A numerical solution to nonlinear lens-shaped membrane-pneumatic systems using the iterative-incremental approach with account for air elastic properties

M F Amoyan*, A Yu Kim, S V Polnikov

Saratov State Technical University named after Yuri Gagarin, 77, Polytechnic street, Saratov, 410054, Russian Federation

E-mail: ezid-007@yandex.ru

Abstract. The authors present an improved methodology for a computer-based static analysis of geometrically and physically nonlinear lens-shaped membrane-pneumatic systems relating the large-span structures coverings using the iterative -incremental technique with the finite element method stepwise application, the universal gas equation, and the improved Euler-Cauchy numerical procedure.

Introduction

Published in the middle of the XX century, the works by D. F. Davidenko “The Parameters Method Variation Application to the Nonlinear Functional Equations Theory” [1] and V. V. Petrov “The Uniformly Distributed Loads Method to the Nonlinear Theory of Plates and Shells” [2] stimulated different versions created to the incremental development approach. These approaches made the foundation for a universal incremental method, which allows various changes in the nonlinear systems parameters.

It was critical to address the challenges related with a lack of the method for solving the nonlinear problems in the construction and scientific workflows.

The idea behind the method of uniformly distributed loads was developed by Prof. V. V. Petrov [2].

Let us assume that the system stress-strain state is described by the nonlinear functional $A(x, y) = 0$, which includes the increment functions continuously dependent on the $x$ system parameters set. These functions reflect the changes in the system rigidity in the course of its deformation, and the required functions $Z$ expressing the displacements, strains and stresses within the system. With this assumption

$$\frac{\partial Z}{\partial x_\mu} = A'_{\nu\mu}(x, y),$$

we write the obtained equation $A(x, y) = 0$ following the method of successive iterations using the numerical Euler procedure:
\Delta Z_{nv} = \Delta x_{n\mu} \cdot A'_\nu (x_{n-1,\mu}, y_{n-1}),

(1)

where \( n \) is the number of the current increment; \( \Delta x_{n\mu} \) is the parameter increment \( x_{\mu} \) at the step \( n \); the value of the \( \mu \) parameter at the end of the previous step \( n-1 \); the value of some incremental function \( y \) at the end of step \( n-1 \).

Early in the XI century, Prof. A. Yu. Kim developed a calculation method for nonlinear lens-shaped membrane-pneumatic systems of medium spans taking into account the air elastic properties \[3, 4\]. In this case, the incremental method was improved due to the incremental-iterative algorithm with a stepwise performance of the Euler-Cauchy procedure to express the third-order accurate scheme. This method includes the following operations sequence:

1. Solving the problem with an initial approximation, i.e. by implementing the incremental method with the first-order-accurate scheme and using the Euler formula

\[ \Delta Z_{nv}^{(1)} = \sum_{\mu=1}^{\mu=n} \Delta x_{n\mu} \cdot A'_\nu (x_{n-1,\mu}, y_{n-1}) . \]

2. Solving the problem with \( c \) approximation in line with the formula

\[ \Delta Z_{nv}^{(c)} = \sum_{\mu=1}^{\mu=n} \Delta x_{n\mu} \cdot A'_\nu (x_{n-1,\mu} + \frac{\Delta x_{n\mu}}{2}, y_{n-1} + \frac{\Delta y_n^{(c-1)}}{2}) , \]

(2)

where \( 2 \leq c \leq 4 \).

Meanwhile, the accumulation function \( \Delta y_n \) increments are averaged using the Runge-Kutta numerical procedure within one step at \( C = 2 \). When \( C = 4 \), the calculation formula (2) of the iterative and incremental procedure shows the error size in the third order accuracy estimates results.

The accumulation functions are related with temperature, kinematic and pneumatic loads. However, only pneumatic types refer to the guiding loads, and on application an incremental step the system acquires the aftereffect, i.e. a subsequent relaxation subjected to stress due to additional air pressure.

The authors are working on improvement of the method developed by A. Yu. Kim in order to solve the problems related to calculating geometrically and physically nonlinear lens-shaped membrane-pneumatic systems for coverings of large span structures using the incremental-iterative technique with a step-by-step application of the finite elements method, the universal gas equation, and the Euler-Cauchy equation of the third accuracy order \[4\].

![Figure 1. A single-span lens-shaped membrane-pneumatic covering of rectangular buildings](image)
Figure 2. The calculation scheme for a single-span lens-shaped membrane-pneumatic system

The authors consider a random step, where the load value is corrected at each iteration procedure. The iterations number is significant to adequately describing the pneumatic system operation in its progress and obtain the required results accuracy. In this case, the Euler-Cauchy procedure with a long-time iterative procedure may require a large number of iterations at $c \geq 5$. Thus, we can take into account the aftereffect phenomenon, which accompanies the changes in the excess air pressure in the lens and creates an internal pneumatic load, which causes efforts redistribution within the pneumatic system components.

The investigated coating system is a spatial system. The spatial Cartesian coordinate system $O_{xyz}$ is applied.

A linearized equations system is formed step-by-step at each step incremental procedure using the connectivity matrix

$$[r_{ab}] \tilde{\eta}_b = (Ra),$$

where $[r_{ab}]$ is the global system stiffness matrix; $\eta_b$ is the column matrix to the required nodal displacements; $(Ra)$ is the column matrix of the absolute terms.

The stiffness matrix $[r_{ab}]$ of the pre-strained combined system under consideration is the sum of matrices, one of which $[r_{ab}^N]$ takes into account the longitudinal pressure impact, whereas the other $[r_{ab}^M]$ takes into account the bending moments impact, i.e.

$$[r_{ab}] = [r_{ab}^N] + [r_{ab}^M].$$

Under address imposition of one matrix over another, the matrices order may differ. The matrix order $[r_{ab}^N]$ equals $K = 3 K^* - d$, where $d$ is the number of designated supporting links in the intermediated nodes. The matrix order $[r_{ab}^M]$ is determined by the number of transverse links superimposed over the enlarged finite element towards the axis $\xi$.

The matrix $[r_{ab}^N]$ of the algorithm for calculating the spatial cable-frame structure consists of the submatrices, each having the form
Each component in the column matrix of the required displacements \( \eta_b \) and absolute terms \( R_a \) consists of the column submatrices

\[
\eta_b = (u_{nb}, v_{nb}, w_{nb})^T, \quad R_a = (-R_{ax}, -R_{ay}, -R_{az})^T,
\]

where \( u_{nb}, v_{nb}, w_{nb} \) are the increments of the node displacements \( b \) at the current stage of the loading directed towards the coordinate axes \( x, y \) and \( z \), whereas \( R_{ax}, R_{ay}, R_{az} \) are the reactions in the links superimposed over the node \( a \) towards the corresponding coordinate axes from the designated load action. At switching from the node numbering to the global node numbering superimposed on the link system, we take into account the following dependence: \( i = 3 \cdot a - 2 ; k = 3 \cdot b - 2 \). The indices \( a_x, a_y \) and \( a_z \) correspond to the indices \( i, i+1, i+2 \), whereas the indices \( b_x, b_y, b_z \) correspond to the indices \( k, k+1, k+2 \).

By calculating the equations system coefficients

\[
\sum_{b=1}^{i'} (r_{ax, bx} \cdot u_{nb} + r_{ax, by} \cdot v_{nb} + r_{ax, bz} \cdot w_{nb}) = -R_{ax} ;
\]

\[
\sum_{b=1}^{k'} (r_{ay, bx} \cdot u_{nb} + r_{ay, by} \cdot v_{nb} + r_{ay, bz} \cdot w_{nb}) = -R_{ay} ;
\]

\[
\sum_{b=1}^{k'} (r_{az, bx} \cdot u_{nb} + r_{az, by} \cdot v_{nb} + r_{az, bz} \cdot w_{nb}) = -R_{az} ;
\]

at \( a = 1, K \), in accordance with the node numbers in the system, we develop a resolving system of equations to the finite elements method

\[
[r_{ik}] \bar{x}_k = (R_i), \text{ where } i = 1, K ; k = 1, K
\]
The air volume increment calculation in the lens \( \Delta V_s \) is conducted based on the new nodes coordinates in the system belts: \( x_a = x_a + u_a, \quad y_a = y_a + v_a \), where \( u_a \) and \( v_a \) are calculated with account for the external impact and non-linearity.

On calculating the volume of the lens \( V_s^{(2)} \) after the loading procedure, and having the data relating the volume of the lens \( V_s^{(1)} \) before loading the system, the lens volume increment is found by the formula
\[
\Delta V_s = V_s^{(2)} - V_s^{(1)}
\]

When calculating the membrane-pneumatic systems, we take into account the elastic properties of the air pumped into hermetically-closed cavity of the pneumatic lens, i.e. we take into account the effect of elastic displacements of the lens-shaped coating belts on the pressure \( p_0 \). We consider the coating made in the form of a single pneumatic lens. The volume, temperature and air pressure of the pneumatic lens are characterized by the parameters \( V, T, P \). The lens \( \Delta V \) volume increment is determined depending on the pressure \( P \) and the air temperature \( T \) in the closed pneumatic cavity at the stage of the structures usage.

From the universal gas equation
\[
\frac{P_i V_i}{T_i} = \frac{P V}{T}
\]
where the parameters \( P_i, V_i, T_i \) characterize the system at the final stage of construction, taking into account the dependencies
\[
P = P_i + \Delta P, \quad V = V_i + \Delta V, \quad T = T_i + \Delta T,
\]
we obtain
\[
\Delta P = \frac{P}{T} \Delta T - \frac{P_i}{V} \Delta V
\]

Using iteration \( c \) we calculate the volumetric increment \( \Delta V \) of the pneumatic lens closed cavity depending on the coating belts vertical deflection at the ransom step \( n \) of the system loading. We express the pressure increment in the closed cavity at the step \( n \) with the increments at the step \( n \) of the temperature \( \Delta T_n \) and the volume \( \Delta V_n \) of the closed cavity (11)
\[
\Delta P_n^{(c)} = p_n^{(c)} = \frac{P_0}{T^0} \Delta T_n - \frac{P_i}{V^0} \Delta V_n.
\]

**Summary**

The membrane-pneumatic structures accurate computation by means of incremental procedures can be applied both with account for the physical and geometric nonlinearity of the membrane material, and elastic properties of the air using the universal gas equation.

The air in the membrane-pneumatic structures lenses is a way to ensure the pressure for the lower and upper membranes, as well as a component in the structure of a membrane or a column which transmits the loads from the stressed to the carrier membrane.

Further perfection of the calculation technology of such systems, with account for elastic properties of air in the lenses, will significantly improve computational accuracy of the given pneumatic structures.

**References**
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