The structure of HCMU metric in a K-Surface

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Abstract

We study the basic structure of a HCMU metric in a K-Surface with prescribed singularities. When the underlying smooth surface is $S^2$, we prove the necessary condition given in [1] for the existence of HCMU metric is also sufficient.

1 Introduction

Let $M$ be any compact, oriented smooth Riemannian surface without boundary, and $M_{\{\alpha_1, \alpha_2, \ldots, \alpha_n\}}$ (where $\alpha_i > 0, \forall i, 1 \leq i \leq n$) denotes a K-Surface associated with $M$. A Riemannian metric $g$ is said to be well defined or smooth in $M_{\{\alpha_1, \alpha_2, \ldots, \alpha_n\}}$ if it satisfies the following two conditions:

1. $g$ is smooth everywhere on $M$ except in a set of singular points $\{p_1, p_2, \ldots, p_n\}$,
2. For any $i (1 \leq i \leq n)$, the metric $g$ has a singular angle of $2\pi \alpha_i$ at the point $p_i$.

Here the condition (2) means that in a small neighborhood of $p_i$, there exists a local complex coordinate chart $(U, z)$ $(z(p_i) = 0)$, s.t.

$$g|_U = h(z, \bar{z}) \frac{1}{|z|^{1-2\alpha_i}} |dz|^2,$$

where $h : U \to R$ is a continuous positive function and smooth on $U \setminus \{0\}$. Two smooth Riemannian metrics on $M_{\{\alpha_1, \alpha_2, \ldots, \alpha_n\}}$ are pointwise conformal to each other if they are related by a multiple of a smooth positive function on $M$.

A natural question is whether or not there exists a “best” metric in every conformal class of a K-Surface. This is an attempt to generalize the classical uniformization theorem to a K-Surface. Recalled that the classical uniformization theorem asserts that in every conformal class of $M$, there must exist a metric with constant scalar curvature. Many papers tried to generalize the uniformization theorem in K-Surfaces. For example, [5] and [3] independently found the sufficient condition under which in a K-Surface, there exists a constant scalar curvature metric. [6] found a necessary condition of the existence of a constant scalar curvature metric in a K-Surface, [9] proved that a uniqueness theorem on constant curvature metric in some K-surfaces. However, there does not exist a constant curvature metric in a K-Surface.

In a serial of papers [1] and [2], the second named author tried to find the “best” metric in a conformal class of a K-Surface, through studying the critical point of the Calabi energy functional. He proposed that two types of metrics can be regarded as candidates of the “best” metric: one is the extremal metric, another is the HCMU metric (Definitions will be given later.).
Let $M\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a $K$-Surface and $g_0$ be a smooth metric in it. Consider the conformal class of $g_0$:

$$S(g_0) = \{ g = e^{2\varphi}g_0, \varphi \in H^2(M) | \int_{M\setminus\{p_1, p_2, \ldots, p_n\}} e^{2\varphi} dg_0 = \int_{M\setminus\{p_1, p_2, \ldots, p_n\}} dg_0 \}.$$ 

Define the Calabi energy functional:

$$E(g) = \int_{M\setminus\{p_1, p_2, \ldots, p_n\}} K^2 dg,$$  

(1)

here $K$ is the scalar curvature of the metric $g$. The Euler-Lagrange equation of $E(g)$ is (cf. [1] [8])

$$\triangle_g K + K^2 = C,$$  

(2)

or equivalently, in a local complex coordinate chart,

$$\frac{\partial}{\partial \bar{z}} K_{zz} = 0,$$  

(3)

where $K_{zz}$ is the 2nd-order $(0, 2)$ type covariant derivatives of $K$.

A metric which satisfies (2) or (3) is called an extremal metric. (3) has two special cases, one is $K \equiv \text{Const}$,

(4)

and the other is

$$K_{zz} = 0, \ K \neq \text{Const}.$$  

(5)

A metric which satisfies (5) is called a HCMU (the Hessian of the Curvature of the Metric is Umbilical) metric. Throughout this paper, we assume that a HCMU metric has finite area and finite Calabi energy.

Let us first quote an Obstruction Theorem from [1].

**Theorem 1.** Let $g$ be a HCMU metric in a $K$-Surface $M\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then the Euler character of the underlying surface should be determined by

$$\chi(M) = \sum_{i=1}^{j} (1 - \alpha_i) + (n - j) + s$$  

(6)

where $s$ is the number of critical points of the Curvature $K$ (excluding the singular points of $g$). Here we assume that $\alpha_1, \alpha_2, \ldots, \alpha_k, (0 \leq k \leq n)$ are the only integers in the set of prescribed angles $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$; and assume that $\{p_{j+1}, p_{j+2}, \ldots, p_k\}$ are the only local extremal points of $K$ in the set of singular points $\{p_j, 0 \leq j \leq k\}$.

The formula (6) is an application of Poincarè-Hopf index theorem. When $g$ is a HCMU metric, the gradient vector field $\nabla K$ of the scalar curvature $K$ is holomorphic. Hence, its real part is a Killing vector field. It was proved in [1] that the singularities of the Killing vector field is a finite set which is the union of the singularities of metric $g$ and the smooth critical points of function $K$. Consequently, any saddle point of $K$ must be the singularities of metric $g$. At these points the index of the vector field is $(1 - \alpha_i)$. Other singularities of this gradient vector field must be local extremal points of $K$ with index 1. Therefore, the Poincarè-Hopf index theorem implies formula (6).

In this paper, we study the following question: whether or not the condition (6) is also sufficient to the existence of HCMU metrics in a K-Surface. Our main result in this paper is:
Theorem A. For $S^2$, given $n$ points $p_1, p_2, \cdots, p_n$ on $S^2$ and $n$ positive numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ with $\alpha_1, \alpha_2, \cdots, \alpha_k$ being the only integers and $\alpha_j \geq 2 (1 \leq j \leq k)$, suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n$ satisfy the following condition: \( \exists j_0 (1 \leq j_0 \leq k) \) and $\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \cdots, \alpha_{\sigma(j_0)}, \sigma(i) \in \{1, 2, \cdots, k\} (1 \leq i \leq j_0)$, s.t.

\[
\sum_{i=1}^{j_0} \alpha_{\sigma(i)} + \chi(M) - n \geq 0. \tag{7}
\]

Then there exists a HCMU metric whose scalar curvature $K$ is not a constant, s.t. the angles of the metric at $p_1, p_2, \cdots, p_n$ are exactly $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $p_{\sigma(1)}, p_{\sigma(2)}, \cdots, p_{\sigma(j_0)}$ are the only saddle points of the scalar curvature $K$.

In fact, we prove for $S^2$ the condition (7) is the necessary and sufficient condition for the existence of a HCMU metric in it.

The simplest HCMU metric in $S^2$ is a football. It only has two extremal points and it is a rotationally symmetric metric. Fixing the area, a HCMU metric is uniquely determined by the ratio of the two angles.

The proof of Theorem A is based on the following Theorem B, which says that any HCMU metric can be divided into a finite number of footballs. Under the condition (7), we can glue some suitable footballs together to obtain a HCMU metric in $S^2$ as desired.

Theorem B. Let $g$ be a HCMU metric on a K-Surface $M$, then there are a finite number of geodesics which connects extremal points and saddle points of the scalar curvature $K$ together. In fact, $M$ can be divided into a finite number of pieces by cutting along these geodesics where each piece is locally isometric to a HCMU metric in some football.

We should point out that in [7], Lin and Zhu use ODE method and geometry of the scalar curvature of HCMU metrics to construct a class of HCMU metrics with finite conical singular angles $2\pi \cdot \text{integers}$ on $S^2$. This kind of HCMU metric is called exceptional HCMU metric where all of its singularities are the saddle points of the scalar curvature $K$. A minimal exceptional HCMU metric is an exceptional HCMU metric with only one minimum point of the scalar curvature $K$. They give an explicit formula for minimal exceptional HCMU metrics. Their theorem shows that a minimal exceptional HCMU metric is determined by three parameters. In comparison, our existence theorem of HCMU metric is more general. Indeed our construction in the proof of Theorem A is actually a minimal exceptional HCMU metrics if all of the singularities are the saddle points of the scalar curvature $K$.

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2 Proof of Theorem B

2.1 Preliminaries

Let $M$ be a compact, oriented smooth Riemannian surface without boundary. $M_{\{\alpha_1, \alpha_2, \cdots, \alpha_n\}}$ denotes its K-Surface. $\{p_1, p_2, \cdots, p_n\}$ is the set of singular points. $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is the corresponding set of singular angles. $\forall p \in M \setminus \{p_1, p_2, \cdots, p_n\}$, assuming that $(U, z)$ is a complex coordinate chart around $p$, $g$ can be written as:

\[
g = e^{2\varphi(z, \bar{z})}|dz|^2.
\]
and

\[ K = -\frac{\Delta \varphi}{e^{2\varphi}}. \]

Equation (5) can be written as:

\[ K_{zz} = \frac{\partial^2 K}{\partial z^2} - 2 \frac{\partial K}{\partial \bar{z}} \frac{\partial \varphi}{\partial z} = 0, \tag{8} \]

which means that the gradient vector field of the scalar curvature \( K \) is holomorphic. The gradient vector field \( \nabla K \) is:

\[ \nabla K = \sqrt{-1} K' \frac{\partial}{\partial z} = \sqrt{-1} e^{-2\varphi} \frac{\partial K}{\partial \bar{z}} \frac{\partial}{\partial \bar{z}}, \]

and its real part is:

\[ \vec{V} = \frac{1}{2} (\sqrt{-1} K' \frac{\partial}{\partial z} - \sqrt{-1} K' \frac{\partial}{\partial \bar{z}}). \]

Then \( \vec{V} \) is a Killing vector field and its integral curve is the level set of the function \( K \).

In fact, by studying the properties of the Killing vector field \( \vec{V} \), the Obstruction Theorem is proved in [1]. We list here the main properties of \( \vec{V} \).

**Proposition 1**[1]. Let \( \text{Sing} \vec{V} \) denote the set of all singular points of \( \vec{V} \) and \( \Omega_p \) denote the set of the integral curves of \( \vec{V} \) which meet \( p \) for \( p \in \text{Sing} \vec{V} \). Then:

1. \( \text{Sing} \vec{V} = \{ \text{smooth critical points of } K \} \cup \{ p_1, p_2, \ldots, p_n \} \) and is a finite set.
2. \( \Omega_p \) is empty or a finite set. Moreover, if \( \Omega_p \neq \emptyset \), \( \Omega_p \) has even number of points and \( g \) has angle \( |\Omega_p|\pi \) at \( p \).
3. \( K \) can be continuously extended to \( M \).
4. \( \text{Sing} \vec{V} \) can be divided into two parts, \( \text{Sing} \vec{V} = S_1 \cup S_2 \), such that
   a. \( S_1 = \{ p \in \text{Sing} \vec{V} | \Omega_p = \emptyset \} \), and if \( p \in S_1 \), then \( p \) is an extremal point of \( K \);
   b. \( S_2 = \{ p \in \text{Sing} \vec{V} | \Omega_p \neq \emptyset \} \), if \( p \in S_2 \), \( p \) is a saddle point of \( K \). and at \( p \) the angle is \( \pi \cdot |\Omega_p| \).

**Proposition 2**[1].

1. Any integral curve of \( \vec{V} \) in the neighborhood of any local extremal of \( K \) point is a topologically circle which contains the point in its interior.
2. If a closed integral curve of \( \vec{V} \) bounds a topological disk in \( M \), which contains only one extremal point of \( K \), then every integral curve of \( \vec{V} \) in this disk is also a topological circle.

**Proposition 3**[1]. At a saddle point of \( K \), the included angle of two adjacent integral curves of \( \nabla K \) is \( \pi \).

### 2.2 Proof of the Theorem B

Let us begin with the study of a football, i.e. a HCMU metric \( g \) in \( S^2 \), which is rationally symmetric and has two extremal points.

According to Proposition 1, the scalar curvature \( K \) is continuous. If \( K \) has only two extremal points \( p \) and \( q \), by Proposition 2.1, in a neighborhood of \( p \), an integral curve \( C \) of \( \vec{V} \) bounds a topological disk \( D \) centered at \( p \). By of Proposition 2.2, the integral curves of \( \vec{V} \) in \( D \) are all topologically concentric circles containing \( p \) in their interiors. Since \( \vec{V} \) is a Killing
vector field, \( g \) is invariant along integral curves of \( \vec{V} \), then \( g \) is rotationally symmetric in \( D \). On the other hand, \( C \) is also a topological circle bounding the disk \( S^2 \setminus D \) which has only one extremal point \( q \), by Proposition 2.2 again, \( g \) is also rotationally symmetric in \( S^2 \setminus D \). Therefore, \( g \) is globally rotationally symmetric. It can be written as:

\[
g = du^2 + f^2(u)d\theta^2 \quad (0 \leq u \leq l, \ 0 \leq \theta \leq 2\pi),
\]

with \( p \) and \( q \) corresponding to \( u = 0 \) and \( l \) respectively, and \( \text{dist}_g(p,q) = l \), see Figure 1.

If we assume the angle of the metric at \( p \) is \( \alpha \), the angle at \( q \) is \( \beta \), \( (\alpha \geq \beta) \), then \( f \) satisfies:

\[
\begin{align*}
& f(0) = f(l) = 0, \\
& f'(0) = \alpha, f'(l) = -\beta, \\
& f(u) > 0, u \in (0, l).
\end{align*}
\]

By (9), the scalar curvature \( K \) is given by:

\[
K = -\frac{f''}{f}.
\]

**Proposition 4.** There is a constant \( c \) such that \( K' = cf \).

*Proof:* Since \( g \) is a HCMU metric, from (8) and (9) we have \( K''f = K'f' \), that is \( K' = cf \). \( \square \)

**Proposition 5.** \( K' \leq 0 \), moreover, if \( K \neq \text{Constant} \), then only when \( u = 0 \) and \( u = l \), \( K' = 0 \).

*Proof:* Define a function: \( F = f'^2 + Kf^2 \). Then \( F(0) = \alpha^2, F(l) = \beta^2, F(l) \leq F(0) \), so there is a \( \xi \in (0,l) \) such that

\[
\frac{F(l) - F(0)}{l} = F'\xi \leq 0.
\]

On the other hand,

\[
F' = 2f'f'' + Kf'^2 + 2Kf f',
\]

by \( K = \frac{-f''}{f} \), we have \( K'(\xi) \leq 0 \). From Proposition 4, \( K' = cf \), and \( f \) is positive on \((0,l)\), if \( K \) is not a constant, \( K' \) does not change its sign from 0 to \( l \). Hence, \( K'(\xi) \leq 0 \) implies \( K'(u) < 0 \), \((\forall u \in (0,l))\). Moreover, \( K' = 0 \) only when \( u = 0 \) and \( l \). \( \square \)

**Remark 1:** From the proof of Proposition 5, we also get \( K \equiv \text{Constant} \) if and only if \( \alpha = \beta \).
In the following, we always assume $K \neq \text{Constant}$. Therefore $K$ decreases monotonically from $p$ to $q$. Substituting $f = \frac{K'}{c}$ into $K = -\frac{f''}{f}$, we get $K''' + K'K = 0$, that is

$$\frac{K'^2}{2} = C_0K - \frac{K^3}{6} + C_1,$$  \hspace{1cm} (12)

here $C_0$ and $C_1$ are two constants. Assuming $K(0) = K_0, K(l) = K_1$ and letting $u = 0$ and $l$ in (12), we know both $K_0$ and $K_1$ are roots of the equation $-\frac{K^3}{6} + C_0K + C_1 = 0$. Then

$$-\frac{K^3}{6} + C_0K + C_1 = \frac{1}{6}(K - K_0)(K - K_1)(K + K_0 + K_1)$$

and

$$K'^2 = \frac{1}{3}(K - K_0)(K - K_1)(K + K_0 + K_1).$$  \hspace{1cm} (13)

We take derivatives of (13) to get:

$$K'' = -\frac{1}{6}[(K - K_0)(K + K_0 + K_1) + (K - K_1)(K + K_0 + K_1) + (K - K_0)(K - K_1)].$$  \hspace{1cm} (14)

Using Proposition 4, we have:

$$cf' = -\frac{1}{6}[(K - K_0)(K + K_0 + K_1) + (K - K_1)(K + K_0 + K_1) + (K - K_0)(K - K_1)].$$  \hspace{1cm} (15)

Let $u = 0$ and $l$ in (15), then by (10):

$$\begin{cases}
\alpha = \frac{(K_1 - K_0)(2K_0 + K_1)}{6c}, \\
\beta = \frac{(K_1 - K_0)(2K_1 + K_0)}{6c}.
\end{cases}$$

Integrate the equation $K' = cf$ from 0 to $l$, we get:

$$K_1 - K_0 = c \int_0^l f(u)du = c \frac{A(g)}{2\pi},$$  \hspace{1cm} (16)

here $A(g)$ denotes the area of the metric $g$. Then we have:

$$\begin{cases}
\alpha = \frac{A(g)}{12\pi}(2K_0 + K_1), \\
\beta = \frac{A(g)}{12\pi}(2K_1 + K_0).
\end{cases}$$  \hspace{1cm} (17)

or:

$$\begin{cases}
K_0 = \frac{4\pi}{A(g)}(2\alpha - \beta), \\
K_1 = \frac{4\pi}{A(g)}(2\beta - \alpha).
\end{cases}$$  \hspace{1cm} (18)

From (18) we see if $\alpha, \beta, A(g)$ are fixed, then $K_0$ and $K_1$ are uniquely determined, and by (13) $K$ is determined, again according to $K' = cf$, $f$ is determined, i.e. the metric $g$ is determined. Therefore we get:
Theorem C. If area and angles at both extremal points are given, there exists a unique rotationally symmetric HCMU metric in $S^2$, which is a football.

Meanwhile we get:

Corollary 1. In a football, assume that $\alpha$ is the angle of the HCMU metric at the local maximum point of $K$, $\beta$ is the angle of the metric at the local minimum point of $K$,

$K_0 = \max K$, $K_1 = \min K$, then

(1) $K_0 > 0$, the sign of $K_1$ is the same as $2\beta - \alpha$.
(2) $K_0 > K_1 > -(K_0 + K_1)$.

Next we consider a HCMU metric $g$ in a K-Surface $M_{(\alpha_1, \alpha_2, \ldots, \alpha_n)}$. Since the integral curves of $\vec{V}$ are the level sets of $K$, $\nabla K \perp \vec{V}$, integral curves of $\nabla K$ are geodesics. If $p$ is a local minimum point of $K$, in a small neighborhood of $p$, the integral curves of $\vec{V}$ are topologically concentric circles and the integral curves of $\nabla K$ are perpendicular to them. Choose an integral curve $c(t)$ $(t \in [0, T])$ of $\vec{V}$. For each $t$, there exists a unique integral curve $C_t$ of $\nabla K$ starting from $p$ and passing through the point $c(t)$. See Figure 2.

![Figure 2](image)

Obviously, $C_t$ must reach some saddle point of $K$ or some local maximum point of $K$. We have the following:

Lemma 1. For $t_1, t_2 \in (0, t_0)$. If both $C_{t_1}$ and $C_{t_2}$ reach local maximum points $q_1$ and $q_2$ directly without passing through any saddle point of $K$, then $\text{dist}_g(p, q_1) = \text{dist}_g(p, q_2) = l$.

Proof: Notice that in a small neighborhood of $p$ (we say $D$), $g$ is rotationally symmetric. Therefore, in $D$, $g$ can be written as $g = du^2 + f^2(u)dt^2$, where $t$ is the parameter of $c(t)$. Hence $K = -\frac{f'}{f}$, $K' = \frac{dK}{du} = cf$. Then by (13)

$$K' = -\sqrt{-\frac{1}{3}(K - K_0)(K^2 + a_0K + a_1)},$$

(19)

here $K_0 = K(p)$, $a_0$ and $a_1$ are constants. On the other hand, if we restrict $K$ at $C_{t_1}$, $K$ is a smooth function of the arc length parameter $s$ ($s = u$) of $C_{t_1}$. Moreover, at $C_{t_1} \cap D$, we have (19). According to ODE theory, we know that (19) holds true at the whole of $C_{t_1}$.
Since $K'(q_1) = 0$ (It is because in a neighborhood of $q_1$, $g$ is also rationally symmetric, we have $K' = \hat{c}\hat{f}$ and $\hat{f}(0) = 0$), we get:

$$K' = -\sqrt{-\frac{1}{3}(K - K_0)(K - K_1)(K + K_0 + K_1)},$$

(20)

here $K_1 = K(q_1)$. At $C_{t_2} \cap D$, we also have

$$K' = -\sqrt{-\frac{1}{3}(K - K_0)(K - K_1)(K + K_0 + K_1)}.$$  

Because along $C_{t_2}$ except at $p$ and $q_2$, there is no point at which $\nabla K = 0$, we get $K(q_2) = K_1$. Furthermore, since

$$\frac{dK}{ds} = -\sqrt{-\frac{1}{3}(K - K_0)(K - K_1)(K + K_0 + K_1)},$$

(21)

$$\frac{ds}{dK} = -\frac{1}{\sqrt{-\frac{1}{3}(K - K_0)(K - K_1)(K + K_0 + K_1)}}.$$  

Hence, the length of a geodesic from $p$ to $q_1$ is:

$$l = \int_{K_1}^{K_2} \frac{dK}{\sqrt{-\frac{1}{3}(K - K_1)(K - K_2)(K + K_1 + K_2)}}.$$  

(22)

The same as above, the length of a geodesic from $p$ to $q_2$ is also $l$. □

**Lemma 2.** Fix $t_0 \in [0, T]$ and suppose $C_{t_0}$ reaches a maximum point $q$ of $K$ directly, then $\exists \varepsilon > 0$, s.t. $\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, $C_t$ reaches the same maximum point $q$ without passing through any saddle point of $K$.

**Proof:** Since the saddle points of $K$ are finite, there exists a small neighborhood of $t_0$ $(t_0 - \varepsilon, t_0 + \varepsilon)$ s.t. each $C_t(t \in (t_0 - \varepsilon, t_0 + \varepsilon))$ does not reach any saddle point. The end points of $C_t$ are continuously dependent on $t$ and the maximum points of $K$ are finite. Therefore, $\exists \varepsilon > 0$ s.t. $\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, $C_t$ reaches the same maximum point. □

**Proof of Theorem B:** By virtue of Lemma 2, $\bigcup_{t \in (t_0 - \varepsilon, t_0 + \varepsilon)} C_t$ is a simply connected domain in $M$. Suppose $F$ is the largest simply connected domain in $M$ which contains $\bigcup_{t \in (t_0 - \varepsilon, t_0 + \varepsilon)} C_t$ and satisfies the following three properties:

1. Any integral curve of $\nabla K$ in $F$ is from $p$ to $q$.
2. Any integral curve of $\nabla K$ in $F$ does not pass through any saddle point of $K$.
3. $\partial F$ are also integral curves of $\nabla K$, but they pass through some saddle points of $K$.

$\partial F$ can be divided into two curves connecting $p$ and $q$, we say $\gamma_1$ and $\gamma_2$. Along $\gamma_1$, there are some saddle points which are connected by geodesic segments (integral curves of $\nabla K$). At each saddle point, by Proposition 3, the included angle of two adjacent geodesics is $\pi$. Hence, $\gamma_1$ is smooth at each saddle point. Therefore, $\gamma_1$ is a smooth geodesic, either is $\gamma_2$. 
On the other hand, in $F$, $g$ is invariant along integral curves of $\vec{V}$, i.e. $g$ is rotationally symmetric in $F$. Therefore, we can parameterize $g$ as: $g = du^2 + f^2(u)d\theta^2$ as before. Hence, $(F, g)$ is isometric to a football.

At a minimum point $p$ of $K$, since there are finite integral curves of $\nabla K$ starting from $p$ and reaching saddle points, we can repeat the above proof to obtain finite pieces of the largest simply connected domains in $M$, which are all isometric to footballs and contain $p$ as a vertex. We can also repeat this operation at each minimum point of $K$ to obtain finite footballs. We claim that every point of $M$ is contained in the union of these footballs. For any point of $M$, there must be an integral curve of $\nabla K$ starting from it or ending at it or passing through it. If there are some saddle points at this integral curve, the point must be on the boundary of some largest domain. If not, this point must be in some largest domain. Therefore, every point of $M$ is contained in the union of these footballs. This completes the proof of Theorem B.

We also have following corollaries.

**Corollary 2.** On HCMU surface, the values of local minimum of scalar curvature $K$ are the same to each other, and the same narration is true for local maximum points of $K$.

**Corollary 3.** Let $g$ be a HCMU metric in a $K$-surface, If scalar curvature $K$ has only extremal points in the Surface, then the metric is a football.

### 3 Proof of Theorem A

We prove following lemma at first.

**Lemma 3.** Two footballs $S^2_{\{\alpha, \beta\}}$ and $S^2_{\{\alpha_1, \beta_1\}}$ can be smoothly glued along their meridians or some segments of the two meridians, iff $\frac{\alpha}{\beta} = \frac{\alpha_1}{\beta_1}$ and $\frac{A(g)}{A(g_1)} = \frac{\alpha}{\alpha_1}$, here $A(g)$ and $A(g_1)$ denote the areas of the two metrics.

**Proof:** Suppose $K_0$ ($\tilde{K}_0$) and $K_1$ ($\tilde{K}_1$) are the maximum and minimum of $K$ ($\tilde{K}$) of football $S^2_{\{\alpha, \beta\}}$ ($S^2_{\{\alpha_1, \beta_1\}}$) respectively.

$(\implies)$ If two footballs can be smoothly glued along their meridians or some segments of the two meridians, we see at each segment being smoothly glued, the arc length is the same, $K$ is the same, so the derivative of $K$, $K'$ is the same. Therefore we get from the equation (13) and Corollary 1, $K_0 = \tilde{K}_0, K_1 = \tilde{K}_1$. Then from the equation (17), we get $\frac{\alpha}{\beta} = \frac{\alpha_1}{\beta_1}$ and $\frac{A(g)}{A(g_1)} = \frac{\alpha}{\alpha_1}$.

$(\impliedby)$ Assume the metric of two football are given by

\[
g = du^2 + \alpha_1^2f^2d\theta^2 (0 \leq u \leq l, 0 \leq \theta \leq \frac{2\pi}{\alpha_1}),
\]

\[
\tilde{g} = du_1^2 + \alpha^2f_1^2d\theta_1^2 (0 \leq u_1 \leq l_1, 0 \leq \theta_1 \leq \frac{2\pi}{\alpha}).
\]

If $\frac{\alpha}{\beta} = \frac{\alpha_1}{\beta_1}$ and $\frac{A(g)}{A(g_1)} = \frac{\alpha}{\alpha_1}$, by the equation (18), we get $K_0 = \tilde{K}_0, K_1 = \tilde{K}_1$. Then by the equation (21) and (20), $u_1 = u, K' = \tilde{K}'$ and by the equation (22), $l = l_1$. Meanwhile, we have $\frac{f_1}{f} = \frac{\alpha_1}{\alpha}$. Hence, if we let $u = u_1$ and $\theta = \theta_1$, then $\alpha_1f = \alpha f_1$, i.e. $g$ and $\tilde{g}$ are locally
isometric. Since the meridians are geodesics, two footballs \( S^2_{\{\alpha, \beta\}} \) and \( S^2_{\{\alpha_i, \beta_i\}} \) can be glued together along their meridians or some segments of meridians. □

Proof of Theorem A.

Claim: there exists a HCMU metric on \( S^2 \) whose singular angles are exactly \( \alpha_1, \alpha_2, \cdots, \alpha_n \) and the angles at the \( j_0 \) saddle points of the scalar curvature \( K \) are exactly \( \alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \cdots, \alpha_{\sigma(j_0)} \).

We will use some suitable footballs to construct \( S^2_{\{\alpha_1, \alpha_2, \cdots, \alpha_n\}} \) which satisfies the condition of the claim. Without loss of generality, we assume \( \alpha_{\sigma(i)} = \alpha_i, \ i = 1, 2, \cdots, j_0 \).

**Step 1: construction of a HCMU metric with one saddle point of angle \( 2\pi \alpha \) \( (i = 1, 2, \cdots, j_0) \).**

Choose \( \alpha_i \) footballs, say \( S^2_{\{x_1, y_1\}}, S^2_{\{x_2, y_2\}}, \cdots, S^2_{\{x_n, y_n\}} \), they satisfy

\[
\frac{x_m}{y_m} = \frac{x_n}{y_n} > 1, \quad \frac{A(y_m)}{A(y_n)} = \frac{x_m}{x_n},
\]

for any \( m, n \in \{1, 2, \cdots, \alpha_i\} \). Then by Lemma 3, two footballs can be glued together. We are going to glue these footballs together. For example, when \( \alpha_i = 3 \), take 3 footballs \( S^2_{\{x_1, y_1\}}, S^2_{\{x_2, y_2\}}, S^2_{\{x_3, y_3\}} \), see Figure 3.

![Figure 3](image)

Along the meridian, cut \( S^2_{\{x_1, y_1\}} \) from \( A \) to \( Q \). \( \hat{AQ} \) becomes two identical arcs: \( B \) and \( B' \).

The same as above, we cut \( S^2_{\{x_2, y_2\}} \) along the meridian and get \( B, B'' \); we cut \( S^2_{\{x_3, y_3\}} \) along the meridian and get \( B', B'' \). Then we glue \( B \) in \( S^2_{\{x_1, y_1\}} \) and \( B \) in \( S^2_{\{x_2, y_2\}} \) together, \( B' \) in \( S^2_{\{x_1, y_1\}} \) and \( B' \) in \( S^2_{\{x_3, y_3\}} \) together, \( B'' \) in \( S^2_{\{x_2, y_2\}} \) and \( B'' \) in \( S^2_{\{x_1, y_1\}} \) together, to obtain a HCMU metric with one saddle point \( A \ (= A' = A'') \) of angle \( 6\pi \). Meanwhile, \( Q = Q' = Q'' \), at which the angle is \( 2\pi(y_1 + y_2 + y_3) \).

Obviously we can glue \( \alpha_i \) footballs together in the same way, to obtain a HCMU metric on \( S^2 \), which has \( \alpha_i \) local maximum points of angles \( 2\pi x_1, 2\pi x_2, \cdots, 2\pi x_{\alpha_i} \), one saddle point of angle \( 2\pi \alpha_i \), and one minimum point of angle \( 2\pi(y_1 + y_2 + \cdots + y_{\alpha_i}) \).

**Step 2: construction of a HCMU metric with \( j_0 \) saddle points.**
We choose $\alpha_1$ footballs $S^2_{\{x_k, y_k\}} (k = 1, 2, \cdots, \alpha_1)$ to construct the first saddle point of angle $2\pi \alpha_1$ like Step 1. Then we choose another meridian on $S^2_{\{x_{\alpha_1}, y_{\alpha_1}\}}$ which is different from the one passing the previous saddle point, cut this meridian like Step 1, choose $\alpha_2 - 1$ footballs $S^2_{\{x_k, y_k\}} (k = \alpha_1 + 1, \cdots, \alpha_1 + \alpha_2 - 1)$, cut them and glue them together with the football $S^2_{\{x_{\alpha_1}, y_{\alpha_1}\}}$ like Step 1, we get the second saddle point of angle $2\pi \alpha_2$. And then we choose $\alpha_3 - 1$ footballs $S^2_{\{x_k, y_k\}} (k = \alpha_1 + \alpha_2, \cdots, \alpha_1 + \alpha_2 + \alpha_3 - 2)$ with $S^2_{\{x_{\alpha_1 + \alpha_2 - 1}, y_{\alpha_1 + \alpha_2 - 1}\}}$ to construct the third saddle point of angle $2\pi \alpha_3$, and so on, finally we have chosen $\sum_{i=1}^{j_0} \alpha_i - (j_0 - 1)$ footballs to construct a HCMU metric on $S^2$ with $j_0$ saddle points, as we desire.

**Step 3: construction of $S^2_{\{\alpha_1, \alpha_2, \cdots, \alpha_n\}}$.**

Denote $N = \sum_{i=1}^{j_0} \alpha_i - (j_0 - 1)$. By Step 2, we have constructed a HCMU metric of $j_0$ saddle points with the angles $2\pi \alpha_1, 2\pi \alpha_2, \cdots, 2\pi \alpha_{j_0}$, $N$ maximum points with the angles $2\pi x_k$ $(k = 1, 2, \cdots, N)$, and one minimum point with the angle $2\pi \sum_{k=1}^{N} y_k$. These angles satisfy:

$$\frac{x_k}{y_k} = \frac{x_l}{y_l} > 1, \ k, l \in \{1, 2, \cdots, N\}. \quad (23)$$

In the following we adjust $x_k, y_k$ to make the metric coincide with what we desire.

Without loss of generality, we assume $\alpha_n = \min_{j_0+1 \le k \le n} \{\alpha_k\}$ and denote $s = \sum_{i=1}^{j_0} \alpha_i - (j_0 - 1) - (n - j_0 - 1)$. By the condition of Theorem A,

$$s = \sum_{i=1}^{j_0} \alpha_i - n + \chi(S^2) \ge 0.$$

**Case 1.** If $s = 0$, we let

$$x_k = \alpha_{j_0 + k} \quad (1 \le k \le N),$$

$$\alpha_n = \sum_{k=1}^{N} y_k.$$

By (23) the angles $y_1, y_2, \cdots, y_N$ must satisfy the following equations:

$$\begin{cases}
\sum_{k=1}^{N} y_k = \alpha_n, \\
\frac{\alpha_{j_0+1}}{y_1} = \frac{\alpha_{j_0+2}}{y_2} = \cdots = \frac{\alpha_{n-1}}{y_{n-j_0-1}}.
\end{cases}$$

The equations have a unique solution

$$y_k = \alpha_{j_0 + k} \frac{\alpha_n}{n-1} \sum_{i=j_0+1}^{n-1} \alpha_i \quad (1 \le k \le N = n - j_0 - 1).$$

Thus we have proved the claim in this case.
Case 2. If $s > 0$, we let

$$
x_k = \alpha_{j_0+k}, \quad (1 \leq k \leq n - j_0 - 1),
$$

$$
x_k = 1 \quad (n - j_0 \leq k \leq N),
$$

$$
\sum_{k=1}^{N} y_k = \alpha_n.
$$

This means that there are $s \ (= N - (j_0 - 1))$ smooth maximum points in the surface, correspondent to angles $x_k \ (k = n - j_0, \cdots, N)$.

The undetermined angles $y_k \ (k = 1, \cdots, N)$ satisfy

$$
\begin{align*}
\sum_{k=1}^{N} y_k &= \alpha_n, \\
\frac{\alpha_{j_0+1}}{y_1} &= \frac{\alpha_{j_0+2}}{y_2} = \cdots = \frac{\alpha_{n-1}}{y_{n-j_0-1}} = \frac{1}{y_{n-j_0}} = \cdots = \frac{1}{y_N}.
\end{align*}
$$

The equations have a unique solution

$$
y_k = \alpha_{j_0+k} \frac{\alpha_n}{s + \sum_{i=j_0+1}^{n-1} \alpha_i} \quad (1 \leq k \leq n - j_0 - 1),
$$

$$
y_k = \frac{\alpha_n}{s + \sum_{i=j_0+1}^{n-1} \alpha_i} \quad (n - j_0 \leq k \leq N).
$$

We also prove the claim in this case.

Till now we have constructed a HCMU metric on $S^2$ with the singular angles $\alpha_1, \alpha_2, \cdots, \alpha_n$ and the correspondent singular points $q_1, q_2, \cdots, q_n$, which may be different from the prescribed points $p_1, p_2, \cdots, p_n$. However, there exists a diffeomorphism $h : S^2 \rightarrow S^2\{q_1, q_2, \cdots, q_n\}$, s.t. $h(p_i) = q_i, \ i = 1, 2, \cdots, n$. Therefore, we can use $h$ to pull back the HCMU metric $g$ constructed by the claim on $S^2$ to obtain the HCMU metric $h^*g$ as Theorem A desires.

**Remark 2:** We must point out that the HCMU metric satisfying the condition in Theorem A is not unique, the metric we construct in the proof has exactly one minimum point of the scalar curvature $K$. Indeed, by Theorem B we know that the construction of a HCMU metric form footballs is a combination problem. If we can find some suitable footballs and glue them together to satisfy the condition of Theorem A, we have a HCMU metric as desired, and the resulting metric may have more than one minimum point of the scalar curvature. For example, we can choose two footballs, either $S^2_{\{\frac{1}{2}, \frac{1}{2}\}}$ and $S^2_{\{1, \frac{1}{2}\}}$, or $S^2_{\{\frac{1}{3}, \frac{1}{2}\}}$ and $S^2_{\{\frac{2}{3}, \frac{1}{2}\}}$, to construct $S^2_{\{2, \frac{1}{2}, \frac{1}{2}\}}$, see following figures for illustration.
In fact, Figure 4 is the same as what we constructed in the proof of Theorem A. However, Figure 5 has two minimum point of the scalar curvature.

**Remark 3:** We should point out that if $\chi(M) \leq 0$, the condition (7) is not sufficient. For example, if $\chi(M) = -2$, we have the following counter example.

**Counter example:** Let $\chi(M) = -2$, $n = 3$, $\alpha_i = 2$, $i = 1, 2, 3$ and $j_0 = 3$. Then $\sum_{i=1}^{j_0} \alpha_{\sigma(i)} + \chi(M) - n = 6 - 2 - 3 = 1 > 0$, but there is no HCMU metric whose scalar curvature is not a constant s.t. three angles of saddle points of $K$ are all $4\pi$. If the metric exists, according to Proposition 1, $K$ is continuous in a K-Surface. It must take its maximum and minimum, but the point at which $K$ takes maximum or minimum can not be a saddle point, so the point is a smooth critical point of $K$. Hence the number of smooth critical points of $K$ is more than 1. However, from the formula

$$\chi(M) = \sum_{i=1}^{j_0} (1 - \alpha_i) + n - j_0 + s,$$

we see $s = 1$. That means the number of smooth critical points of $K$ is 1, a contradiction. □

**Remark 4:** At last we list some problems that might be interesting for future study:

1. For other compact Riemannian surfaces, what is the sufficient and necessary condition for the existence of a HCMU metric?

2. Is the extremal Hermitian metric unique when none of the prescribed angles is an integer multiple of $2\pi$?

3. Given any surface configuration, is the Calabi energy the only factor determining the connected components in the moduli space of HCMU metrics?

4. If we deform the complex structure as well, what is the structure of the moduli space of HCMU metrics? It will be interesting to compare this to the classical Teichmüller space in Riemann surfaces.

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