BEREZINIANS, EXTERIOR POWERS AND RECURRENT SEQUENCES

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Abstract. We study power expansions of the characteristic function of a linear operator $A$ in a $p|q$-dimensional superspace $V$. We show that traces of exterior powers of $A$ satisfy universal recurrence relations of period $q$. 'Underlying' recurrence relations hold in the Grothendieck ring of representations of $\text{GL}(V)$. They are expressed by vanishing of certain Hankel determinants of order $q + 1$ in this ring, which generalizes the vanishing of sufficiently high exterior powers of an ordinary vector space. In particular, this allows to explicitly express the Berezinian of an operator as a rational function of traces. We analyze the Cayley–Hamilton identity in a superspace. Using the geometric meaning of the Berezinian we also give a simple formulation of the analog of Cramer’s rule.

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1. Introduction

1.1. In this paper we study the Berezinians of linear operators in a superspace and in particular the characteristic function $R_A(z) = \text{Ber}(1 + zA)$, where $z$ is a complex variable. Our principal tool is the two power expansions of $R_A(z)$, at zero and at infinity. We also study a similar rational function taking values in a Grothendieck ring. The main results are as follows.
For an arbitrary even linear operator $A$ in a $p|q$-dimensional superspace $V$ we establish universal recurrence relations satisfied by the traces $\text{Tr} \Lambda^k A$ and $\text{Tr} \Sigma^k A$ of the induced action in the exterior powers $\Lambda^k(V)$ and the ‘dual exterior powers’ $\Sigma^k(V) = \text{Ber} V \otimes \Lambda^{p-k} V^*$ (Theorem 1, formulae (3.7) and (3.8)). We obtain similar fundamental recurrence relations satisfied by the spaces $\Lambda^k(V)$ and $\Sigma^n(V)$ in a suitable Grothendieck ring, and underlying the relations for traces (Theorems 4 and 5). In particular, we show how $\text{Tr} \Sigma^k A$, which are rational functions of $A$, can be obtained from the polynomial invariants $\text{Tr} \Lambda^k A$ by a sort of “analytic continuation”. Our considerations lead to effective formulae. For the Berezinian $\text{Ber} A$ we obtain an invariant explicit formula expressing it as the ratio of two Hankel determinants built of $\text{Tr} \Lambda^k A$:

$$\text{Ber} A = \frac{\begin{vmatrix} \text{Tr} \Lambda^{p-q} A & \ldots & \text{Tr} \Lambda^p A \\ \vdots & \ddots & \vdots \\ \text{Tr} \Lambda^p A & \ldots & \text{Tr} \Lambda^{p+q} A \\ \text{Tr} \Lambda^{p-q+2} A & \ldots & \text{Tr} \Lambda^{p+1} A \\ \vdots & \ddots & \vdots \\ \text{Tr} \Lambda^{p+1} A & \ldots & \text{Tr} \Lambda^{p+q} A \end{vmatrix}}{\text{Tr} \Lambda^{p-q+2} A \ldots \text{Tr} \Lambda^{p+1} A \ldots \text{Tr} \Lambda^{p+q} A}.$$ 

One can relate these determinants with characters of polynomial representations of the general linear supergroup corresponding to particular Young diagrams.

Besides this, we discuss two other related topics. For an analog of the Cayley–Hamilton theorem, we analyze the problem of a minimal annihilating polynomial of a linear operator in a superspace and show how it can be obtained from the characteristic function $R_A(z)$. It should be emphasized that in the supercase the rational characteristic function $R_A(z) = \text{Ber}(1 + zA)$ is a more fundamental object than such a ‘characteristic polynomial’, which can be built from it. We also study an analog of Cramer’s rule for the supercase and give for it a geometric proof.

1.2. Motivation and background. Recall that the Berezinian is the analog of the determinant for the $\mathbb{Z}_2$-graded (= super) situation. It was discovered by F. A. Berezin in his studies of second quantization and integration over odd variables. See [1, 3] and references therein. The main feature of $\text{Ber} A$ is that it is not a polynomial in the matrix entries, but a fraction. In the standard definition

$$\text{Ber} A := \det (A_{00} - A_{01}^{-1} A_{10}) (\det A_{11})^{-1},$$

One can relate these determinants with characters of polynomial representations of the general linear supergroup corresponding to particular Young diagrams.
where $A_{00}$, $A_{01}$, $A_{10}$ and $A_{11}$ are the matrix blocks of $A$, the numerator and denominator do not have independent invariant meaning. Exactly because $\text{Ber} A$ is non-polynomial, integration theory in the supercase is non-trivial. In particular, it is well known that the straightforward generalization of the exterior algebra by standard tensor tools transferred to the $\mathbb{Z}_2$-graded situation, is not sufficient, because it is not related with the Berezinian and hence with integration over supermanifolds (for a survey see, e.g., [20, 21]). The simplest objects that one has to consider besides the naive exterior powers $\Lambda^k(V)$ are the ‘dual exterior powers’ $\Sigma^k(V) := \text{Ber} V \otimes \Lambda^{p-k}V^*$ introduced by Bernstein and Leites [5] (when $V$ is the space of covectors on a supermanifold the elements of $\Sigma^k(V)$ are called integral forms).

As we show in this paper, there are surprising “hidden relations” between the naive exterior powers $\Lambda^k(V)$ and the Berezinian, so they are closer than might be expected. This is seen by the comparing of the two expansions of the characteristic function of a linear operator: the expansion at zero gives the traces in $\Lambda^k(V)$, while the expansion at infinity gives the traces in $\Sigma^k(V)$, including the Berezinian. Hence the relations between $\Lambda^k(V)$ and Ber can be perceived as an ‘analytic continuation of a rational function from a neighborhood of zero to the neighborhood of infinity’. (There is an analogy with rational numbers: the ordinary decimal expansion corresponds to an expansion near infinity, while a $p$-adic expansion corresponds to an expansion at zero.) Formal analogs of these expansions yield underlying relations in the Grothendieck ring.

Let us explain the position of these results in comparison with the familiar picture of operators acting in purely even vector spaces. For a vector space $V$ of dimension $n$ all exterior powers starting from $\Lambda^{n+1}(V)$, vanish. Therefore all the traces $\text{Tr} \Lambda^k A$, $k > n$, identically vanish. Also, the top exterior power $\Lambda^n(V)$ is the same as the one-dimensional space $\text{det} V$, and this gives rise to natural isomorphisms $\text{det} V \otimes \Lambda^{n-k}(V^*) \cong \Lambda^k(V)$ (‘duality’). In the $\mathbb{Z}_2$-graded case, for a vector space $V$ of dimension $p|q$, there is an infinite sequence of the exterior powers $\Lambda^k(V)$, which does not terminate. Likewise, there is an infinite sequence of the spaces $\Sigma^k(V) = \text{Ber} V \otimes \Lambda^{p-k}V^*$, stretching to the left, which are now essentially different from $\Lambda^k(V)$. In this paper we establish the following relations in the Grothendieck ring:

$$
\begin{vmatrix}
\Gamma_k & \cdots & \Gamma_{k+q} \\
\cdots & \cdots & \cdots \\
\Gamma_{k+q} & \cdots & \Gamma_{k+2q}
\end{vmatrix} = 0,
$$
where $\Gamma_k = \Lambda^k V - (-\Pi)^q \Sigma^k V$, for all $k \in \mathbb{Z}$. ($\Pi$ is the parity shift functor.) Taken in the range of $k$ where both $\Lambda^k V$ and $\Sigma^k V$ are not zero, it gives the proper replacement for the classical ‘duality isomorphisms’. At the same time, its corollary

$$\begin{vmatrix} \Lambda^k V & \ldots & \Lambda^{k+q} V \\ \ldots & \ldots & \ldots \\ \Lambda^{k+q} V & \ldots & \Lambda^{k+2q} V \end{vmatrix} = 0,$$

for $k \geq p - q + 1$ replaces the vanishing of the sufficiently high exterior powers in the classical case.

The Cayley–Hamilton theorem is closely related with identities for traces. In the classical case, $\chi_A(A) = 0$ for the characteristic polynomial $\chi_A(z) = \det(A - z)$ of a linear operator in an $n$-dimensional space, gives relations for the powers of $A$. It can be deduced from the identity $\text{Tr} \, \Lambda^{n+1} A = 0$ by varying it w.r.t. $A$, and, conversely, it implies identities for traces. Now, in the $\mathbb{Z}_2$-graded case, of course, any even operator satisfies the same polynomial relation as in the classics with $n = p + q$. The trouble, however, is how to give a meaning to the coefficients of this relation as invariants of the operator. This has been a source of confusion of many attempts to generalize the Cayley–Hamilton theorem to the supercase that can be found in the literature.

In this paper we explain how the ‘naive’ Cayley–Hamilton identity (if one forgets about the $\mathbb{Z}_2$-grading) and an identity obtained by varying the relation for traces following from the second formula above, give the same thing. The subordinate role of the ‘Cayley–Hamilton polynomial’ in the supercase as compared to the ‘true’ characteristic function $\text{Ber}(A - z)$ (or the equivalent $R_A(z) = \text{Ber}(1 + zA)$), is clearly seen.

Notice that in the last fifteen years there has been an active work on non-commutative generalizations of determinants initiated by Gelfand and Retakh (see [10, 9]), non-commutative Vieta formulae [7, 6] and related topics of non-commutative geometry. Using the Gelfand–Retakh theory of quasi-determinants, Bergvelt and Rabin in [4] found an analog of Cramer’s formula in the supercase. The situation with Cramer’s rule, i.e., calculating the inverse of a supermatrix, is a bit peculiar. At the first glance one does not expect a role of the Berezinian similar to that of the determinant in the classical case. However, this is true, though not so straightforwardly (e.g., what should be the correct notion of a minor or an adjunct? – see in the main text). We give here a simple direct proof based on the geometrical meaning of the Berezinian.

We would like to stress that our methods throughout this paper are very elementary.
The topics of our paper are intimately related with subtle questions concerning rational and polynomial invariants of operators in super-spaces. As it is known (see below), the distinction between rational and polynomial invariants in the $\mathbb{Z}_2$-graded situation is much sharper than in the classical case. On the other hand, ‘rational’ seems to be intrinsically related with ‘super’. For example, every rational function $R(z)$ such that $R(0) = 1$ can be viewed as the characteristic function of a linear operator, $R(z) = R_A(z)$, its zeros and poles corresponding to the bosonic and fermionic eigenvalues of $A$. A pair of polynomials $P, Q$ of degrees $p$ and $q$ can be viewed as the numerator and denominator of such a characteristic function. One can show that their resultant $\text{Res}(P, Q)$ can be expressed via $\text{Tr} \Lambda^k A$. It is the (super)trace of the representation corresponding to the $p \times q$ rectangular Young diagram $D$, $\text{Res}(P, Q) = \text{Tr} A_D$ (see in the main text).

Note that rational and polynomial invariants of supermatrices were first considered in the pioneer works on representations of Lie superalgebras by Berezin (see references in [3]; some texts of 1975-77 were incorporated into the English version of that posthumous book) and Kac [11]. They showed that all rational invariants of supermatrices $p|q \times p|q$ can be expressed as rational functions of the $p+q$ supertraces $\text{Tr} A, \ldots, \text{Tr} A^{p+q}$. In particular, one can express polynomial invariants, though possibly non-polynomially. It was discovered that not every polynomial on the diagonal matrices separately symmetric in the ‘bosonic’ and ‘fermionic’ eigenvalues can be extended to a polynomial invariant on matrices. In fact, it in general corresponds to a rational invariant function with a denominator of a special appearance [2] (see the English version of [3]). In [2] Berezin gave a criterion for such function to be a polynomial, which was later clarified and extended to other Lie superalgebras by Sergeev [18] (see also [19]). There was an interesting sequel of works by Kantor and Trishin [12, 13, 14], in which the authors were concerned with clarifying the relations in the (infinitely generated) algebra of polynomial invariants. In particular, they found by a method different from ours the relations (3.8) for traces $\text{Tr} \Lambda^k A$ for an arbitrary operator in a $p|q$-dimensional space and came to analogs of the Cayley–Hamilton identity. They did not consider expansions of rational functions. Their main tool was analysis of Young diagrams and the corresponding representations. The coefficients of the expansion at infinity of the characteristic function (which include the Berezinian) did not appear in these papers.

The recurrence relations linking $\Lambda^k V$ and $\Sigma^k V$ that we establish in this paper, both as relations for traces and the relations in the Grothendieck ring, are new. The explicit invariant formula for the
Berezinian as a ratio of two supertraces following from them, is also new. Using these relations more results can be obtained. We hope that our approach allows to reach better clarity of understanding of the Cayley–Hamilton theorem and Cramer’s rule in the supercase.

1.3. Notation. We use standard language of superalgebra and supergeometry. Whenever it could not cause confusion, we drop the prefix ‘super’, writing ‘spaces’, ‘traces’, etc., instead of ‘superspaces’, ‘supertraces’, respectively.

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2. Expansions of the Characteristic Function

Let $A$ be an even linear operator acting in a finite-dimensional superspace $V$. Denote $\dim V = p|q$. Consider the characteristic function of this operator,

$$R_A(z) := \text{Ber}(1 + zA),$$

depending on a complex variable $z$. Here Ber denotes the Berezinian (superdeterminant). If $M$ is an even invertible $p|q \times p|q$ supermatrix, $M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}$, recall that

$$\text{Ber} M = \frac{\det \begin{pmatrix} M_{00} - M_{01}M_{11}^{-1}M_{10} \end{pmatrix}}{\det M_{11}}$$

The Berezinian is a multiplicative function of matrices, hence it is well-defined on linear operators.

Recall that for an even matrix (resp., operator), in the diagonal blocks $M_{00}, M_{11}$ (resp., $A_{00}, A_{11}$) the matrix entries are even and in the antidiagonal blocks $M_{01}, M_{10}$ (resp., $A_{01}, A_{10}$) the entries are odd. In the sequel, when it cannot cause a confusion we do not distinguish sharply operators and the corresponding matrices. Matrix elements can be viewed either as belonging to a given $\mathbb{Z}_2$-graded (super)commutative ring or as free generators. Classically this corresponds to considering an ‘individual’ matrix or a ‘general’ matrix. Strictly speaking one should talk about ‘free modules’ over the ground ring instead of ‘vector spaces’, but we shall not stress this distinction.
Consider the expansion of the rational function $R_A(z)$ at zero:

$$R_A(z) = \sum_{k=0}^{\infty} c_k(A)z^k = 1 + c_1z + c_2z^2 + \ldots.$$  \quad (2.3)

In the ordinary case (where the odd dimension is equal to zero) the function $R_A(z)$ is a polynomial and the expansion (2.3) terminates. It is well known that for a linear operator acting in a $p$-dimensional vector space

$$\det(1 + zA) = 1 + c_1z + \ldots + c_pz^p$$

where $c_k(A) = \text{Tr } \Lambda^kA$ are the traces of the action of the operator $A$ in the exterior powers $\Lambda^kV$. In particular, $c_1(A) = \text{Tr } A$, $c_p(A) = \det A$. For $k > p$, $c_k(A) = 0$ as $\Lambda^kV = 0$.

If the odd dimension of $V$ is not equal to zero, then $\text{Ber}(1 + zA)$ is no longer a polynomial in $z$, but an analog of the formula above still holds:

**Proposition 1.** There is an infinite power expansion

$$\text{Ber}(1 + zA) = \sum_{k=0}^{\infty} c_k(A)z^k \text{ where } c_k(A) = \text{Tr } \Lambda^kA.$$  \quad (2.4)

In (2.4) $\Lambda^kA$ stands for the action of $A$ in the $k$-th exterior power of the superspace $V$, where the exterior algebra $\Lambda(V) = \oplus \Lambda^kV$ is defined as $T(V)/\langle v \otimes u + (-1)^{\bar{v}\bar{u}}u \otimes v \rangle$, $v, u$ being elements of $V$. Parity in $\Lambda(V)$ (the $\mathbb{Z}_2$-grading) is naturally inherited from $V$. There is no “top” power among $\Lambda^kV$, and the Taylor expansion (2.4) is infinite.

We denote the supertrace of a supermatrix by the same symbol as the trace of an ordinary matrix. Recall that for an even supermatrix,

$$\text{Tr } M = \text{Tr } \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix} = \text{Tr } M_{00} - \text{Tr } M_{11}.$$  

Expansion (2.4) can be proved by considering diagonal matrices. As far as we have managed to find out, this formula was first obtained in [17].

The expansion of the characteristic function at infinity leads to traces of the wedge products of the inverse matrix:

$$\text{Ber}(1+zA) = \sum_{k=q-p}^{\infty} c_{-k}^*(A)z^{-k} \text{ where } c_{-k}^*(A) = \text{Ber } A \cdot \text{Tr } \Lambda^{p-q+k}A^{-1}.  \quad (2.5)$$

Formula (2.5) follows from the equalities $\text{Ber}(1+zA) = \text{Ber } A \text{ Ber}(A^{-1}+z) = z^{p-q}\text{Ber } A \text{ Ber}(1+z^{-1}A^{-1})$ and (2.4). The geometric meaning of the expansion (2.5) is as follows. $\text{Ber } A \cdot \text{Tr } \Lambda^{p-k}A^{-1} = \text{Tr } \Sigma^kA$ is the
trace of the representation of \( A \) in the space \( \Sigma^k V := \text{Ber} V \otimes \Lambda^{p-k} V^* \). In the ordinary case, it would be just a “dual” description of the same \( \Lambda^k V \); in the super case these two spaces are essentially different. Hence we get the following proposition.

**Proposition 2.** There is an expansion at infinity

\[
\text{Ber}(1 + zA) = \sum_{k=q-p}^{\infty} c^*_k(A)z^{-k} \quad \text{where} \quad c^*_k(A) = \text{Tr} \Sigma^{q-k} A, \quad (2.6)
\]

which is a Taylor expansion when \( p \leq q \) and a Laurent expansion when \( p > q \). Here \( \text{Tr} \Sigma^{q-k} A = \text{Ber} A \cdot \text{Tr} \Lambda^{p-q+k} A^{-1} \).

Consider the coefficients \( c_k(A) = \text{Tr} \Lambda^k A \). They can be expressed as polynomials via \( s_k(A) = \text{Tr} A^k \). This follows from the Liouville formula (hence basically from the multiplicativity of the Berezinian):

\[
\text{Ber}(1 + zA) = e^{\text{Tr} \ln(1 + zA)} = \exp \left( z \text{Tr} A - \frac{z^2}{2} \text{Tr} A^2 + \frac{z^3}{3} \text{Tr} A^3 + \ldots \right). 
\]

Hence \( c_k(A) \) can be expressed via \( s_k(A) = \text{Tr} A^k \) by the formulae \( c_k(A) = P_k(s_0(A), \ldots, s_k(A)) \), where \( P_k \) are classical Newton’s polynomials. For example, \( c_0 = s_0 = 1 \),

\[
c_1 = s_1, \quad c_2 = \frac{1}{2} (s_1^2 - s_2), \quad c_3 = \frac{1}{6} (s_1^3 - 3s_1s_2 + 2s_3),
\]

etc., where \( c_k = c_k(A) \), \( s_k = s_k(A) \). There is a formula

\[
c_{k+1} = \frac{1}{k+1} (s_1 c_k - s_2 c_{k-1} + \ldots + (-1)^k s_{k+1}). \quad (2.7)
\]

These universal formulae linking \( c_k(A) \) with \( s_k(A) \) are true regardless whether \( V \) is a superspace or ordinary space.

For further considerations it is convenient to define the following polynomials:

\[
\mathcal{H}_k(z) = z^k - c_1 z^{k-1} + c_2 z^{k-2} - \ldots + (-1)^k c_k,
\]

where \( k = 0, 1, 2, \ldots \) We shall refer to them as to the Cayley–Hamilton polynomials. (They appear with the relation to the analog of the Cayley–Hamilton theorem which we discuss later. In the classical case of an \( n \)-dimensional space, \( \pm \mathcal{H}_n(z) \) is the classical characteristic polynomial \( \det(A - z) \) if \( c_k = c_k(A) \).) The following identities are satisfied:

\[
\frac{d c_{k+1}(A)}{dA} = (-1)^k \mathcal{H}_k^A(A) \quad (2.9)
\]

\[
\frac{1}{k+1} \text{Tr}(A \mathcal{H}_k^A(A)) = (-1)^k c_{k+1}(A), \quad (2.10)
\]
Here $\mathcal{H}^A_k(z)$ is the value of the polynomial (2.8) where $c_k = c_k(A)$ at $z = A$. The derivative $\frac{df(A)}{dA}$ of a scalar function of a matrix argument is defined as the matrix which satisfies
$$\langle \frac{df(A)}{dA}, B \rangle = \frac{df(A + tB)}{dt} \bigg|_{t=0}$$
for an arbitrary matrix $B$, where the scalar product of matrices is given by $\langle A, B \rangle = \text{Tr} AB$. Formulae (2.7), (2.9) can be deduced by differentiating the characteristic function $R_A(z) = \text{Ber}(1 + Az)$. Bearing in mind that $\frac{d}{dz} \text{Ber} M = \text{Ber} M \text{Tr}(M^{-1}dM)$, we can come to the following identities:

$$\frac{d}{dz} \log R_A(z) = \text{Tr} (A (1 + Az)^{-1}) = \sum_{k=0}^{\infty} (-1)^k s_{k+1}(A) z^k,$$

$$\frac{d}{dA} \log R_A(z) = (1 + Az)^{-1} z = \sum_{k=0}^{\infty} (-1)^k z^{k+1} A^k.$$

By writing $d \log R_A(z)$ as $(R_A(z))^{-1} dR_A(z)$ and comparing the power series we arrive at (2.7), (2.9).

Unlike the polynomial functions $c_k(A) = \text{Tr} \Lambda^k A$, the coefficients $c^*_k(A) = \text{Tr} \Sigma^{q+k} A = \text{Ber} A \cdot \text{Tr} \Lambda^{p-q-k} A^{-1}$ are rational functions of the matrix entries of $A$. In particular,

$$c^*_{p-q}(A) = \text{Tr} \Sigma^p A = \text{Ber} A.$$

Our task will be to give an expression for $c^*_k(A)$ in terms of polynomial invariants of $A$.

3. Recurrence Relations for Traces of Exterior Powers

Recall that $\Sigma^k A$ denotes the representation of $A$ in the space $\Sigma^k V = \text{Ber} V \cdot \Lambda^{p-k} V^*$, thus $\text{Tr} \Sigma^k A = \text{Ber} A \cdot \text{Tr} \Lambda^{p-k} A^{-1}$.

By definition, $\text{Tr} \Sigma^k A = 0$ when $k > p$ and $\text{Tr} \Lambda^k A = 0$ when $k < 0$.

In the purely even case ($q = 0$, $\dim V = p$), the spaces $\Lambda^k V$ and $\Sigma^k V$ are canonically isomorphic, $\text{Tr} \Lambda^k A = \text{Tr} \Sigma^k A$, and $c_p(A) = \det A$, $c_k(A) = 0$ for $k > p$. We shall find out now what replaces these facts for a general $p|q$-dimensional superspace.

Let us analyze the expansions of the characteristic function $R_A(z)$. One can see that $R_A(z)$ is a fraction of the appearance

$$R_A(z) = \frac{P(z)}{Q(z)} = \frac{1 + a_1 z + a_2 z^2 + \ldots + a_p z^p}{1 + b_1 z + b_2 z^2 + \ldots + b_q z^q}$$

where the numerator is a polynomial of degree $p$ and the denominator is a polynomial of degree $q$. (Consider the diagonal matrices.) In principle the degrees can be less than $p$ and $q$, and the fraction may
be reducible. However, for an operator “in a general position”, this fraction is irreducible and the top coefficients \( a_p, b_q \) can be assumed to be invertible. (A discussion of algebraic problems related with the notion of “general position” in this context can be found in [15]. See also Section 6.) We shall use the notation \( R^+_A(z) \) and \( R^-_A(z) \) for the numerator and denominator of the fraction \( R_A(z) \). Later we shall show how \( R^+_A(z) \) and \( R^-_A(z) \) can be determined from the operator \( A \).

From the well known connection between rational functions and recurrent sequences (see Appendix), one can deduce the following facts:

1. The coefficients \( c_k(A) = \text{Tr} \Lambda^kA \) of the expansion of \( R_A(z) \) at zero (2.4) satisfy the recurrence relation of period \( q \)

\[
b_0 c_{k+q} + \ldots + b_q c_k = 0 \tag{3.1}
\]

for all \( k > p - q \), where \( b_0 = 1 \). In particular, if \( p < q \), then the relation (3.1) holds for all \( c_k \) including the zero values when \( p - q < k < 0 \).

2. The coefficients \( c^*_k(A) = \text{Tr} \Sigma^{q+k}A \) of the expansion of \( R_A(z) \) at infinity (2.6) satisfy the same recurrence relation:

\[
b_0 c^*_k + \ldots + b_q c^*_{k-q} = 0 \tag{3.2}
\]

for all \( k < 0 \). In particular, if \( p < q \), then the relation (3.2) holds for all \( c^*_k \) including the zero values when \( p - q < k < 0 \).

3. If \( p < q \), then \( c_k \) and \( -c^*_k \) can be combined together into a single recurrent sequence, for all \( k \in \mathbb{Z} \):

\[
\hat{c}_k = \begin{cases} 
  c_k & \text{if } k \geq 0 \\
  0 & \text{if } p - q < k < 0 \\
  -c^*_k & \text{if } k \leq p - q
\end{cases} \tag{3.3}
\]

The same holds in general: if one considers \( c_k \) with sufficiently large positive \( k \) and \( -c^*_k \) with sufficiently large negative \( k \), they fit into a single recurrent sequence.

4. Moreover, for arbitrary \( p \) and \( q \) the differences \( \gamma_k = c_k - c^*_k \) satisfy the recurrence relation

\[
b_0 \gamma_{k+q} + \ldots + b_q \gamma_k = 0 \tag{3.4}
\]

for all values of \( k \in \mathbb{Z} \) (notice that \( c_k = 0 \) for \( k < 0 \), \( c^*_k = 0 \) for \( k > p - q \)).

In particular, we have obtained the following fundamental theorem.
**Theorem 1.** For an operator $A$ acting in $p|q$-dimensional vector space the differences

$$
\gamma_k = c_k - c_k^* = \text{Tr } \Lambda^k A - \text{Tr } \Sigma^{q+k} A
$$

form a recurrent sequence with period $q$, for all $k \in \mathbb{Z}$. □

In the classical case of $q = 0$, all terms of the sequence (3.5) are zero and $\text{Tr } \Lambda^k A = \text{Tr } \Sigma^k A$ or $\text{Tr } \Lambda^k A = \det A \cdot \text{Tr } \Lambda^{p-k} A^{-1}$ for any operator $A$ which is a familiar equality. In this case the spaces $\Lambda^k V$ and $\Sigma^k V$ are canonically isomorphic. Theorem 1 actually suggests a relation between spaces $\Lambda^k V$ and $\Sigma^{k+q} V$ for arbitrary $q$ (see details in Section 7).

In (3.5) the terms $c_k = \text{Tr } \Lambda^k A$ and $c_k^* = \text{Tr } \Sigma^{q+k} A$ can be both nonzero only in a finite range, for $k = 0, \ldots, p-q$ when $p > q$. Otherwise $\gamma_k$ equals either $c_k(A)$ (for $k \geq p-q+1$) or $-c_k^*(A)$ (for $k \leq -1$). The relation (3.5) gives us a tool to express terms of the recurrent sequences $c_k^* = \text{Tr } \Sigma^{q+k} A$ and $c_k = \text{Tr } \Lambda^k A$ via each other.

What actually happens, for large $k$, $\gamma_k = c_k$, and they can be continued to the left using (3.4) to obtain $c_k^*(A)$, in particular $c_{p-q}(A) = \text{Ber } A$, as

$$
\text{Ber } A = \text{Tr } \Lambda^{p-q} A - \gamma_{p-q}.
$$

The “continuation to the left” of $c_k(A)$ using the recurrence relation (3.1) corresponds to the analytic continuation of the power series (2.4) representing the rational function $R_A(z)$ near zero.

**Example.** If $p < q$, then $\text{Tr } \Lambda^k A$ and $-\text{Tr } \Sigma^{q+k} A$ make a single recurrent sequence for all $k$, so $\gamma_k = \widehat{c}_k$ in the notation above (3.3). Hence, in particular,

$$
\text{Ber } A = -\widehat{c}_{p-q}.
$$

We give examples of calculations in the next section.

For linear recurrence relations with constant coefficients such as (3.1) or (3.4) it is possible to eliminate the coefficients to obtain the relation “in a closed form”. This is a standard method based on the connection of recurrent sequences and rational functions with infinite Hankel matrices (see, e.g., [8]). Recall that a Hankel matrix is one with the entries $c_{ij} = c_{i+j}$. A recurrence relation for $c_k$ of period $q$ implies the vanishing of Hankel determinants of order $q + 1$.

The statement (3.5) of the Theorem can be reformulated in the following way: the identity

$$
\begin{vmatrix}
\gamma_k(A) & \cdots & \gamma_{k+q}(A) \\
\cdots & \cdots & \cdots \\
\gamma_{k+q}(A) & \cdots & \gamma_{k+2q}(A)
\end{vmatrix} = 0
$$

(3.7)
holds for all \( k \in \mathbb{Z} \).

**Corollary.** The identity

\[
\begin{vmatrix}
    c_k(A) & \ldots & c_{k+q}(A) \\
    \ldots & \ldots & \ldots \\
    c_{k+q}(A) & \ldots & c_{k+2q}(A)
\end{vmatrix} = 0
\]  

holds for all \( k > p - q \).

**Remark.** In works \[2, 11\] appeared a system of equations for \( b_1, \ldots, b_q \) which is our equations (3.1) (for \( p \geq q \)) with \( k = p - q + 1, \ldots, p \), but they did not consider recurrence relations. The recurrence relations for \( c_k = \text{Tr} \Lambda^k A \), in particular the identity (3.8), appeared in \[13\] and was then interpreted in \[14\] by an analysis of Young diagrams. Compared to our work, in \[13, 14\] they came to the recurrence relation for \( c_k \) by pure combinatorics, using an explicit expression of \( c_k \) in terms of symmetric functions of the ‘bosonic’ and ‘fermionic’ eigenvalues for a diagonal matrix, and not from the characteristic function \( R_A(z) \), as we do here. Because of that, in the works \[13, 14\] they never considered the coefficients \( c_k^* \); hence they could not see the general recurrence relations involving both \( c_k \) and \( c_k^* \) that we establish here.

### 4. Berezinian as a Rational Function of Traces

As we established above, the coefficients \( c_k(A) = \text{Tr} \Lambda^k A \) for a linear operator \( A \) in a \( p|q \)-dimensional vector space \( V \) satisfy relations (3.1) making them a \( p|q \)-recurrent sequence (see Appendix for the necessary notions). Basing just on this fact we will give a recurrent procedure for calculating the characteristic function \( R_A(z) = \text{Ber}(1 + zA) \) and the Berezinian of the operator \( A \). Then we will present a closed formula for \( \text{Ber} A \) using the relations (3.7) of Theorem 1.

Let \( \mathbf{c} = \{c_n\}_{n \geq 0} \) be a \( p|q \)-recurrent sequence such that \( c_0 = 1 \). Denote by \( R_{p|q}(z, \mathbf{c}) \) its generating function:

\[
R_{p|q}(z, \mathbf{c}) = \frac{1 + a_1z + \ldots + a_p z^p}{1 + b_1z + \ldots + b_q z^q} = 1 + c_1z + c_2z^2 + \ldots.
\]

The fraction \( R_{p|q}(z, \mathbf{c}) \) is defined by the first \( p + q \) terms \( c_1, c_2, \ldots, c_{p+q} \) of the sequence \( \mathbf{c} \):

\[
R_{p|q}(z, \mathbf{c}) = R_{p|q}(z, c_1, \ldots, c_{p+q}).
\]

In particular, if \( A \) is a \( p|q \times p|q \) matrix and \( \{c_k\} \) is the sequence of the traces of exterior powers of the matrix \( A \) \( (c_k = c_k(A) = \text{Tr} \Lambda^k A) \), then \( R_{p|q}(z, \mathbf{c}) \) coincides with the characteristic function of \( A \):

\[
R_A(z) = R_{p|q}(z, c_1(A), c_2(A), \ldots, c_{p+q}(A)).
\]  

(4.1)
The rational functions $R_{p|q}(z, c) = R_{p|q}(z, c_1, \ldots, c_{p+q})$ have the following properties:

(1) If $p \geq q$, then the sequence $c'$ defined by $c'_k := \frac{c_k + 1}{c_1}$ (assuming that the coefficient $c_1$ is invertible) is a $p-1|q$-recurrent sequence and

$$R_{p|q}(z, c) = 1 + c_1 z R_{p-1|q}(z, c'), \quad (4.2)$$

i.e.,

$$R_{p|q}(z, c_1, \ldots, c_{p+q}) = 1 + c_1 z R_{p-1|q} \left( z, \frac{c_2}{c_1}, \ldots, \frac{c_{p+q}}{c_1} \right). \quad (4.3)$$

(2) The sequence $c'' = \{c''_n\}$ defined according to

$$1 + c''_1 z + c''_2 z^2 + \ldots = \frac{1}{1 + c_1 z + c_2 z^2 + \ldots},$$

for example

$c''_1 = -c_1$, $c''_2 = -c_2 + c_1^2$, $c''_3 = -c_3 + 2c_1 c_2 - c_1^3$, \ldots, \quad (4.4)$

is a $q|p$-recurrent sequence, and

$$R_{p|q}(z, c_1, \ldots, c_{p+q}) = \frac{1}{R_{q|p}(z, c''_1, \ldots, c''_{p+q})}. \quad (4.5)$$

(If $A$ is a $p|q \times p|q$ supermatrix and $A''$ is the parity reversed $q|p \times q|p$ supermatrix, then $c_k(A'') = c_k(A)''.)$

Using these properties one can express the rational function $R_{p|q}$ corresponding to a $p|q$-recurrent sequence via the rational function $R_{0|1}$ corresponding to a $0|1$-recurrent sequence, i.e., a geometric progression. The steps are as follows. If $p < q$, we apply (4.5) to get a $p'|q'$-sequence with $p' > q'$. If $p > q$, we repeatedly apply (4.2) to decrease $p$.

**Example 4.1.** Let $A$ be a $p|1 \times p|1$ matrix. Then it follows from (4.2) and (4.5) that

$$R_A(z) = R_{p|1}(z, c_1(A), c_2(A), \ldots, c_{p+1}(A)) =$$

$$1 + c_1 z R_{p-1|1} \left( z, \frac{c_2}{c_1}, \ldots, \frac{c_{p+1}}{c_1} \right) = \cdots =$$

$$1 + c_1 z + \cdots + c_{p-1} z^{p-1} + c_p z^p R_{0|1} \left( z, \frac{c_{p+1}}{c_p} \right) =$$

$$1 + c_1 z + \cdots + c_{p-1} z^{p-1} + \frac{c_p z^p}{1 - \frac{c_{p+1}}{c_p} z} =$$

$$1 + c_1 z + \cdots + c_{p-1} z^{p-1} + \frac{c_p z^p}{c_p - c_{p+1} z}$$
We can also deduce from here formulae for the Berezinian. One can see from (2.5) that for a $p|q \times p|q$ matrix $A$

$$\text{Ber } A = \lim_{z \to \infty} z^{q-p} R_A(z)$$

(4.6)

Let $c = \{c_n\}, \ n \geq 0$, be an arbitrary $p|q$-recurrent sequence such that $c_0 = 1$ and let $R(z,c)$ be its generating function. Then mimicking (4.6) we define the Berezinian of this sequence by the formula

$$B_{p|q}(c) = \lim_{z \to \infty} z^{q-p} R_{p|q}(z,c).$$

(4.7)

If $c_n = c_n(A) = \text{Tr } A^k$, then $B_{p|q}(c) = \text{Ber } A$. From (4.2) and (4.5) immediately follow relations for $B_{p|q}$:

$$B_{p|q}(c) = B_{p|q}(c_1, \ldots, c_{p+q}) = \begin{cases} c_1 B_{p-1|q}(c') & \text{if } p \geq q + 1 \\ 1 + c_1 B_{p-1|q}(c') & \text{if } p = q \\ \frac{1}{B_{q|p}(c^{\Pi})} & \text{if } p \leq q - 1 \end{cases}$$

(4.8)

where the sequences $c'$ and $c^{\Pi}$ are defined as above.

Using these relations one can calculate the Berezinians of matrices in terms of traces. Note that from these recurrent relations follows that if $p > q$ then for a $p|q$-recurrent sequence $c$, its Berezinian $B_{p|q}$ depends only on the coefficients $c_{p-q}, \ldots, c_p, \ldots, c_{p+q}$.

**Example 4.2.** For a $1|1 \times 1|1$ matrix:

$$\text{Ber } A = B_{1|1}(c_1(A), c_2(A)) = 1 + c_1 B_{0|1}(\frac{c_2}{c_1}) = 1 + \frac{\frac{c_1}{B_{1|0}(\frac{c_2}{c_1})}}{B_{1|0}(\frac{c_2}{c_1})^{\Pi}} =$$

$$1 + \frac{c_1}{c_2} = 1 - \frac{c_1^2}{c_2} = \frac{c_2 - c_1^2}{c_2} = \frac{\text{Tr } A^2 + (\text{Tr } A)^2}{\text{Tr } A^2 - (\text{Tr } A)^2}$$

(we have applied Newton’s formulae to get the last expression).

**Example 4.3.** For a $p|1 \times p|1$ matrix:

$$\text{Ber } A = B_{p|1}(c_{p-1}(A), \ldots, c_p(A), \ldots, c_{p+1}(A)) = c_{p-1} B_{1|1}(\frac{c_p}{c_{p-1}}, \frac{c_{p+1}}{c_{p-1}}) =$$

$$= \frac{c_{p-1}c_{p+1} - c_p^2}{c_{p+1}}$$
Example 4.4. For a $2 \times 2$ matrix:

$$\text{Ber } A = B_{2|2} (c_1(A), c_2(A), c_3(A), c_4(A)) = 1 + c_1 B_{1|2} \left( \frac{c_2}{c_1}, \frac{c_3}{c_1}, \frac{c_4}{c_1} \right) =$$

$$= 1 + \frac{c_1}{B_{2|1} \left( (\frac{c_2}{c_1})^H, (\frac{c_3}{c_1})^H, (\frac{c_4}{c_1})^H \right)} =$$

$$1 + \frac{c_1}{B_{2|1} \left( -\frac{c_2}{c_1}, -\frac{c_3}{c_1}, \left(\frac{c_2}{c_1}\right)^2, -\frac{c_4}{c_1}, + 2 \frac{c_2 c_3}{c_1 c_1} - \left(\frac{c_2}{c_1}\right)^3 \right)} =$$

$$1 - \frac{c_2 B_{1|1} \left( \frac{c_2}{c_2} - \frac{c_2}{c_1}, \frac{c_3}{c_2} - \frac{2c_3}{c_1} + \left(\frac{c_2}{c_1}\right)^2 \right)}{c_2 \left( 1 - \left(\frac{c_2}{c_2} - \frac{2c_3}{c_1} + \left(\frac{c_2}{c_1}\right)^2 \right) \right)} =$$

The last expression can be further simplified, and in principle one can proceed in this way to get the answer for arbitrary $q$, but at this point it is easier to give a general formula. It will reveal an unexpected link with classical algebraic notions.

5. Berezinian and Resultant

Let $A$ be an even linear operator in a $p|q$-dimensional superspace. Consider the relation (5.7) of Theorem 1 for $k = p - q$. Recall that $\gamma_{p-q} = c_{p-q} - c_{p-q}^*$, $\gamma_k = c_k$ for $k \geq p - q + 1$ and $c_{p-q} = \text{Ber } A$. Hence we have the following equalities:

$$\begin{vmatrix} \gamma_{p-q} & \cdots & \gamma_p \\ \cdots & \cdots & \cdots \\ \gamma_p & \cdots & \gamma_{p+q} \end{vmatrix} = \begin{vmatrix} c_{p-q} - \text{Ber } A & \cdots & c_p \\ \cdots & \cdots & \cdots \\ c_p & \cdots & c_{p+q} \end{vmatrix} = \begin{vmatrix} c_{p-q} & \cdots & c_p \\ \cdots & \cdots & \cdots \\ c_p & \cdots & c_{p+q} \end{vmatrix} - \text{Ber } A \begin{vmatrix} c_{p-q+2} & \cdots & c_{p+1} \\ \cdots & \cdots & \cdots \\ c_{p+1} & \cdots & c_{p+q} \end{vmatrix}$$

We arrive at the formula

$$\text{Ber } A = \begin{vmatrix} c_{p-q} & \cdots & c_p \\ \cdots & \cdots & \cdots \\ c_p & \cdots & c_{p+q} \\ c_{p-q+2} & \cdots & c_{p+1} \\ \cdots & \cdots & \cdots \\ c_{p+1} & \cdots & c_{p+q} \end{vmatrix} = \begin{vmatrix} c_{p-q} & \cdots & c_p \\ c_{p-q+2} & \cdots & c_{p+1} \end{vmatrix}_{q+1} \begin{vmatrix} c_{p-q+2} & \cdots & c_{p+1} \end{vmatrix}_q = (5.1)$$

where we used a short notation for Hankel determinants with subscripts denoting their orders. Here as always $c_k = 0$ for $k \leq -1$ and $c_0 = 1.$
Let us make an important observation. By the Schur–Weyl character formula it follows that the Hankel determinants appearing in the numerator and denominator of formula (5.1) are nothing but the traces of the representations of $A$ in the subspaces of tensors corresponding to certain Young diagrams.

Indeed, denote by $D = D_{\lambda_1, \ldots, \lambda_s}$ the Young diagram with $s$ columns, such that the $i$-th column contains $\lambda_i$ cells, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$. Let $V_D$ be an invariant subspace in the tensor power $V^\otimes N$, $N = \lambda_1 + \cdots + \lambda_s$, corresponding to the Young diagram $D = D_{\lambda_1, \ldots, \lambda_s}$, and $A_D$ be the representation of $A$ in $V_D$. Then the Schur–Weyl formula (see [22]) tells that the trace of $A_D$ is expressed via the traces $c_k(A) = \text{Tr} \Lambda^k A$ as the determinant of the following $s \times s$ matrix:

$$a_{ij} = c_{\lambda_i+j-i}(A) = \text{Tr} \Lambda^{\lambda_i+j-i} A,$$

$$\text{Tr} A_D = \det (a_{ij}).$$

It is known that the formula remains valid in the supercase (if trace means supertrace). Let $D(r, s)$ be the rectangular Young diagram with $r$ rows and $s$ columns. So $D(r, s) = D_{\lambda_1, \ldots, \lambda_s}$ with $\lambda_i = r$ for all $i$. One can see that for $D = D(r, s)$ the ‘Schur determinant’ $\text{Tr} A_D$ is equal to the Hankel determinant $|c_{r-s+1}, \ldots, c_r|^s$ of order $s$, with the inverted order of rows. In other words, Hankel determinants appearing in this paper can be interpreted as characters of tensor representations corresponding to rectangular Young diagrams. Hence, in particular, our formula (5.1) for the Berezinian can be rewritten in the following form

$$\text{Ber} A = (-1)^q \frac{\text{Tr} A_{D(p,q+1)}}{\text{Tr} A_{D(p+1,q)}},$$

(5.2)

the sign coming from the change of order of rows in the determinants.

Remark. In the classical situation ($q = 0$) when $c_k(A)$ are the elementary symmetric functions of the eigenvalues of $A$, Schur’s determinants corresponding to Young diagrams (or partitions) when written as functions of these eigenvalues, are special symmetric functions known as Schur functions (see [19]); in the supercase the same Schur determinants when expressed via the eigenvalues are no longer classical symmetric Schur functions but are combinations of functions that are separately symmetric in the ‘bosonic’ and ‘fermionic’ eigenvalues. They should probably be called ‘super Schur functions’.
Example 5.1. For a $2|3 \times 2|3$ matrix we have

$$\text{Ber} A = \begin{vmatrix} 0 & 1 & c_1 & c_2 \\ 1 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & c_4 \\ c_2 & c_3 & c_4 & c_5 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{vmatrix} = -\frac{\text{Tr} A_{D(2,4)}}{\text{Tr} A_{D(3,3)}}.$$ 

The formulae obtained above deserve to be called a theorem.

Theorem 2. The Berezinian of a linear operator $A$ in a $p|q$-dimensional space is equal to the ratio of the traces of the representations in the invariant subspaces of tensors corresponding to the rectangular Young diagrams $D(p,q+1)$ and $D(p+1,q)$

$$\text{Ber} A = \frac{\text{Tr} \Lambda^{p-q} A \ldots \text{Tr} \Lambda^p A}{\text{Tr} \Lambda^{p-q+2} A \ldots \text{Tr} \Lambda^{p+1} A} = \pm \frac{\text{Tr} A_{D(p,q+1)}}{\text{Tr} A_{D(p+1,q)}}. \quad (5.3)$$

Here at the right hand side stand the Hankel determinants of orders $q+1$ and $q$ made of the traces of exterior powers of the operator $A$. □

What is the meaning — as polynomial invariants of $A$ — of the determinants $\text{Tr} A_{D(p,q+1)}$ and $\text{Tr} A_{D(p+1,q)}$ appearing as the numerator and denominator in formula (5.3)?

Definition. Define the following functions of $A$:

$$\text{Ber}^+ A := \lambda_1 \ldots \lambda_p \prod_{i,\alpha} (\lambda_i - \mu_\alpha), \quad (5.4)$$

$$\text{Ber}^- A := \mu_1 \ldots \mu_q \prod_{i,\alpha} (\lambda_i - \mu_\alpha). \quad (5.5)$$

We assume for a moment that $A$ can be diagonalized and $\lambda_i, \mu_\alpha, i = 1, \ldots, p, \alpha = 1, \ldots, q$ stand for its eigenvalues. So

$$\text{Ber} A = \frac{\lambda_1 \ldots \lambda_p}{\mu_1 \ldots \mu_q} = \frac{\text{Ber}^+ A}{\text{Ber}^- A}.$$ 

We shall immediately see that $\text{Ber}^\pm A$ make sense for all $A$.

Denote the product $\prod_{i,\alpha} (\lambda_i - \mu_\alpha)$ by $R$ or $R(A)$. If $R^+_A(z)$ and $R^-_A(z)$ stand for the numerator and denominator of the characteristic function $R_A(z)$, then it is easy to check that $R$ is the classical Silvester’s resultant for the polynomials $R^+_A(z)$ and $R^-_A(z)$, $R = \text{Res}(R^-_A(z), R^+_A(z))$. 

Proposition 3. The resultant of $R_A^+(z)$ and $R_A^-(z)$ can be expressed by the following formula:

$$R = \text{Res}(R_A^+(z), R_A^-(z)) = \prod_{i,\alpha}(\lambda_i - \mu_\alpha) = (-1)^{q(q-1)/2}|c_{p-q+1} \ldots c_p|_q = \text{Tr} A_D(p,q).$$ (5.6)

Proof. The Hankel determinant in the r.h.s.of (5.6) vanishes when $\lambda_i = \mu_\alpha$ for any pair $i, \alpha$. This follows from our recurrence relation (3.8) applied a $(p-1 | q - 1)$-dimensional space. Hence $|c_{p-q+1} \ldots c_p|_q$ is divisible by the resultant. As polynomials in $\lambda_i, \mu_\alpha$ they have the same degree $pq$, hence they must coincide up to a numerical factor, which can be checked, for example, by setting all $\mu_\alpha = 0$. □

It follows that $R = R(A)$ is a polynomial in the matrix entries of $A$.

Note that the statement of Proposition 3 is present in Berezin’s paper [2].

Theorem 3. The following equalities hold:

$$\text{Ber}^+ A = \lambda_1 \ldots \lambda_p \prod_{i,\alpha}(\lambda_i - \mu_\alpha) = |c_{p-q} \ldots c_p|_{q+1}$$ (5.7)

$$\text{Ber}^- A = \mu_1 \ldots \mu_q \prod_{i,\alpha}(\lambda_i - \mu_\alpha) = |c_{p-q+2} \ldots c_{p+1}|_q,$$ (5.8)

i.e., $\text{Ber}^+ A$ and $\text{Ber}^- A$ give exactly the top and bottom of the expression for $\text{Ber} A$ in formula (5.3).

Proof. Indeed, $\lambda_1 \ldots \lambda_p$ and $\mu_1 \ldots \mu_q$ are equal, respectively, to the coefficients $a_p$ and $b_q$ in $R_A^+(z)$ and $R_A^-(z)$. In general, all the coefficients $a_i, b_k$ can be obtained from $c_k$, $k = 1, \ldots, p + q$, by solving simultaneous equations, with the determinant of the system being exactly $R$. Therefore, all coefficients $a_i, b_k$ have the appearance of a polynomial in $c_k$ divided by the same denominator $R = \pm |c_{p-q+1} \ldots c_p|_q = \text{Tr} A_D(p,q)$. Formulae (5.7) and (5.8) follow by a direct application of Cramer’s rule. (In particular, this yields another proof of the expression for the Berezinian [5.3].) □

From the proof, in particular, follows that the polynomials $R^+(z)$ and $R^-(z)$ are defined if the resultant $R = |c_{p-q+1} \ldots c_p|_q$ is invertible.

Notice that the top and bottom of the standard definition of the Berezinian given by fraction (2.2) are non-invariant and non-polynomial functions of the matrix; the products $\lambda_1 \ldots \lambda_p$ and $\mu_1 \ldots \mu_q$ are invariant, but non-polynomial (and defined not explicitly as functions of the matrix entries). The functions $\text{Ber}^\pm A$ are polynomial invariants, and,
as one can see, they are the "minimally possible" modifications of the products of eigenvalues with this property.

We have four remarkable Hankel (or Schur) determinants in this paper: $\text{Tr} A_D(p, q)$, $\text{Tr} A_D(p+1, q)$, $\text{Tr} A_D(p, q+1)$ and $\text{Tr} A_D(p+1, q+1)$; the first being the resultant $R$, the last giving the identity (3.8) of the smallest degree, and the two in the middle arising in the formula for the Berezinian (5.3).

Remark. As a by-product of Proposition 3 we have the following formula for the resultant of two polynomials:

$$\text{Res}(Q, P) = \left| \begin{array}{cccc} c_{p-q+1} & \cdots & c_p \\
\cdots & \cdots & \cdots \\
c_p & \cdots & c_{p+q-1} \end{array} \right|$$

(5.9)

where $P(z) = a_p z^p + \ldots + 1$, $Q(z) = b_q z^q + \ldots + 1$, and the coefficients $c_k = c_k(Q, P)$ are defined as follows:

$$c_k(Q, P) = \sum_{i+j=k} a_i \tau_j (-1)^j$$

(5.10)

where $\tau_j$ are the complete symmetric functions of the roots of $Q$. The r.h.s. of (5.9) can be interpreted as the (super)trace $\pm \text{Tr} A_D(p, q)$, where $A$ is an operator in a $p|q$-dimensional space associated with the pair of polynomials $P, Q$, so that $R_A = \frac{P}{Q}$.

6. Rational and Polynomial Invariants and the Cayley–Hamilton Identity

In the previous section we obtained explicit formulae expressing the Berezinian of a linear operator $A$ as rational function of traces. The Berezinian is an example of a rational invariant function on supermatrices. Let us briefly review general facts concerning such functions. This will be applied to the analysis of the analog of the Cayley–Hamilton theorem.

In the classical case invariant rational functions $F(A)$ on $p \times p$ matrices, $F(A) = F(C^{-1}AC)$, are in a $1-1$ correspondence with rational symmetric functions $f(\lambda_1, \ldots, \lambda_p)$ of $p$ variables, the eigenvalues of $A$. The same is true for polynomial functions, due to the fundamental theorem on symmetric functions and to the fact that the elementary symmetric polynomials $\sigma_k(\lambda)$ (or the power sums $s_k(\lambda)$) are restrictions of the polynomial functions of matrices $\text{Tr} A^k$ (resp., $\text{Tr} A^k$).

This is not the case for $p|q \times p|q$ matrices, where arises a sharp distinction between rational and polynomial invariants.
Every invariant rational function \( F(A) \) on \( p|q \times p|q \) matrices, i.e.,
\[ F(A) = F(C^{-1}AC) \]
for every even invertible matrix \( C \), defines a function \( f(\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q) \) of the eigenvalues of \( A \), with \( \lambda_i \) corresponding to even eigenvectors and \( \mu_\alpha \) to odd eigenvectors, symmetric separately in the variables \( \lambda_1, \ldots, \lambda_p \) and \( \mu_1, \ldots, \mu_q \) (because even and odd eigenvectors cannot be permuted by a similarity transformation).

**Proposition 4.** Every rational \( S_p \times S_q \)-invariant function of \( \lambda_i, \mu_\alpha \) can be expressed as a rational function of the polynomials \( c_1, \ldots, c_{p+q} \) or \( s_1, \ldots, s_{p+q} \), where \( c_k(\lambda, \mu) = \text{Tr} \Lambda^k A \), \( s_k(\lambda, \mu) = \text{Tr} A^k \). (Traces are supertraces).

**Example 6.1.** Consider the \( S_1 \times S_1 \)-invariant polynomial \( f(\lambda, \mu) = \lambda + \mu \). We have
\[ \lambda + \mu = \frac{\lambda^2 - \mu^2}{\lambda - \mu} = \frac{s_2}{s_1} = \frac{c_1^2 - c_2}{c_1}, \]
therefore it corresponds to a rational invariant function on \( 1|1 \times 1|1 \) matrices.

We see that \( S_p \times S_q \)-invariant polynomials do not necessarily extend to invariant polynomials of matrices.

**Proposition 4** (Berezin [2], [3, p. 315], Kac [11]) immediately follows from considerations of the previous section, as all \( S_p \times S_q \)-invariant functions of \( \lambda_i, \mu_\alpha \) are expressed via the elementary symmetric functions of \( \lambda_i \) and \( \mu_\alpha \), i.e., the coefficients \( a_k, b_k \) of the numerator and denominator of the characteristic function \( R_A(z) \), which are rational functions of \( c_1, \ldots, c_{p+q} \). Moreover, for \( S_p \times S_q \)-invariant polynomials \( f(\lambda, \mu) \) it follows that the corresponding rational invariant functions \( F(A) \) can be written as fractions with the numerator being a polynomial invariant function of \( A \) and the denominator being a power of the resultant \( R = R(A) \).

The following non-trivial statement holds.

**Proposition 5** (Berezin, Sergeev). For a \( S_p \times S_q \)-invariant polynomial \( f(\lambda, \mu) \) three conditions are equivalent: (a) the equation
\[ \left( \frac{\partial f}{\partial \lambda_i} + \frac{\partial f}{\partial \mu_j} \right) \bigg|_{\lambda_i = \mu_j} = 0, \]
(6.2)
is satisfied; (b) \( f(\lambda, \mu) \) extends to a polynomial invariant on matrices; (c) \( f(\lambda, \mu) \) can be expressed as a polynomial of a finite number of functions \( c_k(\lambda, \mu), k = 0, 1, 2, 3, \ldots \) (or \( s_k(\lambda, \mu), k = 0, 1, 2, 3, \ldots \)).

The implication (c)⇒(b) is obvious, the implication (b)⇒(a) can be deduced from the invariance condition, the implication (a)⇒(c) is the most technical part. (See [2], [3, p. 294], [18], [19].)
Example 6.2. The $S_1 \times S_1$-invariant polynomial $f(\lambda, \mu) = \mu^N(\lambda - \mu)$ satisfies (6.2) and is in fact equal to the polynomial $(-1)^N c_{N+1}(A)$. It cannot be expressed as a polynomial in $c_1, \ldots, c_k$ if $k \leq N$. On the other hand, in full accordance with Proposition 4, we can express it rationally via $c_1, c_2$:

$$
\mu^N(\lambda - \mu) = (-1)^N c_{N+1}(A) = \frac{c_2^N}{c_1^{N-1}}.
$$

Example 6.2 demonstrates that, differently from the classical case, the algebra of polynomial invariants on supermatrices is not finitely generated (no a priori number of $c_k$ is sufficient) and is not free (the generators $c_k$, $k = 1, 2, \ldots$ satisfy an infinite number of relations (3.8)).

Remark. It would be interesting to describe the class of invariant rational functions on $\lambda_i, \mu_\alpha$ that obey equation (6.2). For example, the characteristic function $R_A(z)$ and the Berezinian $\text{Ber} A$ belong to this class. Hence it contains products of polynomial invariants with arbitrary powers of the Berezinian.

Now let us turn to the Cayley–Hamilton theorem.

For an operator $A$ in a $p|q$-dimensional space it is clear that it annihilates the polynomial $P_A(z) = \prod(\lambda_i - z)(\mu_\alpha - z)$, where $\lambda_i, \mu_\alpha$ stand for the eigenvalues of $A$ as above, and one can see that every polynomial annihilating a generic operator $A$ is divisible by $P_A(z)$, exactly as it is in the classical case. Hence, the polynomial $P_A(z)$ is a minimal polynomial for generic operators. ‘Generic’ means here that all the differences of the eigenvalues, $\lambda_i - \lambda_j, \lambda_i - \mu_\alpha, \mu_\alpha - \mu_\beta$, are invertible. In particular, $R = \text{Res}(R_A^-, R_A^+) = \text{Tr} A_{D(p,q)}$ is invertible and $R_A^\pm(z)$ make sense. This ‘classical characteristic polynomial’ or ‘Cayley–Hamilton polynomial’ of $A$, is expressed in terms of the characteristic function $R_A(z)$ as

$$
P_A(z) = (-z)^{p+q} R_A^+(\frac{-1}{z}) R_A^-(\frac{-1}{z}) =
(\alpha_p - \alpha_{p-1} z + \ldots + (-1)^p z^p) (\beta_q - \beta_{q-1} z + \ldots + (-1)^q z^q).
$$

Since the coefficients of $R_A^\pm(z)$ are rational invariant functions of $A$, with the denominator $R = \text{Res}(R_A^-, R_A^+) = \text{Tr} A_{D(p,q)}$, it follows that the coefficients of $P_A(z)$, too, are rational (not polynomial) invariant functions of $A$, with denominators $R$ or $R^2$.

Example 6.3. Consider a linear operator $A$ in a $p|1$-dimensional vector space $V$. Let us calculate for it the polynomial $P_A(z)$, which is here
\[ P_A(z) = (\lambda_1 - z) \ldots (\lambda_p - z)(\mu - z). \] From Example 4.1 we get

\[ R_A(z) = 1 + c_1 z + \ldots + c_{p-1} z^{p-1} + \frac{c_p}{1 - \frac{c_p+1}{c_p} z} z^p = \]

\[
\left( 1 + \frac{c_1 c_p - c_{p+1}}{c_p} z + \frac{c_2 c_p - c_1 c_{p+1}}{c_p} z^2 + \ldots + \frac{c_p c_p - c_{p-1} c_{p+1}}{c_p} z^p \right) \times \left( 1 - \frac{c_p+1}{c_p} z \right)^{-1}
\]

where \( c_k = c_k(A) = \text{Tr} \Lambda^k A \). Hence

\[ P_A(z) = (-1)^{p+1} \left( z^p - \frac{c_1 c_p - c_{p+1}}{c_p} z^{p-1} + \frac{c_2 c_p - c_1 c_{p+1}}{c_p} z^{p-2} - \ldots + (-1)^p \frac{c_p c_p - c_{p-1} c_{p+1}}{c_p} \right) \left( z + \frac{c_p+1}{c_p} \right) \]

and after simplification using the identity \( c_p c_{p+2} - c_{p+1}^2 = 0 \) we get

\[ P_A(z) = \sum_{k=0}^{p+1} (-1)^{p+1-k} \frac{c_k c_p - 2c_{k-1} c_{p+1} + c_{k-2} c_{p+2}}{c_p} z^{p+1-k} \quad (6.4) \]

where as always \( c_k = 0 \) for \( k < 0 \). Notice that here \( R = c_p \), and it appears in the denominator in the final answer in the first power, not as \( R^2 \) as one might expect, due to identities for \( c_k \). We will see that this is the general case.

By multiplying \( P_A(z) \) by its denominator we can get an annihilating polynomial with the coefficients which are polynomial invariant functions of the matrix entries of \( A \). The advantage of such a polynomial is that it will be an annihilating polynomial for arbitrary operators, not necessarily generic. Notice that a minimal polynomial for generic operators is unique up to a factor \( R^N \).

**Example 6.4.** (Example 6.3 continued.) Multiplying both sides of (6.4) by \( c_p \) we obtain the polynomial

\[ \tilde{P}_A(z) = \sum_{k=0}^{p+1} (-1)^{p+1-k} \frac{c_k c_p - 2c_{k-1} c_{p+1} + c_{k-2} c_{p+2}}{c_p} z^{p+1-k} \quad (6.5) \]

which annihilates an arbitrary operator \( A \) in a \( p \mid 1 \)-dimensional space and whose coefficients are polynomial invariants of \( A \).

Let us show that the ‘naive’ characteristic polynomial discussed above follows also from the recurrence relations of Theorem 1. A
method of constructing a ‘Cayley–Hamilton identity’ from a relation
on traces was given in [14]. Below we shall use that method and then
show that the final answer can be identified with the naive formula
(6.3) up to a factor.

If $A$ is an even linear operator in a $p|q$-dimensional vector space, then,
in particular, the traces of its exterior powers obey relations (3.8) for
all $k > p - q$. For $k = p - q + 1$ we have

$$
\begin{vmatrix}
  c_{p-q+1}(A) & \ldots & c_{p+1}(A) \\
  \ldots & \ldots & \ldots \\
  c_{p+1}(A) & \ldots & c_{p+q+1}(A)
\end{vmatrix} = |c_{p-q+1}(A) \ldots c_{p+1}(A)|_{q+1} = 0. \tag{6.6}
$$

This is a scalar equation valid for any even matrix in a $p|q$-dimensional
space. Hence, by differentiating it one obtains a matrix identity (com-
pare with a formal differential calculus developed in [14]).

In the classical case when $A$ is a linear operator in a $p$-dimensional
vector space ($q = 0$), the relation (6.6) reduces to $c_{p+1}(A) \equiv 0$. Dif-
fentiating this identity gives exactly the vanishing of the Cayley–
Hamilton polynomial $\mathcal{H}_p(z)$ with $c_k = c_k(A)$ at $z = A$, i.e., the classical
Cayley–Hamilton theorem.

For arbitrary $q$, by taking the derivative of (6.6) and applying (2.9),
we get the equality

$$
\sum_{r=p-q+1}^{p+q+1} (-1)^{r-1} F_r^A \mathcal{H}_r^A(A) = 0 \tag{6.7}
$$

where we denote by $F_r$ the partial derivative of the Hankel determinant
$|c_{p-q+1} \ldots c_{p+1}|_{q+1}$,

$$
F_r = \frac{\partial}{\partial c_r} |c_{p-q+1} \ldots c_{p+1}|_{q+1}, \tag{6.8}
$$

and by $F_r^A$ its value when $c_k = c_k(A)$. Define a polynomial in $z$ of
degree $p + q$, with coefficients polynomially depending on $c_k$:

$$
\tilde{P}(z) := \sum_{r=p-q}^{p+q} (-1)^r F_{r+1} \mathcal{H}_r(z). \tag{6.9}
$$

We shall write $\tilde{P}(z) = \tilde{P}_A(z)$ if $c_k = c_k(A)$. It follows that $\tilde{P}_A(z)$ is an
annihilating polynomial for $A$.

**Example 6.5.** Let us make a calculation for $p|1 \times p|1$ matrices. We
have the identity

$$
c_p c_{p+2} - c_{p+1}^2 \equiv 0. \tag{6.10}
$$
By differentiating we get $F_p = c_{p+2}$, $F_{p+1} = -2c_{p+1}$, $F_{p+2} = c_p$. Thus
\[ \tilde{P}_A(z) = (-1)^{p-1}F_p\mathcal{H}_{p-1}(z) + (-1)^pF_{p+1}\mathcal{H}_p(z) + (-1)^{p+1}F_{p+2}\mathcal{H}_{p+1}(z). \]
After substituting the expressions (2.8) for $\mathcal{H}$ terms we immediately get
\[ R \]
which precisely coincides with $R \cdot P_A(z)$ of Example [8,3].

Now we shall prove in general that by differentiating the identity for traces [6,6] one arrives at a multiple of the ‘classical’ characteristic polynomial $P_A(z)$. Indeed, for generic matrices, $P_A(z)$ is a minimal polynomial, and any annihilating polynomial for $A$ is divisible by $P_A(z)$. Consider the polynomial $\tilde{P}_A(z)$ defined in (6.12). Dividing it by $P_A(z)$ we get $\tilde{P}_A(z) = c \cdot P_A(z)$, where $c$ is a constant (as both polynomials are of the same degree). To calculate $c$ compare the top coefficient in $\tilde{P}_A(z)$, which is $(-1)^{p+q}F_{p+q+1}$, with that of $P_A(z)$, which is $(-1)^{p+q}$. We have directly
\[ F_{p+q+1} = \frac{\partial}{\partial c_{p+q+1}} \begin{vmatrix} c_{p-q+1} & \cdots & c_{p+1} \\ \cdots & \cdots & \cdots \\ c_{p+1} & \cdots & c_{p+q+1} \end{vmatrix} = \begin{vmatrix} c_{p-q+1} & \cdots & c_p \\ \cdots & \cdots & \cdots \\ c_p & \cdots & c_{p+q-1} \end{vmatrix} = R. \]
It follows that $c = R$. (We see that remarkably, $R$, not $R^2$, is the common denominator of the fractions that are the coefficients of $P_A(z)$.) We arrive at the following proposition.

**Proposition 6.** The polynomial $\tilde{P}_A(z)$ defined by formula (6.9) where $c_k = c_k(A)$, is an annihilating polynomial for any operator $A$ in a $p|q$-dimensional space. Its coefficients are invariant polynomial functions of $A$. For generic operators, $\tilde{P}_A(z)$ is a minimal polynomial, which divides all annihilating polynomials for $A$. The identity holds:
\[ \tilde{P}_A(z) = R \cdot P_A(z), \]
where $P_A(z) = \prod(\lambda_i - z)(\mu_\alpha - z)$ is the naive characteristic polynomial, with rational coefficients, and $R = \prod(\lambda_i - \mu_\alpha) = \text{Res}(R_A^+, R_A^-)$.

One can call the polynomial $\tilde{P}_A(z)$, with polynomial coefficients, a 'modified characteristic polynomial'. In the classical situation, holds $P_A(z) = \tilde{P}_A(z) = \det(A - z)$. 


7. Recurrence Relations in the Grothendieck Ring

Recurrence relations for the traces of exterior powers of an operator $A$ in a $p|q$-dimensional superspace hold good for any operator, their form being independent of the operator. Such universal relations for traces suggest the existence of underlying relations for the spaces themselves such as in the case of $q = 0$ the equality $\Lambda^k V = 0$ when $k > p$. We shall deduce these relations now.

First of all, let us explain in which sense we may speak about recurrence relations for vector spaces. They hold in a suitable Grothendieck ring. One can consider the Grothendieck ring of the category of all finite-dimensional vector superspaces (i.e., $\mathbb{Z}_2$-graded vector spaces). This ring is isomorphic to $\mathbb{Z}[[\Pi]]/(\Pi^2 - 1)$, which is the ring where dimensions of superspaces take values. An equality in this ring means just the equality of dimensions. Alternatively, one can fix a superspace $V$ and consider the Grothendieck ring of the category of all finite-dimensional superspaces with an action of the supergroup $\text{GL}(V)$, i.e., the Grothendieck ring of the finite-dimensional representations of $\text{GL}(V)$. Equality of two “natural” vector spaces like spaces of tensors over $V$ in this ring should mean the existence of an isomorphism commuting with the action of $\text{GL}(V)$.

As a starting point we use the following relation, which holds for any superspace $V$:

$$\Lambda_z(V) \cdot S_{-z}(V) = 1,$$  \hspace{1cm} (7.1a)

which one might prefer to rewrite as

$$\Lambda_z(V) \cdot \Lambda_{-z}\Pi(\Pi V) = 1$$  \hspace{1cm} (7.1b)

(for a proof it is sufficient to consider one-dimensional spaces). Here $\Lambda_z(V) = \sum z^k \Lambda^k V = 1 + zV + z^2 \Lambda^2 V + \ldots$, etc. These are power series in either of the Grothendieck rings described above. We denote the class of a vector space the same as the space itself. Notice that the unity 1 is the class of the main field. Equalities (7.1) hold in both senses. For example, expanding in $z$ one gets $V - V = 0$, $S^2 V + \Lambda^2 V - V \otimes V = 0$, etc.

Now, for a superspace $V$ we have $V = V_0 \oplus V_1$ where $V_0$ is purely even and $V_1$ is purely odd. We can rewrite this as $V = U \oplus \Pi W$ where both $U$, $W$ are purely even vector spaces. It follows that $\Lambda_z(V) = \Lambda_z(U) \Lambda_z(\Pi W)$, therefore by (7.1b)

$$\Lambda_z(V) = \frac{\Lambda_z(U)}{\Lambda_{-z}\Pi(W)} = \frac{1 + zU + z^2 \Lambda^2 U + \ldots + z^p \Lambda^p U}{1 - z\Pi W + z^2 \Lambda^2 W - \ldots + (-z)^q \Pi^q \Lambda^q W},$$  \hspace{1cm} (7.2)
Note that though $U$ and $W$ with their exterior powers do not belong to the ring of representations of $GL(V)$, they can be thought of as ideal elements that can be adjoined to it, or, which is the same, as elements of the representation ring of the block-diagonal subgroup $GL(U) \times GL(W) \subset GL(V)$. We see that the power series $\Lambda_z(V)$ represents a rational function with the numerator of degree $p$ and denominator of degree $q$. Denote it by $R_V(z)$; it replaces the characteristic function $R_A(z) = \text{Ber}(1 + zA)$ of our previous analysis. $R_A(z)$ can be viewed as the character of $R_V(z)$, for the ring of representations of $GL(V)$.

We can apply to $R_V(z)$ the same reasoning as to $R_A(z)$ above and conclude that the exterior powers $\Lambda^kV$ for a $p|q$-dimensional vector space $V$ satisfy a recurrence relation of period $q$

$$b_0\Lambda^{k+q}V + \ldots + b_q\Lambda^kV = 0 \quad (7.3)$$

for all $k \geq p-q+1$. Here $b_i = (-\Pi)^i\Lambda^iW$. Evidently in the classical case of $q = 0$ this reduces to $\Lambda^kV = 0$ for $k \geq p + 1$. The relations for $c_k(A) = \text{Tr} \Lambda^kA$ then follow from (7.3).

As in Section 3, it is possible to eliminate the coefficients $b_i = (-\Pi)^i\Lambda^iW$ from the recurrence relations (7.3) and express them in a closed form using Hankel determinants. We arrive at the following theorem.

**Theorem 4.** For an arbitrary $p|q$-dimensional vector space $V$ the following Hankel determinants vanish:

$$\begin{vmatrix}
\Lambda^kV & \ldots & \Lambda^{k+q}V \\
\ldots & \ldots & \ldots \\
\Lambda^{k+q}V & \ldots & \Lambda^{k+2q}V
\end{vmatrix} = 0 \quad (7.4)
$$

for all $k \geq p - q + 1$. □

Notice that the expression of the recurrence relation for $\Lambda^kV$ in the form of Hankel’s determinant has an advantage of not using the elements that are not in the ring of representations of $GL(V)$.

**Example 7.1.** Let $\dim V = p|1$. Then (7.4) gives the relation

$$\begin{vmatrix}
\Lambda^kV & \Lambda^{k+1}V \\
\Lambda^{k+1}V & \Lambda^{k+2}V
\end{vmatrix} = 0, \quad (7.5)
$$

i.e., $\Lambda^kV \Lambda^{k+2}V = (\Lambda^{k+1}V)^2$ (product means tensor product) for $k \geq p$. This can be seen directly as follows. $V = U \oplus \Pi W$ where $\dim U = p$, $\dim W = 1$. Hence $\Lambda^kV = \bigoplus_{i+j=k} \Lambda^iU \otimes \Pi^j W$. Note that $S^i W = W^j$. Thus for $k \geq p$ we have $\Lambda^kV = \bigoplus_{i=0}^p \Lambda^iU \otimes (\Pi W)^{k-i}$, therefore $\Lambda^{k+1}V = \Lambda^kV \otimes \Pi W$ (a geometric progression). Obviously, by tensor multiplying $\Lambda^kV$ and $\Lambda^{k+2}V$ we get the isomorphisms $\Lambda^kV \otimes \Lambda^{k+2}V = \ldots$.
\[ \Lambda^k V \otimes \Lambda^{k+1} V \otimes \Pi W = \Lambda^{k+1} V \otimes \Lambda^{k+1} V, \] which is exactly the relation (7.5).

Let us obtain the expansion at infinity for the rational function \( R_V(z) \). For this, we shall rearrange the numerator and denominator in (7.2). Since \( \Lambda^i(U) = \det U \otimes \Lambda^{p-i}(U^*) \) and \( \Lambda^j(W) = \det W \otimes \Lambda^{q-j}(W^*) \), we have

\[
R_V(z) = \frac{\det U}{\det W} \frac{\Lambda^p(U^*) + z\Lambda^{p-1}(U^*) + \ldots + z^p}{\Lambda^q(W^*) - z\Pi\Lambda^{q-1}(W^*) + \ldots + (-z)^q \Pi^q} =
\]

\[
\operatorname{Ber} V (-\Pi)^q z^{p-q} \frac{\Lambda_1(U^*)}{\Lambda_1(W^*)} = \operatorname{Ber} V (-\Pi)^q z^{p-q} \Lambda_1(V^*) =
\]

\[
(-\Pi)^q z^{p-q} \operatorname{Ber} V \sum_{k \leq 0} z^k \Lambda^k(V^*) = (-\Pi)^q \sum_{k \leq 0} z^{k+p-q} \Sigma^{p-k}(V) =
\]

\[
(-\Pi)^q \sum_{k \leq p-q} z^k \Sigma^{k+q}(V).
\]

Hence the rational function \( R_V(z) \) taking values in a Grothendieck ring has the following expansions:

\[ R_V(z) = \sum_{k \geq 0} z^k \Lambda^k(V) \quad \text{(at zero)} \quad (7.6) \]

\[ = \sum_{k \leq p-q} z^k (-\Pi)^q \Sigma^{k+q}(V) \quad \text{(at infinity)} \quad (7.7) \]

In the same way as in Section 4 we arrive at the following theorem.

**Theorem 5.** The sequence in the Grothendieck ring

\[ \Gamma_k = \Lambda^k V - (-\Pi)^q \Sigma^{k+q} V \quad (7.8) \]

for all \( k \in \mathbb{Z} \) is a recurrent sequence of period \( q \).

It very well fits with the equality \( \Lambda^k V = \Sigma^k V \) of the classical case of \( q = 0 \), i.e., \( \Lambda^k V = \det V \otimes \Lambda^{p-k}V^* \), which is a canonical isomorphism compatible with the action of \( \text{GL}(V) \). Theorem 5 implies the vanishing of the Hankel determinants of order \( q + 1 \) made of the elements \( \Gamma_k \).

**Example 7.2.** Consider \( V \) where \( \dim V = 1 | 1 \). Then \( \Lambda^k(V) = 0 \) for \( k < 0, \dim \Lambda^0(V) = 1, \dim \Lambda^k(V) = 1 + \Pi \) for \( k \geq 1 \). In the same way \( \dim \Sigma^{k+1}(V) = 1 + \Pi \) for \( k \leq -1, \dim \Sigma^1(V) = 1, \dim \Sigma^{k+1}(V) = 0 \) for \( k > 0 \). It follows that \( \dim \Lambda^k V - (-\Pi) \dim \Sigma^{k+1}V = 1 + \Pi \) for all \( k \in \mathbb{Z} \), which is a geometric progression with ratio \( \Pi \) infinite in both
directions. This verifies the statement of Theorem 5 for \( V \) at the level of dimensions.

**Example 7.3.** (Continuation of Examples 7.1 and 7.2.) For a super-space \( V \) such that \( \dim V = 1 \mid 1 \) we shall show explicitly an isomorphism \( \varphi: \Lambda^k V \otimes \Lambda^{k+2} V \to \Lambda^{k+1} V \otimes \Lambda^{k+1} V \) commuting with the action of \( \text{GL}(V) \). Let \( e \in V_0, \varepsilon \in V_1 \) be a basis of \( V \). Then \( E_k = \varepsilon \wedge \ldots \wedge \varepsilon \) and \( F_k = e \wedge \varepsilon \wedge \ldots \wedge \varepsilon \) can be taken as a basis in \( \Lambda^k V \) for \( k \geq 1 \). The desired isomorphism \( \varphi \) can be written as follows:

\[
\varphi(E_k \otimes E_{k+2}) = \alpha E_{k+1} \otimes E_{k+1},
\]

\[
\varphi(E_k \otimes F_{k+2}) = \frac{1}{2} \left(-\alpha + \frac{k}{k+1} \beta \right) E_{k+1} \otimes F_{k+1}
\]

\[
+ (-1)^k \frac{1}{2} \left(\alpha + \frac{k}{k+1} \beta \right) F_{k+1} \otimes E_{k+1},
\]

\[
\varphi(F_k \otimes E_{k+2}) = (-1)^k \frac{1}{2} \left(\alpha + \frac{k+2}{k+1} \beta \right) E_{k+1} \otimes F_{k+1}
\]

\[
+ \frac{1}{2} \left(\alpha - \frac{k+2}{k+1} \beta \right) F_{k+1} \otimes E_{k+1},
\]

\[
\varphi(F_k \otimes F_{k+2}) = \beta F_{k+1} \otimes F_{k+1},
\]

where \( \alpha, \beta \) are arbitrary nonzero parameters. In particular, notice that \( \varphi \) is not unique.

**8. Cramer’s Rule in Supermathematics**

In this section we formulate Cramer’s rule in supermathematics basing on the geometrical meaning of the Berezinian. Earlier such a generalization was obtained by Bergveldt and Rabin in [4], who used the ‘hard tools’ of the Gelfand–Retakh quasi-determinants theory (see [10, 9]). Our approach does not use anything but the main properties of the Berezinian.

Let us first formulate the usual Cramer’s rule geometrically. Let \( A \) be a linear operator in an \( n \)-dimensional vector space \( V \). Consider a linear equation

\[
A(x) = y.
\]

Here \( x, y \) are vectors in \( V \). For any volume form \( \rho \) on \( V \) and arbitrary vectors \( v_1, \ldots, v_{n-1} \) we obviously have

\[
\rho(A(x), A(v_1), \ldots, A(v_{n-1})) = \det A \cdot \rho(x, v_1, \ldots, v_{n-1}).
\]

Considering this equation for different vectors \( v_1, \ldots, v_{n-1} \) we can express \( x \) via \( y = A(x) \). Namely, let \( e_1, \ldots, e_n \) be an arbitrary basis
in $V$. Take as $\rho$ the coordinate volume form, i.e., $\rho(e_1, \ldots, e_n) = 1$ and for any other vectors the value of $\rho$ equals the determinant of the matrix consisting of the corresponding coordinate row vectors. Then for the $k$-th coordinate of $x$ we have

$$x^k = \rho(e_1, \ldots, x, \ldots, e_n)$$

($x$ stands at the $k$-th place), hence

$$x^k = \frac{1}{\det A} \rho(A(e_1), \ldots, y, \ldots, A(e_n)) = \frac{1}{\det A} \begin{vmatrix} a_1^1 & \ldots & a_1^n \\ \vdots & \ddots & \vdots \\ y^1 & \ldots & y^n \\ a_n^1 & \ldots & a_n^n \end{vmatrix},$$

where at the r.h.s. the coordinates of $y$ replace the $k$-th row of the matrix of the operator $A$. This is exactly Cramer’s rule. Here we use row vectors rather than columns because it is more convenient in the supercase.

These considerations can be generalized to the supercase as follows.

Let $V$ be a $p|q$-dimensional linear superspace. Consider a volume form $\rho$. Recall that in the supercase a volume form is defined as a function on bases such that a change of basis is equivalent to the multiplying by the Berezinian of the transition matrix. For example, a coordinate volume form associated with a basis $e_1, \ldots, e_{p+q}$ where $e_1, \ldots, e_p$ are even vectors and $e_{p+1}, \ldots, e_{p+q}$ are odd vectors, on vectors $v_1, \ldots, v_{p+q}$ of another basis of the same format equals the Berezinian

$$\text{Ber} \left(\begin{array}{ccc} v_1^1 & \ldots & v_1^{p+q} \\ \vdots & \ddots & \vdots \\ v_{p+q}^1 & \ldots & v_{p+q}^{p+q} \end{array}\right).$$

Here $v_i = v_i^j e_j$. It follows that a volume form is linear in the first $p$ arguments and hence can be extended by linearity to arbitrary vectors (the last $q$ arguments must remain linearly independent odd vectors!). In particular, it is possible to insert an odd vector into one of the first $p$ “even” positions.

As above, for any volume form $\rho$ on $V$ and vectors $v_1, \ldots, v_{p+q-1}$ of the appropriate parity we have

$$\rho(A(v_1), \ldots, A(x), \ldots, A(v_{p+q-1})) = \text{Ber} A \cdot \rho(v_1, \ldots, x, \ldots, v_{p+q-1}),$$

where the vector $x$ stands at the one of the first $p$ “even” places. $A$ is assumed to be an even invertible operator. This leads to a solution of a linear equation

$$A(x) = y.$$  \hspace{1cm} (8.1)
in the superspace $V$ as follows. Take as $\rho$ the coordinate volume form associated with a basis $e_1, \ldots, e_{p+q}$. Then $\rho(e_1, \ldots, x, \ldots, e_{p+q}) = x^k$, if $k = 1, \ldots, p$. Hence the formula for the first $p$ coordinates of $x$ corresponding to the even basis vectors is exactly the same as in the classical case. For $k = 1, \ldots, p$

$$x^k = \frac{1}{\text{Ber} A} \rho(A(e_1), \ldots, y, \ldots, A(e_{p+q})) = \frac{1}{\text{Ber} A} \Delta_k(A, y), \quad (8.2)$$

where

$$\Delta_k(A, y) = \text{Ber} \left( \begin{array}{ccc} a_{11} & \cdots & a_{1p+q} \\ \vdots & \ddots & \vdots \\ y^1 & \cdots & y^{p+q} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{array} \right), \quad (8.3)$$

($y$ inserted at the $k$-th “even” position). To obtain the last $q$ coordinates of $x$ corresponding to the odd basis vectors $e_{p+1}, \ldots, e_{p+q}$, consider the space $\Pi V$ with reversed parity. Let $\rho^\Pi$ be the coordinate volume form on $\Pi V$ corresponding to the basis $e_{p+1}^\Pi, \ldots, e_{p+q}^\Pi, e_1^\Pi, \ldots, e_p^\Pi$.

Now we have

$$\rho^\Pi(e_{p+1}^\Pi, \ldots, x^\Pi, \ldots, e_{p+q}^\Pi, e_1^\Pi, \ldots, e_p^\Pi) = x^k$$

for $k = p + 1, \ldots, p + q$. Introducing the notation

$$\rho^*(v_1, \ldots, v_{p+q}) := \rho^\Pi(v_{p+1}^\Pi, \ldots, v_{p+q}^\Pi, v_1^\Pi, \ldots, v_p^\Pi)$$

and

$$\text{Ber}^* M := \text{Ber} M^\Pi \quad (8.4)$$

for a matrix $M$, we can rewrite this as $x^k = \rho^*(e_1, \ldots, x, \ldots, e_{p+q})$, $k = p + 1, \ldots, p + q$. Hence for $k = p + 1, \ldots, p + q$

$$x^k = \frac{1}{\text{Ber}^* A} \rho^*(A(e_1), \ldots, y, \ldots, A(e_{p+q})) = \frac{1}{\text{Ber}^* A} \Delta^*_k(A, y), \quad (8.5)$$

where

$$\Delta^*_k(A, y) = \text{Ber}^* \left( \begin{array}{ccc} a_{11} & \cdots & a_{1p+q} \\ \vdots & \ddots & \vdots \\ y^1 & \cdots & y^{p+q} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{array} \right), \quad (8.6)$$

($y$ inserted at the $k$-th “odd” position). Formulae (8.2)–(8.6) give a complete solution of the equation (8.1). Recall that the matrix of a linear operator is defined by the formula $A(e_i) = a_{ij} e_j$. Hence $A(x) = A(x^i e_i) = x^i a_{ij} e_j$ if $A$ is even.
Remark. For even invertible matrices the operation $\text{Ber}^*$ is the same as $\text{Ber}^{-1}$. However, for matrices that are not invertible, $\text{Ber}^*$ can make sense, taking a nonzero nilpotent value, while Ber and Ber$^{-1}$ are not defined.

The “super” Cramer’s formulae (8.2)–(8.6) motivate the following definition. Let $D_{ij}(A)$ denote the matrix obtained from an even matrix $A$ by replacing all elements in the $i$-th row by zeros except for the $j$-th element replaced by 1. Notice that $D_{ij}(A)$ may be odd depending on positions of the indices $i,j$.

**Definition.** The $(i, j)$-th cofactor or adjunct of an even $p|q \times p|q$ matrix $A$ is

$$(\text{adj} A)_{ij} := \begin{cases} \text{Ber} D_{ij}(A) & \text{when } i = 1, \ldots, p \\ \text{Ber}^* D_{ij}(A) & \text{when } i = p + 1, \ldots, p + q \end{cases}$$

(8.7)

In the previous notation, $(\text{adj} A)_{ij} = \Delta_i(A, e_j)$ for $i = 1, \ldots, p$ and $(\text{adj} A)_{ij} = \Delta_i^*(A, e_j)$ for $i = p + 1, \ldots, p + q$. Notice that this notion is not symmetrical w.r.t. rows and columns, so it might better be called the “right adjunct”. We have the following formulae for the entries of the inverse matrix:

$$(A^{-1})_{ij} = \begin{cases} (\text{adj} A)_{ji} / \text{Ber } A & \text{when } j = 1, \ldots, p \\ (\text{adj} A)_{ji} / \text{Ber}^* A & \text{when } j = p + 1, \ldots, p + q \end{cases}$$

(8.8)

**Example 8.1.** Consider a $1|1 \times 1|1$ even matrix

$$A = \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}.$$  

Then by formulae (8.7) we get

$$(\text{adj} A)_{11} = \text{Ber} \begin{pmatrix} 1 & 0 \\ \gamma & d \end{pmatrix} = \frac{1}{d}$$

$$(\text{adj} A)_{12} = \text{Ber} \begin{pmatrix} 0 & 1 \\ \gamma & d \end{pmatrix} = -\frac{\gamma}{d}$$

$$(\text{adj} A)_{21} = \text{Ber}^* \begin{pmatrix} a & \beta \\ 1 & 0 \end{pmatrix} = \text{Ber} \begin{pmatrix} 0 & 1 \\ \beta & a \end{pmatrix} = -\frac{\beta}{a^2}$$

$$(\text{adj} A)_{22} = \text{Ber}^* \begin{pmatrix} a & \beta \\ 0 & 1 \end{pmatrix} = \text{Ber} \begin{pmatrix} 1 & 0 \\ \beta & a \end{pmatrix} = \frac{1}{a}$$
Thus for the transpose adjunct matrix we have:

\[ B = \begin{pmatrix} \frac{1}{a} & -\frac{\beta}{a^2} \\ \frac{\gamma}{d} & \frac{1}{a} \end{pmatrix}, \]

and

\[ AB = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \begin{pmatrix} \frac{1}{d} & -\frac{\beta}{a^2} \\ -\frac{\gamma}{a^2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{a}{d} - \frac{\beta \gamma}{d} & 0 \\ 0 & \frac{d}{a} - \frac{\gamma \beta}{a^2} \end{pmatrix} = \begin{pmatrix} \text{Ber} A & 0 \\ 0 & \text{Ber}^* A \end{pmatrix}, \]

as expected.

**Remark.** A different approach to Cramer’s rule was suggested in [12]. They defined certain ‘relative determinants’ of \( A \) polynomially depending on \( A \) and considered the ‘\( \lambda \)-solutions’ \( x \) satisfying \( A(x) = \lambda \cdot y \) instead of \( A(x) = y \), \( \lambda \) being one of the relative determinants. This allowed them to avoid division and to use only polynomial expressions.

**Appendix A. Elementary Properties of Recurrent Sequences**

It is a classical result due to Kronecker that a power series represents a rational function if and only if the infinite Hankel matrix of the coefficients has finite rank. In this section we summarize the relations between recurrent sequences and rational functions used in the main text. We present the material in the form convenient for our purposes. Notice that classical expositions (see [8]) make use of the expansion of a rational function at infinity, while we need to consider simultaneously two expansions, at zero and at infinity.

Let

\[ R(z) = \frac{a_0 + a_1 z + \ldots + a_p z^p}{b_0 + b_1 z + \ldots + b_q z^q} \]  

be a rational function. We assume that the numerator has degree \( p \) and the denominator degree \( q \). The coefficients can be in an arbitrary commutative ring with unit. Consider formal power expansions of the fraction (A.1) at zero and at infinity. Let \( \bar{R}(z) = \sum_{k \geq 0} c_k z^k \) (near zero) and \( \bar{R}(z) = \sum_{k \leq p-q} c_k^* z^k \) (near infinity). Here and below it is convenient to assume that coefficients such as \( a_k, b_k, c_k \), etc., are defined for all values of \( k \in \mathbb{Z} \) but may be equal to zero for some \( k \). Hence we have the equalities

\[ a_n = \sum_{i=0}^q b_i c_{n-i} \]  

(A.2)
for all \( n \), where \( c_k = 0 \) for \( k < 0 \), and
\[
a_n = \sum_{i=0}^{q} b_i c_{n-i}^* \tag{A.3}
\]
for all \( n \), where \( c_k^* = 0 \) for \( k > p - q \). Taking into account that \( a_n = 0 \) for \( n > p \) or \( n < 0 \), we obtain, respectively, that
\[
\sum_{i=0}^{q} b_i c_{n-i} = 0
\]
for all \( n > p \), i.e.,
\[
\sum_{i=0}^{q} b_i c_{k+q-i} = 0 \tag{A.4}
\]
for all \( k > p - q \), and that
\[
\sum_{i=0}^{q} b_i c_{k-i}^* = 0 \tag{A.5}
\]
for all \( k < 0 \). Also, if we subtract (A.3) from (A.2), we obtain that
\[
\sum_{i=0}^{q} b_i \gamma_{n-i} = 0 \tag{A.6}
\]
for all \( k \in \mathbb{Z} \), where \( \gamma_k = c_k - c_k^* \).

It is convenient to introduce the following definition. We say that a sequence \( \{c_k\}_{k \in \mathbb{Z}} \) is right or positive if \( c_k = 0 \) for \( k < 0 \).

**Definition.** A right sequence \( \{c_k\} \) is a \( p|q \)-recurrent sequence or, shortly, a \( p|q \)-sequence if the elements \( c_k \) satisfy a recurrence relation of the form (A.4) for all \( k > p - q \).

It follows that the coefficients \( c_k \) of the power expansion at zero of the fraction (A.1) make a \( p|q \)-recurrent sequence. (The coefficients of the expansion of (A.1) at infinity also make a \( p|q \)-sequence after the re-indexing that makes them a right sequence, \( c_k' := c_{p-q-k}^* \).) The fraction (A.1) is classically referred to as the generating function or the symbol of the recurrent sequence \( \{c_k\} \).

For a sequence \( \{c_k\}_{k \geq 0} \) to be a \( p|q \)-sequence means, if \( p \geq q \), that it satisfies a recurrence relation of period \( q \) except for the \( p - q + 1 \) initial terms \( c_0, \ldots, c_{p-q} \), and if \( p < q \), that it satisfies a recurrence relation of period \( q \) for all terms \( c_k, k \geq 0 \), and can be extended to the left by \( q - p - 1 \) zero terms so that the relation still holds. If we denote the set of all \( p|q \)-sequences by \( S_{p|q} \), then
\[
S_{p|q} \subset S_{p|q+1} \text{ and } S_{p|q} \subset S_{p+1|q}.
\]
Hence we have the following picture for the coefficients of the expansions of the rational function \( A.1 \). The coefficients of the expansions at zero and at infinity satisfy the same recurrence relations of period \( q \).

If \( p < q \), the coefficients \( c_k \) and \( c_k^* \) can be nonzero only in the disjoint ranges \( k \geq 0 \) and \( k \leq p - q \), respectively. The recurrence relation holds for all terms. If \( p \geq q \) (that is, when the fraction is improper), the coefficients \( c_k \) and \( c_k^* \) can be simultaneously nonzero in the finite range \( 0 \leq k \leq p - q \). Separate recurrence relations break down in this range. However, in all cases the sequence \( \gamma_k = c_k - c_k^* \), infinite in both directions and which coincides with either \( c_k \) or \( -c_k^* \) ‘almost everywhere’, satisfies the recurrence relation for all \( k \in \mathbb{Z} \).

If a sequence \( \{c_k\} \) is given, one can consider the associated infinite Hankel matrix with the entries \( a_{ij} = c_{i+j} \). Let \( \{c_k\} \) satisfy a recurrence relation of the form \( A.1 \) for all \( k \geq N \). Assume that \( b_0 \) is invertible. Then the infinite vector \( c_{N+q} = \{c_{q+k}\}_{k \geq N} \) is a linear combination of the vectors \( c_N = \{c_k\}_{k \geq N}, \ldots, c_{N+q-1} = \{c_{q+k-1}\}_{k \geq N} \). Hence their exterior product vanishes. In particular it implies the vanishing of the Hankel minors of order \( q + 1 \):

\[
\begin{vmatrix}
    c_k & \cdots & c_{k+q} \\
    \cdots & \cdots & \cdots \\
    c_{k+q} & \cdots & c_{k+2q}
\end{vmatrix} = 0
\]

where \( k \geq N \). On the other hand, solving a recurrence relation of period \( q \) involves division by a Hankel determinant of order \( q \). There is a vast literature devoted to theoretical and practical aspects of recurrent sequences and Hankel matrices.

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