Nonradial mode excitation as the cause of the Blazhko effect in RR Lyrae stars

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ABSTRACT

A significant fraction of RR Lyrae stars exhibits amplitude and/or phase modulation known as the Blazhko effect. The oscillation spectra suggest that, at least in most of the cases, excitation of nonradial modes in addition to the dominant radial modes is responsible for the effect. Though model calculations predict that nonradial modes may be excited, there are problems with explaining their observed properties in terms of finite amplitude development of the linear instability. We propose a scenario, which like some previous, postulates energy transfer from radial to nonradial modes, but avoids those problems. The scenario predicts lower amplitudes in Blazhko stars. We check this prediction with a new analysis of the Galactic bulge RR Lyrae stars from OGLE-II database. The effect is seen, but the amplitude reduction is smaller than predicted.

1 Introduction

Explaining periodic, or nearly periodic, long term modulation of the light curve seen in many RR Lyrae stars - known as Blazhko effect - constitutes perhaps the greatest challenge to stellar pulsation theory. We cannot claim that we understand these most classical pulsating stars until this effect is explained. The idea that excitation of nonradial mode may be responsible for the effect has been vaguely suggested long time ago (Stellingwerf, 1976). More specific suggestions regarding excitation mechanism came later (Cox, 1993; Kovács; 1993; Van Hoolst & Waelkens; 1995) and these were followed by a detailed stability calculations for a realistic RR Lyrae star model carried by Van Hoolst et al. (1998).

A more direct observational evidence that nonradial modes may be excited in RR Lyrae stars was provided by Olech et al.(1999), who found close peaks in oscillation spectra of certain RRc stars. Similar results, for a large number of RRc and RRab stars, were obtained in subsequent investigations (Moskalik, 2000; Alcock et al. 2000; Olech et al. 2001; Kovács 2001; Moskalik & Porreti, 2003; Mizerski, 2003; Soszyński et al. 2003). The decisive observational confirmation that the Blazhko effect is caused by nonradial mode excitation may come only from analysis of line profile changes. Such an analysis has been undertaken by Kolenberg (2002), but the result was not sufficient for a firm conclusion. More data are needed.

The term Blazhko stars is now used to denote all objects whose oscillation spectra exhibit peaks with frequency spacing much closer than spacing between
consecutive radial modes. The following subtypes have been introduced (e.g. Moskalik & Porreti, 2003): RR-\(\nu\)1 stars with two closely spaced frequencies, RR-\(\nu\)2 stars with three nonequidistant frequencies, and RR-BL stars with three equidistant frequencies. The most common are RR-\(\nu\)1 stars. RR-BL stars, in the time-domain, exhibit periodic modulations in various forms. If amplitudes of the two side peaks are equal, then depending on the phase relation, we either see pure phase or pure amplitude modulation. If the amplitudes are not equal, both phase and amplitude are subject to periodic modulation. Both amplitude and phase are also modulated in RR-\(\nu\)1 stars. A quasi-periodic modulation occurs in RR-\(\nu\)2 case if one of the three peaks has much lower amplitude than the others.

We cannot see any alternative to the nonradial mode excitation as the explanation of RR-\(\nu\)1 and RR-\(\nu\)2. In the case of RR-BL, it is perhaps natural to look first for physical interpretation of the modulation period, rather than the small frequency spacing. In most of such interpretations so far the period was associated with rotation, but those interpretations were never supported by a sound observational evidence nor by a correct model calculations. A resonant coupling between radial modes may, in principle, lead to a periodic limit cycle (Moskalik, 1986), but there is no required resonance in realistic RR Lyrae models. Thus, in our work, we consider only those explanations of Blazhko effect that invoke nonradial mode excitation.

As shown by Van Hoolst et al. (1998) and Dziembowski & Cassisi (1999), there are two ways to excite such modes in RR Lyrae stars. One is through opacity driving and the other is through parametric instability of the radial mode to excitation of nonradial modes at nearly resonant frequency (the 1:1 resonance). In both cases the \(\ell = 1\) modes, with frequencies close to radial modes, are most easily excited. However, the latter mechanism was preferred because the growth rate of nonradial modes is much lower than those of radial modes, which should quickly saturate the linear driving effect.

Nonlinear development of the parametric instability was studied with the amplitude equation formalism by Nowakowski & Dziembowski (2001). After adopting certain approximations, of which the most severe was the use of adiabatic eigenfunctions, they found that the instability leads to pulsation involving radial and nonradial mode(s) with constant amplitude. If there is only one nonradial mode, then the nonlinear synchronization leads to monoperiodic pulsation, so that model does not apply to Blazhko stars. If the coupling involves a pair of modes of opposite azimuthal orders, \(m\), in a rotating star, then synchronization enforces equal distances between the side peaks and the central peak. Thus, this model looked like a plausible explanation of the RR-BL stars, but the other subtypes remained unexplained.

A severe problem for any explanation of the Blazhko effect in terms of nonradial mode excitation was discovered later by the same authors (Nowakowski & Dziembowski, 2003). What they found was that at observed surface amplitude the bona fide nonradial modes become strongly nonlinear in deep interior, and should be damped much more than the linear theory implies. This finding certainly invalidates approximations they adopted in their earlier work. The
question is whether the whole idea of nonradial mode excitation in RR Lyrae stars can survive. This is what we are trying to see in this paper.

2 Consequences of strong nonlinearity of nonradial modes in deep interior

Let us consider a low degree nonradial oscillation of frequency close to that of the fundamental radial mode or the first overtone in an RR Lyrae star envelope. Here we restrict our consideration to the deep part of the envelope, where the Brunt-Väisälä frequency, $N$, is much larger than the oscillation frequency. We begin with the linear case summarizing results of earlier investigations (Van Hoolst et al., 1998; Dziembowski & Cassisi, 1999). Locally in this part of the envelope an oscillation mode is described as a superposition of weakly nonadiabatic gravity waves propagating inward and outward, and the radial displacement is accurately described by the following asymptotic formula

$$\xi_r = \frac{C_+ e^{i\Phi} + C_- e^{-i\Phi}}{r^2 \sqrt{\rho k_r}} Y^m_\ell e^{-i\omega t}. \quad (1)$$

where $C_+$ and $C_-$ are constants, $\omega$ is the complex eigenfrequency,

$$k_r = \frac{d\Phi}{dr} = \frac{\sqrt{l(l+1)} N}{r \omega} (1 + iD) \quad (2)$$

is the radial wave number, $D$ describes local dissipation,

$$\Phi = \int^r dr k_r,$$

and the remaining symbols have their standard meaning. Everything we need to know here about $D$ is that it is negative (dissipation), that $|D| \ll 1$ for low degree modes, and that $D \propto \ell(\ell+1)$. The frequency is given by $\omega_R = \Re(\omega)$ and the growth rate by $\gamma = \Im(\omega)$.

There are many unstable low degree nonradial modes for which the inward and outward waves have nearly equal amplitudes. At $\ell = 1$, all modes in the wide frequency range, extending from below fundamental radial mode to above second overtone, are unstable. Spectra of nonradial modes are very dense as the total phase, $\Re(\Phi)$, is large. There are maxima in the $\gamma(\omega_R)$ dependence near the radial mode frequencies, reflecting the effect of a partial mode trapping in the acoustic cavity. However, owing to higher amplitude in the interior, even the best trapped $\ell = 1$ modes have significantly lower growth rates than the $\ell = 0$ modes.

With increasing $\ell$, the instability ranges shrink in a consequence of increasing damping in the interior. Disparity between the outward and inward wave amplitudes increases. The trapping effect is non-monotonic. It is weakest at $\ell = 2$ and this is why the $\ell = 2$ modes are less likely to be seen in RR Lyrae stars. At
certain $\ell$-values, 7 or 8 in the vicinity of the fundamental mode, and 4 or 5 in the first overtone vicinity, strongly trapped unstable modes appear, for which the outward waves have negligible amplitude. Virtually all the wave energy is then dissipated on the way to the edge of the convective core, where the gravity wave is reflected. Still the total rate of dissipation is much smaller than the rate of energy gain via the opacity mechanism acting in the outer layers. This is due to a large inward amplitude decrease in the evanescent zone separating the acoustic and the gravity wave cavities. The growth rates of such modes are large, but because of cancellation in the disc-averaged intensity variations, such modes are not expected to reach detectable light amplitudes.

For these strongly trapped modes we may ignore the outward wave and use the running wave relation,

$$\frac{\partial \xi}{\partial r} = i k_r \xi,$$

as the inner boundary condition to determine complex eigenfrequencies, $\omega$. This condition may be applied anywhere within the envelope, as long as $N \gg |\omega|$. Formally, we may also calculate in the same way the eigenfrequencies for low degree modes. However, these cannot be true solutions of the linear nonadiabatic oscillation problem for the whole star because the condition demanding $C^+ \approx C^-$ at the edge of the convective core cannot be fulfilled. Nonetheless, we will argue that solutions are applicable if there is a strong nonlinearity deeper in the star interior.

The proper measure of gravity wave nonlinearity is

$$\zeta = |\xi H k_r|$$

where

$$\xi_H = \frac{i k_r r \nabla H \xi_r}{\ell(\ell + 1)},$$

denotes horizontal components of the displacement, which are dominant in this case. If at some place $\zeta \approx 1$, we expect wave breaking and a large energy dissipation. The largest values of $\zeta$ always occur in the hydrogen burning shell, where $N/\omega$ approaches $10^3$ and $r/R \sim 10^{-2}$. The criterion for the strong nonlinearity adopted by Nowakowski & Dziembowski (2003) was $Max(\zeta) = 1$. Using nonadiabatic eigenfunctions, they translated $Max(\zeta) = 1$ to the surface amplitude of radius and luminosity variations, from which they evaluated amplitudes of the bolometric magnitude. The resulting values are mode dependent. The highest surface amplitudes are reached by the modes best trapped in the acoustic cavity. At $\ell = 1$, such modes have somewhat higher frequencies and at $\ell = 2$ slightly lower frequencies than the nearest radial modes. The maximum amplitudes depend on the aspect, $i$, through the $Y^m_l(i,0)$ factor. The values quoted by Nowakowski & Dziembowski (2003) refer to aspect averaged amplitudes. The maximum amplitude at the onset of the strong nonlinearity found in that work were about $5 - 10$ mmag at $\ell = 1$ and by an order of magnitude lower at $\ell = 2$. Even the values at $\ell = 1$ are lower than amplitudes of the secondary peaks found...
in Blazhko stars. The numbers were obtained for one representative model of an RR Lyrae star.

We repeated similar calculations for a sequence of RR Lyrae models with $M = 0.67M_\odot$ and $Z = 0.001$ calculated by Santi Cassisi (sequence S2 in Dziembowski & Cassisi, 1999). One sequence was enough because nonradial mode properties are largely determined by the radial mode periods. The calculated amplitudes, $A_{\text{bol}}$, for $\ell = 1$ are given in Table 1. Side peak amplitudes in many Blazhko stars are often above 100 mmag. The values in Table 1 indicate that the problem with strong nonlinearity may be perhaps avoided in the case of first overtone pulsators at the short period end, but certainly not in most of the cases.

Table 1: Maximum amplitudes of $\ell = 1$ modes near fundamental and first overtone radial modes at the onset of strong nonlinearity in a sequence of RR Lyrae models

| $\log T_{\text{eff}}$ | $\log L$ | $Y_e$ | $P$ | $\Delta \nu/\nu$ | $A_{\text{bol}}$ | $P$ | $\Delta \nu/\nu$ | $A_{\text{bol}}$ |
|---------------------|---------|------|-----|-----------------|-----------------|-----|-----------------|-----------------|
|                     |         |      | FUND. |                 |                 | OVER. |                 |                 |
| 3.805               | 1.700   | 0.687 | 0.638 | 0.014           | 2.2             | 0.473 | 0.012           | 6.9             |
| 3.816               | 1.696   | 0.659 | 0.579 | 0.014           | 3.8             | 0.430 | 0.011           | 10.3            |
| 3.827               | 1.692   | 0.626 | 0.526 | 0.016           | 6.6             | 0.390 | 0.017           | 17.9            |
| 3.838               | 1.688   | 0.589 | 0.479 | 0.012           | 11.6            | 0.355 | 0.010           | 33.2            |
| 3.849               | 1.684   | 0.539 | 0.436 | 0.014           | 18.8            | 0.324 | 0.011           | 55.4            |
| 3.860               | 1.681   | 0.453 | 0.401 | 0.009           | 25.8            | 0.298 | 0.012           | 81.5            |

$L$ - luminosity in the solar units, $P$ - period in days, $\Delta \nu/\nu$ - relative distance between radial mode and the nonradial mode with highest amplitude at the onset of the strong nonlinearity, $A_{\text{bol}}$ - the bolometric amplitude in mmag.

Strong dissipation of the inward propagating g-wave in the shell source justifies use of Eq. (3) as the inner boundary condition, which may be applied in the part of stellar envelope, where $|\omega| \ll N$ and $|\zeta| \ll 1$. This means that the linear asymptotics applies. In realistic models of RR Lyrae stars there is always a zone where both conditions are fulfilled. The damping rates for modes calculated with the boundary condition given in Eq.(3) are large, even for the best trapped $\ell = 1$ modes. They are given in Table 2. We see that, if there is no external source of energy, the modes should decay on the time scale of days. The only available source of energy needed to maintain nonradial oscillations are the radial modes, which draw energy from the radiative flux until they reach amplitude at which driving is saturated. The nonlinear reduction of the driving rate may be modeled with a simple expression

$$\gamma = \gamma_0 \left( 1 - \frac{A^2}{A_s^2} \right),$$

where $\gamma_0$ denotes the linear driving rate, $A$ is the mode amplitude (let it be the relative amplitude of the photospheric radius) and $A_s$ is the saturation
amplitude in the absence of other modes. Consider now the energy budget in the situation, when in addition to an unstable radial mode there is a nearby strongly damped nonradial mode of degree $\ell$. If we ignore frequency difference between the radial and nonradial modes and the influence of the nonradial mode on the driving rate, the condition for energy balance leads to the following relation for the amplitudes

$$A_0^2\gamma_{0,0} I_0 \left(1 - \frac{A_0^2}{A_{0,s}^2}\right) = A_\ell^2\gamma_{\ell,0} I_\ell$$

where

$$I_\ell = \int d^3x \rho |\xi|^2,$$

is mode inertia calculated with the standard normalization $\xi_r(R) = Y_m^\ell R^{-\ell} e^{-i\omega t}$.

If saturation of the radial mode is ignored as well, then the relative amplitude of the nonradial mode at the surface is given by

$$R_0 = \frac{\gamma_{0,0} A_0}{\gamma_{\ell,0} I_\ell}.$$

This quantity is listed in Table 2. Although the ratio refers to displacement, it gives us also a rough assessment of the light amplitude ratio. The relation between relative change of flux and the radial displacement is $\ell$-independent. There is a disc averaging factor of about 0.7 for the $\ell = 1$ mode, but the observed phase relation between temperature and radius in RR Lyrae stars acts in the opposite direction. We see that radial modes may maintain $\ell = 1$ modes with a substantial fractional amplitude. We do not know how the energy is fed to those strongly damped modes. What we know is, that it cannot be in steady state manner because there is no frequency synchronization in RR-$\nu_1$ and RR-$\nu_2$ stars. In this scenario the latter type may occur due to excitation of an $m = \pm 1$ pair.

Inevitable consequence of energy dissipation by the nonradial mode is radial mode amplitude reduction below the saturation level, which must be substantial if the actual amplitude ratio, $R$, approaches maximum values given in Table 2. We have from Eq.(5)

$$\frac{A_0}{A_{0,s}} = \sqrt{1 - \left(\frac{R}{R_0}\right)^2}.$$

There is a cost of feeding nonradial modes. A testable prediction of this scenario is that Blazhko stars should have lower amplitudes than monoperiodic RR Lyrae stars.
Table 2: Maximum values of $\ell = 1$ to $\ell = 0$ mode amplitude ratio for the fundamental and the first overtone doublets

| $\log T_{\text{eff}}$ | $P$ | $\gamma_0$ | $\gamma_1$ | $R_0$ | $P$ | $\gamma_0$ | $\gamma_1$ | $R_0$ |
|------------------------|-----|-------------|-------------|-------|-----|-------------|-------------|-------|
|                        |     |             |             |       |     |             |             |       |
| 3.805                  | 0.670 | 0.021       | -0.101      | 0.420 | 0.472 | 0.087       | -0.106      | 0.779 |
| 3.816                  | 0.579 | 0.020       | -0.110      | 0.387 | 0.430 | 0.094       | -0.115      | 0.783 |
| 3.827                  | 0.526 | 0.015       | -0.123      | 0.318 | 0.390 | 0.088       | -0.143      | 0.680 |
| 3.838                  | 0.479 | 0.007       | -0.141      | 0.212 | 0.355 | 0.065       | -0.199      | 0.492 |
| 3.849                  | 0.436 | 0.001       | -0.157      | 0.054 | 0.324 | 0.011       | -0.264      | 0.319 |
| 3.860                  | 0.401 | -0.005      | -0.175      | -     | 0.298 | 0.010       | -0.332      | 0.146 |

$P$ - period in days, $\gamma_\ell$ - growth rates in d$^{-1}$, $R_0$ - the amplitude ratio

3 Do Blazhko stars have lower radial mode amplitudes?

To resolve this issue, and see if the theoretical predictions can be empirically confirmed, we analyzed RR Lyrae stars from OGLE-II database on the Galactic bulge (Woźniak et al. 2002). We used RR Lyrae catalog derived by Mizerski (2003). His sample contains about 2700 RR Lyrae stars in total, with 550 RRab and about 120 RRc Blazhko stars. Such numerous samples should be sufficient to detect statistically significant effects and draw proper conclusions.

In the first step we analyzed the light curves and found all significant periodicities for every star. It was done in the same way as in Mizerski (2003). As the final result, we obtained frequencies and amplitudes of radial pulsation modes for all RR Lyrae stars, and additionally frequencies and amplitudes of presumably nonradial modes for all Blazhko stars. By amplitude we mean here mathematical definition of amplitude. It was determined as the least squares Fourier fit coefficient. Figure 1 presents nonradial versus radial amplitude diagrams for Blazhko stars. RRab and RRc Blazhko stars are plotted separately.

Triangle-like shape of the diagram for RRab Blazhko stars, with rather uniform distribution, is consistent with the interpretation of the secondary peaks in terms of $\ell = 1$ modes. The observed amplitude for such modes should depend on the aspect angle, $i$, through the factor $\cos i$, if the azimuthal number $m = 0$. Thus, we expect uniform random distribution of their amplitudes at a specified radial mode amplitude. However, we also see objects with abnormally high amplitudes that are difficult to interpret. In the case of RRc stars the picture is less clear, perhaps because of much smaller number of stars. When we compare the amplitudes plotted in Fig. 1 with numbers given in Table 1 we should remember, that numbers quoted there give amplitudes averaged over $i$. The conclusion that the nonradial modes are nonlinear in deep interior seems unavoidable, except perhaps for some of RRc stars.

After identifying all Blazhko stars in our sample, we compared the amplitudes of their radial components with amplitudes in monoperiodic RR Lyrae
stars. The amplitudes strongly depend on the period. Therefore we binned all stars with respect to the pulsation period. Again, RRab and RRc stars were treated separately. As the most of RRab stars in our sample have periods between 0.4 and 0.6 days, we only used stars from this period range. This range was divided into four bins, each 0.05 days wide. Then, in each bin, we computed weighted average of the radial mode amplitude. For Blazhko stars we computed two averages: one average for all Blazhko stars and another average for Blazhko stars with the nonradial component’s amplitude larger than 0.025 magnitude.

The results are presented in Fig. 2. As one can see the presence of the Blazhko effect in RRab stars results in lowering of the radial mode amplitude. This effect can be seen throughout the whole period range. The stars with higher nonradial mode amplitude do not have systematically lower radial mode amplitude than the rest of Blazhko stars. This confirms that indeed most of the difference in nonradial mode amplitudes results from the difference in the aspect. In the case of RRc stars we do not see any significant systematic influence of nonradial mode presence on the radial mode amplitude.

How does the lowering of amplitudes in Blazhko RRab stars seen in Figure 2 compare to the theoretical expectation? The typical reduction is about 10 percent, which according Eq.(7), corresponds to $R = 0.45R_0$. The values of $R_0$ - the maximum nonradial to radial mode amplitude ratio - for fundamental modes are listed in Table 2. They range from 0 at the blue edge of the instability strip to above 0.4 at the lowest temperatures. Let us use the 0.2-0.4 range as
Figure 2: Average amplitudes of radial modes for RRab (top) and RRc (bottom) Blazhko stars. Triangles represent monoperiodic RR Lyrae stars, squares represent all Blazhko stars, pentagons represent Blazhko stars with amplitudes of nonradial component larger than 0.025 and 0.01 magnitude for RRab and RRc classes respectively.

representative for RRab stars. The maximum amplitude ratio, corresponding to \( i = 0^\circ \), is by factor \( \sqrt{1.5} \) higher. In Figure 1, we see stars with \( R \) exceeding the expected maximum value. These objects certainly deserve more careful analysis. For the bulk of RRab stars, the observed amplitude reduction appears somewhat smaller than expected on the basis of Eq.(7) and the data in Figure 1. However, we have to keep in mind that there is uncertainty in calculated values of \( R_0 \), following from uncertainty of \( \gamma_0 \). The values of \( R_0 \) for first overtone pulsations are higher than those for fundamental mode pulsations. This may perhaps explain why we do not see any systematic effect of the amplitude lowering in Blazhko RRc stars.

4 Conclusions

Nonradial mode excitation is the most likely explanation of the Blazhko effect. However, there are problems that this explanation faces. The most severe one is posed by high amplitudes of the supposedly nonradial modes seen in many objects. The high amplitudes imply that the modes become strongly nonlinear and consequently damped in deep interior. We showed that nonradial modes of spherical harmonic degree \( \ell = 1 \) may still be excited if there is a sufficient energy transfer from radial modes due to nonlinear resonant coupling. The condition
sets an upper limit on the amplitude ratio. When this limit is approached, we should see a substantial reduction of the radial mode amplitude. We looked for this effect in observational data.

We presented a new analysis of a large sample of RR Lyrae stars from OGLE-II database on the Galactic bulge. The Blazhko effect is recognized through the presence of close secondary peaks in pulsation spectra. We emphasized that distribution of the secondary peak amplitudes is consistent with the expected distribution for $\ell = 1$ modes due to random distribution of aspects.

The effect of radial mode amplitude reduction is observed only in Blazhko RRab stars, and it appears to be somewhat smaller than predicted. There are cases, both among RRab and RRc stars, where radial and nonradial mode amplitudes are comparable. These cases are difficult to understand within the framework of our scenario. Possible explanation is that we observe a special phase of a nonstationary limit cycle.

What we presented in this paper is only a schematic scenario for maintaining nonradial oscillations. We do not have any detailed models of g-wave breaking in the deep interior and nonlinear interaction between radial and nonradial modes in outer layers. Only with such models, we will know whether we understand how the Blazhko effect arises.

There is certainly more to be learnt from the large sample of Blazhko stars of the Galactic bulge. More extensive analysis of data on these objects, as well as similar data on both Magellanic Clouds, will soon be published (Mizerski, in preparation).

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