Chiral perturbation theory, dispersion relations and final state interactions in $K \rightarrow \pi \pi$

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We discuss the recent literature on the treatment of final state interactions in $K \rightarrow \pi \pi$. Various approaches are compared and particular emphasis is given to the possibility of combining dispersive methods with lattice input. Recent results on the dependence of various quantities on the quark masses at order $p^3$ in the chiral expansion are presented and the relevance for the lattice calculations is discussed.

1. INTRODUCTION

Ideally, lattice QCD calculations should provide numbers that can be directly compared to the experimental ones. In practice, one often needs extra theoretical input and/or treatment before being able to compare to the phenomenology. One important example of this is the $K \rightarrow \pi \pi$ amplitude: until the recent work by Lellouch and Lüscher [1] it was not even known how one could directly calculate this amplitude on the lattice, and the problems of principle related to a naive approach had been identified by Maiani and Testa in a famous no-go theorem [2]. The standard method to work around this no-go theorem consists in calculating on the lattice an unphysical matrix element, $K \rightarrow \pi$, and to use an approximate relation (based on tree-level chiral perturbation theory (CHPT)) to obtain the $K \rightarrow \pi \pi$ amplitude [3]. The latter step is known to represent a rather crude approximation, and sizeable higher order CHPT corrections are expected. Alternative methods to obtain the $K \rightarrow \pi \pi$ amplitude were discussed in [4]. Despite its theoretical cleanliness, the use of the Lellouch–Lüscher method represents a formidable challenge, and is still out of reach for today’s state-of-the-art numerical simulations. Indeed all lattice collaborations [4] that have presented results for the $K \rightarrow \pi \pi$ amplitude at this conference have used either the method of Bernard et al. [3] or one of those proposed in [4].

In view of this, it is worth to ask whether one could devise a better way to obtain the $K \rightarrow \pi \pi$ amplitude from quantities that are calculable today on the lattice. For example, since today’s calculations are all done in the quenched approximation, one may try to find a way to obtain information about the unquenched results from (partially) quenched calculations. Work in this direction has already been done, see e.g. Ref. [6], but I will not touch this issue here. Another very important issue is that of final state interactions (FSI): if one calculates the $K \rightarrow \pi$ matrix element and obtains the $K \rightarrow \pi \pi$ amplitude by using tree level CHPT, one is neglecting FSI. Although these unitarity effects are formally of higher order in the chiral counting, they are known to be particularly large when two pions interact in the $S$ wave in the isospin zero channel, and might be crucial in the final numerical result. The role of FSI has been discussed at length in the recent literature, particularly in connection to the calculations of $\epsilon'/\epsilon$ in the Standard Model [7–11]. The emphasis of the recent literature is on the dispersive treatment of the FSI, but all the groups that calculated $\epsilon'/\epsilon$ making explicit use of the chiral expansion beyond leading order, did also find large corrections due to FSI [12,13]. Earlier treatments of $K \rightarrow \pi \pi$, and in particular of the CP conserving part of the amplitude also discussed the importance of FSI, and its role for the $\Delta I = 1/2$ rule [14,15].

In this talk I will review the recent and the less recent literature on the treatment of FSI in
$K \to \pi\pi$. I will do so by comparing two different approaches: one which explicitly relies on CHPT, and one that uses only dispersion relations. A third possibility, which is also discussed, consists in a dispersion-relation inspired improvement of a chiral calculation. The focus of this discussion will be on the methods, rather than on the numerical results for the $K \to \pi\pi$ amplitude, which I will not touch at all. A second aspect that I will discuss in this talk is the dependence of some quantities on the quark masses, as it is evaluated in CHPT at next-to-next-to-leading order. I will present the analyses which are currently available [14,17], for masses, decay constants and scattering amplitudes. The latter result is directly connected to the problem of FSI: it has been shown that large FSI are also reflected into a very strong dependence on the quark masses [10].

2. THE SCALAR FORM FACTOR: CHPT vs. DISPERSIVE METHODS

In order to compare the chiral expansion and the dispersive approach, it is easier to consider a simpler quantity than the $K \to \pi\pi$ amplitude – if we want two pions in the $S$ wave and isospin zero in the final state, the simplest quantity we can think of is the scalar form factor of the pion:

$$\Gamma(s) = \mathcal{N}(\pi(p_1)\pi(p_2)|uu + \bar{d}d|0), \quad (1)$$

where $s = (p_1 + p_2)^2$ and $\mathcal{N}$ ensures $\Gamma(0) = 1$. In CHPT this has been calculated (and studied numerically) to two loops [18], and has also been analyzed with dispersive methods [19], so that we can immediately compare the two results. We first consider the CHPT calculation, which reads, to one loop [20]

$$\Gamma(s) = 1 + \frac{s}{N}\ln(\frac{4}{\ell_4}) + \frac{1}{2F_\pi^2}(2s - M_\pi^2)\tilde{J}(s) + O(s^2), \quad (2)$$

where $N = 16\pi^2F_\pi^2$, and $\tilde{J}(s)$ is a loop function [20]. In order to compare to the dispersive treatment, it is useful to rewrite this expression in a different way, that makes the $\pi\pi$ phase shift appear explicitly

$$\Gamma(s) = 1 + cs + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta^{(2)}(s')}{s'^2(s' - s)} + O(s^2), \quad (3)$$

where $c$ is a constant related to $\ell_4 \sim \ln M_\pi^2$, and where $\delta^{(2)}$ is the leading order $\pi\pi$ phase shift in the $S$ wave and $I = 0$ channel

$$\delta(s) = \delta^{(2)}(s) + \delta^{(4)}(s) + \ldots$$

$$\delta^{(2)}(s) = \frac{\pi\sigma(s)}{2N}(2s - M_\pi^2), \quad (4)$$

where $\sigma(s) = \sqrt{1 - 4M_\pi^2/s}$. In order to have a numerical representation in CHPT to one loop, one needs to fix one constant, $\ell_4$. Once this is done, e.g. by fixing the derivative of $F(s)$ at $s = 0$ at its physical value (see below), the form factor is as shown by the dashed line in Fig. 1. The same kind of analysis can be repeated to two loops – there, however, one has to fix more constants [18]. In Fig. 1 it is clearly seen that while the chiral expansion behaves nicely up to around $s = (0.4 \text{ GeV})^2$, at higher energies it does not work so well. In particular, the behaviour of the series at $s = M_K^2$ looks as follows:

$$\text{1 (LO)} \to 1.62 \text{ (NLO)} \to 1.67 \text{ (NNLO)}, \quad (5)$$

![Figure 1. Scalar form factor: comparison of the CHPT calculation to leading, next-to-leading and next-to-next-to-leading order and the dispersive one (see legend). The solid curve is the “true form factor”, see text.](image-url)
whereas the “true value” is 1.42. At slightly higher values, the two loop form factor does bend down, as the true form factor does – however, at a quantitative level it does not give a satisfactory representation. It is not difficult to understand why two loops are not yet enough to reproduce correctly the form factor at these energies. The real part of the form factor is proportional to the cosine of the phase

\[ \text{Re} \Gamma(s) \sim \cos(\delta(s)) \sim 1 - \frac{\delta(s)^2}{2} + \ldots, \]  

and since the phase is a quantity of order \( p^2 \) in the chiral expansion, this negative contribution to the real part starts only at NNLO. Numerically, at NNLO the cosine becomes different from one, but it is still larger than the real value by about 20\% – which roughly explains the difference seen in Fig. 1.

The dispersive treatment for the form factor is quite simple: Omnès’ \[21\] tells us that if one knows the phase \( \phi \) of the form factor (on the positive real axis, from \( s = 4M_K^2 \) up to infinity), one knows the form factor in the whole complex plane. Its explicit expression reads

\[ \Gamma(s) = \exp \frac{s}{\pi} \int_{4M_K^2}^{\infty} ds' \frac{\phi(s')}{s'(s'-s)} . \]  

In principle one should worry about the presence of zeros, and consider a corresponding polynomial factor, but for the sake of simplicity we will disregard this possibility. We also assume that the integral in Eq. (7) converges.

Note that what enters in the Omnès function is the phase of the form factor – according to Watson’s theorem, below the inelastic threshold this is equal to the \( \pi\pi \) phase shift in the S wave and isospin zero. Above that, however, it cannot be easily related to observable quantities. Note also that although the inelastic threshold is formally at \( s = 16M_K^2 \), the \( 4\pi \) channel is very weak, and can as well be neglected. Around 1 GeV the much stronger \( 2K \) channel opens up, and one cannot neglect inelastic effects anymore.

In fact what we have indicated until now as “the true form factor” is nothing but the result of a dispersive treatment which explicitly takes into account the contribution of the \( 2K \) channel, in a fully consistent manner (coupled channel analysis), and using the available data \[19\]. In that case, one cannot express the form factor as simply as in Eq. (8). However, even neglecting the inelastic channels, and using the \( \pi\pi \) phase shift, e.g. as given in \[16\], all the way up to a cut-off, one gets a decent description of the form factor, as shown in Fig. 1 where the cut off Omnès function

\[ \Omega(s) = \exp \frac{s}{\pi} \int_{4M_K^2}^{\Lambda^2} ds' \frac{\delta(s')}{s'(s'-s)} , \]  

for \( \Lambda = 1.4 \) GeV is plotted. The precise value of the cut-off does matter for the final result, but not too much, as long as one is not taking it too low.

It is interesting to compare at the algebraic level CHPT to one loop and the Omnès representation: in order to do so we have to expand the latter in a chiral series – this is done by expanding the phase of the form factor, and correspondingly, the exponential. If we do so, we end up exactly with Eq. (7). Note that if we simply insert the leading order chiral phase, Eq. (4), in the dispersive integral (7) we get a divergent result, and therefore are forced to subtract it at least once more, as in Eq. (8). Note also that the the dispersive representation explains also the presence of a chiral logarithm in the constant \( c \) in Eq. (8); with the leading order phase the once-subtracted dispersive integral is infrared divergent and explodes logarithmically as the pion mass goes to zero.

In summary, in the case of the scalar isoscalar form factor at energies around \( s = M_K^2 \) it is clear that the chiral expansion is less economical, and gives a poorer description than the dispersive treatment, even if the latter is done neglecting inelastic effects. This motivates us to apply a dispersive treatment also to the \( K \to \pi\pi \) amplitude.

3. THE \( K \to \pi\pi \) AMPLITUDE

It is easy to see an analogy between the \( K \to \pi\pi \) amplitude and the scalar form factor discussed in the previous section, as far as FSI are concerned. On the other hand, the physical \( K \to \pi\pi \) amplitude does not depend on any kinematic-
cal variable, and the dispersive treatment of the scalar form factor cannot be immediately applied to this amplitude: first we have to extend the definition of the $K \to \pi \pi$ amplitude to make it depend on one (or more) complex variables. There is no unique way to do so, and we will discuss two different options. The chiral calculation, however, does not need this extension, and the corresponding treatment of FSI can be discussed directly for the physical amplitude.

3.1. CHPT treatment of FSI

In CHPT to one loop the $K \to \pi \pi$ amplitude reads as follows

$$A = A^{(2)}\left\{1 + \frac{1}{2F_{\pi}}(2M_{K}^{2} - M_{\pi}^{2})\bar{J}(M_{K}^{2})\right\} - \frac{2M_{K}^{2} - M_{\pi}^{2}}{2N}\left(\ln\frac{M_{\pi}^{2}}{\mu^{2}} + 1 + \ldots\right),$$

where $A^{(2)}$ is the leading order amplitude, and where we have written explicitly only the once subtracted loop function generated by the two-pion exchange, and the accompanying chiral log. The ellipsis stands for other loop functions and counterterms. The problem with this amplitude is that in order to use it for numerical calculations one has to input values for the weak (and strong) counterterms, which are largely unknown, especially if one wants to identify the contributions of individual operators in the weak Hamiltonian. Also, the identification of the contribution of a certain loop diagram is ambiguous, not only because of the presence of the scale $\mu$, but also because the break down into individual loop diagrams depends on the parametrization chosen for the fields in the effective lagrangian. Note, e.g., that logs coming from other loop diagrams have to cancel the term proportional to $M_{K}^{2}\ln M_{\pi}^{2}$ in $A^{(2)}$, so that the full amplitude has a well defined $m_{u} = m_{d} \to 0$ limit. Despite these ambiguities, one may find instructive to evaluate the pion loop contribution in $A^{(2)}$: for $\mu \sim M_{\pi}$ this contribution amounts to about 40%, which is indeed quite large. On this basis, it has often been stated that in CHPT the main one-loop correction to the $K \to \pi \pi$ amplitude comes from $\pi \pi$ FSI [13,14,15]. We stress that, because of the ambiguities mentioned above, this statement cannot be made very precise.

Note the analogy to the case of the scalar form factor: indeed Eq. (9) looks completely analogous to Eq. (8), for $s = M_{K}^{2}$. However, while in that case we could fix the counterterm and the accompanying log (both contained in $\bar{J}$), with the value of the scalar radius of the pion, here we do not have a corresponding quantity to use as input. Numerical statements about this amplitude depend in a crucial way on a theoretical estimate of the low energy constants involved.

3.2. Dispersive treatment: $K$ off-shell

In view of the analogy between Eq. (9) and Eq. (8) for $s = M_{K}^{2}$, it is tempting to make the momentum squared of the kaon the variable on which to construct the dispersion relation. This idea has been proposed by Truong [14], and recently revived by Pallante and Pich [16]. Although the idea is seductively simple, it implies a number of serious problems, all related to the fact that the off-shell extrapolations of a physical hadronic amplitude are arbitrary, in contrast to on-shell amplitudes, which are unambiguous and well-defined. A discussion of the effects of this arbitrariness can be found in Ref. [1], and the interested reader is referred to that paper for further details. Both papers [14,15] ignore this arbitrariness, and choose an unphysical off-shell extrapolation for the kaon. In fact both papers quote for the “off-shell amplitude” at tree level in CHPT

$$A^{(2)}(s) = A(s - M_{K}^{2})^\nu,$$

where $A$ is a constant. Such an extrapolation corresponds to using the kaon field in the effective lagrangian for going off-shell. This choice is unphysical: one could add a term that vanishes at the equations of motion (and hence does not affect any physical observable), and correspondingly change the off-shell amplitude with terms proportional to $s - M_{K}^{2}$. There is no way to prefer one choice over the other as long as one is using the fields in the effective lagrangian: it is well known that these do not have any physical significance and are simply integration variables in the path integral. The calculation of the one-loop corrections to $A^{(2)}$, as done in Ref. [2], makes the sickness of this extrapolation even
more evident: a standard basis of the $O(p^4)$ counterterms, as e.g. given in [22], does not suffice to reabsorb the divergences of the one loop diagrams. Off-shell extrapolations can be unambiguously defined by using quark operators, such as the axial current or the pseudoscalar density – in that case, the off-shell amplitude to leading order is different from [10], and does not change if one adds to the lagrangian terms that vanish at the equations of motion [11]. One-loop corrections can also be unambiguously calculated.

In the approach of Refs. [14,7] introducing the off-shell extrapolation is unnecessary, and, as discussed above, misleading. At tree level, their procedure amounts to multiplying the on-shell tree level amplitude by the Omn`es function evaluated at $s = M_K^2$: $A^{(2)} \rightarrow A^{(2)} \Omega(M_R^2)$. At one loop one can extend this procedure in the following manner [8]:

$$ A^{(2)+(4)} \rightarrow \left[A^{(2)+(4)} - \Delta^{(2)}(M_R^2)\right] \Omega(M_R^2) \ , \ (11) $$

where $\Delta^{(2)}(M_R^2)$ is the $O(p^2)$ part of the Omn`es function

$$ \Delta^{(2)}(s) = \frac{s}{\pi} \int_{4M^2_\pi}^{M^2} \frac{ds'}{s'(s' - s)} \ , \ (12) $$

which needs to be subtracted from the one-loop amplitude to avoid double counting. Note that since the $O(p^2)$ phase shift grows linearly with $s$ [10], the dispersive integral [22] diverges when $\Lambda \rightarrow \infty$ – the final result depends on the choice of the cut-off.

While the effect due to FSI is quite substantial if compared to the tree level chiral amplitude, the exponentiation of the FSI part of the one loop correction gives a much less dramatic increase. If we use a cut-off $\Lambda = 1 \ (1.3) \ \text{GeV}$, we get $\Delta^{(2)}(M_R^2) = 0.3 \ (0.4)$, and $\Omega(M_R^2) = 1.4 \ (1.65)$. The choice of the cut-off gives an uncertainty which is comparable to the increase due to the exponentiation. Moreover, to obtain a numerical estimate from such a calculation, one still needs an estimate for the $O(p^4)$ counterterms [8], as in pure CHPT. The necessity to refer to the chiral expansion and to its many constants is, in my opinion, the main disadvantage of this method. While it is economical and gives a quick estimate of the effect of FSI if one refers to tree level CHPT, it becomes much more involved if one extends it to one loop. In addition, at one loop there remain large uncertainties.

3.3. Dispersive treatment: momentum-carrying Hamiltonian

The necessity to refer to CHPT disappears if one chooses a different extrapolation of the physical amplitude, and allows the weak Hamiltonian to carry momentum, as has been suggested in [10]. The method goes as follows: consider the amplitude

$$ t=0\langle\pi(p_1)\pi(p_2)|H_W^{1/2}(0)|K(q_1)\rangle =: T^+(s, t, u) \ (13) $$

with the Mandelstam variables $s = (p_1+p_2)^2$, $t = (q_1-p_1)^2$, $u = (q_1-p_2)^2$, related by $s + t + u = 2M^2 + M^2_R + q^2$, where $q_2$ is the momentum carried by the weak Hamiltonian. The physical decay amplitude is obtained by setting $q^2 = 0$ ($s = M_K^2$, $t = u = M^2_R$).

The dispersive treatment of a function of three complex variable is rather complicated. However, the problem simplifies considerably if one neglects the contribution of the imaginary parts of $D$ and higher waves. With this approximation, the amplitude can be decomposed into a combination of functions of a single variable:

$$ T^+(s, t, u) = M_0(s) + \left\{ \frac{1}{3} \left[ N_0(t) + 2R_0(t) \right] + \frac{1}{2} \left( s - u - \frac{M^2_\pi\Delta}{t} \right) N_1(t) \right\} + \left\{ (t \leftrightarrow u) \right\} \ (14) $$

where $\Delta = M^2_R - M^2_N$.

In this approximation, the dispersion relation for the full amplitude is transformed into a set of coupled dispersion relations of functions of a single variables. These can be solved numerically, as shown in [10]. An important point concerns the overall number of subtraction constants which is two for the amplitude $T^+$. In order to solve the dispersion relations and calculate the $K \rightarrow \pi \pi$ amplitude, one has to provide as input the two subtraction constants. One of the two can be given in terms of the $K \rightarrow \pi \pi$ amplitude – a soft-pion theorem relates the amplitude at the
soft pion point \((s = u = M_\pi^2, t = M_K^2)\) to the \(K \to \pi\) amplitude up to terms of order \(M_\pi^2\):

\[
-\frac{A(K \to \pi)}{2F_\pi} = a + \frac{N}{3} + O(M_\pi^2),
\]

(15)

where \(N = N_0(M_K^2) + 2R_0(M_K^2)\). Notice that although the process involves a kaon, the relation is based on the use of the \(SU(2)\) symmetry, and therefore suffers from corrections of order \(M_\pi^2\) only. The existence of this soft-pion theorem is a crucial advantage of this approach, because it simplifies the calculation of one of the two subtraction constants. The key of the problem is how to calculate \(b\). This constant is related, e.g., to the derivative in \(s\) of the amplitude \(T^+\) at the soft pion point. In [16] a Ward identity that relates this derivative to a \(K \to \pi\) matrix element was derived. The calculation of this matrix element would determine \(b\). Alternatively, one could try to obtain information on the \(K \to \pi\pi\) amplitude at another unphysical point (i.e. with meson masses at their physical values but with the weak Hamiltonian carrying momentum) and use this input to constrain the second subtraction constant.

In the absence of calculations of \(b\), the numerical results were illustrated by fixing \(b\) at a certain value and then varying it within a fairly wide range. To fix the central value leading order CHPT was used:

\[
b = \frac{3a}{M_K^2 - M_\pi^2}(1 + X + O(M_K^2)) \]

(16)

The size of the correction \(X\) is at the moment unknown, but nothing protects it from being of order \(M_K^2\). In the numerical analysis the generous range \(X = \pm 30\%\) has been used. The results are shown in Fig. 2, where \(|T^+(s, M_K^2 + M_\pi^2 - s, M_\pi^2)|\) versus \(s\) is plotted, comparing the numerical solution of the dispersion relations to the CHPT leading order formula. The latter is what has been used so far whenever a number for the \(K \to \pi\pi\) matrix element extracted from the lattice has been given. The figure shows that large corrections with respect to leading–order CHPT are to be expected. One source of large corrections is the Omnès factor due to \(\pi\pi\) rescattering in the final state. The other potentially dangerous source is represented by \(X\), the next-to-leading order correction to the relation (16) between \(a\) and \(b\). The latter could (depending on the sign) in principle double, or to a large extent reabsorb the correction due to final state interactions.

It is clear that in order to obtain a calculation of \(K \to \pi\pi\) with this method one needs to determine these two constants – and the necessary information is not yet available. On the other hand, it is important to emphasize the advantages of this method with respect to CHPT, or to the Omnès improved version of it: here one does not need to introduce a large number of constants, but only needs two inputs. The relation between these constants and the \(K \to \pi\pi\) amplitude is exact and it is provided by the solution of the dispersion relation. The accuracy of the final result will depend on the accuracy with which the two constants are known and the accuracy in the solution of the dispersion relation. As far as the latter is concerned the prospects of reducing it are quite good: the \(\pi\pi\) phase shifts, which are dominating \([16]\) are known with a remarkable accuracy \([16]\). The numerical analysis performed so far \([16]\) will be improved by taking into account...
the effects of the inelastic contributions in the $s$ channel \cite{23}.

4. CHPT RESULTS ON QUARK MASS DEPENDENCE

In CHPT the dependence on the quark masses is explicit: the low energy constants of the effective lagrangian are quark-mass independent. Therefore, once these are known, one can vary the quark masses and see how any given quantity depends on them. This exercise is particularly interesting from a lattice perspective, since the physical up and down quark masses are still far from the reach of today’s calculations, and one may wish to learn beforehand what to expect, when those values are approached. To make this exercise worthwhile it is important that one goes beyond next-to-leading order: only in this manner can one check how fast the series is converging and where one should stop believing the result of the chiral expansion. In the last few years there have been a few detailed numerical analyses at next-to-next-to-leading order both in the $SU(2) \times SU(2)$ and in the $SU(3) \times SU(3)$ framework, and I will briefly review the results concerning the quark mass dependence.

4.1. Chiral $SU(2) \times SU(2)$

In Ref. \cite{16} the low energy constants appearing in the $\pi\pi$ scattering amplitude were determined by comparing the chiral representation thereof to a phenomenological one that follows from Roy equations. The numerical solution of these \cite{24} provided the necessary input for the determination. It has to be stressed that in this procedure, only the constants that govern the momentum dependence of the amplitude could be determined – those that give the quark mass dependence had to be fixed with theoretical estimates, and as such are subject to larger uncertainties. In Fig. 3 the dependence of the two $S$ wave scattering lengths on the pion mass is shown. The scattering lengths are divided by their leading order value. The figure shows that large deviations from leading order occur rather early, particularly for the $I = 0$ case. This is an effect of the particularly strong attractive interaction in the $I = 0$ channel, as opposed to the much weaker and repulsive one in the $I = 2$ channel. The plot shows that lattice calculations of these quantities should be made down to rather small values of the quark masses if a reliable extrapolation to the physical value has to be made. In comparison, quantities that do not involve two pions in the final state, like the pion mass and decay constant, show a much milder dependence, see Fig. 4. There are calculations of the pion scattering lengths already available in the literature, particularly for the $I = 2$ channel which is considerably easier. Unfortunately they are all done at rather high quark masses. The two most recent ones are in Ref. \cite{25}, where reference to earlier calculations can also be found.

The strong quark mass dependence of the $I = 0$ scattering length sounds as a warning also for future direct calculations of the $K \to \pi\pi$ amplitude. In that case the FSI are even stronger than for the $\pi\pi$ amplitude at threshold (at $s = M_K^2$ the phase in the $I = 0$ channel is around 40°), and the quark mass dependence should presumably be similarly strong – implying again the need for simulations at rather small values of quark masses, before a reliable extrapolation to the physical value can be
Figure 4. Dependence of $F_\pi$ and $M_\pi$ on the average up and down quark masses $M = 2\bar{B}\tilde{m}$.

4.2. Chiral $SU(3) \times SU(3)$

The calculations of quantities at next-to-next-to-leading order in $SU(3) \times SU(3)$ CHPT are much more complicated, and the convergence of the series is certainly slower than for chiral $SU(2) \times SU(2)$. Recently, an analysis of $K_{e4}$ decays at this order has been performed [17]. In this work all the low energy constants of order $p^6$ have been estimated with resonance saturation, whereas the $O(p^4)$ constants have been fitted by comparing to the data on $K_{e4}$ form factors. Again, having fixed all the constants, the authors were also able to consider the quark mass dependence of various quantities, and particularly of meson masses and decay constants. These are shown in Fig. 5 and 6. The results of this paper are puzzling as far as the convergence of the series for masses and decay constants are concerned: the authors find that the $O(p^6)$ contributions are systematically larger than the $O(p^4)$ ones, as one can see from Figs. 5 and 6. It is also surprising how strongly the decay constants depend on the strange quark mass, including the one of the pions. The reader is referred to the original paper [17] for a discussion of the uncertainties involved in the determination of the various low energy constants – as stressed above, those governing the quark mass dependence of meson masses are the most difficult ones to control. In view of this, these plots should rather be viewed as an indication of the qualitative behaviour, rather than precise quantitative statements. It would of course be very interesting if lattice calculations could contribute and calculate the dependence on the strange quark mass of various meson masses and decay constants.

5. SUMMARY

In this talk I have reviewed and critically discussed the recent literature on final state interactions in $K \to \pi\pi$, and a few examples of quark mass dependence of various quantities as calculated in CHPT at next-to-next-to-leading order. I believe that the few issues discussed here are good examples of how important the exchange of information between the lattice and the chiral communities is at present – and of how important it will become in the future.
Figure 6. Dependence on the strange quark mass of the meson decay constants, from [17]

ACKNOWLEDGEMENTS

It is a pleasure to thank M. Büchler, J. Kambor and F. Orellana for a pleasant collaboration on the subject discussed here, and the organizers for the invitation and the perfect organization of the conference. The hospitality at the Institute for Nuclear Theory at the University of Washington (Seattle), where this work was completed, is gratefully acknowledged.

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