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Tome 70, n° 2 (2020), p. 597-619.

<http://aif.centre-mersenne.org/item/AIF_2020__70_2_597_0>
BIFURCATION VALUES OF POLYNOMIAL FUNCTIONS AND PERVERSE SHEAVES

by Kiyoshi TAKEUCHI

Abstract. — We characterize bifurcation values of polynomial functions by using the theory of perverse sheaves and their vanishing cycles. In particular, by introducing a method to compute the jumps of the Euler characteristics with compact support of their fibers, we confirm the conjecture of Némethi–Zaharia in many cases.

1. Introduction

For a polynomial function \( f : \mathbb{C}^n \to \mathbb{C} \) it is well-known that there exists a finite subset \( B \subset \mathbb{C} \) such that the restriction
\[
\mathbb{C}^n \setminus f^{-1}(B) \to \mathbb{C} \setminus B
\]
of \( f \) is a \( C^\infty \) locally trivial fibration. We denote by \( B_f \) the smallest subset \( B \subset \mathbb{C} \) satisfying this condition. Let \( \text{Sing} \, f \subset \mathbb{C}^n \) be the set of the critical points of \( f : \mathbb{C}^n \to \mathbb{C} \). Then by the definition of \( B_f \), obviously we have \( f(\text{Sing} \, f) \subset B_f \). The elements of \( B_f \) are called bifurcation values of \( f \). The determination of the bifurcation set \( B_f \subset \mathbb{C} \) is a fundamental problem and was studied by many mathematicians and from several viewpoints, e.g. [3, 4, 9, 10, 19, 20, 23, 24, 29] and [30]. The essential difficulty consists in the fact that in general \( f \) has a lot of singularities at infinity. Here we study \( B_f \) via the Newton polyhedron of \( f \). We denote by \( \Gamma_\infty(f) \) the convex

Keywords: bifurcation values, perverse sheaves, vanishing cycles.
2020 Mathematics Subject Classification: 14F05, 14F43, 14M25, 32C38, 32S20.
hull of the Newton polytope \( NP(f) \) of \( f \) and the origin in \( \mathbb{R}^n \). We call it the Newton polyhedron at infinity of \( f \). Throughout this paper we assume that \( \dim \Gamma_{\infty}(f) = n \). Recall that \( f \) is said to be \textit{convenient} if \( \Gamma_{\infty}(f) \) intersects the positive part of each coordinate axis. Kouchnirenko [13] proved that if \( f \) is convenient and non-degenerate at infinity (for the definition see Section 3) then \( B_f = f(\operatorname{Sing} f) \). However, in the non-convenient case, Némethi and Zaharia [19] showed that more bifurcation values may occur due to so-called “bad faces”. Let us explain this phenomenon here and refer for details to Section 3.

**Definition 1.1 ([26]).** — We say that a face \( \gamma \prec \Gamma_{\infty}(f) \) is atypical if \( 0 \in \gamma \), \( \dim \gamma \geq 1 \) and the cone \( \sigma(\gamma) \subset \mathbb{R}^n \) which corresponds it in the dual fan of \( \Gamma_{\infty}(f) \) (for the definition see Section 3) is not contained in the first quadrant \( \mathbb{R}^n_+ := \mathbb{R}^n_{\geq 0} \) of \( \mathbb{R}^n \).

This definition is closely related to that of the bad faces of \( NP(f - f(0)) \) in Némethi–Zaharia [19]. See Section 3 for the details and examples. In this paper, we consider the case where \( f \) is not convenient. Let \( \gamma_1, \ldots, \gamma_m \) be the atypical faces of \( \Gamma_{\infty}(f) \). As we see in Theorem 1.2 below, in the generic case where \( f \) is non-degenerate at infinity, the singularities at infinity of \( f \) are produced only from \( \gamma_i \). For \( 1 \leq i \leq m \) let \( K_i = f_{\gamma_i}(\operatorname{Sing} f_{\gamma_i}) \subset \mathbb{C} \) be the set of the critical values of the \( \gamma_i \)-part

\[
(1.2) \quad f_{\gamma_i} : T = (\mathbb{C}^*)^n \longrightarrow \mathbb{C}
\]

of \( f \). Let us set

\[
(1.3) \quad K_f = f(\operatorname{Sing} f) \cup \{f(0)\} \cup \left( \bigcup_{i=1}^m K_i \right).
\]

Then Némethi–Zaharia [19] proved the following fundamental result.

**Theorem 1.2 (Némethi–Zaharia [19]).** — Assume that \( f \) is non-degenerate at infinity. Then we have \( B_f \subset K_f \).

Moreover they proved the equality \( B_f = K_f \) for \( n = 2 \) and conjectured its validity in higher dimensions. The essential problem is to prove the inverse inclusion \( K_f \subset B_f \). This has been a long standing conjecture until now. Later Zaharia [30] proved \( K_f \setminus \{f(0)\} \subset B_f \) for \( n \geq 2 \) under some additional assumptions. In particular, he assumed that \( f \) has isolated singularities at infinity on a fixed smooth toric compactification of \( \mathbb{C}^n \). We can easily see that even if \( f \) is non-degenerate at infinity this condition is not satisfied in general. See (4.13) in the proof of Theorem 4.3 below. Namely his assumption is very strong and moreover depends on the choice
of a particular smooth toric compactification of $\mathbb{C}^n$. In this paper, we overcome this problem by introducing the following intrinsic definition. For $1 \leq i \leq m$ let $L_{\gamma_i} \simeq \mathbb{R}^{\dim \gamma_i}$ be the linear subspace of $\mathbb{R}^n$ spanned by $\gamma_i$ and set $T_i = \text{Spec} (\mathbb{C}[L_{\gamma_i} \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim \gamma_i}$. We regard $f_{\gamma_i}$ as a regular function on $T_i$.

**Definition 1.3.** — We say that $f$ has isolated singularities at infinity over $b \in K_f \setminus [f(\text{Sing} f) \cup \{ f(0) \}]$ if for any $1 \leq i \leq m$ the hypersurface $f_{\gamma_i}^{-1}(b) \subset T_i \simeq (\mathbb{C}^*)^{\dim \gamma_i}$ in $T_i$ has only isolated singular points. We simply say that $f$ has isolated singularities at infinity if it is so over any value $b \in K_f \setminus [f(\text{Sing} f) \cup \{ f(0) \}]$.

With this new definition at hand, by using also the more sophisticated machinery of vanishing cycle functors for constructible sheaves we can eventually work on a singular toric variety. Then we use the theory of perverse sheaves to improve Zaharia’s result. In this way, we prove the inverse inclusion $K_f \setminus \{ f(0) \} \subset B_f$ and confirm the conjecture of [19] in many cases. In particular, for $n = 3$ we obtain the following result.

**Theorem 1.4.** — Let $f : \mathbb{C}^3 \rightarrow \mathbb{C}$ be a non-degenerate polynomial at infinity such that $\dim \Gamma_\infty(f) = 3$. Then, if $f$ has isolated singularities at infinity over $b \in K_f \setminus [f(\text{Sing} f) \cup \{ f(0) \}]$, we have $b \in B_f$. In particular, if $f$ has isolated singularities at infinity, we have $K_f \setminus \{ f(0) \} \subset B_f$.

For $n = 3$ in the generic case, we thus confirm the conjecture of [19]. In fact, to prove Theorem 1.4 we show moreover that the Euler characteristics with compact support of the fibers of $f : \mathbb{C}^n \rightarrow \mathbb{C}$ jump at the point $b$. The jump of Euler characteristics was used as a test for the bifurcation locus in case of “isolated singularities at infinity” (defined in various ways) in many other articles and from different points of view (see [1, 2, 3, 9, 10, 23, 24, 25, 27] etc.). To introduce our results in higher dimensions, we need also the following definition.

**Definition 1.5.** — We say that an atypical face $\gamma_i \prec \Gamma_\infty(f)$ is relatively simple if the cone $\sigma_i := \sigma(\gamma_i) \subset \mathbb{R}^n$ which corresponds to it in the dual fan of $\Gamma_\infty(f)$ is simplicial or satisfies the condition $\dim \sigma_i \leq 3$.

This condition implies that the constant sheaf on the affine toric variety associated to the cone $\sigma_i$ such that $\dim \sigma_i = n - \dim \gamma_i$ is perverse (up to some shift). If $\sigma_i$ is simplicial, then the affine toric variety associated to it is an orbifold and the perversity follows. If $\dim \sigma_i \leq 3$ we can show the corresponding perversity by a result of Fieseler [7] on the intersection cohomology complexes of toric varieties. See Lemma 2.5 below. In higher
dimensions, this perversity is essential in our proof of the inverse inclusion \( K_f \setminus \{f(0)\} \subset B_f \). Note that if \( \dim \gamma_i \geq n - 3 \) we have \( \dim \sigma_i \leq 3 \) and the atypical face \( \gamma_i \) is relatively simple. In particular, if \( n \leq 4 \) this condition is always satisfied. Now we define a function \( \chi_c : \mathbb{C} \to \mathbb{Z} \) on \( \mathbb{C} \) by
\[
(1.4) \quad \chi_c(t) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j_c(f^{-1}(t); \mathbb{C}) \quad (t \in \mathbb{C}).
\]
Let us fix a point \( b \in K_f \setminus [f(\text{Sing}) \cup \{f(0)\}] \subset \bigcup_{i=1}^m K_i \) and define the jump \( E_f(b) \in \mathbb{Z} \) of the function \( \chi_c \) at \( b \) by
\[
(1.5) \quad E_f(b) = (-1)^{n-1} \{ \chi_c(b + \varepsilon) - \chi_c(b) \} \in \mathbb{Z},
\]
where \( \varepsilon > 0 \) is sufficiently small. Recall that for a polytope \( \Delta \) in \( \mathbb{R}^n \) its relative interior \( \text{rel.int}(\Delta) \) is the interior of \( \Delta \) in its affine span \( \text{Aff}(\Delta) \simeq \mathbb{R}^\dim \Delta \) in \( \mathbb{R}^n \). Then we have the following result.

**Theorem 1.6.** — Assume that \( \dim \Gamma_\infty(f) = n \), \( f \) is non-degenerate at infinity and has isolated singularities at infinity over \( b \in K_f \setminus [f(\text{Sing}) \cup \{f(0)\}] \) and for any \( 1 \leq i \leq m \) such that \( b \in K_i \) we have \( \text{rel.int}(\gamma_i) \subset \text{Int}(\mathbb{R}^+_n) \). Assume also that there exists \( 1 \leq i \leq m \) such that \( b \in K_i \) and \( \gamma_i \prec \Gamma_\infty(f) \) is relatively simple. Then we have \( E_f(b) > 0 \) and hence \( b \in B_f \).

If \( n = 4 \), all the atypical faces \( \gamma_i \) are relatively simple and we obtain the following corollary.

**Corollary 1.7.** — Let \( f : \mathbb{C}^4 \to \mathbb{C} \) be a non-degenerate polynomial at infinity such that \( \dim \Gamma_\infty(f) = 4 \). Then, if \( f \) has isolated singularities at infinity over \( b \in K_f \setminus [f(\text{Sing}) \cup \{f(0)\}] \) and for any \( 1 \leq i \leq m \) such that \( b \in K_i \) the condition \( \text{rel.int}(\gamma_i) \subset \text{Int}(\mathbb{R}^+_n) \) is satisfied, then we have \( E_f(b) > 0 \). In particular, if \( f \) has isolated singularities at infinity and \( \Gamma_\infty(f) \setminus \{0\} \subset \text{Int}(\mathbb{R}^+_n) \), we have \( K_f \setminus \{f(0)\} \subset B_f \).

For general \( n \geq 2 \) we have also the following corollary.

**Corollary 1.8.** — Assume that \( \dim \Gamma_\infty(f) = n \) and \( f \) is non-degenerate at infinity. Then, if moreover \( f \) has isolated singularities at infinity, \( \Gamma_\infty(f) \setminus \{0\} \subset \text{Int}(\mathbb{R}^+_n) \) and all the atypical faces \( \gamma_i \) (\( 1 \leq i \leq m \)) are relatively simple, then we have \( K_f \setminus \{f(0)\} \subset B_f \).

Since \( \gamma_i \) is relatively simple if \( \dim \gamma_i \geq n - 3 \), Theorem 1.6 extends the result of Zaharia [30]. Indeed, he assumed the much stronger condition that for any \( 1 \leq i \leq m \) such that \( b \in K_i \) we have \( \dim \gamma_i = n - 1 \) (which implies also \( \text{rel.int}(\gamma_i) \subset \text{Int}(\mathbb{R}^+_n) \)). His assumption means that on a fixed smooth toric compactification of \( \mathbb{C}^n \) compatible with \( \Gamma_\infty(f) \) the function \( f \) has isolated singular points only on \( T \)-orbits at infinity of dimension \( n - 1 \).
over the point $b \in K_f \setminus [f(\text{Sing } f) \cup \{f(0)\}]$. However under our weaker assumption, in the proof of Theorem 1.6 we encounter non-isolated singular points of $f$ at infinity on such a smooth compactification (see (4.13)). We overcome this difficulty by reducing the problem to the case of isolated singular points. To this end, we consider the direct image of the vanishing cycle of a constructible sheaf by a special morphism

\begin{equation}
\pi : X = X_{\Sigma_C} \longrightarrow X_{\Sigma_C}
\end{equation}

of toric varieties. In this way, we can eventually work on the singular toric variety $X_{\Sigma_C}$ canonically associated to $\Gamma_{\infty}(f)$. This is the reason why we can employ our intrinsic definition in Definition 1.3. Then, on $X_{\Sigma_C}$, the function $f$ has only isolated singular points at infinity (over the point $b \in K_f \setminus [f(\text{Sing } f) \cup \{f(0)\}]$). Finally, to finish the proofs of Theorems 1.4 and 1.6, we apply the theory of perverse sheaves and their vanishing cycles. Here we use the perversity of the constant sheaf on the toric variety associated to the cone $\sigma_i$ to obtain the positivity $E_f(b) > 0$. For the moment, it is not clear if we can further relax the assumption on $\sigma_i$ by using the very general formula for vanishing cycle sheaves in Massey [15, Lemma 2.2] etc. Note also that our condition rel. $\text{int}(\gamma_i) \subset \text{Int}(\mathbb{R}_n^+)$ in Theorem 1.6 is equivalent to the one $\sigma_i \cap \mathbb{R}_n^+ = \{0\}$. However in higher dimensions, there still remain some atypical faces for which this condition is not satisfied (see Example 3.5 below). So it is desirable to relax the condition $\sigma_i \cap \mathbb{R}_n^+ = \{0\}$. In this direction, we have only a partial answer in Theorem 4.5 which extends Theorems 1.4 and 1.6 in a unified manner. We hope that we can drop some of the conditions in it in the future.

Acknowledgement. The author would like to express his hearty gratitude to Professor Mihai Tibăr for drawing our attention to this interesting problem. Several discussions with him were very useful. The author thanks him also for his encouragement during the preparation of this paper. Moreover he is very grateful to the referee for many valuable suggestions.

2. Review on constructible and perverse sheaves

In this section, we recall some results on constructible and perverse sheaves. In this paper, we essentially follow the terminology of [5], [11] and [12]. For example, for a topological space $X$ we denote by $D^b(X)$ the derived category whose objects are bounded complexes of sheaves of $\mathbb{C}_X$-modules on $X$. Denote by $D^b_c(X)$ the full subcategory of $D^b(X)$ consisting of constructible objects.
Definition 2.1. — Let $X$ be an algebraic variety over $\mathbb{C}$. Then we say that a $\mathbb{Z}$-valued function $\psi : X \to \mathbb{Z}$ on $X$ is constructible if there exists a stratification $X = \bigsqcup_{\alpha} X_{\alpha}$ of $X$ such that $\psi|_{X_{\alpha}}$ is constant for any $\alpha$. We denote by $F_{\mathbb{Z}}(X)$ the abelian group of constructible functions on $X$.

Let $F \in D^b_{c}(X)$ be a constructible sheaf (complex of sheaves) on an algebraic variety $X$ over $\mathbb{C}$. Then we can naturally associate to it a constructible function $\chi(F) \in F_{\mathbb{Z}}(X)$ on $X$ defined by

$$\chi(F)(x) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j(F) \quad (x \in X).$$

For a constructible function $\psi : X \to \mathbb{Z}$, we take a stratification $X = \bigsqcup_{\alpha} X_{\alpha}$ of $X$ such that $\psi|_{X_{\alpha}}$ is constant for any $\alpha$ as above. We denote the Euler characteristic of $X_{\alpha}$ by $\chi(X_{\alpha})$. Then we set

$$\int_X \psi := \sum_{\alpha} \chi(X_{\alpha}) \cdot \psi(x_{\alpha}) \in \mathbb{Z},$$

where $x_{\alpha}$ is a reference point in $X_{\alpha}$. Then we can easily show that $\int_X \psi \in \mathbb{Z}$ does not depend on the choice of the stratification $X = \bigsqcup_{\alpha} X_{\alpha}$ of $X$. Hence we obtain a homomorphism

$$\int_X : F_{\mathbb{Z}}(X) \to \mathbb{Z}$$

of abelian groups. For $\psi \in F_{\mathbb{Z}}(X)$, we call $\int_X \psi \in \mathbb{Z}$ the topological (Euler) integral of $\psi$ over $X$. More generally, to a morphism $f : X \to Y$ of algebraic varieties over $\mathbb{C}$ we can associate a homomorphism $\int_f : F_{\mathbb{Z}}(X) \to F_{\mathbb{Z}}(Y)$ of abelian groups as follows. For $\psi \in F_{\mathbb{Z}}(X)$ we define $\int_f \psi \in F_{\mathbb{Z}}(Y)$ by

$$\left(\int_f \psi\right)(y) = \int_{f^{-1}(y)} \psi \in \mathbb{Z} \quad (y \in Y).$$

Then for any constructible sheaf $\mathcal{F} \in D^b_{c}(X)$ on $X$ we have the equality

$$\int_f \chi(\mathcal{F}) = \chi(Rf_{*}(\mathcal{F})).$$

Now we recall the following well-known property of Deligne’s vanishing cycle functors. Let $X$ be an algebraic variety over $\mathbb{C}$ and $f : X \to \mathbb{C}$ a non-constant regular function on $X$ and set $X_0 = \{x \in X \mid f(x) = 0\} \subset X$. Then we denote Deligne’s vanishing cycle functor associated to $f$ by

$$\varphi_f : D^b_{c}(X) \to D^b_{c}(X_0)$$

(see [5, Section 4.2] and [12, Section 8.6] etc. for the details).
Proposition 2.2 (cf. [5, Proposition 4.2.11] and [12, Exercise VIII.15] etc.). — Let $\pi : Y \rightarrow X$ be a proper morphism of algebraic varieties over $\mathbb{C}$ and $f : X \rightarrow \mathbb{C}$ a non-constant regular function on $X$. Set $g = f \circ \pi : Y \rightarrow \mathbb{C}$, $X_0 = \{x \in X \mid f(x) = 0\}$ and $Y_0 = \{y \in Y \mid g(y) = 0\}$. Then for any $G \in D^b_c(Y)$ we have an isomorphism
\begin{equation}
\varphi_f(R\pi_*G) \simeq R(\pi|_{Y_0})_*\varphi_g(G),
\end{equation}
where the morphism $\pi|_{Y_0} : Y_0 \rightarrow X_0$ is induced by $\pi$.

Recall that for an algebraic variety $X$ over $\mathbb{C}$ the category $\text{Perv}(X)$ of perverse sheaves on it is a full subcategory of $D^b_c(X)$. Here we use the convention that for smooth $X$ the shifted constant sheaf $\mathbb{C}^X[\dim X] \in D^b_c(X)$ is perverse. The following result is a very special case of [5, Corollary 5.2.17].

Lemma 2.3. — Let $X$ be an algebraic variety $X$ over $\mathbb{C}$ and $Y \subset X$ a hypersurface in it. Set $U = X \setminus Y$ and let $j : U \hookrightarrow X$ be the inclusion map. Then the functors
\begin{equation}
j_!, Rj_* : D^b_c(U) \rightarrow D^b_c(X)
\end{equation}
preserve the perversity.

Now for $F \in D^b_c(X)$ let $S : X = \bigsqcup_{\alpha \in A} X_\alpha$ be a Whitney stratification of $X$ adapted to it. Then for a non-constant regular function $f : X \rightarrow \mathbb{C}$ on $X$ we define a subset $\text{Sing}_S(f) \subset X$ of $X$ by
\begin{equation}
\text{Sing}_S(f) = \bigsqcup_{\alpha \in A} \text{Sing}(f|_{X_\alpha}) \subset X.
\end{equation}
We call it the stratified singular locus of $f$ with respect to $S$ (see [5, Definition 4.2.7]). By the Whitney condition on $S$ it is a closed algebraic subset of $X$. By [5, Proposition 4.2.8] we have
\begin{equation}
\text{supp} \varphi_f(F) \subset X_0 \cap \text{Sing}_S(f).
\end{equation}
Recall also that the shifted vanishing cycle functor
\begin{equation}
p\varphi_f(\cdot) := \varphi_f(\cdot)[-1] : D^b_c(X) \rightarrow D^b_c(X_0)
\end{equation}
preserves the perversity. Then we obtain the following result (see the proofs of [5, Propositions 6.1.1 and 6.1.2]).

Lemma 2.4. — Assume that $F$ is perverse and the dimension of $X_0 \cap \text{Sing}_S(f)$ is zero. Then we have the concentration
\begin{equation}
H^l\{p\varphi_f(F)\} \simeq 0 \quad (l \neq 0).
\end{equation}
Proof. — By our assumption the perverse sheaf $p_\varphi f(\mathcal{F}) \in \text{Perv}(X_0)$ is supported on some points in $X_0$. Then the desired concentration follows immediately from the perversity of $p_\varphi f(\mathcal{F})$ (see [11, Proposition 8.1.22]).

The following lemma will be used in the proofs of our main theorems. Let $\tau$ be a strictly convex rational polyhedral cone in $\mathbb{R}^n$ and $\Sigma_\tau$ the fan in $\mathbb{R}^n$ formed by all its faces. Denote by $X_{\Sigma_\tau}$ the ($n$-dimensional) toric variety associated to $\Sigma_\tau$ (see [8] and [21] etc.).

**Lemma 2.5. —** In the above situation, assume also that $\tau$ is simplicial or satisfies the condition $\dim \tau \leq 3$. Then the constant sheaf $\mathbb{C}_{X_{\Sigma_\tau}}$ on $X_{\Sigma_\tau}$ is perverse (up to some shift).

Proof. — If $\tau$ is simplicial, then $X_{\Sigma_\tau}$ is an orbifold (see [8, p. 34]) and the assertion follows from [11, Proposition 8.2.21]. It is the case when $\dim \tau \leq 2$. Assume that $\dim \tau = 3$. Let $T_\tau \simeq (\mathbb{C}^*)^{n-\dim \tau} \subset X_{\Sigma_\tau}$ be the (minimal) $T$-orbit in $X_{\Sigma_\tau}$ associated to $\tau \in \Sigma_\tau$ and $i_\tau : T_\tau \hookrightarrow X_{\Sigma_\tau}$, $j_\tau : X_{\Sigma_\tau} \setminus T_\tau \hookrightarrow X_{\Sigma_\tau}$ the inclusion maps. Then by Fiesler [7, Theorems 1.1 and 1.2] we obtain

\begin{equation}
H^l i_\tau^{-1} R(j_\tau)_* \mathbb{C}_{X_{\Sigma_\tau} \setminus T_\tau} \simeq \begin{cases} 
\mathbb{C}_{T_\tau} & (l = 0), \\
0 & (l = 1).
\end{cases}
\end{equation}

This implies that we have

\begin{equation}
H^l i_\tau^! C_{X_{\Sigma_\tau}} \simeq 0 \quad (l < 3 = \text{codim } T_\tau).
\end{equation}

Then the assertion follows from [11, Proposition 8.1.22].

3. Some compactifications of $\mathbb{C}^n$

In this section, we recall the constructions of some smooth compactifications of $\mathbb{C}^n$ in Zaharia [30] and Takeuchi–Tibăr [26]. Let $f(x) = \sum_{v \in \mathbb{Z}^n_+} a_v x^v$ be a polynomial on $\mathbb{C}^n$ ($a_v \in \mathbb{C}$).

**Definition 3.1.**

1. We call the convex hull of $\text{supp}(f) := \{v \in \mathbb{Z}^n_+ | a_v \neq 0\} \subset \mathbb{Z}^n_+ \subset \mathbb{R}^n_+$ in $\mathbb{R}^n$ the Newton polytope of $f$ and denote it by $NP(f)$.
2. (see [14] etc.) We call the convex hull of $\{0\} \cup NP(f)$ in $\mathbb{R}^n$ the Newton polyhedron at infinity of $f$ and denote it by $\Gamma_\infty(f)$.
For an element $u \in \mathbb{R}^n$ of (the dual vector space of) $\mathbb{R}^n$ define the supporting face $\gamma_u \prec \Gamma_\infty(f)$ of $u$ in $\Gamma_\infty(f)$ by

$$\gamma_u = \left\{ v \in \Gamma_\infty(f) \left| \langle u, v \rangle = \min_{w \in \Gamma_\infty(f)} \langle u, w \rangle \right. \right\}. \quad (3.1)$$

Then we introduce an equivalence relation $\sim$ on (the dual vector space of) $\mathbb{R}^n$ by $u \sim u' \iff \gamma_u = \gamma_{u'}$. We can easily see that for any face $\gamma \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$ the closure of the equivalence class associated to $\gamma$ in $\mathbb{R}^n$ is an $(n - \dim \gamma)$-dimensional rational convex polyhedral cone $\sigma(\gamma)$ in $\mathbb{R}^n$. Moreover the family $\{ \sigma(\gamma) \mid \gamma \prec \Gamma_\infty(f) \}$ of cones in $\mathbb{R}^n$ thus obtained is a subdivision of $\mathbb{R}^n$. We call it the dual subdivision of $\mathbb{R}^n$ by $\Gamma_\infty(f)$. If $\dim \Gamma_\infty(f) = n$ it satisfies the axiom of fans (see [8] and [21] etc.). We call it the dual fan of $\Gamma_\infty(f)$.

We have the following two classical definitions due to Kouchnirenko:

**Definition 3.2 ([13]).** — Let $\partial f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be the map defined by $\partial f(x) = (\partial_1 f(x), \ldots, \partial_n f(x))$. Then we say that $f$ is tame at infinity if the restriction $(\partial f)^{-1}(B(0; \varepsilon)) \rightarrow B(0; \varepsilon)$ of $\partial f$ to a sufficiently small ball $B(0; \varepsilon)$ centered at the origin $0 \in \mathbb{C}^n$ is proper.

**Definition 3.3 ([13]).** — We say that the polynomial

$$f(x) = \sum_{v \in \mathbb{Z}_n^+} a_v x^v \quad (a_v \in \mathbb{C})$$

is non-degenerate at infinity if for any face $\gamma$ of $\Gamma_\infty(f)$ such that $0 \notin \gamma$ the complex hypersurface $\{ x \in (\mathbb{C}^*)^n \mid f_\gamma(x) = 0 \}$ in $(\mathbb{C}^*)^n$ is smooth and reduced, where we defined the $\gamma$-part $f_\gamma$ of $f$ by $f_\gamma(x) = \sum_{v \in \gamma \cap \mathbb{Z}_n^+} a_v x^v$.

Broughton showed in [3] that if $f$ is non-degenerate at infinity and convenient then it is tame at infinity. This implies that the reduced homology of the general fiber of $f$ is concentrated in dimension $n - 1$. The concentration result was later extended to polynomial functions with isolated singularities with respect to some fiber-compactifying extension of $f$ by Siersma and Tibăr [24] and by Tibăr [28, Theorem 4.6, Corollary 4.7]. In this paper we mainly consider non-convenient polynomials.

**Definition 3.4 ([26]).** — We say that a face $\gamma \prec \Gamma_\infty(f)$ is atypical if $0 \in \gamma$, $\dim \gamma \geq 1$ and the cone $\sigma(\gamma) \subset \mathbb{R}^n$ which corresponds to it in the dual subdivision of $\Gamma_\infty(f)$ is not contained in the first quadrant $\mathbb{R}_+^n$ of $\mathbb{R}^n$.

This definition is related to that of the bad faces of $NP(f - f(0))$ in Némethi–Zaharia [19] as follows. If $\Delta \prec NP(f - f(0))$ is a bad face of $NP(f - f(0))$, then the convex hull $\gamma$ of $\{0\} \cup \Delta$ in $\mathbb{R}^n$ is an atypical
one of $\Gamma_\infty(f)$. Conversely, if $\gamma \prec \Gamma_\infty(f)$ is an atypical face and $\Delta = \gamma \cap NP(f - f(0)) \prec NP(f - f(0))$ satisfies the condition $\dim \Delta = \dim \gamma$ then $\Delta$ is a bad face of $NP(f - f(0))$.

**Example 3.5.** — Let $n = 3$ and consider a non-convenient polynomial $f(x, y, z)$ on $\mathbb{C}^3$ whose Newton polyhedron at infinity $\Gamma_\infty(f)$ is the convex hull of the points $(2, 0, 0), (2, 2, 0), (2, 2, 3) \in \mathbb{R}^3_+$ and the origin $0 = (0, 0, 0) \in \mathbb{R}^3$. Then the line segment connecting the point $(2, 2, 0)$ (resp. $(2, 0, 0))$ and the origin $0 \in \mathbb{R}^3$ is an atypical face of $\Gamma_\infty(f)$. However the triangle whose vertices are the points $(2, 0, 0), (2, 2, 0)$ and the origin $0 \in \mathbb{R}^3$ is not so. Note that for the line segment $\gamma$ connecting $(2, 0, 0)$ and the origin we have $\dim \sigma(\gamma) \cap \mathbb{R}^3_+ = 2$.

From now we recall the smooth compactifications of $\mathbb{C}^n$ in [26] and [30] (for their applications to monodromies at infinity see [6, 17, 18] and [26]). Assume that the polynomial $f(x) = \sum_{v \in \mathbb{Z}^n} a_v x^v \in \mathbb{C}[x_1, \ldots, x_n]$ is “non-convenient” and $\dim \Gamma_\infty(f) = n$. Let $\Sigma_0$ be the dual fan of $\Gamma_\infty(f)$. Assume also that $f$ is non-degenerate at infinity. We consider $\mathbb{C}^n$ as a toric variety associated with the fan $\Xi$ in $\mathbb{R}^n$ formed by all the faces of the first quadrant $\mathbb{R}^n_+ \subset \mathbb{R}^n$. Denote by $T \simeq (\mathbb{C}^*)^n$ the open dense torus in it. Let $\Sigma_1$ be a subdivision of the dual fan $\Sigma_0$ of $\Gamma_\infty(f)$ which contains $\Xi$ as its subfan. Then we can construct a smooth subdivision $\Sigma$ of $\Sigma_1$ without subdividing the cones in $\Xi$ (see e.g. [22, Lemma (2.6), Chapter II, p. 99]). This implies that the toric variety $X_{\Sigma}$ associated with $\Sigma$ is a smooth compactification of $\mathbb{C}^n$. This construction of $X_{\Sigma}$ coincides with the one in Zaharia [30]. Recall that $T$ acts on $X_{\Sigma}$ and the $T$-orbits are parametrized by the cones in $\Sigma$. For a cone $\sigma \in \Sigma$ denote by $T_\sigma \simeq (\mathbb{C}^*)^{n - \dim \sigma}$ the corresponding $T$-orbit. If $\sigma^\perp \simeq \mathbb{R}^{n - \dim \sigma}$ is the orthogonal complement of (the affine span of) $\sigma$ we have $T_\sigma = \text{Spec}(\mathbb{C}[\sigma^\perp \cap \mathbb{Z}^n])$. There exist also natural affine open subsets $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n$ of $X_{\Sigma}$ associated to $n$-dimensional cones $\sigma$ in $\Sigma$ as follows. Let $\sigma$ be an $n$-dimensional (smooth) cone in $\Sigma$ and $\{w_1, \ldots, w_n\} \subset \mathbb{Z}^n$ the set of the (non-zero) primitive vectors on the edges of $\sigma$. Let $\sigma^o$ be the dual cone of $\sigma$. Then by the smoothness of $\sigma$ the semigroup ring $\mathbb{C}[\sigma^o \cap \mathbb{Z}^n]$ is isomorphic to the polynomial ring $\mathbb{C}[y_1, \ldots, y_n]$. This implies that the affine open subset $\mathbb{C}^n(\sigma) := \text{Spec}(\mathbb{C}[\sigma^o \cap \mathbb{Z}^n])$ of $X_{\Sigma}$ is isomorphic to $\mathbb{C}^n_y$. Moreover, on $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$ the function $f(x) = \sum_{v \in \mathbb{Z}^n_+} a_v x^v$ has the following form:

\begin{equation}
(3.2) \quad f(y) = \sum_{v \in \mathbb{Z}^n_+} a_v y_1^{(w_1, v)} \cdots y_n^{(w_n, v)} = y_1^{b_1} \cdots y_n^{b_n} \times f_\sigma(y),
\end{equation}
where we set
\begin{equation}
(3.3) \quad b_i = \min_{v \in \Gamma_{\infty}(f)} \langle w_i, v \rangle \leq 0 \quad (i = 1, 2, \ldots, n)
\end{equation}
and $f_{\sigma}(y)$ is a polynomial on $\mathbb{C}^n(\sigma) \simeq \mathbb{C}_y^n$. In $\mathbb{C}^n(\sigma) \simeq \mathbb{C}_y^n$ the hypersurface $Z := f^{-1}(0) \subset X_\Sigma$ is explicitly written as $\{y \in \mathbb{C}^n(\sigma) \mid f_{\sigma}(y) = 0\}$. By (3.2) we see that $f$ is extended to a meromorphic function on $\mathbb{C}^n(\sigma) \simeq \mathbb{C}_y^n$. The variety $X_\Sigma$ is covered by such affine open subsets. Let $\tau$ be a $d$-dimensional face of the $n$-dimensional cone $\sigma \in \Sigma$. For simplicity, assume that $w_1, \ldots, w_d$ generate $\tau$. Then in the affine chart $\mathbb{C}^n(\sigma) \simeq \mathbb{C}_y^n$ the $T$-orbit $T_\tau$ associated to $\tau$ is explicitly defined by
\begin{equation}
T_\tau = \{(y_1, \ldots, y_n) \in \mathbb{C}^n(\sigma) \mid y_1 = \cdots = y_d = 0, \ y_{d+1}, \ldots, y_n \neq 0\}
\end{equation}
\begin{equation}
\simeq (\mathbb{C}^*)^{n-d}.
\end{equation}
Hence we have
\begin{equation}
X_\Sigma = \bigcup_{\dim \sigma = n} \mathbb{C}^n(\sigma) = \bigsqcup_{\tau \in \Sigma} T_\tau.
\end{equation}
Now $f$ extends to a meromorphic function on $X_\Sigma$, which may still have points of indeterminacy. For simplicity we denote this meromorphic extension also by $f$. From now on, we will eliminate its points of indeterminacy by blowing up $X_\Sigma$ (see [17, Section 3] and [18, Section 3] for the details). For a cone $\sigma$ in $\Sigma$ by taking a non-zero vector $u$ in the relative interior rel. int$(\sigma)$ of $\sigma$ we define a face $\gamma_\sigma$ of $\Gamma_{\infty}(f)$ by $\gamma_\sigma = \gamma_u$. Note that $\gamma_\sigma$ does not depend on the choice of $u \in$ rel. int$(\sigma)$. We call it the supporting face of $\sigma$ in $\Gamma_{\infty}(f)$. Following Libgober–Sperber [14], we say that a $T$-orbit $T_\sigma$ in $X_\Sigma$ (or a cone $\sigma \in \Sigma$) is at infinity if the supporting face $\gamma_\sigma \prec \Gamma_{\infty}(f)$ satisfies the condition $0 \notin \gamma_\sigma$. We can easily see that $f$ has poles on the union of $T$-orbits at infinity as follows. Let $\rho_1, \rho_2, \ldots, \rho_r$ be the $1$-dimensional cones at infinity in $\Sigma$. Then $T_{\rho_1}, T_{\rho_2}, \ldots, T_{\rho_r}$ are the $(n-1)$-dimensional $T$-orbits at infinity in $X_\Sigma$. For any $i = 1, 2, \ldots, r$ the toric divisor $D_i := T_{\rho_i}$ is a smooth hypersurface in $X_\Sigma$. Let us denote the (unique non-zero) primitive vector in $\rho_i \cap \mathbb{Z}^n$ by $u_i$. Then the order $a_i > 0$ of the pole of $f$ along $D_i$ is given by
\begin{equation}
(3.5) \quad a_i = -\min_{v \in \Gamma_{\infty}(f)} \langle u_i, v \rangle.
\end{equation}
From this we see that the poles of $f$ are contained in the normal crossing divisor $D := D_1 \cup \cdots \cup D_r$. Moreover by the non-convenience of $f$, there exist some cones $\sigma \in \Sigma$ such that $\sigma \notin \Xi$ and $0 \in \gamma_\sigma$ i.e. $\gamma_\sigma$ is an atypical face of $\Gamma_{\infty}(f)$. For such $\sigma$ the function $f$ extends holomorphically to a neighborhood of $T_\sigma \subset X_\Sigma \setminus \mathbb{C}^n$. For this reason we call them
horizontal $T$-orbits in $X_\Sigma$ (in the tame case where $f$ is convenient, they do not appear). Note also that by the non-degeneracy at infinity of $f$, for any non-empty subset $I \subset \{1, 2, \ldots, r\}$ the hypersurface $Z = \bigcap_{i \in I} D_i$ intersects $D_I := \bigcap_{i \in I} D_i$ transversally (or the intersection is empty). At such intersection points, $f$ has indeterminacy. We can easily see that the meromorphic extension of $f$ to $X_\Sigma$ has points of indeterminacy in the subvariety $D \cap Z$ of $X_\Sigma$ of codimension two. Now, in order to eliminate the indeterminacy of the meromorphic function $f$ on $X_\Sigma$, we first consider the blow-up $\pi_1 : X^{(1)}_\Sigma \to X_\Sigma$ of $X_\Sigma$ along the $(n-2)$-dimensional smooth subvariety $D \cap Z$. Then the indeterminacy of the pull-back $f \circ \pi_1$ of $f$ to $X^{(1)}_\Sigma$ is improved. If $f \circ \pi_1$ still has points of indeterminacy on the intersection of the exceptional divisor $E_1$ of $\pi_1$ and the proper transform $Z^{(1)}$ of $Z$, we construct the blow-up $\pi_2 : X^{(2)}_\Sigma \to X^{(1)}_\Sigma$ of $X^{(1)}_\Sigma$ along $E_1 \cap Z^{(1)}$. By repeating this procedure $a_1$ times, we obtain a tower of blow-ups

\[
X^{(a_1)}_\Sigma \to \cdots \to X^{(1)}_\Sigma \to X_\Sigma.
\]

For the details see the figures in [17, p. 420]. Then the pull-back of $f$ to $X^{(a_1)}_\Sigma$ has no indeterminacy over $T_{\rho_1}$. It also extends to a holomorphic function on (an open dense subset of) the exceptional divisor of the last blow-up $\pi_{a_1}$. For this reason we call it and its proper transform $F_1$ in the variety $\tilde{X}_\Sigma$ that we construct below horizontal exceptional divisors. Note that for any $t \in \mathbb{C}$ the closure of the hypersurface $f^{-1}(t) \subset \mathbb{C}^n$ in $X^{(a_1)}_\Sigma$ intersects $F_1$ transversally. Moreover it does not intersect the other exceptional divisors.

Next we apply this construction to the proper transforms of $D_2$ and $Z$ in $X^{(a_1)}_\Sigma$. Then we obtain also a tower of blow-ups

\[
X^{(a_1)(a_2)}_\Sigma \to \cdots \to X^{(a_1)(1)}_\Sigma \to X^{(a_1)}_\Sigma
\]

and the indeterminacy of the pull-back of $f$ to $X^{(a_1)(a_2)}_\Sigma$ is eliminated over $T_{\rho_1} \cup T_{\rho_2}$. By applying the same construction to (the proper transforms of) $D_3, D_4, \ldots, D_r$, we finally obtain a proper morphism $\pi : \tilde{X}_\Sigma \to X_\Sigma$ such that $g := f \circ \pi$ has no point of indeterminacy on the whole $\tilde{X}_\Sigma$. Note that the smooth compactification $\tilde{X}_\Sigma$ of $\mathbb{C}^n$ thus obtained is not a toric variety any more. By constructing a blow-up $\tilde{X}_\Sigma \to X_\Sigma$ of $X_\Sigma$ to eliminate the
of holomorphic maps, where \( \iota : \mathbb{C}^n \hookrightarrow \widetilde{X}_\Sigma \) and \( j : \mathbb{C} \hookrightarrow \mathbb{P}^1 \) are the inclusion maps and \( g \) is proper. On \( \widetilde{X}_\Sigma \) we have constructed also \( r \) (smooth) horizontal exceptional divisors \( F_1, F_2, \ldots, F_r \). The other exceptional divisors in \( \widetilde{X}_\Sigma \) are called intermediate exceptional divisors. By our construction of the blow-up \( \pi : \widetilde{X}_\Sigma \rightarrow X_\Sigma \), \( F_1 \cup F_2 \cup \cdots \cup F_r \) is a normal crossing divisor in \( \widetilde{X}_\Sigma \) and for any non-empty subset \( I \subset \{1,2,\ldots,r\} \) and \( t \in \mathbb{C} \) the hypersurface \( g^{-1}(t) \subset \widetilde{X}_\Sigma \) intersects \( F_I := \cap_{i \in I} F_i \) transversally. Moreover \( g^{-1}(t) \) does not intersect intermediate exceptional divisors. For a point \( b \in \mathbb{C} \) define a function \( h : \mathbb{C} \rightarrow \mathbb{C} \) on \( \mathbb{C} \) by \( h(t) = t - b \) so that we have \( h^{-1}(0) = \{b\} \). Then by the above-mentioned property of \( F_i \) and (2.10) the support of the constructible sheaf \( \varphi_{\text{hor}}(\iota_!\mathcal{C}_{\mathbb{C}^n}) \) does not intersect the union of the exceptional divisors in \( \pi : \widetilde{X}_\Sigma \rightarrow X_\Sigma \). Moreover, for the pole divisor \( D = D_1 \cup \cdots \cup D_r \subset X_\Sigma \) of (the meromorphic extension of) \( f \) to \( X_\Sigma \), the support does not intersect \( \pi^{-1}(D) \).

4. Bifurcation sets of polynomial functions

In this section we study the bifurcation values of polynomial functions. Let \( f : \mathbb{C}^n \rightarrow \mathbb{C} \) be a polynomial function. Throughout this section we assume that \( f \) is non-degenerate at infinity and \( \dim \Gamma_\infty(f) = n \). Let \( \Sigma_0 \) be the dual fan of \( \Gamma_\infty(f) \). Let \( \gamma_1, \ldots, \gamma_m \) be the atypical faces of \( \Gamma_\infty(f) \). For \( 1 \leq i \leq m \) let \( K_i \subset \mathbb{C} \) be the set of the critical values of the \( \gamma_i \)-part

\[
f_{\gamma_i} : T = (\mathbb{C}^*)^n \rightarrow \mathbb{C}
\]

of \( f \). We denote by \( \text{Sing} \, f \subset \mathbb{C}^n \) the set of the critical points of \( f : \mathbb{C}^n \rightarrow \mathbb{C} \) and set

\[
K_f = f(\text{Sing} \, f) \cup \{f(0)\} \cup \left( \bigcup_{i=1}^m K_i \right).
\]

Then the following result was obtained by Némethi–Zaharia [19].

**Theorem 4.1** (Némethi–Zaharia [19]). — In the situation above, we have \( B_f \subset K_f \).
Remark 4.2. — If for an atypical face $\gamma_i$ of $\Gamma_\infty(f)$ the face $\Delta = \gamma_i \cap NP(f - f(0)) < NP(f - f(0))$ of $NP(f - f(0))$ is not bad in the sense of Némethi–Zaharia [19], then $dim\,NP(f_{\gamma_i} - f(0)) = dim\,\Delta < dim\,\gamma_i$, $f_{\gamma_i} - f(0)$ is a positively homogeneous Laurent polynomial on $T = (\mathbb{C}^*)^n$ and we have $K_i = \{f(0)\}$. Therefore the above inclusion $B_f \subset K_f$ coincides with the one in [19].

Moreover the authors of [19] proved the equality $B_f = K_f$ for $n = 2$ and conjectured its validity in higher dimensions. Later Zaharia [30] proved it for any $n \geq 2$ but under some supplementary assumptions on $f$. By using the definitions and the notations in Section 1 we can improve his result as follows.

**Theorem 4.3.** — Assume that $f$ has isolated singularities at infinity over $b \in K_f \setminus [f(\text{Sing} f) \cup \{f(0)\}]$ and for any $1 \leq i \leq m$ such that $b \in K_i$ the relative interior $\text{rel. int.}(\gamma_i)$ of $\gamma_i \prec \Gamma_\infty(f)$ is contained in $\text{Int}(\mathbb{R}_n^n)$. Assume also that there exists $1 \leq i \leq m$ such that $b \in K_i$ and $\gamma_i \prec \Gamma_\infty(f)$ is relatively simple. Then we have $E_f(b) > 0$ and hence $b \in B_f$.

**Proof.** — By our assumption, for any $1 \leq i \leq m$ the hypersurface $f_{\gamma_i}^{-1}(b) \subset T_i \simeq (\mathbb{C}^*)^{dim\,\gamma_i}$ in $T_i = \text{Spec}(\mathbb{C}[L_{\gamma_i} \cap \mathbb{Z}^n])$ has only isolated singular points at $p_{i,1}, \ldots, p_{i,n_i}$. Here some $n_i$ can be zero. Obviously we have $n_i > 0$ if and only if $b \in K_i$. From now we shall freely use the smooth compactification $\overline{X}_\Sigma$ of $\mathbb{C}^n$ and the notations related to it in Section 3. Let $\text{Cone}_\infty(f) \subset \mathbb{R}_n^n$ be the cone generated by $\Gamma_\infty(f)$. We define its dual cone $C \subset \mathbb{R}_n^n$ by

$$C = \{u \in \mathbb{R}^n \mid \langle u, v \rangle \geq 0 \text{ for any } v \in \text{Cone}_\infty(f)\}. \tag{4.3}$$

Then a cone $\sigma \in \Sigma$ is at infinity if and only if it is not contained in $C$. We shall prove that the jump $E_f(b) \in \mathbb{Z}$ of the constructible function on $\mathbb{C}$

$$\chi_{C}(t) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j_C(f^{-1}(t); \mathbb{C}) \quad (t \in \mathbb{C}) \tag{4.4}$$

at the point $b \in K_f \setminus [f(\text{Sing} f) \cup \{f(0)\}]$ is positive. For the point $b \in \mathbb{C}$ define a function $h : \mathbb{C} \to \mathbb{C}$ on $\mathbb{C}$ by $h(t) = t - b$ so that we have $h^{-1}(0) = \{b\}$. Then we have

$$E_f(b) = (-1)^{n-1} \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j C(h(R_f \mathbb{C}^n)\,b, \tag{4.5}$$
where $\varphi_h : D^b_c(\mathbb{C}) \rightarrow D^b_c(\{b\})$ is Deligne’s vanishing cycle functor associated to $h$. Since we have $f = g \circ \iota$ on a neighborhood of $b \in K_f \setminus [f(Sing f) \cup \{f(0)\}]$ and $g$ is proper, by Proposition 2.2 we obtain an isomorphism

$$
\varphi_h(Rf_!\mathcal{C}^n) \cong R(g|_{g^{-1}(b)})_* \varphi_{h \circ g}(\iota_!\mathcal{C}^n).
$$

This implies that for the constructible function $\chi\{\varphi_{h \circ g}(\iota_!\mathcal{C}^n)\} \in F_Z(g^{-1}(b))$ on $g^{-1}(b) = (h \circ g)^{-1}(0) \subset X_{\Sigma}$ we have

$$
\sum_{j \in \mathbb{Z}} (-1)^j \dim H^j \varphi_h(Rf_!\mathcal{C}^n)_b = \int_{g^{-1}(b)} \chi\{\varphi_{h \circ g}(\iota_!\mathcal{C}^n)\}.
$$

Hence for the calculation of $E_f(b)$, it suffices to calculate

$$
\chi\{\varphi_{h \circ g}(\iota_!\mathcal{C}^n)\}(p) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j \varphi_{h \circ g}(\iota_!\mathcal{C}^n)_p
$$

at each point $p$ of $g^{-1}(b)$. Let $\Sigma_C$ (resp. $\Sigma_C'$) be the fan formed by all the faces of the cone $C$ (resp. by all the cones in $\Sigma$ contained in $C$) and denote by $X_{\Sigma_C}$ (resp. $X_{\Sigma_C'}$) the possibly singular (resp. smooth) toric variety associated to it. Then $X := X_{\Sigma_C} = \bigcup_{\sigma \subset C} T_{\sigma}$ is an open subset of $X_{\Sigma}$ and there exists a natural proper morphism

$$
\pi : X = X_{\Sigma_C'} \rightarrow X_{\Sigma_C}
$$

of toric varieties. Note that for the pole divisor $D$ of (the meromorphic extension of) $f$ to $X_{\Sigma}$ (see Section 3) we have $X = X_{\Sigma} \setminus D$. Recall also that the centers of the blow-ups in the construction of $X_{\Sigma} \rightarrow X_{\Sigma}$ are above $D = X_{\Sigma} \setminus X$. Hence we can consider $X$ also as an open subset of $X_{\Sigma}$. Since the Newton polytope $NP(f)$ of $f$ is contained in the dual cone $C^0 = Cone_\infty(f)$ of $C$ and

$$
X_{\Sigma_C} = \text{Spec}(\mathbb{C}[C^0 \cap \mathbb{Z}^n]),
$$

we can naturally regard $f$ as regular functions on $X_{\Sigma_C}$ and $X = X_{\Sigma_C'}$. This implies that $X = X_{\Sigma_C'}$ is an open subset of $g^{-1}(C) \cap X_{\Sigma}$. In particular, if $\sigma \in \Sigma_C'$ is not contained in $\mathbb{R}^n_+$ then $T_{\sigma} \subset X \setminus \mathbb{C}^n$ and $f$ extends holomorphically to $T_{\sigma}$. Namely $T_{\sigma}$ is a horizontal $T$-orbit in $X \setminus \mathbb{C}^n$. By our assumption $b \notin f(Sing f)$ and the result at the end of Section 3, we see also that the support of the constructible sheaf $\varphi_{h \circ g}(\iota_!\mathcal{C}^n) \in D^b_c(g^{-1}(b))$ is contained in $(X \setminus \mathbb{C}^n) \cap g^{-1}(b)$. We thus obtain an equality

$$
E_f(b) = (-1)^{n-1} \int_{(X \setminus \mathbb{C}^n) \cap g^{-1}(b)} \chi\{\varphi_{h \circ g}(\iota_!\mathcal{C}^n)\}.
$$

Namely, for the calculation of $E_f(b)$ it suffices to calculate the constructible function $\chi\{\varphi_{h \circ g}(\iota_!\mathcal{C}^n)\}$ only on $T$-orbits in $X \setminus \mathbb{C}^n$ associated to the cones.
\[
\sigma \in \Sigma'_C \subset \Sigma \text{ such that rel. int}(\sigma) \subset C \setminus \mathbb{R}^n_+. \text{ For } \sigma \in \Sigma'_C \subset \Sigma \text{ such that rel. int}(\sigma) \subset \text{Int}(C) \setminus \mathbb{R}^n_+ \text{ we have } \gamma_\sigma = \{0\} \prec \Gamma_\infty(f) \text{ and the restriction of } g|_X : X \to \mathbb{C} \text{ to the } T\text{-orbit } T_\sigma \subset X \text{ is the constant function } f(0) \in \mathbb{C}. \text{ Hence we get } g^{-1}(b) \cap T_\sigma = \emptyset \text{ for the point } b \in K_f \setminus [f(\text{Sing } f) \cup \{f(0)\}].\]

For \(1 \leq i \leq m\) let \(\sigma_i = \sigma(\gamma_i) \in \Sigma_0\) be the cone which corresponds to \(\gamma_i\) in the dual fan \(\Sigma_0\) of \(\Gamma_\infty(f)\). Recall that by the definition of atypical faces we have \(0 \in \gamma_i\) and the face \(\sigma_i \prec C\) of \(C\) is not contained in \(\mathbb{R}^n_+\). For \(\sigma \in \Sigma'_C \subset \Sigma\) such that rel. int(\(\sigma\)) \(\subset \partial C \setminus \mathbb{R}^n_+\) there exists unique \(1 \leq i \leq m\) for which we have rel. int(\(\sigma\)) \(\subset\) rel. int(\(\sigma_i\)). If \(\dim \sigma = \dim \sigma_i\) we have an isomorphism \(T_\sigma \simeq T_i = \text{Spec}(\mathbb{C}[L_{\gamma_i} \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim \gamma_i}\) and the restriction of \(g|_X : X \to \mathbb{C}\) to \(T_\sigma \subset X\) is naturally identified with \(f_{\gamma_i} : T_i \to \mathbb{C}\). This implies that the hypersurface \(g^{-1}(b) \cap T_\sigma \subset T_\sigma \simeq T_i\) has only isolated singular points \(p_{i,1}, \ldots, p_{i,n_i} \in T_\sigma \simeq T_i\) and

\[
T_\sigma \cap \text{supp } \varphi_{\text{hog}}(t_i \mathbb{C}^n) \subset \{p_{i,1}, \ldots, p_{i,n_i}\} \tag{4.12}
\]

in this case. On the other hand, if \(\dim \sigma < \dim \sigma_i\) we have \(\dim T_\sigma > \dim T_i\) and for the hypersurface \(g^{-1}(b) \cap T_\sigma \subset T_\sigma\) there exists an isomorphism

\[
g^{-1}(b) \cap T_\sigma \simeq f_{\gamma_i}^{-1}(b) \times (\mathbb{C}^*)^{\dim T_\sigma - \dim T_i}. \tag{4.13}
\]

This implies that \(g^{-1}(b) \cap T_\sigma \subset T_\sigma\) has non-isolated singular points if \(n_i > 0\). From now on, we shall overcome this difficulty by using Proposition 2.2. For \(1 \leq i \leq m\) let \(\Sigma_i\) be the fan in \(\mathbb{R}^n\) formed by all the faces of \(\sigma_i\) and denote by \(X_{\Sigma_i}\) the (possibly singular) toric variety associated to it. Then \(X_{\Sigma_i}\) is an open subset of \(X_{\Sigma_C}\). Let \(\sigma^\circ_i \subset \mathbb{R}^n\) be the dual cone of \(\sigma_i\) in \(\mathbb{R}^n\). Then \(\sigma_{\gamma_i} \simeq C_i \times \mathbb{R}^{\dim \gamma_i}\) for a proper convex cone \(C_i\) in \(\mathbb{R}^{n-\dim \gamma_i}\) and we have an isomorphism

\[
X_{\Sigma_i} \simeq \text{Spec}(\mathbb{C}[\sigma_{\gamma_i}^\circ \cap \mathbb{Z}^n]). \tag{4.14}
\]

Note that the (minimal) \(T\)-orbit \(T_{\sigma_i}\) in \(X_{\Sigma_i}\) which corresponds to \(\sigma_i \in \Sigma_i\) is naturally identified with \(T_i = \text{Spec}(\mathbb{C}[L_{\gamma_i} \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim \gamma_i}\). More precisely \(X_{\Sigma_i}\) is the product \(X_i \times T_{\sigma_i}\) of the \((n - \dim \gamma_i)\)-dimensional affine toric variety \(X_i = \text{Spec}(\mathbb{C}[C_i \cap \mathbb{Z}^{n-\dim \gamma_i}])\) and \(T_{\sigma_i} \simeq T_i \simeq (\mathbb{C}^*)^{\dim \gamma_i}\). Since \(NP(f) \subset \sigma_i^\circ\) and \(f \in \mathbb{C}[\sigma_{\gamma_i}^\circ \cap \mathbb{Z}^n]\), we can naturally regard \(f\) as a regular function on \(X_{\Sigma_i}\). We denote it by \(f_i : X_{\Sigma_i} \to \mathbb{C}\). For \(1 \leq i \leq m\) let \(\Sigma_i' \subset \Sigma\) be the subfan of \(\Sigma\) consisting of the cones in \(\Sigma\) contained in \(\sigma_i\) and denote by \(X_{\Sigma_i'}\) the smooth toric variety associated to it. Then \(X_{\Sigma_i'}\) is an open subset of \(X \subset \overline{X_{\Sigma}}\) and there exists a proper morphism

\[
\pi_i : X_{\Sigma_i'} \to X_{\Sigma_i} \tag{4.15}
\]
of toric varieties. Moreover we have a commutative diagram

\[
\begin{array}{c}
X_{\Sigma_i'} \longrightarrow X = X_{\Sigma_C} \\
\pi_i \downarrow \quad \quad \quad \downarrow \\
X_{\Sigma_i} \longrightarrow X_{\Sigma_C}
\end{array}
\]

(4.16)

such that \( \pi^{-1}X_{\Sigma_i} = X_{\Sigma_i'} \subset X \), where the horizontal arrows are the inclusion maps. It is also easy to see that the closed subset \((X \setminus \mathbb{C}^n) \cap g^{-1}(b)\) of \(X\) is covered by the affine open subvarieties \(X_{\Sigma_1}', \ldots, X_{\Sigma_m'} \subset X\). Note that for the restriction \(g_i = g|_{X_{\Sigma_i}'} : X_{\Sigma_i'} \longrightarrow \mathbb{C}\) of \(g|_X\) we have \(g_i = f_i \circ \pi_i\). Then by applying Proposition 2.2 to the proper morphism \(\pi_i : X_{\Sigma_i'} \longrightarrow X_{\Sigma_i}\) we obtain an isomorphism

\[
R(\pi_i g_i^{-1}(b)) \ast \mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_i'}} \simeq \mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_i'}} \simeq \mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_C}}.
\]

(4.17)

The advantage to consider \(\mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_i}} \simeq \mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_C}}\) instead of \(\mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_i}}\) is that its support is a discrete subset of \(f^{-1}(b) \subset X_{\Sigma_i} \subset X_{\Sigma_C}\) by our assumption that \(f\) has isolated singularities at infinity over \(b \in K_f \setminus \{f(\text{Sing } f) \cup \{f(0)\}\}\). Set

\[
\mathcal{F}_i = R(\pi_i) \ast (t!\mathcal{C}_n^{\mathbb{C}} \mid_{X_{\Sigma_i}'}) \simeq R(\pi_i)(t!\mathcal{C}_n^{\mathbb{C}} \mid_{X_{\Sigma_i}'}) \in \mathcal{D}^b_c(X_{\Sigma_i}).
\]

Then the topological integral

\[
\int_{g^{-1}(b)} \chi \{\mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_i}'}\} = \int_{(X \setminus \mathbb{C}^n) \cap g^{-1}(b)} \chi \{\mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_i}'}\}
\]

is equal to

\[
\sum_{i=1}^{m} \sum_{j=1}^{n_i} \chi \{\mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_i}'}\}.
\]

(4.19)

If \(b \notin K_i\) (\(\iff n_i = 0\)) we have \(\mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_i}'} \simeq 0\) on a neighborhood of \(T_{\sigma_i} \subset X_{\Sigma_i}\). Let us consider the remaining case where \(b \in K_i\) (\(\iff n_i > 0\)). Then by our assumption \(\text{rel.int} (\gamma_i) \subset \text{Int}(\mathbb{R}^n_+)\) we have \(\sigma_i \cap \mathbb{R}^n_+ = \{0\}\).

This implies that for the embedding \(t_i : T = (\mathbb{C}^*)^n \hookrightarrow X_{\Sigma_i}\) there exists an isomorphism \(\mathcal{F}_i \simeq (t_i)!\mathcal{C}_T\). Hence by Lemma 2.3, \(\mathcal{F}_i\) is a perverse sheaf on \(X_{\Sigma_i}\) (up to some shift). Since the support of \(\mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_i}'}\) is discrete, by Lemma 2.4 we thus obtain the concentration

\[
H^l \mathcal{F}_{\Sigma_i} \mid_{X_{\Sigma_i}'} \simeq 0 \quad (l \neq n - 1)
\]

(4.20)
for any $1 \leq j \leq n_i$. Set $\mu_{i,j} = \dim H^{n-1}_f \varphi_{h_0 f_i}(\mathcal{F}_i)_{p_{i,j}} \geq 0$. Then $E_f(b)$ can be expressed as a sum of non-negative integers as follows:

$$
(4.22) \quad E_f(b) = (-1)^{n-1} \int_{(X \setminus \mathbb{C}^n) \cap g^{-1}(b)} \chi\{\varphi_{h_0 g}(t_i \mathbb{C}_n)\} = \sum_{i=1}^m \sum_{j=1}^{n_i} \mu_{i,j}.
$$

By our assumption there exists $1 \leq i \leq m$ such that $n_i > 0 \iff b \in K_i$ and $\gamma_i \prec \Gamma_\infty(f)$ is relatively simple. Then the cone $\sigma_i \in \Sigma_0$ satisfies the condition $\sigma_i \cap \mathbb{R}_+^n = \{0\}$. For a face $\tau \prec \sigma_i$ of $\sigma_i$ we set $Y_\tau = \overline{T}_\tau \subset X_{\Sigma_i}$ and $f_\tau = f_i|_{Y_\tau} : Y_\tau \to \mathbb{C}$. Note that we have $T_{\sigma_i} = Y_{\sigma_i}$. Then for any $1 \leq j \leq n_i$ we can easily show that $(-1)^{n-1} \mu_{i,j} = \chi\{\varphi_{h_0 f_i}(\mathcal{F}_i)_{p_{i,j}}\} = \chi\{\varphi_{h_0 f_i}(t_i \mathbb{C}_T)_{p_{i,j}}\}$ is equal to the alternating sum

$$
(4.23) \quad \sum_{\tau \prec \sigma_i} (-1)^{\dim \tau} \chi\{\varphi_{h_0 f_\tau}(\mathbb{C}_{Y_\tau})_{p_{i,j}}\}.
$$

Here we used the additivity of the vanishing cycle functor $\varphi_{h_0 f_i}(\cdot)$. Since $\gamma_i$ is relatively simple, by Lemma 2.5 for any face $\tau \prec \sigma_i$ of $\sigma_i$ the constant sheaf $\mathbb{C}_{Y_\tau}$ on $Y_\tau$ is perverse (up to some shift). Moreover by our assumption that $f$ has isolated singularities at infinity over $b \in K_i \setminus \{f(\text{Sing } f) \cup \{f(0)\}\}$, the support of $\varphi_{h_0 f_i}(\mathbb{C}_{Y_\tau})$ is discrete on a neighborhood of $T_{\sigma_i} \subset X_{\Sigma_i}$. By Lemma 2.4 we thus obtain the concentration

$$
(4.24) \quad H^l \varphi_{h_0 f_i}(\mathbb{C}_{Y_\tau})_{p_{i,j}} \simeq 0 \quad (l \neq \dim Y_\tau - 1 = n - \dim \tau - 1)
$$

for any $1 \leq j \leq n_i$ and $\tau \prec \sigma_i$. Set

$$
(4.25) \quad \mu_{i,j,\tau} = \dim H^{n-\dim \tau-1}_f \varphi_{h_0 f_\tau}(\mathbb{C}_{Y_\tau})_{p_{i,j}} \geq 0.
$$

Then $\mu_{i,j} = (-1)^{n-1} \chi\{\varphi_{h_0 f_i}(\mathcal{F}_i)_{p_{i,j}}\} \geq 0$ is expressed as a sum of non-negative integers as follows:

$$
(4.26) \quad \mu_{i,j} = \sum_{\tau \prec \sigma_i} \mu_{i,j,\tau} \geq 0.
$$

Moreover the integer $\mu_{i,j,\sigma_i}$ is positive by the smoothness of $T_{\sigma_i} = Y_{\sigma_i}$. Consequently we get $E_f(b) > 0$. This completes the proof. \hfill \Box

In the generic (Newton non-degenerate) case, for any $1 \leq i \leq m$ and $1 \leq j \leq n_i$ we can explicitly calculate the above integer $\mu_{i,j} \geq 0$ by [16, Theorem 3.4, Corollary 3.6 and Remark 4.3] as follows. First by multiplying a monomial on $T_{\sigma_i} \simeq (\mathbb{C}^*)^{\dim \gamma_i}$ to $f_i$ we may assume that $f_i$ is a regular function on $X_i \times \mathbb{C}^{\dim \gamma_i}$. Next by a translation in $\mathbb{C}^{\dim \gamma_i}$ we reduce the problem to the case $p_{i,j} = 0 \in \mathbb{C}^{\dim \gamma_i}$. Then we can apply [16, Theorem 3.4 and Corollary 3.6] to $\varphi_{h_0 f_i}(\mathcal{F}_i)_{p_{i,j}} \simeq \psi_{h_0 f_i}(\mathcal{F}_i)_{p_{i,j}}$ if $f_i : (X_i \times \mathbb{C}^{\dim \gamma_i}, 0) \to (\mathbb{C}, 0)$ is Newton non-degenerate at $p_{i,j} = 0 \in \mathbb{C}^{\dim \gamma_i}$. In this way, even if $\sigma_i$ is not simplicial we can express the integer $\mu_{i,j} \geq 0$ as an alternating sum of
the normalized volumes of polytopes in \( \mathbb{R}^n_+ \setminus \Gamma_+(f)_{i,j} \), where \( \Gamma_+(f)_{i,j} \subset \mathbb{R}^n_+ \) is the (local) Newton polyhedron of \( f_i \) at \( p_{i,j} \). See [16, Corollary 3.6] for the details. We conjecture that it is positive in our situation. In the case where \( n = 3 \) we have the following stronger result.

**Theorem 4.4.** — Assume that \( n = 3 \) and \( f \) has isolated singularities at infinity over \( b \in K_f \setminus \{ f(\text{Sing } f) \cup \{ f(0) \} \} \). Then we have \( E_f(b) > 0 \) and hence \( b \in B_f \).

**Proof.** — The proof is similar to that of Theorem 4.3. We shall use the notations in it. For any \( 1 \leq i \leq m \) the dimension of the atypical face \( \gamma_i < \Gamma_\infty(f) \) is 1 or 2. If \( \dim \gamma_i = 2 \) and \( n_i > 0 \) we have \( \chi\{ \varphi_{h \circ f_i}(F_i)_{p_{i,j}} \} > 0 \) for any \( 1 \leq j \leq n_i \) by the result of Zaharia [30]. If \( \dim \gamma_i = 1 \) and \( n_i > 0 \) the two-dimensional cone \( \sigma_i \) is simplicial but \( \sigma_i \cap \mathbb{R}^3_+ \) can be bigger than \( \{0\} \). Nevertheless we can show the positivity \( \chi\{ \varphi_{h \circ f_i}(F_i)_{p_{i,j}} \} > 0 \) for any \( 1 \leq j \leq n_i \) by calculating \( F_i \in \mathbb{D}^b(X_{\Sigma_i}) \) very explicitly depending on how \( \sigma_i \) intersects \( \mathbb{R}^3_+ \). First we consider the case where \( \dim \sigma_i = 2 \), \( \dim \sigma_i \cap \mathbb{R}^3_+ = 1 \) and \( \text{rel. int}(\sigma_i \cap \mathbb{R}^3_+) \subset \text{rel. int}(\sigma_i) \). Then for any point \( q \in T_{\sigma_i} \subset X_{\Sigma_i} \) its fiber of the map

\[
(4.27) \quad \pi_i: C^3 \cap X_{\Sigma_i} \to X_{\Sigma_i}
\]
is isomorphic to \( \mathbb{C}^* \). For its cohomology groups with compact support \( H^l_c(\mathbb{C}^*; \mathbb{C}) \) \((l \in \mathbb{Z})\) we have

\[
(4.28) \quad H^l_c(\mathbb{C}^*; \mathbb{C}) \simeq \begin{cases} \mathbb{C} & (l = 1, 2), \\ 0 & (l \neq 1, 2). \end{cases}
\]

Hence for the point \( q \in T_{\sigma_i} \) we have

\[
(4.29) \quad H^l(F_i)_q \simeq \begin{cases} \mathbb{C} & (l = 1, 2), \\ 0 & (l \neq 1, 2). \end{cases}
\]

and \( \chi(F_i)_q = 0 \). Since the two one-dimensional faces \( \rho_{i,1}, \rho_{i,2} \) of \( \sigma_i \) are not contained in \( \mathbb{R}^3_+ \) there exists also an isomorphism

\[
(4.30) \quad F_i|_{X_{\Sigma_i} \setminus T_{\sigma_i}} \simeq (\iota_i)_! \mathbb{C}_T|_{X_{\Sigma_i} \setminus T_{\sigma_i}} = \mathbb{C}_T|_{X_{\Sigma_i} \setminus T_{\sigma_i}}.
\]

It follows from (4.29) and (4.30) we have an equality

\[
(4.31) \quad \chi\{ \varphi_{h \circ f_i}(F_i)_{p_{i,j}} \} = \chi\{ \varphi_{h \circ f_i}((\iota_i)_! \mathbb{C}_T)_{p_{i,j}} \} = \chi\{ \varphi_{h \circ f_i}(\mathbb{C}_T)_{p_{i,j}} \}
\]

for any \( 1 \leq j \leq n_i \). Then for any \( 1 \leq j \leq n_i \) we obtain the positivity

\[
(4.32) \quad \chi\{ \varphi_{h \circ f_i}(F_i)_{p_{i,j}} \} = \chi\{ \varphi_{h \circ f_i}(\mathbb{C}_T)_{p_{i,j}} \} > 0
\]

by the proof of Theorem 4.3. Next we consider the case where \( \dim \sigma_i = 2 \) and \( \sigma_i \cap \mathbb{R}^3_+ \) is one of the two one-dimensional faces \( \rho_{i,1}, \rho_{i,2} \) of \( \sigma_i \). We may
assume that $\sigma_i \cap \mathbb{R}_+^3 = \rho_{i,1}$. For $1 \leq j \leq 2$ we denote by $T_{i,j} \simeq (\mathbb{C}^*)^2$ the $T$-orbit in $X_{\Sigma_i}$ associated to $\rho_{i,j} < \sigma_i$. Then for $Y_{\{2\}} = \overline{T_{i,2}}$ we have an isomorphism $\mathcal{F}_i \simeq \mathbb{C}_{X_{\Sigma_i}\setminus Y_{\{2\}}}$. Since $\mathbb{C}_{X_{\Sigma_i}}$ is a perverse sheaf (up to some shift) and the two-dimensional variety $Y_{\{2\}} = \overline{T_{i,2}}$ is smooth, for any $1 \leq j \leq n_i$ we obtain the positivity

$$\chi\{\varphi_{\#f_i}(\mathcal{F}_i)_{p_i,j}\} = \chi\{\varphi_{\#f_i}(\mathbb{C}_{X_{\Sigma_i}})_{p_i,j}\} - \chi\{\varphi_{\#f_i}(\mathbb{C}_{Y_{\{2\}}} )_{p_i,j}\}$$

$$(4.33) \geq -\chi\{\varphi_{\#f_i}(\mathbb{C}_{Y_{\{2\}}} )_{p_i,j}\} > 0.$$ 

Finally, let us treat the case where $\dim \sigma_i = \dim \sigma_i \cap \mathbb{R}_+^3 = 2$. Since the face $\gamma_i$ is atypical, its dual cone $\sigma_i$ is not contained in $\mathbb{R}_+^3$ and hence we have $\sigma_i \cap \mathbb{R}_+^3 \neq \sigma_i$ in this case. Assume also that $\text{rel. int}(\sigma_i \cap \mathbb{R}_+^3) \subset \text{rel. int}(\sigma_i)$. Then for any point $q \in T_{\sigma_i} \subset X_{\Sigma_i}$ its fiber of the map

$$\pi_i|_{\mathbb{C}^3 \cap X_{\Sigma_i}^i} : \mathbb{C}^3 \cap X_{\Sigma_i}^i \rightarrow X_{\Sigma_i}$$

is isomorphic to the singular algebraic curve $\{(x_1, x_2) \in \mathbb{C}^2 \mid x_1x_2 = 0\} \subset \mathbb{C}^2$. By calculating its Euler characteristic with compact support, we obtain

$$\chi(\mathcal{F}_i)(q) = 1.$$ 

Moreover we have the isomorphism (4.30) in this case. We thus obtain the positivity

$$\chi\{\varphi_{\#f_i}(\mathcal{F}_i)_{p_i,j}\} = \chi\{\varphi_{\#f_i}(\mathbb{C}_{\sigma_i} )_{p_i,j}\} + \chi\{\varphi_{\#f_i}(\mathbb{C}_{T_{\sigma_i}} )_{p_i,j}\} > 0$$

$$(4.35) \text{for any } 1 \leq j \leq n_i.$$ 

Similarly we can prove the non-negativity and the positivity also in the remaining case. This completes the proof. \hfill \Box

We thus confirm the conjecture of [19] for $n = 3$ in the generic case. Similarly, we can improve Theorem 4.3 as follows. In fact, Theorem 4.5 below extends Theorems 4.3 and 4.4 in a unified manner. Note that the condition $\text{rel. int}(\gamma_i) \subset \text{Int}(\mathbb{R}_+^n)$ is equivalent to the one $\sigma_i \cap \mathbb{R}_+^n = \{0\}$ for the cone $\sigma_i = (\sigma_i)^{\gamma_i} \in \Sigma_0$.

**Theorem 4.5.** — Assume that $f$ has isolated singularities at infinity over $b \in K_f \setminus [f(\text{Sing } f) \cup \{f(0)\}]$ and for any $1 \leq i \leq m$ such that $b \in K_i$ the set $\sigma_i \cap \mathbb{R}_+^n$ is a face of $\mathbb{R}_+^n$ of dimension $\leq 2$. Assume also that there exists $1 \leq i \leq m$ such that $b \in K_i$, $\gamma_i \prec \Gamma_\infty(f)$ is relatively simple and moreover in the case $\dim \sigma_i \cap \mathbb{R}_+^n = 2$ the number of the common edges of $\sigma_i \cap \mathbb{R}_+^n$ and $\sigma_i$ is $\leq 1$. Then we have $E_f(b) > 0$ and hence $b \in B_f$.

**Proof.** — The proof is similar to those of Theorems 4.3 and 4.4. We shall use the notations in them. In the proof of Theorem 4.3 we proved for $1 \leq i \leq m$ such that $\sigma_i \cap \mathbb{R}_+^n = \{0\}$ (resp. $\sigma_i \cap \mathbb{R}_+^n = \{0\}$ and $\gamma_i$ is relatively simple) we have $(-1)^{n-1}\chi\{\varphi_{\#f_i}(\mathcal{F}_i)_{p_i,j}\} \geq 0$ (resp. $> 0$) for any $1 \leq j \leq n_i$. Let us consider the remaining cases where $1 \leq \dim \sigma_i \cap \mathbb{R}_+^n \leq 2$. For a face $\tau \prec \sigma_i$ of such $\sigma_i$, by taking a reference point $q \in T_{\tau} \subset X_{\Sigma_i}$
of the $T$-orbit $T_{\tau}$ associated to it we set $e(\tau) = \chi(\mathcal{F}_i)(q)$. Then as in the proof of Theorem 4.4 we can easily show that

\begin{equation}
(4.36) \quad e(\tau) = \begin{cases}
1 & (\dim \tau \cap \mathbb{R}_+^n = \dim \tau), \\
0 & (\dim \tau \cap \mathbb{R}_+^n < \dim \tau).
\end{cases}
\end{equation}

In particular, for the zero-dimensional face $\{0\} \prec \sigma_i$ of $\sigma_i$ we have $T_{\{0\}} = T$, $\mathcal{F}_i|_T \simeq \mathbb{C}_T$ and $e(\{0\}) = 1$. We thus obtain an equality

\begin{equation}
(4.37) \quad (-1)^{n-1} \chi_{\varphi \circ f_i}(\mathcal{F}_i)_{\rho, i, j} = (-1)^{n-1} \sum_{\tau : e(\tau) = 1} \chi_{\varphi \circ f_i}(\mathcal{C}_{T, \tau})_{\rho, i, j}
\end{equation}

for any $1 \leq j \leq n_i$. First let us consider the case where $\dim \sigma_i \cap \mathbb{R}_+^n = 1$. If $\sigma_i \cap \mathbb{R}_+^n$ is not an edge of the cone $\sigma_i$, by (4.37) we have

\begin{equation}
(4.38) \quad (-1)^{n-1} \chi_{\varphi \circ f_i}(\mathcal{F}_i)_{\rho, i, j} = (-1)^{n-1} \chi_{\varphi \circ f_i}(\mathcal{C}_T)_{\rho, i, j}
\end{equation}

for any $1 \leq j \leq n_i$. By the proof of Theorem 4.3 this integer is non-negative. Moreover it is positive if $\gamma_i$ is relatively simple. Let $\rho_1, \rho_2, \ldots, \rho_i, \ldots, \rho_i, \ldots < \sigma_i$ be the edges of $\sigma_i$. For $1 \leq j \leq d_i$ denote by $T_{i, j} \simeq (\mathbb{C}^*)^{n-1}$ the $T$-orbit in $X_{\Sigma_i}$ associated to $\rho_i, j < \sigma_i$. If $\sigma_i \cap \mathbb{R}_+^n$ is an edge $\rho$ of $\sigma_i$, by (4.37) we can easily see that for the remaining edges $\rho_i, j (1 \leq j \leq d_i)$ of $\sigma_i$ satisfying $\rho_i, j \neq \rho$ and the hypersurface $Z_i := \bigcup_{j : \rho_i, j \neq \rho} T_{i, j} \subset X_{\Sigma_i}$ defined by them there exists an isomorphism $\mathcal{F}_i \simeq \mathbb{C}_{X_{\Sigma_i} \setminus Z_i}$. Since the hypersurface complement $X_{\Sigma_i} \setminus Z_i$ is an affine open subset of $X_{\Sigma_i}$, $\mathcal{F}_i$ is perverse (up to some shift) and we obtain the non-negativity

\begin{equation}
(4.39) \quad (-1)^{n-1} \chi_{\varphi \circ f_i}(\mathcal{F}_i)_{\rho, i, j} = (-1)^{n-1} \chi_{\varphi \circ f_i}(\mathcal{C}_{X_{\Sigma_i} \setminus Z_i})_{\rho, i, j} \geq 0
\end{equation}

for any $1 \leq j \leq n_i$. Moreover we can rewrite this integer as follows:

\begin{equation}
(4.40) \quad (-1)^{n-1} \chi_{\varphi \circ f_i}(\mathcal{F}_i)_{\rho, i, j} = (-1)^{n-1} \sum_{\tau : \rho \neq \tau} (-1)^{\dim \tau} \chi_{\varphi \circ f_i}(\mathcal{C}_{T, \tau})_{\rho, i, j}.
\end{equation}

If $\gamma_i$ is relatively simple, the right hand side is a sum of non-negative integers and for a facet $\tau$ of $\sigma_i$ such that $\rho \neq \tau$ the closure $T_{\tau}$ of $T_{\tau}$ is smooth and we have the positivity

\begin{equation}
(4.41) \quad (-1)^{n-1+\dim \tau} \chi_{\varphi \circ f_i}(\mathcal{C}_{T, \tau})_{\rho, i, j} > 0.
\end{equation}

Finally let us consider the case where $\dim \sigma_i \cap \mathbb{R}_+^n = 2$. Assume that $(\sigma_i \cap \mathbb{R}_+^n \setminus \{0\}) \subset \text{rel. int}(\sigma_i)$. Since the case where $\dim \sigma_i = \dim \sigma_i \cap \mathbb{R}_+^n = 2$ was already treated in the proof of Theorem 4.4, here we treat only the case where $\dim \sigma_i > \dim \sigma_i \cap \mathbb{R}_+^n = 2$. Then by (4.37) we obtain the non-negativity

\begin{equation}
(4.42) \quad (-1)^{n-1} \chi_{\varphi \circ f_i}(\mathcal{F}_i)_{\rho, i, j} = (-1)^{n-1} \chi_{\varphi \circ f_i}(\mathcal{C}_T)_{\rho, i, j} \geq 0
\end{equation}
for any $1 \leq j \leq n_i$. Moreover it is positive if $\gamma_i$ is relatively simple. Similarly we can prove the non-negativity and the positivity also in the remaining cases. We omit the details. This completes the proof. □

In the case $n = 4$ we can also partially verify the conjecture of [19] as follows.

**Theorem 4.6.** — Assume that $n = 4$, $f$ has isolated singularities at infinity over $b \in K_f \setminus [f(Sing f) \cup \{f(0)\}]$ and for any $1 \leq i \leq m$ such that $b \in K_i$ and $\dim \sigma_i = \dim \sigma_i \cap \mathbb{R}_+^4 = 3$ there exists no common edge of $\sigma_i$ and $\sigma_i \cap \mathbb{R}_+^4$. Assume also that there exists $1 \leq i \leq m$ such that $b \in K_i$ and in the case $\dim \sigma_i = 3$ and $\dim \sigma_i \cap \mathbb{R}_+^4 = 2$ the number of the common edges of $\sigma_i$ and $\sigma_i \cap \mathbb{R}_+^4$ is $\leq 1$. Then we have $E_f(b) > 0$ and hence $b \in B_f$.

**Corollary 4.7.** — Assume that $n = 4$, $f$ has isolated singularities at infinity over $b \in K_f \setminus [f(Sing f) \cup \{f(0)\}]$ and for any $1 \leq i \leq m$ such that $b \in K_i$ we have $\dim \sigma_i \cap \mathbb{R}_+^4 \leq 1$ or $\dim \sigma_i \leq 2$. Then we have $E_f(b) > 0$ and hence $b \in B_f$.

Since the proof of Theorem 4.6 is similar to those of Theorems 4.3, 4.4 and 4.5, we omit it here.

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Manuscrit reçu le 4 avril 2018,
accepté le 12 mars 2019.

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