Quantum key distribution without reference frame alignment: Exploiting photon orbital angular momentum

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We present a new implementation of the BB84 quantum key distribution protocol that employs a $d$-dimensional Hilbert space spanned by spatial modes of the propagating beam that have a definite value of orbital angular momentum. Each photon carries $\log d$ bits of information, increasing the key generation rate of the protocol. The states used in the transmission part of the protocol are invariant under rotations about the propagation direction, making this implementation independent of the alignment between the reference frames of the sender and receiver. The protocol still works when these reference frames rotate with respect to each other.

Quantum key distribution (QKD) is one of the most developed applications of quantum information theory (QIT). It is based on the properties of quantum states and allows two spatially separated parties to generate a shared secret key. The key generated is secure in the sense that an eavesdropper cannot obtain more than an exponentially small amount of information about the key without being detected.

Photons have been the information carriers of choice for quantum key distribution. Photons can be sent through optical fibers for relatively long distances (the main obstacle being photon absorption), and they can also be sent through open air. A qubit (or bit) of information can be encoded in the polarization degrees of freedom, and the linear superpositions required by QKD can be easily produced with polarization rotators.

However, QKD protocols employing photon polarization have two main restrictions: they only allow transmission of one key bit per photon, and they require the reference frames of the sender and receiver (usually known as Alice and Bob) to be aligned with each other. The latter should be considered an extra resource required by the protocol. It may not seem too strong of a restriction for ground-based stations, but it is important if either Alice or Bob (or both) are based on a moving station such as a satellite. In this case they must continually monitor and control this alignment.

A possible approach to achieve transmission of more than one bit of information per photon is to employ orbital angular momentum (OAM) states of photons, since the Hilbert space spanned by these states is in principle infinite. A lot of attention has been devoted recently to the study of properties of OAM states and to their generation and manipulation. In this letter, we present an implementation of the well-known BB84 protocol for QKD that goes beyond these two restrictions. The protocol encodes the information in different spatial modes of the propagating photon that have a definite value of OAM. By choosing a subset of these modes we can effectively increase the amount of information encoded in each photon. Furthermore, since these states are eigenstates of orbital angular momentum, they are invariant under rotations about the propagation direction of the beam. The QKD protocol can be implemented without alignment of reference frames, and without requiring entangled states of photons.

We can write the electromagnetic vector potential for a linearly polarized laser in the Lorentz gauge propagating in the $\hat{z}$ direction as $\vec{A} = \hat{\mathbf{x}} u(x, y, z) e^{-ikz}$. The spatial modes $u(x, y, z)$ can be obtained by solving the wave equation for this particular ansatz. In the paraxial approximation there are two important families of solutions, known as the Hermite-Gauss (HG) modes, and the Laguerre-Gauss (LG) modes. The respective spatial mode functions are given by

$$u_{nm}^{HG}(x, y, z) = C_{nm}^{HG} \left( \frac{1}{w} \right) e^{-i[k((x^2+y^2)/2)+(n+m+1)]} \times$$

$$\times e^{-\frac{(x^2+y^2)}{w^2}} H_n(x\sqrt{w}) H_m(y\sqrt{w}) \right), \quad (1)$$

$$u_{nm}^{LG}(x, y, z) = C_{nm}^{LG} \left( \frac{1}{w} \right) e^{-i[k(\frac{x^2}{2}+(n+m+1)]} \times$$

$$\times e^{-i(n-m)} \psi(-1)^{min(n,m)} \sqrt{\frac{r}{w}}^{n-m} \times$$

$$\times L_{min(n,m)}^{(n-m)} \left( \frac{2r^2}{w^2} \right), \quad (2)$$

where $n$, $m$ are arbitrary integers, $R(z) = (z_R^2 + z^2)/z$, $k w^2(z) = (z_R^2 + z^2)/z_R$, $\psi(z) = \arctan(z/z_R)$, $z_R$ the Rayleigh range, $H_n(x)$ the Hermite polynomials and $L_p^q(r)$ the generalized Laguerre polynomials. The normalization constants are given by $C_{nm}^{HG} = (2/\pi n! m!)^{\frac{1}{2}} \frac{1}{w^{n+m}}$ and $C_{nm}^{LG} = (2/\pi n! m!)^{\frac{1}{2}} \min(n,m)!$. 

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The phase factor $e^{i(n+m+1)\phi(z)}$ is known as the Gouy phase, and it is the same only for modes of the same order, where the order is defined by $N = n + m$. In our implementation we will need to compensate for this difference in phase. The LG modes represent states of the photon that have orbital angular momentum (OAM) $l = n - m$. A quantum state of the photon can be associated with each of these modes \[8\]. We will use these states in our implementation of the BB84 protocol.

Let $\{|\psi_i\rangle\}_i$ and $\{|\phi_j\rangle\}_j$ be two orthonormal bases of a $d$-dimensional Hilbert space. We say that they are mutually unbiased bases (MUB) if they satisfy $\langle\psi_i|\phi_j\rangle = \delta_{ij}, \ \forall i, j$. This is the main ingredient of the BB84 protocol. Its main consequence is that measuring the state of the system in the “wrong” basis gives absolutely no information about its preparation, since all outcomes are equiprobable. If we want to implement BB84 with a given physical system as the information carrier, we need to select at least two bases that are mutually unbiased. For $d$-dimensional Hilbert spaces, the BB84 protocol can be generalized to use more than 2 MUB \[12\], which increases the security of the protocol \[13\]. In $d$ dimensions there are at most $(d + 1)$ MUB (this bound is known to be tight if $d$ is a prime power).

We will now show how to implement the BB84 protocol with photonic states belonging to a $d$-dimensional Hilbert space. To simplify the presentation we will neglect for now the effects of the Gouy phase. We will come back to it after discussing the implementation of the protocol to show how these phase differences can be corrected. First, we associate a pure quantum state to each spatial mode of the propagating beam. The state vector $|m, n\rangle_{HG}$ represents the state of a photon propagating on the spatial mode given by $u_{nm}^{HG}$, while the state vector $|m, n\rangle_{LG}$ represents a photon on spatial mode $u_{nm}^{LG}$. We will consider a $d$-dimensional subspace spanned by the basis $B_1 = \{|n\rangle_{HG} \equiv |n, n\rangle_{HG}\}_{n=0}^{d-1}$. In the same subspace, we define another basis $B_2 = \{|k\rangle_{HG} = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{i\frac{2\pi}{d}kn} |n\rangle_{HG}\}_{n=0}^{d-1}$. It is easy to see that these two bases are mutually unbiased.

To implement the BB84 protocol using these two bases we need to prescribe how to prepare and how to measure the states in these bases. Let us first consider the measurement problem. For simplicity, let us assume that $d$ is a power of 2. An interferometer that sorts photons according to their HG spatial mode has been proposed in \[14\]. It consists of cascading Mach-Zender interferometers, each one containing an appropriate Fractional Fourier Transform (FRFT) \[15\] applied to one of the arms. This FRFT is applied by sending the signal through a graded-index (GRIN) rod of appropriate length, that has a quadratic index profile $n(r) = n_0 - n_1 r^2$. This interferometer is called a spatial modal interleave (SMI) \[14\], since the input signal is split between the two output ports according to its modal components. A simple diagram of this cascade of SMI is presented in Fig. 1 for the case $d = 4$. It is not difficult to see that the scheme presented in \[14\] can be modified to sort photons that are prepared in the states of $B_1$. Detecting a photon in one of the arms is equivalent to a projective measurement on the basis $B_1$.

The measurement on the $B_2$ basis is a little bit more involved and requires the use of mode analyzers (MODAN) \[16, 17\] that allow us to change the spatial mode of a propagating photon. The setup for this measurement is presented in Fig. 2. The measurement starts by sending the incoming photon through the same interferometer used to measure in the $B_1$. Let the state of the photon be $|k\rangle_{HG}$. If $d = 2^s$, the setup has $s$ stages, with the $j^{th}$ stage consisting of $2^{j-1}$ SMIs. The total number of SMIs is then $2^s - 1$. Each SMI requires an extra input mode in the vacuum state, so we should write the input state as $|\psi\rangle_{\text{input}} = |k\rangle_{HG}|\text{vac}\rangle$, where $|\text{vac}\rangle$ represents the vacuum state of $(2^s - 1)$ modes. After going through the sorter, the state of the system will be

$$|\psi\rangle' = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{i\frac{2\pi}{d}kn} |\text{vac}\rangle \langle n\rangle_{HG} |\text{vac}\rangle,$$

where the modes on the right-hand side represent the different output ports of the interferometer. Note that the spatial mode of the photon is correlated with the output port in which the photon is present, but we know which port corresponds to each spatial mode. Now we apply a mode analyzer (MODAN) to each output port such that the state of the photon is changed into the state.
\( |0\rangle_{HG} \). A MODAN that performs this transformation can be implemented with holographic optical elements \([16]\). After this step, the spatial mode of the photon is the same for all the terms in \( \textbf{3} \), so to simplify the notation, we will indicate this state by \( |1\rangle \), meaning that one photon is present but its spatial mode is not relevant. After the MODAN, the state of the system is now

\[
|\psi''\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{i \frac{2\pi}{d} kn} |\text{vac}\rangle \langle 1|_{\text{\( n\)th mode}} |\text{vac}\rangle,
\]

which represents a superposition, with appropriate phases, of the photon being in each one of the output ports. By means of the Gouy phase, we can perform a Fourier transform to these \( d \) modes, which can be accomplished by using linear optical elements such as mirrors, beam splitters and phase shifters \([18]\). After this step, the state of the system is given by

\[
|\psi\rangle_{\text{output}} = |\text{vac}\rangle \langle 1|_{\text{\( k\)th mode}} |\text{vac}\rangle.
\]

By placing photodetectors in the output modes of the Fourier transform (FT) device we can measure the value of \( k \), which accomplishes a projective measurement in the basis \( B_2 \). The step that employs the MODAN is an example of the quantum erasure effect: by “erasing” the spatial mode information from the photon state, we erase the “which path” information, which allows us to extract the value of \( k \) from the relative phases by using a Fourier transform.

The preparation of states in the two bases can also be solved with the help of MODANs. To prepare a state from \( B_1 \) we will assume that we have a single photon gun that produces a photon in the state \( |0\rangle_{HG} \) which is the usual lowest order Gaussian mode. By sending the photon through an appropriate MODAN, we can in principle transform the spatial mode, thus preparing any other state in \( B_1 \). To prepare a state in \( B_2 \), we can first run the device we used to perform the projective measurement in the basis \( B_2 \) backwards. By appropriately selecting one of the “output” ports of the FT device to send a photon through, we can choose the value of \( k \) for the state that will be output at the other end.

The implementation we have presented so far still requires the sender and the receiver to align their reference systems, because the HG spatial modes given by \([1]\) clearly single out a direction on the plane perpendicular to the direction of propagation. However, we can get rid of this requirement by using the LG modes for the transmission portion of the protocol. This can be accomplished by using a modal converter \([7,11]\) consisting of two cylindrical lenses, that transform the spatial mode \( u^{HG}_{nm}(x, y, z) \) into \( u^{LG}_{nm}(x, y, z) \) and vice versa.

Our implementation of a \( d \)-dimensional BB84 protocol with photonic spatial modes will be the following. First, Alice randomly chooses which one of the bases \( B_1 \) and \( B_2 \) she will use to send the information. Then she randomly chooses a number \( k \) between 0 and \( d - 1 \). If she chose the basis \( B_1 \), she prepares the state \( |k\rangle_{HG} \) which is associated with the Hermite-Gauss spatial mode \( u^{HG}_{d}(x, y, z) \). Then she runs it through a modal converter, that transforms into the state \( |k\rangle_{LG} \), associated with the Laguerre-Gauss spatial mode \( u^{LG}_{d}(x, y, z) \), and then she sends the state to Bob. From \([2]\) we can see that the state \( |k\rangle_{LG} \) has no dependence on the azimuthal angle \( \phi \) (i.e., it has zero orbital angular momentum), and hence it is \textit{invariant under rotations about the propagation direction}. On the other hand, if Alice chooses the basis \( B_2 \), she prepares the state \( \tilde{|k\rangle}_{HG} = \sqrt{\frac{1}{d}} \sum_{n=0}^{d-1} e^{i \frac{2\pi}{d} kn} |n\rangle_{HG} \). She runs it through the modal converter, thus obtaining the state \( \tilde{|k\rangle}_{LG} = \sqrt{\frac{1}{d}} \sum_{n=0}^{d-1} e^{i \frac{2\pi}{d} kn} |n\rangle_{LG} \). As before, we can see each state in this superposition is invariant under rotations about the propagation direction. Then, the state \( |\tilde{k\rangle}_{LG} \) itself is \textit{invariant under these rotations}.

On the other end, Bob sends the photon he receives through a modal converter, that transforms the LG states back into HG states. He then randomly chooses in which basis he wants to measure the photon, and performs the corresponding measurement. The rest of the protocol is the usual BB84. Note that once the photon is sent through the modal converter, a direction in the plane perpendicular to the propagation direction is singled out, since the cylindrical lenses have a preferred direction in that plane. The interferometer employed to do the measurement also has a preferred direction, and this direction must be consistent with the one chosen by the modal converter. However, since the incoming state of the photon is rotationally invariant on that plane, the direction singled out by the cylindrical lenses is \textit{completely arbitrary}. By encoding the information in rotationally invariant states we have effectively decoupled the alignment between the preparation and measuring devices.

We come back now to the problem of the phase differences between propagating modes of different orders introduced by the Gouy phase. This problem does not affect the preparation and measurement of states in the \( B_1 \) basis, since it only adds an overall phase. For states in the \( B_2 \) basis however, the relative phases of different spatial modes are crucial. If we consider the devices that prepare and measure states in the \( B_2 \) basis, since they separate the spatial modes by sending them through different arms of an interferometric device, it is clear that we can compensate any extra phase factors by applying appropriate phase shifts. The problem of dephasing during transmission, where the state is encoded as a superposition of LG modes, can be also fixed by appropriate phase shifters applied within the measuring device. To see this, note that the \( z \) dependence of the Gouy phase comes from the function \( \psi(z) = \arctan(z/z_R) \), where \( z_R \) is the Rayleigh range. Thus, this mode-dependent phase shift can be compensated if we know the distance between the sender and the receiver. Also, this problem simplifies in the limit in which this distance is much larger than the Rayleigh range. Since the modes we use have order \( 2n \), the phase shift in this case takes only the values \( \frac{\pi}{2} \).
Depending on whether \( n \) is even or odd respectively, which can be compensated by phase shifters inserted in the measuring device.

In this implementation, the transmitted state of the photon was encoded in a subspace of spatial modes that have zero angular momentum. We can also implement this protocol by encoding in a subspace with some definite value \( l \) of orbital angular momentum. These states are associated with the LG spatial modes \( u_{(n+l)n}^{LG}(x, y, z) \). Even though these modes are not invariant under rotations about the propagation axis, they only pick up a phase factor \( e^{i\phi_0} \), where \( \phi_0 \) is the angle of rotation. An overall phase factor is irrelevant for a pure state. Moreover, any linear superposition of these states will also pick up the same global phase factor. We can use this subspace of states with \( l \) orbital angular momentum to implement our protocol, and this implementation still does not require alignment between Alice and Bob. Since there are simple ways of sorting states with different orbital angular momentum \( [19, 20] \), we could simultaneously implement QKD on the same channel using different values of OAM (orbital angular momentum multiplexing).

The alignment independence of our implementation can be taken a step further by noting that everything we discussed holds true even when the angle between Alice and Bob’s reference frames is a function of time. The protocol still allows for QKD when the sender and receiver are rotating with respect to each other about the axis of propagation of the signal. However, we should note that this property holds only when we use states with zero OAM for transmission. States with nonzero OAM undergo a frequency shift due to the rotation \( [21, 22] \), which affects the performance of the interferometers used in their measurement. However, if the interferometers are calibrated to compensate for this shift (which in practice implies knowing the relative angular velocity between Alice and Bob), we can still use higher OAM states for QKD.

It is also worth noting that we could implement the BB84 protocol by encoding the information in photon OAM states and their superpositions, that can be analyzed using the techniques described in \[14\]. In this case however, the protocol requires alignment between Alice and Bob. But we can take advantage of this alignment to double the size of the Hilbert space in which the information is encoded, by combining both OAM and polarization degrees of freedom. A detailed description of this implementation will be presented elsewhere \[23\].

In this letter, we introduced a novel implementation of the BB84 quantum key distribution protocol, that encodes information in the spatial modes of propagating photons. This has two main advantages. First, by employing a \( d \)-dimensional Hilbert space we can increase the key generation rate by increasing the bits per photon that can be sent. This gain is only logarithmic in \( d \), but doubling or tripling the key generation rate may be feasible with current technology. On the other hand, this implementation does not require reference frame alignment between Alice and Bob, making it particularly appealing for key distribution between moving stations, such as satellites in space.

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