Random factors in thermal insulation properties of buildings

E A Korol and G A Afanasyev
Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, 129337, Russia

Abstract. We study thermal insulation properties of a multilayer enclosing structures taking into account the random factor in changes in external temperature. The main purpose of the proposed article is to build a mathematical model of heat transmission through the outer layer of multi-layer wall structure with accounting for the randomness of the external temperature and to obtain the estimates of the value of thermal resistance, which guarantees that there is no excess of lost heat over some critical level. Firstly, we construct an algorithm calculating the distribution of the temperature by the thickness of a multi-layered wall structure under assumption that the external temperature is not random. Then an approach taking into account the randomness of this temperature is proposed. It makes possible to estimate the amount of heat released in the external environment from a unit of surface during a unit of time. It is shown that the changes in the external temperature has a significant impact on the amount of the heat lost. The method calculating the thermal resistance of the structure is proposed, thus the heat loss does not exceed some critical level during the heating season with a given probability. The results of the article can be used to carry out the ongoing work on updating the regulations in the field of building heat protection.

1. Introduction
The calculation of a heat loss is the most important stage of the design of a building and its heating system. One of the main tasks in determining the heat loss is to estimate the thermal conductivity factor of the wall’s structures. These estimates are based on the studies on the thermal conductivity of the materials used and their dependencies on time, analysis of their thermal heterogeneities, as well as a number of other properties that require solving rather complex heat physical problems. There is extensive literature on the subject. Here we will note the fundamental works of Gagarin V.G., Pastushkov P.P., Kozlov V.V., K.P., Malyavina E.G., etc. [1-7]). In particular the equation, determining the dependence of the thermal conductivity on the density of the material, was obtained in [1]. It allowed to estimate the factor of the technical quality for the autoclave concrete of modern production. The analysis of the dependency of thermal conductivity on time for polymer insulation materials was carried out in [5], the work of the same authors. These and other works [8-15], show significant progress in the analysis of the insulation properties of materials and have both practical and theoretical significance as well as provide the basis for building more accurate estimates of the required characteristics.

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external temperature and to obtain the estimates of the value of thermal resistance, which guarantees that there is no excess of lost heat over some given critical level.

The actuality of the study is based on the fact that, as far as we know, the mentioned random effect was not covered in the literature as well a significant impact of this factor on the heat loss. The results obtained in the article allow to formulate the conditions for thermal resistance, and therefore on the properties of the insulation layer, for example, on its thickness (at the specified thermal conductivity ratios), so that under the conditions there is a high probability that there will be no excess of some critical value of the amount of heat lost for a long time. The suggested methodology can be useful in updating the regulatory documents in the field of building heat protection.

2. Materials and methods

In this study, it is assumed that the wall structure is a paralelipid with linear dimensions $H_1$, $H_2$, $H_3$, where $H_1$ - thickness. The construction consists of $r$ layers and $δ_i$ - thickness of the $i$-th layer, so that $H_1 = \sum_{i=1}^{r} δ_i = l$. If $μ_1(t)$ and $μ_2(t)$ are the temperatures at the time $t$ of respectively inside and out of the room and if $μ_2(t) < μ_1(t)$, the structure will transfer the heat outside.

To study the process, we will introduce the axis of coordinates so that the OX axis corresponds to the thickness of the structure, and let $u(x, y, z, t)$ be the temperature at the point $(x, y, z)$ at moment $t$, where $0 ≤ x ≤ H_1, 0 ≤ y ≤ H_2, 0 ≤ z ≤ H_3$

![Figure 1. Location of coordinate axles relative to three-layer construction.](image)

It is well known (see for example [16-19]) that if there is no heat exchange through the edges of the structure, the function satisfies the thermal conductivity equation $u(t, x, y, z)$

$$\frac{∂u}{∂t} = a^2 \left( \frac{∂^2 u}{∂x^2} + \frac{∂^2 u}{∂y^2} + \frac{∂^2 u}{∂z^2} \right) + f(t, x, y, z).$$  \hspace{1cm} (1)

Here $a^2$ is the coefficient of the thermal conductivity. It is defined as $a^2 = \frac{λ}{cρ}$, where $λ$ is a thermal conductivity factor, $c$ - specific heat capacity, $ρ$ - density of the material.

$f(t, x, y, z) = \frac{F(t, x, y, z)}{cρ}$, and $F$ - the density of thermal sources at point $(x, y, z)$.

If there are none then it is assumed that $f = 0$.

We consider the case when the structure is uniform in the directions of the planes parallel to ZOY, and nonuniformity arises along the axis OX. So the three dimensional problem becomes one dimensional. The function $u(x, t)$ is the temperature in the point $(x, y, z)$ at moment $t$ in absence of the sources of heat satisfies the equation

$$\frac{∂}{∂x} = \left( \frac{λ}{cρ} \frac{∂u}{∂x} \right) cρ \frac{∂u}{∂t},$$  \hspace{1cm} (2)

where the thermal conductivity factor $λ$, as well as $c$ and $ρ$ may depend on $x$. 


For the enclosing structures under the consideration it is assumed that each of the layers is homogeneous. Let, \( \delta_i, \lambda_i, c_i, \rho_i, i=1, \ldots, r \) - correspondingly the width, thermal conductivity factor, specific heat capacity and the density of the \( i \)-th layer. Inside each layer the function \( u(x, t) \) satisfies thermal conductivity equation. I.e. if \( u_i(x, t) \) is the temperature in the \( i \)-th layer, then

\[
\alpha_i^2 \frac{\partial^2 u_i}{\partial x^2} = \frac{\partial u_i}{\partial t},
\]

where, \( \alpha_i^2 = \frac{\lambda_i}{c_i \rho_i} \), \( i=1,2,\ldots, r \).

The peculiarity of the multilayered structures is that the layers are in close contact with each other, thus are forming a solid environment. It means the continuity of the distribution function of the temperatures \( u(x, t) \) and the continuity of the heat flow at points of \( \{x_i, i=1, \ldots, r-1 \} \) where the layers touch each other. Usually these continuity conditions are called pairing conditions and have the following form

\[
\mu_i(x_i - 0, t) = \mu_{i+1}(x_i + 0, t) - r(x_i), \quad i=1,2,\ldots, r-1.
\]

Finally, we assume that the temperature regime is set on internal and outer surfaces of the structure, which is defined by the initial conditions

\[
u_i(0, t) = \mu_1(t), \ldots, u_r(0, t) = \mu_2(t) \text{ and } u(x, 0) = \varphi(x)
\]

Under the assumptions made, there is a unique solution to the equation (3) and it can be written for each layer \( 1,2,\ldots, r \) (see, for example [7]), so that one can get \( u(t, x) \) in exact form. However, the formulae are quite complex and our main interest is the limit distribution of the temperatures for \( t \to \infty \), i.e. the function \( u(x) = \lim_{t \to \infty} u(x, t) \).

We'll limit ourselves to three-layer wall structures, since the analysis of the general case does not lead to principal difficulties, although the formulae are rather cumbersome.

It is assumed that the temperatures inside \( \mu_1(t) \) and outside of the building \( \mu_2(t) \) are constant, i.e., \( \mu_1(t) = A, \mu_2(t) = B \) and the problem is solved for

\[
\alpha_i^2 \frac{\partial^2 u_i}{\partial x^2} = \frac{\partial u_i}{\partial t}, \quad x \in (x_{i-1}, x_i), i=1,2,3, x_0 = 0
\]

with conditions

\[
u_1(0, t) = A, u_3(l, t) = B, u_i(x, 0) = u_0
\]

and pairing conditions (4).

In each of the layers, the thermal conductivity factors \( a_i, (i=1,2,3) \) are constant, hence the equation (5) has a stationary temperature distribution regime, which is achieved over some time. Since this limit does not depend on time, its derivative on \( t \) in the equation (5) is zero, and the initial distribution of the temperatures (the last of the conditions (6)) does not play a part. For the stationary regime (5) the following form \( \alpha_i^2 \frac{\partial^2 u_i}{\partial x^2} = 0, \quad x \in (x_{i-1}, x_i) \)

Since the second derivatives of the functions \( u_i(x) \) are equal to zero they are linear. The first condition of (4) means continuity in the points \( x_i \) \( (i=1,2) \). So the stationary solution is a piecewise linear function with values \( C \) and \( D \) at the points of \( x_i \), \( i=1 \left( u_1(x_i) = C, u_2(x_2) = D \right) \), where \( x_1 = \delta_1, x_2 = \delta_1 + \delta_2 \)
Temperatures $C$ and $D$ can be found from the second pairing condition. Since $\frac{\partial u_i}{\partial x}$ equals to the tangent of an appropriate region of the piecewise curve, the conditions have the following form

$$\frac{A-C}{\delta_1} \lambda_1 = \frac{C-D}{\delta_2} \lambda_2 = \frac{D-B}{\delta_3} \lambda_3.$$  \hspace{1cm} (7)

The ratio is the thermal resistance $\frac{\delta_i}{\lambda_i} = R_i$ of the $i$-th layer. Hence in these terms the equations (7) has the following form

$$AR_2 - CR_2 = CR_1 - DR_1$$
$$DR_2 - BR_2 = CR_3 - DR_3.$$

The following solution are derived from the above system of the equation

$$C = A \left(1 - \frac{R_1}{R}\right) + B \frac{R_1}{R}, \quad D = A \frac{R_2}{R} + B \left(1 - \frac{R_3}{R}\right).$$ \hspace{1cm} (8)

where $R = R_1 + R_2 + R_3$ - thermal resistance of a three-layer structure.

The amount of heat that goes out in the external environment from a unit of surface during the unit of time is

$$Q = \frac{\lambda_3}{\delta_3} (D - B) = R^{-1} (A - B).$$ \hspace{1cm} (9)

This is the well-known ratio underlying many thermal calculations (see, for example, [7]).

3. Results

The amount of heat emitted by a unit of a surface during a unit of time is set by formula (9), i.e. it is directly proportional to the differences in internal and external temperatures and inversely proportional to the thermal resistance. It is exactly the formula which is usually used in determining the limits for the thermal resistance factor.

By asking the maximum possible amount of heat emitted, you can get a condition on the thermal resistance factor in the form of the inequality

$$R > \frac{A-B}{q_0}.$$ \hspace{1cm} (10)

The internal temperature $A$ in this formula can be considered static ($A=+20$ or $18$ degrees), and for $B$ they often use the average minimum temperature calculated over a long period of meteorological observations in the region.

Suppose that the minimum temperature $B$ for the winter is random, having, for example, a normal distribution with average $-m$ and variance $\sigma^2$. If the maximum allowed heat loss is $q_0$ then minimal is
\[ R_0 = \frac{A+m}{q_0} \]. The probability that the amount of heat lost \( Q \) will exceed the acceptable level \( q_0 = \frac{A+m}{R_0} \)

is determined by the dependency

\[ P(Q > q_0) = P\left(\frac{A-B}{R_0} > q_0\right) = P(-B > R_0q_0 - A) = P(-B > m) = \frac{1}{2}. \]

Thus, if the boundary of the thermal resistance of the structure is calculated on a basis of the average minimum temperature, then with the probability \( \frac{1}{2} \) the actual heat loss will exceed the allowable value \( q_0 \). Accounting for the randomness of the minimal temperature \( B \) can be carried out in the following way. One establishes some level \( 1 - \varepsilon > 0 \) and requires estimating the value of the thermal resistance \( \bar{R} \) so that with the probability not less than \( 1 - \varepsilon \) the amount of the heat loss will be lower the critical level \( q_0 \) during the entire heating season.

I.e., the critical value \( \bar{R} \) is found from the inequality

\[ P\left(\frac{A-B}{\bar{R}} > q_0\right) < \varepsilon. \] (11)

Since \( B \) has a normal distribution, the last inequality is equivalent to the following one

\[ P\left(\frac{A-B}{\bar{R}} > q_0\right) = \Phi\left(\frac{A-m-\bar{R}q_0}{\sigma}\right), \] where \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy. \)

Let \( \theta_\varepsilon \) be a solution to the equation \( \Phi(-\theta_\varepsilon) = \varepsilon \) which can be found from the tables of Laplace function (see, for example [20]). Then the condition (11) leads to a ratio of

\[ \frac{A+m+\sigma\theta_\varepsilon}{q_0}. \]

4. Discussion

Suppose that the thermal resistance is defined by the equation \( R_0q_0 = A + m \). Then \( \bar{R} = R_0 \left( 1 + \frac{\theta_\varepsilon\sigma}{A+m} \right) \). For practical calculations, one can assume that the amount of heat loss does not exceed the allowable level of 0.980. If, for example, statistical observations give the estimates \( m=10 \) and \( \sigma \leq 3 \) then for \( A=20 \) one has \( \bar{R} \leq 1.2R_0 \). Hence, the increase of the thermal resistance by 20% leads to the situation when the quantity of heat loss never exceeds the critical level during the heating season, while for the thermal resistance of \( R_0 \) the heat loss exceeds the critical level in half of the cases.

One can easy calculate by how much the thickness of the insulation layer should be increased, to provide the necessary level of the thermal resistance \( \bar{R} = (1 + \alpha)R_0 \) by using the following formula

\[ R_0 = \frac{\lambda_1 + \delta_1 + \delta_2}{\lambda_1 + \lambda_3}. \] For \( \lambda_1 = \lambda_3 \), the formula has a form \( \bar{\delta}^2 = (1 + \alpha)\delta_2 + \frac{q_0}{\lambda_1}(\delta_1 + \delta_3) \)

If, for example, \( \lambda_1 = \lambda_3 = 0.44 \text{ BTU/m°C}, \delta_1 = 0.10 \text{ m}, \delta_3 = 0.005 \text{ m}, \lambda_2 = 0.10 \text{ BTU/m°C} \), then \( \bar{\delta}^2 = (1 + \alpha)\delta_2 + \frac{0.15}{0.44}\alpha. \)

5. Conclusion

The formula describing the distribution of the temperature across the three layers construction in a stationary regime is derived in the article.

The quantity of the heat \( Q \) released from a unit of the surface of a three-layer structure during a unit of time is established. For a random external temperature the probability of \( Q \) exceeding some critical level is estimated.

The randomness of the external temperature has a significant impact on the amount of the heat released in the environment. If the thermal resistance of the structure is established on a basis of the average external temperature without allowance for its random nature, then the heat loss exceeding the critical value is very likely. An increase in thermal resistance by 15-20% virtually eliminates the possibility of exceeding the critical level. The results of the article can be used to carry out the ongoing work on updating the regulations in the field of building heat protection.
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