Quantum mechanics as an approximation of statistical mechanics for classical fields

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Abstract

We show that, in spite of a rather common opinion, quantum mechanics can be represented as an approximation of classical statistical mechanics. The approximation under consideration is based on the ordinary Taylor expansion of physical variables. The quantum contribution is given by the term of the second order. To escape technical difficulties related to the infinite dimension of phase space for quantum mechanics, we start with a detailed presentation of our approach for the finite dimensional quantum mechanics. We also separate real and complex cases, because the reproduction of the complex structure of quantum mechanics is a special problem which is not related to approximation of classical averages. In our approach quantum mechanics is an approximative theory. It predicts statistical averages only with some precision. In principle, there might be found deviations of averages calculated within the quantum formalism from experimental averages (which are supposed to be equal to classical averages given by our model).

Keywords: quantum and classical averages, von Neumann trace formula, approximation, small parameter, Taylor expansion
1 Introduction

The problem of coupling the quantum probabilistic model (which is based on the Hilbert space calculus) and the classical probabilistic model (which is based on the measure-theoretic calculus) was intensively discussed already by fathers of quantum mechanics, see, e.g., the correspondence between Einstein and Schrödinger [1]. This is the problem of huge complexity, see, e.g., [2]–[5] for different viewpoints and debates.

Now days there is a rather common opinion that the probabilistic structure of quantum mechanics cannot be considered simply as a special mathematical representation of classical (measure-theoretic) probability theory. Such an opinion is based merely on a number of no-go theorems, see appendix 2 for details. On the other hand, there are known various prequantum models that reproduce (at least some) features of quantum mechanics: De Broglie’s double solution theory, Bohmian mechanics, Nelson’s stochastic mechanics, t’ Hooft’s deterministic discrete models, see [6]–[10]. All such models are either nonlocal (as Bohmian mechanics and Nelson’s stochastic mechanics) or reproduce only some (not all) predictions of quantum mechanics. In any event they do not contradict no-go theorems.

In author’s papers [11], [12] there was proposed a new prequantum model: Prequantum Classical Statistical Field Theory, PCSFT. This model is very natural, because this is nothing else than standard classical statistical mechanics: a) state space is phase space; b) variables are functions on phase space; c) statistical states (describing ensembles of systems) are probability measures. One important point is that phase space is infinite-dimensional. Prequantum states (“hidden variables”) can be represented as vector fields, $\psi(x) = (q(x), p(x))$. However, this is merely a technical mathematical feature of PCSFT. The crucial point is a tricky way in which PCSFT is projected onto quantum mechanics. This projection is asymptotic. Quantum mechanics can be considered as an approximative theory: the quantum average given by the von Neumann trace formula appears as the first nontrivial contribution into the classical average (given by the Lebesgue integral).

To clarify the main distinguishing features of our theory, PCSFT, we shall divide its presentation into a few steps. First we consider the finite dimensional case. Here all computations are reduced to asymptotic expansions of simple Gaussian integrals. Then we shall proceed to the infinite-dimensional case where we should consider Gaussian integrals over functional spaces. We also start with consideration of a toy-model of quantum mechanics over reals,
see also [11], [12]. This will simplify the understanding of PCSFT, because
the prequantum phase-space model inducing the quantum model over com-
plex numbers has a rather nontrivial geometric structure. And its presenta-
tion should be separated from asymptotic expansion of Gaussian integrals.

We remark that construction of a prequantum classical statistical model
for finite-dimensional quantum mechanics is interesting not only from purely
mathematical viewpoint. Since electron-spin and photon-polarization can be
described by two dimensional complex spaces, in the finite-dimensional case
our approach shows that that its is possible to construct a pure classical
phase-space models for these quantum phenomena, see appendix 1.

In our approach quantum mechanics is an approximative theory. It pre-
dicts statistical averages only with some precision. In principle, there might
be found deviations of averages calculated within the quantum formalism
from experimental averages (which are supposed to be equal to classical av-
erages given by our model). But at the moment our predictions is not of a
high value for experimentalists, because PCSFT does not predict the mag-
nitude of a small parameter $\alpha$ in the asymptotic representation of classical
averages. In [11], [12] we speculated that the small parameter of PCSFT
$\alpha$ (the dispersion of prequantum fluctuations) can be chosen equal to the
Planck constant $\hbar$. But that speculation was not justified. We notice that
our asymptotic considerations are totally different from the standard con-
siderations on the classical limit of quantum mechanics: obtaining classical
phase space mechanics as the limit of quantum mechanics when the Planck
constant $\hbar$ (which is considered as a small parameter) goes to zero. In our
approach when the small parameter $\alpha$ goes to zero we obtain quantum theory
as the limit case of classical (and not vice versa). In particular, neglecting
by $\hbar$ induces neglecting by “quantum rotation”, spin. But in PCSFT such
degrees of freedom are not neglected, see appendix 1 for details.

2 The Taylor approximation of averages for
functions of random variables

Here we follow chapter 11 of the book [13] of Elena Ventzel. This book was
written in the form of precise instructions which student should follow to
solve a problem:

“in practice we have very often situations in that, although investigated
function of random arguments is not strictly linear, but it differs practically so negligibly from a linear function that it can be approximately considered as linear. This is a consequence of the fact that in many problems fluctuations of random variables play the role of small deviations from the basic law. Since such deviations are relatively small, functions which are not linear in the whole range of variation of their arguments are almost linear in a restricted range of their random changes,” [13], p. 238.

Let \( y = f(x) \). Here in general \( f \) is not linear, but it does not differ so much from linear on some interval \([m_x - \delta, m_x + \delta]\), where \( x = x(\omega) \) is a random variable and
\[
m_x \equiv E x = \int x(\omega) \, dP(\omega)
\]
is its average. Here \( \delta > 0 \) is sufficiently small. Student of a military college should approximate \( f \) by using the first order Taylor expansion at the point \( m_x \):
\[
y(\omega) \approx f(m_x) + f'(m_x)(x(\omega) - m_x). \quad (1)
\]
By taking the average of both sides he obtains:
\[
m_y \approx f(m_x). \quad (2)
\]
The crucial point is that the linear term \( f'(m_x)(x(\omega) - m_x) \) does not give any contribution! Further Elena Ventzel pointed out [13], p. 245: “For some problems the above linearization procedure may be unjustified, because the method of linearization may be not produce a sufficiently good approximation. In such cases to test the applicability of the linearization method and to improve results there can be applied the method which is based on preserving not only the linear term in the expansion of function, but also some terms of higher orders.”

Let \( y = f(x) \). Student now should preserve the first three terms in the expansion of \( f \) into the Taylor series at the point \( m_x \):
\[
y(\omega) \approx f(m_x) + f'(m_x)(x(\omega) - m_x) + \frac{1}{2} f''(m_x)(x(\omega) - m_x)^2. \quad (3)
\]
Hence
\[
m_y \approx f(m_x) + \frac{\sigma_x^2}{2} f''(m_x), \quad (4)
\]
where
\[ \sigma_x^2 = E (x - m_x)^2 = \int (x(\omega) - m_x)^2 \, dP(\omega) \]
is the dispersion of the random variable \( x \).

Let us now consider the special case of symmetric fluctuations:
\[ m_x = 0 \]
and let us restrict considerations to functions \( f \) such that
\[ f(0) = 0. \]

Then we obtain the following special form of (4):
\[ m_y \approx \sigma_x^2 f''(0). \]

We emphasize again that the first derivative does not give any contribution into the average.

Thus at the some level of approximation we can calculate averages not by using the Lebesgue integral (as we do in classical probability theory), but by finding the second derivative. Such a “calculus of probability” would match well with experiment. I hope that reader has already found analogy with the quantum calculus of probabilities. But for a better expression of this analogy we shall consider the multi-dimensional case. Let now \( x = (x_1, \ldots, x_n) \), so we consider a system of \( n \) random variables. We consider the vector average:
\[ m_x = (m_{x_1}, \ldots, m_{x_n}) \]
and the covariance matrix:
\[ B_x = (B^i_j), \quad B^i_j = E (x_i - m_{x_i}) (x_j - m_{x_j}). \]

We now consider the random variable
\[ y(\omega) = f(x_1(\omega), \ldots, x_n(\omega)). \]

By using the Taylor expansion we would like to obtain an algorithm for approximation of the average \( m_y \). We start directly from the second order Taylor expansion:
\[ y(\omega) \approx f(m_{x_1}, \ldots, m_{x_n}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(m_{x_1}, \ldots, m_{x_n})(x_i(\omega) - m_{x_i}) \]
\[ + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} (m_{x_1}, \ldots, m_{x_n})(x_i(\omega) - m_{x_i})(x_j(\omega) - m_{x_j}), \]  
(6)

and hence:

\[ m_y \approx f(m_{x_1}, \ldots, m_{x_1}) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} (m_{x_1}, \ldots, m_{x_1}) B^{ij}. \]  
(7)

By using the vector notations we can rewrite the previous formulas as:

\[ y(\omega) \approx f(m_x) + (f'(m_x), x(\omega) - m_x) + \frac{1}{2} (f''(m_x) (x(\omega) - m_x), x(\omega) - m_x). \]  
(8)

and

\[ m_y \approx f(m_x) + \frac{1}{2} \text{Tr } B_x f''(m_x). \]  
(9)

Let us again consider the special case: \( m_x = 0 \) and \( f(0) = 0 \). We have:

\[ m_y \approx \frac{1}{2} \text{Tr } B_x f''(0). \]  
(10)

We now remark that the Hessian \( f''(0) \) is always a symmetric operator. Let us now represent \( f \) by its second derivative at zero:

\[ f \to A = f''(0). \]

Then we see that, at some level of approximation, instead of operation with Lebesgue integrals, one can use linear algebra:

\[ m_y \approx \frac{1}{2} \text{Tr } B_x A \]  
(11)

We now proceed in mathematically rigorous way, namely, we shall estimate the reminder which was neglected in the approximative formula for average. We also formalize correspondence between classical and quantum statistical models.
3 Classical and quantum statistical models

3.1 Classical statistical model

Classical statistical mechanics on phase space $\Omega_{2n} = \mathbb{R}^n \times \mathbb{R}^n$ can be considered as a special classical statistical model. In general a classical statistical model is defined in the following way:

a). Physical states $\omega$ are represented by points of some set $\Omega$ (state space).

b). Physical variables are represented by functions $f : \Omega \to \mathbb{R}$ belonging to some functional space $V(\Omega)$.

c). Statistical states are represented by probability measures on $\Omega$ belonging to some class $S(\Omega)$.

d). The average of a physical variable (which is represented by a function $f \in V(\Omega)$) with respect to a statistical state (which is represented by a probability measure $\rho \in S(\Omega)$) is given by

$$<f>_{\rho} \equiv \int_{\Omega} f(\omega) d\rho(\omega).$$

(12)

A classical statistical model is a pair

$$M = (S(\Omega), V(\Omega)).$$

In classical statistical mechanics $\Omega = \Omega_{2n}$ is phase space, $V(\Omega_{2n}) = C^\infty(\Omega_{2n})$ is the space of all smooth functions on phase space, $S(\Omega_{2n})$ is the space $PM(\Omega_{2n})$ of all probability measures on phase space and the average is given by the Lebesgue integral on the $\sigma$-algebra of Borel subsets of $\Omega_{2n}$.

Remark 3.1. We emphasize that the space of variables $V(\Omega)$ need not coincide with the space of all random variables $RV(\Omega)$ – measurable functions $\xi : \Omega \to \mathbb{R}$. For example, if $\Omega$ is a differentiable manifold, it is natural to choose $V(\Omega)$ consisting of smooth functions; if $\Omega$ is an analytic manifold, it is natural to choose $V(\Omega)$ consisting of analytic functions and so on. The space of statistical states $S(\Omega)$ need not coincide with the space of all probability measures $PM(\Omega)$. For example, for some statistical model $S(\Omega)$ may consist of Gaussian measures.

1The choice of a concrete functional space $V(\Omega)$ depends on various physical and mathematical factors.
3.2 Real finite-dimensional quantum mechanics

We shall use a toy model of quantum mechanics which based on the real space. Statistical features of the correspondence between a prequantum classical statistical model and quantum mechanics are more evident for this toy model. Denote the algebra of all \((m \times m)\) real matrices by the symbol \(M^{(r)}(m)\). We denote by \(D^{(r)}(m)\) the class of nonnegative symmetric trace-one matrices \(\rho \in M^{(r)}(m)\). We call them “density operators.” We denote by \(L_s^{(r)}(m)\) the class of all symmetric matrices. In the quantum model (for the \(m\)-dimensional real space) statistical states (describing ensembles of systems prepared for measurement) are represented by density matrices and quantum observables by matrices belonging \(L_s^{(r)}(m)\). The \textit{quantum average} of an observable \(A \in L_s^{(r)}(m)\) with respect to a statistical state \(\rho \in D^{(r)}(m)\) is given by the von Neumann trace class formula [4]:

\[
<A>_{\rho} = \text{Tr} \rho A.
\]

(13)

In the operator representation observables and density matrices are corresponding classes of \(\mathbb{R}\)-linear operators. Denote the quantum model by

\[N^{(r)}_{\text{quant}} = (D^{(r)}(m), L_s^{(r)}(m)).\]

If \(m = 1\), then quantum observables are given by real numbers (operators of multiplication by real numbers on the real line) and there is only one statistical state \(\rho = 1\). Here \(<A>_{\rho} = \rho A = A\).

3.3 Complex finite-dimensional quantum mechanics

Denote the algebra of all \((m \times m)\) complex matrices by the symbol \(M^{(c)}(m)\). We denote by \(D^{(c)}(m)\) the class of nonnegative symmetric trace-one matrices \(\rho \in M^{(c)}(m)\). We call them “density operators.” We denote by \(L_s^{(c)}(m)\) the class of all symmetric matrices. In the quantum model (for the \(m\)-dimensional complex space) statistical states (describing ensembles of systems prepared for measurement) are represented by density matrices and quantum observables by matrices belonging \(L_s^{(c)}(m)\). The \textit{quantum average} is given [4] by [13]. In the operator representation observables and density matrices are corresponding classes of \(\mathbb{C}\)-linear operators. Denote the quantum model by

\[N^{(c)}_{\text{quant}} = (D^{(c)}(m), L_s^{(c)}(m)).\]
If $m = 1$, then quantum observables are given by real numbers (operators of multiplication by real numbers on the complex plane) and and there is only one statistical state $\rho = 1$. Here $<A>_{\rho} = \rho A = A$.

### 3.4 Complex quantum mechanics

Denote by $H_c$ a complex (separable and infinite-dimensional) Hilbert space. Denote the algebra of all bounded operators $A : H_c \rightarrow H_c$ by the symbol $\mathcal{L}(H_c)$. The real linear subspace of $\mathcal{L}(H_c)$ consisting of self-adjoint operators is denoted by the symbol $\mathcal{L}_s(H_c)$. We denote by $\mathcal{D}(H_c)$ the class of non-negative trace-one operators $\rho \in \mathcal{L}_s(H_c)$. These are von Neumann density operators. In the quantum model statistical states (describing ensembles\(^2\) of systems prepared for measurement) are represented by density operators and quantum observables by operators from $\mathcal{L}_s(H_c)$. The quantum average is given \(^4\) by \(^{13}\). Denote the quantum model by

$$N_{\text{quant}}(H_c) = (\mathcal{D}(H_c), \mathcal{L}_s(H_c)).$$

### 4 Taylor approximation of classical averages: one dimensional case

States of systems are represented by real numbers, $q \in Q = \mathbb{R}$. Ensembles of such systems are described by probability measures on the real line, statistical states. We consider a special class of preparation procedures. They produce ensembles of systems described by Gaussian probability distributions on $Q$ having the zero mean value and dispersion

$$\sigma^2(\mu) = \alpha + O(\alpha^2),$$

where as always $|O(\alpha^2)| \leq C\alpha^2$ for some constant $C$ and a sufficiently small $\alpha$. The crucial point is that $\alpha$ is a small parameter of our model. Denote this class of probability distributions by the symbol $S^\alpha_G(Q)$.

\(^2\)We follow so called ensemble interpretation of quantum mechanics: Einstein, Margenau, Ballentine, Balian, Nieuwenhuizen and many others, see, e.g., \(^{14}\). By such an interpretation even a pure state (normalized vector of $H_c$) represents an ensemble. In the orthodox Copenhagen interpretation a pure state represents the state of an individual system, e.g., electron.
For a probability \( \mu \in S_G^2(Q) \), we have:

\[
d\mu(q) = \frac{e^{-\frac{q^2}{2(\alpha + O(\alpha^2))}}}{\sqrt{2\pi(\alpha + O(\alpha^2))}}\,dq.
\]

We recall that, for a probability with the zero mean value, its dispersion is given by

\[
\sigma^2(\mu) = \frac{1}{\sqrt{2\pi(\alpha + O(\alpha^2))}} \int_{-\infty}^{\infty} q^2 e^{-\frac{q^2}{2(\alpha + O(\alpha^2))}}\,dq. \tag{16}
\]

As was already pointed out, we consider \( \alpha \) as a small parameter. Therefore Gaussian probability distributions are very sharply concentrated around the point \( q_0 = 0 \). By using the terminology of functional analysis we say that \( \{\mu \equiv \mu(\alpha)\} \) is a \( \delta \)-family: \( \lim_{\alpha \to 0} \mu(\alpha) = \delta \) in the sense of theory of distributions.

In the approximation \( \alpha = 0 \) all systems are located at a single point, namely, \( q_0 \). However, a finer description (in that \( \alpha \) can not be neglected) provides the picture of Gaussian bells concentrated nearby \( q_0 \). We remark that in average a system cannot go far away from \( q_0 \).

By using the Chebyshov inequality one obtain for any \( C > 0 \):

\[
\mu\{q : |q| > C\} \leq \frac{\alpha + O(\alpha^2)}{C^2} \rightarrow O, \alpha \to 0. \tag{17}
\]

But the probabilistic inequality (17) does not exclude the possibility that some system could move far from \( q_0 \) (of course, with a small probability).

We also introduce a class of physical variables in the classical statistical model under consideration:

a) \( f \in C^\infty(\mathbf{R}) \), a smooth function;

b) \( f(0) = 0 \);

c) \( |f^{(4)}(q)| \leq c_f e^{r_f |q|}, c_f, r_f \geq 0 \).

Denote this functional space by the symbol \( \mathcal{V}(Q), Q = \mathbf{R} \).

The restriction to the growth of the fourth derivative will be used when we shall consider the Taylor expansion of \( f \) up two the fourth term. The exponential growth implies integrability with respect to any Gaussian measure.

**Lemma 4.1.** Let \( f \in C^n \) (so it is \( n \) times continuously differentiable) and let its \( n \)th derivative has the exponential growth. Then all derivatives of
orders \( n = 0, \ldots, n - 1 \), also have the exponential growth \( (\) in particular, \( f(q) \) grows exponentially).  

**Proof.** Under these conditions we can use the Taylor expansion with the integral remainder: 

\[
f(q) = f(0) + f'(0)q + \frac{f''(0)q^2}{2} + \frac{f^{(3)}(0)q^3}{3!} + \ldots + \frac{q^n}{n!} \int_0^1 (1 - \theta)^{n-1} f^{(n)}(\theta q) d\theta.
\]  

Since the growth of any polynomial can be compensated by decreasing of the \( e^{-r|q|} \), by using the exponential estimate for the \( n \)th derivative we obtain: 

\[
|f(q)| = C_1 e^{r|q|} + C_2 \frac{q^n}{n!} \int_0^1 (1 - \theta)^{n-1} e^{r|q\theta|} d\theta \leq C e^{r|q|}.
\]  

Here all constants depend on \( f \).

This simple exercise from the course of analysis will be useful in our further considerations. We defined the following classical statistical model on the real line:

A). States of systems are real numbers.

B). Statistical states (ensembles of systems) are represented by Gaussian probabilities having zero average and dispersion \( \sigma^2(\mu) = \alpha + O(\alpha^2), \alpha \to 0 \).

C). Physical variables are smooth functions with exponentially growing fourth derivative which map zero into itself.

We denote this model by \( N^\alpha_{\text{class}} = (S^\alpha_G(Q), \mathcal{V}(Q)) \). As always in classical statistical physics, the average of a physical variable \( f \in \mathcal{V}(Q) \) with respect to an ensemble of systems which is described by a probability \( \mu \in S^\alpha_G(Q) \) is given by the integral: 

\[
< f >_\mu = \frac{1}{\sqrt{2\pi(\alpha + O(\alpha^2))}} \int_{-\infty}^{\infty} f(q) e^{\frac{-q^2}{2(\alpha + O(\alpha^2))}} dq.
\]  

Since \( \alpha \) is a parameter of the model, we can consider averages as functions of \( \alpha : < f >_\mu \equiv < f >_\mu (\alpha) \). We are interested in the asymptotic expansion of averages when \( \alpha \to 0 \). In particular, such an asymptotic expansion will give us the possibility to calculate averages approximately.

**Lemma 4.2.** Let \( f \in \mathcal{V}(Q) \) and let \( \mu \in S^\alpha_G(Q) \). Then 

\[
< f >_\mu (\alpha) = \frac{\alpha}{2} f''(0) + O(\alpha^2).
\]  

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Proof. We start with the scaling of the state variable:

\[ q = \sigma(\mu)x \]  \hspace{1cm} (22)

We have:

\[ <f>_{\mu}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\sigma(\mu)x)e^{-\frac{x^2}{2}} dx. \]  \hspace{1cm} (23)

We now expand \( f(\sigma(\mu)x) \) by using the fourth order Taylor formula with the integral remainder, see Lemma 4.1:

\[ <f>_{\mu}(\alpha) = \sigma^2(\mu) f''(0) + \frac{\sigma^4(\mu)}{4!\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 \left( \int_{0}^{1} (1 - \theta)^3 f^{(4)}(\sigma(\mu)x\theta)d\theta \right) e^{-\frac{x^2}{2}} dx. \]  \hspace{1cm} (24)

We recall that for a Gaussian measure with zero mean value all odd moments are equal to zero. This is an important point of our considerations. This imply that the first nonzero contribution to the classical average is given by the second derivative – quadratic term. Disappearance of the third order term implies the asymptotics \( O(\alpha^2) \). We now estimate the remainder to obtain this asymptotics:

\[ |R(f, \mu)| \leq C\sigma^4(\mu) \int_{-\infty}^{\infty} x^4 \left( \int_{0}^{1} (1 - \theta)^3 e^{r\sigma(\mu)|x|\theta} d\theta \right) e^{-\frac{x^2}{2}} dx. \]

Since we consider \( \alpha \) as a small parameter, we can assume that \( |\sigma(\mu)| \leq 1 \) in the exponential function. Thus:

\[ |R(f, \mu)| \leq C'\sigma^4(\mu) \int_{-\infty}^{\infty} x^4 e^{r|x|-\frac{x^2}{2}} dx. \]

Since \( \sigma^2(\mu) = \alpha + O(\alpha^2) \), we have that \( R(f, \mu) = O(\alpha^2), \alpha \to 0. \)

We consider the dispersion \( \sigma^2(\mu) \) as the intensity of fluctuations in the ensemble of systems. We define the relative average with respect to the intensity of fluctuations by normalizing the average by the main term – namely, \( \alpha \) – in the intensity of fluctuations:

\[ \langle f \rangle_{\mu} = \frac{<f>_{\mu}}{\alpha}. \]
Of course, \( \langle f \rangle_\mu \) is also a function of the parameter \( \alpha \):
\[
\langle f \rangle_\mu(\alpha) = \frac{< f >_\mu(\alpha)}{\alpha}.
\]

**Corollary 4.1.** Let \( f \in V(Q) \) and let \( \mu \in S^\alpha_{G}(Q) \). Then
\[
\langle f \rangle_\mu = \frac{f''(0)}{2} + O(\alpha).
\] (25)

In particular,
\[
\lim_{\alpha \to 0} \langle f \rangle_\mu(\alpha) = \frac{f''(0)}{2}.
\] (26)

**Proposition 4.1.** We have:
\[
\langle f \rangle_\mu = \frac{< f >_\mu}{\sigma^2(\mu)} + O(\alpha).
\] (27)

**Remark 4.1** (About 1/2) The second term in the Taylor formula gives the factor 1/2 which looks rather bothering in our asymptotic formula for the classical average. This factor will disappear in the complex representation and the formula will become nicer.

We have shown that \( \frac{f''(0)}{2} \) gives the approximation of the (classical) relative average. The precision of such an approximation is \( \alpha \). If the level of development of measurement technology is such that all contributions of the magnitude \( \alpha \) are neglected in measurements, then averages can be calculated by using the following simple rule:
\[
\langle f \rangle_\mu^{\text{approx}} = \left[ \frac{< f >_\mu^{\text{approx}}}{\sigma^2(\mu)} \right] = \frac{f''(0)}{2}.
\] (28)

At the first sight such averages have nothing to do with classical averages given by integrals. There could be even presented an interpretation of physics claiming that rules of classical probability theory are violated and relating the exotic rule (28) for calculating of averages to special features of systems under consideration (and not to a special approximation procedure for averages).

Finally, we remark that calculation of averages by (28) is essentially simpler than classical probabilistic averages given by Lebesgue integrals.
5 Taylor approximation of classical averages: multidimensional case

States are vectors \( q \in Q = \mathbb{R}^m \); statistical states are Gaussian distributions with the zero mean value and the dispersion \( \sigma^2(\mu) = \alpha + O(\alpha^2) \). Denote this class of probabilities by the symbol \( S_G^\alpha(Q) \). We introduce the scalar product and norm on \( Q \):

\[
(\xi, q) = \sum_{j=1}^{m} \xi_j q_j, \quad \|q\|^2 = \sum_{j=1}^{m} q_j^2.
\]

If a Gaussian measure \( \mu \) is nondegenerate (so the measure of any open set is positive), then

\[
d\mu(q) = \frac{e^{-\frac{1}{2}(B^{-1}q,q)}}{\sqrt{(2\pi)^m \det B}} dq,
\]

where \( B \) is a positive operator (we consider everywhere only Gaussian measures with zero mean values). If \( \mu \in S_G^\alpha(Q) \) and nondegenerate, then

\[
\sigma^2(\mu) = \frac{1}{\sqrt{(2\pi)^m \det B}} \int_{\mathbb{R}^m} \|q\|^2 e^{-\frac{1}{2}(B^{-1}q,q)} dq = \alpha + O(\alpha^2).
\]

In the general case the easiest way to define a Gaussian measure is to use its Fourier transform:

\[
\tilde{\mu}(\xi) = \int_{\mathbb{R}^m} e^{i(\xi,q)} d\mu(q) = e^{-\frac{1}{2}(Bq,q)},
\]

where \( B = \text{cov } \mu \) is the covariance operator:

\[
(B\xi_1, \xi_2) = \int_{\mathbb{R}^m} (\xi_1, q) (\xi_2, q) d\mu(q).
\]

We remark that by definition a covariance operator is positively defined and symmetric.

**Lemma 5.1.** Let \( \mu \) be a Gaussian measure with the zero mean value and let \( A \) be a symmetric operator. Then

\[
\int_{\mathbb{R}^m} (Aq, q) d\mu(q) = \text{Tr } BA,
\]

where \( B = \text{cov } \mu \).
To prove this lemma we should just expand the quadratic form \((Aq, q)\) with respect to an orthonormal basis.

**Corollary 5.1.** We have

\[
\sigma^2(\mu) = \int_{\mathbb{R}^m} \|q\|^2 \, d\mu(q) = \text{Tr} \, B. \tag{30}
\]

Thus, for \(\mu \in S_G^\alpha(Q)\),

\[
\text{Tr} \, \text{cov} \, \mu = \alpha + O(\alpha^2).
\]

We now define a class of physical variables – \(V(Q)\):

1. \(f \in C^\infty(\mathbb{R}^m)\);
2. \(f(0) = 0\);
3. \(\|f^{(4)}(q)\| \leq c_f e^{r_f \|q\|}, c_f, r_f \geq 0\).

For a function \(f : \mathbb{R}^m \to \mathbb{R}\), its \(n\)th derivative is a (symmetric) \(n\)-linear functional, \(f^{(n)}(q) : \mathbb{R}^m \times \ldots \times \mathbb{R}^m \to \mathbb{R}\). The norm of this functional is given by

\[
\|f^{(n)}(q)\| = \sup_{\|h_j\| = 1} |f^{(n)}(q)(h_1, \ldots, h_n)|.
\]

The norm can be estimated by partial derivatives:

\[
\|f^{(n)}(q)\| \leq \max_{\alpha_1 + \ldots + \alpha_n = n} \left| \frac{\partial^n f(q)}{\partial q_1^{\alpha_1} \ldots \partial q_m^{\alpha_m}} \right|.
\]

It is easy to generalize Lemma 4.1 to the multidimensional case.

Thus we have defined the following classical statistical model: \(N_{\text{class}} = (S_G^\alpha(Q), V(Q))\).

**Lemma 5.2.** Let \(f \in V(Q)\) and let \(\mu \in S_G^\alpha(Q)\). Then

\[
< f >_\mu (\alpha) = \int_{\mathbb{R}^m} f(q) \, d\mu(q) = \frac{\alpha}{2} \text{Tr} \, \rho f''(0) + O(\alpha^2), \tag{31}
\]

where \(\rho\) is a density operator; in fact, \(\rho = \text{cov} \, \mu / \alpha\).

**Proof.** By using the scaling of the state variable and by expanding \(f(\sigma(\mu)x)\) on the basis of the fourth order Taylor formula with the integral remainder we obtain:

\[
< f >_\mu (\alpha) = \frac{\sigma^2(\mu)}{2} \text{Tr} \, \rho f''(0) + \frac{\sigma^4(\mu)}{4!} \int_{\mathbb{R}^m} \left( \int_0^1 (1 - \theta)^3 f^{(4)}(\sigma(\mu)x\theta)(q, q, q, q) \, d\theta \right) \, d\mu_{\text{scal}}(x). \tag{32}
\]
where $\mu_{\text{scal}}$ is a normalized Gaussian measure – the image of $\mu$ under the scaling (22). We now estimate the remainder:

$$|R(f, \mu)| \leq \frac{C'\sigma^4(\mu)}{4!} \int_{\mathbb{R}^m} \|x\|^4 \left( \int_0^1 (1 - \theta)^3 e^{r\sigma(\mu)\|x\|\theta} d\theta \right) d\mu_{\text{scal}}(x).$$

Thus

$$|R(f, \mu)| \leq C'\sigma^4(\mu) \int_{\mathbb{R}^m} \|x\|^4 e^{r\|x\|} d\mu_{\text{scal}}(x).$$

We have that $R(f, \mu) = O(\alpha^2), \alpha \to 0$.

**Corollary 5.2.** Let $f \in \mathcal{V}(Q)$ and let $\mu \in S_{G}^\alpha(Q)$ be nondegenerate. Then

$$\frac{1}{\sqrt{(2\pi)^m \det B}} \int_{\mathbb{R}^m} f(q) e^{\frac{1}{2}(B^{-1}q,q)} dq = \frac{\alpha}{2} \text{Tr} \rho f''(0) + O(\alpha^2). \quad (33)$$

where $\rho = B/\alpha$.

As in the one-dimensional case, we introduce the relative average:

$$\langle f \rangle_{\mu} = \frac{\mu > f}{\alpha} = \frac{\int_{\mathbb{R}^m} f(q) d\mu(q)}{\int_{\mathbb{R}^m} \|q\|^2 d\mu(q)} + O(\alpha).$$

In the case of a nondegenerate Gaussian measure we have:

$$\langle f \rangle_{\mu} = \frac{\int_{\mathbb{R}^m} f(q) e^{\frac{1}{2}(B^{-1}q,q)} dq}{\int_{\mathbb{R}^m} \|q\|^2 e^{\frac{1}{2}(B^{-1}q,q)} dq} + O(\alpha).$$

**Corollary 5.3.** Let $f \in \mathcal{V}(Q)$ and let $\mu \in S_{G}^\alpha(Q)$. Then

$$\langle f \rangle_{\mu} = \frac{1}{2} \text{Tr} \rho f''(0) + O(\alpha). \quad (34)$$

Thus if one neglects by terms of the magnitude $\alpha$, it is possible to use the following approximative calculus of averages:

$$\langle f \rangle_{\mu}^{\text{approx}} = \frac{1}{2} \text{Tr} \rho A, \quad (35)$$

where $A = f''(0)$ and $\rho = \text{cov} \mu_{\text{scal}}$. This is nothing else than the von Neumann trace formula for quantum averages, see (13). To proceed more formally, we consider maps:

$$T : S_{G}^\alpha(Q) \to \mathcal{D}^{(r)}(m), \rho = T(\mu) = \text{cov} \mu_{\text{scal}}; \quad (36)$$

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\[ T : \mathcal{V}(Q) \to \mathcal{L}^{(r)}_s(m), A = T(f) = f''(0) \]  \hspace{1cm} (37)

(we recall that Hessian is always a symmetric matrix).

**Theorem 5.1.** The maps (36), (37) project the classical statistical model \( N_{\text{class}} = (S^\alpha_G(Q), \mathcal{V}(Q)) \) onto the quantum model \( N^{(r)}_{\text{quant}} = (\mathcal{D}^{(r)}(m), \mathcal{L}^{(r)}_s(m)) \) in such a way that classical and quantum averages are coupled by the asymptotic equality:

\[ \langle f \rangle_\mu = \frac{1}{2} < T(f) >_{T(\mu)} + O(\alpha). \]  \hspace{1cm} (38)

### 6 Degenerate Gaussian measures and “pure states”

Consider a Gaussian measure \( \mu \in S^\alpha_G(Q) \) which is concentrated on a linear subspace \( Q_0 \) of \( Q \). So it is nondegenerate on \( Q_0 \). Here \( P : Q \to Q_0 \) is the orthogonal projector onto \( Q_0 \). Denote by \( B_0 \) the covariance matrix of the restriction of \( \mu \) onto \( Q_0 \). Then \( B_0 > 0 \) and \( B = \text{cov} \mu = PB_0P \). Let us now make the scaling (22). Then \( \rho = \text{cov} \mu_{\text{scal}} = P\rho_0P \), where \( \rho_0 = \text{cov} \mu_{\text{scal}}|_{Q_0} \).

Thus for any symmetric matrix \( A \) we have \( \text{Tr} \rho A = \text{Tr} \rho_0(PAP) \). Suppose now that \( \rho_0 = \frac{I}{\dim Q_0} \), where \( I : Q_0 \to Q_0 \) is the unit operator. Then \( \text{Tr} \rho A = \frac{1}{\dim Q_0} \text{Tr} PAP \). We are especially interested in measures concentrated on one dimensional subspaces, \( Q_0 \equiv Q_\Psi = \{ q = c\Psi, c \in \mathbb{R} \} \), where \( \Psi \) has the norm one. Here \( P \equiv P_\Psi \) is the one dimensional projector \( P_\Psi : Q \to Q_\Psi, P_\Psi \phi = (\phi, \Psi)\Psi \), and hence

\[ \text{Tr} \rho A = \text{Tr} P_\Psi A P_\Psi = (A\Psi, \Psi). \]  \hspace{1cm} (39)

Denote a probability \( \mu \in S^\alpha_G(Q) \) which is concentrated on the one dimensional subspace \( Q_\Psi \) by the symbol \( \mu_\Psi \). We obtained the following simple result:

**Proposition 6.1.** For any \( \mu_\Psi \), we have

\[ \langle f \rangle_{\mu_\Psi} = \frac{1}{2} (f''(0)\Psi, \Psi) + O(\alpha). \]  \hspace{1cm} (40)

Thus approximately we have:

\[ \langle f \rangle_{\mu_\Psi} = \frac{1}{2} (A\Psi, \Psi), A = f''(0). \]  \hspace{1cm} (41)
But the right-hand side of this equality is nothing else than the well known quantum formula for the average of the quantum observable $A$ with respect to the pure state $\Psi$, \[4\]. In our approach this quantum formula arose as the approximation of the classical average with respect to a Gaussian ensemble. The only distinguishing feature of such an ensemble is that all systems have states proportional to the vector $\Psi$ (with the probability one). Of course, one can consider the projective space and then all those systems will have the same coordinate. However, real coordinates of systems are different.

**Conclusion.** Quantum averages with respect so called pure states can be easily reproduced as approximations of ordinary ensemble averages with respect to one dimensional Gaussian distributions.

### 7 Prequantum phase space – the two dimensional case

In previous sections we considered the prequantum toy model in that the phase space structure was not taken into account. The corresponding quantum model was over the reals, see also \[11\], \[12\]. On the other hand, physical reality is described by the classical phase space mechanics and the complex quantum mechanics. We shall see that it is possible to create a prequantum phase space model reproducing the complex quantum mechanics. The crucial point is that classical variables and statistical states – functions and measures on phase space – should be invariant with respect to a special group of transformations of phase space.

This fundamental prequantum group is very simple – the special orthogonal group $SO(2)$, the group of rotations of phase space.

States of systems are now represented by points $\psi = (q,p) \in \Omega = Q \times P$, where $Q = P = \mathbb{R}$. Here the $q$ is the position and the $p$ is momentum, so $\Omega$ denotes phase space. Statistical states are represented by Gaussian $SO(2)$-invariant measures having zero mean value and dispersion

$$\sigma^2(\mu) = 2\alpha + O(\alpha^2); \quad (42)$$

physical variables are by $SO(2)$-invariant maps, $f : \Omega \to \mathbb{R}$, which satisfy conditions a), b), c) specifying variables in the real case. Denote these classes of measures and functions, respectively, $S^\alpha_G(\Omega|SO(2))$ and $\mathcal{V}(\Omega|SO(2))$. 
The appearance of the factor 2 has the following motivation: there are two contributions into fluctuations – fluctuations of positions and momenta. We shall see that they are equally distributed. Therefore it is natural to consider as a small parameter of the model the dispersion of e.g. the $q$-fluctuations (which equals to the dispersion of the $p$-fluctuations).

We consider the classical model $N_{\text{class}} = (S_G^\alpha(\Omega|SO(2)), V(\Omega|SO(2)))$. As in the real case, we can obtain the asymptotic expansion of the classical averages, see (34). However, in quantum mechanics we consider the complex structure. We would like to recover it in our classical model. To do this, we shall study in more detail properties of classical probabilities and variables.

A measure $\mu$ is invariant if for any $u \in SO(2)$:
\[
\int_{\mathbb{R}^2} f(uq)d\mu(q) = \int_{\mathbb{R}^2} f(q)d\mu(q). \tag{43}
\]
For a Gaussian measure $\mu$ with the covariance matrix $B$, this is equivalent to the condition:
\[
[u, B] = 0, \quad u \in SO(2). \tag{44}
\]

Let $f$ be a two times differentiable invariant map, so $f(u\psi) = f(\psi)$, for any $u \in SO(2)$. By representing
\[
u = u_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \tag{45}
\]
we have that
\[
f(\cos \theta q - \sin \theta p, \sin \theta q + \cos \theta p) = f(q, p). \tag{46}
\]
This is a rather strong constraint determining a very special class of maps. In particular, we obtain: $u^*\nabla f(u\psi) = \nabla f(\psi)$ and $u^*f''(u\psi)u = f''(\psi)$. Hence $u^*\nabla f(0) = \nabla f(0)$ for any rotation, and thus
\[
\nabla f(0) = 0 \tag{47}
\]
and
\[
[f''(0), u] = 0, \quad u \in SO(2). \tag{48}
\]

It is convenient to introduce the commutator of the set $SO(2)$ in the algebra of all two by two matrices $M(r)(2)$:
\[
SO'(2) = \{ A \in M(r)(2) : [A, u] = 0, u \in SO(2) \}
\]
We remark that a generator of $SO(2)$ can be chosen as the symplectic operator:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Therefore the commutator of $SO'(2)$ coincides with the commutator of $J$:

$$\{J\}' = \{A \in M^{(r)}(2) : [A, J] = 0\}.$$ 

**Proposition 7.1.** Let $\mu \in S_\alpha^\prime(G|SO(2))$ and let $f \in V(\Omega|SO(2))$. Then $B = \text{cov} \, \mu$ and $A = f''(0)$ belong to $SO'(2)$.

**Lemma 7.1.** A matrix $A$ belongs to the commutator $SO'(2)$ iff

$$A = \begin{pmatrix} R & -S \\ S & R \end{pmatrix}.$$ (49)

If $A$ is also symmetric, then it is diagonal: $A = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$. In particular, its trace is given by

$$\text{Tr}A = 2R.$$ (50)

Thus if $\mu \in S_\alpha^\prime(G|SO(2))$, then its covariance matrix is diagonal $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$, where $2b = \alpha + O(\alpha^2)$. Fluctuations of the coordinate $q$ and the momentum $p$ are independent and equally distributed:

$$d\mu(q) = \frac{1}{2\pi b} \exp\left\{-\frac{q^2 + p^2}{2b}\right\} dq.$$

Denote the marginal distributions of $\mu$ by the symbols $\mu_q$ and $\mu_p$, respectively. Then

$$\sigma^2(\mu_q) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{+\infty} q^2 \exp\left\{-\frac{q^2}{2b}\right\} dq = \sigma^2(\mu_p) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{+\infty} p^2 \exp\left\{-\frac{p^2}{2b}\right\} dp.$$

Hence

$$\sigma^2(\mu_q) = \sigma^2(\mu_p) = \frac{1}{2}\sigma^2(\mu) = \alpha + O(\alpha^2).$$

**Proposition 7.2.** Let $f \in V(\Omega|SO(2))$. Then all its odd derivatives at the point $q_0 = 0$ are equal to zero.

**Proof.** For any vector $\phi \in \Omega$, we have $f^{(2n+1)}(u\psi)(u\phi, ..., u\phi) = f^{(2n+1)}(\psi)(\phi, ..., \phi)$, for any rotation $u$. We choose $u = J$. Then:

$$(-1)^{2n+1} f^{(2n+1)}(0)(\phi, ..., \phi) = f^{(2n+1)}(0)(J^2 \phi, ..., J^2 \phi).$$

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\[ f^{(2n+1)}(0)(J\phi, \ldots, J\phi) = f^{(2n+1)}(0)(\phi, \ldots, \phi). \]

Thus \( f^{(2n+1)}(0)(\phi, \ldots, \phi) = 0 \) for any vector \( \phi \in \Omega \).

For a function \( f \in V(\Omega|SO(2)) \), its Hessian has the form \( f''(0) = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \), where \( R \in \mathbb{R} \), and hence:

\[ f(q, p) = \frac{R(q^2 + p^2)}{2} + O(\alpha^2). \]

We remark that, in spite of the coincidence of commutators, the \( SO(2) \)-invariance is not equivalent to the \( J \)-invariance (the later was used as the basis of the theory in [15]).

**Example 7.1.** Let \( f(q, p) = q^3p - qp^3 = qp(q^2 - p^2) \). Then \( f(J\psi) = f(\psi) \).

But take \( \theta = \pi/4 \). Here \( u = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \). Hence \( u \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} (q - p)/2 \\ (q + p)/2 \end{pmatrix} \).

Thus \( f(u\psi) = (q - p)(q + p)(q - p - q - p)(q - p + q + p)/16 = -qp((q^2 - p^2)/4. \)

We are now completely ready to recover the complex structure of quantum mechanics. By Lemma 7.1 any matrix belonging \( SO'(2) \) can be represented in the form: \( A = RI + S(-J) \). By mapping \( I \) into \( 1 \) and \( (-J) \) into \( i \) we obtain a map of the commutator \( SO'(2) \) onto the set of complex numbers \( \mathbb{C} \):

\[ j : SO'(2) \rightarrow \mathbb{C}, z = j(A) = R + iS. \quad (51) \]

This is the isomorphism of two fields.

In particular, a symmetric matrix \( A = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \) is represented by the real number \( j(A) = R \). This is the operator of multiplication by \( R \). The trace of this operator in the one dimensional complex space \( \mathbb{C} \) (with the scalar product, \( (z, w) = z\bar{w} \)) equals \( R \). By (50) we obtain that

\[ \text{Tr } A = 2\text{Tr } j(A), \quad (52) \]

where at the left-hand side we have the real trace and at the right-hand side – the complex trace. Now we can write the basic asymptotic equality for averages in the complex form. In the funny way the Taylor factor \( \frac{1}{2} \) disappears through the transition from the real to complex structure, see [16].
Lemma 7.2. Let \( f \in \mathcal{V}(\Omega|SO(2)) \) and let \( \mu \in S^2_0(\Omega|SO(2)) \). Then

\[
<f>_{\mu}(\alpha) \equiv \int_{\mathbb{R}^2} f(q,p) \, d\mu(q,p) = \alpha j(f''(0)) + O(\alpha^2). 
\]

(53)

Proof. We make the scaling of the state variable:

\[
\psi = \frac{\sigma(\mu)}{\sqrt{2}} \Psi
\]

(54)

Then the image of \( \mu \) is again a Gaussian measure, say \( \mu_{\text{scal}} \), having the dispersion \( \sigma^2(\mu_{\text{scal}}) = 2 \). Set \( D = \text{cov} \ \mu_{\text{scal}} \). In the two dimensional case \( D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \text{Tr} \ D = 2 \). We now have:

\[
<f>_{\mu}(\alpha) = \frac{\sigma^2(\mu)}{4} \text{Tr} \ D f''(0) + O(\alpha^2).
\]

(55)

Thus

\[
<f>_{\mu}(\alpha) = \frac{\sigma^2(\mu)}{2} j(D) j(f''(0)) + O(\alpha^2).
\]

(56)

Finally, we note that in the two dimensional case: \( j(D) = 1 \). Thus we obtain:

\[
<f>_{\mu}(\alpha) = \frac{\sigma^2(\mu)}{2} j(f''(0)) + O(\alpha^2),
\]

(57)

and hence (53).

We recall that in the one dimensional quantum mechanics there is just one “density matrix”, namely, \( \rho = 1 \in \mathbb{R} \).

It is convenient to consider the renormalization of averages by the main term in the intensities of fluctuations of the coordinate and momenta: \( \langle f \rangle_{\mu} = \frac{<f>_{\mu}}{\alpha} \). Then we get:

\[
\langle f \rangle_{\mu}(\alpha) = j(f''(0)) + O(\alpha).
\]

(58)

8 Prequantum phase space – multidimensional case

States of systems are now represented by points \( \psi = (q,p) \in \Omega = Q \times P \), where \( Q = P = \mathbb{R}^m \). Here the \( q = (q_1, \ldots, q_n) \) is the position and the \( p = \ldots \)
(p_1, ..., p_n) is momentum, so Ω denotes phase space. Let us consider the canonical representation of the group SO(2) in the phase space Ω = Q × P:

\[ u = u_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \]

where I is the unit matrix from \( M(r, m) \). The corresponding group of \( \mathbb{R} \)-linear operators (or \( 2m \times 2m \) matrices) we denote by the symbol \( SO_m(2) \).

The classical model \( N_{\text{class}} = (S^*_G(Ω | SO_m(2)), \mathcal{V}(Ω | SO_m(2))) \) in defined in the same way as in the two dimensional case. A Gaussian measure is invariant iff its covariance operator belongs to the commutator \( SO'_m(2) = \{ A \in M^{(r)}(2m) : [A, u] = 0, u \in SO_m(2) \} \). If a smooth function \( f \) is invariant then all its odd derivatives equal to zero and the second derivative belong to the \( SO'_m(2) \). A matrix \( A \in SO'_m(2) \) if it has the form \( (49) \), where \( R, S \in M(r, m) \). In contrast to the two dimensional case a symmetric matrix from \( SO'_m(2) \) can be nondiagonal. It has the form \( (49) \), where \( R^* = R \) and \( S^* = -S \).

There is a natural map (generalizing the map \( j : SO'(2) \rightarrow \mathbb{C} \)) of the commutator \( SO'_m(2) \) onto the set of complex matrices \( M^{(c)}(m) \):

\[ j : SO'_m(2) \rightarrow M^{(c)}(m), z = j(A) = R + iS. \] (60)

This is the isomorphism of two rings. Symmetric matrices are mapped onto symmetric matrices. Let us denote real and complex conjugations by * and \( \star \), respectively. We have \( (R + iS)^* = R^* - iS^* = R + iS \). We also remark that for a symmetric complex matrix:

\[ \text{Tr } j(A) = \text{Tr } (R + iS) = \text{Tr } R = \frac{1}{2} \text{Tr } A. \] (61)

**Lemma 8.1.** Let \( f \in \mathcal{V}(Ω | SO_m(2)) \) and let \( \mu \in S^n_G(Ω | SO_m(2)) \). Then

\[ < f >_\mu (\alpha) = \alpha \text{Tr} \rho \ j(f''(0)) + O(\alpha^2), \] (62)

where \( \rho \in \mathcal{D}^{(c)}(m) \).

**Proof.** We make the scaling and get the \( \mu_{\text{scal}} \) with \( D = \text{cov} \mu_{\text{scal}} \), and \( \text{Tr } D = 2 \). We set \( \rho = j(D) \), here \( \text{Tr } \rho = (\text{Tr } D)/2 = 1 \) and \( \rho \in \mathcal{D}^{(c)}(m) \). Finally

\[ < f >_\mu (\alpha) = \frac{\sigma^2(\mu)}{2} \text{Tr } j(D) j(f''(0)) + O(\alpha^2) \] (63)
implies (62).

We now modify the classical→ quantum projections, (36), (37), to make them consistent with the complex structure:

\[ T : S_G^\alpha(\Omega|SO_m(2)) \to D^{(c)}(m), \rho = T(\mu) = j(\text{cov } \mu_{\text{scal}}); \] (64)

\[ T : V(\Omega|SO_m(2)) \to L^{(c)}_s(m), A = T(f) = j(f''(0)) \] (65)

**Theorem 8.1.** The maps (64), (65) project the classical statistical model

\[ N_{\text{class}} = (S_G^\alpha(\Omega|SO_m(2)), V(\Omega|SO_m(2))) \] onto the quantum model

\[ N_{\text{quant}}^{(c)} = (D^{(c)}(m), L^{(c)}_s(m)) \] in such a way that classical and quantum averages are coupled by the asymptotic equality:

\[ \langle f \rangle_\mu = \langle T(f) \rangle_{T(\mu)} + O(\alpha). \] (66)

### 9 Prequantum phase space

States of systems are now represented by points \( \psi = (q, p) \in \Omega = Q \times P \), where \( Q = P = H \) and \( H \) is a real (separable Hilbert space) with the scalar product \( (\cdot, \cdot) \) and the corresponding norm \( \| \cdot \| \). Here the \( q \in H \) is the position and the \( p \in H \) is momentum, so \( \Omega \) denotes phase space. The real Hilbert space structure on \( \Omega \) is given by the scalar product

\[ (\psi_1, \psi_2) = (q_1, q_2) + (p_1, p_2). \] (67)

In physics \( H = L_2(\mathbb{R}^3) \) is the space of square integrable functions. Thus both position and momentum are functions of \( x \in \mathbb{R}^3 \). These are simply classical fields. A point of such a phase space is a classical vector field \( \psi(x) = (q(x), p(x)) \).

Let us consider the canonical representation of the group \( SO(2) \) in the phase space \( \Omega = Q \times P \), see (59), where \( I : H \to H \) is the unit operator. The corresponding group of continuous \( \mathbb{R} \)-linear operators we denote by the symbol \( SO_H(2) \).

The classical model

\[ N_{\text{class}} = (S_G^{\alpha}(\Omega|SO_H(2)), V(\Omega|SO_H(2))) \]
in the same way as in the finite-dimensional case. We just recall a few basic notions from theory of differentiable functions and Gaussian measures on infinite-dimensional spaces.

Let \( \mu \) be a \( \sigma \)-additive Gaussian measure on the \( \sigma \)-field \( F \) of Borel subsets of \( \Omega \), see [32]. This measure is determined by its covariance operator \( B : \Omega \to \Omega \) and mean value \( m \equiv m_\mu \in \Omega \). For example, \( B \) and \( m \) determines the Fourier transform of \( \rho \):

\[
\tilde{\rho}(y) = \int_\Omega e^{i(y, \psi)} d\mu(\psi) = e^{\frac{i}{2}(B y, y) + i(m, y)}, \quad y \in \Omega.
\]

In what follows we restrict our considerations to Gaussian measures with zero mean value \( m = 0 \), where \( (m, y) = \int_\Omega (y, \psi) d\mu(\psi) = 0 \) for any \( y \in \Omega \).

We recall that the covariance operator \( B \equiv \text{cov} \mu \) is defined by

\[
(B y_1, y_2) = \int_\Omega (y_1, \psi)(y_2, \psi) d\mu(\psi), \quad y_1, y_2 \in \Omega,
\]

and has the following properties: a). \( B \geq 0 \), i.e., \( (By, y) \geq 0, \ y \in \Omega \); b). \( B \) is a bounded self-adjoint operator, \( B \in L_s(\Omega) \); c). \( B \) is a trace-class operator and moreover

\[
\text{Tr} \ B = \int_\Omega ||\psi||^2 d\mu(\psi).
\]

This is dispersion \( \sigma^2(\mu) \) of the probability \( \mu \). Thus \( \sigma^2(\mu) = \text{Tr} \ B \).

We remark that the list of properties of the covariance operator of a Gaussian measure differs from the list of properties of a von Neumann density operator [4] only by one condition: \( \text{Tr} \ \rho = 1 \), for a density operator \( \rho \).

We can easily find the Gaussian integral of a quadratic form (by using expansion with respect to an orthonormal basis and using our previous results on the finite-dimensional Gaussian integrals):

\[
\int_\Omega (A\psi, \psi) d\rho(\psi) = \text{Tr} \ BA,
\]

where \( A \in L_s(\Omega) \).

The differential calculus for maps \( f : \Omega \to \mathbf{R} \) does not differ so much from the differential calculus in the finite dimensional case, \( f : \mathbf{R}^n \to \mathbf{R} \). Instead of the norm on \( \mathbf{R}^n \), one should use the norm on \( \Omega \). We consider so-called Frechet differentiability. Here a function \( f \) is differentiable if it can be represented as

\[
f(\psi_0 + \Delta \psi) = f(\psi_0) + f'(\psi_0)(\Delta \psi) + o(\Delta \psi),
\]

where \( \lim_{||\Delta \psi|| \to 0} \frac{||o(\Delta \psi)||}{||\Delta \psi||} = 0 \). Here at each point \( \psi \) the derivative \( f'(\psi) \) is a continuous linear functional on \( \Omega \); so it can be identified with the element
Then we can define the second derivative as the derivative of the map \( \psi \rightarrow f'(\psi) \) and so on. A map \( f \) is differentiable \( n \)-times iff:

\[
f(\psi_0 + \Delta \psi) = f(\psi_0) + f'(\psi_0)(\Delta \psi) + \frac{1}{2} f''(\psi_0)(\Delta \psi, \Delta \psi) + \ldots + \frac{1}{n!} f^{(n)}(\psi_0)(\Delta \psi, \ldots, \Delta \psi) + o_n(\Delta \psi),
\]

where \( f^{(n)}(\psi_0) \) is a symmetric continuous \( n \)-linear form on \( \Omega \) and

\[
\lim_{\|\Delta \psi\| \to 0} \frac{\|o_n(\Delta \psi)\|}{\|\Delta \psi\|^n} = 0.
\]

For us it is important that the second derivative \( f''(\psi_0) \) can be represented by a self-adjoint operator

\[
f''(\psi_0)(u,v) = (f''(\psi_0)u, v), u, v \in \Omega.
\]

As in the finite-dimensional case the reminder can be represented in the integral form.

A Gaussian measure is invariant iff its covariance operator belongs to the commutator

\[
SO'_{H}(2) = \{ A \in \mathcal{L}(\Omega) : [A, u] = 0, u \in SO_{H}(2) \}.
\]

If a smooth function \( f : \Omega \to \mathbb{R} \) is \( SO'_{H}(2) \)-invariant then all its odd derivatives equal to zero and the second derivative belong to the \( SO'_{H}(2) \). An operator \( A \in SO'_{H}(2) \) if it has the form \( \begin{pmatrix} R & S \\ -S^* & R^* \end{pmatrix} \), where \( R, S \in \mathcal{L}(\mathcal{H}) \). And \( A \in SO'_{H}(2) \cap \mathcal{L}_s(\Omega) \) if it has the form \( \begin{pmatrix} R & S \\ -S^* & R^* \end{pmatrix} \), where \( R^* = R \) and \( S^* = -S \).

Let us now consider the complexification of the Hilbert space \( H : H_c = H \oplus iH \). We denote the algebra of bounded \( \mathbb{C} \)-linear operators \( A : H_c \to H_c \) by the symbol \( \mathcal{L}(H_c) \). The set of self-adjoint operators (with respect to the complex scalar product) we denote by the symbol \( \mathcal{L}_s(H_c) \).

There is a natural map (generalizing the map \( j : SO'_{m}(2) \to \mathbb{C}^m \)) of the commutator \( SO'_{H}(2) \) onto the \( \mathcal{L}(H_c) \):

\[
j : SO'_{H}(2) \to \mathcal{L}(H_c), z = j(A) = R + iS.
\]

This is the isomorphism of two rings. Self-adjoint operators (with respect to the real scalar product) are mapped onto self-adjoint operators (with respect
to the complex scalar product). We also remark that for a self-adjoint trace class operator \( A \in SO_H'(2) \) the equality (61) coupling real and complex traces holds. In the same way as in the finite-dimensional case we prove:

**Lemma 9.1.** Let \( f \in \mathcal{V}(\Omega|SO_H(2)) \) and let \( \mu \in S_\alpha^G(\Omega|SO_H(2)) \). Then the asymptotic equality (62), where \( \rho \in \mathcal{D}(H_c) \), holds.

We now consider the infinite-dimensional generalization of the classical → quantum projections, (64), (65)

\[
T : S_C^\alpha(\Omega|SO_H(2)) \to \mathcal{D}(H_c), \quad \rho = T(\mu) = j(\text{cov} \, \mu_{\text{scal}});
\]

\[
T : \mathcal{V}(\Omega|SO_H(2)) \to \mathcal{L}_s(H_c), \quad A = T(f) = j(f')(0)
\]

Lemma 9.1 implies:

**Theorem 9.1.** The maps (62), (63) project the classical statistical model \( N_{\text{class}} = (S_C^\alpha(\Omega|SO_H(2)), \mathcal{V}(\Omega|SO_H(2))) \) onto the quantum model \( N_{\text{quant}}(H_c) = (\mathcal{D}(H_c), \mathcal{L}_s(H_c)) \) in such a way that classical and quantum averages are coupled by the asymptotic equality (66).

We remark that the idea that the quantum averages can be coupled to integration with respect to the \( \psi \)-function was discussed in a number of papers, see, e.g., [16]–[18] and [19] (so called GAP-measures) as well as extended literature in the last paper. The main distinguishing feature of our approach is elaboration of technique of asymptotic expansion with respect to a small parameter, namely the dispersion of prequantum fluctuations. Comparing with [19] we also mention that we (as well as Bach [16]–[18]) consider the linear space integration and not integration over the unit sphere. This is not simply a technical deviation, but it implies a totally new viewpoint to quantum pure states, see the next section.

### 10 Gaussian measures corresponding to pure quantum states

We now generalize considerations of section 6 to the complex infinite-dimensional case. Let \( \Psi \) be a pure quantum state: \( \Psi \in H_c, \|\Psi\| = 1 \). We define a Gaussian measure \( \mu_\Psi \) which is concentrated on the one dimensional complex space (so the real plane) \( \Pi_\Psi = \{ \psi \in H_c : \psi = c\Psi, c \in \mathbb{C} \} \): the average of the \( \mu_\Psi \) is equal to zero and the complexification of its covariance operator:

\[
j(\rho_{\mu_\Psi}) = \alpha \Psi \otimes \Psi.
\]
The following facts about $\mu_\Psi$ can be obtained through direct computations and Theorem 9.1:

**Proposition 10.1.** For any pure quantum state $\Psi$, we have:

a). $\mu_\Psi \in S_G^0(\Omega|SO_H(2))$;

b). $T(\mu_\Psi) = \Psi \otimes \Psi$;

c). $\langle f \rangle_\mu = \langle j(f''(0))\Psi, \Psi \rangle + O(\alpha)$, $f \in V(\Omega|SO_H(2))$.

## 11 Hamilton-Schrödinger dynamics

States of systems with the infinite number of degrees of freedom - classical fields – are represented by points $\psi = (q,p) \in \Omega$; evolution of a state is described by the Hamiltonian equations. We consider a quadratic Hamilton function: $\mathcal{H}(q,p) = \frac{1}{2}(H_\psi,\psi)$, where $H : \Omega \to \Omega$ is an arbitrary symmetric (bounded) operator; the Hamiltonian equations have the form:

$$\dot{q} = H_{21}q + H_{22}p, \quad \dot{p} = -(H_{11}q + H_{12}p),$$

or

$$\dot{\psi} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = JH\psi \quad (71)$$

Thus quadratic Hamilton functions induce linear Hamilton equations. From (71) we get

$$\psi(t) = U_t\psi, \quad U_t = e^{JHt}.$$

The map $U_t\psi$ is a linear Hamiltonian flow on the phase space $\Omega$. Let us consider a self-adjoint operator $H \in SO'_H(2)$: $H = \begin{pmatrix} R & T \\ -T & R \end{pmatrix}$. This operator defines the quadratic Hamilton function

$$\mathcal{H}(q,p) = \frac{1}{2}[(Rq,p) + 2(Tp,q) + (Rq,q)],$$

where the operator $R$ is symmetric and the operator $T$ is skew symmetric. Corresponding Hamiltonian equations have the form

$$\dot{q} = Rp - Tq, \quad \dot{p} = -(Rq + Tp). \quad (72)$$

We point out that for a $SO_H(2)$-invariant Hamilton function, the Hamiltonian flow $U_t \in SO'_H(2)$. By considering the complex structure on the infinite-dimensional phase space $\Omega$ we write the Hamiltonian equations (71) in the
form of the Schrödinger equation on $H_c$:

$$i\frac{d\psi}{dt} = H\psi.$$  \hspace{1cm} (73)

Its solution has the following complex representation: $\psi(t) = U_t\psi$, $U_t = e^{-iHt}$. We consider the Planck system of units in that $\hbar = 1$. This is the complex representation of flows corresponding to quadratic $SO_{H}(2)$-invariant Hamilton functions.

By choosing $H = L_2(\mathbb{R}^3)$ we see that the interpretation of the solution of this equation coincides with the original interpretation of Schrödinger – this is a classical field

$$\psi(t, x) = (q(t, x), p(t, x)).$$

Example 11.1. Let us consider an important class of Hamilton functions

$$\mathcal{H}(q, p) = \frac{1}{2}[(Rp, p) + (Rq, q)],$$  \hspace{1cm} (74)

where $R$ is a symmetric operator. The corresponding Hamiltonian equations have the form:

$$\dot{q} = Rp, \quad \dot{p} = -Rq.$$  \hspace{1cm} (75)

We now choose $H = L_2(\mathbb{R}^3)$, so $q(x)$ and $p(x)$ are components of the vector-field $\psi(x) = (q(x), p(x))$. We can call fields $q(x)$ and $p(x)$ mutually inducing. The field $p(x)$ induces dynamics of the field $q(x)$ and vice versa, cf. with electric and magnetic components, $q(x) = E(x)$ and $p(x) = B(x)$, of the electromagnetic field, cf. Einstein and Infeld [20], p. 148: “Every change of an electric field produces a magnetic field; every change of this magnetic field produces an electric field; every change of ..., and so on.” We can write the form (74) as

$$\mathcal{H}(q, p) = \frac{1}{2} \int_{\mathbb{R}^6} R(x, y)[q(x)q(y) + p(x)p(y)]dxdy$$  \hspace{1cm} (76)

or

$$\mathcal{H}(\psi) = \frac{1}{2} \int_{\mathbb{R}^6} R(x, y)\psi(x)\bar{\psi}(y)dxdy,$$  \hspace{1cm} (77)

where $R(x, y) = R(y, x)$ is in general a distribution on $\mathbb{R}^6$. We call such a kernel $R(x, y)$ a self-interaction potential for the background field $\psi(x) = (q(x), p(x))$. We point out that $R(x, y)$ induces a self-interaction of each component of the $\psi(x)$, but there is no cross-interaction between components $q(x)$ and $p(x)$ of the vector-field $\psi(x)$.  

29
12 Invariant Gaussian measures of the Hamilton-Schrödinger dynamics and stationary pure states

All Gaussian measures considered in this section are supposed to be \( SO_H(2) \)-invariant. As we have seen, in our approach so called pure states \( \Psi, ||\Psi|| = 1 \), are labels for Gaussian measures concentrated on one dimensional (complex) subspaces \( \Omega_{\Psi} \) of the infinite-dimensional phase-space \( \Omega \). In this section we study the case of so called stationary (pure) states in more detail. The \( \alpha \)-scaling does not play any role in present considerations. Therefore we shall not take it into account. We consider a pure state \( \Psi, ||\Psi|| = 1 \), as the label for the Gaussian measure \( \nu_{\Psi} \) having the zero mean value and the complexification of the covariance operator

\[
j(\rho_{\nu_{\Psi}}) = \Psi \otimes \Psi.
\]

**Theorem 12.1.** Let \( \nu \) be a Gaussian measure (with zero mean value) concentrated on the one-dimensional (complex) subspace corresponding to a normalized vector \( \Psi \). Then \( \nu \) is invariant with respect to the unitary dynamics \( U_t = e^{-itH} \), where \( H: \Omega \to \Omega \) is a bounded self-adjoint operator, iff \( \Psi \) is an eigenvector of \( H \).

**Proof.** A). Let \( H\Psi = \lambda \Psi \). The Gaussian measure \( U_t^*\nu \) has the complexification of the covariance operator

\[
j(\rho_t) = U_t(\Psi \otimes \Psi)U_t^* = U_t\Psi \otimes U_t\Psi = e^{-it\lambda} \Psi \otimes e^{-it\lambda} \Psi = \Psi \otimes \Psi.
\]

Since all measures under consideration are Gaussian, this implies that \( U_t^*\nu = \nu \). Thus \( \nu \) is an invariant measure.

B). Let \( U_t^*\nu = \nu \) and \( \nu = \nu_{\Psi} \) for some \( \Psi, ||\Psi|| = 1 \). We have that \( U_t\Psi \otimes U_t\Psi = \Psi \otimes \Psi \). Thus, for any \( \psi_1, \psi_2 \in \Omega \), we have

\[
\langle \psi_1, U_t\Psi \rangle \langle U_t\Psi, \psi_2 \rangle = \langle \psi_1, \Psi \rangle \langle \Psi, \psi_2 \rangle.
\]

Let us set \( \psi_2 = \Psi \). We obtain: \( \langle \psi_1, c(t)U_t\Psi \rangle = \langle \psi_1, \Psi \rangle \), where \( c(t) = \langle U_t\Psi, \Psi \rangle \). Thus \( c(t)U_t\Psi = \Psi \). We point out that \( c(0) = ||\Psi||^2 = 1 \). Thus \( c'(0)\Psi - iH\Psi = 0 \), or \( H\Psi = -ic'(0)\Psi \). Thus \( \Psi \) is an eigenvector of \( H \) with the eigenvalue \( -ic'(0) \). We remark that \( c'(0) = -i\langle H\Psi, \Psi \rangle \); so \( c'(0) = i\langle H\Psi, \Psi \rangle \). Hence, \( \lambda = -ic'(0) = \langle H, \Psi, \Psi \rangle \).
Conclusion. In PCSFT stationary states of the quantum Hamiltonian (represented by a bounded self-adjoint operator $H$) are labels for Gaussian one-dimensional measures (with the zero mean value) that are invariant with respect to the Schrödinger dynamics $U_t = e^{-itH}$.

We now describe all possible Gaussian measures which are $U_t$-invariant.

Theorem 12.2. Let $H$ be a bounded self-adjoint operator with purely discrete nondegenerate spectrum: $H\Psi_k = \lambda_k \Psi_k$, so $\{\Psi_k\}$ is an orthonormal basis consisting of eigenvectors of $H$. Then any $U_t$-invariant Gaussian measure $\nu$ (with the zero mean value) has the complexification of the covariance operator:

$$j(\rho) = \sum_{k=1}^{\infty} c_k \Psi_k \otimes \Psi_k, c_k \geq 0, \quad (79)$$

and vice versa.

Proof. A). Let $j(\rho)$ has the form (79). Then

$$j(\rho U_t^* \nu) = U_t j(\rho) U_t^* = \sum_{k=1}^{\infty} c_k e^{-i\lambda_k t} \Psi_k \otimes e^{-i\lambda_k t} \Psi_k = j(\rho). \quad (80)$$

Since measures are Gaussian, this implies that $U_t^* \nu = \nu$ for any $t$.

B). Let $U_t^* \nu = \nu$ for any $t$. We remark that the complexification of any covariance operator $\rho$ can be represented in the form:

$$j(\rho) = \sum_{k=1}^{\infty} \langle j(\rho) \Psi_k, \Psi_k \rangle \Psi_k \otimes \Psi_k + \sum_{k \neq l} \langle j(\rho) \Psi_k, \Psi_l \rangle \Psi_k \otimes \Psi_l. \quad (81)$$

We shall show that $\langle j(\rho) \Psi_k, \Psi_j \rangle = 0$ for $k \neq j$. Denote the operator corresponding to $\sum_{k \neq j}$ by $Z$. We have

$$\langle U_t Z U_t^* \psi_1, \psi_2 \rangle = \sum_{k \neq m} \langle j(\rho) \Psi_k, \Psi_m \rangle e^{it(\lambda_m - \lambda_k)} \langle \Psi_k, \psi_2 \rangle \langle \psi_1, \Psi_m \rangle = \langle Z \psi_1, \psi_2 \rangle. \quad (82)$$

Set $\psi_1 = \Psi_j$, $\psi_2 = \Psi_k$. Then

$$\langle U_t Z U_t^* \Psi_m, \Psi_k \rangle = \langle j(\rho) \Psi_k, \Psi_m \rangle e^{it(\lambda_m - \lambda_k)} = \langle j(\rho) \Psi_k, \Psi_m \rangle. \quad (83)$$

Thus $\langle j(\rho) \Psi_k, \Psi_m \rangle = 0, k \neq m$. 

31
13 Stability of hydrogen atom

As we have seen, in PCSFT so called stationary (pure) states of quantum mechanics can play the role of labels for Gaussian measures (which are $SO_H(2)$-invariant and have zero mean value) that are $U_t$-invariant. We now apply our standard $\alpha$-scaling argument and we see that a stationary state $\Psi$ is a label for the Gaussian measure $\mu_\Psi$ with $j(\rho_\mu) = \alpha \Psi \otimes \Psi$. This measure is concentrated on one-dimensional (complex) subspace $\Pi_\Psi$ of phase space $\Omega$. Therefore each realization of an element of the Gaussian ensemble of classical fields corresponding to the statistical state $\mu_\Psi$ gives us the field of the shape $\Psi(x)$, but magnitudes of these fields vary from one realization to another. But by the well known Chebyshov inequality probability that $E(\Psi) = \int_{\mathbb{R}^3} |\Psi(x)|^2 dx$ is large is negligibly small.

Thus in the stationary state we have Gaussian fluctuations of very small magnitudes of the same shape $\Psi(x)$. In PCSFT a stationary quantum state cannot be identified with a stationary classical field, but only with an ensemble of fields having the same shape $\Psi(x)$. Let us now compare descriptions of dynamics of electron in hydrogen atom given by quantum mechanics and our prequantum field theory.

In quantum mechanics stationary bound states of hydrogen atom are of the form:

$$\Psi_{n,l,m}(r, \theta, \phi) = c_{n,l} R^l L_{n+l}^{2l+1}(R) e^{-R/2} Y_l^m(\theta, \phi),$$

(84)

where $R = \frac{2r}{na_0}$, and $a_0 = \frac{\hbar^2}{me^2}$ is a characteristic length for the atom (Bohr radius). We are mainly interested in the presence of the component $e^{-R/2}$.

In PCSFT this stationary bound state is nothing else, but the label for the Gaussian measure $\rho_{\Psi_{n,l,m}}$ which is concentrated on the subspace $\Omega_{\Psi_{n,l,m}}$. Thus PCSFT says that “electron in atom” is nothing else than Gaussian fluctuations of a certain classical field, namely the field $\Psi_{n,l,m}(r, \theta, \phi)$:

$$\psi_{n,l,m}(r, \theta, \phi; \psi) = \gamma(\psi) \Psi_{n,l,m}(r, \theta, \phi),$$

(85)

where $\gamma(\psi)$ is the C-valued Gaussian random variable: $E\gamma = 0, E|\gamma|^2 = \alpha$.

The intensity of the field $\Psi_{n,l,m}(r, \theta, \phi ; \psi)$ varies, but the shape is the same. Therefore this random field does not produce any significant effect for large $R$ (since $e^{-R/2}$ eliminates such effects).

Thus in PCSFT the hydrogen atom stable, since the prequantum random fields $\psi_{n,l,m}(r, \theta, \phi; \psi)$ have a special shape (decreasing exponentially $R \to \infty$).
14 Appendixes

14.1 Classical representation for spin operators

The Pauli matrices are a set of $2 \times 2$ complex Hermitian and unitary matrices. They are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Let $H_c = C^2$ with the complex coordinates $z = (z_1, z_2)$, $z_j = q_j + ip_j$, $j = 1, 2$, and $\Omega = R^2 \times R^2$ with the real coordinates $\omega = (q_1, q_2, p_1, p_2)$. We consider spin operators:

$$\sigma(a) = \sum_{j=1}^{3} a_j \sigma_j : C^2 \rightarrow C^2, \quad a = (a_1, a_2, a_3).$$

Let us consider real matrices $\sigma^{(r)}(a)$:

$$\sigma^{(r)}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma^{(r)}_2 = \begin{pmatrix} 0 & -i \sigma_2 \\ i \sigma_2 & 0 \end{pmatrix}, \quad \sigma^{(r)}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

We remark that these are not Dirac matrices. We set

$$\sigma^{(r)}(a) = j^{-1}(\sigma(a)) = \sum_{j=1}^{3} a_j \sigma^{(r)}_j : R^4 \rightarrow R^4, \quad a = (a_1, a_2, a_3).$$

and consider classical random spin-variables: $f_a(\omega) = \frac{1}{2}(\sigma^{(r)}(a)\omega, \omega)$. Then $T(f_a) = \sigma(a)$ and for any $SO_2(2)$-invariant Gaussian measure $\mu$ on $\Omega = R^2 \times R^2$ with dispersion $\alpha + O(\alpha)$ we have: $\frac{1}{\alpha} \int_{R^4} f_a(\omega)d\mu(\omega) = \text{Tr} j(\rho)\sigma(a) + O(\alpha)$, where $\rho$ is the covariance operator of $\sqrt{\alpha}$-scaling of the Gaussian measure $\mu$. For example,

$$\frac{1}{\alpha} \int_{R^4} (q_1 q_2 + p_1 p_2)d\mu(q_1, q_2, p_1, p_2) = \text{Tr} j(\rho)\sigma_1 + O(\alpha),$$

$$\frac{1}{\alpha} \int_{R^4} (p_1 q_2 - p_2 q_1)d\mu(q_1, q_2, p_1, p_2) = \text{Tr} j(\rho)\sigma_2 + O(\alpha),$$

$$\frac{1}{\alpha} \int_{R^4} (q_1^2 - q_2^2 + p_1^2 - p_2^2)d\mu(q_1, q_2, p_1, p_2) = \text{Tr} j(\rho)\sigma_3 + O(\alpha).$$
We also have: \( \frac{1}{\alpha} \int_{\mathbb{R}^4} (q_1^2 + q_2^2 + p_1^2 + p_2^2) d\mu(q_1, q_2, p_1, p_2) = \text{Tr} j(\rho)I + O(\alpha) = \text{Tr} j(\rho) + O(\alpha) \). By introducing vectors \( \omega_1 = (q_1, p_1) \) and \( \omega_2 = (q_2, p_2) \) we rewrite these asymptotic equalities in shorter way:

\[
\frac{1}{\alpha} \int_{\mathbb{R}^4} (\omega_1, \omega_2) d\mu(\omega_1, \omega_2) = \text{Tr} j(\rho) \sigma_1 + O(\alpha),
\]

\[
\frac{1}{\alpha} \int_{\mathbb{R}^4} (J\omega_1, \omega_2) d\mu(\omega_1, \omega_2) = \text{Tr} j(\rho) \sigma_2 + O(\alpha),
\]

where \( J \) is the symplectic operator and, finally,

\[
\frac{1}{\alpha} \int_{\mathbb{R}^4} (\|\omega_1\|^2 - \|\omega_2\|^2) d\mu(\omega_1, \omega_2) = \text{Tr} j(\rho) \sigma_3 + O(\alpha).
\]

Let us now consider Gaussian measures on \( \Omega = \mathbb{R}^4 \) corresponding to pure quantum states. These are singular Gaussian measures which are concentrated on \( SO_2(2) \)-invariant planes in \( \mathbb{R}^4 \). To determine such a measure, we should find its covariation operator.

**Proposition 14.1.** Let \( \Psi = u + iv, u = (u_1, u_2) \in \mathbb{R}^2, v = (v_1, v_2) \in \mathbb{R}^2 \) be a pure quantum state and let \( \rho_\Psi = \Psi \times \Psi \). Then \( T^{-1}(\rho_\Psi) = \mu_\Psi \), where the Gaussian measure \( \mu_\Psi \) has the covariation operator \( B_\Psi = \alpha D_\Psi \) for

\[
D_\Psi = \begin{pmatrix}
\|g_1\|^2 & (g_1, g_2) & 0 & (Jg_1, g_2) \\
(g_1, g_2) & \|g_2\|^2 & (g_1, Jg_2) & 0 \\
0 & (g_1, Jg_2) & \|g_1\|^2 & (g_1, g_2) \\
(Jg_1, g_2) & 0 & (g_1, g_2) & \|g_2\|^2 
\end{pmatrix}.
\]

Here \( g_1 = (u_1, v_1) \) and \( g_2 = (u_2, v_2) \) are variables which are conjugate to \( \omega_1 = (q_1, p_1) \) and \( \omega_2 = (q_2, p_2) \).

**Proof.** The real space realization of \( \rho_\Psi \) is given by the operator:

\[
j^{-1}(\rho_\Psi) = \begin{pmatrix}
u \otimes u + v \otimes v & v \otimes u - u \otimes v \\
u \otimes v - v \otimes u & u \otimes u + v \otimes v
\end{pmatrix}.
\]

We have in the chosen system of coordinates on the phase space:

\[
u \otimes u = \begin{pmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{pmatrix}, \quad v \otimes v = \begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix}.
\]
Thus:

\[ v \otimes u = \begin{pmatrix} u_1v_1 & u_2v_1 \\ u_1v_2 & u_2v_2 \end{pmatrix}, \quad u \otimes v = \begin{pmatrix} v_1u_1 & v_2u_1 \\ v_1u_2 & v_2u_2 \end{pmatrix}. \]

Hence:

\[ u \otimes u + v \otimes v = \begin{pmatrix} u_1^2 + v_1^2 & u_1u_2 + v_1v_2 \\ u_1u_2 + v_1v_2 & u_2^2 + v_2^2 \end{pmatrix}, \]

\[ v \otimes u - u \otimes v = \begin{pmatrix} 0 & u_2v_1 - v_2u_1 \\ u_1v_2 - v_1u_2 & 0 \end{pmatrix}. \]

To illustrate better correspondence between real and complex state spaces, we now show directly that \( j(D_\Psi) = \Psi \otimes \Psi \) for \( D_\Psi \) given by this Proposition. We have

\[ j(D_\Psi) = \begin{pmatrix} \|g_1\|^2 & (g_1, g_2) \\ (g_1, g_2) & \|g_2\|^2 \end{pmatrix} + i \begin{pmatrix} 0 & (Jg_1, g_2) \\ (Jg_1, g_2) & 0 \end{pmatrix}. \]

This operator acts to a complex vector \( z = (z_1, z_2) \) in the following way:

\[ z_1' = (j(D_\Psi)z)_1 = \|g_1\|^2z_1 + [(g_1, g_2) + i(Jg_1, g_2)]z_2, \]

\[ z_2' = (j(D_\Psi)z)_2 = [(g_1, g_2) + i(g_1, Jg_2)]z_1 + \|g_2\|^2z_2. \]

On the other hand, \( \Psi \otimes \Psi(z) = \langle z, \Psi \rangle = \Psi = (z_1\overline{\Psi}_1 + z_2\overline{\Psi}_2)\overline{\Psi}. \) Here

\[ z_1' = (u_1 + iv_1)(u_1 + iv_1)z_1 + (u_2 - iv_2)(u_1 + iv_1)z_2, \]

\[ z_2' = (u_1 - iv_1)(u_2 + iv_2)z_1 + (u_2 - iv_2)(u_2 + iv_2)z_2. \]

Thus

\[ z_1' = (u_1^2 + v_1^2)z_1 + [(u_1u_2 + v_1v_2) + i(u_2v_1 - u_1v_2)]z_2, \]

\[ z_2' = (u_2^2 + v_2^2)z_2 + [(u_1u_2 + v_1v_2) + i(u_1v_2 - u_2v_1)]z_2. \]

Let us consider Gaussian measures corresponding to pure states for spin up and spin down, \( |1> = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( |0> = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). For vector \( |1> \), we have:

\[ u_1 = 1, u_2 = v_1 = v_2 = 0. \]  Thus \( D_{|1>} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \) and the Fourier transform of the measure \( \mu_{|1>} \) is given by:

\[ \tilde{\mu}_{|1>} (\xi_1, \xi_2, \eta_1, \eta_2) = e^{-\gamma \xi_1^2 + \eta_1^2 / 2}. \]
This is the standard Gaussian measure on the plane \( q_2 = 0, p_2 = 0 \) having the density: 
\[
d\mu_{|1>}(q_1, p_1) = \frac{1}{2\pi\alpha} e^{-\frac{1}{2\alpha}(q_1^2 + p_1^2)}.
\]
In the same way \( \mu_{|0>} \) is the standard Gaussian measure on the plane \( q_1 = 0, p_1 = 0 \) having the density: 
\[
d\mu_{|0>}(q_2, p_2) = \frac{1}{2\pi\alpha} e^{-\frac{1}{2\alpha}(q_2^2 + p_2^2)}.
\]
Let us now consider the Gaussian measure corresponding to superposition of spin up and spin down states: 
\[
\Psi_\theta = \frac{1}{\sqrt{2}}(|0> + e^{i\theta}|1>).
\]
Here \( u_1 = \cos \theta, v_1 = \sin \theta \). Hence 
\[
D_{\Psi_\theta} = \begin{pmatrix}
\cos^2 \theta & \cos \theta \sin \theta & 0 & 0 \\
\cos \theta \sin \theta & \sin^2 \theta & 0 & 0 \\
0 & 0 & \cos^2 \theta & \cos \theta \sin \theta \\
0 & 0 & \cos \theta \sin \theta & \sin^2 \theta
\end{pmatrix}
\]
and the Fourier transform of \( \mu_{\Psi_\theta} \) is given by 
\[
\tilde{\mu}_{\Psi_\theta}(\xi_1, \xi_2, \eta_1, \eta_2) = e^{-\frac{\alpha}{2}[(\cos \theta \xi_1 + \sin \theta \xi_2)^2 + (\cos \theta \eta_1 + \sin \theta \eta_2)^2]}.
\]
Thus pure states \( \Psi_\theta \) correspond to the standard Gaussian measures concentrated on planes obtained by rotations.

14.2 Comparing with no-go theorems of von Neumann, Cohen-Specker and Bell

There are no-go theorems for mathematical attempts to have a map from classical variables to quantum operators which preserves statistics, e.g., theorems of von Neumann, Cohen-Specker and Bell, see [4], [21]–[23]. The no-go theorems say: No such map exists. In this paper we constructed such a map. What goes?

Our construction does not contradict to known no-go theorems, since our map \( T \) does not satisfy some conditions of those theorems. An important condition in all such theorems is that the range of values of a classical variable \( f \) should coincide with the spectrum of the corresponding quantum operator \( T(f) \) — “the range of values postulate.” This postulate is violated in our framework. As we have seen, the classical spin variables are continuous and the quantum spin operators have discrete spectrum. Nevertheless, classical
averages can be approximated by quantum. Our prequantum classical statistical model is not about observations, but about ontic reality (reality as it is when nobody looks at it).

Henry Stapp pointed out [24]: “The problem, basically, is that to apply quantum theory, one must divide the fundamentally undefined physical world into two idealized parts, the observed and observing system, but the theory gives no adequate description of connection between these two parts. The probability function is a function of degrees of freedom of the microscopic observed system, whereas the probabilities it defines are probabilities of responses of macroscopic measuring devices, and these responses are described in terms of quite different degrees of freedom.” Since we do know yet from physics so much about features of classical → quantum correspondence map $T$, we have the freedom to change some conditions which were postulated in the known no-go theorems – for example, the range of values condition. Rejection of this assumption is quite natural, since, as was pointed by Stapp, a classical variable $f$ and its quantum counterpart $T(f)$ depend on completely different degrees of freedom.

### 14.3 Is prequantum classical statistical field theory non-local?

As we have seen, PCSFT does not contradict to the known no-go theorems, in particular, to Bell’s theorem. Therefore this theory might be local. However, it is not easy to formulate the problem of locality/nonlocality in the PCSTF-framework. It is not about observations. Thus we could not apply Bell’s approach [21]–[23] to locality as locality of observations. On the other hand, on the ontic level PCSTF operates not with particles, but with fields. At the first sight, such a theory is nonlocal by its definition, since fields are not localized. But in field theory there was established a different viewpoint to locality and we know that both classical and quantum field theories are local. To formulate the problem of locality for PCSTF, we should proceed in the same way. Therefore we should develop a relativistic version of PCSTF. There are some technical and even ideological problems. As we know, relativistic quantum mechanics is not a well established theory (at least this is a rather common opinion). Thus it is meaningless to develop a relativistic variant of PCSTF which would reproduce relativistic quantum mechanics. The most natural way of development is to construct a kind of PCSTF not
for quantum mechanics, but for quantum field theory and study the problem of locality in such a framework. It is an interesting and complicated problem which will be studied in coming papers of the author.

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