THE WEYL EXTENSION ALGEBRA OF $GL_2(\mathbb{F}_p)$

VANESSA MIEMIETZ AND WILL TURNER

Abstract. We compute the Yoneda extension algebra of the collection of Weyl modules for $GL_2$ over an algebraically closed field of characteristic $p > 0$ by developing a theory of generalised Koszul duality for certain 2-functors, one of which controls the rational representation theory of $GL_2$ over such a field.

1. Intro.
2. Combinatorial description of $w$.
3. Example.
4. Outline.
5. Homological duality for algebraic operators. Algebraic operators.
Homological duality.
Homological duality for operators $P$.
A quasi-isomorphism of operators.
6. Recollections on $GL_2$.
7. The homological dual of $(c, t, t^{-1})$.
The algebra $\Psi$.
Hypotheses of Theorem [15]
Homologically self-dual algebras.
The homological dual of $t$.
The triple.
8. Expressing $w$ via $(\Psi, \star, \star^{-1})$.
9. The algebra $HT_{\Psi}(\star, \star^{-1})$.
A bimodule.
Explicit formulas for the homology of $\star^i$ for $i \leq -1$.
The dg algebra $T_{\Psi}(\star, \star^{-1})$.
The case $p=2$.
Truncating $\Upsilon$.
Polytopal basis for $\Upsilon \leq 1$.
10. Appendix 1: Signs.
11. Appendix 2: Koszul duality.
References

Contents
1. Intro.

This paper belongs to a sequence of papers exploring structural features of the rational representation theory of the algebraic group $GL_2(F)$, where $F$ is an algebraically closed field of characteristic $p > 0$. We denote by

$$\mathcal{W} = \{\text{Sym}^\lambda(V) \otimes \omega^\mu \mid \lambda \in \mathbb{Z}_{\geq 0}, \mu \in \mathbb{Z}\}$$

the set of Weyl modules in the category $G$-$\text{mod}$ of rational representations of $G = GL_2(F)$, where $V$ denotes the natural two dimensional representation of $G$ and $\omega$ the one dimensional determinant representation. By definition, $\text{Sym}^\lambda(V)$ is the space of symmetric tensors powers of $V$ of degree $\lambda$, which is to say the set of fixed points for the symmetric group $S_\lambda$ in its natural action on $V^{\otimes \lambda}$. $\mathcal{W}$ is a complete set of standard objects in the highest weight category $G$-$\text{mod}$ of rational representations of $G$. Our object is to give an explicit description of the Yoneda extension algebra

$$\mathcal{W} = \bigoplus_{\Delta, \Delta' \in \mathcal{W}, k \in \mathbb{Z}} \text{Ext}^k_{G}$$_{$\text{mod}$}\!(\Delta, \Delta')$$

of the collection $\mathcal{W}$. The most important previous discovery in this direction was an algorithm to compute the dimension of $\text{Ext}^k_{G}$$_{$\text{mod}$}\!(\Delta, \Delta')$ for $\Delta, \Delta' \in \mathcal{W}$ and $k \geq 0$, written down by A. Parker ([10], Theorem 5.1). Here we describe the algebra structure. In previous papers, we have described certain 2-functors which control the rational representation theory of $G$ and give a combinatorial description of blocks of rational representations. We have developed a theory of Koszul duality for these operators, allowing us to give an explicit description of the Yoneda extension algebra of the irreducible $G$-modules. In this article, we extend these methods to more general homological dualities and a setting where gradings are not necessarily non-negative.

The category $G$-$\text{mod}$ has countably many blocks, all of which are equivalent. Therefore, the algebra $\mathcal{W}$ is isomorphic to a direct sum of countably many copies of $\mathfrak{w}$, where $\mathfrak{w}$ is the Yoneda extension algebra of the Weyl modules belonging to the principal block of $G$. Our problem is to compute $\mathfrak{w}$.

In the following section of the paper we outline a combinatorial description of $\mathfrak{w}$ as $\lim_q \Omega_F \Omega_Y^q(F[z])$, where $\Omega_Y$ is a certain algebraic operator, and where $Y$ is an algebra with monomial basis indexed by elements of a certain lattice polytope; in the remaining sections we explain why this description is correct. A consequence of this description of $\mathfrak{w}$ is that $\mathfrak{w}$ has a monomial basis indexed by elements of an infinite dimensional polytope. Our analysis proceeds via a more conceptual description of $Y$ involving the tensor algebra over a Koszul algebra $\Psi$ of a certain bimodule $\Psi M \Psi$ (Theorem [11]). This bimodule $M$ satisfies some strong homological properties (Theorem [27]).

We write the names of our algebraic operators in exotic scripts like $\mathfrak{P}$, $\mathfrak{D}$, $\mathfrak{F}$, and $\mathfrak{H}$. These operators are all 2-functors on certain 2-categories, a fact we suppress to avoid frequent checking of axioms; their dominant virtue is their natural behaviour, which means they give a conceptually simple way of encoding complicated homological information.
2. Combinatorial description of \( w \).

Suppose \( \Gamma = \bigoplus_{i,j,k \in \mathbb{Z}} \Gamma_{ijk} \) is a \( \mathbb{Z} \)-trigraded algebra. We have a combinatorial operator \( \mathcal{O}_\Gamma \) which acts on the collection of \( \mathbb{Z} \)-bigraded algebras \( \Sigma \) after the formula

\[
\mathcal{O}_\Gamma(\Sigma)^{ik} = \bigoplus_{j,k_1+k_2=k} \Gamma_{ijk_1} \otimes_F \Sigma^{jk_2},
\]

where we take the super tensor product with respect to the \( k \)-grading.

There is a trigraded algebra \( \Upsilon \) whose structure we partially describe via a polytopal monomial basis; by a monomial basis we mean a basis in which the product of two elements is ± another basis element, or zero. There is a reason for our idleness in giving only a partial definition: in order to describe \( w \) we only need to know what the algebra structure of \( \Upsilon \) looks like when projected onto the subspace \( \Upsilon^{\leq 1} = \bigoplus_{i,j,k \in \mathbb{Z}, r \leq 1} \Upsilon_{ijk} \). The space \( \Upsilon^{\leq 1} \) has basis \( \{m_w\}_{w \in \mathcal{P}_{\Upsilon^{\leq 1}}} \) indexed by a polytope \( \mathcal{P}_{\Upsilon^{\leq 1}} \) in \( \mathbb{Z}^7 \) and product (in its projection onto \( \Upsilon^{\leq 1} \)) given by

\[
m_w m_{w'} = \begin{cases} 
(-1)^{a_j' b_j' + b_o' a_l'} m_w & \text{if } v \in \mathcal{P}_{\Upsilon^{\leq 1}}, v_1 = w_1, w_7 = w_1', w_7' = v_7, \\
0 & \text{and } v_l = w_l + w_l' \text{ for } 2 \leq l \leq 5, \\
\end{cases}
\]

Here we write \( w = (w_1, w_2, w_3, w_4, w_5, w_6, w_7) \in \mathbb{Z}^7 \). The \( ijk \)-degree of a basis element \( m_w \) is \( (w_2, w_3, w_4) \).

For details of how to define the polytope \( \mathcal{P}_{\Upsilon^{\leq 1}} \) and the sign \( (-1)^{a_j' b_j' + b_o' a_l'} \) we refer to section 9. For a more elegant conceptual description of \( \Upsilon \) we also refer to section 9.

Let us consider the field \( F \) as a trigraded algebra concentrated in degree \( (0,0,0) \). We have a natural embedding of trigraded algebras \( F \to \Upsilon \), which sends 1 to \( m_{(1,0,0,0,0,0,1)} \). This embedding lifts to a morphism of operators \( \mathcal{O}_F \to \mathcal{O}_\Upsilon \). We have \( \mathcal{O}_F^2 = \mathcal{O}_F \). Putting these together, we obtain a sequence of operators

\[
\mathcal{O}_F \to \mathcal{O}_F \mathcal{O}_\Upsilon \to \mathcal{O}_F \mathcal{O}_\Upsilon^2 \to ...
\]

which, applied to the bigraded algebra \( F[z, z^{-1}] \) with \( z \) placed in \( jk \) degree \( (1,0) \), gives a sequence of algebra embeddings

\[
\mu_1 \to \mu_2 \to \mu_3 \to ...
\]

where \( \mu_q = \mathcal{O}_F \mathcal{O}_\Upsilon^q(F[z, z^{-1}]) \). Taking the union of the algebras in this sequence gives us an algebra \( \mu \). Our main theorem is the following:

**Theorem 1.** The algebra \( w \) is isomorphic to \( \mu \).

The algebras \( \mu_q \) are isomorphic to Yoneda extension algebras of collections of Weyl modules for certain Schur algebras, and are consequently finite dimensional. The \( k \)-grading on \( \mu \) matches with the natural homological grading on \( w \).

It is straightforward to give a polytopal monomial basis for \( \mu \) as follows: we define the **weight** of a monomial \( m_{w_1} \otimes ... \otimes m_{w_n} \otimes z^{\alpha} \) in \( \Upsilon^{\otimes q} \otimes F[z, z^{-1}] \) to be

\[
(w_2^2 - w_3^1, w_2^3 - w_2^2, ..., w_3^{q-1} - w_3^q, \alpha - w_3^q) \in \mathbb{Z}^{q+1}.
\]

We have a map from the set of monomials in \( \Upsilon^{\otimes q} \otimes F[z, z^{-1}] \) to the set of monomials in \( \Upsilon^{\otimes q+1} \otimes F[z, z^{-1}] \) that sends \( a \) to \( m_{(1,0,0,0,0,0,1)} \otimes a \) and preserves the last \( q + 1 \)
components of the weight. The union over $q$ of monomials in $\Upsilon^q \otimes F[z, z^{-1}]$ of weight zero is a polytopal monomial basis for $\mu$.

3. Example.

The algebra $\Upsilon_q$ is isomorphic to the Yoneda extension algebra of the Weyl modules for a block of a Schur algebra $S(2, r)$ with $p^q$ simple modules. To place our feet on the ground, let us describe an example of such an algebra.

Let $p = 3$. Let $\nu_2$ denote the Yoneda extension algebra of the collection of Weyl modules in a block of a Schur algebra $S(2, r)$ containing 9 simple modules. The Ext$^1$-quiver of this algebra is given by

where the number of bars through the tail of the arrow marks the degree of the arrow in the Ext-grading (aka $k$-grading) and the number labelling the arrow denotes its $j$-grading.

The composition structure of the projectives, with the superscript denoting the $k$-degree and the subscript denoting the $j$-degree, is given by

\[
\begin{array}{cccc}
1^0_0 & 2^0_0 & 3^0_0 & 4^0_0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1^1_1 & 1^{1-1}_1 & 2^1_1 & 2^{1-1}_1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
3^1_2 & 3^{1-2}_2 & 2^2_2 & 1^{3-4}_2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
2^0_2 & 2^1_1 & 3^0_1 & 4^0_1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
3^1_3 & 3^{2-3}_3 & 2^3_3 & 1^{4-5}_3 \\
\end{array}
\]
In section 5 we describe a little theory of algebraic operators, recalling and generalising some previous definitions and results [9]. We then give an overview of Keller’s homological duality, and how it relates to our operators, also in Section 5. We then recall some facts about the representation theory of $GL_2$ and explain how Koszul duality and Keller’s homological duality come into play in this representation theory, giving us various differential graded bimodules to use as input for our algebraic operators (Sections 6 to 8). Specifically, in Section 8 we show that an algebraic operator $O_{HT\Psi}(\varepsilon)$ indexed by the homology of a certain tensor algebra, thought of as a trigraded algebra, can be used to compute the Weyl extension algebra of $GL_2$. We then explicitly compute $HT\Psi(\varepsilon)$ and show it is equal to $\Upsilon$; this is done in Section 9. We have relegated certain generalities to appendices: on signs and Koszul duality.

5. Homological duality for algebraic operators.

Algebraic operators. For our computation of $y_i$, we used trigraded structures. Here, we also use quadragraded structures

$$S = \bigoplus_{d,i,j,k \in \mathbb{Z}} S^{dijk}.$$  

It will be necessary to differentiate between the four gradings; we will call them the $d$-grading, the $i$-grading, the $j$-grading, and the $k$-grading.

As in our previous paper, the $i$ and $j$-gradings will be algebraic. We denote by $\langle 1 \rangle$ a shift by 1 in the $j$-grading, thus $(M \langle n \rangle)_j = M_{j-n}$.

The $k$-grading will always be a homological grading, and differentials always have $k$-degree 1, and $(i, j)$-degree $(0, 0)$. When we speak of a differential (bi-, tri-, quadra-)graded algebra, we mean (bi-, tri-, quadra-)graded algebra which is a differential graded algebra with respect to the $k$-grading. We denote by $H$ the cohomology.
functor, which takes a differential $k$-graded complex $C$ to the $k$-graded vector space $\mathbb{H}C = H^\bullet C$. We denote by $[1]$ a shift by 1 in the $k$-grading.

The $d$-grading will be algebraic, and in our application will form a $\Delta$-grading and will in all relevant cases be positive.

In practice, for the development of our theory, one of the gradings will in fact be obscured, so we only have to consider trigraded structures $S^{ijk}$ or $S^{ijk}$; nevertheless it seems to be important to distinguish between the four types of grading.

There is a fifth grading. When we discuss Koszul duality, we will refer to the grading of a quotient of a quiver algebra by path length, which we call it the $r$-grading. In certain special cases, this $r$-grading coincides with our $j$-grading, but in other cases it does not; this is why we denote it with a different letter. But outside our discussion of Koszul duality, we will not be concerned with the $r$-grading, and we disregard it. Likewise, the sixth and seventh gradings (the $f$ and $h$ gradings).

**Bonded pairs of bimodules.** Let $A$ be a finite dimensional algebra. We say a pair $M = (M, M')$ of $A$-$A$-bimodules are **bonded** if we have homomorphisms $M \otimes_A M' \to A$, $M' \otimes_A M \to A$, such that the resulting pair of maps

$$M \otimes_A M' \otimes_A M \to M$$

are equal, and the resulting pair of maps

$$M' \otimes_A M \otimes_A M' \to M'$$

are equal.

**Lemma 2.** Given a bonded pair $M$ of $A$-$A$-bimodules, the space

$$T_A(M) = (\oplus_{i \geq 1} M^{\otimes_A i}) \oplus A \oplus (\oplus_{i \leq -1} M'^{\otimes_A -i})$$

is a $\mathbb{Z}$-graded algebra, with product given by the natural bimodule homomorphisms

$$M^{\otimes_A i_1} \otimes M'^{\otimes_A -i_2} \to M^{\otimes_A i_1 + i_2}, \quad r_1 \geq r_2,$$

$$M^{\otimes_A i_1} \otimes M'^{\otimes_A -i_2} \to M'^{\otimes_A -i_1 - i_2}, \quad r_2 \geq r_1,$$

$$M'^{\otimes_A -i_1} \otimes M^{\otimes_A i_2} \to M'^{\otimes_A -i_1 - i_2}, \quad r_1 \geq r_2,$$

$$M'^{\otimes_A -i_1} \otimes M^{\otimes_A i_2} \to M^{\otimes_A i_1 + i_2}, \quad r_2 \geq r_1,$$

obtained by applying the maps $M \otimes_A M' \to A$, $M' \otimes_A M \to A$ a number of times; here we write $M^{\otimes_A 0} = M'^{\otimes_A 0} = A$.

**Proof.** We define $A$ to be the tensor algebra of $M \oplus M'$ over $A$, modulo relations implying the product of $M$ and $M'$ lies in $A$, and the product map on these bimodules is given by the maps $M \otimes_A M' \to A$, $M' \otimes_A M \to A$. The algebra $A$ acts naturally on the space $T_A(M)$ via the homomorphisms $M \otimes_A M' \to A$, $M' \otimes_A M \to A$; the fact this is an algebra action follows from the fact that $M$ and $M'$ are bonded. The relations in $A$ imply that $A$ is a quotient of $T_A(M)$ as an $A$-$A$-bimodule. The subspace $A$ of $T_A(M)$ generates $T_A(M)$ as an $A$-module, which implies $T_A(M)$ is a quotient of $A$. Consequently $A$ acts freely on $T_A(M)$, and we can identify $A$ with $T_A(M)$; once we do this we obtain a product on $T_A(M)$ as advertised.

We have a ready supply of bonded pairs of bimodules:
Lemma 3. Suppose $M$ is a differential graded $A$-$A$-bimodule which is projective on the left and right as an $A$-module. Then $M$ and $\Hom_A(M, A)$ are a bonded pair of dg bimodules.

Proof. We have a natural isomorphism of dg bimodules

$$M \to \Hom_A(\Hom_A(M, A), A).$$

Our bonded structure is given by the pair of natural maps

$$M \otimes \Hom_A(M, A) \to A, \quad \Hom_A(M, A) \otimes \Hom_A(M, A), A).$$

We check the natural composition maps

$$M \otimes_A \Hom_A(M, A) \otimes_A \Hom_A(M, A), A)$$

$$\to M \otimes_A A \to M \to \Hom_A(\Hom_A(M, A), A)$$

and

$$M \otimes_A \Hom_A(M, A) \otimes_A \Hom_A(M, A), A)$$

$$\to A \otimes_A \Hom_A(\Hom_A(M, A), A) \to \Hom_A(\Hom_A(M, A), A)$$

are equal. If we identify these with maps

$$\Hom_A(A, M) \otimes_A \Hom_A(M, A) \otimes_A \Hom_A(M, A) \to \Hom_A(\Hom_A(M, A), \Hom_A(A, A))$$

then both send $\alpha \otimes \beta \otimes \gamma \in \Hom_A(A, M) \otimes_A \Hom_A(M, A) \otimes_A \Hom_A(A, M)$ to the morphism sending $\eta \in \Hom_A(M, A)$ to $\alpha \beta \gamma \eta \in \Hom_A(A, A)$. \(\square\)

The collections $\mathcal{T}$ and $\mathcal{U}$. Let $\mathcal{T}$ denote the collection of dg algebras with a pair of bonded dg bimodules

$$\mathcal{T} = \left\{ (A, M) \mid A = \bigoplus_k A^k \text{ a dg algebra, } M = (\bigoplus_k M^k, \bigoplus_k M^k) \right\}$$

 bonded dg $A$-$A$-bimodules.

Let $\mathcal{U}$ denote the collection of differential bigraded algebras with a pair of bonded differential bigraded bimodules

$$\mathcal{U} = \{(A, M) \mid A = \bigoplus_{k \in \mathbb{Z}, d \geq 0} A^{dk}, M = (M, M') = (\bigoplus_{k \in \mathbb{Z}, d \geq 0} M^{dk}, \bigoplus_{k \in \mathbb{Z}, d \geq 0} M'^{dk})\}.$$

We define a Rickard object of $\mathcal{U}$ (or $\mathcal{T}$) to be an object $(A, M)$ of $\mathcal{U}$ (or $\mathcal{T}$), where the homomorphisms $M \otimes_A M' \to A$ and $M' \otimes_A M \to A$ are quasi-isomorphisms. In this case we will call $M$ invertible and often denote $M'$ by $M^{-1}$.

The operator $\mathbb{P}$. We define a $j$-graded object of $\mathcal{U}$ to be an object $(a, m)$ of $\mathcal{U}$, where $a = \bigoplus a^{dk}$ is a differential trigraded algebra, and $m = (m, m') = (\bigoplus m^{dk}, \bigoplus m'^{dk})$ a bonded pair of differential trigraded $a$-$a$-bimodules. Given a $j$-graded object of $\mathcal{U}$, we have an operator

$$\mathbb{P}_{a, m} \circ \mathcal{U}$$

given by

$$\mathbb{P}_{a, m}(A, M) = (\bigoplus a^{dk} \otimes_F M^{\otimes_A j}, (\bigoplus m^{dk} \otimes_F M^{\otimes_A j}, \bigoplus m'^{dk} \otimes_F M'^{\otimes_A j})), $$

where for $j > 0$, $M^{\otimes_A j} = M^{\otimes_A j}$ and for $j < 0$, $M^{\otimes_A j} = M^{\otimes_A -j}$. The algebra structure on $\bigoplus a^{dk} \otimes_F M^{\otimes_A j}$ is the restriction of the algebra structure on the tensor
product of algebras \( \alpha \otimes \mathbb{T}_A(M) \). The \( k \)-grading and differential on the complex \( \bigoplus_{d_0 + d_1 + \ldots + d_j = d} \alpha^{d_0}_{d_1} \otimes M^{d_1} \otimes \ldots \otimes M^{d_j} \) are defined to be the total \( k \)-grading and total differential on the tensor product of complexes. The \( d \)-grading is defined to be the total \( d \)-grading, with the degree \( d \) part given by
\[
\bigoplus_{d_0 + d_1 + \ldots + d_j = d} \alpha^{d_0}_{d_1} \otimes M^{d_1} \otimes \ldots \otimes M^{d_j}.
\]
The bimodule structure, grading and differential on \( \bigoplus_{m} \alpha_{m} \otimes M \otimes \mathbb{T}_A^j(M) \) are defined likewise. We sometimes write
\[
\mathbb{P}_{\alpha,m}(A,M) = (\alpha(A,M), m(A,M)).
\]
If \( \alpha \) and \( \beta \) are differential trigraded algebras, and \( \alpha \otimes \beta \) is a differential trigraded \((\alpha, \beta)\)-bimodule, then we have a differential bigraded \((\alpha(A,M), \beta(A,M))\)-bimodule
\[
x(A,M) := \bigoplus_{d,j,k} x^{d}_{j,k} \otimes M^{d_j} \otimes A^j.
\]

Let \( (A,M) \) and \( (B,N) \) be objects of \( T \), where \( M = (M,M') \) and \( N = (N,N') \).

**Definition 4.** [9, Definition 3] A dg equivalence between objects \( (A,M) \) and \( (B,N) \) of \( T \) is

(i) a dg \( A-B \)-bimodule \( X \) such that \( A X \) belongs to \( A \)-perf, such that \( A X \) generates \( D_d g(A) \), and the natural map
\[
B \rightarrow \text{End}(X)
\]
is a quasi-isomorphism;

(ii) quasi-isomorphisms
\[
X \otimes_B N \rightarrow M \otimes_A X,
\]
\[
X \otimes_B N' \rightarrow M' \otimes_A X,
\]
such that the resulting diagrams of maps
\[
\begin{array}{c}
X \otimes_B N \oplus N' \otimes_B N' \rightarrow M \otimes_A M' \otimes_A X \rightarrow A \otimes_A X \\
X \otimes_B B \rightarrow X \\
X \otimes_B N' \otimes_B N \rightarrow M' \otimes_A M \otimes_A X \rightarrow A \otimes_A X \\
X \otimes_B B \rightarrow X \\
\end{array}
\]
commute.

If there is a dg equivalence between \( (A,M) \) and \( (B,N) \), we write \( (A,M) \triangleright (B,N) \). We then have a derived equivalence
\[
\begin{array}{c}
\text{Hom}_d(X,-) \\
\Downarrow \quad D_d g(A) \quad \downarrow D_d g(B)
\end{array}
\]
and diagrams which commute up to a natural isomorphism:

\[
\begin{array}{ccc}
D_{dg}(B) & \xrightarrow{X \otimes_B \bigotimes} & D_{dg}(A) \\
\downarrow \downarrow & & \downarrow \downarrow \\
D_{dg}(B) & \xrightarrow{X \otimes_B \bigotimes} & D_{dg}(A).
\end{array}
\]

We define a quasi-isomorphism from \((A, M)\) to \((B, N)\) to be a quasi-isomorphism \(A \rightarrow B\), along with compatible quasi-isomorphisms \(A^\dagger \rightarrow B^\dagger\) such that the diagrams

\[
\begin{array}{ccc}
M \otimes_A M^\dagger & \rightarrow & N \otimes_B N^\dagger \\
\downarrow & & \downarrow \\
A & \rightarrow & B
\end{array}
\]

and

\[
\begin{array}{ccc}
M^\dagger \otimes_A M & \rightarrow & N^\dagger \otimes_B N \\
\downarrow & & \downarrow \\
A & \rightarrow & B
\end{array}
\]

commute, where the horizontal maps are given by above quasi-isomorphisms, and vertical maps by the bondings.

The following lemmas are the bonded analogues of results we have established previously [9] and are proved in exactly the same fashion.

**Lemma 5.** [9, Lemma 4] Let \(a\) be a differential bigraded algebra, \(x_a\) and \(y\) differential bigraded modules, \((A, M)\) an object of \(U\). Then

\[
x(A, M) \otimes \alpha_a(A, M) y(A, M) \cong (x \otimes_a y)(A, M).
\]

**Lemma 6.** [9, Lemma 5] Let \(c\) be a differential trigraded algebra, and \((A, M)\) a Rickard object of \(U\). If \(x\) and \(y\) are differential trigraded \(c\)-modules, then we have a quasi-isomorphism

\[
\text{Hom}_c(x, y)(A, M) \rightarrow \text{Hom}_c(A, M)(x(A, M), y(A, M)).
\]

The operators \(\mathbb{P}\) coincide with the operators \(\mathfrak{O}\) from our previous papers [8, 9], if we restrict them to positively graded objects of \(U\), concentrated in \(d\)-degree zero. As the \(d\)-grading is just an extra grading dragged around, which does not interfere with the constructions, the only place we need to take care in extending results from [9] is the extension to the bonded setting. However, using Lemma 2, we find all proofs go through without problem.

**Lemma 7.** [9, Lemma 11] Let \((A, M)\) and \((B, N)\) be objects of \(U\) such that \((A, M) \Rightarrow (B, N)\). Let \((a, m)\) be a \(j\)-graded Rickard object of \(U\). Then \(\mathbb{P}_{a, m}(A, M) \Rightarrow \mathbb{P}_{a, m}(B, N)\).
Lemma 8. \cite[Lemma 12]{ref} Let $(A, M)$ and $(B, N)$ be quasi-isomorphic objects of $\mathcal{U}$. Let $(a, m)$ be a $j$-graded Rickard object of $\mathcal{U}$. Then $\mathbb{P}_{a,m}(A, M)$ and $\mathbb{P}_{a,m}(B, N)$ are quasi-isomorphic objects of $\mathcal{U}$.

The operator $\mathcal{O}$. We now recall the definition of the operator $\mathcal{O}$ as well as the result which asserts that it behaves favourably with respect to taking homology \cite{ref}.

Let $\Gamma = \bigoplus \Gamma^{ijk}$ be a differential trigraded algebra. We have an operator $\mathcal{O}_{\Gamma}$ given by

\begin{equation}
\mathcal{O}_{\Gamma}(\Sigma)^{ij} = \bigoplus_{j+k_1+k_2=k} \Gamma^{ijk_1} \otimes \Sigma^{j,k_2}.
\end{equation}

The algebra structure and differential are obtained by restricting the algebra structure and differential from $\Gamma \otimes \Sigma$. If we forget the differential and the $k$-grading, the operator $\mathcal{O}_{\Gamma}$ is identical to the operator $\mathcal{O}$ defined in the introduction.

Lemma 9. \cite[Lemma 6]{ref} We have

\[ \mathbb{H}\mathcal{O}_{\Gamma} \cong \mathbb{H}\mathcal{O}_{\mathcal{H}\mathcal{R}} \cong \mathcal{O}_{\mathcal{H}\mathcal{R}^2}, \]

for a differential trigraded algebra $\Gamma$.

Comparing $\mathbb{P}$ and $\mathcal{O}$. We have previously made a comparison of operators $\mathcal{O}$ and $\mathcal{O}$ \cite{ref}. We could add a $d$-grading to $\mathcal{O}$ to obtain a direct generalisation of this comparison result. However, in our application, we use the operator $\mathcal{O}$ only in a setting where the $d$-grading is the negative of the $k$-grading, and so to simplify we disregard it. Let $\mathcal{D}$ denote the 2-functor from $\mathcal{U}$ to $\mathcal{T}$ which disregards the $d$-grading. Then, extending the operator $\mathcal{O}$ to the bonded setting, we have $\mathbb{D}\mathbb{P} = \mathcal{O}$. Extending our comparison result in a bonded setting, using Lemma 2, we obtain the following:

Lemma 10. \cite[Corollary 9]{ref} Let $a$ be a differential bigraded algebra and $\mathfrak{m}$ a bonded pair of $a$-$a$ $dg$ bimodules. Then we have an isomorphism of $dg$ algebras

\[ \mathbb{D}\mathbb{P}_{\mathcal{D}}(\mathfrak{m}, (F, F)) \cong \mathcal{O}_{\mathcal{D} \mathcal{O}_{\mathcal{T}_{\mathcal{H}}}}(F[z, z^{-1}]). \]

Homological duality. Keller equivalence. Let $A$ be a finite dimensional algebra with modules $S_1, \ldots, S_f$ which generate the derived category $D(A)$ of $A$. Let $P_l = \bigoplus k P^k_l$ be a projective resolution of $S_l$. Let $\mathcal{F}(A)$ denote the $dg$ algebra

\[ \mathcal{F}(A) = \bigoplus_{k,k'} \operatorname{Hom}_A(\bigoplus_{l=1}^f P^k_l, \bigoplus_{l=1}^f P^{k'}_l). \]

Then $P = \bigoplus_{l=1}^f P_l$ is a differential graded $A$-$\mathcal{F}(A)$-bimodule. There are mutually inverse equivalences

\[ D_{dg}(A) \xrightarrow{\operatorname{Hom}_A(P,-)} D_{dg}(\mathcal{F}(A)) \]
by a theorem of Keller [3, Theorem 3.10]. Since $P$ is projective as an $A$-module, we have a natural isomorphism of functors
\[ \text{Hom}_A(P, -) \cong \text{Hom}_A(P, A) \otimes_A - . \]

The operator $\mathbb{F}$. Let $(A, M)$ be an object of $\mathcal{U}$, where $A^d = 0$ for $d < 0$. Here we define a homological duality operator $\mathbb{F}$ which sends $(A, M)$ to another object of $\mathcal{U}$.

We define $S_1, \ldots, S_f$ to be the direct summands of the degree zero part of $A$ taken with respect to the $d$-grading. Assume $P$ is $djk$-graded. Given a differential graded $A$-$A$-bimodule $M$, denote by $\mathcal{F}(M)$ the dg $\mathcal{F}(A)$-$\mathcal{F}(A)$-bimodule
\[ \mathcal{F}(M) = \text{Hom}_A(P, A) \otimes_A M \otimes_A P. \]
We have $\mathcal{F}(M) \cong \text{Hom}_A(P, M) \otimes_A P$. For a bonded pair $M = (M, M')$ of bimodules, denote by $\mathcal{F}(M)$ the pair $(\mathcal{F}(M), \mathcal{F}(M'))$. Let $\mathbb{F}(A, M) := (\mathcal{F}(A), \mathcal{F}(M))$, with bonded structure defined in the natural way as follows. We have maps
\[ \mathcal{F}(M) \otimes_{\mathcal{F}(A)} \mathcal{F}(M') = \text{Hom}_A(P, A) \otimes_A M \otimes_A P \otimes_{\mathcal{F}(A)} \text{Hom}_A(P, A) \otimes_A M' \otimes_A P \to \mathcal{F}(A) \]
and
\[ \mathcal{F}(M') \otimes_{\mathcal{F}(A)} \mathcal{F}(M) = \text{Hom}_A(P, A) \otimes_A M' \otimes_A P \otimes_{\mathcal{F}(A)} \text{Hom}_A(P, A) \otimes_A M \otimes_A P \to \mathcal{F}(A) \]
which are given by the composition of the natural map $P \otimes_{\mathcal{F}(A)} \text{Hom}_A(P, A) \to A$ and the maps $M \otimes_A M' \to A$ and $M' \otimes_A M \to A$ respectively, the latter maps being given by the bonding. It follows immediately that these maps define a bonding, as the relevant morphisms
\[ \text{Hom}_A(P, A) \otimes_A M \otimes_A P \otimes_{\mathcal{F}(A)} \text{Hom}_A(P, A) \otimes_A M' \otimes_A P \otimes_{\mathcal{F}(A)} \text{Hom}_A(P, A) \otimes_A M \otimes_A P \]
\[ \to \text{Hom}_A(P, A) \otimes_A M \otimes_A P \]
factor through the morphisms
\[ \text{Hom}_A(P, A) \otimes_A M \otimes_A M' \otimes_A M \otimes_A P \to \text{Hom}_A(P, A) \otimes M \otimes P, \]
where we can apply the bonding of $(M, M')$.

Lemma 11. Suppose $(A, M)$ is a Rickard object of $\mathcal{U}$, where $A^d = 0$ for $d < 0$. The bimodule $P$ induces a dg equivalence $(A, M) \triangleright \mathbb{F}(A, M)$.

Proof. The quasi-isomorphisms
\[ P \otimes_{\mathcal{F}(A)} \mathcal{F}(M) \to M \otimes_A P \]
and
\[ P \otimes_{\mathcal{F}(A)} \mathcal{F}(M) \to M' \otimes_A P \]
follow in the same fashion as in [3, Lemma 14]. It remains to be checked that the diagrams
\[ P \otimes_{\mathcal{F}(A)} \mathcal{F}(M) \otimes_{\mathcal{F}(A)} \mathcal{F}(M') \to M \otimes_A M' \otimes_A P \to A \otimes_A P \]
\[ P \otimes_{\mathcal{F}(A)} \mathcal{F}(A) \to P \]
THE WEYL EXTENSION ALGEBRA OF $GL_2(\mathbb{F}_p)$

$\mathcal{F}(M') \otimes_{\mathcal{F}(A)} \mathcal{F}(M) \rightarrow M' \otimes_A M \otimes_A P \rightarrow A \otimes_A P$

\[ \mathcal{F}(A) \rightarrow P \]

commute, which is straightforward. □

**Remark 12** Suppose that $(a, m) = (a, (m, m^{-1}))$ is a $j$-graded Rickard object of $\mathcal{U}$, such that $a$ is concentrated in non-negative $d$-degrees and that a projective resolution $P$ of the degree zero part of $a$ is differential $djk$-trigraded. Then $\mathcal{F}(a)$ inherits a differential $djk$-trigrading. In this case we call $(a, m)$ a *dagger object* of $\mathcal{U}$, and the dg equivalence in Lemma 11 is differential $djk$-trigraded.

**Homological duality for operators $\mathbb{P}$**. We now consider how homological duality behaves towards the operators $\mathbb{P}$.

**Lemma 13.** Let $(A, M)$ be a Rickard object of $\mathcal{U}$. Let $(a, m)$ be a dagger object of $\mathcal{U}$, with homological dual $\mathbb{F}(a, m)$. Assume further that $P(A, M)$ generates $D_{dg}(a(A, M))$. Then we have a differential $dk$-bigraded equivalence

$$\mathbb{P}_{a,m}(A, M) \succ \mathbb{P}_{\mathbb{F}(a,m)}(A, M).$$

**Proof.** Let $P$ be the differential $djk$-trigraded $a$-module inducing the duality between $a$ and $\mathcal{F}(a)$ as in Subsection 5. Notice that $P(A, M)$ is a differential $dk$-bigraded $a(A, M)$-$\mathcal{F}(a)(A, M)$-bimodule.

Now $\mathcal{F}(a)(A, M)$ is by definition $\text{Hom}_a(P, P)(A, M)$, which by Lemma 6 is quasi-isomorphic to $\text{Hom}_a(A, M)(P(A, M), P(A, M))$.

As $P(A, M)$ generates $D_{dg}(a(A, M))$ by assumption, by Keller’s theory $P(A, M)$ induces a dg-equivalence between $D_{dg}(a(A, M))$ and $D_{dg}(\mathcal{F}(a)(A, M))$ [4, Theorem 3.10].

To check the conditions of Definition 4 on bimodules is an easy repeated application of Lemma 5 to the existing diagrams for the equivalence in Lemma 11. □

**Theorem 14.** Let $(A, M)$ be a Rickard object of $\mathcal{U}$. Let $(a, m)$ be a be a dagger object of $\mathcal{U}$. Assume further that $P(A, M)$ generates $D_{dg}(a(A, M))$. We have a chain of differential $dk$-bigraded equivalences

$$\mathbb{P}(\mathbb{P}_{a,m}(A, M)) \prec \mathbb{P}_{a,m}(A, M) \succ \mathbb{P}_{\mathbb{F}(a,m)}(A, M) \succ \mathbb{P}_{\mathbb{F}(a,m)}(\mathbb{F}(A, M)).$$

**Proof.** This follows from Lemmas 11, 13 and □

**Theorem 15.** Assume the following:

Theorem 13 implies we have a differential $dk$-graded equivalence between the objects $\mathbb{P}(\mathbb{P}_{a,m}(A, M))$ and $\mathbb{P}_{\mathbb{F}(a,m)}(\mathbb{F}(A, M))$ of $\mathcal{U}$. We strengthen this as follows:

**Theorem 15.** Assume the following:
(i) Let \((a, m)\) be a dagger object of \(U\) such that \(a\) is concentrated in non-negative \(j\)-degrees.

(ii) Let \((A, M) = (A, (M, M^{-1}))\) be a Rickard object of \(U\), such that

(a) both \(A\) and \(M\) are concentrated in non-negative \(d\)-degrees;

(b) \(M^\otimes j \otimes_A A^{0\bullet} \cong (M^\otimes j)^{0\bullet}\) for all \(j\) such that \(a^{0\bullet} \neq 0\).

(iii) The differential \(d_k\)-bigraded \(a(A, M)^{0\bullet}\)-module \((a(A, M)^{0\bullet})\) generates the derived category \(D_{dg}(a(A, M))\).

Then the chain of equivalences in Theorem 14 lifts to a quasi-isomorphism from \(P_{F(a, m)}(F(A, M))\) to \(F(P_{a, m}(A, M))\).

**Proof.** We denote by \(P_a\) the projective resolution of the \(d\)-degree zero part of \(a\) and by \(P_A\) the projective resolution of the \(d\)-degree zero part of \(A\) and note that we have an isomorphism

\[
F(M)^{\otimes F(A)^{\tau}} \cong \text{Hom}_A(P_A, M^\otimes A^{0\bullet} \otimes A P_A)
\]

and hence an isomorphism

\[
F(a)(F(A, M)) \cong \text{Hom}_A(P_A, F(a)(A, M) \otimes A P_A).
\]

We have a quasi-isomorphism \(P_a(A, M) \to a^{0\bullet}(A, M)\) by \([8, \text{Lemma 15}]\) and quasi-isomorphisms

\[
M^\otimes j \otimes P_A \to M^\otimes j \otimes_A A^{0\bullet} \cong (M^\otimes j)^{0\bullet}
\]

for all \(j\) such that \(a^{0\bullet} \neq 0\) by \([11b, 11a]\). Thanks to \([11]\) and \([11a]\) we also have

\[
(a(A, M)^{0\bullet}) \cong \bigoplus_{j \geq 0, d \geq 0} a^{dj} \otimes_F (M^\otimes j)^{-d\bullet} \cong \bigoplus_{j \geq 0, d \geq 0} a^{dj} \otimes_F (M^\otimes j)^{0\bullet}
\]

and putting the latter two observations together, we obtain that \(P_a(A, M) \otimes_A P_A\) is a projective resolution of \((a(A, M)^{0\bullet})\).

Therefore

\[
F(a(A, M)) \cong \text{End}_{a(A, M)}(P_a(A, M) \otimes_A P_A).
\]

To prove the theorem we need to show that \(F(a(A, M))\) is quasi-isomorphic to \(F(a)(F(A, M))\). We have
\[ \mathcal{F}(a)(\mathcal{F}(A, M)) = \text{Hom}_A(P_A, \mathcal{F}(a)(A, M) \otimes_A P_A) \]

by (2)

\[ = \text{Hom}_A(P_A, \text{Hom}_a(P_a, (A, M) \otimes_A P_A)) \]

by definition of \( \mathcal{F}(a) \)

\[ \cong \text{Hom}_A(P_A, \text{Hom}_{a(A, M)}(P_a(A, M), P_a(A, M)) \otimes_A P_A) \]

by Lemma 6

\[ \cong \text{Hom}_A(P_A, \text{Hom}_{a(A, M)}(P_a(A, M), P_a(A, M) \otimes_A P_A)) \]

by projectivity of \( P_A \)

\[ \cong \text{Hom}_{a(A, M)}(P_a(A, M) \otimes_A P_A, P_a(A, M) \otimes_A P_A) \]

by adjunction

\[ \cong \mathcal{F}(a(A, M)) \]

by (3)

Similarly we have

\[ \mathcal{F}(m)(\mathcal{F}(A, M)) \]

\[ = \text{Hom}_A(P_A, \mathcal{F}(m)(A, M) \otimes_A P_A) \]

\[ = \text{Hom}_A(P_A, \text{Hom}_a(P_a, m \otimes_a P_a)(A, M) \otimes_A P_A) \]

\[ \cong \text{Hom}_A(P_A, \text{Hom}_{a(A, M)}(P_a(A, M), m(A, M) \otimes_{a(A, M)} P_a(A, M) \otimes_A P_A) \]

\[ \cong \text{Hom}_A(P_A, \text{Hom}_{a(A, M)}(P_a(A, M), m(A, M) \otimes_{a(A, M)} P_a(A, M) \otimes_A P_A) \]

\[ \cong \text{Hom}_{a(A, M)}(P_a(A, M) \otimes_A P_A, m(A, M) \otimes_{a(A, M)} P_a(A, M) \otimes_A P_A) \]

\[ \cong \mathcal{F}(m(A, M)) \]

and the analogous chain for \( m^{-1} \).

We need to check compatibility of these quasi-isomorphisms with the bonding.

The bonding on \( \mathcal{F}(m)(\mathcal{F}(A, M)) \) and \( \mathcal{F}(m^{-1})(\mathcal{F}(A, M)) \) is given via

\[ \mathcal{F}(m)(\mathcal{F}(A, M)) \otimes_{\mathcal{F}(a)(\mathcal{F}(A, M))} \mathcal{F}(m^{-1})(\mathcal{F}(A, M)) \cong (\mathcal{F}(m) \otimes_{\mathcal{F}(a)} \mathcal{F}(m^{-1}))(\mathcal{F}(A, M)) \]

from Lemma 6 and the bonding \( \mathcal{F}(m) \otimes_{\mathcal{F}(a)} \mathcal{F}(m^{-1}) \to \mathcal{F}(a) \) which is induced by the evaluation map \( P_a \otimes_{\mathcal{F}(a)} \text{Hom}_a(P_a, a) \to a \) and the bonding on \( m \otimes_a m^{-1} \).

For the bonding on \( \mathcal{F}(m(A, M)) \) and \( \mathcal{F}(m^{-1}(A, M)) \) note that, since \( (a(A, M))^0 \) generates \( D_{dg}(a(A, M)) \), its projective resolution \( P := P_a(A, M) \otimes_A P_A \) contains a progenerator of \( a(A, M) \) so, the evaluation map

\[ P \otimes_{\mathcal{F}(a(A, M))} \text{Hom}_{a(A, M)}(P_a(A, M)) \to a(A, M) \]

is an isomorphism and, using projectivity of \( P \) and, in the last step, Lemma 6 we obtain
\[ \mathcal{F}(m(A, M)) \otimes \mathcal{F}(a(A, M)) = \text{Hom}_{a(A, M)}(P, m(A, M) \otimes a(A, M) P) \]

\[ \cong \text{Hom}_{a(A, M)}(P, a(A, M) \otimes a(A, M) m(A, M)) \]

\[ \cong \text{Hom}_{a(A, M)}(P, a(A, M) m^{-1}(A, M) \otimes a(A, M) P) \]

\[ \cong \text{Hom}_{a(A, M)}(P, a(A, M)) \otimes a(A, M) m(A, M) \]

\[ \cong \text{Hom}_{a(A, M)}(P, a(A, M)) \otimes a(A, M) m^{-1}(A, M) \otimes a(A, M) P \]

which together with the bonding on \( m \otimes a m^{-1} \) induce the bonding. The analogous statements hold for reversed roles of \( m \) and \( m^{-1} \) and our quasi-isomorphisms are hence compatible with the bonding, completing the proof of the theorem. \( \square \)

\section{Recollections on \( GL_2 \).}

We say a \( \mathbb{Z} \)-grading \( A = \oplus_{d \geq 0} A^d \) on a quasi-hereditary algebra is a \( \Delta \)-grading if \( A^d \) is isomorphic to a direct sum of standard modules, for all \( d \).

Let \( Z \) denote the zigzag algebra, generated by the quiver

\[
\begin{array}{cccccc}
\cdots & \eta & \xi & \eta & \xi & \eta & \cdots, \\
\xi & \xi & \xi & \xi & \xi & \xi \\
\eta & \eta & \eta & \eta & \eta & \eta \\
\end{array}
\]

modulo relations \( \xi^2 = \eta^2 = \xi \eta - \eta \xi = 0 \).

The zigzag algebra \( Z \) is a little infinite dimensional algebra with a number of interesting homological properties. For example, it is quasi-hereditary, symmetric, and Koszul. Its derived category admits an action of the braid 2-category, in which braids act faithfully \([5]\). We are especially interested in the quasi-hereditary structure on \( Z \), in which \( \Delta(l) \) has top \( l \) and socle \( l - 1 \) for \( l \in \mathbb{Z} \).

For our application to \( GL_2 \), we will be interested in a finite truncation of \( Z \). Let \( c \) be the finite dimensional subquotient of \( Z \) generated by

\[
\begin{array}{cccccc}
1 & \eta_1 & \eta_2 & \eta_3 & \cdots & p-1 & \eta_{p-1} \\
\xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{p-1} & \xi_p \\
\end{array}
\]

modulo the ideal \( I = (\xi_{l+1} \xi_l, \eta_l \eta_{l+1}, \xi_l \eta_l + \eta_{l+1} \xi_{l+1}, \eta_{p-1} \xi_{p-1} | 1 \leq l \leq p - 2) \).

We will now recall some facts about the rational representation theory of \( G = GL_2(F) \). The category of polynomial representations of \( G \) of degree \( r \) is equivalent to the category \( S(2, r) \)-mod of representations of the Schur algebra \( S(2, r) \) \([3]\). All blocks of \( S(2, r) \)-mod whose number of isomorphism classes of simple objects is \( p^q \) are equivalent. There is a combinatorial way to describe these blocks, which we now describe.
We denote by $\sigma \nu$ the algebra involution of $Z$ which sends vertex $i$ to vertex $p - i$ and exchanges $\xi$ and $\eta$. Let $e_l$ denote the idempotent of $Z$ corresponding to vertex $l \in \mathbb{Z}$. Let

$$t = \sum_{1 \leq l \leq p, 0 \leq m \leq p - 1} e_l Z e_m.$$ 

Then $t$ admits a natural left action by the subquotient $c$ of $Z$. By symmetry, $t$ admits a right action by $c$, if we twist the regular right action by $\sigma \nu$. In this way, $t$ is naturally a $c$-$c$-bimodule.

The algebra $c$ is a quasi-hereditary algebra, and the left restriction $c t$ of $t$ is a full tilting module for $c$. The natural homomorphism $c \to \text{Hom}(c t, c)$ defined by the right action of $c$ on $t$ is an isomorphism, implying that $c$ is Ringel self-dual.

Let $\tilde{t}$ denote a projective resolution of $t$ as a $c$-$c$ bimodule, then $\tilde{t}$ is a two-sided tilting complex, and $\tilde{t} \otimes_c -$ induces a self-equivalence of the derived category $D^b(c)$ of $c$. We also have the adjoint complex $\tilde{t}^{-1} = \text{Hom}_c(\tilde{t}, c)$. We denote by $\mathbf{t}$ the pair of bonded $c$-$c$ dg bimodules $(\tilde{t}, \tilde{t}^{-1})$.

The operator $P_{\mathbf{c}, \mathbf{t}}$ acts on the collection of algebras with a bonded pair of bimodules, such as the pair $(F, F)$ whose algebra is $F$ and whose bonded bimodules $E$ are just the pair of regular bimodules $(F, F)$. The operator $P_{F,0} \mathbb{P}^d_{\mathbf{c}, \mathbf{t}}$ takes an algebra with a bonded pair of bimodules to an algebra, along with a pair of zero bimodules which we disregard. We define $b_j$ to be the category of modules over the $\Delta$-graded algebra $\mathbb{P}_{F,0} \mathbb{P}^d_{\mathbf{c}, \mathbf{t}}(F, F)$. Both $c$ and $t$ are concentrated in non-negative $j$-degrees, and therefore $\tilde{t}^{-1}$ is only present for formal purposes: it is irrelevant for computation. We have an algebra homomorphism $c \to F$ which sends a path in the quiver to $1 \in F$ if it is the path of length zero based at $1$, and $0 \in F$ otherwise. This algebra homomorphism lifts to a morphism of operators $P_{\mathbf{c}, \mathbf{t}} \to P_{F,0}$.

We denote by $b$ the union of these highest weight categories. In a previous paper, we have proved the following:

**Theorem 16.** [8, Corollary 27] Every block of $G$-mod is $\Delta$-equivalent to $b$. Every block of $S(2, r)$-mod whose number of isomorphism classes of simple objects is $p^q$ is $\Delta$-equivalent to $b_q$.

Note that this was formulated only in terms of the operators $\mathcal{O}$. The fact that the operator $\mathbb{P}$ truly gives us the correct $\Delta$-grading also follows from our previous work [7, Theorem 18].

**7. The homological dual of $(\mathbf{c}, \mathbf{t}, \mathbf{t}^{-1})$.**

Towards describing $w$ combinatorially using our algebraic operators, we give a combinatorial description of $\mathcal{F}(\mathbf{c}, \mathbf{t}) = (\mathcal{F}(c), \mathcal{F}(t))$. 
The algebra \( \Psi \). We need to be careful about our gradings: \( c \) is \( djk\)-graded, with \( \eta \) in degree \((1,1,0)\) and \( \xi \) in degree \((0,1,0)\). In order to obtain \( \mathcal{F}(c) \), we need to take a projective resolution of \( c^{0,\infty} \), which is given by \( P = \bigoplus_{1 \leq l \leq p} P_l \) where \( P_l \) is a linear projective resolution of \( \Delta(l) = c^{0,\infty}e_l \). This is given by

\[
P_l = \bigoplus_{k=l}^p ce_k [k-l] \langle k-l \rangle [-(k-l)]
\]

where \([\cdot]\) denotes a shift in the \( d \)-grading. As required, the differential, which is given by right multiplication by \( \eta \), has \( djk \)-grading \((0,0,1)\), turning \( P \) into a differential \( djk \)-trigraded left \( c \)-module. We therefore have

Lemma 17. The object \((c,t) \in U\) is a dagger object and \( c \) is concentrated in non-negative \( d \)- and \( j \)-degrees.

Let \( \Psi \) denote the algebra generated by the quiver

\[
\begin{array}{ccccccc}
1 & \xrightarrow{x} & 2 & \xrightarrow{x} & 3 & \cdots & x & \xrightarrow{x} & \cdots & p-1 & \xrightarrow{x} & p \\
\xi & & \xi & & \xi & & \xi & & \xi & & \xi & & \xi
\end{array}
\]

modulo relations \( \xi x - x \xi = 0, \xi^2 = 0 \), where \( x \) is given \( djk \)-degree \((-1,-1,1)\) and \( \xi \) is given \( djk \)-degree \((0,1,0)\).

The map \( \Psi \to \text{End}_c(P) \) given by

\[
e_l \mapsto (id_{P_l} : P_l \to P_l)
\]

\[
e_{l-1}xe_l \mapsto (P_{l-1} \to P_l) :
\]

\[
\alpha e_{l-1+k} [k] \langle k \rangle [-(k-l)] \mapsto \begin{cases} 0 & \text{if } k = 0, \\
\alpha e_{l-1+k} [k-1] \langle k-1 \rangle [-(k+1)] & \text{if } k > 0,
\end{cases}
\]

which has \( djk \)-degree \((-1,-1,1)\), and

\[
e_{l-1} \xi e_l \mapsto (P_{l-1} \to P_l : \alpha e_{l-1+k} [k] \langle k \rangle [-(k-l)]) \mapsto \alpha \xi e_{l+k} [k] \langle k \rangle [-(k-l)]
\]

which has \( djk \)-degree \((0,1,0)\) is easily seen to give a right \( \Psi \)-action on \( P \), turning \( P \) into a differential \( djk \)-trigraded \( c-\Psi \) bimodule.

The algebra \( \Psi \) is isomorphic to \( \text{Ext}_{\mathfrak{g}}^*(\Delta, \Delta) \) (cf. [6, Example 5.1.1]), which is quasi-isomorphic to \( \text{End}_c(P) \), implying we have a derived equivalence

\[
D(c-\text{dtrigr}_{djk}) \xrightarrow{\text{Hom}_{c}(P,-)} D(\Psi-\text{dtrigr}_{djk})
\]

where \( D(c-\text{dtrigr}_{djk}) \) denotes the derived category of differential \( djk \)-trigraded modules. Given that the \( k \)-grading is exactly the Ext-grading on \( \text{Ext}_{\mathfrak{g}}^*(\Delta, \Delta) \) and the \( d \)-grading on \( \Psi \) is the negative of the \( k \)-grading, we will from now on ignore the \( d \)-grading on \( \Psi \).
Remark 18 The algebra $\Psi$ is Koszul. The algebra $\Psi^!$ is the quadratic dual of $\Psi$, and is generated by the quiver

![Quiver diagram](image)

modulo relations $\xi^* x^* + x^* \xi^* = 0$, $x^* x^* = 0$. We have an algebra isomorphism $\Psi \cong \Psi^!$ which exchanges $e_s$ with $e_{p-s+1}$, exchanges $\xi^*$ with $\xi$, and exchanges $e_s x^* e_{s-1}$ with $(-1)^s e_{p-s+1} \xi e_{p-s+2}$.

Hypotheses of Theorem 15. We now gather the various facts that we need in order to ensure the hypotheses of Theorem 15 are satisfied in our situation.

The $d$-grading on the tilting module. First note that, as the $d$-grading on $t$ does not feature in our algebraic constructions, we are free to choose it as it suits us. We define $t^{d\cdot}$ to be the natural quotient of $t$ which is the direct sum of costandard modules and $t^{d\cdot}$ as the kernel of this surjection. This is obviously a grading on $t$ with respect to which $t$ is concentrated in degree 0 and 1. As $c$ is also non-negatively $d$-graded, a projective $c$-$c$-bimodule resolution $\tilde{t}$ will also be concentrated in non-negative $d$-degrees. Furthermore, as $t \otimes_c \Delta \cong \nabla$, we automatically obtain a quasi-isomorphism $\tilde{t} \otimes_c c^{d\cdot} \cong t \otimes_c c^{d\cdot} \cong t^{d\cdot}$, which is the base step for Condition (iii) of Theorem 15.

Proposition 19. Set $(A_q, M_q) = P^q_{c, t}(F, F)$. Then $(A, M_q)$ is a Rickard object of $\mathcal{U}$ such that

(i) Both $A_q$ and $M_q$ are concentrated in non-negative $d$-degrees.

(ii) We have an isomorphism $M_q \otimes_{A_q} A_q^{d\cdot} \cong M_q^{d\cdot}$.

(iii) The differential $dk$-bigraded $c(A_q, M_q)$-module $(c(A_q, M_q))^{d\cdot}$ generates the derived category $D_{dg}(c(A_q, M_q))$.

Proof. Claim (i) for $A_q$ follows from the fact that the $d$-grading on $P_{F,0}^q P_{c, t}^q P(F, F)$ is just given by the $\Delta$-grading in the quasi-hereditary structure. The same claim for $M_q$ follows from the fact that $c$ and $t$ are concentrated in positive $d$-degrees (as the base step in an induction) and the fact that $t$ is concentrated in positive $j$-degrees, to ensure in the inductive step that only $c$ and $t$ are used in the iterative construction of $M_q$. 
Claim (i) is checked by induction on \( q \). The base step is explained in Subsection 7 and for the inductive step we compute that
\[
M_q \otimes_{A_q} \mathcal{A}_q^0 \cong t(A_q-1, M_{q-1}) \otimes_{c(A_q-1, M_{q-1})} (c(A_q-1, M_{q-1}))^{0*}
\]
\[
\cong t(A_q-1, M_{q-1}) \otimes_{c(A_q-1, M_{q-1})} \left( \bigoplus_{j \geq 0, d \geq 0} c^{0j}* \otimes F (M_q^{0j})^{-d*} \right)
\]
as \( c \) is concentrated in non-negative \( d- \) and \( j- \)degrees
\[
\cong t(A_q-1, M_{q-1}) \otimes_{c(A_q-1, M_{q-1})} \left( \bigoplus_{j \geq 0} c^{0j}* \otimes F (M_q^{0j})^{0*} \right)
\]
as \( M_{q-1} \) is concentrated in non-negative \( d- \)degrees
\[
\cong t(A_q-1, M_{q-1}) \otimes_{c(A_q-1, M_{q-1})} \left( \bigoplus_{j=0,1} c^{0j}* \otimes F (M_{q-1}^{0j})^{0*} \right)
\]
by the induction hypothesis
\[
\cong (t \otimes_{c} c^{00*})(A_q-1, M_{q-1}) \otimes_{A_{q-1}} \mathcal{A}_{q-1}^{0*}
\]
by Lemma 5
\[
\cong \bigoplus_{j=0,1} t^{0j*} \otimes F M_{q-1}^{0j} \otimes \mathcal{A}_{q-1}^{0*}
\]
as \( t^{00*} \) is concentrated in \( j- \)degrees 0,1
\[
\cong \bigoplus_{j=0,1} t^{0j*} \otimes F (M_{q-1}^{0j})^{0*}
\]
by the induction hypothesis
\[
\cong t(A_q-1, M_{q-1})^{0*}
\]
as both \( t \) are concentrated in non-negative \( d- \)degrees
\[
\cong M_q^{0*}.
\]

For Claim (iii) notice that by Theorem 13 \((c(A_q-1, M_{q-1}))^{0*}\) is isomorphic to the sum over all the standard modules in the block \( b_q \) which is well known to be a generator of the derived category. \( \square \)

Proposition 20. Setting \((a, m) = (c, t) \) and \((A, M) = \mathbb{P}_e^q(F, F)\), the assumptions of Theorem 17 are satisfied.

Proof. This is a summary of Lemmas 7 for Condition (i), and Proposition 19 for Conditions (ii) and (iii). \( \square \)

Before we can trace \( t \) through this duality, we need to consider a special case of homological duality. For a general account of Koszul duality, please see Appendix 11.
Homologically self-dual algebras. Throughout the rest of this section, we work in a monoidal 2-category $\mathcal{Dgalg}$ whose objects are differential $k$-graded algebras, whose arrows $A \to B$ are objects in the derived category of dg $A$-$B$-bimodules with product given by derived tensor, whose 2-arrows are morphisms in the derived category of bimodules, and whose monoidal structure is given by the super tensor product $\otimes$ over $F$.

The tensor product of an algebra and its Koszul dual. Suppose $A$ is a Koszul algebra with Koszul dual $A^!$. Let $K_A = A \otimes_{A^0} A^{*}$ denote the left Koszul complex for $A$ and $C_A = A \otimes_{A^0} A^!$ the adjoint of the Koszul complex for $A^!$. Let $A^! K = A^* \otimes_{A^0} A^!$ denote the associated right Koszul complex. Given two $k$-graded algebras $A$ and $B$ we denote $\rho$ the algebra isomorphism $A \otimes B \to B \otimes A$ that sends $a \otimes b$ to $(-1)^{|b||a|} b \otimes a$.

The tensor product $A \otimes A^! = A \otimes_F A^!$ is a Koszul self-dual algebra. The following lemma establishes that Koszul self-duality for $A \otimes A^!$ is homologically dual to $\rho \otimes A^! \otimes A^!$ when $A^!$ is a symmetric algebra.

Lemma 21. Suppose $A^!$ is a symmetric algebra, thus $A \cong A^![s](m)$ for some $s$. Then we have a commutative diagram

\[
\begin{array}{c}
A \otimes A^! \xrightarrow{C_A \otimes A^!} A^! \otimes A^! \\
\downarrow (K_A \otimes K_A)^p \quad \quad \quad \quad \downarrow \rho[s](m) \\
A \otimes A^! \xrightarrow{C_A \otimes A^!} A^! \otimes A^!
\end{array}
\]

in $\mathcal{Dgalg}$.

Proof. We have a diagram

\[
\begin{array}{c}
A \otimes A^! \xrightarrow{C_A \otimes A^!} A^! \otimes A^! \\
\downarrow K_{A} \otimes K_{A^!} \quad \quad \quad \quad \downarrow (A^! K \otimes A^! K_A) \otimes A^!
\end{array}
\]

\[
\begin{array}{c}
A^! \otimes A \xrightarrow{A^! \otimes C_A} A^! \otimes A^! \\
\downarrow \rho \quad \quad \quad \quad \downarrow \rho
\end{array}
\]

\[
\begin{array}{c}
A \otimes A^! \xrightarrow{C_A \otimes A^!} A^! \otimes A^! \\
\downarrow \rho
\end{array}
\]

which commutes because $C_{A A^!} K$ is isomorphic to $A$ and $K_A C_A$ is isomorphic to $A^!$ (in the relevant derived categories of bimodules). We have $A^! K \otimes A K_A \cong A^{*+} \otimes_{A^0} A \otimes_{A^0} A^{*+}$. This is isomorphic to $A^{*+} \otimes_{A^0} C_A^* \otimes_A K_A$ which is isomorphic to $A^{*+}$ since $C_A^*$ and $K_A$ are adjoint equivalences. We have $A^{*+} \cong A[s](m)$ since $A$

is a symmetric algebra. This implies our diagram is equivalent to

\[
A \otimes A' \xrightarrow{C_A \otimes A'} A' \otimes A
\]

\[
A' \otimes A \xrightarrow{\rho \otimes C_A} A' \otimes A
\]

\[
A \otimes A' \xrightarrow{C_A \otimes A'} A' \otimes A
\]

\[\Phi \]

The homological dual of \( t \). The algebra \( \Phi \) is Koszul self-dual, and a homological dual of \( c \). The differential \( jk \)-graded bimodule \( \Phi^\sigma \otimes \Phi^0 \Phi^* \) with differential given by internal multiplication by \( x \otimes \xi + \xi \otimes x \), ie

\[
d(a \otimes b) = (-1)^{|a|} (ax \otimes \xi b + a \xi \otimes xb),
\]

is a twisted version of the Koszul complex for \( \Phi \). Here we show \( \Phi^\sigma \otimes \Phi^0 \Phi^*(1) \) is homologically dual to the \( c \)-\( c \)-bimodule \( t \).

**Lemma 22.** We have a commutative diagram

\[
\begin{array}{c}
\Psi \\
\downarrow \rho \\
\downarrow \rho
\end{array}
\begin{array}{c}
\Phi^\sigma \otimes \Phi^0 \Phi^* \\
\downarrow t(-1) \\
\downarrow t(-1)
\end{array}
\begin{array}{c}
\Phi \\
\downarrow \rho \\
\downarrow \rho
\end{array}
\]

in \( \mathcal{D}_{\text{galg}} \).

**Proof.** To begin with consider the Koszul algebra \( A = S(x), \) with \( x \) in \( jk \)-degree \((-1,1)\). Its Koszul dual is \( A^! = \bigwedge(\eta), \) with \( \eta \) in \( jk \) degree \((1,0)\). We also consider the Koszul algebra \( \bigwedge(\xi) \) with Koszul dual \( S(y), \) where again \( y \) lives in \( jk \)-degree \((-1,1)\) and \( \xi \) in \( jk \) degree \((1,0)\). We have \( \bigwedge(\eta)^* \cong \bigwedge(\eta)[0](-1), \) and thus have a commutative diagram

\[
\begin{array}{c}
S(x) \otimes \bigwedge(\xi) \\
\downarrow (K_x \otimes K_\xi)^* \\
S(y) \otimes \bigwedge(\eta)
\end{array}
\begin{array}{c}
\bigwedge(\eta) \otimes \bigwedge(\xi) \\
\downarrow \rho(-1) \\
\bigwedge(\xi) \otimes \bigwedge(\eta)
\end{array}
\]

where \( K \) and \( C \) denote suitable Koszul complexes and their adjoints.

We record that the differential on \( K_x \otimes K_\xi \) is given by internal multiplication by \( x \otimes \xi + \xi \otimes x \), and thus maps a basis element of the form \( x^n \otimes b \otimes c \otimes d \) for \( b, c, d \) in \( \bigwedge(\eta)^*, \bigwedge(\xi), S(y)^* \) respectively, to \((-1)^n(x^{n+1} \otimes \eta b \otimes c \otimes d + x^n \otimes b \otimes c \xi \otimes yd). \) Here the factor \((-1)^n\) is the sign obtained from the \( k \)-degree of \( x^n \).

We have a natural isomorphism

\[
K_x \otimes K_\xi = S(x) \otimes \bigwedge(\eta)^* \otimes \bigwedge(\xi) \otimes S(y)^* \\
\cong S(x) \otimes \bigwedge(\xi) \otimes \bigwedge(\eta)^* \otimes S(y)^*;
\]
denoting $S(x) \otimes \Lambda(\xi)$ by $B$ we obtain the $B\tilde{B}$-bimodule $B \otimes \tilde{B}^*$ with differential acting on the left, sending an element $a \otimes b$ to $(-1)^{|a|} (ax \otimes \eta b + a\xi \otimes yb$, where again the degree of $a$ is the power of $x$ appearing. (note that the twist with $\rho$ gets swallowed into $\tilde{B}^* = (S(y) \otimes \Lambda(\eta))^* \cong \Lambda(\eta)^* \otimes S(y)^*$). We fix the canonical isomorphism $\vartheta : B \to \tilde{B}$ which takes $x$ to $y$ and $\xi$ to $\eta$. Our commutative diagram now takes the form

$$\begin{array}{ccc}
B & \xrightarrow{C_x \otimes \Lambda(\xi)} & \Lambda(\eta) \otimes \Lambda(\xi) \\
(B \otimes \tilde{B}^*)^\vartheta & \downarrow & (\Lambda(\eta) \otimes \Lambda(\xi))^\vartheta (-1) \\
B & \xrightarrow{(C_x \otimes \Lambda(\eta))^\vartheta (-1)} & \Lambda(\xi) \otimes \Lambda(\eta).
\end{array}$$

This is clearly equivalent to

$$\begin{array}{ccc}
B & \xrightarrow{C_x \otimes \Lambda(\xi)} & \Lambda(\eta) \otimes \Lambda(\xi) \\
(B \otimes \tilde{B}^*)^\vartheta & \downarrow & (\Lambda(\eta) \otimes \Lambda(\xi))^\vartheta (-1) \\
B & \xrightarrow{(C_x \otimes \Lambda(\eta))^\vartheta (-1)} & \Lambda(\xi) \otimes \Lambda(\eta).
\end{array}$$

where $\vartheta^{-1}$ is the isomorphism $\Lambda(\xi) \otimes \Lambda(\eta) \to \Lambda(\eta) \otimes \Lambda(\xi)$ which takes $\xi \otimes 1$ to $\eta \otimes 1$ and $1 \otimes \eta$ to $1 \otimes \xi$. This is again equivalent to

$$\begin{array}{ccc}
B & \xrightarrow{C_x \otimes \Lambda(\xi)} & \Lambda(\eta) \otimes \Lambda(\xi) \\
(B \otimes \tilde{B}^*)^\vartheta & \downarrow & (\Lambda(\eta) \otimes \Lambda(\xi))^\vartheta (-1) \\
B & \xrightarrow{(C_x \otimes \Lambda(\eta))^\vartheta (-1)} & \Lambda(\xi) \otimes \Lambda(\eta).
\end{array}$$

To prove the lemma we string this commutative diagram along a line. In order to do so, we place another grading (called the $f$-grading) on these algebras as follows: $\xi$ and $x$ have degree $-1$, $\eta$ and $y$ have degree $1$. If we denote a shift by 1 in the $f$-grading by $\langle 1 \rangle$ then we have $\Lambda(\eta)^* \cong \Lambda(\eta) \langle -1 \rangle$. The category of $f$-graded modules for $\Lambda(\eta) \otimes \Lambda(\xi)$ is isomorphic to a category of modules over the zigzag algebra $Z$, which is an infinite dimensional quasi-hereditary algebra as described in Section 8. The category of $f$-graded modules for $\tilde{B} = S(x) \otimes \Lambda(\xi)$ is isomorphic to a category of modules over the quasi-hereditary algebra $\Psi_\infty$ with quiver

$$\begin{array}{cccc}
\ldots & 1 & \cdots & \cdots \\
\xi & 2 & \xi & 3 \\
\xi & 4 & \xi & \ldots
\end{array}$$

and relations $\xi x - x \xi = 0, \xi^2 = 0$.

Similarly, the category of graded modules for $\tilde{B} = S(y) \otimes \Lambda(\eta)$ is isomorphic to a category of modules over the quasi-hereditary algebra $\Psi_\infty$ with quiver

$$\begin{array}{cccc}
\ldots & 1 & \cdots & \cdots \\
\eta & 2 & \eta & 3 \\
\eta & 4 & \eta & \ldots
\end{array}$$

and relations $\eta y - y \eta = 0, \eta^2 = 0$. 
We now wish to lift the above diagram to this graded setting. In order to do this, we must specify how \( θ : B \to B \) and \( θ' : \Lambda(η) \otimes \Lambda(ξ) \to \Lambda(ξ) \otimes \Lambda(η) \) lift to the graded setting, by specifying what they do on \( Ψ^0_∞ = \tilde{Ψ}^0_∞ = Z^0 \). We do this by specifying that they send the idempotent \( e_1 \) to \( e_{p+1-l} \) (and denote this endomorphism of \( Ψ^0_∞ = \tilde{Ψ}^0_∞ = Z^0 \) by \( σ_Ψ \)) and denote the resulting algebra isomorphisms by \( σ_Ψ \) and \( σ_Z \) respectively. More precisely \( σ_Ψ : Ψ^∞_∞ \to Ψ^∞_∞ \) sends \( x \) to \( y \) and \( ξ \) to \( η \) and \( σ_Z \) interchanges \( ξ \) and \( η \).

We obtain a diagram

\[
\begin{array}{ccc}
Ψ^∞_∞ & \xrightarrow{Y} & Z \\
\downarrow (Ψ^0_∞ \otimes Ψ^0_∞ \tilde{Ψ}^∞_∞)^{σ_Ψ} & & \downarrow \sigma_Z ν(-1) \\
Ψ^∞_∞ & \xrightarrow{Y} & Z
\end{array}
\]

where \( ν \) denotes the algebra automorphism of \( Z \) sending \( e_1 \) to \( e_{l-1} \), \( η \) to \( η \) and \( ξ \) to \( ξ \) (coming from the fact that \( \Lambda(η)^* \cong \Lambda(η) \langle -1 \rangle \)), and where \( Y \) denotes a \( Ψ^∞_∞ \)-\( Z \)-bimodule whose adjoint has homology \( \oplus_{l∈Z} F_{e_l} \).

The differential on \( Ψ^∞_∞ \otimes Ψ^∞_∞ \tilde{Ψ}^∞_∞ \) is now again given by \( a \otimes b \mapsto (-1)^{|a|}(ax \otimes ηb + aξ \otimes yb) \), where the degree of \( a \) is given by the power of \( x \) appearing.

We now carefully analyse the bimodule \( (Ψ^∞_∞ \otimes Ψ^∞_∞ \tilde{Ψ}^∞_∞)^{σ_Ψ} \). Note that as a \( Ψ^0_∞ \)-\( Ψ \)-bimodule \( (Ψ^∞_∞)^{σ_Ψ} \cong σ(Ψ^∞_∞) \) so \( (Ψ^∞_∞ \otimes Ψ^∞_∞ \tilde{Ψ}^∞_∞)^{σ_Ψ} \cong Ψ^∞_∞ \otimes Ψ^∞_∞ \tilde{Ψ}^∞_∞ \) and our diagram becomes

\[
\begin{array}{ccc}
Ψ^∞_∞ & \xrightarrow{Y} & Z \\
\downarrow (Ψ^∞_∞ \otimes Ψ^∞_∞ \tilde{Ψ}^∞_∞)^{σ_Ψ} & & \downarrow \sigma_Z ν(-1) \\
Ψ^∞_∞ & \xrightarrow{Y} & Z
\end{array}
\]

with the differential on \( Ψ^∞_∞ \otimes Ψ^∞_∞ \tilde{Ψ}^∞_∞ \) given by \( a \otimes b \mapsto (-1)^{|a|}(ax \otimes ξb + aξ \otimes xb) \), where the degree of \( a \) is given by the power of \( x \) appearing.

To truncate this to our finite-dimensional setting, it is more convenient to work with \( Ψ^∞_∞ \otimes Ψ^∞_∞ \tilde{Ψ}^∞_∞ \) than \( Ψ^∞_∞ \otimes Ψ^∞_∞ \tilde{Ψ}^∞_∞ \). Note that the former is right adjoint to \( Ψ^∞_∞ \otimes Ψ^∞_∞ \tilde{Ψ}^∞_∞ \), whereas the latter it left adjoint to the same bimodule. Since we know we are dealing with equivalences, we can choose which dg bimodule we work with. The compatibility of adjunctions with differentials articulated in the appendix on Koszul duality concerning adjunction implies that on all these bimodules the differentials are given by the formula

\[
d(a \otimes b) = (-1)^{|a|}(ax \otimes ξb + aξ \otimes xb).
\]

Under the adjoint equivalences between derived categories of \( Z \) and \( Ψ^∞_∞ \) determined by \( Y \) the subcategory of modules generated by vertices \( l \) with \( l ≥ r \) corresponds to the subcategory of modules given by vertices with \( l ≥ r \). Indeed, since under the functors between \( S(x) \) and \( Λ(η) \) determined by \( C_* \) simple \( Λ(η) \)-modules correspond to injective \( S(x) \)-modules, it follows that under \( Y \) the simple \( Z \)-module indexed by
l corresponds to the $\Psi_\infty$-module with socle $l$ and composition factors $l, l+1, l+2,\ldots$ in ascending radical degrees.

It is a general feature of the theory of quasi-hereditary algebras that cutting at an idempotent corresponding to a set of vertices which forms an ideal in the partial order corresponds to taking a quotient of derived categories in which the simple objects corresponding to the complementary set of vertices are sent to zero. We can consequently cut on both sides at the idempotent $e_{\leq p}$ given by the vertices $\leq p$ and obtain a commutative diagram

$$
\begin{align*}
\Psi_{\leq p}^* \otimes \Psi_0^* & \rightarrow \Psi_{\leq p}^* \otimes \Psi_0^* e_{\leq p} (-1) \\
\Psi_{\leq p}^* \otimes \Psi_0^* e_{\leq p} & \rightarrow \Psi_{\leq p}^* \otimes \Psi_0^* e_{\leq p}(-1) \\
\Psi_{\leq p}^* \otimes \Psi_0^* & \rightarrow \Psi_{\leq p}^* \otimes \Psi_0^* e_{\leq p}(-1) \\
\Psi_{\leq p}^* \otimes \Psi_0^* & \rightarrow \Psi_{\leq p}^* \otimes \Psi_0^* e_{\leq p}(-1)
\end{align*}
$$

Computing $e_{\leq p} \Psi_\infty^* \otimes \Psi_0^* e_{\leq p}$ we see that $e_a \Psi_\infty^* e_k \otimes \Psi_0^* e_k \Psi_\infty^* e_b \neq 0$, for $a, b \leq p$ forces $k \leq p$ and also $p + 1 - k \leq p$, and therefore $k \geq 1$. But $k \geq 1$ implies $a \geq 1$ and $k \geq p$ implies $b \geq 1$, so we have $1 \leq a, b, k \leq p$ and $e_{\leq p} \Psi_\infty^* \otimes \Psi_0^* \sigma \Psi_\infty^* e_{\leq p} \cong \Psi^* \otimes \Psi_0^* \sigma \Psi$. Therefore our diagram becomes

$$
\begin{align*}
\Psi & \rightarrow \Psi_{\leq p} e_{\leq p} \\
\Psi^* \otimes \Psi_0^* & \rightarrow \Psi_{\leq p} e_{\leq p}(-1) \\
\Psi_{\leq p} & \rightarrow \Psi_{\leq p} e_{\leq p}
\end{align*}
$$

Twisting by the anti-automorphisms $\sigma$ of $Z$ and $\Psi_\infty$ which exchange $e_l$ and $e_{p+l-1}$ and fix $\xi, \eta$ and $x$, gives us a commutative diagram

$$
\begin{align*}
\Psi & \rightarrow e_{\geq 1} Z e_{\geq 1} \\
\Psi^* \otimes \Psi_0^* & \rightarrow e_{\geq 1} (Z^\sigma e_{\geq 1}(-1)) \\
\Psi_{\geq 1} & \rightarrow e_{\geq 1} Z e_{\geq 1}
\end{align*}
$$

since $P$ is the opposite of $Y$. Note that upon taking opposites, left differentials on dg bimodules become right differentials, and vice versa.

Computing $e_{\geq 1} (Z^\sigma e_{\geq 1}) e_{\geq 1} = (e_{\geq 1} Z e_{\leq p-1})^\sigma$, which from Section 6 we know to be isomorphic to $t$ as bimodule over $e_{\geq 1} Z e_{\geq 1} / \text{Ann}(e_{\geq 1} (Z^\sigma e_{\geq 1})) \cong c$, so our commutative diagram finally yields

$$
\begin{align*}
\Psi & \rightarrow c \\
\Psi^* \otimes \Psi_0^* & \rightarrow t(-1) \\
\Psi & \rightarrow c
\end{align*}
$$
As above in the infinite case, \( \Psi^\sigma \otimes_{\Psi^0} \Psi^* \) induces a derived self-equivalence of \( \Psi \) and is left adjoint to the equivalence given by \( \Psi \otimes \Psi^* \Psi \), whose right adjoint is \( \Psi^* \otimes_{\Psi^0} \Psi^* \), so it is left to us which adjoint we use in our computations.

\[ \square \]

**Remark 23** As Remark 18 provides us with a \( \Psi \cdot \Psi^0 \)-bimodule isomorphism \( \Psi^! \simeq \Psi^\sigma \), we have a \( \Psi \cdot \Psi \)-bimodule isomorphism between the homological dual of \( \tilde{t} \) given by \( \Psi \otimes_{\Psi^0} \sigma(\Psi^*) \) and the Koszul complex \( \Psi^! \otimes_{\Psi^0} \Psi^* \). Note that however the differentials are *not* the same: In the Koszul complex (after applying our bimodule iso \( \Psi^! \simeq \Psi^\sigma \)), the differential is given by

\[
ae_{p+1-t} \otimes e_l b \mapsto a x e_{p+2-t} \otimes e_{l-1} \xi b + (-1)^t a \xi e_{p+2-t} \otimes e_{l-1} x b
\]

whereas our complex has a differential

\[
ae_{p+1-t} \otimes e_l b \mapsto (-1)^{|a|} (ae_{p+2-t} \otimes e_{l-1} \xi b + a \xi e_{p+2-t} \otimes e_{l-1} x b)
\]

where again the \( k \)-degree of \( a \) is determined by the power of \( x \) appearing rather than by idempotents as in the Koszul complex.

It follows from Lemma 22 that \( F(\tilde{t}) \) is quasi-isomorphic to a twisted version of the Koszul complex for \( \Psi \), shifted in \( j \)-degree by 1, namely to

\[
\mathfrak{K} := \Psi^\sigma \otimes_{\Psi^0} \Psi^* (1)
\]

with differential given by internal multiplication by \( x \otimes \xi + \xi \otimes x \).

**The triple.** Adjunction gives us a quasi-isomorphism

\[
\Psi^\sigma \otimes_{\Psi^0} \Psi^* \otimes_{\Psi^0} \Psi \rightarrow \Psi.
\]

It follows immediately that \( F(\tilde{t}^{-1}) \) is quasi-isomorphic to

\[
\mathfrak{K}^{-1} := \Psi^\sigma \otimes_{\Psi^0} \Psi (-1)
\]

with differential given by internal multiplication by \( x \otimes \xi + \xi \otimes x \). The \( k \)-grading on \( \mathfrak{K}^{-1} \) is identified with the \( k \)-grading on \( \Psi^\sigma \otimes_{\Psi^0} \Psi (-1) \) inherited from the \( k \)-grading on \( \Psi \), which is *different* from the usual homological \( h \)-grading on the Koszul complex; nevertheless the differential has \( k \)-degree 1.

Putting these observations together and setting \( \mathfrak{K} = (\mathfrak{K}, \mathfrak{K}^{-1}) \), we have

**Proposition 24.** The triple \( (\Psi, \mathfrak{K}) \) is quasi-isomorphic to \( F(c, \mathfrak{t}) \).

8. **Expressing \( w \) via \( (\Psi, \mathfrak{K}, \mathfrak{K}^{-1}) \).**

We demonstrate here how the analysis of homological duality of algebraic operators made above can be used to reduce the computation of the Weyl extension algebra \( w \) of the principal block of \( GL_2 \) to the computation of the homology of a certain tensor algebra.

**Proposition 25.** We have \( w_q \cong \mathfrak{D}_F \Omega^q_{\mathfrak{H}^c}(\mathfrak{K})(F[z]). \)
Proof. We have algebra isomorphisms

\[
\begin{align*}
\mathbb{w}_q & \cong \mathbb{H}_{\mathbb{F}_q,0}^{\mathbb{F}_q^q,0}(F,(F,F)) & \text{Theorem 10} \\
& \cong \mathbb{H}_{\mathbb{F}_q,0}^{\mathbb{F}_q^q,0}(F,(F,F)) & \text{Theorem 15 Proposition 20} \\
& \cong \mathbb{H}_{\mathbb{F}_q,0}^{\mathbb{F}_q^q,0}(F,(F,F)) & F(F,(F,F)) = (F,(F,F)) \\
& \cong \mathbb{H}_{\mathbb{F}_q,0}^{\mathbb{F}_q^q,0}(F,(F,F)) & \text{Lemma 10} \\
& \cong \mathbb{H}_{\mathbb{F}_q,0}^{\mathbb{F}_q^q,0}(F,(F,F)) \cap \mathbb{H}(F[z]) = F[z] & \text{Lemma 9} \\
& \cong \mathbb{H}_{\mathbb{F}_q,0}^{\mathbb{F}_q^q,0}(F,(F,F)(F[z],z^{-1})) & \mathbb{H}(F[z]) = F[z] \\
& \cong \mathbb{H}_{\mathbb{F}_q,0}^{\mathbb{F}_q^q,0}(F,(F,F)(F[z],z^{-1})) & \text{Proposition 24}
\end{align*}
\]

\[\square\]

In the following section we study the algebra \(\mathbb{H}_\Psi(\mathfrak{X})\), and show that it is isomorphic to the algebra \(\Psi\) defined in the introduction.

9. The algebra \(\mathbb{H}_\Psi(\mathfrak{X},\mathfrak{X}^{-1})\).

To obtain a combinatorial description of \(\mathbb{H}_\Psi(\mathfrak{X})\) we first analyse its \(\Psi\)-\(\Psi\)-bimodule components, then how to multiply these components, before reducing the structure to a basis given by points in a polytope.

A bimodule. Here we define and study a certain bimodule \(M\) for the algebra \(\Psi\) which is prominent in the homology of \(\mathfrak{X}^t\).

We define \(L_t \in \Psi\)-mod to be the module with basis \(\{x.x^i, \xi.x^i|0 \leq i \leq p-1\}\) where \(e_i\) acts as the identity on \(x.x^{p-i}\) and \(\xi.x^{p-i}\), the generator \(x\) acts on the second component in the obvious way, and \(\xi\) sends \(x.x^i\) to \(x.x^{i+1}\), whilst killing \(\xi.x^{p-i}\). Thus \(L_t\) is a left \(\Psi\)-module of dimension \(2p\), whose Loewy structure is

\[
\begin{array}{c}
p \\
p-1 & p \\
p-2 & p-1 \\
\vdots & \vdots \\
2 & 1 \\
1 \\
1
\end{array}
\]

We define \(L_r \in \text{mod-}\Psi\) to be the module with basis \(\{x^i.x, x^i.\xi|0 \leq i \leq p-1\}\) where \(e_i\) acts as the identity on \(x^{i-1}.x\) and \(x^{i-1}.\xi\), the generator \(x\) acts on the first component in the obvious way and \(\xi\) sends \(x^i.x\) to \(x^{i+1}.\xi\). Thus \(L_r\) is a right \(\Psi\)-module of dimension \(2p\), whose Loewy structure is
The algebra $\Psi$ has a $k$-grading, with the symbol $x$ in degree 1 and the symbol $\xi$ in degree 0. This gives $\Psi$ the structure of a dg algebra with trivial differential. The dg bimodules $\Psi$ and $\Psi^*$ admit natural $k$-gradings, with $x$ in degree 1 and $\xi$ in degree 0.

Both $L_l$ and $L_r$ carry a natural $j$-grading, by placing the symbols $x$ and $\xi$ appearing in the basis elements in degrees $-1$ and 1 respectively. We give $L_l$ and $L_r$ a $k$-grading by placing $\xi.x^i$, $x^i\xi$ in degree $l-1$ and $x.x^l$, $x^l.x$ in degree $l$.

We denote by $K'_l = \Psi^* \mathcal{P} \Psi^*$ the twisted left Koszul complex, with differential given by $d(a \otimes b) = (-1)^{|a|} (ax \otimes \xi b + a\xi \otimes xb)$. We denote by $K'_r = \Psi^* \otimes \Psi$ the twisted right Koszul complex, with differential given by $d(a \otimes b) = (-1)^{|a|} (ax \otimes \xi b + a\xi \otimes xb)$.

**Lemma 26.** (i) We have $L_l^* \cong L_r(p-1)[2-p]$ and $L_r^* \cong L_l(p-1)[2-p]$.

(ii) As ungraded modules, every nonsplit extension of the injective $\Psi^* e_h$ by the projective $\Psi e_h$ is isomorphic to $L_l$; every nonsplit extension of the injective $e_h \Psi^*$ by the projective $e_h \Psi$ is isomorphic to $L_r$.

(iii) $K'_l \otimes_{\Psi} L_l$ is quasi-isomorphic to $L_r(p-1)[1-p]$ and $L_r \otimes_{\Psi} K'_r$ is quasi-isomorphic to $L_r(p-1)[1-p]$.

(iv) As ungraded modules, we have $\text{Ext}_L^d(\Psi, \Psi) = \text{Ext}_\Psi^d(\Psi^*, \Psi) = 0$, for $0 \leq d \leq p-2$, and $\bullet \in \{l, r\}$.

(v) There is a unique nonsplit extension of $L_\bullet$ by an irreducible module for $\bullet \in \{l, r\}$, forming the middle terms of short exact sequences

\[ 0 \to \Psi^0 e_{p-1} \to E_l \to L_l \to 0, \]
\[ 0 \to e_2 \Psi^0 \to E_r \to L_r \to 0. \]

**Proof.** Both (i) and (ii) are proved by easy explicit calculations. The proof of (iii) is a little more subtle as we need to drag $j$ and $k$-gradings through Koszul duality. The algebra $\Psi$ is Koszul self-dual. We consider the image in $D^b(\Psi)$ of modules under the endofunctor $K' \otimes_{\Psi} -$ where $K' = K'_l$ is the twisted Koszul complex for $\Psi$. Here the $k$-grading on the dg $\Psi$-$\Psi$ bimodule $K'$ is inherited from the $k$-grading on $\Psi$, and not the homological $h$-grading on the Koszul complex. Classical Koszul duality sends projectives to simples and simples to injectives. Likewise, here we know that $K' \otimes_{\Psi} \Psi^0 e_1$ is quasi-isomorphic to $\Psi^* e_{p+1-s}$ and $K' \otimes_{\Psi} \Psi e_s$ is quasi-isomorphic to $\Psi^0 e_{p+1-s}$. Applying $K' \otimes_{\Psi} -$ to the exact triangle $\Psi e_p \to L_l \to \Psi^0 e_p \to$ gives us an exact triangle $\Psi^0 e_1 \to K' \otimes_{\Psi} L_l \to \Psi^* e_1 \to$. The unique extension of $\Psi^* e_1$ by
\(\Psi^0 e_1\) is \(L_t\) and thus \(K' \otimes_{\Psi} L_t\) is quasi-isomorphic to \(L_t\). To see the gradings, note that a projective resolution of \(L_t\) is

\[ P_i(-(p-2))[-1] \oplus P_i(-(p-1))[0] \to \cdots \]

\[ \cdots \to P_{p-1}(0)[-1] \oplus P_{p-1}(-2)[0] \to P_p(1)[-1] \oplus P_p(-1)[0]. \]

This has a filtration with sections \(P_{p+k}(k+1)[k-1] \oplus P_{p+k}(k-1)[k]\), for \(k \geq 0, -1, ..., 1 - p\). Tensoring with \(K'\), which is quasi-isomorphic to \(\Psi^{0r}\), we obtain \(L_t(p-1)[1-p]\) as required. The proof for \(L_r\) is similar.

Claim (iv) follows from our projective resolution of \(L_t\) above because applying \(\text{Hom}_{\Psi}(-, \Psi)\) to that linear resolution gives a linear resolution of \(L_r\), whose homology is concentrated in a single degree.

Claim (v) again follows from explicit computation. \(\square\)

We have an endomorphism \(t\) of \(L_t\) of \(jk\)-degree \((2, -1)\) which sends \(x, x^i\) to \(\xi, x^i\), and \(\xi, x^i\) to zero. Writing \(\Lambda\) for the exterior algebra on \(t\), we thus give \(L_t\) the structure of a \(\Psi, \Lambda\)-bimodule. We have an endomorphism \(t\) of \(L_r\) of degree 2 which sends \(x^i, x\) to \(x^i, \xi\), and \(x^i, \xi\) to zero. We thus give \(L_r\) the structure of a \(\Lambda, \Psi\) bimodule.

We define \(M\) to be the \(\Psi, \Psi\)-bimodule \(L_t \otimes_\Lambda L_r\). We define \(\tau\) to be the involution on \(\Psi\) that sends \(x\) to \(x\) and \(\xi\) to \(\xi\). The bimodule \(M\) has some intriguing properties:

**Theorem 27.** (i) We have \(M^\ast \cong M(2p)[-2p+3]\).

(ii) We have \(\Psi M \cong \bigoplus_{h=0}^{p-1} L_t(-1-h)[h]\) and \(M_{\Psi} \cong \bigoplus_{h=0}^{p-1} L_r(-1-h)[h]\).

(iii) We have a short exact sequence of bimodules

\[ 0 \to \Psi(-p-1)[p-1] \to M \to \Psi^\ast(-p+1)[p-2] \to 0. \]

(iv) We have a quasi-isomorphism between \(\mathfrak{K}^i \otimes \Psi M\) and \(M^\tau(i) [i(1-p)]\), for \(i \in \mathbb{Z}\). Similarly we have a quasi-isomorphism between \(M \otimes \mathfrak{K}^i\) and \(M^\tau(i) [i(1-p)]\), for \(i \in \mathbb{Z}\).

(v) We have an exact triangle in the derived category of differential graded \(\Psi, \Psi\)-bimodules

\[ \mathfrak{K}^{-1} \to M^\tau(1)[0] \to \Psi^\tau \otimes_{\Psi^0} \Psi^\ast(1)[-1] \sim, \]

where \(\tau\) is the automorphism that sends \(x\) to \(x\) and \(\xi\) to \(\xi\).

(vi) We have an exact triangle in the derived category of differential graded \(\Psi, \Psi\)-bimodules

\[ \mathfrak{K}^{-2} \to M(-p+1)[p-1] \to \Psi(0)[-1] \sim. \]

Assume \(p \geq 3\).

(vii) We have \(\text{Hom}_{\Psi}(M, M) \cong M(p+1)[1-p]\) and \(M \otimes_{\Psi} M \cong M(1-p)[p-1]\) as \(jk\)-graded \(\Psi, \Psi\)-bimodules.
Proof. 1 We have isomorphisms of bimodules

\[ M^* = \text{Hom}(L_l \otimes_{\Lambda} L_r, F) \]
\[ \cong \text{Hom}_\Lambda(L_l, \text{Hom}(L_r, F)) \]
\[ \cong \text{Hom}_\Lambda(L_l, L_l) \]
\[ \cong \text{Hom}_\Lambda(L_l, L_l \otimes_{\Lambda} \Lambda) \]
\[ \cong L_l \otimes_{\Lambda} \text{Hom}_\Lambda(L_l, \Lambda) \]
\[ \cong L_r \otimes_{\Lambda} L_r \]
\[ \cong M \]

We are using here that \( L_l \) is projective a right \( \Lambda \)-module, some adjunctions, and the fact that \( \Lambda \) is a symmetric algebra. The gradings match up as described: \( M \) is concentrated in \( j \) degrees \( 0 \) down to \( -2p \), thus \( M^* \) is concentrated in degrees \( 2p \) down to \( 0 \); \( M \) is concentrated in \( k \) degrees \( -1 \) up to \( 2p - 2 \), thus \( M^* \) is concentrated in degrees \( 2 - 2p \) up to \( 1 \).

2 This follows directly from the definition.

3 To embed \( \Psi \) in \( M \), we send \( e_h \in \Psi \) to \( x.x^{-h} \otimes x^{h-1}.x \in M \), a map of \( (j, k) \)-degree \( (-p - 1, p - 1) \). The quotient map corresponding to this embedding is then dual to the embedding.

4 As one-sided modules, this follows from Lemma 26. We know that \( K \) is \( K_l \) accompanied by a shift in \( j \)-degree by \( 1 \). It follows that \( K \otimes_{\Psi} L_l = L_l(p)[1 - p] \) and consequently that \( K \otimes_{\Psi} M = M(p)[1 - p] \). The left-sided statement follows from the fact that \( K \otimes_{\Psi} M \cong M' \) as \( \Psi \)-\( \Psi \)-bimodules, the right-sided statement similarly. We do not establish the twist by \( \tau \) here; a detailed analysis confirming the twist appears in Corollary 32 and Lemma 36.

5 The triangle is obtained by tensoring the exact sequence of part 3 on the left with \( \Psi^* \otimes \Psi \) and then shifting in \( j \)-degree by \( p \). We use part 4.

6 The triangle is obtained by tensoring the exact sequence of part 5 on the left with \( \Psi^* \otimes \Psi(-1) \). We again use part 4.

7 First note that applying \( \text{Hom}_\Psi(-, M) \) to the short exact sequence of 3 gives us a long exact sequence

\[ \ldots \leftarrow \text{Ext}^1(\Psi^*, M) \leftarrow \text{Hom}(\Psi(-p-1)[p-1], M) \leftarrow \text{Hom}(M, M) \leftarrow \text{Hom}(\Psi^*, M) \leftarrow 0. \]

By the preceding lemma \( \text{Hom}_\Psi(\Psi^*, M) = \text{Ext}^1_\Psi(\Psi^*, M) = 0 \), and so

\[ \text{Hom}_\Psi(M, M) \cong M(p + 1)[1 - p]. \]

Since the map \( \text{Hom}_\Psi(M, M) \to \text{Hom}_\Psi(\Psi, M) \) is a bimodule homomorphism, we have \( \text{Hom}_\Psi(M, M) \cong M(p + 1)[1 - p] \) as \( jk \)-graded \( \Psi \)-\( \Psi \)-bimodules. By adjunction and part 5 we have

\[ \text{Hom}_F(M \otimes_{\Psi} M, F) \cong \text{Hom}_\Psi(M, M^*) \cong \text{Hom}_\Psi(M, M) \cong M(p + 1)[1 - p]. \]

and so duality gives us \( M \otimes_{\Psi} M \cong M(-p - 1)[p - 1] \). Again, \( jk \)-degrees can be easily checked.  \( \square \)
Lemma 28. We have a bimodule $L_l^0 \otimes \Lambda L_r$ where $\theta$ is the automorphism of $\Lambda$ sending $t$ to $-t$. As a $\Psi\cdot\Psi$-bimodule, $L_l^0 \otimes L_r$ is isomorphic to $M^\tau$.

Proof. On both $L_l^0 \otimes \Lambda L_r$ and $M^\tau$, internal multiplication by $x \otimes \xi + \xi \otimes x$ is zero. Since the spaces have the same dimension, and are quotients $L_l \otimes_F L_r$ they are isomorphic.

It is convenient to truncate $M$ a little. We have a $j$-graded bimodule homomorphism $M \to (\Psi^0 e_p)^\sigma$ which sends $x.1 \otimes 1.\xi \in M$ to $e_p$, and sends all other basis elements of $M$ to zero; we define $\overline{M}$ to be the kernel of this homomorphism.

The projection map $\Psi \otimes \Psi_0 \sigma \Psi^* \to F e_p \otimes F e^*_1$ that factors through $\Psi^0 \otimes \sigma \Psi^0$ is a homomorphism of dg bimodules. We define $\overline{\Psi}$ to be the kernel of this projection; it is quasi-isomorphic to $\overline{\Psi^0} = \oplus_{h=1}^p (\Psi^0 e_h)^\sigma$ as a right and left dg module.

Lemma 29. (i) We have an exact triangle in the derived category of $\Psi\cdot\Psi$-bimodules,

$$\Psi^{-1} \to \overline{M} \langle 1 \rangle \to \overline{\Psi^0} \langle -1 \rangle \to .$$

(ii) The homology of the left $\Psi$-module $\Psi^{-1} e_h$ is isomorphic to $\Psi e_p$ if $h = 1$, to $E_l$ if $h = 2$, and to $L_l \oplus \Psi^0 e_h$ if $h > 2$. The homology of the right $\Psi$-module $e_h \Psi^{-1}$ is isomorphic to $e_1 \Psi$ if $h = p$, to $E_r$ if $h = p - 1$, and to $L_r \oplus e_1 \Psi^0$ if $h < p - 1$.

Assume $p \geq 3$.

(iii) We have

$$\text{Hom}_\Psi(\overline{M}, M) \cong M(p + 1)[1 - p]$$

and

$$\overline{M} \otimes M \cong M \otimes \overline{M} \cong \overline{M} \otimes M \cong M(-p - 1)[p - 1]$$

as $\Psi\cdot\Psi$-bimodules.

Proof. (i) The triangle comes from the triangle in Lemma 27 by cancelling the quotient $(\Psi^0 e_p)^\sigma \to (\Psi^0 e_p)^\sigma$ of $M \to \Psi^0$ that is zero in the derived category.

(ii) We have a quasi-isomorphism between $\Psi^{-1}$ and $\overline{M} \to \overline{\Psi^0} \langle -1 \rangle$.

(iii) We have an exact sequence of bimodules

$$0 \to \overline{M} \to M \to (\Psi^0 e_p)^\sigma \to 0.$$

Applying $\text{Hom}(-, M)$ in the category of left modules gives us a long exact sequence

$$\text{Ext}^1(\Psi^0 e_p, M) \to \text{Hom}(\overline{M}, M) \to \text{Hom}(M, M) \to \text{Hom}(\Psi^0 e_p, M).$$

Since $\text{Hom}(\Psi^0 e_p, M)$ and $\text{Ext}^1(\Psi^0 e_p, M)$ are both zero, we find $\text{Hom}(\overline{M}, M) \cong \text{Hom}(M, M) \cong M$. Again $\text{Hom}_F(M \otimes \overline{M}, F) \cong \text{Hom}_\Psi(\overline{M}, M^\tau) \cong \text{Hom}_\Psi(\overline{M}, M) \cong M$, and dualising (using again $M^\tau \cong M$), we obtain $M \otimes \overline{M} \cong M$. Working instead on the right, we obtain $\overline{M} \otimes M \cong M$. By duality, we have an exact sequence of bimodules

$$0 \to \overline{M} \to M \to (\Psi^0 e_1)^\sigma \to 0.$$
Applying $\text{Hom}(\overline{M}, -)$ in the category of left modules gives us a long exact sequence

$$\text{Ext}^1(\overline{M}, (\Psi^0 e_1)^{\sigma}) \leftrightarrow \text{Hom}(\overline{M}, \overline{M}') \leftrightarrow \text{Hom}(\overline{M}, M) \leftrightarrow \text{Hom}(\overline{M}, (\Psi^0 e_1)^{\sigma}) \leftrightarrow 0$$

Vanishing of the first and last terms of this sequence gives us an isomorphism $\text{Hom}(\overline{M}, \overline{M}') \cong \text{Hom}(\overline{M}, M)$, so $\text{Hom}(\overline{M}, \overline{M}') \cong M(-p)$. Adjunction and $M^* \cong M$ again give us $\overline{M} \otimes \overline{M} \cong M$. All gradings match with those of the previous lemma. □

**Lemma 30.** We have natural homomorphisms of $\Psi, \Psi$-bimodules $\alpha, \beta$, and $\gamma$, that make the triangles

$$
\begin{align*}
M \otimes \Psi \Psi^{0\sigma} & \longrightarrow M e_p (p-1)[1-p] \\
\downarrow \gamma & \quad \beta \\
M (p-1)[1-p] & \quad \gamma \\
\Psi^{i} \Psi^{0\sigma} & \longrightarrow \Psi e_p (-2)[0] \\
\downarrow \gamma & \quad \beta \\
\Psi (-2)[0] & \quad \gamma \\
\overline{M} \otimes \Psi \Psi^{0\sigma} & \longrightarrow \overline{M} e_p (-2)[0] \\
\downarrow \gamma & \quad \beta \\
\Psi (-2)[0] & \quad \gamma \\
\end{align*}
$$

commute.

**Proof.** The arrows $\beta$ are the obvious embeddings. The arrow $\alpha$ in the top left hand diagram sends an element $a \otimes b \otimes e_l \in L_l \otimes L_r \otimes \Psi^{0\sigma}$ to $a \otimes be_p$ for $l = 1$ and to zero for $l > 1$. The arrow $\alpha$ in the top right hand diagram sends an element $e_l \otimes b \otimes c \in \Psi^{0\sigma} \otimes L_l \otimes L_r$ to $e_l b \otimes c$ for $l = 1$ and to zero for $l > 1$. These maps restrict to the arrows $\alpha$ in the bottom diagram. The arrows $\gamma$ are merely compositions of $\alpha$ and $\beta$. □

**Explicit formulas for the homology of $\Psi, \xi$ for $i \leq -1$.** We define $\Upsilon^-$ to be the homology of $T_{\Psi}(\Psi^{i-1})$.

Let $i \leq -1$. We are interested in describing explicit formulas for a basis of $\Upsilon^i$, as a subquotient of

$$\Psi, \xi \cong \Psi^{0\sigma} \otimes_{\Psi^{0\sigma}} \Psi^{0\sigma} \otimes_{\Psi^{0\sigma}} \ldots \otimes_{\Psi^{0\sigma}} \Psi^{0\sigma} \otimes_{\Psi^{0\sigma}} \Psi (i)$$

To simplify notation, in the remains of this section we omit shifts in $jk$-degree.

**The case $i=-1$.** The differential on $\Psi, \xi^{-1} = \Psi^{0\sigma} \otimes \Psi(-1)$ is given by $d(a \otimes b) = (-1)^{|a|b}(ax \otimes x b + a \xi \otimes x b)$. We record that $x$ and $\xi$ super-commute in homology:

**Lemma 31.** In $H(\Psi^{0\sigma} \otimes \Psi)$, we have $x^d e_h \otimes e_h \xi x^d \equiv -x^{d-1} \xi e_h \otimes e_h x^{d+1}$.

**Proof.** This follows immediately from the fact that the image of $x^d e_{h-1} \otimes e_{h-1} x^d$ under the differential is $(-1)^{d-1}(x^d e_h \otimes e_h \xi x^d + x^{d-1} \xi e_h \otimes e_h x^{d+1})$. □

We deduce from this the twist required on our copy of $\overline{M}$ in homology:

**Corollary 32.** The component of $H(\Psi, \xi \otimes \Psi)$ generated by $e_p \otimes e_1$ is isomorphic to $\overline{M}'$ as a $\Psi, \Psi$-bimodule.
Proof. The element $e_p \otimes e_p$ of $\Psi^\sigma \otimes_{\Psi^0} \Psi$ maps to zero under the differential. Since the bimodule $\mathcal{M}$ has simple top $S_p \otimes_{\Psi} S_p^\sigma (-1)$ and we have a unique composition factor of $\Psi^\sigma \otimes \Psi$ isomorphic to $S_p \otimes S_p^\sigma$, we conclude the element $e_p \otimes e_1$ of $\Psi^\sigma \otimes \Psi$ generates the factor $\mathcal{M}$ of $\mathbb{H}(\mathbb{K}^{-1})$, whose basis elements written into pictures of the composition structures for $\mathcal{M}$ look like

\[
\begin{array}{c}
\xi \otimes e_1 \\
\xi e_1 \\
x \otimes e_1 \\
x^2 \otimes e_1 \\
\vdots \\
x^{p-3} \otimes e_1 \\
x^{p-2} \otimes e_1 \\
x^{p-1} \otimes e_1
\end{array}
\]

for $\mathbb{H}(\Psi^\sigma \otimes_{\Psi^0} \Psi)e_1$, and

\[
\begin{array}{c}
\xi \otimes e_1 \\
\xi e_1 \\
x \otimes e_1 \\
x^2 \otimes e_1 \\
\vdots \\
x^{p-3} \otimes e_1 \\
x^{p-2} \otimes e_1 \\
x^{p-1} \otimes e_1 \\
x^{p-2} \otimes e_p \otimes x^{l-1} \\
x^{p-2} \otimes e_p \otimes x^{l-2} \\
\vdots \\
x^{p-2} \otimes e_p \otimes x^{l-2} \\
x^{p-2} \otimes e_p \otimes x^{l-1} \\
x^{p-2} \otimes e_p \otimes x^{l-2} \\
x^{p-2} \otimes e_p \otimes x^{l-1} \\
x^{p-2} \otimes e_p \otimes x^{l-1}
\end{array}
\]

for $\mathbb{H}(\Psi^\sigma \otimes_{\Psi^0} \Psi)e_l$ for $l \geq 2$, where we have used Lemma 31 to move all the $\xi$ to the rear.

To see the twist by $\tau$, note that internal multiplication by $x \otimes \xi$ can be identified with the negative of internal multiplication by $\xi \otimes x$, by Lemma 31.

It is easy to see that $\xi \otimes \xi \in \Psi^\sigma \otimes \Psi$ lies in the kernel of the differential, but not the image, hence contributes to homology; this factor of homology has zero intersection with the factor of $(\Psi e_p)^\sigma \otimes e_1 \Psi$ described above, since the components $\xi e_h \otimes e_{p+1-h} \xi$ of $\xi \otimes \xi$ in homology all belong to a subquotient of $(\Psi e_h)^\sigma \otimes e_{p+1-h} \Psi$ for $1 \leq h \leq p-1$. These components $\xi e_h \otimes e_{p+1-h} \xi$ of $\xi \otimes \xi$ for $1 \leq h \leq p-1$, shifted by $-1$ in $j$-degree, thus give us the factor $\Psi^0$ of homology.

The case $i = -2$. Following the super sign convention, the left differential on the dg $\Psi$-$\Psi$-bimodule $\Psi^\sigma \otimes_{\Psi^0} \Psi^\sigma \otimes_{\Psi^0} \Psi$ is given by

\[
d(a \otimes b \otimes c) = (-1)^{|a|s} (a x \otimes \xi b \otimes c + a \xi \otimes x b \otimes c) + (-1)^{|a|s+|b|s} (a \otimes bx \otimes \xi c + a \otimes b \xi \otimes xc).
\]

for $a, b \in \Psi^\sigma$, $c \in \Psi$.

Lemma 33. $w = (\xi \otimes \xi \otimes 1 - 1 \otimes \xi \otimes \xi) \in \Psi^\sigma \otimes \Psi^\sigma \otimes \Psi$ lies in the kernel of the differential.
Proof. This is straightforward. There is no nontrivial super-commutation involved since $\xi$ has $k$-degree zero. □

The two dimensional subspace $e_p \otimes (e_1 \Psi e_p)^\sigma \otimes e_1$ of $\Psi^\sigma \otimes \Psi^\sigma$, where $\Psi$ is a bimodule, has basis $\{e_p \otimes x^{p-1} \otimes e_1, e_p \otimes x^{p-2} \otimes e_1\}$ maps to zero under the differential, and represents the two composition factors of $\Psi^\sigma \otimes \Psi^\sigma$ isomorphic to $S_p \otimes S_1^{op}$. Since an ungraded bimodule, $M$ has a two dimensional top isomorphic to $(S_p \otimes S_1^{op})^{\otimes 2}$, we conclude that the factor $M$ of homology is generated as a bimodule by $\{e_p \otimes x^{p-1} \otimes e_1, e_p \otimes x^{p-2} \otimes e_1\}$.

We know that $w = (\xi \otimes \xi \otimes 1 - 1 \otimes \xi \otimes \xi) \in \Psi^\sigma \otimes_{\Psi^0} \Psi^\sigma \otimes_{\Psi^0} \Psi$ lies in the kernel of the differential but not the image, hence contributes to homology; this factor of homology has zero intersection with the factor of $(\Psi e_p)^\sigma \otimes (e_1 \Psi e_p)^\sigma \otimes e_1 \Psi$ described above, since the component $\Psi \otimes \xi \otimes \Psi$ and $\Psi^\sigma \otimes (x^{p-1}, x^{p-2} \xi) \otimes \Psi$ have different degrees in the middle tensor. The elements $e_l w e_l$ with $1 \leq l \leq p$ because $e_l F w e_l \cong S_l \otimes S_l^{op}$ as a $\Psi^0, \Psi^0$-bimodule, and all composition factors of the regular bimodule for $\Psi$ outside the top are isomorphic to $S_l \otimes S_m^{op}$ for $l \neq m$.

**Lemma 34.** We have $\xi \otimes \xi \otimes x^{p-1} = (-1)^p x^{p-1} \otimes \xi \otimes \xi$ in $\mathbb{H}((\Psi^\sigma \otimes_{\Psi^0} \Psi^\sigma \otimes_{\Psi^0} \Psi)$, for $1 \leq d \leq p - 1$.

**Proof.** We get the first element by multiplying $w = (\xi \otimes \xi \otimes 1 - 1 \otimes \xi \otimes \xi)$ on the right by $x^{p-1}$, and the second by multiplying $w$ on the left by $(-1)^{p-1} x^{p-1}$. We can show explicitly that $xw = -wx$ in homology, since

$$wx = \xi \otimes \xi \otimes x - 1 \otimes \xi \otimes \xi x, \quad -wx = x \otimes \xi \otimes \xi - x \xi \otimes \xi \otimes 1,$$

and both are equal to $\xi \otimes \xi \otimes x + \xi \otimes x^* \otimes \xi + x \otimes \xi \otimes \xi$ in homology thanks to the following computations of images under the differential acting on the left:

$$d : \xi \otimes 1 \otimes 1 \mapsto (\xi \otimes \xi \otimes x + \xi \otimes x \otimes \xi + \xi x \otimes \xi \otimes 1),$$

$$d : 1 \otimes 1 \otimes \xi \mapsto (\xi \otimes x \otimes \xi + x \otimes \xi \otimes \xi + 1 \otimes \xi \otimes x \xi).$$

The result follows. □

**Lemma 35.** The component of $\mathbb{H}((\Psi^\sigma \otimes_{\Psi^0} \Psi^\sigma \otimes_{\Psi^0} \Psi)$ generated by $w$ is quasi-isomorphic to $\Psi$ as a $\Psi-\Psi$-bimodule.

**Proof.** We have $xw = -wx$ and $\xi w = -w \xi$. So the component of homology is given by $\Psi^\xi$, where $\xi$ is the automorphism that sends $\xi$ to $-\xi$ and $x$ to $-x$. This automorphism is inner, being given by conjugation by $\sum (-1)^l e_l$. Consequently $\Psi^\xi \cong \Psi$ as $\Psi-\Psi$-bimodules. □

**Lemma 36.** The component of $\mathbb{H}((\Psi^\sigma \otimes_{\Psi^0} \Psi^\sigma \otimes_{\Psi^0} \Psi)$ generated by $e_p \otimes x^{p-1} \otimes e_1$ and $e_p \otimes x^{p-2} \xi \otimes e_1$ is isomorphic to $M$ as a $\Psi-\Psi$-bimodule.

**Proof.** We have

$$d(xe_p \otimes e_1 x^{p-2} \otimes e_2) = (-1)^{p-1}(x \otimes x^{p-1} \otimes \xi + x \otimes x^{p-2} \xi \otimes x),$$

$$d(e_1 \otimes e_2 x^{p-2} \otimes e_1 x) = (x \otimes x^{p-2} \otimes x + x \otimes x^{p-1} \otimes x).$$

Thus $x \otimes x^{p-1} \otimes \xi = -x \otimes x^{p-2} \xi \otimes x = \xi \otimes x^{p-1} \otimes x$, so internal multiplication by $x \otimes x \otimes x$ in our copy of $M$ is zero, which means the bimodule $M$ is untwisted. □
The case \( i \leq -3 \).

**Lemma 37.** As a differential \( jk \)-graded \( \Psi \)-\( \Psi \)-bimodule, \( T_\Psi(\mathfrak{M}^{-1}) \) has a filtration with sections

\[
\begin{align*}
\Psi \\
M^\tau(1) &\longrightarrow \Psi^0 \sigma(1) \\
M(-p+1) &\longrightarrow \Psi \\
M(-2p+1) &\longrightarrow M^\tau(1) \longrightarrow \Psi^0 \sigma(1) \\
M(-3p+1) &\longrightarrow M(-p+1) \longrightarrow \Psi \\
&\ldots
\end{align*}
\]

in the \( j \)-grading, and

\[
\begin{align*}
\Psi \\
M^\tau &\longrightarrow \Psi^0 \sigma[-1] \\
M[p-1] &\longrightarrow \Psi[-1] \\
M[2p-2] &\longrightarrow M[-1] \longrightarrow \Psi^0 \sigma[-2] \\
M[3p-3] &\longrightarrow M[p-2] \longrightarrow \Psi[-2] \\
&\ldots
\end{align*}
\]

in the \( k \)-grading

**Proof.** We have separated \( j \)-gradings and \( k \)-gradings for typesetting reasons: the diagrams get too complicated otherwise. In the statement of the lemma, and in similar statements in our script, we take a sequence of arrows

\[
X_0 \to X_1 \to \ldots \to X_l
\]

to represent an object of the derived category with a filtration whose sections are isomorphic to \( X_i[l] \); if we replace \( X_i \) by an isomorphic projective resolution, then the object is isomorphic to a complex formed by taking the total complex of a complex of such resolutions. The rows of the diagram correspond to \( \mathfrak{M}^d, \) for \( i \in \mathbb{Z}_{\leq 0} \). Everything follows from preceding lemmas: we obtain the components of \( \mathfrak{M}^{-2l} \) by applying \( \mathfrak{M}^{-1} \otimes - \) to the \( l+1 \) terms in \( \mathfrak{M}^{1-2l} \) for \( l \geq 1 \). \( \square \)

We now assume \( i \leq -3 \), and write a basis element in \( H((\Psi^\sigma \otimes_{\Psi^0} \Psi)^{-i}) \) as an \(-i+1\)-fold tensor.

**Theorem 38.** (i) For \( i \) even, we set

\[
\mathfrak{x}_{f,-i} = e_p \otimes (\xi \otimes \xi)^{\otimes f} \otimes (x^{p^{-1}})^{\otimes -i-1-2f} \otimes e_1 \\
\equiv \pm e_p \otimes (x^{p^{-1}})^{\otimes -i-1-2f} \otimes (\xi \otimes \xi)^{\otimes f} \otimes e_1 \\
\not\in 0 \in H((\Psi^\sigma \otimes_{\Psi^0} \Psi)^{-i})
\]
and
\[
\mathbf{y}_{f,-i} = e_p \otimes (\xi \otimes \xi) \otimes (x^p-1)^{\otimes -i-2-2f} \otimes x^{p-2} \xi \otimes e_1 \\
\equiv \pm e_p \otimes x^{p-2} \xi \otimes (x^p-1)^{\otimes -i-2-2f} \otimes (\xi \otimes \xi) \otimes e_1 \\
\neq 0 \in H((\Psi^\sigma \otimes \psi \Psi)^{\otimes -i})
\]
for \(0 \leq f \leq \frac{i-2}{2}.

(ii) For \(i\) odd, we set
\[
\mathbf{x}_{f,-i} = e_p \otimes (\xi \otimes \xi) \otimes (x^p-1)^{\otimes -i-1-2f} \otimes e_1 \\
\equiv \pm e_p \otimes (x^p-1)^{\otimes -i-1-2f} \otimes (\xi \otimes \xi) \otimes e_1 \\
\neq 0 \in H((\Psi^\sigma \otimes \psi \Psi)^{\otimes -i})
\]
for \(0 \leq f \leq \frac{i-1}{2},\) and
\[
\mathbf{y}_{f,-i} = e_p \otimes (\xi \otimes \xi) \otimes (x^p-1)^{\otimes -i-2-2f} \otimes x^{p-2} \xi \otimes e_1 \\
\equiv \pm e_p \otimes x^{p-2} \xi \otimes (x^p-1)^{\otimes -i-2-2f} \otimes (\xi \otimes \xi) \otimes e_1 \\
\neq 0 \in H((\Psi^\sigma \otimes \psi \Psi)^{\otimes -i})
\]
for \(0 \leq f \leq \frac{i-3}{2}.

For \(f < \frac{i-2}{2},\) the elements \(\mathbf{x}_{f,-i}\) and \(\mathbf{y}_{f,-i}\) generate the possibly twisted copy of \(M\) in the corresponding homological degree as a bimodule. For \(f = \frac{i-1}{2}\) (and \(i\) odd), \(\mathbf{x}_{f,-i}\) generates a possibly twisted copy of \(M\).

Proof. In all cases the equivalence in homology of the two given representatives follows from Lemma 34. We proceed by induction on \(i\). Cases \(i = -1\) and \(i = -2\) have been examined above. So assume the statement is true for \(-i+1,\) i.e. \(\mathbf{x}_{f,-i+1}\) and \(\mathbf{y}_{f,-i+1}\) are nonzero in \(H((\Psi^\sigma \otimes \psi \Psi)^{\otimes -i+1})\) for \(f \leq \frac{i-2}{2}\) and generate copies of \(M\) as a bimodule. In particular, this means that \(\mathbf{x}_{f,-i+1} x^{p-1}\) and \(\mathbf{y}_{f,-i+1} x^{p-1}\) are nonzero in \(H((\Psi^\sigma \otimes \psi \Psi)^{\otimes -i+1})\). But since \(H((\Psi^\sigma \otimes \psi \Psi)^{\otimes -i}) e_1 = H((\Psi^\sigma \otimes \psi \Psi)^{\otimes -i+1}) e_p \otimes e_1,\) this means that
\[
\mathbf{x}_{f,-i+1} x^{p-1} \otimes e_1 = \mathbf{x}_{f,-i}
\]
and
\[
\mathbf{y}_{f,-i+1} x^{p-1} \otimes e_1 = e_p \otimes (\xi \otimes \xi) \otimes (x^p-1)^{\otimes -i-3-2f} \otimes x^{p-2} \xi \otimes x^{p-1} \otimes e_1 \\
\equiv e_p \otimes (\xi \otimes \xi) \otimes (x^p-1)^{\otimes -i-2-2f} \otimes x^{p-2} \xi \otimes e_1 \\
= \mathbf{y}_{f,-i}
\]
are non-zero in \(H((\Psi^\sigma \otimes \psi \Psi)^{\otimes -i})\). Since \(e_p e_1\) is two-dimensional, we see that they must be the generators of the corresponding copy of \(M\). It remains to consider the cases \(i\) even, \(f = \frac{i-2}{2}\) and \(i\) odd, \(f = \frac{i-1}{2}\). For the first case, we know by the inductive assumption that
\[
x_{f,-i+1} = e_p \otimes (\xi \otimes \xi) \otimes e_1
\]
is non-zero in homology and generates a copy of \(\overline{M}\). In particular this means that \(x_{f,-i+1} x^{p-1}\) and \(x_{f,-i+1} x^{p-2} \xi\) are non-zero and, arguing as above, that \(x_{f,-i+1} x^{p-1} \otimes e_1 = \mathbf{x}_{f,-i}\) and \(x_{f,-i+1} x^{p-2} \xi \otimes e_1 = \mathbf{y}_{f,-i}\) are nonzero in \(H((\Psi^\sigma \otimes \psi \Psi)^{\otimes -i})\) and because \(e_p e_1\) is two dimensional, generate a copy of \(M\). Now consider the case where
is odd and \( f = \frac{1}{2} \), i.e. consider the element \( x_{-i} = e_\sigma \otimes (x \otimes x^{-1}) \otimes e_1 \).

This is obviously in the kernel, but not in the image of the differential and hence non-zero in \( \mathbb{H}((\Psi^\sigma \otimes \xi \Psi)^{(-i)}) \). Since \( e_\sigma \otimes x \otimes e_1 \) is one dimensional, it must be the generator of the corresponding copy of \( \mathbb{H}(\Psi^\sigma \otimes \xi \Psi) \). \( \Box \)

The remaining homology is given as follows:

**Lemma 39.** (i) For \( i \) even, the elements \( e_l w_i \otimes e_1 \) (for \( 1 \leq l \leq p \)) generate, as a \( \Psi \cdot \Psi \)-bimodule the factor isomorphic to \( \Psi \) in \( \mathbb{H}((\Psi^\sigma \otimes \xi \Psi)^{(-i)}) \).

(ii) For \( i \) odd, the elements \( e_l (\xi \otimes \xi) \otimes e_1 \) for \( 1 \leq l \leq p - 1 \) generate the bimodule isomorphic to \( \Psi^{0 \sigma} \) in \( \mathbb{H}((\Psi^\sigma \otimes \xi \Psi)^{(-i)}) \).

**Proof.** For \( i \) even, the elements \( e_l w_i \otimes e_1 \) for \( 1 \leq l \leq p \) are certainly in the kernel of the differential, and the elements are not in the image of differential, for lack of tensor factors \( x \). They must be generators, because \( e_l (Fw_i \otimes e_1) \cong S^1 \otimes S^p \) as an ungraded \( \Psi^0 \cdot \Psi^0 \)-bimodule, and all composition factors of the regular bimodule for \( \Psi \) outside the top are isomorphic to \( S^1 \otimes S^p \) for \( l \neq m \).

For \( i \) odd, the given elements have \( \xi \) in every tensor factor, hence are non-zero in \( \mathbb{H}((\Psi^\sigma \otimes \xi \Psi)^{(-i)}) \). They generate the desired semi-simple bimodule since that is all the homology not accounted for by factors isomorphic to \( M \) and \( \overline{M} \). \( \Box \)

**The dg algebra** \( \mathcal{T}_\Psi(Q, Q^{-1}) \). Suppose \( p \geq 3 \). Here we examine the dg algebra \( \mathcal{T}_\Psi(Q, Q^{-1}) \), whose homology we call \( \Upsilon \). For reasons that are articulated in \[\text{Lemma 43}\] we do not care about the entire algebra structure: all we want to know about the product is what it looks like when composed with projection onto the subspace \( \mathcal{H} \mathcal{T}_\Psi(Q^{(-1)}) \). We define \( \Upsilon^{(-1)} \) to be the homology of \( \mathcal{T}_\Psi(Q^{(-1)}) \).

**Lemma 40.** As \( j \)-graded \( \Psi \cdot \Psi \)-bimodules, \( \Upsilon \) can be identified with the homology of

\[
\begin{array}{cccccccccc}
\Psi & \rightarrow & \Psi^{0 \sigma} & \rightarrow & \Psi^{0 \sigma}[-1] \\
M^j(1) & 0 & \rightarrow & M^j(1) & 0 & \rightarrow & \Psi \\
M^j(-p+1) & 0 & \rightarrow & M^j(-2p+1) & 0 & \rightarrow & M^j(-3p+1) & 0 & \rightarrow & \cdots \\
M^j(-p+1) & 0 & \rightarrow & M^j(-p+1) & 0 & \rightarrow & \Psi \\
\end{array}
\]

As \( k \)-graded \( \Psi \cdot \Psi \)-bimodules, \( \Upsilon \) can be identified with the homology of

\[
\begin{array}{cccccccccc}
\Psi & \rightarrow & \Psi^{0 \sigma} & \rightarrow & \Psi^{0 \sigma}[-1] \\
M^j[0] & 0 & \rightarrow & M^j[-1] & 0 & \rightarrow & \Psi[-1] \\
M^j[p-1] & 0 & \rightarrow & M^j[2p-2] & 0 & \rightarrow & M^j[3p-3] & 0 & \rightarrow & \cdots \\
M^j[p-1] & 0 & \rightarrow & M^j[p-2] & 0 & \rightarrow & \Psi[-2] & \rightarrow & \cdots \\
\end{array}
\]
As an ungraded $\Psi\cdot\Psi$-bimodule $\Upsilon^+$ can be identified with the homology of

\[
\begin{array}{c}
\Psi^* \xrightarrow{0} M \xrightarrow{0} M \\
\Phi^{0\sigma} \xrightarrow{0} M^\tau \xrightarrow{0} M^\tau \\
\Psi^* \xrightarrow{0} M \\
\Phi^{0\sigma} \xrightarrow{} M^\tau \\
\Psi^* \\
\Phi^{0\sigma} \\
\Psi
\end{array}
\]

and

**Proof.** The point is that the rows correspond to $\mathfrak{k}^i$, for $i \in \mathbb{Z}$. Almost everything follows from preceding lemmas and standard facts about Koszul duality. For example, we obtain the components of $\mathfrak{k}^{-2i}$ by applying Koszul duality to the $i+1$ terms in $\mathfrak{k}^{1-2i}$, for $i \geq 1$.

Why does $\Psi$ split off from $M$ in homology in $\mathfrak{k}^{-2i}$? It is generated as a bimodule by $\xi \otimes \xi \otimes 1 - 1 \otimes \xi \otimes \xi$, whereas the copy of $M$ is generated as a bimodule by $e_p \otimes x^{p-1} \otimes e_1$ and $e_p \otimes x^{p-2} \otimes e_1$. Since $p \geq 3$ the middle term in the tensor generating $\Psi$ has different radical degree from the middle term in the tensors generating $M$, and since images of monomials in $x$ and $\xi$ under the differential on $\mathfrak{k}^{-2i}$ have middle terms with identical radical degrees, whilst multiplying such a monomial on the left or right by a monomial in $\Psi$ does not alter the radical degree of the middle term, there is a splitting as required. $\square$

Rewriting the above expressions we see that as $j$-graded $\Psi^0\cdot\Psi^0$-bimodules, $\Upsilon^{\leq 1}$ can be identified with

\[
\begin{array}{c}
\Psi^{0\sigma}(1) \\
\Psi \\
M^{\tau}(1) \oplus \Psi^{0\sigma}(1) \\
M(-p+1) \oplus \Psi \\
M^\tau(-2p+1) \oplus M^{\tau}(1) \oplus \Psi^{0\sigma}(1) \\
M(-3p+1) \oplus M(-p+1) \oplus \Psi \\
\end{array}
\]
whilst as $k$-graded $\Psi^0$-$\Psi^0$-bimodules, $\Upsilon^{\leq 1}$ can be identified with

$$\Psi$$

$$\Psi^0 \otimes \Psi^0$$

$$\Upsilon^\sigma \otimes \Upsilon^\sigma$$

$$\Upsilon^\sigma \otimes \Psi^0$$

$$\Psi \otimes \Psi^0$$

$$M[p - 1]$$

$$M[3p - 3]$$

$$\Upsilon^\sigma [2p - 2]$$

$$\sigma \Upsilon^\sigma$$

$$\sigma \Upsilon^\sigma$$

$$\sigma \Upsilon^\sigma$$

Theorem 41. The algebra $\Upsilon^-$ is isomorphic to the algebra given by

$$\mathbb{T}_\Psi(M(1)u \oplus \Psi^0(1)v) \otimes_F F[w]$$

modulo the relations $v^2 = 0$, $uv = -vu = w$, the relations $xu = -ux$, $xv = -vx$, $\xi u = u \xi$, $\xi v = v \xi$ for $x, \xi \in \Psi$, and relations that ensure products on the generators $\Upsilon^\sigma$ and $\Upsilon^0$ are given by the maps

$$\gamma : \Upsilon^\sigma \otimes \Psi^0 \to \Psi w, \quad \gamma : \Psi^0 \otimes \Upsilon^\sigma \to \Psi w$$

defined in Lemma 31, here $u, v,$ and $w$ are formal variables.

Proof. Again, to simplify notation, we omit shifts in $j$-degree. It is clear from the defining relations, and properties of the bimodules, that the algebra defined by generators and relations is a quotient of the space

$$\Psi$$

$$\Upsilon^\sigma \otimes \Psi^0$$

$$\Upsilon^\sigma \otimes \Upsilon^\sigma$$

$$\Upsilon^\sigma \otimes \Psi^0$$

$$\Psi \otimes \Psi^0$$

$$M[p - 1]$$

$$M[3p - 3]$$

$$\Upsilon^\sigma [2p - 2]$$

$$\sigma \Upsilon^\sigma$$

$$\sigma \Upsilon^\sigma$$

$$\sigma \Upsilon^\sigma$$

as a $\Psi$-$\Psi$-bimodule. Thanks to the relations between $u$, $x$, and $\xi$, we have $Mu^i \cong M^{j-i}$ as $\Psi$-$\Psi$-bimodules. We now observe the existence of an algebra homomorphism from the algebra defined above to $\Upsilon^-$ which identifies $\Upsilon^\sigma \otimes \Psi^0$ with $\Psi^0$, identifies $v$ with $\xi \otimes \xi$, identifies $Mu^2$ with the component $M$ of $H(\mathfrak{g}^-)$, and identifies $w$ with $\xi \otimes \xi \otimes 1 - \xi \otimes \xi \in \mathfrak{g}^{\leq -2}$. Note the relation $wv = -v$ comes from the fact that the sign preceding $1 \otimes \xi \otimes \xi$ in $\xi \otimes \xi \otimes 1 - \xi \otimes \xi \otimes \xi$ is the negative of the sign preceding $\xi \otimes \xi \otimes 1$. The preceding lemma implies that we have an algebra isomorphism from the homology of this algebra to $\Upsilon^-$.}\]
are given by maps $\gamma$: the products
\[ \Psi^w \otimes \Psi^{0\sigma} z \to \Psi^{0\sigma} w, \Psi^{0\sigma} z \otimes \Psi^w \to \Psi^{0\sigma} w, \]
are given by multiplication; and the product of $\Psi^{0\sigma}$ and $\Psi^w$ is zero.

In this section, we have described the product on $\Upsilon^{\leq 1}$ in its entirety: the space is isomorphic to
\[ \Psi^{0\sigma} \]
\[ \Psi \]
\[ M' \oplus \Psi^{0\sigma} \]
\[ M' \oplus M \oplus \Psi \]
\[ M' \oplus \Psi^{0\sigma} \]
\[ M \oplus \Psi \]
\[ M' \oplus M \oplus \Psi \]
and all products on these components are described by the natural maps $M \otimes M \to M$, $M \otimes \Psi \to M$, $\Psi \otimes M \to M$, and $\pm \gamma$: we use here the natural bimodule isomorphism $M' \cong M$ that sends $x^l \otimes x^m \in L_l \otimes L_r^\sigma$ to $(-1)^{l \cdot m} x^l \otimes x^m$. We describe a polytopal version of this in section 9.

**The case $p=2$.** In the preceding section we assumed $p \geq 3$. Here we explain how to adapt the results to the case $p = 2$. This case is exceptional because when $p = 2$ we can extend $L_l$ by a projective $\Psi$-module. We do not need to worry about signs because $1 = -1$ modulo 2.

Let $S^h(x, \xi)$ be the collection of polynomials in $x$ and $\xi$ of degree $h$, a vector space of dimension $h + 1$. We have a natural product map
\[ S^{h_1}(x, \xi) \otimes S^{h_2}(x, \xi) \to S^{h_1+h_2}(x, \xi). \]

For $i \leq 0$, we define $V_{-i}$ to be the $\Psi$-$\Psi$-bimodule
\[ S^{-i-1}(x, \xi) \]
\[ S^{-i}(x, \xi)_l \oplus S^{-i}(x, \xi)_r \]
\[ S^{-i+1}(x, \xi) \]
where $x$ and $\xi$ act by left multiplication sending $S^{-i-1}(x, \xi)$ to $S^{-i}(x, \xi)_l$, sending $S^{-i}(x, \xi)_r$ to $S^{-i+1}(x, \xi)$, sending $S^{-i}(x, \xi)_l$ and $S^{-i+1}(x, \xi)$ to zero; where $x$ and $\xi$ act by right multiplication sending $S^{-i-1}(x, \xi)$ to $S^{-i}(x, \xi)_r$, sending $S^{-i}(x, \xi)_l$ to $S^{-i+1}(x, \xi)$, sending $S^{-i}(x, \xi)_r$ and $S^{-i+1}(x, \xi)$ to zero. We have
\[ S^{-i-1}(x, \xi) = e_2 V_{-i}e_1, \quad S^{-i}(x, \xi)_l = e_1 V_{-i}e_1, \]
\[ S^{-i}(x, \xi)_r = e_2 V_{-i}e_2, \quad S^{-i+1}(x, \xi) = e_1 V_{-i}e_2 \]

**Lemma 42.** Let $p = 2$. The homology of $\mathfrak{F}^i$ is isomorphic to $V_{-i}$, for $i \leq 0$.

**Proof.** The top has a basis of elements of the form $1 \otimes a_1 \otimes \ldots \otimes a_{-i-1} \otimes 1$, where $a_1, \ldots, a_{-i-1}$ is a list of letters consisting of $d$ $x$s and $n-1-d$ $\xi$s. The basis element is independent of the order of the elements in the list. \qed
The product on $\Upsilon^-$ is given by the product on the algebra $S(x, \xi)$ whenever degrees and idempotents match up appropriately, and zero otherwise; for example the product of $S_i^{-1}$ and $S_i^{-2i}$ is zero, whilst the product $S_i^{-i} \otimes S_i^{-2i}$ is given by the natural product on $S(x, \xi)$.

The action of $\Upsilon^1 = (e_1 \otimes e_2, e_2 \otimes e_1)$ on $\Upsilon^i$ for $i \leq 1$ is given here by the maps

$$\Upsilon^1 \otimes \Upsilon^i \rightarrow \Upsilon^{i+1}, \quad \Upsilon^i \otimes \Upsilon^1 \rightarrow \Upsilon^{i+1}$$

which are the identity on a component $S^h(x, \xi) \subset \Upsilon^i$, $\Upsilon^{i+1}$ whenever idempotents match up appropriately, and zero otherwise.

**Truncating $\Upsilon$.** The reason for our intricate analysis of $\Upsilon$ is it enables our description of the algebras $\mathcal{O}_F \mathcal{O}_F^Q(F[z])$, and therefore by Proposition 25 the Weyl extension algebra of $GL_2$, since it is a limit of such algebras. It turns out that to describe the algebra structure on $\mathcal{O}_F \mathcal{O}_F^Q(F[z])$ we only need to know about multiplication on the part of $\Upsilon$ featuring $\mathbb{H}(\mathfrak{h}^i)$ for $i \leq 1$. Here we justify this fact.

Note that for $i \in \mathbb{Z}$,

\begin{equation}
\mathcal{O}^q_{\mathbb{H}{\Psi}(\mathfrak{g}, \mathfrak{g}^{-1})} \mathbb{H}(F[z, z^{-1}])^{i, k} = \bigoplus_{j_1, k_1 + \hat{k}_1 = k} \mathbb{H}(\mathfrak{h}^{j_1, k_1}) \otimes_F (\mathcal{O}^{q-1}_{\mathbb{H}{\Psi}(\mathfrak{g}, \mathfrak{g}^{-1})} \mathbb{H}(F[z, z^{-1}]))^{j_1, \hat{k}_1} \\
= \bigoplus_{j_1, k_1 + \hat{k}_1 = k} \mathbb{H}(\mathfrak{h}^{j_1, k_1}) \otimes_F \left( \bigoplus_{j_2, k_2 + \hat{k}_2 = k_1} \mathbb{H}(\mathfrak{h}^{j_2, k_2}) \otimes_F (\mathcal{O}^{q-2}_{\mathbb{H}{\Psi}(\mathfrak{g}, \mathfrak{g}^{-1})} \mathbb{H}(F[z, z^{-1}]))^{j_2, \hat{k}_2} \right) \\
= \cdots
\end{equation}

where, for $l < 0$ we interpret $\mathfrak{h}^l$ as $(\mathfrak{h}^{-1})^\otimes l$.

A typical direct summand of this looks like

$$\mathbb{H}(\mathfrak{h}^{j_1, k_1}) \otimes_F (\mathbb{H}(\mathfrak{h}^{j_2, k_2}) \otimes_F (\cdots \otimes_F (\mathbb{H}(\mathfrak{h}^{j_q, k_q})))^{j_q, k_q}.$$

**Lemma 43.** For $i \leq 1$ and any $k \in \mathbb{Z}$, no direct summand of $\mathcal{O}^q_{\mathbb{H}{\Psi}(\mathfrak{g}, \mathfrak{g}^{-1})} \mathbb{H}(F[z, z^{-1}])^{i, k}$ involves tensor factors $\mathbb{H}(\mathfrak{h}^{j, k})$ for $i > 1$.

**Proof.** The proof is by induction on $q$. The case $q = 1$ is trivial. Assume it is true for $\mathcal{O}^{q-1}_{\mathbb{H}{\Psi}(\mathfrak{g}, \mathfrak{g}^{-1})} \mathbb{H}(F[z, z^{-1}])^{i, k}$. The first step in (4) together with the observation that for $i \leq 1$, the graded piece $\mathbb{H}(\mathfrak{h}^{i, k})$ is zero for all $j > 1$ then implies the inductive step and the Lemma. \qed
**Polytopal basis for \( \Upsilon \leq 1 \).** We first define monomials in \( \Upsilon \). These form a collection of elements each of which is determined by its \( jk \)-degree, and upon multiplying on the left and right by idempotents \( e_s \) gives an element of a basis of \( \Upsilon \), or zero.

We define a monomial in \( \Upsilon^0 = \Psi \) with representative \( x^l \xi \), for some \( l \). We define a monomial in \( \Upsilon^i \), for \( i \leq -1 \), to be an element of \( \Upsilon^i = HH(\mathbf{R}^{\mathcal{E}}) \) with representative \( x^c \otimes (\xi \otimes \xi)^{\otimes f} \otimes (x^{p-1})^{\otimes i-1-2f} \otimes x^l \) or \( x^c \otimes (\xi \otimes \xi)^{\otimes f} \otimes (x^{p-1})^{\otimes i-1-2f} \otimes x^l \xi \) or \( x^c \xi w^2 \) or \( x^c \xi w^{-2} \) for \( i \) even, or \( \xi^{\otimes 1-i} \) for \( i \) odd. We say there is a unique monomial in \( \Upsilon^1 \), namely 1.

**Theorem 44.**

\[
\dim e_s \Upsilon^{jk} e_t \leq 1
\]

for all \( i \in \mathbb{Z}_{\leq 1} \), \( j, k \in \mathbb{Z} \), \( 1 \leq s, t \leq p \).

**Proof.** Suppose \( p \geq 3 \). \( \Upsilon^i \) is the homology of \( \mathbf{K}^i \), for \( i \in \mathbb{Z}_{\leq 0} \), which is the homology of

\[
M \langle (i + 1)p + 1 \rangle \langle (i + 1)(p - 1) \rangle \oplus M \langle (i + 3)p + 1 \rangle \langle (i + 3)(p - 1) \rangle \oplus \\
\quad \cdots \oplus M \langle (p - 1) \rangle \langle p + 1/2 \rangle \oplus \Psi \langle 1 \rangle^{[1]} \]

for \( i \) even, and the homology of

\[
M^\tau \langle (i + 1)p + 1 \rangle \langle (i + 1)(p - 1) \rangle \oplus M^\tau \langle (i + 3)p + 1 \rangle \langle (i - 1)(p - 1) \rangle \oplus \\
\quad \cdots \oplus M^\tau \langle 1 \rangle^{[1]} \langle i + 1/2 \rangle \rightarrow \Psi^\alpha \langle 1 \rangle^{[i - 1/2]}
\]

for \( i \) odd. In either case, the number of \( \xi \)s appearing in any monomial in \( \Upsilon \) determines which bimodule factor \( M, \Psi, \Psi^\alpha \) that monomial appears in. The number of \( \xi \)s appearing in a monomial can be computed from the \( jk \)-degree of that monomial, since \( \xi \in \Psi \) has \( jk \)-degree \( (1, 0) \) whilst \( x \) has \( jk \)-degree \( (-1, 1) \). Therefore the \( ijk \)-degree of a monomial determines which component \( \Upsilon^i \) and which bimodule factor \( M[t], \Psi[0], \Psi^\alpha \) of that component that monomial appears in. To establish that \( \dim e_s \Upsilon^{jk} e_t \leq 1 \), it is enough for us to observe that a monomial element of \( e_s Me_t \) or \( e_s \Psi e_t \) is determined by its \( jk \)-degree, which is the case since it is determined by the number of \( x \)s and \( \xi \)s appearing.

The case \( p = 2 \) is similar. \( \square \)

**Lemma 45.** The polytope \( \mathcal{P}_\Psi \) corresponding to \( \Psi \) is the set of elements \( (s, j, k, t) \in \mathbb{Z}^4 \) such that \( 1 \leq s \leq t \leq p, 0 \leq j - k \leq 1, \) and \( t - s = j + 2k \).

The polytope \( \mathcal{P}_0 \) corresponding to \( \overline{\Psi^\alpha} \) is the set of elements \( (s, j, k, t) \in \mathbb{Z}^4 \) such that \( t + s = p + 1 \), \( 1 \leq s \leq t \leq p \), minus the element \((p, 0, 0, 1)\), and \( j = k = 0 \).

The polytope \( \mathcal{P}_M \) corresponding to \( M \) is the set of elements \( (s, j, k, t) \in \mathbb{Z}^4 \) such that \( 1 \leq s, t \leq p, j + 2k + 2 = t - 1 - s + p, 0 \leq j + k + 2 \leq 1 \).

The polytope \( \overline{\mathcal{P}_M} \) corresponding to \( \overline{M} \) is the set of elements \( \mathcal{P}_M \) minus the element \((p, 0, -1, 1)\).

**Proof.** We pick out the number of \( \xi \)s in an element of \( M = L_i \otimes L_s \) with \( j + k \), and the number of \( x \)s with \( k \); the total degree is thus \( j + 2k \). For \( M \), we pick out the number of \( \xi \)s with \( j + k + 2 \), and the number of \( x \)s with \( k + 2 \); the total number of \( x \)s and \( \xi \)s is thus \( j + 2k + 4 \); the restrictions are that the number of \( \xi \)s is 0 or 1, the
number of $x$s and $\xi$s is at least 2, and the number of $x$s and $\xi$s is $2 + (t - 1) - (s - p)$; the element $(p, 0, -1, 1)$ corresponds to the element $x \otimes \xi \in M$. This gives us the description of $\mathcal{P}_M$.

To obtain our description of $\mathcal{P}_\Phi$, we similarly pick out the number of $\xi$s with $j + k$ and the total number of $x$s and $\xi$s with $j + 2k$.

The description of $\mathcal{P}_0$ comes since nonzero elements of $e_u \Psi^0 s e_t$ satisfy the constraint $t + s = p + 1$.

Here is a diagram of the polytope for $M$ in case $p = 3$ (we depict its structure as a left module):

$$
\begin{align*}
31_0^{-1} & \ 31_1^{-1} \ 32_1^{-1} \ 32_2^{-1} \ 33_2^{-1} \ 33_3^{-1} \\
21_0^{-1} & \ 21_1^{-1} \ 22_1^{-1} \ 22_2^{-1} \ 23_2^{-1} \ 23_3^{-1} \\
11_1^{-1} & \ 11_2^{-1} \ 12_2^{-1} \ 12_3^{-1} \ 13_3^{-1} \ 13_4^{-1}
\end{align*}
$$

In the diagram an element $(s, j, k, t)$ is written $st^{j_k}$. Here is a diagram of the polytope for $\Psi$ in case $p = 3$:

$$
\begin{align*}
33_0 & \ 33_1 \\
22_0 & \ 23_0 \\
11_0 & \ 12_1 \ 12_2 \ 13_1 \ 13_2
\end{align*}
$$

We introduce integers $a$ and $b$ indexing the powers of $u$ and $v$ appearing in a homogeneous element of $\Psi^{\leq 0}$.

By Theorem 41 we have a basis for $\Psi^{\leq 0}$ indexed by the subset

$$
\{ v = (s, j_0, k_0, a, b, t) \in \mathbb{Z}_0^6 \mid (s, j_0, k_0, t) \in \mathcal{P}_\Phi, a, b \geq 0, a = b \} \cup \\
\{ v = (s, j_0, k_0, a, b, t) \in \mathbb{Z}_0^6 \mid (s, j_0, k_0, t) \in \mathcal{P}_\Phi, a, b \geq 0, a = b - 1 \} \cup \\
\{ v = (s, j_0, k_0, a, b, t) \in \mathbb{Z}_0^6 \mid (s, j_0, k_0, t) \in \mathcal{P}_\Phi, a, b \geq 0, a = b + 1 \} \cup \\
\{ v = (s, j_0, k_0, a, b, t) \in \mathbb{Z}_0^6 \mid (s, j_0, k_0, t) \in \mathcal{P}_M, a, b \geq 0, a \geq b + 1 \}.
$$

The $ijk$-degree of such an element is given by the formulas

- $i = -a - b$;
- $j = j_0 + (a - 1)p + 1$ for $a \geq b + 1$, $j = 0$ for $a = b$, $j = 1$ for $a = b - 1$;
- $k = k_0 + (a - 1)(p - 1)$ for $a \geq b + 1$, $k = 0$ for $a \leq b$.

We define $\mathcal{P}_{\Psi^{\leq 0}}$ to be the corresponding set of elements $(s, i, j, k, a, b, t) \in \mathbb{Z}_7$. We define $\mathcal{P}_{\Psi^{\leq 1}}$ to be $\mathcal{P}_{\Psi^{\leq 0}} \cup \{ (s, -1, 1, -1, 0, 0, p + 1 - s) \in \mathbb{Z}_7 \}$.

Theorem 46 leads us to the following combinatorial description of $\Psi^{\leq 1}$:

**Theorem 46.** $\Psi^{\leq 1}$ has basis $\{ m_w \}_{w \in \mathcal{P}_{\Psi^{\leq 1}}}$ with product given by

$$
m_w m_{w'} = \begin{cases} 
(-1)^{a_j b_j + b_a + b_d} m_v & \text{if } v_1 = w_1, w_7 = w'_7, w'_7 = v_7, v_1 = w_1 + w'_1 \\
0 & \text{for } 2 \leq l \leq 5 \text{ and } v \in \mathcal{P}_{\Psi^{\leq 1}}. 
\end{cases}
$$
10. Appendix 1: Signs.

Super sign convention. Here we record some aspects of the super sign convention that are of relevance for us. A differential graded vector space is a \( \mathbb{Z} \)-graded vector space \( V = \bigoplus_k V^k \) with a graded endomorphism \( d \) of degree 1. We write \( |v| \) for the degree of a homogeneous element of \( V \). We assume \( d \) can act both on the left and the right of \( V \), with the convention \( d(v) = (-1)^{|v|} v d \). A differential graded algebra is a \( \mathbb{Z} \)-graded algebra \( A = \bigoplus_k A^k \) with a differential \( d \) such that
\[
d(ab) = d(a).b + (-1)^{|a|} a.d(b),
\]
or equivalently
\[
(ab)d = a.(b)d + (-1)^{|a|}(a)d.b.
\]
If \( A \) is a differential graded algebra then a differential graded left \( A \)-module is a graded left \( A \)-module \( M \) with differential \( d \) such that
\[
d(a.m) = d(a).m + (-1)^{|a|} a.d(m);
\]
a differential graded right \( A \)-module is a graded right \( A \)-module \( M \) with differential \( d \) such that
\[
d(m.a) = d(m).a + (-1)^{|m|} m.d(a).
\]
If \( A \) and \( B \) are dg algebras then a dg \( A-B \)-bimodule is a graded \( A-B \)-bimodule with a differential which is both a left dg \( A \)-module and a right dg \( B \)-module. If \( A_M \) is a left dg \( A \)-module, then \( \text{End}_A(M) \) is a differential graded algebra which acts on the right of \( M \), giving \( M \) the structure of an \( A-\text{End}_A(M) \)-bimodule, the differential on \( \text{End}_A(M) \) being given by \( m.(\phi)d = ((m)\phi)d - (-1)^{|\phi|}((m)d)\phi \). If \( M_B \) is a right dg \( A \)-module, then \( \text{End}_A(M) \) is a differential graded algebra which acts on the left of \( M \), giving \( M \) the structure of an \( \text{End}_B(M) \)-bimodule, the differential on \( \text{End}_B(M) \) being given by \( d(\phi).m = d(\phi.m) - (-1)^{|\phi|}\phi.d(m) \).

If \( A_M \) and \( B_NC \) are dg bimodules where \( A \), \( B \), and \( C \) are dg algebras, then \( M \otimes_B N \) is a dg \( A-C \)-bimodule with differential
\[
d(m \otimes n) = d(m) \otimes n + (-1)^{|m|} m \otimes d(n).
\]
If \( A_M \) and \( A_NC \) are dg bimodules where \( A \), \( B \), and \( C \) are dg algebras, then \( \text{Hom}_A(M,N) \) is a dg \( B-C \)-bimodule with differential
\[
d(\phi(m)) = d(\phi(m)) + (-1)^{|\phi|}\phi(d(m)).
\]

11. Appendix 2: Koszul duality.

Here we give an account of Koszul duality for Koszul algebras, synthesising the work of Beilinson, Ginzburg, and Soergel, and Keller [1], [4].

In what follows we will exceptionally denote homological degree by \( h \) rather than \( k \), because the homological \( h \)-grading on the Koszul algebras described here is different from the homological \( k \)-grading for the Koszul algebra \( \Psi \) used in the rest of the paper.

Vector space duals. Let \( A^0 \) be a direct product of finitely many fields, thought of as an algebra. We write \( M^* \) for the \( F \)-linear graded dual of a graded vector space over
$F$; the dual of a component in degree $j$ lies in degree $-j$; if $M$ is an $A$-$B$-bimodule, then $M^*$ is a $B$-$A$-bimodule. Given a right $A^0$-module $M$ with dual $M^*$, we have an isomorphism $(Me)^* \cong e(M^*)$ for each primitive idempotent $e$ in $A^0$; we write $\eta_M$ for the sum $\eta : A^0 \to M \otimes_{A^0} M^*$ of units $F \cong A^0 e \to M e \otimes_F eM^*$. If $M$ is an $A^0$-$A^0$-bimodule, then $\eta$ is a homomorphism of $A^0$-$A^0$-bimodules. We have a fixed isomorphism $A^0 \cong A^0$ which sends $1 \in F$ to its dual in $F^*$.

**Quadratic duals.** Let $A = T_{A^0}(A^1)/R$, with $R \subset A^1 \otimes_{A^0} A^1$ be a quadratic algebra whose degree zero part is $A^0$ and whose degree one part $A^1$ is finite dimensional. Let $A^1 = T_{A^0}(A^{1-1})/R^1$ be its quadratic dual, where the $A^0$-$A^0$ bimodules $A^1$ and $A^{1-1}$, and the short exact sequences of $A^0$-$A^0$-bimodules

$$0 \to R \to A^1 \otimes_{A^0} A^1 \to A^2 \to 0$$

$$0 \leftarrow A^{1-2} \leftarrow A^{1-1} \otimes_{A^0} A^{1-1} \leftarrow R^1 \leftarrow 0,$$

are duals of each other. The grading that is implicit here is the radical grading, or $r$-grading: we insist $A$ is generated in $r$-degrees 0 and 1, and $A^1$ is generated in $r$-degrees 0 and $-1$.

**Differential bimodules.** The composition

$$A^0 \xrightarrow{\eta A^1 \otimes_{A^0} A^1} (A^1 \otimes_{A^0} A^1) \otimes_{A^0} (A^{1-1} \otimes_{A^0} A^{1-1}) \xrightarrow{A^2 \otimes_{A^0} A^{2}}$$

is equal to zero, because the first map can be written $\sum_i (b_i \otimes b_i^*) + \sum_j (b_j^* \otimes b_j^*)$ where $\{b_i\}$ is a basis for $R$ with dual basis $\{b_i^*\}$ and $\{b_i^*\}$ is a dual basis for $R^1$ with dual basis $\{b_i^*\}$. Consequently the space $A \otimes_{A^0} A^1$ is a differential bimodule, with differential given by the composition map

$$A \otimes_{A^0} A^1 \xrightarrow{\sim} A \otimes_{A^0} A^0 \otimes_{A^0} A^1 \xrightarrow{1 \otimes_{A^0} 1} A \otimes_{A^0} A^1 \otimes_{A^0} A^{1-1} \otimes_{A^0} A^1 \xrightarrow{d} A \otimes_{A^0} A^1$$

We denote this differential bimodule $C$. This differential acts naturally on the inside of $C$ which is a little awkward notationally: we adopt the convention that this differential applied to $a \otimes \alpha$ is written $d a \otimes \alpha$.

Suppose that $A$ is an $rh$-graded algebra with an $A^0$-$A^0$-bimodule decomposition $A^1 = A^{10} \oplus A^{11}$; thus the $r$-degree 1 part $A^1$ of $A$ decomposes as a direct sum of a $h$-degree 0 part and a $h$-degree 1 part. We have $A^{1-1} = A^{1*}$, and write $A^{1-11} = A^{10*}$, $A^{1-10} = A^{11*}$. Thus $A^{1-1} = A^{1-10} \oplus A^{1-11}$, and $A^1$ are $h$-graded algebras, in such a way that the differential on $C$ has $h$-degree one. If we want to write down the corresponding map on the left or right we apply the super sign convention:

$$d(a \otimes \alpha) = (-1)^{|\alpha|} a \otimes d \alpha$$

$$d(a \otimes \alpha)a = (-1)^{|\alpha|} (a \otimes \alpha)$$

There are only two ways in which we can obtain a left and right differential this way: either $A$ is concentrated in $h$-degree 0, or $A^1$ is concentrated in $h$-degree 0. From now on we assume that one of these is the case. In this way we give $C$ the structure of a differential $h$-graded $A$-$A^1$-bimodule. We denote by $C^l = A^l \otimes_{A^0} A$
the corresponding differential bigraded $A^l$-$A$-bimodule. The left Koszul complex is the differential $rh$-bigraded $A$-$A^l$-bimodule

$$K_l = C \otimes A^l A^{l\ast} \cong A \otimes_{A^0} A^{l\ast}.$$  

**Adjunction.** Given a pair of $rh$-graded modules $M$ and $N$ we define $\text{Hom}(M, N)$ to be the sum of the spaces of $rh$-graded homomorphisms from $M$ to $N$ shifted in degree by $(r, h)$. We have

$$\text{Hom}_A(A \otimes A^0 A^{l\ast}, M) \cong \text{Hom}_{A^0}(A^{l\ast}, M) \cong A^l \otimes A^0 M \cong A^l \otimes_{A^0} A \otimes_A M,$$

by adjunction and the fact that $A^0$ is semi-simple. Consequently, there is an equivalence of functors between categories of differential $rh$-bigraded modules

$$\text{Hom}_{A^l}(K_l, -) \cong C^l \otimes_A - : \text{A-digr}_{rh} \to \text{A'-digr}_{rh},$$

and therefore an adjunction $(K_l \otimes A^l, C^l \otimes A)$; we have a homomorphism of differential $rh$-bigraded bimodules $\rho_l : K_l \otimes A^l C^l \to A$ corresponding to the counit of this adjunction given by the composition of natural maps

$$A \otimes A^0 A^{l\ast} \otimes A^0 A \to A \otimes A^0 A^{l\ast} \otimes A^0 A \to A,$$

we have a homomorphism of differential $rh$-bigraded bimodules $\phi_l : A^l \to C^l \otimes A K_l$ corresponding to the unit of this adjunction given by the composition of natural maps

$$A^l \to \text{Hom}_{A^l}(K_l, K_l) \cong \text{Hom}_{A^l}(A^l, C^l \otimes A K_l) \cong C^l \otimes A K_l.$$

Let us explain why the differentials on $C^l$ and $K_l$ match under the natural isomorphisms. Here is a diagram depicting the counit of the adjunction

$$\begin{array}{ccc}
A & \otimes A^0 & A^{l\ast} \otimes A^l \otimes A^0 A \\
F & & A \\
A & & \\
\end{array}$$

The map sends $a \otimes \alpha \otimes \alpha' \otimes a'$ to $a(\alpha, \alpha')b$. Under the differential on $K_l$ we obtain the map sending $a \otimes \alpha \otimes \alpha' \otimes a'$ to $\sum b(\alpha^* \alpha, \alpha')b$; under the differential on $C^l$ we obtain the map sending $a \otimes \alpha \otimes \alpha' \otimes a'$ to $\sum b(\alpha, \alpha^* \alpha')xb$; these maps are identical by the super sign convention since whenever $\langle \alpha, \alpha' \rangle$ is nonzero for homogeneous $\alpha$ and $\alpha'$, we have $|\alpha|_h + |\alpha'|_h = 0$, and $|x|_h |x^*|_h$ is always zero.

The algebra $A^0$ admits the structure of a differential bigraded $A$-$A^l$-bimodule: we have zero differential, and all elements of strictly positive or strictly negative degrees act as zero. There is a natural homomorphism of differential bigraded $A$-modules $\pi_l : K_l \to A^0$ given by $\pi_l = \rho_l \otimes_A 1_{A^0}$. There is a natural homomorphism of differential bigraded $A$-modules $\iota_l : A^0 \to K_l$ given by $1_{A^0} \otimes_A \phi_l$.

**Koszul algebras.** The algebra $A$ is said to be Koszul if $\pi_l$ is a quasi-isomorphism. This is equivalent to $\iota_l$ being a quasi-isomorphism, or $\rho_l$ being a quasi-isomorphism, or $\phi_l$ being a quasi-isomorphism. Since the Koszul complexes for $A$ and $A^l$ are duals of each other, $A$ is Koszul if and only if $A^l$ is Koszul.
We have a category $A$-dbigr$_r h$ whose objects are differential $r h$-bigraded modules. Localising the quasi-isomorphisms gives us a triangulated category $D(A$-dbigr$_r h$).

If $A$ is Koszul then we have adjoint equivalences of derived categories of differential bigraded modules

$$\text{Hom}_A(K_I,-) \quad D(A$-dbigr$_r h$) \quad D(A^I$-dbigr$_r h$) \quad .$$

We now restrict to the case in which $A$ is concentrated in homological degree 0.

We give $A$ the structure of a differential $r h$-bigraded algebra as follows: $A$ is concentrated entirely in $h$-degree 0, whilst $A^I$ is concentrated in positive degrees; the $h$-grading on $A^I$ is the negative of the $r$-grading.

Writing $C^{r,h} = A^{r+h} \otimes_{A^0} A^{1-h}$, we give $C$ the structure of a differential bigraded $A$-$A^I$-bimodule; the differential has $(r, h)$ degree $(0, 1)$.

The left Koszul complex is the differential bigraded $A$-$A^I$-bimodule $K_I = C \otimes_{A^I} A^{I*} \cong A \otimes_{A^0} A^*$, where the bigrading is given by

$$K_I^{r,h} = A^{r+h} \otimes_{A^0} A^{1-h} = A^{r+h} \otimes_{A^0} A^{I*} ;$$

we denote by $K^{I*}_I = C^I \otimes_A A^*$ the left Koszul complex for $A^I$; the differential has $(r, h)$ degree $(0, 1)$.

There is a natural homomorphism of differential bigraded $A$-modules $\pi_I : K_I \to A^0$ obtained by tensoring $1_C$ on the right with the dual of the embedding $A^0 \to A^I$ and on the left the homomorphism $A \to A^0$ which sends all elements of positive degree to zero. The algebra $A$ is Koszul if $\pi_I$ is a quasi-isomorphism. In that case, the structures described above collapse favourably:

**Theorem 47.** (Beilinson, Ginzburg and Soergel [1], Keller [4]) Suppose $A$ is Koszul. Then the map

$$A^I \to \text{Ext}^*_A(A^0, A^0)$$

induced by the action of $A^I$ on $K_I$ is an isomorphism; if $A^I$ is finite dimensional, then we have adjoint equivalences of bounded derived categories

$$\text{Hom}_A(K_I,-) \quad D^b(A$-gr$_r$) \quad D^b(A^I$-gr$_r$) \quad ,$$

where $A$-gr$_r$ is the category of $r$-graded modules for $A$. We have adjoint equivalences of derived dg categories

$$\text{Hom}_A(K_I,-) \quad D_{dg}(A) \quad D_{dg}(A^I) \quad ,$$

where $D_{dg}(A)$ is the derived category of differential $h$-graded modules for $A$.

**Remark 48** Obviously

$$CC^I CC^I ... CC^I \cong A \otimes_{A^0} A^I \otimes_{A^0} A \otimes_{A^0} ... \otimes_{A^0} A ,$$
where there is one more tensor factor on the right hand side of the isomorphisms than on the left hand side. If $A$ is Koszul, then we have quasi-isomorphisms

$$K_l K_l K_l \ldots K_l K_l \leftarrow A^* \otimes A^0 A^* \otimes A^0 \ldots \otimes A^0 A^*,$$

where there is one fewer tensor factor on the right hand side of the quasi-isomorphism than on the left hand side (here we write $M N$ for a tensor product $M \otimes A N$). The complex $CC^! CC^! \ldots CC^!$ is quasi-isomorphic to the dual of $K_l K_l K_l \ldots K_l K_l$, where there are two more tensor factors in the second term; the complex $CC^! CC^! \ldots CC^!$ is quasi-isomorphic to the dual of $K_l K_l K_l \ldots K_l K_l K_l$, where there are two more tensor factors in the second term.

**Remark 49** There is a right Koszul complex $K_r = A^* \otimes_A C$ which is obtained by tensoring $C$ on the left with $A^*$. We have an adjunction $(C^i \otimes_A -, K_r \otimes_A -)$. We have a homomorphism of dg bimodules $\phi_r : C^i \otimes_A K_r \to A^i$ corresponding to the counit of this adjunction. We have a dg homomorphism $\pi_r : A^0 \to K_r$ obtained by taking the dual of $\pi_l^i$, which is a quasi-isomorphism if and only if $A$ is Koszul. We have an an analogue of Theorem 47 for the right Koszul complex.

**References**

[1] A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. 9 (1996), no. 2, 473-527.

[2] E. Cline, B. Parshall and L. Scott, *Finite-dimensional algebras and highest weight categories*. J. Reine Angew. Math. 391 (1988), 85-99.

[3] J. A. Green, *Polynomial representations of $GL_n$*. Lecture Notes in Mathematics, 830. Springer, Berlin, 1980.

[4] B. Keller, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[5] M. Khovanov, P. Seidel, *Quivers, Floer cohomology, and braid group actions* J. Amer. Math. Soc. 15 (2002) 203-271.

[6] D. Madsen, *On a common generalization of Koszul duality and tilting equivalence*, preprint, arXiv:1007.3282v1, to appear in Adv. Math.

[7] V. Miemietz, W. Turner, *Rational representations of $GL_2$*, Glasgow J. Math. 53 (2011), no.2, 257-275.

[8] V. Miemietz, W. Turner, *Homotopy, Homology and $GL_2$*, Proc. London Math. Soc. (3) 100 (2010), no.2, 585-606.

[9] V. Miemietz, W. Turner, *The Yoneda extension algebra of $GL_2(\mathbb{F}_p)$*, preprint, arXiv:1106.5411v1.

[10] A. Parker, *Higher extensions between modules for $SL_2$*, Adv. Math. 209 (2007), no. 1, 381–405.

[11] J. Rickard, *Derived equivalences as derived functors*. J. London Math. Soc. (2) 43 (1991), no. 1, 37-48.

[12] R. Rouquier, *Derived equivalences and finite dimensional algebras*, Proceedings of the International Congress of Mathematicians (Madrid, 2006 ), vol II, pp. 191-221, EMS Publishing House, 2006.

**Vanessa Miemietz, Will Turner**

School of Mathematics, University of East Anglia, Norwich, NR4 7TJ, UK, v.miemietz@uea.ac.uk

Department of Mathematics, University of Aberdeen, Fraser Noble Building, King’s College, Aberdeen AB24 3UE, UK, w.turner@abdn.ac.uk.