Improved SETH-hardness of unweighted Diameter

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Abstract

We prove that, assuming the Strong Exponential Time Hypothesis, for any \( \delta > 0 \), a \( \frac{5}{3} - \delta \) approximation of the diameter of an undirected unweighted graph with \( n \) vertices needs \( n^{3/2 - o(1)} \) time. This result improves on lower bounds of Backurs, Roditty, Segal, Vassilevska-Williams, and Wein.

1 Introduction

The diameter \( D \) of a graph \( G = (V, E) \) is the maximum shortest path distance between two vertices. A basic algorithmic question, Diameter, is computing or approximating the diameter of a graph. The diameter is a useful measure of the complexity of a graph or network, so efficient algorithms for Diameter are desirable in practice [BE05, PRT12, BCH15, LWCW16]. Throughout, \( n \) denotes the number of vertices and \( m \) denotes the number of edges of a graph \( G \). In this work, we restrict our attention to sparse graphs, i.e., those with \( m = \tilde{O}(n) \).

The fastest exact algorithms for Diameter on sparse graphs run in \( \tilde{O}(m^2) \) time. These algorithms in fact compute the shortest paths between every pair of vertices, solving the All-Pairs Shortest Paths problem (APSP). On the other hand, by running single-source shortest path (breadth-first search in unweighted graphs) from a single vertex \( v \) and returning \( \tilde{D} \), the distance from \( v \) to the furthest vertex, we obtain a 2-approximation in \( \tilde{O}(m) \) time by the triangle inequality.

Algorithms with intermediate runtimes are also known. A line of work [ACIM99, CLR+14, RW13] gave a \( 3/2 \)-approximation algorithm for weighted, directed graphs in time \( \tilde{O}(m^{3/2}) \). Cairo, Grossi and Rizzi [CGR16] generalized this algorithm to give an almost \( (2 - \frac{1}{2\pi}) \)-approximation for Diameter in undirected graphs in time \( \tilde{O}(m^{1+\frac{1}{k+1}}) \).

A natural question is whether these known algorithms are optimal. A line of work based on fine grained complexity gives some partial answers to these questions (see [Wil18] for a survey of fine grained complexity and [RW19] for a survey of fine grained complexity for approximation problems). Assuming the Strong Exponential Time Hypothesis (SETH) [IPZ01], Roditty and Williams [RW13] showed that, for any \( \delta > 0 \), a \( \frac{3}{2} - \delta \)-approximation of diameter needs \( n^{2-o(1)} \) time. This shows that the runtime of the quadratic time exact algorithm cannot be improved to \( O(m^{1.99}) \), assuming SETH. This also shows that the approximation ratio of \( \frac{3}{2} \) achieved by [RW13, CLR+14] in time \( \tilde{O}(m^{3/2}) \) cannot be improved to \( \frac{3}{2} - \delta \), assuming SETH. A later work of Backurs, Roditty, Segal, Vassilevska-Williams, and Wein [BRS+18] showed that the \( \tilde{O}(m^{3/2}) \) runtime of the \( \frac{3}{2} \)-approximation in [RW13, CLR+14] is also optimal. They show that, assuming SETH, a \( (\frac{8}{5} - \delta) \)-approximation of Diameter in unweighted, undirected graphs needs \( n^{3/2-o(1)} \) time [BRS+18, Theorem 1]. Additionally, they show that, assuming SETH, a \( (\frac{5}{4} - \delta) \)-approximation of Diameter in weighted, undirected graphs needs \( n^{3/2-o(1)} \) time [BRS+18, Theorem 19]. Finally, they show that, assuming SETH, for any \( k \geq 3 \), a \( (\frac{2k-7}{3k-4} - \delta) \)-approximation of Diameter in unweighted,
directed graphs needs $n^{1+1/(k-1)-o(1)}$ time, [BRS+18, Theorem 21]. Our main result gives the best of these three hardness results from [BRS+18] with a single hardness result.

**Theorem 1.** Assuming SETH, for all $\delta > 0$ a $(\frac{5}{3} - \delta)$-approximation of Diameter in unweighted, undirected graphs on $n$ vertices needs $n^{3/2-o(1)}$ time.

The best known algorithms and SETH-hardness results for unweighted graphs are illustrated in Figure 1. We observe that Theorem 1 applies to directed graphs and/or weighted graphs as well. The best known hardness results for weighted graphs (directed or undirected) remain unchanged, but the best known hardness results for unweighted graphs now match those for weighted graphs.

The hardness results in [BRS+18] are based on a hard graph construction for a variant of Diameter called ST-Diameter. This construction is a reduction from the Orthogonal Vectors problem. Our main idea is to avoid the “bipartiteness” of the ST-Diameter construction, and instead reduce from Single Set Orthogonal Vectors.

Rubinstein and Williams [RW19] asked two important questions about approximating Diameter, and these questions remain open. First, is the simple near-linear time 2-approximation of Diameter optimal? Can we prove a conditional lower bound for near-linear time algorithms better than $5/3$, or is there a near-linear time $5/3$-approximation algorithm for Diameter? Second, in directed graphs, the basic $2$-approximation algorithm is the best known algorithm even for running time $O(m^{3/2-\varepsilon})$. Is there a $O(m^{3/2-\varepsilon})$ time $(2-\delta)$ approximation algorithm for Diameter in directed graphs, or can we show that a $(2-\delta)$-approximation in directed graphs needs $n^{3/2-o(1)}$ time?

A number of generalizations and extensions of Diameter have been considered and studied, including eccentricities [CLR+14, BRS+18], ST-diameter [BRS+18, DWVV19], bichromatic diameter [DWVV19], roundtrip diameter [RW19], and min-diameter [AWW16, DWV+19].
1.1 Preliminaries

Let \([n] \defeq \{1, \ldots, n\}\). In a graph \(G\), let \(d_G(u, v)\) be the length of the shortest path from \(u\) to \(v\). We omit the subscript \(G\) when it is clear from the context.

The Strong Exponential Time Hypothesis (SETH) states that, for every \(\varepsilon > 0\), there exists a \(k\) such that \(k\text{-SAT}\) on \(n\) variables cannot be solved in \(O(2^{(1-\varepsilon)n})\) time (say, on a word-RAM with \(O(\log n)\)-bit words). The \(3\text{-Orthogonal Vectors (3-OV)}\) problem asks, given sets \(A, B, C \subset \{0, 1\}^d\) of \(n\) vectors each, determine where there exists vectors \(a \in A, b \in B, c \in C\) such that \(a[i] \cdot b[i] \cdot c[i] = 0\) for all \(i \in [d]\). The \(\text{Single-Set 3-Orthogonal Vectors}\) is the 3-OV problem when \(A = B = C\). Williams [Wil05] showed that, assuming SETH, for a large enough constant \(c\), 3-OV needs \(n^{3-o(1)}\) time when \(d = c \log n\). Using a simple reduction from 3-OV to Single-Set 3-OV, we have that, assuming SETH, for a large enough \(c\), Single-Set 3-OV for \(d = c \log n\) also needs \(n^{3-o(1)}\) time.

2 Unweighted, undirected 5 vs. 3 lower bound

2.1 Our improvement over the undirected unweighted 8 vs. 5 lower bound

We first sketch the proof of [BRS+18] that, assuming SETH, a \(8/5 - \delta\)-approximation of Diameter in unweighted, undirected graphs needs \(n^{3/2-o(1)}\) time. We then describe how we improve the construction. The construction of [BRS+18] starts with a 3-OV instance \(\Phi\) with three sets \(A, B,\) and \(C\) of \(n_{OV}\) vectors each. From the instance \(\Phi\), they construct in time \(\tilde{O}(n_{OV})\) a graph \(G\) with vertex set \(S = L_0 \cup L_1 \cup L_2 \cup L_3 = T\), where \(L_i\) only has neighbors in \(L_{i-1}\) and \(L_{i+1}\) for all \(i = 0, 1, 2, 3\). The graph cleverly constructed such that if the instance \(\Phi\) has a solution, there exists vertices \(s \in S\) and \(t \in T\) such that \(d(s, t) \geq 7\), and if the instance \(\Phi\) does not have a solution, then \(d(s, t) \leq 3\) for all vertices \(s \in S\) and \(t \in T\).

This construction immediately gives a SETH-hardness result for ST-Diameter, namely that a \((7/3 - \delta)\)-approximation needs \(n^{3/2-o(1)}\) time. However, it does not yet give a hardness result for Diameter: when the instance \(\Phi\) has a solution, the diameter of \(G\) is indeed at least 7, but when the instance \(\Phi\) has no solution, the diameter of \(G\) could be greater than 3, even though the ST-Diameter is only 3. For example, two vertices in \(S\) could be at distance 6 (the shortest path travels through an arbitrarily vertex in \(T\)), and vertices in \(L_1\) and \(L_2\) could be at distance 5 (the shortest path travels through neighbors in \(S\) and \(T\)). To fix this problem, they add additional vertices and edges, so that any two vertices are at distance 5 when \(\Phi\) has no solution, but there exist two vertices at distance 8 when \(\Phi\) has a solution. This gives the desired hardness construction.

The key idea in our construction is reduce from \(\text{Single-Set 3-OV}\), rather the ordinary 3-OV. There are two reasons to expect this might give an improvement. First, the lower bound of [RW13] that a \((3/2 - \delta)\) approximation of Diameter needs \(n^{2-o(1)}\) times can be proved by reduction from 2-OV or by reduction from Single-Set 2-OV (see e.g., [RW19] for this version of the reduction). The reduction from Single-Set 2-OV is simpler, so we might expect a reduction from Single-Set 3-OV to be easier to work with here too.

Second, reducing from Single-Set 3-OV avoids one of the challenges in turning the ST-Diameter lower bound into a Diameter lower bound in [BRS+18]. In the 3 vs. 7 ST-Diameter construction, the vertices of \(S\) are pairs in \(A \times B\), and the vertices of \(T\) are pairs in \(B \times C\). If \(A = B = C\), then the vertex sets \(S\) and \(T\) look exactly the same. If we contract every pair (vertex) of \(S\) with the corresponding pair (vertex) in \(T\), then, in the “no solution” case, we get for free that every pair of vertices in \(S\) are at distance at most 3. Now, we only have to ensure that the eccentricities of all the middle vertices \(L_1 \cup L_2\) are at most 3 in the “no solution” case, while keeping the diameter in the “solution” case large (at least 5). With a careful choice of edges and an additional trick
of adding (without loss of generality) the all-1s vector to the OV instance, we arrive at our final construction.

2.2 Proof of Theorem 1

Proof of Theorem 1. As SETH implies Single-Set 3-OV needs $n^{3-o(1)}$ time, it suffices to prove that, given an $O(n^{3/2-\varepsilon})$ time algorithm for diameter on undirected, unweighted graphs, we can give a $\tilde{O}(n_{OV}^{3-2\varepsilon})$ time algorithm for Single-Set 3-OV. Start with a Single-Set 3-OV instance $\Phi$ given by a set $A \subset \{0,1\}^d$ with $|A| = n_{OV}$ and $d = c \log n_{NOV}$. We may add the all-1s vector to $A$ without loss of generality, as this does not change whether there is an OV solution or not. We construct a graph with $\tilde{O}(n_{OV}^3)$ vertices and edges from the 3-OV instance such that (1) if $\Phi$ has no solution, any two vertices are at distance 3, and (2) if $\Phi$ has a solution, then there exists two vertices at distance 5. Any $(5/3 - \delta)$-approximation for Diameter distinguishes between graphs of diameter 3 and 5. Thus, any such algorithm running in $\tilde{O}(n_{OV}^{3/2-\varepsilon})$ time solves $\Phi$ in the same time, $\tilde{O}(n_{OV}^{3-2\varepsilon})$ time, as desired.

Construction of the graph  The graph $G$ is illustrated in Figure 2 and constructed as follows. The vertex set $S \cup X \cup Y$ is defined on

$$S = A^2,$$
$$X = \{(a, i, j) \in A \times [d]^2 : a[i] = a[j] = 1\}$$
$$Y = \{(a, i', j) \in A \times [d] : a[i'] = 1\}$$

Throughout, we identify tuples $(a, b) \in A^2, (a, i, j) \in A \times [d]^2, \text{ and } (a, i') = A \times [d]$ with vertices of $G$, and we may denote vertices by $(a, b)_S, (a, i, j)_X, \text{ and } (a, i')_Y$ for disambiguation. The (undirected, unweighted) edges are all of the following.

- Edge between $(a, b)_S \text{ and } (a, i, j)_X \text{ if } a[i] = a[j] = b[i] = 1.$
- Edge between $(a, b)_S \text{ and } (a, i, j)_X \text{ if } a[i] = a[j] = b[j] = 1.$
• Edge between \((a, i, j)_X\) and \((b, i, j)_X\) always.

• Edge between \((a, i, j)_X\) and \((a, i')_Y\) if \(a[i'] = 1\).

• Edge between \((a, b)_S\) and \((a, i')_Y\) if \(a[i'] = b[i'] = 1\).

• Edge between \((a, i')_Y\) and \((c, i')_Y\) always.

Note that each vertex of \(S\) has \(O(d^2)\) neighbors, each vertex of \(X\) has \(O(n_{OV})\) neighbors, and each vertex of \(Y\) has \(O(n_{OV})\) neighbors. The total number of edges and vertices is thus \(O(n_{OV}^2d^2) = \tilde{O}(n_{OV}^2)\). We now show that this construction has diameter 3 when \(\Phi\) has no solution and diameter at least 5 when \(\Phi\) has a solution.

\textbf{3-OV no solution}  Assume that the 3-OV instance has no solution, so that no three (or two) vectors are orthogonal. We show that any pair of vertices have distance at most 3, by casework on which of \(S, X, Y\) the two vertices are in.

• **Both vertices are in \(S\):** Let the vertices be \((a, b)_S\) and \((c, d)_S\). As there is no 3-OV solution, there exists indices \(i\) and \(j\) in \([d]\) such that \(a[i] = b[i] = c[i]\) and \(a[j] = c[j] = d[j]\). Then \((a, b)_S - (a, i, j)_X - (c, i, j)_X - (c, d)_S\) is a valid path.

• **One vertex is in \(S\) and the other vertex is in \(X\):** Let the vertices be \((a, b)_S\) and \((c, i, j)_X\): As there is no 3-OV solution, there exists an index \(i\) such that \(a[i] = b[i] = c[i] = 1\). Then \((a, b)_S - (a, i)_Y - (c, i')_Y - (c, i, j)_X\) is a valid path.

• **One vertex is in \(S\) and the other vertex is in \(Y\):** Let the vertices be \((a, b)_S\) and \((c, i', j)_Y\): As there is no 3-OV solution, there exists an index \(i\) such that \(a[i] = b[i] = c[i] = 1\). Then \((a, b)_S - (a, i, j)_X - (c, i, i)_X - (c, i', j)_Y\) is a valid path.

• **Both vertices are in \(X\):** Let the vertices be \((a, i, j)_X\) and \((c, i', j')_X\): As no two vectors are orthogonal, there exists an index \(i''\) such that \(a[i''] = c[i''] = 1\). Then \((a, i, j)_X - (a, i'')_Y - (c, i'')_Y - (c, i', j')_X\) is a valid path.

• **One vertex in \(X\) and the other vertex in \(Y\):** Let the vertices be \((a, i, j)_X\) and \((c, j')_Y\): Let \(z\) denote the all 1s vector, which is in vector set \(A\) by construction. By definition \(z[i'] = z[i] = z[j] = 1\). Then \((a, i, j)_X - (z, i, j)_X - (z, i')_Y - (c, i')_Y\) is a valid path.

• **Both vertices are in \(Y\):** Let the vertices be \((a, i')_Y\) and \((c, j')_Y\). As no two vectors are orthogonal, there exists an index \(i\) such that \(a[i] = c[i] = 1\). Then \((a, i')_Y - (a, i, i)_X - (c, i, i)_X - (c, j')_Y\) is a path.

\textbf{3-OV has solution}  Now assume that the 3-OV instance has a solution. That is, assume there exists \(a, b, c \in A\) such that \(a[i] \cdot b[i] \cdot c[i] = 0\) for all \(i\). We show there are no paths of length at most 4 from \((a, b)_S\) to \((c, b)_S\). Any such path must use an \(X - X\) edge or a \(Y - Y\) edge or else the first entry \(a\) of the vertex’s identifying tuple does not change. Because of this, the path cannot revisit the set \(S\), as it would otherwise need at least 5 edges. Furthermore, we may assume the path never uses two consecutive \(X - X\) edges, as they can be replaced by a single \(X - X\) edge. Similarly, we may assume the path never uses two consecutive \(Y - Y\) edges. Thus, the path must traverse the sets \(S, X, Y\) in one of the following ways.
1. $S - X - X - S$. The path must be $(a, b)_S - (a, i, j)_X - (c, i, j)_X - (c, b)_S$ for some indices $i, j \in [d]$. The first edge requires that $a[i] = a[j] = 1$ and at least one of the coordinates $b[i]$ or $b[j]$ is 1. The last edge requires that $c[i] = c[j] = 1$. Thus, at least one of $a[i] = b[i] = c[i] = 1$ or $a[j] = b[j] = c[j] = 1$ holds, contradicting orthogonality of $a, b, c$.

2. $S - Y - Y - S$. The path must be $(a, b)_S - (a, i')_Y - (c, i')_Y - (c, b)_S$ for some $i'$. The first edge requires $a[i'] = b[i'] = 1$ and the third edge requires $c[i'] = 1$. Thus, $a[i'] = b[i'] = c[i'] = 1$, contradicting orthogonality.

3. $S - X - X - Y - S$. The path must be $(a, b)_S - (a, i, j)_X - (c, i, j)_X - (c, i')_Y - (c, b)_S$ for some $i, j, i' \in [d]$. The first edge requires $a[i] = a[j] = 1$ and at least one of $b[i], b[j]$ is 1. The vertex $(c, i, j)$ is in the graph only if $c[i] = c[j] = 1$. Thus, at least one of $a[i] = b[i] = c[i] = 1$ or $a[j] = b[j] = c[j] = 1$ holds, contradicting orthogonality of $a, b, c$.

4. $S - Y - X - X - S$. Symmetric to previous case.

5. $S - X - Y - Y - S$. The path must be $(a, b)_S - (a, i, j)_X - (c, i')_Y - (c, i')_Y - (c, b)_Y$. The second edge requires that $a[i'] = 1$ and the third edge requires $b[i'] = c[i'] = 1$. Thus, $a[i'] = b[i'] = c[i'] = 1$, contradicting orthogonality.

6. $S - Y - Y - X - S$. Symmetric to previous case.

This shows that $(a, b)_S$ and $(c, b)_S$ are at distance at least 5, completing the proof. 

\[\square\]

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