Minimax rates in sparse, high-dimensional changepoint detection

Haoyang Liu*, Chao Gao* and Richard J. Samworth†
*University of Chicago and †University of Cambridge

Abstract

We study the detection of a sparse change in a high-dimensional mean vector as a minimax testing problem. Our first main contribution is to derive the exact minimax testing rate across all parameter regimes for \( n \) independent, \( p \)-variate Gaussian observations. This rate exhibits a phase transition when the sparsity level is of order \( \sqrt{p \log \log(8n)} \) and has a very delicate dependence on the sample size: in a certain sparsity regime it involves a triple iterated logarithmic factor in \( n \). We also identify the leading constants in the rate to within a factor of 2 in both sparse and dense asymptotic regimes. Extensions to cases of spatial and temporal dependence are provided.

1 Introduction

The problem of changepoint detection has a long history (e.g. Page, 1955), but has undergone a remarkable renaissance over the last 5–10 years. This has been driven in part because these days sensors and other devices collect and store data on unprecedented scales, often at high frequency, which has placed a greater emphasis on the running time of changepoint detection algorithms (Killick, Fearnhead and Eckley, 2012; Frick, Munk and Sieling, 2014). But it is also because nowadays these data streams are often monitored simultaneously as a multidimensional process, with a changepoint in a subset of the coordinates representing an event of interest. Examples include distributed denial of service attacks as detected by changes in traffic at certain internet routers (Peng et al., 2004) and changes in a subset of blood oxygen level dependent contrast in a subset of voxels in fMRI studies (Aston and Kirch, 2012). Away from time series contexts, the problem is also of interest, for instance in the detection of chromosomal copy number abnormality (Zhang et al., 2010; Wang and Samworth, 2018). Key to the success of changepoint detection methods in such settings is the ability to borrow strength across the different coordinates, in order to be able to detect much smaller changes than would be possible through observation of any single coordinate in isolation.

We initially consider a simple model where, for some \( n \geq 2 \), we observe a \( p \times n \) matrix \( X \) that can be written as

\[
X = \theta + E,
\]

where \( \theta \in \mathbb{R}^{p \times n} \) is deterministic and the entries of \( E \) are independent \( N(0, 1) \) random variables. We wish to test the null hypothesis that the columns of \( \theta \) are constant against the alternative that there exists a time \( t_0 \in \{1, \ldots, n - 1\} \) at which these mean vectors change, in at most \( s \) out of
the \( p \) coordinates. The difficulty of this problem is governed by a signal strength parameter \( \rho^2 \) that measures the squared Euclidean norm of the difference between the mean vectors, rescaled by \( \min(t_0, n - t_0) \); this latter quantity represents the distance of the change from the endpoints of the series and can be interpreted as an effective sample size. The goal is to identify the minimax testing rate in \( \rho^2 \) as a function of the problem parameters \( p, n \) and \( s \), and we denote this by \( \rho^*(p, n, s)^2 \); this is the signal strength at which we can find a test making the sum of the Type I and Type II error probabilities arbitrarily small by choosing \( \rho^2 \) to be an appropriately large multiple of \( \rho^*(p, n, s)^2 \) (where the multiple is not allowed to depend on \( p, n \) and \( s \)), and at which any test has error probability sum arbitrarily close to 1 for a suitably small multiple of \( \rho^*(p, n, s)^2 \).

Our first main contribution, in Theorem 1, is to reveal a particularly subtle form of the exact minimax testing rate in the above problem, namely

\[
\rho^*(p, n, s)^2 \propto \begin{cases} 
\sqrt{p \log \log(8n)} & \text{if } s \geq \sqrt{p \log \log(8n)}, \\
 s \log \left( \frac{ep \log \log(8n)}{s} \right) & \text{if } s < \sqrt{p \log \log(8n)}. 
\end{cases}
\]

This result provides a significant generalization of two known special cases in the literature, namely \( \rho^*(1, n, 1)^2 \) and \( \rho^*(p, 2, s)^2 \); see Section 2.1 for further discussion. Although our optimal testing procedure depends on the sparsity level \( s \), which would often be unknown in practice, we show in Theorem 4 that it is possible to construct an adaptive test that very nearly achieves the same testing rate.

The theorem described above is a finite-sample result, but does not provide information at the level of constants. By contrast, in Section 2.4, we study both dense and sparse asymptotic regimes, and identify the optimal constants to within a factor of \( 2^{3/4} \) in the former case, and a factor of 2 in the latter case. In combination with Theorem 1, then, we are able to provide really quite a precise picture of the minimax testing rate in this problem.

Sections 3 and 4 concern extensions of our results to more general data generating mechanisms that allow for spatial and temporal dependence respectively. In Section 3, we allow for cross-sectional dependence across the coordinates via a non-diagonal covariance matrix \( \Sigma \) for the (Gaussian) columns of \( E \). We identify the sharp minimax testing rate when \( s = p \), though the optimal procedure depends on three functionals of \( \Sigma \), namely its trace, as well as its Frobenius and operator norms. Estimation of these quantities is confounded by the potential presence of the changepoint, but we are able to propose a robust method that retains the same guarantee under a couple of additional conditions. As an example, we consider covariance matrices that are a convex combination of the identity matrix and a matrix of ones; thus, each pair of distinct coordinates has the same (non-negative) covariance. Interestingly, we find here that this covariance structure can make the problem either harder or easier, depending on the sparsity level of the changepoint. In Section 4, we also focus on the case \( s = p \) and allow dependence across the columns of \( E \) (which are still assumed to be jointly Gaussian), controlled through a bound \( B \) on the sum of the contributions of the operator norms of the off-diagonal blocks of the \( np \times np \) covariance matrix. Again, interesting phase transition phenomena in the testing rate occur here, depending on the relative magnitudes of the parameters \( B, p \) and \( n \).
Most prior work on multivariate changepoint detection has proceeded without a sparsity condition and in an asymptotic regime with $n$ growing to infinity with the dimension fixed, including Basuville and Nikiforov (1993), Csörgő and Horváth (1997), Ombao et al. (2005), Aue et al. (2009), Kirch et al. (2015), Zhang et al. (2010) and Horváth and Hušková (2012). Bai (2010) studied the least squares estimator of a change in mean for high-dimensional panel data. Jirak (2015), Cho and Fryzlewicz (2015), Cho (2016) and Wang and Samworth (2018) have all proposed CUSUM-based methods for the estimation of the location of a sparse, high-dimensional changepoint. Aston and Kirch (2018) introduce a notion of efficiency that quantifies the detection power of different statistics in high-dimensional settings. Enikeeva and Harchaoui (2018) study the sparse changepoint detection problem in an asymptotic regime in which $p \to \infty$, and at the same time $s/p \to \infty$ and the sample size not too large, while Xie and Siegmund (2013) develop a mixture procedure to detect such sparse changes. Further related work on high-dimensional changepoint problems include the detection of changes in covariance (e.g. Aue et al., 2009; Cribben and Yu, 2017; Wang et al., 2017) and in sparse dynamic networks (Wang et al., 2018a).

Proofs of our main results are given in Section 5, while auxiliary results appear in Section 6. We close this section by introducing some notation that will be used throughout the paper. For $d \in \mathbb{N}$, we write $[d] := \{1, \ldots, d\}$. Given $a, b \in \mathbb{R}$, we write $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. We also write $a \lesssim b$ to mean that there exists a universal constant $C > 0$ such that $a \leq Cb$; moreover, $a \asymp b$ means $a \lesssim b$ and $b \lesssim a$. For a set $S$, we use $1_S$ and $|S|$ to denote its indicator function and cardinality respectively. For a vector $v = (v_1, \ldots, v_d)^T \in \mathbb{R}^d$, we define the norms $\|v\|_1 := \sum_{\ell=1}^d |v_\ell|$, $\|v\|_2^2 := \sum_{\ell=1}^d v_\ell^2$ and $\|v\|_\infty := \max_{\ell \in [d]} |v_\ell|$, and also define $\|v\|_0 := \sum_{\ell=1}^d 1_{\{v_\ell \neq 0\}}$. Given two vectors $u, v \in \mathbb{R}^d$ and a positive definite matrix $\Sigma \in \mathbb{R}^{d \times d}$, we define $(u, v)_{\Sigma}^{-1} := u^T \Sigma^{-1} v$ and $\|v\|_{\Sigma^{-1}} := (v^T \Sigma^{-1} v)^{1/2}$ and omit the subscripts when $\Sigma = I_d$. More generally, the trace inner product of two matrices $A, B \in \mathbb{R}^{d_1 \times d_2}$ is defined as $(A, B) := \sum_{\ell=1}^{d_1} \sum_{\ell'=1}^{d_2} A_{\ell \ell'} B_{\ell' \ell'},$ while the Frobenius and operator norms of $A$ are given by $\|A\|_F := \sqrt{(A, A)}$ and $\|A\|_{op} := s_{\max}(A)$ respectively, where $s_{\max}(\cdot)$ denotes the largest singular value. The total variation distance between two probability measures $P$ and $Q$ on a measurable space $(\mathcal{X}, \mathcal{A})$ is defined as $TV(P, Q) := \sup_{A \in \mathcal{A}} |P(A) - Q(A)|$. Moreover, if $P$ is absolutely continuous with respect to $Q$, then the Kullback–Leibler divergence is defined as $D(P||Q) = \int_{\mathcal{X}} \log \frac{dP}{dQ} \, dP$, and the chi-squared divergence is defined as $\chi^2(P||Q) := \int_{\mathcal{X}} \left(\frac{dP}{dQ} - 1\right)^2 dQ$. The notation $\mathbb{P}$ and $\mathbb{E}$ are generic probability and expectation operators whose distribution is determined from the context.

## 2 Main results

Recall that we consider a noisy observation of a $p \times n$ matrix $X = \theta + E$, where $n \geq 2$ and each entry of the error matrix $E$ is an independent $N(0, 1)$ random variable. In other words, writing $X_t$ and $\theta_t$ for the $t$th columns of $X$ and $\theta$ respectively, we have $X_t \sim N_p(\theta_t, I_p)$. The goal of our paper is to test whether the multivariate sequence $\{\theta_t\}_{t \in [n]}$ has a changepoint. We define the parameter space of signals without a changepoint by

$$
\Theta_0(p, n) := \{ \theta \in \mathbb{R}^{p \times n} : \theta_t = \mu \text{ for some } \mu \in \mathbb{R}^p \text{ and all } t \in [n] \}.
$$
For \( s \in [p] \) and \( \rho > 0 \), the space consisting of signals with a sparse structural change at time \( t_0 \in [n-1] \) is defined by

\[
\Theta^{(t_0)}(p, n, s, \rho) := \left\{ \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^{p \times n} : \theta_t = \mu_1 \text{ for some } \mu_1 \in \mathbb{R}^p \text{ for all } 1 \leq t \leq t_0, \right. \\
\left. \theta_t = \mu_2 \text{ for some } \mu_2 \in \mathbb{R}^p \text{ for all } t_0 + 1 \leq t \leq n, \right. \\
\|\mu_1 - \mu_2\|_0 \leq s, \ \min(t_0, n - t_0) \|\mu_1 - \mu_2\|_2 \geq \rho^2 \}.
\]

In the definition of \( \Theta^{(t_0)}(p, n, s, \rho) \), the parameters \( p \) and \( n \) determine the size of the problem, while \( t_0 \) is the location of the changepoint. The numbers \( s \) and \( \rho \) parametrize the sparsity level and the magnitude of the structural change respectively. It is worth noting that \( \|\mu_1 - \mu_2\|_2 \) is normalized by the factor \( \min(t_0, n - t_0) \), which plays the role of the effective sample size of the problem. To understand this, consider the problem of testing the changepoint at location \( t_0 \) when \( p = 1 \). Then the natural test statistic is

\[
\frac{1}{t_0} \sum_{t=1}^{t_0} X_t - \frac{1}{n-t_0} \sum_{t=t_0+1}^{n} X_t,
\]

whose variance is \( \frac{n}{t_0(n-t_0)} \sim \frac{1}{\min(t_0, n - t_0)} \). Hence the difficulty of changepoint detection problem depends on the location of the changepoint. Through the normalization factor \( \min(t_0, n - t_0) \), we can define a common signal strength parameter \( \rho \) across different possible changepoint locations. Taking a union over all such changepoint locations, the alternative hypothesis parameter space is given by

\[
\Theta(p, n, s, \rho) := \bigcup_{t_0=1}^{n-1} \Theta^{(t_0)}(p, n, s, \rho).
\]

We will address the problem of testing the two hypotheses

\[
H_0 : \theta \in \Theta_0(p, n), \quad H_1 : \theta \in \Theta(p, n, s, \rho).
\] (2)

To this end, we let \( \Psi \) denote the class of possible test statistics, i.e. measurable functions \( \psi : \mathbb{R}^{p \times n} \rightarrow [0, 1] \). We also define the minimax testing error by

\[
\mathcal{R}(\rho) := \inf_{\psi \in \Psi} \left\{ \sup_{\theta \in \Theta(p, n)} \mathbb{E}_\theta \psi(X) + \sup_{\theta \in \Theta(p, n, s, \rho)} \mathbb{E}_\theta (1 - \psi(X)) \right\},
\]

where we use \( \mathbb{P}_\theta \) or \( \mathbb{E}_\theta \) to denote probabilities and expectations under the data generating process (1). Our goal is to determine the order of the minimax rate of testing in this problem, as defined below.

**Definition 1.** We say \( \rho^* = \rho^*(p, n, s) \) is a minimax rate of testing if the following two conditions are satisfied:

1. For any \( \epsilon \in (0, 1) \), there exists \( C_\epsilon > 0 \), depending only on \( \epsilon \), such that \( \mathcal{R}(C \rho^*) \leq \epsilon \) for any \( C > C_\epsilon \).
2. For any $\epsilon \in (0, 1)$, there exists $c_\epsilon > 0$, depending only on $\epsilon$, such that $\mathcal{R}(c\rho^*) \geq 1 - \epsilon$ for any $c \in (0, c_\epsilon)$.

2.1 Special cases

Special cases of $\rho^*(p, n, s)$ are well understood in the literature. For instance, when $p = s = 1$, we recover the one-dimensional changepoint detection problem. Gao et al. (2019) recently determined that

$$\rho^*(1, n, 1)^2 \asymp \log\log(8n).$$

The rate (3) involves a iterated logarithmic factor, in contrast to a typical logarithmic factor in the minimax rate of sparse signal detection (e.g., Donoho and Jin, 2004; Arias-Castro et al., 2005; Berthet and Rigollet, 2013).

Another solved special case is when $n = 2$. In this setting, we observe $X_1 \sim N_p(\mu_1, I_p)$ and $X_2 \sim N_p(\mu_2, I_p)$, and the problem is to test whether or not $\mu_1 = \mu_2$. Since $X_1 - X_2$ is a sufficient statistic for $\mu_1 - \mu_2$, the problem can be further reduced to a sparse signal detection problem in a Gaussian sequence model. For this problem, Collier et al. (2017) established the minimax detection boundary

$$\rho^*(p, 2, s)^2 \asymp \begin{cases} \sqrt{p} & \text{if } s \geq \sqrt{p} \\ s \log \left( \frac{ep}{s^2} \right) & \text{if } s < \sqrt{p}. \end{cases}$$

(4)

It is interesting to notice the elbow effect in the rate (4). Above the sparsity level of $\sqrt{p}$, one obtains the parametric rate that can be achieved using the test that rejects $H_0$ if $\|X_1 - X_2\|_2^2 > 2p + c\sqrt{p}$ for an appropriate $c > 0$.

It is straightforward to extend both rates (3) and (4) to cases where either $p$ or $n$ is of a constant order. However, the general form of $\rho^*(p, n, s)$ is unknown in the statistical literature.

2.2 Minimax detection boundary

The main result of the paper is given by the following theorem.

**Theorem 1.** The minimax rate of the detection boundary of the problem (2) is given by

$$\rho^*(p, n, s)^2 \asymp \begin{cases} \sqrt{p \log \log(8n)} & \text{if } s \geq \sqrt{p \log \log(8n)} \\ s \log \left( \frac{ep \log \log(8n)}{s^2} \right) \lor \log \log(8n) & \text{if } s < \sqrt{p \log \log(8n)}. \end{cases}$$

(5)

It is important to note that the minimax rate (5) is not a simple sum or multiplication of the rates (3) and (4) for constant $p$ or $n$. The high-dimensional changepoint detection problem differs fundamentally from both its low-dimensional version and the sparse signal detection problem.

We observe that the minimax rate exhibits the two regimes in (5) only when $p \geq \log \log(8n)$, since if $p < \log \log(8n)$, then the condition $s \geq \sqrt{p \log \log(8n)}$ is empty, and (5) has just one regime. Compared with the rate (4), the phase transition boundary for the sparsity $s$ becomes $\sqrt{p \log \log(8n)}$. In fact, the minimax rate (5) can be obtained by first replacing the $p$ in (4) with $p \log \log(8n)$, and then adding the extra term (3).
The dependence of (5) on $n$ is very delicate. Consider the range of sparsity where
\[
\frac{\log \log(8n)}{\log(e \log \log(8n))} \vee \frac{\sqrt{p}}{(\log \log(8n))^C} \lesssim s \lesssim \sqrt{p \log \log(8n)},
\]
for some universal constant $C > 0$. The rate (5) then becomes
\[
\rho^*(p, n, s)^2 \asymp s \log(e \log \log(8n)).
\]
That is, it grows with $n$ at a log log log($\cdot$) rate. To the best of our knowledge, such a triple iterated logarithmic rate has not been found in any other problem before in the statistical literature.

Last but not least, we remark that when $p$ or $n$ is a constant, the rate (5) recovers (3) and (4) as special cases.

**Upper Bound.** To derive the upper bound, we need to construct a testing procedure. We emphasize that the goal of hypothesis testing is to detect the existence of a changepoint; this is in contrast to the problem of changepoint estimation (Cho and Fryzlewicz, 2015; Wang and Samworth, 2018; Wang et al., 2018b), where the goal is to find the changepoint’s location.

If we knew that the changepoint were between $t$ and $n - t + 1$, it would be natural to define the statistic
\[
Y_t := \frac{(X_1 + \ldots + X_t) - (X_{n-t+1} + \ldots + X_n)}{\sqrt{2t}}.
\]
(6)
Note that the definition of $Y_t$ does not use the observations between $t + 1$ and $n - t$. This allows $Y_t$ to detect any changepoint in this range, regardless of its location. The existence of a changepoint implies that $\mathbb{E}_\theta(Y_t) \neq 0$. Since the structural change only occurs in a sparse set of coordinates, we threshold the magnitude of each coordinate $Y_t(j)$ at level $a \geq 0$ to obtain
\[
A_{t,a} := \sum_{j=1}^{p} \{Y_t(j)^2 - \nu_a\} \mathbb{1}_{\{|Y_t(j)| \geq a\}},
\]
where $\nu_a := \mathbb{E}(Z^2 \mid |Z| \geq a)$ is the conditional second moment of $Z \sim N(0, 1)$, given that its magnitude is at least $a$. See Collier et al. (2017) for a similar strategy for the sparse signal detection problem. Note that $A_{t,0} = \sum_{j=1}^{p} \{Y_t(j)^2 - 1\}$ has a centered $\chi^2_p$ distribution under $H_0$.

Since the range of the potential changepoint locations is unknown, a natural first thought is to take a maximum of $A_{t,a}$ over $t \in \lfloor n/2 \rfloor$. It turns out, however, that in high-dimensional settings it is very difficult to control the dependence between these different test statistics at the level of precision required to establish the minimax testing rate. A methodological contribution of this work, then, is the recognition that it suffices to compute a maximum of $A_{t,a}$ over a candidate set $\mathcal{T}$ of locations, because if there exists a changepoint at time $t_0$ and $t_0/2 < \tilde{t} \leq t_0$ for some $\tilde{t} \in \mathcal{T}$, then $\|\mathbb{E}_\theta(Y_{\tilde{t}})\|$ and $\|\mathbb{E}_\theta(Y_{t_0})\|$ are of the same order of magnitude. This observation reflects a key difference between the changepoint testing and estimation problems. To this end, we define
\[
\mathcal{T} := \left\{1, 2, 4, \ldots, 2^{\log_2(n/2)}\right\},
\]
6
so that $|T| = 1 + \lfloor \log_2(n/2) \rfloor$. Then, for a given $r \geq 0$, the testing procedure we consider is given by

$$\psi \equiv \psi_{a,r}(X) := \mathbb{1}_{\{\max_{t \in T} A_{t,a} > r\}}. \quad (7)$$

The theoretical performance of the test (7) is given by the following theorem. We use the notation $r^*(p,n,s)$ for the rate function on the right-hand side of (5).

**Proposition 2.** For any $\epsilon \in (0, 1)$, there exists $C > 0$, depending only on $\epsilon$, such that the testing procedure (7) with $a^2 = 4 \log \left( \frac{ep \log \log(8n)}{s^2} \right) \mathbb{1}_{\{s < \sqrt{p \log \log(8n)}\}}$ and $r = Cr^*(p,n,s)$ satisfies

$$\sup_{\theta \in \Theta_0(p,n)} \mathbb{E}_{\theta}\psi + \sup_{\theta \in \Theta(p,n,s,\rho)} \mathbb{E}_{\theta}(1 - \psi) \leq \epsilon,$$

as long as $\rho^2 \geq 32Cr^*(p,n,s)$.

Just as the minimax rate (5) has two regimes, the testing procedure (7) also uses two different strategies. In the dense regime $s \geq \sqrt{p \log \log(8n)}$, we have $a^2 = 0$ and thus (7) becomes simply $\psi = \mathbb{1}_{\{\max_{t \in T} \|Y_t\|^2 - p > r\}}$. In the sparse regime $s < \sqrt{p \log \log(8n)}$, a thresholding rule is applied at level $a$, where $a^2 = 4 \log \left( \frac{ep \log \log(8n)}{s^2} \right)$. We discuss adaptivity to the sparsity level $s$ in Section 2.3.

**Lower Bound.** We show that the testing procedure (7) is minimax optimal by stating a matching lower bound.

**Proposition 3.** For any $\epsilon \in (0, 1)$, there exists $c > 0$, depending only on $\epsilon$, such that $\mathcal{R}(\rho) \geq 1 - \epsilon$ whenever $\rho^2 \leq cr^*(p,n,s)$.

### 2.3 Adaptation to sparsity

The optimal testing procedure (7) that achieves the minimax detection rate depends on knowledge of the sparsity $s$. In this section, we present an alternative procedure that is adaptive to $s$. We first describe two testing procedures, designed to deal with the dense and sparse regimes respectively. For the dense regime, and for $C > 0$, we consider

$$\psi_{\text{dense}} \equiv \psi_{\text{dense},C} := \mathbb{1}_{\{\max_{t \in T} \|Y_t\|^2 - p > C(\sqrt{p \log \log(8n)} \vee \log \log(8n))\}}. \quad (8)$$

In this dense regime, the cut-off value in (8) is of the same order as that in (7), and does not depend on the sparsity level $s$. For the sparse regime, we consider a slightly different procedure from that used in Proposition 2, namely

$$\psi_{\text{sparse}} \equiv \psi_{\text{sparse},C} := \mathbb{1}_{\{\max_{t \in T} A_{t,a} > C \log \log(8n)\}}.$$

Combining the two tests, we obtain a testing procedure that is adaptive to both regimes, given by

$$\psi_{\text{adaptive}} := \psi_{\text{dense}} \vee \psi_{\text{sparse}}. \quad (9)$$
Theorem 4. For any \( \epsilon \in (0, 1) \), there exists \( C > 0 \), depending only on \( \epsilon \), such that the testing procedure (9) with \( a^2 = 4 \log(ep \log \log(8n)) \) satisfies
\[
\sup_{\theta \in \Theta_0(p,n)} \mathbb{E}_\theta \psi_{\text{adaptive}} + \sup_{\theta \in \Theta(p,n,s,\rho)} \mathbb{E}_\theta (1 - \psi_{\text{adaptive}}) \leq \epsilon,
\]
as long as
\[
\rho^2 \geq \begin{cases} 
32C \left( \sqrt{p \log \log(8n)} \lor \log \log(8n) \right) & \text{if } s \geq \frac{\sqrt{p \log \log(8n)}}{\log(ep \log \log(8n))}, \\
32C \left( s \log(ep \log \log(8n)) \lor \log \log(8n) \right) & \text{if } s < \frac{\sqrt{p \log \log(8n)}}{\log(ep \log \log(8n))}.
\end{cases}
\] (10)

Compared with the minimax rate (5), the rate (10) achieved by the adaptive procedure is nearly optimal except that it misses the factor of \( s^2 \) in the logarithmic term.

2.4 Asymptotic constants

A notable feature of our minimax detection boundary derived in Theorem 1 is that the rate is non-asymptotic, meaning that the result holds for arbitrary \( n \geq 2 \), \( p \in \mathbb{N} \) and \( s \in [p] \). On the other hand, if we are allowed to make a few asymptotic assumptions, we can give explicit constants for the lower and upper bounds. In this subsection, therefore, we let both the dimension \( p \) and the sparsity \( s \) be functions of \( n \), and we consider asymptotics as \( n \to \infty \).

Theorem 5 (Dense regime). Assume that \( s^2/(p \log \log n) \to \infty \) as \( n \to \infty \). Then, with
\[
\rho = \xi \left( p \log \log n \right)^{1/4},
\]
we have \( R(\rho) \to 0 \) when \( \xi > 2 \) and \( R(\rho) \to 1 \) when \( \xi < 2^{1/4} \).

Theorem 6 (Sparse regime). Assume that \( s^2/p \to 0 \) and \( s/\log \log n \to \infty \) as \( n \to \infty \). Then, with
\[
\rho = \xi \sqrt{s \log \left( \frac{p \log \log n}{s^2} \right)},
\]
we have \( R(\rho) \to 0 \) when \( \xi > 2 \) and \( R(\rho) \to 1 \) when \( \xi < 1 \).

These two theorems characterize the asymptotic minimax upper and lower bounds of the change-point detection problem under dense and sparse asymptotics respectively.

3 Spatial dependence

In this section, we consider changepoint detection in settings with cross-sectional dependence in the \( p \) coordinates. To be specific, we now relax our previous assumption on the cross-sectional distribution by supposing only that \( X_t \sim N_p(\theta_t, \Sigma) \) for some general positive definite covariance matrix \( \Sigma \in \mathbb{R}^{p \times p} \); the goal remains to solve the testing problem (2). We retain the notation \( \mathbb{P}_\theta \) and \( \mathbb{E}_\theta \) for probabilities and expectations, with the dependence on \( \Sigma \) suppressed.
Our first result provides the minimax rate of the detection boundary in the dense case where \( s = p \). This sets up a useful benchmark on the difficulty of the problem depending on the covariance structure. Similar to Definition 1, we use the notation \( \rho^*_\Sigma(p, n, p) \) for the minimax rate of testing.

**Theorem 7.** The minimax rate is given by

\[
\rho^*_\Sigma(p, n, p)^2 \asymp \|\Sigma\|_F \sqrt{\log \log(8n)} \lor \|\Sigma\|_{\text{op}} \log \log(8n).
\]

In the special case \( \Sigma = I_p \), Theorem 7 yields

\[
\rho^*_\Sigma(p, n, p)^2 \asymp \sqrt{p \log \log(8n)} \lor \log \log(8n),
\]

which recovers the result of Theorem 1 when \( s = p \).

A test that achieves the optimal rate (11) is given by

\[
\psi := \mathbb{1}_{\left\{ \max_{t \in T} \|Y_t\|^2 - \text{Tr}(\Sigma) > C\left( \|\Sigma\|_F \sqrt{\log \log(8n)} \lor \|\Sigma\|_{\text{op}} \log \log(8n) \right) \right\}},
\]

for an appropriate choice of \( C > 0 \). Though optimal, the procedure (12) relies on knowledge of \( \Sigma \). In fact, one only needs to know \( \text{Tr}(\Sigma), \|\Sigma\|_F \) and \( \|\Sigma\|_{\text{op}} \), rather than the entire covariance matrix \( \Sigma \).

To be even more specific, from a careful examination of the proof, we see that we only need to know \( \text{Tr}(\Sigma) \) up to an additive error that is at most of the same order as the cut-off, whereas knowledge of the orders of \( \|\Sigma\|_F \) and \( \|\Sigma\|_{\text{op}} \), up to multiplication by universal constants, is enough.

We now discuss how to use \( X \) to estimate the three quantities \( \text{Tr}(\Sigma), \|\Sigma\|_F \) and \( \|\Sigma\|_{\text{op}} \). The solution would be straightforward if we knew the location of the changepoint, but in more typical situations where the changepoint location is unknown, this becomes a robust covariance functional estimation problem. We assume that \( n \geq 6 \) and that \( n/3 \) is an integer, since a simple modification can be made if \( n/3 \) is not a integer. We can then divide \( [n] \) into three consecutive blocks \( D_1, D_2, D_3 \), each of whose cardinalities is \( n/3 \geq 2 \). For \( j \in [3] \), we compute the sample covariance matrix

\[
\hat{\Sigma}_{D_j} := \frac{1}{|D_j| - 1} \sum_{t \in D_j} (X_t - \bar{X}_{D_j})(X_t - \bar{X}_{D_j})^T,
\]

where \( \bar{X}_{D_j} := |D_j|^{-1} \sum_{t \in D_j} X_t \). We can then order these three estimators according to their trace, as well as their Frobenius and operator norms, yielding

\[
\text{Tr}(\hat{\Sigma})^{(1)} \leq \text{Tr}(\hat{\Sigma})^{(2)} \leq \text{Tr}(\hat{\Sigma})^{(3)},
\]

\[
\|\hat{\Sigma}\|_F^{(1)} \leq \|\hat{\Sigma}\|_F^{(2)} \leq \|\hat{\Sigma}\|_F^{(3)},
\]

\[
\|\hat{\Sigma}\|_{\text{op}}^{(1)} \leq \|\hat{\Sigma}\|_{\text{op}}^{(2)} \leq \|\hat{\Sigma}\|_{\text{op}}^{(3)}.
\]

The idea is that at least two of the three covariance matrix estimators \( \hat{\Sigma}_{D_1}, \hat{\Sigma}_{D_2}, \hat{\Sigma}_{D_3} \) should be accurate, because there is at most one changepoint location. This motivates us to take the medians \( \text{Tr}(\hat{\Sigma})^{(2)}, \|\hat{\Sigma}\|_F^{(2)} \) and \( \|\hat{\Sigma}\|_{\text{op}}^{(2)} \) with respect to the three functionals as our robust estimators. It is convenient to define \( \Theta(p, n, s, 0) := \Theta_0(p, n) \cup (\cup_{\rho > 0} \Theta(p, n, s, \rho)) \).

**Proposition 8.** Assume \( p \leq cn \) for some \( c > 0 \), and fix an arbitrary positive definite \( \Sigma \in \mathbb{R}^{p \times p} \)
and \( \theta \in \Theta(p,n,p,0) \). Then given \( \epsilon > 0 \), there exists \( C > 0 \), depending only on \( c \) and \( \epsilon \), such that

\[
\left| \text{Tr}(\hat{\Sigma})^{(2)} - \text{Tr}(\Sigma) \right| \leq C \left( \frac{\sqrt{p} \|\Sigma\|_F}{\sqrt{n}} + \frac{p\|\Sigma\|_{op}}{n} \right),
\]

\[
\|\hat{\Sigma}\|_F^{(2)} - \|\Sigma\|_F \leq C\|\Sigma\|_{op}\sqrt{\frac{p^2}{n}}
\]

\[
\|\hat{\Sigma}\|_{op}^{(2)} - \|\Sigma\|_{op} \leq C\|\Sigma\|_{op}\sqrt{\frac{p}{n}},
\]

with \( \mathbb{P}_\theta \)-probability at least \( 1 - \epsilon/4 \).

With the help of Proposition 8, we can plug the estimators \( \text{Tr}(\hat{\Sigma})^{(2)}, \|\hat{\Sigma}\|_F^{(2)}, \|\hat{\Sigma}\|_{op}^{(2)} \) into the procedure (12). This test is adaptive to the unknown covariance structure, and comes with the following performance guarantee.

**Corollary 9.** Assume that \( \sqrt{p}\|\Sigma\|_{op} \leq A\|\Sigma\|_F \) for some \( A > 0 \). Then given \( \epsilon > 0 \), there exist \( c, C > 0 \), depending only on \( A \) and \( \epsilon \), such that if \( p \leq cn \), then the testing procedure

\[
\psi_{\text{Cov}} := \mathbb{1}_{\{\max_{t \in T} \|Y_t\|^2_2 - \text{Tr}(\hat{\Sigma})^{(2)} > C(\|\hat{\Sigma}\|_F^{(2)} \sqrt{\log \log(8n)} \vee \|\hat{\Sigma}\|_{op}^{(2)} \log \log(8n))\}}
\]

satisfies

\[
\sup_{\theta \in \Theta_0(p,n)} \mathbb{E}_\theta \psi_{\text{Cov}} + \sup_{\theta \in \Theta(p,n,s,\rho)} \mathbb{E}_\theta (1 - \psi_{\text{Cov}}) \leq \epsilon,
\]

as long as \( \rho^2 \geq 64C \left( \|\Sigma\|_F \sqrt{\log \log(8n)} \vee \|\Sigma\|_{op} \log \log(8n) \right) \).

**Remark 1.** The conditions \( p \lesssim n \) and \( \sqrt{p}\|\Sigma\|_{op} \lesssim \|\Sigma\|_F \) guarantee that \( \|\hat{\Sigma}\|_F^{(2)} - \|\Sigma\|_F \lesssim \|\Sigma\|_F \) and \( \|\hat{\Sigma}\|_{op}^{(2)} - \|\Sigma\|_{op} \lesssim \|\Sigma\|_{op} \) with high probability, by Proposition 8. Note that \( \sqrt{p}\|\Sigma\|_{op} \lesssim \|\Sigma\|_F \) will be satisfied if all eigenvalues of \( \Sigma \) are of the same order. In fact, it possible to weaken the condition \( \sqrt{p}\|\Sigma\|_{op} \lesssim \|\Sigma\|_F \) using the notion of effective rank (Koltchinskii and Lounici, 2017); however, this greatly complicates the analysis, and we do not pursue this here. Alternatively, Corollary 9 also holds without the \( \sqrt{p}\|\Sigma\|_{op} \leq A\|\Sigma\|_F \) condition but under the stronger dimensionality restriction \( p^2 \leq cn \); this then allows for an arbitrary covariance matrix \( \Sigma \).

To better understand the influence of the covariance structure, consider, for \( \gamma \in [0,1) \), the covariance matrix

\[
\Sigma(\gamma) := (1 - \gamma)I_p + \gamma \mathbf{1}_p \mathbf{1}_p^T,
\]

which has diagonal entries 1 and off-diagonal entries \( \gamma \). The parameter \( \gamma \) controls the pairwise spatial dependence; moreover, \( \|\Sigma(\gamma)\|_F^2 = (1 - \gamma^2)p + p^2\gamma^2 \) and \( \|\Sigma(\gamma)\|_{op} = 1 + (p - 1)\gamma \). By Theorem 7, we have

\[
\rho^*_{\Sigma(\gamma)}(p,n,p)^2 \asymp \sqrt{(1 - \gamma^2)p + p^2\gamma^2} \log \log(8n) \vee \{1 + (p - 1)\gamma\} \log \log(8n).
\]

Thus the spatial dependence significantly increases the difficulty of the testing problem. In particular, if \( \gamma \) is of a constant order, then the minimax rate is \( p \log \log(8n) \), which is much larger than
Remark 2. The testing procedure considered in Theorem 10 can be easily made adaptive to the unknown $\gamma$ required signal strength for testing consistency decreases as $\gamma$ increases. This is in stark contrast to (13) for the same covariance structure when $r \geq 0$, the increased difficulty of testing in this example is just one part of the story. When we consider the sparsity factor $s$, the influence of the covariance structure can be the other way around. To illustrate this interesting phenomenon, we discuss a situation where $s$ is small. Since $X_t \sim N_p(\theta_t, \Sigma(\gamma))$, we have that $Y_t \sim N_p(\Delta_t, \Sigma(\gamma))$ for $t < n/2$, where $\Delta_t := \frac{\langle \theta_{t+1} + \cdots + \theta_n \rangle - \langle \theta_{t-1} + \cdots + \theta_{t-1} \rangle}{\sqrt{2t}}$. Hence

$$Y_t(j) = \Delta_t(j) + \sqrt{\gamma} W_t + \sqrt{1 - \gamma} Z_{tj}, \quad (14)$$

where $W_t, Z_{t1}, \ldots, Z_{tp} \overset{iid}{\sim} N(0, 1)$. When there is no changepoint, we have $\Delta_t = 0$, so $Y_t(j) | W_t \overset{iid}{\sim} N(\sqrt{\gamma} W_t, 1 - \gamma)$ for all $j \in [p]$. When there is a changepoint between $t$ and $n - t + 1$, we have $\| \Delta_t \|_0 \leq s$. In either case, then, we can estimate $\sqrt{\gamma} W_t$ by $\text{Median}(Y_t)$. This motivates a new statistic, defined by

$$\hat{Y}_t := \frac{Y_t - \text{Median}(Y_t) \mathbb{1}_p}{\sqrt{1 - \gamma}}. \quad (15)$$

To construct a scalar summary of $\hat{Y}_t$, we define the functions $f_a(x) := (x - \nu_a) \mathbb{1}_{\{|x| \geq a\}}$ for $x \in \mathbb{R}$ and, for $C' \geq 0$, set

$$g_a(x) \equiv g_{a,C'}(x) := \inf \left\{ f_a(y) : |y - x| \leq C' \frac{\log \log(8n)}{p} \right\}. \quad (16)$$

Note that $g_a(x) = f_a(x)$ when $C' = 0$. The use of a positive $C' > 0$ in (16) is to tolerate the error of $\text{Median}(Y_t)$ as an estimator of $\sqrt{\gamma} W_t$. The new testing procedure is then

$$\psi_{a,r,C'} := \mathbb{1}_{\{ \max_{t \in T} \sum_{j=1}^p g_a(\hat{Y}_t(j)) > r \}}. \quad (17)$$

**Theorem 10.** Assume that $\gamma \in [0, 1)$ and $s \leq (p \log \log(8n))^{1/5}$. Then there exist universal constants $c, C' > 0$ such that if $\frac{\log \log(8n)}{p} \leq c$, then for any $\epsilon \in (0, 1)$, we can find $C > 0$ and $n_0 \in \mathbb{N}$, both depending only on $\epsilon$, such that the testing procedure (17) with $a^2 = 4 \log \left( \frac{\epsilon \log \log(8n)}{s^2} \right)$ and $r = C(1 - \gamma) \left( s \log \left( \frac{\epsilon \log \log(8n)}{s^2} \right) \vee \log \log(8n) \right)$ satisfies

$$\sup_{\theta \in \Theta_0(p,n)} \mathbb{E}_\theta \psi_{a,r,C'} + \sup_{\theta \in \Theta(p,n,s,p)} \mathbb{E}_\theta (1 - \psi_{a,r,C'}) \leq \epsilon,$$

when $n \geq n_0$, as long as $\rho^2 \geq 32C(1 - \gamma) \left( s \log(\epsilon^2 \log \log(8n)) \vee \log \log(8n) \right)$.

Surprisingly, in the sparse regime, the spatial correlation helps changepoint detection, and the required signal strength for testing consistency decreases as $\gamma$ increases. This is in stark contrast to (13) for the same covariance structure when $s = p$.

**Remark 2.** The testing procedure considered in Theorem 10 can be easily made adaptive to the unknown $\gamma$ by taking advantage of Proposition 8. Since $\text{Tr}(\Sigma(\gamma)) = p + (p^2 - p)\gamma$, when $p \geq 2$ the
estimator \( \hat{\gamma} := \frac{\text{Tr}(\hat{\Sigma}) - p}{p^2 - p} \) satisfies

\[
|\hat{\gamma} - \gamma| \lesssim \frac{\sqrt{(1 - \gamma^2)p + p^2 \gamma^2}}{p^{3/2} \sqrt{n}} + \frac{1 + (p - 1)\gamma}{pn},
\]

with probability at least \( 1 - 2e^{-p} \). Then, the procedure with \( \gamma \) replaced by \( \hat{\gamma} \) enjoys the same guarantee of Theorem 10 under mild extra conditions.

To end the section, the next theorem shows that the rate achieved by Theorem 10 is minimax optimal.

**Theorem 11.** Assume that \( \gamma \in [0, 1) \) and \( s \leq \sqrt{p \log \log n} \). Then

\[
\rho^*_\Sigma(p, n, s)^2 \gtrsim (1 - \gamma) \left\{ s \log \left( \frac{ep \log \log (8n)}{s^2} \right) \lor \log \log (8n) \right\}.
\]

(18)

4 Temporal dependence

In this section, we consider the situation where \( X_1, \ldots, X_n \) form a multivariate time series. To be specific, in our model \( X_t = \theta_t + E_t \) for \( t \in [n] \), we now assume that the random vectors \( E_1, \ldots, E_n \) are jointly Gaussian but not necessarily independent. The covariance structure of the error vectors can be parametrized by a covariance matrix \( \Sigma \in \mathbb{R}^{pn \times pn} \), and for \( B \geq 0 \), we write \( \Sigma \in C(p, n, B) \) if:

1. \( \text{Cov}(E_t) = I_p \) for all \( t \in [n] \);
2. \( \sum_{s \in [n] \setminus \{t\}} \|\text{Cov}(E_s, E_t)\|_{op} \leq B \) for all \( t \in [n] \).

Thus the data generating process of \( X \) is completely determined by its mean matrix \( \theta \) and covariance matrix \( \Sigma \in C(p, n, B) \), and we use the notion \( \mathbb{P}_{\theta, \Sigma} \) and \( \mathbb{E}_{\theta, \Sigma} \) for the corresponding probability and expectation. The case \( B = 0 \) reduces to the situation of observations at different time points being independent. Time series dependence in high-dimensional changepoint problems has also been considered by Wang and Samworth (2018); their condition \( \| \sum_{s=1}^n \text{Cov}(E_s, E_t) \|_{op} \leq B \) for all \( t \in [n] \) is only slightly different from ours.

We focus on the case \( s = p \) and do not consider the effect of sparsity. The minimax testing error is defined by

\[
\mathcal{R}(\rho) := \inf_{\psi \in \Psi} \left\{ \sup_{\theta \in \Theta_{\theta}(p, n)} \mathbb{E}_{\theta, \Sigma} \psi + \sup_{\theta \in \Theta_{\theta}(p, n, p, \rho)} \mathbb{E}_{\theta, \Sigma}(1 - \psi) \right\}.
\]

We also define the corresponding minimax rate of detection boundary \( \rho^*(p, n, p, B) \) similar to Definition 1. The testing procedure

\[
\psi_{\text{Temp}} := 1_{\{\max_{t \in T} \|Y_t\|^2 - p > \rho\}}
\]

has the following property:
Theorem 12. For any \( \epsilon \in (0, 1) \), there exists \( C > 0 \), depending only on \( \epsilon \), such that the testing procedure (19) with \( r = C \{ Bp + (1 + B)(\sqrt{p \log \log(8n)} + \log \log(8n)) \} \) satisfies

\[
\sup_{\theta \in \Theta_0(p,n), \Sigma \in \mathcal{C}(p,n,B)} \mathbb{E}_{\theta, \Sigma} \psi_{\text{Temp}} + \sup_{\theta \in \Theta(p,n,p,\rho), \Sigma \in \mathcal{C}(p,n,B)} \mathbb{E}_{\theta, \Sigma}(1 - \psi_{\text{Temp}}) \leq \epsilon,
\]

as long as \( \rho^2 \geq 32C \{ Bp + (1 + B)(\sqrt{p \log \log(8n)} + \log \log(8n)) \} \).

Our final result provides the complementary lower bound.

Theorem 13. Assume that \( B \leq D \sqrt{n/p} \) for some \( D > 0 \), and let

\[
\rho^{*2} \equiv \rho^*(p,n,p,B)^2 := Bp + (1 + B)\{ \sqrt{p \log \log(8n)} \vee \log \log(8n) \}. \tag{20}
\]

Then given \( \epsilon > 0 \), there exist \( c_{\epsilon,D} > 0 \), depending only on \( \epsilon \) and \( D \), and \( p_{\epsilon} \in \mathbb{N} \), depending only on \( \epsilon \), such that \( R(cp^*) \geq 1 - \epsilon \) whenever \( c \in (0, c_{\epsilon,D}) \) and \( p \geq p_{\epsilon} \).

Together, Theorems 12 and 13 reveal the rate of the minimax detection boundary when \( B \leq \sqrt{n/p} \). Observe that when \( B = 0 \), the rate (20) becomes \( \sqrt{p \log \log(8n)} \vee \log \log(8n) \), which matches (5) when \( s = p \). When \( B > 0 \), the rate (20) has an extra multiplicative factor \( 1 + B \) and an extra additive factor \( Bp \), which are present for different reasons. Due to the dependence of the time series, one can think of \( n/(1 + B) \) and \( \rho^2/(1 + B) \) as being the effective sample size and signal strength respectively, instead of \( n \) and \( \rho^2 \) for the independent case, and this leads to the presence of the multiplicative factor \( 1 + B \). On the other hand, the additive term \( Bp \) arises from the fact that \( \mathbb{E}_{\theta, \Sigma} \|Y_t\|^2 - p \) under the null hypothesis is not known completely due to the unknown covariance structure \( \Sigma \in \mathcal{C}(p,n,B) \). When \( B = 0 \), the class \( \mathcal{C}(p,n,B) \) becomes a singleton, and we know that \( \mathbb{E}_{\theta, \Sigma} \|Y_t\|^2 = p \) under the null, so this additional term disappears.

5 Proofs

5.1 Proofs of results in Section 2

Proof of Proposition 2. Fixing \( \epsilon \in (0, 1) \), set \( C = C(\epsilon) := 50C_1/\epsilon \), where the universal constant \( C_1 \geq 1 \) is taken from Lemma 19. We first consider the case where \( s \geq \sqrt{p \log \log(8n)} \). Then \( a = 0 \), so that \( A_{t,a} = \sum_{j=1}^p Y_t(j)^2 - p \). Therefore, for any \( \theta \in \Theta_0(p,n) \), we have \( A_{t,a} \approx \chi_p^2 - p \). Then, by a union bound and Lemma 14, we obtain that with \( x := C_0 \log \log(8n) \),

\[
\mathbb{E}_{\theta, \psi} = \mathbb{P}_{\theta}\left( \max_{t \in T} A_{t,0} > C \sqrt{p \log \log(8n)} \right) \leq \mathbb{P}_{\theta}\left( \max_{t \in T} A_{t,0} > 2 \sqrt{px} + 2x \right) \leq 2 \log(\epsilon n) e^{-x} \leq \frac{\epsilon}{2},
\]

where the final inequality holds because \( C \geq 9 + 9 \log(4/\epsilon) \).

Now suppose that \( \theta \in \Theta(p,n,s,\rho) \). For any \( \theta \in \Theta(p,n,s,\rho) \), there exists some \( t_0 \in [n-1] \) such that \( X_1, \ldots, X_{t_0} \overset{iid}{\sim} N_p(\mu_1, I_p) \) and \( X_{t_0+1}, \ldots, X_n \overset{iid}{\sim} N_p(\mu_2, I_p) \), where the vectors \( \mu_1 \) and \( \mu_2 \) satisfy \( \min(t_0, n - t_0) \|\mu_1 - \mu_2\|^2 \geq \rho^2 \). Without loss of generality, we may assume that \( t_0 \leq n/2 \), since
the case \( t_0 > n/2 \) can be handled by a symmetric argument. By the definition of \( \mathcal{T} \), there exists a unique \( \tilde{t} \in \mathcal{T} \) such that \( t_0/2 < \tilde{t} \leq t_0 \). Now \( A_{t,a} \sim \chi^2_{p,a} - p \), where the non-centrality parameter \( \delta \) satisfies
\[
\delta^2 = \frac{\tilde{t}\|\mu_1 - \mu_2\|^2}{2} \geq \frac{t_0\|\mu_1 - \mu_2\|^2}{4} \geq \rho^2/4.
\]
Therefore, by Chebychev’s inequality,
\[
\mathbb{E}_\theta(1 - \psi) \leq \mathbb{P}_\theta \left( \max_{t \in \mathcal{T}} \|Y_t\|^2 - p \leq \frac{\rho^2}{32} \right) \leq \mathbb{P}_\theta \left( \|Y_{\tilde{t}}\|^2 - p \leq \frac{\delta^2}{8} \right) \leq \frac{2(p + 2\delta^2)}{(7/8)^2\delta^4} \leq \frac{2(p + \rho^2/2)}{(7/32)^2\rho^4} \leq \frac{2}{49C^2\log \log(8n)} + \frac{32}{49C \sqrt{p \log \log(8n)}} \leq \frac{\varepsilon}{2},
\]
where \( C \geq 49/(68\varepsilon) \).

We now consider the case where \( s < \sqrt{p \log \log(8n)} \), and first suppose that \( \theta \in \Theta_0(p,n) \). By Lemma 17 and a union bound, we have
\[
\mathbb{E}_\theta \psi = \mathbb{P}_\theta \left( \max_{t \in \mathcal{T}} A_{t,a} > Cr^* \right) \leq \mathbb{P}_\theta \left( \max_{t \in \mathcal{T}} A_{t,a} > \sqrt{pe^{-a^2/4}x} + x \right) \leq 2 \log(en)e^{-x} \leq \frac{\varepsilon}{2},
\]
where we still take \( x = \frac{C}{9} \log \log(8n) \).

Finally, for \( \theta \in \Theta(p,n,s,\rho) \), we define \( \tilde{t}, \mu_1, \mu_2 \) as in the dense case. Now
\[
\max_{t \in \mathcal{T}} A_{t,a} \geq A_{\tilde{t},a} = \sum_{j=1}^p (Y_{\tilde{t}}(j)^2 - \nu_a)1_{|Y_{\tilde{t}}(j)| \geq a},
\]
where \( Y_{\tilde{t}}(j) \sim N(\Delta_j, 1) \), with \( \Delta_j := \sqrt{\frac{1}{2}\{\mu_1(j) - \mu_2(j)\}} \). By Lemma 18, we have
\[
\mathbb{E}A_{\tilde{t},a} \geq \frac{1}{2} \sum_{j:|\Delta_j| \geq 8a} \Delta_j^2 = \frac{1}{2} \left( \sum_{j=1}^p \Delta_j^2 - \sum_{j:|\Delta_j| < 8a} \Delta_j^2 \right) \geq \frac{1}{2} \left( \delta^2 - 64sa^2 \right) \geq \frac{\delta^2}{4},
\]
where the last inequality uses the fact that \( 4\delta^2 \geq \rho^2 \geq 8Cs^a \). Moreover, by Lemma 19, we have
\[
\text{Var}(A_{\tilde{t},a}) = \sum_{j=1}^p \text{Var}\{ (Y_{\tilde{t}}(j)^2 - \nu_a)1_{|Y_{\tilde{t}}(j)| \geq a} \} \leq C_1 \left( pe^{-a^2/4} + sa^4 + \delta^2 \right).
\]

By Chebychev’s inequality, we deduce that
\[
\mathbb{E}_\theta(1 - \psi) = \mathbb{P}_\theta \left( \max_{t \in \mathcal{T}} A_{t,a} \leq \frac{\rho^2}{32} \right) \leq \mathbb{P}_\theta \left( A_{\tilde{t},a} \leq \frac{\delta^2}{8} \right) \leq \frac{\text{Var}(A_{\tilde{t},a})}{(\mathbb{E}A_{\tilde{t},a} - \delta^2/8)^2} \leq \frac{C_1 \left( pe^{-a^2/4} + sa^4 + \delta^2 \right)}{\delta^4/2^6} \leq \frac{C_1pe^{-a^2/4} + C_1sa^4 + C_1\rho^2/4}{\rho^4/2^{10}} \leq \frac{C_1}{C^2} + \frac{16C_1}{C^2} + \frac{8C_1}{C} \leq \frac{\varepsilon}{2},
\]
as required. The second inequality in (23) is by plugging the definition of \( a \) and the lower bound
on $\rho$.

The proof of Proposition 3 below is based on the lower bound technique that involves bounding the chi-squared divergence.

**Proof of Proposition 3.** By Lemmas 20 and 21, given $\eta > 0$, it suffices to find a probability measure $\nu$ with $\text{supp}(\nu) \subseteq \Theta(p, n, s, \rho)$ and a universal constant $c > 0$ such that

$$
E_\nu(\theta_1, \theta_2) \sim \nu \otimes \nu \exp(\langle \theta_1, \theta_2 \rangle) \leq 1 + \eta,
$$

whenever $\rho = c \rho^*$.

We first consider the case when $s \geq \sqrt{p \log \log(8n)}$. We define $\nu$ to be the distribution of $\theta = (\theta_{j\ell}) \in \Theta(p, n, s, \rho)$ with $\rho := \sqrt{s \beta}$ for some $\beta = \beta(p, n, s)$ to be defined later, and generated according to the following sampling process:

1. Uniformly sample a subset $S \subseteq [p]$ of cardinality $s$;
2. Independently $S$, generate $k \sim \text{Unif}(0, 1, 2, \ldots, \lfloor \log_2(n/2) \rfloor)$;
3. Independently of $(S, k)$, sample $u = (u_1, \ldots, u_p) \in \mathbb{R}^p$, where $u_1, \ldots, u_p \overset{iid}{\sim} \text{Unif}(-1, 1)$;
4. Given the triplet $(S, k, u)$ sampled in the previous steps, define $\theta_{j\ell} := \beta \sqrt{2} k u_j$ for all $(j, \ell) \in S \times [2^k]$ and $\theta_{j\ell} := 0$ otherwise.

Suppose we independently sample triplets $(S, k, u)$ and $(T, l, v)$ from the first three steps and use these two triplets to construct $\theta_1$ and $\theta_2$ according to the fourth step. Then

$$
\langle \theta_1, \theta_2 \rangle = (2^k \land 2^l) \frac{\beta^2}{2^{k+l}} \sum_{j \in S \cap T} u_j v_j = \frac{\beta^2}{2^{|l-k|/2}} \sum_{j \in S \cap T} u_j v_j.
$$

Thus

$$
E_\nu(\theta_1, \theta_2) = E \exp \left( \frac{\beta^2}{2^{|l-k|/2}} \sum_{j \in S \cap T} u_j v_j \right),
$$

where the expectation is over the joint distribution of $(S, k, u, T, l, v)$. But $u_j v_j \overset{iid}{\sim} \text{Unif}(-1, 1)$, so

$$
E_\nu(\theta_1, \theta_2) = \left\{ E \left( \frac{1}{2} e^{\beta^2/2^{|l-k|/2}} + \frac{1}{2} e^{-\beta^2/2^{|l-k|/2}} \right) \right\}^{|S \cap T|} \leq E \exp \left( \frac{|S \cap T|}{2^{|l-k|+1}} \beta^4 \right),
$$

where the final inequality uses the fact that $(e^x + e^{-x})/2 \leq e^{x^2/2}$ for $x \in \mathbb{R}$ and Jensen’s inequality. Note that $|S \cap T|$ is distributed according to the hypergeometric distribution

\[1\text{ The Hyp}(p, s, r) distribution models the number of white balls drawn when sampling r balls without replacement from an urn containing p balls, s of which are white.}
fact that the $\text{Hyp}(p, s, s)$ distribution is no larger, in the convex ordering sense, that the binomial distribution $\text{Bin}(s, s/p)$ (Hoeffding, 1963, Theorem 4), we have

$$
\mathbb{E}\exp\left(\frac{\beta^4}{2|l-k|+1}\{S \cap T\}\right) \leq \left\{\mathbb{E}\left(1 - \frac{s}{p} + \frac{s}{p} e^{\beta^4/2|l-k|+1}\right)^n\right\}^s \\
\leq \mathbb{E}\left\{1 + \frac{s}{2p} \frac{\beta^4}{2|l-k|} e^{\beta^4/2|l-k|+1}\right\}^s =: \mathbb{E}L(l, k), \quad (25)
$$

say, where we have used $e^x - 1 \leq xe^x$ for all $x \geq 0$ and Jensen’s inequality to derive the last inequality above. From now on, we set $\beta := \{c_1 p s^{-2} \log \log(8n)\}^{1/4}$, where $c_1 = c_1(\eta) \in (0, 1/4]$ will be chosen to be sufficiently small. The condition $s \geq \sqrt{p \log \log(8n)}$ ensures that $\beta \leq 1$. We first claim that

$$
\mathbb{E}\{L(l, k) \mathbb{1}_{\{l=k\}}\} \leq \left\{(1 + \eta/4)\mathbb{P}(l = k)\right\} \vee \eta/4,
$$

provided that $c_1 \leq \eta \log (1 + \eta/4)/8$. To see this, first note that for $n \geq \exp(\exp(8/\eta))/8$, we have

$$
\mathbb{E}\{L(l, k) \mathbb{1}_{\{l=k\}}\} \leq \left(1 + \frac{c_1}{s} \log \log(8n)\right)^s \mathbb{P}(l = k) \leq \frac{\log^{1/4}(8n)}{1 + \lceil \log_2(n/2) \rceil} \\
\leq \frac{\eta/8 \log \log(8n)}{\log^{1/4}(8n)} \leq \frac{\eta/4}{1 + \lceil \log_2(n/2) \rceil} \leq \eta/4.
$$

On the other hand, when $n < \exp(\exp(8/\eta))/8$, we have

$$
\mathbb{E}\{L(l, k) \mathbb{1}_{\{l=k\}}\} \leq \log^{c_1}(8n)\mathbb{P}(l = k) \leq \exp(\log^{c_1/\eta}(8n)) \leq \left(1 + \frac{\eta}{4}\right)\mathbb{P}(l = k).
$$

Moreover,

$$
\mathbb{E}\{L(l, k) \mathbb{1}_{0<|l-k|\leq(\eta/8)\log\log(8n)}\} \leq \left(1 + \frac{c_1}{s} \log \log(8n)\right)^s \mathbb{P}(0 < |l-k| \leq \eta/8 \log \log(8n)) \\
\leq \log^{1/4}(8n) \frac{\eta \log \log(8n)}{4(1 + \lceil \log_2(n/2) \rceil)} \leq \frac{\eta}{2}.
$$

(27)

For the third term, we write $a_\eta := \sup_{n \geq 2} \frac{\log \log(8n)}{\log((\eta/8)\log(2\log(8n)))}$. By reducing $\eta > 0$ and $c_1 = c_1(\eta)$ if necessary, we may assume that $c_1 a_\eta \leq \eta/8 \leq 1/2$, so that

$$
\mathbb{E}\{L(l, k) \mathbb{1}_{|l-k|>(\eta/8)\log\log(8n)}\} \leq \left(1 + \frac{c_1 a_\eta}{s}\right)^s \mathbb{P}\{|l-k| > (\eta/8) \log \log(8n)\} \\
\leq (1 + 2c_1 a_\eta)\mathbb{P}\{|l-k| > (\eta/8) \log \log(8n)\} \\
\leq \left(1 + \frac{\eta}{4}\right)\mathbb{P}\{|l-k| > (\eta/8) \log \log(8n)\}.
$$

(28)

From (26), (27) and (28), we conclude that

$$
\mathbb{E}\{L(l, k)\} \leq 1 + \eta,
$$

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which establishes (24) in the case $s \geq \sqrt{p \log \log (8n)}$.

We now consider the case $s < \sqrt{p \log \log (8n)}$ and $s \log \left( \frac{ep \log \log (8n)}{s^2} \right) \geq \log \log (8n)$. The goal is to derive a lower bound with rate $s \log \left( \frac{ep \log \log (8n)}{s^2} \right) \geq \log \log (8n)$. We use the same $\nu$ specified in the previous case except that in the third step, we set $u_j = 1$ for all $j \in S$. With this modification of $\nu$, we have $\langle \theta_1, \theta_2 \rangle = |S \cap T| \frac{\beta^2}{2(|l-k|/2)}$. Again, $|S \cap T|$ is distributed according to the hypergeometric distribution $\text{Hyp}(p, s, s)$, and

$$E_{\langle \theta_1, \theta_2 \rangle \sim \nu \otimes \nu} \exp(\langle \theta_1, \theta_2 \rangle) = E \exp \left( |S \cap T| \frac{\beta^2}{2(|l-k|/2)} \right) \leq \left( E \left( 1 - \frac{s}{p} + \frac{s}{p} e^{\beta^2/2(|l-k|/2)} \right) \right)^s \leq E \left( 1 + \frac{s}{p} e^{\beta^2/2(|l-k|/2)} \right)^s =: ER(l, k),$$

say. We take $\beta := \log^{1/2} \left( \frac{e^p \log \log (8n)}{s^2} \right)$, where $c_2 = c_2(\eta) \in (0, 1/4]$ will be chosen sufficiently small. Parallel to the bounds for $EL(l, k)$, we will split into three terms. For the first term, we have

$$E \{ R(l, k) \mathbb{1}_{\{l=k\}} \} \leq \left( 1 + \frac{c_2^2}{s} \log \log (8n) \right)^s \mathbb{P}(l = k) \leq \left( 1 + \frac{\eta}{4} \right) \mathbb{P}(l = k) \vee \frac{\eta}{4},$$

as before, as long as $c_2 \leq \eta \log (1 + \frac{\eta}{4}) / 8$. For the second term,

$$E \{ R(l, k) \mathbb{1}_{\{0 < |l-k| \leq (\eta/8) \log \log (8n)\}} \} \leq \left( 1 + \frac{\eta}{8} \log \log (8n) \right)^s \mathbb{P} \left( 0 < |l - k| \leq \frac{\eta}{8} \log \log (8n) \right) \leq \left( 1 + \frac{\log \log (8n)}{4s} \right)^s \eta \log \log (8n) \leq \frac{\eta}{2}$$

For the third term, define $b_\eta := \sup_{n \geq 2} \exp \left( \frac{\log \log \log (8n)}{\log (\eta/16) \log^2 (8n)} \right)$. By reducing $c_2 = c_2(\eta)$ if necessary, we may assume that $c_2 \leq \log (1 + \eta/4)/b_\eta$. Then

$$E \{ R(l, k) \mathbb{1}_{\{|l-k| > (\eta/8) \log \log (8n)\}} \} \leq \left( 1 + \frac{s}{p} \exp \left( \frac{\log (c_2 p / s^2) + \log \log \log (8n)}{\log (\eta/16) \log^2 (8n)} \right) \right)^s \mathbb{P} \{|l-k| > (\eta/8) \log \log (8n)\} \leq e^{c_2 b_\eta} \mathbb{P} \{|l-k| > (\eta/8) \log \log (8n)\} \leq \left( 1 + \frac{\eta}{4} \right) \mathbb{P} \{|l-k| > (\eta/8) \log \log (8n)\},$$

which establishes (24) when $s < \sqrt{p \log \log (8n)}$ and $s \log \left( \frac{ep \log \log (8n)}{s^2} \right) \geq \log \log (8n)$.

The final case is $s < \sqrt{p \log \log (8n)}$ and $s \log \left( \frac{ep \log \log (8n)}{s^2} \right) < \log \log (8n)$. Notice that in our definition of the parameter space $\Theta^{(\mu)}(p, n, s, \rho)$, if we restrict $\mu_1$ and $\mu_2$ to agree in all coordinates except perhaps the first, then the testing problem is equivalent to testing between $\Theta_0(1, n)$ and $\Theta(1, n, 1, \rho)$. Therefore, the lower bound construction in Gao et al. (2019) applies directly here and we obtain the rate $\log \log (8n)$. 

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The result follows.

The proof of Theorem 4 uses several arguments from the proof of Proposition 2.

**Proof of Theorem 4.** Fix $\epsilon \in (0, 1)$, and first consider $\theta \in \Theta_{0}(p, n)$. By the same argument as in the proof of Proposition 2, and with $C = C(\epsilon) > 0$ as defined there and $x = C_{1} / \epsilon \log \log(8n)$, we have by Lemma 14 that

$$
\mathbb{E}_{\theta} \psi_{\text{dense}} \leq \mathbb{P}_{\theta} \left( \max_{t \in T} \| Y_{t} \|^{2} - p > C \sqrt{p \log \log(8n)} \right) \\
\leq \mathbb{P}_{\theta} \left( \max_{t \in T} \| Y_{t} \|^{2} - p > 2 \sqrt{p \epsilon x} \right) \leq \frac{\epsilon}{4},
$$

since $C \geq 9 + 9 \log(8/\epsilon)$. For $\psi_{\text{sparse}}$, we apply Lemma 17 with the universal constant $C^{*} > 0$ defined there. Let $C' = C'(\epsilon) > 0$ be such that $C' \geq \log(8 \log(en)/\epsilon) / \log \log(8n)$. By increasing $C = C(\epsilon) > 0$ if necessary, we may assume that

$$
C \log \log(8n) \geq C^{*} \left( \sqrt{\frac{C'}{e}} + C' \log \log(8n) \right) = C^{*} \left( \sqrt{pe^{-a^{2}/4 \epsilon} x} + x \right),
$$

where $x = C' \log \log(8n)$. Then, by a union bound and Lemma 17,

$$
\mathbb{E}_{\theta} \psi_{\text{sparse}} = \mathbb{P}_{\theta} \left( \max_{t \in T} \sum_{j=1}^{p} \{ Y_{t}(j)^{2} - \nu_{a} \} 1_{\{ |Y_{t}(j)| \geq 2 \log^{1/2}(ep \log \log(8n)) \}} > C \log \log(8n) \right) \\
\leq \mathbb{P}_{\theta} \left( \max_{t \in T} \sum_{j=1}^{p} \{ Y_{t}(j)^{2} - \nu_{a} \} 1_{\{ |Y_{t}(j)| \geq 2 \log^{1/2}(ep \log \log(8n)) \}} > C^{*} \left( \sqrt{pe^{-a^{2}/4 \epsilon} x} + x \right) \right) \\
\leq 2 \log(en) e^{-C' \log \log(8n)} \leq \frac{\epsilon}{4}.
$$

Thus

$$
\sup_{\theta \in \Theta_{0}(p, n)} \mathbb{E}_{\theta} \psi_{\text{adaptive}} \leq \sup_{\theta \in \Theta_{0}(p, n)} \mathbb{E}_{\theta} \psi_{\text{dense}} + \sup_{\theta \in \Theta_{0}(p, n)} \mathbb{E}_{\theta} \psi_{\text{sparse}} \leq \frac{\epsilon}{2},
$$

Now we consider $\theta \in \Theta(p, n, s, \rho)$ with $s \geq \sqrt{p \log \log(8n) / \log(ep \log \log(8n))}$. Then,

$$
\mathbb{E}_{\theta}(1 - \psi_{\text{adaptive}}) \leq \mathbb{E}_{\theta}(1 - \psi_{\text{dense}}) \leq \mathbb{P}_{\theta} \left( \max_{t \in T} \| Y_{t} \|^{2} - p \leq \frac{\rho^{2}}{32} \right) \leq \frac{\epsilon}{2},
$$

under the condition on $\rho^{2}$, where the final bound follows from (21).

We finally consider $\theta \in \Theta(p, n, s, \rho)$ with $s < \sqrt{p \log \log(8n) / \log(ep \log \log(8n))}$, which implies that $s < \sqrt{p \log \log(8n)}$. Then, as in the proof of Proposition 2, we have $\rho^{2} \geq 8Cs\sigma^{2}$, so that

$$
\mathbb{E}_{\theta}(1 - \psi_{\text{adaptive}}) \leq \mathbb{E}_{\theta}(1 - \psi_{\text{sparse}}) \leq \mathbb{P}_{\theta} \left( \max_{t \in T} A_{t, a} \leq \frac{\rho^{2}}{32} \right) \leq \frac{\epsilon}{2},
$$

as in (23), where we note that the penultimate inequality in (23) continues to hold with our new
definitions of $\rho$ and $a$. The result follows.

The proofs of Theorem 5 and Theorem 6 are essentially tightening the arguments in the proofs of Proposition 2 and Proposition 3. We highlight only the main differences.

**Proof of Theorem 5.** We first prove the lower bound. Consider the same $\nu$ constructed in the proof of the dense case in Proposition 3, which relies on $\rho^2 \geq s\beta^2$. Then

$$R(\rho) \geq 1 - TV\left(\mathbb{P}_0, \int_{\Theta(p,n,s,\rho)} \mathbb{P}_\theta d\nu(\theta)\right) \geq 1 - \chi^2\left(\int_{\Theta(p,n,s,\rho)} \mathbb{P}_\theta d\nu(\theta) \left\| \mathbb{P}_0 \right\| \right).$$

By Lemma 21 and the bound (25), we need to show that $\limsup_{n \to \infty} \mathbb{E}\left\{L(l,k)\right\} \leq 1$, where $L(l,k)$ is defined in the proof of Proposition 3. Similar to the bounds on $L(l,k)$ obtained in that proof, we see that this is the case whenever $\epsilon_1 < 2$, or equivalently, when $\xi < 2 \epsilon_1^2 / 4$.

For the upper bound, for $\epsilon_1, \epsilon_2 > 0$, consider the test

$$\psi = 1_{\{\max_{t \in T_{\epsilon_2}} \|Y_t\|^2 \geq 2 \sqrt{(1+\epsilon_1)p \log \log(n)}\}},$$

where $T_{\epsilon_2} := \left\{1, \lfloor 1 + \epsilon_2 \rfloor, \lfloor (1 + \epsilon_2)^2 \rfloor, \ldots, \lfloor (1 + \epsilon_2)^{\lfloor \log_{1+\epsilon_2}(n/2) \rfloor} \rfloor \right\}$.

Then, by the same analysis as in the proof of Proposition 2, given $\xi > 2$, there exist $\epsilon_1, \epsilon_2 > 0$, depending only on $\xi$, such that $R(\rho) \to 0$ for $\rho = \xi (p \log \log n)^{1/4}$.

**Proof of Theorem 6.** We first prove the lower bound. Similar to the proof of Theorem 5, we only need to tighten the analysis in the proof of Proposition 3 for the corresponding regime. By Lemma 21 and (30), it suffices to show that $\limsup_{n \to \infty} \mathbb{E}\{R(l,k)\} \leq 1$, where $R(l,k)$ is defined in the proof of Proposition 3. Similar to the bounds on $R(l,k)$ obtained in that proof, we see that this is the case provided that $\epsilon_2 < 1$, or equivalently, when $\xi < 1$.

For the upper bound, for $\epsilon_1, \epsilon_2, C > 0$, we consider the test

$$\psi = 1_{\{\max_{t \in T_{\epsilon_2}} A_{t,a} > C(s+\log \log n)\}},$$

where $T_{\epsilon_2}$ is defined in the proof of Theorem 5, and where

$$a^2 = (2 + \epsilon_1) \log\left(\frac{p \log \log n}{s^2}\right).$$

By scrutinizing the proof of Lemma 17, given any $\epsilon_1 > 0$, we can strengthen the conclusion to

$$\mathbb{P}\left(\sum_{j=1}^{p} (Z_j^2 - \nu_a) 1_{\{|Z_j| \geq a\}} \geq C^* \left(\sqrt{pe^{-a^2/(2+\epsilon_1)}} x + x\right)\right) \leq e^{-x},$$

where $C^*$ might depend on $\epsilon_1$. With the help of this inequality, we can follow the arguments in the proof of Proposition 2 and obtain the conclusion that when $\rho = \xi \sqrt{s \log(s^2 \log \log n)}$ with $\xi > 2$, we have $R(\rho) \to 0$ as $n \to \infty$ provided that $\epsilon_1, \epsilon_2 > 0$, depending only on $\xi$, are sufficiently small, and $C > 0$, depending only on $\xi$ and $\epsilon_1$, is sufficiently large.
5.2 Proofs of results in Section 3

Proof of Theorem 7. For any \( \theta \in \Theta(p,n,p,\rho) \), there exist \( \mu_1, \mu_2 \in \mathbb{R}^p \) and \( t \in [n] \), such that \( X_1, \ldots, X_t \sim N_p(\mu_1, \Sigma) \) and \( X_{t+1}, \ldots, X_n \sim N_p(\mu_2, \Sigma) \). The covariance matrix \( \Sigma \) admits the eigenvalue decomposition \( \Sigma = U \Lambda U^T \) for some orthogonal \( U \in \mathbb{R}^{p \times p} \) and \( \Lambda = \text{diag}(\lambda) \in \mathbb{R}^{p \times p} \), where \( \lambda := (\lambda_1, \ldots, \lambda_p)^T \) and \( \lambda_1 \geq \cdots \geq \lambda_p > 0 \). Then \( U^T X_1, \ldots, U^T X_t \sim N_p(U^T \mu_1, \Lambda) \) and \( U^T X_{t+1}, \ldots, U^T X_n \sim N_p(U^T \mu_2, \Lambda) \). We also have \( \|U^T (\mu_1 - \mu_2)\| = \|\mu_1 - \mu_2\| \), so we can consider a diagonal \( \Sigma \) without loss of generality. From now on, we assume that \( \Sigma = \Lambda \).

We first derive the upper bound. Consider the testing procedure

\[
\psi = 1_{\{\max_{t \in T} \|Y_t\| - \text{Tr}(\Sigma) > r\}},
\]

with \( r = C \left( \sqrt{\|\Sigma\|^2 \log \log(8n) + \|\Sigma\|_{op} \log \log(8n)} \right) \) for some appropriate \( C > 0 \). Then the same argument in the proof of Proposition 2 together with Lemma 14 leads to the desired result.

We now derive the lower bound. We first seek to apply Lemmas 20 and 21 and given \( \eta > 0 \), construct a probability measure \( \nu \) with \( \text{supp}(\mu) \subseteq \Theta(p,n,p,\rho) \) and a universal constant \( c > 0 \) such that

\[
E(\theta_1, \theta_2) \sim \nu \otimes \nu \exp(\langle \theta_1, \theta_2 \rangle \Sigma^{-1}) \leq 1 + \eta,
\]

whenever \( \rho = c \rho_*^2 \). We define \( \nu \) to be the distribution of \( \theta = (\theta_{j\ell}) \in \Theta(p,n,p,\rho) \), sampled according to the following process:

1. Uniformly sample \( k \in \{0, 1, 2, \ldots, \lfloor \log_2(n/2) \rfloor \} \);
2. Independently of \( k \), sample \( u = (u_1, \ldots, u_p)^T \in \mathbb{R}^p \) with independent coordinates, and with \( u_j \sim \text{Unif}(-a_j, a_j) \) for \( j \in [p] \);
3. Given \((k,u)\) sampled in the previous steps, define \( \theta_{j\ell} := 2^{-k/2} u_j \) for all \((j, \ell) \in [p] \times [2^k] \) and \( \theta_{j\ell} := 0 \) otherwise.

If \( \theta \sim \nu \), then \( \theta \in \Theta(p,n,p,\rho) \) with \( \rho^2 = \sum_{j=1}^p a_j^2 \). Suppose that we independently sample \((k,u)\) and \((l,v)\) from the first two steps and use these to construct \( \theta_1 \) and \( \theta_2 \) respectively according to the third step. Then, by direct calculation, we obtain

\[
\langle \theta_1, \theta_2 \rangle_{\Sigma^{-1}} = (2^k \wedge 2^l) \frac{1}{\sqrt{2^{k+l}}} \sum_{j=1}^p \frac{u_j v_j}{\lambda_j} = \frac{1}{2^{k+l/2}} \sum_{j=1}^p \frac{u_j v_j}{\lambda_j}.
\]
Observe that \( u_j v_j \sim \text{Unif}(\{-a_j^2, a_j^2\}) \), so
\[
\mathbb{E}_{(\theta_1, \theta_2) \sim \nu \odot p} \exp(\langle \theta_1, \theta_2 \rangle_{\Sigma^{-1}}) = \mathbb{E} \exp \left( \frac{1}{2} \left( \frac{\sum_{j=1}^{p} u_j v_j}{\lambda_j} \right) \right) = \mathbb{E} \prod_{j=1}^{p} \left\{ \frac{1}{2} \exp \left( \frac{a_j^2}{2 \lambda_j} \right) + \frac{1}{2} \exp \left( - \frac{a_j^2}{2 \lambda_j} \right) \right\} \leq \mathbb{E} \exp \left( \frac{1}{2} \sum_{j=1}^{p} \frac{a_j^4}{\lambda_j^2} \right),
\]
where the last inequality above uses the fact that \((e^x + e^{-x})/2 \leq e^{x^2/2}\). We take \( a_j^2 = \sqrt{c_1 \frac{\lambda_1 \log \log(8n)}{\|\Sigma\|^2}} \) for some sufficiently small \( c_1 = c_1(\eta) > 0 \). Then it can be shown that
\[
\mathbb{E} \exp \left( \frac{1}{2} \sum_{j=1}^{p} \frac{a_j^4}{\lambda_j^2} \right) \leq \mathbb{E} \exp \left( c_1 \frac{\log \log(8n)}{2 \lambda} \right) \leq 1 + \eta
\]
using very similar arguments to those employed in the proof of Proposition 2. We have therefore established (31), which implies the desired lower bound \( \rho^2 = \sum_{j=1}^{p} a_j^2 \leq \frac{\lambda_1 \log \log(8n)}{\|\Sigma\|^2} \).

We also need to prove the lower bound \( \|\Sigma\|_{op} \log \log(8n) \). Recall that we have assumed without loss of generality that \( \Sigma \) is diagonal with non-increasing diagonal elements. Then in our definition of the parameter space \( \Theta(t_0)(p, n, s, \rho) \), if we restrict \( \mu_1 \) and \( \mu_2 \) to agree in all coordinates except perhaps the first, then the testing problem is equivalent to testing between \( \Theta_0(1, n) \) and \( \Theta(1, n, 1, \rho) \) with variance \( \lambda_1 = \|\Sigma\|_{op} \). Therefore, the lower bound construction in Gao et al. (2019) directly applies here and we obtain the desired rate \( \|\Sigma\|_{op} \log \log(8n) \).

**Proof of Proposition 8.** Suppose \( \mathcal{D} \) does not include the changepoint. Then, by Lemma 15, we have that for every \( x > 0 \),
\[
|\text{Tr}(\hat{\Sigma}_D) - \text{Tr}(\Sigma)| \leq 4 \left( \frac{\sqrt{x} \|\Sigma\|_{F}}{\sqrt{n}} + \frac{x \|\Sigma\|_{op}}{n} \right),
\]
with probability at least \( 1 - 2e^{-x} \) (notice that substituting \( n \) for \( n - 1 \) means we multiply the right-hand side by at most 2). We will take \( x = p \log(32/\epsilon) \), which guarantees that \( e^{-x} \leq \epsilon/32 \). Moreover, there exists a universal constant \( \tilde{C} > 0 \), such that for all \( x \geq 1 \)
\[
\|\hat{\Sigma}_D - \Sigma\|_{op} \leq \tilde{C} \|\Sigma\|_{op} \left( \sqrt{\frac{p}{n}} \vee \frac{p}{n} \vee \frac{\sqrt{x}}{n} \vee \frac{x}{n} \right),
\]
with probability at least \( 1 - e^{-x} \) (Koltchinskii and Lounici, 2017, Theorem 1). Here we will take \( x = p \log(16/\epsilon) \). From this we immediately have the error bounds for \( \|\hat{\Sigma}_D\|_{F} \) and \( \|\hat{\Sigma}_D\|_{op} \), because
\[
\|\hat{\Sigma}_D\|_{op} - \|\Sigma\|_{op} \leq \|\hat{\Sigma}_D - \Sigma\|_{op},
\]

\[ \| \hat{\Sigma}_D \|_F - \| \Sigma \|_F \leq \| \hat{\Sigma}_D - \Sigma \|_F \leq \sqrt{p} \| \hat{\Sigma}_D - \Sigma \|_{op}. \]

Since there is only one changepoint, there exists an event of probability at least 1 - \( \epsilon/8 \) on which at least two blocks among \( D_1, D_2, D_3 \) satisfy (32), and an event of probability at least 1 - \( \epsilon/8 \) on which at least two blocks satisfy (33). The desired conclusion therefore follows on taking \( C = 4 \log(32/\epsilon) + \bar{C}(\epsilon^{1/2} \lor 1) \log(16/\epsilon). \]

**Proof of Corollary 9.** Define a set of good events
\[
G := \left\{ \left| \text{Tr}(\hat{\Sigma})^{(2)} - \text{Tr}(\Sigma) \right| \leq (\| \Sigma \|_F + \| \Sigma \|_{op}) / 4, \right.
\left. \| \hat{\Sigma} \|_F^{(2)} - \| \Sigma \|_F \leq \| \Sigma \|_F / 4, \right. \left. \| \hat{\Sigma} \|_{op}^{(2)} - \| \Sigma \|_{op} \leq \| \Sigma \|_{op} / 4 \right\}.
\]

As a direct application of Proposition 8, given \( \epsilon > 0 \), there exists \( c > 0 \), depending only on \( A \) and \( \epsilon \), such that \( P_{\theta}(G^c) \leq \epsilon/4 \) for any \( \theta \in \Theta(p,n,p,0) \). Hence, for \( \theta \in \Theta_0(p,n) \), when \( C \geq 1 \), we have
\[
E_{\theta} \psi_{\text{cov}} \leq P_{\theta} \left( \left\{ \max_{t \in T} \| Y_t \|^2 - \text{Tr}(\hat{\Sigma})^{(2)} > C \left( \| \hat{\Sigma} \|_F \sqrt{\log \log(8n)} + \| \hat{\Sigma} \|_{op} \log \log(8n) \right) \right\} \cap G \right) + P_{\theta}(G^c)
\leq P_{\theta} \left( \max_{t \in T} \| Y_t \|^2 - \text{Tr}(\Sigma) > \frac{C}{2} \left( \| \Sigma \|_F \sqrt{\log \log(8n)} + \| \Sigma \|_{op} \log \log(8n) \right) \right) + \epsilon/4.
\]

Therefore, by Theorem 7, we can choose \( C = C(\epsilon) \geq 1 \) large enough that the error under the null is at most \( \epsilon/2 \). A very similar argument also applies to \( E_{\theta}(1 - \psi_{\text{cov}}) \) for \( \theta \in \Theta(p,n,p,\rho) \) with \( \rho > 0 \): when \( \rho^2 \geq 64C \left( \| \Sigma \|_F \sqrt{\log \log(8n)} \lor \| \Sigma \|_{op} \log \log(8n) \right) \) and after increasing \( C = C(\epsilon) \) if necessary, the error under the alternative is at most \( \epsilon/2 \). This concludes the proof.

**Proof of Theorem 10.** Recalling the representation of \( Y_t(j) \) in (14), we define an oracle version of \( \tilde{Y}_t \) in (15) by
\[
\hat{Y}_t := \frac{Y_t - \sqrt{\gamma} W_t 1_p}{\sqrt{1 - \gamma}}.
\]
Then
\[
\| \hat{Y}_t - Y_t \|_\infty = \frac{|\text{Median}(Y_t) - \sqrt{\gamma} W_t|}{\sqrt{1 - \gamma}}. \tag{34}
\]
By Lemma 22, there exist universal constants \( C_1, C_2, C_3 > 0 \) such that for any \( \theta \in \Theta(p,n,s,0) \), we have
\[
P_{\theta} \left\{ \frac{|\text{Median}(Y_t) - \sqrt{\gamma} W_t|}{\sqrt{1 - \gamma}} > C_1 \left( \frac{s}{p} + \sqrt{\frac{1 + x}{p}} \right) \right\} \leq e^{-C_2 x},
\]
as long as \( C_3 \left( \frac{s}{p} + \sqrt{\frac{1 + x}{p}} \right) \leq 1 \). Using (34) and a union bound argument, we have
\[
\max_{t \in T} \| \hat{Y}_t - Y_t \|_\infty \leq C_4 \sqrt{\frac{\log \log(8n)}{p}}, \tag{35}
\]
22
for some universal constant $C_4 > 0$, with $\mathbb{P}_\theta$-probability at least $1 - 1/\log^2(en)$ for any $\theta \in \Theta(p, n, s, 0)$ under the conditions $s \leq (p \log \log(8n))^{1/5}$ and $\frac{\log \log(8n)}{p} \leq c$. From now on, the event that (35) holds is denoted by $G$.

With the above preparations, we can analyze $E_\theta \psi$ for any $\theta \in \Theta_0(p, n)$. Recalling the definition of $g_\theta(\cdot)$ in (16), we set $C_4'$ in (16) to be $C_4$ in (35). Then, on the event $G$, we have $g_\theta(\tilde{Y}_t(j)) \leq f_\theta(\bar{Y}_t(j))$ for $j \in [p]$, and therefore given $\epsilon > 0$ we can choose $C = C(\epsilon) > 0$ in the definition of $r$ and $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$E_\theta \psi_{a,r,C} \leq E_\theta(1 - \psi_{a,r,C}) + \mathbb{P}_\theta(G^c) \leq \mathbb{P}_\theta\left(\max_{t \in T} \sum_{j=1}^{p} f_\theta(\tilde{Y}_t(j)) > r\right) + \frac{1}{\log^2(en)} \leq \frac{\epsilon}{2},$$

for $n \geq n_0$, where the last inequality is by the same argument as in (22) in the proof of Proposition 2.

Now we analyze $E_\theta(1 - \psi_{a,r,c'})$ for $\theta \in \Theta(p, n, s, \rho)$. Recall from the proof of Proposition 2 that given any $\theta \in \Theta(p, n, s, \rho)$, we may assume there exists $t_0 \in n/2$ such that $t_0 \Vert \mu_1 - \mu_2 \Vert^2 \geq \rho^2$; moreover, there exists a unique $\tilde{t} \in T$ such that $t_0/2 < \tilde{t} \leq t_0$, and

$$\Vert \Delta \tilde{t} \Vert^2 = \frac{t_0 \Vert \mu_1 - \mu_2 \Vert^2}{2} \geq \frac{t_0 \Vert \mu_1 - \mu_2 \Vert^2}{4} \geq \frac{\rho^2}{4}.$$

We introduce a function

$$h_\theta(x) := \inf \left\{ f_\theta(y) : |x - y| \leq \frac{a}{10} \right\} = \begin{cases} 0, & |x| \leq \frac{9}{10} a, \\ a^2 - \nu_a, & \frac{9}{10} a < |x| \leq \frac{11}{10} a, \\ (|x| - \frac{a}{10})^2 - \nu_a, & |x| > \frac{11}{10} a. \end{cases}$$

To gain some intuition, a plot of the functions $h_1(\cdot)$ and $f_1(\cdot)$ are shown in Figure 1. By reducing
c > 0 if necessary, we may assume that $2C' \sqrt{\frac{\log \log(64a)}{p}} \leq \frac{a}{10}$, so we have on the event $G$ that $g_a(\tilde{Y}_t(j)) \geq h_a(\tilde{Y}_t(j))$ for $j \in [p]$. Thus

$$
\mathbb{E}g(1 - \psi_{a,r,C'}) \leq \mathbb{E}_\theta \{ (1 - \psi_{a,r,C'}) \mathbf{1}_G \} + \mathbb{P}_\theta(G^c) \leq \mathbb{P}_\theta \left( \sum_{j=1}^{p} h_a(\tilde{Y}_t(j)) \leq r \right) + \frac{1}{\log^2(en)},
$$

and we now control the first term on the right-hand side. When $\Delta_t(j) = 0$, we have $\mathbb{E}h_a(\tilde{Y}_t(j)) \leq \mathbb{E}f_a(\tilde{Y}_t(j)) = 0$. Moreover, by Lemma 16,

$$
-\mathbb{E}h_a(\tilde{Y}_t(j)) = 2(\nu_a - a^2)\{ \Phi(11a/10) - \Phi(9a/10) \} + 2 \int_{11a/10}^{\infty} \{ \nu_a - (x - a/10) \} \phi(x) \, dx
$$

$$
\leq \frac{4a^3}{5} \phi(9a/10) + 6a^2\{ 1 - \Phi(11a/10) \} \leq e^{-a^2/3}.
$$

Next, for $0 < |\Delta_t(j)| < 8(1 - \gamma)^{1/2}a$, we have $\mathbb{E}h_a(\tilde{Y}_t(j)) \geq - (\nu_a - a^2) \geq -2a^2$, and by Lemma 18, we have $\mathbb{E}h_a(\tilde{Y}_t(j)) \leq \mathbb{E}f_a(\tilde{Y}_t(j)) \leq \frac{64a^2(1)}{1-\gamma} + 1 \lesssim a^2$.

Finally, we handle the case where $|\Delta_t(j)| \geq 8(1 - \gamma)^{1/2}a$, and assume without loss of generality that $\Delta_t(j) \geq 8(1 - \gamma)^{1/2}a$. Observe by Lemma 16 that for $x \geq 4a$, we have

$$(x - a/10)^2 - \nu_a \geq x^2 - \frac{ax}{5} - 3a^2 \geq x^2 - \frac{x^2}{20} - \frac{3x^2}{16} \geq 3x^2/4.$$

Hence

$$
\mathbb{E}h_a(\tilde{Y}_t(j)) \geq \frac{3}{4} \mathbb{E}(\tilde{Y}_t(j)^2 \mathbf{1}_{\{ \tilde{Y}_t(j) \geq 4a \}}) - (\nu_a - a^2)\mathbb{P}(\tilde{Y}_t(j) < 4a)
$$

$$
\geq \frac{3\Delta_t^2(j)}{4(1 - \gamma)} \mathbb{P}(\tilde{Y}_t(j) \geq 4a) - 3a^2\mathbb{P}(\tilde{Y}_t(j) < 4a) \geq \frac{45\Delta_t^2(j)}{128(1 - \gamma)}.
$$

Summarising then, we have

$$
\mathbb{E}h_a(\tilde{Y}_t(j)) \begin{cases} 
\leq 0 \text{ and } \geq -e^{-a^2/3} \text{ if } \Delta_t(j) = 0, \\
\geq -a^2 \text{ and } \lesssim a^2 \text{ if } 0 < |\Delta_t(j)| < 8(1 - \gamma)^{1/2}a, \\
\geq \frac{45\Delta_t^2(j)}{128(1 - \gamma)} \text{ if } |\Delta_t(j)| \geq 8(1 - \gamma)^{1/2}a.
\end{cases}
$$

We now study $\text{Var} \ h_a(\tilde{Y}_t(j))$. When $\Delta_t(j) = 0$, we have

$$
\text{Var} \ h_a(\tilde{Y}_t(j)) \leq \mathbb{E}h_a^2(\tilde{Y}_t(j)) \leq 2 \int_{9a/10}^{\infty} \{ (\nu_a - a^2)^2 \vee ((x - a/10)^2 - \nu_a)^2 \} \phi(x) \, dx
$$

$$
\leq 2 \int_{9a/10}^{\infty} (\nu_a^2 \vee x^2) \phi(x) \, dx \lesssim e^{-a^2/4}.
$$

When $0 < |\Delta_t(j)| < 2(1 - \gamma)^{1/2}a$, assuming that $\Delta_t(j) > 0$ without loss of generality and writing
\( \theta := \Delta_t(j)/(1 - \gamma)^{1/2} \) as shorthand, we have

\[
\mathbb{V} \text{ar } h_a(\tilde{Y}_t(j)) \leq \mathbb{E} h_a^2(\tilde{Y}_t(j)) \\
\leq \left( \int_{-\infty}^{-9a/10} + \int_{9a/10}^{\theta + a} + \int_{\theta + a}^{\infty} \right) \{ (\nu_a - a^2)^2 \vee ((|x| - a/10)^2 - \nu_a) \} \phi(x - \theta) \, dx \\
\lesssim e^{-a^2/4} + a^4 + e^{-a^2/4} \lesssim a^4.
\]

Finally, when \( |\Delta_t(j)| \geq 2(1 - \gamma)^{1/2}a \), Let us define a random variable \( L := 1_{\{\tilde{Y}_t(j) \geq 11a/10\}} \). Then assuming that \( \Delta_t(j) \geq 2(1 - \gamma)^{1/2}a \) without loss of generality, we have that

\[
\mathbb{V} \text{ar } h_a(\tilde{Y}_t(j)) = \mathbb{E} \{ \mathbb{V} \text{ar } h_a(\tilde{Y}_t(j) \mid L) \} + \mathbb{V} \text{ar } \{ \mathbb{E} h_a(\tilde{Y}_t(j) \mid L) \}
\leq \mathbb{P}(L = 0) \mathbb{E} \{ h_a^2(\tilde{Y}_t(j)) \mid L = 0 \} + \mathbb{P}(L = 1) \mathbb{V} \text{ar } \{ (\tilde{Y}_t(j) - a/10)^2 \mid L = 1 \}
\]

Now, similar to the proof of Lemma 19,

\[
|\mathbb{E} \{ h_a(\tilde{Y}_t(j)) \mid L = 0 \}| \leq \nu_a + \mathbb{E} \{ \tilde{Y}_t^2(j) \mid \tilde{Y}_t(j) < 11a/10 \} \lesssim \frac{\Delta_t^2(j)}{1 - \gamma},
\]

\[
\mathbb{E} \{ h_a^2(\tilde{Y}_t(j)) \mid L = 0 \} \leq 2\nu_a^2 + 2\mathbb{E} \{ \tilde{Y}_t^4(j) \mid \tilde{Y}_t(j) < 11a/10 \} \lesssim \frac{\Delta_t^4(j)}{(1 - \gamma)^2},
\]

\[
|\mathbb{E}((\tilde{Y}_t(j) - a/10)^2 - \nu_a \mid L = 1)| \leq \nu_a + \mathbb{E} \{ \tilde{Y}_t^2(j) \mid \tilde{Y}_t(j) \geq 11a/10 \} \lesssim \frac{\Delta_t^2(j)}{1 - \gamma}.
\]

But \( \mathbb{P}(L = 0) = \Phi(11a/10 - \Delta_t(j)/(1 - \gamma)^{1/2}) \leq \Phi(-\frac{9\Delta_t(j)}{20(1 - \gamma)^{1/2}}) \). Finally, we note that

\[
\mathbb{P}(L = 1) \mathbb{V} \text{ar } \{ (\tilde{Y}_t(j) - a/10)^2 \mid L = 1 \} \leq \mathbb{V} \text{ar } \{ (\tilde{Y}_t(j) - a/10)^2 \} \lesssim \frac{\Delta_t^2(j)}{1 - \gamma}.
\]

These observations allow us to deduce that

\[
\mathbb{V} \text{ar } h_a(\tilde{Y}_t(j)) \lesssim \begin{cases} 
  e^{-a^2/4} & \text{if } \Delta_t(j) = 0, \\
  a^4 & \text{if } 0 < |\Delta_t(j)| < 2(1 - \gamma)^{1/2}a, \\
  \frac{\Delta_t^2(j)}{1 - \gamma} & \text{if } |\Delta_t(j)| \geq 2(1 - \gamma)^{1/2}a.
\end{cases}
\]

The bound on the expectation then implies that

\[
\sum_{j: \Delta_t(j) = 0} \left| \mathbb{E} h_a(\tilde{Y}_t(j)) \right| \lesssim p e^{-a^2/3} \lesssim p \log \log(8n) \left( \frac{s^2}{p \log \log(8n)} \right)^{4/3} \leq s,
\]

where we used the condition \( s \leq (p \log \log(8n))^{1/5} \). We deduce similarly to the argument in the
proof of Proposition 2 that
\[
\sum_{j=1}^{p} \mathbb{E} h_a(\bar{Y}_i(j)) \geq \frac{\|\Delta_i\|^2}{4(1 - \gamma)}
\]
provided we choose \( C = C(\epsilon) > 0 \) sufficiently large in the definition of \( \rho \). Moreover,
\[
\sum_{j=1}^{p} \text{Var}(h_a(\bar{Y}_i(j))) \lesssim pe^{-a^2/4} + sa^4 + \frac{\|\Delta_i\|^2}{1 - \gamma}.
\]
By Chebychev’s inequality, we deduce that
\[
\mathbb{P}_\theta\left( \sum_{j=1}^{p} h_a(\bar{Y}_i(j)) \leq r \right) \leq \mathbb{P}_\theta\left( \sum_{j=1}^{p} h_a(\bar{Y}_i(j)) \leq \frac{\|\Delta_i\|^2}{8} \right) \lesssim \frac{pe^{-a^2/4} + sa^4 + \frac{\|\Delta_i\|^2}{1 - \gamma}}{\{\|\Delta_i\|^2/(1 - \gamma)\}^2} \lesssim \frac{pe^{-a^2/4} + sa^4 + \frac{\rho^2}{(1 - \gamma)}}{\rho^2/(1 - \gamma)}.
\]
Hence, by increasing \( n_0 = n_0(\epsilon) \) and \( C = C(\epsilon) > 0 \) if necessary, we may conclude that \( \mathbb{E}_\theta(1 - \psi_{a,r,C'}) \leq \epsilon/2 \), as required.

Proof of Theorem 11. The proof uses similar arguments to those in the proof of Proposition 3, but instead of establishing (24), we need to show that given \( \eta > 0 \), we can find a universal constant \( c > 0 \) such that \( \mathbb{E}_{(\theta_1, \theta_2) \sim \nu \otimes \nu} \exp(\langle \theta_1, \theta_2 \rangle_{\Sigma(\cdot)} \cdot 1) \leq 1 + \eta \) when \( \rho \geq cr_{\Sigma(\cdot)}^* \), where \( r_{\Sigma(\cdot)}^* \) denotes the right-hand side of (18). Since
\[
\Sigma(\cdot)^{-1} = \kappa_1(\gamma) I_p - \kappa_2(\gamma) 1_p 1_p^T,
\]
with \( \kappa_1(\gamma) = \frac{1}{1 - \gamma} \) and \( \kappa_2(\gamma) = \frac{\gamma}{(1 - \gamma)(1 + (p - 1)\gamma)} \), the calculation will be very similar, and essentially our argument replaces \( I_p \) in the proof of Proposition 3 by \( \kappa_1(\gamma) I_p \).

We first consider the case when \( s \leq \sqrt{p \log \log(8n)} \) and \( s \log \left( \frac{ep \log \log(8n)}{s^2} \right) \geq \log \log(8n) \). We define \( \nu \) to be the distribution of \( \theta \), sampled according to the following process:

1. Uniformly sample a subset \( S \) of \( [p] \) of cardinality \( s \);
2. Independently, sample \( k \) according to a uniform distribution on \( \{0, 1, 2, \ldots, \lfloor \log_2(n/2) \rfloor \} \);
3. Given \( (S, k) \) sampled in the previous steps, define \( \theta_{j\ell} := \beta/2^{k/2} \) for all \( (j, \ell) \in S \times [k] \) and \( \theta_{j\ell} := 0 \) otherwise, where \( \beta > 0 \).

Suppose that we generate \( \theta_1 \) and \( \theta_2 \) independently with distribution \( \nu \), where \( \theta_1 \) is generated from \( (S, k) \) and \( \theta_2 \) comes from \( (T, l) \). By (36), we have \( \langle \theta_1, \theta_2 \rangle_{\Sigma(\cdot)} \cdot 1 \leq \frac{\kappa_1(\gamma) \beta^2}{2^{l - k}/2} |S \cap T| \), and thus
\[
\mathbb{E}_{(\theta_1, \theta_2) \sim \nu \otimes \nu} \exp(\langle \theta_1, \theta_2 \rangle_{\Sigma(\cdot)} \cdot 1) \leq \mathbb{E} \exp \left( \frac{\kappa_1(\gamma) \beta^2}{2^{l - k}/2} |S \cap T| \right).
\]

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Note that we obtain the same formula as (29) except that the $\beta^2$ in (29) is replaced by $\kappa_1(\gamma)\beta^2$. This immediately implies that the same argument that bounds (29) can also be applied here and we obtain the lower bound with the desired rate $\kappa_1(\gamma)^{-1}s \log \left(\frac{ep \log \log(8n)}{s^2}\right)$.

Next we consider the case $s \leq \sqrt{p \log \log(8n)}$ and $s \log \left(\frac{ep \log \log(8n)}{s^2}\right) \geq \log \log(8n)$. The sampling process for $\theta \sim \nu$ is now:

1. Sample $k$ according to a uniform distribution on $\{0, 1, 2, \ldots, \lfloor \log_2(n/2) \rfloor\}$;

2. Given $k$, define $\theta_{j\ell} := \beta^2k^{1/2}$ for all $(j, \ell) \in [s] \times [2^k]$ and $\theta_{j\ell} := 0$ otherwise.

Similarly to before, $\langle \theta_1, \theta_2 \rangle_{\Sigma(\gamma)} \leq \frac{\kappa_1(\gamma)\beta^2}{2^\|k\|/2}s$, and thus

$$
\mathbb{E}(\theta_1, \theta_2)_{\sim \nu \otimes \nu} \exp(\langle \theta_1, \theta_2 \rangle_{\Sigma(\gamma)} - 1) \leq \mathbb{E} \exp \left( \frac{\kappa_1(\gamma)\beta^2}{2^\|k\|/2}s \right).
$$

Then, set $\kappa_1(\gamma)\beta^2s = c_1 \log \log(8n)$ for some sufficiently small $c_1 > 0$, and we can follow the exact argument in the proof of Proposition 4.2 in Gao et al. (2019). This leads to the desired lower bound with rate $\rho^2 = s\beta^2 \asymp \kappa_1(\gamma)^{-1}\log \log(8n)$.

5.3 Proofs of Results in Section 4

Proof of Theorem 12. For $t \in \lfloor[n/2]\rfloor$, define $\Gamma_t := \text{Cov}(E_1 + \ldots + E_t) - (E_{n-t+1} + \ldots + E_n)$.

Then

$$
\|Y_t\|^2 \leq \|\Gamma_t\|_{op}\|\Gamma_t^{-1/2}Y_t\|^2 \leq 2t(1 + B)\|\Gamma_t^{-1/2}Y_t\|^2.
$$

Now fix $\theta \in \Theta_0(p, n)$. Given $\epsilon \in (0, 1)$, set $C := C(\epsilon) = 4 + 4\log(4/\epsilon)$. Since $2t\|\Gamma_t^{-1/2}Y_t\|^2 \sim \chi_p^2$, by a union bound and Lemma 14, given $\epsilon > 0$, writing $x = \lfloor 1 + \log(4/\epsilon) \rfloor \log \log(8n)$, we have

$$
\mathbb{E}_{\theta, \Sigma} \psi_{\text{temp}} \leq \mathbb{P}_{\theta, \Sigma} \left( \max_{t \in T} \|Y_t\|^2 - p > Bp + (1 + B)(2\sqrt{\bar{\Sigma}} + 2x) \right)
$$

$$
= \mathbb{P}_{\theta, \Sigma} \left( \max_{t \in T} \|Y_t\|^2 > (1 + B)(p + 2\sqrt{\bar{\Sigma}} + 2x) \right)
$$

$$
\leq |T| \mathbb{P}(\chi_p^2 > p + 2\sqrt{\bar{\Sigma}} + 2x) \leq 2 \log(en) e^{-x} \leq \frac{\epsilon}{2}.
$$

Now, for any $\theta \in \Theta(p, n, s, \rho)$, without loss of generality, we may assume there exists $t_0 \in \lfloor n/2 \rfloor$, such that $t_0\|\mu_1 - \mu_2\|^2 \geq \rho^2$, and a unique $\tilde{t} \in T$ such that $\tilde{t}/2 < \tilde{t} \leq t_0$. Thus $2\tilde{t}\|\Gamma_\tilde{t}^{-1/2}Y_\tilde{t}\|^2 \sim \chi_{\rho, \delta^2}^2$, with

$$
\delta^2 = \frac{t_0\|\mu_1 - \mu_2\|^2}{2} \geq \frac{t_0\|\mu_1 - \mu_2\|^2}{4} \geq \frac{\rho^2}{4}.
$$

Therefore,

$$
\mathbb{E}_{\theta, \Sigma}(\|Y_\tilde{t}\|^2) - p = \delta^2 + \mathbb{E}_{\theta, \Sigma} \left( \left\|\left( E_1 + \ldots + E_{\tilde{t}} - (E_{n-\tilde{t}+1} + \ldots + E_n) \right) / \sqrt{2t} \right\|^2 \right) - p
$$

$$
= \delta^2 + \frac{1}{2t} \text{Tr}(\Gamma_\tilde{t} - 2\tilde{t}I_p) \geq \delta^2 - \frac{p}{2t}\|\Gamma_\tilde{t} - 2\tilde{t}I_p\|_{op} \geq \delta^2 - Bp,
$$

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where the last inequality is by expanding $\Gamma_t$ according to its definition and the condition that $\sum_{j \in [n] \setminus \{i\}} \|\text{Cov}(E_i, E_j)\|_{op} \leq B$ for all $i \in [n]$. Since $C \geq 1/4$, we have $\rho^\ast \geq 8Bp$, and then

$$\mathbb{E}_{\theta, \Sigma}(\|Y_{\hat{t}}\|^2) - p \geq \frac{\delta^2}{2}.$$  

Now write $W_t := Y_t - \mathbb{E}_{\theta, \Sigma}Y_t$, so that $W_t \sim N_p(0, \Gamma_t/(2t))$, and find an orthogonal matrix $Q_t \in \mathbb{R}^{p \times p}$ such that $Q_t^\top\Gamma_tQ_t = D_t$, where $D_t$ is diagonal. Then, with $Z_t \sim N_p(0, I_p)$, we have

$$\text{Var}(\|Y_{\hat{t}}\|^2 - p) = \text{Var}(\|W_{\hat{t}}\|^2 + 2W_{\hat{t}}^\top\mathbb{E}_{\theta, \Sigma}(Y_{\hat{t}}^2)) \leq 2\text{Var}(\|W_{\hat{t}}\|^2) + 4\mathbb{E}_{\theta, \Sigma}(Y_{\hat{t}}^2)\|\Gamma_t\|_{op} = \frac{\|\Gamma_t\|_{op}^2}{t^2} + \frac{4}\|\Gamma_t\|_{op} \leq 4p(1 + B)^2 + 8\delta^2(1 + B).$$

Using Chebychev’s inequality, we therefore have have

$$\mathbb{E}_{\theta, \Sigma}(1 - \psi_{\text{Temp}}) = \mathbb{P}_{\theta, \Sigma}\left(\max_{t \in T} \|Y_t\|^2 - p \leq r \right) \leq \mathbb{P}_{\theta, \Sigma}\left(\|Y_{\hat{t}}\|^2 - p \leq \frac{\rho^2}{32}\right) \leq \mathbb{P}_{\theta, \Sigma}\left(\|Y_{\hat{t}}\|^2 - p \leq \frac{\delta^2}{8}\right) \leq \frac{\rho^4}{2^8p(1 + B)^2 + 2^9\delta^2(1 + B)}.$$  

and we can ensure this final term is bounded above by $\epsilon/2$ by increasing $C = C(\epsilon)$ so that $C \geq 28/\epsilon^{1/2}$.

\textbf{Proof of Theorem 13.} It suffices to prove the result with $\rho^\ast$ replaced with $\rho_1^\ast \vee \rho_2^\ast$, where $\rho_1^\ast := (Bp)^{1/2}$ and $\rho_2^\ast := \sqrt{p}\log\log(8n) \vee \log\log(8n))^{1/2}$. For the lower bound $\rho_1^\ast$, fixing $a \in (0, 1]$, we define a covariance matrix $\Sigma_0 \in \mathbb{R}^{pn \times pn}$, specified by the following three conditions:

1. $\text{Cov}(E_t) = I_p$ for all $t \in [n]$;
2. $\text{Cov}(E_s, E_t) = aI_p$ for all $1 \leq s \neq t \leq n/2$;
3. $\text{Cov}(E_s, E_t) = 0$ for the remaining pairs $s \neq t$.

A sufficient condition for $\Sigma_0 \in C(p, n, B)$ is $na/2 \leq B$. We also define a covariance matrix $\Sigma_1 \in \mathbb{R}^{pn \times pn}$, specified by the following three conditions:

1. $\text{Cov}(E_t) = (a + 1)I_p$ for all $1 \leq t \leq n/2$ and $\text{Cov}(E_t) = I_p$ for all $n/2 < t \leq n$;
2. $\text{Cov}(E_s, E_t) = aI_p$ for all $1 \leq s \neq t \leq n/2$;
3. $\text{Cov}(E_s, E_t) = 0$ for the remaining pairs $s \neq t$.

Let $Z \sim N_p(0, aI_p)$, and let $Q$ denote the conditional distribution of $Z$ given that $\|Z\|^2 \geq pa/2$. In other words,

$$Q(V) = \frac{\int_{\{\mu \in V : \|\mu\|^2 \geq pa/2\}} dN_p(0, aI_p)}{\int_{\{\mu : \|\mu\|^2 \geq pa/2\}} dN_p(0, aI_p)}.$$  

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for any Borel measurable $V \subseteq \mathbb{R}^p$. We then define $\nu$ to be the distribution of the random $p \times n$ matrix $\theta$ that is generated according to the following sampling process:

1. Sample $\mu \sim Q$;
2. Let $\theta_t = \mu$ for all $1 \leq t \leq n/2$ and $\theta_t = 0$ for all $n/2 < t \leq n$.

Then $\text{supp}(\nu) \subseteq \Theta(p, n, p, \rho)$ with $\rho^2 = n p a / 4$. We also define another distribution $\tilde{\nu}$ to be the distribution of $\theta$ when it is generated as follows:

1. Sample $\mu \sim N_p(0, a I_p)$;
2. Let $\theta_t = \mu$ for all $1 \leq t \leq n/2$ and $\theta_t = 0$ for all $n/2 < t \leq n$.

To lower bound $\mathcal{R}(\rho)$, we need to specify several distributions. We define

$$P_0 := P_{0, \Sigma_0}, \quad P_1 := \int_{\text{supp}(\nu)} P_{\theta, I_p} d\nu(\theta).$$

To bridge the relation between $P_0$ and $P_1$, we define

$$\tilde{P}_1 := \int_{\text{supp}(\tilde{\nu})} P_{\theta, I_p} d\tilde{\nu}(\theta).$$

We claim that $\tilde{P}_1 = P_{0, \Sigma_1}$. To see this, first note that if $X \sim \tilde{P}_1$, then $X \overset{d}{=} \theta + E$, where $\theta \sim \tilde{\nu}$, $E$ has independent $N(0, 1)$ entries and $\theta$ and $E$ are independent. Since $\theta$ is a linear transformation of the Gaussian vector $\mu$, we deduce that $X$ is Gaussian. Moreover, $\mathbb{E}_1(X) = 0$ and

$$\text{Cov}(X_s, X_t) = \text{Cov} \left( \mathbb{E}_1(X_s|\mu), \mathbb{E}_1(X_t|\mu) \right) + \mathbb{E}_1 \left\{ \text{Cov}(X_s, X_t|\mu) \right\}$$

$$= \text{Cov}(\theta_s, \theta_t) + \mathbb{E}_1 \left\{ \text{Cov}(X_s, X_t|\mu) \right\}$$

$$= \begin{cases} a I_p 1_{1 \leq s, t \leq n/2} + I_p & \text{if } s = t \\ a I_p 1_{1 \leq s, t \leq n/2} & \text{if } s \neq t. \end{cases}$$

In other words, $\text{Cov}(X) = \Sigma_1$, which establishes our claim. Hence

$$\mathcal{R}(\rho) \geq \inf_{\psi \in \Psi} \{ E_0 \psi + E_1 (1 - \psi) \} = 1 - \text{TV}(P_0, P_1)$$

$$\geq 1 - \text{TV}(P_0, \tilde{P}_1) - \text{TV}(P_1, \tilde{P}_1).$$

Now,

$$\text{TV}(P_0||\tilde{P}_1) = \text{TV}(P_{0, \Sigma_0}||P_{0, \Sigma_1}) \leq \frac{3}{2} \| \Sigma_1^{-1} \Sigma_0 - I_{np} \|_F \leq \frac{3}{2} \| \Sigma_0 - \Sigma_1 \|_F \leq \frac{3}{2} \sqrt{\frac{npa^2}{2}},$$

where the first inequality is by Devroye et al. (2018, Theorem 1.1) and the second inequality is by the fact that the smallest eigenvalue of $\Sigma_1$ is 1.
For the second term, by the data processing inequality (Ali and Silvey, 1966; Zakai and Ziv, 1975), we obtain

\[ \text{TV}(\mathbb{P}_1, \mathbb{P}_2) \leq 2 \int_{\{\mu \mid \mu^2 < p/2\}} dN_p(0, aI_p) = 2p \left( \chi_p^2 < \frac{p}{2} \right) \leq 2e^{-p/16}, \]

where the final inequality follows from Lemma 14. Given \( \epsilon > 0 \), we can therefore let \( a = a^*_B/n \) with \( a^*_B = \sqrt{2\epsilon/(3D)} \), which amounts to choosing \( c_{\epsilon,D} = a^*_B \) and \( p = 16 \log(4/\epsilon) \) to obtain the lower bound \( \rho_2 \).

The lower bound \( \rho_2^2 \) is relatively easier. Without loss of generality, we assume \( n/[B] \) to be an integer. We then divide the set \([n]\) into consecutive blocks \( J_1, J_2, \ldots, J_{n/[B]} \), each of cardinality \([B] \). We define a covariance matrix \( \overline{\Sigma} \in \mathbb{R}^{pn \times pn} \) according to the following two conditions:

1. \( \text{Cov}(E_t) = I_p \) for all \( t \in [n] \);
2. \( \text{Cov}(E_s, E_t) = I_p \) for all \( s \neq t \) in the same block, and otherwise \( \text{Cov}(E_s, E_t) = 0 \).

Since \([B] - 1 \leq B\), we have \( \overline{\Sigma} \in \mathcal{C}(p, n, B) \). Define

\[ \Theta(p, n, p, \rho) := \bigcup_{\ell=0}^{n/[B]} \Theta(\ell[B]) \subseteq \Theta(p, n, p, \rho). \]

Then

\[ \mathcal{R}(\rho) \geq \inf_{\psi \in \Psi} \left( \sup_{\theta \in \Theta(p, n, p, \rho)} \mathbb{E}_{\theta, \Sigma}(\psi) + \sup_{\theta \in \Theta(p, n, p, \rho)} \mathbb{E}_{\theta, \Sigma}(1 - \psi) \right) \]

\[ = \inf_{\psi \in \Psi} \left( \sup_{\theta \in \Theta(p, n, p, [B])} \mathbb{E}_{\theta, t_{\rho[n/[B]]}}(\psi) + \sup_{\theta \in \Theta(p, n, p, [B]/\sqrt{B})} \mathbb{E}_{\theta, t_{\rho[n/[B]]}}(1 - \psi) \right). \]

In other words, we have constructed a covariance structure which leads to a simpler problem with \( n/[B] \) independent observations and signal strength \( \rho^2/[B] \). By Proposition 3, this simpler problem has lower bound \( \rho^2/[B] \geq \sqrt{p \log \log(8n/[B]) \vee \log \log(8n/[B])} \), which is equivalent to the rate \([B] \left( \sqrt{p \log \log(8n/[B]) \vee \log \log(8n/[B])} \right) \) for the original problem. Under the condition \( B \leq D\sqrt{n/p} \), the result follows.

**6 Technical lemmas**

We first state some lemmas that will be used in the proof of Proposition 2, beginning with some standard chi-squared tail bounds.

**Lemma 14** (Lemma 1 of Laurent and Massart (2000)). Let \( Z_1, \ldots, Z_p \overset{iid}{\sim} N(0, 1) \) and let \( \lambda_1 \geq \)
\[ \lambda_2 \geq \cdots \geq \lambda_p \geq 0. \] Then, for any \( x > 0 \), we have
\[
\mathbb{P} \left( \sum_{j=1}^{p} \lambda_j Z_j^2 \geq \sum_{j=1}^{p} \lambda_j + 2 \sqrt{x \sum_{j=1}^{p} \lambda_j^2 + 2 \lambda_1 x} \right) \leq e^{-x},
\]
and
\[
\mathbb{P} \left( \sum_{j=1}^{p} \lambda_j Z_j^2 \leq \sum_{j=1}^{p} \lambda_j - 2 \sqrt{x \sum_{j=1}^{p} \lambda_j^2} \right) \leq e^{-x}.
\]

**Lemma 15.** Let \( X_1, \ldots, X_n \overset{iid}{\sim} N_p(\mu, \Sigma) \), and let
\[
\hat{\Sigma} := \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})(X_i - \bar{X})^T,
\]
where \( \bar{X} := n^{-1} \sum_{i=1}^{n} X_i \). Then for any \( x > 0 \),
\[
\mathbb{P} \left( |\text{Tr}(\hat{\Sigma}) - \text{Tr}(\Sigma)| \geq 2 \|\Sigma\|_F \sqrt{\frac{x}{n-1}} + 2 \|\Sigma\|_{op} \frac{x}{n-1} \right) \leq 2e^{-x}.
\]

**Proof.** After an orthogonal transformation, we may assume without loss of generality that \( \Sigma \) is diagonal, with non-negative diagonal entries \( \lambda_1, \ldots, \lambda_p \), say. Then
\[
\text{Tr}(\hat{\Sigma}) = \frac{1}{n-1} \sum_{j=1}^{p} \sum_{i=1}^{n} \{X_i(j) - \bar{X}(j)\}^2.
\]
Since for \( \lambda_j \neq 0 \), we have \( \sum_{i=1}^{n} \{X_i(j) - \bar{X}(j)\}^2 / \lambda_j \sim \chi^2_{n-1} \), independently for \( j \in [p] \), we have
\[
\text{Tr}(\hat{\Sigma}) \overset{d}{=} \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{j=1}^{p} \lambda_j Z_{ij}^2,
\]
where \( \{Z_{ij}\}_{i \in [n-1], j \in [p]} \overset{iid}{\sim} N(0, 1) \). Then Lemma 14 implies the result. \( \square \)

The next four lemmas are properties of the truncated non-central chi-squared distribution. Recall that \( \nu_a = \mathbb{E}(Z^2 \mid |Z| \geq a) \), where \( Z \sim N(0,1) \).

**Lemma 16.** The function \( a \mapsto \nu_a \) is strictly increasing on \( [0, \infty) \), so that \( \nu_0 \leq \nu_1 \leq 3 \) for all \( a \in [0,1] \). Moreover, the function \( a \mapsto \nu_a / a^2 \) is strictly decreasing on \( (0, \infty) \), so that \( \nu_0 / a^2 \leq \nu_1 \leq 3 \) for all \( a \geq 1 \). Finally, writing \( \gamma_a := \mathbb{E}(Z^4 \mid |Z| \geq a) \), where \( Z \sim N(0,1) \), the function \( a \mapsto \gamma_a / a^4 \) is strictly decreasing on \( (0, \infty) \), so that \( \gamma_0 / a^4 \leq \gamma_1 \leq 11 \) for all \( a \geq 1 \).

**Proof.** First note that \( \nu_a = \int_a^{\infty} x^2 \phi(x) \, dx / \Phi(a) \), where \( \phi \) denotes the standard normal density
function, and where $\tilde{\Phi}(a) := \int_a^\infty \phi(x) \, dx$. Hence, for any $a > 0$, we have

$$
\frac{d}{da} \log \nu_a = -\frac{a^2 \phi(a)}{\int_a^\infty x^2 \phi(x) \, dx} + \frac{\phi(a)}{\Phi(a)} = \frac{\phi(a) \int_a^\infty (x^2 - a^2) \phi(x) \, dx}{\Phi(a) \int_a^\infty x^2 \phi(x) \, dx} > 0,
$$

which proves the first claim. For the second claim, let

$$
g(a) := \log \frac{\nu_a}{a^2} = -2 \log a + \log \int_a^\infty x^2 \phi(x) \, dx - \log \tilde{\Phi}(a).
$$

Then

$$
g'(a) = \frac{-2}{a} - \frac{a^2 \phi(a)}{\int_a^\infty x^2 \phi(x) \, dx} + \frac{\phi(a)}{\Phi(a)}
= \frac{-2 \tilde{\Phi}(a) \int_a^\infty x^2 \phi(x) \, dx - a^3 \phi(a) \tilde{\Phi}(a) + a \phi(a) \int_a^\infty x^2 \phi(x) \, dx}{a \Phi(a) \int_a^\infty x^2 \phi(x) \, dx}.
$$

The denominator of this expression is positive, and, after integrating by parts and writing $h(a) := \phi(a)/\tilde{\Phi}(a)$, the numerator is

$$
-ah(a) - 2 - a^3 h(a) + a^2 h(a)^2 \
\leq -2h(a)^2,
$$

where the final inequality uses the standard Mills ratio bound $h(a) \leq a + 1/a$ for $a > 0$ (Gordon, 1941). This proves the second claim. The final claim follows using a very similar argument, and is omitted for brevity.

**Lemma 17.** Let $Z_1, \ldots, Z_p \overset{iid}{\sim} N(0,1)$. Then there exists a universal constant $C^* > 0$ such that for any $a > 0$ and $x > 0$, we have

$$
P\left(\sum_{j=1}^p (Z_j^2 - \nu_a) 1_{\{|Z_j| \geq a\}} \geq C^* \left(\sqrt{pe^{-a^2/4} x + x}\right)\right) \leq e^{-x}.
$$

**Proof.** By a Chernoff bound, we have for any $u, \lambda > 0$ that,

$$
P\left(\sum_{j=1}^p (Z_j^2 - \nu_a) 1_{\{|Z_j| \geq a\}} \geq u\right) \leq e^{-\lambda u} \left(\mathbb{E}e^{\lambda(Z_j^2 - \nu_a) 1_{\{|Z_j| \geq a\}}\right)^p. \quad (37)
$$

Writing $p(x) := (2\pi)^{-1/2} x^{-1/2} e^{-x/2}$ for the density function of $Z_1^2$, we can bound the moment
generating function above as follows:

\[
\mathbb{E} e^{\lambda (Z^2_1 - \nu_a) \mathbb{1}_{\{|Z_1| \geq a\}}} = \int_0^\infty e^{\lambda (x-\nu_a) \mathbb{1}_{\{x \geq a^2\}}} p(x) \, dx
\]

\[= 1 + \int_{a^2}^\infty \left\{ e^{\lambda (x-\nu_a)} - 1 \right\} p(x) \, dx \]

\[= 1 + \int_{a^2}^\infty \sum_{k=2}^\infty \frac{\lambda^k (x-\nu_a)^k}{k!} p(x) \, dx \]

\[\leq 1 + \frac{\lambda^2}{2} \int_{a^2}^{\nu_a} (\nu_a - x)^2 e^{\lambda (x-\nu_a) p(x)} \, dx + \frac{\lambda^2}{2} \int_{\nu_a}^\infty (x-\nu_a)^2 e^{\lambda (x-\nu_a)} p(x) \, dx. \quad (38)\]

The equality (38) follows because \( \int_{a^2}^\infty (x-\nu_a) p(x) \, dx = 0 \), which can be deduced from the fact that \( \nu_a = \int_{a^2}^\infty xp(x) \, dx / \int_{a^2}^\infty p(x) \, dx \). We now analyze the two integrals on the right-hand side of (39) separately. For the first term, we have

\[
\int_{a^2}^{\nu_a} (\nu_a - x)^2 e^{\lambda (x-\nu_a)} p(x) \, dx \leq (\nu_a - a^2)^2 e^{\lambda (\nu_a - a^2)} \int_{a^2}^\infty p(x) \, dx.
\]

If \( a \geq 1 \), then \( a^2 \leq \nu_a \leq 3a^2 \) by Lemma 16. Hence, for \( \lambda \in (0, 1/4) \), we have that \( (\nu_a - a^2)^2 e^{\lambda (\nu_a - a^2)} \int_{a^2}^\infty p(x) \, dx \leq e^{-a^2/4} \). But \( \nu_a \leq 3 \) for \( a \in [0, 1] \). It follows that for \( a \in [0, 1] \), we still have

\[
(\nu_a - a^2)^2 e^{\lambda (\nu_a - a^2)} \int_{a^2}^\infty p(x) \, dx \leq e^{-a^2/4}
\]

for \( \lambda \in (0, 1/4) \). For the second term of (39) and for \( \lambda \in (0, 1/4) \), we have

\[
\int_{\nu_a}^\infty (x-\nu_a)^2 e^{\lambda (x-\nu_a)} p(x) \, dx \leq 2 \int_{\nu_a}^\infty (x^2 + \nu_a^2) e^{(x-\nu_a)/4} (2\pi)^{-1/2} x^{-1/2} e^{-x/2} \, dx
\]

\[= \sqrt{\frac{2}{\pi}} e^{-\nu_a/4} \left( \int_{\nu_a}^\infty x^{3/2} e^{-x/4} \, dx + \nu_a^2 \int_{\nu_a}^\infty x^{-1/2} e^{-x/4} \, dx \right)
\]

\[\leq e^{-\nu_a/4} \nu_a^2 e^{-\nu_a/4} \leq e^{-\nu_a/4}, \quad (40)\]

where the first inequality in (40) follows from Karamata’s theorem (Bingham et al., 1989, Proposition 1.5.10). Combining the bounds for both terms in (39), we have therefore established that for \( \lambda \in (0, 1/4) \),

\[
\mathbb{E} e^{\lambda (Z^2_1 - \nu_a) \mathbb{1}_{\{|Z_1| \geq a\}}} - 1 \leq \lambda^2 (e^{-a^2/4} + e^{-\nu_a/4}) \leq \lambda^2 e^{-a^2/4}.
\]

Substituting this bound into (37), there exists a universal constant \( C_1 > 0 \) such that for every \( u > 0 \), we have

\[
P\left( \sum_{j=1}^{p} (Z_j^2 - \nu_a) \mathbb{1}_{\{|Z_j| \geq a\}} \geq u \right) \leq \exp\left\{ -\lambda u + p \log(1 + C_1 \lambda^2 e^{-a^2/4}) \right\}
\]

\[\leq \exp\left( -\lambda u + C_1 p \lambda^2 e^{-a^2/4} \right).
\]
Now set \( u = (C_1 + 1)(\sqrt{pe^{-a^2/4}}x + x) \). If \( x \leq pe^{-a^2/4}/64 \), choose \( \lambda = \sqrt{\frac{x}{pe^{-a^2/4}}} \) so that \( \lambda \leq 1/8 \), and then we have
\[
\mathbb{P}\left( \sum_{j=1}^{p} (Z_j^2 - \nu_a) \mathbb{1}_{\{|Z_j| \geq a\}} \geq u \right) \leq e^{-(C_1+1)x+C_1x} = e^{-x}.
\]
If \( x > pe^{-a^2/4}/64 \), choose \( \lambda = 1/8 \), so that
\[
\mathbb{P}\left( \sum_{j=1}^{p} (Z_j^2 - \nu_a) \mathbb{1}_{\{|Z_j| \geq a\}} \geq u \right) \leq e^{-x},
\]
as required.

**Lemma 18.** Let \( Y \sim N(\theta, 1) \). Then there exists a universal constant \( C > 0 \) such that for every \( a \geq 1 \),
\[
\mathbb{E}\left\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \right\} = \begin{cases} 0, & \text{if } \theta = 0, \\ \in [0, C^2a^2 + 1], & \text{if } |\theta| < Ca, \\ \geq \theta^2/2, & \text{if } |\theta| \geq Ca. \end{cases}
\]
In fact, we may take \( C = 8 \).

**Proof of Lemma 18.** The fact that \( \mathbb{E}\left\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \right\} = 0 \) when \( \theta = 0 \) follows by definition of \( \nu_a \). To analyze the case \( |\theta| \geq Ca \), observe that by Cauchy–Schwarz, Lemma 16 and Chebychev’s inequality, for all \( a \in [1, (\theta^2 + 1)^{1/2}] \),
\[
\mathbb{E}\left\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| < a\}} \right\} \leq \sqrt{\mathbb{E}(Y^4) + \nu_a^2} \sqrt{\mathbb{P}(|Y| < a)} \leq \theta^4 + 6\theta^2 + 3 + 9a^4 \sqrt{\mathbb{P}(Y^2 < a^2)} \leq (\theta^2 + 3 + 3a^2) \frac{\sqrt{\var(Y^2)}}{\theta^2 + 1 - a^2} = (\theta^2 + 3 + 3a^2) \frac{\sqrt{2(1 + 2\theta^2)}}{\theta^2 + 1 - a^2}.
\]
Therefore, for all \( a \in [1, (\theta^2 + 1)^{1/2}] \),
\[
\mathbb{E}\left\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \right\} = \theta^2 + 1 - \nu_a - \mathbb{E}\left\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| < a\}} \right\} \geq \theta^2 + 1 - 3a^2 - (\theta^2 + 3 + 3a^2) \frac{\sqrt{2(1 + 2\theta^2)}}{\theta^2 + 1 - a^2}.
\]
Thus, for \( C > 1 \) and \( 1 \leq a \leq |\theta|/C \),
\[
\mathbb{E}\left\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \right\} \geq \theta^2 - \frac{3\theta^2}{C^2} - \frac{(1 + 6/C^2)\theta^2 \times 3|\theta|}{(1 - 1/C^2)\theta^2} \geq \theta^2 \left( 1 - \frac{3}{C^2} - \frac{(1 + 6/C^2)3/C}{(1 - 1/C^2)} \right),
\]
which is at least \( \theta^2/2 \), provided that \( C \geq 8 \). Finally, when \( 0 < |\theta| < Ca \), we have
\[
\mathbb{E}\left\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \right\} \leq \mathbb{E}(Y^2) = \theta^2 + 1 \leq C^2a^2 + 1,
\]
while the fact that \( \mathbb{E}\left\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \right\} \geq 0 \) follows because the expression inside the expectation
is stochastically increasing in $|\theta|$.

\[ \begin{align*}
\text{Lemma 19. Let } Y &\sim N(\theta, 1). \text{ Then there exists a universal constant } C_1 \geq 1 \text{ such that } \\
\text{ } &\text{Var}\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \} \leq \begin{cases} 
C_1 a^2 e^{-a^2/2} & \text{if } \theta = 0, \\
C_1 a^4 & \text{if } 0 < |\theta| < 2a, \\
C_1 \theta^2 & \text{if } |\theta| \geq 2a,
\end{cases} \\
\text{as long as } a \geq 1. 
\end{align*} \]

Proof of Lemma 19. Let $a \geq 1$. When $\theta = 0$, we have

\[ \text{Var}\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \} \leq \mathbb{E}\{ (Y^2 - \nu_a)^2 \mathbb{1}_{\{|Y| \geq a\}} \} = \int_{|x| \geq a} (x^2 - \nu_a)^2 \phi(x) \, dx \leq 2\Phi(a)(\gamma_a - \nu_a^2) \leq 8a^3 e^{-a^2/2}, \]

by Lemma 16. Now, for $\theta \neq 0$, we may write

\[ \begin{align*}
\text{Var}\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \} &= \mathbb{E}\{ \text{Var}\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \mid \mathbb{1}_{\{|Y| \geq a\}} \} \} + \text{Var}\{ \mathbb{E}\{ (Y^2 - \nu_a) \mathbb{1}_{\{|Y| \geq a\}} \mid \mathbb{1}_{\{|Y| \geq a\}} \} \} \\
&= \mathbb{P}(|Y| \geq a)\text{Var}(Y^2 \mid |Y| \geq a) + \mathbb{P}(|Y| < a)\mathbb{P}(|Y| \geq a)\{ \mathbb{E}(Y^2 - \nu_a \mid |Y| \geq a) \}^2. \quad (41)
\end{align*} \]

Moreover, writing $Y = \theta + X$ where $X \sim N(0, 1)$, we have by Lemma 16 and the fact that $\mathbb{E}(X^2 \mid X \geq b) = 1 + b\phi(b)/\Phi(b) \leq 1$ for $b \leq 0$ that

\[ \begin{align*}
\mathbb{E}(Y^2 - \nu_a \mid |Y| \geq a) &\leq 2\mathbb{E}(\theta^2 + X^2 \mid |\theta + X| \geq a) \\
&\leq 2\theta^2 + 2\mathbb{E}(X^2 \mid X \geq a - \theta) \vee 2\mathbb{E}(X^2 \mid X \leq -a - \theta) \\
&\leq 2\theta^2 + 2\mathbb{E}(X^2 \mid X \geq a + |\theta|) \leq 2\theta^2 + 6(a + |\theta|)^2. \quad (43)
\end{align*} \]

Moreover,

\[ \mathbb{E}(\nu_a - Y^2 \mid |Y| \geq a) \leq \nu_a \leq 3a^2. \]

For the first case when $0 < |\theta| < 2a$, the result follows from (41), (42), (43) and Lemma 16. For the second case when $|\theta| \geq 2a$, notice that we also have by Chebyshev’s inequality that $\mathbb{P}(|Y| < a) \leq \frac{2(1 + 2\theta^2)}{(\theta^2 - 1 - a^2)^2} \leq 18/\theta^2$ and the result follows from combining this with (41), (42) and (43).

The following lemma is a direct consequence of results in Tsybakov (2009) and we include the proof for completeness.
Lemma 20. Let $\Theta_0$ and $\Theta_1$ denote general parameter spaces, and consider a family of distribution $\{P_\theta\}_{\theta \in \Theta}$, where $\Theta := \Theta_0 \cup \Theta_1$. Let $\nu_0$ and $\nu_1$ be two distributions supported on $\Theta_0$ and $\Theta_1$ respectively. For $r \in \{0, 1\}$, define $Q_r$ to be the marginal distribution of the random variable $X$ generated hierarchically according to $\theta \sim \nu_r$ and $X|\theta \sim P_\theta$. Then

$$\inf_{\psi \in \Psi} \left\{ \sup_{\theta \in \Theta_0} E_\theta \psi(X) + \sup_{\theta \in \Theta_1} E_\theta (1 - \psi(X)) \right\} \geq \max \left\{ \frac{1}{2} \exp(-\alpha), 1 - \sqrt{\frac{\alpha}{2}} \right\},$$

where $\alpha := \chi^2(Q_0 \| Q_1)$.

Proof. In a slight abuse of notation, we use $E_0$ and $E_1$ to denote expectations with respect to $Q_0$ and $Q_1$ respectively. Then

$$\inf_{\psi \in \Psi} \left\{ \sup_{\theta \in \Theta_0} E_\theta \psi(X) + \sup_{\theta \in \Theta_1} E_\theta (1 - \psi(X)) \right\} \geq \inf_{\psi \in \Psi} \left\{ E_0 \psi(X) + E_1 (1 - \psi(X)) \right\} = 1 - TV(Q_0, Q_1).$$

The result then follows from elementary bounds on the total variation distance given in equations (2.25), (2.26) and Lemma 2.5 of Tsybakov (2009). \square

For Gaussian location mixtures, the chi-squared divergence takes a closed form, which is referred to as the Ingster–Suslina method (Ingster and Suslina, 2012). We again include the proof here for completeness.

Lemma 21. Let $\phi_\Sigma$ denote the density function of the $N_p(0, \Sigma)$ distribution for some positive definite $\Sigma \in \mathbb{R}^{p \times p}$. Define $f_0 := \phi_\Sigma$ and $f_1(\cdot) := \int_{\mathbb{R}^p} \phi_\Sigma(\cdot - \mu) d\nu(\mu)$ for some distribution $\nu$ on $\mathbb{R}^p$. Then

$$\chi^2(f_1 || f_0) = E_{(\mu_1, \mu_2) \sim \nu \otimes \nu} \exp \left( \mu_1^T \Sigma^{-1} \mu_2 \right) - 1.$$

Proof. Then by Fubini’s theorem,

$$\int_{\mathbb{R}^p} \frac{f_1^2}{f_0} = E_{x \sim \phi_\Sigma} \frac{f_1^2(x)}{f_0^2(x)} = E_{x \sim \phi_\Sigma} \frac{\left\{ E_{\mu \sim \nu} \phi_\Sigma(x - \mu) \right\}^2}{\phi_\Sigma^2(x)} = E_{x \sim \phi_\Sigma} E_{(\mu_1, \mu_2) \sim \nu \otimes \nu} \exp \left( - \frac{\|\mu_1\|_{\Sigma_1^{-1}}^2 + \|\mu_2\|_{\Sigma_1^{-1}}^2}{2} + \langle x, \mu_1 + \mu_2 \rangle_{\Sigma_1^{-1}} \right) = E_{(\mu_1, \mu_2) \sim \nu \otimes \nu} E_{x \sim \phi_\Sigma} \exp \left( - \frac{\|\mu_1\|_{\Sigma_1^{-1}}^2 + \|\mu_2\|_{\Sigma_1^{-1}}^2}{2} + \langle x, \mu_1 + \mu_2 \rangle_{\Sigma_1^{-1}} \right) = E_{(\mu_1, \mu_2) \sim \nu \otimes \nu} \exp \left( \mu_1^T \Sigma_1^{-1} \mu_2 \right),$$

and the result follows. \square

The following lemma follows immediately from the proof of Chen et al. (2018, Theorem 2.1).
Lemma 22. Consider independent observations $X_j \sim N(\theta + \delta_j, \sigma^2)$ for $j \in [p]$ and the estimator
\[ \hat{\theta} := \text{Median}(X_1, \ldots, X_p). \]
Assume that $\sum_{j=1}^p 1_{\{\delta_j \neq 0\}} \leq s \leq p/4$. Then there exist universal constants $C_1, C_2, C_3 > 0$, such that
\[ |\hat{\theta} - \theta| \leq C_1 \sigma \left( \frac{s}{p} + \sqrt{\frac{1+x}{p}} \right), \]
with probability at least $1 - e^{-C_2 x}$ for any $x > 0$ such that $C_3 \left( \frac{s}{p} + \sqrt{\frac{1+x}{p}} \right) \leq 1$.

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