1/f noise effect on dissipative dynamics of a LC-shunted qubit

F. T. Vasko

QK Applications, San Francisco, CA 94033, USA

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We consider dissipative dynamics of a flux qubit caused by 1/f noises, which act both on the shunting LC-contour and on the SQUID loop, and modulate the level splitting and the tunnel coupling, respectively. These classical Gaussian noises are partially correlated. The transient evolution of qubit is studied for the regimes: (a) the interwell incoherent tunneling, (b) the relaxation of interlevel population, and (c) the decoherence of the off-diagonal part of a density matrix. For all regimes, the relaxation rates and the frequency renormalization [for the case (c)] are analyzed versus the parameters of qubit and couplings to the noises applied through different channels. The fluctuation effect restricts an averaged description of evolution processes, so that the dissipative dynamics is not valid at tails of relaxation. The results obtained open a way for verification of relaxation parameters and demonstrate a possibility for minimization of coupling between qubit and environment. Under typical level of noises, the contributions considered here are comparable to recent experimental data on the population relaxation and the interwell tunneling.

I. INTRODUCTION

During last years, an essential progress was made towards the implementation of the quantum information protocols, see [1-5] and references therein. A major part of these results are based on different types of the superconducting flux qubits. Whereas the noiseless dynamic properties of these qubits are effectively analyzed with the use of the lumped-element approach, [6-8] both the mechanisms of the qubit-environment interaction and the dissipative dynamics of qubits are not investigated completely. Up to the date, a partial characterization of qubits is performed with the use of the spectroscopy in GHz region, for different regimes of the h.f. excitation and of the readout [9-12]. Beside of this, the incoherent resonant tunneling, both between the anticrossing levels (the Landau-Zener transitions [13]) and between the steady-state levels [14] is employed for study. The experimental data suggest that the qubit-environment interaction is caused by the low-frequency classical noise (described by the near 1/f spectral function) and the high-frequency bosons (described by the quasi-ohmic spectral function), see analysis in [15]. Recent studies of these processes are motivated by the aim to improve the fidelity of the quantum hardware but any consideration based on a simple model of the spin-noise interaction (e.g., see [16, 17] and references therein) is not enough because such results can not be associated to a real device. Thus, it is timely to perform a more study, which is based on the limped-element method and involved a realistic description of a qubit-environment interaction.

In this paper, we perform a complete analysis of the dissipative dynamics for the qubit formed by the SQUID loop shunted by the transmission line (so called fluxmon [15]), see sketch in Fig. 1(a). Parameters of such a qubit are governed by the tilt and control fluxes, \( \varphi \) and \( \psi \), applied through the transmission line and the SQUID loop, respectively. We restrict ourselves by the low-frequency region when interactions with 1/f noises are described by the classical random fluxes, \( \Delta \varphi \) and \( \Delta \psi \), which are added to the tilt and control fluxes. The calculations of transient evolution are based on the simplified model of the qubit incorporated the effective Josephson junction shunted by effective LC contour under external fluxes \( \varphi + \Delta \varphi \) and \( \psi + \Delta \psi \) (blue and red). (c) Modulation of symmetric potential energy (thick curve) via tilt (tl-) and control (j-) channels (red and green, respectively) resulted in variations of level splitting and barrier height, respectively. (d) The same as in panel (c) for the tilted potential, \( \varphi \neq 0 \).

FIG. 1: (a) Sketch of qubit formed by the Josephson junction loop shunted by the transmission line. Tilt and control fluxes, \( \varphi \) and \( \psi \), are shown by blue and red arrows, respectively, with random contributions, \( \Delta \varphi \) and \( \Delta \psi \). (b) Model circuit involved the effective Josephson junction shunted by effective LC contour under external fluxes \( \varphi + \Delta \varphi \) and \( \psi + \Delta \psi \) (blue and red). (c) Modulation of symmetric potential energy (thick curve) via tilt (tl-) and control (j-) channels (red and green, respectively) resulted in variations of level splitting and barrier height, respectively. (d) The same as in panel (c) for the tilted potential, \( \varphi \neq 0 \).

*Electronic address: ftvasko@gmail.com*
ics of the qubit is determined by the correlation functions \( \langle \Delta \varphi_1 \Delta \varphi_1 \rangle \), \( \langle \Delta \psi_1 \Delta \psi_1 \rangle \), and \( \langle \Delta \varphi_1 \Delta \psi_1 \rangle \), which describe noises excited in the transmission line and in the SQUID loop (tl- and j-channels) as well as correlations between these noises (c-channel) \[19\]. These correlations may caused by a common external sources or an inductive coupling between tl- and j-channels. We use the 1/f spectral functions \( \alpha_k / \omega \) with \( k = (tl, j, c) \) so that the dissipative dynamics is determined by the parameters of qubit, the dimensionless noise strengths \( \alpha_k \), and the control fluxes, \( \varphi \) and \( \psi \).

The two lowest electromagnetic modes of the flux qubit are described by the isospin variable and the exact dynamics of qubit is governed by the Bloch equations with a time-dependent rotation frequency. We analyze the transient evolution of the density matrix at \( t > 0 \) for the regimes: (a) of the resonant tunneling between the left and right wells, when the noise-induced broadening exceeds the interlevel energy, (b) of the relaxation of interlevel population, and (c) of the decoherence process of the off-diagonal part of a density matrix. The cases (b) and (c) correspond to the weakly-coupled levels, when the gap frequency exceeds the noise-induced broadening of levels \[20\]. The Bloch equations for the case (a) is solved with the use of the symmetric qubit frame when the eigenstate problem is solved numerically at \( \varphi = 0 \). For the cases (b) and (c) we use the tilt \( \varphi \)-dependent frame, when the basic functions correspond the lower and upper levels. For all cases, it is convenient to transform the Bloch system into the single integro-differential equations with the different kernels and to employ the weak-fluctuation approximation. Within this approach, the averaged responses are written through the Laplace transforms of the averaged kernels and the parameters of relaxation are obtained in the explicit forms. Straightforward calculations of the first-order fluctuating corrections for the cases (a)-(c) give the limitations at tails of relaxation for the average approach outlined above.

Based on this model, we have performed a complete analysis of the tunneling rate, the population relaxation rate, and the decoherence rate, \( \nu_T, \nu_1, \) and \( \nu_2 \), as well as the noise-induced renormalization of the gap frequency. The main results are summarized as follows:

1. Peaks of the relaxation rates between wells or between levels, \( \nu_T \) or \( \nu_1 \) versus tilt \( \varphi \), are due to noise in tl-channel while dip of decoherence rate \( \nu_2 \) near \( \varphi = 0 \) is determined by an interplay of noises in tl- and j-channels. Weak asymmetry of \( \nu_{T,1,2} \) around \( \varphi = 0 \) is determined by c-channel, i.e. by correlations between tl- and j-noises.

2. The half-width of the Gaussian peak of incoherent tunnelling is \( \sqrt{\alpha_0} \), while the amplitude is \( 1/\sqrt{\alpha_0} \). Weak shift of peak is \( \alpha_0 / \alpha_{tl} \) and the direction of shift (to \( \varphi > 0 \) or \( \varphi < 0 \)) is determined by the orientation of fluxes \( \varphi \) and \( \psi \) [parallel or anti-parallel, see Fig. 1(a)]. In addition to the mechanism considered in Ref. 14 and 21, the classical 1/f noise provides essential contribution to the experimental data reported in Ref. 15.

3. The population relaxation rate is \( \propto \alpha_0 \) with tails of peak suppressed as \( \varphi^3 \) and the parameters of peak are comparable to the recent experimental data \[15\]. Due to softness of the barrier, there is no the dependency \( \nu_1 \propto \omega_0^{-1} \) for the case of 1/f noise considered. This disagreement with \[15\] may be compensated by the frequency dispersion of noise caused by the size effect in transmission line \[22\].

4. The decoherence rate \( \nu_2 \) and the renormalization of the gap frequency are increased sharply around \( \varphi = 0 \) and are saturated at tails of the dip. There is the two-mode oscillating decoherence regime with the depth and asymmetry of dip are determined by the ratios \( \alpha_j / \alpha_{tl} \) and \( \alpha_c / \alpha_{tl} \), respectively.

5. The averaged descriptions fail at tails of relaxation because of the rare fluctuations effect. The mean-square-fluctuation of the interwell tunneling increases \( \propto t^2 \) and, for typical parameters, the average description of incoherent tunneling is valid at \( \nu_T t < 3 \). There are time-independent levels of fluctuations for the interlevel population and the decoherence [cases (b) and (c)] with logarithmically enhanced contribution of rare fluctuations. The tilt effect suppresses the fluctuation effects for all regimes.

The paper is organized as follows. The model of the LC-shunted qubit interacted with partially correlated noises in the tilt and control channels is presented in Sec. II. The resonant incoherent tunneling between wells is considered in Sect. III. The population relaxation rate and the decoherence process are analyzed in Sec. IV for the weak-coupling regime. Dynamics of fluctuations is considered in Sect. V for all cases. The concluding remarks, the list of assumptions, and the discussion of current experimental context are given in the last section. The averaged kernels are calculated in Appendix taking into account tl-, j-, and c-channels.

II. QUBIT-NOISE INTERACTION

We start with description of the flux qubit formed by SQUID shunted by LC-contour which are interacted with 1/f noises. Quantum mechanics of such a qubit in the \( \phi \)-representation is described by the Hamiltonian:

\[
\hat{H} = 4E_C \hat{q}^2 + E_L (\phi - \varphi)^2 / 2 + E_J \cos(\psi/2) \cos \phi .
\] (1)

Here \( \hat{q} \) and \( \hat{\varphi} = -i d/d\phi \) are the dimensionless operators of flux and charge, while \( \varphi \) and \( \psi \) are the external tilt and control fluxes penetrated through the LC-contour and the SQUID loop. We use the dimensionless variable \( \phi \) as well as the control fluxes \( \varphi \) and \( \psi \) in units \( \Phi_0 / 2\pi \) where \( \Phi_0 = \pi h/|e| \) is the flux quantum. Eq. (1) involves the charging energy \( \propto E_C = e^2/2C \), the inductive energy \( \propto E_L = (\Phi_0/2\pi)^2/L \), and the effective Josephson energy \( \propto E_J = I_c \Phi_0/2\pi \) which describes the SQUID
loop with the critical current \( I_c \) and can be varied via the factor \( \cos(\psi/2) \). If energy scale is in units \( E_L \), the Hamiltonian (1) is dependent on the external fluxes \( \varphi \) and \( \psi \), as well as on the parameters \( 4E_C/E_L \ll 1 \), and \( \beta_\varphi = E_L \cos(\psi/2)/E_L \approx 1 \). Noise contributions are introduced under the replacements of external factors in Eq. (1) by \( \varphi \to \varphi + \Delta \varphi_t \) and \( \psi \to \psi + \Delta \psi_t \). The Hamiltonian of qubit-noise interaction,

\[
\hat{H}_{\text{int}} = -\Delta \varphi_t E_L \phi(\varphi - \varphi) - \Delta \psi_t E_J \sin(\psi/2)/(\cos \phi)/2, \tag{2}
\]
is written within the linear approach with respect to dimensionless random contributions, \( \Delta \varphi_t \) and \( \Delta \psi_t \). Notice, that energy spectrum determined by the noiseless Hamiltonian (1) is not changed under the replacement \( \varphi \to -\varphi \) (with \( \phi \to -\phi \)) or \( \psi \to -\psi \). In contrast, \( \hat{H}_{\text{int}} \) is symmetric if only \( \varphi \to -\varphi \) and \( \psi \to -\psi \), i.e. the relaxation processes are different for the parallel or antiparallel directions of the external fluxes.

Under averaging over random contributions \( \Delta \varphi_t \) and \( \Delta \psi_t \), we consider the Gaussian random processes taking into account a partial correlation between noises in the \( tl \)- and \( j \)-channels. Introducing the \( 1/f \) spectral function of the \( k \)-th channel as \( \alpha_k/\omega \), one obtains

\[
\langle \Delta \varphi_t \Delta \varphi_{t'} \rangle = \alpha_k \int \frac{d\omega}{\omega} \cos \omega \Delta t \equiv \alpha_k w_{\omega_m} \Delta t,
\]

where \( \alpha_k \) determines a coupling strength in \( k \)-th channel \( (k = tl, j, c) \), \( \Delta t = t - t' \), and we use the cut-off factors \( \omega_{\omega_m} \), which are supposed to be the same in all channels if \( \omega_m^{-1} \ll \Delta t \ll \omega_m \). The explicit expression of the correlator \( w_{\omega_m}(\Delta t) \) is written through the integral cosine which has the logarithmic asymptotic at \( \omega_{\omega_m} \Delta t \ll 1 \) with the Euler’s constant, \( \gamma \approx 0.5772 \). Thus, qubit-environment interaction is characterized by three constants, \( \alpha_k \), and the cut-off frequency \( \omega_m \).

Below we use the basis of the symmetric \( (\varphi = 0) \) eigenstate problem \( \hat{H}_{\varphi=0,\beta} | n \rangle = \varepsilon_n | n \rangle \), where \( \hat{H}_{\varphi,\beta} = \hat{H} - \hat{H}_{\text{int}} \). In the symmetric qubit frame the solutions \( | n \rangle \) and \( \varepsilon_n \) are parametrically dependent on control flux through \( \beta_\varphi \) and \( \varphi = 0 \). The density matrix takes form \( \hat{\rho} = \sum_{n_1, n_2} \rho_{n_1, n_2} | n_1 \rangle \langle n_2 | \), where \( \rho_{n_1, n_2} \) is governed by the standard equation \( i\hbar \partial_t \rho_{n_1, n_2}/\partial t = \{\hat{H}, \hat{\rho} \}_{n_1, n_2} \) with the effective Hamiltonian,

\[
\hat{H}_{nn'} = \varepsilon_n \delta_{nn'} - E_L \phi(\varphi)_{nn'} + \left( \hat{H}_{\text{int}} \right)_{nn'}, \tag{4}
\]

which is written through matrix elements \( \langle \cdot \cdot \cdot \rangle_{nn'} = \langle n \cdot \cdot \cdot | n \cdot \cdot \cdot \rangle \). Note, that a symmetric part of barrier determines \( \varepsilon_n \) while the tilt effect is described here by the off-diagonal matrix element \( \propto (\phi)_{nn'} \). We employ below the two-level approach and consider only the states \( | 0 \rangle \) and \( | 1 \rangle \). Introducing the time-independent gap frequency \( \omega_{10} = (\varepsilon_1 - \varepsilon_0)/\hbar \) and the Pauli matrices \( \sigma \), which are written here with respect to the basis formed by the \( | 0 \rangle \) and \( | 1 \rangle \) states, we arrive to the \( 2 \times 2 \) matrix Hamiltonian

\[
\frac{\hat{H}_t}{\hbar} = \frac{\omega_\varphi \sigma_\varphi}{2}, \quad \omega_t = -2\Delta \varphi_t(0, \omega_{10} - \omega_j \Delta \psi_t), \quad \omega_j = \frac{E_J (\cos \phi)_{11,00} \sin \psi/2}{\hbar}. \tag{5}
\]

Here the frequencies \( \tilde{\omega} \) and \( \omega_j \) determine the additional tunneling mix of the \( | 0 \rangle \) and \( | 1 \rangle \) states due to the total tilt flux and the coupling of qubit with the noise due to barrier fluctuations, respectively. We also omit the contributions \( \propto \text{const} \) and take into account that the wave functions \( \langle \varphi | 0 \rangle \) and \( \langle \varphi | 1 \rangle \) are symmetric and antisymmetric with respect to \( \varphi \to -\varphi \). The matrix elements \( (\cos \phi)_{10} \) and \( (\cos \phi)_{11,00} \) are dependent on \( \beta_\varphi \) and \( \omega_j \) is written through \( \sin(\psi/2) \).

The projection operators on the left/right \((l/r)\) wells in \( \varphi \)-representation with the Hamiltonian (1) are \( \hat{P}_{l/r} = \hat{\theta}(-/+ \varphi) \), so that \( \hat{P}_l + \hat{P}_r = 1 \) and \( \hat{P}_l \cdot \hat{P}_r = 0 \). Within the two-level approach described by the Hamiltonian (5) these operators are given by

\[
\hat{P}_{l/r} = \begin{pmatrix} |11\rangle & |10\rangle & 0 \rangle & |00\rangle \end{pmatrix}_{\varphi < 0/\varphi > 0} \approx 1 - \frac{1}{2} \delta_x, \tag{6}
\]

where \( \langle 1 |1\rangle_{\varphi < 0/\varphi > 0} = 0 \) and \( \langle 0 |0\rangle_{\varphi < 0/\varphi > 0} = 1/2 \) due to symmetry reason. Numerical estimates of the off-diagonal matrix elements with the eigenstates calculated below (see Fig. 2 below) give\( 0.45 < |\langle 1|0\rangle_{\varphi < 0/\varphi > 0}| < 0.5 \) if \( \beta > 1.1 \) and \( |\hat{P}_l \cdot \hat{P}_r| < 0.05 \) instead of the exact zero. This inconsistency is due to we drop the upper states, \( n > 1 \), and below in Sect. III we use the right-hand expression of Eq. (6) which corresponds the weak-tunnel-coupling case.

If tunneling mix increases, it is convenient to use the \emph{tilt qubit frame}. After rotation of \( \hat{H}_t \) around 0Y-axis Eqs. (5) are transformed into

\[
\hat{H}_t = \frac{(\Omega_z \cdot \sigma)}{2}, \quad \Omega_z = \begin{pmatrix} \Delta \varphi_t \omega_{10} + \Delta \psi_t \omega_j \varphi \omega_x \omega_\varphi \left( \sqrt{\omega_{10}^2 + (2\bar{\omega}^2)^2} \right) \end{pmatrix}, \tag{7}
\]

where \( \Omega_x = \sqrt{\omega_{10}^2 + (2\bar{\omega}^2)^2} \) is the level-splitting frequency and random contributions of \( \Omega_x \) give rise to the noise-induced inter-level transitions and the renormalization of frequency \( \Omega_x \). The projection operators on the upper \((+\) \) and lower \((-\) \) levels are determined as \( \hat{P} = (1 \pm \delta_z)/2 \) which is similar to Eq. (6) but \( \hat{P} \) is written in the basis formed by the eigenstates determined by the Hamiltonian (7).

Numerical solutions of the symmetric eigenstate problem \( \hat{H}_{\varphi=0,\beta} | n \rangle = \varepsilon_n | n \rangle \) are obtained after discretization of the Hamiltonian (1) using \( N \sim 10^3 \) points along \( \phi \) axis and diagonalization of \( N \times N \) matrix. In Fig. 2(a) we plot the dimensionless splitting frequencies \( \omega_{10} \) and
system is transformed into the integro-differential equation for redistribution of population between wells. The $2 \times 2$ density matrix takes form $\hat{\rho}_{ij} = (1 + S_{ij} \cdot \vec{\sigma})/2$ with $k = l, r$ and the population in $k$th well $n_k = \text{tr} \hat{P}_k \hat{\rho}_i$ is determined through the projection operators, $\hat{P}_k$, given by Eq. (6). At $t > 0$ the Bloch vector $S_t$ is governed by the standard equation

$$
\frac{dS_t}{dt} = \{\omega \times S_t\}, \quad S_{t=0} = (\mp 1, 0, 0)
$$

with the normalization condition $|S_t| = 1$ and the initial values $S_{t=0x} \equiv S_0 = -1 + 0.1$ or +1 corresponding to qubit which localized in the $l$- or $r$-wells, respectively. In order to reduce the system (8) into equation for $S_{lx}$ we write $S_{ly,z}$ through the integrals $\int_0^t dt' \cdots S_{lx}$ as

$$
\left| \frac{S_{ly}}{S_{lx}} \right| = \int_0^t dt' \cos \theta_{t'} \left| \sin \theta_{t'} \right| \Omega_{t'z} S_{lx}.
$$

After substitution $S_{ly,z}$ into X-component of Eq. (8) one obtains the exact integro-differential equation

$$
\frac{dS_{lx}}{dt} + \omega_{lx} \int_0^t dt' \omega_{lx'} \cos \theta_{t'} S_{lx'} = 0
$$

with the initial condition $S_{t=0,x} = \mp 1$ and the phase factor $\theta_{t'} = \int_0^t dt \omega_{x'}$. Below we separate the averaged part of the Bloch vector, $\langle S_{lx} \rangle$, in Eq. (10) and consider equation

$$
\frac{d}{dt} \langle S_{lx} \rangle + \int_0^t dt' K_{t'} \langle S_{lx'} \rangle = 0, \quad K_{t'} = \langle \omega_{lx} \omega_{lx'} \cos \theta_{t'} \rangle,
$$

where the averaged kernel $K_{t'}$ should be written through the correlators (3). We restrict ourselves by the weak fluctuation regime when $\delta S_t = S_{lx} - \langle S_{lx} \rangle$ can be omitted, see Sect. V A. Straightforward averaging of the kernel $\langle K_{t'} \rangle = K_{\Delta t}$ gives

$$
K_{\Delta t} = e^{-\Gamma_{t} \omega_{lx} (\sin \theta_{lx} B_{\Delta t} - \sin \theta_{lx} B_{\Delta t})},
$$

$$
A_{\Delta t} = \omega_{10}^2 + \omega_{ji}^2 (\alpha_{lx} w_{\omega_{m,\Delta t}} - W_{\Delta t}^2), \quad B_{\Delta t} = 2 \omega_{lx} \omega_{10} W_{\Delta t}
$$

with the noiseless phase $\theta_{\Delta t} = 2\psi \Delta t$ and caused by tilt flux, see Appendix for details. This kernel is dependent on the qubit energies $E_{L,C,J}$ through the frequencies introduced by Eq. (5), the noise strengths $\alpha_{lx,c}$ and $\omega_m$ from Eq. (3), as well as on the external fluxes, $\alpha$ and $\psi$. The decrement of exponential damping, $\Gamma_{\Delta t}$, and the factor $W_{\Delta t}$ in $A_{\Delta t}$ and $B_{\Delta t}$ of Eq. (12) are determined by the integrals

$$
\frac{\Gamma_{\Delta t}}{2 \alpha_{lx}} \int_0^{\Delta t} d\tau \int_0^{\Delta t} d\tau' w_{|\omega_m |\Delta t} = (\omega_{lx} \Delta t) \left( w_{|\omega_{m} \Delta t} | + \frac{3}{2 \pi} \right),
$$

$$
\frac{W_{\Delta t}}{2 |\alpha_{lx}|} \int_0^{\Delta t} d\tau \int_0^{\Delta t} d\tau' w_{|\omega_m |\Delta t} = \omega_{lx} \Delta t \left( w_{|\omega_{m} \Delta t} | + \frac{3}{2 \pi} \right).
$$

FIG. 2: (a) Level splitting frequencies $\omega_{10}$ and $\omega_{21}$ in units $E_L/\hbar$ versus ratio $\beta_\psi = \omega_{10} \sin (\psi/2)/E_L$. Insets show wave functions $\langle \phi(n) \rangle$ for $n = 0$ (black), 1 (red), and 2 (green) at $4E_C/E_L = 0.05$. (b) Matrix element $\langle \phi \rangle_{10}$ versus $\beta_\psi$. (c) Matrix element $\langle \cos \phi \rangle_{11,00}$ versus $\beta_\psi$. Red dashed lines in panels (a) and (c) show exponential asymptotics for the weak coupling regime. In all panels, the dotted, solid, and dashed curves correspond to $4E_C/E_L = 0.01$, 0.005, and 0.0025, respectively.

III. INCOHERENT RESONANT TUNNELING

We evaluate here the rate of the incoherent resonant tunneling between l- and r-wells based on the Bloch equation with random frequency $\omega_t$ given by Eq. (5). This

\[ \omega_{21} \text{ as well as the wave functions for } n = 0, 1, \text{ and } 2 \text{ versus control flux determined by the ratio } \beta_\psi. \text{ For the weak-tunneling regime, if } \beta_\psi > 1.2 \pm 1.3 \text{ depending on } 4E_C/E_L, \text{ the frequency } \omega_{21} \text{ is suppressed exponentially. The dimensionless matrix elements } \langle \phi \rangle_{10} \text{ and } \langle \cos \phi \rangle_{11,00}, \text{ which determine the characteristic frequencies } \omega_j \text{ and } \psi_j, \text{ are plotted in Figs. 2(b) and 2(c), respectively. Dependency of } \langle \phi \rangle_{10} \text{ on } 4E_C/E_L \text{ is negligible, see Fig. 2(b), and the matrix element } \langle \cos \phi \rangle_{11,00} \text{ is exponentially suppressed at } \beta_\psi > 1.2 \pm 1.3, \text{ see asymptotes in Figs. 2(a) and 2(c). We analyze relaxation rates versus } \omega_{10} \text{ and frequencies in Eq. (5); under mapping between } \omega_j \text{- and } \psi_j\text{-variables the non-monotonic dependence of } \omega_j = \text{sign}(\psi)(E_j/2h)(\cos \phi)_{11,00} \sqrt{1 - (\beta_\psi/\beta_0)^2} \text{ on } \psi \text{ may be noticeable: below we use } \psi > 0 \text{ so that } \omega_j > 0. \]
In order to write the explicit expressions here, we use the definition of \( w_{\omega_0} \) given by Eq. (3) and perform integrations by parts over \( \tau \) and \( \tau' \). Under these integrations we replaced the upper limit \( \omega_M \Delta t \) by \( \infty \).

According to Eq. (6), transient evolution of the population in left (or right) well is governed by \( (1 - (S_{tx})/2 \) or \( (1 + (S_{tx})/2) \). The Laplace transform of the averaged Bloch vector, \( S_{tx} \), is obtained from the integro-differential equation (10) with the use of the convolution theorem as follows \( S_{tx} \) takes form:

\[
(S_{tx}) = S_0 \int_C \frac{d\zeta}{2\pi i} \frac{e^{\zeta t}}{\zeta + K_{\zeta}} \propto e^{-\nurt t}, \quad \text{if} \quad \nu rt \gg 1. \tag{14}
\]

The relaxation rate at tail of tunneling decay, \( \nu r \), is obtained as a pole of the complex integral (14), where the contour \( C \) is around complex half-plane \( \text{Im}\zeta < 0 \). Such a pole is determined by the equation \( \zeta + K_{\zeta} = 0 \). As a result, \( \nu r \approx K_{\zeta} \to 0 \) is given by the integral

\[
\nu r \approx \int_0^\infty d\tau e^{-\tau/\nu r} \left( \cos \omega_T \omega_T \Delta t - \sin \omega_T \eta T \right), \tag{15}
\]

where the time scales \( \omega_T \Delta t \leq \text{const}/\sqrt{\omega_T} \) are only essential. Under integration of Eq. (15) we use the logarithmic approach, when the factors determined by Eq. (13) are replaced by the linear and quadratic dependencies, \( W_{\Delta t} \approx 2\alpha_{\Delta t} \omega T \Delta t \) and \( \Gamma_{\Delta t} \approx 2\alpha_{\Delta t} \eta T (\omega_T \Delta t)^2 \). Based on Eq. (3), here we introduced \( \Lambda_T \) according to

\[
w_{\omega m} = \ln(\sqrt{\omega_T} \Lambda_T \omega_T \omega_m) + ct = \Lambda_T, \tag{16}
\]

so that \( \Lambda_T \approx \ln(\sqrt{\omega_T} \omega_T \omega_m) + ct \gg 1 \) with \( ct \sim 1 \). Straightforward integration of Eq. (15) is performed for \( A_{\Delta t} \approx \omega_T^2 \) because \( \alpha_{\Delta t} < \alpha_{tl} \ll 1 \), see estimates below. The explicit formula for the tunneling rate versus tilt flux is given by the Gaussian peak which is modulated by the linear-dependent on \( \varphi \) pre-factor:

\[
\nu r \approx \bar{\nu} \tau e^{-\varphi/\varphi_c} (1 - \varphi/\varphi_c), \tag{17}
\]

\[
\bar{\nu} = \sqrt{\frac{\omega_T^2}{2\varphi_{T}\bar{\omega}}}, \quad \varphi_T = \sqrt{2\alpha_{tl} \Lambda_T}, \quad \varphi_c = \frac{\alpha_{\Delta t} \omega_T}{2\alpha_{\eta T} \Lambda_T}.
\]

Here we introduce the amplitude of peak at symmetric point, \( \bar{\nu} \), the half-width of peak \( \varphi_T \ll 1 \), and the asymmetry parameter \( \varphi_c \). An asymmetry of peak with respect to \( \varphi \rightarrow -\varphi \), at fixed direction of \( \psi \), appears due to correlations between \( tl \)- and \( j \)-channels, \( \varphi_c^{-1} \sim \alpha_c \). This effect is weak enough because \( \varphi_T \ll \varphi_c \); in terms of the shift of peak determined by the requirement \( (d\nu r/d\varphi)_{\varphi=0} = 0 \), we obtain \( \varphi_T \approx \varphi_T^2/2 \varphi_c \approx \varphi_T \).

For the estimates below we use a typical \( 1/1 \) noise level \( \leq 4 \times 10^{-11} \Phi_0^2/\text{Hz} \) determined by the flux-flux spectral function at 1 Hz. [14, 15] so that \( \alpha_{tl, j} \ll \leq 10^{-8} \). The half-width of peak, \( \varphi_T \), is only determined by the level of noise in the \( tl \)-channel and by the ratio \( \bar{\omega}/\omega_m \), through \( \Lambda_T \). Supposing that \( \Lambda_T \sim 10 \div 15 \), one obtains \( \varphi_T \sim (4.5 \div 5.5) \times 10^{-4} \), or in the dimensional units the maximal half-width of peak is \( \sim 70 \div 85 \mu \Phi_0 \). According to Fig. 3 the maximal shift of peak is determined by the ratio

\[
\omega_j/\omega_{10} \sim 15 \div 30, \quad \text{so that} \quad \Phi_0 \approx 2\alpha_{\Delta t} \omega_j/\omega_{10} \leq (3/6) \times 10^{-7} \text{ or } 0.1 \mu \Phi_0.
\]

\[\text{Since} \quad \omega_j \sim 1.2E_L/h \quad \text{is weakly (less than 20%) dependent on the barrier-control variable} \beta_e \text{ or} \omega_{10} \text{[Fig. 2(b)]}, \text{the amplitude} \quad \nu r \quad \text{is approximately proportional to} \omega_{10}, \text{i.e. the rate increases, when overlap of the left and right wave functions increases. At} \quad \omega_{10}/2\pi \approx 0.5 \text{MHz for the device with} E_L/h \approx 250 \text{GHz, one obtains} \quad \nu r \approx 8.4 \times 10^{-13} \text{ms}^{-1} \text{at} \alpha_T \ll 10^{-8} \quad \text{and the maximal tunneling rate decreases with a noise level as} \nu r \propto \alpha_T^{-1/2}. \text{The numerical estimates performed show that the mechanism suggested gives an essential contribution to the resonant tunneling reported in [12] but a shape of peak differs from the experimental data, see discussion in Sect. VI.}

\[\text{IV. WEAK QUBIT-NOISE COUPLING}
\]

We consider below the weak qubit-noise coupling, when the gap frequency \( \Omega \) exceeds the noise-induced modulation of levels and it is convenient to use the tilt qubit frame described by the Hamiltonian \( H \). The 2 x 2 density matrix \( \rho_t = (1 + R_t \cdot \sigma) / 2 \) is written through the Bloch vector \( R_t \) which is governed by the equation \( dR_t/dt = [\Omega_t \times R_t] \) with the normalization condition \( |R_{t=0}| = 1 \). Introducing the circular components \( R_{t\pm} = (R_{tx} \pm iR_{ty})/2 \) we arrive to the system

\[
\left\{ \begin{array}{l}
(d/dt \mp i\Omega_t) R_{t\pm} \pm i\Omega_t R_{t\mp} / 2 = 0 \\
\left( dR_{tx}/dt \pm i\Omega_t R_{ty}(R_{t+} - R_{t-}) = 0 \right)
\end{array} \right., \tag{18}
\]

where \( R_{t-} = R_{t+} \) because \( \rho_t = \rho_t^\dagger \). Below we consider the dissipative dynamics of \( \pm \)-components of \( R_t \) separately. Using the initial condition \( R_{t=0} = 0 \) and expressing \( R_{t\pm} \) through \( R_{tx} \) one obtains the exact integro-differential equation for the \( \pm \)-component of the Bloch vector

\[
\frac{dR_{tx}}{dt} + \Omega_t \int_0^t dt' \Omega_{t'\mp} \cos \Theta_{t't} R_{t'\pm} = 0 \quad \tag{19}
\]

with the phase factor \( \Theta_{t't} = \int_0^{t'} d\tau \Theta_{t'\tau} R_{t'\tau} \). For \( t \rightarrow 0 \) we use the initial condition \( R_{t=0} = 1 \) which corresponds to the initial population localized at the upper level because the population numbers are \( n_{t\pm} = t\hat{P}_{t\pm} \rho_t \) (the
conservation requirement is \( n_{+,t} + n_{-,t} = 1 \). Similarly, using the initial condition \( R_{t=0} = 0 \) and eliminating the \( Z \)-component from the upper Eq. (18), we obtain the exact integro-differential equation for the \( \pm \) components

\[
\left( \frac{d}{dt} + i\Omega_{tx} \right) R_{t\pm} \pm \frac{\Omega_{tx}}{2} \int_0^t dt' \Omega_{tv}(R_{t'\pm} - R_{t'\mp}) = 0 . \tag{20}
\]

As an initial condition we use the normalisation condition at \( t = 0 \) written in the form \( R_{t=0+} + R_{t=0-} = 1/4 \). Here, we restrict ourselves by the weak-fluctuation regime and consider the averaged dynamics determined by Eqs. (19) and (20).

### A. Relaxation of population

We consider the interlevel redistribution of population, \( \Delta n_t \equiv n_{+,t} - n_{-,t} = \text{tr}[\delta \rho] = R_{tx} \), and separate \( R_{tx} \) into the averaged and random parts, \( R_{tx} = \langle \Delta n_t \rangle + \delta n_t \), which are governed by the equations

\[
\frac{d}{dt} \langle \Delta n_t \rangle + \int_0^t dt' N_{t'-t} \langle \Delta n_{t'} \rangle = 0 . \tag{21}
\]

Here we approximate the averaged kernel \( N_{\Delta t} \) neglecting random contributions to the phase \( \Theta_{tv} \to \Omega_{\Delta t} \), see Eq. (A5). Based on Eqs. (3) and (7) we used here the correlator \( \langle \Omega_{tx} \Omega_{tv} \rangle \) as the noise term and neglecting \( \delta f_t \), we find that the \( \varphi \)-dependent contributions to \( X_\varphi \) are weak because \( \omega_j \ll \omega_{\parallel} \). We neglect a weak asymmetry of \( \alpha_{\parallel} \) from the \( \beta_{\parallel} \) contributions. According to Eq. 2 (b) and (c), the rate \( \nu_1 \) takes form

\[
\nu_1 \approx \frac{\tilde{\nu}_1}{[1 + (\varphi/\varphi_1)^j]^{1/2}} , \quad \tilde{\nu}_1 = \frac{2\alpha_\parallel \bar{\omega}^2}{\omega_10} , \quad \varphi_1 = \frac{\omega_10}{2\bar{\omega}} \tag{24}
\]

and, in analogy to the incoherent tunneling regime, the relaxation of population is only governed by the noise in the \( t\ell \)-channel, \( \nu_1 \propto \alpha_\parallel \).

Apart from the tails of the rate \( \nu_1/\tilde{\nu}_1 \leq 10^{-2} \) versus \( \varphi \), where the \( \varphi \) contributions to \( X_\varphi \) are not negligible, the shape of the peak is determined by the amplitude and the half-width, \( \tilde{\nu}_1 \) and \( \varphi_1 \) (we neglect a weak asymmetry of \( \nu_1 \) from \( \alpha_{\parallel} \varphi \) contributions). According to Eq. 2 (b), \( \bar{\omega} \) varies from \( 0.4 \) to \( 1 \) over the weak-coupling region due to the softness of barrier, so that the dependency of \( \tilde{\nu}_1 \) on the gap frequency \( \omega_10 \) differs from \( \omega_{\parallel} \) as it is shown in Fig. 4(a) for the different ratios \( 4E_C/E_L \). The weak-coupling regime takes place under the requirement \( \nu_1/\Omega_{\parallel} \ll 1 \) which is transformed into the condition \( 2\alpha_\parallel (\bar{\omega}/\omega_10)^2 \ll 1 \) at the symmetric point \( \varphi = 0 \). If \( \alpha_\parallel \leq 10^{-8} \) this condition is satisfied and for \( E_L/h \approx 250 \) GHz the maximal rate is \( \tilde{\nu}_1 \approx (0.07 \pm 0.16) \mu s^{-1} \). The half-width of the peak at the same \( E_L/h \) and \( \omega_10/2\pi \) is about \( \varphi_1 \approx 22 \pm 33 \) \( \mu s^{-1} \) and \( \varphi_1 \) shows near-linear dependencies on \( \omega_10 \), as it is plotted in Fig. 4(b).

### B. Decoherence rate

Decoherence of the off-diagonal part of a density matrix in the tilt qubit frame is described by the circular
components of the Bloch vector governed by Eq. (20). Separating rotation with the frequency $\Omega_\varphi$ according to
$R_{\pm} = \exp(\pm i \Omega_\varphi t) r_{\pm}$ and neglecting the high-frequency contributions [i.e., $\exp(\pm i \Omega_\varphi (t + t'))$], we arrive to the closed equations for the slow amplitudes $r_{\pm}$:

$$
\left( \frac{d}{dt} + i \Delta \Omega_\varphi \right) r_{\pm} + \frac{\Omega_\varphi}{2} \int_0^t dt' \Omega_{\varphi \varphi} e^{\pm i \Omega_\varphi (t-t')} r_{\varphi \pm} = 0 .
$$

Because $r_{-} = r_{+}$, below we consider only the + component $r_{+} \equiv r_{+}$. Within the weak-fluctuation regime, we approximate the equations for the averaged and random parts of the amplitude, $\langle r_{+} \rangle = \langle r_{+} \rangle_{\varphi} + \delta r_{+}$, as follows

$$
\frac{d}{dt} \langle r_{+} \rangle - i \langle \Delta \Omega_{t \varphi} \delta r_{+} \rangle + \frac{1}{2} \int_0^t dt' e^{-i \Omega_{\varphi \varphi} (t-t')} \langle \Omega_{\varphi \varphi} \delta r_{+} \rangle \langle r_{+} \rangle_{t} = 0 .
$$

(26)

Here the random part of kernel $\delta N_{\varphi \varphi}$ gives vanishing contribution to $\langle \Delta \Omega_{t \varphi} \delta r_{+} \rangle$, because it contains an average of one or three random factors. As a result, $\langle r_{+} \rangle$ is governed by the equation

$$
\frac{d}{dt} \langle r_{+} \rangle + \int_0^t dt' \langle \Delta \Omega_{t \varphi} \Delta \Omega_{t \varphi} \rangle \langle r_{+} \rangle_{t} \simeq 0
$$

(27)

with the initial condition $\langle r_{t=0} \rangle = r_{0}$ where $r_{0}$ is normalized by the requirement $r_{0}^2 = 1/4$. The first and second integral terms here describe the long- and short-scale contributions to the temporal evolution.

After the Laplace transform of Eq. (27), we obtain the transient solution

$$
r_{+} = r_{0} \left[ \zeta + Z_{\varphi} w_{\varphi} / \Omega_{\varphi}^2 + X_{\varphi} w_{\varphi} + i \Delta \Omega_{t \varphi} / 2 \Omega_{\varphi}^2 \right]^{-1} .
$$

(28)

Here the correlator $\langle \Delta \Omega_{t \varphi} \Delta \Omega_{t \varphi} \rangle = Z_{\varphi} w_{\varphi} \omega_{m \varphi} \Delta t / \Omega_{\varphi}^2$ is written through

$$
Z_{\varphi} = \alpha_{\varphi} (2 \omega_{\varphi}^2)^2 + \alpha_{\varphi} (\omega_{\varphi} \omega_{\varphi 0})^2 - \alpha_{\varphi} \omega_{\varphi 0} (2 \omega_{\varphi})^2 \varphi ,
$$

(29)

which is similarly $\langle \Omega_{t \varphi} \varphi \rangle$ in Eq. (22). We consider below a slow-varied part of $\langle r_{+} \rangle$, when $|\zeta| \ll \nu_{2} \ll \Omega_{\varphi}$ with the decoupling rate $\nu_{2}$, and apply the logarithmic approach, when $w_{\varphi}$ is replaced by $|A_{2} + i \arg (\zeta) / \pi \zeta |$ with $A_{2} \approx \ln (\nu_{2} / \omega_{m \varphi}) + c_{2}$ with $c_{2} \sim 1$. Similarly to Sect. IV A, $w_{\varphi} + i \Delta \Omega_{t \varphi}$ should be replaced by $(1/2 - i \Delta \Omega_{t \varphi}) / \Omega_{\varphi}$ with $A_{1} > A_{2}$ because of $\Omega_{\varphi} \gg \nu_{2}$. Using these replacements, one can search the poles of the solution $r_{+}$ from the quadratic equation:

$$
\zeta^2 + \zeta \nu_{1} (1/2 - i \Lambda_{1} / \pi) + \nu_{2}^2 \Lambda_{2} + i \arg (\zeta) / \pi \simeq 0 ,
$$

(30)

where we introduce $\nu_{\varphi} = \sqrt{Z_{\varphi} / \Omega_{\varphi}}$ and use $\nu_{1} \approx X_{\varphi} / 2 \Omega_{\varphi}^3$ from Sect. IV A. This equation is transformed into $\zeta_{-}

\frac{\omega}{\omega_{\varphi}} (E / \hbar)

0.04

0.02

0.01

0.00

0.03

0.06

0.7 \omega^{0.5}

0.2 \omega^{-0.8}

\begin{align}
\zeta_{+}(\zeta - \zeta_{-}) & \approx 0 \text{ with two poles } \zeta_{\pm} \text{ determined by the relation:} \\
\zeta_{\pm} & = -\nu_{1} (1/2 - i \Lambda_{1} / \pi) / 2 \\
& \pm \sqrt{\nu_{1}^2 (1/2 - i \Lambda_{1} / \pi)^2 / 4 - \nu_{2}^2 \Lambda_{2} / \pi} / \pi
\end{align}

(31)

and it is not an explicit solution because of the factor $\arg (\zeta)$. Within the logarithmic approach, $\Lambda_{1,2} \gg \pi / 2$, the poles are near the imaginary axis and $\arg (\zeta_{\pm}) \approx \pm \pi / 2$.

Thus the poles (31) describe a two mode behavior of the decoherence process. This is because of a different character of the integral contributions in Eq. (27): while the XX-term is cutting off at $t \sim \Omega_{\varphi}^{-1}$, the ZZ-term shows a long-time memory. We obtain the poles as $\zeta_{\pm} \approx i \Delta \Omega_{t \varphi} - \nu_{\pm}$ with the renormalization of the gap frequency $\Delta \Omega_{t \varphi}$ and the decoupling rate $\nu_{\pm}$ given by:

$$
\Delta \Omega_{t \varphi} \approx \nu_{1} \Lambda_{1} / 2 \pi + \sqrt{\nu_{1}^2 (1/2 - i \Lambda_{1} / \pi)^2 / 4 - \nu_{2}^2 \Lambda_{2} / \pi} ,
$$

$$
\nu_{\pm} \approx \frac{\nu_{2}^2 + \nu_{1} \Delta \Omega_{t \varphi}}{4 \sqrt{(\nu_{1} \Lambda_{1} / 2 \pi)^2 + \nu_{2}^2 \Lambda_{2} / \pi}} .
$$

(32)

After the inverse Laplace transform one obtains

$$
\langle r_{t=0} \rangle \approx \nu_{1} \Delta \Omega_{t \varphi} e^{(i \Delta \Omega_{t \varphi} + \nu_{2} / 2 - (i \Delta \Omega_{t \varphi} - \nu_{2} / 2) t} ,
$$

(33)

i.e. an oscillating temporal evolution of $\langle r_{t} \rangle$ is described through $\Delta \Omega_{t \varphi}$ and $\nu_{\pm}$ determined by Eq. (32).

Character of the damping oscillations of (33) is determined by the rates $\nu_{1}$ given by Eq. (24) and $\nu_{\varphi}$ given by

$$
\nu_{\varphi} = \sqrt{\nu_{1} \nu_{2} \omega_{m \varphi}} .
$$

(34)

This rate at symmetric point, $\nu_{\varphi=0} = \sqrt{\nu_{1} \omega_{j}}$, and the tilt dependency of $\nu_{\varphi}$ shown versus $\omega_{10}$ in Figs. 5(a) and 5(b), respectively. In

FIG. 5: Characteristic frequency $\omega_{j}$ (a) and ratio $\zeta / \omega_{j}$ (b) versus gap frequency $\omega_{10}$ for the same conditions as in Fig. 4 (a). Asymptotics are shown by red dots.
contrast to $\nu_1 \propto \alpha_1$, we obtain $\nu_2 \propto \sqrt{\nu_j}$ and the both contributions may be essential even if $\alpha_j \ll \alpha_1$. There are two limiting cases, with ZZ- or XX-correlators dominate in Eq. (27), when the renormalization frequencies and the decoherence rates take forms:

$$\Delta \Omega_\pm \approx \pm \nu_2 \sqrt{\nu_j}, \quad \nu_\pm \approx \frac{\nu_2}{4} \sqrt{\frac{\pi}{\nu_j}} \quad \text{if} \quad \nu_2 \gg \nu_1, \quad (35a)$$

$$\Delta \Omega_+ \approx \nu_1 \frac{\Lambda_1}{\pi}, \quad \nu_+ \approx \frac{\nu_1}{2}, \quad \Delta \Omega_- \approx \nu_- \approx 0, \quad \text{if} \quad \nu_2 \ll \nu_1. \quad (35b)$$

Here the imaginary and real parts of poles are connected as $|\Delta \Omega_\pm| = (4\Lambda_2/\pi)\nu_\pm$ or $\Delta \Omega_+ = (2\Lambda_1/\pi)\nu_+$, i.e. they differ by the scale factors $\propto \Lambda_1/\Lambda_2 \gg 1$. Note, that $\nu_2$ saturates at $\nu_{\text{max}} = \sqrt{\alpha_0 \omega_0}$ and increases $\propto \varphi$ before saturation, if $1 > |\varphi|/\varphi_1 > \omega_j/\omega$. An interplay between these regimes is significant in the region where $\nu_1/\nu_{\text{max}} = \alpha_0 \sqrt{\omega_j} \approx 1$. Fig. 6 compares shapes of the sharp dips of $\nu_2$ and the slow peaks of $\nu_1$ for different levels of noise and different ratios $\nu_1/\nu_2$. For the weakly-correlated noises with $\alpha_c \ll \alpha_{j,t}$, see Figs. 6 (c) and (d), there are near-symmetric dips but for $\alpha_c \sim \alpha_j$ the dips around $\varphi \approx 0$ are asymmetric enough and are shifted to the right if $\omega_j > 0$, see Figs. 6(a) and(b). Moreover, for the fully-correlated identical noises with $\alpha_{tt} = \alpha_j = \alpha_c$, one obtains $\nu_2 = 0$ at $\varphi = (\omega/\omega_j)\varphi_1$, see Fig. 6(a). This peculiarity is caused by the destructive interference of noises in $t$- and $j$-channels and it becomes essential if $\alpha_{jt}/\alpha_{tt} > 0.5$, see Fig. 8 below.

At $\varphi = 0$ the interplay of the long- and short-scale ($\propto \nu_{\text{max}}$ and $\propto \nu_1$) contributions in Eq. (32) gives rise to the dependencies of $\Delta \Omega_\pm$ and $\nu_\pm$ which increase with $\omega_{10}$ as it is shown in Fig. 7. Similarly to Figs. 4 and 5, the growth of $\Delta \Omega_\pm$ and $\nu_\pm$ is dependent on $4E_C/E_L$ and on the noise levels, $\alpha_{tt}$ and $\alpha_j$. Finally, in Fig. 8 we plot $\Delta \Omega_\pm$ and $\nu_\pm$ versus tilt flux, $\varphi/\varphi_1$, for different gap frequencies at $4E_C/E_L = 0.005$; there is the same behavior of dips with numerical variations up to 2 times if $4E_C/E_L$ varies over 0.01–0.0025. Thus, the ratio $\Delta \Omega_\pm/\nu_\pm$ is about ten and contribution of the "−" mode is negligible around $\varphi = 0$. If $\varphi/\varphi_1 > 0.2$, the decoherence rate and $\Delta \Omega_\pm$ increase fast (up to ten times at $\varphi/\varphi_1 \sim 1$, if the gap frequency $\leq 1$ GHz for the parameters used above).

V. FLUCTUATION EFFECT

Finally, we examine the random addendums to the averaged Bloch vectors considered in Sects. III and IV. We analyze the two-point correlations of fluctuations and determine the conditions for applicability of the averaged dynamics in the cases (a–c) considered.
A. Random contributions to tunneling

Here we consider random part of the Bloch vector \( \delta S_t = S_{tx} - \langle S_t \rangle \) governed by the linearized equation

\[
\frac{d\delta S_t}{dt} + \int_0^t dt' K_{\Delta t} \delta S_{t'} = - \int_0^t dt' \delta K_{t'} \langle S_{t'x} \rangle \equiv \delta F_t, \tag{36}
\]

which is obtained after subtraction of the averaged Eq. (11) from the exact Eq. (10). Here a random source \( \delta F_t \) is introduced through the kernel \( \delta K_{t'} = \omega_{t'z} \omega_{t'z} \cos \theta_{t'z} - K_{t'z} \) and a second-order correction \( \propto \delta K_{tt} \delta S_t \) is omitted. Similar to Sect. III, Laplace transform of Eq. (36) gives the solution \( \delta S_t = \delta F_t / (\zeta + K_\zeta) \) and the correlation function \( \langle \delta S_t \delta S_{t'} \rangle \) takes form:

\[
\langle \delta S_t \delta S_{t'} \rangle = \int_C \frac{dz}{2\pi i} \int_C \frac{dz'}{2\pi i} \frac{e^{z(t+t')}}{(\zeta + K_\zeta)} \langle \delta F_t \delta F_{t'} \rangle \tag{37}
\]

\[
\approx \int_0^t dt_1 \int_0^{t_1} dt'_1 e^{-\nu t_t (t+t'-t_1-t'_1)} \langle \delta F_{t_1} \delta F_{t'_1} \rangle.
\]

Here the correlator of random sources is written through \( \langle S_{t,x} \rangle \):

\[
\langle \delta F_{t_1} \delta F_{t'_1} \rangle = \int_0^{t_1} dt_2 \int_0^{t_1} dt'_2 \langle \delta K_{t_1-t_2} \delta K_{t'_1-t'_2} \rangle \langle S_{t,2x} \rangle \langle S_{t',2x} \rangle, \tag{38}
\]

where \( \omega_{t_1z} \omega_{t_1z} \omega_{t'_2z} \omega_{t'_2z} \) is replaced by \( \omega_{t_0z} \) omitting the contributions \( \propto \delta \ldots \).

For the weak fluctuation regime, we approximate \( \langle \cos \theta_{t_1z} \cos \theta_{t'_1z} \rangle \approx \langle \cos \theta_{t_1z} \rangle \langle \cos \theta_{t'_1z} \rangle \) by the first order contribution and using (A.6) we obtain the four-point correlator in (38) as:

\[
\langle \delta K_{t_1-t_2} \delta K_{t'_1-t'_2} \rangle \approx \alpha_{tt_1} (2\omega)^2 \omega_{t_0z} \cos \theta_{t_2z} \cos \theta_{t'_2z}
\times \int_0^{t_2} d\tau \int_0^{t'2} d\tau' \omega_{\tau_1-\tau} \tag{39}
\]

with \( \omega_{\tau_1-\tau} \) given by Eq. (3). At \( \nu t_1, 2 \gg 1 \) one can replace \( \langle S_{t,2} \rangle \) by the expon exp(\(-\nu t_t \)) and the correlator (37) is transformed into

\[
\langle \delta S_t^2 \rangle \approx e^{-2\nu t_t} \alpha_{tt} (2\omega)^2 \omega_{t_0z} \int_0^{t} d\omega \int_0^{t} dt_1 \int_0^{t_1} dt'_1 \int_0^{t_1} dt'_2
\times e^{-\nu (t_2+t'_2-t_1-t'_1)} \cos \theta_{t_2z} \cos \theta_{t'_2z} \left[ \cos \omega (t_1-t'_1) + \cos \omega (t_2-t'_2) - \cos \omega (t_1-t'_2) \right]. \tag{40}
\]

Here we include only the contribution which is logarithmically divergent at \( \omega_{\tau} \rightarrow 0 \). After integrations by parts in Eq. (40) we estimate the rare fluctuations contributions to tunneling as

\[
\langle \delta S_t^2 \rangle \approx \frac{(\omega_{t_0z}/\nu t)^8}{2(1 + \eta_{nu}^2) \nu t} \left( \frac{\omega_{t_0z}}{\nu t} + \frac{\nu t}{2} \right) \tag{41}
\]

\[
\times \left\{ (1 + e^{-\nu t_t})^2, \quad \eta_{nu} < 1, \right.
\]

\[
\left. \left[ \cos (\eta_{nu} \nu t_t) + e^{-\nu t_t} \right]^2, \quad \eta_{nu} \gg 1, \right\}
\]

\[
F_z \equiv x \left\{ (1 - \eta_{nu}^2) \left[ e^{-x} + \cos (\eta_{nu} x) \right] + 2 \eta_{nu} \sin (\eta_{nu} x) \right\},
\]

where the tilt effect is described by the ratio \( \eta_{nu} = 2 \nu \phi / \nu t_t \).

Because of \( \sqrt{\langle \delta S_t^2 \rangle} \propto \Lambda_{\nu}^2 \), the rare fluctuation contributions to lead to the linear growth \( \sqrt{\langle \delta S_t^2 \rangle} \propto \nu t_t \) and restrict the averaged description of tunneling by the condition \( \sqrt{\langle \delta S_t^2 \rangle} < e^{-\nu t_t} \). If \( \omega_{t_0z}/\nu t_t < 1 \) and \( \eta_{nu} \rightarrow 0 \), this requirement is satisfied during the time interval \( \nu t_t < \Lambda \ln (\nu t_0 / \omega_{t_0z}) \) with \( \Lambda \sim 1 \), i.e. the average description fails down starting at \( \nu t_t \sim 3 \div 5 \), for typical parameters. A further evolution, when a fluctuations level should be saturating, is not described in the framework of the approach used. At \( \nu \sim \nu_{\tau} \) when \( \eta_{nu} \gg 1 \), the rare fluctuations are quenched as \( 1/\eta_{nu}^2 \) so that the average description is valid outside of the dip.

B. Fluctuations of population redistribution

Similarly to the averaged population, the solution for it’s fluctuation part \( \delta n_t \) with the zero initial condition is given by \( \delta n_t \approx - \int_0^t dt_1 e^{-\nu_1 (t-t_1)} \delta f_{t_1} \). The correlation function \( \langle \delta n_t \delta n_{t'} \rangle \) is written through the correlator of sources \( \delta f_t \) as follows

\[
\langle \delta n_t \delta n_{t'} \rangle \approx \int_0^t dt_1 \int_0^{t'} dt'_1 e^{-\nu_1 (t-t_1)} \int_0^{t} dt_2 \int_0^{t} dt_2 \delta f_{t_2} \delta f_{t_2}. \tag{42}
\]

Here the correlator of sources is given by

\[
\langle \delta f_t \delta f_{t'} \rangle \approx \int_0^{t_1} dt_2 \int_0^{t_1} dt_2 \cos (\Omega_{t_2} \Delta t_{t_2}) \cos (\Omega_{t_2'} \Delta t_{t_2'})
\times \langle \delta N_{t_2} \delta N_{t_2'} \rangle \tag{43}
\]

and \( \delta N_{t_2} \delta N_{t_2'} \rangle \equiv \langle \Omega_{t_2} \Omega_{t_2'} \Omega_{t_2} \Omega_{t_2'} \rangle - \lambda_{t_2}^2 w_{\tau_1-\tau} \omega_{t_2} \omega_{t_2'} \) is the averaged fluctuations of kernel. The four-point correlator \( \langle \Omega_{t_2} \rangle \ldots \rangle \) is determined by Eq. (A7).

Within the logarithmic approach at \( \nu = 0 \) one obtains \( \langle \delta N_0 \ldots \rangle \approx (2\omega)^4 (\alpha_{\nu t_1} / \pi)^2 \) and the mean-square-fluctuation of population takes form:

\[
\langle \delta n_t^2 \rangle \approx (2\omega)^4 (\alpha_{\nu t_1} / \pi)^2 e^{-2\nu t_t} \times \langle \Re \left[ \int_0^t dt_1 \int_0^{t_1} dt_2 e^{(\nu_{\tau t_2} + \nu_{\omega_{t_1}}) \Delta t_{t_2}} \right]^2 \rangle \approx \int_0^t dt_1 \int_0^{t_1} dt_2 (\nu_{t_2} + \nu_{t_1}) \tag{44}
\]

\[
\approx \left[ \left. \frac{\Re \left[ \int_0^t dt_1 \int_0^{t_1} dt_2 e^{(\nu_{\tau t_2} + \nu_{\omega_{t_1}}) \Delta t_{t_2}} \right]^2 \right] \right].
\]
After straightforward integrations over $t_{1,2}$ and averaging over times $\sim \omega_1^{-1}$, Eq. (44) gives
\[
\langle \delta n \rangle^2 \approx 2k^2 \left[ 1/2 + e^{-2\bar{\nu}t}(1 - \bar{\nu}t)^2 \right],
\]
where $k = (2\bar{\omega}/\omega_1)^2 \alpha_{1r} \Lambda_1/\pi \ll 1$ and $\langle \delta n \rangle^2 \propto \Lambda_1^2$, i.e. the rare fluctuations contribution limits the averaged description of transient evolution.

Similarly to Sect. V A, the averaged description is valid under the condition $\sqrt{\langle \delta n \rangle^2} < e^{-\nu_1 t}$ and at $\varphi = 0$ this requirement is satisfied during the time interval $\bar{\nu}t < \ln(1/k) \gg 1$. If $\varphi = \varphi_1$, we estimate $\Omega_{\varphi_1} \sim \sqrt{2}\omega_1$ and $\Omega_{t_{1x}} \sim \cdot \Delta \varphi_1 \sqrt{2}\omega_1$, so that $\sqrt{\langle \delta n \rangle^2}$ is different from Eqs. (44) and (45) by the numerical factor $\sim 1$. As a result, the above condition for the averaged description remains valid over the peak described by Eq. (24). In addition, $\langle \delta n \rangle_{t_{1x} \gtrless 1} \approx k^2$ gives the steady-state level of fluctuations after redistribution of population.

### C. Random contributions to decoherence

After integration of the lower Eq. (26) with the initial condition $\delta n_{t=0} = 0$, the random part of decoherence is
\[
\delta n \approx i \int_0^t dt_1 2\omega_{1z} \langle \delta t_{1z} \rangle \delta n_{t_{1z}} (t_{1z})
\]
\[
- \frac{1}{2} i \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-i\Delta \omega_{1z}(t_{1z} - t_{2z})} \langle \delta n_{t_{1z}} \delta n_{t_{2z}} (t_{1z}) (t_{2z}) \rangle^r.
\]

Because the averages of $\Delta \Omega_{1z}$ and $\delta N_{1z}$ contributions are separated, the two-point correlation function is transformed into
\[
\langle \delta n \rangle^2 \approx i \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-i\Delta \omega_{1z}(t_{1z} - t_{2z})} \delta N_{t_{1z}} (t_{1z}) \delta n_{t_{2z}} (t_{2z})
\]
\[
+ \frac{1}{4} i \int_0^t dt_1 \int_0^t dt_2 \int_0^{t_1} dt_3 e^{-i\Delta \omega_{1z}(t_{1z} - t_{2z} + t_{3z} - t_{2z})} \delta N_{t_{1z}} (t_{1z}) \delta n_{t_{2z}} (t_{2z}) \delta n_{t_{3z}} (t_{3z})
\]
\[
\times \langle \delta n_{t_{1z}} \delta n_{t_{2z}} \delta n_{t_{3z}} (t_{1z}) (t_{2z}) (t_{3z}) \rangle^r.
\]

where the correlator $\langle \Delta \Omega_{1z} \Delta \Omega_{t_{1z}} \rangle$ is given by Eq. (29). According to Fig. 7, around the symmetric point $\varphi = 0$ only the + mode is essential in Eq. (33) and we use $\langle \delta r \rangle = \gamma_0 \exp[i(\Delta \omega_{1z} - \nu_1 t)]$. Within the logarithmic approach, $D(\ldots)$ is replaced by the constant [see Eq. (44)] and after the straightforward integrations the mean-square fluctuation of amplitude takes form:
\[
\langle \delta n \rangle^2 \approx \gamma_0^2 \left[ \left( \frac{\nu_1 - \nu_2}{\Delta \Omega_{1z}} \right)^2 \frac{\Lambda_2}{\pi} + 2 \left( \frac{\bar{\nu}_1}{\Delta \Omega_{1z}} \right)^2 \left( \frac{\Lambda_1}{\pi} \right)^2 \right]
\]
\[
\times (1 + e^{-2\nu_1 t} - 2e^{-2\nu_1 t} \cos \Delta \Omega_{1z} t).
\]

Here the first and second term in $[\ldots]$ are due to the long- and short-scale contributions to fluctuations [see similar terms in Eq. (27)] and this factor is estimated as logarithmically weak, $[\ldots] \sim 1/\Lambda_{1,2} < 1$, see Eqs. (24), (32), and (34).

Once again, at $\varphi = 0$ the requirement for averaged description of decoherence is $\sqrt{\langle \delta n \rangle^2} < \gamma_0 e^{-\nu_1 t}$. In analogy to the previous section, the results of Sect. IV B are valid if $\nu_1 t < \ln(1/[\ldots])$ and, for a typical parameters, fluctuations are weak over the time interval $\nu_1 t < 3$. A similar condition remains valid over the dip region $|\varphi| < 0.3\varphi_1$ (see Fig.8) where both $X_{\varphi}$ and $Z_{\varphi}$ are changed only due to numerical factors $\sim 1$. Thus, the fluctuations contribution is suppressed only in the saturation region, $|\varphi| \geq \varphi_1$ while the averaged description over the dip is valid during a short enough interval.

### VI. CONCLUDING REMARKS

We present a comprehensive investigation of the dissipative dynamics of a flux qubit, describing interwell and interlevel relaxation as well as decoherence process caused by the 1/f noises passed through SQUID and LC-contour. We analyze the relaxation rates and the renormalization of gap frequency versus the control fluxes and the parameters of noise; this permits one to characterize the qubit-noise interaction. We show how rare fluctuations limit the averaged description at tails of relaxations. Under typical level of noises, the results obtained give contributions comparable to the recent experimental data on the interlevel population relaxation and the interwell tunneling.

Our consideration is based on a several assumptions which are shortly discussed below.

(a) Description of the flux qubit is based on the effective circuit formed by the effective Josephson junction shunted by the LC-contour, instead of the SQUID loop, which is shunted by the transmission line. It is a good approach for the low-frequency region, far below the characteristic frequency of the LC-contour which is $10 \text{ GHz}$ for a device with typical parameters.

(b) Consideration of 1/f noise as a classical random flux is valid for frequencies below $1/2 \text{ GHz}$ and it is a dominant random factor in this region, where the interaction with high-frequency bosons can be omitted. The simplified model with the cut-off frequency at $\omega_m$ used here because the logarithmic character of the cut-off. A possible deviation from 1/f spectrum, e.g. due to the size effect in the transmission line may be described similarly after choosing any specific parameters of device.

(c) The qubit-noise interaction is studied by adding random fluxes to the tilt and control fluxes and by taking into account correlations between these sources. The coupling levels in $t_1$, $j_1$, and $c$-channels are given by the phenomenological parameters $\alpha_{1,j,c}$. Both a microscopic study of a 1/f noise mechanism and a detail description of a noise effect
on the SQUID loop and the LC transmission line are beyond of the scope of the paper and can be performed for specific devices.

(d) The idealized initial conditions were used at \( t = 0 \) without any discussion of a temporal evolution at \( t < 0 \). A more detailed description requires an analysis of the protocol of resonant tunneling and the mechanisms of the ultrafast inter-level excitation, see \([12, 27]\) and references therein. Here we neglect a possible uncertainties during the initialization and readout stages.

(c) We restrict ourselves by the weak-fluctuation regime, when the averaged dissipative dynamics describes an evolution of qubit. Level of fluctuations gives the limitations of the averaged evolution at tails of relaxation. In principle, a chaotic regime contains an additional information on the qubit-environment interaction but an analysis of the two-point spectral functions requires a special consideration.

Now we discuss the current experimental data for the LC-shunted qubits and possibilities for verification of the results obtained. In spite of these qubits were employed for demonstration of the multi-qubit clusters \([11, 2\), \(22]\), the spectroscopic characterization of this type of qubits is not available \([23]\). Recent measurements of the population relaxation rate in the region above 0.5 GHz \([15]\) give the amplitude and half-width of peak in agreement with the results of Sect. IV B but the \( 1/f \) spectral dependency is modified due to the soft barrier effect. The decoherence processes were not analyzed in \([15, 23]\) but such a data are necessary in order to verify of the \( j \)- and \( e \)-channels contributions which determine the depth and asymmetry of the dip. A study of the other peculiarities discussed in Sect. IV C (the two-mode evolution and the renormalization of gap) may be restricted due to the fluctuation-induced suppression of the averaged response. The incoherent tunneling rate reported in \([14, 23]\) was in agreement with the model \([21]\) when the levels are modulated by coupling to the boson thermostat. It lead to the temperature dependent asymmetry of the tunneling peak. Recent measurements \([15]\) show an additional temperature-independent contribution which can be connected with the asymmetric contribution via \( e \)-channel, see Sect. III. In order to confirm this point, one needs to demonstrate a changing of the sign of asymmetry for the parallel and antiparallel directions of \( \varphi \) and \( \psi \), see Fig. 1(a). Thus, the above-discussed partial experimental data does not permit to characterize the qubit-noise interaction completely: it is necessary to perform all possible measurements for the same device and, using an appropriate model of the device, find parameters which will describe all data. Similar program can be develop for other qubits, with an effective circuits different from Fig. 1(b), such as the C-shunted flux qubit \([10]\) or other variants of transmon \([12, 27]\).

To conclude, the obtained results convincingly demonstrate that that transient dynamics of the flux qubit should be analyzed beyond of the simplified two-level model which includes only a noise-induced modulation of interlevel gap. Our consideration, which is based on the lumped-element approach with detailed description of noise effects, opens the way to characterize the flux qubit interacting with low frequency noise and to enhance a fidelity of the device. We believe that a similar study of another types of qubits, the interqubit connections, and the multi-qubit clusters will improve parameters of the quantum hardware.

**Appendix: Averaged kernels**

Here we consider averaging over the random Gaussian noises in \( tl-, j-, \) and \( e \)-channels for the kernels employed in sections III-V. We begin with the kernel \( K_{\Delta t} \) in the averaged integro-differential equation (11). Explicit expression for \( K_{\Delta t} \) is obtained below with the use of \( \omega_{zx} \) and \( \omega_{z} \) given by Eq. (5) and the phase factor \( \theta_{tt'} = \theta_{tt'} + \Delta \theta_{tt'} \):

\[
K_{\Delta t} = \frac{\omega_{10}}{2} \left( \omega_{tt}^2 (\cos \Delta \theta_{tt'} + \omega_{tt}^2 (\Delta \psi_{t} \Delta \psi_{t'} \cos \Delta \theta_{tt'})) \right) - \omega_{f} \omega_{10} \sin \theta_{tt'} \left( (\Delta \psi_{t} + \Delta \psi_{t'}) \sin \Delta \theta_{tt'} \right),
\]

where the unperturbed phase and the noise contribution are given by \( \theta_{\Delta t} = 2/\omega_{10} \Delta t \) and \( \Delta \theta_{tt'} = 2/\omega_{10} \int_{t'}^{t} d\tau \Delta \varphi_{\tau} \), respectively. To carry out the averaging of \( \omega_{10}^2 \) contribution, one should consider all possible pairings in the expansion of cosine and the total number of such pairings in the term of the order \( 2n \) is equal to \((2n)!/2^n n!\), where factor \( 2^n \) is due to symmetry of correlator with respect to \( t \leftrightarrow t' \) and factor \( n! \) gives number of permutations of \( n \) pairs. The infinite sum over \( n \) is transformed to an exponent, \( c. f. \) to the similar averaging over spatial domain \([28]\), so that

\[
\langle \cos \Delta \theta_{tt'} \rangle = \exp \left\{ -2/\omega_{10}^2 \int_{t'}^{t} d\tau \int_{t'}^{t} d\tau' \langle \Delta \varphi_{\tau} \Delta \varphi_{\tau'} \rangle \right\} \equiv e^{-\Gamma_{\Delta t}},
\]

The correlator under integrals over \( \tau \) and \( \tau' \) is defined by Eq. (3) so that \( \Gamma_{\Delta t} \) is determined by Eq. (13a).

After the similar expansion of cosine, the proportional to \( \omega_{t}^2 \) contribution in Eq. (A1) takes form

\[
\langle \Delta \psi_{t} \Delta \psi_{t'} \cos \Delta \theta_{tt'} \rangle = e^{-\Gamma_{\Delta t}} \left[ \langle \Delta \psi_{t} \Delta \psi_{t'} \rangle \right] - (2/\omega_{10})^2 \int_{t'}^{t} d\tau \langle \Delta \psi_{t} \Delta \varphi_{\tau} \rangle \int_{t'}^{t} d\tau' \langle \Delta \psi_{t'} \Delta \varphi_{\tau'} \rangle,
\]

and it involves both \( \cos \alpha_{j} \) and \( \cos \alpha_{e} \) addendums. The last addendum contains the same integrals which are dependent on \( \Delta t \) and are transformed into Eq. (13b). Similar averaging for \( \cos \omega_{z} \omega_{10} \) contribution is performed after the expansion of the sine:

\[
\langle (\Delta \psi_{t} + \Delta \psi_{t'}) \sin \Delta \theta_{tt'} \rangle = 2/\omega_{10} e^{-\Gamma_{\Delta t}} \int_{t'}^{t} d\tau \langle (\Delta \psi_{t} + \Delta \psi_{t'}) \Delta \varphi_{\tau} \rangle \langle (\Delta \psi_{t} + \Delta \psi_{t'}) \Delta \varphi_{\tau} \rangle
\]
and this contribution is written through $B_{\Delta t} \propto W_{\Delta t}$, see Eq. (12).

For the weak-coupling regime, the averaged kernel of Eq. (21) is given by

$$N_{\Delta t} = \cos(\Omega_\Delta t) \left( \Omega_{t_2} \Omega_{t_1} \cos \left( \int_{t_1}^{t_2} d\tau \Delta \Omega_{\tau_2} \right) \right) \quad (A.5)$$

and with $\propto \alpha_-\ldots$ accuracy it can be written as $N_{\Delta t} \approx \cos(\Omega_\Delta t) \left( \Omega_{t_2} \Omega_{t_1} \right)$ with the use $\cos \left( \int_{t_1}^{t_2} \ldots \right) \approx 1$. This approximation gives $\propto \chi_\alpha$ contributions in Sects. IV A and IV B.

Under examination of the fluctuation effect (Sect. V) one needs to calculate the four-point correlators of the random sources. The fluctuations of tunneling [Eq. (38)] are written through the correlator $\langle \cos \theta_{t_1} \cos \theta_{t_2} \rangle$ which is transformed through $\langle \cos(\theta_{t_1} \pm \theta_{t_2}) \rangle$ and using Eq. (A2) one obtains

$$\langle \cos \theta_{t_1} \cos \theta_{t_2} \rangle = \cos \theta_{t_1} \cos \theta_{t_2} e^{-\Gamma_{t_1} t_2 - \Gamma_{t_2} t_1}$$

$$\times \exp \left[ \alpha_{tt} (2\omega)^2 \int_{t_2}^{t_1} d\tau \int_{t_2}^{t_1} d\tau' \omega_{\omega_{in|\tau-\tau'|}} \right] \quad (A.6)$$

The correlator (39) is written after expansion of $\exp[\alpha_{tr} \ldots]$ with respect to $\propto \alpha_{tt}$ contributions. For the weak-coupling regime, the four-point fluctuations of kernels in Eqs. (43) and (47) are transformed as follows:

$$\langle \delta N_{t_1} t_2 \delta N_{t_1'} t_2' \rangle \approx \alpha_{tt} (2\omega)^4 \left( w_{\omega_{in|t_1-t_2}} \right) \left( w_{\omega_{in|t_2-t_1}} \right)$$

$$+ w_{\omega_{in|t_1-t_2}} w_{\omega_{in|t_2-t_1}} \quad (A.7)$$

Within the logarithmic approach, this correlator gives the time-independent factor used in Eqs. (44) and (48).

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