Gravitational Lensing in Weyl Gravity

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Abstract. We calculate the deflection angle of light from a distant source by a galaxy cluster in Weyl’s conformal gravity. The general method of calculation is first applied to calculate the deflection angle in Schwarzschild-de Sitter (Kottler) spacetime in two different coordinate systems. By doing the calculation in two different coordinate systems and obtaining the same result we show that our result is independent both of the coordinate system and of the method used. The deflection angle calculated in Kottler spacetime includes contribution of the cosmological constant, which agrees with one and disagrees with many in the literature. We then calculate the deflection angle in Mannheim-Kazanas spacetime and find a coordinate independent result, which includes contributions from both the cosmological constant and the Mannheim-Kazanas parameter. There are conflicting results on the deflection angle for light in Weyl gravity in the literature. We point out a possible reason for the discrepancy between our work and the others.

Keywords: modified gravity, gravitational lensing
1 Introduction

Bending of light by gravitational field was the most important observation that agreed the prediction of general relativity and made Einstein the most famous physicist overnight. One important consequence of gravitational bending of light is the gravitational lensing phenomenon, in which light from a distant object is bent by an intermediate massive object, that is a gravitational lens, to create multiple images of the source. Gravitational lensing is a successful astronomical tool to obtain great deal of information about the distance of the source, its brightness and perhaps most importantly the mass distribution of the lensing object. If that object is a galaxy cluster, then gravitational lensing, together with X-ray observations, is an indispensable tool to measure the amount of mass in galactic constituents and the intergalactic gas. Those observations gave support to the dark matter paradigm that galaxy halos contain dark matter with much higher mass compared to the luminous baryonic mass making up the stars, as well as galactic and intergalactic gas. Existence of dark matter in galactic halos has also been the standard explanation of the phenomena of flat galactic rotation curves.

Recent years have seen much activity testing the viability of alternative theories of gravity on astrophysical and cosmological phenomena without involving dark sectors. Dark matter paradigm, although very successful to explain diverse astrophysical and cosmological phenomena, has compatibility issues with the particle physics. Dark matter particles should belong to some theory beyond Standard Model, alas there exists no model which is free of theoretical problems and most importantly there are no observations reported by direct or indirect particle physics experiments (for the latest observational status see [1–4]). Phenomenological success of Milgrom’s MOND approach [5, 6]
together with the current (non)-observational situation of dark matter particles make it imperative to search for an explanation of various astrophysical and cosmological phenomena in an alternative theory of gravity.

In a previous work [7] we determined the geometry in the outer region of galaxies in which stars move with almost the same rotational velocity irrespective of their distance from the galactic center. Constancy of the rotational velocity might be seen due to the existence of scale symmetry. Therefore we found this geometry as a solution of Weyl gravity theory, which is the unique local scale symmetric metric theory of gravity. In [7] we also claimed that our solution for the outer region of galaxies should also hold for low density regions up to the scale of galaxy clusters. To check the validity of that claim, in this paper we analyze the gravitational lensing in Weyl gravity by calculating the deflection angle of light from a distant source by a galaxy cluster.

This paper is organized as follows: In the next section we summarize the previous works on the effect of cosmological constant on gravitational lensing and the works on gravitational lensing in Weyl gravity. Then in section (3) we are going to describe the general formalism in two different coordinate systems with two different methods of calculation. We will apply the general formalism to Schwarzschild–de Sitter (Kottler) spacetime in section (4) and show that the result for the deflection angle is independent of the coordinate system and the method used. Afterwards, in section (5) we will obtain the main result of this paper for deflection angle of light in Weyl gravity.

2 Summary of previous works

There are unfinished discussions in the literature on the contribution of the cosmological constant \( \Lambda \) and the Mannheim–Kazanas (MK) parameter \( \gamma \) of MK solution [8, 9] of Weyl gravity to the strong lensing formula. That \( \Lambda \) contributes to bending of light in the Kottler background is first proposed in [10]. Then through a series of papers [11–16] the contribution of \( \Lambda \) to strong lensing is made more precise both conceptually and computationally. It is observed that the null geodesics being independent of \( \Lambda \) does not imply that lensing phenomena are independent of \( \Lambda \). Related to this discussion in an dark matter involving study [17], it is concluded that cold dark matter mass profiles contain information about \( \Lambda \). There have also been some works questioning the validity of these results. In [18] it is argued that any influence coming from \( \Lambda \) should be in higher order terms and lower order influence observed in other works is due to observer’s motion. Arakida also challenged contribution of \( \Lambda \) in [19] by showing that \( \Lambda \) can be absorbed into the definition of impact parameter in the strong lensing formula.

There is also disagreement on the contribution of the MK parameter \( \gamma \) to the strong lensing formula. If one uses the MK solution of Weyl gravity to describe the galactic rotation curves then the parameter \( \gamma \) that multiplies the linear term in the gravitational potential turns out to be very small, but positive [20, 21]. Early applications of MK metric to strong lensing by galaxy clusters reported literally negative results that value of \( \gamma \) should be negative for it to have positive contribution to the bending angle [22, 23]. In a later work [24] Sultana and Kazanas again found a formula which requires \( \gamma \) to be negative, but their result suggested that contribution of \( \gamma \) is rather insignificant compared to the general relativistic contribution. Pireaux claimed [25] that the MK choice of conformal factor
gives an incorrect value for \( \gamma \). Then in the late works \([26–30]\), people used different ideas in definition of the bending angle and found that \( \gamma \) should be positive, getting rid of an apparent paradox.

In this work we are going to contribute to both of these discussions. First it will be observed that our Weyl gravity solution \([7]\) is conformally equivalent to MK solution, as any solution to Weyl gravity field equations should be \([9]\). So we have a different conformal factor compared to MK solution, which makes difference only in the case of massive particle trajectories. Light trajectories do not distinguish conformally equivalent metrics, thus our result for strong lensing is relevant for the discussion on the sign and the value of the MK parameter \( \gamma \). Our result for the deflection angle calculated in the Kottler spacetime also includes a contribution from the cosmological constant \( \Lambda \). This contribution, however, comes out rather differently than the works mentioned above. We believe that how \( \Lambda \) and also \( \gamma \) contribute depends strongly on how and at what point in the calculation the perturbative expansions in various quantities are performed. These quantities are mass \( m \) of the gravitational lens, the cosmological constant \( \Lambda \), and the MK parameter \( \gamma \). In which order the perturbative expansions are made is very important. We find out that expansions first in \( m \), then in \( \gamma \), and finally in \( \Lambda \) is the mathematically correct one, because otherwise one gets higher order terms larger than the lower order terms in perturbation expansions. This type of behavior is physically incorrect. Our result without the MK parameter for the deflection angle (in Kottler spacetime) agrees with the analysis done in \([31]\), which has different result compared to \([11–16]\). With the MK parameter, our result also differs from the ones in the literature, partially agreeing only with the result presented in \([30]\).

3 General formalism

In this section we describe the general formalism on a spherically symmetric static spacetime first in the Schwarzschild-like polar–areal coordinates \([32]\) and then in a conformally equivalent coordinates that we call “Weyl gravity vacuum coordinates.” For the null geodesics the conformal transformation of the metric would not have any effect on the geodesic and thus the deflection angle. By doing the calculation in two different coordinate systems with two different methods and obtaining the same result we show that our result is independent both of the coordinate system \([33]\) and of the method used.

3.1 Polar–areal coordinates

A general Schwarzschild-like spherically symmetric metric in polar–areal coordinates is given by

\[
d s^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\theta^2 + r^2 \sin \theta d\phi^2,
\]

(3.1)

where \( f(r) \) is a general function of the radial coordinate \( r \). In a spherical symmetric static background, the geodesic equation for a null particle can be found using Killing symmetries. Since the metric functions depend only on coordinates \( r \) and \( \theta \), there are two Killing vectors in this background: \( K = \partial_t \) and \( L = \partial_\phi \). These vectors describe the symmetry directions, and thus there are constants of motion associated with them. Those two constants of motion are the total energy, \( E = f(r) \dot{t} \) and the angular momentum, \( L = r^2 \dot{\phi} \). From these definitions we can easily write

\[
r^2 \frac{d\phi}{dt} = f(r) b \quad \text{with} \quad b \equiv \frac{L}{E}.
\]

(3.2)
Now on the metric (3.1) we put the null geodesic condition \( ds^2 = 0 \) and for a null geodesic on the equatorial plane \((\theta = \pi/2)\) we obtain

\[
\frac{du}{d\phi} = \sqrt{\frac{1}{b^2} - u^2 f(u)},
\]

where \( u \equiv \frac{1}{r} \). This equation shows us that the null geodesics depend on \( b \), which is called the impact parameter for flat spacetimes. This is a equation for the null geodesic on equatorial plane that contains only the first derivative of the function \( u(\phi) \). If this equation can be solved for a specific \( f(r) \) and \( u(\phi) \) is determined, one can then evaluate the deflection angle by inverting the function \( u(\phi) \).

If there exists a cosmological horizon in these coordinate system then we are interested to find the coordinate angle difference for the motion of light from the cosmological horizon, \( u_h \), at most to the closest approach distance, \( u_0 \). Otherwise we would be calculating coordinate angle difference for the motion between causally unconnected regions, which would be physically incorrect.

The closest point is defined by \( \frac{du}{d\phi} \bigg|_{u=u_0} = 0 \), and the cosmological horizon is defined by \( f(r_h) = 0 \) with \( u_h = 1/r_h \). Thus the coordinate angle difference is given by

\[
\Delta \phi \equiv \phi(r_h) - \phi(r_0) = \int_{u_h}^{u_0} \frac{du}{\sqrt{\frac{1}{b^2} - u^2 f(u)}}.
\]

From this, one finds the deflection angle as it travels from the source to the observer as

\[
\Delta \alpha = 2 \Delta \phi - \pi,
\]

which should be coordinate independent.

For Schwarzschild spacetime \( f(u) = 1 - 2mu \) and therefore the deflection angle is

\[
\Delta \alpha = 2 \int_{u_h}^{u_0} \frac{du}{\sqrt{\frac{1}{b^2} - u^2 + 2mu^2}} - \pi.
\]

whose exact solution in terms of incomplete elliptic integral of the first kind is given in [31]. In that paper the weak field (equ. (33) of [31]) and strong field limits (equ. (40) of [31]) of deflection angle are also obtained by performing appropriate expansions of the first incomplete elliptic integral.

Square–root in the null geodesic equation (3.3) is the reason of the complicated integral. We can get rid of the square–root by writing the null geodesic equation in the form of a second–order ODE as

\[
\frac{d^2u}{d\phi^2} = -\frac{1}{2} u^2 \frac{df}{du} - uf(u).
\]

Solution of any second order ODE requires specification of two boundary conditions. The boundary conditions that we choose define the point of closest approach of light to the lens: \( u(0) = u_0 \) and \( u'(0) = 0 \). Note that these conditions define the closest point to the lens as the point with coordinates \( \phi = 0 \) and \( r = r_0 = b \).

In the case that the spacetime is non-flat, one has to apply perturbation methods to the flat spacetime solution taking into account the cosmological horizon \( u_h \). Then the angle of deflection will be calculated for a special case of intersection of the null geodesic and the cosmological horizon.
3.2 Weyl gravity vacuum coordinates

Now we find expression for the same deflection angle in a different coordinate system. This will show evidently that the result is coordinate independent. Using the radial coordinate transformation

$$\rho = \frac{r}{\sqrt{f(r)}} \left( \frac{\rho}{\rho_c} \right)^w$$

we obtain a new coordinate system from (3.1) via a conformal transformation

$$ds^2 = \frac{\rho^2}{r^2} \left( -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\theta^2 + r^2 \sin \theta d\phi^2 \right).$$

Therefore we have a new conformal equivalent metric given by

$$ds^2 = -\left( \frac{\rho}{\rho_c} \right)^{2w} dt^2 + \frac{1}{B(\rho)}d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin \theta d\phi^2,$$

with

$$\frac{d\rho^2}{\rho^2 B(\rho)} = \frac{dr^2}{r^2 f(r)}.$$  (3.11)

This is a kind of metric that can be written for the outer region of galaxies [7] where flat rotation curve phenomenon is observed. In that case $\sqrt{w}$ is the rotating speed of a star moving on a circular orbit in the outer region of a galaxy. Thus it is a very small number compared to speed of light, on the order of $10^{-3}$. Therefore light sees the background described by this metric as if $w = 0$. Hence we take this parameter vanishing in the forthcoming calculations and we use the new metric given by

$$ds^2 = -dt^2 + \frac{1}{B(\rho)}d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin \theta d\phi^2.$$  (3.12)

This time from the Killing vector analysis we find

$$\rho^2 \frac{d\phi}{dt} = b \quad \text{with} \quad b \equiv \frac{L}{E},$$

and after applying the null geodesic condition $ds^2 = 0$ on the metric (3.12) we obtain a first–order ODE for a null geodesic on the equatorial plane as

$$\frac{dv}{d\phi} = \sqrt{\left(1 - v^2\right)B(v)},$$

where $v \equiv \frac{b}{\rho}$. Here, again, $b$ is the impact parameter for flat spacetimes.

For the metric of interests in this paper we will observe that there are no cosmological horizon in this new coordinate system. Therefore the angle of deflection will be given by

$$\Delta \alpha = 2 \int_0^1 \frac{dv}{\sqrt{(1 - v^2)B(v)}} - \pi,$$

where 1 in the upper bound of integral corresponds to the turning point $\rho = b$ and 0 to the point at infinity.

We can get rid of the square–root in the null geodesic equation by writing it in the form of a second–order ODE as

$$\frac{d^2v}{d\theta^2} = \frac{1}{2}(1 - v^2)\frac{dB}{dv} - vB(v).$$

The boundary conditions that we choose for the function $v(\theta)$ define the point of closest approach of light to the lens: $v(0) = v_0 = 1$ and $v'(0) = 0$. Note that these conditions define the closest point to the lens as the point with coordinates $\theta = 0$ and $\rho = \rho_0 = b$. 


4 Schwarzschild-de Sitter (Kottler) spacetime

4.1 Polar–areal coordinates

The metric in polar–areal coordinates is as given in (3.1) with

\[ f(r) \equiv 1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2. \]  

(4.1)

Then the null geodesic equation as a second–order ODE becomes

\[ \left( \frac{du}{d\phi} \right)^2 = \frac{1}{b^2} + \frac{\Lambda}{3} - u^2 + 2mu^3, \]

(4.2)

where \( u \equiv \frac{1}{r} \). Before solving this equation we analyze turning points and horizons [34].

Turning points are the points at which \( \frac{du}{d\phi} = 0 \). From this relation one finds that

\[ az^3 - z + 1 = 0, \]  

(4.3)

where \( z = r/2m \) and \( a = 4m^2 \left( \frac{1}{b^2} + \frac{\Lambda}{3} \right) \). There are just three cases to consider. In the following we are going to write just the turning points and ignore negative and complex roots:

1. \( 0 < a < 4/27 \) : there are two turning points given by

\[ z_0 = \frac{2}{\sqrt{3a}} \cos \left( \frac{\pi - \Psi}{3} \right), \]

(4.4)

and

\[ z_- = \frac{2}{\sqrt{3a}} \cos \left( \frac{\pi + \Psi}{3} \right), \]  

(4.5)

where \( \cos^2 \Psi = 27a/4 \).

2. \( a = 4/27 \) : there is only one turning point given as

\[ z_0 = z_- = z_\gamma = \frac{3}{2}, \]  

(4.6)

Two turning points in the previous case approach together to coalesce at the photon sphere, located at \( z_\gamma \).

3. \( a > 4/27 \) : there are no turning points.

Cosmological horizons are located at points where \( f(r) = 0 \). This condition in the present case is equivalent to

\[ yz^3 - z + 1 = 0, \]  

(4.7)

where \( y = 4m^2\Lambda/3 \). Note that parameter \( a \) and \( y \) appear in equations (4.3) and (4.7), respectively, in the same way. Thus for the same range of values there exists cosmological horizon(s):

1. For \( 0 < y < 4/27 \) there are two horizons given by

\[ z_c = \frac{2}{\sqrt{3y}} \cos \left( \frac{\pi - \beta}{3} \right), \]

(4.8)

and

\[ z_h = \frac{2}{\sqrt{3y}} \cos \left( \frac{\pi + \beta}{3} \right), \]  

(4.9)

where \( \cos^2 \beta = 27y/4 \), and we note that \( z_c > z_h \).
2. For \( y = 4/27 \) there is only one horizon, but since it coincides with the photon sphere it is useless for lensing calculations.

Thus we calculate the deflection angle for the first case by using the variables and parameters
\( v = r_0/r, \, \Lambda_0 = \Lambda r_0^2 \) and \( m_0 = m/r_0^2 \). In integral form the deflection angle is
\[
\Delta \alpha = 2 \int_{v_c}^1 \frac{dv}{\sqrt{1 - v^2 + 2m_0v^2 + \Lambda_0/3}} - \pi, \tag{4.10}
\]
where \( v_c = \frac{\sqrt{\Lambda_0}}{2} \cos \left( \frac{x-\beta}{3} \right) \). Here integral can be evaluated after series expansion of the integrand to second order in \( m_0 \) and first order in \( \Lambda_0 \). The result (as previously found in [31]) is
\[
\Delta \alpha = -2\sqrt{\frac{\Lambda_0}{3} + m_0 \left( 4 - 2\sqrt{\frac{\Lambda_0}{3} - 2\frac{\Lambda_0}{3}} \right) + m_0^2 \left( 4\frac{15}{4} \pi - 4 \right) - m_0^2 \left( 3\sqrt{\frac{\Lambda_0}{3} + 2\frac{\Lambda_0}{3}} \right) + \cdots} \tag{4.11}
\]

5 Mannheim-Kazanas spacetime

5.1 Polar–areal coordinates

Weyl gravity solution in polar–areal coordinates were given long time ago by Mannheim and Kazanas (MK) in [9] with
\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{5.1}
\]
where
\[
f(r) = \sqrt{1 - 6m\gamma} - \frac{2m}{r} + \gamma r - kr^2. \tag{5.2}
\]
Here \( m \) is the mass, \( \gamma \) is the MK parameter and \( k \) is related to the cosmological constant by \( k = \Lambda \). The null geodesic equation for this spacetime is given by
\[
\left( \frac{dr}{d\phi} \right)^2 = r^4 \left[ \frac{1}{b^2} + \frac{\Lambda}{3} - \frac{\sqrt{1 - 6m\gamma}}{r^2} + \frac{2m}{r^3} - \frac{\gamma}{r} \right]. \tag{5.3}
\]
Before solving this equation we analyze turning points and horizons as in the Kottler spacetime case.

Turning points are the points at which \( \frac{dr}{d\phi} = 0 \). From this relation one finds that
\[
\left( \frac{\Lambda}{3} + \frac{1}{b^2} \right) r^3 - \gamma r^2 - \sqrt{1 - 6m\gamma r} + 2m = 0. \tag{5.4}
\]
There are again just three cases to consider. In the following we write only the physically meaningful ones:

1. \( 0 < m^2 \left( \Lambda + \frac{3}{b^2} \right) \frac{\gamma^2}{\eta^2} < 1/9 \) : there are three turning points given by
\[
\begin{align*}
    r_0 &= 2 \sqrt{\frac{\Lambda + \frac{3}{b^2}}{\gamma}} \cos \frac{\Psi}{3} + \frac{\gamma}{\Lambda + \frac{3}{b^2}}, \\
    r_+ &= 2 \sqrt{\frac{\Lambda + \frac{3}{b^2}}{\gamma}} \cos \frac{\Psi}{3} + \frac{\gamma}{\Lambda + \frac{3}{b^2}}, \\
    \text{and} \quad r_- &= -2 \sqrt{\frac{\Lambda + \frac{3}{b^2}}{\gamma}} \cos \frac{\Psi}{3} + \frac{\gamma}{\Lambda + \frac{3}{b^2}},
\end{align*}
\]
where \( \cos^2 \Psi = 9m^2 \left( \Lambda + \frac{3}{b^2} \right) \frac{\gamma^2}{\eta^2} \) with \( \epsilon = 1 - \frac{\gamma^2}{3m\left( \Lambda + \frac{3}{b^2} \right)} - \frac{\gamma^2}{4m\sqrt{1 - 6m\gamma}} \) and \( \eta = \sqrt{1 - 6m\gamma} + \frac{\gamma^2}{\Lambda + \frac{3}{b^2}}. \)
2. When \( \Psi = 0 \), limit point of these turning points is the radius of the photon sphere (\( r_\gamma \)) given by

\[
r_0 = r_+ = r_\gamma = \frac{1}{\gamma}(1 - \sqrt{1 - 6m\gamma}).
\]  

This result can be found easily using circular null geodesic conditions [35].

To find locations of cosmological horizons we set \( f(r) = 0 \), which is equivalent to

\[
\frac{\Lambda}{3}r^3 - \gamma r^2 - \sqrt{1 - 6m\gamma}r + 2m = 0.
\]  

The physically meaningful solutions are found for \( 0 < m^2\frac{\Lambda^2}{\xi} < 1/9 \). For this case there are two turning points given by

\[
r_c = 2\sqrt{\frac{\xi}{\Lambda}}\cos\left(\frac{\pi - \delta}{3}\right) + \frac{\gamma}{\Lambda},
\]  

\[
r_h = 2\sqrt{\frac{\xi}{\Lambda}}\cos\left(\frac{\pi + \delta}{3}\right) + \frac{\gamma}{\Lambda},
\]

where \( \cos^2 \delta = 9m^2\frac{\Lambda^2}{\xi} \) with \( \varepsilon = 1 - \frac{9\gamma^2}{32\Lambda^2} - \frac{3}{16\lambda} \sqrt{1 - 6m\gamma} \) and \( \xi = \sqrt{1 - 6m\gamma} + \gamma^2 \).

Geodesic equation can be written in terms of the turning point \( r_0 \) as

\[
\left(\frac{dv}{d\phi}\right)^2 = \sqrt{1 - 6m\gamma_0} (1 - v^2) + \gamma_0 (1 - v) + 2m_0(v^3 - 1)
\]

\[
= 2m_0(1 - v)(v_+ - v)(v - v_-),
\]

where \( v = \frac{m_0}{r} \) and \( v_\pm = \frac{1}{4m_0}\sqrt{1 - 6m\gamma_0} - 2m_0 \pm \sqrt{1 + 2m_0\gamma_0 + 4m_0\sqrt{1 - 6m\gamma_0} - 12m_0^2} \). The deflection angle is then given by

\[
\Delta \alpha = 2\int_{v_c}^{1} \frac{dv}{\sqrt{2m_0(1 - v)(v_+ - v)(v - v_-)}} - \pi
\]

\[
= \frac{4}{\sqrt{2m_0(v_+ - v_-)}} F(p, q) - \pi,
\]

where \( v_c = \frac{m_0}{r_c} \) and \( F(p, q) \) is the elliptic integral of the first kind with \( \sin p = \frac{(v_+ - v_-)(1 - v)}{(1 - v_-(v_+ - v_-))} \) and \( q = \frac{1 - v}{v_+ - v_-} \). Using the expansion of \( q \) in \( \sqrt{m} \) as

\[
q \approx \sqrt{2\gamma} + 4\sqrt{m} - \frac{3(\sqrt{2}\gamma + \sqrt{2})}{\sqrt{\gamma + 2}} m^{3/2} + \cdots,
\]

and the asymptotic expansion of \( F(p, q) \) in \( q \) given by

\[
F(p, q) \approx p + q^2\left(\frac{p}{4} - \frac{1}{8}\sin(2p)\right) + \frac{3}{256}q^4(12p - 8\sin(2p) + \sin(4p)) + \cdots,
\]

we find the deflection angle as

\[
\Delta \alpha = m_0 \left(4 - 2\sqrt{\frac{\Lambda_0}{3}} - 2\frac{\Lambda_0}{3}\right) - 2\sqrt{\frac{\Lambda_0}{3} + \gamma_0\sqrt{\frac{\Lambda_0}{3}}}
\]

\[
+ m_0\gamma_0 \left(2 + \frac{\Lambda_0}{3}\right) + m_0^2\left(\frac{15\pi}{4} - 4 - 3\sqrt{\frac{\Lambda_0}{3} - 2\frac{\Lambda_0}{3}}\right)
\]

\[
+ m_0^2\gamma_0 \left(\frac{15\pi}{4} - 4 - \frac{3}{2}\sqrt{\frac{\Lambda_0}{3}}\right) + \cdots.
\]
5.2 Weyl gravity vacuum coordinates

Any solution of Weyl gravity should be conformally equivalent to the MK solution [9]. To show that the metric

\[ ds^2 = - \left( \frac{\rho}{\rho_c} \right)^{2w} dt^2 + \frac{1}{B(\rho)} d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  
(5.19)

is conformally equivalent to MK metric (5.1), we write conformal equivalence condition as

\[ \frac{\rho^2}{r^2} (-f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2) = - \left( \frac{\rho}{\rho_c} \right)^{2w} dt^2 + \frac{1}{B(\rho)} d\rho^2 + \rho^2 d\Omega^2, \]  
(5.20)

from which we obtain two equations

\[ \frac{\rho^2}{r^2} f = \left( \frac{\rho}{\rho_c} \right)^{2w} \]  
(5.21)

\[ \frac{\rho^2}{r^2} \frac{1}{f(r)} \left( \frac{dr}{d\rho} \right)^2 = \frac{1}{B(\rho)} \]  
(5.22)

From these equations one finds

\[ B(\rho) = \frac{\rho^{2w-4}}{\rho_c^{2w}} \frac{1}{(du/d\rho)^2} \]  
(5.23)

and

\[ 2mu^3 - \sqrt{1-6m\gamma} \ u^2 + \gamma u + \frac{\Lambda}{3} - \frac{\rho^{2w-2}}{\rho_c^{2w}} = 0, \]  
(5.24)

where \( u \equiv \frac{1}{r} \). Solutions of the cubic algebraic equation (5.24) are given by

\[ u_1 = \frac{1}{6m} (h + h^{-1} + \sqrt{1-6m\gamma}), \]  
(5.25)

\[ u_2 = \frac{1}{6m} (e^{i4\pi/3}h + e^{i2\pi/3}h^{-1} + \sqrt{1-6m\gamma}), \]  
(5.26)

\[ u_3 = \frac{1}{6m} (e^{i2\pi/3}h + e^{i4\pi/3}h^{-1} + \sqrt{1-6m\gamma}), \]  
(5.27)

where

\[ h(\rho) = \left[ Q + \sqrt{Q^2 - 1} \right]^{1/3}, \quad Q(\rho) = \left[ - \frac{\Delta_1}{\rho^{2(1-w)}} + \Delta_2 \right], \]  
(5.28)

with

\[ \Delta_1 \equiv \frac{54m^2}{\rho_c^{2w}} \quad \text{and} \quad \Delta_2 \equiv (1 + 3m\gamma) \sqrt{1-6m\gamma} - 54m^2 \frac{\Lambda}{3}. \]  
(5.29)

Now we can find the metric component \( B(\rho) \) from equ. (5.23). After some algebra we find

\[ B_1(\rho) = \frac{3\rho^{2(1-w)}}{8(1-w)^2 \Delta_1} (1 + h^2 + h^{-2})^2, \]  
(5.30)

\[ B_2(\rho) = \frac{3\rho^{2(1-w)}}{8(1-w)^2 \Delta_1} (1 + e^{i2\pi/3}h^2 + e^{i4\pi/3}h^{-2})^2, \]  
(5.31)

\[ B_3(\rho) = \frac{3\rho^{2(1-w)}}{8(1-w)^2 \Delta_1} (1 + e^{i4\pi/3}h^2 + e^{i2\pi/3}h^{-2})^2. \]  
(5.32)

These are the solutions given in [7] (equation (45) of that paper). Note that \( \Delta_{1,2} \) (5.29) correspond to the integration constants in the solutions of the field equations, \( C_{1,2} \) in [7], respectively. We take the parameter \( w \) vanishing in the forthcoming calculations as explained in section (3.2).
We now need to evaluate the deflection angle integral given by

$$\Delta \alpha = 2 \int_0^1 \frac{dv}{\sqrt{B(v)(1-v^2)}} - \pi. \tag{5.33}$$

where $v = \frac{\omega_0}{\rho} = \frac{b}{\rho}$. To evaluate this complicated integral we make further redefinition that $\cos \zeta = Q(\rho)$ and after some algebra we obtain

$$B_i(v) = \frac{1}{144m_0^2v^2}[1 + 2\cos(\frac{2\zeta}{3} + (i - 1)\frac{2\pi}{3})]^2, \tag{5.34}$$

where $m_0 \equiv m_0$. To evaluate the integral (5.33) we use $B_2(v)$. We first expand the integrand of (5.33) perturbatively in terms of $m_0$ and then evaluating the integral we obtain

$$\Delta \alpha = 4m_0 - \pi + \frac{1}{(1+\nu^2)} \left( \frac{45m_0^2\gamma_0^4}{16} - \frac{m_0\gamma_0^3}{2} \right) + \frac{m_0^2\gamma_0^6}{16}\frac{1 + 3\nu^2}{\nu^2(1+\nu^2)^2} + 2\sin^{-1} \frac{1}{\sqrt{1+\nu^2}} + 15m_0^2 \left( \frac{\nu}{2} + \frac{1 + \nu^2}{2}\sin^{-1} \frac{1}{\sqrt{1+\nu^2}} \right) + \cdots, \tag{5.35}$$

where $\gamma_0 \equiv \gamma_0^0$ and $\nu^2 = \frac{\Lambda_0^0}{3} + \frac{\gamma_0^2}{4}$ with $\Lambda_0 \equiv \Lambda_0^0$.

Expanding this expression first in $\gamma$ and then in $\Lambda$ we obtain

$$\Delta \alpha = 4m_0 + \frac{15\pi}{4}m_0^2 - 2\sqrt{\frac{\Lambda_0}{3} + \frac{15\pi}{4}m_0^2\frac{\Lambda_0}{3}} + \cdots \tag{5.36}$$

Note that this result is coordinate independent, because $r_0 = b$ where $b$ is the impact parameter. After coordinate transformation,

$$\rho_0 = \frac{r_0}{\sqrt{f(r_0)}}, \tag{5.37}$$

deflection angle in Mannheim–Kazanas coordinates is found to be

$$\Delta \alpha = m_0 \left( 4 - 2\sqrt{\frac{\Lambda_0}{3} - 2\Lambda_0^0} \right) - 2\sqrt{\frac{\Lambda_0}{3} + \gamma_0 \sqrt{\frac{\Lambda_0}{3}}} + m_0\gamma_0 \left( 2 + \frac{\Lambda_0}{3} \right) + m_0^2 \left( \frac{15\pi}{4} - 4 - 3\sqrt{\frac{\Lambda_0}{3} - 2\Lambda_0^0} \right) + m_0^2\gamma_0 \left( \frac{15\pi}{4} - 4 - \frac{3}{2}\sqrt{\frac{\Lambda_0}{3}} \right) + \cdots, \tag{5.38}$$

where

$$m_0 \equiv \frac{m}{r_0}, \quad \gamma_0 \equiv \gamma r_0, \quad \text{and} \quad \Lambda_0 \equiv \Lambda r_0^2. \tag{5.39}$$

Equation (5.38) for the deflection angle, which is equivalent to equation (5.18) of the previous section is our main result. By doing the calculation in two different coordinate systems and obtaining the same result we thus have shown that this result is independent both of the coordinate system and of the method used. We now look at two special cases: 1) $\gamma = 0$ case to compare our result to a previous one [31] obtained for the Kottler spacetime, which is equivalent to MK spacetime for $\gamma = 0$, and 2) $\Lambda = 0$ case to see the contribution of Weyl gravity to the bending of light in the Schwarzschild geometry.
5.2.1 \( \gamma = 0 \) case

To find the Kottler metric in Weyl gravity vacuum coordinates we take \( \gamma = 0 \) in (5.24) and then the metric function \( B(\rho) \) becomes

\[
B(\nu) = \frac{1}{144m_0^2v^2} [1 + 2 \cos(\frac{2\zeta}{3} + \frac{2\pi}{3})]^2,
\]

(5.40)

where \( \cos \zeta = 1 - 54m_0^2(\Lambda_0/3 + v^2) \) and \( v = \rho_0/\rho \). In this coordinates we also defined \( m_0 = m/\rho_0 \) and \( \Lambda_0 = \rho_0^2\Lambda \). Since the distance of closest approach is \( \rho_0 = b \), the result will be coordinate independent.

The deflection angle (3.15), after series expansion of the integrant to second order in \( m_0 \) and first order in \( \Lambda_0 \), is found to be

\[
\Delta \alpha = -2\sqrt{\frac{\Lambda_0}{3}} + 4m_0 + \frac{15\pi}{4} m_0^2 + \frac{15\pi}{4} m_0^2 \Lambda_0 + \cdots
\]

(5.41)

Using the coordinate transformation

\[
\rho_0 = \frac{\sqrt{\rho}}{\sqrt{\Lambda(\rho_0)}},
\]

(5.42)

we obtain the same result (4.11) as in the Kottler polar–areal coordinates. This result agrees with the equation (55) of [31] for \( \gamma_0 = 0 \).

5.2.2 \( \Lambda = 0 \) case

In the case that \( \Lambda = 0 \), our main result (5.38) for the deflection angle becomes

\[
\Delta \alpha = 4m_0 + 2m_0 \gamma_0 + m_0^2(1 + \gamma_0) \left( \frac{15\pi}{4} - 4 \right),
\]

(5.43)

up to \( m_0^2 \) and \( \gamma_0 \) order.

This result shows that the MK parameter contributes positively to the deflection angle. Comparing our result with the existing ones in the literature we note that we agree, up to this order, with the result of [30] (equation (35) of that paper). In fact there are three kinds of first order corrections in \( \gamma \) to the general relativistic result [28, 30] in the literature: 1) \( \gamma_0 = \gamma\rho_0 \) with negative sign in [23] and with positive sign in [29]. This is clearly a wrong result, because for a gravitational lens the deflection angle diminishes with the impact parameter contrary to what this result suggests [24]. 2) \( m_0^2 \gamma_0 \) of [26–28] exists in our formula, but it is in second order in mass. 3) \( m_0 \gamma_0 \) of [30] also exists in our formula. It is interesting to note that definition of deflection angle in [30] is different than our definition (3.3). Effect of different definitions is not observed in the contribution of MK parameter \( \gamma \), but in the full deflection angle formula (compare equation (37) of [30] with our main result (5.38)). If we also compare the first order correction in \( \gamma \) to the general relativistic result we note that it has a piece independent of the impact parameter and a piece inversely proportional to the impact parameter, which is similar to the Schwarzschild contribution. Hence these contributions increase the lensing effect of a galaxy cluster as expected from an alternative theory to dark matter phenomenology.

6 Conclusions

We utilized our solution of the Weyl gravity [7] to calculate the deflection angle of light from a distant source by a galaxy cluster. We first observed that our Weyl gravity solution is conformally equivalent.
to MK solution, as any solution to Weyl gravity field equations should be. So we have a different conformal factor compared to MK solution, which makes difference only in the case of massive particle trajectories. Light trajectories do not distinguish conformally equivalent metrics, thus our result is relevant for the discussion on the sign and the value of the MK parameter $\gamma$. Our calculation of the deflection angle in the Kottler spacetime included also the contribution of the cosmological constant $\Lambda$. This contribution, however, came out rather differently than the previous works in the literature. The reason of this difference comes from realization that how $\Lambda$ and also $\gamma$ contribute depends strongly on how and at what point in the calculation the perturbative expansions in various quantities are performed. These quantities are mass $m$ of the gravitational lens, the cosmological constant $\Lambda$, and the MK parameter $\gamma$. In which order the perturbative expansions are made is very important. We found out that expansions first in $m$, then in $\gamma$, and finally in $\Lambda$ is the mathematically correct one, because otherwise one gets higher order terms larger than the lower order terms in the expansions. This is against the whole idea of perturbation expansion. We found that our result without MK parameter for the deflection angle agree with the analysis done in [31]. Our result with the MK parameter also differs from the ones in the literature, partially agreeing only with the result presented in [30]. We still have to check these formulas by using different methods such as [36, 37] and then try to analyze observational data to see if our formula agrees at all with the observations without invoking dark matter. These are being two ideas for future research and beyond the scope of this paper.

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