Gradient estimates for a weighted parabolic equation under geometric flow

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Abstract
Let \((M^n, g, e^{-\phi}dv)\) be a weighted Riemannian manifold evolving by geometric flow \(\frac{\partial g}{\partial t} = 2h, \quad \frac{\partial \phi}{\partial t} = \Delta \phi\). In this paper, we obtain a series of space-time gradient estimates for positive solutions of a parabolic partial equation

\[
(\Delta \phi - \partial_t)u(x, t) = q(x, t)u^{a+1}(x, t) + p(x, t)A(u(x, t)), \quad (x, t) \in M \times [0, T],
\]
on a weighted Riemannian manifold under geometric flow. By integrating the gradient estimates, we find the corresponding Harnack inequalities.

Keywords Gradient estimate · Harnack inequality · Parabolic equation · Geometric flow

Mathematics Subject Classification 53C21 · 53E20 · 35K55 · 35B45

1 Introduction

An \(n\)-dimensional smooth weighted Riemannian manifold (or smooth metric measure space ) \((M^n, g, e^{-\phi}dv)\) is an \(n\)-dimensional smooth Riemannian manifold \((M^n, g)\) endowed with a weighted volume element \(e^{-\phi}dv\) such that \(\phi \in C^2(M)\) and \(dv\) is the volume element of \(g\) on \(M\). The weighted Laplacian (or Witten–Laplace operator) \(\Delta \phi = \Delta - \nabla \phi \cdot \nabla\) is a symmetric diffusion operator. In the present paper, we will prove Li–Yau type, local elliptic gradient estimate and another gradient estimates for positive solution of the parabolic equation

\[
(\Delta \phi - \partial_t)u(x, t) = q(x, t)u^{a+1}(x, t) + p(x, t)A(u(x, t)), \quad (x, t) \in M \times [0, T], \quad (1.1)
\]
on a weighted Riemannian manifold \((M^n, g, e^{-\phi}dv)\) evolving by the geometric flow system

\[
\frac{\partial g}{\partial t} = 2h, \quad \frac{\partial \phi}{\partial t} = \Delta \phi. \quad (1.2)
\]
where \((x, t) \in M \times [0, T]\), \(p(x, t), q(x, t)\) are functions on \(M \times [0, T]\) of \(C^2\) in \(x\)-variables and \(C^1\) in \(t\)-variable, \(A(u)\) is a function of \(C^2\) in \(u\), \(a\) is a positive constant, and \(h(x, t)\) is a symmetric \((0, 2)\)-tensor field on \((M, g(t), e^{-\phi} dv)\).

Some examples of geometric flows are the Ricci flow [15] when \(h = -\text{Ric}\) where \(\text{Ric}\) is the Ricci tensor, Yamabe flow [11] when \(h = -\frac{1}{2} R\) where \(R\) is the scalar curvature, Ricci–Bourguignon flow [7] when \(h = -\text{Ric} + \rho R\) where \(\rho\) is constant, and the extended Ricci flow [24] when \(h = -\text{Ric} + \alpha \nabla \phi \otimes \nabla \phi\) where \(\alpha(t)\) is a nonincreasing function and \(\phi\) is a smooth scalar function.

Equation (1.1) is called reaction–diffusion equation which can be found in many mathematical models in physics, chemistry, and biology [28, 31, 33], where \(q u^{a+1} + p A(u)\) and \(\Delta \phi u\) are the reaction and the diffusion terms, respectively. For instance, when \(\phi\) is a constant function, \(a = 0\), and \(A(u) = u \log u\), the nonlinear elliptic equation corresponding to (1.1) is related to the gradient Ricci soliton. When \(\phi\) is a constant function and \(A(u) = 0\) then \((\Delta \phi - \partial_t)u = q u^{a+1}\) which is a simple ecological model for population dynamics.

Gradient and Harnack estimates are powerful tools and important techniques in heat kernel analysis, entropy theory, differential geometry, in particular, in studying solution of parabolic equations from geometry which it developed by P. Li and S.-T. Yau [21]. In fact, they proved the well-known Li–Yau estimate on positive solutions to the heat equation with potential on Riemannian manifold with a fixed Riemannian metric and Ricci curvature bounded from below. Then, they derived Harnack inequalities by integrating the global gradient estimate along the a space-time path which provides a comparison between heat at two different points in space and at different times. After than, this method plays powerful role in study of heat equation, in particular, geometric flows. For instance, Hamilton [16] proved a Harnack estimate for Ricci flow on Riemannian manifolds with weakly positive curvature operator which is used in solving the Poincaré conjecture [6, 27].

In 1993, Hamilton [17] obtained an elliptic type gradient estimate for positive solutions of the heat equations on compact manifolds which was known as the Hamilton type gradient estimate. Then, for complete noncompact manifold, Souplet and Zhang [32] established an elliptic type gradient estimate for bounded solutions of the heat equation by adding a logarithmic correction term. This is called the Souplet–Zhang type gradient estimate. Li–Yau type, Hamilton type, and Souplet–Zhang type gradient estimates have been obtained for other nonlinear parabolic equations on manifolds, for instance see [8–10, 13, 20, 25, 26, 29, 35] and the references therein. On the other hand, many authors used similar techniques to prove gradient estimates and Harnack inequalities for positive solutions of parabolic equations under the geometric flow, for example see [1, 5, 14, 18, 23, 33, 38].

In 2014, Zhu and Li [39] derived Li–Yau estimates for a parabolic equation of the type \((\Delta - q - \partial_t)u = au (\log u)^\alpha\) in \(M \times (0, \infty)\) with a fixed metric where \(a, \alpha\) are constants and \(q \in C^2(M \times (0, \infty))\). In [10], Q. Chen and G. Zhao studied the equation \((\Delta - q - \partial_t)u = A(u)\) with a convection terms on a complete manifold with a fixed metric where \(A(u)\) is a function of \(C^2\) in \(u\). Then, in [38], Zhao obtained Li–Yau type and Hamilton type gradient estimates of equation \((\Delta - q - \partial_t)u = A(u)\) on Riemannian manifold evolving by the geometric flow. In [34], Wu gave a local Li–Yau type gradient estimate for the positive solutions to an nonlinear parabolic equation \(\partial_t u = \Delta \phi u - au \ln u - qu\) in \(M \times [0, T]\), where \(a\) is a real constant and \(q \in C^2(M \times [0, T])\). Also, Wu [34] proved local Hamilton type and Souplet–Zhang type gradient estimates for positive solutions to the equation \(\partial_t u = \Delta \phi u + au \ln u\) with \(a \in \mathbb{R}\) on a smooth metric measure space \((M^n, g, e^{-\varphi} dv)\) with Bakry–Émery Ricci tensor is bounded from below. In 2019, Yang and Zhang [37] proved Li–Yau type, Hamilton
Moreover, for Souplet–Zhang type, and the fourth type gradient estimates for positive solutions of a nonlinear parabolic equation \((\Delta \phi - \partial_t)u = pu + qu^{a+1}\) on smooth metric measure space with a fixed metric.

In this paper, we establish some gradient estimates for bounded positive solution of \((1.1)\) under the geometric flow \((1.2)\), which are richer than \([37, 38]\).

In following we recall some basic definitions on an \(n\)-dimensional weighted Riemannian manifold \((M, g, e^{-\phi} dv)\). The weighted Bochner formula for any smooth function \(f\) is follow as

\[
\frac{1}{2} \Delta \phi |\nabla f|^2 = |\text{Hess} f|^2 + \langle \nabla \Delta \phi f, \nabla f \rangle + \text{Ric}_\phi(\nabla f, \nabla f),
\]

\((1.3)\)

where

\[
\text{Ric}_\phi := \text{Ric} + \text{Hess} \phi
\]

and it is called a Bakry–Émery tensor (see [2]). For any integer \(m > n\), an \((m - n)\)-Bakry–Émery tensor (see [3]) is defined by

\[
\text{Ric}_\phi^{m-n} := \text{Ric} + \text{Hess} \phi - \frac{\nabla \phi \otimes \nabla \phi}{m - n}.
\]

If \(f = \ln u\) then \(\Delta \phi u = u(\Delta \phi f + |\nabla f|^2)\) and \(\partial_t u = u \partial_t f\). Therefore, by \((1.1)\) the function \(f\) satisfies

\[
(\Delta \phi - \partial_t) f = -|\nabla f|^2 + q e^a f + p \hat{A}(f),
\]

\((1.4)\)

where \(\hat{A}(f) = \frac{A(u)}{u}\). Throughout the paper, we assume \(u\) be a positive smooth solution to the general parabolic equation \((1.1)\). We denote by \(n\) the dimension of the manifold \(M\), and by \(d(x, y, t)\) the geodesic distance between \(x, y \in M\) under \(g(t)\). In addition, for any fixed \(x_0 \in M, R > 0\) we define the compact set

\[
Q_{2R, T} := \{(x, t) : d(x, x_0, t) \leq 2R, 0 \leq t \leq T\} \subset M^n \times (-\infty, +\infty).
\]

Let \(f = \ln u\) and \(\hat{A}(f) = \frac{A(u)}{u}\). Then

\[
\hat{A}_f = A'(u) - \frac{A(u)}{u}, \quad \hat{A}_{ff} = u A''(u) - A'(u) + \frac{A(u)}{u}.
\]

Moreover, for \(u > 0\) we define several nonnegative real constants as follows:

\[
\lambda_1 := \sup_{Q_{2R, T}} |\hat{A}|, \quad \Lambda_1 := \sup_{M \times [0, T]} |\hat{A}|,
\]

\[
\lambda_2 := \sup_{Q_{2R, T}} |\hat{A}_f|, \quad \Lambda_2 := \sup_{M \times [0, T]} |\hat{A}_f|,
\]

\[
\lambda_3 := \sup_{Q_{2R, T}} |\hat{A}_{ff}|, \quad \Lambda_3 := \sup_{M \times [0, T]} |\hat{A}_{ff}|,
\]

\[
\gamma_1 := \sup_{Q_{2R, T}} |p|, \quad \Gamma_1 := \sup_{M \times [0, T]} |p|,
\]

\[
\gamma_2 := \sup_{Q_{2R, T}} |\nabla p|, \quad \Gamma_2 := \sup_{M \times [0, T]} |\nabla p|,
\]

\[
\gamma_3 := \sup_{Q_{2R, T}} |\Delta \phi p|, \quad \Gamma_3 := \sup_{M \times [0, T]} |\Delta \phi p|.
\]

and

\[
\sigma_1 := \sup_{Q_{2R, T}} |q|, \quad \Sigma_1 := \sup_{M \times [0, T]} |q|.
\]
\[ \sigma_2 := \sup_{Q_{2R,T}} |\nabla q|, \quad \Sigma_2 := \sup_{M \times [0,T]} |\nabla q|, \]
\[ \sigma_3 := \sup_{Q_{2R,T}} |\Delta_\phi q|, \quad \Sigma_3 := \sup_{M \times [0,T]} |\Delta_\phi q|. \]

Also,
\[ \theta_1 := \sup_{Q_{2R,T}} |\nabla \phi|, \quad \Theta_1 := \sup_{M \times [0,T]} |\nabla \phi|, \]
\[ \theta_2 := \sup_{Q_{2R,T}} |\nabla \Delta_\phi|, \quad \Theta_2 := \sup_{M \times [0,T]} |\nabla \Delta_\phi|. \]

Notice that for any positive solution \( u \) of Eq. (1.1), \( \hat{A} \) is a function of \( C^2 \) in \( u \). Then \( |\hat{A}|, |\hat{A}_f|, \) and \( |\hat{A}_{ff}| \) are continuous functions on compact set \( Q_{2R,T} \). Hence, this function have supremum on \( Q_{2R,T} \). Hence, \( \lambda_i, i = 1, 2, 3 \) are finite. Also, \( \gamma_i, \sigma_i \) for \( i = 1, 2, 3 \) and \( \theta_1, \theta_2 \) are exist and finite. On the other hand, \( \Lambda_i, \Gamma_i, \Sigma_i \) for \( i = 1, 2, 3 \) and \( \Theta_1, \Theta_2 \) are allowed to be infinite, but if \( M \) is compact then these constants must be finite. Since we will obtain some bounds for solutions to equation (1.1), similar as [10, 34, 37, 39], we consider bounded positive solutions to equation (1.1) and we derive some bounds for gradient of these solutions.

The rest of this paper is organized as follows.

In Sect. 2, we give a Li–Yau type gradient estimate for positive solution of (1.1) under the geometric flow (1.2). We firstly prove a local and a global Li–Yau type gradient estimate on complete noncompact weighted Riemannian manifold without boundary (see Theorem 2.1 and Corollary 2.5) and as an immediate consequence of the Corollary 2.5, by integrating the global gradient estimate in space-time we establish the corresponding Harnack inequality (see Corollary 2.6). Then we consider that weighted manifold \( M \) is closed and we obtain global Li–Yau type gradient estimate and its corresponding Harnack inequality for positive solution of (1.1) on \( M \) under geometric flow (1.2) (see Theorem 2.7 and Corollary 2.8). In Sects. 3 and 4, we prove local and global Hamilton type and Souplet-Zhang type gradient estimates on complete noncompact weighted Riemannian manifold without boundary for positive solution of (1.1) on \( M \) under geometric flow (1.2), respectively (see Theorem 3.1, Corollary 3.3, Theorem 4.1, Corollary 4.3). Finally, in Sect. 5, similar as in [23, 37], we obtain a local and a global another type gradient estimate and the corresponding Harnack inequality to global estimate for positive solution of (1.1) under the geometric flow (1.2) on complete noncompact weighted Riemannian manifold without boundary (see Theorem 5.1, Corollary 5.3 and Corollary 5.4).

### 2 Li–Yau type gradient estimates

Firstly, we give a local space-time Li–Yau gradient estimate for (1.1)–(1.2) with conditions of \( \text{Ric}_\phi^{m-n} \) is lower bounded.

**Theorem 2.1** Let \( (M, g(0), e^{-\phi_0}dv) \) be a complete weighted Riemannian manifold, and let \( g(t), \phi(t) \) evolve by (1.2) for \( t \in [0,T] \). Given \( x_0 \) and \( R > 0 \), let \( u \) be a positive solution to (1.1) in \( Q_{2R,T} \) such that \( u^a \leq k \) for some positive constant \( k \). Suppose that there exist constants \( k_1, k_2, k_3, k_4 \) such that
\[ \text{Ric}_\phi^{m-n} \geq -(m-1)k_1g, \quad -k_2g \leq h \leq k_3g, \quad |\nabla h| \leq k_4, \]

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on $Q_{2R,T}$. Then for any $\alpha > 1$ the following estimate holds
\[
\frac{|\nabla u|^2}{u^2} - \alpha qu^\alpha - \alpha p \frac{A(u)}{u} - \alpha \frac{u_t}{u} \leq \frac{\alpha \alpha^2}{2t(1 - \alpha)} + K
\]  
(2.1)
on $Q_{2R,T}$, where $K$ is a constant and depending only on $m$, $k$, $R$, $\alpha$, $k_1$, $k_2$, $k_3$, $\lambda_1$, $\lambda_2$, $\lambda_3$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\gamma_1$, $\gamma_2$, $\theta_1$, $\theta_2$ and $c_0$, $c_1$, $c_2$ where $c_0$, $c_1$, $c_2$ will be defined in the proof of theorem.

**Remark 2.2** If functions $\phi$ and $p$ are constant functions, then we obtain the results of [38]. Also, if the function $p$ is a constant, $a = 0$, and $A(u) = u \log u$, then we derive the results of [34].

For prove our results, we need the following lemmas. From [12] we have

**Lemma 2.3** Let the metric evolves by (1.2). Then for any smooth function $f$, we have
\[
\frac{\partial}{\partial t} |\nabla f|^2 = -2h(\nabla f, \nabla f) + 2(\nabla f, \nabla f_t)
\]
and
\[
(\Delta_\phi f)_t = \Delta_\phi f_t - 2(h, \text{Hess } f) - 2\left(\text{div} h - \frac{1}{2}\nabla (\text{tr}_g h), \nabla f\right)
\]
\[
+ 2h(\nabla \phi, \nabla f) - \langle \nabla f, \nabla \phi_t \rangle
\]
\[
= \Delta_\phi f_t - 2(h, \text{Hess } f) - 2\left(\text{div} h - \frac{1}{2}\nabla (\text{tr}_g h), \nabla f\right)
\]
\[
+ 2h(\nabla \phi, \nabla f) - \langle \nabla f, \nabla \Delta \phi \rangle
\]
where $\text{div}$ is the divergence of $h$.

**Lemma 2.4** Let $(M^n, g, e^{-\phi} dv)$ be a weighted Riemannian manifold, $g(t)$ evolves by (1.2) for $t \in [0, T]$ satisfies the hypotheses of Theorem 2.1. If $f = \ln u$ and $F := t\left(|\nabla f|^2 - \alpha qe^{\alpha f} - \alpha \hat{p} \hat{A}(f) - \alpha f_t\right)$, then for any $t \in (0, \frac{1}{\alpha})$ we have
\[
(\Delta_\phi - \partial_t) F \geq \frac{2(1 - \alpha \alpha t)}{m} \left(|\nabla f|^2 - f_t - q e^{\alpha f} - p \hat{A} \right)^2 - 2(\nabla F, \nabla f) - \frac{F}{t}
\]
\[
+ \alpha t p \hat{A} f \left(|\nabla f|^2 - f_t - q e^{\alpha f} - p \hat{A} \right) - 2t[\alpha (a + 1) - 1] e^{\alpha f} \langle \nabla q, \nabla f \rangle
\]
\[
- \alpha t q e^{\alpha f} \left(|\nabla f|^2 - f_t - q e^{\alpha f} - p \hat{A} \right) - \alpha t \langle \nabla f, \nabla \phi_t \rangle - \alpha t e^{\alpha f} \Delta_\phi q
\]
\[
- 2t ((\alpha - 1) \hat{A} + \alpha \hat{A} f) \langle \nabla f, \nabla p \rangle - \alpha t \Delta_\phi p
\]
\[
- t \left(2(\alpha - 1) p \hat{A} f + \alpha p \hat{A} f f + 2(1 - \alpha \alpha t)(m - 1) k_1 + \alpha k_2 \right)
\]
\[
+ 2(\alpha - 1) q e^{\alpha f} + \alpha A^2 q e^{\alpha f} + 2(\alpha - 1) k_3) |\nabla f|^2
\]
\[
- \frac{\alpha \alpha t}{2 \epsilon} (k_2 + k_3)^2 - 3 \alpha t \sqrt{n} k_4 |\nabla f| - 2 \alpha t k_2 \epsilon |\nabla \phi|^2.
\]  
(2.2)

**Proof** From (1.4) we have
\[
\Delta_\phi f = -\frac{F}{t} - (\alpha - 1) (p \hat{A} + q e^{\alpha f} + f_t).
\]  
(2.3)
By the weighted Bochner formula, (1.4) and Lemma 2.3, we calculate
\[
\Delta_\phi F = 2t|\text{Hess} f|^2 + 2tRic_\phi(\nabla f, \nabla f) + 2t\langle \nabla \Delta_\phi f, \nabla f \rangle - \alpha t \Delta_\phi f_t
-\alpha t e^{af} \Delta_\phi q - \alpha t p \hat{A}_f \Delta_\phi f - \alpha t p \hat{A}_{ff} |\nabla f|^2 - \alpha t a^2 q e^{af} |\nabla f|^2
-\alpha t \hat{A} \Delta_\phi p - 2\alpha t \hat{A}_f \langle \nabla p, \nabla f \rangle - \alpha t q e^{af} \Delta_\phi f
-2t a e^{af} \langle \nabla q, \nabla f \rangle
\]
\[
= 2t|\text{Hess} f|^2 + 2tRic_\phi(\nabla f, \nabla f) + 2t\langle \nabla \Delta_\phi f, \nabla f \rangle - \alpha t (\Delta_\phi f)_t
-2at \langle h, \text{Hess} f \rangle - 2(\text{div} h - \frac{1}{2} \nabla \langle \nabla \phi, h \rangle, \nabla f) + 2ath(\nabla \phi, \nabla f)
-\alpha t \hat{A} \Delta_\phi f - \alpha t e^{af} \Delta_\phi q - \alpha t p \hat{A}_f \Delta_\phi f - \alpha t p \hat{A}_{ff} |\nabla f|^2
-\alpha t \hat{A} \Delta_\phi p - 2\alpha t \hat{A}_f \langle \nabla p, \nabla f \rangle - \alpha t a^2 q e^{af} |\nabla f|^2
-\alpha t q e^{af} \Delta_\phi f - 2t a e^{af} \langle \nabla q, \nabla f \rangle. 
\] (2.4)

By (2.3) we have
\[
\nabla \Delta_\phi f = -\frac{\nabla F}{t} - (\alpha - 1)(p \hat{A}_f \nabla f + \hat{A} \nabla p + e^{af} \nabla q + q e^{af} \nabla f + \nabla f_t) 
\] (2.5)

and
\[
(\Delta_\phi f)_t = \frac{F}{t^2} - \frac{F_t}{t} - (\alpha - 1)(p \hat{A}_f f_t + p_t \hat{A} + q_t e^{af} + a q f_t e^{af} + f_{tt}). 
\] (2.6)

Plugging (2.5) and (2.6) into (2.4), we obtain
\[
\Delta_\phi F = 2t|\text{Hess} f|^2 + 2tRic_\phi(\nabla f, \nabla f) - 2\langle \nabla F, \nabla f \rangle - 2t(\alpha - 1)p \hat{A}_f |\nabla f|^2
-2t(\alpha - 1)\hat{A} \langle \nabla p, \nabla f \rangle - 2t(\alpha - 1)e^{af} \langle \nabla q, \nabla f \rangle - 2t(\alpha - 1)q e^{af} |\nabla f|^2
-2t(\alpha - 1)\langle \nabla f_t, \nabla f \rangle - \alpha \frac{F}{t} + \alpha F_t - \alpha t \hat{A} \Delta_\phi p - 2\alpha t \hat{A}_f \langle \nabla p, \nabla f \rangle
+\alpha t(\alpha - 1)(p \hat{A}_f f_t + p_t \hat{A} + q_t e^{af} + a q f_t e^{af} + f_{tt})
-2at \langle h, \text{Hess} f \rangle - 2at \text{div} h - \frac{1}{2} \nabla \langle \nabla \phi, h \rangle, \nabla f) + 2ath(\nabla \phi, \nabla f)
-\alpha t \hat{A} \Delta_\phi f - \alpha t e^{af} \Delta_\phi q - \alpha t p \hat{A}_f \Delta_\phi f - \alpha t p \hat{A}_{ff} |\nabla f|^2
-\alpha t a^2 q e^{af} |\nabla f|^2 - \alpha t q e^{af} \Delta_\phi f - 2t a e^{af} \langle \nabla q, \nabla f \rangle. 
\] (2.7)

On the other hand, by Lemma (2.3) we derive
\[
F_t = |\nabla f|^2 - \alpha q e^{af} - \alpha p \hat{A}(f) - \alpha f_t
-2h(\nabla f, \nabla f) + 2t(\nabla f, \nabla f_t)
-\alpha t(\alpha - 1)(p \hat{A}_f f_t + p_t \hat{A} + q_t e^{af} + a q f_t e^{af} + f_{tt}). 
\] (2.8)

Applying (2.7) and (2.8), we get
\[
(\Delta_\phi - \partial_t) F = 2t|\text{Hess} f|^2 + 2tRic_\phi(\nabla f, \nabla f) - 2\langle \nabla F, \nabla f \rangle - \frac{F}{t}
-2at \langle h, \text{Hess} f \rangle - 2at \text{div} h - \frac{1}{2} \nabla \langle \nabla \phi, h \rangle, \nabla f) + 2ath(\nabla \phi, \nabla f)
-2t(\alpha - 1)p \hat{A}_f |\nabla f|^2 - 2t(\alpha - 1)\hat{A} \langle \nabla p, \nabla f \rangle
-2t(\alpha - 1)e^{af} \langle \nabla q, \nabla f \rangle - 2t(\alpha - 1)q e^{af} |\nabla f|^2 - \alpha \langle \nabla f, \nabla \phi_t \rangle
\]
−αte^{af} \Delta \phi q + αtp \hat{A} f (|\nabla f|^2 - f_t - qe^{af} - p \hat{A}) - αtp \hat{A}_{ff} |\nabla f|^2

−αt \hat{A} \Delta \phi p - 2αt \hat{A} f (\nabla p, \nabla f) - αtaqe^{af} |\nabla f|^2

−αtaqe^{af} (|\nabla f|^2 - f_t - qe^{af} - p \hat{A})

−2ttae^{af} (\nabla q, \nabla f) - 2(α - 1)th(\nabla f, \nabla f). \quad (2.9)

We can write the boundeness condition on \( h_{ij} \) as 

\[-(k_2 + k_3)g_{ij} \leq h_{ij} \leq (k_2 + k_3)g_{ij} \text{ so that} \]

\[|h|^2 \leq n(k_2 + k_3)^2. \quad (2.10)\]

since \( h_{ij} \) is a symmetric tensor. The Young’s inequality, for any \( \epsilon \in (0, \frac{1}{4\epsilon}) \) implies that 

\[\langle h, \text{Hess} f \rangle \leq \epsilon |\text{Hess} f|^2 + \frac{1}{4\epsilon} |h|^2 \leq \epsilon |\text{Hess} f|^2 + \frac{n}{4\epsilon} (k_2 + k_3)^2. \quad (2.11)\]

Also, we have

\[\left| \text{div} h - \frac{1}{2} \nabla (tr sh) \right| = \left| g^{ij} \nabla_i h_{jl} - \frac{1}{2} g^{ij} \nabla_j h_{ij} \right| \leq \frac{3}{2} |g| |\nabla h| \leq \frac{3}{2} \sqrt{n}k_4. \quad (2.12)\]

Notice also that for any \( m > n \) we derive

\[0 \leq \left( \frac{m - n}{mn} \Delta f + \frac{n}{m(m - n)} \langle \nabla f, \nabla \phi \rangle \right)^2
\]

\[= \left( \frac{1}{n} - \frac{1}{m} \right) (\Delta f)^2 + \frac{2}{m} \Delta f \langle \nabla f, \nabla \phi \rangle + \left( \frac{1}{m - n} - \frac{1}{m} \right) \langle \nabla f, \nabla \phi \rangle^2
\]

\[\leq |\text{Hess} f|^2 - \frac{1}{m} \left( (\Delta f)^2 - 2\Delta f \langle \nabla f, \nabla \phi \rangle + \langle \nabla f, \nabla \phi \rangle^2 \right) + \frac{1}{m - n} \langle \nabla f, \nabla \phi \rangle^2
\]

\[= |\text{Hess} f|^2 - \frac{(\Delta \phi f)^2}{m} + \frac{1}{m - n} \langle \nabla f, \nabla \phi \rangle^2.
\]

Therefore

\[|\text{Hess} f|^2 \geq \frac{(\Delta \phi f)^2}{m} - \frac{1}{m - n} \langle \nabla f, \nabla \phi \rangle^2. \quad (2.13)\]

Substituting (2.11), (2.12), and (2.13) into (2.9) we conclude

\[(\Delta \phi - \partial_t) F \geq \frac{2(1 - \epsilon \alpha)t}{m} \left( |\nabla f|^2 - f_t - qe^{af} - p \hat{A} \right)^2 + 2at \hbar (\nabla \phi, \nabla f)
\]

\[+ 2t(1 - \epsilon \alpha) Ri \epsilon_{\phi}^{m-n} (\nabla f, \nabla f) - 2\langle \nabla F, \nabla f \rangle - \frac{F}{t}
\]

\[+ \text{at} p \hat{A} f (|\nabla f|^2 - f_t - qe^{af} - p \hat{A}) - \text{at} \Delta \phi p
\]

\[-2t[\alpha(a + 1) - 1]e^{af} \langle \nabla q, \nabla f \rangle
\]

\[-\text{at} qe^{af} (|\nabla f|^2 - f_t - qe^{af} - p \hat{A})
\]

\[-\text{at} (\nabla f, \nabla \phi_t) - \text{at} \epsilon^{af} \Delta \phi q - 2t((\alpha - 1) \hat{A} + \alpha \hat{A} f) \langle \nabla f, \nabla p \rangle
\]

\[-t(2(\alpha - 1)p \hat{A} f + \alpha p \hat{A}_{ff} + 2(\alpha - 1)qe^{af} + \alpha \epsilon^{af} \langle \nabla f, \nabla f \rangle
\]

\[-2(\alpha - 1)th(\nabla f, \nabla f) - \frac{atn}{2\epsilon} (k_2 + k_3)^2 - 3at \sqrt{n}k_4 |\nabla f|. \quad (2.14)\]
By Young’s inequality for any $\epsilon \in (0, \frac{1}{a})$, we arrive at
\[
2 \alpha t h(\nabla \phi, \nabla f) \geq -2 \alpha t k_2 (\nabla \phi, \nabla f) \geq -\frac{\alpha t k_2}{2 \epsilon} |\nabla f|^2 - 2 \alpha t k_2 \epsilon |\nabla \phi|^2.
\] (2.15)
Replacing (2.15) into (2.14) and using the assumptions on bounds of $R_{\phi}^{m-n}$ and $h$, we get the inequality (2.2).

\[\textbf{Proof of theorem 2.1}\] Since the Ricci tensor and the evolution of the metric are bounded, then $g(t)$ is uniformly equivalent to the initial metric $g(0)$ (see [11, Corollary 6.11]),
\[e^{-2k_2 T} g(0) \leq g(t) \leq e^{2k_3 T} g(0)\].
Thus the manifold $(M, g(t))$ is also complete for $t \in [0, T]$. Let $\psi(s)$ be a $C^2$-function on $[0, +\infty)$,
\[
\psi(s) = \begin{cases} 
1, & s \in [0, 1], \\
0, & s \in [2, +\infty),
\end{cases}
\]
and it satisfies $\psi(s) \in [0, 1]$, $-c_0 \leq \psi'(s) \leq 0$, $\psi''(s) \geq -c_1$, and $\frac{|\psi''(s)|^2}{\psi(s)} \leq c_1$, where $c_0$ and $c_1$ are absolute constants. Let $R \geq 1$ and define a function
\[\eta(x, t) = \psi \left( \frac{r(x, t)}{R} \right),\]
where $r(x, t) = d(x, x_0, r)$. Using the argument of [4, 21], we can apply maximum principle and invoke Calabi’s trick to assume everywhere smoothness of $\eta(x, t)$ since $\psi(s)$ is in general Lipschitz. Also, we use generalization Laplacian comparison theorem [2, 19, 30, 34] to obtain inequalities of $\eta(x, t)$. Since $R_{\phi}^{m-n} \geq -(m-1)k_1$, the generalization Laplacian comparison theorem implies that
\[\Delta_{\phi} \eta(x) \leq (m-1)\sqrt{k_1} \coth(\sqrt{k_1} r(x))\]
and
\[\Delta_{\phi} \eta = \psi' \frac{\Delta_{\phi} r}{R} + \psi'' \frac{|\nabla r|^2}{R^2} - \frac{c_0}{R} (m-1) \sqrt{k_1} \coth(\sqrt{k_1} r(x)) - \frac{c_1}{R^2}\]
\[\geq - \frac{c_0}{R} (m-1) \left( \sqrt{k_1} + \frac{2}{R} \right) - \frac{c_1}{R^2}.\] (2.16)
Also, we have
\[|\nabla \eta|^2 \eta = \frac{|\psi'|^2 |\nabla r|^2}{R^2 \psi} \leq \frac{c_1}{R^2}.\] (2.17)
Let $G = \eta F$. Fix arbitrary $T_1 \in (0, T]$ and assume that $G$ achieves its maximum at point $(x_0, t_0) \in Q_{2R, T_1}$. If $G(x_0, t_0) \leq 0$, then the result holds trivially and we done. Hence, we may assume that $G(x_0, t_0) > 0$. In this point we have
\[\nabla G = 0, \quad \Delta_{\phi} G \leq 0, \quad \partial_t G \geq 0.\]
Therefore, we conclude
\[\nabla F = -\frac{F}{\eta} \nabla \eta\] (2.18)
and
\[0 \geq (\Delta_{\phi} - \partial_t)G = F(\Delta_{\phi} - \partial_t)\eta + \eta(\Delta_{\phi} - \partial_t)F + 2(\nabla \eta, \nabla F).\] (2.19)
By [33, p. 494], there exist a constant $c_2$ depending only on $c_1$ and $R$ such that
\[ -F \eta_t \geq -c_2k_2 F. \tag{2.20} \]
Replacing (2.16)–(2.18) and (2.20) into (2.19) we get
\[ 0 \geq -\left( \frac{c_0}{R} (m-1) \left( \sqrt{k_1} + \frac{2}{R} \right) + \frac{3c_1}{R^2} + c_2k_2 \right) F + \eta(\Delta \phi - \partial_t) F. \tag{2.21} \]
As in [8, 22, 36], we set
\[ \mu = \frac{|\nabla f|^2(x_0, t_0)}{F(x_0, t_0)} \geq 0, \]
then at point $(x_0, t_0)$, it follows that $|\nabla f| = \sqrt{\mu} F$,
\[ |\nabla f|^2 - f_t - qe^{\alpha f} - p\hat{A} = \left( \mu - \frac{t_0\mu}{t_0\alpha} \right) F, \]
\[ \eta \langle \nabla f, \nabla F \rangle = -F \langle \nabla f, \nabla \eta \rangle \leq \frac{\sqrt{c_1}}{R} \eta^\frac{1}{2} F |\nabla f|, \]
and
\[ 3\alpha \sqrt{nk_4} |\nabla f| \leq 2k_4 |\nabla f|^2 + \frac{9}{8} na^2k_4. \]
Using the above three relation, inequality (2.2), and inequality (2.21) at point $(x_0, t_0)$, we obtain
\[ 0 \geq \frac{2(1 - \epsilon\alpha)t_0}{m} \eta \left( \mu - \frac{t_0\mu}{t_0\alpha} \right)^2 F^2 - \frac{2\sqrt{c_1}}{R} \eta^{\frac{1}{2}} \mu^\frac{1}{2} F^\frac{3}{2} - \frac{\eta F}{t_0} + \alpha t_0\eta p\hat{A}f \left( \mu - \frac{t_0\mu}{t_0\alpha} \right) F - 2t_0\eta [\alpha(a + 1) - 1] e^{\alpha f} \langle \nabla q, \nabla f \rangle - \alpha t_0a q e^{\alpha f} \left( \mu - \frac{t_0\mu}{t_0\alpha} \right) F - \alpha t_0 \eta \langle \nabla f, \nabla \phi_t \rangle - \alpha t_0 e^{\alpha f} \Delta \phi q - 2t_0\eta (\alpha - 1)(\hat{A} + \alpha \hat{A}) f \langle \nabla p, \nabla f \rangle - \alpha t_0 \eta \hat{A} \Delta \phi p - t_0\eta \left( 2(\alpha - 1) - 1 \right) p\hat{A}f + \alpha p\hat{A}_{ff} + 2(1 - \epsilon\alpha)(m - 1)k_1 + \frac{\alpha k_2}{2\epsilon} \]
\[ + 2(\alpha - 1)q e^{\alpha f} + \alpha a^2 q e^{\alpha f} + 2(\alpha - 1)k_3 + 2k_4 \right) \mu F - \frac{\alpha t_0n}{2\epsilon} \eta(k_2 + k_3)^2 - \frac{9}{8} t_0\eta n\alpha^2k_4 - 2\alpha t_0\eta k_2\epsilon |\nabla \phi|^2 - \left( \frac{c_0}{R} (m-1) \left( \sqrt{k_1} + \frac{2}{R} \right) \right) + \frac{3c_1}{R^2} + c_2k_2 \right) F. \tag{2.22} \]
Multiply both sides of (2.22) by $t_0\eta$. By direct computation we conclude that
\[ 0 \geq \frac{2(1 - \epsilon\alpha)}{ma^2} \left( 1 + (\alpha - 1)t_0\mu \right)^2 G^2 - \frac{2\sqrt{c_1}}{R} t_0\mu \frac{1}{2} G^\frac{3}{2} - \eta G + t_0\eta p\hat{A}f G - 2t_0^2 \eta^2 [\alpha(a + 1) - 1] e^{\alpha f} \langle \nabla q, \nabla f \rangle - t_0 a q e^{\alpha f} \left( 1 + (\alpha - 1)t_0\mu \right) G - \alpha t_0^2 \eta^2 \langle \nabla f, \nabla \phi_t \rangle - \alpha t_0^2 \eta^2 e^{\alpha f} \Delta \phi q - 2t_0^2 \eta^2 ((\alpha - 1)(\hat{A} + \alpha \hat{A})f \langle \nabla p, \nabla f \rangle - \alpha t_0^2 \eta^2 \hat{A} \Delta \phi p - t_0^2 \eta ((\alpha - 1)p\hat{A}f + \alpha p\hat{A}_{ff} + 2(1 - \epsilon\alpha)(m - 1)k_1 + \frac{\alpha t_0k_2}{2\epsilon} \]
\[ \cdot \left( \sqrt{k_1} + \frac{2}{R} \right) + \frac{3c_1}{R^2} + c_2k_2 \right) F. \]
\[+2(\alpha - 1)qae^{\alpha f} + \alpha a^2 qe^{\alpha f} + 2(\alpha - 1)k_3 + 2k_4)\mu G\]

\[-\frac{\alpha t_0^2 n}{2\epsilon} \eta^2 (k_2 + k_3)^2 - \frac{9}{8} t_0^2 \eta^2 n\alpha^2 k_4 - 2\alpha t_0^2 \eta^2 k_2 \epsilon |\nabla \phi|^2\]

\[-t_0 \left( \frac{c_0}{R} (m - 1) \left( \frac{\sqrt{k_1} + \frac{2}{R}}{c_2} \right) + \frac{3c_1}{R^2} + c_2 k_2 \right) G.\]  

By Young's inequality, we infer

\[\frac{2\sqrt{c_1}}{R} \mu^2 G^\frac{3}{2} \leq \frac{4(1 - \epsilon \alpha)}{\alpha^2} (\alpha - 1) \mu G^2 + \frac{\alpha a^2 c_1 G}{4(1 - \epsilon \alpha)(\alpha - 1) R^2}.\]

The Cauchy's inequality and Young's inequality imply that

\[2\eta^2 [\alpha(\alpha + 1) - 1]e^{\alpha f} \langle \nabla q, \nabla f \rangle \leq 2[\alpha(\alpha + 1) - 1]k|\nabla q||\nabla f| \leq 2[\alpha(\alpha + 1) - 1]\kappa \sigma_2 \mu^\frac{1}{2} G^\frac{1}{2} \leq \frac{2(1 - \epsilon \alpha)\delta}{m\alpha^2} (\alpha - 1)^2 \mu^2 G^2 + \frac{3}{4} \left( \frac{2\alpha}{1 - \epsilon \alpha}(\alpha - 1)^2 \right) \left[ \alpha(\alpha + 1) - 1 \right]^\frac{1}{2} k^\frac{3}{2} \sigma_2^\frac{3}{2}, \]

and

\[2((\alpha - 1)\hat{A} + \alpha \hat{A}_f) \langle \nabla p, \nabla f \rangle \leq 2((\alpha - 1)\lambda_1 + \alpha \lambda_2)|\nabla p||\nabla f| \leq 2((\alpha - 1)\lambda_1 + \alpha \lambda_2)\gamma_2 \mu^\frac{1}{2} G^\frac{1}{2} \leq \frac{(1 - \epsilon \alpha)(1 - \delta)}{2m\alpha^2} (\alpha - 1)^2 \mu^2 G^2 \]

\[+ \frac{3}{4} \left( \frac{\alpha}{2(1 - \epsilon \alpha)(1 - \delta)(\alpha - 1)^2} \right) \left[ (\alpha - 1)\lambda_1 + \alpha \lambda_2 \right]^\frac{1}{2} \gamma_2^\frac{3}{2}.\]

Also, we have

\[\alpha(\nabla f, \nabla \phi_1) \leq \alpha(\nabla \Delta \phi ||\nabla f || \leq \alpha \theta_2 \mu^\frac{1}{2} G^\frac{1}{2} \leq \frac{(1 - \epsilon \alpha)(1 - \delta)}{2m\alpha^2} (\alpha - 1)^2 \mu^2 G^2 \]

\[+ \frac{3}{4} \left( \frac{\alpha}{2(1 - \epsilon \alpha)(1 - \delta)(\alpha - 1)^2} \right) \left[ (\alpha - 1)\lambda_1 + \alpha \lambda_2 \right]^\frac{1}{2} \theta_2^\frac{3}{2},\]

where \(\delta \in (0, 1)\) is an arbitrary constant. Combining the above inequalities we conclude that

\[0 \geq (1 - \epsilon \alpha)\frac{2}{m\alpha^2} G^2 + \frac{(1 - \epsilon \alpha)(1 - \delta)}{m\alpha^2} (\alpha - 1)^2 \mu^2 G^2 - t_0 \frac{\alpha a^2 c_1 G}{4(1 - \epsilon \alpha)(\alpha - 1) R^2} \]

\[-G - t_0 \gamma_1 \lambda_2 G + \frac{3}{4} t_0^2 \left( \frac{2m\alpha^2}{(1 - \epsilon \alpha)(\alpha - 1)^2} \right)^\frac{1}{2} [\alpha(\alpha + 1) - 1]^\frac{1}{2} k^\frac{3}{2} \sigma_2^\frac{3}{2} \]

\[-t_0 \alpha t_0 k G - \frac{3}{4} t_0^2 \left( \frac{2m\alpha^2}{(1 - \epsilon \alpha)(\alpha - 1)^2} \right)^\frac{1}{2} \alpha^\frac{3}{2} \theta_2^\frac{3}{2} - \alpha t_0 k \sigma_3 \]

\[-\frac{3}{4} t_0^2 \left( \frac{2m\alpha^2}{(1 - \epsilon \alpha)(\alpha - 1)^2} \right)^\frac{1}{2} (\alpha - 1)\lambda_1 + \alpha \lambda_2 \gamma_3^\frac{3}{2} - \alpha t_0^2 \lambda_1 \gamma_3 \]

\[-t_0^2 \frac{1}{2}(\alpha - 1)(\gamma_1 \lambda_2 + 2\alpha t_0 k + 2k_3) + \alpha \gamma_1 \lambda_3 + 2(1 - \epsilon \alpha)(m - 1)k_1 + \frac{\alpha k_2}{2\epsilon}\]
\[
+aa^2\sigma_1 k + 2k_4 + a\sigma_2 k \bigg) \mu G
\]
\[
- \frac{\alpha t_0^2 n}{2\epsilon} (k_2 + k_3)^2 - \frac{9}{8} t_0^2 \mu a^2 \sigma_4 k_4 - 2\alpha t_0^2 k_2 \epsilon \theta_1^2
\]
\[
-t_0 \left( \frac{c_0}{R} (m - 1) \left( \sqrt{k_1 + \frac{2}{R}} + \frac{3c_1}{R^2} + c_2 k_2 \right) G \right).
\]

Set
\[
C_1 := (\alpha - 1)(\gamma_1 \lambda_2 + 2a\sigma_1 k + 2k_3) + \alpha \gamma_1 \lambda_3 + 2(1 - \epsilon \alpha)(m - 1) k_1 \\
+ \frac{\alpha k_2}{2\epsilon} + \alpha a^2 \sigma_1 k + 2k_4 + a\sigma_2 k.
\]

By Young’s inequality, we obtain
\[
C_1 \mu G \leq \frac{(1 - \epsilon \alpha)(1 - \delta)(\alpha - 1)^2}{m \alpha^2} \mu^2 G^2 + \frac{\mu a^2}{4(1 - \epsilon \alpha)(1 - \delta)(\alpha - 1)^2} C_1^2. \tag{2.25}
\]

Plugging (2.25) into (2.24), we deduce
\[
0 \geq \frac{2(1 - \epsilon \alpha)}{m \alpha^2} - G^2
\]
\[
- \left[ 1 + t_0 \left( \frac{m \alpha^2 c_1}{4(1 - \epsilon \alpha)(\alpha - 1) R^2} + \gamma_1 \lambda_2 + a\sigma_1 k \right) \\
+ \frac{c_0}{R} (m - 1) \left( \sqrt{k_1 + \frac{2}{R}} + \frac{3c_1}{R^2} + c_2 k_2 \right) \right] G
\]
\[
- \frac{3}{4} \frac{t_0^2}{\epsilon^2} \left( \frac{2m \alpha^2}{(1 - \epsilon \alpha) \delta (\alpha - 1)^2} \right)^\frac{1}{2} [\alpha(a + 1) - 1]^\frac{3}{2} k_3^\frac{3}{2} \sigma_2^\frac{3}{2} \\
- \frac{3}{4} \frac{t_0^2}{\epsilon^2} \left( \frac{2m \alpha^2}{2(1 - \epsilon \alpha)(1 - \delta)(\alpha - 1)^2} \right)^\frac{1}{2} \alpha^\frac{3}{2} \theta_2^\frac{3}{2} - \alpha t_0^2 k_2 \sigma_3
\]
\[
- \frac{3}{4} \frac{t_0^2}{\epsilon^2} \left( \frac{2m \alpha^2}{2(1 - \epsilon \alpha)(1 - \delta)(\alpha - 1)^2} \right)^\frac{1}{2} \left( (\alpha - 1) \lambda_1 + \alpha \lambda_2 \right)^\frac{3}{2} \gamma_2^\frac{3}{2} - \alpha t_0^2 \lambda_1 \gamma_3
\]
\[
- \frac{m \alpha^2}{4(1 - \epsilon \alpha)(1 - \delta)(\alpha - 1)^2} C_1^2
\]
\[
- \frac{\alpha t_0^2 n}{2\epsilon} (k_2 + k_3)^2 - \frac{9}{8} t_0^2 \mu a^2 k_4 - 2\alpha t_0^2 k_2 \epsilon \theta_1^2. \tag{2.26}
\]

Suppose that
\[
D_1 := 1 + t_0 \left( \frac{m \alpha^2 c_1}{4(1 - \epsilon \alpha)(\alpha - 1) R^2} + \gamma_1 \lambda_2 + a\sigma_1 k \right) \\
+ \frac{c_0}{R} (m - 1) \left( \sqrt{k_1 + \frac{2}{R}} + \frac{3c_1}{R^2} + c_2 k_2 \right),
\]

and
\[
E_1 := \frac{3}{4} \left( \frac{2m \alpha^2}{2(1 - \epsilon \alpha) \delta (\alpha - 1)^2} \right)^\frac{1}{2} [\alpha(a + 1) - 1]^\frac{3}{2} k_3^\frac{3}{2} \sigma_2^\frac{3}{2} \\
+ \frac{3}{4} \left( \frac{m \alpha^2}{2(1 - \epsilon \alpha)(1 - \delta)(\alpha - 1)^2} \right)^\frac{1}{2} \alpha^\frac{3}{2} \theta_2^\frac{3}{2} + \alpha k_3
\]

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\]
Hence we can write (2.26) as

$$0 \geq \frac{2(1 - \epsilon \alpha)}{\alpha} G^2 - D_1 G - t_0^2 E_1.$$ 

For a positive number \(\tilde{a}\) and two nonnegative numbers \(\tilde{b}, \tilde{c}\), the quadratic inequality of the form \(\tilde{a}x^2 - \tilde{b}x - \tilde{c} \leq 0\) implies that \(x \leq \frac{\tilde{b}}{\tilde{a}} + \sqrt{\frac{\tilde{c}}{\alpha}}\). Therefore,

$$G \leq \frac{\alpha}{2(1 - \epsilon \alpha)} D_1 + t_0 \left( \frac{\alpha}{2(1 - \epsilon \alpha)} E_1 \right)^{\frac{1}{2}}.$$

To obtain the required result on \(F(x, t)\), we have \(\eta(x, T_1) = 1\) whenever \(d(x, x_0, T_1) \leq R\). Hence

$$\left( |\nabla f| - \alpha \epsilon a^f - \alpha p \hat{A}(f) - \alpha f_1 \right)(x, T_1) = \frac{F(x, T_1)}{T_1} \leq \frac{G(x_0, t_0)}{T_1}.$$

Since \(T_1\) is arbitrary, we conclude

$$\frac{|\nabla u|^2}{u^2} - \alpha qu^a - \alpha p A(u) - \alpha u_t u - \frac{\alpha}{u} \leq \frac{\alpha}{2(1 - \epsilon \alpha)} + K,$$

where

$$K := \frac{\alpha}{2(1 - \epsilon \alpha)} \left( \frac{\alpha}{2(1 - \epsilon \alpha)(1 - \frac{\alpha}{2})} \right)^\frac{1}{3} \left[ (\alpha + 1) - 1 \right]^\frac{\alpha}{2} + \alpha \epsilon a^f + \alpha k_0 \epsilon a^f,$$

$$E := \frac{3}{4} \left( \frac{2\alpha}{(1 - \epsilon \alpha)(1 - \frac{\alpha}{2})} \right)^\frac{1}{3} \left[ (\alpha + 1) - 1 \right]^\frac{\alpha}{2} + \alpha \epsilon a^f + \alpha k_0 \epsilon a^f,$$

$$+ \frac{3}{4} \left( \frac{\alpha}{2(1 - \epsilon \alpha)(1 - \frac{\alpha}{2})} \right)^\frac{1}{3} \left[ (\alpha + 1) - 1 \right]^\frac{\alpha}{2} + \alpha \epsilon a^f + \alpha k_0 \epsilon a^f,$$

$$+ \frac{3}{4} \left( \frac{\alpha}{2(1 - \epsilon \alpha)(1 - \frac{\alpha}{2})} \right)^\frac{1}{3} \left[ (\alpha + 1) - 1 \right]^\frac{\alpha}{2} + \alpha \epsilon a^f + \alpha k_0 \epsilon a^f,$$

$$+ \frac{3}{4} \left( \frac{\alpha}{2(1 - \epsilon \alpha)(1 - \frac{\alpha}{2})} \right)^\frac{1}{3} \left[ (\alpha + 1) - 1 \right]^\frac{\alpha}{2} + \alpha \epsilon a^f + \alpha k_0 \epsilon a^f,$$

$$+ \frac{3}{4} \left( \frac{\alpha}{2(1 - \epsilon \alpha)(1 - \frac{\alpha}{2})} \right)^\frac{1}{3} \left[ (\alpha + 1) - 1 \right]^\frac{\alpha}{2} + \alpha \epsilon a^f + \alpha k_0 \epsilon a^f.$$
and
\[ C := (\alpha - 1)(\gamma_1 \lambda_2 + 2a \sigma_1 k + 2k_3) + \alpha \gamma_1 \lambda_3 + 2(1 - \epsilon \alpha)(m - 1)k_1 \\
+ \frac{ak_2}{2\epsilon} + \alpha a^2 \sigma_1 k + 2k_4 + a \sigma_2 k. \]

**Corollary 2.5** Let \((M, g(0), e^{-\phi_0} dv)\) be a complete noncompact weighted Riemannian manifold without boundary, and let \(g(t), \phi(t)\) evolve by \((1.1)\) for \(t \in [0, T]\). Let \(u\) be a positive solution to \((1.1)\) in \(M\) such that \(u^a \leq k\) for some positive constant \(k\). Suppose that there exist constants \(k_1, k_2, k_3, k_4\) such that
\[ \text{Ric}^m_g \geq -(m - 1)k_1 g, \quad -k_2 g \leq h \leq k_3 g, \quad |\nabla h| \leq k_4, \]
on \(M\). Then for any \(\alpha > 1\) we have
\[ \frac{|\nabla u|^2}{u^2} - \alpha qu^a - \alpha p \frac{A(u)}{u} - \frac{u_t}{u} \leq \frac{ma^2}{2(1 - \epsilon \alpha)} + K_2 \]
on \(M\), where \(K_2\) is a constant and depending only \(m, k, \alpha, k_1, k_2, k_3, \Lambda_1, \Lambda_2, \Lambda_3, \Sigma_1, \Sigma_2, \Sigma_3, \Gamma_1, \Gamma_2, \Theta_1, \Theta_2\).

**Proof** Since \(g(t)\) is uniformly equivalent to the initial metric \(g(0)\), then \((M, g(t))\) is complete noncompact for \(t \in [0, T]\). For fixe \(\delta \in (0, 1)\), if \(R \to +\infty\) in \((2.1)\) then we obtain inequality \((2.27)\) where
\[ K_2 := \frac{ma^2}{2(1 - \epsilon \alpha)} \left( \Gamma_1 \Lambda_2 + a \Sigma_1 k + c_2 k_2 \right) + \left( \frac{ma^2}{2(1 - \epsilon \alpha)} E_2 \right)^{\frac{1}{2}}, \]
\[ E_2 := \frac{3}{4} \left( \frac{2ma^2}{(1 - \epsilon \alpha)(1 - \delta)(2 - \phi)} \right)^{\frac{1}{2}} [\alpha(a + 1) - 1]^2 k^\alpha \Sigma^3 \]
\[ + \frac{3}{4} \left( \frac{ma^2}{2(1 - \epsilon \alpha)(1 - \delta)(2 - \phi)} \right)^{\frac{1}{2}} \alpha^4 \Theta^2 + ak \Sigma_3 \]
\[ + \frac{3}{4} \left( \frac{ma^2}{2(1 - \epsilon \alpha)(1 - \delta)(2 - \phi)} \right)^{\frac{1}{2}} (\alpha - 1) \Lambda_1 + \alpha \Lambda_2 \right)^2 \Gamma^2 + \alpha \ell_0^2 \Lambda_1 \Gamma_3 \]
\[ + \frac{ma^2}{4(1 - \epsilon \alpha)(1 - \delta)(1 - \phi)} C_2 \]
\[ + \frac{3}{8} \frac{ma^2}{2(1 - \epsilon \alpha)(1 - \delta)(2 - \phi)} k_2 + 2ak_2 \epsilon \Theta_1^2, \]
and
\[ C_2 := (\alpha - 1)(\Gamma_1 \Lambda_2 + 2a \Sigma_1 k + 2k_3) + \alpha \Gamma_1 \Lambda_3 + 2(1 - \epsilon \alpha)(m - 1)k_1 \\
+ \frac{ak_2}{2\epsilon} + \alpha a^2 \Sigma_1 k + 2k_4 + a \Sigma_2 k. \]

As immediate consequence of the global gradient estimates obtained in Corollary 2.5, by integrating the gradient estimates in space-time we obtain the following Harnack inequality. We first introduce the following notation. Given \((y_1, s_1) \in M \times (0, T)\) and \((y_2, s_2) \in M \times (0, T)\) satisfying \(s_1 < s_2\), define
\[ J(y_1, s_1, y_2, s_2) = \inf_{s_1} \int_{y_1}^{s_2} |\xi'(t)|^2_{g(t)} dt, \]
and the infimum is taken over the all smooth curves $\zeta : [s_1, s_2] \to M$ joining $y_1$ and $y_2$.

**Corollary 2.6** With the same assumptions in Corollary 2.6, for $(y_1, s_1) \in M \times (0, T]$ and $(y_2, s_2) \in M \times (0, T]$ such that $s_1 < s_2$, we have

$$u(y_1, s_1) \leq u(y_2, s_2)\left(\frac{s_2}{s_1}\right)^{\frac{ma}{2t(1-\epsilon\alpha)}} \exp\left\{\frac{\alpha J(y_1, s_1, y_2, s_2)}{4}\right\} + (s_2 - s_1)\left(k\Sigma_1 + \Gamma_1\Lambda_1 + \frac{1}{\alpha}K_2\right)\right\}.$$

**Proof** Take the geodesic path $\zeta(t)$ from $y_1$ to $y_2$ with $\zeta(s_1) = y_1$ and $\zeta(s_2) = y_2$. Now consider the path $(\zeta(t), t)$ in space-time. From Corollary 2.5, we have the following gradient estimate

$$-\partial_t(\ln u) \leq k\Sigma_1 + \Gamma_1\Lambda_1 - \frac{1}{\alpha}|
abla(\ln u)|^2 + \frac{ma}{2t(1-\epsilon\alpha)} + \frac{1}{\alpha}K_2. \quad (2.28)$$

Integrating this inequality along $\zeta$, we get

$$\log \frac{u(y_1, s_1)}{u(y_2, s_2)} = -\int_{s_1}^{s_2} \frac{d}{dt}(\ln u(\zeta(t), t))dt$$

$$= -\int_{s_1}^{s_2} \left(\partial_t(\ln u) + \langle \nabla(\ln u)(\zeta(t), t), \dot{\zeta}(t)\rangle\right)dt$$

$$\leq \int_{s_1}^{s_2} \left\{k\Sigma_1 + \Gamma_1\Lambda_1 - \frac{1}{\alpha}|
abla(\ln u)|^2 + \frac{ma}{2t(1-\epsilon\alpha)} + \frac{1}{\alpha}K_2 - \langle \nabla(\ln u), \dot{\zeta}(t)\rangle\right\}dt$$

$$\leq \int_{s_1}^{s_2} \left\{\frac{\alpha}{4}|\dot{\zeta}(t)|^2 + \left(k\Sigma_1 + \Gamma_1\Lambda_1 + \frac{ma}{2t(1-\epsilon\alpha)} + \frac{1}{\alpha}K_2\right)\right\}dt$$

$$\leq \frac{\alpha}{4} \int_{s_1}^{s_2} |\dot{\zeta}(t)|^2 dt + (s_2 - s_1)(k\Sigma_1 + \Gamma_1\Lambda_1 + \frac{1}{\alpha}K_2) + \frac{ma}{2(1-\epsilon\alpha)} \ln \frac{s_2}{s_1},$$

where in the computation above we have used (2.28) to obtain the inequality in the third line and used inequality $-\hat{a}x^2 - \hat{b}x \leq \frac{s^2}{2\alpha}$ to arrive at the inequality in the fourth line. By exponentiation we have

$$u(y_1, s_1) \leq u(y_2, s_2)\left(\frac{s_2}{s_1}\right)^{\frac{ma}{2t(1-\epsilon\alpha)}} \exp\left\{\frac{\alpha}{4} \int_{s_1}^{s_2} |\dot{\zeta}(t)|^2 dt + (s_2 - s_1)(k\Sigma_1 + \Gamma_1\Lambda_1 + \frac{1}{\alpha}K_2)\right\}.$$

We now consider the case that the manifold $M$ is compact and without boundary, i.e., $M$ is closed, and we find a global gradient estimate on a closed weighted Riemannian manifold.

**Theorem 2.7** Let $(M, g(0), e^{-\phi_0}dv)$ be a closed weighted Riemannian manifold, and let $g(t), \phi(t)$ evolve by (1.2) for $t \in [0, T]$. Let $u$ be a positive solution to (1.1) in $M$ such that $u^a \leq k$ for some positive constant $k$. Suppose that there exist constants $k_1, k_2, k_3, k_4$ such that

$$Ric^\phi - n \geq -(m - 1)k_1g, \quad -k_2g \leq h \leq k_3g, \quad |\nabla h| \leq k_4,$$

on $M$. Then for any $\alpha > 1$, the following estimate holds

$$\frac{|
abla u|^2}{u^2} - \alpha qu^a - \alpha p \frac{A(u)}{u} - \alpha \frac{u_t}{u} \leq \frac{ma^2}{2t} + K_3 \quad (2.29)$$
on \( M \), where \( K_3 \) is a constant and depending only \( m, k, \alpha, k_1, k_2, k_3, \Lambda_2, \Lambda_3, \Sigma_1, \Gamma_2, \Theta_1, \Theta_2 \) and \( \epsilon_0, c_1, c_2 \) where \( \epsilon_0, c_1, c_2 \) are defined in the proof of Theorem 2.1.

**Proof** Let \( f = \ln u, F := t \left( |\nabla f|^2 - \alpha q e^{\alpha f} - \alpha p \tilde{A}(f) - \alpha f_i \right) \), and

\[
\tilde{F}(x, t) := F(x, t) - K_3 t.
\]

For any \((x, t) \in M \times [0, T] \) if \( \tilde{F}(x, t) \leq \frac{\alpha q}{2} \) then the result holds trivially. Thus, we consider \( \tilde{F}(x, t) > \frac{\alpha q}{2} \) on \( M \times [0, T] \). Let \((x_1, t_1)\) be a point in \( M \times [0, T] \) at which \( \tilde{F} \) attains its maximum value. Then \( \tilde{F}(x_1, t_1) > \frac{\alpha q}{2} \). As \( \tilde{F}(x_1, 0) = 0 \) then \( t_1 > 0 \). Now using the maximum principle, we conclude that

\[
\nabla \tilde{F}(x_1, t_1) = 0, \quad \Delta_{\phi} \tilde{F}(x_1, t_1) \leq 0, \quad \partial_t \tilde{F}(x_1, t_1) \geq 0.
\]

Therefore at point \((x_1, t_1)\) we deduce

\[
0 \geq (\Delta_{\phi} - \partial_t) \tilde{F} \geq (\Delta_{\phi} - \partial_t) F.
\]

From Lemma 2.4 and identity

\[
|\nabla f|^2 - q e^{\alpha f} - p \tilde{A} - f_i = \frac{1}{\alpha} \frac{F}{t} + \frac{\alpha - 1}{\alpha} |\nabla f|^2,
\]

we obtain

\[
0 \geq \frac{2(1 - \epsilon \alpha) t_1}{m \alpha^2} \left( \frac{F}{t_1} \right)^2 + \frac{4(1 - \epsilon \alpha)(\alpha - 1) t_1}{m \alpha^2} |\nabla f|^2 \frac{F}{t_1} + \frac{2(1 - \epsilon \alpha) t_1}{m \alpha^2} (\alpha - 1)^2 t_1 |\nabla f|^4 \frac{F}{t_1} - \Gamma_1 \Lambda_2 F - \alpha k \Sigma_1 \frac{F}{t_1} \left( 2[\alpha(a + 1) - 1] k \Sigma_2 + 3\alpha \sqrt{n} k_4 + \alpha \Theta_2 \right) |\nabla f| \nonumber
\]

\[
- t_1 \left( (\alpha - 1) \Gamma_1 \Lambda_2 + \alpha \Gamma_1 \Lambda_3 + 2(1 - \epsilon \alpha)(m - 1) k_1 + \frac{\alpha k_2}{2 \epsilon} + (\alpha - 1) \alpha \Sigma_1 \right. 
\]

\[
+ \alpha^2 k \Sigma_1 + 2(\alpha - 1) k_3 \right] |\nabla f|^2 - \alpha t_1 k \Sigma_3 - \frac{\alpha t_1 n}{2 \epsilon} (k_2 + k_3)^2 - 2 \alpha t_1 k_2 \epsilon \Theta_1^2.
\]

Since

\[
\frac{F}{t_1} = \frac{\tilde{F}}{t_1} + K_3 > 0
\]

and

\[
\left( 2[\alpha(a + 1) - 1] k \Sigma_2 + 3\alpha \sqrt{n} k_4 + \alpha \Theta_2 \right) |\nabla f| \nonumber
\]

\[
\leq |\nabla f|^2 + \frac{1}{4} \left( 2[\alpha(a + 1) - 1] k \Sigma_2 + 3\alpha \sqrt{n} k_4 + \alpha \Theta_2 \right)^2,
\]

we get

\[
0 \geq \frac{2(1 - \epsilon \alpha) t_1}{m \alpha^2} \left( \frac{F}{t_1} \right)^2 - \left( 1 + t_1 \Gamma_1 \Lambda_2 + t_1 \alpha k \Sigma_1 \right) \frac{F}{t_1} + \frac{2(1 - \epsilon \alpha) t_1}{m \alpha^2} (\alpha - 1)^2 t_1 |\nabla f|^4
\]

\[
- t_1 \left( (\alpha - 1) \Gamma_1 \Lambda_2 + \alpha \Gamma_1 \Lambda_3 + 2(1 - \epsilon \alpha)(m - 1) k_1 + \frac{\alpha k_2}{2 \epsilon} \right)
\]
\[
+(\alpha - 1)ak\Sigma_1 + \alpha a^2k\Sigma_1 + 2(\alpha - 1)k_3 + 1\right] |\nabla f|^2
\]

\[-\alpha t_1k\Sigma_3 - \frac{\alpha t_1n}{2\epsilon} (k_2 + k_3)^2 - 2\alpha t_1k_2\epsilon \Theta_1^2
\]

\[-\frac{t_1}{4} \left(2[\alpha (a + 1) - 1]k\Sigma_2 + 3\alpha \sqrt{n}k_4 + \alpha \Theta_2\right)^2 .
\]

Set

\[B_1 := (\alpha - 1)\Gamma_1 \Lambda_2 + \alpha \Gamma_1 \Lambda_3 + 2(1 - \epsilon \alpha)(m - 1)k_1 + \frac{\alpha k_2}{2\epsilon}
\]

\[+(\alpha - 1)ak\Sigma_1 + \alpha a^2k\Sigma_1 + 2(\alpha - 1)k_3 + 1,
\]

and

\[B_2 := \alpha k\Sigma_3 + \frac{\alpha n}{2\epsilon} (k_2 + k_3)^2 + 2ak_2\epsilon \Theta_1^2
\]

\[+\frac{1}{4} \left(2[\alpha (a + 1) - 1]k\Sigma_2 + 3\alpha \sqrt{n}k_4 + \alpha \Theta_2\right)^2 .
\]

Since the inequality \(\tilde{a}x^2 - \tilde{b}x \geq -\frac{\tilde{b}^2}{4\tilde{a}}\) holds for \(\tilde{a} > 0\) and \(\tilde{b} \geq 0\), we obtain

\[0 \geq \frac{2(1 - \epsilon \alpha)t_1}{ma^2} \left(\frac{F}{t_1}\right)^2 - (1 + t_1\Gamma_1 \Lambda_2 + t_1ak\Sigma_1) \frac{F}{t_1}
\]

\[-\frac{t_1ma^2}{8(1 - \epsilon \alpha)(\alpha - 1)^2} B_1^2 - t_1B_2.
\]

The inequality \(\tilde{a}x^2 - \tilde{b}x - \tilde{c} < 0\) implies that \(x \leq \frac{1}{2\tilde{a}}(\tilde{b} + \sqrt{\tilde{b}^2 + 4\tilde{a}\tilde{c}})\) holds for a positive constant \(\tilde{a}\) and two nonnegative constants \(\tilde{b}, \tilde{c}\). Then,

\[
\frac{F}{t_1} \leq \frac{ma^2}{4(1 - \epsilon \alpha)t_1} \left\{1 + t_1\Gamma_1 \Lambda_2 + t_1ak\Sigma_1 + \left[1 + t_1\Gamma_1 \Lambda_2 + t_1ak\Sigma_1\right]^2
\]

\[+\frac{8(1 - \epsilon \alpha)t_1^2}{ma^2} \left(\frac{ma^2}{8(1 - \epsilon \alpha)(\alpha - 1)^2} B_1^2 + B_2\right) \right\}^{\frac{1}{2}}.
\]

The inequality \(\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}\) for \(x, y \geq 0\) implies that

\[
\frac{F}{t_1} \leq \frac{ma^2}{4(1 - \epsilon \alpha)t_1} \left\{1 + t_1\Gamma_1 \Lambda_2 + t_1ak\Sigma_1 + \left[1 + t_1\Gamma_1 \Lambda_2 + t_1ak\Sigma_1\right]^2
\]

\[+\frac{4n(1 - \epsilon \alpha)t_1^2}{ma\epsilon} (k_2 + k_3)^2\right\}^{\frac{1}{2}}
\]

\[+\frac{ma^2}{4(1 - \epsilon \alpha)t_1} \left[\frac{8(1 - \epsilon \alpha)t_1^2}{ma^2} \left(\frac{ma^2}{8(1 - \epsilon \alpha)(\alpha - 1)^2} B_1^2 + B_3\right)\right]^{\frac{1}{2}},
\]

where \(B_3 = B_2 - \frac{an}{2\epsilon} (k_2 + k_3)^2\). Let

\[
\epsilon = \frac{t_1(k_2 + k_3)}{1 + t_1\Gamma_1 \Lambda_2 + t_1ak\Sigma_1 + 2t_1(k_2 + k_3) \alpha}
\]
for \( k_2 + k_3 \neq 0 \). If \( k_2 + k_3 = 0 \) the we can choose \( \epsilon = 0 \). Therefore

\[
\begin{align*}
\frac{m\alpha^2}{4(1-\epsilon\alpha)t_1} \left\{ 1 + t_1\Gamma_1\Lambda_2 + t_1ak\Sigma_1 + \left[ (1 + t_1\Gamma_1\Lambda_2 + t_1ak\Sigma_1)^2 \right. \right. \\
+ \left. \left. \frac{4n(1-\epsilon\alpha)t_1^2}{m\alpha\epsilon} (k_2 + k_3)^2 \right \} \right. \\
\leq \frac{m\alpha^2}{2t_1} (1 + t_1\Gamma_1\Lambda_2 + t_1ak\Sigma_1) + m\alpha^2(k_2 + k_3),
\end{align*}
\]

and

\[
\frac{m\alpha^2}{4(1-\epsilon\alpha)t_1} \left[ \frac{8(1-\epsilon\alpha)t_1^2}{m\alpha^2} \right]^{\frac{1}{2}} \leq \sqrt{m\alpha}.
\]

Hence inequality \( \sqrt{x+y} \leq \sqrt{x} + \sqrt{y} \) for \( x, y \geq 0 \) implies that

\[
\frac{F}{t_1} \leq \frac{m\alpha^2}{2t_1} (1 + t_1\Gamma_1\Lambda_2 + t_1ak\Sigma_1) + m\alpha^2(k_2 + k_3) + \frac{m\alpha^2}{2(\alpha - 1)} B_4 + B_5,
\]

where

\[
B_4 := (\alpha - 1)\Gamma_1\Lambda_2 + \alpha\Gamma_1\Lambda_3 + 2(m - 1)k_1 + \frac{\alpha k_2}{2\epsilon} + (\alpha - 1)ak\Sigma_1 + \alpha a^2k\Sigma_1 \\
+ 2(\alpha - 1)k_3 + 1,
\]

and

\[
B_5 := \sqrt{mk\Sigma_3\alpha^2} + \sqrt{2mk_2\Theta_1\alpha^2} \\
+ \frac{\sqrt{m\alpha}}{2} \left( 2[\alpha(a + 1) - 1]k\Sigma_2 + 3\alpha\sqrt{n}k_4 + \alpha\Theta_2 \right).
\]

This implies that \( \bar{F}(x_1, t_1) \leq \frac{m\alpha^2}{2} \) and this is a contradiction. Thus obtain inequality (2.29) where

\[
\begin{align*}
K_3 := \frac{m\alpha^2}{2} (\Gamma_1\Lambda_2 + ak\Sigma_1) + m\alpha^2(k_2 + k_3) \\
+ \frac{m\alpha^2}{2(\alpha - 1)} \left( (\alpha - 1)\Gamma_1\Lambda_2 + \alpha\Gamma_1\Lambda_3 \right) \\
+ \frac{m\alpha^2}{2(\alpha - 1)} \left( 2(m - 1)k_1 + \frac{\alpha k_2}{2\epsilon} + (\alpha - 1)ak\Sigma_1 + \alpha a^2k\Sigma_1 + 2(\alpha - 1)k_3 + 1 \right) \\
+ \sqrt{mk\Sigma_3\alpha^2} + \sqrt{2mk_2\Theta_1\alpha^2} \\
+ \frac{\sqrt{m\alpha}}{2} \left( 2[\alpha(a + 1) - 1]k\Sigma_2 + 3\alpha\sqrt{n}k_4 + \alpha\Theta_2 \right).
\end{align*}
\]

\[\square\]

Similar to Corollary 2.6, integrating the gradient estimates obtained in Theorem 2.7 in space-time we obtain the following Harnack inequality.

\[\square\]
Corollary 2.8 With the same assumptions in Theorem 2.7, for \((y_1, s_1) \in M \times (0, T]\) and \((y_2, s_2) \in M \times (0, T]\) such that \(s_1 < s_2\), we have
\[
u(y_1, s_1) \leq u(y_2, s_2) \left(\frac{s_2}{s_1}\right)^{\frac{m}{2}} \exp \left\{ \alpha J(y_1, s_1, y_2, s_2) + (s_2 - s_1) \left( 3\Sigma_1 + \Gamma_1 \Lambda_1 + \frac{1}{\alpha} K_3 \right) \right\}.
\]

3 Hamilton type gradient estimates

Now we prove Hamilton type gradient estimates for (1.1)–(1.2).

Theorem 3.1 Let \((M, g(0), e^{-\phi(t)} dv)\) be a complete weighted Riemannian manifold, and let \(g(t), \phi(t)\) evolve by (1.2) for \(t \in [0, T]\). Given \(x_0\) and \(R > 0\), let \(u\) be a positive solution to (1.1) in \(Q_{2R,T}\) such that \(k^3 \leq u \leq k^3\) for some positive constants \(k\) and \(k\). Suppose that there exist constants \(k_1, k_2\) such that
\[
\text{Ric}_\phi \geq -(n-1)k_1 g, \quad h \geq -k_2 g,
\]
on \(Q_{2R,T}\). Then, we have the following estimate
\[
\frac{|\nabla u|}{\sqrt{u}} \leq \frac{3 \sqrt{c_1}}{\sqrt{k}} \left\{ \frac{3}{k} \frac{c_1}{R} + \frac{3^4}{4} \left( \frac{2}{3} k^4 \lambda_1 y_1^2 + \frac{2}{3} k^{3\alpha+4} \sigma_2 \right)^{\frac{1}{2}} \right\}
\]
\[
+ \frac{1}{\sqrt{2}} \left[ 2(n-1)k_1 k^2 + \gamma_1 \lambda_1 k^2 + \frac{2}{3} \gamma_1 \lambda_2 k^3 + 2k_2 k + (2a + 1) k^{3\alpha+2} \sigma_1 \right.
\]
\[
+ \left( \frac{c_0}{R} (n-1) \left( \sqrt{k_1} + \frac{2}{R} \right) + \frac{3c_1}{R^2} + c_2 k_2 \right) \left( k_2 \right)^{\frac{1}{2}} \right\},
\]
on \(Q_{2R,T}\), where positive constants \(c_0, c_1,\) and \(c_2\) are introduced in the proof of Theorem 2.1.

Before we prove Theorem 3.1, firstly we derive the following lemma.

Lemma 3.2 Let \((M^n, g, e^{-\phi} dv)\) be a weighted Riemannian manifold, \(g(t)\) evolves by (1.2) for \(t \in [0, T]\) satisfies the hypotheses of Theorem 3.1. If \(v = u^\frac{1}{2}\) and \(H := v|\nabla v|^2\), then we have
\[
(\Delta_\phi - \partial_t) H \geq 4v^{-3} H^2 - 4v^{-1} (\nabla v, \nabla H) - 2(n-1)k_1 H - |p||\hat{A}| H
\]
\[
- \frac{2}{3} v |p||\hat{A}| H - \frac{2}{3} v^2 |\hat{A}| |\nabla p| \sqrt{H} - 2k_2 v^{-1} H
\]
\[
- \frac{2}{3} v^{3\alpha+\frac{5}{2}} |\nabla q| \sqrt{H} - (2a + 1)v^{3\alpha} |q| H.
\]
(3.1)

Proof Since \(u = v^3\), we get \(u_t = 3v^2 v_t\), \(\nabla u = 3v^2 \nabla v\) and \(\Delta_\phi u = 3v^2 \Delta_\phi v + 6v|\nabla v|^2\). From (1.1) we conclude that
\[
(\Delta_\phi - \partial_t)v = -2 \frac{|\nabla v|^2}{v} + \frac{1}{3} v p \hat{A}(v) + \frac{q v^{3\alpha+1}}{3}.
\]
(3.2)

By direct computation, we calculate
\[
\partial_t H = |\nabla v|^2 v_t + 2v(\nabla v_t, \nabla v) - 2h(\nabla v, \nabla v)
\]
Gradient estimates for a weighted. Page 19 of 33

Proof of Theorem 3.1

Choosing \( \psi \) and \( \eta \) as in the proof of Theorem 2.1. For any \( T_1 \in (0, T] \), let \( (x_2, t_2) \in Q_{2R, T_1} \) be a point where \( G(x, t) = \eta(x, t)H(x, t) \) achieve its maximum, and
without loss of generality we can assume $G(x_2, t_2) > 0$, and then $H(x_2, t_2) > 0$. By Lemma 3.2 and a similar argument as in the proof of Theorem 2.1, at point $(x_2, t_2)$ we have

$$0 \geq (\Delta \phi - \partial_t)G = (\Delta \phi - \partial_t)(\eta H)$$

$$= H(\Delta \phi - \partial_t)\eta + \eta(\Delta \phi - \partial_t)H + 2(\nabla \eta, \nabla H)$$

$$\geq \eta(\Delta \phi - \partial_t)H - \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1 + \frac{2}{R}} + \frac{3c_1}{R^2} + c_2k_2\right)\right)H$$

$$\geq 4v^{-3}\eta H^2 - 4\frac{\sqrt{c_1}}{R}v^\frac{3}{2} H^\frac{3}{2} - 2(n - 1)k_1\eta v^3 G - \gamma_1\lambda_1 v^3 \eta G - \frac{2}{3}\gamma_1\lambda_2 v^4 \eta G$$

$$- \frac{2}{3}v^\frac{3}{2} \eta^\frac{3}{2} \lambda_1 \gamma_2 \sqrt{H} - 2k_2 v^{-1} G - \frac{2}{3}v^{3a+\frac{3}{2}} \sigma_2 \eta^\frac{3}{2} \sqrt{H} - (2a + 1)v^{3a + 3} \sigma_1 G$$

$$- \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1 + \frac{2}{R}} + \frac{3c_1}{R^2} + c_2k_2\right)\right)v^3 G$$

$$= 4v^2 G^2 - 4\frac{\sqrt{c_1}}{R}v^\frac{3}{2} G^\frac{3}{2}$$

$$- \left[2(n - 1)k_1 v^2 + \gamma_1\lambda_1 v^2 + \frac{2}{3}\gamma_1\lambda_2 v^3 + 2k_2 v + (2a + 1)v^{3a + 2}\sigma_1\right]v^\eta G$$

$$- \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1 + \frac{2}{R}} + \frac{3c_1}{R^2} + c_2k_2\right)\right)v^3 G$$

$$- \left(\frac{2}{3}v^4 \lambda_1 \gamma_2 + \frac{2}{3}v^{3a + 4}\sigma_2\right)\eta^\frac{3}{2} \sqrt{v^G}. \quad (3.6)$$

at point $(x_2, t_2)$. By Young’s inequality and as $\bar{k}^3 \leq u \leq k^3$, we infer

$$4\frac{\sqrt{c_1}}{R}v^\frac{3}{2} G^\frac{3}{2} \leq v^2 G^2 + 81\frac{c_1^2}{R^4},$$

and

$$\left[2(n - 1)k_1 v^2 + \gamma_1\lambda_1 v^2 + \frac{2}{3}\gamma_1\lambda_2 v^3 + 2k_2 v + (2a + 1)v^{3a + 2}\sigma_1\right]v^\eta G$$

$$- \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1 + \frac{2}{R}} + \frac{3c_1}{R^2} + c_2k_2\right)\right)v^3 G$$

$$\leq v^2 G^2 + \frac{1}{4}\left[2(n - 1)k_1 v^2 + \gamma_1\lambda_1 v^2 + \frac{2}{3}\gamma_1\lambda_2 v^3 + 2k_2 v + (2a + 1)v^{3a + 2}\sigma_1\right.$$  

$$+ \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1 + \frac{2}{R}} + \frac{3c_1}{R^2} + c_2k_2\right)\right)v^2 \right]^2$$

$$\leq v^2 G^2 + \frac{1}{4}\left[2(n - 1)k_1 k^2 + \gamma_1\lambda_1 k^2 + \frac{2}{3}\gamma_1\lambda_2 k^3 + 2k_2 k + (2a + 1)k^{3a + 2}\sigma_1\right.$$  

$$+ \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1 + \frac{2}{R}} + \frac{3c_1}{R^2} + c_2k_2\right)\right)k^2 \right]^2.$$
Using Young’s inequality again, we can write
\[
\left(\frac{2}{3}v^4\lambda_1\gamma_2 + \frac{2}{3}v^{3a+4}\sigma_2\right)\eta^2 \sqrt{vG} \\
\leq v^2G^2 + 3\left(\frac{1}{4}\right)^\frac{1}{3}\left(\frac{2}{3}k^4\lambda_1\gamma_2 + \frac{2}{3}k^{3a+4}\sigma_2\right)^\frac{1}{3} \\
\leq v^2G^2 + 3\left(\frac{1}{4}\right)^\frac{1}{3}\left(\frac{2}{3}k^4\lambda_1\gamma_2 + \frac{2}{3}k^{3a+4}\sigma_2\right)^\frac{1}{3}.
\]
Substituting the above three inequalities into (3.6), we arrive at
\[
v^2G^2 \leq 81\frac{c_1^2}{R^4} + 3\left(\frac{1}{4}\right)^\frac{1}{3}\left(\frac{2}{3}k^4\lambda_1\gamma_2 + \frac{2}{3}k^{3a+4}\sigma_2\right)^\frac{1}{3} \\
\quad + \frac{1}{4}\left[2(n-1)k_1k^2 + \gamma_1\lambda_1k^2 + \frac{2}{3}\gamma_1\lambda_2k^3 + 2k_2k + (2a+1)k^{3a+2}\sigma_1 \\
\quad + \left(c_0\frac{R}{c_1}(n-1)\left(\sqrt{k_1} + \frac{2}{R}\right) + 3c_1\frac{R}{R^2} + c_2k_2\right)k^2\right]^2,
\]
at point \((x_2, t_2)\). Since \(\tilde{k}^3 \leq u\), then \(k_1H^2 \leq v^2G^2\). Therefore
\[
\left(\frac{|\nabla u|^2}{9u}\right)^2 \leq \frac{1}{k^2}\left\{81\frac{c_1^2}{R^4} + 3\left(\frac{1}{4}\right)^\frac{1}{3}\left(\frac{2}{3}k^4\lambda_1\gamma_2 + \frac{2}{3}k^{3a+4}\sigma_2\right)^\frac{1}{3} \\
\quad + \frac{1}{4}\left[2(n-1)k_1k^2 + \gamma_1\lambda_1k^2 + \frac{2}{3}\gamma_1\lambda_2k^3 + 2k_2k + (2a+1)k^{3a+2}\sigma_1 \\
\quad + \left(c_0\frac{R}{c_1}(n-1)\left(\sqrt{k_1} + \frac{2}{R}\right) + 3c_1\frac{R}{R^2} + c_2k_2\right)k^2\right]^2\right\},
\]
at point \((x_2, t_2)\). Hence inequality \(\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}\) for \(x, y \geq 0\) implies that
\[
\frac{|\nabla u|}{\sqrt{u}} \leq \frac{3}{\sqrt{k}}\left\{3\sqrt{\frac{c_1^2}{R^4}} + 3\left(\frac{1}{4}\right)^\frac{1}{3}\left(\frac{2}{3}k^4\lambda_1\gamma_2 + \frac{2}{3}k^{3a+4}\sigma_2\right)^\frac{1}{3} \\
\quad + \frac{1}{\sqrt{2}}\left[2(n-1)k_1k^2 + \gamma_1\lambda_1k^2 + \frac{2}{3}\gamma_1\lambda_2k^3 + 2k_2k + (2a+1)k^{3a+2}\sigma_1 \\
\quad + \left(c_0\frac{R}{c_1}(n-1)\left(\sqrt{k_1} + \frac{2}{R}\right) + 3c_1\frac{R}{R^2} + c_2k_2\right)k^2\right]^\frac{1}{2}\right\},
\]
at point \((x_2, t_2)\). This completes the proof of theorem. \(\square\)

Similar to Corollary 2.5, when \(\langle M, g(0), e^{-\phi_0}dv \rangle\) be a complete noncompact weighted Riemannian manifold without boundary and \(g(t)\) evolve by (1.2), we can conclude a global gradient estimate from Theorem 3.1 by taking \(R \to +\infty\) as follows.

**Corollary 3.3** Let \((M, g(0), e^{-\phi_0}dv)\) be a complete noncompact weighted Riemannian manifold without boundary, and let \(g(t), \phi(t)\) evolve by (1.2) for \(t \in [0, T]\). Let \(u\) be a positive solution to (1.1) in \(M\) such that \(\tilde{k}^3 \leq u \leq k^3\) for some positive constants \(k\) and \(\tilde{k}\). Suppose that there exist constants \(k_1, k_2\) such that
\[
\text{Ric}_\phi \geq -(n-1)k_1g, \quad h \geq -k_2g.
\]
on $M$. Then, the following estimate is true

$$\frac{|\nabla u|}{\sqrt{u}} \leq \frac{3}{\sqrt{k}} \left\{ 3^{\frac{1}{2}} \left( \frac{1}{4} \right)^{\frac{1}{3}} \left( \frac{2}{3} k^4 \Lambda_1 \Gamma_2 + \frac{2}{3} k^{3a+4} \Sigma_2 \right)^{\frac{1}{3}} + \frac{1}{\sqrt{2}} \left[ 2(n-1)k_1 k^2 + \Gamma_1 \Lambda_1 k^2 + \frac{2}{3} \Gamma_1 \Lambda_2 k^3 + 2k_2 k + (2a + 1)k^{3a+2} \Sigma_1 + c_2 k^2 \right]^{\frac{1}{2}} \right\},$$

on $M$, where positive constants $c_0$, $c_1$, and $c_2$ are introduced in the proof of Theorem 2.1.

## 4 Souplet-Zhang type gradient estimate

In this section we obtain a Souplet-Zhang type gradient estimate for (1.1) under (1.2).

**Theorem 4.1** Let $(M, g(0), e^{-\phi_0} dv)$ be a complete weighted Riemannian manifold, and let $g(t), \phi(t)$ evolve by (1.2) for $t \in [0, T]$. Given $x_0$ and $R > 0$, let $u$ be a positive solution to (1.1) in $Q_{2R,T}$ such that $\tilde{k} \leq u \leq k$ for some positive constants $k$ and $\tilde{k}$. Suppose that there exist constants $k_1, k_2$ such that $\text{Ric} \phi \geq -(n-1)k_1 g$, $h \geq -k_2 g$, on $Q_{2R,T}$. Then, we get the following inequality

$$\frac{|\nabla u|}{\sqrt{u}} \leq \left( 1 + \ln \frac{k}{R} \right) \left[ 6^{\frac{1}{2}} \left( \ln \frac{k}{R} \right) \frac{\sqrt{c_1}}{R} + \frac{3}{4} \left( 2\lambda_1 \gamma_2 + 2k^a \sigma_2 \right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \left( \gamma \lambda_2 + (a+1)\sigma_1 k^{a} + (n-1)k_1 + k_2 + \gamma \lambda_1 \right)^{\frac{1}{2}} + \left( c_0 \frac{R}{R} (n-1) \left( \sqrt{k_1} + \frac{2}{R} \right) + \frac{3c_1}{R^{2}} + c_2 k^2 \right)^{\frac{1}{2}} \right],$$

(4.1)
on $Q_{2R,T}$, where positive constants $c_0$, $c_1$, and $c_2$ are introduced in the proof of Theorem 2.1.

Before we prove Theorem 4.1, firstly we establish the following lemma.

**Lemma 4.2** Let $(M^n, g(t), e^{-\phi} dv)$ be a weighted Riemannian manifold, $g(t)$ evolves by (1.2) for $t \in [0, T]$ satisfies the hypotheses of Theorem 4.1. If $f = \ln u$, $\rho := 1 + \ln k$, and $w := |\nabla (\ln (\rho - f))| = 2 \frac{|\nabla u|^2}{(\rho - f)^2}$. Then we have

$$\left( \Delta_\phi - \partial_t \right) w \geq 2\frac{(f - \ln k)}{\rho - f} (\nabla w, \nabla f) + 2(\rho - f) w^2$$

$$- \frac{2}{\rho - f} \left( 2|\nabla p| + e^{af} |\nabla q| \right) \sqrt{w}$$

$$+ 2 \left( p \hat{A} + a q e^{af} - (n-1)k_1 - k_2 + \frac{q e^{af}}{\rho - f} + \frac{p \hat{A}}{\rho - f} \right) w.$$
**Proof** By the weighted Bochner formula, we get

\[
\Delta_{\phi} w = \frac{2(\nabla \Delta_{\phi} f, \nabla f)}{(\rho - f)^2} + \frac{2Ric_{\phi}(\nabla f, \nabla f)}{(\rho - f)^2} + \frac{2|Hess f|^2}{(\rho - f)^2} \\
+ \frac{8Hess(\nabla f, \nabla f)}{(\rho - f)^3} + \frac{2|\nabla f|^2 \Delta_{\phi} f}{(\rho - f)^3} + \frac{6|\nabla f|^4}{(\rho - f)^4}.
\]  

(4.2)

On the other hand

\[
\partial_t w = \frac{2(\nabla f_t, \nabla f)}{(\rho - f)^2} + \frac{2|\nabla f|^2 f_t}{(\rho - f)^3} - \frac{2h(\nabla f, \nabla f)}{(\rho - f)^2}.
\]  

(4.3)

Combining the above two equalities, we infer

\[
(\Delta_{\phi} - \partial_t) w = \frac{2(\nabla (\Delta_{\phi} - \partial_t) f, \nabla f)}{(\rho - f)^2} + \frac{2Ric_{\phi}(\nabla f, \nabla f)}{(\rho - f)^2} + \frac{2|Hess f|^2}{(\rho - f)^2} \\
+ \frac{8Hess(\nabla f, \nabla f)}{(\rho - f)^3} + \frac{2|\nabla f|^2 (\Delta_{\phi} - \partial_t) f}{(\rho - f)^3} + \frac{6|\nabla f|^4}{(\rho - f)^4} + \frac{2h(\nabla f, \nabla f)}{(\rho - f)^2}.
\]

Plugging (1.4) in above equality, we conclude

\[
(\Delta_{\phi} - \partial_t) w = \frac{2|Hess f|^2}{(\rho - f)^2} + \frac{8Hess(\nabla f, \nabla f)}{(\rho - f)^3} + \frac{6|\nabla f|^4}{(\rho - f)^4} - \frac{4Hess(\nabla f, \nabla f)}{(\rho - f)^2} \\
- \frac{2|\nabla f|^4}{(\rho - f)^3} + \frac{2\tilde{A}(\nabla p, \nabla f)}{(\rho - f)^2} + \frac{2\tilde{A}f|\nabla f|^2}{(\rho - f)^2} + \frac{2e^{af}(\nabla q, \nabla f)}{(\rho - f)^2} \\
+ \frac{2aqe^{af}|\nabla f|^2}{(\rho - f)^2} + \frac{2Ric_{\phi}(\nabla f, \nabla f)}{(\rho - f)^2} + \frac{2h(\nabla f, \nabla f)}{(\rho - f)^2} + \frac{2aqe^{af}|\nabla f|^2}{(\rho - f)^3} \\
+ \frac{2p\tilde{A}|\nabla f|^2}{(\rho - f)^3}.
\]

Since

\[
\frac{|Hess f|^2}{(\rho - f)^2} + \frac{2Hess(\nabla f, \nabla f)}{(\rho - f)^3} + \frac{|\nabla f|^4}{(\rho - f)^4} = \frac{1}{(\rho - f)^2} |Hess + \frac{df \otimes df}{\rho - f}|^2 \geq 0
\]

and

\[
\frac{2Hess(\nabla f, \nabla f)}{(\rho - f)^2} + \frac{2|\nabla f|^4}{(\rho - f)^3} = \langle \nabla w, \nabla f \rangle,
\]

we deduce

\[
(\Delta_{\phi} - \partial_t) w \geq \frac{2(\nabla w, \nabla f)}{\rho - f} - 2(\nabla w, \nabla f) + 2\frac{|\nabla f|^4}{(\rho - f)^3} + \frac{2\tilde{A}(\nabla p, \nabla f)}{(\rho - f)^2} + \frac{2p\tilde{A}f|\nabla f|^2}{(\rho - f)^2} \\
+ \frac{2e^{af}(\nabla q, \nabla f)}{(\rho - f)^2} + \frac{2aqe^{af}|\nabla f|^2}{(\rho - f)^2} + \frac{2Ric_{\phi}(\nabla f, \nabla f)}{(\rho - f)^2} + \frac{2h(\nabla f, \nabla f)}{(\rho - f)^2} \\
+ \frac{2aqe^{af}|\nabla f|^2}{(\rho - f)^3} + \frac{2p\tilde{A}|\nabla f|^2}{(\rho - f)^3} \\
\geq \frac{2(f - \ln k)}{\rho - f} \langle \nabla w, \nabla f \rangle + 2(\rho - f)w^2 - \frac{2|\tilde{A}||\nabla p|\sqrt{w}}{\rho - f} + 2p\tilde{A}fw \\
- \frac{2e^{af}|\nabla q|\sqrt{w}}{\rho - f} + 2aqe^{af}w - 2(n - 1)k_1 w - 2k_2 w
\]
This completes the proof of Lemma. □

**Proof of Theorem 4.1** Choosing $\psi$ and $\eta$ as in the proof of Theorems 2.1 and 3.1. For any $T_1 \in (0, T]$, let $(x_3, t_3) \in Q_{2R,T_1}$ be a point where $\mathcal{H}(x, t) = \eta(x, t) w(x, t)$ achieve its maximum, and without loss of generality we can assume $\mathcal{H}(x_3, t_3) > 0$, and then $w(x_3, t_3) > 0$. By Lemma 4.2 and a similar argument as in the proof of Theorem 2.1, at point $(x_3, t_3)$ we have

\[
0 \geq (\Delta \phi - \partial_t) \mathcal{H} = (\Delta \phi - \partial_t) (\eta w)
\]

\[
= w(\Delta \phi - \partial_t) \eta + \eta(\Delta \phi - \partial_t) w + 2(\nabla \eta, \nabla w)
\]

\[
\geq \eta(\Delta \phi - \partial_t) w - \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1} + \frac{2}{R}\right) + \frac{3c_1}{R^2} + c_2 k_2\right) w.
\]

By multiplying both sides of above inequality by $\eta$ we can write

\[
0 \geq \frac{2(f - \ln k)}{\rho - f} \eta^2(\nabla w, \nabla f) + 2(\rho - f) \mathcal{H}^2
\]

\[
- \frac{2}{\rho - f} \eta^2 \left(\hat{A}|\nabla p| + e^{af} |\nabla q|\right) \sqrt{\mathcal{H}}
\]

\[
- 2\left(|p|\hat{A}_f| + a| q e^{af} + (n - 1)k_1 + k_2 + \frac{|q| e^{af}}{\rho - f} + \frac{|p| |\hat{A}|}{\rho - f}\right) \eta \mathcal{H}
\]

\[
- \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1} + \frac{2}{R}\right) + \frac{3c_1}{R^2} + c_2 k_2\right) \mathcal{H}. \quad (4.4)
\]

at point $(x_3, t_3)$. Now we deal with each term of the right-hand side of above inequality. Using Young’s inequality we have

\[
\frac{2(\ln k - f)}{\rho - f} \eta^2(\nabla w, \nabla f) = \frac{2(f - \ln k)}{\rho - f} \mathcal{H}(\nabla \eta, \nabla f) \leq 2 \ln \left(\frac{k}{\rho}\right) \frac{\sqrt{\mathcal{H}}}{R} \mathcal{H}^{\frac{1}{2}}
\]

\[
\leq \frac{1}{4} \mathcal{H}^2 + 6 \left(\ln \frac{k}{\rho}\right)^4 \frac{\sigma^2}{R^4},
\]

\[
\frac{2}{\rho - f} \eta^2 \left(\hat{A}|\nabla p| + e^{af} |\nabla q|\right) \sqrt{\mathcal{H}} \leq 2\left(\lambda_1 \gamma_2 + k^a \sigma_2\right) \sqrt{\mathcal{H}}
\]

\[
\leq \frac{1}{4} \mathcal{H}^2 + \frac{3}{4} \left(2\lambda_1 \gamma_2 + 2k^a \sigma_2\right) \sqrt{\mathcal{H}},
\]

and

\[
2\left(|p|\hat{A}_f| + a| q e^{af} + (n - 1)k_1 + k_2 + \frac{|q| e^{af}}{\rho - f} + \frac{|p| |\hat{A}|}{\rho - f}\right) \eta \mathcal{H}
\]

\[
\leq 2\left(\gamma_1 \lambda_2 + (a + 1) \sigma_1 k^a + (n - 1)k_1 + k_2 + \gamma_1 \lambda_1\right) \eta \mathcal{H}
\]

\[
\leq \frac{1}{4} \mathcal{H}^2 + 4\left(\gamma_1 \lambda_2 + (a + 1) \sigma_1 k^a + (n - 1)k_1 + k_2 + \gamma_1 \lambda_1\right) \eta \mathcal{H}.
\]

Using Young inequality again we get

\[
\left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1} + \frac{2}{R}\right) + \frac{3c_1}{R^2} + c_2 k_2\right) \mathcal{H}
\]
Substituting the above four inequality into (4.4), we obtain

\[ 0 \geq 2(\rho - f)\mathcal{H}^2 - \mathcal{H}^2 - 6 \left( \ln \frac{k}{\bar{k}} \right)^3 \frac{c_1^2}{R^2} - \frac{3}{4} \left( 2\lambda_1 \gamma_2 + 2k^a \sigma_2 \right)^\frac{3}{2} \]

\[ -4 \left( \gamma_1 \lambda_2 + (a + 1)\sigma_1 k^a + (n - 1)k_1 + k_2 + \gamma_1 \lambda_1 \right)^2 \]

\[ -\left( \frac{c_0}{R} (n - 1) \left( \sqrt{k_1} + \frac{2}{R} \right) + \frac{3c_1}{R^2} + c_2 k_2 \right)^2. \]

at point \((x_3, t_3)\). Since \(2\mathcal{H}^2 \leq 2(\rho - f)\mathcal{H}^2\) we get

\[ \mathcal{H}^2 \leq 6 \left( \ln \frac{k}{\bar{k}} \right)^3 \frac{c_1^2}{R^2} + \frac{3}{4} \left( 2\lambda_1 \gamma_2 + 2k^a \sigma_2 \right)^\frac{3}{2} \]

\[ +4 \left( \gamma_1 \lambda_2 + (a + 1)\sigma_1 k^a + (n - 1)k_1 + k_2 + \gamma_1 \lambda_1 \right)^2 \]

\[ +\left( \frac{c_0}{R} (n - 1) \left( \sqrt{k_1} + \frac{2}{R} \right) + \frac{3c_1}{R^2} + c_2 k_2 \right)^2. \]

at point \((x_3, t_3)\). As \(\eta(x, t) = 1\) in \(Q_{2R, T_1}\) and using inequality \(\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}\) for \(x, y \geq 0\), we have

\[ \frac{|\nabla u|}{u(\rho - \ln u)} \leq 6^\frac{1}{2} \left( \ln \frac{k}{\bar{k}} \right) \frac{\sqrt{c_1}}{R} + \left( \frac{3}{4} \right)^{\frac{1}{2}} \left( 2\lambda_1 \gamma_2 + 2k^a \sigma_2 \right)^{\frac{1}{2}} \]

\[ +\sqrt{2} \left( \gamma_1 \lambda_2 + (a + 1)\sigma_1 k^a + (n - 1)k_1 + k_2 + \gamma_1 \lambda_1 \right)^{\frac{1}{2}} \]

\[ +\left( \frac{c_0}{R} (n - 1) \left( \sqrt{k_1} + \frac{2}{R} \right) + \frac{3c_1}{R^2} + c_2 k_2 \right)^{\frac{1}{2}}. \]

The above inequality and \(\rho - \ln u \leq 1 + \ln \frac{k}{\bar{k}}\) imply that (4.1).

Similar to Corollary 2.5, we can deduce a global gradient estimate from Theorem 4.1 by taking \(R \to +\infty\) as follows.

**Corollary 4.3** Let \((M, g(0), e^{-\phi_0} dv)\) be a complete noncompact weighted Riemannian manifold without boundary, and let \(g(t), \phi(t)\) evolve by (1.2) for \(t \in [0, T]\). Let \(u\) be a positive solution to (1.1) in \(M\) such that \(k^3 \leq u \leq \bar{k}^3\) for some positive constants \(k\) and \(\bar{k}\). Suppose that there exist constants \(k_1, k_2\) such that

\[ \text{Ric} \phi \geq -(n - 1)k_1 g, \quad \text{h} \geq -k_2 g. \]

on \(M\). Then, we have

\[ \frac{|\nabla u|}{u} \leq \left( 1 + \ln \frac{k}{\bar{k}} \right) \left[ \left( \frac{3}{4} \right)^\frac{1}{2} \left( 2\Lambda_1 \Gamma_2 + 2k^a \Sigma_2 \right)^{\frac{1}{2}} + \sqrt{c_2 k_2} \right] \]

\[ +\sqrt{2} \left( \Gamma_1 \Lambda_2 + (a + 1)\Sigma_1 k^a + (n - 1)k_1 + k_2 + \Gamma_1 \Lambda_1 \right)^{\frac{1}{2}}, \]

on \(M\), where positive constants \(c_0, c_1,\) and \(c_2\) are introduced in the proof of the Theorem 2.1.
5 The fourth gradient estimate

In this section as in [23, 37] we obtain another type gradient estimate for positive solution of (1.1) under the geometric flow (1.2).

Theorem 5.1 Let \((M^n, g(0), e^{-\phi_0} dv)\) be a complete weighted Riemannian manifold, and let \(g(t), \phi(t)\) evolve by (1.2) for \(t \in [0, T]\). Given \(x_0, R > 0\), let \(u\) be a positive solution to (1.1) in \(Q_{2R,T}\) such that \(\dot{k} \leq u \leq k\) for some positive constants \(k\) and \(\ddot{k}\). Suppose that there exist constants \(k_1, k_2\) such that

\[
Ric_\phi \geq -(m-1)k_1g, \quad -k_2g \leq h \leq k_3g, \quad |\nabla h| \leq k_4,
\]
on \(Q_{2R,T}\). Then we have the following estimate

\[
\frac{|\nabla u|^2}{u^2} - \frac{1}{b} u_t - quu^a - p \hat{A} \leq \frac{4m}{t} + K_4
\]
on \(Q_{2R,T}\), where constant \(K_4\) is depending only on \(m, \alpha, k, \ddot{k}, k_1, k_2, k_3, \lambda_1, \lambda_2, \lambda_3, \sigma_1, \sigma_2, \gamma_1, \gamma_2, \gamma_3, \theta_1, \theta_2\) and \(c_0, c_1, c_2\) where \(c_0, c_1, c_2\) are defined in the proof of Theorem 2.1.

Before we prove Theorem 5.1, firstly we give the following lemma.

Lemma 5.2 Let \((M^n, g(t), e^{-\phi} dv)\) be a weighted Riemannian manifold, \(g(t)\) evolves by (1.2) for \(t \in [0, T]\) satisfies the hypotheses of Theorem 5.1. If \(s = u^{-b}\) and \(S = \frac{|\nabla s|^2}{s^2} + b \frac{s_t}{s} - b^2 q s^{-\frac{\ddot{k}}{b}} - b^2 p \hat{A}\) where \(b\) is a given positive constant, then we have

\[
(\Delta_\phi - \partial_t)S \geq \frac{1}{4m} \left( \frac{S}{b} + |\nabla \ln s|^2 \right)^2 + \frac{2}{5} |\nabla \ln s|^4 + \frac{2}{b} \langle \nabla S, \nabla \ln s \rangle
\]

\[-2(m-1)k_1 |\nabla \ln \dot{s}|^2 - a(a+2s)^{-\frac{(2+\ddot{k})}{b}} q |\nabla s|^2
\]

\[+2abs^{-\frac{(1+\ddot{k})}{b}} \langle \nabla q, \nabla s \rangle + aqs^{-\frac{\ddot{k}}{b}} - b^2 \hat{A}_s - b^2 \hat{A}_p - b^2 ps_\hat{A}_s \left( \frac{S}{b} + |\nabla \ln s|^2 \right)
\]

\[-b^2 p \hat{A}_s |\nabla s|^2 - 2b^2 \hat{A}_s \langle \nabla s, \nabla p \rangle
\]

\[-2(1-b)k_2 |\nabla \ln s|^2 - 2b^2 n(k_2 + k_3)^2
\]

\[-3b \sqrt{nk_4} + 2bk_3 \theta_1 + b \theta_2 |\nabla \ln s|,
\]

(5.2)

for \(0 < b \leq 1\) and if \(b > 1\) then it is enough in the above equality to replace \((1-b)k_2\) with \((b-1)k_3\).

Proof Since \(s = u^{-b}\) we have \(s_t = -bu^{-b-1} u_t, \nabla s = -bu^{-b-1} \nabla u\) and

\[
\Delta_\phi s = -bu^{-b-1} \Delta_\phi u + b(b+1)u^{-b-2} |\nabla u|^2.
\]

Thus, from (1.4) we get

\[
(\Delta_\phi - \partial_t)s = -bps \hat{A} - bqs^{1-\frac{a}{b}} + b + 1 |\nabla s|^2.
\]

(5.3)

By the weighted Bochner formula and Lemma 2.3 we have

\[
\Delta_\phi S = 2s^{-2} |\text{Hess} s|^2 + 2s^{-2} \langle \nabla \Delta_\phi s, \nabla s \rangle + 2s^{-2} Ric_\phi (\nabla s, \nabla s)
\]

\[-8s^{-3} \text{Hesss} (\nabla s, \nabla s) - 2s^{-3} |\nabla s|^2 \Delta_\phi s + 6s^{-4} |\nabla s|^4 - b^2 s^{-\frac{\ddot{k}}{b}} \Delta_\phi q
\]

\[+2abs^{-\frac{(1+\ddot{k})}{b}} \langle \nabla q, \nabla s \rangle - a(a+b)s^{-\frac{(2+\ddot{k})}{b}} q |\nabla s|^2 + abs^{-\frac{(1+\ddot{k})}{b}} q \Delta_\phi s
\]
\(-b^2 \Delta \phi (p \hat{A}) + bs^{-1} (\Delta \phi s) t - bs^{-2} (|\nabla s|^2) t - 2bs^{-2} h(\nabla s, \nabla s) - bs^{-2} s_t \Delta \phi s + 2bs^{-3} s_t |\nabla s|^2 + 2bs^{-1} \langle h, \text{Hesss}\rangle + 2bs^{-1} \left( \text{div} h - \frac{1}{2} \nabla (\text{tr}_g h), \nabla s \right) - 2bs^{-1} h(\nabla \phi, \nabla s) + bs^{-1} \langle \nabla s, \nabla \Delta \phi \rangle. \)

Using again Lemma 2.3 we obtain
\[ \partial_t S = -2s^{-2} h(\nabla s, \nabla s) + 2s^{-2} \langle \nabla (\Delta \phi - \partial_t) s, \nabla s \rangle + 2s^{-2} Ric(s, \nabla s) - 8s^{-3} \text{Hesss}(\nabla s, \nabla s) - 2s^{-3} |\nabla s|^2 (\Delta \phi - \partial_t s) s + 6s^{-4} |\nabla s|^4 - b^2 s^{-\frac{\hat{q}}{2}} (\Delta \phi - \partial_t) q + 2abs^{-1} (1 + \frac{\hat{q}}{2}) (\nabla q, \nabla s) - a(a + b)s^{-2} (\Delta \phi - \partial_t) 2q |\nabla s|^2 + abs^{-1} (1 + \frac{\hat{q}}{2}) q (\Delta \phi - \partial_t) s - b^2 (\Delta \phi - \partial_t) (p \hat{A}) + 2(1 - b)s^{-2} h(\nabla s, \nabla s) + bs^{-1} \partial_t (\Delta \phi - \partial_t) s - bs^{-2} |\nabla s|^2 - bs^{-2} s_t (\Delta \phi - \partial_t) s + 2bs^{-3} s_t |\nabla s|^2 + 2bs^{-1} \left( h, \text{Hesss} \right) + 2bs^{-1} \left( \text{div} h - \frac{1}{2} \nabla (\text{tr}_g h), \nabla s \right) - 2bs^{-1} h(\nabla \phi, \nabla s) + bs^{-1} \langle \nabla s, \nabla \Delta \phi \rangle. \]

Substituting (5.3) in above identity we infer
\[ (\Delta \phi - \partial_t) S = 2s^{-2} \left( |\text{Hesss}|^2 - 2s^{-1} \text{Hesss}(\nabla s, \nabla s) + s^{-2} |\nabla s|^4 \right) + \frac{2}{bs} \langle \nabla S, \nabla s \rangle + 2s^{-2} Ric(s, \nabla s) - a(a + 2)s^{-2} (2 + \frac{\hat{q}}{2}) q |\nabla s|^2 + 2abs^{-1} (1 + \frac{\hat{q}}{2}) \langle \nabla q, \nabla s \rangle + aqbs^{-1} (1 + \frac{\hat{q}}{2}) \langle \nabla s, \nabla \phi \rangle - b^2 p \hat{A} s \langle |\nabla s|^2 - 2b^2 \hat{A} s \langle \nabla s, \nabla p \rangle + 2(1 - b)s^{-2} h(\nabla s, \nabla s) + bs^{-1} \langle h, \text{Hesss} \rangle + 2bs^{-1} \left( \text{div} h - \frac{1}{2} \nabla (\text{tr}_g h), \nabla s \right) - 2bs^{-1} h(\nabla \phi, \nabla s) + bs^{-1} \langle \nabla s, \nabla \Delta \phi \rangle. \]

If \( b \leq 1 \) then from (2.10)–(2.12), for \( \epsilon = \frac{1}{4b} \) we have
\[ 2(1 - b)s^{-2} h(\nabla s, \nabla s) + 2bs^{-1} \langle h, \text{Hesss} \rangle + 2bs^{-1} \left( \text{div} h - \frac{1}{2} \nabla (\text{tr}_g h), \nabla s \right) - 2bs^{-1} h(\nabla \phi, \nabla s) + bs^{-1} \langle \nabla s, \nabla \Delta \phi \rangle \geq -2(1 - b)k_2 |\nabla \ln s|^2 - \frac{1}{2} s^{-2} |\text{Hesss}|^2 - 2b^2 n(k_2 + k_3)^2 - 3bs\sqrt{\theta_k} |\nabla \ln s| - 2bk_3 \theta_1 |\nabla \ln s| - b\theta_2 |\nabla \ln s|. \]

By Cauchy’s inequality, we have
\[ 2s^{-3} \text{Hesss}(\nabla s, \nabla s) \leq \frac{5}{4} s^{-2} |\text{Hesss}|^2 + \frac{4}{5} |\nabla \ln s|^4. \]
Applying (2.1) and (5.5) into (5.4) we obtain

\[
(\Delta \phi - \partial_t) S \geq \frac{1}{4} s^{-2} |\text{Hess} s|^2 + \frac{2}{5} |\nabla \ln s|^4 + \frac{2}{b} (\nabla S, \nabla \ln s) + 2s^{-2} Ric_{\phi}(\nabla s, \nabla s) - a(a + 2)s^{-\frac{4}{5}} q |\nabla s|^2 \\
+ 2ab s^{-\frac{1}{5}} q |\nabla s|^2 - b^2 \hat{A} \Delta \phi p - b^2 p \hat{A} s \Delta \phi s \\
- b^2 p \hat{A} s |\nabla s|^2 - 2b^2 \hat{A} s (\nabla s, \nabla p) \\
- 2(1 - b) k_2 |\nabla \ln s|^2 - 2b^2 n(k_2 + k_3)^2 \\
- 3b \sqrt{n} k_4 |\nabla \ln s| - 2b k_3 \theta |\nabla \ln s| - b \theta_2 |\nabla \ln s|.
\]

(5.7)

Applying (2.1) and \(s^{-1} \Delta \phi s = \frac{s^2}{b} + |\nabla \ln s|^2\) to (5.7), we arrive at (5.2). If \(b > 1\) then it is enough in the above equality to replace \((1 - b) k_2\) with \((b - 1) k_3\). \(\Box\)

**Proof of theorem 5.1** We Choose \(\psi\) and \(\eta\) as in the proof of Theorem 2.1 and define \(W(x, t) = t S(x, t)\). For any \(T_1 \in (0, T]\), let \((x_4, t_4) \in Q_{2R} T_1\) be a point where \(B(x, t) = \eta(x, t) W(x, t)\) achieve its maximum, and without loss of generality we can assume \(b \leq 1\) and \(B(x_4, t_4) > 0\), and then \(W(x_4, t_4) > 0\). By a similar argument as in the proof of Theorem 2.1, at point \((x_4, t_4)\) we have

\[
0 \geq (\Delta \phi - \partial_t) B = (\Delta \phi - \partial_t)(\eta W) \\
= W(\Delta \phi - \partial_t) \eta + \eta (\Delta \phi - \partial_t) W + 2(\nabla \eta, \nabla W) \\
\geq \eta t (\Delta \phi - \partial_t) S - \eta S - \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1} + \frac{2}{R}\right) + \frac{3c_1}{R^2} + c_2 k_2\right) W.
\]

Multiplying the inequality by \(t \eta\) on both sides and using Lemma 5.2 we can write

\[
0 \geq \frac{1}{4m} \eta^2 \left(\frac{W}{b} + t |\nabla \ln s|^2\right)^2 + \frac{2}{5} \eta^2 |\nabla \ln s|^4 + \frac{2(\eta t)^2}{b} (\nabla S, \nabla \ln s) \\
- 2(m - 1) k_1 (\eta t)^2 |\nabla \ln s|^2 - a(a + 2)(\eta t)^2 s^{-\frac{4}{5}} q |\nabla \ln s|^2 \\
+ 2ab (\eta t)^2 s^{-\frac{2}{5}} (\nabla q, \nabla \ln s) + aq (\eta t)^2 s^{-\frac{4}{5}} S - (\eta t)^2 b^2 \hat{A} \Delta \phi p \\
- b^2 \eta^2 t \frac{s \hat{A}}{b} \left(\frac{W}{b} + t |\nabla \ln s|^2\right) - b^2 (\eta t)^2 s p \frac{s^2 \hat{A} \hat{A} s |\nabla \ln s|^2} \\
- 2b^2 (\eta t)^2 s \hat{A} s (\nabla \ln s, \nabla p) - 2(1 - b) k_2 (\eta t)^2 |\nabla \ln s|^2 - 2nb^2 (\eta t)^2 (k_2 + k_3)^2 \\
- (\eta t)^2 \left(3b \sqrt{n} k_4 + 2b k_3 \theta + b \theta_2\right) |\nabla \ln s| - t \eta^2 S \\
- t \eta \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1} + \frac{2}{R}\right) + \frac{3c_1}{R^2} + c_2 k_2\right) W,
\]

(5.8)

at point \((x_4, t_4)\). Now we obtain a lower bound for each term of the right-hand side of above inequality. In point \((x_4, t_4)\), Cauchy’s inequality and Young’s inequality imply that

\[
\frac{2(\eta t)^2}{b} (\nabla S, \nabla \ln s) = -\frac{2nt}{b} W(\nabla \eta, \nabla \ln s) \geq -\frac{2}{b} \eta^2 t \frac{\sqrt{c_1}}{R} W |\nabla \ln s| \\
\geq -\frac{t}{2mb} \eta B |\nabla \ln s|^2 - \frac{2mtc_1}{b R^2} B,
\]

(5.9)

and

\[
\left(2(m - 1) k_1 + a(a + 2)s^{-\frac{4}{5}} q + b^2 ps \hat{A} s + b^2 ps^2 \hat{A} \hat{A} s + 2(1 - b) k_2\right) |\nabla \ln s|^2
\]
\[ \begin{aligned}
&\leq \left(2(m-1)k_1 + a(a+2)\tilde{k}^{-\frac{q}{p}}\sigma_1 + b^2\gamma_1 k \lambda_2 + b^2\gamma_1 k^2 \lambda_3 + 2(1-b)k_2\right)|\nabla \ln s|^2 \\
&\leq \frac{5 + 8m}{80m} |\nabla \ln s|^4 + \frac{20m}{5 + 8m} \left(2(m-1)k_1 + a(a+2)\tilde{k}^{-\frac{q}{p}}\sigma_1 + b^2\gamma_1 k \lambda_2 \\
&+ b^2\gamma_1 k^2 \lambda_3 + 2(1-b)k_2\right)^2.
\end{aligned} \] (5.10)

Using Young’s inequality again we arrive at
\[ 2abs^{-\frac{q}{p}}(\nabla q, \nabla \ln s) \leq 2ab\tilde{k}^{-\frac{q}{p}}\sigma_2|\nabla \ln s| \]
\[ \leq \frac{5 + 8m}{80m} |\nabla \ln s|^4 + \frac{3}{4} \left(\frac{20m}{5 + 8m}\right)^{\frac{1}{3}} (2ab\sigma_2)^{\frac{4}{5}} \tilde{k}^{-\frac{4q}{5p}}. \] (5.11)

Also, we have \( aq(\eta)\gamma_2s^{-\frac{q}{p}}S \geq -a\lambda_1 \eta \gamma k^{-\frac{q}{p}} B, b^2\hat{A}\Delta_{\phi}p \leq b^2\lambda_1 \gamma_3, \) and
\[ b\eta^2tps\hat{A}_s W \leq bt\eta \sigma_1 k \lambda_5 B. \]

According Young’s inequality,
\[ 2b^2s\hat{A}_s(\nabla \ln s, \nabla p) \leq 2b^2\lambda_2 \gamma_2|\nabla \ln s| \]
\[ \leq \frac{5 + 8m}{80m} |\nabla \ln s|^4 + \frac{3}{4} \left(\frac{20m}{5 + 8m}\right)^{\frac{1}{3}} (2b^2\lambda_2 \gamma_2)^{\frac{4}{5}} \] (5.12)

and
\[ \left(3b\sqrt{\eta}k_4 + 2bk_3 \theta_1 + b\theta_2\right)|\nabla \ln s| \]
\[ \leq \frac{5 + 8m}{80m} |\nabla \ln s|^4 + \frac{3}{4} \left(\frac{20m}{5 + 8m}\right)^{\frac{1}{3}} \left(3b\sqrt{\eta}k_4 + 2bk_3 \theta_1 + b\theta_2\right)^{\frac{4}{5}}. \] (5.13)

Substituting (5.9)–(5.13) into (5.8) yields
\[ 0 \geq \frac{1}{4mb^2}B^2 - \left[\frac{2mc_1}{bR^2} + a\sigma_1 \eta \gamma k^{-\frac{q}{p}} + bt\eta \sigma_1 k \lambda_2 + \eta \right. \\
+ t \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1} + \frac{2}{R}\right) + \frac{3c_1}{R^2} + c_2 k_2\right)B \\
- (\eta)\frac{20m}{5 + 8m} \left(2(m-1)k_1 + a(a+2)\tilde{k}^{-\frac{q}{p}}\sigma_1 + b^2\gamma_1 k \lambda_2 \\
+ b^2\gamma_1 k^2 \lambda_3 + 2(1-b)k_2\right)^2 \\
- (\eta)^{\frac{3}{4}} \frac{20m}{5 + 8m} \left(2ab\sigma_2\right)^{\frac{4}{5}} \tilde{k}^{-\frac{4q}{5p}} - (\eta)^2 b^2\lambda_1 \gamma_3 \\
- (\eta)^{\frac{3}{4}} \frac{20m}{5 + 8m} \left(2b^2 \lambda_2 \gamma_2\right)^{\frac{4}{5}} - 2nb^2(\eta^2)(k_2 + k_3)^2 \\
- (\eta)^{\frac{3}{4}} \frac{20m}{5 + 8m} \left(3b\sqrt{\eta}k_4 + 2bk_3 \theta_1 + b\theta_2\right)^{\frac{4}{5}}, \]

at point \((x_4, t_4).\) Set
\[ C_3 = \frac{2mc_1}{bR^2} + a\sigma_1 \eta \gamma k^{-\frac{q}{p}} + b\eta \sigma_1 k \lambda_2 + \left(\frac{c_0}{R} (n - 1) \left(\sqrt{k_1} + \frac{2}{R}\right) + \frac{3c_1}{R^2} + c_2 k_2\right) \]
and
\[
E_3 = \frac{20m}{5+8m} \left( 2(m-1)k_1 + a(a+2)\tilde{k}^{-\frac{a}{2}}\sigma_1 + b^2\gamma_1k_2 + b^2\gamma_1k^2\lambda_3 + 2(1-b)k_2 \right)^2 \\
+ \frac{3}{4} \left( \frac{20m}{5+8m} \right)^\frac{1}{3} (2ab\sigma_2)^{\frac{4}{3}} \tilde{k}^{-\frac{4a}{3\tilde{m}}} + b^2\lambda_1 \gamma_3 \\
+ \frac{3}{4} \left( \frac{20m}{5+8m} \right)^\frac{1}{3} (2b^2\lambda_2\gamma_2)^{\frac{2}{3}} + 2nb(k_2 + k_3)^2 \\
+ \frac{3}{4} \left( \frac{20m}{5+8m} \right)^\frac{1}{3} (3b\sqrt{n}k_4 + 2bk_3\theta_1 + b\theta_2)^{\frac{2}{3}}.
\]
Hence,
\[
0 \geq \frac{1}{4mb^2}B^2 - (\eta + tC_3)B - (\eta t)^2E_3,
\]
at point \((x_4, t_4)\). For a positive number \(\tilde{a}\) and two nonnegative numbers \(\tilde{b}, \tilde{c}\), the quadratic inequality of the form \(\tilde{a}x^2 - \tilde{b}x - \tilde{c} \leq 0\) implies that \(x \leq \frac{-\tilde{b}}{\tilde{a}} + \sqrt{\frac{\tilde{c}}{\tilde{a}}}\). Therefore,
\[
B \leq 4mb^2(\eta + tC_3) + 2b\eta t\sqrt{mE_3},
\]
at point \((x_4, t_4)\). Since \(B = \eta S\) and \(\eta(x, T_1) = 1\), we infer
\[
S \leq \frac{4mb^2}{t} + C_4 + 2b\sqrt{mE_3}
\]
at point \((x_4, t_4)\) where
\[
C_4 = 4mb^2 \left[ \frac{2mc_1}{bR^2} + a\sigma_1k^{-\frac{a}{2}} + b\sigma_1k\lambda_2 + \left( \frac{c_0}{R} \right)(n-1) \left( \sqrt{k_1} + \frac{2}{R} \right) + \frac{3c_1}{R^2} + c_2k_2 \right].
\]
Since \(T_1\) is arbitrary, and using inequality \(\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}\) for \(x, y \geq 0\), we obtain (5.1) where
\[
K_4 = 4m \left[ \frac{2mc_1}{bR^2} + a\sigma_1k^{-\frac{a}{2}} + b\sigma_1k\lambda_2 + \left( \frac{c_0}{R} \right)(n-1) \left( \sqrt{k_1} + \frac{2}{R} \right) + \frac{3c_1}{R^2} + c_2k_2 \right] \\
+ \frac{2m}{b} \sqrt{\frac{20}{5+8m}} \left( 2(m-1)k_1 + a(a+2)\tilde{k}^{-\frac{a}{2}}\sigma_1 + b^2\gamma_1k_2 + b^2\gamma_1k^2\lambda_3 \\
+ 2(1-b)k_2 \right) + \frac{\sqrt{3m}}{b} \left( \frac{20m}{5+8m} \right)^{\frac{1}{6}} (2ab\sigma_2)^{\frac{4}{3}} \tilde{k}^{-\frac{2a}{3\tilde{m}}} + 2\sqrt{m}\lambda_1\gamma_3 \\
+ \frac{\sqrt{3m}}{b} \left( \frac{20m}{5+8m} \right)^{\frac{1}{6}} (2b^2\lambda_2\gamma_2)^{\frac{2}{3}} + 2n\sqrt{2m}(k_2 + k_3) \\
+ \frac{\sqrt{3m}}{b} \left( \frac{20m}{5+8m} \right)^{\frac{1}{6}} (3b\sqrt{n}k_4 + 2bk_3\theta_1 + b\theta_2)^{\frac{2}{3}},
\]
for \(0 < b \leq 1\) and if \(b > 1\) then it is enough in the above equality to replace \((1-b)k_2\) with \((b-1)k_3\). \(\Box\)

Similar to Corollary 2.5, we can deduce a global gradient estimate from Theorem 5.1 by taking \(R \to +\infty\) as follows.
Corollary 5.3 Let \((M, g(0), e^{-\phi} dv)\) be a complete noncompact weighted Riemannian manifold without boundary, and let \(g(t), \phi(t)\) evolve by (1.2) for \(t \in [0, T]\). Let \(u\) be a positive solution to (1.1) in \(M\) such that \(\bar{k} \leq u \leq k\) for some positive constants \(k\) and \(\bar{k}\). Suppose that there exist constants \(k_1, k_2\) such that
\[
\text{Ric}_{\phi} \geq -(n-1)k_1 g, \quad -k_2 g \leq h \leq k_3 g, \quad |\nabla h| \leq k_4
\]
on \(M\). Then there exist positive constants \(c_0, c_1, \text{and} \ c_2\) such that
\[
\frac{|\nabla u|^2}{u^2} - \frac{1}{b} \frac{u_t}{u} - qu'' - p \hat{A} \leq \frac{4m}{t} + K_5
\]
where
\[
K_5 = 4m \left[ a \Sigma_1 k^{-\frac{3}{b}} + b \Sigma_1 k \Lambda_2 + c_2 k_2 \right] + \frac{2m}{b} \sqrt{\frac{20}{5 + 8m}} (2(m - 1)k_1 \\
+ a(a + 2) \bar{k}^{-\frac{3}{b}} \Sigma_1 + b^2 \Gamma_1 k \Lambda_2 + b^2 \Gamma_1 k^2 \Lambda_3 + 2(1 - b)k_2) \\
+ \frac{\sqrt{3m}}{b} \left( \frac{20m}{5 + 8m} \right)^{\frac{1}{6}} (2ab \Sigma_2)^{\frac{1}{2}} \bar{k}^{-\frac{a}{2b}} + 2\sqrt{m\Lambda_1 \Gamma_3} \\
+ \frac{\sqrt{3m}}{b} \left( \frac{20m}{5 + 8m} \right)^{\frac{1}{6}} (2b^2 \Lambda_5 \Gamma_2)^\frac{1}{3} + 2n \sqrt{2m} (k_2 + k_3) \\
+ \frac{\sqrt{3m}}{b} \left( \frac{20m}{5 + 8m} \right)^{\frac{1}{6}} (3b \sqrt{n} k_4 + 2b k_3 \Theta_1 + b \Theta_2)^\frac{5}{3}.
\]

Using the same arguments at in the proof of Corollary 2.6, we can obtain the following Harnack inequality.

Corollary 5.4 With the same assumptions in Corollary 4.3, for \((y_1, s_1) \in M \times (0, T)\) and \((y_2, s_2) \in M \times (0, T)\) such that \(s_1 < s_2\), we have
\[
u(y_1, s_1) \leq \nu(y_2, s_2) \left( \frac{s_2}{s_1} \right)^{4mb} \exp \left( \frac{J(y_1, s_1, y_2, s_2)}{4b} + (s_2 - s_1)b(k \Sigma_1 + \Gamma_1 \Lambda_1 + K_5) \right).
\]

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