WEIGHTED BERGMAN PROJECTION ON THE HARTOGS TRIANGLE

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ABSTRACT. We prove the $L^p$ regularity of the weighted Bergman projection on the Hartogs triangle by using a two-weight inequality on the upper half plane related to the $A^+_p$-condition, where the weight is the product of any power of the absolute value of the variable and the norm square of a non-vanishing holomorphic function of the second component.

1. Introduction

In this paper, we study the $L^p$ regularity of the weighted Bergman projection on the Hartogs triangle and its generalization. The $L^p$ regularity of the ordinary Bergman projection is of considerable interest for many years. Classical result shows that it highly depends on the regularity of the boundary of the underlying domain. For domains having sufficiently smooth boundary, see [8], [21], [15], [19], [16], [17], [6], [1], [14], etc. For the non-smooth case, see [13], [11], [12], [26], etc.

In particular, there are several people recently care about the regularity of the Bergman projection on the Hartogs triangle. In [3] and [4], Chakrabarti and Shaw focus on the $\overline{\partial}$-equation and the corresponding Sobolev regularities on the product domains and the Hartogs triangle. In [7], we show the ordinary Bergman projection is $L^p$ bounded on the Hartogs triangle if and only if $p \in \left(\frac{4}{3}, 4\right)$. In [5], Chakrabarti and Zeytuncu show a sharp result for the $L^p$ regularity of the ordinary Bergman projection on a weighted space of the Hartogs triangle.

A natural question is that, when we look at some weighted space of the Hartogs triangle, can we obtain the $L^p$ regularity of the weighted Bergman projection for a larger range of $p$? The answer is affirmative. We first use the estimates on the punctured disk $D^* = D \setminus \{0\}$, with the weight $\mu(z) = |z|^{s'}$, where $z \in D^*$ and $s' \in \mathbb{R}$.

Theorem 1.1. For $s' \in \mathbb{R}$ with the unique expression $s' = s + 2k$, where $k \in \mathbb{Z}$ and $s \in (0, 2]$, let $B_{s'}$ be the weighted Bergman projection on the space $(D^*, \mu)$, where $\mu(z) = |z|^{s'}$.

1. For $s' \in (0, \infty)$, $B_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in \left(\frac{s+2k+2}{2+k+1}, \frac{s+2k+2}{k+1}\right)$.
2. For $s' \in [-3, 0]$, $B_{s'}$ is $L^p(\mu)$ bounded for $p \in (1, \infty)$.
3. For $s' \in (-4, -3)$, then $k = -2$ and $s \in (0, 1)$, $B_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in \left(2 - s, \frac{2}{s+1}\right)$.
4. When $s' = -4$, $B_{-4}$ is $L^p(\mu)$ bounded for $p \in (1, \infty)$.

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(5) For $s' \in (-\infty, -4)$, $B_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in (\frac{s+2k+2}{k+1}, \frac{s+2k+2}{s+k+1})$.

Now let $\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < |z_2| < 1\}$ be the Hartogs triangle, then by inflation (see Definition 2.4), one obtains the following corollary.

**Corollary 1.2.** Given any $p_0 \in [1, 2)$ with its conjugate exponent $p_0'$, let $\lambda(z) = |z_2|^{-p_0}$, where $z \in \mathbb{H}$, then the weighted Bergman projection on the space $(\mathbb{H}, \lambda)$ is $L^p(\lambda)$ bounded if and only if $p \in (p_0, p_0')$.

If we apply the same technique several times, we will obtain the weighted version of the result in [7]. To be precise, for $j = 1, \ldots, l$, let $\Omega_j$ be a bounded smooth domain in $\mathbb{C}^{m_j}$ with a biholomorphic mapping $\phi_j : \Omega_j \rightarrow \mathbb{B}^{m_j}$ between $\Omega_j$ and the unit ball $\mathbb{B}^{m_j}$ in $\mathbb{C}^{m_j}$. We use the notation $\tilde{z}_j$ to denote the $j$th $m_j$-tuple in $z \in \mathbb{C}^{m_1 + \cdots + m_l}$, that is $z = (\tilde{z}_1, \ldots, \tilde{z}_l)$. Let $N = m_1 + \cdots + m_l + n$, we define the $N$-dimensional Hartogs triangle by

$$
\mathbb{H}^N_{\phi_j} = \{(z, w) \in \mathbb{C}^{m_1 + \cdots + m_l + n} \mid \max_{1 \leq j \leq l} |\phi_j(\tilde{z}_j)| < |w_1| < |w_2| < \cdots < |w_n| < 1\}.
$$

Let $\lambda(w) = |w_1|^{s_1} \cdots |w_n|^{s_n}$, where $s_1, \ldots, s_n \in \mathbb{R}$. We consider the weighted Bergman projection on $(\mathbb{H}^N_{\phi_j}, \lambda)$.

**Corollary 1.3.** Let $(\mathbb{H}^N_{\phi_j}, \lambda)$ be as above, and let $B_{s'}$ be as in Theorem 1.1. Then the weighted Bergman projection on $(\mathbb{H}^N_{\phi_j}, \lambda)$ is $L^p(\lambda)$ bounded if and only if each of the following projections

$$
B_{2(m_1 + \cdots + m_l)} |w_1|^{s_1} B_{2(m_1 + \cdots + m_l)} |w_2|^{s_2} \cdots B_{2(m_1 + \cdots + m_l)} |w_n|^{s_n} \text{ is } L^p \text{ bounded on the corresponding weighted space.}
$$

In other words, assume $p > 1$ and for $j = 1, 2, \ldots, n$ we let $I_j$ be one of the intervals for $p$ in Theorem 1.1, so that the $j$th projection above is $L^p$ bounded if and only if $p \in I_j$, then the weighted Bergman projection on $(\mathbb{H}^N_{\phi_j}, \lambda)$ is $L^p(\lambda)$ bounded if and only if $p \in \cap I_j$.

Another natural question one may ask is that, can we obtain the $L^p$ regularity of the weighted Bergman projection on the Hartogs triangle for a wider class of weights rather than some power of the absolute value of the second component? The answer is partly affirmative.

Follow Zeytuncu’s idea in [20], and use the singular integral approach by Lanzani and Stein in [13], we can use the estimates on the weighted space $(\mathbb{D}^+, \mu)$, where $\mu(z) = |z|^s |g(z)|^2$, $s' \in \mathbb{R}$ and $g$ is a non-vanishing holomorphic function on the unit disk. By applying a Möbius transform, the isolated pole or zero indeed can be any point in the unit disk.

The key observation is that, the weighted Bergman kernel $B_{s'}(z, \zeta)$ on $(\mathbb{D}^+, |z|^{s'})$, can be expressed as a “homotopy” between two weighted Bergman kernels

$$
B_{s'}(z, \zeta) = \frac{s}{2} B_{2k+2}(z, \zeta) + \left(1 - \frac{s}{2}\right) B_{2k}(z, \zeta),
$$

$(z, \zeta) \in \mathbb{D}^+ \times \mathbb{D}^+$, where $s' = s + 2k$, $k \in \mathbb{Z}$ and $s \in (0, 2]$. Therefore, using a biholomorphism $\phi : \mathbb{R}^2_+ \rightarrow \mathbb{D}$, where $\phi(z) = \frac{2z}{1+|z|^2}$, one needs to consider different types of the following two-weight inequality on the upper half plane

$$
(1.1) \quad \int_{\mathbb{R}^2_+} \left| \int_{\mathbb{R}^2_+} \frac{f(w)}{(z-w)^2} d(w) \right|^p \mu_1(z) d(z) \leq C \int_{\mathbb{R}^2_+} |f(z)|^p \mu_2(z) d(z),
$$
where $\mu_1$ and $\mu_2$ are two weights on $\mathbb{R}_+^2$.

In section 4, we will mainly focus on the inequality (1.1), and show the following.

**Theorem 1.4.** For $p > 1$, if the two weights $\mu_1$ and $\mu_2$ satisfy $(\mu_1, \mu_2) \in A_p^+ (\mathbb{R}_+^2)$ and either $\mu_1 \in A_p^+ (\mathbb{R}_+^2)$ or $\mu_2 \in A_p^+ (\mathbb{R}_+^2)$, then (1.1) holds for some $C > 0$.

We postpone the definition of $A_p^+ (\mathbb{R}_+^2)$ to section 4, because it is not very important for us to state the main results. If one compares Theorem 1.4 with Proposition 4.5 in [14], one sees that the condition $\mu_j \in A_p^+ (\mathbb{R}_+^2)$ $j = 1$ or $2$, seems to be redundant. However, up till now, we still can not get rid of this condition, although we believe (1.1) holds even if only assuming $(\mu_1, \mu_2) \in A_p^+ (\mathbb{R}_+^2)$.

**Conjecture 1.5.** For $p > 1$, if the two weights $\mu_1$ and $\mu_2$ satisfy $(\mu_1, \mu_2) \in A_p^+ (\mathbb{R}_+^2)$, then (1.1) holds for some $C > 0$.

By considering the result in [20], we are also interested in the variant of Conjecture 1.5 by assuming the "power-bump" the condition.

**Conjecture 1.6.** For $p > 1$, if the two weights $\mu_1$ and $\mu_2$ satisfy $(\mu_1^r, \mu_2^r) \in A_p^+ (\mathbb{R}_+^2)$ for some $r > 1$, then (1.1) holds for some $C > 0$.

Despite of the fact above, Theorem 1.4 is sufficient for our application.

**Theorem 1.7.** Assume $p > 1$. Let $\mu(z) = |z|^{s'} |g(z)|^2$, where $g$ is a non-vanishing holomorphic function on $\mathbb{D}$ and $s' \in \mathbb{R}$. Suppose the weighted Bergman projection $B_{|g|^2}$ on $(\mathbb{D}, |g|^2)$ is $L^p (|g|^2)$ bounded, and suppose the weighted Bergman projection $B_{s'}$ on $(\mathbb{D}, |z|^{s'})$ is $L^p (|z|^{s'})$ bounded. Then the weighted Bergman projection $B_\mu$ on $(\mathbb{D}, \mu)$ is $L^p (\mu)$ bounded.

Moreover, suppose $B_{|g|^2}$ is $L^p (|g|^2)$ bounded if and only if $p \in (p_0, p_0')$ for some $p_0 \geq 1$ and suppose $B_{s'}$ is $L^p (|z|^{s'})$ bounded if and only if $p \in (p_1, p_1')$ for some $p_1 \geq 1$ as in Theorem 1.1. If $(p_1, p_1') \subset (p_0, p_0')$ properly, then $B_\mu$ is $L^p (\mu)$ bounded if and only if $p \in (p_1, p_1')$.

**Example 1.8.** In [20], if we take $g(z) = (z - 1)^\alpha$ for some $\alpha > 0$, then we see $(p_0, p_0') = (\frac{2\alpha + 2}{\alpha + 2}, \frac{2\alpha + 2}{\alpha + 2})$. By Theorem 1.1, when $s' \in (0, \infty)$, we have $(p_1, p_1') = (s + 2k + 2, s + 2k + 2)$. So $B_\mu$ is $L^p (\mu)$ bounded if $p \in (\frac{2\alpha + 2}{\alpha + 2}, \frac{2\alpha + 2}{\alpha + 2}) \cap (s + 2k + 2, s + 2k + 2)$.

Again by inflation, on the Hartogs triangle, if we let $\lambda(z) = \mu(z_2) |z_2|^{-2}$, where $\mu(z) = |z|^{s'} |g(z)|^2$ for some non-vanishing holomorphic function $g$ on $\mathbb{D}$, then we have the following corollary.

**Corollary 1.9.** Assume $p > 1$. Let $\lambda$ be as above, and let $p_0, p_1$ be as in Theorem 1.7. Then the weighted Bergman projection $B_\lambda$ on $(\mathbb{H}, \lambda)$ is $L^p (\lambda)$ bounded if $p \in (p_0, p_0') \cap (p_1, p_1')$. In addition, if $(p_1, p_1') \subset (p_0, p_0')$ properly, $B_\lambda$ is $L^p (\lambda)$ bounded if and only if $p \in (p_1, p_1')$.

As Chakrabarti and Zeytuncu did in [5], by applying Theorem 1.4, we can also consider the $L^p$ regularity of the weighted Bergman projection mapping from one weighted space to the other. See Corollary 4.18 and the following remarks.

Since the $A_p$ condition is closely related to the maximal function theory, we give some results about the "special" maximal operator $\tilde{M}^+$ in section 5 (see Definition 5.3).
Theorem 1.10. Let $f$ be a measurable function on $\mathbb{R}_+^2$, then for any $0 < q < 1$, the function $\left(\widetilde{M}^+(f)\right)^q$ is in $A^+_1(\mathbb{R}_+^2)$.

Theorem 1.11. Assume $p \geq 1$, suppose $\mu_1$ and $\mu_2$ are two weights on $\mathbb{R}_+^2$, then we have a weak-type $(p, p)$ inequality, namely, there is a $c > 0$ so that
\[\mu_1(\{z \in \mathbb{R}_+^2 | \widetilde{M}^+(f)(z) > \alpha\}) \leq \frac{c}{\alpha^p} \int_{\mathbb{R}_+^2} |f(z)|^p \mu_2(z) d(z)\]
for all $\alpha > 0$, if and only if $(\mu_1, \mu_2) \in A^+_p(\mathbb{R}_+^2)$.

If we consider the following operator
\[\widetilde{B}(f)(z) = \int_{\mathbb{R}_+^2} \frac{1}{|z - w|^2} f(w) d(w),\]
which is the "absolute value" of the Bergman projection on the upper half plane. It is easy to see, for $f \geq 0$ and $z, z' \in \mathbb{R}_+^2$,
\[\widetilde{B}(f)(z') \geq \widetilde{B}(f)(z),\]
whenever $\Re(z') = \Re(z)$ and $\Im(z') \leq \Im(z)$. The special maximal operator $\widetilde{M}^+$ also has the same property above as $\widetilde{B}$. So we hope the results in section 5 will provide some clue to prove Conjecture 1.5 and Conjecture 1.6.

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2. PRELIMINARIES

Let us temporarily consider the general setting for a moment, and suppose $\Omega$ is a domain in $\mathbb{C}^n$.

Definition 2.1. A measurable function $\mu$ is a weight on $\Omega$, if $\mu > 0$ almost everywhere.

As long as we have a weight $\mu$, we can define the weighted $L^p(\mu)$ norm for a measurable function $f$ on $\Omega$ by
\[\|f\|_{L^p(\Omega, \mu)} = \left(\int_{\Omega} |f(z)|^p \mu(z) d(z)\right)^{\frac{1}{p}},\]
and the weighted $L^p(\mu)$ space by $\{f$ measurable $| \|f\|_{L^p(\Omega, \mu)} < \infty\}$, for $p \geq 1$. (Sometimes, we will use a different notation $L^p(\Omega, \mu)$ to emphasize the underlying domain.) Suppose the set of all holomorphic function on $\Omega$ is denoted by $\mathcal{O}(\Omega)$, then we consider the analytic subspace of $L^2(\mu)$ which is denoted by $A^2_\mu(\Omega) = L^2(\mu) \cap \mathcal{O}(\Omega)$.

Definition 2.2. A weight $\mu$ is admissible on $\Omega$, if for any compact subset $K$ of $\Omega$, there exists $C_K > 0$ such that
\[\sup_{z \in K} |f(z)| \leq C_K \|f\|_{L^2(\Omega, \mu)},\]
for all \( f \in A^2_\mu(\Omega) \). For instance, if \( \mu \) is continuous and non-vanishing, then it is admissible.

It is easy to see, if \( \mu \) is admissible on \( \Omega \), then \( A^2_\mu(\Omega) \) is closed in \( L^2(\mu) \).

**Definition 2.3.** For an admissible weight \( \mu \) on \( \Omega \), we define the **weighted Bergman projection** \( B_{\Omega, \mu} \) to be the orthogonal projection from \( L^2(\mu) \) to \( A^2_\mu(\Omega) \). The weighted Bergman projection is an integral operator

\[
B_{\Omega, \mu}(f)(z) = \int_{\Omega} B_{\Omega, \mu}(z, \zeta) f(\zeta) \mu(\zeta) \, d(\zeta),
\]

where \( B_{\Omega, \mu}(z, \zeta) \) is the **weighted Bergman kernel** with \( (z, \zeta) \in \Omega \times \Omega \).

It is not hard to see, every basic property of the ordinary Bergman theory can be moved parallelly to the weighted setting.

**Definition 2.4.** If \( \mu \) is a non-vanishing weight on \( \Omega \), we define the **inflation** \( \tilde{\Omega} \) of \( \Omega \) by

\[
\tilde{\Omega} = \{(z, w) \in \mathbb{C}^{m+n} \mid |z|^2 < \mu(w), w \in \Omega\}.
\]

Note that \( \tilde{\Omega} \) is a Hartogs domain.

Suppose \( \mu \) is a non-vanishing weight on \( \Omega \), and suppose \( \lambda > 0 \) is a function on \( \Omega \) such that it is admissible on \( \Omega \). Then it is easy to see \( \mu^\lambda \) is admissible on \( \Omega \). Throughout this paper, as long as we deal with the weighted Bergman theory, the weight is assumed to be admissible on the corresponding domain.

Before going further, we first give two useful lemmas.

**Lemma 2.5.** Let \( F : X_1 \to X_2 \) be an isometry between two Banach spaces \( X_1 \) and \( X_2 \). Then it induces an isometry \( F^* : \mathfrak{B}(X_1) \to \mathfrak{B}(X_2) \) between the spaces of the bounded operators by \( F^*(T) = F \circ T \circ F^{-1} \), for any \( T \in \mathfrak{B}(X_1) \).

In particular, if \( X_j = H_j \) is a Hilbert space, \( j = 1, 2 \). Let \( S \) be a closed subspace of \( H_1 \), and let \( P : H_1 \to S \) be the orthogonal projection. Then \( F \) induces an orthogonal decomposition \( H_2 = F(S) \oplus F(S^\perp) \), that is \( F(S) \) is closed in \( H_2 \) and \( F(S)^\perp = F(S^\perp) \). Hence, \( F^*(P) : H_2 \to F(S) \) is the orthogonal projection.

**Proof.** The first part of the lemma is straightforward, we only prove the second part. Since \( S \) is closed in \( H_1 \), and since \( F \) is an isometry, it is easy to see \( F(S) \) is closed in \( H_2 \).

To prove the equality \( F(S)^\perp = F(S^\perp) \), we consider the following. For any \( x \in F(S^\perp) \), we have \( F^{-1}(x) \in S^\perp \), then \( F^{-1}(x) \perp S \). Therefore \( x \perp F(S) \), which implies \( x \in F(S)^\perp \). This shows \( F(S^\perp) \subseteq F(S)^\perp \). The other direction follows from the same argument with reverse direction.

In the last statement, for any \( x \in H_2 \), we have a decomposition \( x = y + z \), where \( y \in F(S) \) and \( z \in F(S^\perp) \). Then \( F^{-1}(x) = F^{-1}(y) + F^{-1}(z) \), with \( F^{-1}(y) \in S \) and \( F^{-1}(z) \in S^\perp \). So \( F^*(P)(x) = F(P(F^{-1}(x))) = F(F^{-1}(y)) = y \) is orthogonal. \( \square \)

**Corollary 2.6.** Let \( \Phi : \Omega_1 \to \Omega_2 \) be a biholomorphism between two domains in \( \mathbb{C}^n \). Suppose \( \Omega_j \) is equipped with the weight \( \mu_j \), \( j = 1, 2 \), and \( \mu_2 = \mu_1 \circ \Phi^{-1} \). Then we have the transformation formula for the weighted Bergman kernels

\[
B_{\Omega_1, \mu_1}(z, w) = \det J_{\mathbb{C}} \Phi(z) B_{\Omega_2, \mu_2}(\Phi(z), \Phi(w)) \det J_{\mathbb{C}} \Phi(w),
\]

where \( (z, w) \in \Omega_1 \times \Omega_1 \).
Proof. Let $F : L^2(\Omega_1, \mu_1) \to L^2(\Omega_2, \mu_2)$ be the isometry, by $F(f) = \det J_C(\Phi^{-1}) f \circ \Phi^{-1}$, for any $f \in L^2(\Omega_1, \mu_1)$. Then by above lemma, we have $F^*(B_{\Omega_1, \mu_1}) = B_{\Omega_2, \mu_2}$. By the uniqueness of the weighted Bergman kernel, we obtain the transformation formula above.

**Corollary 2.7.** Let $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphism between two domains in $\mathbb{C}^n$. Suppose $\mathcal{B}_j$ is the weighted Bergman projection for $(\Omega_j, \mu_j)$, $j = 1, 2$, and $\mu_j = |\det J_C(\Phi^{-1})|^2 \mu_1 \circ \Phi^{-1}$. Then for $p \geq 1$, $\mathcal{B}_1$ is $L^p(\mu_1)$ bounded if and only if $\mathcal{B}_2$ is $L^p(\mu_2)$ bounded.

Proof. Let $F : L^p(\Omega_1, \mu_1) \to L^p(\Omega_2, \mu_2)$ be the isometry, by $F(f) = f \circ \Phi^{-1}$, for any $f \in L^p(\Omega_1, \mu_1)$. In particular, when $p = 2$, we have $F^*(\mathcal{B}_1) = \mathcal{B}_2$. Since this $F$ is an isometry for all $p \geq 1$, we see from the first part of above lemma, $\mathcal{B}_1$ is $L^p$-bounded if and only if $\mathcal{B}_2$ is $L^p$-bounded.

**Lemma 2.8.** Suppose we have a weight $\mu_1 > 0$ on $\Omega_1$ and a weight $\mu_2 > 0$ on $\Omega_2$, both non-vanishing. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the integral operators with kernels $T_1(w, \eta)$ on $\Omega_1 \times \Omega_1$ and $T_2(w, \eta)$ on $\Omega_2 \times \Omega_2$, respectively. That is,

$$
\mathcal{T}_1(f)(w) = \int_{\Omega_1} T_1(w, \eta)f(\eta)\mu_1(\eta) \, d\eta, \\
\mathcal{T}_2(f)(w) = \int_{\Omega_2} T_2(w, \eta)f(\eta)\mu_2(\eta) \, d\eta.
$$

Given any $p \in [1, \infty)$, if $\mathcal{T}_1$ is bounded on $L^p(\Omega_1, \mu_1)$ and $\mathcal{T}_2$ is bounded on $L^p(\Omega_2, \mu_2)$, then their product operator $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2$ with kernel $T_1 \otimes T_2$, is bounded on $L^p(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$.

Conversely, assuming $\mathcal{T}_1$ and $\mathcal{T}_2$ both are non-trivial, if one of these two operator is unbounded, then $\mathcal{T}$ is unbounded.

(In fact, the weights can be assumed to be non-negative, i.e. $\mu_1, \mu_2 \geq 0$, if we adopt the convention $0 \cdot \infty = 0$.)

**Proof.** By definition, we have

$$
\mathcal{T}(f)(w_1, w_2) = \int_{\Omega_1 \times \Omega_2} T_1(w_1, \eta_1)T_2(w_2, \eta_2)f(\eta_1, \eta_2)\mu_1(\eta_1)\mu_2(\eta_2) \, d(\eta_1, \eta_2)
= \int_{\Omega_1} T_1(w_1, \eta_1)f(\eta_1, \eta_2)\mu_1(\eta_1) \, d(\eta_1) \int_{\Omega_2} T_2(w_2, \eta_2)\mu_2(\eta_2) \, d(\eta_2)
= \int_{\Omega_2} T_1, \eta_1(f(\eta_1, \eta_2))(w_1)T_2(w_2, \eta_2)\mu_2(\eta_2) \, d(\eta_2)
= T_2, \eta_2(T_1, \eta_1(f(\eta_1, \eta_2))(w_1))(w_2),
$$
where $\mathcal{T}_{1,\eta_1}$ and $\mathcal{T}_{2,\eta_2}$ are operators $\mathcal{T}_1$ and $\mathcal{T}_2$ acting on $\eta_1$ and $\eta_2$ respectively. If $\mathcal{T}_1$ and $\mathcal{T}_2$ both are bounded,

$$
\|T(f)\|_{L_p(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)} = \int_{\Omega_1 \times \Omega_2} |\mathcal{T}_{2,\eta_2}(\mathcal{T}_{1,\eta_1}(f(\eta_1, \eta_2))(w_1))(w_2)|^p \mu_1(w_1) \mu_2(w_2) d(w_1, w_2)
$$

$$
= \int_{\Omega_2} |\mathcal{T}_{2,\eta_2}(\mathcal{T}_{1,\eta_1}(f(\eta_1, \eta_2))(w_1))(w_2)|^p \mu_2(w_2) d(w_2) \int_{\Omega_1} \mu_1(w_1) d(w_1)
$$

$$
\leq c \int_{\Omega_1} |\mathcal{T}_{1,\eta_1}(f(\eta_1, w_2))(w_1)|^p \mu_1(w_1) d(w_1) \int_{\Omega_2} \mu_2(w_2) d(w_2)
$$

$$
= c \int_{\Omega_1} |f(w_1, w_2)|^p \mu_1(w_1) d(w_1) \int_{\Omega_2} \mu_2(w_2) d(w_2)
$$

$$
= \|f\|_{L_p(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)}^p.
$$

Conversely, without loss of generality, we assume $\mathcal{T}_1$ is unbounded, then there is a sequence $\{f_n\} \subset L^p(\Omega_1, \mu_1)$ such that $\|f_n\|_{L_p(\Omega_1, \mu_1)} \leq c < \infty$ for some $c > 0$ and $\|\mathcal{T}_1(f_n)\|_{L_p(\Omega_1, \mu_1)} \to \infty$ as $n \to \infty$.

Since $\mathcal{T}_2$ is non-trivial, there is a function $g \in L^p(\Omega_2, \mu_2)$ such that $g \neq 0$ and $\mathcal{T}_2(g) \neq 0$. If we consider the sequence $(g \otimes f_n)$, we have

$$
\|g \otimes f_n\|_{L_p(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)}^p = \int_{\Omega_1 \times \Omega_2} |f_n(w_1)g(w_2)|^p \mu_1(w_1) \mu_2(w_2) d(w_1, w_2)
$$

$$
= \int_{\Omega_1} |f_n(w_1)|^p \mu_1(w_1) d(w_1) \int_{\Omega_2} |g(w_2)|^p \mu_2(w_2) d(w_2)
$$

and

$$
\lim_{n \to \infty} \|T(g \otimes f_n)\|_{L_p(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)} = \lim_{n \to \infty} \int_{\Omega_1 \times \Omega_2} |T_1(f_n)(w_1)\mathcal{T}_2(g)(w_2)|^p \mu_1(w_1) \mu_2(w_2) d(w_1, w_2)
$$

$$
= \lim_{n \to \infty} \|T_1(f_n)\|_{L_p(\Omega_1, \mu_1)} \|T_2(g)\|_{L_p(\Omega_2, \mu_2)}^p
$$

$$
= \infty.
$$

\[\square\]

**Remark 2.9.** This lemma typically applies to the Bergman projection on product space, since it is easy to see $\mathcal{B}_{\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2} = \mathcal{B}_{\Omega_1, \mu_1} \otimes \mathcal{B}_{\Omega_2, \mu_2}$. Now we are ready to show the inflation theorem, which generalize corollary 4.6 in [26].

**Theorem 2.10.** Let $\Omega \subset \mathbb{C}^n$ be a domain, and $\mu = |g|^2$ for some non-vanishing holomorphic function on $\Omega$. Suppose $\tilde{\Omega} \subset \mathbb{C}^{m+n}$ is the inflation of $\Omega$ via $\mu$, and suppose $\lambda > 0$ is a function on $\Omega$ such that it is admissible on $\tilde{\Omega}$. Then for $p \geq 1$, $\mathcal{B}_{\tilde{\Omega}, \lambda}$ is $L^p(\lambda)$ bounded if and only if $\mathcal{B}_{\Omega, \mu \lambda} = L^p(\mu \lambda)$ bounded.

**Proof.** Since $g$ is holomorphic and non-vanishing, we have the biholomorphism $\Phi : \tilde{\Omega} \to \mathbb{B}^m \times \Omega$ via $\Phi(z, w) = (\frac{w}{\sqrt{|g(z)|}}, w)$, where $\mathbb{B}^m$ is the unit ball in $\mathbb{C}^m$.

A direct computation shows $|\det J_C(\Phi^{-1})|^2 = \mu^m$. So by **Corollary 2.7**, we see $\mathcal{B}_{\tilde{\Omega}, \lambda}$ is $L^p$-bounded if and only if $\mathcal{B}_{\mathbb{B}^m \times \Omega, \mu \lambda}$ is $L^p$-bounded.
But we know that $B_{\mathbb{B}^m \times \Omega_m} = B_{\mathbb{B}^m} \otimes B_{\Omega_m}$, and $B_{\mathbb{B}^m}$ is $L^p$-bounded for all $p \in (1, \infty)$. By Lemma 2.8, we see $B_{\Omega_m}$ is $L^p$-bounded if and only if $B_{\Omega_m \mu_m}$ is $L^p$-bounded. 

\[ \square \]

### 3. The Hartogs triangle and the punctured disk

If we take $\Omega = \mathbb{D}^*$ (the punctured disk) and $\mu(w) = |w|^2$ in Theorem 2.10, then $\overline{\Omega} = \mathbb{H}$ (the Hartogs triangle). So it suggests us to consider the weighted space $(\mathbb{D}^*, |w|^2 \lambda(w))$, for some weight $\lambda$ on $\mathbb{D}^*$. First of all, we look at the special case $|w|^2 \lambda(w) = |w|^{s'}$, for any $s' \in \mathbb{R}$.

**Lemma 3.1.** For $s' \in \mathbb{R}$ with the unique expression $s' = s + 2k$, where $k \in \mathbb{Z}$ and $s \in (0, 2]$, the weighted Bergman kernel $B_{s'}(z, \zeta)$ on $(\mathbb{D}^*, |z|^{s'})$ has a "homotopic" expression

\[
B_{s'}(z, \zeta) = \frac{s}{2} B_{2k+2}(z, \zeta) + (1 - \frac{s}{2}) B_{2k}(z, \zeta)
\]

(3.1)

where $B_{2k}(z, \zeta)$ is the ordinary Bergman kernel on the unit disk and $(z, \zeta) \in \mathbb{D}^* \times \mathbb{D}^*$.

**Proof.** We first determine an orthonormal basis for the space $A^2(\mathbb{D}^*, |z|^{s'})$. Suppose $m, n \in \mathbb{Z}$, by normalizing the volume of $\mathbb{D}$, a direct computation shows, for $m + n + s' + 2 > 0$,

\[
\int_{\mathbb{D}^*} z^{m-n} |z|^{s'} \, d(z) = \begin{cases} 0, & \text{if } n \neq m, \\ \frac{2}{2m+2+s'}, & \text{if } n = m. \end{cases}
\]

Therefore, $\left\{ \sqrt{\frac{2m+2+s'}{2}} z^m \right\}_{m=-(s'+2)}$ is an orthonormal basis. So the weighted Bergman kernel for the space $(\mathbb{D}^*, |z|^{s'})$ is

\[
B_{s'}(z, \zeta) = \sum_{m=-(s'+2)} \frac{2m+2+s'}{2} z^{m-s'-2} \zeta^{-m}
\]

(3.2)

where $t$ is the smallest integer satisfying $t > -\frac{s'}{2}$.

Suppose $s'' = s' + 2$ and $t_1$ is the smallest integer such that $t_1 > -\frac{s''}{2}$, then $t_1 = t - 1$. In this case,

\[
B_{s''}(z, \zeta) = \frac{(t_1 + \frac{s''}{2})(z\overline{\zeta})^{t_1-1} - (t_1 - 1 + \frac{s''}{2})(z\overline{\zeta})^{t_1}}{(1 - z\overline{\zeta})^2}
\]

(3.3)

\[
= \frac{(t_1 + \frac{s''}{2})(z\overline{\zeta})^{t_1-2} - (t - 2 + \frac{s''}{2})(z\overline{\zeta})^{t_1}}{(1 - z\overline{\zeta})^2}
\]

\[
= (z\overline{\zeta})^{-1} B_{s'}(z, \zeta).
\]
Hence, 2 is a "period" of $s'$ for the weighted Bergman kernel $B_s(z, \zeta)$. Let $s' = s \in (0, 2]$, then $t = 0$, and from (3.2) we have

$$B_s(z, \zeta) = \frac{s(z - \zeta)^{-1} + (1 - \frac{s}{2})}{(1 - z\zeta)^2} B_0(z, \zeta) + \frac{s}{2} B_0(z, \zeta).$$

Therefore, combine (3.3) and (3.4), we obtain (3.1). □

Following the idea in [7], we need three lemmas.

**Lemma 3.2 (Schur’s test).** Suppose $X$ is measure space with a positive measure $\mu$. Let $T(x, y)$ be a positive measurable function on $X \times X$, and let $T$ be the integral operator associated to the kernel function $T(x, y)$.

Given $p \in (1, \infty)$ with its conjugate exponent $p'$, if there exists a strictly positive function $h$ a.e. on $X$ and a $M > 0$, such that

1. $\int_X T(x, y) h(y)^p d\mu(y) \leq M h(x)^{p'}$, for a.e. $x \in X$, and
2. $\int_X T(x, y) h(x)^p d\mu(x) \leq M h(y)^{p'}$, for a.e. $y \in X$.

Then $T$ is bounded on $L^p(X, d\mu)$ with $\|T\| \leq M$.

**Proof.** See [7] Theorem 4.1, or [9] for details. □

**Lemma 3.3.** For $-1 < \alpha < 0$ and $\beta > -2$, define

$$I_{\alpha, \beta}(z) = \int_{\mathbb{D}^*} \frac{(1 - |\zeta|^2)^{\alpha} |\zeta|^{\beta}}{|1 - z\zeta|^2} d\zeta,$$

where $z \in \mathbb{D}^*$ (the restrictions $\alpha > -1$ and $\beta > -2$ make the integral convergent). Then we have $I_{\alpha, \beta}(z) \sim (1 - |z|^2)^{\alpha}$, for any $z \in \mathbb{D}^*$.

**Proof.** See [7] Lemma 3.3. □

**Lemma 3.4.** For $p \geq 1$, the sum $A_{n, p} = \sum_{j=1}^{n} j \left( a_j^* - a_{j+1}^* \right)$ diverges when $p = 1$ and converges when $p > 1$, as $n \to \infty$. More precisely, we have

$$\lim_{n \to \infty} A_{n, 1} = \infty,$$

and

$$\lim_{n \to \infty} A_{n, p} \leq c \sum_{j=1}^{\infty} \frac{1}{j^{1+\epsilon}} < \infty,$$

for all $p > 1$, for some $c > 0$ and for sufficiently small $\epsilon > 0$.

**Proof.** See appendix (section 6). □

The proof of **Theorem 1.1** will be essentially the same as the proof of [7] Theorem 1.2. So we will only give a brief outline here. An alternative proof by using a two-weight inequality will be found in section 4.

**Proof of Theorem 1.1 (See the proof of [7] Theorem 1.2 for more details.)**

To prove the boundedness part, by **Lemma 3.1**, we see

$$|B_s(z, \zeta)| \leq |z\zeta|^{-(k+1)} |B_0(z, \zeta)|,$$
This completes the proof.

for some $\nu > 0$. So it suffices to apply Lemma 3.2 to the kernel

$$T(z, \zeta) = |z\zeta|^{-(k+1)} \frac{1}{|1 - z\zeta|^\sigma},$$

with the positive function

$$h(z) = (1 - |z|^2)^\delta |z|^\sigma,$$

for some $\delta, \sigma \in \mathbb{R}$. By Lemma 3.3, we see

$$T(h') (z) \leq Mh(z)^p'$$

if $-1 < \delta p' < 0$, $-2 < \sigma p' + s + k - 1$, and $\sigma p' \leq -(k + 1)$. Similarly,

$$T(h^p) (\zeta) \leq Mh(\zeta)^p$$

if $\delta \in (-\frac{1}{p}, 0)$ and $\sigma \in (-\frac{s+k+1}{p}, -\frac{k+1}{p})$. Therefore, such $\delta$ and $\sigma$ exist when

1. $k \geq 0$, $p \in (\frac{s+2k+2}{s+k+1}, \frac{s+2k+2}{k+1})$;
2. $k = -1$, $p \in (1, \infty)$;
3. $k = -2$ and $1 \leq s \leq 2$, $p \in (1, \infty)$;
4. $k = -2$ and $0 < s < 1$, $p \in (2 - s, \frac{2-s}{1-s})$;
5. $k = -3$ and $s = 2$, $p \in (1, \infty)$;
6. $k \leq -3$ with (5) excluded, $p \in (\frac{s+2k+2}{k+1}, \frac{s+2k+2}{s+k+1})$.

To show the unboundedness part, we only need to look at $p = \frac{s+2k+2}{s+k+1}$. Let $a_j = \left(\frac{1}{j}\right)^j$, $j = 1, 2, 3, \ldots$, and define a function $g$ on $(0, 1]$ such that $g(r) = r^{\frac{1}{j}-(s+k+1)}$, $r \in (a_{j+1}, a_j)$. We consider the sequence

$$f_n(z) = \begin{cases} 
 g(|z|) \left(\frac{\pi}{|z|}\right)^{1+1}, & |z| \in (a_{n+1}, 1], \\
 0, & |z| \in [0, a_{n+1}].
\end{cases}$$

It is easy to see, $\{f_n\} \subset L^2(D^*, |z|^\sigma')$. Since $p = \frac{s+2k+2}{s+k+1} > 1$, by Lemma 3.4, we have

$$\|f_n\|_{L^p(D^*, |z|^\sigma')}^p = \frac{2}{p} A_n \leq c \sum_{j=1}^\infty \frac{1}{j^{1+\epsilon}},$$

for some $\epsilon > 0$ and some $c > 0$.

On the other hand, a direct computation shows

$$B_{s'}(f_n)(z) = sz^{-(k+1)} A_{n,1}.$$ 

It is easy to see

$$s + 2k - (k + 1)p = -2 + \nu,$$

for some $\nu > 0$. So we obtain

$$\|B_{s'}(f_n)\|_{L^p(D^*, |z|^\sigma')} = s \left(\frac{2}{\nu}\right)^{\frac{1}{p}} A_{n,1}.$$ 

Hence, from Lemma 3.4, we see

$$\lim_{n \to \infty} \|B_{s'}(f_n)\|_{L^p(D^*, |z|^\sigma')} = \infty.$$ 

This completes the proof. □
Remark 3.5. The range for $p$ does not change continuously as $s'$ varies. In fact, there are jumps around the integers $s' = 0, 2, 4, \ldots$, and the integers $s' = -4, -6, -8, \ldots$. The range changes continuously only when $s'$ lies between the consecutive integers above or $s' \in (-4, 0]$ (there is no jump around $s' = -2$).

Proof of Corollary 1.2. This is a direct consequence by combining Theorem 2.10 with $\Omega = \mathbb{D}^*$, $\mu(w) = |w|^2$, and Theorem 1.1(2)(3) with $s' = -(2 + p_0)$. □

To show Corollary 1.3, we need the following lemma.

Lemma 3.6. Let $\mathbb{H}^n = \{ z \in \mathbb{C}^n | 0 < |z_1| < \cdots < |z_n| < 1 \}$ be the punctured $n$ dimensional Hartogs triangle. Suppose we have a weight $\lambda(z) = |z_1|^{s_1} \cdots |z_n|^{s_n}$ on $\mathbb{H}^n$, where $s_1, \ldots, s_n \in \mathbb{R}$. Then the weighted Bergman projection $B_{\mathbb{H}^n, \lambda}$ is $L^p(\lambda)$ bounded if and only if each of the following projections

$$B_{s_1}, B_{s_1 + s_2 + 2}, \cdots, B_{s_1 + \cdots + s_n + 2(n-1)}$$

is $L^p$ bounded on the corresponding space.

Proof. As in the proof of Theorem 2.10, we have the biholomorphism $\Phi : \mathbb{H}^n \to (\mathbb{D}^*)^n$ via $\Phi(z) = (\frac{z_1}{z_2}, \ldots, \frac{z_{n-1}}{z_n}, z_n)$. By Corollary 2.7, we see $B_{\mathbb{H}^n, \lambda}$ is $L^p(\lambda)$ bounded if and only if $B_{(\mathbb{D}^*)^n, \lambda}$ is $L^p(\lambda)$-bounded, where $\tilde{\lambda} = |\det J_C \Phi^{-1}|^2 \lambda(\Phi^{-1})$. A direct computation shows

$$|\det J_C \Phi^{-1}(w)|^2 = |w_2 w_3^2 \cdots w_n^{n-1}|^2,$$

where $w \in (\mathbb{D}^*)^n$. So we see

$$B_{(\mathbb{D}^*)^n, \lambda} = B_{s_1} \otimes B_{s_1 + s_2 + 2} \otimes \cdots \otimes B_{s_1 + \cdots + s_n + 2(n-1)},$$

which implies the conclusion by Lemma 2.8. □

Now we are ready to prove Corollary 1.3.

Proof of Corollary 1.3. Iteratively apply Theorem 2.10 $l$ times to $\Omega = \mathbb{H}^n = \{ w \in \mathbb{C}^n | 0 < |w_1| < \cdots < |w_n| < 1 \}$ with the same weight $|w|^2$, then we will arrive at the space

$$\mathbb{H}^N = \{ (z, w) \in \mathbb{C}^{m_1 + \cdots + m_l + n} | \max_{1 \leq j \leq l} |z_j| < |w_1| < |w_2| < \cdots < |w_n| < 1 \}.$$

So the weighted Bergman projection $B_{\mathbb{H}^N, \lambda}$ is $L^p(\lambda)$ bounded if and only if $B_{\mathbb{H}^n, \lambda}$ is $L^p(\lambda)$ bounded, where $\lambda(w) = |w_1|^{s_1} \cdots |w_n|^{s_n}$ and $\tilde{\lambda}(w) = |w_1|^{2(m_1 + \cdots + m_l)} \lambda(w)$. On the other hand, as the proof of Theorem 1.1, we see weighted Bergman projection on $(\mathbb{H}^N, \lambda)$ is $L^p(\lambda)$ bounded if and only if $B_{\mathbb{H}^N, \lambda}$ is $L^p(\lambda)$. Apply Lemma 3.6 to $B_{\mathbb{H}^n, \lambda}$, we obtain the desired result. □

4. The two-weight inequality

In order to deal with a wider class of weights on the space $(\mathbb{H}, \lambda)$, by inflation, it suggests to look at the weighted space $(\mathbb{D}^*, \mu)$. If the absolute value of the weighted Bergman kernel on $(\mathbb{D}^*, \mu)$ is bounded by the absolute value of the product of the ordinary Bergman kernel and some function on $\mathbb{D}^*$, then it suggests us to look at the following two-weight inequality
(4.1) \[
\left| \int_{D^+} \left| \int_{D^+} \frac{f(\eta)}{(1-\zeta\eta)^2} d\eta \right|^p \mu_1(\zeta) d(\zeta) \right| \leq C \int_{D^+} |f(\zeta)|^p \mu_2(\zeta) d(\zeta).
\]

Using the Cayley transform \( \phi : \mathbb{R}^2_+ \to \mathbb{D} \), where \( \phi(z) = \frac{z+i}{z-i} \), (4.1) is nothing but the following two-weight inequality on the upper half plane

(4.2) \[
\left| \int_{D^+} \left| \int_{D^+} \frac{f(w)}{(z-w)^2} d(w) \right|^p \mu_1(z) d(z) \right| \leq C \int_{D^+} |f(z)|^p \mu_2(z) d(z),
\]

where \( \mu_1 \) and \( \mu_2 \) are two weights on \( \mathbb{R}^2_+ \).

Follow Lanzani and Stein’s idea in [13], let us first give the definitions of several variants of the \( A_p \) condition which was introduced by Muckenhoup (see [18] or chapter 5 of [25]).

**Definition 4.1.** For \( p > 1 \), let \( p' \) denote the conjugate exponent of \( p \), we say the two weights \( \mu_1 \) and \( \mu_2 \) are in the \( A_p(\mathbb{R}^2_+) \) class denoted by \((\mu_1, \mu_2) \in A_p(\mathbb{R}^2_+)\), if there is a positive constant \( c \), so that

\[
\frac{1}{|D \cap \mathbb{R}^2_+|} \int_{D \cap \mathbb{R}^2_+} \mu_1(z) d(z) \left( \frac{1}{|D \cap \mathbb{R}^2_+|} \int_{D \cap \mathbb{R}^2_+} \mu_2(z) \frac{d\zeta}{\zeta^p} d(z) \right) \leq c,
\]

for all disk \( D \) centered at \( z \in \mathbb{R}^2_+ \). We say \((\mu_1, \mu_2) \in A^+_p(\mathbb{R}^2_+)\) if the above inequality holds for all special disk \( D \) centered at \( x \in \mathbb{R} \). The class \( A_p(\mathbb{R}^2) \) is defined in the same way as \( A_p(\mathbb{R}^2_+) \) except replacing \( \mathbb{R}^2_+ \) by \( \mathbb{R}^2 \). For some weight \( \mu \), if \((\mu, \mu) \in A^+_p(\mathbb{R}^2_+)\) (resp. \( A_p(\mathbb{R}^2_+ \) and \( A_p(\mathbb{R}^2) \)), we may simply adopt the notation \( \mu \in A^+_p(\mathbb{R}^2_+) \) (resp. \( A_p(\mathbb{R}^2_+ \) and \( A_p(\mathbb{R}^2) \)).

**Remark 4.2.** The class \( A^+_p(\mathbb{R}^2_+) \) is strictly wider than the class \( A_p(\mathbb{R}^2_+) \). For one-weight case see the comments following the definition of \( A^+_p \) in [13]. For two-weight case see Proposition 4.13 below.

**Definition 4.3.** For a weight \( \mu \), a measurable function \( f \), and any measurable set \( W \) with its Lebesgue measure \( |W| \), we may use the notations

\[
\mu(W) = \int_W \mu(z) d(z)
\]

and

\[
\int_W f(z) d(z) = \frac{1}{|W|} \int_W f(z) d(z).
\]

From now on, the symbol \( c \) will denote some positive constant independent of the variables and the functions in the context. So two \( c \)’s in the same equation could be different, but it does not matter at all.

**Definition 4.4.** Let \( \mathcal{B} \) be the ordinary Bergman projection on \( \mathbb{R}^2_+ \), then

\[
\mathcal{B}(f)(z) = \int_{\mathbb{D}} \frac{1}{(z-w)^2} f(w) d(w),
\]

for all \( f \in C_0^\infty(\mathbb{R}^2_+) \). We also consider the "absolute value" operator \( \tilde{\mathcal{B}} \) of \( \mathcal{B} \) defined as

\[
\tilde{\mathcal{B}}(f)(z) = \int_{\mathbb{D}} \frac{1}{|z-w|^2} f(w) d(w),
\]
where we replace the kernel $-\frac{1}{(z-w)^2}$ by its absolute value.

Remark 4.5. We omit the coefficient $\frac{1}{n}$ in front of the integrals, which does not matter at all.

Now, (4.2) becomes
\begin{equation}
\int_{\mathbb{R}_+^2} |B(f)(z)|^p \mu_1(z) \, d(z) \leq C \int_{\mathbb{R}_+^2} |f(z)|^p \mu_2(z) \, d(z).
\end{equation}

We first give a sufficient condition, which will implies Theorem 1.4.

**Theorem 4.6.** For $p > 1$, if the two weights $\mu_1$ and $\mu_2$ satisfy $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$ and either $\mu_1 \in A_p^+(\mathbb{R}_+^2)$ or $\mu_2 \in A_p^+(\mathbb{R}_+^2)$, then the operator $B$ is bounded from $L^p(\mathbb{R}_+^2, \mu_2)$ to $L^p(\mathbb{R}_+^2, \mu_1)$.

The proof of Theorem 4.6 follows the same pattern as in [13] proposition 4.5. We first need to define a standard "tiling" of $\mathbb{R}_+^2$ and the associated averaging operator (or "conditional expectation").

**Definition 4.7.** The standard "tiling" of $\mathbb{R}_+^2$ are the squares $\{S_{j,k}\}$ of form
\[ S_{j,k} = \{z = x + iy \in \mathbb{C} \mid 2^k \leq y \leq 2^{k+1} \text{ and } j \cdot 2^k \leq x \leq (j+1) \cdot 2^{k+1}\} \]
for all $j, k \in \mathbb{Z}$. Note that each $S_{j,k}$ has side-length $2^k$, the interiors of $S_{j,k}$’s are disjoint, and $\mathbb{R}_+^2 = \bigcup_{j,k} S_{j,k}$.

Define the associated **averaging operator** $E$ by
\[ E(f)(z) = \int_{S_{j,k}} f(z) \, d(z), \quad \text{if } z \in S_{j,k}, \]
for any nonnegative measurable function $f$ on $\mathbb{R}_+^2$. Note that $E(f)$ can be infinite.

Remark 4.8. We have the following basic properties of the operator $E$. For any nonnegative measurable functions $f$ and $g$, let $p'$ be the conjugate exponent of $p$, we have
\begin{enumerate}
\item[(a)] \[ \int_{\mathbb{R}_+^2} E(f)(z) g(z) \, d(z) = \int_{\mathbb{R}_+^2} E(f)(z) E(g)(z) \, d(z), \]
\item[(b)] \[ \int_{\mathbb{R}_+^2} (E(f)(z))^p g(z) \, d(z) \leq \int_{\mathbb{R}_+^2} E(f^p)(z) g(z) \, d(z), \]
\item[(c)] \[ E(fg)(z) \leq \left( E(f^p)(z) \right)^{\frac{1}{p}} \left( E(g^{p'})(z) \right)^{\frac{1}{p'}} \text{ for all } z \in \mathbb{R}_+^2. \]
\end{enumerate}
The proof will be found in appendix (section 6).

The key fact is the following proposition whose proof is essentially the same as that of [13] Proposition 4.6, where they only focus on the one weight case. To be self-contained, we will give the details in the appendix.

**Proposition 4.9.** If $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$, then $(E(\mu_1), E(\mu_2)) \in A_p(\mathbb{R}_+^2)$.

**Proof.** See [13] Proposition 4.6 or appendix (section 6). \qed

Now we turn to the two-weight $A_p$ condition.
\textbf{Lemma 4.10.} If \((\mu_1, \mu_2) \in A_p(\mathbb{R}^2)\), then \(\mu_1 \leq c \mu_2\) for some \(c > 0\).

\textit{Proof.} It is easy to see the \(A_p\) condition is equivalent to the following

\[
\mu_1(Q) \left( \int_Q f(z) \, d(z) \right)^p \leq c \int_Q f(z)^p \mu_2(z) \, d(z),
\]

for all \(f \geq 0\) and all squares \(Q\) in \(\mathbb{R}^2\). (See an analogue for one weight case in chapter 5 of [25].)

Let \(f = \chi_Q\) in (4.4), we see \(\mu_1(Q) \leq c \mu_2(Q)\) for all squares \(Q\) in \(\mathbb{R}^2\). But the \(\sigma\)-algebra can be generated from the set of squares, we see \(\mu_1 \leq c \mu_2\) almost everywhere. \(\square\)

We need one more observation from [13].

\textbf{Lemma 4.11.} For any \(f \geq 0\), we have

\[
\overline{B}(f) \leq c E \overline{B}(f).
\]

\textit{Proof.} See the proof of Proposition 4.5 in [13] or appendix (section 6). \(\square\)

\textit{Proof of Theorem 4.6.}

We first assume \(\mu_2 \in A_p^+(\mathbb{R}^2_+)\). It suffices to prove the boundedness of \(\overline{B}\) for \(f \geq 0\). By \textbf{Proposition 4.9}, we have \((E(\mu_1), E(\mu_2)) \in A_p(\mathbb{R}^2_+)\) and \(E(\mu_2) \in A_p(\mathbb{R}^2_+)\). We extend \(E(\mu_1)\) and \(E(\mu_2)\) to \(\mathbb{R}^2\) by reflection about the \(x\)-axis, that is, \(E(\mu_1)(z) = E(\mu_1)(\overline{z})\) for \(z \in \mathbb{R}^2_-, j = 1, 2\). Then, it is easy to see, \((E(\mu_1), E(\mu_2)) \in A_p(\mathbb{R}^2)\) and \(E(\mu_2) \in A_p(\mathbb{R}^2)\). Hence, by \textbf{Lemma 4.10}, \(E(\mu_1) \leq c E(\mu_2)\) almost everywhere, and in particular, it is true on \(\mathbb{R}^2_+\).

Let \(K(z) = \frac{\Psi(\theta)}{r^2} = \frac{\Psi(\theta)}{r^2} \) for \(z \in \mathbb{R}^2 \setminus \{0\}\), where \(z = re^{i\theta}\), \(\Psi\) is smooth with \(\Psi(\theta) = 1\) if \(\theta \in [0, \pi]\), and \(\int_0^{2\pi} \Psi(\theta) \, d(\theta) = 0\). Then \(T(f) = K \ast f\) defines a singular integral operator with the cancellation property \(\int_{|z|=1} K(z) \, d(z) = 0\) and the radial decreasing property \(\left| \frac{\partial}{\partial r} \right|^{\alpha_1} \frac{\partial}{\partial \theta} \right|^{\alpha_2} K(z) \right| \leq c |z|^{-2-\alpha_1}, \) for all \(z \neq 0\) and \(\alpha_1 + \alpha_2 \leq 1\), where \(z = x + iy\). Since \(E(\mu_2) \in A_p(\mathbb{R}^2)\), we have

\[
\int_{\mathbb{R}^2} T(f)(z)^p E(\mu_2)(z) \, d(z) \leq c \int_{\mathbb{R}^2} f(z)^p E(\mu_2)(z) \, d(z),
\]

for all \(f \in C_0^\infty(\mathbb{R}^2)\). (See chapter 5 of [25] for details.) Let \(J(f)(z) = f(\overline{z})\), since \(E(\mu_2)(\overline{z}) = E(\mu_2)(z)\), we have

\[
\int_{\mathbb{R}^2} T(J f)(z)^p E(\mu_2)(z) \, d(z) \leq c \int_{\mathbb{R}^2} J f(z)^p E(\mu_2)(z) \, d(z)
\]

\[
= c \int_{\mathbb{R}^2} f(z)^p E(\mu_2)(z) \, d(z).
\]

In particular, above inequality applies to \(f \in C_0^\infty(\mathbb{R}^2)\). By noting that \(T(J f)(z) = K \ast J f(z) = \overline{B}(f)(z)\) for \(f \in C_0^\infty(\mathbb{R}^2)\) and \(z \in \mathbb{R}^2_+\), we see

\[
\int_{\mathbb{R}^2_+} \overline{B}(f)(z)^p E(\mu_2)(z) \, d(z) \leq \int_{\mathbb{R}^2} T(J f)(z)^p E(\mu_2)(z) \, d(z)
\]

\[
\leq c \int_{\mathbb{R}^2_+} f(z)^p E(\mu_2)(z) \, d(z).
\]
Together with the relation $E(\mu_1) \leq cE(\mu_2)$, we obtain

$$
\int_{\mathbb{R}_+^2} \tilde{B}(f)(z)^p E(\mu_1)(z) \, d(z) \leq c \int_{\mathbb{R}_+^2} \tilde{B}(f)(z)^p E(\mu_2)(z) \, d(z)
$$

(4.6)

$$
\leq c \int_{\mathbb{R}_+^2} f(z)^p E(\mu_2)(z) \, d(z).
$$

Now, for $f \geq 0$, as in [13] Proposition 4.5, by Lemma 4.11 we have

$$
\int_{\mathbb{R}_+^2} \left( \tilde{B}(f)(z) \right)^p \mu_1(z) \, d(z) \leq c \int_{\mathbb{R}_+^2} \left( E \tilde{B}E(f)(z) \right)^p \mu_1(z) \, d(z)
$$

(remark 4.8 (b))

$$
\leq c \int_{\mathbb{R}_+^2} E \left( \tilde{B}E(f)(z) \right)^p E(\mu_1)(z) \, d(z)
$$

(4.7)

(remark 4.8 (a))

$$
= c \int_{\mathbb{R}_+^2} (E(f)(z))^p E(\mu_2)(z) \, d(z).
$$

On the other hand, by remark 4.8 (c), we see

$$
E(f)(z) = E(f \mu_2^{\frac{1}{p}} \cdot \mu_2^{-\frac{1}{p}})(z) \leq \left( E(f^p \mu_2)(z) \right)^{\frac{1}{p}} \left( E(\mu_2^{\frac{p}{p}})(z) \right)^{\frac{1}{p}}.
$$

(4.8)

For any $z = x + iy \in S_{j,k}$, let $x_0$ be the real part of the center of $S_{j,k}$, then $|z - x_0| \leq 2^k + 2^{k+1} \leq 2^{k+2}$. Let $D = D_{2^{k+2}}(x_0)$ be the special disk of radius $2^{k+2}$ centered at $x_0$, so we have $z \in D \cap \mathbb{R}_+^2$, and hence $S_{j,k} \subset D \cap \mathbb{R}_+^2$. Note that $|S_{j,k}| = 2^{2k}$ and $|D \cap \mathbb{R}_+^2| = 2^{2k+3} \pi$, we have $\frac{1}{|S_{j,k}|} = \frac{8\pi}{|D \cap \mathbb{R}_+^2|}$. So for $z \in S_{j,k}$,

$$
E(\mu_2)(z) \left( E(\mu_2^{-\frac{p}{p}})(z) \right)^{\frac{1}{p}} = \int_{S_{j,k}} \mu_2(z) \, d(z) \left( \int_{S_{j,k}} \mu_2(z)^{-\frac{1}{p}} \, d(z) \right)^{\frac{1}{p}}
$$

(4.9)

$$
\leq c \int_{D \cap \mathbb{R}_+^2} \mu_2(z) \, d(z) \left( \int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{1}{p}} \, d(z) \right)^{\frac{1}{p}}
$$

$$
\leq c.
$$
Proposition 4.13. Let $c > 0$ independent of $S_{j,k}$, since $\mu_2 \in A_p^+(\mathbb{R}_+^2)$. Therefore, combine (4.7), (4.8) and (4.9), we see

$$
\int_{\mathbb{R}_+^2} \left( \overline{B}(f)(z) \right)^p \mu_1(z) \, d(z) \leq c \int_{\mathbb{R}_+^2} E(f^p \mu_2)(z) \left( E(\mu_2^{-1/p})(z) \right)^p E(\mu_2)(z) \, d(z)
$$

$$
= c \sum_{j,k} \int_{S_{j,k}} E(f^p \mu_2)(z) \left( E(\mu_2^{-1/p})(z) \right)^p E(\mu_2)(z) \, d(z)
$$

$$
\leq c \sum_{j,k} \int_{S_{j,k}} f(z)^p \mu_2(z) \, d(z)
$$

$$
= c \int_{\mathbb{R}_+^2} f(z)^p \mu_2(z) \, d(z),
$$

which completes the proof of the case $\mu_2 \in A_p^+(\mathbb{R}_+^2)$.

Now assume $\mu_1 \in A_p^+(\mathbb{R}_+^2)$, then we have $E(\mu_1) \in A_p(\mathbb{R}_+^2)$. Almost the same argument shows that (4.7) becomes

$$
\int_{\mathbb{R}_+^2} \left( \overline{B}(f)(z) \right)^p \mu_1(z) \, d(z) \leq c \int_{\mathbb{R}_+^2} (E(f)(z))^p \, d(\mu_1)(z) \, d(z),
$$

and (4.9) becomes

$$
E(\mu_1)(z) \left( E(\mu_2^{-1/p})(z) \right)^p \leq c,
$$

since $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$. This completes the proof.

As an application, we have the following corollary.

**Corollary 4.12.** For $p > 1$, suppose $\mu_1$ and $\mu_2$ are two weights such that $c \mu_1 \geq \mu_2$ for some $c > 0$, then the Bergman projection $B$ on $\mathbb{R}_+^2$ is bounded from $L_p(\mathbb{R}_+^2, \mu_2)$ to $L_p(\mathbb{R}_+^2, \mu_1)$ if and only if $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$.

**Proof.** The sufficiency is immediate. Since $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$ and $c \mu_1 \geq \mu_2$, it is easy to see $\mu_2 \in A_p^+(\mathbb{R}_+^2)$, then the boundedness of $B$ follows from Theorem 4.6.

For necessity, see the proof of Proposition 3.4 in [13] and combine the fact that $c \mu_1 \geq \mu_2$. (See also appendix in section 6.)

Theorem 4.6 and Corollary 4.12 will give an alternative proof of Theorem 1.1. We first have the following observation.

**Proposition 4.13.** For $z \in \mathbb{R}_+^2$, $k \in \mathbb{Z}$, $s \in (0, 2]$ and $p > 1$, suppose

$$
\mu_1(z) = \left| \frac{i - z}{i + z} \right|^{-(k+1)p+s+2k}
$$

and

$$
\mu_2(z) = \left| \frac{i - z}{i + z} \right|^{(1-s-k)p+s+2k},
$$

then $(\mu_1, \mu_2) \notin A_p(\mathbb{R}_+^2)$ for $s \neq 2$. But we have $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$ if and only if $s + 2k + 2 > (k+1)p$ and $p(s + k + 1) > s + 2k + 2$. 
Proof. To show \( (\mu_1, \mu_2) \not\in A_p(\mathbb{R}_+^2) \), we consider any disk \( D_r(i) \) centered at \( i \) with radius \( r < \frac{1}{2} \). For \( z \in D_r(i) \), since \( |i - z| < r < \frac{1}{2} \), we see \( \frac{3}{2} \leq |i + z| \leq \frac{5}{2} \). So we only need to look at

\[
\int_{D_r(i)} |i - z|^{-(k+1)p+s+2k} d(z) \left( \int_{D_r(i)} |i - z|^{-\frac{p}{2}[(1-s-k)p+s+2k]} d(z) \right)^{\frac{p}{2}}.
\]

Assuming both integrands are integrable, we obtain \( r^{(s-2)p} \). But \( s \in (0, 2) \) and \( p > 1 \), we see the quantity above tends to \( \infty \) as \( r \to 0 \).

To show \( (\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2) \), we consider two integrals

\[
I_1 = \frac{1}{|D \cap \mathbb{R}^2_+|} \int_{D \cap \mathbb{R}^2_+} |i - z|^{-(k+1)p+s+2k} \frac{i - z}{i + z} d(z),
\]

and

\[
I_2 = \frac{1}{|D \cap \mathbb{R}^2_+|} \int_{D \cap \mathbb{R}^2_+} |i - z|^{-\frac{p}{2}[(1-s-k)p+s+2k]} \frac{i - z}{i + z} d(z),
\]

where \( D = D_R(x_0) \) is any special disk with radius \( R \) centered at \( x_0 \in \mathbb{R} \). Let \( D_0 = D_{\frac{1}{2}}(i) \) be the disk with radius \( \frac{1}{2} \) centered at \( i \). We separate our arguments into two cases.

Case (I), \( R < \frac{1}{2} \).

It is easy to see \( D \cap D_0 = \emptyset \), then \( |i - z| > \frac{1}{2} \). Note that as \( |z| \to \infty \), \( \frac{i - z}{i + z} \to 1 \), so there is an \( M \) such that when \( |z| > M \), \( 1 \geq \left| \frac{i - z}{i + z} \right| \geq \frac{1}{2} \). But when \( |z| \leq M \), \( |i + z| \leq M + 1 \), so \( 1 \geq \left| \frac{i - z}{i + z} \right| \geq \frac{1}{2(1 + M)} \). Therefore, the integrands in \( I_1 \) and \( I_2 \) are bounded above by some constants that are independent of the special disk \( D \).

Then \( I_1 I_2^p \leq c \), for some \( c > 0 \).

Case (II), \( R \geq \frac{1}{2} \).

We split \( I_1 \) and \( I_2 \) into two integrals respectively, one integrates over \( D \cap \mathbb{R}^2_+ \setminus D_0 \) and the other integrates over \( D \cap \mathbb{R}^2_+ \cap D_0 \). For the same reasoning as in case (I), the parts integrate over \( D \cap \mathbb{R}^2_+ \setminus D_0 \) is bounded. The parts integrate over \( D \cap \mathbb{R}^2_+ \cap D_0 \) are bounded respectively by

\[
\frac{8}{\pi} \int_{D_0} \left| \frac{i - z}{i + z} \right|^{-(k+1)p+s+2k} d(z),
\]

and

\[
\frac{8}{\pi} \int_{D_0} \left| \frac{i - z}{i + z} \right|^{-\frac{p}{2}[(1-s-k)p+s+2k]} d(z).
\]

Since \( |i - z| \leq \frac{1}{2} \) for \( z \in D_0 \), we see \( \frac{3}{2} \leq |i + z| \leq \frac{5}{2} \), so the two integrals above are bounded respectively by

\[
c \int_{D_0} |i - z|^{-(k+1)p+s+2k} d(z),
\]

and

\[
c \int_{D_0} |i - z|^{-\frac{p}{2}[(1-s-k)p+s+2k]} d(z).
\]
The first integral above is bounded by a constant if and only if \(-(k+1)p+s+2k+2 > 0\), and the second is bounded if and only if \(-\frac{p}{p}(1-s-k)p+s+2k+2 > 0\). Solving the two inequalities, we see \(s+2k+2 > (k+1)p\) and \(p(s+k+1) > s+2k+2\).

**Remark 4.14.** From the proof we see, if the weights \(\mu_1\) and \(\mu_2\) only have zeros or poles away from the \(x\)-axis and bounded above and below at \(\infty\), then \((\mu_1, \mu_2) \in A^+_p(\mathbb{R}^2_+)\) if and only if both \(\mu_1\) and \(\mu_2\) are locally integrable.

**Remark 4.15.** Combine the fact \(\mu_1 \geq \mu_2\) and Corollary 4.12 we see that, for the Bergman projection on \(\mathbb{R}^2_+\), the two-weight \(A_p\) condition is not necessary for (4.2) to hold. (Compare to the general Calderón-Zygmund type singular integral.) The reason is that, the Bergman kernel is not singular at all on \(\mathbb{R}^2_+\), and it should be a two dimensional analogue of the so-called Hilbert integral (see [22] and [23]).

Now we give an alternative proof of Theorem 1.1, and then generalize it to Theorem 1.7.

**Alternative Proof of Theorem 1.1.**

For \(s' = s + 2k\), from Lemma 3.1 we know that

\[
B_{s'}(\zeta, \eta) = \frac{s}{2}(\zeta \eta)^{-\alpha}B_0(\zeta, \eta) + (1 - \frac{s}{2})(\zeta \eta)^{-\alpha}B_0(\zeta, \eta)
\]

for \(\zeta, \eta \in \mathbb{D}^*\). So we may write \(B_{s'} = \bar{T}_1 + (1 - \bar{T}_2)\), where \(T_1\) is the operator associated to the kernel \((\zeta \eta)^{-\alpha}B_0(\zeta, \eta)\) with weight \(|\eta|^{s'}\) and \(T_2\) is the operator associated to the kernel \((\zeta \eta)^{-\alpha}B_0(\zeta, \eta)\) with weight \(|\eta|^{s'}\).

For the operator \(T_1\), showing its boundedness is the same as showing

\[
\int_{\mathbb{D}^*} \int_{\mathbb{D}^*} \frac{|\zeta \eta|^{-\alpha}f(\eta)}{(1-\zeta \eta)^{2}} |\zeta|^{s'} d\eta d\zeta \leq C \int_{\mathbb{D}^*} |f(\eta)|^p |\zeta|^{s'} d\zeta.
\]

By the biholomorphism \(\phi : \mathbb{R}^2_+ \to \mathbb{D}\), where \(\phi(z) = \frac{i-z}{1+z}\), we see the above inequality is equivalent to

\[
\int_{\mathbb{R}^2_+ \cap i} \int_{\mathbb{R}^2_+ \cap i} \frac{|\phi(u)|^{-\alpha}f(u)}{(1-\phi(u)^{2}} |\phi(z)|^{s'} d\phi(u) d\phi(z) \leq C \int_{\mathbb{R}^2_+ \cap i} |\phi(z)|^p |\phi(z)|^{s'} d\phi(z),
\]

where \(\phi(z) = f(\phi(z))\). Now, \(\mu_1(z) = 4 \frac{1}{(1+z)^2} \), \(\mu_2(z) = 4 \frac{1}{(1+z)^2} \), and this is exactly (4.3). Note that \(\frac{\phi(z)}{\mu_2(z)} = \frac{(s-2)^2}{1+z} \geq 1\), since \(\frac{1+z}{1+z} \leq 1\) and \(s \leq 2\). So Corollary 4.12 applies, that is, \(T_1\) is bounded if and only if \((\mu_1, \mu_2) \in A^+_p(\mathbb{R}^2_+)\).

For the condition \((\mu_1, \mu_2) \in A^+_p(\mathbb{R}^2_+)\), we first note that \(\sigma(z) = \frac{1}{(1+z)^2} \), and hence \(\sigma \in A^+_p(\mathbb{R}^2_+)\) for all \(p > 1\). To see this, from the classical result \(B_0 = L^p\) bounded for all \(p > 1\), where \(B_0\) is the ordinary Bergman projection on \(\mathbb{D}^*\) which is the same as the ordinary Bergman projection on the unit disk. Then (4.3) holds with \(\mu_1\) and \(\mu_2\) replaced by \(\sigma\), and hence \(\sigma \in A^+_p(\mathbb{R}^2_+)\) for all \(p > 1\).

As in Proposition 4.13, we consider two integrals

\[
I_1 = \frac{1}{|D \cap \mathbb{R}^2_+|} \int_{D \cap \mathbb{R}^2_+} 4 \frac{i-z}{i+z}^{-(k+1)p+s+2k} \left| \frac{1}{(i+z)^2} \right|^{2-p} dz,
\]

\[
I_2 = \frac{1}{|D \cap \mathbb{R}^2_+|} \int_{D \cap \mathbb{R}^2_+} 4 \frac{i-z}{i+z}^{-(k+1)p+s+2k} \left| \frac{1}{(i+z)^2} \right|^{2-p} dz.
\]
and
\[ I_2 = \frac{1}{|D \cap \mathbb{T}^2_+|} \int_{D \cap \mathbb{T}^2_+} \left( 4 \frac{|i - z|^{(1-s-k)p+s+2k}}{|i + z|^2} \cdot \frac{1}{(i + z)^2} \right)^{-\frac{p}{2k}} d(z), \]

where \( D = D_R(x_0) \) is again any special disk with radius \( R \) centered at \( x_0 \in \mathbb{R} \).

For \( R < \frac{1}{2} \), same argument as in the proof of Proposition 4.13 case (I), shows the integrands in \( I_1 \) and \( I_2 \) are bounded above by \( c \sigma \) and \( c \sigma \frac{1}{|z|^2} \) respectively. So this case follows from the fact that \( \sigma \in A_p^+ (\mathbb{T}^2_+) \) for all \( p > 1 \).

For \( R \geq \frac{1}{2} \), same argument as in the proof of Proposition 4.13 case (II) and Remark 4.14, show that we only need to consider whether
\[ |i - z|^{-(k+1)p+s+2k} \text{ and } |i - z|^{-\frac{p}{2k}[(1-s-k)p+s+2k]} \]

are locally integrable away from the \( x \)-axis.

Denoted by \( U_1 = \{ p \in (1, \infty) | s + 2k + 2 > (k + 1)p \text{ and } p(s + k + 1) > s + 2k + 2 \} \) the range for \( p \), it is not difficult to see \( U_1 \) is an interval. So we obtain \( (\mu_1, \mu_2) \in A_p^+ (\mathbb{T}^2_+) \) and hence \( T_1 \) is bounded, if and only if \( p \in U_1 \).

Similarly, \( T_2 \) is bounded if and only if (4.3) holds for \( \mu_1(z) = 4\sigma(z) \left| \frac{i - z}{s + k} \right|^{-kp+s+2k} \)
and \( \mu_2(z) = 4\sigma(z) \left| \frac{i - z}{s + k} \right|^{-(k+1)p+s+2k} \). But \( \mathcal{B}_p(\omega) = \left| \frac{i - z}{s + k} \right|^{-s} \leq 1 \), since \( \frac{i - z}{s + k} \leq 1 \) and \( s > 0 \). So we apply Theorem 4.6 directly, that is, \( T_2 \) is bounded if \( \mu_1 \in A_p^+ (\mathbb{T}^2_+) \) or \( \mu_2 \in A_p^+ (\mathbb{T}^2_+) \).

Following the same argument, we see \( \mu_1 \in A_p^+ (\mathbb{T}^2_+) \) or \( \mu_2 \in A_p^+ (\mathbb{T}^2_+) \) if and only if \( p \in U_2 = \{ p \in (1, \infty) | s + 2k + 2 > pk \text{ and } (s + k + 2)p > s + 2k + 2 \} \) for \( s' \neq -2 \).

But for \( s' = -2 \) we do not need to worry about \( T_2 \). It is not hard to see \( U_2 \) is also an interval and we have \( T_2 \) is bounded if \( p \in U_2 \).

Now if both \( T_1 \) and \( T_2 \) are bounded, then \( \mathcal{B}_{\nu} \) is bounded. Since \( U_1 \subseteq U_2 \), we see \( \mathcal{B}_{\nu} \) is bounded if \( p \in U_1 \). Conversely, for any \( p \notin U_1 \) but \( p \in U_2 \), then \( T_1 \) is unbounded and \( T_2 \) is bounded. Such \( p \) exists, since \( U_1 \subseteq U_2 \) and \( p \) can be either less than the left endpoint of \( U_1 \) or greater than the right endpoint of \( U_1 \). For these \( p \)’s, \( \mathcal{B}_{\nu} \) is unbounded. Hence, by interpolation we see \( \mathcal{B}_{\nu} \) is unbounded for all \( p \notin U_1 \).

Therefore, for \( p > 1 \) \( \mathcal{B}_{\nu} \) is bounded if and only if \( p \in U_1 \). When \( s' \in (0, \infty) \), \( U_1 = \left( \frac{s + 2k + 2}{s + k + 1}, \frac{s + 2k + 2}{k + 1} \right) \). When \( s' \in [-3, 0) \), \( U_1 = (1, \infty) \). When \( s' \in (-4, -3) \), \( U_1 = (2 - s', \frac{s + 2k + 2}{s + k + 1}) \). When \( s' = -4 \), \( U_1 = (1, \infty) \). When \( s' \in (-\infty, -4) \), \( U_1 = (\frac{s + 2k + 2}{k + 1}, \frac{s + 2k + 2}{s + k + 1}) \).

\[ \square \]

Remark 4.16. Besides Corollary 4.12, the analysis for \( T_2 \) here also supports our Conjecture 1.5, since the "effective" bounds for \( p \) is obtained by checking \( (\mu_1, \mu_2) \in A_p^+ (\mathbb{T}^2_+) \).

Now we generalize the arguments above to show Theorem 1.7.

Proof of Theorem 1.7.

For the Bergman kernel for the space \( (\mathcal{D}, |g|^2) \), we have
\[ B_{|g|^2}(\zeta, \eta) = \frac{1}{g(\zeta)g(\eta)} \frac{1}{(1 - \zeta \bar{\eta})^2}. \]

(see [26] Theorem 3.4 or apply Lemma 2.5). So same argument as the proof above shows \( \mathcal{B}_{|g|^2} \) is \( L^p \) bounded if and only if (4.3) holds with \( \mu_1 \) and \( \mu_2 \) replaced by...
\[ \sigma(z) = \left| g(\phi(z)) \cdot \frac{1}{(1+z)^2} \right|^{2-p}. \]

Recall from the statement of the theorem, \( B_{|g|^2} \) is \( L^p(|g|^2) \) bounded if and only if \( p \in (p_0, p'_0) \) for some \( p_0 \geq 1 \). So \( \sigma \in A_+^p(\mathbb{R}^2_+) \) if and only if \( p \in (p_0, p'_0) \).

Recall \( \mu(z) = |z|^{p'}|g(z)|^2 \) on \( \mathbb{D}^* \), the same argument applies for the Bergman projection \( B_\mu \), since for the Bergman kernel we have

\[ B_\mu(\zeta, \eta) = \frac{1}{g(\zeta)g(\eta)} B_{\mu'}(\zeta, \eta). \]

Then using the relation \( B_{\mu'} = \frac{s}{2} T_1 + (1 - \frac{s}{2}) T_2 \) as in the proof above, \( B_\mu \) is \( L^p(\mu) \) bounded if (4.3) holds for the first pair

\[ \mu_1(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{-(k+1)p+s+2k}, \quad \mu_2(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{(1-s-k)p+s+2k} \]

and for the second pair

\[ \mu_1(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{-kp+s+2k}, \quad \mu_2(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{-(s+k)p+s+2k}. \]

Remark 4.17. In fact, the above argument still applies, for any weight \( \mu \) such that the associated Bergman kernel satisfies \( |B_\mu(\zeta, \eta)| \leq \frac{c}{|g(\zeta)g(\eta)|^{1-\frac{1}{2}}}, \) where \( g \) is some holomorphic function. In this case, one needs to deal with the two weights, \( \mu_1(z) = \mu(\phi(z))\left| g(\phi(z)) \right|^{-p}|g'(z)|^{2-p} \) and \( \mu_2(z) = \mu(\phi(z))^{1-p}|g(\phi(z))|^p|g'(z)|^{2-p} \). Note that we have \( \frac{\mu_1}{\mu_2} = \left( \frac{\mu}{|g|^2} \right)^p \), this seems a measurement of how far away the weight \( \mu \) can be a norm-square of some holomorphic function.

As a last application, we can use the two-weight inequality to show the \( L^p \) regularity of the weighted Bergman projection \( B_{\mathbb{H}, \mathbb{H}'} \) on \( \mathbb{H}, |z_2|^{s'} \) mapping from \( L^p(\mathbb{H}, |z_2|^s') \) to \( L^p(\mathbb{H}, |z_2|^k) \) for some \( t \in \mathbb{R} \).

Recall that \( \mathbb{H} = \{(z_1, z_2) | |z_1| < |z_2| < 1 \} \) is the Hartogs triangle, and we have the unique expression \( s' = s + 2k \) for \( s \in (0, 2) \) and \( k \in \mathbb{Z} \). For simplicity, we focus on the case \( k \geq -1 \).

**Corollary 4.18.** Suppose \( B_{\mathbb{H}, \mathbb{H}'} \) is the weighted Bergman projection on the weighted space \( (\mathbb{H}, |z_2|^{s'}) \), assume \( p > 1 \) and \( k \geq -1 \). Then for \( t - s' \leq (2 - s)p \), \( B_{\mathbb{H}, \mathbb{H}'} \) is \( L^p \) bounded from \( L^p(\mathbb{H}, |z_2|^s) \) to \( L^p(\mathbb{H}, |z_2|^t) \) if and only if \( p \in \left( \frac{2k+s+4}{k+s+2}, \frac{t+4}{k+2} \right) \). For \( t - s' > (2 - s)p \), it is bounded if \( p \in \left( \frac{2k+s+4}{k+s+2}, \frac{t+4}{k+2} \right) \).

**Proof.** The boundedness of the mapping is equivalent to

\[ \int_{\mathbb{H}} |B_{\mathbb{H}, \mathbb{H}'}(f)(z)|^p |z_2|^{s'} \, d(z) \leq C \int_{\mathbb{H}} |f(z)|^p |z_2|^s \, d(z). \]
By Corollary 2.6, and consider the biholomorphism \( \Phi : \mathbb{H} \to \mathbb{D} \times \mathbb{D}^* \) via \( \Phi(z) = (\frac{z_1}{z_2}, z_2) \), the inequality above is equivalent to

\[
\int_{\mathbb{D} \times \mathbb{D}^*} \int_{\mathbb{D} \times \mathbb{D}^*} B_0 \otimes B_{\nu'}(\zeta, \eta) |f(\eta)|^2 d(\eta) |\zeta_2|^{1+2-p} d(\zeta) \leq C \int_{\mathbb{D} \times \mathbb{D}^*} |f(\zeta)|^p |\zeta_2|^{s'+2-p} d(\zeta),
\]

where \( B_{\nu'} \) is the weighted Bergman kernel on \((\mathbb{D}^*, |z|^{s'})\). By Lemma 2.8, and the fact that the ordinary Bergman projection on the unit disk is \( L^p \) bounded for all \( p > 1 \), we see the inequality above is equivalent to

\[
\int_{\mathbb{D}^*} \int_{\mathbb{D}^*} B_{\nu'}(\zeta, \eta) |f(\eta)|^2 d(\eta) |\zeta|^{1+2-p} d(\zeta) \leq C \int_{\mathbb{D}^*} |f(\zeta)|^p |\zeta|^{s'+2-p} d(\zeta).
\]

Apply the same argument and consider \( B_{\nu'} = \frac{t}{2}T_1 + (1 - \frac{t}{2})T_2 \) as in the proof above.

We see \( T_1 \) is bounded if (4.3) holds for

\[
\mu_1(z) = 4\sigma(z) \left| \frac{i - z}{i + z} \right|^{-(k+2)p+s+2k+2}, \quad \mu_2(z) = 4\sigma(z) \left| \frac{i - z}{i + z} \right|^{-(k+1)p+t+2}
\]

and \( T_2 \) is bounded if (4.3) holds for

\[
\mu_1(z) = 4\sigma(z) \left| \frac{i - z}{i + z} \right|^{-(k+1)p+t+2}, \quad \mu_2(z) = 4\sigma(z) \left| \frac{i - z}{i + z} \right|^{-(s+k+1)p+s+2k+2}
\]

where \( \sigma(z) = \left| \frac{1}{1 + z^2} \right|^{2-p} \in A^p_+(\mathbb{R}^2_+) \) for all \( p > 1 \).

For \( T_1 \), when \( t - s' \leq (2 - s)p, \) Corollary 4.12 tells us \( T_1 \) is bounded if and only if \( t \in (\frac{k+4}{s+k+3}, \frac{k+4}{k+2}) \).

For \( T_2 \), when \( t - s' + sp \leq 0, \) Corollary 4.12 tells us \( T_2 \) is bounded if and only if \( k > t + 4 \), \( p > \frac{s+k+4}{k+2} \).

So we see, when \( t - s' + sp \leq 0, \) (we must have \( t < s' \) and then \( t - s' \leq (2 - s)p \) trivially), \( B_{\Omega, \nu'} \) is bounded if and only if \( p \in (\frac{2k+2}{k+2}, \frac{k+4}{k+2}) \).

For \( T_3 \), when \( t - s' + sp > 0, \) Theorem 4.6 tells us \( T_2 \) is bounded if \( t + 4 > (k+1)p \) and \( p > \frac{s+k+4}{s+k+3} \).

Then again we see, when \( t - s' + sp > 0 \) and \( t - s' \leq (2 - s)p, \) \( B_{\Omega, \nu'} \) is bounded if and only if \( p \in (\frac{2k+2}{k+2}, \frac{k+4}{k+2}) \). This shows the first part of the corollary.

For \( T_1 \), when \( t - s' > (2 - s)p, \) (we must have \( t > s' \) and then \( t - s' + sp > 0 \) trivially), Theorem 4.6 tells us \( T_1 \) is bounded if \( p \in (\frac{2k+2}{k+2}, \frac{k+4}{k+2}) \). So we see in this case, \( B_{\Omega, \nu'} \) is bounded if \( p \in (\frac{2k+2}{k+2}, \frac{k+4}{k+2}) \). This completes the proof. \[ \square \]

Remark 4.19. Note that when \( k = -1 \) and \( s = 2 \), Corollary 4.18 implies Theorem 1.1 in [5].

Remark 4.20. For the boundedness part, one can also apply the argument in section 3 by using a variant of Schur’s test. However, if we apply the two-weight inequality, we can also consider a wider class of weights, namely, \( \mu(z) = |z_2|^{s'} |g(z_2)|^2 \), where \( z \in \mathbb{H} \) and \( g \) is a non-vanishing holomorphic function on \( \mathbb{D} \). Indeed, one may also consider a weight of the form \( \mu(z) = |z_1|^{-s'} |z_2|^{s'} \). In this case, we have to consider one more the weighted space \( (\mathbb{D}, |z|^{s'}) \).
5. Some analysis for $A_+^2(R_+^2)$

**Definition 5.1.** Two weights $\mu_1$ and $\mu_2$ on $R_+^2$ are in the class $A_+^1(R_+^2)$, denoted by the order pair $(\mu_1, \mu_2) \in A_+^1(R_+^2)$, if there is a $c > 0$, so that for all special disks $D$, we have

$$
\frac{1}{|D \cap R_+^2|} \int_{D \cap R_+^2} \mu_1(\zeta) d(\zeta) \leq c \mu_2(z),
$$

where $z \in D \cap R_+^2$. For some weight $\mu$, if it satisfies $(\mu, \mu) \in A_+^1(R_+^2)$, we simply adopt the notation $\mu \in A_+^1(R_+^2)$.

We first give a simple observation for a special product of two $A_+^1$ weights.

**Proposition 5.2.** Suppose $\mu_1$ and $\mu_2$ are two weights and $\mu_j \in A_+^1(R_+^2)$, $j = 1, 2$. Then for $1 \leq p < \infty$, we have $\mu_1 \mu_2^{1-p} \in A_+^p(R_+^2)$.

**Proof.** By definition, for $j = 1, 2$ we have

$$
\frac{1}{|D \cap R_+^2|} \int_{D \cap R_+^2} \mu_j(z) d(z) \leq c \inf_{z \in D \cap R_+^2} \mu_j(z),
$$

for all special disks $D$. Then we see

$$
(5.1) \quad \frac{1}{|D \cap R_+^2|} \int_{D \cap R_+^2} \mu_1(z) \mu_2(z)^{1-p} d(z) \leq c \inf_{z \in D \cap R_+^2} \mu_1(z) \left( \inf_{z \in D \cap R_+^2} \mu_2(z) \right)^{1-p},
$$

and

$$
(5.2) \quad \left( \frac{1}{|D \cap R_+^2|} \int_{D \cap R_+^2} (\mu_1(z) \mu_2(z)^{1-p})^{-\frac{1}{p'}} d(z) \right)^{\frac{1}{p'}} \leq c \left( \inf_{z \in D \cap R_+^2} \mu_1(z) \right)^{-1} \left( \inf_{z \in D \cap R_+^2} \mu_2(z) \right)^{1-p}.
$$

Combine (5.1) and (5.2), we obtain $\mu_1 \mu_2^{1-p} \in A_+^p(R_+^2)$.

Before going further, we need two definitions and two lemmas.

**Definition 5.3.** For any measurable function $f$ on $R_+^2$, we define the special maximal operator $\tilde{M}^+$ by

$$
\tilde{M}^+(f)(z) = \sup_{z \in D} \frac{1}{|D \cap R_+^2|} \int_{D \cap R_+^2} |f(\zeta)| d(\zeta),
$$

where the supremum is taken over all special disks $D$ centered at the real axis containing $z$. It is clear that $\tilde{M}^+(f)$ is lower semi-continuous.

**Remark 5.4.** It is easy to see a weight $\mu$ belongs to the class $A_+^1(R_+^2)$ is equivalent to $\tilde{M}^+(\mu) \leq c\mu$. The two-weights case is similar.

**Remark 5.5.** It is easy to see, for $z, z' \in R_+^2$,

$$
\tilde{M}^+(f)(z') \geq \tilde{M}^+(f)(z),
$$

whenever $\Re(z') = \Re(z)$ and $\Im(z') \leq \Im(z)$. Note that the "absolute value" of the Bergman projection on the upper half plane

$$
\tilde{B}(f)(z) = \int_{R_+^2} \frac{1}{|z - w|^2} f(w) d(w),
$$
also has the same property above as \( \hat{M}^+ \), for \( f \geq 0 \).

**Definition 5.6.** Let \( \mathcal{S} \) be the collection of all the special squares of form

\[
\tilde{S}_{j,k} = \{ x + iy \in \mathbb{R}_+^2 \mid j \cdot 2^k \leq x \leq (j + 1) \cdot 2^k \text{ and } 0 \leq y \leq 2^k \},
\]

where \( j, k \in \mathbb{Z} \). Given a special square \( \tilde{S}_{j,k} \), we define

\[
\tilde{S}_{j,k}^* = \{ x + iy \in \mathbb{R}_+^2 \mid (j - 2) \cdot 2^k \leq x \leq (j + 3) \cdot 2^k \text{ and } 0 \leq y \leq 5 \cdot 2^k \}.
\]

Then we have an analogue of \( [\text{KS}] \) Lemma 7.

**Lemma 5.7.** Let \( f \geq 0 \) be an integrable function on \( \mathbb{R}_+^2 \), and suppose \( \alpha > 0 \). Then there is a sequence of measurable sets \( \{W_i\} \), and a sequence of special squares \( \{\tilde{S}_i\} \) such that

1. The intersection of different \( W_i \)'s has measure 0.
2. \( \tilde{S}_i \subset W_i \subset \tilde{S}_i^* \).
3. \( \frac{|\tilde{S}_i|}{16} \leq \int_{W_i} f(\zeta) d(\zeta) \).
4. If \( \hat{M}^+(f)(z) > \alpha \), then \( z \in \bigcup W_i \).

**Proof.** Follow the idea in \( [\text{KS}] \), we argue as in the classical Calderón-Zygmund lemma. Since \( \int_{\mathbb{R}_+^2} f(\zeta) d(\zeta) < \infty \), there is a \( k_0 \in \mathbb{Z}^+ \) so that for all \( k \geq k_0 \) and all \( j \in \mathbb{Z} \), we have

\[
(5.3) \quad \frac{1}{|\tilde{S}_{j,k}|} \int_{\tilde{S}_{j,k}} f(\zeta) d(\zeta) \leq \frac{\alpha}{16}.
\]

For the \( k = k_0 - 1 \) level, to each \( j \in \mathbb{Z} \), we have either (5.3) still true or

\[
(5.4) \quad \frac{\alpha}{16} < \frac{1}{|\tilde{S}_{j,k}|} \int_{\tilde{S}_{j,k}} f(\zeta) d(\zeta) \leq \frac{\alpha}{4}.
\]

The right hand side of (5.4) follows from (5.3) in the \( k + 1 \) level. If (5.4) holds for this \( j \), we collect this special square \( \tilde{S}_{j,k} \) into the sequence \( \{\tilde{S}_i\} \), otherwise we continue this process to the \( k - 1 \) level in this \( \tilde{S}_{j,k} \). Therefore, we obtain a sequence of almost disjoint special squares \( \{\tilde{S}_i\} \) satisfying (5.4) with \( \tilde{S}_{j,k} \) replaced by \( \{\tilde{S}_i\} \).

Define \( W_1 = \tilde{S}_1^* \backslash \left( \bigcup_{m \neq 1} \tilde{S}_m \right) \), and successively let

\[
W_l = \tilde{S}_1^* \backslash \left( \bigcup_{m \neq 1} \tilde{S}_m \bigcup \bigcup_{l' < l} W_{l'} \right),
\]

for \( l > 1 \). Properties (1) and (2) are easy to check from this definition. Property (3) follows from \( \tilde{S}_i \subset W_i \) and \( \tilde{S}_i \) satisfies (5.4). If \( \hat{M}^+(f)(z) > \alpha \), then there is a special disk \( D_z = D_r(x_0) \) centered at \( x_0 \in \mathbb{R} \) with radius \( r > 0 \) so that \( z \in D_z \) and

\[
\frac{1}{|D_z \cap \mathbb{R}_+^2|} \int_{D_z \cap \mathbb{R}_+^2} f(\zeta) d(\zeta) > \alpha.
\]

If \( 2^{k_1 - 1} \leq r < 2^{k_1} \) for some \( k_1 \in \mathbb{Z} \), then \( D_z \) intersects at most three special squares \( \tilde{S}_{j,k_1} \)'s and it is contained in the union of these squares. Moreover, we have

\[
|D_z \cap \mathbb{R}_+^2| \leq \frac{\pi}{2} |\tilde{S}_{j,k_1}| < 4 |D_z \cap \mathbb{R}_+^2|.
\]
Therefore, at least one of such special squares, say $\tilde{S}_{j_1,k_1}$, satisfies
\[
\int_{D_{\zeta} \cap \mathbb{R}_+^2 \cap \tilde{S}_{j_1,k_1}} f(\zeta) \, d\zeta > \frac{1}{3} \alpha \left| D_{\zeta} \cap \mathbb{R}_+^2 \right|.
\]
So we obtain
\[
\int_{\tilde{S}_{j_1,k_1}} f(\zeta) \, d\zeta > \frac{\pi}{24} \alpha \left| \tilde{S}_{j_1,k_1} \right| > \frac{\alpha}{16} \left| \tilde{S}_{j_1,k_1} \right|.
\]
From our construction of the sequence $\{\tilde{S}_i\}$, $\tilde{S}_{j_1,k_1}$ cannot be any of those satisfying (5.3), so $\tilde{S}_{j_1,k_1}$ is contained in one of the special squares $\{\tilde{S}_i\}$. Since $\tilde{S}_{j_1,k_1}$ intersects $D_{\zeta}$, we must have $z \in D_{\zeta} \subset \tilde{S}_{j_1,k_1} \subset \tilde{S}_{j_1,k_1}^*$ for some $l_1$. By the definition of $\{W_l\}$, if $z$ is not in $W_1, \ldots, W_t$, then $z$ must be in $\tilde{S}_{m_1}$ for some $m_1$, hence $z \in W_{m_1}$, which implies (4).

\[\square\]

**Lemma 5.8.** Let $f$ be a measurable function on $\mathbb{R}_+^2$, then either $\tilde{M}^+(f)(z) = \infty$ for all $z \in \mathbb{R}_+^2$, or $\tilde{M}^+(f)(z) < \infty$ for all $z \in \mathbb{R}_+^2$.

**Proof.** Assume $\tilde{M}^+(f)(z) = \infty$ for some $z \in \mathbb{R}_+^2$, we show $\tilde{M}^+(f)(z') = \infty$ for any $z' \in \mathbb{R}_+^2$. By definition, there is a sequence of special disks $\{D_n\}$ with $z \in D_n$ and
\[
(5.5) \quad \frac{1}{|D_n \cap \mathbb{R}_+^2|} \int_{D_n \cap \mathbb{R}_+^2} |f(\zeta)| \, d\zeta > n,
\]
for all $n \in \mathbb{Z}^+$. Let $r_n$ be the radius of $D_n$, and let $x_n$ be the center, then $D_n = D_{r_n}(x_n)$. Since $z \in D_n$, we see $r_n > \Re(z) > 0$.

If $\{r_n\}$ is not bounded above, by selecting a subsequence, we may assume $\lim r_n = \infty$. Then given any $z' \in \mathbb{R}_+^2$, we have $r_n \geq |z' - z|$ for $n$ sufficiently large. In this case, it is easy to see $z' \in D_{2r_n}(x_n)$, the special disk centered at $x_n$ with radius $2r_n$. From (5.5), we see
\[
\int_{D_{2r_n}(x_n) \cap \mathbb{R}_+^2} |f(\zeta)| \, d\zeta > \frac{1}{4} n,
\]
for $n$ sufficiently large. This implies $\tilde{M}^+(f)(z') = \infty$.

If $\{r_n\}$ is bounded above, by selecting a subsequence, we may assume $\lim r_n = r$, for some $r$ with $\Re(z) \leq r < \infty$. Note that since $z \in D_n$, we have $\Re(z) \in D_n$, so $D_n \subset D_{r'}(\Re(z))$ for $n$ sufficiently large, where $D_{r'}(\Re(z))$ is a special disk centered at $\Re(z)$ with radius $3r$. Therefore, from (5.5), we see
\[
\int_{D_{r'}(\Re(z)) \cap \mathbb{R}_+^2} |f(\zeta)| \, d\zeta > \frac{1}{2} \pi r^2 \geq cn,
\]
where $c = \frac{1}{2} \pi \Re(z) > 0$, for $n$ sufficiently large. So we obtain
\[
\int_{D_{r'}(\Re(z)) \cap \mathbb{R}_+^2} |f(\zeta)| \, d\zeta = \infty.
\]
Now for any $z' \in \mathbb{R}_+^2$, it is easy to see $z' \in D_{3r+|z' - z|}(\Re(z))$, the special disk centered at $\Re(z)$ with radius $3r + |z' - z|$. From the equality above, we see
\[
\int_{D_{3r+|z' - z|}(\Re(z)) \cap \mathbb{R}_+^2} |f(\zeta)| \, d\zeta = \infty,
\]
which implies $\tilde{M}^+(f)(z') = \infty$. This completes the proof. \[\square\]
Now we are ready to state and prove the following theorems.

**Theorem 5.9.** Let $f$ be a measurable function on $\mathbb{R}^2_+$, then for any $0 < q < 1$, the function $\left(M^+(f)\right)^q$ is in $A^+_1(\mathbb{R}^2_+)$. 

**Proof.** It suffices to show the conclusion for $f \geq 0$. By Lemma 5.8, we can assume $\tilde{M}^+(f)(z) < \infty$ for all $z \in \mathbb{R}^2_+$, otherwise, the conclusion is trivial. If $\tilde{M}^+(f)(z) = 0$ for some $z \in \mathbb{R}^2_+$, then it is easy to see $f = 0$ on $\mathbb{R}^2_+$. In this case, the conclusion is trivial again. So we may assume $0 < \tilde{M}^+(f) < \infty$ on $\mathbb{R}^2_+$.

We use an analogue argument of [28] chapter 5.2. Let $\mu(z) = \left(\tilde{M}^+(f)(z)\right)^q$, we show $\tilde{M}^+(\mu)(z) \leq c\mu(z)$. Then it implies $\mu \in A^+_1(\mathbb{R}^2_+)$. 

Fixed $z \in \mathbb{R}^2_+$, we normalize $f$ by dividing $\tilde{M}^+(f)(z)$, so we may assume $\tilde{M}^+(f)(z) = 1$ and $\mu(z) = 1$. Hence, it suffices to show that there is a $c > 0$ such that 

\[
\frac{1}{|D \cap \mathbb{R}^2_+|} \int_{D \cap \mathbb{R}^2_+} \mu(\zeta) d(\zeta) \leq c,
\]

for any special $D$ containing $z$.

Given a special disk $D = D_R(x_0)$ that contains $z$, let $f_1 = \chi_{D_R(x_0) \cap \mathbb{R}^2_+} f$ and $f_2 = f - f_1$. We first deal with $f_1$. Let $V_\alpha = \{\zeta \in D \cap \mathbb{R}^2_+ \mid \tilde{M}^+(f_1)(\zeta) > \alpha\}$, we have 

\[
\int_{D \cap \mathbb{R}^2_+} \left(\tilde{M}^+(f_1)(\zeta)\right)^q d(\zeta) = \int_0^\infty q\alpha^{q-1} |V_\alpha| d\alpha
\]

\[
= \int_0^1 + \int_1^\infty q\alpha^{q-1} |V_\alpha| d\alpha.
\]

Since $|V_\alpha| \leq |D|$, we see the first integral of (5.7) is bounded by $cR^2$. For the second integral, since $\tilde{M}^+(f)(z) = 1$ and $z \in D_{2R}(x_0) \cap \mathbb{R}^2_+$, we see $f_1$ is integrable on $\mathbb{R}^2_+$. By Lemma 5.7, we have 

\[
|V_\alpha| \leq \sum_l |\mathcal{S}_l| = \sum_l 25 |\mathcal{S}_l|
\]

\[
\leq c \sum_l \frac{1}{\alpha} \int_{W_\alpha} f_1(\eta) d(\eta)
\]

\[
\leq \frac{c}{\alpha} \int_{\mathbb{R}^2_+} f_1(\eta) d(\eta)
\]

\[
= \frac{c}{\alpha} \int_{D_{2R}(x_0) \cap \mathbb{R}^2_+} f(\eta) d(\eta)
\]

\[
\leq \frac{cR^2}{\alpha} \tilde{M}^+(f)(z)
\]

\[
= \frac{cR^2}{\alpha}.
\]

So the second integral of (5.7) is bounded by $cR^2$. Hence, so is (5.7).

Next, we deal with $f_2$. For any $\zeta \in D \cap \mathbb{R}^2_+$, we consider an arbitrary special disk $D'_r$ that contains $\zeta$ and whose radius is $r$. It is easy to see $D'_r \subset D_{2R}(x_0)$. When
2r < R, we have \( D'_r \subset D_{2r+R}(x_0) \subset D_2R(x_0) \). Since \( f_2 \) vanishes on \( D_2R(x_0) \cap \mathbb{R}_+^2 \), we see
\[
\frac{1}{|D'_r \cap \mathbb{R}_+^2|} \int_{D'_r \cap \mathbb{R}_+^2} f_2(\eta) \, d(\eta) = 0 < c,
\]
for any \( c > 0 \). When \( 2r \geq R \), then \((2r + R)^2 \leq 16r^2\), we have
\[
\int_{D'_r \cap \mathbb{R}_+^2} f_2(\eta) \, d(\eta) \leq \int_{D_{2r+R}(x_0) \cap \mathbb{R}_+^2} f(\eta) \, d(\eta) \\
\leq |D_{2r+R}(x_0) \cap \mathbb{R}_+^2| \tilde{M}^+(f)(z) \\
= c(2r + R)^2 \\
\leq cr^2,
\]
since \( z \in D \subset D_{2r+R}(x_0) \). In either case, we obtain
\[
\frac{1}{|D'_r \cap \mathbb{R}_+^2|} \int_{D'_r \cap \mathbb{R}_+^2} f_2(\eta) \, d(\eta) < c.
\]
Since \( D'_r \) is arbitrary, we see \( \tilde{M}^+(f_2)(\zeta) \leq c \), for any \( \zeta \in D \cap \mathbb{R}_+^2 \). Therefore, we obtain
\[
(5.8) \quad \int_{D \cap \mathbb{R}_+^2} \left( \tilde{M}^+(f_2)(\zeta) \right)^q \, d(\zeta) \leq c R^2.
\]
Combine the fact that (5.7) is bounded by \( c R^2 \) and (5.8), we see
\[
\int_{D \cap \mathbb{R}_+^2} \left( \tilde{M}^+(f)(\zeta) \right)^q \, d(\zeta) \leq c R^2,
\]
which implies (5.6). This completes the proof. \( \square \)

**Theorem 5.10.** Assume \( p \geq 1 \), suppose \( \mu_1 \) and \( \mu_2 \) are two weights on \( \mathbb{R}_+^2 \), then we have a weak-type \((p,p)\) inequality, namely, there is a \( c > 0 \) so that
\[
(5.9) \quad \mu_1 \{ \{ z \in \mathbb{R}_+^2 \mid \tilde{M}^+(f)(z) > \alpha \} \} \leq \frac{c}{\alpha^p} \int_{\mathbb{R}_+^2} |f(z)|^p \, \mu_2(z) \, d(z)
\]
for all \( \alpha > 0 \), if and only if \((\mu_1, \mu_2) \in A^+_p(\mathbb{R}_+^2)\).

**Proof.** We use an analogue argument of [18] Theorem 8. For the sufficient part, we only need to show (5.9) for integrable \( f \geq 0 \). To see this, any measurable function \( f \geq 0 \) can be approximated by the increasing sequence \( \{ f_{XD_R} \}_{R > 0} \), where \( D_R = D_R(x_0) \) is a sequence of special disks centered at \( x_0 \in \mathbb{R} \) with radius \( R \). If we prove (5.9) for \( f_{XD_R} \), the monotonic convergent theorem will imply (5.9) for \( f \).

Note that, the set \( \{ z \in \mathbb{R}_+^2 \mid \tilde{M}^+(f)(z) > \alpha \} \) is the union of those of the \( f_{XD_R} \)'s, and these sets of the \( f_{XD_R} \)'s are increasing. But any \( f_{XD_R} \) can be approximated by an increasing sequence of simple functions, since the support of \( f_{XD_R} \) is bounded, these simple functions are integrable. By the same limiting argument, (5.9) for integrable functions will imply (5.9) for \( f_{XD_R} \).

Let \( V_\alpha = \{ z \in \mathbb{R}_+^2 \mid \tilde{M}^+(f)(z) > \alpha \} \), by Lemma 5.7 and argue as (4.9), for \( p > 1 \), we have
\[ \mu_1(V_\alpha) \leq \sum_i \mu_1(W_i) \]
\[ \leq \sum_i \mu_1(W_i) \left( \frac{16}{\alpha |S_i|} \int_{W_i} f(z) d(z) \right)^p \]
\[ \leq \sum_i \frac{c}{\alpha^p} \frac{\mu_1(W_i)}{|S_i|} \int_{W_i} f(z) \mu_2(z) d(z) \left( \frac{1}{|S_i|} \int_{S_i} \mu_2(z)^{-\frac{1}{p}} d(z) \right)^{\frac{p}{p'}} \]
\[ \leq \sum_i \frac{c}{\alpha^p} \int_{W_i} f(z) \mu_2(z) d(z) \]
\[ \leq \frac{c}{\alpha^p} \int_{\mathbb{R}_+^2} f(z) \mu_2(z) d(z). \]

For \( p = 1 \), similarly, we have
\[ \mu_1(V_\alpha) \leq \sum_i \frac{16 \mu_1(W_i)}{\alpha |S_i|} \int_{W_i} f(z) d(z) \]
\[ \leq \sum_i \frac{c}{\alpha} \frac{\mu_1(W_i)}{|S_i|} \int_{W_i} f(z) \mu_2(z) d(z) \left( \inf_{z \in W_i} \mu_2(z) \right)^{-1} \]
\[ \leq \sum_i \frac{c}{\alpha} \int_{W_i} f(z) \mu_2(z) d(z) \frac{\mu_1(S_i^*)}{|S_i^*|} \left( \inf_{z \in S_i^*} \mu_2(z) \right)^{-1} \]
\[ \leq \frac{c}{\alpha} \int_{\mathbb{R}_+^2} f(z) \mu_2(z) d(z). \]

For the necessary part, we first consider \( p > 1 \). Given any special disk \( D \), if \( \int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{1}{p'}} d(z) = \infty \). Then by duality of the space \( L^p(D \cap \mathbb{R}_+^2) \), there is a \( g \in L^p(D \cap \mathbb{R}_+^2) \) so that \( \int_{D \cap \mathbb{R}_+^2} g(z) \mu_2(z)^{-\frac{1}{p'}} d(z) = \infty \). Let \( f = g \mu_2^{-\frac{1}{p'}} \chi_{D \cap \mathbb{R}_+^2} \) on \( \mathbb{R}_+^2 \), then \( \overline{M}^+(f)(z) = \infty \) for all \( z \in D \cap \mathbb{R}_+^2 \). So (5.9) gives \( \mu_1(D \cap \mathbb{R}_+^2) = 0 \), which contradicts to the assumption \( \mu_1 > 0 \) almost everywhere. We also exclude the trivial case \( \mu_2 = \infty \) on \( D \cap \mathbb{R}_+^2 \) to see indeed we have \( \int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{1}{p'}} d(z) < \infty \).

Take \( f = \mu_2^{-\frac{1}{p'}} \chi_{D \cap \mathbb{R}_+^2} \) and \( \alpha = \frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{1}{p'}} d(z) \) in (5.9), we see
\[ \mu_1(D \cap \mathbb{R}_+^2) \leq \frac{c}{\alpha^p} \int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{1}{p'}} \mu_2(z) d(z) \]
\[ = \frac{c |D \cap \mathbb{R}_+^2|}{\alpha^p}, \]
which is equivalent to \( (\mu_1, \mu_2) \in A^+_p(\mathbb{R}_+^2) \).
When \( p = 1 \), given any special disk \( D \), we again exclude the trivial case \( \inf \mu_2 = \infty \), where the infimum is taken over all \( z \in D \cap \mathbb{R}^2_+ \). Then for any \( \epsilon > 0 \), there must be a measurable set \( U \subset D \cap \mathbb{R}^2_+ \) with \( |U| > 0 \), so that \( \mu_2(z) < \epsilon + \inf \mu_2 \) on \( U \).

Take \( f = \chi_U \) and \( \alpha = \frac{|U|}{|D \cap \mathbb{R}^2_+|} \) in (5.9), we see

\[
\mu_1(D \cap \mathbb{R}^2_+) \leq c \frac{|D \cap \mathbb{R}^2_+|}{|U|} \int_U \mu_2(z) d(z) \\
\leq c \frac{|D \cap \mathbb{R}^2_+|}{|U|} \left( \epsilon + \inf_{z \in D \cap \mathbb{R}^2_+} \mu_2(z) \right).
\]

Let \( \epsilon \to 0^+ \), we see the inequality above is equivalent to \((\mu_1, \mu_2) \in A^+_1(\mathbb{R}^2_+)\). This completes the proof. \( \square \)

Remark 5.11. For any measurable \( f \geq 0 \), we do not have the basic inequality \( f \leq \tilde{M}^+(f) \). So we can not follow the classical approach to show the reverse Hölder inequality for a weight in \( A^+_p(\mathbb{R}^2_+) \). Hence, we can not obtain the strong-type \((p, p)\) inequality, nor the factorization of a \( A^+_p \) weight. (Compare to the classical results for \( A_p \) weights, see [25] chapter 5 for details.)

6. Appendix

In this section, we give details of some lemmas from the previous sections. We begin with Lemma 3.4.

Lemma 6.1. For \( p \geq 1 \), the sum \( A_{n, p} = \sum_{j=1}^{n} j \left( a_j^p - a_{j+1}^p \right) \) diverges when \( p = 1 \) and converges when \( p > 1 \), as \( n \to \infty \). More precisely, we have

\[
\lim_{n \to \infty} A_{n, 1} = \infty,
\]

and

\[
\lim_{n \to \infty} A_{n, p} \leq c \sum_{j=1}^{\infty} \frac{1}{j^{1+\epsilon}} < \infty,
\]

for all \( p > 1 \), for some \( c > 0 \) and for sufficiently small \( \epsilon > 0 \).

Proof of Lemma 3.4. We first show the following statement,

\[
\left( \frac{1}{j} \right)^2 \lesssim \frac{1}{j} - \left( \frac{1}{j+1} \right)^{\frac{j+1}{j}} \lesssim \left( \frac{1}{j} \right)^{2-\epsilon'},
\]

for any \( \epsilon' > 0 \), as \( j \to \infty \).

We obtain the first inequality by looking at the limit (applied L’Hôpital’s rule)

\[
\lim_{j \to \infty} \frac{\frac{1}{j} - \left( \frac{1}{j+1} \right)^{\frac{j+1}{j}}}{\left( \frac{1}{j} \right)^2} = \lim_{j \to \infty} -\frac{1}{j^2} + \left( \frac{1}{j+1} \right)^{\frac{j+1}{j}} \left( -\frac{1}{j} \log(j+1) + \frac{1}{j} \right)
\]

\[
= -2 \frac{1}{j^3} + \frac{1}{j} \left( \log(j+1) - \frac{1}{j+1} \right).
\]
Similarly, for the second inequality, we look at the limit (applied L'Hôpital’s rule)

\[
\lim_{j \to \infty} \frac{1 \cdot \frac{1}{j} - \left(\frac{1}{j+1}\right)^\frac{1}{j}}{\log\left(\frac{1}{j+1}\right) + \frac{1}{j+1} \log(j+1)}
\]

\[
= \frac{1}{2} \left[ \lim_{j \to \infty} \frac{1 - \left(\frac{1}{j+1}\right)^\frac{1}{j}}{j} + \lim_{j \to \infty} \left(\frac{1}{j+1}\right)^\frac{1}{j} \log(j+1) \right]
\]

\[
= \frac{1}{2} \left[ \lim_{j \to \infty} \frac{1 - \left(\frac{1}{j+1}\right)^\frac{1}{j}}{j} + 1 + \lim_{j \to \infty} \log(j+1) \right]
\]

(L'Hôpital’s rule)

\[
= \frac{1}{2} \left[ \lim_{j \to \infty} \frac{\left(\frac{1}{j+1}\right)^\frac{1}{j} \left(-\frac{1}{j} \log(j+1) + \frac{1}{j} - \frac{1}{j+1}\right)}{-\frac{1}{j^2}} + 1 + \lim_{j \to \infty} \log(j+1) \right]
\]

\[
= \lim_{j \to \infty} \log(j+1)
\]

\[
= \infty.
\]

Similarly, for the second inequality, we look at the limit (applied L'Hôpital’s rule)

\[
\lim_{j \to \infty} \frac{\frac{1}{j} - \left(\frac{1}{j+1}\right)^\frac{1}{j}}{(\frac{1}{j})^2 - \epsilon^2} = \lim_{j \to \infty} \frac{-\frac{1}{j^2} + \left(\frac{1}{j+1}\right)^\frac{1}{j} \left(-\frac{1}{j} \log(j+1) + \frac{1}{j} - \frac{1}{j+1}\right)}{(\epsilon - 2) \left(\frac{1}{j}\right)^{1-\epsilon'}}
\]

\[
= \frac{1}{2 - \epsilon'} \left[ \lim_{j \to \infty} \frac{1 - \left(\frac{1}{j+1}\right)^\frac{1}{j}}{\left(\frac{1}{j}\right)^{1-\epsilon'}} + \lim_{j \to \infty} \left(\frac{1}{j+1}\right)^\frac{1}{j} \log(j+1) \right]
\]

\[
= \frac{1}{2 - \epsilon'} \left[ \lim_{j \to \infty} \frac{j}{j+1} \cdot \lim_{j \to \infty} \frac{1 - \left(\frac{1}{j+1}\right)^\frac{1}{j}}{\left(\frac{1}{j}\right)^{1-\epsilon'}} + \lim_{j \to \infty} \left(\frac{1}{j+1}\right)^\frac{1}{j} \frac{j}{j+1} \log(j+1) \right]
\]

\[
= \frac{1}{2 - \epsilon'} \left[ \lim_{j \to \infty} \frac{1 - \left(\frac{1}{j+1}\right)^\frac{1}{j}}{\left(\frac{1}{j}\right)^{1-\epsilon'}} + \frac{1}{\left(\frac{1}{j}\right)^{1-\epsilon'}} \right]
\]

(L'Hôpital’s rule)

\[
= \frac{1}{2 - \epsilon'} \left[ \lim_{j \to \infty} \frac{\left(\frac{1}{j+1}\right)^\frac{1}{j} \left(-\frac{1}{j} \log(j+1) + \frac{1}{j} - \frac{1}{j+1}\right)}{-\frac{1}{j^2}} + 0 \right]
\]

\[
= \frac{1}{(2 - \epsilon')(1 - \epsilon')} \lim_{j \to \infty} \left(\frac{1}{j+1}\right)^\frac{1}{j} \left(\frac{\log(j+1)}{j e^{-\epsilon'}} - \frac{j}{(j+1) j e^{-\epsilon'}}\right)
\]

\[
= 0.
\]
This shows the inequality (6.1).

Now, for $p = 1$, we have

$$A_{n,1} = \sum_{j=1}^{n} j \left( a_j^+ - a_{j+1}^+ \right)$$

$$= \sum_{j=1}^{n} j \left[ \left( \frac{1}{j} \right)^{\frac{1}{p}} - \left( \frac{1}{j+1} \right)^{\frac{1}{p}} \right]$$

$$= \sum_{j=1}^{n} j \left[ \frac{1}{j} - \left( \frac{1}{j+1} \right)^{\frac{p+1}{p}} \right]$$

So, from (6.1), we have

$$\lim_{n \to \infty} A_{n,1} \gtrsim \sum_{j=1}^{\infty} j \cdot \left( \frac{1}{j} \right)^2 = \sum_{j=1}^{\infty} \frac{1}{j} = \infty.$$ 

For $p > 1$, we consider the function $\phi(x) = x^p, x \in (0,1]$. By the mean-value theorem, for each $j$, we have

$$\phi \left( \frac{1}{j} \right) - \phi \left( \left( \frac{1}{j+1} \right)^{\frac{1}{p}} \right) = \left[ \frac{1}{j} - \left( \frac{1}{j+1} \right)^{\frac{1}{p}} \right] \phi'(x_j),$$

where $\phi'(x) = px^{p-1}$ and $\left( \frac{1}{j+1} \right)^{\frac{1}{p}} \leq x_j \leq \frac{1}{j}$. Since $p - 1 > 0$, we have

$$x_j^{p-1} \leq \left( \frac{1}{j} \right)^{p-1}.$$ 

So we get

$$\phi \left( \frac{1}{j} \right) - \phi \left( \left( \frac{1}{j+1} \right)^{\frac{1}{p}} \right) \leq \left[ \frac{1}{j} - \left( \frac{1}{j+1} \right)^{\frac{1}{p}} \right] \phi \left( \frac{1}{j} \right)^{p-1}.$$ 

Therefore, from (6.1), we have

$$\lim_{n \to \infty} A_{n,p} = \sum_{j=1}^{\infty} j \left( a_j^+ - a_{j+1}^+ \right)$$

$$= \sum_{j=1}^{\infty} j \left[ \phi \left( \frac{1}{j} \right) - \phi \left( \left( \frac{1}{j+1} \right)^{\frac{1}{p}} \right) \right]$$

$$\leq \sum_{j=1}^{\infty} j \left[ \frac{1}{j} - \left( \frac{1}{j+1} \right)^{\frac{1}{p}} \right] \phi \left( \frac{1}{j} \right)^{p-1}$$

$$\leq \sum_{j=1}^{\infty} j \left( \frac{1}{j} \right)^{2-\epsilon'} \left( \frac{1}{j} \right)^{p-1}$$

$$= \sum_{j=1}^{\infty} \left( \frac{1}{j} \right)^{p-\epsilon'} < \infty,$$

for sufficiently small $\epsilon' > 0$, such that $p - \epsilon' = 1 + \epsilon$ for some $\epsilon > 0$. \qed
The properties of the operator $E$ in Remark 4.8.

**Lemma 6.2.** For any nonnegative measurable functions $f$ and $g$, let $p'$ be the conjugate exponent of $p$, we have

(a) \[
\int_{\mathbb{R}^2_+} E(f)(z)g(z) \, d(z) = \int_{\mathbb{R}^2_+} E(f)(z)E(g)(z) \, d(z),
\]

(b) \[
\int_{\mathbb{R}^2_+} (E(f)(z))^p g(z) \, d(z) \leq \int_{\mathbb{R}^2_+} E(f^p)(z)g(z) \, d(z),
\]

(c) \[
E(fg)(z) \leq \left( E(f^p)(z) \right)^{\frac{1}{p}} \left( E(g^q)(z) \right)^{\frac{1}{q}} \text{ for all } z \in \mathbb{R}^2_+.
\]

**Proof.** For (a), the left hand side of the equality is
\[
\sum_{j,k} \int_{S_{j,k}} E(f)(z)g(z) \, d(z) = \sum_{j,k} E(f)(z)\chi_{S_{j,k}}(z) \int_{S_{j,k}} g(z) \, d(z) = \sum_{j,k} E(f)(z)\chi_{S_{j,k}}(z)E(g)(z) |S_{j,k}|.
\]

On the other hand, the right hand side of the equality is
\[
\sum_{j,k} \int_{S_{j,k}} E(f)(z)E(g)(z) \, d(z) = \sum_{j,k} E(f)(z)E(g)(z)\chi_{S_{j,k}}(z) |S_{j,k}|,
\]
which equals the left hand side.

For (b), we apply Jensen’s inequality to the integral $\int_{S_{j,k}} f(z) \, d(z)$, via the convex function $x^p$. We see
\[
(E(f)(z))^p g(z) \leq E(f^p)(z)g(z),
\]
for all $z \in S_{j,k}$. Integrate over $S_{j,k}$ and sum over all $j, k \in \mathbb{Z}$, then we get the desired inequality.

For (c), it is a direct consequence of Hölder’s inequality. \hfill \square

Next, we prove **Proposition 4.9.**

**Proposition 6.3.** If $(\mu_1, \mu_2) \in A^+_p(\mathbb{R}^2_+)$, then $(E(\mu_1), E(\mu_2)) \in A_p(\mathbb{R}^2_+)$.

**Definition 6.4.** A finite collection of measurable subsets $\{W_i\}_{i=1}^N$ in $\mathbb{R}^2_+$ are comparable to each other if there is a special disk $D_R(x_0)$ centered at $x_0 \in \mathbb{R}$, so that $W_i \subset D_R(x_0) \cap \mathbb{R}^2_+$, while $|D_R(x_0) \cap \mathbb{R}^2_+| \leq a |W_i|$, for some $a \geq 1$, $l = 1, 2, \ldots, N$.

**Lemma 6.5.** If $(\mu_1, \mu_2) \in A^+_p(\mathbb{R}^2_+)$ and if $W_1$ and $W_2$ are comparable, then there is a $C = C(a) \geq 1$, such that $\mu_1(W_1) \leq C \mu_2(W_2)$.

**Proof.** Since $W_1$ and $W_2$ are comparable, we can find a special disk $D_R(x_0)$ so that $W_i \subset D_R(x_0) \cap \mathbb{R}^2_+$ and $|D_R(x_0) \cap \mathbb{R}^2_+| \leq a |W_i|$, $l = 1, 2$. Then
\[
(6.2) \hspace{1cm} \int_{W_1} \mu_1(z) \, d(z) \leq a \int_{D_R(x_0) \cap \mathbb{R}^2_+} \mu_1(z) \, d(z),
\]
and apply Jensen’s inequality to the integral \( \int_{W_2} \mu_2(z) d(z) \), via the convex function \( x^{-\frac{p}{p}} \), we have

\[
(6.3) \quad \left( \int_{W_2} \mu_2(z) d(z) \right)^{-\frac{p}{p}} \leq \int_{W_2} \mu_2(z)^{-\frac{p}{p}} d(z) \leq a \int_{D_R(z_0) \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p}{p}} d(z).
\]

Combine (6.2) and (6.3), since \(|W_1| \leq |D_R(x_0) \cap \mathbb{R}_+^2| \leq a |W_2| \) and since \((\mu_1, \mu_2) \in A^*_p(\mathbb{R}_+^2)\), we see

\[
\mu_1(W_1) \leq Aa^{2+\frac{p}{p}} \mu_2(W_2).
\]

\[\Box\]

Proof of Proposition 4.9/6.3.

For any disk \( D_R(z_0) \) centered at \( z_0 \in \mathbb{R}_+^2 \), let \( x_0 = \Re(z_0) \) and \( y_0 = \Im(z_0) \), denote \( \{W_l\} = \{S_{j,k} \cap D_R(z_0) \cap \mathbb{R}_+^2 \cap S_{j,k} \neq \emptyset\} \). We separate our arguments into two cases.

Case (I), \( y_0 \geq 2R \).

Suppose \( 2^{k_0} \leq y_0 - R \leq 2^{k_0+1} \), for some \( k_0 \in \mathbb{Z} \). Since \( y_0 \geq 2R \), we see \( R \leq 2^{k_0+1} \), \( y_0 \leq 2^{k_0+2} \), and \( y_0 + R \leq 2^{k_0+3} \). Therefore, \( D_R(z_0) \) must be covered by the union of 2 squares of side-length \( 2^{k_0+2} \), 4 squares of side-length \( 2^{k_0+1} \), and 8 squares of side-length \( 2^{k_0} \). So the size of the collection \( \{W_l\} \) is at most \( 2 + 4 + 8 = 14 \).

Now, if \( x + iy \in D_R(z_0) \cap S_{j,k} \), for some \( j, k \in \mathbb{Z} \). Then we have \( y_0 - R \leq y \leq y_0 + R \) and \( 2^k \leq y \leq 2^{k+1} \). So it is easy to see \( \frac{y}{y_0} \leq \frac{2^k}{2^{k_0}} \). For such \( S_{j,k} \), any \( x' + iy' \in S_{j,k} \), we have \( y' \leq 2^{k+1} \leq 3\gamma_0 \), and \( |x' - x_0| \leq R + 2^k \leq 2y_0 \). Then \( x' + iy' \in D_{4y_0}(x_0) \cap \mathbb{R}_+^2 \), and hence \( S_{j,k} \subseteq D_{4y_0}(x_0) \cap \mathbb{R}_+^2 \). But \( |D_{4y_0}(x_0) \cap \mathbb{R}_+^2| = 8\pi y_0^2 \) and \( |S_{j,k}| = (2^k)^2 \geq 4\pi \), we see \( |D_{4y_0}(x_0) \cap \mathbb{R}_+^2| \leq a |S_{j,k}| \) for \( a = 128\pi \). Therefore, we obtain a finite collection \( \{W_l\} \) which are comparable to each other.

Let \( M = \frac{\mu_1(W_1)}{|W_1|} = \max \frac{\mu_1(W_l)}{|W_l|} \) and \( m = \frac{\mu_2(W_2)}{|W_2|} = \min \frac{\mu_2(W_l)}{|W_l|} \). When \( z \in D_R(z_0) \), we must have \( z \in W_l \) for some \( l \), then \( E(\mu_1)(z) = \frac{\mu_1(W_l)}{|W_l|} \leq M \) and \( E(\mu_2)(z) = \frac{\mu_2(W_l)}{|W_l|} \geq m \). Apply Lemma 6.5 to the comparable subcollection \( \{W_1, W_2\} \), we see

\[
\mu_1(W_1) \leq C(a)\mu_2(W_2).
\]

Noting that \( |W_2| \leq a |W_1| \), we obtain \( M \leq C(a)m \), for some \( C(a) > 1 \) independent of \( D_R(z_0) \). Therefore,

\[
\int_{D_R(z_0)} E(\mu_1)(z) d(z) \left( \int_{D_R(z_0)} E(\mu_2)(z)^{-\frac{p}{p}} d(z) \right)^{\frac{p}{p}} \leq Mm^{-1} \leq c.
\]

Case (II), \( y_0 < 2R \).

Let \( S^* = \cup W_l \) be the union of the collection \( \{W_l\} \), then \( D_R(z_0) \cap \mathbb{R}_+^2 \subseteq S^* \subseteq D_{8R}(x_0) \cap \mathbb{R}_+^2 \). To see the second inclusion, we follow a similar argument as case (I). If \( x + iy \in D_R(z_0) \cap \mathbb{R}_+^2 \cap S_{j,k} \), for some \( j, k \in \mathbb{Z} \), then \( y \leq y_0 + R \) and \( 2^k \leq y \).

Since \( y_0 < 2R \), we see \( 2^k < 3R \). For such \( S_{j,k} \), any \( x' + iy' \in S_{j,k} \), we have \( y' \leq 2^{k+1} \leq 6R \), and \( |x' - x_0| \leq R + 2^k < 4R \). So \( x' + iy' \in D_{8R}(x_0) \mathbb{R}_+^2 \), and hence \( S_{j,k} \subseteq D_{8R}(x_0) \mathbb{R}_+^2 \) as desired.
It is easy to see \( \int_{S_{j,k}} E(\mu_1)(z) \, d(z) = \int_{S_{j,k}} \mu_1(z) \, d(z) \) for every \( S_{j,k} \), so we have
\[
\int_{D_R(x_0) \cap \mathbb{R}^2_+} E(\mu_1)(z) \, d(z) \leq \int_{S^*} E(\mu_1)(z) \, d(z)
\]
\[
= \sum_l \int_{W_l} E(\mu_1)(z) \, d(z)
\]
\[
= \sum_l \int_{W_l} \mu_1(z) \, d(z)
\]
\[
= \int_{S^*} \mu_1(z) \, d(z)
\]
\[
\leq \int_{D_R(x_0) \cap \mathbb{R}^2_+} \mu_1(z) \, d(z).
\]

Note that \( |D_R(x_0) \cap \mathbb{R}^2_+| = 32R^2 \pi \) and \( |D_R(x_0) \cap \mathbb{R}^2_+| \geq \frac{R^2}{2} \pi \), so we obtain
\[
(6.4) \quad \int_{D_R(x_0) \cap \mathbb{R}^2_+} E(\mu_1)(z) \, d(z) \leq 64 \int_{D_R(x_0) \cap \mathbb{R}^2_+} \mu_1(z) \, d(z).
\]

Next, for every \( S_{j,k} \), when \( z \in S_{j,k} \), we have \( E(\mu_2)(z)^{-\frac{\nu}{\nu'}} \leq \int_{S_{j,k}} E(\mu_2)(z)^{-\frac{\nu}{\nu'}} \, d(z) \) by Jensen’s inequality applied to the convex function \( x^{-\frac{\nu}{\nu'}} \). So we see \( \int_{S_{j,k}} E(\mu_2)(z)^{-\frac{\nu}{\nu'}} \, d(z) \leq \int_{S_{j,k}} \mu_2(z)^{-\frac{\nu}{\nu'}} \, d(z) \), and hence,
\[
\int_{D_R(x_0) \cap \mathbb{R}^2_+} E(\mu_2)(z)^{-\frac{\nu}{\nu'}} \, d(z) \leq \int_{S^*} E(\mu_2)(z)^{-\frac{\nu}{\nu'}} \, d(z)
\]
\[
= \sum_l \int_{W_l} E(\mu_2)(z)^{-\frac{\nu}{\nu'}} \, d(z)
\]
\[
\leq \sum_l \int_{W_l} \mu_2(z)^{-\frac{\nu}{\nu'}} \, d(z)
\]
\[
= \int_{S^*} \mu_2(z)^{-\frac{\nu}{\nu'}} \, d(z)
\]
\[
\leq \int_{D_R(x_0) \cap \mathbb{R}^2_+} \mu_2(z)^{-\frac{\nu}{\nu'}} \, d(z).
\]

Again, by taking the average, we obtain
\[
(6.5) \quad \int_{D_R(x_0) \cap \mathbb{R}^2_+} E(\mu_2)(z)^{-\frac{\nu}{\nu'}} \, d(z) \leq 64 \int_{D_R(x_0) \cap \mathbb{R}^2_+} \mu_2(z)^{-\frac{\nu}{\nu'}} \, d(z).
\]

Since \( (\mu_1, \mu_2) \in A_p^+(\mathbb{R}^2_+) \), combine (6.4) and (6.5), we see \( (E(\mu_1), E(\mu_2)) \in A_p(\mathbb{R}^2_+) \). \( \square \)

Next, we prove Lemma 4.11.

**Lemma 6.6.** For any \( f \geq 0 \), we have
\[
(6.6) \quad \tilde{B}(f) \leq cE\tilde{B}E(f).
\]
Proof. We first show for any $z, w \in \mathbb{R}^2_+$,
\begin{equation}
E_z E_w \left( \frac{1}{|z - w|^2} \right) \geq \frac{c}{|z - w|^2},
\end{equation}
where $E_z$ and $E_w$ are operators $E$ acting on $z$ and $w$ respectively. Suppose $z \in S_{j,k}$ and $w \in S_{j',k'}$, with $k \geq k'$. We separate our arguments into two cases.

Case (I), $|\Re(z - w)| \leq 2^k$.
Since $z \in S_{j,k}$ and $w \in S_{j',k'}$, we see $|\Im(z - w)| \geq 2^k + 2^k' \geq 2^k$. Then
\[
\frac{1}{|z - w|^2} \leq \frac{1}{|\Im(z - w)|^2} \leq \frac{1}{22^k}.
\]
On the other hand, for any $\zeta \in S_{j,k}$ and $\eta \in S_{j',k'}$, we have
\[
|\Re(\zeta - \eta)| \leq |\Re(\zeta - z)| + |\Re(z - w)| + |\Re(w - \eta)| \leq 2^k + 2^k + 2^k' \leq 3 \cdot 2^k,
\]
and
\[
|\Im(\zeta - \eta)| \leq 2^{k+1} + 2^{k'+1} \leq 2^{k+2}.
\]
So we obtain
\[
\frac{1}{|\zeta - \eta|^2} \geq \frac{1}{(3 \cdot 2^k)^2 + (2^{k+2})^2} = \frac{1}{22^k} \cdot \frac{1}{25}.
\]
Therefore, we have
\[
E_z E_w \left( \frac{1}{|z - w|^2} \right) = \int_{S_{j,k}} \int_{S_{j',k'}} \frac{1}{|\zeta - \eta|^2} d\eta \, d\zeta \geq \frac{1}{25} \cdot \frac{1}{22^k} \cdot \frac{1}{25} = \frac{1}{25} \cdot \frac{1}{|z - w|^2}.
\]

Case (II), $|\Re(z - w)| > 2^k$.
Similarly, we have
\[
\frac{1}{|z - w|^2} \leq \frac{1}{|\Re(z - w)|^2}.
\]
For any $\zeta \in S_{j,k}$ and $\eta \in S_{j',k'}$, again we see
\[
|\Re(\zeta - \eta)| \leq 2^k + 2^k + 2^k' \leq 3 |\Re(z - w)|,
\]
and
\[
|\Im(\zeta - \eta)| \leq 2^{k+1} + 2^{k'+1} \leq 4 |\Re(z - w)|.
\]
So we obtain
\[
\frac{1}{|\zeta - \eta|^2} \geq \frac{1}{(3 |\Re(z - w)|)^2 + (4 |\Re(z - w)|)^2} = \frac{1}{25} \cdot \frac{1}{|\Re(z - w)|^2}.
\]
Therefore, we have
\[
E_z E_w \left( \frac{1}{|z - w|^2} \right) = \int_{S_{j,k}} \int_{S_{j',k'}} \frac{1}{|\zeta - \eta|^2} d\eta \, d\zeta \geq \frac{1}{25} \cdot \frac{1}{|\Re(z - w)|^2} \cdot \frac{1}{25} = \frac{1}{25} \cdot \frac{1}{|z - w|^2}.
\]
For \( k' \geq k \), all arguments remain the same except switching \( k' \) with \( k \), then we have shown (6.7). Now (6.6) follows from the argument below,

\[
E\overline{B}E(f)(z) = E_z \left( \int_{\mathbb{R}^2_+} \frac{1}{|z - w|^2} E(f)(w) \, d(w) \right)
\]

\[
\left( \text{Lemma 6.2 (a)} \right) = E_z \left( \int_{\mathbb{R}^2_+} \frac{1}{|z - w|^2} f(w) \, d(w) \right)
\]

\[
\geq \int_{\mathbb{R}^2_+} c \frac{1}{|z - w|^2} f(w) \, d(w)
\]

\[
= c\overline{B}(f)(z).
\]

\( \square \)

At the end of this section, we show the necessity of \textbf{Corollary 4.12}.

\textbf{Proposition 6.7.} For \( p > 1 \), suppose \( \mu_1 \) and \( \mu_2 \) are two weights such that \( c\mu_1 \geq \mu_2 \) for some \( c > 0 \), then the Bergman projection \( B \) on \( \mathbb{R}^2_+ \) is bounded from \( L^p(\mathbb{R}^2_+, \mu_2) \) to \( L^p(\mathbb{R}^2_+, \mu_1) \) only if \( (\mu_1, \mu_2) \in A^p_+(\mathbb{R}^2_+) \).

\textbf{Proof.} For any special disk \( D = D_R(x_0) \) with \( x_0 \in \mathbb{R} \) and \( R > 0 \), let \( D' = D_R(x_0 + 10Ri) \) be the disk with the same radius but centered at \( (x_0, 10R) \in \mathbb{R}^2_+ \). For this pair \( D \) and \( D' \), we see that if \( z \in D' \) and \( w \in D \), or if \( z \in D \) and \( w \in D' \), we have \( \mathbb{R}(\frac{1}{z - w}) \geq cR^{-2} \) for some \( c > 0 \). If \( B \) is bounded, we see

\[
\int_{D \cap \mathbb{R}^2_+} |B(f)(z)|^p \mu_1(z) \, d(z) \leq c \int_{\mathbb{R}^2_+} |f(z)|^p \mu_2(z) \, d(z).
\]

Take \( f = \chi_{D'} \) in (6.8), we obtain

\[
\mu_1(D \cap \mathbb{R}^2_+) \leq c\mu_2(D') \leq c\mu_1(D').
\]

On the other hand, for any \( f \geq 0 \) on \( \mathbb{R}^2_+ \), we apply (6.8) to the function \( f\chi_{D \cap \mathbb{R}^2_+} \), provided \( f\chi_{D \cap \mathbb{R}^2_+} \in L^p(\mathbb{R}^2_+, \mu_2) \). We see

\[
\int_{D'} |B(f)(z)|^p \mu_1(z) \, d(z) \leq \int_{\mathbb{R}^2_+} |B(f)(z)|^p \mu_1(z) \, d(z) \leq c \int_{D \cap \mathbb{R}^2_+} f(z)^p \mu_2(z) \, d(z),
\]

and for \( z \in D' \),

\[
|B(f)(z)|^p \geq \left( cR^{-2} \int_{D \cap \mathbb{R}^2_+} f(w) \, d(w) \right)^p \geq c \left( \int_{D \cap \mathbb{R}^2_+} f(z) \, d(z) \right)^p.
\]

So we obtain

\[
\mu_1(D') \left( \int_{D \cap \mathbb{R}^2_+} f(z) \, d(z) \right)^p \leq c \int_{D \cap \mathbb{R}^2_+} f(z)^p \mu_2(z) \, d(z),
\]

provided \( f\chi_{D \cap \mathbb{R}^2_+} \in L^p(\mathbb{R}^2_+, \mu_2) \). Therefore, combine (6.9) and (6.10), we have

\[
\mu_1(D \cap \mathbb{R}^2_+) \left( \int_{D \cap \mathbb{R}^2_+} f(z) \, d(z) \right)^p \leq c \int_{D \cap \mathbb{R}^2_+} f(z)^p \mu_2(z) \, d(z),
\]
for some \( c > 0 \), provided \( f \chi_{D \cap \mathbb{R}^2_{+}} \in L^p(\mathbb{R}^2_{+}, \mu_2) \).

To show (6.11) is indeed the \( A^+_p(\mathbb{R}^2_+) \) condition, we argue as the proof of necessary part of Theorem 5.10. If \( \int_{D \cap \mathbb{R}^2_+} \mu_2(z)^{-\frac{p'}{p}} d(z) = \infty \). Then by duality of the space \( L^p(D \cap \mathbb{R}^2_+) \), there is a \( g \in L^p(D \cap \mathbb{R}^2_+) \), so that \( \int_{D \cap \mathbb{R}^2_+} g(z) \mu_2(z)^{-\frac{1}{p'}} d(z) = \infty \). Take \( f = g \mu_2^{-\frac{1}{p'}} \chi_{D \cap \mathbb{R}^2_+} \) in (6.11), then \( \int_{D \cap \mathbb{R}^2_+} f(z) d(z) = \infty \) and \( \int_{D \cap \mathbb{R}^2_+} f(z)^p \mu_2(z) d(z) < \infty \). So (6.11) gives \( \mu_1(D \cap \mathbb{R}^2_+) = 0 \), which contradicts to the assumption \( \mu_1 > 0 \) almost everywhere. So we see \( \int_{D \cap \mathbb{R}^2_+} \mu_2(z)^{-\frac{p'}{p}} d(z) < \infty \).

Now take \( f = (\mu_2)^{-\frac{p'}{p}} \chi_{D \cap \mathbb{R}^2_+} \) in (6.11), and note that
\[
\int_{D \cap \mathbb{R}^2_+} f(z)^p \mu_2(z) d(z) = \int_{D \cap \mathbb{R}^2_+} \mu_2(z)^{-\frac{p'}{p}} d(z) < \infty,
\]

since \( \frac{p'}{p} = p' - 1 \). We see (6.11) is equivalent to the \( A^+_p(\mathbb{R}^2_+) \) condition.

\[\square\]

7. Concluding Remarks

We have shown the \( L^p \) regularity of the weighted Bergman projection on the Hartogs triangle and its generalization. The range for \( p \in (1, \infty) \) is quite sharp. However, we make restriction on the weights, although it belongs to a wide class of functions on the domain. We hope, in the future, we could develop new technique to get around this difficulty. We also shown a two-weight inequality for the Bergman projection on the upper half plane. But we conjecture that the assumption for the weights could be weakened. We provided some clues to apply the special maximal operator to get the desired result. We hope, in the future, we could at least get a weak-type \((p, p)\) inequality for the Bergman projection on the upper half plane.

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