THE RICCI PINCHING FUNCTIONAL ON SOLVMANIFOLDS II

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Abstract. It is natural to ask whether solvsolitons are global maxima for the Ricci pinching functional $F := \frac{\text{scal}^2}{|\text{Ric}|}$ on the set of all left-invariant metrics on a given solvable Lie group $S$, as it is to ask whether they are the only global maxima. A positive answer to both questions was given in a recent paper by the same authors when the Lie algebra $\mathfrak{s}$ of $S$ is either unimodular or has a codimension-one abelian ideal. In the present paper, we prove that this also holds in the following two more general cases: 1) $\mathfrak{s}$ has a nilradical of codimension-one; 2) the nilradical $\mathfrak{n}$ of $\mathfrak{s}$ is abelian and the functional $F$ is restricted to the set of metrics such that $\mathfrak{a} \perp \mathfrak{n}$, where $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is the orthogonal decomposition with respect to the solvsoliton.

1. Introduction

A left-invariant metric on a simply connected solvable Lie group $S$ is called a solvsoliton when its Ricci operator satisfies

$$\text{Ric} = cI + D,$$

for some $c \in \mathbb{R}$, $D \in \text{Der}(\mathfrak{s})$,

where $\mathfrak{s}$ denotes the Lie algebra of $S$ (see [L3]). The definition is a neat combination of geometric and algebraic aspects of a Lie group and the following facts explain very well and from different points of view why these metrics are quite distinguished:

- **Ricci solitons.** Solvsolitons are all Ricci solitons, they are precisely the left-invariant Ricci solitons such that the Ricci flow evolves by just scaling and pullback by automorphisms (see [L3] and [J1, Lemma 5.3]). Moreover, if $S$ is of real type, then any scalar-curvature normalized Ricci flow solution converges in Cheeger-Gromov topology to a non-flat solvsoliton on a possibly different solvable Lie group, which does not depend on the initial metric (see [BL, Theorem A]).

- **Uniqueness.** On a given $S$, there is at most one solvsoliton up to scaling and pullback by automorphisms of $S$ (see [L3, Section 5] and [BL, Corollary 4.3]).

- **Structure.** If $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$ and $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is the orthogonal decomposition with respect to a solvsoliton, then $[\mathfrak{a}, \mathfrak{a}] = 0$ (which already imposes an algebraic constraint on $\mathfrak{s}$, as the Lie algebra $[e_1, e_3] = e_3$, $[e_2, e_4] = e_4$, $[e_1, e_2] = e_5$ shows) and any $\text{ad} Y|_{\mathfrak{n}}$ must be a semisimple operator, although the strongest and least understood obstruction is that $\mathfrak{n}$ has to admit itself a solvsoliton, called a nilsoliton in the nilpotent case (see [L3, Theorem 4.8] and [L2]). We refer to [W, FC] and the references therein for low-dimensional classification results for solvsolitons.

- **Maximal symmetry.** The dimension of the isometry group of a non-flat solvsoliton on $S$ is maximal among all left-invariant metrics on $S$ (see [BL, Corollary C]). A stronger maximality condition holds in the case of a unimodular $S$: the isometry group of a solvsoliton contains all possible isometry groups of left-invariant metrics on $S$ up to conjugation by a diffeomorphism (see [J2, Corollary 1.3], and see [GJ] and [BL, Corollary D] for the (non-unimodular) Einstein case, where such a diffeomorphism is actually an automorphism of $S$).

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• Ricci-pinched. On a given unimodular $S$, solvsolitons are the only global maxima for the Ricci pinching functional

$$F := \frac{\text{scal}^2}{|\text{Ric}|^2},$$

restricted to the set of all left-invariant metrics on $S$ (see [LW1] for nilsolitons and [LW2] Theorem 1.2, (v)] for the general unimodular case). Note that $F$ is measuring in a sense how far is a metric from being Einstein.

We are concerned in this paper with the last property of solvsolitons above. Beyond the unimodular case, solvsolitons were proved in [LW2] to be the only global maxima of $F$ for any almost-abelian $S$ (i.e. $S$ has a codimension-one abelian ideal). Our purpose here is to prove that this also holds among the following two much broader classes of solvable Lie groups.

Theorem 1.1. Let $\mathfrak{s}$ be a solvable Lie algebra with nilradical of codimension-one and assume that $S$ admits a solvsoliton $g$. Then $g$ is a global maximum for the functional $F$ restricted to the set of all left-invariant metrics on $S$. Moreover, any other global maximum $g'$ is also a solvsoliton (i.e. $g' = c\varphi^*g$, for some $c > 0$ and $\varphi \in \text{Aut}(S)$).

Theorem 1.2. Let $\mathfrak{s}$ be a solvable Lie algebra with abelian nilradical $\mathfrak{n}$ and assume that $S$ admits a solvsoliton $g$. If $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is orthogonal with respect to $g$, then the solvsoliton $g$ is a global maximum for the functional $F$ restricted to the set of all left-invariant metrics on $S$ such that $\mathfrak{a} \perp \mathfrak{n}$. Moreover, any other global maximum $g'$ for $F$ on such a set is also a solvsoliton (i.e. $g' = c\varphi^*g$, for some $c > 0$ and $\varphi \in \text{Aut}(S)$).

The proofs of these theorems will be given in Sections 2 and 3, respectively.

2. The functional $F$ on rank-one solvmanifolds

Let $S$ be a solvable Lie group of dimension $n$ such that the nilradical $\mathfrak{n}$ of its Lie algebra $\mathfrak{s}$ is non-abelian and has dimension $n-1$. If we fix a decomposition $\mathfrak{s} = \mathbb{R}Y \oplus \mathfrak{n}$, then the Lie bracket of $\mathfrak{s}$ is determined by the pair $(A, [\cdot, \cdot]_n)$, where $A := \text{ad}Y|_n \in \text{Der}(\mathfrak{n})$ and $[\cdot, \cdot]_n$ is the Lie bracket of $\mathfrak{n}$.

By fixing in addition an inner-product $\langle \cdot, \cdot \rangle$ on $\mathfrak{s}$ such that $Y \perp \mathfrak{n}$ and $|Y| = 1$, each pair $(A, [\cdot, \cdot]_n)$ is identified with the corresponding solvable Lie group $S_{(A,[\cdot,\cdot]_n)}$ endowed with the left-invariant metric defined by $\langle \cdot, \cdot \rangle$. The space of such pairs therefore covers, up to isometry, all left-invariant metrics on Lie groups with a codimension-one nilradical. Indeed, more precisely, the left-invariant metric $(\overline{h}, \overline{\cdot}, \overline{\cdot})$ on the Lie group $S_{(A,[\cdot,\cdot]_n)}$, where, with respect to a fixed orthonormal basis $\{Y, X_1, \ldots, X_{n-1}\}$ of $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$,

$$\overline{h} := \begin{bmatrix} c^{-1} & 0 \\ X & h \end{bmatrix}, \quad c \in \mathbb{R}^*, \quad X \in \mathfrak{n}, \quad h \in \text{GL}(n),$$

is isometric to the pair $(ch(A - \text{ad}_A h^{-1}X)h^{-1}, h \cdot [\cdot, \cdot]_n)$, since $\overline{h}$ is an isomorphism between the corresponding Lie algebras.

It follows from [L3] (25)] that the Ricci curvature of $(A, [\cdot, \cdot]_n)$ is given by,

$$\text{Ric} = \begin{bmatrix} -\text{tr} S(A)^2 & * \\ * & \text{Ric}[\cdot,\cdot]_n + \frac{1}{2}[A,A'] - (\text{tr} A)S(A) \end{bmatrix},$$

where
where $S(A) := \frac{1}{2}(A + A^t)$ and $\text{Ric}_{[, \cdot]}$ denotes de Ricci operator of the nilmanifold $([\cdot, \cdot], \langle \cdot, \cdot \rangle|_{n \times n})$. If $\langle [\cdot, \cdot] \rangle = 2$, which can be assumed up to scaling, then $\text{tr} \text{Ric}_{[, \cdot]} = -1$ and so

$$F(A, [\cdot, \cdot]) = \frac{(\text{tr} S(A)^2 + (\text{tr} A)^2 + 1)^2}{\text{tr} S(A)^2 (\text{tr} S(A)^2 + (\text{tr} A)^2) + \left| \text{Ric}_{[, \cdot]} \right|^2 + \frac{1}{4} \|A, A^t\|^2 + G(A, [\cdot, \cdot])},$$

for some expression $G(A, [\cdot, \cdot]) \geq 0$ that vanishes if $A$ is normal.

Let us suppose that $(A, [\cdot, \cdot])$ is a solvsoliton with $\langle [\cdot, \cdot] \rangle = 2$, i.e. $\text{Ric} + \text{Ric}_{[, \cdot]}^2 I \in \text{Der}(\mathfrak{g})$, which is equivalent by [L3, Theorem 4.8] to

$$[A, A^t] = 0, \quad \text{tr} S(A)^2 = |\text{Ric}_{[, \cdot]}|^2, \quad \text{Ric}_{[, \cdot]} + |\text{Ric}_{[, \cdot]}|^2 I \in \text{Der}(\mathfrak{g}).$$

Consider $(\text{ch}(A - \text{ad}_h h^{-1}X) h^{-1}h \cdot [\cdot, \cdot])$, where $c \neq 0$, $X \in \mathfrak{n}$, $h \in \text{GL}(n)$, and assume (up to scaling) that $\langle h \cdot [\cdot, \cdot] \rangle = 2$. In order to prove Theorem 1.1, it is therefore enough to show that

$$F(\text{ch}(A - \text{ad}_h h^{-1}X) h^{-1}h \cdot [\cdot, \cdot]) \leq F(A, [\cdot, \cdot]).$$

Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k$ be the orthogonal decomposition such that $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_k$ and so on with the rest of the descending central series. Since $A$ is normal, $A^t$ is also a derivation of $\mathfrak{n}$ and thus relative to this decomposition,

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_k \end{bmatrix}, \quad \text{ad}_h h^{-1}X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \ddots & 0 \\ \ast & \ast & 0 \end{bmatrix},$$

where the blocks correspond to each $\mathfrak{n}_i$. This implies that $A$ belongs to the closure of the conjugation class of $A - \text{ad}_h h^{-1}X$ (by conjugating with matrices which are multiples of the identity on each block), and so

$$|h(A - \text{ad}_h h^{-1}X) h^{-1}| \geq |A|,$$

where equality holds if and only if $h(A - \text{ad}_h h^{-1}X) h^{-1}$ is normal. Indeed, recall that $A$ is normal and so it is a minimal vector since the moment map for the conjugation $\text{GL}(n)$-action on $\mathfrak{gl}(n)$ is given by $m(A) = [A, A^t]/|A|^2$ (see [RS, HSS]). On the other hand,

$$\text{tr} S(h(A - \text{ad}_h h^{-1}X) h^{-1})^2 = \frac{1}{2} |h(A - \text{ad}_h h^{-1}X) h^{-1}|^2 + \frac{1}{2} \text{tr} A^2,$$

and we also know that $|\text{Ric}_{h[\cdot, \cdot]}| \geq |\text{Ric}_{[, \cdot]}|$, by using that $[, \cdot]$ is a nilsoliton (see [LW2, Section 3.1]). It therefore follows from (2) that

$$F(\text{ch}(A - \text{ad}_h h^{-1}X) h^{-1}h \cdot [\cdot, \cdot]) \leq \frac{(c^2 T + c^2 (\text{tr} A)^2 + 1)^2}{c^2 T (T + (\text{tr} A)^2) + |\text{Ric}_{[, \cdot]}|^2}
\leq \frac{(c^2 (x + a) + 1)^2}{c^2 (x + b)(x + a) + x_0 + b} =: f(x, c),$$

where $T := \text{tr} S(h(A - \text{ad}_h h^{-1}X) h^{-1})^2$ and

$$x := \frac{1}{2} |h(A - \text{ad}_h h^{-1}X) h^{-1}|^2 \geq \frac{1}{2} |A|^2 =: x_0, \quad b := \frac{1}{2} \text{tr} A^2, \quad a := b + (\text{tr} A)^2.$$

Note that $x_0 + b = |\text{Ric}_{[, \cdot]}|^2 > 0$ and that the value of $F$ at the solvsoliton is given by

$$F(A, [\cdot, \cdot]) = f(x_0, 1) = \frac{x_0 + a + 1}{x_0 + b}.$$

**Lemma 2.1.** $f(x, c) \leq f(x_0, 1)$ for any $x \geq x_0$ and $c \in \mathbb{R}$, where equality holds if and only if $x = x_0$ and $c = \pm 1$. 


Proof. We first note that if we consider the denominator of \( f(x, c) \), given by the parabola
\[
q(x) := c^4(x + b)(x + a) + x_0 + b,
\]
then \( q(x_0) > 0 \) and \( q'(x_0) > 0 \), from which follows that \( q(x) > 0 \) for any \( x \geq x_0 \). It follows that
\[
(6) \quad f(x_0, c) \leq f(x_0, 1), \quad \forall c \in \mathbb{R},
\]
where equality holds if and only if \( c = \pm 1 \), as this is equivalent to
\[
c^4(x_0 + a)^2 + 1 + 2c^2(x_0 + a) \leq (x_0 + a + 1)(c^4(x_0 + a) + 1),
\]
which simplifies to \( 0 \leq (x_0 + a)(c^2 - 1)^2 \).

On the other hand, it is straightforward to show that inequality \( f(x, c) \leq f(x_0, 1) \) can be written as
\[
p(x) = rx^2 + sx + t \geq 0, \quad \forall x \geq x_0,
\]
where \( r = c^4(a - b + 1) > 0 \), \( s = c^4\left(a(a - b + 1) - x_0(a - b) + b\right) - 2c^2(x_0 + b) \) and
\[
t = ac^4(x_0(b - a) + b) - 2ac^2(x_0 + b) + (x_0 + a)(x_0 + b).
\]
It follows from (6) that \( p(x_0) \geq 0 \), where equality holds if and only if \( c = \pm 1 \), if \( p'(x_0) \geq 0 \), then the lemma follows. One can therefore assume that \( p'(x_0) = 2rx_0 + s < 0 \), that is,
\[
(7) \quad c^4(a - b)(x_0 + a) < -c^4(x_0 + a) + c^2(2 - c^2)(x_0 + b),
\]
which is easily seen to imply that \( c^2 < 1 \) by using that \( a > b \). A straightforward computation gives that the discriminant of \( p \) equals
\[
s^2 - 4rt = c^4(a - b)(x_0 + a + 1)\left(c^4(a - b)(x_0 + a + 1) + 4(c^2 - 1)(x_0 + b)\right),
\]
and so it is negative since by (7) and the fact that \( c^2 < 1 \), the factor on the right is smaller than
\[
-c^4(x_0 + a) + c^2(2 - c^2)(x_0 + b) + c^4(a - b) + 4(c^2 - 1)(x_0 + b) = -2(x_0 + b)(c^4 - 3c^2 + 2) < 0.
\]
Thus \( p \) is always positive, concluding the proof.

Alternatively, a simple analytic argument using the partial derivatives of \( f \) gives that \( (x_0, 1) \) is a local maximum of the function \( f \) on the half plane \( \{(x, c) : x \geq x_0\} \) and that the only critical points of \( f \) in this region are \( \{(x, c) : c^2(x + b) = d\} \), with critical values,
\[
f(x, c) = \frac{(c^2(x + a) + 1)^2}{c^2d(x + a) + d} = \frac{c^2(x + a) + 1}{d} = \frac{d + \frac{d}{x+b}(a - b) + 1}{d} \leq f(x_0, 1).
\]
Since
\[
\lim_{x \to \infty} f(x, c) = 1 < f(x_0, 1), \quad \lim_{c \to \infty} f(x, c) = \frac{x + a}{x + b} < 1 + \frac{a - b}{d} < f(x_0, 1),
\]
there are positive numbers \( M_1, M_2 \) such that the value of \( f \) out of the compact region \( \{(x, c) : x \leq x_0, |c| \leq M_2\} \) is always strictly less than \( f(x_0, 1) \). All this implies that \( (x_0, \pm 1) \) are actually the only global maxima of \( f \) on the half plane \( \{(x, c) : x \geq x_0\} \), as desired. \( \square \)

Inequality (4) therefore follows from (5) and the inequality given in Lemma 2.1. On the other hand, if equality holds in (5) and Lemma 2.1 then \( h(A - ad_n h^{-1}X)h^{-1} \) is normal, \( |h(A - ad_n h^{-1}X)h^{-1}| = |A|, c = \pm 1 \) and \( |\text{Ric}_{[\cdot, \cdot]}| = |\text{Ric}_{\{\cdot, \cdot\}}| \). This implies that
\[
\text{tr} S(ch(A - ad_n h^{-1}X)h^{-1})^2 = \frac{1}{2} \text{tr} h(A - ad_n h^{-1}X)h^{-1})^2 + \frac{1}{2} \text{tr} A^2 = \text{tr} S(A)^2
\]
\[
= |\text{Ric}_{\{\cdot, \cdot\}}|^2 = |\text{Ric}_{[\cdot, \cdot]}|^2,
\]
and it follows from [LW2, Section 3.1] that \( h \cdot [\cdot, \cdot]_n \) is a nilsoliton. Thus \( h \cdot [\cdot, \cdot]_n = \tilde{h} \cdot [\cdot, \cdot]_n \) for some \( \tilde{h} \in O(n, \langle \cdot, \cdot \rangle) \) by the uniqueness of nilsolitons (see [L1, Theorem 3.5]) and hence

\[
\text{Ric}_{\tilde{h} \cdot [\cdot, \cdot]_n} + \frac{1}{2} \text{Ric}_{\tilde{h} \cdot [\cdot, \cdot]_n}^2 I = \tilde{h}(\text{Ric}_{\tilde{h} \cdot [\cdot, \cdot]_n} + \frac{1}{2} \text{Ric}_{\tilde{h} \cdot [\cdot, \cdot]_n}^2 I) \tilde{h}^{-1} \in \text{Der}(\tilde{h} \cdot [\cdot, \cdot]_n) = \text{Der}(h \cdot [\cdot, \cdot]_n).
\]

All this implies that \((ch(A - \text{ad}_n h^{-1} X)h^{-1}, h \cdot [\cdot, \cdot]_n)\) is a solvoliton (see (3)), concluding the proof of Theorem 1.1.

3. The functional \( F \) on solvmanifolds with an abelian nilradical

In this section, we consider a solvable Lie group \( S \) of dimension \( n \) such that the nilradical \( n \) of its Lie algebra \( s \) is abelian, say with \( \dim n = n - r \). After fixing a decomposition \( s = a \oplus n \) and a basis \( \{Y_i\} \) of \( a \), the Lie bracket of \( s \) is determined by an \( r \)-tuple \((A_1, \ldots, A_r)\) of linearly independent linear operators of \( n \) such that \([A_i, A_j] = 0\) for all \( i, j \), where \( A_i \coloneqq \text{ad}_{Y_i} \in \text{gl}(n) \), and a bilinear map \( \lambda : a \times a \to n \). We assume that \( S \) admits a solvoliton, hence \( \lambda = 0 \) for the corresponding orthogonal decomposition \( s = a \oplus n \) (see [L3, Theorem 4.8]). By fixing an inner-product \( \langle \cdot, \cdot \rangle \) on \( s \) such that \( a \perp n \) and \( \{Y_i\} \) is orthonormal, each \((A_1, \ldots, A_r)\) is identified with the corresponding solvable Lie group \( S(A_1, \ldots, A_r) \) endowed with the left-invariant metric defined by \( \langle \cdot, \cdot \rangle \). It is easy to see that any left-invariant metric on the Lie group \( S(A_1, \ldots, A_r) \) for which \( a \perp n \) is isometric to some

\[
(h_2(h_1^{-1} A_1)h_2^{-1}, \ldots, h_2(h_1^{-1} A_r)h_2^{-1}), \quad h_1 \in \text{GL}(a), \quad h_2 \in \text{GL}(n),
\]

where \( h_i^{-1} A_i \coloneqq \text{ad} h_i^{-1} Y_i = \sum c_{ij} A_j \) if the matrix of \( h_i^{-1} \) relative to \( \{Y_i\} \) is \([c_{ij}]\).

It follows from [L3, (25)] that the Ricci curvature of \((A_1, \ldots, A_r)\) is given by

\[
\text{Ric} = \begin{bmatrix}
R & 0 \\
0 & \frac{1}{2} \sum [A_i, A'_i] - \sum \langle \text{tr} A_i S(A_i) \rangle
\end{bmatrix}, \quad R_{ij} = -\text{tr} S(A_i) S(A_j).
\]

Up to isometry, it can always be assumed that \( \text{tr} A_2 = \cdots = \text{tr} A_r = 0 \) (i.e. \( H = (\text{tr} A_1) Y_1 \)) by considering in (8) \( h_2 = I \) and a suitable \( h_1 \in O(a, \langle \cdot, \cdot \rangle) \). In that case,

\[
F(A_1, \ldots, A_r) = \frac{(\sum \text{tr} S(A_i)^2 + (\text{tr} A_1)^2)^2}{\sum (\text{tr} S(A_i) S(A_j))^2 + (\text{tr} A_1)^2 \text{tr} S(A_1)^2 + \frac{1}{4} \sum [A_i, A'_i]^2}.
\]

Let us suppose that \((A_1, \ldots, A_r)\) is a solvoliton, that is, \( \text{tr} S(A_i) S(A_j) = \delta_{ij} \) (up to scaling) and \( A_i \) is normal for all \( i \) (see [L3, Theorem 4.8]). We consider an \( r \)-uple as in (8) and assume (up to isometry and scaling) that \( \text{tr} h_1^{-1} A_1 = \text{tr} A_1 \) and \( \text{tr} h_1^{-1} A_2 = \cdots = \text{tr} h_1^{-1} A_r = 0 \) (i.e. \( c_{11} = 1 \) and \( c_{12} = \cdots = c_{1r} = 0 \)). Thus what we must show to prove Theorem 1.2 is that

\[
F(h_2(h_1^{-1} A_1)h_2^{-1}, \ldots, h_2(h_1^{-1} A_r)h_2^{-1}) \leq F(A_1, \ldots, A_r),
\]

for any \( h_1 \in \text{GL}(a) \) and \( h_2 \in \text{GL}(n) \).

By (10), we have that

\[
F(h_2(h_1^{-1} A_1)h_2^{-1}, \ldots, h_2(h_1^{-1} A_r)h_2^{-1}) \leq \frac{(x_1 + \cdots + x_r + a)^2}{x_1^2 + \cdots + x_r^2 + ax_1} = f(x_1, \ldots, x_r),
\]

where

\[
x_i := \text{tr} S(h_2(h_1^{-1} A_i)h_2^{-1})^2, \quad a := (\text{tr} A_1)^2.
\]
Since $h_1^{-1}A_1 = A_1 + c_{21}A_2 + \cdots + c_{r1}A_r$ is normal, it follows from [LW2] (17) that
\[
x_1 = \frac{1}{2} \text{tr} (h_1^{-1}A_1)^2 + \frac{1}{2} h_2(h_1^{-1}A_1)h_2^{-1}\|^2 \geq \frac{1}{2} \text{tr} (h_1^{-1}A_1)^2 + \frac{1}{2} |h_1^{-1}A_1|^2
\]
\[= \text{tr} S(h_1^{-1}A_1)^2 = 1 + \sum_{i=2}^r c_{i1}^2 \geq 1.\]

Note that the value of $F$ at the solvsoliton is given by
\[F(A_1, \ldots, A_r)) = f(1, \ldots, 1) = r + a.\]

**Lemma 3.1.** $f(x_1, \ldots, x_r) \leq f(1, \ldots, 1)$ for any $x_1 \geq 1$, $x_2, \ldots, x_r > 0$, where equality holds if and only if $x_1 = \cdots = x_r = 1$.

**Proof.** An elementary algebraic manipulation gives that the inequality is equivalent to
\[0 \leq \left( (r - 1) \sum x_i^2 - \sum x_ix_j \right) + a^2(x_1 - 1) + a \left( \sum x_i^2 - 2 \sum x_i + rx_1 \right).
\]
Since the first term is $\geq 0$ by the Cauchy-Schwarz inequality $(\sum x_i)^2 \leq r \sum x_i^2$ and the third one is $\geq a \sum (x_i - 1)^2$, one obtains that both the above inequality and the equality condition in the lemma follow. \(\square\)

Since $F$ is invariant under all the assumptions made above up to isometry and scaling, inequality (11) follows from (12) and Lemma 3.1. Moreover, if equality holds, then $h_2(h_1^{-1}A_i)h_2^{-1}$ is normal for all $i$ and
\[
\text{tr} S(h_2(h_1^{-1}A_i)h_2^{-1})S(h_2(h_1^{-1}A_j)h_2^{-1}) = \delta_{ij},
\]
which implies that $(h_2(h_1^{-1}A_1)h_2^{-1}, \ldots, h_2(h_1^{-1}A_r)h_2^{-1})$ is a solvsoliton, concluding the proof of Theorem 1.2.

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