CHARACTERIZATIONS FOR FRACTIONAL HARDY INEQUALITY

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Abstract. We provide a Maz’ya type characterization for a fractional Hardy inequality. As an application, we show that a bounded open set $G$ admits a fractional Hardy inequality if and only if the associated fractional capacity is quasiadditive with respect to Whitney cubes of $G$ and the zero extension operator acting on $C_c(G)$ is bounded in an appropriate manner.

1. Introduction

An open set $\emptyset \neq G \subseteq \mathbb{R}^n$ admits an $(s, p)$-Hardy inequality, for $0 < s < 1$ and $0 < p < \infty$, if there is a constant $C > 0$ such that inequality

$$
\int_G \frac{|u(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \leq C \int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dy \, dx := C[u]_{W^{s,p}(G)}^p
$$

holds for every $u \in C_c(G)$. Sufficient geometric conditions are available for an open set to admit an $(s, p)$-Hardy inequality, e.g., in [2, 3, 4]. However, these conditions are not necessary. Our first result, Theorem 1, is a Maz’ya-type characterization of the $(s, p)$-Hardy inequality in terms of the following $(s, p)$-capacities of compact sets $K \subset G$: we write

$$
cap_{s,p}(K, G) = \inf_{u} [u]_{W^{s,p}(G)}^p,
$$

where the infimum ranges over all real-valued $u \in C_c(G)$, i.e., continuous with compact support in $G$, such that $u(x) \geq 1$ for $x \in K$.

**Theorem 1.** Let $0 < s < 1$ and $0 < p < \infty$. An open set $\emptyset \neq G \subseteq \mathbb{R}^n$ admits an $(s, p)$-Hardy inequality if and only if there is a constant $c > 0$ such that

$$
\int_K \text{dist}(x, \partial G)^{-sp} \, dx \leq c \cap_{s,p}(K, G)
$$

for every compact set $K \subset G$.

A Maz’ya-type characterization for weighted embeddings gives Theorem 1 as a special case, see Proposition 5 with $\omega(x) = \text{dist}(x, \partial G)^{-sp}$. In the proof of this proposition, we adapt the method that is used by Kinnunen and Korte to prove a Maz’ya-type characterization for the non-fractional Hardy inequality, [5, Theorem 2.1]. This method, in turn, is based on a truncation argument in the monograph of Maz’ya, [8, p. 110]. For further information on this type of characterizations, we refer to [8, §2] and [5].

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A close connection between a non-fractional Hardy inequality and a ‘quasiadditivity property of the variational capacity’ was recently found by Lehrbäck and Shanmugalingam, [6]. A Maz’ya-type characterization has a significant role in their work, hence, it is not surprising that Theorem 1 paves our way to the analogous question, namely: what is the connection between an \((s,p)\)-Hardy inequality and quasiadditivity of \(\text{cap}_{s,p}(\cdot,G)\) with respect to a Whitney decomposition \(W(G)\) of an open set \(G\)? In case of bounded open sets, we characterize the \((s,p)\)-Hardy inequality in terms of a (weak) quasiadditivity and a zero extension property, Theorem 2. In order to state this partial extension of [6, Corollary 3.5] we need these definitions.

We say that the \((s,p)\)-capacity \(\text{cap}_{s,p}(\cdot,G)\) is (weakly) \(W(G)\)-quasiadditive, if there is a positive constant \(N > 0\) such that for every compact set \(K \subset G\) (in the weak case for every \(K = \bigcup_{Q \in \mathcal{E}} Q\), where \(\mathcal{E} \subset W(G)\) is finite),

\[
\sum_{Q \in W(G)} \text{cap}_{s,p}(K \cap Q, G) \leq N \text{cap}_{s,p}(K, G).
\]

See [6] for information on the closely related quasiadditivity of variational capacity.

An open set \(G\) is said to admit an \((s,p)\)-zero extension, if there is a constant \(C > 0\) such that the zero extension operator satisfies

\[
|E_G u|_{W^{s,p}([\mathbb{R}^n])} \leq C |u|_{W^{s,p}(G)}
\]

for every function \(u \in C_c(G)\). Here \(E_G u(x) = u(x)\) if \(x \in G\) and \(E_G u(x) = 0\) otherwise. Let us emphasise that only continuous functions with compact support need to have a bounded zero extension, and not all open sets admit an \((s,p)\)-zero extension. We mention in passing that the usual extension problem for \(W^{s,p}(G)\) has been recently solved by Zhou in [11].

Observe that an open set admits an \((s,p)\)-zero extension if, and only if, the following weighted embedding holds, with a constant \(c > 0\) and a weight \(\omega(x) = \int_{\mathbb{R}^n \setminus G} |x - y|^{-n+sp} \, dy\),

\[
(1.3) \quad \int_G |u(x)|^p \omega(x) \, dx \leq c \int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dy \, dx, \quad u \in C_c(G);
\]

Proposition 5 provides a Maz’ya-type characterization for this weighted embedding, which is weaker than the \((s,p)\)-Hardy inequality. Indeed, we have \(\omega(x) \leq \text{dist}(x, \partial G)^{-sp}\) if \(x \in G\).

**Theorem 2.** Let \(0 < s < 1\) and \(1 < p < \infty\) satisfy \(sp < n\). Suppose \(G \neq \emptyset\) is a bounded open set in \(\mathbb{R}^n\). Then the following conditions are equivalent.

1. \(G\) admits an \((s,p)\)-Hardy inequality;
2. \(\text{cap}_{s,p}(\cdot,G)\) is \(W(G)\)-quasiadditive and \(G\) admits an \((s,p)\)-zero extension;
3. \(\text{cap}_{s,p}(\cdot,G)\) is weakly \(W(G)\)-quasiadditive and \(G\) admits an \((s,p)\)-zero extension.

Moreover, the implications \((1) \Rightarrow (2) \Rightarrow (3)\) hold for unbounded open sets \(G \subset \mathbb{R}^n\).

Combined with sufficient conditions for an \((s,p)\)-Hardy inequality, Theorem 2 yields sufficient conditions for the \(W(G)\)-quasiadditivity of \(\text{cap}_{s,p}(\cdot,G)\). Another point-of-view is that the two weighted embeddings (1.1) and (1.3) are equivalent under the \(W(G)\)-quasiadditivity assumption.
The following question remains open to our knowledge. It is motivated by [6], where a positive answer is provided in case of a non-fractional Hardy inequality. Below $\ell(Q)$ stands for the side length of the cube $Q$.

**Question.** Is the condition
\begin{equation}
\text{cap}_{s,p}(\cdot, G) \text{ is weakly } \mathcal{W}(G)\text{-quasiadditive and } \ell(Q)^{n-sp} \lesssim \text{cap}_{s,p}(Q, G) \text{ if } Q \in \mathcal{W}(G)
\end{equation}
equivalent with condition (1) in Theorem 2?

To state this otherwise, is it possible to replace the $(s, p)$-zero extension condition by a testing condition (1.2) restricted to Whitney cubes $K = Q \in \mathcal{W}(G)$? (The lower bound on the capacities of Whitney cubes may be viewed as such.) The fact that $G$ need not admit $(s, p)$-zero extension introduces complications to the treatment of this question, as the boundedness properties for the local maximal operator $M_G$, established by Luiro in [7], are no longer available. And, our proof of Theorem 2 relies on these properties instead of, say, weak Harnack inequalities as in [6].

The structure of this paper is the following. In §2 we present notation and also properties of the local maximal operators. The Maz'ya-type characterization, yielding Theorem 1 in particular, is proven in §3, and the proof of Theorem 2 is divided in sections §4 and §5. In §6, we provide two counterexamples showing that the two conditions occurring in point (2) of Theorem 2 are independent, i.e., neither one of them implies the other one. In fact, the same examples show that the two conditions in either (3) or (4) are also independent.

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## 2. Preliminaries

### 2.1. Whitney cubes.

For an open set $\emptyset \neq G \subseteq \mathbb{R}^n$, we fix its Whitney decomposition $\mathcal{W}(G)$ consisting of closed cubes such that, for each $Q \in \mathcal{W}(G)$,
\begin{equation}
\text{diam}(Q) \leq \text{dist}(Q, \partial G) \leq 4 \text{diam}(Q).
\end{equation}
We have $\sum_{Q \in \mathcal{W}(G)} x_{Q^{**}} \leq C_n x_G$, where $Q^{**} = \frac{2}{3} Q$, and $G = \cup_{Q \in \mathcal{W}(G)} Q$, see [9, VI.1].

### 2.2. Function spaces.

Let us recall the definition of the fractional order Sobolev spaces in open sets $G \subset \mathbb{R}^n$. For $0 < p < \infty$ and $s \in (0, 1)$ we let $W^{s,p}(G)$ be the family of functions $u$ in $L^p(G)$ with
\begin{align*}
\|u\|_{W^{s,p}(G)} := & \|u\|_{L^p(G)} + |u|_{W^{s,p}(G)} \\
:= & \left(\int_G |u(x)|^p \, dx\right)^{1/p} + \left(\int_G \int_G \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dy \, dx\right)^{1/p} < \infty.
\end{align*}
The global fractional Sobolev spaces belong to the well-known scale of Triebel–Lizorkin F-spaces: if \( 1 < p < \infty \) and \( 0 < s < 1 \), then \( W^{s,p}(\mathbb{R}^n) \) coincides with \( F^s_{p,p}(\mathbb{R}^n) \) and the associated norms are equivalent, [10, pp. 6–7]. Let \( G \) be an open set in \( \mathbb{R}^n \), \( 1 < p < \infty \), and \( 0 < s < 1 \). Then

\[
F^s_{p,p}(G) = \{ u \in L^p(G) : \text{there is a } g \in F^s_{p,p}(\mathbb{R}^n) \text{ with } g|_G = u \}
\]

\[
\| u \|_{F^s_{p,p}(G)} = \inf \| g \|_{F^s_{p,p}(\mathbb{R}^n)},
\]

where the infimum is taken over all \( g \in F^s_{p,p}(\mathbb{R}^n) \) such that \( g|_G = u \) pointwise a.e.

2.3. Local maximal operator. Let \( \emptyset \neq G \subseteq \mathbb{R}^n \) be an open set. The local Hardy–Littlewood maximal operator \( M_G \) is defined as follows. For a measurable \( u : G \to \mathbb{R} \),

\[
M_G u(x) = \sup_r \int_{B(x,r)} |u(y)| \, dy,
\]

where the supremum ranges over \( 0 < r < \text{dist}(x, \partial G) \). The following statement is important to us: Luiro has shown that \( M_G \) is bounded on \( F^s_{p,p}(G) \) if \( 1 < p < \infty \) and \( 0 < s < 1 \), we refer to [7, Theorem 3.2]. This yields the following lemma.

**Lemma 3.** Let \( \emptyset \neq G \subseteq \mathbb{R}^n \) be a bounded open set, which admits an \( (s,p) \)-zero extension with \( 0 < s < 1 \) and \( 1 < p < \infty \). Then

\[
|M_G u|_{W^{s,p}(G)} \leq |u|_{W^{s,p}(G)}
\]

for every \( u \in C_c(G) \).

**Proof.** Without loss of generality, we may assume that \( |u|_{W^{s,p}(G)} < \infty \). We have

\[
|M_G u|_{W^{s,p}(G)} \leq \| M_G u \|_{F^s_{p,p}(G)} \\
\leq \| u \|_{F^s_{p,p}(G)} \leq \| E_G u \|_{F^s_{p,p}(\mathbb{R}^n)} \leq \| E_G u \|_{L^p(\mathbb{R}^n)} + \| E_G u \|_{W^{s,p}(\mathbb{R}^n)}.
\]

Since \( G \) admits an \( (s,p) \)-zero extension, the last seminorm is dominated by \( C |u|_{W^{s,p}(G)} \). Let us then fix a compact set \( K \subset \mathbb{R}^n \setminus G \) for which \( |K| > 0 \). By the boundedness of \( G \),

\[
\| E_G u \|_{L^p(\mathbb{R}^n)} = \int_G |u(x)|^p \, dx \leq \int_K \int_G \frac{|E_G u(x) - E_G u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \leq |E_G u|_{W^{s,p}(\mathbb{R}^n)}^p.
\]

The last term is, again, dominated by \( C |u|_{W^{s,p}(G)} \). \( \Box \)

For the convenience of the reader, we provide the proof of the following useful lemma.

**Lemma 4.** Suppose \( G \) is an open set in \( \mathbb{R}^n \) and \( u \in C_c(G) \). Then \( M_G u \) is continuous.

**Proof.** We first observe that the function defined by

\[
\bar{u}(x, r) = \int_{B(x,r)} |u(y)| \, dy
\]

for \( r > 0 \) and \( \bar{u}(x, 0) = |u(x)| \) is continuous on \( \mathbb{R}^n \times [0, \infty) \) (in this definition the function \( |u| \) is zero-extended to the whole \( \mathbb{R}^n \)). Let us fix \( x \in G \) and \( \varepsilon > 0 \). By uniform continuity of the function \( \bar{u} \) on \( \overline{B(x, \text{dist}(x, \partial G))} \times [0, 2 \text{dist}(x, \partial G)] \), there exists \( 0 < \delta < \text{dist}(x, \partial G) \) such that

\[
|\bar{u}(y, s) - \bar{u}(x, t)| < \varepsilon,
\]

for all \( y \in B(x, \delta) \) and \( s, t > 0 \) such that \( s - t < \text{dist}(x, \partial G) \).

For \( r > 0 \) we estimate |

\[
\int_{B(x,r)} \frac{|E_G u(x) - E_G u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy
\]

by |

\[
\int_{B(x,r)} \frac{|E_G u(x) - E_G u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy < \int_{|y-x| < \varepsilon} \frac{|E_G u(x) - E_G u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy
\]

and |

\[
\int_{|y-x| > \varepsilon} \frac{|E_G u(x) - E_G u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy.
\]

The second integral is dominated by \( C |u|_{W^{s,p}(G)}^p \), and the first one is dominated by \( C |u|_{W^{s,p}(G)}^p \) and the uniform continuity of \( \bar{u} \). This completes the proof. \( \Box \)
whenever $|y - x| + |s - t| < \delta$ and $0 \leq s, t \leq 2 \text{dist}(x, \partial G)$. Therefore, if $y \in B(x, \delta/2)$, then for some $r_0 = r_0(y, \epsilon) < \text{dist}(y, \partial G)$

$$M_G u(y) \leq \bar{u}(y, r_0) + \epsilon \leq \bar{u}(x, r_0 \wedge \text{dist}(x, \partial G)) + 2\epsilon \leq M_G u(x) + 2\epsilon,$$

because $|y - x| + |r_0 - r_0 \wedge \text{dist}(x, \partial G)| < \delta$. On the other hand, for some $r_0 \in [0, \text{dist}(x, \partial G))$,

$$M_G u(x) \leq \bar{u}(x, r_0) + \epsilon \leq \bar{u}(y, r_0 \wedge \text{dist}(y, \partial G)) + 2\epsilon \leq M_G u(y) + 2\epsilon.$$

This proves continuity of $M_G u$. \hfill \Box

3. Maz’ya-type characterization

Theorem 1 is implied by the following Maz’ya-type characterization for weighted embeddings, applied with $\omega(x) = \text{dist}(x, \partial G)^{-sp}$. Let us also remark that inequality (1.3) admits a similar characterization with a weight $\omega(x) = \int_{\mathbb{R}^n \backslash G} |x - y|^{-n-sp} \, dy$.

**Proposition 5.** Suppose $G \subset \mathbb{R}^n$ is an open set and $\omega : G \to [0, \infty)$ is a measurable function. The following two conditions are equivalent for $0 < s < 1$ and $0 < p < \infty$.

(A) There is a constant $C > 0$ such that

$$\int_G |u(x)|^p \omega(x) \, dx \leq C |u|_{W^{s,p}(G)}^p, \quad u \in C_c(G).$$

(B) There is a constant $c > 0$ such that, for every compact set $K \subset G$,

$$\int_K \omega(x) \, dx \leq c \text{cap}_{s,p}(K, G).$$

**Proof.** First assume condition (A) holds. Let $u \in C_c(G)$ be such that $u(x) \geq 1$ for every $x \in K$. By (A) we obtain

$$\int_K \omega(x) \, dx \leq \int_G |u(x)|^p \omega(x) \, dx \leq C \int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dy \, dx.$$

Taking infimum over all such functions $u$, we obtain (B) with $c = C$.

Now assume that (B) holds and let $u \in C_c(G)$. For $k \in \mathbb{Z}$ denote

$$E_k = \{x \in G : |u(x)| > 2^k\} \quad \text{and} \quad A_k = E_k \setminus E_{k+1}.$$

Observe that

$$G = \{x \in G : 0 \leq u(x) < \infty\} = \bigcup_{i \in \mathbb{Z}} A_i.$$ 

Hence, by (B) we obtain

$$\int_G |u(x)|^p \omega(x) \, dx \leq \sum_{k \in \mathbb{Z}} 2^{(k+2)p} \int_{A_{k+1}} \omega(x) \, dx$$

$$\leq c 2^{2p} \sum_{k \in \mathbb{Z}} 2^{kp} \text{cap}_{s,p}(A_{k+1}, G).$$
Define $u_k : G \to [0, 1]$ by
$$
u_k(x) = \begin{cases} 
1, & \text{if } |u(x)| \geq 2^{k+1}, \\
\frac{|u(x)|}{2^k} - 1, & \text{if } 2^k < |u(x)| < 2^{k+1}, \\
0, & \text{if } |u(x)| \leq 2^k.
\end{cases}$$

Then $u_k \in C_c(G)$ and it satisfies $u_k = 1$ on $\overline{E}_{k+1} \supset \overline{A}_{k+1}$, hence we may take it as a test function for the capacity. By recalling also (3.1), we obtain that

$$\text{cap}_{s,p}(\overline{A}_{k+1}, G) \leq \int_G \int_G \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+sp}} \, dy \, dx$$

(3.3)

$$\leq 2 \sum_{i \leq k} \sum_{j \geq k} \int_{A_i} \int_{A_j} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+sp}} \, dy \, dx.$$

We observe that $|u_k(x) - u_k(y)| \leq 2^{-k} |u(x) - u(y)|$. Moreover, if $x \in A_i$ and $y \in A_j$, where $i + 2 \leq j$, then $|u(x) - u(y)| \geq |u(y)| - |u(x)| \geq 2^{j-1}$, hence $|u_k(x) - u_k(y)| \leq 1 \leq 2 \cdot 2^{-j} |u(x) - u(y)|$. Therefore,

$$|u_k(x) - u_k(y)| \leq 2 \cdot 2^{-j} |u(x) - u(y)|, \quad (x, y) \in A_i \times A_j,$$

whenever $i \leq k \leq j$. Thus,

$$\sum_{i \leq k} \sum_{j \geq k} \int_{A_i} \int_{A_j} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+sp}} \, dy \, dx$$

$$\leq 2p \sum_{i \leq k} \sum_{j \geq k} 2^{-jp} \int_{A_i} \int_{A_j} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dy \, dx.$$

A similar argument show that

$$\sum_{j \geq k} \int_{A_j} \int_{A_i} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+sp}} \, dy \, dx \leq 2p \sum_{j \geq k} 2^{-jp} \int_{A_j} \int_{A_i} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dy \, dx.$$

Since $\sum_{j=k}^{\infty} 2^{(k-j)p} \leq \frac{1}{2^p}$, we may apply inequalities (3.2) and (3.3) and then change the order of summations to obtain that

$$\int_G |u(x)|^p \omega(x) \, dx \leq \frac{c2^{3p+2}}{1 - 2^{-p}} \int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dy \, dx.$$

Thus condition (A) is satisfied with $C = \frac{c2^{3p+2}}{1 - 2^{-p}}$. \hfill \Box

4. Necessary conditions for Hardy

In this section, we prove the implication (1) $\Rightarrow$ (2) in Theorem 2.

Proposition 6. Suppose $G$ is an open set in $\mathbb{R}^n$ which admits an $(s, p)$-Hardy inequality with $0 < s < 1$ and $0 < p < \infty$. Then $\text{cap}_{s,p}(\cdot, G)$ is $\mathcal{W}(G)$-quasiadditive and $G$ admits an $(s, p)$-zero extension.
To estimate the second series, we first find that
\[
\sum_{\mathcal{W}(G)} \cap_{s,p} (K \cap Q, G) \leq N \inf \int_{G} \int_{G} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dy \, dx,
\]
where the infimum is taken over all \( u \in C_c(G) \) such that \( u \geq 1 \) on \( K \). Fix a function \( u \) for which the infimum is essentially obtained, say, within a factor of 2.

For each Whitney cube \( Q \in \mathcal{W}(G) \) we fix a smooth function \( \phi_Q \) such that \( \chi_Q \leq \phi_Q \leq \chi_Q^* \), where \( Q^* = \frac{12}{10} Q \), and \( |\nabla \phi_Q| \leq \ell(Q)^{-1} \). In particular, we find that \( u_Q := u \phi_Q \) is a test function for \( \cap_{s,p}(K \cap Q, G) \). Hence, with \( Q'' = \frac{8}{5} Q \),
\[
\text{LHS}(4.1) \leq \sum_{\mathcal{W}(G)} \int_{Q} \int_{G} \frac{|u_Q(x) - u_Q(y)|^p}{|x - y|^{n+sp}} \, dy \, dx \\
\leq \sum_{\mathcal{W}(G)} \left\{ \int_{Q'} \frac{|u_Q(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx + \int_{Q''} \int_{Q''} \frac{|u_Q(x) - u_Q(y)|^p}{|x - y|^{n+sp}} \, dy \, dx \right\}
\]
Since \( |u_Q| \leq |u| \) and \( \sum_Q \chi_Q^* \leq 1 \), we may apply the \((s, p)\)-Hardy inequality for
\[
\sum_{\mathcal{W}(G)} \int_{Q'} \frac{|u_Q(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \leq \int_{G} \frac{|u(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \leq \cap_{s,p}(K, G).
\]
To estimate the second series, we first find that
\[
|u_Q(x) - u_Q(y)| = |u(x)\phi_Q(x) - u(x)\phi_Q(y) + u(x)\phi_Q(y) - u(y)\phi_Q(y)| \\
\leq |u(x)||\phi_Q(x) - \phi_Q(y)| + |u(x) - u(y)||\phi_Q(y) \\
\leq |u(x)||x - y|\ell(Q)^{-1} + |u(x) - u(y)|.
\]
Since \( \sum_{\mathcal{W}(G)} \chi_Q^* \leq \chi_G \), we find that
\[
\sum_{\mathcal{W}(G)} \int_{Q''} \int_{Q''} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dy \, dx \leq \cap_{s,p}(K, G).
\]
To estimate the remaining series, we proceed as follows (recall that \( 0 < s < 1 \));
\[
\sum_{\mathcal{W}(G)} \ell(Q)^{-p} \int_{Q''} |u(x)|^p \int_{Q''} \frac{|x - y|^p}{|x - y|^{n+sp}} \, dy \, dx \\
\leq \sum_{\mathcal{W}(G)} \ell(Q)^{-sp} \int_{Q''} |u(x)|^p \, dx \\
\leq \sum_{\mathcal{W}(G)} \int_{Q''} \frac{|u(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \leq \cap_{s,p}(K, G),
\]
and this concludes the proof. \( \square \)
5. SUFFICIENT CONDITIONS FOR HARDY

The implication (3) ⇒ (1) in Theorem 2 is established here. Let us observe that the remaining implication (2) ⇒ (3) is trivial. The outline of the proof below is from [6].

**Proposition 7.** Let \( 0 < s < 1 \) and \( 1 < p < \infty \) satisfy \( sp < n \). Suppose that \( G \subset \mathbb{R}^n \) is a bounded open set such that \( \text{cap}_{s, p}(\cdot, G) \) is weakly \( W(G) \)-quasiadditive and \( G \) admits an \((s, p)\)-zero extension. Then \( G \) admits an \((s, p)\)-Hardy inequality.

**Proof.** By Theorem 1, it suffices to show that

\[
\int_K \text{dist}(x, \partial G)^{-sp} \, dx \lesssim \text{cap}_{s, p}(K, G),
\]

where \( K \subset G \) is compact. We fix a test function \( u \) for \( \text{cap}_{s, p}(K, G) \) such that the infimum in the definition of \((s, p)\)-capacity is obtained within a factor 2. By replacing \( u \) with \( \max\{0, \min\{u, 1\}\} \) we may assume that \( 0 \leq u \leq 1 \). The truncation can be written as \( f \circ u \), where \( f \) is 1-Lipschitz, hence this truncation does not increase the associated seminorm.

Let us split \( W(G) = W_1 \cup W_2 \), where

\[
W_1 = \{Q \in W(G) : \langle u \rangle_Q := \int_Q u < 1/2\}, \quad W_2 = W(G) \setminus W_1.
\]

Write the left-hand side of (5.1) as

\[
(5.2) \quad \left\{ \sum_{Q \in W_1} + \sum_{Q \in W_2} \right\} \int_{K \cap Q} \text{dist}(x, \partial G)^{-sp} \, dx.
\]

To estimate the first series we observe that, for \( x \in K \cap Q \) with \( Q \in W_1 \),

\[
\frac{1}{2} = 1 - \frac{1}{2} < u(x) - \langle u \rangle_Q = |u(x) - \langle u \rangle_Q|.
\]

Thus, by Jensen’s inequality,

\[
\sum_{Q \in W_1} \int_{K \cap Q} \text{dist}(x, \partial G)^{-sp} \, dx \lesssim \sum_{Q \in W_1} \ell(Q)^{-sp} \int_Q |u(x) - \langle u \rangle_Q|^p \, dx
\]

\[
\lesssim \sum_{Q \in W_1} \ell(Q)^{-n-sp} \int_Q \int_{Q} |u(x) - u(y)|^p \, dy \, dx
\]

\[
\lesssim \sum_{Q \in W_1} \int_Q \int_{Q} |u(x) - u(y)|^p \, dy \, dx \lesssim \text{cap}_{s, p}(K, G).
\]

Let us then focus on the remaining series in (5.2), namely, the one over \( W_2 \). We first establish two auxiliary estimates (5.3) and (5.4).

By inequality (2.1), for every \( Q \in W_2 \) and \( x \in \text{int}(Q) \),

\[
(5.3) \quad M_G u(x) \geq \int_Q u(y) \, dy \geq \frac{1}{2}.
\]
The support of $M_G u$ is a compact set in $G$ due to the boundedness of $G$ and the fact that $u \in C_c(G)$. By Lemma 4, we find that $M_G u$ is continuous. Concluding from these facts, we find that there is $\rho > 0$, depending only on $n$, such that $\rho M_G u$ is an admissible test function for $\text{cap}_{s,p}(\bigcup_{Q \in \mathcal{W}_2} Q, G)$.

Another useful estimate for $Q \in \mathcal{W}_2$ is a lower bound for its capacity, namely,

\begin{equation}
\ell(Q)^{n-sp} \leq \text{cap}_{s,p}(Q, G).
\end{equation}

To verify this inequality, let $u_Q \in C_c(G)$ be a test function for $\text{cap}_{s,p}(Q, G)$. A fractional Sobolev embedding theorem, [1, Theorem 6.5] with $p^* = np/(n-sp)$, and the assumption that $G$ admits an $(s, p)$-zero extension yield

$$\ell(Q)^{n-sp} \leq \|E_G u_Q\|_{L_p(\mathbb{R}^n)}^p \leq |E_G u_Q|_{W^{s,p}(\mathbb{R}^n)}^p \leq |u_Q|_{W^{s,p}(G)}^p.$$ 

It remains to infimize the right hand side over functions $u_Q$.

We may continue as follows. By (5.3), (5.4), and the assumed weak quasiadditivity with a finite union $K = \bigcup_{Q \in \mathcal{W}_2} Q$, we find that

$$\sum_{Q \in \mathcal{W}_2} \int_{K \cap Q} \text{dist}(x, \partial G)^{-sp} \, dx \leq \sum_{Q \in \mathcal{W}_2} \ell(Q)^{n-sp} \leq \sum_{Q \in \mathcal{W}_2} \text{cap}_{s,p}(Q, G) \leq N \text{cap}_{s,p}(K, G) \leq N \rho \int_G \frac{|M_G u(x) - M_G u(y)|^p}{|x - y|^{n+sp}} \, dy \, dx .$$

By Lemma 3, the last term is dominated by

$$C N \rho^p |u|^p_{W^{s,p}(G)} \leq \text{cap}_{s,p}(K, G),$$

and this concludes the proof.

\section{Counterexamples}

We provide two domains as counterexamples, showing that neither one of the two conditions in point (2) of Theorem 2 is implied by the other one. In fact, the counterexamples show that the same is true for points (3) and (4) stated in the Introduction.

In both of the constructions here, we rely on computations in [2].

\textbf{Theorem 8.} The cube $G = (0, 1)^n \subset \mathbb{R}^n$ does not admit $(s, p)$-zero extension and $\text{cap}_{s,p}(\cdot, G)$ is $\mathcal{W}(G)$-quasiadditive if $sp < 1$, where $0 < s < 1$ and $1 < p < \infty$.

\textbf{Proof.} By computations in [2, §2], we find that $\text{cap}_{s,p}(K, G) = 0$ for every compact set $K \subset G$. Therefore $\text{cap}_{s,p}(\cdot, G)$ is trivially $\mathcal{W}(G)$-quasiadditive. It is also shown by Dyda that $G$ does not admit an $(s, p)$-Hardy inequality. Hence, by Proposition 7, we find that $G$ does not admit an
(s, p)-zero extension. Alternatively, we may observe that condition (B) in Proposition 5 fails for the weight \( \omega(x) = \int_{\mathbb{R}^n \setminus G} |x - y|^{-n - sp} \, dy \) in (1.3).

**Theorem 9.** Let \( 0 < s < 1 \) and \( 1 < p < \infty \) satisfy \( sp = 1 \). There is a bounded domain \( G \subset \mathbb{R}^2 \) which admits an \((s, p)\)-zero extension and \( \text{cap}_{s, p}(:, G) \) is not weakly \( \mathcal{W}(G)\)-quasiadditive.

**Proof.** Let \( G' \) be the standard Koch snowflake domain in \( \mathbb{R}^2 \), and fix a closed cube \( R \subset G' \). Define \( G = G' \setminus L \), where \( L = x_R + [-\ell(R)/4, \ell(R)/4] \times \{0\} \subset R \) and \( x_R \) is the midpoint of \( R \). The domain \( G' \) admits an \((s, p)\)-Hardy inequality by [3, Theorem 2], and therefore \( G' \) admits an \((s, p)\)-zero extension, see e.g. Proposition 6. Since \( G \subset G' \) and \( |G' \setminus G| = |L| = 0 \), we see that also \( G \) admits an \((s, p)\)-zero extension, see the proof of [4, Theorem 6.5].

It remains to verify that \( \text{cap}_{s, p}(:, G) \) is not weakly \( \mathcal{W}(G)\)-quasiadditive.

Reasoning as in the proof of inequality (5.4) with \( sp = 1 < 2 = n \), we find that

\[
\ell(Q)^{n-sp} \leq \text{cap}_{s, p}(Q, G), \quad Q \in \mathcal{W}(G).
\]

For \( m \in \mathbb{N} \), we let \( \mathcal{W}^m \) denote the family of Whitney cubes \( Q \in \mathcal{W}(G) \) that are contained in \( R \) and satisfy \( \text{diam}(Q) \geq 1/(2m) \). Let \( K_m \subset G \) be the union of cubes in \( \mathcal{W}^m \). Then, for each \( m \),

\[
\int_{K_m} \text{dist}(x, L)^{-1} \, dx \leq \int_{K_m} \text{dist}(x, \partial G)^{-1} \, dx \leq \sum_{Q \in \mathcal{W}^m} \ell(Q)^{n-sp} \leq \sum_{Q \in \mathcal{W}^m} \text{cap}_{s, p}(Q, G) \leq \sum_{Q \in \mathcal{W}(G)} \text{cap}_{s, p}(K_m \cap Q, G).
\]

Observe that \( K_1 \subset K_2 \subset \cdots \) and \( \{L + B(0, \epsilon)\} \setminus L \subset \bigcup_{m = 1}^\infty K_m \) for an appropriate \( \epsilon > 0 \). Thus, the left hand side tends to \( \infty \), as \( m \to \infty \). In particular, we may infer that the last terms, as a function of \( m \), also tends to \( \infty \), as \( m \to \infty \).

We have \( K_m \subset R \setminus (L + B(0, 1/(2m))) =: R \setminus L_m \). Indeed, for every \( Q \in \mathcal{W}^m \),

\[
\frac{1}{2m} \leq \text{diam}(Q) \leq \text{dist}(Q, \partial G) \leq \text{dist}(Q, L).
\]

Thus, in order to finish the proof, it suffices to find functions \( u_m \in C_c(G) \) satisfying \( u_m \geq 1 \) on \( R \setminus L_m \) and \( \sup_m \|u_m\|_{W^{s,p}(G)} < \infty \). In the sequel, we shall restrict ourselves to sufficiently large \( m \) satisfying \( L_m \subset R \). Fix \( v \in C_c^\infty(G') \) such that \( v = 1 \) on \( R \). Fix \( w_m \in C_c(L_m) \) satisfying \( w_m = 1 \) on \( L_m \) and \( \|w_m\|_\infty + m^{-1}\|\nabla w_m\|_\infty \leq 1 \). Now, the function \( u_m = v - w_m \in C_c(G) \) satisfies \( u_m \geq 1 \) on \( R \setminus L_m \). Furthermore,

\[
\|u_m\|_{W^{s,p}(G)}^p \leq \|v\|_{W^{s,p}(G)}^p + \|w_m\|_{W^{s,p}(G)}^p.
\]

Since \( G \) is bounded and \( |v(x) - v(y)| \leq |x - y| \) for every \( x, y \in G \), we find that \( \|v\|_{W^{s,p}(G)}^p < \infty \).
In order to estimate $|w_m|_{W^{s,p}(G)}$, we let $E_m \subset \mathcal{L}$ be a set such that $\mathcal{L} \subset \bigcup_{z \in E_m} B(z, 1/(2m))$ and $\#E_m \leq m = m^{2-sp}$. Now

$$|w_m|_{W^{s,p}(G)} \leq 2 \int_G \int_{L_m} \frac{|w_m(x) - w_m(y)|^p}{|x - y|^{n+sp}} \, dy \, dx =: 2I_m.$$  

Note that $L_m \subset \bigcup_{z \in E_m} B(z, 1/m)$. By writing $B_z := B(z, 1/m)$,

$$I_m \leq \sum_{z \in E_m} \int_G \int_{B_z} \frac{|w_m(x) - w_m(y)|^p}{|x - y|^{n+sp}} \, dy \, dx \leq \sum_{z \in E_m} \sum_{l=0}^\infty \int_{B(z, 2l+2)/m \setminus B(z, 2l/m)} \int_{B_z} \frac{|w_m(x) - w_m(y)|^p}{|x - y|^{n+sp}} \, dy \, dx.$$  

For $l = 0$ we use inequality $|\nabla w_m| \leq m$ and, for the remaining summands with $l = 1, 2, \ldots$, we use inequality $|w_m(x) - w_m(y)| \leq 1$ to conclude that

$$I_m \leq m^{sp-2} \sum_{z \in E_m} \sum_{l=0}^\infty \left( \frac{1}{l+1} \right)^{sp+1} \leq 1.$$  

For further details on the first inequality above, we refer to [2, §2]. □

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