On Wreath Products of One-Class Association Schemes

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Abstract

We give a full description of the algebraic structures of the Bose-Mesner algebra and Terwilliger algebra of the wreath product of one-class association schemes.

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1 Introduction

The wreath product in the theory of association schemes provides a way to construct new (imprimitive) association schemes from old. Recently the Terwilliger algebra of the wreath product of one-class association schemes was described by G. Bhattacharyya, S. Y. Song and R. Tanaka [3]. It was shown that all irreducible modules except for the primary module of the algebra were one-dimensional. There are not many association schemes which have the property: “All non-primary irreducible modules of the Terwilliger algebra are one-dimensional.” In fact, it was proved by R. Tanaka [9] that the class of association schemes coming as the wreath product of one-class association schemes and that of group schemes of finite abelian groups are the only ones that hold this property.

In this paper we revisit the wreath product of one-class association schemes to give a complete structural description of its Bose-Mesner algebra and Terwilliger algebra. Our work is motivated by the work of F. Levenstein, C. Maldonado and D. Penazzi [6] which gives the description of the Terwilliger algebra of the Hamming scheme $H(d, q)$ as symmetric $d$-tensors of the Terwilliger algebra of the one-class association scheme $H(1, q)$ (or $K_q$). It is also motivated by P. Terwilliger [10] and E. Egge [5] in the efforts of determination of an abstract version of the Terwilliger algebra for a given association scheme.
2 Preliminaries and main results

In this section, we briefly recall some basic facts about the Bose-Mesner algebra and Terwilliger algebra of an association scheme, and the definition of the wreath product of association schemes (cf. [1, 8, 10, 11]). Then we introduce our main findings on the structural properties of Bose-Mesner algebra and Terwilliger algebra of the wreath product of one-class association schemes. These findings are formulated as Theorem 2.3, Theorem 2.4, and Theorem 2.6 below.

Let $v$ and $d$ be positive integers. Throughout the paper we will use $[d]$ to denote the set $\{0, 1, 2, \ldots, d\}$ of the first $d + 1$ whole numbers. Let $X$ denote an $v$-element set, and let $M_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra of matrices whose rows and columns are indexed by $X$. Let $R_0, R_1, \ldots, R_d$ be nonempty relations on $X$, and let $A_0, A_1, \ldots, A_d$ be the adjacency matrices of the relations defined by $(A_i)_{xy} = 1$ if $(x, y) \in R_i$; 0 otherwise. The pair $\mathcal{X} = (X, \{R_i\}_{i \in [d]})$ is called a $d$-class (symmetric) association scheme of order $v$ if the following hold:

1. $A_0 = I$,
2. $A_0 + A_1 + \cdots + A_d = J$,
3. $A_i^t = A_i$ for all $i \in [d]$,
4. $A_iA_j = \sum_{h=0}^d p_{ij}^h A_h$ for some nonnegative integers $p_{ij}^h$, for all $h, i, j \in [d]$,

where $I = I_v$ and $J = J_v$ are the $v \times v$ identity matrix and all-one matrix, respectively, and $A^t$ denotes the transpose of $A$. Our association scheme is also commutative: that is, $A_iA_j = A_jA_i$ for all $i, j \in [d]$, since all $A_i$ are symmetric.

The numbers $p_{ij}^h$ are called the intersection numbers and satisfy

\[ p_{ij}^h = |\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|, \]

where $(x, y) \in R_h$. Given an element $x \in X$, let $R_i(x) = \{y \in X : (x, y) \in R_i\}$. Then the intersection number $p_{ii}^0 = |R_i(x)|$, and is called the $i$th-valency of $\mathcal{X}$. The $i$th-valency is denoted $k_i$. It is convenient to represent a given association scheme $\mathcal{X}$ with its adjacency matrices $A_0, A_1, \ldots, A_d$ by the matrix $R(\mathcal{X}) := \sum_{i=0}^d iA_i$. The matrix $R(\mathcal{X})$ is called the association relation matrix of $\mathcal{X}$. The $(d + 1)$-dimensional algebra $\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle$ is a semi-simple algebra known as the Bose-Mesner algebra of $\mathcal{X}$. The algebra admits a second basis $E_0, E_1, \ldots, E_d$ of primitive idempotents.

Given $X$ and $M_X(\mathbb{C})$, by the standard module of $X$, we mean the $v$-dimensional vector space $V = \mathbb{C}^X = \bigoplus_{x \in X} \mathbb{C} \hat{x}$ of column vectors whose coordinates are indexed by $X$. For each $x \in X$, we denote by $\hat{x}$ the column vector with 1 in the $x$th position, and 0 elsewhere. Observe that $M_X(\mathbb{C})$ acts on $V$ by left multiplication. We endow $V$ with the Hermitian inner product. For a given association scheme $\mathcal{X}$, the vector space $V$ can be written as the direct sum of $V_i = E_iV$ where $V_i$ are the maximal common eigenspaces of $A_0, A_1, \ldots, A_d$. Given an element $x \in X$, let $V_i^* = V_i^*(x) = \bigoplus_{y \in R_i(x)} \mathbb{C} \hat{y}$. Both $R_i(x)$ and $V_i^*$ are referred to as the $i$th subconstituent of $\mathcal{X}$ with respect to $x$. Let $E_i^* = E_i^*(x)$ be the orthogonal projection map from
The matrices $E_i^*$, $E_1^*$, $E_d^*$ form a basis for a subalgebra $\mathcal{A}^* = \mathcal{A}^*(x) = \langle E_i^* \rangle_{i \in [d]}$ of $M_X(\mathbb{C})$. The algebra $\mathcal{A}^*$ is a commutative, semi-simple subalgebra of $M_X(\mathbb{C})$. This algebra is called the dual Bose-Mesner algebra of $\mathcal{X}$ with respect to $x$. Let $\mathcal{T} = \mathcal{T}(x)$ denote the subalgebra of $M_X(\mathbb{C})$ generated by the Bose-Mesner algebra $\mathcal{A}$ and the dual Bose-Mesner algebra $\mathcal{A}^*$. We call $\mathcal{T}$ the Terwilliger algebra of $\mathcal{X}$ with respect to $x$.

Terwilliger observed the following relations between triple products $E_i^* A_j E_h^*$ and the intersection numbers of the scheme.

**Proposition 2.1.** [10, Lemma 3.2] For $h, i, j \in [d]$, $E_i^* A_j E_h^* = 0$ if and only if $p_{ij}^h = 0$.

The algebra $\mathcal{T}$ is generated by the set $\{ E_i^* A_j E_h^* : i, j, h \in [d] \}$ of ‘triple products’ as an algebra, but in general, the set $\{ E_i^* A_j E_h^* : i, j, h \in [d] \}$ spans a proper linear subspace of $\mathcal{T}$. However, in [7], A. Munemasa described the combinatorial characteristics of the association schemes for which the set of triple products spans the entire space $\mathcal{T}$ as in the following.

**Proposition 2.2.** [7] The set $\{ E_i^* A_j E_h^* : i, j, h \in [d] \}$ spans $\mathcal{T}$ for each $x \in X$ if and only if $\mathcal{X}$ is triply-regular, that is, the size $p_{ijh}^{lmn}$ of the set $R_i(y) \cap R_j(z) \cap R_h(w)$ is a constant $p_{ijh}$ for all triples $y, z, w \in X$ with $(y, z) \in R_i$, $(y, w) \in R_m$ and $(z, w) \in R_n$.

The wreath product of one-class association schemes that we study in this paper is triply-regular (cf. [2, 3]); and so, the set of triple products spans its Terwilliger algebra with respect to each element of the wreath product.

We now recall the notion of the wreath product of two association schemes. Let $\mathcal{X} = (X, \{ R_i \}_{i \in [d]})$ and $\mathcal{Y} = (Y, \{ S_j \}_{j \in [e]})$ be association schemes of order $|X| = v$ and $|Y| = u$. The **wreath product** $\mathcal{X} \wr \mathcal{Y} = (X \times Y, \{ W_l \}_{l \in [d+e]})$ of $\mathcal{X}$ and $\mathcal{Y}$ is a $(d+e)$-class association scheme, defined by:

\[
W_0 = \{ ((x, y), (x, y)) : (x, y) \in X \times Y \};
\]

\[
W_l = \{ (x_1, y), (x_2, y) : (x_1, x_2) \in R_l, y \in Y \} \quad \text{for} \ 1 \leq l \leq d; \quad \text{and}
\]

\[
W_l = \{ ((x_1, y_1), (x_2, y_2)) : x_1, x_2 \in X, (y_1, y_2) \in S_{l-d} \} \quad \text{for} \ d+1 \leq l \leq d+e.
\]

The association relation matrix of $\mathcal{X} \wr \mathcal{Y}$ is described, in terms of $R(\mathcal{X})$ and $R(\mathcal{Y})$, by

\[
R(\mathcal{X} \wr \mathcal{Y}) = \sum_{l=0}^{d+e} l W_l = I_u \otimes R(\mathcal{X}) + \{ R(\mathcal{Y}) + d(J_u - I_u) \} \otimes J_v.
\]

Let $A_0, A_1, \ldots, A_d$ and $B_0, B_1, \ldots, B_e$ be the adjacency matrices of $\mathcal{X}$ and those of $\mathcal{Y}$, respectively. Then the adjacency matrices $C_i$ of $\mathcal{X} \wr \mathcal{Y}$ are given by

\[
C_0 = B_0 \otimes A_0, \ C_1 = B_0 \otimes A_1, \ldots, \ C_d = B_0 \otimes A_d, \ C_{d+1} = B_1 \otimes J_v, \ldots, \ C_{d+e} = B_e \otimes J_v,
\]
where $\otimes$ denotes the Kronecker product: $A \otimes B = (a_{ij}B)$ of two matrices $A = (a_{ij})$ and $B$. Here and in what follows, we refer to the tensor product $A \otimes B$ of $A \in M_X(\mathbb{C})$ and $B \in M_Y(\mathbb{C})$ as the Kronecker product of $A$ and $B$ in $M_{X \times Y}(\mathbb{C})$.

In order to investigate the algebraic structure of the Bose-Mesner algebra and Terwilliger algebra of the wreath product of one-class association schemes, we shall need the notion of the complex product and subschemes by following [11]. Let $\mathcal{X} = (X, \{R_i\}_{i \in [d]})$ be an association scheme, and let $A_0, A_1, \ldots, A_d$ be the adjacency matrices. For any relations $R_i$ and $R_j$, define

$$R_i R_j := \{ R_h : n^h_{ij} \neq 0 \}.$$ 

Let $\Delta$ be a nonempty subset of $[d]$. Then $\{R_i\}_{i \in \Delta}$ is called a closed subset if $R_h R_j \subseteq \{R_i\}_{i \in \Delta}$ for any $h, j \in \Delta$. If $\{R_i\}_{i \in \Delta}$ is a closed subset, then the $\mathbb{C}$-space with basis $\{A_i\}_{i \in \Delta}$ is a subalgebra of the Bose-Mesner algebra of $\mathcal{X}$, called a Bose-Mesner subalgebra of $\mathcal{X}$, and denoted by $\langle A_i \rangle_{i \in \Delta}$. In this case, for each $x \in X$, let $R_\Delta(x) := \bigcup_{i \in \Delta} R_i(x)$. With the given closed subset $\{R_i\}_{i \in \Delta}$ and $X_\Delta = R_\Delta(x)$ for an arbitrarily fixed $x \in X$, the pair $(X_\Delta, \{R_i\}_{i \in \Delta})$ is an association scheme [11]. This scheme is called the subscheme of $\mathcal{X}$ with respect to $x$ and closed subset $\{R_i\}_{i \in \Delta}$, and is denoted by $\mathcal{X}_\Delta$. Note that the cardinality of the set $X_\Delta$ is $\sum_{i \in \Delta} k_i$, and $\sum_{i \in \Delta} k_i$ divides $\sum_{i \in [d]} k_i$. Furthermore, it is shown that the Bose-Mesner algebra of $\mathcal{X}_\Delta$ is ‘exactly isomorphic’ to $\langle A_i \rangle_{i \in \Delta}$ in the following sense.

Let $\mathcal{X} = (X, \{R_i\}_{i \in [d]})$ be an association scheme, and let $A_0, A_1, \ldots, A_d$ be the adjacency matrices of $\mathcal{X}$. Let $\mathcal{Y} = (Y, \{S_j\}_{j \in [e]})$ be an association scheme, and let $B_0, B_1, \ldots, B_e$ be the adjacency matrices of $\mathcal{Y}$. Let $\{R_i\}_{i \in \Delta}$ be a closed subset of $\mathcal{X}$, and $\{S_j\}_{j \in \Lambda}$ a closed subset of $\mathcal{Y}$. We say that the Bose-Mesner subalgebras $\langle A_i \rangle_{i \in \Delta}$ and $\langle B_j \rangle_{j \in \Lambda}$ are exactly isomorphic if there is a bijection $\pi : \Delta \rightarrow \Lambda$ such that the linear map from $\langle A_i \rangle_{i \in \Delta}$ to $\langle B_j \rangle_{j \in \Lambda}$ induced by $A_i \mapsto B_{\pi(i)}$ is an algebra isomorphism.

We now formulate our main findings. Let $d$ be a positive integer, and $n_1, n_2, \ldots, n_d$ positive integers greater than or equal to 2. Let $K_n$ denote the one-class association scheme of order $n$. Let $\mathcal{X} = (X, \{R_i\}_{i \in [d]}) = K_{n_1} \wr K_{n_2} \cdots \wr K_{n_d}$. Let $\{A_0, A_1, \ldots, A_d\}$ be the basis of Bose-Mesner algebra of $\mathcal{X}$, and $\{E^0_0, E^1_1, \ldots, E^d_d\}$ the basis of the dual Bose-Mesner algebra of $\mathcal{X}$ with respect to a fixed $x \in X$. Let $v = |X| = \prod_{i=1}^d n_i$, and $J = J_v$. The following theorem characterizes the Bose-Mesner algebra of the wreath product of one-class association schemes.

**Theorem 2.3.** The association scheme $\mathcal{X} = (X, \{R_i\}_{i \in [d]}) = K_{n_1} \wr K_{n_2} \cdots \wr K_{n_d}$ has the following properties.

(i) The valencies of $\mathcal{X}$ are

$$k_1 = n_1 - 1, \quad k_i = (k_0 + k_1 + \cdots + k_{i-1})(n_i - 1) = n_1 \cdots n_{i-1}(n_i - 1), \quad \text{for } i = 2, 3, \ldots, d.$$ 

(ii) $A_i A_j = k_i A_j$, \hspace{1em} $0 \leq i < j \leq d$.

(iii) $(A_i)^2 = k_i \left( A_0 + A_1 + \cdots + A_{i-1} + \frac{n_i - 2}{n_i - 1} A_i \right)$, \hspace{1em} $1 \leq i \leq d$. 

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(iv) For any $1 \leq i \leq d$, $\{R_0, R_1, \ldots, R_i\}$ is a closed subset; and so, $\langle A_j \rangle_{j \in [i]}$ forms a Bose-Mesner subalgebra of $A$.

Parts (i) and (iv) of Theorem 2.3 were first observed by G. Bhattacharyya in her Ph.D. dissertation [2]. We derive the following structure theorems for the Terwilliger algebra $T(x)$ of $X$.

**Theorem 2.4.** Let $T(x)$ be the Terwilliger algebra of $X = K_{n_1} \wr K_{n_2} \cdots \wr K_{n_d}$, where $n_i \geq 2$ for all $1 \leq i \leq d$. For any $i, j \in [d]$, let

$$G_{ij} := \begin{cases} k_j^{-1} E_i^*A_j E_j^*, & \text{if } i < j; \\ k_i^{-1} E_i^*A_i E_i^*, & \text{if } i > j; \\ k_i^{-1} E_i^*J E_i^*, & \text{if } i = j. \end{cases}$$

Let $U$ be the $\mathbb{C}$-space spanned by the set $\{G_{ij}\}_{i,j \in [d]}$. Then the following hold.

(i) $U$ is an algebra and isomorphic to $M_{d+1}(\mathbb{C})$.

(ii) $U$ is an ideal of $T(x)$ and the quotient algebra $T(x)/U$ is commutative.

**Corollary 2.5.** [3, Corollary 4.3] If $b$ denotes the number $|\{i \in \{1, 2, \ldots, d\} : n_i = 2\}|$, then

$$T(x) \cong M_{d+1}(\mathbb{C}) \oplus M_1(\mathbb{C})^{|\frac{d(d+1)}{2} - b|}.$$ 

The ideal $U$ in the above theorem is the primary ideal of $T(x)$ related to the primary module in [3, 9]. Each of the $d(d+1)/2 - b$ non-primary ideals is one-dimensional and spanned by a central idempotent. All of these non-primary ideals of $T(x)$ are described in the next theorem.

**Theorem 2.6.** Let $T(x)$ be the Terwilliger algebra of $X = K_{n_1} \wr K_{n_2} \cdots \wr K_{n_d}$, where $n_i \geq 2$ for all $1 \leq i \leq d$. For any $i \in \{1, 2, \ldots, d\}$ and any $h \in [i-1]$, let

$$F_{ih} = \begin{cases} \frac{\sum_{j=0}^{b} E_i^*A_j E_i^*}{\sum_{j=0}^{b} k_j} - \frac{\sum_{j=0}^{b+1} E_i^*A_j E_i^*}{\sum_{j=0}^{b+1} k_j}, & \text{if } h < i - 1; \\ \frac{\sum_{j=0}^{i-1} E_i^*A_j E_i^*}{\sum_{j=0}^{i-1} k_j} - G_{ii}, & \text{if } h = i - 1; \end{cases}$$

Then the set

$$\{F_{ih} : i \in \{1, 2, \ldots, d\}, h \in [i-1]\}$$

has $d(d+1)/2 - b$ nonzero elements, where $b = |\{i \in \{1, 2, \ldots, d\} : n_i = 2\}|$, and each nonzero element is a central idempotent that spans a 1-dimensional non-primary ideal of $T(x)$.

In the following section, we will study the structure of the Bose-Mesner algebra of the wreath product and derive the properties that characterize the wreath product of one-class association schemes. We shall see that Theorem 2.3 is directly deduced from Lemma 3.1 by induction on $d$. We will prove the theorems 2.4 and 2.6 in Section 4. We shall see that our derivation of all these results is chiefly based on a set of equations in adjacency matrices of the association scheme.
3 The Bose-Mesner algebra of $K_{n_1} \wr K_{n_2} \cdots \wr K_{n_d}$

In this section, we give a description of the Bose-Mesner algebras of wreath products of one-class association schemes that will be used in the subsequent section. Let $K_n$ denote the one-class association scheme of order $n$; so, its unique nontrivial relation graph is the complete graph on $n$ vertices.

**Lemma 3.1.** Let $\mathcal{X} = (X, \{R_i\}_{i \in [d]})$ be an association scheme. Suppose $\mathcal{X} = \mathcal{Y} \wr K_n$ for an association scheme $\mathcal{Y} = (Y, \{S_j\}_{j \in [e]})$. Then $e = d - 1$, and by renumbering $R_1, R_2, \ldots, R_d$ if necessary, the following hold.

(i) The set $\{R_i\}_{i \in [d-1]}$ is a closed subset of $\mathcal{X}$ such that the Bose-Mesner subalgebra $\langle A_i \rangle_{i \in [d-1]}$ is exactly isomorphic to the Bose-Mesner algebra of $\mathcal{Y}$.

(ii) For each $i \in \{1, 2, \ldots, d\}$,
\[ A_i A_d = k_i A_d, \quad (3.1) \]
and
\[ (A_d)^2 = \left( \sum_{j=0}^{d-1} k_j \right) \left\{ (n-1) \left( \sum_{j=0}^{d-1} A_j \right) + (n-2) A_d \right\}. \quad (3.2) \]

**Proof.** Let $B_j$ be the adjacency matrix of $S_j$, for $j \in [e]$, and $J_n = J_n - I_n$. Then the adjacency matrices of $\mathcal{Y} \wr K_n$ are
\[ I_n \otimes B_0, \ I_n \otimes B_1, \ldots, I_n \otimes B_e, \ J_n \otimes J_u, \]
where $u = |Y|$. Thus, $e = d - 1$. Let the valency of $S_j$ be $k_j$. Then clearly
\[ (I_n \otimes B_j)(J_n \otimes J_u) = k_j (J_n \otimes J_u), \quad 1 \leq j \leq e, \]
and
\[ (J_n \otimes J_u)^2 = u \left( (n-1)I_n \otimes J_u + (n-2)J_n \otimes J_u \right) \]
\[ = u \left( (n-1) \sum_{j=0}^{e} (I_n \otimes B_j) + (n-2)(J_n \otimes J_u) \right). \]

By renumbering $R_1, R_2, \ldots, R_d$ if necessary, we may assume that the adjacency matrix of $R_i$ is $A_i = I_n \otimes B_i$, $1 \leq i \leq d-1$, and the adjacency matrix of $R_d$ is $A_d = J_n \otimes J_u$. Then the valency of $R_i$ is $k_i$, $1 \leq i \leq d-1$, and (3.1), (3.2) holds. Furthermore, clearly $\{R_1, R_2, \ldots, R_{d-1}\}$ is a closed subset, and the Bose-Mesner subalgebra $\langle A_i \rangle_{i \in [d-1]}$ is exactly isomorphic to the Bose-Mesner algebra of $\mathcal{Y}$. \qed

As a consequence, we have Theorem 2.3, from which we can easily prove the following proposition. We will need this proposition in Section 4.

**Proposition 3.2.** Let $d$ be a positive integer, and let $n_1, n_2, \ldots, n_d$ be positive integers greater than or equal to 2. Let $\mathcal{X} = (X, \{R_i\}_{i \in [d]}) = K_{n_1} \wr K_{n_2} \cdots \wr K_{n_d}$, and let $A_0, A_1, \ldots, A_d$ be the adjacency matrices of $\mathcal{X}$. Then the following hold.
(i) For any \( h \in [d] \),
\[
\left( \sum_{i=0}^{h} A_i \right)^2 = \left( \sum_{i=0}^{h} k_i \right) \left( \sum_{i=0}^{h} A_i \right).
\]

(ii) For any \( g, h \in [d] \) such that \( g \leq h \),
\[
A_g \left( \sum_{i=0}^{h} A_i \right) = k_g \left( \sum_{i=0}^{h} A_i \right).
\]

Proof. (i) We use induction on \( h \). Clearly (i) holds when \( h = 0 \). Assume that \( h > 0 \) and (i) holds for \( h - 1 \). Then we show that (i) holds for \( h \). Recall that for any \( g \in [d] \),
\[
k_0 + k_1 + \ldots + k_g = n_1 n_2 \ldots n_g \text{ and } k_g = (k_0 + k_1 + \ldots + k_{g-1})(n_g - 1); \text{ and so, by Theorem 2.3,}
\]
\[
(A_h)^2 = k_h \left( \sum_{i=0}^{h-1} A_i \right) + \left( \sum_{i=0}^{h-1} k_i \right)(n_h - 2)A_h.
\]

Thus, the induction hypothesis and Theorem 2.3 yield that
\[
\left( \sum_{i=0}^{h} A_i \right)^2 = \left( \sum_{i=0}^{h-1} A_i \right)^2 + 2 \left( \sum_{i=0}^{h-1} A_i \right) A_h + (A_h)^2
\]
\[
= \left( \sum_{i=0}^{h-1} k_i \right) \left( \sum_{i=0}^{h-1} A_i \right) + 2 \left( \sum_{i=0}^{h-1} k_i \right) A_h + k_h \left( \sum_{i=0}^{h-1} A_i \right) + \left( \sum_{i=0}^{h-1} k_i \right)(n_h - 2)A_h
\]
\[
= \left( \sum_{i=0}^{h} k_i \right) \left( \sum_{i=0}^{h} A_i \right).
\]

(ii) Since \( g \leq h \), by Theorem 2.3 we see that
\[
A_g \left( \sum_{i=0}^{h} A_i \right) = A_g \left( J - \sum_{i=h+1}^{d} A_i \right) = A_g J - \sum_{i=h+1}^{d} A_g A_i = k_g J - \sum_{i=h+1}^{d} k_g A_i = k_g \left( \sum_{i=0}^{h} A_i \right).
\]

The next proposition says that the equations in Theorem 2.3(ii) characterize the Bose-Mesner algebra of the wreath product of one-class association schemes.

Proposition 3.3. Let \( \mathcal{X} = (X, \{R_i\}_{i \in [d]}) \) be a commutative association scheme, and \( A_0, A_1, \ldots, A_d \) the adjacency matrices of \( \mathcal{X} \). Assume that
\[
A_i A_j = k_i A_j, \ 0 \leq i < j \leq d.
\]

Then the Bose-Mesner algebra of \( \mathcal{X} \) is exactly isomorphic to the Bose-Mesner algebra of the wreath product of some one-class association schemes.
Proof. It follows from $A_i A_j = k_i A_j$, $0 \leq i < j \leq d$, that $k_0, k_1, \ldots, k_{d-1}$ are the valencies of $\mathcal{X}$, and for any $1 \leq j \leq d$, we have that
\[
\left( \sum_{i=0}^{j-1} A_i \right) A_j = \sum_{i=0}^{j-1} A_i A_j = \left( \sum_{i=0}^{j-1} k_i \right) A_j.
\]
But on the other hand,
\[
\left( \sum_{i=0}^{j-1} A_i \right) A_j = \left( J - \sum_{i=j}^{d} A_i \right) A_j = k_j J - (A_j)^2 - k_j \sum_{i=j+1}^{d} A_i = k_j \sum_{i=0}^{j} A_i - (A_j)^2.
\]
Thus,
\[
(A_j)^2 = k_j \sum_{i=0}^{j-1} A_i + \left( k_j - \sum_{i=0}^{j-1} k_i \right) A_j.
\]
Therefore, for every $i \in [d]$, $\{R_h \}_{h \in [i]}$ is a closed subset of $\mathcal{X}$. In particular, for every $1 \leq i \leq d$, the subset $\{R_h \}_{h \in [i-1]}$ is also a closed subset of $\{R_h \}_{h \in [i]}$. So the valency of the subset $\{R_h \}_{h \in [i-1]}$ divides the valency of the subset $\{R_h \}_{h \in [i]}$. That is, $k_0 + k_1 + \cdots + k_{i-1}$ divides $k_0 + k_1 + \cdots + k_i$. ($k_d$ is the valency of $R_d$.) Hence, $k_0 + k_1 + \cdots + k_{i-1}$ divides $k_i$. Let
\[
n_i = \frac{k_i}{k_0 + k_1 + \cdots + k_{i-1}} + 1, \ 1 \leq i \leq d.
\]
Then each $n_i$ is a positive integer greater than or equal to 2. Clearly $1 + k_1 = n_1$, and for any $2 \leq j \leq d$,
\[
1 + k_1 + \cdots + k_j = n_1 \cdots n_j \quad \text{and} \quad k_j = n_1 \cdots n_{j-1}(n_j - 1).
\]
Hence,
\[
k_j - \sum_{i=0}^{j-1} k_i = \frac{k_j(n_j - 2)}{n_j - 1}, \ 1 \leq j \leq d.
\]
Therefore, by Theorem 2.3 we see that the Bose-Mesner algebra of $\mathcal{X}$ is exactly isomorphic to the Bose-Mesner algebra of the association scheme $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$.

\[\square\]

### 4 The Terwilliger algebra of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$

The purpose of this section is to prove our main results stated in Theorems 2.4 and 2.6 as well as Corollary 2.5. In order to prove them, we shall derive several technical lemmas first. These lemmas also help us to understand the Terwilliger algebra of the wreath product.

Throughout the section, let $\mathcal{X} = (X, \{R_i \}_{i \in [d]}) = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$, where $d, n_1, n_2, \ldots, n_d$ are positive integers greater than or equal to 2. Let $v = |X| = n_1 n_2 \cdots n_d$. Let $\{A_i \}_{i \in [d]}$ be the basis of the Bose-Mesner algebra of $\mathcal{X}$, and let $\{E_{ij}^* \}_{i \in [d]}$ be the basis of the dual Bose-Mesner algebra of $\mathcal{X}$ with respect to an arbitrary (fixed) $x \in X$. Let $J = J_{v}$ be the $v \times v$ all-ones matrix.

The next two lemmas will be used heavily in the sequel. They are found in [3], but the proofs here are different from those in [3]. We show that these results are also derived from Proposition 2.1 and Theorem 2.3.
Lemma 4.1. [3, Theorem 3.5] For any \( i, j, h \in [d] \), the following hold.

(i) If \( i \neq j \), then \( E_i^* A_j E_h^* \neq 0 \) if and only if \( h = \max\{i, j\} \).

(ii) If \( i < j \), then \( E_i^* A_h E_j^* \neq 0 \) if and only if \( h = j \).

(iii) If \( h > i \), then \( E_i^* A_i E_h^* = 0 \).

(iv) If \( h < j \), then \( E_i^* A_h E_j^* \neq 0 \) if and only if \( i = j \).

Proof. (i) Since \( i \neq j \), \( p_{ij}^h \neq 0 \) if and only if \( h = \max\{i, j\} \) by Theorem 2.3. So \( E_i^* A_j E_h^* \neq 0 \) if and only if \( h = \max\{i, j\} \) by Proposition 2.1.

Similarly, (ii) and (iii) follow from Theorem 2.3 and Proposition 2.1, and (iv) follows from (i) and (iii). \( \square \)

Lemma 4.2. [3, Lemma 3.7] For any \( i, j, h \in [d] \), the following hold.

(i) If \( i < j \), then \( E_i^* A_j E_j^* = E_i^* A_j, E_j^* A_j E_i^* = A_j E_i^* \), and \( E_j^* A_i E_i^* = A_i E_j^* \).

(ii) If \( i < j \), then \( E_i^* A_i E_j^* = E_i^* J E_j^* \).

(iii) \( E_i^* J E_i^* = E_i^* (A_0 + A_1 + \cdots + A_d) E_i^* \).

(iv) \( A_i E_i^* = (E_0^* + E_1^* + \cdots + E_d^*) A_i E_i^* \) and \( E_i^* A_i = E_i^* A_i (E_0^* + E_1^* + \cdots + E_i^*) \).

Proof. (i) Since \( i < j \), Lemma 4.1(i) implies that
\[
E_i^* A_j = E_i^* A_j (E_0^* + E_1^* + \cdots + E_d^*) = E_i^* A_j E_j^* \]
and \( A_j E_i^* = (E_0^* + E_1^* + \cdots + E_d^*) A_j E_i^* = E_j^* A_j E_i^* \). Also Lemma 4.1(iv) yields that
\[
A_i E_i^* = (E_0^* + E_1^* + \cdots + E_d^*) A_i E_i^* = E_j^* A_j E_i^*. \]

(ii) Since \( i < j \), Lemma 4.1(ii) yields that \( E_i^* J E_j^* = E_i^* (A_0 + A_1 + \cdots + A_d) E_j^* = E_i^* A_j E_j^* \).

Similarly, (iii) and (iv) follow from Lemma 4.1(i). \( \square \)

For the rest of this section, given a fixed \( x = x_{01} \in X \), the rows and columns of any \( E_i^* \) and \( A_j \), for \( i, j \in [d] \), are indexed by the elements of \( X \) in the following order
\[
x_{01}, x_{11}, \ldots, x_{1k_1}, x_{21}, \ldots, x_{2k_2}, \ldots, x_{d1}, \ldots, x_{dk_d}
\]
such that \((x, x_{hl}) \in R_h, h \in [d] \) and \( 1 \leq l \leq k_h \). We will write any \( E_i^* \) and \( A_j \), for \( i, j \in [d] \), as block matrices such that
\[
E_i^* = \begin{pmatrix}
0 & \cdots & 0 \\
\cdots & 0 & \cdots \\
0 & \cdots & 0 \\
\end{pmatrix}
\]
and
\[
A_j = \begin{pmatrix}
(A_j)_{00} & (A_j)_{01} & (A_j)_{02} & \cdots & (A_j)_{0d} \\
(A_j)_{10} & (A_j)_{11} & (A_j)_{12} & \cdots & (A_j)_{1d} \\
(A_j)_{20} & (A_j)_{21} & (A_j)_{22} & \cdots & (A_j)_{2d} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(A_j)_{d0} & (A_j)_{d1} & (A_j)_{d2} & \cdots & (A_j)_{dd}
\end{pmatrix},
\]
where \((A_j)_{rs}\) is a \( k_r \times k_s \) matrix, called the \((r, s)\)-block of \( A_j \), for \( r, s \in [d] \). We will also write any matrix in the Terwilliger algebra \( T(x) \) as a block matrix in the same way. In particular,
for any $i, j, h \in [d]$, we write $E^*_h A_j E^*_h$ as a block matrix. Thus, the $(i, h)$-block of $E^*_h A_j E^*_h$ is $(A_j)_{ih}$, and any other block of $E^*_h A_j E^*_h$ is zero. We remark that all the results in this paper except for Lemmas 4.3 and 4.4 below are independent of the choice of the order of elements in $X$.

For any positive integers $p$ and $q$, let $J_{p,q}$ be the $p \times q$ matrix whose entries are all 1. The next lemma plays an important role in our discussion.

**Lemma 4.3.** For any $j \in [d]$, 

\[
A_j = \begin{pmatrix}
0 & 0 & \cdots & 0 & J_{1,k_j} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & J_{k_1,k_j} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & J_{k_{j-1},k_j} & 0 & \cdots & 0 \\
J_{k_j,1} & J_{k_j,k_1} & \cdots & J_{k_j,k_{j-1}} & (A_j)_{jj} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & (A_j)_{j+1,j+1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & (A_j)_{dd}
\end{pmatrix}.
\]

**Proof.** Let $i, h \in [d]$. If $i < j$, then $E^*_h A_j E^*_h \neq 0$ if and only if $h = j$ by Lemma 4.1(i), and hence $(A_j)_{ih} \neq 0$ if and only if $h = j$. If $i > j$, then $E^*_h A_j E^*_h \neq 0$ if and only if $h = i$ by Lemma 4.1(i), and hence $(A_j)_{ih} \neq 0$ if and only if $h = i$. Moreover, since $E^*_j A_j E^*_h = 0$ for any $h > j$ by Lemma 4.1(iii), we see that $(A_j)_{jh} = 0$ for any $h > j$. Thus, $A_j$ has the form 

\[
A_j = \begin{pmatrix}
0 & 0 & \cdots & 0 & (A_j)_{0j} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & (A_j)_{1j} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & (A_j)_{j-1,j} & 0 & \cdots & 0 \\
(A_j)_{j0} & (A_j)_{j1} & \cdots & (A_j)_{j,j-1} & (A_j)_{jj} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & (A_j)_{j+1,j+1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & (A_j)_{dd}
\end{pmatrix}.
\]

For any $i < j$, since $E^*_i A_j E^*_j = E^*_i J E^*_j$ by Lemma 4.2(ii), we see that $(A_j)_{ij} = J_{k_i,k_j}$. Also we have $(A_j)_{ji} = J_{k_j,k_i}$ for any $i < j$ since $A_j$ is symmetric. So the lemma holds. 

Motivated by this lemma, we define the following matrices $G_{ij}$, for all $i, j \in [d]$. Let

\[
G_{ij} := \begin{cases}
 k_j^{-1} E^*_i A_j E^*_j, & \text{if } i < j; \\
 k_j^{-1} E^*_i A_j E^*_j, & \text{if } i > j; \\
 k_i^{-1} E^*_i J E^*_j, & \text{if } i = j.
\end{cases}
\]

Clearly, $\{G_{ij} : i, j \in [d]\}$ is a linearly independent subset of $T(x)$. Let $U$ be the $C$-space with basis $\{G_{ij} : i, j \in [d]\}$. Thus, the dimension of $U$ is $(d + 1)^2$.

The next lemma is a direct consequence of Lemma 4.3.
Lemma 4.4. For any $i, j \in [d]$, the $(i, j)$-block of $G_{ij}$ is $k_{j}^{-1}J_{k_{i}, k_{j}}$, and any other block of $G_{ij}$ is zero.

Theorem 2.4(i) follows from the next lemma.

Lemma 4.5. For any $i, j, g, h \in [d]$,

$$G_{ij}G_{gh} = \delta_{jg}G_{ih},$$

(4.1)

where $\delta_{jg}$ is the Kronecker delta.

Proof. Clearly $G_{ij}G_{gh} = 0$ if $j \neq g$. So we only need to show that

$$G_{ij}G_{jh} = G_{ih}.$$

Since the $(i, j)$-block is the only nonzero block of $G_{ij}$, and the $(j, h)$-block is the only nonzero block of $G_{jh}$, we see that every block except for the $(i, h)$-block of $G_{ij}G_{jh}$ is zero. By Lemma 4.4, the $(i, h)$-block of $G_{ij}G_{jh}$ is

$$k_{j}^{-1}J_{k_{i}, k_{j}} \cdot k_{h}^{-1}J_{k_{i}, k_{h}} = k_{h}^{-1}J_{k_{i}, k_{h}}.$$

Thus, $G_{ij}G_{jh} = G_{ih}$ by Lemma 4.4.

Lemma 4.6. For $h, i, j \in [d]$, the following hold.

(i) If $h < i$, then $A_{h}G_{ij} = E_{i}^{*}A_{h}E_{i}^{*}G_{ij}$.

(ii) If $h < j$, then $G_{ij}A_{h} = G_{ij}E_{j}^{*}A_{h}E_{j}^{*}$.

Proof. (i) Since $h < i$, $A_{h}E_{i}^{*} = E_{i}^{*}A_{h}E_{i}^{*}$ by Lemma 4.2(i). Thus,

$$A_{h}G_{ij} = A_{h}E_{i}^{*}G_{ij} = E_{i}^{*}A_{h}E_{i}^{*}G_{ij}.$$

The proof of (ii) is similar.

The first half of Theorem 2.4(ii) will follow from the next lemma.

Lemma 4.7. For any $i, j, h \in [d]$, the following hold.

(i) $A_{h}G_{ij} = \begin{cases} k_{h}G_{ij}, & \text{if } h < i; \\ k_{i}\left(\sum_{r=0}^{i-1}G_{rij}\right) + \left(k_{i} - \sum_{r=0}^{i-1}k_{r}\right)G_{ij}, & \text{if } h = i; \\ k_{i}G_{hij}, & \text{if } h > i. \end{cases}$

(ii) $G_{ij}A_{h} = \begin{cases} k_{h}G_{ij}, & \text{if } h < j; \\ \sum_{r=0}^{j-1}(k_{r}G_{ir}) + \left(k_{j} - \sum_{r=0}^{j-1}k_{r}\right)G_{ij}, & \text{if } h = j; \\ k_{h}G_{ijh}, & \text{if } h > j. \end{cases}$
Proof. (i) If $h < i$, then $A_h G_{ij} = E_i^* A_h E_i^* G_{ij}$ by Lemma 4.6(i). From Lemma 4.3, the $(i, i)$-block of $E_i^* A_h E_i^*$ is $(A_h)_{ii}$, and any other block of $E_i^* A_h E_i^*$ is zero. Note that the sum of entries in any row of $(A_h)_{ii}$ is $k_h$ by Lemma 4.3. So Lemma 4.4 yields that the $(i, j)$-block of $(E_i^* A_h E_i^*) G_{ij}$ is

$$
(A_h)_{ii} \cdot k_j^{-1} J_{k_i, k_j} = k_h k_j^{-1} J_{k_i, k_j},
$$

and any other block of $(E_i^* A_h E_i^*) G_{ij}$ is zero. Thus, $(E_i^* A_h E_i^*) G_{ij} = k_h G_{ij}$ by Lemma 4.4, and (i) holds for $h < i$.

If $h = i$, then by Lemma 4.2(iv),

$$A_h G_{ij} = A_i G_{ij} = A_i E_i^* G_{ij} = \sum_{r=0}^{i-1} E_r^* A_i E_i^* G_{ij} = \sum_{r=0}^{i-1} E_r^* A_i E_i^* G_{ij} + E_i^* A_i E_i^* G_{ij}.
$$

By Lemma 4.5,

$$\sum_{r=0}^{i-1} E_r^* A_i E_i^* G_{ij} = \sum_{r=0}^{i-1} k_i G_{rj} G_{ij} = k_i \sum_{r=0}^{i-1} G_{rj}.
$$

From Lemmas 4.2(iii), 4.6(i), 4.5, and what we have just proved,

$$E_i^* A_i E_i^* G_{ij} = (E_i^* J E_i^* - \sum_{r=0}^{i-1} E_r^* A_i E_i^*) G_{ij} = k_i G_{ii} G_{ij} - \sum_{r=0}^{i-1} A_r G_{ij} = \left(k_i - \sum_{r=0}^{i-1} k_r\right) G_{ij}.
$$

Thus,

$$A_i G_{ij} = k_i \sum_{r=0}^{i-1} G_{rj} + \left(k_i - \sum_{r=0}^{i-1} k_r\right) G_{ij}.
$$

So (i) holds for $h = i$.

If $h > i$, then Lemmas 4.2(i) and 4.5 imply that

$$A_h G_{ij} = A_h E_i^* G_{ij} = E_i^* A_h E_i^* G_{ij} = k_i G_{hi} G_{ij} = k_i G_{hj}.
$$

So (i) holds for $h > i$.

The proof of (ii) is similar.

The next lemma is needed for the proof of the second half of Theorem 2.4(ii) and the proof of Theorem 2.6.

Lemma 4.8. For any $i, g, h \in [d]$ such that $g, h \in [i]$ and at least one of $g$ and $h$ is not equal to $i$, we have that

$$
(E_i^* A_g E_i^*)(E_i^* A_h E_i^*) = E_i^* A_g A_h E_i^*.
$$

Proof. For each $i \in [d]$ and every $g, h \in [i]$, Lemma 4.3 implies that the $(i, i)$-block of $(E_i^* A_g E_i^*)(E_i^* A_h E_i^*)$ is $(A_g)_{ii}(A_h)_{ii}$. If at least one of $g$ and $h$ is not equal to $i$, then by Lemma 4.3, the $(i, i)$-block of $E_i^* A_g A_h E_i^*$ is also $(A_g)_{ii}(A_h)_{ii}$. Thus, $(E_i^* A_g E_i^*)(E_i^* A_h E_i^*) = E_i^* A_g A_h E_i^*$.

Now we are ready to prove Theorem 2.4 and Corollary 2.5.
Proof of Theorem 2.4. (i) For any $i, j, h \in [d]$, $A_h G_{ij} \in \mathcal{U}$ and $G_{ij} A_h \in \mathcal{U}$ by Lemma 4.7, and $E_h^* G_{ij} = \delta_{hi} G_{ij} \in \mathcal{U}$, $G_{ij} E_h^* = \delta_{jh} G_{ij} \in \mathcal{U}$. So $\mathcal{U}$ is an ideal of $\mathcal{T}(x)$. For any $i, j \in [d]$, let $E_{ij}$ be the $(d + 1) \times (d + 1)$ matrix whose $(i, j)$-entry is 1 and whose other entries are all zero. Then the linear map $\varphi : \mathcal{U} \to M_{d+1}(\mathbb{C})$ defined by
\[
\varphi(G_{ij}) = E_{ij}, \quad i, j \in [d],
\]
establishes an isomorphism by (4.1). Thus, $\mathcal{U}$ is isomorphic to $M_{d+1}(\mathbb{C})$.

(ii) Recall that $\mathcal{X} = K_{n_1} \cdots K_{n_d}$ is triply-regular, and hence $\mathcal{T}(x) = \mathcal{T}_0(x)$. Thus,$\mathcal{T}(x)/\mathcal{U} = \{E_i^* A_h E_i^* + \mathcal{U} : i \in [d], \ h \in [i]\}$.

For any $i \in [d]$ and any $g, h \in [i]$, Lemma 4.8 implies that
\[
(E_i^* A_g E_i^*)(E_i^* A_h E_i^*) = E_i^* A_g A_h E_i^* = E_i^* A_h A_g E_i^* = (E_i^* A_h E_i^*)(E_i^* A_g E_i^*), \quad \text{if } g \neq h.
\]
Thus, $\mathcal{T}(x)/\mathcal{U}$ is commutative.

Proof of Corollary 2.5. Since $\mathcal{T}(x)$ is semisimple, and $\mathcal{U}$ is an ideal of $\mathcal{T}(x)$ by Theorem 2.4(ii), we see that $\mathcal{T}(x) = \mathcal{U} \oplus \mathcal{I}$ for some ideal $\mathcal{I}$ of $\mathcal{T}(x)$. Note that $\mathcal{I}$ is also semisimple by Wedderburn-Artin’s Theorem, and $\mathcal{I} \cong \mathcal{T}(x)/\mathcal{U}$ is a commutative algebra by Theorem 2.4(ii). Recall that the dimension of $\mathcal{T}(x)$ is
\[
(d+1)^2 + \frac{d(d+1)}{2} - b,
\]
where $b = |\{i \in \{1, 2, \ldots, d\} : n_i = 2\}|$, and the dimension of $\mathcal{U}$ is $(d+1)^2$. So the dimension of $\mathcal{I}$ is $d(d+1)/2 - b$, and
\[
\mathcal{I} \cong M_1(\mathbb{C})^{\oplus \frac{d(d+1)}{2} - b}.
\]
This completes the proof.

In order to prove Theorem 2.6, we first prove the following two lemmas.

Lemma 4.9. For any $i \in \{1, 2, \ldots, d\}$ and any $h \in [i-1]$, we have that
\[
\left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right)^2 = \left( \sum_{j=0}^{h} k_j \right) \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right). \tag{4.3}
\]

Proof. For any $i \in \{1, 2, \ldots, d\}$ and any $h \in [i-1]$, by Lemma 4.8 we have that
\[
\left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right)^2 = \sum_{l=0}^{h} \sum_{j=0}^{h} (E_i^* A_j E_i^*) (E_i^* A_l E_i^*) = \sum_{l=0}^{h} \sum_{j=0}^{h} E_i^* A_j A_l E_i^* = E_i^*(A_0 + A_1 + \cdots + A_h)^2 E_i^*.
\]
So (4.3) holds by Proposition 3.2(i). \qed
Lemma 4.10. For any \( i, h, g \in [d] \) such that \( i \geq 1 \) and \( h \in [i - 1] \), the following hold.

(i) 
\[
A_g \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) = \begin{cases} 
  k_g \sum_{j=0}^{h} E_i^* A_j E_i^*, & \text{if } g \leq h; \\
  \left( \sum_{j=0}^{h} k_j \right) E_i^* A_g E_i^*, & \text{if } h < g < i; \\
  \left( \sum_{j=0}^{h} k_j \right) A_i E_i^*, & \text{if } g = i; \\
  \left( \sum_{j=0}^{h} k_j \right) E_g^* A_g E_i^*, & \text{if } g > i.
\end{cases}
\]

(ii) 
\[
\left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) A_g = \begin{cases} 
  k_k \sum_{j=0}^{h} E_i^* A_j E_i^*, & \text{if } g \leq h; \\
  \left( \sum_{j=0}^{h} k_j \right) E_i^* A_g E_i^*, & \text{if } h < g < i; \\
  \left( \sum_{j=0}^{h} k_j \right) E_i^* A_i, & \text{if } g = i; \\
  \left( \sum_{j=0}^{h} k_j \right) E_i^* A_g E_g^*, & \text{if } g > i.
\end{cases}
\]

Proof. (i) If \( g \leq h \), then for any \( j \in [h] \), since \( h < i \), by Lemmas 4.2(i) and 4.8, 
\[
A_g E_i^* A_j E_i^* = (E_i^* A_g E_i^*)(E_i^* A_j E_i^*) = E_i^* A_g A_j E_i^*.
\]

Thus, Proposition 3.2(ii) implies that 
\[
A_g \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) = E_i^* \left( A_g \sum_{j=0}^{h} A_j \right) E_i^* = E_i^* \left( k_g \sum_{j=0}^{h} A_j \right) E_i^*.
\]

So (i) holds for \( g \leq h \).

If \( h < g < i \), then similar to the proof of the case \( g \leq h \),
\[
A_g \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) = E_i^* \left( A_g \sum_{j=0}^{h} A_j \right) E_i^* = E_i^* \left( \sum_{j=0}^{h} k_j \right) A_g E_i^*.
\]

So (i) holds for \( h < g < i \).

If \( g = i \), then for any \( l \in [i - 1] \) and any \( j \in [h] \), since \( h < i \), we have that 
\[
E_i^* A_i E_i^* A_j E_i^* = k_j G_{ih} A_j = k_j k_j G_{ih} = k_j E_i^* A_i E_i^*
\]

by Lemmas 4.6(ii) and 4.7(ii), and 
\[
E_i^* A_i E_i^* A_j E_i^* = E_i^* A_i A_j E_i^* = k_j E_i^* A_i E_i^*
\]

by (4.2). Thus, Lemma 4.2(iv) yields that for any \( j \in [h] \),
\[
A_i E_i^* A_j E_i^* = \sum_{l=0}^{i} E_i^* A_i E_i^* A_j E_i^* = k_j \sum_{l=0}^{i} E_i^* A_i E_i^* = k_j A_i E_i^*.
\]

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Therefore,

\[ A_i \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) = \sum_{j=0}^{h} A_i E_i^* A_j E_i^* = \left( \sum_{j=0}^{h} k_j \right) A_i E_i^*. \]

So (i) holds for \( g = i \).

If \( g > i \), then for any \( j \in [h] \), by Lemmas 4.2(i), 4.6(ii), and 4.7(ii),

\[ A_g E_i^* A_j E_i^* = E_g A_g E_i^* A_j E_i^* = k_i G_{ki} A_j = k_j k_i G_{ki} = k_j E_g A_g E_i^*. \]

Thus,

\[ A_g \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) = \sum_{j=0}^{h} A_g E_i^* A_j E_i^* = \left( \sum_{j=0}^{h} k_j \right) E_g A_g E_i^*. \]

So (i) holds for \( g > i \).

The proof of (ii) is similar.

Now we are ready to prove Theorem 2.6, which claims that each nonzero element of the set

\[ \{ F_{ih} : i \in \{1, 2, \ldots, d\}, h \in [i - 1] \} \]

where

\[ F_{ih} = \begin{cases} 
\frac{\sum_{j=0}^{h} E_i^* A_j E_i^*}{\sum_{j=0}^{k_j}} - \frac{\sum_{j=0}^{h+1} E_i^* A_j E_i^*}{\sum_{j=0}^{k_j}}, & \text{if } h < i - 1; \\
\sum_{j=0}^{i-1} E_i^* A_j E_i^* - G_{ii}, & \text{if } h = i - 1;
\end{cases} \]

is a central idempotent that spans a 1-dimensional ideal of \( T(x) \).

**Proof of Theorem 2.6.** Let \( i \in \{1, 2, \ldots, d\} \) and \( h \in [i - 1] \). Clearly \( F_{ih} \neq 0 \) if \( h < i - 1 \).

From Theorem 2.3 and Proposition 2.1 we see that \( E_i^* A_j E_i^* = 0 \) if and only if \( n_i = 2 \) if and only if \( k_i = k_0 + k_1 + \cdots + k_{i-1} \). Hence, \( F_{i,i-1} = 0 \) if and only if \( n_i = 2 \). Therefore, the set \( \{ F_{ih} : i \in \{1, 2, \ldots, d\}, h \in [i - 1] \} \) has \( d(d + 1)/2 - b \) nonzero elements, where \( b = |\{i \in \{1, 2, \ldots, d\} : n_i = 2\}| \).

In the following we show that \( F_{ih}^2 = F_{ih} \), for all \( i \in \{1, 2, \ldots, d\} \) and \( h \in [i - 1] \). First if we assume that \( h < i - 1 \), then by Lemmas 4.2(i) and 4.10(ii),

\[ \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) E_i^* A_{i+1} E_i^* = \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) A_{i+1} = \left( \sum_{j=0}^{h} k_j \right) E_i^* A_{i+1} E_i^*. \]

Thus, Lemma 4.9 implies that

\[ \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) \left( \sum_{j=0}^{h+1} E_i^* A_j E_i^* \right) = \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right)^2 + \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) E_i^* A_{i+1} E_i^* \]

\[ = \left( \sum_{j=0}^{h} k_j \right) \left( \sum_{j=0}^{h+1} E_i^* A_j E_i^* \right). \]
Similarly,
\[
\left( \sum_{j=0}^{h+1} E_i^* A_j E_i^* \right) \left( \sum_{j=0}^{h} E_i^* A_j E_i^* \right) = \left( \sum_{j=0}^{h} k_j \right) \left( \sum_{j=0}^{h+1} E_i^* A_j E_i^* \right).
\]

Therefore, by Lemma 4.9 we see that
\[
F_{ih}^2 = F_{ih}, \quad \text{for any } i \in \{1, 2, \ldots, d\} \text{ and } h \in [i - 2].
\]

Now assume that \( h = i - 1 \). By Lemma 4.7(ii),
\[
\left( \sum_{j=0}^{i-1} E_i^* A_j E_i^* \right) G_{ii} = \sum_{j=0}^{i-1} E_i^* A_j G_{ii} = \left( \sum_{j=0}^{i-1} k_j \right) G_{ii}.
\]

Similarly,
\[
G_{ii} \left( \sum_{j=0}^{i-1} E_i^* A_j E_i^* \right) = \left( \sum_{j=0}^{i-1} k_j \right) G_{ii}.
\]

Thus, Lemmas 4.9 and 4.5 yield that
\[
F_{i,i-1}^2 = F_{i,i-1}, \quad \text{for any } 1 \leq i \leq d.
\]

Therefore, each nonzero element in the set \( \{F_{ih} : i \in \{1, 2, \ldots, d\}, h \in [i - 1]\} \) is an idempotent.

Now we prove that each nonzero element in the set \( \{F_{ih} : i \in \{1, 2, \ldots, d\}, h \in [i - 1]\} \) is a central idempotent that spans a 1-dimensional ideal of the Terwilliger algebra \( \mathcal{T}(x) \). Clearly this is a direct consequence of the following

\[
A_g F_{ih} = F_{ih} A_g = \begin{cases} 
  k_g F_{ih}, & \text{if } g \leq h; \\
  -\left( \sum_{j=0}^{h} k_j \right) F_{ih}, & \text{if } g = h + 1; \\
  0, & \text{if } g > h + 1.
\end{cases} \tag{4.4}
\]

From Lemmas 4.7 and 4.10, we see that (4.4) holds if \( g \leq h \) or \( g > h + 1 \). Now we show that (4.4) is true for \( g = h + 1 \). If \( h < i - 1 \), then by Lemma 4.10(i),
\[
A_{h+1} F_{ih} = E_i^* A_{h+1} E_i^* - \frac{k_{h+1} \sum_{j=0}^{h+1} E_i^* A_j E_i^*}{\sum_{j=0}^{h+1} k_j} = \frac{\left( \sum_{j=0}^{h} k_j \right) \sum_{j=0}^{h+1} E_i^* A_j E_i^* - \left( \sum_{j=0}^{h+1} k_j \right) \sum_{j=0}^{h} E_i^* A_j E_i^*}{\sum_{j=0}^{h+1} k_j} = -\left( \sum_{j=0}^{h} k_j \right) F_{ih}.
\]

Similarly,
\[
F_{ih} A_{h+1} = -\left( \sum_{j=0}^{h} k_j \right) F_{ih}.
\]
Thus, (4.4) holds if $h < i - 1$ and $g = h + 1$. If $h = i - 1$ and $g = h + 1$, then by Lemmas 4.10 and 4.7,

$$A_i F_{i,i-1} = A_i E_i^* - k_i \sum_{j=0}^{i-1} G_{ji} - \left( k_i - \sum_{j=0}^{i-1} k_j \right) G_{ii}$$

$$= A_i E_i^* - \sum_{j=0}^{i-1} E_j^* A_i E_i^* - E_i^* J E_i^* + \left( \sum_{j=0}^{i-1} k_j \right) G_{ii}.$$ 

Since $A_i E_i^* = (E_0^* + E_1^* + \cdots + E_i^*) A_i E_i^*$ by Lemma 4.2(iv) and $E_i^* J E_i^* = E_i^* (A_0 + A_1 + \cdots + A_i) E_i^*$ by Lemma 4.2(iii), we obtain that

$$A_i E_i^* - \sum_{j=0}^{i-1} E_j^* A_i E_i^* - E_i^* J E_i^* = - \sum_{j=0}^{i-1} E_i^* A_j E_i^*.$$ 

Thus,

$$A_i F_{i,i-1} = - \left( \sum_{j=0}^{h} k_j \right) F_{i,i-1}.$$ 

Similarly,

$$F_{i,i-1} A_i = - \left( \sum_{j=0}^{h} k_j \right) F_{i,i-1}.$$ 

So (4.4) holds if $h = i - 1$ and $g = h + 1$. This proves (4.4). Therefore, the theorem holds. 

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