A direct product theorem for bounded-round public-coin randomized communication complexity

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Abstract

A strong direct product theorem for a problem in a given model of computation states that, in order to compute \( k \) instances of the problem, if we provide resource which is less than \( k \) times the resource required for computing one instance of the problem with constant success probability, then the probability of correctly computing all the \( k \) instances together, is exponentially small in \( k \). In this paper, we consider the model of two-party bounded-round public-coin randomized communication complexity. For a relation \( f \subseteq X \times Y \times Z \) (\( X, Y, Z \) are finite sets), let \( R_{t, \text{pub}}^\varepsilon(f) \) denote the two-party \( t \)-message public-coin communication complexity of \( f \) with worst case error \( \varepsilon \). We show that for any relation \( f \) and integer \( k \geq 1 \)

\[
R_{1-2^{-\Omega(k/t^2)}}^{(t), \text{pub}}(f^k) = \Omega\left( \frac{k}{t} \cdot \left( R_{1/3}^{(t), \text{pub}}(f) - O(t^2) \right) \right).
\]

In particular, it implies a strong direct product theorem for the two-party constant-message public-coin randomized communication complexity of all relations \( f \).

Our result for example implies a strong direct product theorem for the pointer chasing problem. This problem has been well studied for understanding round v/s communication trade-offs in both classical and quantum communication protocols \([NW91, Kha00, PRV01, KNTSZ01, JRS02]\).

We show our result using information theoretic arguments. Our arguments and techniques build on the ones used in Jain \([Jai11]\), where a strong direct product theorem for the two-party one-way public-coin communication complexity of all relations is shown (that is the special case of our result when \( t = 1 \)). One key tool used in our work and also in Jain \([Jai11]\) is a message compression technique due to Braverman and Rao \([BR11]\), who used it to show a direct sum theorem for the two-party bounded-round public-coin randomized communication complexity of all relations. Another important tool that we use is a correlated sampling protocol, which for example, has been used in Holenstein \([Hol07]\) for proving a parallel repetition theorem for two-prover games.

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1 Introduction

A fundamental question in complexity theory is how much resource is needed to solve $k$ independent instances of a problem compared to the resource required to solve one instance. More specifically, suppose for solving one instance of a problem with probability of correctness $p$, we require $c$ units of some resource in a given model of computation. A natural way to solve $k$ independent instances of the same problem is to solve them independently, which needs $k \cdot c$ units of resource and the overall success probability is $p^k$. A strong direct product theorem for this problem would state that any algorithm, which solves $k$ independent instances of this problem with $o(k \cdot c)$ units of the resource, can only compute all the $k$ instances correctly with probability at most $p^{-\Omega(k)}$.

In this work, we are concerned with the model of communication complexity which was introduced by Yao [Yao79]. In this model there are different parties who wish to compute a joint relation of their inputs. They do local computation, use public/private coins, and communicate between them to achieve this task. The resource that is counted is the number of bits communicated. The text by Kushilevitz and Nisan [KN96] is an excellent reference for this model. Direct product questions and the weaker direct sum questions have been extensively investigated in different sub-models of communication complexity. A direct sum theorem states that in order to compute $k$ independent instances of a problem, if we provide resources less than $k$ times the resource required to compute one instance of the problem with the constant success probability $p < 1$, then the success probability for computing all the $k$ instances correctly is at most a constant $q < 1$. Some examples of known direct product theorems are: Parnafes, Raz and Wigderson’s [PRW97] theorem for forests of communication protocols; Shaltiel’s [Sha04] theorem for the discrepancy bound (which is a lower bound on the distributional communication complexity) under the uniform distribution; extended to arbitrary distributions by Lee, Shraibman and Špalek [LSv08]; extended to the multiparty case by Viola and Wigderson [VV05]; extended to the generalized discrepancy bound by Sherstov [She11]; Jain, Klauck and Nayak’s [JKN08] theorem for subdistribution bound; Kluck, Špalek, de Wolf’s [KSdW04] theorem for the quantum communication complexity of the set disjointness problem; Klauck’s [Kla10] theorem for the public-coin communication complexity of the set-disjointness problem (which was re-proven using very different arguments in Jain [Jai11]); Ben-Aroya, Regev, and de Wolf’s [BARdW08] theorem for the one-way quantum communication complexity of the index function problem; Jain’s [Jai11] theorem for randomized one-way communication complexity and Jain’s [Jai11] theorem for conditional relative min-entropy bound (which is a lower bound on the public-coin communication complexity). Direct sum theorems have been shown in the public-coin one-way model [JRS03a], public-coin simultaneous message passing model [JRS03a], entanglement-assisted quantum one-way communication model [JRS05], private-coin simultaneous message passing model [JK09] and constant-round public-coin two-way model [BR11]. On the other hand, strong direct product conjectures have been shown to be false by Shaltiel [Sha04] in some models of distributional communication complexity (and of query complexity and circuit depth complexity) under specific choices for the error parameter.

Examples of direct product theorems in others models of computation include Yao’s XOR lemma [Yao82], Raz’s [Raz95] theorem for two-prover games; Shaltiel’s [Sha04] theorem for fair decision trees; Nisan, Rudich and Saks’ [NRS99] theorem for decision forests; Drucker’s [Dru11] theorem for randomized query complexity; Sherstov’s [She11] theorem for approximated polynomial degree and Lee and Roland’s [LR11] theorem for quantum query complexity. Besides their inherent importance, direct product theorems have had various important applications such as in Probabilistically checkable proofs [Raz95], in circuit complexity [Yao82] and in showing time-space tradeoffs [KvdW04, AvdW09, Kla10].

In this paper, we show a direct product theorem for the two-party bounded-round public-coin randomized communication complexity. In this model, for computing a relation $f \subseteq X \times Y \times Z$
Let \( X, Y, Z \) be finite sets, \( f \subseteq X \times Y \times Z \) a relation, \( \varepsilon > 0 \) and \( k, t \geq 1 \) be integers. There exists a constant \( \kappa \) such that,

\[
R^{(t)}_{\varepsilon} \left( f^k \right) = \Omega \left( \frac{\varepsilon \cdot k}{t} \cdot \left( R^{(t)}_{\varepsilon}(f) - \frac{\kappa^2}{\varepsilon^2} \right) \right).
\]

In particular, it implies a strong direct product theorem for the two-party constant-message public-coin randomized communication complexity of all relations \( f \). Our result generalizes the result of Jain [Jai11], which can be regarded as the special case when \( t = 1 \).

As a direct consequence of our result we get a direct product theorem for the pointer chasing problem defined as follows. Let \( n, t \geq 1 \) be integers. Alice and Bob are given functions \( F_A : [n] \to [n] \) and \( F_B : [n] \to [n] \), respectively. Let \( F^t \) represent alternate composition of \( F_A \) and \( F_B \) done \( t \) times, starting with \( F_A \). The parties are supposed to communicate and determine \( F^t(1) \). In the bit version of the problem, the players are supposed to output the least significant bit of \( F^t(s) \). We refer to the \( t \)-pointer chasing problem as FP\(_t^t\) and the bit version as BP\(_t^t\). The pointer chasing problem naturally captures the trade-off between number of messages exchanged and the communication used. There is a straightforward \( t \)-message deterministic protocol with \( t \cdot \log n \) bits of communication for both FP\(_t^t\) and BP\(_t^t\). However if only \( t - 1 \) messages are allowed to be exchanged between the parties, exponentially more communication is required. The communication complexity of this problem has been very well studied both in the classical and quantum models of communication complexity [NW91, Kla00, PRV01, KNTSZ01, JRS02]. The best lower bounds we know so far are as follows (below \( Q^{(t)}(\cdot) \) stands for the \( t \)-message quantum communication complexity).

**Theorem 1.2.** For integer \( t \geq 1 \),

1. [PRV01] \( R^{(t-1)}_{1/3}(\text{FP}_t) \geq \Omega(n \log^{(t-1)} n) \).
2. [PRV01] \( R^{(t-1)}_{1/3}(\text{BP}_t) \geq \Omega(n) \).
3. [JRS02] \( Q^{(t-1)}_{1/3}(\text{FP}_t) \geq \Omega(n \log^{(t-1)} n) \).

As a consequence of Theorem 1.1 we get strong direct product results for this problem. Note that in the descriptions of FP\(_t^t\) and BP\(_t^t\), \( t \) is a fixed constant, not dependent on the input size.

**Corollary 1.3.** For integers \( t, k \geq 1 \),

1. \( R^{(t-1)}_{1-2^{-O(k/\varepsilon^2)}}(\text{FP}_t^k) \geq \Omega \left( \frac{k}{t} \cdot n \log^{(t-1)} n \right) \).
2. \( R^{(t-1)}_{1-2^{-O(k/\varepsilon^2)}}(\text{BP}_t^k) \geq \Omega \left( \frac{k}{t} \cdot n \right) \).

\(^1\)When \( R^{(t)}_{\varepsilon}(f) \) is a constant, then a direct product result can be shown via direct arguments as for example in [Jai11, She11].
Our techniques

We prove our direct product result using information theoretic arguments. Information theory is a versatile tool in communication complexity, especially in proving lower bounds and direct sum and direct product theorems [Cha01, BYJKS02, JRS03a, JRS03b, JRS05, JK09, BBCR10, BR11, Jai11]. The broad argument that we use is as follows. For a given relation $f$, let the communication required for computing one instance with $t$ messages and constant success be $c$. Let us consider a protocol for computing $f^k$ with $t$ messages and communication cost $o(kc)$. Let us condition on success on some $l$ coordinates. If the overall success in these $l$ coordinates is already as small as we want then we are done and stop. Otherwise we exhibit another coordinate $j$ outside of these $l$ coordinates such that the success in the $j$-th coordinate, even conditioned on the success in the $l$ coordinates, is bounded away from 1. This way the overall success keeps going down and becomes exponentially small eventually. We do this argument in the distributional setting where one is concerned with average error over the inputs coming from a specified distribution rather than the worst case error over all inputs. The distributional setting can then be related to the worst case setting by the well known Yao’s principle [Yao79].

More concretely, let $\mu$ be a distribution on $\mathcal{X} \times \mathcal{Y}$, possibly non-product across $\mathcal{X}$ and $\mathcal{Y}$. Let $c$ be the minimum communication required for computing $f$ with $t$-message protocols having error at most $\epsilon$ averaged over $\mu$. Let us consider the inputs for $f^k$ drawn from the distribution $\mu^k$ ($k$ independent copies of $\mu$). Consider a $t$-message protocol $P$ for $f^k$ with communication $o(kc)$ and for the rest of the argument condition on success on a set $C$ of coordinates. If the success probability of this event is as small as we desire then we are done. Otherwise we exhibit a new coordinate $j \not\in C$ satisfying the following conditions: first the distribution of inputs $X_j,Y_j$ (of Alice and Bob respectively) in the $j$-th coordinate is quite close to $\mu$; second the joint distribution $X_jY_jM$ (where $M$ is the message transcript of $P$) can be approximated very well by Alice and Bob using a $t$ message protocol for $f$, when they are given input according to $\mu$, using communication less than $c$. This shows that success in the $j$-th coordinate must be bounded away from one. Since we can simulate each message only approximately, in order to keep the overall error bounded, we are able to make our argument for protocols with a bounded number of message exchanges.

One difficulty that is faced in this argument is that since $\mu$ may be a non-product distribution, Alice and Bob may obtain information about each other’s input in the $j$-th coordinate via their inputs in other coordinates. This is overcome by splitting the distribution $\mu$ into a convex combination of several product distributions. This idea of splitting a non-product distribution into convex combination of product distributions has been used in several previous works to handle non-product distributions in different settings [Raz92, Raz95, BYJKS02, Hol07, BBCR10, BR11, Jai11]. Some important tools that we use in our arguments are a message compression protocol due to Braverman and Rao [BR11] and the correlated sampling protocol that appeared for example in Holenstein [Hol07].

Organization

The rest of the paper is organized as follows. In Section 2, we present some background on information theory and communication complexity. In Section 3, we prove our main result Theorem 1.1 starting with some lemmas that are helpful in building the proof.

2 Preliminaries

Information theory

For integer $n \geq 1$, let $[n]$ represent the set $\{1, 2, \ldots, n\}$. Let $\mathcal{X}$, $\mathcal{Y}$ be finite sets and $k$ be a natural number. Let $\mathcal{X}^k$ be the set $\mathcal{X} \times \cdots \times \mathcal{X}$, the cross product of $\mathcal{X}$ $k$ times. Let $\mu$ be a
(probability) distribution on \( \mathcal{X} \). Let \( \mu(x) \) represent the probability of \( x \in \mathcal{X} \) according to \( \mu \). Let \( X \) be a random variable distributed according to \( \mu \), which we denote by \( X \sim \mu \). We use the same symbol to represent a random variable and its distribution whenever it is clear from the context. The expectation value of some function \( f \) on \( \mathcal{X} \) is denoted as

\[
\mathbb{E}_{x \sim \mathcal{X}} [f(x)] \overset{\text{def}}{=} \sum_{x \in \mathcal{X}} \Pr[X = x] \cdot f(x).
\]

The entropy of \( X \) is defined to be \( H(X) \overset{\text{def}}{=} -\sum_x \mu(x) \cdot \log \mu(x) \). For two distributions \( \mu, \lambda \) on \( \mathcal{X} \), the distribution \( \mu \otimes \lambda \) is defined as \( (\mu \otimes \lambda)(x_1, x_2) \overset{\text{def}}{=} \mu(x_1) \cdot \lambda(x_2) \). Let \( \mu^k \overset{\text{def}}{=} \mu \otimes \cdots \otimes \mu \), \( k \) times. The \( \ell_1 \) distance between \( \mu \) and \( \lambda \) is defined to be half of the \( \ell_1 \) norm of \( \mu - \lambda \); that is

\[
\|\lambda - \mu\|_1 \overset{\text{def}}{=} \frac{1}{2} \sum_x |\lambda(x) - \mu(x)| = \max_{S \subseteq \mathcal{X}} |\lambda_S - \mu_S|,
\]

where \( \lambda_S = \sum_{x \in S} \lambda(x) \). We say that \( \lambda \) is \( \varepsilon \)-close to \( \mu \) if \( \|\lambda - \mu\|_1 \leq \varepsilon \). The relative entropy between distributions \( X \) and \( Y \) on \( \mathcal{X} \) is defined as

\[
S(X \| Y) \overset{\text{def}}{=} \sum_{x \in \mathcal{X}} \Pr[X = x] \cdot \log \frac{\Pr[X = x]}{\Pr[Y = x]}.
\]

The relative min-entropy between them is defined as

\[
S_\infty(X \| Y) \overset{\text{def}}{=} \max_{x \in \mathcal{X}} \left\{ \log \frac{\Pr[X = x]}{\Pr[Y = x]} \right\}.
\]

It is easy to see that \( S(X \| Y) \leq S_\infty(X \| Y) \). Let \( X, Y, Z \) be jointly distributed random variables. Let \( Y_x \) be the distribution of \( Y \) conditioned on \( X = x \). The conditional entropy of \( Y \) conditioned on \( X \) is defined as \( H(Y \| X) \overset{\text{def}}{=} \mathbb{E}_{x \sim \mathcal{X}} [H(Y_x)] = H(XY) - H(X) \). The mutual information between \( X \) and \( Y \) is defined as

\[
I(X; Y) \overset{\text{def}}{=} H(X) + H(Y) - H(XY) = \mathbb{E}_{x \sim \mathcal{X}} [S(X \| Y_x)] = \mathbb{E}_{x \sim \mathcal{X}} [S(Y_x \| Y)].
\]

It is easily seen that \( I(X; Y) = S(XY \| X \otimes Y) \). We say that \( X \) and \( Y \) are independent iff \( I(X; Y) = 0 \). The conditional mutual information between \( X \) and \( Y \), conditioned on \( Z \), is defined as

\[
I(X; Y | Z) \overset{\text{def}}{=} \mathbb{E}_{z \sim Z} [I((X,Y) | Z = z)] = H(X | Z) + H(Y | Z) - H(XY | Z).
\]

The following \textit{chain rule} for mutual information is easily seen,

\[
I(X; Y Z) = I(X; Z) + I(X; Y | Z).
\]

Let \( X, X', Y, Z \) be jointly distributed random variables. We define the joint distribution of \( (X'Z)(YX) \) by

\[
\Pr[(X'Z)(YX) = x, z, y] \overset{\text{def}}{=} \Pr[X' = x, Z = z] \cdot \Pr[Y = y | X = x].
\]

We say that \( X, Y, Z \) is a Markov chain if \( XY Z = (XY)(Z | Y) \) and we denote it by \( X \leftrightarrow Y \leftrightarrow Z \). It is easy to see that \( X, Y, Z \) is a Markov chain if and only if \( I(X; Z | Y) = 0 \). Inou, Linden and Winter [ILW08] showed that if \( I(X; Y | Z) \) is small then \( XY Z \) is close to being a Markov chain.
Lemma 2.1 ([ILW08]). For any random variables $X, Y$ and $Z$, it holds that
\[
I(X; Z|Y) = \min \{S(XYZ|X'Y'Z') : X' \leftrightarrow Y' \leftrightarrow Z'\}.
\]
The minimum is achieved by distribution $X'Y'Z' = (XY)(Z|Y)$.

We will need the following basic facts. A very good text for reference on information theory is [CT91].

**Fact 2.2.** Relative entropy is jointly convex in its arguments. That is, for distributions $\mu, \mu', \lambda, \lambda' \in \mathcal{X}$,
\[
S(\mu (1-p) + \mu (1-p), \lambda + \lambda (1-p)) \leq p \cdot S(\mu, \lambda) + (1-p) \cdot S(\mu', \lambda').
\]

**Fact 2.3.** Relative entropy satisfies the following chain rule. Let $XY$ and $X'Y'$ be random variables on $\mathcal{X} \times \mathcal{Y}$. It holds that
\[
S(X'Y' | XY) = S(X' | X) + \mathbb{E}_{x \sim X} [S(Y' | Y)| X = x] .
\]
In particular, using Fact 2.2
\[
S(X'Y' | X \otimes Y) = S(X' | X) + \mathbb{E}_{x \sim X} [S(Y' | Y)| X = x] \geq S(X' | X) + S(Y' | Y).
\]

**Fact 2.4.** Let $XY$ and $X'Y'$ be random variables on $\mathcal{X} \times \mathcal{Y}$. It holds that
\[
S(X'Y' | X \otimes Y) \geq S(X'Y' | X' \otimes Y') = I(X'; Y').
\]

**Fact 2.5.** For distributions $\lambda$ and $\mu$,
\[
0 \leq \|\lambda - \mu\|_1 \leq \sqrt{S(\lambda, \mu)}.
\]

**Fact 2.6.** Let $\lambda$ and $\mu$ be distributions on $\mathcal{X}$. For any subset $\mathcal{S} \subseteq \mathcal{X}$, it holds that
\[
\sum_{x \in \mathcal{S}} \lambda(x) \cdot \log \frac{\lambda(x)}{\mu(x)} \geq -1.
\]

**Fact 2.7.** The $\ell_1$ distance and relative entropy are monotone non-increasing when subsystems are considered. Let $X, Y, X', Y'$ be random variables, then
\[
\|XY - X'Y'\|_1 \geq \|X - X'\|_1 \quad \text{and} \quad S(XY, X'Y') \geq S(X, X').
\]

**Fact 2.8.** For function $f : \mathcal{X} \times \mathcal{R} \rightarrow \mathcal{Y}$ and random variables $X, Y$ on $\mathcal{X}$ and $R$ on $\mathcal{R}$, such that $R$ is independent of $(XY)$, it holds that
\[
\|X f(X, R) - Y f(Y, R)\|_1 = \|X - Y\|_1.
\]

The following definition was introduced by Holenstein [Hol07]. It plays a critical role in his proof of a parallel repetition theorem for two-prover games.

**Definition 2.9 (Hol07).** For two distributions $(X_0Y_0)$ and $(X_1SY_1T)$, we say that $(X_0, Y_0)$ is $(1 - \varepsilon)$-embeddable in $(X_1S, Y_1T)$ if there exists a probability distribution $R$ over a set $\mathcal{R}$, which is independent of $X_0Y_0$ and functions $f_A : \mathcal{X} \times \mathcal{R} \rightarrow \mathcal{S}$, $f_B : \mathcal{Y} \times \mathcal{R} \rightarrow \mathcal{T}$, such that
\[
\|X_0Y_0, f_A(X_0, R), f_B(Y_0, R) - X_1Y_1ST\|_1 \leq \varepsilon.
\]

The following lemma was shown by Holenstein [Hol07] using a correlated sampling protocol.
Lemma 2.10 ([Hol07]). For random variables $S$, $X$ and $Y$, if
\[
\|SXY - (XY)(S|X)\|_1 \leq \varepsilon
\]
and
\[
\|SXY - (XY)(S|Y)\|_1 \leq \varepsilon,
\]
then $(X,Y)$ is $(1 - 4\varepsilon)$-embeddable in $(XS,YS)$.

We will need the following generalization of the previous lemma.

Lemma 2.11. For joint random variables $(A',B',C')$ and $(A,B)$, satisfying
\[
S(A'B'||AB) \leq \varepsilon
\]
and
\[
\mathbb{E}_{(a,c) \leftarrow A',C'} [S(B'_{a,c}||B_a)] \leq \varepsilon \quad \text{and} \quad \mathbb{E}_{(b,c) \leftarrow B',C'} [S(A'_{b,c}||A_b)] \leq \varepsilon,
\]
it holds that $(A,B)$ is $(1 - 5\sqrt{\varepsilon})$-embeddable in $(A'C',B'C')$.

Proof. Using the definition of the relative entropy, we have the following.
\[
\mathbb{E}_{(a,c) \leftarrow A',C'} [S(B'_{a,c}||B_a)] - \mathbb{E}_{(a,c) \leftarrow A',C'} [S(B'_{a,c}||B_a)] = \mathbb{E}_{(a,b,c) \leftarrow A',B',C'} \left[ \log \frac{\Pr[B' = b|A' = a]}{\Pr[B = b|A = a]} \right] \\
= \mathbb{E}_{a \leftarrow A'} [S(B'_a||B_a)] \geq 0.
\]

This means that
\[
\mathbb{E}_{(a,c) \leftarrow A',C'} [S(B'_{a,c}||B_a)] \leq \mathbb{E}_{(a,c) \leftarrow A',C'} [S(B'_{a,c}||B_a)] \leq \varepsilon. \tag{1}
\]

Then
\[
\mathbb{E}_{(a,c) \leftarrow A',C'} [S(B'_{a,c}||B_a)] = S(A'B'C'||(A'C')(B'|A')) \geq \|A'B'C' - (A'B')(C'|A')\|_1. \tag{2}
\]

Above, Eq. (2) follows from the definition of the relative entropy, Eq. (3) follows because $(A'C')(B'|A')$ and $(A'B')(C'|A')$ are identically distributed, and Eq. (4) follows from Fact 2.5.

Now from Equations (1) and (2) we get
\[
\|A'B'C' - (A'B')(C'|A')\|_1 \leq \sqrt{\varepsilon}.
\]

By similar arguments we get
\[
\|A'B'C' - (A'B')(C'|B')\|_1 \leq \sqrt{\varepsilon}.
\]

The inequalities above and Lemma 2.10 imply that $(A',B')$ is $(1 - 4\sqrt{\varepsilon})$-embeddable in $(A'C',B'C')$.

Furthermore from Fact 2.5 and $S(A'B'||AB) \leq \varepsilon$ we get
\[
\|A'B' - AB\|_1 \leq \sqrt{\varepsilon}.
\]

Finally using the inequality above and Fact 2.8 we get that $(A,B)$ is $(1 - 5\sqrt{\varepsilon})$-embeddable in $(A'C',B'C')$. \qed
Communication complexity

Let \( f \subseteq X \times Y \times Z \) be a relation, \( t \geq 1 \) be an integer and \( \varepsilon \in (0,1) \). In this work we only consider complete relations, that is for every \((x,y) \in X \times Y\), there is some \( z \in Z \) such that \((x,y,z) \in f\). In the two-party \( t \)-message public-coin model of communication, Alice with input \( x \in X \) and Bob with input \( y \in Y \), do local computation using public coins shared between them and exchange \( t \) messages, with Alice sending the first message. At the end of their protocol the party receiving the \( t \)-th message outputs some \( z \in Z \). The output is declared correct if \((x,y,z) \in f\) and wrong otherwise. Let \( R_{\varepsilon}^{(t),\text{pub}}(f) \) represent the two-party \( t \)-message public-coin communication complexity of \( f \) with worst case error \( \varepsilon \), i.e., the communication of the best two-party \( t \)-message public-coin protocol for \( f \) with error for each input \((x,y)\) being at most \( \varepsilon \).

We similarly consider two-party \( t \)-message deterministic protocols where there are no public coins used by Alice and Bob. Let \( \mu \in X \times Y \) be a distribution. We let \( D_{\varepsilon}^{(t),\mu}(f) \) represent the two-party \( t \)-message distributional communication complexity of \( f \) under \( \mu \) with expected error \( \varepsilon \), i.e., the communication of the best two-party \( t \)-message deterministic protocol for \( f \), with distributional error (average error over the inputs) at most \( \varepsilon \) under \( \mu \). Following is a consequence of the min-max theorem in game theory, see e.g., [KN96, Theorem 3.20, page 36].

**Lemma 2.12** (Yao’s principle, [Yao79]). \( R_{\varepsilon}^{(t),\text{pub}}(f) = \max_{\mu} D_{\varepsilon}^{(t),\mu}(f) \).

The following fact about communication protocols can be verified easily.

**Fact 2.13.** Let there be \( t \) messages \( M_1, \ldots, M_t \) in a deterministic communication protocol between Alice and Bob with inputs \( X, Y \) respectively where \( X \) and \( Y \) are independent. Then for any \( s \in [t], X \) and \( Y \) are independent even conditioned on \( M_1, \ldots, M_s \).

## 3 Proof of Theorem 1.1

We start by showing a few lemmas which are helpful in the proof of the main result. The following lemma was shown by Jain [Jai11] and follows primarily from a message compression argument due to Braverman and Rao [BR11].

**Theorem 3.1** ([BR11] [Jai11]). Let \( \delta > 0, c \geq 0 \). Let \( X', Y', N \) be random variables for which \( Y' \leftrightarrow X' \leftrightarrow N \) is a Markov chain and the following holds,

\[
\Pr_{(x,y,m)\sim X',Y',N}[\log \frac{\Pr[N = m | X' = x]}{\Pr[N = m | Y' = y]} > c] \leq \delta.
\]

There exists a public-coin protocol between Alice and Bob, with inputs \( X', Y' \) respectively, with a single message from Alice to Bob of \( c + O(\log(1/\delta)) \) bits, such that at the end of the protocol, Alice and Bob both possess a random variable \( M \) satisfying \( \| X'Y'N - X'Y'M \|_1 \leq 2\delta \).

We will need the following generalization of the above.

**Lemma 3.2.** Let \( c \geq 0, 1 > \varepsilon > 0, \varepsilon' > 0 \). Let \( X', Y', M' \) be random variables for which the following holds,

\[
I(X'; M'|Y') \leq c \quad \text{and} \quad I(Y'; M' | X') \leq \varepsilon.
\]

There exists a public-coin protocol between Alice and Bob, with inputs \( X', Y' \) respectively, with a single message from Alice to Bob of \( \frac{5c\varepsilon}{\varepsilon'} + O(\log \frac{1}{\varepsilon'}) \) bits, such that at the end of the protocol, Alice and Bob both possess a random variable \( M \) satisfying \( \| X'Y'M' - X'Y'M \|_1 \leq 3\varepsilon + 6\varepsilon' \).

**Proof.** Let us introduce a new random variable \( N \) with joint distribution \( X'Y'N \overset{\text{def}}{=} (X'Y')(M'|X') \). Note that \( Y' \leftrightarrow X' \leftrightarrow N \) is a Markov chain. Using Lemma 2.1 we have

\[
S(X'Y'M'|X'Y'N) = I(Y'; M'|X') \leq \varepsilon.
\]

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We bound each term above separately. For the first one, let us define the set

\[ G \]

Applying Fact 2.5, we get that

\[ \Pr\left[ \log \frac{\Pr[N = m | X' = x]}{\Pr[N = m | Y' = y]} \geq \frac{\varepsilon + 5}{\varepsilon'} \right] \leq 3\varepsilon + \sqrt{\varepsilon}. \]

**Proof.** For any \( m, x, y \) it holds that

\[
\log \frac{\Pr[N = m | X' = x]}{\Pr[N = m | Y' = y]} = \log \frac{\Pr[N = m | X' = x, Y' = y]}{\Pr[N = m | Y' = y]}
\]

\[
= \log \frac{\Pr[N = m | X' = x, Y' = y]}{\Pr[M' = m | X' = x, Y' = y]} + \log \frac{\Pr[M' = m | X' = x, Y' = y]}{\Pr[M' = m | Y' = y]}
\]

\[
+ \log \frac{\Pr[M' = m, Y' = y]}{\Pr[N = m, Y' = y]}.
\]

(5)

We bound each term above separately. For the first one, let us define the set

\[ G_1 \]

Consider,

\[
0 \geq - \mathbb{E}_{(x,y) \in M', X', Y'} \left[ S(M'_{xy} || N_{xy}) \right]
\]

\[
= \mathbb{E}_{(x,y) \in M', X', Y'} \left[ \log \frac{\Pr[N = m | X' = x, Y' = y]}{\Pr[M' = m | X' = x, Y' = y]} \right]
\]

\[
= \sum_{(m,x,y) \in G_1} \Pr[M' = m, X' = x, Y' = y] \cdot \log \frac{\Pr[N = m | X' = x, Y' = y]}{\Pr[M' = m | X' = x, Y' = y]}
\]

\[
+ \sum_{(m,x,y) \notin G_1} \Pr[M' = m, X' = x, Y' = y] \cdot \log \frac{\Pr[N = m | X' = x, Y' = y]}{\Pr[M' = m | X' = x, Y' = y]}
\]

\[
\geq \sum_{(m,x,y) \in G_1} \Pr[M' = m, X' = x, Y' = y] \cdot \log \frac{\Pr[N = m | X' = x, Y' = y]}{\Pr[M' = m | X' = x, Y' = y]}
\]

\[
+ \Pr[(M', X', Y') \notin G_1] \cdot \frac{\varepsilon + 1}{\varepsilon'}.
\]

(6)

(7)

(8)

(9)

Above, Eq. (6) and Eq. (8) follow from the definition of the relative entropy, and Eq. (7) follows from the definition of \( G_1 \). To get Eq. (9), we use Fact 2.6. Eq. (9) implies that \( \Pr[(M', X', Y') \notin G_1] \leq \varepsilon' \).

To upper bound the second term let us define

\[ G_2 \]

\[ \text{Claim 3.3.} \]

\[ \text{Proof.} \]

\[ \text{To get Eq. (9), we use Fact 2.6. Eq. (9) implies that} \]

\[ \Pr[(M', X', Y') \notin G_1] \leq \varepsilon'. \]

\[ \text{To upper bound the second term let us define} \]

\[ G_2 \]

\[ \text{9} \]
Consider,
\[
c \geq I(M'; X'|Y')
\]  
(10)
\[
= \mathbb{E}_{(m,x,y) \sim M', X', Y'} \left[ \log \frac{Pr[M' = m|X' = x, Y' = y]}{Pr[M' = m|Y' = y]} \right]
\]  
(11)
\[
= \sum_{(m,x,y) \in G_2} Pr[M' = m, X' = x, Y' = y] \cdot \log \frac{Pr[M' = m|X' = x, Y' = y]}{Pr[M' = m|Y' = y]}
\]  
\[
+ \sum_{(m,x,y) \notin G_2} Pr[M' = m, X' = x, Y' = y] \cdot \log \frac{Pr[M' = m|X' = x, Y' = y]}{Pr[M' = m|Y' = y]}
\]  
\[
\geq \frac{c + 1}{\varepsilon'} \cdot Pr[(M', X', Y') \notin G_2] - 1.
\]  
(12)

Above Eq. (10) is one of the assumptions in the lemma; Eq. (11) follows from the definition of the conditional mutual information; Eq. (12) follows from the definition of \(G_2\) and Fact 2.6.

To bound the last term define
\[G_3 \overset{\text{def}}{=} \left\{ (m, x, y) : \log \frac{Pr[M' = m, Y' = y]}{Pr[N = m, Y' = y]} \leq \frac{c + 1}{\varepsilon'} \right\}.
\]

Consider,
\[
\varepsilon \geq S(X'Y'M'\|X'Y'N)
\]  
(13)
\[
\geq S(Y'M'\|Y'N)
\]
\[
= \mathbb{E}_{(m,x,y) \sim M', X', Y'} \left[ \log \frac{Pr[M' = m, Y' = y]}{Pr[N = m, Y' = y]} \right]
\]  
\[
= \sum_{(m,x,y) \in G_3} Pr[M' = m, X' = x, Y' = y] \cdot \log \frac{Pr[M' = m, Y' = y]}{Pr[N = m, Y' = y]}
\]  
\[
+ \sum_{(m,x,y) \notin G_3} Pr[M' = m, X' = x, Y' = y] \cdot \log \frac{Pr[M' = m, Y' = y]}{Pr[N = m, Y' = y]}
\]  
\[
\geq -1 + Pr[(M', X', Y') \notin G_3] \cdot \frac{c + 1}{\varepsilon'}.
\]  
(14)

Above Eq. (13) follows from Fact 2.7 and Eq. (14) follows from definition of \(G_3\). This implies \(Pr[(M', X', Y') \notin G_3] \leq \varepsilon'\).

On combining the bounds for the three terms, using Eq. (14) and using the union bound we get (recall \(1 > \varepsilon > 0\))
\[
Pr_{(m,x,y) \sim M', X', Y'} \left[ \log \frac{Pr[N = m|X' = x]}{Pr[N = m|Y' = y]} \geq \frac{c + 5}{\varepsilon'} \right] \leq 3\varepsilon'.
\]

Now using \(\|X'Y'M' - X'Y'N\|_1 \leq \sqrt{\varepsilon}\) (as was shown previously), we finally have,
\[
Pr_{(m,x,y) \sim N, X', Y'} \left[ \log \frac{Pr[N = m|X' = x]}{Pr[N = m|Y' = y]} \geq \frac{c + 5}{\varepsilon'} \right] \leq 3\varepsilon' + \sqrt{\varepsilon}.
\]

We will need the following further generalization of the previous lemma.

**Lemma 3.4.** Let \(t \geq 1\) be an integer. Let \(\varepsilon' > 0, c_s \geq 0, 1 > \varepsilon_s > 0\) for each \(1 \leq s \leq t\). Let \(R', X', Y', M'_1, \ldots, M'_s\), be random variables for which the following holds (below \(M'_{<s} \overset{\text{def}}{=} M'_1 \cdots M'_{s-1}\)),
\[
I(X'; M'_{<s}|Y'R'M'_{<s}) \leq c_s, \quad I(Y'; M'_{s}|X'R'M'_{<s}) \leq \varepsilon_s, \quad \text{for odd } s
\]
There exists a public-coin $t$-message protocol $\mathcal{P}_t$ between Alice, with input $X'R'$, and Bob, with input $Y'R'$, with Alice sending the first message. The total communication is

$$\sum_{s=1}^{\lfloor t/2 \rfloor} c_s + 5t + O\left( t \log \frac{1}{\varepsilon'} \right),$$

and at end of the protocol, both Alice and Bob possess random variables $M_1, \ldots, M_t$, satisfying

$$\| R'X'Y'M_1 \cdots M_t - R'X'Y'M'_1 \cdots M'_t \|_1 \leq 3 \sum_{s=1}^{\lfloor t/2 \rfloor} \sqrt{c_s} + 6 \varepsilon t.$$

Proof. We prove the lemma by induction on $t$. For the base case $t = 1$, note that

$$I(X'R'; M'_1 | Y'R') = I(X'; M'_1 | Y'R') \leq c_1$$

and

$$I(Y'R'; M'_1 | X'R') = I(Y'; M'_1 | X'R') \leq \varepsilon_1.$$

Lemma 3.2 implies (by taking $X', Y', M'$ in Lemma 3.2 to be $X'R', Y'R', M'_1$ respectively) that Alice, with input $X'R'$, and Bob, with input $Y'R'$, can run a public-coin protocol with a single message from Alice to Bob of

$$\frac{c_1 + 5}{\varepsilon'} + O(\frac{1}{\varepsilon'})$$

bits and generate a new random variable $M_1$ satisfying

$$\| R'X'Y'M'_1 - R'X'Y'M_1 \|_1 \leq 3 \sqrt{c_1} + 6 \varepsilon'.$$

Now let $t > 1$. Assume $t$ is odd, for even $t$ a similar argument will follow. From the induction hypothesis there exists a public-coin $t - 1$ message protocol $\mathcal{P}_{t-1}$ between Alice, with input $X'R'$, and Bob, with input $Y'R'$, with Alice sending the first message, and total communication

$$\sum_{s=1}^{\lfloor (t-1)/2 \rfloor} c_s + 5(t-1) + O\left( (t-1) \log \frac{1}{\varepsilon'} \right),$$

such that at the end Alice and Bob both possess random variables $M_1, \ldots, M_{t-1}$ satisfying

$$\| R'X'Y'M_1 \cdots M_{t-1} - R'X'Y'M'_1 \cdots M'_{t-1} \|_1 \leq 3 \sum_{s=1}^{\lfloor (t-1)/2 \rfloor} \sqrt{c_s} + 6 \varepsilon'(t-1).$$

Note that

$$I(Y'R'M'_1; M'_1 | X'R'M'_1) = I(Y'; M'_1 | X'R'M'_1) \leq c_t$$

and

$$I(X'R'M'_1; M'_1 | Y'R'M'_1) = I(X'; M'_1 | Y'R'M'_1) \leq \varepsilon_t.$$

Therefore Lemma 3.2 implies (by taking $X', Y', M'$ in Lemma 3.2 to be $X'R'M'_1, Y'R'M'_1, M'_t$ respectively) that Alice, with input $X'R'M'_1$, and Bob, with input $Y'R'M'_1$, can run a public coin protocol $\mathcal{P}$ with a single message from Alice to Bob of

$$\frac{c_t + 5}{\varepsilon'} + O\left( \log \frac{1}{\varepsilon'} \right)$$

11
bits and generate a new random variable $M''_t$ satisfying
\[
\|R'X'Y'M'_t \cdots M'_{t-1} M'_t - R'X'Y'M'_t \cdots M'_{t-1} M''_t\|_1 \leq 3\sqrt{\varepsilon t} + 6\varepsilon'.
\] (18)

Fact 2.8 and Eq. (16) imply that Alice, on input $X'R'M_{\leq t}$ and Bob on input $Y'R'M_{\leq t}$, on running the same protocol $\mathcal{P}$ will generate a new random variable $M_t$ satisfying
\[
\|R'X'Y'M_1 \cdots M_{t-1} M_t - R'X'Y'M_1 \cdots M'_{t-1} M''_t\|_1 = \|R'X'Y'M_1 \cdots M_{t-1} - R'X'Y'M'_1 \cdots M'_{t-1}\|_1 \\
\leq 3 \sum_{s=1}^{t-1} \sqrt{\varepsilon_s} + 6\varepsilon'(t - 1).
\] (19)

Therefore by composing protocol $\mathcal{P}_{t-1}$ and protocol $\mathcal{P}$ and using Equations (16), (17), (18), (19) we get a public-coin $t$-message protocol $\mathcal{P}_t$ between Alice, with input $X'R'$, and Bob, with input $Y'R'$, with Alice sending the first message, and total communication
\[
\sum_{s=1}^{t} c_s + 5t + O\left(t \log \frac{1}{\varepsilon'}\right)
\]
such that at the end Alice and Bob both possess random variables $M_1, \ldots, M_t$ satisfying
\[
\|R'X'Y'M_1 \cdots M_t - R'X'Y'M'_1 \cdots M''_t\|_1 \leq 3 \sum_{s=1}^{t} \sqrt{\varepsilon_s} + 6\varepsilon' t. \quad \Box
\]

Following lemma, obtained from the lemma above, is the one that we will finally use in the proof of our main result.

**Lemma 3.5.** Let random variables $R', X', Y', M'_1, \ldots, M'_t$ and numbers $\varepsilon', c_s, \varepsilon_s$ satisfy all the conditions in Lemma 3.4. Let $\tau > 0$ and let random variables $(X, Y)$ be $(1 - \tau)$-embeddable in $(X'R', Y'R')$. There exists a public-coin $t$-message protocol $\mathcal{Q}_t$ between Alice, with input $X$, and Bob, with input $Y$, with Alice sending the first message, and total communication
\[
\sum_{s=1}^{t} c_s + 5t + O\left(t \log \frac{1}{\varepsilon'}\right)
\]
bits, such that at the end Alice possesses $R_A M_1 \cdots M_t$ and Bob possesses $R_B M_1 \cdots M_t$, such that
\[
\|XYR_A R_B M_1 \cdots M_t - X'Y'R'R'M'_1 \cdots M''_t\|_1 \leq \tau + 3 \sum_{s=1}^{t} \sqrt{\varepsilon_s} + 6\varepsilon' t.
\]

**Proof.** In $\mathcal{Q}_1$, Alice and Bob, using public coins and no communication first generate $R_A, R_B$ such that $\|XYR_A R_B - X'Y'R'\|_1 \leq \tau$. They can do this from the Definition 2.8 of embedding. Now they will run protocol $\mathcal{P}_1$ (as in Lemma 3.4) with Alice’s input being $XR_A$ and Bob’s input being $YR_B$ and at the end both possess $M_1, \ldots, M_t$. From Lemma 3.4 the communication of $\mathcal{Q}_1$ is as desired. Now from Fact 2.8 and Lemma 3.4
\[
\|XYR_A R_B M_1 \cdots M_t - X'Y'R'R'M'_1 \cdots M''_t\|_1 \leq \tau + 3 \sum_{s=1}^{t} \sqrt{\varepsilon_s} + 6\varepsilon' t. \quad \Box
\]

We are now ready to prove our main result, Theorem 1.1. We restate it here for convenience.

**Theorem 1.1.** Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be finite sets, $f \subseteq \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ a relation, $\varepsilon > 0$ and $k, t \geq 1$ be integers. There exists a constant $\kappa$ such that,\[
R_{1-(1-\varepsilon/2)^{1/(\kappa^2/2)}}^{(t),pub}(f^k) = \Omega \left(\frac{\varepsilon \cdot k}{t} \cdot \left( R_{1-(1-\varepsilon/2)^{1/(\kappa^2/2)}}^{(t),pub}(f) - \frac{\kappa t^2}{\varepsilon} \right) \right).
\]
Proof of Theorem 1.1} Let $c \overset{\text{def}}{=} D_{1-\kappa}^t \mu(f) - \kappa \omega^2$ for $\kappa$ to be chosen later. Let $\delta_1 \overset{\text{def}}{=} \frac{\omega^2}{\kappa}$. From Yao’s principle, Lemma 2.12 it suffices to prove that for any distribution $\mu$ on $X \times Y$,

$$D_{1-(1-\varepsilon/2)\delta k}^t \mu^k \geq \delta_1 k \epsilon$$.

Let $XY \sim \mu^k$. Let $Q$ be a $t$-message deterministic protocol between Alice, with input $X$, and Bob, with input $Y$, that computes $f^k$, with Alice sending the first message and total communication $\delta_1 k \epsilon$ bits. We assume $t$ is odd for the rest of the argument and Bob makes the final output (the case when $t$ is even follows similarly). The following Claim 3.6 implies that the success of $Q$ is at most $(1-\varepsilon/2)^{\delta k}$ and this shows the desired.

**Claim 3.6.** For each $i \in [k]$, define a binary random variable $T_i \in \{0,1\}$, which represents the success of $Q$ (that is Bob’s output being correct) on the $i$-th instance. That is, $T_i = 1$ if the $Q$ computes the $i$-th instance of $f$ correctly, and $T_i = 0$ otherwise. Let $k' \overset{\text{def}}{=} \lceil \delta k \rceil$. There exist $k'$ coordinates $\{i_1, \ldots, i_{k'}\}$ such that for each $1 \leq i \leq k' - 1$, either

$$\Pr[T_{i+1} = 0 \mid T_i = 1] \leq (1-\varepsilon/2)^{k'}$$

or

$$\Pr[T_{i+1} = 1 \mid T_i = 1] \leq 1 - \varepsilon/2,$$

where $T^{(r)}_{\overset{\text{def}}{=} \prod_{j=1}^r T_{i_j}}$.

**Proof of Claim 3.6:** For $s \in [t]$, denote the $s$-th message of $Q$ by $M_s$. Define $M \overset{\text{def}}{=} M_1 \cdots M_t$. In the following we assume $1 \leq r < k'$, however same arguments also work when $r = 0$, that is for identifying the first coordinate, which we skip for the sake of avoiding repetition. Suppose we have already identified $r$ coordinates $i_1, \ldots, i_r$ satisfying that $\Pr[T_{i_r} = 1] \leq 1 - \varepsilon/2$ and $\Pr[T_{i_{r+1}} = 1 \mid T_{i_1} = \cdots \mid T_{i_r} = 1] \leq 1 - \varepsilon/2$ for $1 \leq j \leq r - 1$. If $\Pr[T^{(r)} = 1] \leq (1-\varepsilon/2)^{k'}$, we are done.

So from now on, assume $\Pr[T^{(r)} = 1] > (1-\varepsilon/2)^{k'}$. Let $D$ be a random variable uniformly distributed in $\{0,1\}^k$ and independent of $XY$. Let $U_i = X_i$ if $D_i = 0$, and $U_i = Y_i$ if $D_i = 1$. For any random variable $L$, let us introduce the notation: $L^{(r)} \overset{\text{def}}{=} (L \mid T^{(r)} = 1)$. For example, $X^{(r)}Y = (XY \mid T^{(r)} = 1)$. If $L = L_1 \cdots L_k$, define $L_{=i} \overset{\text{def}}{=} L_1 \cdots L_{i-1} L_{i+1} \cdots L_k$, and $L_{<i} \overset{\text{def}}{=} L_1 \cdots L_{i-1}$. Random variable $L_{=i}$ is defined analogously. Let $C \overset{\text{def}}{=} \{i_1, \ldots, i_r\}$. Define $R_i \overset{\text{def}}{=} D_{=i}U_{\setminus i}X_{C \cup \{i-1\}}Y_{C \cup \{i-1\}}$ for $i \in [k]$. We denote an element from the range of $R_i$ by $r_i$.

To prove the claim, we will show that there exists a coordinate $j \notin C$ such that,

1. $(X_j Y_j)$ can be embedded well in $(X^1_j R^1_j, Y^1_j R^1_j)$.
2. Random variables $X^1_j, Y^1_j, M^1_1, \ldots, M^1_t$ satisfy the conditions of Lemma 3.4 with appropriate parameters.

Following is helpful in meeting the first condition.

\[
\delta k > S_{c_X} (X^{1,2}Y^{1,2} \mid XY) \geq S (X^{1,2}Y^{1,2} \mid XY) > \sum_{i \in C} S (X^1_i Y^1_i \mid X_i Y_i),
\]
where Eq. (20) follows from the assumption that $\Pr[T^{(v)} = 1] > 2^{-\delta k}$, and Eq. (21) is from Fact 2.3. Also consider,

$$\delta k > S_\infty \left( X^1 Y^1 D^1 U^1 \parallel XY DU \right) \geq S \left( X^1 Y^1 D^1 U^1 \parallel XY DU \right)$$

$$\geq \sum_{(d,u,x_{C\cup[i-1]} \cup Y_{C\cup[i-1]}) \sim D^1 U^1 : X_{C\cup[i-1]} \cup Y_{C\cup[i-1]} \rightarrow D^1 U^1 : X_{C\cup[i-1]} \cup Y_{C\cup[i-1]}} \mathbb{E} \left[ \mathbb{S} \left( (X^1 Y^1)_{d,u,x_{C\cup[i-1]} \cup Y_{C\cup[i-1]}} \parallel (XY)_{d,u,x_{C\cup[i-1]} \cup Y_{C\cup[i-1]}} \right) \right]$$

(22)

$$= \sum_{i \in C} \mathbb{E} \left[ \mathbb{S} \left( (X^1 Y^1)_{d,u,x_{C\cup[i-1]} \cup Y_{C\cup[i-1]}} \parallel (XY)_{d,u,x_{C\cup[i-1]} \cup Y_{C\cup[i-1]}} \right) \right]$$

(23)

$$= \frac{1}{2} \sum_{i \in C} \mathbb{E} \left[ \mathbb{S} \left( (X^1 Y^1)_{r_i} \parallel (XY)_{r_i} \right) \right] + \frac{1}{2} \sum_{i \in C} \mathbb{E} \left[ \mathbb{S} \left( (X^1 Y^1)_{r_i} \parallel (XY)_{r_i} \right) \right].$$

(24)

(25)

Above, Eq. (22) and Eq. (23) follow from Fact 2.3. Eq. (24) is from the definition of $R_i$. Eq. (25) follows since $D^1_1$ is independent of $R^1_1$ and with probability half $D^1_1$ is 0, in which case $U^1_1 = X^1_1$ and with probability half $D^1_1$ is 1 in which case $U^1_1 = Y^1_1$.

Following calculations are helpful in meeting the second condition.

$$\delta_1 c k \geq |M^1|$$

$$\geq I \left( X^1 Y^1 ; M^1 \mid D^1 U^1 X^1_1 Y^1_1 \right)$$

$$= \sum_{i \in C} I \left( X^1 Y^1 ; M^1 \mid D^1 U^1 X^1_{C\cup[i-1]} Y^1_{C\cup[i-1]} \right)$$

$$= \sum_{i \in C} \sum_{s=1}^{t} I \left( X^1 Y^1 ; M^1 \mid D^1 U^1 X^1_{C\cup[i-1]} Y^1_{C\cup[i-1]} \right)$$

$$= \sum_{i \in C} \sum_{s=1}^{t} \mathbb{E} \left[ \mathbb{S} \left( (X^1 Y^1)_{r_i} \parallel (XY)_{r_i} \right) \right] + \sum_{s \text{ even}} \mathbb{E} \left[ \mathbb{S} \left( (X^1 Y^1)_{r_i} \parallel (XY)_{r_i} \right) \right].$$

(26)

Above we have used the chain rule for mutual information several times. Last inequality follows since $D^1_1$ is independent of $(X^1_1 Y^1_1 R^1_1 M^1)$ and with probability half $D^1_1$ is 0, in which case $U^1_1 = X^1_1$ and with probability half $D^1_1$ is 1 in which case $U^1_1 = Y^1_1$. 
For the following, let \( s \in [t] \) be odd.

\[
\delta k \geq S_\infty (D^1 U^1 X^1 Y^1 M^1_{s>}) D U X Y M_{\leq s} \\
\geq S (D^1 U^1 X^1 Y^1 M^1_{s>}) D U X Y M_{\leq s} \\
\geq \mathbb{E}_{(d,u,x,y,c,m,\leq s)} \left[ S \left( (X^1 Y^1)_d,u,x,y,c,m,\leq s \right) \right] \\
= \sum_{i \notin C} \mathbb{E}_{(d,u,x,y,c,m,\leq s)} \left[ S \left( (X^1 Y^1)_d,u,x,y,c,m,\leq s \right) \right]
\]

\[
\geq \frac{1}{2} \sum_{i \notin C} \mathbb{E}_{(d,u,x,y,c,m,\leq s)} \left[ S \left( (Y^1)_x,i,m,\leq s \right) \right] \\
= \frac{1}{2} \sum_{i \notin C} \mathbb{E}_{(d,u,x,y,c,m,\leq s)} \left[ S \left( (Y^1)_x,i,m,\leq s \right) \right] \\
= \frac{1}{2} \sum_{i \notin C} \mathbb{E}_{(d,u,x,y,c,m,\leq s)} \left[ S \left( (Y^1)_x,i,m,\leq s \right) \right] \\
\geq \frac{1}{2} \sum_{i \notin C} \mathbb{E}_{(d,u,x,y,c,m,\leq s)} \left[ I \left( (Y^1)_x,i,m,\leq s \right) \right] \\
= \frac{1}{2} \sum_{i \notin C} \mathbb{E}_{(d,u,x,y,c,m,\leq s)} \left[ I \left( (Y^1)_x,i,m,\leq s \right) \right]
\]

(27)

(28)

(29)

(30)

Above we have used Fact 2.3 several times. Eq. 27 follows from the definition of \( R_i \); Eq. 28 follows from the fact that \( Y \leftrightarrow X_i R_i M_{\leq s} \leftrightarrow M_s \) for any \( i \), whenever \( s \) is odd; Eq. 29 follows from Fact 2.4.

From a symmetric argument, we can show that when \( s \in [t] \) is even,

\[
\frac{1}{2} \sum_{i \notin C} I \left( (X^1)_i,m,\leq s \right) \leq \delta k.
\]

(31)

Eq. 30 and Eq. 31 together imply

\[
\sum_{i \notin C} \left( \sum_{s \text{ odd}} I \left( (Y^1)_i,M_{\leq s} \right) + \sum_{s \text{ even}} I \left( (X^1)_i,M_{\leq s} \right) \right) \leq 2 \delta k t.
\]

(32)

Combining Equations 21, 25, 26, and 32, and making standard use of Markov’s inequality, we can get a coordinate \( j \notin C \) such that

\[
S \left( (X_j^1 Y_j^1) X_j Y_j \right) \leq 12 \delta,
\]

(33)

\[
\mathbb{E}_{(r_j,x_j) \sim R_j Y_j} \left[ S \left( (Y_j^1)_{r_j,x_j} \right) \right] \leq 12 \delta,
\]

(34)

\[
\mathbb{E}_{(r_j,y_j) \sim R_j Y_j} \left[ S \left( (X_j^1)_{r_j,y_j} \right) \right] \leq 12 \delta,
\]

(35)

\[
\sum_{s \text{ odd}} I \left( (X^1)_j,M_{\leq s} \right) R_j Y_j M_{\leq s} + \sum_{s \text{ even}} I \left( Y^1)_j,M_{\leq s} \right) R_j X_j M_{\leq s} \leq 12 \delta_{c},
\]

(36)

\[
\sum_{s \text{ odd}} I \left( (Y)^1)_j,M_{\leq s} \right) R_j Y_j M_{\leq s} + \sum_{s \text{ even}} I \left( (X^1)_j,M_{\leq s} \right) R_j X_j M_{\leq s} \leq 12 \delta t.
\]

(37)
Set $\varepsilon' \overset{\text{def}}{=} \frac{\varepsilon t^2}{128t}$, and

$$\varepsilon_s \overset{\text{def}}{=} \begin{cases} 1 \{ Y_j^1; M_j^1 \mid R_j^1 X_j^1 M_j^1 \} & s \in [t] \text{ odd,} \\ 1 \{ X_j^1; M_j^1 \mid R_j^1 Y_j^1 M_j^1 \} & s \in [t] \text{ even.} \end{cases}$$

$$c_s \overset{\text{def}}{=} \begin{cases} 1 \{ Y_j^1; M_j^1 \mid R_j^1 Y_j^1 M_j^1 \} & s \in [t] \text{ even,} \\ 1 \{ X_j^1; M_j^1 \mid R_j^1 Y_j^1 M_j^1 \} & s \in [t] \text{ odd.} \end{cases}$$

By (37), $\sum_{s=1}^t \sqrt{\varepsilon_s} \leq \sqrt{128t}$. From Equations (33), (34), (35), and Lemma 2.11 we can infer that $(X_j Y_j)$ is $(1-10\sqrt{\delta})$-embeddable in $(X_j^1 R_j^1; Y_j^1 R_j^1)$. This, combined with Equations (36), (37), and Lemma 3.6 (take $\varepsilon', \varepsilon, c_s$ in the lemma to be as defined above and take $X Y X' Y' R R' M_1 \cdots M_t$ in the lemma to be $X_j Y_j X_j^1 Y_j^1 R_j^1 M_1 \cdots M_t$) imply the following (for appropriate constant $\kappa$). There exists a public-coin $t$-message protocol $Q^1$ between Alice, with input $X_j$, and Bob, with input $Y_j$, with Alice sending the first message and total communication,

$$\frac{12\delta t c + 5t}{\varepsilon' t} + O(t \log \frac{1}{\varepsilon'}) < D^{(t)-\mu}(f),$$

such that at the end Alice possesses $R_A M_1 \cdots M_t$ and Bob possesses $R_B M_1 \cdots M_t$, satisfying

$$\| X_j Y_j R_A R_B M_1 \cdots M_t - X_j^1 Y_j^1 R_j^1 R_j^1 M_1 \cdots M_t \|_1 \leq 10\sqrt{3\delta} + 3\sqrt{128t} + 6\varepsilon' t < \varepsilon/2.$$

Assume for contradiction that $\Pr [ T_j = 1 | T^{(r)} = 1 ] > 1 - \varepsilon/2$. Consider a protocol $Q^2$ (with no communication) for $f$ between Alice, with input $X_j R_j M_1 \cdots M_t$, and Bob, with input $Y_j R_j M_1 \cdots M_t$, as follows. Bob generates the rest of the random variables present in $Y^1$ (not present in his input) himself since, conditioned on his input, those other random variables are independent of Alice’s input (here we use Fact 2.13). Bob then generates the output for the $j$-th coordinate in $Q$, and makes it the output of $Q^2$. This ensures that the success probability of Bob in $Q^2$ is $\Pr [ T_j = 1 | T^{(r)} = 1 ] > 1 - \varepsilon/2$. Now consider protocol $Q^3$ for $f$, with Alice’s input $X_j$ and Bob’s input $Y_j$, which is a composition of $Q^1$ followed by $Q^2$. This ensures, using Fact 2.28, that success probability of Bob (averaged over public coins and the inputs $X_j, Y_j$) in $Q^3$ is larger than $1 - \varepsilon$. Finally by fixing the public coins of $Q^3$, we get a deterministic protocol $Q^4$ for $f$ with Alice’s input $X_j$ and Bob’s input $Y_j$ such that the communication of $Q^4$ is less than $D^{(t)-\mu}(f)$ and Bob’s success probability (averaged over the inputs $X_j, Y_j$) in $Q^4$ is larger than $1 - \varepsilon$. This is a contradiction to the definition of $D^{(t)-\mu}(f)$ (recall that $X_j, Y_j$ are distributed according to $\mu$). Hence it must be that $\Pr [ T_j = 1 | T^{(r)} = 1 ] \leq 1 - \varepsilon/2$. The claim now follows by setting $i_{r + 1} = j$. \hfill \Box

Open problems

Some natural questions that arise from this work are:

1. Can the dependence on $t$ in our direct product theorem be improved?
2. Can these techniques be extended to show direct product theorems for bounded-round quantum communication complexity?

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