A Very Intuitive Geometric Picture of the 24-cell, $E_8$ and $\Lambda_{16}$ Lattices Given by Using the Hopf Maps

Eric Lewin Altschuler$^1$ and Antonio Pérez–Garrido$^2$

$^1$Department of Physical Medicine & Rehabilitation, UMDNJ University Hospital, 150 Bergen Street, B-403, Newark, NJ 07103, USA
email: eric.altschuler@umdnj.edu

$^2$Departamento de Física Aplicada, UPCT Campus Muralla del Mar, Cartagena, 30202 Murcia, Spain
email: Antonio.Perez@upct.es

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Abstract

We use the Hopf fibrillation to give simple and intuitive geometric constructions of the 24-cell, $E_8$ and $\Lambda_{16}$ lattices.

The $n$-dimensional kissing problem asks how many non overlapping unit $n$-dimensional spheres can be placed touching a central unit $n$-dimensional sphere? This question has applications to making efficient codes and other problems [1]. The two dimensional sphere is the disk of points $x^2 + y^2 = 1$, and by using $N$ identical circular coins the reader can easily convince oneself that the kissing number in two dimensions ($K_2$) is six. Similarly, the one dimensional sphere is a line segment, and $K_1 = 2$. Already in three dimensions the problem becomes much more interesting. Indeed, the Seventeenth century featured a dispute between Isaac Newton who believed that $K_3 = 12$, and David Gregory who thought that $K_3 = 13$. Perhaps not surprisingly, Newton was correct but it took more than two and one half centuries to prove this[9]. Using clever linear programming arguments it was proven some decades ago that $K_8 = 240$ and $K_{24} = 196560$ [4] [7] and that $K_4 = 24$ or 25. Recently, a proof that $K_{24} = 24$ using much more extensive and subtle use of linear programming has be presented[6], however, the proof has not yet been fully vetted [8]. There exists a map known as the (first) Hopf map between the surface of a three dimensional sphere and the surface of a four dimensional sphere, which essentially constructs the surface of a four dimensional sphere by considering there to be a circle of points at every point on the surface of a three dimensional sphere. Here we emphasize that the Hopf map also gives a very intuitive way of appreciating the $N = 24$ kissing configuration for $S^3$–known as the 24-cell and then use the second and third Hopf maps to give intuitive descriptions...
of the $E_8/K_8$ lattice, and the so called $\Lambda_{16}$ lattice, currently the best known kissing
configuration in 16 dimensions.

The surface of a four dimensional sphere (a three dimensional locus or manifold
also known as $S^3$) is defined as the points $x^2 + y^2 + z^2 + w^2 = 1$ (as the surface of
a three dimensional sphere ($S^2$) is defined as the points $x^2 + y^2 + z^2 = 1$). The Hopf
map is one between the points on the surface of a four dimensional sphere and the pair
of complex numbers $(w, z)$ with $|w|^2 + |z|^2 = 1$
$$ (w, z) \rightarrow (2wz^*, |z|^2 - |w|^2) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3. \quad (1) $$

One easily checks that:
$$ |2wz|^2 + (|z|^2 - |w|^2)^2 = 4|w|^2|z|^2 + (|z|^2 - |w|^2)^2 = (|z|^2 + |w|^2)^2 = 1. \quad (2) $$

So this does map to the ordinary sphere. If one fixes a point of the ordinary sphere say
$(a, t)$ where $a$ is complex, $t$ is real and $|a|^2 + t^2 = 1$, then its fiber, i.e., the set of all
points which map to it, is a circle
$$ \left( \frac{ae^{i\theta}}{\sqrt{2(1 + t)}}, e^{i\theta} \sqrt{(1 + t)/2} \right). \quad (3) $$

Further details, discussions and proof of the Hopf map from $S^2$ to $S^3$ is given in[3].

The only known configuration on $S^3$ with twenty-four kissing spheres is the so-called 24-cell. In this convex four dimensional polytope all of the faces are octahedra.
The standard way of representing the 24-cell is by the coordinates:
$$ \left( \pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}, 0, 0 \right), \left( \pm \frac{\sqrt{2}}{2}, 0, \pm \frac{\sqrt{2}}{2}, 0 \right), \left( \pm \frac{\sqrt{2}}{2}, 0, 0, \pm \frac{\sqrt{2}}{2} \right) $$

$$ \left( 0, \pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}, 0 \right), \left( 0, \pm \frac{\sqrt{2}}{2}, 0, \pm \frac{\sqrt{2}}{2} \right), \left( 0, 0, \pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2} \right) $$

This configuration is also known as the $E_4$ lattice, and one can form an analogous
lattice in any dimension. By rotating on $S^3$ one can also easily see that these same
twenty-four centers of the kissing spheres can be obtained by lifting, via the (first)
Hopf map from six points on $S^2$ arranged with one at each pole and four arranged at
ninety degree angles around the equator:

Place one point on each pole of a three dimensional sphere and four equally spaced
points on the equator, i.e. the vertex of an octahedron. These points are at the antipodal
points of the three axes of $S^2$. These points can be expressed as $(a, t)$, as stated above:
$$ (0, 1), (0, -1), (1, 0), (-1, 0), (i, 0), (-i, 0) $$

Then four points can be placed on each circle via the Hopf map to give a total of
twenty four points.
$$ (e^{i\theta}, 0), \ \theta = \frac{\pi}{4}(2k + 1), \ k = 0, 1, 2, 3 $$
Figure 1: 24-cell. Parallel projection to a 2D plane. Several views during an $i \cdot 2\pi/15$ rotation in 4D are plotted. Each color corresponds to a different circle on $S^2$. The symmetry of the configuration is evident. The antipodal construction is illustrated as the circles (in $S^2$) are ninety or one hundred eighty degrees from each other. That nearest neighbors are points on different circles (see text for discussion) is illustrated, for example, in subfigure 1 where one sees that the black circle, actually seen as a single point, is surrounded by all other circles but the red circle which is its antipodal.

\[ (0, e^{i\theta}), \quad \theta = \frac{\pi}{4}(2k + 1), \quad k = 0, 1, 2, 3 \]

\[ \left( \frac{e^{i\theta}}{\sqrt{2}}, \frac{e^{i\theta}}{\sqrt{2}} \right), \quad \theta = \frac{\pi}{2}k, \quad k = 0, 1, 2, 3 \]

\[ \left( -\frac{e^{i\theta}}{\sqrt{2}}, \frac{e^{i\theta}}{\sqrt{2}} \right), \quad \theta = \frac{\pi}{2}k, \quad k = 0, 1, 2, 3 \]

\[ \left( i\frac{e^{i\theta}}{\sqrt{2}}, \frac{e^{i\theta}}{\sqrt{2}} \right) \equiv \left( \frac{e^{i(\theta + \pi/2)}}{\sqrt{2}}, \frac{e^{i\theta}}{\sqrt{2}} \right), \quad \theta = \frac{\pi}{2}k, \quad k = 0, 1, 2, 3 \]

It is easy to see that these twenty-four points on $S^3$ (the surface of a unit four-dimensional sphere) are the same as the points of a 24-cell.

The twenty-four points can trivially be seen to be a distance greater or equal than one from each other and thus a kissing configuration in four dimensions of twenty-four points. This is the known configuration illustrating that $K_4$ is $\geq 24$ and is also known as the 24-cell because those points are the vertices of a 4D polytope made of 24 octahedra. 4D polytopes are assemblies of 3D polyhedra (cells) as 3D polyhedra are assemblies of 2D polygons.
Figure 2: $E_8$. Parallel projection to a 2D plane of several views during a rotation by $i \cdot 2\pi/20$ around a plane in 8D are plotted. Each color corresponds to a different circle on $S^4$. Again we see the symmetry inherent in this configuration, and the antipodal construction being highlighted as the circles ($S^3$) are ninety or 180 degrees apart.
Also, the Hopf map gives an extremely simple and somewhat intuitive way to think about the 24-cell: For one point on each equator and four spaced at ninety degrees along the equator then each point in $S^2$ is a distance of ninety degrees away from its nearest neighbor. Now associate with each of these six points a *Hopf circle* placing four points again spaced ninety degrees. The symmetry of the configuration becomes self-evident.

Points on the same circle are ninety degrees—or distance $\sqrt{2}$—apart from each other. Each point has eight nearest neighbors—two points each from the four non-antipodal circles—that are sixty degrees or distance one apart.

Now, in addition to the first Hopf map from $S^3$ to $S^2$, there is a second Hopf map from $S^7$ to $S^4$ (and a third Hopf map from $S^{15}$ to $S^8$). Intuitively the second (third) Hopf map uses quaternions (octonions) to accomplish the map. Using the Cayley-Dickson construction for the normed division algebras, $A_0 = \mathbb{R}$ (real) $A_1 = \mathbb{C}$ (complex) $A_2 = \mathbb{H}$ (quaternions) and $A_3 = \mathbb{O}$ (octonions), we can make $A_n$ from $A_{n-1}$ for $n = 1, 2, 3$. In this construction, an element in $A_n$ is made of a pair of elements $a, b \in A_{n-1}$ with the multiplication:

\[
(a, b)(c, d) = (ac - db^*, a^*d + cb)
\]  

and the conjugation in $A_n$:

\[
(a, b)^* = (a^*, -b)
\]

It is also possible to continue this procedure to get $A_4$ (sedenions) but it is no longer a division algebra. Hopf maps can be compactly defined as a map $h_1$ from $A_n \otimes A_n$ to $A_n \cup \{\infty\}$ followed by a second map $h_2$ from $A_n \cup \{\infty\}$ to $S^{2m}$ (spherical projection), for $n = 0, 1, 2, 3[5]$. $h_1$ and $h_2$ can be stated as:

\[
h_1 : (a, b) \longrightarrow c = ab^{-1}
\]

where $a, b \in A_n$ and $|a|^2 + |b|^2 = 1$, so the point $(a, b)$ is actually a point of $S^m$, being $m = 2^{n+1} - 1$.

\[
h_2 : c \longrightarrow X_i (i = 1, \ldots, 2^n + 1), \quad \sum_{i=1}^{2^n+1} X_i^2 = 1
\]

Now, Dixon (see [2] and refs. therein) seems to have been the first to have appreciated not only that the first Hopf map can be used to generate the 24-cell from points on $S^2$, but that $E_8$ is generated from ten 24-cell’s lifted from $S^4$, and $\Lambda_{16}$ is generated from 18 $E_8$ lattices lifted from $S^8$. Dixon’s discussion was mostly topological. Here we give a remarkably simple and intuitive geometric construction for $E_8$ and $\Lambda_{16}$. We note (see [5] and refs. therein) that such Hopf make constructions may have utility in quantum computers and communication.

$S^7$ is the surface of an eight dimensional sphere. The kissing configuration of eight dimensional spheres on the surface of an eight dimensional sphere as mentioned is known to be 240 points arranged in the $E_8$ lattice. The kissing configuration in five dimensions ($K_5$) which is points arranged on the surface of $S^4$ is thought to be 40 points arranged in an $E_5$ lattice, but there is no proof of this. Initially we wondered
if we could use the second Hopf map to lift from the forty kissing points on $E_5$ six 
points each onto $S^7$ and obtain the $E_8$ lattice/$K_8$ configuration. We have not been 
able to do this, however, we noticed that by taking again the 10 antipodal points from 
the axes on $S^4$, $(\pm 1, 0, 0, 0, 0)$, $(0, \pm 1, 0, 0, 0)$, $(0, 0, \pm 1, 0, 0)$ and 
$(0, 0, 0, 0, \pm 1)$, and lifting 10 24-cells to $S^7$ we get the $E_8$ lattice. This construction 
is illustrated in Figure 2. Again, our construction immediately illustrates that as for 
the 24-cell points are separated by sixty, ninety, 120 or 180 degrees. Our construction 
also intuitive explains why each point has 56 nearest neighbors: eight on "its own" $S^3$
circle, and six (two per orthogonal axis of a 3-dimensional object) on each of the eight 
other non-antipodal circles on $S^4$. 

Similarly, by lifting from the 16 antipodal points of the axes of $S^8$ an $E_8$ lattice 
one gets the $\Lambda_{16}$ lattice! This construction again illustrates why $\Lambda_{16}$ is a kissing con-
figuration with points separated by angles sixty, ninety, 120 or 180 degrees, and why 
each point on $\Lambda_{16}$ has 280 nearest neighbors = 56 on "its own" $S^7$ circle + 14 (two 
points per orthogonal axis of a 7-dimensional object) $\times$ 16, where 16 is the number of 
non-antipodal $S^7$ circles on $S^8$. This construction suggests, but of course in no way 
proves, that $\Lambda_{16}$ may be a configuration of maximum kissing number. Our construction 
give a helpful picture in proving, or disproving this, or in attacking other problems 
such as finding the maximal packing configurations in 4, 8 or 16 dimensions.

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