Effective equations of motion and initial conditions for inflation in quantum cosmology

A.O. Barvinsky\textsuperscript{1} and A.Yu. Kamenshchik\textsuperscript{2}

\textsuperscript{1}Theory Department, Lebedev Physics Institute and Lebedev Research Center in Physics, Leninsky Prospect 53, Moscow 117924, Russia
\textsuperscript{2}L.D. Landau Institute for Theoretical Physics of Russian Academy of Sciences, Kosygina str. 2, Moscow 117334, Russia

Abstract

We obtain effective equations of inflationary dynamics for the mean inflaton and metric fields – expectation values in the no-boundary and tunneling quantum states of the Universe. The equations are derived in the slow roll approximation taking the form of the local Schwiger-DeWitt expansion. In this approximation effective equations follow from the Euclidean effective action calculated on the DeSitter gravitational instanton – the basic element of the no-boundary and tunneling cosmological wavefunctions. Effective equations are applied in the model of the inflaton scalar field coupled to the GUT sector of matter fields and also having a strong nonminimal coupling to the curvature. The inverse of its big nonminimal coupling constant, $-\xi = |\xi| \gg 1$, serves as a small parameter of the slow roll expansion and semiclassical expansion of quantum gravitational effects. As a source of initial conditions for effective equations we use a sharp probability peak recently obtained in the one-loop approximation for the no-boundary and tunneling quantum states and belonging (in virtue of big $|\xi|$) to the GUT energy scale much below the Planck scale. Cosmological consequences of effective equations in the tunneling quantum state predict a finite duration of inflationary stage compatible with the observational status of inflation theory, whereas for the no-boundary state they lead to the infinite inflationary epoch with a constant inflaton field.

\textsuperscript{1}e-mail: barvin@td.lpi.ac.ru
\textsuperscript{2}e-mail: kamen@landau.ac.ru

1. Introduction

It has recently been shown that quantum cosmology with the no-boundary \cite{1, 2, 3} and tunneling \cite{4} quantum states of the Universe can predict initial conditions for the inflationary
scenario [6, 7]. Until very recently this problem was regarded of marginal significance, but now it becomes important in view of raising interest in inflationary models with $\Omega \neq 1$ [7, 8]. In particular, the measure for the pre-inflationary initial conditions (prior probability) is essential for finding the posterior probability of the present value of $\Omega$.

Such an approach suffers from the known problems inherent in the tree-level approximation of quantum cosmology – the lack of normalizability of the cosmological wavefunction and the absence of necessary probability maxima [8, 9]. These problems can be resolved by including the loop effects [11, 12, 13, 14]. They modify the distribution function of the quantum ensemble of inflationary models with different initial values of the inflaton $\varphi$ – a scalar field driving the chaotic inflation with the Hubble constant $H = H(\varphi)$ [15]. In the one-loop approximation the distribution function of this field at the beginning of the Lorentzian quasi-DeSitter evolution has the form [11, 12, 13, 14]

$$\rho_{NB,T}(\varphi) = \text{const} e^{\mp I(\varphi) - \Gamma^{1-\text{loop}}(\varphi)}.$$  \quad (1.1)

Here $I(\varphi)$ is the classical Euclidean action of the model at the gravitational instanton – 4-dimensional (quasi)sphere of the radius $1/H(\varphi)$ and $\Gamma^{1-\text{loop}}(\varphi)$ is the Euclidean one-loop effective action of all quantum fields of the model calculated at this instanton. Important peculiarity of this algorithm is that in contrast with opposite signs of the tree-level part (minus and plus correspond respectively to the no-boundary and tunneling quantum states) the one-loop corrections are the same for both cosmological wave functions [14].

Depending on the anomalous scaling behaviour of the particle physics model, the one-loop term of (1.1) can suppress big values of $\varphi$ making $\rho_{NB,T}(\varphi)$ normalizable in the high energy limit [11]. Moreover, in the model with large nonminimal coupling of the inflaton field to curvature and typical couplings to Higgs, vector gauge and spinor matter fields the distribution function (1.1) has a sharp peak at the grand unification energy scale. For the tunneling quantum state this peak generates the inflationary scenario compatible with the observational status of the inflation theory [3, 5]. In particular, at least in context of closed cosmology considered in [3, 5], it is capable of producing sufficient but finite e-folding number and thus gives rise to intermediate values of $\Omega \neq 1$ without invoking the anthropic considerations of [7] or using exotic supergravity induced inflaton potentials of [8].

These conclusions have been drawn [11, 12, 13, 6] in the assumption that the solutions of classical equations of inflationary dynamics are weighted by the quantum distribution function (1.1). However, the quantum effects qualitatively change the behaviour of the tree-level distribution and, therefore, they are not small. This means that within the same accuracy classical equations of motion should be replaced by the effective equations for the mean inflaton field. Thus, the main goal of this paper will be to derive such equations and infer their consequences at the initial stage of inflation. In particular, we will clarify the most important qualitative aspect of effective dynamics – the direction of evolution from the point of the probability maximum in both quantum states of the Universe.
We consider the cosmological model with the total Lagrangian

\[ L(g_{\mu\nu}, \varphi, \chi, A_\mu, \psi) = L(g_{\mu\nu}, \varphi) + g^{1/2} \left( -\frac{1}{2} \sum_\chi (\nabla \chi)^2 - \frac{1}{4} \sum_A F^2_{\mu\nu}(A) - \sum_\psi \bar{\psi} \nabla \psi \right) + L_{\text{int}}(\varphi, \chi, A_\mu, \psi) \] (1.2)

containing the graviton-inflaton sector

\[ L(g_{\mu\nu}, \varphi) = g^{1/2} \left\{ \frac{m_P^2}{16\pi} R(g_{\mu\nu}) - \frac{1}{2} \xi \varphi^2 R(g_{\mu\nu}) - \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda \varphi^4}{4} \right\}, \] (1.3)

with a big negative nonminimal coupling constant \(-\xi = |\xi| \gg 1\), and generic GUT sector of Higgs \(\chi\), vector gauge \(A_\mu\) and spinor fields \(\psi\) coupled to the inflaton via the interaction term

\[ L_{\text{int}} = \sum_\chi \frac{\lambda_\chi}{4} \chi^2 \varphi^2 + \sum_A \frac{1}{2} g^2 A^2_\mu \varphi^2 + \sum_\psi f_\psi \varphi \bar{\psi} \psi + \text{derivative coupling}. \] (1.4)

This model is of a particular interest for a number of reasons. Firstly, from the phenomenological viewpoint a strong nonminimal coupling allows one to solve the problem of exceedingly small \(\lambda\) (because the observable magnitude of anisotropy \(\Delta T/T \sim 10^{-5}\) is proportional in this model to the ratio \(\sqrt{\lambda}/|\xi|\) \[16\]), and for positive \(\xi\) \[17\] this model is useful for generating inflation with \(\Omega \neq 1\) \[18\]. Secondly, this coupling is inevitable from the viewpoint of renormalization theory. Finally, for a wide class of GUT-type particle physics models \[1.2\] due to a big value \(|\xi|\) there exists a sharp probability peak in \(\rho_{NB,T}(\varphi)\) \[5,6\]. This peak belongs to GUT energy scale – a characteristic value of the effective Hubble constant driving inflation, which is proportional to \(m_P \sqrt{\lambda}/|\xi| \sim 10^{-5} m_P\). This, in its turn, justifies the use of GUT for matter field sector of the model, because this energy scale is much below the supersymmetry and string theory scales.

In the Lagrangian \[1.2\] the inflaton field can be regarded as a component of one of the Higgs multiplets \(\chi\), which has a nonvanishing expectation value in the cosmological quantum state. The inflaton has a quartic selfinteraction and mass term \(m^2 \varphi^2/2\) which for generality can be negative \((m^2 < 0)\), thus, including the case of symmetry breaking. The choice of the interaction Lagrangian \[1.4\] is dictated by the renormalizability of the matter field sector of the theory \[1.2\] and by the requirement of local gauge invariance with respect to arbitrary Yang-Mills group of vector fields \(A_\mu\). The terms of derivative coupling in \[1.4\] should be chosen to guarantee the latter property, but their form is not important. On the contrary, as shown in \[5,6\], the quantum gravitational effects generating the probability peak of the above type crucially depend on the nonderivative part of the interaction Lagrangian.

The organization of the paper is as follows. In Sect.2 we consider the classical inflation dynamics of the graviton-inflaton model \[1.3\] depending on initial conditions generated by the probability peak of \(\rho_{NB,T}(\varphi)\). In Sect.3 the derivation of effective equations for mean fields is outlined on the basis of the Euclidean effective action of the theory. Sects.4 and 5 are devoted to the effective action calculations in the slow-roll approximation equivalent in this
context to the local Schwinger-DeWitt expansion. In Sect. 6 we compare the cosmological consequences of the obtained effective equations for the tunneling and no-boundary quantum states and conclude that phenomenologically the tunneling wavefunction is a more preferable candidate for the initial state of the early Universe. In concluding section we briefly comment on the possibility of extending our results to the case of the open inflation originating from the Hawking-Turok instanton via the no-boundary proposal [8] and the tunnelling proposal of Linde [8]. We also discuss the limitations of the obtained results and their nonperturbative extension and conjecture on the contribution of the quantum mechanical sector of the model (quantum homogeneous mode of the inflaton field) which goes beyond the scope of this paper and will be considered in future publications.

2. Classical equations of motion for chaotic inflation

Here we consider classical equations of inflationary dynamics in the model with the Lagrangian (1.3). For our future purposes we generalize this model to have generic coefficients – functions of the inflaton field – for the effective cosmological term \( V(\varphi) \) and effective gravitational constant \( U(\varphi) \) [19, 20]:

\[
S[g_{\mu\nu}, \varphi] = \int d^4x g^{1/2} \left( U(\varphi) R(g_{\mu\nu}) - \frac{1}{2} (\nabla \varphi)^2 - V(\varphi) \right).
\] (2.1)

With the minisuperspace Robertson-Walker ansatz for the spacetime metric of spatially closed cosmological model (in the cosmic time gauge, \( g_{00} = -1 \), and with the scale factor \( a = a(t) \)) with the spatially homogeneous inflaton field \( \varphi = \varphi(t) \), the equations of motion take the form:

\[
12aU\ddot{a} + 6U\dot{a}^2 + 12aU'\dot{a}\dot{\varphi} + 6a^2U''\varphi^2 + 6a^2U'\dot{\varphi}^2 - 3a^2V = 0,
\] (2.2)

\[
a^3\ddot{\varphi} + 3a^2\dot{a}\dot{\varphi} - 6aU'\ddot{a} - 6a^2U'\dot{a} - 6aU' + a^3V' = 0,
\] (2.3)

\[
a^3 \left( V + \frac{\dot{\varphi}^2}{2} \right) - 6a^3U \left( \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} \right) - 6a^2U'\dot{a}\dot{\varphi} = 0.
\] (2.4)

where the dots denote time derivatives and the prime denotes the derivative of the coefficient functions of the Lagrangian with respect to \( \varphi \). The dynamical equations (2.2) - (2.3) can be solved with respect to second order time derivatives of \( a \) and \( \varphi \) with the result for \( \ddot{\varphi} \):

\[
\ddot{\varphi} = \frac{1}{U + 3U'^2} \left( -\frac{3U\dot{a}\dot{\varphi}}{a} - \frac{6U'^2\dot{\varphi}}{a} + \frac{3UU'\dot{a}^2}{a^2} - 3U'U''\dot{\varphi}^2 - \frac{3U'\dot{\varphi}^2}{4} + \frac{3UU'}{a^2} + \frac{3VV' + UV'}{2} - UV' \right),
\] (2.5)

The constraint equation (2.4) in its turn can be solved with respect to the Hubble “constant” \( \dot{a}/a \) which, when substituted to (2.5), gives

\[
\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} - \frac{1}{U + 3U'^2} \left( 2VV' - UV' - \frac{1}{2} U'\dot{\varphi}^2 - 3U'U''\dot{\varphi}^2 \right) = 0.
\] (2.6)
In the slow-roll regime one can neglect the terms quadratic (and of higher powers) in time derivatives of $\varphi$, so that the system of equations reduces to the expression for the effective Hubble constant
\[
\left(\frac{\dot{a}}{a}\right)^2 = H^2(\varphi) \approx \frac{V(\varphi)}{6U(\varphi)} \tag{2.7}
\]
and the equation of motion for the inflaton field
\[
\ddot{\varphi} + 3H\dot{\varphi} - F(\varphi) = 0, \tag{2.8}
\]
\[
F(\varphi) \approx \frac{2VV' - UV'}{U + 3U^2} = -\frac{U^3}{U + 3U^2} \frac{d}{d\varphi} \left(\frac{V}{U^2}\right), \tag{2.9}
\]
evolving under the action of the rolling force $F(\varphi)$ and the “friction” force $-3H\dot{\varphi}$. For constant $U \equiv m_p^2/16\pi$ the rolling force reduces to the usual gradient of the scalar field potential, while for a nonminimal inflaton it is proportional to the gradient of the modified potential $V(\varphi)/U^2(\varphi)$ renormalized by the nonminimal coupling $\xi$.

Quantum initial conditions for inflation stage crucially depend on the value of the parameter
\[
\delta = -\frac{8\pi |\xi| m^2}{\lambda m_p^2}, \tag{2.10}
\]
characterizing the model (1.3). As shown in [5, 6] the probability peak in the distribution functions of $\varphi$ for the no-boundary and tunneling quantum states exists in complimentary domains of $\delta$: for $\delta < -1$ in the no-boundary case and for $\delta > -1$ for the tunneling one. The parameters of this peak – mean value and relative width – for a large value of the nonminimal coupling $|\xi| \gg 1$ are given in both cases by the same expressions
\[
\varphi_I = m_p \sqrt{8\pi |1 + \delta| |A|}, \quad H(\varphi_I) = m_p \sqrt{\lambda} \frac{\sqrt{2\pi |1 + \delta|}}{3A^2}, \tag{2.11}
\]
\[
\frac{\Delta \varphi}{\varphi_I} \sim \frac{\Delta H}{H} \sim \frac{1}{\sqrt{12A} |\xi|}, \tag{2.12}
\]
where $A$ is the following combination of Higgs, vector gauge boson and Yukawa coupling constants of the GUT-inflaton interaction Lagrangian (1.4)
\[
A = \frac{1}{2\lambda} \left(\sum_x \lambda_x^2 + 16 \sum_A g_A^4 - 16 \sum f_{\psi}^4\right), \tag{2.13}
\]
but correspond to opposite signs of the rolling force generating different inflationary scenarios
\[
F(\varphi) = -\frac{\lambda m_p^2(1 + \delta)}{48\pi \xi^2} \varphi + O(1/|\xi|^3). \tag{2.14}
\]

\[\text{1 The combination } V(\varphi)/U^2(\varphi) \text{ coincides with the inflaton potential in the Einstein frame of the action (2.1) that can be obtained by the conformal transformation of the metric and special reparametrization of the inflaton field [19].} \]
For no-boundary state the maximum of the distribution function $\varphi_I$ belongs to the negative slope of the potential $V(\varphi)/U^2(\varphi)$ and $F(\varphi_I) > 0$, $1 + \delta < 0$. This results in the slow-roll regime in the infinitely long inflationary stage with ever growing inflaton field

$$\dot{\varphi} \simeq \frac{1}{3H(\varphi)} F(\varphi). \quad (2.15)$$

For the tunneling state $\varphi_I$ lies on the positive slope of $V(\varphi)/U^2(\varphi)$ with $F(\varphi_I) < 0$, $1 + \delta > 0$, and the inflationary stage has a slowly decreasing scalar field and finite duration with an approximate e-folding number $\mathbb{E} \mathbb{I}$

$$N \simeq - \int_0^{\varphi_I} d\varphi \frac{H(\varphi)}{\dot{\varphi}} \simeq \frac{48\pi^2}{A}. \quad (2.16)$$

These conclusions are, however, based on classical equations of motion in contrast with the quantum nature of initial conditions originating from the maximum of the quantum distribution function – the quantity drastically different from its tree-level counterpart. The purpose of this paper is to cure this mismatch by replacing the classical equations with the effective equations for expectation values.

### 3. Effective equations for expectation values

Effective equations of motion for expectation values of operators of the total system of fields

$$\phi(x) = \langle \Psi | \hat{\phi}(x) | \Psi \rangle, \quad (3.1)$$

$$\hat{\phi}(x) = \varphi(x), \hat{\chi}(x), \hat{\psi}(x), \hat{A}_\mu(x), \hat{g}_{\mu\nu}(x), ... \quad (3.2)$$

with respect to any quantum state including the no-boundary and tunneling ones, $|\Psi\rangle = |\Psi\rangle_{NB}; |\Psi\rangle_T$, have a generic form

$$\frac{\delta S[\phi]}{\delta \phi(x)} + J^{rad}(x) = 0. \quad (3.3)$$

Here the radiation current $J^{rad}(x)$ accumulates all quantum corrections which begin with the one-loop contribution

$$J^{rad}(x) = \frac{1}{2i} \int dy \, dz \frac{\delta^3 S[\phi]}{\delta \phi(x) \delta \phi(y) \delta \phi(z)} G(z,y) + ... \quad (3.4)$$

*Eq. (2.16)* is valid up to numerical factor $1 + O(\epsilon \ln \epsilon)$, $\epsilon = A/32\pi^2$, slightly different from unity. These corrections correspond to late stages of inflation when lower order terms of $U(\varphi)$ and $V(\varphi)$ come into game. Unfortunately, the references $\mathbb{E} \mathbb{I}$ contain a typo in the leading term of the e-folding number $\mathbb{E} \mathbb{I}$: $8\pi^2$ instead of the correct value $48\pi^2$. This correction raises the upper bound on the universal combination of coupling constants (following from the lower bound on $N$, $N \geq 60$), $A \leq 7.9$, but does not change qualitatively the conclusions of $\mathbb{E} \mathbb{I}$, leaving us with a small parameter $A/32\pi^2 \ll 1$.  

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containing the Wightman function of quantum disturbances in a given quantum state

\[ G(z, y) = \langle \Psi | \Delta \hat{\phi}(z) \Delta \hat{\phi}(y) | \Psi \rangle, \quad \Delta \hat{\phi}(y) \equiv \hat{\phi}(y) - \phi(y). \]  

(3.5)

In the sector of spacetime metric \( \phi(x) = g_{\mu\nu}(x) \) this radiation current coincides, in particular, with the expectation value of the quantum matter stress tensor \( J^{\mu\nu}_{\text{rad}}(x) = \langle \Psi | \hat{T}^{\mu\nu}(x) | \Psi \rangle \).

The calculation of \( J^{\mu\nu}_{\text{rad}}(x) \) even in the one-loop approximation generally presents a hard problem, because the Wightman Green’s function in the external mean field of arbitrary configuration comprises a very complicated nonlocal object that cannot be obtained exactly. Fortunately, the model in question has a number of peculiarities that essentially simplify calculations and look as follows.

To begin with, note that in our model with a large negative constant \( -\xi = |\xi| \gg 1 \) and slowly varying inflaton field the nonminimal coupling efficiently implies a replacement of the Planckian mass parameter \( m_P^2 \) by the effective mass of a much bigger magnitude

\[ m_P^2 \to m_{\text{eff}}^2 = m_P^2 + 8\pi|\xi|\varphi^2 \gg m_P^2. \]  

(3.6)

This essentially improves the semiclassical expansion of quantum gravitational effects, because this expansion goes in inverse powers of \( m_{\text{eff}}^2 \) rather than of \( m_P^2 \).

Big value of \( |\xi| \) has also another important effect caused by the Higgs mechanism for all matter fields interacting with inflaton. As discussed in [5, 6], due to this interaction the corresponding matter particles acquire masses proportional to the background value of the inflaton field, \( m_{\text{part}}^2 \sim \varphi^2 \), but in view of eq.(2.7) (with \( V \) and \( U \) read off the classical Lagrangian (1.3)) the spacetime curvature has an order of magnitude \( R \sim H^2 \sim \lambda\varphi^2/|\xi| \ll \varphi^2 \). Therefore, quantum contribution of matter fields can be expanded in local Schwinger-DeWitt series [21, 22] in powers of the curvature to mass squared ratio

\[ \frac{R}{m_{\text{part}}^2} \sim \frac{\lambda}{|\xi|} \ll 1, \]  

(3.7)

the first few terms giving a dominant contribution polynomial in \( |\xi| \gg 1 \). In the limit of big \( |\xi| \) these terms dominate over contribution of all other fields uncoupled to inflaton and, in particular, over the contribution of the graviton-inflaton sector. Below we show this property by direct calculations in the one-loop approximation. The mechanism of this result is based on the improvement of semiclassical expansion due to the replacement (3.6) and, apparently, holds in multi-loop orders as well.

Finally, in our setting of the problem only two fields have nonvanishing expectation values – spacetime metric and inflaton scalar field. We assume that the slow roll approximation remains applicable also at the quantum level, which means that in the leading order of this approximation the mean spacetime metric corresponds to DeSitter geometry and the mean inflaton field is a spacetime constant scalar. It is also well known that for all massive and/or spatially inhomogeneous modes of fields the no-boundary and tunneling cosmological states turn out to be the Euclidean DeSitter invariant vacuum [23, 24]. Therefore, in the one-loop approximation the Wightman Green’s function (3.5) of such modes \( \Delta \hat{\phi}(y) \) – solutions of
linerized Heisenberg operators – is uniquely fixed by the choice of this vacuum \( \Phi \). The exception from this simple rule are massless scalar fields for which the Euclidean DeSitter vacuum does not exist \( \Phi \) and effectively massless inflaton mode of the graviton-inflaton sector of the model. The quantum state of this mode is not the DeSitter invariant vacuum – the tree-level approximation for the no-boundary and tunneling cosmological states. Rather it is a special state generating due to loop corrections the peak-like distribution function which was obtained in \( \Phi \) where it was shown to be drastically different from that of the DeSitter vacuum. The contribution of this mode to radiation current, violating the DeSitter invariance of effective equations, goes beyond the scope of this paper. It is likely that by the big \( |\xi| \) mechanism of the above type this graviton-inflaton sector does not contribute to the leading order of \( 1/|\xi| \)-expansion, which justifies discarding its peculiarities.

With this reservation, the Wightman Green’s function (3.3) can be regarded the Euclidean DeSitter invariant one. In Lorentzian DeSitter spacetime it can be obtained by a proper analytic continuation from the unique regular Green’s function on the Euclidean section of the DeSitter space \( \Phi \) – a 4-dimensional sphere of the radius \( 1/H(\varphi) \). Taken together with the local Schwinger-DeWitt expansion of \( J^{\text{rad}}(x) \) discussed above this means that the radiation current itself can be obtained by this analytic continuation from the Euclidean radiation current which, in its turn, expresses in terms of the Euclidean effective action

\[
J^{\text{rad}}_E(x) = \frac{\delta \Gamma^{\text{loop}}}{\delta \phi(x)},
\]

\[
\Gamma^{\text{loop}} = \frac{1}{2} \text{Tr} \ln \frac{\delta^2 I[\phi]}{\delta \phi \delta \phi} + ..., \tag{3.9}
\]

where \( I[\phi] \) is the classical Euclidean action of the theory related to the Lorentzian action by standard Wick rotation

\[
I[\phi] = -iS[\phi] \bigg|_{-+++ \rightarrow +++}, \tag{3.10}
\]

Thus the Lorentzian effective equations in the approximation of local Schwinger-DeWitt expansion can be obtained by analytically continuing back to Lorentzian signature from the Euclidean effective equations

\[
\frac{\delta \Gamma}{\delta \phi(x)} = 0 \tag{3.11}
\]

where the Euclidean effective action \( \Gamma = I + \Gamma^{\text{loop}} \) is calculated within this local low-derivative expansion\( \Phi \) or the quasilocal expansion of \( \Phi \) (also considered in the cosmological context in \( \Phi \), \( \Phi \)). For the purposes of our slow-roll approximation we need only the first three terms

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3 The linearized equation of the inflaton mode has a very small mass parameter suppressed by the factor \( O(\lambda/\xi^2) \), and its smallness guarantees the validity of the slow roll approximation – the corner stone of inflation theory.

4 This simple rule of obtaining the Lorentzian effective equations from their Euclidean counterpart is an artifact of two important properties: i) analytic relation between the Green’s functions on the Lorentzian and Euclidean sections of the DeSitter geometry and ii) approximation of local low-derivative expansion. In
of this expansion in powers of derivatives. They reproduce the structures of the classical gravitational action \([2.1]\)

\[
\Gamma[\varphi, g_{\mu\nu}] = \int d^4x g^{1/2} \left( V_{\text{eff}}(\varphi) - U_{\text{eff}}(\varphi) R + \frac{1}{2} G_{\text{eff}}(\varphi) g^{\mu\nu} \varphi,_{\mu} \varphi,_{\nu} + \ldots \right). \tag{3.12}
\]

with the effective coefficient functions of the inflaton field

\[
V_{\text{eff}}(\varphi) = V(\varphi) + V^{\text{loop}}(\varphi),
\]

\[
G_{\text{eff}}(\varphi) = 1 + G^{\text{loop}}(\varphi),
\]

\[
U_{\text{eff}}(\varphi) = U(\varphi) + U^{\text{loop}}(\varphi), \tag{3.13}
\]

modified by quantum terms (the overall sign of this expression differs from \([2.1]\) in view of the Wick rotation to the Euclidean signature). The latter will be built in the next section.

4. One-loop effective action in the slow-roll approximation: massive GUT sector

For massive fields the inverse propagator in \([3.9]\) generically has the form of the covariant differential operator acting in the vector space labeled by their isotopic indices

\[
\frac{\delta^2 I[\phi]}{\delta \phi(x) \delta \phi(y)} = \left( F(\nabla) - m^2 \hat{1} \right) \delta(x - y),
\]

\[
F = \Box + \hat{P} - \frac{1}{6} \hat{1}, \quad \Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu, \tag{4.1}
\]

with the spacetime-dependent potential term \(\hat{P} - \hat{1}R/6\) (in which the curvature scalar term is extracted for reasons of convenience). The matrix form of its coefficients is denoted by hats and \(\hat{1}\) means the unit matrix. The one-loop effective action of such fields

\[
\Gamma^{1-\text{loop}} = \frac{1}{2} \text{Tr} \ln \left( F - m^2 \hat{1} \right) \tag{4.3}
\]

can be decomposed for large mass \(m\) in the local Schwinger-DeWitt series \([21, 22]\)

\[
\Gamma^{1-\text{loop}} = -\frac{1}{32\pi^2} \int d^4x g^{1/2} \text{tr} \left\{ \frac{1}{2} \left( \frac{1}{2 - \omega} - \ln \frac{m^2}{\mu^2} + \frac{3}{2} \right) m^4 \hat{a}_0(x, x) 
+ \left( \frac{1}{2 - \omega} - \ln \frac{m^2}{\mu^2} + 1 \right) m^2 \hat{a}_1(x, x) + \left( \frac{1}{2 - \omega} - \ln \frac{m^2}{\mu^2} \right) \hat{a}_2(x, x) 
+ \sum_{n=1}^{\infty} \frac{(n - 1)!}{m^{2n}} \hat{a}_{n+2}(x, x) \right\}, \quad \omega \rightarrow 2. \tag{4.4}
\]

the approximation of rapidly varying fields nonlocal effective equations for expectation values do not even have the form of the functional derivative of some effective action \([29]\). This is a distinctive feature of the diagrammatic technique for expectation values which is different from the conventional technique for matrix elements between two different states – in and out vacua.
Here $\hat{a}_n(x, x)$ are the Schwinger-DeWitt coefficients that can be systematically calculated for generic theory as spacetime invariants of growing power in spacetime and fibre bundle curvatures, potential term of the operator and their covariant derivatives. For example \cite{21, 30, 22, 31}

\begin{align}
\hat{a}_0(x, x) &= \hat{1}, \\
\hat{a}_1(x, x) &= \hat{P}, \\
\hat{a}_2(x, x) &= \frac{1}{180}(R_{\mu\nu\alpha\beta}^2 - R_{\mu\nu}^2 + \Box R)\hat{1} + \frac{1}{2}\hat{P}^2 + \frac{1}{12}\hat{R}_{\mu\nu}^2 + \frac{1}{6}\Box \hat{P}, \\
\hat{a}_3(x, x) &= \frac{1}{12}(\nabla \hat{P})^2 + \frac{1}{12}(\hat{P} \Box \hat{P} + \Box \hat{P} \hat{P}) + ..., 
\end{align}

where $\hat{R}_{\mu\nu}$ determines the commutator of covariant derivatives acting on fields $\phi$, $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \phi = \hat{R}_{\mu\nu} \phi$ and in $\hat{a}_3(x, x)$ only terms bilinear in $\hat{P}$ with two derivatives are retained.

In (4.4) $\omega$ is half the dimensionality of spacetime serving as a parameter of the dimensional regularization, $\mu_2$ is a parameter reflecting the renormalization ambiguity and tr denotes the matrix (super)traces with respect to isotopic indices of $\hat{a}_n(x, x)$.

In our case the only background field arguments of the effective action consist of slowly varying scalar field and spacetime metric with the curvature satisfying inequality (3.7). From the structure of equations (4.5)-(4.7) it then follows that the renormalized effective action

$$\Gamma^{1-\text{loop}} = \frac{1}{32\pi^2} \int d^4x \, g^{1/2} \text{tr} \left\{ \frac{m^4}{2} \left( \ln \frac{m^2}{\mu^2} - \frac{3}{2} \right) \hat{1} + -m^2 \left( \ln \frac{m^2}{\mu^2} - 1 \right) \hat{a}_1(x, x) + \ln \frac{m^2}{\mu^2} \hat{a}_2(x, x) + \frac{\hat{a}_3(x, x)}{m^2} \right\} + ...$$

is dominated by the contribution of the effective potential due to $\hat{a}_0(x, x) = \hat{1}$ and the gravitational and inflaton kinetic terms of second order in derivatives originating from $\hat{a}_1(x, x)$,

$$\int d^4x \, g^{1/2} \text{tr} \hat{a}_1(x, x) = \int d^4x \, g^{1/2} \left[ u R + g(\nabla \phi)^2 \right]$$

with some coefficients $u$ and $g$ depending on spin, $\hat{a}_2(x, x)$ and $\hat{a}_3(x, x)$. For minimally coupled fields $g = 0$, and $\hat{a}_2(x, x)$ gives only a fourth-order contribution of the derivative expansion. The radiative corrections to the kinetic term of the scalar field originate from the effective $\phi(x)$-dependence of masses $m^2 \sim \phi^2(x)$, and their contribution comes from the terms (4.8) of the third DeWitt coefficient.

As mentioned above, due to Higgs mechanism in the slowly varying inflaton field the GUT particles of the model (1.2) acquire slowly variable masses \cite{13, 14, 15, 16}

$$m^2 = (m^2_\chi, m^2_A, m^2_\psi),$$

$$m^2_\chi = \frac{\lambda \phi^2(x)}{2}, \quad m^2_A = g_A^2 \phi^2(x), \quad m^2_\psi = f_\psi^2 \phi^2(x).$$

This $\phi$-dependence of masses converts the Schwinger-DeWitt series (4.3) into the quasilocal expansion of refs. \cite{26, 27, 28} in which the sums over contributions of such massive particles
of spin 0, spin 1 and spin 1/2 have the form (taking into account the statistics encoded in the supertrace operation $\text{tr}$)

$$\text{tr} \sum \frac{m^4}{2} \hat{1} = \frac{\lambda \varphi^4}{4} A, \quad (4.13)$$

$$\text{tr} \sum \frac{m^4}{2} \ln \frac{m^2}{\mu^2} \hat{1} = \frac{\lambda \varphi^4}{4} \left( A \ln \frac{\varphi^2}{\mu^2} - B \right), \quad (4.14)$$

where $A$ is given by (2.13) and

$$B = -\frac{1}{2\lambda} \left( \sum_{\chi} \lambda_{\chi}^3 \ln(\lambda_{\chi}/2) + 16 \sum_{A} g_{A}^2 \ln g_{A}^2 - 16 \sum_{\psi} f_{\psi}^2 \ln f_{\psi}^2 \right). \quad (4.15)$$

Similarly, in view of the known expressions for $\hat{a}_1(x,x)$ of scalar, vector and spinor fields

$$\text{tr} \sum \frac{m^2}{\mu^2} \hat{a}_1(x,x) = \frac{1}{12} C \varphi^2 R, \quad (4.16)$$

$$\text{tr} \sum m^2 \ln \frac{m^2}{\mu^2} \hat{a}_1(x,x) = \frac{1}{12} \left( C \ln \frac{\varphi^2}{\mu^2} - D \right) \varphi^2 R, \quad (4.17)$$

$$C = \sum_{\chi} \lambda_{\chi} - 4 \sum_{A} g_{A}^2 + 4 \sum_{\psi} f_{\psi}^2, \quad (4.18)$$

$$D = \sum_{\chi} \lambda_{\chi} \ln(\lambda_{\chi}/2) - 4 \sum_{A} g_{A}^2 \ln g_{A}^2 + 4 \sum_{\psi} f_{\psi}^2 \ln f_{\psi}^2. \quad (4.19)$$

Therefore, the one-loop contributions of massive GUT fields to the effective coefficient functions of the potential and curvature terms in (3.12) equal

$$V^{1-\text{loop}}(\varphi) = \frac{\lambda \varphi^4}{128\pi^2} \left( A \ln \frac{\varphi^2}{\mu^2} - B - \frac{3}{2} A \right), \quad (4.20)$$

$$U^{1-\text{loop}}(\varphi) = \frac{\varphi^2}{384\pi^2} \left( C \ln \frac{\varphi^2}{\mu^2} - D - C \right). \quad (4.21)$$

The first two DeWitt coefficients of these GUT fields do not contribute to the kinetic term of the inflaton. The contribution to $G_{\text{eff}}(\varphi)$ in (3.12) comes from $\hat{a}_3(x,x)$ containing the terms (4.8). The $\varphi(x)$-dependent masses generate spacetime dependent $\hat{P}$ with $\nabla_{\mu} \hat{P} \sim -\nabla_{\mu}[m^2(x)] \sim -\nabla_{\mu}[\varphi^2(x)]$ and via these terms lead to finite (renormalization unambiguous) contribution

$$\int d^4x \frac{G^{1/2}}{G} \sum \frac{\hat{a}_3(x,x)}{m^2} = -\frac{1}{6} E \int d^4x g^{1/2}(\nabla \varphi)^2 + ..., \quad (4.22)$$

$$E = \sum_{\chi} \lambda_{\chi} + 8 \sum_{A} g_{A}^2 - 8 \sum_{\psi} f_{\psi}^2, \quad (4.23)$$

A detailed derivation is based on splitting the total potential term of the operator (4.1)-(4.2) into an auxiliary strictly constant mass parameter and a spacetime dependent part $\hat{P}$ with a subsequent perturbation theory in $\hat{P}$.
where ellipses denote the terms of the fourth order in derivatives. Thus, $G^{1-\text{loop}}(\varphi)$ generated by this sector of the theory does not contain renormalization ambiguous logarithms and equals

$$G^{1-\text{loop}}(\varphi) = \frac{1}{6} \frac{E}{32\pi^2}. \quad (4.24)$$

For massless fields or the fields not coupled to the inflaton the local Schwinger-DeWitt expansion does not work and their effective action should be calculated within alternative approximation schemes, like $\zeta$-functional method used in [33, 6]. Comparison of the obtained coefficient functions with the results of [6] shows the coincidence of the dominant logarithmic contribution proportional to $m^4 \ln(m^2/\mu^2)$. Typical masses of particles not coupled to the inflaton are much lower in magnitude than the masses (4.12) at the maximum of the distribution function (1.1), and the ratio of their quantum corrections to those of (4.20) is $m^4/\lambda \varphi^4 \mathcal{A} \sim \lambda \mathcal{A}/(64\pi^2)^2 \ll 1$ (here we assume that with $\delta = O(1)$ it follows from eq. (2.10) that $m^2 \sim m_P^2 (\lambda/8\pi|\xi|$). Thus, such fields can be discarded relative to the distinguished sector of (1.2). The only exception is the graviton-inflaton sector of the model which explicitly involves large parameter $|\xi|$ and, thus, apriori can give a big contribution. In the next section we show that it is actually suppressed by the powers of $1/|\xi|$.

5. One-loop effective action: graviton-inflaton sector

Renormalization of ultraviolet divergences in the theory with the action (2.1) was considered in [19]. Its one-loop divergences have the form

$$\Gamma_{\text{div}}^{1-\text{loop}}[\varphi, g_{\mu\nu}] = \frac{1}{32\pi^2(2 - \omega)} \int d^4x \, g^{1/2} \left\{ \frac{5}{2} \frac{V^2}{U^2} - 2U^2 \left( \frac{\partial V}{\partial \phi} \right)^2 + \frac{1}{2} U^2 \left( \frac{\partial^2 V}{\partial \phi^2} \right)^2 
\right. \\
+ \left[ \left( \frac{45}{2} U^{-3}(U')^2 + U^{-2}G \right) V - 13U^{-2}U'V' \\
- \left( \frac{25}{4} U^{-1}(U')^2 + 2G + \frac{1}{2} U' \frac{d}{d\varphi} \frac{\partial^2 V}{\partial \phi^2} \right) (\nabla \varphi)^2 \\
- \left[ \frac{13}{3} U^{-1}V + \frac{1}{6} U \frac{\partial^2 V}{\partial \phi^2} \right] R \\
\left. + \frac{43}{60} R_{\alpha\beta} + \frac{1}{40} R^2 + O(R\nabla^2 \varphi, \nabla^4 \varphi) \right\}, \quad (5.1)$$

where $O(R\nabla^2 \varphi, \nabla^4 \varphi)$ denotes the terms of the overall fourth order in derivatives – part of them linear in the curvature times second derivatives of the scalar field and the rest with quartic derivatives of the scalar field[^6]. In contrast with the renormalization by matter fields considered above, here the one-loop counterterm also includes kinetic term of the scalar

[^6]: Important correction is in order here. The equation (5.1) above was presented in ref. [19] under the number (2.72) with the erroneous coefficient 5/2 of the first term instead of the correct coefficient 5.
field. The coefficients of this expression – renormalizing the effective potential, kinetic and Einstein terms – are complicated functions of the classical quantities $V(\varphi)$, $G(\varphi)$, $U(\varphi)$ (for generality $G(\varphi)$ is taken here different from 1 also at the classical level). They involve these functions and their derivatives with respect to $\varphi$ denoted by primes, as well as auxiliary inflaton potential

$$\bar{V}(\phi) = \frac{V(\varphi)}{U^2(\varphi)}\bigg|_{\varphi=\varphi(\phi)}$$

in the auxiliary parametrization of the scalar field $\varphi = \varphi(\phi)$ defined by the differential equation

$$\left(\frac{d\phi}{d\varphi}\right)^2 = U^{-2}(\varphi) \left[U(\varphi)G(\varphi) + 3U'^2(\varphi)\right].$$

The derivatives of $\bar{V}(\phi)$ express as

$$\frac{\partial \bar{V}}{\partial \phi} = \frac{UV' - 2U'V}{U^2(UG + 3U'^2)^{1/2}},$$

$$\frac{\partial^2 \bar{V}}{\partial \phi^2} = \frac{1}{(UG + 3U'^2)^2} \left[12U^{-2}(U')^4V - 9U^{-1}(U')^3V' + 3(U')^2V'' - 3U'U''V' + 5U^{-1}(U')^2GV - 2U''GV + UGV'' - \frac{7}{2}U'GV + U'G'V - \frac{1}{2}UG'V'\right],$$

which makes the algorithm (5.1) explicit in terms of the original coefficient functions of the classical Lagrangian. With the tree-level expressions for the latter

$$V = \frac{m^2\varphi^2}{2} + \frac{\lambda\varphi^4}{4}, \quad G = 1, \quad U = \frac{m_P^2}{16\pi^2} + \frac{1}{2}|\xi|\varphi^2,$$

one has the cancellation of the leading powers of $\xi$ and $\varphi$ in (5.4) and (5.3) for large $|\xi| \gg 1$ (till the end of this section we discard in $V(\varphi)$ the subleading term $m^2\varphi^2/2$)

$$\frac{\partial V}{\partial \phi} = O(1/|\xi|^3), \quad \frac{\partial^2 V}{\partial \phi^2} = O(1/|\xi|^3)$$

and from (5.1) obtains the leading behaviour for the divergent parts of the one-loop coefficient functions in the graviton-inflaton effective action

$$V_{div}^{1-\text{loop}} = \frac{1}{32\pi^2(2 - \omega)} \left[\frac{5\lambda^2\varphi^4}{4|\xi|^2} + O(1/|\xi|^3)\right],$$

$$G_{div}^{1-\text{loop}} = \frac{1}{32\pi^2(2 - \omega)} \left[-\frac{7\lambda}{|\xi|} + O(1/|\xi|^2)\right],$$

$$U_{div}^{1-\text{loop}} = \frac{1}{32\pi^2(2 - \omega)} \left[\frac{13\lambda\varphi^2}{6|\xi|} + O(1/|\xi|^2)\right].$$

\footnote{The new scalar field $\phi$ and its potential $\bar{V}(\phi)$ arise in the Einstein frame of the action (2.1), in which there is no nonminimal coupling of $\phi$ to the curvature. Together with the reparametrization of the scalar field the transition to this frame includes the conformal transformation of the metric [19].}
Similarly to heavy fields of the GUT-matter sector, these divergences allow one to estimate the order of magnitude in $|\xi|$ of the renormalized effective functions originating from the graviton-inflaton sector of the model:

\[ V_{\text{grav-infl}}^{1-\text{loop}} = O(1/|\xi|^2), \quad C_{\text{grav-infl}}^{1-\text{loop}} = O(1/|\xi|), \quad U_{\text{grav-infl}}^{1-\text{loop}} = O(1/|\xi|). \] (5.11)

These contributions are much smaller than their matter counterparts (4.20), (4.21) and (4.24). Thus, the graviton-inflaton sector in the model with a large nonminimal coupling gives a negligible contribution. In particular, it generates a very small anomalous scaling behaviour on the DeSitter instanton – $\zeta(0)$ in the zeta-function technique, coinciding with the pole part of the effective action in the dimensional regularization. In the slow-roll approximation of a constant inflaton field on the DeSitter geometry with $R = 2V/U \simeq \lambda \phi^2/|\xi|, R_{\mu\nu} = Rg_{\mu\nu}/4$ it equals

\[ \zeta(0) = 2 \Gamma_{\text{pole}}^{1-\text{loop}} = -\frac{171}{10} + O(1/|\xi|), \] (5.12)

as compared to a very big anomalous scaling of GUT fields quadratic in $|\xi|$ [5, 6].

6. Tunneling vs no-boundary quantum states

According to the discussion of Sect.3 effective equations in the slow roll approximation are given by classical equations (2.7) - (2.9) with classical coefficient functions replaced by their effective counterparts modified by loop corrections (4.20), (4.21) and (4.24). In particular, the quantum rolling force equals

\[ F_{\text{eff}}(\phi) = -\frac{U_{\text{eff}}^3}{U_{\text{eff}} + 3U_{\text{eff}}^2} \frac{d}{d\phi} \left( \frac{V_{\text{eff}}}{U_{\text{eff}}^2} \right). \] (6.1)

These equations are the same for both quantum states – no-boundary and tunneling. Initial conditions are, however, different: initial value of the mean inflaton field $\phi_I$ yields the extremum of the distribution function which is different for these two states [13]. Thus, to sort out the nature of quantum evolution from this probability peak, that is to find the sign of the rolling force at $\phi_I, F_{\text{eff}}(\phi_I)$, we have to consider these two states separately. Consider first the no-boundary case.

In the no-boundary distribution function (1.1) the exponential is determined by the sum of the classical Euclidean action on the DeSitter instanton

\[ I(\phi) = \int_{\text{DS}} d^4x g^{1/2} \left( -U(\phi)R + V(\phi) \right) \bigg|_{\phi=\text{const}} = -\frac{96\pi^2 U^2(\phi)}{V(\phi)} \] (6.2)

8 For heavy massive fields the divergences yield the dominant term of $1/m^2$-expansion [14] by a simple rule – replacement of the pole in spacetime dimensionality by the logarithm of mass, $1/(2-\omega) \to -\ln(m^2/\mu^2)$. For massless fields this rule is not correct, but the order of magnitude of the result is still encoded in the expression for the residue at this pole.

9 In the model with large $|\xi|$ the local extremum of the distribution function is very sharp, so that $\phi_I$ in the leading order coincides with the point of this extremum.
and the one-loop effective action which in the slow-roll approximation on the DeSitter sphere of radius \( H^{-1} = [6U/V]^{1/2} \) coincides with the first order variation of the above expression under the variations \( \delta U = U^{1-\text{loop}} \) and \( \delta V = V^{1-\text{loop}} \)

\[
\Gamma^{1-\text{loop}}(\varphi) = \int d^4x \, g^{1/2} \left( -U^{1-\text{loop}}(\varphi) R + V^{1-\text{loop}}(\varphi) \right) \bigg|_{\varphi=\text{const}} = 96\pi^2 \left( \frac{U^2(\varphi)}{V^2(\varphi)} V^{1-\text{loop}}(\varphi) - \frac{2U(\varphi)}{V(\varphi)} U^{1-\text{loop}}(\varphi) \right). \tag{6.3}
\]

Thus up to higher order terms in powers of quantum corrections the exponential of the no-boundary distribution function coincides with on-shell effective action on the effective DeSitter spacetime of the radius \( H_{\text{eff}}^{-1} = [6U_{\text{eff}}/V_{\text{eff}}]^{1/2} \)

\[
\Gamma(\varphi) = I(\varphi) + \Gamma^{1-\text{loop}}(\varphi) = -\frac{96\pi^2 [U_{\text{eff}}(\varphi)]^2}{V_{\text{eff}}(\varphi)} + O(h^2). \tag{6.4}
\]

Here \( \hbar \) symbolically denotes the quantum terms proportional to either of the following one-loop combinations of coupling constants of Sect. 4, \( \hbar = (1/32\pi^2)(A, B, C, D, E) \). Therefore, in the domain of the slow roll approximation the no-boundary distribution function equals

\[
\rho_{\text{NB}}(\varphi) = \text{const} \exp \left\{ \frac{96\pi^2 [U_{\text{eff}}(\varphi)]^2}{V_{\text{eff}}(\varphi)} + O(h^2) \right\}. \tag{6.5}
\]

Note that this expression is valid irrespective of the concrete form of radiative corrections \( U^{\text{loop}} \) and \( V^{\text{loop}} \), and \( O(h^2) \) here can be regarded as a contribution of multiloop orders.

Now, comparing this expression with (6.1) one can see that in the approximation of the above type the quantum rolling force for the mean inflaton field at its probability maximum is identically zero

\[
F_{\text{eff}}^{\text{NB}}(\varphi_I) \sim \frac{d}{d\varphi} \ln \rho_{\text{NB}}(\varphi) \bigg|_{\varphi=\varphi_I} = 0. \tag{6.6}
\]

This one-loop result holds for any model of matter sources and any structure of their quantum corrections. It implies that the no-boundary model of quantum origin of inflation does not satisfy the requirement of a finite duration of inflation stage: the quantum part of the rolling force cancels its classical part to zero but not reverses its sign to provide the decrease of the inflaton field and gradual exit from inflation (remember that with classical equations of motion the rolling force was of a wrong – positive – sign).

Let us go over to the tunnelling quantum state of the Universe. In contrast with the no-boundary case the exponential of the distribution function contains the difference of the classical Euclidean action and the one-loop effective action. This difference for large \( |\xi| \gg 1 \) generates a probability peak with the field (2.11) in the opposite range of the parameter (2.10), \( \delta > -1 \). According to the detailed discussion of [5, 6] comparison of this peak with the observational restrictions on the inflationary scenario – the lower bound of the classically induced e-folding number (2.16), \( N > 60 \) – leads to the estimate on the universal
combination of coupling constants $A = O(1)$ and similar estimates for the other combinations $(B, C, D, E) = O(1)$ (because such mechanisms as supersymmetry that could provide cancellation of separately big terms of (2.13) do not seem to apply in our model [34]). Therefore, all the quantities that were symbolically denoted above by $\bar{A}$ comprise very small parameters $\frac{A}{32\pi^2} \ll 1$, $\frac{B}{32\pi^2} \ll 1$, etc. With these bounds and in the limit of large nonminimal coupling $|\xi|$ one easily finds the tunneling rolling force by substituting the one-loop expressions of Sect.4 to (6.1). In the vicinity of the tunneling maximum $\varphi_I$ the result reads

$$F_{\text{eff}}^T(\varphi) = -\frac{\lambda m_p^2 (1 + \delta)}{48\pi \xi^2} \varphi \left(1 + \frac{\varphi^2}{\varphi_I^2}\right) + O(1/|\xi|^3), \quad 1 + \delta > 0.$$  

(6.7)

The second term here is the one-loop quantum contribution which obviously doubles the classical force (2.14) at $\varphi = \varphi_I$. Thus it does not qualitatively change the predictions of classical equations of motion: it guarantees the slow decrease of the inflaton field during the inflation and results in its finite duration with slightly different e-folding number

$$N \simeq 3 \int_0^{\varphi_I} d\varphi \frac{[H_{\text{eff}}(\varphi)]^2}{F_{\text{eff}}^T(\varphi)} \simeq \frac{48\pi^2}{A} \ln 2 \left[1 + O(\varepsilon \ln \varepsilon)\right].$$  

(6.8)

Similarly to eq.(2.16) (see the corresponding footnote), we discard here subleading terms coming from subdominant quantum corrections and lower order terms of $U_{\text{eff}}(\varphi)$ and $V_{\text{eff}}(\varphi)$ dominating at later stages of inflation, $\varepsilon = (1/32\pi^2)(A, B, C, D, E) \ll 1$. Thus, the restriction on the minimal duration of inflation $N \geq 60$ at the quantum level does not qualitatively change the classical bound ($A \leq 7.9$)

$$A \leq 5.5.$$  

(6.9)

For completeness let us also present the expression for the rolling force in the vicinity of the probability maximum in the case of the no-boundary state. It can be obtained from eq.(6.1) by inverting the sign of the second term

$$F_{\text{eff}}^\text{NB}(\varphi) = -\frac{\lambda m_p^2 (1 + \delta)}{48\pi \xi^2} \varphi \left(1 - \frac{\varphi^2}{\varphi_I^2}\right) + O(1/|\xi|^3), \quad 1 + \delta < 0.$$  

(6.10)

In accordance with (6.6) it vanishes at $\varphi_I$ and also changes the sign here, which means the stability of this point. This situation is completely similar to the tree-level situation [10] – the Lorentzian DeSitter Universe nucleates from the gravitational (half)instanton in the minimum of the effective potential $V_{\text{eff}}(\varphi)/U_{\text{eff}}^2(\varphi)$ and stays there with a constant stable value of the inflaton field.

7. Conclusions and discussion

The above results add a number of new facets to the dilemma of two wavefunctions – tunneling and no-boundary ones – as candidates for the quantum state of the early Universe.
capable of generating via inflationary scenario the observable large scale structure. They confirm our previous conclusion \[5, 6\] that the tunneling state is more likely to describe its quantum cosmological origin: it predicts a finite inflationary epoch characterized by the e-folding number \(6.8\) and the rolling force (and \(\dot{\phi} \neq 0\)) nonvanishing from the outset of inflation. This guarantees the smallness of density perturbations inverse proportional to \(\dot{\phi}\), which can be important for the inflation models with \(\Omega \neq 1\) \[18, 37\]. The finiteness of inflation stage is a result of a weak breakdown of the De Sitter invariance existing already at the classical level – slow roll of the inflaton field down the hill of the inflaton potential. In the tunneling case one-loop quantum corrections enhance this breakdown and decrease the classical e-folding number by a factor of \(\ln 2\) in \(6.8\).

In the case of the no-boundary quantum state the situation is different. Classically (that is in classical equations of motion) at the maximum of the one-loop distribution function the sign of the deviation from De Sitter invariance is opposite – the inflaton potential has a negative slope. One-loop quantum corrections in effective equations exactly compensate this deviation, so that the mean inflaton field remains constant in a stable minimum of the effective one-loop potential. This conclusion is universal for it does not depend on the form of one-loop corrections. This universality follows from the fact that, unlike the tunneling case, for the no-boundary state one and the same dynamical principle is laid in the foundation of quantum initial conditions (the distribution function \(\rho_{\text{NB}}(\phi)\)) and effective equations for mean fields – the path integral formulation of the no-boundary wavefunction \[1, 2\]. Therefore, one and the same effective inflaton potential enters both the effective equations and probability distribution, which makes the rolling force vanishing in a stable point of the maximum probability. Thus, the no-boundary quantum state is intrinsically more symmetric (in the De Sitter sense) than the tunneling one. This makes it unsatisfactory from the viewpoint of applications in the theory of the early universe\[10\] but renders it very attractive in other problems needing De Sitter invariant vacuum. The exception from this rule exists in the class of models considered in the tree-level approximation \[38, 39, 20\] which can also be ascribed to the no-boundary class\[11\]. In these models with complex inflaton field the probabilistically preferred solutions can describe models with finite inflation stage due to the presence in the inflaton potential of the centrifugal term produced by conserved isotropic charge. For nonminimal inflaton field in these models one can get probability peaks for both no-boundary and tunneling wave functions by means of a proper choice of parameters \[39, 20\].

The preferred nature of the tunneling wavefunction of the above type sounds coherent with recent conclusions of Linde \[8\] who extended the Hawking-Turok no-boundary mecha-

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\[10\] Infinite duration of inflation with constant inflaton, \(\dot{\phi} \approx 0\), and generation of exceedingly large perturbations (inverse proportional to \(\dot{\phi}\)) contradict a generally accepted inflationary scenario with the observable magnitude of microwave background radiation anisotropy, \(\Delta T/T \sim 10^{-5}\).

\[11\] Only by the sign of the exponentiated Euclidean action in the underbarrier domain. From the topological viewpoint this is not a no-boundary state, because it is more likely to describe the underbarrier penetration from one classically allowed region (with small scale factor) to another one with the scale factor infinitely growing to infinity.
nism of creating the open universe \[4\] to the tunneling case. In principle, our results obtained for the closed model can be directly extended to open models by using the Hawking-Turok instanton. This instanton is singular, but according to their calculations \[7\] this singularity is mild enough to retain finite and practically the same value of the instanton action as in (6.2). It is likely that the same will be true also for the anomalous scaling of loop corrections, so that our mechanism of generating the probability peak at GUT scale will also work in the open Universe \[40\]. Then, no anthropic principle – an obvious retreat for theory – should be invoked to reach the conclusions of \[7\] and improve them by finding the value of \(\Omega\) satisfying \(1 > \Omega \gg 0.01\). A similar improvement looks true regarding the implications of tunnelling wavefunction for open inflation by Linde \[8\]. To find \(\Omega\) in this case one should not appeal to special supergravity induced potentials generating inflation only in the limited range of \(\varphi\) \[8\] (which by itself can be regarded as a mild form of the anthropic principle), but should calculate it from the obtained probability maximum. The GUT scale of this peak, in particular, rules out the necessity of considering such potentials which obviously manifest themselves at the much higher supergravitational energy scale.

Here a natural question arises, how seriously should be considered these predictions based only on the one-loop approximation. As it was discussed in \[\text{[6]}\], the main justification of the semiclassical expansion comes from the fact that the energy scale of the system (2.11) is suppressed relative to the Planck scale by the numerical factor \(\sqrt{\lambda}/|\xi|\). In the model with nonminimal inflaton \[16\] this factor determines the magnitude of microwave background anisotropy which is well known from the COBE \[41\] and Relict \[42\] satellite experiments at the level of \(10^{-5}\). This leads to the large value of \(|\xi|\) and, as a consequence, to the large value of the effective Planck mass \(3.6\) and suppression of higher powers of curvatures (3.7) characteristic of multi-loop contributions. The smallness of the graviton-inflaton contribution (2.11), in particular, is a direct consequence of such an improvement of the semiclassical expansion due to the replacement of \(1/m_P^2\) by \(1/(m_P^2 + 8\pi|\xi|\varphi^2)\).

In the GUT sector of the model the smallness of multi-loop orders can be regarded as a consequence of the bound (6.9) and its corollaries\[13\]

\[
\varepsilon \equiv \frac{1}{32\pi^2}(A, B, C, D, E) \ll 1. 
\]

The multi-loop orders roughly proportional to powers of \(\varepsilon\) are thus small. In connection with this observation it is worth explaining the following paradox. The smallness of \(\varepsilon\) obviously implies the smallness of all quantum effects in the system including the one-loop order, so how could the latter qualitatively change the predictions of the tree-level theory – replace

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\text{\[12\]} In contrast with initial conditions, the problem of effective equations for the Hawking-Turok model of open inflation is much harder, because the local Shwinger-DeWitt expansion and slow roll approximation break down near the singularity of the Hawking-Turok instanton \[40\].

\text{\[13\]} As discussed in \[\text{[6]}\], except rather improbable mechanisms of cancellation between different terms of (2.13) the smallness of \(A\) is only guaranteed by small values of all the couplings \(\lambda, g_A, f_\psi\) (as is the case with running gauge coupling constants at the grand unification point). This guarantees small values of other universal combinations of couplings \(B, C, D, E\).
a flat graph of the tree-level distribution by the probability peak advocated in [5, 6]? The explanation is as follows. The smallness of ε indeed leads to small quantum terms of the effective Lagrangian. Large quantum effects of the effective action originate from multiplying the Lagrangian by the 4-volume of the DeSitter instanton, $8\pi^2/3H^4(\varphi)$, which brings to life a large factor of $|\xi|^2$. The classical part of the action, exactly by the same mechanism, is also quadratic in $|\xi|$, but its $\varphi$-dependent part is at most linear in $|\xi| [5, 6]$. This allows one to balance this classical part by the quantum term ($\sim \varepsilon|\xi|^2$) and reach a nontrivial maximum of the probability distribution even though $\varepsilon \ll 1$.

All these arguments have certainly a qualitative nature and remain valid only for fields in the vicinity of the obtained probability maximum. A rigorous proof would require a careful consideration and bookkeeping of multi-loop orders and the nonperturbative analysis of the $\varphi \rightarrow \infty$ limit. At present, however, we have a reliable algorithm for the probability distribution (1.1) only in the one-loop approximation [13]. Its extension to higher orders has not yet been done and might be rather nontrivial in view of intrinsic problems of the Dirac quantization of constrained systems [43, 44]. Therefore, regarding the nonperturbative behaviour we can only put forward a number of hypotheses.

It is likely that for the no-boundary state the exact probability distribution is given by the full Euclidean effective action including all loop corrections. Then in the slow roll approximation it coincides with the expression (6.3) involving full coefficient functions $U_{\text{eff}}(\varphi)$ and $V_{\text{eff}}(\varphi)$ without extra $O(h^2)$ terms. Then the high-energy limit of $\rho_{\text{NB}}(\varphi)$ at $\varphi \rightarrow \infty$ is encoded in the corresponding behaviour of the ratio of these functions $U_{\text{eff}}^2(\varphi)/V_{\text{eff}}(\varphi)$. Their hypothetic asymptotic behaviour can be motivated by the one-loop approximation (6.20) - (6.21) with some effective constants $A_{\text{eff}}$ and $C_{\text{eff}}$ including all multi-loop corrections. Then the hypothetical high-energy behaviour of the no-boundary distribution function looks like

$$\rho_{\text{NB}}(\varphi) \sim \exp \left\{ \frac{C_{\text{eff}}^2}{6A_{\text{eff}}} \ln \varphi \right\}, \quad \varphi \rightarrow \infty, \tag{7.2}$$

and its normalizability (and the requirement of suppression of the high energy scales) imposes the restriction on the effective constant $A_{\text{eff}}$, $A_{\text{eff}} < 0$, quite opposite to the one obtained in our earlier work on the one-loop normalizability of the no-boundary wavefunction [11]. This contradiction should, however, be regarded with a big share of criticism, because it involves too many hypotheses. As regards the tunneling quantum state in the nonperturbative

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14 The starting point for the derivation of the probability algorithm is the solution of the Wheeler-DeWitt equations – quantum Dirac constraints in the coordinate representation of the canonical commutation relations. The physical inner product (generating the probability amplitudes) in this representation is known only in the one-loop approximation [13, 2, 14] – the measure of this inner product apparently has essentially perturbative nature. Thus, the multi-loop extension of the algorithm for probability distribution requires developing the fundamental aspects of quantization formalism for constrained dynamical systems.

15 The generalized renormalization group in the model of graviton nonminimally coupled to inflaton considered in our work [13] leads to the different asymptotics for $V_{\text{eff}}(\varphi)$, $V_{\text{eff}}(\varphi) \sim \varphi^4 \left[ \ln(\varphi^2/\mu^2) \right]^2$, but this behaviour is apparently based on an unappropriate branch of renormalization flow starting from the over-Planckian conformally invariant phase, which does not interpolate between the latter and the GUT phase.
regime, we can say even less than about the no-boundary one. In the one-loop approximation quantum and classical terms enter the game with different signs, so it is tempting to conjecture that the exact answer can again be obtained from semiclassically expanded full effective action by formally inverting the sign of the Planck mass squared, but this conjecture demands a careful check.

Another limitation of the obtained results regards the contribution of the quantum mechanical sector of the homogeneous inflaton mode. This mode, as mentioned in Sect.3, is effectively massless and, thus, does not possess the DeSitter invariant vacuum [25] – the problem which in the quantum cosmological context implies the absence of normalizable no-boundary and tunneling states considered in the tree level approximation. The main achievement of refs. [3, 6] is that beyond this approximation in the model with large $|\xi|$ this problem can have a solution in the form of a sharply peaked distribution with the parameters of the peak (2.11) - (2.12). Therefore, the Green’s function of this mode (3.3), participating in the radiation current of the effective equations, should be defined relative to such a peaked state. Apriori, it is not DeSitter invariant and its contribution to Lorentzian effective equations even in the slow roll approximation does not follow from the Euclidean effective action by Wick rotation (3.10) - (3.11). The method of calculating this contribution is currently under study – it involves actual Hamiltonian reduction of the system to the explicit physical degree of freedom in the homogeneous gravity-inflaton sector of the model and raises important gauge dependence issues. Despite this omission in the effective equations of quantum motion, there is a strong believe that for large $|\xi|$ (that is, in the leading order of the slow roll approximation) this does not affect our conclusions, because all effects of the DeSitter invariance violation, including the contribution of this quantum mechanical sector, are expected to belong to the subleading order of $1/|\xi|$ and $\dot{\varphi}$ expansion. This, however, requires a careful analysis which we postpone till later publication.

The slow roll approximation was a very important ingredient of all our considerations. Being the attribute of the physical setting at the initial stage of inflation it simultaneously served as a means to avoid the problem of nonrenormalizability in local gravity theory: discarding (negligible) higher powers of the curvature in local terms of the action leaves us with the renormalization of only two generalized coupling constants – cosmological term (effective potential $V_{\text{eff}}$) and the gravitational “constant” $16\pi U_{\text{eff}}$. Thus, approximately this brings us to the class of perturbatively renormalizable theories – the so-called renormalization at the threshold [46]. This approximation, however, does not incorporate nonlocal quantum effects that correspond to infinite summation of derivatives in the local Schwinger-DeWitt series [29]. Nonlocal terms of the effective equations for expectation values cannot be obtained by the variational procedure with some nonlocal effective action (see the footnote for eq. (3.11)). These terms include the dissipation effects caused by particle creation in the external

\footnote{The problem of the off-shell gauge dependence of the effective gravitational equations of motion [45, 22] did not arise above, because this gauge dependence (in the natural class of gauges fixing the diffeomorphism invariance) affects only the graviton-inflaton sector of the theory. The latter was discarded because its contribution is suppressed for large nonminimal coupling $|\xi|$.}
(mean) field and become important in the end of inflation at the reheating stage of the
cosmological evolution \[18, 19\]. Their analysis should bring us to the complete cosmological
scenario including the formation of the observable large scale structure and give the answer
if the latter bears the imprint of the quantum cosmological origin from the tunneling or
no-boundary quantum states of the Universe.

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