Numerical simulation of equilibrium plasma configurations in toroidal magnetic traps and their cylindrical analogues

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Abstract. Some basic characteristics investigations of equilibrium magnetoplasma configurations in toroidal traps for plasma confinement often use mathematical models of their straightened into a cylinder analogues. The aim of this work is a comparative analysis of MHD-models in toroidal and cylindrical configurations. We consider the plasma cylinder with the electric current (Z-pinch) as a simpliest trap type and its toroidal analogue and then extend and define more exactly the cylindrical and toroidal models of the 'Galatea-Belt' trap with two current-carrying conductors, immersed into the plasma. Distinctions between configurations in the traps of different geometry are registered and some stability problems are considered.

1. Introduction

Magnetic traps for plasma confinement are a basic element of various plants being elaborated for nuclear fusion research. The widespread traps are toroidal in shape, for example, tokamaks and stellarators. The galatea-traps, proposed by A.I. Morozov [1, 2], where the conductors with electric current, inducing the magnetic field, are immersed into the plasma, have the same geometry. Their simpliest example is the ‘Galatea-Belt’ with two ring-shaped conductors inside the torus [3].

Mathematical modeling and computation are of considerable importance in investigations of equilibrium magnetoplasma configurations in the traps (see [4–7] for example). The models use as a rule magnetohydrodynamic equations. If a trap has any symmetry, its two-dimensional models can be reduced to boundary-value problems with the scalar Grad-Shafranov equation [8–10] for the magnetic flux function. Some conceptual questions, concerning toroidal trap configuration may be explained, solving these problems in the straight cylinder, i.e. in the torus of infinite radius. In this case quantitative distinctions between the toroidal and cylindrical configurations become to be of interest and must be investigated. Some first results concerning numerical simulation of configurations in the ‘Belt’ and in the cylinder with two straight conductors and their comparative analysis are presented in [11,12].

In this paper distinctions between toroidal and cylindrical configurations are considered in the more general form. We investigate some configurations in the plasma cylinder with the electric current (Z-pinch) and its toroidal analogue, that may be regarded as a basis of tokamaks.
Toroidal configurations are removed from the symmetry axis and distorted in comparison with the cylindrical ones. This effect is the stronger, the smaller is the torus radius. When the radius increases, the configurations become nearer the cylindrical ones. Computations are performed for different electric current distributions in the pinch. The configurations considered above are stable with respect to two-dimensional perturbations.

The computer investigations of ‘Galatea-Belt’ configurations, mentioned above, are continued. The mathematical model became more convenient for result interpretation after the more appropriate choice of the magnetic flux function unit of measurement. In the series of computations we have determined the maximum plasma pressure values, that admit stable plasma configurations with given electric current in the conductors.

2. Mathematical models of plasma equilibrium in cylinders

General questions of comparison plasma configurations in torus and their cylindrical analogues are convenient to be considered by means of the example of plasma cylinder and torus, which is corresponding to it. The simplest and most famous example is a round plasma cylinder with electric current parallel to its axis (Z-pinch). We consider its MHD-model as a special case of the model of arbitrary cross-section cylinder, homogeneous in the axial direction. Equilibrium plasma configurations correspond to MHD-equations of plasmastatics \[4, 5, 9\]

\[ \nabla p = \frac{1}{c} \mathbf{j} \times \mathbf{H}; \quad \frac{c}{4\pi} \mathbf{j} = \text{rot} \mathbf{H}; \quad \text{div} \mathbf{H} = 0 \] (1)

for the three variables distributed in space: pressure \( p \), magnetic field strength \( \mathbf{H} \) and electric current density \( \mathbf{j} \). In two-dimensional plane symmetric problems without longitudinal field \( (H_z \equiv 0, \partial / \partial z \equiv 0) \) they are reduced to one scalar second order equation for the magnetic flux function \( \Psi \) - the plane version of Grad-Shafranov equation

\[ \Delta \Psi + 4\pi \frac{dp}{d\Psi} = 0, \] (2)

where \( H_x = \frac{\partial \Psi}{\partial y}, \quad H_y = -\frac{\partial \Psi}{\partial x} \).

Boundary problem with equation (2) should be solved in the region of \((x, y)\) plane, which corresponds the cross-section of the cylinder with the \( z = \text{const} \) plane with given boundary conditions.

The pressure \( p = p(\Psi) \) is the given function of variable \( \Psi \), which describes its distribution between isobaric magnetic surfaces \( \Psi(x, y) = \text{const} \).

Two-dimensional boundary problems are considered in square space regions corresponding to square cross-sections of toroidal traps.

It’s natural to consider problems in round cylinder that are homogeneous in azimuth and therefore one-dimensional in cylindrical coordinates \((r, \varphi, z)\) with the one spatial coordinate \( r \) and unique non-trivial component of magnetic field is \( H_{\varphi} \). The equation (1) became

\[ \frac{dp}{dr} = -\frac{1}{c} \mathbf{j} H; \quad \mathbf{j} = \frac{4\pi H}{r} \frac{dr}{dr}, \] (3)

where \( H = H_{\varphi}, \quad \mathbf{j} = j_z \) in the circle with radius \( R \). Knowing one of the unknown functions \( p, H, \mathbf{j} \), we find two other functions by integration of equations (3). Thus model doesn’t use the equation with Grad-Shafranov type (2), although it is formally valid in the form

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d\Psi}{dr} \right) + 4\pi \frac{dp}{d\Psi} = 0, \]
where the magnetic flux function \( \Psi \) and the dependence \( p(\Psi) \) is concerned with the solutions of equations (3) in the following manner:

\[
H = -\frac{d\Psi}{dr}; \quad \frac{dp}{d\Psi} = \frac{dp}{dr}/\frac{d\Psi}{dr} = -\frac{dp}{dr}/H. \tag{4}
\]

The function \( p(\Psi) \), which was gained here for the round \( Z \)-pinch, is used below in the models of two-dimensional configurations in square cross-section of the cylinder. These models are convenient to be compared with models of toroidal configurations with square cross-section too.

Boundary problems with the eq. (2) are considered in square \( |x| < R, |y| < R \). The bound of the square is considered to be opaque for the magnetic field, thus there is \( H_n = 0 \) on it, where \( n \) is a surface normal to the bound \( \Gamma \). It follows, that boundary condition is \( \Psi = \Psi_{\Gamma} \). The value of constant \( \Psi_{\Gamma} \) may be equal to zero, because it doesn’t matter in the final solution of the problem.

Problems in torus may be homogeneous only in azimuth direction \( \varphi \), as at least two-dimensional. Square cross-sections were chosen because our problems are considered to be solved in cylindrical coordinates, and any other geometry would make the coordinate system and mathematical apparatus much more complicated [9]. The article [11] shows, that configurations in round and square cylinder are insignificantly different only on the periphery.

The ‘Galatea-Belt’ magnetic trap is a torus with two ring-shaped conductors with current immersed into plasma. Its straightened analogue is a cylinder with two conductors, parallel to the axis. Plasma configurations were observed numerically in its round cross-section in [13], and for the reasons above, in its square cross-section in [11,12]. As a continuation and development of recent work we consider square cross-section. Magnetic field, which retains plasma, is formed by conductors with the current. The mathematical model takes this into account in terms of boundary problem with Grad-Shafranov type equation

\[
\Delta \Psi + 4\pi \frac{dp}{d\Psi} + \frac{4\pi}{c} j^{ex} = 0, \tag{5}
\]

where \( j^{ex}(x,y) \) is a given function, which describes the distribution of ‘external’ current. It is located in the places of crossing of conductors with the \( (x,y) \)-plane:

\[
j^{ex}(x,y) = \sum_{k=1}^{2} j_0 \exp \left( -\frac{(x-x_k)^2 + y^2}{r_c^2} \right) \tag{6}
\]

where \( x_1 = x_0, x_2 = -x_0 \) - coordinates of the conductor centres, \( r_c \) is conditional radius of conductors. The coefficient \( j_0 \) is determined from the condition

\[
\int \int j^{ex} dx dy = J_c,
\]

where the integral is evaluated in the vicinity of each conductor, and \( J_c \) is a current value in the conductor [4,11–13].

Isobaric function \( p(\Psi) \) is selected on the basis of requirements to concentrate plasma near the central part of space region restricting it touch the conductors. It must be nonmonotonic with a maximum in the centre of the square and the separatrix \( \Psi = \Psi_{0} \) passing through it. For example, this conditions may be satisfied by the function

\[
p = p_0 \exp \left( -\frac{(\Psi - \Psi_{0})^2}{q^2} \right) \tag{7}
\]

where \( \Psi_{0} = \Psi(0,0) \).

Boundary problems with the eq. (5) are considered in square \( |x| < X, |y| < X \) with condition, as above, \( \Psi = 0 \) on the bound.
3. Plasma equilibrium in toroidal traps

Plasma and magnetic field configurations with toroidal electric current, are homogeneous in azimuth coordinate, are considered, as said above, in torus of square cross-section. The mathematical apparatus of the models is different from the problem above in the coordinate system: cylindrical coordinates \((r, z)\) are used instead of Cartesian ones \((x, y)\).

The Grad-Shafranov equation, proposed in the original papers \([8–10]\) for the axisymmetric problems, is in application for the problem in plasma torus without azimuth field \((H_\phi \equiv 0)\) of the following form

\[
\Delta^* \Psi + 4\pi r^2 \frac{dp}{d\Psi} = 0.
\]

where \(\triangle^* \Psi \equiv r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Psi}{\partial r} \right) + \frac{\partial^2 \Psi}{\partial z^2}, rH_r = -\frac{\partial \Psi}{\partial z}, rH_z = \frac{\partial \Psi}{\partial r}.\)

Boundary problems are considered in the square \(|z| < R, |r - r_0| < R\), where \(r_0\) is a big torus radius. The dependence \(p(\Psi)\) is the same as above, obtained in the one-dimensional problem in the round Z-pinch \((4)\). In the toroidal model of 'Galatea-Belt' the term \(4\pi r j^{ex}\) was added in the left part of equation \((8)\), where

\[
j^{ex}(z, r) = \sum_{k=1}^{2} j_0 \exp \left( - \frac{(r - r_0)^2 + (z - z_k)^2}{r_c^2} \right)
\]

is an 'external' current in ring-shaped conductors with axis \((r_0, z_k)\), \(z_1 = z_0, z_2 = -z_0\). The pressure \(p(\Psi)\) is given by the same formula \((7)\), where parameter \(\Psi_0\) is a value of the desired function in the centre of the configuration and on separatrix, including the singular magnetic field point:

\[
\Psi_0 = \max_r \Psi(r, 0)
\]

The problems are considered in the square \(|r - r_0| < 2z_0, z < 2z_0\), on the bound of it we assume \(\Psi = 0\), like in cylindrical geometry.

4. About methods of solving the problems

Problems, which are considered above, are solved in dimensionless variables: the units of measurement of all values are composed from the given dimensional constants. In both series of problems such constants are characteristic size \(r_u\) - radius \(R\) of pinch or the half of distance between the conductors in 'Belt' \(z_0\) \((x_0\) in plane problems), and characteristic electric current value \(J\) in pinch or in each conductor in 'Belt'. The units which are composed from them are

\[
r_u, H_u = \frac{2J}{cr_u}, j_u = \frac{cH_u}{4\pi r_u}; p_u = \frac{H_u^2}{4\pi}.
\]

Flux function \(\Psi\) has different dimension in plane \((2)\) and toroidal \((8)\) problems, and therefore we assume \(\Psi_u = H_u r_u\) in plane, but in torus we have to add the factor with dimension of length to its unit. In \([11, 12]\) it was assumed, that \(\Psi_u = H_u r_0^2\), and it turned out to be not convenient in the interpretation of the result physical meaning, because dimensionless value of \(\Psi\) is proportional to the big torus radius in this case, but parameter \(q\) in formula \((7)\) measured in the same units remains the same. In this paper

\[
\Psi_u = H_u r_u r_0
\]

has been chosen as a unit of \(\Psi\), \(r_0\) is a torus radius. As a result, the order of dimensionless value of \(\Psi\) remains limited together with the parameter \(q\), when the radius \(r_0\) increases. The comparison of toroidal and cylindrical configurations becomes less complicated.
Figure 1. Equilibrium configurations in the plasma cylinder with 'parabolic’ current a) magnetic field ($\Psi(r, z) = \text{const}$), b) plasma pressure ($p = \text{const}$)

It this case dimensionsless Grad-Shafranov equation has the form

$$\Delta^* \Psi + \left(\frac{r}{r_0}\right)^2 \frac{dp}{d\Psi} + \frac{r}{r_0} j^ex(r, z) = 0.$$ (13)

The difference analogue of Grad-Shafranov type equations is solved numerically by means of iterative relaxation method, which is described and discussed in [4,5,11–13].

5. Computation results

To compare equilibrium plasma and field configurations in a straight plasma cylinder and its toroidal extension, obtained in computer models above, we begin through the following simple example. The given electric current ('parabolic’, dimensionless)

$$j(r) = 3.64(1 - 0.9r^2)$$

is distributed in the circular cylinder $r < 1$. It follows from the equations (3)

$$H(r) = 1.82r(1 - 0.45r^2); \quad \Psi(r) = 0.91(1 - r^2)(0.775 - 0.225r^2);$$

$$p(r) = 3.3(1 - r^2)(0.28 - 0.72r^2 + 0.135r^4)$$

and according to (4)

$$\frac{dp}{d\Psi} = j = 4(\sqrt{3.27\Psi + 1} - 0.91)$$ (14)

The configuration in the plasma cylinder with square cross-section is derived by solving the two-dimensional problem with the equation (2), where the isobaric function $p(\Psi)$ is taken from the one-dimensional problem (14). The same problem is solved in the toroidal geometry with several major axis dimensionless values $r_0$. Computation results are presented by means of magnetic force lines $\Psi = \text{const}$ and isobares $p = \text{const}$ in the straight cylinder and torus at
Figure 2. Equilibrium configurations in the plasma torus with ‘parabolic’ current a) magnetic field ($\Psi(r, z) = \text{const}$), b) plasma pressure ($p = \text{const}$)

Figure 3. Equilibrium configurations in the straightened analogue of ‘Galatea-belt’ a) magnetic field ($\Psi(r, z) = \text{const}$), b) plasma pressure ($p = \text{const}$)

The torus radius value $r_0 = 1$ is chosen as the least admissible one in order that distinctions between the cylinder and the torus would be the most visible. The toroidal configuration is deformed and its magnetic axis is removed at the distance $\delta r = 0.48$. The plasma volume is less than in the cylinder and therefore the maximum values of pressure $p$ and poloidal magnetic flux $\Psi$ are greater. In the computation series with several current distributions $j(r)$ a detailed qualitative and quantitative information about distinctions between magnetoplasma
configurations under consideration is obtained. It allows to evaluate the errors arising when we use straight models instead of toroidal ones.

The mathematical model of equilibrium configurations in the ‘Galatea-Belt’ is released in solving of boundary problems with Grad-Shafranov equation (5) and (13) in the square regions. The results are presented at the fig. 3, 4. The value \( r_0 = 2 \) in toroidal variant is again the least admissible. The plasma and field configurations in the cylinder are symmetric with respect to the coordinate axis. The plasma is concentrated in the central region having a curvilinear quadrangular form with sides convex inwards and thin strips along the field separatrix around the conductors. Because of the more convenient choice of the unit \( \Psi_u \) (12) the values of \( \Psi \) in the cylinder and in the torus have the same order here, that admit easily explain the results. The configurations in torus are deformed as above: the magnetic axis is removed at \( \delta r = 0.52 \). The plasma volume in the torus is rather greater than in the cylinder, then the magnetic flux is less.

It is known, that the model concerned admits the relaxation to the equilibrium only under the restriction \( p_0 < p_{0c}^{cr} \) in (7) for the pressure maximum (dimesionless, i.e. refered to the magnetic pressure). In the computations under the norming (13) we obtained, that the critical value \( p_{0c}^{cr} \) decreases together with the radius \( r_0 \), i.e. when the torus curvature increases. The restriction \( p_0 < p_{0c}^{cr} \) corresponds to the ‘diffusion stability’ of configuration, that is one of the necessary MHD-stability conditions (see [4, 5]).

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