Among Quadratic Hamiltonians, Bogoliubov Transformations and Non-Regular States on CCRs *-Algebra.

I. Pure and Invariant States.
(In the Mood for the Manuceau Verbeure Theorems about Quasi-free States and Automorphisms of the CCR Algebra)

S. A. Chorošavin
March 28, 2022

Abstract

The paper’s features are these:

1) we discuss especially quadratic (alias bilinear) Bose-Hamiltonians, the related Bogoliubov transformations and especially quasi-free-like (alias coherent or Fock-like) states

2) we discuss any quadratic Bose-Hamiltonians and Bogoliubov transformations, whether diagonalizable or not, whether proper or improper, and arbitrary quasi-free-like states, whether regular or non-regular they are

3) we associate notions and terms of the CCRs theory with notions and terms of the indefinite inner product spaces theory. Then, we apply the corresponding ‘bilingual dictionary’ so as to construct invariant states of some of the quadratic Hamiltonians.

1CCRs = Canonical Commutation Relations
1 Introduction

Quadratic (bilinear) Fermi-Hamiltonians have a very attractive property. One can diagonalize them and the way is not unique, as a result, given a Fermi-Hamiltonian, there exists a Fock-like invariant state, and such a state is not unique. Nothing really like this is present in the case of quadratic Bose-Hamiltonians. By the contrast, there are cases, e.g., the case of the repulsive oscillator, where one cannot diagonalize the Bose-Hamiltonian and Fock-like invariant states do not exist at all.

Here we shall say more precisely: there exists no Fock-like invariant state with continuous characteristic function.

But why ‘to diagonalize’ and even why ‘continuous’? One needs firstly invariant states with a definite algebraic structure.

We have tried to compose suitable constructions and here, in this paper, they are presented.\footnote{Fock-like state = even quasi-free state}

\footnote{To be more precise, I must say: this paper is a very large abstract of [Ch1], [Ch2], [Ch3] (and the summarizing [Ch4]), but proofs of theorems omitted.}
2 Prerequisites: The Quadratic Hamiltonians, Canonical Commutation Relations, and Bogoliubov Transformations

This section consists primarily of formal constructions and manipulations as we would like briefly to explain what we mean by ‘Quadratic Hamiltonians, CCRs = Canonical Commutation Relations’ and ‘Bogoliubov (Canonical)Transformations.’

The Quadratic Hamiltonian of an $N$-degree of freedom system is here a formal expression

$$h = \sum_{k,l} \left( s_{kl} a_k^* a_l - \frac{1}{2} t_{kl} a_k a_l - \frac{1}{2} t_{lk} a_l^* a_k^* \right)$$

where

$$s_{kl} = s_{lk} ; \quad t_{kl} = t_{lk}$$

and $a_k^*, a_l; k, l = 1, \ldots, N$ are thought of as elements of an (associative) *-algebra.

We will suppose $a_k^*, a_l; k, l = 1, \ldots, N$ to be subject to the relations

$$[a_k, a_l^*] = (a_k a_l^* - a_l^* a_k) = \delta_{kl} ; \quad [a_k, a_l] = 0 ; \quad [a_l^*, a_k^*] = 0$$

The relations are said to be the Canonical Commutations Relations in the Fock-Dirac form.

Let us write

$$u := (u_1, \ldots, u_N) ; \quad u^+ := (u_1^+, \ldots, u_N^+) ; \quad u^- := (u_1^-, \ldots, u_N^-) ;$$

$$a^+(u) := u_1 a_1^* + \cdots + u_N a_N^*$$

$$a(u) := (a^+(u))^* \equiv u_1^* a_1 \cdots u_N^* a_N$$

$$a^-(u) := a(u^*) \equiv u_1 a_1^* \cdots u_N a_N \equiv a^+(\overline{u})^*$$

(hence $a^+(u) = a(u)^* = a^-(\overline{u})^*$) and in addition we set

$$A(u^+ \oplus u^-) := a^+(u^+) + a^-(u^-) \equiv u_1^* a_1^* + \cdots u_N^* a_N^* + u_1 a_1 + \cdots u_N a_N$$

Formal calculations show that

$$[h, A(u^+ \oplus u^-)] = A(u'^+ \oplus u'^-)$$

where $u'^+ \oplus u'^-$ is defined by

$$\begin{pmatrix} u'^+ \\ u'^- \end{pmatrix} = \begin{pmatrix} S & T \\ \overline{T} & \overline{S} \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

Here $S, T, \overline{T}, \overline{S}$ stand for the operators associated with the matrices $\{s_{kl}\}_{kl}, \{t_{kl}\}_{kl}, \{\overline{t_{kl}}\}_{kl}, \{\overline{s_{kl}}\}_{kl}:

(Su)_k = \sum_l s_{kl} u_l \ , \ (Tv)_k = \sum_l t_{kl} v_l \ , \ etc.$

Note

$$S^* = S ; \quad T^* = \overline{T} ; \quad \overline{S} = \overline{S}$$

Next, we observe that

$$[A(\cdots), A(\cdots)]$$

---

4 Throughout this paper we assume that units of measure are chosen and fixed so that $\hbar = 1$ and so that the momentum quantity, $P$, and the position quantity, $Q$, both become dimensionless quantities.

5 $N$ may be infinite
is scalar-valued (up to the multiplier \( I = \) the unity of the *-algebra):

\[ [A(u^+ \otimes u^-)^*, A(u^+ \otimes u^-)] = < u^+u^-, u^+>_0 - < u^-u^-, u^->_0 \]

where \( < u^+, u^->_0 \) stands for the usual inner product in \( \mathbb{C}^N \):

\[ < u^+, u^->_0 = u_1^1u_1^2 + u_2^2 + \cdots + u_N^Nu_N \]

Motivated by this, we define:

\[ < u^+u^-, u^+u^-> := [A(u^+u^-)^*, A(u^+u^-)] \]

and we note that \( < u^+u^- > \) is an indefinite inner product on the “space of coefficients” of \( a^*, a \).

One can write:

\[ < u^+u^- > = (u^+u^-_0, u^+u^-) \]

where one sets:

\[ J_{a^*a} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \]

Similar formulæ hold for the **Heisenberg-Dirac form of CCRs**:

\[ i [P_k, Q_l] = \delta_{kl} \; i [P_k, P_l] = 0 \; i [Q_k, Q_l] = 0 \; Q^* = Q \; P^* = P \]

We will write

\[ F(x_p \oplus x_q) := x_pP_1 + \cdots + x_pP_N + x_qQ_1 + \cdots + x_qQ_N \]

We use this notation whether \( x_p \oplus x_q \) is a real-valued vector or complex-valued. In addition, we put

\[ P(x_p) := x_pP_1 + \cdots + x_pP_N \quad Q(x_q) := x_qQ_1 + \cdots + x_qQ_N \]

The quadratic Hamiltonian is now a formal expression

\[ \hbar := \frac{1}{2} \sum_{l,m} (M_{lm}P_lP_m - L_{lm}(P_lQ_m + Q_mP_l) + K_{lm}Q_lQ_m) \]

with \( M^T = M, K^T = K \). Then

\[ i \hbar [F(x_p \oplus x_q)] = F(x_p' \oplus x_q') \]

where

\[ \begin{pmatrix} x_p' \\ x_q' \end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p \\ x_q \end{pmatrix} \]

Next, we observe that formally

\[ -i [F(x_p' \oplus x_q'), F(x_p \oplus x_q)] = -(x_p', x_q) + (x_p', x_q) \]

where \((\cdot, \cdot)\) stands for the usual Euclidian-like inner product, alias scalar product of vectors:

\[ (x', x)_0 := x_1'x_1 + x_2'x_2 + \cdots + x_N'x_N \]

So, if we define

\[ s(x_p' \oplus x_q', x_p \oplus x_q) := -i [F(x_p' \oplus x_q'), F(x_p \oplus x_q)] \]

then the \( s \) becomes a \( \mathbb{C} \)-symplectic form, i.e., a bilinear anti-symmetric form on a complex space.

One can write also:
suggests solving the dynamical equations, in particular these formulae as a tenet, as an axiom.

\[ V \]

We will suppose, the solution to the equation exists and let

\[ a^- = \frac{1}{\sqrt{2}}(Q + iP), \quad a^+ = \frac{1}{\sqrt{2}}(Q - iP), \quad Q = \frac{1}{\sqrt{2}}(a^+ + a^-), \quad P = \frac{i}{\sqrt{2}}(a^+ - a^-) \]

Hence,

\[ A(u^+ \oplus u^-) = \frac{1}{\sqrt{2}}F(-iu^+ + iu^- \oplus u^+ + u^-) \]

\[ F(x_p \oplus x_q) = \frac{1}{\sqrt{2}}A(ix_p + x_q \oplus -ix_p + x_q) \]

The formal calculations show \( e^{iF(x_p \oplus x_q)}e^{iF(x_p' \oplus x_q')} = e^{-is(x_p \oplus x_q, x_p' \oplus x_q')}e^{iF(x_p + x_p' \oplus x_q + x_q')} \)

\( (e^{iF(x_p \oplus x_q)})^* = e^{-iF(x_p \oplus x_q)} \)

These formulae are the so-called exponential form of the CCRs. We adopt these formulae as a tenet, as an axiom.

If one restricts himself to the case of real-valued \( x_p, x_q \), then one has the relations

\[ e^{iF(x_p \oplus x_q)}e^{iF(x_p' \oplus x_q')} = e^{-is(x_p \oplus x_q, x_p' \oplus x_q')}e^{iF(x_p + x_p' \oplus x_q + x_q')} \]

\( (e^{iF(x_p \oplus x_q)})^* = e^{-iF(x_p \oplus x_q)} \).

This form of the CCRs is called Weyl \( ^6 \) \( ^7 \) \( ^8 \).

If a Hamiltonian has been given, the standard quantum mechanics practice suggests solving the dynamical equations, in particular

\[ \frac{\partial F(x_p \oplus x_q)(t)}{\partial t} = i[h, F(x_p \oplus x_q)(t)] \]

or “equivalently”

\[ \frac{\partial A(u^+ \oplus u^-)(t)}{\partial t} = i[h, A(u^+ \oplus u^-)(t)] \]

A formal calculation allows one to rewrite this equation as

\[ \frac{\partial}{\partial t} \begin{pmatrix} x_p(t) \\ x_q(t) \end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p(t) \\ x_q(t) \end{pmatrix} \]

respectively

\[ \frac{\partial}{\partial t} \begin{pmatrix} u^+(t) \\ u^-(t) \end{pmatrix} = i \begin{pmatrix} S & T \\ -T^T & -S \end{pmatrix} \begin{pmatrix} u^+(t) \\ u^-(t) \end{pmatrix} \]

We will suppose, the solution to the equation exists and let \( V(t, s) \) denote the corresponding propagator.

Then, \( V(t, s) \) lifts to an *-automorphism \( \alpha_{V(t, s)} \) of CCRs by

\[ \alpha_{V(t, s)}F(x_p \oplus x_q) := F(V(t, s)(x_p \oplus x_q)) \]

---

\( ^6 \) see Appendix A

\( ^7 \) In this case, \( e^{iF(x_p \oplus x_q)} \) is unitary.

\( ^8 \) and \( s(\cdot, \cdot) \) is the usual (pre)symplectic form, i.e., the real-valued anti-symmetric form on real space.

\( ^9 \) It means evolution operator.
α (t, s) e^{iF(x_p \oplus x_q)} := e^{iF(V(t, s)(x_p \oplus x_q))}

This *-automorphism is called either the linear canonical transformation or Bogoliubov transformation or quasi-free automorphism of a Bose system.

Given a “physical system” state \( \omega \), e.g., whether a ground state or a state with an interesting energy distribution or a state with the momentum at given exact value (“plane wave state”) or something like that, and given an observable \( A \), i.e., something like a momentum, energy, position, spin, particles number, etc., we will denote the corresponding expectation value of \( A \) at the state \( \omega \) by \( \omega A \). The functionals

\[ x_p \oplus x_q \mapsto \omega e^{iF(x_p \oplus x_q)} \]

and

\[ u^+ \oplus u^- \mapsto \omega e^{A(u^+ \oplus u^-)} \]

are called the characteristic functionals or characteristic functions of the state \( \omega \). If the function

\[ \lambda \in \mathbb{R} \mapsto \omega e^{iF(x_0^p \oplus x_0^q + \lambda x_p \oplus x_q)} \]

is continuous, then the state \( \omega \) is called regular. As a rule, one assumes therewith that the coefficients \( x_0^p \oplus x_0^q \) are real-valued. If

\[ \omega e^{iF(x_p \oplus x_q)} = e^{-\frac{1}{4}(ax_p^2 + bx_q^2)} , \quad a > 0, b > 0 \]

then \( \omega \) is called even quasi-free.

We will slightly extend the class of these states and choose a definition of the even quasi-free-like states which emphasizes the latter, algebraic, property and partially deemphasizes the continuity property. The main idea behind the states we will discuss is briefly this: Let us suppose that we consider a one-dimensional system and even quasi-free states given by

\[ \omega e^{ix_p P + ix_q Q} = e^{-\frac{1}{4}(ax_p^2 + bx_q^2)} , \quad a > 0, b > 0 \]

The uncertainty principle prescribes \( ab \geq 1 \), and therefore we could not assign \( a := 0 \), whatever real number \( b \) we had chosen. But why not \( a := 0 \), \( b := +\infty \)?” i.e., why not

\[ \omega e^{ix_p P + ix_q Q} := \begin{cases} 1, & \text{if } x_q = 0 \\ 0, & \text{if } x_q \neq 0 \end{cases} \]

Or why not?:

\[ \omega e^{ix_p P + ix_q Q} := \begin{cases} e^{-bx_q^2/4}, & \text{if } x_p = 0 \\ 0, & \text{if } x_p \neq 0 \end{cases} \]

Actually, we may take \( \epsilon > 0 \) and define states \( \omega_\epsilon \) by

\[ \omega_\epsilon e^{ix_p P + ix_q Q} := e^{-\frac{1}{4}(x_p^2/\epsilon + (b + \epsilon)x_q^2)}, \quad b \geq 0 \]

Then we may take limit of \( \omega_\epsilon \) as \( \epsilon \to +0 \) without loss of the main algebraic property of states being positive definite, although we lose partially the continuity property. Thus we obtain just

\[ \omega_{+0} e^{ix_p P + ix_q Q} := \lim_{\epsilon \to +0} \omega_\epsilon e^{ix_p P + ix_q Q} = \begin{cases} e^{-bx_q^2/4}, & \text{if } x_p = 0 \\ 0, & \text{if } x_p \neq 0 \end{cases} \]

and we can refer to such states as abstractions of the usual states, perhaps as artificial abstractions. These states as well as any \( N \)-dimensional analogue, we will call such states the even quasi-free-like or quadratic states.

\[ ^{10} \text{a precise definition see in the next Section} \]
Among Quadratic Hamiltonians

It’s our object.

**COMMENT**

The more detailed description of the notions in this section, one can find in e.g., [Ber], [BR2], and especially [Fey, Stat.Mech.]. Notice that our definition of $P$ and $Q$ slightly differs from the standard. As a rule one sets $P$ and $Q$ so as to

$$a = \frac{1}{\sqrt{2}} (Q + iP), \quad a^* = \frac{1}{\sqrt{2}} (Q - iP), \quad Q = \frac{1}{\sqrt{2}} (a^* + a), \quad P = \frac{i}{\sqrt{2}} (a^* - a)$$
3 Quadratic and Quasi-free States on CCRs Algebra and Quasi-Free-like Atomorphisms

In this section we discuss notions related to that introduced in the previous section.

First, we give an abstract axiomatic definition of the CCRs.

Let \( Z \) be a real vector space, \( s : Z \times Z \to \mathbb{R} \) a bilinear antisymmetric form (\( s \) need not be nondegenerated) and \( V : Z \to Z \) be a linear operator.

Let \( \mathbb{K}, <,> \) be the standard complexification of \( Z, is(,), V \), i.e. \( \mathbb{K} \) is the standard \( \mathbb{R} \)-linear doubleculation of \( Z \):

\[
\mathbb{K} := \mathbb{C}Z = Z \oplus \mathbb{R}Z,
\]

the multiplication with \( i \) is given by

\[
i(f \oplus g) := (-g) \oplus f,
\]

and \( <,> \) is the standard sesquilinear extension of \( is(,) \) and \( V_C \) is the standard \( \mathbb{C} \)-linear extension of \( V \).

Often, we will write \( f + ig \) instead of \( f \oplus g \)

and next the symbol \( C \) will denote the natural complex conjugation in \( \mathbb{K} \):

\[
C(f + ig) := f - ig, \quad (f, g \in Z)
\]

So, \( \mathbb{K}, <,> \) is indefinite inner product space with \( <,> \)-antiunitary involution \( C \):

\[
C^2 = I, \quad < Cf, Cg >= < g, f >, \quad (f, g \in Z)
\]

Remark 1.

We take it as known that \( V \) is a homomorphism of \( s(,) \), i.e.

\[
s(Vf, Vg) = s(f, g) \quad (\forall f, g \in Z)
\]

iff and \( V_C \) is a \( <,> \)-isometric operator. Similarly \( V \) is an automorphism of \( s(,) \) i.e.

\[
V \text{ is bijective and } s(Vf, Vg) = s(f, g) \quad (\forall f, g \in Z)
\]

iff \( V_C \) is a \( <,> \)-unitary operator.

Definition 1.

Abstract Weyl *-algebra, we denote it by \( W_{Z,s} \), is here a free *-algebra on the symbols \( \epsilon \hat{f}, \hat{f} \in Z \) subject to the relations

\[
(\epsilon \hat{f})^* = e^{-i\epsilon} \hat{f}, \quad \epsilon \hat{f} \hat{g} = e^{-is(\hat{f}, \hat{g})/2} \epsilon \hat{f} + \hat{g}.
\]

Remark 2. (cf. e.g., [MV],[BR])

If \( V \) is an automorphism of \( s \), then the correspondence \( \epsilon \hat{f} \mapsto \epsilon^{Vf} \) induces a *-automorphism; this *-automorphism is called quasi-free, often, Bogoliubov *-automorphism, alias Bogoliubov transformation. We will denote it by \( \alpha_V \).

If \( \chi \) is a *-character of the additive group \( Z \) i.e. if

\[
\chi(f + g) = \chi(f)\chi(g), \quad \chi(f)^* = \chi(-f), \quad (\forall f, g \in Z)
\]

then the correspondence \( \epsilon \hat{f} \mapsto \chi(\epsilon^{Vf}) \) extends to a gauge-like *-automorphism, alias coherent *-automorphism; we denote it by \( \alpha_\chi \).

If

\[
\chi(f) = e^{it(f)}
\]
where \( l \) is a real-valued \( \mod 2\pi \)-additive function on \( Z \), then we prefer to write \( \alpha_l \) instead of \( \alpha_\chi \).

The automorphisms of the form
\[
\alpha_{V,\chi} := \alpha_V \alpha_\chi, \\
(\alpha_{\chi,V} := \alpha_\chi \alpha_V)
\]
are called quasi-free-like.

The rest of this section until Example 1 is an extending modification of the Manuceau Verbeure Theory of quasi-free states.

It will be convenient to change (equivalently!) the usual definition of positive quadratic form.

**Definition 2.**
We will say that \( q : Z \to [0, \infty] \) is **quadratic** iff
\[
q(f + g) + q(f - g) = 2[q(f) + q(g)],
\]
\[
q(kf) = k^2 q(f) \quad (f, g \in Z, k \in R)
\]
(hereafter \( 0 \cdot \infty = 0, \infty + \infty = \infty \) and so on)

The set \( Q(q) := \{ f \in Z | q(f) < \infty \} \equiv \{ f \in Z | q(f) \neq \infty \} \) is called the **form domain** or the **domain** of \( q \). If \( Q(q) = Z \), then \( q \) is called **finite**.

Given two quadratic \( q_1, q_2 \), we write
\[
q_1 \leq q_2 \quad \text{iff} \quad q_1(f) \leq q_2(f) \quad (\forall f \in Z)
\]

**Remark 3.**
Given a quadratic \( q \), the form domain of \( q \) is linear, and \( q \) is associated with a unique symmetric bilinear positive (if \( Z \) is over \( \mathbb{R} \)) or symmetric sesquilinear positive form (if \( Z \) is over \( \mathbb{C} \)), we denote it by \( q(\cdot, \cdot) \); this form can be recovered from the \( q \) by the **polarization identity**
\[
q(f, g) = \frac{1}{2}(q(f + g) + q(f - g)) \quad (\text{if } Z \text{ is over } \mathbb{R})
\]
resp.
\[
q(f, g) = \frac{1}{4}(q(f + g) + q(f - g) - i q(f + i g) + i q(f - i g)) \quad (\text{if } Z \text{ is over } \mathbb{C})
\]

**Definition 3.**
We say \( q \) is a quadratic-like **majorant of** \( s \), iff
\[
2|s(f, g)| \leq q(f) + q(g) \quad (f, g \in Z)
\]
and if, of course, \( q \) in itself is quadratic.

**Proposition 1.**

For any majorant \( q \), there exists a minimal quadratic-like majorant, say \( q_0 \), such that \( q_0 \leq q \). Hereafter, we mean by ‘\( q_0 \) is a minimal majorant’ that, if \( q_1 \leq q_0 \) for a quadratic majorant \( q_1 \), then \( q_1 = q_0 \).

**Definition 4.** cf. [Oks]
We say that a linear *-functional \( \omega \) on \( W_{Z,s} \) is **quadratic** (alias even quasi-free-like, generalized even quasi-free) iff
\[
\omega \hat{f} = e^{-q(f)/4} \quad (e^{-\infty} = 0)
\]
Theorem 1.
(i) A quadratic ω is a state iff the associated q is a quadratic-like majorant of s.
(ii) A quadratic ω is a pure state iff the associated q is a minimal quadratic-like majorant of s.

Example 1.
Put \( q(f) := 0 \) at \( f = 0 \) and \( q(f) := +\infty \) otherwise, i.e., define a linear functional \( \delta_0 \) on \( W_{Z,s} \) so that

\[
\delta_0 f := \begin{cases} 
1, & \text{if } f = 0 \\
0, & \text{if } f \neq 0 
\end{cases}
\]

Then, the q is a quadratic-like majorant, called trivial, and \( \delta_0 \) is trivial state. Notice (e.g., [BR2, p.79], EXAMPLE 5.3.2), the \( \delta_0 \) is a trace-state on \( W_{Z,s} \). In addition, this state is invariant under all Bogoliubov transformations.

Definition 5.
Given a quadratic q, we denote its standard complexification by \( q_C \). We define it so:

\[
q_C(f + ig) := q(f) + q(g)
\]

A similar notation is given to the complexification of an arbitrary linear \( T : Z \to Z \):

\[
T_C(f + ig) := Tf + iTg.
\]

Remark 4.
For the complex space case the definition of the sentence ‘a functional, e.g., \( q_C \), is quadratic’ is to be modified:

\[
q_C(kz) = |k|^2 q_C(z).
\]

The rest of the definition remains as before.

Observation 1.
Given a linear \( \tilde{T} : K \to K \), it is of the form \( \tilde{T} = T_C \) for a suitable \( T : Z \to Z \) iff

\[
C \tilde{T} C = \tilde{T}.
\]

Similarly, for any quadratic \( \tilde{q} : K \to [0, \infty] \), there is a quadratic \( q : Z \to [0, \infty] \) such that \( \tilde{q} = q_C \) iff

\[
\tilde{q} C = \tilde{q}.
\]

Theorem 2.
(i) \( q \) is a quadratic-like majorant of \( s \) iff \( q_C \) is a quadratic-like majorant of \( <,> \)
(ii) \( \tilde{q} \) is a minimal quadratic-like majorant of \( s \) iff \( q_C \) is a minimal quadratic-like majorant of \( <,> \)

Remark 5.
We will deal, first and foremost, with quadratic-like majorants. So, if no confusion can occur, we will omit the particle ‘-like’ or the whole word ‘quadratic-like’, although occasionally we will repeat the whole term ‘quadratic-like majorant’ for emphasis.

If a new object is declared, the first question is whether this object does exist. Of course, automorphisms, majorants and invariant (under Bogoliubov transformation) quadratic states, all these objects do exist. Interestingly enough, a finite majorant

---

and this is the unique trace-state if \( s(\cdot,\cdot) \) is nondegenerate
need not exist; a quasi-free *-automorphism need not have an invariant non-trivial quadratic state.

Example 2. ([Bog, p. 62-63, Example 3.2])

Let \( \mathbb{H} \) be the vector space of those doubly infinite numerical sequences where only a finite number of terms with negative index is different from zero, and for \( f = \{ \xi_j \}_{j \in \mathbb{Z}} \in \mathbb{H} \), \( g = \{ \eta_j \}_{j \in \mathbb{Z}} \in \mathbb{H} \) let

\[
\langle f, g \rangle := \sum_{j=-\infty}^{\infty} \xi_j \eta_{j-1}
\]

Then \( \langle \cdot, \cdot \rangle \) cannot have a norm majorant.  

Example 3.

For this Example, let \( S_0 \) denote the linear space of those real-valued sequences \( f : \mathbb{N} \to \mathbb{R} \) such that \( f(n) = 0 \) for all but a finite number of \( n \) and \( S_{\text{all}} \) denote the linear space of all real-valued sequences. Finally, define

\[
Z := S_0 \oplus S_{\text{all}}
\]

and

\[
s(f_1 \oplus f_2, g_1 \oplus g_2) := \sum_n (f_1(n)g_2(n) - f_2(n)g_1(n))
\]

Then the symplectic form \( s(\cdot, \cdot) \) cannot have a norm majorant.

Example 4. ([Ch4])

Let \( R[Z] \) be the free real *-algebra on the symbols \( u[n], n \in \mathbb{Z} \) subject to the relations

\[
\begin{align*}
    u[n]u[m] &= u[n+m], u[n]^* := u[-n] & (n, m \in \mathbb{R}[Z])
\end{align*}
\]

Let \( f \) be a linear functional defined by

\[
f u[n] := e^{\sqrt{|n|}} - e^{-\sqrt{|n|}}
\]

Next, define

\[
K_f := \{ K \in R[Z] \mid (\forall A \in R[Z]) \ f(A^*K) = 0 \}
\]

and

\[
A_f := A + K_f \quad (A \in \mathbb{R}[Z])
\]

Then, the bilinear anti-symmetric form

\[
A, B \mapsto f(A^*B)
\]

lifts to a symplectic form, \( s \), on the quotient space \( \mathbb{R}[Z]/K_f \) and the map

\[
A \in \mathbb{R}[Z] \mapsto u[1]A \in \mathbb{R}[Z] \quad (A \in \mathbb{R}[Z])
\]
lifts to a symplectic automorphism \( V : \mathbb{R}[\mathbb{Z}] / K_f \rightarrow \mathbb{R}[\mathbb{Z}] / K_f \); they are correctly defined by

\[
s(A_f, B_f) := f(A^*B) \quad (A, B \in \mathbb{R}[\mathbb{Z}])
\]

\[
V A_f := (u[1]A)_f \quad (A \in \mathbb{R}[\mathbb{Z}])
\]

Now then, there is no non-trivial \( V \)-invariant majorant of \( s \) and there is no non-trivial quadratic \( \alpha_V \)-invariant state on \( W_{Z,s} \) where \( Z := \mathbb{R}[\mathbb{Z}] / K_f \).

**COMMENT**

A state \( \omega \) is said to be regular iff the function \( x \in \mathbb{R} \rightarrow \omega e^{xf + g} \) is continuous whatever \( f \) and \( g \).

Manuceau and Verbeure [MV] discussed only regular states and therefore only finite quadratic forms and the corresponding states. As for non-regular states, one can confer the approach in this section with one of [FS], [Gru], [LMS], [CMS], and especially with that of [Oks]. Recently Halvorson [Hal] proposed a very interesting standpoint which is reminiscent of some of the papers of Antonets, Shereshevski, first of all [AS].

The ‘non-regular’ part of this section is based entirely on [Ch1], [Ch2], [Ch3] and the summarizing [Ch4].

In Example 4, we have applied a GNS-like construction. For details of such constructions, see e.g., [Schatz] or/and [BD] (or [Ch 2–4], if one deals with the objects discussed in this Section).

\[\text{GNS = Gelfand–Naimark–Segal}\]
4 The Case of Regular Spaces

In the previous section we discussed relatively general spaces and forms. So, the statements were 'in general'.

With stronger hypothesis on \( Z, K \) and forms \( s(\cdot , \cdot) < \cdot , \cdot > \) one can obtain a stronger conclusion. We start with two restricting definitions which one finds among the primary definitions of two different theories. We mean the standard theory of the quasi-free states (e.g., [BR2]) and, as for the second definition, the so-called Krein spaces theory (e.g., [Bog])

**Definition 1.**
\( Z, s \) is said to be **regular** iff there is a linear \( J : Z \rightarrow Z \) such that
1) \( s(Jf, Jg) = s(f, g) \quad \forall f, g \in Z; \)
2) \( J^2 = -I; \)
3) \( s(f, Jf) \geq 0 \quad \forall f \in Z, \)
4) \( Z \) is a real pre-Hilbert space with respect to the scalar product \( f, g \rightarrow s(f, Jg) \quad (f, g \in Z). \)

**Definition 2.** (e.g., [Bog])
Let \( K, <, > \) be an inner product space. \( K, <, > \) is said to be **regular indefinite inner product space** iff there is a linear \( J : K \rightarrow K \) such that
1) \( < Jz, Jw > = < z, w > \quad \forall z, w \in K; \)
2) \( J^2 = I; \)
3) \( < z, Jz > \geq 0 \quad \forall z \in K, \)
4) \( K \) is a pre-Hilbert space with respect to the scalar product \( z, w \rightarrow < z, Jw > \quad (z, w \in K). \)

If \( K \) is complete, then \( K \) is said to be a **Krein space**.

We see one definition is very much like another. We state it in the mathematical terms as an

**Observation 1.**
\( Z, s \) is regular if and only if the corresponding standard complexification of \( Z, s \), i.e., \( K, <, > \) in the sense of the previous section, is a regular indefinite inner product space. For the corresponding \( J \), we may take
\[
\mathfrak{J} := iJ_{\mathcal{C}}, \ \text{recall that} \ J_{\mathcal{C}} := \text{standard complexification of} \ J.
\]

The idea behind the constructions we will discuss is very simple. If we see that some of the primary definitions of two different theories are similar, then we expect it may be well worth stating the similarity between the results of these theories. Thus we need to elaborate a machinery so that we could translate the statements of the one theory into the language of another.

First, we consider what is common in both languages and we start to do it by introducing the general notations of the basic terms.

The symbol \( \mathcal{H} \) will denote a linear space, real or complex, and \( b \) be a bilinear or sesquilinear form respectively. In addition, we suppose that \( b \) is whether symmetric (hermitian for \( \mathcal{C} \)) or antisymmetric (antihermitian for \( \mathcal{C} \)).

The symbol \( \mathfrak{J} \) will denote a linear operator which has either the properties
\[
\mathfrak{J}^2 := -I \quad b(\mathfrak{J}f, \mathfrak{J}g) = b(f, g) \quad \text{case of antisymmetric (antihermitian) } b ,
\]
or
\[
\mathfrak{J}^2 := I \quad b(\mathfrak{J}z, \mathfrak{J}w) = b(z, w) \quad \text{case of symmetric (hermitian) } b.
\]
In addition, we define, unless otherwise specified, that

\[ J^* := -J \quad \text{case of antisymmetric (antihermitian) } b , \]

\[ J^* := J \quad \text{case of symmetric (hermitian) } b . \]

It is unlikely that this definition can produce any confusion: we will deal, typically, with nondegenerated forms \( s, <, > \); in these cases \( J^* \) will coincide with standard s- or \( <, > \) - adjoint of \( J \) respectively.

There are two classes of regular spaces which we will discuss. The first class is given by:

**Example 1.**

Let \( Z_0 \) be a real or complex Hilbert or pre-Hilbert space. Put

\[ Z := Z_0 \oplus Z_0 \]

and

\[ J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \]

This choice of \( Z \) and \( J \) corresponds to the case where we adopt a definition of CCRs phrased in terms of \( P, Q \), i.e., in terms of momentum and position operators.

Another class of regular spaces is:

**Example 2.**

Let \( H_0 \) be a complex Hilbert or pre-Hilbert space. Put \( H_+ := H_0, H_- := H_0 \),

\[ K := H_+ \oplus H_- \]

and

\[ J := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \]

This case corresponds to that, when one adapts himself to the CCRs phrased in terms of \( a^*, a \), i.e., in terms of creation and annihilation operators.

The connection between these classes is simple:

**Observation 2.**

If

\[ J_{a^*a} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad iJ_{pq} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

then

\[ J_{a^*a} \left( \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \right) = \left( \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \right) \left( \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \right) = \left( \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \right) = iJ_{pq} \]

\[ iJ_{pq} \left( \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \right) = \left( \begin{pmatrix} -i & i \\ 1 & -1 \end{pmatrix} \right) = \left( \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \right) \left( \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) = \left( \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \right) \]

We now return to quadratic forms and majorants and henceforth in this section we assume that the above spaces \( Z_0 \) and \( H_0 \) are complete.

**Definition 3.**

We say a form \( q \) is closed if \( q \) is closed as a usual quadratic form on Hilbert space

\[ \overline{Q(q)} := \text{closure of } Q(q) \text{ in } H \text{ with respect to } \| \cdot \| , \]

\[ Q(q) := \text{set of all } (x, y) \in H \times H \text{ such that } q(x, y) \text{ is real and } \| x \| = 1 , \]

\[ \| x \| := \sqrt{q(x, x)} . \]
i.e. is closed in the sense adopted in [RS1].

**Theorem 1.**
Let $q$ be a majorant.
If $q$ is minimal, then $q$ is closed.

**Definition 4.**
We say a form $q$ has an **operator representative** if there exists an $T : D_T \subset H \to H$ such that

$$D_T = Q(q) \text{ and } q(f) = \|Tf\|^2 \quad (f \in Q(q)).$$

**Theorem 2.**
If $q$ is closed, then $q$ has an operator representative.
In addition, there is a unique self-adjoint operator $Q : D_Q \subset \overline{Q(q)} \to \overline{Q(q)}$ such that

$$D_{Q^{1/2}} = Q(q) \text{ and } q_Q := \|Q^{1/2}x\|^2.$$

**Proof.** Straightforward from Definitions 3, 4 and Theorem 1, using [RS1].

**Definition 5.**
In the situation of the Theorem 2, we say $Q$ is the **operator of** $q$ and write $q = q_Q$. Thus, we isolate a class of majorants. We will call these majorants **operator majorants**.
Let $Q$ be the operator of $q$. Then we write

$$P := \text{orthogonal projection of } H \text{ onto } \overline{Q(q)},$$

$$R := (I + Q)^{-1}P.$$

**Remark 1.**
It is evident that

$$0 \leq R \leq I, \quad R = R^*$$

and that $Q$ (and $q$) can be recovered from the $R$ by the formulae

$$Q = R^{-1} - I; D_Q = \text{Ran } R.$$

We now characterize the operator majorants by means of the above $Q$ and $R$.

**Theorem 3.**
(i) $q_Q$ is a majorant iff

$$R + R^* R \leq I;$$

(ii) $q_Q$ is a minimal majorant iff

$$R + R^* R = I.$$

If we handle indefinite inner product space, we can say more:

**Observation 3.**
Let $H = K$, i.e., let us assume $J^* = J$. Then,

$$R + J^* R J = I, \quad 0 \leq R \leq I$$

if and only if

$$R = \frac{1}{2} \begin{pmatrix} I & K^* \\ K & I \end{pmatrix},$$
where \( R \) is thought of as an operator from \( H_+ \oplus H_- \) into \( H_+ \oplus H_- \),
and where \( K \) is an operator such that \( \|K\| \leq 1 \), or,
equivalently,
\[
R = \frac{1}{4} \begin{pmatrix} 2 - K - K^* & iK - iK^* \\ iK - iK^* & 2 + K + K^* \end{pmatrix}
\]
where \( R \) is thought of as an operator from \( Z_0 \oplus Z_0 \) into \( Z_0 \oplus Z_0 \),
and where \( K \) is the same operator as above.

With this Observation 3, Theorem 3 implies

**Corollary 1.**

Let \( H = K \). Then \( q_Q \) is a minimal majorant if and only if
\[
R = \frac{1}{2} \begin{pmatrix} I & K^* \\ K & I \end{pmatrix}
\]
with respect to \( R : H_+ \oplus H_- \to H_+ \oplus H_- \)
and with a \( K \) such that \( \|K\| \leq 1 \) or, equivalently,
\[
R = \frac{1}{4} \begin{pmatrix} 2 - K - K^* & iK - iK^* \\ iK - iK^* & 2 + K + K^* \end{pmatrix}
\]
with respect to \( R : Z_0 \oplus Z_0 \to Z_0 \oplus Z_0 \)
and with the same \( K \).

Finally, we turn to the question: What about complexificated majorants and operators?

The theorems are:

**Theorem 4.** (cf. Observation 3.1)

In the situation described in Example 1, let

\[
K_0 := CZ_0 = \text{the standard complexification of } Z_0
\]

\[
C_0 := \text{the corresponding complex conjugation operator on } K_0, \quad C := C_0 \oplus C_0
\]

Then, a minimal majorant \( \tilde{\varphi} \) on \( K = K_0 \oplus K_0 \) is a complexification of a (minimal majorant) \( \varphi \) on \( Z = Z_0 \oplus Z_0 \) i.e., \( \tilde{\varphi} \) is of the form \( \tilde{\varphi} = q_C \) if and only if
\[
\tilde{\varphi}C = \tilde{\varphi}
\]
or, equivalently,
\[
R = CRC
\]
or, equivalently,
\[
K^* = K := C_0KC_0
\]

**Theorem 5.** (cf. Observation 3.1)

A linear \( \tilde{T} : K_0 \oplus \to K_0 \oplus K_0 \) is of the form \( \tilde{T} = T_C \) for a suitable \( T : Z_0 \oplus Z_0 \to Z_0 \oplus Z_0 \) if and only if
\[
\tilde{T} = CTC
\]
or, equivalently, \( \tilde{T} \) is a **cross-matrix** i.e., \( \tilde{T} \) is of the form
\[
\tilde{T} = \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix}
\]
with respect to the decomposition \( H_+ \oplus H_- \to H_+ \oplus H_- \)

Now, what about **invariant** majorant?

The theorem is:

**Theorem 6.**
Let \( V : \mathcal{H} \rightarrow \mathcal{H} \) be a linear bounded invertible operator, and let \( q \) be an operator majorant of \( b(\cdot,\cdot) \).

The following conditions are equivalent:

(i) \( qV = q \)

(ii) \( qV^{-1} = q \)

(iii) \( (I - R)VR = RV^{**}(1 - R) \)

(iv) \( (I - R)V^{-1}R = RV^{**}(1 - R) \)

In addition, if \( q \) is a minimal majorant, and if \( V \) is an automorphism of \( b(\cdot,\cdot) \), i.e., if

\[
b(Vf, Vg) = b(f, g), \quad \forall f, g \in \mathcal{H},
\]

and if \( b(\cdot,\cdot) \) is symmetric (it means that the situation is the same as one in Example 2), then all conditions (i)-(iv) are equivalent to:

(v) \[
V \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix} V \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix}
\]

Let us discuss the above Theorem, especially, the condition (v) of this Theorem. In discussing them we will indicate at least three factors which have to be taken into account.

First, we observe that the operator

\[
P := \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix}
\]

is a projection operator because \( P^2 = P \). Therefore the condition (v) is a condition for \( PH \) to be a \( V \)-invariant subspace. We emphasize, it is true for any operator \( V \) even if \( V \) is not a Bogoliubov transformation. As for the sort of the subspace, some authors refer to such subspaces as the graph subspaces because one may consider

\[
PH = \{ x_+ \oplus Kx_+ | x_+ \in H_+ \}
\]

as the graph of the operator \( K \). In this case, \( K \) is called the angular operator of \( PH \) with respect to \( H_+ \).

The second factor is that \( \|K\| \leq 1 \). This inequality means, in particular, that whatever \( x \in PH \), the value of \( b(x, x) \) is positive:

\[
b(x, x) = b(x_+ \oplus Kx_+, x_+ \oplus Kx_+) = (x_+ \oplus Kx_+ , J(x_+ \oplus Kx_+)) = \|x_+\|^2 - \|Kx_+\|^2 \geq 0
\]

For such a sort of subspaces, there are special terms:

A subspace, \( L \), is called \( J \)-positive or \( b \)-positive or mere positive if

\[
b(x, x) \geq 0, \quad (\forall x \in L)
\]

Given a positive \( L \), one says ‘\( L \) is maximal positive’, if \( L \) is ‘set’-maximal among positive subspaces. In other words, a maximal positive subspace is a positive subspace \( L \) such that: whatever positive \( L_1 \) is given, \( L_1 \supset L \) implies that \( L_1 = L \).

We can state now: if a quadratic-like majorant is minimal, than the associated subspace is positive. The property of \( L \) being a positive subspace, it, in itself, does not implies that this subspace is of the form

\[
L = \{ x_+ \oplus Kx_+ | x_+ \in H_+ \}, \quad \|K\| \leq 1
\]
but if one replaces ‘being a positive’ by ‘being a maximal positive’, it does.

As the result:
1) if a quadratic-like majorant is minimal, then the associated subspace is maximal positive;
2) every maximal positive subspace is a subspace associated with a unique minimal quadratic-like majorant.

One of the additional factors which have to be taken into account in discussing the above Theorem 6, is that a minimal majorant is regular if and only if $\|K\| < 1$, and this is exactly the case if the corresponding maximal positive subspace is uniformly positive:

\[
(\exists \gamma > 0)(\forall x \in \mathcal{P}H) \quad b(x, x) \geq \gamma b(x, Jx)
\]
or in other words, which are more usual for the Krein spaces theory,

\[
(\exists \gamma > 0)(\forall x \in \mathcal{P}H) \quad < x, x > \geq \gamma \|x\|^2
\]

The third factor which have to be taken into account is that $\mathcal{P}H$ is to be a $V$-invariant subspace.

We will often work with the operator matrix representation of $V$,

\[
V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}
\]

with respect to the decomposition $H_+ \oplus H_- \rightarrow H_+ \oplus H_-$

In this case the mentioned condition (v) will look like this:

\[
\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix}
\]

One can straightforwardly verify that this condition is exactly equivalent to

\[
V_{21} + V_{22}K = K(V_{11} + V_{12}K)
\]

It is the equation which is placed among the most singular equations of the Krein spaces theory and it is just the equation which we will systematically exploit when discussing Examples.

We conclude this section with a dictionary, paralleling the most explicit notions of the Krein spaces theory with the theory of quadratic Bose Hamiltonians.

\footnote{It means that $Q = R^{-1} - I$ exists and is bounded as an operator acting on the whole space $\mathcal{H}$.}

\footnote{If, of course, one is interested in solving invariant subspaces problems}
symplectic operator or form, which has the matrix \((L \ M \\
-K \ -L^T)\) \(\leftrightarrow\) the quadratic Hamiltonian
\[
h := \frac{1}{2} \sum_{l,m} \left( M_{lm} P_l P_m - L_{lm} (P_l Q_m + Q_m P_l) + K_{lm} Q_l Q_m \right)
\]

\(J\)-symmetric operator or form, which has the matrix \((S \ T \\
-T \ -S)\) \(\leftrightarrow\) the quadratic Hamiltonian
\[
h = \sum_{k,l} \left( s_{kl} a_k^* a_l - \frac{i}{2} t_{kl} a_k^* a_l - \frac{1}{2} t_{kl} a_l^* a_k^* \right)
\]

\(J\)-unitary operator with the matrix \((\Phi \ \Psi \\
\Psi \ \Phi)\) \(\leftrightarrow\) invertible Bogoliubov transformation (quasi-free automorphism) with the same matrix

maximal positive subspace with the angular operator \(K\) such that \(K^* = K\) \(\leftrightarrow\) pure quadratic-like state

maximal uniformly positive subspace with the angular operator \(K\) such that \(K^* = K\) \(\leftrightarrow\) regular pure quadratic-like state, i.e., pure even quasi-free state

maximal positive invariant subspace with the angular operator \(K\) such that \(K^* = K\) \(\leftrightarrow\) pure quadratic-like invariant state

Now then, it is time to Examples.

**COMMENT**

About linear canonical transformations and hamiltonians, see e.g., [W1,2,3] for \(\text{dim} < \infty\) and e.g., [Ber], [BR2], [RS2] for the quantum case. Theorem 5 see in [Ber], see also [DK], [K]. The standard point is concentrated on the questions “how diagonalize a given hamiltonian or automorphism?” and “does there exist a regular invariant state?” We interested in any invariant states no matter whether they are regular or not and any hamiltonians no matter whether they are diagonalizable or not.

For terms ‘angular operator’, ‘positive subspace’, ‘maximal positive subspace’ and for other details of the Krein spaces theory, see, e.g., [Bog], [DR].

The approach in this section is based on [Ch1], [Ch2], [Ch3] and the summarizing [Ch4].
5 Examples

5.1 Example 1. Oscillator

In terms of $P, Q$, the Hamiltonian is written as

$$h := \frac{1}{2}P^2 + \frac{1}{2}\Omega_0^2 Q^2$$

Then

$$i[h, F(x_p \oplus x_q)] = F(x'_p \oplus x'_q)$$

where

$$\begin{pmatrix} x'_p \\ x'_q \end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p \\ x_q \end{pmatrix}$$

$$\begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Omega_0^2 & 0 \end{pmatrix}$$

$$V_t := e^{i \begin{pmatrix} 0 \\ -\Omega_0^2 t \end{pmatrix}} = \begin{pmatrix} \cos(\Omega_0 t) & \Omega_0^{-1} \sin(\Omega_0 t) \\ -\Omega_0 \sin(\Omega_0 t) & \cos(\Omega_0 t) \end{pmatrix}$$

In terms of $a^*, a$, the Hamiltonian is rewritten as

$$h := \frac{1}{2}P^2 + \frac{1}{2}\Omega_0^2 Q^2$$

$$= \frac{1}{2} \left( \frac{i}{\sqrt{2}} (a^* - a)^2 + \frac{1}{2}\Omega_0^2 \frac{1}{\sqrt{2}} (a^* + a)^2 \right)$$

$$= \frac{1 + \Omega_0^2}{2} a^* a - \frac{1 - \Omega_0^2}{4} a^2 - \frac{1 - \Omega_0^2}{4} a^2 + \text{const}$$

Formal calculations show that

$$[h, A(u^+ \oplus u^-)] = A(u'^+ \oplus u'^-)$$

where $u^+ \oplus u^-$ is defined by

$$\begin{pmatrix} u'^+ \\ u'^- \end{pmatrix} = \begin{pmatrix} S & T \\ -T & -S \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

$$\begin{pmatrix} S & T \\ -T & -S \end{pmatrix} = \begin{pmatrix} \frac{1 + \Omega_0^2}{2} & \frac{1 - \Omega_0^2}{2} \\ -\frac{1 - \Omega_0^2}{2} & -\frac{1 + \Omega_0^2}{2} \end{pmatrix}$$

$$-\frac{1 - \Omega_0^2}{2} - \frac{1 + \Omega_0^2}{2} K = K \frac{1 + \Omega_0^2}{2} + K \frac{1 - \Omega_0^2}{2} K$$

$$-(1 - \Omega_0^2) = 2K(1 + \Omega_0^2) + (1 - \Omega_0^2)K^2$$

Recall that $\|K\| \leq 1$ and assume $\Omega_0 > 0$. Then

$$K = -\frac{1 - \Omega_0}{1 + \Omega_0}$$

is a unique solution. As for the corresponding $R, Q, q, \omega$, we have in terms of $P, Q$,

$$R = \frac{1}{4} \begin{pmatrix} 2 - K - K^* & iK - iK^* \\ iK - iK^* & 2 + K + K^* \end{pmatrix} = \begin{pmatrix} 1 + \Omega_0 & 0 \\ 0 & \frac{\Omega_0}{1 + \Omega_0} \end{pmatrix}$$

$$Q = \begin{pmatrix} \Omega_0 & 0 \\ 0 & \frac{1}{\Omega_0} \end{pmatrix}$$
Among Quadratic Hamiltonians

\[ \omega e^{ix_p P + ix_q Q} = e^{-\frac{1}{4}\left(\Omega_0 x_p^2 + \frac{1}{\Omega_0} x_q^2\right)} \]

A few details of asymptotic behaviour of \( \alpha_t := \alpha V_t \) are the following: Consider the standard Fock state, i.e., the state, \( \omega_F \), defined by

\[ \omega_F e^{ix_p P + ix_q Q} = e^{-\frac{1}{4}|x_p \oplus x_q|^2} = e^{-\frac{1}{4}(x_p^2 + x_q^2)} \]

Then

\[ \omega_F \alpha_t e^{ix_p P + ix_q Q} \]
\[ = e^{-\frac{1}{4}|V_t(x_p \oplus x_q)|^2} \]
\[ = e^{-\frac{1}{4}\left((\cos(\Omega_0 t) x_p + \Omega_0^{-1} \sin(\Omega_0 t) x_q)^2 + (\Omega_0 \sin(\Omega_0 t) x_p + \cos(\Omega_0 t) x_q)^2\right)} \]

We see, this quantity has no usual limit, neither as \( t \to +\infty \) nor as \( t \to -\infty \), whenever \( \Omega_0 \neq \pm 1 \).
5.2 Example 2. Free Evolution on Line

In terms of $P, Q$, the Hamiltonian is written as
\[ h := \frac{1}{2} P^2 \]

Then
\[ i[h, F(x_p \oplus x_q)] = F(x_p' \oplus x_q') \]

where
\[
\begin{pmatrix}
x_p' \\
x_q'
\end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p \\
x_q
\end{pmatrix}
\]
\[
V_t := e^{i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}
\]

In terms of $a^*, a$, the Hamiltonian is rewritten as
\[ h := \frac{1}{2} P^2 = \frac{1}{2} \sqrt{2} (a^* - a)^2 = \frac{1}{2} a^*a - \frac{1}{4} a^2 + const \]

Formal calculations show that
\[ [h, A(u^+ \oplus u^-)] = A(u'^+ \oplus u'^-) \]

where $u'^+ \oplus u'^-$ is defined by
\[
\begin{pmatrix} u'^+ \\ u'^-
\end{pmatrix} = \begin{pmatrix} S & T \\ -T & -S \end{pmatrix} \begin{pmatrix} u^+ \\ u^-
\end{pmatrix}
\]
\[
\begin{pmatrix} S & T \\ -T & -S \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}
\]
\[-\frac{1}{2} - \frac{1}{2} K = K \frac{1}{2} + K \frac{1}{2} 
-1 = 2K + K^2 \]

Then
\[ K = -1 \]

is a unique solution. As for the corresponding $R, Q, q, \omega$, we have in terms of $P, Q$,
\[ R = \frac{1}{4} \begin{pmatrix} 2 - K - K^* & iK - iK^* \\ iK - iK^* & 2 + K + K^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ \omega e^{ix_P P + ix_Q Q} = e^{-q(x_p \oplus x_q)/4} = e^{-\infty \cdot x_q^2/4} = \begin{cases} 1, \text{ if } x_q = 0 \\ 0, \text{ if } x_q \neq 0 \end{cases} \]

A few details of asymptotic behaviour of \[ \alpha_t := \alpha_{\omega_t} \] are the following: Consider the standard Fock state, i.e., the state, $\omega_F$, defined by
\[ \omega_F e^{ix_P P + ix_Q Q} = e^{-|x_p \oplus x_q|^2/4} = e^{-(x_p^2 + x_q^2)/4} \]
Then
\[
\omega_F \alpha t e^{ix_p P + ix_q Q} = e^{-|V_t(x_p \oplus x_q)|^2/4} = e^{-(x_p + tx_q)^2 + x_q^2)/4}
\]

We see, this quantity has a limit as \( t \to +\infty \) and as \( t \to -\infty \) as well and these limits are equal:

\[
\lim_{t \to \pm \infty} \omega_F \alpha t e^{ix_p P + ix_q Q} = \lim_{t \to \pm \infty} e^{-(x_p + tx_q)^2 + x_q^2)/4} = \begin{cases} e^{-x_p^2/4}, & \text{if } x_q = 0 \\ 0, & \text{if } x_q \neq 0 \end{cases}
\]

Notice,
\[
\lim_{t \to \pm \infty} \omega_F \alpha t e^{ix_p P + ix_q Q}
\]
is not a characteristic functional of a pure state.
5.3 Example 3. \( h := \frac{1}{2}(PQ + QP) \)

In terms of \( P, Q \), the Hamiltonian is written as

\[
h := \frac{1}{2}(PQ + QP)
\]

Then

\[
i[h, F(x_p \oplus x_q)] = F(x'_p \oplus x'_q)
\]

where

\[
\begin{pmatrix} x'_p \\ x'_q \end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p \\ x_q \end{pmatrix}
\]

\[
\begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
V_t := e^{i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}
\]

In terms of \( a^*, a \), the Hamiltonian is rewritten as

\[
h := \frac{1}{2}(PQ + QP)
\]

\[
= \frac{1}{2} \left( \frac{i}{\sqrt{2}}(a^* - a) \cdot \frac{1}{\sqrt{2}}(a^* + a) + \frac{i}{\sqrt{2}}(a^* + a) \cdot \frac{1}{\sqrt{2}}(a^* - a) \right)
\]

\[
= \frac{i}{2} a^* a - \frac{i}{2} a^2
\]

Formal calculations show that

\[
[h, A(u^+ \oplus u^-)] = A(u'^+ \oplus u'^-)
\]

where \( u'^+ \oplus u'^- \) is defined by

\[
\begin{pmatrix} u'^+ \\ u'^- \end{pmatrix} = \begin{pmatrix} S & T \\ -T & -S \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}
\]

\[
\begin{pmatrix} S & T \\ -T & -S \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}
\]

\[-i = K \cdot (-i) \cdot K
\]

Then there are two (!) solutions:

\[
K = K_{+1} = 1
\]

\[
K = K_{-1} = -1
\]
As for the corresponding \( R_{+1}, Q_{+1}, q_{+1}, \omega_{+1} \), and \( R_{-1}, Q_{-1}, q_{-1}, \omega_{-1} \), we have in terms of \( P, Q \),
\[
R_{+1} = \frac{1}{4} \begin{pmatrix} 2 - K_{+1} - K^*_{+1} & iK_{+1} - iK^*_{+1} \\ iK_{+1} - iK^*_{+1} & 2 + K_{+1} + K^*_{+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
\[
\omega_{+1} e^{ix_pP + ix_qQ} = e^{-q_{+1}(x_p \oplus x_q)/4} = e^{-\infty \cdot x_p^2/4} = \begin{cases} 1, & \text{if } x_p = 0 \\ 0, & \text{if } x_p \neq 0 \end{cases}
\]
\[
R_{-1} = \frac{1}{4} \begin{pmatrix} 2 - K_{-1} - K^*_{-1} & iK_{-1} - iK^*_{-1} \\ iK_{-1} - iK^*_{-1} & 2 + K_{-1} + K^*_{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
\[
\omega_{-1} e^{ix_pP + ix_qQ} = e^{-q_{-1}(x_p \oplus x_q)/4} = e^{-\infty \cdot x_q^2/4} = \begin{cases} 1, & \text{if } x_q = 0 \\ 0, & \text{if } x_q \neq 0 \end{cases}
\]

If we confer these expressions with that in Example 1,
\[
\omega e^{ix_pP + ix_qQ} = e^{-q(x_p \oplus x_q)/4} = e^{-(\Omega_0 x_p^2 + 1/\Omega_0 x_q^2)/4}
\]
we can infer that \( \omega_{+1} \) is an approximation of the ground state of an oscillator with the extremely high frequency \( \Omega_0 \) whereas \( \omega_{-1} \) is an approximation of the ground state of an oscillator with the extremely low frequency \( \Omega_0 \).

Finally, we have
\[
\lim_{t \to +\infty} \omega e^{ix_pP + ix_qQ} = \lim_{t \to +\infty} e^{-\frac{e^{-2t x_p^2} + e^{2t x_q^2}}{4}} = \begin{cases} 1, & \text{if } x_q = 0 \\ 0, & \text{if } x_q \neq 0 \end{cases} = \omega_{-1} e^{ix_pP + ix_qQ}
\]
\[
\lim_{t \to -\infty} \omega e^{ix_pP + ix_qQ} = \lim_{t \to -\infty} e^{-\frac{e^{-2t x_p^2} + e^{2t x_q^2}}{4}} = \begin{cases} 1, & \text{if } x_p = 0 \\ 0, & \text{if } x_p \neq 0 \end{cases} = \omega_{+1} e^{ix_pP + ix_qQ}
\]
This situation is typical. Whatever \textbf{regular} quadratic state \( \omega \),
\[
\omega e^{ix_pP + ix_qQ} = e^{-(q_{11} x_p^2 + 2q_{22} x_p x_q + q_{22} x_q^2)/4}
\]
we have choosen, the result is:
\[
\lim_{t \to +\infty} \omega e^{ix_pP + ix_qQ} = \begin{cases} 1, & \text{if } x_q = 0 \\ 0, & \text{if } x_q \neq 0 \end{cases} = \omega_{-1} e^{ix_pP + ix_qQ}
\]
\[
\lim_{t \to -\infty} \omega e^{ix_pP + ix_qQ} = \begin{cases} 1, & \text{if } x_p = 0 \\ 0, & \text{if } x_p \neq 0 \end{cases} = \omega_{+1} e^{ix_pP + ix_qQ}
\]
5.4 Example 4. Repulsive Oscillator

In terms of $P, Q$, the Hamiltonian is written as

$$h := \frac{1}{2}P^2 - \frac{1}{2}\Omega_0^2 Q^2$$

Then

$$i[h, F(x_p \oplus x_q)] = F(x'_p \oplus x'_q)$$

where

$$\begin{pmatrix} x'_p \\ x'_q \end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p \\ x_q \end{pmatrix}$$

$$\begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Omega_0^2 & 0 \end{pmatrix}$$

Formal calculations show that

$$[h, A(u^+ \oplus u^-)] = A(u'^+ \oplus u'^-)$$

where $u'^+ \oplus u'^-$ is defined by

$$\begin{pmatrix} u'^+ \\ u'^- \end{pmatrix} = \begin{pmatrix} S & T \\ -T & -S \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

$$\begin{pmatrix} S & T \\ -T & -S \end{pmatrix} = \begin{pmatrix} \frac{1-\Omega_0^2}{2} & \frac{1+\Omega_0^2}{2} \\ -\frac{1+\Omega_0^2}{2} & \frac{1-\Omega_0^2}{2} \end{pmatrix}$$

$$(1+\Omega_0^2) = 2K(1 - \Omega_0^2) + (1 + \Omega_0^2)K^2$$

Recall that $\|K\| \leq 1$ and assume $\Omega_0 > 0$. Then there are two (!) solutions:

$$K = K_{+1} = -\frac{1 - i\Omega_0}{1 + \Omega_0}$$

$$K = K_{-1} = -\frac{1 + i\Omega_0}{1 - \Omega_0}$$
Among Quadratic Hamiltonians

As for the corresponding $R_{+1}, Q_{+1}, q_{+1}, \omega_{+1}$, and $R_{-1}, Q_{-1}, q_{-1}, \omega_{-1}$, we have in terms of $P, Q,$

$$R_{+1} = \frac{1}{4} \left( \begin{array}{cc} 2 - K_{+1} - K_{+1}^* & iK_{+1} - iK_{+1}^* \\ iK_{+1} - iK_{+1}^* & 2 + K_{+1} + K_{+1}^* \end{array} \right) = \frac{1}{1 + \Omega_0^2} \left( \begin{array}{cc} 1 & \Omega_0 \\ \Omega_0 & \Omega_0^2 \end{array} \right)$$

$$\omega_{+1} e^{ix_p P + ix_q Q} = e^{-q_{+1}(x_p \oplus x_q)/4}$$

$$= e^{-\infty \cdot (-\Omega_0 x_p + x_q)^2/4} = \begin{cases} 1, & \text{if } -\Omega_0 x_p + x_q = 0 \\ 0, & \text{if } -\Omega_0 x_p + x_q \neq 0 \end{cases}$$

$$R_{-1} = \frac{1}{4} \left( \begin{array}{cc} 2 - K_{-1} - K_{-1}^* & iK_{-1} - iK_{-1}^* \\ iK_{-1} - iK_{-1}^* & 2 + K_{-1} + K_{-1}^* \end{array} \right) = \frac{1}{1 + \Omega_0^2} \left( \begin{array}{cc} 1 & -\Omega_0 \\ -\Omega_0 & \Omega_0^2 \end{array} \right)$$

$$\omega_{-1} e^{ix_p P + ix_q Q} = e^{-q_{-1}(x_p \oplus x_q)/4}$$

$$= e^{-\infty \cdot (\Omega_0 x_p + x_q)^2/4} = \begin{cases} 1, & \text{if } \Omega_0 x_p + x_q = 0 \\ 0, & \text{if } \Omega_0 x_p + x_q \neq 0 \end{cases}$$

Finally, one can verify that

$$\lim_{t \to +\infty} \omega F_{\alpha t} e^{ix_p P + ix_q Q} = \omega_{-1} e^{ix_p P + ix_q Q}$$

$$\lim_{t \to -\infty} \omega F_{\alpha t} e^{ix_p P + ix_q Q} = \omega_{+1} e^{ix_p P + ix_q Q}$$

This situation is typical as in the previous Example. Whatever regular quadratic state $\omega,$

$$\omega e^{ix_p P + ix_q Q} = e^{-\left(q_{11} x_p^2 + 2q_{22} x_p x_q + q_{22} x_q^2\right)/4},$$

we have chosen, the result is:

$$\lim_{t \to +\infty} \omega F_{\alpha t} e^{ix_p P + ix_q Q} = \omega_{-1} e^{ix_p P + ix_q Q}$$

$$\lim_{t \to -\infty} \omega F_{\alpha t} e^{ix_p P + ix_q Q} = \omega_{+1} e^{ix_p P + ix_q Q}$$
5.5 Example 5.

consider the approximating (Bogoliubov) Hamiltonian

\[ H'_B = \int dp \left\{ \omega(p)a^*(p)a(p) + \frac{1}{2} \Delta_B(p) [a(p)^*a(-p)^* + a(-p)a(p)] \right\} \]

Formal calculations show that

\[ [H'_B, A(u^+ \oplus u^-)] = A(u^+ + u^-), \]

where \( u^+ \oplus u^- \) is defined by

\[
\begin{pmatrix}
  u^+ \\
u^-
\end{pmatrix} = \begin{pmatrix} S & T \\ -\bar{T} & -\bar{S} \end{pmatrix} \begin{pmatrix} u^+ \\
u^-
\end{pmatrix}
\]

\[
(\begin{array}{cc}
S & T \\
-T & -\bar{S}
\end{array})(p, p') = (\begin{array}{cc}
\omega(p)\delta(p - p') - \Delta_B(p)\delta(p + p') & -\omega(p)\delta(p - p') \\
\Delta_B(p)\delta(p + p') & -\omega(p)\delta(p + p')
\end{array})
\]

\[
(\begin{array}{cc}
S & T \\
-T & -\bar{S}
\end{array}) = (\begin{array}{cc}
\hat{\omega} - \hat{\Delta}_B J_0 \\
J_0 \hat{\Delta}_B - \hat{\omega} J_0 K
\end{array})
\]

\[
J_0 \hat{\Delta}_B - \hat{\omega} K = K\hat{\omega} - K\hat{\Delta}_B J_0 K
\]

\[
-\hat{\Delta}_B + J_0 \hat{\omega} J_0 K = -J_0 K\hat{\omega} + J_0 K\hat{\Delta}_B J_0 K
\]

If we take into account that \( \omega(-p) = \omega(p) \), i.e., \( J_0 \hat{\omega} J_0 = \hat{\omega} \) and if we restrict ourselves to the case where \( J_0 K \) commutes with the multiplications by functions, i.e., if \( K \) is of the form

\[ K(p, p') = \delta(p + p')k_0(p) \]

for a function \( k_0 \), then we obtain:

\[-\Delta_B(p) = -2\omega(p)k_0(p) + \Delta_B(p)k_0(p)^2, \quad |k_0(p)| \leq 1.\]

Thus

\[
k_0(p) = \begin{cases} 
0, & \text{if } p \text{ is such that } \Delta_B(p) = 0, \omega(p) \neq 0 \\
\text{arbitrary}, & \text{if } p \text{ is such that } \\
\frac{\omega(p) - sgn(\omega(p))\sqrt{-\Delta_B(p)^2 + \omega(p)^2}}{\Delta_B(p)}, & \text{if } p \text{ is such that } \\
\frac{\omega(p) - i\epsilon(p)\sqrt{\Delta_B(p)^2 - \omega(p)^2}}{\Delta_B(p)}, & \text{if } p \text{ is such that } \\
\text{where } \epsilon(p)^2 = 1 & -\Delta_B(p)^2 + \omega(p)^2 \geq 0, \Delta(p) \neq 0
\end{cases}
\]

These relationships can be transformed as follows:

\[
k_0(p) = \begin{cases} 
\text{arbitrary}, & \text{if } p \text{ is such that } \\
\frac{\Delta_B(p)}{\omega(p) + sgn(\omega(p))\sqrt{-\Delta_B(p)^2 + \omega(p)^2}}, & \text{if } p \text{ is such that } \\
\frac{\Delta_B(p)}{\omega(p) + i\epsilon(p)\sqrt{\Delta_B(p)^2 - \omega(p)^2}}, & \text{if } p \text{ is such that } \\
\text{where } \epsilon(p)^2 = 1 & -\Delta_B(p)^2 + \omega(p)^2 \geq 0
\end{cases}
\]

In particular, if there are infinitely many \( p \) such that \(-\Delta_B(p)^2 + \omega(p)^2 \leq 0\), then there are infinitely many invariant pure quadratic-like states.
6 Appendix A: The Formal Calculations of $e^{A+B}$

Assume 
\[
[[A, B], B] = 0, \quad [[A, B], A] = 0.
\]

Now let us calculate 
\[
U := e^{t(A+B)}.
\]

The definition of $U$ formally implies 
\[
dU/dt = (A + B)U, \quad U(0) = I.
\]

Let 
\[
U := e^{tA}V
\]

Hence, 
\[
dV/dt = e^{-tA}Be^{tA}V = (B - t[A, B])V, \quad V(0) = I.
\]

Let 
\[
V := e^{tB}C
\]

Then, 
\[
dC/dt = -e^{-tB}t[A, B]e^{tB} = -t[A, B]C, \quad C(0) = I.
\]

Hence 
\[
C = e^{-\frac{1}{2}t[A, B]}
\]

the Result is:
\[
e^{t(A+B)} = e^{tA}e^{tB}e^{-\frac{1}{2}t[A, B]}
\]
\[
e^{i(A+B)} = e^{iA}e^{iB}e^{\frac{i}{2}t[A, B]}
\]
\[
e^{i(A+B)} = e^{iA}e^{iB}e^{\frac{i}{2}[A, B]}
\]
\[
e^{A}e^{B} = e^{\frac{i}{2}[A, B]}e^{A+B}
\]
\[
e^{iA}e^{iB} = e^{-\frac{i}{2}[A, B]}e^{i(A+B)}
\]

Remark .

The usual form of the CCRs is motivated by the Schrödinger representation of the position and momentum operators:
\[
Q = \hat{x}, \quad P = \frac{\hbar}{i} \frac{\partial}{\partial x}.
\]

Hence, 
\[
\frac{i}{\hbar} [P, Q] = 1, \quad [Q, P] = i\hbar, \quad \frac{1}{2} P^2, Q = \frac{1}{2} \cdot 2 \cdot P \cdot \frac{\hbar}{i} = \frac{\hbar}{i} P.
\]
References

[AS] Antonec M.A., Šereševskij I.A.: Kvantovanie Vejlja na kompaktnykh abelevых grupah i kvantovaja mehanika počti-periodičeskij sistem //TMF.1981.T.48,N1,49-59. RŻMAT 1981,11B965

Antonec, M.A.; Šereševski, I.A. Weyl quantization on compact abelian groups and quantum mechanics of almost-periodic systems. (Russian)
Teoret. Mat. Fiz. 48 (1981), no. 1, 49-59. MR 82j:58052

[Arn] V.I. ARNOL'D, Mathematical methods of classical mechanics.
(Matematiceskije metody klassiĉeskoj mehaniki)(Russian) Moskva:Nauka, 1974.

[Ber] F.A. BEREZIN, The Method of Second Quantization, Academic Press, New York, 1966.
F.A. BEREZIN, Methode der zweiten Quantelung. Zweite, neubearbeitete Auflage. (Metod vtoričnogo kvantovanija, 2-e izd.)(Russian), M.: Nauka, 1986,

[Bogn] J. BOGŇAR, Indefinite Inner Product Spaces, Springer-Verlag, Berlin Heidelberg New York, 1974.

[BD] Bonsall Frank F., Duncan John. Complete normed algebras, Berlin, Heidelberg,New York.: Springer-Verlag,1973. 301pp.?
AMS Subject Classification (1970): 46H05 Russ 51 E67

[BR2] O. BRATTELI and D.W. ROBINSON, Operator Algebras and Quantum Statistical Mechanics, Vol. II, Springer-Verlag, New York, Heidelberg and Berlin, 1981.

[Ch81] S.A. CHOROŠAVIN, On Krein spaces and *-algebras.
O svjazi ponjatij teorii prostranstv Krejna i *-algebr. // VINITI 27.04.81, Nr.1916–81 (Russian)

[Ch83] S.A. CHOROŠAVIN, On quadratic states on Weyl *-algebra.
O kvadратичных sostojanjah na *-algebre Vejlja . // VINITI 30.08.83, Nr.4823–83 (Russian)

[Ch84] S.A. CHOROŠAVIN, Quadratic majorants of sesquilinar forms and *- representations.
Kvadратичные mažoranty polutoralinejnych form i *-predstavlenija. // VINITI 09.04.84, Nr.2135–84 (Russian)

[Ch84D] S.A. CHOROŠAVIN, Linear Operators in Indefinite Inner Product Spaces and Quadratic Hamiltonians (Russian) Ph.D. thesis, Voronežh state university, 1984

[DK] L.A. DADAŠEV, V.JU. KULIEV, Diagonalizacija bilineynh boze-gamil-tonianov i asimptotiĉeskoe povedenie poroždaemyh imi gejzenbergovyh polej //TMF.1979.T.39,N3,330-346.

Dadašev,L.A.; Kuliev,V.Ju. Diagonalization of bilinear Bose Hamiltonians and asymptotic behavior of corresponding Heisenberg fields. (Russian)
Teoret. Mat. Fiz. 39 (1979), no. 3, 330–346. MR 80e:81105

[DK2] Dadašev,L.A.; Kuliev,V.Yu. Uninvertible linear canonical transformations.
Rep. Math. Phys. 15 (1979), no. 2, 187–194. MR 80g:81038
Among Quadratic Hamiltonians

[Fey] R.P. FEYNMAN, Statistical Mechanics. A Set of Lectures, W. A. Benjamin, Inc. Advanced Book Program Reading, Massachusetts 1972.

[K] Kuliev, V. Ju. On the general theory of diagonalization of bilinear Hamiltonians. (Russian) Dokl. Akad. Nauk SSSR 253 (1980), no. 4, 860–863. MR 82f:82014

[MV] J. MANUCEAU, A. VERBEURE, Quasi-free states of the CCR-algebra and Bogoliubov transformations, Commun. Math. Phys., 9, (1968), 293–302.

[Oks] A.I. Oksak, Nefokovskie linejnye bozonnye sistemy i ih primenenija v dvumernyh modeljah //TMF.1981. T.48. N3. 297-318.

Oksak, A.I. Non-Fock linear boson systems and their applications in two-dimensional models. (Russian) Teoret. Mat. Fiz. 48 (1981), no. 3, 297–318. MR 84i:81079

[RH] Razumov, A.V.; Hrustalev, O.A.

An application of Bogoljubov’s method to the quantization of boson fields in the neighborhood of the classical solution. (Russian) Teoret. Mat. Fiz. 29 (1976), no. 3, 300–308. MR 56#14327

[RT79] Razumov, A.V.; Taranov, A.Ju.

Dipole interaction of an oscillator with a scalar field. (Russian) Teoret. Mat. Fiz. 38 (1979), no. 3, 355–363. MR 80b:81034

[RS1] M. REED, B. SIMON, Methods of Modern Mathematical Physics, vol 1, Functional analysis, - N.Y.: Academic Press, 1972.

[RS2] M. REED, B. SIMON, Methods of Modern Mathematical Physics, vol 2, Fourier analysis, Self-Adjointness, - N.Y.: Academic Press, 1975.

[RS3] M. REED, B. SIMON, Methods of Modern Mathematical Physics, vol 3, Scattering Theory, - N.Y.: Academic Press, 1979.

[RS4] M. REED, B. SIMON, Methods of Modern Mathematical Physics, vol 4, Analysis of Operators, - N.Y.: Academic Press, 1978.

[Schatz] Schatz, J.A. Representations of Banach Algebras with an Involution. Canad. J. Math. 9 (1957), 435–442.

[W1] John Williamson, On the algebraic problem concerning the normal forms of linear dynamical systems, American Journal of Mathematics, 58 (1936), 141–163.

[W2] John Williamson, On the normal forms of linear canonical transformations in dynamics, American Journal of Mathematics, 59 (1937), 599–617.

[W3] John Williamson, Quasi-unitary matrices, Duke Math. J., 3 (1937), 715–725.

[Z1] Ziman, J.M., Electrons and Phonons. The Theory of Transport Phenomena in Solids, London: Oxford University Press. XIV, 554p. (1960).

Ziman, J.M., Electrons and Phonons, Oxford Classic Text in the Physical Sciences. Oxford: Oxford University Press. xiv, 554p. (2000).

[Z2] Ziman, J.M., Electrons in Metals. A Short Guide to the Fermi Surface, London: Taylor and Frances, Ltd., 80p. (1963).

[Z3] Ziman, J.M., Principles of the Theory of Solids, London: Cambridge University Press 1964. XIV, 360p. (1964).
[Z4] Ziman, J. M., *Elements of Advanced Quantum Theory*, Cambridge: At the University Press 1964, XII, 269 p. (1969).

Electronic Print:

[CMS] mp_arc 98-498
S. Cavallaro, G. Morchio, F. Strocchi A Generalization of the Stone-von Neumann Theorem to Non-Regular Representations of the CCR-Algebra (43K, TeX)

[DR] Michael A. Dritschel and James Rovnyak Operators on Indefinite Inner Product Spaces,
in *Lectures on operator theory and its applications (Waterloo, ON, 1994)*, Fields Institute Monographs, vol. 3, Amer. Math. Soc., Providence, RI, 1996, pp. 141–232.

This document is available via the web in two forms:
http://faraday.clas.virginia.edu/~jlr5m/papers/fields/fieldslectures.ps postscript version (900K)
http://faraday.clas.virginia.edu/~jlr5m/papers/fields/dvi_version.html dvi version (450K)

It has 91 pages, including bibliography and index. Supplementary materials and errata may be found at
http://faraday.clas.virginia.edu/~jlr5m/papers/fields/Supplement.ps postscript version
http://faraday.clas.virginia.edu/~jlr5m/papers/fields/Supplement.dvi dvi version

The Abstract is available via the web in form:
http://www.math.purdue.edu/~mad/pubs/abs10.html

[FS] mp_arc 97-489
Martin Florig, Stephen J. Summers Further Representations of the Canonical Commutation Relations (125K, AmsTex)

LANL E-Print
Paper (*cross-listing*): [math-ph/0006011]
From: Stephen J. Summers <sjs@math.ufl.edu>
Date: Sun, 11 Jun 2000 22:23:37 GMT (35kb)

Title: Further Representations of the Canonical Commutation Relations
Authors: Martin Florig and Stephen J. Summers
Subj-class: Mathematical Physics; Functional Analysis; Operator Algebras
Journal-ref: Proc. Lond. Math. Soc. 80 (2000) 451-490

[Gru] mp_arc 93-329
Grundling H. A Group Algebra for Inductive Limit Groups. Continuity Problems of the Canonical Commutation Relations. (104K, TeX)

[Hal] LANL E-Print
Paper (*cross-listing*): [quant-ph/0110102]
From: Hans Halvorson <hhalvors@princeton.edu>
Date: Tue, 16 Oct 2001 23:51:01 GMT (12kb)

Title: Complementarity of representations in quantum mechanics
Among Quadratic Hamiltonians

Authors: Hans Halvorson
Comments: 14 pages, LaTeX
Subj-class: Quantum Physics; Mathematical Physics

[IT] LANL E-Print
Paper (*cross-listing*): math-ph/0001023
From: Nevena Ilieva <ilieva@pap.univie.ac.at>
Date: Mon, 17 Jan 2000 11:34:22 GMT (7kb)

Title: A mixed mean-field/BCS phase with an energy gap at high $T_c$
Authors: N. Ilieva and W. Thirring
Comments: 7 pages, LaTeX
Report-no: Vienna Preprint UWThPh-2000-2
Subj-class: Mathematical Physics
MSC-class: 81T05, 82B10, 82B23

[LMS] mparc 01-233
J. Loeffelholz, G. Morchio, F. Strocchi Ground state and functional integral representations of the CCR algebra with free evolution (51K, LaTeX 2e)