How to Expand the Zariski Topology

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Abstract

We introduce the notion of a Hu-Liu prime ideal in the context of left commutative rngs, and establish the contravariant functor from the category of left commutative rngs into the category of topological spaces.

It is well known that new points must be introduced in order to expand algebraic geometry over algebraically closed fields into Grothendieck’s scheme theory over commutative rings. We believe that the idea of adding new points to an old space is still essential for the attempt to expand algebraic geometry over algebraically closed fields into a kind of geometry over a class of non-commutative rings. Clearly, whether we can use the natural idea successfully depends on whether we can find new points satisfactorily. Since points in Grothendieck’s scheme theory are prime ideals, the problem of finding satisfactory new points is how to choose some classes of rings to get a satisfactory generalization of prime ideals. The purpose of this paper is to give a solution to the problem. Our solution is based on the notion of the additive halo introduced in [4], which comes from some facts obtained in our attempt to generalize the Lie correspondence between connected linear Lie groups and linear Lie algebras.

After choosing a class of rings called left commutative rngs, we introduce the notion of a Hu-Liu prime ideal, characterize the nil radical by using Hu-Liu prime ideals, and establish the contravariant functor from the category of left commutative rngs into the category of topological spaces.

1 Definitions

Following [2], the word “rng” means a ring which is not assumed to have an identity, and the word “ring” always means a ring with an identity.

We begin this section with the definition of a left commutative rng.
Definition 1.1 A rng \((R, +, \cdot)\) is called a left commutative rng if \(R\) satisfies the following four conditions.

(i) The associative product \(\cdot\) is left commutative; that is,
\[ xyz = yxz \quad \text{for } x, y, z \in R. \]

(ii) There exists a left identity \(1^\ell\) of \(R\) such that
\[ 1^\ell x = x \quad \text{for } x \in R. \]

(iii) There exists a binary operation \(\sharp\) called the local product on the additive halo
\[ h^+(R) := \{ x \mid x \in R \text{ and } x1^\ell = 0 \} \]
such that \((h^+(R), +, \sharp)\) is a commutative ring with an identity \(1^\sharp\). The identity \(1^\sharp\) of the ring \(h^+(R)\) is called the local identity of \(R\).

(iv) The two associative binary operations \(\cdot, \sharp\) satisfy the Hu-Liu triassociative law:
\[ (x\alpha)\sharp \beta = x(\alpha \sharp \beta), \]
where \(\alpha, \beta \in h^+(R)\) and \(x \in R\).

Since the local product \(\sharp\) is commutative, (4) is equivalent to
\[ \alpha \sharp (x\beta) = x(\alpha \sharp \beta) \quad \text{for } \alpha, \beta \in h^+(R) \text{ and } x \in R. \]

The equations (3) and (4) consist of a version of the Hu-Liu triassociative law introduced in [5].

A left commutative rng is sometimes denoted by \((R, +, \cdot, \sharp)\). If \(R\) is a left commutative rng, then the bar-unit set \(h^\times(R)\) of \(R\) is defined by
\[ h^\times(R) := \{ b \mid bx = x \text{ for } x \in R \}. \]

Definition 1.2 Let \(R\) and \(\bar{R}\) be two left commutative rngs with the local identity \(1^\sharp\) and \(\bar{1}^\sharp\), respectively. A map \(\phi : R \to \bar{R}\) is called a left commutative rng homomorphism if
\[ \phi(x + y) = \phi(x) + \phi(y), \]
\[ \phi(xy) = \phi(x)\phi(y), \]
\[ \phi(h^\times(R)) \cap h^\times(\bar{R}) \neq \emptyset, \]
\[ \phi(\alpha \sharp \beta) = \phi(\alpha)\sharp \phi(\beta), \]
\[ \phi(1^\sharp) = \bar{1}^\sharp, \]
where \(x, y \in R\), and \(\alpha, \beta \in h^+(R)\). A bijective left commutative rng homomorphism is called a left commutative rng isomorphism.
Note that (8) is well-defined because of the following fact:

\[ (5), (6) \text{ and } (7) \Rightarrow \phi(h^+(R)) \subseteq h^+(\bar{R}). \]

If \((R, +, \cdot, ^\sharp)\) is a left commutative rng with a left identity \(1^\ell\), then the left identity \(1^\ell\) induces a decomposition of \(R\):

\[ R = R_0 \oplus R_1 \quad (\text{as Abelian groups}), \quad (10) \]

where

\[ R_0 := R1^\ell = \{ x1^\ell \mid x \in R \} \quad \text{and} \quad R_1 := h^+(R) = \{ x \mid x \in R \text{ and } x1^\ell = 0 \}. \]

\(R_0\) and \(R_1\) are called the even part of \(R\) induced by the left identity \(1^\ell\) and the odd part of \(R\), respectively. If \(x \in R\), then

\[ x = x_0 + x_1, \quad x_0 \in R_0 \text{ and } x_1 \in R_1 \]

by (10). \(x_0\) and \(x_1\) are called the even component and the odd component of \(x\) induced by the left identity \(1^\ell\), respectively. We also say that \(x_\varepsilon\) is the \(\varepsilon\)-component of \(x\) induced by the left identity \(1^\ell\), where \(\varepsilon = 0, 1\).

Let \((R, +, \cdot, ^\sharp)\) be a left commutative rng. A subgroup \(I\) of the additive group \((R, +)\) is called an ideal if \(RI \subseteq I, IR \subseteq I\) and \(I \cap h^+(R)\) is an ideal of the ring \((h^+(R), +, ^\sharp)\). If \(h^+(R) \neq 0\), then every left commutative rng \(R\) always has three distinct ideals: 0, \(h^+(R)\) and \(R\). An ideal \(I\) of \(R\) respects to the decomposition (10); that is,

\[ I = I_0 \oplus I_1, \]

where \(I_0 = I \cap R1^\ell\) and \(I_1 = I \cap h^+(R)\) are called the even part and odd part of \(I\) induced by the left identity \(1^\ell\).

Let \(I\) be an ideal of a triring \((R, +, \leftarrow, \rightarrow, ^\sharp)\). We define a binary operation \(\cdot\) on the quotient group

\[ \frac{R}{I} := \{ x + I \mid x \in I \} \]

by

\[ (x + I) \cdot (y + I) := xy + I, \quad (11) \]

where \(x, y \in R\). The well-defined binary operation above makes the quotient group \(\frac{R}{I}\) into a rng with a left identity \(1^\ell + I\), where \(1^\ell\) is a left identity of \(R\). The additive halo of \(\frac{R}{I}\) is given by

\[ h^+\left(\frac{R}{I}\right) = \frac{h^+(R) + I}{I} = \{ \alpha + I \mid \alpha \in h^+(R) \}. \quad (12) \]
We now define a local product on $\mathbb{h}^+(\overline{R})$ by
\[(\alpha + I) \sharp (\beta + I) := \alpha \sharp \beta + I,\] (13)
where $\alpha, \beta \in \mathbb{h}^+(\overline{R})$. One can check that (13) is well-defined, the two binary operations defined by (11) and (13) satisfy Hu-Liu triassociative law, and $\left(\mathbb{h}^+(\overline{R}), +, \sharp\right)$ is a ring with the identity $1^\sharp + I$, where $1^\sharp$ is the local identity of $\mathbb{h}^+(\overline{R})$. Therefore, the quotient group $\overline{R}$ becomes a left commutative rng under (11) and (13), which is called the quotient left commutative rng of $\overline{R}$ with respect to the ideal $I$.

The following definition gives the counterpart of a prime ideal in the context of left commutative rngs.

**Definition 1.3** Let $(\overline{R}, +, \cdot, \sharp)$ be a left commutative rng. An ideal $P$ of $\overline{R}$ is called a Hu-Liu prime ideal if $P \neq \overline{R}$ and the following three conditions are satisfied.

(i) For $x, y \in \overline{R}$, we have
\[xy \in P + \mathbb{h}^+(\overline{R}) \Rightarrow x \in P + \mathbb{h}^+(\overline{R}) \text{ or } y \in P + \mathbb{h}^+(\overline{R}).\] (14)

(ii) For $x, y \in \overline{R}$, we have
\[xy \in P \Rightarrow x \in P + \mathbb{h}^+(\overline{R}) \text{ or } y \in P.\] (15)

(iii) $P \supseteq \mathbb{h}^+(\overline{R})$ or $P \cap \mathbb{h}^+(\overline{R})$ is a prime ideal of the commutative ring $(\mathbb{h}^+(\overline{R}), +, \sharp)$.

Let $\overline{R}$ be a left commutative rng. The set of all Hu-Liu prime ideals of $\overline{R}$ is called the spectrum of $\overline{R}$ and is denoted by $\text{spec}^\sharp \overline{R}$. It is clear that
\[\text{spec}^\sharp \overline{R} = \text{spec}^\sharp_0 \overline{R} \cup \text{spec}^\sharp_1 \overline{R} \quad \text{and} \quad \text{spec}^\sharp_0 \overline{R} \cap \text{spec}^\sharp_1 \overline{R} = \emptyset,
\] where
\[\text{spec}^\sharp_0 \overline{R} := \{ P \mid P \in \text{spec}^\sharp \overline{R} \text{ and } P \supseteq \mathbb{h}^+(\overline{R}) \}\]
is called the even spectrum of $\overline{R}$ and
\[\text{spec}^\sharp_1 \overline{R} := \{ P \mid P \in \text{spec}^\sharp \overline{R} \text{ and } P \nsubseteq \mathbb{h}^+(\overline{R}) \}\]
is called the odd spectrum of $\overline{R}$.

The next proposition gives some equivalent forms of the conditions in Definition 1.3.

**Proposition 1.1** Let $P = P_0 \oplus P_1 \neq \overline{R}$ be an ideal of a left commutative rng $\overline{R}$ with a left identity $1^\ell$, where $P_\varepsilon = P \cap R_\varepsilon$, $R_0 := R_1^\ell$ and $R_1 := \mathbb{h}^+(\overline{R})$. 


(i) $P$ is a Hu-Liu prime ideal of $R$ if and only if
\[ x_0 y_\varepsilon \in P \Rightarrow x_0 \in P_0 \text{ or } y_\varepsilon \in P_\varepsilon \] (16)
and
\[ x_1 y_1 \in P_1 \Rightarrow x_1 \in P_1 \text{ or } y_1 \in P_1, \] (17)
where $x_\varepsilon, y_\varepsilon \in R_\varepsilon$ and $\varepsilon = 0, 1$.

(ii) If $P \supseteq \bar{h}^+(R)$, then $P$ is a Hu-Liu prime ideal of the left commutative rng
$R$ if and only if $\frac{P}{\bar{h}^+(R)}$ is a prime ideal of the commutative ring $\frac{R}{\bar{h}^+(R)}$.

(iii) (14) holds if and only if $P_0$ is a prime ideal of the commutative ring $R_0$.

(iv) (15) holds if and only if the left $\frac{R_0}{P_0}$-module $\frac{R}{P}$ is faithful, where the left
$\frac{R_0}{P_0}$-module action is defined by
\[(x_0 + P_0)(y + P) := x_0 y + P \quad x_0 \in P_0 \text{ and } y \in R.\]

(v) If $P \not\supseteq \bar{h}^+(R)$, then $P_1 = P \cap \bar{h}^+(R)$ is a prime ideal of the commutative
ring $(\bar{h}^+(R), +, \#)$ if and only if $\left(\bar{h}^+ \left(\frac{R}{P}\right), +, \#\right)$ is a domain.

Proof They are direct consequences of Definition 1.3.

2 Nil Radicals

Let $(R, +, \cdot, \#)$ be a left commutative rng with a left identity $1^\ell$ and a local
identity $1^\sharp$.

For $a \in R$ and $\alpha \in \bar{h}^+(R)$, we define the $n$th power $a^n$ and the $n$th local
power $\alpha^{\sharp n}$ as follows:

\[ a^n := \begin{cases} 1^\ell, & \text{if } n = 0; \\ a a \cdots a, & \text{if } n \text{ is a positive integer} \end{cases} \]

and

\[ \alpha^{\sharp n} := \begin{cases} 1^\sharp, & \text{if } n = 0; \\ \alpha \sharp \alpha \sharp \cdots \sharp \alpha, & \text{if } n \text{ is a positive integer.} \end{cases} \]

A product $a^m (\alpha^{\sharp n})$ will be denoted by $a^m \alpha^{\sharp n}$. 
Proposition 2.1 Let $x$ be an element of a left commutative rng $(R, +, \cdot, \sharp)$. Let $1^\ell$ and $1^\ell$ be two left identities of $R$. If $p_1(x)$ and $\overline{p}_1(x)$ are odd components of $x$ induced by $1^\ell$ and $1^\ell$ respectively, then the following are equivalent:

(i) $x^m = 0$ and $p_1(x)\sharp^n = 0$ for some $m, n \in \mathbb{Z}_{>0}$.

(ii) $x^m = 0$ and $\overline{p}_1(x)\sharp^k = 0$ for some $m, k \in \mathbb{Z}_{>0}$.

Proof Since $p_1(x) = x - x1^\ell$ and $\overline{p}_1(x) = x - x1^\ell$, we have

$$p_1(x)^\sharp k = \left(p_1(x) + x(1^\ell - 1^\ell)\right)^\sharp k$$

$$= \sum_{i=0}^{k} \binom{k}{i} p_1(x)^\sharp i \sharp \left(x(1^\ell - 1^\ell)\right)^\sharp (k-i)$$

$$= \sum_{i=0}^{k} \binom{k}{i} p_1(x)^\sharp i \sharp \left(x^{k-i}(1^\ell - 1^\ell)\right)^\sharp (k-i). \quad (18)$$

If (i) is true, then (ii) is true for $s = m$ and $k \geq m + n - 1$ by (18). Similarly, (ii) implies (i).

Definition 2.1 Let $(R, +, \cdot, \sharp)$ be a left commutative rng with a left identity $1^\ell$. An element $x$ of $R$ is called a nilpotent element if

$$x^m = 0 \text{ and } x_1^\sharp n = 0 \text{ for some } m, n \in \mathbb{Z}_{>0},$$

where $x_1 := p_1(x)$ is the old component of $x$ induced by $1^\ell$.

It is clear that Definition 2.1 is independent of the choice of a left identity by Proposition 2.1.

Definition 2.2 The nil radical of a left commutative rng $(R, +, \cdot, \sharp)$ is the set of nilpotent elements of $R$. We shall use nilrad$(R)$ or $\sqrt{0}$ to denote the nil radical of the left commutative rng $R$.

The nil radical of a left commutative rng $R$ can be expressed as

$$\text{nilrad}^\sharp(R) = \text{nilrad}(R1^\ell) \oplus \text{nilrad}(h^+(R)),$$  \quad (19)

where $1^\ell$ is a left identity of $R$, $\text{nilrad}(R1^\ell)$ is the nil radical of the commutative ring $(R1^\ell, +, \cdot)$, and $\text{nilrad}(h^+(R))$ is the nil radical of the commutative ring $(h^+(R), +, \sharp)$.

Proposition 2.2 Let $(R, +, \cdot, \sharp)$ be a left commutative rng.
(i) The nil radical $\mathit{nilrad}^h(R)$ is an ideal of $R$.

(ii) $\mathit{nilrad}^h\left(\frac{R}{\mathit{nilrad}^h(R)}\right) = 0$.

**Proof** (i) By (19), it is enough to prove that

$$R \left( \mathit{nilrad}(h^+(R)) \right) \subseteq \mathit{nilrad}(h^+(R)).$$

(20)

If $x \in R$ and $\alpha \in \mathit{nilrad}(h^+(R))$, then $\alpha^m = 0$ for some $m \in \mathbb{Z}_{>0}$. It follows that

$$(x\alpha)^m = x^m\alpha^m = x^m0 = 0,$$

which implies (20).

(ii) If $x + \mathit{nilrad}^h(R) \in \mathit{nilrad}^h\left(\frac{R}{\mathit{nilrad}^h(R)}\right)$, then

$$(x + \mathit{nilrad}^h(R))^m = \mathit{nilrad}^h(R)$$

(21)

and

$$((x + \mathit{nilrad}^h(R))_1)^n = \mathit{nilrad}^h(R)$$

(22)

for some positive integers $m$ and $n$.

By (21), we get

$$x^m + \mathit{nilrad}^h(R) = \mathit{nilrad}^h(R)$$

$$\Rightarrow x^m \in \mathit{nilrad}^h(R)$$

$$\Rightarrow x^{mu} = (x^m)^u = 0 \text{ for some } u \in \mathbb{Z}_{>0}.$$ (23)

Since $(x + \mathit{nilrad}^h(R))_1 = x_1 + \mathit{nilrad}^h(R)$, we get from (22) that

$$x_1^n + \mathit{nilrad}^h(R) = \mathit{nilrad}^h(R)$$

$$\Rightarrow x_1^n \in \mathit{nilrad}^h(R)$$

$$\Rightarrow x_1^{nv} = \left(x_1^n\right)^v = 0 \text{ for some } v \in \mathbb{Z}_{>0}.$$ (24)

It follows from (23) and (24) that $x \in \mathit{nilrad}^h(R)$. Hence, $x + \mathit{nilrad}^h(R) = \mathit{nilrad}^h(R)$. This proves (ii).

We now characterize the nil radical of a left commutative rng by using Hu-Liu prime ideals.

**Proposition 2.3** The nil radical of a left commutative rng $R$ is the intersection of the Hu-Liu prime ideals of $R$. 

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Proof. In this proof, $R$ denotes a left commutative ring with an left identity $1^\ell$, the $\varepsilon$-component $x_\varepsilon$ of $x \in R$ and the $\varepsilon$-part $I_\varepsilon$ of an ideal $I$ of $R$ are always induced by the left identity $1^\ell$.

Let $x$ be any element of $\text{nilrad}^\varepsilon(R)$. Then $x^m = 0$ and $x_1^{\varepsilon n} = 0$ for some $m$, $n \in \mathbb{Z}_{>0}$. Let $P$ be any Hu-Liu prime ideal of $R$. Since $x^m = 0 \in P + h^+(R)$, we have $x \in P + h^+(R) = P_0 \oplus h^+(R)$ or

$$x_0 \in P_0 \subseteq P.$$  \hspace{1cm} (25)

If $P \nsubseteq h^+(R)$, then $\left( h^+\left(\frac{R}{P}\right), +, \# \right)$ is a domain by Proposition 1.1(v).

Using $x_1^{\varepsilon n} = 0$, we get

$$(x_1 + P)^{\varepsilon n} = x_1^{\varepsilon n} + P = 0 + P = P.$$  

Thus $x_1 + P$ is a nilpotent element of the domain $h^+\left(\frac{R}{P}\right)$, which implies that $x_1 + P$ is the zero element of the domain $h^+\left(\frac{R}{P}\right)$. Hence, we get

$$x_1 \in P.$$  \hspace{1cm} (26)

If $P \supseteq h^+(R)$, then (26) is obviously true.

It follows from (25) and (26) that $x = x_0 + x_1 \in P$. This proves that

$$\text{nilrad}^\varepsilon(R) \subseteq \bigcap_{P \in \text{spec}^\varepsilon R} P.$$  \hspace{1cm} (27)

Conversely, we prove that

$$z \neq \text{nilrad}^\varepsilon(R) \Rightarrow z \neq \bigcap_{P \in \text{spec}^\varepsilon R} P.$$  \hspace{1cm} (28)

Case 1: $z^m \neq 0$ for all $m \in \mathbb{Z}_{>0}$, in which case, $z^m \neq h^+(R)$ for all $m \in \mathbb{Z}_{>0}$. Hence, $z + h^+(R)$ is not a nilpotent element of the commutative ring $R + h^+(R)$; that is,

$$z + h^+(R) \notin \text{nilrad} \left( \frac{R}{h^+(R)} \right) = \bigcap_{P \in \text{spec}^\varepsilon R} \left( \frac{I}{h^+(R)} \right).$$

Hence, there exists a prime ideal $\frac{I}{h^+(R)}$ of the commutative ring $\frac{R}{h^+(R)}$ such that $z \notin I$. Since $I$ is a Hu-Liu prime ideal by Proposition 1.1(ii), (28) holds in this case.
Case 2: $z_1^zn \neq 0$ for all $n \in \mathbb{Z}_{>0}$, in which case, we consider the following set

$$T := \left\{ J \mid J \text{ is an ideal of } R \text{ and } z_1^zn \notin J \text{ for all } n \in \mathbb{Z}_{>0} \right\}.$$ 

Since $\{0\} \in T$, $T$ is nonempty. Clearly, $(T, \subseteq)$ is a partially order set, where $\subseteq$ is the relation of set inclusion. If $\{ J_\lambda \mid \lambda \in \Lambda \}$ is a nonempty totally ordered subset of $T$, then $\cup_{\lambda \in \Lambda} J_\lambda$ is an upper bound of $\{ J_\lambda \mid \lambda \in \Lambda \}$ in $T$. By Zorn’s Lemma, the partially ordered set $(T, \subseteq)$ has a maximal element $P$. We are going to prove that $P$ is a Hu-Liu prime ideal of $R$.

Let $x$ and $y$ be two elements of $R$. First, if $x \notin P + \bar{h}^+(R)$ and $y \notin P + \bar{h}^+(R)$, then $x_0 \notin P_0$ and $y_0 \notin P_0$. Hence, we get

$$P \subset P + x_0R \quad \text{and} \quad P \subset P + y_0R. \quad (29)$$

By (1), we have

$$R(P + x_0R) \subseteq P + x_0R \quad \text{and} \quad (P + x_0R)R \subseteq P + x_0R.$$ 

Moreover, $(P + x_0R) \cap \bar{h}^+(R) = P_1 + x_0R_1$ is an ideal of $(\bar{h}^+(R), +, \sharp)$. This proves that the subgroup $P + x_0R$ is an ideal of the left commutative rng $R$. Similarly, $P + y_0R$ is also an ideal of the $R$. It follows from (29) that

$$P + x_0R \notin T \quad \text{and} \quad P + y_0R \notin T,$$

which imply that

$$z_1^u \in P + x_0R \quad \text{and} \quad z_1^v \in P + y_0R$$

or

$$z_1^u \in P_1 + x_0R_1 \quad \text{and} \quad z_1^v \in P_1 + y_0R_1 \quad \text{for some } u, v \in \mathbb{Z}_{>0}.$$ 

By (3), we have

$$z_1^{(u+v)} = z_1^u \sharp z_1^v \in (P_1 + x_0R_1) \sharp (P_1 + y_0R_1) \subseteq (P_1 + x_0R_1) \sharp (y_0R_1) + (x_0R_1) \sharp (P_1 + y_0R_1)$$

This is a subset of $P$

$$\subseteq P + x_0y_0(R_1 \sharp R_1) \subseteq P + x_0y_0R,$$

which implies that

$$(xy)_0 = x_0y_0 \notin P. \quad (30)$$

Hence, $xy \notin P + \bar{h}^+(R)$. This proves that

$$x \notin P + \bar{h}^+(R) \text{ and } y \notin P + \bar{h}^+(R) \Rightarrow xy \notin P + \bar{h}^+(R). \quad (31)$$
Similarly, we can prove
\[ x \notin P + h^+(R) \text{ and } y \notin P \Rightarrow xy \notin P. \tag{32} \]
and
\[ P_1 = P \cap h^+(R) \text{ is a prime ideal of } (h^+(R), +, \leq). \tag{33} \]

By (31), (32) and (33), \( P \) is a Hu-Liu prime ideal. Since \( z_1 \notin P \), (28) also holds in Case 2.

It follows from (27) and (28) that Proposition 2.3 is true.

If \( I \) is an ideal of a left commutative rng \( R \), then the radical \( \sqrt[\#]{I} \) of \( I \) is defined by
\[ \sqrt[\#]{I} := \{ x \in R \mid x_0^n \in I \cap R1^\ell \text{ and } x_1^n \in I \cap h^+(R) \text{ for some } m, n \in \mathbb{Z}_{>0} \}, \]
where \( x_\epsilon \) is the \( \epsilon \)-component of \( x \) induced by the left identity \( 1^\ell \) of \( R \). Since
\[ \text{nilrad}^\# \left( \frac{R}{I} \right) = \sqrt[\#]{I + I}, \]
\( \sqrt[\#]{I} \) is an ideal of \( R \). An ideal \( I \) of a left commutative rng \( R \) is called a radical ideal if \( \sqrt[\#]{I} = I \).

The next proposition is a corollary of Proposition 2.3.

**Proposition 2.4** If \( I \) is an ideal of a left commutative rng \( R \) and \( I \neq R \), then
\[ \sqrt[\#]{I} = \bigcap_{P \in \text{spec}^\# R \text{ and } P \supseteq I} P. \]

**Proof** By Proposition 2.3 we have
\[ x \in \sqrt[\#]{I} \iff x + I \in \text{nilrad}^\# \left( \frac{R}{I} \right) = \bigcap_{P \in \text{spec}^\# \left( \frac{R}{I} \right)} P \]
\[ \iff x \in \bigcap_{P \in \text{spec}^\# R \text{ and } P \supseteq I} P. \]
3 The Expansion of the Zarisky Topology

Let \((R, +, \cdot, \sharp)\) be a left commutative rng. For an ideal \(I\) of \(R\), we define a subset \(\mathcal{V}(I)\) of \(\text{spec}^\#R\) by

\[
\mathcal{V}(I) := \{ P \mid P \in \text{spec}^\#R \text{ and } P \supseteq I \}.
\]

(34)

\section*{Proposition 3.1}

Let \(R\) be a left commutative rng \(R\).

(i) \(\mathcal{V}(0) = \text{spec}^\#R\) and \(\mathcal{V}(R) = \emptyset\).

(ii) \(\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(I \cap J)\), where \(I\) and \(J\) are two ideals of \(R\).

(iii) \(\bigcap_{\lambda \in \Lambda} \mathcal{V}\left(\frac{\lambda}{I}\right) = \mathcal{V}\left(\sum_{\lambda \in \Lambda} \frac{\lambda}{I}\right)\), where \(\left\{ \frac{\lambda}{I} \mid \lambda \in \Lambda \right\}\) is a set of ideals of \(R\).

\section*{Proof}

Since (i) and (iii) are clear, we need only to prove (ii).

By (34), we have

\[
\mathcal{V}(I) \cup \mathcal{V}(J) \subseteq \mathcal{V}(I \cap J).
\]

(35)

Conversely, we prove that

\[ P \supseteq I \cap J \text{ and } P \not\supseteq I \Rightarrow P \supseteq J. \]

(36)

In the following proof, we assume that the even part and odd part of an ideal are always induced by the same left identity. Since \(P \not\supseteq I\), we have two cases.

\textbf{Case 1:} \(P \not\supseteq I_0\), in which case, there exists \(x_0 \in I_0\) and \(x_0 \not\in P_0\). Hence, \(x_0 J \subseteq I \cap J \subseteq P\), which implies \(P \supseteq J\) by (15).

\textbf{Case 2:} \(P \supseteq I_0\) and \(P \not\supseteq I_1\), in which case, there exists \(x_1 \in I_1\) and \(x_1 \not\in P\). Hence, \(J_0 x_1 \subseteq I \cap J \subseteq P\). This fact and (15) imply that \(J_0 \subseteq P + \bar{h}^+(R)\) or

\[
J_0 \subseteq P_0.
\]

(37)

Also, we have

\[
x_1 \not\in J_1 \subseteq (I \cap J) \subseteq P_1.
\]

(38)

Since \(P \not\supseteq I_1\), \(P \not\supseteq \bar{h}^+(R)\). Thus, \(P_1 = P \cap \bar{h}^+(R)\) is a prime ideal of the commutative ring \((\bar{h}^+(R), +, \sharp)\). Hence, (38) implies that

\[
J_1 \subseteq P_1.
\]

(39)

By (37) and (39), we get \(J = J_0 + J_1 \subseteq P_0 + P_1 = P\). This proves that (36) is also true in Case 2.

Using (36), we have

\[
\mathcal{V}(I) \cup \mathcal{V}(J) \supseteq \mathcal{V}(I \cap J).
\]

(40)
It follows from (35) and (40) that (ii) is true.

Let $R$ be a left commutative rng. By Proposition 3.1, the collection

$$ V := \{ V(I) \mid I \text{ is an ideal of } R \} $$

of subsets of $\text{spec}^2 R$ satisfies the axioms for closed sets in a topological space. The topology on $\text{spec}^2 R$ having the elements of $V$ as closed sets is called the **expanded Zariski topology**. The collection

$$ D := \{ D(I) \mid I \text{ is an ideal of } R \} $$

consists of the open sets of the expanded Zariski topology on $\text{spec}^2 R$, where

$$ D(I) := \text{spec}^2 R \setminus V(I) = \{ P \mid P \in \text{spec}^2 R \text{ and } P \not\supseteq I \}. $$

Since $\text{spec}^2 0 R = \text{V}(h^+(R))$, the even spectrum $\text{spec}^2 0 R$ of $R$ is a closed subspace of $\text{spec}^2 R$ and the odd spectrum $\text{spec}^2 1 R$ of $\text{spec}^2 R$ is an open subspace of $\text{spec}^2 R$.

We define a binary relation $\sim$ on $\text{spec}^2 1 R$ by

$$ P \sim Q \iff P \cap h^+(R) = Q \cap h^+(R) \text{ for } P, Q \in \text{spec}^2 1 R. $$

It is clear that $\sim$ is an equivalence relation. The equivalence class containing $P \in \text{spec}^2 1 R$ is denoted by $[P]$. The quotient topology with respect to the equivalence relation $\sim$ is called the **odd quotient topology** of $\text{spec}^2 1 R$ and is denoted by $\text{spec}^2 1 R/\sim$.

The next proposition gives the basic properties of the closed subspace of $\text{spec}^2 R$ and the odd quotient topology of $\text{spec}^2 1 R$.

**Proposition 3.2** Let $(R, +, \cdot, \sim)$ be a left commutative rng $R$.

(i) The map

$$ \phi^R_0 : \frac{P}{h^+(R)} \mapsto P \text{ for } \frac{P}{h^+(R)} \in \text{spec} \left( \frac{R}{h^+(R)} \right) $$

is a homeomorphism from the spectrum of the commutative ring $\frac{R}{h^+(R)}$ onto the closed subspace $\text{spec}^2 0 R$ of $\text{spec}^2 1 R$.

(ii) The map

$$ \phi^R_1 : [P] \mapsto P \cap h^+(R) \text{ for } [P] \in \text{spec}^2 1 R/\sim $$

is a homeomorphism from the odd quotient topology space $\text{spec}^2 1 R/\sim$ onto the subspace $\{ P \cap h^+(R) \mid P \in \text{spec}^2 1 R \}$ of the spectrum of the commutative ring $(h^+(R), +, \sim)$.  

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Proof Clear.

For any commutative ring \( \tilde{R} \), there exists a left commutative rng \( R \) such that \( \tilde{R} \simeq \frac{R}{h^{+}(R)} \) as rings. It follows from this fact and Proposition 3.2 that the points in the spectrum of a commutative ring appear as the points in the even spectrum of a left commutative rng, and the new points introduced to expand the Zariski topology consist of the odd spectrum of the left commutative rng.

Proposition 3.3 Let \( R \) and \( S \) be left commutative rongs. If \( f : R \to S \) is a left commutative rng homomorphism, then the map

\[ f^\sharp : P \mapsto f^{-1}(P) \text{ for } P \in \text{spec}^\sharp S \]

is continuous from \( \text{spec}^\sharp S \) to \( \text{spec}^\sharp R \), and the restriction map \( f^\sharp|_{\text{spec}^\sharp S} \) is continuous from the subspace \( \text{spec}^\sharp S \) of \( \text{spec}^\sharp S \) to the subspace \( \text{spec}^\sharp R \) of \( \text{spec}^\sharp R \).

Proof First, we prove

\[ P \in \text{spec}^\sharp S \Rightarrow f^{-1}(P) \in \text{spec}^\sharp R. \tag{41} \]

Let \( 1^f_R \) be a left identity of \( R \) such that \( 1^f_S := f(1^f_R) \) is a left identity of \( S \). In the following proof, the \( \varepsilon \)-component of an element of \( R \) are always induced by \( 1^f_R \), and the \( \varepsilon \)-component of an element of \( S \) are always induced by \( 1^f_S \).

For \( x_\varepsilon, y_\varepsilon \in R_\varepsilon \) with \( \varepsilon = 0 \) and 1, we have

\[
\begin{align*}
x_0y_\varepsilon \in (f^{-1}(P))_\varepsilon &= f^{-1}(P) \cap R_\varepsilon \\
\Rightarrow f(x_0)f(y_\varepsilon) &= f(x_0y_\varepsilon) \in P \cap f(R_\varepsilon) \subseteq P \cap S_\varepsilon = P_\varepsilon \\
\Rightarrow f(x_0) \in P_0 \quad \text{or} \quad f(y_\varepsilon) \in P_\varepsilon \\
\Rightarrow x_0 \in f^{-1}(P) \cap R_0 = (f^{-1}(P))_0 \quad \text{or} \quad y_\varepsilon \in f^{-1}(P) \cap R_\varepsilon = (f^{-1}(P))_\varepsilon.
\end{align*}
\]

This proves that

\[ x_0y_\varepsilon \in (f^{-1}(P))_\varepsilon \Rightarrow x_0 \in (f^{-1}(P))_0 \quad \text{or} \quad y_\varepsilon \in (f^{-1}(P))_\varepsilon. \tag{42} \]

If \( f^{-1}(P) \supseteq R_1 \), then (41) is true by (42).

If \( f^{-1}(P) \nsubseteq R_1 \), then \( P \nsubseteq S_1 \); otherwise, \( R_1 \subseteq f^{-1}(S_1) \subseteq f^{-1}(P) \). Hence, \( P \cap S_1 \) is a prime ideal of \( S_1 \). Thus, we have

\[
\begin{align*}
x_1 \neq y_1 \in f^{-1}(P) \cap R_1 \\
\Rightarrow f(x_1) \neq f(y_1) \in P \cap f(R_1) \subseteq P \cap S_1 \\
\Rightarrow f(x_1) \in P \cap S_1 \quad \text{or} \quad f(y_1) \in P \cap S_1 \\
\Rightarrow x_1 \in f^{-1}(P) \cap R_1 \quad \text{or} \quad y_1 \in f^{-1}(P),
\end{align*}
\]

which proves that

\[ f^{-1}(P) \nsubseteq R_1 \Rightarrow f^{-1}(P) \cap R_1 \text{ is a prime ideal of } R_1. \tag{43} \]
By (42) and (43), (41) is also true.

It follows from (41) that the map $f^\sharp$ is well-defined.

Next, we prove that $I$ is an ideal of $R \Rightarrow (f^\sharp)^{-1}(\mathcal{V}(I)) = \mathcal{V}(<f(I)>)$, \hspace{1cm} (44)

where $<f(I)>$ is the ideal of $S$ generated by $f(I)$. Since $Q \in (f^\sharp)^{-1}(\mathcal{V}(I))$

$$\iff f^{-1}(Q) = f^\sharp(Q) \in \mathcal{V}(I)$$

$$\iff f^{-1}(Q) \supseteq I$$

$$\iff Q \supseteq f(I)$$

$$\iff Q \supseteq <f(I)>$$

$$\iff Q \in \mathcal{V}(<f(I)>),$$

we know that (44) is true.

By (44), the pre-image of any closed set in $\text{spec}^S \sharp R$ under $f^\sharp$ is a closed set in $\text{spec}^S \sharp S$. Hence, the map $f^\sharp$ is continuous.

Finally, let $1^\sharp_R$ and $1^\sharp_S$ be the local identity of $R$ and $S$, respectively. If $I$ is an ideal of $R$, then

$$f(I + h^+(R)) = f(I) + f(h^+(R)) \subseteq f(I) + h^+(S) \subseteq <f(I)> + h^+(S)$$

which implies that

$$<f(I + h^+(R))> \subseteq <f(I)> + h^+(S). \hspace{1cm} (45)$$

Conversely, since $1^\sharp_S = f(1^\sharp_R) \in f(h^+(R)) \subseteq f(I + h^+(R))$, we get $h^+(S)$

$$\subseteq <f(I + h^+(R))>.$$ It follows that

$$<f(I + h^+(R))> \supseteq <f(I)> + h^+(S). \hspace{1cm} (46)$$

By (45) and (46), we get

$$<f(I + h^+(R))> = <f(I)> + h^+(S). \hspace{1cm} (47)$$

It follows from (44) and (47) that

$$(f^\sharp)^{-1}(\mathcal{V}(I + h^+(R))) = \mathcal{V}( <f(I)> + h^+(S)), \hspace{1cm}$$

which proves that the restriction map $f^\sharp|_{\text{spec}^S \sharp S}$ is continuous.

By Proposition 3.3, the pair of maps $R \mapsto \text{spec}^S R$, $f \mapsto f^\sharp$ define a contravariant functor from the category of left commutative rings (left commutative rng homomorphisms as morphisms) into the category of topological spaces
(continuous maps as morphisms). Moreover, if \( f : R \to S \) is a left commutative rng homomorphism, then \( f \) induces a commutative ring homomorphism \( \bar{f} : \frac{R}{\hat{h}(R)} \to \frac{S}{\hat{h}(S)} \), and the following two diagrams

\[
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow & & \downarrow \\
\frac{R}{\hat{h}(R)} & \xrightarrow{\bar{f}} & \frac{S}{\hat{h}(S)}
\end{array}
\quad
\text{and}
\quad
\begin{array}{ccc}
\text{spec}^\natural S & \xrightarrow{\bar{f}} & \text{spec}^\natural R \\
\phi^S_0 & & \phi^R_0 \\
\text{spec} \left( \frac{S}{\hat{h}(S)} \right) & \xrightarrow{\bar{f}^*} & \text{spec} \left( \frac{R}{\hat{h}(R)} \right)
\end{array}
\]

are commutative.

Other properties of left commutative rngs can be found in [5].

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