Soliton Solutions for ABS Lattice Equations: II: Casoratians and Bilinearization

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Abstract

In Part I soliton solutions to the ABS list of multi-dimensionally consistent difference equations (except Q4) were derived using connection between the Q3 equation and the NQC equations, and then by reductions. In that work central role was played by a Cauchy matrix. In this work we use a different approach, we derive the \(N\)-soliton solutions following Hirota’s direct and constructive method. This leads to Casoratians and bilinear difference equations. We give here details for the H-series of equations and for Q1; the results for Q3 have been given earlier.

1 Introduction

The analysis of integrability for discrete systems is now in active development. Discreteness introduces many complications in comparison with continuous integrability, mainly due to the lack of Leibniz rule for discrete derivatives. This also includes the fact that space-time itself has many discretizations.

The basic philosophy and definitions have been given in Part I \cite{1}, here we repeat only some essential ingredients. The underlying space is formed by a Cartesian square lattice and the dynamical equation is defined on an elementary square of this lattice. (There are other possible settings but even this case has not been fully analyzed.) As for the definition of integrability we choose “Consistency-around-the cube” (CAC) which is further explained in Section \cite{2,1}. With this choice of integrability many nice properties follow, including the existence of a Lax pair. Furthermore, with mild additional assumption one can classify the integrable models \cite{2} and the “ABS list” is surprisingly short.

In this paper we construct multi-soliton solutions for the H-series of models in the ABS-list, as well as for the Q1 model. Our approach is constructive for each model and is

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based on the approach of Hirota, as explained in Section 2.2. Some work on this problem already exists for the two highest members of the ABS-list, but the Q4 results in [3] is far from explicit and the result for Q3 [4] relied on a specific association with the NQC-model, which has been further elaborated in Part I.

We hope that the detailed analysis of the simpler models, for which everything can be made systematic and explicit, will provide further understanding on the soliton question and that it could be used for other models as well.

In the next section we will give the general background for our approach and then in subsequent sections we will go through the models H1, H2, H3 and Q1.

2 Generalities

2.1 Multidimensional consistency and the ABS list

As in Part I we only consider quadrilateral lattice equations defined by a multi-linear relation on the values at the four corners of an elementary square of Cartesian lattice, see Figure 2.1(a). Given a base point $u_{nm} = u$ we indicate the shifts in the $n,m$ directions by a tilde or a hat as described in Figure 2.1.

Multidimensional consistency is an essential ingredient in the construction of soliton solutions, the new dimensions standing for parameters of the solitons. The consistency is defined as follows: We adjoin a third direction and construct an elementary cube as in Figure 2.1(b), the shifts in the new third direction are denoted by a bar, e.g., $u_{n,m,k+1} = \overline{u}$.

On the base square we have the equation $Q(u, \tilde{u}, \hat{u}, \tilde{\tilde{u}}; p,q) = 0$, and we now introduce the same equation with different variables on the sides and on the top, thus we will have altogether the following set of equations

\begin{align}
Q(u, \tilde{u}, \hat{u}, \tilde{\tilde{u}}; p,q) &= 0, & Q(\overline{u}, \tilde{u}, \hat{u}, \tilde{\tilde{u}}; p,q) &= 0, \\
Q(u, \tilde{u}, \overline{\overline{u}}, \tilde{\tilde{u}}; p,r) &= 0, & Q(\hat{u}, \tilde{u}, \overline{\overline{u}}, \tilde{\tilde{u}}; p,r) &= 0, \\
Q(u, \hat{u}, \overline{\overline{u}}, \tilde{\tilde{u}}; q,r) &= 0, & Q(\tilde{u}, \hat{u}, \overline{\overline{u}}, \tilde{\tilde{u}}; q,r) &= 0.
\end{align}

(2.1a) (2.1b) (2.1c)

Now considering Figure 2.1(b) we can have initial values given at black circles $(u, \tilde{u}, \hat{u}, \overline{u})$, and in order to compute the remaining four values we have six equations. We can use the

![Figure 1](a): The points on which the map is defined, and (b): the consistency cube.
LHS equations in (2.1a), (2.1b), (2.1c) to compute $\hat{u}, \overline{u}, \overline{u}$, respectively, and this leaves the three RHS equations from which we should get the same value for $\overline{u}$, and this implies two consistency conditions.

In [2] a classification of quadrilateral lattice equations was performed following the above definition of integrability, with two additional requirements on the equations: symmetry and the so called “tetrahedron property”, which means that the computed value for $\overline{u}$ should only depend on $\hat{u}, \overline{u}, \overline{u}$ and not on $u$. The ABS list is the following:

\[(u - \hat{u})(\overline{u} - \overline{u}) + q - p = 0, \quad (H1)\]
\[(u - \hat{u})(\overline{u} - \overline{u}) + (q - p)(u + \hat{u} + \overline{u} + \overline{u}) + q^2 - p^2 = 0, \quad (H2)\]
\[p(u\overline{u} + \hat{u}\overline{u}) - q(u\overline{u} + \hat{u}\overline{u}) + \delta(p^2 - q^2) = 0, \quad (H3)\]
\[p(u + \hat{u})(\hat{u} + \overline{u}) - q(u + \overline{u})(\overline{u} + \overline{u}) - \delta^2pq(p - q) = 0, \quad (A1)\]
\[(q^2 - p^2)(u\overline{u}\overline{u} + 1) + q(p^2 - 1)(u\overline{u} + \overline{u}\overline{u}) - p(q^2 - 1)(u\overline{u} + \overline{u}\overline{u}), \quad (A2)\]
\[p(u - \hat{u})(\hat{u} - \hat{u}) - q(u - \overline{u})(\hat{u} - \hat{u}) + \delta^2pq(p - q) = 0, \quad (Q1)\]
\[p(u - \hat{u})(\overline{u} - \overline{u}) - q(u - \hat{u})(\overline{u} - \overline{u}) + pq(p - q)(u + \overline{u} + \hat{u} + \hat{u}) - pq(p - q)(p^2 - pq + q^2) = 0, \quad (Q2)\]
\[(q^2 - p^2)(u\overline{u} + \hat{u}\overline{u}) + q(p^2 - 1)(u\overline{u} + \hat{u}\overline{u}) - p(q^2 - 1)(u\overline{u} + \hat{u}\overline{u}) - \delta^2(p^2 - q^2)(p^2 - 1)(q^2 - 1)/(4pq) = 0, \quad (Q3)\]
\[[h(p)f(q) - h(q)f(p)][(u\overline{u}\overline{u} + 1)f(p)f(q) - (u\overline{u} + \overline{u}\overline{u})] + (f(p)^2f(q)^2 - 1)[(u\overline{u} + \overline{u}\overline{u})f(p) - (u\overline{u} + \overline{u}\overline{u})f(q)] = 0, \quad (Q4)\]

where $h^2 = f^4 + \delta f^2 + 1$. This can be parameterized with Jacobi elliptic functions: $f(x) = k^2\text{sn}(x,k)$, $h(x) = \text{sn}'(x,k)$, $\delta = -(k + 1/k)$. The simpler form of Q4 given above was actually discovered later in [5].

The A-series is auxiliary in the sense that A1 goes to Q1 with $u \rightarrow (-1)^{n+m}u$ and A2 to $Q_{3\delta=0}$ by $u \rightarrow u^{(-1)^{n+m}}$. We will not discuss them here.

### 2.2 Hirota’s bilinear method

Our approach to soliton solutions is based on Hirota’s bilinear method[6]. It has been very successful in the continuous case and it is expected to be equally fruitful in constructing discrete multi-soliton solutions. Hirota’s idea was to make a dependent variable transform into new variables, for which the soliton solution would be given by a polynomial of
exponentials. In terms of these new dependent variables the dynamical equations were quadratic and derivatives appeared only in terms of Hirota’s bilinear derivatives.

But what would be the natural discrete generalization? The essential property of an equation in Hirota’s bilinear form seems to be its gauge invariance (cf., \[7\] in the continuous case) and it has a natural discrete extension, leading us to the following definition:

**Definition 1.** We say an equation is in Hirota bilinear (HB) form if it can be written as

\[
\sum_j c_j f_j(n + \nu_j^+, m + \mu_j^+) g_j(n + \nu_j^-, m + \mu_j^-) = 0 \tag{2.2}
\]

where the index sums \(\nu_j^+ + \nu_j^- = \nu^e, \mu_j^+ + \mu_j^- = \mu^e\) do not depend on \(j\). (The functions \(f, g\) may be the same.)

**Proposition 1.** Equations in HB form are gauge invariant, i.e., if functions \(f_j, g_j\) solve a set of equations in HB form, then so do the gauge transformed functions

\[
f_j'(n, m) = A^n B^m f_j(n, m), \quad g_j'(n, m) = A^n B^m g_j(n, m). \tag{2.3}
\]

**Proof.** We find

\[
f_j'(n + \nu_j^+, m + \mu_j^+) g_j'(n + \nu_j^-, m + \mu_j^-) = A^{2n+\nu_j^++\nu_j^-} B^{2m+\mu_j^++\mu_j^-} f_j(n + \nu_j^+, m + \mu_j^+) g_j(n + \nu_j^-, m + \mu_j^-)
\]

but since the overall factor is the same in each term of the \(j\)-sum in (2.2) it can be taken out. \(\Box\)

For integrable equations in HB form there is a perturbative technique which leads to multi-soliton solutions, more or less algorithmically. This is described in the next section.

### 2.3 Constructing background solutions

First we have to construct the background or vacuum or seed solution, on top of which the soliton solutions are constructed. To do this we use the fixed-point idea [3] in which the consistent equations (2.1) are used with the assumption that \(u = \overline{u}\). However, since some equations are invariant under \(u \rightarrow T(u)\) it is actually sufficient that \(u = T(u)\). We only consider global invariances (that is, the transformation \(T\) is independent of \(n, m\)) and in order to keep the multi-linearity we assume that the transformation is linear fractional:

\[
u_{nm} \rightarrow T(u_{nm}) := \frac{c_1 u_{nm} + c_2}{c_3 u_{nm} + c_4}, \quad c_1 c_4 - c_2 c_3 \neq 0. \tag{2.4}
\]

Depending on the equation there will be conditions on the parameters \(c_i\). It is straightforward to find the invariances of the equations, they are given in Table 1. Note that the special cases with \(\delta = 0\) have a bigger invariance group. For Q4 there are various special cases depending on which limiting case of the elliptic parameterization one chooses.
\[
\begin{array}{|c|c|c|}
\hline
\text{Eq.} & T(u) & T(u) \text{ when } \delta = 0 \\
\hline
H1 & u + c, -u + c & \text{NA} \\
H2 & u & \text{NA} \\
H3 & u, -u & cu, c/u \\
A1 & u, -u & cu, c/u \\
A2 & u, -u, 1/u, -1/u & \text{NA} \\
Q1 & u + c, -u + c & \text{full Möbius} \\
Q2 & u & \text{NA} \\
Q3 & u, -u & cu, c/u \\
Q4 & u, -u, 1/u, -1/u & \text{various} \\
\hline
\end{array}
\]

Table 1: Invariances (2.4) of the equations in the ABS list.

For each invariance \( T \) of a given equation we then need to solve
\[
\begin{align*}
Q(u, \tilde{u}, T(u), T(\tilde{u}); p, r) &= 0, \\
Q(u, \hat{u}, T(u), T(\hat{u}); q, r) &= 0,
\end{align*}
\]
(2.5a) (2.5b)
in order to obtain the corresponding background solution (here \( r \) is a parameter of the solution). Such a solution then automatically solves (2.1a) by virtue of CAC.

It may be possible to construct still further solutions that can be called “background solutions”, but we will not consider them here.

### 2.4 Constructing 1-soliton solutions

Once the background solution is obtained we construct a one-soliton solution (1SS) using the CAC cube once more, with \( \overline{\pi} \) now being the 1SS. This amounts, in fact, to a Bäcklund transformation (BT).

Thus the first task is to solve
\[
\begin{align*}
Q(u, \tilde{\overline{\pi}}, T(u), T(\tilde{\overline{\pi}}); p, r) &= 0, \\
Q(u, \hat{\overline{\pi}}, T(u), T(\hat{\overline{\pi}}); q, r) &= 0,
\end{align*}
\]
(2.6a) (2.6b)
where we take
\[
\overline{\pi} = \overline{\pi}_0 + v.
\]
Here \( \overline{\pi}_0 \) is the background solution and the bar-shift stands for possible modifications to get the \( v \)-equations into a suitable form. Then solving for \( \tilde{v} \) and \( \hat{v} \) from (2.6) we get (for quadratic \( Q \))
\[
\tilde{v} = \frac{Ev}{v + F}, \quad \hat{v} = \frac{Gv}{v + H},
\]
(2.8)
where \( E, F, G, H \) may depend on \( n, m \). The bar-shift modification in \( u_0 \) is to be chosen so that the numerators have no constant term. By introducing \( v = g/f \) and \( \Phi = (g, f)^T \) we can write (2.8) as a matrix equation
\[
\Phi(n + 1, m) = \mathcal{N}(n, m)\Phi(n, m), \quad \Phi(n, m + 1) = \mathcal{M}(n, m)\Phi(n, m),
\]
(2.9)
where
\[ \mathcal{N}(n, m) = \Lambda \begin{pmatrix} E & 0 \\ 1 & F \end{pmatrix}, \quad \mathcal{M}(n, m) = \mathcal{N} \begin{pmatrix} G & 0 \\ 1 & H \end{pmatrix}, \] (2.10)
where \( E, F, G, H \) may depend on \( n, m \). The constants of separation \( \Lambda, \Lambda' \) are to be chosen so that \( \hat{\Phi} = \tilde{\Phi} \). In all cases studied in this paper it turns out that
\[ \mathcal{N}(n, m) = \begin{pmatrix} S_{nm+1} \sigma & 0 \\ U_{nm} & \Delta \end{pmatrix}, \quad \mathcal{M}(n, m) = \begin{pmatrix} T_{nm+1} \tau & 0 \\ U_{nm} & \Omega \end{pmatrix}, \] (2.11)
where \( S, T, \sigma, \tau, \Delta, \Omega \) are constants and \( U_{nm} \) some function of \( n, m \) (sometimes \( U_{nm} = 1 \)).

If we now introduce \( \Psi \) by
\[ \Phi(n, m) = \begin{pmatrix} U_{nm} & 0 \\ 0 & 1 \end{pmatrix} \Psi(n, m), \] (2.12)
then for \( \Psi \) we have
\[ \Psi(n + 1, m) = \begin{pmatrix} S & 0 \\ \sigma & \Delta \end{pmatrix} \Psi(n, m), \quad \Psi(n, m + 1) = \begin{pmatrix} T & 0 \\ \tau & \Omega \end{pmatrix} \Psi(n, m). \] (2.13)
These are compatible if
\[ \frac{\sigma}{S - \Delta} = \frac{\tau}{T - \Omega}, \] (2.14)
and in that case it is easy to derive
\[ \Psi(n, m) = \left( \frac{\tau}{T - \Omega} \right)^n \begin{pmatrix} S^n T^m & 0 \\ \Delta^n \Omega^m \end{pmatrix} \Psi(0, 0). \] (2.15)
From this we can construct \( \Phi(n, m) \) and then \( v \): Introduce
\[ \rho_{n,m} = \left( \frac{S}{\Delta} \right)^n \left( \frac{T}{\Omega} \right)^m \rho_{0,0}, \] (2.16)
and then \( v \) can be written as
\[ v_{n,m} = \frac{U_{nm} \rho_{0,0} / \rho_{n,m}^{0.0}}{1 + \frac{\tau}{T - \Omega} \frac{\rho_{n,m}}{\rho_{0,0} - 1}}. \] (2.17)
Now redefining the constant \( \rho_{0,0} \) we can also write this as
\[ v_{n,m} = \frac{U_{nm} \frac{T - \Omega}{\tau} \rho_{n,m}}{1 + \rho_{n,m}}. \] (2.18)
2.5 N-soliton solutions and Casoratians

Having a 1SS in the form mentioned above allows us to propose a change of dependent variables such that the original equation is given in terms of discrete equations in HB form. These equations will then be shown to have solutions given in Casorati determinant form, corresponding to the Wronskian form solutions for continuous HB equations. We will now discuss some generalities about the Casoratians.

In general the Casorati matrix is constructed as follows: given functions $\psi_i(n,m,l)$ we define the column vector

$$\psi(n,m,l) = (\psi_1(n,m,l), \psi_2(n,m,l), \cdots, \psi_N(n,m,l))^T,$$  \hspace{1cm} (2.19a)

and then the generic $N \times N$ Casorati matrix is composed of such columns with different shifts $l_i$, with the determinant

$$C_{n,m}(\psi; \{l_i\}) = |\psi(n,m,l_1), \psi(n,m,l_2), \cdots, \psi(n,m,l_N)|. \hspace{1cm} (2.19b)$$

Two typical Casoratians that will be used later are

$$C^1_{n,m}(\psi) := |\psi(n,m,0), \psi(n,m,1), \cdots, \psi(n,m,N-1)|$$
$$\equiv |0,1, \cdots, N-1| \equiv |\hat{N}-1|, \hspace{1cm} (2.19c)$$

$$C^2_{n,m}(\psi) := |\psi(n,m,0), \cdots, \psi(n,m,N-2), \psi(n,m,N)|$$
$$\equiv |0,1, \cdots, N-2, N| \equiv |\tilde{N}-2, N|, \hspace{1cm} (2.19d)$$

where we have also introduced the standard shorthand notation \footnote{In the Casoratians used in this work the entries $\psi_i$ in the $N$th-order vector (2.19a) are given by}

$$\psi_i(n,m,l) = g_i^+(a+k_i)^n(b+k_i)^m(c+k_i)^l + g_i^-(a-k_i)^n(b-k_i)^m(c-k_i)^l, \hspace{1cm} (2.20)$$

where $g_i^\pm$ and $k_i$ are parameters (and in some cases $c = 0$). In this form the shifts in $l$ (bar-shifts) are not in any way special and therefore we can equally well define Casoratians where the shifts are in $n$ (tilde-shifts) or in $m$ (hat-shifts). For later use we indicate these three different shifts by the shift operators $E^{\nu}$, $\nu = 1, 2, 3$, respectively, i.e.,

$$E^1\psi \equiv \tilde{\psi}, \hspace{1cm} E^2\psi \equiv \hat{\psi}, \hspace{1cm} E^3\psi \equiv \overline{\psi}.$$ 

Down shifts are denoted by $E_\nu$, $\nu = 1, 2, 3$, respectively.

We can now define a Casorati w.r.t. the three different kinds of shifts,

$$|\hat{N}-1|_{\nu} = |\psi, E^\nu \psi, (E^\nu)^2 \psi, \cdots, (E^\nu)^{N-1} \psi|, \hspace{1cm} (\nu = 1, 2, 3). \hspace{1cm} (2.21)$$

Since

$$(\alpha_\mu - \alpha_\nu)\psi = (E^\mu - E^\nu)\psi, \hspace{1cm} \mu, \nu = 1, 2, 3, \hspace{1cm} (2.22)$$
where
\[ \alpha_1 \equiv a, \quad \alpha_2 \equiv b, \quad \alpha_3 \equiv c, \] (2.23)
we have
\[ (E^\nu)^j \psi = [E^\mu - (\alpha_\mu - \alpha_\nu)]^j \psi, \quad \mu, \nu = 1, 2, 3, \]
and substituting this into (2.21) yields
\[ |\hat{N} - 1|_1 = |\hat{N} - 1|_2 = |\hat{N} - 1|_3. \] (2.24)

When the Casoratians are constructed using \( \psi \) of (2.20) the size on the matrix \( N \) indicates the number of solitons and the set \( \{k_i\}_{i=1}^N \) provides the "velocity" parameters of the solitons, while the parameters \( \{\rho^+, \rho^-\}_1^N \) are related to the locations of the solitons (by gauge invariance only their ratio is significant).

We shall here mention that the bilinear equations we get in the paper are more or less similar to the Hirota-Miwa equation\(^{[11,12]}\)
\[ a(b-c)\tau_{n,m,l+1}\tau_{n+1,m+1,l} + b(c-a)\tau_{n,m+1,l}\tau_{n+1,m,l+1} + c(a-b)\tau_{n+1,m,l}\tau_{n,m+1,l+1} = 0, \] (2.25)
(cf. (5.17,5.20)), or belong to the bilinear equations\(^{[13]}\) which were derived by imposing the transformation\(^{[12]}\)
\[ x_j = \sum_{i=1}^{\infty} l_j a_j^{i} \] (2.26)
on Sato’s bilinear identity. The above transformation, referred to as Miwa transformation, provides a connection between continuous coordinates \( \{x_j\} \) and discrete ones \( \{l_j\} \), and transforms the basic continuous plane wave factor
\[ \exp \left[ -\sum_{jx=1}^{\infty} x_j p_i + \sum_{j=1}^{\infty} x_j q_i \right] \] (2.27)
into the discrete one
\[ \prod_{j=1}^{\infty} \left( \frac{1 - a_j p_i}{1 - a_j q_i} \right)^{l_j}. \] (2.28)
The discrete exponential function (2.28) corresponding to the plane wave factor \( \rho_i \) (I-2.2), plays a central role in the discrete \( \tau \) function in Hirota’s exponential-polynomial form, while in Casoratians its counterpart has the form (2.20).

In most cases a bilinear equation that can be solved by an \( N \times N \) Casoratian (or Casoratians) is reduced to a Laplace expansion of a \( 2N \times 2N \) determinant with zero value. For later convenience we give the following lemmas.

**Lemma 1.**\(^{[9]}\)
\[ \sum_{j=1}^{N} |a_1, \ldots, a_{j-1}, ba_j, a_{j+1}, \ldots, a_N| = \left( \sum_{j=1}^{N} b_j \right) |a_1, \ldots, a_N|, \] (2.29)
where \( a_j = (a_{1j}, \ldots, a_{Nj})^T \) and \( b = (b_1, \ldots, b_N)^T \) are \( N \)-order column vectors, and \( ba_j \) stands for \( (ba_{1j}, \ldots, b_{Na_{Nj}})^T \).
Lemma 2. Suppose that $B$ is an $N \times (N-2)$ matrix and $a, b, c, d$ are $N$-order column vectors, then

$$|B, a, b||B, c, d| - |B, a, c||B, b, d| + |B, a, d||B, b, c| = 0. \quad (2.30)$$

In fact, the l.h.s. of (2.30) is just the Laplace expansion of the following $2N \times 2N$ determinant,

$$\frac{1}{2} \left| \begin{array}{cccc} B & 0 & a & b \\ 0 & B & a & b \\ c & d \\ c & d \end{array} \right| \equiv 0.$$

3 H1

3.1 Background solution and 1-soliton solution

3.1.1 The background solution

The H1 equation is given by

$$H1 \equiv (u - \tilde{u})(\tilde{u} - \tilde{u}) - (p - q) = 0. \quad (3.1)$$

Using the fixed point idea with transformation $\overline{u} = T(u) = u + c$ we get the side-equations of the CAC-cube in the form

$$(\tilde{u} - u)^2 = r + c^2 - p, \quad (\hat{u} - u)^2 = r + c^2 - q.$$  

For convenience we absorb $r$ into $c^2$, and reparameterize $(p, q) \rightarrow (a, b)$ by

$$p = c^2 - a^2, \quad q = c^2 - b^2. \quad (3.2)$$

and then the above equations factorize as

$$(\tilde{u} - u - a)(\tilde{u} - u + a) = 0, \quad (\hat{u} - u - b)(\hat{u} - u + b) = 0, \quad (3.3)$$

Since the factor that vanishes may depend on $n, m$ we actually have to solve

$$\tilde{u} - u = (-1)^\theta a, \quad \hat{u} - u = (-1)^\chi b, \quad (3.4)$$

where $\theta, \chi \in \mathbb{Z}$ may depend on $n, m$. Furthermore, since the value of the exponent is only relevant modulo 2 and $n^2 \equiv n \mod 2$, the exponents must be linear combinations of 1, $n, m, nm$ with coefficients 0 or 1.

The integrability condition for (3.4) leads to

$$b \left[ (-1)^\sigma - (-1)^\tilde{\sigma} \right] = a \left[ (-1)^\theta - (-1)^\tilde{\theta} \right].$$

and since $a, b$ are independent this just means that $\sigma = \sigma(m), \theta = \theta(n)$. Since these are defined modulo 2 we can have $\theta = t_1 n + t_0, \sigma = s_1 m + s_0$ where $t_i, s_i \in \{0, 1\}$. Furthermore, since $a, b$ were only defined up to sign we may take $t_0 = s_0 = 0.$
There are now two essentially different cases for each exponent, either \( \theta(n) = 0 \) or \( n \), and \( \sigma(m) = 0 \) or \( m \). The choice \( \theta = 0 \) leads to a solution \( u = an + \ldots \) while \( \theta = n \) leads to \( u = -\frac{1}{2}(-1)^n a + \ldots \). These solutions can be combined, and thus we get the following set of possibilities for \( u^{0SS} \)

\[
\begin{align*}
&an + bm + \gamma, \quad \text{(3.5a)} \\
&\frac{1}{2}(-1)^n a + bm + \gamma, \quad \text{(3.5b)} \\
&an + \frac{1}{2}(-1)^m b + \gamma, \quad \text{(3.5c)} \\
&\frac{1}{2}(-1)^n a + \frac{1}{2}(-1)^m b + \gamma. \quad \text{(3.5d)}
\end{align*}
\]

We could also try to get a background solution using the fixed point idea with the other transformation mentioned for \( H_1 \) in Table 1, namely \( \overline{u} = T(u) = -u + c \), but it produces the same set: with a reparameterization

\[
p = r + \alpha^2, \quad q = r + \beta^2, \quad u = y + \frac{1}{2}c.
\](3.6)

the side-equations become

\[
(\tilde{y} - y - \alpha)(\tilde{y} - y + \alpha) = 0, \quad (\hat{y} - y - \beta)(\hat{y} - y + \beta) = 0,
\](3.7)

which is as in (3.3). The constant \( c \) can be absorbed into the constant \( \gamma \) and the free parameter \( r \) redefined when comparing (3.2),(3.6), and thus the background solutions are as in (3.5).

It seems that only (3.5a) leads to a soliton type solution so in the following we only consider it.

### 3.1.2 Constructing 1SS using a BT

The BT generating 1SS for \( H_1 \) is

\[
\begin{align*}
(u - \overline{u})(\overline{u} - \overline{\overline{u}}) &= p - \varkappa, \\
(u - \overline{\overline{u}})(\overline{\overline{u}} - \overline{u}) &= \varkappa - q.
\end{align*}
\](3.8)

Here \( u \) is the seed solution (3.5a), \( \varkappa \) is the parameter in the bar-direction, and we search for a new solution \( \overline{u} \) of the form

\[
\overline{u} = \overline{u}_0 + v,
\](3.9)

where \( \overline{u}_0 \) is the bar-shifted background solution (3.5a):

\[
\overline{u}_0 = an + bm + k + \gamma,
\](3.10)

where \( k \) is related to \( \varkappa \) by

\[
\varkappa = c^2 - k^2.
\](3.11)

Substituting (3.5a), (3.9) and (3.10) into (3.8) yields the desired form

\[
\overline{v} = \frac{Ev}{v + F}, \quad \overline{v}_v = \frac{Gv}{v + H},
\](3.12)
where
\[ E = -(a + k), \quad F = -(a - k), \quad G = -(b + k), \quad H = -(b - k). \]

Now matrices \( N, M \) defined in (2.10) are \( n, m \) independent and since \( E - F = G - H = -2k \) they commute and we can take \( \Lambda = \Lambda' = 1 \). Then, following the method presented in Section 2.4, we find
\[
\Phi(n, m) = \left( \frac{E^n G^m}{-2k} \right) \Phi(0, 0),
\]
and if we let
\[
\rho_{n,m} = \left( \frac{E}{F} \right)^n \left( \frac{G}{H} \right)^m \rho_{0,0} = \left( \frac{a + k}{a - k} \right)^n \left( \frac{b + k}{b - k} \right)^m \rho_{0,0},
\]
then we obtain
\[
v_{n,m} = -\frac{2k\rho_{n,m}}{1 + \rho_{n,m}}.
\]

Finally we obtain the 1SS for H1:
\[
u_{n,m}^{1SS} = (an + bm + \gamma) + k + \frac{-2k\rho_{n,m}}{1 + \rho_{n,m}}.
\]

3.2 Multi-soliton solutions

In an explicit form the 1SS given above [(3.16) with (3.14)] is
\[
u_{n,m}^{1SS} = an + bm + \gamma + k + \frac{1 + \rho_{n,m}}{1 + \rho_{n,m}} - \frac{2k\rho_{n,m}}{1 + \rho_{n,m}}.
\]
where we have separated \( \rho_{0,0} \) to \( \rho^{-} / \rho^{+} \).

The above suggests that the basic ingredient in constructing an NSS is \( \psi \) of (2.20) with \( c = 0 \), i.e.,
\[
\psi_i(n, m, l; k_i) = \rho_{i+}^n k_i^n (a + k_i)^m + \rho_{i-}^n (a - k_i)^m + \rho_{i-}^n (a - k_i)^m.
\]

Here the index \( l \) is needed as a column index in the Casorati matrix.

The form of the 1SS (3.17) suggests a generalization to NSS, and after checking explicitly the 2SS and 3SS, we arrive at the following proposition:

**Proposition 2.** \( N \)-soliton solution for H1 is given by
\[
u_{n,m}^{NSS} = an + bm + \gamma - \frac{g}{f},
\]
where \( f = \text{help} - \text{help} \), \( g = \text{help} - \text{help} \), with \( \psi \) given by (3.18).
Note that this solution is the same as (5.29) of Part I with \( A = 0 \). Indeed using (I-2.1.23,2.8c) one finds that
\[
S^{(0,0)} - \sum k_i = \frac{g}{f},
\]
and the parameters \( \varrho_i^\pm \) in (3.18) and \( c_i \) in (I-2.3) are related by
\[
\frac{\varrho_i^+}{\varrho_i^-} = \frac{c_i}{2k_i} \prod_{j \neq i} \frac{k_j - k_i}{k_j + k_i}.
\]
(3.20)

However, the solution (3.19) contains more freedom in the form of the additional parameter \( c \), cf. (3.2) and (I-5.1d).

### 3.2.1 The bilinear form of \( H_1 \)

In order to prove Proposition 2 we write (3.1) in bilinear form. First we introduce a dependent variable transformation
\[
u_{n,m}^{\text{NSS}} = a_n + b_m + \gamma - \frac{g_{n,m}}{f_{n,m}}.
\]
(3.21)

When this is substituted into (3.1) we get a rational expression quartic in \( f, g \). In order to split this expression into bilinear equations we note that if \( f = |\hat{N} - 1|_{\text{par}}, g = |\hat{N} - 2, N|_{\text{par}} \), then for \( N = 2 \) one can quickly scan for possible equations solved by them from the set
\[
\begin{align*}
a_0\hat{g}\hat{f} + a_1\hat{g}\hat{f} + a_2\hat{g}\hat{f} + a_3\hat{g}\hat{f} + a_4\hat{f}\hat{f} + a_5\hat{f}\hat{f} + a_6\hat{g}\hat{g} + a_7\hat{g}\hat{g} &= 0,
\end{align*}
\]
and find the bilinear equations
\[
\mathcal{H}_1 \equiv \hat{g}\hat{f} - \hat{g}\hat{f} + (a - b)(\hat{f}\hat{f} - \hat{f}\hat{f}) = 0,
\]
(3.22a)
\[
\mathcal{H}_2 \equiv \hat{g}\hat{f} - \hat{g} + (a + b)(\hat{f}\hat{f} - \hat{f}\hat{f}) = 0.
\]
(3.22b)

After this it is easy to show that
\[
H_1 \equiv (u - \hat{u})(\hat{u} - \hat{u}) - p + q = -[\mathcal{H}_1 + (a - b)f\hat{f}][\mathcal{H}_2 + (a + b)f\hat{f}]/(f\hat{ff}\hat{f}) + (a^2 - b^2),
\]
and thus the pair (3.22) can be taken as the bilinear form of (3.1).

### 3.2.2 Proof of the Proposition 2

Now it remains to show that the \( f, g \) given in Proposition 2 solve equations (3.22). We prove (3.22a) in its tilde-hat-downshifted version,
\[
\hat{g}\hat{f} - \hat{g}\hat{f} + (a - b)(\hat{f}\hat{f} - \hat{f}\hat{f}) = 0.
\]
(3.23)
Comparing $\psi(n, m, l)$ given by (3.11) and (A.1), we can use the formulas given in Appendix A with $c \equiv 0$, and in this subsection we always have shifts in the $l$-index, i.e., we take $\kappa = 3$ for the formulas in Appendix A. Let us write (3.23) as

$$-(a - b)f \hat{f} + \hat{f}(g + af) - f(\hat{g} + bf) = 0. \quad (3.24)$$

In this formula, $f = \langle N - 1 \rangle_{[3]}$, and for $f$, $f$, $g + af$, $\hat{f}$ and $g + bf$ we use (A.6k) with $\mu = 1$ and $\nu = 2$, (A.6a) with $\mu = 2$, (A.7a) with $\mu = 1$, (A.6a) with $\mu = 1$ and (A.7b) with $\mu = 2$, respectively. Then we have

$$a^{N-2}b^{N-2}[-(a - b)f \hat{f} + \hat{f}(g + af) - f(\hat{g} + bf)]$$

$$= -|\hat{N} - 1|_{[3]}|N - 3, \hat{\psi}(N - 2), \hat{E}^2\psi(N - 2)|_{[3]}$$

$$+ |\hat{N} - 2, \hat{\psi}(N - 2)|_{[3]}|N - 3, N - 1, \hat{\psi}(N - 2)|_{[3]}$$

$$- |\hat{N} - 2, \hat{\psi}(N - 2)|_{[3]}|N - 3, N - 1, \hat{\psi}(N - 2)|_{[3]}$$

$$= 0,$$

where we have made use of Lemma 2 in which $B = (N - 3)$, and $a = \psi(N - 2)$, $b = \psi(N - 1)$, $c = \psi(N - 2)$ and $d = \hat{E}^2\psi(N - 2)$.

For (3.22b) we use its tilde-down-shifted version

$$g \hat{f} - \hat{g}f + (a + b)(f \hat{f} - \hat{f}) = 0, \quad (3.25)$$

and rewrite it as

$$-(a + b)f \hat{f} + \hat{f}(g + af) - f(\hat{g} - bf) = 0,$$

where $f = |\hat{N} - 1|_{[3]}$, and for $\hat{f}$, $\hat{f}$, $g + af$, $\hat{f}$ and $\hat{g} - bf$ we use (A.6i) with $\mu = 2$ and $\nu = 1$, (A.6a) with $\mu = 2$, (A.7a) with $\mu = 1$, (A.6a) with $\mu = 1$ and (A.7b) with $\mu = 2$, respectively. Then we have

$$\frac{1}{|\Omega_z|}a^{N-2}b^{N-2}[-(a + b)f \hat{f} + \hat{f}(g + af) - f(\hat{g} - bf)]$$

$$= -|\hat{N} - 1|_{[3]}|N - 3, \hat{\psi}(N - 2), \hat{E}^2\psi(N - 2)|_{[3]}$$

$$- |\hat{N} - 2, \hat{E}^2\psi(N - 2)|_{[3]}|N - 3, N - 1, \hat{\psi}(N - 2)|_{[3]}$$

$$+ |\hat{N} - 2, \hat{\psi}(N - 2)|_{[3]}|N - 3, N - 1, \hat{E}^2\psi(N - 2)|_{[3]}$$

$$= 0,$$

where we have made use of Lemma 2 in which $B = (N - 3)$, and $a = \psi(N - 2)$, $b = \psi(N - 1)$, $c = \psi(N - 2)$ and $d = \hat{E}^2\psi(N - 2)$. \qed
4 H2

4.1 Background solution

The equation H2 is given by

\[ H^2 \equiv (u - \tilde{u})(\tilde{u} - \hat{u}) - (p - q)(u + \tilde{u} + \hat{u} + p + q) = 0. \]  \hspace{1cm} (4.1)

After reparameterization

\[ p = r - a^2, \quad q = r - b^2, \]  \hspace{1cm} (4.2)

the equations on sides become

\[ (\tilde{u} - u)^2 - 2a^2(\tilde{u} + u) + a^2(a^2 - 2r) = 0, \quad (\tilde{u} - u)^2 - 2b^2(\tilde{u} + u) + b^2(b^2 - 2r) = 0. \]  \hspace{1cm} (4.3)

After the further substitution

\[ u = y^2 - \frac{1}{2}r \]  \hspace{1cm} (4.4)

the equations (4.3) factorize as

\[ (\tilde{y} + y + a)(\tilde{y} + y - a)(\tilde{y} - y + a)(\tilde{y} - y - a) = 0, \]  \hspace{1cm} (4.5a)

\[ (\tilde{y} + y + b)(\tilde{y} + y - b)(\tilde{y} - y + b)(\tilde{y} - y - b) = 0. \]  \hspace{1cm} (4.5b)

Thus we have to solve

\[ \tilde{y} - (-1)^\sigma y = (-1)^\theta a, \quad \tilde{y} - (-1)^\rho y = (-1)^\phi b. \]  \hspace{1cm} (4.6)

where the exponents \( \sigma, \theta, \rho, \phi \in \mathbb{Z} \) are again some linear combination of \( 1, n, m, nm \) with coefficients 0 or 1. Equations (4.6) are satisfied if

\[ \rho = n s_2 + r_1, \quad \sigma = m s_2 + s_1, \quad \theta = m \rho + \sigma + n(t_n + s_1) + t_1, \quad \phi = n \sigma + \rho + m(p_m + r_1) + p_1, \]

but most of the freedom is superfluous: Since the sign of \( y \) was left undetermined in (4.4) we can redefine

\[ y_{n,m} \rightarrow (-1)^{nm} s_2 + n s_1 + m r_1 \ y_{n,m} \]

and then the equations simplify to

\[ \tilde{y} - y = (-1)^{n t_n + t_1} a, \quad \tilde{y} - y = (-1)^{m p_m + p_1} b, \]

already analyzed after (3.4). Thus we find the possible background solutions \( u^{0SS}_{n,m} \) of the form

\[ [a n + b m + \gamma]^2 - \frac{1}{2} r, \]  \hspace{1cm} (4.7a)

\[ \frac{1}{2} (-1)^n a + mb + \gamma]^2 - \frac{1}{2} r; \]  \hspace{1cm} (4.7b)

\[ [n a + \frac{1}{2} (-1)^m b + \gamma]^2 - \frac{1}{2} r; \]  \hspace{1cm} (4.7c)

\[ \frac{1}{2} (-1)^n a + \frac{1}{2} (-1)^m b + \gamma]^2 - \frac{1}{2} r. \]  \hspace{1cm} (4.7d)
4.2 1-soliton solution

Note that H2 is invariant under simultaneous translation of \( p, q \) by \( t \) and \( u \) by \(-\frac{1}{2}t\). We use this freedom to eliminate \( r \) in order to simplify the presentation.

Now we take \((4.7a)\) as the seed solution to construct \( u_{n,m}^{1SS} \) for H2 through its Bäcklund transformation

\[
(u - \tilde{u})(\tilde{u} - \pi) = (p - \kappa)(u + \tilde{u} + \pi + \tilde{\pi} + p + \kappa),
\]

\[
(\tilde{u} - \tilde{\pi})(\tilde{\pi} - \tilde{u}) = (\kappa - q)(u + \tilde{u} + \tilde{\pi} + \kappa + q),
\]

where \( u \) is the seed solution \((4.7a)\) and we search for a new solution \( \pi \) of the form

\[
\pi = \pi_0 + v,
\]

where \( \pi_0 \) is the bar-shifted background solution \((4.7a)\):

\[
\pi_0 = (an + bm + k + \gamma)^2,
\]

with

\[
\kappa = k^2.
\]

Substituting these into \((4.8)\) yields \((2.10)\) with

\[
E = -2(k + a)[(n + 1)a + mb + \gamma], \quad F = -2(k - a)[na + mb + \gamma],
\]

\[
G = -2(k + b)[na + (m + 1)b + \gamma], \quad H = -2(k - b)[na + mb + \gamma].
\]

and the corresponding matrices \( \mathcal{N}, \mathcal{M} \) are compatible if we take

\[
\Lambda = \Lambda' = -1/(2U_{n,m}), \quad U_{n,m} := an + bm + \lambda.
\]

With this \( U \) we obtain \((2.11)\) with

\[
S = a + k, \quad \Delta = a - k, \quad T = b + k, \quad \Omega = b - k, \quad \sigma = \tau = -1/2.
\]

Then defining

\[
\rho_{n,m} = \left( \frac{S}{\Delta} \right)^n \left( \frac{T}{\Omega} \right)^m \rho_{0,0} = \left( \frac{a + k}{a - k} \right)^n \left( \frac{b + k}{b - k} \right)^m \rho_{0,0},
\]

we find

\[
v_{n,m} = \frac{-4kU_{n,m}\rho_{n,m}}{1 + \rho_{n,m}},
\]

and finally we get the 1-soliton for H2 in the form

\[
v_{n,m}^{1SS} = U_{n,m}^2 + 2kU_{n,m} \frac{1 - \rho_{n,m}}{1 + \rho_{n,m}} + k^2.
\]
4.3 Multi-soliton solution

Motivated by the structure of 1SS (4.16) (cf. (3.17)), and after checking 2- and 3-soliton solutions we propose the following Casoratian expression for the NSS:

\[ u_{n,m}^{NSS} = U_{n,m}^2 - 2U_{n,m} \frac{N - 2, N_{[3]}}{|N - 1_{[3]}|} + \frac{N - 3, N - 1, N_{[3]}}{|N - 2, N + 1_{[3]}|}, \]  

where the matrix entries are as for H1 (3.18). In order to prove this we derive a bilinear form of H2. We propose

\[ u_{n,m}^{NSS} = U_{n,m}^2 - 2U_{n,m} \frac{g}{f} + \frac{h + s}{f}, \]  

where \( f, h \) and \( s \) should satisfy

\[ h - s = \gamma f, \]  

where \( \gamma \) is some constant. Indeed, if

\[ f = |N - 1_{[3]}|, g = |N - 2, N_{[3]}|, s = |N - 3, N - 1, N_{[3]}|, h = |N - 2, N + 1_{[3]}|, \]

then noting that \((E^3)^2 \psi_i(n, m, l) = \psi_i(n, m, l) = \psi_i(n, m, l + 2) = k_i^2 \psi_i(n, m, l)\) and using Lemma 1 we have \((\sum_{i=1}^N k_i^2) f = h - s\), i.e. \( \gamma = \sum_{i=1}^N k_i^2 \).

The solution (4.18) is the same as (I-5.26) with \( A = 0 \) and

\[ S^{(0,0)} = \sum_j k_j + \frac{g}{f}, \quad 2S^{(0,1)} - 2(\sum_j k_j) S^{(0,0)} + (\sum_j k_j)^2 = \frac{h + s}{f}, \]

which can be found by using (I-2.1,2.2,2.3,2.8c,5.23).

Under condition (4.19), H2 can be represented through the following bilinear system,

\[ H_1 \equiv \hat{g}\hat{f} - \hat{g}\hat{f} + (a - b)(\hat{f}\hat{f} - \hat{f}\hat{f}) = 0, \]  
\[ H_2 \equiv g\hat{f} - \hat{g}\hat{f} + (a + b)(\hat{f}\hat{f} - \hat{f}\hat{f}) = 0, \]  
\[ H_3 \equiv -(a + b)\hat{f}g + a\hat{f}g + b\hat{f}g + \hat{f}h - \hat{f}h = 0, \]  
\[ H_4 \equiv -(a - b)\hat{f}g + a\hat{f}g - b\hat{f}g + \hat{f}h - \hat{f}h = 0, \]  
\[ H_5 \equiv b(\hat{f}g - f\hat{g}) + \hat{f}h + \hat{f}s - g\hat{g} = 0, \]

in which \( H_1 \) and \( H_2 \) already appeared in (3.22). In terms of the above bilinear equations H2 can be given as

\[ H_2 = \sum_{i=1}^5 H_i P_i, \]
where we have made use of Lemma 2 in which

\begin{align*}
P_1 &= -4(a + b) \left[ (\tilde{U} \tilde{U} - a^2 + b^2) \tilde{f} \tilde{f} - \tilde{U} \tilde{f} \tilde{g} - (a + b) \tilde{f} \tilde{g} \right], \\
P_2 &= -4 \left[ (a - b)(\tilde{U} \tilde{U} - a^2 + b^2) \tilde{f} \tilde{f} + (\tilde{U} \tilde{U} - a^2 + b^2) \tilde{f} \tilde{g} - \tilde{U} \tilde{U} \tilde{f} \tilde{g} - (a - b) \tilde{U} \tilde{f} \tilde{g} \right], \\
P_3 &= 4 \left[ (a - b) \tilde{U} \tilde{f} \tilde{f} + \tilde{U} \tilde{f} \tilde{g} - \tilde{U} \tilde{f} \tilde{g} - \tilde{f} \tilde{h} + \tilde{f} \tilde{h} \right], \\
P_4 &= 4 \left[ (a + b)(\tilde{U} \tilde{f} \tilde{f} - \tilde{f} \tilde{g}) + \tilde{U} (\tilde{f} \tilde{g} - \tilde{f} \tilde{g}) \right], \\
P_5 &= 4(a^2 - b^2) \tilde{f} \tilde{f},
\end{align*}

where \( U \) was defined in (4.12). It remains to prove the following

**Proposition 3.** The Casoratian type determinants \( f, g, h \) and \( s \) given in (4.20) with entries given by (3.18) solve the set of bilinear equations (4.21).

**Proof.** Among (4.21) \( H_1 \) and \( H_2 \) have been proven before. Next we prove (4.21c) in the following form

\[ -(a + b) \tilde{f} g + \tilde{f} (h + a g) - f (\tilde{h} - b \tilde{g}) = 0, \tag{4.23} \]

which is a down-tilde-shifted version of the original one. Since \( \psi_i(n, m, l) \) given by (3.18) is just (A.1) with \( c = 0 \), we use the formulas given in Appendix A with \( c \equiv 0 \) and \( \kappa = 3 \).

In (4.23) \( g = |N - 2, N|_{[3]} \), and for \( \tilde{f}, \tilde{f}, h + a g, f \) and \( \tilde{h} - b \tilde{g} \) we use (A.6l) with \( \mu = 2 \) and \( \nu = 1 \), (A.6c) with \( \mu = 2 \), (A.8a) with \( \mu = 1 \), (A.6a) with \( \mu = 1 \) and (A.8b) with \( \mu = 2 \), respectively. Then we have

\[
\begin{align*}
\frac{1}{|\Omega_2^2|} a^{N-2} b^{N-2} \left[ -(a + b) \tilde{f} g + \tilde{f} (h + a g) - f (\tilde{h} - b \tilde{g}) \right] &= \nonumber \\
&= -|N - 2, N|_{[3]} |N - 3, \psi(N - 2), E^2 \psi(N - 2)|_{[3]} \\
&\quad -|N - 2, E^2 \psi(N - 2)|_{[3]} |N - 3, \psi(N - 2)|_{[3]} \\
&\quad +|N - 2, \psi(N - 2)|_{[3]} |N - 3, N, E^2 \psi(N - 2)|_{[3]} \\
&= 0,
\end{align*}
\]

where we have made use of Lemma 2 in which \( B = (N - 3), a = \psi(N - 2), b = \psi(N), c = \psi(N - 2) \) and \( d = E^2 \psi(N - 2) \).

(4.21d) can be proved similarly after a down-tilde-hat-shift.

Next we prove (4.21c), which can be written as

\[ f (\tilde{h} - b \tilde{g}) - g(\tilde{g} - b \tilde{f}) + \tilde{f} s = 0, \tag{4.24} \]

where \( f = |N - 1|_{[3]}, g = |N - 2, N|_{[3]}, s = |N - 3, N - 1, N|_{[3]} \), and \( \tilde{h} - b \tilde{g}, \tilde{g} - b \tilde{f} \) and \( \tilde{f} \) will be provided by (A.8b) with \( \mu = 2 \), (A.7b) with \( \mu = 2 \) and (A.6a) with \( \mu = 2 \), respectively.
respectively. Then we have

\[
\frac{1}{|\Omega_2|} b^{N-2} \left[ f(\hat{h} - b\hat{g}) - g(\hat{g} - b\hat{f}) + \hat{f}s \right]
\]

\[
= \left| N-1 \right|_{[3]} |N-3, N, \hat{E}^2 \psi(N-2)|_{[3]}
\]

- \left| N-2, N |_{[3]} |N-3, N-1, \hat{E}^2 \psi(N-2)|_{[3]}
\]

+ \left| N-2, \hat{E}^2 \psi(N-2) \right|_{[3]} |N-3, N-1, N|_{[3]}

= 0,
\]

where use has been made of Lemma 2 with \( B = (N-3), a = \psi(N-2), b = \psi(N-1), c = \psi(N) \) and \( d = \hat{E}^2 \psi(N-2) \).

\section*{5 H3}

\subsection*{5.1 Background solution}

H3\( ^\delta \) is given by

\[ H3^\delta \equiv p(u\hat{u} + \hat{u}\hat{u}) - q(u\hat{u} + \hat{u}\hat{u}) - \delta(q^2 - p^2) = 0. \] (5.1)

The side equations for \( T(x) = x \) then read

\[ r(u^2 + \hat{u}^2) - 2pu\hat{u} = \delta(p^2 - r^2), \quad r(u^2 + \hat{u}^2) - 2qu\hat{u} = \delta(q^2 - r^2). \] (5.2)

In this case we reparameterize

\[ p = r \cosh(\alpha'), \quad q = r \cosh(\beta'), \quad u_{nm} = Ae^{y_{nm}} + Be^{-y_{nm}}, \quad AB = -\frac{1}{4}r\delta \] (5.3)

and then the equations (5.2) factorize as

\[ (e^{\phi - y + \phi'} - 1)(e^{\phi - y - \alpha'} - 1)(e^{\phi + y + \phi' - \ln \frac{\phi}{\phi}} - 1)(e^{\phi + y - \alpha' - \ln \frac{\phi}{\phi}} - 1) = 0, \] (5.4a)

\[ (e^{\phi - y + \phi'} - 1)(e^{\phi - y - \beta'} - 1)(e^{\phi + y + \beta' - \ln \frac{\phi}{\phi}} - 1)(e^{\phi + y - \beta' - \ln \frac{\phi}{\phi}} - 1) = 0. \] (5.4b)

Since we only consider real \( u \) the various possibilities can be represented as in (4.6). The analysis is then the same, especially since also here the sign of \( y \) is undetermined in (5.3). Thus the solution for \( y \) is as in (3.5) and for \( u \) we have

\[ u^{0SS} = Ae^{y_{nm}} + Be^{-y_{nm}} = Ae^{\alpha'n + \beta'm + \gamma} + Be^{-\alpha'n - \beta'm - \gamma} = A\alpha^n \beta^m + B\alpha^{-n} \beta^{-m}, \quad AB = -\frac{1}{4}r\delta, \] (5.5)

where \( \alpha = e^{\alpha'}, \beta = e^{\beta'} \).

In the case of \( T(x) = -x \), one can find that the side equations are just as for \( T(x) = x \) except that \( r \to -r \). Since \( r \) is a free parameter this adds nothing new.
5.2 1-soliton solution for H$^3$δ

Now we take (5.5) as the seed solution to construct $u_{n,m}^{1SS}$ for H$^3$δ through its Bäcklund transformation
\[
p(u\tilde{u} + \tilde{u}u) - \varkappa(u\tilde{u} + \tilde{u}u) = \delta(\varkappa^2 - p^2),
\]
\[
\varkappa(u\tilde{u} + \tilde{u}u) - q(u\tilde{u} + \tilde{u}u) = \delta(q^2 - \varkappa^2),
\]
where $u$ is the seed solution (5.5), and we search for the 1SS $u_{n,m}$ of the form
\[
u = \pi_0 + v,
\]
where $\pi_0$ is the bar-shifted background solution (5.5):
\[
u_0 = A\alpha^n\beta^m\kappa + B\alpha^{-n}\beta^{-m}\kappa^{-1},
\]
with $\varkappa$ and $\kappa$ related by
\[
\varkappa = r\frac{1 + \kappa^2}{2\kappa}.
\]

Following again the procedure in Section 2.4 we get $N, M$ in the form (2.11) with
\[
S = r\frac{1 - \alpha^2\kappa^2}{2\alpha\kappa}, \quad \Delta = r\frac{-\alpha^2 + \kappa^2}{2\alpha\kappa}, \quad \sigma = p,
\]
\[
T = r\frac{1 - \beta^2\kappa^2}{2\beta\kappa}, \quad \Omega = r\frac{-\beta^2 + \kappa^2}{2\beta\kappa}, \quad \tau = q,
\]
\[
U_{n,m} = A\alpha^n\beta^m - B\alpha^{-n}\beta^{-m}.
\]

Then defining $\rho$ by
\[
\rho_{n,m} = \left(\frac{S}{\Delta}\right)^n \left(\frac{T}{\Omega}\right)^m \rho_{0,0} = \left(\frac{\alpha^2\kappa^2 - 1}{\alpha^2 - \kappa^2}\right)^n \left(\frac{\beta^2\kappa^2 - 1}{\beta^2 - \kappa^2}\right)^m \rho_{0,0},
\]
we find
\[
v_{n,m} = \frac{U_{n,m}v_{0,0}\rho_{n,m}/\rho_{0,0}}{1 - \kappa^{-2}v_{0,0}\rho_{n,m}/\rho_{0,0}} = \frac{1 - \kappa^{-2}U_{n,m}\rho_{n,m}}{1 + \rho_{n,m}},
\]
and finally
\[
u_{1SS}^{n,m} = \frac{A\alpha^n\beta^m(1 + \kappa^{-2}\rho_{n,m}) + B\alpha^{-n}\beta^{-m}(1 + \kappa^2\rho_{n,m})}{1 + \rho_{n,m}}.
\]

5.3 Bilinear form and Casoratian solutions

5.3.1 N-soliton solution

Noting that $\rho_{n,m}$ given by (5.10) is in a “twisted” form in comparison with (3.14) and (4.14), we first introduce the Möbius transformations for the parameters
\[
\alpha^2 = \frac{a - c}{a + c}, \quad \beta^2 = \frac{b - c}{b + c}, \quad \kappa^2 = \frac{k - c}{k + c}, \quad \Rightarrow \quad p^2 = \frac{r^2c^2}{c^2 - a^2}, \quad q^2 = \frac{r^2c^2}{c^2 - b^2}.
\]
which also contain a new auxiliary parameter $c$. This brings $\rho_{n,m}$ of (5.10) into the canonical form

$$\rho_{n,m} = \left( \frac{a+k}{a-k} \right)^n \left( \frac{b+k}{b-k} \right)^m \rho_{0,0}. \quad (5.13)$$

In terms of the new form of $\rho_{n,m}$ we can write the 1SS (5.11) as

$$u_{n,m}^{1SS} = A\alpha^n \beta^m \frac{\psi(n,m,l+1)}{\psi(n,m,l)} + B\alpha^{-n} \beta^{-m} \frac{\psi(n,m,l-1)}{\psi(n,m,l)}, \quad AB = -\frac{1}{4} r \delta, \quad (5.14)$$

where $\psi$ is as in (2.20).

On the basis of the above 1SS, we propose that the NSS of $H^3\delta$ can be given by

$$u_{n,m}^{NSS} = A\alpha^n \beta^m \frac{\overline{f}}{\overline{f}} + B\alpha^{-n} \beta^{-m} \frac{f}{\overline{f}}, \quad AB = -\frac{1}{4} r \delta, \quad (5.15)$$

where $f = |\overline{N} - 1|_{\pi}$ with entries (2.20). We may consider (5.15) as a dependent variable transformation for the NSS of $H^3\delta$.

The solution (5.15) is the same as (I-5.21) with $B = C = 0$. In fact, let $\frac{1}{r}$ in (5.15) equal to $a$ which is the direction parameter for the bar-shift in Part-I. Then comparing (5.12) with (I-5.1c), using (I-2.32, 5.19) and substituting $\rho_i$ in (I-2.2) by

$$\rho_i = \left( \frac{p+k_i}{p-k_i} \right)^n \left( \frac{q+k_i}{q-k_i} \right)^m \left( \frac{a+k_i}{a-k_i} \right)^l \rho_i^0, \quad (5.16)$$

one finds

$$\psi = \alpha^{-n} \beta^{-m}, \quad V(a) / \prod_j (a - k_j) = \frac{\overline{f}}{f}, \quad V(-a) \times \prod_j (a - k_j) = \frac{\overline{f}}{f},$$

with the same parameter identification as in (3.20).

### 5.3.2 Bilinearization-I

After introducing the two bilinear equations

$$B_1 \equiv 2cf \overline{f} + (a-c)f \overline{f} - (a+c)\overline{f}f = 0, \quad (5.17a)$$

$$B_2 \equiv 2cf \overline{f} + (b-c)f \overline{f} - (b+c)\overline{f}f = 0, \quad (5.17b)$$

we can represent $H^3\delta$ (5.1) as

$$H^3\delta \equiv -\alpha^{4n+2} \beta^{4m+2}(a+c)(b+c)\delta^2 P_1 + 4\alpha^{2n} \beta^{2m} \delta B^2 P_2 + 16(a+c)(b+c)B^4 P_3,$$

$$\frac{32 \alpha^{2n+2} \beta^{2m+2}(a+c)^2(b+c)^2 B^2 \overline{f} \overline{f} \overline{f} \overline{f}}{32 \alpha^{2n+2} \beta^{2m+2}(a+c)^2(b+c)^2 B^2 \overline{f} \overline{f} \overline{f} \overline{f}}$$
respectively. Then we have

\[ \kappa \]

Thus (5.17) can be considered as a bilinearization of \( H^3 \), and the final step in constructing the NSS is

**Proposition 4.** The Casoratian

\[ f = [\widehat{N} - 1]_{\nu}, \quad (\nu = 1, 2 \text{ or } 3), \]

with entries given by \( \psi \) of (2.20), solves the bilinear \( H^3 \) (5.17).

**Proof.** We prove (5.17a) in its down-tilde-shifted version

\[ 2c f f + (a - c) \widehat{f} f - (a + c) \overline{f} f = 0. \]

For this equation we use Casoratians w.r.t. hat shift, i.e., \( \kappa \equiv 2 \) in (A.6). \( f, \bar{f}, \hat{f}, \overline{f} \) and \( \hat{f} \) are given by the formulas (A.6a) with \( \mu = 1 \), (A.6b) with \( \mu = \nu = 3 \), (A.6c) with \( \mu = 3 \), (A.6d) with \( \mu = 1 \) and \( \nu = 3 \), (A.6e) with \( \mu = 3 \) and \( \nu = 1 \), and (A.6a) with \( \mu = 3 \), respectively. Then we have

\[
\begin{align*}
\frac{1}{{|P_3|}}(a-b)^{N-2}(c+b)^{N-2}(c-b)^{N-2}[2c f f + (a - c) \widehat{f} f - (a + c) \overline{f} f] \\
= -|\widehat{N} - 2, \psi(N-2)|_{[2]} |\widehat{N} - 3, \psi(N-2), \hat{E}^3 \psi(N-2)|_{[2]} \\
+ |\widehat{N} - 2, \hat{E}^3 \psi(N-2)|_{[2]} |\widehat{N} - 3, \psi(N-2), \psi(N-2)|_{[2]} \\
+ |\widehat{N} - 2, \psi(N-2)|_{[2]} |\widehat{N} - 3, \psi(N-2), \hat{E}^3 \psi(N-2)|_{[2]} \\
= 0,
\end{align*}
\]

where we have made use of Lemma 2 in which \( B = (\widehat{N} - 3) \), and \( a = \psi(N-2), b = \psi(N-2), c = \psi(N-2) \) and \( d = \hat{E}^3 \psi(N-2) \).

The other bilinear equation (5.17b) can be proved in its down-hat-shifted version in a similar way by taking \( \kappa \equiv 1 \).

### 5.3.3 Bilinearization-II

In fact there is another bilinearization of \( H^3 \) using (5.15). Consider the bilinear system

\[
\begin{align*}
\mathcal{B}_1' & \equiv (b + c) \hat{f} \hat{f} + (a - c) \hat{f} \hat{f} - (a + b) \overline{f} \overline{f} = 0, \quad (5.20a) \\
\mathcal{B}_2' & \equiv (c - b) \hat{f} \hat{f} - (a + c) \hat{f} \hat{f} + (a + b) \overline{f} \overline{f} = 0, \quad (5.20b) \\
\mathcal{B}_3' & \equiv (c - a)(b + c) \hat{f} \hat{f} + (a + c)(b - c) \hat{f} \hat{f} + 2c(a - b) \hat{f} \hat{f} = 0. \quad (5.20c)
\end{align*}
\]
This system is related to $H^3$ through

$$H^3 = \frac{c}{\text{fff}} \left[ A^2 \alpha^{2n} \beta^{2m} \frac{\hat{f} \hat{B}_1 - \hat{f} \hat{B}_2}{(a+c)(b+c)} + B^2 \alpha^{-2n} \beta^{-2m} \frac{\hat{f} \hat{B}_1 - \hat{f} \hat{B}_2}{(a-c)(b-c)} \right] + AB \left( \frac{\hat{f} \hat{B}_2 + \hat{f} \hat{B}_1}{(a+c)(b-c)} - \frac{\hat{f} \hat{B}_1 + \hat{f} \hat{B}_1}{(a-c)(b+c)} - \frac{2(a+b)\hat{f} \hat{B}_1}{a^2 - c^2} \right).$$

The bilinear system (5.20) shares the same Casoratian solutions (5.17):

**Proposition 5.** The Casoratian

$$f = [\overline{N - 1}]_{\nu}, \quad (\nu = 1, 2 \text{ or } 3),$$

with entries given by $\psi$ of (2.20), solves the bilinear $H^3$ (5.20).

**Proof.** We only prove (5.20a) and (5.20c). (5.20b) is similar to (5.20a). We prove (5.20a) in its down-tilde-shifted version, i.e.,

$$(b + c)\hat{f} \hat{f} + (a - c)\hat{f} \hat{f} - (a + b)\hat{f} \hat{f} = 0.$$  

(5.22)

We need to use Casoratians w.r.t. bar shift. So we now fix $\kappa \equiv 3$ in (A.6). For $\hat{f}, \hat{f}, \hat{f}$, $\tilde{f}$ and $\tilde{f}$, we use the formulas (A.6d) with $\mu = 2$, (A.6g) with $\mu = 1$, (A.6b) with $\mu = 1$, (A.6i) with $\mu = 2$ and (A.6n) with $\mu = 2$ and $\nu = 1$, respectively, and $f = [\overline{N - 1}]_{[3]}$. Then we have

$$\frac{1}{|\Gamma|^2} (a - c)^{N-2}(b + c)^{N-2}\left[ (b + c)\hat{f} \hat{f} + (a - c)\hat{f} \hat{f} - (a + b)\hat{f} \hat{f} \right]$$

$$= -[\overline{N - 2}, \hat{E}^2 \psi(N - 1)]_{[3]} [\overline{N - 1}, \psi(N - 1)]_{[3]}$$

$$+ [\overline{N - 2}, \overline{\hat{E}^2 \psi(N - 1)}]_{[3]} [\overline{N - 1}, \psi(N - 1)]_{[3]}$$

$$- [\overline{N - 1}]_{[3]} [\overline{N - 1}, \overline{\hat{E}^2 \psi(N - 1)}]_{[3]}$$

$$= 0,$$

where we have made use of Lemma 2 in which $B = (\overline{N - 2})$, and $a = \psi(0), b = \psi(N - 1)$, $c = \psi(N - 1)$ and $d = \hat{E}^2 \psi(N - 1)$.

For (5.20c), after a down-hat shift we get

$$2c(a - b)\hat{f} \hat{f} - (a - c)(b + c)\hat{f} \hat{f} + (b - c)(a + c)\hat{f} \hat{f} = 0.$$  

(5.23)

In this case we fix $\kappa \equiv 1$ in (A.6). For $\hat{f}, \hat{f}, \tilde{f}, \tilde{f}$ and $\tilde{f}$, we use the formulas (A.6b) with $\mu = 2$, (A.6n) with $\mu = 3$, (A.6n) with $\mu = 3$ and $\nu = 2$, (A.6b) with $\mu = 3$, (A.6b) with $\mu = 3$. 

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with \( \mu = 3 \), and (A.6m) with \( \mu = 2 \) and \( \nu = 3 \), respectively. Then

\[
\frac{1}{|\Gamma_3|}(b - a)^N - 2(c + a)^N - 2(c - a)^N - 2[2c(a - b)f\tilde{f} - (a - c)(b + c)f\tilde{f} + (b - c)(a + c)f\tilde{f}]
\]

= \( -|\overline{N - 2}, \psi(N - 1)|_{[1]}|\overline{N - 2}, \psi(N - 1)|_{\overline{2}}\)

\[+|\overline{N - 2}, \psi(N - 1)|_{[1]}|\overline{N - 2}, \psi(N - 1)|_{[2]}\]

\[+|\overline{N - 2}, \tilde{E}^3\psi(N - 1)|_{[1]}|\overline{N - 2}, \psi(N - 1)|_{\overline{2}}\]

\[=0,
\]

where in Lemma 2 we take this time \( B = (\overline{N - 2}) \), and \( a = \psi(0), b = \overline{\psi}(N - 1), c = \overline{\psi}(N - 1) \) and \( d = \tilde{E}^3\psi(N - 1) \).

\section{Q1 with linear background}

\subsection{Background solution with \( T(x) = x + c \)}

Q1\(^\delta \) is

\[Q1^\delta \equiv p(u - \tilde{u})(\tilde{u} - \overline{u}) - q(u - \tilde{u})(\overline{u} - \tilde{u}) - \delta^2 pq(q - p) = 0.\]  \hfill (6.1)

With the fixed point defined by \( T(x) = x + c \) the side equations for the background are

\[r(u - \tilde{u})^2 = p(c^2 + \delta^2 r(p - r)), \quad r(u - \overline{u})^2 = q(c^2 + \delta^2 r(q - r)).\]  \hfill (6.2)

After the reparameterization \((p, q) \rightarrow (a, b)\) with

\[p = \frac{c^2/r - \delta^2 r}{a^2 - \delta^2}, \quad q = \frac{c^2/r - \delta^2 r}{b^2 - \delta^2}, \quad \alpha := pa, \quad \beta := qb,\]  \hfill (6.3)

equations (6.2) factorize as in (3.3), and thus the 0SS will be as in (3.5), where, however, we should replace \( a \) and \( b \) with \( \alpha \) and \( \beta \), respectively.

\subsection{1-soliton solution}

The BT for constructing the 1SS is

\[p(u - \overline{u})(\overline{u} - \tilde{u}) - \kappa(u - \overline{u})(\overline{u} - \tilde{u}) = \delta^2 pq(q - p),\]  \hfill (6.4a)

\[\kappa(u - \overline{u})(\overline{u} - \overline{u}) - q(u - \overline{u})(\overline{u} - \overline{u}) = \delta^2 pq(q - p).\]  \hfill (6.4b)

Following the usual procedure we take \( u = \alpha n + \beta m + \gamma \) as the 0SS and

\[\overline{u} = \alpha n + \beta m + \gamma + \kappa + v,\]  \hfill (6.5)

as the 1SS, where \( v \) is to be determined. If we now choose

\[\kappa = \frac{c^2/r - \delta^2 r}{k^2 - \delta^2}, \quad \kappa = k\kappa,\]  \hfill (6.6)
then we get (2.10) with
\[ E = -\kappa(a + k), \quad F = -\kappa(a - k), \quad G = -\kappa(b + k), \quad H = -\kappa(b - k). \]

Thus if we define
\[ \rho_{nm} = \left(\frac{a + k}{a - k}\right)^n \left(\frac{b + k}{b - k}\right)^n \rho_{00}, \]
we find the 1SS in the form
\[ u = \alpha n + \beta m + \gamma + \kappa\frac{1 - \rho_{nm}}{1 + \rho_{nm}}, \]
where \( p, q, \kappa \) depend on \( a, b, k \) as given in (6.3), (6.6). This is similar to (3.16) except for the more complicated dependence on the parameters \( a, b, k \).

### 6.3 NSS

After studying the 2SS using Hirota’s perturbative method we propose that the NSS is obtained using the following:

\[ u_{n,m}^\text{NSS} = \alpha n + \beta m + \gamma - \left(\frac{c^2}{r} - \delta^2 r\right)\frac{g}{f}, \quad (6.7) \]

where
\[ f = |\widetilde{N} - 1|_{[3]}, \quad g = | - 1, \widetilde{N} - 1|_{[3]}, \quad (6.8) \]

with \( \psi \) defined by
\[ \psi_i(n, m, l) = \tilde{g}_i^+(a + k_i)^n(b + k_i)^m(\delta + k_i)^l + \tilde{g}_i^-(a - k_i)^n(b - k_i)^m(\delta - k_i)^l, \quad (6.9) \]

The solution (6.7) is consistent with (I-5.11): Taking \( c = 0 \) in (6.7) and \( A = D = 0, B = \delta/2 \) in (I-5.11), then using (I-2.34c,5.1b,5.4) one finds that
\[ a = \delta, \quad S(-a, a) = \sum_j \frac{1}{k_j - a} + \frac{g}{f}. \]

However, the additional parameter \( c \) (6.7) implies more freedom, cf. (6.3) and (I-5.1b).

The functions \( f, g \) satisfy the following bilinear equations (see proposition below) among others
\[ Q_1 \equiv \tilde{f} f(b - \delta) + \tilde{f} f(a + \delta) - \tilde{f} \tilde{f}(a + b) = 0, \quad (6.10a) \]
\[ Q_2 \equiv \tilde{f} f(a - b) + \tilde{f} f(b + \delta) - \tilde{f} \tilde{f}(a + \delta) = 0, \quad (6.10b) \]
\[ Q_3 \equiv -\tilde{f} \tilde{f} + \tilde{f} \tilde{g}(-a + \delta) + \tilde{f} \tilde{f} + \tilde{f} \tilde{g}(b - \delta) + \tilde{f} \tilde{g}(a - b) = 0, \quad (6.10c) \]
\[ Q_4 = \tilde{f} g(a - b) + \tilde{f} g(a + b) - \tilde{f} \tilde{g}(a + b) + \tilde{f} \tilde{g}(-a + b) = 0. \quad (6.10d) \]
We note that $Q_4$ can also be replaced by

$$Q'_4 = Q_3 + Q_4 = \tilde{f}f - \tilde{f}g(a + \delta) - \tilde{f}f + \tilde{f}g(\delta + b) + \tilde{f}g(a - b) = 0,$$

which is similar to $Q_3$. When the dependent variable transformation (6.7) is substituted into $Q1^\delta$ we find that the result can be expressed in terms of the $Q_i$ defined above:

$$Q1^\delta = \frac{(c^2/r - \delta^2 r)^3}{(a^2 - \delta^2)(b^2 - \delta^2)(a - \delta)(b - \delta)} \sum_{i=1}^{4} Q_i P_i,$$

with

$$P_1 = (a - b)[\tilde{f}f g(-a + b) + f(\tilde{f}g - \tilde{f}g)(a + b) + g(\tilde{f}g + \tilde{f}g)(a - b)],$$

$$P_2 = (a + b)[\tilde{f}f g(a - b) + f(\tilde{f}g - \tilde{f}g)(a + b) + g(\tilde{f}g + \tilde{f}g)(a - b)],$$

$$P_3 = (a + b)(a + \delta)[\tilde{f}f g(a - b) + f(\tilde{f}g - \tilde{f}g)(a + \delta)],$$

$$P_4 = f(a + \delta)[\tilde{f}f g(a - b) + (\tilde{f}g - \tilde{f}g)(a - \delta)(b - \delta)].$$

Then it remains to prove the following:

**Proposition 6.** The Casoratian type determinants $f, g$, given in (6.8) with entries given by (6.9) solve the set of bilinear equations (6.10).

**Lemma 3.** By means of Lemma 2, the following formulas are zero, ($\mu = 1, 2$),

$$Y_\mu = f|E_\mu \psi(-1), -1, N - 2|_{[3]} + f|E_\mu \psi(-1), N - 1|_{[3]} - g|E_\mu \psi(-1), N - 2|_{[3]},$$

$$Y_3 = |\tilde{\psi}(-1), -1, N - 2|_{[3]}|\tilde{\psi}(-1), N - 1|_{[3]} - |\tilde{\psi}(-1), -1, N - 2|_{[3]}|\tilde{\psi}(-1), N - 1|_{[3]} + g|\tilde{\psi}(-1), \tilde{\psi}(-1), N - 2|_{[3]};$$

$$Z_\mu = f|\tilde{E}^\mu \psi(-1), -1, N - 2|_{[3]} + f|\tilde{E}^\mu \psi(-1), N - 1|_{[3]} - g|\tilde{E}^\mu \psi(-1), N - 2|_{[3]},$$

$$Z_3 = |\tilde{E}^1 \psi(-1), -1, N - 2|_{[3]}|\tilde{E}^2 \psi(-1), N - 1|_{[3]} - |\tilde{E}^2 \psi(-1), -1, N - 2|_{[3]}|\tilde{E}^1 \psi(-1), N - 1|_{[3]} + g|\tilde{E}^2 \psi(-1), \tilde{E}^1 \psi(-1), N - 2|_{[3]}.$$

**Proof.** For the above formulas, we can use Lemma 2 by respectively taking

$$B = (N - 2), \ (a, b, c, d) = (E_\mu \psi(-1), \psi(-1), \psi(0), \psi(N - 1)),$$

$$B = (N - 2), \ (a, b, c, d) = (\tilde{\psi}(-1), \tilde{\psi}(-1), \tilde{\psi}(-1), \psi(N - 1));$$

$$B = (N - 2), \ (a, b, c, d) = (\tilde{E}^\mu \psi(-1), \psi(-1), \psi(0), \psi(N - 1)),$$

$$B = (N - 2), \ (a, b, c, d) = (\tilde{E}^2 \psi(-1), \tilde{E}^1 \psi(-1), \psi(-1), \psi(N - 1)).$$

\[\square\]
Proof.
Proof for (6.10a): The down-bar shifted (6.10a) is nothing but $B'_2$ of $H^4$ with $c = \delta$.
Proof for (6.10b): By a down-tilde shift (6.10b) is written as
\[
(a - b)\hat{f}f + (\delta + b)\overline{\hat{f}}f - (a + \delta)\overline{\hat{f}}f = 0. \tag{6.16}
\]
We fix $\kappa \equiv 2$ and $c = \delta$ in (A.6). For $\hat{f}$, $\overline{\hat{f}}$, $\overline{\hat{f}}$ and $\overline{\hat{f}}$, we use the formulas (A.6b) with $\mu = 1$, (A.6i) with $\mu = 3$, (A.6d) with $\mu = 3$, (A.6g) with $\mu = 3$ and $\nu = 1$, respectively, and $f = |N - 1|_{[2]}$. Then we have
\[
\frac{1}{|\Gamma_3|}(a - b)^{N-2}(\delta + b)^{N-2}[\hat{f}f + (\delta + b)\overline{\hat{f}}f - (a + \delta)\overline{\hat{f}}f]
= \ |N - 2, \underline{\psi}\hspace{1pt}(N - 1)|_{[2]}|N - 1, \hat{\psi}\hspace{1pt}(N - 1)|_{[2]}
- |N - 2, \hat{\psi}\hspace{1pt}(N - 1)|_{[2]}|N - 1, \hat{\psi}\hspace{1pt}(N - 1)|_{[2]}
- |N - 1|_{[2]}|N - 1, \hat{\psi}\hspace{1pt}(N - 1), \hat{\psi}\hspace{1pt}(N - 1)|_{[2]}
= 0,
\]
where we have made use of Lemma 2 with
\[
B = (\hat{N} - 2, (a, b, c, d) = (\psi(0), \psi(N - 1), \psi(N - 1), \hat{\psi}\hspace{1pt}(N - 1)).
\]
Proof for (6.10c): By a down-tilde-hat shift (6.10c) is written as
\[
-\overline{\hat{f}}\underline{\hat{f}} + (a - \delta)\overline{\hat{f}}g + \hat{f}\overline{\hat{f}}g + (a - b)\overline{\hat{f}}g = 0. \tag{6.17}
\]
This time we fix $\kappa \equiv 3$ and $c = \delta$ in (C.1). For $\hat{f}$, $\overline{\hat{f}}$, $\overline{\hat{f}}$, $\overline{\hat{f}}$ and $\underline{\hat{f}}$, we use the formulas (C.1b) with $\mu = 2$, (C.1a) with $\mu = 1$, (C.1d) with $\mu = 1$, (C.1b) with $\mu = 1$, (C.1c) with $\mu = 2$, and (C.1g), respectively. Then we have
\[
-\overline{\hat{f}}\underline{\hat{f}} + (a - \delta)\overline{\hat{f}}g + \hat{f}\overline{\hat{f}}g + (a - b)\overline{\hat{f}}g
= (a - \delta)^2Y_1 - (b - \delta)^2Y_2 + (a - \delta)^2(b - \delta)^2Z_3
\]
with $Y_j$ defined in (6.14), which are zero in the light of Lemma 3.
Proof for (6.10d): Using (6.10c) to eliminate the term $(a - b)\overline{\hat{f}}g$ from (6.10d), we get
\[
\overline{\hat{f}}\overline{\hat{f}} - (a + \delta)\overline{\hat{f}}g - \hat{f}\overline{\hat{f}}g - (a + \delta)\overline{\hat{f}}g = 0. \tag{6.18}
\]
We fix $\kappa \equiv 3$ and $c = \delta$ in (C.1), and use the formulas (C.1d), (C.1c), (C.1) and (C.1h). Then it turns out that
\[
\overline{\hat{f}}\overline{\hat{f}} - (a + \delta)\overline{\hat{f}}g - \hat{f}\overline{\hat{f}}g + (a + \delta)\overline{\hat{f}}g
= (a + \delta)^2Z_1 - (b + \delta)^2Z_2 - (a + \delta)^2(b + \delta)^2Z_3
\]
with $Z_j$ defined in (6.15), which are zero in the light of Lemma 3.
Thus we have completed the proof for all bilinear equations. □
7 Q1 with power background

7.1 Background solution with $T(x) = -x + c$

With fixed point defined by $T(x) = -x + c$ we use the reparameterization

$$p = \frac{1}{2}r(1 - \cosh(\alpha')) = -\frac{1}{2}r(1 - \alpha)^2/\alpha, \quad (7.1)$$
$$q = \frac{1}{2}r(1 - \cosh(\beta')) = -\frac{1}{2}r(1 - \beta)^2/\beta, \quad (7.2)$$

$$x_{nm} = Ae^{ynm} + B e^{ynm} + c/2,$$ \quad where $AB = \delta^2 r^2/16$, \quad (7.3)

and this leads to equations that factorize as in (5.4). Thus we get power type background solutions for $u$:

$$u = \frac{1}{2}c + A\alpha^n \beta^m + B\alpha^{-n} \beta^{-m}, \quad AB = \delta^2 r^2/16. \quad (7.4)$$

Here $c$ is related to translation freedom and $r$ to scaling freedom; in the following we take $c = 0$.

7.2 1-soliton solution for $Q_1^\delta$

In order to derive the 1SS we use the BT (6.4) with the seed solution (7.4) with

$$\bar{u} = u_0 + v, \quad (7.5)$$

where $u_0$ is the bar-shifted background solution (7.4):

$$u_0 = A\alpha^n \beta^m \kappa + B\alpha^{-n} \beta^{-m} \kappa^{-1}, \quad AB = \delta^2 r^2/16. \quad (7.6)$$

and $\kappa$ is defined through

$$\kappa = -\frac{r}{4}(1 - \kappa)^2/\kappa. \quad (7.7)$$

If we now define

$$U_{n,m} = A\alpha^n \beta^m - B\alpha^{-n} \beta^{-m} \quad (7.8)$$

then it is straightforward to derive (2.11) from (6.4) with

$$S = \frac{r(1 - \alpha)(1 - \kappa)(1 - \alpha \kappa)}{4\alpha \kappa}, \quad \Delta = \frac{r(1 - \alpha)(1 - \kappa)(\alpha - \kappa)}{4\alpha \kappa}, \quad \sigma = p, \quad (7.9)$$

$$T = \frac{r(1 - \beta)(1 - \kappa)(1 - \beta \kappa)}{4\beta \kappa}, \quad \Omega = \frac{r(1 - \beta)(1 - \kappa)(\beta - \kappa)}{4\beta \kappa}, \quad \tau = q. \quad (7.10)$$

On the basis of this we define $\rho$ as usual by

$$\rho_{n,m} = \left(\frac{S}{\Delta}\right)^n \left(\frac{T}{\Omega}\right)^m \rho_{0,0} = \left(\frac{1 - \alpha \kappa}{\alpha - \kappa}\right)^n \left(\frac{1 - \beta \kappa}{\beta - \kappa}\right)^m \rho_{0,0}, \quad (7.11)$$

where $\rho_{0,0}$ is some constant. Then it follows that

$$v_{n,m} = \frac{1 - \kappa^2 U_{n,m} \rho_{n,m}}{1 + \rho_{n,m}}, \quad (7.12)$$
and finally we obtain the 1-soliton for $Q_1^\delta$:

$$
\begin{align*}
u_{n,m}^{1SS} &= \overline{u}_0 + v_{n,m} \\
&= A\alpha^n\beta^m(\kappa + \kappa^{-1}\rho_{n,m}) + B\alpha^{-n}\beta^{-m}(\kappa^{-1} + \kappa\rho_{n,m}) \\
&= \frac{A'\alpha^n\beta^m(1 + \kappa^{-2}\rho_{n,m}) + B'\alpha^{-n}\beta^{-m}(1 + \kappa^2\rho_{n,m})}{1 + \rho_{n,m}},
\end{align*}
$$

where $A'B' = AB = \delta^2 r^2/16$.

### 7.3 Bilinearization

In order to get $\rho$ of (7.11) into a nicer form we use the Möbius transformations

$$
\alpha = \frac{a - c}{a + c}, \quad \beta = \frac{b - c}{b + c}, \quad \kappa = \frac{k - c}{k + c}; \quad \Rightarrow p = \frac{rc^2}{a^2 - c^2}, \quad q = \frac{rc^2}{b^2 - c^2},
$$

which leads to the canonical form

$$
\rho_{n,m} = \left(\frac{a + k}{a - k}\right)^n \left(\frac{b + k}{b - k}\right)^m \rho_{0,0}.
$$

With the above Möbius transformations $Q_1^\delta$ (6.1) can be written as

$$
Q_1^\delta = (b^2 - c^2)(u - \tilde{u})(\tilde{u} - \hat{u}) - (a^2 - c^2)(u - \tilde{u})(\tilde{u} - \hat{u}) - \frac{rc^2c^4(a^2 - b^2)}{(a^2 - c^2)(b^2 - c^2)} = 0,
$$

Using (7.17) and rearranging the parameters $A$ and $B$ we write the 1SS (7.15) as

$$
\begin{align*}
\nu_{n,m}^{1SS} &= A\alpha^n\beta^m \frac{\psi(n, m, l + 2)}{\psi(n, m, l)} + B\alpha^{-n}\beta^{-m} \frac{\psi(n, m, l - 2)}{\psi(n, m, l)}, \quad AB = \delta^2 r^2/16,
\end{align*}
$$

where $\psi$ is as in (2.20).

This structure motivates us to bilinearize $Q_1^\delta$ through the following transformation,

$$
\begin{align*}
u_{n,m}^{NSS} &= A\alpha^n\beta^m \frac{\overline{f}}{f} + B\alpha^{-n}\beta^{-m} \frac{f}{\overline{f}}, \quad AB = \delta^2 r^2/16,
\end{align*}
$$

where $f = |N - 1|_{\nu} (\nu = 1, 2, \text{ or } 3)$ with entries (2.20) and the bar-shift is the shift in the third index $l$, as discussed in Sec. 2.5.

The solution (7.20) with $r = 1$ is the same as (I-5.11) with $B = 0$. In fact, by a comparison (7.16) with (I-3.17) one has

$$
\rho(a) = \alpha^{-n}\beta^{-m}, \quad \rho(-a) = \alpha^n\beta^m;$$

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and substituting $\rho_i$ in (I-2.2) by (5.16) we find from (I-3.17) that

$$(1 + 2aS(-a, -a)) \times (\prod(k_j - a)^2) = \frac{\bar{f}}{f}, \quad (1 - 2aS(a, a)) / (\prod(k_j - a)^2) = \frac{f}{\bar{f}}.$$ 

Surprisingly $Q^1_δ$ can also be bilinearized with the same bilinear equations as $H^3_δ$, namely (5.17), and we have already shown in Proposition 4 that $f = |\hat{N} - 1|_{\nu}$ solves the $B_i$ in (5.17).

The representation of $Q^1_δ$ (7.18) in terms of $B_i$ is as follows:

$$Q^1_δ = \frac{\alpha^{4n+2} \beta^{4m+2} (a + c)^2 (b + c)^2 A^2 P_1 - \alpha^{2n} \beta^{2m} \delta^2 / 4P_2 - (a + c)^2 (b + c)^2 B^2 P_1}{\alpha^{2n+1} \beta^{2m+1} (a + c)^2 (b + c)^2 \bar{f} \hat{f} \tilde{f}}.$$ 

where

$$P_1 = Y \tilde{Y} - X \hat{X}, \quad X = B_1 - 2cf \bar{f}, \quad Y = B_2 - 2cf \hat{f},$$

$$P_2 = -(a + c)(a - c)(b + c)^2 \left( X \hat{X} - 4c^2 f \bar{f} \hat{f} \tilde{f} \right) + 4c^2 (b + c)(b - c) \left( X \hat{X} - 4c^2 f \bar{f} \hat{f} \tilde{f} \right) + (b + c)(b - c)(a + c)^2 \left( Y \tilde{Y} - 4c^2 \tilde{f} \hat{f} \bar{f} \hat{f} \right) + (b + c)(b - c)(a - c)^2 \left( Y \bar{Y} - 4c^2 \tilde{f} \hat{f} \bar{f} \hat{f} \right) - 4c^2 (a + c)(a - c) \left( Y \bar{Y} - 4c^2 \tilde{f} \hat{f} \bar{f} \hat{f} \right).$$

8 Conclusions

In this paper, companion to [1] we have analyzed the soliton solutions to the models $H_1$, $H_2$, $H_3$, and $Q_1$ in the ABS list [2] of partial difference equations. Our method is constructive, progressing in each case from background solution to 1SS to NSS and bilinearization.

Our approach is fairly algorithmic, and as such we hope that it will be usable also for other models with multidimensional consistency. One interesting feature is that in some cases we need several bilinear equations, some of which seem to have the same continuum limit. This is a remainder that there are several ways to discretize a derivative.

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A Casoratians and formulas

We define Casoratians

\[ f_{[\nu]} = |\hat{N} - 1|_{[\nu]}, \quad g_{[\nu]} = |\hat{N} - 2, \hat{N} + 1|_{[\nu]}, \quad h_{[\nu]} = |\hat{N} - 2, \hat{N} + 1|_{[\nu]}, \]
where the matrix entries are given by the function
\[
\psi_{n,m,l} = g_n^+(c + k_i)^{(a + k_i)^n(b + k_i)^m} + g_n^-(c - k_i)^{(a - k_i)^n(b - k_i)^m}. \quad (A.1)
\]
We introduce notations
\[
\Gamma_{\nu,i} = \alpha_\nu^2 - k_i^2, \quad \Gamma_\nu = \text{Diag}(\Gamma_{\nu,1}, \Gamma_{\nu,2}, \cdots, \Gamma_{\nu,N}), \quad (\nu = 1, 2, 3), \quad (A.2)
\]
where
\[
\alpha_1 \equiv a, \quad \alpha_2 \equiv b, \quad \alpha_3 \equiv c, \quad (A.3)
\]
and define the operator \( \hat{E}^\mu \) as
\[
\hat{E}^\mu \psi = \Gamma_{\mu}^{-1} E^\mu \psi, \quad (\nu = 1, 2, 3). \quad (A.4)
\]
The column vector \( \psi \) satisfies
\[
(\alpha_\mu - \alpha_\kappa) \psi = (E^\mu - E^\kappa) \psi, \quad (A.5a)
\]
\[
(\alpha_\mu + \alpha_\nu) \hat{E}^\mu E^\nu \psi = (\hat{E}^\mu + E^\nu) \psi, \quad (A.5b)
\]
\[
E_\mu \psi = [E_\kappa - (\alpha_\mu - \alpha_\kappa)(E_\kappa) + (\alpha_\mu - \alpha_\kappa)^2 E^\mu (E_\kappa)^2] \psi, \quad (A.5c)
\]
\[
E_\mu \psi = [E_\kappa - (\alpha_\mu - \alpha_\kappa)(E_\kappa)^2 + (\alpha_\mu - \alpha_\kappa)^2 E^\mu (E_\kappa)^3 - (\alpha_\mu - \alpha_\kappa)^3 E^\mu (E_\kappa)^3] \psi. \quad (A.5d)
\]
For the Casoratian \( f_{[\nu]} \), we have
\[
-(\alpha_\mu - \alpha_\kappa)^{N-2} E_\mu f_{[\nu]} = |\hat{N} - 2, E_\mu \psi (N - 2)|_{[\nu]}, \quad (A.6a)
\]
\[
(\alpha_\mu - \alpha_\kappa)^{N-1} E_\mu f_{[\nu]} = |\hat{N} - 2, E_\mu \psi (N - 1)|_{[\nu]}, \quad (A.6b)
\]
\[
(\alpha_\mu - \alpha_\kappa)^{N-2} E_\mu f_{[\nu]} = |\hat{N} - 2, \hat{E}^\mu \psi (N - 2)|_{[\nu]}, \quad (\mu \neq \kappa), \quad (A.6c)
\]
\[
(\alpha_\mu - \alpha_\kappa)^{N-1} E_\mu f_{[\nu]} = |\hat{N} - 2, \hat{E}^\mu \psi (N - 1)|_{[\nu]}, \quad (\mu \neq \kappa); \quad (A.6d)
\]
\[
E^\kappa f_{[\nu]} = |\hat{N}||_\kappa, \quad (A.6e)
\]
\[
E_\kappa f_{[\nu]} = |\hat{N} - 2|_{[\nu]}, \quad (A.6f)
\]
\[
-(\alpha_\mu - \alpha_\kappa)^{N-2} E_\mu E^\kappa f_{[\nu]} = |\hat{N} - 1, E_\mu \psi (N - 1)|_{[\nu]}, \quad (A.6g)
\]
\[
(\alpha_\mu - \alpha_\kappa)^{N-1} E_\mu E^\kappa f_{[\nu]} = |\hat{N} - 1, E_\mu \psi (N - 2)|_{[\nu]}, \quad (A.6h)
\]
\[
(\alpha_\mu - \alpha_\kappa)^{N-2} E_\mu E^\kappa f_{[\nu]} = |\hat{N} - 1, \hat{E}^\mu \psi (N - 1)|_{[\nu]}, \quad (\mu \neq \kappa), \quad (A.6i)
\]
\[
(\alpha_\mu - \alpha_\kappa)^{N-1} E_\mu E^\kappa f_{[\nu]} = |\hat{N} - 1, \hat{E}^\mu \psi (N - 2)|_{[\nu]}, \quad (\mu \neq \kappa); \quad (A.6j)
\]
\[
(\alpha_\mu - \alpha_\nu)(\alpha_\mu - \alpha_\kappa)^{N-2} (\alpha_\nu - \alpha_\kappa)^{N-2} E_\mu E_\nu f_{[\nu]} = |\hat{N} - 3, E_\mu \psi (N - 2), E_\nu \psi (N - 2)|_{[\nu]}, \quad (A.6k)
\]
\[
(\alpha_\mu + \alpha_\nu)(\alpha_\mu + \alpha_\kappa)^{N-2} (\alpha_\nu - \alpha_\kappa)^{N-2} E_\mu E_\nu f_{[\nu]} = |\hat{N} - 3, E_\mu \psi (N - 2), \hat{E}^\mu \psi (N - 2)|_{[\nu]}, \quad (\mu \neq \kappa); \quad (A.6l)
\]
\[(\alpha_\mu - \alpha_\nu)(\alpha_\mu - \alpha_\kappa)^{N-2}(\alpha_\nu - \alpha_\kappa)^{N-2}E_\mu E_\nu E^\kappa f_{[\kappa]} = \hat{\Gamma}_\mu \hat{\Gamma}_\nu \hat{\Gamma}_\kappa \hat{\Gamma}(N-2, E_\nu \psi(N-1), E_\mu \psi(N-1))_{[\kappa]}, \quad (A.6m)\]

\[(\alpha_\mu + \alpha_\nu)(\alpha_\mu + \alpha_\kappa)^{N-2}(\alpha_\nu - \alpha_\kappa)^{N-2}E_\mu E_\nu E^\kappa f_{[\kappa]} = \Gamma_\mu \Gamma_\nu \Gamma_\kappa \Gamma(N-2, E_\nu \psi(N-1), \hat{\Gamma}^\mu \psi(N-1))_{[\kappa]}, \quad (\mu \neq \kappa). \quad (A.6n)\]

For \(g_{[\kappa]}\), we have

\[- (\alpha_\mu - \alpha_\kappa)^{N-2}E_\mu [g_{[\kappa]} + (\alpha_\mu - \alpha_\kappa) f_{[\kappa]}] = \hat{\Gamma}_\mu \hat{\Gamma}(N-3, N-1, E_\mu \psi(N-2))_{[\kappa]}, \quad (A.7a)\]

\[(\alpha_\mu + \alpha_\kappa)^{N-2}E_\mu [g_{[\kappa]} - (\alpha_\mu + \alpha_\kappa) f_{[\kappa]}] = \Gamma_\mu \Gamma(N-3, N-1, \hat{\Gamma}^\mu \psi(N-2))_{[\kappa]}, \quad (\mu \neq \kappa) \quad (A.7b)\]

For \(h_{[\kappa]}\),

\[- (\alpha_\mu - \alpha_\kappa)^{N-2}E_\mu [h_{[\kappa]} + (\alpha_\mu - \alpha_\kappa) g_{[\kappa]}] = \hat{\Gamma}_\mu \hat{\Gamma}(N-3, N-1, E_\mu \psi(N-2))_{[\kappa]}, \quad (A.8a)\]

\[(\alpha_\mu + \alpha_\kappa)^{N-2}E_\mu [h_{[\kappa]} - (\alpha_\mu + \alpha_\kappa) g_{[\kappa]}] = \Gamma_\mu \Gamma(N-3, N, \hat{\Gamma}^\mu \psi(N-2))_{[\kappa]}, \quad (\mu \neq \kappa) \quad (A.8b)\]

These formulas, (A.5a)-(A.8a), are valid for \(\mu, \nu, \kappa = 1, 2, 3\) except when otherwise indicated.

**B Proof for the formulas in Appendix A**

The formulas (A.5) can directly be verified. In (A.6)-(A.8), those only containing down shifts \(E_\mu\) and \(E_\nu\) can be proved by using the relations (A.5). Since \(a, b\) and \(c\) appear in \(\psi_i\) equivalently, we take \(\mu = 1, \nu = 2\) and \(\kappa = 3\) in the following proof.

From (A.5a) we have

\[
(a - c)\psi(l) = \psi(l) - \psi(l + 1), \quad (B.1) \\
(b - c)\psi(l) = \psi(l) - \psi(l + 1), \quad (B.2) \\
(a - b)\psi(l) = \psi(l) - \psi(l). \quad (B.3)
\]

Using (B.1) for \(f_{[3]}\) we first find

\[
(a - c) f_{[3]} = [(a - c)\psi(0), \psi(1), \ldots, \psi(N - 1)]_{[3]} = [\psi(0), \psi(1), \ldots, \psi(N - 1)]_{[3]}
\]

and repeating this \(N - 1\) times we get

\[
(a - c)^{N-1} f_{[3]} = [\psi(0), \ldots, \psi(N - 2), \psi(N - 1)]_{[3]} = \hat{\Gamma}(N-2, \psi(N-1))_{[\kappa]}.
\]

This is formula (A.6b). Again using (B.1) and expressing \(\psi(N - 1)\) through \(\psi(N - 2)\) and \(\psi(N - 2)\) in the above formula yields

\[- (a - c)^{N-2} f_{[3]} = \hat{\Gamma}(N-2, \psi(N-2))_{[\kappa]}, \quad (B.4)\]
which is (A.6a). For \( f_{[3]} \) we first using (B.2) from the above formula to obtain
\[
-(b - c)^{N-2}(a - c)^{N-2} f_{[3]} = \Big| N-3, \psi(N-2), \psi(N-2) \Big|_{[3]}
\]
then from (B.3) we get
\[
(a-b)(a - c)^{N-2}(b - c)^{N-2} f_{[3]} = \Big| N-3, \psi(N-2), \psi(N-2) \Big|_{[3]},
\]
which is (A.6k).

For \( g_{[3]} = |N-2, N|_{[3]} \) we find first
\[
(a - c)^{N-2} g_{[3]} = |N-3, \psi(N-2), \psi(N)|_{[3]}
\]
Using (A.5c) to re-write the last column we get
\[
(a - c)^{N-2} g_{[3]} = -|N-3, N-1, \psi(N-2)|_{[3]} + (a - c)|N-2, \psi(N-2)|_{[3]},
\]
which can also be stated as
\[
-(a - c)^{N-2}[g_{[3]} + (a - c) f_{[3]}] = |N-3, N-1, \psi(N-2)|_{[3]},
\]
i.e., (A.7a). Formula (A.8a) can be derived similarly by using (A.5c).

To show how to derive those formulas with up shifts in (A.6)-(A.8), we introduce two auxiliary functions:

\[\text{II} : \quad \phi_i(n, m, l) = g^+_i(c + k_i)^l(a + k_i)^n(b - k_i)^{-m} + g^-_i(c - k_i)^l(a - k_i)^n(b + k_i)^{-m}, \quad (B.8a)\]
\[\text{III} : \quad \omega_i(n, m, l) = g^+_i(c + k_i)^l(a - k_i)^{-n}(b + k_i)^m + g^-_i(c - k_i)^l(a + k_i)^{-n}(b - k_i)^m. \quad (B.8b)\]

Casoratians \( f, g \) and \( h \) (w.r.t. bar-shift) with column vectors \( \phi = (\phi_1, \ldots, \phi_N)^T \) and \( \omega = (\omega_1, \ldots, \omega_N)^T \) are denoted by \( f_{II[3]}, g_{II[3]}, h_{II[3]} \) and \( f_{III[3]}, g_{III[3]}, h_{III[3]} \), respectively. They are related to \( f_{[3]}, g_{[3]} \) and \( h_{[3]} \) through
\[
f_{[3]} = |\Gamma_2|^m f_{II[3]} = |\Gamma_1|^n f_{III[3]}, \quad (B.9a)\]
\[
g_{[3]} = |\Gamma_2|^m g_{II[3]} = |\Gamma_1|^n g_{III[3]}, \quad (B.9b)\]
\[
h_{[3]} = |\Gamma_2|^m h_{II[3]} = |\Gamma_1|^n h_{III[3]}, \quad (B.9c)\]
which follow from \( \psi = \Gamma_2^m \phi = \Gamma_1^n \omega. \)

\( \phi \) satisfies
\[
(a - c)\phi(l) = \frac{\phi(l) - \phi(l + 1)}{c}, \quad (B.10)\]
\[
(b + c)\phi(l) = \frac{\phi(l) + \phi(l + 1)}{c}, \quad (B.11)\]
\[
(a + b)\hat{\phi}(l) = \frac{\hat{\phi}(l) + \hat{\phi}(l)}{c}. \quad (B.12)\]
Similar to the previous case of $\psi$, using these relations we can get

\[
(b + c)^{N-2} \hat{f}_{[a]} = |\hat{N} - 2, \hat{\phi}(N - 2)|_{m}\) \tag{B.13a}
\]

\[
(b + c)^{N-1} \hat{f}_{[a]} = |\hat{N} - 2, \hat{\phi}(N - 1)|_{m}\) \tag{B.13b}
\]

\[-(a + b)(a - c)^{N-2}(b + c)^{N-2} \hat{f}_{[a]} = |\hat{N} - 3, \hat{\phi}(N - 2), \hat{\phi}(N - 2)|_{m}\) \tag{B.13c}
\]

\[
(b + c)^{N-1} \tilde{g}_{[a]} = |\tilde{N} - 1, \hat{\phi}(N - 1)|_{m}\) \tag{B.13d}
\]

\[-(a + b)(a - c)^{N-2}(b + c)^{N-2} \tilde{g}_{[a]} = |\tilde{N} - 2, \hat{\phi}(N - 1), \hat{\phi}(N - 1)|_{m}\) \tag{B.13e}
\]

\[
(b + c)^{N-1} \tilde{f}_{[a]} = |\tilde{N} - 1, \hat{\phi}(N - 1)|_{m}\) \tag{B.13f}
\]

\[
(b + c)^{N-2} \tilde{g}_{[a]} - (b + c) \hat{g}_{[a]} = |\tilde{N} - 3, N - 1, \hat{\phi}(N - 2)|_{m}\) \tag{B.13g}
\]

\[
(b + c)^{N-2} \tilde{f}_{[a]} - (b + c) \hat{f}_{[a]} = |\tilde{N} - 3, N, \hat{\phi}(N - 2)|_{m}\) \tag{B.13h}
\]

Now, noting that $f_{[a]} = |\Gamma_m| f_{[a]}$ and $\psi = \Gamma_m \phi$, we have

\[
(b + c)^{N-2} \hat{f}_{[a]} = (b + c)^{N-2} |\Gamma_m + 1 f_{[a]} = |\Gamma_m | \psi(0), \cdots, \psi(N - 2), \hat{E}^2 \psi(N - 2)|_{m}\) \tag{B.14}
\]

where the operator $\hat{E}^2$ is defined as \(\text{[A.4]}\), The above formula is just \(\text{[A.6c]}\) for $\mu = 2$.

From the rest of \(\text{[B.13]}\) we can get other formulas with an up-hat shift in \(\text{[A.6]}-\text{[A.8]}\), where for \(\text{[A.6]}\) and \(\text{[A.6a]}\) ($\mu = 2, \nu = 1$) we need to use \(\text{[A.5a]}\). Besides, if we take a down-hat shift for \(\text{[B.14]}\) and rewrite the last column by using \(\text{[A.5a]}\) (with $\mu = \nu = 2$), then we get

\[
2b(b + c)^{N-2}(b - c)^{N-2} f_{[a]} = |\Gamma_m | \hat{N} - 3, \psi(N - 2), \hat{E}^2 \psi(N - 2)|_{m},
\]

i.e., the formula \(\text{[A.6]}\) of the case $\mu = \nu = 2$.

Next, using $\omega$ we can derive these formulas with an up-tilde shift in \(\text{[A.6]}-\text{[A.8]}\). $\omega$ satisfies

\[
(a + c)\tilde{\omega}(l) = \omega(l) + \tilde{\omega}(l + 1), \tag{B.15}
\]

\[
(b - c)\omega(l) = \omega(l) - \omega(l + 1), \tag{B.16}
\]

\[
(a + b)\tilde{\omega}(l) = \omega(l) + \tilde{\omega}(l), \tag{B.17}
\]

and from these relations we have

\[
(a + c)^{N-2} \tilde{f}_{[a]} = |\tilde{N} - 2, \tilde{\omega}(N - 2)|_{m}\) \tag{B.18a}
\]

\[
(a + c)^{N-1} \tilde{f}_{[a]} = |\tilde{N} - 2, \tilde{\omega}(N - 1)|_{m}\) \tag{B.18b}
\]

\[
(a + b)(a + c)^{N-2}(b - c)^{N-2} \tilde{f}_{[a]} = |\tilde{N} - 3, \omega(N - 2), \tilde{\omega}(N - 2)|_{m}\) \tag{B.18c}
\]
\[
(a + c)^{N-2} \tilde{f}_{\text{II}[3]} = |N - 1, \tilde{\omega}(N - 1)|_{\text{III}[3]};
\]
(B.18d)
\[
(a + b)(b - c)^{N-2}(a + c)^{N-2} \tilde{f}_{\text{II}[3]} = |N - 2, \omega(N - 1), \tilde{\omega}(N - 1)|_{\text{III}[3]};
\]
(B.18e)
\[
(a + c)^{N-1} \tilde{f}_{\text{III}[3]} = | - 1, N - 3, \tilde{\omega}(N - 2)|_{\text{III}[3]};
\]
(B.18f)
\[
(a + c)^{N-2}[\tilde{g}_{\text{II}[3]} - (a + c) \tilde{f}_{\text{II}[3]}] = |N - 3, N - 1, \tilde{\omega}(N - 2)|_{\text{III}[3]};
\]
(B.18g)
\[
(a + c)^{N-2}[\tilde{h}_{\text{III}[3]} - (a + c) \tilde{g}_{\text{III}[3]}] = |N - 3, N, \tilde{\omega}(N - 2)|_{\text{III}[3]}.
\]
(B.18h)

By using operator \( \hat{E}^1 \), these formulas will generate those of (A.6)-(A.8) with an up-tilde shift.

Thus we can get all formulas for \( f \), \( g \) and \( h \) given by (A.6)-(A.8).

\[ C \] Formulas for \( f = |N - 1|_{[3]} \), \( g = | - 1, N - 1||_{[3]} \)

For the Casoratian \( f_{[\alpha]} \), we have

\[
E_\mu f = E_3 f - (\alpha_\mu - \alpha_3)|E_\mu \psi(-1), N - 2|_{[3]},
\]
(C.1a)
\[
E_\mu E^3 f = f - (\alpha_\mu - \alpha_3)g + (\alpha_\mu - \alpha_3)^2|E_\mu \psi(-1), N - 1|_{[3]},
\]
(C.1b)
\[
E_\mu g = |E_\mu \psi(-1), N - 2|_{[3]} - (\alpha_\mu - \alpha_3)|E_\mu \psi(-1), -1, N - 2|_{[3]},
\]
(C.1c)
\[
\frac{(-1)^N}{|\Gamma_\mu|} E^\mu f = E_3 f - (\alpha_\mu + \alpha_3)|\hat{E}^\mu \psi(-1), N - 2|_{[3]},
\]
(C.1d)
\[
\frac{(-1)^N}{|\Gamma_\mu|} E^\mu E^3 f = f + (\alpha_\mu + \alpha_3)g - (\alpha_\mu + \alpha_3)^2|\hat{E}^\mu \psi(-1), N - 1|_{[3]},
\]
(C.1e)
\[
\frac{(-1)^N}{|\Gamma_\mu|} E^\mu g = -|\hat{E}^\mu \psi(-1), N - 2|_{[3]} - (\alpha_\mu + \alpha_3)|\hat{E}^\mu \psi(-1), -1, N - 2|_{[3]};
\]
(C.1f)
\[
(a - b) \tilde{f}
\]
\[
= (a - b) f - (a - c)^2|\psi(-1), N - 2|_{[3]} + (b - c)^2|\psi(-1), N - 2|_{[3]} \\
+ (a - c)^2(b - c)|\tilde{\psi}(-1), -1, N - 2|_{[3]} - (a - c)(b - c)^2|\tilde{\psi}(-1), -1, N - 2|_{[3]} \\
+ (a - c)^2(b - c)^2|\tilde{\psi}(-1), \tilde{\psi}(-1), N - 2|_{[3]},
\]
(C.1g)
\[
= (a - b) \tilde{f}
\]
\[
= (a - b) f - (a + c)^2|\tilde{E}^1 \psi(-1), N - 2|_{[3]} + (b + c)^2|\tilde{E}^2 \psi(-1), N - 2|_{[3]} \\
- (a + c)^2(b + c)|\tilde{E}^1 \psi(-1), -1, N - 2|_{[3]} + (a + c)(b + c)^2|\tilde{E}^2 \psi(-1), -1, N - 2|_{[3]} \\
- (a + c)^2(b + c)^2|\tilde{E}^2 \psi(-1), \tilde{E}^1 \psi(-1), N - 2|_{[3]}.
\]
(C.1h)
D  Proof for the formulas in Appendix C

Noting that
\[ \psi(l + 1) = \psi(l) - (a - c)\psi(l), \]  
we have
\[ f = [\psi(0), \psi(1), \ldots, \psi(N - 3), \psi(N - 2), \psi(N - 1)]^{\top}_{[3]} \]
\[ = [\psi(0), \psi(1), \ldots, \psi(N - 3), \psi(N - 2) - (a - c)\psi(N - 2)]^{\top}_{[3]} \]
\[ = [\psi(0), \psi(1), \ldots, \psi(N - 3), \psi(N - 2)]^{\top}_{[3]} \]
\[ = \ldots \]
\[ = [\psi(0), \psi(0), \ldots, \psi(N - 4), \psi(N - 3), \psi(N - 2)]^{\top}_{[3]} \]
\[ = f - (a - c)\psi(-1), \bar{N} - 2]^{\top}_{[3]}. \]

This is (C.1a) for \( \mu = 1 \). In a similar way we can prove (C.1b), (C.1c) and (C.1d).

To prove (C.1d), (C.1e), (C.1f) and (C.1h), we consider the Casoratian type determinants
\[ f = |N - 1]_{[3]}, \quad g = | - 1, \bar{N} - 1]_{[3]} \]
with the following entries
\[ IV : \sigma_i(n, m, l) = g_i^+(c + k_i)^l(a - k_i)^-n(b - k_i)^{-m} + g_i^-(c - k_i)^l(a + k_i)^{-n}(b + k_i)^{-m}, \]
and denote such \( f \) and \( g \) by \( f_{IV[3]} \) and \( g_{IV[3]} \). Noting that \( \sigma \) satisfies
\[ (a + c)\hat{\sigma}(l) = \sigma(l) + \hat{\sigma}(l + 1), \]
\[ (b + c)\hat{\sigma}(l) = \sigma(l) + \hat{\sigma}(l + 1), \]
\[ (a - b)\hat{\sigma}(l) = \sigma(l) - \hat{\sigma}(l), \]
we can have
\[ (-1)^N E^\mu f_{IV[3]} = E_3 f_{IV[3]} - (\alpha_\mu + \alpha_3)|E^\mu \sigma(-1), \bar{N} - 2]_{IV[3]}, \]
\[ (D.6a) \]
\[ (-1)^N E^3 f_{IV[3]} = f_{IV[3]} + (\alpha_\mu + \alpha_3)g_{IV[3]} - (\alpha_\mu + \alpha_3)^2|E^\mu \sigma(-1), \bar{N} - 1]_{IV[3]}, \]
\[ (D.6b) \]
\[ (-1)^N E^\mu g_{IV[3]} = -|E^\mu \sigma(-1), \bar{N} - 2]_{IV[3]} \]
\[ -(\alpha_\mu + \alpha_3)|E^\mu \sigma(-1), -1, \bar{N} - 2]_{IV[3]}; \]
\[ (D.6c) \]
\[ (a - b)\hat{f}_{IV[3]} \]
\[ = (a - b)\hat{f}_{IV[3]} - (a + c)^2|\hat{\sigma}(-1), \bar{N} - 2]_{IV[3]} + (b + c)^2|\hat{\sigma}(-1), \bar{N} - 2]_{IV[3]} \]
\[ - (a + c)^2(b + c)|\hat{\sigma}(-1), -1, \bar{N} - 2]_{IV[3]} + (a + c)(b + c)^2|\hat{\sigma}(-1), -1, \bar{N} - 2]_{IV[3]} \]
\[ - (a + c)^2(b + c)^2|\hat{\sigma}(-1), \bar{N} - 2]_{IV[3]} \]
\[ (D.6d) \]

Then, using the relationship
\[ f_{IV[3]} = |\Gamma_1^n \Gamma_2^m f_{IV[3]}, g_{IV[3]} = |\Gamma_1^n \Gamma_2^m g_{IV[3]} \]
and \( \psi = \Gamma_1 \Gamma_2 \Gamma^m \sigma \),
we can finally derive the formulas (C.1d), (C.1e), (C.1f) and (C.1h).