An approach for obtaining integrable Hamiltonians from Poisson-commuting polynomial families

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(Dated: 26 March 2018)

We discuss a general approach permitting the identification of a broad class of sets of Poisson-commuting Hamiltonians, which are integrable in the sense of Liouville. It is shown that all such Hamiltonians can be solved explicitly by a separation of variables Ansatz. The method leads in particular to a proof that the so-called “goldfish” Hamiltonian is maximally superintegrable, and leads to an elementary identification of a full set of integrals of motion. The Hamiltonians in involution with the “goldfish” Hamiltonian are also explicitly integrated. New integrable Hamiltonians are identified, among which some have the property of being isochronous, that is, that all their orbits have the same period. Finally, a peculiar structure is identified in the Poisson brackets between the elementary symmetric functions and the set of Hamiltonians commuting with the “goldfish” Hamiltonian: these can be expressed as products between elementary symmetric functions and Hamiltonians. The structure displays an invariance property with respect to one element, and has both a symmetry and a closure property. The meaning of this structure is not altogether clear to the author, but it turns out to be a powerful tool.

PACS numbers: 45.05.+x, 45.20.Jj
Keywords: integrable systems, polynomials, goldfish equations, Hamiltonian systems

I. INTRODUCTION

In the following, I wish to present a remarkably elementary way of obtaining integrable Hamiltonians by defining suitable families of Poisson-commuting polynomials. As a motivating example, let us consider the following system of ordinary differential equations (ODE’s), discussed initially by Calogero in [2] and further in the book [3]:

$$\ddot{q}_j = 2\dot{q}_j \sum_{k=1}^{N} (\dot{q}_j - \dot{q}_k), \quad j = 1, \ldots, N \tag{1}$$

This system, known as the “goldfish” Hamiltonian, can be solved exactly by the observation that it describes the motion of the zeros of a time dependent monic polynomial $S_N(z, t)$ of degree $N$, given by

$$S_N(z, t) = z^N + \sum_{k=1}^{N} s_k(t) z^{N-k}, \quad s_k(t) = 0 \tag{2}$$

such that $\dot{s}_k(t) = 0$. The way to solving (1) exactly thus consists in evaluating the initial values $s_k(0)$ and $\dot{s}_k(0)$ from $q_k(0)$ and $\dot{q}_k(0)$. One then can readily compute the $s_k(t)$ for all times, and the finding of the $q_k(t)$ is thus reduced to the purely algebraic task of solving for the zeros of $S_N(z, t)$ given the $s_k(t)$.

There is, however, considerably more structure hidden in this system: first, it can be derived from a Hamiltonian. This follows straightforwardly, as remarked in [3], from the fact that the dynamics of the $s_k(t)$ is Hamiltonian (since it is free motion). The corresponding motion of the $q_k(t)$ is thus Hamiltonian as well, since the $q_k(t)$ arise from the $s_k(t)$ by a point transformation. This is obvious for the case of the transformation from the $q_k(t)$ to the $s_k(t)$, which is algebraically straightforward. This Hamiltonian, however, is neither easily described nor analyzed. However, another Hamiltonian was stated in [3], see also [2], given by

$$h_1(p, q) = \sum_{r=1}^{N} e^{p_r} \prod_{s=1(s \neq r)}^{N} (q_r - q_s)^{-1}. \tag{3}$$

Here and in the following, underlined quantities such as $p_r$ will always represent an $N$-component vector $(p_1, \ldots, p_N)$. That this Hamiltonian generates [1] as “Newtonian” equations of motion is a straightforward computation.

Clearly, the solvability of (1) implies the integrability of $h_1(p, q)$. Yet several questions remain: since the system is equivalent to free motion, it must be, in fact, maximally superintegrable, that is, there must exist $2N - 1$ integrals of motion of $h_1(p, q)$, of which $N$ can be chosen in involution. Can these be given in a straightforward way? Can the integration of the Hamiltonian [1] be realised explicitly? And finally, can this be generalised to a broader class of Hamiltonians?

Let us first illustrate the method we shall use on this example: one defines the following non-monic polynomial of degree $N - 1$

$$\mathcal{H}(z|p, q) = \sum_{k=1}^{N} h_k(p, q) z^{N-k} \tag{4}$$

\[a\) Also at Centro Internacional de Ciencias.
by the following conditions:

\[ \mathcal{H}(z|p, q)|_{z=q_k} = e^{p_k}. \] (5)

The polynomial is thereby uniquely defined, and the coefficients \( h_k(p, q) \) can thus be evaluated. It is straightforward to verify that \( h_1(p, q) \) is in fact given by (3), so that the notation is consistent.

It can now be shown rather simply, using the definition of \( \mathcal{H}(z|p, q) \) provided by (5), that

\[ \{ \mathcal{H}(z|p, q), \mathcal{H}(w|p, q) \} = 0 \] (6)

for all \( z \) and \( w \), where the Poisson bracket takes its usual form, see (3). From this follows that all \( h_k(p, q) \) are in involution, thus providing a simple proof of the integrability of \( h_1(p, q) \). \( N \) integrals of motion are now explicitly given, and further analysis easily yields, as we shall see below, another \( N - 1 \). We shall also see that this further provides an approach to solve the Hamiltonian (3) by separation of variables. While we shall not pursue this further here, this might be of interest in the solution of the problem of quantising the goldfish Hamiltonian. While much progress has been made in this direction using Noether symmetries, see for example (2), an explicit approach to quantising (3), allowing for example to display eigenfunctions in configuration space, is still lacking.

As to comparison with earlier works, the following remarks are in order: the most general set of Hamiltonians described in this paper, displayed in (20), have been described in various papers within the context of bi-Hamiltonian and quasi-bi-Hamiltonian systems (3). Results concerning their integrability and their amenability to treatment via separation of variables are obtained there, but the connection to the quite elementary approach presented here is not apparent to the author.

In the following, we show a general way of obtaining results using an approach similar to the one sketched above. In Section II we show an easily obtained, yet quite general result. In particular, a family of Hamiltonians is displayed (20) which is always integrable. It is further shown how separation of variables may always be achieved under these circumstances. In Section III we briefly discuss the behaviour of the total momentum, which generates the group of translations. In Section IV we specify the results to the goldfish Hamiltonian and show a remarkable structure in the Poisson brackets of the elementary symmetric functions with the Hamiltonians from the set of goldfish Hamiltonians. Finally, in Section V we present some conclusions.

II. GENERAL APPROACH

Let us start out with a perfectly general framework of the type sketched in Section I. We describe the polynomial \( \mathcal{A}(z|p, q) \) in the following two ways:

\[ \mathcal{A}(z|p, q) = \sum_{k=1}^{N} a_k(p, q)z^{N-k}, \] (7a)

\[ \mathcal{A}(q_k|p, q) := \mathcal{A}(z|p, q)|_{z=q_k} = a_k(p, q). \] (7b)

Let us here make some notational conventions, to which we shall always adhere. First, the expression \( \mathcal{A}(q_k|p, q) \), as well as all similarly constructed expressions, will always be taken in the meaning of the definition that immediately follows it in (7b). Further, all polynomials, which in the following will always be denoted by capital calligraphic letters, will depend on one or several variables, denoted by \( z \), \( w \) and so on, and parametrically on the variables \( p \) and \( q \), which will denote vectors of dimension \( N \) of variables \( (p_1, \ldots, p_N) \) and \( (q_1, \ldots, q_N) \), which obey the usual Poisson brackets:

\[ \{p_i, q_j\} = \delta_{ij}. \] (8)

The polynomials will always be of degree \( N - 1 \), with no restriction on the values of their coefficients. They are thus uniquely determined by the specification of the values on the \( N \) points \( q_k \), as given in (7b). These values will always be given as (lowercase) Greek letters, whereas the coefficients of the polynomial will always be given as lowercase Latin letters.

The lone exception to the rule that polynomials are all of degree \( N - 1 \), will be the polynomial \( E_N(z|q) \), defined as

\[ E_N(z|q) = \prod_{k=1}^{N} (z - q_k), \] (9)

which is, of course, monic of order \( N \), and is defined by the fact that it vanishes on all values \( z = q_k \). To underline the difference, we denote this polynomial by a capital Latin letter.

A. The Poisson bracket formula for polynomials

Let us consider two polynomials, \( \mathcal{A}(z|p, q) \) and \( \mathcal{B}(w|p, q) \), where \( \mathcal{B}(w|p, q) \) is defined quite analogously to (7), by

\[ \mathcal{B}(w|p, q) = \sum_{k=1}^{N} b_k(p, q)w^{N-k}, \] (10a)

\[ \mathcal{B}(q_k|p, q) = \beta_k(p, q). \] (10b)

We now wish to evaluate \( \mathcal{C}(z, w|p, q) \) defined by

\[ \mathcal{C}(z, w|p, q) = \sum_{r,s=1}^{N} \left\{ a_k(p, q), b_l(p, q) \right\} z^{N-r}w^{N-s} \] (11)

by the simple device of computing its values

\[ \mathcal{C}(q_k, q_l|p, q) = \sum_{r,s=1}^{N} \left\{ a_r(p, q), b_s(p, q) \right\} q_k^{N-r}q_l^{N-s}. \] (12)
This leads, after some straightforward algebra detailed in Appendix A to
\[
C(q_k, q_l | p, q) = \{ \alpha_k(p, q), \beta_l(p, q) \}
\]
\[
+ \left. \frac{\partial \beta_l(p, q)}{\partial w} \right|_{w=q_l} \cdot \left. \frac{\partial \alpha_k(p, q)}{\partial p_k} \right|_{z=q_k}
\]
\[
- \left. \frac{\partial B(w | p, q)}{\partial w} \right|_{w=q_l} \cdot \left. \frac{\partial \alpha_k(p, q)}{\partial p_k} \right|_{z=q_k},
\]
which is the basic equation underlying the entire approach.

The strategy should now be clear: we define polynomials by setting \(\alpha_k(p, q)\) and \(\beta_k(p, q)\) to values that are, in a sense, simple, so that the Poisson brackets are easily evaluated via (13). This yields results which are usually not so trivial for the coefficients \(a_k(p, q)\) and \(b_k(p, q)\).

B. The case of one single self-commuting polynomial

Consider a polynomial \(H(p, q)\) defined as usual by
\[
H(z | p, q) = \sum_{k=1}^{N} h_k(p, q) z^{-k},
\]
(14a)
\[
H(q_k | p, q) = \eta_k(p),
\]
(14b)
and let us assume that
\[
\{ H(z | p, q), H(w | p, q) \} = 0.
\]
(15)
Then the \(h_k(p, q)\) are all in involution, and the Hamiltonians \(h_k(p, q)\) are all integrable in the sense of Liouville.

To generate a set of integrable Hamiltonians, it is therefore sufficient to give a set of functions \(\eta_k(p, q)\) for \(1 \leq k \leq N\), leading to (13) via (13). I am not aware of a general solution to this problem, but the following Ansatz has the required property and yields, as we shall see, a few significant results:
\[
\eta_k(p) = \eta_k(p).
\]
(16)
In that case, it is immediate to verify, using (13), that (15) indeed holds. In this case, the polynomial \(H(z | p, q)\) has the following explicit expression:
\[
H(z | p, q) = \sum_{r=1}^{N} \eta_r(p_r) \prod_{k=1; k \neq r}^{N} \frac{z-q_k}{q_r-q_k}
\]
\[
=- \sum_{r=1}^{N} \eta_r(p_r) \prod_{k=1; k \neq r}^{N} (q_r-q_k) \frac{\partial E_N(z | q)}{\partial q_r},
\]
(17)
where \(E_N(z | q)\) is the (monic) polynomial defined in (15).

We now define a related polynomial \(E(z | q)\) in the usual way:
\[
E(q_k | q) = q_k^N,
\]
(18a)
\[
E(z | q) = z^N - E_N(z | q) := \sum_{k=1}^{N} e_k(q) z^{-k}.
\]
(18b)
The \(e_k(q)\) are therefore (up to an alternating sign) the elementary symmetric functions:
\[
e_k(q) = (-1)^{k-1} \sum_{1 \leq r_1 < r_2 < \cdots < r_k \leq N} \left( \prod_{l=1}^{k} q_{r_l} \right),
\]
(19)
From (17) follows the following explicit expression for \(h_k(p, q)\):
\[
h_k(p, q) = \sum_{r=1}^{N} \eta_r(p_r) \frac{\partial e_k(q)}{\partial q_r} \prod_{l=1; l \neq r}^{N} (q_r-q_l)^{-1}.
\]
(20)
All such Hamiltonians are therefore integrable. Clearly, the Hamiltonian (13) belongs to this class, for \(\eta_k(p) = e^p\) and for \(k = 1\) as do the simplest versions of the Ruijsenaars–Schneider Hamiltonian introduced in (13). We have now explicitly found the expression for the corresponding action variables which are simply given by \(h_k(p, q)\) for \(2 \leq k \leq N\).

C. Separation of variables

Using the above technique, it is also straightforward to solve the Hamilton–Jacobi equation for one of the Hamiltonians of the class (15), or in fact, more generally, for all Hamiltonians satisfying (15).

To this end, one makes the following observation: since all the \(h_k(p, q)\) derived from \(H(z | p, q)\) are constants of the motion for any of them, it follows that, for all the dynamics defined by some \(h_k(p, q)\), one has
\[
H \left[ z | p(t), q(t) \right] = P_0(z),
\]
(21)
where \(P_0(z)\) is the constant polynomial determined by the initial values of the constants of motion \(h_k\). Using (14b) yields, by substituting \(z = q_k(t)\):
\[
P_0[q_k(t)] = \eta_k[p_k(t)].
\]
(22)
Let us now define \(\phi_k(x)\) as the inverse function of \(\eta_k(p)\), that is
\[
\phi_k[\eta_k(p)] = p.
\]
(23)
We will assume in the following that \(\phi_k(x)\) is a uniquely defined function. If this is not the case, special precautions must be taken in each case. If we now define \(S(q)\) so that
\[
p_k = \frac{\partial S}{\partial q_k},
\]
(24)
it follows from (22) that
\[
\frac{\partial S}{\partial q_k} = \phi_k \left[ P_0(q_k) \right].
\]
(25)
This equation has the manifest solution
\[ S(q) = \sum_{k=1}^{N} \int_{0}^{q_k} dy \phi_k[P_0(y)]. \] (26)

The coefficients of the polynomial \( P_0(y) \) provide \( N \) arbitrary constants, which are the initial values of the constants of the motion \( h_i \), for \( 1 \leq i \leq N \), so that we have here a general solution of the Hamilton–Jacobi equation. Let us now define \( h \) as the vector \( (h_1, \ldots, h_N) \). From the general theory, we have that the partial derivative of \( S(q, h) \) with respect to \( h_l \) is also a constant of motion \( \beta_l \):
\[ \beta_l = \frac{\partial S(q, h)}{\partial h_l} = \sum_{k=1}^{N} \int_{0}^{q_k} dy y^{N-l} \frac{d \phi_k[P_0(y)]}{\eta_k[\phi_k[P_0(y)]]}. \] (27)

From this the motion can be recovered, if we remember that, if the dynamics we consider is that of \( h_k \), then \( \beta_k \) corresponds to the time.

This also provides a very elementary way to reduce the Hamiltonian equations of motion, which involve the \( 2N \) variables \( p_k \) and \( q_k \) for \( 1 \leq k \leq N \), to a system of \( N \) first order ordinary differential equations involving the \( q_k \) only. Indeed, Hamilton’s equations yield for the \( q_k \):
\[ \dot{q}_k = \eta_k'(p_k) \prod_{r=1; (r \neq k)}^{N} (q_k - q_r)^{-1}. \] (28)

But according to (22) we may substitute \( p_k \) by \( \phi_k[P_0(q_k)] \). This leads to
\[ \dot{q}_k = \eta_k'(\phi_k[P_0(q_k)]) \prod_{r=1; (r \neq k)}^{N} (q_k - q_r)^{-1}. \] (29)

This result applied to the goldfish case, \( \eta(p) = e^p \), leads to a result obtained in a quite different way by Calogero, namely that the solutions of the system of ODE’s
\[ \dot{q}_k = P_0(q_k) \prod_{r=1; (r \neq k)}^{N} (q_k - q_r)^{-1} \] (30)
for any given fixed polynomial \( P_0(q) \) are also a solution of the goldfish equations (34).

D. An elementary special case

A very elementary example is that in which \( \eta_k(p) = p \). Then we also have \( \phi_k(x) = x \), and hence, from (27)
\[ \beta_l = \frac{1}{N-l+1} \sum_{k=0}^{N} q_k^{N-l+1}. \] (31)

We thus see that for this Hamiltonian, all moments of the \( q_k \) remain constant, except for that corresponding to the Hamiltonian under study, which corresponds to the time. If we wish to know the behavior of \( h_1(p, q) \), then we set
\[ t = \frac{1}{N} \sum_{k=1}^{N} q_k^N, \] (32)
whereas all other moments are constant. As is readily seen, this means that, if one considers the monic polynomial of which the \( q_k \) are the zeros, then its coefficients are all constant except the constant term, which grows proportionally to time.

The equations of motion corresponding to this Hamiltonian, see (20), with \( k = 1 \), are
\[ \dot{q}_j = \prod_{k=1; (k \neq j)}^{N} (q_j - q_k)^{-1}. \] (33)

The same result was found by a completely different approach as Exercise 2.3.4.1-12 in.

III. TRANSLATION INVARIANCE

For the specific case mainly discussed above, in which the functions \( \eta(p, q) \) defined in (10) only depend on the \( N \)-vector \( p \), the issue of translation invariance has interesting ramifications. It is easily seen that the Hamiltonian \( h_1(p, q) \) is, in that case, always invariant under translations, so that, if we set
\[ P = \sum_{k=1}^{N} p_k, \] (34)
one obtains
\[ \{ h_1(p, q), P \} = 0. \] (35)

This can be simply understood: since the values of \( H(q_k|p, q) \) do not depend on \( q_k \), we have, if we define \( a \) as the \( N \)-vector \( (a, \ldots, a) \)
\[ H(z-a|p, q) = H(z|p, q + a). \] (36)

Upon translation of \( z \), by a constant, the leading coefficient of \( H(z|p, q) \) is unchanged, from which the claim follows. On the other hand, the other coefficients are, of course, affected, implying that the Hamiltonians \( h_k(p, q) \) for \( 2 \leq k \leq N \) are not translation invariant, as indeed follows from the explicit expression (20). However, by differentiating (30) with respect to \( a \), one obtains
\[ \{ H(z|p, q), P \} = \frac{\partial H(z|p, q)}{\partial z} \]
\[ = \sum_{k=1}^{N} (N-k) h_k(p, q) z^{N-k-1}. \] (37)
From this follows
\[ \{ h_k(p, q), P \} = (N - k + 1)h_{k-1}(p, q), \quad (38) \]
and therefore, for any \( 1 \leq k \leq N \), one has
\[ \{ h_k(p, q), \{ h_k(p, q), P \} \} = 0. \quad (39) \]
Thus, if we consider the dynamics determined by any given \( h_k(p, q) \), we find
\[ \ddot{P} = 0. \quad (40) \]
We therefore see that for all Hamiltonians, total momentum \( P \) is either conserved or a measure of time.

IV. REMARKABLE PROPERTIES OF THE "GOLDFISH" HAMILTONIAN

The various Hamiltonians we have up to now considered are essentially of the type corresponding to free motion. They correspond to various systems having a type of modified kinetic energy. We do not have integrable cases involving Hamiltonians of the type kinetic plus potential energy, that is, we do not have the possibility of adding purely \( q \) dependent terms while maintaining integrability.

In the case of the "goldfish" Hamiltonian, it is possible to use the same approach to go considerably further. As we shall see, it is possible to obtain results for a rather broad class of combinations between the \( h_k(p, q) \) and the \( e_k(q) \). This is due to the fact that the polynomial \( E(z|q) \) has a Poisson bracket with the polynomial \( H(z|p, q) \) which can in turn be expressed in terms of \( E(z|q) \) and \( H(z|p, q) \). Let us show this now, and then consider the consequences.

A. Poisson brackets for the "goldfish" Hamiltonian

Let us define \( \mathcal{R}(z, w|p, q) \) as
\[ \mathcal{R}(z, w|p, q) = \{ H(p, q), E(z|q) \}. \quad (41) \]
where we are now limiting ourselves to the case of the goldfish Hamiltonian defined by \( \eta_k(p) = e^p \). In Appendix B we show, using (13), that
\[ \mathcal{R}(q_k, q|p, q) = e^{p_k} \delta_{k,l} \prod_{r=1; (r \neq k)}^{N} (q_k - q_r). \quad (42) \]
This has several interesting consequences. First and foremost, it shows that the polynomial \( \mathcal{C}(z, w|p, q) \) is symmetric in \( z \) and \( w \), thereby implying
\[ \{ h_k(p, q), e_1(q) \} = \{ h_k(p, q), e_k(q) \}. \quad (43) \]
Second, we find an explicit expression for \( \mathcal{R}(z, w|p, q) \), namely
\[ \mathcal{R}(z, w|p, q) = \sum_{r=1}^{N} e^{p_r} \left( \prod_{l=1; (l \neq r)}^{N} (z - q_l) \right) \times \left( \prod_{k=1; (k \neq r)}^{N} \frac{w - q_k}{q_r - q_k} \right). \quad (44) \]
From (41) readily follows that the leading term in \( \mathcal{R}(z, w|p, q) \), that is, the polynomial in \( w \) premultiplying \( z^{N-1} \) is given by
\[ \sum_{k=1}^{N} \{ h_1(p, q), e_k(q) \} w^{N-k} = \mathcal{H}(w|p, q), \quad (45) \]
since it satisfies the defining equation (14b). This therefore implies that
\[ \{ h_1(p, q), e_k(q) \} = \{ h_k(p, q), e_1(q) \} = h_k(p, q). \quad (46) \]
Finally, from (42), we immediately observe that
\[ (z - w) \mathcal{R}(z, w|p, q) = 0, \quad (47) \]
where we define the equivalence relation as stating that the left-hand side vanishes for all possible choices of \( z \) and \( w \) among the \( q_k \).

From this one obtains, after some calculations detailed in Appendix C, an explicit expression for the Poisson brackets \( \{ h_k(p, q), e_l(q) \} \) given by
\[ \{ h_k(p, q), e_l(q) \} = \sum_{m=l}^{k+l-2} \left[ e_{k+l-m-1}(q)h_m(p, q) - e_m(q)h_{k+l-m-1}(p, q) \right] + h_{k+l-1}(p, q) \]
where, in the second form of the equation, we adopt the convention suggested in Appendix C that \( e_0(q) = -1 \) and \( h_0(p, q) = 0 \). This relation, which expresses the Poisson brackets of the \( h_k(p, q) \) and the \( e_k(q) \) bilinearly...
in terms of the same variables, will play a basic role in the following. The relation’s mathematical significance is somewhat unclear to me, as it clearly does not correspond to a Lie algebra, but it turns out to be quite powerful.

Summarising, the Poisson bracket of the $e_k$ and the $h_l$ are characterised by 3 fundamental properties:

1. identity:
   \[
   \{ h_k(p, q), e_1(q) \} = h_k(p, q). \tag{49}
   \]

2. symmetry:
   \[
   \{ h_k(p, q), e_j(q) \} = \{ h_l(p, q), e_k(q) \}. \tag{50}
   \]

3. bilinear closure: the Poisson brackets of any $e_k(q)$ with any $h_l(p, q)$ can be expressed as a sum of products of $h_r(p, q)$ and $e_s(q)$.

It is from these three properties alone that all the following results are obtained.

### B. Explicit solution for all Hamiltonians $h_k(p, q)$

To solve for the dynamics of the Hamiltonian $h_k(p, q)$ for any given $k$, it is clearly sufficient to determine the time-dependence of the $e_j(q)$ for $1 \leq j \leq N$. Indeed, the $h_j(p, q)$ are all constant. The $e_j(q)$ are then sufficient to determine the $q_l$ by the algebraic process of computing the roots of $E_N(z(q))$.

The equations obeyed by the $e_j(q)$ are simply given by

\[
\dot{e}_j(q) = \{ h_k(p, q), e_j(q) \}. \tag{51}
\]

Using (48) and the constancy of the $h_j(p, q)$ for all $j$, we obtain the following expression for the equation of motion

\[
\dot{e}_j(q) = \sum_{l=1}^{N} a_{j,l}^{(k)} (h_1, \ldots, h_N) e_l(q) + b_{j}^{(k)} (h_1, \ldots, h_N). \tag{52}
\]

This can be rewritten more simply using the notation introduced in Appendix E that $e_0(q) = -1$. One then has

\[
\dot{e}_j(q) = \sum_{l=0}^{N} a_{j,l}^{(k)} (h_1, \ldots, h_N) e_l(q). \tag{53}
\]

Here the $a_{j,l}^{(k)}$ have replaced the $e_{j}^{(k)}$. Here the $a_{j,l}^{(k)}$ and the $b_{j}^{(k)}$ are linear expressions in the $h_{j}$, where the $h_{j}$ denote the initial values which the integrals of motion $h_{j}(p, q)$ take at the beginning of the evolution.

Some additional notation will allow to formulate this result more compactly: denote by $\epsilon(q)$ the vector $(e_0(q), e_1(q), \ldots, e_N(q))$ and by \( \dot{\epsilon} \) the vector $(\dot{e}_0(q), \dot{e}_1(q), \ldots, \dot{e}_N(q))$. There then exists a matrix $A^{(k)}$ such that

\[
\dot{\epsilon} = A^{(k)}(\epsilon(q)). \tag{54}
\]

The $e_j(q)$ therefore undergo a linear time evolution, the matrix of which depends on the $h_j$. However, since these are constant, they may be viewed as parameters. The matrix $A^{(k)}$ additionally depends on $k$, corresponding to the specific Hamiltonian $h_k(p, q)$ we are considering. Since, however, the flows of the various $h_k(p, q)$ commute, it follows that for all $k$

\[
[A^{(k)}(\mathbf{h}), A^{(l)}(\mathbf{h})] = 0, \tag{55}
\]

leading to the interesting consequence that the $A^{(k)}$ have eigenvectors that can be chosen independent of $k$. Note further that one can find an explicit expression for $A^{(k)}$ using the fundamental relation (48).

Finally, it is also quite easy to rederive the solution of the original “goldfish” Hamiltonian, $h_1(p, q)$. In this case we have

\[
\dot{e}_k(q) = \{ h_1(p, q), e_k(q) \} = h_k(p, q). \tag{56}
\]

Since $h_k$ is conserved, one verifies immediately that the $e_k(q)$ move at a constant velocity, a result which has been derived in many different ways earlier.

### C. Integrals of motion for all Hamiltonians

From the formalism developed above, it is straightforward to derive $2N-1$ integrals of motion for an arbitrary Hamiltonian $h_k(p, q)$. First, we have all the $h_j(p, q)$ for $1 \leq j \leq N$. To obtain the other $N - 1$ integral of motion, we use the solution given by (54) in subsection IV B. Using this, we can write the initial conditions $\epsilon(0)$ as a function of the time $t$ and the $\epsilon(t)$ as follows:

\[
\epsilon(0) = \exp \left( -t A^{(k)} \right) \epsilon(t). \tag{57}
\]

Since the $\epsilon(0)$ are clearly conserved quantities, we have a set of $N$ time-independent conserved quantities. We require a way to eliminate the time-dependence. This is readily done using the observation made in Section III: the total momentum $P$, defined in (34), is proportional to the time. To be precise, one has from (38) that the quantity

\[
\tau = \frac{P}{(N - k + 1) h_{k-1}} \tag{58}
\]

differs from $t$ by an additive constant, so that

\[
\Phi(p, q) = \exp \left[ -\tau(p, q) A^{(k)}(h(p, q)) \right] \epsilon(q) \tag{59}
\]
is a vector of time-independent conserved quantities of $h_k(p, q)$. They are never functionally independent, but
exactly one can always be expressed in terms of all others, thereby leading to the desired result.

The above technique fails in the case of \(h_1(p, q)\), since \(P\) is then a conserved quantity. In that case, a different approach leads more rapidly to an equivalent result: the quantities

\[
\Lambda_{k,l} = e_k(q)h_l(p, q) - e_l(q)h_k(p, q)
\]

all commute with \(h_1(p, q)\), as is readily checked using property (1) of the Poisson bracket, as described in subsection IV A. These are functionally independent of the \(h_j(p, q)\) and the functions \(\Lambda_{k,l}\), for \(2 \leq k \leq N\) are also functionally independent from one another, so that we have found \(2N-1\) integrals of motion.

D. Two other sets of exactly solvable Hamiltonians

The following set of Hamiltonians

\[
\tilde{h}_k(p, q) = h_k(p, q) + \alpha e_k(q)
\]

all commute with each other, as straightforwardly follows from the symmetry property of the Poisson bracket described in subsection IV A. All these Hamiltonians are therefore integrable in the sense of Liouville. Note that this result can also be obtained directly from (13) by showing that the polynomial \(\tilde{H}(p, q)\) defined by

\[
\tilde{H}(q_k, p, q) = e^{p_k} + \alpha q_k^N
\]

indeed defines a Poisson-commuting family, that is, that it satisfies (15). This shows that there is a large variety of possible solutions of (15) beyond the range of the family defined in (20).

Let us now first consider \(\tilde{h}_1(p, q)\). Additionaly to the integrals of motion \(\tilde{h}_k(p, q)\), we find also \(h_k(p, q)/h_1(p, q)\). The integration then becomes entirely straightforward: \(h_1\) satisfies the equation

\[
\dot{h}_1(p, q) = -\alpha h_1(p, q).
\]

If we denote by \(\rho_k\) the constant value \(h_k/h_1\), then the \(h_k\) are determined as \(\rho_k h_1\), whereas, if \(\tilde{h}_k\) denotes the constant value of \(\tilde{h}_k(p, q)\), the \(e_k(q)\) can be obtained directly from the \(\tilde{h}_k\). Note further that, if \(\alpha\) is a pure imaginary number \(i\omega\), then \(\tilde{h}_1(p, q)\) is an isochronous Hamiltonian, since in that case, \(h_1\) is periodic with period \(2\pi/\omega\). The \(e_k(q)\) and the \(h_1(p, q)\) therefore execute a periodic motion, implying that the \(q_k\) do so as well, though possibly with a larger period, see (23).

However, the equations of motion for the Hamiltonians \(\tilde{h}_k(p, q)\) for \(2 \leq k \leq N\), cannot be integrated in the same way, nor have I been able to identify for these a full set of \(2N-1\) sets of integrals of motion.

On the other hand the more general Hamiltonian

\[
H(p, q) = \sum_{k=1}^{N} \lambda_k h_k(p, q) + \mu e_1(q)
\]

can be solved explicitly: once more, the quantities \(h_k(p, q)/h_1(p, q)\) are constants of motion, and further

\[
\dot{h}_1(p, q) = -\mu h_1(p, q),
\]

so that we can compute explicitly all \(h_k\). The equations of motion for the \(e_k\) are

\[
\dot{e}_k(q) = \sum_{l,m=1}^{N} c^{(k)}_{l,m} e_l(q) h_m(p, q),
\]

where the \(c^{(k)}_{l,m}\) are constants, as follows from the closure property. If we now change the time variable to \(\tau\) defined by

\[
d\tau = h_1 dt = h_1(0)e^{-\mu t},
\]

one obtains

\[
\frac{de_k(q)}{d\tau} = \sum_{l,m=1}^{N} c^{(k)}_{l,m} e_l(q).
\]

As a function of \(\tau\), the \(e_k\) are thus the solutions of a linear system of ordinary differential equations, which can be solved in an elementary way. Again, if \(\mu\) is a purely imaginary number \(i\omega\), the system is isochronous with a period of \(2\pi/\omega\).

V. CONCLUSIONS

Summarising, we present an elementary approach to obtaining sets of integrable Hamiltonians. Our main result can be summarised as saying that all Hamiltonians of the form stated in (20) are integrable. The results of Subsection I C show that one can also reduce the problem explicitly to quadratures. This class of Hamiltonians includes, in particular, some forms of the Ruijsenaars–Schneider\textsuperscript{18} Hamiltonian as well as the goldfish Hamiltonian introduced in. For the latter, as shown in Section IV, a large number of additional results can be derived, such as an explicit description of the \(2N-1\) constants of motion corresponding to the maximally superintegrable nature of the dynamics.

ACKNOWLEDGMENTS

I wish to acknowledge frequent conversations with Francesco Calogero, as well as with A.V. Mikhailov and A. Pogrebkov in the framework of the Meeting on Integrable and Quasi-integrable Systems organized by the Centro Internacional de Ciencias from November 14th to December 9th 2016, as well as UNAM–DGAPA–PAPIIT IN103017 and CONACyT 254515 for financial support.
Appendix A: Calculations leading to (13)

Starting from (12), it is readily seen that

\[ C(q_k, q_l|p, q) = \sum_{r,s=1}^{N} \left[ a_r(p, q)q_k^{N-r}, b_s(p, q)q_l^{N-s} \right] \]

\[ - \left( q_k^{N-r}b_s(p, q)q_l^{N-s} \right) a_r(p, q) \]

\[ - \left( a_r(p, q)q_k^{N-r}, q_l^{N-s} \right) b_s(p, q) \]  

(A1)

Here the first term arises from the definitions of \( \alpha_k(p, q) \) and \( \beta_l(p, q) \) as given in (26) and (10b) respectively. The second term is further evaluated by noting that

\[ \left\{ q_k^{N-r}, b_s(p, q)q_l^{N-s} \right\} = - (N - r)q_k^{N-r-1}q_l^{N-s} \frac{\partial b_s(p, q)}{\partial q_k} \]  

(A2)

with a similar result for the third term. We therefore finally find:

\[ C(q_k, q_l|p, q) = \left\{ \alpha_k(p, q), \beta_l(p, q) \right\} \]

\[ + \sum_{r,s=1}^{N} \left[ (N - r)a_r(p, q)q_k^{N-r-1} \frac{\partial b_s(p, q)}{\partial p_k} q_l^{N-s} \right] \]

\[ - (N - s)b_s(p, q)q_l^{N-s-1} \frac{\partial a_r(p, q)}{\partial p_l} q_k^{N-r} \]  

(A3)

which readily yields the final result (13).

Appendix B: Computation of the Poisson bracket (42) of \( \mathcal{E}(z|q) \) with \( \mathcal{H}(w|p, q) \) for the goldfish Hamiltonian

In the following we apply formula (13) to the polynomials \( \mathcal{H}(w|p, q) \) and \( \mathcal{E}(z|q) \), in this order. The first term is given by

\[ \left\{ e^{p_i}, q_k \right\} = Nq_k^{N-1}e^{p_i} \delta_{k,l} \]  

(B1)

The second term vanishes, since

\[ \frac{\partial \mathcal{H}(z|p, q)}{\partial z} \bigg|_{z=q_i} \frac{\partial \mathcal{E}(q)}{\partial p_l} = 0 \]  

(B2)

This recursion allows to express any Poisson bracket between an \( e_k(q) \) and an \( h_l(p, q) \) as a sum of products of \( e_r(q) \) and \( h_s(p, q) \). More explicitly we have

\[ \left\{ h_k(p, q), e_{l+1}(q) \right\} = \sum_{m=l}^{k+l-1} \left[ e_{k+l-m}(q)h_m(p, q) - e_m(q)h_{k+l-m}(p, q) \right] + h_{k+l}(p, q) \]  

(C5)
Note finally that the recursion (C4) in its present form only holds for $1 \leq k, l \leq N - 1$. When, for example, $k = 0$, the expression $\{h_1, e_l\}$ arises, which is equal to $h_l$ and thus not quadratic in the $e$ and $h$. To extend the recursion appropriately, it is sufficient to choose $e_0(q) = -1$ and $h_0(p, q) = 0$. With these values the recursion extends to the range $0 \leq k, l \leq N - 1$ and allows to express any Poisson bracket of the $e$ with the $h$ as a sum of products of $e_k$ with $h_l$, with $0 \leq k, l \leq N$.

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