Matter fields interacting with photons

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Abstract – We have extended the biquaternionic Dirac’s equation to include interactions with photons. The electric field is found to be perpendicular to the matter magnetic field, and the magnetic field is perpendicular to the matter inertial field. Inertial and magnetic masses are found to be conserved separately. The magnetic mass density is a consequence of the coupling between the vector potential and the matter inertial field. The presence of the vector and scalar potentials, and the matter inertial and magnetic fields are found to modify the standard form of the derived Maxwell’s equations. The resulting interacting electrodynamics equations are found to generalize those of axion-like fields of Frank Wilczek or Chern-Simons equations. The axion field satisfies massive Klein-Gordon equation if Lorenz gauge condition is violated. Therefore, axion could be our massive photon. The electromagnetic field vector, \( \vec{F} = \vec{E} + ic \vec{B} \), is found to satisfy massive Dirac’s equation in addition to the fact that \( \vec{\nabla} \cdot \vec{F} = 0 \), where \( \vec{E} \) and \( \vec{B} \) are the electric and magnetic fields, respectively.

Introduction. – The interaction of photons with an electron is coined in the so called quantum electrodynamics (QED). This is formally done by replacing the ordinary partial derivative in Dirac’s equation by the covariant derivative involving the electromagnetic vector and scalar potentials, \( \vec{A} \) and \( \varphi \). Consequently, Dirac’s equation is found to predict the spin of the electron.

In a recent paper we formulated quantum mechanics in which Dirac’s equation is expressed in biquaternionic form \([2]\). This representation is found to allow Dirac’s equation to be expressible in Maxwell-like equations. In this description, the electron is described by vector and scalar wavefunctions, \( \vec{\psi} \) and \( \psi_0 \) \([3]\). There are two fields associated with the mass of the particle. They are functions of \( \vec{\psi} \) and \( \psi_0 \) in an analogous way to the electric and magnetic fields that are associated with the particle charge. We call these two fields the inertial and magnetoinertial fields, \( \vec{E}_m \) and \( \vec{B}_m \). In this new formulation, we have found that quantum mechanics can be visualized as a theory for the evolution of matter fields, in analogy with Maxwell theory that describes the evolution of charge fields.

In the present work, we study the interaction of matter fields with an external electromagnetic field (photon) \([4]\). To this end, we follow a routine that is analogous to that done for quantum electrodynamics. In doing so, we have found that the biquaternionic Dirac’s equation can be described by a symmetrized Maxwell-like equations involving magnetic mass as well.

The concept of magnetic mass has to be presumptively interpreted in the framework of

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this formulation. Both inertial mass and magnetic mass are separately conserved. We treat the mass in quantum mechanics as the charge is treated in electrodynamics. Mass waves are de Broglie waves whereas charge waves are electromagnetic waves. Therefore, they are otherwise analogous to each other. The resulting electrodynamics equations reduce to axions electrodynamics of Frank Wilczek and Chern-Simons [1,5]. It was realized by Weinberg and Wilczek that axions resulted from CP violation in Quantum-Chromodynamic (QCD) [4,6].

In free space the resulting electrodynamic equations are found to be invariant under a duality transformation. This electrodynamics is also invariant under parity and time-reversal transformations. While Wilczek's equations are classical, our present equations are of quantum nature. The electrodynamics we have obtained generalizes Wilczek axions electrodynamics to include magnetic charge and current densities. Moreover, our vacuum electrodynamics is invariant under duality transformation.

This paper proceeds as follows: In Section 2 we present the biquaternionic interacting Dirac's equation. We adopt here the standard approach as done for QED. The resulting equations are that of Maxwellian form with interacting additional fields that are related to the particle inertia (mass)). These additional fields are independent of the energy conservation equation associated with the electromagnetic fields. The mass fields, current are defined in terms of the fundamental constants and the wave functions. From these quantities Maxwell-like equations relating the mass fields are written. The additional fields render the charge (mass) to be conserved. In Section 2 we discuss the symmetries reflected in these equations. As in the original Maxwell's equations, the resulting Maxwell's equations are found to be invariant under duality transformation in vacuum.

In Section 3 we present a comparison between Wilczek axion electrodynamics and our electrodynamics. The comparison reveals that axion electrodynamics emerges as a special case. We define a continuity equation that restores the charge conservation by adding a London’s-like current to the electric current. Our electrodynamics reduces to Maxwell’s equations in a moving frame for special choice of the interacting fields. In Section 4 we how that our electrodynamics can be expressed in a Dirac-like equation if we employ spin-1 matrices, and writing the electric and magnetic fields in a complex form. We end the paper with concluding remarks.

The interacting biquaternionic Dirac's equation. – The ordinary Dirac's equation of a spin-1/2 particle with rest mass $m$ and charge $q$ interacting with a gauge field is expressed as [7]

$$ (p^\mu - qA^\mu)\gamma_\mu \psi = mc \psi, $$

(1)

where $\gamma^\mu$ are the Dirac’s matrices that are expressed in terms of Pauli matrices ($\vec{\sigma}$), $c$ is the speed of light, $\psi$ are the spinors representing the Dirac’s particle, $q$ and $m$ are the particle charge and mass, respectively. $p^\mu$ and $A^\mu$ are the 4-momentum and 4-vector potential, respectively. We anticipate that the Dirac field in the biquaternionic Dirac equation to interact with photons in the way that Dirac equation couples to photons.
In a biquaternionic form (1) is cast in the form

\[(\tilde{P} - q\tilde{A}_g) \tilde{\gamma} \tilde{\Psi} = mc \tilde{\Psi},\]

where \(\tilde{\Psi}, \tilde{P}, \tilde{A}\) and \(\tilde{\gamma}\) are biquaternions defined as

\[
\begin{align*}
\tilde{P} &= \left( \frac{i}{c} E, \vec{p} \right), \\
\tilde{\gamma} &= (i\beta, \vec{\gamma}), \\
\tilde{\Psi} &= \left( \frac{i}{c} \psi_0, \vec{\psi} \right), \\
\tilde{A}_g &= \left( \frac{i}{c} \varphi_g, \vec{A}_g \right),
\end{align*}
\]

where \(\vec{A}_g\) and \(\varphi_g\) are the vector and scalar potentials of the boson interacting with the electron. Recall that a biquaternion is generally represented by a scalar and vector components. In quantum mechanics, the energy and momentum become operators, viz., \(\vec{p} = -\frac{i}{\hbar} \vec{\nabla}\) and \(E = \frac{i}{\hbar} \frac{\partial}{\partial t}\). Note that \(\vec{\gamma} = \beta\vec{\alpha}\), is a \(4 \times 4\) matrix, where \(\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), \(\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}\), and \(\vec{\sigma}\) are the \(2 \times 2\) Pauli matrices.

In a recent paper we have shown that the biquaternionic Dirac equation produces Maxwell-like equations with electric-like and magnetic-like matter fields that are defined from the linear combinations of the biquaternionic matter fields \(\tilde{\Psi}, \psi_0\) and \(\vec{\psi}\) and the \(\vec{\gamma}\) matrices. The biquaternionic free Dirac equation is obtained from (2) by setting \(\vec{A} = 0\), viz., \(\tilde{P} \tilde{\gamma} \tilde{\Psi} = mc \tilde{\Psi}\). While in the ordinary Dirac equation \(\tilde{\Psi}\) are spinors which in 4-dimension has 4-components, \(\tilde{\Psi}\) in our present case is a biquaternion consisting of a scalar and 3-vector totalling to 4-components.

The biquaternion multiplication rule for two biquaternions, \(\tilde{A} = (a_0, \vec{a})\) and \(\tilde{B} = (b_0, \vec{b})\), is given by

\[
\tilde{A} \tilde{B} = (a_0 b_0 - \vec{a} \cdot \vec{b}, a_0 \vec{b} + a \vec{b}_0 + \vec{a} \times \vec{b}),
\]

where in general the scalar and vector parts of each biquaternions are complex.

Now apply (3) and (4) in (2) and then equate the real and imaginary parts into the two sides to each other, to obtain

\[
\tilde{\nabla} \cdot \vec{E}_D = \frac{\rho_D}{\varepsilon_0} - \frac{q c}{\hbar} \vec{A}_g \cdot \vec{B}_D, \quad (5)
\]

\[
\tilde{\nabla} \times \vec{E}_D + \frac{\partial \vec{B}_D}{\partial t} = -\frac{q}{\hbar c} \psi_0 \vec{E}_D - \frac{q c}{\hbar} \vec{A}_g \times \vec{B}_D, \quad (6)
\]

\[
\tilde{\nabla} \times \vec{B}_D - \frac{1}{c^2} \frac{\partial \vec{E}_D}{\partial t} = \mu_0 \vec{J}_D + \frac{q}{\hbar c} \vec{A}_g \times \vec{E}_D - \frac{q c}{\hbar} \varphi_g \vec{B}_D, \quad (7)
\]

and

\[
\tilde{\nabla} \cdot \vec{B}_D = \rho_{m_D}, \quad (8)
\]

where we denote

\[
\vec{E}_D = c^2 \vec{\gamma} \times \vec{\psi}, \quad \vec{B}_D = \vec{\gamma} \psi_0 + c\beta \vec{\psi}, \quad \vec{J}_D = \frac{m c^2}{\mu_0 \hbar} \vec{\psi}, \quad \rho_D = \frac{m \hbar}{\mu_0} \psi_0, \quad \rho_{m_D} = \frac{q}{\hbar c} \vec{A}_g \cdot \vec{E}_D, \quad (9)
\]

provided that\[2\]

\[
\psi_0 = -c\beta \vec{\gamma} \cdot \vec{\psi}. \quad (10)
\]

\[2\text{We can relax this condition by allowing } \Lambda = \beta \psi_0 + \vec{\gamma} \cdot \vec{\psi}. \text{ See Appendix B.}\]
We call here $\vec{E}_D$ and $\vec{B}_D$ the inertial and magnetic matter (inertial) fields, respectively. $\rho_D$ and $\vec{J}_D$ are the matter and current densities, respectively. As in the ordinary electrodynamics, the electromagnetic fields due to a moving charge are perpendicular to each other and the velocity of the charge. This property is also applied to matter inertial fields. This particular plausible linear combinations of the vector and scalar, reproduces Maxwell’s-like equations. This implies that the biquaternionic Dirac equation is very rich. Maxwell’s equations are known to describe spin -1 particle (photon) which is represented by a 3 - vector and a scalar, whereas Dirac’s equation describes spin - $\frac{1}{2}$ particles are represented by 4-component spinors. In our biquaternionic Dirac’s equation, our particle is described by 4 - components (a scalar and a 3-vector).

Employing this formalism, the biquaternionic momentum eigen-value equation is shown to reproduce, the Schrodinger, Dirac and Klein-Gordon equations, where the wavefunction and the momentum operator are generalized to become biquaternions \cite{8}. A different view of a biquaternionic quantum mechanics (QQM) with states defined on a biquaternionic Hilbert space has been developed by Finkelstein et al., Adler and Horwitz \cite{9,11}. It was shown in these papers that the QQM are a far richer theory than the ordinary complex quantum mechanics. However, in our present formulation of quantum mechanics, we express our initial equation as an eigen-value quantum equation but that equation is then reduced to Maxwell-like equations, where we deal with deterministic fields expressed by $\vec{E}_D$ and $\vec{B}_D$ rather than probabilistic wavefunction that the ordinary quantum mechanics works with. The solution of our biquaternionic Dirac equation reduces to solving the Maxwell-like equations.

It is worth mentioning that when $\vec{A}_g$ and $\varphi_g$ are absent, (11) - (15) reduce to the ordinary Maxwell’s-like equations \cite{3}.

\begin{align*}
\nabla \cdot \vec{E}_D &= \frac{\rho_D}{\varepsilon_0}, \quad (11) \\
\nabla \times \vec{E}_D + \frac{\partial \vec{B}_D}{\partial t} &= 0, \quad (12) \\
\nabla \times \vec{B}_D - \frac{1}{c^2} \frac{\partial \vec{E}_D}{\partial t} &= \mu_0 \vec{J}_D, \quad (13)
\end{align*}

and

\begin{equation}
\nabla \cdot \vec{B}_D = 0. \tag{14}
\end{equation}

By employing (9), we see that $\vec{E}_D \cdot \vec{B}_D = 0$ and $\vec{J}_D \cdot \vec{E}_D = 0$, so that the energy conservation equation of the system described by (11) - (14) is

\begin{equation}
\frac{\partial u_D}{\partial t} + \nabla \cdot \vec{S}_D = 0, \quad u_D = \frac{1}{2} \varepsilon_0 E_D^2 + \frac{B_D^2}{2\mu_0}, \quad \vec{S}_D = \frac{E_D \times B_D}{\mu_0}. \tag{15}
\end{equation}

It is interesting that (15) is independent of $\varphi_g$ and $\vec{A}_g$. This implies that these two fields are not dynamical. We see from (9) that magnetic charge (matter) results from interaction of matter fields with the vector potential.

Now take the divergence of (6) and use (5) and (8) to obtain

\begin{equation}
\vec{B}_g \cdot \vec{B}_D - \frac{1}{c^2} \vec{E}_g \cdot \vec{E}_D = \mu_0 (\vec{A}_g \cdot \vec{J}_D - \varphi_g \rho_D), \tag{16}
\end{equation}
where \( \vec{E}_g = -\vec{\nabla} \varphi_g - \frac{\partial \vec{A}_g}{\partial t} \) and \( \vec{B}_g = \vec{\nabla} \times \vec{A}_g \). Taking the divergence of (7) and using (5) and (8) yield

\[
\vec{\nabla} \cdot \vec{J}_D + \frac{\partial \rho_D}{\partial t} = -\frac{q c \varepsilon_0}{\hbar} (\vec{E}_g \cdot \vec{B}_D + \vec{B}_g \cdot \vec{E}_D).
\]

(17)

Equations (16) and (17) show that the electromagnetic fields are coupled to the matter inertial and magnetic fields. The matter current and charge densities are coupled to the vector and scalar potentials, respectively. The continuity equation (mass conservation) dictates that (17) to split into

\[
\vec{\nabla} \cdot \vec{J}_D + \frac{\partial \rho_D}{\partial t} = 0, \quad \vec{E}_g \cdot \vec{B}_D + \vec{B}_g \cdot \vec{E}_D = 0.
\]

(18)

In particular, this equation is satisfied if we take the property that \( \vec{E}_g \) is perpendicular to \( \vec{B}_D \) and \( \vec{B}_g \) is perpendicular to \( \vec{E}_D \).

**Symmetric Dirac equation with magnetic matter density.** – In electromagnetism, one can extend Maxwell’s equations to incorporate magnetic monopoles (magnetic charges) by writing Maxwell’s equations in a symmetric form \([\text{I}]\). This form preserves the duality transformation of the electric and magnetic fields, *viz.*, \( \vec{E} \rightarrow c \vec{B} \) and \( c \vec{B} \rightarrow -\vec{E} \). In the same fashion, we can write (5) - (8) in a symmetric form as

\[
\vec{\nabla} \cdot \vec{E}_D = \frac{\rho'_D}{\varepsilon_0},
\]

(19)

\[
\vec{\nabla} \times \vec{E}_D = -\frac{\partial \vec{B}_D}{\partial t} - \vec{J}_{mD},
\]

(20)

\[
\vec{\nabla} \times \vec{B}_D = \mu_0 \vec{J}_{D}' + \frac{1}{c^2} \frac{\partial \vec{E}_D}{\partial t},
\]

(21)

and

\[
\vec{\nabla} \cdot \vec{B}_D = \rho_{mD},
\]

(22)

where

\[
\rho'_D = \rho_D - \frac{q c \varepsilon_0}{\hbar} \vec{A}_g \cdot \vec{B}_D,
\]

(23)

\[
\vec{J}_{mD} = \frac{q}{\hbar c} (\vec{\psi}_g \vec{E}_D + c^2 \vec{A}_g \times \vec{B}_D),
\]

(24)

\[
\vec{J}_D' = \vec{J}_D - \frac{q c \varepsilon_0}{\hbar} (\vec{\psi}_g \vec{B}_D - \vec{A}_g \times \vec{E}_D).
\]

(25)

We define the quantity \( \rho_{mD} \) as the magnetic matter density. This has to be properly interpreted. It is interesting to observe that the sources of the matter fields are of quantum nature. The solution of our Dirac biquaternionic equation is thus a solution of the system of the Maxwell-like equations, (19) - (22). After solving for \( \vec{E}_D \) and \( \vec{B}_D \) we can then solve for \( \psi_0 \) and \( \vec{\psi} \) using (9) and (10).

The divergence of (24) together with (19), (18) and (22) yield

\[
\vec{\nabla} \cdot \vec{J}_{mD} + \frac{\partial \rho_{mD}}{\partial t} = 0.
\]

(26)
Equations (18) and (26) reveal that inertial and magnetic masses in the presence of interaction are separately conserved.

We conjecture here a possible existence of an electromagnetic system \((e.g.,\) self-interacting photons) that fulfils (5) - (8). For such a system one has

\[
\vec{\nabla} \cdot \vec{E}_g = \frac{\rho}{\varepsilon_0} - \frac{qc}{\hbar} \vec{A}_g \cdot \vec{B}_g, \tag{27}
\]

\[
\vec{\nabla} \times \vec{E}_g + \frac{\partial \vec{B}_g}{\partial t} = -\frac{q}{\hbar c} \left( \varphi_g \vec{E} + c^2 \vec{A}_g \times \vec{B}_g \right), \tag{28}
\]

\[
\vec{\nabla} \times \vec{B}_g - \frac{1}{c^2} \frac{\partial \vec{E}_g}{\partial t} = \mu_0 \vec{J} - \frac{q}{\hbar c} \left( \varphi_g \vec{B} - \vec{A}_g \times \vec{E}_g \right), \tag{29}
\]

and

\[
\vec{\nabla} \cdot \vec{B}_g = \frac{q}{\hbar c} \vec{A}_g \cdot \vec{E}_g. \tag{30}
\]

Notice here \(\vec{J}_{pm} = \frac{q}{\hbar c} (\varphi_g \vec{E}_g + c^2 \vec{A}_g \times \vec{B}_g)\) can be called the photon magnetic current, since it depends on the vector and scalar potential of the photon. This is so because the photon behaves as a magnetic charge (monopole). However, an electric photon electric current can also be defined as \(\vec{J}_{pe} = \frac{q\varepsilon_0}{\hbar} (\vec{A}_g \times \vec{E}_g - \varphi_g \vec{B}_g)\). This is because the photon has an effective charge following its mass. This dual behavior is so because the photon is a particle with mass and charge. The photon once behaves like a magnetic charge \((q_m)\) and then like an electric charge \((q_e)\), but not both simultaneously. These two dual behaviors are related by the uncertainty relation that \(q_m q_e = \hbar/2\) that is coined in the Dirac’s quantization rule \[12\]. It is obvious that the ordinary electrodynamics is restored when \(\vec{A}_g\) and \(\varphi_g = 0\).

**Continuity equation and static fields.** – It is worth to mention that these currents are non-dissipative; since they don’t show in the energy conservation equation of the electromagnetic fields. It is interesting to see that (27) - (30) can be seen as Maxwell’s equations in a background identified by \(\vec{A}_g\) and \(\varphi_g\). Equations similar to (28) and (29) are obtained by Tiwari \[13\].

For a pure electromagnetic system (we omit the subscripts \(g\) and \(D\) and \(m\) will be the mass of the boson field), the continuity equation in (21) will be transformed into

\[
\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = -\frac{2q}{\hbar \mu_0 c} \vec{E} \cdot \vec{B}. \tag{31}
\]

Hence, charge conservation is violated whenever the term, \(\vec{E} \cdot \vec{B} \neq 0\). The presence of this term is found to be associated with the CP (Charge conjugation - Parity) symmetry violation in quantum-chromodynamics \[14\]. It is interesting to see the quantum nature of this violating term. In free space \(\rho = 0\) and \(\vec{J} = 0\), so that \(\vec{E} \cdot \vec{B} = 0\). It is interesting to note that the electrodynamics in (27) - (30) is invariant under parity and time-reversal transformations. It is also invariant under CP transformation. The above system described by (27) - (30) is Lorentz invariant, and gauge invariant, where \(\vec{A}' = \vec{A} - \vec{\nabla} \Lambda, \ \varphi' = \varphi + \frac{\partial \Lambda}{\partial t}\), provided that

\[
\dot{\Lambda} \vec{B} = -\vec{\nabla} \Lambda \times \vec{E}, \quad \dot{\Lambda} \vec{E} = c^2 \vec{\nabla} \Lambda \times \vec{B}, \quad \vec{\nabla} \Lambda \cdot \vec{E} = 0, \quad \vec{\nabla} \Lambda \cdot \vec{B} = 0. \tag{32}
\]
Matter fields interacting with photons

They can be expressed in a covariant form that guarantees Lorentz invariance. Note that the electric and magnetic fields due to a moving charge with velocity \( \vec{v} \) are given by \( \vec{E} = \vec{v} \times \vec{B} \) and \( \vec{B} = -\frac{\vec{v} \times \vec{E}}{c^2} \). Moreover, these fields are perpendicular to the direction of motion. Hence, (32) suggests that \( \Lambda \) should satisfy the relation

\[
\vec{v} = \frac{c^2 \vec{\nabla} \Lambda}{\Lambda}.
\]

It is interesting to note that static fields electrodynamics is described by

\[
\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - \frac{qc}{\hbar} \vec{A} \cdot \vec{B},
\]

\[
\vec{\nabla} \times \vec{E} = -\frac{q}{\hbar c} \left( \varphi \vec{E} + c^2 \vec{A} \times \vec{B} \right),
\]

\[
\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} - \frac{q}{\hbar c} \left( \varphi \vec{B} - \vec{A} \times \vec{E} \right),
\]

and

\[
\vec{\nabla} \cdot \vec{B} = \frac{q}{\hbar c} \vec{A} \cdot \vec{E}.
\]

The electromagnetic currents (the two bracket in the right-hand side in (34) and (35)), are non-dissipative. Notice that in Chern-Simons theory these two currents are of topological origin [3].

**Duality transformation.** Interestingly in free space, \( \vec{J} = 0 \) and \( \rho = 0 \), there exists non-zero effective charge and current densities arising from the interactions. These are \( \rho_{pm} = \frac{q}{\hbar c} \vec{A} \cdot \vec{E} \), \( \rho_{pe} = -\frac{q\varepsilon_0}{\hbar} \vec{A} \cdot \vec{B} \), \( \vec{J}_{pe} \) and \( \vec{J}_{pm} \), respectively. In this case (27) - (30) look like symmetrized Maxwell’s equations. They are invariant under duality transformation, \( \vec{E} \rightarrow c\vec{B} \), \( c\vec{B} \rightarrow -\vec{E} \), \( \varphi \rightarrow \varphi \), \( \vec{A} \rightarrow \vec{A} \), \( \vec{J}_{pe} \rightarrow \varepsilon_0 c \vec{J}_{pm} \), and \( \vec{J}_{pm} \rightarrow -\frac{1}{c\varepsilon_0} \vec{J}_{pe} \). The application of duality transformation in axion electrodynamics is studied recently by Visinelli [15].

**Axion electrodynamics.** – Wilczek has studied the electrodynamics of axions by adding a Lagrangian term to Maxwell’s lagrangian of the form \( \mathcal{L} = \kappa \theta \vec{E} \cdot \vec{B} \), where \( \theta \) is the axion field and \( \kappa \) is a dimensionless constant. He obtained the following equations [4]

\[
\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - c\kappa \vec{\nabla} \theta \cdot \vec{B},
\]

\[
\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0,
\]

\[
\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} + \frac{\kappa}{c} \left( \dot{\theta} \vec{B} + \vec{\nabla} \theta \times \vec{E} \right),
\]

and

\[
\vec{\nabla} \cdot \vec{B} = 0,
\]

where \( \dot{\theta} = \frac{\partial \theta}{\partial t} \). The importance of axions is noted in the field of cosmology, QCD and condensed matter physics.

Interestingly, if we compare (37) - (40) with (27) - (30) then

\[
\dot{\theta} = -\frac{q}{\hbar k} \varphi, \quad \vec{\nabla} \theta = \frac{q}{\hbar k} \vec{A}, \quad \varphi \vec{E} = -c^2 \vec{A} \times \vec{B}, \quad \vec{A} \cdot \vec{E} = 0.
\]
Hence, axion fields coupled to electromagnetic field in an analogous manner photons do. (41) reveals that \( \dot{\theta} \) represents some energy scale of some system, and \( \nabla \theta \) represents its momentum. It is shown by Li et al. that magnetic fluctuations of topological insulators couple to the electromagnetic fields exactly like the axions \[10\]. It is interesting to see that the phase difference
\[
\Delta \theta = \int \nabla \theta \cdot d\ell = \frac{q}{\kappa \hbar} \int A \cdot d\ell = \frac{q}{\kappa \hbar} \phi_B .
\]
If we assume that \( \phi_B = \frac{\hbar}{q} n \), then \( \Delta \theta = \frac{2\pi}{\kappa} n \). This may indicate that \( n/\kappa \) is an integer. The two potentials, \( \vec{A} \) and \( \varphi \) could be different from those ones defined in electromagnetism.

Equation (41) implies that
\[
\vec{E} = \left( \frac{\nabla \theta}{\theta} \right) c^2 \times \vec{B} .
\]
This means that the electric field is perpendicular to the \( \nabla \theta \) and \( \vec{B} \). Since in a moving frame the magnetic field is perpendicular to the velocity vector by the relation \( \vec{E} = \vec{v} \times \vec{B} \), then \( \vec{v} = \left( \frac{\nabla \theta}{\theta} \right) c^2 \). This relation can be compared with the relativistic relation, \( \vec{v} = \frac{\vec{p}}{E} c^2 \), where \( E \) is the total relativistic energy of the particle. This relation implies that, \( \vec{v} \propto \nabla \theta \), i.e., the gradient of \( \theta \) points along the velocity direction of the axion field. The velocity is positive for a decaying (growing) scalar axion field in space and time simultaneously. It is negative otherwise. It is interesting to see that (42) is compatible with (32) if we let \( \theta = \Lambda \). Hence, (32) further implies that \( \vec{E} = \vec{v} \times \vec{B} \) and \( \vec{B} = -\frac{\vec{v}}{\varphi} \times \vec{E} \).

It is remarkable to see that the axion electrodynamics can be obtained from (33) - (36) by gauging them and allowing \( \Lambda = \theta, \vec{A} = 0 \) and \( \varphi = 0 \) such that \( \nabla \Lambda \cdot \vec{E} = 0 \) and \( \Lambda \vec{E} = c^2 \nabla \Lambda \times \vec{B} \). This is consistent with (46) and (47).

Applying (41) in (27) - (30) yields a symmetric Wilczek axion electrodynamics as
\[
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - c \kappa \nabla \theta \cdot \vec{B} ,
\]
\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \kappa \left( -\dot{\theta} \vec{E} + c^2 \nabla \theta \times \vec{B} \right) ,
\]
\[
\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} + \kappa \left( \dot{\theta} \vec{B} + \nabla \theta \times \vec{E} \right) ,
\]
and
\[
\nabla \cdot \vec{B} = \frac{\kappa}{c} \nabla \theta \cdot \vec{E} ,
\]
Differentiating the first equation in (41) and taking the divergence of the second equation in (41) yield
\[
\frac{1}{c^2} \frac{\partial^2 \theta}{\partial t^2} - \nabla^2 \theta + \frac{q}{\hbar \kappa} \left( \nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} \right) = 0 .
\]
If axions satisfy the Klein-Gordon equation then the Lorenz gauge condition will be modified to
\[
\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} = \kappa \left( \frac{m^2 c^2}{\hbar^2} \right) \theta .
\]
Note that the expression $B_c = \frac{m^2 c^2}{\hbar}$ is known as the Schwinger critical field \[17\]. Therefore, the interaction of axions gives rise to a residual magnetic field, $B_c$. Thus, if we know the field the axions create, we can estimate their masses, or vice versa. We see from (54) that the axions field is coupled to the electromagnetic field (photon). We have recently shown that the violation of Lorenz gauge condition leads to interesting consequences \[18\]. Thus, if axions are massless then the Lorenz gauge condition is satisfied. Therefore, the violation of Lorenz gauge condition indicates massive axions. In such a case axions act like massive photon in electrodynamics making the electromagnetic range finite. One can therefore relate the vector and scalar potentials, $\vec{A}$ and $\varphi$ to the axion massive fields. Moreover, axions are coupled to the electromagnetic field as is evident from (32) - (35).

In superconductivity the electromagnetic force is of short range. Thus, if axions exist inside superconductors, they will give rise to effects similar to those induced by massive photons. Hence, the residual magnetic field developed by axions, $B_c$, could be the critical magnetic field observed in superconductors. Moreover, if axions are present today, then the microwave background radiation temperature can set a limit on their masses. The energy density due to axions is $u = \frac{B^2 c}{2 \mu_0}$. This leads to a limit that $m < 4.077 \times 10^{-38} \text{kg} \ (0.023 \text{eV}/c^2)$.

It is interesting to note that despite the presence of interaction, the energy conservation equation of the system in (32) - (35) is the same as that of the ordinary electrodynamics. This implies that axions are not dynamical fields. However, the energy conservation equation of Wilczek equations, (42) - (45), involves spatial and temporal variations of the axions field $\theta$.

Let us now rewrite (48) - (51) as

\[
\vec{\nabla} \cdot \vec{E} = \frac{\bar{\rho}}{\varepsilon_0},
\]

\[
\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{J}_m,
\]

\[
\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J},
\]

and

\[
\vec{\nabla} \cdot \vec{B} = \bar{\rho}_m,
\]

where

\[
\vec{\tilde{E}} = \vec{E} + \kappa c \theta \vec{B},
\]

\[
\vec{\tilde{B}} = \vec{B} - \frac{\kappa}{c} \theta \vec{E},
\]

and

\[
\vec{\tilde{J}} = \vec{J} + \frac{\kappa}{\mu_0 c} \vec{J}_m, \quad \bar{\rho} = \rho + \frac{\kappa}{\mu_0 c} \rho_m, \quad \bar{\rho}_m = -\frac{\kappa}{\varepsilon_0 c} \rho - \frac{\kappa}{\varepsilon_0} \vec{\nabla} \theta \cdot \vec{E}, \quad \vec{\tilde{J}}_m = \frac{\kappa}{c} (\dot{\theta} \vec{E} + c^2 \vec{\nabla} \theta \times \vec{B}).
\]

Notice that $\vec{\tilde{E}}$ and $\vec{\tilde{B}}$ are connected by duality transformation when $\vec{E}$ and $\vec{B}$ are dually transformed. The above equations reduce to the ordinary Maxwell’s equations when $\theta = 0$. However, if $\theta = \text{const.}$, then $\vec{\tilde{J}} = \vec{J}$, $\bar{\rho} = \rho (1 - \kappa^2 \theta)$, $\bar{\rho}_m = -\frac{\kappa}{\varepsilon_0 c} \rho$, $\vec{\tilde{J}}_m = 0$. Hence, (49) - (52) become

\[
\vec{\nabla} \cdot \vec{\tilde{E}} = \frac{(1 - \kappa^2 \theta) \rho}{\varepsilon_0}, \quad \vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \vec{\tilde{B}}}{\partial t},
\]

\[6\] See Appendix A for alternative expression.
and
\[ \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}, \quad \nabla \cdot \vec{B} = -\frac{\kappa \theta}{\varepsilon_0 c} \rho. \]

Charge conservation in (31) can be restored if we expressed (31) and (48) in the form
\[ \nabla \cdot \vec{J}_T + \frac{\partial \rho_T}{\partial t} = 0, \quad (55) \]
where
\[ \vec{J}_T = \vec{J} - \alpha \vec{A}, \quad \rho_T = \rho - \frac{\alpha}{c^2} \varphi, \quad (56) \]
and hence
\[ \theta = -\frac{2q^2 \varepsilon_0}{\kappa n mc^2} \vec{E} \cdot \vec{B}, \quad (57) \]
where \( \alpha \) is a constant. It is thus the total current \( \vec{J}_T \) that is conserved. Notice that in free space (31) implies that charge conservation is restored. In a medium filled with axion field, the electric and magnetic fields are no longer transverse as those in free space. Equation (57) also shows that the source of the axions field is electromagnetic. Moreover, the axions field interacts equally with negative and positive charges. In London’s theory of superconductivity, \( \alpha = \frac{na^2}{m} \), where \( n \) is a number density, and hence (57) can be written as
\[ \theta = -\frac{2 \varepsilon_0}{\kappa n mc} \vec{E} \cdot \vec{B}, \quad (58) \]
Apparently, \( \theta \) is odd under parity and time-reversal transformation. It is even under TP transformation. It is also found to be responsible for the absence of CP violation symmetry in QCD [13]. Equations (58) shows that axions do not occur in empty space where \( \vec{E} \cdot \vec{B} = 0 \), and that the Lorenz gauge condition is restored, as evident from (48).

**Pure magnetic system.** Let us consider now a system in which the electric field vanishes, i.e., \( \vec{E} = 0 \) and \( \varphi = mc^2/q \). Equation (31) shows that the electric charge is conserved. Substituting this in (27) - (30) yields
\[ \rho = \frac{qc \varepsilon_0}{\hbar} \vec{A} \cdot \vec{B}, \quad (59) \]
\[ \frac{\partial \vec{B}}{\partial t} = -\frac{qc}{\hbar} \vec{A} \times \vec{B}, \quad (60) \]
\[ \nabla \times \vec{B} = \mu_0 \vec{J} - \frac{mc}{\hbar} \vec{B}, \quad (61) \]
and
\[ \nabla \cdot \vec{B} = 0. \quad (62) \]
One can associate an angular velocity of (60) for the precession of the magnetic field given by \( \vec{\omega} = \frac{qc}{\hbar} \vec{A} \). Hence, an effective charge density in (59) yields \( \rho = \varepsilon_0 \vec{\omega} \cdot \vec{B} \).

The solution of the above equations yields
\[ \nabla^2 \vec{B} + \left( \frac{mc}{\hbar} \right)^2 \vec{B} = \mu_0 \left( \frac{mc}{\hbar} \vec{J} - \nabla \times \vec{J} \right), \quad (63) \]
This pattern of solution is also obtained by Carroll et al. [19] for static fields arising from stationary neutral source, \( \rho = 0 \), with \( \nabla \cdot \vec{J} = 0 \). Now if \( \nabla \times \vec{J} = \frac{mc}{\hbar} \vec{J} \), then \( \vec{B} \) and \( \vec{J} \) are sinusoidal \((\propto \sin(k \cdot r + \phi))\) with \( k = mc/\hbar \) and \( \phi = \text{constant} \).
Consider a system in which $\vec{B} = 0$ and that $\varphi = mc^2/q$. Equation (31) shows that the electric charge is conserved. In this case (27) - (30) yield

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0},$$

$$\vec{\nabla} \times \vec{E} = -\frac{mc}{\hbar} \vec{E},$$

$$\frac{\partial \vec{E}}{\partial t} = -\frac{\vec{J}}{\varepsilon_0} - \frac{qc}{\hbar} \vec{A} \times \vec{E},$$

and

$$\vec{A} \cdot \vec{E} = 0.$$

Consistency of (64) and (65) requires that $\rho = 0$, and hence (31) yields $\nabla \cdot \vec{J} = 0$, consequently one finds

$$\nabla^2 \vec{E} + \left(\frac{mc}{\hbar}\right)^2 \vec{E} = 0.$$  

(68)

Now if the electric field is static, then the energy conservation equation (or (66)) yields the relation $\vec{J} \cdot \vec{E} = 0$. Equations (63) and (68) are noted by [19] to arise in magnetohydrodynamics. Hence, the electric and magnetic fields in (63), with $\vec{\nabla} \times \vec{J} = \frac{mc}{\hbar} \vec{J}$, and (68) satisfy the Helmholtz equation.

Maxwell’s equations in moving reference frame. – Let us now consider that $\vec{p}' = \vec{p} - q\vec{A} = 0$, or $q\vec{A} = mv$, in (27) - (30). This yields

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - \frac{mc}{\hbar} \vec{v} \cdot \vec{B},$$

$$\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t} - \frac{mc}{\hbar} \left(\vec{E} + \vec{v} \times \vec{B}\right),$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} - \frac{mc}{\hbar} \left(\vec{B} - \vec{v} \times \vec{E}\right),$$

and

$$\vec{\nabla} \cdot \vec{B} = \frac{mc}{\hbar} \vec{v} \cdot \vec{E}.$$  

(72)

Recall that in a moving frame with respect to the charge, the electric and magnetic fields are defined as $\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$ and $\vec{B}' = \vec{B} - \frac{\vec{v}}{c} \times \vec{E}$. Moreover, $\vec{v} \cdot \vec{E} = 0$ and $\vec{v} \cdot \vec{B} = 0$, $\vec{E}' = 0$ and $\vec{B}' = 0$ for transverse field created by a moving charge. In this case (69) - (72) reduce to the ordinary Maxwell’s equations. The same occurs for massless photon ($m = 0$). Hence, (69) - (72) represent Maxwell’s equations including the contribution of the massive photons. The latter fields appeared as quantum corrections to the ordinary Maxwell’s equations. The correction terms vanish for massless photons and accordingly the ordinary Maxwell’s equations are restored.

Equations (65) and (66) suggest two kinds of currents. These are electric and magnetic currents. They can be defined as follows:

$$\vec{J}_m = \frac{mc}{\hbar} \left(\vec{E} + \vec{v} \times \vec{B}\right), \quad \vec{J}_e = \frac{mc}{\mu_0 \hbar} \left(-\vec{B} + \frac{\vec{v}}{c} \times \vec{E}\right).$$  

(A)
They are independent of the charge of the moving particle but depend on its mass. They thus manifest the effect of the quantum inertial mass on the electrodynamics that is normally ignored. Such currents could lead to kinetic inductance exhibited in some electronic systems. Therefore, (A) could have interesting consequences when taken into consideration. We notice that magnetic current and charge are present whenever the velocity of the moving charge makes an angle (not right) with the electric and magnetic fields. Equation (A) suggests a Hall-like transverse current given by $\vec{J}_{tr} = \frac{m}{\hbar c} \vec{v} \times \vec{E}$. This also suggests a transverse conductivity, $\sigma_{tr} = \frac{mv}{\hbar c \mu_0}$.

Let us now consider a stationary particle, i.e., the case when $\vec{v} = 0$. This makes (69) - (72) reduce to

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0},$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \frac{mc}{\hbar} \vec{E},$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} - \frac{mc}{\hbar} \vec{B},$$

and

$$\vec{\nabla} \cdot \vec{B} = 0.$$

The consistency of the above system reveals that $\rho = 0$ and $\vec{\nabla} \cdot \vec{J} = 0$. Upon using (36), one finds that $\vec{E} \cdot \vec{B} = 0$. The solution of (73) - (76) is

$$\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \nabla^2 \vec{E} - \left(\frac{mc}{\hbar}\right)^2 \vec{E} = \frac{\partial}{\partial t} \left( \frac{2mc}{\hbar} \vec{B} - \mu_0 \vec{J} \right).$$

Thus, the electric field satisfies Klein-Gordon equation with an imaginary mass (tachyons) provided that

$$\vec{J} = \frac{2mc}{\mu_0 \hbar} \vec{B}. $$

This relation is normally reflected in a chiral magnetic effect. In this case one can define a magnetic conductivity, $\sigma_c = \frac{2mc}{\mu_0 \hbar}$. This can be written as $\sigma_c = \frac{2mc^2}{\mu_0 \hbar} \sigma_m$.

Substituting (78) in (75) to obtain the corresponding Maxwell’s equations for massive photon

$$\vec{\nabla} \cdot \vec{E} = 0,$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \frac{mc}{\hbar} \vec{E},$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \frac{mc}{\hbar} \vec{B},$$

and

$$\vec{\nabla} \cdot \vec{B} = 0.$$

The above equations reveal that the magnetic field also satisfies Klein-Gordon’s equation with an imaginary mass (tachyons). However, static fields in (79) - (82) yield the two equations,
\[ \nabla \times \vec{E} = -\frac{mc}{\hbar} \vec{E} \quad \text{and} \quad \nabla \times \vec{B} = \frac{mc}{\hbar} \vec{B} \]  
that are solved to give  
\[ \nabla^2 \vec{E} + \left( \frac{mc}{\hbar} \right)^2 \vec{E} = 0 \quad \text{and} \quad \nabla^2 \vec{B} + \left( \frac{mc}{\hbar} \right)^2 \vec{B} = 0. \]  
These are similar to (69) and (72).

Now by defining the electromagnetic field,  
\[ \vec{F} = \vec{E} + c\vec{B} \]  
(27)-(30), for the vacuum case with  \( \varphi = mc^2/q \) and  \( \vec{A} = 0 \), yield

\[ \nabla \cdot \vec{F} = 0, \quad (83) \]

and

\[ i\hbar \frac{\partial \vec{F}}{\partial t} = c\hbar \nabla \times \vec{F} + mc^2 \vec{F}. \quad (84) \]

Equation (84) can be written as

\[ i\hbar \frac{\partial \vec{F}}{\partial t} = -i c\hbar \vec{S} \cdot \nabla \vec{F} + mc^2 \vec{F}, \quad (85) \]

where  \( \vec{S} \) is spin-1 matrices \[22,24\]. This suggests that (85) is a quantum equation for massive photon whose Hamiltonian is defined by

\[ H = c\vec{S} \cdot \vec{p} + mc^2. \quad (86) \]

Moreover, as evident from (85),  \( \vec{F} \) satisfies the Klein-Gordon equation with a source as

\[ \frac{1}{c^2} \frac{\partial^2 \vec{F}}{\partial t^2} - \nabla^2 \vec{F} + \left( \frac{mc}{\hbar} \right)^2 \vec{F} = -2mc \nabla \times \vec{F}. \quad (87) \]

Using (84) this becomes

\[ \frac{1}{c^2} \frac{\partial^2 \vec{F}}{\partial t^2} - \nabla^2 \vec{F} + \frac{2mi}{\hbar} \frac{\partial \vec{F}}{\partial t} - \left( \frac{mc}{\hbar} \right)^2 \vec{F} = 0. \quad (88) \]

It is interesting that (88) is the Dirac’s equation for a free spin-1/2 particle which is normally expressed as a first-order differential equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -i c\vec{\gamma} \cdot \nabla \psi + \beta mc^2 \psi, \quad (89) \]

where  \( \psi \) is the spinor. By taking the second term in the right hand-side in (89) to the other side, and squaring the two sides of the resulting equation, we obtain an equation of the form in (88). Despite the fact that Dirac’s equation describes a spin-1/2 particles and Maxwell’s equations describe spin-1 particle which belong to distinct irreducible representations of the Poincare group, they exhibit a similarity shown above. Thus, when  \( \beta = 1 \),  \( \vec{\gamma} \rightarrow \vec{S} \),  \( \psi \rightarrow \vec{F} \), and  \( m_e \rightarrow m \). This urges us to explore this deep connection. Thus, (88) can be seen as the quantum equation of the massive photon. It is worth to remark that moving the second term in the right - hand side in (89) to the left side and squaring the resulting operator equation yield  \[8\].

\[ \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + \frac{2m\beta i}{\hbar} \frac{\partial \psi}{\partial t} - \left( \frac{mc}{\hbar} \right)^2 \psi = 0. \quad (90) \]

Because of the  \( \beta \) term in (90), we have two solutions (particle-antiparticle), whereas (88) assumes only one solution. This is because the anti-photon is the same as the photon, while the anti-electron is a positron.
The energy conservation equation associated with (84) can be written as

$$\vec{\nabla} \cdot \left( \frac{\vec{F} \times \vec{F}^*}{\mu_0} \right) + \frac{\partial}{\partial t} \left( \frac{-i}{\mu_0 c} \vec{F} \cdot \vec{F}^* \right) = 0.$$  (91)

This shows that the energy conservation equation for massive photon in vacuum is the same as that for massless photon.

**Static fields.** Consider now static electric and magnetic fields. Hence, (69) - (72) reduce to

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - \frac{mc}{q_m \hbar} P_m, \quad P_m = q_m \vec{v} \cdot \vec{B},$$  (92)

$$\vec{\nabla} \times \vec{E} = \frac{mc}{q_e \hbar} \vec{F}_e, \quad \vec{F}_e = q_e \left( \vec{E} + \vec{v} \times \vec{B} \right),$$  (93)

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} - \frac{mc}{q_m \hbar} \vec{F}_m, \quad \vec{F}_m = q_m \left( \vec{B} - \frac{\vec{v}}{c^2} \times \vec{E} \right),$$  (94)

and

$$\vec{\nabla} \cdot \vec{B} = \frac{m}{q_e c \hbar} P_e, \quad P_e = q_e \vec{v} \cdot \vec{E}.$$  (95)

Interestingly, (92) - (95) involve the fields, the force and the power on the moving particle. Taking the dot product of the velocity with (94) and using (92) yield

$$\vec{\nabla} \cdot \left( \vec{E} + \vec{v} \times \vec{B} \right) = \frac{1}{\varepsilon_0} \left( \rho - \frac{\vec{v}}{c^2} \cdot \vec{J} \right),$$  (96)

This can be expressed as Gauss’s law in the moving frame

$$\vec{\nabla} \cdot \vec{E}' = \frac{\rho'}{\varepsilon_0}, \quad \vec{E}' = \vec{E} + \vec{v} \times \vec{B}, \quad \rho' = \rho - \frac{\vec{v}}{c^2} \cdot \vec{J}.$$  (97)

Similarly taking the divergence of (94) and using the fact that \(\vec{\nabla} \cdot \vec{J} = 0\) one obtains

$$\vec{\nabla} \cdot \vec{B}' = 0, \quad \vec{B}' = \vec{B} - \frac{\vec{v}}{c^2} \times \vec{E}.$$  (98)

Taking the cross product of the velocity with (93) and using (92) and (94) yield Ampere’s and Faraday’s equations in the moving frame

$$\vec{\nabla} \times \vec{B}' = \mu_0 \vec{J}', \quad \vec{\nabla} \times \vec{E}' = 0, \quad \vec{J}' = \vec{J} - \rho \vec{v}.$$  (99)

where the two vector identities \(\vec{\nabla} \times (\vec{v} \times \vec{E}) = \vec{v} (\vec{\nabla} \cdot \vec{E}) - (\vec{v} \cdot \vec{\nabla}) \vec{E}\) and \(\vec{v} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{v} \cdot \vec{E}) - (\vec{v} \cdot \vec{\nabla}) \vec{E}\) are employed. Therefore, (92) - (95) are equivalent to Maxwell’s equations in a moving frame with constant velocity where the fields are static.

**Maxwell’s equations inside a medium.** The system of equations, (27) - (30), can be seen as Maxwell’s equations inside a medium with polarization and magnetization vectors, \(\vec{P}\) and \(\vec{M}\), respectively. In this case Maxwell’s equations read (omitting the subscript \(g\))

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \left( \rho - \vec{\nabla} \cdot \vec{P} \right),$$  (100)
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (101) \]
\[ \nabla \times \vec{B} = \mu_0 \left( \vec{J} + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}, \quad (102) \]

and
\[ \nabla \cdot \vec{B} = 0, \quad (103) \]

where
\[ \nabla \cdot \vec{P} = \lambda \vec{A} \cdot \vec{B}, \quad (104) \]
\[ \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} = -\lambda \left( \varphi \vec{B} - \vec{A} \times \vec{E} \right), \quad (105) \]
\[ 0 = \varphi \vec{E} + c^2 \vec{A} \times \vec{B}, \quad (106) \]

and
\[ 0 = \vec{A} \cdot \vec{E}, \quad \lambda = \frac{qc\varepsilon_0}{\hbar}. \quad (107) \]

Taking the dot product of \( \vec{A} \) with (105) and using (104) yield the conservation equation
\[ \nabla \cdot \left( \varphi \vec{P} - \vec{A} \times \vec{M} \right) + \frac{\partial}{\partial t} (\vec{A} \cdot \vec{P}) = -\vec{E} \cdot \vec{P} - \vec{B} \cdot \vec{M}. \quad (108) \]

If we now let \( \vec{M} = 0 \), then (108) reduces to
\[ \nabla \cdot \left( \varphi \vec{P} \right) + \frac{\partial}{\partial t} (\vec{A} \cdot \vec{P}) = -\vec{E} \cdot \vec{P}. \]

The divergence of (105) using (104), (101) and (102), yield
\[ \varepsilon_0 E^2 - \frac{B^2}{\mu_0} = \varphi \rho - \vec{A} \cdot \vec{J}. \quad (109) \]

This equation can be compared with (16). It can be written in a covariant form as
\[ \frac{1}{2\mu_0} F^\mu_\nu F^\nu_\mu = A^\sigma J_\sigma. \quad (110) \]

It is interesting to see that (27) - (30) reduce to ordinary Maxwell’s equations if
\[ \vec{B} = \hat{A} \times \frac{\vec{E}}{c}, \quad \vec{E} = c\hat{A} \times \vec{B}. \quad (111) \]

Hence, the three vectors, \( \vec{A}, \vec{E} \) and \( \vec{B} \) are mutually orthogonal, and that \( \varphi = cA \).

**Concluding remarks.** – We have extended the biquaternionic Dirac’s equation to include interactions with photons. The interactions of the vector and scalar potentials of the photon with the inertial and magnetic fields made the derived Maxwell-like equations to deviate from the free ones. The magnetic mass and current densities arising from the photon interactions with the matter are found to preserve the magnetic mass. The concept of magnetic mass is introduced but no further definition is given. The axions electrodynamics developed by Frank
Wilczek is found to be a special case of the present electrodynamics. In this case the temporal and spatial variations of the axions field are related to the scalar and vector potentials of the photon, respectively.

Axion fields are found to give rise to a residual magnetic field (a Schwinger critical field type) when interacting with the charged particles. Massive axions are found to be analogous to massive photons. The conservation of electric charge is related to the direction in which the electric and magnetic fields make inside the medium. However, the total charge of the system is conserved. The electrodynamics equations in free space are invariant under duality transformation. We have shown that the electromagnetic field vector, \( \vec{F} \), describing massive photons satisfies Dirac’s equation with spin-1 matrices so that \( \nabla \cdot \vec{F} = 0 \).

Moreover, the electrodynamics equations we obtained are shown to express the electromagnetic fields in addition to the electric and magnetic fields developed by the existing charges in the region. The chiral magnetic effect is shown to be associated with the additional current appearing in Ampère’s equation that is proportional to the magnetic field.

It is shown that axions find applications in cosmology particularly it can provide significant contribution to dark matter bewildering astronomers. Besides, phenomena like topological insulator and chiral magnetic effect appearing in the realm of condensed matter can be rigorous investigated in the framework of the present electrodynamics. We undertake to pursue these issues.

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Appendix A: Alternative expression. – Equations (27) - (30) can be written as

\[ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial \tau} = 0, \quad \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial \tau} = \mu_0 \vec{J} \]  \hspace{1cm} (112)

where the operators

\[ \vec{\nabla} \times = \vec{\nabla} \times + \frac{q \varphi_g}{\hbar c}, \quad \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \vec{\omega} \times, \quad \vec{\omega} = \frac{qc}{\hbar} \vec{A}_g, \]  \hspace{1cm} (113)

replaces the usual curl operator. Hence, the above definition suggests that Ampere’s and Faraday’s equations are expressed in a non-inertial frame (rotation frame) having angular velocity \( \vec{\omega} \).

Appendix B: Modified Maxwell’s equations. – If we now let \( \Lambda = \beta \psi_0 + c \vec{g} \cdot \vec{\psi} \) and abandon the condition in (10), then (5) - (8) become

\[ \vec{\nabla} \cdot \vec{E}_D = \frac{\rho_D}{\varepsilon_0} - \frac{qc}{\hbar c} \vec{A}_g \cdot \vec{B}_D - \frac{q}{\hbar} \varphi_g \Lambda, \]  \hspace{1cm} (114)

\[ \vec{\nabla} \cdot \vec{B}_D = \frac{q}{\hbar c} \vec{A}_g \cdot \vec{E}_D + \frac{1}{c} \frac{\partial \Lambda}{\partial t}. \]  \hspace{1cm} (115)

and

\[ \vec{\nabla} \times \vec{E}_D = -\frac{\partial \vec{B}_D}{\partial t} - \frac{q}{\hbar c} \varphi_g \vec{E}_D - \frac{qc}{\hbar} \vec{A}_g \times \vec{B}_D + c \vec{\nabla} \Lambda, \]  \hspace{1cm} (116)

\[ \vec{\nabla} \times \vec{B}_D = \mu_0 \vec{J}_D + \frac{1}{c^2} \frac{\partial \vec{E}_D}{\partial t} - \frac{q}{\hbar c} \varphi_g \vec{B}_D + \frac{q}{\hbar c} \vec{A}_g \times \vec{E}_D - \frac{q}{\hbar} \vec{A}_g \Lambda, \]  \hspace{1cm} (117)

Equations (114) - (117) are also valid for the electromagnetic field by dropping the subscript D from all terms.