Riemann–Hilbert correspondence for irregular holonomic $\mathcal{D}$-modules*

Masaki Kashiwara**

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Abstract. This is a survey paper on the Riemann–Hilbert correspondence on (irregular) holonomic $\mathcal{D}$-modules, based on the 16th Takagi Lectures (2015/11/28). In this paper, we use subanalytic sheaves, an analogous notion to the one of indsheaves.

Keywords and phrases: irregular Riemann–Hilbert problem, irregular holonomic $\mathcal{D}$-modules, ind-sheaves, subanalytic sheaves, Stokes phenomenon

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M. KASHIWARA
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
(e-mail: masaki@kurims.kyoto-u.ac.jp)
Introduction

The classical Riemann–Hilbert problem asks for the existence of a linear ordinary differential equation with regular singularities and a given monodromy on a curve.

Pierre Deligne ([De70]) formulated it as a correspondence between integrable connections with regular singularities on a complex manifold $X$ with a pole on a hypersurface $Y$ and local systems on $X \setminus Y$.

Later the author constructed an equivalence of triangulated categories between $\mathcal{D}_b^h(\mathcal{D}_X)$, the derived category of $\mathcal{D}_X$-modules with regular holonomic cohomologies, and $\mathcal{D}_b^c(C_X)$, the derived category of sheaves on $X$ with $\mathbb{C}$-constructible cohomologies ([Ka80,Ka84]). The equivalence is given by the solution functor

$$\mathcal{I}_X : \mathcal{D}_b^h(\mathcal{D}_X) \xrightarrow{\sim} \mathcal{D}_b^c(C_X)^{\text{op}}.$$ 

Here $\mathcal{I}_X(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Note that $\mathcal{D}_b^h(\mathcal{D}_X)$ is self-dual by the duality functor.
However, it was a long-standing problem to generalize it to the (not necessarily regular) holonomic $\mathcal{D}$-module case. One of the difficulties was that we could not find an appropriate substitute of the target category $\mathcal{D}^b_{\text{hol}}(\mathcal{O}_X)$. Recently, the author solved it jointly with Andrea D’Agnolo ([DK13]) by using an enhanced version of indsheaves.

There are two ingredients for the solution.

One is the notion of indsheaves. This notion was introduced with Pierre Schapira in [KS01] to treat “sheaves” of functions with tempered growth, such as $\mathcal{D}^b_t$ of tempered distributions or $\mathcal{O}_t$ of tempered holomorphic functions.

The other ingredient is adding an extra variable. We consider indsheaves on $M \times \mathbb{R}$, not on the base manifold $M$. This method was originally introduced by Dmitry Tamarkin ([Ta08]) in order to treat non-homogeneous Lagrangian submanifolds of the cotangent bundle in the framework of sheaf theory. In our context, this method affords an appropriate language to capture various growth of solutions at singular points.

Among the results used in the course of the proof is the description of the structure of flat connections due to Takuro Mochizuki ([Mo09,Mo11]) and Kiran S. Kedlaya ([Ke10,Ke11]).

In this survey paper, we explain an outline of the irregular Riemann–Hilbert problem. We use here, instead of the notion of indsheaves, the analogous notion of “subanalytic sheaves”.

For a complex manifold $X$, we construct a triangulated category $\mathcal{D}^b(\mathcal{C}^\text{sub}_X \times \mathbb{R}_\infty)$, called the triangulated category of enhanced subanalytic sheaves, a fully faithful functor $e : \mathcal{D}^b(\mathcal{C}_X) \to \mathcal{D}^b(\mathcal{C}^\text{sub}_X \times \mathbb{R}_\infty)$ and its left quasi-inverse $\text{Hom}^E_{\mathcal{C}^\text{T}_X}(\mathcal{C}^\text{sub}_X, \bullet) : \mathcal{D}^b(\mathcal{C}^\text{sub}_X \times \mathbb{R}_\infty) \to \mathcal{D}^b(\mathcal{C}_X)$. Next we construct $\mathcal{O}^\text{T}_X \subset \mathcal{D}^b(\mathcal{C}^\text{sub}_X \times \mathbb{R}_\infty)$, the enhanced subanalytic sheaf of tempered holomorphic functions such that $\text{Hom}^E_{\mathcal{C}^\text{T}_X}(\mathcal{C}^\text{sub}_X, \mathcal{O}^\text{T}_X) \simeq \mathcal{O}_X$. By using $\mathcal{O}^\text{T}_X$ instead of $\mathcal{O}_X$, we define the enhanced solution functor from the bounded derived category $\mathcal{D}^b(\mathcal{D}_X)$ of $\mathcal{D}_X$-modules to the category $\mathcal{D}^b(\mathcal{C}^\text{sub}_X \times \mathbb{R}_\infty)$ of enhanced subanalytic sheaves by

\[
\text{Sol}^\text{T}_X(\mathcal{M}) := R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^\text{T}_X) \quad \text{for } \mathcal{M} \in \mathcal{D}^b(\mathcal{D}_X).
\]

Restricting it to $\mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$, the subcategory of $\mathcal{D}^b(\mathcal{D}_X)$ consisting of complexes with holonomic cohomologies, we obtain a fully faithful functor

\[\text{Sol}^\text{T}_X : \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X) \to \mathcal{D}^b(\mathcal{C}^\text{sub}_X \times \mathbb{R}_\infty)^{\text{op}}.\]

Furthermore, we have an isomorphism

\[\text{Hom}^E(\text{Sol}^\text{T}_X(\mathcal{M}), \mathcal{O}^\text{T}_X) \simeq \mathcal{M} \quad \text{for any } \mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X).\]

Thus we obtain a quasi-commutative diagram:
This paper is organized as follows. In the first section, we review the local theory of linear ordinary differential equations. In the next sections, we shall review sheaves, $\mathcal{D}$-modules and subanalytic sheaves. After introducing the subanalytic sheaves of tempered distributions and that of tempered holomorphic functions in §4, we define the enhanced version of the de Rham functor and solution functor. Then, in §6, we state our main theorems by using these functors. In the next section §7, we give a very brief outline of the proof of the main theorems by using the results of T. Mochizuki and K.S. Kedlaya.

In the last section §8, we explain how Proposition 1.1 on the Stokes phenomena in the one-dimensional case can be interpreted in terms of the enhanced solution functors.

We refer the reader to [DK13, KS14, KS15, DK15] for a more detailed theory. Remark that the description of the Riemann–Hilbert correspondence in this paper is different from that of loc. cit. in the following points.

(a) We use in loc. cit. indsheaves instead of subanalytic sheaves. Since the category of subanalytic sheaves can be embedded into that of indsheaves, these two descriptions are almost equivalent.
(b) In loc. cit., the category $E^b(I\mathbb{C}_M)$ of enhanced indsheaves is defined as a quotient category of the category $D^b(I\mathbb{C}_M \times \mathbb{R}_\infty)$ of indsheaves on $M \times \mathbb{R}_\infty$. However, $E^b(I\mathbb{C}_M)$ can be also embedded into $D^b(I\mathbb{C}_M \times \mathbb{R}_\infty)$ by the right adjoint $R^E: E^b(I\mathbb{C}_M) \to D^b(I\mathbb{C}_M \times \mathbb{R}_\infty)$ of the quotient functor. In our paper, we use the subanalytic sheaf version of $D^b(I\mathbb{C}_M \times \mathbb{R}_\infty)$ instead of $E^b(I\mathbb{C}_M)$ by using the embedding $R^E$.

1. Linear ordinary differential equations

1.1. One dimensional case

Let us recall the local theory of linear ordinary differential equations. Let $X \subset \mathbb{C}$ be an open subset with $0 \in X$ and let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module such that $\text{SingSupp}(\mathcal{M}) \subset \{0\}$ and $\mathcal{M} \simeq \mathcal{M}(\{0\}) := \mathcal{O}_X(\{0\}) \otimes_{\mathcal{O}_X} \mathcal{M}$. Here
$\mathcal{O}_X(\{0\})$ is the sheaf of meromorphic functions with possible poles at 0. It is equivalent to saying that $\mathcal{M}$ is a $\mathcal{D}_X$-module which is locally isomorphic to $\mathcal{O}_X(\{0\})^r$ for some $r \in \mathbb{Z}_{\geq 0}$ as an $\mathcal{O}_X$-module. Let us take a system of generators $\{u_1, \ldots, u_r\}$ of $\mathcal{M}$ as a free $\mathcal{O}_X(\{0\})$-module on a neighborhood of 0. Then, writing $\vec{u}$ for the column vector with these generators as components, we have

$$\left(1.1\right) \frac{d}{dz} \vec{u} = A(z) \vec{u}$$

for some $A(z) \in \text{Mat}_r(\mathcal{O}_X(\{0\}))$, i.e., for an $(r \times r)$-matrix $A(z)$ whose components are in $\mathcal{O}_X(\{0\})$. Then for any $\mathcal{D}_X$-module $\mathcal{L}$ such that $\mathcal{L} \simeq \mathcal{O}_X(\{0\})^r$, we have

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{L}) = \{ \vec{u} \in \mathcal{L}^r : \vec{u} \text{ satisfies the same differential equation as (1.1)} \},$$

where we associate to $\vec{u}$ the morphism from $\mathcal{M}$ to $\mathcal{L}$ defined by $\vec{u} \mapsto \vec{u}$.

1.2. Regular singularities

If we can choose a system of generators $\{u_1, \ldots, u_r\}$ of $\mathcal{M}$ such that $zA(z)$ has no pole at 0, then we say that 0 is a regular singularity of $\mathcal{M}$, or $\mathcal{M}$ is regular. In such a case, there are $r$ linearly independent solutions of the form

$$\tilde{u}_j = z^{\lambda_j} \sum_{s=0}^{r-1} \tilde{a}_{j,s}(z)(\log z)^s \quad (j = 1, \ldots, r),$$

where $\tilde{a}_{j,s}(z)$ is a vector of holomorphic functions defined on a neighborhood of 0. Hence, after a change of generators $\tilde{v} = D(z)\tilde{u}$ with some invertible matrix $D(z) \in \text{GL}_r(\mathcal{O}_X(\{0\}))$, the new variable $\tilde{v}$ satisfies the equation

$$z \frac{d}{dz} \tilde{v} = C \tilde{v}$$

for some constant matrix $C \in \text{Mat}_r(\mathbb{C})$. Then, by reducing $C$ to a Jordan form, we see that $\mathcal{M}$ is isomorphic to a direct sum of $\mathcal{D}_X$-modules $\mathcal{D}_X(\{0\}) / \mathcal{D}_X(\{0\})(z \frac{d}{dz} - \lambda)^m + 1$ with $\lambda \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$. Note that

$$\mathcal{D}_X(\{0\}) / \mathcal{D}_X(\{0\})(z \frac{d}{dz} - \lambda)^m + 1 \simeq \mathcal{D}_X / \mathcal{D}_X(z \frac{d}{dz} - \lambda - k)^{m+1}$$

for any $k \in \mathbb{Z}$ such that $\lambda + k \not\in \mathbb{Z}_{\geq 0}$. 

Recall that the solution sheaf of $\mathcal{M}$ is defined by
\[ \mathcal{Sol}_X(\mathcal{M}) := \mathcal{RHom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X). \]
Then the local system on $X \setminus \{0\}$
\[ L := \mathcal{Sol}_X(\mathcal{M})|_{X \setminus \{0\}} = \{ \tilde{u} \in (\mathcal{O}_X(\{0\})^r : \frac{d}{dz} \tilde{u} = A(z)\tilde{u} \} \]
has the monodromy $\exp(2\pi \sqrt{-1}C)$. Hence $L$ completely determines $\mathcal{M}$.

1.3. Irregular singularities

In the irregular case, we have the following results on the solutions of the ordinary linear differential equation (1.1):

(i) there exist linearly independent $r$ formal solutions $\tilde{u}_j$ $(j = 1, \ldots, r)$ of (1.1) with the form
\[ \tilde{u}_j = e^{\varphi_j(z)}z^{\lambda_j} \sum_{s=0}^{r-1} \tilde{a}_{j,s}(z)(\log z)^s, \]
where $\varphi_j(z) \in z^{-1/m}\mathbb{C}[z^{-1/m}]$ for some $m \in \mathbb{Z}_{>0}$, $\lambda_j \in \mathbb{C}$, and
\[ \tilde{a}_{j,s}(z) = \sum_{n \in m^{-1}\mathbb{Z}_{\geq 0}} \tilde{a}_{j,s,n}z^n \in \mathbb{C}[[z^{1/m}]]^r \text{ with } \tilde{a}_{j,s,n} \in \mathbb{C}^r, \]

(ii) for any $\theta_0 \in \mathbb{R}$ and each $j = 1, \ldots, r$, there exist an angular neighborhood
\[ D_{\theta_0} = \{ z = re^{i\theta} : |\theta - \theta_0| < \varepsilon \text{ and } 0 < r < \delta \} \]
for sufficiently small $\varepsilon, \delta > 0$ and a holomorphic (column) solution $u_j \in \mathcal{O}_X(D_{\theta_0})^r$ of (1.1) defined on $D_{\theta_0}$ such that
\[ u_j \sim \tilde{u}_j, \]
in the following sense: for any $N > 0$, there exists $C > 0$ such that
\[ |u_j(z) - \hat{u}_j^N(z)| \leq C |e^{\varphi_j(z)}z^{\lambda_j+N}| = C e^{\text{Re}(\varphi_j(z))}|z^{\lambda_j+N}|, \]
where $\hat{u}_j^N(z)$ is the finite partial sum
\[ \hat{u}_j^N(z) = e^{\varphi_j(z)}z^{\lambda_j} \sum_{s=0}^{r-1} \sum_{n \in m^{-1}\mathbb{Z}_{\geq 0}, n \leq N} \tilde{a}_{j,s,n}z^n(\log z)^s. \]
Here we choose branches of $z^{1/m}$ and $\log z$ on $D_{\theta_0}$.
Note that a holomorphic solution $u_j$ is not uniquely determined by the formal solution $\hat{u}_j$. Indeed, $u_j + \sum_{k \neq j} c_k u_k$ also satisfies the same estimate (1.4) whenever

$$\text{Re}(\varphi_k(z)) < \text{Re}(\varphi_j(z)) \text{ on } D_{\theta_0} \text{ if } c_k \neq 0.$$ 

1.4. Stokes phenomena

We choose another sufficiently small angular domain $D_{\theta_1}$ such that $D_{\theta_0} \cap D_{\theta_1} \neq \emptyset$, and, for each $j$, we take a holomorphic solution $u'_j$ defined on $D_{\theta_1}$ and with the asymptotic behavior (1.4) on $D_{\theta_1}$. Then we can write

$$u'_j = \sum a_{j,k} u_k \text{ on } D_{\theta_0} \cap D_{\theta_1}$$

with $a_{j,k} \in \mathbb{C}$. Note that

(1.5) $$\text{Re}(\varphi_k(z)) \leq \text{Re}(\varphi_j(z)) \text{ on } D_{\theta_0} \cap D_{\theta_1} \text{ if } a_{j,k} \neq 0.$$

The matrix $(a_{j,k})_{1 \leq j,k \leq r}$ is called the Stokes matrix. If we cover a neighborhood of $\{0\}$ by such angular domains, then a pair of adjacent angular domains gives a Stokes matrix, and thus we obtain a family of matrices satisfying (1.5).

Conversely, we can find a holonomic $\mathcal{D}$-module $\mathcal{M}$ whose Stokes matrices are a given family of matrices satisfying (1.5).

1.5. Stokes filtrations

Deligne [DMR07] interpreted these results as follows (see also Malgrange [DMR07] and Sabbah [Sa00, Sa13]).

Let $\varpi : \tilde{X} \to X$ be the real blow up of $X$ along $\{0\}$ defined in §7.1 below. Namely,

$$\tilde{X} := \{(r, \zeta) \in \mathbb{R}_{\geq 0} \times \mathbb{C} : |\zeta| = 1, r\zeta \in X\} \text{ and } \varpi (r, \zeta) = r\zeta.$$ 

Recall that $L = (\mathcal{O}_X \mathcal{M})|_{X \setminus \{0\}}$. Let $S := \varpi^{-1}(0)$ and $j : X \setminus \{0\} \to \tilde{X}$ and set

(1.6) $$\tilde{L} = (j_* L)|S.$$

Then $\tilde{L}$ is a local system on $S$ of rank $r$.

For the sake of simplicity, we assume that $m$ in §1.3 (i) is equal to 1.

Set $\Phi = (\mathcal{O}_X (*\{0\})/\mathcal{O}_X)_0$. For $e^{i\theta_0} \in S$ and $\varphi, \psi \in \Phi$, we write $\varphi \preceq e^{i\theta_0} \psi$ if there exists $c \in \mathbb{R}$ such that

$$\text{Re}(\varphi (re^{i\theta})) \leq \text{Re}(\psi (re^{i\theta})) + c \text{ for } 0 < r \ll 1$$

for each $r.e^{i\theta}$.
and $|\theta - \theta_0| \ll 1$ and representatives $\tilde{\varphi}, \tilde{\psi} \in \mathcal{O}_X(*\{0\})_0$ of $\varphi$ and $\psi$. Then $\leq_{e^{i\theta_0}}$ is an order on $\Phi$.

For $\varphi \in \Phi$ and $e^{i\theta} \in S$, we set

$$(F_\varphi)_{e^{i\theta}} = \left\{ u(z) \in (\widetilde{L})_{e^{i\theta}} : |u(z)| \leq C |z^{-M} e^{\varphi(z)}|_{\text{on a neighborhood of }} e^{i\theta} \text{ for some } C > 0 \text{ and } M \in \mathbb{Z}_{>0} \right\}.$$

Then $\{F_\varphi\}_{\varphi \in \Phi}$ satisfies the following conditions by the properties of the solutions explained in §1.3:

(i) $\{F_\varphi\}_{\varphi \in \Phi}$ is a filtration of $\widetilde{L}$, namely,

(a) $F_\varphi$ is a subsheaf of $L$ for any $\varphi \in \Phi$,

(b) $L = \sum_{\varphi \in \Phi} F_\varphi$,

(c) $(F_\varphi)_{e^{i\theta}} \subset (F_\psi)_{e^{i\theta}}$ if $\varphi \leq_{e^{i\theta}} \psi$,

(ii) for any $e^{i\theta_0} \in S$, there exist an open neighborhood $U$ of $e^{i\theta_0}$, a finite subset $I$ of $\Phi$ and a constant subsheaf $H_\varphi$ ($\varphi \in I$) of $L|_U$ such that

(a) $L|_U = \bigoplus_{\varphi \in I} H_\varphi$,

(b) for any $e^{i\theta} \in S$ and $\varphi \in \Phi$, we have

$$(F_\varphi)_{e^{i\theta}} = \bigoplus_{\varphi \in I, \varphi \leq_{e^{i\theta}} \psi} (H_\psi)_{e^{i\theta}}.$$

If the above conditions are satisfied we say that $\{F_\varphi\}_{\varphi \in \Phi}$ is a Stokes filtration of the local system $\widetilde{L}$. Also in case $m > 1$, we can define the notion of Stokes filtration with a suitable modification.

**Proposition 1.1.** The category of holonomic $\mathcal{D}_X$-module $\mathcal{M}$ such that

$$\text{SingSupp}(\mathcal{M}) \subset \{0\} \text{ and } \mathcal{M} \simeq \mathcal{M}(\star \{0\})$$

is equivalent to the category of pairs $(L, \{F_\varphi\})$ of a local system $L$ on $X \setminus \{0\}$ and a Stokes filtration $\{F_\varphi\}$ on $L := (j_*L)|_S$.

In order to generalize this result to holonomic $\mathcal{D}$-modules in the several dimension case, we use enhanced subanalytic sheaves. In the next sections, we shall review sheaves, $\mathcal{D}$-modules and subanalytic sheaves.

### 2. A brief review on sheaves and $\mathcal{D}$-modules

#### 2.1. Sheaves

We refer to [KS90] for all notions of sheaf theory used here. For simplicity, we take the complex number field $\mathbb{C}$ as the base field, although most of the results would remain true when $\mathbb{C}$ is replaced with a commutative ring of finite global dimension.
A topological space is good if it is Hausdorff, locally compact, countable at infinity and has finite flabby dimension.

One denotes by Mod(\(\mathbb{C}_M\)) the abelian category of sheaves of \(\mathbb{C}\)-vector spaces on a good topological space \(M\) and by \(D^b(\mathbb{C}_M)\) its bounded derived category. Note that Mod(\(\mathbb{C}_M\)) has a finite homological dimension.

For a locally closed subset \(A\) of \(M\), one denotes by \(\mathbb{C}_A\) the constant sheaf on \(A\) with stalk \(\mathbb{C}\) extended by 0 on \(X \setminus A\).

One denotes by \(\text{Supp}(F)\) the support of \(F\).

There are many formulas concerning the six operations. For example, we have the formulas below in which \(F, F_1, F_2 \in D^b(\mathbb{C}_M)\), \(G, G_1, G_2 \in D^b(\mathbb{C}_N)\):

\[
R \mathcal{H}om(F_1 \otimes F_2, F) \simeq R \mathcal{H}om(F_1, R \mathcal{H}om(F_2, F)),
R f_* R \mathcal{H}om(f^{-1}G, F) \simeq R \mathcal{H}om(G, R f_* F),
R f'_! (F \otimes f^{-1}G) \simeq (R f'_! F) \otimes G \quad \text{(projection formula)},
\]

\[
f'^{-1} R \mathcal{H}om(G_1, G_2) \simeq R \mathcal{H}om(f^{-1}G_1, f^{-1}G_2),
\]

and for a Cartesian square of good topological spaces,

\[
\begin{array}{ccc}
M' & \xrightarrow{f'} & N' \\
\downarrow g' & \square & \downarrow g \\
M & \xrightarrow{f} & N
\end{array}
\]

we have the base change formulas

\[
g'^{-1} R f'_! \simeq R f'_! g'^{-1} \quad \text{and} \quad g'^{-1} R f_* \simeq R f'_* g'^! .
\]

2.2. \(\mathcal{D}\)-modules

References for \(\mathcal{D}\)-module theory are made to [Ka03]. See also [Ka70, Ka75, Ka78, KK81, Bj93, HTT08]. Here, we shall briefly recall some basic constructions in the theory of \(\mathcal{D}\)-modules.

Let \((X, \mathcal{O}_X)\) be a complex manifold. We denote by

- \(d_X\) the complex dimension of \(X\),
- \(\Omega_X\) the invertible \(\mathcal{O}_X\)-module of differential forms of top degree,
- \(\Omega_{X/Y}\) the invertible \(\mathcal{O}_X\)-module \(\Omega_X \otimes_{\mathcal{O}_Y} f^{-1} \mathcal{O}_Y\) \(f^{-1}(\Omega_{X/Y}^{\otimes -1})\) for a morphism \(f : X \rightarrow Y\) of complex manifolds,
- \(\Theta_X\) the sheaf of holomorphic vector fields,
- \(\mathcal{D}_X\) the sheaf of algebras of finite-order differential operators.
Denote by $\text{Mod}(\mathcal{D}_X)$ the abelian category of left $\mathcal{D}_X$-modules and by $
abla \to \text{Mod}(\mathcal{D}_X)$ that of right $\mathcal{D}_X$-modules. There is an equivalence

$$r : \text{Mod}(\mathcal{D}_X) \xrightarrow{\sim} \text{Mod}(\mathcal{D}_X^\text{op}), \ M \mapsto \mathcal{M} := \Omega_X \otimes_{\mathcal{O}_X} M.$$ 

By this equivalence, it is enough to study left $\mathcal{D}_X$-modules.

The ring $\mathcal{D}_X$ is coherent and one denotes by $	ext{Mod}_{\mathcal{D}}(\mathcal{D}_X)$ the thick abelian subcategory of $\text{Mod}(\mathcal{D}_X)$ consisting of coherent modules.

To a coherent $\mathcal{D}_X$-module $\mathcal{M}$ one associates its characteristic variety $\text{char}(\mathcal{M})$, a closed $\mathbb{C}^\times$-conic co-isotropic (one also says involutive) $\mathbb{C}$-analytic subset of the cotangent bundle $T^* X$. The involutivity property is a central theorem of the theory and is due to [SKK73]. A purely algebraic proof was obtained later in [Gabb81].

If $\text{char}(\mathcal{M})$ is Lagrangian, $\mathcal{M}$ is called holonomic. It is immediately checked that the full subcategory $\text{Mod}_{\text{hol}}(\mathcal{D}_X)$ of $\text{Mod}_{\mathcal{D}}(\mathcal{D}_X)$ consisting of holonomic $\mathcal{D}$-modules is a thick abelian subcategory.

A $\mathcal{D}_X$-module $\mathcal{M}$ is quasi-good if, for any relatively compact open subset $U \subset X$, there is a filtrant family $\{ \mathcal{F}_i \}_i$ of coherent $(\mathcal{E}_X|_U)$-submodules of $\mathcal{M}|_U$ such that $\mathcal{M}|_U = \sum_i \mathcal{F}_i$. Here, a family $\{ \mathcal{F}_i \}_i$ is filtrant if, for any $i, i'$, there exists $i''$ such that $\mathcal{F}_i + \mathcal{F}_{i'} \subset \mathcal{F}_{i''}$.

A $\mathcal{D}_X$-module $\mathcal{M}$ is good if it is quasi-good and coherent. The subcategories of $
abla \to \text{Mod}(\mathcal{D}_X)$ consisting of quasi-good (resp. good) $\mathcal{D}_X$-modules are abelian and thick. Therefore, one has the triangulated categories

- $\nabla \to \text{Mod}_{\text{coh}}(\mathcal{D}_X) = \{ \mathcal{M} \in \nabla \to \text{D}(\mathcal{D}_X) : H^j(\mathcal{M}) \text{ is coherent for all } j \in \mathbb{Z} \}$,
- $\nabla \to \text{Mod}_{\text{hol}}(\mathcal{D}_X) = \{ \mathcal{M} \in \nabla \to \text{D}(\mathcal{D}_X) : H^j(\mathcal{M}) \text{ is holonomic for all } j \in \mathbb{Z} \}$,
- $\nabla \to \text{Mod}_{\text{rh}}(\mathcal{D}_X) = \{ \mathcal{M} \in \nabla \to \text{D}(\mathcal{D}_X) : H^j(\mathcal{M}) \text{ is regular holonomic for all } j \in \mathbb{Z} \}$,
- $\nabla \to \text{Mod}_{\text{q-good}}(\mathcal{D}_X) = \{ \mathcal{M} \in \nabla \to \text{D}(\mathcal{D}_X) : H^j(\mathcal{M}) \text{ is quasi-good for all } j \in \mathbb{Z} \}$,
- $\nabla \to \text{Mod}_{\text{good}}(\mathcal{D}_X) = \{ \mathcal{M} \in \nabla \to \text{D}(\mathcal{D}_X) : H^j(\mathcal{M}) \text{ is good for all } j \in \mathbb{Z} \}$.

One may also consider the unbounded derived categories $\nabla \to \text{D}(\mathcal{D}_X)$, $\nabla \to \text{D}^+(\mathcal{D}_X)$ and $\nabla \to \text{D}^-(\mathcal{D}_X)$ and their full triangulated subcategories consisting of objects with coherent, holonomic, regular holonomic, quasi-good and good cohomologies.

We have the functors

$$\nabla \to \text{R}\text{Hom}_{\mathcal{D}_X}(\bullet, \bullet) : \nabla \to \text{D}(\mathcal{D}_X) \nabla \to \times \nabla \to \text{D}^+(\mathbb{C}_X),$$

$$\nabla \to \text{L}^\otimes_{\mathcal{D}_X} \otimes : \nabla \to \text{D}(\mathcal{D}_X)^\text{op} \times \nabla \to \text{D}(\mathcal{D}_X) \nabla \to \text{D}(\mathbb{C}_X).$$

We also have the functor

$$\nabla \to \text{D}^\otimes \otimes : \nabla \to \text{D}^-(\mathcal{D}_X) \nabla \to \times \nabla \to \text{D}^-(\mathcal{D}_X)$$
constructed as follows. For $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, the tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is endowed with a structure of $\mathcal{D}_X$-module by

$$v(s \otimes t) = (vs) \otimes t + s \otimes (vt) \quad \text{for} \quad v \in \Theta_X, \ s \in \mathcal{M} \text{ and } t \in \mathcal{N}.$$  

The functor $\bullet \otimes \bullet$ is its left derived functor. One defines the duality functor for $\mathcal{D}$-modules by setting

$$\mathbb{D}_X \mathcal{M} = R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\Theta_X} \Omega_X^{\otimes -1})[d_X] \in D^b(\mathcal{D}_X)$$

for $\mathcal{M} \in D^b(\mathcal{D}_X)$.

Now, let $f : X \to Y$ be a morphism of complex manifolds. The transfer bimodule $\mathcal{D}_X \to Y$ is a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule defined as follows. As an $(\Theta_X, f^{-1}\Theta_Y)$-bimodule, $\mathcal{D}_X \to Y = \Theta_X \otimes_{f^{-1}\Theta_Y} f^{-1}\mathcal{D}_Y$. The left $\mathcal{D}_X$-module structure of $\mathcal{D}_X \to Y$ is given by

$$v(a \otimes P) = v(a) \otimes P + \sum_i aa_i \otimes w_i P,$$

where $v \in \Theta_X$ and $\sum_i a_i \otimes w_i$ is its image in $\Theta_X \otimes_{f^{-1}\Theta_Y} f^{-1}\Theta_Y$.

One also uses the opposite transfer bimodule $\mathcal{D}_Y \leftarrow X = f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\Theta_Y} \Omega_X/Y$, an $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule.

Note that for another morphism of complex manifolds $g : Y \to Z$, one has the natural isomorphisms

$$\mathcal{D}_X \to Z \cong \mathcal{D}_X \to Y \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{D}_Z,$$

$$f^{-1}\mathcal{D}_Z \leftarrow X \cong f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\Theta_Y} \mathcal{D}_X \leftarrow X.$$

One can now define the external operations on $\mathcal{D}$-modules by setting:

$$Df^* \mathcal{N} := \mathcal{D}_X \to Y \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N} \quad \text{for} \quad \mathcal{N} \in D^b(\mathcal{D}_Y),$$

$$Df_! \mathcal{M} := Rf_! (\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_X \to Y) \quad \text{for} \quad \mathcal{M} \in D^b(\mathcal{D}_X^\text{op}),$$

and one defines $Df_* \mathcal{M}$ by replacing $Rf_!$ with $Rf_*$ in the above formula. By using the opposite transfer bimodule $\mathcal{D}_Y \leftarrow X$ one defines similarly the inverse image of a right $\mathcal{D}_Y$-module or the direct images of a left $\mathcal{D}_X$-module.

One calls respectively $Df^*$, $Df_*$ and $Df_!$ the inverse image, direct image and proper direct image functors in the category of $\mathcal{D}$-modules.

Note that

$$Df^* \Theta_Y \cong \Theta_X, \quad Df^* \Omega_Y \cong \Omega_X.$$
Also note that the property of being quasi-good is stable by inverse image and tensor product, as well as by direct image by maps proper on the support of the module. The property of being good is stable by duality.

Let \( f : X \to Y \) be a morphism of complex manifolds. One associates the maps

\[
\begin{array}{c}
T^*X & \xleftarrow{f_d} & X \times_Y T^*Y & \xrightarrow{f_\pi} & T^*Y \\
\pi_X & \downarrow & \downarrow & \downarrow & \pi_Y \\
X & \xrightarrow{f} & Y.
\end{array}
\]

One says that \( f \) is non-characteristic for \( \mathcal{N} \in \mathbb{D}_{\text{coh}}(\mathcal{D}_Y) \) if the map \( f_d \) is proper (hence, finite) on \( f_\pi^{-1}(\text{char}(\mathcal{N})) \).

The classical de Rham and solution functors are defined by

\[
\begin{align*}
\mathbb{DR}_X : & \mathbb{D}(\mathcal{D}_X) \to \mathbb{D}(\mathcal{C}_X), & \mathcal{M} & \mapsto \Omega^L_X \otimes_{\mathcal{O}_X} \mathcal{M}, \\
\mathbb{Sol}_X : & \mathbb{D}(\mathcal{D}_X) \to \mathbb{D}(\mathcal{C}_X)^{\text{op}}, & \mathcal{M} & \mapsto R\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X).
\end{align*}
\]

For \( \mathcal{M} \in \mathbb{D}_{\text{coh}}(\mathcal{D}_X) \), one has

\[(2.4) \quad \mathbb{Sol}_X(\mathcal{M}) \simeq \mathbb{DR}_X(\mathbb{D}_X \mathcal{M})[-d_X].\]

Let us list up the relations of the de Rham functors with the inverse and direct image functors.

**Theorem 2.1 (Projection formulas \([Ka03, \text{Theorems 4.2.8, 4.40}]\).** Let \( f : X \to Y \) be a morphism of complex manifolds. For \( \mathcal{M} \in \mathbb{D}(\mathcal{D}_X) \) and \( \mathcal{L} \in \mathbb{D}(\mathcal{D}_Y^{\text{op}}) \), there are natural isomorphisms:

\[
\begin{align*}
\text{D}f_!(\text{D}f^*\mathcal{L} \otimes \mathcal{M}) & \simeq \mathcal{L} \otimes \text{D}f_!\mathcal{M}, \\
\text{R}f_*(\text{D}f^*\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}) & \simeq \mathcal{L} \otimes_{\mathcal{O}_Y} \text{D}f_*\mathcal{M}.
\end{align*}
\]

In particular, there is an isomorphism (commutation of the de Rham functor and direct images)

\[
\text{R}f_!(\mathbb{DR}_X(\mathcal{M})) \simeq \mathbb{DR}_Y(\text{D}f_*\mathcal{M}).
\]

**Theorem 2.2 (Commutation with duality \([Ka03, \text{Sc86}]\).** Let \( f : X \to Y \) be a morphism of complex manifolds.

(i) Let \( \mathcal{M} \in \mathbb{D}_{\text{good}}(\mathcal{D}_X) \) and assume that \( \text{Supp}(\mathcal{M}) \) is proper over \( Y \). Then \( \text{D}f_!\mathcal{M} \in \mathbb{D}_{\text{good}}(\mathcal{D}_Y) \), and \( \mathbb{D}_Y(\text{D}f_!\mathcal{M}) \simeq \text{D}f_!\mathbb{D}_X\mathcal{M} \).

(ii) If \( f \) is non-characteristic for \( \mathcal{N} \in \mathbb{D}_{\text{coh}}(\mathcal{D}_Y) \), then \( \text{D}f^*\mathcal{N} \in \mathbb{D}_{\text{coh}}(\mathcal{D}_X) \) and \( \mathbb{D}_X(\text{D}f^*\mathcal{N}) \simeq \text{D}f^*\mathbb{D}_Y\mathcal{N} \).
Corollary 2.3. Let \( f : X \rightarrow Y \) be a morphism of complex manifolds.

(i) Let \( \mathcal{M} \in \mathbb{D}^b_{\text{good}}(\mathcal{D}_X) \) and assume that \( \text{Supp}(\mathcal{M}) \) is proper over \( Y \). Then we have the isomorphism for \( N \in \mathbb{D}(\mathcal{D}_Y) \):

\[
\text{R} f_\ast \text{R} \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \text{D} f^\ast N) [d_X] \simeq \text{R} \text{Hom}_{\mathcal{D}_Y}(\text{D} f_\ast \mathcal{M}, N) [d_Y].
\]

In particular, with the same hypotheses, we have the isomorphism (commutation of the Sol functor and direct images)

\[
\text{R} f_\ast \text{Sol}_X(\mathcal{M}) [d_X] \simeq \text{Sol}_Y(\text{D} f_\ast \mathcal{M}) [d_Y].
\]

(ii) Let \( N \in \mathbb{D}^b_{\text{coh}}(\mathcal{D}_Y) \) and assume that \( f \) is non-characteristic for \( N \). Then we have the isomorphism for \( \mathcal{M} \in \mathbb{D}(\mathcal{D}_X) \):

\[
\text{R} f_\ast \text{R} \text{Hom}_{\mathcal{D}_X}(\text{D} f^\ast N, \mathcal{M})[d_X] \simeq \text{R} \text{Hom}_{\mathcal{D}_Y}(N, \text{D} f_\ast \mathcal{M})[d_Y].
\]

3. Subanalytic sheaves

3.1. Subanalytic spaces

Let \( M \) be a real analytic manifold. On \( M \) there is the family of subanalytic subsets due to Hironaka ([Hi73]) and Gabrielov ([Gabr68]) (see [BM88,VD98] for an exposition). This family is the smallest family of subsets of \( M \) which satisfies the following properties:

(a) for any real analytic manifold \( N \) and any proper morphism \( f : N \rightarrow M \), the image of \( N \) is subanalytic,

(b) the intersection of two subanalytic subsets is subanalytic,

(c) the complement of a subanalytic subset is subanalytic,

(d) the union of a locally finite family of subanalytic subsets is subanalytic.

This family is a nice family. For example, it is closed by taking the closure and interior; any relatively compact subanalytic subset has finitely many connected components, and each connected component is subanalytic; any closed subanalytic subset is the proper image of a real analytic manifold as in (a).

For real analytic manifolds \( M, N \) and a closed subanalytic subset \( S \) of \( M \), we say that a map \( f : S \rightarrow N \) is subanalytic if its graph is subanalytic in \( M \times N \). One denotes by \( \mathcal{A}_S^\mathbb{R} \) the sheaf of \( \mathbb{R} \)-valued subanalytic continuous maps on \( S \). A subanalytic space \((M, \mathcal{A}_M^\mathbb{R})\), or simply \( M \) for short, is an \( \mathbb{R} \)-ringed space locally isomorphic to \((S, \mathcal{A}_S^\mathbb{R})\) for a closed subanalytic subset \( S \) of a real analytic manifold. In this paper, we assume that a subanalytic space is good, i.e., it is Hausdorff, locally compact, countable at infinity with finite flabby dimension.
A morphism of subanalytic spaces is a morphism of \( \mathbb{R} \)-ringed spaces. Then we obtain the category of subanalytic spaces.

We can define the notion of subanalytic subsets of a subanalytic space.

A sheaf \( F \) on a subanalytic space \( M \) is \( \mathbb{R} \)-constructible if there exists a locally finite family of locally closed subanalytic subsets \( M_j \) (\( j \in J \)) such that \( M = \bigcup_{j \in J} M_j \) and the sheaf \( F|_{M_j} \) is locally constant of finite rank for each \( j \in J \). We denote by \( \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \) the full subcategory of \( \text{Mod}(\mathbb{C}_M) \) consisting of \( \mathbb{R} \)-constructible sheaves. It is a subcategory stable by taking kernels, cokernels and extensions.

One defines the category \( D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \) as the full subcategory of \( D^b(\mathbb{C}_M) \) consisting of objects \( F \) such that \( H^i(F) \) is \( \mathbb{R} \)-constructible for all \( i \in \mathbb{Z} \). It is a triangulated subcategory and equivalent to \( D^b(\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)) \).

### 3.2. Subanalytic sheaves

Subanalytic sheaves are sheaves on a certain Grothendieck topology associated with subanalytic spaces. Here we shall introduce it directly without using the language of Grothendieck topology.

Let \( M \) be a subanalytic space. Let \( \text{Op}_M \) be the category of open subsets. The morphisms are inclusions, that is, \( \text{Hom}_{\text{Op}_M}(U, V) = \text{pt} \) or \( \emptyset \) according to \( U \subset V \) or not. Let \( \text{Op}_{\text{sub}, c}^M \) be the full subcategory of \( \text{Op}_M \) consisting of relatively compact subanalytic open subsets.

Recall that a sheaf is a contravariant functor from \( \text{Op}_M \) to \( \text{Mod}(\mathbb{C}) \) satisfying a certain “patching condition”. By replacing \( \text{Op}_M \) with \( \text{Op}_{\text{sub}, c}^M \) and modifying the “patching condition”, we obtain the notion of subanalytic sheaves introduced in [KS01] (see also [Pr08] for its more detailed study).

**Definition 3.1.** A subanalytic presheaf \( F \) is a contravariant functor from \( \text{Op}_{\text{sub}, c}^M \) to \( \text{Mod}(\mathbb{C}) \). We say that a subanalytic presheaf \( F \) is a subanalytic sheaf if it satisfies:

(i) \( F(\emptyset) = 0 \),

(ii) For \( U, V \in \text{Op}_{\text{sub}, c}^M \), the sequence

\[
0 \longrightarrow F(U \cup V) \xrightarrow{r_1} F(U) \oplus F(V) \xrightarrow{r_2} F(U \cap V)
\]

is exact. Here \( r_1 \) is given by the restriction maps and \( r_2 \) is given by the difference of the restriction maps \( F(U) \rightarrow F(U \cap V) \) and \( F(V) \rightarrow F(U \cap V) \).

Denote by \( \text{Mod}(\mathbb{C}_{\text{sub}}^M) \) the category of subanalytic sheaves. Recall that \( \text{Mod}(\mathbb{C}_M) \) denotes the category of sheaves on \( M \). Since a sheaf is a contravariant functor from \( \text{Op}_M \), the inclusion functor \( \text{Op}_{\text{sub}, c}^M \rightarrow \text{Op}_M \) induces a fully faithful
functor
\[ \iota_M : \text{Mod}(\mathcal{C}_M) \rightarrow \text{Mod}(\mathcal{C}_{M}^{\text{sub}}). \]

For example,
\[ \text{Hom}_{\text{Mod}(\mathcal{C}_{M}^{\text{sub}})}(\iota_M \mathcal{C}_U, F) \simeq F(U) \quad \text{for any } U \in \mathcal{O}_{p_M}^{\text{sub},c}. \]

The functor \( \iota_M \) does not commute with inductive limits (as seen in Example 3.11). We denote by \( \text{``}\lim\rightharpoonup\text{''} \) the inductive limit in \( \text{Mod}(\mathcal{C}_{M}^{\text{sub}}) \) in order to avoid confusion.

Note that
\[ (\text{``}\lim\rightharpoonup\text{''} F_i)(U) \simeq \lim_i F_i(U) \]
for any \( U \in \mathcal{O}_{p_M}^{\text{sub},c} \) and a filtrant inductive system \( \{F_i\}_i \) of subanalytic sheaves.

The functor \( \iota_M \) admits a left adjoint, denoted by \( \alpha_M \). For \( F \in \text{Mod}(\mathcal{C}_{M}^{\text{sub}}) \), the sheaf \( \alpha_M(F) \) is the sheaf given by
\[ \mathcal{O}_{p_M} \ni U \mapsto \lim_{\substack{V \in \mathcal{O}_{p_M}^{\text{sub},c}, V \subseteq U}} F(V). \]

The functor \( \alpha_M \) has a left adjoint \( \beta_M \). For \( F \in \text{Mod}(\mathcal{C}_M) \), \( \beta_M F \) is the subanalytic sheaf associated with the subanalytic presheaf \( \mathcal{O}_{p_M}^{\text{sub},c} \ni U \mapsto F(\overline{U}) \).

Hence we have two pairs of adjoint functors \((\alpha_M, \iota_M)\) and \((\beta_M, \alpha_M)\):

\[ \text{Mod}(\mathcal{C}_M) \xleftarrow{\beta_M} \xrightarrow{\alpha_M} \text{Mod}(\mathcal{C}_{M}^{\text{sub}}). \]

Both \( \text{Mod}(\mathcal{C}_M) \) and \( \text{Mod}(\mathcal{C}_{M}^{\text{sub}}) \) are abelian categories, and \( \alpha_M \) and \( \beta_M \) are exact. The functor \( \iota_M \) is left exact but not right exact. However, we have the following result.

**Proposition 3.2.** The restriction of \( \iota_M \):
\[ \iota_M^{\mathbb{R},c} : \text{Mod}_{\mathbb{R},c}(\mathcal{C}_M) \rightarrow \text{Mod}(\mathcal{C}_{M}^{\text{sub}}) \]

is exact.

In fact, we have a more precise relation of these two categories (see [KS01]).

**Proposition 3.3.** Let \( \text{Mod}_{\mathbb{R},c}(\mathcal{C}_M) \) be the category of \( \mathbb{R} \)-constructible sheaves on \( M \) with compact supports. Then, \( \text{Mod}(\mathcal{C}_{M}^{\text{sub}}) \) is equivalent to \( \text{Ind}(\text{Mod}_{\mathbb{R},c}(\mathcal{C}_M)) \), the category of ind-objects in \( \text{Mod}_{\mathbb{R},c}(\mathcal{C}_M) \).
For ind-objects we refer to [SGA4] or [KS06]. In particular, we have
\[ \text{Hom}_{\text{Mod}(\mathbb{C}_{M}^{\text{sub}})}(t_{M}G, \text{"lim" } F_{i}) \simeq \lim_{i \in I} \text{Hom}_{\text{Mod}(\mathbb{C}_{M}^{\text{sub}})}(t_{M}F_{i}, F_{i}) \]
for any \( G \in \text{Mod}_{\mathbb{C}^{\text{R-c}}(\mathbb{C}_{M})} \) and a filtrant inductive system \( \{ F_{i} \}_{i \in I} \) of subanalytic sheaves.

By the functor \( t_{M}^{\mathbb{R-c}} \), we regard \( \mathbb{R} \)-constructible sheaves as subanalytic sheaves. We can define the restriction functor
\[ \text{Mod}(\mathbb{C}_{M}^{\text{sub}}) \longrightarrow \text{Mod}(\mathbb{C}_{V}^{\text{sub}}) \quad \text{for open subsets } U \text{ and } V \subset U. \]
For \( F \in \text{Mod}(\mathbb{C}_{U}^{\text{sub}}) \), we denote by \( F|_{V} \in \text{Mod}(\mathbb{C}_{V}^{\text{sub}}) \) the image of \( F \) by the restriction functor.

Hence, \( \text{Op}_{M} \ni U \mapsto \text{Mod}(\mathbb{C}_{U}^{\text{sub}}) \) is a prestack on the topological space \( M \).

**Proposition 3.4.** The prestack \( \text{Op}_{M} \ni U \mapsto \text{Mod}(\mathbb{C}_{U}^{\text{sub}}) \) is a stack.

We denote by \( \text{Hom} \) the hom functor as a stack, i.e., for subanalytic sheaves \( F_{1}, F_{2} \) on \( M \), we define
\[ \Gamma(U; \text{Hom}(F_{1}, F_{2})) = \text{Hom}_{\text{Mod}(\mathbb{C}_{U}^{\text{sub}})}(F_{1}|_{U}, F_{2}|_{U}) \]
for any open subset \( U \) of \( M \). It is a sheaf on \( M \).

### 3.3. Bordered spaces

A bordered space \( M = (M, \check{M}) \) is a pair of a good topological space \( \check{M} \) and an open subset \( M \) of \( \check{M} \).

**Notation 3.5.** Let \( M = (M, \check{M}) \) and \( N = (N, \check{N}) \) be bordered spaces. For a continuous map \( f : M \to N \), denote by \( \Gamma_{f} \subset M \times N \) its graph, and by \( \overline{\Gamma}_{f} \) the closure of \( \Gamma_{f} \) in \( \check{M} \times \check{N} \). Consider the projections
\[ \check{M} \xrightarrow{q_{1}} \check{M} \times \check{N} \xrightarrow{q_{2}} \check{N}. \]

Bordered spaces form a category as follows: a morphism \( f : M \to N \) is a continuous map \( f : M \to N \) such that \( q_{1}|_{\overline{\Gamma}_{f}} : \overline{\Gamma}_{f} \to \check{M} \) is proper; the composition of two morphisms is the composition of the underlying continuous maps.

**Remark 3.6.** (i) Let \( f : M \to N \) be a continuous map.

(a) If \( f \) can be extended to a continuous map \( \tilde{f} : \check{M} \to \check{N} \), then \( f \) is a morphism of bordered space from \( M \) to \( N \).
(b) If \( \tilde{N} \) is compact, then \( f \) is a morphism of bordered space from \( M \) to \( N \).

(ii) The forgetful functor from the category of bordered spaces to that of good topological spaces is given by

\[
M = (M, \tilde{M}) \mapsto \tilde{M} := M.
\]

It has a fully faithful left adjoint \( M \mapsto (M, M) \). By this functor, we regard good topological spaces as particular bordered spaces, and denote \((M, M)\) simply by \( M \).

Be aware that \( M = (M, \tilde{M}) \mapsto \tilde{M} \) is not a functor.

(iii) Note that \( M \simeq (M, \overline{M}) \), where \( \overline{M} \) is the closure of \( M \) in \( \tilde{M} \). More generally, for a morphism of bordered spaces \( f : M \to N, M \) is isomorphic to the bordered space \((\Gamma_f, \overline{\Gamma_f})\).

(iv) The category of bordered spaces has an initial object, the empty set. It has also a final object, \( pt \), the topological space consisting of one point. It also admits products:

\[
(M, \tilde{M}) \times (N, \tilde{N}) \simeq (M \times N, \tilde{M} \times \tilde{N}).
\]

Let \( M = (M, \tilde{M}) \) be a bordered space. The morphisms of bordered spaces

\[
(3.2) \quad M \longrightarrow M \xrightarrow{j_M} \tilde{M}
\]

are defined by the continuous maps \( M \xrightarrow{id} M \xleftarrow{\tilde{M}} \tilde{M} \).

**Definition 3.7.** We say that a morphism \( f : M \to N \) is semi-proper if \( q_2|_{\overline{\Gamma_f}} : \overline{\Gamma_f} \to \tilde{N} \) is proper. We say that \( f \) is proper if moreover \( \tilde{f} : \tilde{M} \to \tilde{N} \) is proper.

For example, \( j_M \) is semi-proper.

The class of semi-proper (resp. proper) morphisms is closed under composition.

**Definition 3.8.** A subset \( S \) of a bordered space \( M = (M, \tilde{M}) \) is a subset of \( M \). We say that \( S \) is open (resp. closed, locally closed) if it is so in \( M \). We say that \( S \) is relatively compact if it is contained in a compact subset of \( \tilde{M} \).

As seen by the following obvious lemma, the notion of relatively compact subsets only depends on \( M \) (and not on \( \tilde{M} \)).

**Lemma 3.9.** Let \( f : M \to N \) be a morphism of bordered spaces.

(i) If \( S \) is a relatively compact subset of \( M \), then its image \( \tilde{f}(S) \subset \tilde{N} \) is a relatively compact subset of \( N \).

(ii) Assume furthermore that \( f \) is semi-proper. If \( S \) is a relatively compact subset of \( N \), then its inverse image \( f^{-1}(S) \subset \tilde{M} \) is a relatively compact subset of \( M \).
3.4. Subanalytic sheaves on bordered subanalytic spaces

A bordered subanalytic space is a bordered space \( M = (\tilde{M}, \tilde{\tilde{M}}) \) such that \( \tilde{M} \) is a subanalytic space and \( M \) is a subanalytic open subset of \( \tilde{M} \). Then we can consider the category of bordered subanalytic spaces. A morphism \( M = (M, \tilde{M}) \to N = (N, \tilde{N}) \) of bordered subanalytic spaces is a morphism \( f \) of bordered spaces such that the graph \( \Gamma_f \) is a subanalytic subset of \( \tilde{M} \times \tilde{N} \).

Let \( M = (M, \tilde{M}) \) be a bordered subanalytic space. We denote by \( \text{Op}_{\tilde{M}}^{\text{sub}, c} \) the full subcategory of \( \text{Op}_M \) consisting of open subsets of \( M \) which are subanalytic and relatively compact in \( \tilde{M} \). A subanalytic sheaf on \( M \) is defined as follows.

**Definition 3.10.** A subanalytic presheaf \( F \) on a bordered subanalytic space \( M \) is a contravariant functor from \( \text{Op}_{\tilde{M}}^{\text{sub}, c} \) to \( \text{Mod}(\mathbb{C}) \). We say that a subanalytic presheaf \( F \) is a subanalytic sheaf if it satisfies:

(i) \( F(\emptyset) = 0 \),

(ii) For \( U, V \in \text{Op}_{\tilde{M}}^{\text{sub}, c} \), the sequence

\[
0 \to F(U \cup V) \xrightarrow{r_1} F(U) \oplus F(V) \xrightarrow{r_2} F(U \cap V)
\]

is exact.

We denote by \( \text{Mod}(\mathbb{C}_M^{\text{sub}}) \) the category of subanalytic sheaves on \( M \). We have a canonical fully faithful functor

\[
\iota_M: \text{Mod}(\mathbb{C}_M^{\text{sub}}) \to \text{Mod}(\mathbb{C}_M^{\text{sub}}).
\]

Here \( \text{Mod}(\mathbb{C}_M^{\text{sub}}) \) denotes the category of sheaves on the topological space \( \tilde{M} \). The functor \( \iota_M \) is left exact but not exact.

We say that a sheaf on \( \tilde{M} \) is an \( \mathbb{R} \)-constructible sheaf on \( M \) if it can be extended to an \( \mathbb{R} \)-constructible sheaf on \( \tilde{M} \). Let us denote by \( \text{Mod}_{\mathbb{R}^{-c}}(\mathbb{C}_M) \) the category of \( \mathbb{R} \)-constructible sheaves on \( M \). Then the restriction of \( \iota_M \)

\[
\iota_M^{\mathbb{R}^{-c}}: \text{Mod}_{\mathbb{R}^{-c}}(\mathbb{C}_M) \to \text{Mod}(\mathbb{C}_M^{\text{sub}})
\]

is exact. By this functor, we regard \( \mathbb{R} \)-constructible sheaves on \( M \) as subanalytic sheaves on \( M \).

3.5. Functorial properties of subanalytic sheaves

3.5.1. Tensor product and inner hom

Let \( M = (M, \tilde{M}) \) be a bordered subanalytic space. The category \( \text{Mod}(\mathbb{C}_M^{\text{sub}}) \) has tensor product and inner hom:

\[
\bullet \otimes \bullet : \text{Mod}(\mathbb{C}_M^{\text{sub}}) \times \text{Mod}(\mathbb{C}_M^{\text{sub}}) \to \text{Mod}(\mathbb{C}_M^{\text{sub}})
\]

and

\[
\text{Hom}(\bullet, \bullet) : \text{Mod}(\mathbb{C}_M^{\text{sub}})^{\text{op}} \times \text{Mod}(\mathbb{C}_M^{\text{sub}}) \to \text{Mod}(\mathbb{C}_M^{\text{sub}}).
\]
For $F_1, F_2 \in \text{Mod}(\mathbb{C}^\text{sub}_M)$, their tensor product $F_1 \otimes F_2$ is the subanalytic sheaf associated with the subanalytic presheaf $\text{Op}_{\mathbb{C}^\text{sub}_M} \ni U \mapsto F_1(U) \otimes F_2(U)$. The inner hom $\text{Ihom}(F_1, F_2)$ is given by

$$\text{Op}_{\mathbb{C}^\text{sub}_M} \ni U \mapsto \text{Hom}_{\text{Mod}(\mathbb{C}^\text{sub}_M)}((F_1|_{(U, \tilde{M})}, F_2|_{(U, \tilde{M})}).$$

We have

$$\text{Hom}_{\text{Mod}(\mathbb{C}^\text{sub}_M)}((F_1 \otimes F_2, F_3) \cong \text{Hom}_{\text{Mod}(\mathbb{C}^\text{sub}_M)}(F_1, \text{Ihom}(F_2, F_3))$$

for $F_1, F_2, F_3 \in \text{Mod}(\mathbb{C}^\text{sub}_M)$.

The bifunctor $\bullet \otimes \bullet$ is exact, and $\text{Ihom}(\bullet, \bullet)$ is left exact.

### 3.5.2. Direct images and inverse images

Let $M = (\tilde{M}, M)$ and $N = (\tilde{N}, N)$ be bordered subanalytic spaces and let $f : M \to N$ be a morphism of bordered subanalytic spaces.

For $F \in \text{Mod}(\mathbb{C}^\text{sub}_M)$, its direct image $f_* F \in \text{Mod}(\mathbb{C}^\text{sub}_N)$ is defined by

$$(f_* F)(V) = \text{Hom}_{\text{Mod}(\mathbb{C}^\text{sub}_N)}(C_{f^{-1}V}, F) \text{ for any } V \in \text{Op}_{\mathbb{C}^\text{sub}_N,c}.$$ (3.4)

The functor $f_* : \text{Mod}(\mathbb{C}^\text{sub}_M) \to \text{Mod}(\mathbb{C}^\text{sub}_N)$ has a left adjoint

$$f^{-1} : \text{Mod}(\mathbb{C}^\text{sub}_N) \to \text{Mod}(\mathbb{C}^\text{sub}_M).$$

The functor $f^{-1}$ is called the inverse image functor. For a subanalytic sheaf $G$ on $N$, its inverse image $f^{-1}G$ is the subanalytic sheaf associated with the subanalytic presheaf

$$\text{Op}_{\mathbb{C}^\text{sub}_M} \ni U \mapsto \lim_{V \in \text{Op}_{\mathbb{C}^\text{sub}_N,c}, U \subseteq f^{-1}V} G(V).$$

The functor $f^{-1}$ is exact.

For $F \in \text{Mod}(\mathbb{C}^\text{sub}_M)$, the direct image with proper support $f_{!!} F$ is defined by

$$\Gamma(V ; f_{!!} F) = \lim_{U \subseteq \text{Op}_{\mathbb{C}^\text{sub}_N,c}} \text{Hom}(C_{f^{-1}V} ; F \otimes \mathbb{C}U) \text{ for } V \in \text{Op}_{\mathbb{C}^\text{sub}_N,c}.$$ (3.4)

Here $U$ ranges over the open subsets in $\text{Op}_{\mathbb{C}^\text{sub}_N,c}$ such that $f^{-1}V \cap \overline{U} \to V$ is proper, where $\overline{U}$ denotes the closure of $U$ in $M$. In general, the diagram

$$\begin{array}{ccc}
\text{Mod}(\mathbb{C}_M) & \xrightarrow{f_!} & \text{Mod}(\mathbb{C}^\text{sub}_M) \\
\downarrow f_1 & & \downarrow f_{!!} \\
\text{Mod}(\mathbb{C}_N) & \xrightarrow{f_{!!}} & \text{Mod}(\mathbb{C}^\text{sub}_N)
\end{array}$$

is not commutative, that is why we use the different notation $f_{!!}$. Note that the above diagram commutes if $f$ is semi-proper.
Example 3.11. Let $M = \mathbb{R}_{>0}$, $N = \mathbb{R}$ and let $f : M \to N$ be the canonical inclusion. Then we have

$$f_{!!} \mathbb{C}_M \simeq \lim_{c \to 0+} \mathbb{C}_{\{t>c\}} \quad \text{and} \quad f_! \mathbb{C}_M \simeq \mathbb{C}_{\{t>0\}}.$$ 

They are not isomorphic. Indeed, we have for $U = \{t : 0 < t < 1\} \in \text{Op}_{N}^{\text{sub}, \mathbb{C}}$

$$\Gamma(U; \lim_{c \to 0+} \mathbb{C}_{\{t>c\}}) \simeq \lim_{c \to 0+} \Gamma(U; \mathbb{C}_{\{t>c\}}) \simeq 0 \quad \text{and} \quad \Gamma(U; \mathbb{C}_{\{t>0\}}) \simeq \mathbb{C}.$$ 

Note that the inductive limit of $\mathbb{C}_{\{t>c\}}$ in $\text{Mod}(\mathbb{C}_N)$ is isomorphic to $\mathbb{C}_{\{t>0\}}$.

Recall the morphism $j_M : M \to \tilde{M}$ of bordered subanalytic spaces. We have

$$j_{M!!} j_M^{-1} F \simeq \mathbb{C}_M \otimes F, \quad j_{M*} j_M^{-1} F \simeq \text{Hom}(\mathbb{C}_M, F) \quad \text{for } F \in \text{Mod}(\mathbb{C}_M^{\text{sub}}).$$

Moreover, the functor $j_M^{-1} : \text{Mod}(\mathbb{C}_M^{\text{sub}}) \to \text{Mod}(\mathbb{C}_M^{\text{sub}})$ induces an equivalence of abelian categories:

$$\text{Mod}(\mathbb{C}_M^{\text{sub}}) / \text{Mod}(\mathbb{C}_{M \setminus \tilde{M}}^{\text{sub}}) \simeq \text{Mod}(\mathbb{C}_M^{\text{sub}}).$$

Here $\text{Mod}(\mathbb{C}_{M \setminus \tilde{M}}^{\text{sub}})$ is regarded as a full subcategory of $\text{Mod}(\mathbb{C}_M^{\text{sub}})$ by the fully faithful exact functor $i_* \simeq i_{!!} : \text{Mod}(\mathbb{C}_{M \setminus \tilde{M}}^{\text{sub}}) \to \text{Mod}(\mathbb{C}_M^{\text{sub}})$, where $i : \tilde{M} \setminus M \to \tilde{M}$ is the closed inclusion.

3.6. Derived functors

The fully faithful exact functor

$$i_{M}^{\mathbb{R}-c} : \text{Mod}_{\mathbb{R}-c}(\mathbb{C}_M) \to \text{Mod}(\mathbb{C}_M^{\text{sub}})$$

induces a fully faithful functor $D_{\mathbb{R}-c}(\mathbb{C}_M) \to D(\mathbb{C}_M^{\text{sub}})$ by which we regard $D_{\mathbb{R}-c}(\mathbb{C}_M)$ as a full subcategory of $D(\mathbb{C}_M^{\text{sub}})$.

The functors introduced in the previous subsection have derived functors:

- $\otimes$ : $D(\mathbb{C}_M^{\text{sub}}) \times D(\mathbb{C}_M^{\text{sub}}) \to D(\mathbb{C}_M^{\text{sub}})$,
- $\mathbb{R} \text{Hom}(\cdot, \cdot) : D^{-}(\mathbb{C}_M^{\text{sub}})^{\text{op}} \times D^{+}(\mathbb{C}_M^{\text{sub}}) \to D^{+}(\mathbb{C}_M^{\text{sub}})$,
- $f^{-1} : D(\mathbb{C}_N^{\text{sub}}) \to D(\mathbb{C}_M^{\text{sub}})$,
- $Rf_{*} : D(\mathbb{C}_M^{\text{sub}}) \to D(\mathbb{C}_N^{\text{sub}})$,
- $Rf_{!!} : D(\mathbb{C}_M^{\text{sub}}) \to D(\mathbb{C}_N^{\text{sub}})$. 
The functor $Rf_{!!}$ has a right adjoint:

$$f^! : D^b(\mathbb{C}^\text{sub}_N) \to D^b(\mathbb{C}^\text{sub}_M).$$

If $\tilde{\phi} : \tilde{M} \to \tilde{N}$ is topologically submersive, i.e., $\tilde{\phi}$ is isomorphic to $\tilde{N} \times \mathbb{R}^n \to \tilde{N}$ locally on $\tilde{M}$, then

$$f^! F \simeq \omega_{\tilde{M}/\tilde{N}} \otimes f^{-1} F.$$

Here $\omega_{\tilde{M}/\tilde{N}} : = f^! \mathbb{C}^\text{\textordmasculine}_N \in D^b_{\text{\textordmasculine}-\text{c}}(\mathbb{C}_M) \subset D^b(\mathbb{C}^\text{\textordmasculine}\text{\textunderscore sub}_M)$ is the relative dualizing complex.

These six operations satisfy the properties similar to (2.1) and (2.2) for the Grothendieck’s six operations for sheaves.

### 3.7. Ring actions

Let $M$ be a subanalytic space, and let $\mathcal{A}$ be a sheaf of $\mathbb{C}$-algebras. Let $F$ be a subanalytic sheaf on $M$. We say that $F$ has an action of $\mathcal{A}$, or $F$ is a subanalytic $\mathcal{A}$-module if a homomorphism of sheaves of $\mathbb{C}$-algebras

$$(3.6) \quad \mathcal{A} \to \text{Hom}(F, F)$$

is given. Since $\text{Hom}(F, F) \simeq \alpha_M \text{Hom}(F, F)$, the data (3.6) is equivalent to

$$\beta_M \mathcal{A} \to \text{Hom}(F, F),$$

or $\beta_M \mathcal{A} \otimes F \to F$ with the associativity property. We denote by $\text{Mod}(\mathcal{A}^\text{\textunderscore sub})$ the category of subanalytic $\mathcal{A}$-modules, and by $D^b(\mathcal{A}^\text{\textunderscore sub})$ its bounded derived category.

We have the tensor functor and the hom functor:

$$\otimes_{\mathcal{A}} : D^b(\mathcal{A}^\text{op}) \times D^b(\mathcal{A}^\text{\textunderscore sub}) \to D^-(\mathbb{C}^\text{\textunderscore sub}_M),$$

$$R\text{Hom}_{\mathcal{A}}(\cdot, \cdot) : D^b(\mathcal{A}^\text{op}) \times D^b(\mathcal{A}^\text{\textunderscore sub}) \to D^+(\mathbb{C}^\text{\textunderscore sub}_M).$$

### 4. Subanalytic sheaves of tempered functions

#### 4.1. Tempered distributions

Hereafter, $M$ denotes a real analytic manifold.

An important property of subanalytic subsets is given by the lemma below. (See Lojasiewicz [Lo59] and also [Ma66] for a detailed study of its consequences.)
Lemma 4.1. Let $U$ and $V$ be two relatively compact open subanalytic subsets of $\mathbb{R}^n$. There exist a positive integer $N$ and $C > 0$ such that

$$\text{dist}(x, \mathbb{R}^n \setminus (U \cup V))^N \leq C(\text{dist}(x, \mathbb{R}^n \setminus U) + \text{dist}(x, \mathbb{R}^n \setminus V)).$$

We denote by $\mathcal{D}_M$ the sheaf of Schwartz’s distributions on $M$. Denote by $\mathcal{D}_M(U)$ the image of the restriction map $\Gamma(M; \mathcal{D}_M) \to \Gamma(U; \mathcal{D}_M)$, and call it the space of tempered distributions on $U$.

Using Lemma 4.1, one proves:

Lemma 4.2. The subanalytic presheaf $U \mapsto \mathcal{D}_M(U)$ is a subanalytic sheaf on $M$.

One denotes by $\mathcal{D}_M^\dagger$ this subanalytic sheaf. By the definition, there is a monomorphism

$$\mathcal{D}_M^\dagger \hookrightarrow \iota_M \mathcal{D}_M,$$

and an isomorphism

$$\alpha_M \mathcal{D}_M^\dagger \simeq \mathcal{D}_M.$$

Let us denote by $\mathcal{D}_M$ the sheaf of rings of differential operators with real analytic coefficients. Then, $\mathcal{D}_M^\dagger$ is a subanalytic $\mathcal{D}_M$-module in the sense of §3.7.

4.2. Tempered holomorphic functions

Let $X$ be a complex manifold, and let us denote by $X_{\mathbb{R}}$ the underlying real analytic manifold. We have defined the subanalytic sheaf of tempered distributions $\mathcal{D}_{X,\mathbb{R}}^\dagger$. It is a subanalytic $\mathcal{D}_{X,\mathbb{R}}$-module. Let us consider the Dolbeault complex with coefficients in $\mathcal{D}_{X,\mathbb{R}}^\dagger$:

$$\mathcal{D}_{X,\mathbb{R}}^\dagger \to \Omega^1_{X,\mathbb{C}} \otimes \mathcal{D}_{X,\mathbb{R}}^\dagger \to \cdots \to \Omega_{X,\mathbb{C}}^d \otimes \mathcal{D}_{X,\mathbb{R}}^\dagger.$$

Here $X^c$ is the complex conjugate manifold of $X$. It is a complex in the category $\text{Mod}(\mathcal{D}_{X}^{\text{sub}})$ of subanalytic $\mathcal{D}_X$-modules. Hence we can consider this complex as an object of $\mathcal{D}^b(\mathcal{D}_{X}^{\text{sub}})$, the bounded derived category of $\text{Mod}(\mathcal{D}_{X}^{\text{sub}})$. We denote it by $\mathcal{D}_{X}^{\dagger}$ and call it the subanalytic sheaf of tempered holomorphic functions. Note that its cohomology groups are not concentrated at degree 0 in general.
4.3. Tempered de Rham and solution functors

Setting $\Omega^1_X := \Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^1 \in \mathcal{D}^b((\mathcal{O}_X^\text{op})^\text{sub})$, we define the tempered de Rham and solution functors by

- $\mathcal{D}R^t_X : \mathcal{D}^b(X) \to \mathcal{D}^-(\mathbb{C}_X^\text{sub})$,
- $\mathcal{S}ol^t_X : \mathcal{D}^b(X) \to \mathcal{D}^+(\mathbb{C}_X^\text{sub})^\text{op}$,

where $\mathcal{M} \mapsto \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$ and $\mathcal{M} \mapsto \mathcal{R}\text{Hom}_{\mathcal{D}^b(X)}(\mathcal{M}, \mathcal{O}_X^1)$.

One has $\mathcal{D}R^t_X \cong \alpha_X \circ \mathcal{D}R^t_X$ and $\mathcal{S}ol^t_X \cong \alpha_X \circ \mathcal{S}ol^t_X$.

For $\mathcal{M} \in \mathcal{D}^b_{\text{coh}}(\mathcal{D}^b_Y)$, one has

$$\mathcal{S}ol^t_X(\mathcal{M}) \cong \mathcal{D}R^t_X(D_X \mathcal{M})[-d_X].$$

The next result is a reformulation of a theorem of [Ka84] (see also [KS01, Th. 7.4.1])

**Theorem 4.3.** Let $f : X \to Y$ be a morphism of complex manifolds. There is an isomorphism in $\mathcal{D}^b((f^{-1} \mathcal{D}^Y_{\text{op}})_{\text{sub}})$:

$$\Omega^1_X \otimes_{\mathcal{D}^X} \mathcal{D}_{X \to Y} [d_X] \xrightarrow{\sim} f^* \Omega^1_Y [d_Y].$$

Note that this isomorphism (4.2) is equivalent to the isomorphism

$$\mathcal{D}_{Y \leftarrow X} \xrightarrow{\sim} f^* \mathcal{O}_Y^1 [d_Y] \quad \text{in} \quad \mathcal{D}^b((f^{-1} \mathcal{D}_Y)_{\text{sub}}).$$

**Corollary 4.4.** Let $f : X \to Y$ be a morphism of complex manifolds and let $\mathcal{N} \in \mathcal{D}^b(\mathcal{D}_Y)$. Then (4.2) induces the isomorphism

$$\mathcal{D}R^t_X(Df^* \mathcal{N}) [d_X] \cong f^* \mathcal{D}R^t_Y(\mathcal{N}) [d_Y] \quad \text{in} \quad \mathcal{D}^b(\mathbb{C}_X^\text{sub}).$$

**Corollary 4.5.** For any complex manifold $X$, we have

$$\mathcal{D}R^t_X(\mathcal{O}_X) \cong \mathbb{C}_X [d_X].$$

The next results are a kind of Grauert direct image theorem for tempered holomorphic functions, and its $\mathcal{D}$-module version.

**Theorem 4.6 (Tempered Grauert theorem [KS96, Th. 7.3]).** Let $f : X \to Y$ be a morphism of complex manifolds, let $\mathcal{F} \in \mathcal{D}^b_{\text{coh}}(\mathcal{O}_X)$ and assume that $f$ is proper on $\text{Supp}(\mathcal{F})$. Then there is a natural isomorphism

$$\mathcal{R}f!! (\mathcal{O}_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}) \cong \mathcal{O}_Y^1 \otimes_{\mathcal{O}_Y} \mathcal{R}f_! \mathcal{F}.$$
Proposition 4.7 ([KS01, Th. 7.4.6]). Let \( f : X \to Y \) be a morphism of complex manifolds. Let \( \mathcal{M} \in D^{b}_{q\text{-good}}(\mathcal{D}_X) \) and assume that \( f \) is proper on \( \text{Supp}(\mathcal{M}) \). Then there is an isomorphism in \( D^{b}(\mathbb{C}^{\text{sub}}_Y) \)

\[
\mathcal{D} f^!(\mathcal{D}_f \mathcal{M}) \cong Rf_* \mathcal{D} f^!(\mathcal{M}).
\]

For a closed hypersurface \( S \subset X \), denote by \( \mathcal{O}_X(*S) \) the sheaf of meromorphic functions with poles at \( S \). It is a holonomic \( \mathcal{D}_X \)-module and flat as an \( \mathcal{O}_X \)-module. For \( \mathcal{M} \in D^{b}(\mathcal{D}_X) \) or \( \mathcal{M} \in D^{b}(\mathcal{D}^{\text{sub}}_X) \), set

\[
\mathcal{M}(*S) = \mathcal{M} \otimes \mathcal{O}_X(*S).
\]

Proposition 4.8. Let \( S \) be a closed complex hypersurface in \( X \). There are isomorphisms

\[
\mathcal{O}^{\mathbb{C}}_X(*S) \cong R \text{Hom}(\mathbb{C}^{\text{sub}}_X, \mathcal{O}^{\mathbb{C}}_X) \quad \text{in} \quad D^{b}(\mathcal{D}^{\text{sub}}_X),
\]

\[
\mathcal{O}_X(*S) \cong R \text{Hom}(\mathbb{C}^{\text{sub}}_X, \mathcal{O}^t_X) \quad \text{in} \quad D^{b}(\mathcal{D}_X).
\]

Corollary 4.9. Let \( S \) be a closed complex hypersurface in \( X \). There are isomorphisms in \( D^{b}(\mathcal{D}^{\text{sub}}_X) \)

\[
\mathcal{D} f^!(\mathcal{O}_X(*S)) \cong \mathcal{D} f^!(\mathcal{O}_X(*S)) \cong R \text{Hom}(\mathbb{C}^{\text{sub}}_X, \mathcal{O}_X)[dX].
\]

5. Enhanced subanalytic sheaves

5.1. Enhanced tensor product and inner hom

Consider the 2-point compactification of the real line \( \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\} \). Denote by \( \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\} \) the real projective line. Then \( \overline{\mathbb{R}} \) has a structure of subanalytic space such that the natural map \( \overline{\mathbb{R}} \to \mathbb{P}^1(\mathbb{R}) \) is a morphism of subanalytic spaces.

Notation 5.1. We will consider the bordered subanalytic space

\[
\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}}).
\]

Note that \( \mathbb{R}_\infty \) is isomorphic to \((\mathbb{R}, \mathbb{P}^1(\mathbb{R}))\) as a bordered subanalytic space.

Consider the morphisms of bordered subanalytic spaces

\[
a : \mathbb{R}_\infty \longrightarrow \mathbb{R}_\infty,
\]

\[
\mu, q_1, q_2 : \mathbb{R}_\infty \times \mathbb{R}_\infty \longrightarrow \mathbb{R}_\infty,
\]

where \( a(t) = -t, \mu(t_1, t_2) = t_1 + t_2 \) and \( q_1, q_2 \) are the natural projections.
For a subanalytic space $M$, we will use the same notations for the associated morphisms

$$a : M \times \mathbb{R}_\infty \longrightarrow M \times \mathbb{R}_\infty,$$

$$\mu, q_1, q_2 : M \times \mathbb{R}_\infty \times \mathbb{R}_\infty \longrightarrow M \times \mathbb{R}_\infty.$$

We also use the natural morphisms

$$\pi_M: M \times \mathbb{R}_\infty \longrightarrow M \times \mathbb{R}$$

(5.2)

**Definition 5.2.** The functors

$$\boxplus : \mathcal{D}^b(\mathcal{C}_{\text{sub}}^{M \times \mathbb{R}_\infty}) \times \mathcal{D}^b(\mathcal{C}_{\text{sub}}^{M \times \mathbb{R}_\infty}) \longrightarrow \mathcal{D}^b(\mathcal{C}_{\text{sub}}^{M \times \mathbb{R}_\infty}),$$

$$\text{Ihom}^+: \mathcal{D}^-(\mathcal{C}_{\text{sub}}^{M \times \mathbb{R}_\infty})^\text{op} \times \mathcal{D}^+(\mathcal{C}_{\text{sub}}^{M \times \mathbb{R}_\infty}) \longrightarrow \mathcal{D}^+(\mathcal{C}_{\text{sub}}^{M \times \mathbb{R}_\infty})$$

are defined by

$$K_1 \boxplus K_2 = R\mu!! (q_1^{-1}K_1 \otimes q_2^{-1}K_2),$$

$$\text{Ihom}^+(K_1, K_2) = Rq_1^* R\text{Ihom}(q_2^{-1}K_1, \mu^! K_2).$$

One sets

(5.3) \[ \mathcal{C}_{\{t \geq 0\}} = \mathcal{C}_{\{(x,t) \in M \times \mathbb{R}; t \geq 0\}}. \]

We use similar notation for $\mathcal{C}_{\{t=0\}}, \mathcal{C}_{\{t>0\}}, \mathcal{C}_{\{t\leq 0\}}, \mathcal{C}_{\{t=a\}}$, etc. These are $\mathbb{R}$-constructible sheaves on $M \times \mathbb{R}_\infty$. We also regard them as objects of $\mathcal{D}^b(\mathcal{C}_{\text{sub}}^{M \times \mathbb{R}_\infty})$.

**Lemma 5.3.** For $K \in \mathcal{D}^b(\mathcal{C}_{\text{sub}}^{M \times \mathbb{R}_\infty})$, there are isomorphisms

$$\mathcal{C}_{\{t=0\}} \boxplus K \simeq K \simeq \text{Ihom}^+(\mathcal{C}_{\{t=0\}}, K).$$

More generally, for $a \in \mathbb{R}$, we have

$$\mathcal{C}_{\{t=a\}} \boxplus K \simeq R\mu_a^* K \simeq \text{Ihom}^+(\mathcal{C}_{\{t=-a\}}, K),$$

where $\mu_a : M \times \mathbb{R}_\infty \rightarrow M \times \mathbb{R}_\infty$ is the morphism induced by the translation $t \mapsto t + a$.

**Corollary 5.4.** The category $\mathcal{D}^b(\mathcal{C}_{\text{sub}}^{M \times \mathbb{R}_\infty})$ has a structure of commutative tensor category with $\boxplus$ as tensor product and $\mathcal{C}_{\{t=0\}}$ as unit object.
As seen in the following lemma, the functor $\hom^+$ is the inner hom of the tensor category $\mathcal{D}^b(\mathbb{C}^{\text{sub}}_{M \times \mathbb{R}_\infty})$.

**Lemma 5.5.** For $K_1, K_2, K_3 \in \mathcal{D}^b(\mathbb{C}^{\text{sub}}_{M \times \mathbb{R}_\infty})$ one has

$$\hom_{\mathcal{D}^b(\mathbb{C}^{\text{sub}}_{M \times \mathbb{R}_\infty})}(K_1 \otimes K_2, K_3) \simeq \hom_{\mathcal{D}^b(\mathbb{C}^{\text{sub}}_{M \times \mathbb{R}_\infty})}(K_1, \hom^+(K_2, K_3)).$$

$$\hom^+(K_1 \otimes K_2, K_3) \simeq \hom^+(K_1, \hom^+(K_2, K_3)),$$

$$R\pi_M R\hom(K_1 \otimes K_2, K_3) \simeq R\pi_M R\hom(K_1, \hom^+(K_2, K_3)).$$

We define the outer hom functors on $\mathcal{D}^b(\mathbb{C}^{\text{sub}}_{M \times \mathbb{R}_\infty})$ as follows.

**Definition 5.6.** One defines the hom functor

$$\hom^E: \mathcal{D}^b(\mathbb{C}^{\text{sub}}_{M \times \mathbb{R}_\infty})^{\text{op}} \times \mathcal{D}^b(\mathbb{C}^{\text{sub}}_{M \times \mathbb{R}_\infty}) \longrightarrow \mathcal{D}^+(\mathbb{C}^{\text{sub}}_M),$$

$$\hom^E(K_1, K_2) = R\pi_M R\hom(K_1, K_2),$$

and one sets

$$\hom^E = \alpha_M \circ \hom^E: \mathcal{D}^b(\mathbb{C}^{\text{sub}}_{M \times \mathbb{R}_\infty})^{\text{op}} \times \mathcal{D}^b(\mathbb{C}^{\text{sub}}_{M \times \mathbb{R}_\infty}) \longrightarrow \mathcal{D}^+(\mathbb{C}_M).$$

Note that

$$\hom_{\mathcal{D}^b(\mathbb{C}^{\text{sub}}_{M \times \mathbb{R}_\infty})}(K_1, K_2) \simeq H^0(M; \hom^E(K_1, K_2)).$$

**5.2. Enhanced sheaf of tempered distributions**

Let $M$ be a real analytic manifold. Let $j_M: M \times \mathbb{R}_\infty \rightarrow M \times \mathbb{P}^1(\mathbb{R})$ be the canonical morphism.

Let $t$ be the affine coordinate of $\mathbb{P}^1(\mathbb{R})$. Then, $\partial_t := \partial/\partial t$ is a vector field on $M \times \mathbb{P}^1(\mathbb{R})$, and hence it acts on $\mathcal{A}_{M \times \mathbb{P}^1(\mathbb{R})}^1$.

**Lemma 5.7.** The morphism of subanalytic sheaves

$$\partial_t - 1: \mathcal{A}_{M \times \mathbb{P}^1(\mathbb{R})}^1 \rightarrow \mathcal{A}_{M \times \mathbb{P}^1(\mathbb{R})}^t$$

is an epimorphism.
We define the subanalytic sheaf on $M \times \mathbb{R}_\infty$ by

$$\mathcal{D}_M^T = \text{Ker}(\partial_t - 1 : j_M^{-1} \mathcal{D}_M^{1} \rightarrow j_M^{-1} \mathcal{D}_M^{1} \times P^1(\mathbb{R})).$$

Since any solution of $(\partial_t - 1)u(t, x) = 0$ can be written as $u(t, x) = e^t \varphi(x)$, we have a monomorphism in $\text{Mod}(\mathbb{C}^{\text{sub}}_M \times \mathbb{R}_\infty)$

$$\mathcal{D}_M^T \longrightarrow \pi_-^{-1} t_M \mathcal{D}_M \quad \text{by} \quad u(t, x) \mapsto \varphi(x).$$

Note that $\mathcal{D}_M^T$ is a subanalytic $\pi^{-1}_M \mathcal{D}_M$-module. We call it the enhanced subanalytic sheaf of tempered distributions.

**Proposition 5.8.**

$$\mathcal{D}_M^T \simeq \text{Hom}^+(\mathbb{C}_{\{t \geq a\}}, \mathcal{D}_M) \quad \text{for any} \ a \in \mathbb{R}$$

$$\simeq \text{Hom}^+(\mathbb{C}_M^1[1], \mathcal{D}_M^T).$$

Here we set

$$\mathbb{C}_M^T := \text{“lim”} \ \mathbb{C}_{\{t < c\}} \quad \text{as} \quad c \rightarrow +\infty.$$

The enhanced subanalytic sheaf $\mathbb{C}_M^T$ satisfies

$$\mathbb{C}_M^T[1] \otimes \mathbb{C}_M^T[1] \simeq \mathbb{C}_M^T[1].$$

We can recover $\mathcal{D}_M^T$ and $\mathcal{D}_M$ from $\mathcal{D}_M^T$ as follows:

$$\text{Hom}^E(\mathbb{C}_M^T, \mathcal{D}_M) \simeq \mathcal{D}_M^T,$$

$$\text{Hom}^E(\mathbb{C}_M^T, \mathcal{D}_M^T) \simeq \mathcal{D}_M.$$

**Remark 5.9.** The definition of $\mathcal{D}_M^T$ is slightly different from the one in [DK13, KS15, DK15]. The notation $\mathcal{D}_M^T$ in loc. cit. is equal to $\mathcal{D}_M^T[1]$ in our notation.

### 5.3. Enhanced sheaf of tempered holomorphic functions

Let $X$ be a complex manifold, and let us denote by $X_R$ the underlying real analytic manifold. We have defined the enhanced subanalytic sheaf of tempered distributions $\mathcal{D}_{X_R}^T$. It is a subanalytic $\pi^{-1}_X \mathcal{D}_{X_R}$-module. Let us consider the Dolbeault complex with coefficients in $\mathcal{D}_{X_R}^T$:

$$\mathcal{D}_{X_R}^T \longrightarrow \Omega_{X_R}^1 \otimes \theta_{X_R} \longrightarrow \Omega_{X_R}^2 \otimes \theta_{X_R} \longrightarrow \cdots \longrightarrow \Omega_{X_R}^{d-1} \otimes \theta_{X_R}.$$
Here $X^c$ is the complex conjugate manifold of $X$. It is a complex in the category $\text{Mod}((\pi^{-1}_X D_X)^{\text{sub}})$ of subanalytic $\pi^{-1}_X D_X$-modules, where $D^T_{X_R}$ is situated at degree 0 and $\Omega^d_{X^c} \otimes_{\mathcal{O}_{X^c}} D^T_{X_R}$ at degree $d_X$. Hence we can consider this complex as an object of $D^b((\pi^{-1}_X D_X)^{\text{sub}})$, the bounded derived category of $\text{Mod}((\pi^{-1}_X D_X)^{\text{sub}})$. We denote it by $\mathcal{O}^T_X$ and call it the enhanced sheaf of tempered holomorphic functions. Note that its cohomology groups are not concentrated at degree 0.

**Remark 5.10.** If $X = \text{pt}$, then

$$\mathcal{O}^T_X \simeq D^T_{X_R} \simeq C^T_X := \lim_{c \to +\infty} C_{\{t < c\}}$$

as objects of $D^b(C^{\text{sub}}_{R_{\infty}})$. Indeed, for $-\infty \leq a < b \leq +\infty$, $e^t$ is a tempered distribution on the open interval $(a, b)$ if and only if $(a, b) \subset \{t < c\}$ for some $c \in \mathbb{R}$.

By (5.4), we have

$$\text{Hom}^E(C^T_X, \mathcal{O}^T_X) \simeq \mathcal{O}^T_X \quad \text{and}$$

(5.5)

$$\text{Hom}^E(C^T_X, \mathcal{O}^T_X) \simeq \mathcal{O}_X.$$

### 5.4. Enhanced de Rham and solution functors

We set

$$\Omega^T_X := \pi^{-1}_X \Omega_X \otimes_{\pi^{-1}_X \mathcal{O}_X} \mathcal{O}^T_X \in D^b((\pi^{-1}_X D_X)^{\text{op}}_{\text{sub}}).$$

We define the enhanced de Rham and solution functors

$$(\mathcal{D}^T_X : D^b(D_X) \longrightarrow D^b(C^{\text{sub}}_{X \times \mathbb{R}_{\infty}}),$$

$$\mathcal{T}^T_X : D^b(D_X) \longrightarrow D^b(C^{\text{sub}}_{X \times \mathbb{R}_{\infty}})^{\text{op}}$$

by

$$\mathcal{D}^T_X(M) := \Omega^T_X \otimes_{\pi^{-1}_X \mathcal{O}_X} \pi^{-1}_X M,$$

$$\mathcal{T}^T_X(M) := R \text{Hom}_{\pi_X^{-1} D_X}(\pi^{-1}_X M, \mathcal{O}^T_X).$$

Note that

$$\mathcal{T}^T_X(M) \simeq \mathcal{D}^T_X(D_X M)[-d_X] \quad \text{for} \ M \in D^b_{\text{coh}}(D_X).$$
By (5.5), we have for any \( M \in D^b(D_X) \)
\[
\mathcal{DR}_X^! M \cong \mathcal{Hom}(C_X^T, D^b(D_X)_{\mathbb{C}}) \mathcal{C}T_{X/\mathbb{R}}^X M / \mathcal{DR}_{X/\mathbb{R}}^T X M / \mathcal{Hom}(C_X^T, \mathcal{DR}^T_X M).
\]
(5.6)

For a particular case of holonomic \( D \)-modules, we can calculate explicitly the enhanced de Rham. Let \( Y \subset X \) be a complex analytic hypersurface of a complex manifold \( X \), and set \( U = X \setminus Y \). For \( \varphi \in \mathcal{O}_X(*Y) \), one sets \( D_X e^\varphi = D_X / \{ P \in D_X : Pe^\varphi = 0 \} \), \( e^\varphi_{U|X} = D_X e^\varphi (*Y) \).

Hence \( D_X e^\varphi \) is a \( D_X \)-submodule of \( e^\varphi_{U|X} \), and \( D_X e^\varphi \) as well as \( e^\varphi_{U|X} \) is a holonomic \( D_X \)-module. Note that \( e^\varphi_{U|X} \) is isomorphic to \( \mathcal{O}_X(*Y) \) as an \( \mathcal{O}_X \)-module, and the connection \( \mathcal{O}_X(*Y) \rightarrow \Omega^1_X \otimes \mathcal{O}_X \mathcal{O}_X(*Y) \) is given by \( u \mapsto du + ud\varphi \). We call \( e^\varphi_{U|X} \) the exponential module with exponent \( \varphi \).

Let \( Y \subset X \) be a closed complex analytic hypersurface, and set \( U = X \setminus Y \). For \( \varphi \in \mathcal{O}_X(*Y) \), there are isomorphisms
\[
\mathcal{DR}_X^! (e^\varphi_{U|X}) \cong R \mathcal{Hom}(\pi^{-1}_X C_U, \text{“lim” } C_{\{t < \text{Re } \varphi + c\}}[d_X]) \rightarrow c \to +\infty
\]
The next results are easy consequences of Theorem 4.3, Corollary 4.4, Corollary 4.7.

**Theorem 5.12.** Let \( f : X \rightarrow Y \) be a morphism of complex manifolds. Let \( f_{X/\mathbb{R}} : X \times \mathbb{R}_\infty \rightarrow Y \times \mathbb{R}_\infty \) be the morphism induced by \( f \).

(i) There is an isomorphism in \( D^b((\pi^{-1}_X f^{-1} D_Y)^{\text{sub}}) \)
\[
(f_{X/\mathbb{R}})^! \mathcal{O}_Y[dY] \cong \pi^{-1}_X \mathcal{O}_Y \leftarrow X \otimes_{\pi^{-1}_X \mathcal{O}_X} \mathcal{O}_X^T[dX].
\]
(ii) For any \( \mathcal{N} \in D^b(D_Y) \) there is an isomorphism in \( D^b(C_X^{\text{sub}}) \)
\[
\mathcal{DR}_X^! (D f^* \mathcal{N})[dX] \cong (f_{X/\mathbb{R}})^! D f^* \mathcal{N}[dY].
\]
(iii) Let \( \mathcal{M} \in D^b_{\text{good}}(D_X) \), and assume that \( \text{Supp}(\mathcal{M}) \) is proper over \( Y \). Then, there are isomorphisms in \( D^b(C_Y^{\text{sub}}) \)
\[
\mathcal{DR}_Y^! (D f_* \mathcal{M}) \cong R f_{X/\mathbb{R}}^* \mathcal{DR}_X^! \mathcal{M}.
\]
6. Main theorems

The Riemann–Hilbert correspondence for holonomic $\mathcal{D}$-modules can be stated as follows.

**Theorem 6.1.** There exists a canonical isomorphism functorial with respect to $\mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}X)$:

\[
\mathcal{M}^\mathcal{D} \otimes \mathcal{O}_X^\mathcal{T} \sim \text{Hom}^E(\mathcal{O}_X^\mathcal{T}(\mathcal{M}), \mathcal{O}_X^\mathcal{T}) \text{ in } \mathcal{D}^b(\mathcal{D}^\text{sub}_X).
\]

(6.1)

Applying the functor $\alpha_X$ to (6.1), we obtain

**Theorem 6.2 (Enhanced Riemann–Hilbert correspondence).** There exists a canonical isomorphism functorial with respect to $\mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}X)$:

\[
\mathcal{M} \sim \text{Hom}^E(\mathcal{O}_X^\mathcal{T}(\mathcal{M}), \mathcal{O}_X^\mathcal{T}) \text{ in } \mathcal{D}^b(\mathcal{D}X).
\]

(6.2)

Thus we obtain the quasi-commutative diagram

Here the fully faithful functor $e : \mathcal{D}^b(C_X) \to \mathcal{D}^b(C^\text{sub}_X \times_{\mathbb{R}_\infty})$ is defined by

\[
e(F) := C^\mathcal{T}_X \otimes \pi_X^{-1}F.
\]

Theorem 6.2 shows that $\mathcal{S}^\mathcal{T}_X$ as well as $\mathcal{D}^\mathcal{T}_X$ is faithful. In fact, we can also show the following full faithfulness of the enhanced de Rham functor.

**Theorem 6.3.** For $\mathcal{M}, \mathcal{N} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}X)$, one has an isomorphism

\[
\text{R Hom}_{\mathcal{D}X}(\mathcal{M}, \mathcal{N}) \sim \text{Hom}^E(\mathcal{D}^\mathcal{T}_X \mathcal{M}, \mathcal{D}^\mathcal{T}_X \mathcal{N}).
\]

In particular, the functor

\[
\mathcal{D}^\mathcal{T}_X : \mathcal{D}^b_{\text{hol}}(\mathcal{D}X) \to \mathcal{D}^b(\mathcal{C}^\text{sub}_X \times_{\mathbb{R}_\infty})
\]

is fully faithful.

**Remark 6.4.** Theorems 6.2 and 6.3 due to [DK13, Th. 9.6.1, Th. 9.7.1] are a natural formulation of the Riemann–Hilbert correspondence for irregular $\mathcal{D}$-modules. Theorem 6.1 is due to [KS14, Th. 4.5], which is a generalization of a theorem of J.-E. Björk ([Bj93]).
7. A brief outline of the proof of the main theorems

We reduce the main theorems to the exponential $\mathcal{D}$-module case, using the results of Mochizuki and Kedlaya.

7.1. Real blow up

A classical tool in the study of differential equations is the real blow up.

Recall that $\mathbb{C}^\times$ denotes $\mathbb{C} \setminus \{0\}$ and $\mathbb{R}_{>0}$ the multiplicative group of positive real numbers. Consider the action of $\mathbb{R}_{>0}$ on $\mathbb{C}^\times \times \mathbb{R}$:

$$\mathbb{R}_{>0} \times (\mathbb{C}^\times \times \mathbb{R}) \rightarrow \mathbb{C}^\times \times \mathbb{R}, \quad (a, (z, t)) \mapsto (az, a^{-1}t)$$

and set

$$\tilde{\mathbb{C}}_{\text{tot}} = (\mathbb{C}^\times \times \mathbb{R})/\mathbb{R}_{>0}, \quad \tilde{C} = (\mathbb{C}^\times \times \mathbb{R}_{>0})/\mathbb{R}_{>0}, \quad \tilde{C}^{>0} = (\mathbb{C}^\times \times \mathbb{R}_{>0})/\mathbb{R}_{>0}.$$

One denotes by $\varpi^{\text{tot}}$ the map:

$$\varpi^{\text{tot}}: \tilde{\mathbb{C}}_{\text{tot}} \rightarrow \mathbb{C}, \quad (z, t) \mapsto tz.$$ (7.1)

Then we have

$$\tilde{\mathbb{C}}_{\text{tot}} \supset \tilde{C} \supset \tilde{C}^{>0} \cong \mathbb{C}^\times.$$

Let $X = \mathbb{C}^n \sim \mathbb{C}^r \times \mathbb{C}^{n-r}$ and let $D$ be the divisor $\{z_1 \cdots z_r = 0\}$, where $(z_1, \ldots, z_n)$ is a coordinate system on $X$. Set

$$\tilde{X}_{\text{tot}} = (\tilde{\mathbb{C}}_{\text{tot}})^r \times \mathbb{C}^{n-r}, \quad \tilde{X}^{>0} = (\tilde{C}^{>0})^r \times \mathbb{C}^{n-r}, \quad \tilde{X} = (\tilde{C})^r \times \mathbb{C}^{n-r}.$$

Then $\tilde{X}$ is the closure of $\tilde{X}^{>0}$ in $\tilde{X}_{\text{tot}}$. The map $\varpi^{\text{tot}}$ in (7.1) defines the map

$$\varpi: \tilde{X} \rightarrow X.$$

The map $\varpi$ is proper and induces an isomorphism

$$\varpi|_{\tilde{X}^{>0}}: \tilde{X}^{>0} = \varpi^{-1}(X \setminus D) \cong X \setminus D.$$

We call $\tilde{X}$ the real blow up of $X$ along $D$.

Remark 7.1. The real manifold $\tilde{X}$ (with boundary) as well as the map $\varpi: \tilde{X} \rightarrow X$ may be intrinsically defined for a complex manifold $X$ and a normal crossing divisor $D$, but $\tilde{X}_{\text{tot}}$ is only intrinsically defined as a germ of a manifold in a neighborhood of $\tilde{X}$. 


Definition 7.2. Let $\tilde{A}^X$ be the subsheaf of $j_*(\mathcal{O}_X \setminus D)$ consisting of holomorphic functions tempered at any point of $\tilde{X} \setminus \tilde{X}^{>0} = \varpi^{-1}(D)$. Here, $j : X \setminus D \simeq \tilde{X}^{>0} \hookrightarrow \tilde{X}$ is the open embedding. We set

$$\mathcal{D}_X^A := A^X \otimes_{\varpi^{-1} \mathcal{O}_X} \varpi^{-1} \mathcal{D}_X.$$ 

Then $\tilde{A}^X$ and $\mathcal{D}_X^A$ are sheaves of rings on $\tilde{X}$. We have a commutative diagram

$$\varpi^{-1} \mathcal{O}_X \longrightarrow \varpi^{-1} \mathcal{D}_X$$
$$\downarrow \hspace{1cm} \downarrow$$
$$A^X \longrightarrow \mathcal{D}_X^A.$$ 

We have

$$R\varpi_* A^X \simeq \mathcal{O}_X(*D).$$

For $\mathcal{M} \in D^b(\mathcal{D}_X)$ we set:

$$(7.2) \quad \mathcal{M}^A := \mathcal{D}_X^A \otimes_{\varpi^{-1} \mathcal{D}_X} \varpi^{-1} \mathcal{M} \in D^b(\mathcal{D}_X^A).$$

Then we obtain

$$(7.3) \quad R\varpi_* \mathcal{M}^A \simeq \mathcal{M}(*D).$$

7.2. Normal form

The result in §1.3 for ordinary linear differential equations is generalized to higher dimensions by T. Mochizuki ([Mo09, Mo11]) and K.S. Kedlaya ([Ke10, Ke11]). In this subsection, we collect some of their results that we shall need.

Let $X$ be a complex manifold and $D \subset X$ a normal crossing divisor. We shall use the notations introduced in the previous subsection: in particular the real blow up $\varpi : \tilde{X} \to X$ and the notation $\mathcal{M}^A$ of (7.2).

Definition 7.3. We say that a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ has a normal form along $D$ if

(i) $\mathcal{M} \simeq \mathcal{M}(*D)$,

(ii) $\text{SingSupp}(\mathcal{M}) \subset D$,

(iii) for any $x \in \varpi^{-1}(D) \subset \tilde{X}$, there exist an open neighborhood $U \subset X$ of $\varpi(x)$ and finitely many $\varphi_i \in \Gamma(U; \mathcal{O}_X(*D))$ such that

$$(\mathcal{M}^A)|_V \simeq \left( \bigoplus_i (\mathcal{O}_U^\varphi_{\varphi_i})^A \right)|_V$$

for some open neighborhood $V$ of $x$ with $V \subset \varpi^{-1}(U)$. 

A ramification of $X$ along $D$ on a neighborhood $U$ of $x \in D$ is a finite map

$$p : X' \rightarrow U$$

of the form

$$p(z'_1, \ldots, z'_n) = (z'^{m_1}_1, \ldots, z'^{m_r}_{r}, z'_{r+1}, \ldots, z'_{n})$$

for some $(m_1, \ldots, m_r) \in (\mathbb{Z}_{>0})^r$. Here $(z'_1, \ldots, z'_n)$ is a local coordinate system of $X'$, and $(z_1, \ldots, z_n)$ is a local coordinate system of $X$ such that $D = \{z_1 \cdots z_r = 0\}$.

**Definition 7.4.** We say that a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ has a quasi-normal form along $D$ if it satisfies (i) and (ii) in Definition 7.3, and if for any $x \in D$ there exists a ramification $p : X' \rightarrow U$ on a neighborhood $U$ of $x$ such that $D_{x}(\mathcal{M}|_{U})$ has a normal form along $p^{-1}(D \cap U)$.

**Remark 7.5.** In the above definition, $D_{x}(\mathcal{M}|_{U})$ as well as $D_{x}(\mathcal{M}|_{U})$ is concentrated at degree zero. Moreover, $\mathcal{M}|_{U}$ is a direct summand of $D_{x}(\mathcal{M}|_{U})$.

### 7.3. Results of Mochizuki and Kedlaya

The next result is an essential tool in the study of holonomic $\mathcal{D}$-modules and is easily deduced from the fundamental work of Mochizuki [Mo09, Mo11] (see also Sabbah [Sa00] for preliminary results and see Kedlaya [Ke10, Ke11] for the analytic case).

**Theorem 7.6.** Let $X$ be a complex manifold, $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module and $x \in X$. Then there exist an open neighborhood $U$ of $x$, a closed analytic hypersurface $Y \subset U$, a complex manifold $X'$ and a projective morphism $f : X' \rightarrow U$ such that

(i) $\text{SingSupp}(\mathcal{M}) \cap U \subset Y$,  
(ii) $D := f^{-1}(Y)$ is a normal crossing divisor of $X'$,  
(iii) $f$ induces an isomorphism $X' \setminus D \rightarrow U \setminus Y$,  
(iv) $(D f^* \mathcal{M})(\ast D)$ has a quasi-normal form along $D$.

Remark that, under assumption (iii), $(D f^* \mathcal{M})(\ast D)$ is concentrated at degree zero. Using Theorem 7.6, one easily deduces the next lemma.

**Lemma 7.7.** Let $P_X(\mathcal{M})$ be a statement concerning a complex manifold $X$ and a holonomic object $\mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$. Consider the following conditions.

(a) Let $X = \bigcup_{i \in I} U_i$ be an open covering. Then $P_X(\mathcal{M})$ is true if and only if $P_{U_i}(\mathcal{M}|_{U_i})$ is true for any $i \in I$.  


(b) If $P_X(M)$ is true, then $P_X(M[n])$ is true for any $n \in \mathbb{Z}$.

(c) Let $M' \to M \to M'' \xrightarrow{+1} \xymatrix{ & 1}$ be a distinguished triangle in $D^b_{\text{hol}}(D_X)$. If $P_X(M')$ and $P_X(M'')$ are true, then $P_X(M)$ is true.

(d) Let $M'$ and $M''$ be holonomic $D_X$-modules. If $P_X(M' \oplus M'')$ is true, then $P_X(M')$ is true.

(e) Let $M_0$ and $M_0'$ be holonomic $D_X$-modules. If $P_X(M_0 \otimes M_0')$ is true, then $P_X(M_0) \otimes P_X(M_0')$ is true.

(f) Let $f: X \to Y$ be a projective morphism and $M$ a good holonomic $D_X$-module. If $P_X(M)$ is true, then $P_Y(D_{i*}M)$ is true.

If conditions (a)–(f) are satisfied, then $P_X(M)$ is true for any complex manifold $X$ and any $M \in D^b_{\text{hol}}(D_X)$.

Sketch of the proof of the main theorems in §6. By applying Lemma 7.7, we reduce the assertions to the case of holonomic $D$-modules with a normal form, then to the case of the exponential $D$-modules. □

8. Stokes filtrations and enhanced de Rham functor

In this last section, we explain the relation between the enhanced solution sheaf and the Stokes filtration discussed in §1.5. Let us keep the notations in §1.3. In particular, recall that $0 \in X \subset \mathbb{C}$, $\mathcal{M}$ is a holonomic $D_X$-module, $\varpi: X \to X$ is the projection and $j: X \setminus \{0\} \hookrightarrow \overline{X}$ is the open embedding. We set $X^* := X \setminus \{0\}$. Let $\varpi_R: \overline{X} \times \mathbb{R}_\infty \to X \times \mathbb{R}_\infty$ be the morphism induced by $\varpi$ and let $i: S := \varpi^{-1}(0) \hookrightarrow \overline{X}$ be the closed embedding.

Set

$$\mathcal{M}' = D_X((D_X \mathcal{M})(\ast\{0\}))$$

Then we have a morphism $\mathcal{M}' \to \mathcal{M}$ such that it induces an isomorphism $\mathcal{M}'(\ast\{0\}) \xrightarrow{\sim} \mathcal{M}$.

We set

$$\mathcal{S}^T := \mathcal{S}^T(\mathcal{M}') \simeq R\text{Hom}(\mathbb{C}^*_X \times \mathbb{R}^\infty, \mathcal{S}^T(\mathcal{M})) \in D^b(\mathcal{C}_{X^* \times \mathbb{R}^\infty}^\text{sub})$$

Since $e^t u_j(z)|_{D_{\theta_0} \times \mathbb{R}}$ is tempered on

$$\{t + \text{Re} \varphi_j < c\} := \{(z, t) \in X^* \times \mathbb{R} : t + \text{Re} \varphi_j(z) < c\}$$

for any $c$ (see Proposition 5.11), we have

$$\mathcal{C}_{D_{\theta_0} \times \mathbb{R}} \otimes \mathcal{S}^T \simeq \bigoplus_{1 \leq j \leq r} \mathcal{C}_{D_{\theta_0} \times \mathbb{R}} \otimes \mathcal{S}_{\varphi_j}^T,$$
where

\[(8.1) \quad \mathcal{F}_\varphi^T := \lim_{c \to +\infty} C_{\{t + \text{Re} \widetilde{\varphi} < c\}} \in \text{Mod}(\mathbb{C}_{X \times \mathbb{R}_{\infty}}^{\text{sub}}) \quad \text{for} \quad \varphi \in \Phi.\]

Here \(\widetilde{\varphi} \in \mathcal{O}_X(*\{0\})_0\) is a representative of \(\varphi \in \Phi := \mathcal{O}_X(*\{0\})/\mathcal{O}_X_0\). Note that the right-hand side of (8.1) does not depend on the choice of a representative \(\widetilde{\varphi}\).

Set

\[
\mathcal{F}_\varphi^T := (\mathcal{F}_\varphi)^{-1} \mathcal{I}_\varphi^T \in \mathcal{D}^b(\mathbb{C}_{X \times \mathbb{R}_{\infty}}^{\text{sub}}) \quad \text{and} \quad \mathcal{F}_\varphi^{\mathcal{T}} := (\mathcal{F}_\varphi)^{!} \mathcal{I}_\varphi^{\mathcal{T}} \cong \mathcal{R} \text{Hom}(\mathbb{C}_{X \times \mathbb{R}}, (\mathcal{F}_\varphi)^{-1} \mathcal{I}_\varphi^T) \in \mathcal{D}^b(\mathbb{C}_{X \times \mathbb{R}_{\infty}}^{\text{sub}}).
\]

Then set

\[
K_\varphi := \mathcal{R} \text{Hom}(\mathcal{F}_\varphi^{\mathcal{T}}, \mathcal{I}_\varphi^{\mathcal{T}}) \in \mathcal{D}^b(\mathbb{C}_{X}^{\mathcal{T}}).
\]

Since \(\mathcal{F}_\varphi^T|_{X \times \mathbb{R}_{\infty}} \simeq \mathbb{C}_{X}^T\), we have

\[
K_\varphi|_{X^*} \simeq L := \mathcal{R} \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)|_{X^*}.
\]

Then we obtain a morphism of sheaves on \(S\)

\[
i^{-1} K_\varphi \to i^{-1} j_*(K_\varphi|_{X^*}) \simeq \widetilde{L} := i^{-1} j_* L.
\]

**Lemma 8.1.** The object \(i^{-1} K_\varphi \in \mathcal{D}^b(\mathcal{C}_S)\) is concentrated at degree 0. The above morphism \(i^{-1} K_\varphi \to \widetilde{L}\) is a monomorphism and its image coincides with \(F_\varphi\).

Thus, \(\mathcal{F}^T(\mathcal{M})\) recovers the Stokes filtration \(\{F_\varphi\}_{\varphi \in \Phi}\) on \(\widetilde{L}\).

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