Relationship between the Prime form and the sigma function for some cyclic $(r,s)$ curves.

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Abstract. In this article, we study some cyclic $(r, s)$ curves $X$ given by

$$y^r = x^s + \lambda_1 x^{s-1} + \cdots + \lambda_{s-1} x + \lambda_s.$$ 

We give an expression for the prime form $\mathcal{E}(P, Q)$, where $(P, Q \in X)$, in terms of the sigma function for some such curves, specifically any hyperelliptic curve $(r, s) = (2, 2g+1)$ as well as the cyclic trigonal curve $(r, s) = (3, 4)$,

$$\mathcal{E}(P, Q) = \frac{\sigma_{\sharp_r}(u - v)}{\sqrt{du_1\sqrt{dv_1}},}$$

where $\sharp_r$ is a certain multi-index of differentials. Here $u_1$ and $v_1$ are respectively the first components of $u = w(P)$ and $v = w(Q)$ which are given by the Abel map $w : X \rightarrow \mathbb{C}^g$, where $g$ is the genus of $X$. These explicit formulae are useful in applications, for instance to the problem of constructing classes of Schwarz-Christoffel maps to slit domains.

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$\sigma$-function, prime form
1. Introduction

The Prime form is the fundamental \((-1/2, -1/2)\)-form characterising a compact Riemann surface, which governs its geometrical properties; let us consider a compact Riemann surface \(X\) and its universal covering \(\tilde{X}\). The prime form \(E(P, Q)\) for a pair of points \(P, Q \in \tilde{X}\) is defined by

\[
E(P, Q) := \frac{\theta \left[ \alpha' \alpha'' \right] (\int_{Q}^{P} \nu^I) \sqrt{\zeta(P)} \sqrt{\zeta(Q)}}{\sqrt{\zeta(P)} \sqrt{\zeta(Q)}},
\]

where \([\alpha'\alpha'']\) are a fixed nonsingular odd theta characteristic, \(\theta[\delta](z)\) is a Riemann theta function and \(\zeta(P)\) is

\[
\zeta(P) := \sum_{i=1}^{g} \left( \partial_i \theta \left[ \alpha' \alpha'' \right](0) \right) \hat{\nu}^I_i(P),
\]

where \(\hat{\nu}^I_i(P)\) is the vector of normalized holomorphic one forms over \(X\). Nakayashiki gives an alternative normalisation of the prime form such that

\[
E(P, Q) := e^{t(w(P) - w(Q))\gamma(w(P) - w(Q))} E(P, Q)
\]

where \(w\) is the Abel map, and \(\gamma\) is a certain constant matrix for the curve, which will be specified below.

Fay investigated the relations between the Riemann theta function and the prime form, and found the so-called Fay trisecant formula \([F]\). Nakayashiki has also showed that the Kleinian sigma function for an \((r, s)\) curve is also expressed explicitly in terms of the prime form \([N2]\).

In this article, we express the prime form, in Nakayashiki’s normalisation, in terms of the sigma function, for every hyperelliptic \((2, 2g + 1)\) curve as well as for a cyclic \((3, 4)\) trigonal curve; this was alluded to in the Concluding Remark in \([N1]\).

Recently Nakayashiki \([N2]\), as well as Enolskii, Eilbeck and Gibbons \([EEG]\) have given the characterization of the sigma function in terms of the tau function for an algebraic curve. Since, due to the definition of sigma function in terms of the theta function, we have transformation formulae among theta functions, tau functions and sigma functions, they are basically the same functions. However the sigma and the tau functions can be written in terms of Schur functions, and both are connected to the Sato theory \([M, N2, EEG]\). Further, unlike the theta functions, the sigma functions, as well as Nakayashiki’s definition of the Prime form, are modular invariant for action of \(\text{Sp}(g, \mathbb{Z})\).

Moreover, we will note that as a characterization of sigma, the sigma function could be said to be “algebraic” due to the studies \([B1, B2, B3, BLE1, BLE2, EEG, EEL, EEMOP, KMP, M, MP1, MP2, O1, O2]\). In particular, in contrast to the theta functions, the Taylor series of the sigma function about the origin has coefficients which are \textit{polynomials} in the parameters of the curve.
Historically, the original version of the sigma function was introduced by Weierstrass in order to express symmetric functions on an affine curve associated with a Riemann surface. Klein developed this idea further, to provide the data of the affine coordinate via the Jacobi inversion formulae, generalising the following property of the Weierstrass $\sigma(u)$ function; the genus one $\sigma(u)$ function gives $\varphi(u) = -d^2 \log \sigma(u)/du^2$ and $\varphi(u)' = d\varphi(u)/du$ which recover the differential equation $(\varphi(u)')^2 = 4\varphi(u)^3 + g_2\varphi(u) + g_3$. In other words, by letting $y = 2\varphi(u)'$ and $x = \varphi(u) - c$ with a certain constant $c$, this differential equation is identified with the equation of the affine curve:

$$y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0.$$ 

As in [EEMOP], the addition formulae for the hyperelliptic curves in terms of sigma functions are given by the meromorphic functions on the affine hyperelliptic curve, and they generalise the genus one addition formula:

$$\frac{\sigma(u + v)\sigma(u - v)}{\sigma(u)^2\sigma(v)^2} = \varphi(u) - \varphi(v).$$ (1.1)

Since for genus $g > 1$ case, the dimension of the Jacobi variety differs from the dimension of the curve itself, some properties of the sigma function of genus one cannot survive unchanged in the case of higher genus curves.

However recently we have discovered that for some cyclic $(r,s)$ curves, $X$, given by equations of the form:

$$y^r = x^s + \lambda_1 x^{s-1} + \cdots + \lambda_{s-1} x + \lambda_s,$$

certain higher derivatives of the sigma function provide natural generalisations of important properties of the sigma function of genus one.

(i) The generalized Frobenius-Stickelberger relation in Propositions 6.1 and 6.2 shows a simple connection between the affine coordinate ring $R_g$ and the coordinate of the Jacobi variety $J_g$; this is a natural generalization of a relation found in the elliptic case by Frobenius and Stickelberger [FS].

(ii) As the Jacobi inversion formula, [MP2] shows, $x = -\sigma_{g+1}^{-1}(u)/\sigma_g(u)$ for any $(r,s)$ curve, while further, we have $y = (dx/du_1)/2$ for a hyperelliptic $(2,2g + 1)$ curve, or $y = (dx/du_2)/3$ for a $(3,4)$-curve; these can be understood as direct generalizations of the above genus one relation.

(iii) Further, as in Propositions 7.9 and 7.21, we have a more direct generalization of (1.1). Hence we believe that it is very important to rewrite the formulae of a Riemann surface in terms of sigma functions because they are more directly connected with algebraic properties of the affine ring $R_g$.

‡ The formula which we call the Frobenius-Stickelberger relation seems to have been essentially discovered by Hermite (See [WW, p.458]). However, the paper [FS] is the first which is explicitly written in terms of the $\sigma$ function, and we use its authors’ names.
The paper is organised as follows. In the next section, we will review the theory of cyclic \((r,s)\) curves, introducing the key ideas and fixing our notation. Section 3 generalises Mumford’s construction of meromorphic functions on hyperelliptic curves to general cyclic \((r,s)\) curves; here we introduce the Frobenius-Stickelberger matrices and their determinants. Section 4 reviews the \(\sigma\)-function and its properties. The following section is concerned in particular with the vanishing properties of \(\sigma\) and certain of its derivatives. From these results, it is easy to restate the generalisation of the Frobenius-Stickelberger relation to general hyperelliptic curves, as well as the simplest cyclic trigonal curves, which is done in section 6. In section 7, the Prime form is introduced; the Frobenius-Stickelberger results are then used to give a simple expression for this in terms of derivatives of \(\sigma\). One of the motivations of this study was to permit the explicit construction of Schwartz-Christoffel maps giving formulae for certain reductions of the Benney hierarchy \([BG1, BG2, G]\); we will discuss this briefly in the last section.

Recently Nakayashiki \([N3, NY]\) has also independently generalized the expression of the prime form in terms of the sigma function to every \((r,s)\) curve, though his proof, and the form of his results, differ in detail from those given here. As the general \((r,s)\) curve lacks the cyclic symmetry of the curves studied here, the results in the cyclic case can be expressed in a different, and in some ways simpler, form; our results explicitly use this symmetry. As the results for cyclic curves have applications to problems such as conformal mappings and the reductions of Benney’s equations, it is worth stating these separately.

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2. Cyclic \((r,s)\) curves

We briefly summarise the conventions and notations used in this article. Defining a Riemann surface as in \([ACGH\text{ p.31}]\), we consider a cyclic \((r,s)\) Riemann surface

\[
X := \{(x, y) \mid y^r = f(x)\} \cup \infty
\]

whose finite part is given by an affine equation

\[
y^r = f(x), \quad f(x) := x^s + \lambda_1 x^{s-1} + \cdots + \lambda_{s-1} x + \lambda_s.
\]  

(2.1)

The positive integers \(r\) and \(s\) are coprime, with \(r < s\); the complex numbers \(\lambda_1, \ldots, \lambda_s\) are such that the finite part of \(X\) is smooth and has geometric genus \(g = \frac{(r-1)(s-1)}{2}\). In this article, we consider all hyperelliptic curves, \((r,s) = (2, 2g + 1)\) as well as the cyclic trigonal curve \((r,s) = (3,4)\).
Let \( R_g := \mathbb{C}[x,y]/(y^r - f(x)) \), \( \mathcal{O}_X \) be the sheaf of holomorphic functions over \( X \) and \( \mathcal{J} \) the Jacobian of \( X \). \( R_g = \mathcal{O}_X(*\infty) \) is the ring of meromorphic functions on \( X \) regular outside \( \infty \).

We have the (monic) monomial \( \phi_n \in R_g \) for a non-negative integer \( n \) so that it has order of pole \( N(n) \) at \( \infty \), the \( n \)-th integer in the (increasing) sequence complementary to the Weierstrass gaps: \( \phi_0 = 1, \phi_1 = x, \) etc.; by letting \( t_\infty \) be a local parameter at \( \infty \), the leading term of \( \phi_n \) is proportional to \( t_\infty^{-N(n)} \). By elementary computations, we see that \( N(0) = 0, N(g-1) = 2g - 2, N(g) = 2g \) for \( (r,s) \) curve. Let \( s_n \) and \( r_n \) be such that \( \phi_n = x^{s_n}y^{r_n} \) and then \( N(n) = s_nr + sr_n \).

We introduce the w-degree, \( \deg_w : R_g \to \mathbb{Z} \), which assigns to any element of \( R_g \) the order of its pole at \( \infty \), \( \deg_w(x) = r, \deg_w(y) = s, \deg_w(\phi_n(P)) = N(n) \). Further we also consider the ring \( R_{g,\lambda} := \mathbb{Q}[x,y,\lambda_0,\ldots,\lambda_{s-1}]/(y^r - f(x)) \) by regarding \( \lambda \)'s as indeterminates, and define a \( \lambda \)-degree, \( \deg_{\lambda} : R_{g,\lambda} \to \mathbb{Z} \) as an extension of the w-degree by assigning the degree \( jr \) to each \( \lambda_j \). The defining polynomial \( y^r - f(x) \) of the curve is homogeneous with respect to the \( \lambda \)-degree.

For later convenience, we sometimes denote a point \( P \in X \setminus \infty \) by its affine coordinates \((x,y)\). For the \( k \)-th symmetric product of the curve \( S^k(X) \), its element is sometimes expressed by \((P_1,\ldots,P_k)\) and a divisor \( D = \sum_{i=1}^k P_i \).

For a local parameter \( t \) at \( P \) in \( X \), we denote by \( d_{\geq}(t^\ell) \) the terms of the \( t \)-expansion of a function on \( X \) whose orders of zero at \( P \) are greater than \( \ell \).

For the canonical bundle \( K_X \) a basis \( \{\nu_1^\prime,\ldots,\nu_g^\prime\} \) of \( H^0(X,K_X) \) is given in terms of the \( \phi_i \) following [B1, Ch. VI, §91],

\[
\nu_i^\prime := \frac{\phi_{i-1}(P)dx}{ry^{r-1}}, \quad (i = 1, \ldots, g).
\] (2.2)

For a hyperelliptic curve, these are

\[
\nu_1^\prime = \frac{dx}{2y}, \quad \nu_2^\prime = \frac{x\,dx}{2y}, \quad \ldots, \quad \nu_g^\prime = \frac{x^{g-1}\,dx}{2y},
\] (2.3)

and for the (3, 4) curve case,

\[
\nu_1^\prime = \frac{dx}{3y^2}, \quad \nu_2^\prime = \frac{x\,dx}{3y^2}, \quad \nu_3^\prime = \frac{dx}{3y}.
\] (2.4)

For a one-form \( \nu \) which is expressed by \( (t_\infty^n + d_{>}(t_\infty^n))dx) \) in terms of the local parameter \( t_\infty \) at \( \infty \), we extend the w-degree to one-forms \( \deg_w(\nu) = -n \). Since

\[
\nu_i^\prime = t_\infty^{2g-N(i-1)-2}(1 + d_{>}(t_\infty^n))dt_\infty,
\] (2.5)

we have

\[
\deg_w(\nu_i^\prime) = 2g - N(i-1) - 2,
\]

where \( \deg_{w}(f) = -\deg_{w}(f) \).

We take a homology basis \( \alpha_i, \beta_j \) \((1 \leq i, j \leq g)\) of \( H_1(X,\mathbb{Z}) \) with intersection pairing \( \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0, \alpha_i \cdot \beta_j = \delta_{ij} \). We have the half period matrices,

\[
[\omega' \omega'''] = \frac{1}{2} \left[ \int_{\alpha_i} \nu_i^\prime \int_{\beta_j} \nu_j^\prime \right]_{i,j=1,2,\ldots,g},
\] (2.6)
and the period lattice $\Pi$ generated by $[2\omega', 2\omega'']$. Since the Jacobian $J$ of $X$ is given by $J = \mathbb{C}^g/\Pi$, we have a natural projection $\kappa: \mathbb{C}^g \to J$.

For a positive integer $k$, the Abel map $w: S^k(X) \to \mathbb{C}^g$ is defined by $w: (P_1, \ldots, P_k) \mapsto w(P_1, \ldots, P_k) = \sum_{i=1}^k \int_{P_i}^{P} \nu^I I \in \mathbb{C}^g$ with base-point $\infty$:

$$W^k := \kappa w(S^k(X)) \subset J.$$

(2.7)

We note that the Abel map $w$ has a $\Pi$-ambiguity coming from the choice of path of integration.

For later convenience, we also introduce the normalized periodic matrix $[1 T]$ with $T := \omega'^{-1}\omega''$, the normalized one form $\hat{\nu}^I := \omega'^{-1}\nu^I$, the normalized period lattice $\hat{\Pi}$ generated by $[1 T]$, the normalized Jacobian $\hat{J} := \mathbb{C}^g/\hat{\Pi}$, the natural projection $\hat{\kappa}: \mathbb{C}^g \to \hat{J}$, and the normalized Abel map $\hat{w}(P) := \omega'^{-1}w(P)$. Since $X$ is a non-singular curve, $\omega'$ is a non-singular matrix, and hence the correspondence between $\hat{J}$ and $J$ is a bijection.

We also introduce the singular locus,

$$S^m_n(X) := \{D \in S^n(X) \mid \dim|D| \geq m\},$$

where $|D|$ is the complete linear system $w^{-1}(w(D))$ [ACGH IV.1]. If $n < g$, the singular locus of $S^n(X)$ modulo linear equivalence, or on projecting to the Picard group, is $S^m_1(X)$ [ACGH Ch. IV, Proposition 4.2, Corollary 4.5, where our $S^n(X)$ is $C^0_n$]. We let $W^m_n := \kappa w(S^m_n(X))$.

Since the sigma function is connected with the $\tau$ function in Sato theory due to [N1, EEG] and the $\tau$ function is written for a Young diagram, we construct a Young diagram (cf., e.g., [S, BLE1]) $\Lambda$ from the Weierstrass gap sequence: from the top down, $1 \leq i \leq g$, the rows have length:

$$\Lambda_i = N(g) - N(i - 1) - g + i - 1 = g - N(i - 1) + (i - 1).$$

We define $\Lambda_j \equiv 0$ for all $j > g$. It is known that for any $(r, s)$ curve, we have [S, BLE1]

$$|\Lambda| = \sum_{i=1}^g \Lambda_i = \frac{1}{24}(r^2 - 1)(s^2 - 1).$$

We give two examples: For the case $(r, s) = (2, 9)$ (Table 2.1), and the case $(r, s) = (3, 4)$ (Table 2.2), we have

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $\phi(i)$ | 1 | $x$ | $x^2$ | $x^3$ | $y$ | $x^4$ | $xy$ | $x^5$ |
| $N(i)$ | 0 | 2 | 4 | 6 | 8 | 9 | 10 | 11 |
| $\Lambda_i$ | -4 | 3 | 2 | 1 | - | - | - | - |
For a given Young diagram $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_\ell)$, the length $r$ of the diagonal is called the \textit{rank} of the partition [FH, §4.1, p. 51]. Let $a_i$ and $b_i$ be the number of boxes below and to the right of the $i$-th box of the diagonal, reading from lower right to upper left. Frobenius called $(a_1, \ldots, a_r; b_1, \ldots, b_r)$ the \textit{characteristics of the partition} [FH, §4.1 p. 51]. Here $a_i < a_j$ and $b_i < b_j$ for $i < j$.

**Lemma 2.1.** For $u \in w(P)$, $P \in X$, 

$$\deg_{w^{-1}}(u_i) = N(g) - N(i - 1) - 1 = 2g - N(i - 1) - 1 = \Lambda_i + g - i.$$ 

Here we also extend the degree to $v = w(P_1, \ldots, P_k)$ for points $P_i \in X$ $(i = 1, \ldots, k)$ by $\deg_{w^{-1}}(v_i) = \deg_{w^{-1}}(u_i)$ since by letting each $P_j$ be expressed as a local parameter $t_{\infty,j}$ around $\infty$, we have, formally:

$$v_i = \frac{1}{2g - N(i - 1) - 1} \left( t_{\infty,1}^{2g-N(i-1)-1} + \cdots + t_{\infty,k}^{2g-N(i-1)-1} \right) (1 + d_{>0}(t_{\infty})). \tag{2.8}$$

Here we use the multi-index convention for $\alpha := (\alpha_1, \ldots, \alpha_g)$,

$$t^{\alpha} := t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_g^{\alpha_g}, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_g), \quad |\alpha| := \sum_{i=1}^g \alpha_i,$$

and extend the definition $d_>(t^g)$ to the variables, $t_1, \ldots, t_g$: $d_>(t^g) \in \{\sum_{|\alpha| > \ell} a_\alpha t^\alpha\}$.

It should be noted that $v_g = (t_{\infty,1} + \cdots + t_{\infty,k}) (1 + d_{>0}(t_{\infty}))$ and $\deg_{w^{-1}}(v_i) = 1$.

By Serre duality, $\dim H^0(X, D) = \deg D - g + 1 - \dim H^0(X, (2g - 2)\infty - D)$, we find that the Young diagram is symmetric; the characteristics $a_i = b_i$. We also have the fact that $\deg_{w^{-1}}(u_i) = \Lambda_i + g - i$ is the hooklength (cf. [S Ch. 3]) of the node $(1, i)$ in the Young diagram $\Lambda$.

For later convenience, we introduce ‘truncated Young diagrams’

$$\Lambda^{(k)} := (\Lambda_1, \ldots, \Lambda_k), \quad \Lambda^{[k]} := (\Lambda_{k+1}, \ldots, \Lambda_g). \tag{2.9}$$

**Remark 2.2.** \textbf{Galois action on $X$:} Since there is an action of the cyclic group $\mathbb{Z}_r$ on the curve $X$ such that for $\zeta \in \mathbb{Z}_r$, $\zeta^r : (x, y) \mapsto (x, \zeta^r y)$ for a fixed primitive $r$-th root $\zeta$ of 1. Naturally we have the induced actions: $\zeta^r \phi_n = \zeta^{r_n} \phi_n$, $\zeta^r \nu_n^I = \zeta^{r_n-1+r+1} \nu_n^I = \zeta^{r_n+1} \nu_n^I$ and for $u \in W^k$, $\zeta^r u_n = \zeta_r^{r_n-1} u_n$.

For example, for the case of the $(3,4)$ curve, $(r_0, r_1, r_2) = (0, 0, 1)$ and hence $\zeta^3 u_1 = \zeta^3 u_1$, $\zeta^3 u_2 = \zeta^3 u_2$, $\zeta^3 u_3 = \zeta^3 u_3$. 

| $i$  | 0 | 1 | 2 | 3 |
|------|---|---|---|---|
| $\phi(i)$ | 1 | $x$ | $y$ | $x^2$ |
| $N(i)$ | 0 | 3 | 4 | 6 |
| $\Lambda_i$ | - | 3 | 1 | 1 |
3. Meromorphic function over \((r, s)\) curve

In [MP1], we introduced meromorphic functions on the curve, as a generalization of the polynomial \(U\) in Mumford’s \((U, V, W)\) parameterization of a hyperelliptic Jacobian [Mu Ch. IIIa].

We introduce the Frobenius-Stickelberger (FS) matrix and its determinant following [MP1]. Let \(n\) be a positive integer and \(P_1, \ldots, P_n\) be in \(X\backslash \infty\). We define the \(\ell\)-reduced Frobenius-Stickelberger (FS) matrix by:

\[
\Psi_{i}^{(\ell)}(P_1, P_2, \ldots, P_n) := \begin{pmatrix}
1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_{\ell}(P_1) & \cdots & \phi_n(P_1) \\
1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_{\ell}(P_2) & \cdots & \phi_n(P_2) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \phi_1(P_n) & \phi_2(P_n) & \cdots & \phi_{\ell}(P_n) & \cdots & \phi_n(P_n)
\end{pmatrix},
\]

and \(\psi_{i}^{(\ell)}(P_1, P_2, \ldots, P_n) := |\Psi_{i}^{(\ell)}(P_1, P_2, \ldots, P_n)|\) (a check on top of a letter signifies deletion). It is also convenient to introduce the simpler notation:

\[
\psi_n(P_1, \ldots, P_n) := |\Psi_{1}^{(\ell)}(P_1, \ldots, P_n)|, \quad \Psi_n(P_1, \ldots, P_n) := \Psi_{1}^{(\ell)}(P_1, \ldots, P_n), \quad (3.1)
\]

for the un-bordered matrix. We call this matrix the Frobenius-Stickelberger (FS) matrix and its determinant the Frobenius-Stickelberger (FS) determinant. These become undefined for some tuples in \((X\backslash \infty)^n\).

Further for later convenience, we will define

\[
\Phi_{i}^{(\ell)}(P_1, P_2, \ldots, P_n) := \begin{pmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^{n-1}
\end{pmatrix},
\]

and \(\varphi_n(P_1, \ldots, P_n) := |\Phi_{1}^{(\ell)}(P_1, \ldots, P_n)|\).

For \(n\) points \((P_i)_{i=1, \ldots, n} \in X \backslash \infty\), we find an element \(\mu_n(P) := \mu_n(P; P_1, \ldots, P_n) := \sum_{i=0}^n a_i \phi_i(P)\) of \(R_g\) associated with a point \(P = (x, y)\) in \((X\backslash \infty)\), \(a_i \in \mathbb{C}\) and \(a_n = 1\), such that \(\mu_n(P)\) has a zero at each point \(P_i\) (with multiplicity, if the \(P_i\) are repeated) and has smallest possible order of pole at \(\infty\). We obtain \(\mu_n(P)\) using the FS matrix:

**Proposition 3.1.** For \(P, P_1, \ldots, P_n \in (X\backslash \infty) \times S^n(X\backslash \infty)\), we have \(\mu_n(P)\) by

\[
\mu_n(P) \equiv \mu_n(P; P_1, \ldots, P_n) = \lim_{P_i \to P} \frac{1}{\psi_{n+1}(P_1', \ldots, P_n', P)} \psi_{n+1}(P_1', \ldots, P_n', P),
\]

where the \(P_i'\) are generic and the limit is taken (irrespective of the order) for each \(i\).

We introduce \(\mu_{n,k}(P_1, \ldots, P_n)\) by

\[
\mu_n(P) = \phi_n(P) + \sum_{k=0}^{n-1} (-1)^{n-k} \mu_{n,k}(P_1, \ldots, P_n) \phi_k(P).
\]

Proposition 3.1 gives the following lemma:
Lemma 3.2. Let \( n \) be a positive integer. For \((P_i)_{i=1,\ldots,n} \in S^n(X\backslash\infty)\), the function \( \mu_n \) over \( X \) induces the map:

\[
\alpha_n : S^n(X\backslash\infty) \rightarrow S^{n(n)-n}(X),
\]
i.e., \((P_i)_{i=1,\ldots,n} \in S^n(X\backslash\infty)\) is mapped to an element \((Q_i)_{i=1,\ldots,N(n)-n} \in S^{n(n)-n}(X)\), such that

\[
\sum_{i=1}^{n} P_i - n\infty \sim - \sum_{i=1}^{N(n)-n} Q_i + (N(n) - n)\infty.
\]

For an effective divisor of degree \( n \), \( D \in S^n(X) \), let \( D' \) be the maximal subdivisor of \( D \) which does not contain \( \infty \), \( D = D' + (n - m)\infty \) where \( \deg D' = m(\leq n) \) and \( D' \in S^m(X\backslash\infty) \): we extend the map \( \alpha_n \) to \( S^n(X) \) by defining \( \overline{\alpha}_n(D) = \alpha_m(D') + [N(n) - n - (N(m) - m)]\infty \).

The linear equivalence of Lemma 3.2 gives:

Proposition 3.3. For a positive integer \( n \), the Abel map composed with \( \alpha_n \) induces the map \([-1] : W^n \rightarrow W^{N(n)-n}, (u \mapsto -u)\) satisfying the following commutative diagram:

\[
\begin{array}{ccc}
S^n(X - \infty) & \overset{\alpha_n}{\longrightarrow} & S^{N(n)-n}(X) \\
\downarrow \text{cov} & & \downarrow \text{cov} \\
W^n & \overset{[-1]}{\longrightarrow} & W^{N(n)-n}
\end{array}
\]

Let \( \text{image}([-1]) \) be denoted by \([-1]W^n\).

For \((2,2g+1)\) curve \( y^2 = x^{2g+1} + \lambda_1 x^{2g} + \cdots + \lambda_{2g} x + \lambda_{2g+1} \), the \([-1]\) operation corresponds to the hyperelliptic involution \((x, y) \mapsto (x, -y)\). This correspondence does not hold for cyclic \((r, s)\) curves in general.

For \((3,4)\) curve, we consider \([-1](x_1,y_1)\). By considering the divisor of a meromorphic function of \((x,y)\),

\[
\mu((x,y);(x_1,y_1)) \equiv \begin{vmatrix}
1 & x_1 \\
1 & x
\end{vmatrix} = 0.
\]

This means that

\[
(x_1,y_1) + (x_1,\zeta y_1) + (x_1,\zeta^2 y_1) - 3\infty \sim 0.
\]

In other words, we see that

\[
[-1](x_1,y_1) = (x_1,\zeta y_1) + (x_1,\zeta^2 y_1). \quad (3.2)
\]

Hence, we must have

\[
\int_{\infty}^{(x_1,y_1)} \begin{pmatrix}
\nu_1^f \\
\nu_2^f \\
\nu_3^f
\end{pmatrix} + \int_{\infty}^{(x_1,\zeta y_1)} \begin{pmatrix}
\zeta^2 \nu_1^f \\
\zeta^2 \nu_2^f \\
\zeta^2 \nu_3^f
\end{pmatrix} + \int_{\infty}^{(x_1,\zeta^2 y_1)} \begin{pmatrix}
\zeta \nu_1^f \\
\zeta \nu_2^f \\
\zeta \nu_3^f
\end{pmatrix} \equiv 0, \text{ modulo } \Pi.
\]
4. The $\sigma$-function

As in (2.6), we also introduce the complete Abelian integral of the second kind,
\[
[\eta' \eta''] = \frac{1}{2} \left[ \int_{\alpha_i} \nu^{II}_j \int_{\beta_i} \nu^{II}_j \right]_{i,j=1,2,\ldots,g},
\]  
where $\nu^{II}_j = \nu^{II}_j(x,y)$ ($j = 1,2,\ldots,g$) are differentials of the second kind \cite[Corollary 2.6]{E}, which we defined algebraically in \cite{MP1} after \cite{EEL}.

The following Proposition gives a generalized Legendre relation is given as the following Proposition \cite{B1, BLE2, EEL}.

**Proposition 4.1.** The matrix
\[
M := \begin{bmatrix} 2\omega' & 2\omega'' \\ 2\eta' & 2\eta'' \end{bmatrix},
\]  
satisfies
\[
M \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2\pi\sqrt{-1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.
\]  

By the Riemann bilinear relations \cite{E}, it is known that $\text{Im} (\omega'^{-1}\omega'')$ is positive definite. Referring to Theorem 1.1 in \cite{E}, let
\[
\delta_R := \begin{bmatrix} \delta_R' \\ \delta_R'' \end{bmatrix} \in \left( \frac{1}{2}\mathbb{Z} \right)^{2g},
\]  
be the theta characteristic which gives the Riemann-constant vector $\omega_R = 2\omega''\delta_R' + 2\omega'\delta_R''$ with respect to the base point $\infty$ and the period matrix $[2\omega' 2\omega'']$.

We define the $\sigma$ function as an entire function of $u = \iota(u_1,u_2,\ldots,u_g) \in \mathbb{C}^g$,
\[
\sigma(u) = \sigma(u;M) = \sigma(u_1,u_2,\ldots,u_g;M) = c \exp\left( -\frac{1}{2} \iota\gamma u \theta[\delta_R'] (\frac{1}{2}\omega'^{-1} u; \omega'^{-1}\omega'') \right),
\]  
where $\gamma := \omega'^{-1}\eta'$, $c$ is a constant that depends on the moduli of the curve and $\theta$ is the Riemann $\theta$ function of $\delta := \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \in \left( \frac{1}{2}\mathbb{Z} \right)^{2g},$
\[
\theta[\delta](z;\mathbb{T}) = \sum_{n \in \mathbb{Z}^g} \exp \left\{ -\sqrt{-1}\pi \left[ \iota(n + \delta')(n + \delta') + \frac{1}{2} \iota(n + \delta')(z + \delta') \right] \right\}.
\]

For a given $u \in \mathbb{C}^g$, we introduce $u'$ and $u''$ in $\mathbb{R}^g$ so that $u = 2\omega'u' + 2\omega''u''$. Further for $u, v \in \mathbb{C}^g$, and $\ell = (2\omega'\ell' + 2\omega''\ell'') \in \Pi$, we define
\[
L(u, v) := 2 \iota u(\eta'v' + \eta''v''),
\]
\[
\chi(\ell) := \exp[\pi\sqrt{-1}(2(\ell'\delta_R' - \iota\ell''\delta''') + \iota\ell'\ell'')] \in \{1, -1\}.
\]

Then we have the following quasi-periodic properties of the $\sigma$ function \cite[Proposition 4.3]{MP2};
\[
\sigma(u + \ell) = \sigma(u) \exp(L(u + \frac{1}{2}\ell, \ell))\chi(\ell).
\]  

(4.6)
This quasi-periodic property of $\sigma$ is a straightforward consequence of the similar property holding for the normalized theta function in Chapter VI of [L], namely, for $z \in \mathbb{C}^g$ and $\ell', \ell'' \in \mathbb{Z}^g$,

$$\theta[\delta](z + \ell' + \tau \ell'') = e^{-2\pi \sqrt{-1} \left( \frac{1}{2} \ell' \tau \ell'' + \ell' z + \ell' \delta - \ell'' \delta \right)} \theta[\delta](z).$$

However, we wish to stress that one significant difference between $\theta$ and $\sigma$ is that unlike $\theta$, $\sigma$ is also modular invariant: that is, for all $u \in \mathbb{C}^g$, and $\gamma \in \text{Sp}(2g, \mathbb{Z})$, we have also:

$$\sigma(u; \gamma M) = \sigma(u; M).$$

The vanishing locus of $\sigma$ is:

$$\Theta^{g-1} = (\mathcal{W}^{g-1} \cup [-1] \mathcal{W}^{g-1}) = \mathcal{W}^{g-1}. \quad (4.7)$$

The last equality is due to our choice of base point $\infty$ such that $(2g - 2) \infty = K_X$. Since $\mathcal{W}^k$ is not $[-1]$-invariant in general for $k < g - 1$, we define:

$$\Theta^k := \mathcal{W}^k \cup [-1] \mathcal{W}^k, \quad (4.8)$$

and similarly,

$$\Theta^k_1 := w(\mathcal{S}^k(X)) \cup [-1] w(\mathcal{S}^k(X)). \quad (4.9)$$

For hyperelliptic curves ($r = 2, s = 2g + 1$) with a branch point at $\infty$, $\Theta^k$ equals $\mathcal{W}^k$ for every positive integer $k$, but for general curves it does not.

The Schur function $s_\Lambda(t)$ is given by

$$s_\Lambda(t) := \frac{|t_{\Lambda}^{i+g-i}|_{1 \leq i,j \leq g}}{|t_{\Lambda}^{i} - 1|_{1 \leq i,j \leq g}}, \quad (4.10)$$

and the complete homogeneous symmetric function $h_n^{(\ell_1, \ell_2)} = h_n(t_{\ell_1}, \ldots, t_{\ell_2})$ for positive integers $\ell_1$ and $\ell_2$ ($\ell_1 < \ell_2$) is given by

$$\prod_{i=\ell_1}^{\ell_2} \frac{1}{(1 - z t_i)} = \sum_{n \geq 0} h_n^{(\ell_1, \ell_2)} z^n, \quad h_n^{(\ell_1, \ell_2)} = 0 \text{ for } n < 0.$$

**Proposition 4.2.** [L, Theorem 4.5.1] Using the complete homogeneous symmetric functions $h_n := h_n^{(1,g)}$, we can express $s_\Lambda$ by a $(g \times g)$ Jacobi-Trudi Determinant, $|a_{ij}|_{1 \leq i,j \leq g}$ with $a_{ij} = h_{\Lambda_{i,j}}^{i,j}$:

$$s_\Lambda(t) := |h_{\Lambda_{i,j}}^{i,j}|_{1 \leq i,j \leq g}, \quad h_n = \begin{vmatrix} T_1 & -1 & 0 & \cdots \\ 2T_2 & T_1 & -2 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n-1)T_{n-1} & (n-2)T_{n-2} & (n-3)T_{n-3} & \cdots & 1-n \\ nT_n & (n-1)T_{n-1} & (n-2)T_{n-2} & \cdots & T_1 \end{vmatrix},$$

where $h_0 = 1$, $h_{i<0} = 0$ and $T_k := \frac{1}{k} \sum_{j=1}^{g} t_j^k$. 

\[ \text{Figure 4.2:} \quad \begin{array}{c|cccc}
\hline
\ell_1 & \ell_2 & \ell_3 & \ell_4 \\
\hline
\ell_1 & 1 & 2 & 3 \\
\ell_2 & 4 & 5 & 6 \\
\ell_3 & 7 & 8 & 9 \\
\ell_4 & 10 & 11 & 12 \\
\hline
\end{array} \]
We use the multi-index convention and define a map for $\beta := (\beta_1, \ldots, \beta_g)$,

$$w_\beta := ((2g - N(0) - 1)\beta_1, (2g - N(1) - 1)\beta_2, \ldots, (2g - N(g - 2) - 1)\beta_{g-1}, \beta_g) \in \mathbb{Z}^g.$$ 

By regarding $S_\Lambda(T) := s_\Lambda(t)$ as a function of $T$, the following Proposition was stated by Bukhshtaber, Leykin and Enolskii [BLE1] and proved by Nakayashiki [N1, Theorem 3].

**Proposition 4.3.** The expansion of $\sigma(u)$ at the origin takes the form

$$\sigma(u) = S_\Lambda(T)|_{T=0} + \sum_{w_\beta(\alpha) = |\Lambda|} c_\alpha u^\alpha$$

where $c_\alpha \in \mathbb{Q}[\lambda_j]$ and $S_\Lambda(T)$ is the lowest-order term in the $w$-degree of the $u_i$; $\sigma(u)$ is homogeneous of degree $|\Lambda|$ with respect to the $\lambda$-degrees.

Here we note that the factor $c$ in (1.3) is fixed so that there is no prefactor in $S_\Lambda(T)|_{T=0}$. For the $(3, 4)$ curve case, $\sigma$ is expanded around $\Theta^2$;

$$\sigma(u) = (u_1 - u_3 u_2^2 + \frac{1}{20} u_3^5) + \text{higher weight terms.}$$

**Remark 4.4. Galois action on $\sigma$:** It should be noted that the action of the cyclic group $\mathbb{Z}_r$ on $X$ induces its action on $\mathcal{J}$, by the morphism $S^g(X) \to \mathcal{J}$, and on the coordinate space $\mathbb{C}^g = \kappa^{-1}(\mathcal{J})$ of the $\sigma$-function. Since the lattice $\Pi$ is stable under this action of $\mathbb{Z}_r$, we see that $\sigma(\hat{\zeta}_r u)$ also satisfies (4.6). We shall recall here that the space of the functions satisfying the relation (4.6) is one dimensional (Frobenius’ theorem). Adding the leading terms of the expansion (1.3) and (1.4), we have arrived the formula $\sigma(\hat{\zeta}_r u) = \zeta^a \sigma(u)$ for some integer $a$. Thus we will determine the exponent $a$. For $u^{(i)}_g \in \mathcal{W}^1$ ($i = 1, 2, \ldots, g$), and $t = (u^{(1)}_g, u^{(2)}_g, \ldots, u^{(g)}_g)$, $\sigma$ is expressed by

$$\sigma(u) = s_\Lambda(t) + \sum_{|w_\beta(\alpha)| = |\Lambda| + 1} c_\alpha u^\alpha + \sum_{|w_\beta(\alpha)| = |\Lambda| + 2} c_\alpha u^\alpha + \cdots$$

and the action of $\hat{\zeta}_r$ on $\sigma(u)$ is given through the terms with the same weight,

$$\sigma(\hat{\zeta}_r u) = \zeta^a r s_\Lambda(t) + \zeta^a \sum_{|w_\beta(\alpha)| = |\Lambda| + 1} c_\alpha u^\alpha + \zeta^a \sum_{|w_\beta(\alpha)| = |\Lambda| + 2} c_\alpha u^\alpha + \cdots$$

On the other hand, since $s_\Lambda(\hat{\zeta}_g u) = \zeta^{|\Lambda|(r_g - 1)} s_{\Lambda}(u)$, we have $a = |\Lambda|(r_g - 1)$:

$$\sigma(\hat{\zeta}_g u) = \zeta^{|\Lambda|(r_g - 1)} \sigma(u). \quad (4.11)$$

5. Vanishing of $\sigma$ function

For the truncated Young diagram $\Lambda[k]$ in (2.9) we write the characteristics of the partition by $(a_1, \ldots, a_k; b_1, \ldots, b_k)$ and the cardinality of the set of pairs denoted by $n_k$ [EH §4.1, p. 51], and define

$$N_k := |\Lambda[k]| = \sum_{i=1}^{n_k} (a_i + b_i + 1).$$
We note that \(a_i\) and \(b_i\) depend on \(k\) of \(\Lambda^{[k]}\). In general, \(a_i\) (or \(b_i\)) of \(\Lambda^{[k]}\) differs from \(a_j\) (or \(b_j\)) of \(\Lambda^{[k]}\) if \(i \neq j\). Using them we have the quantities \(a_{n_k-i+1} + b_{n_k-i+1} + 1, i = 1, \ldots, n_k\).

Due to the arguments in [MP2], there exists an integer \(j \in \{k + 1, k + 2, \ldots, g\}\) such that \(a_i + b_i + 1 = \Lambda_j + g - j\), for every \(i = 1, \ldots, n_k\). The correspondence is given by \(L^{[k]}(a_i, b_i) := j\). For \(i = n_k\) case, \(a_{n_k} + b_{n_k} + 1 = \Lambda_{k+1} + g - k - 1\) and \(L^{[k]}(a_{n_k}, b_{n_k}) = k + 1\).

**Definition 5.1.** For \(k = 1, 2, \ldots, g - 1\), and the characteristics of the partition of \(\Lambda^{[k]}\), \((a_1, \ldots, a_{n_k}; b_1, \ldots, b_{n_k})\), we define

\[
\tau_k := \{L^{[k]}(a_1, b_1), L^{[k]}(a_2, b_2), \ldots, L^{[k]}(a_{n_k}, b_{n_k})\},
\]

and

\[
\tau_k^{(i)} := (\tau_k \setminus \{k + 1\}) \bigcup \{i\}, \quad \text{for} \quad i = 1, 2, \ldots, k.
\]

From [MP2, Corollary 5.4], we have

**Proposition 5.2.** For \(u^{[k]} \in \Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1})\), \(u^{[g]} \in \mathbb{C}^g\), \(v \in \mathcal{W}^1\), and \(t \in \mathbb{R} (0 < |t| < 1)\), we have

\[
\begin{align*}
(i) \quad & \left. \frac{\partial^f}{\partial v_{g}} \sigma(tv + u^{[k]}) \right|_{v=0} = 0, \ \ell < N_k; \quad \left. \frac{\partial^N_k}{\partial v_{g}^N_k} \sigma(tv + u^{[k]}) \right|_{v=0} \neq 0, \text{ and} \\
(ii) \quad & \left. \frac{\partial^f}{\partial u_{g}^{[g]}} \sigma(u^{[g]}) \right|_{u^{[g]}=u^{[k]}} = 0, \ \ell < N_k; \quad \left. \frac{\partial^N_k}{\partial u_{g}^{[g]}} \sigma(u^{[g]}) \right|_{u^{[g]}=u^{[k]}} \neq 0.
\end{align*}
\]

Let \(\mathcal{I}\) be the family of all finite sequences made up of unordered sets of numbers between 1 and \(g\). For an element \(I_k \in \mathcal{I}\) and \(u \in \mathbb{C}^g\), define:

\[
\sigma_{I_k} := \left( \prod_{i \in I_k} \frac{\partial}{\partial u_i} \right) \sigma,
\]

\[
\deg_w(I_k) := \sum_{i \in I_k} \deg_w(u_i).
\]

From [MP2, Theorem 5.15], we have

**Proposition 5.3.** Let \(\mathcal{I}_g = \{\emptyset\}\). For each \(k = 1, 2, \ldots, g\), there exists a family of \(\mathcal{I}_k\) sequences consisting of \(1, 2, \ldots, g\) whose element \(I_k\) is such that \(#I_k = n_k\), \(\deg_w(I_k) \geq N_k\), and as a function over \(\kappa^{-1}(\Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1}))\),

\[
\sigma_{I_k} = \begin{cases} 
\neq 0 & \text{for } J_k = I_k \\
= 0 & \text{for } J_k \not\subseteq I_k.
\end{cases}
\]

Moreover, \(\{\tau_k^{(1)}, \tau_k^{(2)}, \ldots, \tau_k^{(k-1)}, \tau_k^{(k)}\} \subset \mathcal{I}_k\) and \(\deg_w(I_k) = N_k\).

**Remark 5.4.** Here we have the convention that for \(v \in \mathcal{W}^k \subset \mathcal{J}\), and \(u \in \mathbb{C}^g\), \(\sigma_{I_k}(v)\) means that it is given by \(\left( \prod_{i \in I_k} \frac{\partial}{\partial u_i} \right) \sigma(u) \bigg|_{u=v}\).
For the case of an \((r, s)\) curve, the action \(\hat{\zeta}_r\) on \(u\) via its Abel preimage \((x_i, y_i) \rightarrow (x_i, \zeta_r y_i), (\zeta_r^r = 1, \zeta_r \neq 1)\), is given by:

\[ \sigma_{2k}(\hat{\zeta}_r u) = \zeta_r^{A(k)(r-1+1)}\sigma_{2k}(u). \]

A non-vanishing theorem for \(\sigma\) over a hyperelliptic \((2, 2g + 1)\) curve \(X\) of genus \(g\) is given in \([O1]\),

\[ \hat{\zeta}_k := \begin{cases} 
{\{g, g-2, \ldots, k+3, k+1\}} & \text{if } g - k \text{ is odd}, \\
{\{g-1, g-3, \ldots, k+3, k+1\}} & \text{otherwise}.
\end{cases} \]

Similar results for \((3, 4)\) and \((3, 5)\) curves are given in \([O2, O3]\).

We show some examples of \(\hat{\zeta}_k\) in Table 1.1.

| \((r, s)\) | \(g\) | \(\hat{\zeta}_1\) | \(\hat{\zeta}_2\) | \(\hat{\zeta}_3\) | \(\hat{\zeta}_4\) | \(\hat{\zeta}_5\) | \(\hat{\zeta}_6\) | \(\hat{\zeta}_7\) |
|-------------|------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \((2, 3)\)   | 1    |                | \(2\)          | \(3\)          | \(4\)          | \(5\)          | \(6\)          | \(7\)          |
| \((2, 5)\)   | 2    |                |                | \(2\)          | \(3\)          | \(4\)          | \(5\)          | \(6\)          |
| \((2, 7)\)   | 3    |                |                |                | \(2\)          | \(3\)          | \(4\)          | \(5\)          |
| \((2, 9)\)   | 4    |                | \(2\)          | \(3\)          | \(4\)          | \(5\)          | \(6\)          | \(7\)          |
| \((2, 11)\)  | 5    | \(2\)          | \(3\)          | \(4\)          | \(5\)          | \(6\)          | \(7\)          | \(8\)          |
| \((2, 13)\)  | 6    | \(2, 3\)       | \(3\)          | \(4\)          | \(5\)          | \(6\)          | \(7\)          | \(8\)          |
| \((2, 15)\)  | 7    | \(2, 3, 5\)    | \(3\)          | \(4\)          | \(5\)          | \(6\)          | \(7\)          | \(8\)          |
| \((2, 17)\)  | 8    | \(2, 3, 5, 7\)| \(3\)          | \(4\)          | \(5\)          | \(6\)          | \(7\)          | \(8\)          |
| \((3, 4)\)   | 3    |                |                | \(2\)          | \(3\)          | \(4\)          | \(5\)          | \(6\)          |
| \((3, 5)\)   | 4    | \(2\)          | \(3\)          | \(4\)          | \(5\)          | \(6\)          | \(7\)          | \(8\)          |

In \([O2]\), the partial derivative over multi-index \(\hat{\zeta}_n\) was given by \(\sigma_{21}(u) = \frac{\partial^2}{\partial u_1^2}\sigma(u) = \sigma_{33}(u)\) but here we have instead defined \(\sigma_{21}(u) = \frac{\partial}{\partial u_2}\sigma(u) = \sigma_2(u)\) because \(\sigma_2(u) = -\sigma_{33}(u)\).

Using them, Theorem 5.24 in \([MP2]\) gives the Jacobi inversion formulae for a stratum in the Jacobian \(J\):

**Theorem 5.5.** For \(k < g\), \((P_1, \ldots, P_k) \in S^k(X \setminus \infty) \setminus (S_1^k(X) \cap S^k(X \setminus \infty))\) and \(u = \pm w(P_1, \ldots, P_k) \in \kappa^{-1}(\Theta^k)\),

\[ \frac{\sigma_{k(i)}(u)}{\sigma_{2k}(u)} = (-1)^{k-i+1} \mu_{k,i-1}(P_1, \ldots, P_k). \]

Since \(\mu_{1,0}(P) = x\) for a certain point \(P = (x, y)\), we have

\[ \frac{\sigma_{k(1)}(u)}{\sigma_{2k}(u)} = -x \quad \text{for} \quad u = w(P). \]

For the case of a cyclic \((3, 4)\) curve, we have \(du_3 = dx/3y\) and thus, \(d(\sigma_{k(1)}(u)/\sigma_{2k}(u))/du_3 = -3y\) as mentioned in Introduction.

6. Frobenius-Stickelberger relation

The prototype of such a formula was first found, for the elliptic case, in \([FS]\). Generalisations of that result to general hyperelliptic curves, and to examples of cyclic
trigonal curves, have been given in [01 Theorem 7.2], [02] and [03]. Here we recall these results, which will be needed below.

In the general hyperelliptic case we have:

**Proposition 6.1.** For the hyperelliptic \((2, 2g + 1)\) curve and a positive integer \(n > 1\), let \((x_1, y_1), \ldots, (x_n, y_n)\) in \(X\), and \(u^{(1)}, \ldots, u^{(n)}\) in \(\kappa^{-1}(\mathcal{W}^1)\) be points such that \(\kappa(u^{(i)}) = \kappa w((x_i, y_i))\). Then the following relation holds:

\[
\frac{\sigma_n \left( \sum_{i=1}^{n} u^{(i)} \prod_{i<j} \sigma_{22}(u^{(i)} - u^{(j)}) \right)}{\prod_{i=1}^{n} \sigma_{21}(u^{(i)})^n} = \epsilon_n \psi_n((x_1, y_1), \ldots, (x_n, y_n)),
\]

where \(\epsilon_n = (-1)^{g+n(n+1)/2}\) for \(n \leq g\) and \(\epsilon_n = (-1)^{(2g-g)(g-1)/2}\) for \(n > g + 1\).

In the case of a trigonal curve, from [02, Theorem 5.3, Lemma 4.3], we have the following Proposition and Corollary:

**Proposition 6.2.** [02 Theorem 5.3] For trigonal \((3, s)\) curve \((s = 4, 5)\), and a positive integer \(n > 1\), let \((x_1, y_1), \ldots, (x_n, y_n)\) in \(X\) and \(u^{(1)}, \ldots, u^{(n)}\) in \(\kappa^{-1}(\mathcal{W}^1)\) be points such that \(\kappa(u^{(i)}) = \kappa w((x_i, y_i))\). Then the following relation holds:

\[
\frac{\sigma_n \left( \sum_{i=1}^{n} u^{(i)} \prod_{i<j} \sigma_{22}(u^{(i)} + \zeta^3 u^{(j)}) \sigma_{22}(u^{(i)} + \zeta^2 u^{(j)}) \right)}{\prod_{i=1}^{n} \sigma_{21}(u^{(i)})^{2n-1}} = \psi_n((x_1, y_1), \ldots, (x_n, y_n)) \varphi_n((x_1, y_1), \ldots, (x_n, y_n)).
\]

**Corollary 6.3.** [02 Theorem 5.3, Lemma 4.3] For a cyclic \((3, 4)\) curve, in particular, let \((x, y)\) and \((x_i, y_i)_{i=1,2}\) be points in \(X \times S^2(X)\) and \((u, v_1, v_2)\) be a point in \(\kappa^{-1}\mathcal{W}^1 \times \kappa^{-1}\mathcal{W}^1 \times \kappa^{-1}\mathcal{W}^1\) such that \(u = w(x, y)\) and \(v_i = w(x_i, y_i)\). Then the following three relations hold:

\[
\frac{\sigma_{22}(u + v_1 + v_2) \prod_{a=1}^{2} \left( \sigma_{22}(u + \zeta^a v_1) \sigma_{22}(u + \zeta^a v_2) \sigma_{22}(v_1 + \zeta^a v_2) \right)}{\sigma_{21}(u)^3 \sigma_{21}(v_1)^5 \sigma_{21}(v_2)^5} = -(x - x_1)(x - x_2)(x_1 - x_2)(y(x_1 - x_2) - y_1(x - x_2) + y_2(x - x_1)),
\]

\[
\frac{\sigma_{22}(v_1 + v_2) \sigma_{22}(v_1 + \zeta v_2) \sigma_{22}(v_1 + \zeta^2 v_2)}{\sigma_{21}(v_1)^3 \sigma_{21}(v_2)^3} = \begin{vmatrix}
1 & x_1 \\
1 & x_2
\end{vmatrix}^2,
\]

\[
\frac{\sigma_{22}(2u)}{\sigma_{21}(u)^4} = 3y^2.
\]

7. The Prime form of an \((r, s)\) curve and the sigma function

7.1. The Prime form of an \((r, s)\) curve

For the universal covering \(\tilde{X}\) of \(X\), let us define the prime form as follows:
Definition 7.1. The prime form $E(P, Q)$ for a point $P, Q \in \tilde{X}$ is defined by

$$E(P, Q) := \frac{\theta \left[ \frac{\alpha'}{\alpha''} \right] (\int_{Q}^{P} \nu^I)}{\sqrt{\zeta(P)} \sqrt{\zeta(Q)}}, \quad (7.1)$$

where $\alpha'$ and $\alpha''$ are a fixed nonsingular odd theta characteristics and $\zeta(P)$ is

$$\zeta(P) := \sum_{i=1}^{g} \left( \partial_{i} \theta \left[ \frac{\alpha'}{\alpha''} \right] (0) \right) \nu_{i}(P).$$

We should note that the prime form is a $(-1/2) \otimes (-1/2)$ form of $\tilde{X}$. Due to the Abel map, for $\hat{\ell} \in \hat{\Pi}$, we have a natural translation action $\gamma_{\hat{\ell}}$ on $\tilde{X}$ such that

$$\hat{w}(\gamma_{\hat{\ell}}(P)) = \hat{w}(P) + \hat{\ell}.$$

In "Tata lectures on theta II" [Mu, 3.210], Mumford shows that

Proposition 7.2. The prime form satisfies the following relations:

(i) $E(P_1, P_2)$ vanishes iff $P_1 = P_2$.

(ii) $E(P_1, P_2)$ has a first order zero along the diagonal $\Delta \subset X \times X$.

(iii) $E(P_1, P_2) = -E(P_2, P_1)$.

(iv) Let $t_i$ be a local parameter at $P_i$ of $X$ such that $\zeta(P_i) = dt_i$, $(i = 1, 2)$,

$$E(P_1, P_2) = \frac{t_1 - t_2}{\sqrt{dt_1} \sqrt{dt_2}} (1 + d_{\geq} (t_1 - t_2)^2).$$

(v) $E(P, Q)$ has the properties for the action $\gamma_{\hat{\ell}}$ for $\hat{\ell} \in \hat{\Pi}$,

$$E(\gamma_{\hat{\ell}} P, Q) = e^{-2\pi \sqrt{-1} \left( \frac{1}{2} \hat{\ell} \tau_{\alpha''} \tau_{\alpha'} + \hat{\ell} \tau_{z} + \hat{\ell} \tau_{\alpha''} - \hat{\ell} \tau_{\alpha'} \right)} E(P, Q).$$

Proposition 7.3.

$$\frac{E(P, Q)}{E(P, Q')} = \exp \int_{P}^{P'} \tau_{Q, Q'}.$$

where $\tau_{Q, Q'}$ is the differential of the third kind, which has residues $+1, -1$ at $Q_1, Q_2$, is regular everywhere else, and is normalized i.e., $\int_{\alpha_i} \tau_{P, Q} = 0$.

Remark 7.4. This fact may be used to construct the Green’s function for the Laplacian on the Riemann surface $X$.

Following Nakayashiki’s definition of the prime form [NI, Definition 3]

Definition 7.5. we define another prime form:

$$\mathcal{E}(P, Q) := e^{i(w(P) - w(Q))\gamma(w(P) - w(Q))} E(P, Q) \quad (7.2)$$

where $\gamma := \omega^{-1} \eta'$.

Remark 7.6. Our definition differs slightly from that in [NI], as here $\mathcal{E}(P, Q)$, like Fay’s $E(P, Q)$, is defined as a $(-1/2, -1/2)$ form.
**Proposition 7.7.** (i) $E(P_1, P_2)$ vanishes iff $P_1 = P_2$.

(ii) $E(P_1, P_2)$ has a first order zero along the diagonal $\Delta \subset X \times X$.

(iii) $E(P_1, P_2) = -E(P_2, P_1)$.

(iv) Let $t_i$ be a local parameter at $P_i$ of $X$ such that $\zeta(P_i) = dt_i$, $(i = 1, 2)$

$$E(P_1, P_2) = \frac{t_1 - t_2}{\sqrt{dt_1}\sqrt{dt_2}}(1 + d \geq ((t_1 - t_2)^2).$$

(v) $E(P, Q)$ has the properties for the action $\gamma_\ell$ on $\tilde{X}$,

$$E(\gamma_\ell P, Q) := \chi(\ell)L \left((w(P) - w(Q)) + \frac{1}{2}(\ell, \ell)\right) E(P, Q).$$

where $w(\gamma_\ell(P)) = w(P) + \ell$.

**Proof.** (i)-(iv) are obvious from the previous Proposition.

From [N1, Proposition 8], we have (v). \hfill \Box

**Proposition 7.8.** If a $(-1/2) \otimes (-1/2)$-form on $\tilde{X} \times \tilde{X}$ has the properties (i), (ii), (iii), (iv) and (v) in Proposition 7.7, it has the form of (7.2).

**Proof.** If there is another $(-1/2) \otimes (-1/2)$-form $E'(P, Q)$ on $\tilde{X} \times \tilde{X}$ with these properties, the quasi-periodic properties (iv) show that $E'(P, Q)/E(P, Q)$ must be a meromorphic function over $\mathbb{C}^g$. Further the vanishing properties (iii) determine that the meromorphic function has no zero and thus must be a constant function. By (iv), this constant is equal to 1. \hfill \Box

This uniqueness theorem will enable us to prove the main results of this paper, the following explicit formulae for the Prime form.

For any hyperelliptic $(2, 2g + 1)$ curve, we have

$$E(P, Q) = \frac{\sigma_{22}(u - v)}{\sqrt{du_1\sqrt{dv_1}}}. $$

Further, for a cyclic $(3, 4)$ curve, we have

$$E(P, Q) = \frac{\sigma(u - v)}{\sqrt{du_1\sqrt{dv_1}}} \equiv \frac{\sigma_{23}(u - v)}{\sqrt{du_1\sqrt{dv_1}}}. $$

We will prove these results in the next two subsections.
7.2. The Prime form - hyperelliptic case

We will prove the theorem for the case of a hyperelliptic curve; first we will establish
several preliminary results.

First, from Proposition 6.1, we directly have the following;

**Proposition 7.9.** Let \((x,y)\) and \((x',y')\) be points in \(X\) and \((u,v)\) \(\in\) \(C^g\times C^g\) such that \(u = w(x,y)\) and \(v = w(x',y')\).

\[
\frac{\sigma_{22}(u + v)\sigma_{22}(u - v)}{\sigma_{21}(u)^2\sigma_{21}(v)^2} = (x - x').
\]

**Remark 7.10.** Proposition 7.9 is a natural generalisation of the addition formula in the \(g = 1\) case (1.1).

Let \((x,y)\) and \((x',y')\) be points in \(X\) and \((u,v)\) \(\in\) \(C^g\times C^g\) such that \(u = w(x,y)\) and \(v = w(x',y')\).

From \([O1, Lemma 9.1]\), we have the relation:

\[
\lim_{u \to v} \frac{\sigma_{22}(u - v)}{u_1 - v_1} = 1.
\]

This implies the following relation:

**Lemma 7.11.** Let \((x,y)\) and \((x',y')\) be points in \(X\) and \((u,v)\) \(\in\) \(C^g\times C^g\) such that \(u = w(x,y)\) and \(v = w(x',y')\).

\[
\frac{\sigma_{22}(u - v)}{\sqrt{du_1\sqrt{dv_1}}} = \frac{u_1 - v_1}{\sqrt{du_1\sqrt{dv_1}}} + \cdots.
\]

**Remark 7.12.** This establishes conditions (ii) and (iv) of 7.8. Further Lemma 4.3 in \([O1]\) shows that \(\sigma\) function is an odd or even function with respect to \(u \in C^g\). Hence this also establishes (iii) of 7.8.

Secondly, from \([O1, Proposition 7.5]\), we have the relation:

**Lemma 7.13.** \([O1, Proposition 7.5]\] Let \((x,y)\) and \((x',y')\) be points in \(X\) and \((u,v)\) \(\in\) \(C^g\times C^g\) such that \(u = w(x,y), v = w(x',y')\) and \(\ell = 2\omega'\ell' + 2\omega''\ell'\) \(\in\) \(\Pi\). Then the following quasi-periodicity relation holds

\[
\frac{\sigma_{22}(u + \ell - v)}{\sqrt{du_1\sqrt{dv_1}}} = \exp(L(u + \frac{1}{2}\ell,\ell))\chi(\ell)\frac{\sigma_{22}(u + \ell - v)}{\sqrt{du_1\sqrt{dv_1}}}.
\]

**Remark 7.14.** This establishes condition (v) of 7.8.

**Lemma 7.15.** Let \((x,y)\) and \((x',y')\) be points in \(X\) and \((u,v)\) \(\in\) \(\kappa^{-1}(\Theta^1 \setminus \Theta^0) \times C^g\) such that \(u = w(x,y)\) and \(v = w(x',y')\). Then the following relation holds

\[
\lim_{v \to 0} \frac{\sigma_{22}(u - v)}{\sqrt{du_1\sqrt{dv_1}}} \neq 0.
\]
Proof. From [MP2 Proposition 5.26] and Proposition 5.2, \( \sigma_{z_2}(u - v) = c_1\sigma_{y_2}(u - v) = c_2\sigma_{y_2}(u)v_2^{-1} + d_2(v_2'), \) where \( c_1 \) and \( c_2 \) are constants. Since the weight of \( v_1 \) is \( 2g - 1 \), we have the relation.

Remark 7.16. Thus \( \frac{\sigma_{z_2}(u - v)}{\sqrt{du_1dv_1}} \) has zeroes only where \( (x, y) = (x', y') \), and by (7.11) these are simple, giving condition (i) of (7.8).

Assembling these results,
\[ \mathcal{E}(P, Q) = \frac{\sigma_{z_2}(u - v)}{\sqrt{du_1dv_1}} \] due to Proposition (7.8). Thus we have proved the following:

**Theorem 7.17.** Let \( (x, y) \) and \( (x', y') \) be points in \( X \) and \( (u, v) \in \mathbb{C}^g \times \mathbb{C}^g \) such that \( u = w(x, y) \) and \( v = w(x', y') \). Then the following relation holds
\[ \mathcal{E}(P, Q) = \frac{\sigma_{z_2}(u - v)}{\sqrt{du_1dv_1}}. \]

Here \( du_1 := \nu_1^I(x, y) \) and \( dv_1 := \nu_1^I(x', y') \).

Remark 7.18. As in [KMP] Proposition 4.9, by letting \( (x_i, y_i) \in X \) \( (i = 1, \ldots, g) \), \( (x'_j, y'_j) \in X \) \( (j = 1, 2) \), \( u \in \mathbb{C}^g, v := v^[1] + v^[2] \in \kappa^{-1}(\mathcal{W}^2) \), and \( v^[j] \in \kappa^{-1}(\mathcal{W}^j) \) \( (j = 1, 2) \) be points such that \( u = w((x_1, y_1), \ldots, (x_g, y_g)) \) and \( v^[j] = w((x'_j, y'_j)) \), \( (j = 1, 2) \), the following relation holds:
\[ \frac{\sigma(u + v)\sigma(u - v)}{\sigma(u)^2\sigma_{z_2}(v)^2} = f(x'_1, x'_2) - 2y'_1y'_2 \frac{(x'_1 - x'_2)^2}{(x'_1 - x'_2)^2} - \sum_{i=1}^g \sum_{j=1}^g \varphi_{ij}(u)x'^{i-1}_1x'^{j-1}_2. \] (7.3)

This corresponds to Fay’s formula [F] (39) using the prime form, which is the basis of “Fay’s trisecant identity”. The relation is also clearly related to the two-dimensional Toda lattice equation [KMP].

7.3. Prime form - a cyclic trigonal curve

Again, before proving our main theorem, we state some preliminary results.

First we consider the action of the Galois group on the \( \sigma \)-function and its derivatives due to [O2 Lemma 4.1, 4.2] and [EEMOP] Lemma 4.2:

**Lemma 7.19.** For \( u \in \mathbb{C}^3 \) and action \( \hat{\zeta}_3 \) on \( u \) via its Abel preimage \( (x_i, y_i) \rightarrow (x_i, \zeta_3y_i) \), corresponding to Remark 2.3, we have
\[ \sigma(\hat{\zeta}_3u) = \zeta_3\sigma(u) \equiv \zeta_3\sigma_{z_3}(u), \quad \sigma_{z_2}(\hat{\zeta}_3u) = \zeta_3^2\sigma_{z_2}(u), \quad \sigma_{z_1}(\hat{\zeta}_3u) = \sigma_{z_1}(u). \] (7.4)

and \( \sigma(-u) = -\sigma(u) \).

Secondly, from [O2 Lemma 6.1], we have
Lemma 7.20. Let \((x, y)\) and \((x', y')\) be points in \(X\) and \((u, v) \in \mathbb{C}^3 \times \mathbb{C}^3\) such that \(u = w(x, y)\) and \(v = w(x', y')\). Then the following relations hold
\[
\lim_{u \to v} \frac{\sigma_{21}(u)(x - x')}{\sigma_{22}(u + \zeta_3 v)} = (\zeta_3 - 1)y, \quad \lim_{u \to v} \frac{\sigma_{21}(u)(x' - x)}{\sigma_{22}(u + \zeta_3 v)} = (\zeta_3^2 - 1)y.
\]
Proof. Noting \(\sigma_{2} = \sigma_{3}\) and \(\sigma_{2}(u) = \sigma_{2}(v)\), from Theorem 5.1 in [MP1] and Theorem 5.23 in [MP2], we have the relation,
\[
\frac{\sigma_{21}(u + v)}{\sigma_{22}(u + \zeta_3 v)} = (-1)^{3+2+1} \frac{y - y'}{x - x'}.
\]
Thus
\[
\lim_{u \to v} \frac{x - x'}{\sigma_{22}(u + \zeta_3 v)} = \lim_{u \to v} \frac{y - \zeta_3 y'}{\sigma_{21}(u + \zeta_3 v)} = \frac{(1 - \zeta_3)(1) y'}{\sigma_{21}((1 + \zeta_3)(1))} = \frac{(1 - \zeta_3)(1) y'}{\sigma_{21}(v)}.
\]
The stated relations then follow.

Next, from the Frobenius-Stickelberger relations, we have the following Proposition:

Proposition 7.21. Let \((x, y)\) and \((x', y')\) be points in \(X\) and \((u, v) \in \mathbb{C}^3 \times \mathbb{C}^3\) such that \(u = w(x, y)\) and \(v = w(x', y')\). Then the following relation holds
\[
\frac{\sigma(w(x, y) - \sigma_{22}(u + v))}{\sigma_{21}(u)^2 \sigma_{21}(v)^2} = \frac{\sigma_{21}(u + \zeta_3 v + \zeta_3^2 v) \sigma_{22}(u + v)}{\sigma_{21}(u)^2 \sigma_{21}(v)^2} = (x - x').
\]
Proof. By letting \(v(a) = w(x, y)\) \((a = 1, 2)\), the left hand side is equal to
\[
\lim_{v(a) \to \zeta_3(v)} \epsilon \psi_3((x, y), (x_1, y_1), (x_2, y_2)) \varphi_3((x, y), (x_1, y_1), (x_2, y_2)) \times \frac{\sigma_{21}(u)^3 \sigma_{21}(v)^3}{(\prod_{a=0}^2 \sigma_{22}(u + \zeta_3^a v))} \frac{\sigma_{21}(v(1))^{2} \sigma_{21}(v(2))^{2}}{\sigma_{21}(v(1))} \frac{\sigma_{12}(v(1) + \zeta_3 v(2)) \sigma_{12}(v(1) + \zeta_3 v(2))}{\sigma_{12}(v(1) + \zeta_3 v(2))}
\]
\[
= (x - x')^3 (\zeta_3 - \zeta_3^2) y' \lim_{u \to \zeta_3 v} \frac{(x_1 - x_2)}{\zeta_3 v(1) - \zeta_3 v(2)} \cdot \frac{1}{(x - x')^2 3(y')^2 3 \zeta_3^2 \sigma_{21}(v(1) + \zeta_3 v(2))}
\]
\[
= (\zeta_3 - \zeta_3^2)(x - x') \frac{\zeta_3 - 1}{3 \zeta_3^2} = (x - x').
\]

Remark 7.22. Proposition 7.21 is the trigonal analogue of Proposition 7.9 and similarly generalises the addition formula in the \(g = 1\) case [11].

We now look at the order of vanishing of \(\sigma_3 \equiv \sigma\):

Lemma 7.23. Let \((x, y)\) and \((x', y')\) be points in \(X\) and \((u, v) \in \mathbb{C}^3 \times \mathbb{C}^3\) such that \(u = w(x, y)\) and \(v = w(x', y')\). Then the following relation holds
\[
\lim_{u \to v} \frac{\sigma_3(u + \zeta_3 v + \zeta_3^2 v)}{u_1 - v_1} = \lim_{u \to v} \frac{\sigma_3(u - v)}{u_1 - v_1} = 1.
\]
Proof. Noting \( \lim_{u \to v} \frac{(x - x')}{u_1 - v_1} = \frac{d}{dv_1} x(v) = 3y(v)^2 \), we have
\[
\lim_{u \to v} \frac{\sigma_{3}\left(u + \zeta_3 v + \zeta_4 v\right)}{u_1 - v_1} = \lim_{u \to v} \frac{\sigma_{21}(u)^2\sigma_{21}(v)^2 (x - x')}{\sigma_{22}(2v)} \frac{u_1 - v_1}{u_1 - v_1} = 1.
\]

Hence we see that:

**Lemma 7.24.** Let \((x, y)\) and \((x', y')\) be points in \(X\) and \((u, v) \in \kappa^{-1}(\Theta^1 \setminus \Theta^0) \times \mathbb{C}^3\) such that \(u = w(x, y)\), and \(v = w(x', y')\). Then the following relation holds
\[
\lim_{v \to 0} \frac{\sigma_{3}(u - v)}{\sqrt{du_1} \sqrt{dv_1}} \neq 0.
\]

**Proof.** Around \(v = 0\), we have \(\sigma(u - v) = \frac{1}{20} \sigma_{33}(u) v_3^2 + \cdots\) whereas \(dv_1 = v_3^2 dv_3 + \cdots\). \(\square\)

Thus, again using Proposition [7.8] we have proved the following:

**Theorem 7.25.** For the cyclic trigonal \((3, 4)\) curve,
\[
\mathcal{E}(P, Q) = \frac{\sigma(u + \zeta_3 v + \zeta_4 v)}{\sqrt{-3} \sqrt{du_1} \sqrt{dv_1}} \equiv \frac{\sigma(u - v)}{\sqrt{du_1} \sqrt{dv_1}} \equiv \frac{\sigma_{3}(u - v)}{\sqrt{du_1} \sqrt{dv_1}}.
\]

8. Applications

8.1. Benney reductions

Part of this work was motivated by the problem of simply describing certain reductions of the Benney hierarchy. As described in [G], this problem reduces geometrically to the construction of Schwartz-Christoffel mappings from the upper half \(p\)-plane to a domain consisting of the upper half \(\lambda\)-plane, minus finitely many straight slits, running from fixed base points on the real axis, to variable end points; the slits should make internal angles which are integer multiples of \(\pi/r\) with one another and with the real axis. The slits are images of disjoint segments of the real \(p\)-axis. In this case the integral formula for the Schwartz-Christoffel map is algebraic; it is a second kind Abelian integral on a cyclic \((r, s)\) curve. The domain and its image are illustrated schematically in the figures below, where we choose \((r, s) = (2, 5)\) for simplicity - this gives the problem of evaluating a hyperelliptic integral of genus 2. In this case there are three slits \(\gamma_i\), perpendicular to the real axis. The respective end points of these slits \(\hat{\lambda}_i\) are the Riemann invariants of the reduced Benney system characterised by this family of mappings. The points \(\hat{p}_i\) are the characteristic velocities of these invariants for the lowest non-trivial flow of the hierarchy.
The problem, in all these cases, reduces to finding a meromorphic function with \( r \) simple poles, as \( p \to \infty \) on each sheet. If the base point of the Abel map is taken as one of the branch points, then the Abel images of the point at infinity on the \( n \)-th sheet will be some point \( \hat{\zeta}^n v \); here \( v \) will depend on the moduli of the curve. In all known cases, the mapping splits into a sum of two terms, one holomorphic, while the other meromorphic term is written in terms of logarithmic derivatives of some derivative of \( \sigma \).

In [YG] this mapping was worked out in the elliptic case, giving the formula

\[
\lambda = k \left( \frac{d}{du} \ln(\sigma(u - v)\sigma(u + v)) \right) - ku + c,
\]

where \( k, c \) are constants.

Further, in [BG1] the analogous map was constructed explicitly for the case of a hyperelliptic curve of genus two, giving the result

\[
\lambda \simeq k \frac{d}{du_1} \ln(\sigma(u - v)\sigma(u + v)),
\]

up to holomorphic terms linear in \( u \).

In [BG2] this approach was extended to higher genus hyperelliptic curves. That result was not in the same form as those for lower genus; further for genus higher than three the resulting formulae make sense only as limits. A more systematic approach, using the expression for the Prime form given above, yields in all these cases an analogous formula:

\[
\lambda \simeq k \left( \frac{\partial}{\partial u_1} \ln(\sigma_{z_2}(u - u_0)\sigma_{z_2}(u + u_0)) \right),
\]
again up to holomorphic terms. This formula clearly reduces to the above two in those cases, consistent with the result given by Crowdy in \[C1\], \[C2\]. There he had used the Schottky-Klein prime function, which he was able to compute numerically for certain hyperelliptic curves.

There are similar problems with general non-hyperelliptic curves, two particular examples of which were studied in \[BG3\] and \[EG\]. In the latter case, it was noted that not only \(\sigma\), but all its first derivatives, vanished for \(u \in \mathcal{W}^1\), a phenomenon which also is found for higher-order hyperelliptic curves. The expression for the reduction given in \[EG\] was thus given in terms of a logarithmic derivative of a second derivative of \(\sigma\).

The methods used in \[MP1\]–\[MP2\] and above, are considerably more powerful; as these have enabled us to show a simple relationship between the Prime form and \(\sigma\), finding a general expression for such reductions should now be comparatively straightforward. The analogous formulae to those given above, in these cases, are expected to reduce to a sum of \(r\) logarithmic derivatives, each having a simple pole on one sheet, as \(p \to \infty\). The approach given in \[G\] suggests the appropriate formula for this mapping in this case is, up to holomorphic terms,

\[
\lambda \simeq k \sum_{i=0}^{r-1} \zeta^i \frac{\partial}{\partial v_1} \ln(\mathcal{E}(u - \hat{\zeta}^{-i}v)),
\]

where \(k\) is a known constant.

For the cyclic (3, 4) case, this lets us write down:

\[
\lambda \simeq k \sum_{i=0}^{2} \zeta^i \frac{\partial}{\partial v_1} \ln(\sigma(u - \hat{\zeta}^{-i}v)),
\]

where \(k\) is a known constant. This should be contrasted with an analogous expression for the momentum \(p\) in terms of \(u\)

\[
p \simeq k \sum_{i=0}^{2} \frac{\partial}{\partial v_1} \ln(\sigma(u - \hat{\zeta}^{-i}v)).
\]

The mapping from \(p\) to \(\lambda\) thus satisfies

\[
\lambda \simeq \zeta^i p + O(1) \quad \text{as } u \to \hat{\zeta}^i v.
\]

8.2. Further applications

Other conformal mapping problems might potentially also become more tractable using this approach. Particular examples include mathematical problems such as the construction of multiply-connected quadrature domains \[CM\], or various related physical problems such as vortex dynamics or Stokes flow in multiply-connected two dimensional regions.
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