\pi\) has maximal rank.

ii) The differential \(\pi_*\) preserves the lengths of horizontal vectors.

Now, let \(\pi : (M, g) \to (B, \tilde{g})\) be a Riemannian submersion. For any \(b \in B\), \(\pi^{-1}(b)\) is closed \(r\)-dimensional submanifold of \(M\). The submanifolds \(\pi^{-1}(b)\) are called \(\text{fibers}\). A vector field tangent to fibers is called \(\text{vertical}\) and a vector field orthogonal to fibers is called \(\text{horizontal}\). If we put \(V_p = \ker(\pi_*)\) at a point \(p \in M\), then it can be obtained an integrable distribution \(V\) corresponding to the foliation of \(M\) determined by the fibres of \(\pi\). The distribution \(V_p\) is called \(\text{vertical space}\) at \(p \in M\).

Let \(\mathcal{H}\) be a complementary distribution of \(V\) determined by the Riemannian metric \(g\). For any \(p \in M\), the distribution \(\mathcal{H}_p = (V_p)^\perp\) is called \(\text{horizontal space}\) on \(M\). Thus, we have the following orthogonal decomposition:

\[ TM = V \oplus \mathcal{H}. \]  

A vector field \(E\) on \(M\) is called \(\text{basic}\) if it is horizontal and \(\pi-\)related to a vector field \(E_\ast\) on \(B\) i.e., \(\pi_\ast E_p = E_{\pi(p)}\) for all \(p \in M\). Furthermore, it is known that if \(E\) and \(F\) are the basic vector fields respectively \(\pi-\)related to \(E_\ast\) and \(F_\ast\), one has

\[ g(E, F) = \tilde{g}(E_\ast, F_\ast) \circ \pi. \]  

Let \(h\) and \(v\) are the projections of \(\Gamma(TM)\) onto \(\Gamma(\mathcal{H})\) and \(\Gamma(V)\), respectively. The \(\text{fundamental tensor fields}\) of \(\pi\), denoted by \(A\) and \(T\), are defined respectively by

\[ A_E F = h \nabla h_E v F + v \nabla h_E h F, \]  
\[ T_E F = h \nabla v_E v F + v \nabla v_E h F. \]
for any \( E, F \in \Gamma(TM) \), where \( \nabla \) is the Levi-Civita connection on \( M \).

Now, let us define the following mappings:

\[
\begin{align*}
T^H : \Gamma(V) \times \Gamma(V) & \to \Gamma(H), \\
(U, V) & \to T^H(U, V) = h\nabla_U V, \\
T^V : \Gamma(V) \times \Gamma(H) & \to \Gamma(V), \\
(U, X) & \to T^V(U, X) = v\nabla_U X,
\end{align*}
\]

and

\[
\begin{align*}
A^H : \Gamma(H) \times \Gamma(V) & \to \Gamma(H), \\
(X, U) & \to A^H(X, U) = h\nabla_X U, \\
A^V : \Gamma(H) \times \Gamma(H) & \to \Gamma(V), \\
(X, Y) & \to A^V(X, Y) = v\nabla_X Y,
\end{align*}
\]

Then, it is clear from (2.4) and (2.5) that \( T^H \) is a symmetric operator on \( \Gamma(V) \times \Gamma(V) \) and \( A^V \) is an anti-symmetric operator on \( \Gamma(H) \times \Gamma(H) \). If (2.4) and (2.5) are taken into account in (2.2), we can write

\[
\begin{align*}
\nabla_U V &= T^H(U, V) + v\nabla_U V, \\
\nabla_V X &= h\nabla_V X + T^V(U, X), \\
\nabla_X U &= A^H(X, U) + v\nabla_X U, \\
\nabla_X Y &= h\nabla_X Y + A^V(X, Y)
\end{align*}
\]

for any \( U, V \in \Gamma(V) \) and \( X, Y \in \Gamma(H) \).

Let \( \{U_1, \ldots, U_r, X_1, \ldots, X_n\} \) be an orthonormal basis on \( T_pM \), where \( \mathcal{V} = \text{Span}\{U_1, \ldots, U_r\} \) and \( \mathcal{H} = \text{Span}\{X_1, \ldots, X_n\} \). The mean curvature vector field \( h(p) \) of any fibre is defined by

\[
h(p) = \frac{1}{r} \sum_{j=1}^r T^H(U_j, U_j).
\]

Note that each fiber is a minimal submanifold of \( M \) if and only if \( h(p) = 0 \) for all \( p \in M \). Furthermore, each fiber is called totally geodesic if both \( T^H \) and \( T^V \) vanish identically and it is called totally umbilical if

\[
T^H(U, V) = g(U, V) h
\]

for all \( U, V \in \Gamma(V) \).

Now we recall the following Theorem [15]:
Theorem 2.1. Let \( \pi : (M, g) \to (B, \bar{g}) \) be a Riemann submersion. Then the horizontal space \( \mathcal{H} \) is an integrable distribution if and only if \( A \) vanishes identically.

Remark 2.2. As a consequence of Theorem 2.1, we see that both \( A^H \) and \( A^V \) are related to integrability of \( \mathcal{H} \), they are identically zero if and only if \( \mathcal{H} \) is integrable.

Let \( R, \tilde{R} \) and \( \hat{R} \) are the curvature tensors on \( M, B \) and be the collection of all curvature tensors on fibers \( \pi^{-1}(b) \) respectively, and \( \tilde{R}(X, Y)Z \) be the horizontal lift of \( \tilde{R}_{\pi(b)}(\pi_{sp}X_b, \pi_{sp}Y_b)Z_b \) at any point \( b \in M \) satisfying

\[
\pi_*(\tilde{R}(X, Y)Z) = \tilde{R}((\pi_*X, \pi_*Y))\pi_*Z.
\]

Then, there exist the following relations between these tensors:

\[
R(U, V, W, G) = \hat{R}(U, V, W, G) + \frac{1}{2} g \left( (T^H(U, G), T^H(V, W))\right.
\]
\[
- g \left( T^H(V, G), T^H(U, W)) \right), \tag{2.11}
\]

\[
R(X, Y, Z, H) = \tilde{R}(X, Y, Z, H) - 2 g (A^V(X, Y), A^V(Z, H))
\]
\[
+ g (A^V(Y, Z), A^V(X, H)) - g (A^V(X, Z), A^V(Y, H)), \tag{2.12}
\]

\[
R(X, V, Y, W) = g ((\nabla_X T)(V, W), Y) + g ((\nabla_V A)(X, Y), W)
\]
\[
- g (T^V(V, X), T^V(W, Y)) + g (A^H(X, V), A^H(Y, W)), \tag{2.13}
\]

for any \( U, V, W, G \in \Gamma(V) \) and \( X, Y, Z, H \in \Gamma(H) \). Note that the above equalities are known as Gauss–Codazzi equations for a Riemannian submersion. With the help of Gauss–Codazzi equations, we get the following relations between the sectional curvatures as follows:

\[
K(U, V) = \hat{K}(U, V) - \|T^H(U, V)\|^2 + g (T^H(U, U), T^H(V, V)), \tag{2.14}
\]

\[
K(X, Y) = \hat{K}(X, Y) + 3\|A^V(X, Y)\|^2, \tag{2.15}
\]

\[
K(X, V) = - g ((\nabla_X T)(V, V), X) + \|T^V(V, X)\|^2 - \|A^H(X, V)\|^2, \tag{2.16}
\]

where \( K, \hat{K} \) and \( \hat{K} \) denote the sectional curvatures in \( M \), any fiber \( \pi^{-1}(b) \) and the horizontal distribution \( \mathcal{H} \), respectively. The scalar curvatures of the vertical and horizontal spaces at a point \( p \in M \) is given respectively by

\[
\hat{\tau}(p) = \sum_{1 \leq i < j \leq r} \hat{K}(U_i, U_j) \tag{2.17}
\]

and

\[
\hat{\tau}(p) = \sum_{1 \leq i < j \leq n} \hat{K}(X_i, X_j). \tag{2.18}
\]

Now, we recall the following definition of [4].
Definition 2.3. Let \( \pi : (M, g) \to (B, \tilde{g}) \) be a Riemann submersion and \( X \) be a horizontal vector field on \( \pi \). Then, horizontal divergence of \( X \) is defined by

\[
div_{\mathcal{H}}(X) = \sum_{i=1}^{n} g(\nabla X_i, X_i).
\] (2.19)

Lemma 2.4. \([15]\) Let \( \pi : (M, g) \to (B, \tilde{g}) \) be a Riemann submersion and \( \{U_1, \ldots, U_r\} \) be any orthonormal basis of \( \Gamma(\mathcal{V}) \). For any \( E \in \Gamma(TM) \) and \( X \in \Gamma(\mathcal{H}) \), we have

\[
g(\nabla_E h, X) = \frac{1}{r} \sum_{j=1}^{r} g((\nabla_{E}) (U_j, U_j), X).
\] (2.20)

As a consequence of Lemma 2.4, we obtain that

\[
div_{\mathcal{H}}(h) = \frac{1}{r} \sum_{i=1}^{r} \sum_{j=1}^{r} g((\nabla_{X_i}) (U_j, U_j), X_i).
\] (2.21)

3 An optimal inequality for Riemannian submersions

We begin this section with the following algebraic lemma:

Lemma 3.1. If \( n > k \geq 2 \) and \( a_1, \ldots, a_n, a \) are real numbers such that

\[
\left( \sum_{i=1}^{n} a_i \right)^2 = (n - k + 1) \left( \sum_{i=1}^{n} a_i^2 + a \right),
\] (3.1)

then

\[
2 \sum_{1 \leq i < j \leq k} a_i a_j \geq a,
\]

with equality holding if and only if

\[
a_1 + a_2 + \cdots + a_k = a_{k+1} = \cdots = a_n.
\]

Proof. By the Cauchy-Schwartz inequality, we have

\[
\left( \sum_{i=1}^{n} a_i \right)^2 \leq (n - k + 1)(a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}^2 + \cdots + a_n^2).
\] (3.2)

From (3.1) and (3.2), we get

\[
\sum_{i=1}^{n} a_i^2 + a \leq (a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}^2 + \cdots + a_n^2.
\]
The above equation is equivalent to
\[ 2 \sum_{1 \leq i < j \leq k} a_i a_j \geq a. \]
The equality holds if and only if \( a_1 + a_2 + \cdots + a_k = a_{k+1} = \cdots = a_n. \)

Let \( \pi : (M, g) \to (B, \bar{g}) \) be a Riemannian submersion between Riemannian manifolds \((M, g)\) and \((B, \bar{g})\). Suppose \( \{U_1, \ldots, U_r, X_1, \ldots, X_n\} \) be an orthonormal basis on \( T_pM \), where \( V = \text{Span}\{U_1, \ldots, U_r\} \) and \( H = \text{Span}\{X_1, \ldots, X_n\} \). Then, we have
\[
\|T_H\|^2 = \sum_{i,j=1}^r g \left( T^H(U_i, U_j), T^H(U_i, U_j) \right), \quad \text{(3.3)}
\]
\[
\|T_V\|^2 = \sum_{i=1}^r \sum_{j=1}^n g \left( T^V(U_i, X_j), T^V(U_i, X_j) \right), \quad \text{(3.4)}
\]
\[
\|A^H\|^2 = \sum_{i=1}^r \sum_{j=1}^n g \left( A^H(X_j, U_i), A^H(X_j, U_i) \right), \quad \text{(3.5)}
\]
\[
\|A^V\|^2 = \sum_{i,j=1}^n g \left( A^V(X_i, X_j), A^V(X_i, X_j) \right). \quad \text{(3.6)}
\]

Putting (2.14), (2.10), (2.21) and (3.3) – (3.6) in
\[
\tau(p) = \sum_{1 \leq i < j \leq n} \left[ K(U_i, U_j) + K(X_i, U_j) + K(X_i, X_j) \right],
\]
we obtain the following lemma:

**Lemma 3.2.** Let \((M, g)\) and \((B, \bar{g})\) be a Riemannian manifolds admitting a Riemannian submersion \( \pi : (M, g) \to (B, \bar{g}) \). For any point \( p \in M \), we have
\[
2\hat{\tau}(p) = 2\check{\tau}(p) + 2\check{\tau}(p) + r^2 \|h(p)\|^2 - \|T^H\|^2 + 3 \|A^V\|^2
- r \text{div}_H(h(p)) + \|T^V\|^2 - \|A^H\|^2. \quad \text{(3.7)}
\]

Now, we are going to give an optimal inequality involving the \( \delta \)–curvature for Riemannian manifolds admitting a Riemannian submersion.

**Theorem 3.3.** Let \( \pi : (M, g) \to (B, \bar{g}) \) be a Riemannian submersion. Then, for each point \( p \in M \) and each \( k \)-plane section \( L_k \subset V_p \) \((r > k \geq 2)\), we have
\[
\delta(k) \leq \hat{\tau}(p) - \hat{\tau}(L_k) + \check{\tau}(p) + \frac{r^2(r-k)}{2(r-k+1)} \|h\|^2 - \frac{r}{2} \text{div}_H(h)
+ \frac{3}{2} \|A^V\|^2 + \frac{1}{2} \|T^V\|^2. \quad \text{(3.8)}
\]
The equality of (3.8) holds at \( p \in M \) if and only if \( A^H \) vanishes identically and the shape operators \( S_{X_1}, \ldots, S_{X_n} \) of \( V_p \) take forms as follows:

\[
S_{X_1} = \begin{pmatrix}
T_{11}^1 & 0 & \cdots & 0 \\
0 & T_{22}^1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & T_{kk}^1
\end{pmatrix},
\]

(3.9)

\[
S_{X_s} = \begin{pmatrix}
T_{11}^s & T_{12}^s & \cdots & T_{1k}^s \\
T_{12}^s & T_{22}^s & \cdots & T_{2k}^s \\
\vdots & \vdots & \ddots & \vdots \\
T_{1k}^s & T_{2k}^s & \cdots & - \sum_{i=1}^{k-1} T_{ii}^s \\
0 & 0 & \cdots & 0
\end{pmatrix}, \quad s \in \{2, \ldots, n\}.
\]

(3.10)

**Proof.** Let \( L_k \) be a \( k \)-plane section of \( V_p \). We choose an orthonormal basis \( \{U_1, \ldots, U_r, X_1, \ldots, X_n\} \) on \( T_p M \) such that \( V = \text{Span}\{U_1, \ldots, U_r\} \) and \( \mathcal{H} = \text{Span}\{X_1, \ldots, X_n\} \). We write

\[
T_{ij}^s = g(T^\mathcal{H}(U_i, U_j), X_s)
\]

(3.11)

for any \( i, j \in \{1, \ldots, r\} \) and \( s \in \{1, \ldots, n\} \). Suppose that the mean curvature vector \( h(p) \) is in the direction of \( X_1 \) and \( X_1, \ldots, X_n \) diagonalize the shape operator \( S_{X_1} \). If we put

\[
\eta = 2\tau(p) - 2\hat{\tau}(p) - \frac{r^2(r-k)}{r-k+1} ||h||^2 + r \text{div}_\mathcal{H}(h(p))
\]

\[
-3 ||A^V||^2 - ||T^V||^2 + ||A^H||^2
\]

in (3.7), it follows that

\[
r^2||h||^2 = (n-k+1)(\eta + ||T^\mathcal{H}||^2).
\]

(3.12)

The equation (3.13) is equivalent to

\[
\left( \sum_{i=1}^{r} T_{ii}^1 \right)^2 = (n-k+1) \left( \eta + \sum_{i=1}^{r} (T_{ii}^1)^2 + \sum_{s=2}^{n} \sum_{i,j=1}^{r} (T_{ij}^s)^2 \right).
\]

(3.14)

Applying Lemma 3.1 to equation (3.14), we get

\[
2 \sum_{1 \leq i < j \leq k} T_{ii}^{n+1} T_{jj}^{n+1} \geq \eta + \sum_{s=2}^{n} \sum_{i,j=1}^{r} (T_{ij}^s)^2.
\]

(3.15)
On the other hand, we have from (3.7) that
\[
\tau(L_k) = \hat{\tau}(L_k) + \sum_{1 \leq i < j \leq k} T_{ij}^1 + \sum_{s=2}^n \sum_{1 \leq i < j \leq k} \left( T_{ij}^s - (T_{ij}^s)^2 \right). \tag{3.16}
\]

From (3.15) and (3.16), we get
\[
\tau(L_k) \geq \hat{\tau}(L_k) + \frac{1}{2} \eta + \sum_{s=2}^n \sum_{j>k} \left\{ (T_{1j}^s)^2 + (T_{2j}^s)^2 + \cdots + (T_{kj}^s)^2 \right\}
+ \frac{1}{2} \sum_{s=2}^n (T_{11}^s + T_{22}^s + \cdots + T_{kk}^s)^2 + \frac{1}{2} \sum_{s=2}^n \sum_{i,j>k} (T_{ij}^s)^2. \tag{3.17}
\]

In view of (3.17), we see that
\[
\tau(\Pi_k) \geq \hat{\tau}(\Pi_k) + \frac{1}{2} \eta. \tag{3.18}
\]

From (3.13) and (3.18), we obtain (3.8).

If the equality case of (3.8) holds, then we have $A^H$ vanishes identically and
\[
\begin{cases}
T_{ij}^1 = T_{2j}^1 = T_{kj}^1 = 0, & j = k + 1, \ldots, r, \\
T_{ij}^s = 0, & i,j = k + 1, \ldots, r, \\
T_{11}^s + T_{22}^s + \cdots + T_{kk}^s = 0
\end{cases} \tag{3.19}
\]

for $s = 2, \ldots, n$. Applying Lemma 3.1, we also have
\[
T_{11}^1 + T_{22}^1 + \cdots + T_{kk}^1 = T_{ll}^1, \quad l = k + 1, \ldots, n. \tag{3.20}
\]

Thus, with respect to a suitable orthonormal basis $\{X_1, \ldots, X_m\}$ on $H_p$, the shape operator of $V_p$ becomes of the form given by (3.9) and (3.10). The proof of the converse part is straightforward.

In particular case of $k = 2$, we have the following:

**Corollary 3.4.** Let $\pi : (M, g) \rightarrow (B, \tilde{g})$ be a Riemannian submersion. Then, for each point $p \in M$ and each plane section $L \subset V_p$, we have
\[
\delta(2) \leq \hat{\tau}(p) - \hat{K}(L) + \hat{\tau}(p) + \frac{\nu^2(r-2)}{2(r-1)} \|h\|^2 - \frac{r}{2} \text{div}_H(h)
+ \frac{3}{2} \|A^V\|^2 + \frac{1}{2} \|T^V\|^2. \tag{3.21}
\]

The equality of (3.21) holds at $p \in M$ if and only if $A^H$ vanishes identically and the shape operators $S_{X_1}, \ldots, S_{X_n}$ of $V_p$ take forms
\[
S_{X_1} = \begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & (a+b)I_{r-2}
\end{pmatrix}, \tag{3.22}
\]

for $s = 2, \ldots, n$. Applying Lemma 3.1, we also have
\[
T_{11}^1 + T_{22}^1 + \cdots + T_{kk}^1 = T_{ll}^1, \quad l = k + 1, \ldots, n. \tag{3.20}
\]
\[
S_{X_s} = \begin{pmatrix}
c_s & d_s & 0 \\
d_s & -c_s & 0 \\
0 & 0 & 0_{r-2}
\end{pmatrix}, \quad s \in \{2, \ldots, n\}. \tag{3.23}
\]

In particular case of \(k = r - 1\), we have the following

**Corollary 3.5.** Let \(\pi: (M, g) \to (B, \tilde{g})\) be a Riemannian submersion. For each vertical unit vector \(U\), we have

\[
\text{Ric}_V(U) \leq \tilde{\text{Ric}}(U) + \frac{r^2}{4}|h|^2 - \frac{r}{2} \text{div}_H(h) + \frac{3}{2} |A^V|^2 + \frac{1}{2} |T^V|^2. \tag{3.24}
\]

The equality case of (3.24) holds for all unit vectors \(U \in \mathcal{V}_p\) if and only if \(A^H\) vanishes identically and we have either

(i) if \(r = 2\), \(\pi\) has totally umbilical fibers at \(p \in M\),

(ii) if \(r \neq 2\), \(\pi\) has totally geodesic fibers at \(p \in M\).

**Proof.** Let \(L_{r-1}\) be a \((r-1)\)-plane section of \(\mathcal{V}_p\). We get from Theorem 3.3 that

\[
\delta(r-1) \leq \tilde{\tau}(p) - \tilde{\tau}(L_{r-1}) + \frac{r^2}{4}|h|^2 - \frac{r}{2} \text{div}_H(h) + \frac{3}{2} |A^V|^2 + \frac{1}{2} |T^V|^2. \tag{3.25}
\]

Now, let \(U\) be a unit vertical vector field such that \(U = U_r\). By a straightforward computation, we obtain (3.24).

The equality of (3.25) holds if and only if the forms of shape operators \(S_{X_s}, s = 1, \ldots, n\), become

\[
S_{X_1} = \begin{pmatrix}
T^1_{11} & 0 & \cdots & 0 & 0 \\
0 & T^1_{22} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & T^1_{(r-1)(r-1)} & 0 \\
0 & 0 & \cdots & 0 & \left(\sum_{i=1}^{r-1} T^1_{ii}\right)
\end{pmatrix}, \tag{3.26}
\]

\[
S_{X_s} = \begin{pmatrix}
T^s_{11} & T^s_{12} & \cdots & T^s_{1(r-1)} & 0 \\
T^s_{12} & T^s_{22} & \cdots & T^s_{2(r-1)} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
T^s_{1(r-1)} & T^s_{2(r-1)} & \cdots & \sum_{i=1}^{r-2} T^s_{ii} & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad r \in \{2, \ldots, n\}. \tag{3.27}
\]
From (3.26) and (3.27), we see that the equality in (3.24) is valid for a unit vertical vector field $U = U_r$ if and only if

$$
\begin{align*}
T^s_{rr} &= T^s_{11} + T^s_{22} + \cdots + T^s_{(r-1)(r-1)} \\
T^s_{1r} &= T^s_{2r} = \cdots = T^s_{(r-1)r} = 0.
\end{align*}
$$

(3.28)

for $s \in \{1, \ldots, n\}$.

Assuming the equality case of (3.24) holds for all unit vertical vector fields, in view of (3.28), for each $s \in \{1, \ldots, n\}$, we have

$$
\begin{align*}
2T^s_{ii} &= T^s_{11} + T^s_{22} + \cdots + T^s_{rr}, \\
T^s_{ij} &= 0, \quad i \neq j
\end{align*}
$$

(3.29)

for all $i \in \{1, \ldots, r\}$ and $s \in \{1, \ldots, n\}$. Thus, we have two cases, namely either $r = 2$ or $r \neq 2$. In the first case we see that $\pi$ has totally umbilical fibers, while in the second case $\pi$ has totally geodesic fibers. The proof of converse part is straightforward. 

Remark 3.6. We note that (3.24) was also proved in [17] (see Theorem 4.1 in [17]). In Theorem 3.5 we gave a new proof for this inequality.

4 Some basic facts on Meteorology

In this section, we shall present some basic facts related to notions the horizontal divergence, the conservation of mass and the continuity equation following [6–8, 19, 21, 25, 27].

This section is going to be handled on three subsections.

4.1 Horizontal divergence and convergence

Definition 4.1. [3] Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_pM$ at $p \in M$. The divergence of a vector field $X$ in $T_pM$ is defined by

$$
\text{div}(X) = \text{trace} (\nabla X) \equiv \sum_{i=1}^{n} g (\nabla_{e_i} X, e_i).
$$

(4.1)

Theorem 4.2. [3] [Divergence Theorem] Let $(M, g)$ be a Riemannian manifold. Suppose that $\mathcal{D}$ to be a compact domain of $M$ with smooth boundary $\partial \mathcal{D}$ and $N$ to be the unit normal vector to $\mathcal{D}$. Then we have

$$
\int_{\mathcal{D}} g(X, N)\nu^\mathcal{D} = \int_{\partial \mathcal{D}} \text{div}(X)\nu^M
$$

(4.2)

for any vector field $X \in \Gamma(\mathcal{D})$. Here, $\nu^M$ is the measure of $(M, g)$. 

10
In fluid kinematics, if a vector field $X$ is considered as velocity of a fluid (a gas), then sign of $\text{div}(X)$ describes the expansion or compression of flow. Therefore, the total expansion or compression of flow can be calculated by the help of divergence theorem so divergence is a useful tool to measuring the net flow of fluid diverging from a point or approaching a point. The first phenomenon is called as horizontal divergence and the other is called as horizontal convergence.

![Figure 4.1: Horizontal Divergence](image1)

![Figure 4.2: Horizontal Convergence](image2)

In meteorology, divergence and convergence are used to measure the change in area of wind pressure or a fluid element. For example, these notions occur in moving air masses. Air masses sometimes move away from a line (or a point) or move towards a line. If a horizontal divergence occurs, then air masses or a fluid element move faster than those behind them and if a horizontal convergence occurs, then they move slower than those behind them.

### 4.2 Conservation of mass

Let $\mathbb{E}^3$ be 3-dimensional Euclidean space with the usual coordinates $(x, y, z)$. Consider an object of mass $m$ which is given by

$$m = \rho v,$$

(4.3)
where \( \rho \) is density and \( v \) is volume of the object. Then we have

\[
\frac{dm}{dt} = \frac{d}{dt}(\rho v) = 0, \tag{4.4}
\]

which implies that

\[
\frac{1}{\rho} \frac{d\rho}{dt} + \frac{1}{v} \frac{dv}{dt} = 0. \tag{4.5}
\]

Putting \( v = \Delta x \Delta y \Delta z \), we get

\[
\frac{1}{v} \frac{dv}{dt} = \Delta x \frac{d(\Delta y \Delta z)}{dt} + \Delta y \frac{d(\Delta x \Delta z)}{dt} + \Delta z \frac{d(\Delta x \Delta y)}{dt}. \tag{4.6}
\]

By a straightforward computation, we see that

\[
\frac{1}{v} \frac{dv}{dt} = \frac{du^1}{dx} + \frac{du^2}{dy} + \frac{du^3}{dz} = \text{div}(U), \tag{4.7}
\]

where \( U = (u^1, u^2, u^3) \) is a vector field on \( \mathbb{R}^3 \). In view of (4.5) and (4.7), we have

\[
\frac{1}{\rho} \frac{d\rho}{dt} + \text{div}(U) = 0. \tag{4.8}
\]

As a consequence of (4.8), we obtain the followings:

i) \( \text{div}(U) < 0 \) if and only if the density increases.

ii) \( \text{div}(U) > 0 \) if and only if the density decreases.

iii) \( \text{div}(U) = 0 \) if and only if the density doesn’t change with time.

### 4.3 Continuity equation

Another important notion in meteorology is the continuity equation. The principle simple states that any matter can either be created or destroyed and implies for the atmosphere that its mass may be redistributed but can never be disappeared. Therefore, this equation gives us that

\[
\text{div}(U) = 0 \tag{4.9}
\]

for any vector field \( U = (u^1, u^2, u^3) \) on \( \mathbb{R}^3 \). It can be written from (4.9) that

\[
\text{div}_H(U) + \frac{\partial u^3}{\partial z} = 0, \tag{4.10}
\]
where \( \text{div}_H(v_H) \) is the horizontal divergence of \( U \) defined by

\[
\text{div}_H(v_H) = \frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y}.
\]

The equation given (4.10) is also known as the *continuity equation* in literature. Integrating (4.10), we have

\[
\omega(p_1, p_0) \equiv u^3(p_1) - u^3(p_0) = -\int_{p_0}^{p_1} \left( \frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y} \right) dz,
\]

where \( p_1 \) and \( p_0 \) is some pressure levels on the atmosphere. If we assume that \( p_0 \) is surface pressure then \( u^3(p_0) = 0 \) and thus we get

\[
\omega(p_1) = -\int_{p_0}^{p_1} \left( \frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y} \right) dz,
\]

This formula tells us that \( w \) at a given pressure level is proportional to the integral of the horizontal divergence. Here, \( \omega(p_1) \) is called the *vertical motion* at \( p_1 \).

**Figure 4.3: Rising and descending motions**

If \( \omega(p) < 0 \) at every point \( p \) then this statement is called *rising motion*, \( \omega(p) > 0 \) at every point \( p \) then this statement is called *descending motion*, (in this case, divergence is called convergence) in meteorology. There is no divergence and it is clear that there is a local maximum or minimum of \( w \) if \( w(p) = 0 \). Therefore, one can consider that convergence and divergence roughly determine where air will be sinking or rising. Rising air can be associated with clouds and precipitation; sinking air can be associated with clear, calm conditions and good weather. Thus, convergence and divergence make the difference between a grey day and a sunny day. The importance of convergence and divergence is related to pressure changes at the surface and height changes of the constant-pressure levels.

As a consequence of the above facts, we can state the followings:
In this section, we shall present a solution way with the help of differential geometry tools for the following natural problem:

"Which conditions should provide to the horizontal divergence or the convergence receives to the maximum value or minimum value?"

To obtain minimum or maximum values of the vertical motion (or horizontal divergence) it can be considered a Riemannian submersion on $E^3$ to $E^2$. Moreover, we can regard to different Riemannian submersions such as a Riemannian submersion on a three dimensional Riemannian manifold to two dimensional Riemannian manifold as

$$
\pi : M^3 \rightarrow N^2.
$$

### Main conclusions

| Case               | Vertical Motion | Horizontal Divergence(Convergence) | Point(s)      |
|--------------------|----------------|-----------------------------------|---------------|
| Rising Motion      | Maximum Value  | Minimum Value                     | Warmest (ideal) |
| Rising Motion      | Minimum Value  | Maximum Value                     | Coolest       |
| Descending Motion  | Maximum Value  | Minimum Value                     | Warmest       |
| Descending Motion  | Minimum Value  | Maximum Value                     | Coolest (ideal) |

Figure 4.4: Minimum or maximum values of vertical motion
It can also be considered globally in high dimensional Riemannian manifolds with taking a Riemannian submersion on $m$-dimensional Riemannian manifold to $n$-dimensional Riemannian manifold.

Taking into account of the continuity equation and \((3.8)\), \((3.21)\) and \((3.24)\) inequalities, we get some result dealing minimum or maximum values of vertical motion for a manifold admitting a Riemannian submersion.

As a consequence of \((3.8)\), we obtain the following:

**Corollary 5.1.** Let $\pi : E^{n+r} \rightarrow E^n$ be a Riemannian submersion. Then we have

$$\frac{r}{2} \omega(p) \geq \delta(k) - \frac{r^2(r-k)}{2(r-k+1)} \|h\|^2 - \frac{3}{2} \|A^V\|^2 - \frac{1}{2} \|T^V\|^2. \quad (5.2)$$

The vertical motion at a point $p$ takes the minimum value if and only if $A^H$ vanishes identically and the matrixes of shape operators of the vertical space of $M$ take the form as \((3.9)\) and \((3.10)\).

As a consequence of \((3.21)\), we obtain the followings:

**Corollary 5.2.** Let $\pi : E^{n+r} \rightarrow E^n$ be a Riemannian submersion with integrable horizontal distribution. Then we have

$$\frac{r}{2} \omega(p) \geq \delta(2) - \frac{r^2(r-2)}{2(r-1)} \|h\|^2 - \frac{1}{2} \|T^V\|^2. \quad (5.3)$$

The vertical motion takes the minimum value if and only if the matrixes of shape operators $S_{x_1}, \ldots, S_{x_n}$ of the vertical space of $M$ take the form as \((3.22)\) and \((3.23)\).

**Corollary 5.3.** Let $\pi : E^{n+r} \rightarrow E^n$ be a Riemannian submersion with totally geodesic leaves and integrable horizontal distribution. Then we have

$$\frac{r}{2} \omega(p) = \delta(2). \quad (5.4)$$

From \((3.21)\), we get the followings:

**Corollary 5.4.** Let $\pi : E^{n+r} \rightarrow E^n$ be a Riemannian submersion. For each vertical unit vector $U$, we have

$$\frac{r}{2} \omega(p) \geq \text{Ric}_V(U) - \frac{r^2}{4} \|h\|^2 - \frac{3}{2} \|A^V\|^2 - \frac{1}{2} \|T^V\|^2. \quad (5.5)$$

The equality case of \((5.6)\) holds for all unit vectors $U \in V_p$ if and only if $A^H$ vanishes identically and we have either

(i) if $r = 2$, $\pi$ has totally umbilical fibers at $p \in M$,

(ii) if $r \neq 2$, $\pi$ has totally geodesic fibers at $p \in M$.

**Corollary 5.5.** Let $\pi : E^{n+r} \rightarrow E^n$ be a Riemannian submersion with totally geodesic fibers. For each vertical unit vector $U$, we have

$$\frac{r}{2} \omega(p) = \text{Ric}_V(U) - \frac{3}{2} \|A^V\|^2. \quad (5.6)$$
References

[1] P. Alegre, B.-Y. Chen, M. I. Munteanu, Riemannian Submersions, δ-Invariants, and Optimal Inequality, Ann. Glob. Anal. Geom. 42 (2010) 317-331.

[2] C. Altafini, Redundant robotic chains on Riemannian submersions, IEEE Transactions on Robotics and Automation, 20(2) (2004), 335-340.

[3] P. Baird and C. J. Wood, Harmonic morphisms between Riemannian manifolds. Oxford University Press, 2003.

[4] A. L. Besse, Einstein Manifolds. Berlin-Heidelberg-New York, Springer-Verlag, 1987.

[5] R. Bhattacharyya and V. Patrangenaru, Nonparametric estimation of location and dispersion on Riemannian manifolds, Journal of Statistical Planning and Inference, 108 (2002), 23-35.

[6] W. H. Brune, How does divergence relate to the air parcel’s area change?, www.e-education.psu.edu/meteo300/node/725, (2015).

[7] W. H. Brune, How is the horizontal divergence/convergence related to vertical motion?, www.e-education.psu.edu/meteo300/node/726, (2015).

[8] W. H. Brune, Why we like conservation?, www.e-education.psu.edu/meteo300/node/707, (2015).

[9] B.-Y. Chen, Some pinching and classification theorems for minimal submanifolds. Arch. Math., 60 (1993) 568-578.

[10] B.-Y. Chen, A Riemannian invariant and its applications to submanifold Theory, Results Math., 27 (1995) 17-28.

[11] B.-Y. Chen, Pseudo-Riemannian geometry, δ-invariants and applications, World Scientific Publishing, Hackensack, NJ, 2011.

[12] B.-Y. Chen, F. Dillen and L. Verstraelen, δ-invariants and their applications to centroaffine geometry, Diff. Geom. Appl., 22 (2005) 341-354.

[13] F. Dillen, J. Fastenakels and J. Van der Veken, A pinching theorem for the normal scalar curvature of invariant submanifolds, J. Geom. Phys., 57 (2007) 833-840.

[14] M. Falcitelli, S. Ianus, A. M. Pastore, M. Visinescu, Some applications of Riemannian submersions in physics, Rev. Roum. Phys., 48 (2003) 627-639.

[15] M. Falcitelli, S. Ianus, A. M. Pastore, Riemannian submersions and related topics, World Scientific Company, 2004.
[16] M. Gülbaşar, E. Kılıç, S. Keleş, A useful orthonormal basis on bi-slant submanifolds of almost Hermitian manifolds, Tamkang J. Math., 47(2), 143-161, 2016.

[17] M. Gülbaşar, Ş. Eken, E. Kılıç, Sharp inequalities involving the Ricci curvature for Riemannian submersions, Kragujevac Journal of Mathematics 42(2), (2017).

[18] S. Hong, K. Matsumoto, M. M. Tripathi, Certain Basic Inequalities for Submanifolds of Locally Conformal Kaehler Space Forms, SUT J. Math. 41 (2005) 75-94.

[19] L. W. Hung, Convergence and divergence, [http://www.weather.gov.hk/m/](http://www.weather.gov.hk/m/), Hong Kong Observatory, June (2010).

[20] L. C. Kennedy, Some results on Einstein metrics on two classes of quotient manifolds, PhD thesis, University of California, 2003.

[21] B. Ketchum, Ocean dumping of industrial wastes. Vol. 12. Springer Science and Business Media, 2013

[22] S. Kobayashi, Submersions of CR-submanifolds, Tohoku Math. J., 89 (1987) 95-100.

[23] F. Memoli, G. Sapiro and P. Thompson, Implicit brain imaging, Neuro Image, 23 (2004) 179-188.

[24] N. Poyraz, E. Yaşar, Chen-like inequalities on lightlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection, Kragujevac Journal of Mathematics 40, (2016) 146-164.

[25] J. Steenburgh, Vertical Motion, Atmos. 5110 SynopticDyanmic Meteorology I.

[26] B. Şahin, Riemannian Submersions from Almost Hermitian Manifolds, Taiwan. J. Math., 17 (2013) 629-659.

[27] R. A. Vaughan, Remote Sensing Applications in Meteorology and Climatology, NATO ASI Series, Series C: Mathematical and Physical Science, 201 (1986).

[28] G. E. Vilcu, On Chen invariants and inequalities in quaternionic geometry, J. Ineq. Appl., (2013) 2013:66.

[29] M. Wang and W. Ziller, Einstein metrics on principal torus bundles, J. Diff. Geo., 31 (1990), 215-248.

[30] H. Zhao, A. R. Kelly, J. Zhou, J. Lu and Y. Y. Yang, Graph attribute embedding via Riemannian submersion learning, Computer Vision and Image Understanding, 115 (2011), 962-975.
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