THE COLORED HADWIGER TRANSVERSAL THEOREM IN $\mathbb{R}^d$

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Abstract. Hadwiger’s transversal theorem gives necessary and sufficient conditions for a family of convex sets in the plane to have a line transversal. A higher dimensional version was obtained by Goodman, Pollack and Wenger, and recently a colorful version appeared due to Arocha, Bracho and Montejano. We show that it is possible to combine both results to obtain a colored version of Hadwiger’s theorem in higher dimensions. The proofs differ from the previous ones and use a variant of the Borsuk-Ulam theorem. To be precise, we prove the following. Let $F$ be a family of convex sets in $\mathbb{R}^d$ in bijection with a family $P$ of points in $\mathbb{R}^{d-1}$. Assume that there is a coloring of $F$ with sufficiently many colors such that any colorful Radon partition of points in $P$ corresponds to a colorful Radon partition of sets in $F$. Then some monochromatic subfamily of $F$ has a hyperplane transversal.

1. Introduction

1.1. Background. Many classical theorems of convexity, such as the theorems of Carathéodory [4], Helly [8], Kirchberger [9], and Tverberg [13], admit remarkable colorful versions. The first of these, the colorful Carathéodory theorem, was discovered by Bárány [2] and its dual version, the colorful Helly theorem, was independently discovered by Lovász (see section 3 of [2]). Apart from their inherent charm, these results also have deep applications which appear to be unaccessible by the classical versions of the theorems (see chapters 8-10 in [10]). Recently a colorful version of Hadwiger’s theorem [7] on common line transversals to families of convex sets in the plane was discovered by Arocha, Bracho, and Montejano [1]. Just as Hadwiger’s theorem has a generalization to hyperplane transversals in any dimension [6, 12], they conjectured that there should also exist a colorful version in higher dimensions. In this note we make the first steps in establishing their conjecture.

1.2. Definitions. Recall Radon’s theorem which states that any set of $k + 2$ points in $\mathbb{R}^k$ can be partitioned into two parts whose convex hulls intersect (see [5]). Moreover, this partition is unique if and only if any $k + 1$ of the points are affinely independent. In general, a Radon partition of a set of points in $\mathbb{R}^k$ is a pair of disjoint subsets whose convex hulls intersect. The combinatorial data which records all the Radon partitions of a set of points in $\mathbb{R}^k$ is an invariant of the point set known as its order-type (see chapter 9.3 in [10] and section 2 in [6]). Radon partitions extend to families of sets in $\mathbb{R}^d$ in a straightforward way. Let $F$ be a family of sets in $\mathbb{R}^d$. A Radon partition of $F$ is a pair of subfamilies $(G_1, G_2)$ of $F$ such that $G_1 \cap G_2 = \emptyset$ and $\text{conv} G_1 \cap \text{conv} G_2 \neq \emptyset$, where $\text{conv} G_i$ denotes the convex hull of the union of the members of $G_i$.

Let $F$ be a family of compact connected sets in $\mathbb{R}^d$. An affine hyperplane which meets every member of $F$ is called a hyperplane transversal. The relationship between hyperplane transversals and Radon partitions comes from the observation that if $F$ has a hyperplane transversal, then the set of all Radon partitions of $F$ “spans” the set of all Radon partitions of a point set $P$ in $\mathbb{R}^k$ for some $k < d$. To see this, simply choose,
from each member of $F$, a point contained in the hyperplane transversal. (Naturally, $F$ may have other Radon partitions as well.) The idea behind the “Hadwiger-type” theorems is that this necessary condition is also sufficient.

**Definition 1.1** ($k$-ordering). Let $F$ be a set or a family of sets. A $k$-ordering of $F$ is a bijection $\varphi : F \rightarrow P$ where $P$ is set of points which affinely span $\mathbb{R}^k$.

**Definition 1.2** (Consistent $k$-ordering). Let $F$ be a family of sets in $\mathbb{R}^d$. A consistent $k$-ordering of $F$ is a $k$-ordering which respects the Radon partitions of $\varphi(F)$. That is,

$$\text{conv} \varphi(F_1) \cap \text{conv} \varphi(F_2) \neq \emptyset \implies \text{conv} F_1 \cap \text{conv} F_2 \neq \emptyset$$

for any pair of subfamilies $F_1$ and $F_2$ of $F$.

The Pollack-Wenger theorem can now be stated as follows.

**Theorem 1.3** (Pollack-Wenger [12]). A family of compact connected sets in $\mathbb{R}^d$ has a hyperplane transversal if and only if $F$ has a consistent $k$-ordering for some $0 \leq k \leq d - 1$.

The history leading up to the Pollack-Wenger theorem starts with the observation that the case $k = 0$ follows from Helly’s theorem in $\mathbb{R}^1$. Next, Hadwiger [7] proved the case $d = 2$ and $k = 1$ under the additional assumption that the members of $F$ are pairwise disjoint. More than two decades later, Katchalski extended Hadwiger’s theorem to arbitrary dimension, still using the condition of pairwise disjointness. In 1988 Goodman and Pollack [6] proved the case for $k = d - 1$ under a condition of separatedness generalizing the disjointness condition. It was not until 1990 that Wenger [14] removed the condition of pairwise disjointness in the case $d = 2$ and $k = 1$, which immediately implies Katchalski’s result as well. Wenger’s discovery showed that the condition of disjointness (and separatedness) was in fact a bit misleading, and the Pollack-Wenger theorem served as a common generalization of the various Hadwiger-type results.

### 1.3. The colorful version

Given a family of sets $F$, an $r$-coloring of $F$ is a partition of $F$ into $r$ non-empty parts $F = F_1 \cup F_2 \cup \cdots \cup F_r$. Each $F_i$ is called a **monochromatic subfamily** of $F$. A **colorful subfamily** of $F$ is a subfamily $G \subset F$ such that $|G \cap F_i| \leq 1$ for all $1 \leq i \leq r$. A **colorful Radon partition** is a Radon partition $(G_1, G_2)$ such that $G_1 \cup G_2$ is colorful.

**Definition 1.4** (Rainbow consistent $k$-ordering). Let $F$ be an $r$-colored family of sets in $\mathbb{R}^d$. A **rainbow consistent $k$-ordering** of $F$ is a $k$-ordering which respects the colorful Radon partitions of $\varphi(F)$. That is,

$$\text{conv} \varphi(F_1) \cap \text{conv} \varphi(F_2) \neq \emptyset \implies \text{conv} F_1 \cap \text{conv} F_2 \neq \emptyset$$

for any pair of subfamilies $F_1$ and $F_2$ of $F$ where $F_1 \cup F_2$ is a colorful subfamily.

Arocha, Bracho, and Montejano [1] discovered the first colorful version of Hadwiger’s theorem, or rather a colorful version of Wenger’s theorem.

**Theorem 1.5.** Let $F$ be a 3-colored family of compact connected sets in $\mathbb{R}^d$. If $F$ has a rainbow consistent $1$-ordering, then some monochromatic subfamily of $F$ has a hyperplane transversal.

As pointed out in [1], it is not hard to formulate the colorful version of the Pollack-Wenger theorem (which they conjectured is true). This leads us to the general “colorful Hadwiger problem”.

**Problem 1.6.** For integers $d$ and $k$, $d > k \geq 0$, determine the smallest integer $r = r(d,k)$ such that if $F$ is an $r$-colored family of compact connected sets in $\mathbb{R}^d$ with a rainbow consistent $k$-ordering, then some monochromatic subfamily of $F$ has a hyperplane transversal.
Note that existence of such and $r$ implies the Pollack-Wenger theorem. To see this, assume that $F$ is a family of compact connected sets in $\mathbb{R}^d$ and take $r$ monochromatic copies $F_1, \ldots, F_r$ of $F$, each of a different color. Clearly, if $F$ has a consistent $k$-ordering then $\bigcup F_i$ has a rainbow consistent $k$-ordering and therefore some $F_i$ has a hyperplane transversal. Since $F_i$ is a copy of $F$, then $F$ also has a hyperplane transversal.

Before getting to our results, let us point out several known bounds for $r(d, k)$.

- $r(d, k) \geq k + 2$. If $F$ is colored by less than $k + 2$ colors, then a bijection $\varphi : F \to P$ where $P$ is any set in general position in $\mathbb{R}^k$ is a rainbow consistent $k$-ordering since the set of colorful Radon partitions of $\varphi(F)$ will be empty.

- $r(d, k) \geq r(d + 1, k)$. If $F$ is an $r$-colored family of compact connected sets in $\mathbb{R}^{d+1}$, let $\pi(F)$ denote the family obtained by projection to $\mathbb{R}^d$. Any rainbow consistent $k$-ordering of $F$ is also a rainbow consistent $k$-ordering of $\pi(F)$, and the preimage of a hyperplane in $\mathbb{R}^d$ is a hyperplane in $\mathbb{R}^{d+1}$.

- $r(1, 0) = 2$. If $F$ is a 2-colored family in $\mathbb{R}^1$, then a rainbow consistent 0-ordering simply means that any two members of $F$ of distinct colors have a point in common. The colorful Helly theorem implies that there is a monochromatic subfamily whose members have a point in common.

- $r(2, 1) = 3$. This is the colorful Hadwiger theorem of Arocha, Bracho, and Montejano [1].

In this note we present an approach which reduces Problem 1.6 to showing that a certain type of subsets are contractible. In the uncolored case these subsets are convex, so we obtain a new proof of the Pollack-Wenger theorem. In the colored case, these subsets correspond to what were called geometric joins in [3], where their topology was studied. Based on results and ideas from [3] we obtain the following.

**Theorem 1.7.** For the function $r(d, k)$ the following bounds hold.

- $r(k + 2, k) \leq \left(\frac{k+2}{2}\right) + 1$.

- $r(4, 2) = 4$.

- $r(k + 1, k) \leq 2(k + 1)^2 + 3$.

Our proof method also gives a new (and simpler) proof of the Arocha-Bracho-Montejano theorem.

2. Proof of Theorem 1.7

It is sufficient to prove Theorem 1.7 for finite families. The general case follows from a standard compactness argument. Moreover, we may assume that the members of $F$ are convex polytopes. To see this, note that a hyperplane meets a compact connected set if and only if it meets the convex hull of the set. Thus we may assume the members of $F$ are convex. Next, we can approximate each convex set $F$ by an inscribed convex polytope $K' \subset K$, forming a new family $F'$ such that the corresponding $k$-ordering, $\varphi' : F' \to P$, is rainbow consistent. This follows from the compactness of the members of $F$, and since the polytopes are inscribed, any hyperplane that intersects $K'$ also intersects $K$. So from here on, we assume $F$ is a finite family of polytopes in $\mathbb{R}^d$.

Let $V$ be the set of vertices of the polytopes in $F$ and $\text{mid}(V)$ the set of midpoints between pairs of points of $V$. For each pair of distinct points $u$ and $v$ in $V \cup \text{mid}(V)$ consider the orthogonal complement $(u - v)^\perp$ which is a hyperplane through the origin in $\mathbb{R}^d$. The set of all such orthogonal complements decomposes $\mathbb{S}^{d-1}$ into a regular antipodal cell complex of dimension $d - 1$ which is denoted by $C$. The cells of $C$ are open and the boundary of $\sigma \subset C$, denoted by $\text{bd}(\sigma)$, is a finite union of cells of $C$. Note that $C$ is homeomorphic to a polytopal complex.
The main step consists in assigning to each cell, $\sigma \in C$, a subset $S(\sigma) \subset \mathbb{R}^m$ with the following properties. 
(The dimension $m$ depends on the values of $d$ and $k$ and will be determined later.)

**Antipodality:** $S(\sigma) = -S(-\sigma)$ for every $\sigma \in C$.

**Monotonicity:** $S(\tau) \subset S(\sigma)$ for every $\tau, \sigma \in C$ with $\tau \subset \text{bd}(\sigma)$.

**Contractibility:** $S(\sigma)$ is contractible for every $\sigma \in C$.

**Lemma 2.1.** Suppose the sets $S(\sigma) \subset \mathbb{R}^m$ satisfy antipodality, monotonicity, and contractibility. If $m < d$, then one of the set $S(\sigma)$ contains the origin.

**Proof.** We construct a continuous, antipodal map, $f : \mathbb{S}^{d-1} \to \mathbb{R}^m$, by building it up inductively on the skeletons of $C$. First define the map $f$ on the 0-skeleton by choosing, for every $v \in \mathbb{S}^{d-1}$ corresponding to a 0-cell of $C$, an arbitrary point $y \in S(v)$, and set $f(v) = y$ and $f(-v) = -y$. Now suppose $f$ has been defined on the $k$-skeleton of $C$ in such a way that for every cell $\tau$, its image $f(\tau)$ is contained in $S(\tau)$. Let $\sigma \in C$ be a $(k+1)$-cell. The function $f$ has already been defined on $\text{bd}(\sigma)$, which is homeomorphic to the $k$-sphere, and the monotonicity property implies that the image $f(\text{bd}(\sigma)) = \bigcup_{v \in \text{bd}(\sigma)} f(v)$ is contained in $S(\sigma)$. Since $S(\sigma)$ is contractible, it is necessarily $k$-connected, and therefore $f$ can be extended continuously on all of $\sigma$ such that its image lies in $S(\sigma)$. Once $f$ has been defined on $\sigma$, extend antipodally on $-\sigma$. We can extend $f$ to the entire $(k+1)$-skeleton by repeating the procedure for every $(k+1)$-cell. We therefore have a continuous, antipodal map, $f : \mathbb{S}^{d-1} \to \mathbb{R}^m$. If $m < d$, then $f$ has a zero by the Borsuk-Ulam theorem (see e.g. [11]) implying that there is a cell $\sigma \in C$ such that $S(\sigma)$ contains the origin.

**Remark 2.2.** The contractibility condition in Lemma 2.1 can be replaced by the following weaker condition: If $\sigma \in C$ is a $k$-cell then $S(\sigma)$ is $(k-1)$-connected.

### 2.1. Construction of $S(\sigma)$

#### 2.1.1. Separated subfamilies

Identify the point $x = (x, t) \in \mathbb{S}^{d-1} \times \mathbb{R}$ with the hyperplane $H(x) = \{v \in \mathbb{R}^d : v \cdot x = t\}$, which should be thought of as an oriented hyperplane, in the sense that $H(x)$ bounds a negative and a positive half-space, where the direction $x \in \mathbb{S}^{d-1}$ points to the positive side. Thus, the hyperplanes corresponding to $x$ and $-x$ determine the same point set but have reverse orientations. The space of all oriented hyperplanes in $\mathbb{R}^d$ is parametrized by $\mathbb{S}^{d-1} \times \mathbb{R}$ and comes equipped with the natural topology. In other words, we consider the space of all oriented hyperplanes as a “$\mathbb{Z}_2$-space”.

For every oriented hyperplane $H \subset \mathbb{R}^d$ there is a corresponding ordered pair of separated subfamilies $(F_1, F_2)$, where $F_1 \subset C$ consists of the members contained in open negative side of $H$, and $F_2 \subset F$ the members in open positive side. The map $x \mapsto (F_1, F_2)$ is a “$\mathbb{Z}_2$-map” in the sense that if $x$ is mapped to $(F_1, F_2)$, then $-x$ is mapped to $(F_2, F_1)$. We write $(F_1, F_2) \subset (F_1', F_2')$ if $F_1 \subset F_1'$ and $F_2 \subset F_2'$, and $(F_1, F_2) = (F_1', F_2')$ in the case of equality.

**Claim 2.3.** Every $x \in \mathbb{S}^{d-1} \times \mathbb{R}$ is contained in an open neighborhood $N(x)$ such that if $x$ corresponds to the separated subfamilies $(F_1, F_2)$ and $x'$ corresponds to the separated subfamilies $(F_1', F_2')$, then $(F_1, F_2) \subset (F_1', F_2')$ for any $x' \in N(x)$.

**Proof.** Each member of $F_1 \cup F_2$ has some positive distance to the hyperplane $H(x)$, and since $|F_1 \cup F_2|$ is finite, a minimum distance is achieved. The distance to each member varies continuously with $x$, and consequently the distance from each member of $F_1 \cup F_2$ to $H(x')$ remains positive for any $x'$ sufficiently close to $x$. 

□
2.1.2. **Central hyperplanes.** We now define a specific hyperplane for every direction \( x \in \mathbb{S}^{d-1} \) as follows. Consider an oriented hyperplane orthogonal to \( x \) such that every member of \( F \) is contained on its positive side. Start translating the hyperplane in the direction \( x \) until the first time its closed negative side contains members of \( F \) of at least \( \lceil \frac{r}{2} \rceil \) distinct colors, and denote this hyperplane \( H_1 \). Similarly, starting with a hyperplane which contains every member of \( F \) on its negative side, we translate it in the direction \( -x \) until the first time its closed positive side contains members of \( F \) of at least \( \lceil \frac{r}{2} \rceil \) distinct colors, and denote this hyperplane \( H_2 \).

The assumption that no monochromatic subfamily of \( F \) has a hyperplane transversal implies that \( H_2 \) is contained in the open positive side of \( H_1 \). If this were not the case, the ordered pair of separated subfamilies \( (F_1, F_2) \) associated with \( H_1 \) would both contain strictly less than \( \lceil \frac{r}{2} \rceil \) colors, implying that \( F_1 \cup F_2 \) contains strictly less than \( r \) colors, hence \( H_1 \) is a hyperplane transversal to some monochromatic subfamily.

For the direction \( x \in \mathbb{S}^{d-1} \), let the **central hyperplane in the direction** \( x \) be the oriented hyperplane which is orthogonal to the direction \( x \) and lies halfway between \( H_1 \) and \( H_2 \).

**Claim 2.4.** For \( x \in \mathbb{S}^{d-1} \), let \( H_x \) be the central hyperplane in the direction \( x \) and \( (F^-_x, F^+_x) \) the associated separated subfamilies. The following hold.

1. The map \( x \mapsto H_x \) is a continuous, antipodal map from \( \mathbb{S}^{d-1} \) to \( \mathbb{S}^{d-1} \times \mathbb{R} \).
2. \( H_x \) passes through a midpoint determined by the vertices of the members of \( F \).
3. \( F^-_x \cup F^+_x \) contains members of every color.
4. \( F^-_x \) and \( F^+_x \) each contain members of at least \( \lceil \frac{r}{2} \rceil \) distinct colors.
5. \( F^-_x = F^+_{-x} \) and \( F^+_x = F^-_{-x} \).

**Proof.** For part (1), continuity follows from the continuity of the distance function, while the antipodality follows from the symmetry in the definition of the hyperplanes \( H_1 \) and \( H_2 \) (in the definition of the central hyperplane). For part (2) we observe that the hyperplanes \( H_1 \) and \( H_2 \) are supporting tangents of members of \( F \), and therefore must each contain at least one vertex of a member of \( F \). Since \( H_x \) lies halfway between \( H_1 \) and \( H_2 \), it must pass through the midpoint of these vertices. Part (3) is just the assumption that no monochromatic subfamily has a hyperplane transversal. Part (4) follows from the observation that \( H_1 \) lies in the open negative side of \( H_x \) while \( H_2 \) lies in the open positive side. Part (5) is a consequence of the antipodality of the map \( x \mapsto H_x \). \(\square\)

**Claim 2.5.** Let \( \sigma \) be a cell of \( C \). Then \( (F^-_x, F^+_x) = (F^-_y, F^+_y) \) for all \( x \) and \( y \) in \( \sigma \).

**Proof.** Notice that a change in \( F^-_x \) (or \( F^+_x \)) occurs only if some member of \( F \) becomes tangent to \( H_x \) (as \( x \) varies continuously). When this happens, \( H_x \) passes through a vertex \( v \in V \), and by Claim 2.4 (2), \( H_x \) also passes through a midpoint \( m \in \text{mid}(V) \). This means that when \( H_x \) becomes tangent to \( v \), the vector \( x \) will enter the orthogonal complement \( (v-m)^+ \), thus leaving the open cell \( \sigma \). \(\square\)

In view of Claim 2.5, the ordered pairs of separated subfamilies may be associated with the cells of \( C \) (rather than the points of \( \mathbb{S}^{d-1} \)). For a cell \( \sigma \in C \) we write \( (F^-_\sigma, F^+_\sigma) \).

**Claim 2.6.** Let \( \tau \) and \( \sigma \) be cells of \( C \). If \( \tau \subset \text{bd}(\sigma) \), then \( (F^-_\tau, F^+_\tau) \subset (F^-_\sigma, F^+_\sigma) \).

**Proof.** For any point \( x \in \tau \) there is an open neighborhood \( N(x) \subset \mathbb{S}^{d-1} \) such that for all \( y \in N(x) \) we have \( y \in \sigma \) for some \( \sigma \in C \) with \( \tau \subset \text{bd}(\sigma) \). The statement now follows by taking the intersection with the open neighborhood from Claim 2.3. \(\square\)
2.1.3. The colorful Radon partitions. Let $P$ be a set of points which affinely spans $\mathbb{R}^k$ and $\varphi : F \to P$ a $k$-ordering. Let $P^+ \in \mathbb{R}^{k+1}$ be the set of points obtained by adding a coordinate with value $1$ to the end of each point in $P$, and let $P^- = -P^+$. For the cell $\sigma \in C$, with associated separated subfamilies $(F^ \sigma_-, F^ \sigma_+)$, define sub-configurations $Q^ \sigma_- \subseteq P^-$ and $Q^ \sigma_+ \subseteq P^+$ as

$$Q^ \sigma_- := \left\{ -\begin{pmatrix} p \\ 1 \end{pmatrix} : \varphi(p) \in F^ \sigma_- \right\} \quad \text{and} \quad Q^ \sigma_+ := \left\{ \begin{pmatrix} p \\ 1 \end{pmatrix} : \varphi(p) \in F^ \sigma_+ \right\}$$

**Lemma 2.7.** Let $Q^ \sigma = Q^ \sigma_- \cup Q^ \sigma_+$. The following hold.

- $Q^ \sigma_+ = -Q^ \sigma_- \sigma \in C$. (Antipodality)
- $Q^ \tau_+ \subseteq Q^ \sigma_+$ for every $\tau, \sigma \in C$ with $\tau \subseteq \text{bd}(\sigma)$. (Monotonicity)

**Proof.** Antipodality follows from Claim 2.4 (5), while monotonicity follows from Claim 2.6.

Since each point in $Q^ \sigma$ corresponds to a unique member of $F$, Claim 2.4 (3) implies that there is a natural $r$-coloring of $Q^ \sigma$. We may therefore speak of the colorful subsets of $Q^ \sigma$. A crucial observation is the following.

**Lemma 2.8.** If $Q$ is a colorful subset of $Q^ \sigma$ and the convex hull of $Q$ contains the origin, then $\varphi : F \to P$ is not rainbow consistent.

**Proof.** Let $Q_1 := -(Q \cap Q^-_\sigma)$ and $Q_2 := Q \cap Q^+_\sigma$. The fact that $0 \in \text{conv}(Q)$ is equivalent to saying that $\text{conv}(Q_1) \cap \text{conv}(Q_2) \neq \emptyset$, hence if $0 \in \text{conv}(Q)$, then this corresponds to a colorful Radon partition $(P_1, P_2)$ of $P$ (by dropping the last coordinate). However, by definition of $Q^ \sigma$, the subfamilies $\varphi^{-1}(P_1) = F^ \sigma_-$ and $\varphi^{-1}(P_2) = F^ \sigma_+$ are strictly separated by a central hyperplane.

2.2. The geometric join. We are now ready to define the sets $S(\sigma)$. This will vary slightly depending on which case of Theorem 1.7 we want to prove.

**Definition 2.9** (Geometric join). Let $A = A_1 \cup A_2 \cup \cdots \cup A_r$ be an $r$-colored set of points in $\mathbb{R}^{k+1}$. The geometric join $\text{GJ}(A)$ of $A$ is the set of all convex combinations of the form $t_1a_1 + t_2a_2 + \cdots + t_ia_r$ where $a_i \in A_i$.

We need the following results from [3].

**Lemma 2.10.** The geometric join of an $r$-colored point set in $\mathbb{R}^{k+1}$ is contractible in the following cases

- $k = 1$ and $r = 3$.
- $k = 2$ and $r = 4$.
- $k \geq 3$ and $r = \binom{k+2}{2} + 1$.

2.2.1. The case $d = k + 2$. We show that $r(k + 2, k) \leq \binom{k+2}{2} + 1$ for $k \geq 3$, and $r(4, 2) = 4$. For every $\sigma \in C$, the set $Q^ \sigma$ is an $r$-colored point set in $\mathbb{R}^{k+1}$. Let $S(\sigma) = \text{GJ}(Q^ \sigma)$. By Lemma 2.7 the sets $S(\sigma)$ satisfy antipodality and monotonicity. Contractibility follows from Lemma 2.10 provided

$$r = \begin{cases} 4 & \text{if } k = 2 \\ \binom{k+2}{2} + 1 & \text{if } k \geq 3 \end{cases}$$

Lemma 2.1 implies that the origin is contained in $S(\sigma)$ for some $\sigma \in C$, and Lemma 2.8 implies that $\varphi : F \to P$ is not rainbow consistent.
2.2.2. The case \(d = k + 1\). We show that \(r(k + 1, k) \leq 2(k + 1)^2 + 3\). For every \(\sigma \in C\), the set \(Q_\sigma\) is an \(r\)-colored point set in \(\mathbb{R}^{k+1}\). Let \(W\) be a hyperplane passing through the origin which strictly separates \(P^-\) and \(P^+\). Let \(S(\sigma) = W \cap \text{GJ}(Q_\sigma)\), note that \(S(\sigma)\) is a subset of \(\mathbb{R}^k\).

**Lemma 2.11.** If \(r = 2(k + 1)^2 + 3\), then \(S(\sigma)\) is star-shaped for every \(\sigma \in C\).

**Proof.** We will apply Krasnoselkii’s theorem (see e.g. Theorem 11.2 in [5]), and since \(S(\sigma) \subset \mathbb{R}^k\) is compact, it suffices to show that every \(k + 1\) boundary points of \(S(\sigma)\) are visible from a common point of \(S(\sigma)\). Let \(x_0, x_1, \ldots, x_k\) be boundary points of \(S(\sigma)\), that is, \(x_i \in W \cap \text{conv}(X_i)\) where \(X_i\) is a colorful subset of \(Q_\sigma\) with \(|X_i| \leq k + 1\). Recall that the points of \(Q_\sigma\) are in bijection with the members of \(F_\sigma \cup F^-\), so by Claim 2.4 (4) it follows that \(Q^+_\sigma\) and \(Q^-_\sigma\) each contain points of at least \((k + 1)^2 + 2\) distinct colors. Since \(|X_0 \cup \cdots \cup X_k| \leq (k + 1)^2\), there exists a point \(p_1 \in Q^+_\sigma\) such that \(X_i \cup \{p_1\}\) is colorful for all \(0 \leq i \leq k\). Similarly, since \(|X_1 \cup \cdots \cup X_d \cup \{p_1\}| \leq (k + 1)^2 + 1\), there exists a point \(p_2 \in Q^-_\sigma\) such that \(X_i \cup \{p_1, p_2\}\) is colorful for all \(0 \leq i \leq k\). Since \(p_1 \in Q^+_\sigma\) and \(p_2 \in Q^-_\sigma\) are strictly separated by \(W\), the segment \(p_1p_2\) intersects \(W\) in a unique point, \(p\), and since \(\text{conv}(X_i \cup \{p_1, p_2\}) \cap W\) is a convex subset contained in \(S(\sigma)\) it follows that all the \(x_i\) are visible from \(p\). □

By Lemmas 2.7 and 2.11 the sets \(S(\sigma)\) satisfy antipodality, monotonicity, and contractibility. So, by Lemma 2.1 there is a \(\sigma \in C\) such that \(S(\sigma)\) contains the origin, and therefore some colorful subset of \(Q_\sigma\) contains the origin in its convex hull. By Lemma 2.8, \(\varphi : F \to P\) is not rainbow consistent.

2.2.3. The case \(d = 2\). We give an alternate proof to the Arocha-Bracho-Montejano theorem which states that \(r(2,1) = 3\). For every \(\sigma \in C\), the set \(Q_\sigma\) is a 3-colored point set in \(\mathbb{R}^2\). Let \(W\) be a line through the origin that strictly separates \(P^-\) and \(P^+\). Let \(S(\sigma)\) be the convex hull of \(W \cap \text{GJ}(Q_\sigma)\).

**Lemma 2.12.** If \(S(\sigma)\) contains the origin, then \(\text{GJ}(Q_\sigma)\) also contains the origin.

**Proof.** Assume that \(S(\sigma)\) contains the origin. There are point \(x_1, x_2 \in Q^+_\sigma\) and \(x_3, x_4 \in Q^-_\sigma\) such that \(\{x_1, x_3\}\) and \(\{x_2, x_4\}\) are both colorful subsets of \(Q_\sigma\) and the segments \(x_1x_3\) and \(x_2x_4\) intersect \(W\) on different sides of the origin. By Claim 2.4 (4), \(Q^+_\sigma\) and \(Q^-_\sigma\) each contain at least 2 distinct colors, and it follows that \(\text{GJ}(Q^+_\sigma)\) and \(\text{GJ}(Q^-_\sigma)\) are both connected. Hence there is a path connecting \(x_1\) to \(x_2\) contained in \(\text{GJ}(Q^+_\sigma)\), and a path connecting \(x_3\) to \(x_4\) in \(\text{GJ}(Q^-_\sigma)\). By combining these paths together with the segments \(x_1x_3\) and \(x_2x_4\), we obtain a closed cycle contained in \(\text{GJ}(Q_\sigma)\) which encloses the origin. Since \(\text{GJ}(Q_\sigma)\) is contractible by Lemma 2.10 with \(k = 1\), \(\text{GJ}(Q_\sigma)\) contains the origin. □

By Lemmas 2.7 and 2.12, \(S(\sigma)\) satisfies the hypothesis of Lemma 2.1. Therefore there is some \(\sigma \in C\) such that \(S(\sigma)\) contains the origin and Lemma 2.8 implies that \(\varphi : F \to P\) is not rainbow consistent.

3. Final remarks

3.1. The case \(d = 3\). The case \(d = k + 1\) is the most interesting one and, by comparing with other colorful theorems, we expect \(r(k + 1, k) = k + 2\). This is in fact true for \(k = 1\) as was shown by Arocha, Bracho and Montejano, however the value of \(r(3,2)\) remains unknown and we were unable to adapt out methods to determine its value. It seems that this problem is strongly connected to the topology of geometric joins.
3.2. Matroids. Theorem 1.7 can be generalized further by using a matroid instead of colors. Let $\mathcal{M}$ be a simple matroid of rank $r$ with rank function $\text{rk}(\cdot)$, whose ground set is the family $F$. The colorful case occurs when $\mathcal{M}$ is a partition matroid. That is, when the elements of $F$ are partitioned into non-empty sets $F = F_1 \cup \cdots \cup F_r$ and a subset $G \subset F$ is independent if and only if $|G \cap F_i| \leq 1$ for all $1 \leq i \leq r$.

The notions of independent Radon partition and independent consistent $k$-ordering can be defined in terms of $\mathcal{M}$ in a natural way. The number $r = r(d, k)$ can be defined as the smallest integer such that if $\mathcal{M}$ is a rank $r$ matroid on a family $F$ of compact connected sets in $\mathbb{R}^d$ with an independent consistent $k$-ordering, then there is a subfamily $G$ of $F$ such that $\text{rk}(F \setminus G) < r$ and $G$ has a hyperplane transversal.

Theorem 3.1. For the function $\mathfrak{r}(d, k)$ the following bounds hold.

1. $\mathfrak{r}(3, 2) = 3$.
2. $\mathfrak{r}(k + 1, k) \leq 2(k + 1)^2 + 3$.

The proof of this theorem is almost identical to the proof of Theorem 1.7, so we omit it.

References

[1] J. L. Arocha, J. Bracho, and L. Montejano, A colorful theorem on transversal lines to plane convex sets. Combinatorica 28 (2008), no. 4, 379–384.
[2] I. Bárány, A generalization of Carathéodory’s theorem. Discrete Math. 40 (1982), no. 2-3, 141–152.
[3] I. Bárány, A. F. Holmsen, and R. Karasev, Topology of geometric joins. Preprint, arXiv:1309.0920, 2013.
[4] C. Carathéodory, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen. Math. Ann. 64 (1907), no. 1, 95–115.
[5] J. Eckhoff, Helly, Radon, and Carathéodory type theorems. Handbook of convex geometry, Vol. A, B, North-Holland, Amsterdam, 1993, pp. 389–448.
[6] J. E. Goodman and R. Pollack, Hadwiger’s transversal theorem in higher dimensions. J. Amer. Math. Soc. 1 (1988), no. 2, 301–309.
[7] H. Hadwiger, Ueber Eibereiche mit gemeinsamer Treffergen. Portugal. Math. 16 (1957), 23–29.
[8] E. Helly, Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten. Monatsh. Math. Phys. 31 (1921), no. 1, 60–91.
[9] P. Kirchberger, Über Tchebychefsche Annäherungsmethoden. Math. Ann. 57 (1903), no. 4, 509–540.
[10] J. Matoušek, Lectures on discrete geometry. Graduate Texts in Mathematics, vol. 212, Springer-Verlag, New York, 2002.
[11] R. Pollack and R. Wenger, Necessary and sufficient conditions for hyperplane transversals. Combinatorica 10 (1990), no. 3, 307–311.
[12] H. Tverberg, A generalization of Radon’s theorem. J. London Math. Soc. 41 (1966), 123–128.

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