A non-order parameter Langevin equation for a bounded Kardar–Parisi–Zhang universality class

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Abstract. We introduce a Langevin equation describing the pinning–depinning phase transition experienced by Kardar–Parisi–Zhang interfaces in the presence of a bounding ‘lower wall’. This provides a continuous description for this universality class, complementary the different and already well documented one for the case of an ‘upper wall’. The Langevin equation is written in terms of a field that is not an order parameter, in contrast to standard approaches, and is studied both by employing a systematic mean field approximation and by means of a recently introduced efficient integration scheme. Our findings are in good agreement with known results from microscopic models in this class, while the numerical precision is improved. This Langevin equation constitutes a sound starting point for further analytical calculations, beyond mean field, needed to shed more light on this poorly understood universality class.

Keywords: kinetic growth processes (theory), nonequilibrium wetting (theory), phase transitions into absorbing states (theory)
1. Introduction

It was shown a few years ago that the introduction of a limiting or bounding wall into a Kardar–Parisi–Zhang (KPZ) interface model [1, 2] leads to quite different phenomenologies depending on whether the wall is an ‘upper’ or a ‘lower’ one [3–6]. This result, the origin of which can be traced back to the absence of height-inversion symmetry in KPZ interfacial dynamics [3, 6], has been verified for several discrete interfacial KPZ-like models [3, 6, 7]. In any of these cases, once a limiting wall is introduced, there are two different phases: a *depinned* one in which the KPZ interface moves freely away from the wall, and a *pinned* one, with a finite stationary average distance from the wall. Separating these two phases there is a non-equilibrium phase transition, whose criticality encompasses also that of synchronizing extended systems [8, 9], non-equilibrium wetting phenomena [10–12], transitions occurring in DNA alignment problems [3, 6], phenomena related to Burgers’ turbulence [3], bounded directed polymers in random media [3], etc. Characterizing and distinguishing between the two above-mentioned pinning–depinning universality classes, with an upper or a lower wall respectively, is therefore a relevant task in many different contexts, as well as a major theoretical problem.

In terms of Langevin equations, which provide an ideal framework in which to discuss universality issues, KPZ-like interfaces are described by the celebrated and profusely studied KPZ equation [1]:

$$\frac{\partial h(r, t)}{\partial t} = a + \lambda (\nabla h(r, t))^2 + D \nabla^2 h(r, t) + \sigma \eta(r, t),$$  \hspace{1cm} (1)

where $h(r, t)$ is the interface height at position $r$ and time $t$, $\lambda$, $D$, and $\sigma$ are constants, and $\eta$ is a Gaussian white noise. In what follows, and without loss of generality, we take $\lambda = +D$. Alternatively, we could also have fixed a given type of limiting wall, for instance, a lower, rigid substrate on top of which the interface grows, and observed the two different classes of depinning transitions under scrutiny depending on the sign of $\lambda^1$.

Let us consider now equation (1) in the presence of an exponential, upper wall, i.e. including the additional term $-b \exp(q h)$, with $b > 0$ and $q > 0$. A transition

$^1$ Indeed, it can be easily shown that a KPZ equation with positive non-linearity and a ‘lower wall’ is equivalent (can be mapped by changing the sign of $h$) to a KPZ equation with a negative non-linearity coefficient and an ‘upper wall’ [3].
A non-order parameter Langevin equation for a bounded Kardar–Parisi–Zhang universality class takes place from a regime characterized by depinned interfaces, flowing to minus infinity for sufficiently negative values of $a$, to one with interfaces pinned to the wall, with exponentially cut off positive values of $h$, above a certain threshold $a_c$. This can also be visualized by performing a Cole–Hopf transformation, $n = \exp(h)$, which leads to
\[
\frac{\partial n(r, t)}{\partial t} = D \nabla^2 n(r, t) + an(r, t) - bn(r, t)^{1+q} + \sigma n(r, t)\eta(r, t).
\] (2)

This is a multiplicative noise equation (interpreted in the Stratonovich sense [13]) defining a, well established by now, phase transition with a corresponding set of critical exponents that characterize the so-called multiplicative noise 1 (MN1) universality class [4]–[6], [14]. At the transition point and in the depinned phase the stationary average value of the order parameter, $\bar{n}$, is zero, while it is non-vanishing above the transition point. The critical exponents and other universal features in this class are not affected by the value of $q$, i.e. by the ‘impenetrability degree’ of the wall; indeed, in microscopic models the wall is typically a hard substrate, corresponding to $q \to \infty$.

On the other hand, on introducing in equation (1) a lower wall, $b \exp(-gh)$, hindering the interface front taking negative values, depinned interfaces flow towards plus infinity. In this case, a natural order parameter, equivalent to that for the upper wall class, going to zero at the transition point, is $m = \exp(-h)$, and the corresponding depinning transition is in the so-called multiplicative noise 2 (MN2) universality class [6,3,7,15,12].

Consequently, we perform the change of variables $m = \exp(-h)$ to obtain [7]
\[
\frac{\partial m(r, t)}{\partial t} = D \nabla^2 m - 2D \frac{(\nabla m)^2}{m} - am - bm^{1-q} + \sigma m\eta(r, t),
\] (3)

where some space and time dependences have been omitted to simplify the notation. Observe that owing to the $(\nabla m)^2/m$ term the equation becomes singular above the transition point, where $\bar{m} = 0$. Equation (3) was studied in detail, both numerically and using mean field approaches, in [7], but no sound result could be obtained owing to the presence of the singular gradient term. Therefore, all the known results for this universality class come from numerical [7,12] as well as some analytical (mean field-like) [15] studies of discrete microscopic models. Let us also stress that direct numerical integrations of KPZ-like Langevin equations (before applying the Cole–Hopf transformation) are uncontrollable due to well documented numerical instabilities [17].

Aimed at filling the gap between discrete and continuous levels of description for the MN2 universality class, in this paper we show that the MN2 phenomenology can be captured by an alternative, multiplicative noise, Langevin equation. To that end, we take the KPZ equation in the presence of a lower wall and perform the change of variables, $n = \exp(h)$, leading to
\[
\frac{\partial n(r, t)}{\partial t} = D \nabla^2 n + an + bn^{1-q} + \sigma n\eta(r, t),
\] (4)

2 In the more general case $\lambda \neq D$, the different coefficient can be reabsorbed by using $n = \exp((\lambda/D)h)$.

3 The sole difference between utilizing the Ito and Stratonovich calculi, in this case, is a trivial shift in $a$ [13].

4 In fact, except for the factor 2 in front of $(\nabla m)^2/m$, equation (3) coincides with the Cole–Hopf transform of $\partial_t h(x, t) = \nabla^2 h + a + b e^{-nh} + \eta(x, t)$, that is the growth of wetting layers toward their equilibrium state [16]. Observe that this is just the equilibrium, Edwards–Wilkinson model, in the presence of a bounding wall. Note also that the factor 2 in equation (3) cannot be reabsorbed by reparametrizing.
which takes a particularly simple form for \( q = 1 \) although, as in the MN1 class, the precise value of \( q > 0 \) is not expected to affect universal features. Note the remarkable difference between equations (2) and (4): while in the former, \( n \) is an order parameter for the MN1 transition, in equation (4) for the MN2 class, it is not, as it diverges at the transition point and in the depinned phase. Therefore equation (4) is not an order parameter Langevin equation and it is \( m = 1/n \) that is to be monitored once the equation for \( n \) is integrated. Typically, Langevin equations are written in terms of an order parameter vanishing at the transition, so that series expansions and truncation of power series to lowest orders can be employed and the applicability of standard perturbative techniques is viable. This is in contrast with equation (4). Furthermore, the noise amplitude diverges at the transition point. This apparent poor behaviour may be why equation (4) has been ignored so far in the literature.

In what follows we study the non-order parameter Langevin equation for the MN2 class, equation (4), using standard mean field approaches and integrating it numerically. Despite the presence of apparent pathologies and divergences at (and above) the transition point, we find that equation (4) reproduces the previously known results for this universality class. In passing, we improve the numerical precision of the corresponding critical exponents. This constitutes a step forward in the general understanding of nonequilibrium phase transitions into absorbing states, allows for a better comparison with the MN1 class, gives a new starting point for future analytical approaches, and serves as an example of a phase transition best characterized in terms of an equation for a diverging field that is not an order parameter.

### 2. Mean field analysis

Standard mean field approaches usually neglect spatial and noise-induced fluctuations. For Langevin equations characterizing spatially extended systems with multiplicative noise, however, this approximation has proved to be too crude and unreliable in any dimension [14] as both spatial structure and noise are relevant features. Therefore, more elaborate methods are required to obtain a sound qualitative picture of the transition [18]–[20]. A mean field approach tailored to account properly for the noise term and the spatially varying order parameter consists in discretizing the Laplacian term as \((1/2d) \sum_j (n_j - n_i)\), where \( d \) is the system dimensionality and the sum is over the nearest neighbours of site \( i \) [19,20]. When these latter are replaced by their average value \( \bar{n} \), a closed Fokker–Planck equation (which involves no approximation in the noise) is obtained for \( P(n, \bar{n}) \). The stationary solution of such an equation is found by imposing the self-consistency requirement [20]

\[
\bar{n} = \frac{\int_0^\infty dn \, nP(n, \bar{n})}{\int_0^\infty dn \, P(n, \bar{n})}.
\]

Note that this approach preserves the crucial role played by the multiplicative noise and includes the spatial coupling even if in an approximate (self-consistent) way. A detailed discussion of the results obtained with this method for the MN1 class can be found in [18]. Applying this procedure to equation (4) one readily obtains

\[
P(n, \bar{n}) \propto n^{2(a-D)/\sigma^2-1} \exp\left\{ -\frac{2b}{\sigma^2 q n^2} - \frac{2D\bar{n}}{\sigma^2 n} \right\},
\]
and after defining $I_p(\tilde{n}) \equiv \int dn\, n^p P(n, \tilde{n})$,

$$\tilde{n} = \frac{I_1(\tilde{n})}{I_0(\tilde{n})}, \quad \tilde{m} = \frac{I_{-1}(\tilde{n})}{I_0(\tilde{n})}.$$  

(7)

To evaluate the scaling behaviour of $I_p$ near the transition, when $\tilde{n}$ becomes large, we first make the substitution $z = 2D\tilde{n}/\sigma^2\tilde{n}$ and then expand the newly generated term \exp\{\(-2b/(\sigma^{2(1-q)}q)\times(z/2D\tilde{n})^q\}\} to first order to obtain

$$I_p(\tilde{n}) \approx A_p\tilde{n}^{p+\gamma} + B_p\tilde{n}^{p+\gamma-q},$$  

(8)

where $\gamma = 2(a-D)/\sigma^2$ and $A_p, B_p$ are expressed using the Gamma function as

$$A_p = \left(\frac{2D}{\sigma^2}\right)^{\gamma+p} \Gamma(-p - \gamma),$$

$$B_p = -\left(\frac{2D}{\sigma^2}\right)^{p+\gamma-q} \frac{2b}{\sigma^2q} \Gamma(-p - \gamma + q).$$

(9)

A direct calculation then yields $\tilde{n} \sim |a + \sigma^2/2|^{-1/q}$ and $\tilde{m} \sim |a + \sigma^2/2|^{1/q}$. Therefore, the order parameter critical exponent is $\beta = 1/q$.

It is instructive to compare these results with those obtained using the same technique for equation (2), $\beta = \max(1/q, \sigma^2/(2D))$ [19,18]. That is, in the MN1 case two possible values of $\beta$, reminiscent of the strong and weak coupling regimes of the KPZ dynamics [2], already appear at this self-consistent mean field level. In contrast, for equation (4), there is no strong coupling regime, which would be characterized by a non-universal, noise-amplitude-dependent value of $\beta$. This fact is related to the presence of a single cut-off in the stationary probability distribution, equation (4), while two different cut-offs, and therefore two different mechanisms controlling the scaling, appear for equation (2) [18]. The implications of this property, as well as its connections with the high dimensional behaviour of the KPZ dynamics, will be analysed elsewhere.

3. Numerical integration of stochastic differential equations

In order to integrate equation (4) as efficiently as possible, we have employed a recently introduced split-step scheme for the integration of Langevin equations with non-additive noise. In this scheme, the Langevin equation under consideration is studied on a lattice and separated into two parts: the first one includes only deterministic terms and is integrated at each time step using any standard Euler scheme: Euler, Runge–Kutta, etc [21] (here we have chosen a simple Euler algorithm). The output of this step is used as the input for integrating (along the same time step) the second part, which may include the linear deterministic term and, more importantly, the noise. This is done by sampling in an exact way the probability distribution function associated with this part.

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of the equation. In the case under study (noise proportional to the field) the second step corresponds to sampling a log-normal distribution solution of $\partial_t n = a n + \sigma n q$ (for more details see\textsuperscript{6} [22]). Therefore the two-step algorithm is finally specified by

$$n(i, t) = n(i, t) + (b n(i, t)^{1-q} + D[n(i + 1, t) + n(i - 1, t) - 2n(i, t)]) \, dt$$

(10)

and

$$n(i, t + dt) = n_1(i, t) \exp\left( a dt + \sigma\sqrt{dt} \eta \right)$$

(11)

where $\eta$ is a random variable extracted from a normal distribution with zero mean and unit variance. Note that the linear deterministic term can be included in either the first or the second step, or partially incorporated in both of them.

We have considered one-dimensional lattices, and fixed $\sigma = 1$, $b = 1$, $D = 0.1$, space mesh $dx = 1$, and time mesh $dt = 0.1$ (note that in this scheme $dt$ can be taken larger than in usual integration algorithms [22]). As the initial condition we take $n(r, t = 0) = 3$. We take $q = 4$ for all simulations except for the results presented in figure 2(a) where we show that asymptotic results do not depend on the value of $q$, as long as $q > 0$. We then iterate the dynamics by employing the previous two-step integration algorithm, using parallel updating, at each site $i$.

A summary of our main findings follows. First, to accurately determine the critical point, we perform decay experiments and average over many independent runs in a system of size $L = 2^{17}$. At criticality, $a_c = -0.143668(3)$, the average density, $\bar{n} = (1/n)$, decays as a power law with an associated exponent $\theta \approx 0.229(5)$ (see figure 1). This is to be compared with the previous estimates $\theta \approx 0.215(15)$ [7] and $\theta \approx 0.228(5)$ [12]. On the other hand, for smaller system sizes, we observe saturation at this value of $a_c$, and the scaling of the saturation values for different system sizes (see the inset of figure 1(a)) gives $\beta/\nu \approx 0.335(5)$ (to be compared with 0.34(2) in [7]).

For other values of $q$ we have verified that, as shown in figure 2, none of the previously reported exponents, or the locations of the critical point, are altered, although $q$-dependent transient effects exist. The invariance of $a_c$ against changes in $q$ can be understood from the fact that $a_c$ corresponds to the value of $a$ for which depinned interfaces, arbitrarily far from the wall, invert their direction of motion; this is not affected by the nature of the wall, i.e. by the value of $q$. Additionally, the order parameter exponent $\beta \approx 0.332(5)$ has been measured for $L = 2^{17}$ (see also figure 2). The previous best estimation was $\beta = 0.32(2)$ [7].

We have also studied the scaling properties of the height field $\bar{h} = -\log(n)$ (see figure 2(b)). An analysis analogous to that presented above for $m$ leads to $\bar{h}(t) \approx e^{0.323(10)}$, $\bar{h} \sim |a - a_c|^{-0.48(3)}$, and $\bar{h}(a_c, L) \sim L^{0.48(2)}$, which define a set of critical exponents for $\bar{h}$ analogous to those for $m$: $\theta_{\bar{h}} \approx -0.323(10)$, $\beta_{\bar{h}} \approx -0.48(3)$, and $\beta_{\bar{h}}/\nu \approx -0.48(2)$ (see figure 2(b)). The previous best estimates for these exponents were $-0.355(15)$, $-0.52(2)$, and $-0.52(2)$ respectively [7]. The minus signs just reflect the fact that $\bar{h}$ diverges at the

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\textsuperscript{6} The part of the Langevin equation to be integrated can be written as $dn_t = a n_t \, dt + \sigma n_t \, dW_t$, where $dW_t$ is a Wiener process. Since this is interpreted in the Stratonovich sense, we can safely perform the change of variables $Y_t = \ln n_t$ and obtain $dY_t = a \, dt + \sigma \, dW_t$. This describes a drifted Brownian motion equation whose solution is given by a normal distribution of mean $y_0 + a \, t$ and variance $\sigma^2 \, t$: $\text{Prob}(Y_{t+dt} = y | Y_t = y_0) = N(y_0 + a \, dt, \sigma^2 \, dt)$. Inverting the change of variables, we are left with a log-normal form, which can be sampled in an exact way by taking: $n(t) = n_0 \exp(a \, dt + \sigma \sqrt{dt} \eta)$. This same expression can also be derived by changing variables in the Langevin equation, performing one time step evolution, and changing the variables back.

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Figure 1. (a) Log–log plot of the time decay of the average order parameter at the critical point, $a_c = -0.143668$, for system sizes $2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11},$ and $2^{17}$, and $q = 3$. In the inset, the average saturation values of the previous curves are plotted as a function of the system size, $L$, on a double-logarithmic scale. (b) Averaged order parameter decay for equation (4) with $q = 0.5, 1, 2, 4$ (from bottom to top) in a system of size $L = 2^{17}$. For any $q > 0$ we observe the same asymptotic decay exponent at the same critical point location.

Figure 2. (a) Log–log plot of the saturation values of $\bar{m}$ for different values of $a$ near to the critical point. From the slope we estimate $\beta = 0.332(5)$. (b) Log–log plot of the time growth of the averaged height $h = -\log(n)$ for the same value of $a$ and same system sizes as in figure 1. In the inset, the average saturation values of the previous curves are plotted as a function of the system size, $L$, on a log–log scale.

transition point. A summary including our best estimates for the critical exponents can be found in table 1. Finally, we stress that our results are compatible with the theoretical predictions $z = 3/2$ and $\nu = 1$, expected to be valid for both MN1 and MN2 [4,3]. Also, all the standard scaling laws connecting exponents are satisfied [3].

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Table 1. Critical exponents for the MN2 class for $d = 1$ as calculated in this paper. Scaling of saturation times gives a $z$ which is consistent with the value $z = \beta/(\theta \nu)$ [3, 4].

| $\theta$ | $\beta$ | $\beta/\nu$ | $\nu$ | $z$ |
|----------|---------|-------------|-------|-----|
| $m$      | 0.229(5)| 0.332(5)    | 0.335(5)| 0.99(3)| 1.46(5)|
| $h$      | -0.323(10) | -0.48(3) | -0.48(2) | 1.0(1) | 1.48(10) |

4. Discussion

We have studied the dynamics of KPZ-like interfaces with $\lambda > 0$ bounded by a lower wall. The results and phenomenology differ from those for upper walls. On performing a Cole–Hopf or logarithmic transformation, the resulting order parameter Langevin equation (3) is singular and no satisfactory result can be derived from it. Instead, the main result of this paper is that a sound Langevin equation (4) can be written in terms of a non-order parameter field which diverges at the transition. For such an equation we have performed (i) a self-consistent mean field analysis leading to the result $\beta = 1/q$, and no trace of any strong coupling regime (noise-dependent $\beta$ exponent value) contrary to what happens for the upper wall case; (ii) extensive numerical integrations of the stochastic equation using a recently introduced very efficient numerical scheme. The critical exponent values obtained are in good agreement with previously known ones measured in simulations of discrete models, and improve the level of accuracy and precision.

In summary, we have shown that an apparently poorly behaved non-order parameter Langevin equation constitutes a sound continuous representation of the pinning–depinning transition experienced by interfaces in the Kardar–Parisi–Zhang class in the presence of a bounding lower wall. Performing further analytical, renormalization group analyses of the present Langevin equation remains a challenging task.

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References

[1] Kardar M, Parisi G and Zhang Y C, 1986 Phys. Rev. Lett. 56 889
[2] Halpin-Healy T and Zhang Y C, 1995 Phys. Rep. 254 215
[3] Muñoz M A, Multiplicative noise in non-equilibrium phase transitions: a tutorial, 2004 Advances in Condensed Matter and Statistical Physics ed E Korutcheva et al (New York: Nova Science Publishers) p 34 [cond-mat/0303650]
[4] Grinstein G, Muñoz M A and Tu Y, 1996 Phys. Rev. Lett. 76 4376
[5] Tu Y, Grinstein G and Muñoz M A, 1997 Phys. Rev. Lett. 78 274
[6] Muñoz M A and Hwa T, 1998 Europhys. Lett. 41 147
[7] Muñoz M A, de los Santos F and Achalbar A, 2003 Braz. J. Phys. 33 443 [cond-mat/0304239]
[8] Ahlers V and Pikovsky A, 2002 Phys. Rev. Lett. 88 254101
[9] Droz M and Lipowski A, 2003 Phys. Rev. E 67 056204
[10] Ginelli F, Ahlers V, Livi R, Mukamel D, Pikovsky A, Politi A and Torcini A, 2003 Phys. Rev. E 68 065102
[11] Muñoz M A and Pastor Satorras R, 2003 Phys. Rev. Lett. 90 204101
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[10] de los Santos F, Telo da Gama M M and Muñoz M A, 2002 Europhys. Lett. 57 803
[11] Hinrichsen H, Livi R, Mukamel D and Politi A, 1997 Phys. Rev. Lett. 79 2710
[12] Kissing T, Kotowitz A, Kurz O, Ginelli F and Hinrichsen H, 2005 J. Stat. Mech. P06002 [cond-mat/0503582]
[13] van Kampen N G, 1981 Stochastic Processes in Physics and Chemistry (Amsterdam: North-Holland)
[14] Genovese W and Muñoz M A, 1999 Phys. Rev. E 60 69
[15] Ginelli F and Hinrichsen H, 2004 J. Phys. A: Math. Gen. 37 11085
[16] Lipowsky R, 1985 J. Phys. A: Math. Gen. 18 L585
[17] Newman T J and Bray A J, 1996 J. Phys. A: Math. Gen. 29 7917
[18] Muñoz M A, Colaiori F and Castellano C, Mean field approach to systems with multiplicative noise, 2005
  Phys. Rev. E at press [cond-mat/0506635]
[19] Birner T, Lippert K, Müller R, Kühnel A and Behn U, 2002 Phys. Rev. E 65 046110
[20] Van den Broeck C, Parrondo J M R, Armero J and Hernández Machado A, 1994 Phys. Rev. E 49 2639
[21] San Miguel M and Toral R, Stochastic effects in physical systems, 1997 Instabilities and Nonequilibrium
  Structures VI ed E Tirapegui and W Zeller (Dordrecht: Kluwer Academic) pp 35–130
  [cond-mat/9707147]
[22] Dornic I, Chaté H and Muñoz M A, 2005 Phys. Rev. Lett. 94 100601
  Dornic I, Chaté H and Muñoz M A, 2005 in preparation
  Pechenik L and Levine H, 1999 Phys. Rev. E 59 3893
  Moro E, 2004 Phys. Rev. E R70 045102

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