WEIGHT MODULES OF DIRECT LIMIT LIE ALGEBRAS

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Abstract. In this article we initiate a systematic study of irreducible weight modules over direct limits of reductive Lie algebras, and in particular over the simple Lie algebras $A(\infty)$, $B(\infty)$, $C(\infty)$ and $D(\infty)$. Our main tool is the shadow method introduced recently in [DMP]. The integrable irreducible modules are an important particular class and we give an explicit parametrization of the finite integrable modules which are analogues of finite-dimensional irreducible modules over reductive Lie algebras. We then introduce the more general class of pseudo highest weight modules. Our most general result is the description of the support of any irreducible weight module.

Key words (1991 MSC): Primary 17B10.

Introduction

The purpose of this paper is to initiate a systematic study of the irreducible weight representations of direct limits of reductive Lie algebras, and in particular of the classical simple direct limit Lie algebras $A(\infty)$, $B(\infty)$, $C(\infty)$ and $D(\infty)$. We study arbitrary, not necessarily highest weight, irreducible weight modules and describe the supports of all such modules. The representation theory of the classical direct limit groups has been initiated in the pioneering works of G. Olshanski [O1] and [O2] and is now in an active phase, see the recent works of A. Habib, [Ha], L. Natarajan, [Na], K.-H. Neeb, [Ne], and L. Natarajan, E. Rodriguez-Carrington and J. A. Wolf, [NRW]. Nevertheless, the structure theory of weight representations of the simple direct limit Lie algebras has until recently been still in its infancy as only highest weight modules have been discussed in the literature, see the works of Yu. A. Bahturin and G. Benkart, [BB], K.-H. Neeb, [Ne], and T. D. Palev, [P].

Our approach is based mainly on the recent paper [DMP] in which a general method for studying the support of weight representations of finite-dimensional Lie algebras (and Lie superalgebras) was developed. We prove first that the shadow of any irreducible weight module $M$ (over any root reductive direct limit Lie algebra, in particular over $A(\infty)$, $B(\infty)$, $C(\infty)$ and $D(\infty)$) is well-defined, which means that for a given root $\alpha$ the intersection of the ray $\lambda + \mathbb{R}_+\alpha$ with the support of $M$, supp$M$, is either finite for all $\lambda \in$ supp$M$ or infinite for all $\lambda \in$ supp$M$. Using this remarkable property of the support, we assign to $M$ a canonical parabolic subalgebra $p_M$ of $\mathfrak{g}$ and then compare $M$ with the irreducible quotient of a certain $\mathfrak{g}$-module induced from $p_M$. In the case of a finite-dimensional Lie algebra, the Fernando-Futorny parabolic induction theorem, see [Fe], [Fu] and [DMP], states that $M$ is always such a quotient and moreover that supp$M$ is nothing but the support of the induced module. In the direct limit case we show that this is no longer true but nevertheless, using the fact that the shadow is well-defined, we obtain an explicit description of the support of any irreducible weight module.
We discuss in more detail the following special cases: when supp\(M\) is finite in all root directions (these are the finite integrable irreducible modules and they are analogues of finite-dimensional irreducible modules over finite-dimensional Lie algebras) and the more general case when supp\(M\) is finite in at least one of each two mutually opposite root directions. An interesting feature in the first case is that \(M\) is not necessarily a highest weight module, i.e. the analogues of finite-dimensional irreducible modules are already outside the class of highest weight modules. We present an explicit parametrization of all finite integrable irreducible modules. The modules corresponding to the second case are by definition pseudo highest weight modules and one of their interesting features is that in general they are not obtained by parabolic induction as in the Fernando-Futorny theorem.

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Acknowledgement. Discussions of various aspects of the subject of this paper with Yu. A. Bahturin, G. Benkart, V. Futorny, Yu. I. Manin, O. Mathieu, G. I. Olshanskii, V. Serganova and J. A. Wolf have been very helpful to us. Special thanks are due to K.-H. Neeb who read an early version of the paper and pointed out some errors. Both authors have been supported in part by an NSF GIG Grant, and I. Penkov acknowledges support from the Louis Pasteur University in Strasbourg where part of this work was done.

1. Generalities on triangular decompositions and weight modules

The ground field is \(\mathbb{C}\). The signs \(\subset\) and \(\supset\) denote semi-direct sum of Lie algebras (if \(\mathfrak{g} = \mathfrak{g}' \subset \mathfrak{g}'',\) then \(\mathfrak{g}'\) is an ideal in \(\mathfrak{g}\) as well as a \(\mathfrak{g}''\)-module). The superscript \(\ast\) always stands for dual space. We set \(\mathbb{R}_+ := \{r \in \mathbb{R} \mid r \geq 0\}, \mathbb{R}_- := -\mathbb{R}_+,\ Z_{\pm} := Z \cap \mathbb{R}_{\pm},\ \delta^ij\) is Kronecker’s delta, and linear span with coefficients in \(\mathbb{C}\) (respectively, \(\mathbb{R}, Z, \mathbb{R}_\pm\)) is denoted by \(< \cdot >\) (resp. by \(\prec \cdot \prec\)). If \(\mathfrak{g}\) is any Lie algebra and \(M\) is a \(\mathfrak{g}\)-module, we call \(M\) integrable iff \(\mathfrak{g}\) acts locally finitely on \(M\); this terminology is introduced by V. Kac, in [BB] integrable modules are called locally finite. If \(\mathfrak{g}\) is a direct sum of Lie algebras, \(\mathfrak{g} = \oplus_{s \in S}\mathfrak{g}^s\), and for each \(s M^s\) is an irreducible \(\mathfrak{g}^s\)-module with a fixed non-zero vector \(m^s \in M\), then \((\otimes_s M^s)(\otimes_s m^s)\) denotes the \(\mathfrak{g}\)-submodule of \(\otimes_s M^s\) generated by \(\otimes_s m^s\). It is easy to check that \((\otimes_s M^s)(\otimes_s m^s)\) is an irreducible \(\mathfrak{g}\)-module.

Let \(\mathfrak{g}\) be a Lie algebra. A Cartan subalgebra of \(\mathfrak{g}\) is by definition a self-normalizing nilpotent Lie subalgebra \(\mathfrak{h} \subset \mathfrak{g}\). We do not assume \(\mathfrak{g}\) or \(\mathfrak{h}\) to be finite-dimensional. A \(\mathfrak{g}\)-module \(M\) is a generalized weight \(\mathfrak{g}\)-module iff as an \(\mathfrak{h}\)-module \(M\) decomposes as the direct sum \(\oplus_{\lambda \in \mathfrak{h}^*} M^\lambda\), where

\[M^\lambda := \{v \in M \mid h - \lambda(h) \text{ acts nilpotently on } v \text{ for every } h \in \mathfrak{h}\}.
\]

We call \(M^\lambda\) the generalized weight space of \(M\) of weight \(\lambda\). Obviously, a generalized weight \(\mathfrak{g}\)-module is integrable as an \(\mathfrak{h}\)-module. The support of \(M\), supp\(M\), consists of all weights \(\lambda\) with \(M^\lambda \neq 0\). A generalized weight module \(M\) is a weight module iff \(\mathfrak{h}\) acts semi-simply on \(M\), i.e. iff each generalized weight space is isomorphic to the direct sum of one-dimensional \(\mathfrak{h}\)-modules.
Henceforth \( \mathfrak{g} \) will denote a Lie superalgebra with a fixed proper Cartan subalgebra \( \mathfrak{h} \) such that \( \mathfrak{g} \) is a generalized weight module, i.e.

\[
\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}^\alpha).
\]

The generalized weight spaces \( \mathfrak{g}^\alpha \) are by definition the root spaces of \( \mathfrak{g} \), \( \Delta := \{ \alpha \in \mathfrak{h}^* \setminus \{0\} | \mathfrak{g}^\alpha \neq 0 \} \) is the set of roots of \( \mathfrak{g} \), and the decomposition \( \overset{\sim}{\mathfrak{g}} \) is the root decomposition of \( \mathfrak{g} \).

Below we recall the definitions of a triangular decomposition of \( \Delta \) and of a Borel subalgebra for an arbitrary Lie algebra \( \mathfrak{g} \) with a root decomposition. For more details see [DP2]. A decomposition of \( \Delta \)

\[
\Delta = \Delta^+ \sqcup \Delta^-
\]

(2)

is a triangular decomposition of \( \Delta \) iff the cone \( \langle \Delta^+ \cup -\Delta^- \rangle_{\mathbb{R}_+} \) (or equivalently, its opposite cone \( -\Delta^+ \cup \Delta^- \rangle_{\mathbb{R}_+} \)) contains no (real) vector subspace. Equivalently, \( \overset{\sim}{\Delta} \) is a triangular decomposition if the following is a well-defined \( \mathbb{R} \)-linear partial order on \( \langle \Delta \rangle_{\mathbb{R}} \):

\[
\eta \geq \mu \Leftrightarrow \eta = \mu + \sum_{i=1}^{n} c_i \alpha_i \text{ for some } \alpha_i \in \Delta^+ \cup -\Delta^- \text{ and } c_i \in \mathbb{R}_+, \text{ or } \mu = \eta.
\]

(A partial order is \( \mathbb{R} \)-linear if it is compatible with addition and multiplication by positive real numbers, and if multiplication by negative real numbers changes order direction. In what follows, unless explicitly stated that it is partial, an order will always be assumed linear, i.e. such that \( \alpha \neq \beta \) implies \( \alpha < \beta \) or \( \alpha > \beta \). In particular, an \( \mathbb{R} \)-linear order is by definition an order which is in addition \( \mathbb{R} \)-linear.) Any regular real hyperplane \( H \) in \( \langle \Delta \rangle_{\mathbb{R}} \) (i.e. a codimension one linear subspace \( H \) with \( H \cap \Delta = \emptyset \)) determines exactly two triangular decompositions of \( \Delta \); we first assign (in an arbitrary way) the sign + to one of the two connected components of \( \langle \Delta \rangle_{\mathbb{R}} \setminus H \), and the sign − to the other. \( \Delta^\pm \) are then by definition the subsets of \( \Delta \) which belong respectively to the ”positive” and the ”negative” connected components of \( \langle \Delta \rangle_{\mathbb{R}} \setminus H \). In general, not every triangular decomposition of \( \Delta \) arises in this way, see [DP]. Nevertheless, it is true that every triangular decomposition is determined by a (not unique) oriented maximal chain of vector suspces in \( \langle \Delta \rangle_{\mathbb{R}} \); see the Appendix where we establish the precise interrelationship between oriented maximal chains in \( \langle \Delta \rangle_{\mathbb{R}} \), \( \mathbb{R} \)-linear orders on \( \langle \Delta \rangle_{\mathbb{R}} \), and triangular decompositions of \( \Delta \).

A Lie subalgebra \( \mathfrak{b} \) of \( \mathfrak{g} \) is by definition a Borel subalgebra of \( \mathfrak{g} \) if \( \mathfrak{b} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha) \) for some triangular decomposition \( \Delta = \Delta^+ \sqcup \Delta^- \). In what follows a Borel subalgebra of \( \mathfrak{g} \) always means a Borel subalgebra containing the fixed Cartan subalgebra \( \mathfrak{h} \). Adopting terminology from affine Kac-Moody algebras, we will call a Borel subalgebra standard iff it corresponds to a triangular decomposition which can be determined by a regular hyperplane \( H \) in \( \langle \Delta \rangle_{\mathbb{R}} \).

If \( \mathfrak{b} \) is a Borel subalgebra and \( \lambda \in \mathfrak{h}^* \), the Verma module \( \tilde{V}_\mathfrak{b}(\lambda) \) with \( \mathfrak{b} \)-highest weight \( \lambda \) is by definition the induced module \( U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_\lambda, \ C_\lambda \) being the one-dimensional \( \mathfrak{b} \)-module on which \( \mathfrak{h} \) acts via \( \lambda \). Any quotient of \( \tilde{V}_\mathfrak{b}(\lambda) \) is by definition a \( \mathfrak{b} \)-highest weight module. Furthermore, \( \tilde{V}_\mathfrak{b}(\lambda) \) has a unique maximal proper \( \mathfrak{g} \)-submodule \( I_\mathfrak{b}(\lambda) \), and \( V_\mathfrak{b}(\lambda) := \tilde{V}_\mathfrak{b}(\lambda) / I_\mathfrak{b}(\lambda) \) is by definition the irreducible \( \mathfrak{b} \)-highest weight \( \mathfrak{g} \)-module with highest weight \( \lambda \).

\[1\text{We use the term cone as a synonym for an } \mathbb{R}_+ \text{-invariant additive subset of a real vector space.}\]
2. Direct limits of reductive Lie algebras

A homomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras with root decomposition is a root homomorphism iff $\varphi(\mathfrak{h}) \subset \mathfrak{h}'$ ($\mathfrak{h}$ and $\mathfrak{h}'$ being the corresponding fixed Cartan subalgebras) and $\varphi$ maps any root space of $\mathfrak{g}$ into a root space of $\mathfrak{g}'$. Let

$$
\mathfrak{g}_1 \xrightarrow{\varphi_1} \mathfrak{g}_2 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_{n-1}} \mathfrak{g}_n \xrightarrow{\varphi_n} \ldots
$$

be a chain of homomorphisms of finite-dimensional Lie algebras and let $\mathfrak{g} := \varinjlim \mathfrak{g}_n$ be the direct limit Lie algebra. We say that $\mathfrak{g}$ is a root direct limit of the system (3) iff $\varphi_n$ is a root homomorphism for every $n$. In the latter case $\mathfrak{h}_n$ denotes the fixed Cartan subalgebra of $\mathfrak{g}_n$. We define a Lie algebra $\mathfrak{g}$ to be a root direct limit Lie algebra iff $\mathfrak{g}$ is a root direct limit of some direct system of the form (3). Furthermore, a non-zero Lie algebra $\mathfrak{g}$ is a root direct limit of the system (3) iff all $\mathfrak{g}_n$ are reductive, and $\mathfrak{g}$ is a root simple direct limit Lie algebra iff $\mathfrak{g}$ is a root direct limit of a system (3) in which all $\mathfrak{g}_n$ are simple.

**Proposition 1.** Let $\mathfrak{g}$ be a root direct limit Lie algebra. Then $\mathfrak{h} := \varinjlim \mathfrak{h}_n$ is a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g}$ has a root decomposition with respect to $\mathfrak{h}$ such that $\Delta = \varinjlim \Delta_n$, where $\Delta_n$ and $\Delta$ are respectively the roots of $\mathfrak{g}_n$ and $\mathfrak{g}$. If all root homomorphisms $\varphi_n$ are embeddings, one has simply $\mathfrak{h} = \varinjlim \mathfrak{h}_n$ and $\Delta = \varinjlim \Delta_n$.

**Proof.** A trivial exercise. 

Every simple finite-dimensional Lie algebra $\mathfrak{g}$ is a root simple direct limit Lie algebra: we set $\mathfrak{g}_n := \mathfrak{g}$ and $\varphi_n := \text{id}_\mathfrak{g}$. To define the simple Lie algebras $A(\infty)$, $B(\infty)$, $C(\infty)$ and $D(\infty)$ it suffices to let $\mathfrak{g}_n$ be the corresponding rank $n$ simple Lie algebra and to request that all $\varphi_n$ be injective root homomorphisms, i.e. root embeddings. Indeed, one can check that in these cases the direct limit Lie algebra does not depend up to isomorphism on the choice of root embeddings $\varphi_n$. More generally, Theorem 4.4 in [3, 4] implies that every infinite-dimensional root simple direct limit Lie algebra is isomorphic to one of the Lie algebras $A(\infty)$, $B(\infty)$, $C(\infty)$ or $D(\infty)$. Note however, that for general root reductive direct limit Lie algebras the direct limit Lie algebra can depend on the choice of root homomorphisms $\varphi_n$ even if they are embeddings. Indeed, set for instance $\mathfrak{g}_n := A(2^n) \oplus B(2^n)$ and let $\varphi_n, \varphi'_n : \mathfrak{g}_n \to \mathfrak{g}_{n+1}$ be root embeddings such that $\varphi_n(A(2^n)) \subset A(2^{n+1})$ and $\varphi'_n(B(2^n)) \subset B(2^{n+1})$ but $\varphi_n(A(2^n) \oplus B(2^n)) \subset B(2^{n+1})$. Then $\mathfrak{g} \simeq A(\infty) \oplus B(\infty)$ but $\mathfrak{g}' \simeq B(\infty)$.

Let $\mathfrak{g}$ be a root reductive direct limit Lie algebra and let $\mathfrak{t}$ be a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{t}$. Then $\Delta_\mathfrak{t} \subset \Delta_\mathfrak{g}$ and we can set $\Delta^{ss}_\mathfrak{t} := \Delta_\mathfrak{t} \cap (-\Delta_\mathfrak{t})$. The Lie subalgebra $\mathfrak{t}^{ss} := [\oplus_{\alpha \in \Delta^{ss}_\mathfrak{t}} \mathfrak{t}^\alpha, \oplus_{\alpha \in \Delta^{ss}_\mathfrak{t}} \mathfrak{t}^\alpha]$ of $\mathfrak{t}$ is an analogue of the semi-simple part of $\mathfrak{t}$ in the case when $\mathfrak{t}$ is not finite-dimensional.

**Theorem 1.** Let $\mathfrak{g}$ be a root reductive direct limit Lie algebra. Then

(i) $\mathfrak{g} \simeq \mathfrak{g}^{ss} \oplus \mathfrak{a}$, $\mathfrak{a}$ being an abelian Lie algebra of finite or countable dimension. Furthermore, $\mathfrak{h} = \mathfrak{h}^{ss} \oplus \mathfrak{a}$, where $\mathfrak{h}^{ss} := \mathfrak{h} \cap \mathfrak{g}^{ss}$.

(ii) $\mathfrak{g}^{ss} \simeq \mathfrak{h}^{ss} \oplus \mathfrak{s}$, $\mathfrak{s}$ being a finite or countable family of root simple direct limit Lie algebras $\mathfrak{g}^s$.

**Proof.** (i) Decompose $\mathfrak{h}$ as $\mathfrak{h} = \mathfrak{h}^{ss} \oplus \mathfrak{a}$ for some vector space $\mathfrak{a}$. Then $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{h}$ (since $\mathfrak{h}$ itself is abelian) of at most countable dimension, and using Proposition 1 one checks that $\mathfrak{g} \simeq \mathfrak{g}^{ss} \oplus \mathfrak{a}$.

(ii) Let $\mathfrak{g} = \varinjlim \mathfrak{g}_n$. Proposition 1 implies immediately that $\mathfrak{g}^{ss} = \varinjlim \mathfrak{g}_n^{ss}$. Furthermore, since a Cartan subalgebra $\mathfrak{h}_n \subset \mathfrak{g}_n$ is fixed for every $n$, there is a canonical
decomposition as
\[ \mathfrak{g}_n = (\bigoplus_{t \in S_n} \mathfrak{g}^t) \oplus \mathbb{Z}_n \]
such that all \( \mathfrak{g}^t \) are simple, \( S_n \) is abelian, and, for every \( \alpha \in \Delta_n \), \( \mathfrak{g}^\alpha \subset \mathfrak{g}^t \) for some \( t \in S_n \). Then, for any \( t \in S_n \), either \( \varphi_n(\mathfrak{g}^t) = 0 \) or there is \( t' \in S_{n+1} \) so that \( \varphi_n(\mathfrak{g}^{t'}) \) is a non-trivial subalgebra of \( \mathfrak{g}^{t'} \) (we assume that the sets \( S_n \) are pairwise disjoint). Put \( S' = \bigcup_n \{ t \in S_n \mid \varphi_n,_{n_1}(\mathfrak{g}^{t}) \neq 0 \text{ for every } n_1 > n \} \), where \( \varphi_{n_1,n_2} := \varphi_{n_2} \circ \ldots \circ \varphi_{n_1} \) for \( n_1 > n_2 \). Introduce an equivalence relation \( \sim \) on \( S' \) by setting \( t_1 \sim t_2 \) for \( t_1 \in S_{n_1}, t_2 \in S_{n_2} \), iff there exists \( n \) such that \( \varphi_{n_1,n}(\mathfrak{g}^{t_1}) \) and \( \varphi_{n_2,n}(\mathfrak{g}^{t_2}) \) belong to the same simple component of \( \mathfrak{g}_n \). Define \( S \) as the set of classes of \( \sim \)-equivalence. For every \( s \in S \) the set \( \{ \mathfrak{g}^t \}_{t \in s} \) is partially ordered by the maps \( \varphi_{n_1,n_2} : \mathfrak{g}^{t_1} \to \mathfrak{g}^{t_2} \). Let \( \mathfrak{g}^s \) be the direct limit Lie algebra of a maximal chain of Lie algebras among \( \{ \mathfrak{g}^t \}_{t \in s} \) with respect to this partial order. Obviously \( \mathfrak{g}^s \) is a root simple direct limit Lie algebra and it does not depend on the choice of the maximal chain. Finally, one checks easily that \( \mathfrak{g}^s = \lim_n \mathfrak{g}^s_n \cong \bigoplus_{s \in S} \mathfrak{g}^s \).

**Example 1.** \( gl(\infty) \) can be defined as the Lie algebra of all infinite matrices \( (a_{ij}), i,j \in \mathbb{Z}_+ \) with finitely many non-zero entries. Then \( gl(\infty) \cong A(\infty) \oplus \mathbb{C} \), but \( gl(\infty) \ncong A(\infty) \oplus \mathbb{C} \) as the center of \( gl(\infty) \) is trivial.

Theorem 3 gives an almost explicit description of all root reductive direct limit Lie algebras. In particular, it implies that any root space of a root reductive direct limit Lie algebra has dimension one. Moreover, if \( \pi : \mathfrak{h}^* \to (\mathfrak{h}^*)^* \) denotes the natural projection, then Theorem 3 implies that \( \pi \) induces a bijection between \( \Delta \) and the set of roots of \( \mathfrak{g}^s \). The exact relationship between irreducible generalized weight \( \mathfrak{g} \)-modules and irreducible generalized weight \( \mathfrak{g}^s \)-modules (all of which turn out to be automatically weight modules) is established in the following proposition.

**Proposition 2.** (i) Every irreducible generalized weight \( \mathfrak{g} \)-module \( M \) is a weight module.

(ii) Every irreducible weight \( \mathfrak{g} \)-module \( M \) is irreducible as a (weight) \( \mathfrak{g}^s \)-module.

(iii) Given any irreducible weight \( \mathfrak{g}^s \)-module \( M^s \), every \( \lambda \in \mathfrak{h}^* \) with \( \pi(\lambda) \in \text{supp} M^s \) defines a unique structure of an irreducible weight \( \mathfrak{g} \)-module on \( M^s \) which extends the \( \mathfrak{g} \)-module structure on \( M^s \). If \( M^s(\lambda) \) denotes the resulting \( \mathfrak{g} \)-module, then \( M^s(\lambda) \cong M^s(\lambda') \) iff \( \pi(\lambda - \lambda') = \sum_i c_i \pi(\alpha_i) \) with \( \alpha_i \in \Delta \) implies \( \lambda - \lambda' = \sum_i c_i \alpha_i \).

**Proof.** (i) Let \( U^0 \) denote the subalgebra of the enveloping algebra \( U(\mathfrak{g}) \) generated by monomials of weight zero. Since \( \mathfrak{h} \) acts semisimply on \( \mathfrak{g} \), the symmetric algebra \( S(\mathfrak{h}) \) belongs to the center of \( U^0 \). Furthermore, any generalized weight space \( M^\Lambda \) is an irreducible \( U^0 \)-module and, by a general version of Schur’s Lemma, \( S(\mathfrak{h}) \) acts via a scalar on \( M^\Lambda \). Therefore \( M \) is a semi-simple \( \mathfrak{h} \)-module, i.e. \( M \) is a weight module.

(ii) Follows immediately from (i) and Theorem 3 (i).

(iii) Let any \( a \in A \) act on the weight space \( M^\mu \) of \( M^s \) via multiplication by \( \lambda(a) + \sum_i \alpha_i(a) \), where \( \mu - \pi(\lambda) = \sum_i c_i \pi(\alpha_i) \). It is straightforward to verify that this equips \( M^s \) with a well-defined \( \mathfrak{g} \)-module structure. The isomorphism criterion is also an easy exercise.

Our main objective in this paper is the study of the irreducible weight modules over root reductive direct limit Lie algebras. The case of finite-dimensional reductive Lie algebras is discussed in particular in \cite{Fe, Fu, DMP}. In the rest of this section we study the structure of the root simple direct limit Lie algebras and \( \mathfrak{g} \) stands for \( A(\infty) \), \( B(\infty) \), \( C(\infty) \) or \( D(\infty) \). If \( \varepsilon_i \) are the usual
linear functions on the Cartan subalgebras of the simple finite-dimensional Lie algebras, see for example [B] or [Hu], one can let \( i \) run from 1 to \( \infty \) and then (using Proposition 3) verify the following list of roots:

\[
A(\infty) : \Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}
\]
\[
B(\infty) : \Delta = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}
\]
\[
C(\infty) : \Delta = \{\pm2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}
\]
\[
D(\infty) : \Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}
\]

Every sequence of complex numbers \((\lambda^n)_{n=1,2,...}\) determines a weight \(\lambda\) of \(\mathfrak{g}\) by setting \(\lambda(\varepsilon_n) := \lambda^n\). For \(\mathfrak{g} = B(\infty), C(\infty)\) or \(D(\infty)\), every weight \(\lambda\) of \(\mathfrak{g}\) recovers the sequence \((\lambda^n := \lambda(\varepsilon_n))_{n=1,2,...}\); for \(\mathfrak{g} = A(\infty)\), the weight \(\lambda\) recovers the sequence \((\lambda^n := \lambda(\varepsilon_n))_{n=1,2,...}\) up to an additive constant only. Furthermore, \((\cdot, \cdot)\) denotes the bilinear form \((\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}\) which extends the usual bilinear forms \((\cdot, \cdot)_n : \mathfrak{h}_n^* \times \mathfrak{h}_n^* \to \mathbb{C}\). If \(\lambda \in \mathfrak{h}^*\) and \(\alpha \in \Delta\), we set \((\lambda, \alpha) := \frac{2(\lambda\alpha)}{(\alpha, \alpha)}\). By definition, \(\lambda \in \mathfrak{h}^*\) is an integral weight of \(\mathfrak{g}\) iff \((\lambda, \alpha) \in \mathbb{Z}\) for every \(\alpha \in \Delta\), and \(\lambda\) is \(\mathfrak{b}\)-dominant, for a Borel subalgebra \(\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha\), iff \((\lambda, \alpha) \geq 0\) for all \(\alpha \in \Delta^+\).

It is proved in [DP2] (Proposition 2) that all Borel subalgebras of \(A(\infty), B(\infty), C(\infty)\) an \(D(\infty)\) are standard. More precisely, for every Borel subalgebra \(\mathfrak{b}\) of \(\mathfrak{g}\) there is a linear function \(\varphi : < \Delta >_\mathbb{R} \to \mathbb{R}\) such that \(\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha\). Define an order (or a partial order) on the set \(\{0\} \cup \{\pm\varepsilon_i\}\) to be \(\mathbb{Z}_2\)-linear iff multiplication by \(-1\) reverses the order. Then Proposition 2 from [DP2] is essentially equivalent to the following statement.

**Proposition 3.** If \(\mathfrak{g} = A(\infty)\), there is a bijection between Borel subalgebras of \(\mathfrak{g}\) and orders on the set \(\{\varepsilon_i\}\).

If \(\mathfrak{g} = B(\infty)\) or \(C(\infty)\), there is a bijection between Borel subalgebras of \(\mathfrak{g}\) and \(\mathbb{Z}_2\)-linear orders on the set \(\{0\} \cup \{\varepsilon_i\}\).

If \(\mathfrak{g} = D(\infty)\), there is a bijection between Borel subalgebras of \(\mathfrak{g}\) and \(\mathbb{Z}_2\)-linear orders on the set \(\{0\} \cup \{\pm\varepsilon_i\}\) with the property that if there is a minimal positive element with respect to this order, this element is of the form \(\varepsilon_i\).

**Proof.** The pull-back via \(\varphi\) of the standard order on \(\mathbb{R}\) determines an order on \(\{0\} \cup \Delta\) which induces an order respectively on \(\{\varepsilon_i\}\) or \(\{0\} \cup \{\pm\varepsilon_i\}\) as desired. Conversely, for every order \(\{\varepsilon_i\}\), or respectively for every \(\mathbb{Z}_2\)-linear order on \(\{0\} \cup \{\pm\varepsilon_i\}\) as in the Proposition, there exists a (non-unique) linear function \(\varphi : < \{\varepsilon_i\} >_\mathbb{R} \to \mathbb{R}\) such that \(\Delta^+ = \varphi^{-1}(\mathbb{R}^+) \cap \Delta\).

The result of Proposition 3 can be found in an equivalent form in [Ne]. The case of \(A(\infty)\) is due to V. Kac (unpublished).

**Example 2.** \(A(\infty)\) is naturally identified with the Lie algebra of traceless infinite matrices \((a_{i,j})_{i,j \in \mathbb{Z}_+}\) with finitely many non-zero entries, as well as with the Lie algebra of traceless double infinite matrices \((a_{i,j})_{i,j \in \mathbb{Z}}\) with finitely many non-zero entries. The respective algebras of upper triangular matrices are non-isomorphic Borel subalgebras of \(A(\infty)\) corresponding respectively to the orders \(\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots\) and \(\ldots > \varepsilon_6 > \varepsilon_4 > \varepsilon_2 > \varepsilon_1 > \varepsilon_3 > \varepsilon_5 > \ldots\).

A parabolic subalgebra of a root reductive direct limit Lie algebra is by definition a Lie subalgebra containing a Borel subalgebra. (The general definition of a parabolic subalgebra of a Lie algebra with root decomposition is given in the Appendix.) Here is an explicit description of all parabolic subalgebras of \(A(\infty), B(\infty), C(\infty)\) and \(D(\infty)\).
Proposition 4. If \( g = A(\infty) \), there is a bijection between parabolic subalgebras of \( g \) and partial orders on the set \( \{ \varepsilon_i \} \).

If \( g = B(\infty) \) or \( C(\infty) \), there is a bijection between parabolic subalgebras of \( g \) and \( \mathbb{Z}_2 \)-linear partial orders on the set \( \{ 0 \} \cup \{ \pm \varepsilon_i \} \).

If \( g = D(\infty) \), there is a bijection between parabolic subalgebras of \( g \) and \( \mathbb{Z}_2 \)-linear partial orders on the set \( \{ 0 \} \cup \{ \pm \varepsilon_i \} \) with the property that if \( \varepsilon_i \) is not comparable with 0 for some \( i \) (i.e. neither \( \varepsilon_i > 0 \) nor \( \varepsilon_i < 0 \)) then \( \varepsilon_j \) is also not comparable with 0 for some \( j \neq i \).

Proof. Given a partial order on \( \{ \varepsilon_i \} \) (respectively, a \( \mathbb{Z}_2 \)-linear partial order on \( \{ 0 \} \cup \{ \pm \varepsilon_i \} \) with the additional property for \( g = D(\infty) \)), it determines a unique partial order \( > \) on the set \( \{ 0 \} \cup \Delta \). Put \( p_\Delta := h \oplus (\oplus_{\alpha > 0 \text{ or } \alpha \text{ not comparable with } 0} \alpha) \). Then \( p_\Delta \) is the parabolic subalgebra corresponding to the initial partial order. Conversely, let \( p \) be a parabolic subalgebra. For \( \alpha \in \Delta \), set \( \alpha > p \) if \( g^\alpha \subset p \) but \( g^{-\alpha} \not\subset p \). Using the explicit form of \( \Delta \) it is easy to verify that this determines a unique partial order on \( \{ \varepsilon_i \} \) (respectively on \( \{ 0 \} \cup \{ \pm \varepsilon_i \} \) as desired).

The next proposition describes \( p^{ss} \) for any parabolic subalgebra \( p \) of \( g = A(\infty) \), \( B(\infty) \), \( C(\infty) \) or \( D(\infty) \), and will be used in Section 3.

Proposition 5. If \( p \subset g \) is a parabolic subalgebra of \( g \), then

(i) \( p^{ss} \) is isomorphic to a direct sum of simple Lie algebras each of which is one of the following:
- \( A(n) \) or \( A(\infty) \), if \( g = A(\infty) \);
- \( A(n) \), \( A(\infty) \), \( B(n) \) or \( B(\infty) \) with at most one simple component isomorphic to \( B(n) \) or \( B(\infty) \), if \( g = B(\infty) \);
- \( A(n) \), \( A(\infty) \), \( C(n) \) or \( C(\infty) \) with at most one simple component isomorphic to \( C(n) \) or \( C(\infty) \), if \( g = C(\infty) \);
- \( A(n) \), \( A(\infty) \), \( D(n) \) or \( D(\infty) \) with at most one simple component isomorphic to \( D(n) \) or \( D(\infty) \), if \( g = D(\infty) \).

(ii) If \( p^{ss} \not\cong 0 \), then \( p^{ss} + h \cong (\oplus_{t \in T} t^i) \oplus Z \), where \( Z \) is abelian and \( t^i \) is isomorphic to \( gl(n) \) or \( gl(\infty) \) for any \( t \in T \) except at most one index \( t_0 \in T \) for which \( t_0^i \) is a root simple direct limit Lie algebra.

Proof. (i) Let, as in Proposition 4, \( >_p \) be the partial order corresponding to \( p \). Partition the set \( \{ \varepsilon_i \} \) (respectively \( \{ 0 \} \cup \{ \pm \varepsilon_i \} \) into subsets in such a way that two elements are comparable with respect to \( >_p \) iff they belong to different subsets. Denote by \( S' \) the resulting set of subsets of \( \{ \varepsilon_i \} \) (respectively of \( \{ 0 \} \cup \{ \pm \varepsilon_i \} \)). Let furthermore \( S \) be a subset of \( S' \) which contains exactly one element of any pair \( s, s' \) of mutually opposite elements of \( S' \), and for every \( s \in S \) define \( g^s \) to be the Lie algebra generated by \( g^a \) where \( a \neq 0 \) belongs to \( s \) or is a sum of any two elements of \( s \). It is straightforward to verify that \( p^{ss} \cong (\oplus_{s \in S, s' \neq 0} g^s) \) is the decomposition of \( p^{ss} \) into a direct sum of root simple direct limit Lie algebras and that this decomposition satisfies (i).

(ii) The main point is to notice that if \( s \neq -s \), then \( g^s \cong A(n) \) for some \( n \) or \( g^s \cong A(\infty) \) and furthermore, that there is at most one \( s \in S \) such that \( s = -s \). To complete the proof it remains to show that each of the Lie subalgebras \( g^s \) for \( s \neq -s \) can be extended to a Lie subalgebra isomorphic to \( gl(n) \) or \( gl(\infty) \). This latter argument is purely combinatorial and we leave it to the reader. \( \square \)
3. THE SHADOW OF AN IRREDUCIBLE WEIGHT MODULE

Let \( \mathfrak{g} \) be a Lie algebra with a root decomposition, \( M \) be an irreducible generalized weight \( \mathfrak{g} \)-module and \( \lambda \) be a fixed point in \( \text{supp} M \). For any \( \alpha \in \Delta \), consider the set \( m^\lambda_\alpha := \{ q \in \mathbb{R} \mid \lambda + q\alpha \in \text{supp} M \} \subset \mathbb{R} \). There are four possible types of sets \( m^\lambda_\alpha \):

- bounded in both directions;
- unbounded in both directions;
- bounded from above but unbounded from below;
- unbounded from above but bounded from below.

It is proved in [DMP] that, if \( \mathfrak{g} \) is finite-dimensional, the type of \( m^\lambda_\alpha \) depends only on \( \alpha \) and not on \( \lambda \), and therefore the module \( M \) itself determines a partition of \( \Delta \) into four mutually disjoint subsets:

\[
\begin{align*}
\Delta^F_M := \{ \alpha \in \Delta \mid m^\lambda_\alpha \text{ is bounded in both directions} \}, \\
\Delta^I_M := \{ \alpha \in \Delta \mid m^\lambda_\alpha \text{ is unbounded in both directions} \}, \\
\Delta^+_M := \{ \alpha \in \Delta \mid m^\lambda_\alpha \text{ is bounded from above and unbounded from below} \}, \\
\Delta^-_M := \{ \alpha \in \Delta \mid m^\lambda_\alpha \text{ is bounded from below and unbounded from above} \}.
\end{align*}
\]

The corresponding decomposition

\[
\mathfrak{g} = (\mathfrak{g}^F_M + \mathfrak{g}^I_M) \oplus \mathfrak{g}^+_M \oplus \mathfrak{g}^-_M,
\]

where \( \mathfrak{g}^F_M := \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta^F_M} \mathfrak{g}^\alpha) \), \( \mathfrak{g}^I_M := \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta^I_M} \mathfrak{g}^\alpha) \) and \( \mathfrak{g}^+_M := (\oplus_{\alpha \in \Delta^+_M} \mathfrak{g}^\alpha) \), is the \( M \)-decomposition of \( \mathfrak{g} \). The triple \((\mathfrak{g}^I_M, \mathfrak{g}^+_M, \mathfrak{g}^-_M)\) is the shadow of \( M \) onto \( \mathfrak{g} \). If \( \mathfrak{g} \) is infinite-dimensional, we say that the shadow of \( M \) onto \( \mathfrak{g} \) is well-defined if it is true that the type of \( m^\lambda_\alpha \) depends only on \( \alpha \) and not on \( \lambda \) and thus the decomposition \((\mathfrak{h}, \mathfrak{g}^I_M, \mathfrak{g}^+_M, \mathfrak{g}^-_M)\) is well-defined.

In the case when \( \mathfrak{g} \) is finite-dimensional and reductive, it is furthermore true that \( \mathfrak{p}_M := (\mathfrak{g}^F_M + \mathfrak{g}^I_M) \oplus \mathfrak{g}^+_M \) is a parabolic subalgebra of \( \mathfrak{g} \) whose reductive part is \( \mathfrak{g}^F_M + \mathfrak{g}^+_M \), and that there is a natural surjection

\[
\varphi_M : U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M \sigma^\mathfrak{p}_M \to M,
\]

where \( M \sigma^\mathfrak{p}_M \) is the irreducible \((\mathfrak{g}^F_M + \mathfrak{g}^I_M)\)-submodule of \( M \) which consists of all vectors in \( M \) annihilated by \( \mathfrak{g}^+_M \). Moreover, \( \text{supp} M \) simply coincides with \( \text{supp}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M \sigma^\mathfrak{p}_M) \). This is the Fernando-Futorny parabolic induction theorem, see [F] and also [DMP], and in particular it provides an explicit description of \( \text{supp} M \).

The main purpose of this paper is to understand analogues of these results for a root reductive direct limit Lie algebra \( \mathfrak{g} \). Very roughly, the situation turns out to be as follows: the shadow of an arbitrary irreducible \( \mathfrak{g} \)-module \( M \) exists and defines a parabolic subalgebra \( \mathfrak{p}_M \) of \( \mathfrak{g} \), however the parabolic induction theorem does not hold. Nevertheless the existence of the shadow and the direct limit structure on \( \mathfrak{g} \) enable us to obtain an explicit description of \( \text{supp} M \) for any \( M \).

Theorem 2. The shadow of \( M \) is well-defined.

Proof. Note first that, for any given \( \lambda \in \text{supp} M \), there exist irreducible \((\mathfrak{g}_n + \mathfrak{h})\)-modules \( M_n \) such that \( \lambda \in \text{supp} M_n \) for every \( n \) and \( \text{supp} M = \cup_n \text{supp} M_n \). Indeed, fix \( m \in M^\lambda \), \( m \neq 0 \), and define \( M_n \) as any irreducible quotient of the \((\mathfrak{g}_n + \mathfrak{h})\)-module \( U(\mathfrak{g}_n + \mathfrak{h}) \cdot m \). One checks immediately that \( \text{supp} M = \cup_n \text{supp} M_n \).
Fix $\alpha \in \Delta$. To prove the Theorem we need to show that if $m_{\alpha}^{\lambda}$ is bounded from above for some $\lambda \in \text{supp} M$ (the proof for the case when $m_{\alpha}^{\lambda}$ is bounded from below is exactly the same) then $m_{\alpha}^{\mu}$ is bounded from above for any other $\mu \in \text{supp} M$. First, let $\lambda' = \lambda + k \alpha$ be the end point of the $\alpha$-string through $\lambda$ in $\text{supp} M$. Then fix $\mu \in \text{supp} M$ and pick $N$ so that $\mu \in \text{supp} M_{N}$, $\lambda' \in \text{supp} M_{N}$ and $\alpha \in \Delta_{N}$. Consider now the $\alpha$-string through $\mu$. It is the union of the $\alpha$-strings through $\mu$ in $\text{supp} M_{n}$ for $n = N, N + 1, \ldots$. The parabolic induction theorem implies that $\text{supp} M_{n}$ (and, in particular, the $\alpha$-string through $\mu$ in $\text{supp} M_{n}$) is contained entirely in an affine half-space in $\lambda^+ < \Delta_{n} \geq \mathbb{R}$ whose boundary (affine) hyperplane $H_{n}$ contains $\lambda'$ and is spanned by vectors in $\Delta_{n}$. Furthermore, $H_{n}' := H_{n} \cap (\lambda^+ < \Delta_{N} \geq \mathbb{R})$ is a hyperplane in $\lambda^+ < \Delta_{N} \geq \mathbb{R}$ which contains $\lambda'$ and is spanned by vectors in $\Delta_{N}$. Since the $\alpha$-string through $\mu$ belongs to $\lambda^+ < \Delta_{N} \geq \mathbb{R}$, the $\alpha$-string through $\mu$ in $\text{supp} M_{n}$ is bounded by $H_{n}'$. But there are only finitely many hyperplanes in $\lambda^+ < \Delta_{N} \geq \mathbb{R}$ passing through $\lambda'$ and spanned by vectors in $\Delta_{N}$ and hence among the hyperplanes $H_{n}'$ for $n = N, N + 1, \ldots$ only finitely many are different. Therefore, the $\alpha$-strings through $\mu$ in $\text{supp} M_{n}$ are uniformly bounded from above, i.e. $m_{\alpha}^{\mu}$ is bounded from above.

The next theorem describes the structure of the $M$-decomposition and is an exact analogue of the corresponding theorem for finite-dimensional reductive Lie algebras.

**Theorem 3.** (i) $\mathfrak{g}_{M}^{F}$, $\mathfrak{g}_{M}^{I}$, $\mathfrak{g}_{M}^{+}$ and $\mathfrak{g}_{M}^{-}$ are Lie subalgebras of $\mathfrak{g}$ and $\mathfrak{h}$ is a Cartan subalgebra for both $\mathfrak{g}_{M}^{F}$ and $\mathfrak{g}_{M}^{I}$.

(ii) $[(\mathfrak{g}_{M}^{F})^{ss}, (\mathfrak{g}_{M}^{I})^{ss}] = 0$ and therefore $\mathfrak{g}_{M}^{F} := \mathfrak{g}_{M}^{F} + \mathfrak{g}_{M}^{I}$ is a Lie subalgebra of $\mathfrak{g}$.

(iii) $\mathfrak{g}_{M}^{+}$ and $\mathfrak{g}_{M}^{-}$ are $\mathfrak{g}_{M}^{F}$-modules.

(iv) $\mathfrak{p}_{M} := \mathfrak{g}_{M}^{F} \oplus \mathfrak{g}_{M}^{+}$ and $\mathfrak{g}_{M}^{F} \oplus \mathfrak{g}_{M}^{-}$ are (mutually opposite) parabolic subalgebras of $\mathfrak{g}$.

**Proof.** (i) Let $\alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$. Lemma 2 in [PS] implies that if $m_{\alpha}^{\lambda}$ and $m_{\beta}^{\lambda}$ are bounded from above, so is $m_{\alpha + \beta}^{\lambda}$. Furthermore, noting that $\text{supp} M = \bigcup_{\lambda} \text{supp} M_{\lambda}$ for some irreducible $(\mathfrak{g}_{n} + \mathfrak{h})$-modules $M_{\lambda}$ (see the proof of Theorem 2) and that the support of every irreducible weight $(\mathfrak{g}_{n} + \mathfrak{h})$-module is convex, we conclude that $\text{supp} M$ is convex. Therefore if $m_{\alpha}^{\lambda}$ and $m_{\beta}^{\lambda}$ are unbounded from above, so is $m_{\alpha + \beta}^{\lambda}$. These two facts imply immediately that all four subspaces $\mathfrak{g}_{M}^{F}$, $\mathfrak{g}_{M}^{I}$, $\mathfrak{g}_{M}^{+}$ and $\mathfrak{g}_{M}^{-}$ are subalgebras of $\mathfrak{g}$. The fact that $\mathfrak{h}$ is a Cartan subalgebra for both $\mathfrak{g}_{M}^{F}$ and $\mathfrak{g}_{M}^{I}$ is obvious.

(ii) If $\alpha' \in \Delta_{M}^{F}$ and $\beta' \in \Delta_{M}^{I}$, then $\alpha' + \beta' \notin \Delta$. Indeed, if $\alpha' + \beta' \in \Delta$ and $m_{\alpha' + \beta'}^{\lambda}$ is bounded from above, then $m_{\alpha'}^{\lambda}$ would be bounded from above because $\beta' = -\alpha' + (\alpha' + \beta')$. If, on the other hand, $\alpha' + \beta' \in \Delta$ and $m_{\alpha' + \beta'}^{\lambda}$ is unbounded from above, then $m_{\alpha'}^{\lambda}$ would be unbounded from above because $\alpha' = -\beta' + (\alpha' + \beta')$. Since both of these conclusions contradict the choice of $\alpha'$ and $\beta'$ we obtain that $\alpha' + \beta' \notin \Delta$ and thus that $[(\mathfrak{g}_{M}^{F})^{ss}, (\mathfrak{g}_{M}^{I})^{ss}] = 0$.

(iii) If $\alpha' \in \Delta_{M}^{F}$, $\beta' \in \Delta_{M}^{+}$ and $\alpha' + \beta' \in \Delta$, then again $m_{\alpha' + \beta'}^{\lambda}$ is bounded from above. Assuming that $m_{\alpha' + \beta'}^{\lambda}$ is bounded from below, we obtain that $m_{\beta'}^{\lambda}$ is bounded from below as well since $\beta' = -(\alpha' + \beta') + \alpha'$, which contradicts the fact that $\beta' \in \Delta_{M}^{+}$. Hence $m_{\alpha' + \beta'}^{\lambda}$ is unbounded from below and $\alpha' + \beta' \in \Delta_{M}$. This proves that $\mathfrak{g}_{M}^{+}$ is a $\mathfrak{g}_{M}^{F}$-module. One shows in a similar way that $\mathfrak{g}_{M}^{-}$ are $\mathfrak{g}_{M}^{F}$- and $\mathfrak{g}_{M}^{I}$-modules.

(iv) is a direct corollary of (i), (ii) and (iii).
We define an irreducible weight \( g \)-module \( M \) to be \textit{cuspidal} iff \( g = g^I_M \).

In the rest of this section we prove that for every parabolic subalgebra \( p \) of \( g \) there is an irreducible \( g \)-module \( M \) so that \( p = p_M \). Indeed, there is the following more general

\begin{theorem} \label{thm:main} Let \( g \) be a root reductive direct limit Lie algebra. For any given splitting \( \Delta = \Delta^F \sqcup \Delta^I \sqcup \Delta^L \sqcup \Delta^C \) with \( \Delta^F = -\Delta^F, \Delta^I = -\Delta^I, \Delta^L = -\Delta^L, \) and such that its corresponding decomposition

\[ g = (g^F + g^I) \oplus g^+ \oplus g^- \]  

satisfies the properties (i) - (iii) of Theorem 3, there exists an irreducible weight module \( M \) for which \( g \) is the \( M \)-decomposition of \( g \).
\end{theorem}

\textbf{Proof.} We start with the observation that it suffices to prove the Theorem for a root simple direct limit Lie algebra. Indeed, let as in Theorem 1 \( g \simeq (\oplus_{s \in S} g^s) \subset A \) and assume that for each \( s M^s \) is an irreducible weight \( g^s \)-module corresponding as in the Theorem to the restriction of the decomposition (3) to \( g^s \). Fix non-zero vectors \( m^s \in M^s \). Then the reader will verify straightforwardly that, for any pair \((M^s) := (\oplus_{s \in S} M^s) \bigoplus (\oplus_{s \in S} m^s), \lambda \) as in Proposition 3 (iii), the \( g \)-module \( M := M^s(\lambda) \) is as required by the Theorem. Therefore in the rest of the proof we will assume that \( g \) is a root simple direct limit Lie algebra. We will prove first that cuspidal modules exist.

\begin{lemma} \label{lem:lemma} Let \( g \) be a simple finite-dimensional Lie algebra. For any given weight \( \mu \in h^* \), there exists a cuspidal \( g \)-module \( M^I \) such that the center \( Z \) of \( U(g) \) acts on \( M \) via the central character \( \chi_{\mu} : Z \to \mathbb{C} \) (obtained by extending \( \mu \) to a homomorphism \( \mu : S(h) \to \mathbb{C} \) and composing with Harish-Chandra’s homomorphism \( Z \to S(h) \)).
\end{lemma}

\textbf{Proof of Lemma 4.} It goes by induction on the rank of \( g \). It is a classical fact that \( sl(2) \) admits a cuspidal module of any given central character, so it remains to make the induction step. Let \( g' \) be a reductive subalgebra of \( g \) which contains \( h \) and such that \( \text{rk} g' = \text{rk} g - 1 \). Let \( M' \) be a cuspidal \( g' \)-module with central character \( \chi_{\mu'} \), where \( \mu' \in h^* \) and \( w'(\mu') - \mu \notin \Delta^L \) for any \( \mu' \in W' \) in the Weyl group \( W' \) of \( g' \). Denote by \( U' \) the subalgebra of \( U(g) \) generated by \( U(g') \) and \( Z \). Since \( U(g) \) is a free \( Z \)-module (Kostant’s Theorem) and \( U(g') \cap Z = \mathbb{C}, U' \) is isomorphic to \( U(g') \otimes_{\mathbb{C}} Z \). Therefore the tensor product \( M'_{\chi_{\mu'}} := M' \otimes_{\mathbb{C}} \chi_{\mu'}, \chi_{\mu'} \) being the \( 1 \)-dimensional \( Z \)-module corresponding to \( \chi_{\mu'} \), is a well-defined \( U' \)-module. Consider now any irreducible quotient \( M \) of the induced \( g \)-module \( U(g) \otimes_{U'} M'_{\chi_{\mu'}} \). Obviously \( M \) is a weight module and we claim that \( M \) is cuspidal. Assuming the contrary, \( M \) would be a quotient of \( U(g) \otimes_{U(p)} M'' \) where \( p \supset g' \) and \( M'' \) is a cuspidal \( g'' \)-module of central character \( \chi_{\mu''} \). Then, since \( U(g) \) is an integrable \( U(g') \)-module and \( M' \) would be a subquotient of \( U(g) \otimes_{U(p)} M'' \), we would have \( w'(\mu') - \mu \notin \Delta^L \) for some \( \mu'' \in W' \), which is contradiction. Therefore \( M \) is cuspidal. \( \square \)

If \( g = g^I \), \( g = \varinjlim g_n \) being an infinite-dimensional root simple direct limit Lie algebra, we can assume that \( \text{rk} g_{n+1} - \text{rk} g_n = 1 \). We then construct \( M^I \) as the direct limit \( \varinjlim M_n \), each \( M_n \) being a cuspidal \( g_n \)-module build by induction precisely as in the Proof of Lemma 3. This proves the Theorem in the cuspidal case.

Let now \( g \neq g^I \). According to Theorem 5, \( g^F \simeq \oplus_{s \in S} g^s \subset A^F \) and \( g^I \simeq \oplus_{t \in T} g^t \subset A^I \), where each of the algebras \( g^r \) for \( r \in S \cup T \) is a root simple direct limit Lie algebra and \( A^F \) and \( A^I \) are respectively abelian subalgebras of \( g^F \).
and \(g'\). Since the Theorem is proved for the cuspidal case, we can choose a cuspidal irreducible \(g'\) module \(M^t\) for each \(t \in T\) and fix a non-zero vector \(m^t \in M^t\). Then \(M^I := (\otimes_{t \in T} M^t)(\otimes m^t)\) is a cuspidal \((g')^{ss}\)-module. Let \(M'\) denote \(M^I\) considered as a \((g^F + g')^{ss}\)-module with trivial action of \((g^F)^{ss}\).

If \(\dim g < \infty\), we can assume furthermore, by Lemma 3, that the \((g')^{ss}\)-module \(\hat{M}^I\) is chosen in such a way that the central character of \(\hat{M}^I\) corresponds to an orbit \(W^I \cdot \eta\) being the Weyl group of \((g')^{ss}\) which contains no weight of the form \(\eta + \kappa\) for an integral weight \(\kappa\) of \((g')^{ss}\). Then it is not difficult to verify (we leave this to the reader) that this condition ensures that if \(MF^I\) denotes \(\hat{M}^I\) with its obvious \(g^F + g'\) and-module structure and if \(M')\) is the irreducible quotient of \(U(g) \otimes_{U(g^F + g')} M^{FI}\), the \(M\)-decomposition of \(g\) is nothing but \((\hat{M})\). Theorem 3 is therefore proved for a finite-dimensional reductive \(g\).

A direct generalization of this argument does not go through for an infinite-dimensional \(g\). Instead, there is the following lemma which allows us to avoid referring to central characters.

**Lemma 2.** If \(g\) is infinite-dimensional, the \((g^F + g')^{ss}\)-module structure on \(M'\) can be extended to a \((g^F + g')\)-module structure in such a way that \((\mu', \alpha) \notin \mathbb{Z}\) for some (and hence any) \(\mu' \in \text{supp} M'\) and any \(\alpha \in \Delta^+ \cup \Delta^-\).

**Proof of Lemma 2.** (iv) implies that \(g^F_M + g'^M = (p_M)^{ss} + b\). We will present the proof in the case when \(g^F_M + g'^M \notin b\). The case when \(g^F_M + g'^M \notin b\) is dealt with in a similar way.

Using Proposition 3, we conclude that \(g^F_M + g'^M \notin b\) implies

\[
\begin{align*}
g^F_M + g'^M_m & \simeq (\oplus_{0 \notin R} g^r) \oplus g^0 \oplus \mathbb{Z},
\end{align*}
\]

where each \(g^r\) is isomorphic to \(gl(n)\) for some \(n\) or to \(gl(\infty), R\) is a finite or countable set which does not contain \(0\) (and may be empty), \(g^0\) is a root simple direct limit Lie algebra, and \(\mathbb{Z}\) is abelian.

Let \(\Delta^s\) be the root system of \(g^s\) for \(s \in R \cup \{0\}\). Set \(\ast \mathbb{Z} := \{k \in (\mathbb{Z}+ \{0\})\}\) there is \(l \in (\mathbb{Z}+ \{0\})\) with \(\varepsilon_k \in \Delta^s\) and \(\# \mathbb{Z} := (\mathbb{Z}+ \{0\})\)\(\cup_{s \in R \cup \{0\}} \ast \mathbb{Z}\). As explained in section 3, any sequence \(\lambda^n = (\lambda^n)_{n=1,2,\ldots} \subset R\) determines a weight \(\lambda\) of \((g^F + g')^{ss}\), however \(\lambda\) reconstructs only the subsequence \(\lambda^n = (\lambda^n)_{n \in (\mathbb{Z}+ \{0\}) \cap \ast \mathbb{Z}}\) up to an additive constant for every index from \(R\). Fix now \(\mu \in \text{supp} M'\) and let \(\{\mu^n\}_{n=1,2,\ldots}\) be a sequence which determines \(\mu\). To prove the Lemma, it suffices to find constants \(\varepsilon_i \in R\) for \(i \in \mathbb{Z}\) with the following properties:

- \(\varepsilon^k = 0\) whenever \(k \in \{0\}\); and the same set \(\ast \mathbb{Z}\);
- \(\varepsilon_k = 0\) for \(k \in \{0\}\);
- \(\mu^k + c^k \notin \mathbb{Z}\) for every \(k \in (\mathbb{Z}+ \{0\})\) for which \(\varepsilon_k \in \Delta^+ \cup \Delta^-\); and \(\varepsilon_k \in \Delta^+ \cup \Delta^-; \mu^k + c^k \notin \mathbb{Z}\) for every \(k \in (\mathbb{Z}+ \{0\})\) for which \(\varepsilon_k \in \Delta^+ \cup \Delta^-\).

Then \(\mu\) will be the weight of \((g^F + g')\) determined by the sequence \(\{\mu^n + c^n\}_{n=1,\infty}\).

Consider the set of sequences \(\{e^k\}_{k \in P \subset (\mathbb{Z}+ \{0\})}\) with the three properties as above. Introduce an order \(<\) on this set by putting \(\{e^k\}_{k \in P} < \{e'^{k'}\}_{k' \in P'}\) if \(P' \subset P''\) and \(e^k = e'^{k'}\) for every \(k \in P'\). The reader will check that every chain (with respect to \(<\)) of sequences is bounded and that for every \(\{e^k\}_{k \in P}\) with \(P \neq (\mathbb{Z}+ \{0\})\) there is a sequence greater than \(\{e^k\}_{k \in P}\). Therefore any maximal element, which exists by Zorn’s lemma, is a sequence \(\{e^k\}\) with the required properties. Lemma 3 is proved.

\[\Box\]
Assuming that \( \mathfrak{g} \) is infinite-dimensional, let now \( M^{F_I} \) be \( M' \) considered as a \((\mathfrak{g}^F + \mathfrak{g}^I) \oplus \mathfrak{g}^+\)-module with trivial action of \( \mathfrak{g}^+ \). Then we define \( M \) as the (unique) irreducible quotient of the \( \mathfrak{g} \)-module \( U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^F + \mathfrak{g}^I) \oplus \mathfrak{g}^+} M^{F_I} \). Using Lemma \( \ref{lemma:irreducible} \)
the reader will verify that the \( M \)-decomposition of \( \mathfrak{g} \) is precisely the decomposition (\( \ref{eq:decomposition} \)). The proof of Theorem \( \ref{thm:decomposition} \) is therefore complete.

4. Integrable modules

Proposition 6. Let \( M \) be integrable. Then

(i) \( \mathfrak{g} = \mathfrak{g}^F_M + \mathfrak{g}^I_M \); furthermore, \( \mathfrak{g} \neq \mathfrak{g}^F_M \) implies \( \dim \lambda^M = \infty \) for any \( \lambda \in \text{supp} \mathcal{M} \).

(ii) If \( \mathfrak{g} \) is finite-dimensional and simple, \( M \) is either cuspidal or \( \mathfrak{g} = \mathfrak{g}^F_M \), and both cases are possible.

Proof. (i) As a consequence of the integrability of \( M \), \( \text{supp} \mathcal{M} \) is \( W \)-invariant. Thus \( \Delta^-_M = \emptyset \) and \( \mathfrak{g} = \mathfrak{g}^F_M + \mathfrak{g}^I_M \). Furthermore, for any \( \alpha \in \Delta \) the subalgebra \( \mathfrak{g}^{\alpha \Delta} \) of \( \mathfrak{g} \) generated by \( \mathfrak{g}^\alpha \) and \( \mathfrak{g}^{-\alpha} \) is isomorphic to \( \mathfrak{sl}(2) \) and the integrability of \( M \) implies that as a \( \mathfrak{g}^{\alpha \Delta} \)-module \( M \) is isomorphic to a direct sum of finite-dimensional modules. The assumption that \( \dim \lambda^M < \infty \) for some \( \lambda \in \text{supp} \mathcal{M} \) would lead us to the conclusion that \( \lambda \) belongs to the support of only finitely many of the finite-dimensional \( \mathfrak{g}^{\alpha \Delta} \)-modules, which would mean that \( \text{supp} \mathcal{M} \) is finite in the direction of both \( \alpha \) and \( -\alpha \). Therefore \( \mathfrak{g} \neq \mathfrak{g}^F_M \) implies \( \dim \lambda^M = \infty \) for any \( \lambda \in \text{supp} \mathcal{M} \).

(ii) If both \( \mathfrak{g}^F_M \neq \mathfrak{h} \) and \( \mathfrak{g}^I_M \neq \mathfrak{h} \), Theorems \( \ref{thm:irreducible} \) and \( \ref{thm:decomposition} \) imply that \((\mathfrak{g}^F_M)^{\mathfrak{h}}\) and \((\mathfrak{g}^I_M)^{\mathfrak{h}}\) are proper ideals in \( \mathfrak{g} \), which is a contradiction. Thus \( \mathfrak{g}^F_M = \mathfrak{h} \) or \( \mathfrak{g}^I_M = \mathfrak{h} \), i.e. \( \mathfrak{g} = \mathfrak{g}^F_M \) or \( \mathfrak{g} = \mathfrak{g}^I_M \).

Clearly, \( \mathfrak{g} = \mathfrak{g}^F_M \) if \( M = \mathfrak{g} \) is the adjoint representation, so the case \( \mathfrak{g} = \mathfrak{g}^I_M \) is obviously possible. To prove that integrable cuspidal modules exist, it is enough to construct a tower of irreducible finite-dimensional \( \mathfrak{g} \)-modules \( M_n, \ldots \rightarrow M_n \rightarrow M_{n+1} \rightarrow \ldots \), such that the support of \( M_n \) is shorter than the support of \( M_{n+1} \) in all root directions of \( \mathfrak{g} \). Obviously, \( M := \lim \mathcal{M}_n \) is then an integrable cuspidal \( \mathfrak{g} \)-module. Here is an explicit example for \( \mathfrak{g} = \mathfrak{A}(\infty) \). Let \( \lambda_n = n\varepsilon_1 - n\varepsilon_2 \) and let \( \mathfrak{b}_n \) be the Borel subalgebra of \( \mathfrak{g}_n \) corresponding to the order \( \varepsilon_2 < \varepsilon_3 < \ldots < \varepsilon_n < \varepsilon_1 \).

Set \( M_n := \mathfrak{V}_{\mathfrak{b}_n}(\lambda_n) \) for \( n \geq 2 \). It is obvious that the support of \( M_{n+1} \) is longer than the support of \( M_n \) in the direction of every root of \( \mathfrak{g}_n \). The reader will also verify that \( M_n \) appears in the decomposition of \( M_{n+1} \) as a direct sum of \( \mathfrak{g}_n \)-modules and hence an embedding \( M_n \rightarrow M_{n+1} \) exists. The proof of Proposition \( \ref{prop:existence} \) is therefore complete.

Obviously, if \( M \) is integrable and is a highest weight module with respect to some Borel subalgebra \( \mathfrak{b} \), then \( \mathfrak{g} = \mathfrak{g}_M^F \). The existence of cuspidal integrable modules is a significant difference with the case of a finite-dimensional Lie algebra, where any integrable irreducible module is finite-dimensional and is thus a highest weight module with respect to every Borel subalgebra. In contrast, for \( \mathfrak{g} = \mathfrak{A}(\infty), \mathfrak{B}(\infty), \mathfrak{C}(\infty) \) and \( \mathfrak{D}(\infty) \), the trivial \( \mathfrak{g} \)-module is the only irreducible \( \mathfrak{g} \)-module which is a highest weight module for all Borel subalgebras. Moreover, as we will see in Example 3 below, the equality \( \mathfrak{g} = \mathfrak{g}_M^F \) does not guarantee the existence of a Borel subalgebra with respect to which \( M \) is a highest weight module. We define \( M \) to be finite integrable iff \( \mathfrak{g} = \mathfrak{g}_M^F \). The rest of this section is devoted to the study of finite integrable irreducible \( \mathfrak{g} \)-modules \( M \) over an arbitrary root reductive direct limit Lie algebra \( \mathfrak{g} \).

The simplest type of finite integrable modules are highest weight integrable modules and they are studied in the recent papers [BR], [NRW] and [Ne]. In [BR] the
highest weights of integrable highest weight modules are computed explicitly. In [NRW] the integrable modules $V^b(\lambda)$ appear in the context of Borel-Weil-Bott’s theorem and are realized as the unique non-zero cohomology groups of line bundles on $G/B$. In [Ne] irreducible highest weight modules with respect to general Borel subalgebras are considered and in particular it is proved that their integrability is equivalent to unitarizability. In Theorem 3 below we discuss highest weight modules and in particular establish a direct limit version of H. Weyl’s character formula for integrable highest weight modules.

First we need to recall the notion of basis of a Borel subalgebra of $g$. A subset $\Sigma \subset \Delta^+$ is a basis of $b = h + (\oplus_{\alpha \in \Delta^+} p^\alpha)$ iff $\Sigma$ is a linearly independent set and every element of $\Delta^+$ is a linear combination of elements of $\Sigma$ with non-negative integer coefficients; the elements of $\Sigma$ are then the simple roots of $b$. Not every Borel subalgebra admits a basis. For root simple direct limit Lie algebras a basis of $b$ is the same as a weak basis in the terminology of [DP2], and in [DP2] all Borel subalgebras admitting a weak basis are described. The result of [DP2] implies that $b$ admits a basis iff the corresponding order on $\{e_i\}$, or respectively on $\{0\} \cup \{\pm e_i\}$, has the following property: for every pair of elements of $\{e_i\}$ (respectively of $\{0\} \cup \{\pm e_i\}$) there are only finitely many elements between them. This latter criterion has been established also by K.-H. Neeb in [Ne].

**Theorem 5.** Let $M = V^b(\lambda)$.

(i) $M$ is integrable iff $\lambda$ is an integral $b$-dominant weight. Furthermore, if $M$ is integrable, then $\text{supp} M = C^\lambda$, where $C^\lambda$ is the intersection of the convex hull of $W \cdot \lambda$ with $\lambda^+ < \Delta > R$.

(ii) $M \simeq V^{b'}(\lambda')$ for given $b'$ and $\lambda'$ iff there exists $w \in W$ for which $\lambda' = w(\lambda)$ and there is a parabolic subalgebra $p$ of $g$ containing both $w(b)$ and $b'$ such that the $p$-submodule of $M$ generated by $M^{\lambda'}$ is one-dimensional.

(iii) $\dim M^{\mu} < \infty$ for all $\mu \in \text{supp} M$ iff $M \simeq V^b(\lambda)$ for some Borel subalgebra $b$ of $g$ which admits a basis.

(iv) If $M$ is integrable, $b$ admits a basis, and $\text{ch} M := \sum_{\mu \in \text{supp} M} \dim M^{\mu} \cdot e^{\mu}$ is the formal character of $M$, we have

$$D \cdot \text{ch} M = \sum_{w \in W} (\text{sgn} w) e^{w(\lambda + \rho_b) - \rho_b},$$

where $D = \prod_{a \in \Delta^+} (1 - e^{-a})$ and $\rho_b \in h^*$ is a weight for which $\rho_b(\alpha) = 1$ for all simple roots of $b$.

**Proof.** (i) An exercise. In [BB] a more general criterion for the integrability of $V^b(\lambda)$ is established for simple direct limit Lie algebras which are not necessarily root simple direct limit Lie algebras.

(ii) Define $p_\lambda \supset b$ as the maximal parabolic subalgebra of $g$ such that the $p_\lambda$-submodule of $M$ generated by $M^\lambda$ is one-dimensional. Then $M \simeq V_{b'}(\lambda)$ iff $b''$ is a subalgebra of $p_\lambda$. Finally, it is an exercise to check that $M \simeq V_{b'}(\lambda')$ iff $\lambda' = w(\lambda)$ and $b'$ is a subalgebra of $p_{\lambda'} = w(p_\lambda)$ for some $w \in W$.

(iii) If $b$ admits a basis, then, using the Poincare-Birkhoff-Witt theorem, one verifies that all weight spaces of the Verma module $V^b(\lambda)$ are finite-dimensional (for any $\lambda \in h^*$), and hence all weight spaces of $V^b(\lambda)$ are finite-dimensional as well.

Conversely, let $M = V^b(\lambda)$ be a highest weight module with finite-dimensional weight spaces. We need to prove the existence of $b$ which admits a basis and such
that $M \simeq V_{b}(\lambda)$. Let $p_{\lambda}$ be as above and let $g \simeq (\oplus_{s \in S} \mathfrak{g}^{s}) \subseteq A$ as in Theorem 1. Clearly, there is a Borel subalgebra $b \subseteq p_{\lambda}$ which admits a basis iff for every $s \in S$ there is a Borel subalgebra $b^{s}$ of $\mathfrak{g}^{s}$ admitting basis such that $b^{s} \subseteq p_{\lambda}^{s} := p_{\lambda} \cap \mathfrak{g}^{s}$. Furthermore, $M^{s} := V_{b^{s}}(\lambda|b^{s})$ (where $b^{s} := b \cap \mathfrak{g}^{s}$ and $b^{s} := \mathfrak{h} \cap \mathfrak{g}^{s}$) is an irreducible $\mathfrak{g}^{s}$-submodule of $M$. Thus it suffices to prove (iii) for the root system direct limit Lie algebras $g^{s}$ and their highest weight modules $M^{s}$. For a finite-dimensional $g^{s}$ (iii) is trivial, so we need to consider only the case when $g^{s} = A(\infty), B(\infty), C(\infty), D(\infty)$.

Let $>_{p_{\lambda}^{s}}$ be the partial order corresponding to $p_{\lambda}^{s}$ and let $\alpha \in (\Delta^{s})^{+} (\Delta^{s}$ being the root system of $\mathfrak{g}^{s}$) be a difference of two elements $\delta_{1}$ and $\delta_{2}$ of $\{\varepsilon_{i}\}$ (respectively of $\{0\} \cup \{\pm \varepsilon_{i}\}$). Then it is not difficult to check that, if there are infinitely many elements of $\{\varepsilon_{i}\}$ (respectively of $\{0\} \cup \{\pm \varepsilon_{i}\}$) between $\delta_{1}$ and $\delta_{2}$ with respect to $>_{p_{\lambda}^{s}}$, the weight space of $M^{s}$ with weight $\lambda - \alpha$ is infinite-dimensional. Therefore, the assumption that all Borel subalgebras $b^{s}$, such that $M^{s} \simeq V_{b^{s}}$, admit no basis is contradictory, i.e. a Borel subalgebra $b^{s}$ exists as required.

(iv) To prove formula (1) it suffices to notice that if $b$ admits a basis, $g$ can be represented as $g = \lim\limits_{n} g_{n}$, where each of the simple roots of $b_{n} = b \cap g_{n}$ is a simple root of $b_{n+1} = b \cap g_{n+1}$. Then $M = \lim\limits_{n} V_{b_{n}}(\lambda|b_{n})$ and in both sides of (1) terms of the form $\varepsilon \cdot e^{\lambda+\mu}$ for $\mu \in < \Delta_{n}, \mathbb{R}$ appear only as they appear in the respective sides of (1) for the $g_{n}$-module $V_{b_{n+1}}(\lambda|b_{n})$. Therefore, Weyl’s original formula implies the infinite version (1).

We now turn our attention to general finite integrable modules $M$. Here is an example of a finite integrable $M$ which is not a highest weight module with respect to any Borel subalgebra of $g$.

**Example 3.** Let $g = A(\infty)$ and $b \subseteq g$ be the Borel subalgebra of $g$ corresponding to the order $\varepsilon_{1} > \varepsilon_{2} > \varepsilon_{3} > \ldots$. Set $\lambda_{n} := \varepsilon_{1} + \cdots + \varepsilon_{n} - n\varepsilon_{n+1}$. Since $\lambda_{n} \in \text{supp} V_{b_{n+1}}(\lambda_{n+1})$ and the weight space $V_{b_{n+1}}(\lambda_{n+1})^{\lambda_{n}}$ is one-dimensional, there is a unique (up to a multiplicative constant) embedding of weight $g_{n}$-modules $V_{b_{n}}(\lambda_{n}) \rightarrow V_{b_{n+1}}(\lambda_{n+1})$. Set $M := \lim\limits_{n} V_{b_{n}}(\lambda_{n})$. Then $g = g_{M}$ and all weight spaces of $M$ are one-dimensional but $M$ is not a highest weight module with respect to any Borel subalgebra of $g$.

The following two theorems provide an explicit parametrization of all finite integrable modules as well as an explicit description of their supports. Let $W_{n} \subseteq W$ denote the Weyl group of $g_{n}$.

**Theorem 6.** Let $M$ be finite integrable.

(i) $M \simeq \lim\limits_{n} M_{n}$ for some direct system of finite-dimensional irreducible $g_{n}$-modules $M_{n}$.

(ii) $\text{supp} M$ determines $M$ up to isomorphism.

(iii) Fix a Borel subalgebra $b$ of $g$. Then $M \simeq \lim\limits_{n} V_{b_{n}}(\lambda_{n}|b_{n})$ for some sequence $\{\lambda_{n}\}$ of integral weights of $g$ such that $\lambda_{n}|b_{n}$ is a $b_{n}$-dominant weight of $g_{n}$ and $\lambda_{n}$ belongs to an edge of the convex hull of $W_{n+1} \cdot \lambda_{n+1}$.

**Proof.** (i) Fix $\lambda \in \text{supp} M$. We claim that there is a unique irreducible finite-dimensional $g_{n}$-module $M_{n}^{\lambda}$ such that $\text{supp} M_{n}^{\lambda} = (\lambda+< \Delta_{n}, \mathbb{Z}) \cap \text{supp} M$. Indeed, the equality $g = g_{M}^{\lambda}$ implies that $(\lambda+< \Delta_{n}, \mathbb{Z}) \cap \text{supp} M$ is a finite set and hence there is $n'$ such that $(\lambda+< \Delta_{n}, \mathbb{Z}) \cap \text{supp} M$ is contained in the support of an irreducible finite-dimensional $g_{n'}$-module $M_{n'}^{\lambda'}$ constructed as in the proof of Theorem 1. Moreover, the construction of $M_{n'}^{\lambda'}$ implies that $(\lambda+< \Delta_{n}, \mathbb{Z}$
\( \cap \text{supp} M_n^\lambda = (\lambda + < \Delta_n >_Z) \cap \text{supp} M \). But, as the reader will check, there is a unique irreducible finite-dimensional \( \mathfrak{g}_n \)-module \( M_n^\lambda \) with \( \text{supp} M_n^\lambda = (\lambda + < \Delta_n >_Z) \cap \text{supp} M_n^\lambda \). Finally \( M_n^\lambda \) is a \( \mathfrak{g}_n \)-submodule of \( M \) according to its construction, and furthermore \( M = \lim_n M_n^\lambda \).

(ii) The crucial point is that (according to its construction) \( M_n^\lambda \) depends only on \( \text{supp} M \) and \( \lambda \in \text{supp} M \). Furthermore, for any \( \lambda' \in \text{supp} M \), there is \( m \) such that \( \lambda' \in \text{supp} M_m^\lambda \) and hence \( M_n^\lambda \simeq M_n^\lambda' \) for \( n > m \). Letting now \( n \) go to \( \infty \), and noting that for any pair \( \lambda, \lambda' \in \text{supp} M \) there is a compatible system of isomorphisms \( M_n^\lambda \simeq M_n^\lambda' \) for all \( n > m \), we conclude that \( M = \lim_n M_n^\lambda \simeq \lim_n M_n^\lambda' \), i.e. that \( \text{supp} M \) determines \( M \) up to isomorphism.

(iii) Since the module \( M_n^\lambda \) defined in (i) is finite-dimensional, \( M_n^\lambda \simeq V_{b_n}(\lambda) \) for some \( \lambda \in \mathfrak{h}_n^* \). There is a unique weight \( \mu \) of \( \mathfrak{g} \) such that \( \mu < < \Delta_n >_\mathbb{R} \) and \( \mu|_{\mathfrak{h}_n} = \lambda|_{\mathfrak{h}_n} - \lambda_n \). Set \( \lambda_n := \lambda - \mu \in \mathfrak{h}_n^* \). Then \( M \simeq \lim_n M_n^\lambda \simeq V_{b_n}(\lambda_n) = \lim_n V_{b_n}(\lambda_n) \) and \( \lambda_n \) belongs to an edge of the convex hull of \( W_{n+1} \cdot \lambda_n + 1 \) because \( \text{supp} M_n^\lambda = (\lambda + < \Delta_n >_Z) \cap \text{supp} M \).

**Theorem 7.** Fix a Borel subalgebra \( \mathfrak{b} \) of \( \mathfrak{g} \). Let \( \{ \lambda_n \} \subset \mathfrak{h}_n^* \) be a sequence of integral weights of \( \lambda_n|_{\mathfrak{b}_n} \) is a \( \mathfrak{b}_n \)-dominant weight of \( \mathfrak{g}_n \) and \( \lambda_n \) belongs to an edge of the convex hull of \( W_{n+1} \cdot \lambda_n + 1 \). Then

(i) \( M \) is finite integrable.

(ii) \( \text{supp} M = \cup_n \mathfrak{C}^\lambda_n \).

(iii) \( M \) is a highest weight module with respect to some Borel subalgebra of \( \mathfrak{g} \) iff there is \( n_0 \) so that \( \lambda_n \in W \cdot \lambda_n + 1 \) for any \( n \geq n_0 \).

(iv) If \( M' \simeq \lim_n V_{b_n}(\lambda'_n|_{\mathfrak{b}_n}) \), \( \{ \lambda'_n \} \) being a sequence of integral weights of \( \mathfrak{g} \), such that \( \lambda'_n|_{\mathfrak{b}_n} \) is a \( \mathfrak{b}_n \)-dominant weight of \( \mathfrak{g}_n \) and \( \lambda'_n \) belongs to an edge of the convex hull of \( W_{n+1} \cdot \lambda'_n + 1 \) \( M \simeq M' \) iff there is \( n_0 \) so that \( \lambda_n = \lambda'_n \) for \( n \geq n_0 \).

**Proof.** The proof is not difficult and is left to the reader. \( \square \)

**Example 4.** Let \( \mathfrak{g} = \mathfrak{g}(\infty) \) and \( M = \mathfrak{g} \) be the adjoint module. If \( \mathfrak{b} \) is the Borel subalgebra corresponding to the order \( \varepsilon_1 > \varepsilon_2 > \ldots \), then \( M \simeq \lim_n V_{\mathfrak{b}_n}(\varepsilon_1 - \varepsilon_n) \). Since \( W \cdot (\varepsilon_1 - \varepsilon_n) = \Delta \), Theorem 7 (iii) implies that there exists a Borel subalgebra \( \mathfrak{b} \) such that the adjoint module is a \( \mathfrak{b} \)-highest weight module. Furthermore, it is not difficult to verify that all such Borel subalgebras \( \mathfrak{b} \) are precisely the Borel subalgebras which correspond to orders on \( \{ \varepsilon_i \} \) for which there exists a pair of indices \( i_0, j_0 \) so that \( \varepsilon_i > \varepsilon_0 \) and \( \varepsilon_i > \varepsilon_{j_0} \) for all \( i \neq i_0, j \neq j_0 \).

5. PSEUDO HIGHEST WEIGHT MODULES

In the case of a finite-dimensional Lie algebra, an irreducible weight module \( M \) with \( \mathfrak{g}_M^F = \mathfrak{h} \) is necessarily a highest weight module for some Borel subalgebra, see [DMP]. As we already know (Example 3) this is no longer true for the direct limit algebras we consider. We define a pseudo highest weight module as an irreducible weight module \( M \) such that \( \mathfrak{g}_M^F = \mathfrak{h} \). Pseudo highest weight modules provide counterexamples also to the obvious extension of the parabolic induction theorem to root reductive direct limit Lie algebras. (The module \( M \) from Example 3 above does not provide such a counterexample since in this case \( \mathfrak{p}_M = \mathfrak{g} \) and \( M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M' \).) Indeed, it suffices to construct \( M \) with \( \mathfrak{g}_M^F = \mathfrak{g}_M^F = \mathfrak{h} \) such that \( M \) is not a highest weight module with respect to \( \mathfrak{h} \otimes \mathfrak{g}_M^F \) and therefore admits no surjection of the form \( U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M' \rightarrow M \) for an irreducible \( \mathfrak{p}_M \)-module \( M' \). Here is such an example.
Example 5. Let $g = A(\infty), B(\infty), C(\infty)$ or $D(\infty)$. Fix a Borel subalgebra $b \subset g$ and let $\lambda \in b^*$ be such that $\langle \lambda, \alpha \rangle \not\in \mathbb{Z}$ for all $\alpha \in \Delta$. Construct $\{\lambda_n\}$ inductively by setting $\lambda_0 := \lambda$, $\lambda_{n+1} := \lambda_n + \alpha_n$, where $\alpha_n$ is a simple root of $b_{n+1}$ which is not a root of $b_n$. There are obvious embeddings $V_{b_n}(\lambda_n|_{b_n}) \to V_{b_{n+1}}(\lambda_{n+1}|_{b_{n+1}})$ so we can set $M := \lim_{\rightarrow} V_{b_n}(\lambda_n|_{b_n})$. Then $g = b \oplus g_M^+ \oplus g_M^-$ where $\mathfrak{h} \oplus g_M^+ = b$. Therefore the only Borel subalgebra with respect to which $M$ could be a highest weight module is $b$. But a direct verification shows that $M$ has no non-zero $b$-highest weight vector and is therefore not a $b$-highest weight module.

For any irreducible root direction of $M$ there is the following natural question: what are the integrable root directions of $M$, i.e. for which roots $\alpha \in \Delta$ is $M$ an integrable $g^\alpha$-module? It is a remarkable fact that if $\Delta_M^{\text{int}}$ is the set of all roots $\alpha$ for which $M$ is $g^\alpha$-integrable, then $\mathfrak{p}_M^{\text{int}} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_M^{\text{int}}} g^\alpha$ is a Lie subalgebra of $g$, and $\mathfrak{p}_M^{\text{int}}$ is nothing but the subset of elements in $g$ which act locally finitely on $M$.

For a finite-dimensional Lie algebra this can be proved using Gabber’s theorem, see Corollary 2.7 in [F] or Proposition 1 in [PS]. For a root direct limit Lie algebra the statement follows immediately from the case of a finite-dimensional Lie algebra.

Proposition 7. (i) $\mathfrak{p}_M^{\text{int}} = (\mathfrak{g}_M^F + (\mathfrak{g}_M^I \cap \mathfrak{p}_M^{\text{int}})) \oplus \mathfrak{g}_M^-$. (ii) If $M$ is a pseudo highest weight module, then $\mathfrak{p}_M^{\text{int}} = \mathfrak{p}_M$; in particular $\mathfrak{p}_M^{\text{int}}$ is a parabolic subalgebra of $g$.

Proof. (i) It is obvious that $\mathfrak{g}_M^F \subset \mathfrak{p}_M^{\text{int}}$ and $\mathfrak{g}_M^- \subset \mathfrak{p}_M^{\text{int}}$. Furthermore, $\mathfrak{p}_M^{\text{int}} \cap \mathfrak{g}_M^- = 0$. Indeed, assuming that $\mathfrak{g}_M^- \cap \mathfrak{p}_M^{\text{int}} \neq 0$ we would have that $g^\alpha$ acts locally nilpotently on $M$ for some $\alpha \in \Delta_M^{\text{int}}$, which is impossible as then $\text{supp}M$ would have to be invariant with respect to the reflection $\sigma_\alpha \in W$. Since $\mathfrak{p}_M^{\text{int}} \supset \mathfrak{h}$, $\mathfrak{p}_M^{\text{int}}$ is a weight submodule of $\mathfrak{g}$ and $\mathfrak{p}_M^{\text{int}} = \mathfrak{p}_M^{\text{int}} \cap ((\mathfrak{g}_M^F + \mathfrak{g}_M^I) \oplus \mathfrak{g}_M^-) = (\mathfrak{g}_M^F + (\mathfrak{g}_M^I \cap \mathfrak{p}_M^{\text{int}})) \oplus \mathfrak{g}_M^-$. (ii) The statement follows immediately from (i) since $\mathfrak{g}_M^- = \mathfrak{h} \subset \mathfrak{g}_M^{\text{int}}$ for a pseudo highest weight module $M$.

The following Proposition provides a more explicit description of $\mathfrak{p}_M = \mathfrak{p}_M^{\text{int}}$ for highest weight modules. It is a version of the main result of [DP2] adapted to highest weight modules with respect to arbitrary Borel subalgebras a root reductive Lie algebra $g$. If $M = V_b(\lambda)$, we say that a root $\alpha \in \Delta^-$ is $M$-simple if $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Delta^-$ implies $\langle \lambda, \beta \rangle = 0$ or $\langle \lambda, \gamma \rangle = 0$.

Proposition 8. Let $M = V_b(\lambda)$ and let $\Sigma_{\lambda, b}^{\text{int}}$ be the set of all $M$-simple roots $\delta \in \Delta^-$ for which $M$ is $g^\delta$-integrable. Then, for any $\alpha \in \Delta^-$, $M$ is $g^\alpha$-integrable (equivalently, $\alpha \in \Delta^+ \cap \Delta_F^M$) iff $\alpha \in < \Sigma_{\lambda, b}^{\text{int}} >_{\mathbb{Z}^+}$.

Proof. Proposition 7(ii) implies that if $\alpha \in < \Sigma_{\lambda, b}^{\text{int}} >_{\mathbb{Z}^+}$ then $M$ is $g^\alpha$-integrable. Let, conversely, $M$ be $g^\alpha$-integrable. If $\alpha$ is not $M$-simple, then $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Delta^-$ with $\langle \lambda, \beta \rangle \neq 0$ and $\langle \lambda, \gamma \rangle \neq 0$. The reader will then check that $| \langle \lambda, \alpha \rangle |$ is strictly bigger than both $| \langle \lambda, \beta \rangle |$ and $| \langle \lambda, \gamma \rangle |$. Choose $n$ big enough so that $\alpha, \beta, \gamma \in \Delta_n$. Applying the main Theorem from [DP2] to $V_{b_n}(\lambda|_{b_n})$, we obtain that $V_{b_n}(\lambda|_{b_n})$ is both $g^\alpha$-integrable and $g^\gamma$-integrable. Therefore $M$ is also $g^\alpha$-integrable as well as $g^\gamma$-integrable. To conclude that $\alpha \in < \Sigma_{\lambda, b}^{\text{int}} >_{\mathbb{Z}^+}$ whenever $V_b(\lambda)$ is $g^\alpha$-integrable one applies now induction on $| \langle \lambda, \alpha \rangle |$. □

Although the parabolic induction theorem does not hold for pseudo highest weight modules, their supports can be described explicitly. In the next section we prove a general theorem describing the support of any irreducible weight module.
An open question about pseudo highest weight modules is whether the statement (ii) of Theorem 7 extends to any pseudo highest weight module, i.e. whether such a module is determined up to an isomorphism by its support.

6. The support of an arbitrary irreducible weight module

Let $M$ be an arbitrary irreducible weight module $M$ with corresponding partition $\Delta = \Delta^F_M \cup \Delta^I_M \cup \Delta^+_M \cup \Delta^-_M$. Define the small Weyl group $W^F$ of $M$ as the Weyl group of $g^F_M$. For $\lambda \in h^*$, set $K^\lambda_M := \langle (W^F \cdot \lambda) + < \Delta^I_M >_Z + < \Delta^-_M >_{Z_+} \text{ and } K^\lambda_M := \langle (W^F \cap W_0) \cdot \lambda > + < \Delta^I_M \cap \Delta_n >_Z + < \Delta^-_M \cap \Delta_n >_{Z_+}$.

**Lemma 3.** For any $\lambda_0 \in \text{supp}M$ and any $n$, there exists $\lambda \in \text{supp}M$ such that $(\lambda_0 + < \Delta_n >_R) \cap \text{supp}M = K^\lambda_M$.

**Proof.** If $\Delta_n \subset \Delta^I_M$, set $\lambda = \lambda_0$. Assume now that $\Delta_n \not\subset \Delta^I_M$. Consider the cone $K := \langle \Delta^I_M \cup \Delta^+_M >_R$. Then $(\lambda_0 + K) \cap (\text{supp}M \cap < \Delta_n >_R)$ is a finite set. As in the proof of Theorem 3, $\text{supp}M = \cup_N \text{supp}M_N$ for some irreducible $g^F_N$-modules $M_N$. Furthermore, there is $N_0$ for which $\text{supp}M_{N_0} \supset (\lambda_0 + K) \cap (\text{supp}M \cap < \Delta_n >_R)$. Let $\lambda \in \text{supp}M_{N_0} \cap < \Delta_n >_R$ be such that $\lambda + \alpha \not\in \text{supp}M_{N_0}$ for any $\alpha \in \Delta_n \cap (\Delta^I_M \cup \Delta^+_M)$. (Such a weight $\lambda$ exists because otherwise we would have $\Delta_n \subset \Delta^I_M$.) Clearly then $(\lambda_0 + < \Delta_n >_R) \cap \text{supp}M = K^\lambda_M$.

Fix now a Borel subalgebra $b \subset (g^F_M + g^I_M) \oplus g^*_M$ of $g$ and let $\lambda \in \text{supp}M$. Then Lemma 3 enables us to construct a sequence $\lambda_3, \lambda_4, \ldots, \lambda_n, \ldots$ such that $\lambda_3 = \lambda$ and $\lambda_n$ is such that
- $-\lambda_n \not\in \text{supp}M$ for any $\alpha \in (\Delta^I_M \cup \Delta^+_M) \cap \Delta_+$;
- $(\lambda_n + < \Delta_n >_R) \cap \text{supp}M = K^\lambda_M$, for $n \geq 4$.

This sequence reconstructs $\text{supp}M$ and is, in a certain sense, assigned naturally to $M$. For a precise formulation, define an equivalence relation $\sim_M$ on $h^*$ by setting $\lambda_1 \sim_M \lambda_2$ iff there exists $w \in W^F$ such that $w(\lambda_1) - \lambda_2 \in < \Delta^I_M >_Z$. Let furthermore $h_M^*$ denote the set of $\sim_M$-equivalence classes and let $p : h^* \rightarrow h^*_M$ be the natural projection.

The following theorem is our most general result describing $\text{supp}M$ for an arbitrary irreducible weight $g$-module $M$. It is a straightforward corollary of the construction of the sequence $\{\lambda_n\}$.

**Theorem 8.** (i) $K^\lambda_1 \subset K^\lambda_2 \subset \ldots$ and $\text{supp}M = \cup_n K^\lambda_n$.

(ii) The sequence $\{p(\lambda_n)\}$ depends on $\lambda$ only, and if $\lambda'$ is another element of $\text{supp}M$, there is $n_0$ so that $p(\lambda_n) = p(\lambda'_n)$ for all $n > n_0$.

(iii) If $\tilde{M}$ is an irreducible weight $g$-module with corresponding sequence $\{\tilde{\lambda}_n\}$, then $\text{supp}M = \text{supp} \tilde{M}$ iff there is $n_0$ such that $p(\lambda_n) = p(\tilde{\lambda}_n)$ for $n > n_0$.

The existence of a sequence of weights $\{\lambda_n\}$ for which $\text{supp}M = \cup_n K^\lambda_n$ follows directly from the fact that $\text{supp}M = \cup_{n \in N} \text{supp}M_n$ for some irreducible $g^*_M$-modules $M_n$ (see the proof of Theorem 3). The main advantage of constructing the sequence $\{\lambda_n\}$ is that $M$ stably determines the sequence $\{p(\lambda_n)\}$. It is an open question to describe explicitly all possible sequences $\{p(\lambda_n)\}$, or equivalently all possible supports of irreducible weight $g$-modules with a given shadow.
APPENDIX. BOREL SUBALGEBRAS OF $\mathfrak{g}$ AND CHAINS OF SUBSPACES IN $<\Delta>\mathbb{R}$

In this Appendix $\mathfrak{g}$ is an arbitrary Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{g}$ admits a root decomposition ($\mathfrak{h}$). Here we establish a precise inter-relation between Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$ (which we call simply Borel subalgebras), $\mathbb{R}$-linear orders on $<\Delta>\mathbb{R}$, and oriented maximal chains of vector subspaces in $<\Delta>\mathbb{R}$. Our motivation is the following. If $\mathfrak{g}$ has finitely many roots (i.e., for instance, if $\mathfrak{g}$ is finite-dimensional), every Borel subalgebra is defined by a (non-unique) regular hyperplane $H$ in $<\Delta>\mathbb{R}$, i.e., by a hyperplane $H$ such that $H \cap \Delta = \emptyset$. If $\mathfrak{g}$ has infinitely many roots this is known to be no longer true (i.e., in general there are Borel subalgebras which do not correspond to any regular hyperplane in $<\Delta>\mathbb{R}$) and in [DP1] we have shown that in the case when $<\Delta>\mathbb{R}$ is finite-dimensional every Borel subalgebra can be defined by a maximal flag of linear subspaces in $<\Delta>\mathbb{R}$. However, when $<\Delta>\mathbb{R}$ is infinite-dimensional, the situation is more complicated and deserves a careful formulation.

We start by stating the relationship between Borel subalgebras $\mathfrak{b}$ with $\mathfrak{b} \supset \mathfrak{h}$ and $\mathbb{R}$-linear orders on $<\Delta>\mathbb{R}$.

**Proposition 9.** Every $\mathbb{R}$-linear order on $<\Delta>\mathbb{R}$ determines a unique Borel subalgebra, and conversely, every Borel subalgebra is determined by an (in general not unique) $\mathbb{R}$-linear order on $<\Delta>\mathbb{R}$.

**Proof.** If $>$ is an $\mathbb{R}$-linear order on $<\Delta>\mathbb{R}$, set $\Delta^\pm := \{\alpha \in \Delta \mid \pm \alpha > 0\}$. Then $\Delta^+ \sqcup \Delta^-$ is a triangular decomposition and thus $>$ determines the Borel subalgebra $\mathfrak{h} \oplus (\oplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha)$. The proof of the converse statement is left to the reader (it is a standard application of Zorn’s Lemma).

In what follows we establish a bijection between $\mathbb{R}$-linear orders on $<\Delta>\mathbb{R}$ and oriented maximal chains of vector subspaces in $<\Delta>\mathbb{R}$. If $V$ is any real vector space, a *chain of vector subspaces* of $V$ is defined as a set of subspaces $F = \{F_\alpha\}_{\alpha \in A}$ of $V$ such that $\alpha \neq \beta$ implies a proper inclusion $F_\alpha \subset F_\beta$ or $F_\beta \subset F_\alpha$. The set $A$ is then automatically ordered. A chain $F$ is a *flag* iff as an ordered set $A$ can be identified with a (finite or infinite) interval in $\mathbb{Z}$. A chain $F$ is *maximal* iff it is not properly contained in any other chain. Maximal chains of linear subspaces may be somewhat counterintuitive as the following example shows that a linear space of countable dimension admits non-countable maximal chains and vice versa.

**Example 6.** Let $V$ be a countable-dimensional space with a basis $\{e_r\}_{r \in \mathbb{Q}}$. The chain of all subspaces $V_t := \{e_r \mid r < t\} \mathbb{R}$ for $t \in \mathbb{R}$, $V'_t := \{e_r \mid r \leq t\} \mathbb{R}$ for $r \in \mathbb{Q}$, $V_{-\infty} := 0$ and $V_{\infty} := V$ is a maximal chain of cardinality continuum. Let $U := \mathbb{C}[\lbrack x \rbrack]$ be the space of formal power series in the indeterminant $x$. The dimension of $U$ is continuum, however $\{0\} \cup F$, $F$ being the flag $\{F_i := x^i W\}_{i \in \mathbb{Z}_+}$, provides an example of a countable maximal chain in $U$.

Furthermore, it turns out that every maximal chain is determined uniquely by a certain subchain. Define a chain $F$ to be *basic* iff it is minimal with the following property: for every $x \in V$ there exists a pair $F_\alpha \subset F_\beta \in F$ with $\dim F_\beta / F_\alpha = 1$ such that $x \in F_\beta \setminus F_\alpha$.

**Lemma 4.** Every maximal chain in $V$ admits a unique basic subchain and conversely, every basic chain is contained in a unique maximal chain.

\[2\text{This is the case for any Kac-Moody algebra which is not finite-dimensional.}\]
There is a bijection between oriented maximal chains in Theorem 9. Let $G = \{F_{\alpha}\}_{\alpha \in A}$ be a maximal chain in $V$. For every non-zero $x \in V$ set $F_x := \bigcup_{\alpha \in F_x} F_\alpha$ and $F'_x := F_x \oplus \mathbb{R}x$. Then $F \cup \{F_x', F'_x\}$ is a chain in $V$ and hence $F_x, F'_x \in F$. Let $G := \bigcup_{x \in V} \{F_x, F'_x\}$. Then $G$ is obviously a basic chain contained in $F$. Noting that $G$ consists exactly of all pairs of subspaces from $F$ with relative codimension one, we conclude that $G$ is the unique basic chain contained in $F$.

Let now $G = \{G_\beta\}_{\beta \in B}$ be a basic chain in $V$ and let $< be the corresponding order on $B$. A subset $C$ of $B$ is a cut of $B$ if $\alpha \in C$ and $\beta < \alpha$ implies $\beta \in C$, and if $\alpha \in C$ and $\beta \notin C$ implies $\beta > \alpha$. The set $\mathcal{B}$ of all cuts of $B$ is naturally ordered. Set $H_\alpha := \bigcup_{\beta \in \alpha} G_\beta$ for $\alpha \in \mathcal{B}$. One checks immediately that $H = \{H_\alpha\}_{\alpha \in \mathcal{B}}$ is a maximal chain in $V$ which contains $G$. On the other hand, given any maximal chain $F = \{F_\alpha\}_{\alpha \in A}$ in $V$ containing $G$, one notices that $F_\alpha = \bigcup_{x \in F_\alpha} G_\beta$. Then the map $\varphi : A \to \mathcal{B}$, $\varphi(\alpha) := \{\beta | G_\beta \subset F_\alpha\}$ is an embedding of $F$ into $H$ and hence $F = H$. The Lemma is proved.

An orientation of a basic chain $G$ is a labeling of the two half-spaces of $G_\beta \setminus G_\alpha$ for every pair of indices $\alpha < \beta$ with $\dim G_\beta / G_\alpha = 1$, by mutually opposite signs $\pm$. An orientation of a maximal chain $F$ is an orientation of its basic subchain. A maximal chain $F$ is oriented iff an orientation of $F$ is fixed.

Theorem 9. There is a bijection between oriented maximal chains in $< \Delta >_\mathbb{R}$ and $\mathbb{R}$-linear orders on $< \Delta >_\mathbb{R}$.

Proof. Lemma 4 implies that it is enough to establish a bijection between oriented basic chains in $< \Delta >_\mathbb{R}$ and $\mathbb{R}$-linear orders on $< \Delta >_\mathbb{R}$.

Let $G$ be an oriented basic chain in $< \Delta >_\mathbb{R}$. We define an $\mathbb{R}$-linear order $>_G$ on $< \Delta >_\mathbb{R}$ by setting $x >_G 0$ or $x <_G 0$ according to the sign of the half-space of $G_\beta \setminus G_\alpha$ to which $x$ belongs. Conversely, given an $\mathbb{R}$-linear order $>_G$ on $< \Delta >_\mathbb{R}$ we build the corresponding oriented basic chain as follows. For a non-zero vector $x \in V$ we set $V_x := \{y \in V \mid (cy + x) > 0 \iff \pm x > 0\}$. Then for every pair of non-zero vectors $x, y \in V$ exactly one of the following is true: $V_x = V_y, V_x \subset V_y$ or $V_x \supset V_y$. Therefore the set of distinct subspaces among $\{V_x\}$ is a chain $F$. The very construction of $F$ shows that it is a basic chain in $< \Delta >_\mathbb{R}$. To complete the proof of the Theorem it remains to orient this chain in the obvious way.

Finally we define a decomposition of $\Delta$,

$$\Delta = \Delta^- \cup \Delta^0 \cup \Delta^+, \tag{8}$$

to be parabolic iff $\Pi(\Delta^-) \cap \Pi(\Delta^+) = \emptyset, 0 \notin \Pi(\Delta \pm)$ and $\Pi(\Delta) \setminus \{0\} = \Pi(\Delta^-) \cup \Pi(\Delta^+)$ is a triangular decomposition of $\Pi(\Delta) \setminus \{0\}$ (i.e. the cone $< \Pi(\Delta^+) \cup -\Pi(\Delta^-) >_{\mathbb{R}^+}$ contains no vector subspace), where $\Pi$ is the projection $< \Delta >_\mathbb{R} \to < \Delta >_\mathbb{R}$. If $\Delta^0 = \emptyset$, a parabolic decomposition is a triangular decomposition. Given a parabolic decomposition $\{\mathfrak{g}^\alpha\}_{\alpha \in \Delta^0}$, its corresponding parabolic subalgebra is by definition $\mathfrak{p} := \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta^0} \mathfrak{g}^\alpha)$.

Given a linear subspace $V'$ in $V$, we define a $V'$-maximal chain of vector subspaces in $V$ to be the preimage in $V$ of a maximal chain in $V/V'$. An $V'$-maximal chain is oriented iff the corresponding maximal chain in $V/V'$ is oriented. Generalizing the corresponding construction for Borel subalgebras, we have

Theorem 10. Any parabolic decomposition $\{\mathfrak{g}^\alpha\}$ is determined by some oriented $< \Delta^0 >_{\mathbb{R}}$-maximal chain of vector subspaces in $V$, and conversely any oriented $V'$-maximal chain in $< \Delta >_{\mathbb{R}}$, for an arbitrary subspace $V'$ of $< \Delta >_{\mathbb{R}}$, defines a unique parabolic decomposition of $\Delta$ with $\Delta^0 = \Delta \cap V'$. 
Proof. An exercise.

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