High frequency dispersive estimates for the Schrödinger equation in high dimensions

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Abstract. We prove optimal dispersive estimates at high frequency for the Schrödinger group for a class of real-valued potentials $V(x) = O(\langle x \rangle^{-\delta})$, $\delta > n - 1$, and $V \in C^k(\mathbb{R}^n)$, $k > k_n$, where $n \geq 4$ and $\frac{n-3}{2} \leq k_n < \frac{n}{2}$. We also give a sufficient condition in terms of $L^1 \to L^\infty$ bounds for the formal iterations of Duhamel’s formula, which might be satisfied for potentials of less regularity.

1 Introduction and statement of results

The purpose of this work is to study the question of finding as large as possible class of real-valued potentials $V \in L^\infty(\mathbb{R}^n)$, $n \geq 4$, for which the Schrödinger propagator $e^{itG} \chi_a(G)$ satisfies optimal (that is, without loss of derivatives) $L^1 \to L^\infty$ dispersive estimates, where $G$ denotes the self-adjoint realization of the operator $-\Delta + V$ on $L^2(\mathbb{R}^n)$, and $\chi_a \in C^\infty(\mathbb{R})$, $\chi_a(\lambda) = 0$ for $\lambda \leq a$, $\chi_a(\lambda) = 1$ for $\lambda \geq a + 1$, $a \gg 1$. To state our results we need to introduce the class $C^k_\delta(\mathbb{R}^n)$, $\delta, k \geq 0$, of all functions $V \in C^k(\mathbb{R}^n)$ satisfying

$$
\|V\|_{C^k_\delta} := \sup_{x \in \mathbb{R}^n} \sum_{0 \leq |\alpha| \leq k_0} \langle x \rangle^\delta |\partial_x^\alpha V(x)| + \nu \sup_{x \in \mathbb{R}^n} \sum_{|\beta| = k_0} \langle x \rangle^\delta \sup_{x' \in \mathbb{R}^n: |x-x'| \leq 1} \left| \frac{\partial^2_x V(x) - \partial^2_x V(x')}{|x-x'|^\nu} \right| < +\infty,
$$

where $k_0 \geq 0$ is an integer and $\nu = k - k_0$ satisfies $0 \leq \nu < 1$.

Theorem 1.1 Given a $\delta > n - 1$, there exists a sequence $\{k_n\}_{n=4}^\infty$, $\frac{n-3}{2} \leq k_n < \frac{n}{2}$, so that if $V \in C^k_\delta(\mathbb{R}^n)$, $k > k_n$, is a real-valued potential, then we have the following high frequency dispersive estimate

$$
\|e^{itG} \chi_a(G)\|_{L^1 \to L^\infty} \leq C|t|^{-n/2}, \quad t \neq 0,
$$

(1.1)

where the constant $C = C(a) > 0$ is independent of $t$.

Remark. It follows from this theorem and the low frequency dispersive estimates proved in [11] that if in addition to the assumptions of Theorem 1.1 (or those of Theorem 1.2 below) we assume that zero is neither an eigenvalue nor a resonance of $G$, then we have the following dispersive estimate

$$
\|e^{itG} P_{ac}\|_{L^1 \to L^\infty} \leq C|t|^{-n/2}, \quad t \neq 0,
$$

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where $P_{ac}$ denotes the spectral projection onto the absolutely continuous spectrum of $G$.

Note that with $k_n = \frac{n}{2}$ the above result follows from [11] where the estimate (1.1) is proved for real-valued potentials $V \in L^\infty(\mathbb{R}^n)$ satisfying

$$|V(x)| \leq C(x)^{-\delta}, \quad \forall x \in \mathbb{R}^n,$$

with constants $C > 0$, $\delta > n - 1$, as well as the condition

$$\hat{V} \in L^1.$$  \hspace{1cm} (1.3)

Previously this has been proved in [9] for potentials satisfying (1.2) with $\delta > n$ and (1.3). Proving (1.1) in dimensions $n \geq 4$ without the condition (1.3), however, turns out to be a difficult problem. Note that the potentials in the above theorem do not satisfy (1.3). On the other hand, the counterexample of [8] shows that the above theorem cannot hold with $k_n < \frac{n}{2}$.

Therefore, it is natural to expect that Theorem 1.1 holds with $\delta > n$ if $n = 4, 5$. Indeed, (1.1) has been proved in [2] when $n = 4, 5$ for potentials $V \in C^k(\mathbb{R}^n)$, $k > \frac{n-3}{2}$, $\delta > 3$ if $n = 4$, $\delta > 5$ if $n = 5$. In [3] an analogue of (1.1) with a logarithmic loss of derivatives has been proved for potentials $V \in C^{\frac{n-3}{2}}(\mathbb{R}^n)$, $\delta > 3$ if $n = 5$, $\delta > 5$ if $n = 7$. It also follows from [5], [17] that (1.1) holds for potentials $V$ satisfying (1.2) with $\delta > n + 2$ and $n = 7$. Note finally that in dimensions one, two and three no regularity of the potential is required in order that (1.1) holds true (see [7], [12], [10], [13], [14], [6]). The same conclusion remains true in dimensions $n \geq 4$ as far as the low and the intermediate frequencies are concerned (see [11], [15]).

To prove (1.1) we make use of the semi-classical expansion of the operator $e^{itG} \psi(h^2G)$ obtained in [1] for potentials satisfying (1.2) with $\delta > \frac{n+2}{2}$, where $\psi \in C^\infty_0((0, +\infty))$ and $0 < h \ll 1$. We thus reduce the problem to estimating uniformly in $h$ the $L^1 \rightarrow L^\infty$ norm of a finite number of operators (denoted by $T_j(t, h)$ below) obtained by iterating the semi-classical Duhamel formula. The advantage is that these operators are defined in terms of the free propagator $e^{itG_0} \psi(h^2G_0)$, where $G_0$ denotes the self-adjoint realization of $-\Delta$ on $L^2(\mathbb{R}^n)$ (see Section 2).

In the present paper we also give a sufficient condition for (1.1) to hold in terms of properties of the formal iterations of Duhamel’s formula defined as follows (for $t > 0$):

$$F_0(t) = e^{itG_0}, \quad F_j(t) = i \int_0^t F_{j-1}(t-\tau)V F_0(\tau) d\tau, \quad j \geq 1.$$

We suppose that there exists a constant $\varepsilon > 0$ such that for all integers $m \geq 1$, $m_1, m_2 \geq 0$, we have the bounds

$$\|F_m(t)\|_{L^1 \rightarrow L^\infty} \leq C m t^{-n/2 + \varepsilon m}, \quad 0 < t \leq 1,$$

$$\left\|\int_{I(\gamma)} F_{m_1}(t-\tau)V F_{m_2}(\tau) d\tau\right\|_{L^1 \rightarrow L^\infty} \leq C m_1, m_2 \gamma^\varepsilon t^{-n/2}, \quad \forall t > 0,$$  \hspace{1cm} (1.4)

(1.5)

where $0 < \gamma \leq 1$, $I(\gamma) \subset [0, t]$ is an interval either of the form $[0, \gamma_1]$ or of the form $[t - \gamma_1, t]$, $\gamma_1 = t/2$ if $t \leq 2\gamma$, $\gamma_1 = \gamma$ if $t \geq 2\gamma$.

**Theorem 1.2** Let $V$ satisfy (1.2) with $\delta > n$ and suppose (1.4) and (1.5) fulfilled. Then, the dispersive estimate (1.1) holds true for all $t > 0$. 

2
It is easy to see that if $V$ satisfies (1.3), then (1.4) and (1.5) hold with $\varepsilon = 1$. However, it might happen that (1.4) and (1.5) hold true for potentials of less regularity. In fact, we expect that (1.4) and (1.5) hold for potentials $V \in C^k_b(\mathbb{R}^n)$ with $\delta > \frac{n+1}{2}, \ k > \frac{n-3}{2}$. Indeed, this has been proved in [2] for $m = 1, m_1 = m_2 = 0$. The problem, however, gets much harder for $m \geq 2, m_1, m_2 \geq 1$.

To prove Theorem 1.2 we take advantage of the analysis carried out in [15] under the only assumption that $V$ satisfies (1.2) with $\delta > \frac{n+2}{2}$ (see Section 5).

2 Reduction to semi-classical dispersive estimates

Set

$$F(t) = i \int_0^t e^{i(t-\tau)G_0} V e^{i\tau G_0} d\tau, \quad t > 0.$$  

It is easy to see that (1.1) is a consequence of the following

**Theorem 2.1** Under the assumptions of Theorem 1.1, the following dispersive estimates hold true for all $0 < h \ll 1, t > 0$:

$$\|F(t)\|_{L^1 \to L^\infty} \leq Ct^{-n/2},$$  

$$\|e^{it G_0} (h^2 G) - e^{it G_0} (h^2 G_0) - F(t) \psi (h^2 G_0)\|_{L^1 \to L^\infty} \leq Ch^\beta t^{-n/2},$$  

with some constants $C, \beta > 0$ independent of $t$ and $h$.

The estimate (2.1) is proved in [2] for potentials $V \in C^k_b(\mathbb{R}^n), \ \delta > n - 1, \ k > \frac{n-3}{2}$. In what follows we will derive (2.2) from the semi-classical expansion obtained in [1] and based on the following semi-classical version of Duhamel’s formula

$$e^{it G_0} (h^2 G) = Q(h) e^{it G_0} \psi_1 (h^2 G_0) \psi (h^2 G) + i \int_0^t Q(h) \psi_1 (h^2 G_0) e^{i(t-\tau)G_0} V e^{i \tau G_0} \psi (h^2 G) d\tau, \quad (2.3)$$

where $\psi_1 \in C^\infty_0((0, +\infty)), \ \psi_1 = 1$ on supp $\psi$, and

$$Q(h) = (1 + \psi_1 (h^2 G_0) - \psi_1 (h^2 G))^{-1}.$$  

Iterating (2.3) $m$ times we get

$$e^{it G_0} (h^2 G) = \sum_{j=0}^m T_j (t, h) + \int_0^t R_m(t-\tau, h) e^{i \tau G_0} \psi (h^2 G) d\tau, \quad (2.4)$$

where the operators $R_j$ are defined as follows

$$R_0(t, h) = iQ(h) e^{iG_0} \psi_1 (h^2 G_0) V,$$

$$R_j(t, h) = \int_0^t R_{j-1}(t-\tau, h) R_0(\tau, h) d\tau, \quad j \geq 1.$$  

The following dispersive estimates are proved in [1] (see Theorem 1.3) (it is easy to see that the $\epsilon$ there can be taken zero).
**Proposition 2.2** Assume that $V$ satisfies (1.2) with $\delta > \frac{n+2}{2}$. Then the following dispersive estimates hold true for all $t > 0$, $0 < h \ll 1$,

$$\|T_j(t, h)\|_{L^1 \to L^\infty} \leq C_j h^{j-n/2} t^{-n/2}, \quad j \geq 1,$$

$$\left\| e^{itG} \psi(h^2 G) - \sum_{j=0}^m T_j(t, h) \right\|_{L^1 \to L^\infty} \leq C_m h^{m+1-n/2} t^{-n/2}, \quad m \geq 1. \tag{2.6}$$

We have (e.g. see Lemma A.1 of [11])

$$\psi_1(h^2 G_0) - \psi_1(h^2 G) = O(h^2) : L^1 \to L^1,$$

so

$$Q(h) = Id + O(h^2) : L^1 \to L^1.$$

Therefore,

$$\left\| T_0(t, h) - e^{itG_0} \psi(h^2 G_0) \right\|_{L^1 \to L^\infty} \leq Ch^2 t^{-n/2}. \tag{2.7}$$

Clearly, the estimate (2.2) follows from combining Proposition 2.2, (2.7) and the following

**Proposition 2.3** Given a $\delta > n - 1$, there exists a sequence $\{k_n\}_{n=1}^\infty$, $\frac{n-3}{2} \leq k_n < \frac{n}{2}$, so that if $V \in C^k_\delta (\mathbb{R}^n)$, $k > k_n$, then we have the estimates

$$\left\| T_1(t, h) - F(t) \psi(h^2 G_0) \right\|_{L^1 \to L^\infty} \leq Ch^\beta t^{-n/2}, \tag{2.8}$$

$$\| T_j(t, h) \|_{L^1 \to L^\infty} \leq Ch^\beta t^{-n/2}, \quad 2 \leq j \leq n/2, \tag{2.9}$$

with some constants $C, \beta > 0$ independent of $h$ and $t$.

### 3 Proof of Proposition 2.3

Set

$$\tilde{T}_1(t, h) = T_1(t, h) - i \int_0^t e^{i(t-\tau)G_0} \psi_1(h^2 G_0) We_{irG_0} \psi(h^2 G_0) d\tau.$$

It is proved in [2] (see Proposition 2.6) that if $V \in C^k_\delta (\mathbb{R}^n)$ with $\delta > n - 1$, $k > (n-3)/2$, then

$$\left\| i \int_0^t e^{i(t-\tau)G_0} \psi_1(h^2 G_0) We_{irG_0} \psi(h^2 G_0) d\tau - F(t) \psi(h^2 G_0) \right\|_{L^1 \to L^\infty} \leq Ch^\beta t^{-n/2} \tag{3.1}$$

with constants $C, \beta > 0$ independent of $h$ and $t$. Therefore, to prove (2.8) it suffices to show that the operator $\tilde{T}_1$ satisfies the estimate

$$\left\| \tilde{T}_1(t, h) \right\|_{L^1 \to L^\infty} \leq Ch^\beta t^{-n/2} \tag{3.2}$$

with constants $C, \beta > 0$ independent of $h$ and $t$. It is easy also to see that the operators $T_j$, $j \geq 2$, are of the form $Q(h) T_j(t, h) \psi(h^2 G)$. Therefore, it suffices to prove (2.9) with $T_j$ replaced by $\tilde{T}_j$. 

4
Let \( \rho \in C^\infty_0(\mathbb{R}^n) \), \( \rho \geq 0 \), be a real-valued function such that \( \int \rho(x)dx = 1 \), and set \( \rho_\theta(x) = \theta^{-n}\rho(x/\theta) \), where \( 0 < \theta \leq 1 \). Let \( V \in C^k_b(\mathbb{R}^n) \) with \( \delta > n - 1 \), where \( k \) will be fixed later on such that \( \frac{n-\delta}{2} < k < \frac{n}{2} \). Set \( \tilde{V}_\theta = V \ast \rho_\theta \). It is easy to see that we have the bounds

\[
|\tilde{V}_\theta(x)| \leq C\langle x \rangle^{-\delta}, \quad \forall x \in \mathbb{R}^n, \quad (3.3)
\]

\[
|\tilde{V}_\theta(x) - V(x)| \leq C\langle x \rangle^{-\delta}, \quad \forall x \in \mathbb{R}^n, \quad (3.4)
\]

\[
|\partial_x^\alpha \tilde{V}_\theta(x)| \leq C\langle x \rangle^{-\delta}, \quad \forall x \in \mathbb{R}^n, |\alpha| \leq k_0, \quad (3.5)
\]

\[
|\partial_x^\alpha \tilde{V}_\theta(x)| \leq C_\alpha \theta^{k-|\alpha|}\langle x \rangle^{-\delta}, \quad \forall x \in \mathbb{R}^n, |\alpha| \geq k_0 + 1, \quad (3.6)
\]

where \( k-1 < k_0 \leq k \) is an integer. Let us also see that

\[
\left\| \tilde{V}_\theta \right\|_{L^1} \leq C \theta^{k-n/2-\epsilon}, \quad \forall 0 < \epsilon \ll 1. \quad (3.7)
\]

Since \( C^k_b(\mathbb{R}^n) \subset H^{k-\epsilon/2}(\mathbb{R}^n) \), \( \forall 0 < \epsilon \ll 1 \), we have \( \langle \xi \rangle^{k-\epsilon/2}\tilde{V}(\xi) \in L^2(\mathbb{R}^n) \). Hence \( \tilde{V} \in L^p(\mathbb{R}^n) \), where \( \frac{1}{p} = \frac{1}{2} + \frac{k}{n+\epsilon} \). We have

\[
\left\| \tilde{V}_\theta \right\|_{L^1} = \left\| \tilde{V}_{\rho_\theta} \right\|_{L^1} \leq \left\| \tilde{V} \right\|_{L^p} \left\| \rho \right\|_{L^q} = C \theta^{-n/q},
\]

where \( \frac{1}{q} = \frac{1}{2} - \frac{k}{n+\epsilon} \), which clearly implies (3.7).

Let \( G_\theta \) denote the self-adjoint realization of \( -\Delta + V_\theta(x) \) on \( L^2(\mathbb{R}^n) \). Denote also by \( Q_\theta(h) \) the operator obtained by replacing in the definition of \( Q(h) \) the operator \( G \) by \( G_\theta \). Define the operators \( \tilde{T}_{j,\theta} \) by replacing \( Q(h) \) and \( V \) by \( Q_\theta(h) \) and \( V_\theta \), respectively, in the definition of \( \tilde{T}_j \). In the case of \( \tilde{T}_1 \) we replace only those \( V \) and \( Q(h) \) staying between the operators \( e^{i(t-\tau)G_0} \) and \( e^{i\tau G_0} \). Using (3.3) and (3.4) we will prove the following

**Proposition 3.1** The following dispersive estimates hold for all \( t > 0, 0 < h \ll 1, 0 < \theta \leq 1 \),

\[
\left\| \tilde{T}_j(t,h) - \tilde{T}_{j,\theta}(t,h) \right\|_{L^1 \rightarrow L^\infty} \leq C \theta^{1/2} h^{j-\epsilon/2} t^{-n/2}, \quad 1 \leq j \leq n/2. \quad (3.8)
\]

**Proof.** We write

\[
\tilde{T}_1(t,h) - \tilde{T}_{1,\theta}(t,h)
\]

\[
= iQ(h) \int_0^t e^{i(t-\tau)G_0} \psi_1(h^2 G_0) (V Q(h) - V_\theta Q_\theta(h)) e^{i\tau G_0} \psi_1(h^2 G_0) d\tau \left( \psi(h^2 G) - \psi(h^2 G_0) \right)
\]

\[
+ i\left( Q(h) - 1 \right) \int_0^t e^{i(t-\tau)G_0} \psi_1(h^2 G_0) (V Q(h) - V_\theta Q_\theta(h)) e^{i\tau G_0} \psi(h^2 G_0) d\tau
\]

\[
+ i \int_0^t e^{i(t-\tau)G_0} \psi_1(h^2 G_0) (V (Q(h) - 1) - V_\theta (Q_\theta(h) - 1)) e^{i\tau G_0} \psi(h^2 G_0) d\tau
\]

\[
= : \sum_{j=1}^3 P_j(t,h). \quad (3.9)
\]

Define the operators \( F_j(t,h), F_{j,\theta}(t,h), j = 0, 1, \ldots \), by

\[
F_0(t,h) = F_{0,\theta}(t,h) = e^{it G_0} \psi_1(h^2 G_0),
\]

5
Therefore, (3.14) follows from combining (3.13), (3.15) and the bound

\[ F_j(t, h) = i \int_0^t F_0(t - \tau, h) VQ(h) F_{j-1}(\tau, h) d\tau, \quad j \geq 1, \]

\[ F_{j,\theta}(t, h) = i \int_0^t F_0(t - \tau, h) V\theta Q\theta(h) F_{j-1,\theta}(\tau, h) d\tau, \quad j \geq 1. \]

Clearly, \( \tilde{T}_j = F_j, \tilde{T}_{j,\theta} = F_{j,\theta} \) for \( j \geq 2 \). We write

\[ F_j(t, h) - F_{j,\theta}(t, h) = i \int_0^t F_0(t - \tau, h) (VQ(h) - V\theta Q\theta(h)) F_{j-1,\theta}(\tau, h) d\tau \]

\[ + i \int_0^t F_0(t - \tau, h) VQ(h) (F_{j-1}(\tau, h) - F_{j-1,\theta}(\tau, h)) d\tau. \] (3.10)

Let us see that (3.8) follows from the following estimates

**Proposition 3.2** For all \( t > 0, 0 < h \ll 1, 0 < \theta \leq 1, 1/2 - \epsilon/2 \leq s \leq (n-1)/2, 0 < \epsilon \ll 1, j \geq 0 \), we have the estimates

\[ \| (x)^{-1/2-s-\epsilon} F_j(t, h) \|_{L^1 \to L^2} \leq C_j h^{j+s-(n-1)/2} t^{-s-1/2}, \] (3.11)

\[ \| (\langle x \rangle)^{-1/2-s-\epsilon} (F_j(t, h) - F_{j,\theta}(t, h)) \|_{L^1 \to L^2} \leq C_j \theta^{1/2} h^{j+s-(n-1)/2} t^{-s-1/2}. \] (3.12)

**Remark.** The estimate (3.11) with \( j = 0 \) holds true for all \( t \neq 0 \). In other words, the adjoint of the operator

\[ A = F_0(t, h)\langle x \rangle^{-1/2-s-\epsilon} : L^2 \to L^\infty \]

satisfies (3.11) with \( j = 0 \), and hence so does \( A \). This will be often used below.

We need the following

**Lemma 3.3** For all \( 0 < h \leq h_0, 0 < \theta \leq 1, 0 \leq s \leq \delta \), we have the bounds

\[ \| (\langle x \rangle)^{-s} Q(h) \langle x \rangle^s \|_{L^2 \to L^2} \leq C, \] (3.13)

\[ \| (\langle x \rangle)^{-s} (Q(h) - Q\theta(h)) \langle x \rangle^s \|_{L^2 \to L^2} \leq C\theta^{1/2}, \] (3.14)

with constants \( C, h_0 > 0 \) independent of \( h \) and \( \theta \).

**Proof.** Clearly, (3.13) follows from the bound

\[ \| (\langle x \rangle)^{-s} \left( \psi_1(h^2 G) - \psi_1(h^2 G_0) \right) \langle x \rangle^s \|_{L^2 \to L^2} \leq C h^2, \] (3.15)

proved in [16] (see Lemma 2.3). To prove (3.14) we write

\[ Q(h) - Q\theta(h) = \left( \psi_1(h^2 G_\theta) - \psi_1(h^2 G) \right) Q(h) + \left( \psi_1(h^2 G_\theta) - \psi_1(h^2 G_0) \right) \left( Q(h) - Q\theta(h) \right). \]

Therefore, (3.14) follows from combining (3.13), (3.15) and the bound

\[ \| (\langle x \rangle)^{-s} \left( \psi_1(h^2 G) - \psi_1(h^2 G_\theta) \right) \langle x \rangle^s \|_{L^2 \to L^2} \leq C\theta^{1/2} h^2. \] (3.16)
To prove (3.16) we will use the Helffer-Sjöstrand formula
\[
\psi_1(h^2 G) = \frac{2}{\pi} \int_C \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) (h^2 G - z^2)^{-1} z L(dz),
\] (3.17)
where \( L(dz) \) denotes the Lebesgue measure on \( C, \tilde{\varphi} \in \mathcal{C}_0^\infty(C) \) is an almost analytic continuation of \( \varphi(\lambda) = \psi_1(\lambda^2) \), supported in a small complex neighbourhood of \( \text{supp} \varphi \) and satisfying
\[
\left| \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) \right| \leq C_N |\text{Im} z|^N, \quad \forall N \geq 1.
\]
In view of (3.17) we can write
\[
\psi_1(h^2 G) - \psi_1(h^2 G_\theta) = \frac{2h^2}{\pi} \int_C \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) (h^2 G_\theta - z^2)^{-1} (V_\theta - V)(h^2 G - z^2)^{-1} z L(dz). \tag{3.18}
\]
It is shown in [16] (see the proof of Lemma 2.3) that the free resolvent satisfies the bound (for \( z \in \text{supp} \tilde{\varphi} \))
\[
\left\| \langle x \rangle^{-s} (h^2 G_0 - z^2)^{-1} \langle x \rangle^s \right\|_{L^2 \to L^2} \leq C_1 |\text{Im} z|^{-q}, \quad \text{Im} z \neq 0, \tag{3.19}
\]
with constants \( C_1, q > 0 \) independent of \( z \) and \( h \). By (3.19) and the identity
\[
(h^2 G - z^2)^{-1} = (h^2 G_\theta - z^2)^{-1} - \frac{h^2}{(h^2 G_\theta - z^2)^{-1}} V (h^2 G_\theta - z^2)^{-1},
\]
we obtain (for \( z \in \text{supp} \tilde{\varphi}, 0 \leq s \leq \delta \))
\[
\left\| \langle x \rangle^{-s} (h^2 G - z^2)^{-1} \langle x \rangle^s \right\|_{L^2 \to L^2} \leq \left\| \langle x \rangle^{-s} (h^2 G_0 - z^2)^{-1} \langle x \rangle^s \right\|_{L^2 \to L^2} + C h^2 \left\| (h^2 G - z^2)^{-1} \right\|_{L^2 \to L^2} \left\| \langle x \rangle^{-s} (h^2 G_0 - z^2)^{-1} \langle x \rangle^s \right\|_{L^2 \to L^2}
\leq C_2 |\text{Im} z|^{-q-1}, \quad \text{Im} z \neq 0. \tag{3.20}
\]
By (3.4), (3.18) and (3.20),
\[
\left\| \langle x \rangle^{-s} \left( \psi_1(h^2 G) - \psi_1(h^2 G_\theta) \right) \langle x \rangle^s \right\|_{L^2 \to L^2} \leq C \theta^{1/2} h^2 \int_C \left| \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) \right| \left\| (h^2 G_\theta - z^2)^{-1} \right\|_{L^2 \to L^2} \left\| \langle x \rangle^{-s} (h^2 G - z^2)^{-1} \langle x \rangle^s \right\|_{L^2 \to L^2} \left( h d(z) \right)
\leq C \theta^{1/2} h^2 \int_C \left| \frac{\partial \tilde{\varphi}}{\partial \bar{z}}(z) \right| |\text{Im} z|^{-q-2} \left( h d(z) \right) \leq C \theta^{1/2} h^2. \tag{3.21}
\]
Using (3.3), (3.4), Lemma 3.3 and (3.11) with \( j = 0 \), we obtain
\[
\left\| \mathcal{P}_\theta^{(3)} (t, h) \right\|_{L^1 \to L^\infty} \leq C \theta^{1/2} \int_0^{t/2} \left\| e^{i(t-\tau)G_0} \psi_1(h^2 G_0) \langle x \rangle^{-n/2-\epsilon} \right\|_{L^2 \to L^\infty} \left\| \langle x \rangle^{-1-\epsilon} e^{i\tau G_0} \psi(h^2 G_0) \right\|_{L^1 \to L^2} d\tau + C \theta^{1/2} \int_{t/2}^t \left\| e^{i(t-\tau)G_0} \psi_1(h^2 G_0) \langle x \rangle^{-1-\epsilon} \right\|_{L^2 \to L^\infty} \left\| \langle x \rangle^{-n/2-\epsilon} e^{i\tau G_0} \psi(h^2 G_0) \right\|_{L^1 \to L^2} d\tau
\]
we have the estimate $C\theta^{1/2}t^{-n/2}h^{1-n/2-\epsilon/2} \int_0^h \tau^{-1+\epsilon/2}d\tau + C\theta^{1/2}t^{-n/2}h^{1-n/2+\epsilon/2} \int_h^\infty \tau^{-1-\epsilon/2}d\tau 
leq C\theta^{1/2}t^{-n/2}h^{1-n/2}.

Clearly, the $L^1 \rightarrow L^\infty$ norm of the operators $\mathcal{P}_\theta^{(j)}$, $j = 1, 2$, can be bounded in the same way. Let now $j \geq 2$. Using (3.10), Proposition 3.2 and Lemma 3.3, we obtain

$$
\|F_j(t, h) - F_j,\theta(t, h)\|_{L^1 \rightarrow L^\infty} 
\leq C\theta^{1/2} \int_0^{t/2} \left\| F_0(t - \tau, h)\langle x\rangle^{-n/2-\epsilon} \right\|_{L^2 \rightarrow L^\infty} \left\| \langle x\rangle^{-1-\epsilon} F_j(t, h) \right\|_{L^1 \rightarrow L^2} d\tau 
+ C\theta^{1/2} \int_{t/2}^t \left\| F_0(t - \tau, h)\langle x\rangle^{-1-\epsilon} \right\|_{L^2 \rightarrow L^\infty} \left\| \langle x\rangle^{-n/2-\epsilon} F_j(t, h) \right\|_{L^1 \rightarrow L^2} d\tau 
+ C \int_0^{t/2} \left\| F_0(t - \tau, h)\langle x\rangle^{-n/2-\epsilon} \right\|_{L^2 \rightarrow L^\infty} \left\| \langle x\rangle^{-1-\epsilon} (F_j(t, h) - F_j,\theta(t, h)) \right\|_{L^1 \rightarrow L^2} d\tau 
+ C \int_{t/2}^t \left\| F_0(t - \tau, h)\langle x\rangle^{-1-\epsilon} \right\|_{L^2 \rightarrow L^\infty} \left\| \langle x\rangle^{-n/2-\epsilon} (F_j(t, h) - F_j,\theta(t, h)) \right\|_{L^1 \rightarrow L^2} d\tau 
\leq C\theta^{1/2}t^{-n/2}h^j\langle x\rangle^{-n/2-\epsilon/2} \int_0^h \tau^{-1+\epsilon/2}d\tau + C\theta^{1/2}t^{-n/2}h^j\langle x\rangle^{-n/2+\epsilon/2} \int_h^\infty \tau^{-1-\epsilon/2}d\tau 
\leq C\theta^{1/2}t^{-n/2}h^j\langle x\rangle^{-n/2}.
\]

\[\Box\]

**Proof of Proposition 3.2.** The estimate (3.11) with $j = 0$ is proved in [15] (see (2.1)). By induction in $j$, it is easy to see that (3.11) for any $j$ follows from this and the following well-known estimate

$$
\left\| \langle x\rangle^{-s} e^{itG_0} \psi_1(h^2G_0)\langle x\rangle^{-s} \right\|_{L^2 \rightarrow L^2} \leq C(t/h)^{-s}, \quad s \geq 0. \tag{3.22}
$$

Similarly, using (3.10) together with Lemma 3.3, one can easily get (3.12). \[\Box\]

### 4 Study of the operators $\widetilde{T}_{j,\theta}$

We will first show that the estimates (2.9) and (3.2) follow from Proposition 3.1 and the following

**Proposition 4.1** Let $V \in C^k_\delta(R^n)$ with $\delta > n - 1$, $\frac{n-3}{2} < k < \frac{n}{2}$. Then, there exist a constant $\varepsilon_0 > 0$ and a sequence $\{p_j\}_{j=1}^\infty$, $p_j > 0$, depending on $\delta$ but independent of $k$, so that for all $0 < h \ll 1$, $0 < \theta \leq 1$, $0 < \varepsilon \ll 1$, $t > 0$, $j \geq 1$, satisfying

$$
h^2 \theta^{k-n/2-\epsilon} \ll 1, \tag{4.1}
$$

we have the estimate

$$
\left\| \widetilde{T}_{j,\theta}(t, h) \right\|_{L^1 \rightarrow L^\infty} \leq C_j h^{\varepsilon_0} t^{-n/2} + C_{j,\epsilon} h^{p_j} \theta^{-j(n/2-k+\epsilon)} t^{-n/2}, \tag{4.2}
$$

where $C_j, C_{j,\epsilon} > 0$ are independent of $t$, $h$ and $\theta$. 

8
Fix an integer $1 \leq j \leq n/2$. Take $\theta = h^{n+1-2j}$ and set
\[ k_n^{(j)} = \frac{n}{2} - \min \left\{ \frac{3}{2}, \frac{p_j}{\beta} \right\} \frac{k}{n + 1 - 2j}. \]

It is easy to see that if $k_n^{(j)} < k < n/2$ and $\epsilon$ is taken small enough, we can arrange (4.1), and the estimates (3.8) and (4.2) imply
\[ \| \tilde{T}_j(t, h) \|_{L^1 \to L^\infty} \leq C h^\beta t^{-n/2}, \tag{4.3} \]
with some $C, \beta > 0$. Thus, taking
\[ k_n = \max_{1 \leq j \leq n/2} k_n^{(j)} \]
we get the desired result.

**Proof of Proposition 4.1.** We need the following

**Lemma 4.2** For all $0 < \theta \leq 1$, $0 < \epsilon \ll 1$, $t \in \mathbb{R}$, we have the estimate
\[ \left\| e^{-itG_0 V_\theta e^{itG_0}} \right\|_{L^1 \to L^1} \leq C \theta^{k-n/2-\epsilon}, \tag{4.4} \]
with a constant $C_\epsilon > 0$ independent of $t$ and $\theta$. Moreover, given any integer $m \geq 1$, the operator $Q_\theta(h)$ can be decomposed as $P_m^{(1)}(h, \theta) + P_m^{(2)}(h, \theta)$, where the operator $P_m^{(1)}$ satisfies the estimate
\[ \left\| e^{-itG_0 P_m^{(1)}(h, \theta)e^{itG_0}} \right\|_{L^1 \to L^1} \leq 2, \tag{4.5} \]
for all $t \in \mathbb{R}$ and all $0 < h \ll 1$, $0 < \theta \leq 1$ such that (4.1) holds, while the operator $P_m^{(2)}$ satisfies the estimate
\[ \left\| (x)^{-s} P_m^{(2)}(h, \theta)(x)^s \right\|_{L^2 \to L^2} \leq C_m h^{2m+2}, \quad 0 \leq s \leq \delta, \tag{4.6} \]
where $C_m > 0$ is independent of $h$ and $\theta$.

**Proof.** The estimate (4.4) follows from (3.7) and the following estimate proved in [9]:
\[ \left\| e^{-itG_0 V_\theta e^{itG_0}} \right\|_{L^1 \to L^1} \leq \left\| \tilde{V}_\theta \right\|_{L^1}. \tag{4.7} \]
To decompose the operator $Q_\theta(h)$ we will use the formula (3.17) together with the resolvent identity
\[
\begin{align*}
(h^2 G_\theta - z^2)^{-1} - (h^2 G_0 - z^2)^{-1} &= \sum_{j=1}^{m} \left( h^2 G_0 - z^2 \right)^{-1} \left( -h^2 V_\theta \left( h^2 G_0 - z^2 \right)^{-1} \right)^j \\
+ (h^2 G_\theta - z^2)^{-1} \left( -h^2 V_\theta \left( h^2 G_0 - z^2 \right)^{-1} \right)^{m+1} := \sum_{\ell=1}^{2} M_m^{(\ell)}(z, h, \theta).
\end{align*}
\]

Set
\[ M_m^{(\ell)}(h, \theta) = \frac{2}{\pi} \int_{C} \frac{\partial \tilde{\varphi}}{\partial z}(z) M_m^{(\ell)}(z, h, \theta) z L(dz), \]

9
\[ P_m^{(1)}(h, \theta) = \left(1 - M_m^{(1)}(h, \theta)\right)^{-1}, \]
\[ P_m^{(2)}(h, \theta) = \left(1 - M_m^{(1)}(h, \theta) - M_m^{(2)}(h, \theta)\right)^{-1} - \left(1 - M_m^{(1)}(h, \theta)\right)^{-1}. \]

By (3.7) and (4.9), we conclude
\[ \left\| e^{-itG_0} M_m^{(1)}(z, h, \theta) e^{itG_0} \right\|_{L^1 \to L^1} \leq \sum_{j=1}^{m} C_j \left\| h^2 \tilde{V}_\theta \right\|_{L^1} \left(1 + \left\| h^2 \tilde{V}_\theta \right\|_{L^1}\right)^{m-1} \]
\[ \leq C_m h^2 \theta^{k-n/2-\epsilon} \leq 1/2, \tag{4.10} \]
provided (4.1) is satisfied. Clearly, (4.5) follows from (4.10). On the other hand, it is easy to see that (4.6) follows from the estimates
\[ \left\| \langle x \rangle^{-s} M_m^{(1)}(h, \theta) \langle x \rangle \right\|_{L^2 \to L^2} \leq C_m h^2, \tag{4.11} \]
\[ \left\| \langle x \rangle^{-s} M_m^{(2)}(h, \theta) \langle x \rangle \right\|_{L^2 \to L^2} \leq C_m h^{2m+2}, \tag{4.12} \]
which in turn follow from (3.19) and (3.20) (which clearly holds with \( G \) replaced by \( G_\theta \)).

Define the operators \( \tilde{T}_{j,\theta} \) by replacing in the definition of \( \tilde{T}_{j,\theta} \) the operator \( Q_\theta(h) \) by \( P_m^{(1)}(h, \theta) \). In precisely the same way as in the proof of (3.8) above, using (4.6) instead of (3.14), we get
\[ \left\| \tilde{T}_{j,\theta}(t, h) - \tilde{T}_{j,\theta}(t, h) \right\|_{L^1 \to L^\infty} \leq Ch^{2m+2j-n/2} t^{-n/2} \leq C h t^{-n/2}, \tag{4.13} \]
provided \( m \) is taken big enough. Therefore, it suffices to prove (4.2) with \( \tilde{T}_{j,\theta} \) replaced by \( \tilde{T}_{j,\theta} \). We will first do so for \( j = 1 \). Let \( 0 < \gamma \ll 1 \) be a parameter to be fixed later on, depending on \( h \). For \( t \geq 2\gamma \), we have
\[ \left\| \tilde{T}_{1,\theta}(t, h) \right\|_{L^1 \to L^1} \leq Ch^2 \left( \int_0^{\gamma} e^{i(t-\tau)G_0} \psi_1(h^2 G_0) V_\theta P_m^{(1)}(h, \theta) e^{i\tau G_0} \psi_1(h^2 G_0) d\tau \right) \]
\[ + C \left( \int_0^{\gamma} e^{i(t-\tau)G_0} \psi_1(h^2 G_0) V_\theta \left( P_m^{(1)}(h, \theta) - 1 \right) e^{i\tau G_0} \psi_1(h^2 G_0) d\tau \right) \]
\[ \leq Ct^{-n/2} \left( \int_0^{\gamma} + \int_{t-\gamma}^{t} \right) \left( \left\| e^{-i\tau G_0} V_\theta P_m^{(1)}(h, \theta) e^{i\tau G_0} \right\|_{L^1 \to L^1} + \left\| e^{-i\tau G_0} V_\theta e^{i\tau G_0} \right\|_{L^1 \to L^1} \right) d\tau \]
where we have used Lemma 4.2 together with (4.11) and (3.11) (with $j = 0$). Clearly, (4.14) still holds for $0 < t \leq 2 \gamma$. Choosing $\gamma$ such that $\gamma^{-n/2-2} = h^{-1/2}$, we deduce the desired estimate from (4.14).

Let now $j \geq 2$. Then $\mathcal{T}_{j,\theta}^2 = F_{j,\theta}^2$, where the operators $F_{j,\theta}^q$, $j = 0, 1, \ldots$, are defined as follows

$$F_{0,\theta}(t, h) = F_0(t, h) = e^{itG_0} \psi_1(h^2 G_0),$$

$$F_{j,\theta}(t, h) = i \int_0^t F_0(t - \tau, h) V_\theta P_m^{(1)}(h, \theta) F_{j-1,\theta}(\tau, h) d\tau, \quad j \geq 1.$$ 

Let $0 < \gamma \ll 1$ be a parameter to be fixed later on, depending on $h$. By Lemma 4.2, for $0 < t \leq 2j\gamma$, $j \geq 1$, we get

$$\left\| F_{j,\theta}^q(t, h) \right\|_{L^1 \rightarrow L^1} \leq C j^{q} \theta^{j(k-n/2-\epsilon)} t^{-n/2} \int_0^{2j \gamma} \left\| e^{-itG_0} F_{j-1,\theta}^q(\tau, h) \right\|_{L^1 \rightarrow L^1} d\tau. \quad (4.15)$$

By induction in $j$, it is easy to see that we have the bound

$$\left\| e^{-itG_0} F_{j,\theta}^q(t, h) \right\|_{L^1 \rightarrow L^1} \leq C j^{q} \theta^{j(k-n/2-\epsilon)}, \quad \forall j \geq 0. \quad (4.16)$$

Clearly, (4.16) is trivial for $j = 0$. Suppose that it holds for $j - 1$. By Lemma 4.2 we have

$$\left\| e^{-itG_0} F_{j,\theta}^q(t, h) \right\|_{L^1 \rightarrow L^1} \leq C j^{q} \theta^{j(k-n/2-\epsilon)} \int_0^t \left\| e^{-itG_0} F_{j-1,\theta}^q(\tau, h) \right\|_{L^1 \rightarrow L^1} d\tau$$

$$\leq C j^{q} \theta^{j(k-n/2-\epsilon)} \int_0^t \tau^{j-1} d\tau = C j^{q} \theta^{j(k-n/2-\epsilon)},$$

which proves (4.16) for $j$. By (4.15) and (4.16), we conclude

$$\left\| F_{j,\theta}^q(t, h) \right\|_{L^1 \rightarrow L^\infty} \leq C j \gamma^j \theta^{j(k-n/2-\epsilon)} t^{-n/2}, \quad 0 < t \leq 2j\gamma; \quad (4.17)$$

with a constant $C_j > 0$ independent of $t$, $h$, $\theta$ and $\gamma$. We would like to obtain a similar estimate when $t \geq 2j\gamma$. To this end, decompose $F_{j,\theta}^q$ as follows

$$F_{j,\theta}^q(t, h) = i \left( \int_0^\gamma + \int_{\gamma}^{t-\gamma} + \int_{t-\gamma}^t \right) F_0(t - \tau, h) V_\theta P_m^{(1)}(h, \theta) F_{j-1,\theta}^q(\tau, h) d\tau =: \sum_{\ell=1}^3 E_{j,\theta}^{(\ell)}(t, h, \gamma).$$

By Lemma 4.2 and (4.16),

$$\left\| E_{j,\theta}^{(1)}(t, h, \gamma) \right\|_{L^1 \rightarrow L^\infty} \leq C j \gamma^j \theta^{j(k-n/2-\epsilon)} t^{-n/2}. \quad (4.18)$$
Clearly, the estimate (3.11) holds with $F_j$ replaced by $F^{\nu}_{j,\theta}$. Using this we obtain

$$
\| E^{(2)}_{j,\theta}(t, h, \gamma) \|_{L^1 \to L^\infty} \leq C \int_{\gamma}^{t/2} \| F_0(t - \tau, h) \langle x \rangle^{-n/2 - \epsilon'} \|_{L^2 \to L^\infty} \| \langle x \rangle^{-n/2 + 1 - \epsilon'} F^{\nu}_{j-1,\theta}(\tau, h) \|_{L^1 \to L^2} d\tau
$$

$$
+ C \int_{t/2}^{t-\gamma} \| F_0(t - \tau, h) \langle x \rangle^{-n/2 + 1 - \epsilon'} \|_{L^2 \to L^\infty} \| \langle x \rangle^{-n/2 + 1 - \epsilon'} F^{\nu}_{j-1,\theta}(\tau, h) \|_{L^1 \to L^2} d\tau
$$

$$
\leq Ch^{j-2+\epsilon'/2 \tau - n/2} \int_{\gamma}^{\infty} \tau^{-n/2 + 1 - \epsilon'} \| C_j \gamma^{-n/2 + 2 - \epsilon'} h^{j-2+\epsilon'/2 \tau - n/2},
$$

(4.19)

with constants $C_j, \epsilon' > 0$ independent of $t, h, \theta$ and $\gamma$ ($\epsilon'$ depending only on $\delta$). Similarly, using (3.22), we also get

$$
\| \langle x \rangle^{-n/2 - \epsilon'} E^{(2)}_{j,\theta}(t, h, \gamma) \|_{L^1 \to L^2} \leq C_j \gamma^{-n/2 + 2 - \epsilon'} h^{n/2 + j-2+\epsilon'/2 \tau - n/2}.
$$

(4.20)

In what follows we will show that the operator $E^{(3)}_{j,\theta}$ satisfies the estimate

$$
\| E^{(3)}_{j,\theta}(t, h, \gamma) \|_{L^1 \to L^\infty} \leq C_j \gamma^{-n/2 + 2 - \epsilon'} h^{j-2+\epsilon'/2 \tau - n/2} + C_j \gamma^{j} \| \theta \|^{(k-n/2-\epsilon')} \tau - n/2, \quad t \geq 2j\gamma.
$$

(4.21)

To this end, it suffices to show that modulo operators satisfying (4.19), the operator $E^{(3)}_{j,\theta}$ is a finite sum of operators of the form

$$
\int_{I_1} \int_{I_2} \ldots \int_{I_j} F_0(t - \tau_1, h) V_\theta P_m^{(1)}(h, \theta) F_0(\tau_1 - \tau_2, h) V_\theta P_m^{(1)}(h, \theta) \ldots V_\theta P_m^{(1)}(h, \theta) F_0(\tau_j, h) d\tau_1 d\tau_2 \ldots d\tau_j,
$$

(4.22)

where $I_\nu, \nu = 1, \ldots, j$, are intervals of length $|I_\nu| = O(\gamma)$. Indeed, by Lemma 4.2 an operator of form (4.22) satisfies (4.18). We will show that given any integer $1 \leq \nu \leq j - 1$, the operator $E^{(3)}_{j,\theta}$ can be written in the form

$$
\int_{\gamma}^{\infty} \ldots \int_{\gamma}^{\infty} F_0(\tau_1, h) V_\theta P_m^{(1)}(h, \theta) \ldots F_0(\tau_{\nu}, h) V_\theta P_m^{(1)}(h, \theta) E^{(3)}_{j-\nu,\theta}(t - \tau_1 - \ldots - \tau_{\nu}, h, \gamma) d\tau_1 \ldots d\tau_{\nu},
$$

(4.23)

modulo operators satisfying (4.19) and operators of the form (4.22). This would imply the desired result because the operator (4.23) with $\nu = j - 1$ is of the form (4.22). We will proceed by induction in $\nu$. Let us see that the claim holds true for $\nu = 1$. We write

$$
E^{(3)}_{j,\theta}(t, h, \gamma) = \sum_{\ell=1}^{3} \int_{0}^{\gamma} F_0(\tau_1, h) V_\theta P_m^{(1)}(h, \theta) E^{(\ell)}_{j-1,\theta}(t - \tau_1, h) d\tau_1.
$$

(4.24)

Clearly, the first operator in the sum in the right-hand side of (4.24) is of the form (4.22), while the third one is of the form (4.23) with $\nu = 1$. On the other hand, using (3.11) with $j = 0$, $s = 1/2 - \epsilon/2$, together with (4.20), it is easy to see that the second one satisfies (4.19). Suppose now that the claim holds true for some $\nu, 1 \leq \nu \leq j - 2$. Then we decompose the operator in (4.23) as follows

$$
\sum_{\ell=1}^{3} \int_{0}^{\gamma} \ldots \int_{0}^{\gamma} F_0(\tau_1, h) V_\theta P_m^{(1)}(h, \theta) \ldots F_0(\tau_{\nu+1}, h) V_\theta P_m^{(1)}(h, \theta)
$$

12
Clearly, the first operator in the sum in (4.25) is of the form (4.22), while the third one is of
the form (4.23) with \( \nu + 1 \). Therefore, to prove the claim it suffices to show that the second one
satisfies (4.19). However, this follows easily from (4.20) and the following consequence of (3.11)
with \( j = 0 \), \( s = 1/2 - \epsilon/2 \), and (3.22):

\[
\left\| F_0(\tau'_1, h) V_\theta P_m^{(1)}(h, \theta) ... F_0(\tau'_{\nu+1}, h) \langle x \rangle^{-1-\epsilon} \right\|_{L^2 \rightarrow L^\infty} \\
\leq C_{\epsilon, \nu} h^{-n/2-\epsilon/2} (\tau'_1)^{-1+\epsilon/2} (\tau'_2/h)^{-1-\epsilon} ... (\tau'_{\nu+1}/h)^{-1-\epsilon},
\]

for every \( 0 < \epsilon \ll 1 \).

By (4.17), (4.18), (4.19) and (4.21), we conclude that the operators \( F_{j, \theta} \), \( j \geq 2 \), satisfy the
estimate

\[
\left\| F_{j, \theta}^2(t, h) \right\|_{L^1 \rightarrow L^\infty} \leq C_j \gamma^{-n/2+2-\epsilon'} h^{-j+2+\epsilon'/2} t^{-n/2} + C_j \gamma^2 \beta^2 \gamma^2 (k-n/2-\epsilon) t^{-n/2}, \forall t > 0.
\]

Choosing \( \gamma \) such that

\[
\gamma^{-n/2+2-\epsilon'} = h^{-j+2-\epsilon'/4},
\]

we deduce the desired estimate from (4.27). \( \square \)

5 Proof of Theorem 1.2

It is easy to see that Theorem 1.2 follows from the following

**Theorem 5.1** Under the assumptions of Theorem 1.2, there exist an integer \( m \) and constants
\( C, \beta > 0 \) such that the following dispersive estimates hold true for all \( 0 < h \ll 1 \), \( t > 0 \):

\[
\| F_j(t) \|_{L^1 \rightarrow L^\infty} \leq C t^{-n/2}, \quad 1 \leq j \leq m,
\]

\[
\left\| e^{itG} \psi(h^2 G) - \sum_{j=0}^{m} F_j(t) \psi(h^2 G) \right\|_{L^1 \rightarrow L^\infty} \leq C h^\beta t^{-n/2}.
\]

**Proof.** Let us first see that (5.1) holds for all \( j \geq 0 \). It is trivial for \( j = 0 \), while when
\( 0 < t \leq 2 \) it follows from (1.4). Let now \( t \geq 2 \). We will proceed by induction in \( j \). Suppose that
(5.1) holds for \( j - 1 \). This implies

\[
\left\| \int_1^{t-1} F_{j-1}(t-\tau) V F_0(\tau) d\tau \right\|_{L^1 \rightarrow L^\infty} \leq \| V \|_{L^1} \int_1^{t-1} \| F_{j-1}(t-\tau) \|_{L^1 \rightarrow L^\infty} \| F_0(\tau) \|_{L^1 \rightarrow L^\infty} d\tau
\]

\[
\leq C \int_1^{t-1} (t-\tau)^{-n/2} \tau^{-n/2} d\tau \leq C t^{-n/2}.
\]

Clearly, (5.1) for \( j \) follows from (5.3) and (1.5) applied with \( \gamma = 1 \), \( m_1 = j - 1 \), \( m_2 = 0 \).

Let \( 0 < \gamma \leq 1 \) be a parameter to be fixed later on, depending on \( h \). Set \( \gamma_1 = \gamma \) if \( t \geq 2 \gamma \),
\( \gamma_1 = t/2 \) if \( t \leq 2 \gamma \). Iterating Duhamel’s formula we obtain the identity

\[
e^{itG} = \sum_{j=0}^{m_1} F_j(t) + i \int_0^t F_{m_1}(t-\tau) V \left( e^{i\gamma \tau} - e^{i\gamma_2 \tau} \right) d\tau
\]
\[= \sum_{j=0}^{m_1} \mathcal{F}_j(t) + i \int_{\gamma_1}^{t} \mathcal{F}_{m_1}(t - \tau) V \left( e^{i\tau G} - e^{i\tau G_0} \right) d\tau \]

\[+ i \sum_{\nu=1}^{m_2-1} \int_{0}^{\tau_1} \mathcal{F}_{m_1}(t - \tau)V\mathcal{F}_{\nu}(\tau)d\tau + i^2 \int_{0}^{\tau_1} \int_{0}^{\tau} \mathcal{F}_{m_1}(t - \tau)V\mathcal{F}_{m_2}(\tau - s)V e^{isG} dsd\tau, \quad (5.4)\]

for all integers \(m_1, m_2 \geq 2\). Hence

\[\left\| e^{itG} \psi(h^2 G) - \sum_{j=0}^{m} \mathcal{F}_j(t)\psi(h^2 G) \right\|_{L^1 \rightarrow L^\infty} \leq \int_{\gamma_1}^{t} \left\| \mathcal{F}_{m_1}(t - \tau) \right\|_{L^1 \rightarrow L^\infty} \left\| V \left( e^{i\tau G} - e^{i\tau G_0} \right) \psi(h^2 G) \right\|_{L^1 \rightarrow L^1} d\tau \]

\[+ C \sum_{\nu=1}^{m_2-1} \int_{0}^{\tau_1} \left\| \mathcal{F}_{m_1}(t - \tau)V\mathcal{F}_{\nu}(\tau)d\tau \right\|_{L^1 \rightarrow L^\infty} + \int_{0}^{\tau_1} \int_{0}^{\tau} \left\| \mathcal{F}_{m_1}(t - \tau)V\mathcal{F}_{m_2}(\tau - s) \right\|_{L^1 \rightarrow L^1} \left\| V e^{isG} \psi(h^2 G) \right\|_{L^1 \rightarrow L^1} d\tau \]

\[\leq C \int_{\gamma_1}^{t} \left\| \mathcal{F}_{m_1}(t - \tau) \right\|_{L^1 \rightarrow L^\infty} \left\| \langle x \rangle^{-n/2 - \epsilon'} \left( e^{i\tau G} - e^{i\tau G_0} \right) \psi(h^2 G) \right\|_{L^1 \rightarrow L^2} d\tau \]

\[+ C \sum_{\nu=1}^{m_2-1} \int_{0}^{\tau_1} \left\| \mathcal{F}_{m_1}(t - \tau)V\mathcal{F}_{\nu}(\tau)d\tau \right\|_{L^1 \rightarrow L^\infty} + C \int_{0}^{\tau_1} \int_{0}^{\tau} \left\| \mathcal{F}_{m_1}(t - \tau) \right\|_{L^1 \rightarrow L^\infty} \left\| \mathcal{F}_{m_2}(\tau - s) \right\|_{L^1 \rightarrow L^\infty} \left\| \langle x \rangle^{-n/2 - \epsilon'} e^{isG} \psi(h^2 G) \right\|_{L^1 \rightarrow L^2} d\tau. \quad (5.5)\]

On the other hand, it is proved in [15] (see Proposition 4.1) for potentials satisfying (1.2) with \(\delta > \frac{n+2}{2}\) that we have the estimate

\[\left\| \langle x \rangle^{-n/2 - \epsilon'} \left( e^{itG} \psi(h^2 G) - e^{itG_0} \psi(h^2 G_0) \right) \right\|_{L^1 \rightarrow L^2} \leq Ch^{1-\epsilon} t^{-n/2}. \quad (5.6)\]

Hence

\[\left\| \langle x \rangle^{-n/2 - \epsilon'} \left( e^{itG} \psi(h^2 G) - e^{itG_0} \psi(h^2 G_0) \right) \right\|_{L^1 \rightarrow L^2} \leq \left\| \langle x \rangle^{-n/2 - \epsilon'} \left( e^{itG} \psi(h^2 G) - e^{itG_0} \psi(h^2 G_0) \right) \right\|_{L^1 \rightarrow L^2} + C \left\| \psi(h^2 G) - \psi(h^2 G_0) \right\|_{L^1 \rightarrow L^1} \left\| e^{itG_0} \right\|_{L^{1} \rightarrow L^\infty} \leq Ch^{1-\epsilon} t^{-n/2} + Ch^2 t^{-n/2} \leq C'h^{1-\epsilon} t^{-n/2}. \quad (5.7)\]

Using (5.7) together with (1.4) and (5.1) we can bound the first integral in the right-hand side of (5.5) by

\[Ch^{1-\epsilon} \left( \int_{\gamma_1}^{t/2} + \int_{t/2}^{t} \right) \left\| \mathcal{F}_{m_1}(t - \tau) \right\|_{L^1 \rightarrow L^\infty} \tau^{-n/2} d\tau \]

\[\leq Ch^{1-\epsilon} t^{-n/2} \int_{\gamma_1}^{t/2} \tau^{-n/2} d\tau + Ch^{1-\epsilon} t^{-n/2} \int_{t/2}^{t} \left\| \mathcal{F}_{m_1}(t - \tau) \right\|_{L^1 \rightarrow L^\infty} d\tau \]
\[ \leq C h^{1-\epsilon} t^{-n/2} \int_{\gamma}^{\infty} \tau^{-n/2} d\tau + C h^{1-\epsilon} t^{-n/2} \int_{0}^{\infty} \|F_{m}(\tau')\|_{L^{1} \to L^{\infty}} d\tau' \]

\[ \leq C h^{1-\epsilon} \gamma^{-(n-2)/2} t^{-n/2} + C h^{1-\epsilon} t^{-n/2} \leq C' h^{1-\epsilon} \gamma^{-(n-2)/2} t^{-n/2}, \]

provided \( m_1 \) is taken big enough. Furthermore, in view of (1.5), each term in the sum in the right-hand side of (5.5) is bounded by \( C \gamma^{t^{-n/2}} \) for all \( t > 0 \). To bound the last integral we will use the following estimate proved in [15] (see Propositions 2.1 and 4.1)

\[ \int_{0}^{\infty} \tau^{-n/2} d\tau \leq C h^{s-(n-1)/2} |t|^{-s-1/2} \quad (5.8) \]

for every \( 0 < \epsilon \ll 1, 1/2 - \epsilon/4 \leq s \leq (n-1)/2, 0 < h \ll 1, \ t \neq 0 \). Using (5.8) together with (1.4) and (5.1) we bound the integral under question by

\[ C h^{-(n-2)/2-\epsilon/4} t^{-n/2} \int_{0}^{\infty} \int_{0}^{t} (\tau - s)^{\epsilon m_2 - n/2} s^{-1+\epsilon/4} ds d\tau \]

\[ \leq C h^{-(n-2)/2-\epsilon/4} t^{-n/2} \int_{0}^{\infty} \tau^{\epsilon m_2 - n/2} d\tau \leq C \gamma^{\epsilon m_2 - n/2} h^{-1} h^{-(n-2)/2} t^{-n/2} \]

provided \( m_2 \) is taken big enough. Summing up the above estimates, we conclude that the left-hand side of (5.5) is bounded by

\[ C h^{1-\epsilon} \gamma^{-(n-2)/2} t^{-n/2} + C \gamma^{t^{-n/2}} + C \gamma^{\epsilon m_2 - n/2} h^{-1} h^{-(n-2)/2} t^{-n/2} \quad (5.9) \]

for all \( t > 0 \) and all \( 0 < \gamma \leq 1 \). Take \( \gamma = h^{1/(n-2)} \) and fix \( m_2 \) so that

\[ \gamma^{\epsilon m_2 - n/2} h^{-1} h^{-(n-2)/2} \leq h. \]

Hence (5.9) is \( O(h^{\beta}) t^{-n/2} \) with some \( \beta > 0 \), which is the desired result. \( \square \)

**Acknowledgements.** We would like to thank William Green for bringing to our attention the papers [4], [5], [17]. The first two authors have been partially supported by the CNPq-Brazil.

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