On the multiparametric $U_q[D_{n+1}^{(2)}]$ vertex model

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Abstract. In this paper we consider families of multiparametric $R$-matrices to make a systematic study of the boundary Yang–Baxter equations in order to discuss the corresponding families of multiparametric $K$-matrices. Our results are indeed a non-trivial generalization of the $K$-matrix solutions of the $U_q[D_{n+1}^{(2)}]$ vertex model when distinct reflections and extra free parameters are admissible.

Keywords: integrable spin chains (vertex models), solvable lattice models, symmetries of integrable models

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1. Introduction

The Yang–Baxter equation is an operator relation for a matrix $R_{ij}(x)$ defined on the tensor product of two $N$-dimensional vector spaces $V_i$ and $V_j$, which reads

$$R_{12}(x_1)R_{13}(x_1x_2)R_{23}(x_2) = R_{23}(x_2)R_{13}(x_1x_2)R_{12}(x_1),$$

(1)

where $x_a$ are arbitrary multiplicative spectral parameters.

Search for solutions of the Yang–Baxter equation is an important subject in the studies of exactly solvable models. Actually, a $R$-matrix satisfying the Yang–Baxter equation generates the Boltzmann weights of a vertex model [1] or the factorizable scattering amplitudes between particles in relativistic field theories [2].

A classification of the solutions of the Yang–Baxter equation for $R$-matrices (with an extra $q$-parameter) was performed by $q$-deformation in a given non-exceptional Lie algebra $G$ [3]–[5]. This quantum group approach permits us to reduce the problem (1) to a linear one, in order to associate a fundamental trigonometric $R$-matrix with each Lie algebra [6, 7] or Lie superalgebra [8]–[10].

The physical understanding of vertex models includes necessarily the exact diagonalization of their transfer matrices, which can provide us with information about the free energy behavior and on the nature of the elementary excitations. This step has been successfully achieved for standard Lie algebras either by the analytical Bethe ansatz [11], a phenomenological technique yielding us solely the transfer matrix eigenvalues, or through the quantum inverse scattering method [12]–[15], an algebraic technique which also gives us the eigenvectors. Usually these systems are studied with periodic boundary conditions, but more general boundaries can also be included in this framework as well. Physical
properties associated with the bulk of the system are not expected to be influenced by boundary conditions in the thermodynamical limit. Nevertheless, there are surface properties such as the interfacial tension where the boundary conditions are of relevance. Moreover, the conformal spectra of lattice models at criticality can be modified by the effect of boundaries [16].

Integrable systems with open boundary conditions can also be accommodated within the framework of the quantum inverse scattering method [17]. In addition to the solution of the Yang–Baxter equation governing the dynamics of the bulk there is another fundamental ingredient, the reflection matrices [18]. These matrices, also referred as $K$-matrices, represent the interactions at the boundaries, and compatibility with the bulk integrability requires these matrices to satisfy the so-called reflection equations [17, 18].

The original formalism of [17] was extended to more general systems in [19], where it is assumed that, for a regular $R$-matrix satisfying the properties of PT-symmetry, unitarity and crossing unitarity, one can derive an integrable open chain Hamiltonian with the right boundaries determined by the solutions of the reflection equation,

$$
R_{21}(x/y)K_2^-(x)R_{12}(xy)K_1^-(y) = K_1^-(y)R_{21}(xy)K_2^-(x)R_{12}(x/y). \tag{2}
$$

A similar equation should also hold for the $K^+$-matrix at the opposite boundary. When these properties are fulfilled one can follow the scheme devised in [19] and the matrix $K^-(x)$ is obtained by solving equation (2) while the matrix $K^+(x)$ can be obtained from the isomorphism

$$
K^-(x) \mapsto K^+(x)^t = K^-(x^{-1} \eta^{-1})V^tV. \tag{3}
$$

Here $t$ stands for the transposition and $\eta$ is a crossing parameter and $V$ a crossing matrix, both being specific to each model.

At the moment, the study of general regular solutions of the reflection equations for vertex models based on $q$-deformed Lie algebras [6, 7] has been successfully accomplished. See for instance [20] and references therein.

In [21], families of $R$-matrices not previously foreseen by the framework of quantum groups were obtained by the baxterization of the representations of the Birman–Wenzl–Murakami algebra. In this paper we are taking into account these results in order to consider the systems which were presented as rather non-trivial extensions of Jimbo’s $U_q[D^{(2)}_{n+1}]$ $R$-matrix. The main interest of this work is to present families of multiparametric $K$-matrices associated with the multiparametric $U_q[D^{(2)}_{n+1}]$ vertex model, by solving its functional reflection equations. $K$-matrices for Jimbo’s $U_q[D^{(2)}_{n+1}]$ vertex model have already been derived in [23]. Here we re-derive these results and also present new solutions for Jimbo’s $U_q[D^{(2)}_{n+1}]$ vertex model as well.

This paper has been organized as follows. In section 2 we present the $R$-matrices, which turn out to be highly non-trivial extensions of the $U_q[D^{(2)}_{n+1}]$ vertex model as presented in [21]. In section 3 we present with some details the corresponding matrix elements of the $K$-matrix solutions. Our conclusions are summarized in section 4. In appendix A we present the solutions with a block diagonal structure compatible with the $n + 1$ distinct $U(1)$ conserved charges of the $U_q[D^{(2)}_{n+1}]$ vertex model [22]. In appendix B we present a more general solution for the $U_q[D^{(2)}_{2}]$ vertex model. In appendix C we present some special solutions.

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2. The multiparametric $U_q[D^{(2)}_{n+1}]$ vertex model

In what follows we will consider the multiparametric $U_q[D^{(2)}_{n+1}]$ $R$-matrix as derived in [21] from the representations of a quotient of the braid-monoid algebra denominated Birman–Wenzl–Murakami algebra [24, 25]. In this situation, the link between braid algebra and the Yang–Baxter equation is made with the help of the baxterization procedure and the permutator,

$$P = \sum_{i,j=1}^{N} e_{i,j} \otimes e_{j,i},$$

where $N = 2n + 2$ and $e_{i,j}$ denotes the standard $N \times N$ Weyl matrices.

In fact, by defining a new matrix $\hat{R}_{ab}(x) = P_{ab}R_{ab}(x)$ one can rewrite the Yang–Baxter equation in the form

$$\hat{R}_{12}(x_1)\hat{R}_{23}(x_1x_2)\hat{R}_{12}(x_2) = \hat{R}_{23}(x_2)\hat{R}_{12}(x_1x_2)\hat{R}_{23}(x_1),$$

and its solution can be rewritten as follows:

$$\hat{R}(x) = \sum_{i,j\neq n+1,n+2} (x^2 - \zeta^2) \left( x^2 - q^2 \right) e_{i,i} \otimes e_{i,i} + q(x^2 - 1)(x^2 - \zeta^2) \sum_{i\neq j, j' \neq n+1, n+2} e_{i,j} \otimes e_{i,j'}$$

$$+ \frac{1}{2} q(x^2 - 1)(x^2 - \zeta^2) \sum_{i,j \neq n+1,n+2} [(1 + \kappa) (e_{i,i} \otimes e_{i,j} + e_{i,j} \otimes e_{i,i})$$

$$+ (1 - \kappa) (e_{i,i} \otimes e_{i,j} + e_{i,j} \otimes e_{i,i})]$$

$$+ \sum_{i,j \neq n+1,n+2} g_{i,j}(x) e_{i,j} \otimes e_{i,j} - (q^2 - 1)(x^2 - \zeta^2)$$

$$\times \left[ \sum_{i,j \neq n+1,n+2} + x^2 \sum_{i,j \neq n+1,n+2} \right] e_{i,j} \otimes e_{i,i} - \frac{1}{2} (q^2 - 1)(x^2 - \zeta^2)$$

$$\times \left[ (x + 1) \left( \sum_{i<n+1,j=n+1,n+2} + x \sum_{j>n+2} \right) (e_{j,j} \otimes e_{i,i} + e_{i',j'} \otimes e_{j',j})$$

$$+ (x - 1) \left( - \sum_{j>n+2} + x \sum_{j>n+2} \right) (e_{j',j} \otimes e_{i,i} + e_{i',i'} \otimes e_{j',j'})$$

$$+ \frac{1}{2} \sum_{i,j \neq n+1,n+2} \left[ b_{ij}^+(x) (e_{i',j} \otimes e_{i,j'} + e_{i',j} \otimes e_{j',j})$$

$$+ b_{ij}^-(x) (e_{i',j} \otimes e_{i,j'} + e_{i,j} \otimes e_{j',j})]$$

$$+ \sum_{i=n+1,n+2} \left[ c_{i,i'}^+(x) e_{i',i} \otimes e_{i,i'} + c_{i,i'}^- (x) e_{i,i} \otimes e_{i,i} + d_{i,i}^+(x) e_{i',i} \otimes e_{i,i}$$

$$+ d_{i,i}^-(x) e_{i',i} \otimes e_{i,i} \right],$$

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where \( i' = N + 1 - i \) and \( \zeta = q^\nu \). The respective Boltzmann weights \( g_{i,j}(x) \), \( b_i^\pm(x) \), \( c_i^\pm(x) \) and \( d_i^\pm(x) \) are given by

\[
g_{i,j}(x) = \begin{cases} (x^2 - 1)(x^2 - \zeta^2) + x^2(q^2 - 1) & i = j, \\ (q^2 - 1)(\zeta^2(x^2 - 1)q^{i' - t_i} - \delta_{i,j'}(x^2 - \zeta^2)) & i < j, \\ (q^2 - 1)x^2 [(x^2 - 1)q^{i' - t_i} - \delta_{i,j'}(x^2 - \zeta^2)] & i > j, \end{cases}
\]

\[
b_i^\pm(x) = \begin{cases} \pm q^i(q^2 - 1)(x^2 - 1)(x \pm \zeta) & i < n + 1, \\ q^i(q^2 - 1)(x^2 - 1)x(x \pm \zeta) & i > n + 2, \end{cases}
\]

\[
c_i^\pm(x) = \pm \frac{1}{2}(q^2 - 1)(\zeta + 1)x(x \mp 1)(x \pm \zeta) + \frac{1}{2}(1 + \nu\kappa)q(x^2 - 1)(x^2 - \zeta^2),
\]

\[
d_i^\pm(x) = \pm \frac{1}{2}(q^2 - 1)(\zeta - 1)x(x \pm 1)(x \pm \zeta) + \frac{1}{2}(1 - \nu\kappa)q(x^2 - 1)(x^2 - \zeta^2),
\]

where \( \kappa = \pm 1 \) and the lower index \( \nu = \pm 1 \) in the weights \( c_i^\pm(x) \) and \( d_i^\pm(x) \) indicates the two possible families of models. The explicit expressions for the variables \( t_i \) and \( \tilde{t}_i \) are

\[
t_i = \begin{cases} i + 1 & i < n + 1, \\ n + \frac{3}{2} & i = n + 1, n + 2, \\ i - 1 & i > n + 2, \end{cases}
\]

\[
\tilde{t}_i = \begin{cases} i + \frac{1}{2} & i < n + 1, \\ i - n - \frac{5}{2} & i > n + 2. \end{cases}
\]

As noted in [21] it is not difficult to recognize that expressions (6)–(12) for the branch \( \kappa = 1 \) and \( \nu = 1 \) indeed reproduce the usual \( U_q[D_n^{(2)}] \) R-matrix [7]. This means that in general the R-matrix should be considered as a non-trivial generalization of Jimbo’s \( U_q[D_n^{(2)}] \) vertex model.

Now, we would like to present some useful properties satisfied by the R-matrix \( R(x) = P\tilde{R}(x) \), where \( \tilde{R}(x) \) refers to the matrix given in equation (6). Besides regularity and unitarity, this R-matrix satisfies the so-called \( PT \) symmetry given by

\[
R_{21}(x) = P_{12}R_{12}(x)P_{12} = [R_{12}]^{t_1 t_2}(x),
\]

where the symbol \( t_k \) denotes the transposition in the space with index \( k \). Yet another property is the crossing symmetry, namely

\[
R_{12}(x) = \frac{\rho(x)}{\rho(\zeta/x)}V_1[R_{12}]^{t_2}(\zeta/x)V_1^{-1},
\]

where \( \rho(x) \) is a convenient normalization

\[
\rho(x) = q(x^2 - 1)(x^2 - \zeta^2),
\]

while the only non-null entries of the normalized matrix \( V \) are the minor diagonal elements \( V_{i,i'} \), namely

\[
V_{i,i'} = \begin{cases} q^{i-1} & i < n + 1, \\ q^{n-1/2} & i = n + 1, n + 2, \\ q^{i-3} & i > n + 2. \end{cases}
\]

\[\text{doi:10.1088/1742-5468/2013/02/P02011}\]
3. The multiparametric $U_q[D^{(2)}_{n+1}]$ $K$-matrix

The purpose of this work is to investigate the general families of regular solutions of the reflection equation (2). Regular solutions mean that the $K$-matrices have the general form

$$K^-(x) = \sum_{i,j=1}^{N} k_{i,j}(x) e_{i,j},$$

such that the condition $k_{i,j}(1) = \delta_{i,j}$ holds for all matrix elements.

The direct substitution of the $K$ and the $R$ matrices in the reflection equation leaves us with a system of $N^4$ functional equations for the entries $k_{i,j}(x)$. In order to solve these equations we shall make use of the derivative method. Thus, by differentiating equation (2) with respect to $y$ and setting $y = 1$, we obtain a set of algebraic equations for the matrix elements $k_{i,j}$ involving the single variable $x$ and $N^2$ parameters

$$\beta_{i,j} = \frac{d k_{i,j}(y)}{d y} \bigg|_{y=1} i, j = 1, 2, \ldots, N.$$ (18)

Although we obtain a large number of equations only a few of them are actually independent, and a direct inspection of these equations, on the lines described in [23], allows us to find the branches of regular solutions. In what follows we shall present our findings for the regular solutions of the reflection equation associated with the multiparametric $U_q[D^{(2)}_{n+1}]$ vertex model. The special cases are presented in the appendices.

3.1. Non-diagonal matrix elements

All families of solutions have a common structure for the non-diagonal matrix elements $k_{i,j}(x)$ with $i \neq j$ and different $k_{n+1,n+2}(x)$ and $k_{n+2,n+1}(x)$. The minor diagonal elements are given by

$$k_{i,i'}(x) = q^{t_{i-2n}}\Gamma(n)^2 \left( \frac{\beta_{i,i'}}{\beta_{1,N}} \right)^2 k_{1,N}(x), \quad i \neq 1, n + 1, n + 2, N,$$ (19)

$$k_{N,1}(x) = q^{2n-3} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right)^2 k_{1,N}(x).$$ (20)

The elements of the first row are given by

$$k_{1,j}(x) = \left( \frac{\beta_{1,j}}{\beta_{1,N}} \right) G(x) k_{1,N}(x), \quad j \neq n + 1, n + 2,$$ (21)

$$k_{1,n+1}(x) = \left( \frac{\beta_{1,N} + x \beta_{1,j}}{\beta_{1,N}} \right) G(x) k_{1,N}(x),$$ (22)

$$k_{1,n+2}(x) = \left( \frac{\beta_{1,N} - x \beta_{1,j}}{\beta_{1,N}} \right) G(x) k_{1,N}(x).$$ (23)

The elements of the first column are given by

$$k_{i,1}(x) = q^{t_{i-3}} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{\beta_{1,i'}}{\beta_{1,N}} \right) G(x) k_{1,N}(x), \quad i \neq n + 1, n + 2,$$ (24)

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Moreover, the remaining non-diagonal matrix elements above the minor diagonal are given by

\begin{align}
  k_{n+1,1}(x) &= q^{n+1-3} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{\beta_+ - \kappa \epsilon x \beta_-}{\beta_{1,N}} \right) G(x)k_{1,1}(x), \\
  k_{n+2,1}(x) &= q^{n+2-3} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{\beta_+ + \kappa \epsilon x \beta_-}{\beta_{1,N}} \right) G(x)k_{1,1}(x).
\end{align}

The elements of the last row are given by

\begin{align}
  k_{N,j}(x) &= \epsilon q^{n-2} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{\beta_1}{\beta_{1,N}} \right) x^2 G(x)k_{1,1}(x), \quad j \neq n+1, n+2, \\
  k_{N,n+1}(x) &= \epsilon q^{n-2} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{x \beta_+ + \epsilon q^n \beta_-}{\beta_{1,N}} \right) x G(x)k_{1,1}(x), \\
  k_{N,n+2}(x) &= \epsilon q^{n-2} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{x \beta_+ - \epsilon q^n \beta_-}{\beta_{1,N}} \right) x G(x)k_{1,1}(x).
\end{align}

The elements of the last column are

\begin{align}
  k_{i,N}(x) &= \epsilon q^{i-n-2} \left( \frac{\beta_{1,i}}{\beta_{1,N}} \right) x^2 G(x)k_{1,1}(x), \quad i \neq n+1, n+2, \\
  k_{n+1,N}(x) &= \epsilon q^{n+1-n-2} \left( \frac{x \beta_+ - \kappa q^n \beta_-}{\beta_{1,N}} \right) x G(x)k_{1,1}(x), \\
  k_{n+2,N}(x) &= \epsilon q^{n+2-n-2} \left( \frac{x \beta_+ + \kappa q^n \beta_-}{\beta_{1,N}} \right) x G(x)k_{1,1}(x).
\end{align}

Moreover, the remaining non-diagonal matrix elements above the minor diagonal are given by

\begin{align}
  k_{i,j}(x) &= q^{i-n-1} \Gamma(n) \left( \frac{\beta_{1,i'}}{\beta_{1,N}} \right) \left( \frac{\beta_{1,j}}{\beta_{1,N}} \right) G(x)k_{1,1}(x), \quad i' > j, \\
  k_{n+1,j}(x) &= q^{n+1-n-1} \Gamma(n) \left( \frac{\beta_{1,j}}{\beta_{1,N}} \right) \left( \frac{\beta_+ - \kappa \epsilon x \beta_-}{\beta_{1,N}} \right) G(x)k_{1,1}(x), \\
  k_{n+2,j}(x) &= q^{n+2-n-1} \Gamma(n) \left( \frac{\beta_{1,j}}{\beta_{1,N}} \right) \left( \frac{\beta_+ + \kappa \epsilon x \beta_-}{\beta_{1,N}} \right) G(x)k_{1,1}(x), \\
  k_{i,n+1}(x) &= q^{i-n-1} \Gamma(n) \left( \frac{\beta_{1,i'}}{\beta_{1,N}} \right) \left( \frac{\beta_+ + x \beta_-}{\beta_{1,N}} \right) G(x)k_{1,1}(x), \\
  k_{i,n+2}(x) &= q^{i-n-1} \Gamma(n) \left( \frac{\beta_{1,i'}}{\beta_{1,N}} \right) \left( \frac{\beta_+ - x \beta_-}{\beta_{1,N}} \right) G(x)k_{1,1}(x),
\end{align}

while the remaining non-diagonal matrix elements below the minor diagonal are given by

\begin{align}
  k_{i,j}(x) &= \epsilon q^{i-2n-1} \Gamma(n) \left( \frac{\beta_{1,i'}}{\beta_{1,N}} \right) \left( \frac{\beta_{1,j}}{\beta_{1,N}} \right) x^2 G(x)k_{1,1}(x), \quad i' < j, \\
  k_{n+1,j}(x) &= \epsilon q^{n+1-2n-1} \Gamma(n) \left( \frac{\beta_{1,j}}{\beta_{1,N}} \right) \left( \frac{x \beta_+ - \kappa q^n \beta_-}{\beta_{1,N}} \right) x G(x)k_{1,1}(x), \\
  k_{n+2,j}(x) &= \epsilon q^{n+2-2n-1} \Gamma(n) \left( \frac{\beta_{1,j}}{\beta_{1,N}} \right) \left( \frac{x \beta_+ + \kappa q^n \beta_-}{\beta_{1,N}} \right) x G(x)k_{1,1}(x).
\end{align}
where we have identified
\[ G(x) = \frac{q^{n-1} + \epsilon}{q^{n-1} + \epsilon x^2}, \quad \Gamma(n) = \frac{q^{n-1} + \epsilon}{q^n}, \]
and
\[ \beta_{\pm} = \frac{1}{2} (\beta_{1,n+1} \pm \beta_{1,n+2}). \]

Here we observe in equations (19)–(43) the \( \kappa \) dependence inherited from the \( R \)-matrix and the new parameter \( \epsilon = \pm 1 \), besides the \( \beta_{i,j} \) parameters.

### 3.2. The block diagonal matrix elements

Here we also identify a common structure for the diagonal matrix elements \( k_{i,i}(x) (i \neq n+1, n+2) \), namely
\[ k_{i,i}(x) = k_{i-1,i-1}(x) + \left( \frac{\beta_{i,i} - \beta_{i-1,i-1}}{\beta_{1,N}} \right) G(x)k_{1,N}(x), \quad i = 2, \ldots, n. \] (45)
\[ k_{j,j}(x) = k_{j-1,j-1}(x) + \left( \frac{\beta_{j,j} - \beta_{j-1,j-1}}{\beta_{1,N}} \right) G(x)x^2k_{1,N}(x), \quad j = n+4, \ldots, N. \] (46)

The central elements \( k_{i,j}(x) \) \((i, j = n + 1, n + 2)\) satisfy a relation slightly different from equation (46):
\[ k_{n+2,n+2}(x) = k_{n+1,n+1}(x) + \left( \frac{\beta_{n+2,n+2} - \beta_{n+1,n+1}}{\beta_{1,N}} \right) xG(x)k_{1,N}(x), \] (47)
and
\[ k_{n+2,n+1}(x) = k_{n+1,n+2}(x) + \left( \frac{\beta_{n+2,n+1} - \beta_{n+1,n+2}}{\beta_{1,N}} \right) xG(x)k_{1,N}(x). \] (48)

At this point we still have to find the remaining matrix elements \( k_{1,1}(x) \), \( k_{n+1,n+1}(x) \), \( k_{n+1,n+2}(x) \) and \( k_{n+3,n+3}(x) \) in terms of \( k_{1,N}(x) \). This issue proved to be a very difficult task, but we managed to solve it completely as follows.

From the non-diagonal elements presented above one can see that all corresponding parameters given by equation (18) are determined in terms of \( \beta_{1,j} \) \((j = 2, \ldots, N)\) and \( \beta_{2,1} \). Taking into account the block diagonal terms, we still have to solve several cumbersome algebraic equations with five unknown and \( 2N \) free parameters. By inspection of these equations we can immediately see that the parameters \( \beta_{1,j} \) for \( j = 2, \ldots, n - 1 \) and \( \beta_{2,1} \) are determined by
\[ \beta_{1,j} = \sigma(-1)^j \frac{\beta_{1,n} \beta_{1,n+3}}{\beta_{1,j'}}, \] (49)
\[ \beta_{2,1} = -\sigma q^{3-2n} \Gamma(n)^2 \frac{\beta_{1,n} \beta_{1,n+3} \beta_{1,n-1}}{\beta_{1,N}^2}, \] (50)
where an $n$-parity given by $\sigma = (-1)^n$ is explicit. After these computations we made the choice

$$k_{1,N}(x) = \frac{1}{2} \beta_{1,N} (x^2 - 1),$$  \hspace{1cm} (51)

in order to simplify our presentation. Here we note from the general solution that $k_{1,N}(x)$ is an arbitrary function satisfying the regularity condition $k_{1,N}(1) = 0$. Therefore, the choice equation (51) does not imply any restriction as compared to the general case.

From this choice follow appropriate expressions for $k_{1,1}(x)$ and $k_{n+3,n+3}(x)$:

$$k_{1,1}(x) = \frac{\sqrt{q} \Gamma(n) G(x)}{2(q + 1) q^n (x^2 + 1) \beta_{1,N}} \times \left\{ -\epsilon \sigma \sqrt{q} (x^2 - \epsilon \sigma) \left[ 2(q^n + \sigma) + \epsilon(q - 1)(x^2 + \epsilon \sigma) \right] \beta_{1,n} \beta_{1,n+3} 
- \kappa(q + 1)(q^n - \epsilon x^2) \left[ (q^n x^2 - \kappa) (\beta_+^2 + \beta_-^2) - (q^n x^2 + \kappa) (\beta_+^2 - \beta_-^2) \right] \right\}$$

and

$$k_{n+3,n+3}(x) = \frac{\sqrt{q} \Gamma(n) G(x) x^2}{2(q + 1) q^n (x^2 + 1) \beta_{1,N}} \times \left\{ \sqrt{q} (x^2 - \epsilon \sigma) \left[ 2(q^n + \sigma) + \epsilon(q - 1)(x^2 + \epsilon \sigma) \right] \beta_{1,n} \beta_{1,n+3} 
- \kappa(q + 1)(q^n - \epsilon x^2) \left[ (q^n - \kappa x^2) (\beta_+^2 + \beta_-^2) - (q^n + \kappa x^2) (\beta_+^2 - \beta_-^2) \right] \right\}.$$  \hspace{1cm} (52)

Now we have several reflection equations involving the diagonal parameters $\beta_{i,i}$ ($i \neq n + 1, n + 2$), which are solved by the recurrence relations

$$\beta_{i,i} = \beta_{i-1,i-1} + \sigma(-1)^i \Gamma(n) \left( \frac{q + 1}{q^{n+1-i}} \right) \left( \frac{\beta_{1,n} \beta_{1,n+3}}{\beta_{1,N}} \right), \hspace{1cm} i = 2, \ldots, n,$$  \hspace{1cm} (54)

and

$$\beta_{j,j} = \beta_{j-1,j-1} - \epsilon \sigma(-1)^j \Gamma(n) \left( \frac{q + 1}{q^{N+1-j}} \right) \left( \frac{\beta_{1,n} \beta_{1,n+3}}{\beta_{1,N}} \right), \hspace{1cm} j = n + 4, \ldots, N,$$  \hspace{1cm} (55)

where $\beta_{1,1}$ and $\beta_{n+3,n+3}$ are determined by equation (18).

Finally, the central terms are determined by

$$k_{n+1,n+1}(x) = \frac{\sqrt{q} \Gamma(n) G(x)}{4 q^n (q + 1) \beta_{1,N}} \left\{ (1 - \epsilon \sigma) \sqrt{q} (q^n + \sigma x^2)(x^2 + 1) \beta_{1,n} \beta_{1,n+3} 
- \kappa(q^n - \epsilon)(q + 1)x^2 \left[ (q^n - \kappa) (\beta_+^2 + \beta_-^2) - (q^n + \kappa) (\beta_+^2 - \beta_-^2) \right] 
- 2 (\kappa \epsilon - 1) q^n (q + 1)x(x^2 - 1) \beta_+ \beta_- \right\};$$

and

$$k_{n+1,n+2}(x) = \frac{\sqrt{q} \Gamma(n) G(x)}{4 q^n (q + 1) \beta_{1,N}} \left( \frac{x^2 - 1}{x^2 + 1} \right) \left\{ (1 + \epsilon \sigma) \sqrt{q} (q^n - \sigma x^2)(x^2 - 1) \beta_{1,n} \beta_{1,n+3} 
+ \kappa(q^n + \epsilon)(q + 1)x^2 \left[ (q^n + \kappa) (\beta_+^2 + \beta_-^2) - (q^n - \kappa) (\beta_+^2 - \beta_-^2) \right] 
- 2 (\kappa \epsilon + 1) q^n (q + 1)x(x^2 + 1) \beta_+ \beta_- \right\}.$$  \hspace{1cm} (56)
from which we can get the central parameters

\[ \beta_{n+2,n+1} = \beta_{n+1,n+2} + 2 (\kappa \epsilon + 1) \sqrt{q} \Gamma(n) \left( \frac{\beta_+ \beta_-}{\beta_{1,N}} \right), \quad (58) \]

\[ \beta_{n+2,n+2} = \beta_{n+1,n+1} + 2 (\kappa \epsilon - 1) \sqrt{q} \Gamma(n) \left( \frac{\beta_+ \beta_-}{\beta_{1,N}} \right), \quad (59) \]

and then we can find the last parameter

\[
\beta_{1,N} = \frac{\sqrt{q} \left( q^n - \epsilon \right) \Gamma(n)}{4q^n(q+1)} \left\{ 2(1-\epsilon \sigma) \sqrt{q} \beta_{1,n} \beta_{1,n+3}
- \kappa(q+1) \left[ (q^n - \kappa) (\beta_+^2 + \beta_-^2) - (q^n + \kappa) (\beta_+^2 - \beta_-^2) \right] \right\}. \quad (60)
\]

At this point almost all equations are satisfied. For the remaining equations to be satisfied is necessary further to impose the constraint

\[ (\nu - 1)\beta_+ \beta_- = 0. \quad (61) \]

4. Concluding remarks

From the results presented above we can identify 12 families of solutions \( S(\nu, \kappa, \epsilon) \) of the multiparametric \( U_q[D^{(2)}_{n+1}] \) vertex model. For \( \nu = 1 \) we obtain four families of solutions (regarding the possibilities of \( \kappa = \pm 1 \) and \( \epsilon = \pm 1 \)) characterized by the \( n + 2 \) free parameters \( \beta_{1,n}, \ldots, \beta_{1,2n+1} \). For \( \nu = -1 \) the solutions are duplicated because of the constraint equation (61) and we obtain eight more families of solutions (regarding the possibilities of \( \kappa = \pm 1, \epsilon = \pm 1 \) and the two possibilities \( \beta_+ = 0 \) or \( \beta_- = 0 \)). In this case the solution presents only \( n + 1 \) free parameters, of course.

We highlight furthermore that the families \( S(1,1,\epsilon) \) correspond to the \( K \)-matrix solutions of Jimbo’s \( U_q[D^{(2)}_{n+1}] \) vertex model. The family \( S(1,1,1) \) was found by Lima-Santos in [23], while the family \( S(1,1,-1) \) represents a new \( K \)-matrix solution for this vertex model.

In [21] the baxterization of the representations of the Birman–Wenzl–Murakami algebra was also used to produce multiparametric solutions of the Yang–Baxter equation invariant by quantum superalgebras. From these representations one can in principle construct new \( K \)-matrices via our study. Thus, new open vertex models could be derived. It would be interesting to know the type of open lattice model with both bosonic and fermionic degrees of freedom that can be obtained. We hope to report on this problem in a future publication.

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Appendix A. The block diagonal solutions

As has already been mentioned in [22], the $U_q[D^{(2)}_{n+1}]$ vertex models have $n+1$ distinct $U(1)$ conserved charges, and the $K$-matrix ansatz compatible with these symmetries is a block diagonal structure.

Looking for the general solution of the corresponding reflection equation, we find that the only possible solution is obtained when the two recurrence relations equations (45) and (46) are degenerated into $k_{1,1}(x)$ and into $k_{N,N}(x)$, respectively. Thus, the block diagonal structure has the form

$$K(x) = \text{diag}(k_{1,1}(x), \ldots, k_{1,1}(x), B(x), k_{N,N}(x), \ldots, k_{N,N}(x)),$$

where $B$ contains the central elements,

$$B(x) = \begin{pmatrix} k_{n+1,n+1}(x) & k_{n+1,n+2}(x) \\ k_{n+2,n+1}(x) & k_{n+2,n+2}(x) \end{pmatrix}. \tag{A.2}$$

The solution can be obtained by the same procedure as described before, and in what follows we only quote the final results.

We have found two solutions for any value of $n$. The first solution is given by

$$k_{N,N}(x) = \left( q^n - \kappa \nu \right) (x^2 + 1) + \beta_{n+1,n+2}(q^n + \kappa \nu)(x^2 - 1) \frac{k_{1,1}(x)}{(q^n - \kappa \nu)(x^2 + 1) - \beta_{n+1,n+2}(q^n + \kappa \nu)(x^2 - 1)}, \tag{A.3}$$

with central elements

$$k_{n+1,n+2}(x) = k_{n+2,n+1}(x) = \frac{\beta_{n+1,n+2}(q^n + \kappa \nu)(x^2 - 1)(q^n - 1)k_{1,1}(x)}{(\kappa \nu x^2 + q^n)(q^n - \kappa \nu)(x^2 + 1) - \beta_{n+1,n+2}(q^n + \kappa \nu)(x^2 - 1)}, \tag{A.4}$$

$$k_{n+1,n+1}(x) = \frac{x(x^2 + 1) [(q^n - 1)x + (x^2 - 1)\Sigma(n)] k_{1,1}(x)}{(\kappa \nu x^2 + q^n)(q^n - \kappa \nu)(x^2 + 1) - \beta_{n+1,n+2}(q^n + \kappa \nu)(x^2 - 1)}, \tag{A.5}$$

$$k_{n+2,n+2}(x) = \frac{x(x^2 + 1) [(q^n - 1)x - (x^2 - 1)\Sigma(n)] k_{1,1}(x)}{(\kappa \nu x^2 + q^n)(q^n - \kappa \nu)(x^2 + 1) - \beta_{n+1,n+2}(q^n + \kappa \nu)(x^2 - 1)}, \tag{A.6}$$

where

$$\Sigma(n) = \epsilon \sqrt{\kappa \nu q^n \left[ (q^n + \kappa \nu)^2 b_{n+1,n+2} - (q^n - \kappa \nu)^2 \right]}, \tag{A.7}$$

The parameter $\epsilon = \pm 1$ indicates the existence of two conjugated solutions and $k_{1,1}(x)$ can be any function that satisfies the regularity condition.

Moreover, we notice that these solutions degenerate into four diagonal solutions when $\beta_{n+1,n+2} = 0$, namely

$$k_{n,n}(x) = \cdots = k_{2,2}(x) = k_{1,1}(x), \tag{A.8}$$

$$k_{n+1,n+1}(x) = \frac{(q^{2n} - 1)x + \epsilon \sqrt{-\kappa \nu q^n (q^n - \kappa \nu)^2 (x^2 - 1)}}{(q^n + \kappa \nu x^2)(q^n - \kappa \nu)} x k_{1,1}(x), \tag{A.9}$$

doi:10.1088/1742-5468/2013/02/P02011
On the multiparametric $U_q[D^{(2)}_{n+1}]$ vertex model

\[ k_{n+2,n+2}(x) = \frac{(q^{2n} - 1)x - \epsilon \sqrt{-\kappa \nu q^n (q^n - \kappa \nu)^2 (x^2 - 1)}}{(q^n + \kappa \nu x^2)(q^n - \kappa \nu)} x k_{1,1}(x), \quad (A.10) \]

\[ k_{N,N}(x) = \cdots = k_{n+3,n+3}(x) = x^2 k_{1,1}(x). \quad (A.11) \]

Here we note that the trivial diagonal solution ($K^-(x) = I$ and $K^+(x) = V^tV$) does not hold for this system [19].

For the second family of block diagonal solutions, in order to simplify our presentation, we made the choice

\[ k_{n+1,n+1}(x) = k_{n+2,n+2}(x) = \frac{1}{2} x(x^2 + 1). \quad (A.12) \]

With this choice, the remaining matrix elements different from zero are given by

\[ k_{1,1}(x) = \frac{1}{2} \frac{(q^n - \kappa \nu x^2)}{(q^n - \kappa \nu)^2} \left\{ -(x^2 - 1) \left[ (q^n + \kappa \nu) \beta_{n+1,n+2} + 2 \Omega(n) \right] + (x^2 + 1)(q^n - \kappa \nu) \right\}, \quad (A.13) \]

\[ k_{N,N}(x) = \frac{1}{2} \frac{x(q^n - \kappa \nu x^2)}{(q^n - \kappa \nu)^2} \left\{ (x^2 - 1) \left[ (q^n + \kappa \nu) \beta_{n+1,n+2} + 2 \Omega(n) \right] + (x^2 + 1)(q^n - \kappa \nu) \right\}, \quad (A.14) \]

and

\[ k_{n+1,n+2}(x) = \frac{1}{2} \frac{(x^2 - 1)}{(q^n - \kappa \nu)^2} \left\{ \beta_{n+1,n+2} \left[ x(q^n + \kappa \nu)^2 - 2 \kappa \nu q^n (x^2 + 1) \right] \\
- (q^n + \kappa \nu) \Omega(n)(x - 1)^2 \right\}, \quad (A.15) \]

\[ k_{n+2,n+1}(x) = \frac{1}{2} \frac{(x^2 - 1)}{(q^n - \kappa \nu)^2} \left\{ \beta_{n+1,n+2} \left[ x(q^n + \kappa \nu)^2 + 2 \kappa \nu q^n (x^2 + 1) \right] \\
+ (q^n + \kappa \nu) \Omega(n)(x + 1)^2 \right\}. \quad (A.16) \]

where

\[ \Omega(n) = \epsilon \sqrt{\kappa \nu q^n (\beta_{n+1,n+2}^2 - 1)}. \quad (A.17) \]

In contrast to the first solution, there is no way to derive a diagonal solution from these four conjugated solutions.

Appendix B. The multiparametric $U_q[D^{(2)}_2]$ vertex model

As mentioned in section 3, the most general $K$-matrix associated with the multiparametric $U_q[D^{(2)}_{n+1}]$ vertex model has $n + 2$ free parameters, namely $\beta_{1,n}$, $\beta_{1,n+1}$, \ldots, $\beta_{1,N-1}$. It is interesting to notice however that only $\beta_{1,n}$, $\beta_{1,n+1}$, $\beta_{1,n+2}$, $\beta_{1,n+3}$ and $\beta_{1,N-1}$ appears explicitly in the solution. For $n = 1$, however, an ambiguity arises due to this choice of writing the solution. Indeed, one can realize that in this case some of these free parameters get confused with others (for instance, we would have $\beta_{1,n+1} = \beta_{1,N-1}$ and $\beta_{1,n+2} = \beta_{1,N}$). Therefore, it is clear that the $U_q[D^{(2)}_2]$ vertex model should be treated separately, in order to find its general solution.

Proceeding in this way, we shall see that the elements out of the diagonals can actually be read directly from equations (19)–(42) of the solution presented in section 3, if we

\[ \text{doi:10.1088/1742-5468/2013/02/P02011} \]
For the elements of the last row we have \( k_n \) and\( k_{n-1} \) and, in the last column, \( k_{1,n} \) and the elements of the first column are given by \( k_{1,1} \). Furthermore, on the main diagonal we have

\[
k_{1,1}(x) = 1 + \frac{\Gamma(1)G(x)k_{1,4}(x)}{4\sqrt{q}(x^2 + 1)\beta_{1,4}} \left\{ (x^2 - 1) (q + 1)^2 \quad k_{1,n-1} \right\}
\]

\[
- [(x^2 - 1)(q + 1) + 4q(x^2 + 1)] (\beta^2_+ - \kappa \beta^-_+) \right\}
\]

and

\[
k_{1,4}(x) = x^2 + \frac{\Gamma(1)G(x)x^2k_{1,4}(x)}{4\sqrt{q}(x^2 + 1)\beta_{1,4}} \left\{ (q + 1) \left[ (x^2 + 1) - 3(x^2 - q) \right] (\beta^2_+ \kappa + \beta^2_-) \right\}
\]

\[
- [(q^2x^2 + 1) + 3(x^2 + q^2)] (\beta^2_+ - 2 \beta^-) \right\}.
\]

On the multiparametric \( U_0[D^{(2)}_{n+1}] \) vertex model
For the central block we can verify that the central matrix elements are still given by equations (47) and (48),

\[ k_{3,2}(x) = k_{2,3}(x) + \left( \frac{\beta_{3,2} - \beta_{2,2}}{\beta_{1,4}} \right) G(x) x k_{1,4}(x), \] (B.13)

\[ k_{3,3}(x) = k_{2,2}(x) + \left( \frac{\beta_{3,3} - \beta_{2,2}}{\beta_{1,4}} \right) G(x) x k_{1,4}(x), \] (B.14)

but now we have

\[ k_{2,2}(x) = \frac{q - x^2}{q - 1} + \frac{\Gamma(1) G(x) k_{1,4}(x)}{4\sqrt{q} \beta_{1,4}^2} \]
\[ \times \left\{ \kappa (x^2 + q) \left[ (q + \kappa)(\beta_{+}^2 - \beta_{-}^2) - (q - \kappa)(\beta_{+}^2 + \beta_{-}^2) \right] \right. \]
\[ - \left. 4qx (\kappa - 1) \beta_+ \beta_- \right\}, \] (B.15)

and

\[ k_{2,3}(x) = \frac{\Gamma(1) G(x) k_{1,4}(x)}{4\sqrt{q} (x^2 + 1) \beta_{1,4}^2} \]
\[ \times \left\{ \kappa x (q + 1) \left[ (q + \kappa)(\beta_{+}^2 + \beta_{-}^2) - 2(q - \kappa)(\beta_{+}^2 - \beta_{-}^2) \right] \right. \]
\[ - \left. 4q(x^2 + 1)(\kappa + 1) \beta_+ \beta_- \right\}. \] (B.16)

After this we get the parameters

\[ \beta_{3,2} = \beta_{2,3} + 2(\kappa + 1) \sqrt{q} \Gamma(1) \left( \frac{\beta_{+}}{\beta_{1,4}} \right), \] (B.17)

\[ \beta_{3,3} = \beta_{2,2} + 2(\kappa - 1) \sqrt{q} \Gamma(1) \left( \frac{\beta_{-}}{\beta_{1,4}} \right), \] (B.18)

and

\[ \beta_{2,1} = \frac{2\sqrt{q}}{q - 1} \left( \frac{\beta_{+} + \beta_{-} \kappa}{\beta_{1,4}} \right) \left[ 1 - \frac{\kappa \sqrt{q} \Gamma(1) (q - 1)}{2} \left( \frac{\beta_{+}^2 - \beta_{-}^2}{\beta_{1,4}^2} \right) \right], \] (B.19)

\[ \beta_{4,1} = \frac{\beta_{1,4} \beta_{2,1}}{\beta_{+} - \kappa \beta_{-}}. \] (B.20)

The remaining reflection equations are also constrained by equation (61) and then we get a solution with three free parameters (\(\beta_{1,2}, \beta_{1,3}\) and \(\beta_{1,4}\)) for \(\nu = 1\) and with only two free parameters for \(\nu = -1\).

Finally, we mention that the block diagonal and the diagonal solutions for the multiparametric \(U_q[D^{(2)}_{n+1}]\) vertex model can be obtained straightforwardly from appendix A, taking into account \(n = 1\).
Appendix C. Special solutions

Again, from the constraint equations (61) we look at the possibility $\beta_+ = \beta_- = 0$. In this case our multiparametric $K$-matrix has the form

$$K^-(x) = \begin{pmatrix} A(x) & 0 & B(x) \\ 0 & E(x) & 0 \\ C(x) & 0 & D(x) \end{pmatrix},$$  \hspace{1cm} (C.1)$$

where the 0 are null matrices, $A(x), B(x), C(x), D(x)$ are $n$ by $n$ matrices and

$$E(x) = \begin{pmatrix} k_{n+1,n+1}(x) & 0 \\ 0 & k_{n+2,n+2}(x) \end{pmatrix}.$$  \hspace{1cm} (C.2)

This results in an $n$ free parameter solution. However, from this special form of the $K$-matrix we have found new solutions, in analogy with the appearance of new $R$-matrices satisfying the Yang–Baxter equations.

Our first special solution does not depend on the parameters $\kappa$ and $\nu$ and its nonzero matrix elements are only in the diagonals. Making $k_{1,N}(x) = \beta_{1,N}(x^2 - 1)/2$, the main diagonal block matrices are

$$A(x) = I_{n \times n}, \quad E(x) = \begin{pmatrix} q - x^2 \\ q - 1 \end{pmatrix}, \quad D(x) = x^2 I_{n \times n}$$ \hspace{1cm} (C.3)

where $I_{n \times n}$ is the $n$ by $n$ unit matrix. The minor diagonal matrix elements are

$$(B)_{i,j} = \frac{1}{2} \beta_{i,j}(x^2 - 1)\delta_{j,i'}, \quad i = 1, 2, \ldots, n,$$

$$(C)_{i,j} = \frac{1}{2} \beta_{i,j}(x^2 - 1)\delta_{j,i'}, \quad i = n + 3, \ldots, N,$$ \hspace{1cm} (C.4)

where the parameters $\beta_{i,j}$ are related by

$$\beta_{i,i'}\beta_{i',i} = \frac{4q}{(q - 1)^2}, \quad i = 1, 2, \ldots, n.$$ \hspace{1cm} (C.5)

This means that we have found a new $n$ free parameter solution which has the form of the letter X.

A second special solution was obtained when $n = 2$, that is, for the multiparametric $U_q[D^{(2)}_3]$ vertex model. In this case it follows from equation (C.1) that the $K$-matrix has the form

$$K^-(x) = \begin{pmatrix} k_{1,1}(x) & k_{1,2}(x) & 0 & 0 & k_{1,5}(x) & k_{1,6}(x) \\ k_{2,1}(x) & k_{2,2}(x) & 0 & 0 & k_{2,5}(x) & k_{2,6}(x) \\ 0 & k_{3,3}(x) & 0 & 0 \\ 0 & 0 & k_{4,4}(x) & 0 \\ k_{5,1}(x) & k_{5,2}(x) & 0 & 0 & k_{5,5}(x) & k_{5,6}(x) \\ k_{6,1}(x) & k_{6,2}(x) & 0 & 0 & k_{6,5}(x) & k_{6,6}(x) \end{pmatrix}.$$  \hspace{1cm} (C.6)
By solving this problem separately we can see that the elements out of the diagonals can be read directly from the solution presented in section 3, provided we set there \( n = 2 \) and \( \epsilon = -1 \). This means we have

\[
k_{1,2}(x) = \frac{\beta_{1,2}G(x)k_{1,6}(x)}{\beta_{1,6}}, \tag{C.7}
\]

\[
k_{2,1}(x) = \frac{\beta_{2,1}G(x)k_{1,6}(x)}{\beta_{1,6}}, \tag{C.8}
\]

\[
k_{1,5}(x) = \frac{\beta_{1,5}G(x)k_{1,6}(x)}{\beta_{1,6}}, \tag{C.9}
\]

\[
k_{2,6}(x) = -\frac{\beta_{1,5}x^2G(x)k_{1,6}(x)}{q\beta_{1,6}}, \tag{C.10}
\]

\[
k_{5,1}(x) = \frac{q\beta_{1,2}\beta_{2,1}G(x)k_{1,6}(x)}{\beta_{1,5}\beta_{1,6}}, \tag{C.11}
\]

\[
k_{6,2}(x) = -\frac{\beta_{1,2}\beta_{2,1}x^2G(x)k_{1,6}(x)}{\beta_{1,5}\beta_{1,6}}, \tag{C.12}
\]

\[
k_{5,6}(x) = -\frac{\beta_{1,2}x^2G(x)k_{1,6}(x)}{\beta_{1,6}}, \tag{C.13}
\]

\[
k_{6,5}(x) = -\frac{\beta_{2,1}x^2G(x)k_{1,6}(x)}{\beta_{1,6}}, \tag{C.14}
\]

where now \( \beta_{2,1} \) is given by

\[
\beta_{2,1} = \left( \frac{\beta_{1,2}\beta_{1,5}}{q\beta_{1,6}} - \frac{2}{q - 1} \right) \beta_{1,5}. \tag{C.15}
\]

On the minor diagonal, we have

\[
k_{2,5}(x) = -\frac{\beta_{2,1}k_{1,6}(x)}{\beta_{1,2}}, \tag{C.16}
\]

\[
k_{5,2}(x) = -\frac{q\beta_{1,2}\beta_{2,1}k_{1,6}(x)}{\beta_{1,2}\beta_{1,5}^2}, \tag{C.17}
\]

\[
k_{6,1}(x) = -\frac{q\beta_{2,1}^2k_{1,6}(x)}{\beta_{1,5}^2}. \tag{C.18}
\]

For the elements of the principal diagonal, the ansatz equations (45) and (46) still hold, so we have

\[
k_{2,2}(x) = k_{1,1}(x) + \left( \frac{\beta_{2,2} - \beta_{1,1}}{\beta_{1,N}} \right) G(x)k_{1,N}(x), \tag{C.19}
\]

\[
k_{6,6}(x) = k_{5,5}(x) + \left( \frac{\beta_{6,6} - \beta_{5,5}}{\beta_{1,N}} \right) x^2G(x)k_{1,N}(x), \tag{C.20}
\]
On the multiparametric $\mathcal{U}_q[D_{n+1}^{(2)}]$ vertex model

where now

$$k_{1,1}(x) = 1 + \frac{\beta_{1,2}\beta_{1,5}G(x)k_{1,6}(x)}{\beta_{1,6}^2},$$

(C.21)

$$k_{5,5}(x) = x^2 + \frac{\beta_{1,2}\beta_{1,5}x^2G(x)k_{1,6}(x)}{\beta_{1,6}^2}.$$  

(C.22)

Finally, for the central block we have

$$k_{4,4}(x) = k_{3,3}(x) = \frac{1}{q - 1} + \left(\frac{q^2 - x^2}{q - x^2}\right)\frac{\beta_{1,2}\beta_{1,5}k_{1,6}(x)}{q\beta_{1,6}^2}.$$  

(C.23)

In this way we obtain a solution which is valid to any value of $\kappa$ and $\nu$ and which is characterized by the free parameters $\beta_{1,2}$, $\beta_{1,5}$ and $\beta_{1,6}$.

To conclude, we mention that this special solution was suggested by the existence of the ‘almost unitary’ solution

$$K^-(x) = \text{diag}(x^{-2}, 1, 1, 1, x^2),$$

(C.24)

previously presented in [22], which is valid only for this particular model as well.

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