THE $K$-THEORY OF FINITELY MANY COMMUTING ENDOMORPHISMS

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Abstract. For a field $k$ we compute the $K$-theory of the exact category of $k[t_1, \ldots, t_n]$-modules that are finite-dimensional over $k$, generalising the work of Kelley and Spanier.

1. Introduction

Let $A$ be a commutative ring and $\mathcal{P}(A)$ the category of finitely-generated projective $A$-modules. Let $\text{End} \, \mathcal{P}(A)$ be the category whose objects are endomorphisms $P \to P$ where $P \in \mathcal{P}(A)$, and whose morphisms $\alpha : (P \to P) \to (Q \to Q)$ are morphisms $\alpha : P \to Q$ in $\mathcal{P}(A)$ such that the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\alpha} & Q \\
\downarrow & & \downarrow \\
P & \xrightarrow{\alpha} & Q
\end{array}
$$

commutes. The category $\text{End} \, \mathcal{P}(A)$ is naturally equivalent to the category of $A[t]$-modules that are finitely generated and projective as $A$-modules via the inclusion map $A \to A[t]$. The category $\text{End} \, \mathcal{P}(A)$ is an exact category and one would like to calculate its Quillen $K$-theory groups. This type of $K$-theory calculation falls under the more general problem of calculating for a ring homomorphism $R \to S$, the $K$-theory of the category of $S$-modules that are finitely generated and projective as $R$-modules.

The calculation of $K_i(\text{End} \, \mathcal{P}(A))$ was given by Kelley and Spanier [KS68] when $A$ is a field and $i = 0$. Almkvist [Alm78] did the calculation when $A$ is an arbitrary commutative ring and $i = 0$, and when $A$ is a field and $i$ is arbitrary. To describe these computations, define the abelian group whose underlying set is

$$
\tilde{A}_0 = \left\{ \frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m} : a_i, b_j \in A \right\}
$$

and whose binary operation is given by the usual multiplication of rational functions. If $f : M \to M$ is an endomorphism of a finitely-generated projective $A$-module, then the characteristic polynomial $\lambda_t(f)$ may be defined by extending $f$ to an endomorphism of a free module, and then defining $\lambda_t(f) = \det(1 + tf)$; see [Alm78] for an alternative definition. We can use this map to describe the results of Kelley and Spanier and Almkvist:

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1.1. Theorem. There is an isomorphism
\[ K_0(\text{End } P(A)) \to K_0(A) \times \tilde A_0 \]
given on generators by \[ [M \xrightarrow{f} M] \mapsto ([M], \lambda_t(f)). \]

In this paper, we generalise this result to the case of finitely many commuting endomorphisms and where \( A = k \) is a field:

1.2. Theorem. The algebraic \( K \)-groups of the exact category \( \text{End}_n P(k) \) are given by
\[ K_i(\text{End}_n P(k)) \cong \bigoplus_M K_i(k[t_1, \ldots, t_n]/M) \]
where \( M \) ranges over all the maximal ideals of the polynomial ring \( k[t_1, \ldots, t_n] \).

We remark that our proof is different than the proofs in [Alm78] and in [KS68]. One feature is that our result for \( i = 0 \) is not phrased in terms of a characteristic polynomial map, which is an advantage in the sense that it is more explicit in terms of the structure of the \( K \)-groups, but a disadvantage in the sense that working with products is more difficult. The result for \( n = 1 \) and higher \( K \)-theory is [Alm78, Theorem 5.2], but again, our proof is entirely different.

2. The Category \( \text{End}_n P(k) \)

Let \( \text{End}_n P(k) \) denote the exact category of \( k[T] := k[t_1, \ldots, t_n] \)-modules that are finitely generated as \( k \)-modules. In this section we calculate \( K_i(\text{End}_n P(k)) \). A finite-dimensional \( k \)-vector space \( V \) with any \( n \) commuting endomorphisms \( f_1, f_2, \ldots, f_n \) of \( V \) may also be considered as a \( k[t_1, \ldots, t_n]/I \)-module where \( I \) is the kernel of the map \( k[t_1, \ldots, t_n] \to k[f_1, \ldots, f_n] \) given by \( t_i \mapsto f_i \); hence:

2.1. Proposition. The category \( \text{End}_n P(k) \) is naturally equivalent to the filtered direct limit
\[ \lim_{\to} (k[T]/I-\text{mod}_{fg}) \]
of exact categories, where \( I \) runs over the set of ideals of \( k[T] \) such that the quotient \( k[T]/I \) is finite-dimensional over \( k \), and if \( I \subseteq J \) are two such ideals, then the functor \( k[T]/J-\text{mod}_{fg} \to k[T]/I-\text{mod}_{fg} \) is the forgetful functor induced by the quotient map \( k[T]/I \to k[T]/J \).

Proof. The limit is indeed filtered, since \( k[T]/(I \cap J) \) is finite-dimensional whenever \( k[T]/I \) and \( k[T]/J \) are finite-dimensional. Define a functor
\[ F : \lim_{\to} (k[T]/I-\text{mod}_{fg}) \to \text{End}_n P(A) \]
as follows. Any element in \( \lim_{\to} (k[T]/I-\text{mod}_{fg}) \) is represented by \( V \in k[T]/I-\text{mod}_{fg} \) for some \( I \). We let \( F(V) \) to be the \( k[T] \) module given by the forgetful functor induced
by the map $k[T] \to k[T]/I$. This functor is well-defined because $k[T]/I$ is finite-

dimensional, and it is easy to see that $F$ gives the required natural equivalence. □

Using this observation and the result that taking $K$-groups of exact categories

commutes with filtered direct limits [Qui72, §2], we obtain the following corollary.

2.2. Corollary. The $K$-theory of the category $\text{End}_n \mathcal{P}(A)$ may be calculated as the
direct limit

$$K_i(\text{End}_n \mathcal{P}(A)) \cong \lim_{I \to J} K_i (k[T]/I - \text{mod}_{fg})$$

For any ring $R$, let $\text{Jac}(R)$ denote the Jacobson radical of $R$. Any surjective

ring homomorphism $R \to S$ induces a surjective ring homomorphism $R/\text{Jac}(R) \to S/\text{Jac}(S)$, and a commutative diagram of rings

\[
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
R/\text{Jac}(R) & \longrightarrow & S/\text{Jac}(S)
\end{array}
\]

In turn, whenever $S$ is finitely generated as an $R$-module, we have a commutative
diagram of forgetful functors

$$R - \text{mod}_{fg} \longleftarrow S - \text{mod}_{fg}$$

\[
\begin{array}{ccc}
R/\text{Jac}(R) - \text{mod}_{fg} & \longleftarrow & S/\text{Jac}(S) - \text{mod}_{fg}
\end{array}
\]

In particular, this applies to the ring homomorphisms $k[T]/I \to k[T]/J$ for $I \subseteq J$
appearing in the direct limit of (2).

2.3. Proposition. The natural transformation of the direct limits of categories in-
duced by the forgetful functors as in (4) for $R = k[T]/I$ induces an isomorphism

algebraic $K$-groups

$$\lim_{I \to J} K_i \left( \frac{k[T]}{\text{Jac}(k[T])/I} - \text{mod}_{fg} \right) \cong \lim_{I \to J} K_i (k[T]/I - \text{mod}_{fg}).$$

Proof. For any Artinian ring $R$, the Jacobson radical $\text{Jac}(R)$ is nilpotent (e.g. [Lam91,

Theorem 4.12]), and so devissage [Qui72, §5, Theorem 4] shows that the inclusion

$R/\text{Jac}(R) - \text{mod}_{fg} \to R - \text{mod}_{fg}$ induces an isomorphism

$$K_i(R/\text{Jac}(R) - \text{mod}_{fg}) \to K_i(R - \text{mod}_{fg})$$

(see [Wei13, Page 439] for more details). In particular, this applies to $R = k[T]/I$

where $k[T]/I$ is finite-dimensional over $k$. □

2.4. Theorem. The algebraic $K$-groups of the exact category $\text{End}_n \mathcal{P}(k)$ are
given by

$$K_i(\text{End}_n \mathcal{P}(k)) \cong \bigoplus_M K_i(k[t_1, \ldots, t_n]/M)$$
where $M$ ranges over all the maximal ideals of the polynomial ring $k[t_1, \ldots, t_n]$.

Proof. We must calculate the limit
\[
\lim_I K_i \left( \frac{k[T]}{\text{Jac}(k[T]/I)} \right) \mod_{fg}
\]
given in Proposition 2.3. So, fix an ideal $I$ such that $k[T]/I$ is finite-dimensional. Then there are finitely many maximal ideals $M_1, \ldots, M_k$ of $k[T]$ that contain $I$ and
\[
k[T]/I \cong \bigoplus_{j=1}^{k} (k[T]/I)_{M_j},
\]
via the obvious map (e.g. [GW10, Theorem 5.20]). Here, by $(k[T]/I)_{M_i}$, we abuse notation and mean the localization of $k[T]/I$ away from the maximal ideal $M_i$ of $k[T]/I$.

If $J$ is an ideal containing $I$, then there is a subset $\{N_1, \ldots, N_\ell\}$ of the maximal ideals $\{M_1, \ldots, M_k\}$ that contain $J$ and the quotient homomorphism $k[T]/I \to k[T]/J$ induces the map
\[
\bigoplus_{j=1}^{k} (k[T]/I)_{M_j} \to \bigoplus_{j=1}^{\ell} (k[T]/J)_{N_j},
\]
which must be the projection map. The induced map on $K$-groups that fits into the direct limit in (6), by is then the map
\[
\bigoplus_{j=1}^{\ell} K_i(k[T]/N_j) \to \bigoplus_{j=1}^{k} K_i(k[T]/M_j),
\]
given by the sum of the inclusion maps. Taking the direct limit gives the stated result, once we note that $k[T]/M$ is finite-dimensional over $k$ for any maximal ideal $M$. □

In particular, we obtain the following amusing, possibly well-known result.

2.5. Corollary. For a field $k$, there is an isomorphism of abelian groups
\[
\tilde{k}_0 := \left\{ \frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m} : a_i, b_j \in k \right\} \cong \bigoplus_{|\text{Spec}(k[T])|} \mathbb{Z}.
\]

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