The modification of the Bel-Robinson energy-momentum

Lau Loi Sh†

Abstract

For describing the non-negative gravitational energy-momentum in terms of a pure Bel-Robinson ‘momentum’ in a quasi-local small sphere limit, the Bel-Robinson tensor $B$ is desirable. However, we found this Bel-Robinson ‘momentum’ can be modified such that it still satisfy the non-spacelike and future pointing requirement. These particular energy-momentum properties can be obtained from a linear combination between $B$ with other tensor $S$ in a small sphere limit. This implies that the Landau-Lifshitz pseudo-tensor is no longer disqualified for this non-spacelike and future pointing requirement. Moreover, we constructed a certain linear combination using tensors $B, S, T$ that gives the dominate energy condition in a small sphere region.

1 Introduction

Bel and Robinson proposed an energy-momentum 4-index tensor since 1958 [1, 2, 3, 4]. This tensor is constructed from the Weyl tensor that analogy with the electromagnetic stress tensor. The Bel-Robinson tensor $B$ is desirable for describing the non-negative gravitational energy in a small sphere limit. According to general relativity, there does not exist unique definition of the local energy of the gravitational field. The Bel-Robinson tensor tensor is desirable for this local energy since the Ricci tensor vanishes in vacuum [5].

It is known, similar with the electromagnetic stress tensor, the Bel-Robinson tensor possesses many nice properties: completely symmetric, divergence free and trace free. Moreover, it satisfies the dominant energy condition [6]. Dominant energy condition automatically fulfills the future pointing and non-spacelike properties, but the converse is not guaranteed. According to the Living Review article, Szabados said (see 4.2.2 in [7]): “Therefore, in vacuum in the leading $r^5$ order any coordinate and Lorentz-covariant quasi-local energy-momentum expression which is non-spacelike and future pointing must be proportional to the Bel-Robinson ‘momentum’ $B_{\mu\lambda\xi\tau}t^\lambda t^\xi t^\tau.$” Note that here $t^\alpha$ is the timelike unit vector and the referred ‘momentum’ means 4-momentum.

Previously, it is believed that the Bel-Robinson ‘momentum’ was a natural choice and unique choice for describing the non-negative gravitational quasi-local energy-momentum expression. In the past, we thought there were only two gravitational energy-momentum expressions contribute the desirable positive definite energy since they give a positive multiple of the Bel-Robinson ‘momentum’ in a small sphere limit. They are the Papapetrou pseudo-tensor [8, 9, 10] and tetrad-teleparallel energy-momentum gauge current expression [11, 12]. We even confidently concluded that the Landau-Lifshitz pseudo-tensors cannot warranty the positivity, but now we claim it was mistaken [10].

Basically, quasi-local methods are not fundamentally different than pseudo-tensor methods [13, 14]. Although pseudo-tensor is an co-ordinates dependent object, it is still a practical way to calculate the work done for an isolated system from an

†email address: s0242010@gmail.com
external universe, e.g., tidal heating through transferring the gravitational field from Jupiter to its satellite Io. More concretely, tidal heating is a real physical observable irreversible process that Jupiter distorts and heats up Io. Purdue used the Landau-Lifshitz pseudotensor to calculate the tidal heating for Io in 1999 [15, 16, 17]. Positive gravitational energy is required for the stability of the spacetime [18] and any quasi-local stress expression which gives the Bel-Robinson ‘momentum’ is the desirable candidate. Moreover, evaluating the quasi-local energy-momentum around a closed 2-surface, we can use the Bel-Robinson ‘momentum’ to test whether the expression can have a chance to give the positivity at the large scale or not. Since negative quasi-local energy guarantees negative for a large scale, while positive quasi-local energy might have a chance for the large scale. Checking the result for the gravitational energy in a small region is an economic way because the positivity energy proof is not easy.

The motivation for reviewing the argument that raised by Szabados is that we suspect there may exist a relaxation such that the desirable physical requirements can be satisfied, i.e., future pointing and non-spacelike. We claim that the verification and explanation given by Szabados is necessary but not sufficient. For example, we find the energy-momentum of the Landau-Lifshitz pseudo-tensor does satisfy the future pointing and non-spacelike requirement, though not a multiple of the pure Bel-Robinson ‘momentum’ in a small sphere limit. Moreover, We claim the Bel-Robinson tensor lost its privilege to achieve the dominate energy condition in a small sphere region, because a certain linear combination of the energy-momentum expression between $B, S, T$ gives the same condition.

2 Technical background

Making use a Taylor series expansion, the metric tensor can be written as

$$g_{\alpha\beta}(x) = g_{\alpha\beta}(0) + \partial_\mu g_{\alpha\beta}(0)x^\mu + \frac{1}{2} \partial_\mu^2 g_{\alpha\beta}(0)x^\mu x^\nu + \ldots.$$  \hspace{1cm} (1)

At the origin in Riemann normal coordinates

$$g_{\alpha\beta}(0) = \eta_{\alpha\beta}, \quad \partial_\mu g_{\alpha\beta}(0) = 0,$$

$$-3\partial_\mu^2 g_{\alpha\beta}(0) = R_{\alpha\beta\mu\nu} + R_{\alpha\mu\beta\nu},$$

$$-3\partial_\nu \Gamma_{\mu\alpha\beta} = R_{\mu\alpha\beta\nu} + R_{\mu\beta\alpha\nu}. \hspace{1cm} (3)$$

In vacuum the Bel-Robinson tensor $B$, and tensors $S$ and $T$ are defined as follows

$$B_{\alpha\beta\mu\nu} := R_{\alpha\mu\xi\nu}R_{\beta\xi\kappa} + R_{\alpha\xi\nu\kappa}R_{\beta\xi\mu} - \frac{1}{8}g_{\alpha\beta}g_{\mu\nu}R_{\rho\tau\lambda\sigma}^2,$$  \hspace{1cm} (4)

$$S_{\alpha\beta\mu\nu} := R_{\alpha\mu\xi\nu}R_{\beta\xi\kappa} + R_{\alpha\xi\nu\kappa}R_{\beta\xi\mu} + \frac{1}{4}g_{\alpha\beta}g_{\mu\nu}R_{\rho\tau\lambda\sigma}^2,$$  \hspace{1cm} (5)

$$T_{\alpha\beta\mu\nu} := -\frac{1}{8}(5g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})R_{\rho\tau\lambda\sigma}^2,$$  \hspace{1cm} (6)

where $R_{\lambda\mu\xi\kappa}$ is the Riemann tensor, $R_{\rho\tau\lambda\sigma}^2 = R_{\rho\tau\lambda\sigma}R_{\rho\tau\lambda\sigma}$, Greek letters mean spacetime and the signature is $+2$. The associated known energy-momentum density are

$$B_{\mu\nu} \delta^{ij} = (E_{\alpha\beta}^2 + H_{\alpha\beta}^2, 2\epsilon_{\alpha\beta}\epsilon_{\mu\nu}E^{\alpha\beta}H^{\mu\nu}), \quad S_{\mu\nu} \delta^{ij} = -\frac{2}{3}T_{\mu\nu} \delta^{ij} = -10(E_{\alpha\beta}^2 - H_{\alpha\beta}^2, \vec{0}),$$  \hspace{1cm} (7)

where Latin denotes spatial indices, $E_{\alpha\beta}^2 = E_{\alpha\beta}E^{\alpha\beta}$ and similarly for $H_{\alpha\beta}^2$. The electric part $E_{\alpha\beta}$ and magnetic part $H_{\alpha\beta}$, are defined in terms of the Weyl curvature [19]:

$$E_{\alpha\beta} = E_{\alpha\beta} + E^{\alpha\beta},$$

$$H_{\alpha\beta} = H_{\alpha\beta} - H^{\alpha\beta}.$$
\[ E_{ab} := C_{amn} t^m t^n \quad \text{and} \quad H_{ab} := *C_{amn} t^m t^n, \] where \( t^m \) is the timelike unit vector and \(*C_{\mu \nu \xi \kappa} \) indicates its dual for the evaluation. Here we emphasize that \( B \) is completely trace free which implies \( B_{\mu 00} = B_{\mu \rho \delta \gamma} \delta^\mu_\rho \delta^\rho_\delta \). The energy component of \( B \) in (7) is non-negative definite for all observers, i.e., positivity. Meanwhile, the 4-momentum of \( B \) possesses the future directed non-spacelike property, i.e.,

\[ B_{000} - |B_{000c}| \geq 0. \quad (8) \]

In a small sphere limit, all of them satisfy the divergence free condition: \( \partial_\alpha (\chi^\alpha_\beta_\mu \nu_\alpha_\beta_\mu x^\mu_\nu) \) is vanishing for all \( \chi \in \{ B, S, W \} \). This condition implies the conservation of energy-momentum. In addition, we list the following

\[ R^2_{\alpha \beta \mu \nu} = 8((E^2_{ab} - H^2_{ab})), \quad R_{\alpha \beta \mu \nu} * R^{\alpha \beta \mu \nu} = 16 E_{ab} H^{ab}. \quad (9) \]

There is a linear combination between \( S \) and \( T \)

\[ W_{\alpha \beta \lambda \sigma} := \frac{3}{2} S_{\alpha \beta \lambda \sigma} + T_{\alpha \beta \lambda \sigma}, \quad (10) \]

such that it possesses zero energy-momentum, i.e., \( W_{\mu 000} = (0, 0, 0, 0) \).

We observe that a certain kind of multiplication between \( E_{ab} \) and \( H_{ab} \) can be classified as the inner and cross products. (i) Inner product: The momentum \( E_{ab} H^{ab} \) can be expressed as an analogy of an inner product

\[
E_{ab} H^{ab} = E_{1a} H^{1a} + E_{2a} H^{2a} + E_{3a} H^{3a} = E_{1a} \cdot \vec{H}_{1b} + E_{2a} \cdot \vec{H}_{2b} + E_{3a} \cdot \vec{H}_{3b}
\]

\[ = |\vec{E}_{1a}| |\vec{H}_{1b}| \cos \theta_1 \vec{n}_1 + |\vec{E}_{2a}| |\vec{H}_{2b}| \cos \theta_2 \vec{n}_2 + |\vec{E}_{3a}| |\vec{H}_{3b}| \cos \theta_3 \vec{n}_3. \quad (11) \]

where \( \theta_1 \) is the angle between vectors \( (E_{11}, E_{12}, E_{13}) \) and \( (H_{11}, H_{12}, H_{13}) \); similarly for \( \theta_2 \) and \( \theta_3 \). Here we defined the 3-dimensional vector \( \vec{E}_{1a} = (E_{11}, E_{12}, E_{13}) \) and its norm \( |\vec{E}_{1a}| = \sqrt{E_{1a}^2} \). Similarly for \( \vec{E}_{2a} \) and \( \vec{E}_{3a} \), etc. It is possible for having different representation since it is legitimated transform the basis vectors from \( \vec{i}, \vec{j}, \vec{k} \) to \( \vec{n}_1, \vec{n}_2, \vec{n}_3 \). Here we consider the absolute value

\[ |E_{ab} H^{ab}| = \sqrt{|\vec{E}_{1a}|^2 |\vec{H}_{1b}|^2 \cos^2 \theta_1 + |\vec{E}_{2a}|^2 |\vec{H}_{2b}|^2 \cos^2 \theta_2 + |\vec{E}_{3a}|^2 |\vec{H}_{3b}|^2 \cos^2 \theta_3} \]

\[ \leq |\vec{E}_{1a}| |\vec{H}_{1b}| + |\vec{E}_{2a}| |\vec{H}_{2b}| + |\vec{E}_{3a}| |\vec{H}_{3b}|. \quad (12) \]

(ii) Cross product: Consider another kind of momentum

\[ \epsilon_{cab} E^{ad} H^{b}_d = (E_{2a} H^{3a} - E_{3a} H^{2a}) \vec{i} + (E_{3a} H^{1a} - E_{1a} H^{3a}) \vec{j} + (E_{1a} H^{2a} - E_{2a} H^{1a}) \vec{k} \]

\[ = \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ E_{11} & E_{12} & E_{13} \\ H_{11} & H_{12} & H_{13} \end{array} \right| + \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ E_{21} & E_{22} & E_{23} \\ H_{21} & H_{22} & H_{23} \end{array} \right| + \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ E_{31} & E_{32} & E_{33} \\ H_{31} & H_{32} & H_{33} \end{array} \right| \]

\[ = |\vec{E}_{1a}| |\vec{H}_{1b}| \sin \theta_1 \vec{n}_1 + |\vec{E}_{2a}| |\vec{H}_{2b}| \sin \theta_2 \vec{n}_2 + |\vec{E}_{3a}| |\vec{H}_{3b}| \sin \theta_3 \vec{n}_3, \quad (13) \]

The absolute magnitude for (13) can be manipulated as

\[ |\epsilon_{cab} E^{ad} H^{b}_d| = \sqrt{|\vec{E}_{1a}|^2 |\vec{H}_{1b}|^2 \sin^2 \theta_1 + |\vec{E}_{2a}|^2 |\vec{H}_{2b}|^2 \sin^2 \theta_2 + |\vec{E}_{3a}|^2 |\vec{H}_{3b}|^2 \sin^2 \theta_3} \]

\[ \leq |\vec{E}_{1a}| |\vec{H}_{1b}| + |\vec{E}_{2a}| |\vec{H}_{2b}| + |\vec{E}_{3a}| |\vec{H}_{3b}|. \quad (14) \]
Thus we have an identity

$$|E_{ab}H^{ab}|^2 + |\epsilon_{cab}E^{ad}H^{bd}|^2 = E^2_{1a}H^2_{1b} + E^2_{2a}H^2_{2b} + E^2_{3a}H^2_{3b}. \quad (16)$$

The Bel-Robinson tensor possesses the dominate energy condition which means it satisfies the future directed non-spacelike property automatically. Perhaps, it may need to bear in mind that $E_{ab}$ or $H_{ab}$ can be assigned any value. Here we illustrate the future directed non-spacelike condition for the Bel-Robinson ‘momentum’:

$$B_{0000} - |B_{000c}| = E^2_{ab} + H^2_{ab} - |2\epsilon_{cab}E^{ad}H^{bd}| \geq (|\vec{E}_{1a}|-|\vec{H}_{1b}|)^2 + (|\vec{E}_{2a}|-|\vec{H}_{2b}|)^2 + (|\vec{E}_{3a}|-|\vec{H}_{3b}|)^2 \geq 0, \quad (17)$$

where $|B_{000c}| \leq 2(|\vec{E}_{1a}||\vec{H}_{1b}| + |\vec{E}_{2a}||\vec{H}_{2b}| + |\vec{E}_{3a}||\vec{H}_{3b}|)$ which is indicated in (15). This result demonstrates that how $B$ possesses the expected future directed non-spacelike and indeed there does not exist any extra room to alter this inequality at a first glance. More concretely, it is definitely forbidden for adding any value of $(E^2_{ab} - H^2_{ab})$ or $E_{ab}H^{ab}$. However, we claim there is a way out to make some modification. Consider the following combination

$$E^2_{ab} + H^2_{ab} - |2\epsilon_{cab}E^{ad}H^{bd}| - |k_1||E_{ab} - H_{ab}| - |k_2||E_{ab}H^{ab}| \geq 0. \quad (18)$$

In order to make the above inequality holds, the unique solution is when both constants $k_1$, $k_2$ vanish simultaneously according to Szabados suggested. This is called the pure Bel-Robinson ‘momentum’ requirement. Actually, we are repeating the same argument with Szabados. To the contrary, we use another point of view to examine the difference between the energy and momentum in (17) again

$$\mathcal{E}^2 - |\vec{P}|^2 = (E^2_{ab} + H^2_{ab})^2 - 4(E^2_{1a}H^2_{1b} + E^2_{2a}H^2_{2b} + E^2_{3a}H^2_{3b}) = 2(E^2_{1a}E^2_{2b} + E^2_{1b}E^2_{2a} + E^2_{2a}E^2_{3b} + H^2_{1a}H^2_{2b} + H^2_{1b}H^2_{2a} + H^2_{1a}H^2_{3b} + H^2_{1b}H^2_{3a} + H^2_{2a}H^2_{3b}) + 2 \left[ E^2_{1a}(H^2_{2b} + H^2_{3b}) + E^2_{2a}(H^2_{1b} + H^2_{3b}) + E^2_{3a}(H^2_{1b} + H^2_{2b}) \right] + (E^2_{1a} - H^2_{1a})^2 + (E^2_{2a} - H^2_{2a})^2 + (E^2_{3a} - H^2_{3a})^2 \geq 0, \quad (19)$$

where the momentum $|\vec{P}| \leq 2\sqrt{E^2_{1a}H^2_{1b} + E^2_{2a}H^2_{2b} + E^2_{3a}H^2_{3b}}$ which is depicted in (14). Now adding the terms of $(E^2_{ab} - H^2_{ab})$ or $E_{ab}H^{ab}$ are no longer impossible. It turns out that we found a different result; one that is strictly forbidden according to the conclusion of Szabados’s article. More precisely, what ranges for constants $k_1$ and $k_2$ may be selected such that the future directed non-spacelike qualities can be kept. For this purpose we use the 5-Petrov types Riemann curvatures for the verification [20]. We obtained the new constraints as follows

$$|k_1| \leq 1, \quad 2|k_1| + |k_2| \leq 2. \quad (20)$$

This means the Bel-Robinson ‘momentum’ is not an unique energy-momentum that satisfies the future directed non-spacelike requirement in a small sphere limit.

3 Small sphere limit

In a small sphere limit, we have proposed $V$ that contribute the same pure Bel-Robinson ‘momentum’ as $B$ does [21]. The detail expression is $V = B + W$. Here
we consider another expression, a certain linear combination between $B$ with $S$, $T$ or $W$ in a small sphere limit. Note that, though the energy-momentum for $W$ are vanishing, it is not zero for the other components within this mentioned region. For example $W_{0011} = 2(E_{a b}^2 - H_{a b}^2 - 3E_{1 a}^2 + 3H_{1 a}^2)$.

Case (i): Consider a simple energy-momentum integral such that within a small sphere limit, we consider a linear combination between $B$ and $S$:

$$B_{a \beta \lambda \sigma} + a_1 S_{a \beta \lambda \sigma}, \quad (21)$$

where $a_1$ is a real number. For constant time $t_0 = 0$, the energy-momentum in vacuum with radius $r$

$$2\kappa \mathcal{P}_\mu = \int_{t_0} (B^0_{\mu ij} + a_1 S^0_{\mu ij}) x^i x^j d^3 x = -\frac{4\pi}{15} r^5 (B_{\mu 000} + a_1 S_{\mu 0 ij} \delta^{ij}), \quad (22)$$

where $\kappa = 8\pi G/c^4$, $G$ is the Newtonian constant and $c$ the speed of light. According to Szabados, the only possibility is when $a_1 = 0$ that satisfies the positivity, future pointing and non-spacelike properties [7]. Explicitly, the pure Bel-Robinson ‘momentum’.

However, we claim that there exists some non-vanishing $a_1$ such that these future directed non-spacelike property is still preserved. Referring to (7), we only vary the energy and without affecting the momentum. Consequently, the energy-momentum for (22) becomes

$$(\mathcal{E}, \mathcal{P}_c) = \frac{2\pi}{15\kappa} r^5 \left[ (E_{a b}^2 + H_{a b}^2) - 10a_1(E_{a b}^2 - H_{a b}^2), 2\epsilon_{a c b}E_d H_{a d} \right]. \quad (23)$$

Generally, the values of $E_{a b}$ and $H_{a b}$ can be arbitrary at a given point, the sign of the energy component of $S$ is uncertain and obviously they should affect the future directed non-spacelike property. Previously, our achievement preferred a multiple of pure Bel-Robinson ‘momentum’ in a small sphere region, and we confidently sure that the result in (23) required $a_1$ vanishes [21]. Nevertheless, we found a certain linear combinations of $B$ and $S$ are legitimate. Referring to (13), we change another angle of view for the comparison

$$\mathcal{E}^2 - |\mathcal{P}|^2 \geq \left[ (E_{a b}^2 + H_{a b}^2) - 10a_1(E_{a b}^2 - H_{a b}^2) \right]^2 - 4(E_{1 a 1 b}^2 + E_{2 a 2 b}^2 + E_{3 a 3 b}^2) - 4 \geq 0, \quad (24)$$

provided that $|a_1| \leq \frac{1}{40}$. Meanwhile, the examination using the 5-Petrov types Riemann curvatures verification serve a more precise value, i.e., $|a_1| \leq \frac{1}{10}$. Here we give a remark: previously we believed both Einstein $t_{a \beta}^E$ and Landau-Lifshitz $t_{a \beta}^{LL}$ pseudo-tensors could not pass the future directed non-spacelike requirement in Riemann normal coordinates [31 [22]:

$$t_{a \beta}^E = \frac{2}{9} \left( B_{a \beta \xi \kappa} - \frac{1}{4} S_{a \beta \xi \kappa} \right) x^\xi x^\kappa, \quad t_{a \beta}^{LL} = \frac{7}{18} \left( B_{a \beta \xi \kappa} + \frac{1}{4} S_{a \beta \xi \kappa} \right) x^\xi x^\kappa. \quad (25)$$

This illustration shows that we are mistaken in the past. Now, the energy-momentum of the Landau-Lifshitz pseudo-tensor (corresponding $|a_1| = \frac{1}{14} < \frac{1}{10}$) is a suitable candidate for fulfilling the future directed non-spacelike requirement. While Einstein pseudo-tensor still failed (associated $|a_1| = \frac{1}{4} > \frac{1}{10}$).

Case (ii): Likewise, replace $S$ by $T$ which is indicated in (6), the combination becomes $B + a_2 T$. The future directed non-spacelike property requires the constant $|a_2| \leq \frac{1}{60}$ for a general comparison. The more precise value, using the 5-Petrov types
Riemann curvatures examination, requires $|a_2| \leq \frac{1}{10}$. Summing up our present result and the previous one, one can simply eliminate away this extra energy and obtain the pure Bel-Robinson ‘momentum’. This means there exists a linear combination, $S + \frac{2}{3} T$ which is denoted in (7) and (10), contributes vanishing energy-momentum.

Case (iii): Dominate energy condition in a small sphere limit. Consider the energy-momentum stress in static

$$t_{\alpha \beta} = \int t_{\alpha \beta ij} x^i x^j d^3 x.$$  

(26)

Suppose the energy $t_{00} = \int t_{00 ij} x^i x^j d^3 x$ is positive definite. The requirement for the dominate energy condition in a small sphere limit is $t_{00} \geq |t_{\alpha \beta}|$ for all $\alpha, \beta$. Here we consider the following combination

$$B_{\alpha \beta \lambda \sigma} + s_1 S_{\alpha \beta \lambda \sigma} + s_2 W_{\alpha \beta \lambda \sigma},$$

(27)

where $s_1, s_2$ are constants. The energy is

$$\int (B_{00 ij} + s_1 S_{00 ij} + s_2 W_{00 ij}) x^i x^j d^3 x \geq 0,$$

(28)

provided that $|s_1| \leq \frac{1}{10}$ and the similar manipulation can be found in (22). For a direct comparison, let $s_2 = 10 s_1$ and without using the 5-Petrov types Riemann curvatures:

$$\int [B_{00 ij} + s_1 (S_{00 ij} + 10 W_{00 ij})] x^i x^j d^3 x \geq \int [B_{0i \beta j} + s_1 (S_{0 \beta ij} + 10 W_{0 \beta ij})] x^i x^j d^3 x \geq 0,$$

(29)

is hold provided $|s_1| \leq \frac{1}{10}$ for all $\alpha, \beta$. Hence, we have a simple combination that satisfies the dominate energy condition in a small sphere limit. For the completeness, as this combination expression possesses the dominate energy condition, it must guarantee the future directed non-spacelike property is hold. We verify this through the following comparing

$$\int [B_{00 ij} + s_1 (S_{00 ij} + 10 W_{00 ij})] x^i x^j d^3 x \geq \int [B_{0c ij} + s_1 (S_{0c ij} + 10 W_{0c ij})] x^i x^j d^3 x \geq 0,$$

(30)

and indeed it is hold when $|s_1| \leq \frac{4}{40}$. The same explanation can be found after (24).

4 Small ellipsoid

Instead of integrate the energy-momentum in a small sphere limit, one can consider small ellipsoid. One of the natural options is the Jupiter-Io system, Jupiter deforms Io from sphere to ellipsoid through the tidal force and vice versa. Consider a simple dimension $(a, b, c) = (\sqrt{1 + \Delta}, 1, 1) a_0$ for non-zero $\Delta > -1$ and $a_0$ finite. In reality, it is slightly deformed and it suits the quasi-local small 2-surface limit. The physical dimension for Io is $(x, y, z) = (3660.0, 3637.4, 3630.6)$ in kilometer. Using our notation: $a = \sqrt{1 + \Delta} a_0$, $b \simeq c \simeq a_0$, where the mean radius $a_0 = 1817$ km and $\Delta = 0.0144$. This kind of deformation is called spheroid. For constant time $t_0 = 0$, the corresponding 4-momentum are

$$2 \kappa P_{\mu} = \int_{t_0} t_{\mu ij} x^i x^j d^3 x = \frac{4 \pi}{15} a_0^5 \left[ t_{\mu 00} + \Delta t^0_{\mu 11} \right] \sqrt{1 + \Delta},$$

(31)
where \( t^0_{\mu ij} \) can be replaced by \( B \) or \( S \). It may be worthwhile to check what is the energy different in a small sphere and ellipsoid limits. In other words, is there any energy change from a small sphere deforms to ellipsoid? Here we use the Schwarzschild metric in spherical coordinates (see §31.2 in [23]) for a simple test:

\[
ds^2 = -(1 - 2Mr) \, dt^2 + (1 - 2Mr)^{-1} \, dr^2 + r^2 (dr^2 + \sin^2 \theta \, d\phi^2),
\]

with the assumption that \( Mr^{-1} \ll 1 \), both the gravitational constant \( G \) and speed of light \( c \) are unity. Certainly, there is no momentum since we are dealing with a static spacetime. The non-vanishing Riemann curvatures are

\[
R_{t\theta\theta\theta} = -R_{\phi\theta\phi\theta} = -2Mr^{-3}
\]

and \( R_{t\theta\theta\phi} = R_{t\phi\theta\phi} = -R_{t\phi\phi\phi} = Mr^{-3} \). The energy for \( B \) and \( S \) are

\[
(B_{0000}, B_{0011}, S_{0000}, S_{0011}) = (6, -2, 12, -28) \frac{M^2}{r^6},
\]

where the value of the Kretschmann scalar \( R_{3\lambda\mu\rho}^2 = 48M^2r^{-6} \).

Case (1): Referring to (31), replace \( t \) by \( B \), the energy-momentum are

\[
2\kappa \mathcal{P}_\mu = \frac{4\pi}{15} a_0^5 \left[ B^0_{\mu 00} + \Delta B^0_{\mu 11} \right] \sqrt{1 + \Delta}.
\]

where

\[
B_{0011} = E_{ab}^2 + H_{ab}^2 - 2E_{1a}^2 - 2H_{1a}^2, \quad B_{0c11} = 2\epsilon_{cab}(E^{ad}H_{bd}^b - 2E_{1a}^aH_{1b}^b)
\]

This result alter the energy and momentum of \( B_{\mu 000} \) simultaneously, i.e., making it analogous with (20): \( k_1 \neq 0 \neq k_2 \). Verify the following quantities

\[
\mathcal{E} - |\vec{\mathcal{P}}| = B_{0000} + \Delta B_{0011} - |B_{000c} + \Delta B_{0c11}|.
\]

Using the 5-Petrov types Riemann curvatures to compare the energy and momentum in (36), the future directed non-spacelike condition requires \( \Delta \in (-1, 3) \). Here we check the energy different, from a small sphere deformed to ellipsoid, for the Jupiter-Io system:

\[
\mathcal{E}_{\text{sphere}} = 6 \left( \frac{4\pi M^2 a_0^5}{15r^6} \right), \quad \mathcal{E}_{\text{ellipsoid}} = 6.01 \left( \frac{4\pi M^2 a_0^5}{15r^6} \right)
\]

where we have used \( \Delta = 0.0144 \). These data indicated that the small ellipsoid absorbs more energy than sphere.

Case (2): According to (31), replace \( t \) by \( B + S \), the energy-momentum become

\[
2\kappa \mathcal{P}_\mu = \frac{4\pi}{15} \left[ (B^0_{\mu ij} + sS^0_{\mu ij})\delta^{ij} + \Delta(B^0_{\mu 11} + sS^0_{\mu 11}) \right] a_0^5 \sqrt{1 + \Delta},
\]

where \( s \) is a constant. The energy-momentum for \( S_{0\mu 11} \)

\[
S_{0011} = -2(E_{ab}^2 - H_{ab}^2 + 2E_{1a}^2 - 2H_{1a}^2),
\]

\[
S_{0c11} = 4(0, E_{1a}H_{3a}^a + E_{3a}H_{1a}^a, -E_{1a}H_{2a}^a - E_{2a}H_{1a}^a).
\]

Based on (25) for \( s = \frac{1}{14} \) which means we choose the Landau-Lifshitz pseudo-tensor as an illustration. Using the 5-Petrov types Riemann curvatures for the verification,
we discovered that when \( \Delta \in \left[-\frac{1}{3}, \frac{1}{5}\right] \) satisfies the future directed non-spacelike requirement. Using the similar technique in Case (1) to check the energy difference, from small sphere deformed to ellipsoid. Here we focus on the Jupiter-Io system as a simple demonstration

\[
E_{\text{sphere}} = 1.71 \left( \frac{4\pi M^2 a_0^5}{15r^6} \right), \quad E_{\text{ellipsoid}} = 1.67 \left( \frac{4\pi M^2 a_0^5}{15r^6} \right)
\]  

(41)

where we have substituted \( \Delta = 0.0144 \) again. These data indicated that the small ellipsoid absorbs less energy than sphere. Naively, apply to the Jupiter-Io tidal heating system using the Landau-Lifshitz pseudo-tensor [17], one may interpret that Io releases energy away when changing the shape from sphere to ellipsoid. Meanwhile, Io absorbs more energy when deforming the shape from ellipsoid to sphere. Eventually, Io does not gain or lose any energy after a complete deformation cycle. This indicates that the interior of Io is in thermal equilibrium [24]. Perhaps, this simple argument might help a little understanding of the real physical situation of Io.

5 Conclusion

To describe the positive quasi-local energy-momentum expression, the Bel-Robinson tensor \( B \) is the most ideal candidate because it gives the Bel-Robinson ‘momentum’ in a small sphere region. In the past, it seems that only this Bel-Robinson ‘momentum’ can manage this specific task: non-spacelike and future pointing. That particular restriction could not allow even a small amount of energy-momentum to be subtracted from this Bel-Robinson ‘momentum’. After some comparison, with or without the 5-Petrov types Riemann curvatures, we discovered that the Bel-Robinson ‘momentum’ implies future directed non-spacelike properties; but the converse is not true. In other words, the Bel-Robinson ‘momentum’ is no longer the unique option for achieving the future pointing non-spacelike requirement. Explicitly, the Bel-Robinson tensor lost its privilege. For example, we thought the Landau-Lifshitz pseudo-tensor was failed meet the future directed non-spacelike requirement, but now we find that it can. Furthermore, we constructed a linear combination, \( B \) with other tensors \( S \) and \( T \), gives the dominate energy condition in a small sphere limit.

Besides the Bel-Robinson tensor, there exists a certain relaxation freedom such that one can still obtain the energy-momentum expression contributes the future directed non-spacelike property. For example, \( B + sS \) in a small sphere and ellipsoid regions. This comparison, in some sense, reflects the reality for the Jupiter-Io system tidal heating through the shape changing of Io. More precisely, the interior of Io is in the thermal equilibrium.

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