Some Remarks on Diametral Dimension and Approximate Diametral Dimension
of Certain Nuclear Fréchet Spaces

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Abstract

The diametral dimension, $\Delta(E)$, and the approximate diametral dimension, $\delta(E)$, of a nuclear Fréchet space $E$ which satisfies $DN$ and $\Omega$, are related to corresponding invariant of power series spaces $\Lambda_1(\varepsilon)$ and $\Lambda_\infty(\varepsilon)$ for some exponent sequence $\varepsilon$. In this article, we examine a question of whether $\delta(E)$ must coincide with that of a power series space if $\Delta(E)$ does the same, and vice versa. In this regard, we first show that this question has an affirmative answer in the infinite type case by showing that $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$ if and only if $\delta(E) = \delta(\Lambda_\infty(\varepsilon))$. Then we consider the question in the finite type case and, among other things, we prove that $\delta(E) = \delta(\Lambda_1(\varepsilon))$ if and only if $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ and $E$ has a prominent bounded subset.

Keywords: Nuclear Fréchet Spaces, Diametral Dimension, Topological Invariants, Prominent Bounded Subsets

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1 Introduction

Power series spaces constitute an important and well studied class in the theory of Fréchet spaces. Linear topological invariants $DN$ and $\Omega$ (see definition below) are enjoyed by many natural nuclear Fréchet spaces appearing in analysis. In particular, spaces of analytic functions, solutions of homogeneous elliptic linear partial differential operators with their natural topologies have the properties $DN$ and $\Omega$, see [13] and [21].

Let $E$ be a nuclear Fréchet space which satisfies $DN$ and $\Omega$. Then it is a well known fact that the diametral dimension $\Delta(E)$ and the approximate diametral dimension $\delta(E)$ of $E$ are set theoretically
between corresponding invariant of power series spaces $\Lambda_1(\varepsilon)$ and $\Lambda_\infty(\varepsilon)$ for some specific exponent sequence $\varepsilon$. Coincidence of diametral dimension and/or approximate diametral dimension of $E$ with that of a power series space yields some structural results. For example, in [3], Aytuna et al. proved that a nuclear Fréchet space $E$ with the properties $DN$ and $\Omega$ contains a complemented copy of $\Lambda_\infty(\varepsilon)$ provided the diametral dimensions of $E$ and $\Lambda_\infty(\varepsilon)$ are equal and $\varepsilon$ is stable. On the other hand, Aytuna [5] characterized tame nuclear Fréchet spaces $E$ with the properties $DN$, $\Omega$ and stable associated exponent sequence $\varepsilon$, as those that satisfies $\delta(E) = \delta(\Lambda_1(\varepsilon))$. These results leads to ask the following question:

**Question 1.1.** Let $E$ be a nuclear Fréchet space with the properties $DN$ and $\Omega$. If diametral dimension of $E$ coincides with that of a power series space, then does this imply that the approximate diametral dimension is also do the same and vice versa?

This article is concerned with this question and the layout is as follows:

In Section 2, we give some preliminary materials. Then in Section 3, we show that Question 1.1 has an affirmative answer when the power series space is of infinite type. In our final section, we search an answer for the Question 1.1 in the finite type case and, in this regard, we first prove that the condition $\delta(E) = \delta(\Lambda_1(\varepsilon))$ always implies $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$. For other direction, the existence of a prominent bounded subset in the nuclear Fréchet space $E$ plays a decisive role for the answer of Question 1.1. Among other things, we prove that $\delta(E) = \delta(\Lambda_1(\varepsilon))$ if and only if $E$ has a prominent bounded set and $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$.

## 2 Preliminaries

In this section, after establishing terminology and notation, we collect some basic facts and definitions that are needed them in the sequel.

We will use the standard terminology and notation of [14] and [13]. Throughout the article, $E$ will denote a nuclear Fréchet space with an increasing sequence of Hilbertian seminorms $(\parallel \cdot \parallel_k)_{k \in \mathbb{N}}$ and the local Hilbert spaces corresponding to the norm $\parallel \cdot \parallel_k$ will be denoted by $E_k$.

For a Fréchet space $E$, $\mathcal{U}(E)$, $\mathcal{B}(E)$ will denote the class of all neighborhoods of zero in $E$ and the class of all bounded sets in $E$, respectively. If $U$ and $V$ are absolutely convex sets of $E$ and $U$ absorbs $V$, that is $V \subseteq CU$ for some $C > 0$, and $L$ is a subspace of $E$, then we set:

$$\delta(V, U, L) = \inf \{t > 0 : V \subseteq tU + L\}.$$  

The $n^{th}$ Kolmogorov diameter of $V$ with respect to $U$ is defined as:

$$d_n(V, U) = \inf \{\delta(V, U, L) : \dim L \leq n\} \quad n = 0, 1, ...$$

and the diametral dimension of $E$ is defined as:

$$\Delta(E) = \left\{(t_n)_{n \in \mathbb{N}} : \forall U \in \mathcal{U}(E) \exists V \in \mathcal{U}(E) \lim_{n \to \infty} t_n d_n(V, U) = 0\right\}$$

$$= \bigcap_{U \in \mathcal{U}(E)} \bigcup_{V \in \mathcal{U}(E)} \Delta(V, U)$$
where $\Delta \left( V, U \right) = \left\{ (t_n)_{n \in \mathbb{N}} : \lim_{n \to \infty} t_n d_n \left( V, U \right) = 0 \right\}$.

Let $U_1 \supset U_2 \supset \cdots \supset U_p \supset \cdots$ be a base of neighborhoods of Fréchet space $E$. Diametral dimension can be represented as

$$\Delta \left( E \right) = \left\{ (t_n)_{n \in \mathbb{N}} : \forall p \in \mathbb{N} \exists q > p \lim_{n \to \infty} t_n d_n \left( U_q, U_p \right) = 0 \right\}.$$ 

The approximate diametral dimension of a Fréchet space $E$ is defined as;

$$\delta \left( E \right) = \left\{ (t_n)_{n \in \mathbb{N}} : \forall U \in \mathcal{U} \left( E \right) \forall B \in \mathcal{B} \left( E \right) \lim_{n \to \infty} t_n d_n \left( B, U \right) = 0 \right\}.$$ 

The concept of the approximative dimension of a linear metric space which is based on $\varepsilon$-capacity of compact sets in the space was introduced by Kolmogorov and Pełczyński, see also [12], [17] and [19]. The relation between invariants introduced above and $\varepsilon$-capacity of compact sets in the space was discovered by Mityagin, [15] and [16]. Among other thing, Mityagin conducted a detailed study of these invariants and used them characterize nuclear locally convex space. The concept of approximate diametral dimension as stated above was given and studied by Bessaga, Pełczyński and Rolewicz, [7]. Demeulenaere et al. [8] showed that the diametral dimension of a nuclear Fréchet space can also be represented as;

$$\Delta \left( E \right) = \left\{ (t_n)_{n \in \mathbb{N}} : \forall p \in \mathbb{N} \exists q > p \lim_{n \to \infty} t_n d_n \left( U_q, U_p \right) = 0 \right\}.$$ 

Let $E$ and $G$ be two Fréchet spaces and $U$ and $V$ be absolutely convex two subsets of space $E$ such that $V \subseteq rU$ for some $r > 0$. If there is a linear map $T : E \to G$, then for all $n \in \mathbb{N}$

$$d_n \left( T \left( V \right), T \left( U \right) \right) \leq d_n \left( V, U \right)$$

holds and so it follows that if $F$ is a subspace or a quotient of $E$, then $\Delta \left( E \right) \subseteq \Delta \left( F \right)$ and $\delta \left( F \right) \subseteq \delta \left( E \right)$. Hence diametral dimension and approximate diametral dimension are invariant under isomorphism, in other words, these are linear topological invariants. For the proof of these and for additional properties of the diametral dimension/approximate diametral dimension, we refer the reader to [7], [16], [18] (Chapter 9), [19] (Chapter 6.5 and 6.6) and [20].

The properties of the canonical topology on diametral dimension of a nuclear Fréchet space:

Let $E$ be a nuclear Fréchet space. Then the diametral dimension

$$\Delta \left( E \right) = \left\{ (t_n)_{n \in \mathbb{N}} : \forall p \in \mathbb{N} \exists q > p \lim_{n \to \infty} t_n d_n \left( U_q, U_p \right) = 0 \right\}$$

where

$$\Delta \left( U_q, U_p \right) = \left\{ (t_n)_{n \in \mathbb{N}} : \lim_{n \to \infty} t_n d_n \left( U_q, U_p \right) = 0 \right\}.$$
is the projective limit of inductive limits of Banach spaces \( \Delta(U_q,U_p) \). Hence \( \Delta(E) \) is a topological space with respect to that topology which will be called as the canonical topology. Furthermore, \( \Delta(E) \) can be considered as a weighted PLB-spaces of continuous functions.

The topological properties of weighted PLB-spaces of continuous functions were studied in [1]. In particular, the following theorem gives an information about the canonical topology of diametral dimension \( \Delta(E) \) and it is a direct consequence of Theorem 3.7 of [1].

**Theorem 2.1.** Let \( E \) be a Fréchet space. The following conditions are equivalent:

1. \( \Delta(E) \) is ultrabornological with respect to the canonical topology.
2. \( \Delta(E) \) is barrelled with respect to the canonical topology.
3. \( \Delta(E) \) satisfies condition \((wQ)\):
   \[
   \forall N \exists M,n \forall K,m, \exists k,S > 0 : \min (d_n(U_n,U_N),d_n(U_k,U_K)) \leq S d_n(U_m,U_M) \quad \forall n \in \mathbb{N}.
   \]

We will use this theorem in the fourth section.

Power series spaces form an important family of Fréchet spaces and they play an significant role in this article. Let \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \) be a non-negative increasing sequence with \( \lim_{n \to \infty} \alpha_n = +\infty \). Throughout this article, all power series spaces are assumed to be nuclear. Recall that a power series space of finite type is defined by

\[
\Lambda_1(\alpha) := \left\{ x = (x_n)_{n \in \mathbb{N}} : \|x\|_k := \sup_{n \in \mathbb{N}} |x_n| e^{-\tau \alpha_n} < +\infty \text{ for all } k \in \mathbb{N} \right\}
\]

and a power series space of infinite type is defined by

\[
\Lambda_\infty(\alpha) := \left\{ x = (x_n)_{n \in \mathbb{N}} : \|x\|_k := \sup_{n \in \mathbb{N}} |x_n| e^{\tau \alpha_n} < +\infty \text{ for all } k \in \mathbb{N} \right\}.
\]

Power series spaces are actually Fréchet spaces equipped with the seminorms \((\|\cdot\|_k)_{k \in \mathbb{N}}\). Diametral dimension and approximate diametral dimension of power series spaces are

\[
\Delta(\Lambda_1(\alpha)) = \Lambda_1(\alpha), \quad \Delta(\Lambda_\infty(\alpha)) = \Lambda_\infty(\alpha)', \quad \delta(\Lambda_1(\alpha)) = \Lambda_1(\alpha)', \quad \text{and} \quad \delta(\Lambda_\infty(\alpha)) = \Lambda_\infty(\alpha),
\]

see [7] and [16].

Other linear topological invariants that are used in this article are \(DN\) and \(\Omega\), see [14] and references therein.

**Definition 2.2.** A nuclear Fréchet space \( E \) is said to have the property \(DN\) and \(\Omega\) when the following conditions hold:

\[
(DN) : \quad \text{There exists a } p \in \mathbb{N} \text{ such that for each } k > p, \text{ an } n > k, \text{ and } 0 < \tau < 1 \text{ and a } C > 0 \text{ exist with }
\]

\[
\|x\|_k \leq C\|x\|_p^{1-\tau}\|x\|_n^\tau \quad \text{for all } x \in E.
\]

\[
(\Omega) : \quad \text{For each } p \in \mathbb{N}, \text{ there exists a } q > p \text{ such that for every } k > q \text{ there exists a } 0 < \theta < 1 \text{ and a } C > 0 \text{ with }
\]

\[
\|y\|_q \leq C\|y\|_p^{1-\theta}\|y\|_k^{\theta} \quad \text{for all } y \in E'.
\]
where
\[ \| y \|_k^* := \sup \{|y(x)| : \|x\|_k \leq 1\} \in \mathbb{R} \cup \{+\infty\} \]
is called the gauge functional of \( U_k^o \) for \( U_k = \{x \in E : \|x\|_k \leq 1\} \).

We end this section by recalling the following result which gives a relation between diametral dimension/approximate diametral dimension of a nuclear Fréchet spaces with the properties \( \text{DN}, \Omega \) and that of a power series spaces \( \Lambda_1(\varepsilon) \) and \( \Lambda_\infty(\varepsilon) \) for some special exponent sequence \( \varepsilon \).

**Proposition 2.3.** [Proposition 1.1, [3]] Let \( E \) be a nuclear Fréchet space with the properties \( \text{DN} \) and \( \Omega \). There exists an exponent sequence (unique up to equivalence) \( (\varepsilon_n)_{n \in \mathbb{N}} \) satisfying:
\[ \Delta(\Lambda_1(\varepsilon)) \subseteq \Delta(E) \subseteq \Delta(\Lambda_\infty(\varepsilon)). \]
Furthermore, \( \Lambda_1(\alpha) \subseteq \Delta(E) \) implies \( \Lambda_1(\alpha) \subseteq \Lambda_1(\varepsilon) \) and \( \Delta(E) \subseteq \Lambda'_\infty(\alpha) \) implies \( \Lambda'_\infty(\varepsilon) \subseteq \Lambda'_\infty(\alpha) \).

The sequence \( \varepsilon \) was called the associated exponent sequence of \( E \) in [3]. The exponent sequence \( \varepsilon \) associated to \( E \) contains some information about the structure of \( E \). We note that \( \Lambda_\infty(\varepsilon) \) is always nuclear provided \( E \) is nuclear, but it may happen that \( \Lambda_1(\varepsilon) \) is not nuclear. Throughout this article, we assume \( \Lambda_1(\varepsilon) \) is nuclear for associated exponent sequence \( \varepsilon \) of a nuclear Fréchet space \( E \). In the proof of the above proposition, Aytuna et al. showed that there is an exponent sequence (unique up to equivalence) \( (\varepsilon_n) \) such that for each \( p \in \mathbb{N} \) and \( q > p \), there exist \( C_1, C_2 > 0 \) and \( a_1, a_2 > 0 \) satisfying \( C_1 e^{-a_1 \varepsilon_n} \leq d_n(U_q, U_p) \leq C_2 e^{-a_2 \varepsilon_n} \) for all \( n \in \mathbb{N} \). From this inequality, it follows
\[ \delta(\Lambda_\infty(\varepsilon)) \subseteq \delta(E) \subseteq \delta(\Lambda_1(\varepsilon)). \]

### 3 Results in the infinite case

The main result in this section is the following theorem which shows that Question 1.1 has an affirmative answer when the power series space is of infinite type.

**Theorem 3.1.** Let \( E \) be a nuclear Fréchet space with properties \( \text{DN} \) and \( \Omega \) and \( \varepsilon = (\varepsilon_n)_{n \in \mathbb{N}} \) be the associated exponent sequence of \( E \). Then \( \Delta(E) = \Delta(\Lambda_\infty(\varepsilon)) \) if and only if \( \delta(E) = \delta(\Lambda_\infty(\varepsilon)) \).

We first need the following lemma for the proof of Theorem 3.1. In [5, Cor. 1.10], Aytuna proved that for a nuclear Fréchet space \( E \) with the properties \( \text{DN} \), \( \Omega \) and associated exponent sequence \( \varepsilon \)
\[ \delta(E) = \delta(\Lambda_1(\varepsilon)) \iff \inf \sup \limsup_{n \in \mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} = 0 \quad (3.1) \]
where \( \varepsilon_n(p,q) = -\log d_n(U_q, U_p) \).

The same characterization can be given for infinite type power series spaces as follows:
Lemma 3.2. Let $E$ be a nuclear Fréchet space with properties $DN$ and $Ω$ and $ε = (εₙ)ₙ∈ℕ$ be the associated exponent sequence of $E$. Then

$$\delta (E) = \delta (Λₜ (ε)) \iff \inf \sup \lim \inf \frac{εₙ(p, q)}{εₙ} = +∞$$

where $εₙ(p, q) = −\log dₙ(,U_q, U_p)$.

Proof. Approximate diametral dimension $δ (E)$ can be written as

$$δ (E) = \bigcup_p \bigcap q ≥ p \delta pq$$

where $δpq = \{ (tₙ)ₙ∈ℕ : \sup_{n∈ℕ} \frac{|tₙ|}{dₙ(U_q, U_p)} < +∞ \}$ is a Banach space with norms $|tₙ|ₚₖₚ = \sup_{n∈ℕ} \frac{|tₙ|}{dₙ(U_q, U_p)}$. Namely, approximate diametral dimension can be equipped with the topological inductive limit of Fréchet spaces. Then, the approximate diametral dimension with this topology is barrelled. On the other hand, the inclusion $δ (E) ⊆ δ (Λₜ (ε)) = Λₜ (ε)$ gives us that the identity mapping $i : δ (E) → Λₜ (ε)$ has a closed graph. Since $δ (E)$ is barrelled, by using Theorem 5 of [11], we conclude that the identity mapping is continuous. Therefore,

$$δ (E) = \bigcup_p \bigcap q ≥ p \delta pq \Rightarrow Λₜ (ε) \text{ is continuous} \iff \forall p \bigcap q ≥ p \delta pq \Rightarrow Λₜ (ε) \text{ is continuous} \iff$$

$$\forall p \forall R > 1 \exists q ≥ p, \ C > 0 \sup_{n∈ℕ} |tₙ| R^{εₙ} ≤ C \sup_{n∈ℕ} \frac{|tₙ|}{dₙ(U_q, U_p)} \forall (tₙ) ∈ δ (E) \iff$$

$$\forall p \forall R > 1 \exists q ≥ p, \ C > 0 \ R^{εₙ} ≤ \frac{C}{dₙ(U_q, U_p)} \forall n ∈ ℕ \iff$$

$$\forall p \sup \lim \inf \frac{εₙ(p, q)}{εₙ} = +∞ \iff \inf \sup \lim \inf \frac{εₙ(p, q)}{εₙ} = +∞.$$  

Now since $δ (E) ≥ δ (Λₜ (ε))$ always holds for the associated exponent sequence $ε$ of $E$, we have

$$δ (E) = δ (Λₜ (ε)) \iff \inf \sup \lim \inf \frac{εₙ(p, q)}{εₙ} = +∞,$$

as desired.

Proof of Theorem 3.1. For the proof of necessity part, assume that $δ (E) = δ (Λₜ (ε))$. By Lemma 3.2,

$$\inf \sup \lim \inf \frac{εₙ(p, q)}{εₙ} = +∞.$$  

Then we have

$$\forall p \forall M > 0 \exists q ≥ p \lim \inf_{n∈ℕ} \frac{εₙ(p, q)}{εₙ} ≥ M$$

and

$$\forall p \forall M > 0 \exists q ≥ p \ dₙ(U_q, U_p) ≤ e^{−Mεₙ} \forall n ∈ ℕ \ (*) .$$

Now, if we take $(xₙ)ₙ∈ℕ ∈ Δ (Λₜ (ε))$, then there exists a $S > 0$ such that $\sup_{n∈ℕ} |xₙ| e^{−Sεₙ} < +∞$ which means that there exists a $C > 0$ such that for every $n ∈ ℕ$

$$|xₙ| ≤ Ce^{Sεₙ}.$$
Now, for a fixed \( p \) and the number \( S \), from \((*)\) we can find a \( q \geq p \) such that for every \( n \in \mathbb{N} \)
\[
|x_n| d_n(U_q, U_p) \leq C e^{S \varepsilon_n} e^{-S \varepsilon_n} = C.
\]
Then, \((x_n)_{n \in \mathbb{N}} \in \Delta(E)\) and so \( \Delta(\Lambda_\infty(\varepsilon)) \subseteq \Delta(E) \). But then since we always have \( \Delta(E) \subseteq \Delta(\Lambda_\infty(\varepsilon)) \), we obtain \( \Delta(E) = \Delta(\Lambda_\infty(\varepsilon)) \).

To prove the sufficiency part, assume \( \Delta(E) = \Delta(\Lambda_\infty(\varepsilon)) \) and \( \delta(E) \neq \delta(\Lambda_\infty(\varepsilon)) \).

\[
\delta(E) \neq \delta(\Lambda_\infty(\varepsilon)) \Leftrightarrow \exists p \sup \liminf_{q \geq p, n \in \mathbb{N}} \frac{\varepsilon_n(p, q)}{\varepsilon_n} < +\infty
\]
\[
\Leftrightarrow \exists p \exists M > 0 \sup \liminf_{q \geq p, n \in \mathbb{N}} \frac{\varepsilon_n(p, q)}{\varepsilon_n} \leq M
\]
\[
\Leftrightarrow \exists p \exists M > 0 \forall q > p \liminf_{n \in \mathbb{N}} \frac{\varepsilon_n(p, q)}{\varepsilon_n} \leq M
\]
\[
\Leftrightarrow \exists p \exists M > 0 \forall q > p \exists I_q \subseteq \mathbb{N} \quad d_n(U_q, U_p) \geq e^{-M \varepsilon_n} \quad \forall n \in I_q
\]

Now since \( \Delta(E) = \Delta(\Lambda_\infty(\varepsilon)) = \Lambda_\infty(\varepsilon)' = \{(x_n)_{n \in \mathbb{N}} : \exists R > 0 \sup_{n \in \mathbb{N}} |x_n| e^{-R \varepsilon_n} < +\infty\} \), for every \( R > 0 \), we have \( e^{R \varepsilon_n} \in \Lambda_\infty(\varepsilon)' = \Delta(E) \). Therefore, for the above \( p \), we can find a \( \tilde{q} > p \), such that
\[
\sup_{n \in \mathbb{N}} e^{R \varepsilon_n} d_n(U_{\tilde{q}}, U_p) < +\infty.
\]

Then for every \( n \in I_\tilde{q} \), we obtain
\[
e^{(R-M)\varepsilon_n} \leq e^{R \varepsilon_n} d_n(U_{\tilde{q}}, U_p) \leq \sup_{n \in \mathbb{N}} e^{R \varepsilon_n} d_n(U_{\tilde{q}}, U_p) < +\infty.
\]

But then if we choose \( R > M \), we have a contradiction. Hence \( \Delta(E) = \Delta(\Lambda_\infty(\varepsilon)) \) implies \( \delta(E) = \delta(\Lambda_\infty(\varepsilon)) \), as desired. \( \square \)

### 4 Results in the finite case

In this section, we turn our attention to the finite type power series case and, as a main result, we prove that Question 1.1 is true in case the nuclear Fréchet space \( E \) has a prominent bounded subset.

We begin this section by giving the following proposition which answers Question 1.1 in one direction.

**Proposition 4.1.** Let \( E \) be a nuclear Fréchet space with properties \( DN \) and \( \Omega \) and \( \varepsilon = (\varepsilon_n)_{n \in \mathbb{N}} \) be the associated exponent sequence of \( E \). Then \( \delta(E) = \delta(\Lambda_1(\varepsilon)) \) implies \( \Delta(E) = \Delta(\Lambda_1(\varepsilon)) \).

**Proof.** Let us assume that \( \delta(E) = \delta(\Lambda_1(\varepsilon)) \). From Corollary 1.10 of [5], we have
\[
\delta(E) = \delta(\Lambda_1(\varepsilon)) \Leftrightarrow \inf \sup \limsup_{p, q \geq p, n \in \mathbb{N}} \frac{\varepsilon_n(p, q)}{\varepsilon_n} = 0
\]
\[
\Leftrightarrow \forall r > 0 \quad \exists p \quad \forall q \geq p \quad \limsup_{n \in \mathbb{N}} \frac{\varepsilon_n(p, q)}{\varepsilon_n} \leq r
\]
\[
\Leftrightarrow \forall r > 0 \quad \exists p \quad \forall q \geq p \quad \exists n_0 \in \mathbb{N} \forall n \geq n_0 \quad d_n(U_q, U_p) \geq e^{-r \varepsilon_n}.
\]

Now, we take \((x_n)_{n \in \mathbb{N}} \in \Delta(E)\) and for the above \( p \), we find a \( \tilde{q} > p \) such that
\[
\sup_{n \in \mathbb{N}} |x_n| d_n(U_{\tilde{q}}, U_p) < +\infty
\]
and from the above inequality, we obtain

\[ |x_n| e^{-\varepsilon_n} \leq \sup_{n \in \mathbb{N}} |x_n| d_n(U_q, U_p) \]

for large \( n \), this means that \( (x_n)_{n \in \mathbb{N}} \in \Delta(\Lambda_1(\varepsilon)) \) and so \( \Delta(E) \subseteq \Delta(\Lambda_1(\varepsilon)) \). But then since \( \Delta(E) \supseteq \Delta(\Lambda_1(\varepsilon)) \), we have \( \Delta(E) = \Delta(\Lambda_1(\varepsilon)) \).

\[ \square \]

**Theorem 4.2.** Let \( E \) be a nuclear Fréchet space with properties DN and \( \Omega \) and \( \varepsilon = (\varepsilon_n)_{n \in \mathbb{N}} \) be the associated exponent sequence of \( E \). If \( \Delta(E) \), with the canonical topology, is barrelled, then \( \Delta(E) = \Delta(\Lambda_1(\varepsilon)) \) if and only if \( \delta(E) = \delta(\Lambda_1(\varepsilon)) \).

**Proof.** The proof of the necessity part follows from Proposition 4.1. To prove the sufficiency part, let \( \Delta(E) = \Delta(\Lambda_1(\varepsilon)) \) and assume that \( \Delta(E) \) with the canonical topology be barrelled. Then since the convergence in \( \Delta(E) \) implies the coordinate-wise convergence, the inclusion \( \Delta(E) \hookrightarrow \Lambda_1(\varepsilon) \) has a closed graph. But then since \( \Delta(E) \) is barrelled, the inclusion map \( \Delta(E) \hookrightarrow \Lambda_1(\varepsilon) \) is continuous by Theorem 5 in [11]. Taking into account that \( \Delta(E) \) is the projective limit of inductive limits of Banach spaces

\[ \bigcap_{p \in \mathbb{N}} \bigcup_{q \geq p+1} \Delta(U_q, U_p), \]

the continuity of the inclusion map \( \bigcap_{p \in \mathbb{N}} \bigcup_{q \geq p+1} \Delta(U_q, U_p) \hookrightarrow \Lambda_1(\varepsilon) \) gives us

\[ \forall t > 0 \ \exists p \ \forall q > p \ \exists C > 0 \ \forall n \in \mathbb{N} \ e^{-\varepsilon_n} \leq C \ d_n(U_q, U_p). \]

This implies \( \inf \sup \limsup_{n \in \mathbb{N}} \frac{\varepsilon_n(p, q)}{\varepsilon_n} = 0 \), so \( \delta(E) = \delta(\Lambda_1(\varepsilon)) \), as desired. \( \square \)

It is worth the note that, by Theorem 2.1, the barrelledness of the canonical topology of \( \Delta(E) \) is equivalent to the following condition (wQ):

\[ \forall N \ \exists M, n \ \forall K, m, \ \exists k, S > 0 : \ \min(d_n(U_n, U_N), d_n(U_k, U_K)) \leq S d_n(U_m, U_M) \ \forall n \in \mathbb{N}. \]

But determining the barrelledness of \( \Delta(E) \) is not easy, in practice. In the following proposition, by posing a condition

**Condition**: \( \forall p, \forall q > p, \exists s > q, \forall k > s, \exists C > 0 : d_n(U_q, U_p) \leq C d_n(U_k, U_s) \ \forall n \in \mathbb{N}. \)

on diameters, we eliminate the barrelledness condition of Theorem 4.2.

**Proposition 4.3.** Let \( E \) be a nuclear Fréchet space with the properties DN and \( \Omega \) and \( \varepsilon \) be the associated exponent sequence of \( E \). If \( E \) satisfies the condition \( \mathbb{A} \) and \( \Delta(E) = \Delta(\Lambda_1(\varepsilon)) \), then \( \delta(E) = \delta(\Lambda_1(\varepsilon)) \).

**Proof.** Suppose that \( E \) satisfies the condition \( \mathbb{A} \) and \( \Delta(E) = \Delta(\Lambda_1(\varepsilon)) \). If \( \delta(E) \neq \delta(\Lambda_1(\varepsilon)) \), then from Corollary 1.10 of [5] we have the following condition:

\[ \exists M > 0 \ \forall p \ \exists q_p > p, I_p \subseteq \mathbb{N} \ d_n(U_q, U_p) < e^{-M\varepsilon_n} \ \forall n \in I_p \quad (4.1) \]

For \( p = 1 \), there exists a number \( q_1 \) and an infinite subset \( I_1 \) so that for all \( n \in I_1 \)

\[ d_n(U_{q_1}, U_p) < e^{-M\varepsilon_n}, \]

and so it follows from the condition \( \mathbb{A} \) that we have a number \( q_2 \) such that for all \( k \geq q_2 \) there exists a \( C > 0 \) so that for all \( n \in \mathbb{N} \)

\[ d_n(U_{q_2}, U_1) \leq C d_n(U_k, U_{q_2}) \]
holds. Then, from the inequality 4.1, there exists a number $q_3$ and an infinite subset $I_2$ so that for all $n \in I_2$

$$d_n (U_{q_1}, U_{q_2}) < e^{-M \varepsilon_n}.$$ 

It follows that there exists a $C_1 > 0$ so that for all $n \in \mathbb{N}$

$$d_n (U_{q_1}, U_{q_2}) \leq C_1 d_n (U_{q_3}, U_{q_4})$$

holds. Now applying the same process for $q_2$ and $q_3$, we can find $q_4, q_5$ and $C_2 > 0$ such that

$$d_n (U_{q_3}, U_{q_4}) \leq C_2 d_n (U_{q_5}, U_{q_6})$$

for all $n \in \mathbb{N}$. Continuing in this way, we can find the sequences $\{q_k\}_{k \in \mathbb{N}}$ and $\{C_k\}_{k \in \mathbb{N}}$ satisfying

$$d_n (U_{q_1}, U_{q_2}) \leq C_1 d_n (U_{q_3}, U_{q_4}) \leq C_2 d_n (U_{q_5}, U_{q_6}) \leq \cdots \leq C_k d_n (U_{q_{2k+1}}, U_{q_{2k}}) \leq \cdots$$ \quad (4.2)

for all $n \in \mathbb{N}$. Moreover, for each $k \in \mathbb{N}$, there exists a $I_k \subseteq \mathbb{N}$ so that

$$d_n (U_{q_{2k+1}}, U_{q_{2k}}) < e^{-M \varepsilon_n}$$ \quad (4.3)

for all $n \in I_k$.

Now, for each $k \in \mathbb{N}$, we define

$$B_k = \left\{ x = (x_n) : \sup_{n \in \mathbb{N}} C_k |x_n|d_n (U_{q_{2k+1}}, U_{q_{2k}}) < +\infty \right\},$$

where $B_k$ is a Banach space under the norm $\|x\|_k = \sup_{n \in \mathbb{N}} C_k |x_n|d_n (U_{q_{2k+1}}, U_{q_{2k}})$ for all $k \in \mathbb{N}$. By the inequality 4.2, we have $B_{k+1} \subseteq B_k$ and $\| \cdot \|_k \leq \| \cdot \|_{k+1}$ for all $k \in \mathbb{N}$. Since $\{q_k\}_{k \in \mathbb{N}}$ is strictly increasing and unbounded, for all $p \in \mathbb{N}$, there exists a $k_0 \in \mathbb{N}$ such that $q_{2k_0} > p$ and this gives us $U_{q_{2k_0}} \subseteq U_p$. For all $n \in \mathbb{N}$

$$d_n (U_{q_{2k_0+1}}, U_p) \leq d_n (U_{q_{2k_0}}, U_{q_{2k_0}}),$$

which means that $\bigcap_{k} B_k \subseteq \Delta (E)$. Moreover, the equality $\Delta (X) = \Lambda_1 (\varepsilon)$ yields a continuous imbedding of the projective limit $\bigcap_{k} B_k$ into $\Lambda_1 (\varepsilon)$. Then since $\bigcap_{k} B_k$ and $\Lambda_1 (\varepsilon)$ are Fréchet spaces and the imbedding map has a closed graph, by Theorem 5 in [11], this map is continuous and so

$$\bigcap_{k} B_k \hookrightarrow \Delta (\Lambda_1 (\varepsilon))$$

is continuous $\iff$ $\forall t > 0 \ \exists k, C \sup_{n \in \mathbb{N}} |x_n|e^{-t \varepsilon_n} \leq C \sup_{n \in \mathbb{N}} |x_n|d_n (U_{q_{2k+1}}, U_{q_{2k}})$

$$\forall (x_n) \in \bigcap_{p} B_p$$

$$\iff \forall t > 0 \ \exists k, C \forall n \in \mathbb{N} \ e^{-t \varepsilon_n} \leq C d_n (U_{q_{2k+1}}, U_{q_{2k}}).$$

But, this is contradictory to the inequality 4.3. Therefore, $\delta (E) = \delta (\Lambda_1 (\varepsilon))$ holds when $\Delta (E) = \Delta (\Lambda_1 (\varepsilon))$ and the condition $\mathfrak{A}$ holds. \hfill $\square$

There could be other diameter conditions as above which yields the same conclusion in Proposition 4.3. For example, by introducing

**Condition $\mathfrak{B}$:** $\forall p \ \forall q_1, q_2, \ldots, q_p, \ \exists 1 \leq s \leq p, \ \exists C > 0 \ \max_{1 \leq i \leq q} d_n (U_i, U_s) \leq C d_n (U_{q_s}, U_{s})$ \quad $\forall n \in \mathbb{N}.$

we have
Proposition 4.4. Let $E$ be a nuclear Fréchet space with the properties $\mathcal{DN}$ and $\Omega$ and $\varepsilon$ be associated exponent sequence of $E$. If $E$ satisfies the condition $\mathfrak{B}$ and $\Delta(E) = \Delta(A_1(\varepsilon))$, then $\delta(E) = \delta(A_1(\varepsilon))$.

The proof is similar to Proposition 4.3 except that the projective limit will be replaced by $\bigcap_k D_k$ where

$$D_k = \left\{ x = (x_n) : \sup_{n \in \mathbb{N}} \max_{1 \leq i \leq p} d_n(U_q, U_i) < +\infty \right\}.$$

For some Fréchet spaces, one can obtain the diametral dimension by using a single bounded subset:

Terzioğlu [22] introduced an absolutely convex bounded subset $B$ of a Fréchet space $E$ as prominent bounded set in case $\lim_{n \to +\infty} x_n d_n(B, U_p) = 0$ for every $p$ implies $(x_n) \in \Delta(E)$. If $E$ has a prominent bounded set $B$, then

$$\Delta(E) = \left\{ (x_n)_{n \in \mathbb{N}} : \forall p, \lim_{n \to +\infty} x_n d_n(B, U_p) = 0 \right\}.$$

In this case one can introduce a natural Fréchet space topology on $\Delta(E)$. Terzioğlu also gave a necessary and sufficient condition for a bounded subset to be prominent [Proposition 3, [21]], namely, $B$ is a prominent set if and only if for each $p$ there is a $q$ and $C > 0$ such that

$$d_n(U_q, U_p) \leq C d_n(B, U_q)$$

hold for all $n \in \mathbb{N}$.

In the following proposition, we prove that having a prominent bounded subset is closely related to Bessaga’s basis free version of Dragilev condition $D_2$, given in [6]:

$$D_2 : \forall p \quad \exists q \geq p + 1 \forall k \geq q + 1 \lim_{n \to +\infty} d_n(U_q, U_p) d_n(U_k, U_q) = 0.$$

Proposition 4.5. Let $E$ be a nuclear Fréchet space. The following are equivalent:

1. $E$ has a prominent bounded set $B$.
2. $E$ has the property $D_2$.
3. For all $p$ there exists $q > p$ such that $\lim_{n \to +\infty} \sup_{l \geq q} d_n(U_q, U_l) d_n(U_p, U_q) = 0$ holds.

We need the following lemma for the proof of Proposition 4.5. As usual, we assume that all semi-norms are Hilbertian.

Lemma 4.6. Let $E$ be a nuclear Fréchet space. Then for all $p, q > p$, there is a $s > q$ such that

$$\lim_{n \to +\infty} d_n(U_s, U_p) d_n(U_q, U_p) = 0.$$

Proof. This is an immediate consequence of Proposition 1.2 in [8].

It is worth noting that, by using Lemma 4.6, the condition $D_2$ can also be stated as follows:

$$D_2 : \forall p \quad \exists q \geq p + 1 \forall k \geq q + 1 \sup_{n \in \mathbb{N}} d_n(U_q, U_p) d_n(U_k, U_q) < +\infty.$$

We are now ready to give the proof of Proposition 4.5.
Proof of Proposition 4.5
1 ⇒ 2: This follows immediately from Lemma 4.6 and the definition of $D_2$.
2 ⇒ 1: Follows from Proposition 5 of [9].
2 ⇔ 3: Suppose $E$ has the condition $D_2$. Then, for all $p$, there exists a $q > p$ such that for all \( k > q \)
\[
\sup_{n \in \mathbb{N}} d_n(U_q, U_p) < \infty \iff \exists M > 0 \quad \forall n \in \mathbb{N} \quad \frac{d_n(U_q, U_p)}{d_n(U_k, U_q)} \leq M
\]
\[
\iff \exists M > 0 \quad \forall n \in \mathbb{N} \quad \varepsilon_n(p, q) \geq -\ln M + \varepsilon_n(q, k)
\]
\[
\iff \limsup_{n \to \infty} \varepsilon_n(p, q) \leq 1.
\]
Hence we obtain for all $p$, there exists a $q > p$ such that
\[
\sup_{l \geq q} \limsup_{n \in \mathbb{N}} \varepsilon_n(q, l) \leq \varepsilon_n(p, q) \leq 1.
\]

As an easy consequence of Proposition 4.5 and (3.1)([5, Cor. 1.10]) above, we obtain the following result which gives a relation between having prominent bounded subset and its approximate diametral dimension of a nuclear Fréchet space with the properties $DN$ and $\Omega$:

**Corollary 4.7.** Let $E$ be a nuclear Fréchet space with the properties $DN$ and $\Omega$ and $\varepsilon$ the associated exponent sequence. $\delta(E) = \delta(\Lambda_1(\varepsilon))$ implies that $E$ has a prominent bounded subset.

The following theorem is the main result of this section which says that Question 1.1 holds true provided $E$ has a prominent bounded subset:

**Theorem 4.8.** Let $E$ be a nuclear Fréchet space with the properties $DN$ and $\Omega$ and $\varepsilon$ the associated exponent sequence. $\delta(E) = \delta(\Lambda_1(\varepsilon))$ if and only if $E$ has a prominent bounded set and $\Delta(E) = \Lambda_1(\varepsilon)$.

**Proof.** Let $E$ be a nuclear Fréchet space with a prominent bounded subset $B$. Then $E$ satisfies condition $D_2$
\[
\forall p \quad \exists q \geq p + 1 \quad \forall k \geq q + 1 \quad \exists C > 0 \quad \lim_{n \to \infty} d_n(U_q, U_p) = 0
\]
and, in particular, if we take $N = p$, $M = n = q$ and $m = k$, we get
\[
\forall N \exists M, n \quad \forall m, \exists S > 0 : \quad d_n(U_n, U_N) \leq S d_n(U_m, U_M) \quad \forall n \in \mathbb{N}.
\]
which means that $E$ satisfy the condition ($wQ$) given in Theorem 2.1 and so $\Delta(E)$ is barrelled with respect to the canonical topology. Hence the result follows from Theorem 4.2.

In the final part of this section we examine the conditions for which the converse of Corollary 4.7 also holds.

For this, we define
\[
\Delta(E) := \left\{ (t_n)_{n \in \mathbb{N}} : \forall p, \forall 0 < \varepsilon < 1, \exists q > p \lim_{n \to +\infty} t_n d_n(U_q, U_p) = 0 \right\}.
\]

The next result provides a condition that implies $\delta(E) = \delta(\Lambda_1(\varepsilon))$ when $E$ has a prominent subset.
Proposition 4.9. Let $E$ be a nuclear Fréchet space with the properties $DN$ and $\Omega$, its associated exponent sequence $\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$. If $E$ has a prominent bounded subset $B$ and $\Delta(E) = \mathbf{\Delta}(E)$, then $\delta(E) = \delta(\Lambda_1(\varepsilon))$.

Proof. Let $B$ be a prominent bounded subset of $E$. Then, for all $p \in \mathbb{N}$, there exists a $q > p$ and a $C > 0$ so that for every $n \in \mathbb{N}$

$$d_n(U_q, U_p) \leq Cd_n(B, U_q)$$

holds. Also, since $B$ is bounded and $\varepsilon_n$ is the associated exponent sequence, then there exist $C_1, C_2, D_1, D_2 > 0$ and $a_1, a_2 > 0$ satisfying

$$D_1e^{\alpha_1\varepsilon_n} \leq C_1d_n(U_q, U_p) \leq d_n(B, U_q) \leq C_2d_n(U_{q+1}, U_q) \leq D_2e^{-\alpha_2\varepsilon_n}$$

for every $n \in \mathbb{N}$. On the other hand, $\Delta(E) = \left\{(x_n)_{n \in \mathbb{N}} : \forall p \limsup_{n \to +\infty} |x_n|d_n(B, U_p) = 0\right\}$ is a Fréchet space since $B$ is a prominent set. Fix $p, q > p$ and $\varepsilon$. Consider the Banach space

$$B_{p,\varepsilon,q} = \left\{t = (t_n)_{n \in \mathbb{N}} : \limsup_{n \to +\infty} |t_n|d_n(U_q, U_p)^\varepsilon = 0\right\}.$$ 

Since $U_{q+1} \subseteq U_q$, we have $d_n(U_{q+1}, U_p) \leq d_n(U_q, U_p)$ for every $n \in \mathbb{N}$ and $B_{p,\varepsilon,q} \subseteq B_{p,\varepsilon,q+1}$. Then we can put the topology on $\Delta(E) \ni (p, \varepsilon) \mapsto B_{p,\varepsilon,q}$ which of the projective limit of inductive limits of Banach spaces $B_{p,\varepsilon,q}$. In view of Grothendieck Factorization theorem ([13], p.225), for all $p, 0 < \varepsilon < 1$ there exists a $q > p$ such that $\Delta(E) \ni B_{p,\varepsilon,q}$ is continous

$$\forall p, \quad 0 < \varepsilon < 1, \quad \exists q > p, \quad C > 0 \quad d_n(U_q, U_p)^\varepsilon \leq d_n(B, U_q) \quad \forall n \in \mathbb{N}.$$ 

Now take $\delta > 0$. Then, for a given $p$, we choose $0 < \varepsilon < 1$ so that $0 < \varepsilon < \frac{\delta}{a_1}$. Then there exists a $\mathcal{C} > 0$ so that for all $n \in \mathbb{N}$,

$$\mathcal{C}e^{-\alpha_1\varepsilon_n} \leq Cd_n(B, U_p)^\varepsilon \leq d_n(U_q, U_p)^\varepsilon \quad \iff \quad \mathcal{C}e^{-\delta\varepsilon_n} \leq d_n(U_q, U_p)^\varepsilon \leq Cd_n(B, U_q)$$

$$\iff \quad \ln \mathcal{C} - \delta\varepsilon_n \leq \ln C + \ln d_n(B, U_q) \leq \ln C + \ln d_n(U_l, U_q)$$

$$\iff \quad -\ln d_n(U_l, U_q) \leq (\ln C - \ln \mathcal{C}) + \delta\varepsilon_n$$

$$\iff \quad \limsup_{n} \frac{\varepsilon_n(q, l)}{\varepsilon_n} \leq \delta.$$ 

Hence, we obtain that for all $\delta > 0$ there is a $q$ so that

$$\sup_{l > q} \limsup_{n} \frac{\varepsilon_n(q, l)}{\varepsilon_n} \leq \delta$$

and

$$\inf \limsup_{n} \frac{\varepsilon_n(q, l)}{\varepsilon_n} = 0,$$

which means that $\delta(E) = \delta(\Lambda_1(\varepsilon))$. \hfill \Box

Note that $\mathbf{\Delta}(E)$ is always an algebra under multiplication. If $(t_n)_{n \in \mathbb{N}} \in \mathbf{\Delta}(E)$, then for all $p, 0 < \varepsilon < 1$, we can choose $q > p$ so that

$$\lim_{n \to +\infty} t_n d_n(U_q, U_p)^{\varepsilon} = 0,$$

which means $(t_n^2) \in \mathbf{\Delta}(E)$. Then for any $(t_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}} \in \mathbf{\Delta}(E)$, we have that $(t_n s_n)_{n \in \mathbb{N}} \in \mathbf{\Delta}(E)$ as

$$|t_n s_n| \leq \frac{|t_n|^2}{2} + \frac{|s_n|^2}{2}$$

for all $n \in \mathbb{N}$. 

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But $\Delta(E)$ need not to be an algebra under multiplication. If it does, then $\Delta(E)$ satisfies the condition "$(t_n) \in \Delta(E)$ implies $(t^2_n) \in \Delta(E)$", vice versa. This condition gives that $(t^m_n) \in \Delta(E)$ for all $m \in \mathbb{N}$.

Now, for a $p$ and $\varepsilon > 0$, we can choose $m \in \mathbb{N}$ so large that $\frac{1}{2m} \leq \varepsilon$ and find a $q$ so that

$$\lim_{n \to \infty} t^m_n d_n (U_q, U_p) = 0 \quad \text{and} \quad \lim_{n \to \infty} t^n d_n (U_q, U_p)^\varepsilon = 0$$

which gives that $\Delta(E) \subseteq \overline{\Delta}(E)$. Since $\overline{\Delta}(E) \subseteq \Delta(E)$ always holds, we have $\overline{\Delta}(E) = \Delta(E)$. Hence we conclude that the followings are equivalent:

1. $\Delta(E) = \overline{\Delta}(E)$
2. $\Delta(E)$ is an algebra under multiplication.
3. $(t_n) \in \Delta(E)$ implies $(t^2_n) \in \Delta(E)$.

We end this paper with the following result which is a generalization of Proposition 4.9:

**Corollary 4.10.** Let $E$ be a nuclear Fréchet space with the properties $\text{DN}$ and $\Omega$, its associated exponent sequence $\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$. $E$ has a prominent bounded subset and $\Delta(E)$ is an algebra if and only if $\delta(E) = \delta(L_1(\varepsilon))$.

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